Cayley Mathematical Papers Boxed Set

Introductory Note by
John Clark

Introduction to the Principles of Grammar

During the evolution of a mind, it is not, nor can be, rational, i.e. cannot function by, or produce, functional grammar systems…

One of the earliest sources which point out the fact humanity will be functionally illiterate is today known, cumulatively, as the Judeo-Christian Scripture. This is possibly the only book in our history that repeatedly tells the reader humanity will not even be able to read that book until a certain time in human history, i.e., a stage in human evolution. Language does separate mankind from the other animals, however, mankind is not yet functionally literate and cannot take advantage of that distinction to the point he will survive.

Proving this is rather simple, unless one is hopelessly illiterate. Ask your computer: *How many parts of speech are possible*? Every moment that a computer is functioning, it produces all the things it can do by simple binary recursion. The Bible put this fact into a metaphor about life:—

Adam and Eve are a Conjugate Binary Pair that, *by Complete Induction and Deduction*, *produce the human race*.

As written, if one cannot read the book, then literacy is the key to opening it, just like any other book. The Key to the House of David, a simple puzzle, is simply literacy, something most of mankind believe that they are capable of, but by example prove to be the opposite.

If the metaphor is too esoteric then one has the *Dialogs* of Plato where Plato called binary recursion *Dialectic*, speech as a binary function. Or one can study Euclid's *Elements*. Or one can refer to the definition of any thing as a relative constrained by correlatives, which again, Plato called nouns and verbs. One can even examine their own body, for any particular sense can abstract either the form, limit, of a thing, or it can abstract the material difference which is again a simple binary distinction.

Binary recursion, when applied to the construction of grammar systems, produce methods of naming, or one can say counting, in order to manage memory. Binary recursion, used for the construction of grammar systems, produce four, and only four, categories of grammar, or four distinct methods of binary recursion: Common Grammar, Arithmetic, Algebra and Geometry. The functionally illiterate cannot comprehend or take advantage of this, of being literate.

These works give a study some material by which to study illiteracy and literacy alike.

Mankind actually believes that one grammar can contradict another, a stance championed by the Einsteins of the world, however, binary recursion is not different from itself, and binary recursion can only produce a binary result. In the words of a famous prophet, “Let your yea’s be yea’s and your nay’s be nay’s.” Which, only the functionally literate can manage.

Your computer is a machine that contradicts you every day, but long before that, your thoughts and actions, have contradicted simple reason.

Language is Universal and Intelligible, i.e. it requires intelligence to grasp it. It cannot be taught, spoken, written, or seen. While Grammar is Particular and Perceptible, which, by the simple binary recursion exampled by Language, produces exactly four methods of binary recursion for human survival: Common Grammar, Arithmetic, Algebra and Geometry. These four are also called Standards of Mental Behavior that produce Standards of Social Behavior. Psychology is commensurate with the principles of Language functionally resident in the mind of man as simple grammar systems.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.
Bombay: E. SEYMOUR HALE.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

SUPPLEMENTARY VOLUME,

CONTAINING TITLES OF PAPERS AND INDEX.

CAMBRIDGE:

AT THE UNIVERSITY PRESS.

1898.

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY J. AND C. F. CLAY,

AT THE UNIVERSITY PRESS.

PREFATORY NOTE.

THE present volume is supplementary to the series of thirteen volumes which contain the collected mathematical papers of the late Professor Cayley.

The first part is a list of the titles of the papers, extracted from the tables of contents of the respective volumes and arranged in the order in which the papers occur.

The second part is an index of subjects and authors. It has been made by my friend, Mr. F. Howard Collins, who most kindly volunteered to do this laborious work; my expectation is that the index will be a useful guide to the papers.

It is now ten years since the printing of the series of volumes began. For about the first seven of the years, Professor Cayley himself acted as Editor; since his death, the duty has fallen to me. My steady wish has been that unnecessary delay in the publication of the volumes should be avoided; that the wish has been realised, is largely due to the staff of the University Press. Everything that could be done in the way of simplifying my task and assisting its progress has been done with zealous good-will and cordial cooperation. To each and to all of them my thanks are given for the help which has enabled me to fulfil the duty I undertook at the request of the Syndics.

A. R. FORSYTH.

31 *January*, 1898.

COMPLETE LIST

OF

TITLES OF THE PAPERS.

[An Asterisk denotes that the paper is not printed in full.]

VOLUME I.

PAGE

1. *On a Theorem in the Geometry of Position* 1
   Camb. Math. Journ. t. II. (1841), pp. 267—271 (1841)

2. *On the Properties of a certain Symbolical Expression* . . 5
   Camb. Math. Journ. t. III. (1843), pp. 62—71 (1842)

3. *On certain Definite Integrals* 13
   Camb. Math. Journ. t. III. (1843), pp. 138—144 (1842)

4. *On certain Expansions, in series of Multiple Sines and Cosines* 19
   Camb. Math. Journ. t. III. (1843), pp. 162—167 (1842)

5. *On the Intersection of Curves* 25
   Camb. Math. Journ. t. III. (1843), pp. 211—213 (1843)

6. *On the Motion of Rotation of a Solid Body* 28
   Camb. Math. Journ. t. III. (1843), pp. 224—232 (1843)

7. *On a class of Differential Equations, and on the Lines of Curvature of an Ellipsoid* 36
   Camb. Math. Journ. t. III. (1843), pp. 264—267 (1843)

8. *On Lagrange's Theorem* 40
   Camb. Math. Journ. t. III. (1843), pp. 283—286 (1843)

9. *Demonstration of Pascal's Theorem* 43
   Camb. Math. Journ. t. IV. (1845), pp. 18—20 (1843)

10. *On the Theory of Algebraical Curves* 46
    Camb. Math. Journ. t. IV. (1845), pp. 102—112 (1844)

11. *Chapters in the Analytical Geometry of* (n) *Dimensions* . . 55
    Camb. Math. Journ. t. IV. (1845), pp. 119—127 (1844)

12. *On the Theory of Determinants* 63
    Camb. Phil. Trans. t. VIII. (1849), pp. 1—16 (1843)

13. *On the Theory of Linear Transformations* 80
    Camb. Math. Journ. t. IV. (1845), pp. 193—209

14. *On Linear Transformations* 95
    Camb. and Dubl. Math. Journ. t. I. (1846), pp. 104—122

PAGE

15. *Note sur deux Formules données par MM. Eisenstein et Hesse* 113
Crelle, t. XXIX. (1845), pp. 54—57

*16. *Mémoire sur les Hyperdéterminants* 117
Crelle, t. XXX. (1846), pp. 1—37

*17. *Note on Mr Bronwin's paper on Elliptic Integrals* . . . 118
Camb. Math. Journ. t. III. (1843), pp. 197, 198

*18. *Remarks on the Rev. B. Bronwin's paper on Jacobi's Theory of Elliptic Functions* 119
Phil. Mag. t. XXII. (1843), pp. 358—368

19. *Investigation of the Transformation of certain Elliptic Functions* 120
Phil. Mag. t. XXV. (1844), pp. 352—354

20. *On certain results relating to Quaternions*. 123
Phil. Mag. t. XXVI. (1845), pp. 141—145

*21. *On Jacobi's Elliptic Functions, in reply to the Rev. B. Bronwin: and on Quaternions* 127
Phil. Mag. t. XXVI. (1845), pp. 208—211

22. *On Algebraical Couples* 128
Phil. Mag. t. XXVII. (1845), pp. 38—40

23. *On the Transformation of Elliptic Functions* 132
Phil. Mag. t. XXVII. (1845), pp. 424—427

24. *On the Inverse Elliptic Functions* 136
Camb. Math. Journ. t. IV. (1845), pp. 257—277

25. *Mémoire sur les Fonctions doublement périodiques* . . . 156
Liouville, t. X. (1845), pp. 385—420

26. *Mémoire sur les Courbes du Troisième Ordre* 183
Liouville, t. IX. (1844), pp. 285—293

27. *Nouvelles remarques sur les Courbes du Troisième Ordre*. . 190
Liouville, t. X. (1845), pp. 102—109

28. *Sur quelques Intégrales Multiples* 195
Liouville, t. X. (1845), pp. 158—168

29. *Addition à la Note sur quelques Intégrales Multiples* . . 204
Liouville, t. X. (1845), pp. 242—244

30. *Mémoire sur les Courbes à double Courbure et les Surfaces développables* 207
Liouville, t. X. (1845), pp. 245—250

31. *Démonstration d'un Théorème de M. Chasles* 212
Liouville, t. X. (1845), pp. 383, 384

PAGE

32. *On some Analytical Formulæ and their application to the Theory of Spherical Coordinates* 213
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 22—33

33. *On the Reduction of $du \div \sqrt{U}$, when U is a Function of the Fourth Order* 224
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 70—73

34. *Note on the Maxima and Minima of Functions of Three Variables* 228
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 74, 75

35. *On Homogeneous Functions of the Third Order with Three Variables* 230
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 97—104

36. *On the Geometrical Representation of the Motion of a Solid Body* 234
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 164—167

37. *On the Rotation of a Solid Body round a Fixed Point* . . 237
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 167—173 and 264—274

38. *Note on a Geometrical Theorem contained in a Paper by Sir W. Thomson* 253
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 207, 208

39. *On the Diametral Planes of a Surface of the Second Order* . 255
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 274—278

40. *On the Theory of Involution in Geometry* 259
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 52—61

41. *On certain Formulæ for Differentiation, with applications to the evaluation of Definite Integrals* 267
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 122—128

42. *On the Caustic by Reflection at a Circle* 273
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 128—130

43. *On the Differential Equations which occur in Dynamical Problems* 276
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 210—219

44. *On a Multiple Integral connected with the Theory of Attractions* 285
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 219—223

45. *On the Theory of Elliptic Functions* 290
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 256—266

46. *Note on a System of Imaginaries* 301
Phil. Mag. t. xxx. (1847), pp. 257, 258

PAGE

47. *Sur la Surface des Ondes* 302
Liouville, t. XI. (1846), pp. 291—296

48. *Note sur les Fonctions de M. Sturm* 306
Liouville, t. XI. (1846), pp. 297—299

49. *Sur quelques Formules du Calcul Intégral* 309
Liouville, t. XII. (1847), pp. 231—240

50. *Sur quelques Théorèmes de la Géométrie de Position*. . . . 317
Crelle, t. XXXI. (1846), pp. 213—227

51. *Problème de Géométrie Analytique* 329
Crelle, t. XXXI. (1846), pp. 227—230

52. *Sur quelques Propriétés des Déterminants Gauches* 332
Crelle, t. XXXII. (1846), pp. 119—123

53. *Recherches sur l'Élimination, et sur la Théorie des Courbes* . 337
Crelle, t. XXXIV. (1847), pp. 30—45

54. *Note sur les Hyperdéterminants*. 352
Crelle, t. XXXIV. (1847), pp. 148—152

55. *Sur quelques Théorèmes de la Géométrie de Position*. . . . 356
Crelle, t. XXXIV. (1847), pp. 270—275

56. *Demonstration of a Geometrical Theorem of Jacobi's* . . . 362
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 48, 49

57. *On the Theory of Elliptic Functions* 364
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 50, 51

58. *Notes on the Abelian Integrals—Jacobi's System of Differential Equations* 366
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 51—54

59. *On the Theory of Elimination* 370
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 116—120

60. *On the Expansion of Integral Functions in a series of Laplace's Coefficients* 375
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 120, 121

61. *On Geometrical Reciprocity* 377
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 173—179

62. *On an Integral Transformation* 383
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 286, 287

63. *Démonstration d'un Théorème de M. Boole concernant des Intégrales Multiples* 384
Liouville, t. XIII. (1848), pp. 245—248

PAGE

64. *Sur la généralisation d'un Théorème de M. Jellett qui se rapporte aux Attractions* 388
Liouville, t. XIII. (1848), pp. 264—268

65. *Nouvelles Recherches sur les Fonctions de M. Sturm* . . 392
Liouville, t. XIII. (1848), pp. 269—274

66. *Sur les Fonctions de Laplace* 397
Liouville, t. XIII. (1848), pp. 275—280

67. *Note sur les Fonctions Elliptiques* 402
Crelle, t. XXXVII. (1848), pp. 58—60

68. *On the application of Quaternions to the Theory of Rotation* . 405
Phil. Mag. t. XXXIII. (1848), pp. 196—200

69. *Sur les Déterminants Gauches* 410
Crelle, t. XXXVIII. (1848), pp. 93—96

70. *Sur quelques Théorèmes de la Géométrie de Position* . . 414
Crelle, t. XXXVIII. (1848), pp. 97—104

71. *Note sur les Fonctions du Second Ordre* 421
Crelle, t. XXXVIII. (1848), pp. 105, 106

72. *Note on the Theory of Permutations* 423
Phil. Mag. t. XXXIV. (1849), pp. 527—529

73. *Abstract of a Memoir by Dr Hesse on the construction of the Surface of the Second Order which passes through nine given points* 425
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 44—46

74. *On the Simultaneous Transformation of Two Homogeneous Functions of the Second Order* 428
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 47—50

75. *On the Attraction of an Ellipsoid* 432
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 50—65

76. *On the Triple Tangent Planes of Surfaces of the Third Order* 445
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 118—132

77. *On the order of certain Systems of Algebraical Equations* . 457
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 132—137

78. *Note on the Motion of Rotation of a Solid of Revolution* . . 462
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 268—270

79. *On a System of Equations connected with Malfatti's Problem, and on another Algebraical System* 465
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 270—275

PAGE

80. *Sur quelques Transmutations des Lignes Courbes* 471
Liouville, t. XIV. (1849), pp. 40—46

81. *Addition au Mémoire sur quelques Transmutations des Lignes Courbes* 476
Liouville, t. XV. (1850), pp. 351—356

82. *On the Triadic Arrangements of Seven and Fifteen Things* . 481
Phil. Mag. t. XXXVII. (1850), pp. 50—53

*83. *On Curves of Double Curvature and Developable Surfaces* . 485
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 18—22

84. *On the Developable Surfaces which arise from two Surfaces of the Second Order* 486
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 46—57

85. *Note on a Family of Curves of the Fourth Order* 496
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 148—152

86. *On the Developable derived from an Equation of the Fifth Order* 500
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 152—159

*87. *Notes on Elliptic Functions (from Jacobi)* 507
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 201—204

88. *On the Transformation of an Elliptic Integral* 508
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 204—206

89. *On the Attraction of Ellipsoids (Jacobi's Method)* 511
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 217—226

90. *Note sur quelques Formules relatives aux Coniques* 519
Crelle, t. XXXIX. (1850), pp. 1—3

91. *Sur le Problème des Contacts* 522
Crelle, t. XXXIX. (1850), pp. 4—13

92. *Note sur un Système de certaines Formules* 532
Crelle, t. XXXIX. (1850), pp. 14, 15

93. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 534
Crelle, t. XXXIX. (1850), pp. 16—22

94. *Note sur l'Addition des Fonctions Elliptiques* 540
Crelle, t. XLI. (1851), pp. 57—65

95. *Note sur quelques Théorèmes de la Géométrie de Position* . . 550
Crelle, t. XLI. (1851), pp. 66—72

96. *Mémoire sur les Coniques inscrites dans une même Surface du Second Ordre* 557
Crelle, t. XLI. (1851), pp. 73—86

PAGE

97. *Note sur la Solution de l'Équation* $x^{257}-1=0$ 564
Crelle, t. XLI. (1851), pp. 81—83

*98. *Note relative à la sixième section du Mémoire sur quelques Théorèmes de la Géométrie de Position* 567
Crelle, t. XLI. (1851), p. 84

99. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 568
Crelle, t. XLI. (1851), pp. 85—92

100. *Note sur la Théorie des Hyperdéterminants* 577
Crelle, t. XLII. (1851), pp. 368—371

Notes and References to papers in Volume I. 581

Volumes II, III, and IV of the *Cambridge Mathematical Journal* have on the title pages the dates 1841, 1843, 1845 respectively, and volume VIII of the *Cambridge Philosophical Transactions* has the date 1849. As each of these volumes extends over more than a single year, I have added the year of publication for the papers 1, 2, ..., 12. In all other cases, the year of publication is shown by the date on the title page of the volume.

VOLUME II.

PAGE

101. *Notes on Lagrange's Theorem* 1
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 37—45

102. *On a Double Infinite Series*. 8
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 45—47

103. *On Certain Definite Integrals* 11
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 136—140

104. *On the Theory of Permutants* 16
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 40—51

105. *Correction to the Postscript to the Paper on Permutants* . . 27
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 97, 98

106. *On the Singularities of Surfaces* 28
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 166—171

107. *On the Theory of Skew Surfaces*. 33
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 171—173

108. *On certain Multiple Integrals connected with the Theory of Attractions* 35
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 174—178

109. *On the Rationalisation of certain Algebraical Equations* . . 40
Camb. and Dubl. Math. Journ. t. VIII. (1853), pp. 97—101

110. *Note on the Transformation of a Trigonometrical Expression* . 45
Camb. and Dubl. Math. Journ. t. IX. (1854), pp. 61, 62

111. *On a Theorem of M. Lejeune-Dirichlet's* 47
Camb. and Dubl. Math. Journ. t. IX. (1854), pp. 163—165

112. *Demonstration of a Theorem relating to the Products of Sums of Squares* 49
Phil. Mag. t. IV. (1852), pp. 515—519

113. *On the Geometrical Representation of the Integral*
$$\int dx \div \sqrt{(x+a)(x+b)(x+c)} \quad . \quad . \quad . \quad . \quad 53$$
Phil. Mag. t. V. (1853), pp. 281—284

PAGE

114. *Analytical Researches connected with Steiner's Extension of Malfatti's Problem* 57
Phil. Trans. t. CXLII. (for 1852), pp. 253—278

115. *Note on the Porism of the In-and-circumscribed Polygon* . . 87
Phil. Mag. t. VI. (1853), pp. 99—102

116. *Correction of two Theorems relating to the In-and-circumscribed Polygon* 91
Phil. Mag. t. VI. (1853), pp. 376, 377

117. *Note on the Integral* $\int dx \div \sqrt{(m-x)(x+a)(x+b)(x+c)}$ . . 93
Phil. Mag. t. VI. (1853), pp. 103—105

118. *On the Harmonic Relation of two Lines or two Points* . . 96
Phil. Mag. t. VI. (1853), pp. 105—107

119. *On a Theorem for the Development of a Factorial* . . . 98
Phil. Mag. t. VI. (1853), pp. 182—185

120. *Note on a Generalisation of the Binomial Theorem* . . . 102
Phil. Mag. t. VI. (1853), p. 185

121. *Note on a Question in the Theory of Probabilities* . . . 103
Phil. Mag. t. VI. (1853), p. 259

122. *On the Homographic Transformation of a Surface of the Second Order into itself* 105
Phil. Mag. t. VI. (1853), pp. 326—333

123. *On the Geometrical Representation of an Abelian Integral* . 113
Phil. Mag. t. VI. (1853), pp. 414—418

124. *On a Property of the Caustic by Refraction of the Circle* . 118
Phil. Mag. t. VI. (1853), pp. 427—431

125. *On the Theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$ 123
Phil. Mag. t. VII. (1854), pp. 40—47

126. *On the Theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$. *Second Part* 131
Phil. Mag. t. VII. (1854), pp. 408, 409

127. *On the Homographic Transformation of a Surface of the Second Order into itself* 133
Phil. Mag. t. VII. (1854), pp. 208—212: continuation of 122

128. *Developments on the Porism of the In-and-circumscribed Polygon* 138
Phil. Mag. t. VII. (1854), pp. 339—345

PAGE

129. *On the Porism of the In-and-circumscribed Triangle, and on an irrational Transformation of two Ternary Quadratic Forms each into itself* 145
Phil. Mag. t. IX. (1855), pp. 513—517

130. *Deuxième Mémoire sur les Fonctions doublement Périodiques* . 150
Liouville, t. XIX. (1854), pp. 193—208: sequel to 25

131. *Nouvelles Recherches sur les Covariants* 164
Crelle, t. XLVII. (1854), pp. 109—125

132. *Réponse à une Question proposée par M. Steiner* . . . 179
Crelle, t. L. (1855), pp. 277, 278

133. *Sur un Théorème de M. Schläfli* 181
Crelle, t. L. (1855), pp. 278—282

134. *Remarques sur la Notation des Fonctions Algébriques* . . 185
Crelle, t. L. (1855), pp. 282—285

135. *Note sur les Covariants d'une Fonction Quadratique, Cubique, ou Biquadratique à deux Indéterminées* 189
Crelle, t. L. (1855), pp. 285—287

136. *Sur la Transformation d'une Fonction Quadratique en elle-même par des Substitutions linéaires* 192
Crelle, t. L. (1855), pp. 288, 289

137. *Recherches Ultérieures sur les Déterminants gauches* . . . 202
Crelle, t. L. (1855), pp. 299—313: continuation of 52 and 69.

138. *Recherches sur les Matrices dont les termes sont des fonctions linéaires d'une seule Indéterminée* 216
Crelle, t. L. (1855), pp. 313—317

139. *An Introductory Memoir on Quantics* 221
Phil. Trans. t. CXLIV. (for 1854), pp. 244—258

140. *Researches on the Partition of Numbers* 235
Phil. Trans. t. CXLV. (for 1855), pp. 127—140

141. *A Second Memoir on Quantics* 250
Phil. Trans. t. CXLVI. (for 1856), pp. 101—126

142. *Numerical Tables Supplementary to Second Memoir on Quantics* 276
Now first published (1889)

143. *Tables of the Covariants* M *to* W *of the Binary Quintic: from the Second, Third, Fifth, Eighth, Ninth and Tenth Memoirs on Quantics* 282
Arranged in the present form (1889)

PAGE

144. *A Third Memoir on Quantics* 310
Phil. Trans. t. CXLVI. (for 1856), pp. 627—647

145. *A Memoir on Caustics* 336
Phil. Trans. t. CXLVII. (for 1857), pp. 273—312

146. *A Memoir on Curves of the Third Order* 381
Phil. Trans. t. CXLVII. (for 1857), pp. 415—446

147. *A Memoir on the Symmetric Functions of the Roots of an Equation* 417
Phil. Trans. t. CXLVII. (for 1857), pp. 489—496

148. *A Memoir on the Resultant of a System of two Equations* . 440
Phil. Trans. t. CXLVII. (for 1857), pp. 703—715

149. *On the Symmetric Functions of the Roots of certain Systems of two Equations* 454
Phil. Trans. t. CXLVII. (for 1857), pp. 717—726

150. *A Memoir on the Conditions for the Existence of given Systems of Equalities among the Roots of an Equation* . . . 465
Phil. Trans. t. CXLVII. (for 1857), pp. 727—731

151. *Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees* 471
Phil. Trans. t. CXLVII. (for 1857), pp. 733—736

152. *A Memoir on the Theory of Matrices* 475
Phil. Trans. t. CXLVIII. (for 1858), pp. 17—37

153. *A Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function* 497
Phil. Trans. t. CXLVIII. (for 1858), pp. 39—46

154. *Supplementary Researches on the Partition of Numbers* . . 506
Phil. Trans. t. CXLVIII. (for 1858), pp. 47—52

155. *A Fourth Memoir on Quantics* 513
Phil. Trans. t. CXLVIII. (for 1858), pp. 415—427

156. *A Fifth Memoir on Quantics* 527
Phil. Trans. t. CXLVIII. (for 1858), pp. 429—460

157. *On the Tangential of a Cubic* 558
Phil. Trans. t. CXLVIII. (for 1858), pp. 461—463

158. *A Sixth Memoir on Quantics* 561
Phil. Trans. t. CXLIX. (for 1859), pp. 61—90

Notes and References to papers in Volume II. 593

VOLUME III.

PAGE

159. *On some Integral Transformations* 1
Quart. Math. Journ. t. I. (1857), pp. 4—6

160. *On a Theorem relating to Reciprocal Triangles* . . . 5
Quart. Math. Journ. t. I. (1857), pp. 7—10

161. *A problem in Permutations* 8
Quart. Math. Journ. t. I. (1857), p. 79

162. *Two letters on Cubic Forms* 9
Quart. Math. Journ. t. I. (1857), pp. 85—87 and 90, 91

163. *On Hansen's Lunar Theory* 13
Quart. Math. Journ. t. I. (1857), pp. 112—125

164. *On Gauss' Theory for the Attraction of Ellipsoids* . . . 25
Quart. Math. Journ. t. I. (1857), pp. 162—166

165. *On some Geometrical Theorems relating to a triangle circumscribed about a Conic* 29
Quart. Math. Journ. t. I. (1857), pp. 169—175

166. *Note on the Homology of Sets* 35
Quart. Math. Journ. t. I. (1857), p. 178

167. *Apropos of Partitions* 36
Quart. Math. Journ. t. I. (1857), pp. 183, 184

*168. *A demonstration of the Fundamental Property of Geodesic Lines* 38
Quart. Math. Journ. t. I. (1857), pp. 185, 186

169. *Eisenstein's Geometrical Proof of the Fundamental Theorem for Quadratic Residues* (*Translated from the Original Memoir,* Crelle, t. XXVIII. (1844), *with an addition by A. Cayley*) . 39
Quart. Math. Journ. t. I. (1857), pp. 186—191

170. *On Schellbach's Solution of Malfatti's Problem* 44
Quart. Math. Journ. t. I. (1857), pp. 222—226

PAGE

171. *Note on Mr Salmon's Equation of the Orthotomic Circle* . 48
Quart. Math. Journ. t. I. (1857), pp. 242—244

172. *Note on the Logic of Characteristics* 51
Quart. Math. Journ. t. I. (1857), pp. 257—259

173. *On Laplace's Method for the Attraction of Ellipsoids* . . 53
Quart. Math. Journ. t. I. (1857), pp. 285—300

*174. *On the Oval of Descartes* 66
Quart. Math. Journ. t. I. (1857), pp. 320—328

175. *On the Porism of the In-and-circumscribed Triangle* . . 67
Quart. Math. Journ. t. I. (1857), pp. 344—354

176. *Note on Jacobi's Canonical Formulæ for Disturbed Motion in an Elliptic Orbit* 76
Quart. Math. Journ. t. I. (1857), pp. 355, 356

177. *Solution of a Mechanical Problem* 78
Quart. Math. Journ. t. I. (1857), pp. 405, 406

178. *On the à posteriori Demonstration of the Porism of the In-and-circumscribed Triangle* 80
Quart. Math. Journ. t. II. (1858), pp. 31—38

179. *On certain Forms of the Equation of a Conic* . . . 86
Quart. Math. Journ. t. II. (1858), pp. 44—48

180. *Note on the Reduction of an Elliptic Orbit to a fixed plane* . 91
Quart. Math. Journ. t. II. (1858), pp. 49—54

181. *On Sir W. R. Hamilton's Method for the Problem of three or more Bodies* 97
Quart. Math. Journ. t. II. (1858), pp. 66—73

182. *On Lagrange's Solution of the Problem of two fixed Centres* . 104
Quart. Math. Journ. t. II. (1858), pp. 76—83

183. *Note on Certain Systems of Circles* 111
Quart. Math. Journ. t. II. (1858), pp. 83—88

184. *A Theorem relating to Surfaces of the Second Order* . . 115
Quart. Math. Journ. t. II. (1858), pp. 140—142

185. *Note on the 'Circular Relation' of Prof. Möbius* . . . 118
Quart. Math. Journ. t. II. (1858), p. 162

186. *On the determination of the value of a certain Determinant* . 120
Quart. Math. Journ. t. II. (1858), pp. 163—166

187. *On the Sums of Certain Series arising from the Equation* $x = u + tfx$ 124
Quart. Math. Journ. t. II. (1858), pp. 167—171

PAGE

188. *On the Simultaneous Transformation of two Homogeneous Functions of the Second Order* 129
Quart. Math. Journ. t. II. (1858), pp. 192—195

189. *Note on a formula in finite Differences* 132
Quart. Math. Journ. t. II. (1858), pp. 198—201

190. *On the System of Conics which pass through the same four points* 136
Quart. Math. Journ. t. II. (1858), pp. 206, 207

191. *Note on the Expansion of the true Anomaly* 139
Quart. Math. Journ. t. II. (1858), pp. 229—232

192. *On the Area of the Conic Section represented by the General Trilinear Equation of the Second Degree* 143
Quart. Math. Journ. t. II. (1858), pp. 248—253

193. *On Rodrigues' Method for the Attraction of Ellipsoids* . . 149
Quart. Math. Journ. t. II. (1858), pp. 333—337

194. *Note on the Theory of Attraction* 154
Quart. Math. Journ. t. II. (1858), pp. 338, 339

195. *Report on the Recent Progress of Theoretical Dynamics* . 156
Report of British Association, 1857, pp. 1—42

196. *Note sur un Problème d'Analyse Indéterminée* . . . 205
Nouvelles Annales de Math. t. XVI. (1857), pp. 161—165

197. *Note on the Theory of Logarithms* 208
Phil. Mag. t. XI. (1856), pp. 275—280

198. *Note on a Result of Elimination* 214
Phil. Mag. t. XI. (1856), pp. 378, 379

199. *Note on the Theory of Elliptic Motion* 216
Phil. Mag. t. XI. (1856), pp. 425—428

200. *On the Cones which pass through a given Curve of the Third Order in Space* 219
Phil. Mag. t. XII. (1856), pp. 20—22

201. *Second Note on the Theory of Logarithms* 222
Phil. Mag. t. XII. (1856), pp. 354—360

202. *Supplementary Remarks on the Porism of the In-and-circumscribed Triangle* 229
Phil. Mag. t. XIII. (1857), pp. 19—30

203. *On the Theory of the Analytical Forms called Trees* . . 242
Phil. Mag. t. XIII. (1857), pp. 172—176

PAGE

204. *On a Problem in the Partition of Numbers* 247
Phil. Mag. t. XIII. (1857), pp. 245—248

205. *Note on the Summation of a Certain Factorial Expression* . 250
Phil. Mag. t. XIII. (1857), pp. 419—423

206. *Note on a Theorem relating to the Rectangular Hyperbola* . 254
Phil. Mag. t. XIII. (1857), p. 423

207. *Analytical Solution of the Problem of Tactions* 255
Phil. Mag. t. XIII. (1857), pp. 507—509

208. *Note on the Equipotential Curve* $\frac{m}{r}+\frac{m'}{r'}=C$ 258
Phil. Mag. t. XIV. (1857), pp. 142—146

209. *A Demonstration of Sir W. R. Hamilton's Theorem of the Isochronism of the Circular Hodograph* 262
Phil. Mag. t. XIV. (1857), pp. 427—430

210. *On the Cubic Transformation of an Elliptic Function* . . 266
Phil. Mag. t. XV. (1858), pp. 363, 364

211. *On a Theorem relating to Hypergeometric Series* 268
Phil. Mag. t. XVI. (1858), pp. 356, 357

212. *A Memoir on the Problem of Disturbed Elliptic Motion* . 270
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 1—29

213. *On the Development of the Disturbing Function in the Lunar Theory* 293
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 69—95

214. *The First part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories* . 319
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 187—215

215. *A Supplementary Memoir on the Problem of Disturbed Elliptic Motion* 344
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 217—234

216. *Tables of the Development of Functions in the Theory of Elliptic Motion* 360
Mem. R. Astron. Soc. t. XXIX. (1861), pp. 191—306

217. *A Memoir on the Problem of the Rotation of a solid Body* . 475
Mem. R. Astron. Soc. t. XXIX. (1861), pp. 307—342

218. *A Third Memoir on the Problem of Disturbed Elliptic Motion* 505
Mem. R. Astron. Soc. t. XXXI. (1863), pp. 43—56

PAGE

219. *On some formulæ relating to the Variation of the Plane of a Planet's Orbit* 516
Monthly Notices R. Astron. Soc. t. XXI. (1861), pp. 43—46

220. *Note on a Theorem of Jacobi's in relation to the Problem of Three Bodies* 519
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 76—78

221. *On the Secular Acceleration of the Moon's Mean Motion* . 522
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 173—230

222. *On Lambert's Theorem for Elliptic Motion* 562
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 238—242

Notes and References to papers in Volume III. 567

VOLUME IV.

PAGE

223. *Note sur un théorème général par rapport à l'élimination* . 1
Tortolini, t. VII. (1856), pp. 454—458

224. *Sur un théorème d'Abel. Note* 5
Tortolini, t. VIII. (1857), pp. 201—203

225. *On a Class of Dynamical Problems* 7
Proc. R. S. t. VIII. (1857), pp. 506—511

226. *On Professor Mac Cullagh's Theorem of the Polar Plane* . . 12
Proc. R. I. A. t. VI. (1858), pp. 481—491

227. *On the Theory of Reciprocal Surfaces* 21
Proc. R. I. A. t. VII. (1862), pp. 20—28

228. *Sur l'intégrale* $\int_0^1 \frac{t^{\mu+\frac{1}{2}}(1-t)^{\mu-\frac{1}{2}}\,dt}{(a+bt-ct^2)^{\mu+1}}$ 28
Liouville, t. II. (1857), pp. 49—51

229. *Note sur une formule pour la réversion des séries* . . . 30
Crelle, t. LII. (1856), pp. 276—284

230. *Note sur la méthode d'élimination de Bezout* 38
Crelle, t. LIII. (1857), pp. 366, 367

231. *Note sur l'équation* $x^2 - Dy^2 = \pm 4$, $D \equiv 5 \pmod{8}$ . . . 40
Crelle, t. LIII. (1857), pp. 369—371

232. *Mémoire sur la forme canonique des fonctions binaires* . . 43
Crelle, t. LIV. (1857), pp. 48—58

233. *Addition au Mémoire sur la forme canonique des fonctions binaires* 53
Crelle, t. LIV. (1857), p. 292

234. *Deuxième Note sur une formule pour la réversion des séries* . 54
Crelle, t. LIV. (1857), pp. 156—161

235. *Sur quelques formules pour la transformation des intégrales elliptiques* 60
Crelle, t. LV. (1858), pp. 15—24

PAGE

236. *Note sur la composition du nombre 47 par rapport aux vingt-troisièmes racines de l'unité* 70
Crelle, t. LV. (1858), p. 192

237. *Théorème sur les déterminants gauches* 72
Crelle, t. LV. (1858), pp. 277, 278

238. *Note sur les normales d'une conique* 74
Crelle, t. LVI. (1859), pp. 182—185

239. *Addition à la Note sur la composition du nombre 47 par rapport aux vingt-troisièmes racines de l'unité* . . . 78
Crelle, t. LVI. (1859), pp. 186, 187

240. *Note on a Theorem in Spherical Trigonometry* . . . 80
Phil. Mag. t. XVII. (1859), p. 151

241. *On Poinsot's four new Regular Solids* 81
Phil. Mag. t. XVII. (1859), pp. 123—128

242. *Second Note on Poinsot's four new Regular Polyhedra* . . 86
Phil. Mag. t. XVII. (1859), pp. 209, 210

243. *On the Theory of Groups as depending on the Symbolic Equation* $\theta^n = 1$. *Third Part* 88
Phil. Mag. t. XVIII. (1859), pp. 34—37. Sequel to 125 and 126

244. *On an Analytical Theorem relating to the distribution of Electricity upon two Spherical Surfaces* 92
Phil. Mag. t. XVIII. (1859), pp. 119—127

245. *On an Analytical Theorem connected with the distribution of Electricity upon two Spherical Surfaces. Second Part* . 99
Phil. Mag. t. XVIII. (1859), pp. 193—202

246. *On Contour and Slope Lines* 108
Phil. Mag. t. XVIII. (1859), pp. 264—268

247. *On the Analytical Forms called Trees. Second Part* . . 112
Phil. Mag. t. XVIII. (1859), pp. 374—378. Continuation of 203

248. *Sketch of a proof of the Theorem that every Algebraic Equation has a Root* 116
Phil. Mag. t. XVIII. (1859), pp. 436—439

249. *Note on Cones of the Third Order* 120
Phil. Mag. t. XVIII. (1859), pp. 439—442

250. *Sur la surface qui est l'enveloppe des plans conduits par les points d'un ellipsoïde perpendiculairement aux rayons menés par le centre* 123
Tortolini, t. II. (1859), pp. 168—179

PAGE

251. *Sur quelques formules pour la différentiation* 135
Tortolini, t. II. (1859), pp. 214—230

252. *Note sur l'équation des différences pour une équation donnée de degré quelconque* 150
Tortolini, t. II. (1859), pp. 365, 366

253. *Sur la courbe parallèle à l'ellipse* 152
Tortolini, t. III. (1860), pp. 311—316

254. *Sur la surface parallèle à l'ellipsoïde* 158
Tortolini, t. III. (1860), pp. 345—352

255. *On a Problem of Double Partitions*. 166
Phil. Mag. t. XX. (1860), pp. 337—341

256. *On a System of Algebraic Equations* 171
Phil. Mag. t. XX. (1860), pp. 341, 342

257. *On the Cubic Centres of a line with respect to three lines and a line* 173
Phil. Mag. t. XX. (1860), pp. 418—423

258. *On a Relation between two Ternary Cubic Forms* 179
Phil. Mag. t. XX. (1860), pp. 512—514

259. *The Problem of Polyhedra* 182
Phil. Trans. t. CXLVII. (for 1857), pp. 183—185

260. *On the Double Tangents of a Plane Curve* 186
Phil. Trans. t. CXLIX. (for 1859), pp. 193—212

261. *On the Conic of Five-pointic Contact at any Point of a Plane Curve* 207
Phil. Trans. t. CXLIX. (for 1859), pp. 371—400

262. *On the Equation of Differences for an Equation of any Order and in particular for the Equations of the Orders two, three, four, and five* 240
Phil. Trans. t. CL. (for 1860), pp. 93—112

263. *Demonstration of a Theorem in Finite Differences* 262
Phil. Trans. t. CL. (for 1860), pp. 321—323

264. *On an Extension of Arbogast's Method of Derivations* . . 265
Phil. Trans. t. CLI. (for 1861), pp. 37—43

265. *Addition to the Memoir on an Extension of Arbogast's Method of Derivations* 272
Now first published, 1891

PAGE

266. *On the Equation for the Product of the Differences of all but one of the Roots of a given Equation* 276
Phil. Trans. t. CLI. (for 1861), pp. 45—59

267. *On the Porism of the In-and-circumscribed Polygon* . . 292
Phil. Trans. t. CLI. (for 1861), pp. 225—239

268. *On a New Auxiliary Equation in the Theory of Equations of the Fifth Order* 309
Phil. Trans. t. CLI. (for 1861), pp. 263—276

269. *A Seventh Memoir on Quantics* 325
Phil. Trans. t. CLI. (for 1861), pp. 277—292

270. *On the Double Tangents of a Curve of the Fourth Order* . 342
Phil. Trans. t. CLI. (for 1861), pp. 357—362

271. *Sur l'invariant le plus simple d'une fonction quadratique bi-ternaire, et sur le résultant de trois fonctions quadratiques ternaires* 349
Crelle, t. LVII. (1860), pp. 139—148

272. *Démonstration d'un théorème de Jacobi par rapport au problème de Pfaff* 359
Crelle, t. LVII. (1860), pp. 273—277

273. *Note sur la transformation de Tschirnhausen* 364
Crelle, t. LVIII. (1861), pp. 259—262

274. *Deuxième Note sur la transformation de Tschirnhausen* . . 368
Crelle, t. LVIII. (1861), pp. 263—269

275. *On Tschirnhausen's Transformation* 375
Phil. Trans. t. CLI. (1861), pp. 561—578

276. *On the Analytical Theory of the Conic* 395
Phil. Trans. t. CLII. (1862), pp. 639—662

277. *On the Wave Surface* 420
Quart. Math. Journ. t. III. (1860), pp. 16—22

278. *Note on the Singular Solutions of Differential Equations* . 426
Quart. Math. Journ. t. III. (1860), pp. 36, 37

279. *On a Theorem relating to Spherical Conics* 428
Quart. Math. Journ. t. III. (1860), p. 53

280. *On the Conics which touch four given lines* 429
Quart. Math. Journ. t. III. (1860), pp. 94—96

281. *Note on the Wave Surface* 432
Quart. Math. Journ. t. III. (1860), pp. 142—144

PAGE

282. *On a particular case of Castillon's Problem* 435
Quart. Math. Journ. t. III. (1860), pp. 157—164

283. *On a Theorem relating to Homographic Figures* . . . 442
Quart. Math. Journ. t. III. (1860), pp. 177—180

284. *On a New Analytical Representation of Curves in Space* . 446
Quart. Math. Journ. t. III. (1860), pp. 225—236

285. *On the System of Conics having double contact with each other* 456
Quart. Math. Journ. t. III. (1860), pp. 246—250

286. *Note on the value of certain Determinants the terms of which are the squared distances of Points in a Plane or in Space* 460
Quart. Math. Journ. t. III. (1860), pp. 275—277

287. *Note on the Equation for the squared differences of the Roots of a Cubic Equation* 463
Quart. Math. Journ. t. III. (1860), pp. 307—309

288. *Note on the Curvature of a plane Curve at a double point, and on the Curvature of Surfaces* 466
Quart. Math. Journ. t. III. (1860), pp. 322—326

289. *On some Numerical Expansions* 470
Quart. Math. Journ. t. III. (1860), pp. 366—369

290. *A Discussion of the Sturmian Constants for cubic and quartic Equations* 473
Quart. Math. Journ. t. IV. (1861), pp. 7—12

291. *On the Demonstration of a Theorem relating to the Moments of Inertia of a Solid Body* 478
Quart. Math. Journ. t. IV. (1861), pp. 25—27

292. *A Theorem in Conics* 481
Quart. Math. Journ. t. IV. (1861), pp. 131—133

293. *On a Certain System of Functional Symbols* 484
Quart. Math. Journ. t. IV. (1861), pp. 225—230

294. *On a New Analytical Representation of Curves in Space* . 490
Quart. Math. Journ. t. V. (1862), pp. 81—86

295. *On the Construction of the ninth point of Intersection of the Cubics which pass through eight given points* . . . 495
Quart. Math. Journ. t. V. (1862), pp. 222—233

296. *On the Conics which pass through the four foci of a given Conic* 505
Quart. Math. Journ. t. V. (1862), pp. 275—280

PAGE

297. *On some Formulæ relating to the Distances of a point from the vertices of a Triangle and to the Problem of Tactions.* 510
Quart. Math. Journ. t. v. (1862), pp. 381—384

298. *Report on the Progress of the Solution of certain Special Problems of Dynamics* 513
Report of the Brit. Assoc. for the Advancement of Science, 1862, pp. 184—252

299. *Mathematics, recent Terminology in* 594
English Cyclopædia, t. v. (1860), pp. 534—542

Notes and References to papers in Volume IV. 609

VOLUME V.

PAGE

300. *Note relative aux droites en involution de M. Sylvester* . . 1
Comptes Rendus, Paris, t. LII. (1861), pp. 1039—1042

301. *Sur les cônes du second ordre qui passent par six points donnés* 4
Comptes Rendus, Paris, t. LII. (1861), pp. 1216—1218

302. *Considérations générales sur les courbes en espace* . . . 7
Comptes Rendus, Paris, t. LIV. (1862), pp. 55—60, 396—400, 672—678

303. *Sur le problème du polygone inscrit et circonscrit. Lettre à M. Poncelet* 21
Comptes Rendus, Paris, t. LV. (1862), pp. 700, 701

304. *Sur un mémoire de Jacobi. Extrait d'une lettre à M. J. Bertrand* 23
Comptes Rendus, Paris, t. LVI. (1863), p. 43

305. *Considérations générales sur les courbes en espace. Courbes du cinquième ordre* 24
Comptes Rendus, Paris, t. LVIII. (1864), pp. 994—1000

306. *Sur les coniques qui touchent des courbes d'ordre quelconque. Extrait d'une lettre à M. Chasles* 31
Comptes Rendus, Paris, t. LIX. (1864), pp. 224, 225

307. *Note sur les fonctions* $\mathrm{al}(x)$, *&c., de M. Weierstrass* . . . 33
Liouville, t. VII. (1862), pp. 137—142

308. *On the* Δ *faced Polyacrons, in reference to the problem of the enumeration of Polyhedra* 38
Manchester Memoirs, t. I. (1862), pp. 248—256

309. *Note on the Theory of Determinants* 45
Phil. Mag. t. XXI. (1861), pp. 180—185

310. *Note on Mr Jerrard's researches on the Equation of the Fifth Order* 50
Phil. Mag. t. XXI. (1861), pp. 210—214

PAGE

311. *On a Theorem of Abel's relating to Equations of the Fifth Order* 55
Phil. Mag. t. XXI. (1861), pp. 257—263

312. *On the Partitions of a Close* 62
Phil. Mag. t. XXI. (1861), pp. 424—428

313. *On a Surface of the Fourth Order* 66
Phil. Mag. t. XXI. (1861), pp. 491—495

314. *On the Curves situate on a Surface of the Second Order* . 70
Phil. Mag. t. XXII. (1861), pp. 35—38

315. *On the Cubic Centres of a Line with respect to three lines and a line* 73
Phil. Mag. t. XXII. (1861), pp. 433—436

*316. *Note on the solution of an Equation of the Fifth Order* . . 77
Phil. Mag. t. XXIII. (1862), pp. 195, 196

317. *Note on the transformation of a certain Differential Equation* 78
Phil. Mag. t. XXIII. (1862), pp. 266, 267

*318. *On a question in the Theory of Probabilities* 80
Phil. Mag. t. XXIII. (1862), pp. 361—365

*319. *Postscript to the paper, On a question in the Theory of Probabilities* 85
Phil. Mag. t. XXIII. (1862), pp. 470, 471

320. *On the Transcendent* $\operatorname{gd} u = \frac{1}{i} \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$ 86
Phil. Mag. t. XXIV. (1862), pp. 19—21

*321. *Final Remarks on Mr Jerrard's theory of Equations of the Fifth Order* 89
Phil. Mag. t. XXIV. (1862), p. 290

322. *On a Skew Surface of the Third Order* 90
Phil. Mag. t. XXIV. (1862), pp. 514—519

323. *On a tactical Theorem relating to the Triads of Fifteen Things* 95
Phil. Mag. t. XXV. (1863), pp. 59—61

324. *Note on a Theorem relating to Surfaces* 98
Phil. Mag. t. XXV. (1863), pp. 61, 62

325. *Note on a Theorem relating to a Triangle, Line, and Conic* . 100
Phil. Mag. t. XXV. (1863), pp. 181—183

PAGE

326. *Theorems relating to the Canonic Roots of a Binary Quantic of an odd order* 103
Phil. Mag. t. XXV. (1863), pp. 206—208

327. *On the Stereographic Projection of the Spherical Conic* . . 106
Phil. Mag. t. XXV. (1863), pp. 350—353

328. *On the delineation of a Cubic Scroll* 110
Phil. Mag. t. XXV. (1863), pp. 528—530

329. *Note on the Problem of Pedal Curves* 113
Phil. Mag. t. XXVI. (1863), pp. 20, 21

330. *On Differential Equations and Umbilici* 115
Phil. Mag. t. XXVI. (1863), pp. 373—379 and 441—452

331. *Analytical Theorem relating to the four Conics inscribed in the same Conic and passing through the same three Points* 131
Phil. Mag. t. XXVII. (1864), pp. 42, 43

332. *Analytical Theorem relating to the sections of a Quadric Surface* 133
Phil. Mag. t. XXVII. (1864), pp. 43, 44

333. *Note on the Nodal Curve of the Developable derived from the Quartic Equation* $(a, b, c, d, e \between t, 1)^4 = 0$ . . . 135
Phil. Mag. t. XXVII. (1864), pp. 437—440

334. *Note on the Theory of Cubic Surfaces* 138
Phil. Mag. t. XXVII. (1864), pp. 493—496

335. *Tables des formes quadratiques binaires pour les déterminants négatifs depuis* $D = -1$ *jusqu'à* $D = -100$, *pour les déterminants positifs non carrés depuis* $D = 2$ *jusqu'à* $D = 99$, *et pour les treize déterminants négatifs irréguliers qui se trouvent dans le premier millier* 141
Crelle, t. LX. (1862), pp. 357—372

336. *Note sur l'élimination* 157
Crelle, t. LX. (1862), pp. 373, 374

337. *Note sur la réalité des racines d'une équation quadratique* . 160
Crelle, t. LXI. (1863), pp. 367, 368

338. *Nouvelles recherches sur l'élimination et la théorie des courbes* 162
Crelle, t. LXIII. (1864), pp. 34—39

339. *On Skew Surfaces, otherwise Scrolls.* 168
Phil. Trans. t. CLIII. (for 1863), pp. 453—483

340. *A Second Memoir on Skew Surfaces, otherwise Scrolls* . . 201
Phil. Trans. t. CLIV. (for 1864), pp. 559—576

PAGE

341. *On the Sextactic Points of a Plane Curve* 221
Phil. Trans. t. CLV. (for 1865), pp. 545—578

342. *On the Conics which pass through three given points and touch a given line* 258
Quart. Math. Journ. t. VI. (1864), pp. 24—30

343. *On the Cusp of the second kind or Nodecusp* 265
Quart. Math. Journ. t. VI. (1864), pp. 74, 75

344. *On Certain Developable Surfaces* 267
Quart. Math. Journ. t. VI. (1864), pp. 108—126

345. *On the Inflexions of the Cubical Divergent Parabolas* . . 284
Quart. Math. Journ. t. VI. (1864), pp. 199—203

346. *Note on an expression for the Resultant of two Binary Cubics* 289
Quart. Math. Journ. t. VI. (1864), pp. 380—382

347. *On the Notion and Boundaries of Algebra* 292
Quart. Math. Journ. t. VI. (1864), pp. 382—384

348. *On the Theory of Involution* 295
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 21—38

349. *On a case of the Involution of Cubic Curves* 313
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 39—80

350. *On the Classification of Cubic Curves* 354
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 81—128

351. *On Cubic Cones and Curves* 401
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 129—144

352. *Suite des recherches sur l'élimination et la théorie des courbes* 416
Crelle, t. LXIV. (1865), pp. 167—171

353. *Note sur la surface du quatrième ordre de Steiner* . . 421
Crelle, t. LXIV. (1865), pp. 172—174

354. *Note sur les singularités supérieures des courbes planes.* . 424
Crelle, t. LXIV. (1865), pp. 369—371

355. *Sur un théorème relatif à huit points situés sur une conique.* 427
Crelle, t. LXV. (1866), pp. 180—184

356. *Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers* 431
Crelle, t. LXV. (1866), pp. 284—291

357. *A Supplementary Memoir on the Theory of Matrices* . . 438
Phil. Trans. t. CLVI. (for 1866), pp. 25—35

PAGE

358. *Addition to the Memoir on Tschirnhausen's Transformation* . 449
Phil. Trans. t. CLVI. (for 1866), pp. 97—100

359. *A Supplementary Memoir on Caustics* 454
Phil. Trans. t. CLVII. (for 1867), pp. 7—16

360. *Note on a Quartic Surface* 465
Phil. Mag. t. XXIX. (1865), pp. 19—22

361. *On Quartic Curves* 468
Phil. Mag. t. XXIX. (1865), pp. 105—108

362. *Note on Lobatschewsky's Imaginary Geometry* 471
Phil. Mag. t. XXIX. (1865), pp. 231—233

363. *On the theory of the Evolute* 473
Phil. Mag. t. XXIX. (1865), pp. 344—350

364. *On a Theorem relating to Five Points in a plane* . . . 480
Phil. Mag. t. XXIX. (1865), pp. 460—464

365. *On the Intersections of a Pencil of four lines by a Pencil of two lines* 484
Phil. Mag. t. XXIX. (1865), pp. 501—503

366. *Note on the Projection of the Ellipsoid* 487
Phil. Mag. t. XXX. (1865), pp. 50—52

367. *On a Triangle in-and-circumscribed to a Quartic Curve* . 489
Phil. Mag. t. XXX. (1865), pp. 340—342

368. *On a problem of Geometrical Permutation* 493
Phil. Mag. t. XXX. (1865), pp. 370—372

369. *On a property of Commutants* 495
Phil. Mag. t. XXX. (1865), pp. 411—413

370. *On the signification of an elementary formula of Solid Geometry* 498
Phil. Mag. t. XXX. (1865), pp. 413, 414

371. *On a Formula for the intersections of a Line and Conic, and on an Integral Formula connected therewith* . . 500
Quart. Math. Journ. t. VII. (1866), pp. 1—6

372. *On the Reciprocation of a Quartic Developable* . . . 505
Quart. Math. Journ. t. VII. (1866), pp. 87—92

373. *On a Special Sextic Developable* 511
Quart. Math. Journ. t. VII. (1866), pp. 105—113

374. *On the Higher Singularities of a Plane Curve* 520
Quart. Math. Journ. t. VII. (1866), pp. 212—223

PAGE

375. *Notes on Polyhedra* 529
Quart. Math. Journ. t. VII. (1866), pp. 304—316

376. *Théorème relatif à l'équilibre de quatre forces* . . . 540
Comptes Rendus, t. LXI. (1865), pp. 829, 830

377. *Note sur la correspondance de deux points sur une courbe* . 542
Comptes Rendus, t. LXII. (1866), pp. 586—590

378. *Report of a Committee appointed by the British Association to consider the formation of a Catalogue of Philosophical Memoirs (A. Cayley, R. Grant, G. G. Stokes)* . . . 546
Report of the British Association (1856), pp. 463, 464

379. *Notices of Communications to the British Association for the Advancement of Science* 549
Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1854 to 1864)

380. *Note on the Rectangular Hyperbola* 554
Oxford, Camb. and Dubl. Messenger of Mathematics, t. I. (1862), p. 77

381. *Note on Bezout's Method of Elimination* 555
Oxford, Camb. and Dubl. Messenger of Mathematics, t. II. (1864), pp. 88, 89

382. *Note on the Tetrahedron* 557
Oxford, Camb. and Dubl. Messenger of Mathematics, t. III. (1866), pp. 8—10

383. *Problems and Solutions* 560
Mathematical Questions with their Solutions from the Educational Times, vols. I. to IV. (1863 to 1865); for contents, see p. 612

Notes and References to papers in Volume V. 613

VOLUME VI.

PAGE

384. *On the Transformation of Plane Curves* 1
Proc. London Math. Society, t. I. (1865—66), No. III. pp. 1—11

385. *On the Correspondence of Two Points on a Curve* . . . 9
Proc. London Math. Society, t. I. (1865—66), No. VII. pp. 1—7

386. *On the Logarithms of Imaginary Quantities* 14
Proc. London Math. Society, t. II. (1866—69), pp. 50—54

387. *Notices of Communications to the London Mathematical Society* 19
Proc. London Math. Society, t. II. (1866—69), pp. 6—7, 25—26, 29, 61—63, 103—104, 123—125

388. *Note on the Composition of Infinitesimal Rotations* . . . 24
Quart. Math. Journ. t. VIII. (1867), pp. 7—10

389. *On a Locus derived from Two Conics* 27
Quart. Math. Journ. t. VIII. (1867), pp. 77—84

390. *Theorem relating to the four Conics which touch the same two lines and pass through the same four points* . . . 35
Quart. Math. Journ. t. VIII. (1867), pp. 162—167

391. *Solution of a Problem of Elimination* 40
Quart. Math. Journ. t. VIII. (1867), pp. 183—185

392. *On the Conics which pass through two given Points and touch two given Lines* 43
Quart. Math. Journ. t. VIII. (1867), pp. 211—219

393. *On the Conics which touch three given Lines and pass through a given Point.* 51
Quart. Math. Journ. t. VIII. (1867), pp. 220—222

394. *On a Locus in relation to the Triangle* 53
Quart. Math. Journ. t. VIII. (1867), pp. 264—277

395. *Investigations in connexion with Casey's Equation* . . . 65
Quart. Math. Journ. t. VIII. (1867), pp. 334—341

PAGE

396. *On a certain Envelope depending on a Triangle inscribed in a Circle* 72
Quart. Math. Journ. t. IX. (1868), pp. 31—41 and 175—176

397. *Specimen Table* $M \equiv a^a b^b$ *(Mod. N) for any prime or composite Modulus* 83
Quart. Math. Journ. t. IX. (1868), pp. 95, 96 and plate

398. *On a certain Sextic Developable and Sextic Surface connected therewith* 87
Quart. Math. Journ. t. IX. (1868), pp. 129—142 and 373—376

399. *On the Cubical Divergent Parabolas* 101
Quart. Math. Journ. t. IX. (1868), pp. 185—189

400. *On the Cubic Curves inscribed in a given Pencil of Six Lines* 105
Quart. Math. Journ. t. IX. (1868), pp. 210—221

401. *A Notation of the Points and Lines in Pascal's Theorem* . 116
Quart. Math. Journ. t. IX. (1868), pp. 268—274

402. *On a Singularity of Surfaces* 123
Quart. Math. Journ. t. IX. (1868), pp. 332—338

403. *On Pascal's Theorem* 129
Quart. Math. Journ. t. IX. (1868), pp. 348—353

404. *Reproduction of Euler's Memoir of* 1758 *on the Rotation of a Solid Body* 135
Quart. Math. Journ. t. IX. (1868), pp. 361—373

405. *An Eighth Memoir on Quantics* 147
Phil. Trans. t. CLVII. (for 1867), pp. 513—554

406. *On the Curves which satisfy given Conditions* 191
Phil. Trans. t. CLVIII. (for 1868), pp. 75—143

407. *Second Memoir on the Curves which satisfy given Conditions; the Principle of Correspondence* 263
Phil. Trans. t. CLVIII. (for 1868), pp. 145—172

408. *Addition to Memoir on the Resultant of a System of two Equations* 292
Phil. Trans. t. CLVIII. (for 1868), pp. 173—180

409. *On the Conditions for the existence of three equal Roots or of two pairs of equal Roots of a Binary Quartic or Quintic* 300
Phil. Trans. t. CLVIII. (for 1868), pp. 577—588

PAGE

410. *A Third Memoir on Skew Surfaces, otherwise Scrolls* . . 312
Phil. Trans. t. CLIX. (for 1869), pp. 111—126

411. *A Memoir on the Theory of Reciprocal Surfaces* . . . 329
Phil. Trans. t. CLIX. (for 1869), pp. 201—229

412. *A Memoir on Cubic Surfaces* 359
Phil. Trans. t. CLIX. (for 1869), pp. 231—326

413. *A Memoir on Abstract Geometry* 456
Phil. Trans. t. CLX. (for 1870), pp. 51—63

414. *On Polyzomal Curves, otherwise the Curves* $\sqrt{U}+\sqrt{V}+\&c.=0$. 470
Trans. R. Soc. Edinburgh, t. XXV. (for 1868), pp. 1—110

415. *Corrections and Additions to the Memoir on the Theory of Reciprocal Surfaces* 577
Phil. Trans. t. CLXII. (for 1872), pp. 83—87

416. *On the Theory of Reciprocal Surfaces* 582
Addition to Salmon's Analytic Geometry of Three Dimensions, 4th ed. (1882), pp. 592—604

Notes and References to papers in Volume VI. 593

Portrait . . . . . *to face Title.*

VOLUME VII.

PAGE

417. *On the Locus of the Foci of the Conics which pass through Four Given Points* 1
Phil. Mag. t. XXXII. (1866), pp. 362—365

418. *A Remark on Differential Equations* 5
Phil. Mag. t. XXXII. (1866), pp. 379—381

419. *A Theorem on Differential Operators* 8
Phil. Mag. t. XXXII. (1866), pp. 461—472

420. *On Riccati's Equation* 9
Phil. Mag. t. XXXVI. (1868), pp. 348—351

421. *Note on the Solvibility of Equations by means of Radicals* . 13
Phil. Mag. t. XXXVI. (1868), pp. 386, 387

422. *On the Geodesic Lines on an Oblate Spheroid* 15
Phil. Mag. t. XL. (1870), pp. 329—340

423. *On the Plane Representation of a Solid Figure* 26
Phil. Mag. t. XLI. (1871), pp. 286—290

424. *On the Attraction of a Terminated Straight Line* 31
Phil. Mag. t. XLI. (1871), pp. 358—360

425. *Note on the Geodesic Lines on an Ellipsoid* 34
Phil. Mag. t. XLI. (1871), pp. 534, 535

*426. *On a supposed New Integration of Differential Equations of the Second Order* 36
Phil. Mag. t. XLII. (1871), pp. 197—199

427. *On Gauss' Pentagramma Mirificum* 37
Phil. Mag. t. XLII. (1871), pp. 311, 312

*428. *Note sur la correspondance de deux points sur une courbe* . 39
Comptes Rendus, t. LXII. (1866), pp. 586—590

*429. *Sur les Coniques déterminées par cinq conditions de contact avec une courbe donnée* 40
Comptes Rendus, t. LXIII. (1866), pp. 9—12

PAGE

430. *Note sur quelques formules de M. E. de Jonquières, relatives aux Courbes qui satisfont à des conditions données* . . 41
Comptes Rendus, t. LXIII. (1866), pp. 666—670

431. *Sur la transformation cubique d'une fonction elliptique* . . 44
Comptes Rendus, t. LXIV. (1867), pp. 560—563

432. *Théorème relatif à la théorie des substitutions* 47
Comptes Rendus, t. LXVII. (1868), pp. 784, 785

433. *Sur les surfaces tétraédrales* 48
Notes to De la Gournerie, *Recherches sur les Surfaces réglées tétraédrales symétriques*, 8vo. Paris, 1867

434. *On Certain Skew Surfaces, otherwise Scrolls* 54
Camb. Phil. Trans. t. XI. Part II. (1869), pp. 277—289

435. *On the Six Coordinates of a Line* 66
Camb. Phil. Trans. t. XI. Part II. (1869), pp. 290—323

436. *On a Certain Sextic Torse* 99
Camb. Phil. Trans. t. XI. Part III. (1871), pp. 507—523

*437. *Démonstration nouvelle du théorème de M. Casey par rapport aux cercles qui touchent à trois cercles donnés* . . . 115
Annali di Matematica, t. I. (1867), pp. 132—134

438. *Note sur quelques torses sextiques* 116
Annali di Matematica, t. II. (1868), pp. 99, 100

439. *Addition à la Note sur quelques torses sextiques* . . . 118
Annali di Matematica, t. II. (1868), pp. 219—221

440. *Note sur une transformation géométrique* 121
Journ. der Mathem. (Crelle), t. LXVII. (1867), pp. 95, 96

441. *Note sur l'algorithme des tangentes doubles d'une courbe du quatrième ordre* 123
Journ. der Mathem. (Crelle), t. LXVIII. (1868), pp. 176—179

442. *Note sur la surface du quatrième ordre douée de seize points singuliers et de seize plans singuliers* 126
Journ. der Mathem. (Crelle), t. LXXIII. (1871), pp. 292, 293

443. *Note on the solution of the Quartic Equation* $\alpha U + 6\beta H = 0$. 128
Math. Ann. t. I. (1869), pp. 54, 55

444. *On the Centro-surface of an Ellipsoid* 130
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 16—18

445. *A Memoir on Quartic Surfaces* 133
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 19—69

PAGE

446. *On the Mechanical Description of a Nodal Bicircular Quartic* 182
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 100—106

447. *On the Rational Transformation between Two Spaces* . . 189
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 127—180

448. *Note on the Cartesian with Two Imaginary Axial Foci* . . 241
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 181, 182

449. *Sketch of recent researches upon Quartic and Quintic Surfaces* 244
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 186—195

450. *Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points* . . 253
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 196—198

451. *A Second Memoir on Quartic Surfaces* 256
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 198—202

452. *On an Analytical Theorem from a new point of view* . . 261
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 220, 221

453. *On a Problem in the Calculus of Variations* 263
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 221, 222

454. *A Third Memoir on Quartic Surfaces* 264
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 234—266

455. *On Plücker's Models of certain Quartic Surfaces* . . . 298
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 281—285

456. *Note on the Discriminant of a Binary Quartic* . . . 303
Quart. Math. Journ. t. X. (1870), p. 23

457. *On the Quartic Surfaces* $(*\between U, V, W)^2 = 0$ 304
Quart. Math. Journ. t. X. (1870), pp. 24—34

458. *On the Anharmonic-Ratio Sextic* 314
Quart. Math. Journ. t. X. (1870), pp. 56, 57

459. *On the Double-Sixers of a Cubic Surface* 316
Quart. Math. Journ. t. X. (1870), pp. 58—71

460. *Note on Mr Frost's paper On the direction of the Lines of Curvature in the neighbourhood of an Umbilicus* . . 330
Quart. Math. Journ. t. X. (1870), pp. 111—113

461. *On the Geometrical Interpretation of the Covariants of a Binary Cubic* 332
Quart. Math. Journ. t. X. (1870), pp. 148, 149

PAGE

462. *A Ninth Memoir on Quantics* 334
Phil. Trans. t. CLXI. (for 1871), pp. 17—50

463. *Note on a Differential Equation* 354
Mem. Manchester Society, t. II. (1865), pp. 111—114

464. *Note on Plana's Lunar Theory* 357
Monthly Notices R. Ast. Society, t. XXIII. (1862—1863), pp. 211—215

465. *Note on the Lunar Theory* 361
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 182—189

466. *Second Note on the Lunar Theory* 367
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 203—207

467. *Expressions for Plana's e, γ in terms of the Elliptic e, γ* . 371
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 265—271

468. *Addition to Second Note on the Lunar Theory* 375
Monthly Notices R. Ast. Society, t. XXVII. (1866—1867), pp. 267—269

469. *On an Expression for the Angular Distance of two Planets* . 377
Monthly Notices R. Ast. Society, t. XXVII. (1866—1867), pp. 312—315

470. *Note on the Attraction of Ellipsoids* 380
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 254—257

471. *Note on the Problem of the Determination of a Planet's Orbit from three observations* 384
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 257—259

472. *Note on Lambert's Theorem for Elliptic Motion* 387
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 318—320

473. *On the Graphical Construction of the Umbral or Penumbral Curve at any instant during a Solar Eclipse* 390
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 162—164

474. *On the Geometrical Theory of Solar Eclipses* 392
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 164—168

475. *On a property of the Stereographic Projection* 397
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 205—207

476. *On the Determination of the Orbit of a Planet from three observations* 400
Mem. R. Ast. Society, t. XXXVIII. (1870), pp. 17—111

477. *On the Graphical Construction of a Solar Eclipse* 479
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 1—17

478. *On the Geodesic Lines on an Ellipsoid* 493
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 31—53

PAGE

479. *The Second Part of a Memoir On the Development of the Disturbing Function in the Lunar and Planetary Theories* 511
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 55—74

480. *On the Expression of Delaunay's l, g, h in terms of his finally adopted Constants* 528
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 8—16

481. *On the Expression of M. Delaunay's h+g in terms of his finally adopted Constants* 534
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), p. 74

482. *Note on a pair of Differential Equations in the Lunar Theory* 535
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 31, 32

483. *On a pair of Differential Equations in the Lunar Theory* . 537
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 201—206

484. *On the variations of the position of the Orbit in the Planetary Theory* 541
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 206—211

485. *Problems and Solutions* 546
Mathematical Questions with their Solutions from the Educational Times, vols. V. to XII. (1866—1869): for contents, see p. 607

Notes and References to papers in Volume VII. 609

Portrait . . . . . *to face Title.*

VOLUME VIII.

PAGE

486. *Note on Dr Glaisher's paper on a theorem in definite integration* 1
Quart. Math. Journ. t. x. (1870), pp. 355, 356

487. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 2
Quart. Math. Journ. t. xi. (1871), pp. 15—25

488. *Note on a relation between two circles* 12
Quart. Math. Journ. t. xi. (1871), pp. 82, 83

489. *On the porism of the in-and-circumscribed polygon, and the* (2, 2) *correspondence of points on a conic* 14
Quart. Math. Journ. t. xi. (1871), pp. 83—91

490. *On a problem of elimination* 22
Quart. Math. Journ. t. xi. (1871), pp. 99—101

491. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 25
Quart. Math. Journ. t. xi. (1871), pp. 111—113

492. *Note on a system of algebraical equations* 29
Quart. Math. Journ. t. xi. (1871), pp. 132, 133

493. *On evolutes and parallel curves* 31
Quart. Math. Journ. t. xi. (1871), pp. 183—200

494. *Example of a special discriminant* 46
Quart. Math. Journ. t. xi. (1871), pp. 211—213

495. *On the envelope of a certain quadric surface* 48
Quart. Math. Journ. t. xi. (1871), pp. 244—246

496. *Tables of the binary cubic forms for the negative determinants* $\equiv 0 \pmod{4}$ *from* -4 *to* -400; *and* $\equiv 1 \pmod{4}$ *from* -3 *to* -99; *and for five irregular negative determinants* . 51
Quart. Math. Journ. t. xi. (1871), pp. 246—261

497. *Note on the calculus of logic* 65
Quart. Math. Journ. t. xi. (1871), pp. 282, 283

PAGE

498. *On the inversion of a quadric surface* 67
Quart. Math. Journ. t. XI. (1871), pp. 283—288

499. *On the theory of the curve and torse* 72
Quart. Math. Journ. t. XI. (1871), pp. 294—317

500. *On a theorem relating to eight points on a conic* . . . 92
Quart. Math. Journ. t. XI. (1871), pp. 344—346

501. *Review. Pineto's tables of logarithms* 95
Quart. Math. Journ. t. XI. (1871), pp. 375, 376

502. *On the surfaces divisible into squares by their curves of curvature* 97
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 8, 9

503. *On the surfaces each the locus of the vertex of a cone which passes through m given points and touches* $6-m$ *given lines* 99
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 11—47

504. *On the mechanical description of certain sextic curves* . . 138
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 105—111

505. *On the surfaces divisible into squares by their curves of curvature* 145
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 120, 121

506. *On the mechanical description of a cubic curve* . . . 147
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 175—178

507. *On the mechanical description of certain quartic curves by a modified oval chuck* 151
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 186—190

508. *On geodesic lines, in particular those of a quadric surface* . 156
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 191—211

509. *Plan of a curve-tracing apparatus* 179
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 345—347

510. *On bicursal curves* 181
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 347—352

511. *Addition to the memoir on geodesic lines, in particular those of a quadric surface* 188
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 368—380

512. *On a correspondence of points in relation to two tetrahedra* . 200
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 396—404

513. *On a bicyclic chuck* 209
Phil. Mag. t. XLIII. (1872), pp. 365—367

PAGE

514. *On the problem of the in-and-circumscribed triangle* . . 212
Phil. Trans. t. CLXI. (for 1871), pp. 369—412

515. *Sur les courbes aplaties* 258
Comptes Rendus, t. LXXIV. (1872), pp. 708—712

516. *Sur une surface quartique aplatie* 262
Comptes Rendus, t. LXXIV. (1872), pp. 1393—1395

517. *Sur les surfaces divisibles en carrés par leurs courbes de courbure et sur la théorie de Dupin* 264
Comptes Rendus, t. LXXIV. (1872), pp. 1445—1449

518. *Sur la condition pour qu'une famille de surfaces données puisse faire partie d'un système orthogonal* 269
Comptes Rendus, t. LXXV. (1872), pp. 177—185, 246—250, 324—330, 381—385, 1800—1803

519. *On curvature and orthogonal surfaces* 292
Phil. Trans. t. CLXIII. (for 1873), pp. 229—251

520. *On the centro-surface of an ellipsoid* 316
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 319—365

521. *On Dr Wiener's model of a cubic surface with 27 real lines; and on the construction of a double-sixer* 366
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 366—383

522. *Note on the theory of invariants* 385
Math. Ann. t. III. (1871), pp. 268—271

523. *On the transformation of unicursal surfaces* 388
Math. Ann. t. III. (1871), pp. 469—474

524. *On the deficiency of certain surfaces* 394
Math. Ann. t. III. (1871), pp. 526—529

525. *An example of the higher transformation of a binary form* . 398
Math. Ann. t. IV. (1871), pp. 359—361

526. *On a surface of the eighth order* 401
Math. Ann. t. IV. (1871), pp. 558—560

527. *On a theorem in covariants* 404
Math. Ann. t. V. (1872), pp. 625—629

528. *On the non-Euclidian geometry* 409
Math. Ann. t. V. (1872), pp. 630—634

529. *A "Smith's Prize" paper* [1868]; *solutions by Prof. Cayley* . 414
Oxford, Camb. and Dubl. Messenger of Mathematics, t. IV. (1868), pp. 201—226

PAGE

530. *Solution of a Senate-House problem* 436
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 24—27

531. *A "Smith's Prize" paper* [1869]; *solutions by Prof. Cayley* . 439
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 41—64

532. *Note on the integration of certain differential equations by series* 458
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 77—82

533. *On the binomial theorem, factorials, and derivations* . . 463
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 102—114

534. *A "Smith's Prize" paper* [1870]; *solutions by Prof. Cayley*. 474
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 182—203

535. *Note on the problem of envelopes* 491
Messenger of Mathematics, t. I. (1872), pp. 3, 4

536. *Note on Lagrange's demonstration of Taylor's theorem* . . 493
Messenger of Mathematics, t. I. (1872), pp. 22—24

537. *Solutions of a Smith's Prize paper for* 1871 496
Messenger of Mathematics, t. I. (1872), pp. 37—47, 71—77, 89—95

538. *Extract from a letter from Prof. Cayley to Mr C. W. Merrifield* 517
Messenger of Mathematics, t. I. (1872), pp. 87, 88

*539. *Further note on Lagrange's demonstration of Taylor's theorem* 519
Messenger of Mathematics, t. I. (1872), pp. 105, 106

540. *On a property of the torse circumscribed about two quadric surfaces* 520
Messenger of Mathematics, t. I. (1872), pp. 111, 112

541. *On the reciprocal of a certain equation of a conic* . . . 522
Messenger of Mathematics, t. I. (1872), pp. 120, 121

*542. *Further note on Taylor's theorem* 524
Messenger of Mathematics, t. I. (1872), p. 137

543. *On an identity in spherical trigonometry* 525
Messenger of Mathematics, t. I. (1872), p. 145

544. *On a penultimate quartic curve* 526
Messenger of Mathematics, t. I. (1872), pp. 178—180

PAGE

545. *On the theory of the singular solutions of differential equations of the first order* 529
Messenger of Mathematics, t. II. (1873), pp. 6—12

546. *Theorems in relation to certain sign-symbols* 535
Messenger of Mathematics, t. II. (1873), pp. 17—20

547. *On the representation of a spherical or other surface on a plane: a Smith's Prize dissertation* 538
Messenger of Mathematics, t. II. (1873), pp. 36, 37

548. *On Listing's theorem.* 540
Messenger of Mathematics, t. II. (1873), pp. 81—89

549. *Note on the maxima of certain factorial functions* . . . 548
Messenger of Mathematics, t. II. (1873), pp. 129, 130

550. *Problem and hypothetical theorems in regard to two quadric surfaces* 550
Messenger of Mathematics, t. II. (1873), p. 137

551. *Two Smith's Prize dissertations* [1872] 551
Messenger of Mathematics, t. II. (1873), pp. 145—149

552. *On a differential formula connected with the theory of confocal conics* 556
Messenger of Mathematics, t. II. (1873), pp. 157, 158

553. *Two Smith's Prize dissertations* [1873] 558
Messenger of Mathematics, t. II. (1873), pp. 161—166

554. *An elliptic-transcendent identity* 564
Messenger of Mathematics, t. II. (1873), p. 179

555. *Notices of Communications to the British Association for the Advancement of Science* 565
Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1870, 1871, 1873).

Notes and References to papers in Volume VIII. 569
Prefatory Note vii
Arthur Cayley: biographical notice by the Editor ix
List of courses of lectures delivered by Professor Cayley . . . xlv
Facsimile of the manuscript of his note on p. 569 . . . *Frontispiece*

VOLUME IX.

PAGE

556. *On Steiner's surface* 1
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 14—25

557. *On certain constructions for bicircular quartics.* 13
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 29—31

558. *A geometrical interpretation of the equations obtained by equating to zero the resultant and the discriminants of two binary quantics* 16
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 31—33

559. [*Note on inversion*] 18
Proc. Lond. Math. Society, t. v. (1873—1874), p. 112

560. [*Addition to Lord Rayleigh's paper "On the numerical calculation of the roots of fluctuating functions"*]. 19
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 123, 124

561. *On the geometrical representation of Cauchy's theorems of root-limitation* 21
Camb. Phil. Trans., t. XII. Part II. (1877), pp. 395—413

562. *On a theorem in maxima and minima: addition* [*to Mr Walton's paper*] *by Professor Cayley* 40
Quart. Math. Journ., t. x. (1870), pp. 262, 263

563. *Note on the transformation of two simultaneous equations.* . 42
Quart. Math. Journ., t. XI. (1871), pp. 266, 267

564. *On a theorem in elimination* 43
Quart. Math. Journ., t. XII. (1873), pp. 5, 6

565. *Note on the Cartesian* 45
Quart. Math. Journ., t. XII. (1873), pp. 16—19

566. *On the transformation of the equation of a surface to a set of chief axes* 48
Quart. Math. Journ., t. XII. (1873), pp. 34—38

PAGE

567. *On an identical equation connected with the theory of invariants* 52
Quart. Math. Journ., t. XII. (1873), pp. 115—118

568. *Note on the integrals* $\int_0^\infty \cos x^2\,dx$ *and* $\int_0^\infty \sin x^3\,dx$ . . . 56
Quart. Math. Journ., t. XII. (1873), pp. 118—126

569. *On the cyclide* 64
Quart. Math. Journ., t. XII. (1873), pp. 148—165

570. *On the superlines of a quadric surface in five-dimensional space* 79
Quart. Math. Journ., t. XII. (1873), pp. 176—180

571. *A demonstration of Dupin's theorem* 84
Quart. Math. Journ., t. XII. (1873), pp. 185—191

572. *Theorem in regard to the Hessian of a quaternary function* . 90
Quart. Math. Journ., t. XII. (1873), pp. 193—197

573. *Note on the* (2, 2) *correspondence of two variables* . . . 94
Quart. Math. Journ., t. XII. (1873), pp. 197, 198

574. *On Wronski's theorem* 96
Quart. Math. Journ., t. XII. (1873), pp. 221—228

575. *On a special quartic transformation of an elliptic function* . 103
Quart. Math. Journ., t. XII. (1873), pp. 266—269

576. *Addition to Mr Walton's paper "On the ray-planes in biaxal crystals"* 107
Quart. Math. Journ., t. XII. (1873), pp. 273—275

577. *Note in illustration of certain general theorems obtained by Dr Lipschitz.* 110
Quart. Math. Journ., t. XII. (1873), pp. 346—349

578. *A memoir on the transformation of elliptic functions* . . 113
Phil. Trans., t. CLXIV. (for 1874), pp. 397—456

579. *Address delivered by the President, Professor Cayley, on presenting the Gold Medal of the [Royal Astronomical] Society to Professor Simon Newcomb* 176
Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 224—233

580. *On the number of distinct terms in a symmetrical or partially symmetrical determinant; with an addition* 185
Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 303—307; p. 335

581. *On a theorem in elliptic motion* 191
Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 337—339

PAGE

582. *Note on the Theory of Precession and Nutation* . . . 194
Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 340—343

583. *On spheroidal trigonometry* 197
Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), p. 92

584. *Addition to Prof. R. S. Ball's paper "Note on a transformation of Lagrange's equations of motion in generalised coordinates, which is convenient in Physical Astronomy"* . 198
Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), pp. 269—271

585. *A new theorem on the equilibrium of four forces acting on a solid body* 201
Phil. Mag., t. XXXI. (1866), pp. 78, 79; Camb. Phil. Soc. Proc., t. I. (1866), p. 235

586. *On the mathematical theory of isomers* 202
Phil. Mag., t. XLVII. (1874), pp. 444—467

587. *A Smith's Prize dissertation* [1873]. 205
Messenger of Mathematics, t. III. (1874), pp. 1—4

588. *Problem [on tetrahedra]* 209
Messenger of Mathematics, t. III. (1874), pp. 50—52

589. *On residuation in regard to a cubic curve* 211
Messenger of Mathematics, t. III. (1874), pp. 62—65

590. *Addition to Prof. Hall's paper "On the motion of a particle toward an attracting centre at which the force is infinite"* . 215
Messenger of Mathematics, t. III. (1874), pp. 149—152

591. *A Smith's Prize paper and dissertation* [1874]; *solutions and remarks* 218
Messenger of Mathematics, t. III. (1874), pp. 165—183; t. IV. (1875), pp. 6—8

592. *On the Mercator's projection of a skew hyperboloid of revolution* 237
Messenger of Mathematics, t. IV. (1875), pp. 17—20

593. *A Sheepshanks' problem* (1866) 241
Messenger of Mathematics, t. IV. (1875), pp. 34—36

594. *On a differential equation in the theory of elliptic functions* . 244
Messenger of Mathematics, t. IV. (1875), pp. 69, 70

595. *On a Senate-House problem* 246
Messenger of Mathematics, t. IV. (1875), pp. 75—78

596. *Note on a theorem of Jacobi's for the transformation of a double integral* 250
Messenger of Mathematics, t. IV. (1875), pp. 92—94

PAGE

597. *On a differential equation in the theory of elliptic functions* . 253
Messenger of Mathematics, t. IV. (1875), pp. 110—113

598. *Note on a process of integration* 257
Messenger of Mathematics, t. IV. (1875), pp. 149, 150

599. *A Smith's Prize dissertation* 259
Messenger of Mathematics, t. IV. (1875), pp. 157—160

600. *Theorem on the n-th Roots of Unity* 263
Messenger of Mathematics, t. IV. (1875), p. 171

601. *Note on the Cassinian* 264
Messenger of Mathematics, t. IV. (1875), pp. 187, 188

602. *On the potentials of polygons and polyhedra* 266
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 20—34

603. *On the potential of the ellipse and the circle* 281
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 38—58

604. *Determination of the attraction of an ellipsoidal shell on an exterior point* 302
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 58—67

605. *Note on a point in the theory of attraction* 312
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 79—81

606. *On the expression of the coordinates of a point of a quartic curve as functions of a parameter* 315
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 81—83

607. *A memoir on prepotentials* 318
Phil. Trans., t. CLXV. (for 1875), pp. 675—774

608. [*Extract from a*] *Report on Mathematical Tables* . . . 424
Brit. Assoc. Report, 1873, pp. 3, 4

609. *On the analytical forms called factions* 426
Brit. Assoc. Report, 1875, Notices of Communications to the Sections, p. 10

610. *On the analytical forms called Trees, with application to the theory of chemical combinations* 427
Brit. Assoc. Report, 1875, pp. 257—305

611. *Report on mathematical tables* 461
Brit. Assoc. Report, 1875, pp. 305—336

612. *Note sur une formule d'intégration indéfinie* 500
Comptes Rendus, t. LXXVIII. (1874), pp. 1624—1629

613. *On the group of points G_4^1 on a sextic curve with five double points* 504
Math. Ann., t. VIII. (1875), pp 359—362

PAGE

614. *On a problem of projection* 508
Quart. Math. Journ., t. XIII. (1875), pp. 19—29

615. *On the conic torus* 519
Quart. Math. Journ., t. XIII. (1875), pp. 127—129

616. *A geometrical illustration of the cubic transformation in elliptic functions* 522
Quart. Math. Journ., t. XIII. (1875), pp. 211—216

617. *On the scalene transformation of a plane curve* . . . 527
Quart. Math. Journ., t. XIII. (1875), pp. 321—328

618. *On the mechanical description of a Cartesian* 535
Quart. Math. Journ., t. XIII. (1875), pp. 328—330

619. *On an algebraical operation* 537
Quart. Math. Journ., t. XIII. (1875), pp. 369—375

620. *Correction of two numerical errors in Sohnke's paper respecting modular equations* 543
Crelle, t. LXXXI. (1876), p. 229

621. *On the number of the univalent radicals* C_nH_{2n+1} . . . 544
Phil. Mag., Ser. 5, t. III. (1877), pp. 34, 35

622. *On a system of equations connected with Malfatti's problem* . 546
Proc. Lond. Math. Society, t. VII. (1876), pp. 38—42

623. *On three-bar motion* 551
Proc. Lond. Math. Society, t. VII. (1876), pp. 136—166

624. *On the bicursal sextic* 581
Proc. Lond. Math. Society, t. VII. (1876), pp. 166—172

625. *On the condition for the existence of a surface cutting at right angles a given set of lines* 587
Proc. Lond. Math. Society, t. VIII. (1877), pp. 53—57

626. *On the general differential equation* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, *where* X, Y *are the same quartic functions of* x, y *respectively* . . 592
Proc. Lond. Math. Society, t. VIII. (1877), pp. 184—199

627. *Geometrical illustration of a theorem relating to an irrational function of an imaginary variable* 609
Proc. Lond. Math. Society, t. VIII. (1877), pp. 212—214

628. *On the circular relation of Möbius* 612
Proc. Lond. Math. Society, t. VIII. (1877), pp. 220—225

629. *On the linear transformation of the integral* $\int\frac{du}{\sqrt{U}}$ . . . 618
Proc. Lond. Math. Society, t. VIII. (1877), pp. 226—229

VOLUME X.

PAGE

630. *On an expression for* $1 \pm \sin(2p+1)u$ *in terms of* $\sin u$ . . 1
Messenger of Mathematics, t. v. (1876), pp. 7, 8

631. *Synopsis of the theory of equations* 3
Messenger of Mathematics, t. v. (1876), pp. 39—49

632. *On Aronhold's integration-formula* 12
Messenger of Mathematics, t. v. (1876), pp. 88—90

*633. *Note on Mr Martin's paper "On the integrals of some differentials"* 15
Messenger of Mathematics, t. v. (1876), p. 163

634. *Theorems in trigonometry and on partitions* 16
Messenger of Mathematics, t. v. (1876), p. 164, p. 188

635. *Note on the demonstration of Clairaut's theorem* . . . 17
Messenger of Mathematics, t. v. (1876), pp. 166, 167

636. *On the theory of the singular solutions of differential equations of the first order* 19
Messenger of Mathematics, t. vi. (1877), pp. 23—27

637. *On a differential equation in the theory of elliptic functions* . 24
Messenger of Mathematics, t. vi. (1877), p. 29

638. *On a q-formula leading to an expression for* E_1 25
Messenger of Mathematics, t. vi. (1877), pp. 63—66

639. *An elementary construction in optics* 28
Messenger of Mathematics, t. vi. (1877), pp. 81, 82

*640. *Further note on Mr Martin's paper* 29
Messenger of Mathematics, t. vi. (1877), pp. 82, 83

641. *On the flexure of a spherical surface* 30
Messenger of Mathematics, t. vi. (1877), pp. 88—90

642. *On a differential relation between the sides of a quadrangle* . 33
Messenger of Mathematics, t. vi. (1877), pp. 99—101

PAGE

643. *On a quartic curve with two odd branches* 36
Messenger of Mathematics, t. VI. (1877), pp. 107, 108

644. *Note on magic squares* 38
Messenger of Mathematics, t. VI. (1877), p. 168

645. *A Smith's Prize Paper*, 1877 39
Messenger of Mathematics, t. VI. (1877), pp. 173—182

646. *On the general equation of differences of the second order* . 47
Quart. Math. Journ., t. XIV. (1877), pp. 23—25

647. *On the quartic surfaces represented by the equation, symmetrical determinant* = 0 50
Quart. Math. Journ., t. XIV. (1877), pp. 46—52

648. *Algebraical theorem* 57
Quart. Math. Journ., t. XIV. (1877), p. 53

649. *Addition to Mr Glaisher's Note on Sylvester's paper "Development of an idea of Eisenstein"* 58
Quart. Math. Journ., t. XIV. (1877), pp. 83, 84

650. *On a quartic surface with twelve nodes* 60
Quart. Math. Journ., t. XIV. (1877), pp. 103—106

651. *On a special surface of minimum area* 63
Quart. Math. Journ., t. XIV. (1877), pp. 190—196

652. *On a sextic torse* 68
Quart. Math. Journ., t. XIV. (1877), pp. 229—235

653. *On a torse depending on the elliptic functions* 73
Quart. Math. Journ., t. XIV. (1877), pp. 235—241

654. *On certain octic surfaces* 79
Quart. Math. Journ., t. XIV. (1877), pp. 249—264

655. *A memoir on differential equations* 93
Quart. Math. Journ., t. XIV. (1877), pp. 292—339

656. *On the theory of partial differential equations* 134
Mathematische Annalen, t. XI. (1877), pp. 194—198

657. *Note on the theory of elliptic integrals* 139
Mathematische Annalen, t. XII. (1877), pp. 143—146

658. *On some formulæ in elliptic integrals* 143
Mathematische Annalen, t. XII. (1877), pp. 369—374

659. *A theorem on groups* 149
Mathematische Annalen, t. XIII. (1878), pp. 561—565

PAGE

660. *On the correspondence of homographies and rotations* . . 153
Mathematische Annalen, t. xv. (1879), pp. 238—240

661. *On the double ϑ-functions* 155
Proc. Lond. Math. Soc., t. IX. (1878), pp. 29, 30

662. *On the double Θ-functions in connexion with a 16-nodal quartic surface* 157
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 210—219

663. *Further investigations on the double ϑ-functions* . . . 166
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 220—233

664. *On the 16-nodal quartic surface* 180
Crelle's Journal der Mathem., t. LXXXIV. (1878), pp. 238—241

665. *A memoir on the double ϑ-functions* 184
Crelle's Journal der Mathem., t. LXXXV. (1878), pp. 214—245

666. *Sur un exemple de réduction d'intégrales abéliennes aux fonctions elliptiques* 214
Comptes Rendus, t. LXXXV. (1877), pp. 265—268, 373—376, 426—429, 472—475

667. *On the bicircular quartic—Addition to Professor Casey's memoir: "On a new form of tangential equation"* . . 223
Phil. Trans., t. 167 (for 1877), pp. 441—460

668. *On compound combinations* 243
Proceedings of the Lit. Phil. Soc. Manchester, t. XVI. (1877), pp. 113, 114; Memoirs, ib., Ser. III., t. VI. (1879), pp. 99, 100

669. *On a problem of arrangements* 245
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 338—342

670. [*Note on Mr Muir's solution of a "problem of arrangement"*] 249
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 388—391

671. *On a sibi-reciprocal surface* 252
Berlin, Akad. Monatsber., (1878), pp. 309—313

672. *On the game of mousetrap* 256
Quart. Math. Journ., t. XV. (1878), pp. 8—10

673. *Note on the theory of correspondence* 259
Quart. Math. Journ., t. XV. (1878), pp. 32, 33

674. *Note on the construction of Cartesians* 261
Quart. Math. Journ., t. XV. (1878), p. 34

675. *On the flecnodal planes of a surface* 262
Quart. Math. Journ., t. XV. (1878), pp. 49—51

PAGE

676. *Note on a theorem in determinants* 265
Quart. Math. Journ., t. xv. (1878), pp. 55—57

677. [*Addition to Mr Glaisher's paper "Proof of Stirling's theorem"*] 267
Quart. Math. Journ., t. xv. (1878), pp. 63, 64

678. *On a system of quadric surfaces* 269
Quart. Math. Journ., t. xv. (1878), pp. 124, 125

679. *On the regular solids* 270
Quart. Math. Journ., t. xv. (1878), pp. 127—131

680. *On the Hessian of a quartic surface* 274
Quart. Math. Journ., t. xv. (1878), pp. 141—144

681. *On the derivatives of three binary quantics* 278
Quart. Math. Journ., t. xv. (1878), pp. 157—168

682. *Formulæ relating to the right line* 287
Quart. Math. Journ., t. xv. (1878), pp. 169—171

683. *On the function* $\operatorname{arc\,sin}(x+iy)$ 290
Quart. Math. Journ., t. xv. (1878), pp. 171—174

684. *On a relation between certain products of differences* . . 293
Quart. Math. Journ., t. xv. (1878), pp. 174, 175

685. *On Mr Cotterill's goniometrical problem* 295
Quart. Math. Journ., t. xv. (1878), pp. 196—198

686. *On a functional equation* 298
Quart. Math. Journ., t. xv. (1878), pp. 315—325; Proc. Lond. Math. Soc., t. ix. (1878), p. 29

687. *Note on the function* $\vartheta(x) = a^2(c-x) \div \{c(c-x) - b^2\}$ . . 307
Quart. Math. Journ., t. xv. (1878), pp. 338—340

688. *Geometrical considerations on a solar eclipse* 310
Quart. Math. Journ., t. xv. (1878), pp. 340—347

689. *On the geometrical representation of imaginary variables by a real correspondence of two planes* 316
Proc. Lond. Math. Soc., t. ix. (1878), pp. 31—39

690. *On the theory of groups* 324
Proc. Lond. Math. Soc., t. ix. (1878), pp. 126—133

691. *Note on Mr Monro's paper "On flexure of spaces"* . . 331
Proc. Lond. Math. Soc., t. ix. (1878), pp. 171, 172

692. *Addition to* [578] *memoir on the transformation of elliptic functions* 333
Phil. Trans., vol. 169, Part II. (for 1878), pp. 419—424

PAGE

693. *A tenth memoir on quantics* 339
Phil. Trans., vol. 169, Part II. (for 1878), pp. 603—661

694. *Desiderata and Suggestions* 401

No. 1. *The theory of groups;*
American Journal of Mathematics, t. I. (1878), pp. 50—52

No. 2. *The theory of groups; graphical representation;*
American Journal of Mathematics, t. I. (1878), pp. 174—176

No. 3. *The Newton-Fourier imaginary problem;*
American Journal of Mathematics, t. II. (1879), p. 97

No. 4. *The mechanical construction of conformable figures;*
American Journal of Mathematics, t. II. (1879), p. 186

695. *A link-work for x^2: extract from a letter to Mr Sylvester* . 407
American Journal of Mathematics, t. I. (1878), p. 386

696. *Calculation of the minimum N.G.F. of the binary seventhic* . 408
American Journal of Mathematics, t. II. (1879), pp. 71—84

697. *On the double ϑ-functions* 422
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 74—81

698. *On a theorem relating to covariants* 430
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 82, 83

699. *On the triple ϑ-functions* 432
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 134—138

700. *On the tetrahedroid as a particular case of the* 16-*nodal quartic surface* 437
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 161—164

701. *Algorithm for the characteristics of the triple ϑ-functions* . 441
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 165—169

702. *On the triple ϑ-functions* 446
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 190—198

703. *On the addition of the double ϑ-functions* 455
Crelle's Journal der Mathem., t. LXXXVIII. (1880), pp. 74—81

704. *A memoir on the single and double theta-functions* . . . 463
Phil. Trans., vol. 171, Part III. (for 1880), pp. 897—1002

705. *Problems and Solutions* 566
Mathematical Questions with their Solutions from the Educational Times, vols. XIV. to LXI. (1871—1894); for contents, see p. 615

VOLUME XI.

PAGE

706. *On the distribution of electricity on two spherical surfaces* . 1
Phil. Mag., Ser. 5, t. v. (1878), pp. 54—60

707. *On the colouring of maps* 7
Geogr. Soc. Proc., t. I. (1879), pp. 259—261

708. *Note sur la théorie des courbes de l'espace* 9
Assoc. Franç., Compt. Rend., t. IX. (1880), pp. 135—139

709. *On the number of constants in the equation of the surface* $PS-QR=0$ 14
Tidsskrift for Mathematik, Ser. 4, t. IV. (1880), pp. 145—148

710. *On a differential equation* 17
Collectanea Mathematica, in memoriam Dominici Chelini, (Milan, Hoepli, 1881), pp. 17—26

711. *On a diagram connected with the transformation of elliptic functions* 26
British Association Report, 1881, p. 534

712. *A partial differential equation connected with the simplest case of Abel's theorem* 27
British Association Report, 1881, pp. 534, 535

713. *Addition to Mr Rowe's "Memoir on Abel's theorem"* . . 29
Phil. Trans., t. CLXXII. (1881), pp. 751—758

714. *Various notes* 37
Messenger of Mathematics, t. VII. (1878), pp. 69 : 115 : 124 : 125

715. *Note on a system of algebraical equations* 39
Messenger of Mathematics, t. VII. (1878), pp. 17, 18

716. *An illustration of the theory of the* ϑ*-functions* 41
Messenger of Mathematics, t. VII. (1878), pp. 27—32

717. *On the triple theta-functions* 47
Messenger of Mathematics, t. VII. (1878), pp. 48—50

PAGE

718. *Addition to Mr Genese's paper "On the theory of envelopes"* . 50
Messenger of Mathematics, t. VII. (1878), pp. 62, 63

719. *Suggestion of a mechanical integrator for the calculation of* $\int(Xdx+Ydy)$ *along an arbitrary path* 52
Messenger of Mathematics, t. VII. (1878), pp. 92—95; British Association Report, 1877, pp. 18—20

720. *Note on Arbogast's method of derivations* 55
Messenger of Mathematics, t. VII. (1878), p. 158

721. *Formulæ involving the seventh roots of unity* 56
Messenger of Mathematics, t. VII. (1878), pp. 177—182

722. *A problem in partitions* 61
Messenger of Mathematics, t. VII. (1878), pp. 187, 188

723. *Various notes* 63
Messenger of Mathematics, t. VIII. (1879), pp. 45, 46: 126: 127

724. *On the deformation of the model of a hyperboloid* . . . 66
Messenger of Mathematics, t. VIII. (1879), pp. 51, 52

725. *New formulæ for the integration of* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$ . . . 68
Messenger of Mathematics, t. VIII. (1879), pp. 60—62

726. *A formula by Gauss for the calculation of* $\log 2$ *and certain other logarithms* 70
Messenger of Mathematics, t. VIII. (1879), pp. 125, 126

727. *Equation of the wave-surface in elliptic coordinates* . . . 71
Messenger of Mathematics, t. VIII. (1879), pp. 190, 191

728. *A theorem in elliptic functions* 73
Proc. Lond. Math. Soc., t. X. (1879), pp. 43—48

729. *On a theorem relating to conformable figures* 78
Proc. Lond. Math. Soc., t. X. (1879), pp. 143—146

730. [*Addition to Mr Spottiswoode's paper "On the twenty-one coordinates of a conic in space"*] 82
Proc. Lond. Math. Soc., t. X. (1879), pp. 194—196

731. *On the binomial equation* $x^9-1=0$; *trisection and quartisection* 84
Proc. Lond. Math. Soc., t. XI. (1880), pp. 4—17

732. *A theorem in spherical trigonometry* 97
Proc. Lond. Math. Soc., t. XI. (1880), pp. 48—50

PAGE

733. *On a formula of elimination* 100
Proc. Lond. Math. Soc., t. XI. (1880), pp. 139—141

734. *On the kinematics of a plane* 103
Quart. Math. Journ., t. XVI. (1879), pp. 1—8

735. *Note on the theory of apsidal surfaces* 111
Quart. Math. Journ., t. XVI. (1879), pp. 109—112

736. *Application of the Newton-Fourier method to an imaginary root of an equation* 114
Quart. Math. Journ., t. XVI. (1879), pp. 179—185

737. *On a covariant formula* 122
Quart. Math. Journ., t. XVI. (1879), pp. 224—226

738. *Note on a hypergeometric series* 125
Quart. Math. Journ., t. XVI. (1879), pp. 268—270

739. *Note on the octahedron function* 128
Quart. Math. Journ., t. XVI. (1879), pp. 280, 281

740. *On certain algebraical identities* 130
Quart. Math. Journ., t. XVI. (1879), pp. 281, 282

741. *On a theorem of Abel's relating to a quintic equation* . . 132
Camb. Phil. Soc. Proc., t. III. (1880), pp. 155—159

742. *On the transformation of coordinates* 136
Camb. Phil. Soc. Proc., t. III. (1880), pp. 178—184

743. *On the Newton-Fourier problem* 143
Camb. Phil. Soc. Proc., t. III. (1880), pp. 231, 232

744. *Table of* $\Delta^m 0^n \div \Pi(m)$ *up to* $m=n=20$ 144
Camb. Phil. Trans., t. XIII. (1883), pp. 1—4

745. *On the Schwarzian derivative, and the polyhedral functions* . 148
Camb. Phil. Trans., t. XIII. (1883), pp. 5—68

*746. *Higher Plane Curves* 217
Salmon's Higher Plane Curves, (3rd ed., 1879), Preface

747. *Note on the degenerate forms of curves* 218
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 383—385

748. *On the bitangents of a quartic*. 221
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 387—389

*749. *Solid Geometry*. 224
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), Preface

PAGE

750. *On the theory of reciprocal surfaces* 225
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), pp. 539—550

751. *Note on Riemann's paper "Versuch einer allgemeinen Auffassung der Integration und Differentiation," Werke, pp.* 331—344 235
Mathematische Annalen, t. XVI. (1880), pp. 81, 82

752. *On the finite groups of linear transformations of a variable; with a correction* 237
Mathematische Annalen, t. XVI. (1880), pp. 260—263; 439, 440

753. *On a theorem relating to the multiple theta-functions* . . 242
Mathematische Annalen, t. XVII. (1880), pp. 115—122

754. *On the connection of certain formulæ in elliptic functions* . 250
Messenger of Mathematics, t. IX. (1880), pp. 23—25

755. *On the matrix* $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, *and in connection therewith the function* $\dfrac{ax+b}{cx+d}$ 252
Messenger of Mathematics, t. IX. (1880), pp. 104—109

756. *A geometrical construction relating to imaginary quantities* . 258
Messenger of Mathematics, t. X. (1881), pp. 1—3

757. *On a Smith's Prize question, relating to potentials* . . . 261
Messenger of Mathematics, t. XI. (1882), pp. 15—18

758. *Solution of a Senate-House problem.* 265
Messenger of Mathematics, t. XI. (1882), pp. 23—25

759. *Illustration of a theorem in the theory of equations* . . 268
Messenger of Mathematics, t. XI. (1882), pp. 111—113

760. *Reduction of* $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ *to elliptic integrals* 270
Messenger of Mathematics, t. XI. (1882), pp. 142, 143

761. *On the theorem of the finite number of the covariants of a binary quantic* 272
Quart. Math. Journ., t. XVII. (1881), pp. 137—147

762. *On Schubert's method for the contacts of a line with a surface* 281
Quart. Math. Journ., t. XVII. (1881), pp. 244—258

763. *On the theorems of the* 2, 4, 8, *and* 16 *squares* . . . 294
Quart. Math. Journ., t. XVII. (1881), pp. 258—276

PAGE

764. *The binomial equation $x^p - 1 = 0$; quinquisection* . . . 314
Proc. Lond. Math. Soc., t. XII. (1881), pp. 15, 16

765. *On the flexure and equilibrium of a skew surface* . . . 317
Proc. Lond. Math. Soc., t. XII. (1881), pp. 103—108

766. *On the geodesic curvature of a curve on a surface* . . . 323
Proc. Lond. Math. Soc., t. XII. (1881), pp. 110—117

767. *On the Gaussian theory of surfaces* 331
Proc. Lond. Math. Soc., t. XII. (1881), pp. 187—192

768. *Note on Landen's theorem* 337
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 47, 48

769. *On a formula relating to elliptic integrals of the third kind* . 340
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 175, 176

770. *On the 34 concomitants of the ternary cubic* 342
American Journal of Mathematics, t. IV. (1881), pp. 1—15

771. *Specimen of a literal table for binary quantics, otherwise a partition table* 357
American Journal of Mathematics, t. IV. (1881), pp. 248—255

772. *On the analytical forms called trees* 365
American Journal of Mathematics, t. IV. (1881), pp. 266—268

773. *On the 8-square imaginaries* 368
American Journal of Mathematics, t. IV. (1881), pp. 293—296

774. *Tables for the binary sextic* 372
American Journal of Mathematics, t. IV. (1881), pp. 379—384

775. *Tables of covariants of the binary sextic* 377
Written in 1894: now first published.

776. *On the Jacobian sextic equation* 389
Quart. Math. Journ., t. XVIII. (1882), pp. 52—65

777. *A solvable case of the quintic equation* 402
Quart. Math. Journ., t. XVIII. (1882), pp. 154—157

778. [*Addition to Mr Hudson's paper "On equal roots of equations"*] 405
Quart. Math. Journ., t. XVIII. (1882), pp. 226—229

779. [*Note on Mr Jeffery's paper "On certain quartic curves, which have a cusp at infinity, whereat the line at infinity is a tangent"*] 408
Proc. Lond. Math. Soc., t. XIV. (1883), p. 85

780. [*Addition to Mr Hammond's paper "Note on an exceptional case in which the fundamental postulate of Professor Sylvester's theory of tamisage fails"*] 409
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 88—91

PAGE

781. *On the automorphic transformation of the binary cubic function* 411
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 103—108

782. *On Monge's "Mémoire sur la théorie des déblais et des remblais"* 417
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 139—142

783. *On Mr Wilkinson's rectangular transformation* . . . 421
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 222—229

784. *Presidential Address to the British Association, Southport, September* 1883 429
British Association Report, 1883, pp. 3—37

785. *Curve* 460
Encyclopædia Britannica, 9th ed., t. VI. (1878), pp. 716—728

786. *Equation* 490
Encyclopædia Britannica, 9th ed., t. VIII. (1878), pp. 497—509

787. *Function* 522
Encyclopædia Britannica, 9th ed., t. IX. (1879), pp. 818—824

788. *Galois* 543
Encyclopædia Britannica, 9th ed., t. X. (1879), p. 48

789. *Gauss* 544
Encyclopædia Britannica, 9th ed., t. X. (1879), p. 116

790. *Geometry (analytical)* 546
Encyclopædia Britannica, 9th ed., t. X. (1879), pp. 408—420

791. *Landen* 583
Encyclopædia Britannica, 9th ed., t. XIV. (1882), p. 271

792. *Locus* 585
Encyclopædia Britannica, 9th ed., t. XIV. (1882), pp. 764, 765

793. *Monge* 586
Encyclopædia Britannica, 9th ed., t. XVI. (1883), pp. 738, 739

794. *Numbers (partition of)* 589
Encyclopædia Britannica, 9th ed., t. XVII. (1884), p. 614

795. *Numbers (theory of)* 592
Encyclopædia Britannica, 9th ed., t. XVII. (1884), pp. 614—624

796. *Series* 617
Encyclopædia Britannica, 9th ed., t. XXI. (1886), pp. 677—682

797. *Surface* 628
Encyclopædia Britannica, 9th ed., t. XXII. (1887), pp. 668—672

798. *Wallis (John)* 640
Encyclopædia Britannica, 9th ed., t. XXIV. (1888), pp. 331, 332

Portrait . . . . . *To face Title.*

VOLUME XII.

PAGE

799. *On curvilinear coordinates* 1
Quart. Math. Journ., t. XIX. (1883), pp. 1—22

800. *Note on the standard solutions of a system of linear equations* 19
Quart. Math. Journ., t. XIX. (1883), pp. 38—40

801. *On seminvariants* 22
Quart. Math. Journ., t. XIX. (1883), pp. 131—138

802. *Note on Captain MacMahon's paper "On the differential equation* $X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0$*"* 30
Quart. Math. Journ., t. XIX. (1883), pp. 182—184

803. *On Mr Anglin's formula for the successive powers of the root of an algebraical equation* 33
Quart. Math. Journ., t. XIX. (1883), pp. 223, 224

804. *On the elliptic-function solution of the equation* $x^3 + y^3 - 1 = 0$. 35
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 106—109

805. *Note on Abel's theorem* 38
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 119—122

806. *Determination of the order of a surface* 42
Messenger of Mathematics, t. XII. (1883), pp. 29—32

807. *A proof of Wilson's theorem* 45
Messenger of Mathematics, t. XII. (1883), p. 41

808. *Note on a form of the modular equation in the transformation of the third order* 46
Messenger of Mathematics, t. XII. (1883), pp. 173, 174

809. *Schröter's construction of the regular pentagon* 47
Messenger of Mathematics, t. XII. (1883), p. 177

810. *Note on a system of equations* 48
Messenger of Mathematics, t. XII. (1883), pp. 191, 192

PAGE

811. *On the linear transformation of the theta-functions* . . 50
Messenger of Mathematics, t. XIII. (1884), pp. 54—60

812. *On Archimedes' theorem for the surface of a cylinder* . . 56
Messenger of Mathematics, t. XIII. (1884), pp. 107, 108

813. [*Note on Mr Griffiths' paper "On a deduction from the elliptic-integral formula* $y = \sin(A + B + C + \ldots)$"] . . 58
Proc. Lond. Math. Soc., t. XV. (1884), p. 81

814. *On double algebra* 60
Proc. Lond. Math. Soc., t. XV. (1884), pp. 185—197

815. *The binomial equation* $x^p - 1 = 0$; *quinquisection. Second part* 72
Proc. Lond. Math. Soc., t. XVI. (1885), pp. 61—63

816. *On the bitangents of a plane quartic* 74
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 93—115; Camb. Phil. Soc. Proc., t. IV. (1883), p. 321

817. *On the sixteen-nodal quartic surface* 95
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 270—272

818. *Note on hyperelliptic integrals of the first order* . . . 98
Crelle's Journal der Mathem., t. XCVIII. (1885), pp. 95, 96

819. *On two cases of the quadric transformation between two planes* 100
Johns Hopkins University Circulars, No. 13 (1882), pp. 178, 179

820. *On a problem of analytical geometry* 102
Johns Hopkins University Circulars, No. 15 (1882), p. 209

821. *On the geometrical representation of an equation between two variables* 104
Johns Hopkins University Circulars, No. 15 (1882), p. 210

822. *On associative imaginaries* 105
Johns Hopkins University Circulars, No. 15 (1882), pp. 211, 212

823. *On the geometrical interpretation of certain formulæ in elliptic functions* 107
Johns Hopkins University Circulars, No. 17 (1882), p. 238

824. *Note on the formulæ of trigonometry* 108
Johns Hopkins University Circulars, No. 17 (1882), p. 241

825. *A memoir on the Abelian and Theta Functions* . . . 109
Chapters I to III, American Journal of Mathematics, t. V. (1882), pp. 137—179; Chapters IV to VII, *ib.*, t. VII. (1885), pp. 101—167

PAGE

826. *Note on a partition series* 217
American Journal of Mathematics, t. VI. (1884), pp. 63, 64

827. *On the non-Euclidian plane geometry* 220
Proc. Roy. Soc., t. XXXVII. (1884), pp. 82—102

828. *A memoir on seminvariants* 239
American Journal of Mathematics, t. VII. (1885), pp. 1—25

829. *Tables of the symmetric functions of the roots, to the degree* 10, *for the form* $1+bx+\frac{cx^2}{1.2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$. 263
American Journal of Mathematics, t. VII. (1885), pp. 47—56

830. *Non-unitary partition tables* 273
American Journal of Mathematics, t. VII. (1885), pp. 57, 58

831. *Seminvariant tables* 275
American Journal of Mathematics, t. VII. (1885), pp. 59—73

832. *Note on an apparent difficulty in the theory of curves, when the coordinates of a point are given as functions of a variable parameter* 290
Messenger of Mathematics, t. XIV. (1885), pp. 12—14

833. *On a formula in elliptic functions* 292
Messenger of Mathematics, t. XIV. (1885), pp. 21, 22

834. *On the addition of the elliptic functions* 294
Messenger of Mathematics, t. XIV. (1885), pp. 56—61

835. *On Cardan's solution of a cubic equation* 299
Messenger of Mathematics, t. XIV. (1885), pp. 96, 97

836. *On the quaternion equation* $qQ-Qq'=0$ 300
Messenger of Mathematics, t. XIV. (1885), pp. 108—112

837. *On the so-called D'Alembert Carnot geometrical paradox* . 305
Messenger of Mathematics, t. XIV. (1885), pp. 113, 114

838. *On the twisted cubics upon a quadric surface.* 307
Messenger of Mathematics, t. XIV. (1885), pp. 129—132

839. *On the matrical equation* $qQ-Qq'=0$ 311
Messenger of Mathematics, t. XIV. (1885), pp. 176—178

840. *On Mascheroni's geometry of the compass* 314
Messenger of Mathematics, t. XIV. (1885), pp. 179—181

841. *On a differential operator* 318
Messenger of Mathematics, t. XIV. (1885), pp. 190, 191

842. *On the value of* $\tan(\sin\theta)-\sin(\tan\theta)$ 319
Messenger of Mathematics, t. XIV. (1885), pp. 191, 192

PAGE

843. *On the quadri-quadric curve in connexion with the theory of elliptic functions* 321
Mathematische Annalen, t. XXV. (1885), pp. 152—156

844. *On a theorem relating to seminvariants* 326
Quart. Math. Journ., t. XX. (1885), pp. 212, 213

845. *On the orthomorphosis of the circle into the parabola* . . 328
Quart. Math. Journ., t. XX. (1885), pp. 213—220

846. *A verification in regard to the linear transformation of the theta-functions* 337
Quart. Math. Journ., t. XXI. (1886), pp. 77—84

847. *On the theory of seminvariants.* 344
Quart. Math. Journ., t. XXI. (1886), pp. 92—107

848. *On the transformation of the double theta-functions* . . 358
Quart. Math. Journ., t. XXI. (1886), pp. 142—178

849. *On the invariants of a linear differential equation* . . . 390
Quart. Math. Journ., t. XXI. (1886), pp. 257—261

850. *On linear differential equations* 394
Quart. Math. Journ., t. XXI. (1886), pp. 321—331

851. *On linear differential equations: the theory of decomposition* . 403
Quart. Math. Journ., t. XXI. (1886), pp. 331—335

852. *Note sur le mémoire de M. Picard "Sur les intégrales de différentielles totales algébriques de première espèce"* . . 408
Bull. des Sciences Math., 2me Sér., t. X. (1886), pp. 75—78

853. *Note on a formula for* $\Delta^n 0^i/n^i$ *when* n, i *are very large numbers* 412
Proc. Roy. Soc. Edin., t. XIV. (1887), pp. 149—153

854. *An algebraical transformation* 416
Messenger of Mathematics, t. XV. (1886), pp. 58, 59

855. *Solution of* $(a, b, c, d) = (a^2, b^2, c^2, d^2)$ 418
Messenger of Mathematics, t. XV. (1886), pp. 59—61

856. *Note on a cubic equation* 421
Messenger of Mathematics, t. XV. (1886), pp. 62—64

857. *Analytical geometrical note on the conic* 424
Messenger of Mathematics, t. XV. (1886), p. 192

858. *Comparison of the Weierstrassian and Jacobian elliptic functions* 425
Messenger of Mathematics, t. XVI. (1887), pp. 129—132

PAGE

859. *On the complex of lines which meet a unicursal quartic curve* 428
Proc. Lond. Math. Soc., t. XVII. (1886), pp. 232—238

860. *On Briot and Bouquet's theory of the differential equation* $F\left(u, \frac{du}{dz}\right)=0$ 432
Proc. Lond. Math. Soc., t. XVIII. (1887), pp. 314—324

861. *Note on a formula relating to the zero-value of a theta-function* 442
Crelle's Journal der Mathem., t. C. (1887), pp. 87, 88

862. *Note on the theory of linear differential equations* . . . 444
Crelle's Journal der Mathem., t. C. (1887), pp. 286—295

863. *Note on the theory of linear differential equations* . . . 453
Crelle's Journal der Mathem., t. CI. (1887), pp. 209—213

864. *On Rudio's inverse centro-surface* 457
Quart. Math. Journ., t. XXII. (1887), pp. 156—158

865. *On multiple algebra* 459
Quart. Math. Journ., t. XXII. (1887), pp. 270—308

866. *Note on Kiepert's L-equations, in the transformation of elliptic functions* 490
Mathematische Annalen, t. XXX. (1887), pp. 75—77

867. *Note on the Jacobian sextic equation* 493
Mathematische Annalen, t. XXX. (1887), pp. 78—84

868. *On the intersection of curves* 500
Mathematische Annalen, t. XXX. (1887), pp. 85—90

869. *On the transformation of elliptic functions* 505
American Journal of Mathematics, t. IX. (1887), pp. 193—224

870. *On the transformation of elliptic functions (sequel)* . . . 535
American Journal of Mathematics, t. X. (1888), pp. 71—93

871. *A case of complex multiplication with imaginary modulus arising out of the cubic transformation in elliptic functions* 556
Proc. Lond. Math. Soc., t. XIX. (1888), pp. 300, 301

*872. *On the finite number of the covariants of a binary quantic* . 558
Mathematische Annalen, t. XXXIV. (1889), pp. 319, 320

873. *System of equations for three circles which cut each other at given angles* 559
Messenger of Mathematics, t. XVII. (1888), pp. 18—21

874. *Note on the Legendrian coefficients of the second kind* . . 562
Messenger of Mathematics, t. XVII. (1888), pp. 21—23

PAGE

875. *On the system of three circles which cut each other at given angles and have their centres in a line* 564
Messenger of Mathematics, t. XVII. (1888), pp. 60—69

876. *On systems of rays* 571
Messenger of Mathematics, t. XVII. (1888), pp. 73—78

877. *Note on the two relations connecting the distances of four points on a circle* 576
Messenger of Mathematics, t. XVII. (1888), pp. 94, 95

878. *Note on the anharmonic ratio equation* 578
Messenger of Mathematics, t. XVII. (1888), pp. 95, 96

879. *Note on the differential equation* $\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0$ . . 580
Messenger of Mathematics, t. XVIII. (1889), p. 90

880. *Note on the relation between the distance of five points in space* 581
Messenger of Mathematics, t. XVIII. (1889), pp. 100—102

881. *On Hermite's H-product theorem* 584
Messenger of Mathematics, t. XVIII. (1889), pp. 104—107

882. *A correspondence of confocal Cartesians with the right lines of a hyperboloid* 587
Messenger of Mathematics, t. XVIII. (1889), pp. 128—130

883. *Analytical formulæ in regard to an octad of points* . . 590
Messenger of Mathematics, t. XVIII. (1889), pp. 149—152

884. *Note sur les surfaces minima et le théorème de Joachimsthal* . 594
Comptes Rendus, t. CVI. (1888), pp. 995, 996

885. *On the Diophantine relation,* $y^2+y'^2=$ *Square* 596
Proc. Lond. Math. Soc., t. XX. (1889), pp. 122—127

886. *On the surfaces with plane or spherical curves of curvature* . 601
American Journal of Mathematics, t. XI. (1889), pp. 71—98; pp. 293—306

887. *On the theory of groups* 639
American Journal of Mathematics, t. XI. (1889), pp. 139—157

VOLUME XIII.

PAGE

888. *On a form of quartic surface with twelve nodes* . . . 1
British Association Report, (1886), pp. 540, 541

889. *On a differential equation and the construction of Milner's lamp* 3
Edinburgh Math. Soc. Proc., t. v (1887), pp. 99—101

890. *Note on the hydrodynamical equations* 6
Proc. Roy. Soc. Edin., t. xv (1888), pp. 342—344

891. *On the binodal quartic and the graphical representation of the elliptic functions* 9
Camb. Phil. Soc. Trans., t. xiv (1889), pp. 484—494

892. *Note on the orthomorphic transformation of a circle into itself* 20
Edinburgh Math. Soc. Proc., t. viii (1890), pp. 91, 92

893. *The bitangents of the quintic* 21
Annals of Mathematics, t. v (1890), pp. 109, 110

894. *An investigation by Wallis of his expression for* π . . . 22
Quart. Math. Journ., t. xxiii (1889), pp. 165—169

895. *A theorem on trees* 26
Quart. Math. Journ., t. xxiii (1889), pp. 376—378

896. *A transformation in elliptic functions* 29
Quart. Math. Journ., t. xxiv (1890), pp. 259—262

897. *Sur les racines d'une équation algébrique* 33
Comptes Rendus, t. cx (1890), pp. 174—176, 215—218

898. *Sur l'équation modulaire pour la transformation de l'ordre* 11 38
Comptes Rendus, t. cxi (1890), pp. 447—449

899. *Sur les surfaces minima* 41
Comptes Rendus, t. cxi (1890), pp. 953, 954

900. *James Joseph Sylvester* 43
(Scientific Worthies in Nature, xxv); *Nature*, vol. xxxix (1889), pp. 217—219

PAGE

901. *Note on the sums of two series* 49
Messenger of Mathematics, t. XIX (1890), pp. 29—31

902. *On the focals of a quadric surface* 51
Messenger of Mathematics, t. XIX (1890), pp. 113—117

903. *On Latin squares* 55
Messenger of Mathematics, t. XIX (1890), pp. 135—137

904. *Note on reciprocal lines* 58
Messenger of Mathematics, t. XIX (1890), pp. 174, 175

905. *On the equation* $x^{17}-1=0$ 60
Messenger of Mathematics, t. XIX (1890), pp. 184—188

906. *Note on Schlaefli's modular equation for the cubic transformation; with a correction* 64
Messenger of Mathematics, t. XX (1891), pp. 59, 60; 120

907. *Note on the ninth roots of unity* 66
Messenger of Mathematics, t. XX (1891), p. 63

908. *On two invariants of a quadri-quadric function* 67
Messenger of Mathematics, t. XX (1891), pp. 68, 69

909. *On a particular case of Kummer's differential equation of the third order* 69
Messenger of Mathematics, t. XX (1891), pp. 75—79

910. *Note on the involutant of two matrices* 74
Messenger of Mathematics, t. XX (1891), pp. 136, 137

911. *On an algebraical identity relating to the six coordinates of a line* 76
Messenger of Mathematics, t. XX (1891), pp. 138—140

912. *On the notion of a plane curve of a given order* 79
Messenger of Mathematics, t. XX (1891), pp. 148—150

913. *On the epitrochoid* 81
Messenger of Mathematics, t. XX (1891), pp. 150—158

914. *On a soluble quintic equation* 88
American Journal of Mathematics, t. XIII (1891), pp. 53—58

915. *On the partitions of a polygon* 93
Proc. Lond. Math. Soc., t. XXII (1891), pp. 237—262

916. [*Note on a theorem in matrices*] 114
Proc. Lond. Math. Soc., t. XXII (1891), p. 458

917. [*Note on the theory of rational transformation*] 115
Proc. Lond. Math. Soc., t. XXII (1891), pp. 475, 476

PAGE

918. *On the substitution-groups for two, three, four, five, six, seven, and eight letters* 117
Quart. Math. Journ., t. XXV (1891), pp. 71—88, 137—155

919. *On the problem of tactions* 150
Quart. Math. Journ., t. XXV (1891), pp. 104—127

920. *On orthomorphosis* 170
Quart. Math. Journ., t. XXV (1891), pp. 203—226

921. *On some problems of orthomorphosis* 191
Crelle's Journal der Mathem., t. CVII (1891), pp. 262—277

922. *Note on the lunar theory* 206
Monthly Notices of the Royal Astronomical Society, t. LII (1892), pp. 2—5

923. *Note on a hyperdeterminant identity* 210
Messenger of Mathematics, t. XXI (1892), pp. 131, 132

924. *On the non-existence of a special group of points* 212
Messenger of Mathematics, t. XXI (1892), pp. 132, 133

925. *On Waring's formula for the sum of the mth powers of the roots of an equation* 213
Messenger of Mathematics, t. XXI (1892), pp. 133—137

926. *Corrected seminvariant tables for the weights* 11 *and* 12 . 217
American Journal of Mathematics, t. XIV (1892), pp. 195—200

927. *On Clifford's paper "On syzygetic relations among the powers of linear quantics"* 224
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 99—104

928. *On the analytical theory of the congruency* 228
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 185—188

929. *Note on the skew surfaces applicable upon a given skew surface* 231
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 217—225

930. *Sur la surface des ondes* 238
Annali di Matematica, Ser. II, t. XX (1892), pp. 1—18

931. *On some formulæ of Codazzi and Weingarten in relation to the application of surfaces to each other* 253
Proc. Lond. Math. Soc., t. XXIV (1893), pp. 210—223

932. *On symmetric functions and seminvariants* 265
American Journal of Mathematics, t. XV (1893), pp. 1—74

PAGE

933. *Tables of pure reciprocants to the weight* 8 333
American Journal of Mathematics, t. XV (1893), pp. 75—77

934. *Note on the so-called quotient G/H in the theory of groups* . 336
American Journal of Mathematics, t. XV (1893), pp. 387, 388

935. *Sur la fonction modulaire* $\chi\omega$ 338
Comptes Rendus, t. CXVI (1893), pp. 1339—1343

936. *Note on uniform convergence* 342
Proc. Roy. Soc. Edin., t. XIX (1893), pp. 203—207

937. *Note on the orthotomic curve of a system of lines in a plane* 346
Messenger of Mathematics, t. XXII (1893), pp. 45, 46

938. *On two cubic equations* 348
Messenger of Mathematics, t. XXII (1893), pp. 69—71

939. *On a case of the involution* $AF+BG+CH=0$, *where* A, B, C, F, G, H *are ternary quadrics* 350
Messenger of Mathematics, t. XXII (1893), pp. 182—186

940. *On the development of* $(1+n^2x)^{\frac{m}{n}}$ 354
Messenger of Mathematics, t. XXII (1893), pp. 186—190

941. *Note on the partial differential equation*
$$Rr+Ss+Tt+U(s^2-rt)-V=0 \quad . \quad . \quad . \quad . \quad 358$$
Quart. Math. Journ., t. XXVI (1893), pp. 1—5

942. *On seminvariants* 362
Quart. Math. Journ., t. XXVI (1893), pp. 66—69

943. *On reciprocants and differential invariants* 366
Quart. Math. Journ., t. XXVI (1893), pp. 169—194, 289—307

944. *On Pfaff-invariants* 405
Quart. Math. Journ., t. XXVI (1893), pp. 195—205

945. *Note on lacunary functions* 415
Quart. Math. Journ., t. XXVI (1893), pp. 279—281

946. *Note on the theory of orthomorphosis* 418
Quart. Math. Journ., t. XXVI (1893), pp. 282—288

947. *On a system of two tetrads of circles: and other systems of two tetrads* 425
Camb. Phil. Soc. Proc., t. VIII (1893), pp. 54—59

PAGE

948. *Report of a Committee appointed for the purpose of carrying on the tables connected with the Pellian equation from the point where the work was left by Degen in* 1817 . . 430
British Association Report, (1893), pp. 73—120

949. *On Halphen's characteristic n, in the theory of curves in space* 468
Crelle's Journal der Mathem., t. CXI (1893), pp. 347—352

950. *On the sextic resolvent equations of Jacobi and Kronecker* . 473
Crelle's Journal der Mathem., t. CXIII (1894), pp. 42—49

951. *Non-Euclidian geometry* 480
Camb. Phil. Soc. Trans., t. XV (1894), pp. 37—61

952. *On the kinematics of a plane, and in particular on three-bar motion: and on a curve-tracing mechanism* . . 505
Camb. Phil. Soc. Trans., t. XV (1894), pp. 391—402

953. *On the nine-points circle* 517
Messenger of Mathematics, t. XXIII (1894), pp. 23—25

954. *On the nine-points circle of a plane triangle* 520
Messenger of Mathematics, t. XXIII (1894), pp. 25—27

955. *The numerical value of* $\Pi(i), = \Gamma(1+i)$ 522
Messenger of Mathematics, t. XXIII (1894), pp. 36—38

956. *On Richelot's integral of the differential equation* $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$ 525
Messenger of Mathematics, t. XXIII (1894), pp. 42—47

957. *Illustrations of Sylow's theorems on groups* 530
Messenger of Mathematics, t. XXIII (1894), pp. 59—62

958. *On the surface of the order n which passes through a given cubic curve* 534
Messenger of Mathematics, t. XXIII (1894), pp. 79, 80

959. *Note on Plücker's equations* 536
Messenger of Mathematics, t. XXIV (1895), pp. 23, 24

960. *On the circle of curvature at any point of an ellipse* . . 537
Messenger of Mathematics, t. XXIV (1895), pp. 47, 48

961. *A trigonometrical identity* 538
Messenger of Mathematics, t. XXIV (1895), pp. 49—51

962. *Co-ordinates versus quaternions* 541
Proc. Roy. Soc. Edin., t. XX (1895), pp. 271—275

PAGE

963. *Note on Dr Muir's Paper "A problem of Sylvester's in Elimination"* 545
Proc. Roy. Soc. Edin., t. XX (1895), pp. 306—308

964. *On the nine-points circle of a spherical triangle* . . . 548
Quart. Math. Journ., t. XXVII (1895), pp. 35—39

965. *On the sixty icosahedral substitutions* 552
Quart. Math. Journ., t. XXVII (1895), pp. 236—242

966. *Note on a memoir in Smith's Collected Papers* . . . 558
Bulletin of the American Math. Soc., 2nd Ser., t. I (1895), pp. 94—96

*967. *An elementary treatise on elliptic functions* 560
First edition, 1876, second edition, 1895

INDEX

TO THE

THIRTEEN VOLUMES.

INDEX.

[*Volumes* are indicated by Roman numerals.
Pages are indicated by Arabic numerals.]

Abbildung: the term, VII, 248; theory, VII, 249—50; (*see also* Transformation of Surfaces).

Abel, N. H.: doubly infinite products, I, 120; inverse elliptic functions, I, 136, 156, 173; modular functions, I, 227; a functional equation, IV, 5—6; elliptic integrals, IV, 63, X, 139; quintic equations, V, 55—61, X, 11, XIII, 88; divergent series, VIII, 494; a theorem of, X, 57; theory of equations, XI, 132—5, 455, 513, 518; elliptic functions, XI, 452; series, XI, 627.

Abelian Functions: Riemann, VI, 2; $p=3$, X, 432—6; early history, XI, 453—4; connected with square roots of sextic and octic functions, XI, 483; the term, XI, 533—4; bitangents of plane quartic, XII, 74; hyperelliptic integrals of first order, XII, 98—9; memoir on theta functions and, XII, 109—216, (introductory, XII, 109; Abel's theorem, XII, 110—20; its proof, XII, 120—31; the major function, XII, 132—48, 149—56; miscellaneous investigations, XII, 157—96; nodal quartic, XII, 196—208; functions T, U, V, Θ, XII, 209—16).

Abelian Integrals: notes on, I, 366—9; and covariants, II, 189—91; Liouville, IV, 546; Riemann, V, 521; reduction to elliptic integrals, X, 214—22; deficiency of ground-curve, XI, 36; first kind of, XII, 38, 408—11; pure theorem for, XII, 110, 112—6, 119—20, 121, 129—30; affected theorem for, XII, 110, 116—7, 121, 130—1, 164—7.

Abel's Theorem: II, 45, 95, XI, 27—8, XII, 30; applied to porisms, IV, 297; Rowe's memoir on, XI, 29—36; note on, XII, 38—41; proof, XII, 120; semi-cubical parabola, XII, 180—6; quadri-quadric curve, XII, 186—9, 292—8; other curves, XII, 189—96.

Absolute: and theory of distance, II, 583—92, 604, V, 550; normals of a conic, IV, 74, 77; theory of evolute, V, 476—9; effect on locus in relation to triangle, VI, 53—64; Cayley's theory of, VIII, xxxvi—vii; evolutes and parallel curves, VIII, 31—44; centro-surface of ellipsoid, VIII, 316, 320; in hypergeometry, VIII, 409—13, XIII, 481—504; the term, XIII, 42; minimal surfaces, XIII, 42.

Abstract Geometry: memoir on, VI, 456—69, VIII, xxxiii, XI, 441—2.

Acceleration: secular (*see* Secular Acceleration).

Acnodal: defined, V, 403, 551, XI, 228.

Acnode: defined, IV, 181, V, 295, 521, VI, 585, XI, 630.

Adams, J. C.: Malfatti's problem, I, 468; moon's mean motion, III, 522, 533—40, 568; lunar theory, VII, 372, VIII, xliii—iv; attraction of ellipsoidal shell, IX, 302; solar eclipses, X, 315.

Addition: of elliptic functions, I, 540—9, 589, XI, 73—7, 454, 530, XII, 294—8; of double theta functions, X, 455—62.

Address: presidential to British Association, XI, 429—59.

Adjoint Curve: IX, 504—7.

Adjoint Linear Form: in quartics, II, 319.

Aggregate: and relation in abstract geometry, VI, 459.
Air: effect on pendulum, IV, 541.
Airy, Sir G. B.: a trigonometrical theorem, IV, 80; roots in algebraic equations, IV, 116, IX, 39.
Algebra: non-commutative, I, 128—31, 301; notion and boundaries, V, 292—4, 620; Cayley founder of modern, VIII, XXX; geometrical illustration of theorems in, IX, 16—7, 21—39; operation connected with covariants, IX, 537—42; expansion theorem, X, 57; system of equations, XI, 39—40; identities, XI, 63—4, 130—1; and time, XI, 443; origin, XI, 445—8; in Greece, XI, 446; and logic, XI, 459, XII, 459; algebraical equations, XI, 506—21; function in, XI, 523—4; Sylvester's work, XIII, 46; Sylvester's principles of universal, XIII, 47; Sylvester on art and, XIII, 48; (*see also* Multiple Algebra).
Algebraic Curves (*see* Curves).
Algebraic Equations (*see* Equations).
Algebraic Theorems: X, 594, 602, 609.
Algebras, Non-commutative: I, 128—31, 301.
Algorithm: for characteristics of triple theta functions, X, 441—5, 452.
Allink: the term, V, 521.
Allotrious: the term, IX, 204.
Alpine Club: Cayley a member, VIII, xi.
Alternant: of operators, XIII, 400—1.
Altitude: of trees, IX, 429—60.
America: Cayley's visit to, VIII, xxi.
American Mathematical Journal: Sylvester's contributions to, XIII, 47.
Ampère, A. M.: inertia, IV, 563, 584; reciprocants, XIII, 366.
Amphigram: the term, VII, 268.
Ampullate: the term, VI, 101.
Anallagmatic: the term, VII, 246.
Analogues: of Pascal's theorem, I, 426, 427.
Analysis: Bernoulli's numbers in, IX, 259—62; (*see also* Combinatory Analysis).
Analytical Geometry (*see* Geometry, analytical).
Analytical Representation of Curves (*see* Representation).
Analytical Theorem: as to Euler's equation, VII, 261—2.
Analytical Theory of conics (*see* Conics).
Anchor Ring: and cyclide, IX, 18.
Angle: interpreted with reference to two points, VI, 497.
Anglin, A. H.: roots of algebraic equations, XII, 33—4.
Angular Distance (*see* Distance).
Anharmonic Ratio: theory of binary quantics, II, 565; sextic equation, VII, 314—5, XII, 578—9.
Anomaly: expansion of the true, III, 139—43; and elliptic motion, IV, 521, 523.
Anti-circle: the term, VIII, 262.
Anti-conic: the term, IX, 65.
Anti-foci: and foci, VII, 567.
Anti-point: the term, VI, 499—500, IX, 65, XIII, 11; problem and solution, VII, 593; orthomorphosis XIII, 184.
Aoust, L.: geodesic curvature, XI, 330.
Aplati (*see* Penultimate Forms).
Apoclastic: the term, XII, 226.
Applicable Surfaces (*see* Skew Surfaces, Surfaces).
Apollonius: tactions, XIII, 152.
Apsidal Surfaces: theory of, XI, 111—3.

Arbelon: the term, XII, 57.
Arbitrary Constants: in mechanical problems, III, 161—5, 200.
Arbogast, L.: method of derivations, II, 257, IV, 265—71, 272—5, 609, XI, 55, 358—60; rule of, VIII, 471, XIII, 267.
Archimedes: and statics, XI, 446; theorem for surface of cylinder, XII, 56—7.
Argand, R.: imaginaries in plane geometry, XII, 460, 468.
Arithmetic: in Greece, XI, 446; Gauss's work, XI, 544.
Arndt, F.: mathematical tables, VIII, 51—5, IX, 490—1.
Aronhold, S. H.: ternary cubic, II, 325, III, 48, IV, 325—30; on lambdaic of binary quartics, II, 550; hyperdeterminants, II, 598—601; incunts, IV, 419; intersection of line and conic, V, 501—4; bitangents of quartic curve, VII, 125; construction of a conic, VII, 593; integration formula, X, 12—4; concomitants of ternary cubic, XI, 342; Abelian function, XII, 157; quadric integral, XII, 162—4, 164—7; cubic transformation, XII, 173—9.
Arrangements: theory of permutations, I, 423—4; triads of seven and fifteen things, I, 481—4, 589, V, 95—7; of numbers, X, 570; latin squares, XIII, 55—7; (*see also* Combinatory Analysis, Permutations, Substitutions).
Art: Cayley's love and practice of, VIII, xxiv; and algebra, XIII, 48.
Askwith, E. H.: substitutions, XIII, 117, 133, 141, 145.
Associative: the term, XII, 461.
Associative Algebra: XII, 303.
Associative Imaginaries: XII, 105—6.
Asteroids: Newcomb on orbits, IX, 176—7.
Astronomy: Cayley's work, VIII, xliii; origin of, XI, 446—7; Gauss's work, XI, 545; transformation of coordinates, XI, 575; Sylvester's work, XIII, 47.
Asymptotes: of algebraic curves, I, 46.
Asymptotic: the term, XIII, 232.
Asyzygetic: covariants and invariants, II, 250; the term, VI, 460—1, VII, 336.
Atomic Theory: Sylvester's work in, XIII, 47.
Attraction and Multiple Integrals: I, 5—12, 13—8, 195—203, 204—6, 285—9, 362—3, 508—10, 586; and theorem of Boole, I, 384—7, 588; and theorem of Jellett, I, 388—91.
Attraction of Ellipsoids: I, 432—4, 511—18, VII, 380—3; Gauss's method, III, 25—8, 149—53; Laplace's, III, 53—65, 567; Rodrigues', III, 149—53.
Attractions: theory, II, 35—9, III, 154—5, XI, 448; of terminated straight line, VII, 31—3; of ellipsoidal shell on exterior point, IX, 302—11; of lens-shaped body, X, 594.
Augmented Equation: XII, 453.
Ausdehnungslehre: Grassmann, XII, 480—9.
Automorphic Transformation: IV, 416, V, 439; of binary cubic function, XI, 411—6.
Autopolar Polyhedra: IV, 185.
Auxesis: XI, 79, 81.
Auxiliars: the term, VI, 156; application to quintics, VI, 186—7.
Auxiliary Equations (*see* Equations).
Axial Systems: of polyhedra, V, 529—39.
Axiom: twelfth of Euclid, XI, 435, XII, 220.
Axis: of inertia, IV, 559—66; cubic surfaces and kinds of, VI, 367; transformation of equation of surface to chief axes, IX, 48—51; radical, XI, 465.
Axonometry: IX, 508—18.

Babbage, C.: homographic function, II, 494; matrices, XI, 252.
Babinet, J.: representation of hemisphere, VIII, 539.

Bacharach, J.: intersection of curves, I, 583, XII, 500.
Baehr, G. F. W.: relative motion, IV, 535, 584.
Ball, Sir R. S.: theory of content, II, 606; roots of quartic, V, 610, VII, 551; dynamical equations, IX, 198—200; non-Euclidian geometry, XIII, 481.
Baltzer, H. R.: book on determinants, IV, 608.
Barriers: the term, X, 320.
Barycentric Calculus: Möbius, XII, 472—3.
Bellavitis, J. L.: equipollences, XII, 473—4.
Beltrami, E.: non-Euclidian plane geometry, XII, 221, 224; orthomorphosis, XIII, 171.
Bernoulli, James: analysis and numbers of, IX, 259—62.
Bertrand, J.: geodesic lines, III, 38; differential equations, III, 164, 189, 203; integrals and mechanical problems, III, 187, 200—3; *Mécanique Analytique*, III, 189—90, 203; wave surface, IV, 432—4; central forces problem, IV, 519—21, 584; problem of two centres, IV, 532, 584; motion of point, IV, 547, 584; problem of three bodies, IV, 547—52, 584, V, 23; curves of curvature on surfaces, VIII, 98; series, XI, 623, 627.
Bessel, F. W.: elliptic motion, IV, 522, 584; precession and nutation, IX, 194—6.
Bezout, E.: determinants, I, 63; elimination, IV, 38—9, V, 159, 555—6; equation of differences, IV, 151, 259.
Bezoutiant: defined, II, 526.
Bezoutic Emanant: defined, II, 525.
Bezoutic Matrix: defined, IV, 607.
Bezoutoid: defined, II, 526.
Biaxial: defined, XIII, 13.
Bibasic: defined, XII, 642.
Bibliography: of covariants and invariants, II, 598—601; of symmetric functions, II, 602—3.
Bicentre: the term, IX, 429—60, X, 599.
Bicircular Quartic, on the: X, 223—42; introductory, X, 223—4; formulæ for fourfold generation, X, 224—6; determination as to reality, X, 226; investigation of elementary *arc* formulæ, X, 226—31; inscribed quadrilateral, X, 231—5.
Bicircular Quartics: and polyzomal curves, VI, 472; foci of, VI, 521—2, 522—3; analytical theory, VI, 528—30; problem and solution, VII, 575, X, 596—8; geometrical construction, IX, 13—5; generation, XIII, 12; transformation of circle, XIII, 185; (*see also* Quartics).
Bicircular Quartic Surfaces: VII, 67, 246; quadric surface inversion, VII, 67.
Bickmore, C. E.: Pellian equation, XIII, 442.
Bicolumn: the term, IX, 28.
Bicorn: the term, VI, 148, 158; equation, VI, 163; form, VI, 164.
Bicursal Curves: VIII, 181—7.
Bicursal Sextic: and binodal quartic, IX, 581.
Bien déterminée: defined, XII, 433.
Bin: the abbreviation for tortuous curves, XIII, 253.
Binary: the term, IV, 604, VI, 466.
Binary Cubics: covariants of, II, 189—91; fifth memoir on quantics, II, 540—5; geometrical representation of covariants of, VII, 332—3; tables of forms, VIII, 51—64; transformation of function, XI, 411—6.
Binary Forms: canonic root, V, 103—5.
Binary Matrices (*see* Matrices).
Binary Quadratic Forms: tables of, V, 141—56, 618.
Binary Quadrics: covariants of, II, 189—91; single, II, 527—9; theories of harmonic relation and of involution, II, 529—40; asyzygetic covariants of, VII, 337—8.

Binary Quantics: canonical form, IV, 43—52, 53; canonical root, V, 103—5; involution, V, 296—301; transformations, VI, 187—90; discriminants of, VII, 303; and covariants, VIII, 566—7, X, 430—1, XI, 272—8; geometrical interpretation, IX, 16—7; factions, IX, 426; derivatives of three, X, 278—86; literal table for, XI, 357—64; finite number of covariants of, XII, 558.

Binary Quartics: covariants of, II, 189—91; fifth memoir on quantics, II, 545—56; theorem of four, VII, 100; generating function of, X, 341.

Binary Quintics: tables of covariants M to W of, II, 282—309; covariants and syzygies of degree 6, VI, 148—53; and sextic, VI, 190; irreducible covariants, VII, 334; tables, VII, 341—6; asyzygetic covariants of, VII, 399—400; $(*)(x, y)^5$, X, 339—400; theorem relating to covariants, X, 430—1; concomitant system for, XI, 272.

Binary Septic: minimum N.G.F., X, 408—21.

Binary Sextic: and quintic, VI, 190; and squared cubic and cubed quadric, XI, 105; concomitant system for, XI, 272; tables for, XI, 372—6, 377—88.

Binet, J.: determinants, I, 63, 64, 581; variation of arbitrary constants, III, 181; relative motion, IV, 535, 584; inertia, IV, 562—3, 584.

Binodal Quartic: transformed to bicursal sextic, IX, 581; graphical representation of elliptic functions, XIII, 9—19; (*see also* Quartics).

Binode: the term, VI, 330, 335, 361, 362, 583—5.

Binomial Equation: theory of numbers, XI, 84—96.

Binomial Theorem: factorials and derivations, II, 101, 102, VIII, 463—73.

Binormal: the term, XIII, 253.

Biography: of Cayley, VIII, ix—xliv.

Bipartite: the term, VI, 464.

Biplanar-node: the term, VI, 360.

Biplanes: the term, VI, 360—1, 362.

Bipoint-locus: the term, VI, 198.

Biquaternions: Clifford, XI, 458, XIII, 481.

Bisection: theory of numbers, XI, 84—96.

Bitangents: of a plane curve, IV, 186—206; of quartic curve, IV, 342—8, VII, 123—4, X, 244, XI, 221—3, XII, 74—94; of quartic, and triple theta functions, X, 444, 446; of curves, XI, 473—4, 480; of quintic, XIII, 21.

Bitetrad: the term, XIII, 551.

Bitrope: the term, VI, 330, 335, 583—5, 591, XI, 228.

Björling, E. G.: root limitation, IX, 39; difference table, XI, 144.

Blissard, J.: factorials, V, 574.

Blunt: the term in seminvariants, XIII, 265, 287, 291, 304—6.

Bodies, Problem of Three: III, 183; Hamilton's method, III, 97—103; Jacobi's theorem, III, 519—21, V, 23.

Body (*see* Solid Body).

Bolyai, J.: hyperbolic geometry, VIII, xxxvii.

Bondset: the term, XII, 642.

Bonnet, Ossian: problem of two centres, IV, 528, 585; imaginary coordinates, VIII, 569; minimal surfaces, XII, 595; curves of curvature, XII, 601, 629—31; skew surfaces, XIII, 231, 237.

Book-keeping: pamphlet by Cayley, VIII, xxiv.

Boole, G.: linear transformations, I, 80, 92, 93—4, 428, 584; multiple integrals, I, 198, 384—7, 588; attractions, I, 285, 289; theorem of Jellett, I, 388; transformation of elliptic integral, I, 508—10; discriminants, I, 584; theory of permutants, II, 26; theory of attractions, II, 35; probabilities, II, 103, 594—8, V, 80—4, 85; hyperdeterminants, II, 598—601; transformation, III, 129; formulæ for differentiation, IV, 135; differential equations, IV, 515, V, 301; involution, V, 301; invariants, VIII, xxx; calculus of logic, VIII, 65—6; integration of differential equations by series, VIII, 458;

prepotentials, IX, 417—23; difference table, XI, 144; matrices, XI, 256; mathematics and logic, XI, 458—9; series, XI, 627.

Booth, J.: rotation of solid body, IV, 577, 585.

Borchardt, C. W.: symmetric functions, II, 417, 421—3; 16-nodal quartic surface, X, 180; theta functions, X, 464, 499; theorem on trees, XIII, 28.

Bordered Skew Determinants: II, 203.

Boron: trees of, IX, 429—60.

Boundaries: of Algebra, V, 292—4, 620.

Bouquet, J. C.: orthogonal surfaces, VIII, 279, 292; periods of elliptic integrals, IX, 618; conformal representation, XI, 80; imaginary variables, XI, 440; elliptic functions, XI, 534; differential equations of first order, XII, 432—41; functions of complex variables, XIII, 190.

Bour, E.: differential equations, III, 164, 197—8, 204, IV, 515, 585; problem of three bodies, IV, 550—2, 585; Gaussian theory of surfaces, XI, 331—6.

Branch: the term, III, 243, X, 36, XI, 476—7; partial, of a curve, V, 425; of polyzomal curves, VI, 474—6, 481—5; main, of trees, IX, 429—60; theorem on trees, XIII, 26—8.

Branch-points: the term, X, 317.

Brassinne, E.: differential equations, III, 186—7, 203.

Bravais, A.: spherical pendulum, IV, 533, 585; on polyhedra, V, 537, 559.

Bretschneider, C. A.: mathematical tables, IX, 486.

Brianchon, C. J.: theorem of, I, 328, V, 4; rectangular hyperbola, III, 254; reciprocal polars, XI, 465.

Briggs, H.: logarithms, XI, 447.

Brill, A.: transformation of plane curves, VI, 593; transformation, and theory of invariants, VIII, 387; sextic curve, IX, 504—7.

Brinkley, J.: formulæ of, IV, 262.

Brioschi, F.: *Sulla variazione* and *Teorema di Mechanica*, III, 190, 203; *degli integrali di un problema di dinamica*, III, 199—200, 203; peninvariants, IV, 246; text-book on determinants, IV, 608; Schwarzian derivative and polyhedral functions, XI, 149, 151; concomitants of ternary cubic, XI, 342; Jacobian sextic equation, XI, 401; theory of equations, XI, 520; elliptic functions and quintic, XII, 493; transformation of elliptic functions, XII, 505—34, 554—5; wave surfaces, XIII, 238; sextic resolvent equations, XIII, 473, 478, 479.

Briot, A.: rotation of solid body, IV, 574, 585; periods of elliptic integral, IX, 618; conformal representation, XI, 80; imaginary variables, XI, 440; elliptic functions, XI, 534; differential equations of first order, XII, 432—41; functions of complex variables, XIII, 190.

British Association: report on Catalogue of Philosophic Memoirs, V, 546—8, 620; communications to, V, 549—53; Cayley president 1883, VIII, xxi—xxii; communications on in-and-circumscribed triangle, VIII, 565—6; correspondence of points and lines in space, VIII, 566; covariants of binary quantic, VIII, 566—7; families of surfaces, VIII, 567; Mercator's projection of surface of revolution, VIII, 567; report on mathematical tables, IX, 461—99; Presidential address, XI, 429—59.

Bronwin, B.: elliptic functions, I, 118, 119, 127, 132.

Buchheim, A.: parallels of, XIII, 481, 489.

Buée, A. Q.: multiple algebra, XII, 467, 471.

Burg, A.: algebraical theorem, X, 57.

Burnside, W. S.: cubic identity, V, 597.

Cagnoli, A.: spherical trigonometry, IV, 80.

Calculus: of logic, VIII, 65—6; and time, XI, 443—4; of functions, XI, 541—2; of forms; (*see also* Covariants, Invariants, Quantics).

Calculus of Variations: Jacobi on, III, 174; problem in, VII, 263.

Cambridge Mathematical Journal: foundation, VIII, xii.

Campaniform: the term, VI, 101, XI, 479.

Canonical Elements: for disturbed motion, III, 77.

Canonical Forms: of quantics, II, 523, 542, 548; of binary quantics, IV, 43—52, 53; the term, IV, 606—7, XIII, 46; quintics, VI, 153—4, X, 355—62, XII, 27; cubic surfaces, VI, 361; Sylvester's work, XIII, 46; (*see also* Formulæ).

Canonisant: defined, II, 523, IV, 45, 53, 606, 607, V, 103—5.

Capacity: of a curve, I, 583; the term, VII, 140; and postulandum, XIII, 115.

Capitation: and seminvariants, XII, 248—50.

Carbon: trees of, IX, 427—60, 544—5.

Cardan, J.: solution of cubic, V, 54, XII, 299.

Cards: game of mousetrap, III, 8, X, 256—8; (*see also* Permutations).

Carey, F. S.: theory of numbers, XII, 73.

Carnot, L.: distances of points, I, 64, 581; on two tetrahedra in perspective, IX, 209—10; geometrical paradox, XII, 305—6.

Cartesians: cusp, I, 589; oval of Descartes, II, 336, 365—6, 370; equation of, II, 370—4, 376; defined, VI, 524, XI, 481; with two imaginary axial foci, VII, 241—3; and cubic curve, VII, 556; problem and solution, VII, 582; note on, IX, 45—7; construction, IX, 317, 535—6, X, 261; invention of coordinates, XI, 449; explained, XI, 461; scalene transformation, IX, 532—4; correspondence of confocal, with right lines of hyperboloid, XII, 587—9.

Casey, J.: equation of, and tangent circles, VI, 65—71; polyzomal curves, VI, 471—2; bicircular quartics, VI, 540, X, 223—42, XIII, 12; tactions, VI, 543, VII, 115, XIII, 153; circle touching three others, VI, 568—73; cyclides and sphero-quartics, VIII, 262—3; cyclide, IX, 64, 75.

Cassinian: note on, IX, 264—5; the word, X, 608; equation of, X, 608.

Castillon, J. F. S. de: problem and porism, IV, 435—41.

Catalan, E.: attractions, I, 288; integral calculus, I, 313; series, III, 127, XI, 627; curve parallel to ellipse, IV, 152; minimal surfaces, XII, 594—5.

Catalecticant: the term, II, 523, IV, 606, 607.

Catalogue of Scientific Papers: issued by Royal Society, V, 546—8, 620.

Catenary: and epitrochoid, XIII, 87.

Cauchy, A. L.: determinants, I, 63, 64, 66; inverse elliptic functions, I, 148, 157, 174; integral calculus, I, 309, 314; permutations, I, 423; partitions, II, 248, V, 48, VII, 577; differential equations, III, 166, 169; theory of logarithms, III, 208—13, 225; polyhedra, IV, 86—7; singularity of function, IV, 105; curve parallel to ellipse, IV, 152; equations of motion, IV, 514, 585; elliptic motion, IV, 524, 585; inertia, IV, 560, 564, 565, 585; geodesic lines on ellipsoid, VII, 498; geometrical representation of root-limitation, IX, 21—39; series, IX, 61, XI, 627; mathematical tables, IX, 475; theory of equations, X, 6, XI, 504; correspondence, X, 290; monogenous function, XI, 537; theory of numbers, XI, 601; roots of algebraic equations, XIII, 35.

Caustics: by reflection at circles, I, 273—5, II, 118—22, 129; memoirs on, II, 336—80, V, 454—65; tracing of, II, 351—2, 362—3, 368—9, 374.

Cavalieri, B.: quadrature of curves, XI, 641.

Cayley, Arthur: portraits of, frontispiece to vols. VI, VII, XI; manuscript of, frontispiece to vol. VIII; biography of, VIII, ix—xliv; mathematical tables, IX, 479, 487, 491.

Cayleyan: the word, I, 586, II, 381.

Central Forces Problem: IV, 517—21.

Centres: problem of two, III, 104—10, IV, 527, 528—9, 589; cubic, and cones, IV, 173—8, 179—81; of curves defined, VI, 522; four points and conic of, VII, 582; of trees, IX, 429—60; the term, X, 599; of three circles, XI, 552.

Centro-curve: kinds of, VIII, 320.

Centro-surface: of ellipsoid, VIII, 8; the term, VIII, 316; Rudio's inverse, XII, 457—8.

Cesser: points of, IV, 130.
Challis, J.: integration of differential equations, VII, 36.
Chance (*see* Probability).
Characteristic Function: of Hamilton, III, 217; for systems of rays, XII, 571.
Characteristics: logic of, III, 51—2; of Chasles, V, 552; theory, VI, 594, XIII, 468—72; of triple theta functions, X, 441—5.
Chartography: surface representation on plane, VIII, 538—9; colouring of maps, XI, 7—8; map projections, XI, 448.
Chasles, M.: intersections of curves, I, 25—7; Pascal's theorem, I, 45; theorem on correspondence, I, 212; a theorem of, demonstrated, I, 355; analogue of Pascal's theorem, I, 427; transformation of curves, I, 478—80; homography, II, 538; cubic curves, IV, 122, 495; inertia, IV, 561, 586; kinematics of solid body, IV, 580, 586; curves on a quadric, V, 11; on a cubic, V, 19; conics touching curves, V, 31—2, 552; scrolls, V, 169, 201, VI, 328; quartic scrolls, V, 201; cubic curves and cones, V, 401; equilibrium of four forces, V, 540—1; correspondence of points in plane curve, V, 542; contact of conics, V, 552; characteristics, V, 552; on united points, VI, 9; curves which satisfy given conditions, VI, 191, 192, 200—26; principle of correspondence, VI, 264, XI, 482, 485—8; foci of conics, VII, 1; six coordinates of a line, VII, 93; attraction of ellipsoids, VII, 380—3; locus *in plano*, VII, 605; cones satisfying six conditions, VIII, 99; penultimate forms of curves, VIII, 258; theory of duality, XI, 467.
Chemistry: Cayley's interest in, VIII, x; application of trees to, IX, 202—4, 427—60, 544—5.
Chessboard: topology of, X, 609—10.
Chord: angle between normal and bisector, X, 576; of two circles, XI, 552—6.
Christie, J. T.: Cayley's law work, VIII, xiii—iv.
Christoffel, E. B.: orthomorphosis, XIII, 180.
Chrystal, G.: uniform convergence, XIII, 343—4.
Chuck: for quartic curves, VIII, 151—5; for curve-tracing, VIII, 179—80; bicyclic, VIII, 209—11.
Circle: Salmon's equation for orthotomic, III, 48—50; and points, V, 560; and ellipse, V, 561; line and parabola, V, 607; envelope of, V, 610; equation of, VI, 501, XI, 558—61; potential of, IX, 290—301; quadrilateral inscribable in, X, 578; orthomorphosis, XII, 328—36, XIII, 20, 182, 202—5; Wallis's π investigation, XIII, 22—5; transformation into bicircular quartic, XIII, 185; and circumference, the terms, XIII, 194; the nine point, XIII, 517—9, 520—1, 548—51; of curvature of an ellipse, XIII, 537.
Circles: powers of, I, 581; systems of, III, 111—4, X, 566; in-and-circumscribed polygon, IV, 303—8; a pair touching three given, VI, 65—71; involution of four, VI, 505—8; relation between two, VIII, 12—3; equal, VIII, 31; minimum enclosing three points, X, 576; system of 15 connected with icosahedron, XI, 208—12; radical axis, XI, 465; radical centre of three, XI, 552; Mascheroni's geometry of the compass, XII, 314—7; system of three which cut each other at given angles, XII, 559—61, 564—70; the two relations connecting the distances of four points on a circle, XII, 576—7; roots of algebraic equation, XIII, 37; problem of tactions, XIII, 150—69; tetrads of, XIII, 425—9; (*see also* Casey, Orthomorphosis).
Circuit: the word, XI, 480.
Circular: the word, XI, 481.
Circular Cubic: and polyzomal curves, VI, 522—8.
Circular Points: at infinity, VIII, 32.
Circular Relation of Möbius: III, 118—9, IX, 612—7.
Circumference: and circle, the terms, XIII, 194.
Cissoid: the term, XI, 461.
Clairaut, A. C.: lunar theory, IV, 518, 586; demonstration of his theorem, X, 17—8; curves of double curvature, XI, 489.

Classes: of curves, II, 569—83, V, 518, 521, VI, 11, XI, 629.

Classification: of curves, V, 613—7; cubics and quartics, VI, 525; quartic surfaces, VII, 244; nodal quartic surfaces, VII, 264—7; mathematical tables, IX, 424—5; cubic curves, XI, 478—80; quartic curves, XI, 480.

Clausen, T.: Castillon's problem, IV, 435—41.

Clebsch, R. F. A.: Steiner's extension of Malfatti's problem, II, 593; Pfaffian equations, IV, 515; singularities of curves, V, 476—7; developable surfaces, V, 518; Abelian integrals, V, 521; transformation of plane curves, VI, 1—8, 593; Casey's equation, VI, 67; binary quintic and sextic, VI, 190; principle of correspondence, VI, 264; reciprocal surfaces, VI, 356; cubic curve in pencil of six lines, VI, 593—4; centro-surface of ellipsoid, VII, 130, VIII, 316; quartic and quintic surfaces, VII, 245—51, 313; bicursal curves, VIII, 182; transformation of unicursal surfaces, VIII, 389; deficiency of surfaces, VIII, 394; covariants, VIII, 404; expression for elliptic integral of second kind, X, 27; concomitants of ternary cubic, XI, 343; tables for binary sextic, XI, 372; Abelian functions, XII, 102, 109; symbolic notation for quantics, XII, 347; seminvariants, XIII, 362; Pfaff-invariants, XIII, 405—14.

Clifford, W. K.: powers of circles and spheres, I, 581; circle and ellipse, V, 561; conic, V, 562; triangles and polygons, V, 589; rational transformation between two spaces, VII, 222—4; quartic surfaces, VII, 246; compound combinations, X, 243; biquaternions, XI, 458; syzygetic relations among the powers of linear quantics, XIII, 224—7; non-Euclidian geometry, XIII, 481.

Close: partitions of a, V, 62—5, 617; defined, V, 63.

Close-planes: the term, VI, 330, 583—5.

Close-points: the term, VI, 330, 339, 341, 583—5.

Cnicnode: the word, VI, 330, 334, 361, 583—5, XI, 228, 631, 633; quartic and quintic surfaces, VII, 245—51.

Cnictrope: the word, VI, 330, 334, 583—5, 591, XI, 228, 631, 633.

Cobezoutiants: defined, II, 524.

Cobezoutoids: defined, II, 524—5.

Cockle, Sir James: resolvent product, IV, 309—13; on quintics, V, 53; theory of equations, XI, 520; invariants, XII, 390—1; criticoids, XIII, 366, 367—8.

Codazzi, D.: application of surfaces, XIII, 253—64.

Coefficients: of Legendre, I, 375—6; development of $(1+n^2x)^{m/n}$, XIII, 354—7.

Cogredient: the term, IV, 607—8, XIII, 46.

Cohen, A.: relative motion, IV, 537, 586; rotation of solid body, IV, 573, 586.

Cole, F. W.: on simple groups, XIII, 533.

Colink: the term, V, 521.

Collins, M.: Lagrange's theorem, II, 3; systems of circles, X, 566—7.

Colour Groups: diagrams representing, X, 328—330, 403—5; the term, XII, 639—41.

Colours: in maps, XI, 7—8.

Colourset: the term, XII, 641.

Columns: the term, XII, 353.

Comberousse, C.: polyhedra, IV, 609.

Combescure, E.: wave surfaces, XIII, 238.

Combinants: of quantics, II, 322; the term, II, 518, IV, 607—8, XIII, 46; and ternary quadratics, IV, 352.

Combinatory Analysis: magic squares, X, 38; compound combinations, X, 243—4; a problem of arrangements, X, 245—8; note on Mr Muir's solution, X, 249—51; the game of mousetrap, X, 256—8; Sylvester's work in, XIII, 47; (*see also* Partitions).

Commutants: the term, I, 584, II, 19, 26, IV, 594, 599—600, XIII, 46; property of, V, 495—7.

Commutative: the term, XII, 461; (*see also* Groups).

Comoment: non-Euclidian geometry, XIII, 481—9.
Compass: Mascheroni's geometry of the, XII, 314—7.
Complex (*see* Surfaces).
Complex Cone: (cubic) defined, V, 402, 404, 551.
Complexes of Lines: IV, 618; through twisted quartic, XII, 428—31.
Complex Multiplication: in elliptic functions, XII, 556—7; (*see also* Multiplication).
Complex Quantities: logarithms of, VI, 14—8.
Complex Variables: and conformal representation, X, 316—23; Newton-Fourier theorem for, X, 405—6; (*see also* Function, Newton-Fourier theorem).
Composition: of quadratic forms, I, 532; of singularities, V, 619; of rotations, VI, 24—6.
Compound Combinations (*see* Combinatory analysis).
Compound Singularities: V, 525.
Conchoid: the term, XI, 460.
Concomitant: the term, IV, 607—8, XIII, 46.
Concomitant-system: of quintic, X, 342.
Cone: touching six lines, VIII, 401—3; formulæ for potentials of, IX, 266—7.
Cones: through cubic curve in space, III, 219—21; note on cubic, IV, 120—2; and cubic centres, IV, 173—8, 179—81; and cubic curves, V, 284—8; kinds of cubic, V, 401—15; and representation of curve, V, 552; circumscribed sextic, VII, 139; satisfying six conditions, VIII, 99—137; the term asymptotic, XIII, 232; characteristic n and theory of curves in space, XIII, 468—72.
Configurations: algebraic, by Hilbert, VI, 596.
Conformal Representation: IX, 609—11, XI, 442, XII, 104; by function $\arcsin(x+iy)$, X, 290—2; mechanical constructions, X, 406; example, XI, 38; theorem, XI, 78—81; and Schwarzian theory, XI, 169—76; imaginary quantities, XI, 258—60; (*see also* Orthomorphosis, Representation, Transformation).
Congregate: the word, X, 339, 345.
Congruences: in *Ency. Brit.*, XI, 628, 634—5; analytical theory, XIII, 228—30.
Conic: theorem of triangle and line, V, 100—2; theorem of eight points on a, V, 427—30; formula for intersections of line and conic, V, 500—4; four points on, V, 571; defined by five conditions, VII, 546, 552; through three points and with double contact, VII, 554; foci of, VII, 571; and four points, VII, 581, 587; construction, VII, 592; (2, 2) correspondence of points on, VIII, 14—21; and cubic, X, 605—7; Monge's differential equation, XII, 393; focals of quadric surface, XIII, 54.
Conic-node: the term, VI, 360.
Conics, Analytical Theory of: IV, 395—419; relating to single conic, IV, 396—402; ditto with point or line, IV, 402—12; ditto with tangent of conic of double contact, IV, 413; relating to two conics, IV, 416—9.
Conics: general theory of, I, 519—21; inscribed in a quadric surface, I, 557—63; in-and-circumscribed polygon, II, 142—4, IV, 295—9; two dimensional geometry, II, 575—83, 586—92; forms of equations of, III, 86—90; area of, and trilinear equation second degree, III, 143—8; normals of, IV, 74—7; of five-pointic contact of plane curve, IV, 207—39; which touch four lines, IV, 429—31; system having double contact, IV, 456—9, VII, 568; theorem in, IV, 481—3; touching curves, V, 31—2, 552; four inscribed in same conic and passing through same three points, V, 131—2; contact of, V, 552; and rectangular hyperbolas, V, 554; problem, V, 562, 582; tangents of, V, 578; intersection of, V, 582; triangle and, V, 593; and cubic, V, 608; drawing of, VI, 19; locus from two, VI, 27—34; theorem of four which touch same two lines and pass through same four points, VI, 35—9; which touch cuspidal cubic, VI, 249—53; contact of third order with given cuspidal cubic and double contact with given cubic, VI, 253—6; Zeuthen's forms for characteristics of conics which satisfy four conditions, VI, 256—8; correspondence, and those which satisfy given conditions, are at least arbitrary, VI, 268—71; five conditions of contact with a given curve, VI, 272—91; foci of, VI, 517—9, VII, 1—4; determined by

five conditions of contact, VII, 40; three, problem and solution, VII, 595; and absolute, VIII, 31—44; theorem of eight points on, VIII, 92—4; cuspidal, of centro-surface, VIII, 352—7; reciprocal of equation, VIII, 522—3; theory of confocal, VIII, 556—7; sets of four points on, X, 569; and lines, X, 602; degenerate forms of curves, XI, 218—20; the term, XI, 460; in *Ency. Brit.*, XI, 561—4; analytical geometrical note on, XII, 424; f and c, XIII, 11—2; non-existence of special group of points, XIII, 212; the nine-point circle, XIII, 517—9.

Conics, Spherical: theorem relating to, IV, 428; and stereographic projection, V, 106—9; (*see also* Polyzomal curves).

Conics which pass through: four points, III, 136—8; four foci of given conic, IV, 505—9; three given points and touch one line, V, 258—64; two given points and touch two given lines, VI, 43—50; two given points and touch given conic, VI, 245—9.

Conic Torus: the, IX, 519—21.

Conjugate Integrals: Hamiltonian, X, 113—5.

Conjugates: table of, and seminvariants, XIII, 303, 307.

Connected Areas: XI, 7.

Connective: of discriminant, II, 529.

Connective Covariant of two Quantics: defined, II, 515.

Conormal Correspondence: of vicinal surfaces, VIII, 301—8.

Constants: number of, in special equations, XI, 14—6.

Constructive Geometry: VII, 27.

Contacts: problem of, I, 522—31; the term, VII, 546.

Content: Ball on theory of, II, 606.

Continuous Function: the term, XI, 539.

Contour: lines, IV, 108—11, 609; defined, V, 63.

Contracovariants: defined, IV, 329.

Contractible Squarewise: the term, XIII, 179.

Contragredient: the term, IV, 607—8, XIII, 46.

Contraprovectant: defined, II, 514.

Contraprovector: the term, II, 514.

Contrasect: the term, XIII, 485.

Contravariant: the term, II, 320, XIII, 46; of ternary cubic, IV, 325.

Convergence: condition of uniform, XIII, 342—5.

Converging Series: product of, IX, 61.

Convertible Matrices (*see* Matrices).

Convolution: the term, VI, 461—2.

Coordinates: in general theory of geometry, II, 604—6; as functions of parameters, VI, 1—2; polyzomal curves, VI, 498—9, 537; trilinear, XI, 467; Plücker, XI, 467; degenerate curves, XI, 488—9; in *Ency. Brit.*, XI, 546—51, 566—7; illustrative of geometry, XI, 552—6; curvilinear, in *Ency. Brit.*, XI, 637; versus quaternions, XIII, 541—4.

Coordinates of a Line: X, 603, XI, 468.

Coordinates of Points: expressions for, V, 517—8; lines and planes, non-Euclidian geometry, XIII, 489—91.

Coordinates, Six of a Line: VII, 66—98, VIII, 401, X, 287, XII, 42—3, 321; introductory, VII, 66; definition and general notions, VII, 67—9; elementary theorems, VII, 69—73; geometrical considerations, VII, 73—5; linear relations between six coordinates, VII, 75—85; geometrical property of an involution of six lines, VII, 85; four given lines and twofold tractor, VII, 85—6; hyperboloid through three given lines, VII, 86—8; six coordinates defined as absolute magnitudes, VII, 88—9, 96—7; statical and kinematical applications, VII, 89—95; transformation of coordinates, VII, 95—6; formulæ of transformation, VII, 97—8.

Coordinates, Spherical: theory of, and systems of equations, I, 213—23.

Coordinates, Transformation of: I, 123—6, IV, 552—9, XI, 136—42; in *Ency. Brit.*, XI, 558—61, 573—6.
Copfaffian: the term, XIII, 406.
Coriolis, G.: motion of three bodies, IV, 541.
Corpus: Sylvester's theory of the, XIII, 47.
Correspondence: on cubic curves, I, 184, 190; homographic figures, I, 212; theory, VI, 263—91, X, 259—60, XI, 482; in-and-circumscribed triangle, VIII, 222—5; vicinal surfaces, VIII, 301—8; of two variables, IX, 94—5, XII, 104; geometrical representation of imaginary variables, X, 316—23; construction of a, XI, 38; quadric transformation between points and planes, XII, 100—1; of Cartesians, and generators of hyperboloid, XII, 587—9.
Correspondence of Points: V, 542—5, VI, 22, VII, 168—70, XI, 440; two on a curve, VI, 9—13; on a conic, VIII, 14—21; on surfaces, VIII, 200—8; and lines, VIII, 566.
Cos-centre: the word, XIII, 551.
Cotes, R.: central forces, IV, 517, 586.
Cotterill, T.: correspondence of points, VI, 22; problem of envelope and two circles, VII, 573; theorem of Geiser, IX, 506; goniometric problem, X, 295—7.
Counter-barriers: the term, X, 320.
Counter Order: the term, XIII, 268.
Couples: algebraic, I, 128—31.
Cournot, A. A.: motion of a body, IV, 583, 586.
Covariantive Forms and Tables: XI, 277—80; M to W of binary quintic, II, 282—309; asyzygetic, to degree 18, VI, 149—52; 34 concomitants of ternary cubic, XI, 342—56; of binary sextic, XI, 372—6, 377—88; theory of tamisage, XI, 409—10.
Covariants: the term, I, 577, 589, II, 224, IV, 594, 605, X, 340, XIII, 46; determined by differential equations, II, 164—78; theory, II, 164—78; of binary cubic, II, 189, 260—2; binary quadratic, II, 189; binary quartic, II, 190, 262—4; asyzygetic, II, 250; binary quantic, II, 269; of cubic, analogous to invariants of quartic, II, 553; bibliography, II, 598—601; of degree 6, VI, 148—53; of binary cubic, geometrical interpretation, VII, 332—3; the terms asyzygetic and irreducible, VII, 336; theory of number of irreducible, VII, 336—7; also new formulæ for asyzygetic, VII, 337—40; also 23 fundamental, VII, 341—8; Gordan's proof for the number, VII, 348—53; theory founded by Cayley, VIII, xxix—xxx; his work, VIII, xxx—xxxii; as transvectants, VIII, 404—8; connected with an algebraical operation, IX, 537—42; derivatives of three binary quantics, X, 278—86; theorem, X, 430—1; a formula, XI, 122—4; formula aud Schwarzian derivative, XI, 184—5; in geometry, XI, 474; Sylvester on, XIII, 47; a hyperdeterminant identity, XIII, 210—11; theory of derivation connected with particular operators, XIII, 329—32; (*see also* Invariants, Linear Transformation, Seminvariants).
Cox, H.: Taylor's theorem, VIII, 493.
Cox, H., Jun.: non-Euclidian geometry, XIII, 481.
Cramer, G.: determinants, I, 63; curve classification, V, 354; transformation of plane curves, VI, 1.
Creedy, C.: tangential of cubic, II, 558; calculations by, III, 361; elliptic motion, IV, 522, 586.
Cremona, L.: on Steiner's quartic surface, V, 423; general theory of correspondence, VI, 22—3; Casey's equation, VI, 66—7; scrolls, VI, 327—8, VII, 245—51; polyzomal curves, VI, 575—6; rational transformation, VII, 189, 200, 207, 222, 253—5, XI, 482, 484; theory of curve and torse, VIII, 72, 76—9, 87—91; geometric transformation, X, 611—2.
Critic Centres (*see* Involution of Cubic Curves).
Criticoids: and invariants, XII, 390; and reciprocants, XIII, 366—7; of Cockle, XIII, 366—7.
Critic Points and Lines: the term, X, 311—5.
Crofton, M. W.: polyzomal curves, VI, 507; Cartesian curves, VII, 582.
Cross-points: the term, X, 317.
Crunodal: the term, V, 402, 551, XI, 228.
Crunode: defined, IV, 181, V, 295, 521, XI, 630.

Crystals: biaxal and ray planes, IX, 107—9.

Cube: axial systems, V, 531—9; automorphic function for, XI, 169, 179—83, 212—6.

Cubic: canonical form of binary, II, 542; equation of differences for, IV, 242, 279; ternary, IV, 325—41; and tables, IV, 333—41; the term, IV, 604; resultant of two binary, V, 289—91; asyzygetic covariants of binary, VII, 338—9; automorphic transformation of binary cubic, XI, 411—6; (*see also* Binary and Ternary Cubics).

Cubic Centres: of lines, V, 73—6.

Cubic Cone: kinds of, V, 401—5, 551, 553; anharmonic property, V, 411—2.

Cubic Curve Classification: V, 354—400, VIII, xxxviii; seven head divisions, V, 355—6; their equations, V, 356—9; thirteen divisions, V, 360—1; notion of group, V, 361; osculating asymptotes, V, 361—3; Newton's classification, V, 364—6, XI, 464; Plücker's, V, 366—8; theory of groups, V, 368—9; groups of hyperbolas Δ, V, 369—70; hyperbolas Δ redundant, V, 370—6; ditto defective, V, 376—88; hyperbolas ⊙, V, 388—9; ditto redundant, V, 389—90; ditto defective, V, 390—1; groups of parabolic hyperbolas, V, 391—4; of central and parabolic hyperbolisms, V, 394; groups of divergent parabolas, V, 395; trident curve and cubical parabola, V, 395; division into species, Newton and Plücker, V, 396—9.

Cubic Curve Memoirs: first, I, 183—9, 586; remarks, I, 190—4, 586; second memoir, II, 381—416, IV, 188; definitions, II, 382—3; theorems relating to conjugate poles, II, 383—5; their proof, II, 385—96; geometrical definition of Quippian, II, 396—7; theorems, II, 397—403; formulæ for intersection of curve and line, II, 404—5; formulæ for satellite point and line, II, 405—9; theorems relating to satellite point, II, 409—12; first polar point of cubic, II, 412—5; recapitulation of geometrical definitions of Pippian, II, 416.

Cubic Curves: tangential of, II, 558—60; cones through, III, 219—21; note on, IV, 120—2; five pointic contact, IV, 231—6; ninth point of intersection of those passing through eight given points, IV, 495—504; twisted, V, 1; sextactic points of plane, V, 233—5; and cones, V, 401—15, 551, 553; inflexions of, V, 493—4, XI, 473; in connexion with quintics and quartics, V, 580; problem, V, 586; derivation of points, VI, 20; intersection of, VI, 20; in pencil of six lines, VI, 105—15, 593—4; nodal, VI, 171—4; foci of circular, VI, 521—2; theory of circular, VI, 526—8; symmetrical circular, VI, 549—50, 550—3; quartic and three, VII, 546; points on, VII, 549; and Cartesian curves, VII, 556; rectangular, VII, 591; mechanical description, VIII, 147—50; residuation in regard to one, IX, 211—4; problem and solution, X, 592—4; equation of, X, 603; and conic, X, 605—7; Abel's theorem applied to, XI, 27—8; degenerate, XI, 220; date of theory, XI, 449; forms and classification, XI, 478—80; circular, XI, 481; systems of, XI, 487; Abel's theorem, XII, 30; elliptic functions, XII, 35—7; as ground-curve in Abel's theorem, XII, 38, 109—216; twisted, on quadric surface, XII, 307—10; notion of, XIII, 79—80; and non-existence of a special group of points, XIII, 212.

Cubic Equations: solution of, II, 542; Tschirnhausen's transformation, IV, 364—7, 377; equation of squared differences, IV, 463—5; Sturmian constants, IV, 473—7; relation between roots, VII, 548; solution by radicals, X, 9; constants of, XI, 556; note on, XII, 421—3; Cardan's solution, XII, 299; on two, XIII, 348—9.

Cubic Forms: letters on, III, 9—12.

Cubic Identity: problem, V, 597.

Cubic Scrolls (*see* Scrolls).

Cubic Seminvariants: generating function, XIII, 306.

Cubic Surfaces Memoir: VI, 359—455, 595—6; Introductory, VI, 359; twenty-three cases, explanations, and tables of singularities, VI, 359—63; determination of number of certain singularities, VI, 364—5; lines and planes of cubic surface, facultative lines, diagrams, VI, 365—6; different kinds of axis, VI, 367; determination of reciprocal equation, VI, 368—70; explanation of sections of memoir, VI, 370—1; equations, I=12, VI, 371—83; $12-C_2$, VI, 383—90; $12-B_3$, VI, 391—6; $12-2C_2$, VI, 397—402; $12-B_4$, VI, 403—7; $12-B_3-C_2$, VI, 407—11; $12-B_5$, VI, 411—8; $12-3C_2$, VI, 418—

22; $12-2B_3$, VI, 422—6; $12-B_4-C_2$, VI, 426—8; $12-B_6$, VI, 429—30; $12-U_6$, VI, 431—3; $12-B_3-2C_2$, VI, 433—6; $12-B_5-C_2$, VI, 437—9; $12-U_7$, VI, 439—40; $12-4C_2$, VI, 441—2; $12-2B-C_2$, VI, 443—4; $12-B_4-2C_2$, VI, 445—6; $12-B_6-C_2$, VI, 447—8; $12-U_8$, VI, 448—9, 451—5; $12-3B_3$, VI, 449—50; synopsis of foregoing, VI, 450; cubic scrolls, VI, 451.

Cubic Surfaces: triple tangent planes, I, 445—56, 589; skew, V, 90—4; delineation of scrolls, V, 110—2; nodal curve of developable from quartic equation, V, 135—7; theory, V, 138—40; five given lines on, VII, 177—8; double sixers, VII, 316—29; and tetrahedra, VII, 607; Wiener's model with 27 real lines, VIII, 366—84; in *Ency. Brit.*, XI, 633.

Cubic Transformation of Elliptic Functions: III, 266—7, VII, 44—6, X, 46, 58, XII, 518—22, 555, 556—7; geometric illustration, IX, 522—6.

Cubi-Cubic Curves: in space, V, 18—9.

Cubinvariants: of binary quartic, I, 94; of quantic, II, 516; the term, IV, 606; of quadri-quadric function, XIII, 67—8.

Cuboid: potential of, IX, 272, 274—5, 278—80.

Cumulant: the word, IV, 600—1.

Cunningham, A.: on number of terms in a determinant, X, 579—80.

Curtate: the term, XI, 155.

Curvature: lines of, on ellipsoid, I, 36—9; of plane curve at double point, IV, 466—9; of surfaces, IV, 466—9; geodesic, XI, 323—30; (*see also* Curves of Curvature, Orthogonal Surfaces).

Curves: and developables, I, 207—11, 485, 586—7, 589; and two dimensional geometry, II, 569—83; partial branch of, V, 425; reciprocation, V, 505—10; representation by cone and monoid surface, V, 552; nodal, spinode and cuspidal, of cubic surfaces, VI, 450, 595; and space of m dimensions, VI, 456—7; correspondence of two points on, VII, 39; graduation, VII, 426; mechanical description, of, VIII, 138—44, 147—50, 151—5, X, 576; bicyclic chuck for, VIII, 209—11; penultimate forms, VIII, 258—61, 262—3; property of curve and torse, VIII, 520—1; coordinates and equations, X, 546; degenerate forms, XI, 218—20, 487—9; abstract geometry, XI, 441—2; in *Ency. Brit.*, XI, 460—89, 572—3, 579—80; and theory of equations, XI, 501; and function, XI, 540—1; and solid geometry, XI, 569; quadrature of, XI, 641—2; minimal surfaces, XIII, 41; orthotomic, of a system of lines in a plane, XIII, 346—7; (*see also* Correspondence, Cubic Curves, Nodal Curves, Polyzonal Curves).

Curves, Algebraic: I, 46—54, 584.

Curves, Bicursal: VIII, 181—7.

Curves, Classification of: V, 613—7; (*see also* Cubic Curve Classification).

Curves, Excubo-quartic: V, 282.

Curves in Space: analytical representation, IV, 446—55, 490—5, 616—8, VII, 66, XI, 9—13; defined by conoid and monoid surfaces, V, 7—20, 552, 553, 613; quartic, V, 11—5; quintic, V, 15—6, 24—30, 552, 553, 613; quadri-cubic, V, 16; quadri-quartic, V, 17; cubi-cubic, V, 18—9; Halphen's characteristic n in theory of, XIII, 468—72.

Curves, Intersections of: I, 25—7, 583, XII, 500—4; real, IX, 21.

Curves of Curvature: near umbilicus, VII, 330—1; on surfaces, VIII, 97—8, 145—6, 264—8; *Ency. Brit.*, XI, 628, 635—6; wave surface, XII, 249; surfaces with spherical, XII, 601—38.

Curves of Striction: I, 234.

Curves, Opposite: V, 468.

Curves, Parallel: envelopes and surfaces, IV, 123—33, 152—7, 158—65; and evolutes, VIII, 31—5; theory of, X, 260; the critic, in solar eclipses, X, 311—5.

Curves, Pedal: V, 113—4.

Curves, Penultimate Quartic: VIII, 526—8.

Curves, Penumbral: geometrical theory of projection, VII, 483, 488—9, 489—92.

Curves, Plane: double tangents of, IV, 186—206; conic of five-pointic contact of, IV, 207—39;

curvature at double point, IV, 466—9; higher singularities, V, 424—6, 520—8, 619; correspondence of points on, V, 542—5; transformation, VI, 1—8, 593, VIII, 387; notion of, of given order, XIII, 79—80.

Curves, Plane, sextactic points of: V, 221—57, 618—9; condition for point, V, 222—5; notations and remarks, V, 225—6; first transformation, V, 226; second, V, 227—8; third, V, 228—9; fourth and final form, V, 229—33; application to cubic, V, 233—5; proof of identities, V, 235—7; Jacobian formula, V, 237—8; proofs of equations and identities, V, 239—47; appendix, V, 247.

Curves, Poloid: IV, 571.

Curves, Rhizic: IX, 34.

Curves, Serpoloid: IV, 571.

Curves, Sextic: VII, 256—7, VIII, 138—44, X, 612.

Curves, Symmetric: I, 473.

Curves, Theory of: and elimination, I, 337—51, V, 162—7, 416—20; evolution, XI, 449—51, XII, 102—3, 290—1.

Curves, Theory of, and Torse: VIII, 72—91; explanations and notation, VIII, 72—4; Plücker-Cayley equations, VIII, 74, 75—6, 80—1; Salmon-Cremona equations, VIII, 74, 76—9, 87—91; geometrical theory of foregoing relations, VIII, 79—80; tables, VIII, 81—4; nodal curve x, VIII, 84—7.

Curves, Three-bar: IX, 551—80, 585.

Curves, Transformations of: I, 471—5, 476—80; scalene, IX, 527—34.

Curves, Triangular: VII, 59.

Curves, Twisted Quartic: XII, 428—31.

Curves which satisfy given conditions: VI, 191—262, 594, VII, 40; Introductory, VI, 191; previous memoirs, VI, 191—2; quasi-geometrical representation of conditions, VI, 193—200; Chasles' and Zeuthen's researches, VI, 200—26; extensions of de Jonquières, VI, 226—42; form of equation of curves of a series of given index, VI, 242—3; line-pairs which pass through three given points and touch a given conic, VI, 244; conics which pass through two given points and touch given conic, VI, 245—9; conics which touch cuspidal cubic, VI, 249—53; conics which have contact of third order with given cuspidal cubic and double contact with given cubic, VI, 253—6; Zeuthen's forms for characteristics of conics which satisfy four conditions, VI, 256—8; question from de Jonquières' formula, VI, 258—62; the principle of correspondence, VI, 263—91; (introductory, VI, 263—4; correspondence of two points on a curve, VI, 264—8; application to conics which satisfy given conditions, one at least arbitrary, VI, 268—71; five conditions of contact with a given curve, VI, 272—91).

Curve Tracing: Cayley's liking for, VIII, xxxix; mechanism, VIII, 179—80, XIII, 515—6; importance, XI, 461; order of, XI, 461.

Curvilinear Coordinates: XI, 330, XII, 1—18; surfaces divisible into squares, VIII, 146; geodesic lines, VIII, 156—67; curves of curvature, VIII, 264—8; orthogonal surfaces, VIII, 269—91.

Cusp: of Cartesian at circular points at infinity, I, 589; synonymous with spinode, II, 28, IV, 22, 27; of second kind or node-cusp, V, 265—6, 618; order of plexus for, V, 309—12; the term, XI, 468.

Cuspidal: defined, V, 403, 551, VII, 244.

Cuspidal Conic: of centro-surface, VIII, 352—7.

Cuspidal Cubic: VII, 561.

Cuspidal Curves: and cubic surfaces, VI, 450; (*see also* Cubic Surfaces, Surfaces).

Cuspidal Isochronic: the term, VII, 473.

Cuspidal Locus: in singular solutions, VIII, 533.

Cyc: the abbreviation in groups, XIII, 119.

Cyclide: of Dupin, V, 467, XII, 615; the term, VII, 246, VIII, 262, IX, 64—5; and anchor ring, IX, 18; on, IX, 64—78; the parabolic, IX, 73—8; in *Ency. Brit.*, XI, 634.

Cycloid: the term, XI, 447.

Cyclotomy: (Kreistheilung), XI, 58, 86.
Cylinder: in *Ency. Brit.*, XI, 572—3; Archimedes' theorem for surface, XII, 56—7.

d'Alembert, J. le R.: rotation of solid body, IV, 567, 586; geometric paradox, XII, 305—6.
Dandelin, G. P.: theory of, and on caustics, II, 339—40; wave surface, IV, 433—4.
Darboux, G.: powers of circles and spheres, I, 581; the torus, VII, 247; quartic surfaces, VIII, 262; cyclide, IX, 64; continuous function, XI, 539; curves of curvature, XII, 615; quartics, XIII, 13.
Davis, W. Barrett: calculations by, III, 361, IV, 376; quantic covariants, VII, 335; Sohnke's modular equations, IX, 543.
Déblais: theory of, XI, 417—20, 449, 587.
de Bruno, Faà: invariants of degree 12 belonging to quintic, II, 314; symmetric functions, II, 602; canonical forms, IV, 52; elimination, IV, 608.
Decadianome: the term, VII, 134; and symmetroid, VII, 256.
Decapitation: and seminvariants, XII, 248—50.
Decomposition: linear differential equations and theory of, XII, 402, 403—7.
Dedekind, J. W. R.: probabilities, II, 594—8, V, 80; modular function, XIII, 338—41.
Def: the abbreviation in groups, XIII, 120.
Deficiency: and genus of curve, V, 467, 517, 619; of curves, V, 618, VIII, 391, XI, 450; the term, VI, 2; and transformation, VI, 3, XI, 482—5; the term applied to surface, VI, 356; of certain surfaces, VIII, 394—7, XI, 230; surfaces of negative, VIII, 397; of sextic curve, IX, 504—7; of curve and Abelian integrals, XI, 30—6.
Definite Integrals: with complex variables, I, 181—2, 310; differentiation and evaluation, I, 267—72, 587; on a, IV, 28—9; note on Glaisher's paper, VIII, 1; note on two, IX, 56—63; (*see also* Attractions, Potentials, Prepotentials).
Definitions (*see* the word desired).
Deformation: the term, I, 234; of hyperboloid, XI, 66—7; of skew surfaces, XI, 317—22, 331; (*see also* Surfaces).
Degeneracy: of surfaces, V, 98—9; of scrolls, V, 201—3; of curves, XI, 218—20.
Degen's Mathematical Tables: IV, 40, IX, 478—9, X, 586; report of British Association Committee, XIII, 430—67.
Deg-order: the term, X, 339.
Degrees: of quantics defined, II, 221.
Degrees, Honorary: conferred on Cayley, VIII, xx—xxi; conferred on Sylvester, XIII, 43.
de Jonquières, E.: cubic curves, I, 586, IV, 496; on curves, IV, 454; curves which satisfy given conditions, VI, 191, 192, VII, 41—3; form of equation of curves of a series of given index, VI, 242—3; question from formula, VI, 258—62; points on cubic curve, VII, 550, 553; correspondence, XI, 486.
de la Goupillière, H.: inertia, IV, 566.
de la Gournerie, M.: tetrahedral scrolls, VII, 48—53; scrolls, VII, 54; quartic and quintic surfaces, VII, 246, 247, 251; torus, VIII, 25; octic surfaces, X, 81.
Delambre, J. B. J.: Tables du Soleil, III, 474.
Delaunay, C. E.: lunar theory, VII, 357, 372, 376, 528—33, 534, IX, 180, XIII, 206—7.
de Morgan, A.: root in every algebraic equation, IV, 116—9; root limitation, IX, 39; series, XI, 623, 627; and Sylvester, XIII, 45.
Denumerant: illustrated, IV, 169—70.
Denumerate: defined, IV, 241.
Departure Point: in lunar theory, III, 19, 270, 295.
Derivation: of points of cubics, VI, 20; and Übereinanderschiebung, VII, 348; and seminvariants, XIII, 362—5.

Derivational Function: the term, I, 63.

Derivations: extension of Arbogast's method, II, 257, IV, 265—71, 272—5, 609, XI, 55; binomial theorem and factorials, VIII, 463—73.

Derivatives: and hyperdeterminants, I, 95; of point on cubic, IV, 231; of three binary quantics, X, 278—86; and covariants, X, 340, 377—94; of a function, X, 590—2; in binary forms, XI, 272; (*see also* Schwarz).

de St Laurent, T.: caustic by reflexion, I, 273—5.

Desboves, A.: planetary perturbation, III, 185, 203; problem of two centres, IV, 532, 586.

Descartes, R.: ovals of, and transformation of curves, I, 478, 479—80, 589; oval of, III, 66; formulæ in *Epistolæ*, IV, 512; geometry of, XI, 437; (*see also* Cartesians).

Determinants: applied to distances of points, I, 1—4, 581, IV, 510—2; Pascal's theorem, I, 43—5; the term, I, 63; theory of, I, 63—79; theory of linear transformations, I, 80—94, 584; of vis viva, I, 284; note on hyperdeterminants, I, 352—5, 588; geometrical reciprocity, I, 377—82; "skew" and "symmetric," I, 410—3; history, I, 581; multiplication, I, 581, XI, 495; value of certain, III, 120—3, IV, 460—2; the term, IV, 594, 596—9; and Pfaffian, IV, 600; development of, V, 45—9; tables of binary cubic forms for negative, VIII, 51—64; Smith's Prize dissertation, VIII, 551—5; symmetrical, IX, 185—90, X, 579; notation, X, 95—7; theorem in, X, 265—6; in *Ency. Brit.*, XI, 490—7; decomposition, XI, 495—6; theory of numbers, XI, 604—9; (*see also* Hyperdeterminants, Skew Determinants).

Determinator: defined, II, 59.

Determinirende: (indicial), and differential equations, XII, 398, 401, 453.

Developables: and curves, I, 207—11, 586—7; the term, I, 486, XI, 573; from two quadrics, I, 486—95; from quintic curve, I, 500—6; planar, I, 505; from quartic, V, 135—7; prohessians, V, 267—83; quartics, V, 268—71; general theory, V, 271—2; special quintic, V, 272—8; special sextic, V, 279—83; reciprocation of quartic developable, V, 505—10; a special sextic, V, 511—9; sextic, and sextic surfaces, VI, 87—100; focals of a quadric surface, XIII, 51—4; (*see also* Torse).

Development: of factorial, II, 98—101; coefficients in powers of $(1+n^2x)^{m/n}$, XIII, 354—7.

Dew-Smith, A. G.: portrait of Cayley, XI (frontispiece).

Diagonals: and partitions of a polygon, XIII, 93—113.

Diagrams: the term, VII, 405; of planet's orbit from three observations, 5 plates, VII, to face 478; solar eclipse, VII, to face 492; geodesic lines on ellipsoid, VII, 510; coloured, representing groups, X, 328—30; transformation of elliptic functions, XI, 26; seminvariants, and solution by square-, XIII, 288—98; (*see also* Tables).

Diameter: as used by Newton, V, 362.

Diametral planes (*see* Planes).

Dianome: the term, VII, 133, 148; (*see also* Quartic Surfaces).

Diaphoric: the term, XI, 156.

Dickinson, L.: portraits of Cayley, VI (frontispiece), VII (frontispiece), VIII, XX.

Differences: equation of squared, for cubic, IV, 463—5; relation between certain products of, X, 293—4; on a functional equation, X, 298—306; (*see also* Equation of Differences).

Differential Equation Memoir: X, 93—133; introductory, X, 93—4, 94—5; notations, X, 95—7; dependence of functions, X, 97; general differential system, X, 98—102; the Multiplier, X, 102—5; Pfaffian theorem, X, 106; Hamiltonian system, derived from general system, X, 106—7; Poisson-Jacobi theorem, an identity in regard to functions (H, Θ), X, 108—9; peculiar to Hamiltonian system, X, 110—3; conjugate integrals of Hamiltonian system, X, 113—5; Hamiltonian system—the function V, X, 115—8; partial differential equation $H=$constant, X, 119—25; examples, X, 125—32; partial differential equation containing the dependent variable, reduction to standard form, X, 132—3.

Differential Equations: and lines of curvature of ellipsoid, I, 36—9; dynamical, I, 276—84; Jacobi's

system of, I, 366—9; Jacobi on theory, III, 174; theorem of Jacobi on Pfaff's problem, IV, 359—63; singular solutions, IV, 426—7; transformation, IV, 574, V, 78—9; umbilici, V, 115—30; solution when algebraical, VII, 5—7; supposed new integration, VII, 36; note on one, VII, 354—6; pair in lunar theory, VII, 535—6, 537—40; integration by series, VIII, 458—62; Euler's, IX, 592—608; and theory of elliptic functions, X, 20; and sides of quadrangle, X, 33—5; theory of partial, X, 134—8; elliptic and single theta functions, X, 422—9; hypergeometric series, XI, 17—25; Abel's theorem, XI, 27—8; new formulæ for integration of Euler's equation, XI, 68—9; mathematics and physics, XI, 449; connected with elliptic functions, XII, 30—2; Briot and Bouquet's theory, XII, 432—41; of circular functions, XII, 580; a diophantine relation, XII, 596—600; and construction of Milner's lamp, XIII, 3—5; Kummer's, of third order, XIII, 69—73; on a partial, XIII, 358—61; Richelot's integral of Euler's, XIII, 525—9; (*see also* Partial Differential Equations, Riccati, Schwarz, Singular Solutions).

Differential Equations, Linear: invariants of one, XII, 390—3; general theory, XII, 394—403, 444—52, 453—6; theory of decomposition, XII, 403—7.

Differential Equations of First Order: theory of singular solutions, VIII, 529—34, X, 19—24.

Differential Invariants (*see* Invariants).

Differential Operators: VII, 8.

Differential Relations: of double theta-functions, X, 559—65.

Differentiation: evaluation of definite integrals, I, 267—72, 587; formulæ for, IV, 135—49; fractional, XI, 235—6.

Dimensions in Geometry (*see* Geometry).

Dimidiate: the term, XIII, 119.

Dimidiation: the term, XIII, 122.

Diophantine Differential Relation: XII, 596—600.

Diptich: the term, XII, 596.

Dirichlet, G. L. (*see* Lejeune-Dirichlet).

Director, Nodal (*see* Nodal Director).

Directrix: and scrolls, VII, 60; and the absolute, XIII, 481—9, 501; kinematics of a plane, XIII, 505—6.

Discriminant: and invariant, I, 584; defined, II, 176, IV, 603, VI, 466—7; of quantics, II, 320; the sign □, II, 528; special, connected with curve, V, 163; of quintic, problem, V, 592; of binary quantic, VII, 303, IX, 16—7; example of a special, VIII, 46—7; (*see also* Quantics).

Discriminant Locus: the term, VI, 198.

Displacement: the term in Abel's theorem, XII, 110, 157—62.

Distance: general theory of, II, 561, 583—92, 604—6, V, 550; notion of, in analytical geometry, V, 550; the term, VI, 497; angular, of two planets, VII, 377—9; Cayley and Klein on theory, VIII, xxxvi—vii; general notion, VIII, 31; Euclidian geometry, XI, 435—7; non-Euclidian geometry, XIII, 480—504; (*see also* Points).

Distribution of Electricity: on spherical surfaces, IV, 92—8, 99—107, XI, 1—6.

Distributively: the term, VI, 459.

Disturbing Function: in lunar theory, III, 293—308, 319—43; in rotation of solid body, III, 486.

Divisors: tables of, IX, 462—70.

Dodecahedron: construction, IV, 82—3; axial systems, V, 531—9; as regular solid, X, 270—3; automorphic function for, XI, 169, 179—83, 184, 212—6.

Donkin, W. F.: expansions in multiple sines, I, 583; differential equations, dynamical, III, 191—7, 203—4 344; transformation of trigonometric series, III, 567; attractions, III, 567; a definite integral, IV, 29 formulæ for differentiation, IV, 135—49; central forces problem, IV, 521; spherical pendulum, IV, 534, 536, 586; dynamical problems, IV, 547, 586; elimination of nodes in three bodies, IV, 551, 586; rotation of solid body, IV, 578, 586.

Dostor, G.: polyhedra, IV, 609.

Dots: notation for lines and planes of cubic surfaces, VI, 365—6, 373—449; and seminvariants, XIII, 267.
Double Algebra: XII, 465.
Double Contact: conics having with each other, IV, 456—9.
Double Point: the term, VI, 1; on ground-curve, XII, 110, 129.
Double Pyramid (*see* Polygons).
Double-Sixer: and cubic surfaces, VI, 372, VII, 316—29; construction, VIII, 366—84.
Double Tangents (*see* Bitangents).
Double Theta Functions: X, 155—6, 166—79, 180, 422—9, 474—5, 497, 565; in connexion with 16-nodal quartic surface, X, 157—65; memoir on, X, 184—213; (Part I, preliminary investigations, X, 184—9; Part II, the double theta functions, X, 189—213); addition of, X, 455—62; evolution, XI, 454; transformation, XII, 358—89; (*see also* Theta functions).
Doubly Infinite Products: I, 120—2, 132—5, 136—55, 156—82, 585, 586, X, 492—4, XI, 46, XII, 50—5; and doubly periodic functions, II, 150—63; and definite integrals, IX, 60; transformation of, X, 494—7.
Doubly Periodic Functions: I, 156—82; and doubly infinite products, II, 150—63; and definite integrals, IX, 61; the term, XI, 530.
Drawing: geometrical, VI, 19; of quartic curves mechanically, VIII, 151—5; curves generally, VIII, 179—80; (*see also* Representation).
Droop, H. R.: isochronism of circular hodograph, III, 265; central forces problem, IV, 520, 587.
Duality: in geometry, II, 561—2, 568, XI, 450, 467.
Du Bois-Reymond, P. L.: uniform convergence, XIII, 343.
Dumas, W.: spherical pendulum, IV, 534, 587.
Dupin, C.: cyclide of, V, 467, IX, 64, XII, 615; quartic and quintic surfaces, VII, 246; theorem of, VIII, 264—8, 562, IX, 84—9.
Duplication of Groups: X, 149—52.
Durège, H.: Landen's theorem, XI, 339.
Durfee, W. P.: symmetric functions, II, 602—3.
Dynamics: differential equations of, I, 276—84; a class of problems, IV, 7—11; similarity of two dynamical systems, VIII, 558—63; Lagrange's general equation in, IX, 110—2, 198—200; general equations in, IX, 215—7; and time, XI, 444; transformation of coordinates, XI, 575.
Dynamics, Recent Progress in Theoretical: III, 156—204, IV, 514; introduction, III, 156—7; Lagrange, *Mécanique Analytique*, III, 157—8, 201, 202; Lagrange, equations of motion, III, 158, 200; lunar theory, III, 158—9; Poisson, planetary theory, III, 159, 201; Laplace's theory, III, 159, 201; Lagrange's planetary theory, III, 159—61, 162—3, 201; Lagrange, variation of arbitrary constants in mechanical problems, III, 161—5, 200; also Poisson, III, 163—5, 200, 201, 202; Cauchy, differential equations, III, 166; Hamiltonian method of dynamics, III, 166—74, 200, 202; its relations to Lagrange's, III, 171—3, 200; and Poisson, III, 173—4, 200; Jacobi, calculus of variations and differential equations, III, 174—82, 200, 202; *De Motu Puncti Singularis*, III, 182—3, 202; problem of three bodies, III, 183; Jacobi, *Theoria Novi Multiplicatoris*, III, 183—5; Jacobi, theory of ideal coordinates, III, 185; Liouville, equations of motion, III, 185; Desboves, planetary perturbation, III, 185, 203; Serret, integration of differential equations, III, 185—6, 203; Sturm, integration of dynamical equations, III, 186, 203; Ostrogradsky, dynamical equations, III, 186, 203; Brassinne, differential equations, III, 186—7, 203; Bertrand, integrals to mechanical problems, III, 187, 203; and integration of differential equations, III, 188—9, 203; and *Mécanique Analytique*, III, 189—90, 203; Brioschi, *Sulla Variazione*, and *Teorema di Meccanica*, III, 190, 203; Liouville, integration of differential equations, III, 191—2, 203; Donkin, dynamical differential equations, III, 191—7, 203—4; Bour, integration of differential equations of analytical mechanics, III, 197—8, 204; Liouville on Bour's memoir, III, 199, 204; Brioschi, *Degli Integrali di un Problema di Dinamica*, III, 199—200, 203; Bertrand, integrals of several mechanical problems, III, 200, 203; summary, III, 200.
Dynamics, Report on Progress of Solution of Certain Problems: IV, 513—93; introductory, IV,

513—5; rectilinear motion, IV, 515—6; central forces, IV, 516—26; elliptic motion, IV, 521—4; problem of two centres, IV, 524—32; spherical pendulum, IV, 532—4; motion as affected by the Earth and relative motion generally, IV, 534—7; motion of single particle, IV, 537—8; motion of three mutually attracting bodies in a right line, IV, 538—40; motion of three bodies, IV, 540—1; motion in resisting medium, IV, 541; integration of equations of motion, IV, 542—6; memoirs by Jacobi, Bertrand and Donkin, IV, 546—7; problem of three bodies, IV, 548—52; transformation of coordinates, IV, 552—9; principal axes and moments of inertia, IV, 559—66; rotation of solid body, IV, 566—80; kinematics of solid body, IV, 580—2; rotation round fixed point, IV, 582—3; other cases of the motion of a solid body, IV, 583—4.

Earth, The: rotation of, III, 485, IV, 534—7; (*see also* Gravity).

Eclipses (*see* Solar Eclipses).

Edge: defined, V, 63.

Eindeutig: (Uniform function), XII, 433.

Eisenstein, F. G.: linear transformations, I, 90, 101, 111, 113—6, 585; hyperdeterminants, I, 353, II, 598—601; elliptic functions, I, 586; cubic forms, III, 9; quadratic residues, III, 39—43; finite differences, IV, 263; mathematical tables, IX, 492—3; "development of an idea of," X, 58—9; development of $(1+n^2x)^{m/n}$, XIII, 357.

Elastic Strings: problem with, III, 78—9.

Electricity: distribution on spherical surfaces, IV, 92—8, 99—107, X, 299, 307, XI, 1—6.

Elements: Jacobi's canonical, III, 77; of arc, X, 235—7; a reduction to elliptic integrals, X, 239—42.

Eliminant: the term, IV, 597; of two quantics, XI, 100—2.

Elimination: and theory of curves, I, 337—51, V, 162—7, 416—20; from connected equations, I, 370—4; and linear transformations, I, 457; theorem of Schläfli, II, 181—4; a result of, III, 214—5; general theorem, IV, 1—4; Bezout's method, IV, 38—9, V, 555—6; of nodes in three bodies, IV, 551; the term, IV, 594; text-books on, IV, 608; note on, V, 157—9; problem, VI, 40—2, VIII, 22—4; the resultant of a system of two equations, VI, 292—9; theorem, IX, 43—4; formula of, XI, 100—2; theory of equations, XI, 490; a problem of Sylvester's, XIII, 545—7.

Ellipse: curves parallel to, IV, 123—33, 152—7; and circle, V, 561; foci of, V, 586; and quadrilateral, V, 604; circles of curvature, VII, 555, XIII, 537; potential, IX, 281—90; negative pedals, X, 576; cubic curves, XI, 478; in *Ency. Brit.*, XI, 561—4; focals of quadric surface, XIII, 54; and epitrochoid, XIII, 82—7; orthomorphosis into a circle, XIII, 188—9, 422—4.

Ellipsoid, Attraction of: I, 388—91, 432—44, 582, VII, 380—3, XI, 448; Jacobi's method, I, 511—18; Gauss's method, III, 25—8, 149—53; Laplace's method, III, 53—65; Rodrigues' method, III, 149—53; theory of, III, 154—5; and terminated straight line, VII, 31—3.

Ellipsoid, Centro-Surface of, memoir: VIII, 316—65; introductory, VIII, 316—7; the ellipsoid, VIII, 317—20; sequential and concomitant centro-curves, VIII, 320; expressions for coordinates of point on centro-surface, VIII, 320—4; discussion by means of equations, principal sections, &c., VIII, 324—30; generation of surface considered geometrically, VIII, 330—1; nodal curve, VIII, 332—52; eight cuspidal conics, VIII, 352—7; centro-surface as envelope of quadric, VIII, 357—8; another generation of centro-surface, VIII, 359—61; a third generation of centro-surface, VIII, 361—2; reciprocal surface, VIII, 363; delineation of centro-surface for particular case, VIII, 363—5.

Ellipsoid, Geodesic Lines on, memoir: VII, 493—510; introductory, VII, 493—4; course of the lines, VII, 494—5; lines through an umbilicus, VII, 495—501; formulæ, VII, 501—3; umbilicar geodesics, VII, 503; tables, VII, 504—6; projection on umbilicar plane, VII, 507; elliptic function formulæ, VII, 507—10; diagram, VII, to face 510.

Ellipsoids: lines of curvature, I, 36—9; surface parallel to, IV, 123—33, 158—65, X, 575; the momental, IV, 560; of gyration, IV, 560; central, IV, 564; projection, V, 487—8; geodesic lines on, VII, 34—5; centro-surface and sextic torse, VII, 113—4; centro-surface, VII, 130—2; geodesic lines, VIII, 174—8;

attraction of ellipsoidal shell on exterior point, IX, 302—11; negative pedals, X, 576; in *Ency. Brit.*, XI, 576—9.

Elliptic Coordinates: equation of wave surface in, XI, 71—2.

Elliptic Functions: Bronwin on, I, 118, 119; of Jacobi, I, 127, 507, 586; integral calculus, I, 383; multiplication of, I, 534—9, 568—76, 589; addition of, I, 540—9, 589, XII, 294—8; connected with theory of numbers, II, 48; system of modular symbols, IV, 484—9; Weierstrass, V, 33—7; treatise by Cayley, VIII, xviii, xxviii, XIII, 560; a general differential equation, IX, 592—608; a differential equation in theory of, X, 24; and integration, X, 25—7; torse depending on, X, 73—8; reduction of Abelian integrals to, X, 214—22; and single theta functions, X, 422—9, 463, 472; certain algebraic identities, XI, 130—1; evolution of, XI, 451—5; and quartic function, XI, 483; kinds of, XI, 529; symmetrical differential equation and, XII, 30—2; solution of $x^3+y^3-1=0$, XII, 35—7; Weierstrassian and Jacobian compared, XII, 425—7; Kiepert's L-equations, XII, 490—2; graphical representation, XIII, 9—19; and sextic resolvent equations, XIII, 473—9; theta and omega functions, XIII, 558—9; (*see also* Gudermannian, Theta functions).

Elliptic Functions Formulæ: Serret's, III, 3; for geodesic lines on special ellipsoid, VII, 507—10; one, XI, 65, XII, 292—3; connexion of certain, XI, 250—1; geometrical interpretation of certain, XII, 107.

Elliptic Functions, Inverse: I, 136—55, 156—82, 586; and definite integrals, II, 3.

Elliptic Functions, Memoir on Transformation of: IX, 113—75; introductory, IX, 113—4; the general problem, IX, 114—7; Ωk modular equations, IX, 117—8; equation-systems, IX, 119—20; Ωk form, IX, 121—6; modular equation, IX, 126—37; tables, IX, 128—35, 163; multiplier equation, IX, 138—40; multiplier as rational function of u, v, IX, 140—4; multiplication of elliptic functions, IX, 144—7; transformations, IX, 147—55; general theory of q-transcendents, IX, 155—69; four forms of modular equation and curves represented thereby, IX, 169—75.

Elliptic Functions, Theorems in: XI, 73—7; Landen's, XI, 337—9, 584; Hermite's H-product, XII, 584—6.

Elliptic Functions, Theory of: I, 290—300, 364—5, 402—4, 587, 589; and quadri-quadric curve, XII, 321—5.

Elliptic Functions, Transformation of: I, 120—2, 132—5, 585, IX, 543, X, 333—8, 611, XI, 26, XII, 416—7, 505—34, 535—55, XIII, 29—32; cubic, III, 266—7, VII, 44—6, 244—5, 253—6, XII, 46, 556—7, XIII, 64—5; special quartic, IX, 103—6; geometric illustration of cubic, IX, 522—6; orthomorphosis, XIII, 191—205.

Elliptic Integrals: reduction of $\frac{du}{\sqrt{U}}$, I, 224—7; transformation, I, 508—10, IX, 618—21; geometrical representation, II, 53—6, 113—7; discussion, II, 93—5; and covariants, II, 189—92; transformation formulæ, IV, 60—9, 609; expression for second kind of, X, 25—7; note on theory, X, 139—42; some formulæ in, X, 143—8; of third kind, X, 489—92; problem, X, 614; theta functions, XI, 41—6; note on, XI, 64; reduction of an integral to, XI, 270—1; of third kind, formulæ, XI, 340—1.

Elliptic Motion: expansion of true anomaly, III, 139—42, 567; trilinear equation of second degree, III, 143—8; theory of, III, 216—8; tables of functions in theory, III, 360—474, VII, 516; Lambert's theorem, III, 562—5, VII, 387—9; and dynamical progress, IV, 521—4; a theorem, IX, 191—3; and body let fall at equator, IX, 241—3.

Elliptic Motion, Disturbed: memoirs on, III, 270—92, 344—59, 505—15.

Elliptic Space: and non-Euclidian geometry, XIII, 481.

Elliptic-Transcendent Identity: VIII, 564.

Ellis, R. L.: orthogonal surfaces, VIII, 272; differential equations, VIII, 458; Dupin's theorem, IX, 88.

Emanants: of quantics, II, 321; theory, II, 518; Bezoutoidal, II, 525, 526; the term, IV, 604, XIII, 46.

Emanation: theory of, II, 321.

Encke, J. F.: *über die speciellen Störungen*, III, 179—80; fluctuating functions, IX, 19; roots of numerical equations, X, 5.

Encyclopædia Britannica, articles from on:—Curve, XI, 460—89; Equation, XI, 490—521; Function, XI, 522—42; Galois, XI, 543; Gauss, XI, 544—5; Geometry (analytical), XI, 546—82; Landen, XI, 583—4; Locus, XI, 585; Monge, XI, 586—8; Partition of numbers, XI, 589—91; Theory of numbers, XI, 592—616; Series, XI, 617—27; Surface, XI, 628—39; John Wallis, XI, 640—3.

Endecadic Transformation: in elliptic functions, IX, 152—5.

Endoscopic: the term, I, 588.

Ennead: the term, VII, 256, VIII, 566.

Enneadianome: the term, VII, 134.

Enneagon: in-and-circumscribed, IV, 298—303.

Envelopes: developable of two equations, I, 486; parallel curves and surfaces, IV, 123—33, 152—7, 158—65; defined, IV, 458, VI, 467; of circle, V, 610, VII, 591; and locus in regard to triangle, VI, 72—82; depending on two circles, problem and solution, VII, 573; of plane curve, VII, 606; of a certain quadric surface, VIII, 48—50; locus in singular solutions, VIII, 533; problem of, VIII, 491—2; of family of quadrics, X, 589; theory, XI, 50—1; of variable curves, XI, 475—6.

Epicycloid: and caustic, II, 345.

Epispheric integrals: Gauss-Jacobi theory, IX, 410—7.

Epitrochoid: XIII, 81—7.

Equal: applied to circles, VIII, 31.

Equality: among roots of an equation, II, 465—70, VI, 300—12; idea of, XI, 431.

Equation of Differences: for equation of any degree, IV, 150—1; for equation of any order, IV, 240—61; tables, IV, 246—56; of all but one, of roots of given equation, IV, 276—91; and quintic equation, IV, 309—24, 609—16; and cubic equation, IV, 463—5.

Equation, Pellian (*see* Pellian Equation).

Equations: systems of spherical coordinates, I, 213—23; with quantics, defined, II, 221; auxiliary for quintics, IV, 309—24; determination of reciprocal, with cubic surfaces, VI, 368—70; the term, VI, 466; solubility by radicals, VII, 13—4; system of, problem and solution, VII, 578, 581, X, 601; transformation, IX, 42, 48—51; on a functional equation, X, 298—306; Cassinian, problem, X, 608; Jacobian sextic, XI, 389—401, XII, 493—9; equal roots of, XI, 405—7; of curves, XI, 462—4; of Plücker, XI, 469—73, XIII, 536; in *Ency. Brit.*, XI, 490—521, (introductory, XI, 490; determinant, XI, 490—7; imaginary, XI, 502—6); of right line and circle, XI, 558—61; of conics, XI, 563; seminvariants, XII, 19—21; fundamental, and deformation of surfaces, XI, 331; note on system of, XII, 48—9; for three circles which cut each other at given angles, XII, 559—61, 564—70; anharmonic ratio, XII, 578—9; hydrodynamical, XIII, 6—8; Sylvester on ternary cubic-form, XIII, 47; on soluble quintic, XIII, 88—92; Waring's formula for sum of *m*th powers of roots of, XIII, 213—6; sextic resolvent of Jacobi and Kronecker, XIII, 473—9.

Equations, Algebraic: rationalization, II, 40—4; theory, II, 124; theorem that every one has a root, IV, 116—9; system of, IV, 171—2, VIII, 29—30; in *Ency. Brit.*, XI, 506—21; Anglin's formula for successive powers of the root of, XII, 33—4; roots of one, XIII, 33—7.

Equations, Cubic (*see* Cubic equations).

Equations, Modular: for transformation of order 11, XIII, 38—40; for cubic transformation, XIII, 64—5; (*see also* Transformation of Elliptic Functions).

Equations of Motion: in lunar theory, XIII, 206.

Equations, Solutions of: $x^{257}-1=0$, I, 564—6; $\theta^n=1$, and theory of groups, II, 123—30, 131—2, IV, 88—91, X, 610; $x^p-1=0$, XI, 314—6, XII, 72—3; elliptic function solution of $x^3+y^3-1=0$, XII, 35—7; the quaternion $qQ-Qq'=0$, XII, 300—4, 311—3; $(abcd)=(a^2b^2c^2d^2)$, XII, 418—20; $x^{17}-1=0$, XIII, 60—3.

Equations, Theory of: synopsis, X, 3—11; Newton-Fourier method, and imaginary root, XI, 114—21, 143; theorem of Abel's and quintic equation, XI, 132—5; theorem in, XI, 268—9; evolution, XI, 455; in *Ency. Brit.*, XI, 497—521.

Equator: action of gravity at the, IX, 241—3.
Equilibrium: of four forces, V, 540—1, IX, 201; of skew surface, XI, 317—22.
Equimomental Surfaces (*see* Surfaces).
Equipollences: Bellavitis, XII, 473—4.
Equipollent: the term, XII, 473.
Equipotential Curve: III, 258—61.
Essential Singularity of Function: IV, 150.
Eta-Functions: product, XII, 584—6; (*see also* Theta Functions).
Euclid: space of, XI, 434—7; evolution of geometry, XI, 446; proof of I, 47, XI, 557.
Euler, L.: rotation of solid body, I, 237, IV, 566, 567—9, 587, VI, 135—46; involution, I, 259; elliptic functions, I, 366; skew determinants, II, 214; transformation of coordinates, II, 497, IV, 553—7, 587; sums of series, III, 127; indeterminate equations, III, 205—7; polyhedra, IV, 84, 86—7, V, 62—5, 617; *Determinatio Orbitæ Cometæ*, IV, 519, 587; problem of two centres, IV, 525—7, 587; three mutually attracting bodies in right line, IV, 538—9, 587; motion of three bodies, IV, 540, 587; inertia, IV, 562; kinematics of solid body, IV, 580, 587; rotation formulæ, V, 537; differential equation of, VII, 261—2, IX, 592—608, XI, 68—9; binomial theorem, VIII, 463; mathematical tables, IX, 463—6, 471—2, 477—8, 481, 487; theorem on sums of squares, XI, 294; partitions, XI, 360, XII, 219; intersections of cubic curves, XI, 449; gamma function, XI, 535—6; eight-squares theorem, XII, 465; Latin squares, XIII, 55; differential equation of, integrated by Richelot, XIII, 525—9.
Evans, A. B.: Degen's tables, X, 586.
Evectant: of quantics, II, 321.
Evector: of quantics, II, 321.
Evolutes: theory of, V, 473—9; and parallel curves, VIII, 31—45; nodes of, VIII, 329, 351.
Evolution: of geometry, XI, 445—8.
Ewing, J. A.: curve-tracing mechanism, XIII, 505.
Excuboquartic: defined, V, 10, VII, 99; curves, VI, 87—8, XI, 9—13.
Exoscopic: the term, I, 588.
Expansions: in multiple sines and cosines, I, 19—24, 583; in Laplace's coefficients, I, 375—6; of true anomaly, III, 139—42; numerical, IV, 470—2.
Expectation: problem and solution in, X, 587; (*see also* Probability).
Experience and Cognition: XI, 431.
Exponential Functions: and double theta functions, X, 184—5; the term, XI, 524—7.
Extension: in conformal representation, XI, 78.
Extent: the term in seminvariants, XIII, 269, 363.
Extraordinaries: and non-commutative algebras, I, 128—31, 301; the term, XII, 60, 461.

Facients: defined, II, 221, IV, 604, VI, 464.
Factions: the term, IX, 426.
Factorial Expressions: summation of, III, 250—3.
Factorials: developments of, II, 98—101, 594; problems, V, 574, VII, 597; binomial theorem and derivations, VIII, 463—73; maxima of certain functions, VIII, 548—9.
Factors, Special: the term, I, 337.
Facultative: the term, VI, 156, 365; lines of cubic surfaces, VI, 450.
Facultative Points: of Sylvester, XIII, 46.
Family of Quadrics: envelope of, X, 589.
Family of Surfaces: part of orthogonal system, VIII, 269—91.
Fermat, P. de: theorem of, XI, 457, 597, 611, 615—6.
Ferrers, N. M.: conjugate partitions, II, 419; area of conic, III, 143—8; correspondence, X, 290; Legendrian coefficients, XII, 563.

Fiedler, W.: symmetric functions, II, 602.
Figures: for Pascal's theorem, VI, 116—23; mechanical construction of conformable, X, 406; use of arabic, XI, 446.
Finite Differences: formulæ in, III, 132—5, XII, 412—5; electricity on spherical surfaces, IV, 92—8, 99—107; theorem and demonstration, IV, 262—4; general equation of differences of second order, X, 47—9; Stirling's theorem, X, 267—8; table of, XI, 144—7.
Finite Groups (*see* Groups).
Finiteness: of concomitant system of quantic, VII, 334, XI, 272—80.
First Kind: of Abelian integrals, XII, 408—11.
First Order: of differential equations, XII, 432—41; (*see also* Differential Equations).
Five-dimensional Geometry: IX, 79; (*see also* Hypergeometry).
Five-pointic Contact: conic of, IV, 207—39.
Flat: the term in covariants, VIII, 406—8.
Flat-cone: the term, VIII, 102.
Flecnodal Curve: VI, 342; and torse, VI, 345, 582—5.
Flecnode: defined, II, 28—32.
Fleflecnodal Planes: of a surface, X, 262—4.
Fleflecnode: defined, II, 28—32, IX, 264; of curve in transformation of elliptic functions, IX, 170—1.
Flex: the term, V, 521.
Flexure of Skew Surface: XI, 317—22; (*see also* Surfaces).
Floquet, G.: linear differential equations, XII, 394, 402.
Fluctuating Functions: addition to Lord Rayleigh's paper, IX, 19—20.
Fluxions: and Landen, XI, 583.
Focals: of a quadric surface, XIII, 51—4.
Foci: of conics, IV, 505—9, VII, 1—4, 571; theory of, VI, 515—34; and antifoci, problem and solution, VII, 567; locus of, problem and solution, VII, 568.
Focus: the term, VI, 515, IX, 552, XI, 481.
Foot: non-Euclidian geometry, XIII, 483—4.
Forcenex, D. de: multiple algebra, XII, 466.
Forces: equilibrium of, V, 540—1, VII, 91—5; general equation of virtual velocities, IX, 205—8; resultant, X, 589; (*see also* Dynamics).
Forms: cubic, III, 9—12; quadratic, III, 11—12; theory, XI, 604—9.
Formulæ: Jacobi's canonical, for disturbed motion, III, 76—7; in finite differences, III, 132—5; for differentiation, IV, 135—49; distances of point, and tactions, IV, 510—2; signification of elementary one in solid geometry, V, 498—9; integrals for intersections of line and conic, V, 500—4; canonical form of quantics, VI, 153—4; of two sets each of four concyclic points, VI, 509—11, 512—5; focal, and polyzomal curves, VI, 547, 549; of de Jonquières, VII, 41—3; transformation of coordinates, VII, 95—6, 97—8; geodesic lines on ellipsoid, VII, 501—3, 507—10; trigonometric, XII, 108.
Forsyth, A. R.: biographical notice of Cayley, VIII, ix—xliv; addition of elliptic functions, XII, 294.
Foucault, J. B. L.: the earth's rotation, IV, 535, 536, 588.
Fouché, M.: polyhedra, IV, 609.
Fourier, J. B. J.: theorem as to roots of equations, X, 5; theory of equations, XI, 500; (*see also* Newton-Fourier Theorem).
Fourth Dimension: Cayley on, VIII, xxxiii—v.
Fraction-Theorem: Jacobi's, XII, 123—5.
Francais, J. F.: imaginaries, XII, 468.
Franklin, F.: quantics, XIII, 47.
Fresnel, A. J.: wave-surface, IV, 420, XI, 449; wave and tetrahedroid surfaces, X, 252; Sylvester on the optical theory of, XIII, 44.

Frobenius, G.: linear differential equations, XII, 394.

Frost, P.: curves of curvature near umbilicus, VII, 330—1.

Fuchs, L.: Schwarzian derivative and polyhedral functions, XI, 149; linear differential equations, XII, 394, 453.

Function: the term "derivational," I, 63; transformation of bipartite quadric, II, 497—505; relation among derivatives of, X, 590—2; octahedron, XI, 128—9; general theory of, XI, 439—41; linear, XI, 492; in *Ency. Brit.*, XI, 522—42, (introductory, XI, 522—3; known functions, XI, 523—37; functions in general, XI, 537—41; calculus of, XI, 541—2); two invariants of quadri-quadric, XIII, 67—8; on the modular $\chi\omega$, XIII, 338—41.

Functional Determinant: the term, II, 319, IV, 607.

Functional Equation: theorem of Abel, IV, 5—6.

Functions: doubly-periodic, II, 150—63; notation of algebraic, II, 185—8; al(x) of Weierstrass, V, 33—7; homotypical, V, 50; rhizic, IX, 34; tests for dependence of, X, 97; early history of theory, XI, 451—5; values of symmetric, XIII, 318—21; on lacunary, XIII, 415—7; (*see also* Schwarzian Derivative, Generating Functions, Symmetric Functions).

Fundamental Notions: in Mathematics, XI, 434—8, 442—5.

Fuss, P. H. v.: porism formula, II, 90; in-and-circumscribed polygon, II, 140, V, 21—2.

Galileo: and dynamics, XI, 447.

Galois, E.: groups and permutations, II, 134; groups, XI, 133, XIII, 533; theory of numbers, XI, 457, 593, 614; theory of equations, XI, 518—9, 520, 521; biographical notice, XI, 543.

Gamma Function: theory, I, 309—16, 588; a double infinite series, II, 8; the term, XI, 534; (*see also* Definite Integrals).

Gaultier, L.: systems of circles, III, 113; radical axis, XI, 465.

Gauss, J. K. F.: determinants, I, 64; linear transformations, I, 585; attraction of ellipsoids, III, 25—8, 149—53; central forces problem, IV, 520, 588; relative motion, IV, 534; binary quadratic forms, V, 618; *pentagramma mirificum*, VII, 37—8; *Theoria Motus*, VII, 414; geodesic lines on quadric surface, VIII, 156—61; potential of ellipse, IX, 281—2; epispheric integrals, IX, 321, 410—17; mathematical tables, IX, 466—7, 470, 472, 475—7, 488; roots of unity, XI, 60; lemniscate, XI, 64; calculation of log 2, XI, 70; geodesic curvature, XI, 323—4; his theory of surfaces, XI, 331—6; imaginary variables, XI, 439; attractions, XI, 448; theory of equations, XI, 455, 504, 516; theory of numbers, XI, 455, 599, 603; gamma function, XI, 534; biographical notice, XI, 544—5; multiple algebra, XII, 471—2; roots of algebraic equation, XIII, 35; orthomorphosis, XIII, 191; application of surfaces to each other, XIII, 253—64.

Geiser, C. F.: quartic and quintic surfaces, VII, 252; theorem of Cotterill, IX, 506.

Generating Functions: of symmetric functions, Borchardt's, II, 417, 421—3; connected with covariants, IX, 537—42; of quintic, X, 339—400; of quartic, X, 341; of sextic, and binary sextic, X, 394—400; of binary septic, X, 408—21; seminvariants of a given degree, XIII, 306—8.

Generation: of bicircular quartic, X, 223—6.

Generator: the term, V, 169—70, 173—9, 181; nodal, of scrolls, V, 169—70, 179—81.

Generatrix: and the absolute, XIII, 481—9, 501.

Genese, R. W.: theory of envelopes, XI, 50—1.

Genus of Curve: (Geschlecht), after Riemann, V, 467, 517, 619.

Geodesic Curvature: XI, 323—30.

Geodesic Lines: property of, III, 38; on oblate spheroid, VII, 15—25; on ellipsoid, VII, 34—5; in *Ency. Brit.*, XI, 628, 636—7; of pseudosphere, XII, 220—38; wave surfaces, XIII, 252; (*see also* Ellipsoids).

Geodesic Lines, in particular on quadric surface, memoir: VIII, 156—78, 188—99; preliminary formulæ, VIII, 156—8; general theory of them on a surface, VIII, 159—62; circular curves are geodesics, VIII, 162; chief lines not in general geodesics, VIII, 163; special form of geodesic equation, VIII, 163—4;

geodesics on quadric surface, VIII, 164—70; formulæ for position of point, VIII, 170—4; ellipsoid and skew hyperboloid, VIII, 174—8, 188—99; tables, VIII, 196—9.

Geometrical Construction: in optics, X, 28; of heptagon, X, 609.

Geometrical Representation: of elliptic functions, III, 3; of imaginary variables, X, 316—23; of an equation between two variables, XII, 104.

Geometry: of *n* dimensions, I, 55—62; reciprocity, I, 377—82; of quantics, II, 222; of one and two dimensions defined, II, 561—2; of one dimension, II, 563—9, 583—96; of two dimensions, II, 569—83, 586—92; relations of, metrical and descriptive, II, 592; non-Euclidian and hyper-, II, 604—6, VIII, XXXIII—V, 409—13, XII, 220—38; Lobatschewsky's imaginary, V, 471; problem of permutation, V, 493—4; signification of elementary formula, V, 498—9; notion of absolute, V, 550; drawings in, VI, 9; constructive, VII, 26—30; transformation, VII, 121—2; Cayley and Klein on metrical, VIII, XXXVI—VII; hyperbolic, elliptic and parabolic, VIII, XXXVII; Cayley's work in analytical, VIII, XXXVIII; formulæ relating to right line, X, 287—9; considerations on solar eclipse, X, 310—5; interpretation of algebraic equations, X, 581; solid, XI, 224; Schubert's numerative, XI, 281—93; Mill on, XI, 432—4; Euclidian, XI, 434—7; Cartesian, XI, 437—9; abstract, XI, 441—2; origin, XI, 445—8; in Greece, XI, 446; evolution of descriptive, XI, 448—9; date of extensions in, XI, 449—51; plane and solid, XI, 450—1; function in, XI, 522—3; interpretation of elliptic function formulæ, XII, 107; d'Alembert Carnot paradox, XII, 305—6; of the compass, XII, 314—7; algebra and logic, XII, 459; (*see also* Hypergeometry: for General Theory, *see* Quantics, sixth memoir).

Geometry, Abstract, Memoir on: VI, 456—69, 596; introductory, VI, 456—7; space, VI, 456—7; general explanations, VI, 457—62, 596; omal relation, order, VI, 463; parametric relations, VI, 463—4; quantics, notations, etc., VI, 464—6; resultant, discriminant, VI, 466—7; consecutive points, tangent omals, VI, 467—9.

Geometry, Analytical, in Ency. Brit.: XI, 546—82; introductory, XI, 546; Part I, pure analytical, XI, 546—67; is descriptive, XI, 552—6; metrical theory, XI, 556—7; equations of right line and circle—transformation of coordinates, XI, 558—61; the conics, XI, 561—4; tangent, normal, circle and radius of curvature, XI, 564—5; coordinates, XI, 566—7; Part II, solid analytical geometry, introductory, XI, 567—9; metrical theory, XI, 570; line, plane, and sphere, XI, 571—2; cylinders, cones, ruled surfaces, XI, 572—3; transformation of coordinates, XI, 573—6; quadric surfaces (paraboloids, ellipsoids, and hyperboloids), XI, 576—9; curves: tangent, osculating plane, curvature, XI, 579—80; surfaces: tangent lines and plane, curvature, XI, 580—2; (*see also* Hypergeometry).

Geometry of Position: theorems in, I, 317—28, 356—61, 414—20, 550—6, 567, 588.

Gergonne, J. D.: caustics, II, 118, 339, 341, 368; polyzomal curves, VI, 520.

Geschlecht: (genus) of curve, after Riemann, V, 467, 517, 619.

Glaisher, J. W. L.: notation for elliptic functions, I, 548; definite integration, VIII, 1; centro-surface of ellipsoid, VIII, 364; report on mathematical tables, IX, 461—99; development of an idea of Eisenstein, X, 58—9; proof of Stirling's theorem, X, 267—8; quadrilateral inscribable in circle, X, 578; log 2, XI, 70; elliptic functions, XI, 73; least factors of numbers, XI, 430; modular function $\chi\omega$, XIII, 338—41; theta and omega functions, XIII, 558—9.

Glide: the term, I, 236.

Glover, J. W.: on theory of groups, XIII, 533.

Goniometry: Cotterill's problem in, X, 295—7.

Göpel, A.: theory of numbers, IV, 41; theta functions, VIII, XLII, X, 464, 499, XII, 363—4; double theta functions and 16-nodal quartic surface, X, 157, 162, 172, 173, 175, 180—1; table of tetrads, 508, 549—51; double theta functions, XI, 454.

Gordan, P.: binary quintic and sextic, VI, 190; irreducible covariants of binary quantic, VII, 334, 341, 348—53; covariants of binary quantic, VIII, 566; finiteness of concomitant systems, X, 286; derivatives, X, 340, 377; Schwarzian derivative and polyhedral functions, XI, 149, 199; finite groups, XI, 237—41; covariantive forms and tables, XI, 272; concomitants of ternary cubic, XI, 343; Abelian functions, XII, 102, 109; icosahedral substitutions, XIII, 552.

Goursat, E.: Kummer's differential equation, XIII, 69—73.
Graduation Curve: the term, VII, 426.
Graham, A.: Pellian equation, XIII, 442.
Grant, R.: report on Catalogue of Memoirs, V, 546—8, 620.
Graphical Construction: in solar eclipses, VII, 390—1, 479—92; geodesic lines on ellipsoid, VII, 507, 510; theory of groups, X, 403—5.
Grassmann, H.: multiple algebra, XII, 465, 480—9.
Gravelius, H.: on non-Euclidian geometry, XIII, 481.
Graves, J. T.: algebraic couples, I, 128; geometry of position, I, 319, 414—20; imaginaries, I, 586.
Gravity: and relative motion, IV, 534—7; action at equator, IX, 241—3; Clairaut's theorem, X, 17—8; effects of theory, XI, 447—8.
Greatheed, S. S.: expansions in multiple sines, I, 583; elliptic motion, IV, 522, 588.
Greek: Cayley's knowledge of, VIII, xxiv; geometry, arithmetic, and algebra, XI, 446.
Green, G.: attractions of ellipsoids, I, 582; attraction of terminated straight line, VII, 32; potentials of polygons and polyhedra, IX, 279; integration of prepotential equation, IX, 320—1, 343, 393—404; attractions, XI, 448.
Greenhill, A. G.: deformation and flexure of surfaces, XI, 66—7.
Greenians: the term, IX, 393.
Greer, H. R.: locus, envelope, and triangle, VI, 72.
Gregory, D. F.: differential and integral calculus, VIII, 272.
Griffiths, J.: series of triangles, VII, 599; curve of sixth order, X, 612; deduction from $y=\sin(A+B+C+\ldots)$, XII, 58—9.
Ground-curve: and Abel's theorem, XII, 38, 109—216.
Groups: of lines and points, I, 317—28, 356—61, 414—20, 550—6; depending on symbolic equation $\delta^n=1$, II, 123—30, 131—2, IV, 88—91; theory of, IV, 88—91, X, 324—30, XII, 639—56; the term, IV, 594, 596, VII, 123; rotations of polyhedra, V, 529, 559; Cayley's work at, VIII, xxxiii; theorems on, X, 149—52, 153—4; desiderata and suggestions on theory, X, 401—6; partitions and theory of, XI, 62; homographic transformations, XI, 189—90, 196—208, 237—41; linear transformation of a variable, XI, 237—41; Jacobian sextic, XI, 389, 393—6; the notion, XI, 509—10; Latin squares, XIII, 55—7; substitution groups for two to eight letters, XIII, 117—49; of points, non-existence of a special, XIII, 212; quotient G/H in theory of, XIII, 336—7; illustrations of Sylow's theorems on, XIII, 530—3; of sixty icosahedral substitutions, XIII, 552, 556.
Grunert, J. A.: difference-table, XI, 144.
Gudermann, C.: elliptic integrals, I, 224; transformation of an integral, I, 383; transformation of elliptic functions, III, 2; spherical pendulum, IV, 534, 588; logarithms, VII, 414.
Gudermannian: V, 86—8, 617; Lobatschewsky's imaginary geometry, V, 472; tables of, V, 617.
Gundelfinger, S.: concomitants of ternary cubic, XI, 342—3.

Hall, A.: motion of particle towards attracting centre, IX, 215—7.
Halphen, G. H.: inverse elliptic functions, I, 586; curves in space, V, 613—7; higher singularities of plane curves, V, 619; curves satisfying given conditions, VI, 594—5; classification of curves, XI, 451; invariants of differential equations, XII, 392—3; transformation in elliptic functions, XIII, 29; Sylvester on reciprocants, XIII, 48; reciprocants, XIII, 333, 366, 368—81, 381—98; characteristic n in the theory of curves in space, XIII, 468—72.
Halsted, G. B.: hyperspace and non-Euclidian geometry, II, 606.
Hamilton, Sir W. R.: quaternions, I, 123—6, 238, 335, 586; form of equations of motion, I, 284; homographic transformation of quadrics, II, 105, 133; focal relations, II, 143; problem of three or more bodies, III, 97—103; differential equations, III, 164, IV, 514; method of dynamics, III, 166—74, 200, 202; equations of motion, III, 186; isochronism of circular hodograph, III, 262—5;

essential singularity of function, IV, 105; central forces problem, IV, 520; hodograph, IV, 520; transformation of coordinates, IV, 558—9, 588; ray systems, VIII, 504, XII, 571—5; surface orthogonal to set of lines, IX, 587; system of differential equations, X, 113—8; equations of central orbit, X, 613; on mathematics, XI, 431; algebra and time, XI, 443; conical refraction, XI, 449; multiple algebra, XII, 460, 466, 474—5; Sylvester on Hamiltonian numbers, XIII, 48; (*see also* Differential Equations).

Hammond, J.: theory of tamisage, XI, 409—10; seminvariants, XII, 253; Sylvester's reciprocants, XIII, 47—8, 381, 388; on Hamiltonian numbers, XIII, 48.

Hansen, P. A.: lunar theory, III, 13—24, 291—2; elliptic orbit, III, 95; expansion of true anomaly, III, 140; planetary theory, III, 268—9, IX, 180—3; disturbed elliptic motion, III, 270—1; disturbing function in lunar theory, III, 293, 319—43; variation of plane of planet's orbit, III, 516—8; elliptic motion, IV, 522, 523, 588; relative motion, IV, 536, 588; pendulum, IV, 541, 588; spheroidal trigonometry, IX, 197.

Hargreave, C. J.: on differential equations, VIII, 458.

Harley, R.: equation of differences, IV, 241, 245; symmetric products and quintics, IV, 310—13; quintics, V, 53; a differential equation, VII, 354; theory of equations, XI, 520; invariants, XII, 390—1.

Harmoconic: defined, V, 342.

Harmonic Relations: of two lines or points, II, 96—7; theory of, and two or more quadrics, II, 529—40.

Harmonics: symmetric, II, 555; inscribed, III, 113; reciprocal lines, XIII, 58—9; and non-Euclidian geometry, XIII, 482—9.

Harriot, T.: mathematical discoveries, XI, 437.

Hart, A. S.: cubic curves, IV, 499; relative motion, IV, 535; triple tangent planes, VI, 372, 375; nine-point circle, XIII, 548.

Haughton, S.: inertia, IV, 564—5, 588.

Haupttangenten: (inflexional tangents), VIII, 157.

Heal, W. E.: bitangents of quintic, XIII, 21.

Hearn, G. W.: on a geometrical locus, I, 496; quartic curves, I, 496; quadric curves, V, 262.

Heath, R. S.: non-Euclidian geometry, XIII, 481, 499.

Helmholtz, H. von: hydrodynamical equations, XIII, 6—8.

Hemihedron: the word, X, 328.

Hemipolyhedron: the word, X, 328.

Hensley, P. J.: foci of conics, IV, 505—9.

Heptacron (*see* Polyacra).

Heptagon: construction, X, 609.

Hermite, C.: homographic transformation of quadric into itself, II, 107; elliptic integral and covariants of quartic, II, 191; law of reciprocity, II, 232, 234; skew invariant of quintic, II, 233; transformation of quadric function, II, 499; hyperdeterminants, II, 598—601; elliptic integrals, IV, 68—9; ternary cubics, IV, 326, 330; Tschirnhausen's transformation, IV, 364—7, 375, VI, 165, 170; automorphic transformation, IV, 416; elliptic functions and solution of quintic, IV, 484—9; matrices, V, 438, XII, 367—70, 386; quantics, VI, 147; quintic equation, VI, 170; nodal cubic, VI, 174—6; canonical form of quintic, VI, 177—83; transformation of elliptic functions, VII, 44, IX, 113, XII, 337, 416—7, XIII, 31, 39; reduction of Abelian integrals, X, 214; concomitants of ternary cubic, XI, 342; elliptic functions, XI, 452; theory of equations, XI, 520; Abelian functions, XII, 98; transformation of double theta functions, XII, 358; H-product theorem, XII, 584—6; cubic equations, XIII, 349; omega functions, XIII, 558.

Herschel, Sir J. F. W.: finite differences, IV, 95, 107, 262; Brinkley's formulæ, X, 58—9; difference table, XI, 144.

Hesse, L. O.: linear transformations, I, 87, 113—6, 232—3, 584, 585, VI, 22—3, 73; cubic curves, I, 194, II, 399; involution, I, 259; hyperdeterminants, I, 354; abstract of memoir on quadric surfaces, I, 425—7; inflexions of cubic, I, 584, II, 29, III, 48; on elimination, IV, 3; on inflexions, IV, 186, XI, 473; double tangents, IV, 186—7, 343—6; cubic forms, IV, 353; geometric transformation, VII, 121—2; bitangents of quartic curve, VII, 123—4; double theta functions, X, 177; triple theta functions, X, 446, 448, 451; the thirty-four concomitants of ternary cubic, XI, 342; bitangents of a plane quartic, XII, 76.

Hessians: and Eisenstein's theorem, I, 585; defined, II, 319, IV, 607, XI, 471, 474; and Pippian, II, 381—2, 383—95, 416; the sign for, II, 541; the quadricovariant, II, 545; of quaternary functions, IX, 90—3; of a quartic surface, X, 274—7; (*see also* Quantics).

Hexagon: and conic, problem, V, 576; theorem of inscribed, XI, 556.

Hexagram (*see* Pascal's theorem).

Hexahedron: edges of, problem, X, 613; automorphic function for, XI, 184.

Hierholzer, C.: cones satisfying six conditions, VIII, 99; surface of eighth order, VIII, 401.

Higher Singularities of Curves (*see* Singularities).

Hilbert, D.: curves in space, V, 614; abstract geometry, VI, 596.

Hills: altitude, and roots of algebraic equation, XIII, 33—7.

Hirsch, Meyer: algebra of, II, 417, 440; partitions, VII, 577.

Hirst, T. A.: negative pedals, IV, 164.

History: importance of mathematical, VIII, xii.

Hodograph: isochronism of circular, III, 262—5; Hamilton's, IV, 520; and pedal curves, V, 113.

Hölder, O.: theory of groups, XIII, 336, 533.

Holditch, H.: caustics, II, 363.

Holomorphic: the term, XI, 81.

Homaloids: prepotentials of, IX, 408—9.

Homographic Figures: matrices, II, 219; theorem, IV, 442—5.

Homographic Function: distribution of electricity, XI, 2—6; and matrix of the second order, XI, 252—7.

Homographic Relations: and theory of numbers, IX, 613; function of, X, 298—306; powers of, X, 305—6, 307—9.

Homographic Transformation: of quadric surfaces, II, 105—112; of single theta functions, XII, 337—43; (*see also* Groups, Transformations).

Homographies: correspondence with rotations, X, 153—4, XI, 237—8.

Homography: different classes of, II, 219—220; and quadrics, II, 536—40; ternary quantics and theory of, II, 565—9, 578; of ranges and pencils, II, 578.

Homology: of conics, I, 523, 557; the term, I, 557—8; of sets, III, 35.

Homotypic: the term, VII, 123.

Homotypical: the term, V, 50.

Hopkins, W.: attractions and multiple integrals, I, 582.

Horner, W. G.: theory of equations, XI, 499, 502.

Hoüel, J.: equations of motion, III, 169, 170.

Hudson, C.: equal roots of equations, XI, 405—7.

Hydrodynamics: note on equations of, XIII, 6—8.

Hydrogen: trees of, IX, 427—60, 544—5.

Hyperbola: rectangular, III, 254, V, 554; classification, V, 354—400, VIII, xxxviii; arc of, XI, 337—9; and cubic curves, XI, 478; in *Ency. Brit.*, XI, 561—4; focals of quadric surface, XIII, 54; and epitrochoid, XIII, 82—7; tactions, XIII, 150—69; (*see also* Conics).

Hyperboloid: coordinates of, V, 72; and scrolls, VII, 65; and tractor, VII, 73—5; passing through three given lines, VII, 86—8; Mercator's projection, VIII, 568, IX, 237—40; deformation, XI, 66—7; in *Ency. Brit.*, XI, 576—9; confocal Cartesians and right lines of, XII, 587—9.

Hyperdeterminants: the term, I, 81, 95, 114, 585; note on, I, 352—5, 588; a system of certain formulæ, I, 533; theory, I, 577—9; theory of permutants, II, 19; theory of intermutants, II, 26; quantics, II, 225; theory of seminvariants, XII, 344; Sylvester's work in, XIII, 46; an identity, XIII, 210—11; (*see also* Covariants, Invariants).

Hyperdimensional Space: quadrics in, IX, 79—83; (*see also* Hypergeometry, Hyperspace).

Hyperelliptic Functions: trisection of, VI, 594; and theta functions, X, 162—5, 166—79, 184—214, 551—5; and triple theta functions, X, 432—6; addition-theorem, X, 455—62; the term, XI, 533—4; and nodal quartics, XII, 196—208; (*see also* Theta-Functions).

Hyperelliptic Integrals: of first order, XII, 98—9.

Hypergeometric Series: summation of a certain factorial expression, III, 250—3; theorem, III, 268—9; differential equations, XI, 17—25; note on, XI, 125—7; and Schwarzian derivative, XI, 176—9.

Hypergeometry: of n dimensions, I, 55—62; a branch of mathematics, VIII, xxxiii—v; five-dimensional, IX, 79—83; and quadric surfaces, IX, 246—9; 21 coordinates of conic in space, XI, 82—3; Sylvester's work in, XIII, 46; (*see also* Hyperspace, Prepotentials).

Hyperspace: and quantics, II, 222; and non-Euclidian geometry, II, 606; representation by means of, VI, 198; of four dimensions, special theorem, IX, 246—9; (*see also* Hypergeometry).

Icosahedra: construction, IV, 81—2; axial systems, V, 531—9; Klein on rotations of, X, 153; as regular solids, X, 270—3; automorphic function, XI, 169, 179—83, 185, 212—6.

Icosahedral Substitutions (*see* Substitutions).

Ideal: the term, VI, 483.

Ideal Numbers: XI, 456.

Idem: defined, XII, 66.

Idempotent: the term, XII, 61.

Identities: cubic, V, 597; trigonometrical, VIII, 525, XI, 38, XIII, 538—40; elliptic transcendent, VIII, 564; a transcendental, XI, 37; algebraic, XI, 63—4, 130—1, XIII, 76—8; a hyperdeterminant, XIII, 210—11.

Imaginaries: on an octuple system of, I, 301; eight-square, XI, 368—71, XII, 465; the term, XI, 439; theory of equations, XI, 502—6; and function, XI, 523; associative, XII, 61, 105—6; perpendicularity, XII, 466—72; roots of equation, XIII, 36; Sylvester's work at, XIII, 46; quaternions, XIII, 542.

Imaginary Quantities: logarithms, VI, 14—8; geometrical construction relating to, XI, 258—60.

Immit: defined, IV, 109.

Improper: conditions for curves, VI, 193.

Increment: the term, VI, 468.

Indefinite: applied to integration, IX, 500—3; the term, XIII, 290.

Indeterminate Equations: problem in indeterminate analysis, III, 205—7.

Index: to philosophic memoirs, report on, V, 546—8, 620.

Indicial Equation: of differential equation, XII, 398, 453.

Indicial Function: of differential equation, XII, 398, 401.

Inertia: axes and moments of, IV, 478—80, 559—66.

Ineunt: defined, II, 574, V, 521, VI, 469; non-Euclidian geometry, XIII, 489.

Infinitesimal Rotations: VI, 24—6.

Infinity: in geometry, XI, 464.

Inflexional Tangents: and geodesic lines, VIII, 157; (*see also* Tangents).

Inflexions: of cubical divergent parabolas, V, 284—8; of cubic curve, I, 584, III, 48; Hesse on, IV, 186, V, 493—4, XI, 473; of curves, XI, 471—3, 480.

Integral Calculus: some formulæ of, I, 309—16, 588; transformation, I, 383; Picard's memoir on, XII, 408—11.

Integral Functions: Legendre's coefficients, I, 375—6; the term, IV, 603—4, XI, 523; prepotential surface, IX, 321—30, 330—4, 352—9; potential solid, IX, 334—7; epispheric, IX, 410—17; reduction

of transcendental, x, 214—22; hyperelliptic, of first order, XII, 98—9; regular, of differential equation, XII, 395—6; number of, XII, 399; subregular of differential equation, XII, 444—52; (*see also* Abelian Integrals, Definite Integrals, Elliptic Integrals, Transformations).

Integrals: transformation of double, IX, 250—2; of differential equations of first order, X, 19.

Integration: a supposed new, VII, 36; theorem of, VII, 588; by series of differential equations, VIII, 458—62; a process of, IX, 257—8, X, 15, 29; indefinite, IX, 500—3; Aronhold's formula, X, 12—14; of Euler's equation, XI, 68—9.

Integrator: mechanical, XI, 52—4.

Intercalation: root-limitation, IX, 22—7; for right line, IX, 28—33; Sylvester's theory of, XIII, 46.

Intermediates: of two quantics, defined, II, 515; of binary quartic, II, 549; and ternary cubics, IV, 326.

Intermutants: the term, II, 19, 26, IV, 594, 600.

Interpolation: Smith's Prize dissertation, VIII, 551—5.

Intersect-developable: of two quadrics, I, 486—95.

Intersections: the term, VII, 546; of two curves, IX, 21, XII, 117—20; of cubic and line, XII, 100.

Invariable Plane: and rotation of solid body, I, 237, VI, 142.

Invariants: the term, I, 577, 589, II, 176, 224, IV, 594, 605, XIII, 46; and discriminants, I, 584; determined by differential equations, II, 164—78; and roots, II, 176; differential equation satisfied by, II, 176—8; and binary quantics, II, 266—8; of quartic, and covariants of cubic, analogous, II, 553; bibliography, II, 598—601; of biternary quantics, IV, 349—58; 18-thic of quintic in terms of roots, VI, 154—6; Cayley founder of, VIII, xxviii—xxx; his work, VIII, xxx—xxxii; and transformation of quantics, VIII, 385—7; quadratic transformation of a binary form, VIII, 398—400; identical equation connected with theory, IX, 52—5; Hessian of quaternary function, IX, 90—3; minimum N. G. F. of binary septic, X, 408—21; stereographic projection, XI, 187—9; in geometry, XI, 474; of a linear differential equation, XII, 390—3; Sylvester's work at, XIII, 46, 47; two, of quadri-quadric function, XIII, 67—8; differential, and reciprocants, XIII, 366—404; Pfaff-, XIII, 405—14; (*see also* Covariants, Linear Transformations, Seminvariants).

Invariants and Covariants: XII, 22—9; standard solutions of system of linear equations, XII, 19—21; finite number of the covariants of a binary quantic, XII, 558.

Inversion: of quadric surface, VIII, 67—71; note on, IX, 18.

Inverts: quadric function of, XI, 153—6.

Involutant: of two binary matrices, XIII, 74—5.

Involution: theory of geometrical, I, 259—66, 587; and two or more quadrics, II, 529—40; of six lines, IV, 582, VII, 66, 85, 95; lines in, V, 1—3; theory, V, 295—313; the term, VI, 460; of four circles, VI, 505—8; and ternary quadrics, XIII, 350—3.

Involution of Cubic Curves, Memoir: V, 313—53, VII, 238; explanations, definitions, and results, V, 314—8; general formulæ for critic centres, V, 318—9; twofold and one-with-twofold centre, V, 319—24; tangents at a node, V, 325—8; triangle of critic centres, V, 328—9; the three-centre conic, V, 329—36, 337—8; transformation equation of cubic, V, 339—41; cubic locus, harmoconics, and harmonic conic, V, 341—5; miscellaneous, V, 345—53.

Irrational: and subrational, IX, 315.

Irreducible: the term, VII, 336, XII, 23.

Irreducible Concomitants: of quintic, X, 342.

Irreducible Covariants: and invariants, II, 250.

Irreducible Syzygies (*see* Syzygies).

Irregular: the term, VI, 457, 459.

Iseccentric Lines: and planet's orbit, VII, 468.

Isobaric: the term, XIII, 266.

Isobarism: of covariants, II, 233.

Isochronic: the term, nodal and cuspidal, VII, 473.

Isochronism: of circular hodograph, III, 262—5.
Isomers: Mathematical theory of, IX, 202—4.
Isoparametric Lines: and planet's orbit, VII, 467.
Isoperimetrical Problem: VII, 263.
Isothermals: of Meyer, XIII, 175.

Jacobi, K. G. J.: theory of algebraic curves, I, 53; determinants, I, 63, 64, 66; quaternions, I, 126, 127, 586; inverse elliptic functions, I, 132, 136, 152, 156, 162, 180; rotation of solid body, I, 238, IV, 575, 576—7, 579; involution, I, 259, 263; definite integral, I, 270—1; dynamical differential equations, I, 276—9; elliptic functions, I, 290—300, 507, 587, XI, 452; demonstration of theorem on focal lines, I, 362—3; differential equations of Abelian functions, I, 366—9; skew determinants, I, 411; simultaneous linear transformations, I, 428; transformation of integrals, I, 440, 442; attraction of ellipsoids, I, 511—8; solution of equation $x^{2\sigma}-1=0$, I, 564; Lagrange's theorem, II, 7; geometrical representation of elliptic integral, II, 56; in-and-circumscribed polygon, II, 141; partition of numbers, II, 248; lunar theory, III, 13; canonical formulæ for disturbed motion, III, 76—7; problem of three or more bodies, III, 102; transformation, III, 129; finite differences, III, 132—5; complete integral of partial differential equation, III, 166; planetary theory, III, 173; calculus of variations and differential equations, III, 174—84, 200, 202; *De Motu Puncti singularis*, III, 182—3, 202; problem of three bodies, III, 519—21, IV, 541, 548—51, 589, V, 23, VI, 183; *Theoria Novi Multiplicatoris*, III, 183—5; theory of ideal coordinates, III, 185; Encke's memoir *über die speciellen Störungen*, III, 179—80; in-and-circumscribed triangle, III, 236; disturbed elliptic motion, III, 270—1; canonical system of formulæ, III, 290; reversion of series, IV, 30—7; transformation of elliptic integrals, IV, 60, 64; double tangents, IV, 187; conics, IV, 207; Pfaff's problem, IV, 359—63; central forces problem, IV, 520, 589; *Nova Methodus*, IV, 515, 521, 589; elliptic motion, IV, 522, 589; problem of two centres, IV, 530, 532, 589; motion of a single particle, IV, 537—8, 589; motion in resisting medium, IV, 541, 589; motion of a point, IV, 547, 589; elimination of nodes, IV, 551, 589; transformation of coordinates, IV, 554, 557, 589; Weierstrass's function al (x), V, 34—5; transformation of elliptic functions, V, 472, IX, 103, 113—75, XII, 59, 505—34; *Canon Arithmeticus*, VI, 83—6, XI, 85, 86; the Jacobian relation, VI, 467; geodesic lines on ellipsoid, VII, 493; transformation of double integral, IX, 250—2, 254; epispheric integrals, IX, 321, 410—17; mathematical tables, IX, 472—3, 484—5; series, X, 25—7; Poisson's theorem, X, 108—9, 110—3; theta functions, X, 156, 473, 478, 490, 496—7, XI, 41—6, XIII, 559; roots of unity, XI, 58—60; Schwarzian derivative and polyhedral functions, XI, 149; hypergeometric series, XI, 178; Landen's theorem, XI, 339; Abelian functions, XI, 454; theory of numbers, XI, 602; theorem in simultaneous equations, XII, 39; fraction theorem, XII, 123—5; Weierstrassian and Jacobian elliptic functions, XII, 425—7; sextic equation, XI, 389—401, XII, 493—9; sums of two series, XIII, 50; modular equations, XIII, 64; characteristic n and curves in space, XIII, 469; sextic resolvent equations, XIII, 473—9.
Jacobian: defined, II, 319, IV, 607; of two quantics, II, 517; theory of, and polyzomal curves, VI, 566—8; the extended notion, VII, 134; of surfaces, VII, 134—6; of two curves, IX, 21; of quadric surfaces, X, 568; of six points, X, 613; rational transformation, XIII, 116; (*see also* Quantics).
Jefferson, T.: founder of Virginian mathematical professorship, XIII, 43.
Jeffery, H. M.: on quartic curves, XI, 408.
Jellett, J. H.: theorem of, on attractions, I, 388—91.
Jenkins, M.: Arbogast's method of derivations, IV, 609; problem in factorials, VII, 597.
Jerrard, G. B.: quintics, IV, 391—4, V, 50—4, 77, 89; theory of equations, XI, 520.
Joachimsthal, F.: theory of covariants, II, 177, 234; theorem of, II, 521, XII, 594—5, 601, 629; normals of a conic, IV, 74—7; attraction of terminated straight line, VII, 33.
Johnson, W. W.: matrices, XI, 252.

Jordan, C.: trees, x, 599, xi, 366; substitutions, xi, 455; theory of equations, xi, 520—1; uniform convergence, xiii, 343—4.

Joubert, P.: transformation of elliptic functions, ix, 113, 138.

Kant, I.: cognition and space, xi, 431; negative magnitude, xi, 434.

Kelland, P.: quaternions, xii, 475.

Kelvin, Lord: equimomental surfaces, i, 253—4; definite integral, i, 270; differentiation and evaluation of definite integrals, i, 587; relative motion, iv, 535, 593; inertia, iv, 565, 566, 593; attraction of terminated straight line, vii, 33; report on mathematical tables, ix, 461—99; distribution of electricity, xi, 6.

Kempe, A. B.: theory of groups, xii, 639.

Kenogram: the term, ix, 202—3.

Kepler, J.: problem of, iv, 521; and ellipse, xi, 447.

Key, T. H.: Professor of Mathematics, University of Virginia, xiii, 43.

Kiepert, L.: transformation of elliptic functions, xii, 490—2.

Kinematics: of solid body, iv, 580—2; six coordinates of a line, vii, 89—95; of a plane, xi, 103—110, xiii, 505—16.

Kirkman, T. P.: schoolgirl problem, i, 483, 589; geometry of position, i, 550—6; sums of squares, ii, 49—52; double summation of factorial expression, iii, 250—3; autopolar polyhedra, iv, 85, 182; enumeration of polyhedra, v, 38; partitions of a polygon, xiii, 93.

Klein, F.: theory of distance, ii, 604, viii, xxxvi—vii; coordinates in non-Euclidian geometry, ii, 604—6; non-Euclidian geometry, viii, 409—13, xiii, 481; correspondence of homographies and rotations, x, 153; a system of quadric surfaces, x, 269; octahedral function, xi, 128; Schwarzian derivative and polyhedral functions, xi, 149, 151, 179, 183, 185; his classes of geometry, xi, 436; transformation of elliptic functions, xii, 29, 46, 490, 547.

Knots: of trees, iii, 243, ix, 429—60, xi, 365—7, xiii, 26—8; in topography, defined, iv, 104.

Königsberger, L.: transformation of elliptic functions, ix, 113, 140; theta functions, x, 499.

Korndörfer, G.: quartic and quintic surfaces, vii, 246, 249, 252.

Kowalski, M.: orbit of Neptune, ix, 180.

Kreistheilung: (cyclotomy), xi, 58, 86.

Kronecker, L.: equation of differences, iv, 609—16; quintics, v, 55; theory of equations, xi, 520; sextic resolvent equations, xiii, 473—9.

Kummer, E. E.: solution of equation $x^{257}-1=0$, i, 564; wave surfaces, i, 587; composition of numbers, iv, 70—1, 78—9; Steiner's quartic surface, v, 423; 16-nodal quartic surfaces, v, 431—7, vii, 126—7, x, 437; quartic surfaces, vii, 134, 176, 313, ix, 71; quintic surfaces, vii, 245, 247, 252; nodal quartic surfaces, vii, 264—297; surface of, viii, 69; prepotentials, ix, 364; mathematical tables, ix, 494; double theta functions and 16-nodal quartic surface, x, 158, 161; octic surface, x, 252; theta functions, x, 499; table of hexads, x, 506, 538, 552; hypergeometric series, xi, 17—25, 177; Schwarzian derivative and polyhedral functions, xi, 149; theory of numbers, xi, 456; systems of rays, xii, 573; differential equation of third order, xiii, 69—73.

Lachlan, R.: circles and spheres, i, 581, xiii, 13.

Lacroix, S. F.: transformation of coordinates, iv, 557; imaginaries, xii, 468.

Lacunary Functions: xiii, 415—7.

Ladd, Christine: the Pascal hexagram, vi, 594.

Lagrange, J. L.: theorem of expansion, i, 40—2, 584, ii, 1—7, iii, 141; determinants, i, 64; variation of parameters, i, 243; dynamical differential equations, i, 279; elliptic functions, i, 366; distances of points, i, 581; attractions and multiple integrals, i, 581—2; caustics, ii, 353; problem of two fixed centres, iii, 104—10, iv, 527, 528—9, 589; sums of series, iii, 124; *Mécanique Analytique*, iii,

157—8, 201, 202; equations of motion, III, 158, 200, IX, 198—200; planetary theory, III, 159—61, 162—3, 201; variation of arbitrary constants in mechanical problems, III, 161—5, 200, 201; coefficient (a, b) of, III, 163; Hamilton's method of dynamics, III, 171—3, 200; disturbed elliptic motion, III, 271—2; equations of differences, IV, 240, 252; resolvents, IV, 309; central forces problem, IV, 519—20, 589; elliptic motion, IV, 521—2, 589; expansion of anomalies, IV, 522; spherical pendulum, IV, 532—3, 589; rotation of solid body, IV, 566, 569, 589; homotypical functions, V, 50; invariable plane, VI, 142; invariants, VIII, XXX; demonstration of Taylor's theorem, VIII, 493—5, 519; virtual velocities, IX, 205—8; prime roots of unity, XI, 57; Schwarzian derivative, XI, 149; theory of equations, XI, 455, 498—500, 514—5, 517, 520; envelopes, XI, 475; minimal surfaces, XI, 638; five points in space, XII, 581—3; theorem of expansion and partitions of polygon, XIII, 113; Waring's formula, XIII, 215—6; reciprocants, XIII, 366; Richelot's integral of Euler's equation, XIII, 526.

Laguerre, E.: theory of matrices, II, 604.

Lamb, H.: conformal representation, X, 290.

Lambdaic: defined, II, 523, IV, 49, 53; of binary quartic, II, 550.

Lambert, J. H.: theorem on circular hodograph of, III, 262—5; theorem for elliptic motion, III, 562—5, VII, 387—9; central forces problem, IV, 519, 520, 589; equation of planet's orbit from three observations, VII, 412—5.

Lamé, G.: curvilinear coordinates, VIII, 146, XI, 637, XII, 17; orthogonal surfaces, VIII, 280, 291, 292.

Lamp, Milner's: a differential equation, and construction of, XIII, 3—5.

Lancret, M. A.: curves of curvature, XII, 601.

Landen, J.: theorem of, in elliptic functions, XI, 337—9; biographical notice, XI, 583—4.

Languages: Cayley's knowledge of, VIII, xxiii.

Laplace, P. S.: on Lagrange's theorem, I, 42, II, 7; determinants, I, 63; functions of, I, 397—401, 588; attraction of ellipsoids, I, 581, III, 53—65, 567; planetary theory, III, 159, 201; disturbed elliptic motion, III, 505, 510—11; on secular variation, III, 568; elliptic motion, IV, 524, 589; relative motion, IV, 534, 536, 589; motion of three bodies, IV, 540—1, 589; prepotentials, IX, 393; finite differences, XII, 412.

Last Multiplier: IV, 530, 547, 551, 590.

Latin Squares: XIII, 55—7.

Latitude: parametric, VII, 16, IX, 238.

Lattice: in theory of numbers, III, 40.

Laverty, W. H.: systems of equations, VII, 578.

Law, The: Cayley's work at, VIII, xiii—xv, xix.

Lebesgue, V. A.: determinants, I, 63.

Lectures: delivered by Cayley, VIII, xvi—xvii, xlv.

Lefort, F.: elliptic motion, IV, 522, 589.

Left-handed: circuits in root-limitation, IX, 22—3.

Legendre, A. M.: elliptic functions, I, 136, 156, 507, V, 618, XI, 452, 537, 584, XII, 35—7; elliptic integrals, I, 224; coefficients of, I, 375—6; attraction of ellipsoid, I, 432—7, 442; functions of, IV, 99, 106; rectilinear motion, IV, 516, 590; central forces problem, IV, 521, 590; problem of two centres, IV, 530, 590; rotation of solid body, IV, 570, 590; geodesic lines on oblate spheroid, VII, 15—25; reduced latitude, VII, 16; orbit of planet from three observations, VII, 414; mathematical tables, IX, 467—8, 478, 487; Landen's theorem, XI, 339; theory of numbers, XI, 455, 602—4, 616, XII, 35—7; second kind of coefficients, XII, 562—3; gamma function, XIII, 49.

Lejeune-Dirichlet, P. G.: multiple integrals, I, 195, 582—3; integration, I, 588; theorem of, II, 10, 47—8; binary quadratic forms, V, 141; prepotentials, IX, 321, 417—23; attractions, XI, 448; theory of numbers, XI, 456, 616.

Lemniscate Function: XI, 65; and orthomorphosis, XIII, 191—205.

Letters: substitution groups for two to eight, XIII, 117—49.

Leverrier, U. J. J.: disturbing function in planetary theory, III, 321, VII, 511—27; elliptic motion, III, 361, 362, IV, 523, 590; position of orbit in planetary theory, VII, 545.

Lévy, M.: orthogonal surfaces, VIII, 269, 569—70; Dupin's theorem, IX, 85.

Light, Polarized: MacCullagh's theorem, IV, 12—20.

Limaçon of Pascal: I, 480, XI, 477.

Lindemann, F.: non-Euclidian geometry, XIII, 481.

Linear: and omal relations in abstract geometry, VI, 463.

Linear Differential Equations: invariants of, XII, 390—3; general theory, XII, 394—402, 444—52; decomposition, XII, 403—7.

Linear Equations: and determinants, XI, 490; standard solutions of system of, XII, 19—21.

Linear Function: the term, XI, 492.

Linear Quantics (*see* Quantics).

Linear Substitutions: note on a function in, X, 307—9.

Linear Transformations: theory of, I, 80—94, 95—112, 117, 584, 585; Eisenstein's and Hesse's formulæ, I, 113—6, 585; homogeneous functions of third order with three variables, I, 230—3; hyperdeterminants, I, 352—5, 577—9, 588, 589; theory of permutations, I, 423—4; simultaneous, of two homogeneous functions of second order, I, 428—31; theory of permutants, II, 19—23; the term, IV, 594, 605; of elliptic integrals, IX, 618—21; of theta functions, XII, 50—5; Sylvester's work in, XIII, 46; (*see also* Covariants, Invariants, Quantics).

Line-geometry: and congruences, XIII, 228—30; (*see also* Coordinates, Lines).

Lineo-linear: the term, II, 517, IV, 604, VI, 464.

Lineo-linear Transformation: between planes, VII, 215—6, 236—8.

Line-pairs: the term, VI, 206, 209, 210, 211; through three given points and touching given conic, VI, 201, 244, 594.

Line-pair-point: the term, VI, 202, 210, 211, 269, 594—5.

Lines: on cubic surfaces, I, 445—56, VIII, 371—6; harmonic relation of two, II, 96—7; of cubic curve, II, 382; satellite, II, 383, V, 359; formulæ, II, 405—9; line, plane and point, defined, II, 561—2; contour and slope, IV, 108—11, 609; cubic centres and cones, IV, 173—8, 179—81; geometry of, IV, 446—55, 616—8; involution, IV, 582, V, 1—3; cubic centres of three lines and a line, V, 73—6; theorem of conic and triangle, V, 100—2; intersections of pencils of four and two, V, 484—6; formulæ for intersections of line and conic, V, 500—4; circle and parabola, problem, V, 607; notation in Pascal's theorem, VI, 116—23; facultative, VI, 365—6, 450; dot-notation for, and planes and cubic surfaces, VI, 365—6, 373—449; twenty-seven on cubic surface, VI, 371—87; attraction of terminated straight, VII, 31—3; five on cubic surface, VII, 177—8; homographic transformation, VII, 193—7; spherogram and isoparametric, VII, 467—8; iseccentric and *e*-spherogram, VII, 468—70; isochronic and time spherogram, VII, 470—7; Cayley's work on six coordinates of, VIII, XXXV; potentials of, IX, 278—80; formulæ relating to right, X, 287—9; and points, X, 570; and conics, X, 602; contact with a surface, XI, 281—93; Mill on, XI, 432—3; non-Euclidian geometry, XI, 437, XIII, 480—504; evolution of theory of curves, XI, 450—1; singularities of curves, XI, 468; in *Ency. Brit.*, XI, 548, 571—2; equation of right, XI, 558—61; and surface, XI, 629; Mascheroni's geometry of the compass, XII, 314—7; reciprocal, XIII, 58—9, 481; identity relating to six coordinates of a line, XIII, 76—8; and notion of plane curve of given order, XIII, 79—80; syzygetic relations, XIII, 224—7; of striction, on skew surface, XIII, 232—7; system of in a plane, and their orthotomic circle, XIII, 346—7; and point, distance, XIII, 495—7; theory of two lines, XIII, 497—504; (*see also* Coordinates, Curvature, Geodesic Lines).

Line Systems: two-dimensional geometry, II, 569—83.

Link: the term, V, 521, VII, 183, XIII, 506.

Linkage: the MacMahon, XIII, 265, 292, 293, 298—301.

Link-work: X, 407.

Liouville, J.: integration of differential equations, III, 180, 191—2; equations of motion, III, 185; Bour's memoir, III, 199, 204; a definite integral, IV, 28—9; elliptic motion, IV, 522, 590; problem of two centres, IV, 530; motion of three bodies, IV, 541, 590; point moving in plane, IV, 542—6, 590; Abelian integrals, IV, 546, 590; on roots of equations, IX, 39; indefinite integration, IX, 500—3; geodesic curvature, XI, 323, 328.

Lipschitz, R.: on Lagrange's equations, IX, 110—2.

Listing, J. B.: partitions of a close, V, 617, VI, 22; theorem of, VIII, 540—7.

Lobatschewsky, N. I.: geometry of, V, 471—2, VIII, xxxvii, XI, 436, XII, 220—38.

Locus: defined, IV, 458, VI, 458, 460; from two conics, VI, 27—34; in relation to triangle, VI, 53—64; and envelope in regard to triangle, VI, 72—82; *in solido*, problem and solution, VII, 599; *in plano*, problem and solution, VII, 599; the terms nodal-, cuspidal-, envelope-, tac-, VIII, 533—4; in *Ency. Brit.*, XI, 585.

Logarithms: theory of, III, 208—13, 222—8; of imaginary quantities, VI, 14—8; Pineto's tables, VIII, 95—6; calculation of log 2, XI, 70; origin, XI, 447; function in, XI, 526—7.

Logic: of characteristics, III, 51—2; calculus of, VIII, 65—6; Mill on, XI, 432—4; recent developments, XI, 458—9; geometry and algebra, XII, 459.

Logistic: and algebra, V, 293—4.

London Mathematical Society: Sylvester's connexion with, XIII, 45.

Lottner, C. L. E.: motion of solid body, IV, 583, 590.

Lubbock, Sir J. W.: theory of the Moon, III, 309—10.

Lunar Theory: Hansen's, III, 13—24; disturbed elliptic motion, III, 270—92; development of disturbing function, III, 293—318, 319—43, VII, 511—27; tables, III, 299—308; theorem of Jacobi, III, 519—21; Plana's, VII, 357—60, 361—6, 367—70, 371—4, 375—6; Delaunay's l, g, h, and $h+g$, VII, 528—33, 534; pair of differential equations in, VII, 535—6, 537—40; Newcomb's work, IX, 179—80; note on, XIII, 206—9; (*see also* Elliptic Motion, Disturbed).

Lüroth, J.: six coordinates of a line, VII, 66.

Luther, E.: theory of equations, XI, 520.

MacAulay, A.: quaternions and hydrodynamical equations, XIII, 8.

MacClintock, E.: quintics, IV, 609—16.

MacCullagh, J.: polar plane theorem, IV, 12—20; spherical conics, IV, 428; inertia, IV, 564—5, 588; rotation of solid body, IV, 574, 588.

MacDowell, J.: triangle and circle, V, 564.

Maclaurin, C.: cubic curves, I, 586; attraction of ellipsoids, III, 27, 62, 155; pedal curves, V, 113—4.

MacMahon, P. A.: symmetric functions, II, 603; seminvariants and symmetric functions, VIII, xxxi, XIII, 286—8, 290, 292, 293, 302; a differential equation, XII, 30—2; seminvariants, XII, 239, 254, 261, 275, 349, XIII, 265; a differential operator, XII, 318; Waring's formula for form of equation adopted by, XIII, 214; reciprocants and differential invariants, XIII, 399—404.

Magic Squares: X, 38.

Magnetism: Sabine's work, XI, 430; Gauss's work, XI, 544.

Maillard, M. S.: penultimate forms of curves, VIII, 258; degenerate forms of curves, XI, 220; systems of curves, XI, 487.

Major Function in Abel's theorem: XII, 111, 132—56.

Malet, J. C.: invariants, XII, 390—1.

Malfatti, G. F.: sextic resolvent equations, XIII, 473.

Malfatti's Problem: and system of equations, I, 465—70; Steiner's extension, II, 57—86, 593; Schellbach's solution, III, 44—7; system of equations connected with, IX, 546—50.

Malus, É. L.: systems of rays, XII, 573.

Mannheim, A.: epitrochoid, XIII, 81.

Manuscript of Cayley: facsimile of, on frontispiece, VIII.
Maps: surface representation on plane, VIII, 538—9; colouring, XI, 7—8; projections, XI, 448.
Mars: Newcomb on observations of, IX, 177.
Marsano, G. B.: mathematical tables, IX, 481—3.
Martin, A.: integration, X, 15, 29; probabilities, X, 600; Pellian equation, XIII, 467.
Mascheroni, L.: geometry of the compass, XII, 314—7.
Masères, F.: algebraic equations, IV, 171.
Mathematical Society of London: Sylvester's connexion with, XIII, 45.
Mathematical Tables (*see* Tables).
Mathematics: recent terminology, IV, 594—608; Mill on, XI, 432—4; relation to physics, XI, 444, 449; extent, XI, 449; Sylvester on its relation to music, XIII, 45.
Matrices: notation, II, 185—8, XI, 243; constituents being linear functions of one variable, II, 216—20; unity, II, 477; theory of, II, 475—96, 604, V, 438—48; which satisfy algebraic equation of own order, II, 483; convertible, II, 488; rectangular, II, 494—6; in automorphic linear transformation, II, 497—505; a linear system, II, 604; for reduction of quintic to Jerrard's form, IV, 392; the term, IV, 594, 601—2; Bezoutic, IV, 607; Cayley's method of verification, VIII, XXVII; Cayley the founder of theory, VIII, XXXII—III; transformation of coordinates, XI, 136—42; of order two and the homographic function, XI, 252—7; quaternions, XII, 303, 311, 479; the equation, $qQ-Qq'=0$, XII, 311—3; and theta function transformation, XII, 367—72, 386—9; quintic, XII, 376—80; Sylvester's theory of the corpus, XIII, 47; involutant of two binary, XIII, 74—5; six coordinates of a line, XIII, 76—8; note on a theorem in, XIII, 114; and sixty icosahedral substitutions, XIII, 552—7.
Maurice, F.: variation of arbitrary constants, III, 166, 202.
Maxima: of certain factorial functions, VIII, 548—9.
Maxima and Minima: of functions of three variables, I, 228—9; theorem in, IX, 40—1; theory of *déblais* and *remblais*, XI, 417—20.
Maximum Indicator: VI, 83.
Maxwell, J. C.: contour lines, IV, 609; quartic and quintic surfaces, VII, 246, 252; on Cayley, VIII, XX.
Mean Motion (*see* Motion).
Mechanical Construction: curve tracing, VIII, 179—80, XIII, 515—6; of Cartesians, IX, 317, 535—6; of conformable figures, X, 406.
Mechanics: variation of arbitrary constants in, III, 161—5, 200; construction of conformable figures, X, 406; integrator, XI, 52—4; and time, XI, 444; function in, XI, 522—3; curve tracing, XIII, 515—6.
Mehler, F. G.: attractions of polyhedra, IX, 266.
Memoirs: list of, on theoretical dynamics, IV, 584—593; (*see also Ency. Brit. and* the word desired).
Mention, M.: in-and-circumscribed polygon, IV, 294, 303—5.
Mercator's Projection: VIII, 567, IX, 237—40.
Merrifield, C. W.: letter to, VIII, 517—8.
Mersenne, M.: the cycloid, XI, 447.
Metageometry (*see* Hypergeometry, Prepotentials).
Method of Derivations: Arbogast's, II, 257, IV, 265—71, 272—5, 609, XI, 55; binomial theorem and factorials, VIII, 463—73.
Methyl: trees of, IX, 544—5.
Metrical Geometry: II, 592.
Metrical Theory: pure analytical geometry, XI, 556—7; solid geometry, XI, 570.
Meunier, J. B. M. C.: theorem of, III, 38.
Meyer, F.: history of quantics, VIII, XXXI; orthomorphosis, XIII, 175, 187.
Mill, J. S.: logic and mathematics, XI, 432—4.
Miller, W. J. C.: triangles, V, 566; conics, V, 582; negative pedals of ellipsoid and ellipse, X, 576—7; geometrical interpretation, X, 604.

Milner's Lamp: differential equation and construction of, XIII, 3—5.
Minima (*see* Maxima and Minima).
Minimal Surfaces (*see* Surfaces).
Minor: the term, XI, 496.
Minor Function: in Abel's theorem, XII, 111.
Mirrors: systems of rays, XII, 571—5.
Möbius, A. F.: geometry of position, I, 360; reciprocal figures, I, 415; developable from quintic curve, I, 500; tortuous curves, I, 500; circular relation, III, 118—9, IX, 612—7, XI, 188; cubic curves, IV, 120, XI, 479; in-and-circumscribed triangle, IV, 439—41; cubic curves and cones, V, 401, 551; opposite curves, V, 468; equilibrium of four forces, V, 540—1; coordinates of a line, VII, 66, 93; multiple algebra, XII, 472, 473.
Models: Plücker's, of quartic surfaces, VII, 298—302; Wiener's, of cubic surface, VIII, 366—84.
Modular: the term, XIII, 559.
Modular Equations: in elliptic functions, IX, 117—8, 126—37, 169—75, XII, 507—34; errors in Sohnke's paper, IX, 543; for cubic transformation, XII, 46; quintic transformation, XII, 416.
Modular Functions: system of symbols, IV, 484—9; $\chi\omega$, XIII, 338—41.
Modulus: of transformation, the term, IV, 605; table for any prime or composite, VI, 83—6.
Moment: non-Euclidian geometry, XIII, 481—9.
Moment of Inertia: of solid body, IV, 478—80, 559—66.
Monge, G.: transformation of coordinates, IV, 557; theory of *déblais* and *remblais*, XI, 417—20; descriptive geometry, XI, 448—9; reciprocal polars, XI, 465; biographical notice, XI, 586—8; non-Euclidian plane geometry, XII, 221; differential equation of conic, XII, 393.
Monodromic: the term, XII, 432.
Monogenic Function: XI, 80, 537.
Monoid Surfaces: and curves in space, V, 8, 552; and quintic curves in space, V, 24—30, 552, 553, 613.
Monotropic: the term, XII, 432.
Monro, C. J.: flexure of spaces, X, 331—2.
Montucla, J. F.: on Wallis, XI, 642.
Moon: secular acceleration of mean motion, III, 522—61, 568; (*see also* Lunar Theory, Solar Eclipse).
Morley, F.: topology of chessboard, X, 609—10; systems of circles and spheres, XIII, 13.
Motion: of solid body, I, 28—35, 583; secular acceleration of moon's, III, 522—61, 568; Lagrange's equations of, IX, 198—200; on three-bar, IX, 551—80, 585, XIII, 505—16; Sylvester on recent discoveries in mechanical conversion, XIII, 44; (*see also* Dynamics, Elliptic Motion, Kinematics, Lunar Theory).
Motte, A.: problem of tactions, XIII, 151.
Moulton, J. F.: matrices, XI, 256.
Mountains: altitude, and roots of algebraic equation, XIII, 33—7.
Mourey, C. V.: imaginaries, XII, 468.
Mousetrap: the game of, III, 8, X, 256—8.
Moutard, T.: quartic surfaces, VII, 246, VIII, 262.
Muir, T.: history of determinants, I, 581; problem of arrangements, X, 249—51; elimination, XIII, 545—7.
Multiform Series: defined, IV, 456.
Multilinear Operator of MacMahon: XIII, 399.
Multiple Algebra: on, XI, 446, XII, 60—71, 459—89; associative imaginaries, XII, 105—6.
Multiple Integrals: and attractions, I, 5—12, 13—8, 195—203, 204—6, 285—9, 438—44, 581, 586, II, 35—9; demonstration of a theorem of Boole, I, 384—7, 588.
Multiple Sines: X, 1—2.
Multiple Theta-functions (*see* Theta-functions).
Multiplication: of elliptic functions, I, 534—9, 568—76, 589, IX, 138—47, XII, 507; of determinants,

XI, 495; of extraordinaries, XII, 461—2; complex, in elliptic functions, XII, 556—7; (*see also* Transformation).

Multiplier: Jacobi's theory of, I, 276, 279; theory of, in differential equations, X, 102—6; in elliptic integrals, X, 139.

Multiplier Equations: in elliptic functions, IX, 138—47, XII, 507.

Murdoch, P.: *Newtoni Genesis Curvarum per Umbras*, V, 284, 288; curve classification, V, 354; cubic curves and cones, V, 402; the simplex cubical parabola, VI, 101.

Murphy, H.: four points in plane or space, VII, 585.

Murphy, R.: Legendre's coefficients, I, 376.

Music and Mathematics: Sylvester on, XIII, 45.

Napier, J.: logarithms, XI, 447.

Natani, L.: Pfaffian equations, IV, 515.

Nature: notice on Sylvester, XIII, 43—8.

Neg: the abbreviation in groups, XIII, 119.

Negative: the rule of signs, IV, 595—6, XI, 492.

Negative Deficiency: VIII, 397.

Neptune: Newcomb's astronomical work, IX, 180—4.

Neutral: the term, VI, 101.

Newcomb, S.: astronomical work of, IX, 176—84.

Newton, Sir I.: cubic curves, IV, 122; rectilinear motion, IV, 515, 590; central forces, IV, 515, 590; parabolas, V, 284; curve classification, V, 354, 364—6, 396—9; diameter, V, 362; cubic curves, V, 401, 551, XI, 464; conics, V, 562; forms of cubical parabolas, VI, 101; theorem as to roots of equations, X, 5; *Principia*, XI, 447—8; branches of curves, XI, 477; theory of equations, XI, 500, 502; roots of algebraic equation, XIII, 35; Sylvester's work at rule of, XIII, 46; tactions, XIII, 151—69; Newton-circle, the term, XIII, 152.

Newton-Fourier Theorem: imaginary roots, X, 405—6, XI, 143, XIII, 36; extension to complex variables, X, 405—6; theory of equations, XI, 114—21, 122.

Nexal: the term, VIII, 73.

Nil: the term, XII, 66.

Nilfactum: and quantic, VI, 466.

Nilpotent: the term, XII, 61.

Nilvalent: the word, IX, 202.

Nine-point Circle: XIII, 517—9, 520—1, 548—51.

Nitrogen: tree of, IX, 430.

Nivellators: Sylvester's theory of, XIII, 47.

Nodal: the term, VII, 244.

Nodal Anallagmatic: the term, VIII, 67.

Nodal Bicircular Quartic: mechanical description, VII, 182—8.

Nodal Cubic: VI, 171—4, VII, 555.

Nodal Curves: of developable from quartic, V, 135—7; of cubic surface, VI, 450; centro-surface of ellipsoid, VIII, 332—52.

Nodal Director: the term, V, 169—70.

Nodal Generator of Scrolls: V, 169—70, 179—81.

Nodal Isochronic: the term, VII, 473.

Nodal Locus: in singular solutions, VIII, 533.

Nodal Quartic: defined, V, 10; mechanical description of bicircular, VII, 182—8; and hyperelliptic functions, XII, 196—208.

Nodal Quartic Surfaces (*see* Quartic Surfaces).

Nodal Residue of Scrolls: v, 169—70, 181—3, 184, 187.
Nodal Total of Scrolls: v, 169—70, 183—9.
Node-couple: defined, II, 29, IV, 22, XI, 227; curve, and plane, and torse, VI, 355, 582—5; torse, VI, 601.
Node-cusp: v, 265—6, 618.
Node-form: the term, VII, 274.
Nodes: the term, II, 28, IV, 22, 27, 181, V, 295, XI, 468; elimination of, in three bodies, IV, 551, V, 23; number on quartic surface, VII, 133—81; quartic surface with twelve, XIII, 1—2.
Node-tangent: defined, II, 29—32.
Node-triplet: the term, II, 30.
Nodo-focus: of bicircular quartic, VI, 522—3, 523—6; the term, IX, 264.
Non-commutative Algebra (*see* Algebra).
Non-Euclidian Geometry, Memoir on: introduction, XIII, 480—1; geometrical notions, XIII, 481—9; point, line, and plane coordinates, general formulæ, XIII, 489—91; the absolute, XIII, 491—5; distance of a point and line, XIII, 495—7; distance of a plane and line, XIII, 497; theory of two lines, XIII, 497—504.
Non-Euclidian Geometry: VIII, 409—13, XII, 220—38; (*see also* Hypergeometry).
Non-facultative Space: VI, 156.
Non-scrolar Surfaces: quartic and quintic, VII, 245.
Non-unitariants: the term, XIII, 265.
Non-unitary Symmetric Functions: and seminvariants, XII, 239, 275, XIII, 267—98; tables, XII, 273—4.
Norm: and polyzomal curves, VI, 474, 573—5.
Normal Elementary Integral: of differential equation, XII, 396—7, 444.
Normals: in *Ency. Brit.*, XI, 564—5; (*see also* Conics).
Normal Variables: in dynamics, IX, 111.
Notation: algebraic functions, II, 185—8; matrices, II, 185—8; quantics, II, 223; for disturbing function compared, III, 310—8; quantics and abstract geometry, VI, 464—6; differential equations, X, 95—7; for double theta functions, X, 497; theta functions, XI, 47—9, 243—5; umbral, XIII, 301—6.
Nöther, M.: curves in space, V, 613—7; rational transformation, VII, 255; deficiency of surfaces, VIII, 395; sextic curve, IX, 504—7; classification of curves, XI, 451; Abelian function, XII, 149.
Novel-reading: at Cambridge by Cayley, VIII, x—xi, xxiii.
Nullity: Sylvester's theory of, XIII, 47.
Number: time, and space, V, 292, 620, XI, 442—4; theory of equations, XI, 502.
Numbers: a theorem of Lejeune-Dirichlet's, II, 47—8; tables of binary cubic forms, VIII, 51—64; use of Bernoulli's, in analysis, IX, 259—62; arrangements of, X, 570; Sylvester and Hammond on Hamiltonian, XIII, 48; (*see also* Partition of Numbers).
Numbers, Theory of, in *Ency. Brit.*: XI, 592—616; ordinary and complex theories, XI, 592—3; ordinary theory, XI, 594—609, 615—6; theory of forms, XI, 604—9; complex theories, XI, 609—16.
Numbers, Theory of: Pellian equation, IV, 40—2, IX, 477—8, XI, 615, XIII, 430—67; composition of, IV, 70—1, 78—9; specimen table, VI, 83—6; $x^p-1=0$, trisection and quartisection, XI, 84—96; $x^p-1=0$, and quinquisection, XI, 314—6, XII, 72—3; H. J. S. Smith on, XI, 429; imaginaries, XI, 444—5; evolution, XI, 455—6; Wilson's theorem, XII, 45; Sylvester on, XIII, 47; (*see also* Partition of Numbers).
Numerative Geometry: Schubert's, XI, 281—93.
Numerical Equations: X, 3—6.
Numerical Expansions: IV, 470—2.
Numerical Generating Function: X, 339, 408.
Nutation: note on theory of, IX, 194—6.

Obliquity: the term, XIII, 234.
Observations (*see* Orbits, Planet's Orbit, Solar Eclipse).
Octacron: enumeration of polyhedra, V, 38—44.
Octad: the term, I, 586, VII, 133, 152, XII, 590.
Octadic-quartic Surfaces: X, 51.
Octagon: theorem of eight points on a conic, VIII, 92—4.
Octahedral Function: XI, 128—9.
Octahedron: axial system, V, 531—9; automorphic function for, XI, 169, 179—83, 212—6.
Octaves: elliptic functions, I, 127, 586; imaginaries, I, 301.
Octavic Surface: VIII, 401—3.
Octic Function: and Abelian function, XI, 483.
Octics: and twisted cubics, XII, 310.
Octic Surface: on a sibi-reciprocal, X, 252—5; (*see also* Surfaces).
Octo-dianome: the term, VII, 134.
Octo-hexahedron: the term, X, 328.
Odd Branch of Curve: X, 36.
Off-planes: the term, VI, 330, 577, 583—5.
Off-points: the term, VI, 330, 338, 577, 583—5.
Olbers, W.: orbits of asteroids, IX, 177.
Olivier, T.: conics inscribed in quadric surface, I, 557.
Omal: the word, VI, 194, 463, 467—9.
Omega Functions: the term, XI, 453; note on Smith's memoir, XIII, 558—9.
Omphali: the term, VIII, 326.
Operandator: defined, III, 242.
Operations: and substitutions, XIII, 530.
Operators: differential, VII, 8; and seminvariants, XIII, 322—32; MacMahon's multilinear, XIII, 399.
Optics: MacCullagh's theorem in polarized light, IV, 12—20; geometrical construction in, X, 28.
Orbits: Jacobi's canonical formulæ for disturbed motion, III, 76—7; reduction to fixed plane, III, 91—6; variation in plane of planet's, III, 516—8; central forces problem, IV, 516—21; position of, in planetary theory, VII, 541—5; of asteroid, and Newcomb, IX, 176—7; Hamiltonian equations of central, X, 613; Sylvester's work at, XIII, 47; (*see also* Planetary Theory, Planet's Orbit).
Order: of system of equations, I, 457—61, 589; of quantics, defined, II, 221; of curve, II, 569—83, XI, 462; in abstract geometry, defined, VI, 463; of curve and surface, XI, 629.
Ordinary Point for Differential Equations: XII, 394.
Oriani, B.: elliptic motion, III, 474, IV, 528.
Orr, W. McF.: tetrads of circles, XIII, 425.
Orthocentre: the term, XIII, 550.
Orthogonal Surfaces: VIII, 269—91, 292, 569—70; Smith's Prize dissertation, VIII, 558—63.
Orthogonal Surfaces and Curvature, Memoir on: VIII, 292—315; introductory, VIII, 292—3; curvature of surfaces, VIII, 293—300; conormal correspondence of vicinal surfaces, VIII, 301—8; condition that the two surfaces may belong to orthogonal system, VIII, 309—11; family of surfaces, VIII, 312—5.
Orthomorphic Transformation (*see* Orthomorphosis).
Orthomorphosis: of circle into parabola, V, 618, XII, 328—36; of a circle into itself, XIII, 20; general theory, XIII, 170—90; some problems, XIII, 191—205; note on theory, XIII, 418—24; (*see also* Conformal Representation).
Orthotomic: the term, IX, 13.
Orthotomic Circles: and polyzomal curves, VI, 501; and Jacobians, VI, 568.
Orthotomic Curve: of a system of lines in a plane, XIII, 346—7.
Orthotomic Surfaces: in *Ency. Brit.*, XI, 637—8.

Oscnode: defined, II, 28—32.
Oscular: the term, VI, 334, 361, 362.
Ostrogradsky, M. A.: dynamic equations, III, 186, 203; transformation of differential equations, IV, 514; virtual velocities, IX, 207.
Outcrops: the term, VIII, 326, 351.
Oval Chuck for Quartic Curves: VIII, 151—5.
Ovals: of Descartes, I, 479, II, 118, 336, III, 66; and quartic curves, V, 468—70; twice-indented, X, 318; and functions, XI, 540; in *Ency. Brit.*, XI, 549—51; roots of algebraic equations, XIII, 37; Sylvester's work at, XIII, 47; orthomorphosis, XIII, 185—6, 202.
Oxygen: trees of, IX, 427—60.

Π: Wallis's investigation for, XIII, 22—5.
Pagani, G. M.: central forces problem, IV, 520, 590; motion of solid body, IV, 583, 590.
Painvin, L.: last multiplier, IV, 551, 590.
Parabola: inflexions of cubical divergent, V, 284—8, VI, 101—4; classification, V, 356, 395, VI, 101; line and circle, problem, V, 607; polyzomal curves, VI, 542; cubic curves, XI, 478; in *Ency. Brit.*, XI, 548—51, 561—4; orthomorphosis of circle into, XII, 328—36; and epitrochoid, XIII, 86—7; orthomorphosis into circle, XIII, 421—2.
Parabolic Cyclide: IX, 73—8.
Paraboloids: in *Ency. Brit.*, XI, 576—9.
Paradox: the d'Alembert-Carnot geometrical, XII, 305—6.
Paraffins: trees of, IX, 427—60.
Parallel Curves: envelopes and surfaces, IV, 123—33, 152—7, 158—65; and evolutes, VIII, 31—5; theory of, X, 260; the critic in solar eclipses, X, 311—5.
Parallels: and non-Euclidian geometry, XIII, 480—1, 481—9; the terms right and left, XIII, 488, 502.
Parallel Surfaces: of paraboloid, VIII, 7; of ellipsoid, VIII, 9, IX, 591; in *Ency. Brit.*, X, 637—8.
Parametric Class and Order: of systems of cones, V, 552.
Parametric Latitude: VII, 16, IX, 238.
Parametric Relation: VI, 463—4; of triple orthogonal system, VIII, 292—315.
Parazome: the word, VI, 477.
Partial Differential Equations: integral of, III, 166; system of, VIII, 517—8; Jacobi's, in transformation of elliptic functions, XII, 530—3; on a, XIII, 358—61.
Particle: under central forces, X, 575; (*see also* Dynamics).
Partition of Numbers: II, 218, 235—49, V, 48; and quantics, II, 265; supplementary researches, II, 506—12; a problem in, III, 247—9; tactical, V, 294, XI, 443.
Partitions: conjugate, due to Ferrers, II, 419; formulæ in, III, 36—7; problem of double, IV, 166—70; of a close, V, 62—5, 617; problems, VII, 575, X, 611, XI, 61—2; tables, IX, 480—3, XI, 357—64; theorems in trigonometry and, X, 16; in *Ency. Brit.*, XI, 589—91; note on a partition-series, XII, 217—9; non-unitary partition, XII, 273—4; Sylvester's constructive theory of, XIII, 47; of a polygon, XIII, 93—113; and seminvariants, XIII, 269.
Pascal, B.: hexagram of, I, 356; limaçon of, I, 480; some theorems of geometry of position, I. 550—6; lines of, I, 551, 588; curves, XI, 447; inscribed hexagon, XI, 556.
Pascal's Theorem: intersection of curves, I, 25—7; demonstration, I, 43—5; Chasles' form of, I, 45; on, I, 322—8, 414, VI, 129—34, 594; generalized, V, 4; notation of points and lines, VI, 116—22, 594.
Peacock, G.: multiple algebra, XII, 460, 467, 469, 470—1.
Peaucellier, A.: mechanical construction of Cartesian by his cell, IX, 317; cell of, and scalene transformation, IX, 527—34; Sylvester on his discoveries, XIII, 44.
Pedal Curves: Maclaurin on, V, 113—4.

Peirce, B.: orbit of Neptune and Uranus, IX, 180, 182; multiple algebra, XI, 457—8, XII, 60—71; associative algebras, XII, 106, 459, 465; imaginaries, XII, 303.
Pellian Equation: IV, 40—2; tables, IX, 477—80; and theory of numbers, XI, 615; report of committee on, XIII, 430—67.
Pencils: defined, II, 577; homography of, II, 578; intersections of four- and two- lined, V, 484—6; of six lines and cubic curves, VI, 105—15, 593—4.
Pendulum, Spherical: IV, 532—4, 535—7, 541.
Peninvariants: and seminvariants, IV, 241; the term, IV, 606.
Pentagon: a theorem relating to, I, 318—9; Gauss's *Pentagramma Mirificum*, VII, 37—8; Schröter's construction of regular, XII, 47; (*see also* Polygons).
Pentagramma Mirificum: of Gauss, VII, 37—8.
Pentagraph: illustrating a function, XI, 440; curve tracing, XIII, 515—6.
Penultimate Forms: of curves, VIII, 258—61; of surfaces, VIII, 262—3.
Penultimate Quartic Curve: VIII, 526—8.
Penumbral Curve (*see* Solar Eclipse).
Periodic Functions: the term, XI, 529; (*see also* Doubly Periodic Functions).
Periods: of elliptic integrals, IX, 618; of theta functions, X, 467—9.
Permissive Points: in differential equations, XII, 434—41.
Permutants: theory of, II, 16—26, 27; defined, II, 17, IV, 594, 596, 600; Sylvester on, II, 26—7.
Permutations: theory of, I, 423—4; idea of group, II, 124; problem of geometric, V, 493—4; commutants, V, 495—7; colours on faces of polyhedra, V, 539; (*see also* Arrangements, Combinatory Analysis).
Perott, J.: binary quadratic forms, V, 618.
Perpendicular: in non-Euclidian geometry, XIII, 480—1, 481—9.
Perpendicularity: and imaginaries, XII, 466—72.
Perpetuants: and seminvariants, XII, 250—7, XIII, 266; sextic, XII, 257—62; reducible seminvariants, XIII, 308—13; Strohian theory of, XIII, 314—8.
Perspective: of triangles, III, 5; five points in a plane, V, 480—3; theory, XI, 442.
Pfaff, J. F.: problem, IV, 359—63.
Pfaffian Differentials: XIII, 361, 405—14.
Pfaffian Equations: Natani and Clebsch, IV, 515.
Pfaffians: and skew determinants, I, 411, II, 203; the term, I, 589, II, 19, IV, 594, 600; and differential equations, X, 96—7, 106.
Pfaff-invariants: XIII, 405—14.
Physics: relation to mathematics, XI, 444, 449; (*see also* Dynamics, Electricity, Light).
Picard, É.: integral calculus, XII, 408—11.
Pinch-planes: the term, VI, 330, 335, 583—5, X, 53—6.
Pinch-points: the term, VI, 123, 330, 335, 582—5, X, 53—6, XI, 227.
Pineto, S.: tables of logarithms (review), VIII, 95—6.
Pippian: defined, I, 586, II, 381, 397—400, 400—3; and Hessian, II, 383—95; geometrical definition, II, 416.
Pirie, B.: inertia, IV, 564.
Plana, G. A. A.: lunar theory, III, 536, 568, VII, 357—60, 361—6, 367—70, 371—4, 375—6; distribution of electricity, IV, 92, 100—7.
Planar Developables: the term, I, 505.
Planarity: of developables, V, 517.
Plane Curves (*see* Curves, Plane).
Plane-integral: prepotential, IX, 337—43.
Plane of Orbit: variation in, III, 516—8.

Plane Representation of Solids: VII, 26—30.

Planes: diametral, of quadric surface, I, 255—8; point and line defined, II, 561—2; geometry of two dimensions, II, 569—83; MacCullagh's theorem of polar, IV, 12—20; lines and dots of cubic surfaces, VI, 365—6, 373—449; rational transformation, VII, 197—213, 216—9; quadric transformation, VII, 213—6; also lineo-linear, VII, 215—6; determined by point and three lines, VII, 571; flecnodal, of a surface, X, 262—4; kinematics of, XI, 103—10, XIII, 505—16; in *Ency. Brit.*, XI, 571—2; osculating and normal, XI, 579—80; and surface, XI, 629; non-Euclidian geometry, XIII, 481—504; and line distance, XIII, 497.

Planetary Theory: Desboves', III, 185, 203; development of disturbing function, III, 319—43, VII, 511—27; variation in plane of orbit, III, 516—8, VII, 541—5; theorem of Jacobi, III, 519—21; Newcomb's work, IX, 180—4.

Planets: angular distance of two, VII, 377—9.

Planet's Orbit from Three Observations: VII, 384—6, 400—78; introductory, VII, 400—1; the general theory, VII, 401—6; determination of orbit from given trivector, VII, 406—12; time formulæ, Lambert's equation, VII, 412—5; formulæ for transformation between two sets of rectangular axes, VII, 415—7; intersection of orbit plane by single ray, VII, 417—26; trivector and orbit, VII, 426—8; special symmetrical system of three rays, VII, 428—9; Planogram No. 1, meridian 90°—270°, VII, 429, 430—40; No. 2, meridian 0°—180°, VII, 429, 441—51; No. 3, orbit-pole at point *A*, VII, 429, 452—4; No. 4, orbit-pole in ecliptic, VII, 429, 455—9; No. 5, orbit-pole on a separator, VII, 429, 459—67; spherogram and isoparametric lines, VII, 467—8; *e*-spherogram and iseccentric lines, VII, 468—70; time-spherogram and isochronic lines, VII, 470—7.

Planogram: the term, VII, 404; three plates, VII, to face 478; meridian 90°—270°, VII, 429, 430—40; 0°—180°, VII, 429, 441—51; orbit-pole at point *A*, VII, 429, 452—4; in ecliptic, VII, 429, 455—9; on separator, VII, 429, 459—67.

Plates (*see* Diagrams, *also* Tables).

Plato: and geometry, XI, 446.

Playfair, J.: on twelfth axiom of Euclid, XI, 435.

Plerogram: the term, IX, 202.

Plexus: the term, IV, 603, VI, 458; Sylvester's term, XIII, 46.

Plücker, J.: theory of algebraic curves, I, 53, 54; curves and developables, I, 207, 208, 210, 586—7; involution, I, 259, 261; elimination, and theory of curves, I, 344; geometry of position, I, 356, 553—6; geometrical reciprocity, I, 380; reciprocal figures, I, 418; quadric surfaces, I, 421; cubic surfaces, I, 446; transformation of curves, I, 478; singularities of plane curves, I, 586, V, 520—2, 619, XI, 450; cubic curves and cones, IV, 173—8; double tangents, IV, 186; points of six-pointic contact on cubic, IV, 207; cubic curves, IV, 495, 617, V, 402; line geometry, IV, 616—8; hyperboloid coordinates, V, 72; node-cusp, V, 265—6; curve classification, V, 354—400; numbers for singularities of plane curves, V, 424, 476, 517; higher singularities of plane curves, V, 426, 619; pencil intersections, V, 484; numbers of, VI, 68, VIII, 41—5, XI, 469—73; species of cubical parabola, VI, 101; focus, VI, 515, XI, 481; six coordinates of a line, VII, 66; quartic surface models, VII, 298—302; construction of a conic, VII, 592; hypergeometry, VIII, xxxv; theory of curve and torse, VIII, 74, 75—6, 80—1; theory of curves, XI, 467; envelopes, XI, 475—6; note on equations of, XIII, 536.

Pohlke, K.: theorem in axonometry, IX, 508.

Poincaré, H.: lacunary functions, XIII, 415.

Poinsot, L.: polygons and polyhedra, IV, 81—5, 86—7, 609; inertia, IV, 563, 590—1; rotation of solid body, IV, 571—3, 577, 591; kinematics of solid body, IV, 580, 581, 591.

Point: of cubic curve, II, 382; satellite, II, 383; formulæ, II, 405—9; theorems, II, 409—12; plane and line defined, II, 561—2; and ineunt of a curve, II, 574; lattice, III, 40; distances of, from triangle and formulæ, IV, 510—2; tritom, V, 138; the term polar of, V, 570; and abstract

geometry, VI, 458; potential of, IX, 278—80; singularities of curves, XI, 468; coordinates of, as functions of parameter, XII, 290—1; and line distance, XIII, 495—7; two-way, XIII, 507; for-forwards, and back-backwards, XIII, 510.

Point-pairs: the term, II, 564—5, VI, 202, 206—7, 208, 210, 269, 594—5; degenerate forms of curves, XI, 218.

Points: distances of, I, 1—4, 581; some theorems in geometry of position, I, 317—28; of inflexion, I, 345—9, 354; of osculation, I, 349—51; harmonic relation of two, II, 96—7; of cesser, defined, IV, 130; critical defined, IV, 130; five in a plane, V, 480—3; correspondence on plane curve of, V, 542—5; and circle, problem, V, 560; correspondence of two on a curve, VI, 9—13, 264—8, VII, 39; notation of, in Pascal's theorem, VI, 116—23; abstract geometry, VI, 463; consecutive, VI, 467—9; system of 16, and polyzomal curves, VI, 501—3, 504—5; problem of random, VII, 585; problem and solution of four in plane or space, VII, 585; four and conic, VII, 587; on particular sextic curve, IX, 504—7; branch- and cross-, X, 317; and lines, problem and solution, X, 570; on a circle, function of, XI, 130; double- and pinch-, XI, 227; Mill on, XI, 432—3; representation on plane, XI, 442; evolution theory of curves, XI, 450—1; at infinity, XI, 464; relation between the distance of five in space, XII, 581—3; analytical formulæ in regard to octad of, XII, 590—3; Sylvester's facultative, XIII, 46; non-existence of a special group, XIII, 212; syzygetic relations, XIII, 224—7; non-Euclidian geometry, XIII, 480—504; coordinates of, and non-Euclidian geometry, XIII, 489—91; (*see also* Orthomorphosis).

Point-systems: and one-dimensional geometry, II, 563—9, 583—86; and two-dimensional geometry, II, 569—83, 586—92.

Poisson, S. D.: attraction of ellipsoids, III, 155; planetary theory, III, 159, 201; variation of arbitrary constants in mechanical problems, III, 163—5, 200, 201, 202; coefficient (a, b) of, III, 163; Hamilton's method of dynamics, III, 173—4, 200; integration of differential equations, III, 180; distribution of electricity, IV, 92—5, 100—7, X, 299, XI, 1; elliptic motion, IV, 522; relative motion, IV, 535, 591; motion of projectile, IV, 541, 591; inertia, IV, 563, 591; rotation of solid body, IV, 566, 569, 573, 591; rotation round fixed point, IV, 582, 591; motion of solid body, IV, 583, 591; attraction of ellipsoidal shell, IX, 302; Jacobi's theorem, X, 108—9, 110—3.

Polar: of point, V, 570, X, 54, XI, 465.

Polar Conjugate: of curve of third class, II, 383.

Polar Reciprocal: I, 230, 378, 416.

Polarization: MacCullagh's theorem, IV, 12—20.

Poles: conjugate, of cubic curve, II, 382—5; two-dimensional geometry, II, 579—83, 586—92; the term, XI, 465.

Pollock, Sir F.: on circumscribed triangle, III, 29—34.

Poloid Curve: IV, 571—2.

Polyacra: triangle-faced, and enumeration of polyhedra, V, 38—44.

Polygons: in-and-circumscribed, II, 87—9, 91—2, 138—44, 145—9, IV, 292—308, 435—41, V, 21—2, VIII, 14—21, 212; partitions of close-, V, 62—5, 617; and triangles, problem, V, 589; potential of, IX, 266—80; automorphic function for, XI, 169, 179—83, 212—6; partitions of, XIII, 93—113.

Polyhedra: Poinsot's four new regular solids, IV, 81—5, 86—7, 609; the problem of, IV, 182—5, 609; autopolar, IV, 185; enumeration of, and triangle-faced polyacra, V, 38—40; partitions of close-, V, 62—5, 617; axial properties, V, 529—39; potential of, IX, 266—80.

Polyhedral Functions (*see* Hypergeometric Series, Schwarzian Derivative).

Polyzomal Curves, Memoir on: VI, 470—576, VII, 115; introductory, VI, 470—2; Part I, polyzomal curves in general, VI, 473—97; definitions and preliminary remarks, VI, 473—4; the branches, VI, 474—6; points common to two branches, VI, 476—8; singularities of a ν zomal, VI, 478—9; zomals with common point or points, VI, 479—81; depression of order of ν zomal curve from

ideal factor of branch or branches, VI, 481—5; the trizomal and tetrazomal, VI, 485; intersection of two ν zomals having same zomal curve, VI, 486—7; theorem of decomposition of tetrazomal, VI, 487—9; application to trizomal, VI, 489—94; tetrazomal curve, VI, 494; variable zomal of trizomal curve, resumed, VI, 494—7; Part II, subsidiary investigations, VI, 497—515; preliminary remarks, VI, 497—8; circular points at infinity; rectangular and circular coordinates, VI, 498—9; antipoints; definition and fundamental properties, VI, 499—500; antipoints of circle, VI, 500; antipoints and pair of orthotomic circles, VI, 500; forms of equation of circle, VI, 501; system of 16 points, VI, 501—3; property in regard to four confocal conics, VI, 503—4; system of sixteen points, the axial case, VI, 504—5; involution of four circles, VI, 505—8; locus connected with foregoing, VI, 508—9; formulæ of two sets each of four concyclic points, VI, 509—11; ditto further properties, VI, 512—5; Part III, theory of foci, VI, 515—34; the general theory, VI, 515—7; foci of conics, VI, 517—9; variable zomal applied to conic, VI, 519—21; foci of circular cubic and bicircular quartic, VI, 521—2; centre of circular cubic, and nodo-foci, etc., of bicircular quartic, VI, 522—3; circular cubic and bicircular quartic; symmetrical case, VI, 523; ditto, singular forms, VI, 523—6; analytical theory for circular cubic, VI, 526—8; ditto, for bicircular quartic, VI, 528—30; property that points of contact of tangents from pair of concyclic foci lie in a circle, VI, 530—34; Part IV, trizomal and tetrazomal curves where the zomals are circles, VI, 534—66; the trizomal curve-tangents at *I*, *J*, etc., VI, 534—7; foci of conic represented by equation in areal coordinates, VI, 537; theorem of variable zomal, VI, 539—41; relation between conic and circle, VI, 541—2; case of double contact, Casey's equation in problem of tactions, VI, 543; intersections of conic and orthotomic circle on set of four concyclic foci, VI, 543—4; construction of symmetrical curve, VI, 544—6; focal formulæ for general curve, VI, 547; circular cubic, VI, 548—9; focal formulæ for symmetrical curve, VI, 549; symmetrical circular cubic, VI, 549—50; general ditto, VI, 550—3; transformation to new set of concyclic foci, VI, 553; tetrazomal curve, decomposable or indecomposable, VI, 553—4; cases of indecomposable, VI, 554—5; ditto, centres being in line, VI, 555—6; the decomposable curve, VI, 556—7; ditto, centres not in a line, VI, 557—61; ditto, centres in a line, VI, 561—5; ditto, transformation to a different set of concyclic foci, VI, 565—6; theory of Jacobian, VI, 566—8; Casey's theorem for circle touching three given circles, VI, 568—73; a norm when the centres are in line, VI, 573—5; trizomal curves with cusp or two nodes, VI, 575—6.

Poncelet, J. V.: harmonic relations, II, 96; porism of in-and-circumscribed triangle, III, 80—5; rectangular hyperbola, III, 254; in-and-circumscribed polygon, V, 21—2; reciprocal polars, XI, 466.

Pontécoulant, G. de: *Système du Monde*, III, 309—10; *Lunar Theory*, III, 521, VII, 357.

Porism: homographic, defined, III, 74, 84; allographic defined, III, 75, 85; of polygon and correspondence, IX, 94.

Porism of in-and-circumscribed Polygon: II, 87—9, 91—2, 97, 138—44, 145—9, IV, 292—308, VIII, 14—21, 212.

Porism of in-and-circumscribed Triangle: II, 56, 87—90, 91, III, 67—75, 80—5, 229—41, V, 549—50, 579, VIII, 212—57.

Portraits of Cayley: frontispiece to vols. VI, VII, XI.

Pos: the abbreviation in groups, XIII, 119.

Positive: the rule of signs, IV, 595—6, XI, 492.

Postulandum of Curve: the term, I, 583, VII, 140, XII, 501; and capacity, XIII, 115.

Postulation: the term, I, 583, VII, 140, 225, VIII, 394; of curve, XII, 501.

Potential: and attractions, I, 195.

Potentials: of polygons and polyhedra, IX, 266—80; of ellipse and circle, IX, 281—301; Smith's Prize question on, XI, 261—4.

Potential-solid: prepotential, IX, 346—7.

Potential-surface: prepotential, IX, 343—6.

Potenzkreis of Steiner: III, 113.

Power: of a matrix, II, 492—4; of homographic function, XI, 253—7.
Power-enders: the term, XIII, 267, 270, 295; and reciprocants, XIII, 333.
Powers: successive, of homographic function, X, 305—6, 307—9; of roots of algebraical equations, XII, 33—4.
Precession: note on theory of, IX, 194—6.
Prepotentials, Memoir on: IX, 318—423; introductory, IX, 318—21; prepotential plane, theorem A, IX, 319, 337—43; potential surface, theorem C, IX, 320, 343—6; potential solid, theorem D, IX, 320, 346—7; the prepotential surface integral, IX, 321—30; its continuity, IX, 330—4; potential solid integral, IX, 334—7; examples of foregoing, IX, 347—50; surface and volume of sphere, IX, 351—2; integral, IX, 352—9; prepotentials of uniform spherical shell and solid sphere, IX, 359—79; examples, theorem A, IX, 379—93; Green's integration of prepotential equation, IX, 393—404; examples, theorem C, IX, 404—7; examples, theorem D, IX, 407—8; prepotentials of homaloids, IX, 408—9; Gauss-Jacobi theory of epispheric integrals, IX, 410—7; methods of Lejeune-Dirichlet and Boole, IX, 417—23.
Prepotentials: Smith's Prize question, XI, 261.
Presidential Address: to British Association, XI, 429—59.
Prime Numbers: B.A. report on tables of, IX, 462—70.
Prime Roots: tables, IX, 471—7.
Prn: the abbreviation for tortuous curves, XIII, 253.
Principal System of Sextic Curve: VII, 236—8.
Principia: solution of problem, Book I, Sect. V, Lemma XXVII, VII, 30; (*see also* Newton).
Principiants: and reciprocants, XIII, 388—98.
Probabilities: questions in theory of, II, 103—4, 594—8, V, 80—5, X, 588, 600—1, 614.
Problems: mechanical, III, 78—9; a class of dynamical, continuous impact, IV, 7—11; (*see also* Dynamics, Smith's Prize Papers, Three Bodies).
Problems and Solutions from the Educational Times: V, 560—612; table of contents, V, 612; VII, 546—607; table of contents, VII, 607—8; X, 566—614; table of contents, X, 615—6; (*see also* Smith's Prize Papers).
Product: resolvent, IV, 309—13.
Product-theorem: for theta functions, X, 464, 474, 509—46.
Progress of Theoretical Dynamics (*see* Dynamics).
Prohessians: defined, V, 267; and developables, V, 513—4.
Prohibitive Points: in differential equations, XII, 434—41.
Projectile: effect of resisting medium, IV, 541.
Projection: stereographic, of spherical conic, V, 106—9; of ellipsoid, V, 487—8; plane representation of solid figure, VII, 26—30; stereographic, VII, 397—9, XI, 187—9, 569; blank, VII, 482; of surface on plane, VIII, 538—9; Mercator's, VIII, 567; Mercator's, of skew hyperboloid of revolution, IX, 237—40; a problem of, IX, 508—18; map, XI, 448.
Prolusions: Sylvester's Astronomical, XIII, 47.
Provectant: defined, II, 514.
Provector: defined, II, 514.
Pseudosphere: the term, XII, 220.
Ptolemy: stereographic projection, XI, 448.
Puiseux, V.: algebraic functions, III, 225; spherical pendulum, IV, 533, 591; motion of a body, IV, 583, 591.
Pyramid (*see* Polygons, Polyhedra).

Quadrangle: in-and-circumscribed, IV, 307—8; differential relation between sides, X, 33—5.
Quadratic Equation: roots, V, 160—1; and geometrical interpretation, XI, 258—60.
Quadratic Form: composition, I, 532; transformation of, into itself, II, 192—201, 215; tables, V, 141—56, IX, 480—3, 486—93.

Quadratic Residues: Eisenstein's geometrical proof, III, 39—43.
Quadratic Transformation of Binary Form: VIII, 398—400; (*see also* Transformation).
Quadratics: resultant of three ternary, and invariant of biternary, IV, 349—58.
Quadric Cones: of six given points, V, 4—6; through given points, X, 575.
Quadric Curves: V, 70—2.
Quadric Equations: transformation of two, I, 428—31, III, 129—31; automorphic linear transformation of, II, 497—505; solution by radicals, X, 9; two related, XI, 37.
Quadric Integral: due to Aronhold, XII, 162—9.
Quadricone: the term, VI, 334, 585, VII, 264.
Quadricovariant: of quantic, II, 520; or Hessian, II, 545; the term, IV, 606.
Quadrics: through nine points, I, 425—7; developable from two, I, 486—95; homographic transformation into itself, II, 105—12, 117, 133—7; theorem on surfaces, III, 115—7; equation of differences for, IV, 242; the term, IV, 604; sections of, V, 133—4; through three lines, VII, 177; in hyperdimensional space, IX, 79—83; covariants of, IX, 537—42; envelope of family of, X, 589; correspondence of confocal Cartesians with right line of a hyperboloid, XII, 587—9; (*see also* Binary Quadrics).
Quadric Seminvariants: generating functions of, XIII, 306.
Quadric Surfaces: n-dimensional geometry, I, 62; diametral planes of, I, 255—8; centres of similitude, I, 329—31; note, I, 421—2, 589; abstract of memoir by Hesse, I, 425—7; conics inscribed in a, I, 557—63; envelope of certain, VIII, 48—50; inversion, VIII, 67—71; problem, and hypothetical theorems, VIII, 550; and four-dimensional space, IX, 246—9; a system of, X, 269; Jacobian of, X, 568; in *Ency. Brit.*, XI, 576—9, 632; twisted cubics on, XII, 307—10; focals of, XIII, 51—4; reciprocal lines, XIII, 58—9; (*see also* Geodesic Lines).
Quadric Transformation: between planes, VII, 213—5, 219—21, XII, 100—1; (*see also* Transformation).
Quadri-cubic Curves in Space: V, 16.
Quadricuspidal: the word, VII, 51.
Quadrifactions: the term, IX, 426.
Quadrilateral: and ellipse, V, 604; inscribed in bicircular quartic, X, 231—5; inscribable in circle, X, 578.
Quadrinvariant: of binary quartic, first occurrence, I, 93; of quantic, II, 516; the term, IV, 606; of quadriquadric function, XIII, 67—8.
Quadriquadric: the term and kinds, V, 10, VII, 99.
Quadriquadric Curves: in space, V, 17; on, V, 282; sextic torse for cuspidal edge having, X, 68—72; Abel's theorem, XII, 186—9, 292—8, 321—5; and elliptic functions, XII, 292—8, 321—5.
Quadriquadric Function: two invariants of, XIII, 67—8.
Quadriquadric Transformation: between spaces, VII, 229—30; (*see also* Transformation).
Quadrispinal: the term, VII, 65.
Quantics, Introductory Memoir: II, 221—34, 598—601.
Quantics, Second Memoir: II, 250—75; numerical tables, II, 276—81.
Quantics, Third Memoir: II, 310—35.
Quantics, Fourth Memoir: II, 513—26; definitions, II, 513—5; covariants and invariants of degrees, two, three, four, II, 515—20; calculation of discriminant, II, 520—2; the catalecticant, lambdaic, and canonisant, II, 522—3; bezoutiants, cobezoutiants, II, 524—6.
Quantics, Fifth Memoir: II, 527—57, 604—6; the single quadric, II, 527—9; two or more, and theories of harmonic relation and involution, II, 529—40; cubics, II, 540—5; quartics, II, 545—56.
Quantics, Sixth Memoir: analytical theory of binary and ternary, II, 561—83; general theory of distance, II, 583—92; its style, VIII, xxvii.
Quantics, Seventh Memoir: chiefly ternary cubics, IV, 325—41; tables, IV, 333—41.
Quantics, Eighth Memoir: VI, 147—90; introductory, IV, 147—8; binary quintic, covariants and syzygies of degree 6, VI, 148—53; formulæ for canonical form, VI, 153—4; 18-thic invariant, VI,

154—6; character of equation, auxiliars, facultative and non-facultative space, VI, 156—8; application to quartic equation, VI, 158—61; characters of quintic equation, VI, 161—5; Tschirnhausen's transformation, VI, 165—9; Hermite's application of Tschirnhausen's transformation to quintic, VI, 170; nodal cubic, VI, 171—4; Hermite's criteria, VI, 174—6; his canonical form of quintic, VI, 177—83; imaginary linear transformations, VI, 183—6; application to auxiliars of quintic, VI, 186—7; theorem of binary quantic, VI, 187—90; the binary quintic and sextic, VI, 190.

Quantics, Ninth Memoir: VII, 334—53, IX, 537—42; introductory, VII, 334—5; theory of number of irreducible covariants, VII, 336—7; new formulæ for number of asyzygetic covariants, VII, 337—40; the 23 fundamental covariants, VII, 341—8; tables, VII, 341—6; Gordan's proof for the complete system of 23, and concomitants of quintic, VII, 348—53.

Quantics, Tenth Memoir: X, 339—400; introductory, X, 339—40; numerical and real generating functions, X, 341—8; table 96, X, 349—55; theory of the canonical form, X, 355—62; table 97, X, 362—9; table 98, X, 370—6; derivatives and tables, X, 377—94; numerical generating functions, N.G.F. of a sextic, X, 394—6; table, X, 397—400.

Quantics: defined, II, 221, IV, 594, 604; resultant of, II, 320; discriminants, II, 320; notation of abstract geometry, VI, 464—6; and nilfactum, VI, 466; character of the ten memoirs, VIII, XXX—XXXI; transformable into each other, VIII, 385—7; eliminant of two, XI, 100—2; Sylvester's work in, XIII, 47; syzygetic relations among the powers of linear, XIII, 224—7; and seminvariants, XIII, 363; (*see also* Binary Quantics, Quadratics).

Quarterly Journal of Pure and Applied Mathematics: VIII, xii.

Quartic Curves: transformation, I, 476—80, 589; special family of, I, 496—9; bitangents of, IV, 342—8, VII, 123—4, X, 244, XI, 221—3, 474; cuspidal defined, V, 10; in space, V, 11—5; and ovals, V, 468—70; triangle in-and-circumscribed to a, V, 489—92; with three double points, V, 550, 553; in connexion with cubic and quintic, problem, VI, 580; problem, V, 596; and sextic torse, VII, 99—100; tricuspidal, problem, VII, 589; mechanical description, VIII, 151—5; a penultimate, VIII, 526—8; construction of bicircular, IX, 13—5; and functions of a single parameter, IX, 315—7; with two odd branches, X, 36—7; bicircular, X, 223—42; triple theta functions, X, 446—54; problem and solution, X, 582—6; trinodal, problem, X, 602; singular tangents of, problem, X, 603; degenerate, XI, 220; with cusp at infinity, XI, 408; forms and classification, XI, 480; circular, XI, 481; ground curve in Abel's theorem, XII, 38, 109—216; bitangents of plane-, XII, 74—94; twisted, XII, 428—31; (*see also* Bicircular, Binary, Binodal, *and* Nodal Quartics).

Quartic Developables: and developable surfaces, V, 268—71; reciprocation of, V, 505—10.

Quartic Equations: conditions for systems of equal roots, II, 467—8; evolution, II, 547; Tschirnhausen's transformation, IV, 368—74, V, 449; Sturmian constants, IV, 473—7; nodal curve of developable from, V, 135—7; and quantics, VI, 158—61; solution of $aU+6\beta H=0$, VII, 128—9; roots, VII, 551, X, 575; solution by radicals, X, 10.

Quartic Matrix: Hermite's, XII, 367—72.

Quartics: canonical form, II, 548; equation of differences for, IV, 243, 279; the term, IV, 604; roots of, problem, V, 610; conditions for existence of systems of equal roots, VI, 300—12; and three cubics, problem, VII, 546; reality of roots, problem, X, 608.

Quartic Scroll (*see* Scrolls).

Quartic Seminvariants: XII, 20; generating functions, XIII, 306; and perpetuants, XIII, 316.

Quartic Surfaces, First Memoir: VII, 133—81, 609—10; introductory, VII, 133—4; Jacobian surfaces, VII, 134—6; surface by equating to zero a symmetrical determinant, VII, 136—7; surfaces $F(P, Q)=0$, etc., VII, 138; nodes of quartic surface, VII, 138—40; number of constants contained in a surface, VII, 140—1; general theory of quartic surface with given nodes, VII, 141—4; Jacobian surface of six given points, VII, 144—5; ditto of seven, or an octad of points, VII, 145—8; the dianodal surface, VII, 148—52; octadic surfaces with 9 or 10 nodes, VII, 152—5; dianomes with 9 or 10 nodes, VII, 155; dianodal curve of 8 points, VII, 155—6; ten nodes, VII, 156; dianodal

centres of 9 points, VII, 156; result as to dianomes, VII, 156; the symmetroid, (lineo-linear correspondence of quartic surfaces), VII, 157—9; ditto and Jacobian, VII, 160—3; symmetroid with given nodes, VII, 163—6, 259; Jacobian with given lines, VII, 167; correspondence on the Jacobian, VII, 168—70; further investigations as to Jacobian, VII, 171—5; persymmetrical case: Hessian of a cubic, VII, 175; quartics with 11 or more nodes, VII, 176—7; quadric surface through three given lines, VII, 177; condition that five given lines may lie in a cubic surface, VII, 177—8; condition that seven given lines may lie in a quartic, VII, 178; Jacobian of 6 points, VII, 178—9; locus of vertex of quadric cone which touches each of six given lines, VII, 180—1.

Quartic Surfaces, Second Memoir: VII, 256—60, 609—10.

Quartic Surfaces, Third Memoir: VII, 264—97, 609—10; preliminary considerations and classification, VII, 264—7; sextic curves, VII, 267—71; nodal determination, VII, 271—3; quartic surfaces resumed, VII, 273—4; enumeration of the cases, VII, 274—80; notation for cases afterwards considered, VII, 280—1; 16-nodal surface and table, VII, 281—4; 15-nodal surface and table, VII, 285—8; equation of ditto, VII, 288—9; 14-nodal surface and table, VII, 289—92; 13-nodal surface and table, VII, 293—7.

Quartic Surfaces: on, V, 66—9; Steiner, V, 421—3, IX, 1—2; 16-nodal, V, 431—7, VII, 126—7, X, 157—65, 180—3, 604, XII, 95—7; note on, V, 465—7; recent researches, VII, 244—52; Plücker's models, VII, 298—302; some special, VII, 304—13, VIII, 2—11, 25—8; surface and sphere, problem, VII, 589; section of surface, problem, VII, 593; penultimate forms of, VIII, 262—3; symmetrical determinant = 0, X, 50—6; 12-nodal, X, 60—2, XIII, 1—2; Hessian of, X, 274—7; tetrahedroid as 16-nodal, X, 437—40; equation of, X, 609; in *Ency. Brit.*, XI, 633—4; (*see also* Cyclide).

Quartic Syzygy: and elliptic integrals, II, 191, IV, 68—9, 609.

Quartic Transformation: of elliptic functions, IX, 103—6.

Quartinvariants: of quantic, II, 516, 520.

Quartisection: theory of numbers, XI, 84—96.

Quasi-inversion: and orthomorphosis, XIII, 192—3.

Quasi-minima: the term, XIII, 42.

Quasi-normal: the term, XIII, 228.

Quaternary: the term, VI, 464.

Quaternary Function: Hessian of, IX, 90—3.

Quaternions: certain results, I, 123—6, 127; algebraic couples, I, 128—31; rotation, I, 405—9, 589, V, 537; formulæ of, II, 107; transformation of quadrics, II, 135; skew determinants, II, 214; transformation of coordinates, IV, 559; the equation $qQ-Qq'=0$, XII, 300—4, 311—3; matrices, XII, 303; multiple algebra, XII, 474; hydrodynamical equations, XIII, 8; versus coordinates, XIII, 541—4.

Quet, J. A.: relative motion, IV, 536, 592.

Quetelet, M. A.: theory of Gergonne, and on caustics, II, 339; wave surface, IV, 433—4.

Quinquisection: theory of numbers, XI, 314—6, XII, 72—3.

Quintic Curves: and developables, I, 500—6; in space, V, 15—6, 20, 24—30, 552, 553, 613; in connexion with cubic and quartic, V, 580.

Quintic Developables: and surfaces, V, 272—8, 518.

Quintic Equations: conditions for systems of equal roots, II, 468—70; equation of differences, IV, 150—1, 246—61, 276—91; Tschirnhausen's transformation, IV, 375—94; tables, IV, 379—80, 387—90; Jerrard's researches, V, 50—4, 77, 89; character of, VI, 161—5; solvibility by radicals, VII, 13—4, X, 11; theorem of Abel, XI, 132—5; solvable case of, XI, 402—4; and elliptic functions, XIII, 473; their sextic resolvents, XIII, 473—9.

Quintic Matrix: XII, 376—80.

Quintics: auxiliary equation for, IV, 309—24; Jerrard's form, IV, 392; soluble by elliptic functions,

IV, 484; the term, IV, 604; MacClintock on, IV, 609—16; theorem of Abel as to soluble, V, 55—61, XI, 402—4; discriminant of, problem, V, 592; conditions for existence of systems of equal roots, VI, 300—12; concomitant system of, X, 342; syzygies among covariants of, X, 346—55; canonical form, X, 355—62; resolvent sextic of, XI, 396; bitangents of the, XIII, 21; and seminvariants, XIII, 363—5; (*see also* Binary Quintics).

Quintic Seminvariants: and perpetuants, XIII, 309.

Quintic Surfaces: and developables, V, 272—8, 518; recent researches, VII, 244—52.

Quintic Transformation: of elliptic functions, IX, 122, 148, XII, 522—5.

Quippian: the word, II, 381, 396—7.

Quotient: G/H in theory of groups, XIII, 336—7.

Raabe, J. L.: summation of series, II, 15.

Radials: the term, XI, 259, XIII, 179.

Radicals: and solvibility of equations, VII, 13—4, X, 8—11; theory of equations, XI, 511—20, 521; Galois and theory of, XI, 543; Weierstrassian cubic transformation, XIII, 31.

Radicals (Chemical): number of univalent, IX, 544—5.

Radii: the term link- and bar-, VII, 183; curvature of wave-surface, XIII, 248.

Range: defined, II, 577; homography, II, 578.

Rank of Seminvariants: XII, 22.

Ratio: and abstract geometry, VI, 457—62.

Rational Functions: the term, IV, 603—4.

Rationalisation: of algebraic equations, II, 40—4.

Rational Transformation (*see* Transformation, Rational).

Ray, M. N.: solution of equations, X, 610—11.

Ray Planes: and biaxal crystals, IX, 107—9.

Rayleigh, Lord: fluctuating functions, IX, 19—20.

Rays: special symmetrical system of three, VII, 428—9; the term, X, 55; systems of, XII, 571—5.

Real Generating Function: X, 339.

Real Intersections of Curves: IX, 21.

Reciprocal Figures: I, 415—20.

Reciprocal Matrix: II, 481.

Reciprocal Polars: I, 416, 421; Monge, XI, 465; Poncelet, XI, 466.

Reciprocals: of quartic scrolls, VI, 317—27; of cubic surfaces, VI, 368—455; of quartic surfaces, VII, 305; of centro-surface of ellipsoids, VIII, 363; equation of conic, VIII, 522—3; reciprocal lines, XIII, 58—9.

Reciprocal Surfaces, Theory of: VI, 329—58, 577—81, 582—91, 596—601; extension of Salmon's fundamental equations, VI, 329—31; developments, VI, 331—4; new singularities, VI, 334—41; application to a class of surfaces, VI, 341—2; flecnodal curve, VI, 342; surfaces of revolution, in connexion with spinodal and flecnodal curve, VI, 342—4; flecnodal torse, VI, 345; general surface of order n without singularities, VI, 345—6; formula for β', VI, 347—53; recapitulation, VI, 353—5; addition, VI, 355—8; Zeuthen, VI, 596—601; theory, XI, 225—34.

Reciprocants: of quantic, II, 320; the term, IV, 607, XIII, 366; and sextactic points, V, 618; of cubic, VI, 73; and invariants, XII, 393; Sylvester on, XIII, 47—8; tables of pure, to weight eight, XIII, 333—5; and differential invariants, XIII, 366—404; Halphen on, XIII, 366, 368—81, 381—98; Cockle, XIII, 366, 367—8; Ampère and Lagrange, XIII, 366; Sylvester, XIII, 366, 379—81, 381—98; MacMahon, XIII, 399—404.

Reciprocation of Quartic Developable: V, 505—10.

Reciprocity: geometrical, I, 377—82; and quantics, II, 232, 234; law of, for invariants, II, 516; and homography, II, 578.

Rectangle: potential of, IX, 278—80.
Rectilinear Motion: IV, 515—6.
Reduced Latitude: VII, 16, IX, 238.
Reducible Seminvariants: and perpetuants, XIII, 308—13.
Reducible Syzygies (*see* Syzygies).
Reduction: of transcendental integrals, X, 214—22.
Reech, F.: contour lines, IV, 609.
Reflection: caustics by, I, 273—5, II, 118—22, 129.
Region: the term, IX, 331.
Regular: the term, VI, 457, 459.
Regulator: the term, VII, 402.
Regulus: the term, XI, 573, 632.
Rehorovsky, W.: symmetric functions, II, 602.
Relation: and abstract geometry, VI, 457—62; omal, VI, 463; parametric, VI, 463—4; a discriminant, VI, 467; Jacobian, VI, 467.
Relink: the term, V, 521.
Remblais: theory of, XI, 417—20, 449, 587.
Reports: on progress of theoretical dynamics, III, 156—204; on progress in solution of certain problems in dynamics, IV, 513—93; on Pellian equation, XIII, 430—67.
Representation: analytical, of curves in space, IV, 446—55, 490—5, XI, 83; of solid figure in plane, VII, 26—30; of surfaces on a plane, VIII, 538; of variables by correspondence of planes, X, 316—23; conformal, XI, 78—81; graphical, of binodal quartic and the elliptic functions, XIII, 9—19; Sylvester on graphical, XIII, 47; (*see also* Orthomorphosis, Transformation).
Réseau: the term, VII, 253.
Residuation: of cubic curve, IX, 211—4, XII, 115—6; of curves, XII, 502; Sylvester's theory of, XIII, 47.
Residues: Cauchy's theorem on, I, 148, 174; Eisenstein's geometrical proof of quadratic, III, 39—43; nodal, of scrolls, V, 169—70, 181—3, 184, 187.
Resisting Medium: motion in, IV, 541.
Resolvent Equations: sextic, of Jacobi and Kronecker, XIII, 473—9.
Resolvents: after Lagrange, IV, 309; of quintics, XI, 396.
Resultant: the term, I, 63, 337, IV, 597, 602—3, VI, 466—7; of quantics, II, 320; of two equations, II, 440—53, VI, 292—9; of two binary quantics, IV, 1—4, IX, 16—7; of three ternary quadratics, IV, 349—58; of two binary cubics, V, 289; of forces, X, 589.
Resultor: defined, II, 59.
Reversion: of series, IV, 30—7, 54—9.
Reuschle, K. G.: mathematical tables, IX, 468—9, 473, 485, 494—9, XI, 95—6; theory of numbers, XI, 85—6, 315, 612.
Rhamphoid Cusp: V, 265—6, 618.
Rhizic Theory: root-limitation, IX, 34—8.
Ribaucour, C. R.: orthogonal surfaces, VIII, 569—70.
Riccati, J. F.: solution of equation, VII, 9—12.
Richelot, F. J.: Abelian integrals, I, 366, 367; solution of equation $x^{257}-1=0$, I, 564; porism formula, II, 90; in-and-circumscribed triangle, III, 237—41; spherical pendulum, IV, 534, 592; rotation of solid body, IV, 577—8, 592; rotation round fixed point, IV, 583, 592; two quartic curves, X, 584; integral of Euler's differential equation, XIII, 525—9.
Richmond, H. W.: Pascal's theorem, VI, 594.
Riemann, G. F. B.: doubly infinite series, II, 593; genus of curve, V, 476—7, 517; Abelian integrals, V, 521, XI, 30; Abelian functions, VI, 2, 264, 593; elliptic geometry, VIII, xxxvii; transformation and theory of invariants, VIII, 387; surface of, and correspondence, X, 317, 323; bitangents of

quartic, XI, 221—3; fractional differentiation, XI, 235—6; notion of space, XI, 435—7; correspondence of points, XI, 440; deficiency of curves, XI, 450; hyperelliptic functions, XI, 454—5; transformation, XI, 482; elliptic functions, XI, 534, 537, 539, 546; theory of numbers, XI, 616; series, XI, 627; minimal surface, XI, 639; Abelian functions and plane quartics, XII, 74, 87, 95; linear differential equations, XII, 396; orthomorphosis, XIII, 180, 189, 204.

Right-handed Root-limitation: IX, 22—3.

Roberts, C. A.: Pellian equation, XIII, 467.

Roberts, M.: geodesic lines on ellipsoid, VII, 34.

Roberts, S.: description of nodal bicircular quartic, VII, 182—8; points on cubic curve, VII, 549; three-bar curve, IX, 551; symmetrical determinant, X, 579—80; theorems of squares, XI, 294; curves, XI, 481; kinematics of a plane, XIII, 505.

Roberts, W.: transformations of curves, I, 471—5, 478; surface parallel to ellipsoid, IV, 158—65.

Roberts, W. R. W.: cyclide, IX, 75.

Rodrigues, O.: motion of solid body, I, 28—35, 124, 405, 583; quaternions, I, 124, 586; rotation of solid body, I, 237; skew determinants, I, 335; expansions in multiple sines, I, 583; attraction of ellipsoids, III, 149—53; transformation of coordinates, IV, 558, 559, 592, XI, 575; kinematics of solid body, IV, 581, 592; on rotation formulæ, V, 537; correspondence of homographies and rotations, X, 153.

Rohn, K.: quartic surfaces, VII, 609—10; quartic surface with twelve nodes, XIII, 2.

Roof: the term in non-Euclidian geometry, XIII, 484.

Root-limitation: geometrical representation, IX, 21—39; general theory, IX, 22—7; intercalation theory for right line, IX, 28—33; rhizic theory, IX, 34—8.

Roots: in forms called trees, III, 243, XI, 365—7; of algebraic equations, IV, 116—9, XI, 506—21, XII, 33—4, XIII, 33—7; quadratic equations, V, 160—1; of equations, Cauchy's theorem, IX, 21—39; of unity, IX, 263; of quartic, reality of, X, 608; imaginary, of equations, XI, 114—21, 502—6; equal, of equations, XI, 405—7; theory of real equations, XI, 497—502; Sylvester's work at, XIII, 46; ninth, of unity, XIII, 66; Waring's formula for sum of mth powers of, of an equation, XIII, 213—6; of a quantic, symmetric functions of, XIII, 271—85.

Roots of Unity: prime, XI, 56—60.

Rosenhain, J. G.: theta functions, VIII, xlii, X, 464, 499; double theta functions and 16-nodal quartic surface, X, 158, 162; theory of numbers, XI, 60; double theta functions, XI, 454.

Rotation: of solid body, I, 28—35, 237, 583, II, 107, III, 475—504, IV, 566—80, 592; quaternions and theory of, I, 405—9, 589; formula of, I, 586; and elliptic motion, III, 475; of earth, III, 485; infinitesimal, V, 498—9, VI, 24—6; of group of polyhedra, V, 529, 559; Euler's memoir of 1758, VI, 135—46; equilibrium of, VII, 91—5; and homography, X, 153—4; in conformal representation, XI, 78.

Roulette: the term, XI, 447.

Route: the term, XII, 640.

Rowe, R. C.: memoir on Abel's theorem, XI, 29—36; partitions of a polygon, XIII, 93, 112.

Royal Society: Cayley elected a fellow in 1852, VIII, xiii; Croonian lecture founded, VIII, xv; medals bestowed on Cayley, VIII, xxi.

Rudio, F.: inverse centro-surfaces, XII, 457—8.

Rueb, A. S.: motion of solid body, I, 464; spherical pendulum, IV, 534; rotation of solid body, IV, 573—4, 592.

Rule of Signs: and determinants, XI, 492.

Sabine, Sir E.: death of, XI, 429—30.

Sadleir, Lady Mary: endowments by, VIII, xv.

Sadlerian Professorship: Cayley appointed to, VIII, xvi.

Safford, T. H.: orbits of Neptune and Uranus, IX, 183.

St Laurent, M.: on caustics, II, 118, 121, 122, 347, 355, 368.

Salmon, G.: cubic surfaces and triple tangents, I, 446, 456, 589; linear transformations and elimination, I, 457—61; singular contact, I, 486; curves and developables, I, 492, 587; developable from quintic curve, I, 500—1, 505; systems of equations, I, 533; geometry of position, I, 555; hyperdeterminants, I, 579, II, 598—601; on a plane touching a surface, II, 29; triple tangent planes of third order, II, 29; invariant of ternary cubic, II, 325; quippian, II, 381; tables of covariants, II, 536—7; binary quartics, II, 549; tangential of cubic, II, 558; equation of orthotomic circle, III, 48—50; reciprocal surfaces, IV, 21—7, VI, 329—58, 359, 582—91; surface parallel to ellipsoid, IV, 158—65; double tangents, IV, 187—206, 343, XI, 473—4; cubic curves, IV, 188; conics and five-pointic contact, IV, 207—39; higher algebra, IV, 608; curves in space, V, 9—20, 614; quartic surfaces, V, 66, VII, 136; cubic surfaces, V, 140, VI, 359; scrolls, V, 168—9, 193, 200; prohessian, V, 267; involution, V, 301; higher singularities of plane curves, V, 620; plane curves, VI, 2; invariants, VI, 108; quintics, VI, 154; hyperspace, VI, 191; elimination, and curves which satisfy given conditions, VI, 192; extension of his fundamental equations, VI, 329—31; polyzomal curves, VI, 472, 531, 560; tetrahedral scrolls, VII, 52, 65; sextic torse, VII, 113, 114; centro-surface of ellipsoid, VII, 130, VIII, 316, 320, 323; rational transformation between two spaces, VII, 226, 237; bicircular quartic, VII, 575; locus *in plano*, VII, 606; correspondence with Cayley, VIII, XV; on Cayley, VIII, XXV; evolutes and parallel curves, VIII, 33; theory of curve and torse, VIII, 72, 76—9, 87—91; theory of invariants, VIII, 386; transformation of unicursal surfaces, VIII, 390, 391; residuation, IX, 211; triple theta functions, X, 444; tortuous curves, XI, 9; higher plane curves, XI, 217; Gaussian theory of surfaces, XI, 332; concomitants of ternary cubic, XI, 342; tables for binary sextic, XI, 377; Jacobian sextic equation, XI, 390, 400; equal roots of equations, XI, 407; works on geometry, XI, 546; minimal surfaces, XI, 639; bitangents of quintics, XIII, 21; wave surfaces, XIII, 252.

Satellite Line: II, 383, V, 359.

Scalars and Quaternions: XIII, 541.

Scalene Transformation of Plane Curve: IX, 527—34.

Schellbach, C. H.: solution of Malfatti's problem, III, 44—7.

Schläfli, L.: discriminants, I, 584; elimination, II, 181—4, 404; symmetric functions, II, 454; hyperdeterminants, II, 598—601; resultants, IV, 2—4; numerical expansions, IV, 471; cubic surfaces, VI, 359, 361, 362, 372, VII, 250; quartic surfaces, VII, 308; modular equation for cubic transformation, XIII, 64—5.

Schlömilch, O.: attractions, I, 288; a definite integral, IV, 29.

Schoolgirl Problem: I, 483, 589, V, 95—7.

Schottky, F.: theta functions, XI, 242—9.

Schröter, H.: Steiner's quartic surface, V, 423; construction of regular pentagon, XII, 47.

Schubert, H.: elliptic motion, III, 473, 474, IV, 523; *abzählende Geometrie*, XI, 281—93, 459.

Schwarz, H. A.: inverse elliptic functions, I, 586; developable surfaces, V, 517—9; deficiency, VI, 2; scrolls, VI, 312; quintic scrolls, VII, 250, 252; projections, IX, 508; surface of minimum area, X, 63; hypergeometric series, XI, 125; orthomorphosis, XII, 328, XIII, 188, 191, 192, 193, 202; Kummer's differential equation, XIII, 69.

Schwarzian Derivative and Polyhedral Functions, Memoir: XI, 148—216; introductory, XI, 148—51; Part I, XI, 151—79; the derivative, XI, 151—3; quadric function of three or more inverts, XI, 153—6; functions P, Q, R, XI, 156—7; table ditto, XI, 158—9; differential equations involving (x, z) and (s, x), XI, 160—9; Schwarzian theory, XI, 169—76; connexion with differential equation for hypergeometric series, XI, 176—9; Part II, the polyhedral functions, XI, 179—216; origin and properties, XI, 179—83; covariantive formulæ, XI, 184—5; the forms of $f5$ and $h5$, XI, 185—6; stereographic projection, XI, 187—9; groups of homographic transformations, XI, 189—90, 196—208; the regular

polyhedra, XI, 190—5; system of fifteen circles, XI, 208—12; regular polyhedra as solid figures, XI, 212—6.

Schwarzian Derivative, and Polyhedral Functions: XI, 149, 173, 177; and Kummer's differential equation, XIII, 69; and reciprocants, XIII, 366.

Schwarzian Function: Sylvester on the, XIII, 381.

Scientific Memoirs Catalogue: report on, V, 546—8, 620.

Scrolar: the term, VI, 334.

Scrolls, First Memoir: V, 168—200.

Scrolls, Second Memoir: V, 201—20, VI, 360; degeneracy, V, 201—3; with two directrix lines, V, 203—5; twofold directrix line, V, 205—7; scroll equations, V, 207—10; cubic scrolls, V, 210—3; quartic scrolls, V, 214—9; general theory, V, 219—20.

Scrolls, Third Memoir: VI, 312—28; quartic scrolls, VI, 312—4, 314—7, 328; reciprocals, VI, 317—25, 325—7.

Scrolls: cubic, V, 90—4, 110—2; quartic, VI, 451; tetrahedral, VII, 48—53; on certain, VII, 54—65; recent researches, VII, 250—1; and octic surfaces, X, 79—92; flexure and equilibrium, XI, 317—22; the term, XI, 573; applicable upon a given skew surface, XIII, 231—7.

Secondary Caustics (*see* Caustics).

Secular Acceleration of Moon's Mean Motion: III, 522—561.

Seeber, L.: mathematical tables, IX, 491—2.

Segar, H. W.: development of $(1+n^2x)^{m/n}$, XIII, 354—6.

Segner, J. A. v.: inertia, IV, 561—2, 592.

Segregates: the term, X, 339, 345; table of, X, 349—55.

Semi-cubical Parabola: and Abel's theorem, XII, 180—6.

Seminvariants, memoir: XII, 239—62; introductory, XII, 239—40; multiplication of two symmetric functions, XII, 240—8; capitation and decapitation, XII, 248—50; perpetuants, etc., XII, 250—7; sextic perpetuants and sextic syzygies, syzygants, XII, 257—62.

Seminvariants: the term, IV, 241, 606; theory of, XII, 22—9, 344—57, XIII, 362—5; tables, XII, 275—89, XIII, 217—23; theorem relating to, XII, 326—7; (*see also* Covariants, Invariants).

Seminvariants and Symmetric Functions, Memoir: XIII, 265—332; introductory, XIII, 265—6; the coefficients $(a, b, c, \ldots)$, XIII, 266—71; symmetric functions of the roots, XIII, 271—85; MacMahon form of equation, XIII, 286—8; the *I-and-F* problem, and solution by square diagrams, XIII, 288—98; MacMahon linkage, XIII, 298—301; umbral notation, Stroh's theory, XIII, 301—6; symmetric functions of a given degree: generating functions, XIII, 306—8; reducible seminvariants—perpetuants, XIII, 308—13; Strohian theory resumed: application to perpetuants, XIII, 314—8; investigation of the values of the foregoing functions, XIII, $\Pi_{10}(x+y)\,\Pi_{15}(x+y)$ and $\Pi_{10}(x+y+z)$, XIII, 318—21; the operators, XIII, 322—32.

Senate House Problems: IX, 246—9, XI, 265—7, XIII, 538—40; (*see also* Smith's Prize Papers).

Separations of Partitions: II, 603; (*see also* Partitions).

Separator: the term, VII, 402.

Septic, Binary: generating functions of, X, 408—21.

Septic Transformation in Elliptic Functions: IX, 148—52, X, 333—8.

Series: a double infinite, II, 8—10, 593; sums of, III, 124—8, XIII, 49—50; reversion, IV, 30—7; formula for reversion, IV, 54—9; in integration of differential equations, VIII, 458—62; hypergeometric, XI, 17—25, 125—7; in *Ency. Brit.*, XI, 617—27 (finite, XI, 617—20; infinite, XI, 620—7).

Serpoloid Curve: IV, 571—2.

Serret, J. A.: attractions, I, 288; integral calculus, I, 313; elliptic functions, III, 3; integration of differential equations, III, 185—6, 203; problem of two centres, IV, 528, 531, 592; theory of substitutions, VII, 47; curves of curvature, VII, 331, XII, 601—38; orthogonal surfaces, VIII, 279, 292; indefinite integration, IX, 500—3; transformation of cubic function, XI, 411; minimal surfaces, XII, 595.

Serret, P.: syzygetic relations, XIII, 224.
Servois, J. F.: multiple algebra, XII, 468.
Sets: homology of, III, 35.
Sextactic Points: and plane curves, V, 545, VI, 217; and reciprocants, V, 618.
Sextactic Reciprocant: XIII, 387.
Sextic: binary, and quintics, VI, 190; unicursal, VI, 248; and cubic curves in pencil of six lines, VI, 593—4; the anharmonic-ratio, VII, 314—5; bicursal, IX, 551, 581—6; tricircular, IX, 562—70; numerical generating function, X, 394—6.
Sextic Cone: circumscribed to quartic surface, VII, 265; and nodes, table, VII, 291.
Sextic Curves: foci of conics, VII, 1—4; rational transformation, VII, 236—8; and nodes, VII, 256—7; and quartic surfaces, VII, 267—71; mechanical description, VIII, 138—44; with five double points, IX, 504—7.
Sextic Developable: V, 279—83, 511—9, VI, 87—100.
Sextic Function: and Abelian functions, XI, 483.
Sextic Resolvent: Jacobian, IV, 310, XI, 389—401, XII, 493—9.
Sextic Resolvent Equations: of Jacobi and Kronecker, XIII, 473—9.
Sextic Seminvariants: and perpetuants, XIII, 317.
Sextic, Spherical: and oval, V, 469.
Sextic Syzygies: XII, 257—62, 273.
Sextic Torse (*see* Torse, on a certain Sextic, *also* Torses).
Shanks, W.: log 2, XI, 70.
Sharp: the term, VIII, 406—8, XIII, 265, 291, 304—6, 362.
Sharp-cone: the term, VIII, 102.
Sheets: roots in algebraic equations, IV, 116—9; cubic curves, IV, 120—2.
Shell: formulæ for potential of, IX, 266—7; attraction of ellipsoidal, on exterior point, IX, 302—11.
Sibi-reciprocal Surfaces: VI, 21, X, 252—5.
Siebeck, F. H.: binodal quartic and graphical representation of the elliptic functions, XIII, 10.
Signs: rule of, IV, 595—6, XI, 492.
Sign Symbols: theorems, VIII, 535—7.
Simple Cone: defined, V, 402, 404, 551.
Simple Groups: XIII, 533.
Simultaneous Equations: Jacobi's theorem in, XII, 39.
Simultaneous Roots of Two Equations: Jacobi's theorem in, XII, 123—5.
Sines, Multiple: X, 1—2.
Single Theta Functions: memoir, X, 473, 476—97; and double theta functions, X, 155—6, 186—9; and elliptic functions, XI, 250—1; linear transformation, XII, 50—5.
Singular Curve: on surface, VII, 244.
Singularities: of curves and developables, I, 208; of tortuous curves, I, 500; of surfaces, II, 28—32, IV, 21—7, VI, 123—8, 334—41, 354, 582—5, VIII, 394—8, XI, 225—6, 630—1; of plane curves, V, 424—6, 476—7, 520—8, 619, VI, 3, XI, 30—6; of curves and torses, V, 516; compound, V, 525; of curves in space, V, 613; of cubic surfaces, VI, 363; reciprocal surfaces, VI, 596—601; of curves, XI, 486—70.
Singular Point: for differential equations, XII, 395; integrals in domain of, XII, 395—402.
Singular Solutions of Differential Equations: IV, 426—7; of first order, VIII, 529—34 X, 19—24.
Six Coordinates of a Line (*see* Coordinates, Six of a Line).
Six Lines: Sylvester's involution of, VII, 66.
Six-pointic Contact: on cubic, IV, 207.
Sixteen-nodal Quartic Surfaces: I, 587, V, 431—7, VI, 126—7, 281—4, X, 157—65, 180—3, 437—40, 464, 548—51, 604, XII, 95—7.

Skew: the word, I, 332.
Skew Antipoint: the term, IX, 65—6.
Skew Convertible Matrices: II, 489.
Skew Covariants: II, 233.
Skew Cubics: demonstration of Chasles' theorem, I, 212.
Skew Curvature: the term, I, 234.
Skew Determinants: the term, and some properties of, I, 332—6, 410; on, I, 410—3, 589; researches, II, 202—15, IV, 72—3; and transformation, II, 497; a theorem, IV, 72—3.
Skew Hyperboloids: geodesic lines, VIII, 174—8, 188—99; of revolution, projection of, IX, 237—40.
Skew Matrix: IV, 602.
Skew Polars: I, 378.
Skew Reciprocals: the term, I, 415.
Skew Surfaces (*see* Scrolls).
Slope Lines: IV, 108—11, 609.
Smith, H. J. S.: higher singularities of plane curves, V, 619; transformation of elliptic functions, IX, 174—5; report on mathematical tables, IX, 461—99; death of, XI, 429; theory of numbers, XI, 612; theta and omega functions, XII, 50; theta functions, XII, 337; transformation, XIII, 38; predecessor of Sylvester in Oxford chair, XIII, 44; on a memoir by, XIII, 558—9.
Smith, P.: on Lagrange's solution of caustic, II, 353.
Smith's Prize Papers: VIII, 414—35, 436—8, 439—57, 474—90, 496—516, 538—9, 551—5, 558—63; infinitesimal rotation, VI, 24—6; general equation for virtual velocities, IX, 205—8; solutions and remarks, IX, 218—36; Bernoulli's numbers in analysis, IX, 259—62; problems and solutions 1877, X, 39—46; question on theory of equations, XI, 115; on potentials, XI, 261—4.
Sohnke, L. A.: motion in resisting medium, IV, 541; transformation of elliptic functions, IX, 113, 114; ditto table, IX, 128—35; modular equations, IX, 543.
Solar Eclipse, Graphical Construction: VII, 390—1, 479—92; geometrical theory, VII, 392—6, X, 310—5; general explanation, VII, 479—82; modification for single blank projection, VII, 482—4; construction of relative orbits, VII, 484—7; geometrical theory of projection of penumbral curve, VII, 488—9; details and application to eclipse 21—22/12/70, VII, 489—92.
Solid Body: motion of, I, 28—35, 583; geometrical representation of motion, I, 234—6; rotation round fixed point, I, 237—52, 336; rotation of, I, 462—4, III, 475—504, IV, 577, 592; four forces acting on, IX, 201.
Solid Integral Prepotential: IX, 334—7.
Solid of Revolution: attraction of, I, 508.
Solids: Poinsot's four new regular, IV, 81—5, 86—7; plane representation of, VII, 26—30.
Soluble Quintics: IV, 484, V, 55—61, XI, 402—4.
Solutions (*see* Problems, Problems and Solutions, Smith's Prize Papers).
Somoff, J.: rotation of solid body, IV, 577, 592.
Space: of any number of dimensions, and quantics, II, 222; facultative and non-facultative, VI, 156; dimensions and abstract geometry, VI, 456—7; rational transformation, VII, 189—240; multiple, VIII, xxxiii—v; points and lines, correspondence in, VIII, 566; flexure, X, 331—2; the term in five-dimensional geometry, IX, 79; theories of, XI, 434—7; curves in, XI, 489; elliptic, and non-Euclidian geometry, XIII, 481; (*see also* Hyperspace).
Special Conditions for Curves: VI, 193.
Species: of Quartic Scrolls, V, 201, VI, 328; twenty-three, of cubic surfaces, VI, 359—60.
Specific: the term, XIII, 290.
Sphere: powers of, I, 581; and polyhedron, V, 531; problem and solution, VII, 563; prepotentials, IX, 351—2, 359—79; in *Ency. Brit.*, XI, 571—2.
Spherical Conics: theorem, IV, 428; stereographic projection, V, 106—9; (*see also* Polyzomal Curves).

Spherical Curves of Curvature: surfaces with, XII, 601—38.
Spherical Pendulum: IV, 532—4, 535—7, 541.
Spherical Triangle: theorem of, IV, 80; nine-point circle of, XIII, 548—51.
Spherogram: the term, VII, 404; and isoperimetric lines, VII, 467—8; *e*- and iseccentric lines, VII, 468—70; time- and isochronic lines, VII, 470—7; two plates, VII, to face 478.
Spheroidal Trigonometry: IX, 197.
Spheroid, Oblate: geodesic lines on, VII, 15—25.
Spheroquartic: the term, VIII, 262.
Spinodal Curve: VI, 342—4, 583—5.
Spinode: synonymous with cusp, II, 28, IV, 22, 27; the term, V, 521; plane and torse, VI, 355, 583—5, 601.
Spinode Curves: and cubic surfaces, VI, 450, 595.
Spottiswoode, W.: ineunts, IV, 419; determinants, IV, 608; text-book on determinants, IV, 608; intersection of line and conic, V, 500—4; twenty-one coordinates of conic in space, XI, 82—3; death of, XI, 430; Sylvester's researches, XIII, 44.
Square Diagrams: seminvariants and solution by, XIII, 288—98.
Squares: products of sums of, II, 49—52; surfaces divisible by curves of curvature into, VIII, 97—8, 145—6, 264—8; theorems of 2-, 4-, 8-, 16-, XI, 294—313; imaginaries of 8-, XI, 368—71; Mill on, XI, 432—3; Latin-, XIII, 55—7; orthomorphosis of, into circles, XIII, 191—202.
Squares of Roots: equations of, IV, 242—3.
Squarewise Contractible: the term, XIII, 179.
Stader, P.: central forces problem, IV, 520, 592.
Standard Solutions: of system of linear equations, XII, 19—21.
Statics: six coordinates of a line, VII, 89—95; and time, XI, 444; and Archimedes, XI, 446.
Stativity: the term, VIII, 213.
Staudt, K. G. C. v.: distances of points, I, 581; theory of distance, II, 605; theorem of, on Bernoulli's numbers, IX, 261.
Steiner, J.: Pascal's theorem, I, 322—8; geometry of position, I, 356, 550—6; transformation of curves, I, 474; conics inscribed in a quadric surface, I, 557; cubic surfaces and triple tangent planes, I, 589; extension of Malfatti's problem, II, 57—86, 593; porism formula, II, 90; harmonic relation of two lines or points, II, 96; in-and-circumscribed polygon, II, 141; problems on quadrics, II, 179—80; pippian, II, 381, 391; systems of circles, III, 113; double tangents, IV, 187; conics, IV, 207; point of six-pointic contact on cubic, IV, 207; conics touching curves, V, 31—2; generalized theorem, V, 100—2; pedals, V, 113; quartic surface, V, 421—3; theorem of eight points on a conic, V, 427—30; Casey's equation, VI, 67; locus, envelope, and triangle, VI, 72; bitangents of quartic curve, VII, 124; quartic surface of, VII, 247, VIII, 389, IX, 1—2, X, 607; attraction of ellipsoidal shell, IX, 302.
Stellated Polyhedra: IV, 82, 609.
Stereographic Projection: of spherical conic, V, 106—9; geodesic lines on oblate spheroid, VII, 24—5; property of, VII, 397—9, XI, 187—9, 569.
Stereoscope: and cubic curves, IV, 122.
Stern, M. A.: sums of certain series, III, 126.
Stirling, J.: curve classification, V, 354; proof of his theorem, X, 267—8.
Stockwell, J. N.: determinants, V, 45—9.
Stokes, Sir G. G.: surfaces, envelopes, and parallel curves, IV, 123—33; resisting medium, IV, 541; Catalogue of Scientific Memoirs, V, 546—8, 620; report on mathematical tables, IX, 461—99.
Streblosis: XI, 79, 81.
Striction, Curve of: the term, I, 234.
Striction, Lines of: on skew surfaces, XIII, 232—7.

Stroh, E: perpetuants, XIII, 266, 301—6, 314—8.
Study, E.: Zeuthen on, VI, 594.
Sturm, J. C. F.: integration of dynamical equations, III, 186, 203.
Sturm, R.: homography, VIII, 200; correspondence of points and tetrahedra, VIII, 200—8; root-limitation, IX, 39; theory of equations, X, 4—5, XI, 498—9, 505.
Sturmian Constants: for cubic and quartic equations, IV, 473—7; for quantics, VI, 159.
Sturmian Functions: note on, I, 306—8, 588; new researches, I, 392—6; endoscopic and exoscopic expressions, I, 588; tables for equations from second to fifth degrees, II, 471—4.
Sturm's Theorem: VI, 159, 161; Sylvester's work at, XIII, 46.
Subinvariants: the term, XII, 251, 273.
Subrational: the term, IX, 315.
Sub-regular Integrals: of differential equations, XII, 444—52.
Substitutions: theorem relative to theory, VII, 47; arising from a problem in arrangements, X, 247—8; and theory of groups, X, 324—30, 401—6; and permutations, X, 574; evolution, XI, 455; the notion of, XI, 509—10, 521; Latin squares, XIII, 55—7; groups for two to eight letters, XIII, 117—49; Sylow's theorems on groups, XIII, 530—3; sixty icosahedral, XIII, 552—7.
Subsurface: the term, IX, 79.
Summit: defined, V, 63, XIII, 507.
Sums: of squares, II, 49—52; the term, X, 186, 192; of two series, XIII, 49—50.
Sun: and moon's mean motion, III, 522—61; Newcomb on parallax, IX, 177—8; (*see also* Solar Eclipse).
Supercurve: the term, IX, 79.
Superlines: in hyperspace, IX, 79—83.
Supp: the term, VI, 263.
Supplement: the term, VI, 263.
Suremain-de-Missery, A.: imaginaries, XII, 467.
Surface, Congruence, Complex, in *Ency. Brit.*: XI, 628—39; introductory, XI, 628—9; surfaces in general: torses, XI, 629—32; surfaces of orders 2, 3, and 4, XI, 632—4; congruences and complexes, XI, 634—5; curves of curvature: asymptotic lines, XI, 635—6; geodesic lines, XI, 636—7; curvilinear coordinates, XI, 637; orthotomic surfaces: parallel surfaces, XI, 637—8; minimal surface, XI, 638—9.
Surface-integral: prepotential, IX, 321—30.
Surface of Centres: for wave-surface, XIII, 248; (*see also* Ellipsoid, Centro-surface of).
Surface of Cylinder: Archimedes' theorem for, XII, 56—7.
Surface of Revolution: and Mercator's projection, VIII, 567.
Surfaces: equimomental, I, 253—4; wave (tetrahedroid), I, 302—5, 587; confocal, I, 362—3; singularities, II, 28—32, IV, 22—7; theory of skew, II, 33—4; envelopes and parallel curves, IV, 123—33, 152—7, 158—65; curvature of, IV, 466—9; theorem on degenerate, V, 98—9; developable, and prohessians, V, 267—83; planar, V, 578; sibi-reciprocal, VI, 21, X, 252—5; sextic, VI, 87—100; singularity of, VI, 123—8; tetrahedral, VII, 48—53; on certain skew, VII, 54—65; Steiner's, VII, 247; intersection of two, VII, 563; divisible into squares by curves of curvature, VIII, 97—8, 145—6, 264—8; correspondence, transformation, and deficiency, VIII, 200—8; penultimate forms of, VIII, 262—3; transformation of unicursal, VIII, 388—93; deficiency of certain, VIII, 394—7; reciprocal, VIII, 394; of eighth order, VIII, 401—3; representation on plane, VIII, 538—9; families of, VIII, 567; transformation of equation of, to chief axes, IX, 48—51; the term, in five-dimensional geometry, IX, 79; orthogonal to set of lines, IX, 587—91; flexure of spherical, X, 30—2; of minimum area, X, 63—7, XIII, 41—2; octic, X, 79—92; on a sibi-reciprocal (octic), X, 252—5; fleflecnodal planes, X, 262—4; Jacobian of six points, X, 281—93; flexure of, X, 331—2; distribution of electricity on two spherical, XI, 1—6; general theory, XI, 14—6, 224; deformation and flexure of, XI, 66—7, 317—22; theory of apsidal, XI, 111—3; theory of reciprocal, XI, 225—34; contact of line with, XI, 281—93; geodesic curvature of

curve on a, XI, 323—30; Gaussian theory, XI, 331—6; and solid geometry, XI, 569; ruled, in *Ency. Brit.*, XI, 572—3; in *Ency. Brit.*, XI, 580—2; general theory of curvilinear coordinates, XII, 1—18; determination of order of surface, XII, 42—4; minimal, and Joachimsthal's theorem, XII, 594—5; with plane or spherical curves of curvature, XII, 601—38; quasi-minimal, XIII, 42; the absolute, XIII, 42; applicable to each other, XIII, 253—64; and systems of tetrads of circles, XIII, 425—9; of order n which pass through given cubic curve, XIII, 534—5; (*see also* Developables, Monoid, Orthogonal, Parallel, Reciprocal, and Wave Surfaces, Scrolls).

Switzerland: Cayley's visits to, VIII, xi.

Syllogism: theory of, VIII, 65—6.

Sylow, L.: theorems on groups, XIII, 530—3.

Sylvester, J. J.: special factors, I, 337; Sturmian functions, I, 392, II, 471—4; schoolgirl problem, I, 483; theory of hyperdeterminants, I, 577, 589; commutants, I, 584; endoscopic, I, 588; theory of permutants, II, 23, 26, 27; rationalization of algebraical equations, II, 42; matrices, II, 219, 604; law of reciprocity, II, 232, 234; partitions, II, 248—9, 506, XII, 217; contravariants, II, 320; combinants, II, 322; cubic curves, II, 405; symmetric functions, II, 465; canonical forms, II, 523; cobezoutiants, II, 524; bezoutiants, II, 526; hyperdeterminants, II, 598—601; logic of characteristics, III, 52; a special determinant, III, 122; elimination, III, 214—5; independent variables in differential calculus, III, 246; reversion of series, IV, 36, 37, 54—9; canonical form of binary quantics, IV, 43—52, 53; double partitions, IV, 166—70; conics and five-pointic contact, IV, 231; on derivative of point on cubic, IV, 231; finite differences, IV, 263; equation of differences, IV, 277; invariants, IV, 349; Tschirnhausen's transformation, IV, 391; volume of tetrahedron, IV, 462; involution of six lines, IV, 582, 593, VII, 66; lines in involution, V, 1—3; quadric cones, V, 6; quartic surfaces, V, 69; canonic root of binary quantic, V, 103—5; discriminant of quintic, V, 592; conic and cubic, V, 608; derivation of points of cubic curve, VI, 20; quintics, VI, 147—8; on roots of algebraical equation, VI, 147; bicorn, VI, 158; foci of conics, VII, 1—4; differential operators, VII, 8; cubic transformation of elliptic functions, VII, 44; Cartesian curves and cubic curve, VII, 556; spherical problem, VII, 563; discussions with, on covariants, VIII, xv; theory of matrices, VIII, xxxii—iii; root-limitation, IX, 22, 39; elimination, IX, 43, XIII, 545—7; residuation, IX, 211; quartic curves and functions of a single parameter, IX, 315—7; scalene transformation, IX, 527, 534; development of idea of Eisenstein, X, 58—9; numerical generating function, X, 339; linkwork, X, 407; N. G. F. of binary septic, X, 408—9; theorem relating to covariants, X, 430; on trees, X, 598—600; theory of tamisage, XI, 409—10; partitions, XII, 217; perpetuants, XII, 251, 252, 253; non-unitary partition tables, XII, 273; d'Alembert-Carnot geometrical paradox, XII, 305—6; umbræ, XII, 347; invariants and reciprocants, XII, 393; a Diophantine relation, XII, 596; *Nature*, notice in, XIII, 43—8; syzygetic relations, XIII, 224; reciprocants, XIII, 333—5, 366, 379—81; lectures on theory of reciprocants, XIII, 381—98.

Symbolical Forms: of hyperdeterminants, I, 80—94; of covariants, I, 577, 585.

Symbols: modular functions, IV, 484—9.

Symmetric: the term, I, 410.

Symmetrical: the term, IV, 599, 604, VI, 524—5, XI, 496.

Symmetric Covariants: II, 233.

Symmetric Curve: and system of equations, I, 473.

Symmetric Functions: of roots of an equation, II, 417—39, 602—3; partitions, II, 418; tables, II, 423—39; resultant of a system of two equations, II, 440—53; tables, II, 445—53, VI, 292—9; of the roots of certain systems of two equations, II, 454—64, VI, 292—9; conditions for existence of given systems of equalities among roots of an equation, II, 465—70, 603—4, VI, 300—12; tables, II, 467; conditions for existence of systems of equal roots of binary quartic or quintic, VI, 300—12; and theory of equations, X, 6—8; non-unitary, and seminvariants, XII, 239—48, 275; tables of roots, XII, 263—72, 273—4; a differential operator, XII, 318; seminvariants, XIII, 265—332; (*see also* Seminvariants).

Symmetroid: the term, VII, 134, 280; lineo-linear correspondence of quartic surfaces, VII, 157—9; and Jacobian, VII, 160—3, 175; with given nodes, VII, 163—6; and decadianome, VII, 256, 259; and circumscribed cone, VII, 258—9; theory, VII, 264.

Symmetry: Sylvester on, XIII, 45.

Symptose: the term, I, 523, 529, 557—8.

Syntypic: the term, VII, 123.

Système Linéaire: of Laguerre is a matrix, II, 604.

System of Equations: order of, I, 457—61, 589; connected with Malfatti's problem, I, 465—70; note, I, 532—3, 589; algebraical, XI, 39—40.

Syzygant: the term, XII, 251; and seminvariants, XII, 257—62.

Syzygies: of degree six, VI, 148—53; of binary quintic connected, VII, 334; for binary cubic, IX, 55; of quintic, X, 346—55; of sextic, XII, 257—62; of binary quartic, and elliptic integrals, XIII, 32; Sylvester's work in, XIII, 46; syzygetic relations among powers of linear quantics, XIII, 224—7; and seminvariants, XIII, 310.

Tables, Brit. Assoc. Report on Mathematical: IX, 461—99; introductory, IX, 461—2; of divisors and prime numbers, IX, 462—70; prime roots, IX, 471—7; Pellian equation, IX, 477—80; partitions, IX, 480—3; quadratic forms, IX, 484—6; binary, ternary, quadratic, and higher forms, IX, 486—93; complex theories, IX, 493—9.

Tables: linear transformations, I, 108; of covariants for quadratic, cubic, quartic, quintic, II, 276—81, II, 310—35; of covariants M to W of binary quintic, II, 282—309; covariants for sextic, II, 314—5; for septimic, II, 315—6; for octavic, II, 316—8; for nonic, II, 318—9; of concomitants of ternary quadric, II, 322—3; of ternary cubic, II, 323—9, 331—5; of symmetric functions of roots of equation, II, 423—39; of resultants of two equations, II, 449—53; Sturmian functions for equations from second to fifth degrees, II, 471—4; disturbing function in lunar theory, III, 299—308, 311—8, VII, 516, 519—24, 525—7; of functions in theory of elliptic motion, III, 360—474; Degen's, for Pellian equation, IV, 40; equation of differences, IV, 246—56, 280—91; Arbogast's method of derivations, IV, 274—5; concomitants of ternary cubics, IV, 333—41; Tschirnhausen's transformation for quartics, IV, 373—4, 379—80; and for quintics, IV, 387—90; numerical expansions, IV, 470; polyacra, V, 44; binary quadratic forms, V, 141—56, 618; properties of scrolls, V, 171—2; axial systems of polyhedra, V, 532—9; curves in space, V, 616; for prime or composite modulus, VI, 83—6; asyzygetic covariants, VI, 149—152; quantics, VI, 167—8; resultant of a system of two equations, VI, 292—9; conditions for existence of systems of equal roots of quartic or quintic, VI, 300—12; singularities of cubic surfaces, VI, 363; also lines and planes, VI, 373; Legendre's elliptic functions, VII, 20; geodesic lines on oblate spheroid, VII, 23; rational transformation between two spaces, VII, 210—3, 224; nodal quartic surfaces, VII, 283, 287, 291, 296; quartic surfaces, VII, 310, 609—10; irreducible covariants of binary quintic, VII, 341—6; planogram No. 1, VII, 439—40; ditto No. 2, VII, 450—1; geodesic lines on ellipsoid, VII, 504—6; binary cubic forms, VIII, 51—64; theory of curve and torse, VIII, 81—4; Pineto's of logarithms (review), VIII, 95—6; cones satisfying six conditions, VIII, 100; geodesic lines, particularly on quadric surface, VIII, 196—9; in-and-circumscribed triangle, VIII, 214—21; centro-surface of ellipsoid, VIII, 365; Steiner's surface, IX, 7; transformation of elliptic functions, IX, 128—35, 163; Newcomb's planetary, IX, 181—4; projection of skew hyperboloid of revolution, IX, 240; classification for mathematical, IX, 424—5; report on mathematical, IX, 424—5; chemical trees, IX, 436—43, 446—8, 450—60, 544—5; double theta functions, X, 168—9, 171, 172—3; regular solids, X, 270—3; concomitants of quintic, X, 349—55, 362—9, 370—6, 377—94, 397—400; transvectants for quintic, X, 378—394; Kummer hexads, X, 506; theta functions, X, 507—10, 513—28, 530—6, 540—2, 544—6; theory of numbers, trisection, XI, 89; ditto quartisection, XI, 94; Reuschle's, of prime roots, XI, 95—6; of finite differences, XI, 144—7; connected with polyhedral function, XI, 158—9, 192; covariantive, XI, 272—80; Schubert's

numerative geometry, XI, 286; theorems of squares, XI, 299—313; theory of numbers, XI, 316; concomitants of ternary cubic, XI, 345—7; literal, for binary quantics, otherwise a partition table, XI, 357—64; for binary sextic, XI, 372—6, 377—88; covariantive, XI, 409—10; Plücker's equations, XI, 472; of Gauss, XI, 545; symmetric functions of roots of an equation, XII, 263—72, XIII, 272—4, 288; non-unitary partition, XII, 273—4; seminvariant, XII, 275—89; orthomorphosis of circle into parabola, XII, 336; of groups, orders two to twelve, XII, 643—56; Wallis's investigation for π, XIII, 23—5; quadrinvariant and cubinvariant of quadri-quadric function, XIII, 68; partitions of a polygon, XIII, 95, 112; theory of rational transformation, XIII, 116; substitution groups for two to eight letters, XIII, 118—49; corrected seminvariant for weights 11 and 12, XIII, 217—23; of conjugates, XIII, 303; seminvariants and symmetric functions, XIII, 311, 313, 331—2; of pure reciprocants to weight 8, XIII, 333—5; report of British Association committee on Pellian equation, and tables, XIII, 430—67; omega and theta functions, XIII, 558—9.

Tacinvariant: the term, IV, 607.

Tac-locus: in singular solutions, VIII, 533.

Tacnode: defined, II, 28—32, V, 266.

Tactic: and algebra, V, 293—4; the term, XI, 443; (*see also* Arrangements, Groups).

Tactinvariant: of two quantics, II, 320; the term, V, 305.

Tactions: analytical solution, III, 255—7; formulæ, IV, 510—2; Casey's equation, VI, 543; and trizomal curves, VI, 575; problem of, XIII, 150—69.

Tait, P. G.: arrangements, X, 245; quaternions, XII, 303, 475, XIII, 541—4; finite differences, XII, 412.

Talbot, W. H. F.: curve of, IV, 123.

Tamisage: Sylvester's theory of, XI, 409—10.

Tangential: defined, II, 558; of a curve, IV, 188.

Tangent Lines: and surface, XI, 630, 632—4.

Tangent Omals: VI, 467—9.

Tangent Planes: and surface, XI, 630, 632—4.

Tangents: and two-dimensional geometry, II, 575; inflexional and chief, VIII, 157, 294; singular of a quartic, X, 603; in *Ency. Brit.*, XI, 564—5, 579—80; (*see also* Bitangents).

Tannery, J.: linear differential equations, XII, 394.

Tantipartite: the term, I, 584, II, 517, IV, 464, 604.

Taylor, C.: locus *in plano* problem, VII, 599; general theory of surfaces, XII, 42—4.

Taylor, H. M.: inversion, IX, 18; partitions of a polygon, XIII, 93, 112.

Taylor's Theorem: Lagrange's demonstration, VIII, 493—5, 519; note on, VIII, 524.

Tchebycheff, P.: theory of numbers, XI, 616.

Terminology: recent mathematical, IV, 594—608.

Terms: in symmetrical determinant, IX, 185—90.

Ternary: the term, IV, 604, VI, 464.

Ternary Cubics: relation between two, IV, 79—81; memoir on quantics, IV, 325—41; form problem, VII, 548; the 34 concomitants, XI, 342—56; canonical form, XI, 343.

Ternary Quadratics: resultant of three, IV, 349—58.

Ternary Quadrics: and involution, XIII, 350—3.

Ternary Quantics: and bitangents of plane curve, IV, 188; involution, V, 301—9.

Tetrads: the term, XII, 590; systems of, XIII, 425—9.

Tetrahedra: reciprocals, III, 7; axial systems, V, 531—9; note on, V, 557—9; and cubic surfaces, VII, 607; correspondence of points in relation to two, VIII, 200—8; Steiner's surface, IX, 1—12; in perspective, IX, 209—10; automorphic function for, XI, 169, 179—83, 184, 212—6.

Tetrahedral Surfaces: VII, 48—53, 54—65.

Tetrahedroid: and wave surface, I, 302—5, 587; 16-nodal quartic surfaces, V, 431—7; the term, VI, 21, X, 252; and scrolls, VII, 245; as particular case of 16-nodal quartic surface, X, 437—40.

Tetrazomal (*see* Polyzomal Curves).

Text-Books: on determinants, elimination and higher algebra, IV, 608.

Theory of Equations (*see* Equations, Theory of).

Theory of Groups (*see* Groups).

Theory of Numbers (*see* Numbers, Theory of).

Theta Functions, Memoir on Single and Double: X, 463—565; historical, X, 463—4; Part I, X, 464—76; definitions, X, 464—5; allied functions, X, 465—6; even-integer alteration of characters, X, 466; odd ditto, X, 466; even and odd functions, X, 467; quarter-periods unity, X, 467—8; conjoint quarter quasi-periods, X, 468—9; product-theorem, X, 469—71; résumé of ulterior theory of the single functions, X, 471—3; ditto, double functions, X, 474—5; remark as to notation, X, 475—6; Part II, X, 476—97; notation, X, 476; constants of the theory, X, 477—8; product theorem, X, 478—80; the square-set, X, 481—2; relation between the constants, X, 482—3; product-sets, X, 483—4; comparison with Jacobi's formulæ, X, 485; the square set, X, 485—7, 488—9; elliptic integrals of third kind, X, 489—90; addition formulæ, X, 491—2; doubly infinite product forms, X, 492—4; transformation q to r, X, 494—7; Part III, the double theta functions, X, 497—565; product-theorem, X, 497—506; tables, X, 506—8; product-theorem and its results, X, 509—39; tables, X, 513—39; the first set, X, 539; second ditto, X, 540; third ditto, X, 541; fourth ditto, X, 542; considerations, X, 543—8; résumé, X, 548; 16-nodal quartic surfaces, X, 548—51; x, y expressions of theta functions, X, 551—5; further results of product-theorem, X, 555—7; differential relations connecting theta and quotient functions, X, 557—9; differential relations of theta functions, X, 559—61; ditto, u, v, x, y, X, 561—5.

Theta Functions: of Jacobi, I, 136, 290; and elliptic integrals, XI, 41—6; theory of multiple, XI, 242—9; notation, XI, 243—5; evolution, XI, 451—5; the term, XI, 532; linear transformation, XII, 337—43; formula relating to zero value of, XII, 442—3; Smith's memoir, XIII, 558—9; (*see also* Abelian, Double Theta, Elliptic, Single Theta, and Triple Theta, Functions).

Third Class: curves of, II, 395—6.

Thomae, J.: linear differential equations, XII, 394, 396, 444; theta functions, XII, 442.

Thomson, F. D.: tangents of conic, V, 578.

Thomson, J.: mechanical integrator, XI, 53.

Thomson, W. (*see* Kelvin, Lord).

Three-bar Motion: IX, 551—80, XI, 481, XIII, 505—16.

Three Bodies: problem of, III, 97—103, 183, IV, 548—552; in a line, IV, 538—40; other cases, IV, 540—1.

Time and Number: XI, 442—4.

Tissot, A.: spherical pendulum, IV, 534, 593.

Titus, Colonel: arithmetical problem, IV, 171—2.

Todhunter, I.: conics, IV, 481; Taylor's theorem, VIII, 493—5; q-squares, X, 27; probabilities, X, 600.

Topography: contour and slope lines, IV, 108—11, 609.

Topology: of space, VI, 22; of chessboard, X, 609.

Torsal: the term, VI, 334, 336, 341, 355, 582—5.

Torse, on a Certain Sextic: VII, 99—114; introductory, VII, 99—100; theorem of four binary quartics, VII, 100; standard equation of unicursal quartic, VII, 101; tangent line and osculating plane of unicursal quartic, VII, 101; its final form, VII, 102; determination of sextic torse, VII, 102—3; principal sections of ditto, VII, 103—5; partial determination of equation, VII, 105; determination of the unknown coefficients, VII, 106—11; equation of sextic torse, VII, 112; ditto, and centro-surface of ellipsoid, VII, 113—4.

Torses: the term, V, 182, XI, 573; and scrolls, V, 199—200; and curves, V, 505—10; a special sextic developable, V, 511—9; singularities, VI, 601; on some sextic, VII, 116—7, 118—20; circumscribed to two quadrics, VIII, 520—1; on a sextic, X, 68—72; depending on elliptic functions, X, 73—8;

and certain octic surfaces, x, 79—92; kinds of, xi, 227; in *Ency. Brit.*, xi, 628, 629—32; and surfaces, xi, 632; and non-Euclidian plane geometry, xii, 222; (*see also* Developables).

Torsion: the term, i, 234, xiii, 232, 234.

Tortolini, B.: envelopes, parallel curves and surfaces, iv, 123—33; parallel surfaces of ellipsoid, iv, 133.

Tortuous Curves (*see* Curves).

Torus: the term, vii, 246, viii, 25; paper by Darboux, vii, 247; the conic-, ix, 519—21.

Townsend, R.: inertia, iv, 566, 593; confocal quadrics, viii, 520.

Tractor: the term, vii, 73—5, x, 269; six coordinates of a line, vii, 85—6, 93—5.

Trajectories: root-limitation, ix, 22—7; and orthomorphosis, xiii, 170.

Transcendental Analysis (*see* Function).

Transcendental Function: the term, xi, 524.

Transcendental Integrals (*see* Abelian Integrals).

Transcendent, Gudermannian: v, 86—8, 617.

Transformation: of quadratic forms, ii, 145—9, 192—201, 215; of two quadric functions, iii, 129—31; the term modulus of, iv, 605; plane curves, vi, 1—8, 593, viii, 387; Cremona's, vi, 22—3; polyzomal curves, vi, 553, 565—6; two quantics into each other, viii, 385—7; unicursal surfaces, viii, 388—93; binary quadratic form, viii, 398—400; doubly infinite products, x, 494—7; theories, xi, 482; Landen's, xi, 584; double theta functions, xii, 358—89; of order 11, and modular equation, xiii, 38—40; modular equation for cubic, xiii, 64—5; (*see also* Special Headings below).

Transformation, Automorphic: iv, 416, v, 439; of binary cubic function, xi, 411—6.

Transformation, Cubic: in elliptic functions, ix, 522—6.

Transformation, Geometric: vii, 121—2.

Transformation, Homographic: xi, 189—90, 196—208.

Transformation, Linear: ii, 225, xi, 237—41; imaginary linear, vi, 183—6; lineo-linear, vii, 215—6, 236—8; of theta functions, xii, 337—43.

Transformation of Coordinates: i, 123—6, 586, iv, 552—9, vii, 95, 415—7, xi, 136—42, 558—61; formulæ, vii, 97—8.

Transformation of Elliptic Functions: i, 120—2, 585, v, 472, ix, 103—6, 244—5, x, 333—8, 611, xi, 26, xii, 416—7, 535—54, xiii, 29—32, 490—2, 505—34, 535—55, 556—7.

Transformation of Elliptic Integrals: i, 508—10, iv, 60—9, 609.

Transformation of Equations: ix, 42, 48—51; of differential, v, 78—9.

Transformation of Integrals: i, 383, iii, 1—4, 438—44, ix, 250—2.

Transformation of Tschirnhausen: vi, 165—9, xi, 396; for cubics, iv, 364—7, xiii, 421; quartics, iv, 368—74; quartics and quintics, iv, 375—94, v, 449—53; theory of equations, xi, 509.

Transformation, Orthomorphic: of a circle into itself, xiii, 20.

Transformation, Quadric: of elliptic functions, xii, 58; between two planes, xii, 100—1.

Transformation, Rational, between Two Spaces, Memoir: vii, 189—240; introductory, vii, 189—90; general principle, vii, 190—3; homographic transformation between two lines, vii, 193—7; rational ditto between two planes, vii, 197—213, 216—21; tables, vii, 210—3; quadric transformation between two planes, vii, 213—6; quadric transformation any number of times repeated, vii, 219—21; reduction of general rational transformation to a series of quadric transformations, vii, 222—4; rational transformation between two spaces, vii, 224—9, 238—40; ditto quadri-quadric, vii, 229—30; ditto quadri-cubic, vii, 230—3; ditto cubo-cubic, vii, 233—8; this principal system consists of six lines, vii, 234—6; principal system of a proper sextic curve—the lineo-linear transformation between two spaces, vii, 236—8.

Transformation, Rational: of plane curves, vi, 1—8; does not alter deficiency, vi, 3; between two planes and special systems of points, vii, 253—5; note on a theory of, xiii, 115—6.

Transformation, Rectangular: xi, 421—8.

Transformation, Scalene: of plane curve, ix, 527—34.

Transformation, Septic: of elliptic functions, x, 333—8, xii, 535—54.
Transformation, Special Quartic: of elliptic functions, ix, 103—6.
Transmutant: defined, ii, 515.
Transpose: the term, ii, 493.
Transvectant: form of covariants, viii, 404—8; (*see also* Derivatives).
Trees: analytical forms called, iii, 242—6, iv, 112—5, xi, 365—7; curves which satisfy given conditions, vi, 260; application to chemistry, ix, 202—4, 427—60, 544—5; problem and solution, x, 598—600; a theorem on, xiii, 26—8.
Triads: of seven and fifteen things, i, 481—4, 589; of fifteen things, v, 95—7.
Triangle: harmonic relation of point and line, ii, 96—7; reciprocal triangles, iii, 5—7; circumscribed about conic, properties of, iii, 29—34; theorem of line and conic, v, 100—2; problems, v, 564, 566, 593, vii, 581, 599, x, 575; locus in relation to, vi, 53—64; locus and envelope, vi, 72—82; solution of problem in *Principia*, Bk I. Sec. V. Lemma xxvii, vii, 30; potential of, ix, 270—1; non-Euclidian, xiii, 482—3; nine-point circle of a plane, xiii, 520—1.
Triangle, In-and-circumscribed, the Problem of, Memoir: viii, 212—57; introductory, viii, 212—3; tables, viii, 214—21; principle of correspondence, viii, 222—5; locus of a free angle, viii, 225—7; application of theory to locus, viii, 227—8; solutions for 52 cases, viii, 228—51; the case 52, viii, 251—57.
Triangle, In-and-circumscribed: ii, 87—90, 91—2, 138—44, 145—9, iii, 67—75, 229—41, iv, 435—41, v, 489—92, 549—50, 553, viii, 565—6; a posteriori demonstration of porism, iii, 80—5; (*see also* Porism).
Triangle, Spherical: theorem of, iv, 80, xi, 97—9; nine-point circle of, xiii, 548—51.
Tricircular Sextic: ix, 562—70.
Trident Curve: classification, v, 355—69, 395.
Trigonometry: transformation of an expression, ii, 45—6; multiple sines, x, 1—2; theorem in partitions and, x, 16; identities, xi, 38, xiii, 538—40; formulæ, xii, 108; an expansion, xii, 319—20.
Trigonometry, Spherical: theorem, iv, 80, xi, 97—9; identity, viii, 525; foundation, xi, 570.
Trigonometry, Spheroidal: ix, 197.
Trihedral Pair: the term, vi, 374.
Trinodal Quartic: x, 602; (*see also* Quartic Curves).
Tripair: the term, x, 450—1.
Tripartite: the term, vi, 464.
Triple Tangent Planes: of cubic surfaces, i, 445—56, 589, vi, 372, 376.
Triple Theta Functions: x, 432—6, xi, 47—9; algorithm for characteristics of, x, 441—5; and quartic curves, x, 446—54.
Tris: the abbreviation in groups, xiii, 119.
Trisection: in theory of numbers, xi, 84—96.
Tritom: and point, v, 138.
Trivector: the term, vii, 400, 401; planet's orbit from, vii, 406—12, 426—8.
Trizomal (*see* Polyzomal Curves).
Trope: the term, vi, 330, viii, 73, x, 54—5.
Tropical Point: the term, xii, 433.
Truel, H. D.: imaginaries, xii, 467.
Tschirnhausen's Transformation: vi, 165—9, xi, 396; for cubics, iv, 364—7, xiii, 421; quartics, iv, 368—74, 375—82, v, 449—53; quintics, iv, 382—94; theory of equations, xi, 509.
Tucker, R.: geometrical interpretation, x, 581.
Twisted: the term, vi, 524—5.
Two Centres Problem: iv, 524—32.
Two-way Point: the term, xiii, 507.

Ueberschiebung: the word, I, 585; (*see also* Derivations).
Ultra-elliptic Functions (*see* Hyperelliptic Functions).
Umbilicar Centres: the term, VIII, 326, 351.
Umbilici: and differential equations, V, 115—30; curves of curvature near, VII, 330—1; on surface of nth order, VIII, 320; the term, XI, 581; (*see also* Geodesic Lines).
Umbral: theory of seminvariants, XIII, 266; notation of, and seminvariants, XIII, 301—6.
Umbræ: the term, XII, 347.
Unibasic: the term, XII, 642.
Unicursal Curves: VI, 2.
Unicursal Octics: XII, 310.
Unicursal Quartics: standard equation, VII, 101; tangent line and osculating plane, VII, 101; its final form, VII, 102.
Unicursal Surfaces: transformation of, VIII, 388—93.
Unicursal Twisted Quartic: XII, 428—31.
Uniform Convergence: XIII, 342—5.
Uniform Function: XII, 433.
Uniform Series: defined, IV, 457.
Unipartite: the term, VI, 464.
Uniplanar-node: the term, VI, 361.
United Points: in correspondence, VI, 9.
Unity: prime roots, XI, 56—60; imaginary roots, IX, 263; ninth roots, XIII, 66.
Universal Algebra: Sylvester's theory of, XIII, 47.
Unode: the term, VI, 362.
Uranus: Newcomb's work, IX, 180—4.

Vacuity: Sylvester's theory of, XIII, 47.
Valentiner, H.: curves in space, V, 613—7; theory of surfaces, XI, 14—6.
Values: principal, of complex expression, I, 309; of $\Pi i = \Gamma(1+i)$, XIII, 522—4.
Vandermonde, A. T.: solution by radicals, X, 11; theory of equations, XI, 513.
Variables: (2, 2) correspondence of two, IX, 94—5; normal in dynamics, IX, 111; imaginary, XI, 439—41.
Variation: of parameters in rotation of solid body, I, 242; of arbitrary constants, III, 161—200; of planet's orbit, III, 516—8, VII, 541—5.
Variations, Calculus of: Jacobi on, III, 174; problem in, VII, 263.
Velocities, Virtual: general equation, IX, 205—8.
Veronese, G.: Pascal's theorem, VI, 594; four-dimensional space, XI, 442.
Vertices of Cones (*see* Cones).
Vicinal Surfaces: conormal correspondence of, VIII, 301—8; (*see also* Surfaces).
Vieta, F.: tactions, XIII, 152.
Virginia: Key and Sylvester, professors at, XIII, 43.
Virtual Velocities: general equation, IX, 205—8.

Walker, S. C.: the anharmonic-ratio sextic, VII, 314—5; orbit of Neptune, IX, 180.
Wallis, John: biographical notice, XI, 640—3; multiple algebra, XII, 466; his expression for π, XIII, 22—5.
Walton, W.: root-limitation, IX, 39; maxima and minima, IX, 40—1; transformation of equations, IX, 42; integration and definite integrals, IX, 56—63; ray planes and biaxal crystals, IX, 107—9.
Waring, E.: equation of differences, IV, 240, 252; sum of mth powers of the roots of an equation, XIII, 213—6.
Warren, J.: on binary cubics V, 289; curvilinear coordinates, XII, 1—18; multiple algebra, XII, 460, 468.

Wave Surfaces: tetrahedroid, I, 302—5, 587, VI, 21; on, IV, 420—6, 432—4, XIII, 238—52; equation of, in elliptic coordinates, XI, 71—2; evolution of Fresnel's, XI, 449.
Weber, H. H.: triple theta functions, X, 444, 446—54; bitangents of quartic, XI, 221—3; elliptic functions, XIII, 559.
Weddle, M.: cubic curves, IV, 497; quadric cones, V, 4.
Weierstrass, K.: doubly infinite product, I, 586; function $Al(x)$, I, 587; *al*-functions, V, 33—7; Steiner's quartic surface, V, 423; infinite products, VIII, xl; triple theta functions, X, 432, 434; theta functions, X, 499; elliptic integrals, XI, 64; theta functions, XI, 242; theory of functions, XI, 451—2, 454; function of, XI, 540; Abelian functions, XII, 98; and Jacobian elliptic functions, XII, 425—7; transformation in elliptic functions, XIII, 29, 31.
Weight: and partitions of a polygon, XIII, 110.
Weingarten, J.: application of surfaces to each other, XIII, 253—64.
Whewell, W.: dynamics, IV, 518; mathematics, XI, 431—2; number and time, XI, 442.
Whitworth, W. A.: triangles and conics, V, 593.
Wiener, C.: model of cubic surface with twenty-seven real lines, VIII, 366—84.
Wilbraham, H.: probabilities, II, 594—8, V, 85.
Wilkinson, M. M. U.: Taylor's theorem, VIII, 519; chances, X, 588; rectangular transformation, XI, 421—8.
Wilkinson, T. T.: circle and points, V, 560.
Wilson, J.: theorem of, XI, 598; and proof, XII, 45.
Wolstenholme, J.: relation among derivatives of a function, X, 590—2; conic and cubic, X, 605—7.
Women: Cayley and higher education of, VIII, xix.
Woolhouse, W. S. B.: theorem of integration, problem, VII, 588; algebraical theorem, X, 594—6.
Worms, H.: rotation of the Earth, IV, 537, 593.
Woven: the term, XIII, 121.
Wright, E.: Mercator's projection, XI, 448.
Wright, T. C.: on Cayley as a law-student, VIII, xiv.
Writing of Cayley: frontispiece, VIII.
Wronski, H.: theorem of, IX, 96—102.

Young, G. P.: soluble quintic equations, XIII, 88; theory of groups, XIII, 336, 533.
Young, J. R.: sums of squares, II, 52; theorems of squares, XI, 294, 301.

Zech, P.: wave surface, IV, 420—5, 432—4.
Zero-values: of theta functions, X, 499—500, XII, 442—3; (*see also* Theta Functions).
Zeuthen, H. G.: curves and developables, I, 587; sextactic points, V, 545; curves which satisfy given conditions, VI, 191, 192, 200—26, 594; capitals, VI, 280; reciprocal surfaces, VI, 577—81, 591, 596—601, XI, 234; cubic surfaces, VI, 595—6; correspondence of two points on a curve, VII, 39; theory of conics, VII, 552—4; theory of curve and torse, VIII, 72; table of singularities of torse, VIII, 81—2; degenerate forms of curves, XI, 220; quartic curves, XI, 480; systems of curves, XI, 486—7.
Zolotareff, G.: elliptic integrals, X, 143.
Zomal: defined, VI, 473; (*see also* Polyzomal Curves).
Zornow, A. R.: mathematical tables, IX, 486.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY & SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. I.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1889

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

PREFACE.

RATHER more than a year ago I was requested by the Syndics of the University Press to allow my mathematical papers to be reprinted in a collected form: I had great pleasure in acceding to a request so complimentary to myself, and I willingly undertook the work of superintending the impression of them, and of adding such notes and references as might appear to me desirable.

The present volume contains one hundred papers (numbered 1, 2, 3,...,100) originally published in the years 1841 to 1853. They are here reproduced nearly, but not exactly, in chronological order: and as nearly as may be in their original forms; but in a few cases where the paper is controversial, or where it is a translation (into French or English) of an English or French paper, only the title is given: there are in some few cases omissions which are indicated where they occur. The number is printed in the upper inside corner of the page; it is intended that the numbers shall run consecutively through all the volumes; and thus a paper can be referred to simply by its number.

I have of course corrected obvious typographical errors, and in particular have freely altered punctuation, but I have not attempted to verify formulæ. Additions are made in square brackets [ ]; to avoid confusion with these, square brackets occurring in the original papers have in general been changed into twisted ones { }, but where they occur in a formula it was not always possible to make the alteration. The addition in a square bracket is very frequently that of a date: it appears to me that the proper reference to a serial work is by the number of the volume, accompanied by the date on the title page: the date is always useful, and, in the case of two or more series of a Journal or set of Transactions, we avoid the necessity of a reference to the series; Liouville t. I. (1850) is better than Liouville, Série 2, t. I. I regret that this rule has not been strictly followed as regards the titles of some of the earlier papers, see the remark at the end of the Contents.

A. CAYLEY.

January 23, 1889.

CONTENTS.

[An Asterisk denotes that the Paper is not printed in full.]

PAGE

1. *On a Theorem in the Geometry of Position* 1
   Camb. Math. Jour. t. II. (1841), pp. 267—271 (1841)

2. *On the Properties of a certain Symbolical Expression* . . 5
   Camb. Math. Jour. t. III. (1843), pp. 62—71 (1842)

3. *On certain Definite Integrals* 13
   Camb. Math. Jour. t. III. (1843), pp. 138—144 (1842)

4. *On certain Expansions, in series of Multiple Sines and Cosines* 19
   Camb. Math. Jour. t. III. (1843), pp. 162—167 (1842)

5. *On the Intersection of Curves* 25
   Camb. Math. Jour. t. III. (1843), pp. 211—213 (1843)

6. *On the Motion of Rotation of a Solid Body* 28
   Camb. Math. Jour. t. III. (1843), pp. 224—232 (1843)

7. *On a class of Differential Equations, and on the Lines of Curvature of an Ellipsoid* 36
   Camb. Math. Jour. t. III. (1843), pp. 264—267 (1843)

8. *On Lagrange's Theorem* 40
   Camb. Math. Jour. t. III. (1843), pp. 283—286 (1843)

9. *Demonstration of Pascal's Theorem* 43
   Camb. Math. Jour. t. IV. (1845), pp. 18—20 (1843)

10. *On the Theory of Algebraical Curves* 46
    Camb. Math. Jour. t. IV. (1845), pp. 102—112 (1844)

11. *Chapters in the Analytical Geometry of* (n) *Dimensions* . . 55
    Camb. Math. Jour. t. IV. (1845), pp. 119—127 (1844)

12. *On the Theory of Determinants* 63
    Camb. Phil. Trans. t. VIII. (1849), pp. 1—16 (1843)

PAGE

13. *On the Theory of Linear Transformations* 80
Camb. Math. Jour. t. IV. (1845), pp. 193—209

14. *On Linear Transformations* 95
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 104—122

15. *Note sur deux Formules données par MM. Eisenstein et Hesse* . 113
Crelle, t. XXIX. (1845), pp. 54—57

*16. *Mémoire sur les Hyperdéterminants* 117
Crelle, t. XXX. (1846), pp. 1—37

*17. *Note on Mr Bronwin's paper on Elliptic Integrals* . . . 118
Camb. Math. Jour. t. III. (1843), pp. 197, 198

*18. *Remarks on the Rev. B. Bronwin's paper on Jacobi's Theory of Elliptic Functions* 119
Phil. Mag. t. XXII. (1843), pp. 358—368

19. *Investigation of the Transformation of certain Elliptic Functions* 120
Phil. Mag. t. XXV. (1844), pp. 352—354

20. *On certain results relating to Quaternions* 123
Phil. Mag. t. XXVI. (1845), pp. 141—145

*21. *On Jacobi's Elliptic Functions, in reply to the Rev. B. Bronwin: and on Quaternions* 127
Phil. Mag. t. XXVI. (1845), pp. 208—211

22. *On Algebraical Couples* 128
Phil. Mag. t. XXVII. (1845), pp. 38—40

23. *On the Transformation of Elliptic Functions* 132
Phil. Mag. t. XXVII. (1845), pp. 424—427

24. *On the Inverse Elliptic Functions* 136
Camb. Math. Jour. t. IV. (1845), pp. 257—277

25. *Mémoire sur les Fonctions doublement périodiques* . . . 156
Liouville, t. X. (1845), pp. 385—420

26. *Mémoire sur les Courbes du Troisième Ordre* 183
Liouville, t. IX. (1844), pp. 285—293

27. *Nouvelles remarques sur les Courbes du Troisième Ordre* . . 190
Liouville, t. X. (1845), pp. 102—109

28. *Sur quelques Intégrales Multiples* 195
Liouville, t. X. (1845), pp. 158—168

29. *Addition à la Note sur quelques Intégrales Multiples* . . 204
Liouville, t. X. (1845), pp. 242—244

PAGE

30. *Mémoire sur les Courbes à double Courbure et les Surfaces développables* 207
Liouville, t. x. (1845), pp. 245—250

31. *Démonstration d'un Théorème de M. Chasles* 212
Liouville, t. x. (1845), pp. 383, 384

32. *On some Analytical Formulæ and their application to the Theory of Spherical Coordinates* 213
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 22—33

33. *On the Reduction of* $du \div \sqrt{U}$, *when* U *is a Function of the Fourth Order* 224
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 70—73

34. *Note on the Maxima and Minima of Functions of Three Variables* 228
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 74, 75

35. *On Homogeneous Functions of the Third Order with Three Variables* 230
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 97—104

36. *On the Geometrical Representation of the Motion of a Solid Body* 234
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 164—167

37. *On the Rotation of a Solid Body round a Fixed Point* . . 237
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 167—173 and 264—274

38. *Note on a Geometrical Theorem contained in a Paper by Sir W. Thomson* 253
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 207, 208

39. *On the Diametral Planes of a Surface of the Second Order* . 255
Camb. and Dubl. Math. Jour. t. I. (1846), pp. 274—278

40. *On the Theory of Involution in Geometry* 259
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 52—61

41. *On certain Formulæ for Differentiation, with applications to the evaluation of Definite Integrals* 267
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 122—128

42. *On the Caustic by Reflection at a Circle* 273
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 128—130

43. *On the Differential Equations which occur in Dynamical Problems* 276
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 210—219

PAGE

44. *On a Multiple Integral connected with the Theory of Attractions* 285
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 219—223

45. *On the Theory of Elliptic Functions* 290
Camb. and Dubl. Math. Jour. t. II. (1847), pp. 256—266

46. *Note on a System of Imaginaries* 301
Phil. Mag. t. XXX. (1847), pp. 257, 258

47. *Sur la Surface des Ondes* 302
Liouville, t. XI. (1846), pp. 291—296

48. *Note sur les Fonctions de M. Sturm* 306
Liouville, t. XI. (1846), pp. 297—299

49. *Sur quelques Formules du Calcul Intégral* 309
Liouville, t. XII. (1847), pp. 231—240

50. *Sur quelques Théorèmes de la Géométrie de Position* . . 317
Crelle, t. XXXI. (1846), pp. 213—227

51. *Problème de Géométrie Analytique* 329
Crelle, t. XXXI. (1846), pp. 227—230

52. *Sur quelques Propriétés des Déterminants Gauches* . . . 332
Crelle, t. XXXII. (1846), pp. 119—123

53. *Recherches sur l'Élimination, et sur la Théorie des Courbes* . 337
Crelle, t. XXXIV. (1847), pp. 30—45

54. *Note sur les Hyperdéterminants* 352
Crelle, t. XXXIV. (1847), pp. 148—152

55. *Sur quelques Théorèmes de la Géométrie de Position* . . 356
Crelle, t. XXXIV. (1847), pp. 270—275

56. *Demonstration of a Geometrical Theorem of Jacobi's* . . 362
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 48, 49

57. *On the Theory of Elliptic Functions* 364
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 50, 51

58. *Notes on the Abelian Integrals—Jacobi's System of Differential Equations* 366
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 51—54

59. *On the Theory of Elimination* 370
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 116—120

60. *On the Expansion of Integral Functions in a series of Laplace's Coefficients* 375
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 120, 121

PAGE

61. *On Geometrical Reciprocity* 377
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 173—179

62. *On an Integral Transformation* 383
Camb. and Dubl. Math. Jour. t. III. (1848), pp. 286, 287

63. *Démonstration d'un Théorème de M. Boole concernant des Intégrales Multiples* 384
Liouville, t. XIII. (1848), pp. 245—248

64. *Sur la généralisation d'un Théorème de M. Jellett qui se rapporte aux Attractions* 388
Liouville, t. XIII. (1848), pp. 264—268

65. *Nouvelles Recherches sur les Fonctions de M. Sturm* . . 392
Liouville, t. XIII. (1848), pp. 269—274

66. *Sur les Fonctions de Laplace* 397
Liouville, t. XIII. (1848), pp. 275—280

67. *Note sur les Fonctions Elliptiques* 402
Crelle, t. XXXVII. (1848), pp. 58—60

68. *On the application of Quaternions to the Theory of Rotation*. 405
Phil. Mag. t. XXXIII. (1848), pp. 196—200

69. *Sur les Déterminants Gauches* 410
Crelle, t. XXXVIII. (1848), pp. 93—96

70. *Sur quelques Théorèmes de la Géométrie de Position* . . 414
Crelle, t. XXXVIII. (1848), pp. 97—104

71. *Note sur les Fonctions du Second Ordre*. 421
Crelle, t. XXXVIII. (1848), pp. 105, 106

72. *Note on the Theory of Permutations* 423
Phil. Mag. t. XXXIV. (1849), pp. 527—529

73. *Abstract of a Memoir by Dr Hesse on the construction of the Surface of the Second Order which passes through nine given points* 425
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 44—46

74. *On the Simultaneous Transformation of Two Homogeneous Functions of the Second Order* 428
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 47—50

75. *On the Attraction of an Ellipsoid* 432
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 50—65

PAGE

76. *On the Triple Tangent Planes of Surfaces of the Third Order* 445
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 118—132

77. *On the order of certain Systems of Algebraical Equations* . 457
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 132—137

78. *Note on the Motion of Rotation of a Solid of Revolution* . . 462
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 268—270

79. *On a System of Equations connected with Malfatti's Problem, and on another Algebraical System* 465
Camb. and Dubl. Math. Jour. t. IV. (1849), pp. 270—275

80. *Sur quelques Transmutations des Lignes Courbes* . . . 471
Liouville, t. XIV. (1849), pp. 40—46

81. *Addition au Mémoire sur quelques Transmutations des Lignes Courbes* 476
Liouville, t. XV. (1850), pp. 351—356

82. *On the Triadic Arrangements of Seven and Fifteen Things* . 481
Phil. Mag. t. XXXVII. (1850), pp. 50—53

*83. *On Curves of Double Curvature and Developable Surfaces* . 485
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 18—22

84. *On the Developable Surfaces which arise from two Surfaces of the Second Order* 486
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 46—57

85. *Note on a Family of Curves of the Fourth Order* . . . 496
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 148—152

86. *On the Developable derived from an Equation of the Fifth Order* 500
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 152—159

*87. *Notes on Elliptic Functions (from Jacobi)* 507
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 201—204

88. *On the Transformation of an Elliptic Integral* 508
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 204—206

89. *On the Attraction of Ellipsoids (Jacobi's Method)* . . . 511
Camb. and Dubl. Math. Jour. t. V. (1850), pp. 217—226

90. *Note sur quelques Formules relatives aux Coniques* . . . 519
Crelle, t. XXXIX. (1850), pp. 1—3

91. *Sur le Problème des Contacts* 522
Crelle, t. XXXIX. (1850), pp. 4—13

PAGE

92. *Note sur un Système de certaines Formules* 532
Crelle, t. XXXIX. (1850), pp. 14, 15

93. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 534
Crelle, t. XXXIX. (1850), pp. 16—22

94. *Note sur l'Addition des Fonctions Elliptiques* 540
Crelle, t. XLI. (1851), pp. 57—65

95. *Note sur quelques Théorèmes de la Géométrie de Position*. . 550
Crelle, t. XLI. (1851), pp. 66—72

96. *Mémoire sur les Coniques inscrites dans une même Surface du Second Ordre*. 557
Crelle, t. XLI. (1851), pp. 73—86

97. *Note sur la Solution de l'Équation* $x^{257}-1=0$ 564
Crelle, t. XLI. (1851), pp. 81—83

*98. *Note relative à la sixième section du Mémoire sur quelques Théorèmes de la Géométrie de Position* 567
Crelle, t. XLI. (1851), p. 84

99. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 568
Crelle, t. XLI. (1851), pp. 85—92

100. *Note sur la Théorie des Hyperdéterminants* 577
Crelle, t. XLII. (1851), pp. 368—371

Notes and References 581

Volumes II, III, and IV of the *Cambridge Mathematical Journal* have on the title pages the dates 1841, 1843, 1845 respectively, and volume VIII of the *Cambridge Philosophical Transactions* has the date 1849. As each of these volumes extends over more than a single year, I have added, see p. vii, the year of publication for the papers 1, 2, ... 12. In all other cases, the year of publication is shown by the date on the title page of the volume.

CLASSIFICATION.

GEOMETRY

Intersection of curves, 5
Motion of solid body, 6, 36, 37, 68, 78
Theory of algebraical curves, 10
Analytical geometry of n dimensions, 11
Curves and developables, 30, 83
Geometrical involution, 40
Geometry of position, 50, 55, 70, 95, 98
Theory of Curves, 53
Reciprocity, 61
Transmutations of curves, 80, 81

Distances of points, 1
Lines of curvature of ellipsoid, 7
Pascal's theorem, 9
Transformation of coordinates, 20
Cubic curves, 26, 27
Skew Cubics, theorem of Chasles', 31
Spherical coordinates, 32
Theorem of Sir W. Thomson, Equimomental surfaces, 38
Diametral planes of quadric surface, 39
Caustic by reflexion of circle, 42
Wave surface (Tetrahedroid), 47
Centres of similitude of quadric surfaces, 51
Theorem of Jacobi, Confocal surfaces, 56
Theorem of Jellett, Attraction and surface of ellipsoid, 64
Quadric surfaces, 71
Abstract of memoir by Hesse, Quadric surfaces, 73
Cubic surfaces, triple tangent planes, 76
Developable from two quadric surfaces, 84
Quartic curves, 85
Developable from quintic curve, 86
Formulæ for two conics, 90
Problem of contacts, 91
Conics inscribed in a quadric surface, 96

ANALYSIS

Determinants, 1, 9, 12, 52, 61, 69
Attractions and multiple integrals, 2, 3, 28, 29, 41, 44, 56, 63, 64, 88
Attraction of Ellipsoids, Legendre, 75; Jacobi, 89
Linear transformations, &c. 12, 13, 14, 15, 16, 33, 34, 35, 39, 53, 54, 71, 74, 77, 100
Elliptic functions, 17, 18, 19, 21, 23, 24, 25, 33, 45, 57, 58, 62, 67, 87, 88, 93, 94, 99

Expansions in multiple sines and cosines, 4
Differential equations, 7, 43, 58
Lagrange's theorem, 8
Quaternions, 20, 68
Octaves, 21, 46
Algebraical couples, 22
Maxima and minima, 34
Differentiation, 41
Sturmian functions, 48, 65
Integral Calculus, 49, 62
Elimination, 53, 59
Legendre's coefficients, 60
Laplace's functions, 66
Arrangements, 72, 82
Systems of equations, 32, 77, 79, 90, 92, 96
Equation $x^{257} - 1 = 0$, 97

1.

ON A THEOREM IN THE GEOMETRY OF POSITION.

[From the *Cambridge Mathematical Journal*, vol. II. (1841), pp. 267—271.]

WE propose to apply the following (new?) theorem to the solution of two problems in Analytical Geometry.

Let the symbols

$$|\,\alpha\,|\,, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}, \quad \begin{vmatrix} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ \&c.}$$

denote the quantities

$$\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma, \text{ \&c.}$$

(the law of whose formation is tolerably well known, but may be thus expressed,

$$|\,\alpha\,| = \alpha, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} = \alpha\,|\,\beta'\,| - \alpha'\,|\,\beta\,|\,,$$

$$\begin{vmatrix} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \alpha \begin{vmatrix} \beta', & \gamma' \\ \beta'', & \gamma'' \end{vmatrix} + \alpha' \begin{vmatrix} \beta'', & \gamma'' \\ \beta\,, & \gamma \end{vmatrix} + \alpha'' \begin{vmatrix} \beta\,, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \text{ \&c.}$$

the signs + being used when the number of terms in the side of the square is odd, and + and − alternately when it is even.)

Then the theorem in question is

$$\begin{vmatrix} \rho\,\alpha + \sigma\,\beta + \tau\,\gamma\,.., & \rho\,\alpha' + \sigma\,\beta' + \tau\,\gamma'\,.., & \rho\,\alpha'' + \sigma\,\beta'' + \tau\,\gamma''\,.. \\ \rho'\,\alpha + \sigma'\,\beta + \tau'\,\gamma\,.., & \rho'\,\alpha' + \sigma'\,\beta' + \tau'\,\gamma'\,.., & \rho'\,\alpha'' + \sigma'\,\beta'' + \tau'\,\gamma''\,.. \\ \rho''\,\alpha + \sigma''\,\beta + \tau''\,\gamma\,.., & \rho''\,\alpha' + \sigma''\,\beta' + \tau''\,\gamma'\,.., & \rho''\,\alpha'' + \sigma''\,\beta'' + \tau''\,\gamma''\,.. \\ \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} \rho\,, & \sigma\,, & \tau\,.. \\ \rho', & \sigma', & \tau'\,.. \\ \rho'', & \sigma'', & \tau''\,.. \\ \vdots & \vdots & \vdots \end{vmatrix} \begin{vmatrix} \alpha'', & \beta'', & \gamma''\,.. \\ \alpha', & \beta', & \gamma'\,.. \\ \alpha\,, & \beta\,, & \gamma\,.. \\ \vdots & \vdots & \vdots \end{vmatrix}$$

(This theorem admits of a generalisation which we shall not have occasion to make use of, and which therefore we may notice at another opportunity.)

To find the relation that exists between the distances of five points in space.

We have, in general, whatever x_1, y_1, z_1, w_1, &c. denote

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2+w_1^2, & -2x_1, & -2y_1, & -2z_1, & -2w_1, & 1 \\ x_2^2+y_2^2+z_2^2+w_2^2, & -2x_2, & -2y_2, & -2z_2, & -2w_2, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2+w_5^2, & -2x_5, & -2y_5, & -2z_5, & -2w_5, & 1 \\ 1, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & w_1, & x_1^2+y_1^2+z_1^2+w_1^2 \\ 1, & x_2, & y_2, & z_2, & w_2, & x_2^2+y_2^2+z_2^2+w_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & x_5, & y_5, & z_5, & w_5, & x_5^2+y_5^2+z_5^2+w_5^2 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \overline{x_1-x_1}^2+\overline{y_1-y_1}^2+\overline{z_1-z_1}^2+\overline{w_1-w_1}^2, & & & & & \\ & \overline{x_1-x_2}^2+\ldots, & \overline{x_1-x_3}^2+\ldots, & \overline{x_1-x_4}^2+\ldots, & \overline{x_1-x_5}^2+\ldots, & 1 \\ \overline{x_2-x_1}^2+\ldots, & & & & & \\ & \overline{x_2-x_2}^2+\ldots, & \overline{x_2-x_3}^2+\ldots, & \overline{x_2-x_4}^2+\ldots, & \overline{x_2-x_5}^2+\ldots, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overline{x_5-x_1}^2+\ldots, & & & & & \\ & \overline{x_5-x_2}^2+\ldots, & \overline{x_5-x_3}^2+\ldots, & \overline{x_5-x_4}^2+\ldots, & \overline{x_5-x_5}^2+\ldots, & 1 \\ & 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

Putting the w's equal to 0, each factor of the first side of the equation vanishes, and therefore in this case the second side of the equation becomes equal to zero. Hence x_1, y_1, z_1, x_2, y_2, z_2, &c. being the coordinates of the points 1, 2, &c. situated arbitrarily in space, and $\overline{12}^2$, $\overline{13}^2$, &c. denoting the squares of the distances between these points, we have immediately the required relation

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is easily expanded, though from the mere number of terms the process is somewhat long.

Precisely the same investigation is applicable to the case of four points in a plane, or three points in a straight line. Thus the former gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The latter gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0;$$

or expanding,

$$\overline{12}^4 + \overline{13}^4 + \overline{23}^4 - 2 \,.\, \overline{12}^2\,\overline{13}^2 - 2 \,.\, \overline{13}^2\,\overline{23}^2 - 2 \,.\, \overline{12}^2\,\overline{23}^2 = 0;$$

which may be derived immediately from the equation

$$\pm \overline{12} \pm \overline{13} = \pm \overline{23},$$

and is the simplest form under which this equation, cleared of the ambiguous signs, can be put.

(The above result may be deduced so elegantly from the general theory of elimination, that notwithstanding its simplicity it is perhaps worth mentioning.)

Let $$x_{,,} - x_{,,,} = \alpha, \quad x_{,,,} - x_{,} = \beta, \quad \overline{x_{,} - x_{,,}} = \gamma;$$

then $$\overline{12}^2 = \gamma^2, \quad \overline{23}^2 = \alpha^2, \quad \overline{31}^2 = \beta^2, \quad \text{and} \quad \alpha + \beta + \gamma = 0;$$

from which α, β, γ are to be eliminated. Multiplying the last equation by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and reducing by the three first,

$$\begin{aligned} 0 \,.\, \alpha + \overline{12}^2 \,.\, \beta + \overline{31}^2 \,.\, \gamma + \quad \alpha\beta\gamma &= 0, \\ \overline{12}^2 \,.\, \alpha + \quad 0 \,.\, \beta + \overline{23}^2 \,.\, \gamma + \quad \alpha\beta\gamma &= 0, \\ \overline{31}^2 \,.\, \alpha + \overline{23}^2 \,.\, \beta + \quad 0 \,.\, \gamma + \quad \alpha\beta\gamma &= 0, \\ \alpha + \quad \beta + \quad \gamma + 0 \,.\, \alpha\beta\gamma &= 0; \end{aligned}$$

from which, eliminating α, β, γ, $\alpha\beta\gamma$ by the general theory of simple equations,

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The (additional) equation that exists between the distances of five points on a sphere or four points in a circle, has such a remarkable analogy with the preceding, that they almost require to be noticed at the same time.

If α, β, γ, r be the coordinates of the centre, and the radius of the sphere, and $\delta = \alpha^2 + \beta^2 + \gamma^2 - r^2$, we have immediately

$$\begin{matrix} x_1^2 + y_1^2 + z_1^2 - 2\alpha x_1 - 2\beta y_1 - 2\gamma z_1 + \delta = 0, \\ \vdots \\ x_5^2 + y_5^2 + z_5^2 - 2\alpha x_5 - 2\beta y_5 - 2\gamma z_5 + \delta = 0; \end{matrix}$$

whence eliminating α, β, γ, δ,

$$\begin{vmatrix} x_1^2 + y_1^2 + z_1^2, & -2x_1, & -2y_1, & -2z_1, & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_5^2 + y_5^2 + z_5^2, & -2x_5, & -2y_5, & -2z_5, & 1 \end{vmatrix} = 0;$$

whence, multiplying by

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & x_1^2 + y_1^2 + z_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1, & x_5, & y_5, & z_5, & x_5^2 + y_5^2 + z_5^2 \end{vmatrix}$$

we have immediately

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0.$$

Forming the analogous equation for four points in a circle, and expanding, we readily deduce

$$\overline{14}^4\,\overline{23}^4 + \overline{12}^4\,\overline{34}^4 + \overline{13}^4\,\overline{24}^4 - 2\,.\,\overline{12}^2\,\overline{34}^2\,\overline{13}^2\,\overline{24}^2 - 2\,.\,\overline{14}^2\,\overline{23}^2\,\overline{13}^2\,\overline{24}^2 - 2\,.\,\overline{14}^2\,\overline{23}^2\,\overline{12}^2\,\overline{34}^2 = 0,$$

which is the rational, and therefore analytically the most simple form of

$$\overline{12}\;\overline{34} + \overline{14}\;\overline{23} = \overline{13}\;\overline{24}.$$

Euclid, B. vi., last proposition.

(It may be remarked that the two factors we have employed in the preceding eliminations, only differ by a numerical factor.)

2.

ON THE PROPERTIES OF A CERTAIN SYMBOLICAL EXPRESSION.

[From the *Cambridge Mathematical Journal*, vol. III. (1841), pp. 62—71.]

THE series

$$S_{p\,0}^{\infty} \cdot \zeta_p (a^2 + b^2 \ldots \text{n terms})^{p+i} \left(\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \cdots\right)^p \frac{1}{\{(1+l)\,a^2 + (1+m)\,b^2 \ldots\}^i}$$

$$\left(\zeta_p = \frac{1}{2^{2p+1}\,1\,.\,2 \ldots p\,.\,i\,(i+1) \ldots (i+p)}\right) \ldots\ldots (\psi),$$

possesses some remarkable properties, which it is the object of the present paper to investigate. We shall prove that the symbolical expression (ψ) is independent of a, b, &c., and equivalent to the definite integral

$$\int_0^1 \frac{x^{2i-1}\,dx}{\{(1+lx^2)\,(1+mx^2) \ldots\}^{\frac{1}{2}}},$$

a property which we shall afterwards apply to the investigation of the attractions of an ellipsoid upon an external point, and to some other analogous integrals. The demonstration of this, which is one of considerable complexity, may be effected as follows:

Writing the symbol $\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \cdots$ under the form

$$\left(\frac{d^2}{da^2} + \frac{d^2}{db^2} \cdots\right) - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \cdots\right) = \Delta - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \cdots\right) \text{ suppose,}$$

let the pth power of this quantity be expanded in powers of Δ. The general term is

$$(-1)^q \cdot \frac{p\,(p-1) \ldots (p-q+1)}{1\,.\,2 \ldots q} \cdot \Delta^{p-q} \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \cdots\right)^q,$$

which is to be applied to $\dfrac{1}{\{(1+l)\,a^2 \ldots\}^i}$.

Considering the expression

$$\left(\frac{1}{1+l}\cdot\frac{d^2}{da^2}\cdots\right)^q \frac{1}{\{(1+l)\,a^2\ldots\}^i};$$

if for a moment we write

$$(1+l)\,a^2 = a_1^2,\ \&c.;\quad \Delta_1 = \frac{d^2}{da_1^2}+\frac{d^2}{db_1^2}\cdots;\quad \rho_1 = a_1^2+b_1^2\ldots,$$

this becomes

$$\Delta_1^q \frac{1}{\rho_1^i}.$$

Now it is immediately seen that $\Delta_1 \dfrac{1}{\rho_1^{i'}} = \dfrac{2i'(2i'+2-\mathfrak{n})}{\rho_1^{i+1}}$;

from which we may deduce

$$\Delta_1^q \frac{1}{\rho_1^i} = \frac{2i\,(2i+2)\ldots(2i+2q-2)\,(2i+2-\mathfrak{n})\ldots(2i+2q-\mathfrak{n})}{\rho_1^{i+q}},$$

or, restoring the value of ρ_1, and forming the expression for the general term of (ψ), this is

$$\zeta_p\cdot\rho^{p+1}\left\{\begin{array}{l} \Delta^p \dfrac{1}{(a^2+b^2\ldots+la^2+mb^2+\&c.)^i} \\ -\dfrac{p}{1}\,2i\,(2i+2-\mathfrak{n})\,\Delta^{p-1}\dfrac{1}{(a^2+b^2+\ldots+la^2+mb^2+\ldots)^i} \\ +\&c. \end{array}\right.$$

ρ representing the quantity $a^2+b^2+\&c.$

Hence, selecting the terms of the sth order in l, m, &c. the expression for the part of (ψ) which is of the sth order in l, m &c. may be written under the form

$$S_{p\,0}^{\ s}\frac{(-1)^s\rho^{p+1}\zeta_p}{1\,.\,2\ldots s}$$

multiplied by

$$\left\{\begin{array}{l} i\,(i+1)\ldots(i+s-1)\,\Delta^p\dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1}\,2i\,(2i+2-\mathfrak{n})\,(i+1)\ldots\quad(i+s)\,\Delta^{p-1}\dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,(p-1)}{1\,.\,2}\,2i\,(2i+2)\,(2i+2-\mathfrak{n})\,(2i+4-\mathfrak{n})\,(i+2)\ldots(i+s+1)\,\Delta^{p-2}\dfrac{U}{\rho^{i+s+2}} \\ -\&c.\qquad [la^2+mb^2\ldots = U \text{ suppose}] \end{array}\right.$$

which for conciseness we shall represent by

$$\frac{(-1)^s}{1\,.\,2\ldots s}S_{p\,0}^{\ s}\rho^{p+1}\cdot\zeta_p\left\{\begin{array}{l} \alpha_s\,\Delta^p\dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1}\,\beta_s\,\Delta^{p-1}\dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,(p-1)}{1\,.\,2}\,\gamma_s\,\Delta^{p-2}\dfrac{U}{\rho^{i+s+2}} \\ -\&c. \end{array}\right.$$

$= S$ suppose.

Now U representing any homogeneous function of the order $2s$, it is easily seen that

$$\Delta \frac{U}{\rho^i} = \frac{\Delta U}{\rho^i} + 2i\,(2i + 2 - 4s - \mathfrak{n})\,\frac{U}{\rho^{i+1}};$$

and repeating continually the operation Δ, observing that ΔU, $\Delta^2 U$, &c. are of the orders $2\,(s-1)$, $2\,(s-2)$, &c. we at length arrive at

$$\begin{aligned}
\Delta^q . \frac{U}{\rho^i} &= \Delta^q\, U . \frac{1}{\rho^i} \\
&\quad + \frac{q}{1}\, 2i\,(2i + 2q - 4s - \mathfrak{n})\, \Delta^{q-1}\, U . \frac{1}{\rho^{i+1}} \\
&\quad + \frac{q\,(q-1)}{1\,.\,2}\, 2i\,(2i+2)\,(2i + 2q - 4s - \mathfrak{n})\,(2i + 2q - 4s - \mathfrak{n} - 2)\, \Delta^{q-2}\, U . \frac{1}{\rho^{i+2}} \\
&\quad \vdots \\
&\quad + 2i\,(2i+2) \ldots (2i+2q)\,(2i + 2q - 4s - \mathfrak{n}) \ldots \qquad (2i + 2 - 4s - \mathfrak{n})\, U . \frac{1}{\rho^{i+q}}.
\end{aligned}$$

Changing i into $s + i + i'$, we have an equation which we may represent by

$$\Delta^q \frac{U}{\rho^{s+i+i'}} = A_{q,\,i'} \frac{\Delta^q\, U}{\rho^{s+i+i'}} + {}^1A_{q,\,i'} \frac{\Delta^{q-1}\, U}{\rho^{s+i+i'+1}} \ldots + {}^qA_{q,\,i'} \frac{U}{\rho^{s+i+i'+q}} \ldots \qquad (\alpha),$$

where in general

$$\begin{aligned}
{}^rA_{q,\,i'} &= \frac{q\,(q-1) \ldots (q-r+1)}{1\,.\,2 \ldots r} \\
&\quad \times (2s + 2i + 2i')\,(2s + 2i' + 2i + 2) \ldots (2s + 2i' + 2i + 2r - 2) \\
&\quad \times (2i + 2i' + 2q - 2s - \mathfrak{n}) \ldots (2i + 2i' + 2q - 2s - \mathfrak{n} - 2r + 2).
\end{aligned}$$

Now the value of S, written at full length, is

$$\frac{(-1)^s}{1\,.\,2 \ldots s} \left\{ \begin{aligned}
&\zeta_s \rho^{s+1} \left(\alpha_s \Delta^s \frac{U}{\rho^{s+1}} - \frac{s}{1} \beta_s \Delta^{s-1} \frac{U}{\rho^{s+i+1}} \ldots \right. \\
&+ \zeta_{s-1}\, \rho^{s+i-1} \left(\alpha_s \Delta^{s-1} \frac{U}{\rho^{s+1}} - \frac{s-1}{1} \beta_s \Delta^{s-2} \frac{U}{\rho^{s+i+1}} + \ldots \right. \\
&+ \&c.
\end{aligned} \right.$$

and substituting for the several terms of this expansion the values given by the equation (α), we have

$$S = \frac{(-1)^s}{1\,.\,2 \ldots\ldots s} \left(k_0 \Delta^s U + k_1 \frac{1}{\rho} \Delta^{s-1} U \ldots\ldots + k_s \frac{1}{\rho^s}\, U \right)$$

where in general

$$\begin{aligned}
k_{\mathrm{x}} &= \alpha_s \left({}^{\mathrm{x}}A_{s,\,0}\, \zeta_s + {}^{\mathrm{x}-1}A_{s-1,\,0}\, \zeta_{s-1} \ldots + A_{s-\mathrm{x},\,0}\, \zeta_{s-\mathrm{x}} \right) \\
&\quad - \beta_s \left(\frac{s}{1}\, {}^{\mathrm{x}-1}A_{s-1,\,1}\, \zeta_s \ldots + \frac{(s - \mathrm{x} + 1)}{1} A_{s-\mathrm{x},\,1}\, \zeta_{s-\mathrm{x}+1} \right) \\
&\quad \vdots \\
&\quad \pm \lambda_s \left(\frac{s\,(s-1) \ldots (s - \mathrm{x} + 1)}{1\,.\,2 \ldots \mathrm{x}} A_{s-\mathrm{x},\,\mathrm{x}}\, \zeta_s \right),
\end{aligned}$$

λ_s being the $(\mathrm{x}+1)^{\text{th}}$ of the series $\alpha_s, \beta_s \ldots$

Substituting for the quantities involved in this expression, and putting, for simplicity $2i+2-\mathrm{n}=2\gamma$, we have, without any further reduction, except that of arranging the factors of the different terms, and cancelling those which appear in the numerator and denominator of the same term,

$$\frac{(-1)^s k_{\mathrm{x}}}{1.2\ldots\ldots s}=\frac{(-1)^{s-\mathrm{x}}(1-\gamma)(2-\gamma)\ldots(\mathrm{x}-\gamma)}{2^{2s+1}.1.2\ldots s.1.2\ldots(s-\mathrm{x}).1.2\ldots\mathrm{x}}$$

multiplied by the series

$$(i+s+1)\ldots(i+s+\mathrm{x}-1) \text{ into}$$

$$\left\{1+\frac{\gamma}{1}\frac{\mathrm{x}}{\mathrm{x}-\gamma}+\frac{\gamma(\gamma+1)}{1.2}\frac{\mathrm{x}(\mathrm{x}-1)}{(\mathrm{x}-\gamma)(\mathrm{x}-1-\gamma)}+\ldots \qquad (\mathrm{x}+1) \text{ terms}\right\}$$

$$-\frac{(i+s)\ldots(i+s+\mathrm{x}-2)}{1-\gamma} \text{ into}$$

$$\left\{\mathrm{x}+\frac{\gamma}{1}\frac{\mathrm{x}(\mathrm{x}-1)}{\mathrm{x}-\gamma}+\frac{\gamma(\gamma+1)}{1.2}\frac{\mathrm{x}(\mathrm{x}-1)(\mathrm{x}-2)}{(\mathrm{x}-\gamma)(\mathrm{x}-1-\gamma)}+\ldots \qquad \mathrm{x} \text{ terms}\right\}$$

$$\vdots$$

$$+(-1)^r\frac{(i+s-r+1)\ldots(i+s+\mathrm{x}-r-1)}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)} \text{ into}$$

$$\left\{\mathrm{x}(\mathrm{x}-1)\ldots(\mathrm{x}-r+1)+\frac{\gamma}{1}\frac{\mathrm{x}(\mathrm{x}-1)\ldots(\mathrm{x}-r)}{\mathrm{x}-\gamma}+\ldots(\mathrm{x}+r-1) \text{ terms}\right\}$$

to $r=\mathrm{x}$.

Now it may be shown that

$$\frac{1}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)}$$

$$\left\{\mathrm{x}(\mathrm{x}-1)\ldots(\mathrm{x}-r+1)+\frac{\gamma}{1}\frac{\mathrm{x}(\mathrm{x}-1)\ldots(\mathrm{x}-r)}{\mathrm{x}-\gamma}+\&\text{c.}\ldots(\mathrm{x}+1-r) \text{ terms}\right\}$$

$$=\frac{\mathrm{x}(\mathrm{x}-1)\ldots(r+1).\mathrm{x}(\mathrm{x}-1)\ldots(\mathrm{x}-r+1)}{(1-\gamma)(2-\gamma)\ldots(\mathrm{x}-\gamma)},$$

which reduces the expression for k_{x} to the form

$$\frac{(-1)^s k_{\mathrm{x}}}{1.2\ldots s}=\frac{(-1)^{s+\mathrm{x}}}{2^{2s+1}.1.2\ldots s.1.2\ldots(s-\mathrm{x})}\left\{\begin{array}{l}(i+s+1)\ldots(i+s+\mathrm{x}-1)\\ -\frac{\mathrm{x}}{1}(i+s)\ldots \qquad (i+s+\mathrm{x}-2)\\ +\frac{\mathrm{x}(\mathrm{x}-1)}{1.2}(i+s-1)\ldots(i+s+\mathrm{x}-3)\\ \pm\ \&\text{c. } (\mathrm{x}+1) \text{ terms};\end{array}\right.$$

from which it may be shown, that except for $\mathrm{x}=0$, $k_{\mathrm{x}}=0$.

The value $\mathrm{x}=0$, observing that the expression

$$(i+s+1)(i+s+2)\ldots(i+s-1)$$

represents $\frac{1}{i+s}$, gives

$$\frac{(-1)^s k_0}{1.2\ldots s} = \frac{(-1)^s}{2^{2s}(1.2\ldots s)^2.(2i+2s)};$$

or we have simply

$$S = \frac{(-1)^s}{2^{2s}(1.2\ldots s)^2.(2i+2s)}\Delta^s U,$$

where $$\Delta = \frac{d^2}{da^2} + \frac{d^2}{db^2} + \ldots, \quad U = (la^2 + mb^2 \ldots)^s.$$

Consider the term $$\frac{1.2\ldots s}{1.2\ldots\lambda.1.2\ldots\mu.\,\&c.} a^{2\lambda} b^{2\mu} \ldots l^\lambda m^\mu \ldots;$$

with respect to this, Δ^s reduces itself to

$$\frac{1.2\ldots s}{1.2\ldots\lambda.1.2\ldots\mu.\,\&c.}\left(\frac{d}{da}\right)^{2\lambda}\ldots$$

and the corresponding term of S is

$$\frac{(-1)^s}{2^{2s}(2i+2s)(1.2\ldots\lambda.1.2\ldots\mu.\ \&c.)^2} 1.2\ldots 2\lambda.1.2\ldots 2\mu.\ \&c.\ l^\lambda m^\mu \ldots$$

$$= \frac{(-1)^s.1.3\ldots(2\lambda-1).1.3\ldots(2\mu-1).\ \&c.}{(2i+2s)\,2.4\ldots 2\lambda.2.4\ldots 2\mu.\ \&c.} l^\lambda m^\mu \ldots$$

which, omitting the factor $\frac{1}{2i+2s}$, and multiplying by x^{2s}, is the general term of the sth order in $l, m, \ldots$ of

$$\frac{1}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}}.$$

The term itself is therefore the general term of

$$\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}};$$

or taking the sum of all such terms for the complete value of S, and the sum of the different values of S for the values 0, 1, 2 ... of the variable s, we have the required equation

$$\psi = \int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}}.$$

Another and perhaps more remarkable form of this equation may be deduced by writing $\frac{a^2}{1+l}$, $\frac{b^2}{1+m}$, &c. for a^2, b^2, &c., and putting $\frac{a^2}{1+l} + \frac{b^2}{1+m} + \&c. = \eta^2$, $l\eta^2 = \alpha^2$, $m\eta^2 = \beta^2$, &c.: we readily deduce

$$\eta^{n-2i}\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(\eta^2+\alpha^2x^2)(\eta^2+\beta^2x^2)\ldots\}}}$$

$$= S_p \sum_0^\infty \frac{1}{2^{2p+1}.1.2\ldots p.i(i+1)\ldots(i+p)}\left(\alpha^2\frac{d^2}{da^2} + \beta^2\frac{d^2}{db^2}\ldots\right)^p \frac{1}{(a^2+b^2\ldots)^i},$$

η being determined by the equation

$$\frac{a^2}{\eta^2+\alpha^2}+\frac{b^2}{\eta^2+\beta^2}\dots=1;$$

or, as it may otherwise be written,

$$\eta^2=a^2+b^2+\dots-\frac{a^2\alpha^2}{\eta^2+\alpha^2}-\frac{b^2\beta^2}{\eta^2+\beta^2}-\text{\&c.}$$

$\mathfrak{n}$, it will be recollected, denotes the number of the quantities a, b, &c.

Now suppose

$$V=\iint\dots\phi(a-x,\ b-y,\dots)\,dx\,dy\dots$$

(the integral sign being repeated $\mathfrak{n}$ times) where the limits of the integral are given by the equation

$$\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\text{\&c.}=1;$$

and that it is permitted, throughout the integral to expand the function $\phi(a-x,\dots)$ in ascending powers of x, y, &c. (the condition for which is apparently that of ϕ not becoming infinite for any values of x, y, &c., included within the limits of the integration): then observing that any integral of the form $\iint\dots x^p y^q\dots dx\,dy$ &c.... where any one of the exponents p, q, &c.... is odd, when taken between the required limits contains equal positive and negative elements and therefore vanishes, the general term of V assumes the form

$$\frac{1}{1.2\dots 2r.1.2\dots 2s\dots}\left(\frac{d}{da}\right)^{2r}\left(\frac{d}{db}\right)^{2s}\dots\phi(a,\ b\dots)\iint\dots x^{2r}y^{2s}\dots dx\,dy\dots$$

Also, by a formula quoted in the eleventh No. of the *Mathematical Journal*, the value of the definite integral $\iint\dots x^{2r}y^{2s}\dots dx\,dy\dots$ is

$$h^{2r+1}h_{,}^{2s+1}\dots\frac{\Gamma(r+\frac{1}{2})\,\Gamma(s+\frac{1}{2})\dots}{\Gamma(r+s+\dots+\frac{1}{2}\mathfrak{n}+1)},$$

(observing that the value there given referring to positive values only of the variables, must be multiplied by $2^{\mathfrak{n}}$): or, as it may be written

$$h^{2r+1}h_{,}^{2s+1}\dots\pi^{\frac{1}{2}\mathfrak{n}}.\frac{1}{2^{r+s\dots}}\frac{1.3\dots(2r-1).1.3\dots(2s-1)\dots}{\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\dots(\frac{1}{2}\mathfrak{n}+r+s\dots)\,\Gamma(\frac{1}{2}\mathfrak{n})}:$$

hence the general term of V takes the form

$$\frac{hh_{,}\dots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\frac{1}{\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\dots(\frac{1}{2}\mathfrak{n}+r+s\dots)}\cdot\frac{1}{2^{2r+2s\dots}}\frac{1}{1.2.3\dots r.1.2\dots s\dots}$$
$$\times\left(h^2\frac{d^2}{da^2}\right)^r\left(h_{,}^2\frac{d^2}{db^2}\right)^s\dots\phi(a,\ b,\dots);$$

and putting $r+s+\text{\&c.}=p$, and taking the sum of the terms that answer to the same value of p, it is immediately seen that this sum is

$$=\frac{hh_{,}\dots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\cdot\frac{1}{2^{2p}.1.2\dots p.\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\dots(\frac{1}{2}\mathfrak{n}+p)}\left(h^2\frac{d^2}{da^2}+h_{,}^2\frac{d^2}{db^2}\dots\right)^p\phi(a,\ b\dots).$$

Or the function $\phi(a-x,\ b-y\ldots)$ not becoming infinite within the limits of the integration, we have

$$\iint\ldots\phi(a-x,\ b-y\ldots)\,dx\,dy\ldots$$

$$=\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\mathcal{S}_{p\,0}^{\infty}\frac{1}{2^{2p+1}.1.2\ldots p.\frac{1}{2}\mathfrak{n}(\frac{1}{2}\mathfrak{n}+1)\ldots(\frac{1}{2}\mathfrak{n}+p)}\left(h^2\frac{d^2}{da^2}+h_{,}^2\frac{d^2}{db^2}\ldots\right)^p\phi(a,\ b\ldots),$$

the integral on the first side of the equation extending to all real values of x, y, &c., subject to $\frac{x^2}{h^2}+\frac{y^2}{h^2}+\ldots<1$.

Suppose in the first place $\quad \phi(a,\ b\ldots)=\dfrac{1}{(a^2+b^2\ldots)^{\frac{1}{2}\mathfrak{n}}}$.

By a preceding formula the second side of the equation reduces itself to

$$\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^1\frac{x^{\mathfrak{n}-1}\,dx}{\sqrt{\{(\eta^2+h^2x^2)(\eta^2+h_{,}^2x^2)\ldots(\mathfrak{n}\text{ factors})\}}},$$

η being given by $\qquad \dfrac{a^2}{\eta^2+h^2}+\dfrac{b^2}{\eta^2+h_{,}^2}\ldots=1.$

Hence the formula

$$\iint\ldots\mathfrak{n}\text{ times }\frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}}$$

$$=\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^1\frac{x^{\mathfrak{n}-1}.dx}{\sqrt{\{(\eta^2+h^2x^2)(\eta^2+h_{,}^2x^2)\ldots(\mathfrak{n}\text{ factors})\}}};$$

where the integral on the first side of the equation extends to all real values of x, y, &c. satisfying $\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\text{\&c.}\ldots<1$; η^2, as we have seen, is determined by

$$\frac{a^2}{\eta^2+h^2}+\frac{b^2}{\eta^2+h_{,}^2}+\text{\&c.}=1;$$

and finally, the condition of $\phi(a-x,\ b-y\ldots)$ not becoming infinite within the limits of the integration, reduces itself to $\frac{a^2}{h^2}+\frac{b^2}{h_{,}^2}+\ldots>1$, which must be satisfied by these quantities.

Suppose in the next place that the function $\phi(a,\ b\ldots)$ satisfies $\frac{d^2\phi}{da^2}+\frac{d^2\phi}{db^2}+\text{\&c.}=0$. The factor $\left(h^2\frac{d^2}{da^2}+\text{\&c.}\right)$ may be written under the form

$$(h_{,}^2-h^2)\frac{d^2}{db^2}+(h_{,,}^2-h^2)\frac{d^2}{dc^2}+\text{\&c.}+h^2\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots\right)=(h_{,}^2-h^2)\frac{d^2}{db^2}+\text{\&c.}$$

since, as applied to the function ϕ, $\frac{d^2}{da^2}+\frac{d^2}{db^2}+\&c.$ is equivalent to 0; we have in this case

$$\iint\ldots\phi(a-x,\ b-y\ldots)\,dx\,dy\ldots$$

$$=\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}S_{p\ 0}^{\ \infty}\frac{1}{2^{2p+1}.1.2\ldots p.\frac{1}{2}\mathfrak{n}\ldots(\frac{1}{2}\mathfrak{n}+p)}\left\{(h_{,}^2-h^2)\frac{d^2}{db^2}+\ldots\right\}^p\phi(a,\ b\ldots);$$

or the first side divided by $hh_{,}\ldots$ has the remarkable property of depending on the differences $h_{,}^2-h^2$, &c. only; this is the generalisation of a well-known property of the function V, in the theory of the attraction of a spheroid upon an external point.

If in this equation we put $\phi(a,\ b\ldots)=\frac{a}{(a^2+b^2\ldots)^{\frac{1}{2}\mathfrak{n}}}$, which satisfies the required condition $\frac{d^2\phi}{da^2}+\&c.=0$, then transferring the factor a to the left-hand side of the sign S, and putting in a preceding formula, $\alpha^2=0$, $\beta^2=h_{,}^2-h^2$, &c. and η^2+h^2 for η^2, we obtain

$$\iint\ldots(\mathfrak{n}\text{ times})\frac{(a-x)\,dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}}$$

$$=\frac{2hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}a}{\sqrt{(\eta^2+h^2)}.\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^1\frac{x^{\mathfrak{n}-1}\,dx}{\sqrt{[\{\eta^2+h^2+(h_{,}^2-h^2)x^2\}\{\eta^2+h^2+(h_{,,}^2-h^2)x^2\}\ldots(\mathfrak{n}-1)\text{ factors}]}},$$

where, as before, the integrations on the first side extend to all real values of x, y, &c., satisfying $\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}\ldots<1$; η^2 is determined by $\frac{a^2}{\eta^2+h^2}+\&c.=1$; and a, $b_{,}\ldots h$, $h_{,}$, &c. are subject to $\frac{a^2}{h^2}+\frac{b^2}{h_{,}^2}+\&c.\ldots>1$.

For $\mathfrak{n}=3$, this becomes,

$$\iiint\frac{(a-x)\,dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}}$$

$$=\frac{4\pi hh_{,}h_{,,}a}{\sqrt{(h^2+\eta^2)}}\int_0^1\frac{x^2dx}{\sqrt{[\{\eta^2+h^2+(h_{,}^2-h^2)x^2\}\{\eta^2+h^2+(h_{,,}^2-h^2)x^2\}]}}:$$

the integrations on the first side extending over the ellipsoid whose semiaxes are h, $h_{,}$, $h_{,,}$, and the point whose coordinates are a, b, c, being exterior to this ellipsoid; also $\frac{a^2}{\eta^2+h^2}+\frac{b^2}{\eta^2+h_{,}^2}+\frac{c^2}{\eta^2+h_{,,}^2}=1$: a known theorem.

3.

ON CERTAIN DEFINITE INTEGRALS.

[From the *Cambridge Mathematical Journal*, vol. III. (1841), pp. 138—144.]

In the first place, we shall consider the integral

$$V = \iint \ldots (\mathfrak{n} \text{ times}) \frac{dx\,dy\ldots}{\{(a-x)^2 + (b-y)^2 \ldots\}^{\frac{1}{2}\mathfrak{n}-1}},$$

the integration extending to all real values of the variables, subject to the condition

$$\frac{x^2}{h^2} + \frac{y^2}{h_{,}^2} + \ldots < \text{ or } = 1,$$

and the constants a, b, &c. satisfying the condition

$$\frac{a^2}{h^2} + \frac{b^2}{h_{,}^2} \ldots > 1.$$

We have

$$\frac{dV}{da} = -(\mathfrak{n}-2) \iint \ldots (\mathfrak{n} \text{ times}) \frac{(a-x)\,dx\,dy\ldots}{\{(a-x)^2 + (b-y)^2 \ldots\}^{\frac{1}{2}\mathfrak{n}}},$$

$$= -(\mathfrak{n}-2) \frac{2hh_{,}\ldots \pi^{\frac{1}{2}\mathfrak{n}} a}{\sqrt{(\xi+h^2)}\,.\,\Gamma(\frac{1}{2}\mathfrak{n})} \int_0^1 \frac{x^{\mathfrak{n}-1}\,dx}{\sqrt{[\{\xi+h^2+(h_{,}^2-h^2)\,x^2\}\,\{\xi+h^2+(h_{,,}^2-h^2)\,x^2\}\ldots]}},$$

ξ being determined by the equation

$$\frac{a^2}{\xi+h^2} + \frac{b^2}{\xi+h_{,}^2} \ldots = 1,$$

by a formula [see p. 12] in a paper, [2], "On the Properties of a Certain Symbolical Expression," in the preceding No. of this Journal: ξ having been substituted for the η^2 of the formula.

Let the variable x, on the second side of the equation, be replaced by ϕ, where

$$x^2 = \frac{\xi + h^2}{\xi + h^2 + \phi};$$

we have without difficulty

$$\frac{dV}{da} = -(\mathfrak{n}-2)\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \frac{d\phi}{(\xi+h^2+\phi)\sqrt{\Phi}},$$

where $$\Phi = (\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots$$

and similarly

$$\frac{dV}{db} = -(\mathfrak{n}-2)\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}b}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \frac{d\phi}{(\xi+h_{,}^2+\phi)\sqrt{\Phi}},$$

&c. .....

From these values it is easy to verify the equation

$$V = \frac{(\mathfrak{n}-2)hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty\left(1 - \frac{a^2}{\xi+h^2+\phi} - \frac{b^2}{\xi+h_{,}^2+\phi}\cdots\right)\frac{d\phi}{\sqrt{\Phi}}.$$

For this evidently verifies the above values of $\frac{dV}{da}$, $\frac{dV}{db}$, &c. if only the term $\frac{dV}{d\xi}\,d\xi$ vanishes; and we have

$$\frac{dV}{d\xi} = \frac{(\mathfrak{n}-2)hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty d\phi\cdot\frac{d}{d\xi}\left(1 - \frac{a^2}{\xi+h^2+\phi}\cdots\right)\frac{1}{\sqrt{\Phi}};$$

or, observing that

$$\frac{d}{d\xi}\left(1 - \frac{a^2}{\xi+h^2+\phi} - \cdots\right)\frac{1}{\sqrt{(\Phi)}} = \frac{d}{d\phi}\left(1 - \frac{a^2}{\xi+h^2+\phi}\cdots\right)\frac{1}{\sqrt{(\Phi)}},$$

and taking the integral from 0 to ∞,

$$\frac{dV}{d\xi} = -\frac{(\mathfrak{n}-2)hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\left(1 - \frac{a^2}{\xi+h^2} - \frac{b^2}{\xi+h_{,}^2}\cdots\right)\frac{1}{\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}}, = 0,$$

in virtue of the equation which determines ξ.

No constant has been added to the value of V, since the two sides of the equation vanish as they should do for a, $b\ldots$ infinite, for which values ξ is also infinite and the quantity

$$\left(1 - \frac{a^2}{\xi+h^2+\phi}\cdots\right)\frac{1}{\sqrt{(\Phi)}},$$

which is always less than $\frac{1}{\sqrt{(\Phi)}}$, vanishes.

Hence, restoring the values of V and Φ,

$$\iint\ldots(\mathfrak{n}\text{ times})\frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}-1}}$$

$$= \frac{(\mathfrak{n}-2)hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty\left(1 - \frac{a^2}{\xi+h^2+\phi} - \frac{b^2}{\xi+h_{,}^2+\phi}\cdots\right)\frac{d\phi}{\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots\}}}$$

the limits of the first side of the equation, and the condition to be satisfied by a, b, &c., also the equation for the determination of ξ, being as above.

The integral

$$V' = \iint \ldots (\mathfrak{n} \text{ times}) \frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}},$$

between the same limits, and with the same condition to be satisfied by the constants, has been obtained [see p. 11] in the paper already quoted. Writing ξ instead of η^2, and $x^2 = \dfrac{\xi}{\xi+\phi}$, we have

$$V' = \frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_0^\infty \frac{d\phi}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots\}}},$$

where

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi^2+h_{,}^2}\ldots = 1.$$

Let $\nabla = \dfrac{d^2}{da^2}+\dfrac{d^2}{db^2}+\ldots$ Then by the assistance of a formula,

$$\nabla^q \frac{1}{(a^2+b^2\ldots)^i} = 2i(2i+2)\ldots(2i+2q-2)(2i+2-\mathfrak{n})\ldots(2i+2q-\mathfrak{n}).\frac{1}{(a^2+b^2\ldots)^{i+q}}$$

given in the same paper [see p. 6], in which it is obvious that a, $b\ldots$ may be changed into $a-x$, $b-y$, &c. ...; also putting $i=\frac{1}{2}\mathfrak{n}$; we have

$$\iint \ldots (\mathfrak{n} \text{ times}) \frac{dx\,dy\ldots}{\{(a-x)^2+\ldots\}^{\frac{1}{2}\mathfrak{n}+q}} = \frac{hh_{,}\ldots\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q}.1.2\ldots q.\Gamma(\frac{1}{2}\mathfrak{n}+q)}\int_0^\infty d\phi.\nabla^q \frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\ldots\}}}.$$

Now in general, if $\chi\xi$ be any function of ξ,

$$\nabla\chi\xi = \chi'\xi\left(\frac{d^2\xi}{da^2}+\frac{d^2\xi}{db^2}\ldots\right)+\chi''\xi\left\{\left(\frac{d\xi}{da}\right)^2+\left(\frac{d\xi}{db}\right)^2\ldots\right\} = \chi'\xi\,\Sigma\left(\frac{d^2\xi}{da^2}\right)+\chi''\xi\,\Sigma\left(\frac{d\xi}{da}\right)^2, \text{ suppose.}$$

But from the equation

$$\Sigma\frac{a^2}{(\xi+h^2)} = 1,$$

we obtain

$$\frac{2a}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d\xi}{da} = 0,$$

whence

$$\Sigma\left(\frac{d\xi}{da}\right)^2 = \frac{4}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

Also

$$\frac{2}{\xi+h^2}-4\frac{a}{(\xi+h^2)^2}\frac{d\xi}{da}+2\left\{\Sigma\frac{a^2}{(\xi+h^2)^3}\right\}\left(\frac{d\xi}{da}\right)^2-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\frac{d^2\xi}{da^2} = 0;$$

whence taking the sum Σ, and observing that

$$-4\Sigma\frac{a}{(\xi+h^2)^2}\frac{d\xi}{da} = -8\frac{\Sigma\dfrac{a^2}{(\xi+h^2)^3}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}} = -2\Sigma\frac{a^2}{(\xi+h^2)^3}.\Sigma\left(\frac{d\xi}{da}\right)^2,$$

$$2\Sigma\frac{1}{\xi+h^2}-\left\{\Sigma\frac{a^2}{(\xi+h^2)^2}\right\}\Sigma\left(\frac{d^2\xi}{da^2}\right) = 0;$$

we find
$$\Sigma\left(\frac{d^2\xi}{da^2}\right)=\frac{2\Sigma\dfrac{1}{\xi+h^2}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}};$$

and we hence obtain
$$\nabla\chi\xi=\frac{2\chi'\xi\,\Sigma\dfrac{1}{\xi+h^2}+4\chi''\xi}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}.$$

Hence the function
$$\int_0^\infty d\phi\,.\,\nabla\frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\ldots\}}}$$
(observing that differentiation with respect to ξ is the same as differentiation with respect to ϕ) becomes integrable, and taking the integral between the proper limits, its value is
$$-\frac{2\chi_0\xi\,\Sigma\dfrac{1}{\xi+h^2}+4\chi_0'\xi}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}};$$
where
$$\chi_0\xi=\frac{1}{\xi\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}}.$$

We have immediately
$$\frac{\chi_0'\xi}{\chi_0\xi}=-\tfrac{1}{2}\left(\frac{2}{\xi}+\Sigma\frac{1}{\xi+h^2}\right);$$
or
$$2\chi_0\xi\,\Sigma\left(\frac{1}{\xi+h^2}\right)+4\chi_0'\xi=-4\frac{\chi_0\xi}{\xi};$$
whence
$$\int_0^\infty d\phi\,.\,\nabla\frac{1}{(\xi+\phi)\sqrt{\{(\xi+h^2+\phi)\ldots\}}}=\frac{4}{\xi^2\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_{,}^2)^2}+\ldots\right\}}.$$

Hence restoring the value of ∇, and of the first side of the equation,
$$\iint\ldots(\mathfrak{n}\text{ times})\frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}+q}}$$
$$=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{2^{2q-2}.1.2\ldots q.\Gamma(\frac{1}{2}\mathfrak{n}+q)}\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots\right)^{q-1}\frac{1}{\xi^2\sqrt{\{(\xi+h^2)(\xi+h_{,}^2)\ldots\}}\left\{\dfrac{a^2}{(\xi+h^2)^2}+\dfrac{b^2}{(\xi+h_{,}^2)^2}+\ldots\right\}},$$
with the condition
$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}\ldots=1;$$
from which equation the differential coefficients of ξ, which enter into the preceding result, are to be determined.

In general if u be any function of ξ, a, b ...

$$\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\cdots\right)u=\frac{4\dfrac{d^2u}{d\xi^2}+2\dfrac{du}{d\xi}\Sigma\dfrac{1}{\xi+h^2}+4\dfrac{d^2u}{d\xi da}\Sigma\dfrac{a}{\xi+h^2}}{\Sigma\dfrac{a^2}{(\xi+h^2)^2}}+\Sigma\frac{d^2u}{da^2},$$

from which the values of the second side for $q=1$, $q=2$, &c. may be successively calculated.

The performance of the operation $\left(\frac{d}{da}\right)^p\left(\frac{d}{db}\right)^q\left(\frac{d}{dc}\right)^r\ldots$, upon the integral V', leads in like manner to a very great number of integrals, all of them expressible algebraically, for a single differentiation renders the integration with respect to ϕ possible. But this is a subject which need not be further considered at present.

We shall consider, lastly, the definite integral

$$U=\iint\ldots(\mathfrak{n}\text{ times})\frac{(a-x)f\left(\dfrac{x^2}{h^2}+\dfrac{y^2}{h_{,}^2}+\ldots\right)dxdy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}},$$

limits, &c. as before. This is readily deduced from the less general one

$$\iint\ldots(\mathfrak{n}\text{ times})\frac{(a-x)\,dxdy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}}.$$

For representing this quantity by $F(h, h_{,}\ldots)$, it may be seen that

$$U=\int_0^1 f(m^2)\frac{d}{dm}F(mh,\ mh_{,}\ldots)\,dm;$$

but in the value of $F(h, h_{,}\ldots)$, changing h, $h_{,}\ldots$ into mh, $mh_{,}\ldots$ also writing $m^2\phi$ instead of ϕ, and $m^2\xi'$ for ξ, we have

$$F(mh,\ mh_{,}\ldots)=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\int_0^\infty\frac{d\phi}{(\xi'+h^2+\phi)\sqrt{(\Phi')}};$$

where
$$\Phi'=(\xi'+h^2+\phi)(\xi'+h_{,}^2+\phi)\ldots$$

and
$$\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_{,}^2}+\ldots=m^2.$$

Hence
$$\frac{d}{dm}F(mh,\ mh_{,}\ldots)=\frac{d\xi'}{dm}\frac{d}{d\xi'}F(mh,\ mh_{,}\ldots),$$

$$=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\frac{d\xi'}{dm}\int_0^\infty d\phi\frac{d}{d\xi'}\frac{1}{(\xi'+h^2+\phi)\sqrt{(\Phi')}},$$

or, observing that $\frac{d}{d\xi'}$ is equivalent to $\frac{d}{d\phi}$, and effecting the integration between the proper limits,

$$\frac{d}{dm}F(mh,\ mh_{,}\ldots)=-\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma(\frac{1}{2}\mathfrak{n})}a\frac{1}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\ldots\}}}.$$

Substituting this value, also $f\left\{\frac{a^2}{\xi'+h^2}+\frac{b^2}{\xi'+h_{,}^2}+\ldots\right\}$ for $f(m^2)$, in the value of U, and observing that $m=0$ gives $\xi'=\infty$, $m=1$ gives $\xi'=\xi$, where ξ is a quantity determined as before by the equation

$$\frac{a^2}{\xi+h^2}+\frac{b^2}{\xi+h_{,}^2}+\ldots=1,$$

we have
$$U=-\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_{\infty}^{\xi}\frac{f\left\{\frac{a^2}{\xi'+h^2}+\ldots\right\}d\xi'}{(\xi'+h^2)\sqrt{\{(\xi'+h^2)(\xi'+h_{,}^2)\ldots\}}},$$

or writing $\phi+\xi$ for ξ', $d\xi'=d\phi$, the limits of ϕ are 0, ∞; or, inverting the limits and omitting the negative sign,

$$U=\frac{hh_{,}\ldots\pi^{\frac{1}{2}\mathfrak{n}}a}{\Gamma(\frac{1}{2}\mathfrak{n})}\int_{0}^{\infty}\frac{f\left\{\frac{a^2}{\xi+h^2+\phi}+\frac{b^2}{\xi+h_{,}^2+\phi}+\ldots\right\}d\phi}{(\xi+h^2+\phi)\sqrt{\{(\xi+h^2+\phi)(\xi+h_{,}^2+\phi)\ldots\}}};$$

which, in the particular case of $\mathfrak{n}=3$, may easily be made to coincide with known results. The analogous integral

$$\iint\ldots(\mathfrak{n}\text{ times})\frac{f\left\{\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\ldots\right\}dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}\mathfrak{n}}}$$

is apparently not reducible to a single integral.

4.

ON CERTAIN EXPANSIONS, IN SERIES OF MULTIPLE SINES AND COSINES.

[From the *Cambridge Mathematical Journal*, vol. III. (1842), pp. 162—167.]

In the following paper we shall suppose ϵ the base of the hyperbolic system of logarithms; e a constant, such that its modulus, and also the modulus of $\frac{1}{e}\{1-\sqrt{1-e^2}\}$, are each of them less than unity; $\chi\{\epsilon^{u\sqrt{(-1)}}\}$ a function of u, which, as u increases from 0 to π, passes continuously from the former of these values to the latter, without becoming a maximum in the interval, $f\{\epsilon^{u\sqrt{(-1)}}\}$ any function of u which remains finite and continuous for values of u included between the above limits. Hence, writing

$$\chi\{\epsilon^{u\sqrt{(-1)}}\} = m \quad \text{.....(1)},$$

and considering the quantity

$$\frac{\sqrt{1-e^2}\, f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} \quad \text{..... (2)},$$

as a function of m, for values of m or u included between the limits 0 and π, we have

$$\frac{\sqrt{1-e^2}\, f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} = \frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^{\pi}\frac{\sqrt{1-e^2}\, f\{\epsilon^{u\sqrt{(-1)}}\}\cos rm\, dm}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} \ldots(3),$$

(Poisson, *Mec.* tom. I. p. 650); which may also be written

$$\frac{\sqrt{1-e^2}\, f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} = \frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^{\pi}\frac{\sqrt{1-e^2}\, f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}\, du}{1-e\cos u} \ldots(4);$$

and if the first side of the equation be generally expansible in a series of multiple cosines of m, instead of being so in particular cases only, its expanded value will always be the one given by the second side of the preceding equation.

Now, between the limits 0 and π, the function

$$f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}$$

will always be expansible in a series of multiple cosines of u; and if by any algebraical process the function $f\rho\cos r\chi\rho$ can be expanded in the form

$$f\rho\cos r\chi\rho = \Sigma_{-\infty}^{\infty}\alpha_s\rho^s, \quad (\alpha_s = \alpha_{-s}) \quad\text{.....}(5);$$

we have, in a convergent series,

$$f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\} = \alpha_0 + 2\Sigma_1^{\infty}\alpha_s\cos su \quad\text{.....}(6).$$

Again, putting

$$\frac{1}{e}\{1-\sqrt{1-e^2}\} = \lambda \quad\text{.....}(7),$$

we have

$$\frac{\sqrt{1-e^2}}{1-e\cos u} = 1 + 2\Sigma_1^{\infty}\lambda^p\cos pu \quad\text{.....}(8).$$

Multiplying these two series, and effecting the integration, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\frac{\sqrt{1-e^2}f\{\epsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\epsilon^{u\sqrt{(-1)}}\}\,du}{1-e\cos u} = 2\{\tfrac{1}{2}\alpha_0 + \Sigma_1^{\infty}(\alpha_s\lambda^s)\} \quad\text{.....}(9),$$

and the second side of this equation being obviously derived from the expansion of $f\lambda\cos r\chi\lambda$ by rejecting negative powers of λ and dividing by 2, the term independent of λ may conveniently be represented by the notation

$$\overbrace{2f\lambda\cos r\chi\lambda} \quad\text{.....}(10);$$

where in general, if $\Gamma\lambda$ can be expanded in the form

$$\Gamma\lambda = \Sigma_{-\infty}^{\infty}(A_s\lambda^s), \qquad [A_{-s} = A_s] \quad\text{.....}(11),$$

we have

$$\overbrace{\Gamma\lambda} = \tfrac{1}{2}A_0 + \Sigma_1^{\infty}A_s\lambda^s \quad\text{.....}(12).$$

(By what has preceded, the expansion of $\Gamma\lambda$ in the above form is always possible in a certain sense; however, in the remainder of the present paper, $\Gamma\lambda$ will always be of a form to satisfy the equation $\Gamma\left(\frac{1}{\lambda}\right) = \Gamma\lambda$, except in cases which will afterwards be considered, where the condition $A_{-s} = A_s$ is unnecessary.)

Hence, observing the equations (4), (9), (10),

$$\frac{\sqrt{1-e^2}f\{\epsilon^{u\sqrt{(-1)}}\}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)} = \Sigma_{-\infty}^{\infty}\cos rm\,\overbrace{2\cos r\chi\lambda f\lambda} \quad\text{.....}(13);$$

from which, assuming a system of equations analogous to (1), and representing by $\Pi(\Phi)$ the product $\Phi_1\Phi_2\ldots$, it is easy to deduce

$$\Pi\left\{\frac{\sqrt{1-e^2}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}\right\}f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}}\ldots\}$$

$$= \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\Pi\cos rm\,\overbrace{\Pi(2\cos r\chi\lambda)f(\lambda_1, \lambda_2\ldots)} \quad\text{.....}(14),$$

where $\Gamma(\lambda_1, \lambda_2 \ldots)$ being expansible in the form

$$\Gamma(\lambda_1, \lambda_2 \ldots) = \Sigma_{-\infty}^{\infty} \Sigma_{-\infty}^{\infty} \ldots A_{s_1, s_2 \ldots} \lambda_1^{s_1} \lambda_2^{s_2} \ldots [A_{s_1, s_2 \ldots} = A_{-s_1, -s_2 \ldots}] \ldots\ldots (15),$$

$$\Gamma(\lambda_1, \lambda_2 \ldots) = \Sigma_0^{\infty} \Sigma_0^{\infty} \ldots \frac{1}{2^N} A_{s_1, s_2 \ldots} \lambda_1^{s_1} \lambda_2^{s_2} \ldots, \quad \ldots\ldots\ldots\ldots (16),$$

N being the number of exponents which vanish.

The equations (13) and (14) may also be written in the forms

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm \, \overbrace{2 \cos r\chi\lambda \frac{\sqrt{-1}\, \chi'\lambda \{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}}{\sqrt{1 - e^2}}} f\lambda \ldots\ldots (17),$$

$$f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}} \ldots\}$$

$$= \Sigma_{-\infty}^{\infty} \Sigma_{-\infty}^{\infty} \ldots \Pi (\cos rm) \, \overbrace{\Pi \left\{2 \cos r\chi\lambda \frac{\sqrt{-1}\, \chi'\lambda \{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}}{\sqrt{1 - e^2}}\right\} f(\lambda_1, \lambda_2 \ldots)} \ldots (18).$$

As examples of these formulæ, we may assume

$$\chi\{\epsilon^{u\sqrt{(-1)}}\} = m = u - e \sin u \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

Hence, putting

$$\lambda^r \epsilon^{-\frac{re}{2}(\lambda - \lambda^{-1})} + \lambda^{-r} \epsilon^{\frac{re}{2}(\lambda - \lambda^{-1})} = \Lambda_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and observing the equation

$$\sqrt{-1}\, \epsilon^{u\sqrt{(-1)}} \chi'\{\epsilon^{u\sqrt{(-1)}}\} = 1 - e \cos u \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

the equation (17) becomes

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^{\infty} \cos rm \Lambda_r \overbrace{\frac{\{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}^2}{\sqrt{(1 - e^2)}}} f\lambda \ldots\ldots\ldots\ldots (22).$$

Thus, if

$$\theta - \varpi = \cos^{-1} \frac{\cos u - e}{1 - e \cos u} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23),$$

assuming

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \frac{\cos u - e}{1 - e \cos u} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

$$\cos(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \frac{1}{\sqrt{1 - e^2}} \cos rm \, \overbrace{\{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\} \{\tfrac{1}{2}(\lambda + \lambda^{-1}) - e\}} \Lambda_r \ldots (25),$$

the term corresponding to $r = 0$ being

$$\frac{1}{2\sqrt{1 - e^2}} \{2\lambda - 2e - e(\lambda^2 + 1) + 2e^2\lambda\}, \; = -e \ldots\ldots\ldots\ldots\ldots (26).$$

Again, assuming

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \frac{d\theta}{dm} = \frac{\sqrt{1 - e^2}}{(1 - e \cos u)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

and integrating the resulting equation with respect to m,

$$\theta - \varpi = \Sigma_{-\infty}^{\infty} \frac{\sin rm}{r} \Lambda_r = m + 2\Sigma_1^{\infty} \frac{\sin rm}{r} \Lambda_r \quad \text{.....} (28),$$

a formula given in the fifth No. of the *Mathematical Journal*, and which suggested the present paper.

As another example, let

$$f\{\epsilon^{u\sqrt{(-1)}}\} = \cos(\theta - \varpi) \frac{d\theta}{dm} = \frac{\sqrt{1-e^2}\,(\cos u - e)}{(1 - e\cos u)^2} \quad \text{.....} (29).$$

Then integrating with respect to m, there is a term

$$2m \frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})} \quad \text{.....} (30),$$

which it is evident, *à priori*, must vanish. Equating it to zero, and reducing, we obtain

$$\frac{e}{1-e^2} = \frac{\lambda + \lambda^{-1}}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})} \quad \text{.....} (31),$$

that is

$$\frac{e}{1-e^2} = \lambda + \frac{e}{2}(\lambda^2 + 1) + \frac{e^2}{4}(\lambda^3 + 3\lambda) + \frac{e^3}{8}(\lambda^4 + 4\lambda^2 + 3) + \ldots \quad \text{.....} (32),$$

a singular formula, which may be verified by substituting for λ its value: we then obtain

$$\sin(\theta - \varpi) = 2\Sigma_1^{\infty} \frac{\sin rm}{r} \Lambda_r \frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}(\lambda + \lambda^{-1})} \quad \text{.....} (33).$$

The expansions of $\sin k(\theta - \varpi)$, $\cos k(\theta - \varpi)$, are in like manner given by the formulæ

$$\cos k(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \Lambda_r L' \cos kL \cos rm \quad \text{.....} (34),$$

$$\sin k(\theta - \varpi) = \Sigma_{-\infty}^{\infty} \Lambda_r \frac{1}{kr} \cos kL \frac{\sin rm}{r} \quad \text{.....} (35),$$

where, to abbreviate, we have written

$$\cos^{-1}\left\{\frac{\frac{1}{2}(\lambda + \lambda^{-1}) - e}{1 - \frac{1}{2}e(\lambda + \lambda^{-1})}\right\} = L \quad \text{.....} (36),$$

$$\frac{\{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}^2}{\sqrt{1-e^2}} = L' \quad \text{.....} (37).$$

Forming the analogous expressions for

$$\cos k(\theta' - \varpi'), \qquad \sin k(\theta' - \varpi'),$$

substituting in

$$\cos k(\theta-\theta') = \cos k(\varpi-\varpi')\{\cos k(\theta-\varpi)\cos k(\theta'-\varpi') + \sin k(\theta-\varpi)\sin k(\theta'-\varpi')\}$$
$$-\sin k(\varpi-\varpi')\{\sin k(\theta-\varpi)\cos k(\theta'-\varpi') - \sin k(\theta'-\varpi')\cos k(\theta-\varpi)\},$$

and reducing the whole to multiple cosines, the final result takes the very simple form

$$\cos k(\theta-\theta') = \Sigma_{-\infty}^{\infty}\cos\{r'm'-rm+k(\varpi-\varpi')\}\,\overbrace{\Lambda_r\Lambda'_{r'}\cos kL\cos kL'\left(L-\frac{1}{kr}\right)\left(L'-\frac{1}{kr'}\right)}\ \ldots(38).$$

Again, formulæ analogous to (14), (18), may be deduced from the equation

$$\Gamma(m_1, m_2\ldots)$$
$$= \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{array}{l}\cos(r_1m_1+r_2m_2\ldots)\displaystyle\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\cos(r_1m_1+r_2m_2\ldots)\,\Gamma(m_1, m_2\ldots)\\ +\sin(r_1m_1+r_2m_2\ldots)\displaystyle\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\sin(r_1m_1+r_2m_2\ldots)\,\Gamma(m_1, m_2\ldots)\end{array}\right. \quad (39),$$

which holds from $m_1=0$ to $m_1=2\pi$, &c., but in many cases universally. In this case, writing for $\Gamma(m_1, m_2\ldots)$ the function

$$\Pi\left\{\frac{1}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}}\,\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{1-e\cos u}\right\}f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}}\ldots\}\ldots(40);$$

and observing

$$\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{1-e\cos u} = \frac{1+\lambda\epsilon^{-u\sqrt{(-1)}}}{1-\lambda\epsilon^{-u\sqrt{(-1)}}} = 1+2\Sigma_1^{\infty}\{\cos su-\sqrt{-1}\sin su\}\lambda^s\ldots\ldots(41),$$

an exactly similar analysis, (except that in the expansion

$$\Gamma(\lambda_1, \lambda_2\ldots) = \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots A_{s_1, s_2}\ldots\lambda_1^{s_1}\lambda_2^{s_2}\ldots,$$

the supposition is not made that $A_{s_1, s_2}\ldots = A_{-s_1, -s_2}\ldots$), leads to the result

$$f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}}\ldots\}\,\Pi\left\{\frac{\sqrt{1-e^2}-e\sin u\sqrt{-1}}{\sqrt{-1}\,\epsilon^{u\sqrt{(-1)}}\,\chi'\{\epsilon^{u\sqrt{(-1)}}\}(1-e\cos u)}\right\}$$
$$= \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{array}{l}\cos(r_1m_1+r_2m_2\ldots)\,\overbrace{2^n\cos(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1, \lambda_2\ldots)}\\ +\sin(r_1m_1+r_2m_2\ldots)\,\overbrace{2^n\sin(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1, \lambda_2\ldots)}\end{array}\right.\ldots(42),$$

(n) being the number of variables $u_1, u_2\ldots$. Hence also $f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}}\ldots\}$

$$= \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\left\{\begin{array}{l}\cos(r_1m_1+\ldots)\,\overbrace{\cos(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{-1}\,\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1, \lambda_2\ldots)}\\ +\sin(r_1m_1+\ldots)\,\overbrace{\sin(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{-1}\,\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1, \lambda_2\ldots)}\end{array}\right.$$
$$\ldots\ldots\ldots(43).$$

By choosing for $f\{\epsilon^{u_1\sqrt{(-1)}}, \epsilon^{u_2\sqrt{(-1)}} \ldots\}$, functions expansible without sines, or without cosines, a variety of formulæ may be obtained: we may instance

$$\frac{(\lambda-\lambda^{-1})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda_r}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44),$$

Λ_r having the same meaning as before.

Also,
$$\frac{\{\frac{1}{2}(\lambda+\lambda^{-1})-e\}\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda'_r}{\sqrt{1-e^2}-\frac{1}{2}e(\lambda-\lambda^{-1})}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

where
$$\Lambda'_r=\lambda^r\epsilon^{-\frac{re}{2}(\lambda-\lambda^{-1})}-\lambda^{-r}\epsilon^{\frac{re}{2}(\lambda-\lambda^{-1})} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

Again,
$$\frac{\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}(\lambda-\lambda^{-1})\Lambda'_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}+\frac{2}{r}\Delta_r=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

and
$$\frac{\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\frac{1}{2}e\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}=\{1-\tfrac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\tfrac{1}{2}e\}\Lambda_r, \ldots(48);$$

or, what is the same thing,

$$\frac{(\lambda-\lambda^{-1})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\{(\lambda+\lambda^{-1})-\frac{1}{2}e\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}=0 \quad\ldots\ldots\ldots\ldots\ldots(49);$$

or, comparing with (44),

$$\frac{(\lambda^2-\lambda^{-2})\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}\Lambda_r}{1-\frac{1}{2}\frac{e}{\sqrt{1-e^2}}(\lambda-\lambda^{-1})}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

which are all obtained by applying the formula (43) to the expansion of $\frac{\sin}{\cos}(\theta-\varpi)$, and comparing with the equations (25), (33).

5.

ON THE INTERSECTION OF CURVES.

[From the *Cambridge Mathematical Journal*, vol. III. (1843), pp. 211—213.]

THE following theorem is quoted in a note of Chasles' Aperçu Historique &c., *Memoires de Bruxelles*, tom. XI. p. 149, where M. Chasles employs it in the demonstration of Pascal's theorem: "If a curve of the third order pass through eight of the points of intersection of two curves of the third order, it passes through the ninth point of intersection." The application in question is so elegant, that it deserves to be generally known. Consider a hexagon inscribed in a conic section. The aggregate of three alternate sides may be looked upon as forming a curve of the third order, and that of the remaining sides, a second curve of the same order. These two intersect in nine points, viz. the six angular points of the hexagon, and the three points which are the intersections of pairs of opposite sides. Suppose a curve of the third order passing through eight of these points, viz. the aggregate of the conic section passing through the angular points of the hexagon, and of the line joining two of the three intersections of pairs of opposite sides. This passes through the ninth point, by the theorem of Chasles, i.e. the three intersections of pairs of opposite sides lie in the same straight line, (since obviously the third intersection does *not* lie in the conic section); which is Pascal's theorem.

The demonstration of the above property of curves of the third order is one of extreme simplicity. Let $U=0$, $V=0$, be the equations of two curves of the third order, the curve of the same order which passes through eight of their points of intersection (which may be considered as eight perfectly arbitrary points), and a ninth arbitrary point, will be perfectly determinate. Let U_0, V_0, be the values of U, V, when the coordinates of this last point are written in place of x, y. Then $UV_0 - U_0V = 0$, satisfies the above conditions, or it is the equation to the curve required; but it is an equation which is satisfied by all the nine points of intersection of the two curves, i.e. any curve that passes through eight of these points of intersection, passes also through the ninth.

Consider generally two curves, $U_m = 0$, $V_n = 0$, of the orders m and n respectively, and a curve of the rth order (r not less than m or n) passing through the mn points of intersection. The equation to such a curve will be of the form

$$U = u_{r-m} U_m + v_{r-n} V_n = 0,$$

u_{r-m}, v_{r-n}, denoting two polynomes of the orders $r-m$, $r-n$, with all their coefficients complete. It would at first sight appear that the curve $U=0$ might be made to pass through as many as $\{1+2\ldots+(r-m+1)\}+\{1+2\ldots(r-n+1)\}-1$, arbitrary points, i.e.

$$\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2)-1;$$

or, what is the same thing,

$$\tfrac{1}{2}r(r+3)-mn+\tfrac{1}{2}(r-m-n+1)(r-m-n+2)$$

arbitrary points, such being apparently the number of disposable constants. This is in fact the case as long as r is not greater than $m+n-1$; but when r exceeds this, there arise, between the polynomes which multiply the disposable coefficients, certain linear relations which cause them to group themselves into a smaller number of disposable quantities. Thus, if r be not less than $m+n$, forming different polynomes of the form $x^\alpha y^\beta V_n$ $[\alpha+\beta = \text{or} < m]$, and multiplying by the coefficients of $x^\alpha y^\beta$ in U_m and adding, we obtain a sum $U_m V_n$, which might have been obtained by taking the different polynomes of the form $x^\gamma y^\delta U_m$ $[\gamma+\delta = \text{or} < n]$, multiplying by the coefficients of $x^\gamma y^\delta$ in V_n, and adding: or we have a linear relation between the different polynomes of the forms $x^\alpha y^\beta V_n$, and $x^\gamma y^\delta U_m$. In the case where r is not less than $m+n+1$, there are two more such relations, viz. those obtained in the same way from the different polynomes $x^\alpha y^\beta . xV_n$, $x^\gamma y^\delta . xU_m$, and $x^\alpha y^\beta . yV_n$, $x^\gamma y^\delta . yU_m$, &c.; and in general, whatever be the excess of r above $m+n-1$, the number of these linear relations is

$$1+2\ldots(r-m-n+1)=\tfrac{1}{2}(r-m-n+1)(r-m-n+2).$$

Hence, if r be not less than $m+n$, the number of points through which a curve of the rth order may be made to pass, in addition to the mn points which are the intersections of $U_m=0$, $V_n=0$, is simply $\tfrac{1}{2}r(r+3)-mn$. In the case of $r=m+n-1$, or $r=m+n-2$, the two formulæ coincide. Hence we may enunciate the theorem

"A curve of the rth order, passing through the mn points of intersection of two curves of the mth and n^{th} orders respectively, may be made to pass through $\tfrac{1}{2}r(r+3)-mn+\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$ arbitrary points, if r be not greater than $m+n-3$: if r be greater than this value, it may be made to pass through $\tfrac{1}{2}r(r+3)-mn$ points only."

Suppose r not greater than $m+n-3$, and a curve of the rth order made to pass through

$$\tfrac{1}{2}r(r+3)-mn+\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

arbitrary points, and

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the mn points of intersection above. Such a curve passes through $\frac{1}{2}r(r+3)$ given points, and though the $mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$ latter points are not perfectly arbitrary, there appears to be no reason why the relation between the positions of these points should be such as to prevent the curve from being *completely determined* by these conditions. But if it be so, then the curve must pass through the remaining $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ points of intersection, or we have the theorem

"If a curve of the rth order (r not less than m or n, not greater than $m+n-3$) pass through

$$mn-\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

of the points of intersection of two curves of the mth and n^{th} orders respectively, it passes through the remaining

$$\tfrac{1}{2}(m+n-r-1)(m+n-r-2)$$

points of intersection."

6.

ON THE MOTION OF ROTATION OF A SOLID BODY.

[From the *Cambridge Mathematical Journal*, vol. III. (1843), pp. 224—232.]

In the fifth volume of Liouville's Journal, in a paper "Des lois géometriques qui régissent les déplacemens d'un système solide," M. Olinde Rodrigues has given some very elegant formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only. The principal object of the present paper is to apply these to the problem of the rotation of a solid body; but I shall first demonstrate the formulæ in question, and some others connected with the same subject which may be useful on other occasions.

Let Ax, Ay, Az; $Ax_{,}$, $Ay_{,}$, $Az_{,}$, be any two sets of rectangular axes passing through the point A: x, y, z, $x_{,}$, $y_{,}$, $z_{,}$, being taken for the points where these lines intersect the spherical surface described round the centre A with radius unity. Join $xx_{,}$, $yy_{,}$, $zz_{,}$, by arcs of great circles, and through the central points of these describe great circles cutting them at right angles: these are easily seen to intersect in a certain point P. Let $Px=f$, $Py=g$, $Pz=h$; then also $Px_{,}=f$, $Py_{,}=g$, $Pz_{,}=h$: and $\angle xPx_{,}=\angle yPy_{,}=\angle zPz_{,}=\theta$ suppose, θ being measured from xP towards yP, yP towards zP, or zP towards xP. The cosines of f, g, h, are of course connected by the equation

$$\cos^2 f+\cos^2 g+\cos^2 h=1.$$

Let α, β, γ; α', β', γ'; α'', β'', γ'', represent the cosines of $x_{,}x$, $y_{,}x$, $z_{,}x$; $x_{,}y$, $y_{,}y$, $z_{,}y$; $x_{,}z$, $y_{,}z$, $z_{,}z$: these quantities are to be determined as functions of f, g, h, θ.

Suppose for a moment,

$$\angle yPz=\mathrm{x}, \qquad \angle zPx=\mathrm{y}, \qquad \angle xPy=\mathrm{z};$$

then
$$\begin{aligned}
\alpha &= \cos^2 f \qquad\qquad + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g + \sin f \sin g \cos(\mathrm{z}-\theta),\\
\alpha'' &= \cos f \cos h + \sin f \sin h \cos(\mathrm{y}+\theta),\\
\beta &= \cos g \cos f + \sin g \sin f \cos(\mathrm{z}+\theta),\\
\beta' &= \cos^2 g \qquad\qquad + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h + \sin g \sin h \cos(\mathrm{x}-\theta),\\
\gamma &= \cos h \cos f + \sin h \sin f \cos(\mathrm{y}-\theta),\\
\gamma' &= \cos h \cos g + \sin h \sin g \cos(\mathrm{x}+\theta),\\
\gamma'' &= \cos^2 h \qquad\qquad + \sin^2 h \cos\theta.
\end{aligned}$$

Also
$$\begin{aligned}
\sin g \sin h \cos \mathrm{x} &= -\cos g \cos h,\\
\sin h \sin f \cos \mathrm{y} &= -\cos h \cos f,\\
\sin f \sin g \cos \mathrm{z} &= -\cos f \cos g,
\end{aligned}$$

and
$$\begin{aligned}
\sin g \sin h \sin \mathrm{x} &= \cos f,\\
\sin h \sin f \sin \mathrm{y} &= \cos g,\\
\sin f \sin g \sin \mathrm{z} &= \cos h.
\end{aligned}$$

Substituting,
$$\begin{aligned}
\alpha &= \cos^2 f \qquad\qquad\qquad + \sin^2 f \cos\theta,\\
\alpha' &= \cos f \cos g\,(1-\cos\theta) + \cos h \sin\theta,\\
\alpha'' &= \cos f \cos h\,(1-\cos\theta) - \cos g \sin\theta,\\
\beta &= \cos g \cos f\,(1-\cos\theta) - \cos h \sin\theta,\\
\beta' &= \cos^2 g \qquad\qquad\qquad + \sin^2 g \cos\theta,\\
\beta'' &= \cos g \cos h\,(1-\cos\theta) + \cos f \sin\theta,\\
\gamma &= \cos h \cos f\,(1-\cos\theta) + \cos g \sin\theta,\\
\gamma' &= \cos h \cos g\,(1-\cos\theta) - \cos f \sin\theta,\\
\gamma'' &= \cos^2 h \qquad\qquad\qquad + \sin^2 h \cos\theta.
\end{aligned}$$

Assume $\lambda = \tan\frac{1}{2}\theta \cos f$, $\mu = \tan\frac{1}{2}\theta \cos g$, $\nu = \tan\frac{1}{2}\theta \cos h$, and $\sec^2\frac{1}{2}\theta = 1 + \lambda^2 + \mu^2 + \nu^2 = \kappa$;

then
$$\begin{aligned}
&\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, && \kappa\alpha' = 2(\lambda\mu + \nu), && \kappa\alpha'' = 2(\nu\lambda - \mu),\\
&\kappa\beta = 2(\lambda\mu - \nu), && \kappa\beta' = 1 + \mu^2 - \nu^2 - \lambda^2, && \kappa\beta'' = 2(\mu\nu + \lambda),\\
&\kappa\gamma = 2(\nu\lambda + \mu), && \kappa\gamma' = 2(\mu\nu - \lambda), && \kappa\gamma'' = 1 + \nu^2 - \lambda^2 - \mu^2;
\end{aligned}$$

which are the formulæ required, differing only from those in Liouville, by having λ, μ, ν, instead of $\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}p$; and α, α', α''; β, β', β''; γ, γ', γ'', instead of a, b, c; a', b', c'; a'', b'', c''. It is to be remarked, that β', β'', β; γ'', γ, γ', are deduced from α, α', α'', by writing μ, ν, λ; ν, λ, μ, for λ, μ, ν.

Let $1 + \alpha + \beta' + \gamma'' = \upsilon$; then $\kappa\upsilon = 4$, and we have
$$\begin{aligned}
&\lambda\upsilon = \beta'' - \gamma', && \mu\upsilon = \gamma - \alpha'', && \nu\upsilon = \alpha' - \beta,\\
&\lambda^2\upsilon = 1 + \alpha - \beta' - \gamma'', && \mu^2\upsilon = 1 - \alpha + \beta' - \gamma'', && \nu^2\upsilon = 1 - \alpha - \beta' - \gamma''.
\end{aligned}$$

Suppose that Ax, Ay, Az, are referred to axes AX, AY, AZ, by the quantities l, m, n, k, analogous to λ, μ, ν, κ, these latter axes being referred to $Ax_{,}$, $Ay_{,}$, $Az_{,}$, by the quantities $l_{,}$, $m_{,}$, $n_{,}$, $k_{,}$.

Let a, b, c; a', b', c'; a'', b'', c''; $a_{,}$, $b_{,}$, $c_{,}$; $a_{,}'$, $b_{,}'$, $c_{,}'$; $a_{,}''$, $b_{,}''$, $c_{,}''$, denote the quantities analogous to α, β, γ; α', β', γ'; α'', β'', γ''. Then we have, by spherical trigonometry, the formulæ

$$\begin{aligned}
\alpha &= a\,a_{,} + b\,a_{,}' + c\,a_{,}'', & \beta &= a\,b_{,} + b\,b_{,}' + c\,b_{,}'', & \gamma &= a\,c_{,} + b\,c_{,}' + c\,c_{,}'';\\
\alpha' &= a'\,a_{,} + b'\,a_{,}' + c'\,a_{,}'', & \beta' &= a'\,b_{,} + b'\,b_{,}' + c'\,b_{,}'', & \gamma' &= a'\,c_{,} + b'\,c_{,}' + c'\,c_{,}'';\\
\alpha'' &= a''\,a_{,} + b''\,a_{,}' + c''\,a_{,}'', & \beta'' &= a''\,b_{,} + b''\,b_{,}' + c''\,b_{,}'', & \gamma'' &= a''\,c_{,} + b''\,c_{,}' + c''\,c_{,}''.
\end{aligned}$$

Then expressing a, b, c; a', b', c'; a'', b'', c''; $a_{,}$, $b_{,}$, $c_{,}$; $a_{,}'$, $b_{,}'$, $c_{,}'$; $a_{,}''$, $b_{,}''$, $c_{,}''$, in terms of l, m, n; $l_{,}$, $m_{,}$, $n_{,}$, after some reductions we arrive at

$$\begin{aligned}
kk_{,}\nu &= 4\,(1 - ll_{,} - mm_{,} - nn_{,})^2, \quad = 4\Pi^2 \text{ suppose},\\
kk_{,}(\beta'' - \gamma) &= 4\,(l + l_{,} + n_{,}m - nm_{,})\,\Pi,\\
kk_{,}(\gamma - \alpha') &= 4\,(m + m_{,} + l_{,}m - lm_{,})\,\Pi,\\
kk_{,}(\alpha' - \beta'') &= 4\,(n + n_{,} + m_{,}n - mn_{,})\,\Pi;
\end{aligned}$$

and hence

$$\begin{aligned}
\Pi &= 1 - ll_{,} - mm_{,} - nn_{,}, & \Pi\lambda &= l + l_{,} + n_{,}m - nm_{,},\\
\Pi\mu &= m + m_{,} + l_{,}m - lm_{,}, & \Pi\nu &= n + n_{,} + m_{,}n - mn_{,},
\end{aligned}$$

which are formulæ of considerable elegance for exhibiting the combined effect of successive displacements of the axes. The following analogous ones are readily obtained:

$$\begin{aligned}
P &= 1 + \lambda l + \mu m + \nu n, & Pl_{,} &= \lambda - l - \nu m + \mu n,\\
Pm_{,} &= \mu - m - \lambda n + \nu l, & Pn_{,} &= \nu - n - \mu l + \lambda m:
\end{aligned}$$

and again,

$$\begin{aligned}
P_{,} &= 1 + \lambda l_{,} + \mu m_{,} + \nu n_{,}, & P_{,}l &= \lambda - l_{,} + \nu m_{,} - \mu n_{,},\\
P_{,}m &= \mu - m_{,} + \lambda n_{,} - \nu l_{,}, & P_{,}n &= \nu - n_{,} + \mu l_{,} - \lambda m_{,}.
\end{aligned}$$

These formulæ will be found useful in the integration of the equations of rotation of a solid body.

Next it is required to express the quantities p, q, r, in terms of λ, μ, ν, where as usual

$$p = \gamma\frac{d\beta}{dt} + \gamma'\frac{d\beta'}{dt} + \gamma''\frac{d\beta''}{dt},$$

$$q = \alpha\frac{d\gamma}{dt} + \alpha'\frac{d\gamma'}{dt} + \alpha''\frac{d\gamma''}{dt},$$

$$r = \beta\frac{d\alpha}{dt} + \beta'\frac{d\alpha'}{dt} + \beta''\frac{d\alpha''}{dt}.$$

Differentiating the values of $\beta\kappa$, $\beta'\kappa$, $\beta''\kappa$, multiplying by γ, γ', γ'', and adding,

$$\kappa p = 2\lambda'(\gamma\mu - \gamma'\lambda + \gamma'') + 2\mu'(\gamma\lambda - \gamma'\mu + \gamma''\nu) + 2\nu'(-\gamma - \gamma'\nu + \gamma''\mu),$$

where λ', μ', ν', denote $\frac{d\lambda}{dt}$, $\frac{d\mu}{dt}$, $\frac{d\nu}{dt}$. Reducing, we have

$$\kappa p = 2(\lambda' + \nu\mu' - \nu'\mu):$$

from which it is easy to derive the system

$$\begin{aligned}\kappa p &= 2(\quad \lambda' + \nu\mu' - \nu'\mu),\\ \kappa q &= 2(-\nu\lambda' + \mu' + \nu'\lambda),\\ \kappa r &= 2(\quad \mu\lambda' - \lambda\mu' + \nu');\end{aligned}$$

or, determining λ', μ', ν', from these equations, the equivalent system

$$\begin{aligned}2\lambda' &= (1+\lambda^2)p + (\lambda\mu - \nu)q + (\nu\lambda + \mu)r,\\ 2\mu' &= (\lambda\mu + \nu)p + (1+\mu^2)q + (\mu\nu - \lambda)r,\\ 2\nu' &= (\nu\lambda - \mu)p + (\mu\nu + \lambda)q + (1+\nu^2)r.\end{aligned}$$

The following equation also is immediately obtained,

$$\kappa' = \kappa(\lambda p + \mu q + \nu r).$$

The subsequent part of the problem requires the knowledge of the differential coefficients of p, q, r, with respect to λ, μ, ν; λ', μ', ν'. It will be sufficient to write down the six

$$\begin{aligned}\kappa\frac{dp}{d\lambda'} &= 2, & \kappa\frac{dp}{d\lambda} + 2p\lambda &= 0,\\ \kappa\frac{dq}{d\lambda'} &= -2\nu, & \kappa\frac{dq}{d\lambda} + 2q\lambda &= 2\nu',\\ \kappa\frac{dr}{d\lambda'} &= 2\mu, & \kappa\frac{dr}{d\lambda} + 2r\lambda &= -2\mu',\end{aligned}$$

from which the others are immediately obtained.

Suppose now a solid body acted on by any forces, and revolving round a fixed point. The equations of motion are

$$\begin{aligned}\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda} &= \frac{dV}{d\lambda},\\ \frac{d}{dt}\frac{dT}{d\mu'} - \frac{dT}{d\mu} &= \frac{dV}{d\mu},\\ \frac{d}{dt}\frac{dT}{d\nu'} - \frac{dT}{d\nu} &= \frac{dV}{d\nu};\end{aligned}$$

where $\qquad T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$; $\quad V = \Sigma\,[\int(Xdx + Ydy + Zdz)]\,dm$;

or if $Xdx + Ydy + Zdz$ is not an exact differential, $\frac{dV}{d\lambda}$, $\frac{dV}{d\mu}$, $\frac{dV}{d\nu}$, are independent symbols standing for

$$\Sigma\left(X\frac{dx}{d\lambda} + Y\frac{dy}{d\lambda} + Z\frac{dz}{d\lambda}\right)dm, \ldots\ldots$$

see *Mécanique Analytique*, Avertissement, t. I. p. v. [Ed. 3, p. VII.]: only in this latter case V stands for the disturbing function, the principal forces vanishing.

Now, considering the first of the above equations

$$\frac{dT}{d\lambda'} = Ap\frac{dp}{d\lambda} + Bq\frac{dq}{d\lambda} + Cr\frac{dr}{d\lambda}, \quad = \frac{2}{\kappa}(Ap - \nu Bq + \mu Cr);$$

whence, writing $\qquad p',\ q',\ r',\ \kappa'$, for $\dfrac{dp}{dt},\ \dfrac{dq}{dt},\ \dfrac{dr}{dt},\ \dfrac{d\kappa}{dt}$,

$$\frac{d}{dt}\frac{dT}{d\lambda'} = \frac{2}{\kappa}(Ap' - \nu Bq' + \mu Cr') - \frac{2}{\kappa}Bq\nu' + \frac{2}{\kappa}Cr\mu' - \frac{2\kappa'}{\kappa^2}(Ap - \nu Bq + \mu Cr).$$

Also $\quad \dfrac{dT}{d\lambda} = Ap\dfrac{dp}{d\lambda} + Bq\dfrac{dq}{d\lambda} + Cr\dfrac{dr}{d\lambda}, \quad = -\dfrac{2\lambda}{\kappa}(Ap^2 + Bq^2 + Cr^2) + \dfrac{2}{\kappa}Bq\nu' - \dfrac{2}{\kappa}Cr\mu';$

and hence $\frac{1}{2}\left(\dfrac{d}{dt}\dfrac{dT}{d\lambda'} - \dfrac{dT}{d\lambda}\right)$

$$= \frac{1}{\kappa}(Ap' - \nu Bq' + \mu Cr') - \frac{2}{\kappa}Bq\nu' + \frac{2}{\kappa}Cr\mu' + \frac{\lambda}{\kappa}(Ap^2 + Bq^2 + Cr^2) - \frac{\kappa'}{\kappa^2}(Ap - \nu Bq + \mu Cr).$$

Substituting for $\lambda',\ \mu',\ \nu',\ \kappa'$, after all reductions,

$$\tfrac{1}{2}\left(\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda}\right) = \frac{1}{\kappa}[\{Ap' + (C-B)qr\} - \nu\{Bq' + (A-C)rp\} + \mu\{Cr + (B-A)pq\}];$$

and, forming the analogous quantities in μ, ν, and substituting in the equations of motion, these become

$$\{Ap' + (C-B)qr\} - \nu\{Bq' + (A-C)rp\} + \mu\{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\nu\{Ap' + (C-B)qr\} + \{Bq' + (A-C)rp\} - \lambda\{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\mu},$$

$$\mu\{Ap' + (C-B)qr\} + \lambda\{Bq' + (A-C)rp\} + \{Cr' + (B-A)pq\} = \tfrac{1}{2}\kappa\frac{dV}{d\nu};$$

or eliminating, and replacing $p',\ q',\ r'$, by $\dfrac{dp}{dt},\ \dfrac{dq}{dt},\ \dfrac{dr}{dt}$, we obtain

$$A\frac{dp}{dt} + (C-B)qr = \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu + \nu)\frac{dV}{d\mu} + (\nu\lambda - \mu)\frac{dV}{d\nu}\right\},$$

$$B\frac{dq}{dt} + (A-C)rp = \tfrac{1}{2}\left\{(\lambda\mu - \nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu + \lambda)\frac{dV}{d\nu}\right\},$$

$$C\frac{dr}{dt} + (B-A)pq = \tfrac{1}{2}\left\{(\nu\lambda + \mu)\frac{dV}{d\lambda} + (\mu\nu - \lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\};$$

to which are to be joined

$$\kappa p = 2\left(\quad \frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt}\right),$$

$$\kappa q = 2\left(-\nu\frac{d\lambda}{dt} + \quad\frac{d\mu}{dt} + \lambda\frac{d\nu}{dt}\right),$$

$$\kappa r = 2\left(\quad \mu\frac{d\lambda}{dt} - \lambda\frac{d\mu}{dt} + \quad\frac{d\nu}{dt}\right);$$

where it will be recollected

$$\kappa = 1 + \lambda^2 + \mu^2 + \nu^2;$$

and on the integration of these six equations depends the complete determination of the motion.

If we neglect the terms depending on V, the first three equations may be integrated in the form

$$p^2 = p_1^2 - \frac{C-B}{A}\phi, \qquad q^2 = q_1^2 - \frac{A-C}{B}\phi, \qquad r_2 + r_1^2 - \frac{B-A}{C}\phi,$$

$$2t = \int \frac{d\phi}{\sqrt{\left\{\left(p_1^2 - \frac{C-B}{A}\phi\right)\left(q_1^2 - \frac{A-C}{B}\phi\right)\left(r_1^2 - \frac{B-A}{C}\phi\right)\right\}}};$$

and considering p, q, r as functions of ϕ, given by these equations, the three latter ones take the form

$$\frac{\kappa}{4qr} = \quad\frac{d\lambda}{d\phi} + \nu\frac{d\mu}{d\phi} - \mu\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4rp} = -\nu\frac{d\lambda}{d\phi} + \quad\frac{d\mu}{d\phi} + \lambda\frac{d\nu}{d\phi},$$

$$\frac{\kappa}{4pq} = \quad\mu\frac{d\lambda}{d\phi} - \lambda\frac{d\mu}{d\phi} + \quad\frac{d\nu}{d\phi};$$

of which, as is well known, the equations following, equivalent to two independent equations, are integrals,

$$\begin{aligned}
\kappa g &= \quad Ap\,(1 + \lambda^2 - \mu^2 - \nu^2) + 2Bq\,(\lambda\mu - \nu) &&+ 2Cr\,(\nu\lambda + \mu),\\
\kappa g' &= 2Ap\,(\lambda\mu + \nu) &&+ \quad Bq\,(1 + \mu^2 - \lambda^2 - \nu^2) + 2Cr\,(\mu\nu - \lambda),\\
\kappa g'' &= 2Ap\,(\nu\lambda - \mu) &&+ 2Bq\,(\mu\nu + \lambda) \quad + \quad Cr\,(1 + \nu^2 - \lambda^2 - \mu^2);
\end{aligned}$$

where g, g', g'', are arbitrary constants satisfying

$$g^2 + g'^2 + g''^2 = A^2p_1^2 + B^2q_1^2 + C^2r_1^2.$$

To obtain another integral, it is apparently necessary, as in the ordinary theory, to revert to the consideration of the invariable plane. Suppose $g' = 0$, $g'' = 0$,

then $$g'' = \sqrt{(A^2p_1^2 + B^2q_1^2 + C^2r_1^2)}, = k \text{ suppose.}$$

We easily obtain, where λ_0, μ_0, ν_0, κ_0 are written for λ, μ, ν, κ, to denote this particular supposition,

$$\kappa_0 Ap = 2(\nu_0\lambda_0 - \mu_0)k,$$
$$\kappa_0 Bq = 2(\mu_0\nu_0 + \lambda_0)k,$$
$$\kappa_0 Cr = (1 + \nu_0^2 - \lambda_0^2 - \mu_0^2)k;$$

whence, and from $\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2$, $\kappa_0 Cr = (2 + 2\nu_0^2 - \kappa_0)k$, we obtain

$$\kappa_0 = \frac{(2+2\nu_0^2)k}{k+Cr}, \qquad \nu_0\lambda_0 - \mu_0 = \frac{(1+\nu_0^2)Ap}{k+Cr}, \qquad \mu_0\nu_0 + \lambda_0 = \frac{(1+\nu_0^2)Bq}{k+Cr}.$$

Hence, writing $h = Ap_1^2 + Bq_1^2 + Cr_1^2$, the equation

$$\frac{d\nu_0}{d\phi} = \frac{1}{4pqr}\{(\nu_0\lambda_0 - \mu_0)p + (\mu_0\nu_0 + \lambda_0)q + (1+\nu^2)r\}$$

reduces itself to
$$\frac{4}{1+\nu_0^2}\frac{d\nu_0}{d\phi} = \frac{h+kr}{(k+Cr)pqr},$$

or, integrating,
$$4\tan^{-1}\nu_0 = \int\frac{(h+kr)\,d\phi}{(k+Cr)pqr}.$$

The integral takes rather a simpler form if p, q, ϕ be considered functions of r, and becomes

$$2\tan^{-1}\nu_0 = \int\frac{h+kr}{k+Cr}\,\frac{C\sqrt{(AB)}\,dr}{\sqrt{[\{k^2 - Bh + (B-C)Cr^2\}\{Ah - k^2 + (C-A)Cr^2\}]}};$$

and then, ν_0 being determined, λ_0, μ_0 are given by the equations

$$\lambda_0 = \frac{\nu_0 Ap + Bq}{k+Cr}, \qquad \mu_0 = \frac{\nu_0 Bq - Ap}{k+Cr}.$$

Hence l, m, n, denoting arbitrary constants, the general values of λ, μ, ν, are given by the equations

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$
$$P_0\lambda = l + \lambda_0 + m\nu_0 - n\mu_0,$$
$$P_0\mu = m + \mu_0 + n\lambda_0 - l\nu_0,$$
$$P_0\nu = n + \nu_0 + l\mu_0 - m\lambda_0.$$

In a following paper I propose to develope the formulæ for the variations of the arbitrary constants p_1, q_1, r_1, l, m, n, when the terms involving V are taken into account.

Note. It may be as well to verify independently the analytical conclusion immediately deducible from the preceding formulæ, viz. if λ, μ, ν, be given by the differential equations,

$$\kappa p = \frac{d\lambda}{dt} + \nu\frac{d\mu}{dt} - \mu\frac{d\nu}{dt},$$

$$\kappa q = -\nu\frac{d\lambda}{dt} + \frac{d\mu}{dt} + \lambda\frac{d\nu}{dt},$$

$$\kappa r = \mu\frac{d\lambda}{dt} - \lambda\frac{d\mu}{dt} + \frac{d\nu}{dt},$$

where $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, and p, q, r, are any functions of t. Then if λ_0, μ_0, ν_0, be particular values of λ, μ, ν, and l, m, n, arbitrary constants, the general integrals are given by the system

$$P_0 = 1 - l\lambda_0 - m\mu_0 - n\nu_0,$$
$$P_0\lambda = l + \lambda_0 + m\nu_0 - n\mu_0,$$
$$P_0\mu = m + \mu_0 + n\lambda_0 - l\nu_0,$$
$$P_0\nu = n + \nu_0 + l\mu_0 - m\lambda_0.$$

Assuming these equations, we deduce the equivalent system,

$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, l = \lambda - \lambda_0 + \nu_0\mu - \nu\mu_0,$$
$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, m = \mu - \mu_0 + \lambda_0\nu - \lambda\nu_0,$$
$$(1 + \lambda\lambda_0 + \mu\mu_0 + \nu\nu_0)\, n = \nu - \nu_0 + \mu_0\lambda - \mu\lambda_0.$$

Differentiate the first of these and eliminate l, the result takes the form $0 =$

$$-(\mu_0^2 + \nu_0^2)(\lambda' + \nu\mu' - \nu'\mu) - (\nu_0 - \lambda_0\mu_0)(-\nu\lambda' + \mu' + \lambda\nu') + (\mu_0 + \lambda_0\nu_0)(\mu\lambda' - \lambda\mu' + \nu') + \kappa_0\lambda',$$
$$+ (\mu^2 + \nu^2)(\lambda_0' + \nu_0\mu_0' - \nu_0'\mu_0) + (\nu - \lambda\mu)(-\nu_0\lambda_0' + \mu_0' + \lambda_0\nu_0') - (\mu + \lambda\nu)(\mu_0\lambda_0' - \lambda_0\mu_0' + \nu_0') - \kappa\lambda_0',$$

where λ', &c. denote $\frac{d\lambda}{dt}$, &c. and $\kappa_0 = 1 + \lambda_0^2 + \mu_0^2 + \nu_0^2$.

Reducing by the differential equations in λ, μ, ν; λ_0, μ_0, ν_0, this becomes

$$\kappa_0\{\lambda' + \tfrac{1}{2}p(\mu^2 + \nu^2) + \tfrac{1}{2}q(\nu - \lambda\mu) - \tfrac{1}{2}r(\mu + \lambda\nu)\}$$
$$-\kappa\{\lambda_0' + \tfrac{1}{2}p(\mu_0^2 + \nu_0^2) + \tfrac{1}{2}q(\nu_0 - \lambda_0\mu_0) - \tfrac{1}{2}r(\mu_0 + \lambda\nu_0)\} = 0;$$

or substituting for λ', λ_0', we have the identical equation

$$\tfrac{1}{2}p(\kappa_0\kappa - \kappa\kappa_0) = 0:$$

and similarly may the remaining equations be verified.

7.

ON A CLASS OF DIFFERENTIAL EQUATIONS, AND ON THE LINES OF CURVATURE OF AN ELLIPSOID.

[From the *Cambridge Mathematical Journal*, vol. III. (1843), pp. 264—267.]

CONSIDER the primitive equation

$$fx + gy + hz + \ldots\ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

between n variables x, y, z, the constants f, g, h being connected by the equation

$$H(f, g, h\ldots\ldots) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

H denoting a homogeneous function. Suppose that f, g, h..... are determined by the conditions

$$\begin{array}{l} fx_1 + gy_1 + hz_1 \ \ldots\ldots = 0 \\ \vdots \qquad \vdots \qquad \vdots \\ fx_{n-2} + gy_{n-2} + hz_{n-2} \ldots\ldots = 0. \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

Then writing

$$X = \begin{vmatrix} y & , z & , \ldots \\ y_1 & , z_1 & , \ldots \\ \vdots & \vdots & \\ y_{n-2}, & z_{n-2}, & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

with analogous expressions for y, z.....; the equations (3) give f, g, h,..... proportional to x, y, z,..... or eliminating f, g, h..... by the equation (2),

$$H(X, Y, Z\ldots\ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Conversely the equation (5), which contains, in appearance, $n(n-2)$ arbitrary constants, is equivalent to the system (1), (2). And if H be a rational integral function of the order r, the first side of the equation (5) is the product of r factors, each of them of the form given by the system (1), (2).

Now from the equation (1), we have the system

$$\begin{array}{llll} fx & +gy & +hz & \ldots\ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6), \\ fdx & +gdy & +hdx & \ldots\ldots = 0, \\ \vdots & \vdots & \vdots & \\ fd^{n-2}x & +gd^{n-2}y & +hd^{n-2}z & \ldots\ldots = 0, \end{array}$$

or writing

$$X = \begin{vmatrix} y & , z & , \ldots \\ dy & , dz & , \ldots \\ \vdots & \vdots & \\ d^{n-2}y, & d^{n-2}z, & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

with analogous expressions for Y, Z.....; then from the equations (6), f, g, h..... are proportional to X, Y, Z.....: or, eliminating by (2),

$$H(X,\ Y,\ Z\ldots\ldots) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Conversely the integral of the equation (8) of the order $(n-2)$ is given either by the system of equations (1), (2), in which it is evident that the number of arbitrary constants is reduced to $(n-2)$; or, by the equation (5), which contains in appearance $n(n-2)$ arbitrary constants, but which we have seen is equivalent in reality to the system (1), (2).

Thus, with three variables, the integral of

$$H(ydz - zdy,\ zdx - xdz,\ xdy - ydx) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

may be expressed in the form

$$H(yz_1 - y_1z,\ \ zx_1 - z_1x,\ \ xy_1 - x_1y) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and also in the form

$$fx + gy + hz = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where

$$H(f,\ g,\ h) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The above principles afford an elegant mode of integrating the differential equation for the lines of curvature of an ellipsoid. The equation in question is

$$(b^2 - c^2)\,xdydz + (c^2 - a^2)\,ydzdx + (a^2 - b^2)\,zdxdy = 0 \ldots\ldots\ldots\ldots (13),$$

where x, y, z are connected by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14);$$

writing

$$\frac{x^2}{a^2} = u,\quad \frac{y^2}{b^2} = v,\quad \frac{z^2}{c^2} = w \ \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

these become

$$(b^2 - c^2)\,udvdw + (c^2 - a^2)\,vdwdu + (a^2 - b^2)\,wdudv = 0 \ldots\ldots\ldots\ldots (16),$$

$$u + v + w = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Multiplying by

$$-\{(vdu-udv)(wdv-vdw)(udw-wdu)\}^{-1},$$

the first of these becomes

$$\frac{-a^2du}{(vdu-udv)(udw-wdu)}+\frac{-b^2dv}{(wdv-vdw)(vdu-udv)}+\frac{-c^2dw}{(udw-wdu)(wdv-vdw)}=0\ldots\ (18);$$

but writing (17) and its derived equations under the form

$$u+(v+w)=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (19),$$

$$du+(dv+dw)=0,$$

we deduce $$-du(v+w)+u(dv+dw)=-du\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

i.e. $$-du=-(vdu-udv)+(udw-wdu)\ \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

and similarly

$$-dv=-(wdv-vdw)+(vdu-udv),$$

$$-dw=-(udw-wdu)+wdv-vdw).$$

Substituting,

$$\frac{b^2-c^2}{wdv-vdw}+\frac{c^2-a^2}{udw-wdu}+\frac{a^2-b^2}{vdu-udv}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(22);$$

the integral of which may be written in the form

$$\frac{b^2-c^2}{ww_1-vw_1}+\frac{c^2-a^2}{uw_1-wu_1}+\frac{a^2-b^2}{vu_1-v_1u}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ (23),$$

where, on account of (17),

$$u_1+v_1+w_1=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24);$$

and also in the form $$fu+gv+hw=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (25),$$

where f, g, h are connected by

$$\frac{b^2-c^2}{f}+\frac{c^2-a^2}{g}+\frac{a^2-b^2}{g}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26);$$

this last equation is satisfied identically by

$$f=\frac{b^2-c^2}{B^2-C^2},\qquad g=\frac{c^2-a^2}{C^2-A^2},\qquad h=\frac{a^2-b^2}{A^2-B^2}\ \ldots\ldots\ldots\ldots\ (27).$$

Restoring x, y, z, x_1, y_1, z_1 for u, v, w, u_1, v_1, w_1, the equations to a line of curvature passing through a given point x_1, y_1, z_1, on the ellipsoid, are the equation (14) and

$$\frac{(b^2-c^2)}{a^2(y_1^2z^2-y^2z_1^2)}+\frac{(c^2-a^2)}{b^2(z_1^2x^2-z^2x_1^2)}+\frac{(a^2-b^2)}{c^2(x_1^2y^2-x^2y_1^2)}=0\ \ldots\ldots\ldots\ldots\ (28),$$

or again, under a known form, they are the equation (14) and

$$\frac{(b^2-c^2)}{B^2-C^2}\frac{x^2}{a^2}+\frac{c^2-a^2}{C^2-A^2}\frac{y^2}{b^2}+\frac{a^2-b^2}{A^2-B^2}\frac{z^2}{c^2}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(29).$$

From the equations (14), (29) it is easy to prove the well-known form

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1 \quad \text{.....} (30);$$

in fact, multiplying (29) by m, and adding to (14), we have the equation (30), if the equations

$$\frac{1}{a^2}+m\frac{b^2-c^2}{B^2-C^2}\frac{1}{a^2}=\frac{1}{a^2+\theta}, \quad \text{.....} (31),$$

$$\frac{1}{b^2}+m\frac{c^2-a^2}{C^2-A^2}\frac{1}{b^2}=\frac{1}{b^2+\theta},$$

$$\frac{1}{c^2}+m\frac{a^2-b^2}{A^2-B^2}\frac{1}{c^2}=\frac{1}{(c^2+\theta)},$$

are satisfied.

But on reduction, these take the form

$$(B^2-C^2)\,\theta+(b^2-c^2)\,m\theta+ma^2(b^2-c^2)=0, \quad \text{.....}(32),$$

$$(C^2-A^2)\,\theta+(c^2-a^2)\,m\theta+mb^2(c^2-a^2)=0,$$

$$(A^2-B^2)\,\theta+(a^2-b^2)\,m\theta+mc^2(a^2-b^2)=0,$$

and since, by adding, an identical equation is obtained, m and θ may be determined to satisfy these equations. The values of θ, m are

$$\theta=\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{a^2(B^2-C^2)+b^2(C^2-A^2)+c^2(A^2-B^2)} \quad \text{.....} (33),$$

$$m=\frac{b^2c^2(B^2-C^2)+c^2a^2(C^2-A^2)+a^2b^2(A^2-B^2)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)} \quad \text{.....}(34).$$

8.

ON LAGRANGE'S THEOREM.

[From the *Cambridge Mathematical Journal*, vol. III. (1843), pp. 283—286.]

THE value given by Lagrange's theorem for the expansion of any function of the quantity x, determined by the equation

$$x = u + hfx \quad \text{.....(1)},$$

admits of being expressed in rather a remarkable symbolical form. The *à priori* deduction of this, independently of any expansion, presents some difficulties; I shall therefore content myself with showing that the form in question satisfies the equations

$$\frac{d}{du} \cdot \int F'x\, fx\, dx = \frac{d}{dh} \cdot \int F'x\, dx \quad \text{.....(2)},$$

$$Fx = Fu \text{ for } h = 0 \quad \text{.....(3)},$$

deduced from the equation (1), and which are sufficient to determine the expansion of Fx, considered as a function of u and h in powers of h.

Consider generally the symbolical expression

$$\phi\left(h\frac{d}{dh}\right) \Xi h \quad \text{.....(4)},$$

$\phi\left(h\frac{d}{dh}\right)$ involving in general symbols of operation relative to any of the other variables entering into Ξh. Then, if Ξh be expansible in the form

$$\Xi h = \Sigma\,(Ah^m) \quad \text{.....(5)},$$

it is obvious that

$$\phi\left(h\frac{d}{dh}\right) \Xi h = \Sigma\,\{\phi m\,.\,(Ah^m)\} = \Sigma\,\{(\phi m\,.\,A)\,h^m\} \quad \text{..... (6)}.$$

For instance, u representing a variable contained in the function Ξh, and taking a particular form of $\phi\left(h\frac{d}{dh}\right)$,

$$\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h = \Sigma\left(\frac{d^m A}{du^m}h^m\right) \quad \text{.....} \quad (7).$$

From this it is easy to demonstrate

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\{h\Xi h\} \quad \text{.....} \quad (8),$$

$$\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\{h\Xi' h\} \quad \text{.....} \quad (9),$$

where $\Xi' h$ denotes $\frac{d}{dh}\Xi h$, as usual. Hence also

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi' h\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\Xi h\right\} \quad \text{.....} \quad (10),$$

of which a particular case is

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}F'u\, fu\, e^{hfu}\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}F'u\, e^{hfu}\right\} \quad \text{.....} \quad (11).$$

Also,

$$\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) = Fu \text{ for } h=0 \quad \text{.....} \quad (12).$$

Hence the form in question for Fx is

$$Fx = \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) \quad \text{.....} \quad (13);$$

from which, differentiating with respect to u, and writing F instead of F',

$$\frac{Fx}{1-hf'x} = \left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\, e^{hfu}) \quad \text{.....} \quad (14),$$

a well-known form of Lagrange's theorem, almost equally important with the more usual one. It is easy to deduce (13) from (14). To do this, we have only to form the equation

$$\frac{-hFx\,f'x}{1-hf'x} = -h\left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\,f'u\, e^{hfu}) \quad \text{.....} \quad (15),$$

deduced from (14) by writing $Fxf'x$ for fx, and adding this to (14),

$$\begin{aligned} Fx &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\, e^{hfu}) - h\left(\frac{d}{du}\right)^{h\frac{d}{dh}}(Fu\,f'u\, e^{hfu}) \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}\left\{\frac{d}{du}(Fu\, e^{hfu}) - h\,f'u\,Fu\, e^{hfu}\right\} \\ &= \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\, e^{hfu}) \quad \text{.....} \quad (16). \end{aligned}$$

In the case of several variables, if

$$x = u + hf(x, x_1 \ldots), \quad x_1 = u_1 + h_1 f_1(x, x_1 \ldots), \text{ \&c.} \ldots\ldots\ldots\ldots\ldots (17),$$

writing for shortness

$$F, f, f_1 \ldots \text{ for } F(u, u_1 \ldots), \quad f(u, u_1 \ldots), \quad f_1(u, u_1 \ldots), \ldots$$

then the formula is

$$\frac{F(x, x_1 \ldots)}{\{1 - hf'(x)\}\{1 - h_1 f_1'(x_1) \ldots\}} = \left(\frac{d}{du}\right)^h \frac{d}{dh} \left(\frac{d}{du_1}\right)^{h_1} \frac{d}{dh_1} \ldots (Fe^{hf + h_1 f_1 + \cdots}) \ldots\ldots (18),$$

$$\{\text{where } f'(x) \text{ is written to denote } \frac{d}{dx} f(x, x_1 \ldots), \text{ \&c.}\}$$

or the coefficient of $h^n h_1^{n_1} \ldots\ldots$ in the expansion of

$$\frac{F(x, x_1 \ldots\ldots)}{\{1 - hf'(x)\}\{1 - h_1 f_1'(x_1)\} \ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

is

$$= \frac{1}{1 \,.\, 2 \ldots n \,.\, 1 \,.\, 2 \ldots n_1} \left(\frac{d}{du}\right)^n \left(\frac{d}{du_1}\right)^{n_1} \ldots F f^n f_1^{n_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

From the formula (18), a formula may be deduced for the expansion of $F(x, x_1 \ldots\ldots)$, in the same way as (13) was deduced from (14), but the result is not expressible in a simple form by this method. An apparently simple form has indeed been given for this expansion by Laplace, *Mécanique Celeste*, [Ed. 1, 1798] tom. I. p. 176; but the expression there given for the general term, requires first that certain differentiations should be performed, and then that certain changes should be made in the result, quantities $z, z' \ldots\ldots$, are to be changed into $z^n, z_1^{n_1} \ldots\ldots$; in other words, the general term is not really expressed by known symbols of operation only. The formula (18) is probably known, but I have not met with it anywhere.

9.

DEMONSTRATION OF PASCAL'S THEOREM.

[From the *Cambridge Mathematical Journal*, vol. IV. (1843), pp. 18—20.]

LEMMA 1. Let $U = Ax + By + Cz = 0$ be the equation to a plane passing through a given point taken for the origin, and consider the planes

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = 0, \quad U_4 = 0, \quad U_5 = 0, \quad U_6 = 0;$$

the condition which expresses that the intersections of the planes (1) and (2), (3) and (4), (5) and (6) lie in the same plane, may be written down under the form

$$\begin{vmatrix} A_1, & A_2, & A_3, & A_4 & . & . \\ B_1, & B_2, & B_3, & B_4 & . & . \\ C_1, & C_2, & C_3, & C_4 & . & . \\ . & . & A_3, & A_4, & A_5, & A_6 \\ . & . & B_3, & B_4, & B_5, & B_6 \\ . & . & C_3, & C_4, & C_5, & C_6 \end{vmatrix} = 0.$$

LEMMA 2. Representing the determinants

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \text{ \&c.}$$

by the abbreviated notation $\overline{123}$, &c.; the following equation is identically true:

$$\overline{345} \,.\, \overline{126} - \overline{346} \,.\, \overline{125} + \overline{356} \,.\, \overline{124} - \overline{456} \,.\, \overline{123} = 0.$$

This is an immediate consequence of the equations

$$\begin{vmatrix} . & . & x_3, & x_4, & x_5, & x_6 \\ . & . & y_3, & y_4, & y_5, & y_6 \\ . & . & z_3, & z_4, & z_5, & z_6 \\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \\ z_1, & z_2, & z_3, & z_4, & z_5, & z_6 \end{vmatrix} = \begin{vmatrix} . & . & x_3, & x_4, & x_5, & x_6 \\ . & . & y_3, & y_4, & y_5, & y_6 \\ . & . & z_3, & z_4, & z_5, & z_6 \\ x_1, & x_2, & . & . & . & . \\ y_1, & y_2, & . & . & . & . \\ z_1, & z_2, & . & . & . & . \end{vmatrix} = 0.$$

Consider now the points 1, 2, 3, 4, 5, 6, the coordinates of these being respectively x_1, y_1, z_1.....x_6, y_6, z_6. I represent, for shortness, the equation to the plane passing through the origin and the points 1, 2, which may be called the plane $\overline{12}$, in the form

$$x\,\overline{12}_x + y\,\overline{12}_y + z\,\overline{12}_z = 0\,;$$

consequently the symbols $\overline{12}_x$, $\overline{12}_y$, $\overline{12}_z$ denote respectively $y_1z_2 - y_2z_1$, $z_1x_2 - z_2x_1$, $x_1y_2 - x_2y_1$, and similarly for the planes $\overline{13}$, &c. If now the intersections of $\overline{12}$ and $\overline{45}$, $\overline{23}$ and $\overline{56}$, $\overline{34}$ and $\overline{61}$ lie in the same plane, we must have, by Lemma (1), the equation

$$\begin{vmatrix} 12_x, & 45_x, & 23_x, & 56_x, & . & . \\ 12_y, & 45_y, & 23_y, & 56_y, & . & . \\ 12_z, & 45_z, & 23_z, & 56_z, & . & . \\ . & . & 23_x, & 56_x, & 34_x, & 61_x \\ . & . & 23_y, & 56_y, & 34_y, & 61_y \\ . & . & 23_z, & 56_z, & 34_z, & 61_z \end{vmatrix} = 0.$$

Multiplying the two sides of this equation by the two sides respectively of the equation

$$\begin{vmatrix} x_6, & x_1, & x_2, & . & . & . \\ y_6, & y_1, & y_2, & . & . & . \\ z_6, & z_1, & z_2, & . & . & . \\ . & . & . & x_3, & x_4, & x_5 \\ . & . & . & y_3, & y_4, & y_5 \\ . & . & . & z_3, & z_4, & z_5 \end{vmatrix} = \overline{612}\,.\,\overline{345},$$

and observing the equations

$$x_6\,\overline{12}_x + y_6\,\overline{12}_y + z_6\,\overline{12}_z = \overline{612}, \quad \overline{112} = 0, \text{ \&c.}$$

this becomes

$$\begin{vmatrix} 612 & . & . & . & . & . \\ 645, & 145, & 245, & . & . & . \\ 623, & 123, & . & . & 423, & 523 \\ . & 156, & 256, & 356, & 456, & . \\ . & . & . & . & . & 534 \\ . & . & . & 361, & 461, & 561 \end{vmatrix} = 0,$$

reducible to

$$\overline{612}\ \overline{534} \begin{vmatrix} \overline{145}, & \overline{245}, & . & . \\ 123, & . & . & 423 \\ 156, & 256, & 356, & 456 \\ . & . & 361, & 461 \end{vmatrix} = 0;$$

or, omitting the factor $\overline{612}\,.\,\overline{534}$ and expanding,

$$\overline{145}\,.\,\overline{256}\,.\,\overline{423}\,.\,\overline{361} + \overline{245}\,.\,\overline{123}\,.\,\overline{456}\,.\,\overline{361} - \overline{245}\,.\,\overline{123}\,.\,\overline{356}\,.\,\overline{461} - \overline{245}\,.\,\overline{156}\,.\,\overline{423}\,.\,\overline{361} = 0.$$

Considering for instance x_6, y_6, z_6 as variable, this equation expresses evidently that the point 6 lies in a cone of the second order having the origin for its vertex, and the equation is evidently satisfied by writing x_6, y_6, $z_6 = x_1$, y_1, z_1, or x_3, y_3, z_3, or x_4, y_4, z_4, or x_5, y_5, z_5, and thus the cone passes through the points 1, 3, 4, 5. For $x_6, y_6, z_6 = x_2, y_2, z_2$, the equation becomes, reducing and dividing by $\overline{245}\,.\,\overline{123}$,

$$\overline{452}\,.\,\overline{321} - \overline{352}\,.\,\overline{421} + \overline{152}\,.\,\overline{423} = 0,$$

which is deducible from Lemma (2), by writing x_6, y_6, $z_6 = x_2$, y_2, z_2, and is therefore identically true. Hence the cone passes through the point (2), and therefore the points 1, 2, 3, 4, 5, 6 lie in the same cone of the second order, which is Pascal's Theorem. I have demonstrated it in the cone, for the sake of symmetry; but by writing throughout unity instead of z, the above applies directly to the case of the theorem in the plane.

The demonstration of Chasles' form of Pascal's Theorem (viz. that the anharmonic relation of the planes $\overline{61}$, $\overline{62}$, $\overline{63}$, $\overline{64}$ is the same with that of $\overline{51}$, $\overline{52}$, $\overline{53}$, $\overline{54}$), is very much simpler; but as it would require some preparatory information with reference to the analytical definition of the similarity of anharmonic relation, I must defer it to another opportunity.

10.

ON THE THEORY OF ALGEBRAIC CURVES.

[From the *Cambridge Mathematical Journal*, vol. IV. (1843), pp. 102—112.]

SUPPOSE a curve defined by the equation $U=0$, U being a rational and integral function of the mth order of the coordinates x, y. It may always be assumed, without loss of generality, that the terms involving x^m, y^m, both of them appear in U; and also that the coefficient of y^m is equal to unity: for in any particular curve where this was not the case, by transforming the axes, and dividing the new equation by the coefficient of y^m, the conditions in question would become satisfied. Let H_m denote the terms of U, which are of the order m, and let $y-\alpha x$, $y-\beta x \ldots y-\lambda x$ be the factors of H_m. If the quantities α, $\beta \ldots \lambda$ are all of them different, the curve is said to have a number of asymptotic directions equal to the degree of its equation. Such curves only will be considered in the present paper, the consideration of the far more complicated theory of those curves, the number of whose asymptotic directions is less than the degree of their equation, being entirely rejected. Assuming, then, that the factors of H_m are all of them different, we may deduce from the equation $U=0$, by known methods, the series

$$y = \alpha x + \alpha' + \frac{\alpha''}{x} + \ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

$$y = \beta x + \beta' + \frac{\beta''}{x} + \ldots\ldots,$$

$$\vdots$$

$$y = \lambda x + \lambda' + \frac{\lambda''}{x} + \ldots\ldots,$$

and these being obtained, we have, identically,

$$U = (y - \alpha x - \alpha' - \ldots)(y - \beta x - \beta' - \ldots) \ldots (y - \lambda x - \lambda' - \ldots) \ldots\ldots\ldots (2),$$

the negative powers of x on the second side, in point of fact, destroying each other. Supposing in general that fx containing positive and negative powers of x, Efx denotes the function which is obtained by the rejection of the negative powers, we may write

$$U = E\left(y - \alpha x \ldots - \frac{\alpha^{(m)}}{x^{m-1}}\right)\left(y - \beta x \ldots - \frac{\beta^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x \ldots - \frac{\lambda^{(m)}}{x^{m-1}}\right) \ldots\ldots(3),$$

the symbol E being necessary in the present case, because, when the series are continued only to the power x^{-m+1}, the negative powers no longer destroy each other.

We may henceforward consider U as originally given by the equation (3), the $m(m+1)$ quantities α, $\alpha' \ldots \alpha^{(m)}$, β, $\beta' \ldots \beta^{(m)}, \ldots \lambda$, $\lambda' \ldots \lambda^{(m)}$ satisfying the equations obtained from the supposition that it is possible to determine the following terms $\alpha^{(m+1)}$, $\beta^{(m+1)} \ldots \lambda^{(m+1)}, \ldots$ so that the terms containing negative powers of x, on the second side of equation (2), vanish. It is easily seen that α, $\beta \ldots \lambda$, α', $\beta' \ldots \lambda'$ are entirely arbitrary, α'', $\beta'' \ldots \lambda''$ satisfy a single equation involving only the preceding quantities, α''', $\beta''' \ldots \lambda'''$ two equations involving the quantities which precede them, and so on, until $\alpha^{(m)}$, $\beta^{(m)} \ldots \lambda^{(m)}$, which satisfy $(m-1)$ relations involving the preceding quantities. Thus the $m(m+1)$ quantities in question satisfy $\frac{1}{2}m(m-1)$ equations, or they may be considered as functions of $m(m+1) - \frac{1}{2}m(m-1) = \frac{1}{2}m(m+3)$ arbitrary constants. Hence the value of U, given by the equation (3), is the most general expression for a function of the mth order. It is to be remarked also that the quantities $\alpha^{(m+1)}$, $\beta^{(m+1)} \ldots \lambda^{(m+1)}$, ... are all of them completely determinable as functions of α, $\beta \ldots \lambda, \ldots \alpha^{(m)}$, $\beta^{(m)} \ldots \lambda^{(m)}$.

The advantage of the above mode of expressing the function U, is the facility obtained by means of it for the elimination of the variable y from the equation $U = 0$, and any other analogous one $V = 0$. In fact, suppose V expressed in the same manner as U, or by the equation

$$V = E\left(y - Ax \ldots - \frac{A^{(n)}}{x^{n-1}}\right)\left(y - Bx \ldots - \frac{B^{(n)}}{x^{n-1}}\right)\ldots\left(y - Kx \ldots - \frac{K^{(n)}}{x^{n-1}}\right) \ldots\ldots\ldots(4),$$

n being the degree of the function V. It is almost unnecessary to remark, that A, $B \ldots K, \ldots A^{(n)}$, $B^{(n)} \ldots K^{(n)}$ are to be considered as functions of $\frac{1}{2}n(n+3)$ arbitrary constants, and that the subsequent $A^{(n+1)}$, $B^{(n+1)} \ldots K^{(n+1)} \ldots$ can be completely determined as functions of these. Determining the values of y from the equation (3), viz. the values given by the equations (2); substituting these successively in the equation

$$V = (y - Ax - \ldots)(y - Bx - \ldots)\ldots(y - Kx - \ldots) = 0 \ \ldots\ldots\ldots\ldots \ (5),$$

analogous to (2), and taking the product of the quantities so obtained, also observing that this product must be independent of negative powers of x, the result of the elimination may be written down under the form

$$E\left(\begin{array}{c} \left\{(\alpha - A)x \ldots + \dfrac{\alpha^{(mn)} - A^{(mn)}}{x^{mn-1}}\right\} \ldots \left\{(\alpha - Kx) \ldots + \dfrac{\alpha^{(mn)} - K^{(mn)}}{x^{mn-1}}\right\} \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \times\left\{(\lambda - A)x \ldots + \dfrac{\lambda^{(mn)} - A^{(mn)}}{x^{mn-1}}\right\} \ldots \left\{(\lambda - Kx) \ldots + \dfrac{\lambda^{(mn)} - K^{(mn)}}{x^{mn-1}}\right\} \end{array}\right) \ldots (6),$$

the series in { } being continued only to x^{-mn+1}, because the terms after this point produce in the whole product only terms involving negative powers of x. It is for the same reason that the series in ( ), in the equations (3) and (4), are only continued to the terms involving x^{-m+1}, x^{-n+1} respectively.

The first side of the equation (6) is of the order mn, in x, as it ought to be. But it is easy to see, from the form of the expression, in what case the order of the first side reduces itself to a number less than mn. Thus, if n be not greater than m, and the following equations be satisfied,

$$\begin{array}{lll} A=\alpha, & A^{(1)}=\alpha^{(1)}\ldots A^{(r-1)}=\alpha^{(r-1)}, & r \not> n \ldots\ldots\ldots\ldots\ldots\ldots (7), \\ B=\beta, & B^{(1)}=\beta^{(1)}\ldots B^{(s-1)}=\beta^{(s-1)}, & s \not> r, \\ \vdots & & \\ K=\kappa, & K^{(1)}=\kappa^{(1)}\ldots K^{(v-1)}=\kappa^{(v-1)}, & v \not> u, \end{array}$$

the degree of the equation (6) is evidently $mn-r-s\ldots-v$, or the curves $U=0$, $V=0$ intersect in this number only of points. If $mn-r-s\ldots-v=0$, the curves $U=0$ and $V=0$ do not intersect at all, and if $mn-r-s-v$ be negative, $=-\omega$ suppose, the equation (6) is satisfied identically; or the functions U, V have a common factor, the number ω expressing the degree of this factor in x, y.

Supposing the function V given arbitrarily, it may be required to determine U, so that the curves $U=0$, $V=0$ intersect in a number $mn-k$ points. This may in general be done, and done in a variety of ways, for any value of k from unity to $\frac{1}{2}m(m+3)$. I shall not discuss the question generally at present, nor examine into the meaning of the quantity $mn-\frac{1}{2}m(m+3)\{=\frac{1}{2}m(2n-m-3)\}$ becoming negative, but confine myself to the simple case of U and V, both of them functions of the second order. It is required, then, to find the equations of all those curves of the second order which intersect a given curve of the second order in a number of points less than four.

Assume in general

$$V=E\left(y-Ax-A'-\frac{A''}{x}\right)\left(y-Bx-B'-\frac{B''}{x}\right),$$

then A'', B'' satisfy $A''+B''=0$, and putting $B''=\dfrac{K}{A-B}$, and therefore $A''=-\dfrac{K}{A-B}$, and reducing, we obtain

$$V=(y-Ax-A')(y-Bx-B')+K.$$

Similarly assume $$U=E\left(y-\alpha x-\alpha'-\frac{\alpha''}{x}\right)\left(y-\beta x-\beta'-\frac{\beta''}{x}\right),$$

then α'', β'' satisfy $\alpha''+\beta''=0$, and putting $\beta''=\dfrac{k}{\alpha-\beta}$, and therefore $\alpha''=-\dfrac{k}{\alpha-\beta}$, and reducing, we obtain

$$U=(y-\alpha x-\alpha')(y-\beta x-\beta')+k.$$

Suppose

(1) $U=0$, $V=0$ intersect in three points, we must have $\alpha=A$, or the curve $U=0$ must have one of its asymptotes parallel to one of the asymptotes of $V=0$.

(2) The curves intersect in two points. We must have $\alpha=A$, $\alpha'=A'$, or else $\alpha=A$, $\beta=B$; i.e. $U=0$ must have one of its asymptotes coincident with one of the asymptotes of the curve $V=0$, or else it must have its two asymptotes parallel to those of $V=0$. The latter case is that of similar and similarly situated curves.

(3) Suppose the curves intersect in a single point only. Then either $\alpha=A$, $\alpha'=A'$, $\alpha''=A''$, which it is easy to see gives

$$U=(y-Ax-A')(y-\beta x-\beta')+K\frac{A-\beta}{A-B},$$

or else $\alpha=A$, $\alpha'=A'$, $\beta=B$, which is the case of one of the asymptotes of the curve $U=0$, coinciding with one of those of the curve $V=0$, and the remaining asymptotes parallel. As for the first case, if a, a_1 are the transverse axes, θ, θ_1 the inclinations of the two asymptotes to each other, these four quantities are connected by the equation

$$\frac{a^2}{a_1^2}=\frac{\tan\theta\cos^2\frac{1}{2}\theta}{\tan\theta_1\cos^2\frac{1}{2}\theta_1};$$

and besides, one of the asymptotes of the first curve is coincident with one of the asymptotes of the second. This is a remarkable case; it may be as well to verify that $U=0$, $V=0$ do intersect in a single point only. Multiplying the first equation by $y-Bx-B'$, the second by $y-\beta x-\beta'$, and subtracting, the result is

$$(A-\beta)(y-Bx-B')-(A-B)(y-\beta x-\beta')=0,$$

reducible to

$$y-Ax=\frac{A(B'-\beta')+B\beta'-B'\beta}{B-\beta},\quad \text{i.e. } y-Ax-C=0.$$

Combining this with $V=0$, we have an equation of the form $y-Bx-D=0$. And from this and $y-Ax-C=0$, x, y are determined by means of a simple equation.

(4) Lastly, when the curves do not intersect at all. Here $\alpha=A$, $\alpha'=A'$, $\beta=B$, $\beta'=B'$, or the asymptotes of $U=0$ coincide with those of $V=0$; i.e. the curves are similar, similarly situated, and concentric: or else $\alpha=A$, $\alpha'=A'$, $\alpha''=A''$, $\beta=B$; here

$$U=(y-Ax-A')(y-Bx-\beta')+K,$$

or the required curve has one of its asymptotes coincident with one of those of the proposed curve; the remaining two asymptotes are parallel, and the magnitudes of the curves are equal.

In general, if two curves of the orders m and n, respectively, are such that r asymptotes of the first are parallel to as many of the second, s out of these asymptotes

being coincident in the two curves, the number of points of intersection is $mn-r-s$; but the converse of this theorem is not true.

In a former paper, *On the Intersection of Curves*, [5], I investigated the number of arbitrary constants in the equation of a curve of a given order ρ subjected to pass through the mn points of intersection of two curves of the orders m and n respectively. The reasoning there employed is not applicable to the case where the two curves intersect in a number of points less than mn. In fact, it was assumed that, $W=0$ being the equation of the required curve, W was of the form $uU+vV$; u, v being polynomials of the degrees $\rho-m$, $\rho-n$ respectively. This is, in point of fact, true in the case there considered, viz. that in which the two curves intersect in mn points; but where the number of points of intersection is less than this, u, v may be assumed polynomials of an order *higher* than $\rho-m$, $\rho-n$, and yet $uU+vV$ reduce itself to the order ρ. The preceding investigations enable us to resolve the question for every possible case.

Considering then the functions U, V determined as before by the equations (3), (4), suppose, in the first place, we have a system of equations

$$\alpha=A,\quad \beta=B\ldots\ldots\theta=H\ (t \text{ equations})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Assume
$$P=(y-\alpha x-\ldots)(y-\beta x-\ldots)\ldots(y-\theta x-\ldots),$$
$$Q=(y-Ax-\ldots)(y-Bx-\ldots)\ldots(y-Hx-\ldots);$$
$$\Upsilon=(y-\iota x-\ldots)\ldots(y-\kappa x-\ldots),$$
$$\Psi=(y-\mathrm{I}x-\ldots)\ldots(y-Kx-\ldots);$$

whence
$$U=P\Upsilon,\qquad V=Q\Psi.$$

Suppose
$$\Upsilon=E\Upsilon+\Delta\Upsilon,\qquad \Psi=E\Psi+\Delta\Psi,$$

$$\begin{aligned}E\Psi\,.\,U-E\Upsilon\,.\,V&=E\Psi\,.\,P\Upsilon-E\Upsilon\,.\,Q\Psi,\\&=E\Psi\,.\,P\,(E\Upsilon+\Delta\Upsilon)-E\Upsilon\,.\,Q\,(E\Psi+\Delta\Psi),\\&=E\Upsilon\,.\,E\Psi\,.\,(P-Q)+E\Psi\,.\,P\,.\,\Delta\Upsilon-E\Upsilon\,.\,Q\,.\,\Delta\Psi,\\&=E\,\{E\Upsilon\,.\,E\Psi\,.\,(P-Q)+E\Psi\,.\,P\Delta\Upsilon-E\Upsilon\,.\,Q\,.\,\Delta\Psi\},\\&=\Pi \text{ suppose.}\end{aligned}$$

In this expression $E\Upsilon$, $E\Psi$ are of the degrees $m-t$, $n-t$, $\Delta\Upsilon$, $\Delta\Psi$ of the degree -1, and P, Q, $P-Q$ of the degrees t, t, $t-1$ respectively. The terms of Π are therefore of the degrees $m+n-t-1$, $m-1$, $n-1$ respectively, and the largest of these is in general $m+n-t-1$. Suppose, however, that $m+n-t-1$ is equal to $m-1$ (it cannot be inferior to it), then $t=n$; Ψ becomes equal to unity, or $\Delta\Psi$ vanishes. The remaining two terms of Π are $E\Upsilon\,(P-Q)$, $P\Delta\Upsilon$, which are of the degrees $m-1$, $n-1$ respectively. Π is still of the degree $m-1$, supposing $m>n$. If $m=n$, the term $P\Delta\Upsilon$ vanishes. Π is still of the degree $m-1$. Hence in every case the degree of Π is $m+n-t-1$: assuming always that $P-Q$ does not reduce itself to a degree lower than $t-1$, (which is always the case as long as the equations

$\alpha' = A'$, $\beta' = B' \ldots\ldots \theta' = H'$ are not all of them satisfied simultaneously). It will be seen presently that we shall gain in symmetry by wording the theorem thus: the degree of Π is equal to the greatest of the two quantities $m+n-t-1$, $m-1$.

Suppose next, in addition to the equations (8), we have

$$\alpha' = A', \quad \beta' = B' \ldots\ldots \zeta' = F', \quad t' \text{ equations}, \; t' \not> t \ldots (8').$$

Then, taking Υ', Ψ', P', Q' the analogous quantities to Υ, Ψ, P, Q, and putting

$$E\Psi' . U - E\Upsilon' . V = \Pi',$$

we have, as before,

$$\Pi' = E\{E\Upsilon' . E\Psi' . (P' - Q') + E\Psi' . P'\Delta\Upsilon' - E\Upsilon' . Q'\Delta\Psi'\}.$$

The degree of $P' - Q'$ is $t' - 2$ (unless simultaneously $\alpha'' = A''$, $\beta'' = B'' \ldots\ldots \zeta'' = F''$, in which case the degree may be lower). The degrees, therefore, of the terms of Π' are $m+n-t'-2$, $n-1$, $m-1$. Or we may say that the degree of Π' is equal to the greatest of the quantities $m+n-t'-2$, $m-1$; though to establish this proposition in the case where $t' = n-1$ would require some additional considerations.

Continuing in this manner until we come to the quantity $\Pi^{(k-1)}$, the degree of this quantity is the greatest of the two numbers $m+n-t^{(k-1)}-k$, $m-1$. And we may suppose that none of the equations $\alpha^{(k)} = A^{(k)} \ldots\ldots$ are satisfied, so that the series Π, $\Pi' \ldots\ldots \Pi^{(k-1)}$ is not to be continued beyond this point.

Considering now the equation of the curve passing through the $mn - t - t' \ldots - t^{(k-1)}$ points of intersection of $U=0$, $V=0$, we may write

$$W = uU + vV + p\Pi + p'\Pi' \ldots\ldots + p^{(k-1)}\,\Pi^{(k-1)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

for the required equation; the dimensions of u, v, p, $p' \ldots p^{(k-1)}$ being respectively

$$\rho - m, \quad \rho - n; \quad \rho - m - n + t + 1 \text{ or } \rho - m + 1;$$
$$\rho - m - n + t' + 2 \text{ or } \rho - m + 1; \ldots\ldots \rho - m - n + t^{(k-1)} + k \text{ or } \rho - m + 1,$$

the lowest of the two numbers being taken for the dimensions of p, $p' \ldots p^{(k-1)}$. Also, if any of these numbers become negative, the corresponding term is to be rejected. In saying that the degrees of p, $p' \ldots\ldots p^{(k-1)}$ have these actual values, it is supposed that the degrees of Π, $\Pi' \ldots\ldots \Pi^{k-1}$ actually ascend to the greatest of the values

$$\rho - m - n + t + 1, \text{ or } m-1; \quad m+n-t'-2, \text{ or } m-1; \quad -m+n-t^{(k-1)}+k, \text{ or } m-1.$$

The cases of exception to this are when several of the consecutive numbers t, $t' \ldots t^{(k-1)}$ are equal. In this case the corresponding terms of the series Π, $\Pi' \ldots\ldots \Pi^{(k-1)}$, are also equal. Suppose for instance t, t' were equal, Π, Π' would also be equal. A term of p of an *order higher* by unity than $\rho - m - n + t + 1$, or $\rho - m + 1$, which is the highest term admissible, produces in $p\Pi$ a term identical with one of the terms of $p'\Pi$; so that nothing is gained in generality by admitting such terms into p. The equation (9), with the preceding values for the dimensions of p, $p' \ldots\ldots p^{(k-1)}$, may be

employed, therefore, even when several consecutive terms of the series $t, t' \ldots\ldots t^{(k-1)}$ are equal. It will be convenient also to assume that $\rho - m$ is not negative, or at least for greater simplicity to examine this case in the first place.

u, U, and v, V, contain terms of the form $x^\alpha y^\beta U$, $x^\gamma y^\delta V$, $\alpha + \beta \not> \rho - m$, $\gamma + \delta \not> \rho - n$; $p\Pi$ contains terms of this form, and in addition terms for which $\alpha + \beta = \rho + 1 - m$, $\gamma + \delta = \rho + 1 - n$. It is useless to repeat the former terms, so that we may assume for p, a *homogeneous* function of the order $\rho - m - n + t + 1$, or $\rho - m + 1$; in which case $p\Pi$ consists only of terms for which $\alpha + \beta = \rho + 1 - m$, $\gamma + \delta = \rho + 1 - n$. And the general expression of p contains $\rho - m - n + t + 2$, or $\rho - m + 2$, arbitrary constants. Similarly $p'\Pi'$ contains terms of the form of those in uU, vV, $p\Pi$, and also terms for which $\alpha + \beta = \rho + 2 - m$, $\gamma + \delta = \rho + 2 - n$; the latter terms only need be considered, or p' may be assumed to be a homogeneous function of the order $\rho - m - n + t' + 2$, or $\rho - m + 1$, containing therefore $\rho - m - n - t' + 3$, or $\rho - m + 2$ arbitrary constants.

Similarly $p^{(k-1)}$ contains $\rho - m - n + t^{(k-1)} + k + 1$ or $\rho - m + 2$ arbitrary constants. Assume

$$\nabla = \binom{\rho - m - n + t + 2}{\rho - m + 2} + \binom{\rho - m - n + t' + 3}{\rho - m + 2} \ldots + \binom{\rho - m - n + t^{(k-1)} + k + 1}{\rho - m + 2} \ldots (10),$$

where, in forming the value of ∇ the least of the two quantities in ( ) is to be taken; this value also, if negative, being replaced by zero. The number of arbitrary constants in $p, p' \ldots p^{(k-1)}$ is consequently equal to ∇.

The numbers of arbitrary constants in u, v, are respectively

$$\{1 + 2 \ldots\ldots + (\rho - m + 1)\} \text{ and } \{1 + 2 \ldots\ldots + (\rho - n + 1)\}$$

i.e.

$$\tfrac{1}{2}(\rho - m + 1)(\rho - m + 2), \text{ and } \tfrac{1}{2}(\rho - n + 1)(\rho - n + 2);$$

thus the whole number of arbitrary constants in W, diminished by unity (since nothing is gained in generality, by leaving the coefficient (for instance of y^ρ) indeterminate, instead of supposing it equal to unity) becomes

$$\tfrac{1}{2}(\rho - m + 1)(\rho - m + 2) + \tfrac{1}{2}(\rho - n + 1)(\rho - n + 2) + \nabla - 1,$$

reducible to

$$\tfrac{1}{2}\rho(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - mn + \nabla.$$

By the reasonings contained in the paper already referred to, if $\rho + k - m - n + 1$ be positive, to find the number of really disposable constants in W, we must subtract from this number a number $\frac{1}{2}(\rho + k - m - n + 1)(\rho + k - m - n + 2)$. Hence, calling ϕ the number of disposable constants in W, we have

$$\phi = \tfrac{1}{2}\rho(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 1)(\rho - m - n + 2) - mn + \nabla - \Lambda \ldots\ldots (11),$$

where

$$\Lambda = 0, \text{ if } \rho + k - m - n + 1 \text{ be negative or zero} \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

$$\Lambda = \tfrac{1}{2}(\rho + k - m - n + 1)(\rho + k - m - n + 2),$$

if $\rho + k - m - n + 1$ be positive; and ∇ is given by the equation (9).

Also, if θ be the number of points through which the curve $W=0$ can be drawn, *including* the points of intersection of the curves $U=0$, $V=0$, then

$$\theta = \phi + (mn - t - t' \ldots\ldots - t^{(k-1)}) \text{ or}$$

$$\theta = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2) + \nabla - \Lambda - t - t' \ldots - t^{(k-1)} \ldots\ldots (13).$$

Any particular cases may be deduced with the greatest facility from these general formulæ. Thus, supposing the curves to intersect in the complete number of points mn, we have

$$\phi = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(1-\delta)(\rho-m-n+1)(\rho-m-n+2) - mn,$$

δ being zero or unity according as $\rho < m+n-1$ or $\rho > m+n-1$. Reducing, we have, for $\rho \not> m+n-3$,

$$\phi = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2) - mn,$$
$$\theta = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2);$$

and for $\rho > m+n-3$,

$$\phi = \tfrac{1}{2}\rho(\rho+3) - mn,$$
$$\theta = \tfrac{1}{2}\rho(\rho+3).$$

Suppose, in the next place, the curves have t parallel pairs of asymptotes, none of these pairs being coincident. Then

$$\rho \not> m+n-t-2,$$
$$\phi = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2) - mn,$$
$$\theta = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2) - t;$$

$$\rho > m+n-t-2, \quad \rho \not> m+n-2,$$
$$\phi = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+2)(\rho-m-n+3) - mn + t,$$
$$\theta = \tfrac{1}{2}\rho(\rho+3) + \tfrac{1}{2}(\rho-m-n+2)(\rho-m-n+3),$$

$$\rho > m+n-2,$$
$$\phi = \tfrac{1}{2}\rho(\rho+3) - mn + t,$$
$$\theta = \tfrac{1}{2}\rho(\rho+3).$$

In these formulæ, if t be equal to 2 or greater than 2, the limiting conditions are more conveniently written

$$\rho \not> m+n-t-2; \quad \rho \not> m+n-t-2 > m+n-4; \quad \rho > m+n-4.$$

Similarly may the solution of the question be explicitly obtained when the curves have t asymptotes parallel, and t' out of these coincident, but the number of separate formulæ will be greater.

In conclusion, I may add the following references to two memoirs on the present subject: the conclusions in one point of view are considerably less general even than those of my former paper, though much more so in another. Jacobi, Theoremata de punctis intersectionis duarum curvarum algebraicarum; *Crelle's Journal*, vol. XV. [1836, pp. 285—308]; Plücker, Théorèmes généraux concernant les equations à plusieurs variables, d'un degré quelconque entre un nombre quelconque d'inconnues. D° vol. XVI. [1837, pp. 47—57].

Addition.

As an exemplification of the preceding formulæ, and besides as a question interesting in itself, it may be proposed to determine the asymptotic curves of the rth order of a given curve having all its asymptotic directions distinct,—r being any number less than the degree of the equation of the given curve.

DEFINITION. A curve of the rth order, which intersects a given curve of the mth order in a number of points, $= mr - \frac{1}{2}r(r+3)$, is said to be an asymptotic curve of the rth order to the curve in question. Suppose, as before, $U=0$ being the equation to the given curve,

$$U = E\left(y - \alpha x - \ldots - \frac{\alpha^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x \ldots - \frac{\lambda^{(m)}}{x^{m-1}}\right);$$

and let $\theta, \phi \ldots \omega$ denote any combination of r terms out of the series $\alpha \ldots \lambda$, and $\theta', \phi' \ldots \omega'$, &c. ... the corresponding terms out of $\alpha' \ldots \lambda'$, &c. Then, writing

$$V = E\left\{\left(y - \theta x \ldots - \frac{\theta^{(m)}}{x^{m-1}}\right)\left(y - \phi x \ldots - \frac{\phi^{(m-1)}}{x^{m-2}} - \frac{\Phi^{(m)}}{x^{m-1}}\right) \times \ldots\ldots\right.$$

$$\left.\left(y - \psi x \ldots - \frac{\psi^{(m-2)}}{x^{m-3}} - \frac{\Psi^{(m-1)}}{x^{m-2}} - \frac{\Psi^{(m)}}{x^{m-1}}\right)\ldots\left(y - \omega x - \Omega' \ldots - \frac{\Omega^{(m)}}{x^{m-1}}\right)\right\},$$

(where the quantities $\Phi^{(m)}$, $\Psi^{(m-1)}$, $\Psi^{(m)}$, $\Omega' \ldots \Omega^{(m)}$ are entirely determinate, since, by what has preceded, $\theta', \phi' \ldots \Omega'$ satisfy a certain equation, $\theta'', \phi'', \ldots \Omega''$ two equations..... $\theta^{(m)}$, $\Phi^{(m)}$, ... $\Omega^{(m)}$ $(m-1)$ equations), we have $V=0$ for the required equation of the asymptotic curve. It is obvious that the whole number of asymptotic curves of the order r, is $n(n-1)\ldots(n-r+1)$, viz. $1.2\ldots r$ curves for each combination of $\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$ asymptotes. Some particular instances of asymptotic curves will be found in a memoir by M. Plücker, *Liouville's Journal*, vol. I. [1836, pp. 229—252], Enumération des courbes du quatrième ordre, &c. The general theory does not seem to be one to which much attention has been paid.

11.

CHAPTERS IN THE ANALYTICAL GEOMETRY OF (n) DIMENSIONS.

[From the *Cambridge Mathematical Journal*, vol. IV. (1843), pp. 119—127.]

CHAP. 1. *On some preliminary formulæ.*

I TAKE for granted all the ordinary formulæ relating to determinants. It will be convenient, however, to write down a few, relating to a certain system of determinants, which are of considerable importance in that which follows: they are all of them either known, or immediately deducible from known formulæ.

Consider the series of terms

$$\begin{matrix} x_1\,, & x_2 \ldots\ldots x_n \\ A_1\,, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1\,, & K_2 \ldots\ldots K_n \end{matrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

the number of the quantities $A \ldots\ldots K$ being equal to q $(q < n)$. Suppose $q+1$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in $\dfrac{n\,(n-1)\ldots\ldots(q+2)}{1\,.\,2\ldots\ldots n-q-1}$ different ways. The system of determinants so obtained will be represented by the notation

$$\left\| \begin{matrix} x_1\,, & x_2 \ldots\ldots x_n \\ A_1\,, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1\,, & K_2 \ldots\ldots K_n \end{matrix} \right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and the system of equations, obtained by equating each of these determinants to zero, by the notation

$$\left\| \begin{matrix} x_1\,, & x_2 \ldots\ldots x_n \\ A_1\,, & A_2 \ldots\ldots A_n \\ \vdots & \vdots \\ K_1 & K_2 \ldots\ldots K_n \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The $\dfrac{n(n-1)\ldots\ldots(q+2)}{1.2\ldots\ldots(n-q+1)}$ equations represented by this formula reduce themselves to $(n-q)$ independent equations. Imagine these expressed by

$$(1)=0,\quad (2)=0\ldots\ldots(n-q)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

any one of the determinants of (2) is reducible to the form

$$\Theta_1(1)+\Theta_2(2)\ldots+\Theta_{n-q}(n-q)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $\Theta_1, \Theta_2\ldots\Theta_{n-q}$ are coefficients independent of $x_1, x_2\ldots x_n$. The equations (3) may be replaced by

$$\left\|\begin{matrix} \lambda_1x_1+\lambda_2x_2+\ldots\lambda_nx_n, & \mu_1x_1+\ldots, & \ldots\tau_1x_1+\ldots \\ \lambda_1A_1+\lambda_2A_2+\ldots\lambda_nx_n, & \mu_1A_1+\ldots, & \ldots\tau_1A_1+\ldots \\ \vdots & & \\ \lambda_1K_1+\lambda_2K_2+\ldots\lambda_nK_n, & \mu_1K_1+\ldots, & \tau_1K_1+\ldots \end{matrix}\right\|=0\ldots\ldots\ldots\ldots (6),$$

and conversely from (6) we may deduce (3), unless

$$\left|\begin{matrix} \lambda_1, & \lambda_2,\ldots\lambda_n \\ \mu_1, & \mu_2,\ldots\mu_n \\ \vdots & \\ \tau_1, & \tau_2,\ldots\tau_n \end{matrix}\right|=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

(The number of the quantities $\lambda, \mu\ldots\tau$ is of course equal to n.) The equations (3) may also be expressed in the form

$$\left\|\begin{matrix} x_1 & , & x_2, & \ldots & x_n \\ \lambda_1A_1+\ldots\omega_1K_1, & & \lambda_1A_2+\ldots\omega_1K_2, & \ldots & \lambda_1A_n\ldots+\omega_1K_n \\ \vdots & & \vdots & & \\ \lambda_qA_1+\ldots\omega_qK_1, & & \lambda_qA_2+\ldots\omega_qK_2, & \ldots & \lambda_qA_n\ldots+\omega_qK_n \end{matrix}\right\|\ldots\ldots\ldots\ldots (8),$$

the number of the quantities $\lambda, \mu\ldots\omega$ being q.

And conversely (3) is deducible from (8), unless

$$\left|\begin{matrix} \lambda_1,\ldots\omega_1 \\ \vdots \\ \lambda_q,\ldots\omega_q \end{matrix}\right|=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

CHAP. 2. *On the determination of linear equations in $x_1, x_2,\ldots x_n$ which are satisfied by the values of these quantities derived from given systems of linear equations.*

It is required to find linear equations in $x_1,\ldots x_n$ which are satisfied by the values of these quantities derived—1. from the equations $\mathfrak{A}'=0$, $\mathfrak{B}'=0\ldots\mathfrak{G}'=0$; 2. from the equations $\mathfrak{A}''=0$, $\mathfrak{B}''=0\ldots\mathfrak{N}''=0$; 3. from $\mathfrak{A}'''=0$, $\mathfrak{B}'''=0\ldots\mathfrak{K}'''=0$, &c. &c., where

$$\mathfrak{A}'=A_1x_1+A_2x_2\ldots+A_nx_n,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\mathfrak{B}'=B_1x_1+B_2x_2\ldots+B_nx_n,$$
$$\vdots$$

and similarly $\mathfrak{A}'', \mathfrak{B}'',\ldots, \mathfrak{A}''', \mathfrak{B}''',\ldots$, &c. are linear functions of the coordinates $x_1, x_2,\ldots x_n$.

Also r', r''... representing the number of equations in the systems (1), (2)... and k the number of these given systems,

$$(n-r')+(n-r'')+\ldots \not> n-1 \text{ or } (k-1)n+1 \not> r'+r''+\ldots$$

Assume

$$0=\lambda'\mathfrak{A}'+\mu'\mathfrak{B}'+\ldots,$$
$$\lambda'\mathfrak{A}'+\mu'\mathfrak{B}'+\ldots=\lambda''\mathfrak{A}''+\mu''\mathfrak{B}''+\ldots=\lambda'''\mathfrak{A}'''+\mu'''\mathfrak{B}'''+\ldots=\&c. \ldots\ldots (2),$$

the latter equations denoting the equations obtained by equating to zero the terms involving x_1, those involving x_2, &c.... separately. Suppose, in addition to these, a set of linear equations in λ', μ'... λ'', μ''... so that, with the preceding ones, there is a sufficient number of equations for the elimination of these quantities. Then, performing the elimination, we thus obtain equations $\Psi=0$, where Ψ is a function of x_1, x_2... which vanishes for the values of these quantities derived from the equations (1) or (2)... &c. The series of equations $\Psi=0$ may be expressed in the form

$$\left\|\begin{array}{llllll}
\mathfrak{A}', & \mathfrak{B}', \ldots \mathfrak{C}', & & & & \\
A_1', & B_1', \ldots G_1', & A_1'', \ldots O''_1, & & & \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots & & & \\
A_n', & B_n', \ldots G_n', & A_n'', \ldots O_n'', & & & \\
 & & A_1'', \ldots O_1'', & A_1''', \ldots R_1''', & & \\
 & & \vdots \quad \vdots & \vdots \quad \vdots & & \\
 & & A_n'', \ldots O_n'', & A_n''', \ldots R_n''', & & \\
 & & \vdots \quad \vdots & \vdots \quad \vdots & \vdots & \vdots
\end{array}\right\| = 0 \ldots\ldots (3).$$

CHAP. 3. *On reciprocal equations.*

Consider a system of equations

$$\begin{array}{l}
A_1x_1+A_2x_2\ldots+A_nx_n=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1), \\
\vdots \\
K_1x_1+K_2x_2\ldots+K_nx_n=0,
\end{array}$$

(r in number).

The reciprocal system with respect to a given function (U) of the second order in x_1, x_2... x_n, is said to be

$$\left\|\begin{array}{lll}
d_{x_1}U, & d_{x_2}U, \ldots & d_{x_n}U \\
A_1, & A_2, \ldots & A_n \\
\vdots & \vdots & \vdots \\
K_1, & K_2, \ldots & K_n
\end{array}\right\| = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

($n-r$ in number).

It must first be shown that the reciprocal system to (2) is the system (1), or that the systems (1), (2) are reciprocals of each other.

Consider, in general, the system of equations

$$\begin{aligned} &\alpha_1 d_{x_1} U + \alpha_2 d_{x_2} U \ldots + \alpha_n d_{x_n} U = 0 \quad \ldots\ldots\ldots\ldots (3). \\ &\vdots \\ &\lambda_1 d_{x_1} U + \lambda_2 d_{x_2} U \ldots + \lambda_n d_{x_n} U = 0. \end{aligned}$$

Suppose $\quad 2U = \Sigma (\alpha^2) x_\alpha^2 + 2\Sigma (\alpha\beta) x_\alpha x_\beta$, so that $d_{x_\alpha} U = \Sigma (s\alpha) x_\alpha \ldots\ldots\ldots\ldots (4), (5).$

The equations (3) may be written

$$x_1 \{\alpha_1 (1^2) + \alpha_2 (12) \ldots + \alpha_n (1n)\} + \ldots + x_n \{\alpha_1 (n1) + \alpha_2 (n2) \ldots + \alpha_n (n^2)\} = 0 \ldots\ldots\ldots (6),$$

&c.

and forming the reciprocals of these, also replacing $d_{x_1} U$, $d_{x_2} U \ldots$ by their values, we have

$$\left\| \begin{matrix} x_1 (1^2) + x_2 (12) + \ldots x_n (1n), \ldots x_1 (n1) + x_2 (n2) \ldots + x_n (n^2) \\ \alpha_1 (1^2) + \alpha_2 (12) + \ldots \alpha_n (1n), \ldots \alpha_1 (n1) + \alpha_2 (n2) \ldots + \alpha_n (n^2) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \lambda_1 (1^2) + \lambda_2 (12) + \ldots \lambda_n (1n), \ldots \lambda_1 (n1) + \lambda_2 (n2) \ldots + \lambda_n (n^2) \end{matrix} \right\| = 0 \ldots\ldots (7).$$

From these, assuming

$$\left| \begin{matrix} (1^2), & (12), \ldots (1n) \\ (21), & (2^2), \ldots (2n) \\ \vdots & \\ (n1), & (n2), \ldots (n^2) \end{matrix} \right| \neq 0 \quad \ldots\ldots\ldots\ldots (8)$$

we obtain, for the reciprocal system of (3),

$$\left\| \begin{matrix} x_1, & x_2, \ldots x_n \\ \alpha_1, & \alpha_2, \ldots \alpha_n \\ \vdots & \\ \lambda_1, & \lambda_2, \ldots \lambda_n \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots\ldots (9).$$

Now, suppose the equations (3) represent the system (2); their number in this case must be $n - r$. Also if θ represent any one of the quantities $\alpha, \beta \ldots \lambda$, we have

$$\begin{aligned} &A_1\theta_1 + A_2\theta_2 \ldots + A_n\theta_n = 0 \quad \ldots\ldots\ldots\ldots (10), \\ &\vdots \\ &K_1\theta_1 + K_2\theta_2 \ldots + K_n\theta_n = 0. \end{aligned}$$

By means of these equations, the system (9) may be reduced to the form

$$\left\| \begin{matrix} A_1 x_1 + A_2 x_2 \ldots + A_n x_n, \ldots K_1 x_1 + K_2 x_2 \ldots + K_n x_n, & x_{r+1}, & x_{r+2}, \ldots x_n \\ 0 \qquad , \ldots \qquad 0 \qquad , & \alpha_{r+1}, & \alpha_{r+2}, \ldots \alpha_n \\ \vdots & \vdots & \\ 0 \qquad , \ldots \qquad 0 \qquad , & \lambda_{r+1}, & \lambda_{r+2}, \ldots \lambda_n \end{matrix} \right\| = 0 \ldots (11),$$

which are satisfied by the equations (1). Hence the reciprocal system to (2) is (1), or (1), (2) are reciprocals to each other.

THEOREM. Consider the equations

$$(\mathfrak{A}' = 0,\ \mathfrak{B}' = 0 \ldots \mathfrak{G}' = 0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$
$$(\mathfrak{A}'' = 0,\ \mathfrak{B}'' = 0 \ldots \mathfrak{N}'' = 0),$$
$$(\mathfrak{A}''' = 0,\ \mathfrak{B}''' = 0 \ldots \mathfrak{K}''' = 0),$$
&c.

of Chap. 2. The equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \ldots & d_{x_n}U \\ A_1', & A_2', \ldots & A_n' \\ \vdots & & \vdots \\ G_1', & G_2', \ldots & G_n' \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \ldots & d_{x_n}U \\ A_1'', & A_2'', \ldots & A_n'' \\ \vdots & & \vdots \\ O_1'', & O_2'', \ldots & O_n'' \end{matrix} \right\| = 0, \ldots (13),$$
&c.

which are the reciprocals of these systems, represent taken conjointly the reciprocal of the system of equations (3) of the same chapter.

Let this system, which contains $n - \{(n-r) + (n-r') + \ldots\}$ equations, be represented by

$$\alpha_1 x_1 + \alpha_2 x_2 \ldots + \alpha_n x_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$
$$\beta_1 x_1 + \beta_2 x_2 \ldots + \beta_n x_n = 0.$$
$$\vdots$$
$$\zeta_1 x_1 + \zeta_2 x_2 \ldots + \zeta_n x_n = 0.$$

The reciprocal system is

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \ldots & d_{x_n}U \\ \alpha_1, & \alpha_2, \ldots & \alpha_n \\ \vdots & & \\ \zeta_1, & \zeta_2, \ldots & \zeta_n \end{matrix} \right\| = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

containing $(n-r) + (n-r') +$ &c. ... equations.

Also, by the formulæ in Chap. 2,

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \lambda_1'\mathfrak{A}' + \mu_1'\mathfrak{B}' + \ldots \sigma_1'\mathfrak{G}' \ (\lambda,\ \mu \ldots \sigma,\ r' \text{ in number}).$$
$$\beta_1 x_1 + \ldots + \beta_n x_n = \lambda_2'\mathfrak{A}' + \mu_2'\mathfrak{B}' + \ldots \sigma_2'\mathfrak{G}'$$
$$\vdots$$
$$\zeta_1 x_1 \quad \ldots + \zeta_n x_n = \lambda_\theta'\mathfrak{A}' + \mu_\theta'\mathfrak{B}' + \ldots \sigma_\theta'\mathfrak{G} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

writing $\theta = n - \{(n-r) + (n-r') + \ldots\}$.

Also, assuming any arbitrary quantities $\eta_1,\ \eta_2 \ldots \eta_n \ldots \phi_1,\ \phi_2 \ldots \phi_n$ (the number of sets being $(r' - \theta)$,) such that

$$\eta_1 x_1 \ldots + \eta_n x_n = \lambda_{\theta+1}'\mathfrak{A}' + \mu_{\theta+1}'\mathfrak{B}' + \ldots \sigma_{\theta+1}\mathfrak{G}' \ldots\ldots\ldots\ldots\ldots (17),$$
$$\vdots$$
$$\phi_1 x_1 \ldots + \phi_n x_n = \lambda_{r'}'\ \mathfrak{A}' + \mu_{r'}'\ \mathfrak{B}' + \ldots \sigma_{r'}'\ \mathfrak{G}'.$$

From the equations (15) we deduce the $(n-r)$ equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \dots d_{x_n}U \\ \eta_1, & \eta_2, \dots \eta_n \\ \vdots & \\ \phi_1, & \phi_2, \dots \phi_n \end{matrix} \right\| = 0 \quad \dots\dots\dots\dots \quad (18).$$

Hence, writing

$$\begin{aligned} \eta &= \lambda_1' A + \mu_1' B + \dots \sigma_1' G \quad \dots\dots\dots\dots \quad (19). \\ &\vdots \\ \phi &= \lambda_r' A + \mu_r' B + \dots \sigma_r' G, \end{aligned}$$

and reducing, by the formula (8) of Chap. 1, we have

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U, \dots d_{x_n}U \\ A_1', & A_2', \dots A_n' \\ \vdots & \\ G_1', & G_2', \dots G_n' \end{matrix} \right\| = 0 \quad \dots\dots\dots\dots \quad (20);$$

and similarly may the remaining formulæ of (13) be deduced from the equation (15). Hence the required theorem is demonstrated, a theorem which may be more clearly stated as follows:—

The reciprocals of several systems of equations form together the reciprocal of the equation which is satisfied by the values of the variables which satisfy each of the original systems of equations. (The theorem requires that the number of all the reciprocal equations shall be less than the number of variables.)

Conversely, consider several systems of equations, the whole number of the equations being less than the number of variables. These systems, taken conjointly, have for their reciprocal, the equation which is satisfied by the values satisfying the reciprocal system of each of the given systems.

CHAP. 4. *On some properties of functions of the second order.*

Suppose, as before, U denotes the general function of the second order, or

$$2U = \Sigma\,(\alpha^2)\, x_\alpha^2 + 2\Sigma\,(\alpha\beta)\, x_\alpha x_\beta \quad \dots\dots\dots\dots (21).$$

Also let V denote a function of the second order of the form

$$V = H\left(\left\| \begin{matrix} x_1, & x_2, \dots x_n \\ \alpha_1, & \alpha_2, \dots \alpha_n \\ \vdots & \\ \rho_1, & \rho_2, \dots \rho_n \end{matrix} \right\| \right) \quad \dots\dots\dots\dots \quad (22),$$

(H being the symbol of a homogeneous function of the second order, and the number r of the quantities $\alpha, \beta \dots \rho$, being less than $n-1$). [Observe that $\alpha_1, \beta_1, \dots \rho_1, \dots \alpha_n, \beta_n, \dots \rho_n$, or say the suffixed quantities $\alpha, \beta, \dots \rho$ (r in number) are used to denote coefficients: α, β, without suffixes, are any two numbers in the series of suffixes $1, 2, 3, \dots n$.] Then $2U - 2kV$, k arbitrary, is of the form

$$\Sigma\,[\alpha^2]\, x_\alpha^2 + 2\Sigma\,[\alpha\beta]\, x_\alpha x_\beta \quad \dots\dots\dots\dots (23).$$

Suppose $X_1, X_2, \dots X_n$ determined by the equations

$$\begin{aligned} &[1^2]X_1 + [12]X_2 \dots + [1n]X_n = 0 \quad \dots\dots\dots\dots (24), \\ &[21]X_1 + [2^2]X_2 \dots + [2n]X_n = 0, \\ &\vdots \\ &[n1]X_1 + [n2]X_2 \dots + [n^2]X_n = 0; \end{aligned}$$

equations that involve the condition that k satisfies an equation of the order $n-r$, as will be presently proved.

Then shall $x_1 = X_1 \dots x_n = X_n$ satisfy the system of equations, which is the reciprocal of

$$\left\| \begin{matrix} x_1, & x_2, \dots x_n \\ \alpha_1, & \alpha_2, \dots \alpha_n \\ \vdots & \\ \rho_1, & \rho_2, \dots \rho_n \end{matrix} \right\| = 0 \quad \dots\dots\dots\dots (25).$$

To prove these properties, in the first place we must find the form of V. Consider the quantities $\xi_A, \xi_B, \dots \xi_L$, $(n-r)$ in number, of the form

$$\begin{aligned} &\xi_A = A_1 x_1 + A_2 x_2 \dots + A_n x_n, \quad \dots\dots\dots\dots (26), \\ &\xi_B = B_1 x_1 + B_2 x_2 \dots + B_n x_n, \\ &\vdots \\ &\xi_L = L_1 x_1 + L_2 x_2 \dots + L_n x_n; \end{aligned}$$

where, if Θ represent any of the quantities $A, B \dots L$,

$$\begin{aligned} &\alpha_1\Theta_1 + \alpha_2\Theta_2 \dots + \alpha_n\Theta_n = 0, \quad \dots\dots\dots\dots (27), \\ &\beta_1\Theta_1 + \beta_2\Theta_2 \dots + \beta_n\Theta_n = 0, \\ &\vdots \\ &\rho_1\Theta_1 + \rho_2\Theta_2 \dots + \rho_n\Theta_n = 0, \end{aligned}$$

$$2V = (A^2)\xi_A^2 + (B^2)\xi_B^2 + \dots + 2(AB)\xi_A\xi_B + \dots = \Sigma(A^2)\xi_A^2 + 2\Sigma(AB)\xi_A\xi_B.$$

Hence, if $\quad 2V = \Sigma\{\alpha^2\}x_\alpha^2 + 2\Sigma\{\alpha\beta\}x_\alpha x_\beta \dots\dots\dots\dots (28),$

we have for the coefficients of this form

$$\{1^2\} = \Sigma(A^2)A_1^2 + 2\Sigma(AB)A_1B_1, \quad \{12\} = \Sigma(A^2)A_1A_2 + \Sigma(AB)(A_1B_2 + A_2B_1),$$
$$\vdots \qquad\qquad\qquad \vdots$$

and consequently the coefficients of $2U - 2kV$ are

$$[1^2] = (1^2) - k\{1^2\}, \qquad [12] = (12) - k\{12\}.$$
$$\vdots \qquad\qquad\qquad \vdots$$

Hence, θ representing any of the quantities $\alpha, \beta \dots \rho$,

$$\begin{aligned} &\theta_1\{1^2\} + \theta_2\{12\} \dots + \theta_n\{1n\} = 0 \quad \dots\dots\dots\dots (29), \\ &\vdots \\ &\theta_1\{n1\} + \theta_2\{n2\} \dots + \theta_n\{n^2\} = 0; \end{aligned}$$

whence also
$$\theta_1[1^2]+\dots\theta_n[1n]=\theta_1(1^2)+\dots\theta_n(1n),$$
$$\vdots$$
$$\theta_1[n1]+\dots\theta_n[n^2]=\theta_1(n1)+\dots\theta_n(n^2).$$

Hence, the equations for determining $X_1,\dots X_n$ may be reduced to

$$X_1[\alpha_1(1^2)+\dots\alpha_n(1n)]+X_2[\alpha_1(21)\dots+\alpha_n(2n)]\dots+X_n[\alpha_1(n1)\dots+\alpha_n(n^2)]=0\dots(30),$$
$$X_1[\beta_1(1^2)+\dots\beta_n(1n)]+X_2[\beta_1(21)\dots+\beta_n(2n)]\dots+X_n[\beta_1(n1)\dots+\beta_n(n^2)]=0,$$
$$\vdots$$
$$X_1[\rho_1(1^2)+\dots\rho_n(1n)]+X_2[\rho_1(21)\dots+\rho_n(2n)]\dots+X_n[\rho_n(n1)\dots+\rho_n(n^2)]=0.$$
$$X_1[r+1,\ 1]+\quad X_2[r+1,\ 2]\dots+\quad X_n[r+1,\ n]\quad=0,$$
$$\vdots$$
$$X_1[n,\ 1]+\quad X_2[n,\ 2]\dots+\quad X_n[n^2]\quad=0.$$

Eliminating $X_1\dots X_n$, since the first r equations do not contain k, the equation in this quantity is of the order $n-r$.

Next form the reciprocals of the equations (25). These are

$$\left\|\begin{matrix} d_{x_1}U, & d_{x_2}U,\dots d_{x_n}U \\ A_1, & A_2,\dots A_n \\ \vdots & \\ L_1, & L_2,\dots L_n \end{matrix}\right\|=0\ \dots\dots\dots\dots\dots\dots\dots\dots\ (31).$$

From which we may deduce

$$\left\|\begin{matrix} \alpha_1 d_{x_1}U\dots+\alpha_n d_{x_n}U, & \beta_1 d_{x_1}U\dots+\beta_n d_x U,\dots\rho_1 d_{x_1}U\dots\rho_n d_{x_n}U, & d_{x_{r+1}}U,\dots d_{x_n}U \\ 0\quad, & 0\quad,\dots\quad 0\quad, & A_{r+1},\dots A_n \\ \vdots & & \\ 0\quad, & 0\quad,\dots\quad 0\quad, & L_{r+1},\dots L_n \end{matrix}\right\|=0\dots(32),$$

which are evidently satisfied by $x_1=X_1,\ x_2=X_2\dots x_n=X_n$.

In the case of four variables, the above investigation demonstrates the following properties of surfaces of the second order.

I. If a cone intersect a surface of the second order, three different cones may be drawn through the curve of intersection, and the vertices of these lie in the plane which is the polar reciprocal of the vertex of the intersecting cone.

II. If two planes intersect a surface of the second order through the curve of intersection, two cones may be drawn, and the vertices of these lie in the line which is the polar reciprocal of the line of intersection of the two planes.

Both these theorems are undoubtedly known, though I am not able to refer for them to any given place.

12.

ON THE THEORY OF DETERMINANTS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. VIII. (1843), pp. 1—16.]

THE following Memoir is composed of two separate investigations, each of them having a general reference to the Theory of Determinants, but otherwise perfectly unconnected. The name of "Determinants" or "Resultants" has been given, as is well known, to the functions which equated to zero express the result of the elimination of any number of variables from as many linear equations, without constant terms. But the same functions occur in the resolution of a system of linear equations, in the general problem of elimination between algebraic equations, and particular cases of them in algebraic geometry, in the theory of numbers, and, in short, in almost every part of mathematics. They have accordingly been a subject of very considerable attention with analysts. Occurring, apparently for the first time, in Cramer's *Introduction à l'Analyse des Lignes Courbes*, 1750: they are afterwards met with in a Memoir *On Elimination*, by Bezout, *Mémoires de l'Académie*, 1764; in two Memoirs by Laplace and Vandermonde in the same collection, 1774; in Bezout's *Théorie générale des Equations algebriques* [1779]; in Memoirs by Binet, *Journal de l'École Polytechnique*, vol. IX. [1813]; by Cauchy, *ditto*, vol. X. [1815]; by Jacobi, *Crelle's Journal*, vol. XXII. [1841]; Lebesgue, *Liouville*, [vol. II. 1837], &c. The Memoirs of Cauchy and Jacobi contain the greatest part of their known properties, and may be considered as constituting the general theory of the subject. In the first part of the present paper, I consider the properties of certain derivational functions of a quantity U, linear in two separate sets of variables (by the term "Derivational Function," I would propose to denote those functions, the nature of which depends upon the form of the quantity to which they refer, with respect to the variables entering into it, e.g. the differential coefficient of any quantity is a derivational function. The theory of derivational functions is apparently one that would admit of interesting developments). The particular functions of this class which are here considered, are closely connected with the theory of the reciprocal polars of surfaces of the second order, which latter is indeed a particular case of the theory of these functions.

In the second part, I consider the notation and properties of certain functions resolvable into a series of determinants, but the nature of which can hardly be explained independently of the notation.

In the first section I have denoted a determinant, by simply writing down in the form of a square the different quantities of which it is made up. This is not concise, but it is clearer than any abridged notation. The ordinary properties of determinants, I have throughout taken for granted; these may easily be learnt by referring to the Memoirs of Cauchy and Jacobi, quoted above. It may however be convenient to write down the following fundamental property, demonstrated by these authors, and by Binet.

$$\begin{vmatrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \\ \vdots & \end{vmatrix} \begin{vmatrix} \rho, & \sigma, \ldots \\ \rho', & \sigma', \\ \vdots & \end{vmatrix} = \begin{vmatrix} \rho\alpha + \sigma\beta \ldots, & \rho\alpha' + \sigma\beta' \ldots, \ldots \\ \rho'\alpha + \sigma'\beta \ldots, & \rho'\alpha' + \rho'\sigma' \ldots, \\ \vdots & \end{vmatrix} \quad \ldots\ldots (\odot),$$

an equation, particular cases of which are of very frequent occurrence, e.g. in the investigations on the forms of numbers in Gauss' *Disquisitiones Arithmetica* [1801], in Lagrange's *Determination of the Elements of a Comet's Orbit* [1780], &c. I have applied it in the *Cambridge Mathematical Journal* [1] to Carnot's problem, of finding the relation between the distances of five points in space, and to another geometrical problem. With respect to the notation of the second section, this is so fully explained there, as to render it unnecessary to say anything further about it at present.

§ 1. On the properties of certain determinants, considered as Derivational Functions.

Consider the function

$$\begin{aligned} U = x(\alpha\xi + \beta\eta + \ldots) + & \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1), \\ x'(\alpha'\xi + \beta'\eta + \ldots) + & \\ \vdots & \end{aligned}$$

(n lines, and n terms in each line);

and suppose

$$KU = \begin{vmatrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \ldots \\ \vdots & \vdots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

(The single letter κ being employed instead of KU, in cases where the quantity KU, rather than the functional symbol K, is being considered.) And write

$$FU = -\begin{vmatrix} & \mathrm{A}x + \mathrm{A}'x' + \ldots, & \mathrm{B}x + \mathrm{B}'x' + \ldots, \ldots \\ \mathrm{R}\,\xi + \mathrm{S}\,\eta + \ldots, & \alpha & , \beta & , \ldots \\ \mathrm{R}'\xi + \mathrm{S}'\eta + \ldots, & \alpha' & , \beta' & , \ldots \\ \vdots & & & \end{vmatrix} \quad \ldots\ldots\ldots (3).$$

$$\mathcal{T}U = -\begin{vmatrix} & \mathrm{R}x + \mathrm{R}'x' + \ldots, & \mathrm{S}x + \mathrm{S}'x' + \ldots, \ldots \\ \mathrm{A}\,\xi + \mathrm{B}\,\eta + \ldots, & \alpha & , \beta & , \ldots \\ \mathrm{A}'\xi + \mathrm{B}'\eta + \ldots, & \alpha' & , \beta' & , \ldots \\ \vdots & & & \end{vmatrix} \quad \ldots\ldots\ldots (4).$$

The symbols K, F, $\mathcal{T}$ possess properties which it is the object of this section to investigate.

Let $A, B, \ldots, A', B', \ldots, \ldots$ be given by the equations:

$$A = \begin{vmatrix} \beta', & \gamma', \ldots \\ \beta'', & \gamma'', \\ \vdots & \end{vmatrix}, \quad B = \pm \begin{vmatrix} \gamma', & \delta', \ldots \\ \gamma'', & \delta'', \\ \vdots & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

$$A' = \pm \begin{vmatrix} \beta'', & \gamma'', \ldots \\ \beta''', & \gamma''', \\ \vdots & \end{vmatrix}, \quad B' = \begin{vmatrix} \gamma'', & \delta'', \ldots \\ \gamma''', & \delta''', \\ \vdots & \end{vmatrix}$$

(The upper or lower signs according as n is odd or even.)

These quantities satisfy the double series of equations,

$$\begin{aligned} &A\alpha + B\beta + \ldots = \kappa, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6). \\ &A\alpha' + B\beta' + \ldots = 0, \\ &\vdots \\ &A'\alpha + B'\beta + \ldots = 0, \\ &A'\alpha' + B'\beta' + \ldots = \kappa, \\ &\vdots \\ &\&c. \end{aligned}$$

$$\begin{aligned} &A\alpha + A'\alpha' + \ldots = \kappa, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7), \\ &A\beta + A'\beta' + \ldots = 0, \\ &\vdots \\ &B\alpha + B'\alpha' + \ldots = 0, \\ &B\beta + B'\beta' + \ldots = \kappa, \\ &\vdots \\ &\&c. \end{aligned}$$

the second side of each equation being 0, except for the rth equation of the rth set of equations in the systems.

Let $\lambda, \mu, \ldots$ represent the $r^{\text{th}}, \overline{r+1}^{\text{th}}, \ldots$ terms of the series $\alpha, \beta, \ldots$; $L, M, \ldots$ the corresponding terms of the series $A, B \ldots$, where r is any number less than n, and consider the determinant

$$\begin{vmatrix} A & , \ldots L \\ \vdots & \\ A^{(r-1)}, & \ldots L^{(r-1)} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

which may be expressed as a determinant of the nth order, in the form

$$\begin{vmatrix} A & , \ldots L & , & 0, & 0, \ldots \\ \vdots & & & & \\ A^{(r-1)}, & \ldots L^{(r-1)}, & & 0, & 0, \\ 0 & , \quad 0 & , & 1, & 0, \\ 0 & , \quad 0 & , & 0, & 1, \\ \vdots & & & & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Multiplying this by the two sides of the equation

$$\kappa = \begin{vmatrix} \alpha, & \beta, \dots \\ \alpha', & \beta', \\ \vdots & \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10),$$

and reducing the result by the equation ($\odot$), and the equations (6), the second side becomes

$$\begin{vmatrix} \kappa, & 0, \dots & & & \\ 0, & \kappa, & & & \\ \vdots & & \ddots & & \\ & & \kappa, & 0 \;, & 0 \;, \dots \\ & & 0, & \mu^{(r)} \;, & \nu^{(r)} \;, \\ & & 0, & \mu^{(r+1)}, & \nu^{(r+1)}, \\ & & \vdots & & \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots (11),$$

which is equivalent to

$$\kappa^r \begin{vmatrix} \mu^{(r)} \;, & \nu^{(r)} \;, \dots \\ \mu^{(r+1)}, & \nu^{(r+1)}, \\ \vdots & \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (12),$$

or we have the equation

$$\begin{vmatrix} A \quad, \dots L \\ \vdots \\ A^{(r-1)}, \dots L^{(r-1)} \end{vmatrix} = \kappa^{r-1} \begin{vmatrix} \mu^{(r)} \;, & \nu^{(r)} \;, \dots \\ \mu^{(r+1)}, & \nu^{(r+1)}, \\ \vdots & \end{vmatrix} \quad \dots\dots\dots\dots\dots (13),$$

which in the particular case of $r = n$, becomes

$$\begin{vmatrix} A\,, & B, \dots \\ A', & B', \\ \vdots & \end{vmatrix} = \kappa^{r-1} \quad \dots\dots\dots\dots\dots\dots\dots\dots (14),$$

which latter equation is given by M. Cauchy in the memoirs already quoted; the proof in the "*Exercises*," being nearly the same with the above one of the more general equation (13). The equation (13) itself has been demonstrated by Jacobi somewhat less directly. Consider now the function FU, given by the equation (3). This may be expanded in the form

$$\begin{aligned} FU = {} & (\mathrm{R}\,\xi + \mathrm{S}\,\eta + \dots)\,[A\;(\mathrm{A}x + \mathrm{A}'x' + \dots) + B\;(\mathrm{B}x + \mathrm{B}'x' + \dots) + \dots] + \quad \dots\dots (15), \\ & (\mathrm{R}'\xi + \mathrm{S}'\eta + \dots)\,[A'\,(\mathrm{A}x + \mathrm{A}'x' + \dots) + B'\,(\mathrm{B}x + \mathrm{B}'x' + \dots) + \dots] + \\ & \vdots \end{aligned}$$

which may be written

$$\begin{aligned} FU = {} & x\;(\mathrm{A}\,\xi + \mathrm{B}\,\eta + \dots) + \quad \dots\dots\dots\dots\dots\dots\dots (16), \\ & x'\,(\mathrm{A}'\xi + \mathrm{B}'\eta + \dots) + \\ & \vdots \end{aligned}$$

by putting

$$\begin{aligned}
\mathrm{A} &= \mathrm{a}\,(\mathrm{R}A + \mathrm{R}'A' + \ldots) + \mathrm{b}\,(\mathrm{R}B + \mathrm{R}'B' + \ldots) + \ldots, \\
\mathrm{B} &= \mathrm{a}\,(\mathrm{S}A + \mathrm{S}'A' + \ldots) + \mathrm{b}\,(\mathrm{S}B + \mathrm{S}'B' + \ldots) + \ldots, \\
&\vdots \\
\mathrm{A}' &= \mathrm{a}'(\mathrm{R}A + \mathrm{R}'A' + \ldots) + \mathrm{b}'\,(\mathrm{R}B + \mathrm{R}'B' + \ldots) + \ldots, \\
\mathrm{B}' &= \mathrm{a}'\,(\mathrm{S}A + \mathrm{S}'A' + \ldots) + \mathrm{b}'\,(\mathrm{S}B + \mathrm{S}'B' + \ldots) + \ldots, \\
&\vdots
\end{aligned} \qquad \ldots\ldots\ldots (17).$$

Hence,

$$KFU = \begin{vmatrix} \mathrm{A}, & \mathrm{B}, \ldots \\ \mathrm{A}', & \mathrm{B}', \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$$= \begin{vmatrix} \mathrm{A}, & \mathrm{B}, \ldots \\ \mathrm{A}', & \mathrm{B}', \\ \vdots & \end{vmatrix} \begin{vmatrix} \mathrm{R}, & \mathrm{S}, \ldots \\ \mathrm{R}', & \mathrm{S}', \\ \vdots & \end{vmatrix} \begin{vmatrix} A, & B, \ldots \\ A', & B', \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (19),$$

or observing the equation (14), and writing

$$\begin{vmatrix} \mathrm{A}, & \mathrm{B}, \ldots \\ \mathrm{A}', & \mathrm{B}', \\ \vdots & \end{vmatrix} = J \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

$$\begin{vmatrix} \mathrm{R}, & \mathrm{S}, \ldots \\ \mathrm{R}', & \mathrm{S}', \\ \vdots & \end{vmatrix} = \mathcal{J} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

this becomes

$$KFU = J\mathcal{J}\,.\,(KU)^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Hence likewise

$$K\mathcal{F}U = J\mathcal{J}\,.\,(KU)^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Consider next the equation

$$\mathcal{F}FU = -\begin{vmatrix} & \mathrm{R}x + \mathrm{R}'x' + \ldots, & \mathrm{S}x + \mathrm{S}'x' + \ldots, \ldots \\ \mathrm{A}\,\xi + \mathrm{B}\,\eta + \ldots, & \mathrm{A} & , \mathrm{B} & , \\ \mathrm{A}'\xi + \mathrm{B}'\eta + \ldots, & \mathrm{A}' & , \mathrm{B}' & , \\ \vdots & & & \end{vmatrix} \ldots\ldots\ldots (24).$$

$$= -\begin{vmatrix} 1, & \\ & \mathrm{A}, \ \mathrm{B}, \ldots \\ & \mathrm{A}', \ \mathrm{B}', \\ & \end{vmatrix} \begin{vmatrix} 1, & \\ & \mathrm{R}, \ \mathrm{S}, \ldots \\ & \mathrm{R}', \ \mathrm{S}', \\ & \end{vmatrix} \begin{vmatrix} & \xi, & \eta, \ldots \\ x, & A, & B, \\ x', & A', & B', \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots\ldots (25).$$

$$= -J\mathcal{J} \begin{vmatrix} & \xi, & \eta, \ldots \\ x, & A, & B, \\ x', & A', & B', \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

If the two sides of this equation are multiplied by the two sides of the equation (2), written under the form

$$\kappa = \begin{vmatrix} 1, & & \\ & \alpha, & \beta, \ldots \\ & \alpha', & \beta', \\ & \vdots & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

the second side is reduced to

$$-\mathcal{J}\mathcal{f} \begin{vmatrix} & \alpha\xi + \beta\eta \ldots, & \alpha'\xi + \beta'\eta \ldots, \ldots \\ x, & \kappa \quad , & . \quad , \\ x', & . \quad , & \kappa \quad , \\ \vdots & & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

$$= -\mathcal{J}\mathcal{f} \,.\, \kappa^{n-1} \,.\, U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29),$$

and hence

$$\mathcal{F}FU = \mathcal{J}\mathcal{f} \,.\, (KU)^{n-2} \,.\, U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

Similarly

$$F\mathcal{F}U = \mathcal{J}\mathcal{f} \,.\, (KU)^{n-2} \,.\, U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31);$$

also combining these with the equations (22), (23),

$$\frac{\mathcal{F}FU}{KFU} = \frac{F\mathcal{F}U}{K\mathcal{F}U} = \frac{U}{KU} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

It may be remarked here that if U, V are functions connected by the equation

$$FU = cFV, \quad \text{or } \mathcal{F}U = c\mathcal{F}V, \quad \ldots\ldots\ldots\ldots\ldots\ldots (33),$$

then in general

$$U = c^{\frac{1}{n-1}} V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

To prove this, observing that the first of the equations (33) may be written

$$FU = F(c^{\frac{1}{n-1}} V) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

we have

$$\mathcal{F} \,.\, FU = \mathcal{F} \,.\, F(c^{\frac{1}{n-1}} V) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36),$$

or

$$\mathcal{J}\mathcal{f} \,.\, (KU)^{n-2} U = \mathcal{J}\mathcal{f} \,[K(c^{\frac{1}{n-1}} V)]^{n-2} c^{\frac{1}{n-1}} V \ldots\ldots\ldots\ldots\ldots (37).$$

If neither $\mathcal{J}$, $\mathcal{f}$ nor (KU) vanish, this equation is of the form

$$U = kV \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38),$$

whence substituting in (33),

$$k^{n-1} = c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

which demonstrates the equation (34); and this equation might be proved in like manner from the second of the equations (33). If however, $J=0$, or $f=0$, the above proof fails, and if $KU=0$, the proof also fails, unless at the same line $n=2$. In all these cases probably, certainly in the case of $KU=0$, $n \neq 2$, the equation (34) is not a necessary consequence of (33). In fact FU, or $\mathcal{T}U$ may be given, and yet U remain indeterminate.

Let $U_{,}$, $\alpha_{,}$, $\beta_{,}, \ldots A_{,}$, $B_{,}$, &c.... be analogous to U, α, $\beta \ldots$, $A_{,}$, B, &c.... and consider the equation

$$K(KU_{,}.FU+gKU.FU_{,}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40),$$

$$= \begin{vmatrix} \kappa_{,}A+g\kappa A_{,}, & \kappa_{,}B+g\kappa B_{,}, \ldots \\ \kappa_{,}A'+g\kappa A_{,}', & \kappa_{,}B+g\kappa B_{,}', \\ \vdots & \end{vmatrix}$$

Multiply the two sides by the two sides of the equation (2), the second side becomes, after reduction,

$$\begin{vmatrix} \kappa_{,}\kappa+g\kappa(A_{,}\alpha+B_{,}\beta+\ldots), & g\kappa(A_{,}'\alpha+B_{,}'\beta+\ldots), \ldots \\ g\kappa(A_{,}\alpha'+B_{,}\beta'+\ldots), & \kappa_{,}\kappa+g\kappa(A_{,}'\alpha'+B_{,}'\beta'+\ldots), \\ \vdots & \end{vmatrix} \ldots\ldots (41).$$

Multiplying by the two sides of the analogous equation

$$\kappa_{,} = \begin{vmatrix} \alpha_{,}, & \alpha_{,}', \ldots \\ \beta_{,}, & \beta_{,}, \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42),$$

and reducing, the second side becomes

$$\begin{vmatrix} \kappa\kappa_{,}(\alpha_{,}+g\alpha), & \kappa\kappa_{,}(\beta_{,}+g\beta), \ldots \\ \kappa\kappa_{,}(\alpha_{,}'+g\alpha'), & \kappa\kappa_{,}(\beta_{,}'+g\beta'), \\ \vdots & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (43),$$

$$= \kappa^{n}.\kappa_{,}^{n}.K(U_{,}+gU) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44),$$

whence $$K(KU_{,}.FU+gKU.FU_{,}) = (KU)^{n-1}(KU_{,})^{n-1}K(U_{,}+gU) \ldots\ldots (45),$$

and similarly

$$K(KU_{,}.\mathcal{T}U+gKU.\mathcal{T}U_{,}) = (KU)^{n-1}(KU_{,})^{n-1}K(U_{,}+gU) \ldots\ldots (46).$$

In a similar manner is the following equation to be demonstrated,

$$\mathcal{T}(KU_{,}.FU+gKU.FU_{,}) = F(KU_{,}.\mathcal{T}U+gKU.\mathcal{T}U_{,}) = \ldots\ldots\ldots\ldots (47),$$

$$-Jf.(KU)^{n-2}(KU_{,})^{n-2} \times \begin{vmatrix} & \alpha_{,}x+\alpha_{,}'x' \ldots, & \beta_{,}x+\beta_{,}'x' \ldots, \ldots \\ \alpha\xi+\beta\eta \ldots, & \alpha_{,}+g\alpha \ldots, & \beta_{,}+g\beta \ldots, \\ \alpha'\xi+\beta'\eta \ldots, & \alpha_{,}'+g\alpha' \ldots, & \beta_{,}'+g\beta_{,}' \ldots, \\ \vdots & & \end{vmatrix}$$

Suppose $$\overline{U} = \Sigma\,(\rho\xi + \sigma\eta + \ldots)\,(ax + a'x' + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (48),$$

this expression being the abbreviation of

$$\begin{aligned} \overline{U} = {} & (\rho\,\xi + \sigma\,\eta + \ldots)\,(a\,x + a'x' + \ldots) + \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (49), \\ & (\rho_{,}\xi + \sigma_{,}\eta + \ldots)\,(a_{,}x + a_{,}'x' + \ldots) + \\ & + \\ & \vdots \end{aligned}$$

[$(n-1)$ lines, or a smaller number].

then $$K\overline{U} = \begin{vmatrix} \Sigma a\,\rho, & \Sigma a\,\sigma, \ldots \\ \Sigma a'\rho, & \Sigma a'\sigma, \\ \vdots & \end{vmatrix} \text{ is } = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

which follows from the equation ($\odot$).

Conversely, whenever $K\overline{U} = 0$, $\overline{U}$ is of the above form.

Also $$F\overline{U} = -\begin{vmatrix} & \mathrm{A}x + \mathrm{A}'x' + \ldots, & \mathrm{B}x + \mathrm{B}'x' + \ldots, \ldots \\ \mathrm{R}\,\xi + \mathrm{S}\,\eta + \ldots, & \Sigma a\,\rho \quad , & \Sigma a\,\sigma \quad , \\ \mathrm{R}'\xi + \mathrm{S}'\eta + \ldots, & \Sigma a'\rho \quad , & \Sigma a'\sigma \quad , \\ \vdots & & \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (51),$$

which may be transformed into

$$F\overline{U} = \begin{vmatrix} \mathrm{A}x + \mathrm{A}'x' \ldots, & \mathrm{B}x + \mathrm{B}'x' \ldots, \ldots \\ \rho \quad , & \sigma \quad , \\ \vdots & \end{vmatrix} \begin{vmatrix} \mathrm{R}\xi + \mathrm{S}\eta \ldots, & \mathrm{R}'\xi + \mathrm{S}'\eta \ldots, \ldots \\ a \quad , & a' \quad , \\ \vdots & \end{vmatrix} \ldots (52),$$

(for shortness, I omit the demonstration of this equation).

And similarly,

$$\mathcal{T}\overline{U} = \begin{vmatrix} \mathrm{R}x + \mathrm{R}'x' + \ldots, & \mathrm{S}x + \mathrm{S}'x' + \ldots, \ldots \\ \rho \quad , & \sigma \quad , \\ \vdots & \end{vmatrix} \begin{vmatrix} \mathrm{A}\xi + \mathrm{B}\eta + \ldots, & \mathrm{A}'\xi + \mathrm{B}'\eta + \ldots, \ldots \\ a \quad , & a' \quad , \\ \vdots & \end{vmatrix} \ldots (53),$$

where it is obvious that if the sum Σ contain fewer than $(n-1)$ terms, $FU = 0$, $\mathcal{T}U = 0$.

The equations (52), (53) express the theorem, that whenever $K\overline{U} = 0$, the functions $F\overline{U}$, $\mathcal{T}\overline{U}$ are each of them the product of two determinants.

If next $$U_{,} = U + \overline{U},$$

then in (45) taking $g=-1$ [the Numbers (56) &c. which follow are as in the original memoir]

$$K\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}=K\{K(U+\overline{U}).\mathcal{T}U-KU.\mathcal{T}(U+\overline{U})\}\ \ldots\ldots\ldots(56),$$
$$=(KU)^{n-1}.(K(U+\overline{U}))^{n-1}.K\overline{U},$$

or observing the equation (50),

$$K\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}=K\{K(U+\overline{U}).\mathcal{T}U-KU.\mathcal{T}(U+\overline{U})\}=0\ \ldots\ldots(57).$$

Hence $F\{(K(U+\overline{U}).\mathcal{T}U-KU.\mathcal{T}(U+\overline{U})\}=\mathcal{T}\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}$ are each of them the product of two determinants. But this result admits of a further reduction: we have

$$F\{K(U+\overline{U}).\mathcal{T}U-KU.\mathcal{T}(U+\overline{U})\}=\mathcal{T}\{K(U+\overline{U}).FU-KU.F(U+\overline{U})\}\ldots\ldots\ldots(58)$$

$$=-J\Gamma(KU)^{n-2}.(K(U+\overline{U}))^{n-2}\begin{vmatrix} & \alpha_{,}x+\alpha_{,}'x'+\ldots, & \beta_{,}x+\beta_{,}'x'+\ldots,\ldots \\ \alpha\xi+\beta\eta+\ldots, & \alpha_{,}-\alpha\quad, & \beta_{,}-\beta\quad, \\ \alpha'\xi+\beta'\eta+\ldots, & \alpha_{,}'-\alpha'\quad, & \beta_{,}'+\beta'\quad, \\ \vdots & & \end{vmatrix},$$

substituting $\alpha_{,}=\alpha+\Sigma\rho a$, &c. ...., also observing that if the second line be multiplied by x, the third by x', ... and the sum subtracted from the first line, the value of the determinant is not altered, and that the effect of this is simply to change $\alpha_{,}$, $\alpha_{,}'$ ... into α, α' ... in the first line, and introduce into the corner place a quantity $-U$, which in the expansion of the determinant is multiplied by zero: this may be written in the form

$$-J\Gamma(KU)^{n-2}(K(U+\overline{U}))^{n-2}\begin{vmatrix} & \alpha x+\alpha'x'+\ldots, & \beta x+\beta'x'+\ldots,\ldots \\ \alpha\xi+\beta\eta+\ldots, & \Sigma\rho a\quad, & \Sigma\sigma a\quad, \\ \alpha'\xi+\beta'\eta+\ldots, & \Sigma\rho\alpha'\quad, & \Sigma\sigma\alpha'\quad, \\ \vdots & & \end{vmatrix}\ldots(59),$$

which may be reduced to

$$J\Gamma.(KU)^{n-2}.(K(U+\overline{U}))^{n-2}\times\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60),$$

$$\begin{vmatrix} \alpha x+\alpha'x'+\ldots, & \beta x+\beta'x'+\ldots,\ldots \\ \rho\quad, & \sigma\quad, \\ \vdots & \end{vmatrix}\begin{vmatrix} \alpha\xi+\beta\eta+\ldots, & \alpha'\xi+\beta'\eta\ldots,\ldots \\ a\quad, & \alpha'\quad, \\ \vdots & \end{vmatrix}$$

If each of these determinants are multiplied by the quantity $(KU)^{n-1}$, expressed under the two forms

$$\begin{vmatrix} A, & B,\ldots \\ A', & B', \\ \vdots & \end{vmatrix},\quad \begin{vmatrix} A, & A',\ldots \\ B, & B', \\ \vdots & \end{vmatrix}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

they would become respectively

$$KU.\begin{vmatrix} x & , & x' & ,\dots \\ A\rho+B\sigma+\dots, & & A'\rho+B'\sigma+\dots, & \\ \vdots & & & \end{vmatrix}, \quad KU.\begin{vmatrix} \xi & , & \eta & ,\dots \\ Aa+A'a'+\dots, & & Ba+B'a'+\dots, & \\ \vdots & & & \end{vmatrix} \dots (62),$$

so that finally

$$F\{K(U+\overline{U}).\mathcal{T}U-KU.\mathcal{T}(U+\overline{U})\}=\mathcal{T}\{K(U+\overline{U})FU-KU.F(U+\overline{U})\} \dots\dots\dots (63).$$

$$=J\Gamma\cdot\left(\frac{K(U+\overline{U})}{KU}\right)^{n-2}\times\begin{vmatrix} x & , & x' & ,\dots \\ A\rho+B\sigma+\dots, & & A'\rho+B'\sigma+\dots, & \\ \vdots & & & \end{vmatrix}\begin{vmatrix} \xi & , & \eta & ,\dots \\ Aa+A'a'+\dots, & & Ba+B'a'+\dots, & \\ \vdots & & & \end{vmatrix}$$

The second side of this may be written under the forms

$$\left(\frac{K(U+\overline{U})}{KU}\right)^{n-2}\begin{vmatrix} \mathrm{A}x+\mathrm{A}'x'+\dots & , & \mathrm{B}x+\mathrm{B}'x'+\dots & ,\dots \\ \mathrm{A}(A\rho+B\sigma..)+\mathrm{A}'(A'\rho+B'\sigma..)+\dots, & & \mathrm{B}(A\rho+B\sigma..)+\mathrm{B}'(A'\rho+B'\sigma..)+\dots, & \\ \vdots & & & \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} R\xi+S\eta+\dots & , & R'\xi+S'\eta+\dots & ,\dots \\ R(Aa+A'a'..)+S(Ba+B'a'..)+\dots, & & R'(Aa+A'a'..)+S'(Ba+B'a'..)+\dots, & \\ \vdots & & & \end{vmatrix} \dots (64).$$

And

$$\left(\frac{K(U+\overline{U})}{KU}\right)^{n-2}\begin{vmatrix} Rx+R'x'+\dots & , & Sx+S'x'+\dots & ,\dots \\ R(A\rho+B\sigma..)+R'(A'\rho+R'\sigma..)+\dots, & & S(A\rho+B\sigma..)+S'(A'\rho+B'\sigma..)+\dots, & \\ \vdots & & & \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} A\xi+B\eta+\dots & , & A'\xi+B'\eta+\dots & ,\dots \\ A(Aa+A'a'..)+B(Ba+B'a'..)+\dots, & & A'(Aa+A'a'..)+B'(Ba+B'a'..)+\dots, & \\ \vdots & & & \end{vmatrix} \dots (65).$$

And again, by the equations (52), (53), in the new forms

$$\left(\frac{K(U+\overline{U})}{KU}\right)^{n-2}F.\Sigma\{[(A\rho+B\sigma\ \dots)(\mathrm{A}\xi+\mathrm{B}\eta\ \dots)+(A'\rho+B'\sigma\dots)(\mathrm{A}'\xi+\mathrm{B}'\eta\dots)\dots]$$
$$\times[(Aa+A'a'\dots)(\mathrm{R}x+\mathrm{R}'x'\dots)+(Ba+B'a'\dots)(\mathrm{S}x+\mathrm{S}'x'\dots)\dots]\}\dots\dots(66),$$

$$\left(\frac{K(U+\overline{U})}{(KU)}\right)^{n-2}\mathcal{T}.\Sigma\{[(A\rho+B\sigma\ \dots)(\mathrm{R}\xi+\mathrm{S}\eta\ \dots)+(A'\rho'+B'\sigma\dots)(\mathrm{R}'\xi+\mathrm{S}'\eta\dots)\dots]$$
$$\times[(Aa+A'a'\dots)(\mathrm{A}x+\mathrm{A}'x'\dots)+(Ba+B'a'\dots)(\mathrm{B}x+\mathrm{B}'x'\dots)\dots]\}\dots(67).$$

Comparing these latter forms with the two equivalent quantities forming the first side of (53), and observing (33), (34), it would appear at first sight that

$$K(U+\bar{U}).\mathcal{F}U-KU.\mathcal{F}(U+\bar{U})$$
$$=\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}}\{\Sigma\,[(A\rho+B\sigma\ldots)(\mathrm{A}\xi+\mathrm{B}\eta\ldots)+(A'\rho+B'\sigma\ldots)(\mathrm{A}'\xi+\mathrm{B}'\eta\ldots)\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{R}x+\mathrm{R}'x'\ldots)+(Ba+B'a'\ldots)(\mathrm{S}x+\mathrm{S}'x'\ldots)\ldots]\},$$

$$K(U+\bar{U}).FU-KU.F(U+\bar{U})$$
$$=\left(\frac{K(U+\bar{U})}{KU}\right)^{\frac{n-2}{n-1}}\Sigma\,\{[(A\rho+B\sigma\ldots)(\mathrm{R}\xi+\mathrm{S}\eta\ldots)+(A'\rho+B'\sigma\ldots)(\mathrm{R}'\xi+\mathrm{S}'\eta\ldots)\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{A}x+\mathrm{A}'x'\ldots)+(Ba+B'a'\ldots)(\mathrm{B}x+\mathrm{B}'x'\ldots)\ldots]\},$$

which however are not true, except for $n=2$, on account of the equation (57). In the case of $n=2$, these equations become

$$K(U+\bar{U}).\mathcal{F}U-KU.\mathcal{F}(U+\bar{U})$$
$$=[(A\rho+B\sigma\ldots)(\mathrm{A}\xi+\mathrm{B}\eta\ldots)+(A'\rho+B'\sigma\ldots)(\mathrm{A}'\xi+\mathrm{B}'\eta\ldots)+\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{R}x+\mathrm{R}'x'\ldots)+(Ba+B'a'\ldots)(\mathrm{S}x+\mathrm{S}'x'\ldots)\ldots]\ldots\ldots\ldots(68),$$

$$K(U+\bar{U})FU-KU.F(U+\bar{U})$$
$$=[(A\rho+B\sigma\ldots)(\mathrm{R}\xi+\mathrm{S}\eta\ldots)+(A'\rho+B'\sigma\ldots)(\mathrm{R}'\xi+\mathrm{S}'\eta\ldots)\ldots]$$
$$\times[(Aa+A'a'\ldots)(\mathrm{A}x+\mathrm{A}'x'\ldots)+(Ba+B'a'\ldots)(\mathrm{B}x+\mathrm{B}'x'+\ldots)\ldots]\ldots\ldots(69),$$

and it is remarkable that these equations ((68), (69)) are true whatever be the value of n, provided Σ contains a single term only. The demonstration of this theorem is somewhat tedious, but it may perhaps be as well to give it at full length. It is obvious that the equation (69) alone need be proved, (68) following immediately when this is done.

I premise by noticing the following general property of determinants. The function

$$\begin{vmatrix} \alpha+\Sigma\rho a, & \beta+\Sigma\sigma a,\ldots \\ \alpha'+\Sigma\rho a', & \beta'+\Sigma\sigma' a, \\ \vdots & \end{vmatrix}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

(where $\Sigma\rho a=\rho_1 a_1+\rho_2 a_2\ldots+\rho_s a_s$), contains no term whose dimension in the quantities a, $a'\ldots$, or in the other quantities ρ, $\sigma\ldots$, is higher than s. (Of course if the order of the determinant be less than s or equal to it, this number becomes the limit of the dimension of any term in a, $a'\ldots$ or ρ, $\sigma\ldots$, and the theorem is useless.) This is easily proved by means of a well-known theorem,

$$\begin{vmatrix} \Sigma\rho a, & \Sigma\sigma a,\ldots \\ \Sigma\rho' a, & \Sigma\sigma' a, \\ \vdots & \end{vmatrix}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(71),$$

whenever s is less than the number expressing the order of the determinant. Hence in the formula (70), if Σ contain a single term only, the first side of the equation is linear in $\rho, \sigma, \ldots$ and also in $a, a', \ldots$, i.e. it consists of a term independent of all these quantities, and a second term linear in the products $\rho a, \rho a', \ldots \sigma a, \sigma a', \ldots$ This is therefore the form of $K(U+\overline{U})$.

Consider the several equations

$$\begin{aligned} \kappa = KU &= A\alpha + B\beta + \ldots \\ &= A'\alpha' + B'\beta' + \ldots \\ &= \&c. \end{aligned} \qquad \ldots\ldots\ldots\ldots (72),$$

it is easy to deduce

$$\begin{aligned} \kappa_{,} = K(U+\overline{U}) = KU &+ A\rho a + B\sigma a + \\ &+ A'\rho a' + B'\sigma a' + \\ &\vdots \end{aligned} \qquad \ldots\ldots\ldots\ldots (73).$$

To find the values of A, B, &c. corresponding to $U+\overline{U}$, we must write

$$\begin{aligned} A &= \mathrm{M}'\beta + \mathrm{N}'\gamma' + \\ &= \mathrm{M}''\beta + \mathrm{N}''\gamma'' + \\ &= \&c. \end{aligned} \qquad \ldots\ldots\ldots\ldots (74),$$

where

$$\mathrm{M}' = \begin{vmatrix} \gamma'', & \delta'', \ldots \\ \gamma''', & \delta''', \\ \vdots & \end{vmatrix}, \quad \mathrm{N}' = \pm \begin{vmatrix} \delta'', & \epsilon'', \ldots \\ \delta''', & \epsilon''', \\ \vdots & \end{vmatrix} \qquad \ldots\ldots\ldots\ldots (75),$$

$$\mathrm{M}'' = \pm \begin{vmatrix} \gamma''', & \delta''', \ldots \\ \gamma'''', & \delta'''', \\ \vdots & \end{vmatrix}, \quad \mathrm{N}'' = \begin{vmatrix} \delta''', & \epsilon''', \ldots \\ \delta'''', & \epsilon'''', \\ \vdots & \end{vmatrix}, \ \&c.$$

the order of each of these determinants being $\overline{n-2}$, and the upper or lower signs being used according as $\overline{n-1}$ is odd or even, i.e. as n is even or odd. Hence

$$\begin{aligned} A_{,} = A &+ \mathrm{M}'\sigma a' + \mathrm{N}'\tau a' + \ldots \\ &+ \mathrm{M}''\sigma a'' + \mathrm{N}''\tau a'' + \ldots \\ &\vdots \end{aligned} \qquad \ldots\ldots\ldots\ldots (76),$$

and therefore

$$\begin{aligned} \kappa_{,}A - \kappa A_{,} = \quad & A^2\rho a + (AB \qquad)\sigma a + (AC \qquad)\tau a + \ldots \\ &+ AA'\rho a' + (AB' - \kappa\mathrm{M}')\sigma a' + (AC' - \kappa\mathrm{N}')\tau a' + \ldots \\ &+ AA''\rho a'' + (AB'' - \kappa\mathrm{M}'')\sigma a'' + (AC'' - \kappa\mathrm{N}'')\tau a'' + \\ &\vdots \end{aligned} \qquad \ldots\ldots\ldots (77),$$

the additional quantities C, τ having been introduced for greater clearness. Now the equations

$$\begin{aligned} &AB' - \kappa M' = A'B, \quad AC' - \kappa N' = A'C, \ldots \\ &AB'' - \kappa M'' = A''B, \quad AC'' - \kappa N'' = A''C, \\ &\vdots \end{aligned} \qquad \ldots\ldots\ldots\ldots (78),$$

written under the form

$$AB' - A'B = \kappa M', \quad AC' - A'C = \kappa N', \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(79),$$
$$AB'' - A''B = \kappa M'', \quad AC'' - A''C = \kappa N'',$$
$$\vdots$$

are particular cases of the equation (13), and are therefore identically true. Hence, substituting in (77),

$$\begin{aligned} \kappa_{,}A - \kappa A_{,} = \quad & A^2\rho a + A\,B\sigma a + A\,C\tau a \ldots + \quad \ldots\ldots\ldots\ldots\ldots\ldots (80), \\ & + A\,A'\rho a' + A'\,B\sigma a' + A'\,C\tau a' \ldots + \\ & + A''A\rho a'' + A''B\sigma a'' + A''C\tau a'' \ldots + \\ & \vdots \\ & = (\rho A + \sigma B + \ldots)(Aa + A'a' + \ldots). \end{aligned}$$

Forming in a similar manner, the combinations $\kappa_{,}B - \kappa B_{,}, \ldots \kappa_{,}A' - \kappa A_{,}', \ \kappa_{,}B - \kappa B_{,}', \ldots$, multiplying by the products of the different quantities $Ax + A'x' \ldots$, $Bx + B'x' \ldots, \ldots$ $R\xi + S\eta \ldots$, $R'\xi + S'\eta \ldots, \ldots$ and adding so as to form the function $K(U + \bar{U}) . FU - KU . F(U + \bar{U})$, we obtain the required formula, viz. that the value of this quantity is

$$\begin{aligned} & = [(\rho A + \sigma B \ \ldots)(\mathrm{R}\xi + \mathrm{S}\eta \ \ldots) + (A'\rho + B'\sigma \ldots)(\mathrm{R}'\xi + \mathrm{S}'\eta \ldots) + \ldots] \ldots\ldots (81); \\ & \times [(Aa + A'a' \ldots)(\mathrm{A}x + \mathrm{A}'x' \ldots) + (Ba + B'a' \ldots)(\mathrm{B}x + \mathrm{B}'x' \ldots) + \ldots] \end{aligned}$$

with this theorem, I conclude the present section,—noticing only, as a problem worthy of investigation, the discovery of the forms of the second sides of the equations (68), (69), in the case of Σ containing more than a single term.

§ 2. On the notation and properties of certain functions resolvable into a series of determinants.

Let the letters $\quad r_1, \ r_2, \ldots r_k \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

represent a permutation of the numbers

$$1, \ 2, \ldots k \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then in the series (1), if one of the letters succeeds mediately or immediately a letter representing a higher number than its own, for each time that this happens there is said to be a "derangement" or "inversion." It is to be remarked that if any letter succeed s letters representing higher numbers, this is reckoned for the same number s of inversions.

Suppose next the symbol

$$\pm_r \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

denotes the sign + or −, according as the number of inversions in the series (1) is even or odd.

This being premised, consider the symbol

$$\left\{\begin{matrix} A\rho_1\sigma_1\ldots(n) \\ \vdots \\ \rho_k\sigma_k\ldots \end{matrix}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

denoting the sum of all the different terms of the form

$$\pm_r \pm_s \ldots A\rho_{r_1}\sigma_{s_1}\ldots \quad \ldots A\rho_{r_k}\sigma_{s_k}\ldots \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

the letters $\quad r_1,\ r_2\ldots r_k;\ s_1,\ s_2\ldots s_k;$ &c. $\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$

denoting any permutations whatever, the same or different, of the series of numbers (2) [and the several combinations of $\rho\sigma\ldots$ being understood as denoting suffixes of the A's]. The number of terms represented by the symbol (5) is evidently

$$(1\,.\,2\ldots k)^n\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

In some cases it will be necessary to leave a certain number of the vertical rows ρ, $\sigma\ldots$ unpermuted. This will be represented by writing the mark (†) immediately above the rows in question. So that for instance

$$\left\{\begin{matrix} & & \overset{\dagger}{\theta_1}\overset{\dagger}{\phi_1} & & \\ A\rho_1\sigma_1 & \ldots & & \ldots & (n) \\ \vdots & & & & \\ \rho_k\sigma_k & \ldots & \theta_k\phi_k & & \end{matrix}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

the number of rows with the (†) being x, denotes the sum of the

$$(1\,.\,2\ldots k)^{n-x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

terms, of the form

$$\pm_r \pm_s \ldots A\rho_{r_1}\sigma_{s_1}\ldots\theta_1\phi_1\ldots A\rho_{r_k}\sigma_{s_k}\ldots\theta_k\phi_k\ldots \quad\ldots\ldots\ldots\ldots \quad (10).$$

Then it is obvious, that if all the rows have the mark (†) the notation (8) denotes a single product only, and if the mark (†) be placed over all but one of the rows the notation (8) belongs to a determinant. It is obvious also that we may write

$$\left\{\begin{matrix} A\rho_1\sigma_1\ldots\theta_1\phi_1\ldots(n) \\ \vdots \\ \rho_k\sigma_k\ldots\theta_k\phi_k \end{matrix}\right\} = \Sigma\pm_u\pm_v\ldots\left\{\begin{matrix} A\rho_1\sigma_1\ldots\overset{\dagger}{\theta_{u_1}}\overset{\dagger}{\phi_{v_1}}\ldots(n) \\ \vdots \\ \rho_k\sigma_k\ldots\theta_{u_k}\phi_{v_k} \end{matrix}\right\}\ldots\ldots \quad (11),$$

where Σ refers to the different permutations,

$$u_1,\ u_2,\ldots u_k;\ v_1,\ v_2,\ldots v_k;\ \&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12),$$

which can be formed out of the numbers (2). The equation (11) would still be true, if the mark (†) were placed over any number of the columns ρ, $\sigma\ldots$

Suppose in this equation a single column only is left without the mark (†) on the second side of the equation; the first side is then expressed as the sum of a number

$$(1\,.\,2\ldots k)^{n-1},\ \text{or generally}\ (1\,.\,2\ldots k)^{n-x-1}\ \ldots\ldots\ldots\ldots\ldots \quad (13),$$

of determinants, according as we consider the symbol (4) or the more general one (8). And this may be done in n or $(n-x)$ different ways respectively.

It may be remarked, that the symbol (8) is the same in form as if a single column only had the mark (†) over it; the number n being at the same time reduced from n to $(n-x+1)$: for the marked columns of symbols may be replaced by a single marked column of new symbols. Hence, without loss of generality, the theorems which follow may be stated with reference to a single marked column only.

Suppose the letters

$$\rho_1,\quad \rho_2, \dots \rho_k;\quad \sigma_1,\quad \sigma_2, \dots \sigma_k;\ \&c. \quad \dots\dots\dots\dots\dots\dots\dots \quad (14)$$

denote certain permutations of

$$\alpha_1,\quad \alpha_2, \dots \alpha_k;\quad \beta_1,\quad \beta_2, \dots \beta_k;\ \&c. \quad \dots\dots\dots\dots\dots\dots\dots \quad (15),$$

in such a manner that

$$\rho_1 = \alpha_{g_1},\quad \rho_2 = \alpha_{g_2}, \dots \rho_k = \alpha_{g_k};\quad \sigma_1 = \beta_{h_1},\quad \sigma_2 = \beta_{h_2}, \dots \sigma_k = \beta_{h_k} \dots \quad \dots\dots(16).$$

Then the two following theorems may be proved:

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\sigma_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \end{matrix}\right\} = \pm_g \pm_h \dots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \quad \dots\dots\dots\dots\dots\dots \quad (17),$$

if n be even: but in the contrary case

$$\left\{\begin{matrix} \overset{\dagger}{A}\rho_1\sigma_1 \dots (n) \\ \vdots \\ \rho_k\sigma_k \end{matrix}\right\} = + \pm_g \dots \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \quad \dots\dots\dots\dots\dots \quad (18).$$

By means of these, and the equation (11), a fundamental property of the symbol (3) may be demonstrated. We have

$$\left\{\begin{matrix} A\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} = \Sigma \pm_g \left\{\begin{matrix} \overset{\dagger}{A}\rho_1\beta_1 \dots (n) \\ \vdots \\ \rho_k\beta_k \end{matrix}\right\} \quad \dots\dots\dots\dots\dots\dots\dots \quad (19),$$

which when n is even, reduces itself by (17) to

$$\left\{\begin{matrix} A\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\} \Sigma (\pm_g \pm_g . 1) \dots\dots\dots\dots\dots \quad (20)$$

$$= 1 . 2 \dots k \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\}.$$

But when n is odd, from the equation (18),

$$\left\{\begin{matrix} A\alpha_1, & \beta_1 \dots (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} = \left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \dots (n) \\ \vdots \\ \alpha_k, \beta_k \end{matrix}\right\} \Sigma(\pm_g 1) = 0 \dots\dots\dots\dots\dots (21),$$

since the number of negative and positive values of $\pm_g$ are equal.

From the equation (20), it follows that when n is even, the value of a symbol of the form

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1, & (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (22)$$

is the same, over whichever of the columns α, $\beta \dots$ the mark (†) is placed. To denote this indifference, the preceding quantity is better represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1, & \beta_1 \dots (n) \\ \vdots & \\ \alpha_k\beta_k & \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (23),$$

this last form being never employed when n is odd, in which case the same property does not hold. Hence also an ordinary determinant is represented by

$$\left\{\begin{matrix} \overset{\dagger}{A}\alpha_1\beta_1 \\ \vdots \\ \alpha_k\beta_k \end{matrix}\right\}, \quad \left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \\ \vdots\,\vdots \\ k\,k \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (24),$$

the latter form being obviously equally general with the former one.

It is obvious from the equations (17), (18), that the expression (22) vanishes, in the case of n even whenever any two of the symbols α are equivalent, or any two of the symbols β, &c.; but if n be odd, this property holds for the symbols β, &c., but not for the marked ones α. In fact, the interchange of the two equal symbols, in each case, changes the sign of the expression (22), but they evidently leave it unaltered, i.e. the quantity in question must be zero.

Consider now the symbol

$$\left\{\begin{matrix} \overset{\dagger}{A}\,1\,1 \dots (2p) \\ \vdots\,\vdots \\ k\,k \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (25),$$

which, for shortness, may be denoted by

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (26).$$

I proceed to prove a theorem, which may be expressed as follows:

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\}\,.\,\{\overset{\dagger}{B}\,.\,k\,.\,2q\} = \{\overline{AB}|\,k\,.\,2p+2q-2\} \quad \text{.....(27)},$$

where

$$\overline{AB}|_{rs\ldots xy\ldots} = S\,.\,A_{rs\ldots l}\,B_{xy\ldots l} \quad \text{..... (28)},$$

the number of the symbols $r, s, \ldots$ being obviously $2p-1$, and that of $x, y, \ldots$ being $2q-1$. The summatory sign S refers to l, and denotes the sum of the several terms corresponding to values of l from $l=1$ to $l=k$. Also the theorem would be equally true if l had been placed in any position whatever in the series $r, s \ldots l$; and again, in any position whatever in the series $x, y \ldots l$, instead of at the end of each of these. With a very slight modification this may be made to suit the case of an odd number instead of one of the two even numbers $2p$, $2q$; (in fact, it is only necessary to place the mark (†) in $\{\overline{AB}|\ldots\}$ over the column corresponding to the marked column in $\{A\ldots\}$, $\{A\ldots\}$ being the symbol for which the number of columns is odd), but it is inapplicable where the two numbers are odd. Consider the second side of (27); this may be expanded in the form

$$\Sigma + \pm_s \ldots \pm_x \pm_y \ldots \overline{AB}|_{1\,s_1\ldots x_1 y_1\ldots} \,.\, \overline{AB}|_{2\,s_2\ldots x_2 y_2\ldots} \ldots \overline{AB}|_{k\,s_k\ldots x_k y_k\ldots} \quad \text{..... (29)},$$

where Σ refers to the different quantities $s, \ldots, x, y, \ldots$ as in (11).

Substituting from (28), this becomes

$$\Sigma\,.\,S_{l_1}\ldots S_{l_k}\,(+\pm_s \ldots \pm_x \pm_y A_{1\,s_1\ldots l_1}\ldots A_{k\,s_k\ldots l_k}\ldots B_{x_1 y_1\ldots l_1}\ldots B_{x_k y_k\ldots l_k})\ldots \text{(30)}.$$

Effecting the summation with respect to $x, y \ldots$ this becomes

$$\Sigma\,.\,S_{l_1}\ldots S_{l_k} + \pm_s \ldots A_{1\,s_1\ldots l_1}\ldots A_{k\,s_k\ldots l_k} \left\{\begin{matrix} \overset{\dagger}{B}1\,1\ldots l_1 \\ \vdots \\ k\,k\ldots l_k \end{matrix}\right\} \text{..... (31)},$$

Σ now referring to $s, \ldots$ only. The quantity under the sign Σ vanishes if any two of the quantities l are equal, and in the contrary case, we have

$$\left\{\begin{matrix} \overset{\dagger}{B}1\,1\ldots l_1 \\ \vdots \\ k\,k\ldots l_k \end{matrix}\right\} = \pm_l \{\overset{\dagger}{B}\,.\,k\,.\,2q\} \quad \text{.....(32)},$$

which reduces the above to

$$\{\overset{\dagger}{B}\,.\,k\,.\,2q\}\,.\,\Sigma + \pm_s \ldots \pm_l A_{l\,.\,s_1\ldots l_1}\ldots A_{k\,.\,s_k\ldots l_k} \quad \text{.....(33)},$$

Σ referring to the quantities $s\ldots$, and also to the quantities l. And this is evidently equivalent to

$$\{\overset{\dagger}{A}\,.\,k\,.\,2p\}\,\{\overset{\dagger}{B}\,.\,k\,.\,2q\} \quad \text{..... (34)},$$

the theorem to be proved. It is obvious that when $p=1$, $q=1$, the equation (27), coincides with the theorem (⊙), quoted in the introduction to this paper.

13.

ON THE THEORY OF LINEAR TRANSFORMATIONS.

[From the *Cambridge Mathematical Journal*, vol. IV. (1845), pp. 193—209.]

THE following investigations were suggested to me by a very elegant paper on the same subject, published in the *Journal* by Mr Boole. The following remarkable theorem is there arrived at. If a rational homogeneous function U, of the nth order, with the m variables $x, y \ldots$, be transformed by linear substitutions into a function V of the new variables, $\xi, \eta \ldots$; if, moreover, θU expresses the function of the coefficients of U, which, equated to zero, is the result of the elimination of the variables from the series of equations $d_x U = 0$, $d_y U = 0$, &c., and of course θV the analogous function of the coefficients of V: then $\theta V = E^{n\alpha} . \theta U$, where E is the determinant formed by the coefficients of the equations which connect $x, y \ldots$ with $\xi, \eta \ldots$,[1] and $\alpha = (n-1)^{m-1}$. In attempting to demonstrate this very beautiful property, it occurred to me that it might be generalised by considering for the function U, not a homogeneous function of the nth order between m variables, but one of the same order, containing n sets of m variables, and the variables of each set entering linearly. The form which Mr Boole's theorem thus assumes is $\theta V = E_1^{\alpha} . E_2^{\alpha} \ldots E_n^{\alpha} . \theta U$. This it was easy to demonstrate would be true, if θU satisfied a certain system of partial differential equations. I imagined at first that these would determine the function θU, (supposed, in analogy with Mr Boole's function, to represent the result of the elimination of the variables from $d_{x_1} U = 0$, $d_{y_1} U = 0, \ldots d_{x_2} U = 0$, &c.): this I afterwards found was not the case; and thus I was led to a class of functions, including as a particular case the function θU, all of them possessed of the same characteristic property. The system of partial differential equations was without difficulty replaced by a more fundamental system of equations, upon which, assumed as definitions, the theory appears to me naturally to depend; and it is this view of it which I intend partially to develope in the present paper.

[1] The value of α was left undetermined, but Mr Boole has since informed me, he was acquainted with it at the time his paper was written; and has given it in a subsequent paper.

I have already employed the notation

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta, \dots \\ \alpha', & \beta', & \gamma', & \delta', \\ \alpha'', & \beta'', & \gamma'', & \delta'', \\ \vdots & & & \end{matrix} \right\| \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

(where the number of horizontal rows is less than that of vertical ones) to denote the series of determinants,

$$\left| \begin{matrix} \alpha, & \beta, & \gamma, \dots \\ \alpha', & \beta', & \gamma', \\ \alpha'', & \beta'', & \gamma'', \\ \vdots & & \end{matrix} \right| \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (2),$$

which can be formed out of the above quantities by selecting any system of vertical rows; these different determinants not being connected together by the sign $+$, or in any other manner, but being looked upon as perfectly separate.

The fundamental theorem for the multiplication of determinants gives, applied to these, the formula

$$\left\| \begin{matrix} A, & B, & C, & D, \dots \\ A', & B', & C', & D', \\ A'', & B'', & C'', & D'', \\ \vdots & & & \end{matrix} \right\| = E \left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta, \dots \\ \alpha', & \beta', & \gamma', & \delta', \\ \alpha'', & \beta'', & \gamma'', & \delta'', \\ \vdots & & & \end{matrix} \right\| \quad \dots\dots\dots\dots (3),$$

where

$$\left.\begin{matrix} A = \lambda\alpha + \lambda'\alpha' + \lambda''\alpha'' + \dots \\ B = \lambda\beta + \lambda'\beta' + \lambda''\beta'' + \dots \\ \vdots \\ A' = \mu\alpha + \mu'\alpha' + \mu''\alpha'' + \dots \\ B' = \mu\beta + \mu'\beta' + \mu''\beta'' + \dots \\ \vdots \\ \&c. \end{matrix}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (4),$$

$$E = \left| \begin{matrix} \lambda, & \mu, \dots \\ \lambda', & \mu', \\ \vdots & \end{matrix} \right| \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (5),$$

and the meaning of the equation is, that the terms on the first side are equal, each to each, to the terms on the second side.

This preliminary theorem being explained, consider a set of arbitrary coefficients, represented by the general formula

$$rst\dots \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6),$$

in which the number of symbolical letters $r, s, \dots$ is n, and where each of these is supposed to assume all integer values, from 1 to m inclusively.

Let $$\alpha s_{,}' t_{,}' \ldots, \quad \alpha s_{,}'' t_{,}'' \ldots, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

represent the whole series, taken in any order, in which the first symbolical letter is α. Similarly,

$$r_{,,}' \alpha t_{,,}' \ldots, \quad r_{,,}'' \alpha t_{,,}'', \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

the whole series of those in which the second symbolical letter is α, and so on.

Imagine a function u of the coefficients, which is simultaneously of the forms

$$u = H_p \left\| \begin{matrix} 1 s_{,}' t_{,}' \ldots, & 1 s_{,}'' t_{,}'' \ldots, \ldots \\ 2 s_{,}' t_{,}' \ldots, & 2 s_{,}'' t_{,}'' \ldots, \\ \vdots & \end{matrix} \right\| \qquad \ldots\ldots\ldots\ldots\ldots\ldots (A),$$

$$u = H_p \left\| \begin{matrix} r_{,,}' 1 t_{,,}' \ldots, & r_{,,}'' 1 t_{,,}'' \ldots, \ldots \\ r_{,,}' 2 t_{,,}' \ldots, & r_{,,}'' 2 t_{,,}'' \ldots, \\ \vdots & \end{matrix} \right\|$$

&c.; in which H_p denotes a rational homogeneous function of the order p. The function H is not necessarily supposed to be the same in the above equations, and in point of fact it will not in general be so. The number of equations is of course $= n$.

The function u, whose properties we proceed to investigate, may conveniently be named a "Hyperdeterminant." Any function satisfying any of the equations (A), without satisfying all of them, will be an "Incomplete Hyperdeterminant." But, considering in the first place such as are complete—

Let $\dot{r}st \ldots$ be a new set of coefficients connected with the former ones by a system of equations of the form

$$\dot{r}st \ldots = \lambda_1^r 1st \ldots + \lambda_2^r 2st \ldots + \lambda_m^r mst \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

(where the r in $\lambda_1^r \ldots$ is not an exponent, but an affix).

Suppose $\dot{u}$ is the same function of these new coefficients that u was of the former ones. Then consider the first of the equations (A) and the equation (3), and writing

$$L = \left| \begin{matrix} \lambda_1^1, & \lambda_1^2, \ldots \\ \lambda_2^1, & \lambda_2^2, \\ \vdots & \end{matrix} \right| \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

we have immediately the equation

$$\dot{u} = L^p . u \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Consider the new set of coefficients

$$\dot{r}\dot{s}t \ldots = \mu_1^s \dot{r}1t \ldots + \mu_2^s \dot{r}2t \ldots + \ldots + \mu_m^s \dot{r}mt \ldots \qquad \ldots\ldots\ldots\ldots\ldots (12),$$

and $\ddot{u}$ the analogous function of these; then, from the second of the equations (A) and the equation (3), and writing

$$M = \begin{vmatrix} \mu_1^1, & \mu_1^2, \dots \\ \mu_2^1, & \mu_2^2, \\ \vdots & \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots \quad (13),$$

$$\ddot{u} = M^p \,.\, \dot{u} = L^p M^p \,.\, u \quad \dots\dots\dots\dots\dots\dots \quad (14).$$

In like manner, considering the new coefficients $\dot{r}\dot{s}\dot{t}\dots$, where

$$\dot{r}\dot{s}\dot{t}\dots = \nu_1^t\, \dot{r}\dot{s}1 \dots + \nu_2^t\, \dot{r}\dot{s}2 \dots \quad \dots + \nu_n^t\, \dot{r}\dot{s}m \dots \quad \dots\dots\dots \quad (15),$$

the new function $\dddot{u}$, and the quantity N given by

$$N = \begin{vmatrix} \nu_1^1, & \nu_1^2, \dots \\ \nu_2^1, & \nu_2^2, \\ \vdots & \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots \quad (16),$$

we have, as before,

$$\dddot{u} = N^p\, \ddot{u} = L^p M^p N^p\, u \dots\dots\dots \quad \dots\dots\dots\dots\dots\dots (17),$$

or

$$\dddot{u} = L^p M^p N^p u \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (18);$$

whence, generally, denoting the last result by u',

$$u' = L^p M^p N^p \dots u \quad \dots\dots\dots\dots\dots\dots \quad (\text{B, } 1).$$

Consider now the function

$$\Sigma\Sigma\Sigma \dots (rst \dots x_r y_s z_t \dots) \quad \dots\dots\dots\dots\dots\dots \quad (19),$$

where the Σ's refer successively to $r, s, t, \dots$, and denote summations from 1 to m inclusively. If u be looked upon as a derivative from the above function, we may write

$$u = \Theta \,.\, \Sigma\Sigma\Sigma \dots (rst \dots x_r y_s z_t \dots) \dots\dots\dots\dots\dots\dots (20).$$

Assume

$$\left.\begin{aligned} x_r &= \lambda_r^1\, \dot{x}_1 + \lambda_r^2\, \dot{x}_2 \dots + \lambda_r^m\, \dot{x}_m \\ y_s &= \mu_s^1\, \dot{y}_1 + \mu_s^2\, \dot{y}_2 \dots + \mu_s^m\, \dot{y}_m \\ &\vdots \end{aligned}\right\} \dots\dots\dots\dots\dots\dots \quad (21).$$

It is easy to obtain

$$\Sigma\Sigma\Sigma \dots (rst \dots x_r y_s z_t \dots) = \Sigma\Sigma\Sigma \dots (\dot{r}st \dots \dot{x}_r y_s z_t \dots), \dots = \Sigma\Sigma\Sigma \dots (\dot{r}\dot{s}\dot{t} \dots \dot{x}_r \dot{y}_s \dot{z}_t \dots) \dots \quad \dots (22),$$

and the formula for (u) becomes

$$\Theta\, \Sigma\Sigma\Sigma \dots (\dot{r}\dot{s}\dot{t} \dots \dot{x}_r \dot{y}_s \dot{z}_t) \dots = L^p M^p N^p \dots \Theta\, \Sigma\Sigma\Sigma \dots (rst \dots x_r y_s z_t) \dots\dots\dots (\text{B, } 2).$$

Proceeding to obtain the expression of the coefficients $\dot{r}\dot{s}\dot{t}\dots$ in terms of the coefficients $rst\dots$, we have

$$\dot{r}\dot{s}\dot{t} \dots = \Sigma\Sigma\Sigma \dots (\lambda_f^r \mu_g^s \nu_h^t \dots fgh \dots) \quad \dots\dots\dots\dots\dots\dots \quad (\text{C}),$$

where the Σ's refer successively to $f, g, h, \ldots$, denoting summations from 1 to m inclusively. Having this equation, it is perhaps as well to retain

$$u' = L^p M^p N^p \ldots u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(B, 1)},$$

instead of (B, 2), that form being principally useful in showing the relation of the function u to the theory of the transformation of functions.

It may immediately be seen, that in the equations (B), (C) we may, if we please, omit any number of the marks of variation $(')$, omitting at the same time the corresponding signs Σ, and the corresponding factors of the series $L, M, N \ldots$

Also, if u be such as only to satisfy some of the equations (A); then, if in the same formulæ we omit the corresponding marks $(')$, summatory signs, and terms of the series $L, M, N \ldots$, the resulting equations are still true.

From the formulæ (A) we may obtain the partial differential equations

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) u = 0, \text{ or } pu, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(D)},$$

$$\Sigma\Sigma \ldots \left(rat \ldots \frac{d}{dr\beta t \ldots}\right) u = 0, \text{ or } pu,$$

according as α is not equal, or is equal, to β;

and so on: the summatory signs referring in every case to those of the series $r, s, t, \ldots$, which are left variable, and extending from 1 to m inclusively.

To demonstrate this, consider the general form of u, as given by the first of the equations (A). This is evidently composed of a series of terms, each of the form

$$cPQR \ldots (p \text{ factors}),$$

in which

$$P = \left| \begin{array}{ll} 1s_{,}'t_{,}' \ldots, & 1s_{,}''t_{,}'' \ldots, \ldots (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}' \ldots, & \alpha s_{,}''t_{,}'' \ldots, \\ \vdots & \\ \beta s_{,}'t_{,}' \ldots, & \beta s_{,}''t_{,}'' \ldots, \\ \vdots & \end{array} \right|$$

Q, R, &c. being of the same form; and we have

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) u = cQR \ldots \Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P + \&c. + \&c.,$$

and

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P = \left| \begin{array}{ll} 1s_{,}'t_{,}' \ldots, & 1s_{,}''t_{,}'' \ldots, \ldots (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}' \ldots, & \alpha s_{,}''t_{,}'' \ldots, \ldots \\ \vdots & \\ \alpha s_{,}'t_{,}' \ldots, & \alpha s_{,}''t_{,}'' \ldots, \ldots \\ \vdots & \end{array} \right| = 0;$$

so that all the terms on the second side of the equation vanish. If, however, $\beta = \alpha$,

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P = P;$$

whence, on the second side, we have

$$\begin{aligned} cQR \ldots P + cPR \ldots Q + \&c. &= p \,.\, cPQR \ldots + \&c. + \&c. \\ &= pU, \end{aligned}$$

or the theorem in question is proved.

In the case of an incomplete hyperdeterminant, the corresponding systems of equations are of course to be omitted. In every case it is from these equations that the form of the function u is to be investigated; they entirely replace the system (A).

A very important case of the general theory is, when we suppose the coefficients $rst \ldots$ to have the property $r's't' \ldots = r''s''t'' \ldots$, whenever $r's't' \ldots$ and $r''s''t'' \ldots$ denote the same combination of letters; and also that the coefficients λ are equal to the coefficients μ, $\nu \ldots$, each to each. In this case the coefficients $\dot{r}\dot{s}\dot{t} \ldots$ have likewise the same property, viz. that $\dot{r}''\dot{s}''\dot{t}'' \ldots = \dot{r}'\dot{s}'\dot{t}' \ldots$, whenever $r's't' \ldots$ and $r''s''t'' \ldots$ denote the same combination of letters.

The equations (B, 1), (B, 2), become in this case

$$u' = L^{np} \,.\, u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B, 3)},$$

$$\Theta\, \Sigma\Sigma\Sigma \ldots \left\{\frac{[n]^n}{[\alpha]^\alpha\, [\beta]^\beta \ldots} \dot{r}\dot{s}\dot{t} \ldots \dot{x}_r \dot{x}_s \dot{x}_t \ldots\right\} = L^{np} \,.\, \Theta \left\{\frac{[n]^n}{[\alpha]^\alpha\, [\beta]^\beta \ldots} rst \ldots x_r x_s x_t \ldots\right\} \ldots\ldots \text{(B, 4)},$$

where only different combinations of values are to be taken for r, s, $t, \ldots$ and α, $\beta, \ldots$ express how often the same number occurs in the series. In the equation (C), μ, ν must be replaced by λ, the equations (D) are no longer satisfied; the equations (A) reduce themselves to a single one, (so that there can be no question here of incomplete hyperdeterminants): but this is no longer sufficient to determine the function sought after. For this reason, the particular case, treated separately, would be far more difficult than the general one; but the formulæ of the general case being first established, these apply immediately to the particular one[1]. The case in question may be defined as that of symmetrical hyperdeterminants, (a denomination already adopted for ordinary determinants). It would be easily seen what on the same principle would be meant by partially symmetrical hyperdeterminants.

I have not yet succeeded in obtaining the general expression of a hyperdeterminant; the only cases in which I can do so are the following: I. $p = 1$, n even, (if n be odd, there only exist incomplete hyperdeterminants). II. $p = 2$, $m = 2$, n even. III. $p = 3$, $m = 2$, $n = 4$.

I. The first case is, in fact, that of the functions considered at the termination of a paper in the *Cambridge Philosophical Transactions*, vol. VIII. part I. [12]; though at that time I was quite unacquainted with the general theory.

[1] See concluding paragraph of this paper.

Using the notation there employed, we have

$$u = \left\{ \begin{array}{l} \overset{\dagger}{1\,1 \ldots (n)} \\ 2\,2 \\ \vdots\,\vdots \\ mm \end{array} \right\},$$

a complete hyperdeterminant when n is even; and when n is odd the functions

$$\left\{ \begin{array}{l} \overset{\dagger}{1\,1 \ldots (n)} \\ 2\,2 \\ \vdots\,\vdots \\ mm \end{array} \right\}, \qquad \left\{ \begin{array}{l} \overset{\dagger}{1\,1 \ldots (n)} \\ 2\,2 \\ \vdots\,\vdots \\ mm \end{array} \right\}$$

are each of them incomplete hyperdeterminants.

(A) In the case of $n = 2$, the complete hyperdeterminant is simply the ordinary determinant

$$\left| \begin{array}{ll} 1\,1, & 1\,2, \ldots 1m \\ 2\,1, & 2\,2, \ldots 2m \\ \vdots & \\ m1, & m2, \ldots mm \end{array} \right|.$$

Stating the general conclusion as applied to this case, which is a very well known one,

"If the function

$$U = \Sigma\Sigma\,(rs\,.\,x_r y_s)$$

be transformed into a similar function

$$\Sigma\Sigma\,(\dot{r}\dot{s}\,.\,\dot{x}_r \dot{y}_s),$$

by means of the substitutions

$$\dot{x}_r = \lambda_r^{\,1}\,\dot{x}_1 + \lambda_r^{\,2}\,\dot{x}_2 \ldots + \lambda_r^{\,m}\,\dot{x}_m,$$

$$\dot{y}_s = \mu_s^{\,1}\,\dot{y}_1 + \mu_s^{\,2}\,\dot{y}_2 \ldots + \mu_s^{\,m}\,\dot{y}_m;$$

then

$$\left| \begin{array}{ll} \dot{1}\dot{1}, & \dot{1}\dot{2}, \ldots \\ \dot{2}\dot{1}, & \dot{2}\dot{2}, \\ \vdots & \end{array} \right| = \left| \begin{array}{ll} \lambda_1^{\,1}, & \lambda_2^{\,1}, \ldots \\ \lambda_1^{\,2}, & \lambda_2^{\,2}, \\ \vdots & \end{array} \right| \left| \begin{array}{ll} \mu_1^{\,1}, & \mu_2^{\,1}, \ldots \\ \mu_1^{\,2}, & \mu_2^{\,2}, \\ \vdots & \end{array} \right| \left| \begin{array}{ll} 11, & 12, \ldots \\ 21, & 22, \\ \vdots & \end{array} \right|.$$

Also, by what has preceded,

$$\dot{r}\dot{s} = \Sigma\Sigma\,(\lambda_f^{\,r}\,.\,\mu_g^{\,s}\,.\,fg);$$

so that the theorem is easily seen to amount to the following one—"If the terms of a determinant of the mth order be of the form $\Sigma_r\Sigma_s\,(rs\,.\,x_{r\rho}\,y_{s\sigma})$, r, s extending as before, from 1 to m inclusively, the determinant itself is the product of three determinants; the first formed with the coefficients rs, the second with the quantities x, and the third with the quantities y."

In a following number of the *Journal* I shall prove, and apply to the theories of Maxima and Minima and of Spherical Coordinates, (I may just mention having obtained, in an elegant form, the formulæ for transforming from one oblique set of coordinates to another oblique one) the more general theorem,

"If k be the order of the determinant formed as above, the determinant itself is a quadratic function, its coefficients being determinants formed with the coefficients rs, its variables being determinants formed respectively with the variables x and the variables y; and the number of variables in each set being the number of combinations of k things out of m, ($=1$ if $k=m$; if $k>m$ the determinant vanishes)."

I shall give in the same paper the demonstration of a very beautiful theorem, rather relating, however, to determinants than to quadratic functions, proved by Hesse in a Memoir in *Crelle's Journal*, vol. XX., "De curvis et superficiebus secundi ordinis;" and from which he has deduced the most interesting geometrical results. Another Memoir, by the same author, *Crelle*, vol. XXVIII., "Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiter Grade mit zwei Variabeln," though relating in point of fact rather to functions of the third order, contains some most important results. A few theorems on quadratic functions, belonging, however, to a different part of the subject, will be found in my paper already quoted in the *Cambridge Philosophical Transactions* [12]; and likewise in a paper in the *Journal*, Chapters in the Algebraical Geometry of n dimensions [11].

I shall, just before concluding this case, write down the particular formula corresponding to three variables, and for the symmetrical case. It is, as is well known, the theorem,

"If $$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy$$

be transformed into

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\theta^2 + 2\mathfrak{F}\eta\theta + 2\mathfrak{G}\xi\theta + 2\mathfrak{H}\xi\eta$$

by means of
$$\begin{aligned} x &= \alpha\ \xi + \beta\ \eta + \gamma\ \theta, \\ y &= \alpha'\ \xi + \beta'\ \eta + \gamma'\ \theta, \\ z &= \alpha''\xi + \beta''\eta + \gamma''\theta, \end{aligned}$$

then
$$(\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{F}^2 - \mathfrak{B}\mathfrak{G}^2 - \mathfrak{C}\mathfrak{H}^2 + 2\mathfrak{F}\mathfrak{G}\mathfrak{H}) =$$
$$(\alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma)^2 (ABC - AF^2 - BG^2 - CH^2 + 2FGH)."$$

(*B*) Let $n=3$, and for greater simplicity $m=2$; write

$$\begin{aligned} a &= 111, & e &= 112, \\ b &= 211, & f &= 212, \\ c &= 121, & g &= 122, \\ d &= 221, & h &= 222, \end{aligned}$$

so that $$U = a\,x_1y_1z_1 + b\,x_2y_1z_1 + c\,x_1y_2z_1 + d\,x_2y_2z_1 + e\,x_1y_1z_2 + f\,x_2y_1z_2 + g\,x_1y_2z_2 + h\,x_2y_2z_2.$$

There is no complete hyperdeterminant (i.e. for $p=1$), and the incomplete ones are

$$ah-bg-cf+de=u_{,}\ , \text{ suppose,}$$
$$ah-de-bg+cf=u_{,,},$$
$$ah-cf-de+bg=u_{,,,}.$$

Thus, suppose the transforming equations are

$$x_1=\lambda_1^1\,\dot{x}_1+\lambda_1^2\,\dot{x}_2\,,$$
$$x_2=\lambda_2^1\,\dot{x}_1+\lambda_2^2\,\dot{x}_2\,;$$
$$y_1=\mu_1^1\,\dot{y}_1+\mu_1^2\,\dot{y}_2\,,$$
$$y_2=\mu_2^1\,\dot{y}^1+\mu_2^2\,\dot{y}_2\,;$$
$$z_1=\nu_1^1\,\dot{z}_1+\nu_1^2\,\dot{z}_2\,,$$
$$z_2=\nu_2^1\,\dot{z}_1+\nu_2^2\,\dot{z}_2\,;$$

then

$$\ddot{u}_{,}=(\mu_1^1\,\mu_2^2-\mu_2^1\mu_1^2)\,(\nu_1^1\,\nu_2^2-\nu_2^1\,\nu_1^2)\,u_1\ , \text{ where } y,\ z \text{ are changed}$$
$$\ddot{u}_{,,}=(\nu_1^1\,\nu_2^2-\nu_2^1\,\nu_1^2)\,(\lambda_1^1\,\lambda_2^2-\lambda_2^1\,\lambda_1^2)\,u_{,,}\ , \quad \text{,,} \quad z,\ x \quad \text{,,}$$
$$\ddot{u}_{,,,}=(\lambda_1^1\,\lambda_2^2-\lambda_2^1\,\lambda_1^2)\,(\mu_1^1\,\mu_2^2-\mu_2^1\,\mu_1^2)\,u_{,,,}, \quad \text{,,} \quad x,\ y \quad \text{,,}$$

We might also have assumed

$$u_{,}=ad-bc, \text{ or } eh-gf,$$
$$u_{,,}=af-be, \text{ or } ch-dg,$$
$$u_{,,,}=ag-ce, \text{ or } bh-df,$$

but these are ordinary determinants.

(C) $n=4,\ m=2.$

$$\begin{array}{ll} a=1111, & i=1112, \\ b=2111, & j=2112, \\ c=1211, & k=1212, \\ d=2211, & l=2212, \\ e=1121, & m=1122, \\ f=2121, & n=2122, \\ g=2211, & o=2212, \\ h=2221, & p=2222. \end{array}$$

$$\begin{aligned} U= \quad & a\,x_1y_1z_1w_1+b\,x_2y_2z_1w_1+c\,x_1y_2z_1w_1+d\,x_2y_2z_1w_1 \\ & +e\,x_1y_1z_2w_1+f\,x_2y_1z_2w_1+g\,x_2y_2z_1w_1+h\,x_2y_2z_2w_1 \\ & +i\,x_1y_1z_1w_2+j\,x_2y_1z_1w_2+k\,x_1y_2z_1w_2+l\,x_2y_2z_1w_2 \\ & +m\,x_1y_1z_2w_2+n\,x_2y_1z_2w_2+o\,x_1y_2z_1w_2+p\,x_2y_2z_2w_2, \end{aligned}$$

we have

$$u= \quad ap-bo-cn+dm-el+fk+gj-hi\,;$$

so that, with the same sets of transforming equations as above, and the additional one,

$$w_1 = \rho_1^1 \dot{w}_1 + \rho_1^2 \dot{w}_2,$$
$$w_n = \rho_2^1 \dot{w}_1 + \rho_2^2 \dot{w}_2,$$

we have $$\ddot{u} = (\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)(\mu_1^1\mu_2^2 - \mu_2^1\mu_1^2)(\nu_1^1\nu_2^2 - \nu_2^1\nu_1^2)(\rho_1^1\rho_2^2 - \rho_2^1\rho_1^2).u;$$

this is important when viewed in reference to a result which will presently be obtained.

If we take the symmetrical case, we have

$$U = \alpha x^4 + 4\beta x^3 y + 6\gamma x^2 y^2 + 4\delta x y^3 + \epsilon y^4;$$

which is transformed into

$$U' = \alpha' x'^4 + 4\beta' x'^3 y' + 6\gamma' x'^2 y'^2 + 4\delta' x' y'^3 + \epsilon' y'^4,$$

by means of

$$x = \lambda x' + \mu y',$$
$$y = \lambda_, x' + \mu_, y';$$

then, if

$$u = \alpha\epsilon - 4\beta\delta + 3\gamma^2,$$
$$u' = \alpha'\epsilon' - 4\beta'\delta' + 3\gamma'^2,$$

we have $$u' = (\lambda\mu_, - \lambda_,\mu)^4 . u.$$

II. Where $p = 2$, $m = 2$, n is odd.

The expression

$$u = \begin{vmatrix} \overset{\dagger}{\left\{\begin{matrix}1111\ldots(n)\\1222\end{matrix}\right\}}, & \overset{\dagger}{\left\{\begin{matrix}1111\ldots(n)\\2222\end{matrix}\right\}}, \\ \overset{\dagger}{\left\{\begin{matrix}1111\ldots(n)\\2222\end{matrix}\right\}}, & \overset{\dagger}{\left\{\begin{matrix}2111\ldots(n)\\2222\end{matrix}\right\}}, \end{vmatrix}$$

is a complete hyperdeterminant; and that over whichever row the mark (†) of nonpermutation is placed. The different expressions so obtained are not, however, all of them independent functions: thus, in the following example, where $n = 3$, the three functions are absolutely identical.

(*A*). $n = 3$, notation as in I. (*B*).

$$u = a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde + 4adfg + 4bech,$$

and then $$\ddot{u} = (\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)^2 (\mu_1^1\mu_2^2 - \mu_2^1\mu_1^2)^2 (\nu_1^1\nu_2^2 - \nu_2^1\nu_1^2)^2 u.$$

This is in many respects an interesting example. We see that the function u may be expressed in the three following forms:

$$u = (ah - bg - cf + de)^2 + 4(ad - bc)(fg - eh) \ldots\ldots\ldots\ldots (1),$$
$$u = (ah - bg - de + cf)^2 + 4(af - be)(dg - ch) \ldots\ldots\ldots\ldots (2),$$
$$u = (ah - cf - de + bg)^2 + 4(ag - ce)(df - bh) \ldots\ldots\ldots\ldots (3),$$

which are indeed the direct results of the general form above given, the sign (†) being placed in succession over the different columns: and the three forms, as just remarked, are in this case identical.

We see from the first of these that u is of the second or third, from the second that u is of the first or third, from the third that u is of the first or second of the three following forms:

$$u = H_2 \left\| \begin{matrix} a, & b, & c, & d \\ e, & f, & g, & h \end{matrix} \right\|, \quad u = H_2 \left\| \begin{matrix} a, & b, & e, & f \\ c, & d, & g, & h \end{matrix} \right\|, \quad u = H_2 \left\| \begin{matrix} a, & c, & e, & g \\ b, & d, & f, & h \end{matrix} \right\|,$$

which is as it should be.

The following is a singular property of u.

Let $$a' = \tfrac{1}{2}\frac{du}{da}, \quad b' = \tfrac{1}{2}\frac{du}{db}, \ \ldots\ldots\ h' = \tfrac{1}{2}\frac{du}{dh},$$

then, u' being the same function of these new coefficients that u is of the former ones,

$$u' = u^3.$$

To prove this, write

$$p = ah - bg - cf + de, \quad q = (ad - bc), \quad r = eh - fg;$$

$$\begin{aligned} a_{,} &= ap - 2qe, & e_{,} &= -2ra + pe, \\ b_{,} &= bp - 2qf, & f_{,} &= -2rb + pf, \\ c_{,} &= cp - 2qg, & g_{,} &= -2rc + pg, \\ d_{,} &= dp - 2qh, & h_{,} &= -2rd + ph; \end{aligned}$$

we have, as a particular case of the general formula just obtained,

$$u_{,} = (p^2 - 4qr)^2\, u = u^2 \,.\, u, \ = u^3.$$

Also

$$\begin{aligned} a_{,} &= \ \ h', & e_{,} &= \ \ d', \\ b_{,} &= -g', & f_{,} &= -c', \\ c_{,} &= -f', & g_{,} &= -b', \\ d_{,} &= \ \ e', & h_{,} &= \ \ a'; \end{aligned}$$

whence $u_{,} = u'$, that is $u' = u^3$.

There is no difficulty in showing also, that if a'', $b'', \ldots h''$ are derived from a', $b' \ldots h'$, as these are from a, b, $\ldots h$, then

$$a'' = u^2 a, \quad b'' = u^2 b, \ldots h'' = u^2 h.$$

The particular case of this theorem, which corresponds to symmetrical values of the coefficients, is given by M. Eisenstein, *Crelle*, vol. XXVII. [1844], as a corollary to his researches on the cubic forms of numbers.

Considering this symmetrical case

$$U = \alpha x^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3,$$
$$u = \alpha^2\delta^2 - 6\alpha\delta\beta\gamma - 3\beta^2\gamma^2 + 4\beta^3\delta + 4\alpha^3\gamma,$$

so that if U be transformed into

$$U' = \alpha' x'^3 + 3\beta' x'^2 y' + 3\gamma' x' y'^2 + \delta' y'^3,$$

by means of

$$x = \lambda\, x' + \mu\, y',$$
$$y = \lambda_{,} x' + \mu_{,} y',$$

then if

$$u' = \alpha'^2\delta'^2 - 6\alpha'\delta'\beta'\gamma' - 3\beta'^2\gamma'^2 + 4\beta'^3\delta' + 4\alpha'^3\gamma',$$

we have

$$u' = (\lambda\mu_{,} - \lambda_{,}\mu)^6 \,.\, u.$$

III. $$p = 3, \quad m = 2, \quad n = 4.$$

Notation as in I. (C),

$$u = A\,(\mathfrak{A} + 3\mathfrak{B} + 3\mathfrak{C} + 6\mathfrak{D} + 6\mathfrak{E}) - B\,(\mathfrak{C} + \mathfrak{D} - 3\mathfrak{E} - \mathfrak{F} + 2\mathfrak{G} + 3\mathfrak{H}),$$

where A, B are arbitrary constants, and $\mathfrak{A}$, $\mathfrak{B}$, &c. ... $\mathfrak{H}$ are functions of the coefficients, given as follows:—

$$\mathfrak{A} = a^3p^3 - b^3o^3 - c^3n^3 + d^3m^3 - e^3l^3 + f^3k^3 + g^3j^3 - h^3i^3.$$

$$\begin{aligned}\mathfrak{B} = &- a^2p^2bo + b^2o^2ap + c^2n^2dm - d^2m^2cn + e^2l^2fk - f^2k^2el - g^2j^2hi + h^2i^2gj\\
&- a^2p^2cn + b^2o^2dm + c^2n^2ap - d^2m^2bo + e^2l^2gj - f^2k^2hi - g^2j^2el + h^2i^2fk\\
&- a^2p^2el + b^2o^2fk + c^2n^2gj - d^2m^2hi + e^2l^2ap - f^2k^2bo - g^2j^2cn + h^2i^2dm\\
&- a^2p^2hi + b^2o^2gj + c^2n^2fk - d^2m^2el + e^2l^2dm - f^2k^2cn - g^2j^2bo + h^2i^2ap.\end{aligned}$$

$$\begin{aligned}\mathfrak{C} = &+ a^2p^2dm - b^2o^2cn - c^2n^2bo + d^2m^2ap - e^2l^2hi + f^2k^2gj + g^2j^2fk - h^2i^2el\\
&+ a^2p^2fk - b^2o^2el - c^2n^2hi + d^2m^2gj - e^2l^2bo + f^2k^2ap + g^2j^2dm - h^2i^2cn\\
&+ a^2p^2gj - b^2o^2hi - c^2n^2el + d^2m^2fk - e^2l^2cn + f^2k^2dm + g^2j^2ap - h^2i^2bo.\end{aligned}$$

$$\begin{aligned}\mathfrak{D} = &\, apbocn - apbodm - apcndm + bocndm - elfkgj + elfkhi + elgjhi - higjfk\\
&+ apboel - apbofk - cndmgj + dmcnhi - elfkap + elfkbo + gjhicn - higjdm\\
&- apbogj + apbohi + cndmel - dmcnkf + elfkcn - elfkdm - gjhiap + higjbo\\
&+ apcnel - bodmfk - cnapgj + dmbohi - elgjap + fkhibo + gjelcn - hifkdm\\
&- apcnfk + bodmel + cnaphi - dmboel + elgjbo - fkhiap - gjeldm + hifkcn\\
&- apdmel + bocnkf + cmbogj - dmaphi + elhiap - fkgjbo - gjfkcn + hidmel.\end{aligned}$$

$$\mathfrak{E} = apdmfk - bocnel - bocnih + dmapgj - elhibo + fkapgj + gjfkdm - hielcn.$$

$$\begin{aligned}\mathfrak{F} = &\, a^2phjo - b^2goip - c^2nflm + d^2emkn - e^2dlkn + f^2ckml + g^2bjpi - h^2aijo\\
&- i^2phbg + j^2ogah + k^2nfde - l^2mecf + m^2ldcf - n^2ckde - o^2bjah + p^2aibg\\
&+ a^2phkn - b^2golm - c^2nfip + d^2emjo - e^2dljo + f^2ckip + g^2bjpl - h^2aink\\
&- i^2phcf + j^2ogde + k^2nfah - l^2mebg + m^2ldbg - n^2ckah - o^2bjde + p^2aicf\\
&+ a^2phlm - b^2gokn - c^2nfjo + d^2emip - e^2ldpi + f^2ckjo + g^2bjnk - h^2aiml\\
&- i^2phde + j^2gocf + k^2nfbg - l^2meah + m^2dlah - n^2ckbg - o^2bjcf + p^2aide\\
&+ a^2pdno - b^2ompo - c^2bpmn + d^2aomn - e^2hjkl + f^2gilk + g^2flij - h^2ekji\\
&- i^2lfgh + j^2kehg + k^2jhef - l^2igfe + m^2pbcd - h^2aodc - o^2ndab + p^2mbca\\
&+ a^2plng - b^2hkmo - c^2ejpn + d^2fiom - e^2cpjl + f^2doik + g^2anlj - h^2bmki\\
&- i^2odfh + j^2pceg + k^2mhbf - l^2nage + m^2khbd - n^2lgac - o^2ifbd + p^2ieca\\
&+ a^2plfo - b^2ekpo - c^2hjmn + d^2gimn - e^2bpkl + f^2aolk + g^2dnij - h^2cmji\\
&- i^2ndgh + j^2mchg + k^2pbef - l^2oafe + m^2jhcd - n^2igcd - o^2lfab + p^2keba.\end{aligned}$$

$$\begin{aligned}\mathfrak{G} = {} & apbgkn - boalhm - cndiep + dmcjfo - elfocj + fkpied + gjhmla - ignbk \\ & - ihjocf + gjidep + kflamh - lekbng + mdnbkg - ncmhal - obpied + paofjc \\ & + apbglm - boahkn - cndejo + dmcfip - elcfip + fkedoj + gjhakn - hibgml \\ & - ihjode + gijpcf + kflmbg - leknah + mdknah - ncmlbg - obpicf + paojde \\ & + apcflg - bojehk - cnjehk + dmilgf - elmpbc + fkadno + gjadno - hipmcb \\ & - ihadno + gjbmcp + kfcbpm - eladno + mdjehk - ncilfg - obilfg + pahejk \\ & + apcflm - bodkne - cnahjo + dmbgip - elbgip + fkahjo + gjednk - hicfml \\ & - ihknde + mdahjo + kfipbg - bocfml + gjcfml - ncpigb - leahjo + paknde \\ & + apidng - bojcmh - cnbkpe + dmaflo - elmhjc - fkidng + gjaflo - ihbkpe \\ & - ihaflo + gjbkpe + fkcjhm - elidng + mdbkpe - cnaflo - boidng + pahmcj \\ & + apidfo - bojcep - nbkhm + cdmalng - elmnbk - fkalng + gjidfo - ihpecj \\ & - ihalng + gjkbmh + fkcjpe - elidfo + mdjcep - ncidfo - obalng + pahmbk.\end{aligned}$$

$$\begin{aligned}\mathfrak{H} = {} & a^2honl - b^2pgmk - c^2pfmj + d^2onie - e^2dpkj + f^2ilco + g^2blni - h^2amkj \\ & - i^2pdfg + j^2oech + k^2nbeh - l^2mafg + m^2blch - n^2kadg - o^2jadf + p^2ibec:\end{aligned}$$

we have, as usual,

$$\dddot{u} = (\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2)^3 (\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2)^3 (\nu_1{}^1\nu_2{}^2 - \nu_2{}^1\nu_1{}^2)^3 (\rho_1{}^1\rho_2{}^2 - \rho_1{}^2\rho_2{}^1)^3 \,.\, u.$$

Particular forms of U are

$A = 1, \quad B = 0:$

$u = \mathfrak{A} + 3\mathfrak{B} + 3\mathfrak{C} + 6\mathfrak{D} + 6\mathfrak{E} = (ap - bo - cn + dm - el + fk + gj - hi)^3, \; = \nu$ suppose.

$A = 1, \quad B = 9:$

$u = \mathfrak{A} + 3\mathfrak{B} - 6\mathfrak{C} - 3\mathfrak{D} + 33\mathfrak{E} + 9\mathfrak{F} - 18\mathfrak{G} - 27\mathfrak{H}, \; = \theta U$ suppose,

where $\theta U = 0$ is the result of the elimination of the variables from the equations $d_{x_1}U = 0$, $d_{y_1}U = 0$, $d_{z_1}U = 0$, $d_{w_1}U = 0$, $d_{x_2}U = 0$, $d_{y_2}U = 0$, $d_{z_2}U = 0$, $d_{w_2}U = 0$. In fact, by an investigation similar to Mr Boole's, applied to a function such as U, it is shown that θU has the characteristic property of the function u: also in the present case u is the most general function of its kind, so that θU is obtained from U by properly determining the constant. This has been effected by comparing the value of u, in the symmetrical case, with the value of θU, in the same case, the expanded expression of which is given by Mr Boole in the *Journal*, vol. IV. p. 169. Assuming $A = 1$, the result was $B = 9$. [Incorrect: the result of the elimination is not $\theta U = 0$, but an equation of a higher degree.]

The general form of u now becomes

$$u = \alpha\nu^3 + \beta\theta U,$$

in which α, β, are indeterminate.

We have $$\dddot{u} = \alpha\dddot{\nu}^3 + \beta\theta\dddot{U} = M(\alpha\nu^3 + \beta\theta U),$$

where $$M = (\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2)^3 (\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2)^3 (\nu_1{}^1\nu_2{}^2 - \nu_2{}^1\nu_1{}^2)^3 (\rho_1{}^1\rho_2{}^2 - \rho_2{}^1\rho_1{}^2)^3.$$

and thence $$\ddddot{\nu}^3 = M\nu^3,$$

which coincides with a previous formula, and

$$\theta\dddot{U} = M\theta U:$$

whence, eliminating M,

$$\frac{\theta\dddot{U}}{\ddddot{\nu}^3} = \frac{\theta U}{\nu^3},$$

an equation which is remarkable as containing only the constants of U and $\ddddot{U}$: it is an equation of condition which must exist among the constants of $\dddot{U}$ in order that this function may be derivable by linear substitutions from U.

In the symmetrical case, or where

$$U = \alpha x^4 + 4\beta x^3 y + 6\gamma x^2 y^2 + 4\delta x y^3 + \epsilon y^4,$$

it has been already seen that ν is given by

$$\nu = \alpha\epsilon - 4\beta\delta + 3\gamma^2.$$

Proceeding to form θU, we have

$$\begin{aligned}
\mathfrak{A} &= \alpha^3\epsilon^3 - 4\beta^3\delta^3 + 3\gamma^6,\\
\mathfrak{B} &= 4\,(\alpha\epsilon\beta^2\delta^2 - \alpha^2\epsilon^2\beta\delta + 3\gamma^2\beta^2\delta^2 - 3\beta\delta\gamma^4),\\
\mathfrak{C} &= 3\,(\alpha^2\epsilon^2\gamma^2 + 2\gamma^6 + \alpha\epsilon\gamma^4 - 4\beta^3\delta^3),\\
\mathfrak{D} &= 6\,(\alpha\epsilon\beta^2\delta^2 - 2\alpha\epsilon\beta\delta\gamma^2 + 3\beta^2\gamma^2\delta^2 - 2\beta\delta\gamma^4),\\
\mathfrak{E} &= 3\alpha\epsilon\gamma^4 - 4\beta^3\delta^3 + \gamma^4,\\
\mathfrak{F} &= 6\,(\alpha^2\delta^2\gamma\epsilon + \epsilon^2\beta^2\gamma\alpha - 2\beta^2\epsilon\gamma\delta - 2\delta^2\alpha\beta\gamma - 4\beta^2\gamma^2\delta^2 . + 4\,.\,\beta\delta\gamma^4 + \gamma^2\beta^2\epsilon + \gamma^2\alpha\delta^2),\\
\mathfrak{G} &= 12\,(\alpha\epsilon\beta\delta\gamma^2 - \alpha\beta\gamma\delta^3 - \epsilon\gamma\delta\beta^3 + \gamma^3\alpha\delta^2 + \gamma^3\epsilon\beta^2 + \beta\delta\gamma^4 - 2\beta^2\gamma^2\delta^2),\\
\mathfrak{H} &= (\alpha^2\delta^4 + \epsilon^2\beta^4 - 4\beta^2\epsilon\gamma^2 - 4\alpha\gamma^2\delta^2 + 6\beta^2\gamma^2\delta^2),
\end{aligned}$$

and these values give

$$\begin{aligned}
\theta U = \alpha^3\epsilon^3 &- 6\alpha\beta^2\delta^2\epsilon - 12\alpha^2\beta\delta\epsilon^2 - 18\alpha^3\gamma^2\epsilon^2 - 27\alpha^2\delta^4 - 27\beta^4\epsilon^2 + 36\beta^2\gamma^2\delta^2 + 54\,\alpha^2\gamma\delta^2\epsilon + 54\,\alpha\beta^2\gamma\epsilon^2\\
&- 54\alpha\gamma^3\delta^2 - 54\beta^2\gamma^3\epsilon - 64\beta^3\delta^3 + 81\alpha\gamma^4\epsilon + 108\alpha\beta\gamma\delta^3 + 108\beta^3\gamma\delta\epsilon - 180\alpha\beta\gamma^2\delta\epsilon,
\end{aligned}$$

so that this function, divided by $(\alpha\epsilon - 4\beta\delta + 3\gamma^2)^3$, is invariable for all functions of the fourth order which can be deduced one from the other by linear substitutions. The function $\alpha\epsilon - 4\beta\delta + 3\gamma^2$ occurs in other investigations: I have met with it in a problem relating to a homogeneous function of two variables, of any order whatever, α, β, γ, δ, ϵ signifying the fourth differential coefficients of the function. But this is only remotely connected with the present subject.

Since writing the above, Mr Boole has pointed out to me that in the transformation of a function of the fourth order of the form $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$,—besides his function θu, and my quadratic function $ae - 4bd + 3c^2$,—there exists a function

of the third order $ace - b^2e - ad^2 - c^3 + 2bdc$, possessing precisely the same characteristic property, and that, moreover, the function θu may be reduced to the form

$$(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - eb^2 - c^3 + 2bdc)^2\,;$$

the latter part of which was verified by trial; the former he has demonstrated in a manner which, though very elegant, does not appear to be the most direct which the theorem admits of. In fact, it may be obtained by a method just hinted at by Mr Boole, in his earliest paper on the subject, *Mathematical Journal*, vol. II. p. 70. The equations $d_x^2u = 0$, $d_xd_yu = 0$, $d_y^2u = 0$, imply the corresponding equations for the transformed function: from these equations we might obtain two relations between the coefficients, which, in the case of a function of the fourth order, are of the orders 3 and 4 respectively: these imply the corresponding relations between the coefficients of the transformed function. Let $A = 0$, $B = 0$, $A' = 0$, $B' = 0$, represent these equations; then, since $A = 0$, $B = 0$, imply $A' = 0$, we must have $A' = \Lambda A' + \mathrm{M}B$, Λ, M, being functions of λ, λ', μ, &c. μ': but B being of the fourth order, while A, A' are only of the third order in the coefficients of u, it is evident that the term $\mathrm{M}B$ must disappear, or that the equation is of the form $A' = \Lambda A$. The function A is obviously the function which, equated to zero, would be the result of the elimination of x^2, xy, y^2, considered as independent quantities from the equations $ax^2 + 2bxy + cy^2 = 0$, $bx^2 + 2cxy + dy^2 = 0$, $cx^2 + 2dxy + ey^2 = 0$, viz. the function given above. Hence the two functions on which the linear transformation of functions of the fourth order ultimately depend are the very simple ones

$$ae - 4bd + 3c^2, \quad ace - ad^2 - eb^2 - c^3 + 2bdc,$$

the function of the sixth order being merely a derivative from these. The above method may easily be extended: thus for instance, in the transformation of functions of any even order, I am in possession of several of the transforming functions; that of the fourth order, for functions of the sixth order, I have actually expanded: but it does not appear to contain the complete theory. Again, in the particular case of homogeneous functions of two variables, the transforming functions may be expressed as symmetrical functions of the roots of the equation $u = 0$, which gives rise to an entirely distinct theory. This, however, I have not as yet developed sufficiently for publication. There does not appear to be anything very directly analogous to the subject of this note, in my general theory: if this be so, it proves the absolute necessity of a distinct investigation for the present case, that which I have denominated the symmetrical one.

14.

ON LINEAR TRANSFORMATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 104—122.]

IN continuing my researches on the present subject, I have been led to a new manner of considering the question, which, at the same time that it is much more general, has the advantage of applying directly to the only case which one can possibly hope to develope with any degree of completeness, that of functions of two variables. In fact the question may be proposed, "To find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." By Derivative I understand a function deduced in any manner whatever from the given functions, and I give the name of Hyperdeterminant Derivative, or simply of Hyperdeterminant, to those derivatives which have the property just enunciated. These derivatives may easily be expressed explicitly, by means of the known method of the separation of symbols. We thus obtain the most general expression of a hyperdeterminant. But there remains a question to be resolved, which appears to present very great difficulties, that of determining the *independent* derivatives, and the relation between these and the remaining ones. I have only succeeded in treating a very particular case of this question, which shows however in what way the general problem is to be attacked.

Imagine p series each of m variables

$$x_1,\quad y_1, \ldots \text{\&c.} \qquad x_2,\quad y_2, \ldots \text{\&c.} \qquad x_p,\quad y_p, \ldots \text{\&c.},$$

where p is at least as great as m.

Similarly p' series each of m' variables

$$x_1',\quad y_1', \ldots \text{\&c.} \qquad x_2',\quad y_2', \ldots, \text{\&c.} \qquad \ldots x_{p'}',\quad y_{p'}', \ldots \text{\&c.},$$

$p^{\backprime}$ at least as great as $m^{\backprime}$, and so on. Let the analogous variables $\dot{x}$, $\dot{y}$... be connected with these by the equations

$$\begin{aligned} x &= \lambda \dot{x} + \mu \dot{y} + \dots, \\ y &= \lambda' \dot{x} + \mu' \dot{y} + \dots, \\ &\vdots \\ x^{\backprime} &= \lambda^{\backprime} \dot{x}^{\backprime} + \mu^{\backprime} \dot{y}^{\backprime} + \dots, \\ y^{\backprime} &= \lambda^{\backprime\backprime} \dot{x}^{\backprime} + \mu^{\backprime\backprime} \dot{y}^{\backprime} + \dots, \\ &\vdots \end{aligned}$$

where x, y,... stand for x_1, y_1,... or x_2, y_2,... or x_p, y_p,...; $x^{\backprime}$, $y^{\backprime}$,... stand for $x_1^{\backprime}$, $y_1^{\backprime}$,... or $x_2^{\backprime}$, $y_2^{\backprime}$,... or $x_p^{\backprime}$, $y_p^{\backprime}$, &c.... The coefficients $\lambda, \mu, \dots, \lambda', \mu', \dots$ &c.; $\lambda^{\backprime}, \mu^{\backprime}, \dots, \lambda^{\backprime\backprime}, \mu^{\backprime\backprime}, \dots$ remain the same in all these systems. Suppose next,

$$\xi = \delta_x, \quad \eta = \delta_y,$$

i.e.

$$\xi_1 = \delta_{x_1}, \quad \eta_1 = \delta_{y_1}, \dots, \dot{\xi}_1 = \delta_{\dot{x}_1}, \dots$$

(where δ_x, δ_y... are the symbols of differentiation relative to x, y, &c.). Then evidently

$$\begin{aligned} \dot{\xi} &= \lambda \xi + \lambda' \eta + \dots, \\ \dot{\eta} &= \mu \xi + \mu' \eta + \dots, \\ &\vdots \end{aligned}$$

with similar equations for $\dot{\xi}^{\backprime}$, $\dot{\eta}^{\backprime}$,... Suppose

$$\|\Omega\| = \left\| \begin{matrix} \xi_1, & \xi_2, \dots \xi_p \\ \eta_1, & \eta_2, \dots \eta_p \\ \vdots & \end{matrix} \right\|, \quad \|\Omega^{\backprime}\| = \left\| \begin{matrix} \xi_1^{\backprime}, & \xi_2^{\backprime}, \dots \xi_p^{\backprime} \\ \eta_1^{\backprime}, & \eta_2^{\backprime}, \dots \eta_p^{\backprime} \\ \vdots & \end{matrix} \right\|,$$

that is to say $\|\Omega\|$ is the series of determinants formed by choosing any m vertical columns to compose a determinant, and similarly $\|\Omega^{\backprime}\|$, &c. Suppose, besides,

$$E = \left| \begin{matrix} \lambda, & \mu, \dots \\ \lambda', & \mu', \dots \\ \vdots & \end{matrix} \right|, \quad E^{\backprime} = \left| \begin{matrix} \lambda^{\backprime}, & \mu^{\backprime}, \dots \\ \lambda^{\backprime\backprime}, & \mu^{\backprime\backprime}, \dots \\ \vdots & \end{matrix} \right|.$$

Then, by the known properties of determinants,

$$\|\dot{\Omega}\| = E \|\Omega\|, \quad \|\dot{\Omega}^{\backprime}\| = E^{\backprime} \|\Omega^{\backprime}\| \text{ \&c.},$$

i.e. the terms on the one side are respectively equal to the terms on the other. Hence if

$$\square = F(\|\Omega\|^{f}, \quad \|\Omega^{\backprime}\|^{f^{\backprime}}, \dots),$$

i.e. $\square$ a rational and integral function, homogeneous of the order f in the quantities of the series $\|\Omega\|$, homogeneous of the order $f^{\backprime}$ in the quantities of the series $\|\Omega^{\backprime}\|$, &c., we have immediately

$$\dot{\square} = E^{f} E^{\backprime f^{\backprime}} \dots \square;$$

or if U be any function whatever of the variables $x, y \ldots$ which is transformed by the linear substitutions above into U, then

$$\dot{\square}\dot{U} = E^f E'^{f'} \ldots \square U;$$

or the function

$$\square U$$

is by the above definition a hyperdeterminant derivative. The symbol $\square$ may be called "symbol of hyperdeterminant derivation," or simply "hyperdeterminant symbol."

Let $A, B, \ldots$ represent the different quantities of the series $\|\Omega\|$,—$A', B', \ldots$ those of the series $\|\Omega'\|$, &c. ..., then $\square$ may be reduced to a single term, and we may write

$$\square = A^\alpha B^\beta \ldots A'^{\alpha'} B'^{\beta'} \ldots$$

Also U may be supposed of the form

$$U = \Theta\Phi \ldots$$

where Θ, Φ are functions of the variables of one of the sets $x, y, \ldots$, of one of the sets $x', y', \ldots$, &c., thus Θ is of the form

$$F(x_1, y_1, \ldots x_1', y_1', \ldots),$$

and so on. The functions $\Theta, \Phi \ldots$ may be the same or different. It may be supposed after the differentiations that several of the sets $x, y, \ldots$ or of the sets $x', y', \ldots$ become identical: in such cases it will always be assumed that the functions $\Theta, \ldots$ into which these sets of variables enter, are similar; so that they become absolutely identical, when the variables they contain are made so. Thus the general expression of a hyperdeterminant is

$$\square U = A^\alpha B^\beta \ldots A'^{\alpha'} B'^{\beta'} \ldots \Theta\Phi \ldots$$

in which, after the differentiations, any number of the sets of variables are made equal. For instance, if all the sets $x, y \ldots$ and all the sets $x', y' \ldots$ are made equal, the hyperdeterminant refers to a single function $F(x, y \ldots x', y' \ldots)$. In any other case it refers not to a single function but to several.

What precedes, is the general theory: it might perhaps have been made clearer by confining it to a particular case: and by doing this from the beginning, it will be seen that it presents no real difficulties. Passing at present to some developments, to do this, I neglect entirely the sets $x', y' \ldots$ and I assume that the number m of variables in each of the sets $x, y \ldots$ reduces itself to two; so that I consider functions of two variables x, y only. The functions Θ, Φ, &c. reduce themselves to functions $V_1, V_2 \ldots V_p$ of the variables x_1, y_1, or $x_2, y_2 \ldots$ or x_p, y_p. Writing also

$$\xi_1\eta_2 - \xi_2\eta_1 = \overline{12}, \text{ \&c.}$$

the symbols $A, B \ldots$ reduce themselves to $\overline{12}, \overline{13} \ldots$. Hence for functions of two variables, there results the following still tolerably general form

$$\square U = \overline{12}^\alpha\ \overline{13}^\beta\ \overline{14}^\gamma \ldots \overline{23}^{\beta'}\ \overline{24}^{\gamma'} \ldots \overline{34}^{\gamma''} \ldots V_1 V_2 V_3 V_4 \ldots$$

The functions V_1, $V_2 \ldots$ may be the same or different: but they will be supposed the same whenever the corresponding variables are made equal. This equality will be denoted by writing, for instance,

$$\square VV'VV \ldots$$

to represent the value assumed by

$$\square V_1 V_2 V_3 V_4 \ldots$$

when after the differentiations

$$x_1,\ y_1 = x_3,\ y_3 = x_4,\ y_4 = x,\ y\,;$$
$$x_2,\ y_2 = x',\ y'\,;$$
&c.

It is easy to determine the general term of $\square U$. To do this, writing for shortness

$$\alpha + \beta + \gamma \ldots = f_1,$$
$$\alpha + \beta' + \gamma' \ldots = f_2,$$
$$\beta + \beta' + \gamma'' \ldots = f_3,$$
&c.

$$N = (-)^{r+s+t\ldots+r'+s'\ldots t'\ldots} \frac{[\alpha]^r}{[r]^r} \frac{[\beta]^s}{[s]^s} \frac{[\gamma]^t}{[t]^t} \cdots \frac{[\beta']^{s'}}{[r']^{r'}} \frac{[\gamma']^{t'}}{[s']^{t'}} \cdots \frac{[\gamma'']^{t''}}{[t'']^{t''}} \cdots$$

$$\xi^{f-r}\eta^r V \text{ or } \delta_x{}^{f-r}\delta_y{}^r V = \overset{f}{V}{}_{,}{}^r \text{ or } V_{,}{}^r,$$

the general term is

$$N \overset{f_1}{V}_{1,}{}^{r+s+t\cdots} \overset{f_2}{V}_{2,}{}^{\alpha-r+s'+t'\cdots} \overset{f_3}{V}_{3,}{}^{\beta-s+\beta'-s'+t'\cdots}$$

where r, s, $t, \ldots s'$, $t', \ldots t'', \ldots$ extend from 0 to α, β, $\gamma \ldots \beta'$, $\gamma', \ldots \gamma'' \ldots$ respectively. It would be easy to change this general term in a way similar to that which will be employed presently for the particular case of $\square V_1 V_2 V_3$.

If several of the functions become identical, and for these some of the letters f are equivalent, it is clear that the derivative $\square U$ refers to a certain number of functions V_1, $V_2 \ldots$ the same or different, of the variables x, y; x', y'; ... and besides that this derivative is homogeneous, of the degrees θ_1, $\theta_1', \ldots$ with respect to the differential coefficients of the orders f_1, $f_1', \ldots$ &c. of V_1, (consequently homogeneous of the order $\theta_1 + \theta_1' + \ldots$ with respect to these differential coefficients collectively), homogeneous and of the degrees θ_2, $\theta_2', \ldots$ with respect to the differential coefficients of the orders f_2, $f_2' \ldots$ of V_2, (consequently of the order $\theta_2 + \theta_2' \ldots$ with respect to these collectively), and so on. The degree with respect to all the functions is of course $\theta_1 + \theta_1' \ldots + \theta_2 + \theta_2' + \ldots, = p$ suppose. In general, only a single function will be considered,

and it will be assumed that $\square U$ only contains the differential coefficients of the fth order. In this case, the derivative is said to be of the degree p and of the order f. The most convenient classification is by degrees, rather than by orders.

Commencing with the simplest case, that of functions of the second order (and writing V, W instead of V_1, V_2), we have

$$\square VW = \overline{12}^{\alpha} VW,$$

(where ξ_1, η_1 apply to V and ξ_2, η_2 to W). This will be constantly represented in the sequel by the notation

$$\overline{12}^{\alpha} VW = B_\alpha(V,\ W).$$

Hence, writing $$\delta_x{}^{\alpha} V = V^{,0},\quad \delta_x{}^{\alpha-1}\delta_y V = V^{,1}\ \dots,$$

we have $$B_\alpha(V,\ W) = V^{,0}W^{,\alpha} - \frac{[\alpha]^1}{[1]^1} V^{,1}W^{,\alpha-1} + \dots;$$

and in particular, according as α is odd or even,

$$B_\alpha(V,\ V) = 0,$$

$$\tfrac{1}{2}B_\alpha(V,\ V) = V^{,0}V^{,\alpha} - \frac{\alpha}{1} V^{,1}V^{,\alpha-1} + \dots,$$

continued to the term which contains $V^{,\frac{1}{2}\alpha}V^{,\frac{1}{2}\alpha}$, the coefficient of this last term being divided by two.

Thus, for the functions $\frac{1}{2}(ax^2 + 2bxy + cy^2)$, $\frac{1}{24}(ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)$, &c., if α be made equal to 2, 4, &c. respectively, we have the *constant* derivatives

$$\begin{aligned}
&ac - b^2,\\
&ae - 4bd + 3c^2,\\
&ag - 6bf + 15ce - 10d^2,\\
&ai - 8bh + 28cg - 56df + 35e^2,\\
&\ \vdots
\end{aligned}$$

which have all of them the property of remaining unaltered, *à un facteur près*, when the variables are transformed by means of $x = \lambda\dot{x} + \mu\dot{y}$, $y = \lambda'\dot{x} + \mu'\dot{y}$. Thus, for instance, if these equations give

$$ax^2 + 2bxy + cy^2 = \dot{a}\dot{x}^2 + 2\dot{b}\dot{x}\dot{y} + \dot{c}\dot{y}^2,$$

then $$\dot{a}\dot{c} - \dot{b}^2 = (\lambda\mu' - \lambda'\mu)^2 . (ac - b^2),$$

and so on. This is the general property, which we call to mind for the case of these constant derivatives.

The above functions may be transformed by means of the identical equation

$$B_a(V,\ W) = \overline{12}^{a-k} B_k(V,\ W),$$

to make use of which, it is only necessary to remark the general formula

$$\xi_1^\lambda \eta_1^\mu \xi_2^\rho \eta_2^\sigma B_k(V,\ W) = B_k(\xi^\lambda \eta^\mu V,\ \xi^\rho \eta^\sigma W).$$

Thus, if $k = 1$, we obtain for the above series, the new forms

$$\begin{aligned}
&ac - b^2,\\
&(ae - bd) - 3(bd - c^2),\\
&(ag - bf) - 5(bf - ce) + 10(ce - d^2),\\
&(ai - bh) - 7(bh - cg) + 21(cg - df) - 35(df - e^2),\\
&\&c.,
\end{aligned}$$

the law of which is evident. This shows also that these functions may be linearly expressed by means of the series of determinants

$$\begin{Vmatrix} a, & b \\ b, & c \end{Vmatrix} \quad \begin{Vmatrix} a, & b, & c \\ b, & c, & d \end{Vmatrix} \ \&c.$$

We may also immediately deduce from them the derivatives B which relate to two functions. For example, for functions of the sixth order this is

$$ag' + a'g - 6(bf' + b'f) + 15(ce' + c'e) - 20dd',$$

which has an obvious connection with

$$ag - 6bf + 15ce - 10d^2;$$

and the same is the case for functions of any order.

The following theorem is easily verified; but I am unacquainted with the general theory to which it belongs.

"If U, V are any functions of the second order, and $W = \lambda U + \mu V$; then

$$B_2'[B_2(W,\ W),\ \ B_2(W,\ W)] = 0$$

(where B_2' relates to λ, μ) is the same that would be obtained by the elimination of x, y between $U = 0$, $V = 0$." (See Note[1].)

In fact this becomes

$$4(ac - b^2)(a'c' - b'^2) - (ac' + a'c - 2bb')^2 = 0,$$

which is one of the forms under which the result of the elimination of the variables from two quadratic equations may be written. This is a result for which I am indebted to Mr Boole.

[1] Not given with the present paper.

Passing to the third degree, we may consider in particular the derivatives

$$\square UVW = \overline{23}^{\alpha}\,\overline{31}^{\alpha}\,\overline{12}^{\alpha}\,UVW = C_\alpha(U,\ V,\ W):$$

writing for shortness

$$A_r = \frac{[\alpha]^r}{[r]^r},\quad \delta_x^{2\alpha-r}\delta_y{}^r U = U^{\cdot r},$$

we have the general term

$$C_\alpha(U,\ V,\ W) = \Sigma\,\{(-)^{r+s+t}\,A_r A_s A_t U^{\cdot\alpha+t-s} V^{\cdot\alpha+r-t} W^{\cdot\alpha+s-r}\},$$

where r, s, t extend from 0 to α. By changing the suffixes r, s the following more convenient formula

$$C_\alpha(U,\ V,\ W) = \Sigma\Sigma\,\{(-)^{\sigma+\rho}U^{\cdot\rho}V^{\cdot\sigma}W^{\cdot 3\alpha-\rho-\sigma}\Sigma\,[(-)^t\,A_{\rho-t}A_{\sigma+t-\alpha}A_t]\},$$

where t extends from 0 to 2α: ρ, σ, and $3\alpha-\rho-\sigma$ must be positive and not greater than 2α.

In particular, according as α is odd or even,

$$C_\alpha(U,\ U,\ U) = 0,$$

$$C_\alpha(U,\ U,\ U) = 6\Sigma\Sigma\,\{(-)^{\rho+\sigma}U^{\cdot\rho}U^{\cdot\sigma}U^{\cdot 3\alpha-\rho-\sigma}\Sigma\,[(-)^t\,A_{\rho-t}A_{\sigma+t-\alpha}A_t]\},$$

omitting therein those values of ρ, σ for which $\rho > \sigma$ or $\sigma > 3\alpha-\rho-\sigma$, and dividing by two the terms in which $\rho = \sigma$ or $\sigma = 3\alpha-\rho-\sigma$, and by six the term for which

$$\rho = \sigma = 3\alpha-\rho-\sigma,\ = \alpha.$$

In particular, for functions of the fourth or eighth orders we have the constant derivatives

$$ace - ad^2 - b^2e - c^3 + 2bcd\,;$$

$$aei - 4ibd - 4afh + 3ag^2 + 3ic^2 + 12beh - 8chd - 8bgf - 22ceg + 24cf^2 + 24d^2g - 36def + 15e^3;$$

the first of which is a simple determinant. Thus we have been led to the functions $ae - 4bd + 3c^2$ and $ace - ad^2 - eb^2 - c^3 + 2bcd$, which occur in my "Note sur quelques formules &c." (*Crelle*, vol. XXIX. [1845] [15]), and in the forms which M. Eisenstein has given for the solutions of equations of the first four degrees.

Let U be a function of the order 4α: the derivative C may be expressed by means of the derivatives B.

For, consider the function

$$B_{4\alpha}[U,\ B_{2\alpha}(V,\ W)]\,;$$

paying attention to the signification of B, this may be written

$$\overline{1\theta}^{4\alpha}\,\overline{23}^{2\alpha}\,UVW,$$

where the symbols ξ_θ, η_θ refer to the two systems $x_2, y_2 : x_3, y_3$. Thus it is easily seen that we may write

$$\xi_\theta = \xi_2 + \xi_3, \quad \eta_\theta = \eta_2 + \eta_3, \quad \text{or} \quad \overline{1\theta} = \overline{12} + \overline{13} = \overline{12} - \overline{31},$$

whence the function becomes

$$(\overline{12} - \overline{31})^{4\alpha}\, \overline{23}^{2\alpha}\, UVW,$$

of which all the terms vanish except

$$\frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}}\, \overline{12}^{2\alpha}\, \overline{23}^{2\alpha}\, \overline{31}^{2\alpha}\, UVW.$$

Hence putting

$$K = \frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}} = \frac{2^{4\alpha}\, 1 \,.\, 3 \ldots (4\alpha - 1)}{2 \,.\, 4 \ldots 4\alpha},$$

we have

$$B_{4\alpha}[U,\ B_{2\alpha}(V,\ W)] = KC_\alpha(U,\ V,\ W),$$

or in particular

$$B_{4\alpha}[U,\ B_{2\alpha}(U,\ U)] = KC_\alpha(U,\ U,\ U).$$

Thus for example, neglecting a numerical factor,

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2$$
$$= (ac - b^2)\, x^4 + 2\,(ad - bc)\, x^3y + (ae + 2bd - 3c^2)\, x^2y^2 + 2\,(be - cd)\, xy^3 + (ce - d^2)\, y^4,$$

and then

$$e\,(ac - b^2) - 4d\,\tfrac{2}{4}\,(ad - bc) + 6c\,\tfrac{1}{6}\,(ae + 2bd - 3c^2) - 4b\,\tfrac{2}{4}\,(be - cd) + a\,(ce - d^2)$$
$$= 3\,(ace - ad^2 - b^2e - c^3 + 2bcd).$$

We have likewise the singular equation

$$B_{2\alpha}(V,\ W) = K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}} \ldots + y^{4\alpha}\frac{d}{da_0}\right) C_\alpha(U,\ V,\ W)$$

where

$$U = \frac{1}{[4\alpha]^{4\alpha}}\left(a_0x^{4\alpha} - \frac{[4\alpha]^1}{1}\, a_1x^{4\alpha-1}y \ldots + a_{4\alpha}y^{4\alpha}\right), \text{ \&c.}$$

If however $U = V = W$, we must write

$$B_{2\alpha}(U,\ U) = \tfrac{1}{3}\, K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}} \ldots + y^{4\alpha}\frac{d}{da_0}\right) C_\alpha(U,\ U,\ U),$$

the reason of which is easily seen. This subject will be resumed in the sequel.

The functions C may be transformed in the same way as the functions B have been. In fact

$$C_\alpha(U,\ V,\ W) = \overline{12}^{\alpha-k}\ \overline{23}^{\alpha-k}\ \overline{31}^{\alpha-k}\ C_k(U,\ V,\ W);$$

if in particular $k=1$, then

$$C_1(U,\ V,\ W) = \begin{vmatrix} U^{,0} & U^{,1} & U^{,2} \\ V^{,0} & V^{,1} & V^{,2} \\ W^{,0} & W^{,1} & W^{,2} \end{vmatrix}, \quad U^{,0} \text{ for } \overset{2}{U}{}^{,0}, \text{ \&c.}$$

but in general

$$\xi_1^{\rho'}\eta_1^{\rho}\xi_2^{\sigma'}\eta_2^{\sigma}\xi_3^{\tau'}\eta_3^{\tau}C_1(U,\ V,\ W), \text{ where } \rho+\rho'=\sigma+\sigma'=\tau+\tau'=2\alpha-2,$$

$$= C_1(\overset{2\alpha-2}{U}{}^{,\rho}\ \overset{2\alpha-2}{V}{}^{,\sigma}\ \overset{2\alpha-2}{W}{}^{,\tau}) = \begin{vmatrix} U^{,\rho} & U^{,\rho+1} & U^{,\rho+2} \\ V^{,\sigma} & V^{,\sigma+1} & V^{,\rho+2} \\ W^{,\tau} & W^{,\tau+1} & W^{,\rho+2} \end{vmatrix}, \quad U^{,\rho} \text{ for } \overset{2\alpha}{U}{}^{,\rho}, \text{ \&c.}$$

whence $$C_\alpha(U,\ V,\ W) = \Sigma\Sigma\,\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & V^{,\sigma-1} & W^{,3\alpha-\rho-\sigma-2} \\ U^{,\rho+1} & V^{,\sigma} & W^{,3\alpha-\rho-\sigma-1} \\ U^{,\rho+2} & V^{,\sigma+1} & W^{,3\alpha-\rho-\sigma} \end{vmatrix} \Sigma\,[(-)^t A_t' A'_{\rho-t} A'_{\sigma-\alpha+t}]\},$$

where $A_t' = \dfrac{[\alpha-1]^t}{[t]^t}$; t extends from 0 to $\overline{\alpha-1}$; ρ, $\sigma-1$, and $3\alpha-\rho-\sigma-2$ may have each of them any positive values not greater than $2\alpha-2$.

In particular

$$C_\alpha(U,\ U,\ U) = 6\Sigma\Sigma\,\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & U^{,\sigma-1} & U^{,3\alpha-\rho-\sigma-2} \\ U^{,\rho+1} & U^{,\sigma} & U^{,3\alpha-\rho-\sigma-1} \\ U^{,\rho+2} & U^{,\sigma+1} & U^{,3\alpha-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A'_t A'_{\rho-t} A'_{\sigma-\alpha-t}]\},$$

where ρ, σ need only have such values that $\rho<\sigma-1$, $\sigma-1<3\alpha-\rho-\sigma-2$.

In particular the derivative $aei-\ldots+15e^3$ may be transformed into

$$\begin{vmatrix} a, & d, & g \\ b, & e, & h \\ c, & f, & i \end{vmatrix} - 3 \begin{vmatrix} a, & e, & f \\ b, & f, & g \\ c, & g, & h \end{vmatrix} - 3 \begin{vmatrix} b, & c, & g \\ c, & d, & h \\ d, & e, & i \end{vmatrix} + 6 \begin{vmatrix} b, & d, & f \\ c, & e, & g \\ d, & f, & h \end{vmatrix} - 15 \begin{vmatrix} c, & d, & e \\ d, & e, & f \\ e, & f, & g \end{vmatrix}$$

in which form it is obviously a linear function of the determinants

$$\left\| \begin{matrix} a, & b, & c, & d, & e, & f, & g \\ b, & c, & d, & e, & f, & g, & h \\ c, & d, & e, & f, & g, & h, & i \end{matrix} \right\|,$$

which is true generally.

Omitting for the present the theory of derivatives of the form

$$\square UVW = \overline{23}^{\alpha}\, \overline{31}^{\beta}\, \overline{12}^{\gamma}\, UVW,$$

we pass on to the derivatives of the fourth degree, considering those forms in which all the differential coefficients are of the same order. We may write

$$\square UVWX = (\overline{12}\,.\,\overline{34})^{\alpha}\,(\overline{13}\,.\,\overline{42})^{\beta}\,(\overline{14}\,.\,\overline{23})^{\gamma}\, UVWX = D_{\alpha,\beta,\gamma}(U,\ V,\ W,\ X) = D_{\alpha,\beta,\gamma};$$

or if for shortness

$$\overline{12}\,.\,\overline{34} = \mathfrak{A},\quad \overline{13}\,.\,\overline{42} = \mathfrak{B},\quad \overline{14}\,.\,\overline{23} = \mathfrak{C},$$

we have

$$D_{\alpha,\beta,\gamma} = \mathfrak{A}^{\alpha}\mathfrak{B}^{\beta}\mathfrak{C}^{\gamma}\,.\,UVWX.$$

Suppose $U = V = W = X$, and consider the derivatives which correspond to the same value f of $\alpha+\beta+\gamma$. The question is to determine how many of these are independent, and to express the remaining ones in terms of these. Since the functions become equal after the differentiations, we are at liberty before the differentiations to interchange the symbolic numbers 1, 2, 3, 4 in any manner whatever. We have thus

$$D_{\alpha,\beta,\gamma} = D_{\beta,\gamma,\alpha} = D_{\gamma,\alpha,\beta} = (-)^f D_{\alpha,\gamma,\beta} = (-)^f D_{\gamma,\beta,\alpha} = (-)^f D_{\beta,\alpha,\gamma};$$

but the identical equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

multiplied by $\mathfrak{A}^a\mathfrak{B}^b\mathfrak{C}^c$ and applied to the product $UVWX$, gives

$$D_{a+1,b,c} + D_{a,b+1,c} + D_{a,b,c+1} = 0;$$

whence if $a+b+c = f-1$, we have a set of equations between the derivatives $D_{\alpha,\beta,\gamma}$ for which $\alpha+\beta+\gamma = f$. Reducing these by the conditions first found, suppose Θf is the number of divisions of an integer f into three parts, zero admissible, but permutations of the same three parts rejected. The number of derivatives is Θf, and the number of relations between them is $\Theta(f-1)$. Hence $\Theta f - \Theta(f-1)$ of these derivatives are independent: only when f is even, one of these is $D_{f,0,0}$, i.e. $\overline{12}^f\,\overline{34}^f\,.\,UVWX$, i.e. $\overline{12}^f UV\,.\,\overline{34}^f WX$, or $B_f(U,\ V)\,B_f(X,\ W)$, i.e. $[B_f(U,\ U)]^2$; rejecting this, the number of independent derivatives, when f is even, is $\Theta f - \Theta(f-1) - 1$. Let $E\left(\frac{a}{b}\right)$ be the greatest integer contained in the fraction $\frac{a}{b}$; the number required may be shown to be

$$E\frac{f}{6} \text{ or } E\frac{f+3}{6},$$

according as f is even or odd. Giving to f the six forms

$$6g, \quad 6g+1, \quad 6g+2, \quad 6g+3, \quad 6g+4, \quad 6g+5,$$

the corresponding numbers of the independent derivatives are

$$g, \quad g, \quad g, \quad g+1, \quad g, \quad g+1;$$

thus there is a single derivative for the orders 3, 5, 6, 7, 8, 10, ... two for the orders 9, 11, 12, 13, 14, 16, ... &c.

When f is even, the terms $D_{f-3,3,0}$, $D_{f-6,6,0}$ ..., and when f is odd, the terms $D_{f-1,1,0}$, $D_{f-4,4,0}$, $D_{f-7,7,0}$, &c. may be taken for independent derivatives: by stopping immediately before that in which the second suffix exceeds the first, the right number of terms is always obtained. Thus, when $f=9$ the independent derivatives are D_{810}, D_{540}, and we have the system of equations

$$\begin{aligned}
&D_{900}+D_{810}+D_{801}=0, && D_{621}+D_{531}+D_{522}=0,\\
&D_{810}+D_{720}+D_{711}=0, && D_{540}+D_{450}+D_{441}=0,\\
&D_{720}+D_{630}+D_{621}=0, && D_{531}+D_{441}+D_{432}=0,\\
&D_{711}+D_{621}+D_{612}=0, && D_{522}+D_{432}+D_{423}=0,\\
&D_{630}+D_{531}+D_{540}=0, && D_{432}+D_{342}+D_{333}=0,
\end{aligned}$$

which are to be reduced by

$$D_{900}=-D_{900}=0, \qquad D_{801}=-D_{810}, \text{ \&c.}$$

It is easy to form the table

$D_{200}= \quad B_2^2,$	$D_{500}=0,$	$D_{700}=0,$
$D_{110}=-\frac{1}{2}B_2^2,$	$D_{410},$	$D_{610},$
	$D_{320}=-D_{410},$	$D_{520}=-D_{610},$
$D_{300}=0,$	$D_{311}=0,$	$D_{511}=0,$
$D_{210},$	$D_{221}=0,$	$D_{430}= \quad D_{610},$
$D_{111}=0,$		$D_{421}=0,$
	$D_{600}= \quad B_6^2,$	$D_{331}=0,$
$D_{400}= \quad B_4^2,$	$D_{510}= \quad -\frac{1}{2}B_6^2,$	$D_{222}=0,$
$D_{310}=-\frac{1}{2}B_4^2,$	$D_{420}=-\frac{2}{3}D_{330}+\frac{1}{6}B_6^2,$	
$D_{220}= \quad \frac{1}{2}B_4^2,$	$D_{411}= \quad \frac{2}{3}D_{330}+\frac{1}{3}B_6^2,$	
$D_{211}=0,$	$D_{330},$	
	$D_{321}=-\frac{1}{3}D_{330}-\frac{1}{6}B_6^2,$	
	$D_{222}= \quad \frac{2}{3}D_{330}+\frac{1}{3}B_6^2,$	

$$
\begin{aligned}
D_{800} &= B_8^2, & D_{900} &= 0,\\
D_{710} &= -\tfrac{1}{2} B_8^2, & D_{810}&,\\
D_{620} &= -\tfrac{2}{3} D_{530} + \tfrac{1}{6} B_8^2, & D_{720} &= - D_{810},\\
D_{611} &= \tfrac{2}{3} D_{530} + \tfrac{1}{3} B_8^2, & D_{711} &= 0,\\
D_{530}&, & D_{630} &= \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540},\\
D_{521} &= -\tfrac{1}{3} D_{530} - \tfrac{1}{12} B_8^2, & D_{621} &= \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540},\\
D_{440} &= -\tfrac{16}{15} D_{530} - \tfrac{1}{30} B_8^2, & D_{540}&,\\
D_{431} &= \tfrac{1}{15} D_{530} - \tfrac{1}{30} B_8^2, & D_{531} &= -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540},\\
D_{422} &= \tfrac{4}{15} D_{530} + \tfrac{2}{15} B_8^2, & D_{522} &= 0,\\
D_{332} &= -\tfrac{2}{15} D_{530} - \tfrac{1}{15} B_8^2, & D_{441} &= 0,\\
& & D_{432} &= \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540},\\
& & D_{333} &= 0.
\end{aligned}
$$

Whatever be the value, all the tables except the three first commence thus, according as f is even or odd,

$$
\begin{aligned}
D_{f,0,0} &= B_f^2, & \text{or}\quad D_{f,0,0} &= 0,\\
D_{f-1,1,0} &= -\tfrac{1}{2} B_f^2, & D_{f-1,1,0}&,\\
D_{f-2,2,0} &= -\tfrac{4}{3} D_{f-3,3,0} + \tfrac{1}{6} B_f^2, & D_{f-2,2,0} &= - D_{f-1,1,0},\\
D_{f-2,1,1} &= \tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{3} B_f^2, & D_{f-2,1,1} &= 0,\\
D_{f-3,3,0} & & \vdots &\\
\vdots & &&
\end{aligned}
$$

but beyond this I am not acquainted with the law.

To give some formulæ for the transformation of these derivatives; we have, for example,

$$D_{f-1,1,0} = (\overline{12}\,.\,\overline{34})^{f-1}\,\overline{13}\,.\,\overline{42}\; UUUU = \overline{13}\,.\,\overline{42}\; B_{f-1}(U,\ U)\, B_{f-1}(U,\ U).$$

But
$$\overline{13}\,.\,\overline{42} = \xi_1\eta_2\eta_3\xi_4 - \xi_1\xi_2\eta_3\eta_4 - \eta_1\eta_2\xi_3\xi_4 + \eta_1\xi_2\xi_3\eta_4,$$

and
$$\begin{aligned}
\xi_1\eta_2\eta_3\xi_4\, B_{f-1}(U,\ U)\, B_{f-1}(U,\ U) &= B_{f-1}(\xi U,\ \eta U)\, B_{f-1}(\eta U,\ \xi U)\\
&= B_{f-1}(U^{\cdot 0}\, U^{\cdot 1})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 0}),\ \&c.
\end{aligned}$$

(where $U^{\cdot 0}$, $U^{\cdot 1}$ stand for $\overset{1}{U}{}^{\cdot 0}\,\overset{1}{U}{}^{\cdot 1}$, &c.); or

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 0})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 1}) - B_{f-1}(U^{\cdot 0}\, U^{1})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 0})\},$$

which reduces itself to

$$\begin{aligned}
D_{f-1,1,0} &= -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 1})\}^2,\\
D_{f-1,1,0} &= -2\,\{B_{f-1}(U^{\cdot 0}\, U^{\cdot 0})\, B_{f-1}(U^{\cdot 1}\, U^{\cdot 1}) - [B_{f-1}(U^{\cdot 0}\, U^{\cdot 1})]^2\},
\end{aligned}$$

according as f is even or odd.

For example, for the orders 3, 5, 7, 9, we have

$$D_{210} = -2\{4(ac-b^2)(bd-c^2)-(ad-bc)^2\},$$

$$D_{410} = -2\{4(ae-4bd+3c^2)(bf-4ce+3d^2)-(af-3be+2cd)^2\},$$

$$D_{610} = -2\{4(ag-6bf+15ce-10d^2)(bh-6cg+15df-10e^2)-(ah-5bg+9cf-5de)^2\},$$

$$D_{810} = -2\{4(ai-8bh+28cg-56df+35e^2)(bj-8ci+28dh-56eg+35f^2) - (aj-7bi+20ch-28dg+14ef)^2\}.$$

The derivatives D will be presently calculated in a completely expanded form up to the ninth order. We have, therefore, still to find the derivatives of the sixth and eighth orders, and a second derivative of the ninth order. For the sixth order, the simplest method is to make use of D_{222}, which is easily seen to be equal to

$$24\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}.$$

For the two others we have the general formulæ

$$D_{f-2,2,0} = 2\{B_{f-2}(U'^0 U'^0)B_{f-2}(U'^2 U'^2) - 4B_{f-2}(U'^0 U'^1)B_{f-2}(U'^2 U'^1) + B_{f-2}(U'^0 U'^2)B_{f-2}(U'^2 U'^0) + 2[B_{f-2}(U'^1 U'^1)]^2\},$$

where U'^0, U'^1, U'^2 have been written for $\overset{2}{U}{}'^0$, $\overset{2}{U}{}'^1$, $\overset{2}{U}{}'^2$; a formula which is demonstrated in precisely the same way as that for $D_{f-1,1,0}$.

$$D_{f-3,3,0} = -2\{ B_{f-3}(U'^0 U'^3)B_{f-3}(U'^3 U'^0) - 6B_{f-3}(U'^0 U'^1)B_{f-3}(U'^3 U'^2) + 6B_{f-3}(U'^0 U'^2)B_{f-3}(U'^3 U'^1) + 9B_{f-3}(U'^1 U'^1)B_{f-3}(U'^2 U'^2) - 9B_{f-3}(U'^1 U'^2)B_{f-3}(U'^2 U'^1) - B_{f-3}(U'^0 U'^3)B_{f-3}(U'^3 U'^0)\},$$

(in which U'^0, &c. stand for $\overset{0}{U}{}'^0$, &c.). In particular

$$D_{620} = 2\{4(ag-6bf+15ce-10d^2)(ci-6dh+15eg-10f^2) - 4(ah-5bg+9cf-5de)\times (bi+5ch+9dg-5ef)+(ai-6bh+16cg-26df+15e^2)^2+8(bh-6cg+15df-10e^2)^2\},$$

$$D_{630} = -2\{4(ag-6bf+15ce-10d^2)(dj-6ei+15fh-10g^2)-6(ah-5bg+9cf-5de)\times (cj-5di+9eh-5fg)+6(ai-6bh+16cg-26df+15e^2)(bj-6ci+16dh-26eg+15f^2) + 36(bh-6cg+15df-10e^2)(ci-6dh+15eg-10f^2)-9(bi-5ch+9dg-5ef)^2 - (aj-6bi+15ch-19dg+9ef)^2\}.$$

Hence we have all the elements necessary for the calculation of the following table of the independent constant derivatives of the fourth degree, up to the ninth order. [I have arranged the terms alphabetically and in tabular form as in my Memoirs on Quantics, and have corrected some inaccuracies];

$D_{210} = 2 \times$

a^2d^2	$+1$
$abcd$	-6
ac^3	$+4$
b^3d	$+4$
b^2c^2	-3

± 9

$D_{410} = 2 \times$

a^2f^2	$+1$
$abef$	-10
$acdf$	$+4$
ace^2	$+16$
ad^2e	-12
b^2df	$+16$
b^2e^2	$+9$
bc^2f	-12
$bcde$	-76
bd^3	$+48$
c^3e	$+48$
c^2d^2	-32

± 142

$D_{222} = 24 \times$

$aceg$	$+1$
acf^2	-1
ad^2g	-1
$adef$	$+2$
ae^3	-1
b^2eg	-1
b^2f^2	$+1$
$bcdg$	$+2$
$bcef$	-2
bd^2f	-2
bde^2	$+2$
c^3g	-1
c^2df	$+2$
c^2e^2	$+1$
cd^2e	-3
d^4	$+1$

± 12

$D_{610} = 2 \times$

a^2h^2	$+1$
$abgh$	-14
$acfh$	$+18$
acg^2	$+24$
$adeh$	-10
$adfg$	-60
ae^2g	$+40$
b^2fh	$+24$
b^2g^2	$+25$
$bceh$	-60
$bcfg$	-234
bd^2h	$+40$
$bdeg$	$+50$
bdf^2	$+360$
be^2f	-240
c^2eg	$+360$
c^2f^2	$+81$
cd^2g	-240
$cdef$	-990
ce^3	$+600$
d^3f	$+600$
d^2e^2	$+375$

± 2223

$D_{620} = 2 \times$

a^2i^2	$+1$
$abhi$	-16
$acgi$	$+36$
ach^2	$+20$
$adfi$	-52
$adgh$	-60
ae^2i	$+30$
$aefh$	$+20$
aeg^2	$+60$
af^2g	-40
b^2gi	$+20$
b^2h^2	$+44$
$bcfi$	-60
$bcgh$	-388
$bdei$	$+20$
$bdfh$	$+696$
bdg^2	$+180$
be^2h	-340
$befg$	-460
bf^3	$+240$
c^2ei	$+60$
c^2fh	$+180$
c^2g^2	$+544$
cd^2i	-40
$cdeh$	-460
$cdfg$	-2596
ce^2g	$+2340$
cef^2	-420
d^3h	$+240$
d^2eg	-420
d^2f^2	$+12876$
de^2f	-3280
e^4	$+1025$

± 8632

$D_{810} = 2 \times$ | $D_{621} = 4 \times$

a^2j^2	$+1$	.....
$abij$	-18	.....
$achj$	$+40$	$+2$
aci^2	$+32$	-2
$adgj$	-56	-7
$adhi$	-112	$+7$
$aefj$	$+28$	$+5$
$aegi$	$+224$	$+22$
aeh^2	.....	-27
af^2i	-140	-25
$afgh$	.....	$+45$
ag^3	.....	-20
b^2hj	$+32$	-2
b^2i^2	$+49$	$+2$
$bcgj$	-112	$+7$
$bchi$	-536	-7
$bdfj$	$+224$	$+22$
$bdgi$	$+392$	-74
bdh^2	$+896$	$+52$
be^2j	-140	-25
$befi$	-196	$+73$
$begh$	-1792	-23
bf^2h	$+1120$	-70
bfg^2	.....	$+45$
c^2fj	.....	-27
c^2gi	$+896$	$+52$
c^2h^2	$+400$	-25
$cdej$	.....	$+45$
$cdfi$	-1792	-23
$cdgh$	-4256	-22
ce^2i	$+1120$	-70
$cefh$	$+560$	$+127$
ceg^2	$+6272$	-32
cf^2g	-3920	-25
d^2j	.....	-20
d^2fh	$+6272$	-32
d^2g^2	$+784$	$+47$
d^2ei	.....	$+45$
de^2h	-3920	-25
$defg$	-13328	-85
df^3	$+7840$	$+50$
e^3g	$+7840$	$+50$
e^2f^2	-4704	-30
	± 35022	± 698

$$D_{330} = \tfrac{3}{2} D_{222} + \tfrac{1}{2} B_6^2,$$
$$D_{530} = -\tfrac{3}{2} D_{620} + \tfrac{1}{4} B_8^2,$$
$$D_{540} = 2 D_{621} + D_{810},$$

equations which give the functions D_{330}, D_{530}, D_{540}, by means of which the others have been expressed.

We may now proceed to demonstrate an important property of the derivatives of the fourth degree, analogous to the one which exists for the third degree. Let U, V, W, X be functions of any order f: then, investigating the value of the expression

$$B_{2f-2\alpha}[B_\alpha(U, V), \quad B_\alpha(W, X)],$$

this reduces itself in the first place to

$$\overline{\theta\phi}^{2f-2\alpha}\,\overline{12}^{\alpha}\,\overline{34}^{\alpha}\,UVWX,$$

where ξ_θ, η_θ refer to U and V, and ξ_ϕ, η_ϕ to W and X: this comes to writing $\xi_\theta = \xi_1 + \xi_2$, $\eta_\theta = \eta_1 + \eta_2$, and $\xi_\phi = \xi_3 + \xi_4$, $\eta_\phi = \eta_3 + \eta_4$; whence

$$\overline{\theta\phi} = \overline{13} + \overline{14} + \overline{23} + \overline{24},$$

or the function in question is

$$(13 + 14 + 23 + 24)^{2f-2\alpha}\,\overline{12}^{\alpha}\,\overline{34}^{\alpha}\,UVWX.$$

But all the terms of this where the sum of the indices of ξ_1, η_1 or ξ_2, η_2 or ξ_3, η_3 or ξ_4, η_4, exceed f, vanish: whence it is only necessary to consider those of the form

$$K_r(\overline{13}\,.\,\overline{42})^r\,(\overline{14}\,.\,\overline{23})^{f-\alpha-r}\,(\overline{12}\,.\,\overline{34})^{\alpha}\,UVWX,$$

where K_r denotes the numerical coefficient

$$\frac{(-)^r\,[2f-2\alpha]^{2f-2\alpha}}{[r]^r\,[r]^r\,[f-\alpha-r]^{f-\alpha-r}\,[f-\alpha-r]^{f-\alpha-r}},$$

or $$B_{2f-2\alpha}[B_\alpha(U, V), \quad B_\alpha(W, X)] = \Sigma\,\{K_r D_{\alpha, r, f-\alpha-r}(U, V, W, X)\}.$$

In particular, if $U = V = W = X$, writing also B_α for $B_\alpha(U, U)$,

$$B_{2f-2\alpha}(B_\alpha, B_\alpha) = \Sigma\,(K_r\,D_{\alpha, r, f-\alpha-r}).$$

If α is odd, this becomes

$$0 = \Sigma\,(K_r\,D_{\alpha, r, f-\alpha-r}),$$

an equation which must be satisfied identically by the relations that exist between the quantities D: if, on the contrary, α is even, we see that there are as many independent functions of the form

$$B_{2f-2\alpha}(B_\alpha, B_\alpha)$$

as there are of the form D; and that these two systems may be linearly expressed, either by means of the other. Thus, for the orders 3, 5, 7, the derivatives D are respectively equal, neglecting a numerical factor, to

$$B_6(U^2, U^2), \quad B_{10}(U^2, U^2), \quad B_{14}(U^2, U^2);$$

for the sixth order they may be linearly expressed by means of

$$B_{12}(U^2, U^2), \quad B_6^2,$$

and so on. All that remains to complete the theory of the fourth degree is to find the general solution of this system of equations, as also of the system connecting the derivatives D.

Passing on to a more general property; let $U_1, U_2, \dots U_p$ be functions of the orders $f_1, f_2 \dots f_p$; and suppose

$$\Theta(U_2 \dots U_p), \; = \square U_2 \dots U_p,$$

a function of the degree f_1 in the variables: suppose that $\Theta(U_2 \dots U_p)$ contains the differential coefficients of the order r_2 for U_2, r_3 for U_3, &c., so that $f_1 = (f_2 - r_2) + \dots (f_p - r_p)$. Consider the expression

$$B_{f_1}\{U_1, \Theta(U_2 \dots U_p)\},$$

which reduces itself in the first place to

$$(\overline{12} + \overline{13} \dots + \overline{1p})^{f_1} \square U_1 U_2 \dots U_p,$$

then to

$$K(\overline{12}^{f_2 - r_2} \overline{13}^{f_3 - r_3} \dots \overline{1p}^{f_p - r_p} \square U_1 U_2 \dots U_p;$$

where for shortness

$$K = \frac{[f_1]^{f_1}}{[f_2 - r_2]^{f_2 - r_2} \dots [f_p - r_p]^{f_p - r_p}}.$$

For if one of the indices were smaller another would be greater, for instance that of $\overline{12}$: and the symbols ξ_2, η_2 in $\overline{12}^{f_2 - r_2 - \lambda} \square$ would rise to an order higher than f_2, or the term would vanish. Hence, writing

$$\square' = \overline{12}^{f_2 - r_2} \overline{13}^{f_3 - r_3} \dots \overline{1p}^{f_p - r_p},$$

and

$$\Theta'(U_1, U_2 \dots, U_p) = \square' U_1 U_2 \dots U_p,$$

we have

$$B_{f_1}\{U_1, \Theta(U_2, \dots U_p)\} = K\Theta'(U_1, U_2 \dots U_p);$$

i.e. the first side is a constant derivative of $U_1, U_2 \dots U_p$.

Suppose

$$U_1 = \frac{1}{[f_1]^{f_1}}(a_0 x^{f_1} + \dots),$$

$$\Theta(U_2, \dots U_p) = \frac{1}{[f_1]^{f_1}}(A_0 x^{f_1} + \dots),$$

then

$$K\Theta'(U_1 \dots U_p) = a_0 A_{f_1} - \frac{f_1}{1} a_1 A_{f_1 - 1} + \dots;$$

i.e.

$$A_{f_1} = K \frac{d}{da_0} \Theta'(U_1 \dots U_p), \quad \frac{f_1}{1} A_{f_1 - 1} = K \frac{d}{da_1} \Theta'(U_1, U_2 \dots U_p) \dots,$$

or finally,

$$\Theta(U_2, \dots U_p) = \frac{K}{[f_1]^{f_1}} \left(x^{f_1} \frac{d}{da_{f_1}} - x^{f_1 - 1} y \frac{d}{da_{f_1 - 1}} + \dots \right) \Theta'(U_1, \dots U_p),$$

an equation which holds good (changing, however, the numerical factor,) when several of the functions $U_1 \dots U_p$ become identical. Hence the theorem: if U be a function given by

$$U = \frac{1}{[f]^f}(a_0 x^f + a_1 x^{f-1} y + \dots),$$

and Θ be any constant derivative whatever of U, then

$$\left(x^f \frac{d}{da_f} - x^{f-1}y \frac{d}{da_{f-1}} + \ldots\right)\Theta$$

is a derivative of U, and its value, neglecting a numerical factor, may be found by omitting in the symbol $\square$, which corresponds to the derivative Θ, the factors which contain any one, no matter which, of the symbolic numbers.

If, for example,

$$-\tfrac{1}{2} D_{216} = \Theta = 6abcd - 4ac^3 - 4bd^3 + 3b^2c^2 - a^2d^2,$$

or

$$\square = \overline{12}^3 . \overline{34}^3 . \overline{13} . \overline{42};$$

then

$$\left(x^3 \frac{d}{dd} - x^2y \frac{d}{dc} + xy^2 \frac{d}{db} - y^3 \frac{d}{da}\right)\Theta$$

reduces itself, omitting a numerical factor, to

$$\overline{12}^2\, \overline{13}\, UUU = -\tfrac{1}{2} B_1 \{U,\ B_1(U,\ U)\}.$$

This may be compared with some formulæ of M. Eisenstein's (*Crelle*, vol. XXVII. [1844, pp. 105, 106]; adopting his notation, we have

$$\Phi = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

$$F = \tfrac{1}{36} B_2(\Phi,\ \Phi) = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

$$\Phi_1 = -\tfrac{1}{2}\left(x^3 \frac{d}{da} - x^2y \frac{d}{dc} + xy^2 \frac{d}{db} - y^3 \frac{d}{da}\right) D,$$

where D is the same as Θ. Hence to the system of formulæ which he has given, we may add the two following:

$$\Phi_1 = \quad \tfrac{1}{3}\left(\frac{d\Phi}{dx}\ \frac{dF}{dy} - \frac{d\Phi}{dy}\ \frac{dF}{dx}\right),$$

$$\Phi_1 = -\tfrac{1}{216}\left\{\frac{d^3\Phi}{dx^3}\ \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dy} - \frac{d^3\phi}{dx^2\, dy}\left(2\frac{d^2\Phi}{dx\, dy}\ \frac{d\Phi}{dy} + \frac{d^2\Phi}{dy^2}\ \frac{d\Phi}{dx}\right)\right.$$
$$\left. + \frac{d^3\Phi}{dx\, dy^2}\left(2\frac{d^2\Phi}{dx\, dy}\ \frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dy}\right) - \frac{d^3\Phi}{dy^3}\ \frac{d^2\Phi}{dx^2}\ \frac{d\Phi}{dx}\right\},$$

the first of which explains most simply the origin of the function Φ_1.

It will be sufficient to indicate the reductions which may be applied to derivatives of the form

$$C_{\alpha,\beta,\gamma}(U,\ V,\ W) = \overline{23}^\alpha . \overline{31}^\beta . \overline{12}^\gamma\, UVW,$$

where U, V, W are *homogeneous* functions. In fact, if

$$\xi x + \eta y = \Xi,$$

the above becomes, neglecting a numerical factor,

$$(\Xi_1 . \overline{23})^\alpha . (\Xi_2 . \overline{31})^\beta . (\Xi_3 . \overline{12})^\gamma\, UVW,$$

where the symbols ξ, η are supposed not to affect the x, y which enter into the expressions Ξ. But we have identically

$$\Xi_1 \overline{23} + \Xi_2 \overline{31} + \Xi_3 \overline{12} = 0,$$

an equation which gives rise to reductions similar to those which have been found for the derivatives $D_{\alpha,\beta,\gamma}$, but which require to be performed with care, in order to avoid inaccuracies with respect to the numerical factors. It may, however, be at once inferred, that the number of independent derivatives $C_{\alpha,\beta,\gamma}$ is the same with that of the independent derivatives $D_{\alpha,\beta,\gamma}$ for the same value of $\alpha+\beta+\gamma$.

From similar reasonings to those by which $B\{U, B(U, U)\}$ has been found, the following general theorem may be inferred.

"The derivative of any number of the derivatives of one or more functions, or even of any number of functions of these derivatives, is itself a derivative of the original functions."

For the complete reduction of these double derivatives, it would be sufficient, theoretically, to be able to reduce to the smallest number possible, the derivatives of any given degree whatever. This has been done for the derivatives of the third degree $C_{\alpha,\beta,\gamma}$, and for those of the fourth degree, in which all the differentiations rise to the same order $(D_{\alpha,\beta,\gamma})$: it seems, however, very difficult to extend these methods even to the next simplest cases,—extensive researches in the theory of the division of numbers would probably be necessary. Important results might be obtained by connecting the theory of hyperdeterminants with that of elimination, but I have not yet arrived at anything satisfactory upon this subject. I shall conclude with the remark, that it is very easy to find a series, or rather a series of series of hyperdeterminants of all degrees, viz. the determinants

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \text{ \&c.}$$

$$\begin{vmatrix} . & a, & 2b, & c \\ . & b, & 2c, & d \\ a, & 2b, & c, & . \\ b, & 2c, & d, & . \end{vmatrix}, \quad \begin{vmatrix} . & a, & 4b, & 6c, & 4d, & e \\ . & b, & 4c, & 6d, & 4e, & f \\ . & c, & 4d, & 6e, & 4f, & g \\ a, & 4b, & 6c, & 4d, & e, & . \\ b, & 4c, & 6d, & 4e, & f, & . \\ c, & 4d, & 6e, & 4f, & g, & . \end{vmatrix} \text{ \&c.} \quad \begin{vmatrix} . & a, & 3b, & 3c, & d \\ . & b, & 3c, & 3d, & e \\ . & a, & 3b, & 3c, & d, & . \\ . & b, & 3c, & 3d, & e, & . \\ a, & 3b, & 3c, & d, & . & . \\ b, & 3c, & 3d, & e, & . & . \end{vmatrix} \text{ \&c.}$$

[I have inserted in these determinants the numerical coefficients which were by mistake omitted.]

However, these functions are not all independent; e.g. the last may be linearly expressed by the square of the second and the cube of $(ae-4bd+3c^2)$; nor do I know the symbolical form of these hyperdeterminant determinants.

15.

NOTE SUR DEUX FORMULES DONNEES PAR M. M. EISENSTEIN ET HESSE.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle) vol. XXIX. (1845), pp. 54—57.]

MR. EISENSTEIN a donné [*Journal*, t. XXVII. (1844), pp. 105—106] cette formule assez remarquable :

$$(a^2d^2 - 3b^2c^2 + 4ac^3 + 4bd^3 - 6abcd)^2 = A^2D^2 - 3B^2C^2 + 4AC^3 + 4DB^3 - 6ABCD,$$

où A, B, C, D sont des fonctions données de a, b, c, d. Cela peut se généraliser comme suit.

Soit

$$u = a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde + 4adfg + 4bceh,$$

et de plus

$$A = \tfrac{1}{2}\frac{du}{da}, \qquad B = \tfrac{1}{2}\frac{du}{db}, \qquad C = \tfrac{1}{2}\frac{du}{dc}, \ldots \qquad H = \tfrac{1}{2}\frac{du}{dh}.$$

Représentons par U la même fonction de A, B, C, ... H, que la fonction u l'est des quantités a, b, c, ... h, l'on a l'équation

$$U = u^3.$$

C'est un cas particulier d'une propriété générale de la fonction u, que l'on peut énoncer ainsi. Imaginons la fonction

$$ax_1y_1z_1 + bx_1y_1z_2 + cx_1y_2z_1 + dx_1y_2z_2 + ex_2y_1z_1 + fx_2y_1z_2 + gx_2y_2z_1 + hx_2y_2z_2,$$

transformée en

$$a'x'_1y'_1z'_1 + b'x'_1y'_1z'_2 + \ldots \qquad\qquad + h'x'_2y'_2z'_2,$$

par les substitutions

$$x_1 = \lambda_1 x_1' + \lambda_2 x_2', \quad y_1 = \mu_1 y_1' + \mu_2 y_2', \quad z_1 = \nu_1 z_1' + \nu_2 z_2',$$
$$x_2 = \lambda_1' x_1' + \lambda_2' x_2', \quad y_2 = \mu_1' y_1' + \mu_2' y_2', \quad z_2 = \nu_1' z_1' + \nu_2' z_2';$$

et soit u', la fonction analogue à u, des nouveaux coefficients a', b', c', ... h': alors

$$u' = (\lambda_1 \lambda_2' - \lambda_2 \lambda_1')^2 (\mu_1 \eta_2' - \mu_2 \mu_1')^2 (\nu_1 \nu_2' - \nu_2 \nu_1')^2 . u.$$

En échangeant seulement les z, ceci donne

$$u' = (\nu_1 \nu_2' - \nu_2 \nu_1')^2 . u,$$

ou

$$a' = \nu_1 a + \nu_1' e, \quad b' = \nu_1 b + \nu_1' f, \quad c' = \nu_1 c + \nu_1' g, \quad d' = \nu_1 d + \nu_1' h,$$
$$e' = \nu_2 a + \nu_2' e, \quad f' = \nu_2 b + \nu_2' f, \quad g' = \nu_2 c + \nu_2' g, \quad h' = \nu_2 d + \nu_2' h.$$

Soit

$$\nu_1 = ah - bg - cf - de,$$
$$\nu_1' = -2(ad - bc),$$
$$\nu_2 = -2(eh - fg),$$
$$\nu_2' = ah - bg - cf - de \; (= \nu_1),$$

on trouve d'abord

$$(\nu_1 \nu_2' - \nu_2 \nu_2') = u^2, \quad \text{ou} \quad u' = u^3,$$

et puis, en ayant égard aux valeurs de A, B, C, ... H:

$$a' = H, \quad b' = -G, \quad c' = -F, \quad d' = E,$$
$$e' = D, \quad f' = -B, \quad g' = -C, \quad h' = A,$$

de manière que $u' = U$, d'où enfin

$$U = u^3.$$

La propriété de la fonction u que je viens d'énoncer, se rapporte à une théorie assez générale, d'une nouvelle espèce de fonctions algébriques dont je m'occupe actuellement, et lesquelles à cause de leur analogie avec les déterminantes, on pourrait comme je crois nommer "Hyperdéterminantes." Je me propose de poser les premiers fondemens de cette théorie dans un mémoire qui va paraître dans le prochain No. du *Cambridge Mathematical Journal* (No. XXIII.) [13].

A présent je passe à une formule de Mr. Hesse [*Journal*, t. XXVIII. (1844), p. 88], qui se rapporte aussi à la même théorie. Soit V, une fonction homogène du troisième ordre, et à trois variables x, y, z. Soit ∇U, la déterminante formée avec les coefficients du second ordre de V, arrangés en cette forme:

$$\frac{d^2V}{dx^2}, \quad \frac{d^2V}{dxdy}, \quad \frac{d^2V}{dxdz},$$
$$\frac{d^2V}{dxdy}, \quad \frac{d^2V}{dy^2}, \quad \frac{d^2V}{dydz},$$
$$\frac{d^2V}{dzdx}, \quad \frac{d^2V}{dzdy}, \quad \frac{d^2V}{dz^2};$$

soit a une quantité constante quelconque et soient A et B deux autres constantes déterminés: Mr. Hesse à démontré l'équation remarquable

$$\nabla(U + a\nabla U) = AU + B\nabla U;$$

mais sans donner la forme des coefficients A et B, ce qui parait être très difficile à effectuer. En considérant le cas d'une fonction homogène de deux variables seulement, mais d'ailleurs d'un ordre quelconque, on parvient à un théorème analogue qui m'a paru intéressant.

"Soit U une fonction homogène et de l'ordre ν des deux variables x, y, et ∇U la déterminante $\frac{d^2V}{dx^2} \cdot \frac{d^2V}{dy^2} - \left(\frac{d^2V}{dxdy}\right)^2$, l'on a

$$(\nu-2)(\nu-3)^3 . \nabla(U + a\nabla U) = \{-\nu(\nu-1)(\nu-3)^2 Ja + \nu(\nu-1)(2\nu-5)^2 Ia^2\} U$$
$$+ \{(\nu-2)(\nu-3)^3 + (\nu-2)(\nu-3)(2\nu-5) Ja^2\} \nabla U.$$

En représentant par i, j, k, l, m, les coefficients différentiels du quatrième ordre de U, on a

$$I = ikm - il^2 - jm^2 - k^3 + 2jkl,$$
$$J = 4jl - 3k^2 - mi,$$

de manière que I, J sont des fonctions de x, y des ordres $3(\nu-4)$ et $2(\nu-4)$ respectivement."

Ce qu'il y a de remarquable, c'est la forme de ces deux quantités I, J. On voit d'abord que la fonction I est la déterminante formée avec les termes

$$\begin{matrix} i, & j, & k \\ j, & k, & l \\ k, & l, & m. \end{matrix}$$

Mais de plus les deux fonctions I, J ont la propriété suivante. Si l'on imagine une fonction du quatrième ordre

$$l\xi^4 + 4j\xi^3\eta + 6k\xi^2\eta^2 + 4l\xi\eta^3 + m\eta^4,$$

transformée en

$$l'\xi'^4 + 4j'\xi'^3\eta' + 6k'\xi'^2\eta'^2 + 4l'\xi'\eta'^3 + m'\eta'^4,$$

au moyen de

$$\xi = \lambda\xi' + \mu\eta',$$
$$\eta = \lambda'\xi' + \mu'\eta',$$

en représentant par I', J', les mêmes fonctions de i', j', k', l', m', on a

$$J' = (\lambda\mu' - \lambda'\mu)^6 . J, \quad I' = (\lambda\mu' - \lambda'\mu)^4 . I.$$

L'on parviendrait, je crois, à des résultats plus simples en considérant une fonction U, homogène en x, y, et aussi en x_1, y_1, et en posant

$$\nabla U = \frac{d^2V}{dxdx_1} \cdot \frac{d^2V}{dydy_1} - \frac{d^2V}{dxdy_1} \cdot \frac{d^2V}{dydx_1}.$$

Par exemple, si U est du second ordre en x, y et aussi en x_1, y_1, de manière que

$$\begin{aligned} U = \quad & x_1^2 \ (A\,x^2 + 2B\,xy + C\,y^2) \\ & + 2x_1y_1\,(A'x^2 + 2B'xy + C'y^2) \\ & + \ y_1^2 \ (A''x^2 + 2B''xy + C''y^2); \end{aligned}$$

on a simplement

$$\nabla\nabla U = \mathfrak{A}\,U,$$

en représentant par $\mathfrak{A}$ la déterminante formée avec les coefficients

$$\begin{matrix} A\,, & B\,, & C\,; \\ A'\,, & B'\,, & C'\,; \\ A''\,, & B''\,, & C''\,; \end{matrix}$$

et multipliée par 2^3. Mais il faudrait développer tout cela beaucoup plus loin.

P.S. Je ne sais pas si l'on a jamais discuté la question suivante, qui doit conduire, il me semble, à des résultats importants. Soient L, M, L', M', U, V des fonctions de x. En éliminant cette variable entre les équations $LU + MV = 0$ et $L'U + M'V = 0$, et en représentant l'équation ainsi obtenue par $[LU + MV,\ L'U + M'V] = 0$, et de même par $[U,\ V] = 0$ l'équation obtenue en éliminant x entre $U = 0$, $V = 0$, il est clair que $[LU + MU,\ L'U + M'V]$ contiendra $[U,\ V]$ comme facteur: quelles sont maintenant les propriétés de l'autre facteur qu'on peut écrire sous les trois formes: $[LU + MV,\ L'U + M'V] : [U,\ V]$, $[LU + MV,\ LM' - L'M] : [L,\ M]$ et $[L'U + M'V,\ LM' - L'M] : [L',\ M']$)?

16.

MÉMOIRE SUR LES HYPERDÉTERMINANTS (TRADUCTION D'UN MÉMOIRE ANGLAIS INSÉRÉ DANS LE "CAMBRIDGE MATHEMATICAL JOURNAL" AVEC QUELQUES ADDITIONS DE L'AUTEUR).

[From the *Journal für die reine und angewandte Mathematik* (Crelle), vol. XXX. (1846), pp. 1—37.]

This is a translation of the foregoing papers, [13] and [14], On Linear Transformations, the additions are inconsiderable: in date of publication it was subsequent to [13] but I think preceded [14].

17.

NOTE ON MR BRONWIN'S PAPER ON ELLIPTIC INTEGRALS.

[From the *Cambridge Mathematical Journal*, vol. III. (1843), pp. 197—198.]

This Note was in answer to objections raised by Mr Bronwin in his paper "On Elliptic Functions" Camb. Math. Jour. vol. III. (1843) pp. 123—131, to some of the formulæ of transformation in the *Fundamenta Nova*. There is in it a serious error which was afterwards pointed out by Mr Bronwin. The Note is omitted, as are also the other papers, [18] and part of [21], which refer to the same subject.

18.

REMARKS ON THE REV. B. BRONWIN'S PAPER ON JACOBI'S THEORY OF ELLIPTIC FUNCTIONS.

[From the *Philosophical Magazine*, vol. XXII. (1843), pp. 358—368.]

This paper is omitted: see [17].

19.

INVESTIGATION OF THE TRANSFORMATION OF CERTAIN ELLIPTIC FUNCTIONS.

[From the *Philosophical Magazine*, vol. XXV. (1844), pp. 352—354.]

THE function sinam u (ϕu for shortness) may be expressed in the form

$$\phi u = u\,\Pi\left(1+\frac{u}{2mK+2m'K'i}\right)\div\Pi\left(1+\frac{u}{2mK+(2m'+1)\,K'i}\right)\text{.....(1)}$$

where m, m' receive any integer, positive or negative, values whatever, omitting only the combination $m=0$, $m'=0$ in the numerator (Abel, *Œuvres*, t. I. p. 212, [Ed. 2, p. 343] but with modifications to adapt it to Jacobi's notation; also the positive and negative values of m, m' are not collected together as in Abel's formulæ). We deduce from this

$$\frac{\phi\,(u+\theta)}{\phi\theta}=\Pi\left(1+\frac{u}{2mK+2m'K'i+\theta}\right)\div\Pi\left(1+\frac{u}{2mK+(2m'+1)\,K'i+\theta}\right)\text{...(2).}$$

Suppose now $K=aH+a'H'i$, $K'i=bH+b'H'i$, a, b, a', b' integers, and $ab'-a'b$ a positive number ν. Also let $\theta=fH+f'H'i$; f, f' integers such that $af'-a'f$, $bf'-b'f$, ν, have not all three any common factor. Consider the expression

$$v=\frac{\phi u\,\phi\,(u+2\omega)\,\ldots\,\phi\,(u+2\,(\nu-1)\,\omega)}{\phi\,(2\omega)\ldots\phi\,(2\,(\nu-1)\,\omega)}\text{ .... (3),}$$

from which

$$v=u\Pi\left(1+\frac{u}{2mK+2m'K'i+2r\theta}\right)\div\Pi\left(1+\frac{u}{2mK+(2m'+1)\,K'i+2r\theta}\right)\text{..... (4)}$$

where r extends from 0 to $\nu-1$ inclusively, the single combination $m=0$, $m'=0$, $r=0$ being omitted in the numerator. We may write

$$mK+m'K'i+r\theta=\mu H+\mu'H'i,$$

μ, μ' denoting any integers whatever. Also to given values of μ, μ' there corresponds only a single system of values of m, m', r. To prove this we must show that the equations

$$ma + m'b + rf = \mu,$$
$$ma' + m'b' + rf' = \mu',$$

can always be satisfied, and satisfied in a single manner only. Observing the value of ν,

$$\nu m + r(b'f - bf') = \mu b' - \mu' b;$$

then if ν and $b'f - bf'$ have no common factor, there is a single value of r less than ν, which gives an integer value for m. This being the case, $m'b$ and $m'b'$ are both integers, and therefore, since b, b' have no common factor (for such a factor would divide ν and $b'f - bf'$), m' is also an integer. If, however, ν and $b'f - bf'$ have a common factor c, so that $\nu = ab' - a'b = c\phi$, $b'f - bf' = c\phi'$; then $(af' - a'f)\,b' = c\,(\phi f' - \phi' f)$, or since no factor of c divides $af' - a'f$, c divides b', and consequently b. The equation for ν may therefore be divided by c. Hence, putting $\frac{\nu}{c} = \nu_{,}$, we may find a value of r, say $r_{,}$, less than $\nu_{,}$, which makes m an integer; and the general value of r less than ν which makes m an integer, is $r = r_{,} + s\nu_{,}$, where s is a positive integer less than c. But m being integral, bm', $b'm'$, and consequently cm' are integral; we have also

$$c\nu_{,}m' + (r_{,} + s\nu_{,})(af' - a'f) = a\mu' - a'\mu;$$

and there may be found a single value of s less than c, giving an integer value for m'. Hence in every case there is a single system of values of m, m', r, corresponding to any assumed integer values whatever of μ, μ'. Hence

$$U = u\Pi\left(1 + \frac{u}{2\mu H + 2\mu' H'i}\right) \div \Pi\left(1 + \frac{u}{2\mu H + (2\mu' + 1)\,H'i}\right) = \phi_{,}u \;\ldots\ldots\ldots (5)$$

$\phi_{,}u$ being a function similar to ϕu, or $\sin \operatorname{am} u$, but to a different modulus, viz. such that the complete functions are H, H' instead of K, K'. We have therefore

$$\phi_{,}u = \frac{\phi u \phi(u + 2\omega) \ldots \phi(u + 2(\nu - 1)\omega)}{\phi(2\omega) \ldots \phi(2(\nu - 1)\omega)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Expressing ω in terms of K, K', we have $\nu H = b'K - a'K'i$, $-\nu H'i = bK - aK'i$, and therefore $\nu\omega = (b'f - bf')\,K - (a'f - af')\,K'i$. Let g, g' be any two integer numbers having no common factor, which is also a factor of ν, we may always determine a, b, a', b', so that $\nu\omega = gK - g'K'i$. This will be the case if $g = b'f - bf'$, $g' = a'f - af'$. One of the quantities f, f' may be assumed equal to 0. Suppose $f' = 0$, then $g = b'f$, $g' = a'f$; whence $ag - bg' = \nu f$. Let k be the greatest common measure of g, g', so that $g = kg_{,}$, $g' = kg'_{,}$; then, since no factor of k divides ν, k must divide f, or $f = kf_{,}$, but $g_{,} = b'f$, $g'_{,} = af'$, and a', b' are integers, or f must divide g, g'; whence $f_{,} = 1$, or $f = k$. Also $ag_{,} - bg'_{,} = \nu$, where $g_{,}$ and $g'_{,}$ are prime to each other, so that integer values may always be found for a and b; so that in the equation (1) we have

$$\omega = \frac{gK - g'K'i}{\nu} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

g, g' being any integer numbers such that no common factor of g, g' also divides ν.

The above supposition, $f' = 0$, is, however, only a particular one; omitting it, the conditions to be satisfied by a, b, a', b', may be written under the form

$$\left.\begin{aligned} ab' - a'b &= \nu, \\ ag - bg' &\equiv 0 \ [\text{mod. } \nu], \\ a'g - b'g' &\equiv 0 \ [\text{mod. } \nu], \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

to which we may join the equations before obtained,

$$\left.\begin{aligned} \nu H &= b'K - a'K'i, \\ -\nu H'i &= bK - aKi, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

which contain the theory of the modular equation. This, however, involves some further investigations, which are not sufficiently connected with the present subject to be attempted here.

20.

ON CERTAIN RESULTS RELATING TO QUATERNIONS.

[From the *Philosophical Magazine*, vol. XXVI. (1845), pp. 141—145.]

IN his last paper on Quaternions [*Phil. Mag.* vol. XXV. (1844), p. 491] Sir William R. Hamilton has alluded to a paper of mine on the Analytical Geometry of (n) Dimensions, in the *Cambridge Mathematical Journal* [11], as one that might refer to the same subject. It may perhaps be as well to notice that the investigations there contained have no reference whatever to Sir William Hamilton's very beautiful theory; a more correct title for them would have been, a Generalization of the Analysis which occurs in ordinary Analytical Geometry.

I take this opportunity of communicating one or two results relating to quaternions; the first of them does appear to me rather a curious one.

Observing that

$$(A+Bi+Cj+Dk)^{-1}=(A-Bi-Cj-Dk)\div(A^2+B^2+C^2+D^2) \quad \text{.....(1)}$$

it is easy to form the equation

$$\left.\begin{array}{l}(A+Bi+Cj+Dk)^{-1}(\alpha+\beta i+\gamma j+\delta k)(A+Bi+Cj+Dk)\\ =\dfrac{1}{A^2+B^2+C^2+D^2}\cdot\\ \left\{\begin{array}{lll}\alpha(A^2+B^2+C^2+D^2) & & \\ +i\,[\,\beta(A^2+B^2-C^2-D^2) & +2\gamma(BC+AD) & +2\delta(BD-AC)\quad]\\ +j\,[2\beta(BC-AD) & +\gamma(A^2-B^2+C^2-D^2) & +2\delta(CD+AB)\quad]\\ +k\,[2\beta(BD+AC) & +2\gamma(CD-AB) & +\delta(A^2-B^2-C^2+D^2)]\end{array}\right.\end{array}\right\}\ldots(2)$$

which I have given with these letters for the sake of reference; it will be convenient to change the notation and write

$$\left.\begin{array}{l}(1+\lambda i+\mu j+\nu k)^{-1}(ix+jy+kz)(1+\lambda i+\mu j+\nu k)\\ =\dfrac{1}{1+\lambda^2+\mu^2+\nu^2}\\ \left\{\begin{array}{lll} i\,[\,x(1+\lambda^2-\mu^2-\nu^2)+2y(\lambda\mu+\nu) & & +2z(\lambda\nu-\mu)\quad] \\ +j\,[2x(\lambda\mu-\nu) & +\ y(1-\lambda^2+\mu^2-\nu^2)+2z(\mu\nu+\lambda) & \quad] \\ +k\,[2x(\lambda\nu+\mu) & +2y(\mu\nu-\lambda) & +\ z(1-\lambda^2-\mu^2+\nu^2)] \end{array}\right.\end{array}\right\}\ldots\ (3)$$

$$\left.\begin{array}{l} =\ i(\alpha x+\alpha' y+\alpha'' z)\\ \quad +j(\beta x+\beta' y+\beta'' z)\\ \quad +k(\gamma x+\gamma' y+\gamma'' z)\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

suppose. The peculiarity of this formula is, that the coefficients α, β, ... are precisely such that a system of formulæ

$$\left.\begin{array}{l} x_{,}=\alpha x+\alpha' y+\alpha'' z\\ y_{,}=\beta' x+\beta' y+\beta'' z\\ z_{,}=\gamma x+\gamma' y+\gamma'' z\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (5)$$

denotes the transformation from one set of rectangular axes to another set, also rectangular. Nor is this all, the quantities λ, μ, ν may be geometrically interpreted. Suppose the axes Ax, Ay, Az could be made to coincide with the axes $Ax_{,}$, $Ay_{,}$, $Az_{,}$ by means of a revolution through an angle θ round an axis AP inclined to Ax, Ay, Az, at angles f, g, h then

$$\lambda=\tan\tfrac{1}{2}\theta\cos f,\qquad \mu=\tan\tfrac{1}{2}\theta\cos g,\qquad \nu=\tan\tfrac{1}{2}\theta\cos h.$$

In fact the formulæ are precisely those given for such a transformation by M. Olinde Rodrigues Liouville, t. v., "Des lois géometriques qui régissent les déplacemens d'un système solide" (or *Camb. Math. Journal*, t. iii. p. 224 [6]). It would be an interesting question to account, *à priori*, for the appearance of these coefficients here.

The ordinary definition of a determinant naturally leads to that of a quaternion determinant. We have, for instance,

$$\begin{vmatrix}\varpi, & \phi\\ \varpi', & \phi'\end{vmatrix}=\varpi\phi'-\varpi'\phi,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (6),$$

$$\begin{vmatrix}\varpi, & \phi, & \chi\\ \varpi', & \phi', & \chi'\\ \varpi'', & \phi'', & \chi''\end{vmatrix}=\varpi(\phi'\chi''-\phi''\chi')+\varpi'(\phi''\chi-\phi\chi'')+\varpi''(\phi\chi'-\phi'\chi),$$

&c., the same as for common determinants, only here the order of the factors on each term of the second side of the equation is essential, and not, as in the other case, arbitrary. Thus, for instance,

$$\begin{vmatrix}\varpi, & \varpi'\\ \varpi, & \varpi'\end{vmatrix}=\varpi\varpi'-\varpi\varpi',\ =0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (7),$$

but
$$\begin{vmatrix} \varpi, & \varpi \\ \varpi', & \varpi' \end{vmatrix} = \varpi\varpi' - \varpi'\varpi, \neq 0 \quad \text{.....(8)},$$

that is, a quaternion determinant does not vanish when two vertical rows become identical. One is immediately led to inquire what the value of such determinants is. Suppose

$$\varpi = x + iy + jz + kw, \quad \varpi' = x' + iy' + jz' + kw', \text{ \&c.},$$

it is easy to prove

$$\begin{vmatrix} \varpi & \varpi \\ \varpi' & \varpi' \end{vmatrix} = -2 \begin{vmatrix} i, & j, & k \\ x, & y, & z \\ x', & y', & z' \end{vmatrix} \quad \text{..... (9)},$$

$$\begin{vmatrix} \varpi, & \varpi, & \varpi \\ \varpi', & \varpi', & \varpi' \\ \varpi'', & \varpi'', & \varpi'' \end{vmatrix} = -2 \begin{vmatrix} 3, & i, & j, & k \\ x, & y, & z, & w \\ x', & y', & z', & w' \\ x'', & y'', & z'', & w'' \end{vmatrix} \quad \text{..... (10)},$$

$$\begin{vmatrix} \varpi, & \varpi, & \varpi, & \varpi \\ \varpi', & \varpi', & \varpi', & \varpi' \\ \varpi'', & \varpi'', & \varpi'', & \varpi'' \\ \varpi''', & \varpi''', & \varpi''', & \varpi''' \end{vmatrix} = 0 \quad \text{..... (11)},$$

or a quaternion determinant vanishes when four or more of its vertical rows become identical.

Again, it is immediately seen that

$$\begin{vmatrix} \varpi, & \phi \\ \varpi', & \phi' \end{vmatrix} + \begin{vmatrix} \phi, & \varpi \\ \phi', & \varpi' \end{vmatrix} = \begin{vmatrix} \varpi, & \varpi \\ \phi', & \phi' \end{vmatrix} - \begin{vmatrix} \varpi', & \varpi' \\ \phi, & \phi \end{vmatrix} \quad \text{..... (12)}$$

&c. for determinants of any order, whence the theorem, if any four (or more) adjacent vertical columns of a quaternion determinant be transposed in every possible manner, the sum of all these determinants vanishes, which is a much less simple property than the one which exists for the horizontal rows, viz. the same that in ordinary determinants exists for the horizontal or vertical rows indifferently. It is important to remark that the equations

$$\begin{vmatrix} \varpi, & \phi \\ \varpi', & \phi' \end{vmatrix} = 0 \text{ or } \begin{vmatrix} \varpi, & \varpi' \\ \phi, & \phi' \end{vmatrix} = 0, \text{ \&c. .....(13)}$$

i.e.
$$\varpi\phi' - \varpi'\phi = 0, \text{ or } \varpi\phi' - \phi\varpi' = 0, \text{ \&c.}$$

are none of them the result of the elimination of Π, Φ, from the two equations

$$\begin{aligned} \varpi\Pi + \phi\Phi &= 0, \quad \text{..... (14).} \\ \varpi'\Pi + \phi'\Phi &= 0, \end{aligned}$$

On the contrary, the result of this elimination is the very different equation

$$\varpi^{-1}.\phi - \varpi'^{-1}.\phi' = 0 \quad \text{.....} \quad (15),$$

equivalent of course to four independent equations, one of which may evidently be replaced by

$$M\varpi.M\phi' - M\varpi'.M\phi = 0 \quad \text{.....} \quad (16),$$

if $M\varpi$, &c. denotes the modulus of ϖ, &c. An equation analogous to this last will undoubtedly hold for any number of equations, but it is difficult to say what is the equation analogous to the one immediately preceding this, in the case of a greater number of equations, or rather, it is difficult to give the result in a symmetrical form independent of extraneous factors.

I may just, in conclusion, mention what appears to me a possible application of Sir William Hamilton's interesting discovery. In the same way that the circular functions depend on infinite products, such as

$$x\Pi\left(1 + \frac{x}{m\pi}\right), \text{ \&c.} \quad \text{.....} \quad (17),$$

$\{m$ any integer from ∞ to $-\infty$, omitting $m = 0\}$

and the inverse elliptic functions on the doubly infinite products

$$x\Pi\left(1 + \frac{x}{mw + n\varpi i}\right), \text{ \&c.} \quad \text{.....} \quad (18)$$

$\{m$ and n integers from ∞ to $-\infty$, omitting $m = 0$, $n = 0\}$,

may not the inverse ultra-elliptic functions of the next order of complexity depend on the quadruply infinite products

$$x\Pi\left(1 + \frac{x}{mw + n\varpi i + o\phi j + p\psi k}\right)? \quad \text{.....} \quad (19)$$

$\{m, n, o, p$ integers from ∞ to $-\infty$, omitting $m = 0$, $n = 0$, $o = 0$, $p = 0\}$.

It seems as if some supposition of this kind would remove a difficulty started by Jacobi (Crelle, t. ix.) with respect to the multiple periodicity of these functions. Of course this must remain a mere suggestion until the theory of quaternions is very much more developed than it is at present; in particular the theory of quaternion exponentials would have to be developed, for even in a product, such as (18), there is a certain singular exponential factor running through the theory, as appears from some formulæ in Jacobi's *Fund. Nova* (relative to his functions Θ, H), the occurrence of which may be accounted for, *à priori*, as I have succeeded in doing in a paper to be published shortly in the *Cambridge Mathematical Journal* [24].

21.

ON JACOBI'S ELLIPTIC FUNCTIONS, IN REPLY TO THE REV. B. BRONWIN; AND ON QUATERNIONS.

[From the *Philosophical Magazine*, vol. XXVI. (1845), pp. 208, 211.]

The first part of this Paper is omitted, see [17]: only the Postscript on Quaternions, pp. 210, 211, is printed.

IT is possible to form an analogous theory with seven imaginary roots of (-1) (? with $\nu = 2^n - 1$ roots when ν is a prime number). Thus if these be $i_1, i_2, i_3, i_4, i_5, i_6, i_7$, which group together according to the types

$$123, \quad 145, \quad 624, \quad 653, \quad 725, \quad 734, \quad 176,$$

i.e. the type 123 denotes the system of equations

$$\begin{array}{lll} i_1 i_2 = i_3, & i_2 i_3 = i_1, & i_3 i_1 = i_2, \\ i_2 i_1 = -i_3, & i_3 i_2 = -i_1, & i_1 i_3 = -i_2, \end{array}$$

&c. We have the following expression for the product of two factors:

$$\begin{aligned} &(X_0 + X_1 i_1 + \dots X_7 i_7)(X'_0 + X'_1 i_1 + \dots X'_7 i_7) \\ &\quad = X_0 X'_0 - X_1 X'_1 - X_2 X'_2 \dots - X_7 X'_7 \\ &\qquad + [\overline{23} + \overline{45} + \overline{76} + (01)]\, i_1 \\ &\qquad + [\overline{31} + \overline{46} + \overline{57} + (02)]\, i_2 \\ &\qquad + [\overline{12} + \overline{65} + \overline{47} + (03)]\, i_3 \\ &\qquad + [\overline{51} + \overline{62} + \overline{47} + (04)]\, i_4 \\ &\qquad + [\overline{14} + \overline{36} + \overline{72} + (05)]\, i_5 \\ &\qquad + [\overline{24} + \overline{53} + \overline{17} + (06)]\, i_6 \\ &\qquad + [\overline{25} + \overline{34} + \overline{61} + (07)]\, i_7 \end{aligned}$$

where $\qquad (01) = X_0 X'_1 + X_1 X'_0 \dots; \quad \overline{12} = X_1 X'_2 - X_2 X'_1$ &c.;

and the modulus of this expression is the product of the moduli of the factors. The above system of types requires some care in writing down, and not only with respect to the combinations of the letters, but also to their order; it would be vitiated, e.g. by writing 716 instead of 176. A theorem analogous to that which I gave before, for quaternions, is the following:—If $\Lambda = 1 + \lambda_1 i_1 \dots + \lambda_7 i_7$, $X = x_1 i_1 \dots + x_7 i_7$: it is immediately shown that the possible part of $\Lambda^{-1} X \Lambda$ vanishes, and that the coefficients of $i_1, \dots i_7$ are linear functions of $x_1, \dots x_7$. The modulus of the above expression is evidently the modulus of X; hence "we may determine seven linear functions of $x_1 \dots x_7$, the sum of whose squares is equal to $x_1^2 + \dots + x_7^2$." The number of arbitrary quantities is however only seven, instead of twenty-one, as it should be.

22.

ON ALGEBRAICAL COUPLES.

[From the *Philosophical Magazine*, vol. XXVII. (1845), pp. 38—40.]

It is worth while, in connection with the theory of quaternions and the researches of Mr Graves (*Phil. Mag.* [Vol. XXVI. (1845), pp. 315—320]), to investigate the properties of a couple $ix+jy$ in which i, j are symbols such that

$$\begin{aligned} i^2 &= \alpha i + \beta j, \\ ij &= \alpha' i + \beta' j, \\ ji &= \gamma i + \delta j, \\ j^2 &= \gamma' i + \delta' j. \end{aligned}$$

If
$$\overline{ix+jy}\,\overline{ix_1+jy_1} = iX + jY,$$

then
$$\begin{aligned} X &= \alpha xx_1 + \alpha' xy_1 + \gamma x_1 y + \gamma' yy_1, \\ Y &= \beta' xx_1 + \beta' xy_1 + \delta' x_1 y + \delta' yy_1. \end{aligned}$$

Imagine the constants α, β... so determined that $ix+jy$ may have a modulus of the form $K(x+\lambda y)(x+\mu y)$; there results one of the four following essentially independent systems

A.
$$\begin{aligned} i^2 &= \frac{1}{\lambda\mu}(\delta\lambda\mu + \gamma\overline{\lambda+\mu})\, i - \frac{\gamma}{\lambda\mu} j, \\ ij &= ji = \gamma i + \delta j, \\ j^2 &= -\lambda\mu\delta i + (\gamma + \overline{\lambda+\mu}\delta)\, j, \end{aligned}$$

$$\left\{ \begin{aligned} X + \lambda Y &= \frac{1}{\lambda}(\gamma+\lambda\delta)(x+\lambda y)(x_1+\lambda y_1), \\ X + \mu Y &= \frac{1}{\mu}(\gamma+\mu\delta)(x+\mu y)(x+\mu y_1). \end{aligned} \right.$$

The couple may be said to have the *two linear moduli*,

$$\frac{1}{\lambda}(\gamma+\lambda\delta)(x+\lambda y), \quad \frac{1}{\mu}(\gamma+\mu\delta)(x+\mu y);$$

as well as the quadratic one,

$$\frac{1}{\lambda\mu}(\gamma+\lambda\delta)(\gamma+\mu\delta)(x+\lambda y)(x+\mu y),$$

the product of these, which is the modulus, and the only modulus in the remaining systems.

B.

$$i^2 = -\delta i + \frac{1}{\lambda\mu}(\gamma + \delta\overline{\lambda+\mu})j,$$

$$ij = ji = \gamma i + \delta j,$$

$$j^2 = (\overline{\lambda+\mu}\,\gamma + \lambda\mu\delta)\,i - \gamma j,$$

$$\begin{cases} X + \lambda Y = \dfrac{1}{\mu}(\gamma+\lambda\delta)(x+\mu y)(x_1+\mu y_1), \\ X + \mu Y = \dfrac{1}{\lambda}(\gamma+\mu\delta)(x+\lambda y)(x_1+\lambda y_1). \end{cases}$$

C.

$$i^2 = \frac{1}{\lambda\mu}(\delta\lambda\mu + \gamma\,\overline{\lambda+\mu})\,i - \frac{\gamma}{\lambda\mu}j,$$

$$ij = \left(\frac{\mu^2+\mu\lambda+\lambda^2}{\mu\lambda}\gamma + \overline{\mu+\lambda}\,\delta\right)i + \left(-\delta - \frac{\lambda+\mu}{\lambda\mu}\gamma\right)j,$$

$$ji = \gamma i + \delta j,$$

$$j^2 = (\overline{\lambda+\mu}\gamma + \lambda\mu\delta)\,i - \gamma j,$$

$$\begin{cases} X + \lambda Y = \dfrac{1}{\lambda}(\gamma+\lambda\delta)(x+\lambda y)(x_1+\mu y_1), \\ X + \mu Y = \dfrac{1}{\mu}(\gamma+\mu\delta)(x+\mu y)(x_1+\lambda y_1). \end{cases}$$

D.

$$i^2 = -\delta i + \frac{1}{\lambda\mu}(\gamma + \delta\,\overline{\lambda+\mu})j,$$

$$ij = (-\gamma - \overline{\lambda+\mu}\,\delta)\,i + \left(\frac{\lambda+\mu}{\lambda\mu}\gamma + \frac{\lambda^2+\lambda\mu+\mu^2}{\lambda\mu}\delta\right)j,$$

$$ji = \gamma i + \delta j,$$

$$j^2 = -\lambda\mu\delta i + (\gamma + \overline{\lambda+\mu}\,\delta)\,j,$$

$$\begin{cases} X + \lambda Y = \dfrac{1}{\mu}(\gamma+\lambda\delta)(x+\mu y)(x_1+\lambda y_1), \\ X + \mu Y = \dfrac{1}{\lambda}(\gamma+\mu\delta)(x+\lambda y)(x_1+\mu y_1). \end{cases}$$

The formulæ are much simpler and not essentially less general, if $\mu = -\lambda$. They thus become

A′.
$$\begin{aligned} i^2 &= \delta i + \frac{\gamma}{\lambda^2} j, \\ ij = ji &= \gamma i + \delta j, \\ j^2 &= \lambda^2 \delta i + \gamma j, \end{aligned}$$
$$X \pm \lambda Y = \pm \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \pm \lambda y)(x_1 \pm \lambda y_1).$$

(Two linear moduli.)

B′.
$$\begin{aligned} i^2 &= -\delta i - \frac{\gamma}{\lambda^2} j, \\ ij = ji &= \gamma i + \delta j, \\ j^2 &= -\lambda^2 \delta i - \gamma j, \end{aligned}$$
$$X \pm \lambda Y = \mp \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \mp \lambda y)(x_1 \mp \lambda y_1).$$

C′.
$$\begin{aligned} i^2 &= -\delta i + \frac{\gamma}{\lambda^2} j, \\ ij &= -\gamma i - \delta j, \\ ji &= \gamma i + \delta j, \\ j^2 &= -\lambda^2 \delta i - \gamma j, \end{aligned}$$
$$X \pm \lambda Y = \pm \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \pm \lambda y)(x_1 \mp \lambda y_1).$$

D′.
$$\begin{aligned} i^2 &= -\delta i - \frac{\gamma}{\lambda^2} j, \\ ij &= -\gamma i - \delta j, \\ ji &= \gamma i + \delta j, \\ j^2 &= \lambda^2 \delta i + \gamma j, \end{aligned}$$
$$X \pm \lambda Y = \mp \frac{1}{\lambda} (\gamma \pm \lambda\delta)(x \mp \lambda y)(x_1 \pm \lambda y).$$

There is a system more general than (A.) having a *single* linear modulus $q(\theta X + Y)$: this is

E.
$$\begin{aligned} i^2 &= \alpha (i - \theta j) + \theta^2 q j, \\ ij &= \alpha' (i - \theta j) + \theta q j, \\ ji &= \gamma (i - \theta j) + \theta q j, \\ j^2 &= \gamma' (i - \theta j) + q j, \end{aligned}$$
$$\theta X + Y = q(\theta x + y)(\theta x_1 + y_1);$$

or, without real loss of generality,

E′.

$$i^2 = \alpha\, i,$$
$$ij = \alpha' i,$$
$$ji = \gamma\, i,$$
$$j^2 = \gamma' i + qj,$$
$$Y = qyy_1.$$

To complete the theory of this system, one may add the identical equation

$$X + \frac{1}{\theta - qM} Y = \frac{q\,(\theta^2 - M\alpha)}{\theta - qM} \left(x + \frac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'} y\right) \left(x_1 + \frac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma} y_1\right),$$

where
$$M = \frac{\theta\,(\gamma - \theta\gamma') - (\alpha - \theta\alpha')}{\alpha'\gamma - \alpha\gamma'}.$$

By determining the constants, so that $\frac{1}{\theta - qM} = \frac{\alpha' - \theta\gamma}{\alpha - \theta\alpha'} = \frac{\alpha' - \theta\gamma'}{\alpha - \theta\gamma}$, the system would reduce itself to the form A.

23.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Philosophical Magazine and Journal of Science*, vol. XXVII. (1845), pp. 424—427.]

IN a former paper [19] I gave a proof of Jacobi's theorem, which I suggested [21] would lead to the resolution of the very important problem of finding the relation between the complete functions. This is in fact effected by the formulæ there given, but there is an apparent indeterminateness in them, the cause of which it is necessary to explain, and which I shall now show to be inherent in the problem. For the sake of supplying an omission, for the detection of which I am indebted to Mr Bronwin, I will first recapitulate the steps of the demonstration.

If $\frac{1}{2}\omega$, $\frac{1}{2}\upsilon$ [1] be the complete functions corresponding to ϕ, x, then this function is expressible in the form

$$\phi x = x\Pi\left(1+\frac{x}{m\omega+n\upsilon}\right) \div \Pi\left(1+\frac{x}{m+\frac{1}{2}\omega+n+\frac{1}{2}\upsilon}\right).$$

Let p be any prime number, μ, ν integers not divisible by p, and

$$\theta = \frac{\mu\omega+\nu\upsilon}{p}.$$

The function

$$\phi_1 x = \phi x \frac{\phi(x+2\theta)}{\phi 2\theta}\,\frac{\phi(x+4\theta)}{\phi 4\theta}\cdots\frac{\phi(x+2(p-1)\theta)}{\phi(2(p-1)\theta)}$$

is always reducible to the form

$$x\Pi\left(1+\frac{x}{m'\omega'+n'\upsilon'}\right) \div \Pi\left(1+\frac{x}{m'+\frac{1}{2}\omega'+n'+\frac{1}{2}\upsilon'}\right);$$

[1] Analogous to the K, $K'i$ of M. Jacobi.

and thus $\phi_1 x$ is an inverse function, the complete functions of which are $\frac{1}{2}\omega'$, $\frac{1}{2}\upsilon'$: and ω', υ' are connected with ω, υ by the equations

$$\omega' = \frac{1}{p}(\alpha\omega + \beta\upsilon),$$

$$\upsilon' = \frac{1}{p}(\alpha'\omega + \beta'\upsilon),$$

α, β, α', β' being any integers subject to the conditions that α, β' are odd and α', β even; also

$$\alpha\beta' - \alpha'\beta = p,$$
$$\mu\beta' - \nu\alpha' = l'p,$$
$$\mu\beta - \nu\alpha = lp,$$

l, l' being any integers whatever. In fact, to prove this, we have only to consider the general form of a factor in the numerator of $\phi_1 x$. Omitting a constant factor, this is

$$\left(1 + \frac{x}{m\omega + n\upsilon + 2r\theta}\right) \quad [r < p];$$

and it is to be shown that we can always satisfy the equation

$$m\omega + n\upsilon + 2r\theta = m'\omega' + n'\upsilon',$$

or the equations

$$pm + 2r\mu = m'\alpha + n'\alpha',$$
$$pn + 2r\nu = m'\beta + n'\beta';$$

and also that to each set of values of m, n, r, there is a unique set of values of m', n', and *vice versâ*. This is done in the paper referred to. Moreover, with the suppositions just made as to the numbers α, β' being odd and α', β even, it is obvious that m' is odd or even, according as m is, and n' according as n is, which shows that we can likewise satisfy

$$\overline{m + \tfrac{1}{2}}\,\omega + \overline{n + \tfrac{1}{2}}\,\upsilon + 2r\theta = \overline{m' + \tfrac{1}{2}}\,\omega' + \overline{n' + \tfrac{1}{2}}\,\upsilon';$$

and thus the denominator of $\phi_1 x$ is also reducible to the required form.

Now proceeding to the immediate object of this paper, α, β, α', β', and consequently ω', υ' are to a certain extent indeterminate. Let A, B, A', B' be a particular set of values of α, β, α', β', and O, P the corresponding values of ω', υ'. We have evidently A, B' odd and A', B even. Also

$$AB' - A'B = p,$$
$$\mu B' - \nu A' = L'p,$$
$$\mu B - \nu A = Lp,$$
$$O = \frac{1}{p}(A\omega + B\upsilon),$$
$$U = \frac{1}{p}(A'\omega + B'\upsilon).$$

By eliminating ω, v from these equations and the former system, it is easy to obtain

$$\omega' = a\,O + b\,U,$$
$$v' = a'O + b'U,$$

where
$$a = \frac{1}{p}(\alpha B' - \beta A'), \quad b = -\frac{1}{p}(\alpha B - \beta A),$$
$$a' = \frac{1}{p}(\alpha' B' - \beta' A'), \quad b' = -\frac{1}{p}(\alpha' B - \beta' A).$$

The coefficients a, b, a', b' are integers, as is obvious from the equation $\mu(\alpha' B - \beta' A) = p(L'\alpha - lA')$, and the others analogous to it; moreover, a, b' are odd and a', b are even, and

$$ab' - a'b = \frac{1}{p^2}(A'B - A'B)(\alpha\beta' - \alpha'\beta);$$

that is
$$ab' - a'b = 1.$$

Hence the theorem,—"The general values ω', v' of the complete functions are linearly connected with the particular system of values O, U by the equations, $\omega' = aO + bU$, $v' = a'O + b'U$, in which a, b' are odd integers and a', b even ones, satisfying the condition $ab' - a'b = 1$."

With this relation between O, U and ω', v', it is easy to show that the function $\phi_1 x$ is precisely the same, whether O, U or ω', v' be taken for the complete functions. In fact, stating the proposition relatively to ϕx, we have,—"The inverse function ϕx is not altered by the change of ω, v into ω', v', where $\omega' = \alpha\omega + \beta v$, $v' = \alpha'\omega + \beta' v$, and α, β, α', β' satisfy the conditions that α, β' are odd, α', β even, and $\alpha\beta' - \alpha'\beta = 1$." This is immediately shown by writing

$$m\omega + nv = m'\omega' + n'v',$$

or
$$m = m'\alpha + n'\alpha',$$
$$n = m'\beta + n'\beta'.$$

It is obvious that to each set of values of m, n there is a unique set of values of m', n', and *vice versâ*: also that odd or even values of m, m' or n, n' always correspond to each. It is, in fact, the preceding reasoning applied to the case of $p = 1$.

Hence finally the theorem,—"The only conditions for determining ω', v' are the equations

$$\omega' = \frac{1}{p}(\alpha\omega + \beta v), \qquad v' = \frac{1}{p}(\alpha'\omega + \beta' v),$$

where α, β' are odd and α', β even, and

$$\alpha\beta' - \alpha'\beta = p, \quad \mu\beta' - \nu\alpha' = l'p, \quad \mu\beta - \nu\alpha = lp,$$

l and l' arbitrary integers: and it is absolutely indifferent what system of values is adopted for ω', v', the value of $\phi_1 x$ is precisely the same."

We derive from the above the somewhat singular conclusion, that the complete functions are not absolutely determinate functions of the modulus; notwithstanding that they are given by the apparently determinate conditions,

$$\tfrac{1}{2}\,\omega = \int_0^{\frac{1}{c}} \frac{dx}{\sqrt{(1-c^2x^2)\,(1+e^2x^2)}},$$

$$\tfrac{1}{2}\,\upsilon = \int_0^{\frac{1}{e}} \frac{dx}{\sqrt{(1+c^2x^2)\,(1-e^2x^2)}}.$$

In fact definite integrals are in many cases really indeterminate, and acquire different values according as we consider the variable to pass through real values, or through imaginary ones. Where the limits are real, it is tacitly supposed that the variable passes through a succession of real values, and thus ω, υ may be considered as completely determined by these equations, but only in consequence of this tacit supposition. If c and e are imaginary, there is absolutely no system of values to be selected for ω, υ in preference to any other system. The only remaining difficulty is to show from the integral itself, independently of the theory of elliptic functions, that such integrals contain an indeterminateness of two arbitrary integers; and this difficulty is equally great in the simplest cases. Why, *à priori*, do the functions

$$\sin^{-1}x = \int_0 \frac{dx}{\sqrt{1-x^2}}, \qquad \text{or } \log x = \int_0 \frac{dx}{x}$$

contain a single indeterminate integer?

Obs. I am of course aware, that in treating of the properties of such products as $\Pi\left(1+\dfrac{x}{m\omega+n\upsilon}\right)$, it is absolutely necessary to pay attention to the relations between the infinite limiting values of m and n; and that this introduces certain exponential factors, to which no allusion has been made. But these factors always disappear from the quotient of two such products, and to have made mention of them would only have been embarrassing the demonstration without necessity.

24.

ON THE INVERSE ELLIPTIC FUNCTIONS.

[From the *Cambridge Mathematical Journal*, t. IV. (1845), pp. 257—277.]

THE properties of the inverse elliptic functions have been the object of the researches of the two illustrious analysts, Abel and Jacobi. Among their most remarkable ones may be reckoned the formulæ given by Abel (*Œuvres*, t. I. p. 212 [Ed. 2, p. 343]), in which the functions $\phi\alpha$, $f\alpha$, $F\alpha$, (corresponding to Jacobi's $\sin\mathrm{am}\,.\,\alpha$, $\cos\mathrm{am}\,.\,\alpha$, $\Delta am\,.\,\alpha$, though not precisely equivalent to these, Abel's radical being $[(1-c^2x^2)(1+e^2x^2)]^{\frac{1}{2}}$, and Jacobi's, like that of Legendre's $[(1-x^2)(1-k^2x^2)]_{\frac{1}{2}}$), are expressed in the form of fractions, having a common denominator; and this, together with the three numerators, resolved into a doubly infinite series of factors; i.e. the general factor contains two independent integers. These formulæ may conveniently be referred to as "Abel's double factorial expressions" for the functions ϕ, f, F. By dividing each of these products into an infinite number of partial products, and expressing these by means of circular or exponential functions, Abel has obtained (pp. 216—218) two other systems of formulæ for the same quantities, which may be referred to as "Abel's first and second single factorial systems." The theory of the functions forming the above numerators and denominator, is mentioned by Abel in a letter to Legendre (*Œuvres*, t. II. p. 259 [Ed. 2, p. 272]), as a subject to which his attention had been directed, but none of his researches upon them have ever been published. Abel's double factorial expressions have nowhere anything analogous to them in Jacobi's *Fund. Nova;* but the system of formulæ analogous to the first single factorial system is given by Jacobi (p. 86), and the second system is implicitly contained in some of the subsequent formulæ. The functions forming the numerator and denominator of $\sin\mathrm{am}\,.\,u$, Jacobi represents, omitting a constant factor, by $\mathrm{H}(u)$, $\Theta(u)$; and proceeds to investigate the properties of these new functions. This he principally effects by means of a very remarkable equation of the form

$$l\Theta(u) = \tfrac{1}{2}Au^2 + B\textstyle\int_0 du\,.\int_0 du \sin^2\mathrm{am}\,u,$$

(*Fund. Nova*, pp. 145, 133), by which $\Theta(u)$ is made to depend on the known function $\sin \mathrm{am}.u$. The other two numerators are easily expressed by means of the two functions H, Θ.

From the omission of Abel's double factorial expressions, which are the only ones which display clearly the real nature of the functions in the numerators and denominators; and besides, from the different form of Jacobi's radical, which complicates the transformation from an impossible to a possible argument, it is difficult to trace the connection between Jacobi's formulæ; and in particular to account for the appearance of an exponential factor which runs through them. It would seem therefore natural to make the whole theory depend upon the definitions of the new transcendental functions to which Abel's double factorial expressions lead one, even if these definitions were not of such a nature, that one only wonders they should never have been assumed *à priori* from the analogy of the circular functions sin, cos, and quite independently of the theory of elliptic integrals. This is accordingly what I have done in the present paper, in which therefore I assume no single property of elliptic functions, but demonstrate them all, from my fundamental equations. For the sake however of comparison, I retain entirely the notation of Abel. Several of the formulæ that will be obtained are new.

The infinite product

$$x\Pi\left(1+\frac{x}{m\omega}\right) \quad\text{.....}\quad (1),$$

where m receives the integer values $\pm 1, \pm 2, \ldots \pm r$, converges, as is well known, as r becomes indefinitely great to a determinate function $\sin\frac{\pi x}{\omega}$ of x; the theory of which might, if necessary, be investigated from this property assumed as a definition. We are thus naturally led to investigate the properties of the new transcendant

$$u = x\Pi\Pi\left(1+\frac{x}{m\omega+n\upsilon i}\right) \quad\text{.....}\quad (2):$$

m and n are integer numbers, positive or negative; and it is supposed that whatever positive value is attributed to either of these, the corresponding negative one is also given to it. $i=\sqrt{(-1)}$, ω and υ are real positive quantities. (At least this is the standard case, and the only one we shall explicitly consider. Many of the formulæ obtained are true, with slight modifications, whatever ω and υ represent, provided only $\omega : \upsilon i$ be not a real quantity; for if it were so, $m\omega+n\upsilon i$ for some values of m, n would vanish, or at least become indefinitely small, and u would cease to be a determinate function of x.)[1]

Now the value of the above expression, or, as for the sake of shortness it may be written, of the function

$$u = x\Pi\Pi\left\{1+\frac{x}{(m,\ n)}\right\} \quad\text{.....}\quad (3),$$

[1] I have examined the case of impossible values of ω and υ in a paper which I am preparing for *Crelle's Journal*. [The paper here referred to is [25], actually published in *Liouville's Journal*].

depends in a remarkable manner on the mode in which the superior limits of m, n are assigned. Imagine m, n to have any positive or negative integer values satisfying the equation

$$\phi(m^2, n^2) < T \quad \text{.....} \quad (4).$$

Consider, for greater distinctness, m, n as the coordinates of a point; the equation $\phi(m^2, n^2) = T$ belongs to a certain curve symmetrical with respect to the two axes. I suppose besides that this is a continuous curve without multiple points, and such that the minimum value of a radius vector through the origin continually increases as T increases, and becomes infinite with T. The curve may be *analytically* discontinuous, this is of no importance. The condition with respect to the limits is then that m and n must be integer values denoting the coordinates of a point *within* the above curve, the whole system of such integer values being successively taken for these quantities.

Suppose, next, u' denotes the same function as u, except that the limiting condition is

$$\phi'(m^2, n^2) < T' \quad \text{.....} \quad (5).$$

The curve $\phi'(m^2, n^2) = T'$ is supposed to possess the same properties with the other limiting curve, and, for greater distinctness, to lie entirely outside of it; but this last condition is nonessential.

These conditions being satisfied, the ratio $u' : u$ is very easily determined in the limiting case of T and T' infinite. In fact

$$\frac{u'}{u} = \Pi\Pi \left\{1 + \frac{x}{(m, n)}\right\} \quad \text{.....} \quad (6),$$

or

$$l\frac{u'}{u} = \Sigma\Sigma l \left\{1 + \frac{x}{(m, n)}\right\} \quad \text{.....} \quad (7),$$

the limiting conditions being

$$\phi(m^2, n^2) > T \quad \text{.....} \quad (8),$$
$$\phi'(m^2, n^2) < T'.$$

Now

$$l\left\{1 + \frac{x}{(m, n)}\right\} = \frac{x}{(m, n)} - \tfrac{1}{2} \cdot \frac{x^2}{(m, n)^2} + \text{.....} \quad (9),$$

$$l\frac{u'}{u} = x \,.\, \Sigma\Sigma \frac{1}{(m, n)} - \tfrac{1}{2} x^2 \,.\, \Sigma\Sigma \frac{1}{(m, n)^2} + \text{.....} \quad (10),$$

or, the alternate terms vanishing on account of the positive and negative values destroying each other,

$$l\frac{u'}{u} = -\tfrac{1}{2} x^2 \,.\, \Sigma\Sigma \frac{1}{(m, n)^2} - \tfrac{1}{4} x^4 \,.\, \Sigma\Sigma \frac{1}{(m, n)^4} - \text{.....} \quad (11).$$

In general

$$\Sigma\Sigma\psi(m, n) = \iint \psi(m, n)\, dm dn + P \quad \text{.....} \quad (12),$$

P denoting a series the first term of which is of the form $C\psi(m, n)$, and the remaining ones depending on the differential coefficients of this quantity with respect to m and n. The limits between which the two sides are to be taken, are identical.

In the present case, supposing T and T' indefinitely great, it is easy to see that the first term of the expression for $l\frac{u'}{u}$ is the only one which is not indefinitely small; and we have

$$l\frac{u'}{u} = -\tfrac{1}{2}Ax^2, \text{ or } u' = u\epsilon^{-\frac{1}{2}Ax^2} \quad \text{(13)},$$

where

$$A = \iint \frac{dmdn}{(m, n)^2} = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{(14)};$$

the limits of the integration being given by

$$\phi\,(m^2, n^2) > T \quad \text{(15)},$$
$$\phi'\,(m^2, n^2) < T'.$$

Some particular cases are important. Suppose the limits of u' are given by

$$m^2\omega^2 < T^2, \quad n^2v^2 < T^2 \quad \text{(16)},$$

and those of u, by

$$m^2\omega^2 + n^2v^2 < T^2 \quad \text{(17)};$$

we have

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{(18)},$$

$$= -\frac{1}{\omega}\int dn\left\{\frac{1}{T+nvi} - \frac{1}{\sqrt{(T^2-n^2v^2)}+nvi} - \frac{1}{-\sqrt{(T^2-n^2v^2)}+nvi} - \frac{1}{-T+nvi}\right\}$$

$$= -\frac{2}{\omega}\int dn\left\{\frac{T}{T^2+n^2v^2} - \frac{\sqrt{(T^2-n^2v^2)}}{T^2}\right\} \quad (nv = -T, \quad nv = T)$$

$$= -\frac{2}{\omega v}\int_{-1}^{1} d\theta\left\{\frac{1}{1+\theta^2} - \sqrt{(1-\theta^2)}\right\} = -\frac{2}{\omega v}(\pi - \pi) = 0;$$

or, in this case,

$$u' = u \quad \text{(19)}.$$

Again, let the limits of u' be

$$m^2\omega^2 < R'^2, \quad n^2v^2 < S'^2 \quad \text{(20)},$$

and those of u,

$$m^2\omega^2 < R^2, \quad n^2v^2 < S^2 \quad \text{(21)}.$$

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad \text{(22)},$$

$$= -\frac{1}{\omega}\int dn\left\{\frac{1}{R'+nvi} - \frac{1}{R+nvi} + \frac{1}{-R+nvi} - \frac{1}{-R'+nvi}\right\},$$

where the limits are $n^2v^2 < S'^2$, for the terms containing R', $n^2v^2 < S^2$, for the terms containing R,

$$= -\frac{2}{\omega v i} l \frac{R' + S'i}{R' - S'i} \frac{R - Si}{R + Si} \quad \text{.....(23)},$$

$$= -\frac{4}{\omega v}(\lambda' - \lambda), \quad \text{if } \lambda' = \tan^{-1}\frac{S'}{R'}, \quad \lambda = \tan^{-1}\frac{S}{R},$$

the arcs λ, λ' being included between the limits 0, $\frac{1}{2}\pi$. Hence

$$u' = u\epsilon^{2(\lambda'-\lambda)\frac{x^2}{\omega v}} \quad \text{.....(24)}.$$

In particular if $\frac{S'}{R'} = \frac{S}{R}$, $u' = u$. If $\frac{S'}{R'} = 0$, $\frac{S}{R} = 1$, $u' = u\epsilon^{-\frac{1}{2}\beta x^2}$; if $\frac{S'}{R'} = \infty$, $\frac{S}{R} = 1$, $u' = u\epsilon^{\frac{1}{2}\beta x^2}$: where $\beta = \frac{\pi}{\omega v}$, for which quantity it will continue to be used.

We may now completely define the functions whose properties are to be investigated. Writing, for shortness,

$$(m, n) = m\omega + nvi \quad \text{.....}(A),$$

$$(\overline{m}, n) = (m + \tfrac{1}{2})\,\omega + nvi,$$

$$(m, \overline{n}) = m\omega + (n + \tfrac{1}{2})\,vi,$$

$$(\overline{m}, \overline{n}) = (m + \tfrac{1}{2})\,\omega + (n + \tfrac{1}{2})\,vi\,;$$

we may put

$$\gamma x = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\} \quad \text{.....}(B),$$

$$gx = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, n)}\right\},$$

$$Gx = \Pi\Pi\left\{1 + \frac{x}{(m, \overline{n})}\right\},$$

$$\mathfrak{G}x = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, \overline{n})}\right\};$$

the limits being given respectively by the equations

$$\text{mod.}\,(m, n) < T, \quad \text{mod.}\,(\overline{m}, n) < T, \quad \text{mod.}\,(m, \overline{n}) < T, \quad \text{mod.}\,(\overline{m}, \overline{n}) < T,$$

T being finally infinite. The system of values $m = 0$, $n = 0$, is of course omitted in γx.

The functions γx, gx, Gx, $\mathfrak{G}x$, are all of them real finite functions of x, possessing properties analogous to that of u. Thus, representing any one of them by Jx, we have

$$Jx = \epsilon^{\pm\frac{1}{2}\beta x^2} J_{\pm\beta}\, x \quad \text{.....}(C),$$

where $J_{\pm\beta}x$ is the same as Jx, only for $J_\beta x$ the limits are given by $m^2\omega^2$ or $(m+\frac{1}{2})^2\omega^2 < R^2$, n^2v^2 or $(n+\frac{1}{2})^2 v^2 < S$, ($R$, S, and $\frac{R}{S}$ infinite), and for $J_{-\beta}x$, by the same formulæ, (R, S, and $\frac{S}{R}$ infinite). It is to this equation that the most characteristic properties of the functions Jx are due.

The following equations are deduced immediately from the above definitions:

$$\gamma(-x) = -\gamma x, \quad g(-x) = gx, \quad G(-x) = Gx, \quad \mathfrak{G}(-x) = \mathfrak{G}x \text{ ..... } (D),$$
$$\gamma(0) = 0, \quad g(0) = 1, \quad G(0) = 1, \quad \mathfrak{G}(0) = 1,$$
$$\gamma'(0) = 1.$$

Suppose $\gamma_1 x$, $g_1 x$, $G_1 x$, $\mathfrak{G}_1 x$, are the values that would have been obtained for γx, gx, Gx, $\mathfrak{G}x$ by interchanging ω and v,—then changing x into xi, and interchanging m and n, by which means the limiting equations are the same in the two cases, we obtain the following system of equations:

$$\gamma_1(xi) = i\gamma x \text{ ..... } (E),$$
$$g_1(xi) = Gx,$$
$$G_1(xi) = gx,$$
$$\mathfrak{G}_1(xi) = \mathfrak{G}x;$$

or otherwise,

$$\gamma(xi) = i\gamma_1 x \text{ ..... } (F),$$
$$g(xi) = G_1 x,$$
$$G(xi) = g_1 x,$$
$$\mathfrak{G}(xi) = \mathfrak{G}_1 x,$$

equations which are useful in transforming almost any other property of the functions J.

The functions $J_\beta x$ are changed one into another, except as regards a constant multiplier, by the change of x into $x + \frac{\omega}{2}$. This will be shown in a Note, or it may be seen from some formulæ deduced immediately from the definitions of the functions $J_\beta x$, which will be given in the sequel[1]. Observing the relation between Jx and $J_\beta x$, we have in particular

$$\gamma\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} A gx \text{ ..... } (G),$$

$$g\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\omega\beta x} B\gamma x,$$

$$G\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} C\mathfrak{G}x,$$

$$\mathfrak{G}\left(x + \frac{\omega}{2}\right) = \epsilon^{\frac{1}{2}\beta\omega x} DGx,$$

[1] Not given in the present paper. [The Note *was* given, see p. 154, and the formulæ referred to must have been the formulæ (M) p. 144.]

where A, B, C, D, are most simply determined by writing $x=0$, $x=-\frac{\omega}{2}$. Putting at the same time $\epsilon^{\beta\omega^2}=\epsilon^{\frac{\pi\omega}{v}}=q_1^{-1}$,

$$A=\gamma\left(\frac{\omega}{2}\right)\text{.....}(H),$$

$$B=-q_1^{-\frac{1}{4}}\div\gamma\left(\frac{\omega}{2}\right),$$

$$C=G\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div\mathfrak{G}\left(\frac{\omega}{2}\right),$$

$$D=\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div G\left(\frac{\omega}{2}\right);$$

whence also

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\text{ .... (25).}$$

Similarly, the functions $J_{-\beta}x$ are changed one into the other by the change of x into $x+\frac{1}{2}vi$. We have in the same way

$$\gamma\left(x+\frac{vi}{2}\right)=\epsilon^{-\frac{1}{2}\beta vxi}A'Gx\text{.....}(I),$$

$$g\left(x+\frac{vi}{2}\right)=\epsilon^{-\frac{1}{2}\beta vxi}B'\mathfrak{G}x,$$

$$G\left(x+\frac{vi}{2}\right)=\epsilon^{-\frac{1}{2}\beta vxi}C'\gamma x,$$

$$\mathfrak{G}\left(x+\frac{vi}{2}\right)=\epsilon^{-\frac{1}{2}\beta vxi}D'gx.$$

Whence

$$A'=\gamma\left(\frac{vi}{2}\right)\text{.....}(J).$$

$$B'=g\left(\frac{vi}{2}\right)=q^{-\frac{1}{4}}\div\mathfrak{G}\left(\frac{vi}{2}\right),$$

$$C'=-q^{-\frac{1}{4}}\div\gamma\left(\frac{vi}{2}\right),$$

$$D'=\mathfrak{G}\left(\frac{vi}{2}\right)=q^{-\frac{1}{4}}\div g\left(\frac{vi}{2}\right);$$

where $\epsilon^{\beta v^2}=\epsilon^{\frac{\pi v}{\omega}}=q^{-1}$. It is obvious that the relation between q and q_1 is $lq\,.\,lq_1=-\pi^2$.

We obtain from the above

$$g\left(\frac{vi}{2}\right)\mathfrak{G}\left(\frac{vi}{2}\right)=q^{-\frac{1}{4}}\text{ ..... (26).}$$

Also, by making $x=\frac{vi}{2}$ in the expression for $\gamma\left(x+\frac{\omega}{2}\right)$ and $x=\frac{\omega}{2}$ in that for $\gamma\left(x+\frac{vi}{2}\right)$, we have

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{vi}{2}\right) = -i\gamma\left(\frac{vi}{2}\right) G\left(\frac{\omega}{2}\right) \quad \text{.....} \quad (27),$$

and the same or an equivalent one would have been obtained from the functions g, G, $\mathfrak{G}$.

By combining the above systems, we deduce one of the form

$$\gamma\left(x+\frac{\omega}{2}+\frac{vi}{2}\right) = \epsilon^{\frac{1}{2}\beta x(\omega-vi)} A''\mathfrak{G}x \quad \text{.....} \quad (K),$$

$$g\left(x+\frac{\omega}{2}+\frac{vi}{2}\right) = \epsilon^{\frac{1}{2}\beta x(\omega-vi)} B''Gx,$$

$$G\left(x+\frac{\omega}{2}+\frac{vi}{2}\right) = \epsilon^{\frac{1}{2}\beta x(\omega-vi)} C''gx,$$

$$\mathfrak{G}\left(x+\frac{\omega}{2}+\frac{vi}{2}\right) = \epsilon^{\frac{1}{2}\beta x(\omega-vi)} D''\gamma x;$$

and, observing the equation $\epsilon^{\beta\omega vi}=\epsilon^{\pi i}=(-1)$, with the following values for the coefficients,

$$A''=(-1)^{\frac{1}{2}}.\gamma\left(\frac{\omega}{2}\right) g\left(\frac{vi}{2}\right) \quad \text{.....} \quad (L),$$

$$B''=-(-1)^{\frac{1}{2}}.q_1^{-\frac{1}{4}}.\gamma\left(\frac{vi}{2}\right)\div\gamma\left(\frac{\omega}{2}\right),$$

$$C''=(-1)^{\frac{1}{2}}.\mathfrak{G}\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{vi}{2}\right),$$

$$D''=-(-1)^{\frac{1}{2}}.q^{-\frac{1}{4}}.\mathfrak{G}\left(\frac{\omega}{2}\right)\div\gamma\left(\frac{vi}{2}\right).$$

Collecting the formulæ which connect $\gamma\left(\frac{\omega}{2}\right)$, $\gamma\left(\frac{vi}{2}\right)$, ... these are

$$g\left(\frac{\omega}{2}\right)=0 \quad \text{.....} \quad (L\ bis),$$

$$G\left(\frac{vi}{2}\right)=0,$$

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}},$$

$$g\left(\frac{vi}{2}\right)\mathfrak{G}\left(\frac{vi}{2}\right)=q^{-\frac{1}{4}},$$

$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{vi}{2}\right)=-i\gamma\left(\frac{vi}{2}\right) G\left(\frac{\omega}{2}\right).$$

And by the assistance of these

$$B''C'' \div A''D'' = B'D' \div A'C' = CD \div AB = -1 \quad \text{.....(28)},$$

$$A'B' \div C'D' = -A''B'' \div C''D'' = -\gamma^2\left(\frac{\nu i}{2}\right) \div \mathfrak{G}^2\left(\frac{\nu i}{2}\right),$$

$$A''C'' \div B''D'' = -A'C' \div B'D' = \gamma^2\left(\frac{\omega}{2}\right) \div \mathfrak{G}^2\left(\frac{\omega}{2}\right),$$

$$A\,D \div B\,C = -A'D' \div B'C' = \gamma^3\left(\frac{\omega}{2}\right) \div G^2\left(\frac{\omega}{2}\right) = -\gamma^2\left(\frac{\nu i}{2}\right) \div g^2\left(\frac{\nu i}{2}\right),$$

which will be required presently.

It is now easy to proceed to the general systems of formulæ,

$$\Theta = (-1)^{mn}\, \epsilon^{\beta x(m\omega - n\nu i)}\, q_1^{-\frac{1}{2}m^2}\, q^{-\frac{1}{2}n^2} \quad \text{.....} (M),$$

$$\gamma\{x + (m,\ n)\} = (-1)^{m+n} \,.\, \Theta\gamma x\,,$$

$$g\{x + (m,\ n)\} = (-1)^{m} \,.\, \Theta g x\,,$$

$$G\{x + (m,\ n)\} = (-1)^{n} \,.\, \Theta G x\,,$$

$$\mathfrak{G}\{x + (m,\ n)\} = \Theta\mathfrak{G}x.$$

$$\Phi = (-1)^{n(m+\frac{1}{2})}\, \epsilon^{\beta x[(m+\frac{1}{2})\omega - n\nu i]}\, q_1^{-\frac{1}{2}m^2 - \frac{1}{8}m}\, q^{-\frac{1}{2}n^2}.$$

$$\gamma\{x + (\overline{m},\ n)\} = (-1)^{m+n} \,.\, \Phi A g x\,,$$

$$g\{x + (\overline{m},\ n)\} = (-1)^{m} \,.\, \Phi B\gamma x\,,$$

$$G\{x + (\overline{m},\ n)\} = (-1)^{n} \,.\, \Phi C\mathfrak{G}x,$$

$$\mathfrak{G}\{x + (\overline{m},\ n)\} = \Phi D G x.$$

$$\Psi = (-1)^{m(n+\frac{1}{2})}\, \epsilon^{\beta x[m\omega - (n+\frac{1}{2})\nu i]}\, q_1^{-\frac{1}{2}m^2}\, q^{-\frac{1}{2}n^2 - \frac{1}{8}n}$$

$$\gamma\{x + (m,\ \overline{n})\} = (-1)^{m+n} \,.\, \Psi A' G x,$$

$$g\{x + (m,\ \overline{n})\} = (-1)^{m} \,.\, \Psi B'\mathfrak{G}x,$$

$$G\{x + (m,\ \overline{n})\} = (-1)^{n} \,.\, \Psi C'\gamma x\,,$$

$$\mathfrak{G}\{x + (m,\ \overline{n})\} = \Psi D' g x\,.$$

$$\Omega = (-1)^{mn + \frac{1}{2}m + \frac{1}{2}n}\, \epsilon^{\beta x[(m+\frac{1}{2})\omega - (n+\frac{1}{2})\nu i]}\, q_1^{-\frac{1}{2}m^2 - \frac{1}{8}m}\, q^{-\frac{1}{2}n^2 - \frac{1}{8}n}.$$

$$\gamma\{x + (\overline{m},\ \overline{n})\} = (-1)^{m+n} \,.\, \Omega A''\mathfrak{G}x,$$

$$g\{x + (\overline{m},\ \overline{n})\} = (-1)^{m} \,.\, \Omega B'' G x\,,$$

$$G\{x + (\overline{m},\ \overline{n})\} = (-1)^{n} \,.\, \Omega C'' g x\,,$$

$$\mathfrak{G}\{x + (\overline{m},\ \overline{n})\} = \,.\, \Omega D''\gamma x\,.$$

Suppose $x=0$, we have the new systems,

$$\Theta_0 = (-1)^{mn} q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2} \text{ ..... } (M\,bis).$$

$$\gamma(m,\ n) = 0, \qquad \gamma'(m,\ n) = (-1)^{m+n}\Theta_0,$$

$$g(m,\ n) = (-1)^m \Theta_0,$$

$$G(m,\ n) = (-1)^n \Theta_0,$$

$$\mathfrak{G}(m,\ n) = \Theta_0.$$

$$\Phi_0 = (-1)^{n(m+\frac{1}{2})} q_1^{-\frac{1}{2}m^2-\frac{1}{2}m} q^{-\frac{1}{2}n^2}.$$

$$\gamma(\bar{m},\ n) = (-1)^{m+n}\Phi_0 A,$$

$$g(\bar{m},\ n) = 0, \qquad g'(\bar{m},\ n) = (-1)^m \Phi_0 B,$$

$$G(\bar{m},\ n) = (-1)^n \Phi_0 C,$$

$$\mathfrak{G}(\bar{m},\ n) = \Phi_0 D.$$

$$\Psi_0 = (-1)^{m(n+\frac{1}{2})} q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2-\frac{1}{2}n}.$$

$$\gamma(m,\ \bar{n}) = (-1)^{m+n}\Psi_0 A',$$

$$g(m,\ \bar{n}) = (-1)^m \Psi_0 B',$$

$$G(m,\ \bar{n}) = 0, \qquad G'(m,\ \bar{n}) = (-1)^n \Psi_0 C',$$

$$\mathfrak{G}(m,\ \bar{n}) = \Psi_0 D'.$$

$$\Omega_0 = (-1)^{mn+\frac{1}{2}m+\frac{1}{2}n} q_1^{-\frac{1}{2}m^2-\frac{1}{2}m} q^{-\frac{1}{2}n^2-\frac{1}{2}n}.$$

$$\gamma(\bar{m},\ \bar{n}) = (-1)^{m+n}\Omega_0 A'',$$

$$g(\bar{m},\ \bar{n}) = (-1)^m \Omega_0 B'',$$

$$G(\bar{m},\ \bar{n}) = (-1)^n \Omega_0 C'',$$

$$\mathfrak{G}(\bar{m},\ \bar{n}) = 0, \qquad \mathfrak{G}'(\bar{m},\ \bar{n}) = \Omega_0 D''.$$

We obtain immediately, by taking the logarithmic differentials of the functions γx, gx, Gx, $\mathfrak{G}x$, the equations

$$\gamma' x \div \gamma x = \Sigma\Sigma\,\{x-(m,\ n)\}^{-1}, \quad m=0, \quad n=0 \text{ admissible, } \ldots\ldots (N),$$

$$g' x \div gx = \Sigma\Sigma\,\{x-(\bar{m},\ n)\}^{-1},$$

$$G' x \div Gx = \Sigma\Sigma\,\{x-(m,\ \bar{n})\}^{-1},$$

$$\mathfrak{G}' x \div \mathfrak{G}x = \Sigma\Sigma\,\{x-(\bar{m},\ \bar{n})\}^{-1},$$

the limits being the same as in the case of the factorial expressions.

Consider an equation

$$gx\,Gx \div \gamma x\,\mathfrak{G}x = \Sigma\Sigma\,[\mathfrak{A}\,\{x-(m,\ n)\}^{-1} + \mathfrak{B}\,\{x-(\bar{m},\ \bar{n})\}^{-1}] \text{ .....} (29),$$

we have $$\mathfrak{A} = g(m, n)\, G(m, n) \div \gamma'(m, n)\, \mathfrak{G}(m, n) = 1 \quad \text{.....(30)},$$

$$\mathfrak{B} = g(\bar{m}, \bar{n})\, G(\bar{m}, \bar{n}) \div \gamma(\bar{m}, \bar{n})\, \mathfrak{G}'(\bar{m}, \bar{n}) = B''C'' \div A''D'' = -1 \quad \text{... (31)}.$$

(The application of the ordinary method of decomposition into partial fractions, which is in general exceedingly precarious when applied to transcendental functions, is justified here by a theorem of Cauchy's, which will presently be quoted.) We have thus

$$gx\,Gx \div \gamma x\,\mathfrak{G}x = (\gamma' x \div \gamma x) - (\mathfrak{G}' x \div \mathfrak{G} x),$$

and similarly

$$\begin{aligned}
gx\,\mathfrak{G}x \div \gamma x\;\; Gx &= (\gamma' x \div \gamma x) - (G' x \div Gx), \quad \text{.....(O)},\\
Gx\,\mathfrak{G}x \div \gamma x\;\; gx &= (\gamma' x \div \gamma x) - (g' x \div gx),\\
-b^2\gamma x\,\mathfrak{G}x \div gx\;\; Gx &= (g' x \div gx) - (G' x \div Gx),\\
e^2\gamma x\;\; gx \div Gx\,\mathfrak{G}x &= (G' x \div Gx) - (\mathfrak{G}' x \div \mathfrak{G} x),\\
c^2\gamma x\,Gx \div \mathfrak{G}x\;\; gx &= (\mathfrak{G}' x \div \mathfrak{G} x) - (g' x \div gx);
\end{aligned}$$

in which we have written

$$\gamma\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}\left(\frac{\upsilon i}{2}\right) = \frac{i}{e} \quad \text{..... (32)},$$

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},$$

$$\gamma\left(\frac{\omega}{2}\right) \div G\left(\frac{\omega}{2}\right) = -i\left\{\gamma\left(\frac{\upsilon i}{2}\right) \div g\left(\frac{\upsilon i}{2}\right)\right\} = \frac{1}{b}.$$

Eliminating the derived coefficients,

$$\begin{aligned}
G^2x - \mathfrak{G}^2x &= e^2\gamma^2x, \quad \text{..... (33)},\\
g^2x - G^2x &= -b^2\gamma^2x,\\
\mathfrak{G}^2x - g^2x &= c^2\gamma^2x.
\end{aligned}$$

Adding these equations, $b^2 = e^2 + c^2$, or $b = \sqrt{(e^2 + c^2)}$, in which sense it will continue to be used.

Also,

$$\begin{aligned}
g^2x &= \mathfrak{G}^2x - c^2\gamma^2x, \quad \text{..... (P)}.\\
G^2x &= \mathfrak{G}^2x + e^2\gamma^2x.
\end{aligned}$$

Suppose

$$\begin{aligned}
\phi x &= \gamma x \div \mathfrak{G}x, \quad \text{.....(Q)},\\
fx &= gx \div \mathfrak{G}x,\\
Fx &= Gx \div \mathfrak{G}x,
\end{aligned}$$

then

$$\begin{aligned}
f^2x &= 1 - c^2\phi^2x, \quad \text{..... (R)},\\
F^2x &= 1 + e^2\phi^2x,
\end{aligned}$$

and also
$$\phi' x = fx\, Fx\ , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (S),$$
$$f' x = -c^2 \phi x\, Fx,$$
$$F' x = e^2 \phi x f x\ .$$

Hence, putting for fx, Fx, their values,
$$1 = \frac{\phi' x}{\sqrt{(1 - c^2\phi^2 x)(1 + e^2\phi^2 x)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (T);$$
or writing $\phi x = y$, and integrating,
$$x = \int_0^{\phi x} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (U),$$
or
$$\phi^{-1} y = \int_0^y \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}},$$
which shows that ϕ is an inverse elliptic function.

The equations which are the foundation of the theory of the functions ϕ, f, F, are deduced immediately from the equations (S). (Abel, *Œuvres*, tom. I. p. 143 [Ed. 2, p. 268.]) These are
$$\phi(x+y) = \frac{\phi x\, fy\, Fy + \phi y\, fx\, Fx}{1 + e^2c^2\, \phi^2 x\, \phi^2 y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (V),$$
$$f(x+y) = \frac{fx\, fy - c^2\, \phi x\, \phi y\, Fx\, Fy}{1 + e^2c^2\, \phi^2 x\, \phi^2 y},$$
$$F(x+y) = \frac{Fx\, Fy + e^2\, \phi x\, \phi y\, fx\, fy}{1 + e^2c^2\, \phi^2 x\, \phi^2 y};$$
so that from this point we may take for granted any properties of these functions. We see, for instance, immediately,
$$\phi\left(\frac{\upsilon i}{2}\right) = \frac{i}{e}, \quad \phi\left(\frac{\omega}{2}\right) = \frac{1}{c};$$
whence
$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (W),$$
$$\frac{\upsilon i}{2} = \int_0^{\frac{i}{e}} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}}, \quad \text{or} \quad \frac{\upsilon}{2} = \int_0^{\frac{1}{e}} \frac{dy}{\sqrt{(1 + c^2y^2)(1 - e^2y^2)}} \ldots\ldots\ldots (X),$$
which give the values of ω, υ in terms of c, e; values which may be developed in a variety of ways, in infinite series. We may also express $\gamma\left(\frac{\omega}{2}\right)$, &c., and consequently A, B... &c., by means of the quantities c, e. We have only to combine the equations
$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c}, \quad \gamma\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}\,\frac{\upsilon i}{2} = \frac{i}{e}, \quad G\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{b}{c}, \quad g\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}\left(\frac{\upsilon i}{2}\right) = \frac{b}{e} \ldots (34),$$

with the former relations between these quantities, and we have

$$\gamma\left(\frac{\omega}{2}\right)=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}},\qquad \gamma\left(\frac{\upsilon i}{2}\right)=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}},\ \ldots\ldots\ldots\ldots\ldots\ldots(Y),$$

$$g\left(\frac{\omega}{2}\right)=0,\qquad g\left(\frac{\upsilon i}{2}\right)=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}},$$

$$G\left(\frac{\omega}{2}\right)=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}},\qquad \mathfrak{G}\left(\frac{\upsilon i}{2}\right)=0,$$

$$\mathfrak{G}\left(\frac{\omega}{2}\right)=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}},\qquad \mathfrak{G}\left(\frac{\upsilon i}{2}\right)=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}.$$

$$\begin{aligned}
A&=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & A'&=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}, & A''&=(-1)^{\frac{1}{2}}c^{-\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ , & \ldots\ (Z).\\
B&=-b^{\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & B'&=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}\ , & B''&=-(-1)^{\frac{1}{2}}ic^{\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}, & \\
C&=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & C'&=-ib^{\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}\ , & C''&=(-1)^{\frac{1}{2}}c^{-\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ , & \\
D&=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}\ , & D'&=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}\ , & D''&=-(-1)^{\frac{1}{2}}ic^{\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}\ , &
\end{aligned}$$

which are to be substituted in any formulæ into which these quantities enter.

The following is Cauchy's Theorem, (*Exercises de Math.* t. II. p. 289).

"If in attributing to the modulus r of the variable

$$z=r\{\cos p+\sqrt{(-1)}\sin p\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

infinitely great values, these can be chosen so that the two functions

$$\frac{fz+f(-z)}{2},\quad \frac{fz-f(-z)}{2z},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

sensibly vanish, whatever be the value of p, or vanish in general, though ceasing to do so and obtaining *finite* values for certain particular values of p; then

$$fx=\sum\frac{\{(fz)\}}{x-z}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (37),$$

the integral residue being reduced to its principal value."

To understand this, it is only necessary to remark that the integral residue in question is the series of fractions that would be obtained by the ordinary process of decomposition; and by the principal value is meant, that *all* those roots are to be taken, the modulus of which is not greater than a certain limit, this limit being afterwards made infinite.

Suppose now fx is a fraction, the numerator and denominator of which are monomials of the form $(\gamma x)^l(gx)^m\ldots$, l, $m\ldots$ being positive integers, and of course no common factor being left in the numerator and denominator.

Let λ be the excess of the degree of the denominator over that of the numerator. Suppose the modulus r of (z) has any value not the same with any of the moduli of

$$(m,\ n),\quad (\bar{m},\ n),\quad (m,\ \bar{n}),\quad (\bar{m},\ \bar{n})\ \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots(38).$$

Then we have

$$r(\cos p + i \sin p) = m\omega + nvi + \theta \quad \text{.....(39)},$$

θ being a finite quantity, such that none of the functions $J\theta$ vanish. m and n are the greatest integer values which allow the possible part of θ and the coefficient of its impossible part to remain positive. We have therefore

$$m^2\omega^2 + n^2v^2 = r^2 - M \quad \text{.....(40)},$$

M being finite; or when r is infinite, at least one of the values m, n is infinite. The function fz reduces itself to the form

$$q_1^{\frac{1}{2}\lambda m^2} q^{\frac{1}{2}\lambda n^2} \epsilon^{mA+nB} F \quad \text{.....(41)},$$

where F is finite. Hence q_1 and q being always less than unity, fz, and consequently both $\frac{1}{2}\{fz + f(-z)\}$ and $\frac{1}{2z}\{fz - f(-z)\}$ vanish for $r = \infty$, as long as λ is positive.

In the case of $\lambda = 0$, the conditions are still satisfied, if we suppose fx to denote an uneven function of x: for when $\lambda = 0$, the index of exponential in the above expression vanishes, or fz is constantly finite. But fz being an odd function of z, $fz + f(-z) = 0$. And $\frac{1}{2z}\{fz - f(-z)\}$ vanishes for z infinite, on account of the z in the denominator: hence the expansion is admissible in this case. But it is certainly so also, in a great many cases at least, where fz is an even function of z; for these may be deduced from the others by a simple change in the value of the variable. For instance, from the expansion of $\gamma x \div gx$, which is an odd function, by writing $x + \frac{vi}{2}$ for x, we obtain that of $Gx \div \mathfrak{G}x$, which is even.

A case of some importance is when the function is of the above form, multiplied by an exponential $\epsilon^{\frac{1}{2}ax^2+bx}$. Here writing $z = m\omega + nvi + \theta$, the admissibility of the formula depends on the evanescence of

$$\epsilon^{\frac{1}{2}a(m\omega+nvi)^2} q_1^{\frac{1}{2}\lambda m^2} q^{\frac{1}{2}\lambda n^2} \quad \text{.....(42)};$$

or, if $a = h + ki$, this becomes, omitting a finite factor,

$$\epsilon^{-\frac{1}{2}m^2(\lambda\beta - h)\omega^2 - \frac{1}{2}n^2(\lambda\beta + h)v^2 - kmn\omega v} \quad \text{.....(43)},$$

which vanishes if $h^2 + k^2 < \lambda^2\beta^2$, i.e. the modulus of a is less than $\lambda\beta$. The limiting case is admissible when the series is convergent.

We obtain in this way a very great variety of formulæ. For instance,

$$\epsilon^{\frac{1}{2}ax^2+bx} \div \gamma x = \Sigma\Sigma\,[(-1)^{-mn-m-n}\, \epsilon^{\frac{1}{2}a(m,\,n)^2+b(m,\,n)}\, q_1^{\frac{1}{2}m^2} q^{\frac{1}{2}n^2} \{x - (m,\ n)\}^{-1}] \quad \text{.....}(A'),$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div gx = -b^{\frac{1}{2}} c^{-\frac{1}{2}} \Sigma\Sigma\,[(-1)^{-mn-m-\frac{1}{2}n}\, \epsilon^{\frac{1}{2}a(\overline{m},\,n)^2+b(\overline{m},\,n)}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2} q^{\frac{1}{2}n^2} \{x - (\overline{m},\ n)\}^{-1}],$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div Gx = ib^{-\frac{1}{2}} e^{-\frac{1}{2}} \Sigma\Sigma\,[(-1)^{-mn-\frac{1}{2}m-n}\, \epsilon^{\frac{1}{2}a(m,\,\overline{n})^2+b(m,\,\overline{n})}\, q_1^{\frac{1}{2}m^2} q^{\frac{1}{2}(n+\frac{1}{2})^2} \{x - (m,\ \overline{n})\}^{-1}],$$

$$\epsilon^{\frac{1}{2}ax^2+bx} \div \mathfrak{G}x = ic^{-\frac{1}{2}} e^{-\frac{1}{2}} \Sigma\Sigma\,[(-1)^{-(m+\frac{1}{2})(n+\frac{1}{2})}\, \epsilon^{\frac{1}{2}a(\overline{m},\,\overline{n})^2+b(\overline{m},\,\overline{n})}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2} q^{\frac{1}{2}(n+\frac{1}{2})^2} \{x - (\overline{m},\ \overline{n})\}^{-1}],$$

in which the modulus of a must not exceed β: in the limiting cases, for $a=\beta$, b must be entirely impossible, and for $a=-\beta$, b must be entirely real. The formulæ for γx are

$$\epsilon^{\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q^{n^2}\, \epsilon^{b\,(m,\,n)}\, \{x-(m,\,n)\}^{-1} \quad \text{..... (44)},$$

$$\epsilon^{-\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma\,(-1)^{m-n}\, q_1^{m^2}\, \epsilon^{b\,(m,\,n)}\, \{x-(m,\,n)\}^{-1};$$

and for $b=0$,

$$\epsilon^{\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q^{n^2}\, \{x-(m,\,n)\}^{-1} \quad \text{..... (45)},$$

$$\epsilon^{-\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma\,(-1)^{-m-n}\, q_1^{m^2}\, \{x-(m,\,n)\}^{-1}.$$

Next the system,

$$\mathfrak{C}x \div \gamma x = \Sigma\Sigma\,(-1)^{m+n}\, \{x-(m,\,n)\}^{-1} \quad \text{..... } (B').$$

$$gx \div \gamma x = \Sigma\Sigma\,(-1)^{m}\, \{x-(m,\,n)\}^{-1},$$

$$Gx \div \gamma x = \Sigma\Sigma\,(-1)^{n}\, \{x-(m,\,n)\}^{-1};$$

$$\gamma x \div gx = -b^{-1}c^{-1}\,\Sigma\Sigma\,(-1)^{n}\, \{x-(\bar{m},\,n)\}^{-1},$$

$$Gx \div gx = -c^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\, \{x-(\bar{m},\,n)\}^{-1},$$

$$\mathfrak{C}x \div gx = -b^{-1}\,\Sigma\Sigma\,(-1)^{m}\, \{x-(\bar{m},\,n)\}^{-1};$$

$$\gamma x \div Gx = -b^{-1}e^{-1}\,\Sigma\Sigma\,(-1)^{m}\, \{x-(m,\,\bar{n})\}^{-1},$$

$$gx \div Gx = ie^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\, \{x-(m,\,\bar{n})\}^{-1},$$

$$\mathfrak{C}x \div Gx = ib^{-1}\,\Sigma\Sigma\,(-1)^{n}\, \{x-(m,\,\bar{n})\}^{-1};$$

$$\gamma x \div \mathfrak{C}x = ic^{-1}e^{-1}\,\Sigma\Sigma\,(-1)^{m+n}\, \{x-(\bar{m},\,\bar{n})\}^{-1},$$

$$gx \div \mathfrak{C}x = e^{-1}\,\Sigma\Sigma\,(-1)^{m}\, \{x-(\bar{m},\,\bar{n})\}^{-1},$$

$$Gx \div \mathfrak{C}x = ic^{-1}\,\Sigma\Sigma\,(-1)^{n}\, \{x-(\bar{m},\,\bar{n})\}^{-1},$$

which is partially given by Abel.

We may obtain, in like manner, expressions for the functions

$$\frac{1}{\gamma x\, gx},\ \frac{1}{\gamma x\, Gx},\ \ldots \text{ (six terms of this form) } \ldots\ldots\ldots\ldots\ldots\ldots (C'),$$

$$\frac{Gx}{\gamma x\, gx},\ \ldots \qquad \text{(twelve)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (D'),$$

$$\frac{\gamma x\, gx}{\mathfrak{C}x\, Gx},\ \ldots \qquad \text{(six)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (E'),$$

$$\frac{1}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (F'),$$

$$\frac{\mathfrak{C}x}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (G'),$$

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{C}x},\ \ldots \qquad \text{(one)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (H');$$

each of them, except (E'), (the system for which, admitting no exponential, has already been given,) multiplied by an exponential $\epsilon^{\frac{1}{2}ax^2+bx}$, the limits of a being $\pm 2\beta$, $\pm\beta$, $\pm\beta$, $\pm 3\beta$, $\pm 2\beta$, $\pm 4\beta$. For the limiting values, b must be entirely impossible for the superior limit, and entirely possible for the inferior one.

Thus the last case is

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{C}\mathfrak{q}x}\,\epsilon^{\frac{1}{2}ax^2+bx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (H'),$$

$$\begin{aligned}
&= \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(m,\,n)^2+b\,(m,\,n)}\, q_1^{2m^2}\quad q^{2n^2}\quad \{x-(m,\,n)\}^{-1}]\\
&- \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(\overline{m},\,n)^2+b\,(\overline{m},\,n)}\, q_1^{2\,(m+\frac{1}{2})^2}\, q^{2n^2}\quad \{x-(\overline{m},\,n)\}^{-1}]\\
&+ \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(m,\,\overline{n})^2+b\,(m,\,\overline{n})}\, q_1^{2m^2}\quad q^{2\,(n+\frac{1}{2})^2}\,\{x-(m,\,\overline{n})\}^{-1}]\\
&- \Sigma\Sigma\,[\epsilon^{\frac{1}{2}a\,(\overline{m},\,\overline{n})^2+b\,(\overline{m},\,\overline{n})}\, q_1^{2\,(m+\frac{1}{2})^2}\, q^{2\,(n+\frac{1}{2})^2}\,\{x-(\overline{m},\,\overline{n})\}^{-1}];
\end{aligned}$$

in particular

$$\begin{aligned}
\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{C}\mathfrak{q}x}\,\epsilon^{a\beta x^2} &= \Sigma\Sigma q_1^{4m^2}\quad \{x-(m,\,n)\}^{-1} \quad\ldots\ldots\ldots\ldots\ldots(46),\\
&- \Sigma\Sigma\, q_1^{(2m+1)^2}\,\{x-(\overline{m},\,n)\}^{-1}\\
&+ \Sigma\Sigma\, q_1^{4m^2}\quad \{x-(m,\,\overline{n})\}^{-1}\\
&+ \Sigma\Sigma\, q_1^{(2m+1)^2}\,\{x-(\overline{m},\,\overline{n})\}^{-1},
\end{aligned}$$

or the analogous formula obtained by changing β, q_1, m into $-\beta$, q, n.

The function $\phi^2 x$, which is even, and for which $\lambda = 0$, cannot be expanded entirely in a series of partial fractions: but $(x-a)^{-1}\phi^2 x$ may be so expanded. Multiply by $(x-a)$, the second side has for its general term

$$(x-a)\,(Mx+N)\,\{x-(\overline{m},\,\overline{n})\}^{-2},$$

equivalent to

$$K' + (M'x+N')\,\{x-(\overline{m},\,\overline{n})\}^{-2}.$$

Summing all the K''s, we have an equation of the form

$$\phi^2 x = A + \Sigma\Sigma\,[L\,\{x-(\overline{m},\,\overline{n})\}^{-2} + M\,\{x-(\overline{m},\,\overline{n})\}^{-1}] \quad\ldots\ldots\ldots\ldots\ldots (47).$$

To determine the coefficients as simply as possible, change x into $x+\frac{1}{2}\omega+\frac{1}{2}n\omega i$,

$$-e^{-2}c^{-2}(\phi x)^{-2} = A + \Sigma\Sigma\,[L\,\{x-(m,\,n)\}^{-2} + M\,\{x-(m,\,n)\}^{-1}] \ldots\ldots\ldots (48),$$

$$L = -e^{-2}c^{-2}\,[\{x-(m,\,n)\}^2\,(\phi x)^{-2}],\ x=(m,\,n) \quad\ldots\ldots\ldots\ldots(49),$$

$$M = -e^{-2}c^{-2}\,\partial_x\,[\{x-(m,\,n)\}^2\,(\phi x)^{-2}],$$

or writing $x+(m,\,n)$ for x, and therefore $x=0$ in the values of L and M,

$$L = -e^{-2}c^{-2}\,\{x^2\,(\phi x)^{-2}\} = e^{-2}c^{-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

$$M = -e^{-2}c^{-2}\,\partial_x\,\{x^2\,(\phi x)^{-2}\} = 0\,;$$

whence

$$\phi^2 x = A - e^{-2}c^{-2}\,\Sigma\Sigma\,\{x-(\overline{m},\,\overline{n})\}^{-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51).$$

Integrating this last equation twice,

$$\int_0 dx \int_0 dx\, \phi^2 x = \tfrac{1}{2} A x^2 + e^{-2} c^{-2} \Sigma\Sigma l \{x - (\overline{m}, \overline{n})\} \quad \text{.....} (52),$$

or

$$\mathfrak{G}x = \epsilon^{-\frac{1}{2} e^2 c^2 A x^2 + e^2 c^2 \int_0 dx \int_0 dx\, \phi^2 x} \quad \text{.....} (53),$$

an equation from which it is easy to determine the coefficient A.

Suppose for a moment $\phi_{,} x = \int_0 \phi^2 x\, dx$, $\phi_{,,} x = \int_0 \phi_{,} x\, dx$; then, since $\phi^2(x+\omega) - \phi^2 x = 0$,

$$\phi_{,}(x+\omega) - \phi x = \phi_{,}\omega, \quad \phi_{,,}(x+\omega) - \phi_{,} x = \phi_{,,}\omega + x\phi_{,}\omega.$$

But similarly $\phi^2 x - \phi^2(\omega - x) = 0$; whence

$$\phi_{,} x + \phi_{,}(\omega - x) = \phi_{,}\omega, \quad \phi_{,,} x - \phi_{,,}(\omega - x) + \phi_{,,}\omega = x\phi_{,}\omega;$$

whence, writing $x = \frac{\omega}{2}$,

$$\phi_{,}\omega = 2\phi_{,}\left(\frac{\omega}{2}\right), \quad \phi_{,,}\omega = \omega\phi_{,}\left(\frac{\omega}{2}\right), \quad \text{or} \quad \phi_{,,}(x+\omega) - \phi_{,,} x = \phi_{,}\left(\frac{\omega}{2}\right)(2x+\omega).$$

Hence

$$\mathfrak{G}(x+\omega) = \epsilon^{-\frac{1}{2} e^2 c^2 \{A - \frac{1}{2}\omega\phi_{,}(\frac{1}{2}\omega)\}(2\omega x + \omega^2)}\, \mathfrak{G}x \quad \text{.....} (54).$$

But

$$\mathfrak{G}(x+\omega) = \epsilon^{\beta\omega x} q_1^{-\frac{1}{2}}\, \mathfrak{G}x = \epsilon^{\frac{1}{2}\beta(2\omega x + \omega^2)}\, \mathfrak{G}x \quad \text{.....} (55);$$

or, comparing these,

$$-e^2 c^2 \left\{A - \frac{2}{\omega}\phi_{,}\left(\frac{\omega}{2}\right)\right\} = \beta \quad \text{.....} (56),$$

$$-\tfrac{1}{2} e^2 c^2 A = \tfrac{1}{2}\beta - \frac{e^2 c^2}{\omega}\phi_{,}\left(\frac{\omega}{2}\right) \quad \text{.....} (57),$$

or writing

$$M = \frac{e^2 c^2}{\omega}\int_0^{\frac{1}{2}\omega} \phi^2 x\, dx \quad \text{.....} (58),$$

then

$$\mathfrak{G}x = \epsilon^{(\frac{1}{2}\beta - M)x^2 + e^2 c^2 \int_0 dx \int_0 dx\, \phi^2 x} \quad \text{.....} (I');$$

which is the formula corresponding to the one of Jacobi's referred to at the beginning of this paper. Analogous formulæ may be deduced from it by writing $x + \frac{\omega}{2}$, or $x + \frac{\nu i}{2}$, or $x + \frac{\omega}{2} + \frac{\nu i}{2}$, instead of x.

The following formulæ, making the necessary changes of notation, are taken from Jacobi. We have

$$\phi^2(x+a) - \phi^2(x-a) = \frac{4\phi a\, fa\, Fa\, \phi x\, fx Fx}{(1 + e^2 c^2 \phi^2 a\, \phi^2 x)^2} \quad \text{.....} (59),$$

whence $$\int_0 \{\phi^2(x+a) - \phi^2(x-a)\}\,dx = \frac{2\phi a\, fa\, Fa\, \phi^2 x}{1 + e^2c^2\phi^2 a\phi^2 x} \quad \text{.....(60)},$$

the first side of which is

$$\int_{-a} \phi^2(x+a)\,dx - \int_a \phi^2(x-a)\,dx - 2\int_0 \phi^2 a\,da \quad \text{.....(61)}.$$

Hence, multiplying by e^2c^2, and observing the value of $\mathfrak{G}x$,

$$\frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} - \frac{\mathfrak{G}'(x-a)}{\mathfrak{G}(x-a)} - 2\frac{\mathfrak{G}'a}{\mathfrak{G}a} = \frac{2e^2c^2\, fa\, Fa\, \phi a\, \phi^2 x}{1 + e^2c^2\phi^2 a\phi^2 x} \quad \text{.....(62)}.$$

If in this case we interchange x, a and add,

$$\frac{\mathfrak{G}'x}{\mathfrak{G}x} + \frac{\mathfrak{G}'a}{\mathfrak{G}a} - \frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} = e^2c^2\,\phi a\,\phi x\,\phi(a+x) \quad \text{.....(63)}.$$

[By subtracting, we should have obtained an equation only differing from the above in the sign of a.]

Integrating the last equation but one, with respect to a,

$$l\mathfrak{G}(x+a) + l\mathfrak{G}(x-a) - 2l\mathfrak{G}x - 2l\mathfrak{G}a = l(1 + e^2c^2\phi^2 x\phi^2 a),$$

the integral being taken from $a = 0$. Hence

$$\mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2 x\,\mathfrak{G}^2 a\,(1 + e^2c^2\phi^2 x\phi^2 a) \quad \text{.....(64)};$$

or

$$\left.\begin{aligned} \mathfrak{G}(x+a)\,\mathfrak{G}(x-a) &= \mathfrak{G}^2 x\,\mathfrak{G}^2 a + e^2c^2\gamma^2 x\gamma^2 a, \\ \text{whence also} \quad & \\ \gamma(x+a)\;\gamma(x-a) &= \gamma^2 x\,\mathfrak{G}^2 a - \gamma^2 a\,\mathfrak{G}^2 x, \\ g(x+a)\;g(x-a) &= g^2 x\,\mathfrak{G}^2 a - c^2 g^2 a\,\mathfrak{G}^2 x, \\ G(x+a)\;G(x-a) &= G^2 x\,\mathfrak{G}^2 a + e^2 G^2 a\,\mathfrak{G}^2 x, \end{aligned}\right\} \quad \text{.....}(J),$$

these equations being obtained from the first by the change of x into $x + \frac{\omega}{2}$, $x + \frac{\nu i}{2}$, $x + \frac{\omega}{2} + \frac{\nu i}{2}$. They form a most important group of formulæ in the present theory. By integrating the same formulæ with respect to x, and representing by $\Pi(x, a)$ the integral $\int_0 \frac{-e^2c^2\,\phi a\, fa\, Fa\, \phi^2 x dx}{1 + e^2c^2\phi^2 a\phi^2 x}$, Jacobi obtains

$$\Pi(x, a) = \tfrac{1}{2} l\frac{\mathfrak{G}(x-a)}{\mathfrak{G}(x+a)} + x\frac{\mathfrak{G}'a}{\mathfrak{G}a}:$$

an equation which conducts him almost immediately to the formulæ for the addition of the argument or of the parameter in the function Π. This, however, is not very

closely connected with the present subject. For some formulæ also deduced from (63), by which $\frac{\mathfrak{G}(x-a)\,\mathfrak{G}(y-a)\,\mathfrak{G}(x+y+a)}{\mathfrak{G}(x+a)\,\mathfrak{G}(y+a)\,\mathfrak{G}(x+y-a)}$ is expressed in terms of the function ϕ, see Jacobi.

NOTE.—We have

$$\gamma_\beta x = x\Pi\Pi\left(1+\frac{x}{(m,\ n)}\right).$$

$$g_\beta x = \ \Pi\Pi\left(1+\frac{x}{(\bar{m},\ n)}\right),$$

the limits of n being $\pm q$, and those of m being $\pm p$, in the first case, and p, $-p-1$, in the second case. Also $\frac{p}{q}=\infty$.

We deduce immediately

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=\left(x+\frac{\omega}{2}\right)\Pi\Pi\left\{1+\frac{\left(x+\frac{\omega}{2}\right)}{(m,\ n)}\right\}=\Pi\Pi\left(1+\frac{x}{(\bar{m},\ n)}\right)\div\frac{\omega}{2}\,\Pi\Pi\frac{(m,\ n)}{(\bar{m},\ n)}$$

(paying attention to the omission of $(m=0,\ n=0)$ in $\gamma_\beta x$, and supposing that this value enters into the numerator of the expression just obtained, but not into its denominator). This is of the form

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=A\Pi\Pi\left(1+\frac{x}{(\bar{m},\ n)}\right);$$

but the limits are not the same in this product and in $g_\beta x$. In the latter m assumes the value $-p-1$, which it does not in the former; hence

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)\div g_\beta x=A\div\Pi_n\left(1+\frac{x}{-(p+\frac{1}{2})\,\omega+nvi}\right),$$

and the above product reduces itself to unity in consequence of all the values assumed by n being indefinitely small compared with the quantity $(p+\frac{1}{2})$; we have therefore

$$\gamma_\beta\left(x+\frac{\omega}{2}\right)=Ag_\beta x \quad\text{.....(65),}$$

and similar expressions for the remaining functions. To illustrate this further, suppose we had been considering, instead of $\gamma_\beta x$, the function $\gamma_{-\beta} x$, given by the same formula, but with $\frac{p}{q}=0$, instead of $\frac{p}{q}=\infty$. We have in this case also

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)\div g_{-\beta} x=A'\div\Pi_n\left(1+\frac{x}{(-p+\frac{1}{2})\,\omega+nvi}\right),$$

A' different from A on account of the different limits. The divisor of the second side takes the form

$$\{x-(p+\tfrac{1}{2})\,\omega\}\,\Pi\left(1+\frac{x-(p+\frac{1}{2})\,\omega}{nvi}\right)\div(-p+\tfrac{1}{2})\,\omega\Pi\left(1+\frac{(p+\frac{1}{2})\,\omega}{nvi}\right),$$

where the extreme values of n are infinite as compared with p. This may be reduced to

$$-\sin\frac{\pi}{vi}\{x-(p+\tfrac{1}{2})\,\omega\}\div\sin(p+\tfrac{1}{2})\,\omega,$$

$$=\epsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\,\omega-x]}\div\epsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\,\omega]}=\epsilon^{-\frac{\pi x}{v}},$$

neglecting the exponentials whose indices are infinitely great and negative. Observing the value of β this becomes $\epsilon^{-\omega\beta x}$, and we have

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)=\epsilon^{\beta\omega x}\,.\,A'g_{-\beta}\,.\,x:$$

a result of the form of that which would be deduced from the equations $\gamma_{-\beta}x=\epsilon^{\beta x^2}\gamma_\beta x$, $g_{-\beta}x=\epsilon^{\beta x^2}g_\beta x$, $\gamma_\beta\left(x+\frac{\omega}{2}\right)=Ag_\beta x$. It is scarcely necessary to remark that $\gamma_{-\beta}x$ has the same relations to the change of x into $x+\frac{vi}{2}$ as $\gamma_\beta x$ has to that of x into $x+\frac{\omega}{2}$.

25.

MÉMOIRE SUR LES FONCTIONS DOUBLEMENT PÉRIODIQUES.

[From the *Journal des Mathématiques* (Liouville), tom. x. (1845), pp. 385—420.]

Un des plus beaux résultats des recherches de l'illustre Abel, dans la théorie des fonctions elliptiques, consiste dans les expressions qu'il obtint pour les fonctions inverses $\phi\alpha$, $f\alpha$, $F\alpha$ (équivalentes à-peu-près à $\sin \operatorname{am}\alpha$, $\cos \operatorname{am}\alpha$, $\Delta \operatorname{am}\alpha$) en forme de fractions avec un dénominateur commun: ce dénominateur et les trois numérateurs étant chacun le produit d'une suite infinie double de facteurs. On ne sait pas à quel point Abel avait poussé l'investigation des propriétés de ces nouvelles fonctions; on trouve seulement, dans une Lettre à Legendre, imprimée parmi ses Œuvres [t. II. p. 259, Ed. 2, p. 274], qu'il s'en était occupé. Depuis, les fonctions H, Θ, qui sont essentiellement les mêmes que ces fonctions d'Abel, ont été l'objet des savantes recherches de M. Jacobi, à qui l'on doit, en particulier, la belle formule

$$\log \Theta(\alpha) - \log \Theta(0) = \tfrac{1}{2}\alpha^2\left(1 - \frac{E'}{K}\right) - k^2 \int_0^\alpha d\alpha \int_0^\alpha d\alpha \sin^2 \operatorname{am}\alpha,$$

qui est vraiment fondamentale, et sur laquelle on peut dire que sa théorie est basée. Mais les expressions qu'obtient M. Jacobi pour les fonctions H, Θ, sont sous la forme d'un produit d'une suite infinie simple de facteurs, ce qui ne met pas à beaucoup près si bien en évidence la vraie nature de ces fonctions que les expressions d'Abel; celles-ci sont, en outre, si analogues aux formules en produits infinis des fonctions circulaires, que l'on est seulement étonné que personne ne se soit avisé jusqu'ici de les poser, à priori, comme les définitions les plus simples des fonctions doublement périodiques, pour en déduire la théorie de ces fonctions. C'est de cette manière que je me propose de traiter ici la question. Je prends pour définitions les formules d'Abel, en supposant, pour plus de généralité, que les fonctions complètes Ω, Υ (K, K' de M. Jacobi) sont chacune de la forme $A + B\sqrt{-1}$ (ce qui donne lieu à quelques intégrations assez délicates). Et de ces seules équations, sans me servir en rien de la

théorie des fonctions elliptiques, je déduis les propriétés fondamentales des fonctions en question, et de là des fonctions elliptiques. On a ainsi quatre fonctions à considérer au lieu des deux H, Θ, dont l'une est pour ainsi dire analogue à un sinus et les autres à des cosinus. Mais ce qu'il y a de remarquable, c'est l'apparition d'un facteur exponentiel qui entre presque partout. On le prévoyait d'après les formules de M. Jacobi, mais ces formules n'expliquent pas, ce me semble, pourquoi ce facteur s'y rencontre: mon analyse le fait voir de la manière la plus satisfaisante. Ce facteur résulte, en effet, de ce que, pour les produits infinis doubles, il ne suffit pas, pour obtenir un résultat déterminé, d'attribuer aux deux entiers variables des valeurs quelconques depuis $-\infty$ jusqu'à ∞, même en supposant l'égalité des valeurs positives et négatives; il faut, en outre, établir une relation entre les valeurs infinies que reçoivent les deux variables. Mais dans les produits que je considère, on démontre qu'en supposant toujours cette égalité des valeurs positives et négatives, quelque liaison que l'on établisse entre les valeurs infinies, il résulte toujours la même valeur du produit, à un facteur exponentiel près, dont l'indice est le carré de x, multiplié par une constante dont la valeur s'exprime au moyen d'une intégrale définie double, et qui dépend de la liaison établie entre les valeurs infinies des variables. C'est-à-dire qu'en multipliant par un facteur exponentiel de cette forme, convenablement choisi, on peut changer à volonté la relation en question sans affecter la valeur du produit. Voilà l'idée fondamentale du Mémoire qui suit.

En me servant de quelques formules de M. Cauchy, relatives à la décomposition des fonctions en fractions simples, j'établis d'une manière rigoureuse des relations entre les trois quotients de mes quatre fonctions, qui sont les mêmes par lesquelles Abel démontre les propriétés fondamentales des fonctions ϕ, f, F. Ces formules une fois obtenues, on peut supposer connue toute la théorie des fonctions elliptiques. Ces théorèmes de M. Cauchy me conduisent, en outre, à un grand nombre de nouvelles formules qui contiennent des suites infinies doubles. Parmi celles-ci, il y en a une qui me fournit la démonstration du théorème déjà cité de M. Jacobi, théorème duquel il déduit une foule de résultats intéressants. Je finis en citant ceux qui se rapportent de plus près aux fonctions dont je parle. J'espère reprendre une autre fois la considération d'une autre partie de la théorie, dans laquelle j'entrevois des conclusions intéressantes.

Soient Ω, Υ des quantités finies quelconques, assujetties à la seule condition que la fraction $\Omega : \Upsilon$ ne soit pas réelle. En représentant par m, n des entiers positifs ou négatifs quelconques, mettons, pour abréger,

$$m\Omega + n\Upsilon = (m, n) \quad \text{.....} \quad (1),$$

et considérons une expression de cette forme

$$u = x\Pi \left\{1 + \frac{x}{(m, n)}\right\} \quad \text{.....} \quad (2),$$

où le symbole Π dénote, comme à l'ordinaire, le produit d'un nombre infini de facteurs que l'on obtient en donnant à m, n des valeurs entières quelconques, depuis $-\infty$ jusqu'à

$+\infty$, en excluant seulement la combinaison ($m=0$, $n=0$). Pour qu'une telle expression soit finie, il faut que pour chaque combinaison de valeurs de m, n, il y en ait une autre des mêmes valeurs avec les signes contraires. Cependant, comme on l'a déjà expliqué, cela ne suffit pas pour rendre déterminée la valeur de u. Soit ϕ une fonction de m, n qui ne change pas en changeant à la fois les signes de ces deux quantités; et imaginons que l'équation

$$\phi = T$$

représente une courbe fermée, dont tous les points s'éloignent, au cas limite de $T=\infty$, d'une distance infinie de l'origine. Cela posé, en donnant à m, n des valeurs entières qui satisfassent à cette condition $\phi \overset{=}{<} T$, et puis faisant $T=\infty$, on obtient pour u une valeur parfaitement déterminée, qui dépend de la forme de la fonction ϕ. Soit u' ce que devient la fonction u en changeant seulement l'équation aux limites dans l'équation analogue

$$\phi' = T;$$

on peut, pour simplifier, supposer que la courbe représentée par cette équation soit située entièrement en dehors de celle que représente l'équation

$$\phi = T,$$

mais cela n'est pas essentiel. Il est facile de trouver une relation très-simple qui existe entre ces deux expressions u' et u. En effet,

$$u' : u = \Pi \left\{1 + \frac{x}{(m,\ n)}\right\},$$

en donnant à m, n des valeurs qui satisfassent à la fois aux deux conditions $\phi > T$, $\phi' < T$. Donc, en considérant toujours ces valeurs,

$$\log u' - \log u = \log \Pi \left\{1 + \frac{x}{(m,\ n)}\right\} = \Sigma \log \left\{1 + \frac{x}{(m,\ n)}\right\} = x \Sigma \frac{1}{(m,\ n)} - \tfrac{1}{2}x^2 \Sigma \frac{1}{(m,\ n)^2} + \ldots$$

Dans cette expression, les termes qui contiennent les puissances impaires x s'évanouissent, à cause des valeurs égales positives et négatives. Mais puisque, à la limite, m, n ne reçoivent que des valeurs infinies, on peut négliger les termes multipliés par x^4, &c. Donc

$$\log u' - \log u = -\tfrac{1}{2}x^2 \Sigma \frac{1}{(m,\ n)^2},$$

ou enfin,

$$\log u' - \log u = -\tfrac{1}{2}Ax^2, \quad u' = u\epsilon^{-\frac{1}{2}Ax^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

où j'ai représenté par ϵ la base du système hyperbolique de logarithmes, et où A est donné par l'équation

$$A = \Sigma \frac{1}{(m,\ n)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Il est facile de voir que l'on peut changer la sommation en intégration double, et écrire

$$A = \iint \frac{dm\,dn}{(m,\ n)^2} \quad \text{.....} \quad (5),$$

entre les mêmes limites qu'auparavant, c'est-à-dire que m, n doivent satisfaire aux deux conditions $\phi > T$, $\phi' < T$.

Prenons par exemple, pour limite supérieure,

$$(m^2 = \mathrm{m}^2, \quad n^2 = \mathrm{n}^2),$$

et pour limite inférieure,

$$(m^2 + n^2 = T^2);$$

m, n sont censés contenir T comme facteur, de manière qu'ils deviennent infinis avec cette quantité; on suppose aussi $\mathrm{m} > T$, $\mathrm{n} > T$, mais cela est seulement pour la clarté.

Il devient nécessaire à ce point de définir de plus près les valeurs de Ω, Υ; nous écrirons

$$\Omega = \omega + \omega' i, \quad \Upsilon = \upsilon + \upsilon' i \quad \text{.....} (6),$$

où $i = \sqrt{-1}$; ω, ω', υ, υ' sont des quantités réelles, telles que $\omega\upsilon' - \omega'\upsilon$ ne s'évanouit pas; c'est la condition pour que $\Omega : \Upsilon$ contienne une partie imaginaire.

Avec les coordonnées polaires

$$A = \iint \frac{dr\,d\theta}{r(\Omega\cos\theta + \Upsilon\sin\theta)^2} = \int \frac{d\theta\,(\log \mathrm{r} - \log T)}{(\Omega\cos\theta + \Upsilon\sin\theta)^2} \quad \text{.....} (7),$$

en représentant par r ce que devient r à la limite supérieure; l'intégrale doit être prise depuis $\theta = 0$ jusqu'à $\theta = 2\pi$. On voit tout de suite que la partie qui contient $\log T$ s'évanouit; donc

$$A = \int_0^{2\pi} \frac{\log \mathrm{r}\,d\theta}{(\Omega\cos\theta + \Upsilon\sin\theta)^2}.$$

Soit α un angle positif plus petit que $\frac{1}{2}\pi$, tel que $\tan\alpha = \frac{\mathrm{m}}{\mathrm{n}}$, on a évidemment

$$\mathrm{r} = \frac{\pm\,\mathrm{m}}{\cos\theta},$$

depuis $\theta = -\alpha$ jusqu'à $\theta = +\alpha$, ou depuis $\theta = \pi - \alpha$ jusqu'à $\theta = \pi + \alpha$, et

$$\mathrm{r} = \frac{\pm\,\mathrm{n}}{\cos\theta},$$

depuis $\theta = \alpha$ jusqu'à $\theta = \pi - \alpha$, ou depuis $\theta = \pi + \alpha$ jusqu'à $\theta = 2\pi - \alpha$ (le signe ambigu, de manière que r soit toujours positif). En réunissant les parties opposées de l'intégrale, on obtient

$$A = 2\int_{-\alpha}^{\alpha} \frac{(\log \mathrm{m} - \log\cos\theta)\,d\theta}{(\Omega\cos\theta + \Upsilon\sin\theta)^2} + 2\int_{\alpha}^{\pi-\alpha} \frac{(\log \mathrm{n} - \log\sin\theta)\,d\theta}{(\Omega\cos\theta + \Upsilon\sin\theta)^2},$$

ou, en mettant dans la seconde intégrale $\frac{1}{2}\pi - \alpha = \alpha'$, et $\frac{1}{2}\pi - \theta$ au lieu de θ,

$$A = 2\int_{-\alpha}^{\alpha} \frac{(\log \mathrm{m} - \log\cos\theta)\,d\theta}{(\Omega\cos\theta + \Upsilon\sin\theta)^2} + 2\int_{-\alpha'}^{\alpha} \frac{(\log \mathrm{n} - \log\cos\theta)\,d\theta}{(\Upsilon\cos\theta + \Omega\sin\theta)^2}.$$

La première intégrale se réduit à

$$-2(\log \mathrm{m} - \log\cos\theta)\frac{1}{\Upsilon(\Omega + \Upsilon\tan\theta)} \text{ (entre les limites)}$$

$$+\frac{2}{\pi}\int_{-\alpha}^{\alpha} \frac{\tan\theta\,d\theta}{(\Omega + \Upsilon\tan\theta)}$$

$$= \frac{4(\log \mathrm{m} - \log\cos\alpha)\tan\alpha}{\Omega^2 - \Upsilon^2\tan^2\alpha} - 4\int_0^{\alpha} \frac{\tan^2\theta\,d\theta}{\Omega^2 - \Upsilon^2\tan^2\theta}$$

$$= \frac{4(\log \mathrm{m} - \log\cos\alpha)\tan\alpha}{\Omega^2 - \Upsilon^2\tan^2\alpha} + \frac{4}{\Omega^2 + \Upsilon^2}\int_0^{\alpha} d\theta\left[1 - \frac{\Omega^2(1 + \tan^2\theta)}{\Omega^2 - \Upsilon^2\tan^2\theta}\right].$$

L'intégrale, dans cette formule

$$\int_0^{\alpha} \frac{(1 + \tan^2\theta)\,d\theta}{\Omega^2 - \Upsilon^2\tan^2\theta} = \int_0^{\tan\alpha} \frac{dx}{\Omega^2 - \Upsilon^2 x^2},$$

s'exprime tout de suite par les logarithmes, mais il faut apporter une attention particulière à la manière de déterminer quelle valeur doit être attribuée à ce logarithme, dans le cas où la partie réelle en est négative. Je renvoie cette discussion à une Note. Voici le résultat auquel j'arrive.

En représentant par Lu la valeur principale du logarithme de u, quand il y a une valeur principale (c'est-à-dire quand la partie réelle de u est positive), et écrivant

$$L_{\pm\pi i}u = Lu, \text{ ou } L(-u) \pm \pi i,$$

selon qu'il y a une valeur principale de $\log u$, ou de $\log(-u)$, on a

$$\int_0^{\tan\alpha} \frac{dx}{\Omega^2 - \Upsilon^2 x^2} = \frac{1}{2\Omega\Upsilon} L_{\pm\pi i}\left(\frac{\Omega + \Upsilon\tan\alpha}{\Omega - \Upsilon\tan\alpha}\right) \quad \text{.....(8)},$$

en prenant le signe supérieur pour $\omega v' - \omega' v$ positif, et le signe inférieur pour $\omega v' - \omega' v$ négatif.

La première partie de A se réduit donc à

$$\frac{4(\log \mathrm{m} - \log\cos\alpha)\tan\alpha}{\Omega^2 - \Upsilon^2\tan^2\alpha} + \frac{4}{\Omega^2 + \Upsilon^2}\left(\alpha - \frac{1}{2}\frac{\Omega}{\Upsilon} L_{\pm\pi i}\frac{\Omega + \Upsilon\tan\alpha}{\Omega - \Upsilon\tan\alpha}\right),$$

ou, en rétablissant la valeur de α, à

$$\frac{4\mathrm{mn}\log\sqrt{\mathrm{m}^2 + \mathrm{n}^2}}{\mathrm{m}^2\Omega^2 - \mathrm{n}^2\Upsilon^2} + \frac{4}{\Omega^2 + \Upsilon^2}\left(\arctan\frac{\mathrm{n}}{\mathrm{m}} - \frac{1}{2}\frac{\Omega}{\Upsilon} L_{\pm\pi i}\frac{\mathrm{m}\Omega + \mathrm{n}\Upsilon}{\mathrm{m}\Omega - \mathrm{n}\Upsilon}\right).$$

Pour avoir la seconde partie de A, il faut changer m, n, Ω, Υ en n, m, Υ, Ω. En faisant cela, on change le signe de $\omega v' - \omega' v$; ainsi il faut écrire $L_{\mp\pi i}$ au lieu de $L_{\pm\pi i}$. En réunissant les deux parties, et mettant $\frac{1}{2}\pi$ au lieu de arc tan $\frac{\mathrm{n}}{\mathrm{m}}$ + arc tan $\frac{\mathrm{m}}{\mathrm{n}}$, on obtient enfin

$$A = \frac{2}{\Omega^2 + \Upsilon^2}\left\{\pi + \frac{\Omega}{\Upsilon} L_{\pm\pi i}\left(\frac{\mathrm{m}\Omega + \mathrm{n}\Upsilon}{\mathrm{m}\Omega - \mathrm{n}\Upsilon}\right) - \frac{\Upsilon}{\Omega} L_{\pm\pi i}\left(\frac{\mathrm{n}\Upsilon + \mathrm{m}\Omega}{\mathrm{n}\Upsilon - \mathrm{m}\Omega}\right)\right\} \quad \text{.....(9)},$$

formule que l'on pourrait rendre plus simple en distinguant les deux cas où le premier ou le second logarithme a une valeur principale; mais il vaut mieux la laisser sous cette forme.

On aurait pu croire que la valeur de A pourrait s'obtenir plus facilement en réduisant A à la différence de deux intégrales définies, les conditions pour les limites étant données, dans la première, par $m^2 < \mathrm{m}^2$, $n^2 < \mathrm{n}^2$, et, pour la seconde, par $m^2 + n^2 < T^2$, en désignant généralement les coordonnées par m, n. Cependant, de cette manière, on admet dans les deux intégrales les valeurs $m = 0$, $n = 0$, qui rendent infinie la fonction à intégrer. Ainsi la valeur de l'intégrale dépendrait du choix des variables, et l'on obtiendrait des résultats inexacts. Autrement dit, on obtiendrait de cette façon la valeur de A, à une quantité ∇ près, qui est la différence de deux intégrales de la même forme, entre des limites $m^2 < \mu^2$, $n^2 < \nu^2$ ou $m^2 + n^2 < \tau^2$, μ, ν, τ des quantités infiniment petites. On voit tout de suite que ∇ n'est pas pour cela infiniment petit (en effet, sa valeur dépend du rapport $\mu : \nu$ et nullement des valeurs absolues de ces quantités), et, pour en trouver la valeur, il faudrait se servir de l'analyse précédente.

Soit, comme exemple, $\frac{\mathrm{m}}{\mathrm{n}} = \infty$,

$$A = \frac{2}{\Omega^2 + \Upsilon^2}\left\{\pi - \frac{\Omega}{\Upsilon} L_{\pm\pi i}(1) - \frac{\Upsilon}{\Omega} L_{\mp\pi i}(-1)\right\}$$

$$= \frac{2\pi}{\Omega^2 + \Upsilon^2}\left(1 \pm \frac{\Upsilon i}{\Omega}\right) = \frac{2\pi}{\Omega(\Omega \mp \Upsilon i)} = -\frac{2\pi i}{\Omega\Upsilon} \cdot \frac{\Upsilon}{\Omega \mp \Upsilon i}.$$

De même, pour $\frac{\mathrm{n}}{\mathrm{m}} = \infty$,

$$A = \frac{2}{\Omega^2 + \Upsilon^2}\left\{\pi - \frac{\Omega}{\Upsilon} L_{\pm\pi i}(-1) - \frac{\Pi}{\Upsilon} L_{\mp\pi i}(1)\right\}$$

$$= \frac{2\pi}{\Omega^2 + \Upsilon^2}\left(1 \mp \frac{\Omega i}{\Upsilon}\right) = \pm \frac{2\pi i}{\Upsilon(\Omega \mp \Upsilon)} = \mp \frac{2\pi i}{\Omega\Upsilon} \cdot \frac{\Omega}{\Omega \mp \Upsilon i}.$$

En représentant par A', A'' ces deux valeurs de A, on a

$$A' - A'' = \frac{\mp 2\pi i}{\Omega\Upsilon} = -2\mathcal{E} \quad \text{.....(10)},$$

où j'écris

$$\mathcal{E} = \pm \frac{\pi i}{\Omega\Upsilon} \quad \text{.....(11)},$$

en prenant toujours le signe supérieur pour $\omega v' - \omega' v$ positif, et le signe inférieur pour $\omega v' - \omega' v$ négatif.

Soient u_ε ce que devient u en prenant $\frac{m}{n} = \infty$, $u_{-\varepsilon}$ ce que devient cette même fonction en prenant $\frac{n}{m} = \infty$; on a évidemment

$$u_{-\varepsilon} : u_\varepsilon = \epsilon^{-\frac{1}{2}(A'-A'')x^2} = \epsilon^{\varepsilon x^2}.$$

On peut donc s'imaginer une fonction U donnée par les équations

$$U = \epsilon^{-\frac{1}{2}\varepsilon x^2} u_{-\varepsilon} = \epsilon^{\frac{1}{2}\varepsilon x^2} u_\varepsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12).$$

On verra dans la suite que u_ε est analogue à la fonction $H(u)$ de M. Jacobi. Il convient cependant, pour la symétrie, de considérer, au lieu de u_ε, la nouvelle fonction U qui vient d'être définie. Quoique suffisamment donnée par ces équations, nous allons en chercher encore une autre définition. Pour cela, il faut trouver la valeur de l'intégrale définie qui détermine A, prise depuis la même limite inférieure jusqu'à celle donnée par l'équation

$$\text{mod}\,(m, n) = T.$$

Mettons, comme auparavant,

$$m = r\cos\theta, \qquad n = r\sin\theta;$$

soient aussi

$$\Omega_1 = \omega - \omega' i, \quad \Upsilon_1 = v - v' i.$$

L'équation pour la limite supérieure se réduit à

$$r^2(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega_1\cos\theta + \Upsilon_1\sin\theta) = T^2,$$

et celle pour la limite inférieure, à

$$r = T.$$

On trouve tout de suite

$$\left.\begin{aligned}
A &= -\tfrac{1}{2}\int_0^{2\pi} \frac{\log(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega_1\cos\theta + \Upsilon_1\sin\theta)\,d\theta}{(\Omega\cos\theta + \Upsilon\sin\theta)^2} \\
&= \frac{1}{2\Upsilon}\frac{1}{(\Omega + \Upsilon\tan\theta)}\log(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega_1\cos\theta + \Upsilon_1\sin\theta)\ (\text{entre } \theta = 0,\ \theta = 2\pi) \\
&\quad - \frac{1}{2\Upsilon}\int_0^{2\pi}\frac{d\theta}{\Omega + \Upsilon\tan\theta}\left(\frac{\Upsilon - \Omega\tan\theta}{\Omega + \Upsilon\tan\theta} + \frac{\Upsilon_1 - \Omega_1\tan\theta}{\Omega_1 + \Upsilon_1\tan\theta}\right) \\
&= -\frac{1}{\Upsilon}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{d\theta}{\Omega + \Upsilon\tan\theta}\left(\frac{\Upsilon - \Omega\tan\theta}{\Omega + \Upsilon\tan\theta} + \frac{\Upsilon_1 - \Omega_1\tan\theta}{\Omega_1 + \Upsilon_1\tan\theta}\right)
\end{aligned}\right\} (13).$$

D'abord

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega + \Upsilon \tan\theta} \frac{\Upsilon_1 - \Omega_1 \tan\theta}{\Omega_1 + \Upsilon_1 \tan\theta} = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left(\frac{M}{\Omega + \Upsilon \tan\theta} + \frac{M_1}{\Omega_1 + \Upsilon_1 \tan\theta} \right) d\theta,$$

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega + \Upsilon \tan\theta} = 2\Omega \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\Omega^2 - \Upsilon^2 \tan^2\theta} = \frac{2\Omega}{\Omega^2 + \Upsilon^2} \int_0^{\frac{1}{2}\pi} \left[1 + \frac{\Upsilon^2 (1 + \tan^2\theta)}{\Omega^2 - \Upsilon^2 \tan^2\theta} \right] d\theta$$

$$= \frac{2\Omega}{\Omega^2 + \Upsilon^2} \left\{ \theta + \frac{\Upsilon}{2\Omega} L_{\pm\pi i} \left(\frac{\Omega + \Upsilon \tan\theta}{\Omega - \Upsilon \tan\theta} \right) \right\} \quad (\text{entre } \theta = 0, \ \theta = \tfrac{1}{2}\pi)$$

$$= \frac{2\Omega}{\Omega^2 + \Upsilon^2} \left(\tfrac{1}{2}\pi \pm \frac{\Upsilon}{2\Omega} \pi i \right) = \frac{\pi}{\Omega \mp \Upsilon i}.$$

On a de même, en remarquant qu'en changeant Ω, Υ en Υ, Ω, on change le signe de $\omega v' - \omega' v$,

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega_1 + \Upsilon_1 \tan\theta} = \frac{\pi}{\Omega_1 \pm \Upsilon_1 i}.$$

D'un autre côté,

$$M = \frac{\Upsilon\Upsilon_1 + \Omega\Omega_1}{\Upsilon\Omega_1 - \Upsilon_1\Omega}, \qquad M_1 = -\frac{\Upsilon_1^2 + \Omega_1^2}{\Upsilon\Omega_1 - \Upsilon_1\Omega};$$

cela donne

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega + \Upsilon \tan\theta} \frac{\Upsilon_1 - \Omega_1 \tan\theta}{\Omega_1 + \Upsilon_1 \tan\theta} = \frac{\pi}{\Upsilon\Omega_1 - \Upsilon_1\Omega} \left\{ \frac{\Upsilon\Upsilon_1 + \Omega\Omega_1}{\Omega \mp \Upsilon i} - (\Omega \mp \Upsilon i) \right\}$$

$$= \frac{\pi}{\Upsilon\Omega_1 - \Upsilon_1\Omega} \frac{\Upsilon\Upsilon_1 + \Omega\Omega_1 - (\Omega \mp \Upsilon i)(\Omega_1 \pm \Upsilon_1 i \mp 2\Upsilon_1 i)}{\Omega \mp \Upsilon i} = \pm \frac{\pi i}{\Omega \mp \Upsilon i} \pm \frac{2\Upsilon_1 \pi i}{\Upsilon\Omega_1 - \Upsilon_1\Omega},$$

ou enfin

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega + \Upsilon \tan\theta} \frac{\Upsilon_1 - \Omega_1 \tan\theta}{\Omega_1 + \Upsilon_1 \tan\theta} = \pm \frac{\pi i}{\Omega \mp \Upsilon i} \pm \frac{\pi \Upsilon_1}{\omega v' - \omega' v};$$

et de même

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta\,(\Upsilon - \Omega \tan\theta)}{(\Omega + \Upsilon \tan\theta)^2} = \frac{\pm \pi i}{\Omega \mp \Upsilon i},$$

en omettant seulement le dernier terme de l'autre intégrale ; ce que l'on peut vérifier, au reste, en différentiant par rapport à Ω, Υ l'équation

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\theta}{\Omega + \Upsilon \tan\theta} = \frac{\pi}{\Omega \mp \Upsilon i}.$$

On a donc, en ajoutant les deux parties qui composent l'intégrale,

$$A = \mp \frac{2\pi i}{\Omega\Upsilon} \frac{\Omega}{\Omega \mp \Upsilon i} \mp \frac{\Upsilon_1 \pi}{\Upsilon(\omega v' - \omega' v)} \quad \text{.....} \quad (14).$$

Il est facile de voir, en écrivant

$$A = \mp \frac{2\pi i}{\Omega\Upsilon} \frac{\Omega}{\Omega \mp \Upsilon i} \pm \frac{\pi \Upsilon_1 i}{\Upsilon(\Upsilon\Omega_1 - \Upsilon_1\Omega)},$$

que cette expression ne change pas de valeur en mettant Ω_1, Υ_1, Ω, Υ au lieu de Ω, Υ, Ω_1, Υ_1, pourvu qu'on change aussi le signe de i; cela doit évidemment être ainsi et peut servir de vérification.

Soit, pour un moment, u_1 ce que devient u en prenant pour limite l'équation

$$m^2 + n^2 = T^2,$$

et supposons que dans la fonction u on ait pour limite l'équation

$$\operatorname{mod}(m, n) = T.$$

En retenant la valeur de A, qui vient d'être trouvée,

$$u = \epsilon^{-\frac{1}{2}Ax^2} u_1;$$

mais aussi

$$U = \epsilon^{\mp\frac{1}{2}\frac{\pi i}{\Omega\Upsilon}x^2} u_{-\xi} = \epsilon^{\left(\mp\frac{1}{2}\frac{\pi i}{\Omega\Upsilon} \pm \frac{\pi i}{\Omega\Upsilon}\frac{\Omega}{\Omega \mp \Upsilon i}\right)x^2} u_1,$$

à cause de l'équation, conséquence facile des résultats précédents,

$$u_{-\xi} = \epsilon^{\pm\frac{\pi i}{\Omega\Upsilon}\frac{\Omega}{\Omega \mp \Upsilon i}x^2} u_1.$$

En éliminant u_1, on obtient, entre u, U, une équation de la forme

$$U = \epsilon^{-\frac{1}{2}Bx^2} u \text{.....} (15),$$

dans laquelle

$$B = -A \pm \frac{\pi i}{\Omega\Upsilon} \mp \frac{2\pi i}{\Omega \mp \Upsilon i} = \pm \frac{\pi i}{\Omega\Upsilon} \pm \frac{\Upsilon_1 \pi}{\Upsilon(\omega v' - \omega' v)}$$

$$= \frac{\pm \pi}{\Omega\Upsilon(\omega v' - \omega' v)} [2(\omega v' - \omega' v) + (\omega + \omega' i)(v - v' i)],$$

ou enfin

$$B = \frac{\pm \pi(\omega v + \omega' v')}{\Omega\Upsilon(\omega v' - \omega' v)} = \frac{\pi(\omega v' + \omega' v')}{\Omega\Upsilon \operatorname{mod}(\omega v' - \omega' v)} \text{.....} (16).$$

En rassemblant tous ces résultats, on a le système de formules

$$
(\mathrm{A})\quad \left\{
\begin{array}{l}
U = \epsilon^{-\frac{1}{2}Bx^2}\, u = \epsilon^{-\frac{1}{2}\mathcal{E}x^2}\, u_{-\mathcal{E}} = \epsilon^{\frac{1}{2}\mathcal{E}x^2}\, u_{\mathcal{E}}, \\
u = \epsilon^{\frac{1}{2}(B-\mathcal{E})\,x^2}\, u_{-\mathcal{E}} = \epsilon^{\frac{1}{2}(B+\mathcal{E})\,x^2}\, u_{\mathcal{E}}, \\
\mathcal{E} = \dfrac{\pi i}{\Omega\Upsilon}\, \dfrac{(\omega\upsilon' - \omega'\upsilon)}{\mathrm{mod}\,(\omega\upsilon' - \omega'\upsilon)}, \\
B = \dfrac{\pi\,(\omega\upsilon + \omega'\upsilon')}{\mathrm{mod}\,(\omega\upsilon' - \omega'\upsilon)}, \\
u,\ u_{\mathcal{E}},\ u_{-\mathcal{E}} \text{ des fonctions de la forme} \\
\qquad x\Pi\left\{1 + \dfrac{x}{(m,\ n)}\right\}, \\
\text{mais avec des limites différentes, savoir :} \\
\mathrm{mod}\,(m,\ n) = T,\quad T = \infty,\quad \text{pour la fonction } u\,; \\
m^2 = \mathrm{m}^2,\ n^2 = \mathrm{n}^2,\quad \mathrm{m} = \infty,\quad \mathrm{n} = \infty,\quad \dfrac{\mathrm{m}}{\mathrm{n}} = \infty,\ \text{pour } u_{\beta}\,; \\
m^2 = \mathrm{m}^2,\ n^2 = \mathrm{n}^2,\quad m = \infty,\quad \mathrm{n} = \infty,\quad \dfrac{\mathrm{n}}{\mathrm{m}} = \infty,\ \text{pour } u_{-\mathcal{E}}.
\end{array}
\right.
$$

A présent, il importe de remarquer que la fonction $u_{\mathcal{E}}$ est périodique à l'égard de Ω, de la manière d'un sinus ou d'un cosinus, mais ne l'est pas à l'égard de Υ; de même $u_{-\mathcal{E}}$ est périodique à l'égard de Υ, mais non à l'égard de Ω. Quant à U, u, elles ne sont simplement périodiques ni à l'égard de Ω, ni à l'égard de Υ. Pour démontrer cela, considérons, par exemple, l'équation

$$u_{\mathcal{E}} = x\,\Pi\left(1 + \frac{x}{(m,\ n)}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

[m depuis $-\mathrm{m}$ jusqu'à m, n depuis $-\mathrm{n}$ jusqu'à n, $\mathrm{m} = \infty$, $\mathrm{n} = \infty$, $\dfrac{\mathrm{m}}{\mathrm{n}} = \infty$, le système $(m = 0,\ n = 0)$ toujours exclu]; en représentant par $u'_{\mathcal{E}}$ ce que devient $u_{\mathcal{E}}$ quand on écrit $x + \Omega$ au lieu de x, on a évidemment

$$
\begin{aligned}
u'_{\mathcal{E}} &= (x + \Omega)\,\Pi\left(1 + \frac{x + \Omega}{(m,\ n)}\right) \\
&= -\,x\,\Pi\left(1 + \frac{1}{(m+1,\ n)}\right) : \Pi\,\frac{(m+1,\ n)}{(m,\ n)},
\end{aligned}
$$

en admettant dans le premier produit la combinaison $(m = 0,\ n = 0)$, mais excluant $(m + 1 = 0,\ n = 0)$, et excluant l'une et l'autre du second produit. Cette équation est de la forme

$$u'_{\mathcal{E}} = A'x\Pi\left(1 + \frac{x}{(m+1,\ n)}\right),$$

avec les mêmes limites que dans u; donc

$$u'_{\mathcal{E}} : u_{\mathcal{E}} = A'\Pi\left(1 + \frac{x}{(\mathrm{m}+1,\ n)}\right) : \Pi\left(1 + \frac{x}{(-\mathrm{m},\ n)}\right),$$

où Π se rapporte à la seule variable n qui doit s'étendre depuis $-\mathrm{n}$ jusqu'à n. Puisqu'ainsi n ne reçoit jamais des valeurs qui soient comparables à m, chacun de ces produits se réduit à l'unité, et l'on obtient

$$u'_{\mathfrak{g}} : u_{\mathfrak{g}} = A',$$

ou enfin

$$u'_{\mathfrak{g}} = -u_{\mathfrak{g}} \quad \text{.....} \quad (18),$$

en posant

$$x = -\tfrac{1}{2}\Omega,$$

et remarquant que u est fonction impaire de x, ce qui donne

$$A' = -1.$$

Soit de même $u''_{\mathfrak{g}}$ ce que devient $u_{\mathfrak{g}}$ en changeant x en $x + \Upsilon$. On a pareillement

$$u''_{\mathfrak{g}} : u_{\mathfrak{g}} = A''\Pi\left(1 + \frac{x}{(m,\ \mathrm{n}+1)}\right) : \Pi\left(1 + \frac{x}{(m,\ -\mathrm{n})}\right),$$

où Π se rapporte à la seule variable m. Mais ici les produits ne se réduisent point à l'unité; en effet,

$$\Pi\left(1 + \frac{x}{(m,\ \mathrm{n}+1)}\right) \text{ ou } \Pi\left(1 + \frac{x}{(m,\ \mathrm{n})}\right)$$

$$= (x + \mathrm{n}\Upsilon)\,\Pi\left(1 + \frac{x - \mathrm{n}\Upsilon}{m\Omega}\right) : \mathrm{n}\Upsilon\Pi\left(1 + \frac{\mathrm{n}\Upsilon}{m\Omega}\right)$$

$$= \sin\frac{\pi}{\Omega}(x + \mathrm{n}\Upsilon) : \sin\frac{\pi}{\Omega}\mathrm{n}\Upsilon.$$

Mais quand la partie imaginaire de θ est infinie, on trouve

$$\sin\theta = \frac{1}{2i}(\epsilon^{\theta i} - \epsilon^{-\theta i}) = \pm\frac{1}{2i}\epsilon^{\pm\theta i},$$

selon que la partie réelle de θi est positive ou négative. Or la partie réelle de $\dfrac{i\Upsilon}{\Omega}$ est de signe contraire à $\omega\nu' - \omega'\nu$; on a donc

$$\Pi\left(1 + \frac{x}{(m,\ \mathrm{n}+1)}\right) = \epsilon^{\mp\frac{\pi i}{\Omega}x} = \epsilon^{-\mathfrak{g}\Upsilon x};$$

de même,

$$\Pi\left(1 + \frac{x}{(m,\ -\mathrm{n})}\right) = \epsilon^{\mathfrak{g}\Upsilon x},$$

et de là

$$u''_{\mathfrak{g}} = A''\,\epsilon^{-2\mathfrak{g}\Upsilon x}\,u_{\mathfrak{g}} \quad \text{.....} \quad (19),$$

équation de la même forme que celle que l'on obtiendrait en posant

$$u_{\mathfrak{g}} = \epsilon^{-\mathfrak{g}x^2}\,u_{-\mathfrak{g}},$$

et remarquant que $u_{-\mathfrak{e}}$ est périodique à l'égard de Υ. On voit aussi qu'en sommant par rapport à m, il est permis d'exprimer $u_{\mathfrak{e}}$ par un produit infini simple, dont chaque facteur est de la forme

$$\sin\frac{\pi}{\Omega}(x+n\Upsilon),$$

mais qu'on n'arrive pas à un tel résultat, en sommant par rapport à n. En effet, l'équation

$$u_{\mathfrak{e}} = \epsilon^{-\mathfrak{e}x^2} u_{-\mathfrak{e}}$$

fait voir que $u_{\mathfrak{e}}$ s'exprime par un tel produit où chaque terme est de la forme

$$\sin\frac{\pi}{\Upsilon}(x+m\Omega),$$

multiplié par le facteur exponentiel $\epsilon^{-\mathfrak{e}x^2}$. On établit de cette façon l'identité, à un facteur constant près, de $u_{\mathfrak{e}}$ à $H(u)$; mais, à présent, pour éviter les longueurs, je ne me propose de considérer aucun de ces produits infinis simples.

Il faut maintenant, pour le développement de la théorie, introduire les trois fonctions qui correspondent au cosinus. Mettant, pour abréger,

$$\bar{m} = m + \tfrac{1}{2}, \quad \bar{n} = n + \tfrac{1}{2} \quad \text{.....(20)},$$

j'écris

$$\text{(B)} \quad \left\{\begin{array}{ll} \mathrm{y}x = \epsilon^{-\frac{1}{2}Bx^2} x \Pi \left\{1 + \dfrac{x}{(m,\ n)}\right\}. & \text{Limites. } \mathrm{mod}\ (m,\ n) = T,\ \ T = \infty. \\[2ex] \mathrm{g}x = \epsilon^{-\frac{1}{2}Bx^2}\ \Pi \left\{1 + \dfrac{x}{(\bar{m},\ n)}\right\}. & \mathrm{mod}\ (\bar{m},\ n) = T. \\[2ex] Gx = \epsilon^{-\frac{1}{2}Bx^2}\ \Pi \left\{1 + \dfrac{x}{(m,\ \bar{n})}\right\}. & \mathrm{mod}\ (m,\ \bar{n}) = T. \\[2ex] Zx = \epsilon^{-\frac{1}{2}Bx^2}\ \Pi \left\{1 + \dfrac{x}{(\bar{m},\ \bar{n})}\right\}. & \mathrm{mod}\ (\bar{m},\ \bar{n}) = T. \end{array}\right.$$

En représentant par $\mathrm{y}_{\mathfrak{e}}x$, $\mathrm{y}_{-\mathfrak{e}}x$, &c., ce que deviennent ces fonctions, et prenant pour limites

$$(m^2 = \mathrm{m}^2,\ n^2 = \mathrm{n}^2), \quad (\bar{m}^2 = \bar{\mathrm{m}}^2,\ n^2 = \mathrm{n}^2),$$
$$(m^2 = \mathrm{m}^2,\ \bar{n}^2 = \bar{\mathrm{n}}^2), \quad (\bar{m}^2 = \bar{\mathrm{m}}^2,\ \bar{n}^2 = \bar{\mathrm{n}}^2),$$

où $\dfrac{\mathrm{m}}{\mathrm{n}} = \infty$ pour $\mathrm{y}_{\mathfrak{e}}x$, &c., $\dfrac{\mathrm{n}}{\mathrm{m}} = \infty$ pour $\mathrm{y}_{-\mathfrak{e}}x$, &c.,

on a, en général, en mettant J au lieu de l'une quelconque des lettres y, g, G, Z,

$$Jx = \epsilon^{\frac{1}{2}\mathfrak{e}x^2} J_{\mathfrak{e}}x = \epsilon^{-\frac{1}{2}\mathfrak{e}x^2} J_{-\mathfrak{e}}x \text{.....(21)},$$

où $J_{\mathfrak{e}}x$ est périodique à l'égard de Ω, et $J_{-\mathfrak{e}}x$ à l'égard de Υ. C'est cette équation remarquable qui définit la loi de périodicité des fonctions J, et de laquelle se déduisent presque toutes les propriétés de ces fonctions.

Il est clair qu'en changeant entre elles les quantités Ω, Υ (quantités que nous désignerons comme les *fonctions complètes*), on ne change pas les fonctions yx, Zx, tandis que gx se change en Gx, et Gx en gx. On change de cette manière $\mathcal{E}$ en $-\mathcal{E}$ et B en $-B$ (par exemple, $y_{\mathcal{E}}x$ se change en $y_{-\mathcal{E}}x$, &c.). Cette considération fait voir que toute propriété des fonctions J est double, et donne le moyen le plus facile pour passer d'une propriété quelconque à la propriété correspondante.

On tire immédiatement des définitions mêmes les équations

$$\text{(C)} \qquad \left\{\begin{array}{llll} y(-x) = -yx, & g(-x) = gx, & G(-x) = Gx, & Z(-x) = Zx; \\ y0 = 0, & g0 = 1, & G0 = 1, & Z0 = 1; \\ y'0 = 1. & & & \end{array}\right.$$

Il est facile de démontrer, de la même manière dont nous avons prouvé la périodicité de $J_{\mathcal{E}}x$ à l'égard de Ω, que ces fonctions se changent l'une en l'autre, à un facteur constant près, en changeant x en $x+\frac{1}{2}\Omega$. Faisant donc attention à l'équation qui lie ensemble Jx et $J_{\mathcal{E}}x$, on obtient le système de formules

$$\left.\begin{array}{l} y\,(x+\frac{1}{2}\,\Omega) = \epsilon^{\frac{1}{2}\beta\Omega x}\,Agx, \\ g\,(x+\frac{1}{2}\,\Omega) = \epsilon^{\frac{1}{2}\beta\Omega x}\,Byx, \\ G\,(x+\frac{1}{2}\,\Omega) = \epsilon^{\frac{1}{2}\beta\Omega x}\,CZx, \\ Z\,(x+\frac{1}{2}\,\Omega) = \epsilon^{\frac{1}{2}\beta\Omega x}\,DZx. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (22)$$

Pour déterminer A, B, C, D, posons

$$x = 0 \quad \text{ou} \quad x = -\tfrac{1}{2}\,\Omega.$$

En écrivant, pour abréger,

$$\text{(D)} \qquad \epsilon^{\mathcal{E}\Omega^2} = \epsilon^{\pm\frac{\pi\Omega i}{\Upsilon}} = q_1^{-1},$$

on trouve

$$\left.\begin{array}{l} A = y\,(\tfrac{1}{2}\Omega), \\ B = -\,q_1^{-\frac{1}{4}} : y\,(\tfrac{1}{2}\,\Omega), \\ C = G\,(\tfrac{1}{2}\,\Omega) = q_1^{-\frac{1}{4}} : Z\,(\tfrac{1}{2}\,\Omega), \\ D = Z\,(\tfrac{1}{2}\,\Omega) = q_1^{-\frac{1}{4}} : G\,(\tfrac{1}{2}\,\Omega); \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (23)$$

d'où l'on déduit cette équation de condition,

$$G\,(\tfrac{1}{2}\,\Omega)\,Z\,(\tfrac{1}{2}\,\Omega) = q_1^{-\frac{1}{4}} \quad \ldots\ldots\ldots\ldots \quad (24).$$

On a de même le système

$$\left.\begin{array}{l} y\,(x+\frac{1}{2}\,\Upsilon) = \epsilon^{-\frac{1}{2}\mathcal{E}\Upsilon x}\,A'Gx, \\ g\,(x+\frac{1}{2}\,\Upsilon) = \epsilon^{-\frac{1}{2}\mathcal{E}\Upsilon x}\,B'\,Zx, \\ G\,(x+\frac{1}{2}\,\Upsilon) = \epsilon^{-\frac{1}{2}\mathcal{E}\Upsilon x}\,C'\,yx, \\ Z\,(x+\frac{1}{2}\,\Upsilon) = \epsilon^{-\frac{1}{2}\mathcal{E}\Upsilon x}\,D'Gx, \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (25).$$

De là, si

$$\epsilon^{-\epsilon\Upsilon^2} = \epsilon^{\pm\frac{\pi\Upsilon i}{\Omega}} = q^{-1},$$

ce qui donne

$$\text{(E)} \qquad \log q \log q_1 = -\pi^2,$$

il résulte

$$\left.\begin{aligned} A' &= \mathrm{y}\left(\tfrac{1}{2}\Upsilon\right), \\ B' &= \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right) = q^{-\frac{1}{4}} : Z\left(\tfrac{1}{2}\Upsilon\right), \\ C' &= -q^{-\frac{1}{4}} : \mathrm{y}\left(\tfrac{1}{2}\Upsilon\right), \\ D' &= Z\left(\tfrac{1}{2}\Upsilon\right) : \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right); \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (26);$$

d'où nous tirons l'équation de condition

$$\mathrm{g}\left(\tfrac{1}{2}\Upsilon\right) Z\left(\tfrac{1}{2}\Upsilon\right) = q^{-\frac{1}{4}} \quad \ldots\ldots\ldots\ldots (27).$$

En posant $x = \frac{1}{2}\Upsilon$ dans l'équation pour $\mathrm{y}(x + \frac{1}{2}\Omega)$ et $x = \frac{1}{2}\Omega$ dans l'équation pour $\mathrm{y}(x + \frac{1}{2}\Upsilon)$, on obtient aussi

$$\mathrm{y}\left(\tfrac{1}{2}\Omega\right) \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right) = -i\mathrm{y}\left(\tfrac{1}{2}\Upsilon\right) G\left(\tfrac{1}{2}\Omega\right) \quad \ldots\ldots\ldots\ldots (28),$$

et l'on déduirait cette même équation, ou une équation équivalente, des autres formules, de manière qu'on ne peut pas trouver d'autres équations de condition.

Enfin, en rapprochant ces systèmes, on obtient

$$\left.\begin{aligned} \mathrm{y}\left(x + \tfrac{1}{2}\Omega + \tfrac{1}{2}\Upsilon\right) &= \epsilon^{\frac{1}{2}\beta x(\Omega - \Upsilon)} A'' Zx, \\ \mathrm{g}\left(x + \tfrac{1}{2}\Omega + \tfrac{1}{2}\Upsilon\right) &= \epsilon^{\frac{1}{2}\beta x(\Omega - \Upsilon)} B'' gx, \\ G\left(x + \tfrac{1}{2}\Omega + \tfrac{1}{2}\Upsilon\right) &= \epsilon^{\frac{1}{2}\beta x(\Omega - \Upsilon)} C'' Gx, \\ Z\left(x + \tfrac{1}{2}\Omega + \tfrac{1}{2}\Upsilon\right) &= \epsilon^{\frac{1}{2}\beta x(\Omega - \Upsilon)} D'' \mathrm{y}x, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (29),$$

avec les équations suivantes pour les coefficients,

$$\left.\begin{aligned} A'' &= (-1)^{\frac{1}{4}} \mathrm{y}\left(\tfrac{1}{2}\Omega\right) \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right), \\ B'' &= -(-1)^{\frac{1}{4}} q_1^{-\frac{1}{4}} \mathrm{y}\left(\tfrac{1}{2}\Upsilon\right) : \mathrm{y}\left(\tfrac{1}{2}\Omega\right), \\ C'' &= (-1)^{\frac{1}{4}} R\left(\tfrac{1}{2}\Omega\right) Z\left(\tfrac{1}{2}\Upsilon\right), \\ D'' &= -(-1)^{\frac{1}{4}} q^{-\frac{1}{4}} Z\left(\tfrac{1}{2}\Omega\right) : \mathrm{y}\left(\tfrac{1}{2}\Upsilon\right). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (30).$$

En rassemblant les équations entre $J(\frac{1}{2}\Omega)$, $J(\frac{1}{2}\Upsilon)$, on a

$$\left.\begin{aligned} \mathrm{g}\left(\tfrac{1}{2}\Omega\right) &= 0, \\ G\left(\tfrac{1}{2}\Upsilon\right) &= 0, \\ G\left(\tfrac{1}{2}\Omega\right) Z\left(\tfrac{1}{2}\Omega\right) &= q_1^{-\frac{1}{4}}, \\ \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right) Z\left(\tfrac{1}{2}\Upsilon\right) &= q^{-\frac{1}{4}}, \\ \mathrm{y}\left(\tfrac{1}{2}\Omega\right) \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right) &= -i\mathrm{y}\left(\tfrac{1}{2}\Upsilon\right) G\left(\tfrac{1}{2}\Omega\right), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (31),$$

d'où l'on conclut les formules

$$\left.\begin{array}{l}
B''C'' : A''D'' = B'D' : A'C' = CD : AB = -1, \\
A'B' : C'D' = -A''B'' : C''D'' = -\mathrm{y}^2(\tfrac{1}{2}\Upsilon) : Z^2(\tfrac{1}{2}\Upsilon). \\
A''C'' : B''D'' = -A'C' : B'D' = \mathrm{y}^2(\tfrac{1}{2}\Omega) : Z^2(\tfrac{1}{2}\Omega), \\
A\,D : B\,C = -A'D' : B'C' = \mathrm{y}^2(\tfrac{1}{2}\Omega) : G^2(\tfrac{1}{2}\Omega), \\
\qquad\qquad\qquad\qquad\qquad\qquad = -\mathrm{y}^2(\tfrac{1}{2}\Upsilon) : \mathrm{g}^2(\tfrac{1}{2}\Upsilon),
\end{array}\right\} \ldots\ldots\ldots(32),$$

dont nous aurons bientôt besoin.

On peut passer maintenant à ce système général de formules, où j'écris $(-)^m$ au lieu de $(-1)^m$:

$$(\mathrm{F})\quad \left\{\begin{array}{ll}
\Theta = (-)^{mn}\,\epsilon^{\xi x\,(m,\,-n)}\,q_1^{-\frac{1}{2}m^2}\,q^{-\frac{1}{2}n^2}, & \\
\text{là, comme toujours, } (m,\,-n) = m\Omega - n\Upsilon\,; & \\
\mathrm{y}\,\{x+(m,\,n)\} = (-)^{m+n}\,\Theta\mathrm{y}x, & \\
\mathrm{g}\,\{x+(m,\,n)\} = (-)^{m}\,\Theta\mathrm{g}x, & \\
G\,\{x+(m,\,n)\} = (-)^{n}\,\Theta Gx, & \\
Z\,\{x+(m,\,n)\} = \Theta Zx\,; & \\
\Phi = (-)^{nm+\frac{1}{2}n}\,\epsilon^{\xi x\,(\overline{m},\,-n)}\,q_1^{-\frac{1}{2}m^2-\frac{1}{2}m}\,q^{-\frac{1}{2}n^2}\,; & \\
\mathrm{y}\,\{x+(\overline{m},\,n)\} = (-)^{m+n}\,\Phi A\mathrm{g}x, & \\
\mathrm{g}\,\{x+(\overline{m},\,n)\} = (-)^{m}\,\Phi B\mathrm{y}x, & \\
G\,\{x+(\overline{m},\,n)\} = (-)^{n}\,\Phi CZx, & \\
Z\,\{x+(\overline{m},\,n)\} = \Phi DGx\,; & \\
\Psi = (-)^{mn+\frac{1}{2}m}\,\epsilon^{\xi x\,(m,\,-\overline{n})}\,q_1^{-\frac{1}{2}m^2}\,q^{-\frac{1}{2}n^2-\frac{1}{2}n}\,; & \\
\mathrm{y}\,\{x+(m,\,\overline{n})\} = (-)^{m+n}\,\Psi A'Gx, & \\
\mathrm{g}\,\{x+(m,\,\overline{n})\} = (-)^{m}\,\Psi B'Zx, & \\
G\,\{x+(m,\,\overline{n})\} = (-)^{n}\,\Psi C'\mathrm{y}x, & \\
Z\,\{x+(m,\,\overline{n})\} = \Psi D'\mathrm{g}x\,; & \\
\mathrm{X} = (-)^{mn+\frac{1}{2}m+\frac{1}{2}n}\,\epsilon^{\xi x\,(\overline{m},\,-\overline{n})}\,q_1^{-\frac{1}{2}m^2-\frac{1}{2}m}\,q^{-\frac{1}{2}n^2-\frac{1}{2}n}\,; & \\
\mathrm{y}\,\{x+(\overline{m},\,\overline{n})\} = (-)^{m+n}\,\mathrm{X}A''Zx, & \\
\mathrm{g}\,\{x+(\overline{m},\,\overline{n})\} = (-)^{m}\,\mathrm{X}B''Gx, & \\
G\,\{x+(\overline{m},\,\overline{n})\} = (-)^{n}\,\mathrm{X}C''\mathrm{g}x, & \\
Z\,\{x+(\overline{m},\,\overline{n})\} = \mathrm{X}D''\mathrm{y}x. &
\end{array}\right.$$

Soit $x = 0$:

$$
(\mathrm{G})\quad \left\{
\begin{array}{ll}
\Theta_0 = (-)^{mn}\, q_1^{-\frac{1}{2}m^2}\, q^{-\frac{1}{2}n^2}; & \\
\mathrm{y}\,(m,\ n) = \quad 0, & \mathrm{y}'\,(m,\ n) = (-)^{m+n}\,\Theta_0, \\
\mathrm{g}\,(m,\ n) = (-)^m\,\Theta_0, & \\
G\,(m,\ n) = (-)^n\,\Theta_0, & \\
Z\,(m,\ n) = \quad \Theta_0; & \\
\Phi_0 = (-)^{mn+\frac{1}{2}m}\, q_1^{-\frac{1}{2}m^2-\frac{1}{2}m-\frac{1}{2}n^2}; & \\
\mathrm{y}\,(\overline{m},\ n) = (-)^{m+n}\,\Phi_0 A, & \\
\mathrm{g}\,(\overline{m},\ n) = \quad 0, & \mathrm{g}'\,(\overline{m},\ n) = (-)^m\,\Phi_0 B, \\
G\,(\overline{m},\ n) = (-)^n\,\Phi_0 C, & \\
Z\,(\overline{m},\ n) = \quad \Phi_0 D; & \\
\Psi_0 = (-)^{mn+\frac{1}{2}n}\, q_1^{-\frac{1}{2}m^2}\, q^{-\frac{1}{2}n^2-\frac{1}{2}n}; & \\
\mathrm{y}\,(m,\ \overline{n}) = (-)^{m+n}\,\Psi_0 A', & \\
\mathrm{g}\,(m,\ \overline{n}) = (-)^m\,\Psi_0 B', & \\
G\,(m,\ \overline{n}) = \quad 0, & G'\,(m,\ \overline{n}) = (-)^n\,\Psi_0 C', \\
Z\,(m,\ \overline{n}) = \quad \Psi_0 D'; & \\
\mathrm{X}_0 = (-)^{mn+\frac{1}{2}m+\frac{1}{2}n}\, q_1^{-\frac{1}{2}m^2-\frac{1}{2}m}\, q^{-\frac{1}{2}n^2-\frac{1}{2}n}; & \\
\mathrm{y}\,(\overline{m},\ \overline{n}) = (-)^{m+n}\,\mathrm{X}_0 A'', & \\
\mathrm{g}\,(\overline{m},\ \overline{n}) = (-)^m\,\mathrm{X}_0 B'', & \\
G\,(\overline{m},\ \overline{n}) = (-)^n\,X_0 C'', & \\
Z\,(\overline{m},\ \overline{n}) = \quad 0, & Z'\,(\overline{m},\ \overline{n}) = X_0 D''.
\end{array}
\right.
$$

On a de suite, en prenant les différentielles des logarithmes des fonctions Jx,

$$
(\mathrm{H})\quad \left\{
\begin{array}{l}
\mathrm{y}'x\ :\ \mathrm{y}x = -Bx + \sum \{x - (m,\ n)\}^{-1}, \\
\mathrm{g}'x\ :\ \mathrm{g}x = -Bx + \sum \{x - (\overline{m},\ n)\}^{-1}, \\
G'x\ :\ Gx = -Bx + \sum \{x - (m,\ \overline{n})\}^{-1}, \\
Z'x\ :\ Zx = -Bx + \sum \{x - (\overline{m},\ \overline{n})\}^{-1}.
\end{array}
\right.
$$

(B est le coefficient de x^2, dans l'exponentielle $\epsilon^{-\frac{1}{2}Bx^2}$ des équations (B); il n'y a pas à craindre de le confondre avec le B qui entre dans les équations pour $\mathrm{y}\,(\overline{m},\ n)$, &c.: c'est par hasard que j'ai pris deux fois la même lettre.)

Réduisons en fractions simples la fonction

$$\mathrm{g}x\, Gx \,:\, \mathrm{y}x\, Zx.$$

En écrivant

$$\mathrm{g}x Gx \,:\, \mathrm{y}x Zx = \sum [L\,\{x-(m,\ n)\}^{-1} + M\,\{x-(\overline{m},\ \overline{n})\}^{-1}],$$

on obtient

$$L = \mathrm{g}\,(m,\ n)\, G\,(m,\ n) \,:\, \mathrm{y}'\,(m,\ n)\, Z\,(m,\ n) = 1,$$

$$M = \mathrm{g}\,(\overline{m},\ \overline{n})\, G\,(\overline{m},\ \overline{n}) \,:\, \mathrm{y}\,(\overline{m},\ \overline{n})\, Z'\,(\overline{m},\ \overline{n}) = B''C'' - A''D'' = 1.$$

Donc

$$\mathrm{g}x\, Gx \,:\, \mathrm{y}x\, Zx = \sum \{x-(m,\ n)\}^{-1} - \sum \{x-(\overline{m},\ \overline{n})\}^{-1},$$

c'est-à-dire

$$\mathrm{g}x\, Gx \,:\, \mathrm{y}x\, Zx = (\mathrm{y}'x \,:\, \mathrm{y}x) - (Z'x \,:\, Zx).$$

(Nous allons bientôt justifier, au moyen d'un théorème de M. Cauchy, l'emploi de cette méthode de décomposition en fractions simples qui ne s'applique pas toujours aux fonctions transcendantes.)

On obtient de même

$$(\mathrm{I})\quad \left\{\begin{array}{l}
\mathrm{g}x\, Zx \,:\, \mathrm{y}x\, Gx = (\mathrm{y}'x \,:\, \mathrm{y}x) - (G'x \,:\, Gx),\\
Gx\, Zx \,:\, \mathrm{y}x\, \mathrm{g}x = (\mathrm{y}'x \,:\, \mathrm{y}x) - (\mathrm{g}'x \,:\, \mathrm{g}x),\\
-b^2\mathrm{y}zZx \,:\, \mathrm{g}x\, Gx = (\mathrm{g}'x \,:\, \mathrm{g}x) - (G'x \,:\, Gx),\\
e^2\mathrm{y}x\mathrm{g}x \,:\, Gx\, Zx = (G'x \,:\, Gx) - (Z'x \,:\, Zx),\\
c^2\mathrm{y}xGx \,:\, Zx\, \mathrm{y}x = (Z'x \,:\, Zx) - (\mathrm{y}'x \,:\, \mathrm{y}x),
\end{array}\right.$$

en mettant, pour abréger,

$$(\mathrm{J})\quad \left\{\begin{array}{l}
\mathrm{y}\,(\tfrac{1}{2}\,\mathrm{T}) \,:\, Z\,(\tfrac{1}{2}\,\Omega) = \dfrac{i}{e},\ (^1)\\[2ex]
\mathrm{y}\,(\tfrac{1}{2}\,\Omega) \,:\, Z\,(\tfrac{1}{2}\,\Omega) = \dfrac{1}{c},\\[2ex]
\mathrm{y}\,(\tfrac{1}{2}\,\Omega) \,:\, G\,(\tfrac{1}{2}\,\Omega) = -i\mathrm{y}\,(\tfrac{1}{2}\,\mathrm{T}) \div \mathrm{g}\,(\tfrac{1}{2}\,\mathrm{T}) = \dfrac{1}{b}.
\end{array}\right.$$

Donc, en éliminant les fonctions dérivées,

$$\begin{aligned}
G^2x - Z^2x &= \quad e^2\mathrm{y}^2x,\\
\mathrm{g}^2x - G^2x &= -\,b^2\mathrm{y}^2x,\\
Z^2x - \mathrm{g}^2x &= \quad c^2\mathrm{y}^2x,
\end{aligned}$$

[1] J'ai écrit $\frac{i}{e}$ au lieu de $\frac{1}{e}$ pour me conformer à la notation d'Abel ; il est à peine nécessaire de remarquer que c, e sont, en général, l'une et l'autre des quantités imaginaires.

ce qui donne, en ajoutant,

$$\text{(K)} \qquad b^2 = e^2 + c^2, \text{ ou } b = \sqrt{e^2 + c^2},$$

en nous retiendrons désormais la lettre b dans cette signification. Puis

$$\text{(L)} \qquad \begin{cases} g^2 x = Z^2 x - c^2 y^2 x, \\ G^2 x = Z^2 x + e^2 y^2 x. \end{cases}$$

Soient maintenant

$$\text{(M)} \qquad \begin{cases} \phi x = \mathrm{y} x : Zx, \\ fx = \mathrm{g} x : Zx, \\ Fx = Gx : Zx. \end{cases}$$

On obtient

$$\text{(N)} \qquad \begin{cases} f^2 x = 1 - c^2 \phi^2 x, \\ F^2 x = 1 + e^2 \phi^2 x; \end{cases}$$

$$\text{(O)} \qquad \begin{cases} \phi' x = fx Fx, \\ f' x = -c^2 \phi x Fx, \\ F' x = e^2 \phi x fx. \end{cases}$$

Ces équations sont précisément les équations fondamentales d'Abel (*Œuvres*, t. I., p. 143 [Ed. 2, p. 268]), et il en déduit

$$\text{(P)} \qquad \begin{cases} \phi(x+y) = \dfrac{\phi x fy\, Fy + \phi y fx\, Fx}{1 + e^2 c^2 \phi^2 x \phi^2 y}, \\ f(x+y) = \dfrac{fx fy - c^2 \phi x \phi y\, Fx Fy}{1 + e^2 c^4 \phi^2 x \phi^2 y}, \\ F(x+y) = \dfrac{Fx Fy + e^2 \phi x \phi y fx fy}{1 + e^2 c^2 \phi^2 x \phi^2 y}, \end{cases}$$

qui sont les formules connues pour l'addition des fonctions elliptiques.

En effet, on peut écrire l'équation

$$\phi' x = fx\, Fx$$

sous la forme

$$1 = \frac{\phi' x}{\sqrt{(1 - c^2 \phi^2 x)(1 + e^2 \phi^2 x)}},$$

ou, en mettant $y = \phi x$,

$$\text{(Q)} \qquad x = \int_0^{\phi x} \frac{dy}{\sqrt{(1 - c^2 y^2)(1 + e^2 y^2)}};$$

on peut donc de ce point supposer connues toutes les propriétés des fonctions elliptiques. On a, par exemple,

$$\phi\left(\tfrac{1}{2}\Upsilon\right)=\frac{i}{e}, \quad \phi\left(\tfrac{1}{2}\Omega\right)=\frac{1}{c},$$

et de là

$$\text{(R)} \quad \begin{cases} \tfrac{1}{2}\Omega=\displaystyle\int_0^{\frac{1}{c}}\frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}}, \\ \tfrac{1}{2}\Upsilon=\displaystyle\int_0^{\frac{i}{e}}\frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}}, \end{cases}$$

qui déterminent Ω, Υ en fonction de c, e. Il paraît au premier abord que Υ, Ω soient des fonctions parfaitement déterminées de c, e. J'ai des raisons pour croire que cela n'est pas précisément le cas, et que la question admettrait des développements intéressants; mais je réserve ce sujet pour une autre occasion.

On peut, à l'aide des équations entre $\mathrm{y}\left(\tfrac{1}{2}\Omega\right)$, &c., et les quantités c, e, exprimer cette suite de fonctions en termes de c, e. On déduit:

$$\text{(S)} \quad \begin{cases} \mathrm{y}\left(\tfrac{1}{2}\Omega\right)=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & \mathrm{y}\left(\tfrac{1}{2}\Upsilon\right)=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}; \\ \mathrm{g}\left(\tfrac{1}{2}\Omega\right)=0, & \mathrm{g}\left(\tfrac{1}{2}\Upsilon\right)=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}; \\ G\left(\tfrac{1}{2}\Omega\right)=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & G\left(\tfrac{1}{2}\Upsilon\right)=0; \\ Z\left(\tfrac{1}{2}\Omega\right)=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}, & Z\left(\tfrac{1}{2}\Upsilon\right)=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}; \end{cases}$$

et de là

$$\text{(T)} \quad \begin{cases} A=b^{-\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & A'=ib^{-\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}, & A''=(-)^{\frac{1}{4}}c^{-\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}; \\ B=-b^{\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}, & B'=b^{\frac{1}{2}}e^{-\frac{1}{2}}q^{-\frac{1}{8}}, & B''=-(-)^{\frac{1}{4}}ic^{\frac{1}{2}}e^{-\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}; \\ C=b^{\frac{1}{2}}c^{-\frac{1}{2}}q_1^{-\frac{1}{8}}, & C'=-ib^{\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}, & C''=(-)^{\frac{1}{4}}c^{-\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}; \\ D=b^{-\frac{1}{2}}c^{\frac{1}{2}}q_1^{-\frac{1}{8}}, & D'=b^{-\frac{1}{2}}e^{\frac{1}{2}}q^{-\frac{1}{8}}, & D''=-(-)^{\frac{1}{4}}ic^{\frac{1}{2}}e^{\frac{1}{2}}q_1^{-\frac{1}{8}}q^{-\frac{1}{8}}; \end{cases}$$

qu'on doit introduire dans toutes les formules où entrent les quantités A, B, &c.

Voici le théorème de M. Cauchy (*Exercices de Mathématiques*, tome II. page 289):

"Si, en attribuant au module r de la variable

$$z=r(\cos p+i\sin p)$$

des valeurs infiniment grandes, on peut les choisir de manière que les deux fonctions

$$\frac{fz+f(-z)}{2}, \quad \frac{fz-f(-z)}{2z}$$

deviennent sensiblement nulles, quel que soit l'angle p, ou du moins que chacune de ces fonctions reste toujours finie ou infiniment petite, et ne cesse d'être infiniment

petite, en demeurant finie, que dans le voisinage de certaines valeurs particulières de p, on aura

$$fx = \mathcal{L}\,\frac{((fz))}{x-z},$$

pourvu qu'on réduise le résidu intégral à sa valeur principale."

On se rappelle que cela veut dire qu'en supposant ces conditions satisfaites, la fonction fx peut s'exprimer de la manière ordinaire comme la somme d'une suite de fractions simples, mais qu'il faut étendre d'abord la sommation aux racines de l'équation

$$\frac{1}{fx} = 0,$$

dont les modules sont inférieurs à une certaine limite qu'on fait alors infinie.

Soit, par exemple,

$$fx = \epsilon^{\frac{1}{2}ax^2+bx}\,\frac{T(x)}{T_1(x)} \quad\text{.....} \quad (33),$$

où Tx, T_1x ne contiennent que des facteurs qui sont des puissances entières des fonctions yx, gx, Gx, Zx. En supposant que r n'est d'aucune des formes

$$\text{mod}\,(m,\ n),\quad \text{mod}\,(\bar{m},\ n),\quad \text{mod}\,(m,\ \bar{n}),\quad \text{mod}\,(\bar{m},\ \bar{n}),$$

mais, d'ailleurs, une quantité infinie quelconque, on peut toujours écrire

$$z = r(\cos p + i\sin p) = (m,\ n) + \theta \quad\text{.....} \quad (34),$$

où θ est une quantité finie telle qu'aucune des fonctions $y\theta$, $g\theta$, $G\theta$, $Z\theta$ ne s'évanouit, m, n sont des entiers dont l'un au moins est infini, mais qui varient depuis $-\infty$ jusqu'à ∞, avec l'angle p. Soit 8 le degré de Tx, 8_1 celui de T_1x à l'égard de yx, &c.; soit aussi $\lambda = 8_1 - 8$; on a, en général, une équation de cette forme,

$$fz = \epsilon^{\frac{1}{2}a\,(m,\,n)^2}\, q_1^{\frac{1}{2}\lambda m^2}\, q^{\frac{1}{2}\lambda n^2}\, \epsilon^{Im+Jn}\, F \quad\text{.....} \quad (35),$$

où F est fini. En supposant donc que la partie réelle de

$$a\,(m,\ n)^2 - \lambda\mathcal{E}\Omega^2 m^2 + \lambda\mathcal{E}\Upsilon^2 n^2 \quad\text{.....} \quad (36)$$

est négative quelles que soient les valeurs de m, n, on a

$$fz = 0,$$

et de même

$$\frac{fz + f(-z)}{2} = 0, \qquad \frac{fz - f(-z)}{2z} = 0.$$

Si, au contraire, cette partie réelle est positive, ces trois fonctions sont toujours infinies. Il y a cependant un cas particulier à considérer, savoir, celui où cette partie réelle se réduit à Pm^2 ou Qn^2, P, Q étant des quantités positives. Il faut ici que le

coefficient de n ou de m dans la partie réelle de $Im+Jn$ s'évanouisse. Enfin, si les parties réelles s'évanouissent entièrement, ce qui ne peut arriver que pour

$$a=0, \quad b=0, \quad \lambda=0,$$

fx est fini. On a donc pour fx fonction impaire,

$$\frac{fz+f(-z)}{2}=0, \qquad \frac{fz-f(-z)}{2z}=0$$

(la seconde équation à cause du dénominateur infini z). Il est cependant certain que, dans plusieurs cas pour lesquels fx est fonction paire, on peut réduire fx en une suite de fractions simples: par exemple, $yx : gx$ est une fonction impaire que l'on peut, par ce qui précède, développer en suite de fractions simples; en écrivant $x+\frac{1}{2}\Upsilon$ au lieu de x, on déduit un pareil développement pour $Gx : Zx$ qui est fonction paire.

Remarquons que quand la partie réelle de

$$a(m, n)^2 - \lambda\xi\Omega^2 m^2 + \lambda\xi\Upsilon^2 n^2$$

est négative pour toute valeur de m ou n, la suite pour fx est toujours convergente. En effet, les numérateurs des fractions simples contiennent ce facteur de fz,

$$\epsilon^{\frac{1}{2}a(m,\,n)^2} q_1^{\frac{1}{2}\lambda m^2} q_2^{\frac{1}{2}\lambda n^2},$$

qui s'évanouit pour les valeurs infiniment grandes de m, n. Dans le cas où la partie réelle est positive, le développement ne peut jamais être vrai; je crois qu'en général il est vrai, dans les cas limites, quand la suite que l'on obtient est convergente. Il y a des exceptions cependant; on en verra une en développant $\phi^2 x$.

Avant de passer aux exemples, développons la condition pour que la partie réelle en question soit toujours négative.

En supposant

$$a=h+ki,$$

on obtient pour cette quantité l'expression

$$\left.\begin{aligned} & m^2\left\{h(\omega^2-\omega'^2)-2k\omega\omega'-\frac{\lambda\pi \bmod (\omega\upsilon'-\omega'\upsilon)}{\upsilon^2+\upsilon'^2}\right\} \\ + \quad & n^2\left\{h(\upsilon^2-\upsilon'^2)-2k\upsilon\upsilon'-\frac{\lambda\pi \bmod (\omega\upsilon'-\omega'\upsilon)}{\omega^2+\omega'^2}\right\} \\ + & 2mn\left\{h(\omega\upsilon-\omega'\upsilon')-k(\omega\upsilon'+\omega'\upsilon)\right\}, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

qui doit toujours rester négative. Cela donne, après quelques réductions, la condition

$$\left.\begin{aligned} & h^2+k^2+\frac{2\lambda\pi(\omega\upsilon+\omega'\upsilon')}{(\omega^2+\omega'^2)(\varpi^2+\varpi'^2)\bmod(\omega\upsilon'-\omega'\upsilon)}\left[(\omega\upsilon-\omega'\upsilon')h+(\omega\upsilon'+\omega'\upsilon)k\right] \\ & -\frac{\lambda^2\pi^2}{(\omega^2+\omega'^2)(\upsilon^2+\upsilon'^2)}<0. \end{aligned}\right\} \ldots (38).$$

Les valeurs $$h = 0, \quad k = 0$$

satisfont à cette condition, laquelle du reste (en considérant h, k comme les coordonnées d'un point) est satisfaite pour tout point situé en dedans d'un certain cercle qui inclut l'origine, et dont on aurait l'équation en remplaçant le signe $<$ par le signe $=$ dans la formule (38).

On obtient de cette façon une grande variété de formules particulières. Par exemple celles-ci :

$$(\mathrm{U})\left\{\begin{array}{ll}
\epsilon^{\frac{1}{2}ax^2+bx} : \mathrm{y}x = & \sum [(-)^{-mn-m-n}\, \epsilon^{\frac{1}{2}a\,(m,\,n)^2+b\,(m,\,n)}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}n^2}\, \{x-(m,\,n)\}^{-1}], \\
\epsilon^{\frac{1}{2}ax^2+bx} : \mathrm{g}x = -b^{-\frac{1}{2}}\, c^{-\frac{1}{2}} & \sum [(-)^{-mn-m-\frac{1}{2}n}\, \epsilon^{\frac{1}{2}a\,(\overline{m},\,n)^2+b\,(\overline{m},\,n)}\, q_1^{\frac{1}{2}\,(m+\frac{1}{2})^2}\, q^{\frac{1}{2}n^2}\, \{x-(\overline{m},\,n)\}^{-1}], \\
\epsilon^{\frac{1}{2}ax^2+bx} : Gx = ib^{-\frac{1}{2}}\, c^{-\frac{1}{2}} & \sum [(-)^{-mn-\frac{1}{2}m-n}\, \epsilon^{\frac{1}{2}a\,(m,\,\overline{n})^2+b\,(m,\,\overline{n})}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}\,(n+\frac{1}{2})^2}\, \{x-(m,\,\overline{n})\}^{-1}], \\
\epsilon^{\frac{1}{2}ax^2+bx} : Zx = ic^{-\frac{1}{2}}\, e^{-\frac{1}{2}} & \sum [(-)^{-\,(m+\frac{1}{2})\,(n+\frac{1}{2})}\, \epsilon^{\frac{1}{2}a\,(\overline{m},\,\overline{n})^2+b\,(\overline{m},\,\overline{n})}\, q_1^{\frac{1}{2}\,(m+\frac{1}{2})^2}\, q^{\frac{1}{2}\,(n+\frac{1}{2})^2}\, \{x-(\overline{m},\,\overline{n})\}^{-1}],
\end{array}\right.$$

dans lesquelles, pour trouver les limites de a, il faut faire $\lambda = 1$ dans la formule (38).

On a ensuite ce système de formules sans exponentielles,

$$(\mathrm{V})\left\{\begin{array}{ll}
\mathrm{g}x : \mathrm{y}x = & \sum [(-)^{n}\, \{x-(m,\,n)\}^{-1}], \\
Gx : \mathrm{y}x = & \sum [(-)^{m}\, \{x-(m,\,n)\}^{-1}], \\
Zx : \mathrm{y}x = & \sum [(-)^{m+n}\, \{x-(m,\,n)\}^{-1}]; \\
\mathrm{y}x : \mathrm{g}x = -b^{-1}c^{-1} & \sum [(-)^{n}\, \{x-(\overline{m},\,n)\}^{-1}], \\
Gx : \mathrm{g}x = -\quad c^{-1} & \sum [(-)^{m+n}\, \{x-(\overline{m},\,n)\}^{-1}], \\
Zx : \mathrm{g}x = -\quad b^{-1} & \sum [(-)^{m}\, \{x-(\overline{m},\,n)\}^{-1}]; \\
\mathrm{y}x : Gx = -b^{-1}e^{-1} & \sum [(-)^{m}\, \{x-(m,\,\overline{n})\}^{-1}], \\
\mathrm{g}x : Gx = \quad ie^{-1} & \sum [(-)^{m+n}\, \{x-(m,\,\overline{n})\}^{-1}], \\
Zx : Gx = \quad ib^{-1} & \sum [(-)^{n}\, \{x-(m,\,\overline{n})\}^{-1}]; \\
\mathrm{y}x : Zx = \quad ic^{-1}e^{-1} & \sum [(-)^{m+n}\, \{x-(\overline{m},\,\overline{n})\}^{-1}], \\
\mathrm{g}x : Zx = \quad e^{-1} & \sum [(-)^{m}\, \{x-(\overline{m},\,\overline{n})\}^{-1}], \\
Gx : Zx = \quad ic^{-1} & \sum [(-)^{n}\, \{x-(\overline{m},\,\overline{n})\}^{-1}],
\end{array}\right.$$

dont quelques-unes ont été données par Abel. Il faut remarquer que celles de ces équations où la fonction est impaire sont justifiées par le théorème de M. Cauchy, et que les autres se déduisent de celles-ci. On peut trouver de même le développement des fonctions

$$\frac{1}{\mathrm{y}x\,\mathrm{g}x}, \ldots, \quad \frac{Gx}{\mathrm{y}x\,\mathrm{g}x}, \ldots, \quad \frac{\mathrm{y}x\,\mathrm{g}x}{Zx\,Gx}$$

(celles-ci, qui sont toutes impaires, ont été déja considérées; nous venons de faire voir que le développement est admissible);

$$\frac{1}{\mathrm{y}x\,\mathrm{g}x\,Gx}, \cdots, \quad \frac{Zx}{\mathrm{y}x\,\mathrm{g}x\,Gx}, \cdots, \quad \frac{1}{\mathrm{y}x\,\mathrm{g}x\,Zx}, \cdots,$$

chacune, excepté celles de la suite $\frac{\mathrm{y}x\,\mathrm{g}x}{Gx\,Zx}$, multipliée par un facteur exponentiel $e^{\frac{1}{2}ax^2+bx}$. Par exemple,

$$\text{(W)} \quad \left\{ \begin{aligned} \frac{1}{\mathrm{y}x\,\mathrm{g}x\,Gx\,Zz} = & \sum [e^{\frac{1}{2}a(m,\,n)^2+b(m,\,n)}\, q_1^{2m^2} q^{2n^2} \qquad \{x-(m,\,n)\}^{-1}] \\ & - \sum [e^{\frac{1}{2}a(\overline{m},\,n)^2+b(\overline{m},\,n)}\, q_1^{2(m+\frac{1}{2})^2} q^{2n^2} \qquad \{x-(\overline{m},\,n)\}^{-1}] \\ & + \sum [e^{\frac{1}{2}a(m,\,\overline{n})^2+b(m,\,\overline{n})}\, q_1^{2m^2} q^{2(n+\frac{1}{2})^2} \qquad \{x-(m,\,\overline{n})\}^{-1}] \\ & - \sum [e^{\frac{1}{2}a(\overline{m},\,\overline{n})^2+b(\overline{m},\,\overline{n})}\, q_1^{2(m+\frac{1}{2})^2} q^{2(n+\frac{1}{2})^2} \{x-(\overline{m},\,\overline{n})\}^{-1}]. \end{aligned} \right.$$

Mais on est conduit à un résultat beaucoup plus important en considérant, par exemple, le développement de $\phi^2 x$. Cette fonction contient non-seulement une suite de fractions simples, mais aussi un terme constant; nous écrirons

$$\phi^2 x = J + \sum [L\,\{x-(\overline{m},\,\overline{n})\}^{-2} + M\,\{x-(\overline{m},\,\overline{n})\}^{-1}] \;\ldots\ldots\ldots\ldots\ldots\; (39).$$

Pour déterminer L, M de la manière la plus simple, changeons x en $x+\frac{1}{2}\Omega+\frac{1}{2}\Upsilon$. En faisant attention à la valeur de $\phi x = \mathrm{y}x : Zx$, on obtient

$$-e^{-2}c^{-2}(\phi x)^{-2} = J + \sum [L\,\{x-(m,\,n)\}^{-2} + M\,\{x-(m,\,n)\}^{-1}];$$

de là

$$L = -e^{-2}c^{-2}\,\{x-(m,\,n)\}^2\,(\phi x)^{-2},$$

$$M = -e^{-2}c^{-2}\frac{d}{dx}[\{x-(m,\,n)\}^2\,(\phi x)^{-2}], \quad x=(m,\,n),$$

ou enfin

$$L = -e^{-2}c^{-2}\,x^2\,(\phi x)^{-2}, \quad \text{pour } x=0,$$

$$M = -e^2c^2\frac{d}{dx}[x^2\,(\phi x)^{-2}],$$

ce qui donne

$$\left.\begin{aligned} & L = -e^{-2}c^{-2}, \quad M=0, \\ & \phi^2 x = J - e^{-2}c^{-2}\sum \{x-(\overline{m},\,\overline{n})\}^{-2}. \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (40).$$

En intégrant deux fois

$$\int_0^x dx \int_0^x \phi^2 x\,dx = \tfrac{1}{2}Jx^2 + e^{-2}c^{-2}\sum \log\{x-(\overline{m},\,\overline{n})\} \;\ldots\ldots\ldots\ldots\; (41),$$

ou, à cause de $\log Zx = -\frac{1}{2}Bx^2 + \sum \log\{x - (\bar{m}, \bar{n})\}$,

$$\int_0^x dx \int_0^x \phi^2 x\, dx = \tfrac{1}{2} I x^2 + e^{-2} c^{-2} \log Zx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42),$$

en mettant, pour abréger,

$$I = J - e^{-2} c^{-2} B$$

(on n'a pas ajouté de constante arbitraire, parce que Zx est fonction paire de x qui se réduit à l'unité pour $x = 0$). Puis, on a

$$Zx = \epsilon^{-\frac{1}{2}e^2 c^2 I x^2 + e^2 c^2 \int_0^x dx \int_0^x \phi^2 x\, dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43).$$

De là il est facile de déterminer la valeur de I. Soit, pour un moment,

$$\phi_{,} x = \int_0^x \phi^2 x\, dx, \quad \phi_{,,} x = \int_0^x \phi_{,} x\, dx.$$

Puisque

$$\phi^2 (x + \Omega) - \phi^2 x = 0,$$

on a

$$\phi_{,} (x + \Omega) = \phi_{,} x - \phi_{,} \Omega,$$
$$\phi_{,,} (x + \Omega) - \phi_{,,} x = x \phi_{,} \Omega.$$

Mais de l'équation

$$\phi^2 x - \phi^2 (\Omega - x) = 0$$

on déduit

$$\phi_{,} x + \phi_{,} (\Omega - x) = \phi_{,} \Omega, \quad \phi_{,,} x - \phi_{,,} (\Omega - x) = x \phi_{,} \Omega,$$

ou, en faisant $x = \frac{1}{2}\Omega$,

$$\phi_{,} \Omega = 2 \phi_{,} (\tfrac{1}{2}\Omega), \quad \phi_{,,} \Omega = \Omega \phi_{,} \tfrac{1}{2}\Omega,$$

c'est-à-dire

$$\phi_{,,} (x + \Omega) - \phi_{,,} x = (2x + \Omega)\, \phi_{,} \tfrac{1}{2}\Omega\,;$$

et de là

$$Z(x + \Omega) = \epsilon^{-\frac{1}{2}e^2 c^2 \left[I - \frac{2}{\Omega} \phi_{,} (\frac{1}{2}\Omega) \right] 2 (\Omega x + \Omega^2)} Zx.$$

Mais on a déjà

$$Z(x + \Omega) = \epsilon^{\beta \Omega x} q_1^{-\frac{1}{2}} Zx = \epsilon^{\frac{1}{2}\beta (2\Omega x + \Omega^2)} Zx\,;$$

donc enfin

$$-e^2 c^2 \left[I - \frac{2}{\Omega} \phi_{,} (\tfrac{1}{2}\Omega) \right] = \beta,$$

c'est-à-dire

$$-\tfrac{1}{2} e^2 c^2 I = \tfrac{1}{2}\beta - \frac{e^2 c^2}{\Omega} \phi_{,} (\tfrac{1}{2}\Omega) = \tfrac{1}{2}\beta - \frac{e^2 c^2}{\Omega} \int_0^{\frac{1}{2}\Omega} \phi^2 x\, dx \ldots\ldots\ldots\ldots (44).$$

Soit, pour abréger,

$$M = \frac{e^2c^2}{\Omega}\int_0^{\frac{1}{2}\Omega} \phi^2 x\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45);$$

on a cette équation,

$$\text{(X)} \qquad Zx = \varepsilon^{(\frac{1}{2}\beta - M)x^2 + e^2c^2\int_0^x dx \int_0^x \phi^2 x\, dx},$$

qui exprime la fonction Zx au moyen de ϕx. C'est, en effet, la formule remarquable de M. Jacobi, que nous avons citée dans l'introduction de ce Mémoire.

En changeant seulement la notation, on a, d'après M. Jacobi,

$$\phi^2(x+a) - \phi^2(x-a) = \frac{4(\phi a\, fa\, Fa\, \phi x\, fx\, Fx)}{(1+e^2c^2\,\phi^2 a\,\phi^2 x)^2} \ldots\ldots\ldots\ldots\ldots (46),$$

$$\int_0^x \{\phi^2(x+a) - \phi^2(x-a)\}\, dx = \frac{2\phi a\, fa\, Fa\, \phi^2 x}{1+e^2c^2\,\phi^2 a\,\phi^2 x} \ldots\ldots\ldots\ldots\ldots\ldots (47),$$

ou, ce qui est la même chose,

$$\int_{-a}^x \phi^2(x+a)\, dx - \int_a^x \phi^2(x-a)\, dx - 2\int_0^x \phi^2 a\, da = \frac{2\phi a\, fa\, Fa\, \phi^2 x}{1+e^2c^2\,\phi^2 a\,\phi^2 x} \ldots\ldots (48).$$

De là, en multipliant par e^2c^2, et faisant attention à la valeur de Zx,

$$\frac{Z'(x+a)}{Z(x+a)} - \frac{Z'(x-a)}{Z(x-a)} - \frac{2Z'a}{Za} = \frac{2e^2c^2\,\phi a\, fa\, Fa\, \phi^2 x}{1+e^2c^2\,\phi^2 a\,\phi^2 x} \ldots\ldots\ldots\ldots\ldots (49).$$

Écrivant dans cette équation a, x au lieu de x, a, et ajoutant, on obtient

$$\frac{Z'x}{Zx} + \frac{Z'a}{Za} - \frac{Z'(x+a)}{Z(x+a)} = e^2c^2\,\phi a\,\phi x\,\phi(a+x) \ldots\ldots\ldots\ldots\ldots\ldots (50).$$

En intégrant la même équation par rapport à a,

$$\left.\begin{aligned} &\log Z(x+a) + \log Z(x-a) - 2\log Zx - 2\log Za \\ &\quad = \log(1+e^2c^2\,\phi^2 a\,\phi^2 x) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (51),$$

c'est-à-dire

$$Z(x+a)\,Z(x-a) = Z^2x\,Z^2a\,(1+e^2c^2\,\phi^2 a\,\phi^2 x) \ldots\ldots\ldots\ldots\ldots\ldots (52),$$

ou, ce qui est la même chose,

$$Z(x+a)\,Z(x-a) = Z^2x\,Z^2a + e^2c^2\,\mathrm{y}^2x\,\mathrm{y}^2a \ldots\ldots\ldots\ldots\ldots\ldots (53),$$

de laquelle on déduit facilement, en écrivant $x+\frac{1}{2}\Omega$, $x+\frac{1}{2}\Upsilon$, $x+\frac{1}{2}\Omega+\frac{1}{2}\Upsilon$ au lieu de x, les équations complémentaires

$$\text{(Y)} \ldots\ldots\ldots\ldots\ldots\ldots \left\{\begin{aligned} &\mathrm{y}(x+a)\,\mathrm{y}(x-a) = \mathrm{y}^2x\,Z^2a - \mathrm{y}^2a\,Z^2x, \\ &g(x+a)\,g(x-a) = g^2x\,Z^2a - c^2g^2a\,Z^2x, \\ &G(x+a)\,G(x-a) = G^2x\,Z^2a + e^2G^2a\,Z^2x. \end{aligned}\right.$$

Quoiqu'elle ne soit pas liée très-étroitement avec la théorie actuelle, on peut ajouter ici cette autre formule de M. Jacobi, qu'il obtient en intégrant par rapport à x, au lieu de a,

$$\Pi(x, a) = \int_0^x \frac{-e^2c^2\,\phi a\, fa\, Fa\, \phi^2 x\, dx}{1 + e^2c^2\,\phi^2 a\,\phi^2 x} = \tfrac{1}{2}\log\frac{Z(x-a)}{Z(x+a)} + x\frac{Z'a}{Za} \quad \ldots\ldots\ldots \quad (54),$$

au moyen de laquelle il déduit des formules pour l'addition des arguments ou des paramètres des fonctions de la troisième espèce. On trouve aussi, dans les *Fund. Nova*, quelques formules déduites de l'équation (49) pour exprimer $\dfrac{Z(x-a)\,Z(y-a)\,Z(x+y+a)}{Z(x+a)\,Z(y+a)\,Z(x+y-a)}$ au moyen de la fonction ϕ; il serait facile de déduire des équations semblables pour les autres fonctions y, g, G.

Note sur une intégrale définie.

Soient k_1, k des quantités réelles dont la première est la plus grande; $\Omega = \omega + \omega' i$, $\Upsilon = \upsilon + \upsilon' i$ des quantités quelconques, telles que $\omega\upsilon' - \omega'\upsilon$ ne s'évanouisse pas, et écrivons

$$u = \int_k^{k_1} \frac{dx}{\Omega + \Upsilon x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (55).$$

L'intégrale u a toujours une valeur finie et déterminée, puisque le dénominateur ne devient jamais zéro. On a évidemment

$$u = \int_0^\infty dx \left[\frac{1}{\Omega + \Upsilon(x+k)} - \frac{1}{\Omega + \Upsilon(x+k_1)}\right] \ldots\ldots\ldots\ldots\ldots\ldots \quad (56),$$

et l'intégrale indéfinie est

$$u = -\frac{1}{\Upsilon}\log\left\{\pm\frac{\Omega + \Upsilon(x+k)}{\Omega + \Upsilon(x+k_1)}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (57);$$

il faut passer de là à l'intégrale définie, entre les limites 0, ∞. Soit

$$\frac{\Omega + \Upsilon(x+k)}{\Omega + \Upsilon(x+k_1)} = A_x + iB_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (58).$$

Il est facile de voir qu'en faisant abstraction d'un dénominateur toujours positif, A_x est une fonction du second degré en x, et B_x se réduit à la quantité constante $(\omega\upsilon' - \omega'\upsilon)(k_1 - k)$. Le signe de B_x est donc toujours le même que celui de $\omega\upsilon' - \omega'\upsilon$; quant à celui de A_x, puisque évidemment $A_\infty = 1$, il est clair que si A_0 est positif, A_x reste toujours positif, ou change deux fois de signe quand x passe de 0 à ∞. Si, au contraire, A_0 est négatif, A_x est négatif depuis $x = 0$ jusqu'à une certaine valeur positive de x, $x = \alpha$, et positif depuis $x = \alpha$ jusqu'à $x = \infty$. Considérons d'abord ce dernier cas. En représentant par Lx la valeur principale de $\log x$ (cela suppose que la partie réelle de x est positive), on obtient cette valeur de u,

$$u = \frac{1}{\Upsilon} L\left[-\frac{\Omega + \Upsilon k_1}{\Omega + \Upsilon k}\right] - \frac{1}{\Upsilon} L\left[-\frac{\Omega + \Upsilon(\alpha + k_1 - \epsilon)}{\Omega + \Upsilon(\alpha + k - \epsilon)}\right] - \frac{1}{\Upsilon} L\left[\frac{\Omega + \Upsilon(\alpha + k_1 + \epsilon)}{\Omega + \Upsilon(\alpha + k + \epsilon)}\right],$$

où ϵ est supposé une quantité infiniment petite positive. La somme des derniers termes se réduit à

$$\frac{i}{\Upsilon}\left(-\operatorname{arc\,tan}\frac{B_\alpha}{A_{\alpha-\epsilon}}+\operatorname{arc\,tan}\frac{A_\alpha}{B^{\alpha+\epsilon}}\right) \quad \text{(59)},$$

où, comme à l'ordinaire, arc tan x doit être situé entre les limites $\pm\frac{1}{2}\pi$.

Dans cette formule, $A_{\alpha-\epsilon}$ est une quantité infiniment petite et négative; $A_{\alpha+\epsilon}$ est une quantité infiniment petite et positive; donc, quand B_α est négatif, les arcs se réduisent à $\frac{1}{2}\pi$, $-\frac{1}{2}\pi$, et si B_α est positif, à $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$. On a donc

$$u=\frac{1}{\Upsilon}L\left(-\frac{\Omega+\Upsilon k_1}{\Omega+\Upsilon k}\right)\pm\frac{1}{\Upsilon}\pi i \quad \text{(60)},$$

où il faut se servir du signe supérieur ou inférieur selon que $\omega v'-\omega' v$ est positif ou négatif. Dans le cas où A_0 est positif, si A_x reste toujours positif, on obtient tout de suite

$$u=\frac{1}{\Upsilon}L\left(\frac{\Omega+\Upsilon k_1}{\Omega+\Upsilon k}\right) \quad \text{(61)}.$$

Si A_x change deux fois de signe, par exemple pour $x=\alpha$ et $x=\beta$, il faut introduire les corrections

$$\frac{i}{\Upsilon}\left(-\operatorname{arc\,tan}\frac{B_\alpha}{A_{\alpha-\epsilon}}+\operatorname{arc\,tan}\frac{B_\alpha}{A_{\alpha+\epsilon}}\right)+\frac{i}{\Upsilon}\left(-\operatorname{arc\,tan}\frac{B_\beta}{A_{\beta-\epsilon}}+\operatorname{arc\,tan}\frac{B_\beta}{A_{\beta+\epsilon}}\right),$$

qui se détruisent l'une l'autre, en sorte que l'on a le même résultat que si A_x était toujours positif. En se servant de la notation employée dans le Mémoire, on a dans tous les cas

$$u=L_{\pm\pi i}\left(\frac{\Omega+\Upsilon k_1}{\Omega+\Upsilon k}\right) \quad \text{(62)},$$

où le signe se détermine d'après celui de $\omega v'-\omega' v$.

En particulier,

$$\int_{-k}^{k}\frac{dx}{\Omega+\Upsilon x}=\Omega\int_0^k\frac{dx}{\Omega^2-\Upsilon^2x^2}=\frac{1}{\Upsilon}L_{\pm\pi i}\left(\frac{\Omega+\Upsilon k}{\Omega-\Upsilon k}\right),$$

ou

$$\int_0^{\tan\alpha}\frac{dx}{\Omega^2-\Upsilon^2x^2}=\frac{1}{2\Omega\Upsilon}L_{\pm\pi i}\left(\frac{\Omega+\Upsilon\tan\alpha}{\Omega-\Upsilon\tan\alpha}\right) \quad \text{(63)}.$$

Je ne sais pas si l'on a cherché avant moi la valeur exacte de cette intégrale définie, qui est cependant la plus simple qu'on puisse imaginer, et qui devrait, je crois, trouver place dans les livres élémentaires.

NOTA. Une partie de ces recherches a été déjà imprimée dans le *Cambridge Mathematical Journal* [24]; je me suis borné au cas où Ω, Υ sont de la forme ω, vi, ce qui simplifie beaucoup la détermination des intégrales définies doubles; mais la forme générale des résultats en est très-peu affectée.

26.

MÉMOIRE SUR LES COURBES DU TROISIÈME ORDRE.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tome IX. (1844), pp. 285—293.]

CONSIDÉRONS d'abord la surface du troisième ordre qui passe par les six arêtes d'un tétraèdre quelconque. Cette surface sera touchée selon chaque arête par un seul plan; je dis que: "les plans tangents selon les arêtes opposées se rencontrent en trois droites qui sont dans le même plan, et chacune de ces droites est située entièrement sur la surface."

En effet, si nous représentons par $P=0$, $Q=0$, $R=0$, $S=0$, les équations des quatre plans du tétraèdre, et par α, β, γ, δ des constantes arbitraires, l'équation de la surface ne peut avoir que la forme

$$\alpha QRS + \beta PRS + \gamma PQS + \delta PQR = 0.$$

Soient $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$ ce que deviennent les quantités P, Q, R, S, quand on change les coordonnées x, y, z en de nouvelles variables ξ, η, ζ; l'équation du plan tangent se réduit facilement à la forme

$$\begin{aligned}(\mathfrak{P} - P)(\beta RS + \gamma QS + \delta QR) + (\mathfrak{Q} - Q)(\alpha RS + \gamma PS + \delta PR)\\ + (\mathfrak{R} - R)(\alpha QS + \beta PS + \delta PQ) + (\mathfrak{S} - S)(\alpha QR + \beta PR + \gamma PQ) = 0.\end{aligned}$$

Soient $P=0$, $Q=0$; cette équation devient

$$\beta\mathfrak{P} + \alpha\mathfrak{Q} = 0;$$

et de même, si $R=0$, $S=0$, l'équation devient

$$\delta\mathfrak{R} + \gamma\mathfrak{S} = 0.$$

De ces équations on déduit les deux suivantes:

$$\alpha\mathfrak{Q}\mathfrak{R}\mathfrak{S} + \beta\mathfrak{P}\mathfrak{R}\mathfrak{S} + \gamma\mathfrak{P}\mathfrak{Q}\mathfrak{S} + \delta\mathfrak{P}\mathfrak{Q}\mathfrak{R} = 0,$$

$$\frac{1}{\alpha}\mathfrak{P} + \frac{1}{\beta}\mathfrak{Q} + \frac{1}{\gamma}\mathfrak{R} + \frac{1}{\delta}\mathfrak{S} = 0.$$

On conclut de la première, que la droite d'intersection des deux plans tangents est située sur la surface; et de la seconde, que cette droite est dans un même plan, quelles que soient les deux arêtes opposées par lesquelles on a mené les deux plans tangents. Ainsi le théorème est démontré.

En considérant une section quelconque de cette surface, ou même en supposant que P, Q, R, S ne contiennent que deux variables x, y, nous avons ce théorème:

"Etant donnée une courbe du troisième ordre qui passe par les six points d'intersection de quatre droites, les tangentes à la courbe en ces six points se rencontrent deux à deux en trois points qui sont les points d'intersection de la courbe par une droite. Les tangentes qui doivent être prises ensemble sont les tangentes en deux points A, A', tels que A est l'intersection de deux des quatre lignes données, et A' l'intersection des deux autres lignes, points que l'on peut nommer *opposés*.

"De même, si une courbe du troisième ordre passe par cinq de ces points, de telle manière que l'intersection des tangentes à la courbe en deux points opposés soit située sur la courbe, la courbe passe par le sixième point, et par les points d'intersection des tangentes menées par les deux autres paires de points opposés."

A présent, posons une courbe quelconque du troisième ordre et deux points A, A' sur la courbe, tels que les tangentes en ces deux points se rencontrent sur la courbe (cela suppose que la courbe est de la sixième ou quatrième classe, et non pas de la troisième). Un autre point B sur la courbe étant pris à volonté, les droites AB, $A'B$ rencontrent la courbe en H et h; et les droites Ah, $A'H$ se rencontrent en un point B' situé sur la courbe. Les tangentes en B, B', et de même les tangentes en H, h, se rencontrent sur la courbe en deux points qui sont en ligne droite avec le point d'intersection des tangentes en A et A'.

On peut dire que les deux points B, B', ou les deux points H, h, sont une paire de points correspondante à la paire A, A'. Les deux paires B, B' et H, h sont évidemment correspondantes l'une à l'autre, et, de plus, A, A' correspond à ces deux paires, de la même manière que H, h correspond à A, A', et B, B'; de sorte que l'on peut dire que les trois paires A, A'; B, B'; H, h sont supplémentaires l'une aux deux autres.

Soit C, C' une autre paire de points correspondante à A, A'; je dis que *B, B' et C, C' sont des paires correspondantes l'une à l'autre;* et, de plus, *si d'un point P quelconque de la courbe, l'on mène des lignes aux points A, A', B, B', C, C', ces lignes forment un faisceau en involution.*

En effet, considérons six points quelconques A, A', B, B', C, C' dans le même plan, et représentons par f, g, h les points d'intersection de BC' et $B'C$, CA' et $C'A$, AB' et $A'B$, et par F, G, H les points d'intersection de CB et $C'B'$, CA et $C'A'$, AB et $A'B'$. Nous allons faire voir que le lieu d'un point P qui se meut de telle manière que les lignes menées aux points A, B, C, A', B', C' forment toujours un faisceau en involution, est une courbe du troisième ordre qui passe par ces six points, et aussi par les points f, g, h, F, G, H.

Soient α, α', β, β', γ, γ' les tangentes trigonométriques des inclinaisons des lignes PA, PA', PB, PB', PC, PC', sur une ligne fixe quelconque que l'on peut prendre pour l'axe des x. Pour que ces lignes forment un faisceau en involution, nous devrions avoir la relation

$$\alpha\alpha'(\beta+\beta'-\gamma-\gamma')+\beta\beta'(\gamma+\gamma'-\alpha-\alpha')+\gamma\gamma'(\alpha+\alpha'-\beta-\beta')=0,$$

ou, ce qui revient à la même chose, l'une quelconque des quatre équations

$$\begin{aligned}
&(\alpha-\beta')(\beta-\gamma')(\gamma-\alpha')+(\alpha'-\beta)(\beta'-\gamma)(\gamma'-\alpha)=0,\\
&(\alpha'-\beta')(\beta-\gamma')(\gamma-\alpha)+(\alpha-\beta)(\beta'-\gamma)(\gamma'-\alpha')=0,\\
&(\alpha-\beta)(\beta'-\gamma')(\gamma-\alpha')+(\alpha'-\beta')(\beta-\gamma)(\gamma'-\alpha)=0,\\
&(\alpha'-\beta)(\beta'-\gamma')(\gamma-\alpha)+(\alpha-\beta')(\beta-\gamma)(\gamma'-\alpha')=0.
\end{aligned}$$

Soient x_P, y_P, x_A, y_A, etc., les coordonnées de P, A, etc.

$$\alpha=\frac{y_P-y_A}{x_P-y_A},$$

$$\alpha-\beta'=\frac{y_P-y_A}{x_P-x_A}-\frac{y_P-y_{B'}}{x_P-x_{B'}}=-\frac{1}{(x_P-x_A)(x_P-x_{B'})}(PAB'),$$

en représentant par (PAB'), etc., les quantités telles que

$$x_P(y_A-y_{B'})+y_P(x_A-x_{B'})+x_Ay_{B'}-x_{B'}y_A.$$

L'équation de la courbe peut donc se mettre sous l'une quelconque des quatre formes

$$\begin{aligned}
&PAB'\,.\,PBC'\,.\,PCA'+PA'B\,.\,PB'C\,.\,PC'A=0,\\
&PA'B'\,.\,PBC'\,.\,PCA+PAB\,.\,PB'C\,.\,PC'A'=0,\\
&PAB\,.\,PB'C'\,.\,PCA'+PA'B'\,.\,PBC\,.\,PC'A=0,\\
&PA'B\,.\,PB'C'\,.\,PCA+PAB'\,.\,PBC\,.\,PC'A'=0,
\end{aligned}$$

qui sont du troisième ordre. La première fait voir que la courbe cherchée passe par les neuf points A, B, C, A', B', C', f, g, h. La troisième ou la quatrième fait voir qu'elle passe de plus par le point F; la quatrième ou la seconde, qu'elle passe par le point G; la seconde ou la troisième, qu'elle passe par le point H. Ainsi la courbe passe par les douze points A, B, C, A', B', C', f, g, h, F, G, H.

On peut remarquer qu'une courbe du troisième ordre qui passe par dix quelconques de ces points, ou même par neuf quelconques, pourvu que nous exceptions les combinaisons de A, B, C, A', B', C', avec f, g, h, ou f, G, H, ou F, g, H, ou F, G, h, ne peut être que la courbe que nous venons de trouver. On peut aussi remarquer en passant que si A, A', B, B', C, C' sont les points d'intersection de quatre droites, l'équation ci-devant trouvée est satisfaite identiquement, de sorte que la position du point P est absolument arbitraire. Cela étant connu, je ne m'arrête pas pour le démontrer.

Revenons au cas d'une courbe donnée, avec trois paires de points A, A', B, B', C, C', comme auparavant, tels que B, B' et C, C' sont des paires correspondantes à A, A'. Les

points $A, B, C, A', B', C', G, g, H, h$ sont situés sur la courbe; ainsi F, f sont aussi sur la courbe (c'est-à-dire que B, B' et C, C' sont des paires correspondantes), et les lignes menées par un point P quelconque de la courbe et les points A, B, C, A', B', C' forment un faisceau en involution: théorème ci-devant énoncé.

Considérons une courbe du troisième ordre, de la quatrième classe (c'est-à-dire, telle que d'un point quelconque on ne peut lui mener que quatre tangentes). D'un point sur la courbe, indépendamment de la tangente en ce point, on ne peut mener que deux tangentes. Prenons les deux points K, L sur la courbe, et menons les tangentes KA, KA', LB, LB' touchant la courbe en A, A', B, B'. Il est clair qu'en ce cas A, A' et B, B' sont des paires correspondantes de points. Mais le cas général où la courbe est de la sixième classe est moins simple. Considérons, en effet, pour une telle courbe, les huit points A_1, A_2, A_3, A_4 et B_1, B_2, B_3, B_4 de contact des tangentes menées par les deux points K et L. En choisissant A, A' et B de quelque manière que ce soit, parmi les points A_1, A_2, A_3, A_4 et B_1, B_2, B_3, B_4 respectivement, le point B', qui doit entrer dans cette dernière série, ne peut pas être choisi à volonté, mais est parfaitement déterminé. En désignant convenablement les points B, on peut toujours supposer que les paires correspondantes soient A_1A_2 ou A_3A_4 avec B_1B_2 ou B_3B_4; A_1A_3 ou A_2A_4 avec B_1B_3 ou B_2B_4; A_1A_4 ou A_2A_3 avec B_1B_4 ou B_2B_3. Cela suppose que

$$A_1B_1,\quad A_2B_2,\quad A_3B_3,\quad A_4B_4;\qquad A_1B_2,\quad A_2B_1,\quad A_3B_4,\quad A_4B_3;$$
$$A_1B_3,\quad A_3B_1,\quad A_2B_4,\quad A_4B_2;\qquad A_1B_4,\quad A_4B_1,\quad A_2B_3,\quad A_3B_2,$$

se rencontrent, chaque système, dans le même point. Il faut expliquer avec plus d'evidence la corrélation de ces huit points.

Imaginons les six points A, A'; B, B'; C, C', tels que AA', BB', CC' se rencontrent dans le même point. Formons, comme auparavant, le système de points f, g, h, F, G, H. Les propriétés de ce système de douze points sont très-nombreuses. Non-seulement F, G, H; F, g, h; f, G, h; f, g, H sont en ligne droite; mais, en outre, les trois droites de chacun des onze systèmes que voici se rencontrent dans le même point:

$$\begin{array}{llllllllll}
Af, & Bg, & Ch; & A'f, & B'g, & C'h; & & & & \\
AA', & Gg, & Hh; & BB', & Hh, & Ff; & CC', & Ff, & Gg; \\
A'f, & BG, & CH; & B'g, & CH, & AF; & C'h, & AF, & BG; \\
Af, & B'G, & C'H; & Bg, & C'H, & A'F; & Ch, & A'F, & B'G.
\end{array}$$

(Cela est connu, je crois; au reste, pour le démontrer, considérons un parallélipipède, tel que gf, GF, AB, $A'B'$ en soient quatre arêtes parallèles, les autres arêtes parallèles étant gA, $A'G$, fB, $B'F$; gA', $A'G$, Bf, $B'F$. Soient H le point de rencontre, à l'infini, du premier système d'arêtes parallèles; C' et C les points de rencontre, à l'infini, des arêtes des second et troisième systèmes respectivement; h le point de rencontre des quatre diagonales du parallélipipède; faisons la perspective de ce système, désignant chaque point de la projection par la même lettre: l'on voit sans peine que la figure plane, ainsi obtenue, a les propriétés en question.)

A présent, en examinant la figure, on voit qu'il est permis de prendre pour A_1, A_2, A_3, A_4 les points A, A', F, f, et pour B_1, B_2, B_3, B_4 les points B, B', G, g, pour que les systèmes A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4 aient la corrélation ci-devant trouvée. Le système C, C', H, h est supplémentaire à ces deux-ci.

Toutes les propriétés que je viens de trouver sont telles qu'il y a à chacune une propriété correspondante que l'on peut obtenir par la théorie des polaires réciproques. Nous avons ainsi des propriétés non moins intéressantes des courbes de la troisième classe et du sixième ou quatrième ordre.

En prenant pour la courbe du troisième ordre l'ensemble d'une section conique et d'une droite, pour que les tangentes en A, A' se rencontrent sur la courbe, il faut que A, A' soient situées sur la section conique de telle manière que AA' passe par le pôle de la droite à l'égard de la conique. Alors B, B'; C, C' étant pris de la même manière, BC', $B'C$ et BC, $B'C'$ se rencontrent sur la droite.

Les lignes tirées d'un point P quelconque de la conique, ou de la droite à A, B, C; A', B', C', forment un faisceau en involution.

Le premier théorème et la première partie du second sont très-bien connus; je ne sais s'il en est de même de la dernière partie de ce théorème.

ADDITION.

La droite PP′, *menée par deux points* P, P′ *d'une courbe du troisième ordre, qui correspondent toujours à une paire correspondante donnée de points* A, A′, *est toujours tangente à une certaine courbe de la troisième classe.*

Pour démontrer ce théorème, imaginons une paire fixe B, B' de points qui corresponde à la paire donnée A, A'. Soient $P=0$, $Q=0$, $R=0$, $S=0$ les équations des lignes qui joignent A, A' avec B, B'; l'équation de la courbe donnée du troisième ordre peut se mettre sous la forme

$$\frac{\alpha}{P}+\frac{\beta}{Q}+\frac{\gamma}{R}+\frac{\delta}{S}=0 \quad\text{.....(1).}$$

On peut toujours supposer, sans perte de généralité, que l'équation

$$P+Q+R+S=0 \quad\text{.....(2)}$$

soit identiquement vraie, car chaque équation de la forme $P=0$ peut être censée contenir une constante arbitraire qui en multiplie tous les termes.

Donc, en faisant

$$P+R=\theta, \qquad P=\theta-R,$$

on peut toujours supposer

$$Q+S=-\theta, \qquad Q=-\theta-S,$$

et l'équation de la courbe se transforme en

$$\frac{\alpha}{\theta - R} - \frac{\beta}{\theta + S} + \frac{\gamma}{R} + \frac{\delta}{S} = 0 \quad \text{.....(3)},$$

ce que l'on peut écrire aussi sous la forme

$$\frac{\theta + (\lambda\mu + \lambda + \mu) S}{(\theta + \lambda R)(\theta + \mu R)} + \frac{-\theta + (\nu\rho + \nu + \rho) R}{(-\theta + \nu S)(-\theta + \rho S)} = 0 \quad \text{.....(4)},$$

en déterminant convenablement les constantes λ, μ, ν, ρ. En effet, en réduisant, les deux équations deviennent identiques au moyen des quatre conditions

$$\left.\begin{aligned} &\delta = -k\lambda\mu, \qquad \gamma = -k\nu\rho, \\ &\delta - \beta = k\lambda\mu(\nu\rho + \nu + \rho), \\ &\gamma - \alpha = k\nu\rho(\lambda\mu + \lambda + \mu), \end{aligned}\right\} \quad \text{.....(5)},$$

où k est une quantité arbitraire, de manière que des quantités λ, μ, ν, ρ, il y en a une seule qui peut être prise à volonté.

De l'équation (4) l'on déduit tout de suite cette autre forme,

$$\frac{1}{\mu - \lambda}\left[\frac{\mu(\lambda + 1)}{\theta + \lambda R} - \frac{\lambda(\mu + 1)}{\theta + \mu R}\right] + \frac{1}{\rho - \nu}\left[\frac{\rho(\nu + 1)}{-\theta + \nu S} - \frac{\nu(\rho + 1)}{-\theta + \rho S}\right] = 0 \quad \text{.....(6)};$$

et de là nous voyons que les points donnés par les deux systèmes d'équations

$$\left.\begin{aligned} &(\theta + \lambda R = 0, \quad -\theta + \nu S = 0), \\ &(\theta + \mu R = 0, \quad -\theta + \rho S = 0), \end{aligned}\right\} \quad \text{.....(7)}$$

correspondent toujours l'un à l'autre et aux points donnés par les deux systèmes

$$\left.\begin{aligned} &(\theta = 0, \quad R = 0), \\ &(\theta = 0, \quad S = 0), \end{aligned}\right\} \quad \text{.....(8)},$$

c'est-à-dire aux points A, A'.

On trouve assez facilement pour l'équation de la droite menée par les points déterminés par les deux systèmes (7), points que l'on peut prendre pour P et P',

$$(\mu\nu - \rho\lambda)\theta + \lambda\mu(\nu - \rho) R + \nu\rho(\lambda - \mu) S = 0 \quad \text{.....(9)},$$

ou

$$C\theta + AR + BS = 0 \quad \text{.....(10)},$$

en faisant

$$\left.\begin{aligned} &pC = \mu\nu - \rho\lambda, \\ &pA = \lambda\mu(\nu - \rho), \\ &pB = \nu\rho(\lambda - \mu). \end{aligned}\right\} \quad \text{.....(11)}.$$

Éliminons des équations (5) et (11) les cinq quantités λ, μ, ν, ρ, p. Pour opérer de la manière la plus élégante, formons d'abord les équations identiques

$$\left.\begin{aligned}&[\mu\nu-\rho\lambda-\nu\rho\ (\lambda-\mu)](\nu-\rho)(\lambda\mu+\lambda+\mu)\\&+[\mu\nu-\rho\lambda+\lambda\mu(\nu-\rho)](\lambda-\mu)(\nu\rho+\nu+\rho)\\&\qquad=(\nu\lambda-\mu\rho)[\lambda\mu(\nu-\rho)-\nu\rho(\lambda-\mu)+2(\mu\nu-\rho\lambda)],\\&\lambda\mu(\nu-\rho)^2-\nu\rho(\lambda-\mu)^2=(\mu\nu-\rho\lambda)(\lambda\nu-\mu\rho),\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(12),$$

ce qui donne

$$\left.\begin{aligned}&A(C-B)(\gamma-\alpha)+B(C+A)(\delta-\beta)\\&\qquad=\frac{k}{p}\lambda\mu\nu\rho(\nu\lambda-\mu\rho)(2C+A-B),\\&A^2\gamma-B^2\delta=-\frac{k}{p}\lambda\mu\nu\rho(\nu\lambda-\mu\rho)C,\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

et de là

$$\left.\begin{aligned}&C[A(C-B)(\gamma-\alpha)+B(C+A)(\delta-\beta)]\\&\qquad+(2C+A-B)(A^2\gamma-B^2\delta)\end{aligned}\right\}=0\ \ldots\ldots\ldots\ldots\ldots(14),$$

ou, toute réduction faite,

$$\left.\begin{aligned}&A(A+C)(A+C-B\gamma)+B(C-B)(A+B-C\delta)\\&\qquad-AC(C-B\alpha)-BC(C+A\beta)\end{aligned}\right\}=0\ \ldots\ldots\ldots(15);$$

et puisque cette équation est du troisième ordre à l'égard des quantités A, B, C, il est évident que la ligne donnée par l'équation (10) est toujours tangente à une certaine courbe de la troisième classe. En supposant $\gamma=0$, $\delta=0$, ce qui réduit la courbe du troisième ordre aux lignes $R=0$, $S=0$, $(\beta-\alpha)\theta-\beta R-\alpha S=0$, l'équation (15) se partage dans les deux équations

$$C=0,\quad A(C-B)\alpha+B(C+A)\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

et la courbe de la troisième classe se réduit au point ($R=0$, $S=0$) et à une conique. Cela est un théorème connu, car l'on sait bien que si A, A' sont des points quelconques sur deux droites données α, α', et B, B' deux autres points déterminés, sur ces droites, de manière que l'intersection des lignes AB', $A'B$ soit toujours sur une ligne droite donnée, les deux lignes α, α' sont divisées homographiquement, B, B' étant deux points correspondants; et de plus, que la ligne qui unit les points correspondants de deux lignes divisées homographiquement, est toujours tangente à une certaine conique. Quant au point d'intersection des lignes α, α', que nous venons de trouver comme formant avec la conique une courbe de la troisième classe, on observera que, dans notre théorie, non-seulement les points des lignes α, α' se correspondent, mais que le point d'intersection des lignes α, α' correspond à tous les points de la troisième droite. La conique touche les deux droites α, α'; cela n'a pas, je crois, d'analogie dans la théorie générale.

27.

NOUVELLES REMARQUES SUR LES COURBES DU TROISIÈME ORDRE.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. X. (1845), pp. 102—109.]

JE me propose de développer ici quelques conséquences de la théorie que j'ai donnée, il y a quelques mois, dans ce Journal[1], [26], des points *correspondants* des courbes du troisième ordre. Rappelons d'abord la signification de ce terme et ajoutons-y quelques nouvelles définitions.

On dit que les points A, A' sont correspondants quand les tangentes à la courbe en ces points se rencontrent sur la courbe. En considérant les points correspondants A, A' et les deux autres points correspondants B, B', on dit que ces paires correspondent, quand les points d'intersection H, h de AB', $A'B$ ou AB, $A'B'$ sont situés sur la courbe. Les trois paires A, A', B, B', H, h sont nommées paires supplémentaires ou système supplémentaire. On dirait de même que les deux systèmes AA', BB', Hh et $A_1A'_1$, $B_1B'_1$, H_1h_1 sont des systemes supplémentaires correspondants, si, par exemple, A, A' et A_1, A'_1, &c., formaient des paires correspondantes.

On peut nommer quadrilatère inscrit le système de quatre droites qui passent par les points supplémentaires AA', BB', Hh, et conique d'involution chaque conique tangente à ces quatre droites, ou, en d'autres termes, inscrite dans un quadrilatère inscrit. Il est inutile d'expliquer ce que veulent dire coniques d'involution correspondantes, ou quadrilatères correspondants, ou l'expression conique correspondante à une paire donnée de points correspondants, &c.

Démontrons les théorèmes suivants :

THÉORÈME I. "Les tangentes menées par un point P de la courbe à trois coniques d'involution correspondantes forment un faisceau en involution."

[1] Voyez page 285 du tome IX.

Chaque conique peut se réduire à une paire de points qui correspondent aussi aux coniques (ce qui justifie la dénomination que nous avons donnée à ces coniques). Considérons un quadrilatère inscrit AA', BB', Hh, la conique d'involution tangente aux côtés de ce quadrilatère, et deux paires LL', MM' de points correspondants, ces paires étant correspondantes l'une à l'autre et à la conique d'involution. On sait que les lignes menées d'un point quelconque, et ainsi du point P de la courbe, par les points A, A', B, B', forment avec les tangentes à la conique menées par ce même point, un faisceau en involution. Mais PA, PA', PB, PB', PL, PL', et de même PB, PB', PL, PL', PM, PM', forment aussi des faisceaux en involution; donc les deux tangentes forment, avec PL, PL' et PM, PM', un faisceau en involution. De même, avec une conique correspondante à la première, PL, PL', PM, PM' et les deux tangentes forment un faisceau en involution; donc PL, PL' et les quatre tangentes forment un faisceau en involution, et de même en introduisant la troisième conique.

THÉORÈME II. "On peut circonscrire à une conique d'involution donnée une infinité de quadrilatères inscrits correspondants au premier."

Soient A, A', B, B', H, h comme auparavant; et A_1, A'_1 une paire de points correspondants qui correspondent à ceux-ci; en menant par A_1, A'_1 des tangentes à la conique qui se rencontrent en B_1, B'_1, H_1, h_1, on voit d'abord que les lignes PA, PA', PA_1, PA'_1 et les tangentes à la conique forment un faisceau en involution; et réciproquement, chaque point P qui satisfait à cette condition appartient à la courbe. Mais en prenant, par exemple, pour P le point B_1, les lignes PA_1, PA'_1 deviennent identiques avec les deux tangentes, ce qui satisfait à la condition d'involution. Donc B_1, B'_1, H_1, h_1 appartiennent à la courbe, ou A_1, A'_1, B_1, B'_1, H_1, h_1 forment un quadrilatère inscrit correspondant au premier ou à la conique.

THÉORÈME III. "Les centres d'homologie de deux coniques d'involution correspondantes forment un quadrilatère inscrit correspondant aux coniques."

Considérons les deux coniques et une troisième conique quelconque. Chaque point P pour lequel les six tangentes forment un faisceau en involution appartient à la courbe. En prenant pour P un centre d'homologie des deux premières coniques, les deux paires de tangentes deviennent identiques, ce qui satisfait à la condition d'involution; donc les six centres d'homologie sont sur la courbe, ou ces six points sont les sommets d'un quadrilatère inscrit qui correspond aussi aux coniques.

Réciproquement,

THÉORÈME IV. "Le lieu d'un point P qui se meut de manière que les tangentes menées par ce point à trois coniques données quelconques, forment toujours un faisceau en involution, est une courbe du troisième ordre qui passe par les dix-huit centres d'homologie des coniques prises deux à deux, ces centres formant six à six des quadrilatères inscrits correspondants."

La démonstration analytique de la première partie de ce théorème, quoique longue, me paraît assez intéressante pour trouver place ici. Pour plus de symétrie, prenons $\frac{\xi}{\zeta}$, $\frac{\eta}{\zeta}$ pour les coordonnées indéfinies d'un point, et représentons par

$$\Upsilon = 0, \quad \Upsilon' = 0, \quad \Upsilon'' = 0,$$

les équations des trois coniques, Υ, &c. étant des fonctions homogènes de la forme

$$\Upsilon = A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta, \text{ \&c.}$$

Soient $\frac{x}{z}$, $\frac{y}{z}$ les coordonnées du point P; mettons, pour abréger,

$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy,$$
$$W = Ax\xi + By\eta + Cz\zeta + F(y\zeta + \eta z) + G(z\xi + \zeta x) + H(x\eta + y\xi)$$

(de manière que $W = 0$ serait l'équation de la ligne polaire du point P). Nous avons

$$U\Upsilon - W^2 = 0,$$

pour l'équation des deux tangentes menées par le point P à la conique. Cela est probablement connu. Il est clair d'abord que cette équation appartient à une conique qui a un double contact avec la conique donnée, et par la forme à laquelle nous allons réduire cette équation, on voit ensuite qu'elle appartient à un système de deux droites qui passent par le point P. En développant et écrivant

$$BC - F^2 = a, \quad CA - G^2 = b, \quad AB - H^2 = c,$$
$$GH - AF = f, \quad HF - BG = g, \quad FG - CH = h,$$

on trouve, en effet,

$$a(y\zeta - z\eta)^2 + b(z\xi - x\zeta)^2 + c(x\eta - y\xi)^2$$
$$+ 2f(z\xi - x\zeta)(x\eta - y\xi) + 2g(x\eta - y\xi)(y\zeta - z\eta) + 2h(y\zeta - z\eta)(z\xi - x\zeta) = 0,$$

et de là, en posant

$$\mathfrak{A} = bz^2 + cy^2 - 2fyz, \quad \mathfrak{F} = ayz - gxy - hxz + fx^2,$$
$$\mathfrak{B} = cx^2 + az^2 - 2gxz, \quad \mathfrak{G} = bzx - hyz - fxy + gy^2,$$
$$\mathfrak{C} = ay^2 + bx^2 - 2hxy, \quad \mathfrak{H} = cxy - fxz - gyz + hz^2,$$

on obtient

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 - 2\mathfrak{F}\eta\zeta - 2\mathfrak{G}\xi\zeta - 2\mathfrak{H}\xi\eta = 0,$$

ou, en transportant l'origine au point P,

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 - 2\mathfrak{H}\xi\eta = 0;$$

et de même, pour l'équation des tangentes, par ce même point aux deux autres coniques

$$\mathfrak{A}'\xi^2 + \mathfrak{B}'\eta^2 - 2\mathfrak{H}'\xi\eta = 0,$$
$$\mathfrak{A}''\xi^2 + \mathfrak{B}''\eta^2 - 2\mathfrak{H}''\xi\eta = 0.$$

On obtient donc, pour la condition que ces lignes forment un faisceau en involution,

$$\begin{bmatrix} \mathfrak{A}, & \mathfrak{B}, & \mathfrak{H} \\ \mathfrak{A}', & \mathfrak{B}', & \mathfrak{H}' \\ \mathfrak{A}'', & \mathfrak{B}'', & \mathfrak{H}'' \end{bmatrix} = 0,$$

en représentant de cette manière le déterminant formé avec ces neuf quantités. Formons d'abord la fonction

$$\mathfrak{A}'\mathfrak{B}'' - \mathfrak{A}''\mathfrak{B}'.$$

Cela se reduit à

$$z\,[-2\,(fc)\,x^2y - 2\,(cg)\,xy^2 + (ca)\,y^2z - 2\,(fa)\,yz^2 - 2\,(bg)\,xz^2 + (bc)\,x^2z + 4\,(fg)\,xyz + (ba)\,z^3],$$

où $(fc) = (f'c'' - f''c')$, &c.

Multipliant par $\mathfrak{H}$, formant ensuite les quantités analogues et ajoutant; écrivant aussi

$$c\,(fg) + c'\,(f'g') + c''\,(f''g'') = (cfg),$$

c'est-à-dire (cfg) pour le déterminant formé avec les neuf quantités

$$c,\ f,\ g,\quad c',\ f',\ g',\quad c'',\ f'',\ g'',$$

on obtient d'abord les termes

$$-4\,(cfg)\,x^2y^2z^2,\quad +2\,(fcg)\,x^3y^2z^2,\quad +2\,(gfc)\,x^2y^3z^2,$$

qui se détruisent; les autres termes contiennent z^3 comme facteur, et en écartant cette quantité, l'on obtient en dernière analyse l'équation

$$\begin{aligned}&(cbf)\,x^3 + (acg)\,y^3 + (bah)\,z^3 + 4\,(fgh)\,xyz\\ &\quad + [2\,(agf) + (cah)]\,y^2z + [2\,(bhg) + (abf)]\,z^2x + [2\,(cfh) + (bcg)]\,x^2y\\ &\quad + [2\,(afh) + (abg)]\,yz^2 + [2\,(bgf) + (bch)]\,zx^2 + [2\,(chg) + (caf)]\,xy^2 = 0\end{aligned}$$

(où l'on peut, si l'on veut, écrire $z = 1$). C'est l'équation d'une courbe du troisième ordre.

Observation. Cette méthode peut être utile dans l'investigation d'autres problèmes relatifs aux coniques. Par exemple, la question de déterminer les centres d'homologie de deux coniques données revient à celle-ci: satisfaire identiquement à l'équation

$$\begin{aligned}&\mathfrak{A}\,\xi^2 + \mathfrak{B}\,\eta^2 + \mathfrak{C}\,\zeta^2 + 2\mathfrak{F}\,\eta\zeta + 2\mathfrak{G}\,\xi\zeta + 2\mathfrak{H}\,\xi\eta\\ &k\,(\mathfrak{A}'\xi^2 + \mathfrak{B}'\eta^2 + \mathfrak{C}'\zeta^2 + 2\,\mathfrak{F}'\eta\zeta + 2\mathfrak{G}'\xi\zeta + 2\mathfrak{H}'\xi\eta) = 0,\end{aligned}$$

parce que, en effectuant cela, il est facile de voir que $\frac{x}{z}$, $\frac{y}{z}$ sont les coordonnées du centre cherché. Écrivons

$$a + ka' = \mathrm{a},\quad c + kb' = \mathrm{b},\quad \&c.,$$

l'on obtient les six équations

$$\begin{aligned}&\mathrm{b}z^2 + \mathrm{c}y^2 - 2\mathrm{f}yz = 0, &&\mathrm{a}yz - \mathrm{g}xy - \mathrm{h}xz + \mathrm{f}x^2 = 0,\\ &\mathrm{c}x^2 + \mathrm{a}z^2 - 2\mathrm{g}zx = 0, &&\mathrm{b}zx - \mathrm{h}yz - \mathrm{f}yx + \mathrm{g}y^2 = 0,\\ &\mathrm{a}y^2 + \mathrm{b}x^2 - 2\mathrm{h}xy = 0, &&\mathrm{c}xy - \mathrm{f}zx - \mathrm{g}zy + \mathrm{h}z^2 = 0,\end{aligned}$$

dont les trois dernières se déduisent des autres. On peut, de ces six équations, éliminer les six quantités x^2, y^2, z^2, yz, zx, xy, considérées comme indépendantes; on obtient ainsi, toute réduction faite,

$$(\mathrm{abc} - \mathrm{af}^2 - \mathrm{bg}^2 - \mathrm{ch}^2 + 2\mathrm{fgh})^2 = 0,$$

équation qui détermine la quantité k; les trois premières équations deviennent alors équivalentes à deux, qui suffisent pour déterminer les rapports $\frac{x}{z}$, $\frac{y}{z}$. Il serait facile de rapprocher cette solution de celle que l'on déduit de la théorie des polaires reciproques. Par exemple, l'équation qui vient d'être obtenue entre les quantités a, b, ... est précisément celle qui exprime que la fonction

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}xz + 2\mathrm{h}xy$$

se divise en facteurs linéaires. Remarquons encore que, dans la géométrie solide, l'équation

$$U\Upsilon - W^2 = 0$$

appartient au cône, ayant le point P pour sommet, et circonscrit à une surface du second ordre qui a pour équation $\Upsilon = 0$. C'est un cas particulier d'une autre formule que voici :

"En représentant par $\beth$ une fonction linéaire des trois variables ξ, η, ζ (ou de quatre variables sans termes constants), et par P la même fonction de x, y, z, l'équation

$$\beth^2 U - 2\beth P W + P^2 \Upsilon = 0$$

appartient au cône ayant pour sommet le point dont les coordonnées sont x, y, z, et passant par la courbe d'intersection du plan $P = 0$, et de la surface du second ordre $\Upsilon = 0$. Et de même pour deux variables."

Je finirai en citant un théorème de géométrie dû à M. Hesse[1], qui a quelques rapports avec le sujet que je viens de traiter :

"Le lieu d'un point P, qui se meut de manière que ses polaires par rapport à trois coniques données se rencontrent dans le même point, est une courbe du troisième ordre; et encore cette courbe ne change pas quand on remplace les coniques données

$$U = 0, \quad U' = 0, \quad U'' = 0,$$

par trois nouvelles coniques de la forme

$$\lambda U + \lambda' U' + \lambda'' U'' = 0."$$

Cette courbe du troisième ordre est tout à fait distincte de celle que j'ai considérée.

[1] *Voyez* le Journal de M. Crelle, tome XXVIII.

28.

SUR QUELQUES INTÉGRALES MULTIPLES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tome X. (1845), pp. 158—168.]

J'AI donné, il y a trois ans, dans le *Cambridge Mathematical Journal*, [2], une formule assez singulière pour l'intégrale multiple

$$\int \ldots dx_1\, dx_2 \ldots \phi\,(a_1 - x_1,\quad a_2 - x_2 \ldots),$$

prise entre les limites données par l'équation

$$\frac{x_1^2}{h_1^2} + \frac{x_2^2}{h_2^2} + \ldots = 1,$$

la fonction ϕ étant seulement assujettie à la condition de ne pas devenir infinie entre les limites de l'intégration. L'expression que j'obtiens est une suite infinie, dont le terme général est de cette forme,

$$A_p \left(h_1^2 \frac{d^2}{da_1^2} + h_2^2 \frac{d^2}{da_2^2} + \ldots\right)^p \phi\,(a_1,\ a_2 \ldots).$$

En appliquant ce résultat à un cas particulier, j'ai obtenu l'intégrale à n variables analogue à celle qui exprime le *potentiel* d'un ellipsoïde homogène, pour un point extérieur. J'ai depuis cherché à étendre ces résultats au cas d'une densité variable et égale à une fonction rationnelle et entière des coordonnées, et d'une loi d'attraction selon une puissance quelconque de la distance (toujours à n variables); mais quoique j'aie réussi à effectuer cette généralisation, mes formules étaient si confuses et si inférieures à celles que donne la belle analyse de M. Lejeune-Dirichlet, que je ne les ai jamais publiées. Cependant, en revenant il y a quelques jours sur ce sujet, en

me fondant sur une intégrale plus générale, j'ai trouvé que la question était à peine plus difficile que dans le cas d'une densité constante, et se laissait traiter exactement de la même manière. J'ai réussi de cette façon à exprimer l'intégrale cherchée au moyen d'une seule intégrale définie *abélienne*, et de ses coefficients différentiels relatifs aux constantes qui y entrent; et il m'a paru que les formules que j'ai ainsi obtenues pourraient n'être pas tout à fait indignes de l'attention des géomètres.

Considérons l'intégrale multiple à n variables V, donnée par l'équation

$$V = \int dx_1 \ldots x_1^{2\alpha_1+1} \ldots (f \text{ termes})\ x_{f+1}^{2\alpha_{f+1}} \ldots (n-f \text{ termes})\ \phi(a_1 - x_1 t, \ldots),$$

où les variables $x_1, \ldots$ doivent recevoir des valeurs réelles quelconques, positives ou négatives, qui satisfassent à la condition

$$\frac{x_1^2}{h_1^2} + \ldots \leqq 1;$$

et l'on suppose de plus qu'il est permis de développer (sous le signe intégral) la fonction ϕ suivant les puissances ascendantes des variables $x_1, \ldots$.

En faisant ce développement, il est clair que les termes qui contiennent des puissances impaires d'une ou de plusieurs des variables se détruisent par l'intégration; donc, en ne faisant attention qu'aux termes qui contiennent seulement des puissances paires, on a ce terme général,

$$\frac{(-)^f t^{2p+f}}{[2r_1+1]^{2r_1+1} \ldots [2r_{f+1}]^{2r_{f+1}} \ldots} \left(\frac{d}{da_1}\right)^{2r_1+1} \ldots \left(\frac{d}{da_{f+1}}\right)^{2r_{f+1}} \ldots \phi(a_1, \ldots)$$

$$\times \int dx_1 \ldots x_1^{2\alpha_1+2r_1+2} \ldots x_{f+1}^{2\alpha_{f+1}+2r_{f+1}} \ldots,$$

où

$$p = r_1 + \ldots;$$

il est à peine nécessaire de remarquer que, dans l'expression

$$[2r_1+1]^{2r_1+1} \ldots [2r_{f+1}]^{2r_{f+1}} \ldots,$$

il faut prendre f termes tels que $[2r_1+1]^{2r_1+1}$, et $\overline{n-f}$ termes de l'autre forme $[2r_{f+1}]^{2r_{f+1}}$, et ainsi dans tous les cas semblables.

L'intégrale qui entre dans cette formule a été trouvée, comme on sait, par M. Dirichlet. Sa valeur est

$$\frac{\Gamma(\alpha_1 + r_1 + \frac{3}{2}) \ldots \Gamma(\alpha_{f+1} + r_{f+1} + \frac{1}{2}) \ldots}{\Gamma(p + k + f + \frac{1}{2}n + 1)} h_1^{2\alpha_1+2r_1+3} \ldots h_{f+1}^{2\alpha_{f+1}+2r_{f+1}+1},$$

où

$$k = \alpha_1 + \ldots.$$

Le terme entier devient, après quelques réductions très-simples,

$$\frac{(-)^f \pi^{\frac{1}{2}n} t^{2p+f}}{2^{2p+k+f}\Gamma(p+k+f+\frac{1}{2}n+1)}$$

$$\times \frac{L_1 \ldots M_{f+1} \ldots}{[r]^{r_1} \ldots} h_1^{2\alpha_1+2r_1+3} \ldots h_{f+1}^{2\alpha_{f+1}+2r_{f+1}+1} \left(\frac{d}{da_1}\right)^{2r_1+1} \ldots \left(\frac{d}{da_{f+1}}\right)^{2r_{f+1}} \ldots \phi(a_1, \ldots),$$

où l'on écrit, pour un moment,

$$L_1 = (2r_1 + 3)(2r_1 + 5) \ldots (2r_1 + 2\alpha_1 + 1),$$
$$\vdots$$
$$M_{f+1} = (2r_{f+1}+1)(2r_{f+1}+3) \ldots (2r_{f+1}+2\alpha_{f+1}-1);$$
$$\vdots$$

puis on le transforme en

$$\frac{(-)^f \pi^{\frac{1}{2}n} t^{2p+f}}{2^{2p+k+f}\Gamma(p+k+f+\frac{1}{2}n+1)}$$

$$\times \left(\frac{d}{da_1} \ldots \frac{d}{da_f}\right)\left(h_1^3 \frac{d}{dh_1}\right)^{\alpha_1} \ldots h_1^3 \ldots h_{f+1} \ldots \frac{1}{[r_1]^{r_1}}\left(h_1^2 \frac{d^2}{da_1^2}\right)^{r_1} \ldots \phi(\alpha_1, \ldots).$$

En effet, les symboles $\frac{d}{da_1}$ et $h_1^3 \frac{d}{dh_1}$ étant conversibles, on a

$$\frac{d}{da_1}\left(h_1^3 \frac{d}{dh_1}\right)^{\alpha_1} h_1^{2r_1+3}\left(\frac{d}{da_1}\right)^{2r_1} = L_1 h_1^{2r_1+2\alpha_1+3}\left(\frac{d}{da_1}\right)^{2r_1+1};$$

et ainsi de suite.

En prenant la somme de tous les termes qui correspondent à une même valeur de p, on a

$$S \frac{1}{[r_1]^{r_1}}\left(h_1^2 \frac{d^2}{da_1^2}\right)^{2r_1} \ldots = \frac{1}{[p]^p} \nabla^p.$$

En posant

$$\nabla = \left(h_1^2 \frac{d^2}{da_1^2} + \ldots\right);$$

puis, en observant que p doit s'étendre depuis 0 jusqu'à ∞, on a, pour l'intégrale cherchée,

$$V = \frac{(-)^f \pi^{\frac{1}{2}n}}{2^{k+f}}\left(\frac{d}{da_1} \ldots \frac{d}{da_f}\right)\left(h_1^3 \frac{d}{dh_1}\right)^{\alpha_1} \ldots$$

$$\times \left[h_1^3 \ldots h_{f+1} \ldots t^f S_{p=0}^{p=\infty}\left(\frac{t^{2p}}{2^{2p}[p]^p \Gamma(p+k+f+\frac{1}{2}n+1)} \nabla^p \phi(a_1, \ldots)\right)\right],$$

ce qui fait voir que l'intégrale V dépend de cette seule expression,

$$U = S_0^\infty\left(\frac{t^{2p}}{2^{2p}[p]^p \Gamma(p+k+f+\frac{1}{2}n+1)} \nabla^p \phi(a_1, \ldots)\right).$$

On peut remarquer, en passant, que cette quantité satisfait à cette équation différentielle,

$$\frac{1}{t}\frac{d}{dt}\left(t^{-2k-2f-n+1}\frac{d}{dt}(t^{2k+2f+n}U)\right)-\nabla U=0,$$

ou

$$\frac{d^2U}{dt^2}+(2k+2f+n+1)\frac{1}{t}\frac{dU}{dt}-\nabla U=0.$$

M. G. Boole, de Lincoln, a déduit une équation semblable de mes formules dans le *Mathematical Journal*. C'est à lui qu'on doit l'introduction dans l'intégrale proposée de cette quantité t, ce qui, au reste, n'est pas d'une grande importance ici; mais j'ai cru devoir la conserver, à cause de cette équation même, qui pourrait conduire à des résultats intéressants.

A présent, soit

$$\phi(a_1-x_1t,\ldots)=\frac{1}{\{(a_1-x_1t)^2+\ldots\}^{\frac{1}{2}n-s}},$$

où s est un nombre entier; je pose, pour abréger,

$$\tfrac{1}{2}n-s=i,\quad k+f+s=\sigma,$$

ce qui donne

$$\Gamma(p+k+f+\tfrac{1}{2}n+1)=\Gamma(p+i+\sigma+1)=[p+i+\sigma]^{p+1}\Gamma(i+\sigma);$$

et ainsi

$$U=\frac{1}{\Gamma(i+\sigma)}S_0^\infty\left(\frac{t^{2p}}{2^{2p}[p]^p[p+i+\sigma]^{p+1}}\nabla^p\frac{1}{(a_1^2+\ldots)^i}\right).$$

Le cas de $\sigma=0$, qui est le plus simple, est, en effet, celui du Mémoire cité, et à ce cas on peut réduire celui de σ entier négatif; il y a même deux manières d'effectuer cette réduction. Représentons, en effet, la valeur de U par cette notation plus complète U_i; on obtient tout de suite

$$U_i^{-\sigma}=2^{-2\sigma}t^{2\sigma-2i}\left(\frac{1}{t}\frac{d}{dt}\right)^\sigma(t^{2i}U_i^0),$$

et

$$U_i^{-\sigma}=\frac{1}{2^{2\sigma}[1-i]^\sigma[1-\frac{1}{2}n]^\sigma}\left(\frac{d^2}{da_1^2}+\ldots\right)^\sigma U_{i-\sigma}^0,$$

la seconde desquelles équations se déduit de cette formule facile à démontrer,

$$\left(\frac{d^2}{da_1^2}+\ldots\right)^\sigma\frac{1}{(a_1^2+\ldots)^{i-\sigma}}=2^{2\sigma}[i-1]^\sigma[i-\tfrac{1}{2}n]^\sigma\frac{1}{(a_1^2+\ldots)^i}.$$

Nous pouvons donc, dans la suite, ne considérer que le cas de σ entier positif.

Commençons par opérer la transformation que voici; en déterminant ζ par l'équation

$$\frac{a_1^2}{\zeta+h_1^2}+\ldots=t^2,$$

je pose
$$l_1 = \frac{h_1^2}{\zeta}, \ldots, \quad a_1^2 = (1+l_1)\, t^2 \alpha_1^2, \ldots,$$

ce qui donne
$$\zeta = \alpha_1^2 + \ldots,$$
$$a_1^2 + \ldots = t^2 [(1+l_1)\, \alpha_1^2 + \ldots],$$
$$\nabla = \frac{\zeta}{t^2}\left(\frac{l_1}{1+l_1}\right) \frac{d^2}{d\alpha_1^2} + \ldots),$$

où il faut remarquer que ce ζ, contenu en ∇, ne doit pas être affecté des symboles $\frac{d}{d\alpha_1}, \ldots$ de ∇, de manière qu'il faut écrire

$$\nabla^p = t^{-2p} \zeta^p \left(\frac{l_1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right)^p = t^{-2p} (\alpha_1^2 + \ldots)^p \left(\frac{l_1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right)^p,$$

$$U = \frac{1}{\Gamma(i)\, t^{2i} (\alpha_1^2 + \ldots)^i}$$

$$\times \mathfrak{S}_0^\infty \left(\frac{1}{2^{2p}[p+i+\sigma]^{p+\sigma+1}[p]^p} (\alpha_1^2 + \ldots)^{p+i} \left(\frac{l_1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right)^p \frac{1}{\{\alpha_1^2(1+l_1) + \ldots\}^i}\right),$$

formules qui se prêtent mieux aux réductions, quoique plus compliquées en apparence.

Écrivons d'abord
$$\frac{l_1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots = \left(\frac{d^2}{d\alpha_1^2} + \ldots\right) - \left(\frac{1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right) = \Delta - \left(\frac{1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right).$$

En développant la $p^{ième}$ puissance de ce symbole selon les puissances de Δ, on a
$$(-)^{p-q} \frac{[p]^p}{[p-q]^{p-q}[q]^q} \Delta^q \left(\frac{1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right)^{p-q},$$

qui doit s'appliquer à $\frac{1}{\{\alpha_1^2(1+l_1) + \ldots\}^i}$.

Considérons l'expression
$$\left(\frac{l_1}{1+l_1} \frac{d^2}{d\alpha_1^2} + \ldots\right)^{p-q} \frac{1}{\{\alpha_1^2(1+l_1) + \ldots\}^i},$$

et mettons, pour un moment,
$$(1+l_1)\, \alpha_1^2 = \alpha_1'^2 \ldots;$$

cette expression se trouve réduite à
$$\left(\frac{d^2}{d\alpha'^2} + \ldots\right)^{p-q} \frac{1}{(\alpha_1'^2 + \ldots)^i},$$

laquelle, par une formule déjà citée, devient
$$2^{2p-2q}[i+p-q-1]^{p-q}[i+p-q-\tfrac{1}{2}n]^{p-q} \frac{1}{(\alpha_1'^2 + \ldots)^{i+p-q}},$$

c'est-à-dire $$2^{2p-2q}[i+p-q-1]^{p-q}[i+p-q-\tfrac{1}{2}n]^{p-q}\frac{1}{\{\alpha_1^2(1+l_1)+\ldots\}^{i+p-q}}.$$

Le terme général de U, en faisant abstraction du facteur $\dfrac{1}{\Gamma(i)\,t^{2i}(\alpha_1^2+\ldots)^i}$, se réduit à

$$\frac{(-)^{p-q}(\alpha_1^2+\ldots)^{p+i}}{2^{2q}[p+i+\sigma]^{\sigma+q+1}[p-q]^{p-q}[q]^q}[i+p-q-\tfrac{1}{2}n]^{p-q}\,\Delta^q\frac{1}{\{\alpha_1^2(1+l_1)+\ldots\}^{i+p-q}};$$

considérons la partie de ce terme qui est de l'ordre θ en $l_1, \ldots$, elle devient

$$\frac{(-)^{p-q+\theta}(\alpha_1^2+\ldots)^{p+1}}{2^{2q}[p-q]^{p-q}[q]^q[s]^s}[i+p-q+\theta-1]^{\theta-\sigma-q-1}$$
$$\times[i+p-q-\tfrac{1}{2}n]^{p-q}\,\Delta^q\frac{(l_1\alpha_1^2+\ldots)^\theta}{(\alpha_1^2+\ldots)^{i+p-q+\theta}}.$$

Soit, en général, Θ une fonction homogène de l'ordre 2θ des lettres $\alpha_1, \ldots$; on a

$$\Delta\frac{\Theta}{(\alpha_1^2+\ldots)^i}=\frac{\Delta\Theta}{(\alpha_1^2+\ldots)^i}+2^2i\,(i+1-2\theta-\tfrac{1}{2}n)\frac{\Theta}{(\alpha_1^2+\ldots)^{i+1}};$$

et de là, en répétant toujours l'opération Δ, et faisant attention à ce que $\Delta\Theta$, $\Delta^2\Theta, \ldots$ sont des fonctions homogènes des ordres $\overline{2\theta-1}$, $\overline{2\theta-2}$, &c., on obtient

$$\Delta^q\frac{\Theta}{(\alpha_1^2+\ldots)^i}=\frac{[q]^q}{[\lambda]^\lambda[q-\lambda]^{q-\lambda}}2^{2q-2\lambda}[i+q-\lambda-1]^{q-\lambda}$$
$$\times[i+q-2\theta-\tfrac{1}{2}n]^{q-\lambda}\,\Delta^\lambda\Theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{i+q-\lambda}},$$

ou $$\Delta^q\frac{(l_1\alpha_1^2+\ldots)^\theta}{(\alpha_1^2+\ldots)^{i+p-q+\theta}}=\frac{[q]^q}{[\lambda]^\lambda[q-\lambda]^{q-\lambda}}2^{2q-2\lambda}[i+p+\theta-\lambda-1]^{q-\lambda}$$
$$\times[i+p-\theta-\tfrac{1}{2}n]^{q-\lambda}\,\Delta^\lambda(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{i+p+\theta-\lambda}},$$

depuis $\lambda=0$ jusqu'à $\lambda=q$.

Puis, pour le terme général de U,

$$\frac{(-)^{p-q+\theta}}{2^{2\lambda}[p-q]^{p-q}[\theta]^\theta[\lambda]^\lambda[q-\lambda]^{q-\lambda}}[i+p+\theta-\lambda-1]^{\theta-\sigma-\lambda-1}$$
$$\times[i+p-q-\tfrac{1}{2}n]^{p-q}[i+p-\theta-\tfrac{1}{2}n]^{q-\lambda}\,\Delta^\lambda(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{\theta-\lambda}},$$

ou $$\frac{(-)^{-q}}{[p-q]^{p-q}[q-\lambda]^{q-\lambda}}[i+p-q-\tfrac{1}{2}n]^{p-q}[i+p-\theta-\tfrac{1}{2}n]^{q-\lambda}$$
$$\times\frac{(-)^{p+\theta}}{2^{2\lambda}[\theta]^\theta[\lambda]^\lambda}[i+p+\theta-\lambda-1]^{\theta-\sigma-\lambda-1}\,\Delta^\lambda(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{\theta-\lambda}};$$

le dernier facteur étant indépendant de q, q doit s'étendre depuis 0 jusqu'à p. Mais à cause de $[q-\lambda]^{q-\lambda}$ qui devient infini pour $q<\lambda$, on peut faire étendre q depuis $q=\lambda$ jusqu'à $q=p$; ou, en écrivant

$$q-\lambda=q', \quad p-\lambda=p',$$

q' s'étend depuis 0 jusqu'à p'. Le facteur à sommer est donc

$$(-)^\lambda \left\{ \frac{(-)^{q'}}{[p'-q']^{p'-q'}\,[q']^{q'}} [C-q']^{p'-q'}\,[M]^{q'} \right\},$$

si, pour un moment, l'on pose

$$C=i+p'-\tfrac{1}{2}n, \quad M=i+p-\theta-\tfrac{1}{2}n.$$

Cette somme se réduit à

$$(-)^\lambda \frac{[C-M]^{C-M-p'}}{[C-M-p']^{C-M-p'}},$$

c'est-à-dire à

$$(-)^\lambda \frac{[\theta-\lambda]^{\theta-p}}{[\theta-p]^{\theta-p}},$$

et nous avons ainsi le terme général

$$\frac{(-)^{\lambda+p+\theta}\,[\theta-\lambda]^{\theta-p}\,[i+\theta-\lambda-1+p]^{\theta-\sigma-\lambda-1}}{2^{2\lambda}\,[\theta]^\theta\,[\lambda]^\lambda\,[\theta-p]^{\theta-p}}\,\Delta^\lambda\,(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{\theta-\lambda}}.$$

Écrivons

$$p=\theta-p';$$

cela devient

$$\frac{(-)^{p'}\,[M]^{p'}\,[C-p']^{C-A}}{[p']^{p'}}\,\frac{(-)^\kappa}{2^{2\lambda}\,[\theta]^\theta\,[\lambda]^\lambda}\,\Delta^\lambda\,(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{\theta-\lambda}},$$

où
$$M=\theta-\lambda, \quad C=i+2\theta-\lambda-1, \quad C-A=\theta-\sigma-\lambda-1;$$

p' doit s'étendre depuis $-\infty$ jusqu'à θ. Mais à cause de $[p']^{p'}$, qui devient infini, pour p' négatif, on peut l'étendre seulement depuis 0 jusqu'à θ, ou même seulement depuis 0 jusqu'à M, à cause du facteur $[M]^{p'}$. La somme se réduit à

$$[C-M]^{C-M-A}\,[C-A]^M=[i+\theta-1]^{-\sigma-1}\,[\theta-\sigma-\lambda-1]^{\theta-\lambda},$$

et le terme général devient

$$\frac{(-)^\lambda}{2^{2\lambda}\,[\theta]^\theta\,[\lambda]^\lambda}\,[i+\theta-1]^{-\sigma-1}\cdot[\theta-\sigma-\lambda-1]^{\theta-\lambda}\,\Delta^\lambda\,(l_1\alpha_1^2+\ldots)^\theta\cdot\frac{1}{(\alpha_1^2+\ldots)^{\theta-\lambda}}.$$

Écrivons enfin

$$\theta=\lambda+\kappa,$$

le terme devient

$$\frac{(-)^\lambda}{2^{2\lambda}[\lambda]^\lambda[\lambda+\kappa]^{\lambda+\kappa}[i+\lambda+\kappa+\sigma]^{\sigma+1}}[\kappa-\sigma-1]^\kappa\,\Delta^\lambda(l_1\alpha_1^2+\ldots)^{\lambda+\kappa}\cdot\frac{1}{(\alpha_1^2+\ldots)^\lambda}.$$

Il faut que κ soit toujours positif, car autrement le terme s'évanouit à cause de $\Delta^\lambda(l_1\alpha_1^2+\ldots)^{\lambda+\kappa}$; mais pour κ plus grand que σ, le facteur $[\kappa-\sigma-1]^\kappa$ s'évanouit, donc κ s'étend seulement depuis 0 jusqu'à σ.

Soit $k_1+\ldots=\kappa$, et considérons les termes de $\Delta^\lambda(l_1\alpha_1^2+\ldots)^{\lambda+\kappa}$ qui contiennent $\alpha_1^{2k_1},\ldots$; ces termes seront de la forme

$$\frac{[\lambda]^\lambda}{[q_1]^{q_1}\ldots}\,\frac{[\lambda+\kappa]^{\lambda+\kappa}}{[q_1+k_1]^{q_1+k_1}\ldots}\left(\frac{d^2}{d\alpha_1^2}\right)^{2q_1}\ldots l_1^{q_1+k_1}\alpha_1^{2q_1+2k_1}\ldots,$$

où

$$q_1+\ldots=\lambda,$$

c'est-à-dire

$$\frac{[\lambda]^\lambda[\lambda+\kappa]^{\lambda+\kappa}[2q_1+2k_1]^{2q_1}\ldots}{[q_1]^{q_1}\ldots[q_1+k_1]^{q_1+k_1}\ldots}\,l_1^{q_1+k_1}\ldots\alpha_1^{2k_1}\ldots,$$

ou, en réduisant,

$$2^{2\lambda}[\lambda]^\lambda[\lambda+\kappa]^{\lambda+\kappa}\frac{\alpha_1^{2k_1}\ldots}{[k_1]^{k_1}\ldots}\,\frac{[q_1+k_1-\frac{1}{2}]^{q_1}\ldots}{[q_1]^{q_1}\ldots}\,l_1^{q_1+k_1}\ldots;$$

et cela donne pour U,

$$\frac{(-)^\lambda}{[i+\kappa+\lambda+\sigma]^{\sigma+1}}\,\frac{[q_1+k_1-\frac{1}{2}]^{q_1}\ldots}{[q_1]^{q_1}\ldots}\,l_1^{q_1+k_1}\ldots\frac{[\kappa-\sigma-1]^\kappa.\alpha_1^{2k_1}\ldots}{[k_1]^{k_1}}\,\frac{1}{(\alpha_1^2+\ldots)^\kappa}.$$

Soit

$$\frac{1}{\{(1+l_1u)\ldots\}^{\frac{1}{2}}}=(0)+(1)\,u\ldots+(\lambda)\,u^\lambda+\ldots;$$

on a

$$(\lambda)=(-)^\lambda\frac{[q_1-\frac{1}{2}]^{q_1}\ldots}{[q_1]^{q_1}\ldots}\,l_1^{q_1}\ldots+\&c.,\quad q_1+\ldots=\lambda;$$

et de là,

$$\frac{1}{[k_1-\frac{1}{2}]^{k_1}\ldots}\,\frac{1}{l_1^{\frac{1}{2}}\ldots}\left(l_1^2\frac{d}{dl_1}\right)^{k_1}\ldots l_1^{\frac{1}{2}}(\lambda)=(-)^\lambda\frac{[q_1+k_1-\frac{1}{2}]^{q_1}\ldots}{[q_1]^{q_1}\ldots}\,l^{q_1+k_1}\ldots,$$

de manière que le terme de U devient

$$\frac{1}{l_1^{\frac{1}{2}}\ldots}\left(l_1^2\frac{d}{dl_1}\right)^{k_1}\ldots l_1^{\frac{1}{2}}\left(\frac{1}{[i+\kappa+\lambda+\sigma]^{\sigma+1}}(\lambda)\right)\frac{[\kappa-\sigma-1]^\kappa\,\alpha^{2k_1}\ldots}{[2k_1]^{2k_1}\ldots}\,\frac{1}{(\alpha_1^2+\ldots)^\kappa}.$$

Prenant la somme pour λ, à l'aide de

$$[\sigma]^\sigma\int_0^1(1-u)^\sigma\,u^{i+\kappa+\lambda-1}\,du=\frac{1}{[i+\kappa+\sigma+\lambda]^{\sigma-1}}$$

(ce qui suppose $l+u>0$), on obtient

$$\frac{2^{2\kappa}[\sigma]^{\sigma}[\kappa-\sigma-1]^{\kappa}\,\alpha_1^{2k_1}\ldots}{[2k_1]^{2k_1}\ldots(\alpha_1^2+\ldots)^{\kappa}}\,\frac{1}{\sqrt{l_1\ldots}}\left(l_1^2\frac{d}{dl_1}\right)^{k_1}\ldots\sqrt{l_1\ldots}\int_0^1\frac{(1-u)^{\sigma}\,u^{i+\kappa-1}\,du}{\{(1+l_1u)\ldots\}^{\frac{1}{2}}},$$

ou, mettant $l_1=\dfrac{h_1^2}{\zeta},\ldots,$

$$2^{\kappa}[\sigma]^{\sigma}[\kappa-\sigma-1]^{\kappa}\,\frac{\alpha_1^{2k_1}\ldots}{[2k_1]^{2k_1}\ldots}\,\zeta^{\frac{1}{2}n-2\kappa}\times\frac{1}{h_1\ldots}\left(h_1^3\frac{d}{dh_1}\right)^{k_1}\ldots h_1\ldots\int_0^1\frac{(1-u)^{\sigma}\,u^{i+\kappa-1}\,du}{\{(\zeta+h_1^2u)\ldots\}^{\frac{1}{2}}};$$

en rétablissant le facteur constant $\dfrac{1}{\Gamma(i)\,t^{2i}\,\zeta^{i}}$ de U, le terme général de cette quantité est

$$\frac{2^{\kappa}[\sigma]^{\sigma}[\kappa-\sigma-1]^{\kappa}}{t^{2i}\,\Gamma(i)}\,\frac{\alpha_1^{2k_1}\ldots}{[2k_1]^{2k_1}\ldots}\,\zeta^{\frac{1}{2}n-2\kappa+i}\times\frac{1}{h_1\ldots}\left(h_1^3\frac{d}{dh_1}\right)^{k_1}\ldots h_1\ldots\int_0^1\frac{(1-u)^{\sigma}\,u^{i+\kappa-1}\,du}{\{(\zeta+h_1^2u)\ldots\}^{\frac{1}{2}}},$$

où k_1, &c., sont des entiers positifs quelconques qui satisfont à

$$k_1+\ldots=\kappa,$$

et κ peut s'étendre depuis 0 jusqu'à σ. Il faut observer qu'en différentiant par $\dfrac{d}{dh_1}$, &c., on ne doit pas considérer ζ comme fonction de $h_1\ldots$, ce qui est cependant nécessaire dans l'équation entre V et U.

29.

ADDITION A LA NOTE SUR QUELQUES INTÉGRALES MULTIPLES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tome x. (1845), pp. 242—244.]

On démontre, au moyen des formules que j'ai données sur ce sujet, [28], une propriété remarquable de l'intégrale multiple

$$V = \int dx_1 \ldots dx_n x_1^{2\alpha_1+1} \ldots x_{f+1}^{2\alpha_{f+1}} \ldots \phi(a_1 - x_1 t, \ldots),$$

prise entre les limites

$$\frac{x_1^2}{h_1^2} + \ldots + \frac{x_n^2}{h_n^2} \leqq 1.$$

En effet, en supposant toujours que l'on peut développer la fonction ϕ selon les puissances entières et positives de t, l'intégrale V peut s'exprimer sous la forme

$$V = \frac{(-)^f \pi^{\frac{1}{2}n} t^f}{2^{k+f}} \left(\frac{d}{da_1} \ldots \frac{d}{da_f}\right) \left(h_1^2 \frac{d}{dh_1}\right)^{\alpha_1} \ldots \left(h_n^2 \frac{d}{dh_n}\right)^{\alpha_n} h_1^3 \ldots h_{f+1} \ldots U,$$

où

$$k = \alpha_1 + \ldots + \alpha_n,$$

$$U = S_{p=0}^{p=\infty} \left(\frac{t^{2p}}{2^{2p}\Gamma(p+1)\Gamma(p+k+f+\frac{1}{2}n+1)} \nabla^p \phi(a_1, \ldots)\right),$$

$$\nabla = h_1^2 \frac{d^2}{da_1^2} + \ldots + h_n^2 \frac{d^2}{da_n^2}.$$

Soit

$$W = \int dx_1 \ldots \phi(a_1 - x_1 t T, \ldots)$$

entre les mêmes limites que V. On a

$$W = \pi^{\frac{1}{2}n} h_1 \dots h_n \mathbf{S}_{p=0}^{p=\infty} \left(\frac{T^{2p} t^{2p}}{2^{2p} \Gamma(p+1) \Gamma(p+\frac{1}{2}n+1)} \nabla^p \phi(a_1, \dots) \right).$$

En multipliant par

$$\frac{2}{\Gamma(k+f)} T^{n+1} (1-T^2)^{k+f} dT,$$

et intégrant depuis $T=0$ jusqu'à $T=1$, on obtient

$$\frac{2}{\Gamma(k+f)} \int_0^1 T^{n+1} (1-T^2)^{k+f} W dT$$

$$= \pi^{\frac{1}{2}n} h_1 \dots h_n \mathbf{S}_{p=0}^{p=\infty} \left(\frac{t^{2p}}{2^{2p} \Gamma(p+1) \Gamma(p+k+f+\frac{1}{2}n+1)} \nabla^p \phi(a_1, \dots) \right),$$

puisque en général

$$\frac{2}{\Gamma(k+f)} \int_0^1 T^{2p+n+1} (1-T^2)^{k+f} dT = \frac{\Gamma(p+\frac{1}{2}n+1)}{\Gamma(p+k+f+\frac{1}{2}n+1)}.$$

On a donc l'équation

$$\frac{2}{\Gamma(k+f)} \int_0^1 T^{n+1} (1-T^2)^{k+f} W dT = \pi^{\frac{1}{2}n} h_1 \dots h_n U.$$

Mettons la valeur de U qui en résulte, dans l'équation entre V et U; faisons aussi $t=1$, ce qui ne nuit pas à la généralité. On a, en rassemblant les formules,

$$V = \int dx_1 \dots dx_n \dots x_1^{2a_1+1} \dots x_{f+1}^{2a_{f+1}} \dots \phi(a_1 - x_1, \dots),$$

$$W = \int dx_1 \dots dx_n \dots \phi(a_1 - x_1 T, \dots),$$

$$V = \frac{(-)^f}{2^{k+f-1} \Gamma(k+f)} \left(\frac{d}{da_1} \dots \frac{d}{da_f} \right) \left(h_1^3 \frac{d}{dh_1} \right)^{a_1} \dots \left(h_n^3 \frac{d}{dh_n} \right)^{a_n} h_1^2 \dots h_f^2 \int_0^1 T^{n+1} (1-T^2)^{k+f} W dT;$$

ce qui établit ce théorème: La suite des intégrales V s'exprime au moyen des coefficients différentiels par rapport à $h_1 \dots h_n$ et $a_1 \dots a_f$ de la seule intégrale

$$\int_0^1 T^{n+1} (1-T^2)^{k+f} \int dx_1 \dots dx_n \dots \phi(a_1 - x_1 T, \dots).$$

En supposant que les indices dans V soient tous pairs, on a $f=0$. Les symboles $\frac{d}{da_1}, \dots$ n'entrent plus dans l'expression de V. En changeant la fonction ϕ, on a ces formules plus simples,

$$\left\{\begin{aligned} V &= \int dx_1 \ldots dx_n\, x_1^{2\alpha_1} \ldots x_n^{2\alpha_n}\, \phi(x_1, \ldots, x_n), \\ W &= \int dx_1 \ldots dx_n\, \phi(Tx_1, \ldots, Tx_n), \\ V &= \frac{1}{2^{k-1}\,\Gamma(k)} \left(h_1^3 \frac{d}{dh_1}\right)^{\alpha_1} \ldots \left(h_n^3 \frac{d}{dh_n}\right)^{\alpha_n} \int_0^1 T^{n+1}(1-T^2)^k\, W dT, \end{aligned}\right.$$

où V s'exprime au moyen des coefficients différentiels par rapport à $h_1, \ldots, h_n$ de l'intégrale

$$\int_0^1 T^{n+1}(1-T^2)^k \int dx_1 \ldots dx_n\, \phi(Tx_1, \ldots, Tx_n),$$

de manière que nous avons trouvé des relations entre des intégrales définies qui contiennent une fonction indéterminée, assujettie à la seule condition d'être développable dans les limites de l'intégration selon les puissances entières et positives des variables[1], mais dans lesquelles les limites sont données par

$$\frac{x_1^2}{h_1^2} + \ldots + \frac{x_n^2}{h_n^2} < 1.$$

[1] Cette condition est supposée dans la démonstration, mais il me paraît, d'après la forme du résultat, qu'elle n'est pas essentielle.

30.

MÉMOIRE SUR LES COURBES À DOUBLE COURBURE ET LES SURFACES DÉVELOPPABLES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tome x. (1845), pp. 245—250.

On trouve, dans la *Théorie des courbes algébriques* de M. Plücker, de très-belles recherches sur le nombre des différentes singularités (points d'inflexion, tangentes doubles, &c.) des courbes planes. Les mêmes principes peuvent s'appliquer au cas des courbes à trois dimensions. Pour cela, considérons une suite continue de points dans l'espace, les lignes qui passent par deux points consécutifs, et les plans qui passent par trois points consécutifs; ou, en envisageant autrement la même figure, une suite de lignes dont chacune rencontre la ligne consécutive, les points d'intersection de deux lignes consécutives, et les plans qui contiennent ces lignes; ou encore de cette manière: une suite de plans, les lignes d'intersection de deux plans consécutifs, les points d'intersection de trois plans consécutifs. C'est ce que l'on peut nommer *un système simple.* Ce système est évidemment formé d'une courbe à double courbure et d'une surface développable; la courbe est l'arête de rebroussement de la surface, la surface est l'osculatrice développable de la courbe. Les points du système sont des points dans la courbe, les lignes du système sont les tangentes à la courbe, les plans du système sont les plans osculateurs de la courbe. De même, les plans sont les plans tangents de la surface, les lignes sont les génératrices de la surface; pour les points, on peut les nommer les points de rebroussement de la surface. J'entendrai dans la suite par les termes *ligne par deux points, ligne dans deux plans,* une ligne menée par deux points quelconques (non consécutifs en général) du système, et la ligne d'intersection de deux plans quelconques (non consécutifs en général) du système. Enfin, chaque ligne du système est rencontrée, en général, par un certain nombre d'autres lignes non consécutives du système. Je nommerai le point de rencontre d'une telle paire de lignes *point dans deux lignes,* et le plan qui contient une telle paire *plan par deux lignes.*

Supposons qu'un plan donné contient, *en général*, m points du système, qu'une ligne donnée rencontre, *en général*, r lignes du système, qu'un point donné est situé, *en général*, dans n plans du système. Le système est dit être de l'ordre m, du rang r, de la classe n. On voit tout de suite que l'ordre de la courbe est égal à l'ordre du système, ou à m; la classe de la courbe au rang du système, ou à r. Et de même, l'ordre de la surface au rang du système, ou à r; la classe de la surface à la classe du système, ou à n.

Cela posé, les singularités proprement dites (ouvrage cité, page 202) sont les deux suivantes, analogues aux points d'inflexion et de rebroussement dans les courbes planes:

1. Quand quatre points consécutifs sont situés dans le même plan, ou, autrement dit, quand trois lignes consécutives sont situées dans le même plan, ou quand deux plans consécutifs deviennent identiques; je dirai qu'il y a alors un *plan stationnaire*, et je représenterai par la lettre α le nombre de ces plans.

2. Quand quatre plans consécutifs se rencontrent dans le même point, ou, autrement dit, quand trois lignes se rencontrent au même point, ou quand deux points consécutifs deviennent identiques; je dirai qu'il y a alors un *point stationnaire*, et je représenterai par β le nombre de ces points.

Dans le premier cas, il y a un point d'inflexion sphérique dans la courbe, et une ligne d'inflexion dans la surface. On n'a pas donné de noms à ce qui arrive dans la courbe et la surface dans le second cas; et puisque la singularité est suffisamment distinguée déjà en la nommant *point stationnaire*, il n'est pas necessaire de suppléer à cette omission. On peut dire que ces deux cas sont les singularités simples d'un système. Il y a des singularités d'un ordre plus élevé dont on n'a pas besoin ici. Ensuite, il y a des singularités d'une autre espèce, en quelque sorte analogues aux points et tangentes doubles, mais qui ont rapport à un point ou plan indéterminé (hors du système); savoir:

3. Un plan donné peut contenir, *en général*, un nombre g de *lignes dans deux plans*.

4. Un point donné peut être situé, *en général*, dans un nombre h de *lignes par deux points*.

5. Un plan donné peut contenir, *en général*, un nombre x de *points dans deux lignes*.

6. Un point donné peut être situé, *en général*, dans un nombre y de *plans par deux lignes*.

Ces quatre cas sont les singularités impropres simples du système.

Il faut maintenant chercher les relations qui ont lieu entre les nombres m, r, n, α, β, g, h, x, y.

Citons d'abord les formules de M. Plücker pour les courbes planes (page 211), en changeant seulement les lettres, pour éviter la confusion. En représentant par μ l'ordre d'une courbe, par ν sa classe, par ξ le nombre de ses points doubles, η de ses points

de rebroussement, a de ses tangentes doubles, b de ses points d'inflexion, l'on aura les six équations

$$\nu = \mu \,.\, \overline{\mu - 1} - (2\xi + 3\eta),$$

$$b = 3\mu \,.\, \overline{\mu - 2} - (6\xi + 8\eta),$$

$$a = \tfrac{1}{2}\,\mu \,.\, \overline{\mu - 2}\,\overline{\mu^2 - 9} - (2\xi + 3\eta)(\mu \,.\, \overline{\mu - 1} - 6) + 2\xi \,.\, \overline{\xi - 1} + \tfrac{9}{2}\eta \,.\, \overline{\eta - 1} + 6\xi\eta\,;$$

$$\mu = \nu \,.\, \overline{\nu - 1} - (2a + 3b),$$

$$\eta = 3\nu \,.\, \overline{\nu - 2} - (6a + 8b),$$

$$\xi = \tfrac{1}{2}\,\nu \,.\, \overline{\nu - 2}\,\overline{\nu^2 - 9} - (2a + 3b)(\nu \,.\, \overline{\nu - 1} - 6) + 2a \,.\, \overline{a - 1} + \tfrac{9}{2} b \,.\, \overline{b - 1} + 6ab,$$

dont les trois dernières se dérivent des trois premières, et *vice versâ*.

Considérons ensuite un plan donné quelconque en conjonction avec le système. Ce plan coupe la surface suivant une courbe plane. Les points de cette courbe sont les points de rencontre du plan avec les lignes du système, les tangentes de cette courbe sont les lignes de rencontre du plan avec les plans du système. Il est clair que la courbe est de l'ordre r et de la classe n. Chaque fois que le plan contient un point en deux lignes, la courbe a un point double; aux points où le plan rencontre la courbe à double courbure, il y a dans la courbe un rebroussement (car, dans ce cas, il y a deux lignes du système qui coupent le plan en un même point, c'est-à-dire qu'il y a dans la courbe d'intersection un point stationnaire ou de rebroussement). Quand le plan contient une ligne en deux plans, il y a dans la courbe une tangente double; et enfin, pour chaque plan stationnaire du système, il y a dans la courbe une tangente stationnaire ou une inflexion. Donc nous avons, dans la courbe, r l'ordre, n la classe, x le nombre de points doubles, m de points de rebroussement, g de tangentes doubles, α de points d'inflexion, ce qui donne les six équations (équivalentes à trois):

$$n = r \,.\, \overline{r - 1} - (2x + 3m),$$

$$\alpha = 3r \,.\, \overline{r - 2} - (6x + 8m),$$

$$g = \tfrac{1}{2}\, r \,.\, \overline{r - 2}\,\overline{r^2 - 9} - (2x + 3m)(r \,.\, \overline{r - 1} - 6) + 2x \,.\, \overline{x - 1} + \tfrac{9}{2} m \,.\, \overline{m - 1} + 6xm;$$

$$r = n \,.\, \overline{n - 1} - (2g + 3\alpha),$$

$$m = 3n \,.\, \overline{n - 2} - (6g + 8\alpha),$$

$$x = \tfrac{1}{2}\, n \,.\, \overline{n - 2}\,\overline{n^2 - 9} - (2g + 3\alpha)(n \,.\, \overline{n - 1} - 6) + 2g \,.\, \overline{g - 1} + \tfrac{9}{2}\alpha \,.\, \overline{\alpha - 1} + 6g\alpha$$

(où l'on peut remarquer la symétrie à l'égard des combinaisons n, α, g, et r, m, x).

De même, en considérant un point donné quelconque en conjonction avec le système, ce point détermine, avec la courbe à double courbure, une surface conique, et, par des raisonnements pareils, on fait voir que, dans cette surface conique, l'ordre est m,

la classe r, le nombre des lignes doubles n, des lignes de rebroussement $\mathcal{E}$, des plans tangents doubles y, des lignes d'inflexion h, ce qui donne les équations (dont trois seulement sont indépendantes):

$$r = m \,.\, \overline{m-1} - (2h + 3\mathcal{E}),$$

$$n = 3m \,.\, \overline{m-2} - (6h + 8\mathcal{E}),$$

$$y = \tfrac{1}{2} m \,.\, \overline{m-2}\, \overline{m^2-9} - (2h + 3\mathcal{E})\,(m \,.\, \overline{m-1} - 6) + 2h \,.\, \overline{h-1} + \tfrac{3}{2}\mathcal{E} \,.\, \overline{\mathcal{E}-1} + 6h\mathcal{E}\,;$$

$$m = r \,.\, \overline{r-1} - (2y + 3n),$$

$$\mathcal{E} = 3r \,.\, \overline{r-2} - (6y + 8n),$$

$$h = \tfrac{1}{2} r \,.\, \overline{r-2}\, \overline{r^2-9} - (2y + 3n)\,(r \,.\, \overline{r-1} - 6) + 2y \,.\, \overline{y-1} + \tfrac{3}{2} n \,.\, \overline{n-1} + 6yn$$

(dans lesquelles on remarque la correspondance r, n, y; m, $\mathcal{E}$, h: et, en les comparant avec les autres six équations, la correspondance m, r, n, α, $\mathcal{E}$, g, h, x, y; n, r, m, $\mathcal{E}$, α, h, g, y, x).

En considérant une courbe à double courbure d'un ordre donné m, on peut attribuer à h, $\mathcal{E}$ des valeurs quelconques (entre certaines limites), et l'on a alors, pour déterminer les autres quantités, les équations suivantes:

$$r = m \,.\, \overline{m-1} - (2h + 3\mathcal{E}),$$

$$n = 3m \,.\, \overline{m-2} - (6h + 8\mathcal{E}),$$

$$y = \tfrac{1}{2} m \,.\, \overline{m-2}\, \overline{m^2-9} - (2h + 3\mathcal{E})\,(m \,.\, \overline{m-1} - 6) + 2h \,.\, \overline{h-1} + \tfrac{3}{2}\mathcal{E} \,.\, \overline{\mathcal{E}-1} + 6h\mathcal{E}\,;$$

$$n = r \,.\, \overline{r-1} - (2x + 3m),$$

$$\alpha = 3r \,.\, \overline{r-2} - (6x + 8m),$$

$$g = \tfrac{1}{2} r \,.\, \overline{r-2}\, \overline{r^2-9} - (2x + 3m)\,(r \,.\, \overline{r-1} - 6) + 2x \,.\, \overline{x-1} + \tfrac{3}{2} m \,.\, \overline{m-1} + 6xm.$$

Au cas d'une courbe plane les trois premières équations continuent d'être vraies, mais les trois autres n'ont pas de sens. Le cas le plus simple est celui d'une courbe du troisième ordre dans l'espace. On a, dans ce cas, $m=3$, $n=1$, $\mathcal{E}=0$; et de là le système

m,	n,	r,	α,	$\mathcal{E}$,	g,	h,	x,	y;
3,	3,	4,	0,	0,	1,	1,	0,	0;

c'est-à-dire l'osculatrice développable d'une courbe du troisième ordre est une surface du quatrième ordre, &c. On obtient aussi un système très-simple, en écrivant $m=4$, $h=2$, $\mathcal{E}=1$, ce qui donne

m,	n,	r,	α,	$\mathcal{E}$,	g,	h,	x,	y;
4,	4,	5,	1,	1,	2,	2,	2,	2.

Mais cela n'appartient pas, à ce que je crois, aux courbes les plus générales du quatrième ordre.

Le problème de classifier les courbes à double courbure au moyen des surfaces que l'on peut faire passer par ces courbes, ou de trouver la nature d'une courbe qui est l'intersection de deux surfaces données, paraît appartenir plutôt à la théorie des surfaces qu'à celle des courbes. Je n'ai rien de complet à offrir sur cela. Seulement je crois pouvoir dire que quand la courbe d'intersection de deux surfaces des ordres μ, ν est de l'ordre $\mu\nu$ (ce qui est le cas général), on a toujours

$$2h = \mu\nu \,.\, \overline{\mu - 1}\, \overline{\nu - 1},$$

de manière que la classe de la courbe est au plus $mn\,(m + n - 2)$. Mais j'espère revenir une autre fois sur cette question. Il est presque inutile de remarquer que pour un système qui est réciproque polaire d'un système donné, il faut seulement changer $m, r, n, \alpha, \beta, g, h, x, y$ en $n, r, m, \beta, \alpha, h, g, y, x$. Par exemple, les deux systèmes qui viennent d'être considérés ont des réciproques de la même forme.

31.

DEMONSTRATION D'UN THÉORÈME DE M. CHASLES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tome x. (1845), pp. 383—384.]

"Soient P, P' des points correspondants de deux figures homographiques; si la droite PP' passe toujours par un point fixe O, les points P sont situés sur une courbe du troisième degré, qui passe par ce même point."

Soient $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$ les coordonnées de P; $\frac{x'}{w'}$, $\frac{y'}{w'}$, $\frac{z'}{w'}$ celles de P'. En supposant que x', y', z', w' sont des fonctions linéaires (sans terme constant) de x, y, z, w, les deux figures seront homographiques.

Soient, de même, $\frac{\alpha}{\delta}$, $\frac{\beta}{\delta}$, $\frac{\gamma}{\delta}$ les coordonnées de O; $\frac{\lambda}{\varpi}$, $\frac{\mu}{\varpi}$, $\frac{\nu}{\varpi}$ les coordonnées d'un point quelconque T.

Puisque P, P', O sont sur la même droite, on peut faire passer un plan par les quatre points P, P', O, T. Cela donne tout de suite l'équation

$$\left\{\begin{matrix} \lambda, & \mu, & \nu, & \varpi \\ \alpha, & \beta, & \gamma, & \delta \\ x, & y, & z, & w \\ x', & y', & z', & w' \end{matrix}\right\} = 0$$

(en representant de cette manière le déterminant formé avec les quantités λ, μ, ν, &c.), équation qui doit être satisfaite quels que soient λ, μ, ν, ϖ, et qui équivaut ainsi aux deux conditions

$$\left\{\begin{matrix} \alpha, & \beta, & \gamma \\ x, & y, & z \\ x', & y', & z' \end{matrix}\right\} = 0, \qquad \left\{\begin{matrix} \alpha, & \beta, & \delta \\ x, & y, & w \\ x', & y', & w' \end{matrix}\right\} = 0.$$

Ces deux dernières équations sont du second degré par rapport aux quantités $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$. Le point P est donc situé à l'intersection de deux surfaces du second ordre. Mais ces surfaces ont en commun la droite représentée par les équations

$$\alpha y - \beta x = 0, \quad \alpha x' - \beta y' = 0.$$

Donc elles se coupent de plus suivant une courbe du troisième degré qui passe évidemment par le point O, parce qu'on satisfait aux équations en écrivant

$$x : \alpha = y : \beta = z : \gamma = w : \delta.$$

32.

ON SOME ANALYTICAL FORMULÆ, AND THEIR APPLICATION TO THE THEORY OF SPHERICAL COORDINATES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 22—33.]

SECTION 1.

THE formulæ in question are only very particular cases of some relating to the theory of the transformation of functions of the second order, which will be given in a following paper. But the case of three variables, here as elsewhere, admits of a symmetrical notation so much simpler than in any other case (on the principle that with three quantities a, b, c, functions of b, c; of c, a; and of a, b, may symmetrically be denoted by A, B, C, which is not possible with a greater number of variables) that it will be convenient to employ here a notation entirely different from that made use of in the general case, and by means of which the results will be exhibited in a more compact form. There is no difficulty in verifying by actual multiplication, any of the equations here obtained.

It will be expedient to employ the abbreviation of making a single letter stand for a system of quantities. Thus for instance, if $\delta = \theta, \phi, \psi$, this merely means that $\Phi(\delta)$ is to stand for $\Phi(\theta, \phi, \psi)$, $k\delta$ for $k\theta$, $k\phi$, $k\psi$, &c.

Suppose then

$$\omega = \xi, \eta, \zeta, \quad \text{.....(1)},$$

$$\omega' = \xi', \eta', \zeta',$$

$$\vdots$$

$$Q = A, B, C, F, G, H \quad \text{.....(2)},$$

$$W(\omega, \omega', Q) = A\xi\xi' + B\eta\eta' + C\zeta\zeta' + F(\eta\zeta' + \eta'\zeta) + G(\zeta\xi' + \zeta'\xi) + H(\xi\eta' + \xi'\eta) \ldots (3),$$

the function W satisfies a remarkable equation, as follows:

write
$$\mathfrak{A} = BC - F^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2.$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$
$$\mathfrak{Q} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$\overline{\omega\omega'} = \eta\zeta' - \eta'\zeta, \quad \zeta\xi' - \zeta'\xi, \quad \xi\eta' - \xi'\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

we have
$$W(\omega_1, \omega_2, Q)\, W(\omega_3, \omega_4, Q) - W(\omega_1, \omega_3, Q)\, W(\omega_2, \omega_4, Q) = W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ \ldots (7);$$
of which we may notice also the particular cases
$$W(\omega_1, \omega_2, Q)\, W(\omega_3, \omega_3, Q) - W(\omega_1, \omega_3, Q)\, W(\omega_2, \omega_3, Q) = W(\overline{\omega_1\omega_3}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ \ldots (8),$$
$$W(\omega_1, \omega_1, Q)\, W(\omega_2, \omega_2, Q) - \{W(\omega_1, \omega_2, Q)\}^2 = W(\overline{\omega_1\omega_2}, \overline{\omega_1\omega_2}, \mathfrak{Q}) \ \ldots (9).$$

To these we may join the following formulæ, for the transformation of the function W.

Suppose
$$\omega_1 = \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1, \quad \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''z_1, \quad \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1 \ \ldots\ldots\ldots (10),$$
$$\omega_2 = \mathrm{a}x_2 + \mathrm{a}'y_2 + \mathrm{a}''z_2, \quad \mathrm{b}x_2 + \mathrm{b}'y_2 + \mathrm{b}''z_2, \quad \mathrm{c}x_2 + \mathrm{c}'y_2 + \mathrm{c}''z_2,$$
$$\vdots$$

then, writing
$$g = \mathrm{a}, \quad \mathrm{b}, \quad \mathrm{c} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$
$$g' = \mathrm{a}', \quad \mathrm{b}', \quad \mathrm{c}',$$
$$g'' = \mathrm{a}'', \quad \mathrm{b}'', \quad \mathrm{c}'',$$
$$p_1 = x_1, \ y_1, \ z_1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$
$$p_2 = x_2, \ y_2, \ z_2,$$
$$\vdots$$
$$\Theta = W(g, g, Q), \ W(g', g', Q), \ W(g'', g'', Q), \ W(g', g'', Q), \ W(g'', g, Q), \ W(g, g', Q) \ (13),$$

we have
$$W(\omega_1, \omega_2, Q) = W(p_1, p_2, \Theta) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Similarly, writing
$$\Psi = W(\overline{g'g''}, \overline{g'g''}, \mathfrak{Q}), \quad W(\overline{g''g}, \overline{g''g}, \mathfrak{Q}), \quad W(\overline{gg'}, \overline{gg'}, \mathfrak{Q})$$
$$W(\overline{gg'}, \overline{g''g}, \mathfrak{Q}), \quad W(\overline{g'g''}, \overline{gg'}, Q), \quad W(\overline{g''g}, \overline{g'g''}, \mathfrak{Q}), \ \ldots\ldots (15),$$
we have
$$W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \mathfrak{Q}) = W(\overline{p_1p_4}, \overline{p_2p_3}, \Psi) \ \ldots\ldots\ldots\ldots\ldots\ldots (16),$$
in which equations $\mathfrak{Q}$ may obviously be changed into Q.

SECTION 2. *Geometrical Applications.*

Consider any three axes Ax, Ay, Az, and let λ, μ, ν be the cosines of the inclinations of these lines to each other.

Let Λ, M, N be the inclinations of the coordinate planes to each other; l, m, n the inclination of the axes to the coordinate planes. Suppose, besides,

$$\begin{aligned} \mathfrak{a} &= 1 - \lambda^2 \quad \ldots\ldots (17), \\ \mathfrak{b} &= 1 - \mu^2, \\ \mathfrak{c} &= 1 - \nu^2, \\ \mathfrak{f} &= \mu\nu - \lambda, \\ \mathfrak{g} &= \nu\lambda - \mu, \\ \mathfrak{h} &= \lambda\mu - \nu, \\ k &= 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu \ldots\ldots (18); \end{aligned}$$

we have the following systems of equations:

$$\begin{aligned} &\sqrt{(\mathfrak{bc})}\cos\Lambda = -\mathfrak{f}, \quad \sqrt{(\mathfrak{bc})}\sin\Lambda = \sqrt{(k)}, \quad \sqrt{(\mathfrak{a})}\sin l = \sqrt{(k)} \ldots\ldots (19). \\ &\sqrt{(\mathfrak{ca})}\cos\mathrm{M} = -\mathfrak{g}, \quad \sqrt{(\mathfrak{ca})}\sin\mathrm{M} = \sqrt{(k)}, \quad \sqrt{(\mathfrak{b})}\sin m = \sqrt{(k)} \\ &\sqrt{(\mathfrak{ab})}\cos\mathrm{N} = -\mathfrak{h}, \quad \sqrt{(\mathfrak{ab})}\sin\mathrm{N} = \sqrt{(k)}, \quad \sqrt{(\mathfrak{c})}\sin n = \sqrt{(k)}. \end{aligned}$$

$$\begin{aligned} \mathfrak{a} + \nu\mathfrak{h} + \mu\mathfrak{g} &= k, \quad \ldots\ldots (20). \\ \nu\mathfrak{a} + \mathfrak{h} + \lambda\mathfrak{g} &= 0, \\ \mu\mathfrak{a} + \lambda\mathfrak{h} + \mathfrak{g} &= 0. \end{aligned}$$

$$\begin{aligned} \mathfrak{h} + \nu\mathfrak{b} + \mu\mathfrak{f} &= 0, \quad \ldots\ldots (21). \\ \nu\mathfrak{h} + \mathfrak{b} + \lambda\mathfrak{f} &= k, \\ \mu\mathfrak{h} + \lambda\mathfrak{b} + \mathfrak{f} &= 0. \end{aligned}$$

$$\begin{aligned} \mathfrak{g} + \nu\mathfrak{f} + \mu\mathfrak{c} &= 0, \quad \ldots\ldots (22). \\ \nu\mathfrak{g} + \mathfrak{f} + \lambda\mathfrak{c} &= 0, \\ \mu\mathfrak{g} + \lambda\mathfrak{f} + \mathfrak{c} &= k. \end{aligned}$$

$$\begin{aligned} \mathfrak{bc} - \mathfrak{f}^2 &= ka \quad \ldots\ldots (23). \\ \mathfrak{ca} - \mathfrak{g}^2 &= kb, \\ \mathfrak{ab} - \mathfrak{h}^2 &= kc, \\ \mathfrak{gh} - \mathfrak{af} &= kf, \\ \mathfrak{hf} - \mathfrak{bg} &= kg, \\ \mathfrak{fg} - \mathfrak{ch} &= kh, \end{aligned}$$

$$\mathfrak{abc} - \mathfrak{af}^2 - \mathfrak{bg}^2 - \mathfrak{ch}^2 + 2\mathfrak{fgh} = k^2 \ldots\ldots (24).$$

Imagine now a line AO, and let α, β, γ be the cosines of its inclinations to the three axes. Suppose also, θ, ϕ, χ being its inclinations to the coordinate planes, we write

$$a = \frac{\sin\theta}{\sqrt{(\mathfrak{a})}}, \quad b = \frac{\sin\phi}{\sqrt{(\mathfrak{b})}}, \quad c = \frac{\sin\chi}{\sqrt{(\mathfrak{c})}} \quad \text{.....} \quad (25).$$

If we consider a point P on the line AO, at a distance unity from the origin, we see immediately, by considering the projections in the directions perpendicular to the coordinate planes, that the coordinates of this point are a, b, c. By projecting on the three axes and on the line AO, we then obtain the equations

$$\alpha = a + \nu b + \mu c \quad \text{.....} \quad (26),$$

$$\beta = \nu a + b + \lambda c,$$

$$\gamma = \mu a + \lambda b + c,$$

$$1 = \alpha a + \beta b + \gamma c \quad \text{.....} \quad (27),$$

from which we obtain

$$ka = \mathfrak{a}\alpha + \mathfrak{h}\beta + \mathfrak{g}\gamma \quad \text{.....} \quad (28),$$

$$kb = \mathfrak{h}\alpha + \mathfrak{b}\beta + \mathfrak{f}\gamma,$$

$$kc = \mathfrak{g}\alpha + \mathfrak{f}\beta + \mathfrak{c}\gamma,$$

$$1 = \alpha a + \beta b + \gamma c \quad \text{.....} \quad (29),$$

and hence

$$1 = a^2 + b^2 + c^2 + 2\lambda bc + 2\mu ac + 2\nu ab \quad \text{.....} \quad (30),$$

$$k = \mathfrak{a}\alpha^2 + \mathfrak{b}\beta^2 + \mathfrak{c}\gamma^2 + 2\mathfrak{f}\beta\gamma + 2\mathfrak{g}\alpha\gamma + 2\mathfrak{h}\alpha\beta \quad \text{.....} \quad (31).$$

Hence writing

$$a,\ b,\ c = t \quad \text{.....} \quad (32),$$

$$\alpha,\ \beta,\ \gamma = \tau \quad \text{.....} \quad (33),$$

$$1,\ 1,\ 1,\ \lambda,\ \mu,\ \nu = \mathrm{q} \quad \text{.....} \quad (34),$$

$$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{f},\ \mathfrak{g},\ \mathfrak{h} = \mathfrak{q} \quad \text{.....} \quad (35),$$

we have the equations

$$1 = W(t,\ t,\ \mathfrak{q}) \quad \text{.....} \quad (36),$$

$$k = W(\tau,\ \tau,\ \mathfrak{q}) \quad \text{.....} \quad (37).$$

Let AO' be any other line, and δ its inclination to AO: α', β', γ', a', b', c', the quantities corresponding to α, β, γ, a, b, c, and similarly t', τ' to t, τ. We have of course

$$1 = W(t',\ t',\ \mathrm{q}) \quad \text{.....} \quad (38),$$

$$k = W(\tau',\ \tau',\ \mathfrak{q}) \quad \text{.....} \quad (39).$$

We have besides, by projecting on the line AO', the equation

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad \text{.....} \quad (40),$$

or the analogous one

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad \text{.....} \quad (41).$$

From either of which we deduce

$$\cos\delta = aa' + bb' + cc' + \lambda(bc' + b'c) + \mu(ca' + c'a) + \nu(ab' + a'b) \quad \text{.....} \quad (42),$$

$$k\cos\delta = \mathfrak{a}\alpha\alpha' + \mathfrak{b}\beta\beta' + \mathfrak{c}\gamma\gamma' + \mathfrak{f}(\beta\gamma' + \beta'\gamma) + \mathfrak{g}(\gamma\alpha' + \gamma'\alpha') + \mathfrak{h}(\alpha\beta + \alpha'\beta) \quad \text{.....} \quad (43);$$

which may otherwise be written

$$\cos\delta = W(t, t', \mathrm{q}) \quad \text{.....} \quad (44),$$

$$k\cos\delta = W(\tau, \tau', \mathfrak{q}) \quad \text{.....} \quad (45);$$

or again, observing the equations which connect the quantities t, τ,

$$\cos\delta = \frac{W(t, t', \mathrm{q})}{\sqrt{\{W(t, t, \mathrm{q})\, W(t', t', \mathrm{q})\}}} \quad \text{.....} \quad (46),$$

$$\cos\delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\tau, \tau, \mathfrak{q})\, W(\tau', \tau', \mathfrak{q})\}}} \quad \text{.....} \quad (47),$$

forms which, though more complicated, have certain advantages; for instance, we derive immediately from them the new equations

$$\sin\delta = \frac{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathrm{q})\}}}{\sqrt{\{W(t, t, \mathrm{q})\, W(t', t', \mathrm{q})\}}} \quad \text{.....} \quad (48),$$

$$\sin\delta = \frac{\sqrt{\{kW(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathfrak{q})\}}}{\sqrt{\{W(\tau, \tau, \mathfrak{q})\, W(\tau', \tau', \mathfrak{q})\}}} \quad \text{.....} \quad (49).$$

written more simply thus

$$\sin\delta = W(\underline{\overline{tt'}}, \underline{\overline{tt'}}, \mathrm{q}) \quad \text{.....} \quad (50),$$

$$\sqrt{k}\sin\delta = \sqrt{\{W(\underline{\overline{\tau\tau'}}, \underline{\overline{\tau\tau'}}, \mathrm{q})\}} \quad \text{.....} \quad (51);$$

to these we may join

$$\cot\delta = \frac{W(t, t', \mathrm{q})}{\sqrt{\{W(\underline{\overline{tt'}}, \underline{\overline{tt'}}, \mathrm{q})\}}} \quad \text{.....} \quad (52),$$

$$\sqrt{k}\cot\delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\underline{\overline{\tau\tau'}}, \underline{\overline{\tau\tau'}}, \mathrm{q})\}}} \quad \text{.....} \quad (53).$$

SECTION 3. *On Spherical Coordinates.*

Consider the points X, Y, Z, on the surface of a sphere, as the intersections of the three axes of the preceding section, with a sphere having its centre in the origin. It is evident that λ, μ, ν are the cosines of the sides of the spherical triangle XYZ,

Λ, M, N are its sides, l, m, n are the perpendiculars from the angles upon the opposite sides. Let P be the point where the line AO intersects the sphere: the position of the point P may be determined by means of the ratios $\xi : \eta : \zeta$, supposing ξ, η, ζ denote quantities proportional to the α, β, γ of the preceding section, i.e.

$$\xi : \eta : \zeta = \cos PX : \cos PY : \cos PZ \quad \text{.....} (54);$$

or again, by means of the ratios $x : y : z$, supposing x, y, z denote quantities proportional to the a, b, c of the preceding section, i.e.

$$x : y : z = \frac{\sin Px}{\sin X} : \frac{\sin Py}{\sin Y} : \frac{\sin Pz}{\sin Z} \quad \text{.....} (55),$$

(Px, Py, Pz are the perpendiculars from P on the sides of the spherical triangle XYZ).

These last equations may be otherwise written,

$$\frac{x \sin X}{y \sin Y} = \frac{\sin PZY}{\sin PZX} \quad \text{.....} (56).$$

$$\frac{y \sin Y}{z \sin Z} = \frac{\sin PXZ}{\sin PXY},$$

$$\frac{z \sin Z}{x \sin X} = \frac{\sin PYX}{\sin PYZ}.$$

The ratios $\xi : \eta : \zeta$, or $x : y : z$, are termed the spherical coordinate ratios of the point P. The two together may be termed conjoint systems: the first may be termed the cosine system, and the second the sine system. The coordinates of the two systems are evidently connected by

$$\xi : \eta : \zeta = x + \nu y + \mu z : \nu x + y + \lambda z : \mu x + \lambda y + z \quad \text{.....} (57),$$

or

$$x : y : z = \mathfrak{a}\xi + \mathfrak{h}\eta + \mathfrak{g}\zeta : \mathfrak{h}\xi + \mathfrak{b}\eta + \mathfrak{f}\zeta : \mathfrak{g}\xi + \mathfrak{f}\eta + \mathfrak{c}\zeta \quad \text{.....} (58).$$

The systems may conveniently be represented by the single letters

$$\omega = \xi,\ \eta,\ \zeta \quad \text{.....} (59),$$

$$p = x,\ y,\ z \quad \text{.....} (60).$$

Fundamental formula of spherical coordinates; distance of two points.

Let P, P' be the points, δ their distance, ω, p the conjoint coordinate systems of the first point, ω', p' of the second; we have obviously

$$\cot \delta = \frac{W(p,\ p',\ \mathfrak{q})}{\sqrt{\{W(p,\ p,\ \mathfrak{q})\ W(p',\ p',\ \mathfrak{q})\}}} \quad \text{.....} (61),$$

$$\sin \delta = \frac{\sqrt{\{W(\overline{pp'},\ \overline{pp'},\ \mathfrak{q})\}}}{\sqrt{\{W(p,\ p,\ \mathfrak{q})\ W(p',\ p',\ \mathfrak{q})\}}},$$

$$\cot \delta = \frac{W(p,\ p',\ \mathfrak{q})}{\sqrt{\{W(\overline{pp'},\ \overline{pp'}\ \mathfrak{q})\}}};$$

or
$$\cos\delta = \frac{W(\omega,\ \omega',\ \mathfrak{q})}{\sqrt{\{W(\omega,\ \omega,\ \mathfrak{q})\ W(\omega',\ \omega',\ \mathfrak{q})\}}} \quad\text{.....(62).}$$

$$\frac{1}{\sqrt{k}}\sin\delta = \frac{\sqrt{\{W(\overline{\omega\omega'},\ \overline{\omega\omega'},\ \mathrm{q})\}}}{\sqrt{\{W(\omega,\ \omega,\ \mathfrak{q})\ W(\omega',\ \omega',\ \mathfrak{q})\}}},$$

$$\sqrt{k}\,\{\cot\delta = \frac{W(\omega,\ \omega',\ \mathfrak{q})}{\sqrt{\{W(\overline{\omega\omega'},\ \overline{\omega\omega'},\ \mathrm{q})\}}}.$$

Equation of a great Circle.

Let the conjoint coordinate systems of the pole be

$$e = a,\ b,\ c \quad\text{.....(63),}$$

$$\epsilon = \alpha,\ \beta,\ \gamma \quad\text{.....(64),}$$

then, expressing that the distance of any point P in the locus from the pole is equal to 90°, we have immediately the equations

$$W(p,\ e,\ \mathrm{q}) = 0 \quad\text{..... (65),}$$

$$W(\omega,\ \epsilon,\ \mathfrak{q}) = 0 \quad\text{..... (66),}$$

which may otherwise be written in the forms

$$a\xi + b\eta + c\zeta = 0 \quad\text{..... (67),}$$

$$\alpha x + \beta y + \gamma\zeta = 0 \quad\text{..... (68),}$$

or the equation of a great circle is linear in either coordinate system. Conversely, any linear equation belongs to a great circle.

Suppose the equation given in the form

$$A\xi + B\eta + C\zeta = 0 \quad\text{.....(69);}$$

or by an equation between cosine coordinate ratios:—the sine system for the pole is given by

$$e = A,\ B,\ C \quad\text{..... (70),}$$

and the cosine system by

$$\epsilon = A + \nu B + \mu C,\quad \nu A + B + \lambda C,\quad \mu A + \lambda B + C \quad\text{.....(71).}$$

Suppose the circle given by an equation between sine coordinates, or in the form

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 \quad\text{.....(72),}$$

the cosine system of coordinates for the pole is given by

$$\epsilon = \mathrm{A},\ \mathrm{B},\ \mathrm{C} \quad\text{.....(73),}$$

and the sine system by

$$e = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C},\quad \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C},\quad \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \quad\text{.....(74).}$$

It is hardly necessary to observe, that if

$$A\xi + B\eta + C\zeta = 0 \quad\text{.....(75)},$$

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 \quad\text{.....(76)},$$

represent the same great circle,

$$\mathrm{A} : \mathrm{B} : \mathrm{C} = A + \nu B + \mu C : \nu A + B + \lambda C : \mu A + \lambda B + C \quad\text{.....(77)},$$

$$A : B : C = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C} : \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C} : \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \quad\text{.....(78)}.$$

Inclination of two great Circles.

Let the equations of these be

$$\begin{cases} A\xi + B\eta + C\zeta = 0 & \text{.....(79)}, \\ \text{or } \mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 & \text{.....(80)}, \end{cases}$$

$$\begin{cases} A'\xi + B'\eta + C'\zeta = 0 & \text{.....(81)}, \\ \text{or } \mathrm{A}'x + \mathrm{B}'y + \mathrm{C}'z = 0 & \text{.....(82)}, \end{cases}$$

and let e, ϵ, have the same values as above, and e', ϵ', corresponding ones. To obtain the inclination of the two circles, we have only, in the formulæ given above for the distance of two points, to change p, p', ω, ω', into e, e', ϵ, ϵ'.

The distance of a point from a given circle may be found with equal facility; for this is evidently the complement of the distance of the point from the pole of the circle. In like manner we may find the condition that two great circles intersect at right angles, &c.

There are evidently a whole class of formulæ, not by any means peculiar to the present system of coordinates, such as

$$Ax + By + Cz - s(A'x + B'y + C'z) \quad\text{..... (83)},$$

for the equation of a great circle subjected to pass through the points of intersection of

$$Ax + By + Cz = 0, \quad A'x + B'y + C'z = 0.$$

Again,

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0 \quad\text{.....(84)},$$

for the equation of the great circle which passes through the points given by the sine systems $a : b : c$ and $a' : b' : c'$, &c., and which are obtained so easily that it is not worth while writing down any more of them.

Transformation of Coordinates.

Let X_1, Y_1, Z_1, be the new points of reference, and suppose X_1 is given by the conjoint systems $e = a$, b, c, $\epsilon = \alpha$, β, γ; and similarly Y_1, Z_1, by the analogous systems e', ϵ'; e'', ϵ''.

Suppose, as before, P is given by one of the systems ω, p; and let ω_1, p_1 be the new systems which determine the position of P with reference to X_1, Y_1, Z_1.

In the first place, λ_1, μ_1, ν_1, are given by the formulæ

$$\lambda_1 = \frac{W(e', e'', \mathrm{q})}{\sqrt{\{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})\}}} = \frac{W(\epsilon', \epsilon'', \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \quad \ldots\ldots (85),$$

$$\mu_1 = \frac{W(e'', e, \mathrm{q})}{\sqrt{\{W(e'', e'', \mathrm{q})\, W(e, e, \mathrm{q})\}}} = \frac{W(\epsilon'', \epsilon, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\nu_1 = \frac{W(e, e', \mathrm{q})}{\sqrt{\{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})\}}} = \frac{W(\epsilon, \epsilon', \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

The system ω_1 is evidently given immediately by

$$\xi_1 : \eta_1 : \zeta_1 = \frac{W(e, p, \mathrm{q})}{\sqrt{\{W(e, e, \mathrm{q})\}}} : \frac{W(e', p, \mathrm{q})}{\sqrt{\{W(e', e', \mathrm{q})\}}} : \frac{W(e'', p, \mathrm{q})}{\sqrt{\{W(e'', e'', \mathrm{q})\}}} \quad \ldots\ldots\ldots (86)$$

$$= \frac{W(\epsilon, \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}} : \frac{W(\epsilon', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}} : \frac{W(\epsilon'', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \quad \ldots\ldots\ldots (87),$$

and from these we may obtain the system p_1, by means of the formulæ

$$x_1 : y_1 : z_1 = \mathfrak{a}_1\xi_1 + \mathfrak{h}_1\eta_1 + \mathfrak{g}_1\zeta_1 : \mathfrak{h}_1\xi_1 + \mathfrak{b}_1\eta_1 + \mathfrak{f}_1\zeta_1 : \mathfrak{g}_1\xi_1 + \mathfrak{f}_1\eta_1 + \mathfrak{c}_1\zeta_1 \ldots\ldots (88).$$

This requires some further development however. We must in the first place form the system $\mathfrak{a}_1$, $\mathfrak{b}_1$, $\mathfrak{c}_1$, $\mathfrak{f}_1$, $\mathfrak{g}_1$, $\mathfrak{h}_1$: this is done immediately from the formulæ of Sect. 2, and we have

$$\mathfrak{a}_1 = \frac{W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q})}{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})} = \frac{kW(\overline{\epsilon'\epsilon''}, \overline{\epsilon'\epsilon''}, \mathrm{q})}{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (89),$$

$$\mathfrak{b}_1 = \frac{W(\overline{e''e}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', \mathrm{q})\, W(e'', e, \mathrm{q})} = \frac{kW(\overline{\epsilon''\epsilon}, \overline{\epsilon''\epsilon}, \mathrm{q})}{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})},$$

$$\mathfrak{c}_1 = \frac{W(\overline{ee'}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})} = \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})},$$

$$\mathfrak{f}_1 = \frac{W(\overline{e''e}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\sqrt{\{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})\}}} = \frac{kW(\overline{\epsilon''\epsilon}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}},$$

$$\mathfrak{g}_1 = \frac{W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})}{W(e', e', \mathrm{q})\sqrt{\{W(e'', e'', \mathrm{q})\, W(e, e, \mathrm{q})\}}} = \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon'\epsilon''}, \mathrm{q})}{W(\epsilon', \epsilon', \mathfrak{q})\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\mathfrak{h}_1 = \frac{W(\overline{e'e''}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', \mathrm{q})\sqrt{\{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})\}}} = \frac{kW(\overline{\epsilon'\epsilon''}, \overline{\epsilon''\epsilon}, \mathrm{q})}{W(\epsilon'', \epsilon'', \mathfrak{q})\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, \mathfrak{q})\}} \times \quad \text{.....} \quad (90),$$

$$\{W(e, p, \mathfrak{q})\, W(\overline{\underline{e'e''}}, \overline{\underline{e'e''}}, \mathfrak{q}) + W(e', p, \mathfrak{q})\, W(\overline{\underline{e'e''}}, \overline{\underline{e''e}}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\, W(\overline{\underline{e'e''}}, \overline{\underline{ee'}}, \mathfrak{q})\}$$

$$: \sqrt{\{W(e', e', \mathfrak{q})\}} \times$$

$$\{W(e, p, \mathfrak{q})\, W(\overline{\underline{e''e}}, \overline{\underline{e'e''}}, \mathfrak{q}) + W(e', p, \mathfrak{q})\, W(\overline{\underline{e''e}}, \overline{\underline{e''e}}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\, W(\overline{\underline{e''e}}, \overline{\underline{ee'}}, \mathfrak{q})\}$$

$$: \sqrt{\{W(e'', e'', \mathfrak{q})\}} \times$$

$$\{W(e, p, \mathfrak{q})\, W(\overline{\underline{ee'}}, \overline{\underline{e'e''}}, \mathfrak{q}) + W(e', p, \mathfrak{q})\, W(\overline{\underline{ee'}}, \overline{\underline{e''e}}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\, W(\overline{\underline{ee'}}, \overline{\underline{ee'}}, \mathfrak{q})\};$$

these may be reduced to the very simple form

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, \mathfrak{q})\}}\, W(\overline{\underline{e'e''}}, \omega, \mathfrak{q}) \quad \text{.....} \quad (91),$$

$$: \sqrt{\{W(e', e', \mathfrak{q})\}}\, W(\overline{\underline{e''e}}, \omega, \mathfrak{q}),$$

$$: \sqrt{\{W(e'', e'', \mathfrak{q})\}}\, W(\overline{\underline{ee'}}, \omega, \mathfrak{q}).$$

and in like manner we obtain

$$x_1 : y_1 : z_1 = \sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}\, W(\overline{\underline{\epsilon'\epsilon}}, p, \mathfrak{q}) \quad \text{.....} \quad (92),$$

$$: \sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}\, W(\overline{\underline{\epsilon''\epsilon}}, p, \mathfrak{q}),$$

$$: \sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}\, W(\overline{\underline{\epsilon\epsilon'}}, p, \mathfrak{q}).$$

It will be as well to indicate the steps of this reduction. Consider the quantity in { } in the first line of the equation which gives the ratios $x_1 : y_1 : z_1$; and suppose for a moment $\overline{\underline{e'e''}} = l, m, n$, &c.: then, selecting the portion of the expression which is multiplied by $\mathfrak{a}$, this is

$$= \mathfrak{a}l\,\{l\,(a\xi + b\eta + c\zeta) + l'\,(a'\xi + b'\eta + c'\zeta) + l''\,(a''\xi + b''\eta + c''\zeta)\},$$

or, since

$$la + l'a' + l''a'' = \overline{\underline{ee'e''}}, \quad lb + l'b' + l''b'' = 0, \quad lc + l'c' + l''c'' = 0,$$

this reduces itself to $\overline{\underline{ee'e''}}.\,\mathfrak{a}l\xi$, which is a term of

$$\overline{\underline{ee'e''}}\, W(\overline{\underline{e'e''}}, \omega, \mathfrak{q});$$

and by comparing the remaining terms in the same manner, it would be seen that the whole reduces itself to

$$\overline{\underline{ee'e''}}\, W(\overline{\underline{e'e''}}, \omega, \mathfrak{q});$$

whence the formulæ in question.

The formulæ (86), (87), (91), (92), completely resolve the problem of the transformation of coordinates; they determine respectively p_1 from p or ω, ω_1 from p or ω.

To complete the present part of the subject we may add the following formulæ.

Suppose

$$x_1 : y_1 : z_1 = \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1 \quad \text{.....} \quad (93),$$

$$: \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''y_1,$$

$$: \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1,$$

which we see from the preceding formulæ is the form of the relation between the systems p_1 and p. And suppose, as before, λ_1, μ_1, ν_1 are the cosines of the distances of the new points of reference X_1, Y_1, Z_1.

We can immediately determine the relations that must exist between these coefficients, in order that they may actually denote such a transformation. For this purpose write

$$\begin{aligned} \mathrm{a}\,,\ \mathrm{b}\,,\ \mathrm{c} &= j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (94),\\ \mathrm{a}',\ \mathrm{b}',\ \mathrm{c}' &= j',\\ \mathrm{a}'',\ \mathrm{b}'',\ \mathrm{c}'' &= j''. \end{aligned}$$

Then the distance between the point P and any other point P' is given by the formula

$$\cos\delta = \frac{W(p,\ p',\ \mathrm{q})}{\sqrt{\{W(p,\ p,\ \mathrm{q})\ W(p',\ p',\ \mathrm{q})\}}} = \frac{W(p_1,\ p_1',\ \Theta)}{\sqrt{\{W(p_1,\ p_1,\ \Theta)\ W(p_1,\ p_1',\ \Theta)\}}} \ldots (95),$$

where

$$\Theta = W(j, j, \mathrm{q}),\quad W(j', j', \mathrm{q}),\quad W(j'', j'', \mathrm{q}),\quad W(j', j'', \mathrm{q}),\quad W(j'', j, \mathrm{q}),\quad W(j, j', \mathrm{q}) \ldots (96).$$

But we must evidently have

$$\cos\delta = \frac{W(p_1,\ p_1',\ \mathrm{q}_1)}{\sqrt{\{W(p_1,\ p_1,\ \mathrm{q}_1)\ W(p_1',\ p_1',\ \mathrm{q}_1)\}}} \ldots\ldots\ldots\ldots\ldots\ldots (97),$$

or the quantities Θ must be proportional to the quantities q, i.e.

$$\begin{aligned} &W(j,\ j,\ \mathrm{q}) : W(j',\ j',\ \mathrm{q}) : W(j'',\ j'',\ \mathrm{q}) : W(j',\ j'',\ \mathrm{q}) : W(j'',\ j,\ \mathrm{q}) : W(j'',\ j,\ \mathrm{q})\\ &= 1 \quad : \quad 1 \quad : \quad 1 \quad : \quad \lambda_1 \quad : \quad \mu_1 \quad : \quad \nu_1 \ldots (98). \end{aligned}$$

And in precisely the same manner, if instead of x, y, z, x_1, y_1, z_1, in the above formulæ, we had had ξ, η, ζ : ξ_1, η_1, ζ_1, the result would have been

$$\begin{aligned} &W(j,\ j,\ \mathfrak{q}) : W(j',\ j',\ \mathfrak{q}) : W(j'',\ j'',\ \mathfrak{q}) : W(j',\ j'',\ \mathfrak{q}) : W(j'',\ j,\ \mathfrak{q}) : W(j'',\ j,\ \mathfrak{q})\\ &= \mathfrak{a} \quad : \quad \mathfrak{b} \quad : \quad \mathfrak{c} \quad : \quad \mathfrak{f} \quad : \quad \mathfrak{g} \quad : \quad \mathfrak{h} \ldots (99). \end{aligned}$$

It is hardly necessary to remark, that throughout the preceding formulæ an expression, such as $W(p,\ p',\ \mathrm{q})$, is proportional to either of the quantities

$$x\xi' + y\eta' + z\zeta' \text{ or } x'\xi + y'\eta + z'\zeta,$$

and may be changed into one of these multiplied by an arbitrary constant, which may be always made to disappear by a corresponding change in another quantity of the same form: thus, for instance,

$$\frac{W(p,\ p',\ \mathrm{q})}{W(p,\ p,\ \mathrm{q})} = \frac{x'\xi + y'\eta + z'\zeta}{x\xi + y\eta + z\zeta} \ldots\ldots\ldots\ldots\ldots\ldots (100);$$

but these forms being unsymmetrical, it is better in general not to introduce them.

All the preceding expressions simplify exceedingly, reducing themselves in fact to the ordinary formulæ for the transformation of rectangular coordinates in Geometry of three dimensions, for the case where the triangle XYZ has its sides and angles right angles. As this presents no difficulty, I shall not enter upon it at present.

33.

ON THE REDUCTION OF $\frac{du}{\sqrt{U}}$, WHEN U IS A FUNCTION OF THE FOURTH ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 70—73.]

It is well known that the transformation of this differential expression into a similar one, in which the function in the denominator contains only even powers of the corresponding variable, is the first step in the process of reducing $\int \frac{du}{\sqrt{U}}$ to elliptic integrals. And, accordingly, the different modes of effecting this have been examined, more or less, by most of those who have written on the subject. The simplest supposition, that adopted by Legendre, and likewise discussed in some detail by Gudermann, is that u is a fraction, the numerator and denominator of which are linear functions of the new variable. But the theory of this transformation admits of being developed further than it has yet been done, as regards the equation which determines the modulus of the elliptic function. This may be effected most easily as follows.

Suppose

$$U = a + 4bu + 6cu^2 + 4du^3 + eu^4,$$
$$P = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Also let

$$P' = a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4$$

be what P becomes after writing

$$x = \lambda x' + \mu y',$$
$$y = \lambda_{,} x' + \mu_{,} y':$$

and let

$$U' = a' + 4b'u' + 6c'u'^2 + 4d'u'^3 + e'u'^4.$$

Suppose, moreover,

$$\begin{cases} k = \lambda\mu, & -\lambda_{,}\mu_{,}, \\ I = ae & -4bd + 3c^2, \\ I' = a'e' & -4b'd' + 3c'^2, \\ J = ace & -ad^2 - be^2 - c^3 + 2bcd, \\ J' = a'c'e' & -a'd'^2 - b'e'^2 - c'^3 + 2b'c'd'; \end{cases}$$

we have evidently

$$xdy - ydx = k(x'dy' - y'dx'),$$

or

$$\frac{xdy - ydx}{\sqrt{P}} = k\frac{x'dy' - y'dx'}{\sqrt{P'}}.$$

Hence writing

$$u = \frac{y}{x}, \qquad u' = \frac{y'}{x'};$$

and therefore

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = \frac{du}{U^{\frac{1}{2}}}, \qquad \frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}} = \frac{du'}{U'^{\frac{1}{2}}},$$

we obtain

$$\frac{du}{\sqrt{U}} = k\frac{du'}{\sqrt{U'}},$$

the equation between u and u' being

$$u = \frac{\lambda + \mu u'}{\lambda_{,} + \mu_{,}u'}.$$

Next, to determine the relations between the coefficients of U and U'. Since P, P' are obtained from each other by linear transformations (*Math. Journal*, vol. IV. p. 208), [13, p. 94], we have between the coefficients of these functions and of the transforming equations, the relations

$$I' = k^4 I,$$
$$J' = k^6 J;$$

whence also

$$\frac{J'^2}{I'^3} = \frac{J^2}{I^3}.$$

Suppose now

$$U' = a'(1 + pu'^2)(1 + qu'^2),$$

or

$$b' = 0, \quad d' = 0, \quad 6c' = a'(p + q), \quad e' = a'pq;$$

whence also

$$I' = \tfrac{1}{12}a'^2(p^2 + q^2 + 14pq),$$
$$J' = \tfrac{1}{216}a'^3(p + q)(34pq - p^2 - q^2);$$

$$p^2 + q^2 + 14pq = 12\,\frac{k^4}{a'^2}\,I,$$

$$(p+q)(34pq - p^2 - q^2) = 216\,\frac{k^6}{a'^3}\,J\,;$$

therefore

$$\frac{(p+q)^2(34pq - p^2 - q^2)^2}{(p^2+q^2+14pq)^3} = \frac{27J^2}{I^3},$$

whence also

$$\frac{108pq\,(p-q)^4}{(p^2+q^2+14pq)^3} = 1 - \frac{27J^2}{I^3},$$

which determines the relation between p and q. Also

$$\frac{k}{\sqrt{a'}} = \left(\frac{p^2+q^2+14pq}{12I}\right)^{\frac{1}{4}},$$

so that

$$\frac{du}{\sqrt{U}} = \left(\frac{p^2+q^2+14pq}{12I}\right)^{\frac{1}{4}} \frac{du'}{\sqrt{\{(1+pu'^2)(1+qu'^2)\}}}.$$

If in particular $p = -1$, writing also $-q$ for q,

$$\frac{du}{\sqrt{U}} = \left(\frac{q^2+14q+1}{12I}\right)^{\frac{1}{4}} \frac{du'}{\sqrt{\{(1-u'^2)(1-qu'^2)\}}},$$

where

$$\frac{108q\,(1-q)^4}{(q^2+14q+1)^3} = 1 - \frac{27J^2}{I^3}.$$

Suppose, for shortness,

$$M = \tfrac{27}{4}\cdot\frac{1}{\left(1 - \dfrac{27J^2}{I^3}\right)}, \quad \text{or } \tfrac{1}{108}\left(1 - \frac{27J^2}{I^3}\right) = \frac{1}{16M},$$

then
$$(q^2+14q+1)^3 - 16Mq\,(q-1)^4 = 0,$$

i.e.
$$\left(q + \frac{1}{q} + 14\right)^3 - 16M\left(q^{\frac{1}{2}} - \frac{1}{q^{\frac{1}{2}}}\right)^4 = 0.$$

Let
$$q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \frac{4}{(\theta-1)^{\frac{1}{2}}},$$

then
$$\theta^3 - M(\theta - 1) = 0,$$

which determines θ. And then

$$q = \frac{7 + \theta + 4\,(3+\theta)^{\frac{1}{2}}}{\theta - 1}.$$

Suppose $q=\alpha$ is one of the values of q; the equation becomes

$$\frac{(q^2+14q+1)^3}{q(q-1)^4}=\frac{(\alpha^2+14\alpha+1)^3}{\alpha(\alpha-1)^4}, \quad =\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}, \quad \text{if } \alpha=\beta^4.$$

Now if $$q=\left(\frac{1-\beta}{1+\beta}\right)^4,$$

then $$(q^2+14q+1)=\frac{16(\beta^8+14\beta^4+1)}{(1+\beta)^8}, \quad q-1=-\frac{8\beta(1+\beta^2)}{(1+\beta)^4},$$

which values satisfy the above equation: hence also, identically,

$$(q^2+14q+1)^3-q(q-1)^4\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}$$

$$=(q-\beta^4)\left(q-\frac{1}{\beta^4}\right)\left\{q-\left(\frac{1-\beta}{1+\beta}\right)^4\right\}\left\{q-\left(\frac{1+\beta}{1-\beta}\right)^4\right\}\left\{q-\left(\frac{1-\beta i}{1+\beta i}\right)^4\right\}\left\{q-\left(\frac{1+\beta i}{1-\beta i}\right)^4\right\};$$

or the values of q take the form

$$\beta^4, \quad \frac{1}{\beta^4}, \quad \left(\frac{1-\beta}{1+\beta}\right)^4, \quad \left(\frac{1+\beta}{1-\beta}\right)^4, \quad \left(\frac{1-\beta i}{1+\beta i}\right)^4, \quad \left(\frac{1+\beta i}{1-\beta i}\right)^4.$$

(Comp. *Abel. Œuv.* tom. I. p. 310 [Ed. 2, p. 459].)

The equation $$\theta^3-M\theta+M=0$$

has its three roots real if $27-4M$ is negative, and only a single real root if $27-4M$ is positive. Writing the equation under the form

$$(\theta+3)^3-9(\theta+3)^2+(27-M)(\theta+3)-(27-4M)=0,$$

we see that in the former case θ has two values greater than -3, and a single value less than -3. Writing the equation under the form

$$(\theta-1)^3+3(\theta-1)^2+(3-M)(\theta-1)+1=0, \ (3-M \text{ is negative})$$

the positive roots are both greater than 1. Hence, in this case, q has four positive values and two imaginary ones. In the second case θ has a single real value, which is greater than -3 and less than 1. Hence q has two negative values and four imaginary ones. In the former case, I^3-27J^2 is positive, and the function U has either four imaginary factors or four real ones. In the second case, I^3-27J^2 is negative, or the function U has two real and two imaginary factors.

34.

NOTE ON THE MAXIMA AND MINIMA OF FUNCTIONS OF THREE VARIABLES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 74, 75.]

If A, B, C, F, G, H, be any real quantities, such that

$$BC+CA+AB-F^2-G^2-H^2,$$

and

$$(A+B+C)(ABC-AF^2-BG^2-CH^2+2FGH)$$

are positive; the six quantities

$$BC-F^2,\ CA-G^2,\ AB-H^2,\ AK,\ BK,\ CK,$$

(where $K=ABC-AF^2-BG^2-CH^2+2FGH$) are all of them positive. It is unnecessary to point out the connection of this property with the theory of maxima and minima.

To demonstrate this, writing as usual

$$\begin{aligned} BC-F^2&=A', & GH-AF&=F', \\ CA-G^2&=B', & HF-BG&=G', \\ AB-H^2&=C', & FG-CH&=H', \end{aligned}$$

and K as above: then if A'', B'', C'', F'', G'', H'', K' be formed from A', B', C', F', G', H', as these and K are from A, B, C, F, G, H, we have the well-known formulæ

$$\begin{aligned} A''&=KA, & F''&=KF, & K'&=K^2. \\ B''&=KB, & G''&=KG, \\ C''&=KC, & H''&=KH, \end{aligned}$$

It is required to show that if $A'+B'+C'$ and $A''+B''+C''$ are positive, A', B', C', A'', B'', C'' are so likewise.

Consider the cubic equation

$$(A'-k)(B'-k)(C'-k)-(A'-k)F'^2-(B'-k)G'^2-(C'-k)H'^2+2F'G'H'=0,$$

the roots of which are all real. By the formulæ just given this may be written

$$k^3-k^2(A'+B'+C')+k(A''+B''+C'')-K^2=0;$$

and the terms of this equation are alternately positive and negative; i.e. the roots are all positive. Hence the roots of the limiting equation

$$(B'-k)(C'-k)-F'^2=0$$

are positive, i.e. $B'+C'$ and $B'C'$ are positive: but from the second condition B', C' are of the same sign: consequently they are of the same sign with $B'+C'$, or positive. Also $A''=B'C'-F'^2$ is positive. Similarly, considering the other limiting equations, A', B', C', A'', B'', C'' are all of them positive.

In connection with the above I may notice the following theorem. The roots of the equation

$$(A-ka)(B-kb)(C-ck)-(A-ka)(F-kf)^2-(B-kb)(G-kg)^2-(C-kc)(H-kh)^2$$
$$+2(F-kf)(G-kg)(H-kh)=0,$$

are all of them real, if either of the functions

$$Ax^2+By^2+Cz^2+2Fyz+2Gxz+2Hxy,$$
$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy,$$

preserve constantly the same sign. The above form parts of a general system of properties of functions of the second order.

35.

ON HOMOGENEOUS FUNCTIONS OF THE THIRD ORDER WITH THREE VARIABLES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 97—104.]

THE following problem corresponds to the geometrical question of determining the polar reciprocal of a plane curve of the third order: the solution of it is also important, with reference to the linear transformations of homogeneous functions of three variables of the third order; reasons for which it has appeared to me worth while to obtain the completely developed result.

Let

$$3U = ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y + 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz \quad \text{.....(1).}$$

It is required to eliminate x, y, z, λ from the equations

$$U = 0 \quad \text{.....(2),}$$

$$\left.\begin{aligned} \frac{dU}{dx} + \lambda\xi &= 0, \\ \frac{dU}{dy} + \lambda\eta &= 0, \\ \frac{dU}{dz} + \lambda\zeta &= 0, \end{aligned}\right\} \quad \text{.....(3).}$$

From the equations (2), (3), we obtain immediately

$$\Theta = \xi x + \eta y + \zeta z = 0 \quad \text{.....(4);}$$

and thence

$$\Theta x = 0, \quad \Theta y = 0, \quad \Theta z = 0 \quad \text{.....(5);}$$

so that a single equation more, such as

$$\Phi = 0 \text{.....} (6),$$

where Φ is homogeneous and of the second order in x, y, z, would, in conjunction with the equations (3) and (5), enable us to eliminate linearly the seven quantities x^2, y^2, z^2, yz, zx, xy, λ. Such an equation may be thus obtained.

Let L, M, N, R, S, T, be the second differential coefficients of U, each of them divided by two. The equations (3) may be written

$$\begin{aligned} Lx + Ty + Sz + \lambda\xi &= 0, \text{.....} (7). \\ Tx + My + Rz + \lambda\eta &= 0, \\ Sx + Ry + Nz + \lambda\zeta &= 0. \end{aligned}$$

And joining to these the equation (4),

$$\xi x + \eta y + \zeta z \qquad = 0,$$

we have, by the elimination of x, y, z, in so far as they explicitly appear, and λ, an equation $\Phi = 0$ of the required form. Hence we may write

$$\Phi = -\begin{vmatrix} L, & T, & S, & \xi \\ T, & M, & R, & \eta \\ S, & R, & N, & \zeta \\ \xi, & \eta, & \zeta, & \end{vmatrix} \text{.....} (8);$$

or substituting for L, M, N, R, S, T, and expanding,

$$\Phi = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy \text{.....} (9),$$

where the values of A, B, C, F, G, H are .....(10).

[I omit these values (10), and the values in the subsequent equations (13), (14), (20): the values (10) and (13) serve for the calculation of (14), FU, the expression for which with the letters $(a, b, c, f, g, h, i, j, k, l)$ in place of $(a, b, c, i, j, k, i_1, j_1, k_1, l)$ is reproduced in my "Third Memoir on Quantics," *Phil. Trans.* vol. CXLVI. (1856) pp. 627—647: the values (20) serve for finding that of (21), $K(U)$, but the developed expression of this has not been calculated.]

Performing the elimination indicated, the result may be represented by

$$\mathsf{F}U = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k, & \xi \\ k, & b, & i_1, & i, & l, & k_1, & \eta \\ j_1, & i, & c, & i_1, & j, & l, & \zeta \\ 2\xi, & . & . & . & \zeta, & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A & B & C & F & G & H & . \end{vmatrix} = 0 \text{.....} (11).$$

Partially expanding,

$$\mathsf{F}U = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h} \text{.....} (12).$$

The values of the coefficients a, b, c, f, g, h may be useful on other occasions: they are as follows. .....(13).

Substituting these values the result after all reductions becomes

$$0 = F(U) = \text{.....}(14).$$

It would be desirable, in conjunction with the above, to obtain the equation

$$K(U) = 0,$$

which results from the elimination of x, y, z from the equations

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0 \text{ .....}(15),$$

(i.e. the condition of a curve of the third order having a multiple point), but to effect this would be exceedingly laborious. The following is the process of the elimination as given by Dr Hesse, *Crelle*, t. XXVIII. (and which applies also to the case of any three equations of the second order). Forming the function ∇U, of the third order in x, y, z, by means of the equation

$$\nabla U = \begin{vmatrix} L, & T, & S \\ T, & M, & R \\ S, & R, & N \end{vmatrix} \text{ .....}(16),$$

(L, M, N, R, S, T, the same as before).

Then, in consequence of the equations (15), we have not only

$$\nabla U = 0 \text{.....}(17),$$

which is very easily proved to be the case, but also

$$\frac{d}{dx}\nabla U = 0, \quad \frac{d}{dy}\nabla U = 0, \quad \frac{d}{dz}\nabla U = 0 \text{.....}(18),$$

as will be shown in a subsequent paper "On Points of Inflection." [I think never written.]

And from the six equations (15), (18), the six quantities x^2, y^2, z^2, yz, zx, xy, may be linearly eliminated: we have

$$\nabla U = Ax^3 + By^3 + Cz^3 + 3Iy^2z + 3Jz^2x + 3Kx^2y + 3I_1yz^2 + 3J_1zx^2 + 3K_1xy^2 + 6\Lambda xyz \text{...}(19),$$

where the values of the coefficients A, B, ... Λ are .....(20),

and the result of the elimination is

$$K(U) = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k \\ k, & b, & i_1, & i, & l, & k_1 \\ j_1, & i, & c, & i_1, & j, & l \\ A, & K_1, & J, & L, & J_1, & K \\ K, & B, & I_1, & I, & L, & K_1 \\ J_1, & I, & C, & I_1, & J, & L \end{vmatrix} = 0 \text{ .....}(21).$$

{$K(U)$ is consequently, as is well known, a function of the twelfth order in a, b, c, i, j, k, i_1, j_1, k_1, l}.

The equation $$\nabla U = 0,$$

combined with that of the curve, determine, as Dr Hesse has demonstrated in the paper quoted, the points of inflection of the curve. It may be inferred from this, that if U reduce itself to the form

$$U = (\alpha x^2 + \beta y^2 + \gamma z^2 + 2\iota yz + 2\kappa xz + 2\lambda xy)\, P = VP \quad \text{.....(22)},$$

P a linear function of x, y, z: then ∇U takes the form

$$\nabla U = P\,(\rho V + \sigma P^2) \quad \text{.....(23)},$$

where ρ is of the second order in the coefficients of P, and also in the coefficients α, β, γ, ι, κ, λ: and σ is equal to the determinant

$$\begin{vmatrix} \alpha, & \lambda, & \kappa \\ \lambda, & \beta, & \iota \\ \kappa, & \iota, & \gamma \end{vmatrix} \quad \text{.....(24)},$$

multiplied by a numerical factor. If U is of the form

$$U = PQR \quad \text{.....(25)},$$

then $$\nabla U = \rho PQR = \rho U \quad \text{.....(26)},$$

and this equation is consequently the condition of the function U being resolvable into linear factors. The equation in question resolves itself into

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{I}{i} = \frac{J}{j} = \frac{K}{k} = \frac{I_1}{i_1} = \frac{J_1}{j_1} = \frac{K_1}{k_1} = \frac{\Lambda}{l} \quad \text{.....(27)};$$

a system which must contain three independent equations only. It would be interesting to verify this *à posteriori*.

36.

ON THE GEOMETRICAL REPRESENTATION OF THE MOTION OF A SOLID BODY.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 164—167.]

LET P, Q, R, ... be consecutive generating lines of a skew surface, and on these take points p', p; q', q; r', r ..... such that pq', qr' ..... are the shortest distances between P and Q, Q and R, &c. Then for the generating line P, the ratio of the inclination of the lines P, Q to the distance pq' is said to be "the torsion," the angle $q'pq$ is said to be the deviation, and the ratio of the inclination of the planes Qpq' and Qqr' to the inclination of P and Q is said to be the "skew curvature." And similarly for any other generating line; so that the torsion and deviation depend on the position of the consecutive line, and the skew curvature on the position of the two consecutive lines. The curve pqr ..... is said to be the minimum distance curve [or curve of striction]. {When the skew surface degenerates into a developable surface, the torsion is infinite, the deviation a right angle, the skew curvature proportional to the curvature of the principal section, i.e. it is the distance of a point from the edge of regression, multiplied into the reciprocal of the radius of curvature, a product which is evidently constant along a generating line. Also the curve of minimum distance becomes the edge of regression.} A skew surface, considered independently of its position in space, is determined when for each generating line we know the torsion, deviation, and skew curvature. For, assuming arbitrarily the line P and the point p, also the plane in which pq' lies, the position of Q is completely determined from the given torsion and deviation; and then Q being known, the position of R is completely determined from the skew curvature for P, and the torsion and deviation for Q; and similarly the consecutive generating lines are to be determined.

Two skew surfaces are said to be "deformations" of each other, when for corresponding generating lines the torsion is always the same. Thus a surface will be deformed if considering the elements between the successive generating lines P, Q ..... as rigid, these

elements be made to revolve round the successive generating lines P, Q ..... and to slide along them. {They are "transformations", when not only the torsions but also the deviations are equal at corresponding generating lines: thus, if the sliding of the elements along P, Q ..... be omitted, the new surface will be, not a deformation, but a transformation of the other.} No two skew surfaces can be made to roll and slide one upon the other, so that their successive generating lines coincide, unless one of them is a deformation of the other: and when this is the case, the rolling and sliding motions are *completely determined*. In fact the angular velocity of the generating line is the angular velocity round this line, into the difference of the skew curvatures of the two surfaces; the velocity of translation of the generating line in its own direction is to the angular velocity of the generating line, as the difference of the deviations is to the torsion. {This includes also the case in which one surface is a transformation of the other, where the motion is evidently a rolling one.} A skew surface moving in this manner upon another of which it is the deformation, may be said to "glide" upon it. We may now state the kinematical theorem:

"Any motion whatever of a solid body in space may be represented as the 'gliding' motion of one skew surface upon another fixed in space, and of which it is the deformation."

a theorem which is to be considered as the generalization of the well-known one—

"Any motion of a solid body round a fixed point may be represented as the rolling motion of a conical surface upon a second conical surface fixed in space."

and of the supplementary theorem—

"The angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of curvatures of the two cones at any point in the same line, to the reciprocal of the distance of the point from the vertex."

The analytical demonstration of this last theorem is rather interesting: it depends on the following formulæ. Forming two determinants, the first with the angular velocities round three axes fixed in space, and the first and second derived coefficients of these velocities with respect to the time; the other in the same way with the angular velocities round axes fixed in the body; the difference of these determinants is equal to the fourth power of the angular velocity into the square of the angular velocity of the instantaneous axis.

To show this, let p, q, r be the angular velocities round the axes fixed in the body; u, v, w those round axes fixed in space; ω the angular velocity round the instantaneous axis; ∇, Ω the two determinants: the theorem comes to

$$\nabla - \Omega = M,$$

where $$M = \omega^2 (p'^2 + q'^2 + r'^2 - \omega'^2), \text{ or } \omega^2 (u'^2 + v'^2 + w'^2 - \omega'^2).$$

Here

$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, \\ v &= \alpha' p + \beta' q + \gamma' r, \\ w &= \alpha'' p + \beta'' q + \gamma'' r; \end{aligned}$$

whence
$$u' = \alpha\, p' + \beta\, q' + \gamma\, r',$$
$$v' = \alpha'\, p' + \beta'\, q' + \gamma'\, r',$$
$$w' = \alpha'' p' + \beta'' q' + \gamma'' r',$$

(the remaining terms vanishing as is well known); and therefore

$$vw' - v'w = \alpha\ (qr' - q'r) + \beta\ (rp' - r'p) + \gamma\ (pq' - p'q),$$
$$wu' - w'u = \alpha'\ (qr' - q'r) + \beta'\ (rp' - r'p) + \gamma'\ (pq' - p'q),$$
$$uv' - u'v = \alpha''\,(qr' - q'r) + \beta''\,(rp' - r'p) + \gamma''\,(pq' - p'q).$$

Hence

$$vw'' - v''w = \alpha\ (qr'' - q''r) + \beta\ (rp'' - r''p) + \gamma\ (pq'' - p''q) + u'\omega^2 - u\omega\omega',$$
$$wu'' - w''u = \alpha'\ (qr'' - q''r) + \beta'\ (rp'' - r''p) + \gamma'\ (pq'' - p''q) + v'\omega^2 - v\omega\omega',$$
$$wv'' - u''v = \alpha''\,(qr'' - q''r) + \beta''\,(rp'' - r''p) + \gamma''\,(pq'' - p''q) + w'\omega^2 - w\omega\omega',$$

and multiplying these by u', v', w', and adding, the required equation is immediately obtained.

In fact, if r be the distance of a point in the instantaneous axis from the vertex, and ρ, σ the radii of curvature of the two cones at that point, then

$$\frac{r}{\rho} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\Omega, \quad \frac{r}{\sigma} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\nabla,$$

as may be shown without difficulty: and the angular velocity of the instantaneous axis is given by the equation $\varpi = \frac{M^{\frac{1}{2}}}{\omega^2}$; hence the relation between the two angular velocities is

$$\omega : \varpi = \frac{1}{\rho} - \frac{1}{\sigma} : \frac{1}{r}.$$

37.

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 167—173 and 264—274.]

THE difficulty of completing elegantly the solution of this problem, in the case where no forces act upon the body, arises from the complexity and want of symmetry of the ordinary formulæ for determining the position of one set of rectangular axes with respect to another set; in consequence of which it has hitherto been considered necessary to make a particular supposition relative to the position of the fixed axes in space, viz. that one of them shall be perpendicular to the "invariable plane" of the rotating body. But some formulæ for the above purpose, given also by Euler, are entirely free from these objections. Imagine two sets of axes Ax, Ay, Az, $Ax_{,}$, $Ay_{,}$, $Az_{,}$. The former set can be made to coincide with the second set, by a rotation θ round a certain axis AR, inclined to Ax, Ay, Az at angles f, g, h. (As usual f, g, h are the angles RAx, RAy, RAz considered as positive, and the rotation is in the same direction as a rotation round Az from x towards y.) This axis may be termed the resultant axis, and the angle θ the resultant rotation. The formulæ of Euler express the coefficients of the transformation in terms of the resultant rotation and of the position of the resultant axis, i.e. in terms of θ and of the angles f, g, h, whose cosines are connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

This idea was improved upon by M. Rodrigues (*Liouv.* tom. v. p. 404), who introduced the quantities

$$\tan \tfrac{1}{2}\theta \cos f, \quad \tan \tfrac{1}{2}\theta \cos g, \quad \tan \tfrac{1}{2}\theta \cos h,$$

(quantities which will be represented by λ, μ, ν) by means of which he expressed the

coefficients as fractions, the numerators of which are very simple rational functions of the second order of λ, μ, ν, and which have the common denominator $(1+\lambda^2+\mu^2+\nu^2)$. These quantities may conveniently be termed the "coordinates of the resultant rotation," and the denominator or the square of the secant of the semi-angle of resultant rotation will be the "modulus" of the rotation. The elegance of these results led me to apply them to the mechanical question, and I gave in the *Journal* (vol. III. p. 224), [6], the differential equations of motion obtained in terms of λ, μ, ν: which I integrated as in the common theory, by supposing one of the fixed axes to be perpendicular to the invariable plane. Though my attention was again called to the subject, by the connexion of some of these formulæ with Sir William Hamilton's theory of quaternions, no other way of performing the integration occurred to me. The grand discovery however of Jacobi, of the possibility of reducing to quadratures the two final differential equations of any mechanical problem, when the remaining integrals are known, induced me to resume the problem, and at least attempt to bring it so far as to obtain a differential equation of the first order between two variables only, the multiplier of which could be obtained theoretically by Jacobi's discovery. The choice of two new variables to which the equations of the problem led me, enabled me to effect this with the greatest simplicity; and the differential equation which I finally obtained, turned out to be integrable *per se*, so that the laborious process of finding the multiplier became unnecessary. The new variables Ω, v have the following geometrical interpretations, $\Omega = k \tan \frac{1}{2}\theta \cos I$, where k is the principal moment, θ as before the angle of resultant rotation, and I is the inclination of the resultant axis to the perpendicular upon the invariable plane, and $v = k^2 \cos^2 \frac{1}{2} J$; where, if we imagine a line AQ having the same position relatively to the axes in fixed space that the perpendicular upon the invariable plane has to the principal axes of the rotating body, then J is the inclination of this line to the above perpendicular. To the choice of these variables I was led by the analysis only. It will be seen that p, q, r are functions of v only, while λ, μ, ν contain besides the variable Ω. In obtaining these relations a singular equation $\Omega^2 = \kappa v - k^2$ occurs (equation 13), which may also be written $1 + \tan^2 \frac{1}{2}\theta \cos^2 I = \sec^2 \frac{1}{2}\theta \cos^2 \frac{1}{2} J$, in which form the interpretation of the quantities I, J has just been given. The equation (17), it may be remarked, is self-evident: it expresses that the inclination of the resultant axis to the normal of the invariable plane, is equal to the inclination of the same axis to the line AQ. Now the resultant axis having the same inclination to the axes fixed in space as it has to the principal axes, and the line AQ the same inclinations to these fixed axes that the normal to the invariable plane has to the principal axes, the truth of the proposition becomes manifest. The correspondence in form between the systems (10) and (14) is also worth remarking. The final results at which I arrive are, that the time and the arc whose tangent is $\Omega \div k$, are each of them expressible as the integrals of certain algebraical functions of v. The notation throughout is the same as that made use of in the paper already quoted.

The equations of rotatory motion are

$$dt = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\lambda}{\Lambda} = \frac{d\mu}{M} = \frac{d\nu}{N} \quad \text{.....} (1),$$

where

$$\left.\begin{aligned} P &= \frac{1}{A}\left[(B-C)\,qr + \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu+\nu)\frac{dV}{d\mu} + (\lambda\nu-\mu)\frac{dV}{d\nu}\right\}\right], \\ Q &= \frac{1}{B}\left[(C-A)\,rp + \tfrac{1}{2}\left\{(\mu\lambda-\nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu+\lambda)\frac{dV}{d\nu}\right\}\right], \\ R &= \frac{1}{C}\left[(A-B)\,pq + \tfrac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda} + (\mu\nu-\lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\}\right], \end{aligned}\right\} \quad \ldots\ldots (2).$$

$$\left.\begin{aligned} \Lambda &= \tfrac{1}{2}\{(1+\lambda^2)\,p + (\lambda\mu-\nu)\,q + (\lambda\nu+\mu)\,r\}, \\ \mathrm{M} &= \tfrac{1}{2}\{(\mu\lambda+\nu)\,p + (1+\mu^2)\,q + (\mu\nu-\lambda)\,r\}, \\ \mathrm{N} &= \tfrac{1}{2}\{(\nu\lambda-\mu)\,p + (\mu\nu+\lambda)\,q + (1+\nu^2)\,r\}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

[whence also $\lambda\Lambda + \mu\mathrm{M} + \nu\mathrm{N} = \frac{1}{2}\kappa\,(\lambda p + \mu q + \nu r)$ ..... (3 *bis*)].

In the case where the forces vanish, the first three equations become simply

$$\left.\begin{aligned} P &= \frac{1}{A}\,(B-C)\,qr, \\ Q &= \frac{1}{B}\,(C-A)\,rp, \\ R &= \frac{1}{C}\,(A-B)\,pq, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and here the usual four integrals of the system are

$$Ap^2 + Bq^2 + Cr^2 = h \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\left.\begin{aligned} Ap\,(1+\lambda^2-\mu^2-\nu^2) + 2Bq\,(\lambda\mu-\nu) + 2Cr\,(\nu\lambda+\mu) &= \mathrm{a}\,(1+\lambda^2+\mu^2+\nu^2), \\ 2Ap\,(\lambda\mu+\nu) + Bq\,(1+\mu^2-\nu^2-\lambda^2) + 2Cr\,(\mu\nu-\lambda) &= \mathrm{b}\,(1+\lambda^2+\mu^2+\nu^2), \\ 2Ap\,(\nu\lambda-\mu) + 2Bq\,(\mu\nu+\lambda) + Cr\,(1+\nu^2-\lambda^2-\mu^2) &= \mathrm{c}\,(1+\lambda^2+\mu^2+\nu^2), \end{aligned}\right\} \quad \ldots\ldots (6),$$

or as they may also be written,

$$\left.\begin{aligned} \mathrm{a}\,(1+\lambda^2-\mu^2-\nu^2) + 2\mathrm{b}\,(\lambda\mu+\nu) + 2\mathrm{c}\,(\nu\lambda-\mu) &= Ap\,(1+\lambda^2+\mu^2+\nu^2), \\ 2\mathrm{a}\,(\lambda\mu-\nu) + \mathrm{b}\,(1+\mu^2-\nu^2-\lambda^2) + 2\mathrm{c}\,(\mu\nu+\lambda) &= Bq\,(1+\lambda^2+\mu^2+\nu^2), \\ 2\mathrm{a}\,(\nu\lambda+\mu) + 2\mathrm{b}\,(\mu\nu-\lambda) + \mathrm{c}\,(1+\nu^2-\lambda^2-\mu^2) &= Cr\,(1+\lambda^2+\mu^2+\nu^2), \end{aligned}\right\} \quad \ldots\ldots (6\ bis);$$

to which we may add,

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

where

$$k^2 = \mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Introducing the quantities κ, Ω, (the former of which has been already made use of) given by the equations

$$\left.\begin{aligned} \kappa &= 1 + \lambda^2 + \mu^2 + \nu^2, \\ \Omega &= \lambda Ap + \mu Bq + \nu Cr, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

The equations (6) may be written under the form

$$\left.\begin{aligned} 2\lambda\Omega + 2\mu Cr - 2\nu Bq &= \kappa(Ap + \mathrm{a}) - 2Ap, \\ -2\lambda Cr + 2\mu\Omega + 2\nu Ap &= \kappa(Bq + \mathrm{b}) - 2Bq, \\ 2\lambda Bq - 2\mu Ap + 2\nu\Omega &= \kappa(Cr + \mathrm{c}) - 2Cr, \end{aligned}\right\} \quad \ldots\ldots(10),$$

whence also, multiplying by Ap, Bq, Cr, and adding,

$$2\Omega^2 = \kappa\{k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\} - 2k^2 \ldots\ldots(11),$$

or writing

$$k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c}) = 2v \ldots\ldots(12),$$

this becomes

$$\Omega^2 = \kappa v - k^2 \ldots\ldots(13);$$

an equation, the geometrical interpretation of which has already been given.

From the equations (10) we deduce the inverse system

$$\left.\begin{aligned} \mathrm{a}\Omega - \mathrm{b}Cr + \mathrm{c}Bq &= 2\lambda v - \Omega Ap, \\ \mathrm{a}Cr + \mathrm{b}\Omega - \mathrm{c}Ap &= 2\mu v - \Omega Bq, \\ -\mathrm{a}Bq + \mathrm{b}Ap + \mathrm{c}\Omega &= 2\nu v - \Omega Cr, \end{aligned}\right\} \quad \ldots\ldots(14),$$

which are easily verified by multiplying by Ω, Cr, $-Bq$; or by $-Cr$, Ω, Ap; or Bq, $-Ap$, Ω: adding and reducing, by which means the equations (10) are re-obtained. Hence also if for shortness

$$\left.\begin{aligned} \Phi &= \mathrm{a}p + \mathrm{b}q + \mathrm{c}r, \\ \nabla &= \mathrm{a}qr(B - C) + \mathrm{b}rp(C - A) + \mathrm{c}pq(A - B), \end{aligned}\right\} \quad \ldots\ldots(15),$$

we have, multiplying by p, q, r, and adding,

$$\Omega\Phi - \nabla = 2v(\lambda p + \mu q + \nu r) - \Omega h \ldots\ldots(16).$$

To these may be added the equation

$$\Omega = \mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu \ldots\ldots(17),$$

which follows immediately from either of the systems (10) or (14).

We may also put the equations (10) under this other form,

$$\left.\begin{aligned} 2\lambda\Omega - 2\mu\mathrm{c} + 2\nu\mathrm{b} &= \kappa(Ap + \mathrm{a}) - 2\mathrm{a}, \\ 2\lambda\mathrm{c} + 2\mu\Omega - 2\nu\mathrm{a} &= \kappa(Bq + \mathrm{b}) - 2\mathrm{b}, \\ -2\lambda\mathrm{b} + 2\mu\mathrm{a} + 2\nu\Omega &= \kappa(Cr + \mathrm{c}) - 2\mathrm{c}, \end{aligned}\right\} \quad \ldots\ldots(10\ bis).$$

It may be remarked now, that p, q, r are functions of v; since we have to determine these quantities, the three equations

$$\left.\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= h, \\ A^2p^2 + B^2q^2 + C^2r^2 &= k^2, \\ Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c} &= 2v - k^2, \end{aligned}\right\} \quad \ldots\ldots(18).$$

Also λ, μ, ν are given by the equations (14) as functions of p, q, r, Ω, i.e. of v, Ω; so that every thing is prepared for the investigation of the differential equation between v, Ω. To find this we have immediately

$$dv = \tfrac{1}{2}(A\mathrm{a}dp + B\mathrm{b}dq + C\mathrm{c}dr) = \tfrac{1}{2}\nabla dt \quad \text{.....(19)},$$

from the equations (4) and (15). ∇ is of course to be considered as a given function of v. Again,

$$\Omega d\Omega = \tfrac{1}{2}(\kappa dv + v d\kappa) \quad \text{.....(20)},$$

where

$$d\kappa = 2(\lambda d\lambda + \mu d\mu + \nu dv) \quad \text{.....(21)};$$

or from the equations (1), (3), [and (3 *bis*)],

$$d\kappa = \kappa(\lambda p + \mu q + \nu r)\, dt \quad \text{.....(22)}.$$

Hence, from (16),

$$2v d\kappa = \kappa\{\Omega(h + \Phi) - \nabla\}\, dt \quad \text{.....(23)};$$

or

$$2(v d\kappa + \kappa dv) = \kappa\Omega(h + \Phi)\, dt \quad \text{.....(24)},$$

whence

$$d\Omega = \tfrac{1}{4}\kappa(h + \Phi)\, dt,$$

$$= \tfrac{1}{4}\frac{\Omega^2 + k^2}{v}(h + \Phi)\, dt \quad \text{.....(25)},$$

and therefore, from (19),

$$\frac{2d\Omega}{\Omega^2 + k^2} = \frac{h + \Phi}{v\nabla}\, dv \quad \text{.....(26)},$$

the required differential equation, in which Φ, ∇ are given functions of v, i.e. they are functions of p, q, r by the equations (15), and these quantities are functions of v by (18). The variables in (26) are therefore separated, and we have the integral equation

$$2\tan^{-1}\frac{\Omega}{k} = \delta + k\int\frac{(h + \Phi)\, dv}{v\nabla} \quad \text{.....(27)},$$

where δ is the constant of integration. The equation (19) gives also

$$t - \epsilon = 2\int\frac{dv}{\nabla} \quad \text{.....(28)};$$

and thus the solution of the problem is completely effected. The integrals may be taken from any particular value v_0 of v. The variable Ω may be exhibited as the integral of an *explicit* algebraical function, by recurring to the variable ϕ of the paper quoted.

Thus if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h,$$

$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$

$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2v_0 - k^2;$$

then the values of p, q, r are respectively

$$\sqrt{\left\{p_0^2 - \frac{1}{A}(C-B)\phi\right\}}, \quad \sqrt{\left\{q_0^2 - \frac{1}{A}(A-C)\phi\right\}}, \quad \sqrt{\left\{r_0^2 - \frac{1}{C}(B-A)\phi\right\}},$$

where

$$dt = \tfrac{1}{2}\frac{d\phi}{pqr} = \frac{2d\nu}{\nabla}, \quad \text{or } \frac{d\nu}{\nabla} = \tfrac{1}{4}\frac{d\phi}{pqr};$$

and then

$$4\tan^{-1}\frac{\Omega}{k} = 2\delta + k\int_0 \frac{(h + \mathrm{a}p + \mathrm{b}q + \mathrm{c}r)\,d\phi}{(k^2 + A\mathrm{p}\mathrm{a} + B\mathrm{q}\mathrm{b} + C\mathrm{r}\mathrm{c})pqr},$$

in which form it is exactly analogous to the equation there obtained, p. 230, [6, p. 34]

$$4\tan^{-1}\nu_0 = \int\frac{(h + kr)\,d\phi}{(k + Cr)\,pqr}.$$

On the Variation of the Constants, when the body is acted upon by Forces.

The dynamical equations of a problem being expressed in the form

$$\frac{d}{dt}\frac{dT}{d\lambda'} - \frac{dT}{d\lambda} = \frac{dV}{d\lambda},$$

$$\frac{d}{dt}\frac{dT}{d\mu'} - \frac{dT}{d\mu} = \frac{dV}{d\mu},$$

$$\frac{d}{dt}\frac{dT}{d\nu'} - \frac{dT}{d\nu} = \frac{dV}{d\nu},$$

suppose the equations obtained from these by neglecting the function V, are integrated; each of the six integrals may be expressed in the form

$$a = f(\lambda, \mu, \nu, \lambda', \mu', \nu', t),$$

where a denotes any one of the arbitrary constants. Assume

$$\frac{dT}{d\lambda'} = u, \quad \frac{dT}{d\mu'} = v, \quad \frac{dT}{d\nu} = w;$$

then λ', μ', ν' may be expressed in terms of λ, μ, ν, u, v, w, and the integrals may be reduced to the form

$$a = F(\lambda, \mu, \nu, u, v, w, t).$$

These equations may be considered as the integrals of the proposed system, taking into account the terms involving V, provided the constants [say a, b, c, d, e, f] be sup-

posed to become variable. We have, in this case, by Lagrange's theory of the variation of the arbitrary constants, the formulæ

$$\frac{da}{dt} = (a, b)\frac{dV}{db} + (a, c)\frac{dV}{dc} + (a, d)\frac{dV}{dd} + (a, e)\frac{dV}{de} + (a, f)\frac{dV}{df};$$

where

$$(a, b) = \left(\frac{da}{du}\frac{db}{d\lambda} - \frac{da}{d\lambda}\frac{db}{du}\right) + \left(\frac{da}{dv}\frac{db}{d\mu} - \frac{da}{d\mu}\frac{db}{dv}\right) + \left(\frac{da}{dw}\frac{db}{d\nu} - \frac{da}{d\nu}\frac{db}{dw}\right),$$

and in which V is supposed to be expressed as a function of a, b, c, d, e, f, t.

Thus the solution of the problem requires the calculation of thirty coefficients (a, b), or rather of fifteen only, since evidently $(a, b) = -(b, a)$. It is known that these coefficients are functions of a, b, c, d, e, f, without t; so that, in calculating them, any assumed arbitrary value, e.g. $t = 0$, may be given to the time.

In practice, it often happens that one of the arbitrary constants, e.g. a, may be expressed in the form

$$a = F(\lambda, \mu, \nu, u, v, w, t, b, c, d, e, f),$$

where b, c, d, e, f are given functions of $\lambda, \mu, \nu, u, v, w, t$. In this case, it is easily seen that we may write

$$(a, b) = \{(a, b)\} + (c, b)\frac{da}{dc} + (d, b)\frac{da}{dd} + (e, b)\frac{da}{de} + (f, b)\frac{da}{df},$$

where, in the calculation of $\{(a, b)\}$, the differentiations upon a are performed without taking into account the variability of b, c.....

In the particular problem in question, the following are the values of the new variables u, v, w (*Math. Journal*, memoir already quoted, [6]),

$$u = \frac{2}{\kappa}(\quad Ap - \nu Bq + \mu Cr), \quad \text{.....} \quad (29),$$

$$v = \frac{2}{\kappa}(\quad \nu Ap + \quad Bq - \lambda Cr),$$

$$w = \frac{2}{\kappa}(-\mu Ap + \lambda Bq + \quad Cr);$$

equations which may also be expressed in the form

$$2Ap = (1 + \lambda^2)u + (\lambda\mu + \nu)v + (\nu\lambda - \mu)w \quad \text{.....}(30),$$

$$2Bq = (\lambda\mu - \nu)u + (1 + \mu^2)v + (\mu\nu + \lambda)w,$$

$$2Cr = (\nu\lambda + \mu)u + (\mu\nu - \lambda)v + (1 + \nu^2)w,$$

or putting for shortness

$$\lambda u + \mu v + \nu w = \varpi \quad \text{.....}(31),$$

these become

$$2Ap = \lambda\varpi + u + \nu v - \mu w, \quad \text{(32)}.$$
$$2Bq = \mu\varpi - \nu u + v + \lambda w,$$
$$2Cr = \nu\varpi + \mu u - \lambda v + w,$$

whence also

$$2\Omega = \kappa\varpi \quad \text{(33)}.$$

Substituting the values of Ap, Bq, Cr, given by (30) in the equations (6), we deduce

$$2\mathrm{a} = \lambda\varpi + u - \nu v + \mu w \quad \text{(34)},$$
$$2\mathrm{b} = \mu\varpi + \nu u + v - \lambda w,$$
$$2\mathrm{c} = \nu\varpi - \mu u + \lambda v + w,$$

whence also

$$2(\mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu) = \kappa\varpi \quad \text{(35)},$$

which in fact follows from (33) and (17). And likewise the inverse system,

$$u = \frac{2}{\kappa}(\mathrm{a} + \nu\mathrm{b} - \mu\mathrm{c}) \quad \text{(36)}.$$
$$v = \frac{2}{\kappa}(-\nu\mathrm{a} + \mathrm{b} + \lambda\mathrm{c}),$$
$$w = \frac{2}{\kappa}(\mu\mathrm{a} - \lambda\mathrm{b} + \mathrm{c}).$$

It is easy to deduce

$$k^2 = \tfrac{1}{4}\kappa\,[u^2 + v^2 + w^2 + \varpi^2] \quad \text{(37)},$$
$$\upsilon = \tfrac{1}{4}\,[(u^2 + v^2 + w^2) + (1 + \kappa)\,\varpi^2] \quad \text{(38)}.$$

Again, from the equations (10 *bis*),

$$\begin{aligned}\kappa\,(\mathrm{b}Cr - \mathrm{c}Bq) &= -2\lambda\,(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2) + 2\mathrm{a}\,(\lambda\mathrm{a} + \mu\mathrm{b} + \nu\mathrm{c}) + 2\,(\mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -2\lambda k^2 + 2\,(\mathrm{a} + \mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -2\lambda k^2 + \kappa u\Omega\,;\end{aligned}$$

and, forming also the similar expressions for $\kappa\,(\mathrm{c}Ap - \mathrm{a}Cr)$, and $\kappa\,(\mathrm{a}Bq - \mathrm{b}Ap)$, we thus obtain

$$\Omega u - \frac{2}{\kappa}k^2\lambda = \mathrm{b}Cr - \mathrm{c}Bq \quad \text{(39)},$$
$$\Omega v - \frac{2}{\kappa}k^2\mu = \mathrm{c}Ap - \mathrm{b}Cr,$$
$$\Omega w - \frac{2}{\kappa}k^2\nu = \mathrm{a}Bq - \mathrm{c}Ap\,;$$

to which many others might probably be joined.

The constants of the problem are a, b, c, h, ϵ, δ. Of these a, b, c are given as functions of λ, μ, ν, u, v, w, by the equations (34); in which ϖ is to be considered as

standing for $\lambda u + \mu v + \nu w$. {These determine k^2, which is however given immediately by (37).} As for h, we have

$$h = \frac{1}{A}(Ap)^2 + \frac{1}{B}(Bq)^2 + \frac{1}{C}(Cr)^2,$$

where Ap, Bq, Cr are given as functions of λ, μ, ν, u, v, w by (32), in which also ϖ stands for $\lambda u + \mu v + \nu w$. Again,

$$\epsilon = t - \tfrac{1}{2}\int\frac{dv}{\nabla},$$

$$\delta = 2\tan^{-1}\frac{\kappa\varpi}{2k} - \tfrac{1}{4}k\int\frac{(h+\Phi)\,dv}{v\nabla};$$

in each of which ∇, Φ are functions of v, and of a, b, c, h, partly as entering explicitly into these functions, partly as contained implicitly in p, q, r, which enter into ∇, Φ, and are functions of v, h, k given by (18). After the integration v is to be considered a function of λ, μ, ν, u, v, w given by (38). Both of the integrals may be supposed taken from a certain value v_0 of v, which may be considered as an absolutely invariable arbitrary constant, since without it we have the right number, six, of arbitrary constants.

First to find (a, b), (b, c), and (c, a). From (34) we have

$$\begin{aligned}(\mathrm{a}, \mathrm{b}) = \tfrac{1}{4}\{ & (1+\lambda^2)(\mu u - w) - (\lambda u + \varpi)(\lambda\mu + \nu) \\ & + (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1+\mu^2) \\ & + (\nu\lambda + \mu)(u + \mu w) - (\lambda w - v)(\mu\nu - \lambda)\} \\ = \tfrac{1}{2}(\mu u - \lambda v - w - \nu\varpi) = & -\tfrac{1}{2}2\mathrm{c} = -\mathrm{c};\end{aligned}$$

whence the system

$$(\mathrm{b}, \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c}, \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a}, \mathrm{b}) = -\mathrm{c} \quad \text{.....(40).}$$

Also we may add $\quad (k, \mathrm{a}) = \frac{\mathrm{a}}{k}(\mathrm{a}, \mathrm{a}) + \frac{\mathrm{b}}{k}(\mathrm{b}, \mathrm{a}) + \frac{\mathrm{c}}{k}(\mathrm{c}, \mathrm{a}) = 0,$

or

$$(k, \mathrm{a}) = 0, \quad (k, \mathrm{b}) = 0, \quad (k, \mathrm{c}) = 0 \quad \text{..... (41),}$$

which will be useful in calculating some of the following coefficients.

Proceeding to calculate (a, h), (b, h), (c, h). It is seen immediately that

$$(\mathrm{a}, h) = 2\{p(\mathrm{a}, Ap) + q(\mathrm{a}, Bq) + r(\mathrm{a}, Cr)\},$$

where Ap, Bq, Cr, are given by the equations (32), so that

$$\begin{aligned}(\mathrm{a}, Ap) = \tfrac{1}{4}\{ & (1+\lambda^2)(\lambda u + \varpi) - (1+\lambda^2)(\lambda u + \varpi) \\ & + (\lambda\mu - \nu)(\lambda v - w) - (\lambda\mu + \nu)(\lambda v + w) \\ & + (\nu\lambda + \mu)(v + \lambda w) - (\nu\lambda - \mu)(-v + \lambda w)\}\end{aligned}$$

i.e. $\quad (\mathrm{a}, Ap) = 0. \quad \text{.....(42).}$

Similarly $$(\mathrm{a},\ Bq) = \tfrac{1}{4}\{\ (1+\lambda^2)(\mu u + w) - (\lambda u + \varpi)(\lambda\mu - \nu)$$
$$+ (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1 + \mu^2)$$
$$+ (\nu\lambda + \mu)(\mu w - u) - (-v + \lambda w)(\mu\nu + \lambda)\}$$

i.e. $$(\mathrm{a},\ Bq) = 0, \qquad \text{.....(43)},$$

and similarly $$(\mathrm{a},\ Cr) = 0\,;$$

whence $$(\mathrm{a},\ h) = 0, \text{ and therefore } (\mathrm{b},\ h) = 0,\ (\mathrm{c},\ h) = 0 \qquad \text{..... (44)};$$

also $$(k,\ h) = 0, \qquad \text{..... (45)}.$$

Next we have to determine $(\mathrm{a},\ \epsilon)$, $(\mathrm{b},\ \epsilon)$, $(\mathrm{c},\ \epsilon)$. Here ϵ being a function of u, v, w, λ, μ, ν, a, b, c, h, we must write

$$(\mathrm{a},\ \epsilon) = \{(\mathrm{a},\ \epsilon)\} + (\mathrm{a},\ \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{a},\ \mathrm{c})\frac{d\epsilon}{d\mathrm{c}} + (\mathrm{a},\ h)\frac{d\epsilon}{dh},$$

i.e. $$(\mathrm{a},\ \epsilon) = \{(\mathrm{a},\ \epsilon)\} + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

But $$\epsilon = t - 2\int\frac{dv}{\nabla}\,; \text{ and thence } \{(\mathrm{a},\ \epsilon)\} = -\frac{2}{\nabla}(\mathrm{a},\ v),$$

and v is given immediately as a function of λ, μ, ν, u, v, w, by the equation (38). Hence

$$(\mathrm{a},\ v) = \tfrac{1}{4}[\ (1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\}$$
$$+ (\lambda\mu - \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v + w)\{v + \mu(1+\kappa)\varpi\}$$
$$+ (\nu\lambda + \mu)\{(1+\kappa)w\varpi + \nu\varpi^2)\} - (-v + \lambda w)\{w + \nu(1+\kappa)\varpi\}]$$
$$= \tfrac{1}{4}\{(1+\kappa)\varpi u - \lambda(1+\kappa)\varpi^2 + \lambda\kappa - \lambda(u^2 + v^2 + w^2) - u\varpi\}$$
$$= \tfrac{1}{4}\{\kappa u\varpi - \lambda\varpi^2 - \lambda(u^2 + v^2 + w^2)\}$$
$$= \tfrac{1}{4}\kappa u\varpi - \frac{k^2\lambda}{\kappa} = \tfrac{1}{2}\left(\Omega u - \frac{2k^2\lambda}{\kappa}\right) \text{ [by (37) and (33)]},$$
$$= \tfrac{1}{2}(\mathrm{b}Cr - \mathrm{c}Bq) \qquad \text{..... (46)};$$

whence $$\{(\mathrm{a},\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq),$$

$$(\mathrm{a},\ \epsilon) = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

The terms $\mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}$ are evidently of the form $F(v) - F(v_0)$.

If therefore we suppose $v = v_0$, we have

$$(\mathrm{a},\ \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0) \qquad \text{..... (47)},$$

if p_0, q_0, r_0, ∇_0 refer to the value v_0 of v, i.e. if

$$A\,p_0^2 + B\,q_0^2 + C\,r_0^2 = h \quad\text{.....(48)},$$
$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$
$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2v_0 - k^2.$$

{This implies evidently

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = \frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0),$$

an equation which it is interesting to verify. In fact, from the value of ϵ

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = -2\int dv\left(\mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}\right)\frac{1}{\nabla} = 2\int dv\frac{1}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right);$$

or we have to show that

$$\frac{d}{dv}\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) = \frac{2}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right) = \frac{2}{\nabla^2}\delta\nabla;$$

if for shortness

$$\delta = \mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}.$$

Now ∇ containing a, b, c explicitly, and also as involved in p, q, r, we have

$$\delta\nabla = \mathrm{b}pq\,(A-B) - crp\,(C-A) + \frac{d\nabla}{dp}\delta p + \frac{d\nabla}{dq}\delta q + \frac{d\nabla}{dr}\delta r = \mathrm{b}pq\,(A-B) - crp\,(C-A) + \delta'\nabla$$

suppose. The equation to be verified becomes

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) - (\mathrm{b}Cr - \mathrm{c}Bq)\frac{d\nabla}{dv} = 2\{\mathrm{b}pq\,(A-B) - crp\,(C-A) + \delta'\nabla\}.$$

Now, observing that $\delta k = 0$, we have

$$A\,p\delta p + B\,q\delta q + C\,r\delta r = 0,$$
$$A^2p\delta p + B^2q\delta q + C^2r\delta r = 0,$$
$$A\,\mathrm{a}\delta p + B\,\mathrm{b}\delta q + C\,\mathrm{c}\delta r = -(\mathrm{b}Cr - \mathrm{c}Bq).$$

Also,

$$A\,p\frac{dp}{dv} + B\,q\frac{dq}{dv} + C\,r\frac{dr}{dv} = 0,$$

$$A^2p\frac{dp}{dv} + B^2q\frac{dq}{dv} + C^2r\frac{dr}{dv} = 0,$$

$$A\,\mathrm{a}\frac{dp}{dv} + B\,\mathrm{b}\frac{dq}{dv} + C\,\mathrm{c}\frac{dr}{dv} = 2,$$

whence evidently

$$\frac{dp}{dv}=\frac{-2}{\mathrm{b}Cr-\mathrm{c}Bq}\delta p,\quad \frac{dq}{dv}=\frac{-2}{\mathrm{b}Cr-\mathrm{c}Bq}\delta q,\quad \frac{dr}{dv}=\frac{-2}{\mathrm{b}Cr-\mathrm{c}Bq}\delta r,$$

or

$$\frac{d\nabla}{dv}=\frac{-2}{\mathrm{b}Cr-\mathrm{c}Bq}\delta'\nabla;$$

or the equation to be verified is simply

$$\nabla\left(\mathrm{b}C\frac{dr}{dv}-\mathrm{c}B\frac{dq}{dv}\right)=2\{\mathrm{b}pq(A-B)-\mathrm{c}rp(C-A)\};$$

which follows immediately from the three equations just given for the determination of $\frac{dp}{dv}, \frac{dq}{dv}, \frac{dr}{dv}$ }.

From these values also

$$(k,\ \epsilon)=0 \quad\text{.....}\quad (49).$$

Next, to calculate $(h,\ \epsilon)$,

$$(h,\ \epsilon)=\{(h,\ \epsilon)\}+(h,\ \mathrm{a})\frac{d\epsilon}{d\mathrm{a}}+(h,\ \mathrm{b})\frac{d\epsilon}{d\mathrm{b}}+(h,\ \mathrm{c})\frac{d\epsilon}{d\mathrm{c}};$$

but the three last terms being evidently such as to vanish for $v=v_0$, we may neglect them, and consider $(h,\ \epsilon)$ as the value which $\{(h,\ \epsilon)\}$ assumes for this value of v.

Now
$$\{(h,\ \epsilon)\}=2p\{(Ap,\ \epsilon)\}+2q\{(Bq,\ \epsilon)\}+2r\{(Cr,\ \epsilon)\},$$

where
$$\{(Ap,\ \epsilon)\}=-\frac{2}{\nabla}(Ap,\ v),$$

and

$$\begin{aligned}(Ap,\ v)=\tfrac{1}{4}[\ &(1+\lambda^2)\{(1+\kappa)u\varpi+\lambda\varpi^2\}-(\lambda u+\varpi)\{u+\lambda(1+\kappa)\varpi\}\\ &+(\lambda\mu+\nu)\{(1+\kappa)v\varpi+\mu\varpi^2\}-(\lambda v-w)\{v+\mu(1+\kappa)\varpi\}\\ &+(\nu\lambda-\mu)\{(1+\kappa)w\varpi+\nu\varpi^2\}-(v+\lambda w)\{w+\nu(1+\kappa)\varpi\}]\\ =\tfrac{1}{4}&\{(1+\kappa)\varpi u+\lambda\kappa\varpi^2-\lambda(1+\kappa)\varpi^2-\lambda(u^2+v^2+w^2)-\varpi u\}=(\mathrm{a},\ v)\\ =\tfrac{1}{2}&(\mathrm{b}Cr-\mathrm{c}Bq),\quad\text{.....}\quad(50),\end{aligned}$$

whence
$$\{(Ap,\ \epsilon)\}=-\frac{1}{\nabla}(\mathrm{b}Cr-\mathrm{c}Bq)\text{.....}\quad(51),$$

and therefore
$$\{(Bq,\ \epsilon)\}=-\frac{1}{\nabla}(\mathrm{c}Ap-\mathrm{a}Cr),$$

$$\{(Cr,\ \epsilon)\}=-\frac{1}{\nabla}(\mathrm{a}Bq-\mathrm{b}Ap),$$

whence
$$\{(h,\ \epsilon)\} = -2,$$
and therefore
$$(h,\ \epsilon) = -2 \quad\text{.....}\quad (52).$$

Next, to find (a, δ), (b, δ), (c, δ), (h, δ),
$$\delta = 2\tan^{-1}\frac{\kappa\varpi}{2k} - k\int\frac{(h+\Phi)\,dv}{v\nabla},$$
$$= \delta' + \delta'' \text{ suppose,}$$
$$(\mathrm{a},\ \delta) = (\mathrm{a},\ \delta') + (\mathrm{a},\ \delta''),$$
$$(\mathrm{a},\ \delta') = \frac{k}{\kappa v}(\mathrm{a},\ \kappa\varpi) + (\mathrm{a},\ k)\frac{d\delta'}{dk},$$
{observing
$$\kappa^2\varpi^2 + 4k^2 = 4(\Omega^2 + k^2) = 4\kappa v\}$$
$$= \frac{k}{\kappa v}(\mathrm{a},\ \kappa\varpi),$$
where
$$\begin{aligned}(\mathrm{a},\ \kappa\varpi) = \tfrac{1}{2}\{\ &(1+\lambda^2)(\kappa v + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu - \nu)(\kappa v + 2\mu\varpi) - (\lambda v + w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (-v + \lambda w)\kappa\nu\ \}\end{aligned}$$
$$= \tfrac{1}{2}\kappa(u + \lambda\varpi) = Ap - \nu Bq + \mu Cr + \lambda\Omega = \tfrac{1}{2}(\mathrm{a} + Ap)\kappa \quad\text{.....}(53),$$
by equations (29), (33), and (10);

that is
$$(\mathrm{a},\ \delta') = \frac{k}{2v}(\mathrm{a} + Ap).$$

Also
$$(\mathrm{a},\ \delta'') = -k\frac{h+\Phi}{v\nabla}(\mathrm{a},\ v) + (\mathrm{a},\ \mathrm{b})\frac{d\delta''}{d\mathrm{b}} + \&\mathrm{c}.$$
$$= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + Fv - Fv_0,$$
whence
$$(\mathrm{a},\ \delta) = \frac{k}{2v}\{\mathrm{a} + Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\} + Fv - Fv_0,$$
or putting $v = v_0$,
$$(\mathrm{a},\ \delta) = \frac{k}{2v_0}\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\} \quad\text{.....}(54),$$
and therefore also
$$(\mathrm{b},\ \delta) = \frac{k}{2v_0}\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\},$$
$$(\mathrm{c},\ \delta) = \frac{k}{2v_0}\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - bAp_0)\}.$$

Again,
$$(h,\ \delta) = 2p\,(Ap,\ \delta) + 2q\,(Bq,\ \delta) + 2r\,(Cr,\ \delta),$$
$$(Ap,\ \delta) = (Ap,\ \delta') + (Ap,\ \delta''),$$
$$(Ap,\ \delta') = \frac{k}{\kappa\upsilon}(Ap,\ \kappa\varpi) + (Ap,\ k)\frac{d\delta'}{dk}$$
$$= \frac{k}{\kappa\upsilon}(Ap,\ \kappa\varpi),$$
$$\begin{aligned}(Ap,\ \kappa\varpi) = \tfrac{1}{2}\{\ & (1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\,\kappa\lambda \\ & + (\lambda\mu + \nu)(\kappa v + 2\mu\varpi) - (\lambda v - w)\,\kappa\mu \\ & + (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (v + \lambda w)\,\kappa\nu\ \} \\ = \tfrac{1}{2}\kappa\,(u + \lambda\varpi) = \tfrac{1}{2}\kappa\,(\mathrm{a} + Ap) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55);\end{aligned}$$

therefore
$$(Ap,\ \delta') = \frac{k}{2u}(\mathrm{a} + Ap) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$
$$(Ap,\ \delta'') = -k\frac{h+\Phi}{\upsilon\nabla}(Ap,\ \upsilon) + \&\mathrm{c.}$$
$$= -\tfrac{1}{2}k\,\frac{h+\Phi}{\upsilon\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + F\upsilon - F\upsilon_0,$$
$$(Ap,\ \delta) = \frac{k}{2\upsilon}\{\mathrm{a} - Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\} + F\upsilon - F\upsilon_0,$$

and similarly for $(Bq,\ \delta)$, $(Cr,\ \delta)$. Substituting, and neglecting the terms which vanish for $\upsilon = \upsilon_0$,
$$(h,\ \delta) = \frac{k}{\upsilon}\left(\Phi + h - \frac{\Phi + h}{\nabla}\nabla\right),$$

i.e.
$$(h,\ \delta) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57).$$

Lastly, to find $(\epsilon,\ \delta)$,
$$(\epsilon,\ \delta) = \{(\epsilon,\ \delta)\} + (\mathrm{a},\ \delta)\frac{d\epsilon}{d\mathrm{a}} + (\mathrm{b},\ \delta)\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{c},\ \delta)\frac{d\epsilon}{d\mathrm{c}},$$

where, in $\{(\epsilon,\ \delta)\}$, the differentiations upon ϵ are supposed not to affect the constants a, b, c. Neglecting the terms which vanish for $\upsilon = \upsilon_0$,
$$(\epsilon,\ \delta) = \{(\epsilon,\ \delta)\},$$
$$\{(\epsilon,\ \delta)\} = \{(\epsilon,\ \delta')\} + \{(\epsilon,\ \delta'')\},$$
$$\{(\epsilon,\ \delta')\} = [\{(\epsilon,\ \delta')\}] + (\epsilon,\ k)\frac{d\delta'}{dk} = [\{(\epsilon,\ \delta')\}];$$

where, in $[\{(\epsilon,\ \delta')\}]$, the differentiations upon ϵ and δ do not affect the constants.
$$\{(\epsilon,\ \delta'')\} = [\{(\epsilon,\ \delta'')\}] + (\epsilon,\ \mathrm{a})\frac{d\delta''}{d\mathrm{a}} + \&\mathrm{c.}$$

i.e.
$$\{(\epsilon,\ \delta'')\} = [\{(\epsilon,\ \delta'')\}]:$$

neglecting the terms which vanish for $v = v_0$,

therefore
$$(\epsilon,\ \delta) = [\{(\epsilon,\ \delta')\}] + [\{(\epsilon,\ \delta'')\}]$$
$$= [\{(\epsilon,\ \delta')\}];$$

since
$$[\{(\epsilon,\ \delta'')\}] = (v,\ v)\frac{d\epsilon}{dv}\frac{d\delta''}{dv} = 0.$$

Hence
$$(\epsilon,\ \delta) = -\tfrac{1}{2}\frac{k}{\kappa\nabla v}(v,\ \kappa\varpi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58),$$

$$(v,\ \kappa\varpi) = \tfrac{1}{2}\left(\begin{array}{l} \{u + (1+\kappa)\lambda\varpi\}(2\lambda\varpi + \kappa u) - \{\lambda\varpi^2 + (1+\kappa)\varpi u\}\kappa\lambda \\ + \{v + (1+\kappa)\mu\varpi\}(2\mu\varpi + \kappa v) - \{\mu\varpi^2 + (1+\kappa)\varpi v\}\kappa\mu \\ + \{w + (1+\kappa)\nu\varpi\}(2\nu\varpi + \kappa w) - \{\nu\varpi^2 + (1+\kappa)\varpi w\}\kappa\nu \end{array}\right)$$
$$= \tfrac{1}{2}\{2\varpi^2 + \kappa(u^2+v^2+w^2) + 2(1+\kappa)(\kappa-1)\varpi^2 + \kappa(1+\kappa)\varpi^2$$
$$- \kappa(\kappa-1)\varpi^2 - \kappa(\kappa+1)\varpi^2\}$$
$$= \tfrac{1}{2}\kappa\{(\kappa+1)\varpi^2 + (u^2+v^2+w^2)\} = \tfrac{1}{2}4\kappa v = 2\kappa v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (59),$$

therefore
$$(\epsilon,\ \delta) = -\frac{k}{\nabla_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60).$$

Hence, recapitulating,

$$\left.\begin{array}{l}
(\mathrm{b},\ \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c},\ \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a},\ \mathrm{b}) = -\mathrm{c}, \\
(\mathrm{a},\ h) = 0, \quad (\mathrm{b},\ h) = 0, \quad (\mathrm{c},\ h) = 0, \\
(\mathrm{a},\ \epsilon) = -\dfrac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0), \\
(\mathrm{b},\ \epsilon) = -\dfrac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0), \\
(\mathrm{c},\ \epsilon) = -\dfrac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0), \\
(h,\ \epsilon) = -2, \\
(\mathrm{a},\ \delta) = \dfrac{k}{2v_0}\{\mathrm{a} + Ap_0 - \dfrac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\}, \\
(\mathrm{b},\ \delta) = \dfrac{k}{2v_0}\{\mathrm{b} + Bq_0 - \dfrac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\}, \\
(\mathrm{c},\ \delta) = \dfrac{k}{2v_0}\{\mathrm{c} + Cr_0 - \dfrac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\}, \\
(h,\ \delta) = 0, \\
(\epsilon,\ \delta) = -\dfrac{k}{\nabla_0}
\end{array}\right\} \ldots\ldots\ldots\ldots (61),$$

and therefore

$$\left.\begin{aligned}
\frac{d\mathrm{a}}{dt} &= -\mathrm{c}\frac{dV}{d\mathrm{b}} + \mathrm{b}\frac{dV}{d\mathrm{c}} - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\epsilon} + \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\delta},\\
\frac{d\mathrm{b}}{dt} &= -\mathrm{a}\frac{dV}{d\mathrm{c}} + \mathrm{c}\frac{dV}{d\mathrm{a}} - \frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\frac{dV}{d\epsilon} + \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\delta},\\
\frac{d\mathrm{c}}{dt} &= -\mathrm{b}\frac{dV}{d\mathrm{a}} + \mathrm{a}\frac{dV}{d\mathrm{b}} - \frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\epsilon} + \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\delta},\\
\frac{dh}{dt} &= -2\frac{dV}{d\epsilon},\\
\frac{d\epsilon}{dt} &= \frac{1}{\nabla_0}\left\{(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\mathrm{a}} + (\mathrm{c}Ap_0 - \mathrm{a}Bq_0)\frac{dV}{d\mathrm{b}} + (\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\mathrm{c}}\right\} + 2\frac{dV}{dh} - \frac{k}{\nabla_0}\frac{dV}{d\delta},\\
\frac{d\delta}{dt} &= -\frac{k}{2v_0}\left[\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\mathrm{a}}\right.\\
&\qquad + \left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\mathrm{b}}\\
&\qquad \left. + \left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\mathrm{c}}\right] + \frac{k}{\nabla_0}\frac{dV}{d\epsilon},
\end{aligned}\right\} \quad (62),$$

to which we may join

$$\frac{dk}{dt} = \frac{dV}{d\delta} \quad \text{.....} (63).$$

We have thus the complete system of formulæ.

38.

NOTE ON A GEOMETRICAL THEOREM CONTAINED IN A PAPER BY SIR W. THOMSON.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 207, 208.]

It is easily shown that if three confocal surfaces of the second order pass through a point P, then the square of the distance of this point from the origin is equal to the sum of the squares of three of the axes, no two of which are parallel or belong to the same surface (the squares of one or two of the axes of the hyperboloids being considered negative); i.e. if

$$\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

$$\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1;$$

then

$$x^2+y^2+z^2=a^2+b^2+c^2+h+k+l.$$

In fact these equations give

$$x^2=\frac{(a^2+h)(a^2+k)(a^2+l)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2=\frac{(b^2+h)(b^2+k)(b^2+l)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2=\frac{(c^2+h)(c^2+k)(c^2+l)}{(c^2-a^2)(c^2-b^2)},$$

and adding these and reducing, we have the relation in question; which is also immediately obtained by forming the cubic whose roots are h, k, l.

From this property may be deduced the theorem given by Mr Thomson in the preceding memoir ["On the Principal Axes of a Solid Body," pp. 127—133 and 195—206, see p. 205]. In fact, writing

$$r^2 = x^2 + y^2 + z^2,$$

and

$$k = -a^2 - b^2 - c^2 + h,$$

we see that in consequence of these relations the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{r^2 + a^2 - h} + \frac{y^2}{r^2 + b^2 - h} + \frac{z^2}{r^2 + c^2 - h} = 1,$$

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1,$$

are equivalent to two independent equations, i.e. the third can be deduced from the two first. Now the first equation is that of an ellipsoid (or generally a surface of the second order, since a, b, c are not necessarily real). The second is that of what may be called a conjugate equimomental surface, defining the term as follows: "The conjugate equimomental surfaces of an ellipsoid (or surface of the second order) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, are the equimomental surfaces derived in the usual manner from any surface of the second order $\frac{x^2}{h - a^2} + \frac{y^2}{h - b^2} + \frac{z^2}{h - c^2} = 1$, which is confocal with the *conjugate* surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ of the given ellipsoid," viz. by measuring along any line through the centre distances equal to the axes of the section by a plane through the centre perpendicular to this line, and taking the locus of the points so determined for the equimomental surface. The third equation is that of a surface confocal with the given ellipsoid; hence the theorem, "The curves of curvature of a given ellipsoid lie upon a system of conjugate equimomental surfaces."

But since the first and second equations are evidently satisfied by the combination of the first equation with the relation $r^2 = h$, which is that of a sphere, we have also, "The curve of intersection of the ellipsoid with any one of the conjugate equimomental surfaces, is composed of the line of curvature and a spherical conic." And these two curves being each of them of the fourth order make up the complete curve of intersection, which should obviously be of the eighth order.

It would be an interesting question to determine the relations existing between the curve of curvature and the spherical conic, which have been thus brought into connection by means of the conjugate equimomental surfaces; i.e. between the two curves obtained by combining the equation of the ellipsoid with

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1,$$

$$r^2 = a^2 + b^2 + c^2 + k,$$

respectively: but it will be sufficient at present to have suggested the problem.

39.

ON THE DIAMETRAL PLANES OF A SURFACE OF THE SECOND ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. I. (1846), pp. 274—278.]

LET $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$, be the equation of a surface of the second order referred to its centre, and let $\alpha x + \alpha' y + \alpha'' z = 0$ be the equation of one of its diametral planes; then, as usual

$$\begin{aligned} (A-u)\,\alpha + \quad H\alpha' + \quad G\alpha'' &= 0, \\ H\alpha + (B-u)\,\alpha' + \quad F\alpha'' &= 0, \\ G\alpha + \quad F\alpha' + (C-u)\,\alpha'' &= 0, \end{aligned}$$

which are equivalent to two independent equations, and consequently capable of determining the ratios $\alpha : \alpha' : \alpha''$, provided that u satisfy the cubic equation that is obtained by eliminating α, α', α'' from the three equations.

We have from the second and third, from the third and first, and from the first and second equations respectively,

$$\alpha : \alpha' : \alpha'' = \mathfrak{A}_{,} : \mathfrak{H}_{,} : \mathfrak{G}_{,} = \mathfrak{H}_{,} : \mathfrak{B}_{,} : \mathfrak{F}_{,} = \mathfrak{G}_{,} : \mathfrak{F}_{,} : \mathfrak{C}_{,};$$

where, if

$$\begin{aligned} \mathfrak{A} &= BC - F^2, \\ \mathfrak{B} &= CA - G^2, \\ \mathfrak{C} &= AB - H^2, \\ \mathfrak{F} &= GH - AF, \\ \mathfrak{G} &= HF - BG, \\ \mathfrak{H} &= FG - CH, \end{aligned}$$

$\mathfrak{A}_{,}$, $\mathfrak{B}_{,}$, $\mathfrak{C}_{,}$, $\mathfrak{F}_{,}$, $\mathfrak{G}_{,}$, $\mathfrak{H}_{,}$ are what these become when A, B, C are changed into

$A-u$, $B-u$, $C-u$, so that

$$\begin{aligned}
\mathfrak{A}_{,} &= \mathfrak{A} - (B+C)\,u + u^2,\\
\mathfrak{B}_{,} &= \mathfrak{B} - (C+A)\,u + u^2,\\
\mathfrak{C}_{,} &= \mathfrak{C} - (A+B)\,u + u^2,\\
\mathfrak{F}_{,} &= \mathfrak{F} + Fu\,,\\
\mathfrak{G}_{,} &= \mathfrak{G} + Gu\,,\\
\mathfrak{H}_{,} &= \mathfrak{H} + Hu.
\end{aligned}$$

Hence the equation $\alpha x + \alpha' y + \alpha'' z = 0$ may be written in the three forms

$$\begin{aligned}
\mathfrak{A}_{,}x + \mathfrak{H}_{,}y + \mathfrak{G}_{,}z &= 0,\\
\mathfrak{H}_{,}x + \mathfrak{B}_{,}y + \mathfrak{F}_{,}z &= 0,\\
\mathfrak{G}_{,}x + \mathfrak{F}_{,}y + \mathfrak{C}_{,}z &= 0;
\end{aligned}$$

or, what comes to the same thing, as follows,

$$\begin{aligned}
\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + u\,(Ax + Hy + Gz) + vx &= 0,\\
\mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z + u\,(Hx + By + Fz) + vy &= 0,\\
\mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z + u\,(Gx + Fy + Cz) + vz &= 0,
\end{aligned}$$

in which for shortness v has been written instead of

$$u^2 - (A+B+C)\,u.$$

The elimination of u, v from these equations gives a result $\Theta = 0$, where Θ is a homogeneous function of the third order in x, y, z; and this equation, it is evident, must belong to the three diametral planes jointly, i.e. Θ must be the product of three linear factors, each of which equated to zero would correspond to a diametral plane. Thus the system of diametral planes is given by

$$\Theta = \begin{vmatrix} \mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z, & Ax + Hy + Gz, & x \\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z, & Hx + By + Fz, & y \\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z, & Gx + Fy + Cz, & z \end{vmatrix} = 0,$$

or developing the determinant, as follows,

$$\begin{aligned}
\Theta = {} & (G\mathfrak{H} - H\mathfrak{G})\,x^3 + (H\mathfrak{F} - F\mathfrak{H})\,y^3 + (F\mathfrak{G} - G\mathfrak{F})\,z^3\\
& + \{\ G\,(\mathfrak{C} - \mathfrak{B}) - \mathfrak{G}\,(C - B) - (H\mathfrak{F} - F\mathfrak{H})\}\,yz^2\\
& + \{\ H\,(\mathfrak{A} - \mathfrak{C}) - \mathfrak{H}\,(A - C) - (F\mathfrak{G} - G\mathfrak{F})\}\,zx^2\\
& + \{\ F\,(\mathfrak{B} - \mathfrak{A}) - \mathfrak{F}\,(B - A) - (G\mathfrak{H} - H\mathfrak{G})\}\,xy^2\\
& + \{-H\,(\mathfrak{C} - \mathfrak{B}) + \mathfrak{H}\,(C - B) + (F\mathfrak{G} - G\mathfrak{F})\}\,y^2z\\
& + \{-F\,(\mathfrak{A} - \mathfrak{C}) + \mathfrak{F}\,(A - C) + (G\mathfrak{H} - H\mathfrak{G})\}\,z^2x\\
& + \{-G\,(\mathfrak{B} - \mathfrak{A}) + \mathfrak{G}\,(B - A) + (H\mathfrak{F} - F\mathfrak{H})\}\,x^2y\\
& + (C\mathfrak{B} - B\mathfrak{A} + \mathfrak{A}\mathfrak{C} - C\mathfrak{A} + B\mathfrak{A} - A\mathfrak{B})\,xyz\,;
\end{aligned}$$

or reducing

$$\begin{aligned}\Theta = \quad & \{F(G^2-H^2)-GH(C-B)\}\,x^3 \\ & +\{G(H^2-F^2)-HF(A-C)\}\,y^3 \\ & +\{H(F^2-G^2)-FG(B-A)\}\,z^3 \\ & +\{G(A-B)(B-C)+FH(A+B-2C)+G(F^2+G^2-2H^2)\}\,yz^2 \\ & +\{H(B-C)(C-A)+GF(B+C-2A)+H(G^2+H^2-2F^2)\}\,zx^2 \\ & +\{F(C-A)(A-B)+GH(C+A-2B)+F(H^2+F^2-2G^2)\}\,xy^2 \\ & +\{H(B-C)(C-A)+FG(C+A-2B)+H(H^2+F^2-2G^2)\}\,y^2z \\ & +\{F(C-A)(A-B)+GH(A+B-2C)+F(F^2+G^2-2H^2)\}\,z^2x \\ & +\{G(A-B)(B-C)+HF(B+C-2A)+G(G^2+H^2-2F^2)\}\,x^2y \\ & -\{(A-B)(B-C)(C-A)+(B-C)F^2+(C-A)G^2+(A-B)H^2\}\,xyz.\end{aligned}$$

In the case of *curves* of the second order, the result is much more simple; we have

$$\Theta = \begin{vmatrix} Ax+Hy, & x \\ Hx+By, & y \end{vmatrix} = 0,$$

i.e.

$$\Theta = H(y^2-x^2)+(A-B)\,xy = 0,$$

for the equation of the two diameters.

The above formulæ may be applied to the question of finding the diametral planes of the cone circumscribed about a given surface of the second order, (or of the lines bisecting the angles made by two tangents of a curve of the second order). Considering the latter question first: if

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the equation of the curve, and α, β the coordinates of the point of intersection of the two tangents, the equation of the pair of tangents is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2=0;$$

or making the point of intersection the origin,

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}\right)^2=0,$$

i.e.

$$(\beta x-\alpha y)^2-(b^2x^2+a^2y^2)=0;$$

whence $A=\beta^2-b^2$, $B=\alpha^2-a^2$, $H=-\alpha\beta$, and the equation to the lines bisecting the angles formed by the tangents is

$$\alpha\beta(x^2-y^2)-\{\alpha^2-\beta^2-(a^2-b^2)\}\,xy=0,$$

which is the same for all confocal ellipses; whence the known theorem,

"If there be two confocal ellipses, and tangents be drawn to the second from any point P of the first, the tangent and normal of the first conic at the point P, bisect the angles formed by the two tangents in question."

In the case of surfaces, the equation of the circumscribing cone referred to its vertex as origin, is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}\right)^2=0\,;$$

whence

$$\begin{aligned}
A &= \beta^2c^2+\gamma^2b^2-b^2c^2,\\
B &= \gamma^2a^2+\alpha^2c^2-a^2c^2,\\
C &= \alpha^2b^2+\beta^2a^2-b^2a^2,\\
F &= -\,a^2\beta\gamma,\\
G &= -\,b^2\gamma\alpha\,,\\
H &= -\,c^2\alpha\beta\,.
\end{aligned}$$

Hence, omitting the factor $b^2c^2\alpha^2+c^2a^2\beta^2+a^2b^2\gamma^2-a^2b^2c^2$, we have

$$\begin{aligned}
\mathfrak{A} &= \alpha^2-a^2,\\
\mathfrak{B} &= \beta^2-b^2,\\
\mathfrak{C} &= \gamma^2-c^2,\\
\mathfrak{F} &= \beta\gamma,\\
\mathfrak{G} &= \gamma\alpha\,,\\
\mathfrak{H} &= \alpha\beta;
\end{aligned}$$

and the equation of the system of diametral planes becomes

$$\begin{aligned}
\Theta=0= \quad & \alpha^2\beta\gamma\,(c^2-b^2)\,x^3+\beta^2\gamma\alpha\,(a^2-c^2)\,y^3+\gamma^2\alpha\beta\,(b^2-a^2)\,z^3\\
+\ & \gamma\alpha\,\{\alpha^2(c^2-b^2)+\beta^2(b^2+c^2-2a^2)-\gamma^2(b^2-a^2)+(b^2-a^2)(c^2-b^2)\}\,yz^2\\
+\ & \alpha\beta\,\{-\alpha^2(c^2-b^2)+\beta^2(a^2-c^2)+\gamma^2(c^2+a^2-2b^2)+(c^2-b^2)(a^2-c^2)\}\,zx^2\\
+\ & \gamma\alpha\,\{\alpha^2(a^2+b^2-2c^2)-\beta^2(a^2-c^2)+\gamma^2(b^2-a^2)+(a^2-c^2)(b^2-a^2)\}\,xy^2\\
-\ & \alpha\beta\,\{\alpha^2(c^2-b^2)-\beta^2(a^2-c^2)-\gamma^2(b^2+c^2-2a^2)-(a^2-c^2)(c^2-b^2)\}\,y^2z\\
-\ & \beta\gamma\,\{-\alpha^2(c^2+a^2-2b^2)+\beta^2(a^2-c^2)-\gamma^2(b^2-a^2)-(b^2-a^2)(a^2-c^2)\}\,z^2x\\
-\ & \gamma\alpha\,\{-\alpha^2(c^2-b^2)-\beta^2(a^2+b^2-2c^2)+\gamma^2(b^2-a^2)-(c^2-b^2)(b^2-a^2)\}\,x^2y\\
+\ & \{(a^2-b^2)(b^2-c^2)(c^2-a^2)+\\
& \quad(\alpha^4+\beta^2\gamma^2)(b^2-c^2)-(\beta^4+\gamma^2\alpha^2)(c^2-a^2)-(\gamma^4+\alpha^2\beta^2)(a^2-b^2)+\\
& \quad\alpha^2(b^2-c^2)(2a^2-b^2-c^2)+\beta^2(c^2-a^2)(2b^2-c^2-a^2)+\gamma^2(a^2-b^2)(2c^2-a^2-b^2)\}\,xyz\,;
\end{aligned}$$

and since this is a function of a^2-b^2, b^2-c^2, and c^2-a^2, the equation is the same for all confocal ellipsoids; whence the known theorem, "The axes of the circumscribing cone having its vertex in a given point P, are tangents to the curves of intersection of the three surfaces, confocal with the given surface, which pass through the point P."

40.

ON THE THEORY OF INVOLUTION IN GEOMETRY.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 52—61.]

WHEN three conics have the same points of intersection, any transversal intersects the system in six points, which are said to be in involution. It appears natural to apply the term to the conics themselves; and then it is easy to generalize the notion of involution so as to apply it to functions of any number of variables. Thus, if U, V... be homogeneous functions of the same order of any number of variables x, y ..., a function Θ, which is a linear function of U, V ..., is said to be in involution with these functions. More generally Θ may be said to be in involution with any system of factors of these functions: or if U, V ... be given functions of x, y, z ..., homogeneous of the degrees m, n ..., and u, v, ... arbitrary homogeneous functions of the degrees $r-m$, $r-n$ ...; then, if $\Theta = uU + vV + \ldots$, Θ is a function of the degree r, which is in involution with U, V.... The question which immediately arises, is to find the degree of generality of Θ, or the number of arbitrary constants which it contains. And this is a question which, from the variety and interest of its geometrical interpretations, has very frequently been treated of by geometers, though never, I believe, in quite so general a form, (the number r has almost always had particular values given to it, except in a short paper of my own, on the particular case of two curves, in the *Journal*, vol. III. p. 211 [5]).[1] There is also an analytical application of

[1] The first suggestion of the problem is contained in a memoir of Euler's—"Sur une contradiction apparente dans la doctrine des lignes courbes," *Mém. de Berlin*, t. IV. [1748] p. 219. It is noticed also in Cramer's *Introduction à l'analyse des lignes courbes* [1750]. The following memoirs also have been published on the subject: Plücker, "Recherches sur les courbes algébriques de tous les degrés," *Gerg. Ann.* t. XIX. [1828—29] p. 97; "Recherches sur les surfaces algébriques de tous les degrés," p. 129; (a great number of memoirs on particular applications of the theory are contained in Gergonne;) Jacobi, "De relationibus quæ locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum dati ordinis, simul cum enodatione paradoxi algebraici," *Crelle*, t. XV. [1836]; Plücker, "Théorèmes généraux concernant les équations d'un degré quelconque entre un nombre quelconque d'inconnues," *Crelle*, t. XVI. [1837], (but this last must be read with caution, as several of the theorems are incorrect, or at least stated without the proper limitations); and the *Einleitende Betrachtungen*, in Plücker's "Theorie der algebraischen Curven" [1839]. The following memoirs of Hesse, containing developments relative to the case of three surfaces of the second order, may likewise be mentioned, "De curvis et superficiebus secundi gradus," *Crelle*, t. XX. [1840] p. 285; and "Ueber die lineare Construction des achten Schnitt-punctes dreier Oberflächen zweiter Ordnung, wenn sieben Schnitt-puncte derselben gegeben sind," *Crelle*, t. XXVI. [1843] p. 147.

the theory, of considerable interest, to the problem of elimination between any number of equations containing the same number of variables. Suppose, for instance, two equations, $U=0$, $V=0$, when U, V are homogeneous functions of x, y of the degrees m, n respectively. To eliminate the variables it is sufficient to multiply the first equation by x^{n-1}, $x^{n-2}y\ldots$, y^{n-1}, and the second by $x^{m-1}\ldots$, y^{m-1}, and from the equations so obtained to eliminate linearly the $(m+n)$ quantities x^{m+n-1}, $x^{m+n-2}y\ldots$, y^{m+n-1}. But in the case of a greater number of equations it is not at first obvious how many new equations should be obtained; and when a number apparently sufficiently great have been found, it may happen that the equations so obtained are not independent, and that the elimination cannot be performed. But in showing the connexion that exists between these different equations, the theory of involution explains in what manner a system is to be formed, which includes all the really independent equations, and gives the means of detecting the extraneous factors which appear in the result of the linear elimination of the different terms; but I do not see at present any mode of obtaining the final result at once in its reduced form free from any extraneous factors.

Let X, Y, ... be given homogeneous functions of the same degree of any number of variables, and suppose

$$\Theta = \alpha X + \beta Y + \ldots,$$

α, β... being constants, and the number of terms in the series being g; Θ contains therefore g arbitrary constants. If however, by giving to α, β ... particular values α_1, β_1 ..., or α_2, β_2 ..., and representing by Θ_1, Θ_2 ... the corresponding values of Θ, we have *identically*

$$\Theta_1 = 0, \quad \Theta_2 = 0, \ldots (h \text{ equations});$$

then the constants in Θ group themselves together into a smaller number $g-h$ of arbitrary constants. This supposes, however, that the last mentioned equations are linearly independent; if there are a certain number k of equations

$$\Phi_1 = 0, \quad \Phi_2 = 0 \ldots,$$

(where Φ_1, Φ_2, ... are linear functions of Θ_1, Θ_2...) which are identically satisfied, independently of the h equations, then the equations in question are equivalent to $h-k$ equations, and the function Θ contains $g-(h-k)$ or $g-h+k$, arbitrary constants. Similarly if the functions Φ are not independent; so that the number of arbitrary constants really contained in Θ is always

$$N = g - h + k - \&c. \ldots$$

Consider now the case of a function Θ, homogeneous of the rth degree in the variables x, y...{$(\theta+1)$ in number}. Let U, V... be functions of the degrees m, n ..., and suppose

$$\Theta = uU + vV + \ldots$$

where u, v ... are arbitrary functions of the degrees $r-m$, $r-n$, ... {r is supposed throughout greater than m, n ...}. Suppose for shortness that the number of terms in the complete function of θ variables, and of the order ρ, i.e. the quotient $\frac{[\rho+\theta]^\theta}{[\theta]^\theta}$, is

represented by $[\rho, \theta]$; then the function Θ contains apparently a number

$$([r-m, \theta]+[r-n, \theta]+\ldots)$$

of arbitrary constants.

But since we should have identically $\Theta=0$ by assuming $u=LV$, $v=-LU$, $w=0$, &c.... (L the general function of the order $r-m-n$), or $u=MW$, $v=0$, $w=-MU$ (M the general function of the order $r-m-p$) &c., the number N must be diminished by

$$[r-m-n, \theta]+[r-m-p, \theta]+[r-n-p, \theta]+\ldots;$$

but the equations just obtained are themselves not linearly independent, and in consequence of this the number of arbitrary constants has to be increased by

$$[r-m-n-p, \theta]+\ldots;$$

and so on. Hence finally the whole number of arbitrary constants in the function θ is

$$\begin{aligned} N=&[r-m, \theta]+[r-n, \theta]+[r-p, \theta]+\ldots \\ &-[r-m-n, \theta]-[r-m-p, \theta]-[r-n-p, \theta]-\ldots \\ &\qquad +[r-m-n-p, \theta]+\ldots \pm \text{ \&c. \&c. } \ldots\ldots\ldots\ldots\ldots \text{(A)}. \end{aligned}$$

This however supposes that all the numbers $r-m$, $r-n\ldots$, $r-m-n\ldots$, are positive: whenever this is not the case for any one of them, the corresponding term is obviously to be omitted. With this convention the equation (A) gives always the correct number of arbitrary constants in Θ: it will be convenient to represent it in the abbreviated form

$$N=\{r : m, n, p, \ldots : \theta\}.$$

An expression analogous to this, for the particular case of $r=m$, but incorrect on account of the omission of all the terms after the second line, has been given by M. Plücker (*Crelle*, tom. XVI. p. 55), and even some of his particular formulæ are incorrect. But proceeding to examine some particular cases: if $r>m+n+p+\ldots-\theta-1$, then in the expression (A) either no terms are to be omitted, or else the terms to be omitted reduce themselves to zero, so that N is given by this formula continued to its last term. It will be subsequently shown that in this case

$$\{r : m, n, p \ldots : \theta\}=[r, \theta]-mnp \ldots;$$

or in the case of two or three variables, we have the theorem, "If a curve or surface of the order r be determined to pass through the mn points of intersection of two curves of the orders m and n, or the mnp points of intersection of three surfaces of the orders m, n, p; then if $r>m+n-3$, or $r>m+n+p-4$, the curve or surface contains precisely the same number of arbitrary constants as if the mn or mnp points were perfectly arbitrary."

This is natural enough; the peculiarity is in the case where $r \not> m+n-3$, or $r \not> m+n+p-4$. For instance, for two curves, $r \not> m+n-3$, we have

$$\{r : m, n : 2\}=[r-m, 2]+[r-n, 2]=[r, 2]-mn+[r-m-n, 2],$$

or the new curve contains $\frac{1}{2}[m+n-r-1]^2$ more arbitrary constants than it would do if the mn points, through which it was made to pass, had been perfectly arbitrary; a result given before in the *Journal*, [5].

In the case of surfaces, if $r \not> m+n+p-4$. Then assuming $r > m+n-4$, $m+p-4$, or $n+p-4$, we have

$$\begin{aligned}\{r : m, n, p : 3\} &= [r-m, 3]+[r-n, 3]+[r-p, 3] \\ &\quad -[r-m-n, 3]-[r-m-p, 3]-[r-n-p, 3] \\ &= [r, 3]-mnp-[r-m-n-p, 3];\end{aligned}$$

or the surface contains $\frac{1}{6}[m+n+p-r-1]^3$ more arbitrary constants than it would do if the mnp points, through which it was made to pass, had been perfectly arbitrary. Similarly, in the case where r is not greater than one or more of the quantities $m+n-4$, $m+p-4$, $n+p-4$. Thus in particular, if r be not greater than the least of these quantities

$$\begin{aligned}\{r : m, n, p : 3\} = [r, 3]-mnp+[r-n-p, 3]+[r-m-p, 3] \\ +[r-m-n, 3]-[r-m-n-p, 3];\end{aligned}$$

or the surface contains

$$\tfrac{1}{6}[m+n+p-r-1]^3-\tfrac{1}{6}[n+p-r-1]^3-\tfrac{1}{6}[m+p-r-1]^3-\tfrac{1}{6}[m+n-r-1]^3$$

more arbitrary constants than it would otherwise have done. Again, for a surface of the rth order, subjected to pass through the curve of intersection of two surfaces of the orders m, n,

$$\{r : m, n, 3\} = [r-m, 3]+[r-n, 3]-[r-m-n, 3];$$

in which the last term, or $\frac{1}{6}[m+n-r-1]^3$, is to be omitted when $r \not> m+n-4$.

The function of the rth order, which is satisfied by the systems of values which satisfy the equations of the orders m, n... contains, we have seen, $[r, m, n, p \ldots \theta]$ arbitrary constants; hence it may be determined so as to pass through this number, diminished by unity, of arbitrary points. But the equation being determined in general by the condition of being satisfied by $[r, \theta]-1$ systems of variables, it will be completely determined if, in addition to the above number of arbitrary systems, we suppose it to be satisfied by a number $N=[r, \theta]-\{r; m, n, p \ldots : \theta\}$ of systems satisfying the equations above. Hence the theorem

"The equation of the rth order which is satisfied by a number

$$N=[r, \theta]-\{r; m, n, p \ldots : \theta\}$$

of systems satisfying the equations of the orders m, n, p... is satisfied by any systems whatever which satisfy these equations."

In particular—"The surface of the rth order which passes through a number

$$[r, \theta]-\{r : m, n : \theta\}$$

of points in the curve of intersection of two surfaces of the orders m, n,—or through $[r, \theta] - \{r : m, n, p : \theta\}$ of the mnp points of intersection of three surfaces of the orders m, n, p,—passes through the curve of intersection, or through the mnp points of intersection."

Thus a surface of the second order which passes through eight points of the curve of intersection of two surfaces of the second order passes through this curve; and any surface of the second order which passes through seven of the points of intersection of three surfaces of the second order passes through the eighth point. (The first theorem obviously fails if the eight points have the relation in question, i.e. if they are the eight points of intersection of three surfaces of the second order.)

Again—"The curve of the rth order which passes through $[r, \theta] - \{r : m, n : \theta\}$ of the points of intersection of two curves of the orders mn, passes through the remaining points of intersection." e.g. "Any curve of the third order which passes through eight of the points of intersection of two curves of the third order, passes also through the ninth point."

Consider next the following question, which [as regards particular cases] has been treated of by Jacobi in the memoir already quoted (*Crelle*, tom. XV.). "To find the number of relations which must exist between $K(\theta+1)$ variables, forming K systems, each of which satisfies simultaneously equations of the orders m, n, p... respectively; the number ϕ of these equations being anything less than θ; or ϕ being equal to θ, provided at the same time $K = mnp$ ...."

Suppose $m \nless n$, $n \nless p$... and write

$$[m, \theta] - \{m : m, n, p \ldots : \theta\} = N,$$
$$[n, \theta] - \{n : n, p \ldots\ldots : \theta\} = N',$$
&c.

Imagine the equations of the orders n, p... given. Any function of the mth order which is satisfied by N of the systems of values which satisfy the given equations, and any particular equation of the mth order, is satisfied by the remaining $K-N$ systems of values. Hence assuming N systems, satisfying the equations of the orders n, p ... but otherwise arbitrary, the remaining systems must satisfy these equations, and a completely determinate equation of the mth order; i.e. there must be ϕ relations between the variables of each system, and consequently $\phi(K-N)$ relations in all. Similarly, if the equations of the orders p ... were given, N' systems of variables might be assumed satisfying these equations, but otherwise arbitrary; the remaining $N-N'$ systems satisfy $(\phi-1)$ determinate equations, or the number of relations between the variables is $(\phi-1)(N-N')\ldots$; continuing in the same manner the total number of relations between the variables is

$$\phi(K-N) + (\phi-1)(N-N') + (\phi-2)(N'-N'') + \ldots$$

in which however any term $(\phi-1)(N-N')$ or $(\phi-2)(N-N')$ ... &c., which becomes negative, *must be omitted*. It is obvious that we may write more simply

$$N = [m, \theta] - 1 - \{m; n, p \ldots \theta\},$$
$$N' = [n, \theta] - 1 - \{n; p \ldots \theta\}, \text{ \&c.}$$

In particular, to find the relations which must exist between the coordinates of mn points in order that they may be the points of intersection of two curves of the orders m, n respectively: here $K=mn$, $N=\frac{1}{2}[m+2]^2-\frac{1}{2}[m-n+2]^2=\frac{1}{2}(2mn-n^2+3n)$, $N'=\frac{1}{2}(n^2+3n+2)$, so that $N-N'=m(m-n)-1$ which becomes negative when $m=n$; hence in general the required number of conditions is $mn-3n+1$, but when $m=n$, the number in question becomes $(n-1)(n-2)$.

Passing to the case of surfaces; to determine the number of relations which must exist between the coordinates of mnp points, in order that they may be the points of intersection of surfaces of the orders m, n, p respectively. The number required is

$$3(mnp-N)+2(N-N')+(N'-N''),$$

where

$$N=[m,\ 3]-1-[m-n,\ 3]-[m-p,\ 3]+[m-n-p,\ 3]$$

(this last term to be omitted when $m<n+p-3$),

$$N'=[n,\ 3]-1-[n-p,\ 3],$$
$$N''=[p,\ 3]-1.$$

If, for instance, $m>n+p-3$, so as to retain the term $[m-n-p,\ 3]$, and $n>p$, so as to retain the term $N'-N''$, the number becomes, after all reductions,

$$2mnp+np^2-4np-2p^2-\tfrac{1}{3}(p-1)(p-2)(p-3),$$

a formula given by Jacobi. If, however, $n=p$, this number must be augmented by unity. Again, for $m<n+p-3$, the required number is

$$2mnp+np^2-4np-2p^2-\tfrac{1}{3}(p-1)(p-2)(p-3)$$
$$-\tfrac{1}{6}(n+p-m-1)(n+p-m-2)(n+p-m-3),$$

which however must be augmented by unity if $m=n$ or $n=p$, and by 3 if $m=n=p$. But without entering into further details about this part of the subject, which has been sufficiently illustrated by the examples that have been given, I pass on to notice the application of the above theory to the problem of elimination. Imagine $(\theta+1)$ equations between the $(\theta+1)$ variables, the first sides of these being, as before, rational and integral homogeneous functions of the variables of the orders m, n, p ... respectively. Writing $m+n+p\ldots-\theta=r$, and multiplying the first equation by all the terms of the form $x^\alpha y^\beta\ldots$ of the degree $r-m$, the second equation by all the terms of the same form, of the degree $r-n$, and so on, there result a certain number of equations, containing all the terms $x^\alpha y^\beta\ldots$ of the degree r. But these equations are not independent; and the reasoning in the former part of the present paper shows that the number of independent equations is given by the symbol $\{r:m,\ n,\ p\ldots:\theta\}$; the number of terms $x^\alpha y^\beta\ldots$ is evidently $[r,\ \theta]$; and it will be shown immediately that for the actual value of r,

$$[r,\ \theta]-\{r;\ m,\ n,\ p\ \ldots:\theta\}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{B});$$

so that the number of quantities to be linearly eliminated is precisely equal to the number of equations, or the elimination is always possible. I may mention also that,

supposing the coefficients of all the equations to be of the order unity, the order of the result, free from extraneous factors, may be shown to be

$$[r-m,\ \theta]+\ldots-2\{[r-m-n,\ \theta]+\ldots\}+3\{[r-m-n-p,\ \theta]+\ldots\}-\&c.$$
$$=mn\ldots+mp\ldots+np\ldots+\&c. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C),$$

(the equality of which will be presently proved) a result which agrees with that deduced from the theory of symmetrical functions; but I am not in possession of any mode of directly obtaining the final result in this its most simplified form. My method, which it is not necessary to explain here more particularly, leads me to the formation of a set of functions

$$P,\ Q,\ \ldots\ldots\ X,\ Y,\ Z,$$

θ in number, such that Z divides Y, this quotient divides X, and so on until we have a certain quotient which divides P, and this quotient equated to zero is the result of the elimination freed from extraneous factors. It only remains to demonstrate the formulæ (A), (B), and (C). Suppose in general that (k) denotes the sum of all the terms of the form $m^a n^b \ldots$, which can be formed with a given combination of k letters out of the ϕ letters $m,\ n,\ p\ldots$; and let $\Sigma(k)$ denote the sum of all the series (k) obtained by taking all the possible different combinations of k letters. It is evident that $\Sigma(k)$ is a multiple of (ϕ), $\{(\phi)$ denoting of course the sum of all the terms $m^a n^b \ldots,\ m,\ n\ldots$ being any letters whatever out of the series $m,\ n,\ p\ldots\}$. Let g be the number of exponents $a,\ b,\ \ldots$, then (ϕ) contains $[\phi]^g$ terms, also (k) contains $[k]^g$ terms, and the number of terms such as (k) in the sum $\Sigma(k)$ is $[\phi]^{\phi-k} \div [\phi-k]^{\phi-k}$. Hence evidently

$$\Sigma(k)=\frac{[\phi-g]^{\phi-k}}{[\phi-k]^{\phi-k}}(\phi),$$

or, what comes to the same thing,

$$\Sigma(\phi-k)=\frac{[\phi-g]^k}{[k]^k}(\phi).$$

Let A be an indeterminate coefficient, σ a summatory sign referring to different systems of exponents; then

$$\Sigma\sigma A(\phi-k)=\sigma\frac{[\phi-g]^k}{[k]^k}A(\phi),$$

or, giving to k the values $1,\ 2\ldots\phi$, multiplying each equation by an arbitrary coefficient, and adding, putting also for shortness $\sigma A(\phi-k)=U_{\phi-k}$, we have

$$\alpha_\phi U_\phi+\alpha_{\phi-1}\Sigma U_{\phi-1}+\ldots=\sigma\left(\alpha_\phi+\alpha_{\phi-1}\frac{[\phi-g]^1}{[1]^1}+\ldots\right)A(\phi);$$

whence in particular,

$$U_\phi-\Sigma U_{\phi-1}+\ldots=\sigma\{0^{\phi-g}A(\phi)\},$$
$$\Sigma U_{\phi-1}-2\Sigma U_{\phi-2}+\ldots=\sigma\{(\phi-g)\,0^{\phi-g-1}A(\phi)\},$$

which are still equations of considerable generality. If now $\phi = \theta$ and U_θ is a function of $m+n+p+\ldots$ of the order θ, the quantity $\sigma\{0^{\theta-g} A(\theta)\}$ reduces itself to the single term of U_θ which contains the product $mnp\ldots$. Hence, if

$$U_\theta = [\alpha + m + n + p \ldots, \theta]$$

in which afterwards $\alpha = r - m - n - p - \ldots$ we have the formula (A). Again, if $\phi = \theta + 1$, and $U_{\theta+1}$ is a function of $m+n+p\ldots$ of the order θ, the sum $\sigma\{0^{\theta+1-g} A(\phi)\}$ vanishes; whence writing $U_{\theta+1} = [m+n+p\ldots - \theta, \theta]$, we have the formula (B). Similarly, if in the second formula $\phi = \theta + 1$, and $U_{\theta+1}$ is a function of $m+n+p\ldots$ of the degree θ, then

$$\sigma\{(\theta + 1 - g)\, 0^{\theta-g} A(\theta+1)\},$$

reduces itself to the term which contains $mn\ldots + np\ldots + mp\ldots +$ &c.; whence, if

$$U_{\theta+1} = [m + n + p + \ldots - \theta, \theta],$$

we have the formula (C).

41.

ON CERTAIN FORMULÆ FOR DIFFERENTIATION WITH APPLICATIONS TO THE EVALUATION OF DEFINITE INTEGRALS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 122—128.]

In attempting to investigate a formula in the theory of multiple definite integrals (which will be noticed in the sequel), I was led to the question of determining the $(i+1)^{\text{th}}$ differential coefficient of the $2i^{\text{th}}$ power of $\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}$; the only way that occurred for effecting this was to find the successive differential coefficients of this quantity, which may be effected as follows. Assume

$$U_{k,i}=\{(x+\lambda)(x+\mu)\}^{\frac{1}{2}k}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i},$$

then

$$\begin{aligned}\frac{1}{U_{k,i}}\frac{d}{dx}U_{k,i}&=\tfrac{1}{2}k\frac{2x+\lambda+\mu}{(x+\lambda)(x+\mu)}-\frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}\\&=\tfrac{1}{2}k\frac{\{\sqrt{(x+\lambda)}+\sqrt{(x+\mu)}\}^2-2\sqrt{\{(x+\lambda)(x+\mu)\}}}{(x+\lambda)(x+\mu)}-\frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}\\&=\tfrac{1}{2}k\frac{(\lambda-\mu)^2}{\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^2(x+\lambda)(x+\mu)}-\frac{k+i}{\sqrt{\{(x+\lambda)(x+\mu)\}}};\end{aligned}$$

or, attending to the signification of $U_{k,i}$,

$$\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k(\lambda-\mu)^2U_{k-2,i-1}-(k+i)U_{k-1,i}.$$

Hence
$$-\frac{1}{i}\frac{d}{dx}U_{0,i}=U_{-1,i}$$

$$\frac{1}{i}\frac{d^2}{dx^2}U_{0,i}=\tfrac{1}{2}(\lambda-\mu)^2U_{-2,i-1}+(i-1)U_{-2,i},$$

&c.

from which the law is easily seen to be of the form

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta K_{r,\theta}(\lambda-\mu)^{2r-2-2\theta} U_{-2r+1+\theta,\, i-r+1+\theta}$$

(where the extreme values of θ are 0 and $(r-1)$ respectively) and $K_{r,\theta}$ is determined by

$$K_{r+1,\theta+1} = (r-1-\tfrac{1}{2}\theta) K_{r,\theta+1} + (i-3r+2+2\theta) K_{r,\theta}.$$

This equation is satisfied by

$$K_{r,\theta} = \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma\frac{1}{2}\,\Gamma(\theta+1)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)};$$

for in the first place this gives

$$(r-1-\tfrac{1}{2}\theta) K_{r,\theta+1} = \frac{(r-1-\frac{1}{2}\theta)}{\Gamma\frac{1}{2}\,\Gamma(\theta+2)} \frac{\Gamma(r-\frac{3}{2}-\theta)\,\Gamma(2r-2-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(2r-3-2\theta)\,\Gamma(i-r+1)}$$

$$= \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-2-2\theta)\,\Gamma(i-r+1)},$$

and hence the second side of the equation reduces itself to

$$\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}\{2(r-1-\theta)(i-r+\theta+1)+(\theta+1)(i-3r+2-2\theta)\},$$

where the quantity within brackets reduces itself to $(i-r)(2r-1-\theta)$, so that the above value reduces itself to $K_{r+1,\theta+1}$, which verifies the equation in question. Also by comparing the first few terms, it is immediately seen that the above is the correct value of $K_{r,\theta}$, so that

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+1)}(\lambda-\mu)^{2r-2-\theta} U_{-2r+1+\theta,\, i-r+\theta+1} \quad \ldots(1),$$

θ extending as before from 0 to $(r-1)$. In particular if i be integer and $r=i+1$,

$$\frac{(-)^{i+1}}{i}\left(\frac{d}{dx}\right)^{i+1}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i} = \frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i}\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}} \quad \ldots\ldots\ldots(2),$$

(since the factor $\Gamma(i-r+\theta+1) \div \Gamma(i-r+1)$ vanishes except for $\theta=0$ on account of $\Gamma(i-r+1)=\infty$). Thus also, if r be greater than $(i+1)$, $=i+1+s$ suppose, then

$$(-)^s\left(\frac{d}{dx}\right)^s \frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}}$$

$$= S_\theta \frac{\Gamma(i+s+\frac{1}{2}-\theta)\,\Gamma(2i+2s+1-\theta)\,\Gamma(\theta-s)}{\Gamma(i+\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2i+2s+1-2\theta)\,\Gamma(-s)}(\lambda-\mu)^{2s-2\theta} U_{-2i-2s-1+\theta,\, -s+\theta} \quad \ldots\ldots(3),$$

where θ extends only from $\theta=0$ to $\theta=s$, on account of the factor $\Gamma(\theta-s)\div\Gamma(-s)$, which vanishes for greater values of θ: a rather better form is obtained by replacing this factor by

$$(-)^\theta \frac{\Gamma(1+s)}{\Gamma(1+s-\theta)}.$$

The above formulæ have been deduced on the supposition of i being an integer; assuming that they hold generally, the equation (2) gives, by writing $(i-\frac{1}{2})$ for i,

$$\frac{(-)^{i+\frac{1}{2}}}{i-\frac{1}{2}}\left(\frac{d}{dx}\right)^{i+\frac{1}{2}}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i-1}=\frac{\Gamma i}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i-1}\frac{1}{\{(x+\lambda)(x+\mu)\}^i},$$

or integrating $(i+\frac{1}{2})$ times by means of the formula

$$\int_0^\infty x^{i-\frac{1}{2}} fx\,dx=\frac{\Gamma(i+\frac{1}{2})}{(-)^{i+\frac{1}{2}}}\left(\int_\infty d\alpha\right)^{i+\frac{1}{2}} f\alpha,\quad \alpha=0;$$

this gives

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{\{(x+\lambda)(x+\mu)\}^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}}\text{.....(4),}^1$$

whence also

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(x+\lambda)^{i+1}(x+\mu)^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i}\sqrt{\lambda}}\text{ .....(5);}$$

and from these, by simple transformations,

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{1}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\frac{(\alpha-\beta)^i}{(\sqrt{m}+1)^{2i}}\text{ .....(6),}$$

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{3}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{(\alpha-\beta)^{i-1}}{(\sqrt{m}+1)^{2i-1}}\text{ .....(7).}$$

These last two formulæ are connected also by the following general property:

"If $$(a,\ b,\ i)=\int_\beta^\alpha \frac{(\alpha-x)^{a-1}(x-\beta)^{b-1}dx}{\{(\alpha-x)+m(x-\beta)\}^i},$$

then $$(a,\ b,\ i)=\frac{\Gamma a\Gamma b}{\Gamma(a+b-i)\,\Gamma i}(\alpha-\beta)^{b-i}(a+b-i,\ i,\ \beta)\text{"....(8),}$$

which I have proved by means of a multiple[2] integral. From (6) we may obtain for $\gamma<1$,

$$\int_{-1}^1 \frac{(1-x^2)^{i-\frac{1}{2}}dx}{(1-2\gamma x+\gamma^2)^i}=\frac{\Gamma(\frac{1}{2})\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\text{ .....(9),}$$

[1] This is immediately transformed into

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(ax^2+bx+c)^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\frac{1}{\{b+2\sqrt{(ac)}\}^{i-\frac{1}{2}}},$$

which is a particular case of a formula which will be demonstrated in a subsequent paper. [I am not sure to what this refers.]

[2] [The triple integral $\iiint u^{i-1}x^{a-1}y^{\beta-1}e^{-(x+my)u-x-y}\,dx\,dy\,du$.]

which however is only a particular case of

$$\int_{-1}^{1} dx\,(1-x^2)^{i-\frac{1}{2}}\,(1-2\,\gamma x+\gamma^2)^{-i}\,\frac{d}{d\beta}\left[\beta^i\left(1-2\frac{\beta}{\gamma}x+\frac{\beta^2}{\gamma^2}\right)^{-i}\right]$$
$$=\frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\beta^{i-1}\,(1-\beta)^{-2i}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

which supposes γ and $\frac{\beta}{\gamma}$ each less than unity. This formula was obtained in the case of $(i+\frac{1}{2})$ an integer, from a theorem, *Leg. Cal. Int.*, tom. II. p. 258, but there is no doubt that it is generally true.

From (9), by writing $x=\cos\theta$, we have

$$\int_0^\pi \frac{\sin^{2i}\theta\,d\theta}{(1-2\gamma\cos\theta+\gamma^2)^i}=\frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

which may also be demonstrated by the common equation in the theory of elliptic functions $\sin(\phi-\theta)=\gamma\sin\phi$, as was pointed out to me by Mr [Sir W.] Thomson. It may be compared with the following formula of Jacobi's, *Crelle*, tom. XV. [1836] p. 7,

$$\int_0^\pi \frac{\sin^{2i-1}\theta\,d\theta}{(1-2\gamma\cos\theta+\gamma^2)^i}=\frac{1}{\Gamma(i+\frac{1}{2})}\int_0^\pi \frac{\cos(i-\frac{1}{2})\,\theta\,d\theta}{\sqrt{(1-2\gamma\cos\theta+\gamma^2)}}\ \ldots\ldots\ldots\ldots(12).$$

Consider the multiple integral

$$W=\int\frac{dx\,dy\ldots}{\{(x-a)^2+\ldots u^2\}^i}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

the number of variables being $(2i+1)$ (not necessarily odd), and the equation of the limits being

$$x^2+y^2\ldots=\xi;$$

then, as will presently be shown, W may be expanded in the form

$$W=\pi^{i+\frac{1}{2}}\,S_\lambda\,\frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}\left(\frac{d}{du}\right)^{2\lambda}\int_0^\infty \xi^{i-\frac{1}{2}}\,(\xi+u^2)^{-i}d\xi\ \ldots\ldots\ldots(14),$$

where $A=a^2+b^2+\ldots\ldots$ and λ extends from 0 to ∞. Suppose next

$$V=\int\frac{dx\,dy\ldots}{\{(x-a)^2\ldots+u^2\}^i\,(x^2+\ldots v^2)^{i+1}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15):$$

the number of variables as before, and the limits for each variable being $-\infty$, ∞. We have immediately

$$V=\int_0^\infty \frac{1}{(\xi+v^2)^{i+1}}\,\frac{dW}{d\xi}\,d\xi;$$

W as before, i.e.

$$V=\pi^{i+\frac{1}{2}}\,S_\lambda\,\frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}\left(\frac{d}{du}\right)^{2\lambda}\int_0^\infty \frac{\xi^{i-\frac{1}{2}}\,d\xi}{(\xi+u^2)^i\,(\xi+v^2)^{i+1}}.$$

But writing u^2, v^2 for λ, μ in the formula (5) (u and v being supposed positive), the integral in this formula is

$$\frac{\sqrt{\pi}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\frac{1}{v(u+v)^{2i}};$$

hence, after a slight reduction,

$$V=\frac{\pi^{i+1}}{v\Gamma(i+1)}\,S\,\frac{(-)^\lambda\,\Gamma(i+\lambda+1)}{\Gamma(i+1)\,\Gamma(\lambda+1)}\frac{A^\lambda}{\{(u+v)^2\}^\lambda};$$

or finally

$$V=\frac{\pi^{i+1}}{\Gamma(i+1)}\frac{1}{v\,\{(u+v)^2+A\}^i}\quad\text{.....(16)},$$

a remarkable formula, the discovery of which is due to Mr Thomson. It only remains to prove the formula for W. Out of the variety of ways in which this may be accomplished, the following is a tolerably simple one. In the first place, by a linear transformation corresponding to that between two sets of rectangular axes, we have

$$W=\int\frac{dx\,dy\ldots}{\{(x-\sqrt{A})^2+y^2\ldots+u^2\}^i};$$

or expanding in powers of A, and putting for shortness $R=x^2+y^2\ldots+u^2$, the general term of W is

$$(-)^\sigma A^\lambda\frac{\Gamma(i+\lambda+\sigma)}{\Gamma i\,\Gamma(\lambda-\sigma+1)\,\Gamma(2\sigma+1)}2^{2\sigma}\int x^{2\sigma}R^{-i-\lambda-\sigma}\,dx\,dy\ldots$$

the limits being as before $x^2+y^2+\ldots=\xi$. To effect the integrations, write $\sqrt{\xi}\sqrt{x}$, $\sqrt{\xi}\sqrt{y}$, &c. for x, y ... so that the equation of the limits becomes $x+y+\ldots=1$. Also restricting the integral to positive values, we must multiply it by 2^{2i+1}: the integral thus becomes

$$\xi^{\sigma+i+\frac{1}{2}}\int x^{\sigma-\frac{1}{2}}y^{-\frac{1}{2}}\ldots\{\xi(x+y\ldots)+u^2\}^{-i-\lambda-\sigma}\,dx\,dy\ldots$$

equivalent to

$$\xi^{\sigma+i+\frac{1}{2}}\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})}\int_0^1\theta^{i+\sigma-\frac{1}{2}}(\xi\theta+u^2)^{-i-\lambda-\sigma}\,d\theta;$$

i.e. to

$$\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})}\int_0\xi^{i+\sigma-\frac{1}{2}}(\xi+u^2)^{-i-\lambda-\sigma}\,d\xi.$$

Hence, after a slight reduction, the general term of W is

$$\frac{\pi^{i+\frac{1}{2}}}{\Gamma i}(-)^{\lambda+\sigma}A^\lambda\frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})}\int_0\xi^{i+\sigma-\frac{1}{2}}(\xi+u^2)^{-i-\lambda-\sigma}\,d\xi,$$

where σ may be considered as extending from 0 to λ inclusively, and then λ from 0 to ∞. But by a formula easily proved

$$\left(\frac{d}{du}\right)^{2\lambda}(\xi+u^2)^{-1}=\frac{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}{\Gamma i}\times$$

$$S\,(-)^\sigma\frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})}\xi^\sigma(\xi+u^2)^{-i-\lambda-\sigma},$$

where σ extends from 0 to λ. Hence, substituting and prefixing the summatory sign,

$$W = \pi^{i+\frac{1}{2}} S \frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)\, d\xi,$$

where λ extends from 0 to ∞, the formula required.

[I annex the following Note added in MS. in my copy of the *Journal*, and referring to the formula, *ante* p. 267; a is written to denote $\lambda - \mu$.

N.B.—It would be worth while to find the general differential coefficient of $U_{k,i}$.

$$\partial_x U_{k,i} = -(k+i)\, U_{k-1,i} + \tfrac{1}{2} k a^2\, U_{k-2,i-1},$$

from which it is easy to see that

$$\begin{aligned} \partial_x^r U_{k,i} = (-)^r \quad & K_{r,0} \quad U_{k-r,i} \\ & \vdots \\ + (-)^{r-\theta} & K_{r,\theta}\, a^{2\theta}\, U_{k-r-\theta,i-\theta} \\ & \vdots \\ + \quad & K_{r,r}\, a^{2r}\, U_{k-2r,i-r}. \end{aligned}$$

The general term of $\partial_x^{r+1} U_{k,i}$ is

$$\begin{aligned} (-)^{r-\theta} \quad & K_{r,\theta}\, a^{2\theta} [\tfrac{1}{2}(k-r-\theta)\, a^2 U_{k-r-\theta-2,i-\theta-1}] \\ + (-)^{r-\theta-1} & K_{r+\theta+1}\, a^{2\theta+2} [-(k+i-2\theta-2-r)\, U_{k-r-\theta-2,i-\theta-1}], \end{aligned}$$

which must be equal to

$$(-)^{r-\theta} \quad K_{r+1,\theta+1}\, a^{2\theta+2}\, U_{k-r-\theta-2,i-\theta-1},$$

therefore

$$K_{r+1,\theta+1} = (k+i-r-2\theta-2)\, K_{r,\theta+1} + \tfrac{1}{2}(k-r-\theta)\, K_{r,\theta}.$$

In particular

$$\begin{aligned} & K_{r+1,0} \quad -(k+i-r) \quad K_{r,0} = 0, \\ & K_{r+1,1} \quad -(k+i-r-2)\, K_{r,1} = \tfrac{1}{2}(k-r)\, K_{r,0}, \\ & \vdots \\ & K_{r+1,r+1} - (\tfrac{1}{2}k - r) \quad K_{r,r} = 0, \end{aligned}$$

whence

$$\begin{aligned} K_{r,0} \quad & = [k+i]^r \\ K_{r,1} \quad & = \tfrac{1}{2} r \{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\} [k+i-2]^{r-2}, \\ & \vdots \\ K_{r,r} \quad & = [\tfrac{1}{2}k]^r, \end{aligned}$$

which appears to indicate a complicated general law.

Even the verification of $K_{r,1}$ is long, thus the equation becomes

$$\overline{r+1}\, [k+i-2]^{r-1} \{k^2 + (i-r-1)\, k - \tfrac{1}{2} r i\} - (k+i-r-2)\, r \{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\} [k+i-2]^{r-2} = (k-r)\, [k+i]^r,$$

or

$$\overline{r+1}\, (k+i-r) \{k^2 + (i-r-1)\, k - \tfrac{1}{2} r i\} - r\, (k+i-r-2) \{k^2 + (i-r)\, k - \tfrac{1}{2}(r-1)\, i\} = (k-r)\, (k+i)\, (k+i-1),$$

which is identical, as may be most easily seen by taking first the coefficient of k^3, and then writing $k = r$, $k = -i$, $k = -i-1$.]

42.

ON THE CAUSTIC BY REFLECTION AT A CIRCLE.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 128—130.]

THE following solution of the problem is that given by M. de St-Laurent (*Annales de Gergonne*, t. XVII. [1826] pp. 128—134); the process of elimination is somewhat different.

The centre of the circle being taken for the origin, let k be its radius; a, b the coordinates of the luminous point; ξ, η those of the point at which the reflection takes place; x, y those of any point in the reflected ray: we have in the first place

$$\xi^2+\eta^2=k^2 \quad \text{.....(1).}$$

There is no difficulty in finding the equation of the reflected ray[1]; this is

$$(b\xi-a\eta)(\xi x+\eta y-k^2)+(y\xi-x\eta)(a\xi+b\eta-k^2)=0 \quad \text{.....(2),}$$

[1] To do this in the simplest way, write

$$\rho^2=(\xi-x)^2+(\eta-y)^2, \quad \sigma^2=(\xi-a)^2+(\eta-b)^2,$$

then, by the condition of reflection,

$$\rho+\sigma=\text{min.},$$

ρ, σ being considered as functions of the variables ξ, η, which are connected by the equation (1). Hence

$$\frac{\xi-x}{\rho}+\frac{\xi-a}{\sigma}+\lambda\xi=0,$$

$$\frac{\eta-y}{\rho}+\frac{\eta-b}{\sigma}+\lambda\eta=0;$$

or, eliminating λ,

$$\frac{\eta x-\xi y}{\rho}+\frac{\eta a-\xi b}{\sigma}=0,$$

whence

$$(\eta x-\xi y)^2[(\xi-a)^2+(\eta-b)^2]=(\eta a-\xi b)^2[(\xi-x)^2+(\eta-y)^2].$$

This may be written

$$\{(\eta x-\xi y)(\xi-a)-(\eta a-\xi b)(\xi-x)\}[(\eta x-\xi y)(\xi-a)+(\eta a-\xi b)(\xi-x)]$$
$$+\{(\eta x-\xi y)(\eta-b)-(\eta a-\xi b)(\eta-y)\}[(\eta x-\xi y)(\eta-b)+(\eta a-\xi b)(\eta-y)]=0;$$

the factors in { } reduce themselves respectively to ξP and ηP, where $P=\xi(b-y)-\eta(a-x)+ay-bx$; omitting the factor P, (which equated to zero, is the equation of the line through (a, b) and (ξ, η),) and replacing $\xi(\xi-a)+\eta(\eta-b)$ and $\xi(\xi-x)+\eta(\eta-y)$ by $k^2-a\xi-b\eta$ and $k^2-\xi x-\eta y$, respectively, we have the equation given above.

or, arranging the terms in a more convenient order,

$$(bx+ay)(\xi^2-\eta^2)+2(by-ax)\xi\eta-k^2(b+y)\xi+k^2(a+x)\eta=0 \quad\text{.....(2')}.$$

Hence, considering ξ, η as indeterminate parameters connected by the equation (1), the locus of the curve generated by the continued intersections of the lines (2) will be found by eliminating ξ, η, λ from these equations and the system

$$\xi[\lambda+2(bx+ay)]+\eta[2(by-ax)]-k^2(b+y)=0 \quad\text{.....(3)},$$

$$\xi[2(by-ax)]+\eta[\lambda-2(bx+ay)]+k^2(a+x)=0 \quad\text{.....(4)},$$

and from these, multiplying by ξ, η, adding and reducing by (2), we have

$$-\xi(b+y)+\eta(a+x)-\lambda=0 \quad\text{.....(5)},$$

which replaces the equation (2) or (2'). Thus the equations from which ξ, η, λ are to be eliminated are (1), (3), (4), (5).

From (3), (4), (5), by the elimination of ξ, η, we have

$$\begin{aligned}-\lambda\{\lambda^2-4(bx+ay)^2\}-4k^2(by-ax)(a+x)(b+y)\\ -k^2(a+x)^2[\lambda+2(bx+ay)]\\ -k^2(b+y)^2[\lambda-2(bx+ay)]\\ +4\lambda(by-ax)^2=0 \quad\text{.....(6)},\end{aligned}$$

or, reducing,

$$\begin{aligned}-\lambda^3+\lambda\{4(a^2+b^2)(x^2+y^2)-k^2[(a+x)^2+(b+y)^2]\}\\ -2k^2(bx-ay)(x^2+y^2-a^2-b^2)=0 \quad\text{.....(7)};\end{aligned}$$

which may be represented by

$$-\lambda^3+\lambda Q-2R=0 \quad\text{.....(7')}.$$

Again, from the equations (4), (3), transposing the last terms and adding the squares, also reducing by (1),

$$\begin{aligned}k^4[(a+x)^2+(b+y)^2]=k^2\lambda^2+4k^2(a^2+b^2)(x^2+y^2)\\ +4\lambda[(\xi^2-\eta^2)(bx+ay)+2\xi\eta(by-ax)] \quad\text{.....(8)};\end{aligned}$$

but from the same equations, multiplying by ξ, η and adding, also reducing by (1),

$$k^2\lambda+2(bx+ay)(\xi^2-\eta^2)+4\xi\eta(by-ax)+k^2[-\xi(b+y)+\eta(a+x)]=0 \quad\text{.....(9)},$$

or reducing by (5) and dividing by two,

$$k^2\lambda+(bx+ay)(\xi^2-\eta^2)+2\xi\eta(by-ax)=0 \quad\text{.....(10)}.$$

Using this to reduce (8),

$$k^2[(a+x)^2+(b+y)^2]=4(a^2+b^2)(x^2+y^2)+3\lambda^2 \quad \text{.....(11)},$$

or, from the value of P,

$$-3\lambda^2+Q=0 \quad \text{.....(12)},$$

which singularly enough is the derived equation of (7′) with respect to λ: so that the equation of the curve is obtained by expressing that two of the roots of the equation (7′) are equal. Multiplying (12) by λ and reducing by (7′),

$$-\lambda Q+3R=0,$$

or, combining this with (12),

$$27R^2-Q^3=0\,;$$

whence, replacing R, Q by their values, we find

$$27k^4(bx-ay)^2(x^2+y^2-a^2-b^2)^2-\{4(a^2+b^2)(x^2+y^2)-k^2[(a+x)^2+(b+y)^2]\}^3=0,$$

the equation of M. de St-Laurent.

43.

ON THE DIFFERENTIAL EQUATIONS WHICH OCCUR IN DYNAMICAL PROBLEMS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 210—219.]

JACOBI, in a very elaborate memoir, "Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi"[1], has demonstrated a remarkable property of an extensive class of differential equations, namely, that when all the integrals of the system except a single one are known, the remaining integral can always be determined by a quadrature. Included in the class in question are, as Jacobi proceeds to show, the differential equations corresponding to any dynamical problem in which neither the forces nor the equations of condition involve the velocities; i.e. in all ordinary dynamical problems, when all the integrals but one are known, the remaining integral can be determined by quadratures. In the case where the forces and equations of condition are likewise independent of the time, it is immediately seen that the system may be transformed into a system in which the number of equations is less by unity than in the original one, and which does not involve the time, which may afterwards be determined by a quadrature[2]; and, Jacobi's theorem applying to this new system, he arrives at the proposition "In any dynamical problem where the forces and equations of condition contain only the coordinates of the different points of the system, when all the integrals but two are determined, the remaining integrals may be found by quadratures only." In the following paper, which contains the demonstrations of these propositions, the analysis employed by Jacobi has been considerably varied in the details, but the leading features of it are preserved.

[1] *Crelle*, t. XXVII. [1844], pp. 199—268 and t. XXIX. [1845], pp. 213—279 and 333—376. Compare also the memoir in *Liouville*, t. X. [1845], pp. 337—346.

[2] For, representing the velocities by x', y' ... the dynamical system takes the form

$$dt : dx : dy \ldots : dx' : dy' \ldots = 1 : x' : y' \ldots : X : Y \ldots,$$

and the system in question is simply $dx : dy \ldots : dx' : dy' \ldots = x' : y' \ldots : X : Y \ldots$.

§ 1. Let the variables $x, y, z, \ldots$ &c. be connected with the variables $u, v, w, \ldots$ by the same number of equations, so that the variables of each set may be considered as functions of those of the other set. And assume

$$dx\,dy\ldots = \nabla\,du\,dv\ldots;$$

if from the functions which equated to zero express the relations between the two sets of variables we form two determinants, the former with the differential coefficients of these functions with respect to $u, v, \ldots$ and the latter with the differential coefficients of the same functions with respect to $x, y, \ldots$ the quotient with its sign changed obtained by dividing the first of these determinants by the second is, as is well known, the value of the function ∇.

Putting for shortness

$$\frac{dx}{du}=\alpha,\quad \frac{dy}{du}=\beta,\ldots;\quad \frac{dx}{dv}=\alpha',\quad \frac{dy}{dv}=\beta',\ldots \text{ \&c.}$$

and

$$\frac{du}{dx}=A,\quad \frac{du}{dy}=B,\ldots;\quad \frac{dv}{dx}=A',\quad \frac{dv}{dy}=B',\ldots$$

∇ is the reciprocal of the determinant formed with $A, B, \ldots; A', B', \ldots$, &c.; or it is the determinant formed with $\alpha, \beta, \ldots \alpha', \beta', \ldots$, &c.

From the first of these forms, i.e. considering ∇ as a function of $A, B, \ldots$

$$\frac{d\nabla}{dA}=-\nabla\alpha,\quad \frac{d\nabla}{dB}=-\nabla\beta,\ldots\quad \frac{d\nabla}{dA'}=-\nabla\alpha',\quad \frac{d\nabla}{dB'}=-\nabla\beta',\ldots$$

where the quantities $\alpha, \beta, \ldots \alpha', \beta', \ldots$ and $A, B, \ldots A', B', \ldots$ may be interchanged provided $-\nabla$ be substituted for ∇. (Demonstrations of these formulæ or of some equivalent to them will be found in Jacobi's memoir "De determinantibus functionalibus," *Crelle*, t. XXII. [(1841) pp. 319—359].)

Hence

$$\frac{1}{\nabla}d\nabla+\alpha dA+\beta dB+\ldots+\alpha' dA'+\beta' dB'+\ldots=0,$$

or reducing by

$$\frac{dA}{dy}=\frac{dB}{dx},\ldots;\quad \frac{dA'}{dy}=\frac{dB'}{dx},\ldots \text{ \&c.}$$

this becomes

$$\left.\begin{aligned}\frac{1}{\nabla}d\nabla+\alpha\left(\frac{dA}{dx}dx+\frac{dB}{dx}dy+\ldots\right)+\beta\left(\frac{dA}{dy}dx+\frac{dB}{dy}dy+\ldots\right)\ldots\\ +\alpha\left(\frac{dA'}{dx}dx+\frac{dB'}{dx}dy+\ldots\right)+\beta'\left(\frac{dA'}{dy}dx+\frac{dB}{dy}dy+\ldots\right)\ldots\\ \vdots\end{aligned}\right\}=0;$$

or, reducing,

$$\frac{1}{\nabla}\,d\nabla+\left(\frac{dA}{du}+\frac{dA'}{dv}+\ldots\right)dx+\left(\frac{dB}{du}+\frac{dB'}{dv}+\ldots\right)dy+\ldots=0\,;$$

whence separating the differentials and replacing A, A', ...; B, B', ...; by their values

$$\frac{1}{\nabla}\frac{d\nabla}{dx}+\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\ldots=0,$$

$$\frac{1}{\nabla}\frac{d\nabla}{dy}+\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\ldots=0,$$

$$\vdots$$

(in which $-\nabla$, u, v ...; x, y ... may be substituted for ∇, x, y ...; u, v ...).

§ 2. Let X, Y ... be any functions of the variables x, y, ... and assume

$$U=X\frac{du}{dx}+Y\frac{du}{dy}+\ldots\,;$$

$$V=X\frac{dv}{dx}+Y\frac{dv}{dy}+\ldots\,;$$

U, V, ... being expressed in terms of u, v, ... Then

$$\frac{dU}{du}+\frac{dV}{dv}+\ldots=X\left(\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\ldots\right)+Y\left(\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\ldots\right)\ldots$$

$$+\left(\frac{dX}{du}\cdot\frac{du}{dx}+\frac{dX}{dv}\cdot\frac{dv}{dx}+\ldots\right)+\left(\frac{dY}{du}\cdot\frac{du}{dy}+\frac{dY}{dv}\cdot\frac{dv}{dy}+\ldots\right)\ldots$$

i.e.

$$\nabla\left(\frac{dU}{du}+\frac{dV}{dv}+\ldots\right)=-\left(X\frac{d\nabla}{dx}+Y\frac{d\nabla}{dy}+\ldots\right)+\nabla\left(\frac{dX}{dx}+\frac{dY}{dy}+\ldots\right).$$

Also, whatever be the value of M,

$$U\frac{dM\nabla}{du}+V\frac{dM\nabla}{dv}+\ldots=X\frac{dM\nabla}{dx}+Y\frac{dM\nabla}{dy}+\ldots\,;$$

and from these two properties,

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}+\ldots=\nabla\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\ldots\right).$$

§ 3. Consider the system of differential equations

$$dx:dy:dz\ldots=X:Y:Z\ldots$$

(where, for greater clearness, an additional letter z has been introduced). From these we deduce the equivalent system

$$du:dv:dw\ldots=U:V:W\ldots.$$

Suppose that u and v continue to represent arbitrary functions of $x, y, z, \ldots$ but that the remaining functions $w, \ldots$ are such as to satisfy $W=0, \ldots$ (so that $w, \ldots$ may be considered as the constants introduced by obtaining all the integrals but one of the system of differential equations in $x, y, z, \ldots$), we have

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=\nabla\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots\right).$$

Also the only one of the transformed equations which remains to be integrated is

$$du : dv = U : V, \quad \text{or } Vdu - Udv = 0,$$

(in which it is supposed that U and V are expressed by means of the other integrals in terms of u and v).

Suppose M can be so determined that

$$\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots=0,$$

(M is what Jacobi terms the multiplier of the proposed system of differential equations): then

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=0,$$

or $M\nabla$ is the multiplier of $Vdu - Udv = 0$, so that

$$\int M\nabla\,(Vdu - Udv) = \text{const.}$$

Hence the theorem:—"Given a multiplier of the system of equations

$$dx : dy : dz, \ldots = X : Y : Z \ldots$$

(the meaning of the term being defined as above), then if all the integrals but one of this system are known, the remaining integral depends upon a quadrature."

Jacobi proceeds to discuss a variety of different systems of equations in which it is possible to determine the multiplier M. Among the most important of these may be considered the system corresponding to the general problem of Dynamics, which may be discussed under three different forms.

§ 4. Lagrange's first form[1].

Let the *whole series* of coordinates, each of them multiplied by the square root of the corresponding mass, be represented by $x, y, \ldots$ and in the same way the whole series of forces, each of them multiplied by the square root of the corresponding mass, by $P, Q, \ldots$; then the equations of motion are

$$\frac{d^2x}{dt^2}=X, \quad \frac{d^2y}{dt^2}=Y, \ldots;$$

[1] I have slightly modified the form so as to avoid the introduction of the masses, and to allow x (for instance) to stand for any one of the coordinates of any of the points, instead of standing for a coordinate parallel to a particular axis.

where

$$X = P + \lambda \frac{d\Theta}{dx} + \mu \frac{d\Phi}{dx} + \dots,$$

$$Y = Q + \lambda \frac{d\Theta}{dy} + \mu \frac{d\Phi}{dy} + \dots,$$

$$\vdots$$

where $\Theta = 0$, $\Phi = 0, \dots$ are the equations of condition connecting the variables, and λ, $\mu, \dots$ coefficients to be determined by substituting the values of $\frac{d^2x}{dt^2}$, &c. in the equations $\frac{d^2\Theta}{dt^2} = 0$, $\frac{d^2\Phi}{dt^2} = 0$, &c. It is supposed that as well P, Q, ... as Θ, Φ, ... are independent of the velocities.

In order to reduce these to an analogous form to that previously employed, we have only to write

$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \dots$$

which gives

$$\begin{aligned} dt : dx : dy : dz \dots : dx' : dy' : dz' \dots \\ = 1 : x' : y' : z' \dots : X : Y : Z \dots \end{aligned}$$

Supposing that M is independent of x', y', z', ... the equation on which it depends becomes immediately

$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \dots\right) = 0,$$

where for shortness

$$\delta = \frac{d}{dt} + x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} + \dots.$$

To reduce this we must first determine the values of λ, μ ..., and for this we have

$$\frac{d^2\Theta}{dt^2} = \delta^2\Theta + \frac{d\Theta}{dx}\frac{d^2x}{dt^2} + \frac{d\Theta}{dy}\frac{d^2y}{dt^2} + \dots \qquad = 0, \text{ \&c.}$$

i.e.

$$\delta^2\Theta + P\frac{d\Theta}{dx} + Q\frac{d\Theta}{dy} + \dots + a\lambda + h\mu + g\nu + \dots = 0,$$

$$\delta^2\Phi + P\frac{d\Phi}{dx} + Q\frac{d\Phi}{dy} + \dots + h\lambda + b\mu + f\nu + \dots = 0,$$

$$\delta^2\Psi + P\frac{d\Psi}{dx} + Q\frac{d\Psi}{dy} + \dots + g\lambda = f\mu + c\nu + \dots = 0.$$

$$\vdots$$

where for greater clearness an additional letter of the series Θ, Φ ... has been introduced, and where

$$a = \left(\frac{d\Theta}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \dots,$$

$$b = \left(\frac{d\Phi}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \dots,$$

$$\vdots$$

$$h = \left(\frac{d\Theta}{dx}\frac{d\Phi}{dx} + \frac{d\Theta}{dy}\frac{d\Phi}{dy}\right) + \dots.$$

$$\vdots$$

Hence differentiating with respect to x',

$$2\delta\frac{d\Theta}{dx} + a\frac{d\lambda}{dx'} + h\frac{d\mu}{dx'} + g\frac{d\nu}{dx'}\dots = 0,$$

$$2\delta\frac{d\Phi}{dx} + h\frac{d\lambda}{dx'} + b\frac{d\mu}{dx'} + f\frac{d\nu}{dx'}\dots = 0,$$

$$y\delta\frac{d\Psi}{dx} + g\frac{d\lambda}{dx'} + f\frac{d\mu}{dx'} + c\frac{d\nu}{dx'}\dots = 0;$$

$$\vdots$$

or representing by K the determinant formed with the quantities $a, h, g, \dots h, b, f, \dots g, f, c, \dots$ and by $A, H, G, \dots H, B, F, \dots G, F, C, \dots$ the inverse system of coefficients, we have

$$2\left(A\delta\frac{d\Theta}{dx} + H\delta\frac{d\Phi}{dx} + G\delta\frac{d\Psi}{dx}\dots\right) + K\frac{d\lambda}{dx'} = 0,$$

$$2\left(H\delta\frac{d\Theta}{dx} + B\delta\frac{d\Phi}{dx} + F\delta\frac{d\Psi}{dx}\dots\right) + K\frac{d\mu}{dx'} = 0,$$

$$2\left(G\delta\frac{d\Theta}{dx} + F\delta\frac{d\Phi}{dx} + C\delta\frac{d\Psi}{dx}\dots\right) + K\frac{d\nu}{dx'} = 0,$$

$$\vdots$$

whence, multiplying by $\frac{d\Theta}{dx}$, $\frac{d\Phi}{dx}$, $\frac{d\Psi}{dx}$, ... and adding,

$$A\delta\left(\frac{d\Theta}{dx}\right)^2 + B\delta\left(\frac{d\Phi}{dx}\right)^2 \dots + 2H\delta\frac{d\Theta}{dx}\frac{d\Phi}{dx}\dots + K\frac{dX}{dx'} = 0,$$

and, forming the similar equations with the remaining variables and adding,

$$A\delta a + B\delta b + C\delta c \dots + 2F\delta f + 2G\delta g + 2H\delta h + \dots + K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'}\dots\right) = 0;$$

i.e.

$$\delta K + K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \dots\right) = 0.$$

Thus the equation in M reduces itself to

$$K\delta M - M\delta K = 0,$$

which is satisfied by $M = K$. It may be remarked that K reduces itself to the sum of the squares of the different functional determinants formed with the differential coefficients of Θ, Φ ... with respect to the different combinations of as many variables out of the series x, y ....

§ 5. Lagrange's second form.

Here the equations of motion are assumed to be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} - P = 0,$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} - Q = 0,$$

$$\frac{d}{dt}\frac{dT}{dz'} - \frac{dT}{dz} - R = 0,$$

$$\vdots$$

where $2T$ represents the vis viva of the system, x, y, z, ... are the independent variables on which the solution of the problem depends, and x', y', z', ... their differential coefficients with respect to the time. It is assumed as before P, Q, R ... do not contain x', y', z', ...

Suppose these equations give

$$\begin{aligned} dt : dx : dy : dz \ldots : dx' : dy' : dz' \ldots \\ = 1 : x' : y' : z' \ldots : X : Y : Z \ldots; \end{aligned}$$

then the equation which determines the multiplier M takes as before the form

$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \ldots\right) = 0.$$

To reduce this equation, substituting for T its value which is of the form

$$T = \tfrac{1}{2}(ax'^2 + by'^2 + cz'^2 \ldots + 2fy'z' + 2gz'x' + 2hx'y' \ldots),$$

and putting for shortness

$$L = ax' + hy' + gz' \ldots,$$

$$M = hx' + by' + fz' \ldots,$$

$$N = gx' + fy' + cz' \ldots$$

$$\vdots$$

the equations which determine X, Y, Z ... are

$$aX + hY + gZ \ldots + \delta L - \frac{dT}{dx} - P = 0,$$

$$hX + bY + fZ \ldots + \delta M - \frac{dT}{dy} - Q = 0,$$

$$gX + fY + cZ \ldots + \delta N - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

Hence, differentiating with respect to x',

$$a\frac{dX}{dx'} + h\frac{dY}{dx'} + g\frac{dZ}{dx'} \ldots + \delta a \qquad\qquad = 0,$$

$$h\frac{dX}{dx'} + b\frac{dY}{dx'} + f\frac{dZ}{dx'} \ldots + \delta h + \frac{dM}{dx} - \frac{dL}{dy} = 0,$$

$$g\frac{dX}{dx'} + f\frac{dY}{dx'} + c\frac{dZ}{dx'} \ldots + \delta g + \frac{dN}{dx} - \frac{dL}{dz} = 0;$$

$$\vdots$$

or representing by K the determinant formed with $a, h, g, \ldots h, b, f, \ldots g, f, c, \ldots$ and by $A, H, G, \ldots H, B, F, \ldots G, F, C \ldots$ the inverse system of coefficients, we have

$$K\frac{dX}{dx'} + A\delta a + H\delta h + G\delta g \ldots + \quad * \quad + H\left(\frac{dM}{dx} - \frac{dL}{dy}\right) + G\left(\frac{dN}{dx} - \frac{dL}{dz}\right) \ldots = 0,$$

and similarly

$$K\frac{dY}{dy'} + H\delta h + B\delta b + F\delta f \ldots + H\left(\frac{dL}{dy} - \frac{dM}{dx}\right) + \quad * \quad + F\left(\frac{dN}{dy} - \frac{dM}{dz}\right) \ldots = 0,$$

$$K\frac{dZ}{dz'} + G\delta g + F\delta f + C\delta c \ldots + G\left(\frac{dL}{dz} - \frac{dN}{dx}\right) + F\left(\frac{dM}{dz} - \frac{dN}{dy}\right) + \quad * \quad \ldots = 0.$$

$$\vdots$$

Hence, adding,

$$K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right) + A\delta a + B\delta b + C\delta c \ldots + 2F\delta f + 2G\delta g + 2H\delta h \ldots = 0;$$

and thus we have as before, though with symbols bearing an entirely different signification,

$$K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \ldots\right) + \delta K = 0;$$

and thence $K\delta M - M\delta K = 0$, and $M = K$.

(The value of K *in this section* may I think be conveniently termed "the determinant of the vis viva," with respect to the variables $x, y, z, \ldots$. It may be remarked that "the determinant of the vis viva" with respect to any other system of variables $u, v, w, \ldots$ is $= \nabla^2 K$, ∇ as before.)

§ 6. Third form of the equations of motion. [Hamiltonian form.]

Here writing

$$\frac{dT}{dx'} = \xi, \ \frac{dT}{dy'} = \eta, \ldots,$$

and taking $t, x, y, \ldots \xi, \eta, \ldots$ for the variables of the problem [and considering T to be expressed as a function of these variables: to denote this change it would have been proper to use instead of T a new letter H] the equations of motion reduce themselves to

$$\left\{\begin{array}{ll} \dfrac{d\xi}{dt} = -\dfrac{dT}{dx} + P, & \dfrac{dx}{dt} = \dfrac{dT}{d\xi}, \\[2ex] \dfrac{d\eta}{dt} = -\dfrac{dT}{d\eta} + Q, & \dfrac{dy}{dt} = \dfrac{dT}{d\eta}; \\[2ex] \vdots & \vdots \end{array}\right.$$

or putting for shortness

$$\left\{\begin{array}{ll} P - \dfrac{dT}{dx} = X, & \dfrac{dT}{d\xi} = \Xi, \\[2ex] Q - \dfrac{dT}{d\eta} = Y, & \dfrac{dT}{d\eta} = \mathrm{H}, \\[2ex] \vdots & \vdots \end{array}\right.$$

they become

$$\begin{aligned} &dt : dx : dy : dz \ldots : d\xi : d\eta : d\zeta \ldots \\ &= 1 : \Xi : \mathrm{H} : \Omega \ldots : X : Y : Z \ldots. \end{aligned}$$

Hence writing the equation in M under the form

$$\delta M + M\left(\frac{d\Xi}{dx} + \frac{d\mathrm{H}}{dy} + \ldots + \frac{dX}{d\xi} + \frac{dY}{d\eta} + \ldots\right) = 0;$$

$$\left(\text{where } \delta = \frac{d}{dt} + \Xi\frac{d}{dx} + \mathrm{H}\frac{d}{dy} \ldots + X\frac{d}{d\xi} + Y\frac{d}{dy} + \ldots\right),$$

we see immediately that (P, Q ... being as before independent of the velocities, and consequently of $\xi, \eta, \zeta, \ldots$),

$$\frac{d\Xi}{dx} + \frac{dX}{d\xi} = 0, \ \frac{d\mathrm{H}}{dy} + \frac{dY}{d\eta} = 0, \ \&c.$$

Hence $\delta M = 0$, which is satisfied by $M = 1$.

44.

ON A MULTIPLE INTEGRAL CONNECTED WITH THE THEORY OF ATTRACTIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 219—223.]

MR BOOLE [in the Memoir "On a Certain Multiple definite Integral" Irish Acad. Trans. vol. XXI. (1848), pp. 40—150] has given for the integral with n variables

$$V=\int\frac{\phi\left(\frac{x^2}{f^2}+\frac{y^2}{g^2}+\dots\right)dx\,dy\dots}{[(a-x)^2+(b-y)^2\dots+u^2]^{\frac{1}{2}n+q}}\quad\dots\dots\dots\dots\dots\dots\dots(1);$$

limits
$$\frac{x^2}{f^2}+\frac{y^2}{g^2}+\dots\overset{=}{<}1,$$

the following formula, or one which may readily be reduced to that form[1],

$$V=\frac{fg\dots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_\eta^\infty\frac{Ss^{-q-1}\,ds}{\sqrt{\{(s+f^2)(s+g^2)\dots\}}}\quad\dots\dots\dots\dots\dots\dots\dots(2),$$

where

$$S=\frac{(1-\sigma)^{-q}}{\Gamma(-q)}\int_0^1 t^{-q-1}\phi\{\sigma+t(1-\sigma)\}\,dt\quad\dots\dots\dots\dots\dots\dots(3);$$

in which

$$\sigma=\frac{a^2}{f^2+s}+\frac{b^2}{g^2+s}\dots+\frac{u^2}{s}\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(4)$$

and η is determined by

$$1=\frac{a^2}{f^2+\eta}+\frac{b^2}{g^2+\eta}\dots+\frac{u^2}{\eta}.$$

[1] See note at the end of this paper.

Suppose $f = g = \ldots = \infty$; also assume

$$\phi(\lambda) = \frac{1}{(f^2\lambda + v^2)^{\frac{1}{2}n+q'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

then the integral becomes

$$U = \int \frac{dx\,dy\ldots}{(x^2+y^2\ldots+v^2)^{\frac{1}{2}n+q'}\{(x-a)^2+(y-b^2)\ldots+u^2\}^{\frac{1}{2}n+q}} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

the limits for each variable being $-\infty$, ∞.

Now, writing $f^2 s$ for s and $f^2\eta$ for η, the new value of η reduces itself to zero, and

$$U = \frac{f^{-2q}\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_0^\infty \frac{S\,ds}{(1+s)^{\frac{1}{2}n}}.$$

Also $\sigma = 0$; but

$$f^2\sigma = \frac{r^2}{1+s} + \frac{u^2}{s},$$

where $r^2 = a^2 + b^2 + \ldots$ whence also putting $\frac{t}{f^2}$ for t, $\phi\{\sigma + t(1-\sigma)\}$ becomes

$$\frac{1}{\{f^2\sigma + t(1-\sigma) + v^2\}^{\frac{1}{2}n+q'}};$$

i.e.

$$\frac{1}{(t+A)^{\frac{1}{2}n+q'}},$$

if for a moment

$$A = \frac{r^2}{1+s} + \frac{u^2}{s} + v^2.$$

Hence

$$S = \frac{f^{2q}}{\Gamma(-q)}\int_0^\infty \frac{t^{-q-1}\,dt}{(t+A)^{\frac{1}{2}n+q'}}$$

$$= \frac{\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q')}\frac{f^{2q}}{A^{\frac{1}{2}n+q+q'}};$$

or substituting in U, and replacing A by its value,

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')}\int_0^\infty \frac{s^{-q-1}\,ds}{(1+s)^{\frac{1}{2}n}\left(\frac{r^2}{1+s}+\frac{u^2}{s}+v^2\right)^{\frac{1}{2}n+q+q'}};$$

or, what comes to the same,

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')}\int_0^\infty \frac{s^{\frac{1}{2}n+q'-1}(1+s)^{q+q'}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n+q+q'}} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

where

$$j = u^2 + v^2 + r^2.$$

The only practicable case is that of $q' = -q$, for which

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)} \int_0^\infty \frac{s^{\frac{1}{2}n-q-1}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n}} \quad \text{.....(8).}$$

Consider the more general expression

$$\Theta = \int_0^\infty s^{-q-1}\,\phi\left(\frac{v^2s^2+js+u^2}{s}\right) ds \quad \text{.....(9);}$$

by writing

$$2u\sqrt{s} = \sqrt{(s'+4uv)} \pm \sqrt{s'},$$

the upper sign from $s = \infty$ to $s = \frac{u}{v}$, and the lower one from $s = \frac{u}{v}$ to $s = 0$, it is easy to derive

$$\Theta = (2v)^{2q} \int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q} + \{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,\phi(s+j+2uv)\,ds \quad \text{..... ..(10).}$$

Now, by a formula which will presently be demonstrated,

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q} + \{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,e^{-\theta s}\,ds$$

$$= \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\,\theta^{-q} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,e^{-\theta s}\,ds \quad \text{.....(11);}$$

whence

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q} + \{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,Fs\,.\,ds$$

$$= \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q} Fs\,.\,ds \quad \text{.....(12).}$$

Thus, by merely changing the function,

$$\Theta = \frac{2^{2q+1}\,v^{2q}\,\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q} \phi(s+j+2uv)\,ds \quad \text{.....(13);}$$

and hence in the particular case in question

$$U = \frac{2^{2q+1}\,v^{2q}\,\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)} \int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,(s+j+2uv)^{-\frac{1}{2}n+q}\,ds \quad \text{.....(14),}$$

by means of the formula

$$\left(-\frac{d}{ds}\right)^{-q} (s+\alpha)^{-\frac{1}{2}n} = \frac{\Gamma(\frac{1}{2}n-q)}{\Gamma(\frac{1}{2}n)}\,(s+\alpha)^{-\frac{1}{2}n+q}.$$

But as there may be some doubt about this formula, which is not exactly equivalent either to Liouville's or Peacock's expression for the general differential coefficient of a

power, it is worth while to remark that, by first transforming the $\frac{1}{2}n^{\text{th}}$ power into an exponential, and then reducing as above (thus avoiding the general differentiation), we should have obtained

$$U = \frac{2^{2q+1} v^{2q} \pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)} \int_0^\infty d\theta \int_0^\infty ds\, \theta^{\frac{1}{2}n-q-1} e^{-\theta(s+j+2uv)} s^{-\frac{1}{2}-q} (s+4uv)^{-\frac{1}{2}-q} e^{-\theta s},$$

which reduces itself to the equation (14) by simply performing the integration with respect to θ; thus establishing the formula beyond doubt[1]. The integral may evidently be effected in finite terms when either q or $q-\frac{1}{2}$ is integral. Thus for instance in the simplest case of all, or when $q=-\frac{1}{2}$,

$$U = \frac{\pi^{\frac{1}{2}(n-1)}}{v\Gamma\frac{1}{2}(n+1)} \frac{1}{(j+2uv)^{\frac{1}{2}(n-1)}} = \int_{-\infty}^{\infty} \frac{dx\,dy\ldots}{(x^2+y^2\ldots+v^2)^{\frac{1}{2}(n+1)}\,\{(x-a)^2+\ldots u^2\}^{\frac{1}{2}(n-1)}},$$

a formula of which several demonstrations have already been given in the *Journal*.

The following is a demonstration, though an indirect one, of the formula (11): in the first place

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\sqrt{(s+4uv)}} e^{-\theta s}\,ds$$

$$= \frac{2\Gamma(\frac{1}{2}-q)\,\theta^q e^{2uv\theta}}{\sqrt{\pi}\,(4uv)^{2q}} \int_0^\infty (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x}\,dx \quad \ldots\ldots\ldots\ldots(16),$$

(where as usual $i=\sqrt{-1}$): to prove this, we have

$$\int_{-\infty}^{\infty} (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x}\,dx = \frac{1}{\Gamma(\frac{1}{2}-q)} \int_{-\infty}^{\infty} dx \int_0^\infty dt\, t^{-\frac{1}{2}-q} e^{-t(4u^2v^2+x^2)+i\theta x}$$

$$= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)} \int_0^\infty dt\, t^{-1-q} e^{-4u^2v^2t-\frac{\theta^2}{4t}};$$

or, putting $4uv\sqrt{t}=\sqrt{(s+4uv)}\pm\sqrt{s}$ (which is a transformation already employed in the present paper), the formula required follows immediately.

Now, by a formula due to M. Catalan, but first rigorously demonstrated by M. Serret,

$$\int_0^\infty \frac{\cos ax\,dx}{(1+x^2)^n} = \frac{\pi}{(\Gamma n)^2} \int_0^\infty e^{-(a+2z)} (z+a)^{n-1} z^{n-1}\,dz,$$

(*Liouville*, t. VIII. [1843] p. 1), and by a slight modification in the form of this equation

$$\int_{-\infty}^{\infty} (4u^2v^2+x^2)^{q-\frac{1}{2}} e^{i\theta x}\,dx = \frac{\pi e^{-2uv\theta}(4uv)^{2q}}{\theta^{2q}\,\Gamma^2(\frac{1}{2}-q)} \int_0^\infty s^{-q-\frac{1}{2}} (s+4uv)^{-q-\frac{1}{2}} e^{-\theta s}\,ds,$$

which, compared with (16), gives the required equation.

[1] A paper by M. Schlömilch "Note sur la variation des constantes arbitraires d'une Integrale definie," *Crelle*, t. XXXIII. [1846], pp. 268—280, will be found to contain formulæ analogous to some of the preceding ones.

NOTE.—One of the intermediate formulæ of Mr Boole [in the Memoir referred to] may be written as follows:

$$S = \frac{1}{\pi} \int_0^1 d\alpha \int_0^\infty dv\, v^q \cos\left[(\alpha - \sigma) v + \tfrac{1}{2} q\pi\right] \phi\alpha,$$

or what comes to the same thing, putting $i = \sqrt{-1}$, and rejecting the impossible part of the integral,

$$S = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \int_0^1 d\alpha \int_0^\infty dv\, v^q e^{2iv(\alpha-\sigma)} \phi\alpha,$$

i.e.

$$S = \int_0^1 I\phi\alpha\, d\alpha, \quad I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \int_0^\infty dv\, v^q e^{2iv(\alpha-\sigma)}.$$

Now $(\alpha - \sigma)$ being positive,

$$I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \Gamma(q+1)\, e^{\frac{1}{2}(q+1)\pi i} (\alpha - \sigma)^{-q-1};$$

i.e.

$$I = \frac{1}{\pi} e^{(q+\frac{1}{2})\pi i} \Gamma(q+1) (\alpha - \sigma)^{-q-1},$$

or, retaining the real part only,

$$I = -\frac{1}{\pi} \sin q\pi\, \Gamma(q+1) (\alpha - \sigma)^{-q-1};$$

i.e.

$$I = \frac{1}{\Gamma(-q)} (\alpha - \sigma)^{-q-1}.$$

But $(\alpha - \sigma)$ being negative,

$$I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \Gamma(q+1)\, e^{-\frac{1}{2}(q+1)\pi i} (\sigma - \alpha)^{-q-1};$$

i.e.

$$I = \frac{1}{\pi} e^{-\frac{1}{2}\pi i} \Gamma(q+1) (\sigma - \alpha)^{-q-1},$$

or, retaining the real part only, $I = 0$.

Hence

$$S = \frac{1}{\Gamma(-q)} \int_\sigma^1 (\alpha - \sigma)^{-q-1} \phi\alpha\, d\alpha;$$

or putting $\alpha = \sigma + t(1-\sigma)$, or $\alpha - \sigma = t(1-\sigma)$,

$$S = \frac{(1-\sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1} \phi\left[\sigma + t(1-\sigma)\right] dt;$$

the expression in the text. Mr Boole's final value is

$$S = \left(-\frac{d}{d\sigma}\right)^q \phi(\sigma),$$

which, though simpler, appears to me to be in some respects less convenient.

45.

ON THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), pp. 256—266.]

ADOPTING the notation of the *Fund. Nova*, except that for shortness sn u, cn u, dn u are written instead of sinam u, cosam u, ∇ am u, let the functions $\Theta(u)$, $\mathrm{H}(u)$ be defined by the equations

$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\frac{1}{2}u^2\left(1-\frac{E}{K}\right)-k^2\int_0 du\int_0 du\,\mathrm{sn}^2 u} \quad \text{.....(1)},$$

$$\mathrm{H}u = -ie^{-\frac{\pi(K'-2iu)}{4K}}\,\Theta(u+iK') \quad \text{.....(2)},$$

it is required from these equations to express sn u in terms of the functions $\mathrm{H}(u)$, $\Theta(u)$. To accomplish this we have

$$\begin{aligned}\frac{d^2}{du^2}\log \mathrm{sn}\, u &= \frac{1}{\mathrm{sn}\, u}\frac{d^2}{du^2}\mathrm{sn}\, u - \frac{1}{\mathrm{sn}^2 u}\left(\frac{d}{du}\mathrm{sn}\, u\right)^2 \\ &= -(1+k^2)+2k^2\,\mathrm{sn}^2 u - \left\{\frac{1}{\mathrm{sn}^2 u}-(1+k^2)+k^2\,\mathrm{sn}^2 u\right\} \\ &= k^2\,\mathrm{sn}^2 u - \frac{1}{\mathrm{sn}^2 u};\end{aligned}$$

whence also

$$\frac{d^2}{du^2}\log \mathrm{sn}\, u = k^2\,\mathrm{sn}^2 u - k^2\,\mathrm{sn}^2(u+iK').$$

If for a moment

$$\psi_{,} u = \int_0 du\,\mathrm{sn}^2 u, \qquad \psi_{,,} u = \int_0 du\int_0 du\,\mathrm{sn}^2 u,$$

then

$$\log \mathrm{sn}\, u = k^2\psi_{,,} u - k^2\psi_{,,}(u+iK') + Au + B;$$

or writing $-u$ for u and subtracting, $\psi_{\prime\prime}u$ being an even function,

$$2Au = \pi i - k^2\psi_{\prime\prime}(iK' - u) + k^2\psi_{\prime\prime}(iK' + u),$$

or putting $u = K$,

$$2AK = \pi i - k^2\psi_{\prime\prime}(iK' - K) + k^2\psi_{\prime\prime}(iK' + K).$$

Now
$$\operatorname{sn}^2(u + K) - \operatorname{sn}^2(u - K) = 0,$$

and therefore
$$\psi_{\prime}(u + K) - \psi_{\prime}(u - K) = 2\psi_{\prime}K,$$
$$\psi_{\prime\prime}(u + K) - \psi_{\prime\prime}(u - K) = 2u\psi_{\prime}K;$$

or
$$\psi_{\prime\prime}(iK' + K) - \psi_{\prime\prime}(iK' - K) = 2iK'\psi_{\prime}K.$$

Also
$$E(u) = u - k^2\psi_{\prime}u,$$

or
$$E = K - k^2\psi_{\prime}K, \quad \text{i.e.} \quad \psi_{\prime}K = \frac{K}{k^2}\left(1 - \frac{E}{K}\right).$$

Hence

$$A = iK'\left(1 - \frac{E}{K}\right) + \frac{\pi i}{2K},$$

$$\log \operatorname{sn} u = k^2\psi_{\prime\prime}u - k^2\psi_{\prime\prime}(u + iK') + uiK'\left(1 - \frac{E}{K}\right) + \frac{\pi ui}{2K} + B$$

$$= k^2\psi_{\prime\prime}u - k^2\psi_{\prime\prime}(u + iK') + \tfrac{1}{2}\left[(u + iK')^2 - u^2\right]\left(1 - \frac{E}{K}\right) + \frac{\pi ui}{2K} + B',$$

i.e.
$$\log \operatorname{sn} u = \log \Theta(u + iK') - \log \Theta u + \frac{\pi ui}{2K} + B',$$

or, changing the constant,

$$\operatorname{sn} u = Ce^{\frac{\pi ui}{2K}} \frac{\Theta(u + iK')}{\Theta u}.$$

Now, to determine C, write $u - iK'$ for u; this gives

$$\frac{1}{k \operatorname{sn} u} = Ce^{\frac{\pi i}{2K}(u - iK')} \frac{\Theta u}{\Theta(u - iK')};$$

and again changing u into $-u$,

$$-\operatorname{sn} u = Ce^{-\frac{\pi ui}{2K}} \frac{\Theta(u - iK')}{\Theta u};$$

whence, multiplying these last two equations,

$$C^2 = -\frac{1}{k} e^{-\frac{\pi K'}{2K}},$$

or
$$C = \frac{1}{i\sqrt{k}} e^{-\frac{\pi K'}{4K}};$$

whence
$$\operatorname{sn} u = \frac{1}{i\sqrt{k}} e^{-\frac{\pi(K'-2iu)}{4K}} \frac{\Theta(u+iK')}{\Theta u},$$

i.e.
$$\sqrt{k}\operatorname{sn} u = \frac{\mathrm{H}(u)}{\Theta(u)} \quad \text{.....} \quad (3);$$

and the equations (1), (2) and (3) may be considered as comprehending the theory of the functions $\mathrm{H}(u)$, $\Theta(u)$. The preceding process is, in fact, the converse of that made use of in the *Fund. Nova;* Jacobi having obtained for $\operatorname{sn} u$ an expression in the form of a fraction, takes the numerator of it for $\mathrm{H}(u)$ and the denominator for $\Theta(u)$, and thence deduces the equations (1), (2), the intermediate steps of the demonstration being conducted by means of infinite series; the necessity of which is avoided by the preceding investigation.

I proceed to investigate certain results relating to these functions, and to the theory of elliptic functions which have been given by Jacobi in two papers, "Suite des notices sur les fonctions elliptiques," *Crelle,* t. III. [1828] p. 306, and t. IV. [1829] p. 185, but without demonstration.

In the first place, the equation
$$\frac{d^2\Sigma}{du^2} - 2u\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2kk'^2\frac{d\Sigma}{dk} = 0 \quad \text{.....} (4)$$

is satisfied by $\Sigma = \Theta(u)$ or $\Sigma = \mathrm{H}(u)$. It will be sufficient to prove this for $\Sigma = \Theta(u)$, since a similar demonstration may easily be found for the other value. The following preliminary formulæ will be required:
$$k\frac{dK}{dk} = \frac{E}{k'^2} - K, \qquad k\frac{dE}{dk} = E - K,$$
$$k\frac{dK'}{dk} = -\frac{E'}{k'^2} + \frac{k^2K}{k'^2}, \qquad KK' - EK' - E'K = -\tfrac{1}{2}\pi,$$

which are all of them known.

Now, writing $\Theta(u)$ under the slightly more convenient form
$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\int_0 du \int_0 du\, \mathrm{dn}^2 u - \frac{1}{2}u^2\frac{E}{K}},$$

we have
$$\frac{d\,\Theta u}{du} = \left(\int_0 du\, \mathrm{dn}^2 u - \frac{E}{K}u\right)\Theta u = \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, \mathrm{cn}^2 u\right\}\Theta u,$$
$$\frac{d^2\,\Theta u}{du^2} = \left[\mathrm{dn}^2 u - \frac{E}{K} + \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, \mathrm{cn}^2 u\right\}^2\right]\Theta u,$$
$$\frac{d\,\Theta u}{dk} = \left[\frac{1}{2Kk'}\frac{dKk'}{dk} - \tfrac{1}{2}u^2\frac{d}{dk}\frac{E}{K} + \int_0 du \int_0 du\, \frac{d}{dk}\mathrm{dn}^2 u\right]\Theta u.$$

The success of the process depends upon a transformation of the double integral

$$\int_0 du \int_0 du \frac{d}{dk} \operatorname{dn}^2 u\,;$$

to effect this we have

$$\frac{d}{dk} \operatorname{dn}^2 u = -2k \operatorname{sn} u \left(\operatorname{sn} u + k \frac{d}{dk} \operatorname{sn} u\right);$$

but, by a known formula,

$$k'^2 \frac{d}{dk} \operatorname{sn} u = -k \operatorname{cn} u \operatorname{dn} u \int_0 \operatorname{cn}^2 u\, du + k \operatorname{cn}^2 u \operatorname{sn} u\,;$$

whence

$$\operatorname{sn} u + k \frac{d}{dk} \operatorname{sn} u = \frac{1}{k'^2} \operatorname{sn} u \operatorname{dn}^2 u - k^2 \operatorname{cn} u \operatorname{dn} u \int_0 du \operatorname{cn}^2 u,$$

or

$$\frac{d}{dk} \operatorname{dn}^2 u = -\frac{2k}{k'^2} (\operatorname{sn}^2 u \operatorname{dn}^2 u - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \int_0 du \operatorname{cn}^2 u)$$

$$= -\frac{2k}{k'^2} \left\{\operatorname{sn}^2 u \operatorname{dn}^2 u + \tfrac{1}{2} k^2 \left(\frac{d}{du} \operatorname{cn}^2 u\right) \int_0 du \operatorname{cn}^2 u\right\};$$

whence

$$\int_0 du \int_0 du \frac{d}{dk} \operatorname{dn}^2 u = -\frac{2k}{k'^2} \left\{\int_0 du \int_0 du \operatorname{sn}^2 u \operatorname{dn}^2 v + \tfrac{1}{2} k^2 \int_0 du (\operatorname{cn}^2 u \int du \operatorname{cn}^2 u - \int du \operatorname{cn}^4 u)\right\}$$

$$= -\frac{k}{k'^2} \left\{\int_0 du \int_0 du (2 \operatorname{sn}^2 u \operatorname{dn}^2 u - k^2 \operatorname{cn}^4 u) + \tfrac{1}{2} k^2 (\int_0 du \operatorname{cn}^2 u)^2\right\}.$$

But

$$\frac{d^2}{du^2} \operatorname{sn}^2 u = 2 (\operatorname{cn}^2 u \operatorname{dn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u) = 2 (k'^2 - 2 \operatorname{sn}^2 u \operatorname{dn}^2 u + k^2 \operatorname{cn}^4 u)\,;$$

or, integrating,

$$\operatorname{sn}^2 u = k'^2 u^2 - 2 \int_0 du \int_0 du (2 \operatorname{sn}^2 u \operatorname{dn}^2 u - k^2 \operatorname{cn}^4 u)\,;$$

whence at length

$$\int_0 du \int_0 du \frac{d}{dk} \operatorname{dn}^2 u = -\tfrac{1}{2} k u^2 + \tfrac{1}{2} \frac{k}{k'^2} \operatorname{sn}^2 u + \frac{k^3}{2k'^2} (\int_0 du \operatorname{cn}^2 u)^2.$$

Also

$$\frac{d}{dk} Kk' = \frac{E - K}{kk'}, \quad \frac{d}{dk} \frac{E}{K} = \frac{1}{kk'^2} \left\{k'^2 \left(\frac{2E}{K} - 1\right) - \frac{E^2}{K^2}\right\},$$

so that

$$\frac{d\,\Theta u}{dk} = \frac{1}{2kk'^2} \left\{\frac{E}{K} - \operatorname{dn}^2 u - u^2 \left(k'^2 - \frac{E}{K}\right)^2 - k^4 (\int du \operatorname{cn}^2 u)^2\right\} \Theta u.$$

Substituting these values of $\frac{d}{du}\Theta u$, $\frac{d^2}{du^2}\Theta u$ and $\frac{d}{dk}\Theta u$ in the equation (4) in the place of the corresponding differential coefficients of Σ, all the terms vanish, or the equation is satisfied by $\Sigma = \Theta(u)$, and similarly it would be satisfied by $\Sigma = \mathrm{H}(u)$.

Assume now

$$\omega = \frac{\pi K'}{K}, \quad v = \frac{\pi u}{2K};$$

then observing the equation

$$\frac{d}{dk}\frac{K'}{K} = \frac{1}{K^2kk'^2}(KK' - KE' - K'E) = -\frac{\pi}{2K^2kk'^2},$$

we have

$$\frac{d\Sigma}{du} = \frac{\pi}{2K}\frac{d\Sigma}{dv}, \quad \frac{d^2\Sigma}{du^2} = \frac{\pi^2}{4K^2}\frac{d^2\Sigma}{dv^2},$$

$$\frac{d\Sigma}{dk} = \frac{v}{kk'^2}\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{dv} - \frac{\pi^2}{2K^2kk'^2}\frac{d\Sigma}{d\omega};$$

whence, substituting in the equation (4), this becomes

$$\frac{d^2\Sigma}{dv^2} - 4\frac{d\Sigma}{d\omega} = 0 \quad \text{.....(5)},$$

which is of course satisfied as before by $\Sigma = \Theta(u)$, or $\Sigma = \mathrm{H}(u)$, an equation demonstrated in a different manner (by means of expansions) by Jacobi in the Memoirs referred to.

Consider next the equation

$$\frac{d^2\Sigma}{du^2} - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2nkk'^2\frac{d\Sigma}{dk} = 0 \quad \text{.....(6)},$$

(n being any positive integer number). Then, by assuming

$$\omega = n\frac{\pi K'}{K}, \quad v = \frac{n\pi u}{K},$$

we should be led as before to the equation (5). Hence, considering Θu or $\mathrm{H}u$ as functions of u and $\frac{K'}{K}$, the equation (6) is satisfied by assuming for Σ a corresponding function of nu and $\frac{nK'}{K}$. Let λ be the modulus corresponding to a transformation of the nth order; then Λ, Λ' being the complete functions corresponding to this modulus, $\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}$, so that the equation (6) will be satisfied by assuming $\Sigma = \Theta_{,}(nu)$ or $\Sigma = \mathrm{H}_{,}(nu)$, where $\Theta_{,}$, $\mathrm{H}_{,}$ correspond to the new modulus λ.

Assume now in the equation (6),

$$\Sigma = \left(\frac{\pi}{2}\right)^{\frac{1}{2}(n-1)} (Kk')^{-\frac{1}{2}(n-1)} \Theta^n u \,.\, z.$$

Hence, substituting,

$$\frac{d^2}{du^2}(\Theta^n u \,.\, z) - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d}{du}(\Theta^n u \,.\, z) + 2nkk'^2 (Kk')^{\frac{1}{2}(n-1)} \frac{d}{dk}\left[(Kk')^{-\frac{1}{2}(n-1)} \Theta^n u \,.\, z\right] = 0\,;$$

but
$$(Kk')^{\frac{1}{2}(n-1)} \frac{d}{dk}\left[(Kk')^{-\frac{1}{2}(n-1)} \Theta^n u \,.\, z\right] = \frac{d}{dk}(\Theta^n u \,.\, z) - \frac{n-1}{2Kk'}\frac{dKk'}{dk}\Theta^n u \,.\, z,$$

or effecting the differentiation, and eliminating $\frac{d\Theta u}{dk}$ by means of the equation obtained from (4) by writing $\Sigma = \Theta u$,

$$(Kk')^{\frac{1}{2}(n-1)} \frac{d}{dk}\left[(Kk')^{-\frac{1}{2}(n-1)} \Theta^n u \,.\, z\right]$$
$$= \Theta^n u \left[\frac{dz}{dk} - \frac{nz}{2kk'^2 \Theta u}\left\{\frac{d^2 \Theta u}{du^2} - 2\left(k'^2 - \frac{E}{K}\right)\frac{d\Theta u}{du}\right\} + \frac{n-1}{2kk'^2}\left(1 - \frac{E}{K}\right) z\right].$$

Substituting in (6) and reducing,

$$\frac{d^2 z}{du^2} + 2n\left[\frac{1}{\Theta n}\frac{d\Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2n\,kk'^2 \frac{dz}{dk}$$
$$+ n(n-1)\left\{\left[\frac{1}{\Theta^2 u}\left(\frac{d\Theta u}{du}\right)^2 - \frac{1}{\Theta u}\frac{d^2 \Theta u}{du^2}\right] + \left(1 - \frac{E}{K}\right)\right\} z = 0,$$

i.e.
$$\frac{d^2 z}{du^2} + 2n\left[\frac{d \log \Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2n\,kk'^2 \frac{dz}{dk}$$
$$+ n(n-1)\left[-\frac{d^2 \log \Theta u}{du^2} + \left(1 - \frac{E}{K}\right)\right] z = 0.$$

But

$$\frac{d \log \Theta u}{du} = u\left(k'^2 - \frac{E}{K}\right) = k^2 \int_0 du \,\mathrm{cn}^2 u,$$

$$\frac{d^2 \log \Theta u}{du^2} = 1 - \frac{E}{K} - k^2 \,\mathrm{sn}^2 u\,;$$

whence

$$\frac{d^2 z}{du^2} + 2nk^2 \left(\int_0 du \,\mathrm{cn}^2 u\right)\frac{dz}{du} + 2nkk'^2 \frac{dz}{dk} + n(n-1)\,k^2 \,\mathrm{sn}^2 u \,.\, z = 0 \ldots\ldots\ldots\ldots (7)\,;$$

which is therefore satisfied by

$$z = \left(\frac{2Kk'}{\pi}\right)^{\frac{1}{2}(n-1)} \frac{\Theta nu}{\Theta^n u}, \quad z = \left(\frac{2Kk'}{\pi}\right)^{\frac{1}{2}(n-1)} \frac{\mathrm{H} nu}{\Theta^n u}:$$

and each of these values is an algebraical function of $\operatorname{sn} u$, (viz. either a rational function or a rational function multiplied by $\operatorname{cn} u \operatorname{dn} u$). Also, in the transformation of the nth order,

$$\sqrt{\lambda}\operatorname{sn}_{,} u = \frac{\mathrm{H}_{,}(nu)}{\Theta_{,}(nu)};$$

so that it is clear that the above values of z may be taken for the denominator and numerator respectively of $\sqrt{\lambda}\operatorname{sn}_{,} u$; i.e. these quantities each of them satisfy the equation (7).

By assuming

$$x = \sqrt{k}\operatorname{sn} u, \quad \alpha = k + \frac{1}{k},$$

this becomes

$$n(n-1)x^2 z + (n-1)(\alpha x - 2x^3)\frac{dz}{dx} + (1 - \alpha x^2 + x^4)\frac{d^2 z}{dx^2} - 2n(\alpha^2 - 4)\frac{dz}{d\alpha} = 0 \quad \ldots\ldots\ldots(8);$$

which is therefore satisfied by assuming for z either the numerator or the denominator of $\sqrt{\lambda}\operatorname{sn}_{,} u$ (the transformation of the nth order), which is the form in which the property is given by Jacobi.

In the case where n is odd, the denominator is of the form

$$B_0 + B_{,} x^2 \ldots + B_{\frac{1}{2}(n-1)} x^{n-1},$$

and then the numerator is

$$x(B_{\frac{1}{2}(n-1)} \ldots + B_{,} x^{n-3} + B_0 x^{n-1}),$$

where

$$B_0 = \sqrt{\left(\frac{\lambda'}{k'M}\right)}, \quad B_{\frac{1}{2}(n-1)} = \sqrt{\left(\frac{\lambda\lambda'}{kk'M^2}\right)};$$

and all the remaining coefficients may be determined from these, the modular equation being supposed known. But the principal use of the formula is for the multiplication of elliptic functions, which it is well known corresponds to the case where n is a square number. Writing $n = \nu^2$, when ν is odd, the denominator is

$$1 + B_2 x^4 \ldots + B_{\frac{1}{2}(\nu^2-3)} x^{\nu^2-3} \pm \nu x^{\nu^2-1},$$

(the $\pm$ sign according as $\nu = (4p+1)$ or $(4p-1)$); and the numerator is obtained from this by multiplying by x and reversing the order of the coefficients. When ν is even the denominator is

$$1 + B_2 x^4 \ldots \pm B_2 x^{\nu^2-4} \pm x^{\nu^2},$$

($+$ or $-$, according as $\nu = 4p$ or $\nu = 4p+2$), so that there are only half as many coefficients to be determined; but then the numerator must be separately investigated.

In general, by leaving n indeterminate, and integrating in the form of a series arranged according to ascending powers of x^2; then, whenever n is a square number, the series terminates and gives the denominator of the corresponding formula of multiplication; but the general form of the coefficients has not hitherto been discovered.

By writing $\frac{x}{\sqrt{n}}$ instead of x, and then making n infinite, the equation (8) takes the form

$$x^2 z + \alpha x \frac{dz}{dx} + \frac{d^2 z}{dx^2} - 2(\alpha^2 - 4)\frac{dz}{d\alpha} = 0 \quad\text{.....(9):}$$

and it is worth while, before attempting the solution of the general case, to discuss this more simple one[1].

Assume

$$z = 1 + C_1 \frac{x^2}{1.2} \dots + C_r \frac{x^{2r}}{1.2 \dots 2r} + \dots;$$

then it is easy to obtain

$$C_{r+2} = -(2r+1)(2r+2)C_r - (2r+2)\alpha C_{r+1} + 2(\alpha^2 - 4)\frac{dC_{r+1}}{d\alpha}.$$

The general form may be seen to be

$$C_r = (-)^{r+1}\{2^{2r-3} C_r^1 \alpha^{r-2} + 2^{2r-6} C_r^2 \alpha^{r-4} + \dots\},$$

and then

$$C_{r+1}^p - pC_r^p = -r(2r-1)C_{r-1}^{p-1} + 16(r+2-2p)C_r^{p-1}.$$

The complete value of C_r^p (assuming $C_r^0 = 0$) is given by an equation of the form

$$C_r^p = {}^0C_r^p + {}^1C_r^p\, 2^r + {}^2C_r^p\, 3^r \dots + {}^{p-1}C_r^p\, p^r,$$

where ${}^0C_r^p$, ${}^1C_r^p$, ... are algebraical functions of r of the degrees $2p-2$, $2p-4$, &c. respectively; but as I am not able completely to effect the integration, and my only object being to give an idea of the law of the successive terms, it will be sufficient to consider the first or algebraical term ${}^0C_r^p$, which is determined by the same equation as C_r^p, and is moreover completely determined by this equation and the single additional

[1] Writing $(\beta+2)$ for α, and putting $z = e^{\frac{1}{2}x^2}\rho$, this becomes

$$\frac{d^2\rho}{dx^2} - \rho = \beta x^2 \rho - \beta x \frac{d\rho}{dx} + (8\beta + 2\beta^2)\frac{d\rho}{d\beta};$$

and if $\rho = \Sigma Z_n \beta^n$,

$$\frac{d^2 Z_n}{dx^2} - (8n+1)Z_n = \left(x^2 + 2n - 2 - x\frac{d}{dx}\right) Z_{n-1};$$

from which the successive values of Z_0, Z_1, &c. might be calculated.

relation $C_r^1 = 1$, since the arbitrary constants of the integration affect only the terms multiplied by 2^r, 3^r, &c.

Assume C_r^p

$$= \frac{1}{[p-1]^{p-1}} \{2^{p-1} L^p [r-2]^{2p-2} + 2^{p-2} M^p [r-3]^{2p-3} + \ldots 2^{-p+1} X^p [r-2p]^0\};$$

and substituting this value,

$$\begin{aligned}
(1-p) L^p &= (1-p) \{L^{p-1} \qquad\qquad\qquad\qquad\}, \\
(1-p) M^p - 2p(2-2p) L^p &= (1-p) \{M^{p-1} - 11 L^{p-1} \qquad\qquad\}, \\
(1-p) N^p - 2p(3-2p) M^p &= (1-p) \{N^{p-1} - 7M^{p-1} + 12 L^{p-1}\}, \\
(1-p) O^p - 2p(4-2p) N^p &= (1-p) \{O^{p-1} - 3N^{p-1} + 30 M^{p-1}\}, \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

the law of which is obvious, the coefficients on the second side in the qth line being 1, $4q-19$, and $(2q-3)(2q-2)$ respectively. By successive integrations and substitutions

$$\begin{aligned}
L^p - L^{p-1} &= 0, & L^p &= 1, \\
M^p - M^{p-1} &= 4p - 11, & M^p &= (p-1)(2p-7), \\
N^p - N^{p-1} &= -8p^3 + 26p^2 + 49p - 114; & &\vdots \\
&\vdots
\end{aligned}$$

(the constants determined by $M^1 = 0$, $N^1 = 0$, $O^2 = 0$, $P^2 = 0$, ... so as to make C_r^p contain positive powers only of r).

The following are a few of the complete values of C_r^p, the constants determined so as to satisfy $C_{p+1}^p = 0$ (except $C_2^1 = 1$), and the factorials being partially developed in powers of r, viz.

$$\begin{aligned}
C_r^1 &= 1, \\
C_r^2 &= (r-3)(2r-7), \\
C_r^3 &= \tfrac{1}{2}(r-4)(r-5)(4r^2 - 24r + 51), \\
C_r^4 &= \tfrac{1}{6}\{(r-5)(r-6)(r-7)(8r^3 - 60r^2 + 286r + 63) + 384(9r^2 - 93r + 242 - 2 . 4^{r-5})\}, \\
&\text{\&c.}
\end{aligned}$$

(it is curious that C_5^4, C_6^4, C_7^4, all three of them vanish). It seems hopeless to continue this investigation any further.

Returning to the equation (8), and assuming for z an expression of the same form as before, we have, corresponding to the equations before found for the coefficients C_r,

$$C_{r+2} = -(2r+1)(2r+2)(n-2r)(n-2r-1) C_r - (2r+2)(n-2r-2) \alpha C_{r+1} + 2n(\alpha^2 - 4) \frac{dC_{r+1}}{d\alpha}.$$

The case corresponding to the denominator in the multiplication of elliptic functions is that of $C_0=1$, $C_1=0$. It is easy to form the table—

$$
\begin{aligned}
C_0 &= 1,\\
C_1 &= 0,\\
C_2 &= -2n(n-1),\\
C_3 &= 8n(n-1)(n-4)\alpha,\\
C_4 &= -4n(n-1)(n-4)[n+75]\\
&\quad -32n(n-1)(n-4)(n-9)\alpha^2,\\
C_5 &= 96n(n-1)(n-4)(n-9)\quad[n+44]\alpha\\
&\quad +128n(n-1)(n-4)(n-9)(n-16)\alpha^3,\\
C_6 &= -24n(n-1)(n-4)(n-9)[17n^2+403n+9000]\\
&\quad -960n(n-1)(n-4)(n-9)(n-16)\quad[n+41]\alpha^2\\
&\quad -512n(n-1)(n-4)(n-9)(n-16)(n-25)\quad\alpha^4,\\
C_7 &= +96n(n-1)(n-4)(n-9)(n-16)[79n^2+2825n+36180]\alpha\\
&\quad +7168n(n-1)(n-4)(n-9)(n-16)(n-25)\quad[n+42]\alpha^3\\
&\quad +2048n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\quad\alpha^5,\\
C_8 &= -48n(n-1)(n-4)(n-9)[283n^4-26978n^3+277827n^2-5491932n+127764000]\\
&\quad -3840n(n-1)(n-4)(n-9)(n-16)(n-25)\quad[23n^2+1069n+23436]\alpha^2\\
&\quad -15360n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\quad[3n+133]\alpha^4\\
&\quad -8192n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)(n-49)\quad\alpha^6,
\end{aligned}
$$

&c.

in which of course the coefficient of the highest power of n, in the successive coefficients C_r, is the value of C_r obtained from the equation (8). With regard to the law of these coefficients I have found that

$$
\begin{aligned}
C_r = (-)^{r+1}\, &2^{2r-3}\, n(n-1^2)\ldots\{n-(r-1)^2\}\, C_r^1\, \alpha^{r-2}\\
&+2^{2r-6}\, n(n-1^2)\ldots\{n-(r-2)^2\}\, C_r^2\, \alpha^{r-4}\\
&+2^{2r-9}\, n(n-1^2)\ldots\{n-(r-3)^2\}\, C_r^3\, \alpha^{r-6}\\
&+\&c.
\end{aligned}
$$

(where however the next term does not contain, as would at first sight be supposed, the factor $n(n-1^2)\ldots\{n-(r-4)^2\}$). And then

$$
\begin{aligned}
C_r^1 &= 1,\\
C_r^2 &= (r-3)[n(2r-7)+(r-1)(8r-7)],\\
C_r^3 &= \tfrac{1}{2}(r-4)(r-5)[\quad n^2(4r^2-24r+51)\\
&\qquad +n(32r^3-220r^2+412r-255)\\
&\qquad +2(r-1)(r-2)(32r^2-88r+51)].
\end{aligned}
$$

In conclusion may be given the following results, in which, recapitulating the notation

$$x = \sqrt{k}\,\mathrm{sn}\,u, \quad \alpha = k + \frac{1}{k}, \quad \Delta x = \sqrt{(1 - \alpha x^2 + x^4)},$$

$$\sqrt{k}\,\mathrm{sn}\,2u = \frac{2x\,\Delta x}{1 - x^4},$$

$$\sqrt{k}\,\mathrm{sn}\,3u = \frac{x\,(3 - 4\alpha x^2 + 6x^4 - x^8)}{1 - 6x^4 + 4\alpha x^6 - 3x^8},$$

$$\sqrt{k}\,\mathrm{sn}\,4u = \frac{4x\,\Delta x\,(1 - x^4)\,(1 - 2\alpha x^2 + 6x^4 - 2\alpha x^6 + x^8)}{1 - 20x^4 + 32\alpha x^6 - (26 + 16\alpha^2)\,x^8 + 32\alpha x^{10} - 20x^{12} + x^{16}},$$

$$\sqrt{k}\,\mathrm{sn}\,5u = x\,\frac{\{5 - 20\alpha x^2 + (62 + 16\alpha^2)\,x^4 - 80\alpha x^6 - 105x^8 + 360\alpha x^{10} - (300 + 240\alpha^2)\,x^{12} + (368\alpha + 64\alpha^3)\,x^{14} - (125 + 160\alpha^2)\,x^{16} + 140\alpha x^{18} - 50x^{20} + x^{24}\}}{\{1 - 50x^4 + 140\alpha x^6 - (125 + 160\alpha^2)\,x^8 + (368\alpha + 64\alpha^3)\,x^{10} - (300 + 240\alpha^2)\,x^{12} + 360\alpha x^{14} - 105x^{16} - 80\alpha x^{18} + (62 + 16\alpha^2)\,x^{20} - 20\alpha x^{22} + 5x^{24}\}}$$

&c.

Thus, writing $-x^2$ for x^2, $k = 1$, and therefore $\alpha = 2$,

$$\tan 3u = x\,\frac{(3 + 8x^2 + 6x^4 - x^8)}{1 - 6x^4 - 8x^6 - 3x^8} = \frac{x\,(3 - x^2)\,(1 + x^2)^2}{(1 - 3x^2)\,(1 + x^2)^3} = \frac{x\,(3 - x^2)}{1 - 3x^2},$$

where $x = \tan u$. (And in general in reducing $\tan nu$ the extraneous factor in the numerator and denominator is $(1 + x^2)^{\frac{1}{2}n(n-1)}$.)

46.

NOTE ON A SYSTEM OF IMAGINARIES.

[From the *Philosophical Magazine*, vol. XXX. (1847), pp. 257—258.]

THE octuple system of imaginary quantities i_1, i_2, i_3, i_4, i_5, i_6, i_7, which I mentioned in a former paper [21], (and the conditions for the combination of which are contained in the symbols

$$123, \quad 246, \quad 374, \quad 145, \quad 275, \quad 365, \quad 167,$$

i.e. in the formulæ

$$i_2 i_3 = i_1, \quad i_3 i_1 = i_2, \quad i_1 i_2 = i_3, \qquad i_3 i_2 = -i_1, \quad i_1 i_3 = -i_2, \quad i_2 i_1 = -i_3,$$

with corresponding formulæ for the other triplets $i_5 i_6$, &c.,) possesses the following property; namely, if i_α, i_β, i_γ be any three of the seven quantities which do *not* form a triplet, then

$$(i_\alpha i_\beta) \,.\, i_\gamma = -i_\alpha \,.\, (i_\beta i_\gamma).$$

Thus, for instance,

$$(i_3 i_4) \,.\, i_5 = -i_7 \,.\, i_5 = -i_2;$$

but

$$i_3 \,.\, (i_4 i_5) = \quad i_3 \,.\, i_1 = \quad i_2,$$

and similarly for any other such combination. When i_α, i_β, i_γ form a triplet, the two products are equal, and reduce themselves each to -1, or each to $+1$, according to the order of the three quantities forming the triplet. Hence in the octuple system in question neither the commutative nor the distributive law holds, which is a still wider departure from the laws of ordinary algebra than that which is presented by Sir W. Hamilton's quaternions.

I may mention, that a system of coefficients, which I have obtained for the rectangular transformation of coordinates in n dimensions (Crelle, t. XXXII. [1846] "*Sur quelques propriétés des Déterminans gauches*" [52]), does not appear to be at all connected with any system of imaginary quantities, though coinciding in the case of $n = 3$ with those mentioned in my paper "On Certain Results relating to Quaternions," *Phil. Mag.* Feb. 1845, [20].

47.

SUR LA SURFACE DES ONDES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XI. (1846), pp. 291—296.]

LA surface des ondes est un cas particulier d'une autre surface à laquelle on peut donner le nom de *tétraédroïde*, et dont voici la propriété fondamentale :

"Le tétraédroïde est une surface du quatrième ordre, qui est coupée par les plans d'un certain tétraèdre suivant des paires de coniques par rapport auxquelles les trois sommets du tétraèdre dans ce plan sont des points conjugués. De plus : les seize points d'intersection des quatre paires de coniques sont des points singuliers de la surface, c'est-à-dire des points où, au lieu d'un plan tangent, il y a un cône tangent du second ordre."

On verra plus loin que la surface eût été suffisamment définie en disant qu'un seul de ces points était un point singulier ; l'existence des quinze autres points singuliers est donc une propriété assez remarquable. Mais, avant d'aller plus loin, il convient de rappeler la signification des points conjugués par rapport à une conique : cela veut dire que la polaire de chacun des trois points passe par les deux autres. Parmi les propriétés d'un tel système, on peut citer celle-ci :

"Les points d'intersection de deux coniques, par rapport auxquelles les mêmes trois points sont des points conjugués, sont situés deux à deux sur six droites, lesquelles passent deux à deux par ces trois points."

De là : "Les quatre points singuliers compris dans chaque face du tétraèdre sont situés deux à deux sur trois paires de droites, lesquelles passent par les trois sommets dans cette face."

Autrement dit, les seize points singuliers sont situés deux à deux sur vingt-quatre droites, lesquelles passent six à six par les sommets et sont situées six à six dans les plans du tétraèdre. Donc, en considérant les six droites qui passent par le même

sommet, par ces droites combinées deux à deux, il passe quinze plans, y compris trois plans du tétraèdre; en excluant ceux-ci, il y a pour chaque sommet douze plans, chacun desquels passe par quatre points singuliers; donc la courbe d'intersection d'un de ces plans avec la surface a quatre points doubles, ce qui ne peut pas arriver pour une courbe du quatrième ordre, à moins qu'elle ne se réduise à deux coniques.

Donc: "Il y a, outre les plans du tétraèdre, quarante-huit plans, douze par chaque sommet, lesquels rencontrent la surface suivant des paires de coniques."

On pourrait de même chercher combien il y a de plans qui rencontrent la surface suivant des courbes avec trois points doubles, &c. Passons aux cônes circonscrits; on démontre facilement par l'analyse cette propriété:

"Les cônes circonscrits à la surface ayant pour sommets les sommets du tétraèdre, se réduisent à des paires de cônes du second ordre, lesquelles la touchent suivant les deux coniques de la face opposée."

De plus: "Les seize plans qui touchent quatre à quatre ces paires de cônes sont des plans tangents singuliers, dont chacun touche la surface suivant une conique."

Il y a de plus, entre ces plans, des relations analogues à celles qui existent entre les points singuliers, de manière que l'on déduit de même le théorème:

"Il y a quarante-huit points, douze à douze dans les quatre plans du tétraèdre, pour chacun desquels les cônes circonscrits se réduisent à des paires de cônes du second ordre."

Ajoutons que ces quarante-huit points correspondent d'une manière particulière aux quarante-huit plans, et que les cônes circonscrits touchent la surface suivant les coniques situées dans les plans correspondants.

La réciproque d'une surface du quatrième ordre est, en général, de l'ordre trente-six; mais ici, à cause des seize points singuliers, cet ordre se réduit de trente-deux, savoir, à quatre. Et les propriétés qui viennent d'être énoncées par rapport aux cônes circonscrits montrent que la réciproque étant de cet ordre est nécessairement une surface de la même espèce; c'est-à-dire:

"La réciproque d'un tétraédroïde est aussi un tétraédroïde."

Donc le tétraédroïde est surface de la quatrième classe. Puisque cette surface n'a pas de lignes doubles ou de lignes de rebroussement, il n'y a pas de réduction dans le nombre qui exprime *le rang* de la surface, lequel est ainsi égal à douze, c'est-à-dire le cône circonscrit est ordinairement de l'ordre douze. Mais nous venons de voir que ce cône est seulement de la quatrième classe (en effet, la classe du cône est la même chose que celle de la surface); donc il y a réduction de cent vingt-huit dans la classe du cône. Les seize points singuliers de la surface donnent lieu à autant de lignes doubles dans le cône, ce qui effectue une réduction de trente-deux; il y a encore une réduction à effectuer de quatre-vingt-seize, qui doit avoir lieu à cause des lignes doubles ou des lignes de rebroussement du cône. En supposant qu'il y a y de celles-ci et x de celles-là (outre les seize lignes doubles dont on a fait mention), il faut que l'on ait

$$2x + 3y = 96,$$

où $x+y$ ne doit pas être plus grand que trente-neuf; mais cela ne suffit pas pour déterminer ces deux nombres.

Nous pouvons encore ajouter que les seize cônes qui touchent la surface aux points singuliers sont circonscrits, quatre à quatre, à quatre surfaces du second ordre, et que les seize courbes de contact des plans singuliers sont situées quatre à quatre sur quatre surfaces du second ordre.

Je vais donner maintenant une idée de la théorie analytique. En représentant par

$$x=0, \quad y=0, \quad z=0, \quad w=0$$

les équations des quatre faces du tétraèdre, et par

$$U=0$$

l'équation de la surface, U sera une fonction homogène du quatrième degré de x, y, z, w, laquelle, en faisant évanouir une quelconque des variables, doit se diviser en deux facteurs du second degré, *fonctions paires* des trois autres variables; de plus, à cause de la condition par rapport aux points singuliers, U ne peut pas contenir de terme $xyzw$. On doit donc avoir

$$U = Ax^4 + By^4 + Cz^4 + 2Fy^2z^2 + 2Gz^2x^2 + 2Hx^2y^2 + 2Lx^2w^2 + 2My^2w^2 + 2Nz^2w^2 + Pw^4,$$

où il faut que les coefficients A, B, C, P, F, G, H, L, M, N aient un système inverse[1] de la forme 0, 0, 0, 0, f, g, h, l, m, n. Donc, en formant l'inverse de ce système, on obtient, toute réduction accomplie,

$$\begin{aligned} U = {} & mnfx^4 + nlgy^4 + lmhz^4 + fghw^4 \\ & + (\ \ lf - mg - nh)(\ ly^2z^2 + fx^2w^2) \\ & + (-lf + mg - nh)(mz^2x^2 + gy^2w^2) \\ & + (-lf - mg + nh)(nx^2y^2 + hz^2w^2) = 0. \end{aligned}$$

En effet, en écrivant

$$\begin{aligned} \lambda &= \ \ lf - mg - nh, \quad \mu = -lf + mg - nh, \\ \nu &= -lf - mg + nh, \quad \omega = \ \ lf + mg + nh, \\ \nabla &= l^2f^2 + m^2g^2 + n^2h^2 - 2mngh - 2nlhf - 2lmfg, \end{aligned}$$

l'équation de la courbe pourra s'écrire sous les quatre formes

$$\begin{aligned} (2mnfx^2 + \ \ n\nu y^2 + \ \ m\mu z^2 + \ \ f\lambda w^2)^2 &= \nabla\,(n^2y^4 + m^2z^4 + f^2w^4 - 2mfz^2w^2 - 2fnw^2y^2 - 2nmy^2z^2), \\ (\ \ n\nu x^2 + 2nlgy^2 + \ \ l\lambda z^2 + \ \ g\mu w^2)^2 &= \nabla\,(l^2z^4 + g^2w^4 + n^2x^4 - 2gnw^2x^2 - 2nl\,x^2z^2 - 2lg\ \ z^2w^2), \\ (\ \ m\mu x^2 + \ \ l\lambda y^2 + 2lmhz^2 + \ \ h\nu w^2)^2 &= \nabla\,(h^2w^2 + m^2x^4 + l^2y^4 - 2ml\,x^2y^2 - 2lh\,y^2w^2 - 2hmw^2x^2), \\ (\ \ f\lambda x^2 + \ \ g\mu y^2 + \ \ h\nu z^2 + 2fghw^2)^2 &= \nabla\,(f^2x^4 + g^2y^4 + h^2z^4 - 2gh\ y^2z^2 - 2hf\ z^2x^2 - 2fg\ \ x^2y^2); \end{aligned}$$

[1] En général, quand deux systèmes de quantités sont exprimés linéairement les uns au moyen des autres, on dit que les deux systèmes de coefficients sont des systèmes inverses; on passe facilement de là à l'idée du système inverse des coefficients d'une fonction du second ordre, et de tels systèmes se rencontrent si souvent, que l'on doit avoir un terme pour exprimer sans circonlocution cette relation.

ce qui met en évidence les équations des sections de la surface par les quatre plans du tétraèdre, et aussi celles des points singuliers, lesquelles sont, en effet,

$$ny \pm mz \pm fw = 0, \quad lz \pm gw \pm nx = 0,$$
$$hw \pm mx \pm ly = 0, \quad fx \pm gy \pm hz = 0;$$

et ces plans touchent la surface suivant les courbes d'intersection avec les surfaces

$$2mnfx^2 + n\nu y^2 + m\mu z^2 + f\lambda w^2 = 0, \quad \&c., \quad \&c.,$$

ce qui démontre le théorème énoncé par rapport aux seize courbes de contact des plans singuliers, et de là celui pour les seize cônes tangents aux points singuliers.

Pour déduire de là la forme ordinaire de l'équation de la surface des ondes, écrivons

$$l = \alpha\beta\gamma(b\gamma - c\beta), \quad m = \alpha\beta\gamma(c\alpha - a\gamma), \quad n = \alpha\beta\gamma(a\beta - b\alpha),$$
$$f = ka\alpha(b\gamma - c\beta), \quad g = kb\beta(c\alpha - a\gamma), \quad h = kc\gamma(a\beta - b\alpha),$$

équations qui suffisent pour déterminer les rapports

$$a : b : c : \alpha : \beta : \gamma : k$$

au moyen de l, m, n, f, g, h. De cette manière, l'équation de la surface se réduit à

$$\alpha\beta\gamma(ax^2 + by^2 + cz^2)(\alpha x^2 + \beta y^2 + \gamma z^2)$$
$$- ka\alpha(b\gamma + c\beta)x^2w^2 - kb\beta(c\alpha + a\gamma)y^2w^2 - kc\gamma(a\beta + b\alpha)z^2w^2 + k^2abcw^4 = 0,$$

laquelle se réduit à la surface des ondes en écrivant

$$\frac{x}{w}\sqrt{\frac{\alpha}{k}} = \xi, \quad \frac{y}{w}\sqrt{\frac{\beta}{k}} = \eta, \quad \frac{z}{w}\sqrt{\frac{\gamma}{k}} = \zeta,$$

ξ, η, ζ étant des coordonnées rectangulaires, c'est-à-dire en faisant la transformation homographique du tétraédroïde, de manière que l'un des plans du tétraèdre passe à l'infini, et que les trois autres deviennent rectangulaires. De plus, en particularisant la transformation de manière que trois des coniques d'intersection se réduisent à des cercles, cette surface rentre dans la surface des ondes. Il va sans dire que, dans le cas général, plusieurs des points ou des plans dont nous avons parlé sont nécessairement imaginaires; l'énumération de tous les cas différents aurait été d'une longueur effrayante. Il y aurait beaucoup à dire sur les cas particuliers où quelques-unes des coniques se réduisent à des paires de droites (réelles ou imaginaires). Je me contente d'énoncer cette propriété, très-facile à démontrer, de la surface ordinaire des ondes: au cas où $c = 0$, cette surface peut être engendrée par un cercle (ayant le centre de la surface pour centre, et dans un plan passant par l'axe des z), lequel se meut de manière à passer par la conique

$$a^2x^2 + c^2y^2 = a^2b^2.$$

On trouve, dans le *Cambridge and Dublin Mathematical Journal*, t. I. [1846], p. 208, [38] la démonstration d'une autre propriété de la surface des ondes par rapport aux lignes de courbure des surfaces du second ordre.

48.

NOTE SUR LES FONCTIONS DE M. STURM.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XI. (1846), pp. 297—299.]

On sait que la suite des fonctions

$$\begin{aligned} fx &= (x-a)(x-b)(x-c)(x-d)\dots, \\ f_1x &= \Sigma(x-b)(x-c)(x-d)\dots, \\ f_2x &= \Sigma(a-b)^2(x-c)(x-d)\dots, \\ f_3x &= \Sigma(a-b)^2(b-c)^2(c-a)^2(x-d)\dots, \end{aligned}$$

est de la plus grande utilité dans la théorie de la résolution numérique des équations. En effet, on obtient tout de suite à leur moyen le nombre de racines réelles comprises entre deux limites quelconques. Il était donc intéressant de chercher la manière d'exprimer ces fonctions par les coefficients de fx.

Soit, pour cela, m un nombre quelconque, pas plus grand que le degré n de cette fonction. En prenant k pour $m^{ième}$ racine de la suite $a, b, c, \dots$, et mettant, pour abréger,

$$P = (-)^{\frac{1}{2}m(m-1)}(a-b)(a-c)\dots(a-k)(b-c)\dots(b-k)\dots(j-k),$$

cela donne

$$f_m x : fx = \Sigma \frac{P^2}{(x-a)(x-b)\dots(x-k)},$$

dans laquelle expression

$$P = \begin{vmatrix} 1, & a, \dots, & a^{m-1}, \\ 1, & b, \dots, & b^{m-1}, \\ \dots & \dots & \dots \\ 1, & k, \dots, & k^{m-1}, \end{vmatrix}$$

et, de plus,

$$\frac{P}{(x-a)(x-b)\dots(x-k)} = (-)^{\frac{1}{2}m(m-1)}\left[\frac{(b-c)\dots(b-k)\dots(j-k)}{(x-a)}+\dots\right],$$

dans laquelle le coefficient de x^{-r} est égal à

$$(-)^{\frac{1}{2}m(m-1)}\left[a^{r-1}(b-c)\dots(b-k)\dots(j-k)+\dots\right],$$

c'est-à-dire à

$$\begin{vmatrix} 1, & a, \dots, & a^{m-2}, & a^{r-1} \\ 1, & b, \dots, & b^{m-2}, & b^{r-1} \\ \dots & \dots & \dots & \dots \\ 1, & k, \dots, & k^{m-2}, & k^{r-1} \end{vmatrix}.$$

Donc enfin le coefficient de x^{-r} dans $f_m x : fx$ est égal à

$$\Sigma \begin{vmatrix} 1, & a, \dots, & a^{m-1} \\ \dots & \dots & \dots \\ 1, & k, \dots, & k^{m-1} \end{vmatrix} \times \begin{vmatrix} 1, & a, \dots, & a^{m-2}, & a^{r-1} \\ \dots & \dots & \dots & \dots \\ 1, & k, \dots, & k^{m-2}, & k^{r-1} \end{vmatrix}$$

où, au moyen d'une propriété connue des déterminants, en représentant comme à l'ordinaire par S_q la somme des $q^{\text{ièmes}}$ puissances de toutes les racines, ce coefficient devient égal à

$$\begin{vmatrix} S_0, & S_1, \dots, & S_{m-2}, & S_{r-1} \\ S_1, & S_2, \dots, & S_{m-1}, & S_r \\ \dots & \dots & \dots & \dots \\ S_{m-1}, & S_m, \dots, & S_{2m-3}, & S_{r+m-2} \end{vmatrix}.$$

De là, en mettant

$$T_q = \Sigma \frac{a^q}{x-a},$$

$$f_m x : fx = \begin{vmatrix} S_0, & S_1, \dots, & S_{m-2}, & T_0 \\ S_1, & S_2, \dots, & S_{m-1}, & T_1 \\ \dots & \dots & \dots & \dots \\ S_{m-1}, & S_m, \dots, & S_{2m-3}, & T_{m-1} \end{vmatrix}$$

en multipliant par fx, et mettant

$$Q_{m,r} = S_{m+r-1} - p_1 S_{m+r-2} \dots + (-)^r p_r S_{m-1},$$

où l'on suppose

$$fx = x^n - p_1 x^{n-1} \dots + (-)^n p_n,$$

on obtient

$$f_m x = \Sigma_r x^{n-m-r} \begin{vmatrix} S_0, & S_1, \dots, & S_{m-2}, & Q_{m,r} \\ S_1, & S_2, \dots, & S_{m-1}, & Q_{m+1,r} \\ \dots & \dots & \dots & \dots \\ S_{m-1}, & S_m, \dots, & S_{2m-3}, & Q_{2m-1,r} \end{vmatrix}$$

où r peut ne s'étendre que depuis 0 jusqu'à $n-m$, puisque $f_m x$ est fonction entière. Au moyen des relations connues qui existent entre les quantités S_q, on a

$$Q_{m+s,r} = (-)^r p_{r+1} S_{m+s-2} \cdots + (-)^{n-1} p_n S_{r+m+s-n-1} \cdots \quad [r+m+s > n],$$

$$Q_{m+s,r} = (-)^r p_{r+1} S_{m+s-2} \cdots + (-)^{r+m+s-3} p_{r+m+s-2} S_1$$
$$+ (-)^{r+m+s-2} p_{r+m+s-1} (r+m+s-1) \ldots\ldots\ldots\ldots \left[r+m+s \overset{=}{<} n\right],$$

et de là, en posant

$$Q'_{m+s,r} = (-)^{r+m-1} p_{r+m} S_{s-1} \cdots + (-)^{n-1} p_n S_{r+m+s-n-1} \cdots [r+m+s > n],$$

$$Q'_{m+s,r} = (-)^{r+m-1} p_{r+m} S_{s-1} \cdots + (-)^{r+m+s-3} p_{r+m+s-2} S_1$$
$$+ (-)^{r+m+s-2} p_{r+m+s-1} (r+m+s-n-1) \ldots \left[r+m+s \overset{=}{<} n\right],$$

on peut, par les propriétés des déterminants, réduire $Q_{m+s,r}$ à $Q'_{m+s,r}$ dans l'expression de $f_m x$. Nous avons donc exprimé cette fonction au moyen des coefficients $p_1, p_2, \ldots$ et des sommes $S_1, S_2, \ldots, S_{2m-3}$, lesquelles s'expriment facilement par ces mêmes coefficients (et pour calculer $f_1 x, f_2 x, \ldots, f_n x$, on aurait seulement besoin de calculer ces sommes une fois pour toutes jusqu'à S_{2n-3}); il serait donc facile de former des tables de ces fonctions, pour les équations d'un degré quelconque, ce qui pourrait à peine s'effectuer d'aucune autre manière.

49.

SUR QUELQUES FORMULES DU CALCUL INTÉGRAL.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XII. (1847), pp. 231—240.]

SOIT $x+y\sqrt{-1}$ ou $x+iy$ une quantité imaginaire quelconque; faisons

$$\rho=\sqrt{x^2+y^2}, \qquad \theta=\text{arc tan}\frac{y}{x} \quad \text{.....} \quad (1),$$

ρ étant une quantité positive, et θ un arc compris entre les limites $\frac{1}{2}\pi$, $-\frac{1}{2}\pi$. Cela posé, écrivons

$$\left.\begin{array}{ll}(x+iy)^m=\rho^m e^{im\theta} & (x \text{ positif}),\\ (x+iy)^m=\rho^m e^{im(\theta\pm\pi)} & (x \text{ négatif});\end{array}\right\} \text{.....} (2),$$

(dans la seconde de ces formules, il faut prendre le signe supérieur ou inférieur, selon que y est positif ou négatif.) Au cas de x positif, la valeur du second membre sera ce que M. Cauchy a appelé *valeur principale* de $(x+iy)^m$. Au cas de x négatif, on peut aussi, à ce qu'il me semble, nommer cette valeur *valeur principale*. Cela paraît contraire à la théorie de M. Cauchy (*Exercices de Mathématiques*, t. I. [1826] p. 2); mais la démonstration que l'on y trouve de l'impossibilité d'une valeur principale pour x négatif ne s'applique qu'au cas où l'on suppose que le signe $\pm$ est toujours le même sans avoir égard au signe de y. Seulement, selon nos définitions, il importe de remarquer qu'il n'y a pas de valeur principale pour x négatif, au cas particulier où $y=0$; ou plutôt dans ce cas, et dans ce cas seulement, la valeur principale devient indéterminée.

Soit, en particulier, $x=0$; les deux formules conduisent au même résultat, savoir

$$(iy)^m=(\pm y)^m e^{\pm\frac{1}{2}m\pi i} \text{.....} (3),$$

le signe comme auparavant. Car en considérant x comme infiniment petit positif, on obtient

$$\theta = \pm \tfrac{1}{2}\pi,$$

et, en considérant x comme infiniment petit négatif,

$$\theta = \mp \tfrac{1}{2}\pi,$$

et de là

$$\theta \pm \pi = \pm \tfrac{1}{2}\pi.$$

Ainsi cette formule est toujours vraie, sans qu'il soit nécessaire de considérer iy comme limite de $x+iy$, x positif ou x négatif.

Remarquons encore que cette fonction $(iy)^m$ reste continue quand y passe par zéro, ce qui a lieu aussi pour $(x+iy)^m$, x positif, mais non pas pour $(x+iy)^m$, x négatif.

Les mêmes remarques s'appliquent aux valeurs principales des logarithmes, lesquelles doivent se définir d'une manière analogue par les équations

$$\left.\begin{array}{ll} \log(x+iy) = \log\rho + i\theta & (x \text{ positif}), \\ \log(x+iy) = \log\rho + i(\theta \pm \pi) & (x \text{ négatif}), \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

le signe ambigu, comme auparavant.

On démontre sans difficulté que ces valeurs principales satisfont en tout cas aux équations

$$\left.\begin{array}{l} (x+iy)^m (x'+iy')^m = [(x+iy)(x'+iy')]^m, \\ \log(x+iy)\log(x'+iy') = \log[(x+iy)(x'+iy')] \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \quad (5);$$

seulement ces équations deviennent indéterminées au cas où, l'une des quantités x, x', $xx'-yy'$ étant négative, la quantité correspondante y, y', $xy'+x'y$ s'évanouit.

Au moyen de cette définition de la valeur principale d'un logarithme, on obtient

$$\int_\beta^\alpha \frac{dx}{A+Bx} = \log\left(\frac{A+B\alpha}{A+B\beta}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

où α, β sont réels, et A, B sont assujettis à la seule condition qu'au cas où α, β seraient de signes contraires, la partie imaginaire de $\dfrac{A}{B}$ ne s'évanouisse pas. En effet, dans ce cas, l'intégrale et la valeur principale du logarithme deviennent toutes les deux indéterminées. Sans doute il y a une valeur que l'on peut appeler *principale* de l'intégrale, mais cette valeur n'est égale à *aucun* des logarithmes de $\dfrac{A+B\alpha}{A+B\beta}$, et les notions des valeurs principales d'une intégrale et d'un logarithme n'ont pas de rapport ensemble. Ce résultat s'accorde avec celui que j'ai trouvé dans mon Mémoire "*Sur les fonctions doublement périodiques,*" [25].

Je passe à quelques autres applications de ces principes, qui ont rapport à la théorie des fonctions Γ. Soit d'abord r un nombre positif plus petit que l'unité, et écrivons

$$U = \int_{-\infty}^{\infty} (ix)^{r-1} e^{ix}\, dx;$$

on obtient

$$U = \int_0^{\infty} (ix)^{r-1} e^{ix} dx + \int_0^{\infty} (-ix)^{r-1} e^{-ix}\, dx,$$

ou enfin, puisque x est positif dans ces deux intégrales,

$$U = e^{\frac{1}{2}(r-1)\pi i} \int_0^{\infty} x^{r-1} e^{ix}\, dx + e^{-\frac{1}{2}(r-1)\pi i} \int_0^{\infty} x^{r-1} e^{-ix}\, dx;$$

au moyen de la formule connue,

$$\int_0^{\infty} x^{r-1} e^{\pm ix}\, dx = e^{\pm \frac{1}{2} r\pi i}\, \Gamma r,$$

on en conclut

$$U = 2\cos(r - \tfrac{1}{2})\pi \,.\, \Gamma r = 2 \sin r\pi \,.\, \Gamma r,$$

savoir

$$\sin r\pi \,.\, \Gamma r = \tfrac{1}{2} \int_{-\infty}^{\infty} (ix)^{r-1} e^{ix}\, dx.$$

On obtiendrait de même

$$0 = \tfrac{1}{2} \int_{-\infty}^{\infty} (-ix)^{r-1} e^{ix}\, dx.$$

Au premier coup d'œil, ces équations pourraient paraître en contradiction l'une avec l'autre : mais cela n'est pas ainsi, parce qu'il n'est pas vrai que $(-ix)^{r-1}$ soit égal à $(-1)^{r-1}(ix)^{r-1}$, $(-1)^{r-1}$ étant facteur invariable; mais au contraire $(-ix)^{r-1} = e^{\pm(r-1)\pi i}(ix)^{r-1}$, selon que x est positif ou négatif.

Dans la première de ces équations, on peut remplacer ix par $c + ix$, et dans la seconde, $-ix$ par $c - ix$, c étant positif; mais cela n'est pas permis pour c négatif, à cause de la discontinuité de valeur de $(c+ix)^{r-1}$ ou $(c-ix)^{r-1}$ dans ce cas, quand x passe par zéro. En multipliant la première équation par e^{-c}, et différentiant un nombre quelconque de fois, en admettant, pour les valeurs négatives de r, l'équation

$$\Gamma(r+1) = r\Gamma r,$$

la formule ne change pas de forme, et l'on obtient

$$\sin r\pi \Gamma r \,.\, e^{-c} = \tfrac{1}{2} \int_{-\infty}^{\infty} (c+ix)^{r-1} e^{ix}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7);$$

et de même, au moyen de la seconde équation,

$$0 = \tfrac{1}{2} \int_{-\infty}^{\infty} (c-ix)^{r-1} e^{ix}\, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

dans lesquelles formules c est positif, et r est un nombre quelconque entre 1, $-\infty$, sans exclure la limite supérieure; seulement, pour $r=0$, le facteur $\sin r\pi\Gamma r$ se réduit à π. Au cas où r est plus grand que zéro, on peut, si l'on veut, écrire aussi $c=0$. Dans tous les cas, on peut remplacer $\sin r\pi\Gamma r$ par $\frac{\pi}{\Gamma(1-r)}$. Il importe de remarquer que ces mêmes intégrales sont absolument inexprimables au cas où c est négatif: en effet, en écrivant $-c$ au lieu de c (c positif), on obtiendrait

$$\int_{-\infty}^{\infty}(-c+ix)^{r-1}e^{ix}\,dx = e^{(r-1)\pi i}\int_0^{\infty}(c-ix)^{r-1}e^{ix}\,dx + e^{-(r-1)\pi i}\int_{-\infty}^{0}(c-ix)^{r-1}e^{ix}\,dx.$$

Mais, par la seconde des équations dont il s'agit,

$$0=\int_0^{\infty}(c-ix)^{r-1}e^{ix}\,dx+\int_{-\infty}^{0}(c-ix)^{r-1}e^{ix}\,dx;$$

donc

$$\int_{-\infty}^{\infty}(-c+ix)^{r-1}e^{ix}\,dx=-2i\sin r\pi\int_0^{\infty}(c-ix)^{r-1}e^{ix}\,dx.$$

Or l'intégrale au second membre ne peut pas s'exprimer par les transcendantes connues, à moins que r ne soit entier. Écrivons encore, c étant toujours positif,

$$\left.\begin{aligned} I &= \int_0^{\infty}(c+ix)^{r-1}e^{ix}\,dx,\\ I_1 &= \int_0^{\infty}(c-ix)^{r-1}e^{ix}\,dx,\\ I_2 &= \int_0^{\infty}(c+ix)^{r-1}e^{-ix}\,dx,\\ I_3 &= \int_0^{\infty}(c-ix)^{r-1}e^{-ix}\,dx.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (9).$$

Toutes les fonctions

$$\int(\pm c\pm ix)^{r-1}e^{\pm ix}\,dx,$$

entre les limites 0, ∞, ou $-\infty$, 0, ou $-\infty$, ∞, s'expriment facilement au moyen de I, I_1, I_2, I_3. Mais ces quatre fonctions ne sont pas connues; seulement, au moyen des équations qui viennent d'être trouvées, on obtient

$$\left.\begin{aligned} 2\sin r\pi\,\Gamma r\,.\,e^{-c} &= I+I_3,\\ 0 &= I_1+I_2.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (10).$$

On déduit encore de ces mêmes formules:

$$\left.\begin{aligned} &\sin r\pi\,\Gamma r\,.\,e^{-c}\\ &=\int_{-\infty}^{\infty}(c+ix)^{r-1}\cos x\,dx = \quad i\int_{-\infty}^{\infty}(c+ix)^{r-1}\sin x\,dx,\\ &=\int_{-\infty}^{\infty}(c-ix)^{r-1}\cos x\,dx=-i\int_{-\infty}^{\infty}(c-ix)^{r-1}\sin x\,dx.\end{aligned}\right\}\quad\ldots\ldots\ldots\ (11).$$

En supposant que $r-1$ est entier négatif, l'intégrale

$$\int_0^\infty (c^2+x^2)^{r-1}\cos x dx$$

se décompose facilement dans une suite d'intégrales de la forme

$$\int_{-\infty}^{\infty} (c \pm ix)^{p-1}\cos x dx\,;$$

et, en prenant la somme de celles-ci, on obtient la formule

$$\int_0^\infty (c^2+x^2)^{r-1}\cos x dx = \frac{\pi e^{-c}(2c)^{2r-1}}{\Gamma^2(1-r)}\int_0^\infty \theta^{-r}(\theta+2c)^{-r}e^{-\theta}d\theta \;\ldots\ldots\ldots\; (12),$$

laquelle est due à M. Catalan. Cependant, ni cette démonstration ni celle de M. Catalan ne s'appliquent au cas où r n'est pas entier; la formule subsiste encore dans ce cas, comme M. Serret l'a démontré rigoureusement. En essayant de la vérifier, je suis tombé sur cette autre formule:

$$\left.\begin{aligned}&\int_0^\infty (c^2+x^2)^{r-1}\cos x dx\\ &= \frac{\sqrt{\pi}e^{-c}(2c)^{2r-1}}{2\Gamma(1-r)}\int_0^\infty \frac{(\sqrt{\theta+2c}+\sqrt{\theta})^{1-2r}+(\sqrt{\theta+2c}-\sqrt{\theta})^{1-2r}}{\sqrt{\theta}\sqrt{\theta+2c}}e^{-\theta}d\theta\end{aligned}\right\}\ldots\ldots (13),$$

ce qui suppose, comme auparavant, que $\overline{r-1}$ soit négatif; en comparant les deux valeurs, on est conduit au résultat singulier (en écrivant α au lieu de $2c$, et $\frac{1}{2}(1-p)$ pour r):

$$\left.\begin{aligned}&\int_0^\infty \frac{(\sqrt{\theta+\alpha}+\sqrt{\theta})^p+(\sqrt{\theta+\alpha}-\sqrt{\theta})^p}{\sqrt{\theta}\sqrt{\theta+\alpha}}e^{-\theta}d\theta\\ &= \frac{2\sqrt{\pi}}{\Gamma\frac{1}{2}(p+1)}\int_0^\infty \theta^{\frac{1}{2}(p-1)}(\theta+\alpha)^{\frac{1}{2}(p-1)}e^{-\theta}d\theta\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (14),$$

lequel peut se démontrer sans difficulté, quand p est entier positif impair, en développant les deux membres suivant les puissances de α; cela se fait au moyen de

$$\frac{(\sqrt{1+x}+1)^p+(\sqrt{1+x}-1)^p}{\sqrt{1+x}} = \mathcal{S}_r\left[x^r\frac{2^{p-2r}\Gamma(p-r)}{\Gamma(p-2r)\Gamma(r+1)}\right]\ldots\ldots\ldots\ldots(15),$$

où r s'étend depuis 0 jusqu'à $\frac{1}{2}(p-1)$. J'ai déduit de cette formule (14) des formules assez remarquables qui se rapportent aux attractions, lesquelles paraîtront dans un numéro prochain du *Cambridge and Dublin Mathematical Journal*, [41]. On peut encore démontrer cette formule singulière:

$$\Gamma(r+a-1) = \frac{1}{2\sin r\pi}\int_{-\infty}^{\infty}(ix)^{r-1}(-ix)^{a-1}e^{ix}dx \;\ldots\ldots\ldots\ldots\ldots\; (16)\,;$$

en effet, pour la vérifier, il suffit de réduire l'intégrale, d'abord à

$$\int_0^\infty (ix)^{r-1}(-ix)^{a-1}e^{ix}\,dx + \int_0^\infty (-ix)^{r-1}(ix)^{a-1}e^{-ix}\,dx,$$

puis à

$$e^{\frac{1}{2}(r-a)\pi i}\int_0^\infty x^{r+a-2}e^{ix}\,dx + e^{\frac{1}{2}(a-r)\pi i}\int_0^\infty x^{r+a-2}e^{-ix}\,dx,$$

dont les deux parties s'obtiennent au moyen d'une formule donnée ci-dessus. Si, au lieu de $(-ix)^{a-1}$, l'on avait $(ix)^{a-1}$, l'intégrale se réduirait à zéro; mais cela rentre dans une formule plus simple.

J'ajouterai encore ces deux formules-ci,

$$\left.\begin{aligned} 2\pi c^a e^{-c} &= \int_{-\infty}^{\infty}\frac{(-ix)^a}{c+ix}e^{ix}\,dx\,, \\ 0 &= \int_{-\infty}^{\infty}\frac{(ix)^a}{c+ix}e^{-ix}\,dx, \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (17),$$

dont je supprime la démonstration. En écrivant dans la dernière $-x$ au lieu de x, puis ajoutant, on a

$$\pi c^{a-1}e^{-c} = \int_{-\infty}^{\infty}\frac{(-ix)^a}{c^2+x^2}e^{ix}\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18);$$

d'où l'on déduit tout de suite cette formule de M. Cauchy

$$\tfrac{1}{2}\pi c^{a-1}e^{-c} = \int_0^\infty \frac{x^a}{c^2+x^2}\cos\left(\tfrac{1}{2}a\pi - x\right)dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

La formule (7) peut être considérée comme une définition de la fonction Γr au cas de r négatif; et, à ce point de vue, elle a, ce me semble, quelques avantages sur celle que M. Cauchy a donnée au moyen des intégrales extraordinaires. Je passe à la définition, au moyen d'une intégrale définie, de la seconde intégrale eulérienne,

$$\mathfrak{F}(m,\ n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (20),$$

quand m ou n, ou tous les deux, sont négatifs. Soit d'abord m et n tous les deux positifs, mais n plus petit que l'unité; on obtient au moyen de la formule (7) et de la définition ordinaire des fonctions Γ,

$$\Gamma m\,\Gamma n = \frac{1}{2\sin n\pi}\int_0^\infty dx\int_{-\infty}^{\infty}dy\,x^{m-1}(iy)^{n-1}e^{-x+iy},$$

et de là, en mettant

$$y = \alpha x,\quad dy = x\,d\alpha,$$

et intégrant par rapport à x,

$$\mathfrak{F}(m,\ n) = \frac{1}{2\sin n\pi}\int_{-\infty}^{\infty}\frac{(i\alpha)^{n-1}\,d\alpha}{(1-i\alpha)^{m+n}},$$

ou en écrivant $k+\alpha i$ au lieu de αi, k étant positif,

$$\mathfrak{F}(m, n) = \frac{1}{2\sin n\pi}\int_{-\infty}^{\infty}\frac{(k+\alpha i)^{n-1}\,d\alpha}{(1-k-\alpha i)^{m+n}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

formule qui est vraie, même pour les valeurs négatives de n. En effet, si cette formule est vraie pour une valeur quelconque particulière de n, elle sera aussi vraie pour la valeur $n+p$, p étant entier positif quelconque; ce qui se démontre au moyen des formules de réduction: donc il ne s'agit que de la démontrer au cas où n est positif et plus petit que l'unité, et dans ce cas, puisque la fonction à intégrer est toujours finie, on peut écrire sans crainte $k+\alpha i$ au lieu de αi, ce qui la réduit à la formule qui vient d'être démontrée; donc cette formule (21) est la définition cherchée, au cas où n est négatif, ou positif et plus petit que l'unité.

Si, de plus, m est négatif, ou positif et plus petit que l'unité, $l-m$ sera positif, et l'on déduira

$$\mathfrak{F}(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \frac{\Gamma(1-m-n)\,\Gamma n}{\Gamma(1-m)}\,\frac{\sin(m+n)\pi}{\sin m\pi},$$

c'est-à-dire

$$\mathfrak{F}(m, n) = \frac{\sin(m+n)\pi}{\sin m\pi}\,\mathfrak{F}(1-m-n, n);$$

ou enfin, par l'équation (21),

$$\mathfrak{F}(m, n) = \frac{\sin(m+n)\pi}{2\sin m\pi\sin n\pi}\int_{-\infty}^{\infty}(k+i\alpha)^{n-1}(1-k-i\alpha)^{m-1}\,d\alpha,$$

laquelle suppose seulement que $m+n$ soit négatif, ou positif et plus petit que l'unité. Elle présente une analogie assez frappante avec l'équation ordinaire

$$\mathfrak{F}(m, n) = \int_0^1 \alpha^{n-1}(1-\alpha)^{m-1}\,d\alpha$$

qui correspond aux valeurs positives de m, n; de même que l'équation (21) est analogue à cette autre forme

$$\mathfrak{F}(m, n) = \int_0^{\infty}\frac{\alpha^{n-1}\,d\alpha}{(1+\alpha)^{m-n}},$$

qui correspond aussi aux valeurs positives de m et n.

On peut se proposer de vérifier l'équation (m et n positifs et plus petits que l'unité)

$$\Gamma m\,\Gamma n = \frac{1}{4\sin m\pi\sin n\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,(ix)^{m-1}(iy)^{n-1}e^{i(x+y)},$$

en transformant le second membre au moyen de $x=\alpha y$. Pour cela, on distinguera quatre cas, selon que x et y sont tous les deux positifs ou négatifs, ou l'un positif et l'autre négatif. En mettant dans les deux premiers

$$x=\alpha y, \quad dx = y\,d\alpha,$$

et en intégrant par rapport à y, on trouvera que les deux portions correspondantes de l'intégrale double se réuniront en

$$-\Gamma(m+n).\,2\cos(m+n)\pi\int_0^\infty \frac{\alpha^{m-1}\,d\alpha}{(1+\alpha)^{m+n}},$$

c'est-à-dire à

$$-2\cos(m+n)\pi\,\Gamma m\,\Gamma n.$$

Pour les deux autres portions de l'intégrale double, en écrivant

$$x=-\alpha y,$$

on verra qu'il faut encore distinguer les deux cas $\alpha<1$ et $\alpha>1$. Les quatre intégrales ainsi obtenues se réuniront cependant dans les deux

$$2\cos n\pi\,\Gamma(m+n)\int_0^1 \frac{\alpha^{m-1}\,d\alpha}{(1-\alpha)^{m+n}},\qquad 2\cos m\pi\int_1^\infty \frac{\alpha^{m-1}\,d\alpha}{(\alpha-1)^{m+n}},$$

savoir, après quelques réductions faciles, dans celles-ci,

$$\frac{2\cos n\pi\sin m\pi}{\sin(m+n)\pi}\,\Gamma m\,\Gamma n,\qquad \frac{2\cos m\pi\sin m\pi}{\sin(m+n)}\,\Gamma m\,\Gamma n,$$

lesquelles se réuniront en

$$\cos(m-n)\pi\,\Gamma m\,\Gamma n.$$

On obtient donc enfin l'équation identique

$$4\sin m\pi\sin n\pi=2\cos(m-n)\pi-2\cos(m+n)\pi;$$

ce qui suffit pour la vérification dont il s'agit.

50.

SUR QUELQUES THÉORÈMES DE LA GÉOMÉTRIE DE POSITION.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tome XXXI. (1846), pp. 213—227.]

§ I.

En prenant pour donné un système quelconque de points et de droites, on peut mener par deux points donnés des nouvelles droites, ou trouver des points nouveaux, savoir les points d'intersection de deux des droites données; et ainsi de suite. On obtient de cette manière un nouveau système de points et de droites, qui peut avoir la propriété que plusieurs des points sont situés dans une même droite, ou que plusieurs des droites passent par le même point; ce qui donne lieu à autant de théorèmes de géométrie de position. On a déjà étudié la théorie de plusieurs de ces systèmes; par exemple de celui de quatre points; de six points, situés deux à deux sur trois droites qui se rencontrent dans un même point; de six points trois à trois sur deux droites, ou plus généralement, de six points sur une conique (ce dernier cas, celui de l'hexagramme mystique de Pascal, n'est pas encore épuisé; nous y reviendrons dans la suite), et même de quelques systèmes dans l'espace. Cependant il existe des systèmes plus généraux que ceux qui ont été examinés, et dont les propriétés peuvent être aperçues d'une manière presque intuitive, et qui, à ce que je crois, sont nouveaux. Commençons par le cas le plus simple. Imaginons un nombre n de points situés d'une manière quelconque dans l'espace, et que nous désignerons par 1, 2, 3, ... n. Qu'on fasse passer par toutes les combinaisons de deux points des droites, et par toutes les combinaisons de trois points des plans; puis coupons ces droites et ces plans par un plan quelconque, les droites selon des points, et les plans selon des droites. Soit $\alpha\beta$ le point qui correspond à la droite menée par les deux points α, β; soit de même $\beta\gamma$ le point qui correspond à celle menée par les points β, γ, et ainsi de suite. Soit de plus $\alpha\beta\gamma$ la droite qui correspond au plan passant par les trois points α, β, γ, etc. Il est clair que les trois points $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$ seront situés dans la droite $\alpha\beta\gamma$. Donc en représentant par

N_2, N_3, ... les nombres des combinaisons de n lettres prises deux à deux, trois à trois etc. à la fois, on a le théorème suivant:

THÉORÈME I. *On peut former un système de N_2 points situés trois à trois sur N_3 droites: savoir en représentant les points par* 12, 13, 23, *etc. et les droites par* 123, *etc., les points* 12, 13, 23 *seront situés sur la droite* 123, *et ainsi de suite.*

Pour $n=3$, ou $n=4$, cela est tout simple; on aura trois points sur une droite, ou six points trois à trois sur quatre droites; il n'en résulte aucune propriété géométrique. Pour $n=5$ on a dix points, trois à trois sur autant de droites, savoir les points

12, 13, 14, 15, 23, 24, 25, 34, 35, 45

et les droites

123, 124, 125, 134, 135, 145, 234, 235, 245, 345.

Les points 12, 13, 14, 23, 24, 34 sont les angles d'un quadrilatère quelconque[1], le point 15 est tout à fait arbitraire, le point 25 est situé sur la droite passant par les points 12 et 15, mais sa position sur cette droite est arbitraire. On déterminera depuis les points 35, 45; 35 comme point d'intersection des droites passant par 13 et 15 et par 23 et 25, c'est-à-dire des droites 135 et 235, et de même 45 comme point d'intersection des lignes 145 et 245. Les points 35 et 45 auront la propriété géométrique d'être en ligne droite avec 34, ou bien tous les trois seront dans une même droite 345.

Étudions de plus près la figure que nous venons de former. En prenant le cinq numéros dans un ordre déterminé, par exemple dans l'ordre naturel 1, 2, 3, 4, 5, les cinq points 12, 23, 34, 45, 51 pourront être considérés comme formant un pentagone que nous représenterons par la notation (12345). Les côtés de ce pentagone sont évidemment 123, 234, 345, 451, 512. De même les points 13, 35, 52, 24, 41 peuvent être considérés comme formant le pentagone (13524) dont les côtés sont 135, 352, 524, 241, 413. Ce pentagone est circonscrit au premier, car ses côtés passent évidemment par les angles 15, 23, 45, 12, 34 du premier: mais il est de même inscrit à celui-ci, car ses angles sont situés respectivement dans les côtés 123, 345, 512, 234, 451 de ce même pentagone. Donc les pentagones

(12345), (13524)

sont à la fois circonscrits et inscrits l'un à l'autre, donc:

THÉORÈME II. *La figure composée de dix points, trois à trois dans dix droites, peut être considérée (même de six manières différentes) sous la forme de deux pentagones, inscrits et circonscrits l'un à l'autre.*

Ou encore

THÉORÈME III. *Étant donné un pentagone quelconque, on peut toujours trouver un autre pentagone qui y est à la fois circonscrit et inscrit. Ce second pentagone peut satisfaire à une seule condition donnée quelconque.*

[1] Il faut avoir égard toujours à la différence entre *quadrilatère* et *quadrangle;* chaque quadrilatère a quatre côtés et six angles, chaque quadrangle a quatre angles et six côtés.

Si par exemple le second pentagone doit avoir un de ses angles sur un point donné d'un côté du premier, la construction se déduit tout de suite de ce qui précède.

Ces paires correspondantes de pentagones forment une figure connue. On en trouve la construction dans une note de M. [J. T.] Graves dans le *Philosophical Magazine* [vol. xv. 1839], mais la même figure est encore mieux connue sous un autre point de vue. En effet, considérons le point 12, et les droites 123, 124, 125 qui passent par ce point; puis les triangles dont les angles sont 13, 14, 15 et 23, 24, 25. Les côtés de ces mêmes triangles sont 134, 135, 145 et 234, 235, 245, et les côtés correspondants se rencontrent dans les points 34, 35, 45 qui sont en ligne droite. Donc le théorème sur les pentagones est le suivant:

"*Si les angles de deux triangles sont situés deux à deux dans trois droites qui se rencontrent dans un point, leurs côtés homologues se coupent dans trois points en ligne droite.*"

Remarquons aussi que ce théorème particulier (en n'empruntant rien des trois dimensions de l'espace) reproduit le théorème général relatif au nombre n. Il n'y a pour cela qu'à considérer n droites passant par le même point, et qui peuvent être désignées par 1, 2, 3, ... n. En choisissant d'abord les points 12, 13, tout triangle dont les trois angles sont situés dans les droites 1, 2, 3, pendant que deux de ses côtés passent par 12, 13, a la propriété que le troisième côté passe par un point déterminé 23 situé dans la droite passant par 12, 13. En prenant arbitrairement le point 14, on obtient avec les droites 1, 3, 4 ou 1, 2, 4 les nouveaux points 34, 24 qui sont en ligne droite avec 23, et ainsi de suite.

Passons au cas $n = 6$. Il existe ici quinze points situés trois à trois sur vingt droites, ou bien vingt droites qui se coupent quatre à quatre en quinze points. Il n'y a point ici des systèmes d'hexagones, mais il existe un système de neuf points qui est assez remarquable. Divisons d'une manière quelconque les numéros 1, 2, 3, 4, 5, 6 en deux suites par trois, par exemple en 1, 3, 5 et 2, 4, 6, et considérons les neuf points

$$\begin{array}{lll} 12, & 14, & 16, \\ 32, & 34, & 36, \\ 52, & 54, & 56. \end{array}$$

Les droites qui passent par 12 et 32, 14 et 34, 16 et 36, savoir 132, 134, 136, se rencontrent dans le même point 13. De même les droites qui passent par 32 et 52, 34 et 54, 36 et 56 se rencontrent dans 35, et les droites qui passent par 12 et 52, 14 et 54, 16 et 56 se rencontrent dans 15. Les points 13, 15 et 35 sont sur la même droite 135. En considérant les points 12, 14, 16 comme formant un triangle, et de même les points 32, 34, 36 et 52, 54, 56, cela revient à dire que les droites menées par les angles homologues des triangles prises deux à deux, se rencontrent trois à trois dans trois points situés dans la même droite. Ou bien, ce que l'on savait déjà par le théorème 3: les côtés homologues des triangles se rencontrent trois à trois dans trois points situés en ligne droite. En effet, les côtés des triangles sont 124, 126, 146 pour la première, et 324, 326, 346 et 524, 526, 546 pour les deux autres. Les trois premiers côtés se

rencontrent dans 24, les autres dans 26 et 46, et ces trois points sont dans la droite 246. Maintenant tout cela arrive également en combinant les colonnes verticales, ou en considérant les neuf points comme formant les trois autres triangles dont les angles sont 12, 32, 52; 14, 34, 54; 16, 36, 56. Cela donne lieu au théorème suivant:

THÉORÈME IV. *Le système de quinze points, situés trois à trois sur vingt droites, contient (et cela même de dix manières différentes) un système de neuf points qui ont la propriété de former de deux manières différentes trois triangles, tels, que les droites qui passent par leurs angles homologues, prises deux à deux, se rencontrent dans trois points qui sont en ligne droite, tandis que les côtés homologues des triangles se coupent trois à trois en trois autres points qui sont aussi en ligne droite. Dans la seconde manière de former les triangles, ces deux systèmes de trois points en ligne droite sont seulement échangés.*

Il ne reste qu'à savoir combien il y en a d'arbitraires dans le système de quinze points situés trois à trois sur vingt droites. En supposant le système formé pour le nombre cinq, on peut prendre arbitrairement 16 et 26 sur la droite 126 qui est déterminée par les points 12 et 16. Donc 12, 13, 14, 15 et 16 sont arbitraires et 23, 24, 25, 26 sont arbitrairement situés sur des droites données. L'existence des droites 345, 346, 356, 456 constitue autant de théorèmes géométriques; c'est-à-dire, chacune de ces droites est déterminée par *trois points.*

En essayant d'approfondir la théorie de six points sur la même conique, on rencontrera un système de neuf points, tel que ceux que nous venons d'examiner; mais il est moins général. Il existe des relations entre les points qui n'ont pas lieu dans le système général. Je renvoie cette discussion à une section séparée de ce mémoire, et je passe au cas de $n = 7$.

Pour ce cas on a tout de suite le théorème suivant:

THÉORÈME V. *Le système de vingt et un points situés trois à trois sur trente-cinq droites, peut être considéré (même de cent vingt manières différentes) comme composé de trois heptagones, le premier circonscrit au second, le second au troisième et le troisième au premier. Les heptagones par exemple peuvent être* (1234567), (1357246), (1526374).

Dans ce système 12, 13, 14, 15, 16, 17 sont arbitraires, et 23, 24, 25, 26, 27 le sont sur des droites données; les droites 345, 346, 347, 356, 357, 367, 456, 457, 467, 567 sont déterminées chacune par trois points. Dans le cas général 12, 13 ... 1n sont arbitraires, et 23 ... 2n le sont sur des droites données. Il existe $\frac{1}{6}(n-2)(n-3)(n-4)$ droites dont chacune est déterminée par trois points. Un théorème analogue à celui-ci a lieu quand n est un nombre premier: savoir le suivant:

THÉORÈME VI. *Le système de N_2 points, situés trois à trois sur N_3 droites, peut être considéré* $\left(\text{même de } \frac{1.2\ldots(n-2)}{n-1} \text{ manières}\right)$ *comme composé de $\frac{1}{2}(n-1)$ n-gones, le premier circonscrit au second, le second au troisième, etc., et le dernier au premier.*

Je ne connais pas d'autres cas où l'idée des nombres premiers se présente dans la géométrie. Il sera peut-être possible de trouver des théorèmes analogues à III, IV, V, pour toutes les formes du nombre n, mais je n'ai pas encore examiné cela.

Le théorème général I, peut être considéré comme l'expression d'un fait analytique, qui doit également avoir lieu en considérant quatre coordonnées au lieu de trois. Ici une interprétation géométrique a lieu, qui s'applique aux points *dans l'espace.* On peut en effet, *sans recourir à aucune notion métaphysique à l'égard de la possibilité de l'espace à quatre dimensions,* raisonner comme suit (tout cela pourra aussi être traduit facilement en langue purement analytique): En supposant quatre dimensions de l'espace, il faudra considérer des *lignes* déterminées par deux points, des *demi-plans* déterminés par trois points, et des *plans* déterminés par quatre points; (deux plans se coupent alors suivant un demi-plan, etc.). L'espace ordinaire doit être considéré comme plan, et il coupera un plan selon un plan ordinaire, un demi-plan selon une ligne ordinaire, et une ligne selon un point ordinaire. Tout cela posé: en considérant un nombre n de points, et les combinant deux à deux, trois à trois, et quatre à quatre par des lignes, des demi-plans et des plans, puis coupant le système par l'espace considéré comme plan, on obtient le théorème suivant de géométrie à trois dimensions:

THÉORÈME VII. *On peut former un système de N_2 points, situés trois à trois dans N_3 droites qui elles-mêmes sont situées quatre à quatre dans N_4 plans. En représentant les points par* 12, 13, *etc., les points situés dans la même droite sont* 12, 13, 23; *et les droites étant représentées par* 123 *etc. comme auparavant, les droites* 123, 124, 134, 234 *sont situées dans le même plan* 1234.

En coupant cette figure par un plan, on obtient le théorème suivant de géométrie plane:

THÉORÈME VIII. *On peut former un système de N_3 points situés quatre à quatre dans N_4 droites. Les points doivent être représentées par la notation* 123, *etc. et les droites par* 1234, *etc. Alors* 123, 124, 134, 234 *sont dans la même droite désignée par* 1234.

De même, en considérant un espace à $p+2$ dimensions, on obtient la proposition suivante, encore plus générale:

THÉORÈME IX. *On peut former dans l'espace un système de N_p points, qui passent $p+1$ à $p+1$ par N_{p+1} droites, situées $p+2$ à $p+2$ dans N_{p+2} plans, ou bien pour la géométrie plane, un système de N_p points, situés $p+1$ à $p+1$ dans N_{p+1} droites.*

Des théorèmes analogues à IV et V seraient probablement très nombreux et très compliqués.

Les réciproques polaires auront évidemment lieu pour tous ces théorèmes; on pourrait aussi les démontrer directement d'une manière analogue.

§ II.

SUR LE THÉORÈME DE PASCAL.

En considérant six points sur la même conique, et les prenant dans un ordre déterminé, pour en former un hexagone, on sait que les côtés opposés se rencontrent dans trois points situés en ligne droite. En prenant les points dans un ordre quelconque, on en peut former soixante hexagones, à chacun desquels correspond une droite; il s'agit maintenant de trouver les relations entre ces droites.

M. Steiner a prouvé dans son ouvrage *Systematische Entwickelungen u. s. w.* [1832], que ces soixante droites passent trois à trois par vingt points, et il ajoute que ces vingt points sont situés quatre à quatre sur quinze droites. La première partie de ce théorème peut être démontrée assez facilement, comme nous le verrons: mais pour la seconde partie, je n'ai pas réussi à trouver les combinaisons de quatre points qui doivent être situés en ligne droite, et il me paraît même qu'il est impossible de les trouver[1].

Cherchons les combinaisons des droites qui doivent passer trois à trois par le même point.

Soient 1, 2, 3, 4, 5, 6 les six points situés sur la même conique. Considérons d'abord l'hexagone 123456 que l'on obtient en prenant les points dans un ordre déterminé. Suivant le théorème de Pascal les trois points

$$12.45, \qquad 23.56, \qquad 34.61$$

(où 12.45 désigne le point d'intersection des lignes passant par les points 1, 2 et 4, 5) sont situés en ligne droite. Considérons les six hexagones

$$\begin{array}{c} 1\ 2\ 3\ 4\ 5\ 6 \\ 1\ 4\ 3\ 6\ 5\ 2 \\ 1\ 6\ 3\ 2\ 5\ 4 \\ \hline 1\ 4\ 3\ 2\ 5\ 6 \\ 1\ 2\ 3\ 6\ 5\ 4 \\ 1\ 6\ 3\ 4\ 5\ 2 \end{array}$$

qu'on tire du premier en permutant les nombres 2, 4, 6 correspondants aux sommets alternés de l'hexagone. Pour les trois premiers on fait les permutations cycliques de ces nombres (savoir 246, 462, 624), pour les trois autres on fait d'abord une inversion 426,

[1] Je ne sais pas s'il existe une démonstration de la seconde partie du théorème; je n'ai pu la trouver nulle part. Au cas que cette partie du théorème n'était pas correcte, il paraît que l'on devra peut-être lui substituer la proposition suivante: "Les vingt points déterminent deux à deux dix lignes qui passent trois à trois par dix points." On verra dans ce qui suit, de quelle manière il faudrait combiner ces points. [Voir 55.]

puis les permutations cycliques (426, 264, 642). En écrivant les combinaisons des points qui doivent être situés en ligne droite, on a

12.45	23.56	34.61
36.12	56.14	52.34
45.36	14.23	16.52
14.25	43.56	32.61
36.14	56.12	54.42
25.36	12.34	61.54

Suivant cette table les points sur la même horizontale sont en ligne droite.

On remarquera d'abord que les trois premières droites passent par les angles des triangles dont les côtés sont 36, 45, 12 et 14, 23, 56. Les côtés homologues de ces triangles se rencontrent en 36.14, 45.23, 12.56 qui sont en ligne droite, c'est-à-dire, par un théorème déjà cité: les trois lignes passent par un même point. On aurait été conduit au même résultat en observant que les trois premières droites passent par les triangles dont les côtés sont 14, 23, 56 et 52, 16, 34 ou enfin 52, 16, 34 et 36, 45, 12. De même les trois dernières droites passent par le même point. Donc il a été démontré ce qui suit:

THÉORÈME X. *En considérant les trois hexagones qu'on obtient en permutant cycliquement les angles alternés du premier, les trois droites qui y correspondent se rencontrent dans un même point. Les soixante lignes passent donc trois à trois par vingt points.*

Ajoutons qu'aux trois hexagones de ce théorème correspondent d'une manière particulière trois autres hexagones, ou que les vingt points doivent se combiner deux à deux d'une manière particulière.

Mais on se formera une idée plus claire du système en remarquant que les neuf droites

36, 45, 12
14, 23, 56
25, 61, 34

ont entre elles une relation qui est polaire réciproque de celle entre les neuf points du théorème IV. Pour faciliter cette comparaison, je prendrai d'abord le théorème analogue pour les tangentes d'une conique.

THÉORÈME XI. Soient 1, 3, 5 et 2, 4, 6 des tangentes à une même conique et 12, etc. les points d'intersection de ces droites: les neuf points

36, 45, 12
14, 23, 56
25, 61, 34

peuvent être déterminés au moyen de six points de l'espace A, B, C, α, β, γ, de manière que $A\alpha$, etc. représente le point d'intersection de la droite passant par A, α avec le plan de la figure. Les points sont correspondants entre eux de cette manière:

$$\begin{array}{lll} A\alpha, & A\beta, & A\gamma \\ B\alpha, & B\beta, & B\gamma \\ C\alpha, & C\beta, & C\gamma \end{array}$$

seulement les points 36, 23, 34 etc. sont en ligne droite, ce qui n'aurait pas lieu pour les points $A\alpha$, $B\beta$, $C\gamma$, si la position de A, B, C, α, β, γ était arbitraire. On est donc conduit à ce problème:

Trouver six points A, B, C, α, β, γ dans l'espace, tels, qu'en représentant par $A\alpha$, etc. l'intersection de la droite menée par $A\alpha$ avec un plan donné, les combinaisons des points

$$\begin{array}{l} (A\alpha,\ B\beta,\ C\gamma) \\ (A\beta,\ B\gamma,\ C\alpha) \\ (A\gamma,\ B\alpha,\ C\beta) \\ (A\alpha,\ B\gamma,\ C\beta) \\ (A\beta,\ B\alpha,\ C\gamma) \\ (A\gamma,\ B\beta,\ C\alpha) \end{array}$$

soient en ligne droite.

Pour le théorème de Pascal, cela donne:

THÉORÈME XII. *Soient* 1, 3, 5 *et* 2, 4, 6 *des points d'une conique, les neuf lignes*

$$\begin{array}{lll} 36, & 45, & 12 \\ 14, & 23, & 56 \\ 25, & 61, & 34 \end{array}$$

peuvent être considérées comme les projections des lignes

$$\begin{array}{lll} A\alpha, & A\beta, & A\gamma \\ B\alpha, & B\beta, & B\gamma \\ C\alpha, & C\beta, & C\gamma \end{array}$$

sur le plan de la figure, où A, B, C, α, β, γ sont six plans, dont la relation reste encore à déterminer.

En effectuant la solution du problème que j'ai indiquée on aurait, à ce qu'il me semble, un point de vue tout à fait nouveau d'envisager les coniques.

Je vais ajouter encore quelques réflexions sur la manière de chercher les relations qui existent entre les vingt points. En écrivant seulement les angles alternés des hexagones, on a cette table:

1.2.3
1.2.4
1.2.5
1.2.6
1.3.4
1.3.5
1.3.6
1.4.5
1.4.6
1.5.6

A chaque symbole correspondent six hexagones, qui, à ce que nous avons vu, se partagent en deux paires de trois hexagones, et à chaque combinaison de trois, il correspond un point. Il y a donc deux points qui correspondent au symbole 1.3.5, deux qui correspondent au symbole 1.3.6, deux au symbole 1.5.6 etc. En représentant donc par $\overline{35}$, $\overline{36}$, $\overline{56}$, les droites passant par ces paires de points, il me paraît probable que ces droites aient ensemble les relations du théorème I, (savoir que $\overline{35}$, $\overline{36}$, $\overline{56}$ se rencontrent dans un point etc.), ce qui donnerait lieu au théorème hypothétique que j'ai énoncé dans une note. Voilà, à ce que je puis apercevoir, la seule manière symétrique de combiner les droites. Mais au moins les symboles

1.3.5
1.3.6
1.5.6

ont entre eux des rapports singuliers. En effet, écrivons pour chacun les neuf points du théorème XII, on a ce tableau:

36,	45,	12
14,	23,	56
25,	61,	34
35,	64,	12
14,	23,	56
26,	15,	34
35,	46,	12
14,	25,	36
26,	13,	54

qui ne contient que quatorze points. Cela mérite des recherches ultérieures.

Démonstration analytique du théorème de Pascal, et de la première partie de celui de M. Steiner. Formules relatives au même sujet.

Soient $P=0$, $Q=0$, $R=0$ les équations des lignes 12, 34, 56. On démontrera assez facilement que les équations des lignes 45, 61, 23 peuvent être représentées par

$$\begin{aligned} P+\nu Q+\mu R&=0,\\ \nu P+\ \ Q+\lambda R&=0,\\ \mu P+\lambda Q+\ \ R&=0. \end{aligned}$$

En effet les six points 1, 2, 3, 4, 5, 6 seront situés dans la conique

$$P^2+Q^2+R^2+\lambda+\frac{1}{\lambda}QR+\mu+\frac{1}{\mu}PR+\nu+\frac{1}{\nu}PQ=0\,;$$

car en faisant dans cette équation $P=0$, l'équation se réduit à

$$\frac{1}{\lambda}(Q+\lambda R)(\lambda Q+R)=0\,;$$

c'est-à-dire, la conique contient les points déterminés par

$$\begin{aligned} (\ P=0,\quad & \nu P+Q+\lambda R=0\ ),\\ (\ P=0,\quad & \mu P+\lambda Q+R=0\ ), \end{aligned}$$

ou bien les points 1, 2; et de même elle contient les autres points 3, 4, 5, 6. Les fonctions P, Q, R sont censées contenir chacune deux constantes arbitraires; donc on a neuf constantes arbitraires dans ce système, qui par conséquent est tout-à-fait général. On peut former le système suivant d'équations:

$$\begin{array}{lrrrl}
12. & P & & & =0,\\
13. & \lambda\mu P+ & Q+ & \lambda R & =0,\\
14. & \lambda P+ & \mu Q+ & \lambda\mu R & =0,\\
15. & P+ & \nu Q+ & \nu\lambda R & =0,\\
16. & \nu P+ & Q+ & \lambda R & =0,\\
23. & \mu P+ & \lambda Q+ & R & =0,\\
24. & P+ & \mu\lambda Q+ & \mu R & =0,\\
25. & \lambda P+ & \nu\lambda Q+ & \nu R & =0,\\
26. & \nu\lambda P+ & \lambda Q+ & R & =0,\\
34. & & Q & & =0,\\
35. & \mu P+ & \mu\nu Q+ & R & =0,\\
36. & \mu\nu P+ & \mu Q+ & \nu R & =0,\\
45. & P+ & \nu Q+ & \mu R & =0,\\
46. & \nu P+ & Q+ & \nu\mu R & =0,\\
56. & & & R & =0.
\end{array}$$

Écrivons les équations des lignes comprises dans la table de neuf points ci-dessus donnés. On a d'abord

$$\begin{array}{lll} \mu\nu P + \mu Q + \nu R = 0, & P + \nu Q + \mu R = 0, & P = 0, \\ \lambda P + \mu Q + \lambda\mu R = 0, & \mu P + \lambda Q + R = 0, & R = 0, \\ \lambda P + \nu\lambda Q + \nu R = 0, & \nu P + Q + \lambda R = 0, & Q = 0. \end{array}$$

En combinant la seconde et la troisième colonne verticale du tableau, on obtient pour les trois points d'intersection des côtés opposés de l'hexagone 123456, les équations

$$\begin{array}{ll} (\ P = 0, & \mu Q + \nu R = 0\ ), \\ (\ R = 0, & \lambda P + \mu Q = 0\ ), \\ (\ Q = 0, & \lambda P + \nu R = 0\ ), \end{array}$$

qui appartiennent à trois points situés sur la droite

$$\lambda P + \mu Q + \nu R = 0,$$

ce qui suffit pour démontrer le théorème de Pascal.

On obtient de même, en combinant les autres paires de colonnes verticales, deux systèmes de trois points, respectivement situés dans les droites

$$\frac{P}{\lambda} + \frac{Q}{\mu} + \frac{R}{\nu} = 0 \quad \text{et}$$

$$\left(\frac{P}{\lambda} + \frac{Q}{\mu} + \frac{R}{\nu}\right)\lambda\mu\nu + \lambda P + \mu Q + \nu R = 0,$$

lesquelles, avec la droite qu'on vient de trouver,

$$\lambda P + \mu Q + \nu R = 0,$$

se rencontrent évidemment dans un même point, déterminé par les deux équations

$$\lambda P + \mu Q + \nu R = 0 \quad \text{et}$$

$$\frac{P}{\lambda} + \frac{Q}{\mu} + \frac{R}{\nu} = 0.$$

Voilà une démonstration de la première partie du théorème de M. Steiner. Les équations que nous venons de trouver appartiennent au point d'intersection des trois droites qui correspondent au premier des trois hexagones du symbole 1.3.5. Pour trouver l'autre point correspondant de la même manière à ce symbole, il faut combiner les colonnes horizontales, ce qui donne pour les coordonnées de ce point :

$$(\lambda - \mu\nu)\, P = (\mu - \nu\lambda)\, Q = (\nu - \lambda\mu)\, R.$$

En cherchant de même les expressions des points qui correspondent aux symboles 1.3.6 et 1.5.6, on obtient des résultats moins élégants, mais qui valent peut-être la peine d'être énoncés ici.

Je forme cette table complète :

Systèmes de trois lignes, qui se rencontrent dans un point.

Pour 1.3.5

$$\text{I.}\quad \begin{cases} \lambda P + \mu Q + \nu R = 0, \\ \dfrac{P}{\lambda} + \dfrac{Q}{\mu} + \dfrac{R}{\nu} = 0, \\ \lambda P + \mu Q + \nu R + \lambda\mu\nu \left(\dfrac{P}{\lambda} + \dfrac{Q}{\mu} + \dfrac{R}{\nu}\right) = 0; \end{cases}$$

$$\text{II.}\quad \begin{cases} (\lambda - \mu\nu) P = (\nu - \lambda\mu) R, \\ (\nu - \lambda\mu) R = (\mu - \nu\lambda) Q, \\ (\mu - \nu\lambda) Q = (\lambda - \mu\nu) P; \end{cases}$$

Pour 1.3.6

$$\text{I.}\quad \begin{cases} \mu P + \lambda Q + \lambda\mu\nu R = 0, \\ \nu\lambda P + \mu\nu Q + R = 0, \\ (\mu P + \lambda Q + \lambda\mu\nu R) + (\nu\lambda P + \mu\nu Q + R) = 0; \end{cases}$$

$$\text{II.}\quad \begin{cases} (\mu - \nu\lambda) P + (1 - \mu\nu\lambda) R = 0, \\ (1 - \nu\lambda\mu) R + (\lambda - \nu\mu) Q = 0, \\ (\lambda - \mu\nu) Q - (\mu - \nu\lambda) P = 0; \end{cases}$$

Pour 1.5.6

$$\text{I.}\quad \begin{cases} (\mu\nu - \lambda\mu^2\nu^2 - \lambda + \lambda\mu^2) P + (\mu - \nu\lambda) Q + (\mu^2\nu - \lambda\mu\nu^2) R = 0, \\ (\mu\nu - \lambda\mu^2\nu^2 - \lambda + \lambda\nu^2) P + (\mu\nu^2 - \lambda\mu^2\nu) Q + (\nu - \lambda\mu) R = 0, \\ (\lambda\mu^2 - \lambda\nu^2) P + (\mu - \nu\lambda - \mu\nu^2 + \lambda\mu^2\nu) Q + (\mu^2\nu - \lambda\mu\nu^2 - \nu + \lambda\mu) R = 0; \end{cases}$$

$$\text{II.}\quad \begin{cases} (\lambda - \lambda\nu^2 + \mu\nu - \lambda\mu^2) P + (\mu - \lambda\mu^2\nu) Q + (\nu - \lambda\mu\nu^2) R = 0, \\ (\lambda\nu^2 - \lambda\mu^2\nu^2 - \mu\nu + \lambda\mu^2) P + (\nu\lambda - \mu\nu^2) Q + (\lambda\mu - \mu^2\nu) R = 0, \\ (\lambda - \lambda\mu^2\nu^2) P + (\nu\lambda - \mu\nu^2 + \mu - \lambda\mu^2\nu) Q + (\lambda\mu - \mu^2\nu + \nu - \lambda\mu\nu^2) R = 0. \end{cases}$$

Note sur le théorème de M. Brianchon.

On peut donner une démonstration semblable de ce théorème, en prenant pour les équations des six tangentes celles-ci :

$$\begin{aligned} &1.\quad P = 0, \\ &3.\quad Q = 0, \\ &5.\quad R = 0, \\ &4.\quad \alpha P + \beta Q + \gamma R = 0, \\ &6.\quad \alpha' P + \beta' Q + \gamma' R = 0, \\ &2.\quad \alpha'' P + \beta'' Q + \gamma'' R = 0, \end{aligned}$$

et en cherchant la relation entre les coefficients qui est nécessaire pour que ces six équations appartiennent aux tangentes d'une même conique. On obtient facilement

$$(\alpha\gamma' - \alpha'\gamma)(\beta'\alpha'' - \beta''\alpha')(\gamma''\beta - \gamma\beta'') = (\alpha'\gamma'' - \alpha''\gamma')(\beta''\alpha - \beta\alpha'')(\gamma\beta' - \gamma'\beta),$$

ce qui exprime aussi la condition pour que les trois diagonales se rencontrent dans un même point.

51.

PROBLÈME DE GÉOMÉTRIE ANALYTIQUE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXI. (1846), pp. 227—230.]

Trouver explicitement les coordonnées des centres de similitude de deux surfaces du second ordre, dont chacune est circonscrite à une même surface de cet ordre.

LEMME.

Soit

$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Lxw + 2Myw + 2Nzw + Pw^2 \ldots\ldots (1),$$

l'expression générale d'une fonction homogène du second ordre à quatre variables, et considérons les dérivées

$$KU = \begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & P \end{vmatrix}, \qquad F_{po}\,U = \begin{vmatrix} & \xi, & \eta, & \rho, & \omega \\ \alpha, & A, & H, & G, & L \\ \beta, & H, & B, & F, & M \\ \gamma, & G, & F, & C, & N \\ \delta, & L, & M, & N, & P \end{vmatrix} \ldots\ldots\ldots\ldots (2),$$

où dans $F_{po}\,U$ les lettres o, p écrites en bas servent à indiquer les variables α, β, γ, δ et ξ, η, ρ, ω qui doivent entrer dans l'expression de cette fonction; par exemple $F_{pp}U$ est ce que devient $F_{po}U$, en écrivant ξ, η, ρ, ω au lieu de α, β, γ, δ.

Cela posé, soit

$$U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Lxw + 2Myw + 2Nzw + 2Pw^2, \ldots (3),$$

$$V = \alpha x + \beta y + \gamma z + \delta w, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

on aura cette équation identique :

$$F_{pp}(U+V^2)\,KU - F_{pp}(U)\,K\,(U+V^2) = (F_{op}U)^2 \ldots\ldots\ldots\ldots (5),$$

qui subsiste même pour un nombre quelconque de variables.

L'expression analytique du théorème consiste en effet en ce que les réciproques de deux surfaces du second ordre, circonscrites l'une à l'autre, sont deux surfaces du second ordre qui ont cette même relation. Car en prenant $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$ pour les coordonnées d'un point, les équations de deux surfaces circonscrites l'une à l'autre sont

$$\left.\begin{aligned} U &= 0, \\ U+V^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Les équations de leurs réciproques polaires (par rapport à $x^2+y^2+z^2+w^2=0$) sont

$$F_{pp}U = 0, \quad F_{pp}(U+V^2) = 0,$$

c'est-à-dire, et vertu du théorème qui vient d'être posé :

$$F_{pp}U = 0, \qquad K\,(U+V^2)\,F_{pp}U + (F_{op}U)^2 = 0 \ldots\ldots\ldots\ldots (7),$$

où $K(U+V^2)$ est constant ; c'est-à-dire les premières parties des équations ne diffèrent entre elles que par le carré de la fonction linéaire $(F_{op}U)$; ce qui prouve le théorème en question.

SOLUTION.

Soient $$U+V_1^2 = 0 \text{ et } U+V_2^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

les équations des deux surfaces, dont chacune est circonscrite à $U=0$. Les expressions de V_1 et V_2 sont

$$\left.\begin{aligned} V_1 &= \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 w, \\ \text{et} \quad V_2 &= \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 w, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

et les lettres o_1, o_2 écrites en bas se rapporteront à α_1, β_1, γ_1, δ_1 et à α_2, β_2, γ_2, δ_2 respectivement. Mettons de plus, pour abréger,

$$\left.\begin{aligned} K(U+V_1^2) &= K_1, \\ K(U+V_2^2) &= K_2, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Les polaires des deux surfaces ont pour équations

$$\left.\begin{aligned} K_1 F_{pp}(U) + (F_{o_1 p}U)^2 &= 0, \\ K_2 F_{pp}(U) + (F_{o_2 p}U)^2 &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

et ces surfaces polaires se rencontrent évidemment selon les courbes situées dans les plans exprimés par les équations

$$\sqrt{K_2}F_{o_1 p}U \pm \sqrt{K_1}F_{o_2 p}U = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12) ;$$

équations qu'on peut écrire sous cette forme très simple :

$$F_{o'p}(U) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

en mettant

$$\left.\begin{aligned} \alpha' &= \sqrt{K_2}\,\alpha_1 \pm \sqrt{K_1}\,\alpha_2, \\ \beta' &= \sqrt{K_2}\,\beta_1 \pm \sqrt{K_1}\,\beta_2, \\ \gamma' &= \sqrt{K_2}\,\gamma_1 \pm \sqrt{K_1}\,\gamma_2, \\ \delta' &= \sqrt{K_2}\,\delta_1 \pm \sqrt{K_1}\,\delta_2, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

et supposant que o' se rapporte à α', β', γ', δ'. On a enfin, en se servant de la notation complète des déterminants,

$$\begin{vmatrix} & \xi, & \eta, & \rho, & \omega \\ \alpha', & A, & H, & G, & L \\ \beta', & H, & B, & F, & M \\ \gamma', & G, & F, & C, & N \\ \delta', & L, & M, & N, & P \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

équation qui est double, à cause des doubles valeurs de α', β', γ', δ'. Les pôles de ces plans sont les centres de similitude des deux surfaces données. Soit donc identiquement

$$\begin{vmatrix} & \xi, & \eta, & \rho, & \omega \\ \alpha', & A, & H, & G, & L \\ \beta', & H, & B, & F, & M \\ \gamma', & G, & F, & C, & N \\ \delta', & L, & M, & N, & P \end{vmatrix} = \mathfrak{A}\xi + \mathfrak{B}\eta + \mathfrak{C}\rho + \mathfrak{D}\omega \ \ldots (16),$$

on a

$$x : y : z : w = \mathfrak{A} : \mathfrak{B} : \mathfrak{C} : \mathfrak{D} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

pour les coordonnées $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$ des deux centres de similitude. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ sont données par l'équation (16), savoir par

$$\mathfrak{A} = \begin{vmatrix} . & 1 & . & . & . \\ \alpha', & A, & H, & G, & L \\ \beta', & H, & B, & F, & M \\ \gamma', & G, & F, & C, & N \\ \delta', & L, & M, & N, & P \end{vmatrix} \quad \text{ou } \mathfrak{A} = -\begin{vmatrix} \alpha', & H, & G, & L \\ \beta', & B, & F, & M \\ \gamma', & F, & C, & N \\ \delta', & M, & N, & P \end{vmatrix}, \text{ \&c.} \quad (18),$$

α', β', γ', δ' sont donnés par (14), et K_1, K_2 représentent ce que devient le déterminant

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & P \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

en écrivant $A+\alpha_1^2$, $B+\beta_1^2$, $C+\gamma_1^2$, $P+\delta_1^2$, $F+\beta_1\gamma_1$, $G+\gamma_1\alpha_1$, $H+\alpha_1\beta_1$, $L+\alpha_1\delta_1$, $M+\beta_1\delta_1$, $N+\gamma_1\delta_1$, ou $A+\alpha_2^2$, &c. au lieu de A, B, C, P, F, G, H, L, M, N.

52.

SUR QUELQUES PROPRIÉTÉS DES DÉTERMINANTS GAUCHES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXII. (1846), pp. 119—123.]

I.

JE donne le nom de *déterminant gauche*, à un déterminant formé par un système de quantités $\lambda_{r.s}$ qui satisfont aux conditions

$$\lambda_{r.s} = -\lambda_{s.r}\,[r \neq s] \quad \text{.....} \quad (1).$$

J'appelle aussi un tel système, *système gauche*. On obtiendra des formules plus simples (quoique cette supposition ne soit pas essentielle), en considérant seulement les systèmes pour lesquels on a aussi

$$\lambda_{r.r} = 1 \quad \text{.....} \quad (2).$$

Je suppose dans tout ce qui va suivre, que le déterminant est de l'ordre n, et que par conséquent les suffixes variables r, s, &c., s'étendent toujours depuis l'unité jusqu'à n.

En posant les équations

$$\Sigma_r \lambda_{r.s} x_r = P_s, \quad \Sigma_s \lambda_{r.s} x_s = Q_r \quad \text{.....} \quad (3),$$

j'exprime les systèmes inverses qui déterminent les P, Q par les x, de la manière suivante:

$$Kx_r = \Sigma_s \Lambda_{r.s} P_s, \quad \text{et} \quad Kx_s = \Sigma_r \Lambda_{r.s} Q_r \quad \text{.....} \quad (4),$$

où K désigne le déterminant formé avec les quantités $\lambda_{r.s}$, et $\Lambda_{r.s}$ le coefficient différentiel de K par rapport à $\lambda_{r.s}$; bien entendu, que la différentiation doit être

effectuée avant d'avoir particularisé ces quantités par les équations (1), (2). On sait que ces fonctions Λ satisfont aux conditions

$$\left.\begin{array}{l} \Sigma_r \lambda_{r.s} \Lambda_{r.s'} = 0 \ , \ s \neq s', \\ \Sigma_r \lambda_{r.s} \Lambda_{r.s} = K, \\ \Sigma_r \lambda_{r.s} \Lambda_{r'.s} = 0 \ , \ r \neq r', \\ \Sigma_s \lambda_{r.s} \Lambda_{r.s} = K. \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (5).$$

Je tire des équations (4), en échangeant r et s dans la dernière de ces équations:

$$\Sigma_s \Lambda_{r.s} P_s = \Sigma_s \Lambda_{s.r} Q_s \quad \ldots\ldots\ldots\ldots \quad (6),$$

et de là, en multipliant respectivement les différentes équations de ce système par $\lambda_{s'.r}$, et prenant la somme de ces produits:

$$\Sigma_s (\Sigma_r \lambda_{s'.r} \Lambda_{r.s}) P_s = \Sigma_s (\Sigma_r \lambda_{s'.r} \Lambda_{s.r}) Q_s \ldots\ldots\ldots\ldots (7).$$

On a d'abord par les équations (5)

$$\Sigma_s (\Sigma_r \lambda_{s'.r} \Lambda_{s.r}) Q_s = K Q_{s'} \ldots\ldots\ldots\ldots (8);$$

puis par les équations (1) et (2)

$$\Sigma_r \lambda_{s'.r} \Lambda_{r.s} = 2\Lambda_{s'.s} - \Sigma_r \lambda_{r.s'} \Lambda_{r.s} \ldots\ldots\ldots\ldots (9),$$

c'est-à-dire par les équations (5):

$$\left.\begin{array}{l} \Sigma_r \lambda_{s'.r} \Lambda_{r.s} = 2\Lambda_{s'.s}; \ s' \neq s \\ \Sigma_r \lambda_{s'.r} \Lambda_{r.s'} = 2\Lambda_{s'.s'} - K \end{array}\right\} \ldots\ldots\ldots\ldots (10);$$

et

ce qui donne

$$\Sigma_s (\Sigma_r \lambda_{s'.r} \Lambda_{r.s}) P_s = 2(\Sigma_s \Lambda_{s'.s} P_s) - K P_{s'} \ldots\ldots\ldots\ldots (11).$$

Substituant les équations (8) et (11) dans la formule (7), on obtiendra, en écrivant r au lieu de s':

$$K Q_r = 2(\Sigma_s \Lambda_{r.s} P_s) - K P_r \ldots\ldots\ldots\ldots (12),$$

et également

$$K P_s = 2(\Sigma_r \Lambda_{r.s} Q_r) - K Q_s \ldots\ldots\ldots\ldots (13).$$

Posant maintenant

$$\left.\begin{array}{l} K\alpha_{r.s} = 2\Lambda_{r.s}; \ r \neq s \\ K\alpha_{r.r} = 2\Lambda_{r.r} - K \end{array}\right\} \ldots\ldots\ldots\ldots (14):$$

et

les formules (12), (13) se changeront en

$$Q_r = \Sigma_s \alpha_{r.s} P_s \ \text{et} \ P_s = \Sigma_r \alpha_{r.s} Q_r \ldots\ldots\ldots\ldots (15):$$

équations qui sont nécessairement équivalentes. On a donc identiquement

$$\left.\begin{array}{ll}(1) & \left\{\begin{array}{l}\Sigma_r\, \alpha_{r.s}\alpha_{r.s'} = 0\,;\ s \neq s',\\ \Sigma_r\, \alpha_{r.s}\alpha_{r.s} = 1,\end{array}\right.\\ (2) & \left\{\begin{array}{l}\Sigma_s\, \alpha_{r.s}\alpha_{r'.s} = 0\,;\ r \neq r',\\ \Sigma_s\, \alpha_{r.s}\alpha_{r.s} = 1:\end{array}\right.\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

c'est-à-dire, on a trouvé un système de n^2 quantités $\alpha_{r.s}$, fonctions explicites et rationnelles d'un nombre $\frac{1}{2}n(n-1)$ de variables indépendantes, qui satisfont identiquement aux formules (16, 1) et (16, 2). On sait qu'en géométrie cela veut dire que pour $n=2$ ou $n=3$ de tels systèmes donnent les coefficients propres à effectuer la transformation de deux systèmes de coordonnées rectangulaires; nous dirons par analogie, que des systèmes qui satisfont aux équations (16) pour une valeur quelconque de n, sont propres à effectuer la transformation entre deux systèmes de coordonnées rectangulaires. On a donc le théorème suivant:

Les coefficients propres à la transformation de coordonnées rectangulaires, peuvent être exprimés rationnellement au moyen de quantités arbitraires $\lambda_{r.s}$, soumises aux conditions

$$\lambda_{s.r} = -\lambda_{s.r}\ [r \neq s]\,;\quad \lambda_{r.r} = 1.$$

Pour développer les formules, il faut d'abord former le déterminant K de ce système, puis le système inverse $\Lambda_{r.s}, \ldots$ et écrire

$$K\alpha_{r.s} = 2\Lambda_{r.s}\,[r \neq s]\,;\quad K\alpha_{r.r} = 2\Lambda_{r.r} - K\,;$$

ce qui donne le système cherché.

Soit par exemple $n=3$. Écrivons pour le système des quantités $\lambda_{r.s}$:

$$\left.\begin{array}{rrr} 1, & \nu, & -\mu,\\ -\nu, & 1, & \lambda,\\ \mu, & -\lambda, & 1,\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17),$$

ce qui donne $K = 1 + \lambda^2 + \mu^2 + \nu^2$, et pour le système des fonctions $\Lambda_{r.s}$

$$\left.\begin{array}{lll} 1+\lambda^2, & \lambda\mu + \nu\,, & \nu\lambda - \mu,\\ \lambda\mu - \nu\,, & 1+\mu^2, & \mu\nu + \lambda,\\ \nu\lambda + \mu, & \mu\nu - \lambda\,, & 1+\nu^2.\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

De là on obtient pour le système de coefficients $\alpha, \beta, \gamma;\ \alpha', \beta', \gamma';\ \alpha'', \beta'', \gamma''$:

$$\left.\begin{array}{lll} K\alpha = 1+\lambda^2-\mu^2-\nu^2, & K\alpha' = 2(\lambda\mu+\nu) & , \quad K\alpha'' = 2(\nu\lambda-\mu) \quad ,\\ K\beta = 2(\lambda\mu-\nu) \quad , & K\beta' = (1+\mu^2-\nu^2-\lambda^2), & K\beta'' = 2(\mu\nu+\lambda) \quad ,\\ K\gamma = 2(\nu\lambda+\mu) \quad , & K\gamma' = 2(\mu\nu-\lambda) \quad , & K\gamma'' = (1+\nu^2-\lambda^2-\mu^2)\,;\end{array}\right\} \ldots (19),$$

ce qui se rapporte à la transformation

$$\left.\begin{array}{ll} x = \alpha\, x_1 + \beta\, y_1 + \gamma\, z_1, & x_1 = \alpha x + \alpha' y + \alpha'' z\,,\\ y = \alpha' x_1 + \beta' y_1 + \gamma' z_1, & y_1 = \beta x + \beta' y + \beta'' z,\\ z = \alpha'' x_1 + \beta'' y_1 + \gamma'' z_1, & z_1 = \gamma x + \gamma' y + \gamma'' z,\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

de deux systèmes de coordonnées rectangulaires. En effet, les coefficients λ, μ, ν ont une signification géométrique: Les axes Ax_1, Ay_1, Az_1 vont coïncider avec les axes Ax, Ay, Az, par la rotation θ autour d'un certain axe AR (qu'on peut nommer "Axe résultant"). En prenant f, g, h pour les inclinaisons de cet axe à Ax, Ay, Az, on a $\lambda = \operatorname{tang}\frac{1}{2}\theta\cos f$, $\mu = \operatorname{tang}\frac{1}{2}\theta\cos g$, $\nu = \operatorname{tang}\frac{1}{2}\theta\cos h$. Cette expression de l'axe AR est due à Euler; les quantités λ, μ, ν ont été introduites pour la première fois, par M. Olinde Rodrigues, dans un mémoire "Sur les lois géométriques qui régissent les déplacements d'un système solide" (Liouville, tom. v. [1840]), où il donne [des expressions semblables à celles] qu'on vient de trouver ici, pour les coefficients de la transformation, en termes de λ, μ, ν. Ces mêmes quantités λ, μ, ν (il y a à remarquer cela en passant) sont liées de la manière la plus étroite avec celles de la belle théorie de Sir W. Hamilton sur les Quaternions. Je les ai appliquées à la théorie de la rotation d'un corps solide. Avant de donner une idée des résultats auxquels je suis parvenu, je passe aux formules de transformation qui se rapportent au cas de $n=4$. Je prends ici pour le système des quantités λ:

$$\left.\begin{array}{rrrr} 1, & a, & b, & c, \\ -a, & 1, & -h, & g, \\ -b, & h, & 1, & -f, \\ -c, & -g, & f, & 1, \end{array}\right\} \dots\dots\dots\dots (21),$$

ce qui donne, en mettant pour abréger, $af+bg+ch=\theta$,

$$K = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \theta^2 \dots\dots\dots\dots (22),$$

et puis pour les quantités $\Lambda_{r.s}$ le système

$$\left.\begin{array}{llll} 1+f^2+g^2+h^2, & f\theta+a+bh-cg, & g\theta+b+cf-ah, & h\theta+c+ag-bf, \\ -f\theta-a+bh-cg, & 1+f^2+b^2+c^2, & -c\theta-h+fg-ab, & b\theta+g+hf-ca, \\ -g\theta-b+cf-ah, & c\theta+h+fg-ab, & 1+g^2+c^2+a^2, & -a\theta-f+gh-bc, \\ -h\theta-c+ag-bf, & -b\theta-g+hf-ca, & a\theta+f+gh-bc, & 1+h^2+a^2+b^2, \end{array}\right\} \dots (23),$$

de manière que pour

$$\left.\begin{array}{llll} K\alpha, & K\alpha', & K\alpha'', & K\alpha''', \\ K\beta, & K\beta', & K\beta'', & K\beta''', \\ K\gamma, & K\gamma', & K\gamma'', & K\gamma''', \\ K\delta, & K\delta', & K\delta'', & K\delta''', \end{array}\right\} \dots\dots\dots\dots (24),$$

on obtient le système suivant:

$$\begin{array}{llll} 1+f^2+g^2+h^2-a^2-b^2-c^2, & 2(f\theta+a+bh-cg), & 2(g\theta+b+cf-ah), & 2(h\theta+c+ag-bf), \\ 2(-f\theta-a+bh-cg), & 1+f^2+b^2+c^2-g^2-h^2-a^2, & 2(-c\theta-h+fg-ab), & 2(b\theta+g+hf-ca), \\ 2(-g\theta-b+cf-ah), & 2(c\theta+h+fg-ab), & 1+g^2+c^2+a^2-f^2-h^2-b^2, & 2(-a\theta-f+gh-bc), \\ 2(-h\theta-c+ag-bf), & 2(-b\theta-g+hf-ca), & 2(a\theta+f+gh-bc), & 1+h^2+a^2+b^2-f^2-g^2-c^2, \end{array}$$

et ainsi de suite pour des valeurs quelconques de n.

II.

Maintenant je vais citer les formules que j'ai présentées dans le *Journal de Cambridge*, t. III. (1843) p. 225, [6], pour la rotation d'un corps solide autour d'un point fixe. Mettant, comme à l'ordinaire, les vitesses angulaires autour des axes principaux p, q, r, les *moments* du corps pour ces mêmes axes $=A$, B, C, et la fonction des forces $=V$: les équations citées pourront être écrites sous la forme

$$dt = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\lambda}{\Lambda} = \frac{d\mu}{M} = \frac{d\nu}{N} \quad \text{.....} \quad (26),$$

$$\left.\begin{aligned}
P &= \frac{1}{A}\left((B-C)\,qr + \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu+\nu)\frac{dV}{d\mu} + (\lambda\nu-\mu)\frac{dV}{d\nu}\right\}\right),\\
Q &= \frac{1}{B}\left((C-A)\,rp + \tfrac{1}{2}\left\{(\mu\lambda-\nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu+\lambda)\frac{dV}{d\nu}\right\}\right),\\
R &= \frac{1}{C}\left((A-B)\,pq + \tfrac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda} + (\mu\nu-\lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\}\right),\\
\Lambda &= \tfrac{1}{2}\{(1+\lambda^2)\,p + (\lambda\mu-\nu)\,q + (\lambda\nu+\mu)\,r\},\\
M &= \tfrac{1}{2}\{(\mu\lambda+\nu)\,p + (1+\mu^2)\,q + (\mu\nu-\lambda)\,r\},\\
N &= \tfrac{1}{2}\{(\nu\lambda-\mu)\,p + (\nu\mu+\lambda)\,q + (1+\nu^2)\,r\}.
\end{aligned}\right\} \cdots (27).$$

En effet, pour obtenir ces formules, il n'y a qu'à chercher au moyen de λ, μ, ν, et de leurs dérivées par rapport aux temps λ', μ', ν', l'expression de la fonction $T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$ {qui exprime la demi-somme des forces vives}: cela fait, les formules générales que Lagrange a données pour la solution des problèmes de dynamique, conduisent immédiatement aux équations en question. Dans le mémoire cité j'ai intégré ces équations pour le cas où la fonction V est zéro, et en prenant, comme dans la théorie ordinaire, le plan invariable pour plan des deux axes. Ce n'est que dernièrement que j'ai trouvé la manière convenable de traiter ce système d'équations; je le fais au moyen de deux nouvelles variables Ω, v, entre lesquelles je trouve une équation différentielle dont les variables sont séparées, et j'exprime en termes de celles-ci les autres variables du problème, y compris le temps; et cela sans aucune supposition particulière, relative à la position des axes des coordonnées par rapport au plan invariable. Le développement de cette théorie paraîtra dans le prochain No. du *Cambridge and Dublin Mathematical Journal*, [37]. Je m'occupe aussi de la recherche des formules pour les variations des constantes arbitraires relatives aux perturbatrices. Il serait bien intéressant (comme problème d'analyse pure) d'étendre ces recherches au cas d'une valeur quelconque de n; il faudrait pour cela, chercher les valeurs des quantités analogues à p, q, r, former une fonction T, en prenant la somme des carrés de chacune de ces quantités, chaque carré multiplié par un coefficient constant, et puis former les équations $\frac{d}{dt}\cdot\frac{dT}{d\lambda'} - \frac{dT}{d\lambda} = 0$, &c., analogues aux équations de Dynamique. Mais je n'ai encore rien trouvé sur ce sujet.

53.

RECHERCHES SUR L'ÉLIMINATION, ET SUR LA THÉORIE DES COURBES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tome XXXIV. (1847), pp. 30—45.]

En désignant par U, V, W, ... des fonctions homogènes des ordres m, n, p, &c. et d'un nombre égal de variables respectivement, et en supposant que ces fonctions soient les plus générales possibles, c'est-à-dire que le coefficient de chaque terme soit une lettre indéterminée: on sait que les équations $U=0$, $V=0$, $W=0$, ... offrent une relation $\Theta=0$, dans laquelle les variables n'entrent plus, et où la fonction Θ, que l'on peut nommer *Résultant complet des équations*, est homogène et de l'ordre np ... par rapport aux coefficients de U, de l'ordre mp ... par rapport à ceux de V, et ainsi de suite, tandis qu'elle n'est pas décomposable en facteurs. Cela posé, supposons que les coefficients de U, V, ..., au lieu d'être tous indéterminés, soient des fonctions quelconques d'un certain nombre de quantités arbitraires. Substituant ces valeurs dans la fonction Θ, cette fonction sera toujours le *Résultant complet* des équations. Mais Θ peut quelquefois être décomposable en facteurs, dont quelques-uns doivent être éliminés. En effet les coefficients de U, V, ... peuvent contenir des quantités ξ, η, ... censées comme variables, et d'autres quantités α, β, ... qui sont constantes, et il peut s'agir de la relation entre ξ, η, ... qui est nécessaire pour que les équations $U=0$, $V=0$, &c. puissent subsister conjointement. Dans ce cas tout facteur Λ du résultant complet Θ, qui ne contient pas les coefficients variables ξ, η, ..., doit être rejeté. En supprimant ces facteurs, et exprimant par Φ le facteur qui reste, cette fonction est alors ce que nous nommerons *Résultant réduit*. Cependant les facteurs Λ, proprement dits, ne sont jamais des facteurs étrangers, et ce n'est qu'à cause du point de vue particulier, auquel on a envisagé la question, qu'ils ont été rejetés; d'un autre point de vue ces facteurs auraient pu devenir le *Résultant réduit*. Nous les nommerons donc, à l'exemple de M. Sylvester, *Facteurs spéciaux*.

Pour faire voir tout cela avec plus de clarté, prenons pour exemple le problème suivant de Géométrie analytique:

"Trouver les équations du système des lignes tirées par un point donné aux points d'intersection de deux courbes données."

Soient $U=0$, $V=0$ les équations des deux courbes, U, V étant des fonctions homogènes des variables x, y, z, des ordres m et n respectivement, ce qui revient à prendre $x:z$ et $y:z$ pour coordonnées d'un point. De même il faut exprimer par (α, β, γ) le point dont les coordonnées sont $\alpha:\gamma$ et $\beta:\gamma$; et ainsi pour tous cas semblables. Il s'entend, qu'on suppose partout que les coefficients de U, V, ou de U, restent absolument indéterminés. Représentons par (α, β, γ) le point donné, et par (ξ, η, ζ) un point quelconque d'une des lignes dont il s'agit. En éliminant x, y, z entre les équations

$$U=0,\ V=0,\ \text{et}\ \ x(\beta\zeta-\gamma\eta)+y(\gamma\xi-\alpha\zeta)+z(\alpha\eta-\beta\xi)=0 \quad \ldots\ldots\ldots\ldots(1).$$

on obtient l'équation cherchée $\Theta=0$. Ici Θ est une fonction homogène de l'ordre mn par rapport à $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$ et $\alpha\eta-\beta\xi$; de l'ordre n par rapport aux coefficients de U; et de l'ordre n par rapport à ceux de V; de plus cette fonction est décomposable en mn facteurs linéaires par rapport à ξ, η, ζ, dont les coefficients sont des fonctions irrationnelles de α, β, γ et des coefficients de U et V (en effet toute fonction homogène de $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$, $\alpha\eta-\beta\xi$ est douée de cette propriété, qui subsiste encore en échangeant entre eux ξ, η, ζ et α, β, γ). Chaque facteur linéaire, égalé à zéro, appartient à une des lignes en question. Voilà pourquoi $\Theta=0$ est considérée comme équation du système des lignes.

Soit maintenant proposé le problème:

"Trouver l'équation du système des tangentes tirées d'un point fixe à une courbe donnée."

Il y a ici à éliminer x, y, z entre les équations

$$\left.\begin{aligned} &U=0,\\ &\alpha\frac{dU}{dx}+\beta\frac{dU}{dy}+\gamma\frac{dU}{dz}=0\ \text{et}\\ &x(\beta\zeta-\gamma\eta)+y(\gamma\xi-\alpha\zeta)+z(\alpha\eta-\beta\xi)=0. \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Le résultant complet est une fonction homogène de l'ordre $n(n-1)$ par rapport à $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$ et $\alpha\eta-\beta\xi$ [n représente ici l'ordre de la fonction U], de l'ordre n par rapport à α, β, γ, et de l'ordre $2n-1$ par rapport aux coefficients de U. Mais il existe dans ce cas un facteur spécial U_0 qui est ce que devient U en écrivant α, β, γ à la place de x, y, z. En le mettant de côté, le résultant réduit Φ est fonction de l'ordre $n(n-1)$ par rapport à $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$ et $\alpha\eta-\beta\xi$, et de l'ordre $2(n-1)$ par rapport aux coefficients de U, et l'équation $\Phi=0$ correspond au système de tangentes.

En mettant x, y, z à la place de $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$, $\alpha\eta-\beta\xi$, Φ devient une fonction de l'ordre $n(n-1)$ par rapport à x, y, z, et de l'ordre $2(n-1)$, comme elle l'était ci-dessus, par rapport aux coefficients de U. Nous désignerons cette nouvelle valeur de Φ par $\mathsf{F}U$; c'est-à-dire nous représenterons par $\mathsf{F}U$ le résultant réduit des équations

$$\left.\begin{array}{l} U=0, \\ \alpha\dfrac{dU}{dx}+\beta\dfrac{dU}{dy}+\gamma\dfrac{dU}{dz}=0, \text{ et} \\ x\mathrm{x}+y\mathrm{y}+z\mathrm{z}=0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$\mathsf{F}U$ étant une fonction des ordres $n(n-1)$ et $2(n-1)$ par rapport à x, y, z, et aux coefficients de U. On sait que l'équation $\mathsf{F}U=0$ est celle de la polaire réciproque de la courbe, par rapport à la conique auxiliaire $x^2+y^2+z^2=0$.

Or les équations (2) peuvent être écrites aussi sous la forme

$$\left.\begin{array}{l} U=0, \\ \alpha\dfrac{dU}{dx}+\beta\dfrac{dU}{dy}+\gamma\dfrac{dU}{dz}=0, \\ \xi\dfrac{dU}{dx}+\eta\dfrac{dU}{dy}+\zeta\dfrac{dU}{dz}=0. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Ici le résultant complet est de l'ordre $n(n-1)$ par rapport à α, β, γ, ou à ξ, η, ζ (car ce résultant complet doit être comme ci-dessus fonction de ce même ordre de $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$ et $\alpha\eta-\beta\xi$) et de l'ordre $(n-1)(3n-1)$ par rapport aux coefficients de U. Le facteur spécial dans ce cas est donc une fonction de l'ordre $3(n-1)^2$ des coefficients de U, et il est facile de trouver sa forme; car on satisferait aux équations (4) en posant

$$\frac{dU}{dx}=0, \quad \frac{dU}{dy}=0, \quad \frac{dU}{dz}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

le résultant de ces équations, que nous désignerons toujours par KU, doit donc se présenter comme facteur spécial du résultant complet du système. Mais KU étant une fonction des coefficients de l'ordre $3(n-1)^2$, elle est précisément le facteur spécial dont il s'agit. (Il est clair que l'équation $KU=0$ serait la condition nécessaire, pour que la courbe pût avoir un point multiple.)

Reprenons le premier système. On satisfait à la dernière équation en écrivant

$$x=\alpha l+\xi m, \quad y=\beta l+\eta m, \quad z=\gamma l+\zeta m \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Soient $[U]$, $[V]$ ce que deviennent U, V par cette substitution. En éliminant l, m entre les deux équations

$$[U]=0, \quad [V]=0,$$

on obtiendra le même résultant Θ que ci-dessus. En effet, le résultant de ces deux équations est des ordres n et m par rapport aux coefficients de U et V, et de l'ordre $2mn$ par rapport à α, β, γ, ξ, η, ζ: donc il faut qu'il soit égal à Θ, à un facteur numérique près. On a donc le théorème suivant:

THÉORÈME I. L'équation du système des droites menées par un point donné (α, β, γ) aux points d'intersection des deux courbes $U=0$, $V=0$, se trouve en éliminant les nouvelles variables l, m entre les deux équations $[U]=0$ et $[V]=0$, où $[U]$, $[V]$ sont ce que deviennent U et V par les substitutions

$$x=\alpha l+\xi m, \quad y=\beta l+\eta m, \quad z=\gamma l+\zeta m \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

En opérant également sur le second système d'équations, on obtient directement le résultant réduit, sans que l'opération soit embarrassée par aucun facteur spécial. En effet, on peut remplacer le système par

$$\left.\begin{aligned} &\alpha\frac{dU}{dx}+\beta\frac{dU}{dy}+\gamma\frac{dU}{dz}=0,\\ &\xi\frac{dU}{dx}+\eta\frac{dU}{dy}+\zeta\frac{dU}{dz}=0,\\ &x(\beta\zeta-\gamma\eta)+y(\gamma\xi-\alpha\zeta)+z(\alpha\eta-\beta\xi)=0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

et en faisant les substitutions (6) dans la dernière équation, les deux autres équations se changent en

$$\frac{d[U]}{dl}=0 \quad \text{et} \quad \frac{d[U]}{dm}=0,$$

où $[U]$ est ce que devient U par cette substitution. Le résultant de ces équations est de l'ordre $2n(n-1)$ par rapport à α, β, γ, ξ, η, ζ {c'est-à-dire une fonction de $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$, $\alpha\eta-\beta\xi$, de l'ordre $n(n-1)$}, et de l'ordre $2(n-1)$ par rapport aux coefficients de U; donc il n'y a plus de facteur spécial. De là suit:

THÉORÈME II. L'équation du système de tangentes menées du point donné (α, β, γ) à la courbe $U=0$, se trouve en éliminant l, m entre les équations

$$\frac{d[U]}{dl}=0, \quad \frac{d[V]}{dl}=0,$$

$[U]$ étant ce que devient U par les substitutions $x=\alpha l+\xi m$, $y=\beta l+\eta m$ et $z=\gamma l+\zeta m$. En représentant l'équation par $\Phi=0$, Φ est une fonction de $\beta\zeta-\gamma\eta$, $\gamma\xi-\alpha\zeta$, $\alpha\eta-\beta\xi$, et en remplaçant ces fonctions par x, y, z, on obtient l'équation de la polaire réciproque (par rapport à $x^2+y^2+z^2=0$) de la courbe donnée.

Ce beau théorème est dû à M. Joachimsthal, qui me l'a communiqué l'été passé pendant mon séjour à Berlin, avec une démonstration.

On déduit de là, comme cas très particulier, une forme du résultant des deux équations $ax^2+2bxy+cy^2=0$ et $a'x^2+2b'xy+c'y^2=0$, citée dans mon mémoire sur les hyperdéterminants (t. XXX. de ce journal, [16]). En effet, soit $x^2=\mathrm{z}$, $xy=-\mathrm{y}$, $y^2=\mathrm{x}$, et $U=\mathrm{xz}-\mathrm{y}^2$, on aura évidemment $U=0$,

$$a\frac{dU}{d\mathrm{x}}+b\frac{dU}{d\mathrm{y}}+c\frac{dU}{d\mathrm{z}}=0, \quad \text{et} \quad a'\frac{dU}{d\mathrm{x}}+b'\frac{dU}{d\mathrm{y}}+c'\frac{dU}{d\mathrm{z}}=0,$$

donc en cherchant le résultant de ces équations de la manière indiquée dans le théorème, on obtient la formule en question $4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0$. Cependant la véritable généralisation de cette formule, à ce que je crois, reste encore à trouver.

Il suit des principes développés dans le mémoire cité, que le résultant des deux équations $L=0$, $M=0$ (où L, M sont des fonctions homogènes des deux variables l, m) peut toujours être présenté sous la forme $\Theta=\nabla LL\ldots MM\ldots$, où ∇ est une composition d'expressions symboliques, telles que $(12)^\alpha(13)^\beta\ldots$, dans lesquelles $(12)=\partial_{l_1}\partial_{m_2}-\partial_{m_1}\partial_{l_2}$, &c. Par exemple pour $L=al^2+2blm+cm^2$, $M=a'l^2+2b'lm+c'm^2$, on a

$$\Theta=\{(12)^2(34)^2-(13)^2(24)^2\}\,LL\,MM,$$

{c'est-à-dire, comme ci-dessus, $\Theta=4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2$}. En appliquant cette théorie à l'élimination des inconnues entre les équations $[U]=0$, $[V]=0$, on obtient

$$\partial_l=\alpha\partial_x+\beta\partial_y+\gamma\partial_z,\ \text{ et }\ \partial_m=\xi\partial_x+\eta\partial_y+\zeta\partial_z,$$

et en faisant

$$\beta\zeta-\gamma\eta=\mathrm{x},\quad \gamma\xi-\alpha\zeta=\mathrm{y},\quad \alpha\eta-\beta\xi=\mathrm{z},$$

et
$$\partial_{y_1}\partial_{z_2}-\partial_{z_2}\partial_{y_1}=(12)',\quad \partial_{z_1}\partial_{x_2}-\partial_{x_2}\partial_{z_1}=(12)'',\quad \partial_{x_1}\partial_{y_2}-\partial_{x_2}\partial_{y_1}=(12)''',$$

cela donne

$$(12)=\mathrm{x}\,(12)'+\mathrm{y}\,(12)''+\mathrm{z}\,(12)''':$$

équation qui peut être présentée sous la forme abrégée

$$(12)=(P\,12).$$

On obtient le résultant cherché en faisant cette substitution dans tous les symboles que contient ∇, et en introduisant U, V au lieu de $[U]$, $[V]$.

Les mêmes remarques peuvent être appliquées au cas d'une élimination entre les deux équations $\frac{dL}{dl}=0$, $\frac{dL}{dm}=0$, car le résultant sera exprimé ici sous la même forme $\Theta=\nabla LL\ldots$. Cherchons par exemple l'équation de la polaire réciproque d'une courbe du second ordre: en observant que le résultant des équations $\frac{dL}{dl}=0$, $\frac{dL}{dm}=0$ (où L est une fonction du second degré) sera exprimé sous la forme $\Theta=(12)^2\,LL$, on obtient immédiatement pour la réciproque de la courbe du second ordre $U=0$, l'équation

$$\mathsf{F}U=(P12)^2\,U\,U=0,$$

laquelle se réduit en effet à la forme connue.

Passons au cas d'une courbe du *troisième ordre*. Comme pour une fonction de deux variables, le résultant des équations

$$\frac{dL}{dl}=0,\ \frac{dL}{dm}=0,$$

aura la forme

$$\Theta = (12)^2 (34)^2 (13) (42) \, LLLL,$$

on a pour la polaire de $U = 0$,

$$-2\mathsf{F}U = (P12)^2 (P34)^2 (P13) (P42) \, UUUU = 0,$$

et il serait facile de calculer par là le coefficient d'une puissance ou d'un produit quelconque des variables. Par exemple le coefficient de z^6 se réduit à

$$\{(12)'''\}^2 \{(34)'''\}^2 \{(13)'''\}^2 \{(42)'''\} \, UUUU,$$

ou, toute réduction faite, et en supposant

$$6U = ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y + 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz$$

à: $2(6bcii_1 - 4i^3c - 4i_1^3b + 3i^2i_1^2 - b^2c^2)$. L'équation complètement développée, que j'ai donnée pour cette polaire dans le *Cambridge and Dublin Mathematical Journal*, t. I. [1846], p. 97 [35], et que j'ai obtenue par une élimination directe, pourra ainsi être vérifiée.

Nous allons passer maintenant à la théorie des *points d'inflexion* et des *tangentes doubles* de la courbe $U = 0$. Ces singularités peuvent être traitées par une analyse semblable, en remarquant que parmi les $n(n-1)$ tangentes, menées à la courbe d'un point P situé sur la courbe, la tangente en ce point se présente généralement deux fois, et trois fois, selon que le point P est un point d'inflexion, ou un point de contact d'une tangente double.

Désignons comme ci-dessus par (U) ce que devient U en écrivant $lx + m\xi$, $ly + m\eta$, $lz + m\zeta$ à la place de x, y et z. En mettant $\xi\partial_x + \eta\partial_y + \zeta\partial_z = \partial$, on a évidemment

$$[U] = l^n U + l^{n-1} m \, \partial U + \frac{1}{1.2} l^{n-2} m^2 \, \partial^2 U + \ldots;$$

et en éliminant l, m entre $\frac{d[U]}{dl} = 0$, $\frac{d[U]}{dm} = 0$, on obtient un résultant $\Theta = 0$, où Θ est une fonction de U, ∂U, $\partial^2 U$, ... $\partial^n U$, et qui a, comme le remarque M. Joachimsthal, la propriété dont il s'agit. En écrivant $U = 0$, Θ contient le facteur $(\partial U)^2$, et en mettant de côté ce facteur et écrivant $\partial U = 0$, Θ contient le facteur $(\partial^2 U)^2$; et ainsi de suite. Nous avons supposé que le point P, auquel appartiennent les coordonnées x, y et z, est un point de la courbe; de manière que l'on a actuellement $U = 0$. En faisant donc cette supposition et en éliminant le facteur $(\partial U)^2$, l'équation $\Theta = 0$ prend la forme $X\partial U + Y(\partial^2 U)^2 = 0$ [puisqu'en mettant $\partial U = 0$, l'équation contiendra à gauche le facteur $(\partial^2 U)^2$]. Dans le cas où P est un point d'inflexion, ou un point de contact d'une tangente double, cette équation contiendra ∂U comme facteur: donc il faut que $Y(\partial^2 U)^2$ contienne ce facteur, c'est-à-dire: ou $\partial^2 U$, ou Y, contiendra le facteur ∂U. Dans le premier cas il s'agit d'un point d'inflexion, dans le second cas d'un point de contact d'une tangente double.

Considérons d'abord les points d'inflexion. Comme $\partial^2 U$ contient ∂U comme facteur, il faut que cette fonction devienne zéro pour toutes les valeurs de ξ, η, ζ qui font

évanouir ∂U. Désignons par L, M, N les coefficients différentiels du premier ordre de U, de manière que $\partial U = L\xi + M\eta + N\zeta$. Cette quantité s'évanouit identiquement en faisant $\xi = \beta N - \gamma M$, $\eta = \gamma L - \alpha N$, $\zeta = \alpha M - \beta L$: donc il faut que $\partial^2 U$ s'évanouisse par la substitution de ces valeurs, quelles que soient les quantités α, β, γ. En désignant par D ce que devient le symbole D par cette substitution, c'est-à-dire en faisant

$$D = \alpha(M\partial_x - N\partial_y) + \beta(N\partial_x - L\partial_z) + \gamma(L\partial_y - M\partial_x),$$

la condition d'un point d'inflexion se réduit tout simplement à

$$D^2 U = 0:$$

équation dans laquelle les symboles ∂_x, ∂_y, ∂_z ne doivent pas contenir L, M, N. Nous reviendrons sur cette équation dans une note; pour le moment il suffit de remarquer, qu'en vertu de relations qui existent entre L, M, N et les dérivées a, b, c, f, g, h du second ordre, on a identiquement

$$(n-1)^2 D^2 U = n(n-1)\Psi . U - (\alpha x + \beta y + \gamma z)^2 (\nabla U),$$

où Ψ est une fonction de α, β, γ et des dérivées a, b, c, f, g, h, dont la forme sera donnée dans la suite, et

$$\nabla U = abc - af^2 - bg^2 - ch^2 + 2fgh.$$

Donc à cause de $U = 0$, la seule condition pour déterminer les points d'inflexion est l'équation

$$\nabla U = 0,$$

qui est de l'ordre $3(n-2)$ par rapport aux variables, et de l'ordre 3 par rapport aux coefficients de U. Cela est déjà connu par les recherches de M. Hesse. J'ai donné ici cette équation pour faire voir la liaison qui existe entre cette question et celle de trouver les tangentes doubles, à laquelle je vais passer maintenant.

On obtient l'équation qui détermine les points de contact de ces tangentes, en faisant les mêmes substitutions $\xi = \beta N - \gamma M$, &c. dans la fonction Y, et en égalant à zéro les coefficients des différentes puissances ou produits de α, β, γ. Remarquons que cette fonction Y s'obtient par une fonction de l'ordre $n^2 - n$ par rapport à x, y, z, ou à ξ, η, ζ, et de l'ordre $2(n-1)$ par rapport aux coefficients, en éliminant les facteurs $(\partial U)^2$ et $(\partial^2 U)^2$, qui ensemble montent au degré $4n-6$ par rapport à x, y, z, à 6 par rapport à ξ, η, ζ, et à 4 par rapport aux coefficients. Donc Y est des degrés $n^2 - n - 6$, $n^2 - 5n - 6$ et $2n - 6$ par rapport à x, y, z, à ξ, η, ζ, et aux coefficients de U, respectivement. En substituant donc ξ, η, ζ, cette fonction devient de l'ordre $n^2 - n - 6$ par rapport à α, β, γ, de l'ordre $n^3 - 2n^2 - 10n + 12$ par rapport à x, y, z {savoir $(n^2 - 5n + 6) + (n-1)(n^2 - n - 6)$}, et de l'ordre $n^2 + n - 12$ par rapport aux coefficients {savoir $(2n-6) + (n^2 - n - 6)$}. Mais on sait que les conditions de l'évanouissement de Y doivent se réduire à une seule équation; et cela ne peut arriver que de la même manière dont la réduction analogue a eu lieu pour les points d'inflexion. En écrivant

donc $[Y]$ à la place de ce que devient Y après la substitution, il faut que l'on ait identiquement:

$$[Y] = \Lambda\, U + N(\Pi U),$$

N étant fonction de α, β, γ du degré $n^2 - n - 6$. Il paraît de plus probable que cette fonction aura la même forme que celle pour les points d'inflexion, savoir $N = (\alpha x + \beta y + \gamma z)^{n^2-n-6}$. ($\Pi U$, comme ci-dessus ∇U, est censé exprimer non pas un produit, mais une dérivée de U: de même plus bas PU et QU.) Cela étant, ΠU sera du degré $(n-2)(n^2-9)$ par rapport à x, y, z {c'est-à-dire $n^3 - 2n^2 - 10n + 12 - (n^2 - n - 6)$}, et du degré $n^2 + n - 12$ par rapport aux coefficients. On a donc le théorème suivant:

THÉORÈME. On trouve les points de contact de tangentes doubles, en combinant avec l'équation de la courbe une équation $\Pi U = 0$ de l'ordre $(n-2)(n^2-9)$ par rapport aux variables, et de l'ordre $n^2 + n - 12$ par rapport aux coefficients; c'est-à-dire, puisqu'il correspond deux points de contact à une tangente double, le nombre de ces tangentes est égal à $\frac{1}{2}n(n-2)(n^2-9)$: théorème démontré indirectement par M. Plücker.

Cherchons maintenant les équations du système des tangentes aux points d'inflexion et du système des tangentes doubles.

Pour trouver la première équation, il faut éliminer x, y, z entre les trois équations

$$U = 0, \quad \nabla U = 0, \quad \mathrm{x}\frac{dU}{dx} + \mathrm{y}\frac{dU}{dy} + \mathrm{z}\frac{dU}{dz} = 0.$$

Le résultant complet sera du degré $3n(n-2)$ par rapport à x, y et z, et de l'ordre $9n^2 - 18n + 6$ {savoir $3(n-2)(n-1) + 3n(n-1) + 3n(n-2)$} par rapport aux coefficients; mais il existe ici le facteur spécial $(KU)^2$, et ce facteur étant éliminé, on obtient un résultant réduit $QU = 0$, du degré $3n(n-2)$ par rapport aux variables, et du même degré $3n(n-2)$ par rapport aux coefficients.

De même on aura l'équation du système des tangentes doubles, en éliminant x, y, z, entre les trois équations

$$U = 0, \quad \Pi U = 0, \text{ et } \mathrm{x}\frac{dU}{dx} + \mathrm{y}\frac{dU}{dy} + \mathrm{z}\frac{dU}{dz} = 0.$$

Le résultant complet est ici du degré $n(n-2)(n^2-9)$ par rapport à x, y, z, et du degré $3n^4 - 5n^3 - 29n^2 + 57n - 18$ {savoir $(n-1)(n-2)(n^2-9) + n(n-2)(n^2-9) + n(n-1)(n^2+n-12)$} par rapport aux coefficients. Mais il existe de même dans ce cas un facteur spécial $(KU)^{n^2-n-6}$ {du degré $3(n-1)^2(n^2-n-6) = 3n^4 - 9n^3 - 9n^2 + 33n - 18$}, et en l'éliminant, le résultant réduit sera du degré $4n(n-2)(n-3)$ par rapport aux coefficients. Mais le terme de cette équation à gauche sera évidemment un *carré;* on aura donc pour l'équation du système des tangentes doubles, la relation $PU = 0$, où PU est une fonction du degré $\frac{1}{2}n(n-2)(n^2-9)$ par rapport aux variables, et du degré $2n(n-2)(n-3)$ par rapport aux coefficients. Donc on pourra former le tableau suivant des degrés des différentes équations obtenues:

	Degrés par rapport aux variables.	Degrés par rapport aux coefficients.
Equation de la courbe $U=0$, ...	n	1
Condition d'un point multiple $KU=0$, ...	0	$3(n-1)^2$
Polaire réciproque $\mathsf{F}U=0$, ...	$n(n-1)$	$2(n-1)$
Courbe des inflexions $\nabla U=0$, ...	$3n(n-2)$	3
Courbes des contacts des tangentes doubles $\Pi U=0$,	$(n-2)(n^2-9)$	$(n+4)(n-3)$
Systèmes des tangentes aux points d'inflexion $QU=0$,	$3n(n-2)$	$3n(n-2)$
Système des tangentes doubles $PU=0$, ...	$\frac{1}{2}n(n-2)(n^2-9)$	$2n(n-2)(n-3)$.

La polaire de la polaire réciproque $\mathsf{FF}U$ sera évidemment du degré $(n^2-n)(n^2-n-1)$ par rapport aux variables, et du degré $4(n-1)(n^2-n-1)$ par rapport aux coefficients. Cette polaire de la polaire contiendra, comme on le sait par la théorie géométrique développée par M. Plücker, les facteurs U, $(PU)^2$ et $(QU)^3$; il faut y ajouter encore le facteur constant KU, et l'on aura définitivement l'équation

$$\mathsf{FF}U=(KU)(PU)^2(QU)^3 \,.\, U,$$

dans laquelle il sera facile à vérifier que les deux côtés sont des mêmes degrés par rapport aux variables et aux constantes. En effet on a

$$4(n-1)(n^2-n-1)=3(n-1)^2+4n(n-2)(n-3)+9n(n-2)+1,$$

et

$$n(n-1)(n^2-n-1)=n(n-2)(n^2-9)+9n(n-2)+n.$$

Mon but a été ici de donner une idée précise des théorèmes à démontrer, pour former une théorie toute analytique des polaires réciproques; je n'ai fait qu'avancer ces théorèmes (sans chercher à les démontrer), pour faire voir leur liaison avec la théorie de l'élimination et avec celle des hyperdéterminants; c'est à cette dernière en particulier qu'il faut, je crois, recourir pour démontrer la formule donnée ci-dessus $[Y]=\Lambda \,.\, U+(\alpha x+\beta y+\gamma z)^{n^2-n-6}(\Pi U)$, et pour trouver définitivement la forme de la dérivée ΠU, au moyen de laquelle on déterminera les points de contact des tangentes doubles. Je serais bien aise que ces recherches puissent de quelque manière faciliter la solution du problème des réciproques des *surfaces:* objet, qui est resté encore dans une complète obscurité.

Note sur les points d'inflexion.

Je vais d'abord rassembler plusieurs formules qui se rapportent au système des coefficients dans le développement de D^2U. On a

$$D^2U=\mathrm{a}\alpha^2+\mathrm{b}\beta^2+\mathrm{c}\gamma^2+2\mathrm{f}\beta\gamma+2\mathrm{g}\gamma\alpha+2\mathrm{h}\alpha\beta,$$

où

$$\mathrm{a}=M^2c-2MNf+N^2b\,,$$
$$\mathrm{b}=N^2a-2NLg+L^2c\,,$$
$$\mathrm{c}=L^2b-2LMh+M^2a,$$

$$\begin{aligned}
\mathrm{f} &= -MNa + NLh + LMg - L^2 f,\\
\mathrm{g} &= +MNh - NLb + LMf - M^2 g,\\
\mathrm{h} &= +MNg + NLf - LMc - N^2 h.
\end{aligned}$$

Nous écrirons de plus, pour abréger,

$$\begin{array}{lll}
bc - f^2 = \mathfrak{A}, & gh - af = \mathfrak{F}, & \nabla = abc - af^2 - bg^2 - ch^2 + 2fgh,\\
ca - g^2 = \mathfrak{B}, & hf - bg = \mathfrak{G}, & \\
ab - h^2 = \mathfrak{C}, & fg - ch = \mathfrak{H}, & \\
\mathrm{bc} - \mathrm{f}^2 = (\mathfrak{A}), & \&c. & \\
\&c. & &
\end{array}$$

et enfin

$$\Phi = \mathfrak{A}L^2 + \mathfrak{B}M^2 + \mathfrak{C}N^2 + 2\mathfrak{F}MN + 2\mathfrak{G}NL + 2\mathfrak{H}MN.$$

On obtient identiquement

$$\begin{aligned}
L\mathrm{a} + M\mathrm{h} + N\mathrm{g} &= 0,\\
L\mathrm{h} + M\mathrm{b} + N\mathrm{f} &= 0,\\
L\mathrm{g} + M\mathrm{f} + N\mathrm{c} &= 0;
\end{aligned}$$

$$\begin{array}{lll}
(\mathfrak{A}) = L^2\Phi\ , & (\mathfrak{B}) = M^2\Phi\ , & (\mathfrak{C}) = N^2\Phi\ ,\\
(\mathfrak{F}) = MN\Phi, & (\mathfrak{G}) = NL\Phi, & (\mathfrak{H}) = LM\Phi;
\end{array}$$

$$\begin{aligned}
f\mathrm{f} - MN\mathfrak{F} &= (Nh - Lf)(Lf - Mg),\\
g\mathrm{g} - NL\mathfrak{G} &= (Lf - Mg)(Mg - Nh),\\
h\mathrm{h} - LM\mathfrak{H} &= (Mg - Nh)(Nh - Lf);
\end{aligned}$$

$$\begin{aligned}
4L^2MN\mathfrak{F} &+ \mathrm{b}(L^2 b + M^2 a) + \mathrm{c}(N^2 a + L^2 c) - \mathrm{bc} - 2L^2 a\mathrm{a} = (L^2 b - M^2 a)(N^2 a + L^2 c),\\
4M^2NL\mathfrak{G} &+ \mathrm{c}(M^2 c + N^2 b) + \mathrm{a}(L^2 b + M^2 a) - \mathrm{ca} - 2M^2 b\mathrm{b} = (M^2 c - N^2 b)(L^2 b - M^2 a),\\
4N^2LM\mathfrak{H} &+ \mathrm{a}(N^2 a + L^2 c) + \mathrm{b}(M^2 c + N^2 b) - \mathrm{ab} - 2N^2 c\mathrm{c} = (N^2 a - L^2 c)(M^2 c - N^2 b);
\end{aligned}$$

$$\begin{aligned}
4L^2MN\mathfrak{A} - \mathrm{a}^2 + 2\mathrm{a}(M^2 c + N^2 b) &= -(M^2 c - N^2 b)^2,\\
4M^2NL\mathfrak{B} - \mathrm{b}^2 + 2\mathrm{b}(N^2 a + L^2 c) &= -(N^2 a - L^2 c)^2,\\
4N^2LM\mathfrak{C} - \mathrm{c}^2 + 2\mathrm{c}(L^2 b + M^2 a) &= -(L^2 b - M^2 a)^2;
\end{aligned}$$

$$\begin{aligned}
L^2MN\mathfrak{A} &+ N\mathrm{g}(+Lf + Mg - Nh) + M\mathrm{h}(Lf - Mg + Nh) - \mathrm{gh} = -MN(Nh - Mg)^2,\\
M^2NL\mathfrak{B} &+ L\mathrm{h}(-Lf + Mg + Nh) + N\mathrm{f}(Lf + Mg - Nh) - \mathrm{hf} = -NL(Lf - Nh)^2,\\
N^2LM\mathfrak{C} &+ M\mathrm{f}(+Lf - Mg + Nh) + L\mathrm{g}(Lf + Mg + Nh) - \mathrm{fg} = -LM(Mg - Lf)^2;
\end{aligned}$$

$$\begin{aligned}
2\nabla L^2 &= -a^2\mathrm{a} + (ab - 2h^2)\mathrm{b} + (ca - 2g^2)\mathrm{c} + 2(af - 2gh)\mathrm{f} - 2ag\ \mathrm{g} - 2ah\ \mathrm{h},\\
2\nabla M^2 &= +(ab - 2h^2)\mathrm{a} - b^2\mathrm{b} + (bc - 2f^2)\mathrm{c} - 2bf\mathrm{f} + 2(bg - 2hf)\mathrm{g} - 2bh\ \mathrm{h},\\
2\nabla N^2 &= +(ca - 2g^2)\mathrm{a} + (bc - 2f^2)\mathrm{b} - c^2\mathrm{c} - 2cf\mathrm{f} - 2cg\mathrm{g} + 2(ch - 2fg)\mathrm{h};
\end{aligned}$$

$$2\nabla MN = +(af-2gh)\,\mathrm{a} - bf\,\mathrm{b} - cf\,\mathrm{c} - 2bc\,\mathrm{f} - 2ch\,\mathrm{g} - 2bg\,\mathrm{h},$$

$$2\nabla NL = -ag\,\mathrm{a} + (bg-2hf)\,\mathrm{b} - cg\,\mathrm{c} - 2ch\,\mathrm{f} - 2ca\,\mathrm{g} - 2af\,\mathrm{h},$$

$$2\nabla LM = -ah\,\mathrm{a} - bh\,\mathrm{b} + (ch-2fg)\,\mathrm{c} - 2bg\,\mathrm{f} - 2af\,\mathrm{g} - 2ab\,\mathrm{h}.$$

Dans ces équations, si $Lx+My+Nz$ était facteur de $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, on aurait évidemment $\mathrm{a}=0$, $\mathrm{b}=0$, $\mathrm{c}=0$, $\mathrm{f}=0$, $\mathrm{g}=0$, $\mathrm{h}=0$, et de là $\nabla=0$, $\Phi=$ un carré, ce qui donnerait des formules plus simples, et auxquelles on pourrait encore donner plusieurs autres formes. Par exemple on tire d'un de ces systèmes, $-1 \div MN\mathfrak{F} = (Mg-Nh) \div \Theta$, &c., où $\Theta = (Mg-Nh)(Nh-Lf)(Lf-Mg)$, et de là les expressions

$$\frac{L}{\mathfrak{F}} + \frac{M}{\mathfrak{G}} + \frac{N}{\mathfrak{H}} = 0,$$

$$\frac{L^2 f}{\mathfrak{F}} + \frac{M^2 g}{\mathfrak{G}} + \frac{N^2 h}{\mathfrak{H}} = 0,$$

$$\frac{L^3 f^2}{\mathfrak{F}} + \frac{M^3 g^2}{\mathfrak{G}} + \frac{N^3 h^3}{\mathfrak{H}} = LMN,$$

auxquelles on pourrait ajouter encore plusieurs systèmes analogues, ce qui se ferait sans la moindre difficulté.

Jusqu'ici $L, M, N, a, b, c, f, g, h$ ont été des quantités quelconques. En supposant qu'elles soient les dérivées du premier et du second ordre d'une fonction U, on a

$$(n-1)\,L = ax+hy+gz,$$

$$(n-1)\,M = hx+by+fz,$$

$$(n-1)\,N = gx+fy+cz,$$

$$n\,(n-1)\,U = ax^2+by^2+cz^2+2fyz+2gxz+2hxy,$$

et la substitution de ces valeurs donne la formule du texte:

$$(n-1)^2\,D^2U = n\,(n-1)\,\Psi\,.\,U - (\alpha x+\beta y+\gamma z)^2\,(\nabla U),$$

(∇U) étant ici $=\nabla$, et Ψ étant donnée par l'équation

$$\Psi = \mathfrak{A}\alpha^2 + \mathfrak{B}\beta^2 + \mathfrak{C}\gamma^2 + 2\mathfrak{F}\beta\gamma + 2\mathfrak{G}\gamma\alpha + 2\mathfrak{H}\alpha\beta.$$

Cela peut être vérifié facilement par la substitution actuelle: mais nous allons le démontrer par la théorie des hyperdéterminants. En effet, soit (123) le déterminant symbolique formé avec $\partial_{x_1}, \partial_{y_1}, \partial_{z_1}$; $\partial_{x_2}, \partial_{y_2}, \partial_{z_2}$; $\partial_{x_3}, \partial_{y_3}, \partial_{z_3}$, $(A23)$ le déterminant symbolique formé avec α, β, γ; $\partial_{x_2}, \partial_{y_2}, \partial_{z_2}$; $\partial_{x_3}, \partial_{y_3}, \partial_{z_3}$, et ainsi de suite; puis soient T_1, T_2, ... les fonctions $x\partial_{x_1}+y\partial_{y_1}+z\partial_{z_1}$, $x\partial_{x_2}+y\partial_{y_2}+z\partial_{z_2}$, &c. et $\rho = \alpha x+\beta y+\gamma z$, on aura identiquement:

$$\rho\,(123) = -T_1\,(A23) - T_2\,(A31) - T_3\,(A12),$$

et en élevant au carré les deux membres de cette équation, ajoutant le facteur $U_1U_2U_3$, et réduisant les variables après avoir différentié suivant x, y, z, puis en tenant compte des équations

$$(123)^2\, U_1U_2U_3 = 6\nabla U,$$

$$T_1^2\, A\,(23)^2\, U_1U_2U_3 = 2n\,(n-1)\, U\,.\,\Psi,$$

$$T_1T_2A\,(23)\,A\,(31)\, U_1U_2U_3 = -\,(n-1)^2\, D^2U,$$

on obtiendra le résultat dont il s'agit.

Il sera aussi facile de déduire de cette formule quelques propriétés de la courbe $\nabla U = 0$, dans les cas d'un point double ou d'un point de rebroussement de la courbe $U = 0$. En effet, pour un point double, les dérivées L, M, N de U, et par suite D^2U, et ses dérivées du premier ordre, s'évanouissent. De plus, pour un point de rebroussement on a $\Psi = 0$. Donc, pour un point double on a $\nabla U = 0$, où la courbe exprimée par cette équation passe par chaque point double (y compris les points de rebroussement) de la courbe $U = 0$. Prenons la dérivée de l'équation en question, en affectant du symbole $\partial = \xi\partial_x + \eta\partial_y + \zeta\partial_z$ ses deux membres. Supprimant les termes qui se réduisent à zéro aux points doubles, cela donne

$$0 = (\alpha x + \beta y + \gamma z)^2\, \partial\nabla U;$$

c'est-à-dire qu'il y aura aussi à chacun de ces points un point double de la courbe $\nabla U = 0$. Passant aux dérivées du second ordre, on aura

$$(n-1)^2\, \partial^2 D^2 U = n\,(n-1)\, \Psi\partial^2 U - (\alpha x + \beta y + \gamma z)^2\, \partial^2\nabla U.$$

Or ici

$$\partial^2 D^2 U = \alpha^2\, \partial^2\mathrm{a} + \beta^2\partial^2\mathrm{b} + \gamma^2\, \partial^2\mathrm{c} + 2\beta\gamma\partial^2\mathrm{f} + 2\gamma\alpha\partial^2 g + 2\alpha\beta\partial^2\mathrm{h},$$

et $\partial^2\mathrm{a}$, &c. se réduisent à

$$2\,(a\partial M^2 - 2f\partial M\partial N + b\partial N^2),\ \&c.$$

savoir (en mettant $a\xi + h\eta + g\zeta$, $h\xi + b\eta + f\zeta$, $g\xi + f\eta + c\zeta$ à la place de ∂L, ∂M, ∂N, et en ayant égard à la condition $\nabla U = 0$) à $2\partial^2 U\,.\,\mathfrak{A}$, &c.; on aura donc $\partial^2 Du = 2\partial^2 U\,.\,\Psi$, ou enfin

$$(n-1)\,(n-2)\,\partial^2 U\,.\,\Psi = -\,(\alpha x + \beta y + \gamma z)^2\, \partial^2\nabla U,$$

c'est-à-dire: les deux courbes $U = 0$, $\nabla U = 0$, se toucheront dans les points doubles. Enfin pour un point de rebroussement $\partial^2\nabla U$ s'évanouit, c'est-à-dire, il existe un point triple dans la courbe $\nabla U = 0$. Mais il peut être démontré que deux branches de la courbe se touchent en ce point, et qu'elles touchent aussi la courbe $U = 0$; c'est-à-dire qu'il y aura dans la courbe $\nabla U = 0$ un point de rebroussement, et une autre branche de la courbe qui passe par ce point. Pour cela il faudra passer aux dérivées du troisième ordre. Cela donne, en supprimant les termes qui s'évanouissent:

$$(n-1)^3\, \partial^3\, D^2 U = 3n\,(n-1)\,\partial\Psi\,.\,\partial^2 U - (\alpha x + \beta y + \gamma z)^2\, \partial^3\nabla U.$$

Ici on a

$$\partial^3 D^2 U = \alpha^2 \partial^3 \mathrm{a} + \&\mathrm{c.},$$

$$\partial^3 \mathrm{a} = 6\,(\partial c \partial M^2 - 2\partial f \partial M \partial N + \partial b \partial N^2) + 6\,\{c\partial M \partial^2 M - f(\partial M \partial^2 N + \partial N \partial^2 M) + b\partial N \partial^2 N\},\ \&\mathrm{c.},$$

et les deux lignes de cette expression se réduisent à $6\,\partial^2 U\,.\,\partial\mathfrak{A}$, et à zéro, respectivement. En effet, en remplaçant ∂M, ∂N par leurs valeurs, le coefficient de ξ^2 dans la première ligne devient $6\,(h^2\partial c - 2gh\partial f + g^2\partial b) = 6a\,(b\partial c + c\partial b - 2f\partial f)$ (et à cause de $\mathfrak{B} = 0$, $\mathfrak{C} = 0$, $\mathfrak{F} = 0$) $= 6a\partial\mathfrak{A}$; et également pour les autres termes. De même, le coefficient de ξ^2 dans la seconde ligne devient $6\,\{ch\partial h - f(g\partial h + h\partial g) + bg\partial g\}$. En y ajoutant $3\,(h^2\partial c - 2gh\partial f + g^2\partial b)$, savoir $3a\partial\mathfrak{A}$, la somme se réduit à $3\partial\,(ch^2 - 2fgh + bg^2) = 3\partial\,(a\mathfrak{A} - \nabla)$ $= 3a\partial\mathfrak{A}$; donc le coefficient en question s'évanouit. En cherchant de la même manière les autres coefficients, on trouvera les valeurs dont il s'agit, et ainsi $\partial^3 \mathrm{a} = 6\partial^2 U\,.\,\partial\mathfrak{A}$, &c. et de là $\partial^3 D^2 U = 6\,\partial^2 U \partial\Psi$; donc enfin

$$3\,(n-1)\,(n-2)\,\partial^2 U \partial\Psi = -\,(\alpha x + \beta y + \gamma z)^2\,\partial^3 \nabla U\,;$$

ce qui suffit pour démontrer le théorème, qui peut être énoncé comme suit:

THÉORÈME. "Il existe un point double de la courbe $\nabla U = 0$, pour chaque point double de la courbe $U = 0$, et les deux courbes ont des tangentes communes dans ces points. De plus, il existe un point triple de la courbe $\nabla U = 0$, pour chaque point de rebroussement de la courbe $U = 0$, savoir un point de rebroussement dont les deux branches touchent la tangente de la courbe $U = 0$, et encore une troisième branche, qui passe par le point de rebroussement."

Il suit de là que dans le cas d'un point double, ce point doit être considéré comme la réunion de six points d'intersection, et dans celui d'un point de rebroussement, de huit points d'intersection. C'est de cette manière que l'on se rend compte du théorème de M. Plücker qui dit que l'effet de ces deux singularités est de faire disparaître respectivement six ou huit points d'inflexion de la courbe donnée.

Examinons, en finissant, la théorie des points *d'osculation*. Il est facile de voir que la condition d'un tel point (savoir d'un point dans lequel la tangente coupe la courbe en quatre points consécutifs) consiste en ce que $\partial^3 U$ contient ∂U comme facteur, ou, autrement dit, que l'équation $D^3 U = 0$ est *identiquement* vraie. On obtient ainsi *dix* conditions, qui se réduisent assez facilement à *six*; mais pour les réduire à *une seule* condition, il faut prendre la dérivée, avec le symbole D de la valeur donnée ci-dessus de $D^3 U$. On doit cependant ne pas oublier que $D\,.\,D^2 U$, outre le terme $D^3 U$, contient aussi des termes que l'on obtient en faisant opérer les symboles ∂_x, ∂_y, ∂_z sur les quantités L, M, N qui entrent dans $D^2 U$. Car il est convenu que les symboles ∂_x, ∂_y, ∂_z dans $D^3 U$, ne doivent pas affecter les lettres L, M, N. Cependant il est remarquable que dans le cas actuel, ces termes se détruisent entièrement. En effet, en les désignant par Ω, on obtient

$$\Omega = 2D\,[(M\gamma - N\beta)\,(a\partial_L + h\partial_M + g\partial_N) + (N\alpha - L\beta)\,(h\partial_L + b\partial_M + f\partial_N)$$
$$+ (L\beta - M\gamma)\,(g\partial_L + f\partial_M + c\partial_N)]\,.\,DU,$$

ou il faut d'abord effectuer les opérations ∂_L, ∂_M, ∂_N qui se rapportent à D, et ensuite rapporter les ∂_x, ∂_y, ∂_z à la fonction U. Cela donne

$$\Omega = 2\,[\alpha\,(N\partial_y - M\partial_z) + \beta\,(L\partial_z - N\partial_y) + \gamma\,(M\partial_x - L\partial_y)] \times$$

$$\left\{\begin{array}{l} \quad\alpha^2\,[N\,(f\partial_y - b\partial_z) - M(c\partial_y - f\partial_z)] \\ +\beta^2\,[L\,(g\partial_z - c\partial_x) - N(a\partial_z - g\partial_x)] \\ +\gamma^2\,[M\,(h\partial_x - a\partial_y) - L\,(b\partial_x - h\partial_y)] \\ +\beta\gamma\,[-L\,(h\partial_z + g\partial_y - 2f\partial_x) + M\,(a\partial_z - g\partial_x) + N\,(a\partial_y - h\partial_x)] \\ +\gamma\alpha\,[-M\,(f\partial_x + h\partial_z - 2g\partial_y) + N\,(b\partial_x - h\partial_y) + L\,(b\partial_z - f\partial_y)] \\ +\alpha\beta\,[-N\,(g\partial_y + f\partial_x - 2h\partial_z) + L\,(c\partial_y - f\partial_z) + M\,(c\partial_x - g\partial_z)] \end{array}\right\} U,$$

où ∂_x, ∂_y, ∂_z se rapportent seulement à U. Or tous les termes de cette expression *s'évanouissent.* Par exemple le coefficient de α^3 devient

$$(N\partial_y - M\partial_z)\,[N\,(f\partial_y - b\partial_z) - M\,(c\partial_y - f\partial_z)]\,U$$
$$= [N^2\,(f\partial_y^2 - b\partial_z\partial_y) + M^2\,(c\partial_y\partial_z - f\partial_z^2) - MN\,(c\partial_y^2 - f\partial_y\partial_z + f\partial_y\partial_z - b\partial_z^2)\,U = 0,$$

et ainsi pour les autres termes; donc on a $\Omega = 0$.

Donc, en transportant à l'autre côté de l'équation les termes qui contiennent U, DU ou ∇U, on obtient cette formule très simple:

$$(n-1)^2\,D^3U = -(\alpha x + \beta y + \gamma z)^2\,D\nabla U,$$

ou la seule condition d'un point d'osculation (en ayant égard à l'équation $\nabla U = 0$) se réduit à $D\nabla U = 0$. Savoir les dérivées $\partial_x\nabla U$, $\partial_y\nabla U$, $\partial_z\nabla U$ doivent être proportionnelles à $\partial_x U$, $\partial_y U$, $\partial_z U$ (ce qui équivaut à une seule condition, en vertu de $U = 0$, $\nabla U = 0$; comme on le voit facilement). Cela donne le théorème suivant.

THÉORÈME. "Dans le cas d'un point *d'osculation*, les deux courbes $U = 0$, $\nabla U = 0$ se touchent."

Il n'y a presque pas de doute que la dérivée $\nabla\nabla U$ ne se réduise *toujours* à la forme $R\,.\,U + S\,.\,\nabla U$. En effet, M. Hesse l'a démontré pour les fonctions de trois variables et du troisième degré, et moi, je l'ai vérifié pour les fonctions de deux variables d'un degré quelconque. Cela étant, les points d'inflexion de la courbe $\nabla U = 0$ sont situés aux points d'intersection avec $U = 0$, et au cas où les deux courbes se touchent, ce point de contact doit être considéré comme la réunion de trois points d'intersection: donc,

"Tout point *d'osculation* peut être envisagé comme point de réunion de *trois* points *d'inflexion.*"

Nous démontrerons encore, d'une manière conforme à celle dont nous avons trouvé l'expression de D^2U, l'expression qui vient d'être donnée pour D^3U à moyen de la formule

$$\rho^2\,(123)^2 = \{T_1\,(A23) + T_2\,(A31) + T_3\,(A12)\}^2.$$

En multipliant les deux membres par $(A14)+(A24)+(A34)$, le terme à gauche peut être présenté sous la forme $\rho^2(A\theta 4)(123)^2$, où ∂_{x_θ}, ∂_{y_θ}, ∂_{z_θ} se rapportent à tous les systèmes de variables. En y appliquant le produit $U_1U_2U_3U_4$ (les variables identiques après les différentiations), on obtiendra $6\rho^2(A\theta 4)U_4\nabla U = -6\,D\nabla U\,.\,\rho^2$. Pour la droite de l'équation on a d'abord trois termes comme $T_1^2(A14)(A23)^2\,U_1U_2U_3U_4$, lesquels se détruisent évidemment, à cause de $(A14)\,U_1U_4=0$; puis six termes de la forme $(A14)(A23)(A31)\,U_1U_2U_3U_4$, qui se détruisent aussi, puisqu'en échangeant 1,3 et 2,4, le terme change de signe; puis trois termes comme $T_1^2\,\{(A24)+(A34)\}\,(A23)^2\,U_1U_2U_3U_4$, &c. savoir $T_1^2\,U_1\,\{(A24)+(A34)\}\,(A23)^2\,U_2U_3U_4 = -2n\,(n-1)\,U\,.\,D\Psi$; c'est-à-dire tous ces termes sont $6n\,(n-1)\,U\,D\Psi$; enfin trois termes $2T_1T_2\,(A23)\,(A31)\,(A34)\,U_1U_2U_3U_4 = 2\,(n-1)^2\,D^3U$, ou, tous pris ensemble, $6\,(n-1)^2D^3U$. Donc, en supprimant le terme $-6n\,(n-1)\,U\,D\Psi$, à cause de $U=0$, on obtient la même équation que ci-dessus, savoir:

$$(n-1)^2D^3U = -(\alpha x+\beta y+\gamma z)^2D\nabla U.$$

On pourrait croire qu'il y a une équation analogue $(n-1)^2D^4U = -(\alpha x+\beta y+\gamma z)^2D^2\nabla U$ pour les dérivées du *quatrième* degré, mais cela n'est pas. En effet, il est facile de voir que pour un point d'osculation, $D^2\nabla U$ se réduit à $\nabla\nabla U$, à un facteur près, c'est-à-dire l'on aurait $D^2\nabla U=0$, puisque $\nabla\nabla U$ s'évanouit aux points d'osculation: donc on aurait généralement pour un point d'osculation $D^4U=0$; mais cela est seulement le caractère des points d'osculation d'un plus haut degré, savoir de ceux où la tangente rencontre la courbe en cinq points consécutifs. Pour le quatrième degré le problème devient trop compliqué pour être traité de cette manière.

54.

NOTE SUR LES HYPERDÉTERMINANTS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tome XXXIV. (1847), pp. 148—152.]

I. SOIT V une fonction homogène de x, y de $2p^{\text{ième}}$ ordre. En égalant à zéro les coefficients différentiels du $p^{\text{ième}}$ ordre de cette fonction, pris par rapport à x, y, et en éliminant ces variables, on obtiendra entre les coefficients de la fonction un nombre p d'équations. Or parmi ces équations il y aura toujours une seule du second ordre, savoir

$$B(U, U) = 0$$

(suivant la notation dans mon mémoire sur les hyperdéterminants, t. XXX. [(1846), 16]. Par exemple, en écrivant t au lieu de $x : y$ on a identiquement

$$\begin{aligned} & c(at+b) \\ & -b(bt+c) \\ & = (ac-b^2)\,t, \end{aligned}$$

$$\begin{aligned} & (\quad et \qquad)(at^2+2bt+c) \\ & -(4dt+2e)(bt^2+2ct+d) \\ & +(3ct+2d)(ct^2+2dt+e) \\ & = (ae-4bd+3c^2)\,t^3, \end{aligned}$$

$$\begin{aligned} & (\quad gt^2 \qquad\qquad)(at^3+3bt^2+3ct+d) \\ & -(\ 6ft^2+\ 3gt \qquad)(bt^3+3ct^2+3dt+e) \\ & +(15et^2+18ft+6g)(ct^3+3dt^2+3et+f) \\ & -(10dt^2+15et+6f)(dt^3+3et^2+3ft+g) \\ & = (ag-6bf+15ec-10d^2)\,t^5, \end{aligned}$$

et ainsi de suite: il ne reste qu'à déterminer la loi des coefficients numériques des facteurs à gauche. Pour cela, représentons par $A, B, C, \ldots$ les coefficients de $(1+s)^{2p}$, et par $A', B', C', \ldots$ les coefficients de $(1-s)^{-p}$. On aura pour ces nombres le système suivant:

$$\begin{array}{llll} +A'A, & & & \\ -A'B, & -B'A, & & \\ +A'C, & +B'B, & +C'A, & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \pm I, & \pm K, & \pm L \ldots . & \end{array}$$

Les nombres $I, K, L \ldots$ de la dernière ligne seront à déterminer au moyen d'une autre règle: il faut faire évanouir les sommes des nombres dans la même ligne verticale, ces nombres étant pris avec leurs signes actuels. Je suis parvenu par *induction* à ces formules, mais il ne serait pas, je crois, très difficile de les démontrer directement.

Il me paraît possible que tous les hyperdéterminants qui se rapportent à la fonction V, puissent être trouvés en éliminant entre les équations (en nombre de p) dont il s'agit, et cela dans le cas où U est de l'ordre $2p$ ou $2p+1$; au moins cette règle se vérifie pour les fonctions de deuxième, troisième et quatrième ordres, et cela paraissait (à priori) moins probable pour les dérivées d'un degré plus élevé que pour celles du second degré, pour lesquelles, comme on vient de le voir, il est effectivement vrai.

Cela étant, il y aura seulement un nombre p de dérivées indépendantes pour les fonctions du $2p^{\text{ième}}$ et du $(2p+1)^{\text{ième}}$ ordre: conclusion que je ne puis pas démontrer.

II. Soit $\nabla = 6abcd + 3b^2c^2 - a^2d^2 - 4ac^3 - 4b^3d$, et représentons par ∇_1 le déterminant formé avec les coefficients différentiels du second ordre de ∇ par rapport à a, b, c, d, on aura

$$\nabla_1 = 3\nabla^2$$

(propriété qui a un rapport singulier avec celle qu'a démontrée M. Eisenstein par rapport aux coefficients du premier ordre de la même fonction ∇). La démonstration que je puis donner de ce théorème est à la vérité assez compliquée, mais je ne vois pas d'autre. En mettant

$$p = \tfrac{2}{3}(bd - c^2), \quad q = \tfrac{1}{3}(bc - ad), \quad r = \tfrac{2}{3}(ac - b^2),$$

on obtient

$$\nabla_1 = 81 \begin{vmatrix} a^2, & ab, & ac - 3r, & ad + 9q \\ ba, & b^2 + 2r, & bc - q, & bd - 3p \\ ca - 3r, & cb - q, & c^2 + 2p, & cd \\ da + 9q, & db - 3p, & dc, & d^2 \end{vmatrix}$$

où le déterminant ne contient que les termes du quatrième ou troisième degré en p, q, r, et en développant on a

$$\begin{aligned} \nabla_1 = 81\,\{&9(pr - q^2)^2 - 2a^2p^3 - 2d^3r^3 - 12q(abp^2 + cdr^2) - 18q^2(b^2p + c^2r) \\ &- 6(pr + q^2)(acp + bdr) - 2adq(3pr - q^2) - 18bcq(pr + q^2)\}, \end{aligned}$$

d'où, en réduisant au moyen des expressions

$$ap = -(2bq + cr), \quad dr = -(2cq + bp), \quad ad = bc - 3q,$$

on tire $\nabla_1 = 81\,\{9\,(pr - q^2)^2 + 4\,(b^2p + 2bcq + c^2r)\,(pr + q^2) + 6q^2\,(3pr - q^2)\}$,

ou enfin, au moyen de $2\,(b^2p + 2bcq + c^2r) = -3pr$:

$$\nabla_1 = 243\,(pr - q^2)^2 = 3\nabla^2;$$

voilà l'équation qu'il s'agissait à démontrer.

III. En considérant $a : d$, $b : d$, $c : d$ comme représentants les trois coordonnées d'un point, ou, si l'on veut, des fonctions linéaires de ces coordonnées, l'équation $\nabla = 0$ appartient évidemment à une surface développable (de quatrième ordre). Mais la condition pour que l'équation $U = 0$ (V étant une fonction homogène de quatre variables) appartienne à une surface développable, est, que le déterminant formé avec les coefficients différentiels du second ordre de la fonction, s'évanouisse (théorème de M. Hesse, t. XXVIII. [(1844) pp. 97—107, "Ueber die Wendepunkte der Curven dritter Ordnung"]). Donc il faut que ∇_1 s'évanouisse au moyen de $\nabla = 0$, c'est-à-dire, il faut que ∇_1 contienne le facteur ∇; ce qui s'accorde parfaitement avec l'équation qui vient d'être présentée. Mais il ne peut être prouvé de cette manière que l'autre facteur doit aussi se réduire à ∇, et même cela n'est pas vrai si ∇_1 vient d'une fonction d'un plus haut degré que le quatrième.

Il suit de cela qu'en supposant toujours que les coefficients soient des fonctions linéaires des coordonnées, le résultat $\Theta = 0$ de l'élimination de x, y entre $\frac{dU}{dx} = 0$, $\frac{dU}{dy} = 0$ appartient toujours à une surface développable. De même l'élimination de x, y entre les trois équations $\frac{d^2U}{dx^2} = 0$, $\frac{d^2U}{dx\,dy} = 0$, $\frac{d^2U}{dy^2} = 0$ conduit aux équations de l'arête de rebroussement de la surface, et de plus, en éliminant entre les équations $\frac{d^3U}{dx^3} = 0$, $\frac{d^3U}{dx^2\,dy} = 0$, $\frac{d^3U}{dx\,dy^2} = 0$, $\frac{d^3U}{dy^3} = 0$, on obtient les points de rebroussement de l'arête de rebroussement. Cela conduit à quelques résultats remarquables.

Par exemple, la surface développable dont l'équation est

$$\nabla = 6abcd + 3b^2c^2 - 4ac^3 - 4b^3d - a^2d^2,$$

a pour arête de rebroussement la courbe dont les équations (équivalentes à deux équations seulement) sont $bd - c^2 = 0$, $ad - bc = 0$, $ac - b^2 = 0$, ce qui est une *courbe du troisième ordre* seulement. Car en considérant deux quelconques de ces trois équations, par exemple celles-ci: $bd - c^2 = 0$, $ad - bc = 0$, ces équations appartiennent à deux surfaces du second ordre qui ont en commun la droite $d = 0$, $c = 0$: cela s'accorde avec un résultat que j'ai donné dans mon mémoire sur les surfaces développables dans le journal de M. Liouville [t. X. (1845), 30].

Egalement, en considérant une équation de quatrième degré en t, on obtient une surface développable.

$$(ae - 4bd + 3c^2)^3 - 27\,(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0,$$

qui a pour arête de rebroussement la courbe du sixième ordre exprimée par les équations $ae - 4bd + 3c^2 = 0$, $ace + 2bcd - ad^2 - b^2e - c^3 = 0$. Je n'ai pas complètement réussi à expliquer

pourquoi cette courbe du sixième ordre a une osculatrice développable, seulement du sixième ordre, mais cette réduction s'opère en partie au moyen des points de rebroussement de la courbe, qui se trouvent au moyen des équations $at+b=0$, $bt+c=0$, $ct+d=0$, $dt+e=0$. Pour savoir à combien de points ces équations correspondent, il faut remarquer que, a, b, c, d, e étant des fonctions linéaires des coordonnées, on aura toujours entre ces quantités une équation linéaire telle que

$$Aa+Bb+Cc+Dd+Ee=0,$$

où A, B, ... sont des constantes. Donc, en éliminant a, b, c, d, e, on obtient $A-Bt+Ct^2-Dt^3+Et^4=0$, équation du quatrième ordre, et à chaque valeur de t il correspond un des points dont il s'agit; donc la courbe du sixième ordre a quatre points de rebroussement.

Également la surface développable qui correspond à une équation du $m^{\text{ième}}$ ordre, est de l'ordre $2(m-1)$; l'arête de rebroussement est de l'ordre $3(m-2)$, et il y a dans cette courbe un nombre $4(m-3)$ de points de rebroussement. Il faut toujours se rappeler que ces surfaces développables ne sont pas les surfaces développables les plus générales qui existent de l'ordre $2(m-1)$, excepté dans le cas des surfaces développables du quatrième ordre.

IV. Il vaut peut-être la peine de donner en passant une démonstration de ce théorème de M. Chasles: "Le plan qui passe par trois points qui se meuvent avec des vitesses uniformes dans trois droites quelconques, enveloppe une surface développable du quatrième degré." En effet, en supposant que $\alpha:\delta$, $\beta:\delta$, $\gamma:\delta$; $\alpha':\delta'$, $\beta':\delta'$, $\gamma':\delta'$; $\alpha'':\delta''$, $\beta'':\delta''$, $\gamma'':\delta''$, soient des fonctions linéaires du temps ((δ, δ', δ'') peuvent être constants, ou, si l'on veut, des fonctions linéaires du temps, ce qui correspond à un cas un peu plus général que celui de M. Chasles), on peut prendre ces valeurs pour coordonnées des trois points mobiles. Donc, en prenant $x:w$, $y:w$, $z:w$ pour coordonnées d'un point quelconque du plan, on obtient l'équation de ce plan, en égalant à zéro le déterminant formé avec les valeurs x, y, z, w; α, β, γ, δ; α', β', γ', δ'; α'', β'', γ'', δ''; ce qui donne une équation de la forme $a+3bt+3ct^2+dt^3=0$, a, b, c, d étant des fonctions linéaires de x, y, z, w; et cela suffit pour démontrer le théorème dont il s'agit.

V. En finissant j'indiquerai un principe de classification des courbes à double courbure qui me paraît être de quelque importance; savoir, on pourra distinguer les courbes qui ne peuvent pas être l'intersection *complète* de deux surfaces, de celles qui peuvent l'être. Par exemple, en faisant passer par une courbe donnée du troisième ordre deux surfaces du second ordre, la courbe n'est pas l'intersection complète des deux surfaces; celles-ci se coupent dans cette courbe et dans une certaine droite. Quel est le théorème analogue pour les courbes du $n^{\text{ième}}$ ordre? Peut-on, par exemple, toujours combiner avec une courbe donnée d'un ordre quelconque, une autre courbe qui est l'intersection complète de deux surfaces, de manière que l'ensemble des deux courbes soit une intersection complète de deux surfaces? Et si non: de quelle manière trouvera-t-on les équations générales d'une courbe de $n^{\text{ième}}$ ordre? Quel est le degré de généralité de ces équations? Il y a une foule d'autres questions qu'on pourrait ici proposer. J'ai proposé une question analogue dans le point de vue analytique, mais elle est restée sans réponse.

55.

SUR QUELQUES THÉORÈMES DE LA GÉOMÉTRIE DE POSITION.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tome XXXIV. (1847), pp. 270—275. Continued from t. XXXI. p. 227, **50**.]

§ III.

LORSQUE j'avais sous la plume la première partie de ce mémoire, je ne savais pas que la dernière partie du théorème de M. Steiner sur l'hexagramme de Pascal (savoir que les vingt points d'intersection des soixante droites sont situés, par quatre, dans quinze droites) avait déjà été démontrée d'une manière aussi simple qu'élégante par M. Plücker dans son mémoire, "*Über ein neues Princip der Geometrie*" (t. v. [1828] p. 269). En supposant maintenant cette démonstration connue, je veux examiner de plus près la corrélation de ces vingt points, en adoptant une notation plus commode.

Soit $\alpha\beta\gamma\delta\epsilon\zeta$ une permutation quelconque des nombres 1, 2, 3, 4, 5, 6: cette permutation peut être nommée *directe* ou *inverse*, selon qu'elle est formée par un nombre pair ou impair d'inversions. Des six permutations $\alpha\beta\gamma\delta\epsilon\zeta$, $\alpha\delta\gamma\zeta\epsilon\beta$, $\alpha\zeta\gamma\beta\epsilon\delta$; $\alpha\delta\gamma\beta\epsilon\zeta$, $\alpha\beta\gamma\zeta\epsilon\delta$, $\alpha\zeta\gamma\delta\epsilon\beta$, les trois premières ou les trois dernières sont *directes*. Nous représenterons les trois permutations directes par $(\alpha\gamma\epsilon)$. Les trois droites que donne le théorème de Pascal, appliqué aux hexagones correspondants à ce symbole, se coupent dans un des vingt points dont il s'agit: point qui peut être représenté par la même notation $(\alpha\gamma\epsilon)$. En supposant que $\alpha\beta\gamma\delta\epsilon\zeta$ est une permutation directe, le point $\alpha\gamma\epsilon$ correspond au point $\begin{pmatrix}\alpha\gamma\epsilon\\ \beta\delta\zeta\end{pmatrix}$ de M. Plücker. Partout dans cette section on pourra changer les mots "directe" et "inverse."

Les six permutations des lettres α, γ, ϵ ne donnent qu'un seul point; de manière que les vingt points, dont il s'agit, sont

123,	124,	125,	126,	134,	135,	136,	145,	146,	156,
234,	235,	236,	245,	246,	256,	345,	346,	356,	456.

Or, pour trouver comment il faudra combiner ces points, j'écris

$$\begin{matrix} \alpha\gamma\epsilon \\ \beta\delta\zeta \end{matrix}$$

et je tire de là le système

$$\frac{\alpha\gamma\epsilon}{\beta\delta\zeta}, \quad \frac{\alpha\delta\zeta}{\beta\gamma\epsilon}, \quad \frac{\beta\gamma\zeta}{\alpha\delta\epsilon}, \quad \frac{\beta\delta\epsilon}{\alpha\gamma\zeta}.$$

Les quatre points

$$\alpha\gamma\epsilon, \quad \alpha\delta\zeta, \quad \beta\gamma\zeta, \quad \beta\delta\epsilon$$

seront situés sur la même droite, que l'on peut représenter par $\alpha\beta.\gamma\delta.\epsilon\zeta$. Les quinze combinaisons, quatre à quatre, des vingt points, seront

	135,	146,	236,	245	dans	la	droite	12.34.56
	136,	154,	234,	256	„	„	„	12.35.64
	134,	165,	235,	264	„	„	„	12.36.45
	146,	152,	342,	356	„	„	„	13.45.62
	142,	165,	345,	362	„	„	„	13.46.25
	145,	126,	346,	325	„	„	„	13.42.56
	152,	163,	453,	462	„	„	„	14.56.23
A.	153,	126,	456,	423	„	„	„	14.52.36
	156,	132,	452,	436	„	„	„	14.53.62
	163,	124,	564,	523	„	„	„	15.62.34
	164,	132,	562,	534	„	„	„	15.63.42
	162,	143,	563,	542	„	„	„	15.64.23
	124,	135,	625,	634	„	„	„	16.23.45
	125,	143,	623,	645	„	„	„	16.24.53
	123,	154,	624,	653	„	„	„	16.25.34,

où les droites s'obtiennent en permutant dans 12.34.56 d'abord les derniers trois numéros et puis dans ces trois permutations les derniers cinq numéros. Par là la manière de trouver les droites est claire.

Cette énonciation des points et des droites, dont il s'agit, en même temps qu'elle est parfaitement symétrique, est la seule qui se présente naturellement. Cependant la

symétrie en est si compliquée et si peu manifeste qu'il sera bon d'adopter une autre notation. Pour cela, je forme le tableau auxiliaire suivant, dont l'arrangement est assez clair :

135	*ace*	351	*cea*	513	*eac*
246	*bdf*	246	*fbd*	246	*dfb*
146	*acb*	346	*cef*	546	*ead*
235	*edf*	251	*abd*	213	*cfb*
236	*acd*	256	*ceb*	216	*eaf*
145	*bef*	341	*fad*	543	*dcb*
245	*acf*	241	*ced*	243	*eab*
136	*bde*	356	*fba*	516	*dfc*

De là, en écrivant

$$B.\quad \begin{cases} 123 = bcf\,, & 234 = abe\,, \\ 124 = cde\,, & 235 = def\,, \\ 125 = abd\,, & 236 = acd\,, \\ 126 = aef\,, & 245 = acf\,, \\ 134 = adf\,, & 246 = bdf\,, \\ 135 = ace\,, & 256 = bce\,, \\ 136 = bde\,, & 345 = bcd\,, \\ 145 = bef\,, & 346 = cef\,, \\ 146 = abc\,, & 356 = abf\,, \\ 156 = cdf\,, & 456 = ade\,; \end{cases}$$

et de plus

$$C.\quad \begin{cases} 12.34.56 = ac\,, & 13.45.62 = ab\,, & 14.56.23 = bd\,, \\ 12.35.64 = be\,, & 13.46.25 = cd\,, & 14.52.36 = ae\,, \\ 12.36.45 = df\,, & 13.42.56 = ef\,, & 14.53.62 = cf\,, \\ & 15.62.34 = de\,, & 16.23.45 = ce\,, \\ & 15.63.42 = bc\,, & 16.24.53 = ad\,, \\ & 15.64.23 = ef\,, & 16.25.34 = bf\,; \end{cases}$$

savoir (en représentant les points 123, 124, &c., par *bcf, cde,* &c., et les droites 12.34.56, 12.35.64, &c., par *ac, be,* &c.) on verra dans le tableau (*A*) que les points situés dans la droite *ac* sont *ace, acb, acd, acf,* que les points dans la droite *be* sont *bed, bef, bea, bec,* et ainsi de suite, de manière que ce système des vingt points et des quinze droites est précisément le système *réciproque* de celui des quinze points et des vingt

droites que nous avons considéré dans la première section de ce mémoire; ou, autrement dit, que les vingt points et les quinze droites sont les projections sur un plan, des points et des droites d'intersection de six plans dans l'espace. Seulement la figure plane, ainsi formée, contient quatorze quantités arbitraires, tandis que le système de six points sur une conique n'en contient que onze, de sorte qu'il doit y avoir des relations entre ces six plans. On obtient donc la forme suivante plus complète du théorème de M. Steiner (théorème qui en même temps est le complément du théorème XII. § 1 de ce mémoire).

THÉORÈME XIV. "Les soixante droites correspondantes aux hexagones formés par six points d'une conique se coupent trois à trois dans vingt points qui peuvent être considérés comme les projections des points d'intersection d'un système de six plans (dont d'ailleurs la liaison reste encore à chercher)."

Également

THÉORÈME XIII. "Les soixante points correspondants aux hexagones formés par six tangentes d'une conique sont situés trois à trois sur vingt droites déterminées par des plans qui passent par trois points quelconques d'un système de six points dans l'espace (la liaison de ce système de six points étant encore à chercher)."

§ IV.

Soient a, f; b, g; c, h les points correspondants d'un système de points situés sur la même droite, et en involution. Nommons "*faisceau*" les trois côtés d'un quadrilatère qui se coupent dans un même point et "*triangle*" les trois côtés qui ne se coupent pas dans un même point (de sorte que dans tout quadrilatère il y aura quatre *faisceaux* et quatre *triangles*). Les quadrilatères dont les côtés passent par les six points en involution, peuvent être classés en deux systèmes: Dans le premier les faisceaux passeront par f, g, h; f, b, c; g, c, a; h, a, b, et les triangles par a, b, c; a, g, h; b, h, f; c, f, g; dans l'autre il en sera le contraire. Deux quadrilatères qui appartiennent à ces deux systèmes respectivement, peuvent être dits "en rapport inverse l'un à l'autre." Soient $ABCD$, $A'B'C'D'$ deux quadrilatères non situés dans le même plan, et soumis à la condition que les côtés

$$DA,\ B'C';\ DB,\ C'A';\ DC,\ A'B';\ BC,\ D'A';\ CA,\ D'B';\ AB,\ D'C'$$

coupent la droite dans les points

$$f,\ g,\ h,\ a,\ b,\ c$$

respectivement. Les deux tétraèdres $A'BCD$, $AB'C'D'$, et également les tétraèdres $AB'CD$ et $A'BC'D'$; $ABC'D$ et $A'B'CD'$; $ABCD'$ et $A'B'C'D$ seront respectivement inscrits et circonscrits l'un à l'autre. Car en considérant par exemple ces deux-ci: $A'BCD$, $AB'C'D'$: A' est dans le plan $B'C'D'$, B est dans le plan $C'D'A$ (parce que les droites AB, $C'D'$ se rencontrent); C est dans le plan $D'AB'$ (parce que AC et $B'D'$ se rencontrent), et D est dans le plan $AB'C'$ (parce que AD et $B'C'$ se rencontrent). Également A, B', C', D'

sont situés dans les plans BCD, CDA', $DA'B$ et $A'BC$ respectivement; et cela vérifie la relation dont il s'agit. Ce théorème est dû à M. Möbius qui l'a obtenu par son Calcul Barycentrique (ce Journal, t. III. [1828], p. 273), et en considérant un système polaire dans lequel le plan réciproque d'un point quelconque passe toujours par le point même (Statik, c. VI. § 86, et ce Journal t. X. [1833], p. 317). On trouve aussi quelques remarques sur ce sujet dans l'ouvrage "*Systematische Entwickelungen u. s. w.*" de M. Steiner, p. 247. Je ne croyais pas inutile d'en faire voir la relation avec la théorie de l'involution. Remarquons aussi que non seulement les quadrilatères $ABCD$ et $A'B'C'D'$, mais aussi ceux-ci $ABC'D'$ et $A'B'CD$, $ACB'D'$ et $A'C'BD$, $ADB'C'$ et $A'D'BC$ sont en rapport inverse entre eux. Par cela la symétrie de la figure est complétée; mais on n'en tire pas de nouveaux systèmes de tétraèdres inscrits et circonscrits.

M. Möbius a démontré qu'il n'existe pas des quadrilatères réels simples, à quatre côtés et quatre angles, inscrits et circonscrits. Mais en considérant les points imaginaires, l'existence en est possible, et on trouve des systèmes de cette sorte parmi les neuf points d'inflexion d'une courbe de troisième ordre. Je renvoie cette discussion à une autre occasion, § V.

Je me bornerais, sans examiner de plus près la figure qui en résulte, à démontrer le théorème suivant: "Si un point et n droites sont donnés, les points d'intersection de chaque droite avec la polaire du point, relative aux autres $n-1$ droites, sont situés sur une même droite polaire du point, relative au système des droites." Je prends un système de droites, considéré comme formant une courbe, et j'entends par polaire ou droite polaire, la dernière des polaires successives du point, relative à la courbe. En représentant analytiquement la courbe par $V=0$, V est une fonction homogène d'un ordre quelconque en x, y, z; et si $\alpha:\beta:\gamma$ sont les valeurs de $x:y:z$ relatives au point, l'équation

$$(\alpha\partial_x+\beta\partial_y+\gamma\partial_z)^{p-1}U=0$$

est celle de l'une quelconque des polaires successives.

Soit $p=0$, $q=0$, $r=0$, ... les équations des droites, p, q, r, ... seront des fonctions linéaires de x, y, z. Soient comme plus haut $x:y:z=\alpha:\beta:\gamma$ les équations qui déterminent le point: l'équation de la droite polaire du point, relative aux n droites, est

$$(\alpha\partial_x+\beta\partial_y+\gamma\partial_z)^{n-1}pqr\ldots=0.$$

Soient a, b, c, ... ce que deviennent p, q, r, ... en écrivant α, β, γ, ... au lieu de x, y, z, ... , on obtient aisément

$$\alpha\partial_x+\beta\partial_y+\gamma\partial_z=a\partial_p+b\partial_q+\ldots;$$

et de là on tire

$$(a\partial_p+b\partial_q+c\partial_r+\ldots)^{n-1}pqr\ldots=0,$$

pour la polaire cherchée. Les différentiations étant effectuées selon p, q, r, ... , comme variables indépendantes, on obtient

$$pbc\ldots+aqc\ldots+\ldots=0$$

ou, ce qui est la même chose:

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}+\ldots=0.$$

De même la polaire du point, relative aux droites $q=0$, $r=0$, ... , est

$$\frac{q}{b}+\frac{r}{c}+\ldots=0,$$

et par cette raison l'intersection de cette droite avec $p=0$ est évidemment située sur la droite polaire du point, relative à toutes les droites; ce qui prouve la proposition dont il s'agit.

Par exemple, en considérant les droites BC, CA, AB qui passent par trois points A, B, C: la polaire d'un point O, relative aux droites AB, AC, est une droite $A\alpha$ telle que AB, AC; AO, $A\alpha$ forment un faisceau harmonique. Soit α le point d'intersection de $A\alpha$ et BC, et supposons le même pour les points β, γ: les points α, β, γ seront situés (comme on le sait) sur une même droite, qui est celle que je nomme *polaire* de O, relative aux côtés du triangle, et que M. Plücker a nommé "*harmonicale*." On sait de plus que les droites $A\alpha$, $B\beta$, $C\gamma$ peuvent être construites comme suit: en prenant a, b, c pour les points d'intersection de OA, OB, OC avec BC, CA, AB respectivement, les droites bc, BC; ca, CA; ab, AB se rencontreront dans les points α, β, γ; ce qui offre une règle facile pour construire la polaire d'un point, relative à un nombre quelconque de droites.

Remarquons en finissant que la conique qui passe par les points A, B, C, et qui touche deux des trois droites $A\alpha$, $B\beta$, $C\gamma$, touche aussi la troisième et est effectivement la polaire conique du point, relative aux trois côtés du triangle. En combinant cela avec la propriété connue que la $r^{\text{ième}}$ polaire de O', relative à la même courbe, passe par O, si la $(n-r)^{\text{ième}}$ polaire d'un point O, relative à une courbe du $n^{\text{ième}}$ ordre, passe par un point O', on obtiendra le théorème 22 de M. Steiner (ce Journal, t. IV. [1828] p. 209).

56.

DEMONSTRATION OF A GEOMETRICAL THEOREM OF JACOBI'S.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 48—49.]

THE theorem in question (*Crelle*, t. XII. [1834] p. 137) may be thus stated:

"If a cone be circumscribed about a surface of the second order, the focal lines of the cone are generating lines of a surface of the second order confocal to the given surface and which passes through the vertex of the cone."

Let (α, β, γ) be the coordinates of the given point,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

the equation to the given surface. The equation of the circumscribed cone referred to its vertex is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}\right)^2=0,$$

whence it is easily seen that the equation of the supplementary cone (i.e. the cone generated by lines through the vertex at right angles to the tangent planes of the cone in question) is

$$(\alpha x+\beta y+\gamma z)^2-a^2x^2-b^2y^2-c^2z^2=0. \text{ (1)}$$

Suppose we have identically

$$(\alpha x+\beta y+\gamma z)^2-a^2x^2-b^2y^2-c^2z^2-h\,(x^2+y^2+z^2)=(lx+my+nz)\,(l'x+m'y+n'z);$$

$lx+my+nz=0$ will determine the direction of one of the cyclic planes of the supplementary cone, and hence taking the centre for the origin the equations of the focal lines of the circumscribed cone are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

[1] $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ being the equation of the first cone, that of the supplementary cone is $\mathfrak{A}x^2+\mathfrak{B}y^2+\mathfrak{C}z^2+2\mathfrak{F}yz+2\mathfrak{G}zx+2\mathfrak{H}xy=0$, these letters [denoting the inverse coefficients $BC-F^2$, &c.].

It only remains therefore to determine the values of l, m, n from the last equation but one. The condition which expresses that the first side of this equation divides itself into factors is easily reduced to

$$\frac{\alpha^2}{a^2+h}+\frac{\beta^2}{b^2+h}+\frac{\gamma^2}{c^2+h}=1 \quad\text{.....(A).}$$

Next, since the equation is identical, write

$$x=\frac{\alpha}{a^2+h},\quad y=\frac{\beta}{b^2+h},\quad z=\frac{\gamma}{c^2+h},$$

we deduce

$$0=\frac{l\alpha}{a^2+h}+\frac{m\beta}{b^2+h}+\frac{n\gamma}{c^2+h} \quad\text{..... (B).}$$

Again, putting

$$x=\frac{l}{a^2+h},\quad y=\frac{m}{b^2+h},\quad z=\frac{n}{c^2+h},$$

the whole equation divides by

$$\left(\frac{ll'}{a^2+h}+\frac{mm'}{b^2+h}+\frac{nn'}{c^2+h}+1\right),$$

a factor whose value is easily seen to be $=-1$. And rejecting this, we have

$$\frac{l^2}{a^2+h}+\frac{m^2}{b^2+h}+\frac{n^2}{c^2+h}=0 \quad\text{..... (C).}$$

Thus of the three equations (A), (B), and (C), the first determines h, and the remaining two give the ratios $l : m : n$. It is obvious that

$$\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1$$

is the equation of the surface confocal with the given surface which passes through the point (α, β, γ). The generating lines at this point are found by combining this equation with that of the tangent plane at the same point, viz.

$$\frac{\alpha x}{a^2+h}+\frac{\beta y}{b^2+h}+\frac{\gamma z}{c^2+h}=1;$$

and since these two equations are satisfied by $x=\alpha+lr$, $y=\beta+mr$, $z=\gamma+nr$, if l, m, n are determined by the equations above, it follows that the focal lines of the cone are the generating lines of the surface, the theorem which was to be demonstrated. It is needless to remark that of the three confocal surfaces, the hyperboloid of one sheet has alone real generating lines; this is as it should be, since a cone has six focal lines, of which four are always imaginary.

57.

ON THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 50—51.]

WE have seen [45] that the equation

$$n(n-1)x^2z+(n-1)(\alpha x-2x^3)\frac{dz}{dx}+(1-\alpha x^2+x^4)\frac{d^2z}{dx^2}-2n(\alpha^2-4)\frac{dz}{d\alpha}=0$$

is integrable, in the case of n an odd number, in the form $z=B_0+B_1x^2\ldots+B_{\frac{1}{2}(n-1)}x^{n-1}$; and the coefficients at the beginning of the series have already been determined; to find those at the end of it, the most convenient mode of writing the series will be

$$z=\mu\Sigma\frac{(-)^r D_r}{1.2\ldots(2r+1)}x^{n-1-2r},$$

and then the coefficients D_r are determined by

$$D_{r+2}=(2r+3)(n-2r-3)D_{r+1}\alpha-2n(\alpha^2-4)\frac{dD_{r+1}}{d\alpha}$$
$$-(2r+3)(2r+2)(n-2r-2)(n-2r-1)D_r.$$

The first coefficients then are

$$D_0=1,$$

$$D_1=(n-1)\alpha,$$

$$D_2=2(n-1)(n+6)+(n-1)(n-9)\alpha^2,$$

$$D_3=6(n-1)(n-9)(n+10)\alpha+(n-1)(n-9)(n-25)\alpha^3,$$

$$\begin{aligned}D_4=&-36(n-1)(n^3-13n^2+36n+420)\\&+12(n-1)(n-9)(n-25)(n+14)\alpha^2\\&+(n-1)(n-9)(n-25)(n-49)\alpha^4,\end{aligned}$$

$$\begin{aligned} D_5 = &-12(n-1)(n-9)(47n^3-355n^2+3188n+31500)\alpha \\ &+20(n-1)(n-9)(n-25)(n-49)(n+18)\alpha^3 \\ &+(n-1)(n-9)(n-25)(n-49)(n-81)\alpha^5, \end{aligned}$$

$$\begin{aligned} D_6 = &-24(n-1)(n-9)(\ 23n^4+2375n^3-14638n^2+116100n+\ 693000) \\ &-12(n-1)(n-9)(493n^4-8882n^3+70317n^2-361641n-7276500)\alpha^2 \\ &+30(n-1)(n-9)(n-25)(n-49)(n-81)(n+22)\alpha^4 \\ &+(n-1)(n-9)(n-25)(n-36)(n-49)(n-81)(n-121)\alpha^6, \\ &\&c. \end{aligned}$$

And, in general,

$$\begin{aligned} D_r = &(n-1)(n-9)\ldots\{n-(2r-1)^2\}\alpha^r \\ &+r(r-1)(n-1)(n-9)\ldots\{n-(2r-3)^2\}(n+4r-2)\alpha^{r-2}, \\ &\&c. \end{aligned}$$

(where however the next term does not contain the factor $(n-1)(n-9)\ldots\{n-(2r-5)^2\}$).

In the case when $n=\nu^2$, then in order that the constant term may reduce itself to unity, we must assume

$$\mu = (-)^{\frac{1}{2}(\nu-1)}\nu\,;$$

this is evident from what has preceded.

58.

NOTES ON THE ABELIAN INTEGRALS.—JACOBI'S SYSTEM OF DIFFERENTIAL EQUATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 51—54.]

THE theory of elliptic functions depends, it is well known, on the differential equation $\frac{dx}{\sqrt{(fx)}}+\frac{dy}{\sqrt{(fy)}}=0$, ($fx$ denoting a rational and integral function of the fourth order), the integral of which was discovered by Euler, though first regularly derived from the differential equation by Lagrange. The theory of the Abelian integrals depends in like manner, as is proved by Jacobi, in the memoir "Considerationes generales de transcendentibus Abelianis" (*Crelle*, t. IX. [1832] p. 394) to depend, upon the system of equations

$$\Sigma\frac{dx}{\sqrt{(fx)}}=0,\quad \Sigma\frac{xdx}{\sqrt{(fx)}}=0,\ \ldots\ldots\ \Sigma\frac{x^{n-2}\,dx}{\sqrt{(fx)}}=0\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where fx is a rational and integral function of the order $2n-1$ or $2n$, and the sums Σ contain n terms.

The integration of this system of equations is of course virtually comprehended in Abel's theorem; the problem was to obtain $(n-1)$ integrals each of them containing a single independent arbitrary constant. One such integral was first obtained by Richelot (*Crelle*, t. XXIII. [1842] p. 354), "Ueber die Integration eines merkwürdigen Systems Differentialgleichungen." by a method founded on that of Lagrange for the solution of Euler's equation; and a second integral very ingeniously deduced from it. A complete system of integrals in the required form is afterwards obtained, not by direct integration, but by means of Abel's theorem: there is this objection to them, however, that any one of them contains two roots of the equation $fx=0$. The next paper on the subject is one by Jacobi, "Demonstratio Nova theorematis Abeliani" (*Crelle*, t. XXIV. [1842] p. 28), in which a complete system of equations is deduced by direct integration,

each of which contains only a single root of the equation $fx=0$. But in Richelot's second memoir "Einige neue Integralgleichungen des Jacobischen Systems Differentialgleichungen" (*Crelle*, t. XXV. [1843] p. 97), the equations are obtained by direct integration in a form not involving any of the roots of this equation; the method employed in obtaining them being in a great measure founded upon the memoir just quoted of Jacobi's. The following is the process of integration.

Denoting the variables by $x_1, x_2 \dots x_n$, and writing

$$Fa = (a-x_1)(a-x_2)\dots(a-x_n),$$

so that

$$F'x_1 = (x_1-x_2)\dots(x_1-x_n),$$
&c.

then the system of differential equations is satisfied by assuming that $x_1, x_2 \dots x_n$ are functions of a new variable t, determined by the equations

$$\frac{dx_1}{dt} = \frac{\sqrt{(fx_1)}}{F'x_1}, \text{ \&c.}$$

(In fact these equations give $\Sigma \dfrac{dx}{\sqrt{(fx)}} = dt\, \Sigma \dfrac{1}{F'x} = 0$, &c.)

From these we deduce, by differentiation,

$$\frac{d^2x_1}{dt^2} = \tfrac{1}{2}\frac{d}{dx_1}\frac{fx_1}{(F'x_1)^2} + \frac{\sqrt{(fx_1)}}{F'x_1}\Sigma'\frac{\sqrt{(fx)}}{(x_1-x)F'x}$$

(where Σ' refers to all the roots except x_1) and a set of analogous equations for $x_2, x_3 \dots x_n$.

Dividing this by $a-x_1$, where a is arbitrary, and reducing by

$$\frac{1}{(a-x_1)(x_1-x)} = \frac{1}{2(a-x)(a-x_1)}\left(1-\frac{x+x_1-2a}{x_1-x}\right),$$

we have

$$\frac{1}{a-x_1}\frac{d^2x_1}{dt^2} = \frac{1}{2(a-x_1)}\frac{d}{dx_1}\frac{fx_1}{(F'x_1)^2}$$
$$+\tfrac{1}{2}\frac{\sqrt{(fx_1)}}{(a-x_1)F'x_1}\Sigma'\frac{\sqrt{(fx)}}{(a-x)F'x} - \tfrac{1}{2}\Sigma'\frac{\sqrt{(fx)}\sqrt{(fx_1)}}{F'x\,F'x_1}\frac{(x_1+x-2a)}{(a-x)(a-x_1)(x_1-x)},$$

that is

$$\frac{1}{a-x_1}\frac{d^2x_1}{dt^2} = \frac{1}{2(a-x_1)}\frac{d}{dx_1}\frac{fx_1}{(F'x_1)^2}$$
$$+\tfrac{1}{2}\frac{\sqrt{(fx_1)}}{(a-x_1)F'x_1}\Sigma\frac{\sqrt{(fx)}}{(a-x)F'x} - \tfrac{1}{2}\frac{fx_1}{(a-x_1)^2(F'x_1)^2}$$
$$-\tfrac{1}{2}\Sigma'\frac{\sqrt{(fx)}\sqrt{(fx_1)}}{F'x\,F'x_1}\frac{(x_1+x-2a)}{(a-x)(a-x_1)(x_1-x)};$$

and taking the sum of all the equations of this form, the last term disappears on account of the factor $x_1 - x$ in the denominator, and the result is

$$\Sigma \frac{1}{\alpha - x} \frac{d^2x}{dt^2} = \tfrac{1}{2}\Sigma \frac{1}{\alpha - x} \frac{d}{dx} \frac{fx}{(F'x)^2} + \tfrac{1}{2}\left\{\Sigma \frac{\sqrt{(fx)}}{(\alpha - x) F'x}\right\}^2 - \tfrac{1}{2}\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2}.$$

This being premised, assume

$$y = \sqrt{(F\alpha)},$$

which, by differentiation, gives

$$\frac{dy}{dt} = -\tfrac{1}{2}y\Sigma \frac{1}{\alpha - x} \frac{dx}{dt} = -\tfrac{1}{2}y\Sigma \frac{\sqrt{(fx)}}{(\alpha - x) F'x},$$

and thence

$$\frac{d^2y}{dt^2} = -\tfrac{1}{2}\frac{dy}{dt}\Sigma \frac{1}{\alpha - x}\frac{dx}{dt} + \tfrac{1}{2}y\Sigma \frac{1}{(\alpha - x)^2}\left(\frac{dx}{dt}\right)^2 - \tfrac{1}{2}y\Sigma \frac{1}{(\alpha - x)}\frac{d^2x}{dt^2},$$

that is
$$\frac{d^2y}{dt^2} = \tfrac{1}{4}y\left(\Sigma \frac{\sqrt{(fx)}}{(\alpha - x) F'x}\right)^2 - \tfrac{1}{2}y\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} - \tfrac{1}{2}y\Sigma \frac{1}{\alpha - x}\frac{d^2x}{dt^2}.$$

Substituting the preceding value of

$$\Sigma \frac{1}{\alpha - x}\frac{d^2x}{dt^2},$$

we have
$$\frac{d^2y}{dt^2} = -\tfrac{1}{4}y\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} - \tfrac{1}{4}y\Sigma \frac{1}{\alpha - x}\frac{d}{dx}\frac{fx}{(F'x)^2};$$

that is
$$4\frac{d^2y}{dt^2} + y\left\{\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} + \Sigma \frac{1}{(\alpha - x)}\frac{d}{dx}\frac{fx}{(F'x)^2}\right\} = 0.$$

Now the fractional part of $\dfrac{f\alpha}{(F\alpha)^2}$ is equal to

$$\Sigma \frac{fx}{(\alpha - x)^2 (F'x)^2} + \Sigma \frac{1}{\alpha - x}\frac{d}{dx}\frac{fx}{(F'x)^2}.$$

Also if L be the coefficient of x^{2n} in $f\alpha$, the integral part is simply equal to L, (since $(F\alpha)^2$ is a function of the order $2n$, in which the coefficient of α^{2n} is unity). Hence the coefficient of y in the last equation is simply

$$\frac{f\alpha}{(F\alpha)^2} - L, \quad = \frac{f\alpha}{y^4} - L;$$

or we have

$$4\frac{d^2y}{dt^2} + y\left(\frac{f\alpha}{y^4} - L\right) = 0,$$

viz. multiplying by the factor $2\frac{dy}{dt}$, and integrating,

$$4\left(\frac{dy}{dt}\right)^2 - \frac{f\alpha}{y^2} - Ly^2 = C.$$

Hence replacing y and $\frac{dy}{dt}$ by their values

$$\sqrt{(F\alpha)} \text{ and } -\tfrac{1}{2}\sqrt{(F\alpha)}\,\Sigma\frac{\sqrt{(fx)}}{(\alpha-x)\,F'x},$$

we have

$$F\alpha\left\{\Sigma\frac{\sqrt{(fx)}}{(\alpha-x)\,F'x}\right\}^2 - \frac{f\alpha}{F\alpha} - LF\alpha = C,$$

for one of the integrals of the proposed system of equations: and since α is arbitrary, the complete system is obtained by giving any $(n-1)$ particular values to α, and changing the value of the constant of integration C; or by expanding the first side of the equation in terms of α, and equating the different coefficients to arbitrary constants. The *à posteriori* demonstration that all the results so obtained are equivalent to $(n-1)$ independent equations would probably be of considerable interest.

59.

ON THE THEORY OF ELIMINATION.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 116—120.]

SUPPOSE the variables X_1, $X_2 \ldots$, g in number, are connected by the h linear equations

$$\Theta_1 = \alpha_1 X_1 + \alpha_2 X_2 \ldots = 0,$$
$$\Theta_2 = \beta_1 X_1 + \beta_2 X_2 \ldots = 0,$$
$$\vdots$$

these equations not being all independent, but connected by the k linear equations

$$\Phi_1 = \alpha_1' \Theta_1 + \alpha_2' \Theta_2 + \ldots = 0,$$
$$\Phi_2 = \beta_1' \Theta_1 + \beta_2' \Theta_2 + \ldots = 0,$$
$$\vdots$$

these last equations not being independent, but connected by the l linear equations

$$\Psi_1 = \alpha_1'' \Phi_1 + \alpha_2'' \Phi_2 + \ldots = 0,$$
$$\Psi_2 = \beta_1'' \Phi_1 + \beta_2'' \Phi_2 + \ldots = 0,$$
$$\vdots$$

and so on for any number of systems of equations.

Suppose also that $g - h + k - l + \ldots = 0$; in which case the number of quantities X_1, $X_2, \ldots$ will be equal to the number of really independent equations connecting them, and we may obtain by the elimination of these quantities a result $\nabla = 0$.

To explain the formation of this final result, write

$$\nabla = \begin{array}{|ll|ll|l}
\hline
\alpha_1, & \beta_1 \ldots & & & \\
\alpha_2, & \beta_2 & & & \\
\vdots & & & & \\
\hline
\alpha_1', & \alpha_2' \ldots & \alpha_1'', & \beta_1'' \ldots & \\
\beta_1', & \beta_2' & \alpha_2'', & \beta_2'' & \\
\vdots & & \vdots & & \\
\hline
 & & \alpha_1''', & \alpha_2''' \ldots & \\
 & & \beta_1''', & \beta_2''' & \\
 & & \vdots & & \\
\hline
\end{array}$$

which for shortness may be thus represented,

$$\nabla = \begin{array}{|c|c|l}
\hline
\Omega & & \\
\hline
\Omega' & \Omega'' & \\
\hline
 & \Omega''' & \\
\hline
\end{array}$$

where Ω, Ω', Ω'', Ω''', Ω'''', ... contain respectively h, h, l, l, n, n, ... vertical rows, and g, k, k, m, m, p, ... horizontal rows.

It is obvious, from the form in which these systems have been arranged, what is meant by speaking of a certain number of the vertical rows of Ω' and the *supplementary* vertical rows of Ω; or of a certain number of the horizontal rows of Ω'' and the *supplementary* horizontal rows of Ω', &c.

Suppose that there is only one set of equations, or $g = h$: we have here only a single system Ω, which contains h vertical and h horizontal rows, and ∇ is simply the determinant formed with the system of quantities Ω. We may write in this case $\nabla = Q$.

Suppose that there are two sets of equations, or $g = h - k$: we have here two systems Ω, Ω', of which Ω contains h vertical and $h - k$ horizontal rows, Ω' contains h vertical and k horizontal rows. From any k of the h vertical rows of Ω' form a determinant, and call this Q'; from the supplementary $h - k$ vertical rows of Ω form a determinant, and call this Q: then Q' divides Q, and we have $\nabla = Q \div Q'$.

Suppose that there are three sets of equations, or $g=h-k+l$: we have here three systems, Ω, Ω', Ω'', of which Ω contains h vertical and $h-k+l$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows. From any l of the k horizontal rows of Ω'' form a determinant, and call this Q''; from the $k-l$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h-k+l$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q'' divides Q', this quotient divides Q, and we have $\nabla = Q \div (Q' \div Q'')$.

Suppose that there are four sets of equations, or $g=h-k+l-m$: we have here four systems, Ω, Ω', Ω'', and Ω''', of which Ω contains h vertical and $h-k+l-m$ horizontal rows, Ω' contains h vertical and k horizontal rows, Ω'' contains l vertical and k horizontal rows, and Ω''' contains l vertical and m horizontal rows. From any m of the l vertical rows of Ω''' form a determinant, and call this Q'''; from the $l-m$ supplementary vertical rows of Ω'', choosing the horizontal rows at pleasure, form a determinant, and call this Q''; from the $k-l+m$ supplementary horizontal rows of Ω', choosing the vertical rows at pleasure, form a determinant, and call this Q'; from the $h-k+l-m$ supplementary vertical rows of Ω form a determinant, and call this Q: then Q''' divides Q'', this quotient divides Q', this quotient divides Q, and $\nabla = Q \div \{Q' \div (Q'' \div Q''')\}$. The mode of proceeding is obvious.

It is clear, that if all the coefficients α, $\beta, \ldots$ be considered of the order unity, ∇ is of the order $h-2k+3l-$ &c.

What has preceded constitutes the theory of elimination alluded to in my memoir "On the Theory of Involution in Geometry," *Journal*, vol. II. p. 52—61, [40]. And thus the problem of eliminating any number of variables x, $y \ldots$ from the same number of equations $U=0$, $V=0, \ldots$ (where U, $V, \ldots$ are homogeneous functions of any orders whatever) is completely solved; though, as before remarked, I am not in possession of any method of arriving *at once* at the final result in its most simplified form; my process, on the contrary, leads me to a result encumbered by an extraneous factor, which is only got rid of by a number of successive divisions less by two than the number of variables to be eliminated.

To illustrate the preceding method, consider the three equations of the second order,

$$U = a\ x^2 + b\,y^2 + c\ z^2 + l\ yz + m\ zx + n\ xy = 0,$$
$$V = a'\ x^2 + b'y^2 + c'\ z^2 + l'\ yz + m'\ zx + n'\ xy = 0,$$
$$W = a''x^2 + b''y^2 + c''z^2 + l''yz + m''zx + n''xy = 0.$$

Here, to eliminate the fifteen quantities x^4, y^4, z^4, y^3z, z^3x, x^3y, yz^3, zx^3, xy^3, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, we have the eighteen equations

$$x^2U = 0,\quad y^2U = 0,\quad z^2U = 0,\quad yzU = 0,\quad zxU = 0,\quad xyU = 0,$$
$$x^2V = 0,\quad y^2V = 0,\quad z^2V = 0,\quad yzV = 0,\quad zxV = 0,\quad xyV = 0,$$
$$x^2W = 0,\quad y^2W = 0,\quad z^2W = 0,\quad yzW = 0,\quad zxW = 0,\quad xyW = 0,$$

equations, however, which are not independent, but are connected by

$$a''x^2V + b''y^2V + c''z^2W + l''yzV + m''zxV + n''xyV$$
$$-(a'x^2W + b'y^2W + c'z^2W + l'yzW + m'zxW + n'xyW) = 0,$$

$$ax^2W + by^2W + cz^2W + lyzW + mzxW + nxyW$$
$$-(a''x^2U + b''y^2U + c''z^2U + l''yzU + m''zxU + n''xyU) = 0,$$

$$a'x^2U + b'y^2U + c'z^2U + l'yzU + m'zxU + n'xyU$$
$$-(ax^2V + by^2V + cz^2V + lyzV + mzxV + nxyV) = 0.$$

Arranging these coefficients in the required form, we have the following value of ∇.

a						a'						a''					
	b						b'						b''				
		c						c'						c''			
	l		b				l'		b'				l''		b''		
		m		c				m'		c'				m''		c''	
n					a	n'					a'	n''					a''
		l	c					l'	c'					l''	c''		
m				a		m'				a'		m''				a''	
	n				b		n'				b'		n''				b''
	c	b	l				c'	b'	l'				c''	b''	l''		
c		a		m		c'		a'		m'		c''		a''		m''	
b	a				n	b'	a'				n'	b''	a''				n''
l			a	n	m	l'			a'	n'	m'	l''			a''	n''	m''
	m		n	b	l		m'		n'	b'	l'		m''		n''	b''	l''
		n	m	l	c			n'	m'	l'	c'			n''	m''	l''	c''

						a''	b''	c''	l''	m''	n''	$-a'$	$-b'$	$-c'$	$-l'$	$-m'$	$-n'$
$-a''$	$-b''$	$-c''$	$-l''$	$-m''$	$-n''$							a	b	c	l	m	n
a'	b'	c'	l'	m'	n'	$-a$	$-b$	$-c$	$-l$	$-m$	$-n$						

which may be represented as before by

$$\nabla = \left|\begin{array}{c} \Omega \\ \hline \Omega' \end{array}\right| .$$

Thus, for instance, selecting the first, second, and sixth lines of Ω' to form the determinant Q', we have $Q' = a''(a'b'' - a''b')$; and then Q must be formed from the third, fourth, fifth, seventh, &c.... eighteenth lines of Ω. (It is obvious that if Q' had been formed from the first, second, and third lines of Ω', we should have had $Q' = 0$; the corresponding value of Q would also have vanished, and an illusory result be obtained; and similarly for several other combinations of lines.)

60.

ON THE EXPANSION OF INTEGRAL FUNCTIONS IN A SERIES OF LAPLACE'S COEFFICIENTS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 120, 121.]

SUPPOSE

$$S = A\mu^s + A_1\mu^{s-1} + \dots,$$
$$= \alpha Q_s + \alpha_1 Q_{s-1} + \dots \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(1),$$

where, as usual, Q_0, Q_1, &c. are the coefficients of the successive powers of p in $(1 - 2\mu p + p^2)^{-\frac{1}{2}}$. Assume

$$S = \frac{\sqrt{(1+\Delta^2)}}{\sqrt{(1 - 2\mu\Delta + \Delta^2)}}\, C \dots\dots\dots\dots\dots\dots\dots\dots\dots(2),$$

where Δ refers to C; then expanding this expression, first in powers of μ, and then in a series of terms of the form $\sqrt{(1+\Delta^2)}\,.\,Q$, and comparing these with the preceding values of S,

$$A_{s-q} = \frac{1\,.\,3 \dots 2q-1}{2\,.\,4 \dots 2q} \left(\frac{2\Delta}{1+\Delta^2}\right)^q . C,$$

$$\alpha_{s-r} = \sqrt{(1+\Delta^2)}\,.\,\Delta^r\,.\,C \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3).$$

Assume $\dfrac{2\Delta}{1+\Delta^2} = \delta$, that is,

$$\Delta = \frac{1}{\delta}\{1 - \sqrt{(1-\delta^2)}\}, \quad \sqrt{(1+\Delta)^2} = \frac{\sqrt{2}}{\delta}\{1 - \sqrt{(1-\delta^2)}\}^{\frac{1}{2}} \dots\dots\dots\dots\dots(4).$$

Then

$$\delta^q C = \frac{2\,.\,4 \dots 2q}{1\,.\,3 \dots 2q-1} A_{s-q},$$

$$\alpha_{s-r} = \frac{\sqrt{2}}{\delta^{r+1}}\{1 - \sqrt{(1-\delta^2)}\}^{r+\frac{1}{2}}\,.\,C \dots\dots\dots\dots\dots\dots\dots\dots (5);$$

or, expanding this last equation in powers of δ,

$$\alpha_{s-r} = \frac{(2r+1)}{2^r}\delta^r\left(\frac{1}{2r+1} + \frac{1}{2}\frac{\delta^2}{2^2} + \frac{(2r+7)}{2\,.\,4}\frac{\delta^4}{2^4} + \frac{(2r+9)(2r+11)}{2\,.\,4\,.\,6}\frac{\delta^4}{2^6} + \ldots\right) C \ldots (6);$$

or, replacing the successive terms of the form $\delta^2 . C$ by their respective values,

$$\alpha_{s-r} = (2r+1)\left\{\frac{2\,.\,4\ldots\quad 2r}{3\,.\,5\ldots(2r+1)\,2^r} A_{s-r} + \frac{4\,.\,6\ldots(2r+4)}{3\,.\,5\ldots(2r+3)\,2^{r+2}} A_{s-r-2}\ldots\right.$$

$$\left. + \frac{(2k+2)(2k+4)\ldots\quad(2r+4k)}{3\quad.\quad 5\quad\ldots(2r+2k+1)\,2^{r+2k}} A_{s-r-2k}\ldots\right\}\ldots\ldots\ldots\ldots(7).$$

Thus, if $S = \mu^s$, so that $A_{s-r} = 0$, except in the particular case $A = 1$,

$$\alpha_{2k-1} = 0,$$

$$\alpha_{2k} = \frac{2s-4k+1}{2^s}\,\frac{(2k+2)(2k+4)\ldots\quad 2s}{3\quad.\quad 5\quad\ldots(2s-2k+1)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

or

$$\mu^s = \frac{1}{2^s}\Sigma\left\{(2s-4k+1)\frac{(2k+2)(2k+4)\ldots\quad 2s}{3\quad.\quad 5\quad\ldots(2s-2k+1)} Q_{s-2k}\right\}\ldots\ldots\ldots\ldots(9),$$

which of course includes the preceding case. By substituting the expanded values of the coefficients Q, or again, by determining the value of $(1-\mu)^s$ in terms of these coefficients, and equating it with that given in Murphy's *Electricity*, [8°. Cambridge, 1833], p. 10, or in a variety of other ways, a series of identical equations involving sums of factorials may readily be obtained. The mode of employing the general theory of the separation of symbols made use of in the preceding example, may easily be applied to the solution of analogous questions.

61.

ON GEOMETRICAL RECIPROCITY.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 173—179.]

THE fundamental theorem of reciprocity in plane geometry may be thus stated.

"The points and lines of a plane P may be considered as corresponding to the lines and points of a plane P' in such a manner that to a set of points in a line in the first figure, there corresponds a set of lines through a point in the second figure, (namely through the point corresponding to the line); and to a set of lines through a point in the first figure, there corresponds a set of points in a line in the second figure, (namely in the line corresponding to the point)."

And from this theorem, without its being in any respect necessary further to particularize the nature of the correspondence, or to consider in any manner the relative position of the two planes, an endless variety of propositions and theories may be deduced, as, for instance, the duality of all theorems which relate to the purely descriptive properties of figures, the theory of the singular points and tangents of curves, &c.

Suppose, however, that the two planes coincide, so that a point may be considered indifferently as belonging to the first or to the second figure: an entirely independent series of propositions (which, properly speaking, form no part of the general theory of reciprocity) result from this particularization. In general, the line in the second figure which corresponds to a point considered as belonging to the first figure, and the line in the first figure which corresponds to the same point considered as belonging to the second figure, will not be identical; neither will the point in the second figure which corresponds with a line considered as belonging to the first figure, and the point in the first figure which corresponds to the same line considered as belonging to the second figure, be identical.

In the particular case where these lines and points are respectively identical (the identity of the lines implies that of the points and *vice versâ*) we have the theory of "reciprocal polars." Here, where it is unnecessary to define whether the points or lines belong to the first or second figures, the line corresponding to a point and the point corresponding to a line are spoken of as the polar of the point and the pole of the line, or as reciprocal polars.

"The points which lie in their respective polars are situated in a conic, to which the polars are tangents." Or, stating the *same* theorem conversely,

"The lines which pass through their respective poles are tangents to a conic, the points of contact being the poles."

To determine the polar of a point, let two tangents be drawn through this point to the conic, the points of contact are the poles of the tangents; hence the line joining them is the polar of the point of intersection of the tangents, that is, "The polar of a point is the line joining the points of contact of the tangents which pass through the point."

Conversely, and by the same reasoning,

"The pole of a line is the intersection of the tangents at the points where the line meets the conic."

The actual geometrical constructions in the several cases where the point is within or without the conic, or the line does or does not intersect the conic, do not enter into the plan of the present memoir.

Passing to the general case where the lines and points in question are not identical, which I should propose to term the theory of "Skew Polars" (Polaires Gauches), we have the theorem,

"Considering the points in the first figure which are situated in their respective corresponding lines in the second figure, or the points in the second figure which are situated in their respective corresponding lines in the first figure, in either case the points are situated in the same conic (which will be spoken of as the 'pole conic'), and the lines are tangents to the same conic (which will be spoken of as the 'polar conic'), and these two conics have a double contact." This theorem is evidently identical with the converse theorem.

The corresponding lines to a point in the pole conic are the tangents through this point to the polar conic; viz. one of these tangents is the corresponding line when the point is considered as belonging to the first figure, and the other tangent is the corresponding line when the point is considered as belonging to the second figure.

The corresponding points to a tangent of the polar conic are the points where this line intersects the pole conic; viz. one of these points is the corresponding point when the line is considered as belonging to the first figure, and the other is the corresponding point when the line is considered as belonging to the second figure.

Let i be a point in the pole conic, and when i is considered as belonging to the first figure, let iI_1 be considered as the corresponding line in the second figure (I_1 being the point of contact on the polar conic).

Then if j be another point in the pole conic, in order to determine which of the tangents is the line in the second figure which corresponds to j considered as a point of the first figure, let iI_2 be the other tangent through I: the points of contact of the tangents through j may be marked with the letters J_1, J_2, in such order that I_1J_2, I_2J_1 meet in the line of contact of the two conics, and then jJ_1 is the required corresponding line. Again, I and i, as before, if B be a tangent to the polar conic, then, marking the point of contact as J_1, let J_2 be so determined that I_1J_2, I_2J_1 meet in the line of contact of the conics: the tangent to the polar conic at J_2 will meet the pole conic in one of the points where it is met by the line B, and calling this point j, B considered as belonging to the second figure will have j for its corresponding point in the first figure. Similarly, if the point of contact had been marked J_2, J_1 would be determined by an analogous construction, and the tangent at J_1 would meet the pole conic in one of the points where it is met by the line B (viz. the other point of intersection); and representing this by j', B considered as belonging to the first figure would have j' for its corresponding point in the second figure, that is, considered as belonging to the second figure, it would have j for its corresponding point in the first figure (the same as before).

Similar considerations apply in the case where a tangent A of the polar conic, considered as belonging to one of the figures, has for its corresponding point in the other figure one of its points of intersection with the polar conic; in fact, if A represents the line iI_1, then A, considered as belonging to the second figure has i for its corresponding point in the first figure, which shows that this question is identical with the former one.

To appreciate these constructions it is necessary to bear in mind the following system of theorems, the third and fourth of which are the polar reciprocals of the first and second.

If there be two conics having a double contact, such that K is the line joining the points of contact, and k the point of intersection of the tangents at the points of contact:

1. If two tangents to one of the conics meet the other in i, i_1 and j, j_1 respectively, then, properly selecting the points j, j_1, the lines ij_1, i_1j meet in K. And

2. The line joining the points of intersection of the tangents at i, j_1, and of the tangents at i_1, j passes through k. Also

3. If from two points of one of the conics, tangents be drawn touching the other in the points I, I_1 and J, J_1, then, properly selecting the points J, J_1, the lines IJ_1, I_1J meet in K. And

4. The line joining the points of intersection of the tangents at I_1J_1 and of the tangents at I_1, J passes through k.

These theorems are in fact particular cases of two theorems relating to two conics having a double contact with a given conic.

It may be remarked also that the corresponding points to a tangent of the pole conic are the points of contact of the tangents to the polar conic which pass through the point of contact of the given tangent, and the corresponding lines to a point of the polar conic are the tangents to the pole conic at the points where it is intersected by the tangent at the point in question.

We have now to determine the corresponding lines to a given point and the corresponding points to a given line, which is immediately effected by means of the preceding results.

Thus, if the point be given,

"Through the point draw tangents to the polar conic, meeting the pole conic in A_1, A_2 and B_1, B_2 (so that A_1B_2 and A_2B_1, intersect on the line joining the points of contact of the conics), then A_2B_2 and A_1B_1 are the required lines."

In fact A_1, B_1 and A_2, B_2 are pairs of points corresponding to the two tangents, so that A_1B_1 and A_2B_2 are the lines which correspond to their point of intersection, that is, to the given point, and similarly for the remaining constructions. Again,

"Through the point draw tangents to the pole conic, and from the points of contact draw tangents to the polar conic, touching it in α_1, α_2 and β_1, β_2 (so that $\alpha_1\beta_2$ and $\alpha_2\beta_1$ intersect on the line joining the points of contact of the conic), then $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are the required lines."

So that A_1, B_1, α_1, β_1 are situated in the same line, and also A_2, B_2, α_2, β_2.

Again, if the line be given,

"Through the points where the line meets the pole conic draw tangents to the polar conic C_1, C_2 and D_1, D_2 (so that the points C_1D_2 and C_2D_1 lie on a line passing through the intersection of the tangents at the points of contact of the tangents), then C_1D_1 and C_2D_2 are the required points."

Again,

"At the points where the line meets the polar conic draw tangents meeting the pole conic, and let γ_1, γ_2 and δ_1, δ_2 be the tangents to the pole conic at these points (so that the points $\gamma_1\delta_2$ and $\gamma_2\delta_1$ lie on a line through the intersection of the tangents at the points of contact of the conics), then γ_1, δ_1 and γ_2, δ_2 are the required points"; so that C_1, D_1, γ_1, δ_1 pass through the same point and also C_2, D_2, γ_2, δ_2.

"The preceding constructions have been almost entirely taken from Plücker's "System der Analytischen Geometrie," § 3, Allgemeine Betrachtungen über Coordinatenbestimmung. I subjoin analytical demonstrations of some of the theorems in question.

Using x, y, z to determine the position of a variable point, and putting for shortness

$$\xi = ax + a'y + a''z,$$
$$\eta = bx + b'y + b''z,$$
$$\zeta = cx + c'y + c''z.$$

then if the position of a point be determined by the coordinates α, β, γ, the equation of one of the corresponding lines is

$$\alpha\xi + \beta\eta + \gamma\zeta = 0,$$

(that of the other is obtainable from this by writing a, b, c; a', b', c'; a'', b'', c'', for a, a', a''; b, b', b''; c, c', c''). Hence if the point lies in the corresponding line, this equation must be satisfied by putting α, β, γ for x, y, z; or, substituting x, y, z in the place of α, β, γ, the point must lie in the conic

$$U = ax^2 + b'y^2 + c''z^2 + (b'' + c')\,yz + (c + a'')\,zx + (a' + b)\,xy = 0,$$

(which equation is evidently not altered by interchanging the coefficients, as above). Again, determining the curve traced out by the line $\alpha\xi + \beta\eta + \gamma\zeta = 0$, when α, β, γ are connected by the equation into which $U = 0$ is transformed by the substitution of these letters for x, y, z; we obtain

$$V = -\begin{vmatrix} & \xi\ , & \eta\ , & \zeta \\ \xi, & 2a\ , & a' + b, & a'' + c \\ \eta, & a' + b, & 2b'\ , & b'' + c' \\ \zeta, & a'' + c, & b'' + c', & 2c'' \end{vmatrix} = 0,$$

which is also a conic. It only remains to be seen that the conics $U = 0$, $V = 0$ have a double contact. Writing for shortness

$$\nabla = \begin{vmatrix} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

it may be seen by expansion that the following equation is identically true,

$$V = 4\nabla U - [x\,(ab'' - a''b + a'c - ac') + y\,(b'c - bc' + b''a' - b'a'') + z\,(c''a' - c'a'' + cb'' - c''b)]^2,$$

which proves the property in question.

Suppose the equations of the two conics to be given, and let it be required to determine the corresponding lines to the point defined by the coordinates α, β, γ.

Writing, to abbreviate,

$$\begin{cases} U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy, \\ U_0 = A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta, \\ W = A\alpha x + B\beta y + C\gamma z + F\,(\beta z + \gamma y) + G\,(\gamma x + \alpha z) + H\,(\alpha y + \beta x), \\ P = lx + my + nz, \\ P_0 = l\alpha + m\beta + n\gamma, \\ K = ABC - AF^2 - BG^2 - CH^2 + 2FGH, \end{cases}$$

$$\mathfrak{A} = BC - F^2\,,\quad \mathfrak{B} = CA - G^2\,,\quad \mathfrak{C} = AB - H^2\,,$$
$$\mathfrak{F} = GH - AF,\quad \mathfrak{G} = HF - BG,\quad \mathfrak{H} = FG - CH,$$

$$\Theta = \mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + 2\mathfrak{H}lm,$$

$$\begin{aligned}\square = {} & (\mathfrak{A}l + \mathfrak{H}m + \mathfrak{G}n)(\gamma y - \beta z)\\ & + (\mathfrak{H}l + \mathfrak{B}m + \mathfrak{F}n)(\alpha z - \gamma x)\\ & + (\mathfrak{G}l + \mathfrak{F}m + \mathfrak{C}n)(\beta x - \alpha y)\,;\end{aligned}$$

suppose $U = 0$ represents the equation of the polar conic, $U - P^2 = 0$ that of the pole conic. The two tangents drawn to the polar conic are represented by $UU_0 - W^2 = 0$, and by determining k in such a way that $UU_0 - W^2 - k(U - P^2)$ may divide into factors the equation

$$UU_0 - W^2 - k(U - P^2) = 0,$$

represents the lines passing through the points of intersection of the tangents with the pole conic. Thus if $k = U_0$, the equation reduces itself to $U_0P^2 - W^2 = 0$, or $W = \pm\sqrt{(U_0)}\,P$, the equation of two straight lines each of which passes through the point of intersection of the lines $P = 0$, $W = 0$, (that is, of the line of contact of the conics, and the ordinary polar of the point with respect to the polar conic); these are in fact the lines A_1B_2, A_2B_1 intersecting in the line of contact. The remaining value of k is not easily determined, but by a somewhat tedious process I have found it to be

$$= K(U_0 - P_0^2) \div (\Theta - K).$$

In fact, substituting the value, it may be shown that

$$\square^2 + K(PP_0 - W)^2 = K(U_0 - P_0^2)(U - P^2) + (\Theta - K)(UU_0 - W^2),$$

which is an equation of the required form. To verify this, we have, by a simple reduction,

$$\Theta(UU_0 - W^2) - \square^2 = K(UP_0^2 - 2WPP_0 + U_0P^2)\,;$$

or, writing for shortness

$$\gamma y - \beta z = \xi,\quad \alpha z - \gamma x = \eta,\quad \beta x - \alpha y = \zeta,$$

$$\begin{aligned}&(\mathfrak{A}l^2 + \mathfrak{B}m^2 + \mathfrak{C}n^2 + 2\mathfrak{F}mn + 2\mathfrak{G}nl + \mathfrak{H}lm)(\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\zeta^2 + 2\mathfrak{F}\eta\zeta + 2\mathfrak{G}\zeta\xi + 2\mathfrak{H}\xi\eta)\\ &\qquad - [\mathfrak{A}l\xi + \mathfrak{B}m\eta + \mathfrak{C}n\zeta + \mathfrak{F}(n\eta + m\zeta) + \mathfrak{G}(l\zeta + n\xi) + \mathfrak{H}(m\xi + l\eta)]^2\\ &= K\{A(m\zeta - n\eta)^2 + B(n\xi - l\zeta)^2 + C(l\eta - m\xi)^2\\ &\qquad + 2F(n\xi - l\zeta)(l\eta - m\xi) + 2G(l\eta - m\xi)(m\zeta - n\eta) + 2H(m\zeta - n\eta)(n\xi - l\zeta)\},\end{aligned}$$

which is easily seen to be identically true.

62.

ON AN INTEGRAL TRANSFORMATION.

[From the *Cambridge and Dublin Mathematical Journal*, vol. III. (1848), pp. 286—287.]

THE following transformation, given for elliptic functions by Gudermann (*Crelle*, t. XXIII. [1842], p. 330) is useful for some other integrals.

If
$$y = \frac{dbc - dba - dca + abc - (bc - ad)\,z}{(bc - ad) + (d - b - c + a)\,z},$$
then, putting
$$K = (bc - ad) + (d - b - c + a)\,z,$$
we have, supposing $a < b < c < d$, so that $(b-a)$, $(c-a)$, $(d-b)$, $(d-c)$ are positive,
$$\begin{aligned}
K(y-a) &= (b-a)(c-a)(d-z),\\
K(y-b) &= (b-a)(d-b)(c-z),\\
K(y-c) &= (c-a)(d-c)(b-z),\\
K(y-d) &= (d-b)(d-c)(a-z),\\
K^2\,dy &= -(b-a)(c-a)(d-b)(d-c)\,dz.
\end{aligned}$$

In particular, if $\alpha + \beta + \gamma + \delta = -2$,
$$(y-a)^\alpha (y-b)^\beta (y-c)^\gamma (y-d)^\delta\,dy = -M(z-a)^\delta (z-b)^\gamma (z-c)^\beta (z-d)^\alpha\,dz,$$
where
$$M = (b-a)^{\alpha+\beta+1}(c-a)^{\alpha+\gamma+1}(d-b)^{\beta+\delta+1}(d-c)^{\gamma+\delta+1}.$$

Thus, if $\alpha = \beta = \gamma = \delta = -\frac{1}{2}$,
$$\frac{dy}{\{-(y-a)(y-b)(y-c)(y-d)\}^{\frac{1}{2}}} = \frac{-dz}{\{-(z-a)(z-b)(z-c)(z-d)\}^{\frac{1}{2}}}.$$

In any case when $y = a$, $y = b$, the corresponding values of z are $z = d$, $z = c$; the last formula becomes by this means
$$\int_a^b \frac{dy}{\{-(y-a)(y-b)(y-c)(y-d)\}^{\frac{1}{2}}} = \int_c^d \frac{dy}{\{-(z-a)(z-b)(z-c)(z-d)\}^{\frac{1}{2}}}.$$

63.

DÉMONSTRATION D'UN THÉORÈME DE M. BOOLE CONCERNANT DES INTÉGRALES MULTIPLES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIII. (1848), pp. 245—248.]

THÉORÈME. "Soient P, Q des fonctions de n variables $x, y, \ldots$; lesquelles fonctions satisfont à la condition

$$\int_{-\infty}^{\infty} \ldots dxdy \ldots e^{-(Pv+Qw)i} = \frac{\pi^{\frac{1}{2}n} e^{-\frac{1}{2}n\pi i} e^{-G(v,w)i}}{\sqrt{H(v, w)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

où, comme à l'ordinaire, $i = \sqrt{-1}$; $G(v, w)$, $H(v, w)$ sont des fonctions homogènes de v, w des ordres 1 et n respectivement (on verra, dans la suite, qu'il y a plusieurs fonctions P, Q qui satisfont à une équation de cette forme).

Cela étant, posons

$$V = \int \ldots dxdy \ldots \frac{f(P)}{Q^{\frac{1}{2}n+q}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

les limites de l'intégration étant données par la condition $P = 1$; et soient

$$G\left(1, \frac{1}{s}\right) = \sigma, \quad H\left(1, \frac{1}{s}\right) = s^{-n}\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

On aura pour l'intégrale V cette formule,

$$V = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)} \int_0^{\infty} \frac{Ss^{-q-1}\, ds}{\sqrt{\phi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

dans laquelle

$$S = \frac{(1-\sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1} f[\sigma + t(1-\sigma)]\, dt \quad \ldots\ldots\ldots\ldots\ldots (5)."$$

Ce théorème remarquable est dû à M. Boole, qui me l'a communiqué sous une forme un peu différente[1], en me priant d'y suppléer la démonstration et d'en faire part aux géomètres; il ne m'a fallu, pour le prouver, que modifier un peu le procédé dont s'est servi M. Boole même, dans son Mémoire: "On a certain Multiple Definite Integral," *Irish Transactions*, t. XXI. [1848].

Je vais donc reproduire cette démonstration en l'appliquant au problème dont il s'agit.

On démontre par une analyse semblable à peu près à celle par laquelle se démontre le théorème de Fourier, que l'expression

$$\frac{1}{\pi\Gamma(\frac{1}{2}n+q)}\int_0^1 d\alpha\int_0^\infty dv\int_0^\infty dw\, e^{[(\alpha-P)v-Qw+\frac{1}{2}(\frac{1}{2}n+q)\pi]i}\ldots w^{\frac{1}{2}n+q-1}f\alpha \;\ldots\ldots\ldots\; (6),$$

se réduit (en n'y faisant attention qu'à la partie réelle) à $\frac{fP}{Q^{\frac{1}{2}n+q}}$ ou à zéro, selon que la quantité P se trouve ou ne se trouve pas comprise entre les limites 0, 1. Donc, en substituant cette intégrale triple dans l'expression de V, on peut étendre depuis $-\infty$ jusqu'à ∞ les intégrations par rapport aux variables $x, y, \ldots$; de cette manière, et en réduisant par l'équation (1), on obtient tout de suite

$$V=\frac{\pi^{\frac{1}{2}n-1}e^{\frac{1}{2}q\pi i}}{\Gamma(\frac{1}{2}n+q)}\int_0^1 d\alpha\int_0^\infty dv\int_0^\infty dw\,\frac{e^{[\alpha v-G(v,w)]i}\ldots w^{\frac{1}{2}n+q-1}}{\sqrt{H(v,w)}}f\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Donc, en écrivant

$$w=\frac{v}{s},\qquad dw=-\frac{vds}{s^2}$$

{ce qui donne, par les équations (3),

$$G(v,w)=v\sigma,\quad H(v,w)=v^n s^{-n}\phi\},$$

les limites par rapport à la nouvelle variable s seront ∞, 0, et l'on obtiendra, en changeant l'ordre des intégrations,

$$V=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_0^\infty \frac{ds\,Ss^{-q-1}}{\sqrt{\phi}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8),$$

dans laquelle expression

$$S=\frac{1}{\pi}e^{\frac{1}{2}q\pi i}\int_0^1 d\alpha\int_0^\infty dv\, v^q\, e^{(\alpha-\sigma)vi}f\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9);$$

[1] M. Boole écrit

$$S=\left(-\frac{d}{d\sigma}\right)^q f\sigma,$$

expression à la vérité plus simple, mais qui donne lieu, ce me semble, à quelques difficultés.

et il ne s'agit plus que de faire voir l'identité de cette valeur avec celle qui est donnée par l'équation (5). Pour cela, je remarque que l'on aura

$$\left.\begin{aligned}\int_0^\infty dv\, v^q\, e^{(\alpha-\sigma)\,vi} &= \Gamma(q+1)\, e^{\frac{1}{2}(q+1)\pi i}\ (\alpha-\sigma)^{-q-1},\\ \text{ou}\qquad &= \Gamma(q+1)\, e^{-\frac{1}{2}(q+1)\pi i}\,(\sigma-\alpha)^{-q-1},\end{aligned}\right\}\ldots\ldots\ldots\ldots\ (10),$$

selon que $(\alpha-\sigma)$ est positif ou négatif; les valeurs correspondantes de $\frac{1}{\pi}e^{\frac{1}{2}q\pi i}\int_0^\infty dv\, v^q\, e^{(\alpha-\sigma)vi}$ sont

$$\frac{1}{\pi}e^{(q+\frac{1}{2})\pi i}\,\Gamma(q+1)(\alpha-\sigma)^{-q-1}\ \text{ et }\ \frac{1}{\pi}e^{-\frac{1}{2}\pi i}\,\Gamma(q+1)(\sigma-\alpha)^{-q-1}\ \ldots\ldots\ldots\ (11);$$

or, en ne faisant attention qu'aux parties réelles, et en réduisant par une propriété connue des fonctions Γ, ces valeurs se réduisent à $\frac{1}{\Gamma(-q)}(\alpha-\sigma)^{-q-1}$ et zéro respectivement; d'après cela, l'équation (9) se réduit à

$$S=\frac{1}{\Gamma(-q)}\int_\sigma^1(\alpha-\sigma)^{-q-1}f\alpha\, d\alpha\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (12),$$

et enfin, en écrivant

$$\alpha=\sigma+t\,(1-\sigma),\quad d\alpha=(1-\sigma)\,dt,$$

on obtient pour S la valeur donnée par l'équation (5), de manière que la formule dont il s'agit se trouve complétement démontrée.

Il paraît difficile de trouver les formes générales de P, Q {rien n'étant, je crois, connu sur la solution des équations telles que l'équation (1)}; mais des formes particulières se présentent assez facilement. Ainsi, en ne considérant que les exemples que m'a donnés M. Boole (lesquels j'ai depuis vérifiés), soit

$$P=2\,(lx+my+\ldots),\quad Q=v^2+x^2+y^2+\ldots\ \ \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

En substituant ces valeurs dans l'équation (1), les intégrations s'effectuent sans difficulté et sous la forme nécessaire, et l'on obtient

$$G(v,\ w)=wv^2-\frac{(l^2+m^2+\ldots)\,v^2}{w},\quad H(v,\ w)=w^n,$$

ou enfin

$$\sigma=\frac{v^2}{s}-(l^2+m^2+\ldots)\,s,\quad \phi=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Soit encore (ce qui comprend, comme cas particulier, le problème des attractions)

$$P=\frac{x^2}{f^2}+\frac{y^2}{g^2}+\ldots,\quad Q=v^2+(a-x)^2+(b-y)^2+\ldots\ \ \ldots\ldots\ldots\ldots\ (15);$$

on obtient sans plus de difficulté

$$\phi = \frac{(s+f^2)(s+g^2)\ldots}{f^2g^2}, \quad \sigma = \frac{v}{s} + \frac{a^2}{f^2+s} + \frac{b^2}{g^2+s}\ldots \quad \ldots\ldots\ldots\ldots (16).$$

Soit encore, pour dernier exemple,

$$P = l^2x^2 + \frac{L^2}{x^2} + \ldots, \quad Q = v^2 + \lambda^2x^2 + \frac{\Lambda^2}{x^2} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (17);$$

on aura, pour ce cas,

$$\left.\begin{aligned} \phi &= (l^2s + \lambda^2)(L^2s + \Lambda^2)\ldots, \\ \sigma &= \frac{1}{s}\{v^2 + 2\sqrt{(l^2s+\lambda^2)(L^2s+\Lambda^2)} + \ldots\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (18);$$

les formules qui se rapportent à cet exemple aussi bien qu'au premier sont, je crois, entièrement nouvelles.

64.

SUR LA GÉNÉRALISATION D'UN THÉORÈME DE M. JELLETT, QUI SE RAPPORTE AUX ATTRACTIONS.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIII. (1848), pp. 264—268.]

LES formules qu'a données M. Jellett pour exprimer les attractions d'un ellipsoïde au moyen de l'expression de la surface de l'ellipsoïde réciproque (t. XI. de ce Journal, [1846], page 92) peuvent s'étendre au cas d'un nombre quelconque de variables.

Pour démontrer cela, je pars de cette formule

$$\left.\begin{aligned}&\int \frac{\phi\left(\frac{x^2}{f^2}+\frac{y^2}{g^2}\dots\right)dx\,dy\dots}{[(a-x)^2+(b-y)^2\dots+u^2]^{\frac{1}{2}n+q}}\quad\left(\text{limites } \frac{x^2}{f^2}+\frac{y^2}{g^2}\dots=1\right)\\ &=\frac{fg\dots\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_\eta^\infty \frac{Ss^{-q-1}\,ds}{\sqrt{(s+f^2)(s+g^2)\dots}},\end{aligned}\right\}\dots\dots\dots\dots(1),$$

dans laquelle n est le nombre des variables $x, y, \dots$, et où

$$S=\frac{(1-\sigma)^{-q}}{\Gamma(-q)}\int_0^1 t^{-q-1}\phi[\sigma+t(1-\sigma)]\,dt\dots\dots\dots\dots\dots\dots\dots(2),$$

$$\sigma=\frac{a^2}{s+f^2}+\frac{b^2}{s+g^2}\dots+\frac{u^2}{s},$$

$$1=\frac{a^2}{\eta+f^2}+\frac{b^2}{\eta+g^2}\dots+\frac{u^2}{\eta},$$

formule due à M. Boole qui l'a démontrée sous une forme un peu différente (*Irish Transactions*, t. XXI.). La modification que j'y ai introduite se trouve démontrée dans le *Cambridge and Dublin Mathematical Journal*, t. II. p. 223[1] [44].

[1] Cette formule peut d'ailleurs se déduire comme cas particulier de la formule très-générale de M. Boole que M. Cayley a démontrée dans le cahier précédent. (J. L.); [63].

On déduit de là, en écrivant fx, gy, ... au lieu de x, y, ..., en réduisant à zéro les quantités a, b, ... , u (ce qui donne aussi $\eta = 0$), et en donnant une forme convenable à la fonction ϕ,

$$\int \frac{(x^2+y^2\ldots)^{\frac{5}{2}-\frac{1}{2}n}\,dx\,dy\ldots}{(f^2x^2+g^2y^2\ldots)^2} = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-2)}\int_0^\infty \frac{s^{\frac{1}{2}n-3}\,ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots(3)$$

(les limites de l'intégrale au premier membre de cette équation étant données par $x^2+y^2\ldots=1$).

Donc, en écrivant

$$\Sigma = f^2g^2\ldots\int \frac{(x^2+y^2\ldots)^{\frac{5}{2}-\frac{1}{2}n}\,dx\,dy\ldots}{(f^2x^2+g^2y^2\ldots)^2},$$

on aura

$$\Sigma = \frac{2\pi^{\frac{1}{2}n}f^2g^2\ldots}{\Gamma(\frac{1}{2}n-2)}\int_0^\infty \frac{s^{\frac{1}{2}n-3}\,ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots(4).$$

Soit Σ' ce que devient Σ en écrivant $\frac{1}{f}$, $\frac{1}{g}$, ... au lieu de f, g, ... : en écrivant en même temps $\frac{1}{s}$ au lieu de s, on obtient

$$\Sigma' = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-2)}\frac{1}{fg\ldots}\int_0^\infty \frac{s\,ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots(5);$$

et de là, en écrivant

$$\frac{1}{f^2}\frac{d}{df}\Sigma' f = F, \quad \frac{1}{g^2}\frac{d}{dg}\Sigma' g = G, \ldots \quad \ldots\ldots\ldots\ldots(6),$$

on déduit

$$F = \frac{-2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-2)}\frac{1}{fg\ldots}\int_0^\infty \frac{s}{s+f^2}\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}, \text{ \&c.} \quad \ldots\ldots(7).$$

Cela étant, remarquons que l'intégrale

$$\int_0^\infty \frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots(8)$$

est fonction homogène de l'ordre $(2-n)$ par rapport aux quantités f, g, ... (en effet, cela se voit tout de suite en faisant $s = f^2\theta$). Donc

$$\int_0^\infty \left(-\frac{f^2}{s+f^2}-\frac{g^2}{s+g^2}\ldots\right)\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} = (2-n)\int_0^\infty \frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}},$$

ou, en écrivant $\frac{s}{s+f^2}-1$, &c., au lieu de $-\frac{f^2}{s+f^2}$, &c.,

$$\int_0^\infty \left(\frac{s}{s+f^2}+\frac{s}{s+g^2}+\ldots\right)\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} = 2\int_0^\infty \frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots(9),$$

et de là aussi

$$\int_0^\infty \left(\frac{-s}{s+f^2}+\frac{s}{s+g^2}+\ldots\right)\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}=2f^2\int_0^\infty \frac{1}{s+f^2}\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}, \text{ \&c. } (10).$$

Donc enfin

$$\left.\begin{aligned}&\int_0^\infty \left(1-\frac{a^2}{s+f^2}\ldots\right)\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}\\ &=\tfrac{1}{2}\int_0^\infty \left\{\begin{aligned}&\left(\frac{s}{s+f^2}+\frac{s}{s+g^2}+\ldots\right)\\ &-\frac{a^2}{f^2}\left(\frac{-s}{s+f^2}+\frac{s}{s+g^2}+\ldots\right)-\frac{b^2}{g^2}\left(\frac{s}{s+f^2}+\frac{-s}{s+g^2}\ldots\right)-\ldots\end{aligned}\right\}\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}\end{aligned}\right\}(11),$$

c'est-à-dire

$$\left.\begin{aligned}&\int_0^\infty \left(1-\frac{a^2}{s+f^2}-\frac{b^2}{s+g^2}\ldots\right)\frac{ds}{\sqrt{(s+f^2)(s+g^2)\ldots}}\\ &=-\frac{\Gamma(\frac{1}{2}n-2)\,.\,fg\ldots}{4\pi^{\frac{1}{2}n}}\left\{\begin{aligned}&(\ F+G+\ldots)\\ &-\frac{a^2}{f^2}(-F+G+\ldots)-\frac{b^2}{g^2}(F-G+\ldots)-\ldots\end{aligned}\right\}\end{aligned}\right\}\ldots\ldots(12).$$

En particularisant d'une manière convenable la formule (1), on obtient, pour le cas de $\frac{a^2}{f^2}+\frac{b^2}{g^2}\ldots>1$, cette formule connue

$$\left.\begin{aligned}V&=\int\frac{dx\,dy\ldots}{[(a-x)^2+(b-y)^2\ldots]^{\frac{1}{2}n-1}}\\ &=\frac{\pi^{\frac{1}{2}n}fg\ldots}{\Gamma(\frac{1}{2}n-1)}\int_0^\infty\left(1-\frac{a^2}{s+f^2}-\ldots\right)\frac{ds}{\sqrt{(s+f^2)\ldots}}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(13)$$

$\left(\text{l'équation des limites étant, comme auparavant, } \frac{x^2}{f^2}+\frac{y^2}{g^2}+\ldots=1\right)$;

et de là, vu la formule (12), résulte

$$V=-\frac{f^2g^2\ldots}{2(n-2)}\left[(F+G\ldots)-\frac{a^2}{f^2}(-F+G+\ldots)-\frac{b^2}{g^2}(F-G+\ldots)\ldots\right]\ldots\ldots(14).$$

L'expression de Σ, en écrivant $r\cos\alpha$, $r\cos\beta,\ldots$ au lieu de x, $y,\ldots$, remplaçant $dx\,dy\ldots$ par $r^{n-1}\,dr\,dS$, et intégrant depuis $r=0$ jusqu'à $r=1$, se réduit à

$$\Sigma=f^2g^2\ldots\int\frac{dS}{(f^2\cos^2\alpha+g^2\cos^2\beta+\ldots)^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

de sorte qu'au cas de $n=3$ cette fonction se réduit à l'expression qu'a donnée M. Jellett pour la surface d'un ellipsoïde. Donc, en se rappelant que les attractions

sont représentées par $\frac{dV}{da}$, $\frac{dV}{db}$, $\frac{dV}{dc}$, on voit que l'équation (14) équivaut, pour ce cas, aux formules de M. Jellett.

Remarquons, qu'en transformant l'intégrale (4) de la même manière dont nous avons transformé l'intégrale (8), on obtient

$$\Sigma = \frac{\pi^{\frac{1}{2}n} f^2 g^2 \ldots}{\Gamma(\frac{1}{2}n-1)} \int_0^\infty \left(\frac{1}{s+f^2} + \frac{1}{s+g^2} \ldots\right) \frac{s^{\frac{1}{2}n-2}\, ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots(16),$$

ce qui donne pour $n=3$ cette expression très-simple de la surface de l'ellipsoïde aux demi-axes f, g, h,

$$\Sigma = \pi f^2 g^2 h^2 \int_0^\infty \left(\frac{1}{s+f^2} + \frac{1}{s+g^2} + \frac{1}{s+h^2}\right) \frac{s^{-\frac{1}{2}}\, ds}{\sqrt{(s+f^2)(s+g^2)(s+h^2)}} \quad \ldots\ldots\ldots\ldots(17),$$

formule qui se vérifie tout de suite au cas de $f=g=h$.

L'expression encore plus simple que donne l'équation (4), savoir,

$$\Sigma = -\pi f^2 g^2 h^2 \int_0^\infty \frac{s^{-\frac{3}{2}}\, ds}{\sqrt{(s+f^2)(s+g^2)(s+h^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

n'est pas exempte de difficulté à cause de la valeur apparemment infinie du second membre de l'équation.

Au cas d'une sphère, cela se réduit à

$$\Sigma = -\pi f^2 \int_0^\infty \frac{s^{-\frac{3}{2}}\, ds}{(1+s)^{\frac{3}{2}}},$$

ce qui serait, en effet, exact si la formule

$$\int_0^\infty \frac{s^{m-1}\, ds}{(1+s)^{m+n}} = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

subsistait pour les valeurs négatives de m. Cela nous apprend que les intégrales de la forme

$$\int_0^\infty \frac{s^{-q-1}\, ds}{\sqrt{(s+f^2)(s+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

ne sont pas à rejeter au cas des valeurs positives de q; il est même facile, en répétant continuellement le procédé de réduction que nous venons d'employer, de présenter ces intégrales sous une forme où il n'y a plus de terme infini. En m'aidant de l'analogie de quelques formules qui se trouvent dans mon Mémoire *Sur quelques formules du calcul intégral* (t. XI. de ce Journal, p. 231 [49]), je crois même pouvoir avancer que cette intégrale doit se remplacer par

$$\frac{-1}{2\sin q\pi} \int_{-\infty}^\infty \frac{(k+si)^{-q-1}\, ds}{\sqrt{(k+si+f^2)(k+si+g^2)\ldots}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

où, comme à l'ordinaire, $i=\sqrt{-1}$, et où k dénote une quantité quelconque dont la partie réelle ne s'évanouit pas. Mais je renvoie cette discussion à une autre occasion.

65.

NOUVELLES RECHERCHES SUR LES FONCTIONS DE M. STURM.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIII. (1848), pp. 269—274.]

En développant une remarque faite par M. Sylvester dans un Mémoire publié il y a huit ou neuf ans dans le *Philosophical Magazine*, j'ai trouvé des expressions assez simples des fonctions de M. Sturm, composées au moyen des coefficients mêmes; ce qui convient mieux que d'exprimer ces fonctions, comme je l'ai déjà fait dans le t. XI. de ce Journal, p. 297, [48], par les sommes des puissances. D'ailleurs il ne m'est plus nécessaire de parler des expressions de M. Sylvester, ou même des divisions successives de M. Sturm; mais ma méthode fait voir directement que les fonctions que je vais définir sont douées de la propriété fondamentale sur laquelle se repose la théorie de M. Sturm, à savoir que, en considérant trois fonctions successives, la première et la dernière fonction sont de signe contraire pour toute valeur de la variable qui fait évanouir la fonction intermédiaire; cependant je n'ai pas encore réussi à démontrer dans toute sa généralité l'équation identique d'où dépend cette propriété.

Soient d'abord V, V' des fonctions du même degré n,

$$V = ax^n + bx^{n-1} + \ldots,$$
$$V' = a'x^n + b'x^{n-1} + \ldots,$$

et écrivons

$$-F_1 = \begin{vmatrix} V, & V' \\ a & a' \end{vmatrix},$$

$$F_2 = \begin{vmatrix} xV, & V, & xV', & V' \\ a & . & a' & . \\ b & a & b' & a' \\ c & b & c' & b' \end{vmatrix},$$

$$-F_3 = \begin{vmatrix} x^2V, & xV, & V, & x^2V', & xV', & V' \\ a & . & . & a' & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \end{vmatrix}, \text{ \&c.}$$

(ce qui suffit pour faire voir la loi de ces fonctions successives F_1, F_2, ...). Il résulte des propriétés élémentaires des déterminants que ces fonctions sont des ordres $\overline{n-1}$, $\overline{n-2}$, $\overline{n-3}$, &c., respectivement, par rapport à la variable x. En effet, dans F_1 le coefficient de x^n se réduit à $\begin{vmatrix} a, & a' \\ a & a' \end{vmatrix}$, savoir, à zéro; de même, dans F_2, les coefficients de x^n et x^{n-1} se réduisent chacun à zéro; et ainsi de suite.

Soient encore

$$P_1 = \begin{vmatrix} a, & a' \\ b & b' \end{vmatrix}, \qquad P_1' = \begin{vmatrix} a, & a' \\ c & c' \end{vmatrix},$$

$$P_2 = \begin{vmatrix} a, & . & a', & . \\ b & a & b' & a' \\ c & b & c' & b' \\ d & c & d' & c' \end{vmatrix}, \qquad P_2' = \begin{vmatrix} a, & . & a', & . \\ b & a & b' & a' \\ c & b & c' & b' \\ e & d & e' & d' \end{vmatrix},$$

$$P_3 = \begin{vmatrix} a, & . & . & a', & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \\ f & e & d & f' & e' & d' \end{vmatrix}, \text{ \&c.}, \qquad P_3' = \begin{vmatrix} a, & . & . & a', & . & . \\ b & a & . & b' & a' & . \\ c & b & a & c' & b' & a' \\ d & c & b & d' & c' & b' \\ e & d & c & e' & d' & c' \\ g & f & e & g' & f' & e' \end{vmatrix}, \text{ \&c.}$$

(ce qui suffit pour indiquer la loi). On aura entre ces différentes fonctions F, P, P' cette suite remarquable d'équations identiques,

$$P_1^2 F_3 + (xP_1P_2 + P_1P_2' + P_1'P_2)\, F_2 + P_2^2 F_1 = 0,$$
$$P_2^2 F_4 + (xP_2P_3 + P_2P_3' + P_2'P_3)\, F_3 + P_3^2 F_2 = 0,$$
$$\text{\&c.},$$

lesquelles équations, dans ce Mémoire, seront prises pour vraies. Cela étant, il est évident que F_1 et F_3 seront de signe contraire pour toute valeur de x qui fait évanouir F_2; F_2 et F_4 seront de signe contraire pour toute valeur de x qui fait évanouir F_3; et ainsi de suite.

Ces formules renferment le cas où les deux fonctions V, V' ne sont pas du même degré (en effet, pour les y adapter, on n'a besoin que de faire évanouir quelques-uns des premiers coefficients de V ou de V'). Il est donc permis de supposer que V' soit la dérivée de V. Dans ce cas, $F_1 = aV'$, et on verra dans un moment que les fonctions F_2, F_3, ... contiennent chacune le facteur a^2, de manière qu'il convient d'écrire $F_2 = a^2 V_2$, $F_3 = a^2 V_3$, ... Ce facteur a^2 peut être évidemment écarté, et ce sera de même avec le facteur a de F, pourvu, ce que je supposerai dans la suite, que a soit positif. On aura de cette manière les fonctions V, V', V_2, V_3, ... douées des propriétés des fonctions de M. Sturm. En effet, elles seront précisément les fonctions fx, f_1x, f_2x, ... du Mémoire déjà cité, ce qui cependant pourrait être difficile à démontrer à priori.

On déduit tout de suite des expressions de F_2, F_3, ... ,

$$aV_2 = -\begin{vmatrix} V, & xV', & V' \\ a & na & . \\ b & \overline{n-1}b & na \end{vmatrix}, \quad aV_3 = \begin{vmatrix} xV, & V, & x^2V', & xV', & V' \\ a & . & na & . & . \\ b & a & \overline{n-1}b & na & . \\ c & b & \overline{n-2}c & \overline{n-1}b & na \\ d & c & \overline{n-3}d & \overline{n-2}c & \overline{n-1}b \end{vmatrix}, \text{ \&c.}$$

Ces formules se simplifient au moyen des propriétés connues des déterminants, et en écrivant

$$xV' - nV = -U,$$

cela donne

$$aV_2 = \begin{vmatrix} V, & U, & V' \\ a & . & . \\ b & b & na \end{vmatrix}, \quad aV_3 = \begin{vmatrix} xV, & V, & xU, & U, & V' \\ a & . & . & . & . \\ b & a & b & . & . \\ c & b & 2c & b & na \\ d & c & 3d & 3c & \overline{n-1}b \end{vmatrix}, \text{ \&c.};$$

ou enfin, et en écrivant un autre terme de la suite, afin de mieux faire voir la loi,

$$V_2 = -\begin{vmatrix} U, & V' \\ b & na \end{vmatrix}, \quad V_3 = -\begin{vmatrix} V, & xU, & U, & V' \\ a & b & . & . \\ b & 2c & b & na \\ c & 3d & 2c & \overline{n-1}b \end{vmatrix},$$

$$V_4 = -\begin{vmatrix} xV, & V, & x^2U, & xU, & V' \\ a & . & b & . & . \\ b & a & 2c & b & . \\ c & b & 3d & 2c & na \\ d & c & 4e & 3d & \overline{n-1}b \\ e & d & 5f & 4e & \overline{n-2}c \end{vmatrix}, \text{ \&c.,}$$

formules dans lesquelles

$$\begin{cases} V = ax^n + bx^{n-1} + cx^{n-2} + \dots, \\ V' = nax^{n-1} + \overline{n-1}\,bx^{n-2} + \dots, \\ U = bx^{n-1} + 2cx^{n-2} + \dots, \end{cases}$$

et où, en substituant ces valeurs, on peut commencer pour V_2 avec le terme qui contient x^{n-2}, pour V_3 avec le terme qui contient x^{n-3}, et ainsi de suite, puisque les termes des ordres plus hauts s'évanouissent identiquement. Voilà, je crois, les expressions les plus simples des fonctions de M. Sturm.

Je donnerai en conclusion ces formes développées des fonctions jusqu'à V_4.

$$\begin{aligned}
V &= ax^n + bx^{n-1} + cx^{n-2} + \dots; \\
V' &= nax^{n-1} + \overline{n-1}\,bx^{n-2} + \overline{n-2}\,cx^{n-3} + \dots; \\
V_2 &= -na\,\{2cx^{n-2} + 3dx^{n-3} + \dots \\
&\quad + b\,\{\overline{n-1}\,bx^{n-2} + \overline{n-2}\,cx^{n-3} + \dots; \\
V_3 &= [2nabc - \overline{n-1}\,b^3 \qquad\qquad]\,\{dx^{n-3} + ex^{n-4} + \dots \\
&\quad + [-2n^2ac + \overline{n-1}\,ab^2 \qquad\qquad]\,\{4ex^{n-3} + 5fx^{n-4} + \dots \\
&\quad + [3na^2d - (3n-2)\,abc + (n-1)\,b^3]\,\{3dx^{n-3} + 4ex^{n-4} + \dots \\
&\quad + [-3abd + 4ac^2 - b^2c \qquad\qquad]\,\{(n-2)\,cx^{n-3} + (n-3)\,dx^{n-4} + \dots; \\
V_4 &= A\,\{fx^{n-4} + gx^{n-5} + \dots \\
&\quad + B\,\{ex^{n-4} + fx^{n-5} + \dots \\
&\quad + C\,\{6gx^{n-4} + 7hx^{n-5} + \dots \\
&\quad + D\,\{5fx^{n-4} + 6gx^{n-5} + \dots \\
&\quad + E\,\{4ex^{n-4} + 5fx^{n-5} + \dots \\
&\quad + F\,\{\overline{n-3}\,dx^{n-4} + \overline{n-4}\,ex^{n-5} + \dots;
\end{aligned}$$

dans cette dernière expression j'ai mis, pour abréger,

$$\begin{aligned}
A &= 9na^3bd^2 - 8na^3bce + (4n-4)\,ab^3e - (10n-12)\,ab^2cd \\
&\quad + (4n-8)\,abc^3 - (n-2)\,b^3c^2 + (2n-2)\,b^4d, \\
B &= 10na^3bcf - 12na^3bde - 16na^2c^2e + 18na^2cd^2 - (5n-5)\,ab^3f \\
&\quad + (18n-16)\,ab^2ce + (3n-9)\,ab^2d^2 - (24n-36)\,abc^2d + (8n-16)\,ac^4 \\
&\quad - (3n-3)\,b^4e + (5n-7)\,b^3cd - (2n-4)\,b^2c^3, \\
C &= 8na^3ce - 9na^3d^2 - (4n-4)\,a^2b^2e + (10n-12)\,a^2bcd \\
&\quad - (4n-8)\,a^2c^3 - (2n-2)\,ab^3d + (n-2)\,ab^2e^2, \\
D &= -10na^3cf + 12na^3de + (5n-5)\,a^2b^2f - (2n-8)\,a^2bce \\
&\quad - (12n-9)\,a^2bd^2 + (4n-12)\,a^2c^2d - (n-1)\,ab^3e + (9n-9)\,ab^2cd \\
&\quad - (4n-8)\,abe^3 - (2n-2)\,b^4d + (n-2)\,b^3c^2,
\end{aligned}$$

$$\begin{aligned} E = \; & 15na^3df - 16na^3e^2 - (15n-10)\,a^2bcf + (17n-12)\,a^2bde \\ & + (16n-16)\,a^2c^2e - (15n-18)\,a^2cd^2 + (5n-5)\,ab^2f \\ & - (17n-18)\,ab^2ce + (14n-24)\,abc^2d - (n-3)\,ab^2d^2 \\ & - (4n-8)\,ac^4 + (3n-3)\,b^4e - (3n-5)\,b^3cd + (n-2)\,b^2c^3, \end{aligned}$$

$$\begin{aligned} F = \; & -15a^2bdf + 16a^2be^2 + 20a^2c^2f - 27a^2d^3 - 48a^2cde \\ & - 5ab^2cf + 7ab^2de - 4abc^3e - 18abcd^2 + 4ac^3d \\ & - 3b^3ce + 4b^3d^2 - b^2c^3d^2. \end{aligned}$$

Il serait évidemment inutile de vouloir pousser plus loin ces calculs.

[MS. addition in my copy of Liouville.

Quoique ces expressions soient ce qu'il y a de plus simple pour le calcul numérique des fonctions de M. Sturm, cependant sous le point de vue analytique il convient de modifier un peu la forme de ces expressions. En effet en écrivant

$$V = ax^n + \frac{n}{1}\,bx^{n-2} + \ldots,$$

mettons

$$P = ax^{n-1} + \frac{n-1}{1}\,bx^{n-2} + \ldots, \; = ax^{n-1} + \theta_1\,bx^{n-2} + \ldots,$$

$$Q = bx^{n-1} + \frac{n-1}{1}\,cx^{n-2} + \ldots, \; = bx^{n-1} + \theta_1\,cx^{n-2} + \ldots,$$

l'on aura

$$\begin{aligned} V' &= nP, \\ U' &= nQ, \\ V &= Px + Q; \end{aligned}$$

et cela donne après une réduction légère

$$V_2 = \begin{vmatrix} P, & Q \\ a & b \end{vmatrix}, \qquad V_3 = -\begin{vmatrix} xP, & P, & xQ, & Q \\ a & . & b & . \\ \theta_1 b & a & \theta_1 c & b \\ \theta_2 c & \theta_1 b & \theta_2 d & \theta_1 c \end{vmatrix},$$

$$V_4 = \begin{vmatrix} x^2P, & xP, & P, & x^2Q, & xQ, & Q \\ a & . & . & b & . & . \\ \theta_1 b & a & . & \theta_1 c & b & . \\ \theta_2 c & \theta_1 b & a & \theta_2 d & \theta_1 c & b \\ \theta_3 d & \theta_2 c & \theta_1 b & \theta_3 e & \theta_2 d & \theta_1 c \\ \theta_4 e & \theta_3 d & \theta_2 c & \theta_4 f & \theta_3 e & \theta_2 d \end{vmatrix}$$

et ainsi de suite.]

66.

SUR LES FONCTIONS DE LAPLACE.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIII. (1848), pp. 275—280].

J'AI réussi à étendre à un nombre quelconque de variables la théorie des fonctions de Laplace, en me fondant sur le théorème que voici:

"Les coefficients $l, m, \ldots, l', m', \ldots$ étant assujettis aux conditions

$$\left.\begin{aligned} l^2 + m^2 + \ldots = 0, \\ l'^2 + m'^2 + \ldots = 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

et les limites de l'intégration étant données par

$$x^2 + y^2 + \ldots = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

l'on aura pour toutes valeurs entières et positives de s, s', excepté pour $s = s'$,

$$\int (lx + my + \ldots)^s (l'x + m'y + \ldots)^{s'} dxdy \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

et pour $s = s'$,

$$\int (lx + my + \ldots)^s (l'x + m'y + \ldots)^s dxdy \ldots = N_s (ll' + mm' + \ldots)^s \ldots\ldots\ldots\ldots (4),$$

en faisant, pour abréger,

$$N_s = \frac{\pi^{\frac{1}{2}n}\, \Gamma(s+1)}{2^s\, \Gamma(\frac{1}{2}n + s + 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

(où n dénote le nombre des variables)."

En admettant d'abord comme vrai ce théorème qui sera démontré plus bas, je remarque qu'il est permis d'écrire $\frac{d}{da}, \frac{d}{db}, \ldots, \frac{d}{da'}, \frac{d}{db'}, \ldots$, au lieu de $l, m, \ldots, l', m', \ldots$, où ces nouveaux symboles se rapportent à de certaines fonctions f, f' de $a, b, \ldots$ et de $a', b', \ldots$ respectivement. De là ce nouveau théorème:

"Les fonctions f, f' étant assujetties aux conditions

$$\left.\begin{aligned}\frac{d^2f}{da^2}+\frac{d^2f'}{db^2}+\ldots=0,\\ \frac{d^2f'}{da'^2}+\frac{d^2f'}{db^2}+\ldots=0,\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

on aura (entre les mêmes limites qu'auparavant), excepté pour $s=s'$,

$$\int\left(x\frac{d}{da}+y\frac{d}{db}+\ldots\right)^s f\,.\left(x\frac{d}{da'}+y\frac{d}{db'}+\ldots\right)^{s'} f'\,.\,dxdy\ldots=0\ldots\ldots\ldots\ldots(7),$$

et pour $s=s'$,

$$\left.\begin{aligned}&\int\left(x\frac{d}{da}+y\frac{d}{db}+\ldots\right)^s f\,.\left(x\frac{d}{da'}+y\frac{d}{db'}+\ldots\right)^{s} f'\,.\,dxdy\ldots\\ &=N_s\left(\frac{d}{da}\frac{d}{da'}+\frac{d}{db}\frac{d}{db'}+\ldots\right)^s ff'.\end{aligned}\right\}\ldots\ldots\ldots\ldots(8)."$$

Il est facile de voir que les expressions

$$\left(x\frac{d}{da}+y\frac{d}{db}+\ldots\right)^s f,\quad \left(x\frac{d}{da'}+y\frac{d}{db'}+\ldots\right)^{s'} f'$$

satisfont à l'équation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\ldots=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

et, de plus, qu'elles sont les fonctions entières et homogènes, des degrés s et s' respectivement, les plus générales qui puissent satisfaire à cette équation. On a donc ce théorème:

"Soient V_s, $W_{s'}$ les fonctions entières et homogènes des degrés s et s' respectivement, les plus générales qui satisfassent à l'équation (9); on aura toujours, excepté au cas de $s=s'$,

$$\int V_s\,W_{s'}\,dxdy\ldots=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

(les limites étant les mêmes qu'auparavant)."

Écrivons à présent

$$f=(a^2+b^2+\ldots)^{-\frac{1}{2}n+1},$$

valeur qui satisfait à la première des équations (8), et nous obtiendrons par la différentiation successive, en faisant attention à la seconde de ces mêmes équations,

$$\left.\begin{aligned}&\left(\frac{d}{da}\frac{d}{da'}+\frac{d}{db}\frac{d}{db'}+\ldots\right)^s(a^2+b^2+\ldots)^{-\frac{1}{2}n+1}f'\\ &=(-)^s\,2^s\,\frac{\Gamma(\frac{1}{2}n+s-1)}{\Gamma(\frac{1}{2}n-1)}\left(a\frac{d}{da'}+b\frac{d}{db'}+\ldots\right)^s f'(a^2+b^2+\ldots)^{-\frac{1}{2}n-s+1}\end{aligned}\right\}\ldots(11).$$

En représentant, comme auparavant, par W_s la fonction

$$\left(x\frac{d}{da'}+y\frac{d}{db'}+\ldots\right)^s f',$$

soit W_s' ce que devient W_s en écrivant a, b, ... au lieu de x, y, ... , c'est-à-dire écrivons

$$W_s'=\left(a\frac{d}{da'}+b\frac{d}{db'}+\ldots\right)^s f'.$$

On déduit de là, et au moyen de l'équation (11), en substituant dans l'équation (8), la formule

$$\left.\begin{aligned}&\frac{(-)^s}{\Gamma(s+1)}\int\left(x\frac{d}{da}+y\frac{d}{db}+\ldots\right)^s(a^2+b^2+\ldots)^{-\frac{1}{2}n+1}W_s\,dxdy\ldots\\&\qquad=M_s(a^2+b^2+\ldots)^{-\frac{1}{2}n-s+1}W_s'\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(12),$$

en faisant, pour abréger,

$$\Gamma(s+1)M_s=\frac{\Gamma\,2^s(\frac{1}{2}n+s-1)}{\Gamma(\frac{1}{2}n-1)}N_s,$$

ou bien

$$M_s=\frac{4\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-1)(n+2s)(n+2s-2)}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Soit maintenant

$$\frac{1}{[(a-x)^2+\ldots]^{\frac{1}{2}n-1}}=\frac{Q_0}{(a^2+b^2+\ldots)^{\frac{1}{2}n-1}}\cdots+\frac{Q_s}{(a^2+b^2+\ldots)^{\frac{1}{2}n+s-1}}\quad\ldots\ldots\ldots\ldots(14),$$

ou, autrement dit, soit

$$Q_s=\frac{(-)^s}{\Gamma(s+1)}\left(x\frac{d}{da}+y\frac{d}{db}+\ldots\right)^s(a^2+b^2+\ldots)^{-\frac{1}{2}n+1}\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

l'équation (12) devient

$$\int Q_sW_s\,dxdy\ldots=M_sW_s'\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

{où la valeur de M_s est donnée par l'équation (13)}. Les deux équations (10) et (16) contiennent la théorie des fonctions W_s, Q_s, lesquelles comprennent évidemment, comme cas particulier, les fonctions de Laplace.

Pour démontrer le théorème exprimé par les équations (1), (2), (3), (4), (5), je fais d'abord abstraction des équations (1), et j'écris

$$\begin{aligned}x&=\ l\xi+\ l'\eta+\ l''\zeta\ldots,\\y&=m\xi+m'\eta+m''\zeta\ldots,\end{aligned}$$

où les coefficients sont tels que l'équation

$$x^2+y^2+\ldots=p\xi^2+2q\xi\eta+p'\eta^2+p''\zeta^2+\ldots$$

soit identiquement vraie. Cela suppose que les valeurs de p, p', q soient respectivement $l^2+m^2+\ldots$, $l'^2+m'^2+\ldots$, et $ll'+mm'+\ldots$, et que les sommes de produits telles que $ll''+mm''+\ldots$, $l'l''+m'm''+\ldots$ se réduisent chacune à zéro. De là

$$\begin{aligned} dxdy\ldots &= \sqrt{pp'-q^2}\,\sqrt{p''}\ldots d\xi\, d\eta\, d\zeta\ldots, \\ lx+my+\ldots &= p\xi+q\eta, \\ l'x+m'y+\ldots &= q\xi+p'\eta. \end{aligned}$$

En représentant par I l'intégrale au premier membre de l'équation (3), cela donne

$$I=\sqrt{pp'-q^2}\,\sqrt{p''}\ldots\int(p\xi+q\eta)^s(q\xi+p'\eta)^{s'}\,d\xi\,d\eta\,d\zeta\ldots,$$

l'équation des limites étant

$$p\xi^2+2q\xi\eta+p'\eta^2+p''\zeta^2+\ldots=1.$$

Cette intégrale se simplifie en écrivant

$$\xi\sqrt{p}+\frac{q\eta}{\sqrt{p}}=\xi_{,},\quad \eta\sqrt{p'-\frac{q^2}{p}}=\eta_{,},\quad \zeta\sqrt{p''}=\zeta_{,},\ldots;$$

car alors

$$\begin{aligned} &\sqrt{pp'-q^2}\,\sqrt{p''}\ldots d\xi\,d\eta\,d\zeta\ldots=d\xi_{,}\,d\eta_{,}\,d\zeta_{,}\ldots, \\ &p\xi+q\eta=\sqrt{p}\,\xi_{,}, \\ &q\xi+p'\eta=\frac{1}{\sqrt{p}}(q\xi+\sqrt{pp'-q^2}\,\eta_{,}), \end{aligned}$$

et de là

$$I=p^{\frac{1}{2}(s-s')}\int\xi_{,}^{s}(q\xi_{,}+\sqrt{pp'-q^2}\,\eta_{,})^{s'}\,d\xi_{,}\,d\eta_{,}\,d\zeta_{,}\ldots,$$

l'équation des limites étant

$$\xi_{,}^2+\eta_{,}^2+\ldots=1.$$

En supposant $s>s'$, l'intégrale s'évanouit pour $p=0$, et de même quand $s'>s$, elle s'évanouit pour $p'=0$. Donc, en écrivant $p=0$, $p'=0$, on aura toujours, excepté pour $s=s'$, l'équation $I=0$; ce qui revient à l'équation (3). Au cas de $s=s'$, en écrivant de même $p=0$, $p'=0$, on trouve

$$I=q^s\int\xi_{,}^{s}(\xi_{,}+i\eta_{,})^s\,d\xi_{,}\,d\eta_{,}\,d\zeta_{,}\ldots$$

(où, comme à l'ordinaire, $i=\sqrt{-1}$). En faisant attention à la valeur de q, et en comparant avec l'équation (4), cette dernière équation sera démontrée en vérifiant la formule

$$N_s=\int\xi_{,}^{s}(\xi_{,}+i\eta_{,})^s\,d\xi_{,}\,d\eta_{,}\,d\zeta_{,}\ldots.$$

Soient pour cela

$$\xi_{,}=\rho\cos\theta,\qquad \eta_{,}=\rho\sin\theta.$$

Cela donne (en omettant la partie imaginaire qui s'évanouit évidemment)

$$N_s = \int \rho^{2s+1} \cos^s \theta \cos s\theta \, d\rho \, d\theta \, d\zeta, \ldots.$$

En effectuant d'abord l'intégration par rapport à $\zeta, \ldots$, les limites de ces variables sont données par

$$\zeta^2 + \ldots = 1 - \rho^2,$$

et l'on trouve tout de suite

$$N_s = \frac{\pi^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n)} \int \rho^{2s+1} (1-\rho^2)^{\frac{1}{2}n-1} \cos^s \theta \cos s\theta \, d\rho \, d\theta.$$

Cette formule doit être intégrée depuis $\rho = 0$ à $\rho = 1$, et depuis $\theta = 0$ à $\theta = 2\pi$; mais en multipliant par quatre, on peut n'étendre l'intégration par rapport à θ que depuis $\theta = 0$ jusqu'à $\theta = \frac{1}{2}\pi$. De là

$$N_s = \frac{4\pi^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n)} \int_0^1 \rho^{2s+1} (1-\rho^2)^{\frac{1}{2}n-1} \, d\rho \int_0^{\frac{1}{2}\pi} \cos^s \theta \cos s\theta \, d\theta;$$

et enfin, au moyen des formules connues

$$\int_0^1 \rho^{2s+1} (1-\rho^2)^{\frac{1}{2}n-1} \, d\rho = \frac{\Gamma(s+1)\,\Gamma(\frac{1}{2}n)}{2\,\Gamma(\frac{1}{2}n+s+1)},$$

$$\int_0^{\frac{1}{2}\pi} \cos^s \theta \cos s\theta \, d\theta = \frac{\pi}{2^{s+1}},$$

on retrouve la formule (5), laquelle il s'agissait de démontrer. Ainsi le théorème fondamental est complétement établi.

67.

NOTE SUR LES FONCTIONS ELLIPTIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXVII. (1848), pp. 58—60.]

SOIT $x = \sqrt{k} \sin \operatorname{am} u$ et $\alpha = k + \frac{1}{k}$: la fonction $\sqrt{k} \sin \operatorname{am} nu$ (où n est un entier), peut être exprimée sous forme d'une fraction dont le dénominateur est une fonction rationnelle et entière par rapport à x et α. En exprimant ce dénominateur par z, on aura

$$n^2(n^2-1)x^2z + (n^2-1)(\alpha x - 2x^3)\frac{dz}{dx} + (1 - \alpha x^2 + x^4)\frac{d^2z}{dx^2} - 2n^2(\alpha^2-4)\frac{dz}{d\alpha} = 0 \ldots \quad (1).$$

Cette équation est due à M. Jacobi, (voyez les deux mémoires "Suite des notices sur les fonctions elliptiques," t. III. [1828] p. 306 et t. IV. [1829] p. 185.)

En essayant d'intégrer cette équation à moyen d'une suite ordonnée suivant les puissances de x, et en considérant en particulier les cas $n = 2$, 3, 4 et 5, j'ai trouvé que les différentes puissances de α se présentent et disparaissent d'une manière assez bizarre: (voyez mon mémoire "On the theory of elliptic functions," *Cambridge and Dublin Math. Journal*, t. II. [1847], p. 256, [45].) J'ai reconnu depuis que cela vient de ce que la valeur de z est composée de plusieurs séries indépendantes; une quelconque de ces séries ordonnée selon les puissances descendantes de α va à l'infini; mais, en combinant les différentes séries, les termes qui contiennent les puissances négatives de α se détruisent. Par rapport à x chacune de ces séries ne contient que des puissances paires et positives, car les puissances négatives qui y entrent apparemment, se réduisent toujours à zéro. En effet, on satisfait à l'équation (1) en supposant pour z une expression de la forme

$$z_s = 2^{2s(n-2s)}\alpha^{s(n-2s)}Z_0 \ldots + (-1)^q 2^{2s(n-2s)-2q}\alpha^{s(n-2s)-q}.Z_q \ldots \quad \ldots\ldots\ldots\ldots \quad (2)$$

(où s est arbitraire). Cela donne pour Z_q l'équation aux différences mêlées:

$$\left[x^2\frac{d^2}{dx^2}-(n^2-1)\,x\frac{d}{dx}+2n^2s\,(n-2s)-2n^2q\right]Z_q$$
$$+\left[\left\{n^2(n^2-1)\,x^2+x^4\frac{d^2}{dx^2}-(2n^2-2)\,x^3\frac{d}{dx}\right\}+\frac{d^2}{dx^2}\right]4Z_{q-1}$$
$$-128n^2\left[s\,(n-2s)-q+2\right]Z_{q-2}=0.$$

Pour intégrer cette équation, supposons

$$Z_q=\Sigma\frac{Z_q^{\sigma}}{\Gamma(\sigma+1)\,\Gamma(q-\sigma+1)}\,x^{2ns+2q-4\sigma},$$

où la sommation se rapporte à σ et s'étend depuis $\sigma=0$ jusqu'à $\sigma=q$. Toute réduction faite, et ayant mis pour plus de simplicité $n^2-2ns=\lambda$, $2ns=\mu$, on obtient pour Z_q^{σ} l'expression

4.
$$\begin{aligned}&[-(q-\sigma)\,\lambda-\sigma\mu+(q-2\sigma)^2]\,Z_q^{\sigma}\\&+(q-\sigma)\,(\lambda-2q+4\sigma+2)\,(\lambda-2q+4\sigma+1)\,Z_{q-1}^{\sigma}\\&+\sigma\,(\mu+2q-4\sigma+2)\,(\mu+2q-4\sigma+1)\,Z_{q-1}^{\sigma-1}\\&+16\sigma\,(q-\sigma)\,[\lambda\mu-2\,(q-2)\,(\lambda+\mu)]\,Z_{q-2}^{\sigma-1}=0.\end{aligned}$$

En supposant la valeur de Z_0^0 égale à l'unité, les valeurs de Z_q^{σ} se trouvent complétement déterminées; malheureusement la loi des coefficients n'est pas en évidence, excepté dans le cas de $\sigma=0$, ou $\sigma=q$. En calculant les termes successifs, on obtient

$$\begin{aligned}z_s=\;&(4\alpha)^{s(n-2s)}\,.\,x^{2ns}\\&-(4\alpha)^{s(n-2s)-1}\left\{\begin{aligned}&\frac{1}{1}\lambda\,.\,x^{2ns+2}\\&+\frac{1}{1}\mu\,.\,x^{2ns-2}\end{aligned}\right.\\&+(4\alpha)^{s(n-2s)-2}\left\{\begin{aligned}&\frac{1}{1.2}\qquad\lambda\,(\lambda-3)\,x^{2ns+4}\\&+\frac{1}{1.1}\left(\lambda\mu+2-\frac{10\lambda\mu}{\lambda+\mu}\right)x^{2ns}\\&+\frac{1}{1.2}\qquad\mu\,(\mu-3)\,x^{2ns-4}\end{aligned}\right.\\&-(4\alpha)^{s(n-2s)-3}\left\{\begin{aligned}&\frac{1}{1.2.3}\qquad\lambda\,(\lambda-4)\,(\lambda-5)\,x^{2ns+6}\\&+\frac{1}{1.2.1}\,\lambda\left(\mu\,(\lambda-3)+40-\frac{20\lambda\mu}{\lambda+\mu}\right)x^{2ns+2}\\&+\frac{1}{1.1.2}\,\mu\left(\lambda\,(\mu-3)+40-\frac{20\lambda\mu}{\lambda+\mu}\right)x^{2ns-2}\\&+\frac{1}{1.2.3}\qquad\mu\,(\mu-4)\,(\mu-5)\,x^{2ns-6}\end{aligned}\right.\\&+\;\&c.\end{aligned}$$

On aurait une valeur assez générale de z en multipliant les différentes fonctions z_s chacune par une constante arbitraire, et en sommant les produits; mais dans le cas actuel où z dénote le dénominateur de $\sqrt{k}\sin\operatorname{am} nu$, la valeur convenable de z se réduit à

$$z = z_0 \pm z_1 \ldots \pm z_s \ldots,$$

où s est un nombre entier et positif, entre zéro et $\frac{1}{2}n$ ou $\frac{1}{2}(n-1)$. On aura par exemple dans le cas $n=5$ (les signes étant tous positifs si n est impair, et alternativement positifs et négatifs si n est pair), en supprimant les puissances négatives de α (lesquelles s'entredétruisent):

$$\begin{aligned}
z_0 &= 1,\\
z_1 &= 64\alpha^3x^{10} - \alpha^2(160x^8 + 240x^{12}) + \alpha(140x^6 + 368x^{10} + 360x^{14})\\
&\qquad - (50x^4 + 125x^8 + 300x^{12} + 275x^{16}),\\
z_2 &= 16\alpha^2x^{20} - \alpha(80x^{18} + 20x^{22}) + (170x^{16} + 62x^{20} + 5x^{24}),
\end{aligned}$$

et de là:

$$\begin{aligned}
z = 1 &- 50x^4 + 140\alpha x^5 - (125 + 160\alpha^2)x^8 + (368\alpha + 64\alpha^3)x^{10} - (300 + 240\alpha^2)x^{12}\\
&+ 360\alpha x^{14} - 105x^{16} - 80\alpha x^{18} + (62 + 16\alpha^2)x^{20} - 20\alpha x^{22} + 5x^{24};
\end{aligned}$$

ce qui est effectivement la valeur de z que j'ai trouvée dans le mémoire cité pour ce cas particulier.

68.

ON THE APPLICATION OF QUATERNIONS TO THE THEORY OF ROTATION.

[From the *Philosophical Magazine*, vol. XXXIII. (1848), pp. 196—200.]

IN a paper published in the *Philosophical Magazine*, February 1845, [20], I showed how some formulæ of M. Olinde Rodrigues relating to the rotation of a solid body might be expressed in a very simple form by means of Sir W. Hamilton's theory of quaternions. The property in question may be thus stated. Suppose a solid body which revolves through an angle θ round an axis passing through the origin and inclined to the axes of coordinates at angles a, b, c. Let

$$\lambda = \tan \tfrac{1}{2}\theta \cos a, \qquad \mu = \tan \tfrac{1}{2}\theta \cos b, \qquad \nu = \tan \tfrac{1}{2}\theta \cos c,$$

and write

$$\Lambda = 1 + i\lambda + j\mu + k\nu;$$

let x, y, z be the coordinates of a point in the body previous to the rotation, x_1, y_1, z_1 those of the same point after the rotation, and suppose

$$\Pi = ix + jy + kz,$$
$$\Pi_1 = ix_1 + jy_1 + kz_1;$$

then the coordinates after the rotation may be determined by the formula

$$\Pi_1 = \Lambda \Pi \Lambda^{-1};$$

viz., developing the second side of this equation,

$$\begin{aligned}\Pi_1 = i\,&(\alpha\, x + \beta\, y + \gamma z\ )\\ +j\,&(\alpha' x + \beta' y + \gamma' z)\\ +k\,&(\alpha'' x + \beta'' y + \gamma'' z),\end{aligned}$$

where, putting to abbreviate $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, we have

$$\begin{aligned}
&\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, \quad \kappa\alpha' = 2(\lambda\mu + \nu) \quad , \quad \kappa\alpha'' = 2(\lambda\nu - \mu) \quad , \\
&\kappa\beta = 2(\lambda\mu - \nu) \quad , \quad \kappa\beta' = 1 - \lambda^2 + \mu^2 - \nu^2, \quad \kappa\beta'' = 2(\mu\nu + \lambda) \quad , \\
&\kappa\gamma = 2(\lambda\nu + \mu) \quad , \quad \kappa\gamma' = 2(\mu\nu - \lambda) \quad , \quad \kappa\gamma'' = 1 - \lambda^2 - \mu^2 + \nu^2 ;
\end{aligned}$$

these values satisfying identically the well-known system of equations connecting the quantities α, β, γ, α', β', γ', α'', β'', γ''.

The quantities a, b, c, θ being immediately known when λ, μ, ν are known, these last quantities completely determine the direction and magnitude of the rotation, and may therefore be termed the coordinates of the rotation; Λ will be the quaternion of the rotation. I propose here to develope a few of the consequences which may be deduced from the preceding formulæ.

Suppose, in the first place, $\Pi = \Lambda - 1$, then $\Pi_1 = \Lambda - 1$, which evidently implies that the point is on the axis of rotation. The equation $\Pi_1 = \Pi$ gives the identical equations

$$\begin{aligned}
&\lambda(\alpha - 1) + \mu\,\beta + \nu\,\gamma = 0, \\
&\lambda\,\alpha' + \mu(\beta' - 1) + \nu\,\gamma' = 0, \\
&\lambda\,\alpha'' + \mu\,\beta'' + \nu(\gamma'' - 1) = 0 ;
\end{aligned}$$

from which, by changing the signs of λ, μ, ν, we derive

$$\begin{aligned}
&\lambda(\alpha - 1) + \mu\,\alpha' + \nu\,\alpha'' = 0, \\
&\lambda\,\beta + \mu(\beta' - 1) + \nu\,\beta'' = 0, \\
&\lambda\,\gamma + \mu\,\gamma' + \nu(\gamma'' - 1) = 0.
\end{aligned}$$

Hence evidently, whatever be the value of Π,

$$\Lambda\Pi\Lambda^{-1} - \Pi = 0,$$

if after the multiplication i, j, k are changed into λ, μ, ν, a property which will be required in the sequel.

By changing the signs of λ, μ, ν, we also deduce

$$\begin{aligned}
\Lambda^{-1}\Pi\Lambda = i\ &(\alpha x + \alpha' y + \alpha'' z) \\
+ j\ &(\beta x + \beta' y + \beta'' z) \\
+ k\ &(\gamma x + \gamma' y + \gamma'' z),
\end{aligned}$$

where α, β, γ, α', β', γ', α'', β'', γ'' are the same as before.

Let the question be proposed to compound two rotations (both axes of rotation being supposed to pass through the origin). Let L be the first axis, Λ the quaternion of rotation, L' the second axis, which is supposed to be fixed in space, so as not to alter its direction by reason of the first rotation, Λ' the corresponding quaternion of rotation. The combined effect is given at once by

$$\Pi_1 = \Lambda'(\Lambda\Pi\Lambda^{-1})\,\Lambda'^{-1},$$

that is,

$$\Pi_1 = \Lambda'\Lambda\Pi\,(\Lambda'\Lambda)^{-1} \quad ;$$

or since (if Λ_1 be the quaternion for the combined rotation) $\Pi_1 = \Lambda_1 \Pi \Lambda_1^{-1}$, we have clearly

$$\Lambda_1 = M_1 \Lambda' \Lambda,$$

M_1 denoting the reciprocal of the real part of $\Lambda'\Lambda$, so that

$$M_1^{-1} = 1 - \lambda\lambda' - \mu\mu' - \nu\nu'.$$

Retaining this value, the coefficients of the combined rotation are given by

$$\begin{aligned} \lambda_1 &= M_1 (\lambda + \lambda' + \mu'\nu - \mu\nu'), \\ \mu_1 &= M_1 (\mu + \mu' + \nu'\lambda - \nu\lambda'), \\ \nu_1 &= M_1 (\nu + \nu' + \lambda'\mu - \lambda\mu'); \end{aligned}$$

to which may be joined [if $\kappa_1 = 1 + \lambda_1^2 + \mu_1^2 + \nu_1^2$],

$$\kappa_1 = M_1^2 \kappa\kappa',$$

κ, κ', κ_1 as before. Λ or Λ' may be determined with equal facility in terms of Λ', Λ_1, or Λ, Λ_1. These formulæ are given in my paper on the rotation of a solid body (*Cambridge Mathematical Journal*, vol. III. p. 226, [6]).

If the axis L' be fixed in the body and moveable with it, its position after the first rotation is obtained from the formula $\Pi_1 = \Lambda\Pi\Lambda^{-1}$ by writing $\Pi = \Lambda' - 1$. Representing by $\Lambda'' - 1$ the corresponding value of Π_1, we have $\Lambda'' = \Lambda\Lambda'\Lambda^{-1}$, which is the value to be used instead of Λ' in the preceding formula for the combined rotation, thus the quaternion of rotation is proportional to $\Lambda\Lambda'\Lambda^{-1}\Lambda$, that is to $\Lambda\Lambda'$. Hence here

$$\Lambda_1 = M_1 \Lambda\Lambda',$$

which only differs from the preceding in the order of the quaternion factors. If the fixed and moveable axes be mixed together in any order whatever, the fixed axes taken in order being L, L', ... and the moveable axes taken in order being L_0, L_0' ... then the combined effect of the rotations is given by

$$\Lambda_1 = M \ldots \Lambda''\Lambda'\Lambda\Lambda_0\Lambda_0' \ldots,$$

M being the reciprocal of the real term of the product of all the quaternions.

Suppose next the axes do not pass through the same point. If α, β, γ be the coordinates of a point in L, and

$$\Gamma = \alpha i + \beta j + \gamma k,$$

then the formula for the rotation is

$$\Pi_1 - \Gamma = \Lambda (\Pi - \Gamma) \Lambda^{-1},$$

or

$$\Pi_1 = \Lambda\Pi\Lambda^{-1} - (\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

where the first term indicates a rotation round a parallel axis through the origin, and the second term a translation.

For two axes L, L' fixed in space,

$$\Pi_1 = \Lambda'\Lambda\Pi\,(\Lambda'\Lambda)^{-1} - (\Lambda\Gamma'\Lambda'^{-1} - \Gamma') - \Lambda'\,(\Lambda\Gamma\Lambda^{-1} - \Gamma)\,\Lambda'^{-1};$$

and so on for any number, the last terms being always a translation. If the two axes are parallel, and the rotations equal and opposite,

$$\Lambda = \Lambda'^{-1},$$

whence

$$\Pi_1 = \Pi + \Lambda'\,(\Gamma - \Gamma')\,\Lambda'^{-1}\,(\Gamma - \Gamma');$$

or there is only a translation. The constant term vanishes if i, j, k are changed into λ', μ', ν', which proves that the translation is in a plane perpendicular to the axes.

Any motion of a solid body being represented by a rotation and a translation, it may be required to resolve this into two rotations. We have

$$\Pi_1 = \Lambda_1\Pi\Lambda_1^{-1} + T,$$

where T is a given quaternion whose constant term vanishes. Hence, comparing this with the general formula just given for the combination of two rotations,

$$\Lambda_1 = M_1\Lambda'\Lambda,$$
$$T = -(\Lambda'\Gamma'\Lambda'^{-1} - \Gamma') - \Lambda'\,(\Lambda\Gamma\Lambda^{-1} - \Gamma)\,\Lambda'^{-1},$$

the second of which equations may be simplified by putting $\Lambda'^{-1}T\Lambda' = S$, by which it may be reduced to

$$S = (\Lambda'^{-1}\Gamma'\Lambda' - \Gamma') - (\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

which, with the preceding equation $\Lambda_1 = M_1\Lambda'\Lambda$, contains the solution of the problem. Thus if Λ or Λ' be given, the other is immediately known; hence also S is known. If in the last equation, after the multiplication is completely effected, we change i, j, k into λ, μ, ν, or λ', μ', ν', we have respectively,

$$S = \Lambda'^{-1}\Gamma'\Lambda' - \Gamma', \quad S = -(\Lambda\Gamma\Lambda^{-1} - \Gamma),$$

which are equations which must be satisfied by the coefficients of Γ' and Γ respectively. Thus if the direction of one axis is given, that of the other is known, and the axes must lie in certain known planes. If the position of one of the axes in its plane be assumed, the equation containing S divides itself into three others (equivalent to two independent equations) for the determination of the position in its plane of the other axis. If the axes are parallel, λ, μ, ν are proportional to λ', μ', ν'; or changing i, j, k into λ, μ, ν, or λ', μ', ν', we have $S = 0$; or what is the same thing, $T = 0$, which shows that the translation must be perpendicular to the plane of the two axes.

If p, q, r have their ordinary signification in the theory of rotation, then from the values in the paper in the *Cambridge Mathematical Journal* already quoted,

$$\kappa\,(ip + jq + kr) = 2\frac{d\Lambda}{dt}\Lambda + \frac{d\kappa}{dt};$$

but I have not ascertained whether this formula leads to any results of importance. It may, however, be made use of to deduce the following property of quaternions, viz. if $\Lambda_1 = M_1\Lambda'\Lambda$, M_1 as before, then

$$\frac{1}{\kappa_1}\left(2\frac{d\Lambda_1}{dt}\Lambda_1 + \frac{d\kappa_1}{dt}\right) = \frac{1}{\kappa}\left(2\frac{d\Lambda}{dt}\Lambda + \frac{d\kappa}{dt}\right),$$

in which the coefficients of Λ' are considered constant.

To verify this *à posteriori*, if in the first place we substitute for κ_1 its value $M_1^2\kappa\kappa'$, we have

$$\frac{d\kappa_1}{dt} = M_1^2\kappa'\frac{d\kappa}{dt} + \frac{2}{M_1}\frac{dM_1}{dt}\kappa_1,$$

and thence

$$\frac{d\Lambda_1}{dt}\Lambda_1 + \frac{1}{M_1}\frac{dM_1}{dt}\kappa_1 = M^2\kappa'\frac{d\Lambda}{dt}\Lambda.$$

Also

$$\frac{d\Lambda_1}{dt}\Lambda_1 = \left(\frac{1}{M_1}\frac{dM_1}{dt}\Lambda_1 + M_1\Lambda'\frac{d\Lambda}{dt}\right)\Lambda_1 = \frac{1}{M_1}\frac{dM_1}{dt}\Lambda_1^2 + M_1^2\Lambda'\frac{d\Lambda}{dt}\Lambda'\Lambda,$$

which reduces the equation to

$$\frac{1}{M_1}\frac{dM_1}{dt}(\Lambda_1^2 + \kappa_1) + M_1^2\Lambda'\frac{d\Lambda}{dt}\Lambda'\Lambda = M_1^2\kappa_1\frac{d\Lambda}{dt}\Lambda.$$

Hence observing that

$$\Lambda_1^2 + \kappa_1 = 2\Lambda_1 = 2M_1\Lambda'\Lambda,$$

and omitting the factor Λ from the resulting equation,

$$\frac{2}{M_1^2}\frac{dM_1}{dt}\Lambda' + \Lambda'\frac{d\Lambda}{dt}\Lambda' = \kappa'\frac{d\Lambda}{dt};$$

or since

$$\frac{1}{M_1} = 1 - \lambda\lambda' - \mu\mu' - \nu\nu',$$

substituting and dividing by Λ', we obtain

$$2\left(\lambda'\frac{d\lambda}{dt} + \mu'\frac{d\mu}{dt} + \nu'\frac{d\nu}{dt}\right) = \kappa'\Lambda'^{-1}\frac{d\Lambda}{dt} - \frac{d\Lambda}{dt}\Lambda',$$

or finally,

$$2\left(\lambda'\frac{d\lambda}{dt} + \mu'\frac{d\mu}{dt} + \nu'\frac{d\nu}{dt}\right) = (1 - i\lambda' - j\mu' - k\nu')\left(i\frac{d\lambda}{dt} + j\frac{d\mu}{dt} + k\frac{d\nu}{dt}\right)$$

$$- \left(i\frac{d\lambda}{dt} + j\frac{d\mu}{dt} + k\frac{d\nu}{dt}\right)(1 + i\lambda' + j\mu' + k\nu')$$

$$= -(i\lambda' + j\mu' + k\nu')\left(i\frac{d\lambda}{dt} + j\frac{d\mu}{dt} + k\frac{d\nu}{dt}\right) - \left(i\frac{d\lambda}{dt} + j\frac{d\mu}{dt} + k\frac{d\nu}{dt}\right)(i\lambda' + j\mu' + k\nu'),$$

which is obviously true.

69.

SUR LES DÉTERMINANTS GAUCHES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXVIII. (1848), pp. 93—96: continued from t. XXXII. p. 123, **52**.]

J'AI nommé "*Déterminant gauche*" un déterminant formé par un système de quantités $\lambda_{r.s}$, qui satisfont aux conditions

$$\lambda_{s.r} = -\lambda_{r.s} \, (r \neq s) \quad \text{.....} \quad (1),$$

(où les valeurs de r, s s'étendent depuis l'unité jusqu'à n).

Or ces déterminants peuvent facilement être exprimés par un système de déterminants pareils, dont les termes satisfont à ces conditions même dans le cas où les valeurs de r et s deviennent égales, ou pour lesquels on a

$$\lambda_{r.s} = -\lambda_{s.r} \, (r \neq s) \, ; \quad \lambda_{r.r} = 0 \quad \text{.....} \quad (2) \, ;$$

ces déterminants peuvent être nommés "*gauches* et *symétriques*."

En effet, soit Ω le déterminant gauche dont il s'agit, cette fonction peut être présentée sous la forme

$$\Omega = \Omega_0 + \Omega_1 \lambda_{1.1} + \Omega_2 \lambda_{2.2} \ldots + \Omega_{12} \lambda_{1.1} \lambda_{2.2} \ldots \quad \text{.....} (3),$$

où Ω_0 est ce que devient Ω si $\lambda_{1.1}$, $\lambda_{2.2}$, &c. sont réduits à zéro; Ω_1 est ce que devient le coefficient de $\lambda_{1.1}$ sous la même condition, et ainsi de suite; c'est-à-dire: Ω_0 est le déterminant formé par les quantités $\lambda_{r.s}$ en supposant que ces quantités satisfassent aux conditions (2), et en donnant à r les valeurs 1, 2, 3, ... n; Ω_1 est le determinant formé pareillement en donnant à r, s les valeurs 2, 3, ... n; Ω_2 s'obtient en donnant à r, s les valeurs 1, 3, ... n, et ainsi de suite; cela est aisé de voir si l'on range les quantités $\lambda_{r.s}$ en forme de carré.

Or les déterminants gauches et symétriques (savoir les déterminants dont les termes satisfont aux conditions (2)) se réduisent à zéro pour un n *impair*, et pour un n *pair* aux carrés des fonctions que M. Jacobi a traitées dans son mémoire "Ueber die Pfaff'sche Integrations-Methode" (t. II. [1827], p. 354, de ce journal) et dans le mémoire "Theoria novi multiplicatoris æquationum differentialium" t. XXIX. [1845], p. 236, &c.

En effet, on voit d'abord (par ce que dit M. Jacobi) que le déterminant s'évanouit pour un n *impair*, et que pour un n pair il aura pour *facteur* la fonction dont il s'agit; mais je ne sais pas si l'on a déjà remarqué que l'autre facteur se réduit à la même fonction.

On obtient ces fonctions (dont je reprends ici la théorie), par les propriétés générales d'un déterminant, défini de la manière que voici: en exprimant par $(12\ldots n)$ une fonction quelconque dans laquelle entrent les nombres symboliques $1, 2, \ldots n$, et par $\pm$, le signe correspondant à une permutation quelconque de ces nombres, la fonction

$$\Sigma \pm (12\ldots n)$$

(où Σ désigne la somme de tous les termes qu'on obtient en permutant ces nombres d'une manière quelconque) est ce qu'on nomme *Déterminant*. On pourrait encore généraliser cette définition en admettant plusieurs systèmes de nombres $1, 2, \ldots n$; $1', 2', \ldots n'$; ... qui alors devraient être permutés indépendamment les uns des autres; on obtiendrait de cette manière une infinité d'autres fonctions, mentionnées (t. XXX. [1846] p. 7). Dans le cas des déterminants ordinaires, auquel je ne m'arrêterai pas ici, on aura $(12\ldots n) = \lambda_{\alpha.1}\lambda_{\beta.2}\ldots\lambda_{\kappa.n}$. Pour les cas des fonctions dont il s'agit (les fonctions de M. Jacobi), on supposera n *pair*, et l'on écrira

$$(12\ldots n) = \lambda_{1.2}\lambda_{3.4}\ldots\lambda_{n-1,n},$$

où $\lambda_{r.s}$ sont des quantités quelconques qui satisfont aux équations (1). La fonction sera composée d'un nombre $1.2\ldots n$ de termes; mais parmi eux il n'y aura que $1.3\ldots(n-1)$ termes différents qui se trouveront répétés $2^{\frac{1}{2}n}(1.2\ldots\frac{1}{2}n)$ fois, et qu'on obtiendra en permutant cycliquement d'abord les $n-1$ derniers nombres, puis les $n-3$ derniers nombres de chaque permutation, et ainsi de suite; le signe étant toujours $+$. Il pourra être démontré, comme pour les déterminants, que ces fonctions changent de signe en permutant deux quelconques des nombres symboliques, et qu'elles s'évanouissent si deux de ces nombres deviennent identiques. De plus, en exprimant par $[12\ldots n]$ la fonction dont il s'agit, la règle qui vient d'être énoncée, donnera pour la formation de ces fonctions:

$$[12\ldots n] = \lambda_{1.2}[34\ldots n] + \lambda_{1.3}[4\ldots n-2]\ldots + \lambda_{1.n}[23\ldots(n-1)].$$

Cela posé, revenons aux déterminants gauches et symétriques; et soit d'abord n un nombre impair. Alors le déterminant sera composé de plusieurs termes, chacun multiplié par le produit de l'une des quantités $\lambda_{1.2}, \lambda_{1.3}, \ldots \lambda_{1.n}$ par une des quantités $\lambda_{2.1}, \lambda_{3.1}, \ldots \lambda_{n.1}$. Il est facile de voir que pour chaque terme, multiplié par $\lambda_{1.\alpha}\lambda_{\beta.1}\,(\alpha \neq \beta)$, il existera un terme *égal* et de signe contraire, multiplié par $\lambda_{1.\beta}\lambda_{\alpha.1}$. Or $\lambda_{1.\alpha}\lambda_{\beta.1} = \lambda_{\alpha.1}\lambda_{1.\beta}$: donc

ces deux termes se *détruiront*. Restent les termes multipliés par $\lambda_{1.\alpha}\lambda_{\alpha.1}$: le coefficient d'un terme de cette forme sera un déterminant gauche de l'ordre $n-2$, mais précisément de la forme du déterminant même de l'ordre n dont il s'agit. Or n étant impair, $n-2$ le sera également: donc tout déterminant gauche et symétrique s'évanouira, si cela a lieu pour les déterminants pareils d'un ordre inférieur de deux unités. Or pour $n=3$ on a évidemment $\lambda_{1.2}\lambda_{2.3}\lambda_{3.1}+\lambda_{2.1}\lambda_{3.2}\lambda_{1.3}=0$, donc:

"Tout déterminant gauche et symétrique d'un ordre *impair* est zéro."

Soit maintenant n un nombre *pair*. Considérons le déterminant gauche et symétrique plus général qu'on obtient en donnant à r les valeurs α, 2, 3, ... n, et à s les valeurs β, 2, 3, ... n. En développant cette fonction comme on vient de le faire dans le cas d'un n *impair*, il se présentera d'abord un terme multiplié par $\lambda_{\alpha.\beta}$. Mais ce terme sera un déterminant gauche et symétrique de l'ordre $n-1$ (savoir celui que l'on obtiendrait en donnant à r, s les valeurs 2, 3, ... n), et comme $n-1$ est impair, ce terme s'évanouira. Puis le coefficient de $-\lambda_{\alpha.\alpha'}\lambda_{\beta'.\beta}$ (ou de $\lambda_{\alpha.\alpha'}\lambda_{\beta.\beta'}$) sera le déterminant gauche et symétrique qu'on obtient en donnant à r les valeurs 2, 3, ... n (α excepté), et à s les valeurs 2, 3, ... n (β' excepté). En admettant que ce déterminant gauche se réduit à $[(\alpha'+1)\ldots n\,2\ldots(\alpha'-1)]\,[(\beta'+1)\ldots n\,2\ldots(\beta'-1)]$, on aura

$$\lambda_{\alpha.\alpha'}[(\alpha'+1)\ldots(\alpha'-1)].\lambda_{\beta.\beta'}[(\beta'+1)\ldots(\beta'-1)]$$

pour le terme général, et la somme de tous ces termes se réduira à

$$\{\lambda_{\alpha.2}[34\ldots n]+\lambda_{\alpha.3}[4\ldots n2]+\ldots\}.\{\lambda_{\beta.2}[34\ldots n]+\lambda_{\beta.3}[4\ldots n2]+\ldots\},$$

ou enfin à $[\alpha 23\ldots n]\;[\beta 23\ldots n]$. Or si le théorème en question a lieu pour $n-2$, il aura lieu aussi pour n. Pour $n=2$ le déterminant est $\lambda_{\alpha.\beta}\lambda_{2.2}-\lambda_{2.\beta}\lambda_{\alpha.2}$, c'est-à-dire $\lambda_{\alpha.2}\lambda_{\beta.2}$ ou $[\alpha 2].[\beta 2]$: donc la même chose a toujours lieu et on obtient le théorème que voici:

"Le déterminant gauche et symétrique qu'on obtient en donnant à r les valeurs α, 2, 3, ... n, et à s les valeurs β, 2, 3, ... n (où n est *pair*), se réduit à

$$[\alpha 23\ldots n].[\beta 23\ldots n];$$

et en particulier, en donnant à r, s les valeurs 1, 2, ... n, ce déterminant se réduit à $[12\ldots n]^2$."

Appliquons ces théorèmes à la réduction de l'équation (3). En supposant pour plus de simplicité $\lambda_{r.r}=1$, on aura pour un n *pair*:

$$\begin{aligned}\Omega=[123\ldots n]^2+[34\ldots n]^2+[56\ldots n]^2+\ldots+1\,;\\ \vdots\\ +[24\ldots n]^2\\ \vdots\end{aligned}$$

et pour un n *impair:*

$$\begin{aligned}\Omega = \quad & [23\ldots n]^2 + [45\ldots n]^2 + \ldots + 1.\\ & \qquad\qquad\qquad\vdots\\ & + [13\ldots n]^2\\ & \quad\vdots\end{aligned}$$

Particulièrement pour $n=4$ on obtient:

$$\begin{aligned}\Omega &= [1234]^2 + [12]^2 + [13]^2 + [14]^2 + [23]^2 + [34]^2 + [42]^2 + 1,\\ &= (\lambda_{1.2}\lambda_{3.4} + \lambda_{1.3}\lambda_{2.4} + \lambda_{1.4}\lambda_{2.3})^2 + \lambda_{1.2}^2 + \lambda_{1.3}^2 + \lambda_{1.4}^2 + \lambda_{2.4}^2 + \lambda_{4.2}^2 + \lambda_{2.3}^2 + 1,\end{aligned}$$

et pour $n=3$:

$$\begin{aligned}\Omega &= [23]^2 + [31]^2 + [12]^2 + 1,\\ &= \lambda_{2.3}^2 + \lambda_{3.1}^2 + \lambda_{1.2}^2 + 1:\end{aligned}$$

résultats qui s'accordent parfaitement avec ceux que j'ai donnés dans mon premier mémoire sur ce sujet.

Il serait possible de trouver pour les quantités $\Lambda_{r,s}$ (mémoire cité) des expressions analogues à celles que nous venons de donner pour Ω: mais elles seraient beaucoup plus compliquées.

70.

SUR QUELQUES THÉORÈMES DE LA GÉOMÉTRIE DE POSITION.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXVIII. (1848), pp. 97—104: continued from t. XXXIV. p. 275, 55.]

§ V.

LE théorème de Pascal appliqué au cas où une conique se réduit à deux droites, donne lieu à un système de neuf points situés trois à trois dans neuf droites qui passent réciproquement trois à trois par les neuf points. Je représenterai ces points par

1, 4, 7; 2, 5, 8; 3, 6, 9,

et les droites par

135, 426, 789; 129, 483, 756; 186, 459, 723.

On peut prendre pour la conique dont il s'agit, deux quelconques de ces droites qui ne se rencontrent pas dans un des neuf points, ou ce qui revient au même, deux quelconques des droites qui appartiennent à un des trois systèmes dans lesquels les droites viennent d'être divisées; alors la troisième droite sera celle qui contient les points de rencontre des côtés opposés de l'hexagone. Par exemple, en prenant pour conique les deux droites 135, 246, l'hexagone sera 123456, et les points de rencontre des côtés opposés, savoir de 12 et 45, de 23 et 56, et de 34 et 61, seront respectivement 9, 7, 8; c'est-à-dire des points situés dans la droite 789; de manière que:

THÉORÈME XV. "La figure formée par neuf points situés 3 à 3 dans 9 droites peut être considérée, de neuf manières différentes, comme résultante du théorème de Pascal appliqué au cas où la conique se réduit à deux droites."

Il y a, comme l'a remarqué M. Graves dans le mémoire cité du *Philosophical Magazine*, une autre manière assez singulière d'envisager la figure. Considérons pour cela les trois triangles 126, 489, 735, qui peuvent être dérivés assez simplement de

l'arrangement 135, 426, 789. Le premier de ces triangles est circonscrit au second; car les côtés 26, 61, 12 contiennent respectivement les points 4, 8, 9; également le second est circonscrit au troisième; et le troisième au premier. On tire encore du même arrangement 135, 426, 789 un autre système pareil de triangles; savoir 189, 435, 726. Pareillement les arrangements 129, 483, 756 et 186, 459, 723 donnent lieu chacun à deux systèmes analogues. Donc:

THÉORÈME XVI. La figure formée par neuf points situés 3 à 3 dans 9 droites peut être considérée, de six manières différentes, comme composée de trois triangles circonscrits cycliquement l'un à l'autre.

Représentons maintenant par 1, 2, 3, 4, 5, 6, 7, 8, 9 les neuf points d'inflexion d'une courbe du troisième ordre (six de ces points seront nécessairement imaginaires). Ces points sont, comme on sait, situés 3 à 3 dans douze droites qui passent réciproquement quatre à quatre par les neuf points, et qui peuvent être représentées par

$$135,\ 426,\ 789;\quad 129,\ 483,\ 756;\quad 186,\ 459,\ 723;\quad 147,\ 258,\ 369;$$

de sorte que ce système est un cas particulier de celui que nous venons de considérer. En considérant deux quelconques des droites qui ont un point en commun, par exemple 147, 186, et les deux autres droites qui passent par ce même point 135, 129, on obtient par là deux quadrilatères 4876 et 3952 qui ont pour centre commun le point 7 et dont chacun est circonscrit à l'autre. On aurait pu obtenir par ces mêmes droites 147, 186, 135, 129 des autres systèmes pareils; donc

THÉORÈME XVII. Huit points quelconques, parmi neuf points d'inflexion d'une courbe du troisième ordre, peuvent être considérés de trois manières différentes comme formant deux quadrilatères circonscrits l'un à l'autre. Les diagonales de ces six quadrilatères se réduisent à quatre droites qui passent par le neuvième point d'inflexion.

§ VI. *Sur les figures réciproques.*

Soient $\xi:\omega$, $\eta:\omega$, $\rho:\omega$ les coordonnées d'un point. Les deux plans définis par les équations

$$\left.\begin{array}{l}(a\ \xi+b\ \eta+c\ \rho+d\ \omega)\ x\\ +(a'\ \xi+b'\ \eta+c'\ \rho+d'\ \omega)\ y\\ +(a''\ \xi+b''\ \eta+c''\ \rho+d''\ \omega)\ z\\ +(a'''\xi+b'''\eta+c'''\rho+d'''\omega)\ w\end{array}\right\}=0,\ \text{et}\ \left.\begin{array}{r}(a\xi+a'\eta+a''\rho+a'''\omega)\ x\\ +(b\xi+b'\eta+b''\rho+b'''\omega)\ y\\ +(c\xi+c'\eta+c''\rho+c'''\omega)\ z\\ +(d\xi+d'\eta+d''\rho+d'''\omega)\ w\end{array}\right\}=0,$$

seront les *plans réciproques* du point donné.

Il peut arriver que le plan réciproque d'un point quelconque passe par ce point même; ce qui implique aussi l'identité des deux plans réciproques. Ce cas particulier a été l'objet des recherches de M. Möbius. Je parlerai dans la suite des réciproques de cette espèce en les appelant *Réciproques gauches.*

Mais généralement, pour que les plans réciproques d'un point passent par ce point même, il faut que le point soit situé sur une certaine surface du second ordre; et

alors tous les réciproques des points situés sur cette surface seront des plans tangents d'une autre surface du second ordre. Ces deux surfaces pourront être identiques; ce qui implique aussi l'identité des deux plans réciproques. Ce cas particulier constitue en effet la théorie connue des *Polaires réciproques.*

Je me propose ici d'examiner la théorie du cas général, où les deux surfaces ne sont point identiques. On pourra démontrer que dans ce cas les deux surfaces ont nécessairement en commun quatre droites: ces droites se rencontrent de manière à former un quadrilatère gauche que je représenterai par $ABCD$. Il est évident que les deux surfaces se touchent aux points A, B, C, D. En effet, le plan DAB est le plan tangent de l'une et de l'autre de ces surfaces au point A; et de même ABC, BCD et CDA sont respectivement les plans tangents aux points B, C et D. Les deux réciproques du point A se réduisent à ce même plan DAB, et il en est également pour les points B, C, D: il suit de là que les droites AC et BD sont réciproques l'une à l'autre, tandis que les droites AB, BC, CD, DA sont respectivement réciproques chacune à elle-même.

Les réciproques d'un point quelconque passent par la droite qui est l'intersection du plan polaire du point par rapport à la première surface, et de la réciproque gauche du point déterminé d'une manière qui sera expliquée tout à l'heure. Donc les réciproques d'un point quelconque de la première surface passent par la droite qui est l'intersection du plan tangent de la première surface au point dont il s'agit, et de la réciproque gauche du même point; de manière que si cette droite d'intersection était connue, les réciproques d'un point de la première surface seraient les plans tangents de la seconde surface menés par cette droite d'intersection; ou, pour trouver les réciproques d'un point de la première surface, on n'a qu'à chercher la réciproque gauche dont je viens de parler.

Dans ce système de réciproques gauches, les réciproques gauches des points A, B, C, D sont (comme dans le système des réciproques que nous considérons) les plans DAB, ABC, BCD et CAD, et de là les droites AC et DB sont réciproques gauches l'une à l'autre, tandis que les droites AB, BC, CD, DA sont réciproques gauches chacune à elle-même. Mais cela ne suffit pas pour déterminer le système des réciproques gauches. En effet, le système des réciproques que nous considérons n'est pas complètement déterminé au moyen des deux surfaces; il contient encore une quantité arbitraire (on peut facilement se satisfaire de cela). Or remarquons que la réciproque gauche d'un plan quelconque, passant par la droite AB (ou par l'une quelconque des droites AB, BC, CD, DA), est située dans cette même droite. Considérons un plan donné quelconque pAB passant par cette droite AB; la réciproque gauche de ce plan sera un point P de la droite AB, dont la position pourra être prise à volonté. Au moyen de ce plan pAB et de sa réciproque gauche P, sur la droite AB, on pourra facilement construire le système complète des réciproques gauches. Car soit q un point quelconque, et représentons par Q la réciproque gauche du plan qAB (Q sera aussi un point de la droite AB); considérons les quatre plans DAB, CAB, pAB, qAB qui se rencontrent selon la droite AB, et qui ont respectivement pour réciproques gauches les points A, B, P, Q (situés sur cette même droite AB): le rapport anharmonique des quatre

plans sera égal au rapport anharmonique des quatre points; ce qui suffit pour déterminer le point Q, puisque les quatres plans et les trois autres points sont donnés, et la construction graphique pour déterminer ce point Q est parfaitement connue. Il est d'ailleurs évident que la droite menée par un point donné, de sorte qu'elle rencontre des droites réciproques gauches l'une à l'autre, est située dans la réciproque gauche de ce point. Donc, en menant par le point q la droite qui rencontre les deux droites AC et BD, le plan passant par cette droite et par le point Q, est la réciproque gauche du point q qu'il s'agissait de trouver[1]. Également, on pourrait construire la réciproque gauche d'un plan donné.

Donc enfin, pour trouver les réciproques d'un point de la première surface:

(A) "Construisez le plan tangent à ce point de la première surface, et construisez de la manière expliquée ci-dessus, la réciproque gauche du point. Par la droite d'intersection de ces deux plans menez deux plans tangents à la seconde surface: ces deux plans seront les réciproques qu'il s'agissait de trouver."

On pourra construire d'une manière analogue les réciproques d'un plan tangent de la première surface. En effet:

(B) "En construisant les réciproques du point de contact avec la première surface du plan dont il s'agit, les points de contact avec la seconde surface, de ces deux plans, seront les réciproques dont il s'agissait."

Également, pour trouver les réciproques d'un plan tangent donné de la seconde surface:

(C) "Construisez le point de contact de ce plan avec la seconde surface, et construisez la réciproque gauche de ce même plan: la droite menée par ces deux points rencontrera la première surface dans deux points qui seront les réciproques que l'on désirait."

Et pour trouver les réciproques d'un point de la seconde surface:

(D) "Construisez les réciproques du plan tangent passant par le point donné de la seconde surface: les plans tangents à la première surface passant par ces deux points, seront les réciproques qu'il s'agissait de trouver."

En effet les théorèmes (C, D) ne sont que des transformations des théorèmes (A, B) au moyen de la théorie des polaires réciproques.

Enfin, pour trouver les réciproques d'un point quelconque, menez par ce point trois plans tangents ou à la première ou à la seconde surface, et construisez les réciproques de ces plans au moyen du théorème (B ou C): les deux plans menés par ces points réciproques, pris trois et trois ensemble, et combinés de manière que l'intersection des deux plans coïncide avec l'intersection de la polaire par rapport à la première surface et de la réciproque gauche du point donné, selon la remarque ci-dessus, seront les réciproques cherchées; et de la même manière pour les réciproques d'un plan quelconque.

[1] Il y a un cas très simple qui mérite d'être considéré; savoir celui où le point P coïncide avec A. Dans ce cas Q coïncide aussi avec A, et la réciproque gauche de q se réduit au plan qAC. De même, si les points P, B coïncident, la réciproque gauche de q se réduit au plan qBD.

Je n'essaierai pas d'énumérer ici le grand nombre de relations descriptives qui pourraient être tirées des constructions qu'on vient d'expliquer. L'on remarquera sans peine l'analogie parfaite qui existe pour toute cette théorie et la théorie correspondante de la géométrie à deux dimensions, telle que M. Plücker l'a exposée, "System der analytischen Geometrie," [4°, Berlin, 1835], pp. 78—83. On pourra aussi consulter sur ce point un mémoire [61] que je viens de composer pour le *Cambridge and Dublin Mathematical Journal*, et qui paraîtra prochainement.

La vérification analytique de ces théorèmes donne lieu à des développements assez intéressants. Pour obtenir l'équation de la première des surfaces du second ordre, il suffit d'écrire ξ, η, ρ, ω au lieu de x, y, z, w, dans l'équation de l'un ou de l'autre des plans réciproques. En supposant que $x=0$, $y=0$, $z=0$, $w=0$ soient des plans conjugués par rapport à la surface, et en remarquant que chacune de ces coordonnées peut être censée contenir un facteur constant, l'équation de la première surface peut être écrite sous la forme

$$x^2+y^2+z^2+w^2=0.$$

On aura alors pour a, b, c, d; a', b', c', d'; a'', b'', c'', d''; a''', b''', c''', d''' un système de la forme $l, -h, g, -a$; $h, l, -f, -b$; $-g, f, l, -c$; a, b, c, l, et les équations des plans réciproques deviendront

$$(\xi x+\eta y+\rho z+\omega w)\pm\left\{\begin{array}{l} x(\quad .\ -h\eta+g\rho-a\omega) \\ +y(\ h\xi\ .\quad -f\rho-b\omega) \\ +z(-g\xi+f\eta\quad .\ -cw) \\ +w(\ a\xi+b\eta+c\rho\quad .\ )\end{array}\right\}=0;$$

ce qui prouve le théorème énoncé ci-dessus; savoir que les deux réciproques passent par la droite d'intersection de la polaire et de la réciproque gauche du point. On obtient sans difficulté, pour l'équation de la seconde surface:

$$(x-hy+gz-aw)^2+(hx+y-fz-bw)^2+(-gx+fy+z-cw)^2+(ax+by+cz+w)^2=0,$$

ou sous une autre forme plus commode:

$$\begin{aligned}&(x^2+y^2+z^2+w^2)\\ &\quad+(.\ -hy+gz-aw)^2+(hx.-fz-bw)^2+(-gx+fy.-cw)^2+(ax+by+cz.)^2=0.\end{aligned}$$

Les coordonnées des points A, B, C, D seront déterminées au moyen des expressions

$$\frac{.\ -hy+gz-aw}{x}=\frac{hx.-fz-bw}{y}=\frac{-gx+fy.-cw}{z}=\frac{ax+by+cz.}{w},$$

ou, en introduisant la quantité indéterminée s:

$$\begin{aligned} sx-hy+gz-aw&=0,\\ hx+sy-fz-bw&=0,\\ -gx+fy+sz-cw&=0,\\ ax+by+cz+sw&=0.\end{aligned}$$

En effet, on démontrera aisément, non seulement que les points définis par ces équations sont situés dans l'intersection des deux surfaces, mais aussi que les deux surfaces se touchent dans ces points-ci; ce qui fait voir qu'en combinant convenablement les quatre points, les deux surfaces doivent se couper (comme nous l'avons déjà avancé), suivant les droites AB, BC, CD, DA.

Je vais examiner encore de plus près le système d'équations qui déterminent ces points. On en tire tout de suite, pour déterminer s, l'équation

$$s^4 + \mu s^2 + \Theta^2 = 0,$$

où l'on a fait pour abréger:

$$\mu = a^2 + b^2 + c^2 + f^2 + g^2 + h^2,$$
$$\Theta = af + bg + ch.$$

Supposons encore

$$s^2 f + a\Theta = A, \quad s^2 g + b\Theta = B, \quad s^2 h + c\Theta = C$$

et

$$s^2 a + f\Theta = F, \quad s^2 b + g\Theta = G, \quad s^2 c + h\Theta = H:$$

l'équation qui détermine s, pourra être écrite sous cette autre forme:

$$AF + BG + CH = 0.$$

J'ai trouvé que les équations des droites AC et BD peuvent être présentées sous les formes assez élégantes

$$\left.\begin{aligned} -Hy + Gz - Aw &= 0, \\ Hx \quad . \quad - Fz - Bw &= 0, \\ -Gx + Fy \quad . \quad - Cw &= 0, \\ Ax + By + Cz \quad . \quad &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} -Cy + Bz - Fw &= 0, \\ Cx \quad . \quad - Az - Gw &= 0, \\ -Bx + Ay \quad . \quad - Hw &= 0, \\ Fx + Gy + Hz \quad . \quad &= 0, \end{aligned}\right\}$$

où chacun de ces systèmes d'équations n'est équivalent qu'à un système de deux équations. En entrechangeant les racines de l'équation en s^2, c'est-à-dire en écrivant $\frac{\Theta^2}{s^2}$ au lieu de s^2, les deux systèmes ne font que s'entrechanger; comme en effet cela doit être.

Enfin, les équations des plans ACD, BAC, ou des plans ABD, BDC peuvent être exprimées sous les formes

$$PQ = (\,. - Hy + Gz - Aw)^2 + (Hx. - Fz - Bw)^2 + (-Gx + Fy\,. - Cw)^2 + (Ax + By + Cz\,.)^2 = 0,$$
$$RS = (\,. - Cy + Bz - Fw)^2 + (Cx\,. - Az - Gw)^2 + (-Bx + Ay\,. - Hw)^2 + (Fx + Gy + Hz\,.)^2 = 0\,;$$

équations desquelles on tire les relations identiques

$$PQ + RS = -s^2(\mu^2 - 4\Theta^2)(x^2 + y^2 + z^2 + w^2),$$
$$\Theta^2 PQ + s^4 RS = s^4(\mu^2 - 4\Theta^2)$$
$$\times [(\,. - hy + gz - aw)^2 + (hx \ldots fz - bw)^2 + (-gx - fy\,. - cw)^2 + (ax + by + cz\,.)^2],$$

qui mettent en évidence ce que l'on savait déjà, savoir, que les deux surfaces contiennent les droites AB, BC, CD, DA.

Mettons pour un moment, pour abréger:

$$\begin{aligned} &. \; -h\eta + g\rho - a\omega = l\,, \\ &h\xi \;\; . \; -f\rho - b\omega = m, \\ -g\xi - f\eta \;\; &. \; -c\omega = n\,, \\ a\xi + b\eta + c\rho \;\; &. \; = p\,, \end{aligned}$$

de sorte que l'équation de la réciproque gauche du point $(\xi, \eta, \rho, \omega)$ devient

$$lx + my + nz + pw = 0\,;$$

supposons de plus que, en combinant cette équation avec les équations des droites AC et BD respectivement, on obtient $x : y : z : w = X : Y : Z : W$ et $x : y : z : w = X_{,} : Y_{,} : Z_{,} : W_{,}$ respectivement. De là on tire

$$\left.\begin{aligned} X &= \;\; . \; -Cm + Bn - Fp\,, \\ Y &= Cl \;\; . \; -An - Gp\,, \\ Z &= -Bl + Am \;\; . \; -Hp, \\ W &= Fl + Gm + Hn \;\; . \end{aligned}\right\} \qquad \left.\begin{aligned} X_{,} &= \;\; . \; -Hm + Gn - Ap, \\ Y_{,} &= Hl \;\; . \; +Fn - Bp, \\ Z_{,} &= -Gl + Fm \;\; . \; -Cp\,, \\ W_{,} &= Al + Bm + Cn \;\; . \end{aligned}\right\}$$

et de là, identiquement:

$$\begin{aligned} \Theta(\Theta^2 - s^4)\,\xi &= \Theta X - s^2 X_{,}\,, \\ \Theta(\Theta^2 - s^4)\,\eta &= \Theta Y - s^2 Y_{,}\,, \\ \Theta(\Theta^2 - s^4)\,\rho &= \Theta Z - s^2 Z_{,}\,, \\ \Theta(\Theta^2 - s^4)\,\omega &= \Theta W - s^2 W_{,}\,; \end{aligned}$$

ce qui correspond en effet au théorème énoncé ci-dessus, savoir que la réciproque gauche d'un point rencontre les droites AC, BD en deux points tels que la droite passant par ces deux points passe par le point même. Les démonstrations des différents théorèmes d'analyse dont je me sers, n'ont point de difficultés.

71.

NOTE SUR LES FONCTIONS DU SECOND ORDRE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXVIII. (1848), pp. 105—106.]

SOIENT $x, y, \ldots$ des variables dont le nombre est $2n$ ou $2n+1$; représentons par $\xi, \eta, \ldots$ un nombre égal de fonctions linéaires des variables $x, y, \ldots$ et soit

$$\begin{aligned} U &= \xi x + \eta y + \ldots \\ &= Ax^2 + By^2 + \ldots + 2Hxy + \ldots \end{aligned}$$

et

$$V = \begin{vmatrix} \cdot & \xi, & \eta \ldots \\ \xi, & A, & H \\ \eta, & H, & B \\ \vdots & & \end{vmatrix}.$$

J'ai trouvé que les deux fonctions U et V peuvent être réduites à la forme

$$\begin{aligned} U &= \lambda\,\Omega + \mu\,\Theta, \\ V &= \lambda'\Omega + \mu'\Theta, \end{aligned}$$

où $\lambda, \mu, \lambda', \mu'$ sont des coefficients constants, Ω est une fonction du second ordre des n variables (fonctions linéaires de $x, y, \ldots$), et Θ est une fonction de n ou de $n+1$ variables (fonctions linèaires de $x, y, \ldots$); selon que le nombre des variables $x, y, \ldots$ est $2n$ ou $2n+1$.

Par exemple pour trois variables x, y, z, on a

$$\begin{aligned} U &= \lambda x^2 + \mu yz, \\ V &= \lambda' x^2 + \mu' yz; \end{aligned}$$

et les coniques représentées par les équations $U=0$, $V=0$ ont entre elles un contact double. Cela se rapporte à la théorie des réciproques dans la géométrie plane, telle que M. Plücker l'a présentée dans son "System der analytischen Geometrie" [4°, Berlin, 1835].

De même pour quatre variables x, y, z, w on a

$$U = \lambda xy + \mu zw,$$
$$V = \lambda' xy + \mu' zw,$$

et les surfaces représentées par les équations $U=0$, $V=0$ se coupent dans quatre droites, ou bien se touchent aux quatre sommets d'un quadrilatère gauche. Cela se rapporte également à la théorie des réciproques dans l'espace: théorie dont j'ai parlé dans mon Mémoire "Sur quelques théorèmes de la géométrie de position," § VI, [70].

Il y a à remarquer que dans le cas où les coefficients de x, y, ... dans les fonctions ξ, η, ... forment un système symétrique, c'est-à-dire, où

$$\xi = Ax + Hy + \dots,$$
$$\eta = Hx + By + \dots,$$
$$\vdots$$

on a $\lambda : \mu = \lambda' : \mu'$, et de là $V = KU$, K étant donné par l'équation

$$K = \begin{vmatrix} A, & H, \dots \\ H, & B, \\ \vdots & \end{vmatrix}$$

ce qui est connu.

Remarquons enfin que dans le cas général où

$$\xi = ax + by + \dots,$$
$$\eta = a'x + b'y + \dots,$$
$$\vdots$$

la fonction V ne change pas de valeur en écrivant au lieu de ces expressions:

$$\xi = ax + a'y + \dots,$$
$$\eta = bx + b'y + \dots,$$
$$\vdots$$

propriété qui se rapporte aussi à la théorie des réciproques.

72.

NOTE ON THE THEORY OF PERMUTATIONS.

[From the *Philosophical Magazine*, vol. XXXIV. (1849), pp. 527—529.]

It seems worth inquiring whether the distinction made use of in the theory of determinants, of the permutations of a series of things all of them different, into positive and negative permutations, can be made in the case of a series of things not all of them different. The ordinary rule is well known, viz. permutations are considered as positive or negative according as they are derived from the primitive arrangement by an even or an odd number of inversions (that is, interchanges of two things); and it is obvious that this rule fails when two or more of the series of things become identical, since in this case any given permutation can be derived indifferently by means of an even or an odd number of inversions. To state the rule in a different form, it will be convenient to enter into some preliminary explanations. Consider a series of n things, all of them different, and let $abc\ldots$ be the primitive arrangement; imagine a symbol such as $(xyz)\ (u)\ (vw)\ldots$ where x, y, &c., are the entire series of n things, and which symbol is to be considered as furnishing a rule by which a permutation is to be derived from the primitive arrangement $abc\ldots$ as follows, viz. the (xyz) of the symbol denotes that the letters x, y, z in the primitive arrangement $abc\ldots$ are to be interchanged x into y, y into z, z into x. The (u) of the symbol denotes that the letter u in the primitive arrangement $abc\ldots$ is to remain unaltered. The (vw) of the symbol denotes that the letters v, w in the primitive arrangement are to be interchanged v into w and w into v, and so on. It is easily seen that any permutation whatever can be derived (and derived in one manner only) from the primitive arrangement by means of a rule such as is furnished by the symbol in question[1]; and moreover that the number of inversions requisite in order to obtain the permutation by means of the rule in question, is always the smallest number of

[1] See on this subject Cauchy's "Mémoire sur les Arrangemens &c.", *Exercises d'Analyse et de Physique Mathématique*, t. III. [1844], p. 151.

inversions by which the permutation can be derived. Let α, β ... be the number of letters in the components (xyz), (u) (vw), &c., λ the number of these components. The number of inversions in question is evidently $\overline{\alpha-1}+\overline{\beta-1}+$&c., or what comes to the same thing, this number is $(n-\lambda)$. It will be convenient to term this number λ the exponent of irregularity of the permutation, and then $(n-\lambda)$ may be termed the supplement of the exponent of irregularity. The rule in the case of a series of things, all of them different, may consequently be stated as follows: "a permutation is positive or negative according as the supplement of the exponent of irregularity is even or odd." Consider now a series of things, not all of them different, and suppose that this is derived from the system of the same number of things *abc* ... all of them originally different, by supposing for instance $a=b=$&c., $f=g=$&c. A given permutation of the system of things not all of them different, is of course derivable under the supposition in question from several different permutations of the series *abc* .... Considering the supplements of the exponents of irregularity of these last-mentioned several permutations, we may consider the given permutation as positive or negative according as the *least of these numbers* is even or odd. Hence we obtain the rule, "a permutation of a series of things not all of them different, is positive or negative according as the minimum supplement of irregularity of the permutation is even or odd, the system being considered as a particular case of a system of the same number of things all of them different, and the given permutation being successively considered as derived from the different permutations which upon this supposition reduce themselves to the given permutation." This only differs from the rule, "a permutation of a series of things, not all of them different, is positive or negative according as the minimum number of inversions by which it can be obtained is even or odd, the system being considered &c.," inasmuch as the former enunciation is based upon and indicates a direct method of determining the minimum number of inversions requisite in order to obtain a given permutation; but the latter is, in simple cases, of the easier application. As a very simple example, treated by the former rule, we may consider the permutation 1212 derived from the primitive arrangement 1122. Considering this primitive arrangement as a particular case of *abcd*, there are four permutations which, on the suppositions $a=b=1$, $c=d=2$, reduce themselves to 1212, viz. *acbd, bcad, adbc, bdac,* which are obtained by means of the respective symbols $(a)(bc)(d)$; $(abc)(d)$; $(a)(bdc)$; $(abdc)$, the supplements of the exponents of irregularity being therefore 1, 2, 2, 3, or the permutation being negative; in fact it is obviously derivable by means of an inversion of the two mean terms.

73.

ABSTRACT OF A MEMOIR BY DR HESSE ON THE CONSTRUCTION OF THE SURFACE OF THE SECOND ORDER WHICH PASSES THROUGH NINE GIVEN POINTS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 44—46.]

THE construction to be presently given of the surface of the second order which passes through nine given points, is taken from a memoir by Dr Hesse (*Crelle*, t. XXIV. [1842], p. 36). It depends upon the following lemma, which is there demonstrated.

LEMMA. The polar plane of a fixed point P with respect to any surface of the second order passing through seven given points, passes through a fixed point Q (which may be termed the harmonic pole of the point P with respect to the system of surfaces of the second order).

PROBLEM. Given the seven points 1, 2, 3, 4, 5, 6, 7, and a point P, to construct the harmonic pole Q of the point P with respect to the system of surfaces of the second order passing through the seven points.

The required point Q may be considered as the intersection of the polar planes of the point P with respect to any three hyperboloids, each of which passes through the seven given points; any such hyperboloid may be considered as determined by means of three of its generating lines. These considerations lead to the construction following.

1. Connecting the points 1 and 2, and also the points 3 and 4, by two straight lines, and determining the three lines, each of which passes through one of the points 5, 6, 7, and intersects both of the first-mentioned lines, the three lines so determined are generating lines of a hyperboloid passing through the seven points.

Two other systems of generating lines (belonging to two new hyperboloids) are determined by the like construction, interchanging the points 1, 2, 3, 4. And by interchanging all the seven points we obtain 105 systems of generating lines (belonging to as many different hyperboloids, unless some of these hyperboloids are identical).

2. It remains to be shown how the polar plane of the point P with respect to one of the 105 hyperboloids may be constructed. Drawing through the point P three lines, each of which passes through two of the three given generating lines of the hyperboloid in question, the points of intersection of the lines so determined with the generating lines which they respectively intersect, are points of the hyperboloid. Hence, constructing upon each of the three lines in question the harmonic pole of the point P with respect to the two points of intersection, the plane passing through the three harmonic poles is the polar plane of P with respect to the hyperboloid. Hence, constructing the polar planes of P with respect to any three of the 105 hyperboloids, the point of intersection of these three polar planes is the required point Q.

PROBLEM. To construct the polar plane of a point P with respect to the surface of the second order which passes through nine given points 1, 2, 3, 4, 5, 6, 7, 8, 9.

Consider any seven of the nine points, e.g. the points 1, 2, 3, 4, 5, 6, 7, and construct the harmonic pole of the point P with respect to the system of surfaces of the second order passing through these seven points. By permuting the different points we obtain 36 different points Q, all of which lie in the same plane. This plane (which is of course determined by any three of the thirty-six points) is the required polar plane. Hence we obtain the solution of

PROBLEM. To construct the surface of the second order which passes through nine given points 1, 2, 3, 4, 5, 6, 7, 8, 9.

Assuming the point P arbitrarily, construct the polar plane of this point with respect to the surface of the second order passing through the nine points. Join the point P with any one of the nine points, e.g. the point 1, and on the line so formed determine the harmonic pole R of the point 1 with respect to the point P, and the point where the line $P1$ is intersected by the polar plane. R is a point of the required surface of the second order, which surface is therefore determined by giving every possible position to the point P.

This construction is the complete analogue of Pascal's theorem *considered as a construction for describing the conic section which passes through five given points.* And it would appear that the principles by means of which the construction is obtained ought to lead to the analogue of Pascal's theorem considered in its ordinary form, that is, *as a relation between six points of a conic,* or in other words to the solution of the problem to determine the relation between ten points of a surface of the second order; but this problem, one of the most interesting in the theory of surfaces of the second order, remains as yet unsolved. The problem last mentioned was proposed as a prize question by the Brussels Academy, which subsequently proposed the more general

question to determine the analogue of Pascal's theorem for surfaces of the second order. This of course admitted of being answered in a variety of different ways, according to the different ways of viewing the theorem of Pascal. Thus, M. Chasles, considering Pascal's theorem as a property of a conic intersected by the three sides of a triangle, discovered the following very elegant analogous theorem for surfaces of the second order.

"The six edges of a tetrahedron may be considered as intersecting a surface of the second order in twelve points lying three and three upon four planes, each one of which contains three points lying on edges which pass through the same angle of the tetrahedron; these planes meet the faces opposite to these angles in four straight lines which are generating lines (of the same species) of a certain hyperboloid."

It is hardly necessary to remark that all the properties involved in the present memoir are such as to admit of being transformed by the theory of reciprocal polars.

74.

ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 47—50.]

THE theory of the simultaneous transformation by linear substitutions of two homogeneous functions of the second order has been developed by Jacobi in the memoir "De binis quibuslibet functionibus &c., *Crelle*, t. XII. [1834], p. 1; but the simplest method of treating the problem is the one derived from Mr Boole's Theory of Linear Transformations, combined with the remark in his "Notes on Linear Transformations," in the *Cambridge Mathematical Journal*, vol. IV. [1845], p. 167. As I shall have occasion to refer to the results of this theory in the second part of my paper "On the Attraction of Ellipsoids," in the present number of the *Journal* [75], I take this opportunity of developing the formula in question; considering for greater convenience the case of three variables only.

Suppose that by a linear transformation,

$$\begin{aligned} x &= \alpha\, x_1 + \beta\, y_1 + \gamma\, z_1, \\ y &= \alpha'\, x_1 + \beta'\, y_1 + \gamma'\, z_1, \\ z &= \alpha'' x_1 + \beta'' y_1 + \gamma'' z_1, \end{aligned}$$

we have identically,

$$\begin{aligned} ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy &= a_1x_1^2 + b_1y_1^2 + c_1z_1^2 + 2f_1y_1z_1 + 2g_1z_1x_1 + 2h_1x_1y_1, \\ Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy &= A_1x_1^2 + B_1y_1^2 + C_1z_1^2 + 2F_1y_1z_1 + 2G_1z_1x_1 + 2H_1x_1y_1. \end{aligned}$$

Of course, whatever be the values of a, b, c, f, g, h, the same transformation gives

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}zx + 2\mathrm{h}xy = \mathrm{a}_1x_1^2 + \mathrm{b}_1y_1^2 + \mathrm{c}_1z_1^2 + 2\mathrm{f}_1y_1z_1 + 2\mathrm{g}_1z_1x_1 + 2\mathrm{h}_1x_1y_1,$$

provided that we have

$$\begin{cases} \mathrm{a}_1 = \mathrm{a}\alpha^2 + \mathrm{b}\alpha'^2 + \mathrm{c}\alpha''^2 + 2\mathrm{f}\alpha'\alpha'' + 2\mathrm{g}\alpha''\alpha + 2\mathrm{h}\alpha\alpha', \\ \mathrm{b}_1 = \mathrm{a}\beta^2 + \mathrm{b}\beta'^2 + \mathrm{c}\beta''^2 + 2\mathrm{f}\beta'\beta'' + 2\mathrm{g}\beta''\beta + 2\mathrm{h}\beta\beta', \\ \mathrm{c}_1 = \mathrm{a}\gamma^2 + \mathrm{b}\gamma'^2 + \mathrm{c}\gamma''^2 + 2\mathrm{f}\gamma'\gamma'' + 2\mathrm{g}\gamma''\gamma + 2\mathrm{h}\gamma\gamma', \\ \mathrm{f}_1 = \mathrm{a}\beta\gamma + \mathrm{b}\beta'\gamma' + \mathrm{c}\beta''\gamma'' + \mathrm{f}(\beta'\gamma'' + \beta''\gamma') + \mathrm{g}(\beta''\gamma + \beta\gamma'') + \mathrm{h}(\beta\gamma' + \beta'\gamma), \\ \mathrm{g}_1 = \mathrm{a}\gamma\alpha + \mathrm{b}\gamma'\alpha' + \mathrm{c}\gamma''\alpha'' + \mathrm{f}(\gamma'\alpha'' + \gamma''\alpha') + \mathrm{g}(\gamma''\alpha + \gamma\alpha'') + \mathrm{h}(\gamma\alpha' + \gamma'\alpha), \\ \mathrm{h}_1 = \mathrm{a}\alpha\beta + \mathrm{b}\alpha'\beta' + \mathrm{c}\alpha''\beta'' + \mathrm{f}(\alpha'\beta'' + \alpha''\beta') + \mathrm{g}(\alpha''\beta + \alpha\beta'') + \mathrm{h}(\alpha\beta' + \alpha'\beta). \end{cases}$$

Representing for a moment the equations between the pairs of functions of the second order by

$$u = u_1, \quad U = U_1, \quad v = v_1,$$

we have, whatever be the value of λ,

$$\lambda u + U + v = \lambda u_1 + U_1 + v_1.$$

Hence, if

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \Pi; \text{ then}$$

$$\begin{vmatrix} \lambda a_1 + A_1 + \mathrm{a}_1, & \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda g_1 + G_1 + \mathrm{g}_1 \\ \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda b_1 + B_1 + \mathrm{b}_1, & \lambda f_1 + F_1 + \mathrm{f}_1 \\ \lambda g_1 + G_1 + \mathrm{g}_1, & \lambda f_1 + F_1 + \mathrm{f}_1, & \lambda c_1 + C_1 + \mathrm{c}_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} \lambda a + A + \mathrm{a}, & \lambda h + H + \mathrm{h}, & \lambda g + G + \mathrm{g} \\ \lambda h + H + \mathrm{h}, & \lambda b + B + \mathrm{b}, & \lambda f + F + \mathrm{f} \\ \lambda g + G + \mathrm{g}, & \lambda f + F + \mathrm{f}, & \lambda c + C + \mathrm{c} \end{vmatrix}.$$

Hence, since a, b, c, f, g, h, are arbitrary,

$$\begin{vmatrix} \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix}$$

which determine the relations which must subsist between the coefficients of the functions of the second order. We derive

$$\begin{vmatrix} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

and by comparing the coefficients of a, &c., if we write for shortness,

$$\mathfrak{A} = \begin{vmatrix} 1, & . & . \\ ., & \lambda b + B, & \lambda f + F \\ ., & \lambda f + F, & \lambda c + C \end{vmatrix} \text{ \&c., then}$$

$$\mathfrak{A}_1\alpha^2 + \mathfrak{B}_1\beta^2 + \mathfrak{C}_1\gamma^2 + 2\mathfrak{F}_1\beta\gamma + 2\mathfrak{G}_1\gamma\alpha + 2\mathfrak{H}_1\alpha\beta = \Pi^2\mathfrak{A},$$
$$\mathfrak{A}_1\alpha'^2 + \mathfrak{B}_1\beta'^2 + \mathfrak{C}_1\gamma'^2 + 2\mathfrak{F}_1\beta'\gamma' + 2\mathfrak{G}_1\gamma'\alpha' + 2\mathfrak{H}_1\alpha'\beta' = \Pi^2\mathfrak{B},$$
$$\mathfrak{A}_1\alpha''^2 + \mathfrak{B}_1\beta''^2 + \mathfrak{C}_1\gamma''^2 + 2\mathfrak{F}_1\beta''\gamma'' + 2\mathfrak{G}_1\gamma''\alpha'' + 2\mathfrak{H}_1\alpha''\beta'' = \Pi^2\mathfrak{C},$$

$$\mathfrak{A}_1\alpha'\alpha'' + \mathfrak{B}_1\beta'\beta'' + \mathfrak{C}_1\gamma'\gamma'' + \mathfrak{F}_1(\beta'\gamma'' + \beta''\gamma') + \mathfrak{G}_1(\gamma'\alpha'' + \gamma''\alpha') + \mathfrak{H}_1(\alpha'\beta'' + \alpha''\beta') = \Pi^2\mathfrak{F},$$
$$\mathfrak{A}_1\alpha''\alpha + \mathfrak{B}_1\beta''\beta + \mathfrak{C}_1\gamma''\gamma + \mathfrak{F}_1(\beta''\gamma + \beta\gamma'') + \mathfrak{G}_1(\gamma''\alpha + \gamma\alpha'') + \mathfrak{H}_1(\alpha''\beta + \alpha\beta'') = \Pi^2\mathfrak{G},$$
$$\mathfrak{A}_1\alpha\alpha' + \mathfrak{B}_1\beta\beta' + \mathfrak{C}_1\gamma\gamma' + \mathfrak{F}_1(\beta\gamma' + \beta'\gamma) + \mathfrak{G}_1(\gamma\alpha' + \gamma'\alpha) + \mathfrak{H}_1(\alpha\beta' + \alpha'\beta) = \Pi^2\mathfrak{H},$$

each of which virtually contains three equations on account of the indeterminate quantity λ. A somewhat more elegant form may be given to these equations; thus the first of them is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, \\ \alpha, & \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \beta, & \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \gamma, & \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} 1, & . & . \\ ., & \lambda b + B, & \lambda f + F \\ ., & \lambda f + F, & \lambda c + C \end{vmatrix}$$

from which the form of the whole system is sufficiently obvious. The actual values of the coefficients α, β, &c. can only be obtained in the particular case where $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$. If we suppose besides (which is no additional loss of generality) that $a_1 = b_1 = c_1 = 1$, then the whole system of formulæ becomes

$$(A_1 + \lambda)(B_1 + \lambda)(C_1 + \lambda) = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix};$$

$$1 = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ or } \Pi^2 = \kappa^{-1} \text{ suppose; and then}$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha^2 + (C_1 + \lambda)(A_1 + \lambda)\beta^2 + (A_1 + \lambda)(B_1 + \lambda)\gamma^2 = \frac{1}{\kappa}\mathfrak{A},$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha'^2 + (C_1 + \lambda)(A_1 + \lambda)\beta'^2 + (A_1 + \lambda)(B_1 + \lambda)\gamma'^2 = \frac{1}{\kappa}\mathfrak{B},$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha''^2 + (C_1 + \lambda)(A_1 + \lambda)\beta''^2 + (A_1 + \lambda)(B_1 + \lambda)\gamma''^2 = \frac{1}{\kappa}\mathfrak{C},$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha'\alpha'' + (C_1 + \lambda)(A_1 + \lambda)\beta'\beta'' + (A_1 + \lambda)(B_1 + \lambda)\gamma'\gamma'' = \frac{1}{\kappa}\mathfrak{F},$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha''\alpha + (C_1 + \lambda)(A_1 + \lambda)\beta''\beta + (A_1 + \lambda)(B_1 + \lambda)\gamma''\gamma = \frac{1}{\kappa}\mathfrak{G},$$

$$(B_1 + \lambda)(C_1 + \lambda)\alpha\alpha' + (C_1 + \lambda)(A_1 + \lambda)\beta\beta' + (A_1 + \lambda)(B_1 + \lambda)\gamma\gamma' = \frac{1}{\kappa}\mathfrak{H},$$

where, writing down the expanded values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$,

$$\begin{aligned}
(\lambda b + B)(\lambda c + C) - (\lambda f + F)^2 &= \mathfrak{A}, \\
(\lambda c + C)(\lambda a + A) - (\lambda g + G)^2 &= \mathfrak{B}, \\
(\lambda a + A)(\lambda b + B) - (\lambda h + H)^2 &= \mathfrak{C}, \\
(\lambda g + G)(\lambda h + H) - (\lambda a + A)(\lambda f + F) &= \mathfrak{F}, \\
(\lambda h + H)(\lambda f + F) - (\lambda b + B)(\lambda g + G) &= \mathfrak{G}, \\
(\lambda f + F)(\lambda g + G) - (\lambda c + C)(\lambda h + H) &= \mathfrak{H}.
\end{aligned}$$

By writing successively $\lambda = -A_1$, $\lambda = -B_1$, $\lambda = -C_1$, we see in the first place that A_1, B_1, C_1 are the roots of the same cubic equation, and we obtain next the values of α^2, β^2, γ^2, &c. in terms of these quantities A_1, B_1, C_1, and of the coefficients a, b, &c., A, B, &c. It is easy to see how the above formulæ would have been modified if a_1, b_1, c_1, instead of being equal to unity, had one or more of them been equal to unity with a negative sign. It is obvious that every step of the preceding process is equally applicable whatever be the number of variables.

75.

ON THE ATTRACTION OF AN ELLIPSOID.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 50—65.]

PART I.—ON LEGENDRE'S SOLUTION OF THE PROBLEM OF THE ATTRACTION OF AN ELLIPSOID ON AN EXTERNAL POINT.

I PROPOSE in the following paper to give an outline of Legendre's investigation of the attraction of an ellipsoid upon an exterior point, ["Mémoire sur les Intégrales Doubles," Paris, Mem. Acad. Sc. for 1788, published 1791, pp. 454—486], one of the earliest and (notwithstanding its complexity) most elegant solutions of the problem. It will be convenient to begin by considering some of the geometrical properties of a system of cones made use of in the investigation.

§ 1. The equation of the ellipsoid referred to axes parallel to the principal axes, and passing through the attracted point, may be written under the form

$$l(x-a)^2+m(y-b)^2+n(z-c)^2-k=0,$$

(where $\sqrt{\frac{k}{l}}$, $\sqrt{\frac{k}{m}}$, $\sqrt{\frac{k}{n}}$ are the semiaxes, and a, b, c are the coordinates of the attracted point referred to the principal axes). Or putting $la^2+mb^2+nc^2-k=\delta$, this equation becomes

$$lx^2+my^2+nz^2-2(lax+mby+ncz)+\delta=0.$$

The cones in question are those which have the same axes and directions of circular section as the cone having its vertex in the attracted point and circumscribed about the ellipsoid. The equation of the system of cones (containing the arbitrary parameter ω) is

$$(lx^2+my^2+nz^2)\,\delta-(lax+mby+ncz)^2+\omega^2(x^2+y^2+z^2)=0\,;$$

or as it may also be written,

$$(\omega^2+l\delta-l^2a^2)\,x^2+(\omega^2+m\delta-m^2b^2)\,y^2+(\omega^2+n\delta-n^2c^2)\,z^2-2mnbcyz-2nlcazx-2lmabxy=0.$$

For $\omega = 0$, the cone coincides with the circumscribed cone; as ω increases, the aperture of the cone gradually diminishes, until for a certain value, $\omega = \Omega$, the cone reduces itself to a straight line (the normal of the confocal ellipsoid through the attracted point). It is easily seen that Ω^2 is the positive root of the equation

$$\frac{l^2a^2}{\Omega^2 + l\delta} + \frac{m^2b^2}{\Omega^2 + m\delta} + \frac{n^2c^2}{\Omega^2 + n\delta} = 1,$$

a different form of which may be obtained by writing $\Omega^2 = \frac{k\delta}{\xi}$, ξ being then determined by means of the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1;$$

that is,

$$\sqrt{\left(\frac{k}{l} + \xi\right)}, \quad \sqrt{\left(\frac{k}{m} + \xi\right)}, \quad \sqrt{\left(\frac{k}{n} + \xi\right)}$$

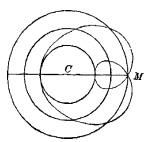
are the semiaxes of the confocal ellipsoid through the attracted point.

In the case where ω remains indeterminate, it is obvious that the cone intersects the ellipsoid in the curve in which the ellipsoid is intersected by a certain hyperboloid of revolution of two sheets, having the attracted point for a focus, and the plane of contact of the ellipsoid with the circumscribed cone (that is the polar plane of the attracted point) for the corresponding directrix plane: also the excentricity of the hyperboloid is $\frac{1}{\omega}\sqrt{(l^2a^2 + m^2b^2 + n^2c^2)}$, which suffices for its complete determination. For $\omega = 0$, the hyperboloid reduces itself to the plane of contact of the ellipsoid with the circumscribed cone, and for $\omega = \Omega$, the hyperboloid and the ellipsoid have a double contact, viz. at the points where the ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point.

If ω remains constant while k is supposed to vary, that is, if the ellipsoid vary in magnitude (the position and proportion of its axes remaining unaltered), the locus of the intersection of the cone and the ellipsoid is a surface of the fourth order defined by the equation

$$(lx^2 + my^2 + nz^2 - lax - mby - ncz)^2 = \omega^2(x^2 + y^2 + z^2),$$

and consisting of an exterior and an interior sheet meeting at the attracted point, which is a conical point on the surface, viz. a point where the tangent plane is



replaced by a tangent cone. The general form of this surface is easily seen from the figure, in which the ellipsoid has been replaced by a sphere, and the surface in question

is that generated by the revolution of the curve round the line CM. The surface of the fourth order being once described for any particular value of ω, the cone corresponding to any one of the series of similar, similarly situated, and concentric ellipsoids is at once determined by means of the intersection of the ellipsoid in question with the surface of the fourth order. It is clear too that there is always one of these ellipsoids which has a double contact with the surface of the fourth order, viz. at the points where this ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point; thus there is always an ellipsoid for which the cone corresponding to a given value of ω reduces itself to a straight line.

Consider the attracting ellipsoid, which for distinction may be termed the ellipsoid S, and the two cones C, C', which correspond to the values ω, $\omega - d\omega$ of the variable parameter. Legendre shows that the attraction of the portion of the ellipsoid S included between the two cones C, C' is independent of the quantity k, which determines the magnitude of the ellipsoid: that is, if there be any other ellipsoid T similarly situated and concentric to and with the ellipsoid S, and two cones D, D', which for the ellipsoid T correspond to the same values ω, $\omega - d\omega$ of the variable parameter; then the attraction of the portion of the ellipsoid S, included between the two cones C, C', is equal to the attraction of the portion of the ellipsoid T included between the two cones D and D'. By taking for the ellipsoid T the ellipsoid for which the cone D reduces itself to a straight line, the aperture of the cone D' is indefinitely small, and the attraction of the portion of the ellipsoid T included within the cone D' is at once determined; and thus the attraction of the portion of the ellipsoid S included between the cones C, C' is obtained in a finite form. Hence the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$ corresponding to the values $\omega_{,}$, $\omega_{,,}$ of the variable parameter, is expressed by means of a single integral, and by extending the integration from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, the attraction of the whole ellipsoid is obtained in the form of a single integral readily reducible to that given by the ordinary solutions. It is clear too that the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$, is equal to that of the portion of the ellipsoid T included between the corresponding cones $D_{,}$ and $D_{,,}$. Hence also, assuming for the ellipsoid T, that for which the cone $D_{,,}$ reduces itself to a straight line, and supposing that the cones $C_{,}$ and $D_{,}$ coincide with the circumscribing cones, the attraction of the portion of the ellipsoid S exterior to the cone $C_{,,}$ is equal to the attraction of the entire ellipsoid T. More generally, the attraction of the portion of the ellipsoid S included between the cones $C_{,}$ and $C_{,,}$ is equal to the attraction of the shell included between the surfaces of the two ellipsoids, for which the cones $D_{,}$ and $D_{,,}$ respectively reduce themselves to straight lines.

§ 2. Proceeding to the analytical solution, and resuming the equation of the ellipsoid

$$lx^2 + my^2 + nz^2 - 2(lax + mby + ncz) + \delta = 0,$$

and that of the cone

$$(lx^2 + my^2 + nz^2)\,\delta - (lax + mby + ncz)^2 + \omega^2 (x^2 + y^2 + z^2) = 0\,;$$

consider a radius vector on the conical surface such that the cosines of its inclinations to the axes are

$$\frac{P}{\Theta}, \frac{Q}{\Theta}, \frac{R}{\Theta}, \{\Theta = \sqrt{(P^2 + Q^2 + R^2)}\},$$

P, Q, R and Θ being functions of the parameter ω, and of another variable ϕ, which determines the position of the radius vector upon the conical surface. Also let ρ be the length of the portion of the radius vector which lies within the ellipsoid; then representing by dS the spherical angle corresponding to the variations of ω and ϕ, the attraction in the direction of the axis of x is given by the formula

$$A = \iint \rho \frac{P}{\Theta} dS.$$

Also by a known formula

$$dS = \frac{1}{\Theta^2}\left\{P\left(\frac{dQ}{d\phi}\frac{dR}{d\omega} - \frac{dR}{d\phi}\frac{dQ}{d\omega}\right) + Q\left(\frac{dR}{d\phi}\frac{dP}{d\omega} - \frac{dR}{d\omega}\frac{dP}{d\phi}\right) + R\left(\frac{dP}{d\phi}\frac{dQ}{d\omega} - \frac{dP}{\delta\omega}\frac{dQ}{d\phi}\right)\right\},$$

and it is easy to obtain

$$\rho = \frac{2\omega\Theta^2}{lP^2 + mQ^2 + nR^2}.$$

The quantities P, Q, R have now to be expressed as functions of ω, ϕ, so that their values substituted for x, y, z, may satisfy identically the equation of the cone. This may be done by assuming

$$P = p,$$

$$Q = mb\,(\omega^2 + n\delta)\left(la + \frac{D}{U}\cos\phi\right) - \frac{D\sqrt{p}}{U} nc \sin\phi,$$

$$R = nc\,(\omega^2 + m\delta)\left(la + \frac{D}{U}\cos\phi\right) + \frac{D\sqrt{p}}{U} mb \sin\phi,$$

where

$$p = (\omega^2 + m\delta)(\omega^2 + n\delta) - m^2b^2(\omega^2 + n\delta) - n^2c^2(\omega^2 + m\delta),$$

$$U^2 = m^2b^2(\omega^2 + n\delta) + n^2c^2(\omega^2 + m\delta),$$

$$D^2 = (\omega^2 + l\delta)(\omega^2 + m\delta)(\omega^2 + n\delta)\left\{\frac{l^2a^2}{\omega^2 + l\delta} + \frac{m^2b^2}{\omega^2 + m\delta} + \frac{n^2c^2}{\omega^2 + n\delta} - 1\right\};$$

a system of values which, in point of fact, depend upon the following geometrical considerations: by treating x as a constant in the equation of the cone, that is, in effect by considering the sections of the cone by planes parallel to that of yz, the equation of the cone becomes that of an ellipse; transforming first to a set of axes through the centre and then to a set of conjugate axes, one of which passes through the point where the plane of the ellipse is intersected by the axis of x, then the equation takes the form $\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} = 1$, and is satisfied by $\xi = A\cos\phi$, $\eta = B\sin\phi$, and $\frac{y}{x}$, $\frac{z}{x}$ being of course linear functions of these values, the preceding expressions may be obtained.

The substitution of the above values of P, Q, R (a somewhat tedious one which does not occur in the process actually made use of by Legendre) gives the very simple result,

$$dS = \frac{P^{\frac{1}{2}}\omega\, d\omega\, d\phi}{\Theta};$$

and the formula for the attraction becomes

$$A = 2\iint \frac{P^{\frac{3}{2}}\omega^2\, d\omega\, d\phi}{lP^2 + mQ^2 + nR^2},$$

which is of the form $A = 2\int I\omega^2 d\omega$, where

$$I = \int \frac{P^{\frac{3}{2}} d\phi}{lP^2 + mQ^2 + nR^2},$$

which last integral, taken between the limits $\phi = 0$ and $\phi = 2\pi$, and multiplied by $2\omega^2 d\omega$, expresses the attraction of the portion of the ellipsoid included between two consecutive cones. The integration is evidently possible, but the actual performance of it is the great difficulty of Legendre's process. The result, as before mentioned, is independent of the quantity k, or, what comes to the same thing, of the quantity δ: assuming this property (an assumption which in fact resolves itself into the consideration of the ellipsoid for which the cone reduces itself to a straight line, as before explained), the integral is at once obtained by writing $\delta = \Delta$ where Δ represents the positive root of the equation

$$\frac{l^2a^2}{\omega^2 + l\Delta} + \frac{m^2b^2}{\omega^2 + m\Delta} + \frac{n^2c^2}{\omega^2 + n\Delta} - 1 = 0.$$

This gives

$$P = \frac{l^2a^2(\omega^2 + m\Delta)(\omega^2 + n\Delta)}{\omega^2 + l\Delta},$$

$$Q = lmab(\omega^2 + n\Delta),$$

$$R = lnac(\omega^2 + m\Delta),$$

values independent of ϕ, or the value of I is found by multiplying the quantity under the integral sign by 2π: and hence we have

$$A = 4\pi la \int \frac{\omega^2(\omega^2 + l\Delta)^{\frac{1}{2}}(\omega^2 + m\Delta)^{\frac{3}{2}}(\omega^2 + n\Delta)^{\frac{3}{2}}\, d\omega}{l^2a^2(\omega^2 + m\Delta)^2(\omega^2 + n\Delta)^2 + m^3b^2(\omega^2 + n\Delta)^2(\omega^2 + l\Delta)^2 + n^3c^2(\omega^2 + n\Delta)^2(\omega^2 + l\Delta)^2},$$

where of course Δ is to be considered as a function of ω. By integrating from $\omega = \omega_{,}$ to $\omega = \omega_{,,}$, we have the attraction of the portion of the ellipsoid included between any two of the series of cones, and to obtain the attraction of the whole ellipsoid we must integrate from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, where ξ is determined as before by the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1;$$

and it is obvious that for this value of ω we have $\Delta = \delta$. The expression for the attraction is easily reduced to a known form by writing $y = \frac{k\Delta}{\omega^2}$; this gives

$$A = 4\pi la \int \frac{k^{\frac{1}{2}}(k+ly)^{\frac{1}{2}}(k+my)^{\frac{3}{2}}(k+ny)^{\frac{3}{2}}\omega d\omega}{l^2a^2(k+my)^2(k+ny)^2 + m^2b^2(k+ny)^2(k+ly)^2 + n^2c^2(k+ly)^2(k+my)^2}.$$

Also

$$\omega^2 = k\left(\frac{l^2a^2}{k+ly} + \frac{m^2b^2}{k+my} + \frac{n^2c^2}{k+ny}\right);$$

whence

$$\omega d\omega = -\frac{k\left[l^2a^2(k+my)^2(k+ny)^2 + m^2b^2(k+ny)^2(k+ly)^2 + n^2c^2(k+ly)^2(k+my)^2\right]}{2(k+ly)^2(k+my)^2(k+ny)^2},$$

and thus
$$A = 2\pi k^{\frac{3}{2}} la \int \frac{dy}{(k+ly)^{\frac{3}{2}}(k+my)^{\frac{3}{2}}(k+ny)^{\frac{3}{2}}},$$

where for the entire ellipsoid the integral is to be taken from $y = \xi$ to $y = \infty$. A better known form is readily obtained by writing $x^2 = \frac{k+l\xi}{k+ly}$, in which case the limits for the entire ellipsoid are $x = 0$, $x = 1$.

It may be as well to indicate the first step of the reduction of the integral I, viz. the method of resolving the denominator into two factors. We have identically,

$$(\Delta - \delta)(lP^2 + mQ^2 + nR^2) = \omega^2(P^2 + Q^2 + R^2) + \Delta(lP^2 + mQ^2 + nR^2) - (laP + mbQ + ncR)^2,$$

and the second side of this equation is resolvable into two factors independently of the particular values of P, Q, R. Representing this second side for a moment in the notation of a general quadratic function, or under the form

$$AP^2 + BQ^2 + CR^2 + 2FQR + 2GRP + 2HPQ,$$

we have the required solution,

$lP^2 + mQ^2 + nR^2 =$

$$\frac{1}{A}\left[AP + \{H + \sqrt{(-\mathfrak{C})}\}Q + \{G + \sqrt{(-\mathfrak{B})}\}R\right]\left[AP + \{H - \sqrt{(-\mathfrak{C})}\}Q + \{G - \sqrt{(-\mathfrak{B})}\}R\right];$$

where, as usual, $\mathfrak{B} = CA - G^2$, $\mathfrak{C} = AB - H^2$, and the roots must be so taken that $\sqrt{(-\mathfrak{B})}\sqrt{(-\mathfrak{C})} = \mathfrak{F}\ \{\mathfrak{F} = (GH - AF)\}$.

I have purposely restricted myself so far to the problem considered by Legendre: the general transformation, of which the preceding is a particular case, and also a simpler mode of effecting the integration, are given in the next part of this paper.

PART II.—ON A FORMULA FOR THE TRANSFORMATION OF CERTAIN MULTIPLE INTEGRALS.

Consider the integral

$$V = \int F(x, y, \ldots)\, dx\, dy \ldots,$$

where the number of variables $x, y, \ldots$ is equal to n, and $F(x, y, \ldots)$ is a homogeneous function of the order μ.

Suppose that $x, y, \ldots$ are connected by a homogeneous equation $\psi(x, y, \ldots) = 0$ containing a variable parameter ω (so that ω is a homogeneous function of the order zero in the variables $x, y, \ldots$). Then, writing

$$r^2 = x^2 + y^2 + \ldots, \quad x = r\alpha, \quad y = r\beta, \ldots$$

the quantities $\alpha, \beta, \ldots$ are connected by the equations

$$\alpha^2 + \beta^2 + \ldots = 1, \quad \psi(\alpha, \beta, \ldots) = 0,$$

and we may therefore consider them as functions of ω and of $(n-2)$ independent variables θ, ϕ, &c.; whence

$$dx\, dy \ldots = r^{n-1} \nabla\, dr\, d\omega\, d\theta \ldots,$$

where

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \ldots \\ \dfrac{d\alpha}{d\omega}, & \dfrac{d\beta}{d\omega}, & \\ \dfrac{d\alpha}{d\theta}, & \dfrac{d\beta}{d\theta}, & \\ \vdots & & \end{vmatrix}$$

Also

$$F(x, y, \ldots) = r^\mu F(\alpha, \beta \ldots),$$

and therefore

$$V = \int r^{\mu+n-1} F(\alpha, \beta, \ldots) \nabla\, dr\, d\omega\, d\theta \ldots,$$

or, integrating with respect to r,

$$\int r^{\mu+n-1}\, dr = \frac{1}{\mu+n} r^{\mu+n},$$

which, taken between the proper limits, is a function of $\alpha, \beta, \ldots$, equal $f(\alpha, \beta, \ldots)$ suppose; this gives

$$V = \int f(\alpha, \beta, \ldots) F(\alpha, \beta, \ldots) \nabla\, d\omega\, d\theta \ldots,$$

in which I shall assume that the limits of ω are constant. If, in order to get rid of the condition $\alpha^2 + \beta^2 \ldots = 1$, we assume

$$\alpha = \frac{p}{\rho}, \quad \beta = \frac{q}{\rho}, \ldots, \quad \rho^2 = p^2 + q^2 + \ldots$$

the preceding expression for V becomes

$$V=\int f\left(\frac{p}{\rho}, \frac{q}{\rho}, \ldots\right) F(p, q, \ldots) \frac{1}{\rho^{\mu+n}} D\, d\omega\, d\theta \ldots,$$

in which

$$D=\begin{vmatrix} p, & q, & \ldots \\ \frac{dp}{d\omega}, & \frac{dq}{d\omega}, & \\ \frac{dp}{d\theta}, & \frac{dq}{d\theta}, & \\ \vdots & & \end{vmatrix}$$

Assume

$$p = P\xi + P'\eta + P''\zeta \ldots$$
$$q = Q\xi + Q'\eta + Q''\zeta \ldots$$
$$\vdots$$

where the number of variables $\xi, \eta, \zeta \ldots$ (functions in general of ω, θ, &c.) is n, and where the coefficients P, Q, &c. are supposed to be functions of ω only. We have

$$\frac{dp}{d\theta} = P\frac{d\xi}{d\theta} + P'\frac{d\eta}{d\theta} + P''\frac{d\zeta}{d\theta} + \ldots,$$

$$\frac{dq}{d\theta} = Q\frac{d\xi}{d\theta} + Q'\frac{d\eta}{d\theta} + Q''\frac{d\zeta}{d\theta} + \ldots,$$

and, substituting these values as well as those of p, q, &c., but retaining the terms $\frac{dp}{d\omega}$, $\frac{dq}{d\omega}$, &c. in their original form, the determinant D resolves itself into the sum of a series of products,

$$\begin{vmatrix} \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \ldots \\ P', & Q', \\ P'', & Q'', \\ \vdots & \end{vmatrix} \begin{vmatrix} 1 & . & . & \ldots \\ . & \eta, & \zeta, & \\ . & \frac{d\eta}{d\theta}, & \frac{d\zeta}{d\theta}, & \\ \vdots & & & \end{vmatrix}.$$

Let Ψ be the function to which $\psi\,(p, q, \ldots)$ is changed by the substitution of the above values of $p, q, \ldots$ so that Ψ is a homogeneous function of $\xi, \eta, \zeta, \ldots$ and we have the relation $\Psi = 0$. (It will be convenient to consider $\xi, \eta, \zeta, \ldots$ as functions of ω, θ, &c., such as to satisfy identically this last equation.) We deduce

$$\frac{1}{X}\begin{vmatrix} 1, & . & . & \ldots \\ . & \eta, & \zeta, & \\ . & \frac{d\eta}{d\theta}, & \frac{d\zeta}{d\theta}, & \\ \vdots & & & \end{vmatrix} = \frac{1}{Y}\begin{vmatrix} . & 1, & . & \ldots \\ \xi, & . & \zeta, & \\ \frac{d\xi}{d\theta}, & . & \frac{d\zeta}{d\theta}, & \\ \vdots & & & \end{vmatrix} = \&c. = S \text{ suppose},$$

where for shortness $X=\frac{d\Psi}{d\xi}$, $Y=\frac{d\Psi}{d\eta}$, &c. The substitution of these values gives

$$D=\begin{vmatrix} & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \dots \\ X, & P\ , & Q\ , \\ Y, & P', & Q', \\ Z, & P'', & Q'', \\ \vdots & & \end{vmatrix} S,$$

where it will be remarked that the successive horizontal lines (after the first) of the determinant are the differential coefficients of Ψ, p, q, ... with respect to ξ, with respect to η, &c. In general, if ψ denote any function of p, q, ... these quantities being themselves functions of ω, ξ, η, ..., and ξ, η, ... containing ω; also if Ψ be what ψ becomes when for p, q, ... we substitute their values in ω, ξ, η, ...; then we have identically

$$\begin{vmatrix} \frac{d\psi}{d\omega}, & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \dots \\ X\ , & P\ , & Q\ , \\ Y\ , & P', & Q', \\ Z\ , & P'', & Q'', \\ \vdots & & \end{vmatrix} = 0. \text{ (1)}$$

In the present case however, writing for shortness $\psi(p, q, \dots)=\psi$, this function ψ contains ω explicitly as well as implicitly through p, q, &c. The formula is still true if for $\frac{d\psi}{d\omega}$ we substitute $\frac{d(\psi)}{d\omega}-\frac{d\psi}{d\omega}$, $\frac{d\psi}{d\omega}$ on the second side denoting a partial differential coefficient taken only so far as ω is explicitly contained in ψ. And considering p, q, ... as functions of ω, ξ, η, ... (ξ, η, ... themselves functions of ω and of other variables which need not here be considered), ψ or Ψ vanishes identically, and we have $\frac{d(\psi)}{d\omega}=0$. Hence, in the last formula, we have to write $-\frac{d\psi}{d\omega}$ instead of $\frac{d\psi}{d\omega}$, and we thus derive

$$\frac{d\psi}{d\omega}\begin{vmatrix} P, & Q, \dots \\ P', & Q', \\ \vdots & \end{vmatrix} = \begin{vmatrix} & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \dots \\ X, & P\ , & Q\ , \\ Y, & P', & Q', \\ Z, & P'', & Q'', \\ \vdots & & \end{vmatrix},$$

[1] This formula, or one equivalent to it, is given in Jacobi's memoir "De Determinantibus Functionalibus," *Crelle*, t. XXII. [1841] p. 319.

whence D becomes

$$D = \begin{vmatrix} P, & Q, \ldots \\ P', & Q', \\ \vdots & \end{vmatrix} \frac{d\psi}{d\omega} S,$$

which is the value to be made use of in the equation

$$V = \int f\left(\frac{p}{\rho}, \frac{q}{\rho} \ldots\right) F(p, q, \ldots) \frac{1}{\rho^{\mu+n}} D \, d\omega \, d\theta \ldots .$$

The principal use of the formula is where ψ is a homogeneous function of the second order of $p, q, \ldots$. Thus, suppose

$$\psi = \tfrac{1}{2}(Ap^2 + Bq^2 \ldots + 2Hpq + \ldots),$$

also, for the sake of conformity to the usual notation in the theory of transformation of quadratic functions, writing $\alpha, \alpha', \ldots \beta, \beta', \ldots$ instead of $P, P', \ldots Q, Q', \ldots$ and putting after the differentiations $\xi = 1$, we have

$$\begin{aligned} p &= \alpha + \alpha'\eta + \alpha''\zeta + \ldots, \\ q &= \beta + \beta'\eta + \beta''\zeta + \ldots, \\ &\vdots \end{aligned}$$

values which we may assume to give rise to the equation

$$(Ap^2 + Bq^2 \ldots + 2Hpq \ldots) = (1 - \eta^2 - \zeta^2 - \ldots),$$

(where $\eta, \zeta, \ldots$ are taken to be functions of θ, &c. such as to satisfy identically the equation $1 = \eta^2 + \zeta^2 + \ldots$).

Hence, by a well-known property, if

$$\begin{vmatrix} A, & H, \ldots \\ H, & B, \\ \vdots & \end{vmatrix} = \kappa,$$

we have

$$\begin{vmatrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \\ \vdots & \end{vmatrix} = \sqrt{\left\{\frac{(-)^{n-1}}{\kappa}\right\}};$$

so that, observing that in the present case $X = 1$, and therefore

$$S = \begin{vmatrix} \eta, & \zeta, \ldots \\ \frac{d\eta}{d\theta}, & \frac{d\zeta}{d\theta}, \\ \vdots & \end{vmatrix}$$

we have

$$V = \int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}} f\left(\frac{p}{\rho}, \frac{q}{\rho}, \ldots\right) F(p, q, \ldots) \frac{1}{\rho^{\mu+n}} \frac{d\psi}{d\omega} S \, d\omega \, d\theta \ldots .$$

The remainder of the process of integration may in many cases be effected by the method made use of by Jacobi in the memoir "De binis quibuslibet functionibus &c." *Crelle*, t. XII. [1834] p. 1, viz. the coefficients α, α', &c., β, &c., may in addition to the conditions which they are already supposed to satisfy, be so determined as to reduce any homogeneous function of p, q, r, ... entering into the integral to a form containing the squares only of the variables. This method is applied in the memoir in question to the integrals of n variables, analogous to those which give the attraction of an ellipsoid; and that directly without effecting an integration with respect to the radius vector. I proceed to show how the preceding investigations lead to Legendre's integral, and how the method in question effects with the utmost simplicity the integration which Legendre accomplished by means of what Poisson has spoken of as inextricable calculations.

Consider in particular the formula

$$V = \int \frac{(x,\ y \ldots)^h\, dx\, dy \ldots}{(x^2 + y^2 \ldots)^{\frac{h}{2}-i}},$$

the number of variables being as before n, and $(x,\ y, \ldots)^h$ denoting a homogeneous function of the order h. The equation for the limits is assumed to be

$$l\,(x-a)^2 + m\,(y-b)^2 + \ldots = k.$$

Assume

$$\psi\,(x,\ y\,, \ldots) = \omega^2\,(x^2 + y^2 + \ldots) + \delta\,(lx^2 + my^2 + \ldots) - (lax + mby + \ldots)^2,$$

(where δ, $= la^2 + mb^2 \ldots - k$, is taken to be positive); or more simply,

$$\psi\,(x,\ y \ldots) = (\omega^2 + l\delta - l^2a^2)\,x^2 + (\omega^2 + m\delta - m^2b^2)\,y^2 + \ldots - 2lm\,ab\,xy - \ldots$$

Here $\mu = h + 2i - 3$. Also, putting for shortness

$$lap + mbq + \ldots = \Lambda, \quad lp^2 + mq^2 + \ldots = \Phi,$$

it is easy to obtain

$$f\left(\frac{p}{\rho},\ \frac{q}{\rho}, \ldots\right) = \frac{1}{h + 2i + n - 3}\ \frac{\rho^{h+2i+n-3}\left[(\Lambda + \omega\rho)^{h+2i+n-3} - (\Lambda - \omega\rho)^{h+2i+n-3}\right]}{\Phi^{h+2i+n-3}}.$$

Also
$$F(p,\ q, \ldots) = \frac{(p,\ q, \ldots)^h}{\rho^{3-2i}}, \qquad \frac{d\psi}{d\omega} = 2\omega\rho^2,$$

$$\kappa = (\omega^2 + l\delta)\,(\omega^2 + m\delta) \ldots \left(1 - \frac{l^2a^2}{\omega^2 + l\delta} - \frac{m^2b^2}{\omega^2 + m\delta} - \&c.\right),$$

values which give

$$V = \frac{2}{h + 2i + n - 3} \int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}}\ \frac{\omega\rho^{2i-1}\left[(\Lambda + \omega\rho)^{h+2i+n-3} - (\Lambda - \omega\rho)^{h+2i+n-3}\right](p,\ q, \ldots)^h\, S d\omega\, d\theta \ldots}{\Phi^{h+2i+n-3}},$$

where it will be remembered that $p, q, \ldots$ are linear functions (with constant terms) of $(n-1)$ variables $\eta, \zeta, \ldots$, these last mentioned quantities being themselves functions of $(n-2)$ variables θ, &c. such that $1-\eta^2-\zeta^2-\ldots=0$ identically. If besides we suppose that $\Phi = lp^2+mq^2+\ldots$ reduces itself to the form $\frac{1}{P}-\frac{\eta^2}{Q}-$ &c., we have, by the formula of the paper "On the Simultaneous Transformation of two Homogeneous Equations of the Second Order," [74],

$$\left(1-\frac{\lambda}{P}\right)\left(1-\frac{\lambda}{Q}\right)\ldots=$$

$$\frac{1}{\kappa}(\omega^2+l\delta-l\lambda)(\omega^2+m\delta-m\lambda)\ldots\left(1-\frac{l^2a^2}{\omega^2+l\delta-l\lambda}-\frac{m^2b^2}{\omega^2+m\delta-m\lambda}-\ldots\right),$$

which is true, whatever be the value of λ.

It seems difficult to proceed further with the general formula, and I shall suppose $n=3$, $i=0$, $h=1$, $(x, y\ldots)^h=x$, or write

$$V=\int\frac{x\,dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

the equation of the limits being

$$l(x-a)^2+m(y-b)^2+n(z-c)^2=k.$$

Here we may assume $\eta=\cos\theta$, $\zeta=\sin\theta$, (values which give $S=1$). And we have

$$V=2\int\frac{\omega^2 d\omega}{\kappa^{\frac{1}{2}}}\int\frac{(\alpha+\alpha'\cos\theta+\alpha''\sin\theta)\,d\theta}{\frac{1}{P}-\frac{\cos^2\theta}{Q}-\frac{\sin^2\theta}{R}},$$

from $\theta=0$ to $\theta=2\pi$; or, what comes to the same thing,

$$V=8\int\frac{\omega^2 d\omega}{\kappa^{\frac{1}{2}}}\int\frac{\alpha\,d\theta}{\frac{1}{P}-\frac{\cos^2\theta}{Q}-\frac{\sin^2\theta}{R}},$$

from $\theta=0$ to $\theta=\frac{1}{2}\pi$. Hence

$$V=4\pi\int\frac{\alpha\omega^2 P\sqrt{(QR)}\,d\omega}{\kappa^{\frac{1}{2}}\sqrt{\{(P-Q)(P-R)\}}};$$

we have from the formulæ of the paper before quoted,

$$\alpha^2=\frac{QR}{\kappa}\,\frac{(B-mP)(C-nP)-F^2}{(P-Q)(P-R)},$$

B, C, F, being the coefficients of y^2, z^2, yz in $\psi(x, y, z)$, viz.

$$B=\omega^2+m\delta-m^2b^2,\quad C=\omega^2+n\delta-n^2c^2,\quad F=mnbc;$$

and consequently

$$V=4\pi\int\omega^2\,d\omega\,\frac{PQR}{\kappa}\,\frac{\{(B-mP)(C-nP)-F^2\}^{\frac{1}{2}}}{(P-Q)(P-R)}.$$

Also from the equation

$$\left(1-\frac{\lambda}{P}\right)\left(1-\frac{\lambda}{Q}\right)\left(1-\frac{\lambda}{R}\right)=$$

$$\frac{1}{\kappa}(\omega^2+l\delta-l\lambda)(\omega^2+m\delta-m\lambda)(\omega^2+n\delta-n\lambda)\left(1-\frac{l^2a^2}{\omega^2+l\delta-l\lambda}-\frac{m^2b^2}{\omega^2+m\delta-m\lambda}-\frac{n^2c^2}{\omega^2+n\delta-n\lambda}\right);$$

differentiating with respect to λ, and writing $\lambda=P$,

$$\frac{-\frac{1}{P}\left(1-\frac{P}{Q}\right)\left(1-\frac{P}{R}\right)}{(\omega^2+l\delta-lP)(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)}$$

$$=-\frac{1}{\kappa}\left\{\frac{l^2a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^2b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^2c^2}{(\omega^2+n\delta-nP)^2}\right\},$$

or, as this may be written,

$$\frac{PQR}{\kappa(P-Q)(P-R)}$$

$$=\frac{1}{(\omega^2+l\delta-lP)(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)\left\{\frac{l^2a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^2b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^2c^2}{(\omega^2+n\delta-nP)^2}\right\}},$$

and from the values first written down, for B, C, F, we obtain $(B-mP)(C-nP)-F^2$

$$=(\omega^2+m\delta)(\omega^2+n\delta)-m^2b^2(\omega^2+n\delta)-n^2c^2(\omega^2+m\delta)-mP(\omega^2+n\delta-n^2c^2)-nP(\omega^2+m\delta-m^2b^2)+mnP^2$$

$$=(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)-m^2b^2(\omega^2+n\delta-nP)-n^2c^2(\omega^2+m\delta-mP)$$

$$=\frac{l^2a^2(\omega^2+m\delta-mP)(\omega^2+n\delta-nP)}{\omega^2+l\delta-lP},$$ the last reduction being effected by means of the equation

$$1-\frac{l^2a^2}{\omega^2+l\delta-lP}-\frac{m^2b^2}{\omega^2+m\delta-mP}-\frac{n^2c^2}{\omega^2+n\delta-nP}=0.$$

Hence

$$\frac{PQR}{\kappa}\frac{\{(B-mP)(C-nP)-F^2\}^{\frac{1}{2}}}{(P-Q)(P-R)}$$

$$=\frac{la}{(\omega^2+l\delta-lP)^{\frac{3}{2}}(\omega^2+m\delta-mP)^{\frac{1}{2}}(\omega^2+n\delta-nP)^{\frac{1}{2}}\left\{\frac{l^2a^2}{(\omega^2+l\delta-lP)^2}+\frac{m^2b^2}{(\omega^2+m\delta-mP)^2}+\frac{n^2c^2}{(\omega^2+n\delta-nP)^2}\right\}}.$$

Substituting this value, and multiplying out the fractions in the denominator,

$$V=4\pi la\times$$

$$\int\frac{\omega^2(\omega^2+l\delta-lP)^{\frac{1}{2}}(\omega^2+m\delta-mP)^{\frac{3}{2}}(\omega^2+n\delta-nP)^{\frac{3}{2}}\,d\omega}{l^2a^2(\omega^2+m\delta-mP)^2(\omega^2+n\delta-nP)^2+m^2b^2(\omega^2+n\delta-nP)^2(\omega^2+l\delta-lP)^2+n^2c^2(\omega^2+l\delta-lP)^2(\omega^2+m\delta-mP)^2}$$

the reduction of which integral has been already treated of in the former part of this present memoir.

76.

ON THE TRIPLE TANGENT PLANES OF SURFACES OF THE THIRD ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 118—132.]

A SURFACE of the third order contains in general a certain number of straight lines. Any plane through one of these lines intersects the surface in the line and in a conic, that is in a curve or system of the third order having two double points. Such a plane is therefore a double tangent plane of the surface, the double points (or points where the line and conic intersect) being the points of contact. By properly determining the plane, the conic will reduce itself to a pair of straight lines. Here the plane intersects the surface in three straight lines, that is in a curve or system of the third order having three double points, and the plane is therefore a triple tangent plane, the three double points or points of intersection of the lines taken two and two together being the points of contact. The number of lines and triple tangent planes is determined by means of a theorem very easily demonstrated, viz. that through each line there may be drawn five (and only five) triple tangent planes. Thus, considering any triple tangent plane, through each of the three lines in this plane there may be drawn (in addition to the plane in question) four triple tangent planes: these twelve new planes give rise to twenty-four new lines upon the surface, making up with the former three lines, twenty-seven lines upon the surface. It is clear that there can be no lines upon the surface besides these twenty-seven; for since the three lines upon the triple tangent plane are the complete intersection of this plane with the surface, every other line upon the surface must meet the triple tangent plane in a point upon one of the three lines, and must therefore lie in a plane passing through one of these lines, such plane (since it meets the surface in two lines and therefore in a third line) being obviously a triple tangent plane. Hence the whole number of lines upon the surface is twenty-seven; and it immediately follows that the number of triple tangent planes is forty-five. The number of lines upon the surface may also be obtained by the following method, which has the advantage of not assuming

à priori the existence of a line upon the surface. Imagine the cone having for its vertex a given point not upon the surface and circumscribed about the surface, every double tangent plane of the cone is also a double tangent plane of the surface, and therefore intersects the surface in a straight line (and a conic). And, conversely, if there be any line upon the surface, the plane through this line and the vertex of the cone will be a double tangent plane of the cone. Hence the number of double tangent planes of the cones is precisely that of the lines upon the surface. By the theorems in Mr [Dr] Salmon's paper "On the degree of a surface reciprocal to a given one," *Journal*, vol. II. [1847] p. 65, the cone is of the sixth order and has no double lines and six cuspidal lines: hence by the formula in Plücker's "Theorie der algebraischen Curven," [1839] p. 211, stated so as to apply to cones instead of plane curves, viz. n being the order, x the number of double lines, y that of the cuspidal lines, u that of the double tangent planes, then

$$u = \tfrac{1}{2}n(n-2)(n^2-9) - (2x+3y)(n^2-n-6) + 2x(x-1) + 6xy + \tfrac{3}{2}y(y-1),$$

the number of double tangent planes is twenty-seven, which is therefore also the number of lines upon the surface.

Suppose the equation of one of the triple tangent planes to be $w=0$, and let $x=0$, $y=0$, be the equation of *any* two triple tangent planes intersecting the plane $w=0$ in two of the lines in which it meets the surface. Let $z=0$ be the equation of a triple tangent plane meeting $w=0$ in the remaining line in which it intersects the surface. The equation of the surface of the third order is in every case of the form $wP+kxyz=0$, P being a function of the second order, but of the four different planes which the equation $z=0$ may be supposed to represent, one of them such that the function P resolves itself into the product of a pair of factors, and for the remaining three this resolution into factors does not take place. This will be obvious from the sequel: at present I shall suppose that the plane $z=0$ is of the latter class, or that $P=0$ represents a proper surface of the second order. Since $x=0$, $y=0$, $z=0$, are treble tangent planes of the surface, each of these planes must be a tangent plane of the surface of the second order $P=0$, and this will be the case if we assume

$$P = x^2+y^2+z^2+w^2$$
$$+yz\left(mn+\frac{1}{mn}\right)+zx\left(nl+\frac{1}{nl}\right)+xy\left(lm+\frac{1}{lm}\right)+xw\left(l+\frac{1}{l}\right)+yw\left(m+\frac{1}{m}\right)+zw\left(n+\frac{1}{n}\right);$$

and considering x, y, z and w as each of them implicitly containing an arbitrary constant, this is the most general function which satisfies the conditions in question.

We are thus led to the equation of the surface of the third order:

$$U = w\left\{x^2+y^2+z^2+w^2+\right.$$
$$\left. yz\left(mn+\frac{1}{mn}\right)+zx\left(nl+\frac{1}{nl}\right)+xy\left(lm+\frac{1}{lm}\right)+xw\left(l+\frac{1}{l}\right)+yw\left(m+\frac{1}{m}\right)+zw\left(n+\frac{1}{n}\right)\right\}+kxyz=0.$$

I have found that by expressing the parameter k in the particular form

$$k=\frac{p^2-\left(lmn-\frac{1}{lmn}\right)^2}{2\left(p-lmn-\frac{1}{lmn}\right)},$$

or, as this equation may be more conveniently written,

$$k=\frac{p^2-\beta^2}{2(p-\alpha)};\quad \alpha=lmn+\frac{1}{lmn},\quad \beta=lmn-\frac{1}{lmn},\text{[1]}$$

the equations of all the planes are expressible in a rational form. These equations are in fact the following: [I have added, here and in the table p. 450, the reference numbers 12′, 23′, &c. constituting a different notation for the lines and planes.]

(w)	$w=0.$	12′
(θ)	$lx+my+nz+w\left[1+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0,$	23′
$(\bar{\theta})$	$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}+w\left[1+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0,$	31′
(x)	$x=0,$	12 . 34 . 56
(y)	$y=0,$	42′
(z)	$z=0,$	14′
(ξ)	$x+\frac{1}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)w=0,$	1′2
(η)	$y+\frac{1}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)w=0,$	2′3
(ζ)	$z+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)w=0,$	3′1
(f)	$lx+\frac{y}{m}+\frac{z}{n}+w=0,$	41′
(g)	$\frac{x}{l}+my+\frac{z}{n}+w=0,$	34′
(h)	$\frac{x}{l}+\frac{y}{m}+nz+w=0,$	13 . 24 . 56

[1] A somewhat more elegant form is obtained by writing $p=2q+\alpha$; this gives

$$k=\frac{2}{q}\cdot(q+lmn)\left(q+\frac{1}{lmn}\right),\ \&c.$$

($\bar{f}$)	$\frac{x}{l}+my+nz+w=0,$	24′
($\bar{g}$)	$lx+\frac{y}{m}+nz+w=0,$	14 . 23 . 56
($\bar{h}$)	$lx+my+\frac{z}{n}+w=0,$	43′
(x)	$x+\frac{l(p-\alpha)+2mn}{p+\beta}w=0,$	12 . 35 . 46
(y)	$y+\frac{m(p-\alpha)+2nl}{p+\beta}w=0,$	52′
(z)	$z+\frac{n(p-\alpha)+2lm}{p+\beta}w=0,$	15′
($\bar{x}$)	$x+\frac{\frac{1}{l}(p-\alpha)+\frac{2}{mn}}{p-\beta}w=0,$	12 . 36 . 45
($\bar{y}$)	$y+\frac{\frac{1}{m}(p-\alpha)+\frac{2}{nl}}{p-\beta}w=0,$	62′
($\bar{z}$)	$z+\frac{\frac{1}{n}(p-\alpha)+\frac{2}{lm}}{p-\beta}w=0,$	16′
(l)	$-\frac{2n}{m(p-\alpha)}x+\frac{1}{m}y+nz+w=0,$	56′
(m)	$lx-\frac{2l}{n(p-\alpha)}y+\frac{1}{n}z+w=0,$	45′
(n)	$\frac{1}{l}x+my-\frac{2m}{l(p-\alpha)}z+w=0,$	4′6
($\bar{l}$)	$-\frac{2m}{n(p-\alpha)}x+my+\frac{1}{n}z+w=0,$	15 . 26 . 34
($\bar{m}$)	$\frac{1}{l}x-\frac{2l}{l(p-\alpha)}y+nz+w=0,$	16 . 24 . 35
($\bar{n}$)	$lx+\frac{1}{m}y-\frac{2l}{m(p-\alpha)}z+w=0,$	14 . 25 . 36
($l_{,}$)	$-\frac{n(p-\alpha)}{2m}x+\frac{y}{m}+nz+w=0,$	65′
($m_{,}$)	$lx-\frac{l(p-\alpha)}{2n}y+\frac{1}{n}z+w=0,$	46′
($n_{,}$)	$\frac{1}{l}x+my-\frac{m(p-\alpha)}{2l}z+w=0,$	4′5

$$(\bar{\mathrm{l}}_{,})\quad -\frac{m(p-\alpha)}{2n}x+my+\frac{1}{n}z+w=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 16.25.34$$

$$(\bar{\mathrm{m}}_{,})\quad \frac{1}{l}x-\frac{n(p-\alpha)}{2l}y+nz+w=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 15.24.36$$

$$(\bar{\mathrm{n}}_{,})\quad lx+\frac{1}{m}y-\frac{l(p-\alpha)}{2m}z+w=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 14.26.35$$

$$(\mathrm{p})\quad -\frac{2x}{p-\alpha}+ny+mz+[mn(p-\alpha)-2l(1-m^2-n^2)]\frac{w}{p+\beta}=0, \quad \ldots\ldots\ldots\ldots 51'$$

$$(\mathrm{q})\quad nx-\frac{2y}{p-\alpha}+lz+[nl(p-\alpha)-2m(1-n^2-l^2)]\frac{w}{p+\beta}=0, \quad \ldots\ldots\ldots\ldots 35'$$

$$(\mathrm{r})\quad mx+ly-\frac{2z}{p-\alpha}+[lm(p-\alpha)-2n(1-l^2-m^2)]\frac{w}{p+\beta}=0, \quad \ldots\ 13.25.46$$

$$(\bar{\mathrm{p}})\quad -\frac{2x}{p-\alpha}+\frac{1}{n}y+\frac{1}{m}z+\left[\frac{1}{mn}(p-\alpha)-\frac{2}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)\right]\frac{w}{p-\beta}=0, \quad \ldots\ldots\ 26'$$

$$(\bar{\mathrm{q}})\quad \frac{1}{n}x-\frac{2y}{p-\alpha}+\frac{1}{l}z+\left[\frac{1}{nl}(p-\alpha)-\frac{2}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)\right]\frac{w}{p-\beta}=0, \ldots 16.23.45$$

$$(\bar{\mathrm{r}})\quad \frac{1}{m}x+\frac{1}{l}y-\frac{2z}{p-\alpha}+\left[\frac{1}{lm}(p-\alpha)-\frac{2}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)\right]\frac{w}{p-\beta}=0, \quad \ldots\ldots\ldots 63'$$

$$(\mathrm{p}_{,})\quad -\frac{p-\alpha}{2}x+\frac{y}{n}+\frac{z}{m}-lmn\left[\frac{1}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)(p-\alpha)-\frac{2}{mn}\right]\frac{w}{p+\beta}=0, \quad \ldots\ldots\ldots 25'$$

$$(\mathrm{q}_{,})\quad \frac{x}{n}-\frac{p-\alpha}{2}y+\frac{z}{l}-lmn\left[\frac{1}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)(p-\alpha)-\frac{2}{nl}\right]\frac{w}{p+\beta}=0, \ldots 15.23.46$$

$$(\mathrm{r}_{,})\quad \frac{x}{m}+\frac{y}{l}-\frac{p-\alpha}{2}z-lmn\left[\frac{1}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)(p-\alpha)-\frac{2}{lm}\right]\frac{w}{p+\beta}=0, \quad \ldots\ldots\ldots 53'$$

$$(\bar{\mathrm{p}}_{,})\quad -\frac{p-\alpha}{2}x+ny+mz-\frac{1}{lmn}[l(1-m^2-n^2)(p-\alpha)-2mn]\frac{w}{p-\beta}=0, \quad \ldots\ldots\ldots 61'$$

$$(\bar{\mathrm{q}}_{,})\quad nx-\frac{p-\alpha}{2}y+lz-\frac{1}{lmn}[m(1-n^2-l^2)(p-\alpha)-2nl]\frac{w}{p-\beta}=0, \quad \ldots\ldots\ldots 36'$$

$$(\bar{\mathrm{r}}_{,})\quad mx+ly-\frac{p-\alpha}{2}z-\frac{1}{lmn}[n(1-l^2-m^2)(p-\alpha)-2lm]\frac{w}{p-\beta}=0. \ldots 13.26.45$$

In fact, representing the several functions on the left-hand side of these equations respectively by the letters placed opposite to them respectively, the function U is expressible in the sixteen forms following

$$\begin{aligned} U &= w\mathrm{f}\bar{\mathrm{f}} + k\xi yz, \\ &= w\mathrm{g}\bar{\mathrm{g}} + k\eta zx, \\ &= w\mathrm{h}\bar{\mathrm{h}} + k\zeta xy, \\ &= w\theta\bar{\theta} + k\xi\eta\zeta, \end{aligned}$$

$$
\begin{aligned}
&= w\ \mathrm{l}\bar{\mathrm{l}}_{,} + k\mathrm{y}\bar{\mathrm{z}}x,\\
&= w\mathrm{m}\bar{\mathrm{m}}_{,} + kz\bar{\mathrm{x}}y,\\
&= w\ \mathrm{n}\bar{\mathrm{n}}_{,} + k\mathrm{x}\bar{\mathrm{y}}z,\\
&= w\ \mathrm{l}_{,}\mathrm{l} + k\bar{\mathrm{y}}\mathrm{z}x,\\
&= w\mathrm{m}_{,}\bar{\mathrm{m}} + k\bar{z}\mathrm{x}y,\\
&= w\ \mathrm{n}_{,}\bar{\mathrm{n}} + k\bar{\mathrm{x}}\mathrm{y}z,\\
&= w\ \mathrm{pp}_{,} + k\xi\mathrm{yz},\\
&= w\ \mathrm{qq}_{,} + k\eta\mathrm{zx},\\
&= w\ \mathrm{rr}_{,} + k\zeta\mathrm{xy},\\
&= w\ \mathrm{p}\bar{\mathrm{p}}_{,} + k\xi\bar{\mathrm{y}}\bar{\mathrm{z}},\\
&= w\ \mathrm{q}\bar{\mathrm{q}}_{,} + k\eta\bar{z}\bar{\mathrm{x}},\\
&= w\ \mathrm{r}\bar{\mathrm{r}}_{,} + k\zeta\bar{\mathrm{x}}\bar{\mathrm{y}},
\end{aligned}
$$

(being the forms containing w, out of a complete system of one hundred and twenty different forms).

The forty-five planes pass five and five through the twenty-seven lines in the following manner:

$$
\begin{array}{lll}
(a_1)\ (w,\ x,\ \xi,\ \mathrm{x},\ \bar{\mathrm{x}})\ldots 12, & (a_4)\ (x,\ \mathrm{g},\ \bar{\mathrm{h}},\ \mathrm{l},\ \bar{\mathrm{l}}_{,})\ldots 34, & (a_7)\ (\mathrm{x},\ \mathrm{m}_{,},\ \mathrm{n},\ \mathrm{q}_{,},\ \mathrm{r})\ldots 46,\\
(b_1)\ (w,\ y,\ \eta,\ \mathrm{y},\ \bar{\mathrm{y}})\ldots 2', & (b_4)\ (y,\ \mathrm{h},\ \bar{\mathrm{f}},\ \bar{\mathrm{m}},\ \bar{\mathrm{m}}_{,})\ldots 24, & (b_7)\ (\mathrm{y},\ \mathrm{n}_{,},\ \mathrm{l},\ \mathrm{r}_{,},\ \mathrm{p})\ldots 5,\\
(c_1)\ (w,\ z,\ \zeta,\ \mathrm{z},\ \bar{\mathrm{z}})\ldots 1, & (c_4)\ (z,\ \mathrm{f},\ \bar{\mathrm{g}},\ \bar{\mathrm{n}},\ \bar{\mathrm{n}}_{,})\ldots 14, & (c_7)\ (\mathrm{z},\ \mathrm{l}_{,},\ \mathrm{m},\ \mathrm{p}_{,},\ \mathrm{q})\ldots 5',\\
(a_2)\ (\xi,\ \bar{\mathrm{f}},\ \theta,\ \bar{\mathrm{p}},\ \mathrm{p}_{,})\ldots 2, & (a_5)\ (x,\ \bar{\mathrm{g}},\ \mathrm{h},\ \mathrm{l},\ \mathrm{l}_{,})\ldots 56, & (a_8)\ (\bar{\mathrm{x}},\ \bar{\mathrm{m}}_{,},\ \bar{\mathrm{n}},\ \bar{\mathrm{q}}_{,},\ \bar{\mathrm{r}})\ldots 36,\\
(b_2)\ (\eta,\ \bar{\mathrm{g}},\ \theta,\ \bar{\mathrm{q}},\ \mathrm{q}_{,})\ldots 23, & (b_5)\ (y,\ \bar{\mathrm{h}},\ \mathrm{f},\ \mathrm{m},\ \mathrm{m}_{,})\ldots 4, & (b_8)\ (\bar{\mathrm{y}},\ \bar{\mathrm{n}}_{,},\ \bar{\mathrm{l}},\ \bar{\mathrm{r}}_{,},\ \bar{\mathrm{p}})\ldots 26,\\
(c_2)\ (\zeta,\ \bar{\mathrm{h}},\ \theta,\ \bar{\mathrm{r}},\ \mathrm{r}_{,})\ldots 3', & (c_5)\ (z,\ \bar{\mathrm{f}},\ \mathrm{g},\ \mathrm{n},\ \mathrm{n}_{,})\ldots 4', & (c_8)\ (\bar{\mathrm{z}},\ \bar{\mathrm{l}}_{,},\ \bar{\mathrm{m}},\ \bar{\mathrm{p}}_{,},\ \bar{\mathrm{q}})\ldots 16,\\
(a_3)\ (\xi,\ \mathrm{f},\ \bar{\theta},\ \mathrm{p},\ \bar{\mathrm{p}}_{,})\ldots 1', & (a_6)\ (\mathrm{x},\ \mathrm{m},\ \mathrm{n}_{,},\ \mathrm{q},\ \mathrm{r}_{,})\ldots 35, & (a_9)\ (\bar{\mathrm{x}},\ \mathrm{m},\ \mathrm{n}_{,},\ \bar{\mathrm{q}},\ \bar{\mathrm{r}}_{,})\ldots 45,\\
(b_3)\ (\eta,\ \mathrm{g},\ \bar{\theta},\ \mathrm{q},\ \bar{\mathrm{q}}_{,})\ldots 3, & (b_6)\ (\mathrm{y},\ \mathrm{n},\ \mathrm{l}_{,},\ \mathrm{r},\ \mathrm{p}_{,})\ldots 25, & (b_9)\ (\bar{\mathrm{y}},\ \mathrm{n},\ \mathrm{l}_{,},\ \bar{\mathrm{r}},\ \bar{\mathrm{p}}_{,})\ldots 6,\\
(c_3)\ (\zeta,\ \mathrm{h},\ \bar{\theta},\ \mathrm{r},\ \bar{\mathrm{r}}_{,})\ldots 13, & (c_6)\ (z,\ \mathrm{l},\ \mathrm{m}_{,},\ \mathrm{p},\ \mathrm{q}_{,})\ldots 15, & (c_9)\ (\bar{z},\ \mathrm{l},\ \mathrm{m}_{,},\ \bar{\mathrm{p}},\ \bar{\mathrm{q}}_{,})\ldots 6',
\end{array}
$$

where each line may be represented by the letter placed oppositive to the system of planes passing through it. The twenty-seven lines lie three and three upon the forty-five planes in the following manner:

$$
\begin{array}{llllllll}
(w) & a_1b_1c_1, & (\mathrm{f}) & a_3b_5c_4, & (\mathrm{l}) & a_5b_7c_9, & (\mathrm{p}) & a_3b_7c_6,\\
(\theta) & a_2b_2c_2, & (\mathrm{g}) & b_3c_5a_4, & (\mathrm{m}) & b_5c_7a_9, & (\mathrm{q}) & b_3c_7a_6,\\
(\theta) & a_3b_3c_3, & (\mathrm{h}) & c_3a_5b_4, & (\mathrm{n}) & c_5a_7b_9, & (\mathrm{r}) & c_3a_7b_6,
\end{array}
$$

(x)	$a_1a_4a_5$,	$(\bar{f})$	$a_2b_4c_5$,	$(\bar{l})$	$a_4b_2c_5$,	$(\bar{p})$	$a_2b_5c_9$,
(y)	$b_1b_4b_5$,	$(\bar{g})$	$b_2c_4a_5$,	$(\bar{m})$	$b_4c_2a_5$,	$(\bar{q})$	$b_2c_5a_9$,
(z)	$c_1c_4c_5$,	$(\bar{h})$	$c_2a_4b_5$,	$(\bar{n})$	$c_4a_2b_5$,	$(\bar{r})$	$c_2a_5b_9$,
(ξ)	$a_1a_2a_3$,	(x)	$a_1a_6a_7$,	$(l_{,})$	$a_2b_3c_7$,	$(p_{,})$	$a_2b_6c_7$,
(η)	$b_1b_2b_3$,	(y)	$b_1b_6b_7$,	$(m_{,})$	$b_3c_9a_7$,	$(q_{,})$	$b_2c_6a_7$,
(ζ)	$c_1c_2c_3$,	(z)	$c_1c_6c_7$,	$(n_{,})$	$c_3a_9b_7$,	$(r_{,})$	$c_2a_6b_7$,
		$(\bar{x})$	$a_1a_8a_9$,	$(\bar{l}_{,})$	$a_4b_6c_8$,	$(\bar{p}_{,})$	$a_3b_9c_8$,
		$(\bar{y})$	$b_1b_8b_9$,	$(\bar{m}_{,})$	$b_4c_6a_8$,	$(\bar{q}_{,})$	$b_3c_9a_8$,
		$(\bar{z})$	$c_1c_8c_9$,	$(\bar{n}_{,})$	$c_4a_6b_8$,	$(\bar{r}_{,})$	$c_3a_9b_8$.

The preceding method was the one that first occurred to me, and which appears to conduct most simply to the actual analytical expressions for the forty-five planes; but it is worth noticing that the relations between the lines and planes might have been obtained almost without algebraical developments, if we had supposed that P, instead of representing a proper surface of the second order, had represented a pair of planes. This would have conducted at once to one of the one hundred and twenty forms U, e.g. $U = w\theta\bar{\theta} + k\xi\eta\zeta$. Or changing the notation so as to include k in one of the linear functions, $U = ace - bdf$, and it is indeed obvious *à priori*, by merely reckoning the number of arbitrary constants, that any function of the third order can be put under this form. If we suppose $a = \mu b$ to be the equation of one of the triple tangent planes through the intersection of the planes a and b, the plane $a = \mu b$ meets the surface in the same lines in which it meets the hyperboloid $\mu ce - df = 0$, that is, the two lines in the plane are generating lines of different species, and consequently one of them meets the pair of lines cd and ef, and the other of them meets the pair of lines cf and de (where cd represents the line of intersection of the planes $c = 0$, $d = 0$, &c.). This suggests a notation for the lines in question, viz. each line may be represented by the three lines which it meets, or by the symbols $ab.cd.ef$ and $ab.cf.de$. Or observing that μ has three values, and that the same considerations apply *mutatis mutandis* to the planes through bc and ca, the whole system of lines may be represented by the notation,

ab,	ad,	af,
cb,	cd,	cf,
eb,	ed,	ef,
$(ab.cd.ef)_1$,	$(ab.cd.ef)_2$,	$(ab.cd.ef)_3$,
$(ad.cf.eb)_1$,	$(ad.cf.eb)_2$,	$(ad.cf.eb)_3$,
$(af.cb.ed)_1$,	$(af.cb.ed)_2$,	$(af.cb.ed)_3$,
$(ab.cf.ed)_1$,	$(ab.cf.ed)_2$,	$(ab.cf.ed)_3$,
$(ad.cb.ef)_1$,	$(ad.cb.ef)_2$,	$(ad.cb.ef)_3$,
$(af.cd.eb)_1$,	$(af.cd.eb)_2$,	$(af.cd.eb)_3$,

where the last eighteen lines have been divided into two systems of nine each. The five planes through $(ab.cd.ef)_1$ may be considered as cutting the surface in

$$\begin{gathered} ab;\ (ab.cf.ed)_1, \\ cd;\ (af.cd.eb)_1, \\ ef;\ (ad.cb.ef)_1, \\ (ad.cf.eb)_2;\ (af.cb.ed)_3, \\ (ad.cf.eb)_3;\ (af.cb.ed)_2, \end{gathered}$$

(which supposes however that the distinguishing suffixes 1, 2, 3, are added to the different planes according to a certain rule). And similarly for the lines in the planes through the other lines represented by symbols of the like form. The five planes through ab intersect the surface in the lines

$$\begin{array}{cc} cb, & eb, \\ ad, & af, \\ (ab.cd.ef)_1, & (ab.cf.ed)_1, \\ (ab.cd.ef)_2, & (ab.cf.ed)_2, \\ (ab.cd.ef)_3, & (ab.cf.ed)_3, \end{array}$$

and similarly for the planes through the other lines represented by symbols of a like form.

Observing that ξ, η, ζ, correspond to b, d, f, and w, θ, $\bar{\theta}$, to a, c, e respectively, ab corresponds to the intersection of w and ξ, i.e. to a_1, &c.; also $(ab.cd.ef)_1$, $(ab.cd.ef)_2$, $(ab.cd.ef)_3$ correspond to three lines meeting a_1, b_2, and c_3, i.e. to a_5, a_7, a_9, &c.; and the system of the twenty-seven lines as last written down corresponds to the system,

$$\begin{array}{ccc} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \\ a_5, & a_7, & a_9, \\ b_5, & b_7, & b_9, \\ c_5, & c_7, & c_9, \\ a_4, & a_6, & a_8, \\ b_4, & b_6, & b_8, \\ c_4, & c_6, & c_8. \end{array}$$

The investigations last given are almost complete in themselves as the geometrical theory of the subject: there is however some difficulty in seeing *à priori* the nature of the correspondence between the planes which determines which are the planes which ought to be distinguished with the same one of the symbolic numbers, 1, 2, 3.

There is great difficulty in conceiving the complete figure formed by the twenty-seven lines, indeed this can hardly I think be accomplished until a more perfect notation is discovered. In the mean time it is easy to find theorems which partially exhibit the properties of the system. For instance, any two lines, a_1, b_2, which do not meet are intersected by five other lines, a_2, b_1, a_5, a_7, a_9, (no two of which meet). Any four of these last-mentioned lines are intersected by the lines a_1, b_2 and no other lines, but any three of them, e.g. a_5, a_7, a_9, are intersected by the lines a_1, b_2, and by some third line (in the case in question the line c_3). Or generally any three lines, no two of which meet, are intersected by three other lines, no two of which meet. Again, the lines which do not meet any one of the lines a_5, a_7, a_9, are a_2, a_3, b_3, b_1, c_1, c_2: these lines form a hexagon, the pairs of the opposite sides of which, a_3, b_1; a_3, c_1; b_3, c_2, are met by the pairs a_1, b_2; c_3, a_1 and a_1, b_2, respectively, viz. by pairs out of the system of three lines intersecting the system a_5, a_7, a_9. And the lines a_5, a_7, a_9 may be considered as representing any three lines no two of which meet. Again, consider three lines in the same triple tangent plane, e.g. a_1, b_1, c_1, and the hexahedron formed by any six triple tangent planes passing two and two through these lines, e.g. the planes x, y, z, ξ, η, ζ. These planes contain (independently of the lines a_1, b_1, c_1) the twelve lines a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5, c_2, c_3, c_4, c_5. Consider three contiguous faces of the hexahedron, e.g. x, y, z, the lines in these planes, viz. a_4, b_5, c_4, a_5, b_4, c_5, form a hexagon the opposite sides of which intersect in a point, or in other words these six lines are generating lines of a hyperboloid. The same property holds for the systems x, η, ζ; ξ, y, ζ; ξ, η, z. But for the system ξ, η, ζ, the six lines are a_2, b_2, c_2, and a_3, b_3, c_3, which form two triangles, and similarly for the systems ξ, y, z; x, η, z; and x, y, ζ; so that the twelve lines form four hexagons (the opposite sides of which intersect) circumscribed round four of the angles of the hexahedron, and four pairs of triangles about the opposite four angles of the hexahedron. The number of such theorems might be multiplied indefinitely, and the number of different combinations of lines or planes to which each theorem applies is also very considerable.

Consider the four planes x, ξ, x, $\bar{\text{x}}$, and represent for a moment the equations of these planes by $x+Aw=0$, $x+Bw=0$, $x+Cw=0$, $x+Dw=0$, so that

$$A=0,\quad B=\frac{1}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right),\quad C=\frac{l(p-\alpha)+2mn}{p+\beta},\quad D=\frac{\frac{1}{l}(p-\alpha)+\frac{2}{mn}}{p-\beta}.$$

By the assistance of

$$B-C=\frac{-1}{lmn(p^2-\beta^2)}[ln(p-\alpha)+2m][lm(p-\alpha)+2n],$$

it is easy to obtain

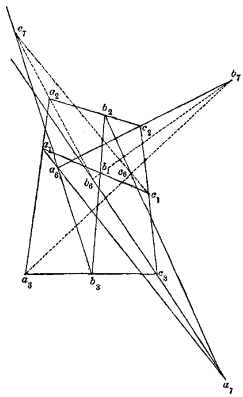
$$\frac{(A-C)(D-B)}{(A-D)(B-C)}=lmn\frac{p-\beta}{p+\beta}\frac{[l(p-\alpha)+2mn][m(p-\alpha)+2nl][n(p-\alpha)+2lm]}{[mn(p-\alpha)+2l][nl(p-\alpha)+2m][lm(p-\alpha)+2n]},$$

which remains unaltered for cyclical permutations of l, m, n, i.e. the anharmonic ratio of x, ξ, x, $\bar{\mathrm{x}}$ is the same as that of y, η, y, $\bar{\mathrm{y}}$, or z, ζ, z, $\bar{\mathrm{z}}$; there is of course no correspondence of x to y or ξ to η, &c., the correspondence is by the general properties of anharmonic ratios, a correspondence of the system x, ξ, x, $\bar{\mathrm{x}}$, to any one of the systems $(y, \eta, \mathrm{y}, \bar{\mathrm{y}})$, or $(\eta, y, \bar{\mathrm{y}}, \mathrm{y})$, or $(\mathrm{y}, \bar{\mathrm{y}}, y, \eta)$, or $(\bar{\mathrm{y}}, \mathrm{y}, \eta, y)$, indifferently. The theorems may be stated generally as follows: "Considering two lines in the same triple tangent plane, the remaining triple tangent planes through these two lines respectively are homologous systems."

Suppose the surface of the third order intersected by an arbitrary plane. The curve of intersection is of course one of the third order, and the positions upon this curve of six of the points in which it is intersected may be arbitrarily assumed. Let these points be the points in which the plane is intersected by the lines a_1, b_1, a_6, b_6, c_6, a_8; or as we may term them, the points a_1, b_1, a_6, b_6, c_6, a_8.[1] The point c_1 is of course the point in which the line a_1b_1 intersects the curve. The straight lines $a_4b_5c_8$, $b_4c_6a_8$, $c_4a_5b_8$, and $a_4b_5c_6$, $b_4c_8a_6$, $c_4a_8b_6$, show that c_4 and b_4 are the points in which a_5b_6, and a_8c_6 intersect the curve, and then b_8 and c_8 are determined as the intersections of a_6c_4, a_5b_4 with the curve. The intersection of the lines b_5c_8 and b_6c_6 (which is known to be a point upon the curve by the theorem, every curve of the third order passing through eight of the points of intersection of two curves of the third order passes through the ninth point of intersection) is the point a_4. The systems a_4, b_4, c_4; a_8, b_6, c_6; a_8, b_8, c_8, determine the conjugate system a_5, b_5, c_5; a_7, b_7, c_7; a_9, b_9, c_9; by reason of the straight lines $a_1a_4a_5$, $b_1b_4b_5$, $c_1c_4c_5$; $a_1a_6a_7$, $b_1b_6b_7$, $c_1c_6c_7$; $a_1a_8a_9$, $b_1b_8b_9$, $c_1c_8c_9$, viz. a_5 is the point where a_1a_4 intersects the curve, &c. The relations of the systems $(a_4, b_4, c_4; a_5, b_5, c_5)$, $(a_6, b_6, c_6; a_7, b_7, c_7)$, $(a_8, b_8, c_8; a_9, b_9, c_9)$ to the system a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are precisely identical. It is only necessary to show how the points a_2, b_2, c_2; a_3, b_3, c_3 of the latter system are determined by means of one of the former systems, suppose the system a_6, b_6, c_6; a_7, b_7, c_7; and to discover a compendious statement of the relation between the two systems. The points a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3; a_6, b_6, c_6; a_7, b_7, c_7, are a system of fifteen points lying on the fifteen straight lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $a_1a_6a_7$, $b_1b_6b_7$, $c_1c_6c_7$, $a_3b_7c_6$, $b_3c_7a_6$, $c_3a_7b_6$, $a_2b_6c_7$, $b_2c_6a_7$, $c_2a_6b_7$, viz. the nine points a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the points of intersection of the three lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$ with the three lines $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, and the remaining six points form a hexagon $a_6b_7c_6a_7b_6c_7$, of which the diagonals a_6a_7, b_6b_7, c_6c_7 pass through the points a_1, b_1, c_1, respectively, the alternate sides a_6b_7, c_6a_7, and b_6c_7 pass through the points c_3, b_3, a_3 respectively, and the remaining alternate sides b_7c_6, a_7b_6, and c_7a_6 pass through the three points a_3, b_3, c_3 respectively. The fifteen points of such a system do not necessarily lie upon a curve of the third order, as will presently be seen: in the actual case however where all the points lie upon a given curve of the third order, and the points a_1, b_1, c_1; a_6, b_6, c_6; a_7, b_7, c_7 are known, a_2, b_2, c_2; a_3, b_3, c_3 are the intersections of the curve with b_6c_7, c_6a_7, a_6b_7, b_7c_6, c_7a_6, a_7b_6 respectively, and the fact

[1] In general, the point in which any line upon the surface intersects the plane in question may be represented by the symbol of the line, and the line in which any triple tangent plane intersects the plane in question may be represented by the symbol of the triple tangent plane: thus, a_1, b_1, c_1 are points in the line $a_1b_1c_1$, or in the line w, &c.

of the existence of the lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ is an immediate consequence of the theorem quoted above with respect to curves of the third order—a theorem from which the entire system of relations between the twenty-seven points on the curve might have been deduced *à priori*. But returning to the system of fifteen points, suppose the lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, and $a_1a_2a_3$, $b_1b_2b_3$, and also the point a_6 to be given arbitrarily. The point a_7 lies on the line a_1a_6, suppose its position upon this line to be arbitrarily assumed (in which case, since the ten points a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, are sufficient to determine a curve of the third order, there is no curve of the third order through these points and the point a_7). If the points



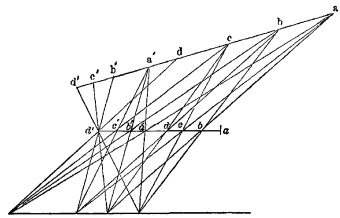
b_6, c_6, b_7, c_7 can be so determined that the sides of the quadrilateral $b_6b_7c_6c_7$, viz. b_6b_7, b_7c_6, c_6c_7, c_7b_6 pass through the points b_1, a_3, c_1, a_2 respectively, while the angles b_6, b_7, c_6, c_7 lie upon the lines a_7c_3, a_6c_2, a_7b_2 and a_6b_2 respectively, the required conditions will be satisfied by the fifteen points in question; and the solution of this problem is known. I have not ascertained whether in the case of an arbitrary position as above of the point a_7, it is possible to determine a complete system of twenty-seven points lying three and three upon forty-five lines in the same manner as the twenty-seven points upon the curve of the third order; but it appears probable that this is the case, and to determine whether it be so or not, presents itself as an interesting problem for investigation.

Suppose that the intersecting plane coincides with one of the triple tangent planes. Here we have a system of twenty-four points, lying eight and eight in three lines; the twenty-four points lie also three and three in thirty-two lines, which last-mentioned lines therefore pass four and four through the twenty-four points. If we represent by a, b, c, d, a', b', c', d' and a, b, c, d, a′, b′, c′, d′, the eight points, and eight points

which lie upon two of the three lines (the order being determinate), the systems of four lines which intersect in the eight points of the third line are

$$\begin{array}{llll} (aa, & bb, & cc, & dd), \\ (ab', & ba', & c'd, & d'c), \\ (ac', & ca', & d'b, & b'd), \\ (ad', & da', & b'c, & c'b), \end{array} \qquad \begin{array}{llll} (a'a', & b'b', & c'c', & d'd'); \\ (a'b, & b'a, & cd', & dc'); \\ (a'c, & c'a, & bd', & db'); \\ (a'd, & d'a, & bc', & cb'): \end{array}$$

the principle of symmetry made use of in this notation (which however represents the actual symmetry of the system very imperfectly) being obviously entirely different from that of the case of an arbitrary intersecting plane. The transition case where the



intersecting plane passes through one of the lines upon the surface (and is thus a double tangent plane) would be worth examining. It should be remarked that the preceding theory is very materially modified when the surface of the third order has one or more conical points; and in the case of a double line (for which the surface becomes a ruled surface) the theory entirely ceases to be applicable. I may mention in conclusion that the whole subject of this memoir was developed in a correspondence with Mr Salmon, and in particular, that I am indebted to him for the determination of the number of lines upon the surface and for the investigations connected with the representation of the twenty-seven lines by means of the letters a, c, e, b, d, f, as developed above.

77.

ON THE ORDER OF CERTAIN SYSTEMS OF ALGEBRAICAL EQUATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 132—137.[1]]

SUPPOSE the variables $x, y \ldots$ so connected that any one of the ratios $x : y : z, \ldots$ or, more generally, any determinate function of these ratios, depends on an equation of the μth order. The variables $x, y, z \ldots$ are said to form a system of the μth order.

In the case of two variables x, y, supposing that these are connected by an equation $U = 0$ (U being a homogeneous function of the order μ) the variables form a system of the μth order; and, conversely, whenever the variables form a system of the μth order, they are connected by an equation of the above form.

In the case of a greater number of variables, the question is one of much greater difficulty. Thus with three variables x, y, z; if μ be resolvable into the factors μ', μ'', then, supposing the variables to be connected by the equations $U = 0$, $V = 0$, U and V being homogeneous functions of the orders μ', μ'', respectively, they will it is true form a system of the μth order, but the converse proposition does not hold: for instance, if μ is a prime number, the only mode of forming a system of the μth order would on the above principle be to assume $\mu' = \mu$, $\mu'' = 1$, that is to suppose the variables connected by an equation of the μth order and a linear equation; but this is far from being the most general method of obtaining such a system. In fact, systems not belonging to the class in question may be obtained by the introduction of subsidiary

[1] This memoir was intended to appear at the same time with Mr Salmon's "Note on a Result of Elimination," (*Journal*, vol. III. p. 169) with which it is very much connected.

variables to be eliminated: the simplest example is the following: suppose a, b, a', b', a'', b'' to be linear functions (without constant terms) of x, y, z, and write

$$\begin{cases} a\xi + b\eta = 0, \\ a'\xi + b'\eta = 0, \\ a''\xi + b''\eta = 0; \end{cases}$$

equations from which, by the elimination of ξ, η, two relations may be obtained between the variables x, y, z.

Suppose, however, from these three equations x, y, z are first eliminated: the ratio $\xi : \eta$ will evidently be determined by a cubic equation; and assuming $\xi : \eta$ to be equal to one of the roots of this, any two of the three equations may be considered as implying the third; and will likewise determine linearly the ratios $x : y : z$. Hence any determinate function of these ratios depends on a cubic equation only, or the system is one of the third order. But the order of the system may be obtained by means of the equations resulting from the elimination of ξ, η; and since this will explain the following more general example (in which the corresponding process is the only one which readily offers itself), it will be convenient to deduce the preceding result in this manner. Thus, performing the elimination, we have

$$L = (a'b'' - a''b') = 0, \quad L' = (a''b - ab'') = 0, \quad L'' = (ab' - a'b) = 0.$$

Here the equations $L = 0$, $L' = 0$, $L'' = 0$, are each of them of the second order, and any two of them may be considered as implying the third. For we have identically,

$$aL + a'L' + a''L'' = 0, \text{ (1)}$$

so that $L = 0$, $L' = 0$, gives $a''L'' = 0$, or $L'' = 0$. Nevertheless the system is imperfectly represented by means of two equations only. For instance, $L = 0$, $L' = 0$ do, of themselves, represent a system which is really of the fourth order. In fact, these equations are satisfied by $a'' = 0$, $b'' = 0$, (which is to be considered as forming a system of the first order), but these values do not satisfy the remaining equation $L'' = 0$. In other words, the equations $L = 0$, $L' = 0$ contain an extraneous system of the first order, and which is seen to be extraneous by means of the last equation L'': the system required is the system of the third order which is common to the three equations $L = 0$, $L' = 0$, $L'' = 0$.

Suppose, more generally, that x, y, z are connected by $\overline{p+1}$ equations, involving p variables $\xi, \eta, \zeta \ldots$,

$$\begin{aligned} a\xi + b\eta + c\zeta + \ldots &= 0, \\ a'\xi + b'\eta + c'\zeta + \ldots &= 0; \\ &\vdots \end{aligned}$$

[1] Also $bL + b'L' + b''L'' = 0$: but since by the elimination of L'', taking into account the actual values of L and L', we obtain an identical equation, these two relations may be considered as equivalent to a single one.

or what comes to the same by the equations (equivalent to two independent relations)

$$\left\| \begin{matrix} a, & a', & a'', \ldots a^{(p)} \\ b, & b', & b'', \ldots b^{(p)} \\ \vdots & & \end{matrix} \right\| = 0;$$

(where the number of horizontal rows is p). Consider x, y, z, as connected by the two equations

$$\left| \begin{matrix} a, \ldots a^{(p-2)}, & a^{(p-1)} \\ b, \ldots b^{(p-2)}, & b^{(p-1)} \\ \vdots & \end{matrix} \right| = 0, \quad \left| \begin{matrix} a, \ldots a^{(p-2)}, & a^{p} \\ b, \ldots b^{(p-2)}, & b^{p} \\ \vdots & \end{matrix} \right| = 0:$$

these form a system of the order p^2, but they involve the extraneous system

$$\left\| \begin{matrix} a, & b, & \ldots\ldots \\ \vdots & & \\ a^{(p-2)}, & b^{(p-2)}, & \ldots \end{matrix} \right\| = 0.$$

Suppose $\phi(p)$ is the order of the system in question, then the order of this last system is $\phi(p-1)$ and hence $\phi(p) = p^2 - \phi(p-1)$: observing that $\phi(2) = 3$, this gives directly $\phi(p) = \frac{1}{2}p(p+1)$. Hence the order of the system is $\frac{1}{2}p(p+1)$.

Suppose x, y, z, connected by equations of the form $U=0$, $V=0$, $W=0$; U, V, W being linear in x, y, z, and homogeneous functions of the orders m, n, p respectively in ξ, η. By eliminating x, y, z, the ratio $\xi : \eta$ will be determined by an equation of the order $m+n+p$; and since when this is known the ratios $x : y : z$ are linearly determinable, we have $m+n+p$ for the order μ of the system.

Thus, if $m = n = p = 2$, selecting the particular system

$$a\xi^2 + 2b\xi\eta + c\eta^2 = 0,$$
$$b\xi^2 + 2c\xi\eta + d\eta^2 = 0,$$
$$c\xi^2 + 2d\xi\eta + e\eta^2 = 0,$$

it is possible in this case to obtain two resulting equations of the orders two and three respectively, and which consequently constitute the system of the sixth order, without containing any extraneous system. In fact, from the identical equation

$$\begin{aligned} & e\xi \,.\, (a\xi^2 + 2b\xi\eta + c\eta^2) \\ & - (4d\xi + 2e\eta)(b\xi^2 + 2c\xi\eta + d\eta^2) \\ & + (3c\xi + 2d\eta)(c\xi^2 + 2d\xi\eta + e\eta^2) \\ & \qquad = (ae - 4bd + 3c^2)\,\xi^3, \end{aligned}$$

and
$$\begin{aligned}&(ce-d^2)(a\xi^2+2b\xi\eta+c\eta^2)\\&+(cd-be)(b\xi^2+2c\xi\eta+d\eta^2)\\&+(bd-c^2)(c\xi^2+2d\xi\eta+e\eta^2)\\&\qquad=(ace-ad^2-b^2e-c^3+2bcd)\,\xi^2,\end{aligned}$$

we deduce
$$\begin{cases}ae-4bd+3c^2=0,\\ ace-ad^2-b^2e-c^3+2bcd=0;\end{cases}$$

which form the system in question, and may for shortness be represented by $I=0$, $J=0$.

The three equations in ξ, η may be considered as expressing that
$$a\xi^3+3b\xi^2\eta+3c\xi\eta^2+d\eta^3=0,$$
$$b\xi^3+3c\xi^2\eta+3d\xi\eta^2+e\eta^3=0,$$

have a pair of equal roots in common; in other words, that it is possible to satisfy identically
$$(A\xi+B\eta)(a\xi^3+3b\xi^2\eta+3c\xi\eta^2+d\eta^3)+(A'\xi+B'\eta)(b\xi^3+3c\xi^2\eta+3d\xi\eta^2+e\eta^3)=0.$$

Equating to zero the separate terms of this equation, and eliminating A, B, A', B', we obtain
$$\left\|\begin{matrix}\cdot & a, & 3b, & 3c, & d\\ \cdot & b, & 3c, & 3d, & e\\ a, & 3b, & 3c, & d, & \cdot\\ b, & 3c, & 3d, & e, & \cdot\end{matrix}\right\|=0.$$

It is not at first sight obvious what connection these equations have with the two, $I=0$, $J=0$, but by actual expansion they reduce themselves to the following five,
$$\begin{aligned}3\,[2(ce-d^2)\,I-3eJ]&=0,\\ 3\,[(be-cd)\,I-3dJ]&=0,\\ [-(ae+2bd-3c^2)\,I+9cJ]&=0,\\ 3\,[(ad-bc)\,I-3bJ]&=0,\\ 3\,[2(ac-b^2)\,I-3aJ]&=0;\end{aligned}$$

which are satisfied by $I=0$, $J=0$. By the theorem above given, the equations are to be considered as forming a system of the tenth order; the system must therefore be considered as composed of the system $I=0$, $J=0$, and of a system of the fourth order. The system of the fourth order may be written in the form
$$\begin{aligned}&2(ac-b^2) : ad-bc : ae+2bd-3c^2 : be-cd : 2(ce-d^2) : 3J\\ =\ & a : b : 3c : d : e : I:\end{aligned}$$

but to justify this, it must be shown first that these equations reduce themselves to two independent equations; and next that system is really one of the fourth order. We may remark in the first place, that if

$$u, \; = a\xi^4 + 4b\xi^3\eta + 6c\xi^2\eta^2 + 4d\xi\eta^3 + e\eta^4,$$

is a perfect square, the coefficients will be proportional to those of $\frac{d^2u}{d\xi^2}\frac{d^2u}{d\eta^2} - \left(\frac{d^2u}{d\xi d\eta}\right)^2$. [1] Thus the conditions requisite in order that u may be a perfect square, are given by the system

$$\begin{array}{ccccccccc} & 2(ac-b^2) & : & (ad-bc) & : & ae+2bd-3c^2 & : & be-cd & : & 2(ce-d^2) \\ = & a & : & b & : & 3c & : & d & : & e, \end{array}$$

or these equations are equivalent to two independent equations only (this may be easily verified *à posteriori*); and by writing $3J$ in the form

$$e(ac-b^2) - 2d(ad-bc) + c(ae+2bd-3c^2) - 2b(be-cd) + a(ce-d^2),$$

the remaining equations of the complete system (3) are immediately deduced; thus the latter system contains only two independent equations. (The preceding reasoning shows that the system (3) expresses the conditions in order that the equations

$$a\xi^3 + 3b\xi^2\eta + 3c\xi\eta^2 + d\eta^3 = 0, \quad b\xi^3 + 3c\xi^2\eta + 3d\xi\eta^2 + e\eta^3 = 0,$$

may have a pair of unequal roots in common: we have already seen that the equations $I=0$, $J=0$ represent the conditions in order that these two equations may have a pair of equal roots in common.) Finally, to verify *à posteriori* the fact of the system (3) being one only of the fourth order, me may, as Mr Salmon has done in the memoir above referred to, represent the system by the two equations

$$a(ce-d^2) - e(ac-b^2) = 0, \quad e(ad-bc) - 2b(ce-d^2) = 0,$$

that is, by

$$ad^2 - eb^2 = 0, \quad 2bd^2 - 3bce + ade = 0.$$

These equations contain the extraneous system ($a=0$, $b=0$) and the extraneous systems ($b=0$, $d=0$) and ($d=0$, $e=0$), each of which last, as Mr Salmon has remarked from geometrical considerations, counts double, or the system is one of the 4th order only.

[1] More generally whatever be the order of u, if u contain a square factor, this square factor may easily be shown to occur in $\frac{d^2u}{d\xi^2}\frac{d^2u}{d\eta^2} - \left(\frac{d^2u}{d\xi d\eta}\right)^2$.

78.

NOTE ON THE MOTION OF ROTATION OF A SOLID OF REVOLUTION.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 268—270.]

USING the notation employed in my former papers on the subject of rotation (*Cambridge Math. Journal*, vol. III. pp. 224—232, [6]; and *Cambridge and Dublin Math. Journal*, vol. I. pp. 167—264, [37]), suppose $B = A$, then r is constant, equal to n suppose; and writing

$$\frac{(A-C)n}{A} = \nu, \quad \text{or } C = \left(1 - \frac{\nu}{n}\right) A;$$

also putting

$$\theta = \nu t + \gamma,$$

(where γ is an arbitrary constant) the values of p, q, r are easily seen to be given by the equations

$$\begin{cases} p = M \sin\theta, \\ q = M \cos\theta, \\ r = n, \end{cases}$$

(where M is arbitrary). And consequently

$$h = A\ \{M^2 + n(n-\nu)\},$$
$$k^2 = A^2\{M^2 + \ (n-\nu)^2\}.$$

Also, since $\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 = k^2$, we may write

$$\mathrm{a} = -k \sin i \cos j,$$
$$\mathrm{b} = \ k \cos i \cos j,$$
$$\mathrm{c} = \ k \sin j;$$

k having the value above given, and the angles i, j being arbitrary.

From the equations (12) and (15) in the second of the papers quoted, we deduce

$$2v = k\{k + A(n-\nu)\sin j + MA\cos j\cos(\theta+i)\},$$
$$\Phi = k\{n\sin j + M\cos j\cos(\theta+i)\},$$
$$\nabla = -\nu kMA\cos j\sin(\theta+i),$$

(values which verify as they should do the equation (19)). Hence, from the equation (27), writing $\frac{2dv}{\nabla} = dt = \frac{1}{\nu}d\theta$, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{1}{\nu}\int d\theta\,\frac{h + kn\sin j + kM\cos j\cos(\theta+i)}{k + A(n-\nu)\sin j + MA\cos j\cos(\theta+i)}.$$

This is easily integrated; but the only case which appears likely to give a simple result is when the quantity under the integral sign is constant, or

$$A(h + kn\sin j) = k\{k + A(n-\nu)\sin j\},$$

or

$$Ah - k^2 + Ak\nu\sin j = 0;$$

that is,

$$A(n-\nu) + k\sin j = 0:$$

whence

$$\sin j = -\frac{A(n-\nu)}{k},\quad \cos j = \frac{AM}{k},\quad \text{or } \tan j = \frac{-(n-\nu)}{M} = -\frac{C}{A}\,\frac{n}{M}.$$

Observing that $\frac{1}{2}\pi - j$ is the inclination of the axis of z to the normal to the invariable plane, this equation shows that the supposition above is not any restriction upon the generality of the motion, but amounts only to supposing that the axis of z (which is a line fixed in space) is taken upon the surface of a certain right cone having for its axis the perpendicular to the invariable plane. Resuming the solution of the problem, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{k}{\nu A}\theta,$$

which may also be written under the form

$$2\tan^{-1}\frac{\Omega}{k} = \delta_1 + \frac{kt}{A},$$

$\left(\text{where } \delta_1 = \delta + \frac{k\gamma}{\nu A}\right)$. And hence

$$\Omega = k\tan\tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right) = k\tan\psi,$$

where

$$\psi = \tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right).$$

Substituting these values,

$$\mathrm{a} = -MA\sin i, \quad \mathrm{b} = MA\cos i, \quad \mathrm{c} = -A(n-\nu),$$

$$2\upsilon = A^2M^2\{1+\cos(\theta+i)\} = 2M^2A^2\cos^2\tfrac{1}{2}(\theta+i);$$

and substituting in the equations (14) the values of λ, μ, ν reduce themselves to

$$\begin{cases} \lambda = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\sin\tfrac{1}{2}(\theta-i) - A(n-\nu)\cos\tfrac{1}{2}(\theta-i)\}, \\ \mu = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\cos\tfrac{1}{2}(\theta-i) + A(n-\nu)\sin\tfrac{1}{2}(\theta-i)\}, \\ \nu = \quad \tan\tfrac{1}{2}(\theta+i); \end{cases}$$

where, recapitulating, $\theta = \nu t + \gamma$, $2\psi = \dfrac{kt}{A} + \delta_1$.

I may notice, in connexion with the problem of rotation, a memoir, "Specimen Inaugurale de motu gyratorio corporis rigidi &c.," by A. S. Rueb (Utrecht, 1834), which contains some very interesting developments of the ordinary solution of the problem, by means of the theory of elliptic functions.

79.

ON A SYSTEM OF EQUATIONS CONNECTED WITH MALFATTI'S PROBLEM, AND ON ANOTHER ALGEBRAICAL SYSTEM.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IV. (1849), pp. 270—275.]

CONSIDER the equations

$$\begin{aligned} by^2 + cz^2 + 2fyz &= \theta^2 a\,(bc - f^2)\,, \\ cz^2 + ax^2 + 2gzx &= \theta^2 b\,(ca - g^2)\,, \\ ax^2 + by^2 + 2hxy &= \theta^2 c\,(ab - h^2)\,; \end{aligned}$$

or, as they may be more conveniently written,

$$\begin{aligned} by^2 + cz^2 + 2fyz &= \theta^2 a\mathfrak{A}, \\ cz^2 + ax^2 + 2gzx &= \theta^2 b\mathfrak{B}, \\ ax^2 + by^2 + 2hxy &= \theta^2 c\mathfrak{C}. \end{aligned}$$

The second and third equations give

$$(g^2\mathfrak{C} - h^2\mathfrak{B})\,x^2 - b^2\mathfrak{B}y^2 + c^2\mathfrak{C}z^2 + 2cg\mathfrak{C}zx - 2bh\mathfrak{B}xy = 0\,,$$

hence $\{(g^2\mathfrak{C} - h^2\mathfrak{B})\,x - bh\mathfrak{B}y + cg\mathfrak{C}z\}^2 - \mathfrak{B}\mathfrak{C}\,(-bgy + chz)^2 = 0$, and consequently

$$(g^2\mathfrak{C} - h^2\mathfrak{B})\,x - b\sqrt{\mathfrak{B}}\,(g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}})\,y + c\sqrt{\mathfrak{C}}\,(g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}})\,z = 0:$$

dividing this by $g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}}$, and writing down the system of equations to which the equation thus obtained belongs,

$$\begin{aligned} (g\sqrt{\mathfrak{C}} - h\sqrt{\mathfrak{B}})\,x - \quad & b\sqrt{\mathfrak{B}}\;y + \quad c\sqrt{\mathfrak{C}}\;z = 0, \\ a\sqrt{\mathfrak{A}}\;x + (h\sqrt{\mathfrak{A}} - f\sqrt{\mathfrak{C}})\,y - \quad & c\sqrt{\mathfrak{C}}\;z = 0, \\ -a\sqrt{\mathfrak{A}}\;x + \quad & b\sqrt{\mathfrak{B}}\;y + (f\sqrt{\mathfrak{B}} - g\sqrt{\mathfrak{A}})\,z = 0. \end{aligned}$$

Hence also

$$\begin{aligned}
&(\ \ h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} - f\sqrt{\mathfrak{C}})\, y - (\ \ g\sqrt{\mathfrak{A}} - f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})\, z = 0,\\
&(-g\sqrt{\mathfrak{A}} + f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})\, z - (\ \ a\sqrt{\mathfrak{A}} + h\sqrt{\mathfrak{B}} - g\sqrt{\mathfrak{C}})\, x = 0,\\
&(\ \ a\sqrt{\mathfrak{A}} - h\sqrt{\mathfrak{B}} + g\sqrt{\mathfrak{C}})\, x - (-h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} + f\sqrt{\mathfrak{C}})\, y = 0\,;
\end{aligned}$$

these equations may be written

$$\begin{aligned}
&\frac{1}{K}\{\ \ \mathfrak{F}\sqrt{\mathfrak{A}} - \mathfrak{G}\sqrt{\mathfrak{B}} - \mathfrak{H}\sqrt{\mathfrak{C}} + \sqrt{(\mathfrak{A}\mathfrak{B}\mathfrak{C})}\}\ [\{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\}\, y - \{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}\, z] = 0,\\
&\frac{1}{K}\{-\mathfrak{F}\sqrt{\mathfrak{A}} + \mathfrak{G}\sqrt{\mathfrak{B}} - \mathfrak{H}\sqrt{\mathfrak{C}} + \sqrt{(\mathfrak{A}\mathfrak{B}\mathfrak{C})}\}\ [\{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}\, z - \{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\, x] = 0,\\
&\frac{1}{K}\{-\mathfrak{F}\sqrt{\mathfrak{A}} - \mathfrak{G}\sqrt{\mathfrak{B}} + \mathfrak{H}\sqrt{\mathfrak{C}} + \sqrt{(\mathfrak{A}\mathfrak{B}\mathfrak{C})}\}\ [\{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\, x - \{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\}\, y] = 0\,;
\end{aligned}$$

where, as usual,

$$\begin{aligned}
&\mathfrak{F} = gh - af, \quad \mathfrak{G} = hf - bg, \quad \mathfrak{H} = fg - ch,\\
&K = abc - af^2 - bg^2 - ch^2 + 2fgh\,;
\end{aligned}$$

(in fact the coefficient of y in the first equation is

$$\frac{1}{K}\{(\mathfrak{F}\mathfrak{G} - \mathfrak{C}\mathfrak{H})\sqrt{\mathfrak{A}} + (\mathfrak{A}\mathfrak{C} - \mathfrak{G}^2)\sqrt{\mathfrak{B}} - (\mathfrak{G}\mathfrak{H} - \mathfrak{A}\mathfrak{F})\sqrt{\mathfrak{C}}\}, = h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} - f\sqrt{\mathfrak{C}},$$

as it should be, and similarly for the coefficients of the remaining terms). We have therefore

$$\{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\, x = \{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\}\, y = \{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}\, z\,;$$

or, what comes to the same thing,

$$\begin{aligned}
yz &= \tfrac{1}{2}\, s\, \{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\},\\
zx &= \tfrac{1}{2}\, s\, \{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\},\\
xy &= \tfrac{1}{2}\, s\, \{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}.
\end{aligned}$$

Now

$$\begin{aligned}
a\,\{\mathfrak{G} + \sqrt{(\mathfrak{A}\mathfrak{C})}\}\,\{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\} &= \{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\,\{abc - fgh + f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\},\\
b\,\{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}\,\{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\} &= \{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\}\,\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} + g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\},\\
c\,\{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\,\{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\} &= \{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\}\,\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} + h\sqrt{(\mathfrak{A}\mathfrak{B})}\},
\end{aligned}$$

{as readily appears by writing the first of these equations under the form

$$a\,\{\mathfrak{G} + \sqrt{(\mathfrak{A}\mathfrak{C})}\}\,\{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\} = \{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\}\,\{a\mathfrak{A} - f\mathfrak{F} + f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\},$$

and comparing the rational term and the coefficients of $\sqrt{(\mathfrak{B}\mathfrak{C})}$, $\sqrt{(\mathfrak{C}\mathfrak{A})}$, $\sqrt{(\mathfrak{A}\mathfrak{B})}$}.

Hence, observing the values of yz, zx, xy, we find

$$x^2 = \frac{s}{2a}\{abc - fgh + f\sqrt{(\mathfrak{BC})} - g\sqrt{(\mathfrak{CA})} - h\sqrt{(\mathfrak{AB})}\},$$

$$y^2 = \frac{s}{2b}\{abc - fgh - f\sqrt{(\mathfrak{BC})} + g\sqrt{(\mathfrak{CA})} - h\sqrt{(\mathfrak{AB})}\},$$

$$z^2 = \frac{s}{2c}\{abc - fgh - f\sqrt{(\mathfrak{BC})} - g\sqrt{(\mathfrak{CA})} - h\sqrt{(\mathfrak{AB})}\}.$$

Hence, forming the value of any one of the functions $by^2 + cz^2 + 2fyz$, $cz^2 + ax^2 + 2gxy$, $ax^2 + by^2 + 2hxy$, we obtain $s = \theta^2$; or we have

$$\begin{cases} x^2 = \dfrac{\theta^2}{2a}\{abc - fgh + f\sqrt{(\mathfrak{BC})} - g\sqrt{(\mathfrak{CA})} - h\sqrt{(\mathfrak{AB})}\}, \\ y^2 = \dfrac{\theta^2}{2b}\{abc - fgh - f\sqrt{(\mathfrak{BC})} + g\sqrt{(\mathfrak{CA})} - h\sqrt{(\mathfrak{AB})}\}, \\ z^2 = \dfrac{\theta^2}{2c}\{abc - fgh - f\sqrt{(\mathfrak{BC})} - g\sqrt{(\mathfrak{CA})} + h\sqrt{(\mathfrak{AB})}\}, \\ yz = \frac{1}{2}\theta^2\{\mathfrak{F} + \sqrt{(\mathfrak{BC})}\}, \\ zx = \frac{1}{2}\theta^2\{\mathfrak{G} + \sqrt{(\mathfrak{CA})}\}, \\ xy = \frac{1}{2}\theta^2\{\mathfrak{H} + \sqrt{(\mathfrak{AB})}\}. \end{cases}$$

It may be remarked that the equations

$$b\ y^2 + c\ z^2 + 2f\ yz = L,$$
$$c'\ z^2 + a'x^2 + 2g'\ yz = M,$$
$$a''x^2 + b''y^2 + 2h''xy = N,$$

in which the coefficients are supposed to be such that the functions

$$M(a''x^2 + b''y^2 + 2h''xy) - N(c'\ z^2 + a'\ x^2 + 2g'\ yz),$$
$$N(b\ y^2 + c\ z^2 + 2f\ yz) - L(a''x^2 + b''y^2 + 2h''yz),$$
$$L(c'\ z^2 + a'x^2 + 2g'yz) - M(b\ y^2 + c\ z^2 + 2f\ yz),$$

are each of them decomposable into linear factors, may always be reduced to a system of equations similar to those which have just been solved.

Suppose

$$f = g = h = \frac{1}{\theta^2} = \sqrt{\left(\frac{abc}{a+b+c}\right)} = r,$$

and write $\sqrt{X}$, $\sqrt{Y}$, $\sqrt{Z}$ instead of x, y, z. The equations to be solved become

$$bY + cZ + 2r\sqrt{(YZ)} = (b+c)\,r,$$
$$cZ + aX + 2r\sqrt{(ZX)} = (c+a)\,r,$$
$$aX + bY + 2r\sqrt{(XY)} = (a+b)\,r,$$

where $r^2 = \dfrac{abc}{a+b+c}$, and the solution is

$$\begin{aligned}
\mathrm{x} &= \frac{r}{2a}\{a+b+c-r+\sqrt{(r^2+a^2)}-\sqrt{(r^2+b^2)}-\sqrt{(r^2+c^2)}\},\\
\mathrm{y} &= \frac{r}{2b}\{a+b+c-r+\sqrt{(r^2+a^2)}+\sqrt{(r^2+b^2)}-\sqrt{(r^2+c^2)}\},\\
\mathrm{z} &= \frac{r}{2c}\{a+b+c-r-\sqrt{(r^2+a^2)}-\sqrt{(r^2+b^2)}+\sqrt{(r^2+c^2)}\},\\
\sqrt{(\mathrm{yz})} &= \tfrac{1}{2}\{r-a+\sqrt{(r^2+a^2)}\},\\
\sqrt{(\mathrm{zx})} &= \tfrac{1}{2}\{r-b+\sqrt{(r^2+b^2)}\},\\
\sqrt{(\mathrm{xy})} &= \tfrac{1}{2}\{r-c+\sqrt{(r^2+c^2)}\},
\end{aligned}$$

a system of formulæ which contain the solution of the problem "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle." In fact, if r denote the radius of the inscribed circle, and a, b, c the distances of the angles of the triangle from the points where the sides are touched by the inscribed circle (quantities which it is well known satisfy the condition $r^2 = \dfrac{abc}{a+b+c}$), also if x, y, z denote the radii of the required circles, there is no difficulty whatever in obtaining for the determination of x, y, z, the above system of equations. The problem in question was first proposed and solved by an Italian geometer named Malfatti, and has been called after him Malfatti's problem. His solution, dated 1803, and published in the 10th volume of the Transactions of the Italian Academy of Sciences, appears to have consisted in showing that the values first found for the radii of the three circles satisfy the equations given above, without any indication of the process of obtaining the expressions for these radii. Further information as to the history of the problem may be found in the memoir "Das Malfattische Problem neu gelöst von C. Adams," Winterthur, 1846.

In connexion with the preceding investigations may be considered the problem of determining l and m from the equations

$$\begin{aligned}
B(l+\theta)^2 - 2H(l+\theta)m + (A+1)m^2 &= 0,\\
A(m+\theta)^2 - 2H(m+\theta)l + (B+1)l^2 &= 0;
\end{aligned}$$

which express that the function

$$\theta^2 U + (lx+my)^2, \qquad (U = Ax^2 + 2Hxy + By^2),$$

has for one of its factors a factor of $U+x^2$, and for the other of its factors a factor of $U+y^2$. There is no difficulty in solving these equations; and if we write

$$K = AB - H^2, \quad \varpi_1 = \sqrt{(-K-B)}, \quad \varpi_2 = \sqrt{(-K-A)},$$

the result is easily shown to be

$$l : m : \theta = A(B+H+\varpi_1) : B(A+H+\varpi_2) : (H+\varpi_1)(H+\varpi_2) - AB.$$

But the problem may be considered as the problem for two variables, analogous to that of determining the conic having a double contact with a given conic, and touching three conics each of them having a double contact with the given conic; and in this point of view I was led to the following solution. If we assume

$$Bl - Hm = u, \quad -Hl + Am = v,$$

or, what is the same thing,

$$Kl = Au + Hv, \quad Km = Hu + Bv,$$

then putting

$$\nabla = Au^2 + 2Huv + Bv^2,$$

the two equations become after some reduction

$$(u - K\theta)^2 = -\varpi_1^2\left(K\theta^2 + \frac{1}{K}\nabla\right),$$

$$(v - K\theta)^2 = -\varpi_2^2\left(K\theta^2 + \frac{1}{K}\nabla\right).$$

Hence, writing $K\theta^2 + \frac{1}{K}\nabla = -s^2$, we have

$$u = K\theta + \varpi_1 s, \quad v = K\theta + \varpi_2 s, \quad \nabla + K^2\theta^2 + Ks^2 = 0;$$

and substituting these values of u, v in the last equation,

$$A(K\theta + \varpi_1 s)^2 + 2H(K\theta + \varpi_1 s)(K\theta + \varpi_2 s) + B(K\theta + \varpi_2 s)^2 + K^2\theta^2 + Ks^2 = 0,$$

or reducing,

$$K^2\theta^2(A + 2H + B + 1) + 2K\theta s\{(A+H)\varpi_1 + (H+B)\varpi_2\} + s^2(A\varpi_1^2 + 2H\varpi_1\varpi_2 + B\varpi_2^2 + K) = 0;$$

whence

$$\begin{aligned}
&[K\theta(A + 2H + B + 1) + s\{(A+H)\varpi_1 + (H+B)\varpi_2\}]^2 \\
&= s^2[\{(A+H)\varpi_1 + (H+B)\varpi_2\}^2 - (A + 2H + B + 1)(A\varpi_1^2 + 2H\varpi_1\varpi_2 + B\varpi_2^2 + K)], \\
&= s^2\{-K(\varpi_1 - \varpi_2)^2 - (A\varpi_1^2 + 2H\varpi_1\varpi_2 + B\varpi_2^2) - K(A + 2H + B + 1)\}, \\
&= s^2\{-(A+K)\varpi_1^2 - (B+K)\varpi_2^2 - K(A + 2H + B + 1) + 2(K-H)\varpi_1\varpi_2\}, \\
&= s^2\{2\varpi_1^2\varpi_2^2 - K(A + 2H + B + 1) + 2(K-H)\varpi_1\varpi_2\}, \\
&= s^2(\varpi_1\varpi_2 + K - H)^2:
\end{aligned}$$

and therefore

$$K\theta(A+2H+B+1)+s\{(A+H)\varpi_1+(H+B)\varpi_2\}=s(\varpi_1\varpi_2+K-H),$$

giving

$$s=\frac{K\theta(A+2H+B+1)}{\varpi_1\varpi_2-(A+H)\varpi_1-(H+B)\varpi_2+K-H}.$$

But

$$Kl=Au+Hv=(A+H)K\theta+(A\varpi_1+H\varpi_2)s,\quad Km=(H+B)K\theta+(H\varpi_1+B\varpi_2)s,$$

and substituting the above value of s, we obtain, after some simple reductions,

$$\begin{aligned} l:m:\theta=(\varpi_1\varpi_2+K-H)(A+H-\varpi_2)&:(\varpi_1\varpi_2+K-H)(B+H-\varpi_2)\\ &:\{\varpi_1\varpi_2-(A+H)\varpi_1-(B+H)\varpi_2+K-H\},\end{aligned}$$

a result which presents itself in a very different form from the one previously obtained: if, however, the terms of this proportion be multiplied by the factor

$$\frac{H\varpi_1\varpi_2-K\varpi_1-K\varpi_2+AB-KH}{(A+2H+B+1)K},$$

they become (as they ought to do) identical with those of the former proportion, and the identical equations to which this process gives rise are not without interest.

80.

SUR QUELQUES TRANSMUTATIONS DES LIGNES COURBES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIV. (1849) pp. 40—46.]

JE suppose qu'une courbe quelconque soit représentée par une équation homogène entre trois variables x, y, z, et je me propose d'examiner (en ne faisant attention qu'aux droites et aux coniques) ce que signifient les transmutations

I. $$\xi = \sqrt{x}, \quad \eta = \sqrt{y}, \quad \zeta = \sqrt{z},$$

II. $$\xi = x^2, \quad \eta = y^2, \quad \zeta = z^2,$$

III. $$\xi = \frac{1}{x}, \quad \eta = \frac{1}{y}, \quad \zeta = \frac{1}{z},$$

ces quantités étant prises dans chaque cas pour de nouvelles coordonnées. Les théorèmes auxquels on se trouve conduit comprennent les théorèmes qu'obtient M. William Roberts par sa méthode générale de transmutation en écrivant $n = \frac{1}{2}$, $n = 2$. (*Voir* sa Note sur ce sujet, t. XIII. [1848] de ce Journal, page 209).[1]

I. Soit

$$\xi = \sqrt{x}, \quad \eta = \sqrt{y}, \quad \zeta = \sqrt{z}.$$

Je supposerai ici et partout dans ce mémoire que PQR soit le triangle formé par les droites

$$x = 0, \quad y = 0, \quad z = 0.$$

[1] M. William Roberts a bien voulu me parler l'été passé à Dublin, avant que cette Note eût paru, des théorèmes qui devaient s'y trouver: cela me suggéra, et je lui communiquai, peu de jours après, la deuxième et la troisième de mes méthodes de transmutation. On verra dans la suite que ce sont la première et la deuxième de mes méthodes qui correspondent aux cas $n = \frac{1}{2}$, $n = 2$, respectivement, mais que la troisième est aussi très étroitement liée avec le cas $n = \frac{1}{2}$.

Cela posé, en considérant les nouvelles coordonnées ξ, η, ζ, on voit tout de suite qu'un système d'équations telles que

$$\xi : \eta : \zeta = \alpha : \beta : \gamma$$

correspond à un point; qu'une équation linéaire quelconque

$$A\xi + B\eta + C\zeta = 0$$

correspond à une conique qui touche les trois côtés du triangle PQR (ou, si l'on veut, inscrite à ce triangle); et qu'une équation quelconque du second ordre

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\zeta\eta + 2g\xi\zeta + 2h\eta\xi = 0$$

correspond à une courbe du quatrième ordre qui touche les trois côtés du triangle PQR, chaque côté deux fois. Et réciproquement, de telles coniques et de telles courbes du quatrième ordre peuvent toujours se représenter par une équation linéaire ou par une équation du second ordre entre les coordonnées ξ, η, ζ.

On déduit de là cette propriété générale:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à des coniques, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des coniques qui touchent chacune trois droites fixes, et à des courbes du quatrième ordre qui touchent chacune ces trois droites fixes, deux fois chaque droite."

En supposant que les coefficients g, h se réduisent à zéro, l'équation

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta = 0$$

est celle d'une conique tangente aux deux droites PQ, PR, et la courbe du quatrième ordre se réduit à deux coniques égales et superposées l'une à l'autre. De là:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des coniques qui touchent chacune trois droites fixes, et à une conique qui touche deux de ces droites fixes."

En particulier, en supposant que les deux droites fixes que la conique touche soient les deux droites tangentes à cette conique qui passent par un des foyers, et que la troisième droite fixe soit à l'infini:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des paraboles qui ont pour foyer commun un point fixe, et à une conique qui a aussi ce point fixe pour un de ses foyers."

II. Soit à présent

$$\xi = x^2, \quad \eta = y^2, \quad \zeta = z^2.$$

Un système d'équations telles que

$$\xi : \eta : \zeta = \alpha : \beta : \gamma$$

correspond à quatre points qui formeront ce que l'on peut nommer *un système symétrique par rapport aux points* P, Q, R, ou tout simplement *un système symétrique.* On voit sans peine que les quatre points d'un même système symétrique sont tels, que le quadrilatère formé par ces quatre points a pour points de concours des diagonales et des côtés opposés les points P, Q, R. Pour ne pas interrompre la suite des raisonnements, il convient d'entrer dans quelques détails relatifs à ce sujet. Nous nommerons *courbe symétrique par rapport aux points* P, Q, R, ou simplement *courbe symétrique,* toute courbe lieu d'un système symétrique. Cela posé, il y a une infinité de coniques symétriques; savoir, toute conique par rapport à laquelle les points P, Q, R sont des points conjugués (cela veut dire par rapport à laquelle l'un quelconque de ces points a pour polaire la droite menée par les deux autres) est une conique symétrique. Remarquons qu'une courbe symétrique du quatrième ordre peut se réduire à un système de deux coniques telles, que les points de chaque système symétrique de la courbe soient partagés deux à deux sur les deux coniques. En effet, considérons une conique telle, que, par rapport à cette conique, le point P ait pour polaire la droite QR. Il est facile de construire une autre conique telle, que l'ensemble des deux coniques soit une courbe symétrique. Pour cela, menons une transversale quelconque PMN qui rencontre la conique donnée aux points M, N: soient M', N' les points de rencontre des droites QM, NR et des droites QN, MR; le lieu des deux points M', N' (lesquels seront en ligne droite avec le point P) sera la conique dont il s'agit. Nous dirons que les deux coniques sont des coniques supplémentaires.

En revenant à notre but actuel, une équation linéaire quelconque

$$A\xi + B\eta + C\zeta = 0$$

correspond à une conique symétrique. Une équation du second ordre quelconque

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0$$

correspond à une courbe symétrique du quatrième ordre. Et réciproquement, toute conique symétrique, ou toute courbe symétrique du quatrième ordre, peut se représenter par une équation linéaire, ou du second ordre, avec les coordonnées ξ, η, ζ. De là cette propriété générale:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à des coniques, conduit à un théorème qui se rapporte d'une manière analogue à des systèmes symétriques (par rapport à trois points fixes), à des coniques symétriques, et à des courbes symétriques du quatrième ordre."

Dans ce théorème on peut, si l'on veut, considérer l'un des points d'un système symétrique comme représentant le système, et substituer aux mots *systèmes symétriques* (par rapport à trois points fixes), le mot *points.*

En supposant que dans la courbe du quatrième ordre on ait à la fois

$$ac - g^2 = 0, \quad ab - h^2 = 0,$$

il est facile de voir que la courbe du quatrième ordre se réduit à deux coniques supplémentaires.

En effet, ces conditions étant remplies, en rétablissant les valeurs de ξ, η, ζ, on obtient une équation qui se divise en deux équations, telles que

$$\alpha x^2 + \beta y^2 + \gamma z^2 - \delta yz = 0, \quad \alpha x^2 + \beta y^2 + \gamma z^2 + \delta yz = 0,$$

qui appartiennent, comme on le voit sans peine, à une paire de coniques supplémentaires.

De là, en remarquant qu'il est permis de faire abstraction de l'une de ces coniques:

"Tout théorème qui se rapporte à des points, à des droites, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des coniques symétriques (par rapport à trois points fixes), et à une conique telle, que, par rapport à cette conique, l'un des trois points fixes a pour polaire la droite menée par les deux autres points fixes."

Supposons en particulier que les deux points fixes dont nous venons de parler soient les points où la droite à l'infini est rencontrée par un cercle qui a pour centre le troisième point fixe; ce troisième point fixe sera le centre tant des coniques symétriques par rapport à ces trois points fixes, que de la conique par rapport à laquelle le troisième point fixe a pour polaire la droite menée par les deux autres points fixes. De plus, les asymptotes de l'une quelconque des coniques symétriques formeront avec les droites imaginaires asymptotes du cercle un faisceau harmonique, et de là les asymptotes de la conique dont il s'agit seront à angle droit, ou, autrement dit, cette conique sera une hyperbole équilatère. De là enfin:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des hyperboles équilatères et concentriques, et à une conique concentrique avec les hyperboles."

III. Soit enfin

$$\xi = \frac{1}{x}, \quad \eta = \frac{1}{y}, \quad \zeta = \frac{1}{z}.$$

Cette supposition conduit à l'une des méthodes de transmutation données par M. Steiner parmi les observations générales qui forment la conclusion de son ouvrage intitulé: *Systematische Entwickelung* &c., [Berlin, 1832], méthode qu'obtient M. Steiner au moyen de la théorie de l'hyperboloïde gauche. Je la reproduis ici tant pour en faire voir la théorie analytique qu'à cause de son analogie avec les méthodes I. et II.

Il est évident d'abord qu'un système d'équations telles que

$$\xi : \eta : \zeta = \alpha : \beta : \gamma$$

correspond à un point; de même, une équation linéaire quelconque, telle que

$$A\xi + B\eta + C\zeta = 0,$$

correspond à une conique qui passe par les trois points P, Q, R, et une équation quelconque du second ordre, telle que

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0,$$

correspond à une courbe du quatrième ordre qui a les trois points P, Q, R pour points doubles. Réciproquement, de telles coniques et de telles courbes du quatrième ordre peuvent toujours se représenter par des équations linéaires et par des équations du second ordre. De là:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à des coniques, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des coniques qui passent par trois points fixes, et à des courbes du quatrième ordre qui ont ces trois points fixes pour points doubles."

En supposant que l'on ait à la fois $a=0$, $b=0$, la courbe du quatrième ordre se réduit à une conique qui passe par les deux points Q, R (ou, si l'on veut, au système formé de cette conique et des droites PQ, PR). De là:

"Tout théorème descriptif qui se rapporte à des points, à des droites, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des points, à des coniques qui passent par trois points fixes, et à une conique qui passe par deux de ces points fixes."

On ne peut pas particulariser ce théorème de manière à obtenir des théorèmes intéressants. Mais, en prenant les polaires réciproques des deux théorèmes qui viennent d'être obtenus, on a ces propriétés nouvelles:

"Tout théorème descriptif qui se rapporte à des droites, à des points, et à des coniques, conduit à un théorème qui se rapporte d'une manière analogue à des droites, à des coniques qui touchent chacune trois droites fixes, et à des courbes de la *quatrième classe* qui touchent chacune ces mêmes droites fixes, deux fois chaque droite;"

et:

"Tout théorème qui se rapporte à des droites, à des points, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des droites, à des coniques qui touchent chacune trois droites fixes, et à une conique qui touche deux de ces droites fixes."

De là, comme dans la méthode I.:

"Tout théorème qui se rapporte à des droites, à des points, et à une conique, conduit à un théorème qui se rapporte d'une manière analogue à des droites, à des paraboles qui ont pour foyer commun un point fixe, et à une conique qui a ce même point fixe pour un de ses foyers;"

théorème que l'on doit comparer avec le troisième théorème que donne la méthode I.

Pour compléter cette théorie, nous aurions dû ajouter les deux théorèmes polaires réciproques du premier et du deuxième théorème que donne la méthode I.; mais cela se fait sans la moindre peine. Quant au premier et au deuxième théorème que donne la méthode II., les polaires réciproques de ces deux théorèmes ne conduisent à rien de nouveau.

81.

ADDITION AU MÉMOIRE SUR QUELQUES TRANSMUTATIONS DES LIGNES COURBES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XV. (1850), pp. 351—356: continued from t. XIV. p. 46, **80**.]

JE me propose de résumer ici la théorie des courbes du quatrième ordre, auxquelles donne lieu la première de mes méthodes de transmutation appliquée à une conique quelconque.

L'équation d'une telle courbe est de la forme

$$ax + by + cz + 2f\sqrt{yz} + 2g\sqrt{zx} + 2h\sqrt{xy} = 0,$$

et nous avons déjà vu que cette courbe a pour tangentes doubles les trois droites $x=0$, $y=0$, $z=0$ (droites que nous avons représentées par QR, RP, PQ). Pour trouver les autres propriétés de la courbe, mettons l'équation sous la forme

$$\left(a - \frac{gh}{f}\right)x + \left(b - \frac{hf}{g}\right)y + \left(c - \frac{fg}{h}\right)z + \left(\sqrt{\frac{gh}{f}}\sqrt{x} + \sqrt{\frac{hf}{g}}\sqrt{y} + \sqrt{\frac{fg}{h}}\sqrt{z}\right)^2 = 0;$$

puis, en écrivant, pour plus de simplicité, $\frac{fx}{gh}$, $\frac{gy}{hf}$, $\frac{fz}{gh}$ au lieu de x, y, z, cette équation devient

$$\left(\frac{af}{gh} - 1\right)x + \left(\frac{bg}{hf} - 1\right)y + \left(\frac{ch}{fg} - 1\right)z + (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 = 0;$$

et de là, en mettant

$$\left(\frac{af}{gh} - 1\right)x + \left(\frac{bg}{hf} - 1\right)y + \left(\frac{ch}{fg} - 1\right)z = -w,$$

l'équation de la courbe prend cette forme très simple

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{w}=0,$$

en se souvenant toujours que les quantités x, y, z, w satisfont à l'équation linéaire qui vient d'être donnée. Je représenterai dans la suite cette équation linéaire par

$$\alpha x+\beta y+\gamma z+\delta w=0.$$

Il est évident que la droite $w=0$, de même que les droites QR, RP, PQ, est tangente double de la courbe. De plus, ces quatre droites sont le système complet des tangentes doubles, car la courbe a, comme nous allons le voir, trois points doubles: en effet, la forme rationnelle de l'équation est

$$(x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw)^2-64xyzw=0,$$

et, au moyen de l'identité

$$(x+y)^2(z+w)^2-16xyzw=(x-y)^2(z+w)^2+(x+y)^2(z-w)^2-(x-y)^2(z-w)^2,$$

cette équation rationnelle se transforme en

$$[(x-y)^2-(z-w)^2]^2-4(x+y-z-w)[(x+y)(z-w)^2-(z+w)(x-y)^2]=0,$$

laquelle fait voir que le point $(x=y,\ z=w)$ est un point double; de là aussi les points $(x=z,\ w=y)$, $(x=w,\ y=z)$ sont des points doubles. Remarquons en passant qu'en supposant que les coefficients α, β, γ, δ restent indéterminés, les droites $x=y$, $x=z$, $x=w$ seront des droites quelconques par les points $(x=0,\ y=0)$, $(x=0,\ z=0)$, $(x=0,\ w=0)$ respectivement, et ces droites une fois connues, les droites $y=z$, $z=w$, $w=y$ seront déterminées, la première au moyen des points $(y=0,\ z=0)$, $(y=x,\ z=x)$, la deuxième au moyen des points $(z=0,\ w=0)$, $(z=x,\ w=x)$, et la troisième au moyen des points $(w=0,\ y=0)$, $(w=x,\ y=x)$; et les trois droites ainsi déterminées se couperont nécessairement dans un même point. Cela revient au théorème suivant:

"Les trois points doubles d'une courbe du quatrième ordre avec trois points doubles sont les centres d'un quadrangle dont les côtés passent par les angles du quadrilatère formé par les tangentes doubles de la courbe."

Cette propriété des courbes du quatrième ordre dont il s'agit (je veux dire celle d'avoir trois points doubles) aurait dû faire partie du théorème général donné auparavant pour cette première méthode de transmutation.

En supposant que la conique à transmuter passe par le point P, on aura

$$\alpha+\delta=0,$$

et il suit de là que le point double $(x=w,\ y=w)$, identique dans ce cas avec le point P, se change en point de rebroussement, et en même temps que les droites PQ, PR ne sont plus des tangentes doubles proprement dites, mais se réduisent à des

tangentes simples qui passent par le point de rebroussement. Ajoutons que la tangente au point de rebroussement est la droite $x = w$.

En supposant que la conique à transmuter passe à la fois par les deux points P et Q, nous aurons

$$\alpha + \delta = 0, \quad \beta + \delta = 0.$$

Ici les deux points doubles $(x = w,\ y = z)$, $(y = w,\ x = z)$, identiques dans ce cas avec les points P, Q, deviennent des points de rebroussement, les droites QR, RP, PQ ne sont plus des tangentes doubles proprement dites, mais les droites RP, PQ sont les tangentes simples qui passent par les deux points de rebroussement respectivement, et la droite PQ est la droite menée par les deux points de rebroussement. Ajoutons que les tangentes aux deux points de rebroussement respectivement sont les droites $x = w$ et $y = w$.

On sait qu'un cercle quelconque peut s'envisager comme conique qui passe par deux points fixes, savoir [les points circulaires à l'infini, c'est-à-dire] les points où l'infini, considéré comme droite, est rencontré par les deux droites imaginaires auxquelles se réduit un cercle évanouissant quelconque. En nommant ces droites les *axes imaginaires* de leur point d'intersection, prenons pour les droites PQ, PR les axes imaginaires d'un point quelconque P, et pour la droite QR l'infini. Cela étant, un cercle quelconque sera transmuté dans une courbe du quatrième ordre ayant deux points de rebroussement aux points où l'infini est rencontré par les axes imaginaires du point P, ou, ce qui est la même chose, d'un point quelconque, et ayant de plus un point double. Et le point P, comme point d'intersection de deux axes imaginaires tangents de la courbe, est un foyer de la courbe (voyez le Mémoire de M. Plücker: *Ueber solche Punkte die bei Curven höhern Ordnung als der zweiten den Brennpunkten der Kegelschnitte entsprechen*, Journal de M. Crelle, t. x. [1833] pp. 84—91). Cela suffit pour faire voir que la courbe est un limaçon de Pascal ayant le point P pour le foyer qui n'est pas le point double. En effet, prenant pour vrai le théorème: "Les ovales de Descartes ont deux points de rebroussement aux points où l'infini est rencontré par les axes imaginaires d'un point quelconque[1]," comme cela revient à huit conditions, et qu'un ovale de Descartes peut être déterminé de manière à satisfaire à six conditions (ce qui fait en tout quatorze conditions, nombre des conditions qui déterminent une courbe du quatrième ordre), toute courbe du quatrième ordre avec deux points de rebroussement, tels que nous venons de les mentionner, sera un ovale de Descartes, et si, de plus, la courbe du quatrième ordre a un point double, elle se réduira en limaçon (cas particulier, comme on sait, des ovales de Descartes). Donc, en résumé, tout cercle est transmuté dans un limaçon ayant un point fixe pour le foyer qui n'est pas le point double, théorème qui se rapporte à la méthode de M. Roberts pour le cas $n = \frac{1}{2}$. L'on doit cependant remarquer que cette méthode est due à M. Chasles. En effet, on trouve dans la Note citée de l'*Aperçu historique*, non-seulement la propriété des droites de se transmuter en des paraboles, mais aussi

[1] M. Chasles a remarqué en passant (Note XXI. de l'*Aperçu historique* [Bruxelles, 1839]) que les ovales de Descartes ont deux points conjugués imaginaires à l'infini, théorème moins complet que celui que je viens d'énoncer. Pour la démonstration du théorème complet, voyez la Note à la suite de ce mémoire.

celle des cercles de se transmuter en des ovales de Descartes (seulement M. Chasles paraît ne pas avoir remarqué que ces ovales étaient nécessairement des limaçons), et c'est la lecture de cette Note qui m'a appris cette théorie de la transmutation des cercles.

En supposant que la conique à transmuter passe par les trois points P, Q, R, nous aurons

$$\alpha+\delta=0,\quad \beta+\delta=0,\quad \gamma+\delta=0;$$

les points doubles, identiques (dans ce cas) avec les points P, Q, R, deviennent des points de rebroussement, et les droites QR, RP, PQ, au lieu d'être des tangentes doubles, sont tout simplement les droites qui passent chacune par deux points de rebroussement. Ajoutons que les tangentes de la courbe, aux trois points de rebroussement respectivement, sont les droites $x-w=0$, $y-w=0$, $z-w=0$.

Il y a encore un cas particulier à considérer, savoir celui où la conique à transmuter est telle que, par rapport à cette conique, les points Q et R sont situés chacun dans la polaire de l'autre; on a alors $f=0$, cas qui échappe à l'analyse ci-devant employée. On voit sans peine que les deux points doubles $(y=w,\ x=z)$, $(z=w,\ y=z)$ deviennent ici identiques, ce qui donne lieu à un point d'osculation. La droite QR et la droite $w=0$ ne sont plus des tangentes doubles proprement dites, mais ces droites deviennent l'une et l'autre identiques avec la tangente au point d'osculation.

Note sur les ovales de Descartes.

De l'équation de ces ovales,

$$\sqrt{(x-a)^2+y^2}=m\sqrt{x^2+y^2}+n,$$

on tire d'abord

$$(1-m^2)(x^2+y^2)-2ax+a^2-n^2=2mn\sqrt{x^2+y^2},$$

puis

$$\begin{aligned}(1-m^2)^2(x^2+y^2)^2-4a(1-m^2)x(x^2+y^2)\\ +2[a^2(1-m^2)-n^2(1+m^2)](x^2+y^2)+4a^2x^2-4a(a^2-n^2)x+(a^2-n^2)^2=0.\end{aligned}$$

Pour trouver la nature de la courbe à l'infini, mettons $x+yi=\xi$, $x-yi=\eta$, $i=\sqrt{-1}$, et introduisons la quantité ζ de manière à rendre l'équation homogène. Cela donne

$$\begin{aligned}(1-m^2)^2\xi^2\eta^2-2a(1-m^2)(\xi+\eta)\xi\eta\zeta\\ +2[a^2(1-m^2)-n^2(1+m^2)]\xi\eta\zeta^2+a^2(\xi+\eta)^2\zeta^2-2a(a^2-n^2)(\xi+\eta)\zeta^3+(a^2-n^2)^2\zeta^4=0;\end{aligned}$$

ce qui fait voir, sans la moindre peine, qu'il y a des points de rebroussement aux points $(\xi=0,\ \zeta=0)$, $(\eta=0,\ \zeta=0)$, savoir, aux points où l'infini, considéré comme droite, est rencontré par les droites $x+yi=0$, $x-yi=0$, qui sont les axes imaginaires du point $x=0$, $y=0$.

Nous pouvons remarquer, en passant, que l'équation

$$(1-m^2)(x^2+y^2)-2ax+(a^2-n^2)=2mn\sqrt{x^2+y^2},$$

conduit, avec beaucoup de facilité, à une autre propriété, donnée par M. Chasles dans la Note déjà citée. En effet, en mettant

$$a'=\frac{n^2-a^2}{a(m^2-1)},\qquad m'=\frac{m^2(n^2-1)}{a(m^2-1)},\qquad n'=\frac{n}{a},$$

cette équation se transforme en

$$(1-m'^2)(x^2+y^2)-2a'x+(a'^2-n'^2)=2m'n'\sqrt{x^2+y^2},$$

et par là on voit que l'équation primitive peut se transformer en

$$\sqrt{(x-a')^2+y^2}=m'\sqrt{x^2+y^2}+n',$$

c'est-à-dire, il y a toujours un troisième foyer de la courbe.

Il ne reste qu'à démontrer que la transformée (selon la méthode de M. Chasles) d'un cercle est toujours un limaçon. Soit, pour cela,

$$r^2-2ar\cos\theta+\delta=0$$

l'équation du cercle; en mettant $\sqrt{mr}$ au lieu de r, et $\frac{1}{2}\theta$ au lieu de θ, cette équation devient $mr-2a\sqrt{mr}\cos\frac{1}{2}\theta+\delta=0$, ce qui donne $(mr+\delta)^2=2a^2mr(1+\cos\theta)$, ou, en mettant $r\cos\theta=x$ et en réduisant à une forme rationnelle,

$$[m^2(x^2+y^2)+\delta^2-2a^2mx]^2-4m^2(\delta-a^2)(x^2+y^2)=0,$$

ce qui appartient évidemment à un ovale de Descartes. En mettant $y=0$, l'équation devient

$$[m^2x^2+2m(\delta-2a^2)x+\delta^2](mx-\delta)^2=0,$$

c'est-à-dire le point $mx-\delta=0$, $y=0$ est point double, ou la courbe est le limaçon de Pascal.

82.

ON THE TRIADIC ARRANGEMENTS OF SEVEN AND FIFTEEN THINGS.

[From the *Philosophical Magazine*, vol. XXXVII. (1850), pp. 50—53.]

THERE is no difficulty in forming with seven letters, a, b, c, d, e, f, g, a system of seven triads containing every possible duad; or, in other words, such that no two triads of the system contain the same duad. One such system, for instance, is

$$abc,\quad ade,\quad afg,\quad bdf,\quad beg,\quad cdg,\quad cef;$$

and this is obviously one of six different systems obtained by permuting the letters a, b, c. We have therefore six different systems containing the triad abc; and there being the same number of systems containing the triads abd, abe, abf and abg respectively, there are in all thirty-five different systems, each of them containing every possible duad. It is deserving of notice, that it is impossible to arrange the thirty-five triads formed with the seven letters into five systems, each of them possessing the property in question. In fact, if this could be done, the system just given might be taken for one of the systems of seven triads. With this system we might (of the systems of seven triads which contain the triad abd) combine either the system

$$abd,\quad acg,\quad aef,\quad bce,\quad bfg,\quad dcf,\quad deg,$$

or the system

$$abd,\quad acf,\quad aeg,\quad bcg,\quad bef,\quad dce,\quad dfg;$$

(but any one of the other abd systems would be found to contain a triad in common with the given abc system, and therefore cannot be made use of: for instance, the system abd, acg, aef, bcf, beg, dce, dfg contains the triad beg in common with the given abc system): and whichever of the two proper abd systems we select to combine with the given abc system, it will be found that there is no abe system which does not contain some triad in common, either with the abc system or with the abd system.

The order of the letters in a triad has been thus far disregarded. There are some properties which depend upon considering the triads obtained by cyclical permutations of the three letters as identical, but distinct from the triads obtained by a permutation of two letters, or inversion. Thus *abc, bca, cab* are to be considered as identical *inter se*, but distinct from the triads *acb, cba, bac*, which are also identical *inter se*. I write down the system (equivalent, as far as the mere combination of the letters is concerned, to the system at the commencement of this paper)

ade, afg, bdf, bge, cdg, cef, cba,

derived, it is to be observed, from a pair of triads, *ade, afg*, by a cyclical permutation of the *e, f, g*, and by successively changing the *a* into *b* and into *c*, the remaining triad of the system being the letters *a, b, c* taken in an inverse order. Let it be proposed to derive the system in the same manner from any other two triads of the system; for instance, from the triads *acb, ade*. The process of derivation gives

acb, ade, gcd, geb, fce, fbd, fga, [1]

which is, in fact, the original system. But attempt to derive the system from the two triads *ebg, efc*, the process of derivation gives

ebg, efc, dbf, dcg, abc, agf, ade,

which is not the original system, inasmuch as the triads *dbf, dcg, abc, agf* are inversions of the triads *bdf, cdg, cba, afg* of the original system. The point to be attended to, however, is, that *both* triads of the pair *dbf, dcg*, or of the pair *abc, agf*, are inversions of the triads of the corresponding pair in the original system; the pair is either reproduced (as the pair *efc, dbf*), or there is an inversion of both triads. Where there is no such inversion of the triads of a pair, the system may be said to be properly reproduced; and where there is inversion of the triads of one or more pairs, to be improperly reproduced. There is no difficulty in seeing that the system is properly reproduced from a pair of triads containing in common any one of the letters *a, b, c* or *d*, and improperly reproduced from pairs of triads containing in common any one of the letters *d, e* or *f*. It is owing to the reproduction, proper or improper, of the system from any pair of duads that it is possible to form a system of "octaves" analogous to the quaternions of Sir William Hamilton; the impossibility of a corresponding system of fifteen imaginary quantities arises from the circumstance of there being always, in whatever manner the system of triads is formed, an inversion of a single triad of some one or more pairs of triads containing a letter in common. When the system is considered as successively derived from different pairs, the system is not according to the previous definition reproduced either properly or improperly. A system of triads having the necessary properties with respect to the mere combination of the letters (viz. that $\alpha\beta\gamma$ and $\alpha\delta\epsilon$ being any two

[1] The order of the letters *f, g* is selected so as to reproduce the original system so far as the mere combination of the letters is concerned.

triads having a letter in common, there shall be triads such as $\zeta\beta\delta$, $\zeta\gamma\epsilon$, and $\eta\beta\epsilon$, $\eta\gamma\delta$) may easily be found; the system to be presently given of the triads of fifteen things would answer the purpose. And so would many other systems.

Dropping the consideration of the order of the letters which form a triad, I pass to the case of a system of fifteen letters, *a, b, c, d, e, f, g, h, i, j, k, l, m, n, o.* It is possible in this case, not only to form systems of thirty-five triads containing every possible duad, but this can be done in such manner that the system of thirty-five triads can be arranged in seven systems of five triads, each of these systems containing the fifteen letters[1]. My solution is obtained by a process of derivation from the arrangements *ab . cf . dg . eh* and *ab . cd . ef . gh* as follows; viz. the triads are

iab	*jac*	*kaf*	*lad*	*mag*	*nae*	*oah*
icf	*jfb*	*kbc*	*lce*	*mch*	*ncd*	*ocg*
idg	*jde*	*kdh*	*lgb*	*mbd*	*ngf*	*ofd*
ieh	*jhg*	*kge*	*lhf*	*mfe*	*nhb*	*obe*

and a system formed with *i, j, k, l, m, n o,* which are then arranged in the form

klo	*ino*	*jmo*	*ilm*	*jln*	*ijk*	*kmn*
iab	*jac*	*lad*	*nae*	*kaf*	*mag*	*oah*
ncd	*mdb*	*kbc*	*ocg*	*mch*	*lce*	*icf*
mef	*keg*	*ieh*	*jfb*	*obe*	*ofd*	*jde*
jgh	*lhf*	*nfg*	*khd*	*idg*	*nhb*	*lbg*

an arrangement, which, it may be remarked, contains eight different systems (such as have been considered in the former part of this paper) of seven letters such as *i, j, k, l, m, n, o*; and seven of other seven letters, such as *i, j, k, a, b, c, f*[2]. The theory of the arrangement seems to be worth further investigation.

Assuming that the four hundred and fifty-five triads of fifteen things can be arranged in thirteen systems of thirty-five triads, each system of thirty-five triads containing every possible duad, it seems natural to inquire whether the thirteen systems can be obtained from any one of them by cyclical permutations of thirteen letters. This is, I think, impossible. For let the cyclical permutation be of the letters *a, b, c, d, e, f, g, h, i, j, k, l, m.* Consider separately the triads which contain the letter *n* and the letter *o*; neither of these systems of triads contains the letter, whatever it is, which forms a triad with *n* and *o*. Hence, omitting the letters *n, o,* we have two different sets, each of them of six duads, and composed of the same twelve letters. And each of these systems of duads ought, by the cyclical permutation

[1] The problem was proposed by Mr Kirkman, and has, to my knowledge, excited some attention in the form "To make a school of fifteen young ladies walk together in threes every day for a week so that each two may walk together." It will be seen from the text that I am uncertain as to the existence of a solution to the further problem suggested by Mr Sylvester, "to make the school walk every week in the quarter so that each three may walk together."

[2] [I have somewhat altered this sentence so as to express more clearly what appeared to be the meaning of it.]

in question, to produce the whole system of the seventy-eight duads of the thirteen letters. Hence arranging the duads of the thirteen letters in the form

$$\begin{array}{l}
ab\,.\,bc\,.\,cd\,.\,de\,.\,ef\,.\,fg\,.\,gh\,.\,hi\,.\,ij\,.\,jk\,.\,kl\,.\,lm\,.\,ma \\
ac\,.\,bd\,.\,ce\,.\,df\,.\,eg\,.\,fh\,.\,gi\,.\,hj\,.\,ik\,.\,jl\,.\,km\,.\,la\,.\,mb \\
ad\,.\,be\,.\,cf\,.\,dg\,.\,eh\,.\,fi\,.\,gj\,.\,hk\,.\,il\,.\,jm\,.\,ka\,.\,lb\,.\,mc \\
ae\,.\,bf\,.\,cg\,.\,dh\,.\,ei\,.\,fj\,.\,gk\,.\,hl\,.\,im\,.\,ja\,.\,kb\,.\,lc\,.\,md \\
af\,.\,bg\,.\,ch\,.\,di\,.\,ej\,.\,fk\,.\,gl\,.\,hm\,.\,ia\,.\,jb\,.\,kc\,.\,ld\,.\,me \\
ag\,.\,bh\,.\,ci\,.\,dj\,.\,ek\,.\,fl\,.\,gm\,.\,ha\,.\,ib\,.\,jc\,.\,kd\,.\,le\,.\,mf
\end{array}$$

and consequently the duads of each set ought to be situated one duad in each line. Suppose the sets of duads are composed of the letters *a, b, c, d, e, f, g, h, i, j, k, l,* it does not appear that there is any set of six duads composed of these letters, and situated one duad in each line, other than the single set *al, bk, cj, di, eh, fg*; and the same being the case for any twelve letters out of the thirteen, the derivation of the thirteen systems of thirty-five triads by means of the cyclical permutations of thirteen letters is impossible. And there does not seem to be any obvious rule for the derivation of the thirteen systems from any one of them, or any *prima facie* reason for believing that the thirteen systems do really exist, it having been already shown that such systems do not exist in the case of seven things.

83.

ON CURVES OF DOUBLE CURVATURE AND DEVELOPABLE SURFACES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 18—22.]

This is translated with some slight alterations from the Memoir in Liouville, t. x. (1845) pp. 245—250, 30.

84.

ON THE DEVELOPABLE SURFACES WHICH ARISE FROM TWO SURFACES OF THE SECOND ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 46—57.]

ANY two surfaces considered in relation to each other give rise to a curve of intersection, or, as I shall term it, an Intersect and a circumscribed Developable[1] or Envelope. The Intersect is of course the edge of regression of a certain Developable which may be termed the Intersect-Developable, the Envelope has an edge of regression which may be termed the Envelope-Curve. The order of the Intersect is the product of the orders of the two surfaces, the class of the Envelope is the product of the classes of the two surfaces. When neither the Intersect breaks up into curves of lower order, nor the Envelope into Developables of lower class, the two surfaces are said to form a proper system. In the case of two surfaces of the second order (and class) the Intersect is of the fourth order and the Envelope of the fourth class. Every proper system of two surfaces of the second order belongs to one of the following three classes:—*A*. There is no contact between the surfaces; *B*. There is an ordinary contact; *C*. There is a singular contact. Or the three classes may be distinguished by reference to the conjugates (conjugate points or planes) of the system. *A*. The four conjugates are all distinct; *B*. Two conjugates coincide; *C*. Three conjugates coincide.

To explain this it is necessary to remark that in the general case of two surfaces of the second order not in contact (that is for systems of the class *A*) there is a certain tetrahedron such that with respect to either of the surfaces (or more generally with respect to any surface of the second order passing through the Intersect of the system

[1] The term 'Developable' is used as a substantive, as the reciprocal to 'Curve,' which means curve of double curvature. The same remark applies to the use of the term in the compound Intersect-Developable. For the signification of the term 'singular contact,' employed lower down, see Mr Salmon's memoir 'On the Classification of Curves of Double Curvature,' [same volume] p. 32.

or inscribed in the Envelope) the angles and faces of the tetrahedron are reciprocals of each other, each angle of its opposite face, and *vice versâ*. The angles of the tetrahedron are termed the conjugate points of the system, and the faces of the tetrahedron are termed the conjugate planes of the system, and the term conjugates may be used to denote indifferently either the conjugate planes or the conjugate points. A conjugate plane and the conjugate point reciprocal to it are said to correspond to each other. Each conjugate point is evidently the point of intersection of the three conjugate planes to which it does not correspond, and in like manner each conjugate plane is the plane through the three conjugate points to which it does not correspond.

In the case of a system belonging to the class B, two conjugate points coincide together in the point of contact forming what may be termed a double conjugate point, and in like manner two conjugate planes coincide in the plane of contact (that is the tangent plane through the point of contact) forming what may be termed a double conjugate plane. The remaining conjugate points and planes may be distinguished as single conjugate points and single conjugate planes. It is clear that the double conjugate plane passes through the three conjugate points, and that the double conjugate point is the point of intersection of the three conjugate planes: moreover each single conjugate plane passes through the single conjugate point to which it does not correspond and the double conjugate point; and each single conjugate point lies on the line of intersection of the single conjugate plane to which it does not correspond and the double conjugate plane.

In the case of a system belonging to the class (C), three conjugate points coincide together in the point of contact forming what may be termed a triple conjugate point, and three conjugate planes coincide together in the plane of contact forming a triple conjugate plane. The remaining conjugate point and conjugate plane may be distinguished as the single conjugate point and single conjugate plane. The triple conjugate plane passes through the two conjugate points and the triple conjugate point lies on the line of intersection of the two conjugate planes; the single conjugate plane passes through the triple conjugate point and the single conjugate point lies on the triple conjugate plane.

Suppose now that it is required to find the Intersect-Developable of two surfaces of the second order. If the equations of the surfaces be $\Upsilon = 0$, $\Upsilon' = 0$ (Υ, Υ' being homogeneous functions of the second order of the coordinates ξ, η, ζ, ω), and x, y, z, w represent the coordinates of a point upon the required developable surface: if moreover U, U' are the same functions of x, y, z, w that Υ, Υ' are of ξ, η, ζ, ω and X, Y, Z, W; X', Y', Z', W' denote the differential coefficients of U, U' with respect to x, y, z, w; then it is easy to see that the equation of the Intersect-Developable is obtained by eliminating ξ, η, ζ, ω between the equations

$$\Upsilon = 0, \quad \Upsilon' = 0,$$

$$X\xi + Y\eta + Z\zeta + W\omega = 0,$$

$$X'\xi + Y'\eta + Z'\zeta + W'\omega = 0.$$

If, for shortness, we suppose

$$\bar{F} = YZ' - Y'Z, \quad \bar{L} = XW' - X'W,$$
$$\bar{G} = ZX' - Z'X, \quad \bar{M} = YW' - Y'W,$$
$$\bar{H} = XY' - X'Y, \quad \bar{N} = ZW' - Z'W,$$

(values which give rise to the identical equation

$$\bar{L}\bar{F} + \bar{M}\bar{G} + \bar{N}\bar{H} = 0),$$

then, λ, μ, ν, ρ denoting any indeterminate quantities, the two linear equations in ξ, η, ζ, ω are identically satisfied by assuming

$$\begin{aligned}
\xi &= \quad . \quad \bar{N}\mu - \bar{M}\nu + \bar{F}\rho\,, \\
\eta &= -\bar{N}\lambda \quad . \quad + \bar{L}\nu + \bar{G}\rho\,, \\
\zeta &= \bar{M}\lambda - \bar{L}\mu \quad . \quad + \bar{H}\rho, \\
\omega &= -F\lambda - \bar{G}\mu - \bar{H}\nu \quad .
\end{aligned}$$

and, substituting these values in the equations $\Upsilon = 0$, $\Upsilon' = 0$, we have two equations:

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu + 2L\lambda\rho + 2M\mu\rho + 2N\nu\rho = 0,$$
$$A'\lambda^2 + B'\mu^2 + C'\nu^2 + 2F'\mu\nu + 2G'\nu\lambda + 2H'\lambda\mu + 2L'\lambda\rho + 2M'\mu\rho + 2N'\nu\rho = 0,$$

which are of course such as to permit the four quantities λ, μ, ν, ρ to be simultaneously eliminated. The coefficients of these equations are obviously of the fourth order in x, y, z, w.

Suppose for a moment that these coefficients (instead of being such as to permit this simultaneous elimination of λ, μ, ν, ρ) denoted any arbitrary quantities, and suppose that the indeterminates λ, μ, ν, ρ were besides connected by two linear equations,

$$a\lambda + b\mu + c\nu + d\rho = 0,$$
$$a'\lambda + b'\mu + c'\nu + d'\rho = 0\,;$$

then, putting

$$bc' - b'c = f, \quad ad' - a'd = l\,,$$
$$ca' - c'a = g, \quad bd' - b'd = m,$$
$$ab' - a'b = h, \quad cd' - c'd = n\,,$$

(values which give rise to the identical equation $lf + mg + nh = 0$), and effecting the elimination of λ, μ, ν, ρ between the four equations, we should obtain a final equation $\square = 0$, in which $\square$ is a homogeneous function of the second order in each of the systems of coefficients A, B, &c., and A', B', &c., and a homogeneous function of the fourth order (indeterminate to a certain extent in its form on account of the identical

equation $lf + mg + nh = 0$) in the coefficients f, g, h, l, m, n [1]. But re-establishing the actual values of the coefficients A, B, &c., A', B', &c. (by which means the function $\square$ becomes a function of the sixteenth order in x, y, z, w) the quantities f, g, h, l, m, n ought, it is clear, to disappear of themselves; and the way this happens is that the function $\square$ resolves itself into the product of two factors M and Ψ, the latter of which is independent of f, g, h, l, m, n. The factor M is a function of the fourth order in these quantities, and it is also of the eighth order in the variables x, y, z, w: the factor Ψ is consequently of the eighth order in x, y, z, w. And the result of the elimination being represented by the equation $\Psi = 0$, the Intersect-Developable in the general case, or (what is the same thing) for systems of the class (A), is of the eighth order. In the case of a system of the class (B) the equation obtained as above contains as a factor the square, and in the case of a system of the class (C) the cube, of the linear function which equated to zero is the equation of the plane of contact. The Intersect-Developable of a system of the class (B) is therefore a Developable of the sixth order, and that of a system of the class (C) a Developable of the fifth order. The elimination is in every case most simply effected by supposing two of the quantities λ, μ, ν, ρ to vanish (e.g. ν and ρ): the equations between which the elimination has to be effected then are

$$A\lambda^2 + B\mu^2 + 2H\lambda\mu = 0,$$
$$A'\lambda^2 + B'\mu^2 + 2H'\lambda\mu = 0;$$

and the result may be presented under the equivalent forms

$$(AB' + A'B - 2HH')^2 - 4(AB - H^2)(A'B' - H'^2) = 0,$$

and

$$(AB' - A'B)^2 + 4(AH' - A'H)(BH' - B'H) = 0,$$

the latter of which is the most convenient. These two forms still contain an extraneous factor of the eighth order in x, y, z, w, of which they can only be divested by substituting the actual values of A, B, H, A', B', H'.

A. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0,$$

[1] I believe the result of the elimination is

$$\square = 4(PR - Q^2) = 0,$$

where, if we write $uA + u'A' = A$, &c., the quantities P, Q, R are given by the equation (identical with respect to u, u')

$$\begin{aligned} Pu^2 + 2Quu' + Ru'^2 &= (Aa^2 + \ldots)(Aa'^2 + \ldots) - (Aaa' + \ldots)^2 \\ &= u^2\{(BC - F^2)f^2 + \ldots\} + uu'\{(BC' + B'C - 2FF')f^2 + \ldots\} + u'^2\{(B'C' - F'^2)f^2 + \ldots\} \end{aligned}$$

a theorem connected with that given in the second part of my memoir 'On Linear Transformations' (*Journal*, vol. I. p. 109) [see 14, p. 100]. I am not in possession of any verification *à posteriori* of what is subsequently stated as to the resolution into factors of the function $\square$ and the forms of these factors.

where $x=0$, $y=0$, $z=0$, and $w=0$, are the equations of the four conjugate planes. There is no particular difficulty in performing the operations indicated by the general process given above; and if we write, in order to abbreviate,

$$\begin{aligned} bc'-b'c&=f, & ad'-a'd&=l, \\ ca'-c'a&=g, & bd'-b'd&=m, \\ ab'-a'b&=h, & cd'-c'd&=n, \end{aligned}$$

(values which satisfy the identical equation $lf+mg+nh=0$), the result after all reductions is

$$\begin{aligned} &l^2f^4y^4z^4+m^2g^4z^4x^4+n^2h^4x^4y^4+l^4f^2x^4w^4+m^4g^2y^4w^4+n^4h^2z^4w^4 \\ &+2mng^2h^2x^4y^2z^2+2nlh^2f^2y^4z^2x^2+2lmf^2g^2z^4x^2y^2 \\ &-2m^2n^2ghw^4y^2z^2-2n^2l^2hfw^4z^2x^2-2l^2m^2fgw^4x^2y^2 \\ &+2fmg^2l^2x^4z^2w^2+2gnh^2m^2y^4x^2w^2+2hlf^2n^2z^4y^2w^2 \\ &-2fnh^2l^2x^4y^2w^2-2glf^2m^2y^4z^2w^2-2hmg^2n^2z^4x^2w^2 \\ &+2(mg-nh)(nh-lf)(lf-mg)x^2y^2z^2w^2=0, \end{aligned}$$

which is therefore the equation of the Intersect-Developable for this case. The discussion of the geometrical properties of the surface will be very much facilitated by presenting the equation under the following form, which is evidently one of a system of six different forms,

$$\begin{aligned} &\{m(gx^2+nw^2)(hy^2-gz^2+lw^2)-l(-fy^2+nw^2)(-hx^2+fz^2+mw^2)\}^2 \\ &\qquad -4fglmx^2y^2(hy^2-gz^2+lw^2)(-hx^2+fz^2+mw^2)=0. \end{aligned}$$

B. Two surfaces forming a system belonging to this class may be represented by equations such as

$$\begin{aligned} ax^2+by^2+cz^2+2nzw&=0, \\ a'x^2+b'y^2+c'z^2+2n'zw&=0, \end{aligned}$$

in which $x=0$, $y=0$ are the equations of the single conjugate planes, $z=0$ that of the double conjugate plane or plane of contact, $w=0$ that of an indeterminate plane through the two single conjugate points. If we write

$$\begin{aligned} bc'-b'c&=f, & an'-a'n&=p, \\ ca'-c'a&=g, & bn'-b'n&=q, \\ ab'-a'b&=h, & cn'-c'n&=r, \end{aligned}$$

(values which satisfy the identical equation $pf+qg+rh=0$), the result after all reductions is

$$\begin{aligned} &r^4h^2z^6+2pr^2h(rh-qg)z^4x^2-2qr^2h(pf-rh)z^4y^2 \\ &+4p^2qr^2hz^3x^3w-4pq^2r^3hz^3y^3w \\ &+p^2(rh-qg)^2z^2x^4+q^2(pf-rh)^2z^2y^4+2pq(4r^2h^2-fgpq)z^2x^2y^2 \\ &+4p^3q(rh-qg)zx^4w+4pq^3zy^4w-4p^2q^2(qg-pf)zx^2y^2w \\ &+4p^4q^2x^4w^2+4p^2q^4y^4w^2+8p^3q^3x^2y^2w^2+4p^3qrh^2x^4y^2+4pq^2rh^2x^2y^4=0, \end{aligned}$$

which is therefore the equation of the Intersect-Developable for systems of the case in question. The equation may also be presented under the form

$$\{q(px^2+rz^2)(hy^2-gz^2+2pzw)-p(qy^2+rz^2)(-hx^2+fz^2+2qzw)\}^2$$
$$+4p^2q^2x^2y^2(hy^2-gz^2+2pzw)(-hx^2+fz^2+2qzw)=0,$$

which it is to be remarked contains the extraneous factor z^2. The following is also a form of the same equation,

$$\{r(qg-pf)z^2-fp^2zx^2+gq^2zy^2+2pq(px^2+qy^2)w\}^2$$
$$-4pq(px^2+qy^2+rz^2)\{r(hy^2-gz^2)(fz^2-hx^2)+2pq(gx^2-fy^2)zw\}=0.$$

C. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$a\,x^2+b\,y^2+2f\,yz+2n\,zw=0,$$
$$a'x^2+b'y^2+2f'yz+2n'zw=0,$$

in which $bn'-b'n=0$, $af'-a'f=0$. In these equations $x=0$ is the equation of a properly chosen plane passing through the two conjugate points, $y=0$ is the equation of the single conjugate plane, $z=0$ that of the triple conjugate plane, and $w=0$ is the equation of a properly chosen plane passing through the single conjugate point. Or without loss of generality, we may write

$$\alpha\,(x^2-2yz)+\beta\,(y^2-2zw)=0,$$
$$\alpha'(x^2-2yz)+\beta'(y^2-2zw)=0,$$

where x, y, z and w have the same signification as before[1]. The result after all reductions is

$$4z^3w^2+12y^2z^2w+9y^4z-24x^2yzw-4x^2y^3+8x^4w=0,$$

which may also be presented under the forms

$$z(y^2-2zw)^2-4y(y^2-2zw)(x^2-2yz)+8w(x^2-2yz)^2=0,$$

and

$$z(3y^2+2zw)^2-4x^2(y^3-2x^2w+6yzw)=0.$$

[In these three equations and in the last two equations of p. 495 as originally printed, there was by mistake, an interchange of the letters x and y.]

[1] Of course in working out the equation of the Intersect-Developable, it is simpler to employ the equations $x^2-2yz=0$, $y^2-2zw=0$. These equations belong to two cones which pass through the Intersect and have their vertices in the triple conjugate point and single conjugate point respectively. I have not alluded to these cones in the text, as the theory of them does not come within the plan of the present memoir, the immediate object of which is to exhibit the equations of certain developable surfaces—but these cones are convenient in the present case as furnishing the easiest means of defining the planes $x=0$, $w=0$. If we represent for a moment the single conjugate point by S and the triple conjugate point by T (and the cones through these points by the same letters), then the point T is a point upon the cone S, and the triple conjugate plane which touches the cone S along the line TS touches the cone T along some generating line TM. Let the other tangent plane through the line TS to the cone T be TM', where M' may represent the point where the generating line in question meets the cone S; and we may consider M as the point of intersection of the line TM with the tangent plane through the line SM' to the cone S: then the plane TMM' is the plane represented by the equation $x=0$, and the plane SMM' is that represented by the equation $w=0$. We may add that $y=0$ is the equation of the plane TSM', and $z=0$ that of the plane TSM.

Proceeding next to the problem of finding the envelope of two surfaces of the second order, this is most readily effected by the following method communicated to me by Mr Salmon. Retaining the preceding notation, the equation $U+kU'=0$ belongs to a surface of the second order passing through the Intersect of the two surfaces $U=0$, $U'=0$. The polar reciprocal of this surface $U+kU'=0$ is therefore a surface inscribed in the envelope of the reciprocals of the two surfaces $U=0$, $U'=0$, and consequently this envelope is the envelope (in the ordinary sense of the word) of the reciprocal of the surface $U+kU'=0$, k being considered as a variable parameter. It is easily seen that the reciprocal of the surface $U+kU'=0$ is given by an equation of the form

$$\mathrm{A}+3\mathrm{B}k+3\mathrm{C}k^2+\mathrm{D}k^3=0,$$

in which A, B, C, D are homogeneous functions of the second order in the coordinates x, y, z, w. Differentiating with respect to k, and performing the elimination, we have for the equation of the envelope in question,

$$(\mathrm{AD}-\mathrm{BC})^2-4(\mathrm{AC}-\mathrm{B}^2)(\mathrm{BD}-\mathrm{C}^2)=0\,;$$

or the envelope is, in general or (what is the same thing) for a system of the class (A), a developable of the eighth order. For a system of the class (B) the equation contains as a factor, the square of the linear function which equated to zero is the equation of the plane of contact; or the envelope is in this case a Developable of the sixth order. And in the case of a system of the class (C) the equation contains as a factor the cube of this linear function; or the envelope is a developable of the fifth order only.

A. We may take for the two surfaces the reciprocals (with respect to $x^2+y^2+z^2+w^2=0$) of the equations made use of in determining the Intersect-Developable. The equations of these reciprocals are

$$bcdx^2+cday^2+dabz^2+abcw^2=0\,,$$
$$b'c'd'x^2+c'd'a'y^2+d'a'b'z^2+a'b'c'w^2=0\,;$$

and it is clear from the form of them (as compared with the equations of the surfaces of which they are the reciprocals) that $x=0$, $y=0$, $z=0$, $w=0$, are still the equations of the conjugate planes. We have, introducing the numerical factor 3 to avoid fractions,

$$3\,\{(b+kb')(c+kc')(d+kd')x^2+(c+kc')(d+kd')(a+ka')y^2$$
$$+(d+kd')(a+ka')(b+kb')z^2+(a+ka')(b+kb')(c+kc')w^2\}$$

$=\mathrm{A}+3\mathrm{B}k+3\mathrm{C}k^2+\mathrm{D}k^3$, which determine the values of A, B, C, D.

We have in fact

$$\mathrm{A}=3\,(bcdx^2+cday^2+dabz^2+abcw^2)$$
$$\mathrm{B}=\ (b'cd+bc'd+bcd')\,x^2+\ldots\ldots$$
$$\mathrm{C}=\ (bc'd'+b'cd'+b'c'd)\,x^2+\ldots\ldots$$
$$\mathrm{D}=3\,(b'c'd'x^2+c'd'a'y^2+d'a'b'z^2+a'b'c'w^2)$$

and these values give (with the same signification as before of f, g, h, l, m, n)

$$2(\mathrm{AC}-\mathrm{B}^2) = Aa^2 + Bb^2 + Cc^2 + 2Fbc + 2Gca + 2Hab + 2Lad + 2Mbd + 2Ncd + Dd^2,$$
$$2(\mathrm{BD}-\mathrm{C}^2) = Aa'^2 + Bb'^2 + Cc'^2 + 2Fb'c' + 2Gc'a' + 2Ha'b' + 2La'd' + 2Mb'd' + 2Nc'd' + Dd'^2,$$
$$\mathrm{AD}-\mathrm{BC} = Aaa' + Bbb' + Ccc' + F(bc'+b'c) + G(ca'+c'a) + H(ab'+a'b)$$
$$+ L(ad'+a'd) + M(bd'+b'd) + N(cd'+c'd) + Ddd',$$

where
$$\begin{aligned}
\mathrm{A} &= n^2y^4 + m^2x^4 + f^2w^4 + 2fnz^2w^2 - 2fny^2w^2 + 2nmy^2z^2,\\
\mathrm{B} &= l^2z^4 + n^2x^4 + g^2w^4 + 2gnx^2w^2 - 2glz^2w^2 + 2lnz^2x^2,\\
\mathrm{C} &= m^2x^4 + l^2y^4 + h^2w^4 + 2hly^2w^2 - 2hmx^2w^2 + 2mlx^2y^2,\\
\mathrm{D} &= f^2x^4 + g^2y^4 + h^2z^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgy^2z^2,\\
\mathrm{F} &= l^2y^2z^2,\\
\mathrm{G} &= m^2z^2x^2,\\
\mathrm{H} &= n^2x^2y^2,\\
\mathrm{L} &= f^2x^2w^2,\\
\mathrm{M} &= g^2y^2w^2,\\
\mathrm{N} &= h^2z^2w^2;
\end{aligned}$$

and then $4(\mathrm{AC}-\mathrm{B}^2)(\mathrm{BD}-\mathrm{C}^2)-(\mathrm{AD}-\mathrm{BC})^2 =$

$$\begin{aligned}
&(BC-F^2)f^2 + (CA-G^2)g^2 + (AB-H^2)h^2 + (AD-L^2)l^2 + (BD-M^2)m^2 + (CD-N^2)n^2\\
&\quad + 2(GH-AF)gh + 2(HF-BG)hf + 2(FG-CH)fg\\
&\quad - 2(MN-DF)mn - 2(NL-DG)nl - 2(LM-DH)lm\\
&\quad + 2(AM-LH)lh + 2(BN-NF)mf + 2(CL-NG)ng\\
&\quad - 2(AN-LG)lg - 2(BL-MH)mh - 2(CM-NF)nf\\
&\quad + 2(NH-MG)lf + 2(LF-NH)mg + 2(MG-LF)nh.
\end{aligned}$$

Substituting the values of A, B, &c., in this expression, the result after all reductions is

$$\begin{aligned}
&f^2m^2n^2x^8 + g^2n^2l^2y^8 + h^2l^2m^2z^8 + f^2g^2h^2w^8\\
&+ 2gl^2n(mg-nh)y^6z^2 + 2hm^2l(nh-lf)z^6x^2 + 2fn^2m(lf-mg)x^6y^2\\
&- 2h^2m(mg-nh)y^2z^6 - 2fm^2n(nh-lf)z^2x^6 - 2gn^2l(lf-mg)x^2y^6\\
&+ 2f^2mn(mg-nh)x^6w^2 + 2g^2nl(nh-lf)y^6w^2 + 2h^2lm(lf-mg)z^6w^2\\
&- 2f^2gh(mg-nh)x^2w^6 - 2fg^2h(nh-lf)y^2w^6 - 2fgh^2(lf-mg)z^2w^6\\
&+ f^2(l^2f^2-6ghmn)w^4x^4 + g^2(m^2g^2-6hfnl)w^4y^4 + h^2(n^2h^2-6lmfg)w^4z^4\\
&+ l^2(l^2f^2-6ghmn)y^4z^4 + m^2(m^2g^2-6hfnl)z^4x^4 + n^2(n^2h^2-6lmfg)x^4y^4\\
&+ 2gh(ghmn-3f^2l^2)w^4y^2z^2 + 2hf(hfnl-3g^2m^2)w^4z^2x^2 + 2fg(fglm-3h^2n^2)w^4x^2y^2\\
&+ 2hm(ghmn-3f^2l^2)z^4x^2w^2 + 2fn(hfnl-3g^2m^2)x^4y^2w^2 + 2gl(fglm-3h^2n^2)y^4z^2w^2\\
&- 2gn(ghmn-3f^2l^2)y^4x^2w^2 - 2hl(hfnl-3g^2m^2)z^4y^2w^2 - 2fm(fglm-3h^2n^2)x^4z^2w^2\\
&- 2mn(ghmn-3f^2l^2)x^4y^2z^2 - 2nl(hfnl-3g^2m^2)y^4z^2x^2 - 2lm(fglm-3h^2n^2)x^4z^2y^2\\
&- 2(mg-nh)(nh-lf)(lf-mg)x^2y^2z^2w^2 = 0,
\end{aligned}$$

which is therefore the equation of the envelope for this case. The equation may also be presented under the form

$$w^6\Theta + (mnx^2 + nly^2 + lmz^2)^2 (f^2x^4 + g^2y^4 + h^2x^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgx^2y^2) = 0\,;$$

and there are probably other forms proper to exhibit the different geometrical properties of the surface, but with which I am not yet acquainted.

B. Here taking for the two surfaces the reciprocals of the equations made use of in determining the Intersect-Developable, the equations of these reciprocals are

$$n^2\, b\, x^2 + n^2\, a\, y^2 - a\, b\, c\, w^2 + 2n\, a\, b\, zw = 0,$$
$$n'^2 b' x^2 + n'^2 a' y^2 - a' b' c' w^2 + 2n' a' b' zw = 0,$$

which are similar to the equations of the surfaces of which they are reciprocal, only z and w are interchanged, so that here $x = 0$, $y = 0$ are the single conjugate planes, $z = 0$ is an indeterminate plane passing through the single conjugate points, and $w = 0$ is the equation of the double conjugate plane or plane of contact.

The values of A, B, C, D are

$$\begin{aligned}
A &= 3\,(n^2bx^2 \;\; + n^2ay^2 - abcw^2 + 2nabzw),\\
B &= \;\; (2nn'b \;\; + n^2b')\, x^2 + \ldots\ldots\\
C &= \;\; (2nn'b' + n'^2b)\, x^2 + \ldots\ldots\\
D &= 3\,(n'^2b'x^2 + n'^2a'y^2 - a'b'c'w^2 + 2n'a'b'zw).
\end{aligned}$$

Hence, using f, g, h, p, q, r in the same sense as before, we have for $2\,(AC - B^2)$, $2\,(BD - C^2)$, $(AD - BC)$ expressions of the same form as in the last case (p, q, r being written for l, m, n), but in which

$$\begin{aligned}
A &= \;\; f^2w^4 + 4q^2z^2w^2 + 8qry^2w^2 - 4fqzw^3,\\
B &= \;\; g^2w^4 + 4p^2z^2w^2 + 8prx^2w^2 + gpzw^3,\\
C &= \;\; h^2w^4,\\
D &= \;\; 2q^2x^4 + 2p^2y^4 + 4h^2z^2w^2 + 4pqx^2y^2 - 8qhx^2zw + 8phy^2zw + 2ghy^2w^2 + 2fhx^2w^2,\\
F &= -\,p^2y^2w^2,\\
G &= -\,q^2x^2w^2,\\
H &= \;\; 0,\\
L &= \;\; 2pqy^2zw,\\
M &= \;\; 2pqx^2zw,\\
N &= -\,2h^2zw^3\;.
\end{aligned}$$

The substitution of these values gives after all reductions the result

$$
\begin{aligned}
& f^2g^2h^2w^6 + 4(pf-qg)fgh^2zw^5 \\
& + 4(r^2h^2 - 6pqgf)h^2z^2w^4 + 2(q^2g^2 + 2prfh)fhx^2w^4 + 2(p^2f^2 + 2qrgh)ghy^2w^4 \\
& - 16(pf-qg)z^3w^3 - 4(q^2g^2 - 4p^2f^2 - 6pqfg)qhx^2zw^3 - 4(p^2f^2 - 4q^2g^2 - 6pqfg)phy^2zw^3 \\
& + 16p^2q^2h^2z^4w^2 - 8(pf+4qg)q^2phx^2z^2w^2 - 8(qg+4pf)pq^2hy^2z^2w^2 \\
& + (q^2g^2 + 8prfh)q^2x^4w^2 + (p^2f^2 + 8qrgh)p^2y^4w^2 + 2(10r^2h^2 - pqfg)pqx^2y^2w^2 \\
& - 16p^2q^3hx^2z^3w + 16p^3q^2hy^2z^3w \\
& + 4(4pf+5qg)pq^3x^4zw - 4(4qg+5pf)p^3qy^4zw - 4(pf-qg)p^2q^2x^2y^2zw \\
& + 4p^3q^4x^4z^2 + 4p^4q^2y^4z^2 + 8p^3q^3x^2y^2z^2 + 4pq^4rx^6 + 4p^4qry^6 + 12p^3q^2rx^4y^2 + 12p^3q^2rx^2y^4 = 0\,;
\end{aligned}
$$

which is therefore the equation of the envelope for this case. This equation may be presented under the form

$$w\Psi + 4pq(qx^2 + py^2)^2(qrx^2 + rpy^2 + pqz^2) = 0,$$

and there are probably other forms which I am not yet acquainted with.

C. The reciprocals of the two surfaces made use of in determining the Intersect-Developable, although in reality a system of the same nature with the surfaces of which they are reciprocals, are represented by equations of a somewhat different form. There is no real loss of generality in replacing the two surfaces by the reciprocals of the cones $x^2 = 2yz$, $y^2 = 2zw$; or we may take the two conics

$$(x^2 - 2yz = 0,\ w = 0) \text{ and } (y^2 - 2zw = 0,\ x = 0),$$

for the surfaces of which the envelope has to be found, these conics being, it is evident, the sections by the planes $w = 0$ and $x = 0$ respectively of the cones the Intersect-Developable of which was before determined. The process of determining the envelope is however essentially different: supposing the plane $\xi x + \eta y + \zeta z + \omega w = 0$ to be the equation of a tangent plane to the two conics (that is, of a plane passing through a tangent of each of the conics) then the condition of touching the first conic gives $\xi^2 - 2\eta\zeta = 0$, and that of touching the second conic gives $\eta^2 - 2\zeta\omega = 0$. We have therefore to find the envelope (in the ordinary sense of the word) of the plane $\xi x + \eta y + \zeta z + \omega w = 0$, in which the coefficients ξ, η, ζ, ω are variable quantities subject to the conditions

$$\xi^2 - 2\eta\zeta = 0, \quad \eta^2 - 2\zeta\omega = 0.$$

The result which is obtained without difficulty by the method of indeterminate multipliers, [or more easily by writing $\xi : \eta : \zeta : w = 2\theta^3 : 2\theta^2 : \theta^4 : 2$] is

$$8y^4z - 32y^2z^2w + 32z^3w^2 - 27x^4w + 27x^2yzw - 4x^3y^3 = 0,$$

which may also be written under the form

$$8z(y^2 - 2zw)^2 - x^2\{4y^3 + 9(3x^2 - 8yz)w\} = 0.$$

[Another form, containing the factor w, is $4(y^2 + 2zw)^3 - (2y^3 + 27x^2w - 36yzw)^2 = 0$.]

85.

NOTE ON A FAMILY OF CURVES OF THE FOURTH ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 148—152.]

THE following theorem, in a slightly different and somewhat less general form, is demonstrated in Mr Hearn's "Researches on Curves of the Second Order, &c.", Lond. 1846: "The locus of the pole of a line, $u+v+w=0$, with respect to the conics passing through the angles of the triangle $(u=0,\ v=0,\ w=0)$, and touching a fixed line $\alpha u+\beta v+\gamma w=0$, is the curve of the fourth order,

$$\sqrt{\{\alpha u\,(v+w-u)\}}+\sqrt{\{\beta v\,(w+u-v)\}}+\sqrt{\{\gamma w\,(u+v-w)\}}=0\,;"$$

the difference in fact being, that with Mr Hearn the indeterminate line $u+v+w=0$ is replaced by the line ∞, so that the poles in question become the centres of the conics.

Previous to discussing the curve of the fourth order, it will be convenient to notice a property of curves of the fourth order with three double points. Such curves contain eleven arbitrary constants: or if we consider the double points as given, then five arbitrary constants. From each double point may be drawn two tangents to the curve; any five of the points of contact of these tangents determine the curve, and consequently determine the sixth point of contact. The nature of this relation will be subsequently explained; at present it may be remarked that it is such that, if three of the points of contact (each one of such points of contact corresponding to a different double point) lie in a straight line, the remaining three points of contact also lie in a straight line. A curve of the fourth order having three given double points and besides such that the points of contact of the tangents from the double points lie three and three in two straight lines, contains therefore four arbitrary constants. Now it is easily seen that the curve in question has three double points, viz. the points given by the equations

$$(u=0,\ v-w=0),\quad (v=0,\ w-u=0),\quad (w=0,\ u-v=0),$$

points which may be geometrically defined as the projections from the angles of the triangle $(u=0,\ v=0,\ w=0)$ upon the opposite sides, of the point $(u=v=w)$ which is the harmonic with respect to the triangle of the given line $u+v+w=0$. Moreover,

the lines $u=0$, $v=0$, $w=0$ (being lines which, as we have seen, pass through the double points) touch the curve in three points lying in a line, viz. the given line $\alpha u+\beta v+\gamma w=0$. Hence the curve in question is a curve with three double points, such that the points of contact of the tangents from the double points lie three and three in two straight lines. Considering the double points as given, the functions u, v, w contain two arbitrary ratios, and the ratios of the quantities α, β, γ being arbitrary, the equation of the curve contains four arbitrary constants, or it represents the general curve of the class to which it has been stated to belong.

As to the investigation of the above-mentioned theorem with respect to curves of the fourth order with three double points, the general form of the equation of such a curve is

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0,$$

where the double points are the angles of the triangle ($x=0$, $y=0$, $z=0$). It may be remarked in passing, that the six tangents at the double point touch the conic

$$ax^2+by^2+cz^2-2fyz-2gxz-2hxy=0.$$

To determine the tangents through ($y=0$, $z=0$), we have only to write the equation to the curve under the form

$$\left(\frac{a}{x}+\frac{h}{y}+\frac{g}{z}\right)^2+\frac{C}{y^2}+\frac{B}{z^2}-\frac{2F}{yz}=0:$$

the points of contact are given by the system

$$\frac{a}{x}+\frac{h}{y}+\frac{g}{z}=0,$$

$$\frac{C}{y^2}+\frac{B}{z^2}-\frac{2F}{yz}=0;$$

the latter equation (which evidently belongs to a pair of lines) determining the tangents. The former equation is that of a conic passing through the angles of the triangle $x=0$, $y=0$, $z=0$: since the tangents pass through the point ($y=0$, $z=0$) they evidently each intersect the conic in one other point only. The equation of the tangents shows that these lines are the tangents through the point $y=0$, $z=0$ to the conic whose equation is

$$aA^2x^2+bB^2y^2+cC^2z^2+2fBCyz+2gCAzx+2hABxy=0.$$

To complete the construction of the points of contact it may be remarked, that the equations which determine the coordinates of these points may be presented under the form

$$Ax=\{A\qquad\qquad\}\,x,$$

$$By=\left\{H-g\sqrt{\left(\frac{-K}{a}\right)}\right\}x,$$

$$Cz=\left\{G+h\sqrt{\left(\frac{-K}{a}\right)}\right\}x,$$

whence also

$$aAx + hBy + gCz = \qquad Kx,$$

$$hAx + bBy + fCz = \quad G\sqrt{\left(\frac{-K}{a}\right)}\,x,$$

$$gAx + fBy + cCz = -H\sqrt{\left(\frac{-K}{a}\right)}\,x:$$

or writing for shortness ξ, η, ζ instead of the linear functions forming the first sides of these equations respectively, we have

$$\frac{\xi}{\sqrt{(-aK)}} = \frac{\eta}{G} = -\frac{\zeta}{H};$$

from which it follows at once that the equation to the line forming the two points of contact is

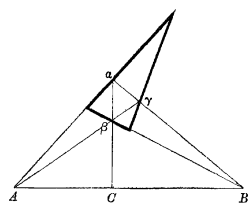
$$\frac{\eta}{G} + \frac{\zeta}{H} = 0;$$

and this may again be considered as the line joining the points $(\xi = 0,\ \frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0)$ and $(\eta = 0,\ \zeta = 0)$.

Now $\xi = 0$, $\eta = 0$, $\zeta = 0$, are the polars of the double points (or angles of the triangle $x = 0$, $y = 0$, $z = 0$) with respect to the last-mentioned conic. The equation of the line joining the point $(y = 0,\ z = 0)$ with the point $(\eta = 0,\ \zeta = 0)$, is easily seen to be $GBy - CHz = 0$; from which it follows, that the lines forming each double point with the intersection of the polars of the other two double points meet in a point, the coordinates of which are given by

$$x : y : z = \frac{1}{AF} : \frac{1}{BG} : \frac{1}{CH},$$

and the polar of this point is the line $\frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0$. The construction of this line is thus determined; and we have already seen how this leads to the construction of



the lines joining the points of contact of the tangents from the double points. In fact, in the figure, if $\alpha\beta\gamma$ be the triangle whose sides are $\xi = 0$, $\eta = 0$, $\zeta = 0$, and A, B, C the points of intersection of the sides of this triangle with the line

$$\frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0,$$

the lines in question are $A\alpha$, $B\beta$, $C\gamma$. Moreover, the points of contact upon $A\alpha$ are harmonics with respect to A, α, and in like manner the points of contact on $B\beta$, $C\gamma$ are respectively harmonics with respect to B, β and to C, γ. This proves that if three of the points of contact are in the same straight line, the remaining three are also in the same straight line; in fact, we may consider three of the points of contact as given by

$$\begin{aligned} \xi : \eta : \zeta &= \sqrt{(-aK)} : \quad G \quad : -H, \\ \xi : \eta : \zeta &= \quad -F \quad : \sqrt{(-bK)} : \quad H, \\ \xi : \eta : \zeta &= \quad F \quad : \quad -G \quad : \sqrt{(-cK)}; \end{aligned}$$

and the condition that these may be in the same line is

$$\sqrt{(-a)}\sqrt{(-b)}\sqrt{(-c)} + GH\sqrt{(-a)} + HF\sqrt{(-b)} + FG\sqrt{(-c)} = 0,$$

which remains unaltered when the signs of all the roots are changed. The equation just obtained may be considered as the condition which must exist between the coefficients of

$$\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} + \frac{2f}{yz} + \frac{2g}{zx} + \frac{2h}{xy} = 0,$$

in order that this curve may be transformable into the form

$$\sqrt{\{\alpha u(v+w-u)\}} + \sqrt{\{\beta v(w+u-v)\}} + \sqrt{\{\gamma w(u+v-w)\}} = 0.$$

86.

ON THE DEVELOPABLE DERIVED FROM AN EQUATION OF THE FIFTH ORDER.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 152—159.]

MÖBIUS, in his "Barycentrische Calcul," [Leipzig, 1827], has considered, or rather suggested for consideration, the family of curves of double curvature given by equations such as $x : y : z : w = A : B : C : D$, where A, B, C, D are rational and integral functions of an indeterminate quantity t. Observing that the plane $Ax + By + Cz + Dw = 0$ may be considered as the polar of the point determined by the system of equations last preceding, the reciprocal of the curve of Möbius is the developable, which is the envelope of a plane the coefficients in the equation of which are rational and integral functions of an indeterminate quantity t, or what is equivalent, homogeneous functions of two variables ξ, η. Such an equation may be represented by $U = a\xi^n + n\xi^{n-1}\eta + \ldots = 0$, (where a, b, &c. are linear functions of the coordinates); and we are thus led to the developables noticed, I believe for the first time, in my "Note sur les Hyper-determinants," *Crelle*, t. XXXIV. p. 148, [54]. I there remarked, that not only the equation of the developable was to be obtained by eliminating ξ, η from the first derived equations of $U = 0$; but that the second derived equations conducted in like manner to the edge of regression, and the third derived equations to the cusps or stationary points of the edge of regression. It followed that the order of the surface was $2(n-1)$, that of the edge of regression $3(n-2)$, and the number of stationary points $4(n-3)$. These values lead at once, as Mr Salmon pointed out to me, to the table,

$$\begin{aligned}
m &= 3(n-2),\\
n &= n,\\
r &= 2(n-1),\\
\alpha &= 0,\\
\beta &= 4(n-3),\\
g &= \tfrac{1}{2}(n-1)(n-2),\\
h &= \tfrac{1}{2}(9n^2 - 53n + 80),\\
x &= 2(n-2)(n-3),\\
y &= 2(n-1)(n-3),
\end{aligned}$$

where the letters in the first column have the same signification as in my memoir in Liouville, [30], translated in the last number of the *Journal*. The order of the nodal line is of course $2(n-2)(n-3)$; Mr Salmon has ascertained that there are upon this line $6(n-3)(n-4)$ stationary points and $\frac{4}{3}(n-3)(n-4)(n-5)$ real double points, (the stationary points lying on the edge of regression, and with the stationary points of the edge of regression forming the system of intersections of the nodal line and edge of regression, and the real double points being triple points upon the surface). Also, that the number of apparent double points of the nodal line is

$$(n-3)(2n^3-18n^2+57n-65).$$

The case of U a function of the second order gives rise to the cone $ac-b^2=0$. When U is a function of the third order, we have the developable

$$4(ac-b^2)(bd-c^2)-(ad-bc)^2=0,$$

which is the general developable of the fourth order having for its edge of regression the curve of the third order,

$$ac-b^2=0,\quad bd-c^2=0,\quad ad-bc=0,$$

which is likewise the most general curve of this order: there are of course in this case no stationary points on the edge of regression. In the case where U is of the fourth order we have the developable of the sixth order,

$$(ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-b^2e-c^3)^2=0;$$

having for its edge of regression the curve of the sixth order,

$$ae-4bd+3c^2=0,\quad ace+2bcd-ad^2-b^2e-c^3=0,$$

with four stationary points determined by the equations

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}.$$

The form exhibiting the nodal line of the surface has been given in the *Journal* by Mr Salmon. I do not notice it here, but pass on to the principal subject of the present paper, which is to exhibit the edge of regression and the stationary points of this edge of regression for the developable obtained from the equation of the fifth order,

$$U=a\xi^5+5b\xi^4\eta+10c\xi^3\eta^2+10d\xi^2\eta^3+5e\xi\eta^4+f\eta^5=0;$$

viz. that represented by the equation

$$\square=0=a^4f^4+160a^2cef^2+\ldots-4000b^2c^3e^3,$$

[I do not reproduce here this expression for the discriminant of the binary quintic] a result for which I am indebted to Mr Salmon.

To effect the reduction of this expression, consider in the first place the equations which determine the stationary points of the edge of regression. Writing instead of $\xi : \eta$ the single letter t, these equations are

$$\begin{aligned} at^2 + 2bt + c &= 0, \\ bt^2 + 2ct + d &= 0, \\ ct^2 + 2dt + e &= 0, \\ dt^2 + 2et + f &= 0, \end{aligned}$$

write for shortness

$$\begin{aligned} A &= 2\,(bf - 4ce + 3d^2), \\ B &= af - 3be + 2cd, \\ C &= 2\,(ae - 4bd + 3c^2), \end{aligned}$$

and let α, 3β, 3γ, δ represent the terms of

$$\left\| \begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{matrix} \right\|,$$

viz.

$$\begin{aligned} \alpha &= bdf - be^2 + 2cde - c^2f - d^3, \\ 3\beta &= adf - ae^2 - bcf + bde + c^2e - cd^2, \\ 3\gamma &= acf - ade - b^2f + bd^2 + bce - c^2d, \\ \delta &= ace - ad^2 - b^2e + 2bcd - c^3; \end{aligned}$$

it is obvious at first sight that the result of the elimination of t from the four quadratic equations is the system (equivalent of course to three equations),

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0.$$

The system in question may however be represented under the more simple form (which shows at once that the number of stationary points is, as it ought to be, eight),

$$A = 0, \quad B = 0, \quad C = 0;$$

this appears from the identical equations,

$$\begin{aligned} &(2ct + 3d)\,(bt^2 + 2ct + d) \\ &- (2bt + 4c)\,(ct^2 + 2dt + e) \\ &\qquad + b\,(dt^2 + 2et + f) = \tfrac{1}{2}A; \end{aligned}$$

$$\begin{aligned} &(2ct + 3d)\,(at^2 + 2bt + c) \\ &\qquad - c\,(bt^2 + 2ct + d) \\ &- (2at + 3b)\,(ct^2 + 2dt + e) \\ &\qquad + a\,(dt^2 + 2et + f) = B; \end{aligned}$$

$$\begin{aligned}&(2bt+3c)(at^2+2bt+c)\\ -&(2at+4b)(bt^2+2ct+d)\\ &+a(ct^2+2dt+e)=\tfrac{1}{2}C,\end{aligned}$$

(formulæ the first and third of which are readily deduced from an equation given in the Note on Hyperdeterminants above quoted). The connexion between the quantities A, B, C and α, β, γ, δ, is given by

$$\begin{aligned}Aa-2Bb+Cc&=-6\delta,\\ Ab-2Bc+Cd&=-6\gamma,\\ Ac-2Bd+Ce&=-6\beta,\\ Ad-2Be+Cf&=-6\alpha.\end{aligned}$$

The theory of the stationary points being thus obtained, the next question is that of finding the equations of the edge of regression. We have for this to eliminate t from the three cubic equations,

$$\begin{aligned}at^3+3bt^2+3ct+d&=0,\\ bt^3+3ct^2+3dt+e&=0,\\ ct^3+3dt^2+3et+f&=0:\end{aligned}$$

treating the quantities t^3, t^2, t^1, t^0 as if they were independent, we at once obtain

$$\beta t+\alpha=0,\quad \delta t+\gamma=0,\quad \gamma t^3-\alpha=0,\quad \delta t^2-\beta=0;$$

or as this system may be more conveniently written,

$$\beta t+\alpha=0,\quad \gamma t+\beta=0,\quad \delta t+\gamma=0.$$

But the most simple forms are obtained from the identical equations,

$$\begin{aligned}&ft(at^3+3bt^2+3ct+d),\\ -&(3et+f)(bt^3+3ct^2+3dt+e),\\ +&(2dt+e)(ct^3+3dt^2+3et+f)=t^3(Bt+A);\end{aligned}$$

$$\begin{aligned}&(bt+c)(at^3+3bt^2+3ct+d),\\ -&(at+3b)(bt^3+3ct^2+3dt+e),\\ +&\quad a\quad(ct^3+3dt^2+3et+f)=\quad Ct+B;\end{aligned}$$

equations which, combined with those which precede, give the complete system

$$\beta t+\alpha=0,\quad \gamma t+\beta=0,\quad \delta t+\gamma=0,\quad Bt+A=0,\quad Ct+B=0:$$

or the equations of the edge of regression are given by the system (equivalent of course to two equations),

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & A, & B \\ \beta, & \gamma, & \delta, & B, & C \end{matrix} \right\| = 0.$$

The simplest mode of verifying *à posteriori* that the edge of regression is only of the ninth order, appears to be to consider this curve as the common intersection of the three surfaces of the seventh order:

$$A^3a - 3A^2Bb + 3AB^2c - B^3d = 0,$$
$$A^3b - 3A^2Bc + 3AB^2d - B^3e = 0,$$
$$A^3c - 3A^2Bd + 3AB^2e - B^3f = 0,$$

(which are at once obtained by combining the equation $Bt + A = 0$ with the cubic equations in t). It is obvious from a preceding equation that if the equations first given are multiplied by fA, $-3eA + fB$, $2dA - eB$, and added, an identical result is obtained. This shows that the curve of the forty-ninth order, the intersection of the first two surfaces, is made up of the curve in question, the curve of the fourth order $A = 0$, $B = 0$ (which reckons for thirty-six, as being a triple line on each surface), and the curve which is common to the two surfaces of the seventh order and the surface $2dA - eB = 0$. The equations of this last curve may be written,

$$e(af - 3be + 2cd) - 4d(bf - 4ce + 3d^2) = 0,$$
$$e^3a - 6e^2db + 12ed^2c - 8d^4 = 0,$$
$$e^3b - 6e^2dc + 4ed^3 = 0;$$

or, observing that these equations are

$$f(ae - 4bd) - 3(be^2 - 6ced + 4d^3) = 0,$$
$$e^2(ae - 4bd) - 2d(be^2 - 6ced + 4d^3) = 0,$$
$$e(be^2 - 6ced + 4d^3) = 0;$$

the last-mentioned curve is the intersection of

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$

where the second surface contains the double line $e = 0$, $d = 0$, which is also a single line upon the first surface. Omitting this extraneous line, the intersection is of the *fourth* order; and we may remark that, in passing, it is determined (exclusively of the double line) as the intersection of the three surfaces

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$
$$a^2d - 6abc + 4b^3 = 0,$$

being in fact of the species IV. 4, of Mr Salmon's paper, "On the Classification of Curves of Double Curvature" [*Journal*, vol. V. (1850), pp. 23—46]. But returning to the question in hand; since $49 = 9 + 4 + 36$, we see that the curve common to the three surfaces of the seventh order, or the edge of regression, is, as it ought to be, of the ninth order. It only remains to express the equation of the developable surface in terms of the functions A, B, C, α, β, γ, δ, which determine the stationary points and edge of regression; I have satisfied myself that the required formula is

$$\square = (AC - B^2)^2 - 1152\,\{A\,(\beta\delta - \gamma^2) - B\,(\alpha\delta - \beta\gamma) + C\,(\alpha\gamma - \beta^2)\} = 0,$$

where the quantities α, β, γ, δ may be replaced by their values in A, B, C; and it will be noticed that when this is done, the terms of $\square$ are each of them at least of the third order in the last-mentioned functions.

I propose to term the family of developables treated of in this paper, 'planar developables.' In general, the coefficients of the generating plane of a developable being algebraical functions of a variable parameter t, the equation rationalized with respect to the parameter belongs to a system of n different planes; the developable which is the envelope of such a system may be termed a 'multiplanar developable,' and in the particular case of n being equal to unity, we have a planar developable. It would be very desirable to have some means of ascertaining from the equation of a developable what the degree of its 'planarity' is.

P.S.—At the time of writing the preceding paper I was under the impression that the only surface of the fourth order through the edge of regression was that given by the equation $AC - B^2 = 0$; but Mr Salmon has since made known to me an entirely new form of the function $\square$, the component functions of which, equated to zero, give six different surfaces of the fourth order, each of them passing through the edge of regression. The form in question is

$$3\,\square = LL' + 64MM' - 64NN',$$

where

$$\begin{aligned}
L &= a^2f^2 + 225b^2e^2 - 32ace^2 - 32b^3df + 480bd^3 + 480c^3e - 34abef + 76acdf - 12bc^2f \\
&\qquad - 12ad^2e - 820bcde - 320c^2d^2, \\
L' &= 3a^2f^2 - 45b^2e^2 + 64ace^2 + 64b^3df - 22abef - 12acdf - 36bc^2f \\
&\qquad - 36ad^2e + 20bcde, \\
M &= 10bcd^2 - 18ad^3 - 15bc^2e + 32acde + 6b^2cf - 9ac^2f + 2abdf + a^2ef - 9abe^2, \\
M' &= 10c^2de - 18c^3f - 15bd^2e + 32bcdf + 6ade^2 - 9b^2ef + 2acef + abf^2 - 9ad^2f, \\
N &= 10b^2d^2 - 15b^3ce - 12acd^2 + 18ac^2e + abde - 2a^2e^2 + 6b^3f - 9abcf + 3a^2df, \\
N' &= 10c^2e^2 - 15bde^2 - 12c^2df + 18bd^2f + bcef - 2b^2f^2 + 6ae^3 - 9adef + 3acf^2,
\end{aligned}$$

where it should be noticed that

$$L + 3L' = -10\,(AC - B^2).$$

The expressions of L, L', M, M', N, N' as linear functions of A, B, C (also due to Mr Salmon) are

$$L = (11\,ae + 28bd - 39c^2)\,A + (\ af - 75be + 74cd)\,B + (\ 11bf + 28ce - 39d^2)\,C,$$

$$L' = (-7ae + \ 4bd + \ 3c^2)\,A + (3af + 15be - 18cd)\,B + (-7bf + \ 4ce + \ 3d^2)\,C,$$

$$M = 3\,(bc - ad)\,A + 3\,(ae - c^2)\,B + \ (cd - af)\,C,$$

$$M' = \ (cd - af)\,A + 3\,(bf - d^2)\,B + 3\,(de - cf)\,C,$$

$$N = 3\,(b^2 - ac)\,A + 3\,(ad - bc)\,B + \ (bd - ae)\,C,$$

$$N' = \ (ce - bf)\,A + 3\,(cf - de)\,B + 3\,(e^2 - df)\,C.$$

I propose resuming the subject of these forms, and the general theory, in a subsequent paper. [This was never written.]

87.

NOTES ON ELLIPTIC FUNCTIONS (FROM JACOBI).

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 201—204.]

THESE Notes are mere translations from Jacobi's "Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algèbre," [*Crelle*, t. VII. (1831) pp. 41—43], and from the addition to the notice by him of the third supplement to Legendre's "Théorie des fonctions elliptiques" [*Crelle*, t. VIII. (1832) pp. 413—417].

88.

ON THE TRANSFORMATION OF AN ELLIPTIC INTEGRAL.

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 204—206.]

THE following is a demonstration of a formula proved incidentally by Mr Boole (*Journal*, vol. II. [1847] p. 7), in a paper "On the Attraction of a Solid of Revolution on an External Point."

Let
$$U = \int_{-1}^{1} \frac{dx}{\sqrt{[(1-x^2)\{1-(mx+n)^2\}]}};$$

then, assuming
$$ix = \frac{\alpha + iy}{1 - i\alpha y},$$

(so that $x = \pm 1$ gives $y = \pm 1$), we obtain
$$1 - x^2 = \frac{(1+\alpha^2)(1-y^2)}{(1-i\alpha y)^2},$$

$$mx + n = \frac{(n - im\alpha) + (m - in\alpha)\, y}{1 - i\alpha y}.$$

Assume therefore
$$i\alpha + (n - im\alpha)(m - in\alpha) = 0,$$

whence
$$-i\alpha = \frac{(1 - m^2 - n^2) + \Delta}{2mn} \qquad (\Delta^2 = 1 + m^4 + n^4 - 2m^2 - 2n^2 - 2m^2n^2),$$

we find

$$1-(mx+n)^2=\frac{1-(n-im\alpha)^2}{(1-i\alpha y)^2}\{1-(m-in\alpha)^2y^2\},$$

and also

$$dx=\frac{(1+\alpha^2)\,dy}{(1-i\alpha y)^2},$$

whence

$$U=\sqrt{\left\{\frac{1+\alpha^2}{1-(n-im\alpha)^2}\right\}}\int_{-1}^{1}\frac{dy}{\sqrt{[(1-y^2)\{1-(m-in\alpha)^2y^2\}]}},$$

that is

$$U=2\sqrt{\left\{\frac{1+\alpha^2}{1-(n-im\alpha)^2}\right\}}\int_{0}^{1}\frac{dy}{\sqrt{[(1-y^2)\{1-(m-in\alpha)^2y^2\}]}}.$$

But since

$$n-im\alpha=\frac{1-m^2+n^2+\Delta}{2n},$$

$$m-in\alpha=\frac{1+m^2-n^2+\Delta}{2m},$$

we have

$$1-(n-im\alpha)^2=-\frac{\Delta}{2n^2}\quad(\Delta+1-m^2+n^2),$$

$$1+\alpha^2=-\frac{\Delta}{2m^2n^2}(\Delta+1-m^2-n^2);$$

and therefore

$$\frac{1+\alpha^2}{1-(n-im\alpha)^2}=\frac{1}{m^2}\,\frac{\Delta+1-m^2-n^2}{\Delta+1-m^2+n^2}$$

$$=\frac{1}{m^2}\,\frac{(1-m^2-n^2+\Delta)(1-m^2+n^2-\Delta)}{(1-m^2+n^2+\Delta)(1-m^2+n^2-\Delta)}=\frac{2(1+m^2-n^2+\Delta)}{4m^2};$$

consequently

$$U=\frac{1}{m}\sqrt{\{2(1+m^2-n^2+\Delta)\}}\int_0^1\frac{dy}{\sqrt{\left[(1-y^2)\left\{1-\left(\frac{1+m^2-n^2+\Delta}{2m}\right)^2y^2\right\}\right]}}.$$

Write

$$k=\frac{1+m^2-n^2+\Delta}{2m},\qquad \lambda^2=\frac{4m}{(1+m)^2-n^2};$$

then

$$U=\frac{4\sqrt{k}}{\lambda}\,\frac{1}{\sqrt{\{(1+m)^2-n^2\}}}\int_0^1\frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}};$$

where λ and k are connected by the relation that exists for the transformation of the second order, viz.

$$\lambda=\frac{2\sqrt{k}}{1+k},$$

as may be immediately verified; hence, assuming

$$y = \frac{\lambda z}{\sqrt{k}}\sqrt{\left(\frac{1-z^2}{1-\lambda^2 z^2}\right)},$$

which gives

$$\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} = \frac{\lambda}{\sqrt{k}}\int_0^1 \frac{dz}{\sqrt{\{(1-z^2)(1-\lambda^2z^2)\}}},$$

we find

$$U = \frac{4}{\sqrt{\{(1+m)^2-n^2\}}}\int_0^1 \frac{dz}{\sqrt{\left\{(1-z^2)\left(1-\frac{4m}{(1+m)^2-n^2}z^2\right)\right\}}};$$

that is

$$\int_{-1}^{1} \frac{dx}{\sqrt{[(1-x^2)\{1-(mx+n)\}^2]}} = 4\int_0^1 \frac{dz}{\sqrt{(1-z^2)[\{(1+m)^2-n^2\}-4mz^2]}}.$$

Writing here

$$x = \cos\theta, \quad z = \cos\tfrac{1}{2}\phi,$$

then

$$\int_0^\pi \frac{d\theta}{\sqrt{\{1-(m\cos\theta+n)^2\}}} = \int_0^\pi \frac{d\phi}{\sqrt{(1+m^2-n^2-2m\cos\phi)}};$$

or if

$$m = \frac{r}{a}, \qquad n = -\frac{iz}{a},$$

then finally

$$\int_0^\pi \frac{d\theta}{\sqrt{\{a^2+(z+ir\cos\theta)^2\}}} = \int_0^\pi \frac{d\phi}{\sqrt{(a^2+r^2+z^2-2ar\cos\phi)}},$$

the formula in question.

89.

ON THE ATTRACTION OF ELLIPSOIDS (JACOBI'S METHOD).

[From the *Cambridge and Dublin Mathematical Journal*, vol. v. (1850), pp. 217—226.]

IN a letter published in 1846 in *Liouville's Journal* (t. XI. p. 341) Jacobi says, "Il y a quatorze ans, je me suis posé le problème de chercher l'attraction d'un ellipsoïde homogène exercée sur un point extérieur quelconque par une méthode analogue à celle employée par Maclaurin par rapport aux points situés dans les axes principaux. J'y suis parvenu par trois substitutions consécutives. La première est une transformation de coordonnées; par la seconde le radical $\sqrt{(1-m^2\sin^2\beta\cos^2\psi-n^2\sin^2\beta\sin^2\psi)}$ qui entre dans la double intégrale transformée est rendu rationnel au moyen de la double substitution

$$m\sin\beta\cos\psi=\sin\eta\cos\theta,\quad m\sin\beta\sin\psi=\sin\eta\sin\theta;$$

la troisième est encore une transformation de coordonnées. La recherche du sens géométrique de ces trois substitutions m'a conduit à approfondir la théorie des surfaces confocales par rapport auxquelles je découvris quantité de beaux théorèmes dont je communiquai quelques-uns des principaux à M. Steiner. Considérons l'ellipsoïde confocal mené par le point attiré P et le point p de l'ellipsoïde proposé, conjugué à P. Soient Q et q deux autres points conjugués quelconques situés respectivement sur l'ellipsoïde extérieur et intérieur. Menons de P un premier cône tangent à l'ellipsoïde intérieur, de p un second cône tangent à l'ellipsoïde extérieur. Ce dernier, tout imaginaire qu'il est, a ses trois axes réels (ainsi que ses deux droites focales). La première substitution ramène les axes de l'ellipsoïde à ceux du premier cône (c'est la substitution employée par Poisson, mais que j'avais antérieurement traitée et même étendue à un nombre quelconque de variables dans le mémoire *De binis Functionibus homogeneis &c.* [*Crelle*, t. XII. (1834) pp. 1—69]). Par la seconde substitution les angles que la droite Pq forme avec les axes du premier cône sont ramenés aux angles que la droite pQ forme avec les axes du second. Par la dernière substitution, on retourne de ces axes aux axes de l'ellipsoïde. La seconde substitution répond à un théorème de géométrie remarquable, savoir que 'Les cosinus des angles que la droite Pq forme avec deux des axes du premier cône sont en raison constante avec les cosinus des angles que la droite

pQ forme avec deux des axes du second cône; ces deux axes sont les tangents situés respectivement dans les sections de plus grande et de moindre courbure de chaque ellipsoïde, le troisième axe étant la normale à l'ellipsoïde.' Tout cela semble difficile à établir par la synthèse."

The object of this paper is to develope the above method of finding the attraction of an ellipsoid.

Consider an exterior ellipsoid, the squared semiaxes of which are $f+u$, $g+u$, $h+u$; and an interior ellipsoid, the squared semiaxes of which are $f+\bar{u}$, $g+\bar{u}$, $h+\bar{u}$. Let u, p, q be the elliptic coordinates of a point P on the exterior ellipsoid, the elliptic coordinates of the corresponding point $\bar{P}$ on the interior ellipsoid will be $\bar{u}$, p, q, and if a, b, c and $\bar{a}$, $\bar{b}$, $\bar{c}$ represent the ordinary coordinates of these points (the principal axes being the axes of coordinates), we have

$$a^2 = \frac{(f+u)(f+q)(f+r)}{(f-g)(f-h)}, \qquad \bar{a}^2 = \frac{(f+\bar{u})(f+q)(f+r)}{(f-g)(f-h)},$$

$$b^2 = \frac{(g+u)(g+q)(g+r)}{(g-h)(g-f)}, \qquad \bar{b}^2 = \frac{(g+\bar{u})(g+q)(g+r)}{(g-h)(g-f)},$$

$$c^2 = \frac{(h+u)(h+q)(h+r)}{(h-g)(h-f)}, \qquad \bar{c}^2 = \frac{(h+\bar{u})(h+q)(h+r)}{(h-f)(h-g)}.$$

I form the systems of equations

$$a_1^2 = \frac{(u+f)(u+g)(u+h)}{(u-q)(u-r)}, \qquad \bar{a}_1^2 = \frac{(\bar{u}+f)(\bar{u}+g)(\bar{u}+h)}{(\bar{u}-q)(\bar{u}-r)},$$

$$b_1^2 = \frac{(q+f)(q+g)(q+h)}{(q-r)(q-u)}, \qquad \bar{b}_1^2 = \frac{(q+f)(q+g)(q+h)}{(\bar{q}-r)(q-\bar{u})},$$

$$c_1^2 = \frac{(r+f)(r+g)(r+h)}{(r-u)(r-q)}, \qquad \bar{c}_1^2 = \frac{(r+f)(r+g)(r+h)}{(r-\bar{u})(r-q)}.$$

$$\alpha = \frac{a_1 a}{f+u}, \quad \beta = \frac{a_1 b}{g+u}, \quad \gamma = \frac{a_1 c}{h+u},$$

$$\alpha' = \frac{b_1 a}{f+q}, \quad \beta' = \frac{b_1 b}{g+q}, \quad \gamma' = \frac{b_1 c}{h+q},$$

$$\alpha'' = \frac{c_1 a}{f+r}, \quad \beta'' = \frac{c_1 b}{g+r}, \quad \gamma'' = \frac{c_1 c}{h+r},$$

$$\bar{\alpha} = \frac{\bar{a}_1 \bar{a}}{f+\bar{u}}, \quad \bar{\beta} = \frac{\bar{a}_1 \bar{b}}{g+\bar{u}}, \quad \bar{\gamma} = \frac{\bar{a}_1 \bar{c}}{h+\bar{u}},$$

$$\bar{\alpha}' = \frac{\bar{b}_1 \bar{a}}{f+q}, \quad \bar{\beta}' = \frac{\bar{b}_1 \bar{b}}{g+q}, \quad \bar{\gamma}' = \frac{\bar{b}_1 \bar{c}}{h+q},$$

$$\bar{\alpha}'' = \frac{\bar{c}_1 \bar{a}}{f+r}, \quad \bar{\beta}'' = \frac{\bar{c}_1 \bar{b}}{g+r}, \quad \bar{\gamma}'' = \frac{\bar{c}_1 \bar{c}}{h+r}.$$

And then writing

$$\begin{aligned} X &= \alpha X_1 + \alpha' Y_1 + \alpha'' Z_1, \\ Y &= \beta X_1 + \beta' Y_1 + \beta'' Z_1, \\ Z &= \gamma X_1 + \gamma' Y_1 + \gamma'' Z_1, \end{aligned} \qquad \begin{aligned} \bar{X} &= \bar{\alpha}\bar{X}_1 + \bar{\alpha}'\bar{Y}_1 + \bar{\alpha}''\bar{Z}_1, \\ \bar{Y} &= \bar{\beta}\bar{X}_1 + \bar{\beta}'\bar{Y}_1 + \bar{\beta}''\bar{Z}_1, \\ \bar{Z} &= \bar{\gamma}\bar{X}_1 + \bar{\gamma}'\bar{Y}_1 + \bar{\gamma}''\bar{Z}_1, \end{aligned}$$

if X, Y, Z are the cosines of the inclinations of a line $P\bar{Q}$ to the principal axes of the ellipsoids, X_1, Y_1, Z_1 will be the cosines of the inclinations of this line to the principal axes of the cone having P for its vertex, and circumscribed about the interior ellipsoid. In like manner, X, $\bar{Y}$, $\bar{Z}$ being the cosines of the inclinations of a line $\bar{P}Q$ to the principal axes of the ellipsoids, $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ will be the cosines of the inclinations of this line to the principal axes of the cone having $\bar{P}$ for its vertex and circumscribed about the exterior ellipsoid. Assuming that the points Q, $\bar{Q}$ are situated upon the exterior and interior ellipsoids respectively, suppose that X_1, Y_1, Z_1 and $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ are connected by the equivalent systems of equations,

$$X_1 = \sqrt{(u-\bar{u})}\sqrt{\left(\frac{\bar{X}_1^2}{u-\bar{u}} + \frac{\bar{Y}_1^2}{u-q} + \frac{\bar{Z}_1^2}{u-r}\right)},$$

$$Y_1 = \sqrt{\left(\frac{\bar{u}-q}{u-q}\right)}\,\bar{Y}_1,$$

$$Z_1 = \sqrt{\left(\frac{\bar{u}-r}{u-r}\right)}\,\bar{Z}_1,$$

$$\bar{X}_1 = \sqrt{(\bar{u}-u)}\sqrt{\left(\frac{X_1^2}{\bar{u}-u} + \frac{Y_1^2}{\bar{u}-q} + \frac{Z_1^2}{\bar{u}-r}\right)},$$

$$\bar{Y}_1 = \sqrt{\left(\frac{u-q}{\bar{u}-q}\right)}\,Y_1,$$

$$\bar{Z}_1 = \sqrt{\left(\frac{u-r}{\bar{u}-r}\right)}\,Z_1;$$

then it will presently be shown that the points Q, $\bar{Q}$ are corresponding points, which will prove the geometrical theorem of Jacobi. Before proceeding further it will be convenient to notice the formulæ

$$1 - \frac{a^2}{f+\bar{u}} - \frac{b^2}{g+\bar{u}} - \frac{c^2}{h+\bar{u}} = \frac{\bar{u}-u}{\bar{a}_1^2},$$

$$\frac{Xa}{f+\bar{u}} + \frac{Yb}{g+\bar{u}} + \frac{Zc}{h+\bar{u}} = \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X_1 a_1}{\bar{u}-u} + \frac{Y_1 b_1}{\bar{u}-q} + \frac{Z_1 c_1}{\bar{u}-r}\right),$$

$$\left(\frac{Xa}{f+\bar{u}} + \frac{Yb}{g+\bar{u}} + \frac{Zc}{h+\bar{u}}\right)^2 + \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X^2}{f+\bar{u}} + \frac{Y^2}{g+\bar{u}} + \frac{Z^2}{h+\bar{u}}\right)$$

$$= \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X_1^2}{\bar{u}-u} + \frac{Y_1^2}{\bar{u}-q} + \frac{Z_1^2}{\bar{u}-r}\right) = \frac{\bar{X}_1^2}{\bar{a}_1^2};$$

and the corresponding ones

$$1-\frac{\bar{a}^2}{f+u}-\frac{\bar{b}^2}{g+u}-\frac{\bar{c}^2}{h+u}=\frac{u-\bar{u}}{a_1^2},$$

$$\frac{\bar{X}\bar{a}}{f+u}+\frac{\bar{Y}\bar{b}}{g+u}+\frac{\bar{Z}\bar{c}}{h+u}=\frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}_1\bar{a}_1}{u-\bar{u}}+\frac{\bar{Y}_1\bar{b}_1}{u-q}+\frac{\bar{Z}_1\bar{c}_1}{u-r}\right),$$

$$\left(\frac{\bar{X}\bar{a}}{f+u}+\frac{\bar{Y}\bar{b}}{g+u}+\frac{\bar{Z}\bar{c}}{h+u}\right)^2+\frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}^2}{f+u}+\frac{\bar{Y}^2}{g+u}+\frac{\bar{Z}^2}{h+u}\right)$$

$$=\frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}_1^2}{u-\bar{u}}+\frac{\bar{Y}_1^2}{u-q}+\frac{\bar{Z}_1^2}{u-r}\right)=\frac{X_1^2}{a_1^2}.$$

The coordinates of the point $\bar{Q}$ are obviously $a+\rho X$, $b+\rho Y$, $c+\rho Z$ (where $\rho=P\bar{Q}$); substituting these values in the equation of the interior ellipsoid, we obtain

$$\rho^2\left(\frac{X^2}{f+\bar{u}}+\frac{Y^2}{g+\bar{u}}+\frac{Z^2}{h+\bar{u}}\right)+2\rho\left(\frac{Xa}{f+\bar{u}}+\frac{Yb}{g+\bar{u}}+\frac{Zc}{h+\bar{u}}\right)+\left(\frac{a^2}{f+\bar{u}}+\frac{b^2}{g+\bar{u}}+\frac{c^2}{h+\bar{u}}-1\right)=0:$$

reducing the coefficients of this equation by the formulæ first given, and omitting a factor $\frac{\bar{u}-u}{\bar{a}_1^2}$, we obtain

$$\left\{\frac{\bar{a}_1^2\bar{X}^2}{(\bar{u}-u)^2}-\left(\frac{X_1a_1}{\bar{u}-u}+\frac{Y_1b_1}{\bar{u}_1-q}+\frac{Z_1c_1}{\bar{u}-r}\right)^2\right\}\rho^2+2\rho\left(\frac{X_1a_1}{\bar{u}-u}+\frac{Y_1b_1}{u-q}+\frac{Z_1c_1}{\bar{u}-r}\right)-1=0,$$

that is,

$$\frac{\bar{a}_1^2\bar{X}_1^2}{(\bar{u}-u)^2}\rho^2=\left\{\rho\left(\frac{X_1a_1}{\bar{u}-u}+\frac{Y_1b_1}{\bar{u}-q}+\frac{Z_1c_1}{\bar{u}-r}\right)-1\right\}^2,$$

or

$$\rho=\frac{1}{\dfrac{X_1a_1-\bar{X}_1\bar{a}_1}{\bar{u}-u}+\dfrac{Y_1a_1}{\bar{u}-q}+\dfrac{Z_1c_1}{\bar{u}-r}};$$

which is easily transformed into

$$\rho=\frac{\bar{u}-u}{a_1X_1-\bar{a}_1\bar{X}_1+\dfrac{(f+u)\,b_1Y_1-(f+\bar{u})\,\bar{b}_1\bar{Y}_1}{f+q}+\dfrac{(f+u)\,c_1Z_1-(f+\bar{u})\,\bar{c}_1\bar{Z}_1}{f+r}}:$$

and this form remaining unaltered when u and $\bar{u}$ are interchanged, it follows that if $\bar{P}Q=\bar{\rho}$, then $\rho=\bar{\rho}$, which is a known theorem. The value of ρ or $\bar{\rho}$ may however be expressed in a yet simpler form; for, considering the expression

$$\frac{X}{\sqrt{(f+\bar{u})}}-\frac{\bar{X}}{\sqrt{(f+u)}}=\frac{a}{\sqrt{(f+\bar{u})}}\left\{\frac{a_1X_1}{f+u}+\frac{b_1Y_1}{f+q}+\frac{c_1Z_1}{f+r}\right\}-\frac{\bar{a}}{\sqrt{(f+u)}}\left\{\frac{\bar{a}_1\bar{X}_1}{f+\bar{u}}+\frac{\bar{b}_1\bar{Y}_1}{f+q}+\frac{\bar{c}_1\bar{Z}_1}{f+r}\right\}$$

$$=\frac{-1}{\bar{u}-u}\left\{\frac{a}{\sqrt{(f+\bar{u})}}-\frac{\bar{a}}{\sqrt{(f+u)}}\right\}$$

$$\times\left\{a_1X_1-\bar{a}_1\bar{X}_1+\frac{(f+u)\,b_1Y_1-(f+\bar{u})\,\bar{b}_1\bar{Y}_1}{f+q}+\frac{(f+u)\,c_1Z_1-(f+\bar{u})\,\bar{c}_1\bar{Z}_1}{f+r}\right\},$$

we see that

$$\frac{X}{\sqrt{(f+\bar{u})}}-\frac{\bar{X}}{\sqrt{(f+u)}}=-\frac{1}{\rho}\left(\frac{a}{\sqrt{(f+\bar{u})}}-\frac{\bar{a}}{\sqrt{(f+u)}}\right),$$

and similarly

$$\frac{Y}{\sqrt{(f+\bar{u})}}-\frac{\bar{Y}}{\sqrt{(g+u)}}=-\frac{1}{\rho}\left(\frac{b}{\sqrt{(g+\bar{u})}}-\frac{\bar{b}}{\sqrt{(g+u)}}\right),$$

$$\frac{Z}{\sqrt{(h+\bar{u})}}-\frac{\bar{Z}}{\sqrt{(h+u)}}=-\frac{1}{\rho}\left(\frac{c}{\sqrt{(h+\bar{u})}}-\frac{\bar{c}}{\sqrt{(h+u)}}\right);$$

which are in fact the equations which express that Q and $\bar{Q}$ are corresponding points.

It is proper to remark that supposing, as we are at liberty to do, that P, $\bar{P}$ are situate in corresponding octants of the two ellipsoids, then if the curve of contact of the circumscribed cone having P for its vertex divide the surface of the interior ellipsoid into two parts $\bar{M}$, $\bar{N}$, of which the former lies contiguous to $\bar{P}$: also if the curve of intersection of the tangent plane at $\bar{P}$ divide the surface of the exterior ellipsoid into two parts M, N, of which M lies contiguous to the point P; then the different points of M, $\bar{M}$ correspond to each other, as do also the different points of N, $\bar{N}$.

We may now pass to the integral calculus problem. The Attraction parallel to the axis of x is

$$A=\int\frac{x\,dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

the limits of the integration being given by

$$\frac{(x+a)^2}{f+\bar{u}}+\frac{(y+b)^2}{g+\bar{u}}+\frac{(z+c)^2}{h+\bar{u}}=1;$$

or putting

$$x=rX,\quad y=rY,\quad z=rZ,$$

where X, Y, Z have the same signification, as before, we have

$$dx\,dy\,dz=r^2dr\,dS,$$

and then

$$A=\int Xdr\,dS=\int\rho X\,dS,$$

where ρ has the same signification as before: it will be convenient to leave the formula in this form, rather than to take at once the difference of the two values of ρ, but of course the integration is as in the ordinary methods to be performed so as to extend to the whole volume of the ellipsoid. The expression dS denotes the differen-

tial of a spherical surface radius unity, and if θ, ϕ are the parameters by which the position of ρ is determined, we have

$$dS = \begin{vmatrix} X, & Y, & Z \\ \frac{dX}{d\theta}, & \frac{dY}{d\theta}, & \frac{dZ}{d\theta} \\ \frac{dX}{d\phi}, & \frac{dY}{d\phi}, & \frac{dZ}{d\phi} \end{vmatrix} d\theta\, d\phi.$$

In the present case

$$dS = dS_1 = \begin{vmatrix} X_1, & Y_1, & Z_1 \\ \frac{dX_1}{d\bar{Y}_1}, & \frac{dY_1}{d\bar{Y}_1}, & \frac{dZ_1}{d\bar{Y}_1} \\ \frac{dX_1}{d\bar{Z}_1}, & \frac{dY_1}{d\bar{Z}_1}, & \frac{dZ_1}{d\bar{Z}_1} \end{vmatrix} d\bar{Y}_1 d\bar{Z}_1,$$

or from the values of X_1, Y_1, Z_1 in terms of $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ {observing that $\bar{X}_1$ must be replaced by its value $\sqrt{(1-\bar{Y}_1^2-\bar{Z}_1^2)}$} we deduce

$$dS = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \frac{1}{\bar{X}_1} d\bar{Y}_1 d\bar{Z}_1.$$

But $d\bar{S} = d\bar{S}_1 = \dfrac{1}{\bar{X}_1} d\bar{Y}_1 d\bar{Z}_1$,

whence

$$dS = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \frac{\bar{X}_1 d\bar{S}}{\bar{X}_1},$$

which shows that the corresponding elements of the spheres whose centres are P, $\bar{P}$, projected upon the tangent planes at P and $\bar{P}$ respectively, are in a constant ratio. It may be noticed also that if μ, $\bar{\mu}$ are the masses of the ellipsoids, the ratio in question

$$= \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} = \frac{\bar{\mu}\alpha_1}{\mu\bar{\alpha}_1}.$$

We have thus

$$A = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \int \frac{\bar{\rho}\bar{X}_1 X d\bar{S}}{\bar{X}_1},$$

that is

$$A = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \int \frac{\bar{\rho}(\alpha X_1 + \alpha' Y_1 + \alpha'' Z_1)\bar{X}_1 d\bar{S}}{X_1}.$$

The value which it will be convenient to use for $\bar{\rho}$ is that derived from the equation

$$\bar{\rho}^2\left(\frac{\bar{X}^2}{f+u} + \frac{\bar{Y}^2}{g+u} + \frac{\bar{Z}^2}{h+u}\right) + 2\bar{\rho}\left(\frac{\bar{X}\bar{a}}{f+u} + \frac{\bar{Y}\bar{b}}{g+u} + \frac{\bar{Z}\bar{c}}{h+u}\right) + \frac{\bar{u}-u}{a_1^2} = 0,$$

with only the transformation of expressing the radical in terms of X_1, viz.

$$\rho = \frac{\dfrac{\bar{X}\bar{a}}{f+u} + \dfrac{\bar{Y}\bar{b}}{g+u} + \dfrac{\bar{Z}\bar{c}}{h+u} + \dfrac{1}{a_1}X_1}{\dfrac{\bar{X}^2}{f+u} + \dfrac{\bar{Y}^2}{g+u} + \dfrac{\bar{Z}^2}{h+u}};$$

substituting these values and observing that Y_1 and Z_1 are rational functions of $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, but that $\bar{X}_1$ is a radical, and that in order to extend the integration to the whole ellipsoid, the values corresponding to the opposite signs of X_1 will require to be added, the quantity to be integrated (omitting for the moment the exterior constant factor) is

$$= \frac{\left\{\alpha\left(\dfrac{\bar{X}\bar{a}}{f+u} + \dfrac{\bar{Y}\bar{b}}{g+u} + \dfrac{\bar{Z}\bar{c}}{h+u}\right) + \dfrac{1}{a_1}(\alpha' Y_1 + \alpha'' Z_1)\right\}\bar{X}_1 d\bar{S}}{\dfrac{\bar{X}^2}{f+u} + \dfrac{\bar{Y}^2}{g+u} + \dfrac{\bar{Z}^2}{h+u}},$$

the integration to be extended over the spherical area $\bar{S}$. Consider the quantity within { }, this is

$$\alpha\left(\frac{\bar{X}\bar{a}}{f+u} + \frac{\bar{Y}\bar{b}}{g+u} + \frac{\bar{Z}\bar{c}}{h+u}\right) + \frac{\alpha'}{a_1}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)}(\bar{\alpha}'\bar{X} + \bar{\beta}'\bar{Y} + \bar{\gamma}'\bar{Z}) + \frac{\alpha''}{a_1}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}(\bar{\alpha}''\bar{X} + \bar{\beta}''\bar{Y} + \bar{\gamma}''\bar{Z}).$$

The coefficients of $\bar{Y}$ and $\bar{Z}$ vanish, in fact that of $\bar{Y}$ is

$$\frac{a_1 a}{f+u}\frac{\bar{b}}{g+u} + \frac{b_1 a}{a_1(f+q)}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)}\frac{\bar{b}_1\bar{b}}{g+q} + \frac{c_1 a}{a_1(f+r)}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}\frac{\bar{c}_1\bar{b}}{g+r}$$

$$= \frac{a\bar{b}}{a_1}\left\{\frac{a_1^2}{(f+u)(g+u)} + \frac{b_1\bar{b}_1}{(f+q)(g+q)}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)} + \frac{c_1\bar{c}_1}{(f+r)(g+r)}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}\right\}$$

$$= \frac{ab_1}{a_1}\left\{\frac{a_1^2}{(f+u)(g+u)} + \frac{b_1^2}{(f+q)(g+q)} + \frac{c_1^2}{(f+r)(g+r)}\right\} = 0;$$

and similarly for the coefficient of $\bar{Z}$.

The coefficient of $\bar{X}$ is in like manner shown to be

$$\frac{a\bar{a}}{a_1}\left\{\frac{a_1^2}{(f+u)^2} + \frac{b_1^2}{(f+q)^2} + \frac{c_1^2}{(f+r)^2}\right\} = \frac{a\bar{a}}{a_1}\frac{(f-g)(f-h)}{(f+u)(f+q)(f+r)} = \frac{a\bar{a}}{a^2 a_1} = \frac{\bar{a}}{a a_1};$$

or the quantity in question is simply

$$\frac{\bar{a}}{a a_1}\bar{X}.$$

Multiplying this by $\bar{X}_1, = \bar{\alpha} X + \bar{\alpha}' \bar{Y} + \bar{\alpha}'' \bar{Z}$, the terms containing $\bar{X}\bar{Y}$, $\bar{X}\bar{Z}$ vanish after the integration, and we need only consider the term $\frac{\bar{a}\,\bar{\alpha}}{a a_1}\bar{X}^2$, or what is the same $\frac{\bar{a}^2 \bar{a}_1}{a a_1 (f+\bar{u})}\bar{X}^2$. Hence

$$A = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u+q)(u-r)}\right\}} \frac{\bar{a}^2 \bar{a}_1}{a a_1 (f+\bar{u})} \int \frac{\bar{X}^2 d\bar{S}}{\frac{\bar{X}^2}{f+u} + \frac{\bar{Y}^2}{g+u} + \frac{\bar{Z}^2}{h+u}}.$$

The value of the corresponding function $\bar{A}$ (that is, the attraction of the exterior ellipsoid upon $\bar{P}$) is

$$\bar{A} = \frac{\bar{a}}{f+u} \int \frac{\bar{X}^2 d\bar{S}}{\frac{\bar{X}^2}{f+u} + \frac{\bar{Y}^2}{g+u} + \frac{\bar{Z}^2}{h+u}},$$

the limits being the same, whence

$$A \div \bar{A} = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \frac{f+u}{f+\bar{u}} \frac{\bar{a}\bar{a}_1}{a a_1} = \frac{\sqrt{(\bar{u}+g)}\sqrt{(\bar{u}+h)}}{\sqrt{(u+g)}\sqrt{(u+h)}};$$

or we have

$$A = \frac{\sqrt{(\bar{u}+g)}\sqrt{(\bar{u}+h)}}{\sqrt{(u+g)}\sqrt{(u+h)}}\bar{A}, \quad B = \frac{\sqrt{(\bar{u}+h)}\sqrt{(\bar{u}+f)}}{\sqrt{(u+h)}\sqrt{(\bar{u}+f)}}\bar{B}, \quad C = \frac{\sqrt{(\bar{u}+f)}\sqrt{(\bar{u}+g)}}{\sqrt{(u+f)}\sqrt{(u+g)}}\bar{C},$$

formulæ which constitute in fact Ivory's theorem.

Let K, $\bar{K}$ denote the attractions in the directions of the normals at P, $\bar{P}$, we have

$$K = \frac{\bar{\mu} a_1}{\mu \bar{a}_1} \int \bar{X}_1 d\bar{S}, \quad \bar{K} = \int \bar{X}_1 d\bar{S},$$

or

$$K = \frac{\bar{\mu} a_1}{\mu \bar{a}_1} K;$$

and it is important to remark that this is true not only for the entire ellipsoids; but if $\mathfrak{M}$, $\mathfrak{N}$ denote the attractions of the cones standing on the portions $\bar{M}$, $\bar{N}$ of the surface of the interior ellipsoid, and $\bar{\mathfrak{M}}$, $\bar{\mathfrak{N}}$ the attractions of the portions of the exterior ellipsoid bounded by the tangent plane at $\bar{P}$, and the portions M, N of the surface of the exterior ellipsoid, then

$$\mathfrak{M} = -\frac{\bar{\mu} a_1}{\mu a_1}\bar{\mathfrak{M}}, \quad \mathfrak{N} = \frac{\bar{\mu} a_1}{\mu a_1}\bar{\mathfrak{N}},$$

where obviously

$$K = \mathfrak{N} - \mathfrak{M}, \quad \bar{K} = \bar{\mathfrak{N}} + \bar{\mathfrak{M}};$$

this theorem is so far as I am aware new.

90.

NOTE SUR QUELQUES FORMULES RELATIVES AUX CONIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXIX. (1850), pp. 1—3.]

SOIT, comme à l'ordinaire:

$$\left.\begin{aligned}
\mathfrak{A} &= BC - F^2, \\
\mathfrak{B} &= CA - G^2, \\
\mathfrak{C} &= AB - H^2, \\
\mathfrak{F} &= GH - AF, \\
\mathfrak{G} &= HF - BG, \\
\mathfrak{H} &= FG - CH, \\
K &= ABC - AF^2 - BG^2 - CH^2 + 2FGH,
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

et désignons, pour abréger, les fonctions

$$\begin{aligned}
&Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy, \\
&A\alpha x + B\beta y + C\gamma z + F(\beta z + \gamma y) + G(\gamma x + \alpha z) + H(\alpha y + \beta x), \\
&A(\beta z - \gamma y)^2 + B(\gamma x - \alpha z)^2 + C(\alpha y - \beta x)^2 + 2F(\gamma x - \alpha z)(\alpha y - \beta x) \\
&\qquad\qquad + 2G(\alpha y - \beta x)(\beta z - \gamma y) + 2H(\beta z - \gamma y)(\gamma x - \alpha z); \\
&\&c.
\end{aligned}$$

par $\qquad Ax^2 + \ldots; \quad A\alpha x + \ldots; \quad A(\beta z - \gamma y)^2 + \ldots; \quad \&c.$

Cela posé, soient $\mathfrak{A} + \mathrm{a}$, $\mathfrak{B} + \mathrm{b}$, $\mathfrak{C} + \mathrm{c}$, $\mathfrak{F} + \mathrm{f}$, $\mathfrak{G} + \mathrm{g}$, $\mathfrak{H} + \mathrm{h}$, $K + \mathrm{k}$ ce que deviennent $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, K, en écrivant $A + \alpha^2$, $B + \beta^2$, $C + \gamma^2$, $F + \beta\gamma$, $G + \gamma\alpha$, $H + \alpha\beta$ au lieu de A, B, C, F, G, H. Alors on aura d'abord

$$\mathrm{k} = \mathfrak{A}\alpha^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

et les quantités a, b, c, f, g, h seront données par l'équation

$$\mathrm{a}x^2 + \ldots = A\,(\beta z - \gamma y)^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

ou, si l'on veut, par celle-ci :

$$\mathrm{a}xx_1 + \ldots = A\,(\beta z - \gamma y)\,(\beta z_1 - \gamma y_1) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

(savoir, en considérant ces équations comme identiques par rapport à x, y, z et x_1, y_1, z_1 respectivement).

On obtient sans difficulté les équations identiques :

$$(A\alpha\alpha_1 + \ldots)\,(Axx_1 + \ldots) - (A\alpha x + \ldots)\,(A\alpha_1 x_1 + \ldots) = \mathfrak{A}\,(\beta z_1 - \gamma y_1)\,(\beta_1 z - \gamma_1 y) + \ldots, \quad (5),$$

$$(\mathfrak{A}\alpha\alpha_1 + \ldots)\,(\mathfrak{A}xx_1 + \ldots) - (\mathfrak{A}\alpha x + \ldots)\,(\mathfrak{A}\alpha_1 x_1 + \ldots) = K\,[A\,(\beta z_1 - \gamma y_1)\,(\beta_1 z - \gamma_1 y) + \ldots] \quad (6).$$

Comme on sait, la condition sous laquelle la droite $lx + my + nz = 0$ touche la conique $U = Ax^2 + \ldots = 0$ peut être présentée sous la forme $\mathfrak{A}l^2 + \ldots = 0$. Donc la condition pour que cette droite touche la conique $U + (\alpha x + \beta y + \gamma z)^2 = 0$, est

$$\mathfrak{A}l^2 + \ldots + A\,(\gamma m - \beta n)^2 + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

En réduisant au moyen de l'équation $(\mathfrak{A}\alpha^2 + \ldots)\,(\mathfrak{A}l^2 + \ldots) = K\,[A\,(\gamma m - \beta n)^2 + \ldots]$ (laquelle n'est qu'un cas particulier de l'équation (6)), cette condition devient :

$$(K + \mathfrak{A}\alpha^2 + \ldots)\,(\mathfrak{A}l^2 + \ldots) - (\mathfrak{A}\alpha l + \ldots)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Pour trouver la condition sous laquelle les coniques

$$U + (\alpha x + \beta y + \gamma z)^2 = 0, \quad U + (\alpha_1 x + \beta_1 y + \gamma_1 z)^2 = 0$$

se touchent, on n'a qu'à remarquer que l'équation de la tangente commune est, ou

$$(\alpha - \alpha_1)\,x + (\beta - \beta_1)\,y + (\gamma - \gamma_1)\,z = 0, \qquad \text{ou} \qquad (\alpha + \alpha_1)\,x + (\beta + \beta_1)\,y + (\gamma + \gamma_1)\,z = 0.$$

En ne considérant que la première de ces droites, on a pour la condition cherchée :

$$\mathfrak{A}\,(\alpha - \alpha_1)^2 + \ldots + A\,(\beta\gamma_1 - \beta_1\gamma)^2 + \ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

ou, en réduisant au moyen du même cas particulier de l'équation (6), on obtient cette condition sous la forme

$$(K + \mathfrak{A}\alpha^2 + \ldots)\,(K + \mathfrak{A}\alpha_1^2 + \ldots) - (K + \mathfrak{A}\alpha\alpha_1 + \ldots)^2 = 0 \ldots\ldots\ldots\ldots (10).$$

On sait que les coordonnées X, Y, Z du pôle de la droite $lx + my + nz = 0$, par rapport à la conique $U = 0$, peuvent être trouvées par l'équation

$$\lambda X + \mu Y + \nu Z = \mathfrak{A}l\lambda + \ldots,$$

(considérée comme identique par rapport à λ, μ, ν). De là les coordonnées X, Y, Z du pôle de cette même droite par rapport à la conique

$$U + (\alpha x + \beta y + \gamma z)^2 = 0$$

se trouveront par l'équation

$$\lambda X + \mu Y + \nu Z = \mathfrak{A} l\lambda + \ldots + A(\gamma m - \beta n)(\gamma\mu - \beta\nu) \ldots \quad \ldots\ldots\ldots \quad (11)$$

(considérée comme identique par rapport à λ, μ, ν). Cette équation peut aussi être présentée sous la forme

$$K(\lambda X + \mu Y + \nu Z) = (K + \mathfrak{A}\alpha^2 + \ldots)(\mathfrak{A} l\lambda + \ldots) - (\mathfrak{A}\alpha\lambda + \ldots)(\mathfrak{A}\alpha l + \ldots) \quad \ldots (12),$$

ce qui peut être démontré facilement au moyen d'un cas particulier de l'équation (6).

Si les deux droites $lx + my + nz = 0$, $l'x + m'y + n'z = 0$, touchent la conique $U = 0$, l'équation de la droite qui passe par les points de contact sera $A(mn' - m'n)x + \ldots = 0$. Donc: si ces deux droites touchent la conique $U + (\alpha x + \beta y + \gamma z)^2 = 0$, l'équation de la droite qui passe par les deux points de contact sera

$$A(mn' - m'n)x + \ldots + (\alpha x + \beta y + \gamma z)[\alpha(mn' - m'n) + \beta(nl' - n'l) + \gamma(lm' - l'm)] = 0 \ \ldots (13).$$

Les formules obtenues seront utiles pour la solution du problème du mémoire suivant, [91]. Je les ai rapprochées ici pour ne pas interrompre cette solution.

91.

SUR LE PROBLÈME DES CONTACTS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXIX. (1850), pp. 4—13.]

JE me propose ici la solution analytique du problème suivant:

"Étant données trois coniques inscrites à une même conique: trouver une autre conique, aussi inscrite à cette conique, qui touche les trois coniques inscrites; et tirer de là les constructions géométriques ordinaires."

Je commence par récapituler quelques-unes des propriétés d'un système de trois coniques inscrites à la même conique.

Un système de six droites qui passent trois à trois par quatre points, s'appelle *quadrangle*. Les points de rencontre des côtés opposés sont les *centres* du quadrangle; les *côtés* du triangle formé par ces trois centres sont les *axes* du quadrangle. De même, un système de six points situés trois à trois sur quatre droites, s'appelle *quadrilatère*. Les droites qui passent par les angles opposés sont les *axes;* et les angles du triangle formé par les trois axes sont les *centres* du quadrilatère.

Deux coniques quelconques se coupent en quatre points qui forment un quadrangle inscrit aux deux coniques. Elles ont quatre tangentes communes qui forment un quadrilatère circonscrit aux deux coniques. Le quadrangle inscrit et le quadrilatère circonscrit ont les même centres et les mêmes axes.

Si deux coniques sont circonscrites ou inscrites l'une à l'autre, la droite qui passe par les deux points de contact s'appelle *chorde de contact*, et le point de rencontre des deux tangentes communes s'appelle *centre de contact*.

Cela posé: les coniques circonscrites à deux coniques données, peuvent être divisées en trois classes: une conique circonscrite appartient à une quelconque de ces trois classes, selon que les points de rencontre des chordes de contact de la conique

circonscrite et de chacune des deux coniques données coïncide avec un quelconque des trois centres du quadrangle inscrit, ou du quadrilatère circonscrit; ou, si l'on veut, selon que la droite qui passe par les centres de contact de la conique circonscrite et des deux coniques données, coïncide avec un quelconque des trois axes du quadrangle inscrit, ou du quadrilatère circonscrit.

En considérant les deux coniques données, et une conique circonscrite, nous dirons que les deux côtés du quadrangle inscrit qui se coupent dans le point d'intersection des deux chordes de contact, sont les *axes de symptose* des deux coniques données, et que les deux angles du quadrilatère circonscrit, situés sur la droite qui passe par les deux centres de contact, sont les *centres d'homologie* des deux coniques données.

Soient maintenant inscrites trois coniques à la même conique. En combinant deux à deux ces trois coniques, les six axes de symptose se couperont trois à trois en quatre points que nous nommerons *centres de symptose*, et les six centres d'homologie seront situés trois à trois sur quatre droites que nous appelerons *axes d'homologie*.

Soient

$$U + V_1^2 = 0, \quad U + V_2^2 = 0, \quad U + V_3^2 = 0$$

les équations des trois coniques inscrites, où

$$\begin{aligned} U &= Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy, \\ V_1 &= \alpha_1 x + \beta_1 y + \gamma_1 z, \\ V_2 &= \alpha_2 x + \beta_2 y + \gamma_2 z, \\ V_3 &= \alpha_3 x + \beta_3 y + \gamma_3 z. \end{aligned}$$

Si

$$lx + my + nz = 0$$

est l'équation d'une tangente commune aux coniques $U + V_1^2 = 0$, $U + V_2^2 = 0$, les formules de la "Note sur quelques formules &c." [90], en adoptant la notation de cette note, donneront les équations

$$\begin{aligned} (K + \mathfrak{A}\alpha_1^2 + \ldots)(\mathfrak{A}l^2 + \ldots) - (\mathfrak{A}\alpha_1 l + \ldots)^2 &= 0, \\ (K + \mathfrak{A}\alpha_2^2 + \ldots)(\mathfrak{A}l^2 + \ldots) - (\mathfrak{A}\alpha_2 l + \ldots)^2 &= 0, \end{aligned}$$

qui serviront à déterminer les valeurs de l, m, n; on obtient par là l'éxpression

$$\sqrt{(K + \mathfrak{A}\alpha_2^2 + \ldots)(\mathfrak{A}\alpha_1 l + \ldots)} - \sqrt{(K + \mathfrak{A}\alpha_1^2 + \ldots)(\mathfrak{A}\alpha_2 l + \ldots)},$$

qui fait voir que l'équation $lx + my + nz = 0$ est satisfaite en écrivant

$$\begin{aligned} x : y : z = {} & \sqrt{(K + \mathfrak{A}\alpha_1^2 + \ldots)(\mathfrak{A}\alpha_1 + \mathfrak{H}\beta_1 + \mathfrak{G}\gamma_1)} - \sqrt{(K + \mathfrak{A}\alpha_2^2 + \ldots)(\mathfrak{A}\alpha_2 + \mathfrak{H}\beta_2 + \mathfrak{G}\gamma_2)} \ldots (1) \\ : {} & \sqrt{(K + \mathfrak{A}\alpha_1^2 + \ldots)(\mathfrak{H}\alpha_1 + \mathfrak{B}\beta_1 + \mathfrak{F}\gamma_1)} - \sqrt{(K + \mathfrak{A}\alpha_2^2 + \ldots)(\mathfrak{H}\alpha_2 + \mathfrak{B}\beta_2 + \mathfrak{F}\gamma_2)} \\ : {} & \sqrt{(K + \mathfrak{A}\alpha_1^2 + \ldots)(\mathfrak{G}\alpha_1 + \mathfrak{F}\beta_1 + \mathfrak{C}\gamma_1)} - \sqrt{(K + \mathfrak{A}\alpha_2^2 + \ldots)(\mathfrak{G}\alpha_2 + \mathfrak{F}\beta_2 + \mathfrak{C}\gamma_2)}, \end{aligned}$$

et ces équations, qui peuvent aussi être présentées sous la forme plus simple

$$\begin{aligned} Ax+Hy+Gz : Hx+By+Fz : Gx+Fy+Cz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2), \\ = \sqrt{(K+\mathfrak{A}\alpha_1^2\ldots)}\,\alpha_2-\sqrt{(K+\mathfrak{A}\alpha_2^2\ldots)}\,\alpha_1 \\ : \sqrt{(K+\mathfrak{A}\alpha_1^2\ldots)}\,\beta_2-\sqrt{(K+\mathfrak{A}\alpha_2^2\ldots)}\,\beta_1 \\ : \sqrt{(K+\mathfrak{A}\alpha_1^2\ldots)}\,\gamma_2-\sqrt{(K+\mathfrak{A}\alpha_2^2\ldots)}\,\gamma_1, \end{aligned}$$

correspondent à un centre d'homologie des deux coniques $U+V_1^2=0$, $U+V_2^2=0$.

En mettant, pour abréger,

$$\sqrt{(K+\mathfrak{A}\alpha_1^2+\ldots)}=p_1,\quad \sqrt{(K+\mathfrak{A}\alpha_2^2+\ldots)}=p_2,\quad \sqrt{(K+\mathfrak{A}\alpha_3^2+\ldots)}=p_3 \ldots\ldots\ldots (3),$$

on obtient facilement, pour un des axes d'homologie des trois coniques, l'équation

$$\begin{vmatrix} & p_1, & p_2, & p_3 \\ Ax+Hy+Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx+By+Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx+Fy+Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

et celles des trois autres axes d'homologie en peuvent être tirées en changeant les signes de p_1, p_2, p_3.

Remarquons que l'équation

$$\begin{vmatrix} & 1\,, & 1\,, & 1 \\ Ax+Hy+Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx+By+Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx+Fy+Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

est celle de la polaire d'un des centres de symptose des trois coniques par rapport à la conique circonscrite $U=0$, savoir de celle qui est donnée par le système $V_1=V_2=V_3$, et que les équations des trois autres polaires correspondantes se trouveront en changeant les signes de α_1, β_1, γ_1, ou de α_2, β_2, γ_2, ou de α_3, β_3, γ_3.

Cherchons le pôle de l'axe d'homologie dont nous venons de trouver l'équation, par rapport à une quatrième conique inscrite $U+V^2=0$ $(V=\alpha x+\beta y+\gamma z)$. En exprimant cette équation par $lx+my+nz=0$, on obtiendra les coordonnées de ce pôle au moyen de l'équation

$$K(\lambda X+\mu Y+\nu Z)=p^2(\mathfrak{A}l\lambda+\ldots)-(\mathfrak{A}\alpha\lambda+\ldots)(\mathfrak{A}\alpha l+\ldots) \ldots\ldots\ldots\ldots (6),$$

où, pour abréger, on a mis $p=\sqrt{(K+\mathfrak{A}\alpha^2+\ldots)}$. (Mémoire cité; équation (12).) Mais ici on a

$$lx+my+nz=\begin{vmatrix} & p_1, & p_2, & p_3 \\ Ax+Hy+Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx+By+Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx+Fy+Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}$$

ce qui donne immédiatement

$$\mathfrak{A}l\lambda + \ldots = K \begin{vmatrix} & p_1, & p_2, & p_3 \\ \lambda, & \alpha_1, & \alpha_2, & \alpha_3 \\ \mu, & \beta_1, & \beta_2, & \beta_3 \\ \nu, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}, \quad \mathfrak{A}\alpha l + \ldots = K \begin{vmatrix} & p_1, & p_2, & p_3 \\ \alpha, & \alpha_1, & \alpha_2, & \alpha_3 \\ \beta, & \beta_1, & \beta_2, & \beta_3 \\ \gamma, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix};$$

et de là on obtient

$$\lambda X + \mu Y + \nu Z = p^2 \begin{vmatrix} & p_1, & p_2, & p_3 \\ \lambda, & \alpha_1, & \alpha_2, & \alpha_3 \\ \mu, & \beta_1, & \beta_2, & \beta_3 \\ \nu, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} - (\mathfrak{A}\alpha\lambda + \ldots) \begin{vmatrix} & p_1, & p_2, & p_3 \\ \alpha, & \alpha_1, & \alpha_2, & \alpha_3 \\ \beta, & \beta_1, & \beta_2, & \beta_3 \\ \gamma, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} \ldots\ldots(7);$$

savoir, en considérant cette équation comme identique par rapport à λ, μ, ν, on obtient les coordonnées X, Y, Z du point dont il s'agit. En prenant particulièrement ce pôle par rapport à la conique $U + V_2^2 = 0$, cette équation se réduit à

$$\frac{1}{p_1}(\lambda X + \mu Y + \nu Z) = p_1 \begin{vmatrix} & p_1, & p_2, & p_3 \\ \lambda, & \alpha_1, & \alpha_2, & \alpha_3 \\ \mu, & \beta_1, & \beta_2, & \beta_3 \\ \nu, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} - (\mathfrak{A}\alpha_1\lambda + \ldots) \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} \ldots\ldots (8),$$

où, si l'on veut, X, Y, Z seront déterminés par les expressions

$$\left.\begin{aligned} \alpha_1 X + \beta_1 Y + \gamma_1 Z &= -p_1^2 + \mathfrak{A}\alpha_1^2 + \ldots, = -K \\ \alpha_2 X + \beta_2 Y + \gamma_2 Z &= -p_1 p_2 + \mathfrak{A}\alpha_1\alpha_2 + \ldots, \\ \alpha_3 X + \beta_3 Y + \gamma_3 Z &= -p_1 p_3 + \mathfrak{A}\alpha_1\alpha_3 + \ldots, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Le facteur $\frac{1}{p_1}$ a été supprimé. De là on obtient aussi l'équation de la droite menée par ce point, savoir par le pôle de l'axe d'homologie par rapport à la conique $U + V_1^2 = 0$, et par le centre de symptose $V_1 = V_2 = V_3$. En effet, cette équation est

$$\begin{vmatrix} V_1, & V_2, & V_3 \\ 1, & 1, & 1 \\ p_1^2 - \mathfrak{A}\alpha_1^2 - \ldots, & p_1 p_2 - \mathfrak{A}\alpha_1\alpha_2 - \ldots, & p_1 p_3 - \mathfrak{A}\alpha_1\alpha_3 - \ldots \end{vmatrix} = 0 \ldots\ldots (10).$$

On pourrait également chercher le point d'intersection de la polaire du centre de symptose $V_1 = V_2 = V_3$ par rapport à $U + V_1^2 = 0$, et de l'axe d'homologie; ce point serait évidemment le pôle de la droite exprimée par la dernière équation, par rapport à $U + V_1^2 = 0$.

Ces résultats seront utiles pour l'interprétation de la formule relative à la conique qui touche les trois coniques données et que nous irons chercher maintenant.

Représentons par $U + V^2 = 0$ l'équation de cette conique; l'équation (10) du mémoire cité donnera les expressions

$$\left.\begin{aligned} \mathfrak{A}\alpha\alpha_1 + \ldots &= -K + pp_1, \\ \mathfrak{A}\alpha\alpha_2 + \ldots &= -K + pp_2, \\ \mathfrak{A}\alpha\alpha_3 + \ldots &= -K + pp_3, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

auxquelles nous ajouterons l'équation qui donne la valeur de p, savoir

$$\mathfrak{A}\alpha^2 + \ldots = -K + p^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Il n'y a qu'à substituer dans cette dernière équation les valeurs de α, β, γ que donnent les trois autres. Par là on obtient, pour déterminer p, une équation du second degré et de la forme

$$K^2L - 2KpM + p^2N = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13);$$

c'est-à-dire, en faisant $M^2 - NL = \Omega^2$, $\Lambda = \dfrac{L}{\Omega + M}$, on aura $p = \Lambda K$ et de là

$$\left.\begin{aligned} \mathfrak{A}\alpha\alpha_1 + \ldots &= -K(\Lambda p_1 - 1), \\ \mathfrak{A}\alpha\alpha_2 + \ldots &= -K(\Lambda p_2 - 1), \\ \mathfrak{A}\alpha\alpha_3 + \ldots &= -K(\Lambda p_3 - 1), \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

où Λ est une quantité connue, dont la valeur sera donnée dans la suite. Pour le moment il suffit de remarquer qu'en changeant à la fois les signes de p_1, p_2, p_3, Ω, cette quantité Λ ne change que de signe. Au lieu de chercher les valeurs de α, β, γ, il vaut mieux éliminer ces quantités entre ces dernières équations et l'équation $V = \alpha x + \beta y + \gamma z = 0$. Cela donne, pour trouver V, l'équation

$$\begin{vmatrix} V, & K(\Lambda p_1 - 1), & K(\Lambda p_2 - 1), & K(\Lambda p_3 - 1) \\ x, & \mathfrak{A}\alpha_1 + \mathfrak{H}\beta_1 + \mathfrak{G}\gamma_1, & \mathfrak{A}\alpha_2 + \mathfrak{H}\beta_2 + \mathfrak{G}\gamma_2, & \mathfrak{A}\alpha_3 + \mathfrak{H}\beta_3 + \mathfrak{G}\gamma_3 \\ y, & \mathfrak{H}\alpha_1 + \mathfrak{B}\beta_1 + \mathfrak{F}\gamma_1, & \mathfrak{H}\alpha_2 + \mathfrak{B}\beta_2 + \mathfrak{F}\gamma_2, & \mathfrak{H}\alpha_3 + \mathfrak{B}\beta_3 + \mathfrak{F}\gamma_3 \\ z, & \mathfrak{G}\alpha_1 + \mathfrak{F}\beta_1 + \mathfrak{C}\gamma_1, & \mathfrak{G}\alpha_2 + \mathfrak{F}\beta_2 + \mathfrak{C}\gamma_2, & \mathfrak{G}\alpha_3 + \mathfrak{F}\beta_3 + \mathfrak{C}\gamma_3 \end{vmatrix} = 0 \ldots (15),$$

qui peut aussi être écrite comme suit:

$$\begin{vmatrix} V, & \Lambda p_1 - 1, & \Lambda p_2 - 1, & \Lambda p_3 - 1 \\ Ax + Hy + Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx + By + Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx + Fy + Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (16).$$

En mettant $V = 0$, on aura l'équation de la chorde de contact de la conique cherchée et de la conique circonscrite $U = 0$. En remarquant que l'équation de la

conique cherchée peut être mise sous la forme $V = \sqrt{-U}$, l'équation de cette conique se présentera sous la forme très simple:

$$\begin{vmatrix} \sqrt{-U}, & \Lambda p_1 - 1, & \Lambda p_2 - 1, & \Lambda p_3 - 1 \\ Ax + Hy + Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx + By + Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx + Fy + Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (17),$$

ou enfin, si l'on veut, sous la forme plus usitée:

$$\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}^2 U + \begin{vmatrix} & \Lambda p_1 - 1, & \Lambda p_2 - 1, & \Lambda p_3 - 1 \\ Ax + Hy + Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx + By + Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx + Fy + Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}^2 = 0 \ldots (18);$$

la premiere de ces deux formes est peut-être la plus élégante.

Les propriétés géométriques sont absolument indépendantes de la valeur de la quantité Λ: mais pour compléter la solution, je vais donner l'expression de cette quantité. Pour cela, remarquons qu'en mettant pour abréger:

$$\Pi = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$l_1 = \beta_2\gamma_3 - \beta_3\gamma_2, \quad l_2 = \beta_3\gamma_1 - \beta_1\gamma_3, \ldots$$
$$m_1 = \gamma_2\alpha_3 - \gamma_3\alpha_2,$$
$$\vdots$$
$$\lambda = l_1(pp_1 - K) + l_2(pp_2 - K) + l_3(pp_3 - K),$$
$$\vdots$$

on aura d'abord l'équation identique

$$-K\Pi(\alpha x + \beta y + \gamma z) = \begin{vmatrix} & pp_1 - K, & pp_2 - K, & pp_3 - K \\ Ax + Hy + Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx + By + Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx + Fy + Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix},$$

ou, ce qui est la même chose,

$$K\Pi(\alpha x + \beta y + \gamma z) = \lambda(Ax + Hy + Gz) + \mu(Hx + By + Fz) + \nu(Gx + Fy + Cz).$$

Cela donne

$$\Pi(\mathfrak{A}\alpha^2 + \ldots) = \lambda\alpha + \mu\beta + \nu\gamma, \quad K\Pi(\lambda\alpha + \mu\beta + \nu\gamma) = (A\lambda^2 + \ldots),$$

c'est-à-dire

$$K\Pi^2(\mathfrak{A}\alpha^2+\ldots)=A\lambda^2+\ldots,$$

et, en vertu de cette expression, l'équation qui sert à déterminer p se réduit à

$$K\Pi^2+A\lambda^2+\ldots-K\Pi^2p^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20).$$

En la comparant avec l'équation $K^2L-2KpM+p^2N=0$, on obtient

$$\left.\begin{aligned} L &= \Pi^2+[A(l_1+l_2+l_3)^2+\ldots],\\ M &= [A(l_1+l_2+l_3)(l_1p_1+l_2p_2+l_3p_3)+\ldots],\\ N &= -K\Pi^2+[A(l_1p_1+l_2p_2+l_3p_3)^2+\ldots],\end{aligned}\right\}\ldots\ldots\ldots\ldots(21),$$

et de là, par une transformation déjà employée,

$$M^2-NL=\Omega^2=\Pi^2\left\{\begin{aligned} &-\ [A(l_1p_1+l_2p_2+l_3p_3)^2+\ldots]\\ &+K[A(l_1+l_2+l_3)^2+\ldots]+K\Pi^2\\ &-\ [\mathfrak{A}\{p_1(\alpha_2-\alpha_3)+p_2(\alpha_3-\alpha_1)+p_3(\alpha_1-\alpha_2)\}^2+\ldots]\end{aligned}\right\}\ldots(22);$$

mais l'interpretation de ce résultat paraît être difficile.

En revenant sur l'équation trouvée pour la conique qui touche les trois coniques données, remarquons que les signes de α_1, β_1, γ_1, ou de α_2, β_2, γ_2, ou de α_3, β_3, γ_3, peuvent être changés conjointement. Cela revient en effet à écrire $-V_1$, ou $-V_2$, ou $-V_3$ au lieu de $+V_1$, ou de $+V_2$, ou de $+V_3$, ce qui ne change pas les coniques inscrites. Mais il est facile de voir qu'en changeant à la fois les signes de V_1, V_2, V_3, on ne change pas l'équation de la conique dont il s'agit; cette équation ne change non plus, en changeant à la fois les signes de p_1, p_2, p_3, Ω; de manière que l'équation trouvée correspond réellement à 32 coniques différentes. En distinguant ces 32 coniques par des symboles de la forme

$$(\pm V_1,\ \pm V_2,\ \pm V_3,\ \pm p_1,\ \pm p_2,\ \pm p_3,\ \pm\Omega):$$

quatre symboles tels que

$$\begin{aligned} (\ \ V_1,&\ \ V_2,\ \ V_3,\ \ p_1,\ \ p_2,\ \ p_3,\ \ \Omega),\\ (-V_1,&\ -V_2,\ -V_3,\ \ p_1,\ \ p_2,\ \ p_3,\ \ \Omega),\\ (\ \ V_1,&\ \ V_2,\ \ V_3,\ -p_1,\ -p_2,\ -p_3,\ -\Omega),\\ (-V_1,&\ -V_2,\ -V_3,\ -p_1,\ -p_2,\ -p_3,\ -\Omega),\end{aligned}$$

ne se rapporteront qu'à une seule conique. Nous appelerons *paires de coniques* deux coniques quelconques exprimées par des symboles de la forme

$$(V_1,\ V_2,\ V_3,\ p_1,\ p_2,\ p_3,\ \pm\Omega);$$

donc les 32 coniques forment 16 paires, groupées quatre à quatre de deux manières différentes: savoir, pour former un groupe, quatre paires telles que

$$(\pm V_1,\ \pm V_2,\ V_3,\ p_1,\ p_2,\ p_3,\ \pm\Omega),$$

ou quatre paires telles que

$$(V_1,\ V_2,\ V_3,\ \pm p_1,\ \pm p_2,\ p_3,\ \pm \Omega)$$

peuvent être combinées. Ces deux espèces de groupes peuvent être distinguées par les noms *groupes par rapport à un axe d'homologie*, et *groupes par rapport à un centre de symptose*. En effet: considérons une paire de coniques, par exemple celle qui est représentée par les symboles $(V_1,\ V_2,\ V_3,\ p_1,\ p_2,\ p_3,\ \pm\Omega)$. Les équations des deux coniques de la paire sont les mêmes aux valeurs de Λ près; il est donc évident que les chordes de contact de ces deux coniques avec la conique circonscrite $U=0$ doivent se rencontrer au point d'intersection des droites

$$\begin{vmatrix} & 1, & 1, & 1 \\ Ax+Hy+Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx+By+Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx+Fy+Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = 0, \quad \begin{vmatrix} & p_1, & p_2, & p_3 \\ Ax+Hy+Gz, & \alpha_1, & \alpha_2, & \alpha_3 \\ Hx+By+Fz, & \beta_1, & \beta_2, & \beta_3 \\ Gx+Fy+Cz, & \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = 0.$$

La première de ces équations se rapporte à la polaire d'un des centres de symptose des trois coniques inscrites par rapport à la conique circonscrite; la seconde se rapporte à un des axes d'homologie. On a donc le théorème suivant:

"Les points de rencontre des 16 chordes de contact des paires de coniques sont les points de rencontre des polaires des quatre centres de symptose par rapport à la conique circonscrite, avec les quatre axes d'homologie."

Cela suffit pour expliquer la manière dont les deux espèces de groupes ont été distinguées.

Cherchons l'équation de la droite menée par les points de contact d'une des coniques inscrites (par exemple celle que donne l'equation $U+V_1^2=0$) avec deux coniques de la même paire. En représentant par

$$U+(\alpha x+\beta y+\gamma z)^2=0 \quad \text{et} \quad U+(\alpha' x+\beta' y+\gamma' z)^2=0$$

les équations de ces deux coniques: les équations des tangentes communes seront

$$(\alpha-\alpha_1)\,x+(\beta-\beta_1)\,y+(\gamma-\gamma_1)\,z=0 \quad \text{et} \quad (\alpha'-\alpha_1)\,x+(\beta'-\beta_1)\,y+(\gamma'-\gamma_1)\,z=0,$$

et l'équation de la droite qui passe par les points de contact de ces deux droites avec la conique $U+V_1^2=0$ est:

$$\begin{aligned} &A\,(\beta\gamma'-\beta'\gamma)\,x+\ldots+A\,(\gamma_1(\beta'-\beta)-\beta_1(\gamma'-\gamma))\,x+\ldots \\ &\quad+(\alpha_1 x+\beta_1 y+\gamma_1 z)\,[\alpha_1(\beta\gamma'-\beta'\gamma)+\beta_1(\gamma\alpha'-\gamma'\alpha)+\gamma_1(\alpha\beta'-\alpha'\beta)]=0 \;\ldots\ldots\; (23). \end{aligned}$$

Pour réduire cette équation, en exprimant par Π, l_1, m_1 &c. les valeurs plus haut, mettons $\lambda=l_1(\Lambda p_1-1)+l_2(\Lambda p_2-1)+l_3(\Lambda p_3-1)$ &c., et soient λ', μ', ν' ce que deviennent les valeurs de λ, μ, ν en écrivant Λ' au lieu de Λ. On aura

$$\Pi\alpha=A\lambda+H\mu+G\nu, \text{ \&c.}$$

et des expressions pareilles des valeurs de λ', μ', ν'. Delà on tire

$$\Pi^2(\beta\gamma'-\beta'\gamma)=(\mu\nu'-\mu'\nu)\,\mathfrak{A}+(\nu\lambda'-\nu'\lambda)\,\mathfrak{H}+(\lambda\mu'-\lambda'\mu)\,\mathfrak{G},$$

&c.,

$$\mu\nu'-\mu'\nu=(m_2n_3-m_3n_2)\,\{(\Lambda p_2-1)(\Lambda' p_3-1)-(\Lambda' p_2-1)(\Lambda p_3-1)\}+\&c.$$
$$=\Pi(\Lambda'-\Lambda)\,[\alpha_1(p_2-p_3)+\alpha_2(p_3-p_1)+\alpha_3(p_1-p_2)],$$

&c.

c'est-à-dire, en supprimant le facteur commun $\dfrac{\Lambda'-\Lambda}{\Pi}$:

$$\begin{aligned}\beta\gamma'-\beta'\gamma=\ & (\mathfrak{A}\alpha_1+\mathfrak{H}\beta_1+\mathfrak{G}\gamma_1)(p_2-p_3)\\ &+(\mathfrak{A}\alpha_2+\mathfrak{H}\beta_2+\mathfrak{G}\gamma_2)(p_3-p_1)\\ &+(\mathfrak{A}\alpha_3+\mathfrak{H}\beta_3+\mathfrak{G}\gamma_3)(p_1-p_2),\end{aligned}$$

&c.

Aussi on aura, en supprimant le même facteur:

$$\begin{aligned}\gamma_1(\beta'-\beta)-\beta_1(\gamma'-\gamma)=\ & (l_1p_1+l_2p_2+l_3p_3)(\gamma_1H-\beta_1G)\\ &+(m_1p_1+m_2p_2+m_3p_3)(\alpha_1G-\gamma_1A)\\ &+(n_1p_1+n_2p_2+n_3p_3)(\beta_1A-\alpha_1H).\end{aligned}$$

De là on obtient immédiatement

$$A(\beta\gamma'-\beta'\gamma)x+\ldots=K[V_1(p_2-p_3)+V_2(p_3-p_1)+V_3(p_1-p_2)],$$

et, par une réduction un peu plus difficile,

$$\begin{aligned}A(\gamma_1(\beta'-\beta)-\beta_1(\gamma'-\gamma))x+\ldots\\ =\ & (l_1p_1+l_2p_2+l_3p_3)[z(\mathfrak{H}\alpha_1+\mathfrak{B}\beta_1+\mathfrak{F}\gamma_1)-y(\mathfrak{G}\alpha_1+\mathfrak{F}\beta_1+\mathfrak{C}\gamma_1)]\\ &+(m_1p_1+m_2p_2+m_3p_3)[x(\mathfrak{G}\alpha_1+\mathfrak{F}\beta_1+\mathfrak{C}\gamma_1)-z(\mathfrak{A}\alpha_1+\mathfrak{H}\beta_1+\mathfrak{G}\gamma_1)]\\ &+(n_1p_1+n_2p_2+n_3p_3)[y(\mathfrak{A}\alpha_1+\mathfrak{H}\beta_1+\mathfrak{G}\gamma_1)-x(\mathfrak{H}\alpha_1+\mathfrak{B}\beta_1+\mathfrak{F}\gamma_1)]\end{aligned}$$

$$\begin{aligned}=V_1[p_2(\mathfrak{A}\alpha_1\alpha_3+\ldots)-p_3(\mathfrak{A}\alpha_1\alpha_2+\ldots)]+V_2[p_3(\mathfrak{A}\alpha_1^2+\ldots)-p_1(\mathfrak{A}\alpha_1\alpha_3+\ldots)]\\ +V_3[p_1(\mathfrak{A}\alpha_1\alpha_2+\ldots)-p_2(\mathfrak{A}\alpha_1^2+\ldots)];\end{aligned}$$

et enfin

$$\begin{aligned}(\alpha_1x+\beta_1y+\gamma_1z)[\alpha_1(\beta\gamma'-\beta'\gamma)+\beta_1(\gamma\alpha'-\gamma'\alpha)+\gamma_1(\alpha\beta'-\alpha'\beta)]\\ =V_1[\mathfrak{A}\alpha_1^2+\ldots)(p_2-p_3)+(\mathfrak{A}\alpha_1\alpha_2+\ldots)(p_3-p_1)+(\mathfrak{A}\alpha_1\alpha_3+\ldots)(p_1-p_2)].\end{aligned}$$

Donc, en réunissant ces expressions des trois parties de l'équation dont il s'agit, cette équation se réduit à

$$\begin{aligned}&V_1[(K+\mathfrak{A}\alpha_1^2+\ldots)(p_2-p_3)-(\mathfrak{A}\alpha_1\alpha_2+\ldots)p_1+(\mathfrak{A}\alpha_1\alpha_3+\ldots)p_1]\\ &+V_2[(K+\mathfrak{A}\alpha_1^2+\ldots)\,p_3-(K+\mathfrak{A}\alpha_1^2+\ldots)p_1]\\ &+V_3[(K+\mathfrak{A}\alpha_1\alpha_2+\ldots)\,p_1-(K+\mathfrak{A}\alpha_1^2+\ldots)p_2]=0\ \ldots\ldots\ldots\ldots\ldots\ldots(24).\end{aligned}$$

En mettant p_1^2 au lieu de $K + \mathfrak{A}\alpha_1^2 + \ldots$ et en supprimant le facteur commun p_1, on aura

$$V_1[p_1p_2 - (\mathfrak{A}\alpha_1\alpha_2 + \ldots) - p_1p_3 + (\mathfrak{A}\alpha_1\alpha_3 + \ldots)] + V_2[p_1p_3 - (\mathfrak{A}\alpha_1\alpha_3 + \ldots) - K]$$
$$+ V_3[K - p_1p_2 + (\mathfrak{A}\alpha_1\alpha_2 + \ldots)] = 0,$$

et en remettant $p_1^2 - (\mathfrak{A}\alpha_1^2 + \ldots) = K$, on obtient pour l'équation dont il s'agit:

$$\begin{vmatrix} V_1, & V_2, & V_3 \\ 1\ , & 1\ , & 1 \\ p_1^2 - (\mathfrak{A}\alpha_1^2 + \ldots), & p_1p_2 - (\mathfrak{A}\alpha_1\alpha_2 + \ldots), & p_1p_3 - (\mathfrak{A}\alpha_1\alpha_3 + \ldots) \end{vmatrix} = 0 \ldots (25).$$

Cette équation est celle de la droite menée par les points de contact de deux coniques de la même paire avec la conique $U + V_1^2 = 0$. Elle est la même qui a été déjà obtenue pour la droite résultante d'une certaine construction géométrique; on est donc arrivé au théorème connu suivant:

"La droite menée par le pôle d'un axe d'homologie par rapport à une des trois coniques inscrites, et par un centre de symptose, rencontre cette même conique en deux points qui sont les points de contact de cette conique avec deux coniques de la même paire;"

ou, ce qui revient au même:

"Le point de rencontre de la polaire d'un centre de symptose, par rapport à une des trois coniques inscrites, et d'un axe d'homologie, est le point de rencontre des tangentes communes de cette même conique et de deux coniques de la même paire."

92.

NOTE SUR UN SYSTÈME DE CERTAINES FORMULES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXIX. (1850), pp. 14, 15.]

LES formules dont il s'agit se rapportent à la théorie de la composition des formes quadratiques. Je les présente ici pour faire voir la relation qui existe entre elles et quelques formules de mon mémoire sur les hyperdéterminants (t. XXX. p. 1), [16]. En adoptant la notation de ce mémoire, et en mettant

$$\left.\begin{array}{lll} 2A = \overset{\dagger}{\begin{Bmatrix}111\\122\end{Bmatrix}}, & 2B = \overset{\dagger}{\begin{Bmatrix}111\\222\end{Bmatrix}}, & 2C = \overset{\dagger}{\begin{Bmatrix}211\\222\end{Bmatrix}}, \\ 2A' = \overset{\dagger}{\begin{Bmatrix}111\\212\end{Bmatrix}}, & 2B' = \overset{\dagger}{\begin{Bmatrix}111\\222\end{Bmatrix}}, & 2C' = \overset{\dagger}{\begin{Bmatrix}121\\222\end{Bmatrix}}, \\ 2A'' = \overset{\dagger}{\begin{Bmatrix}111\\221\end{Bmatrix}}, & 2B'' = \overset{\dagger}{\begin{Bmatrix}111\\222\end{Bmatrix}}, & 2C'' = \overset{\dagger}{\begin{Bmatrix}112\\222\end{Bmatrix}}, \end{array}\right\} \quad \text{..... (1),}$$

savoir, en mettant pour abréger,

$$\left.\begin{array}{ll} a = 111, & e = 112, \\ b = 211, & f = 212, \\ c = 121, & g = 122, \\ d = 221, & h = 222; \end{array}\right\} \quad \text{..... (2);}$$

$$\left.\begin{array}{lll} A = ag - ce, & 2B = ah + bg - de - cf, & C = bh - df, \\ A' = af - be, & 2B' = ah + cf - bg - de, & C' = ch - dg, \\ A'' = fg - eh, & 2B'' = ah + de - bg - cf, & C'' = bc - ad, \end{array}\right\} \quad \text{.....(3),}$$

on aura identiquement:

$$\left.\begin{array}{l} AA' = A''a^2 + \quad 2B''ae \quad + C''e^2\,, \\ AB' = A''ac + B''(ag + ce) + C''eg\,, \\ AC' = A''c^2 + \quad 2B''cg \quad + C''g^2\,; \end{array}\right\} \quad \text{..... (4);}$$

$$\left.\begin{aligned} BA' &= A''ab + B''(af + be) + C''ef\,, \\ BB' + \Theta &= A''ad + B''(ah + de) + C''eh\,, \\ BB' - \Theta &= A''be + B''(bg + cf) + C''fg\,, \\ BC' &= A''cd + B''(ch + dg) + C''gh\,; \end{aligned}\right\} \quad \text{.....(5)}\,;$$

$$\left.\begin{aligned} CA' &= A''b^2 + \quad 2B''bf \quad + C''f^2\,, \\ CB' &= A''bd + B''(bh + df) + C''fh\,, \\ CC' &= A''d^2 + \quad 2B''dh \quad + C''h^2\,; \end{aligned}\right\} \quad \text{.....(6)}\,;$$

$$\Theta = B^2 - AC = B'^2 - A'C' = B''^2 - A''C'' \quad \text{.....(7)}$$

$$= \tfrac{1}{4}\{a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2 - 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde + 4adfg + 4bech\}.$$

En regardant la seconde et la troisième des équations (5) comme équivalentes avec la seule équation $2BB' = A''(ad + be) + B''(ah + de + bg + cf) + C''(eh + fg)$, on trouvera que les systèmes (4, 5, 6) répondent à la transformation

$$A''z_1^2 + 2B''z_1z_2 + C''z_2^2 = (Ax_1^2 + 2Bx_1x_2 + Cx_2^2)(A'y_1^2 + 2B'y_1y_2 + C'y_2^2) \text{.....(8)},$$

$$z_1 = ax_1x_2 + by_1x_2 + cx_1y_2 + dy_1y_2,$$

$$z_2 = ex_1x_2 + fy_1x_2 + gx_1y_2 + hy_1y_2,$$

qui appartient à une théorie dont celle des transformations linéaires n'est qu'un cas particulier.

Je profite de cette occasion pour donner une addition à la "Note sur les hyperdéterminants" (t. XXXIV.), [54]. J'y ai dit (§ III.) que je ne pouvais pas expliquer la raison de ce que la courbe du sixième ordre donnée par les équations $ae - 4bd + 3c^2 = 0$ et $ace + 2bcd - ad^2 - eb^2 - c^3 = 0$, ait une osculatrice développable qui n'est que du sixième ordre, mais que cette réduction s'opérait en partie au moyen des quatre points de rebroussement de la courbe. En effet, cette courbe, considérée comme l'intersection de deux surfaces, l'une du second et l'autre du troisième ordre, a six droites que dans le mémoire [30] cité dans cette note j'ai nommé *droites par deux points:* cela suffit pour compléter la réduction dont il s'agit. M. Salmon, à qui je dois cette remarque, m'a fait voir aussi que l'expression que j'ai donnée pour le nombre des points de rebroussement de la courbe, dans le cas d'une équation du $m^{\text{ième}}$ ordre, combinée avec les formules du mémoire mentionné, suffit pour former le tableau complèt des singularités de la famille de surfaces développables dont il s'agissait, savoir:

m,	n,	r,	α,	β,	g,	h,
$3(m-2)$,	m,	$2(m-1)$,	0,	$4(m-3)$,	$\frac{1}{2}(m-1)(m-2)$,	$\frac{1}{2}(9m^2 - 53n + 80)$,

x,	y,
$2(n-2)(n-3)$,	$2(-1)(n-3)$.

Ici, dans la ligne supérieure, m, n, r, α, β, g, h, x, y ont les mêmes significations que dans le mémoire dont je viens de parler; et dans la ligne inférieure, m est le degré de l'équation primitive.

93.

NOTE SUR QUELQUES FORMULES QUI SE RAPPORTENT À LA MULTIPLICATION DES FONCTIONS ELLIPTIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XXXIX. (1850), pp. 16—22.]

LES fonctions

$$\{\lambda,\ \mu,\ x,\ y\} = P_{0,0} - \left(\frac{1}{1} P_{1,0}\, x + \frac{1}{1} P_{0,1}\, y\right) + \left(\frac{1}{1.2} P_{2,0}\, x^2 + \frac{1}{1.1} P_{1,1}\, xy + \frac{1}{1.2} P_{0,2}\, y^2\right) - \&c.,$$

où $P_{0,0} = 1$ et les autres coefficients sont donnés par l'équation à différences

$$\begin{aligned} &[-l\lambda - m\mu + (l-m)^2]\, P_{l,m} \\ &+ l\,(\lambda - 2l + 2m + 2)\,(\lambda - 2l + 2m + 1)\, P_{l-1,m} \\ &+ m\,(\mu + 2l - 2m + 2)\,(\mu + 2l - 2m + 1)\, P_{l,m-1} \\ &- 16lm\,[\lambda\mu - (2l + 2m - 4)\,(\lambda + \mu)]\, P_{l-1,m-1} = 0, \end{aligned}$$

jouent, comme je crois, un rôle important dans la théorie des fonctions elliptiques[1].

[1] La fonction $\{\lambda,\ \mu,\ x,\ y\}$ satisfait à l'équation

$$\begin{aligned} &-[\lambda\,(\lambda-1)\,x + \mu\,(\mu-1)\,y + 16\lambda\mu\, xy]\, u \\ &+[-(\lambda-1) + (4\lambda-6)\,x + (4\mu+2)\,y + 32\,(\lambda+\mu)\, xy]\, y\,\frac{du}{dx} \\ &+[-(\mu-1) + (4\lambda+2)\,x + (4\mu-6)\,y + 32\,(\lambda+\mu)\, xy]\, y\,\frac{du}{dy} \\ &\qquad + (1 - 4x - 4y)\left(x^2\frac{d^2u}{dx^2} - 2xy\,\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2}\right) = 0, \end{aligned}$$

qui peut être tirée de l'équation

$$n\,(n-1)\,x^2 u + (n-1)\,(ax - 2x^3)\,\frac{du}{dx} + (1 - ax^2 + x^4)\,\frac{d^2u}{dx^2} - 2n\,(a^2 - 4)\,\frac{du}{da} = 0$$

(voyez le mémoire cité plus bas), mais qu'on obtient plus facilement au moyen de l'équation à différences à laquelle satisfont les coefficients $P_{l,m}$.

En effet, en faisant $x = \sqrt{k} \sin \operatorname{am} u$, $\alpha = k + \frac{1}{k}$ et en représentant par z le dénominateur de la fonction $\sqrt{k} \sin \operatorname{am} nu$ (où n est un entier positif quelconque), on aura

$$z = z_1 + z_2 + \ldots + z_s + \ldots,$$

cette série étant continuée jusqu'au terme $z_{\frac{1}{2}n}$ ou $z_{\frac{1}{2}(n-1)}$, selon que n est pair ou impair, et la fonction z étant donnée par l'équation

$$z_s = (-1)^{(n+1)s} (4\alpha)^{s(n-2s)} x^{2ns} \left\{ n^2 - 2ns,\ 2ns,\ \frac{x^2}{4\alpha},\ \frac{1}{4\alpha x^2} \right\},$$

où cependant les termes qui contiennent des puissances négatives de α doivent être négligés. Ces formules reviennent à celles que j'ai présentées dans la "Note sur les fonctions elliptiques" (t. XXXVII.), [67].

En revenant aux fonctions $\{\lambda, \mu, x, y\}$, j'ai trouvé les deux formules

$$\begin{aligned} P_{l,0} &= \lambda [\lambda - l - 1]^{l-1}, \\ P_{l,1} &= \mu\lambda [\lambda - l - 1]^{l-1} \\ &\quad + l\lambda [\lambda - l - 1]^{l-3} \{(18l - 16)\lambda - (16\lambda^2 - 10l - 4)\} \\ &\quad - \frac{10l\lambda\mu}{\lambda + \mu} \lambda [\lambda - l]^{l-2} \end{aligned}$$

(où selon la notation de Vandermonde la factorielle $p(p-1)\ldots(p-q+1)$ est exprimée par $[p]^q$). De là, et en calculant la valeur de $P_{2,2}$ à l'aide de l'équation à différences, on obtient:

$$\begin{aligned} P_{0,0} &= 1, \\ P_{1,0} &= \lambda, \\ P_{0,1} &= \mu, \\ P_{2,0} &= \lambda(\lambda - 3), \\ P_{1,1} &= \left(\lambda\mu + 2 - \frac{10\lambda\mu}{\lambda + \mu}\right), \\ P_{0,2} &= \mu(\mu - 3), \\ P_{3,0} &= \lambda(\lambda - 4)(\lambda - 5), \\ P_{2,1} &= \lambda\left((\lambda - 3)\mu + 40 - \frac{20\lambda\mu}{\lambda + \mu}\right), \\ P_{1,2} &= \mu\left((\mu - 3)\lambda + 40 - \frac{20\lambda\mu}{\lambda + \mu}\right), \\ P_{0,3} &= \mu(\mu - 4)(\mu - 5), \end{aligned}$$

$$P_{4,0} = \lambda(\lambda-5)(\lambda-6)(\lambda-7),$$

$$P_{3,1} = \lambda\left(\mu(\lambda-4)(\lambda-5) + 114\lambda - 330 - \frac{30\lambda\mu}{\lambda+\mu}(\lambda-3)\right),$$

$$P_{2,2} = \left(\lambda(\lambda-3)\mu(\mu-3) + 152\lambda\mu + 336 - \frac{40\lambda^2\mu^2 + 1156\lambda\mu}{\lambda+\mu} + \frac{200\lambda^2\mu^2}{(\lambda+\mu)^2}\right),$$

$$P_{1,3} = \mu\left(\lambda(\mu-4)(\mu-5) + 114\mu - 330 - \frac{30\lambda\mu}{\lambda+\mu}(\mu-3)\right),$$

$$P_{0,4} = \mu(\mu-5)(\mu-6)(\mu-7);$$

la première partie de cette table se trouve dans la note citée.

Nous remarquerons en passant que pour $y=0$, on a $\{\lambda, \mu, x, 0\} = (\frac{1}{2} + \sqrt{(\frac{1}{4} - x)})^\lambda$. On sait que la théorie de la multiplication des fonctions circulaires depend de la fonction $(x + \sqrt{(x^2-1)})^\lambda$ ou, en faisant $\frac{1}{4}x^{-2} = \mathrm{x}$, de la fonction $(\frac{1}{2} + \sqrt{(\frac{1}{4} - \mathrm{x})})^\lambda$. Cela fait espérer que l'on parviendra par les fonctions $\{\lambda, \mu, x, y\}$ à la théorie complète de la multiplication des fonctions elliptiques.

J'ai calculé les valeurs qui servent à trouver les dénominateurs z de sin am nu, où n est un quelconque des entiers 1, 2, 3, 4, 5, 6, 7. Les voici:

$$n=1, \quad z_0 = 1, \qquad z = z_0,$$

$$n=2, \quad z_0 = 1, \quad -z_1 = -x^4, \qquad z = z_0 - z_1,$$

$$n=3, \quad z_0 = 1, \quad +z_1 = 4\alpha x^6 - (6x^4 + 3x^8), \qquad z = z_0 + z_1,$$

$$n=4, \quad z_0 = 1, \quad -z_1 = -16\alpha^2 x^8 + 4\alpha(8x^{10} + 8x^6) - (20x^{12} + 26x^8 + 20x^4),$$

$$z_2 = x^{16}, \qquad z = z_0 - z_1 + z_2,$$

$$n=5, \quad z_0 = 1, \qquad z = z_0 + z_1 + z_2,$$

$$\begin{aligned} z_1 = {} & 64\alpha^3 \quad x^{10} \\ & - 16\alpha^2(15x^{12} + 10x^8) \\ & + 4\alpha(90x^{14} + 92x^{10} + 35x^6) \\ & - (275x^{16} + 300x^{12} + 125x^6 + 50x^4), \end{aligned}$$

$$\begin{aligned} z_2 = {} & 16\alpha^2 \quad x^{20} \\ & - 4\alpha(5x^{22} + 20x^{18}) \\ & + (5x^{24} + 62x^{20} + 170x^{16}), \end{aligned}$$

$$n=6, \quad +z_0 = 1, \qquad z = z_0 - z_1 + z_2 - z_3,$$

$$\begin{aligned} -z_1 = {} & -256\alpha^4 \quad x^{12} \\ & + 64\alpha^3(24x^{14} + 12x^{10}) \\ & - 16\alpha^2(252x^{16} + 210x^{12} + 54x^8) \\ & + 4\alpha(1520x^{18} + 1584x^{14} + 576x^{10} + 112x^6) \\ & - (5814x^{20} + 7704x^{16} + 2400x^{12} + 444x^8 + 105x^4), \end{aligned}$$

$$
\begin{aligned}
+z_2 = \;& 256\alpha^4 \quad x^{24} \\
& - 64\alpha^3 (12x^{26} + 24x^{22}) \\
& + 16\alpha^2 (54x^{28} + 210x^{24} + 252x^{20}) \\
& - 4\alpha (112x^{30} + 576x^{26} + 1584x^{22} + 1520x^{18}) \\
& + (105x^{32} + 444x^{28} + 2400x^{24} + 7704x^{20} + 5814x^{16}), \\
-z_3 = \;& -x^{36},
\end{aligned}
$$

$$
\begin{aligned}
& n = 7, \qquad\qquad z = z_0 + z_1 + z_2 + z_3. \\
& z_0 = 1, \\
& z_1 = 1024\alpha^5 \quad x^{14} \\
& \quad - 256\alpha^4 (35x^{16} + 14x^{12}) \\
& \quad + 64\alpha^3 (1120x^{18} + 196x^{14} + 77x^{10}) \\
& \quad - 16\alpha^2 (5425x^{20} + 5040x^{16} + 1575x^{12} + 210x^8) \\
& \quad + 4\alpha (35525x^{22} + 41300x^{18} + 14934x^{14} + 2604x^{10} + 294x^6) \\
& \quad - (166257x^{24} + 260750x^{20} + 220395x^{16} + 14756x^{12} + 1304x^8 + 196x^4), \\
& z_2 = 4096\alpha^6 \quad x^{28} \\
& \quad - 1024\alpha^5 (21x^{30} + 28x^{26}) \\
& \quad + 256\alpha^4 (189x^{32} + 470x^{28} + 350x^{24}) \\
& \quad - 64\alpha^3 (952x^{34} + 3192x^{30} + 4550x^{26} + 2576x^{22}) \\
& \quad + 16\alpha^2 (2940x^{36} + 11200x^{32} + 21750x^{28} + 25452x^{24} + 12397x^{20}) \\
& \quad - 4\alpha (5733x^{38} + 22064x^{34} + 44324x^{30} + 82488x^{26} + 96761x^{22} + 40964x^{18}) \\
& \quad + (7007x^{40} + 59388x^{36} + 35231x^{32} + 41132x^{28} + 278173x^{24} + 302918x^{20} + 94962x^{16}), \\
& z_3 = 64\alpha^3 \quad x^{42} \\
& \quad - 16\alpha^2 (7x^{44} + 42x^{40}) \\
& \quad + 4\alpha (14x^{46} + 236x^{42} + 819x^{38}) \\
& \quad - (7x^{48} + 308x^{44} + 4053x^{40} + 9842x^{36}).
\end{aligned}
$$

Pour rassembler tous mes résultats, je veux citer ceux que j'ai donné dans le *Cambridge and Dublin Mathematical Journal* [vol. II. (1847), 45, and vol. III. (1848), 57]. En écrivant la lettre p au lieu de n (symbole qui représente le carré du nombre n de ce mémoire), et en changeant les signes des termes alternatifs, on aura pour solution particulière de l'équation

$$p(p-1)x^2 z + (p-1)(\alpha x - 2x^2)\frac{dz}{dx} + (1 - \alpha x^2 + x^4)\frac{d^2 z}{dx^2} - 2p(\alpha^2 - 4)\frac{dz}{d\alpha} = 0$$

(savoir pour la solution qui pour $p = n^2$ se réduit au dénominateur de $\sin \operatorname{am} u$) la valeur

$$z = 1 - C_2 \frac{x^4}{1.2.3.4} + C_6 \frac{x^6}{1.2.3.4.5.6} - \&c.,$$

les coefficients étant déterminés au moyen de l'expression

$$C_{r+2} = -(2r+1)(2r+2)(p-2r)(p-2r-1)C_r + (2r+2)(p-2r-2)\alpha C_{r+1} - 2p(\alpha^2-4)\frac{dC_{r+1}}{d\alpha}.$$

Cela donne les valeurs particulières suivantes:

$$C_2 = 2p(p-1),$$
$$C_3 = 8p(p-1)(p-4)\alpha,$$
$$\&c.$$

[viz. with the change referred to, these are the values of C_2, C_3, ... C_8 given *ante* p. 299].

On remarquera que dans ces formules le premier terme de C_8 ne contient pas, comme on pourrait l'attendre, le facteur $(p-25)$. Cela vient de ce que le coefficient C_8 est composé des coefficients des termes correspondants de z_0 et z_1, tandis que les coefficients C_7, &c., sont tout simplement des coefficients de z_0. La suite des coefficients C offre plusieurs discontinuités de cette sorte. Par exemple on obtient généralement

$$C_r = (-)^{r+1} \begin{cases} 2^{2r-8}p(p-1^2)\dots(p-(r-1)^2)C_r^1\alpha^{r-2} \\ +2^{2r-6}p(p-1^2)\dots(p-(r-2)^2)C_r^2\alpha^{r-4} \\ +2^{2r-2}p(p-1^2)\dots(p-(r-3)^2)C_r^3\alpha^{r-6} \\ +\&c.; \end{cases}$$

mais le terme suivant ne contient pas le facteur $p(p-1^2)\dots(p-(r-4)^2)$. Quant à la loi des coefficients C_r^1, C_r^2, C_r^3, on a

$$C_r^1 = 1,$$
$$C_r^2 = (r-3)\{n(2r-7)+(r-1)(8r-7)\},$$
$$C_r^3 = (r-4)(r-5)\{n^2(4r^2-24r+51)+n(32r^3-220r^2+412r-255) + 2(r-1)(r-2)(32r^2-88r+51)\}.$$

Également, en ordonnant la série suivant les puissances descendantes de x, la quantité z étant la solution particulière qui pour $p = n^2$ (n impair) se réduit au dénominateur de $\sin \operatorname{am} nu$, on aura

$$z = (-1)^{\frac{1}{2}(\sqrt{p}-1)} . \sqrt{p}\left(x^{p-1} - D_1\frac{x^{p-3}}{1.2.3} + D_2\frac{x^{p-5}}{1.2.3.4.5} - \&c.\right),$$

où les coefficients D sont donnés par l'expression

$$D_{r+2} = -(2r+3)(2r+2)(p-2r-2)(p-2r-1)D_r$$
$$+(2r+3)(p-2r-3)\alpha D_{r+1} - 2p(\alpha^2-4)\frac{dD_{r+1}}{d\alpha},$$

ce qui donne les valeurs particulières suivantes:

$$D_1 = \quad (p-1)\alpha,$$
$$D_2 = \quad 2(p-1)(p+6)$$
$$+ \quad (p-1)(p-9)\alpha^2,$$
&c.

[viz. with the change referred to, these are the values of D_1, $D_2, \ldots D_6$ given *ante* pp. 364, 365].

Les mêmes remarques sont applicables aux coefficients D; seulement la discontinuité a lieu ici dès le coefficient D_4. Il paraît que c'est cause de cette discontinuité que le signe négatif se présente aux premiers termes des coefficients D_4, &c. En effet, on a généralement:

$$D_r = \quad (p-1)(p-9)\ldots(p-(2r-1)^2)\alpha^r$$
$$+(p-1)(p-9)\ldots(p-(2r-3)^2)r(r-1)(p+4r-2)\alpha^{r-2}$$
$$+ \text{ \&c.};$$

ici la discontinuité se présente déjà dans le terme suivant, qui ne contient pas le facteur $(p-1)(p-9)\ldots(p-(2r-5)^2)$. Et c'est précisément le terme suivant qui devient négatif dans les expressions de D_4, D_5 et D_6. Mais tout cela est moins important que la théorie des séries partielles z_s, sur lesquelles les recherches ultérieures seront à fonder.

PROBLÈME.

Donner la solution de l'équation à différences

$$(-l\lambda - m\mu + (l-m)^2)P_{l,m}$$
$$+\ l\ (\lambda - 2l + 2m + 2)(\lambda - 2l + 2m + 1)P_{l-1,m}$$
$$+ m\ (\mu + 2l - 2m + 2)(\mu + 2l - 2m + 1)P_{l,m-1}$$
$$-16lm\ (\lambda\mu - (2l + 2m - 4)(\lambda + \mu))P_{l-1,m-1} = 0,$$

dans laquelle $P_{0,0} = 1$. (Voyez la "Note sur quelques formules &c.")

94.

NOTE SUR L'ADDITION DES FONCTIONS ELLIPTIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), pp. 57—65.]

SOIT, pour observer autant que possible la symétrie:

$$Su = \sqrt{k}\sin\operatorname{am}\frac{u}{\sqrt{k}},$$

$$Cu = \cos\operatorname{am}\frac{u}{\sqrt{k}},$$

$$Gu = \Delta\operatorname{am}\frac{u}{\sqrt{k}},$$

$$\alpha = k + \frac{1}{k}, \quad \tfrac{1}{2}v = \frac{K}{\sqrt{k}}, \quad \tfrac{1}{2}S = \frac{K + iK'}{\sqrt{k}};$$

et soit pour abréger:

$$Su = x, \quad Sv = y, \text{ \&c.}$$

$$Cu = \sqrt{\left(1 - \frac{x^2}{k}\right)} = X_{,},$$

$$Gu = \sqrt{(1 - kx^2)} = X_{,,},$$

$$Cu\,Gu = \sqrt{(1 - \alpha x^2 + x^4)} = X_{,}X_{,,} = X.$$

Cela posé, les méthodes d'Abel donnent les expressions suivantes de sin am d'une somme quelconque d'arcs: savoir

$$S(u + v + \ldots) = (-1)^{n-1}\frac{[\theta,\ \theta^3, \ldots \theta^{2n-1},\ \Theta\ ,\ \theta^2\Theta, \ldots \theta^{2n-4}\Theta]}{[1,\ \theta^2, \ldots \theta^{2n-2},\ \theta\Theta,\ \theta^3\Theta, \ldots \theta^{2n-3}\Theta]},$$

pour un nombre *impair* $2n-1$ d'arcs, et

$$S(u+v+\ldots)=-\frac{[1,\ \theta^2,\ldots\theta^{2n},\ \theta\Theta,\ \theta^3\Theta,\ldots\theta^{2n-3}\Theta]}{[\theta,\ \theta^3,\ldots\theta^{2n-1},\ \Theta,\ \theta^2\Theta,\ldots\theta^{2n-2}\Theta]}$$

pour un nombre pair $2n$ d'arcs. Dans ces expressions les symboles dans lesquelles entrent les lettres θ, Θ, sont censés représenter les déterminants, dont on obtient les termes en changeant successivement ces lettres en x, X; y, Y; &c.

J'ai trouvé qu'on a aussi

$$C(u+v+\ldots)=\frac{[\Theta_{,},\ \theta^2\Theta_{,},\ldots\theta^{2n-2}\Theta_{,},\ \theta\Theta_{,,},\ \theta^3\Theta_{,,},\ldots\theta^{2n-3}\Theta_{,,}]}{[1,\ \theta^2,\ \ldots\theta^{2n-2},\ \theta\Theta\ ,\ \theta^3\Theta\ ,\ldots\theta^{2n-3}\Theta\ ]}$$

pour un nombre impair $2n-1$ d'arcs, et

$$C(u+v+\ldots)=\frac{[\theta\Theta_{,,},\ \theta^3\Theta_{,},\ldots\theta^{2n-1}\Theta_{,},\ \Theta_{,,},\ \theta^2\Theta_{,,},\ldots\theta^{2n-2}\Theta_{,,}]}{[\ \theta,\ \theta^3,\ \ldots\ \theta^{2n-1},\ \Theta,\ \theta^2\Theta\ ,\ldots\theta^{2n-2}\Theta\ ]}$$

pour un nombre pair $2n$ d'arcs. Les valeurs correspondantes de $G(u+v+\ldots)$ se trouvent en échangeant les symboles $\Theta_{,}$ et $\Theta_{,,}$.

Particulièrement pour la somme de trois arcs on a :

$$S(u+v+w)=\frac{-[\theta,\ \theta^3,\ \Theta]}{[1,\ \theta^2,\ \theta\Theta]},$$

$$C(u+v+w)=\frac{[\Theta_{,},\ \theta^2\Theta_{,},\ \theta\Theta_{,,}]}{[1,\ \theta^2,\ \theta\Theta]},$$

$$G(u+v+w)=\frac{[\Theta_{,,},\ \theta^2\Theta_{,,},\ \theta\Theta_{,}]}{[1,\ \theta^2,\ \theta\Theta]}.$$

Pour réduire ces expressions à une forme qui soit encore applicable au cas où deux quelconques des quantités u, v, w sont égales, il n'y a qu'à multiplier les termes des fractions à droite par

$$\Omega=\frac{-(xY+yX)(yZ+zY)(zX+xZ)}{(x^2-y^2)(y^2-z^2)(z^2-x^2)};$$

cela donne, après une réduction un peu difficile :

$$-\Omega\,[\,\theta,\quad\theta^3,\quad\Theta\,]=(xYZ+yZX+zXY)-xyz\,(\alpha-x^2-y^2-z^2+x^2y^2z^2),$$

$$\Omega\,[\Theta_{,},\ \theta^2\Theta_{,},\ \theta\Theta_{,,}]=(1-kx^2y^2z^2)\,X_{,}Y_{,}Z_{,}-\frac{1}{k}(yzX+zxY+xyZ)\,X_{,,}Y_{,,}Z_{,,},$$

$$\Omega\,[\Theta_{,,},\ \theta^2\Theta_{,,},\ \theta\Theta_{,}]=\left(1-\frac{1}{k}x^2y^2z^2\right)X_{,,}Y_{,,}Z_{,,}-k\,(yzX+zxY+xyZ)\,X_{,}Y_{,}Z_{,},$$

$$\Omega\,[\ 1,\quad\theta^2,\quad\theta\Theta]=\ 1-y^2z^2-z^2x^2-x^2y^2+\alpha x^2y^2z^2-xyz\,(xYZ+yZX+zXY),$$

de manière qu'en écrivant

$$M = 1 - x^2y^2 - y^2z^2 - z^2x^2 + \alpha x^2y^2z^2 - xyz\,(xYZ + yZX + zXY),$$

on a

$$S(u+v+w) = \frac{xYZ + yZX + zXY - xyz\,(\alpha - x^2 - y^2 - z^2 + x^2y^2z^2)}{M},$$

$$C(u+v+w) = \frac{(1 - kx^2y^2z^2)\,X_{,}Y_{,}Z_{,} - \frac{1}{k}(yzX + zxY + xyZ)\,X_{,,}Y_{,,}Z_{,,}}{M},$$

$$G(u+v+w) = \frac{\left(1 - \frac{1}{k}x^2y^2z^2\right)X_{,,}Y_{,,}Z_{,,} - k\,(yzX + zxY + xyZ)\,X_{,}Y_{,}Z_{,}}{M}.$$

Les mêmes formules peuvent être trouvées plus simplement en écrivant $u + \frac{1}{2}v + \frac{1}{2}\delta$, $v + \frac{1}{2}v + \frac{1}{2}\delta$ au lieu de u, v. La somme $u+v+w$ se change par là en $u+v+w+(v+\delta)$, et les fonctions $S(u+v+w)$, $C(u+v+w)$, $G(u+v+w)$ deviennent $S(u+v+w)$, $-C(u+v+w)$, $-G(u+v+w)$. De plus x, $X_{,}$, $X_{,,}$, X et y, $Y_{,}$, $Y_{,,}$, Y se changent en $-\frac{1}{x}$, $\frac{-i}{\sqrt{k}}\frac{X_{,,}}{x}$, $i\sqrt{k}\frac{X_{,}}{x}$, $\frac{X}{x^2}$ et $-\frac{1}{y}$, $\frac{-i}{\sqrt{k}}\frac{Y_{,,}}{y}$, $i\sqrt{k}\frac{Y_{,}}{y}$, $\frac{Y}{y^2}$, et l'on a

$$S(u+v+w) = -\begin{vmatrix} x^2, & 1, & -xX \\ y^2, & 1, & -yY \\ z, & z^2, & Z \end{vmatrix} \div \begin{vmatrix} x^3, & x, & -X \\ y^3, & y, & -Y \\ 1, & z^2, & zZ \end{vmatrix}$$

$$C(u+v+w) = \frac{1}{k}\begin{vmatrix} x^2X_{,,}, & X_{,,}, & kxX_{,} \\ y^2Y_{,,}, & Y_{,,}, & kyY_{,} \\ Z_{,}, & z^2Z_{,}, & zZ_{,,} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & -X \\ y^3, & y, & -Y \\ 1, & z^2, & zZ \end{vmatrix}$$

$$G(u+v+w) = k\begin{vmatrix} x^2X_{,}, & X_{,}, & \frac{1}{k}xX_{,,} \\ y^2Y_{,}, & Y_{,}, & \frac{1}{k}yY_{,,} \\ Z_{,,}, & z^2Z_{,,}, & zZ_{,} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & -X \\ y^3, & y, & -Y \\ 1, & z^2, & zZ \end{vmatrix}$$

Ces formules conduisent aux formes réduites que l'on obtient en multipliant par le facteur beaucoup plus simple

$$\Omega_{,} = -\frac{xY + yX}{x^2 - y^2}.$$

En passant, il y a à noter les équations identiques

$$\frac{(yZ + zY)(zX + xZ)}{(y^2 - z^2)(z^2 - x^2)}\begin{vmatrix} x, & x^3, & X \\ y, & y^3, & Y \\ z, & z^3, & Z \end{vmatrix} = \begin{vmatrix} x^3, & x, & -X \\ y^3, & y, & -Y \\ 1, & z^2, & zZ \end{vmatrix}, \text{ \&c.}$$

auxquelles conduit la méthode qui vient d'être expliquée. Aussi en multipliant les valeurs de $C(u+v+w)$, $G(u+v+w)$ on obtient l'équation

$$C(u+v+w)\,G(u+v+w)=\frac{\Psi XYZ+\Lambda yzX+\mathrm{M}zxY+\mathrm{N}xyZ}{\{1-y^2z^2-z^2x^2-x^2y^2+\alpha x^2y^2z^2-xyz\,(xYZ+yZX+zXY)\}^2},$$

dans laquelle

$$\begin{aligned}
\Psi &= 1 \;+y^2z^2+z^2x^2+x^2y^2-4\alpha x^2y^2z^2+x^2y^2z^2\,(x^2+y^2+z^2)+x^4y^4z^4,\\
\Lambda &= \Pi+2x^2Y^2Z^2,\\
\mathrm{M} &= \Pi+2y^2Z^2X^2,\\
\mathrm{N} &= \Pi+2z^2X^2Y^2,\\
\Pi &= -\alpha+2\,(x^2+y^2+z^2)-\alpha\,(y^2z^2+z^2x^2+x^2y^2)+(2\alpha^2-4)\,x^2y^2z^2\\
&\qquad +2x^2y^2z^2\,(y^2z^2+z^2x^2+x^2y^2)-\alpha x^2y^2z^2\,(x^2+y^2+z^2)-\alpha x^4y^4z^4.
\end{aligned}$$

Pour le cas de quatre arcs, je n'ai trouvé que le sin am de la *somme*. En effet on a

$$S(u+v+w+p)=-\begin{vmatrix} 1, & x^2, & x^4, & xX\\ 1, & y^2, & y^4, & yY\\ 1, & z^2, & z^4, & zZ\\ 1, & t^2, & t^4, & tT \end{vmatrix} \div \begin{vmatrix} x, & x^3, & X, & x^2X\\ y, & y^3, & Y, & y^2Y\\ z, & z^3, & Z, & z^2Z\\ t, & t^3, & T, & t^2T \end{vmatrix}$$

où les termes de la fraction sont à multiplier par

$$\Omega=-\frac{(xY+yX)\,(xZ+zX)\,(xT+tX)\,(yZ+zY)\,(zT+tZ)\,(tY+yT)}{(x^2-y^2)\,(x^2-z^2)\,(x^2-t^2)\,(y^2-z^2)\,(z^2-t^2)\,(t^2-y^2)}.$$

Mais il est plus simple de se servir de la forme

$$S(u+v+w+p)=-\begin{vmatrix} x^4, & x^2, & 1, & -xX\\ y^4, & y^2, & 1, & -yY\\ 1, & z^2, & z^4, & zZ\\ 1, & t^2, & t^4, & tT \end{vmatrix} \div \begin{vmatrix} x^3, & x, & -x^2X, & -X\\ y^3, & y, & -y^2Y, & -Y\\ z, & z^3, & Z, & z^2Z\\ t, & t^3, & T, & t^2T \end{vmatrix}$$

que l'on obtient de la même manière que la forme analogue pour trois arcs. Ici le facteur est

$$\Omega_1=\frac{(yX+xY)\,(zT+tZ)}{(x^2-y^2)\,(z^2-t^2)},$$

et l'on obtient, toute réduction faite,

$$S(u+v+w+p)=\frac{\mathfrak{N}}{\mathfrak{D}},$$

$$\begin{aligned}\mathfrak{N} = (1 - x^2y^2z^2t^2)(xYZT + yZTX + zTXY + tXYZ)\\ - \{(\alpha - x^2 - y^2 - z^2 - t^2 + y^2z^2t^2 + z^2t^2x^2 + t^2x^2y^2 + x^2y^2z^2 - \alpha x^2y^2z^2t^2)\\ \times (Xyzt + Yztx + Ztxy + Txyz)\},\end{aligned}$$

$$\begin{aligned}\mathfrak{D} = 1 - x^2y^2 - x^2z^2 - x^2t^2 - y^2z^2 - z^2t^2 - t^2y^2 - (x^2y^2 + x^2z^2 + x^2t^2 + y^2z^2 + z^2t^2 + t^2y^2)\, x^2y^2z^2t^2\\ + x^4y^4z^4t^4 + \alpha\,(x^2y^2z^2 + y^2z^2t^2 + z^2t^2x^2 + t^2x^2y^2) + \alpha\,(x^2 + y^2 + z^2 + t^2)\, x^2y^2z^2t^2\\ + (2 - 2\alpha^2)\, x^2y^2z^2t^2\\ - (x^2Y^2 + y^2X^2)\, ztZT - (x^2Z^2 + z^2X^2)\, ytYT - (x^2T^2 + t^2X^2)\, yzYZ\\ - (y^2Z^2 + z^2Y^2)\, xtXT - (z^2T^2 + t^2Z^2)\, xyXY - (t^2Y^2 + y^2T^2)\, xzXZ.\end{aligned}$$

Il y a à remarquer qu'en employant la première valeur de $S(u+v+w+p)$ et le facteur correspondant, on aurait trouvé le même numérateur, et aussi le même dénominateur, ce qui donne lieu à des équations identiques, semblables à celles qui ont lieu pour le cas de trois arcs.

Revenons à l'expression

$$\begin{vmatrix} x, & x^3, & X \\ y, & y^3, & Y \\ z, & z^3, & Z \end{vmatrix} \frac{(yZ + zY)(zX + xZ)(xY + yX)}{(y^2 - z^2)(z^2 - x^2)(x^2 - y^2)}$$

qui donne le numérateur de $S(u+v+w)$. En mettant $x^2 = a$, $\frac{1}{k}X = A$, &c. on voit qu'il s'agit d'effectuer la division de

$$\begin{vmatrix} 1, & a, & A \\ 1, & b, & B \\ 1, & c, & C \end{vmatrix} (B + C)(C + A)(A + B)$$

par le produit $(b-c)(c-a)(a-b)$, les fonctions A, B, C denotant des racines carrées de fonctions rationnelles d'une forme particulière. Or, en supposant toujours que les carrés de A, B, C soient des fonctions rationnelles, et d'ailleurs d'une forme quelconque, cela peut se faire dans tous les cas particuliers au moyen de l'équation identique

$$\begin{vmatrix} 1, & a, & A \\ 1, & b, & B \\ 1, & c, & C \end{vmatrix} (B + C)(C + A)(A + B)$$

$$= \begin{vmatrix} 1, & a, & A^2 \\ 1, & b, & B^2 \\ 1, & c, & C^2 \end{vmatrix} (A^2 + B^2 + C^2 + BC + CA + AB) - \begin{vmatrix} 1, & a, & A^4 \\ 1, & b, & B^4 \\ 1, & c, & C^4 \end{vmatrix}.$$

De même le dénominateur de $S(u+v+w)$ depend de l'équation analogue

$$\begin{vmatrix} 1, & a, & aA \\ 1, & b, & bB \\ 1, & c, & cC \end{vmatrix} (B+C)(C+A)(A+B)$$

$$= \begin{vmatrix} 1, & a, & aA^2 \\ 1, & b, & bB^2 \\ 1, & c, & cC^2 \end{vmatrix} (A^2+B^2+C^2+BC+CA+AB) - \begin{vmatrix} 1, & a, & aA^4 \\ 1, & b, & bB^4 \\ 1, & c, & cC^4 \end{vmatrix};$$

et le numérateur et le dénominateur de $S(u+v+w+p)$ dependent de l'équation

$$\begin{vmatrix} 1, & a, & a^2, & aA \\ 1, & b, & b^2, & bB \\ 1, & c, & c^2, & cC \\ 1, & d, & d^2, & dD \end{vmatrix} (A+B)(A+C)(A+D)(B+C)(B+D)(C+D)$$

$$= M \begin{vmatrix} 1, & a, & a^2, & aA^2 \\ 1, & b, & b^2, & bB^2 \\ 1, & c, & c^2, & cC^2 \\ 1, & d, & d^2, & dD^2 \end{vmatrix} - N \begin{vmatrix} 1, & a, & a^2, & aA^4 \\ 1, & b, & b^2, & bB^4 \\ 1, & c, & c^2, & cC^4 \\ 1, & d, & d^2, & dD^4 \end{vmatrix} + P \begin{vmatrix} 1, & a, & a^2, & aA^6 \\ 1, & b, & b^2, & bB^6 \\ 1, & c, & c^2, & cC^6 \\ 1, & d, & d^2, & dD^6 \end{vmatrix},$$

dans laquelle

$$M = 2ab^2c^2 + \ldots + a^3b^2 + \ldots + a^2bc + \ldots + 3a^2bcd + \ldots,$$

$$N = (a+b+c+d)(a^2+b^2+c^2+d^2) + (abc+bcd+cda+dab),$$

$$P = a+b+c+d,$$

et de l'équation

$$\begin{vmatrix} 1, & a, & A, & aA \\ 1, & b, & B, & bB \\ 1, & c, & C, & cC \\ 1, & d, & D, & dD \end{vmatrix} (A+B)(A+C)(A+D)(B+C)(B+D)(C+D)$$

$$= (a^2b^2 + \ldots + abc^2 + \ldots + 2abcd) \begin{vmatrix} 1, & a, & A^2, & aA^2 \\ 1, & b, & B^2, & bB^2 \\ 1, & c, & C^2, & cC^2 \\ 1, & d, & D^2, & dD^2 \end{vmatrix} - \begin{vmatrix} 1, & a, & A^4, & aA^4 \\ 1, & b, & B^4, & bB^4 \\ 1, & c, & C^4, & cC^4 \\ 1, & d, & D^4, & dD^4 \end{vmatrix};$$

mais je n'ai pas encore trouvé la loi générale de ces équations.

Pour faciliter l'usage des symboles S, C, G, je veux exprimer par cette notation les propriétés les plus simples des fonctions elliptiques. Cela me donnera aussi l'opportunité d'arranger d'une manière particulière les formules qui se rapportent à la somme ou à la différence de deux arcs. On a d'abord

$$C^2u = 1 - \frac{1}{k} \cdot S^2u,$$

$$G^2u = 1 - k \cdot S^2u,$$

$$S'u = Cu \cdot Gu,$$

$$C'u = -\frac{1}{k} \cdot Su \cdot Gu,$$

$$G'u = -k \cdot Su \cdot Cu,$$

$$S(0) = 0, \qquad C(0) = 1, \qquad G(0) = 1,$$

$$S'(0) = 1, \qquad C'(0) = 0, \qquad G'(0) = 0,$$

$$S(-u) = -S(u), \qquad C(-u) = C(u), \qquad G(-u) = G(u),$$

$$S(\tfrac{1}{2}\upsilon) = \sqrt{k}, \qquad S(\tfrac{1}{2}\delta) = \frac{1}{\sqrt{k}}, \qquad S(\tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = \infty,$$

$$C(\tfrac{1}{2}\upsilon) = 0, \qquad C(\tfrac{1}{2}\delta) = \frac{ik'}{k}, \qquad C(\tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = \infty,$$

$$G(\tfrac{1}{2}\upsilon) = k', \qquad G(\tfrac{1}{2}\delta) = 0, \qquad G(\tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = \infty,$$

$$S(u + \tfrac{1}{2}\upsilon) = \frac{\sqrt{k} \cdot Cu}{Gu}, \qquad S(u + \tfrac{1}{2}\delta) = \frac{1}{\sqrt{k}} \cdot \frac{Gu}{Cu},$$

$$C(u + \tfrac{1}{2}\upsilon) = \frac{-k'}{\sqrt{k}} \cdot \frac{Su}{Gu}, \qquad C(u + \tfrac{1}{2}\delta) = \frac{ik'}{k} \cdot \frac{1}{Cu},$$

$$G(u + \tfrac{1}{2}\upsilon) = \frac{k'}{Gu}, \qquad G(u + \tfrac{1}{2}\delta) = \frac{-ik'}{\sqrt{k}} \cdot \frac{Su}{Gu},$$

$$S(u + \tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = -\frac{1}{Su},$$

$$C(u + \tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = \frac{-i}{\sqrt{k}} \cdot \frac{Gu}{Su},$$

$$G(u + \tfrac{1}{2}\upsilon + \tfrac{1}{2}\delta) = i\sqrt{k} \cdot \frac{Cu}{Su},$$

$$S(u + m\upsilon + n\delta) = (-1)^{m+n} \cdot Su,$$

$$C(u + m\upsilon + n\delta) = (-1)^{m} \cdot Cu,$$

$$G(u + m\upsilon + n\delta) = (-1)^{n} \cdot Gu.$$

Dans ces équations les symboles S, C, G, v, δ, k, k', i et S, G, C, δ, v, $\frac{1}{k}$, $\frac{ik'}{k}$, $-i$ peuvent être échangés les uns d'avec les autres. Les formules fondamentales qui se rapportent à deux arcs sont

$$S(u+v)=\frac{Su\,.\,Cv\,.\,Gv+Sv\,.\,Cu\,.\,Gu}{1-S^2u\,.\,S^2v},$$

$$C(u+v)=\frac{Cu\,.\,Cv-\frac{1}{k}\,.\,Su\,.\,Gu\,.\,Sv\,.\,Gv}{1-S^2u\,.\,S^2v},$$

$$G(u+v)=\frac{Gu\,.\,Gv-k\,.\,Su\,.\,Cu\,.\,Sv\,.\,Cv}{1-S^2u\,.\,S^2v},$$

auxquelles on ajoutera:

$$C(u+v)\,G(u+v)=\frac{(1+S^2u\,.\,S^2v)\,(Cu\,.\,Gu\,.\,Cv\,.\,Gv)-Su\,.\,Sv\,(\alpha-2S^2u-2S^2v+\alpha\,S^2u\,.\,S^2v)}{(1-S^2u\,.\,S^2v)^2}.$$

Mais pour trouver toutes les formes différentes de ces équations, mettons pour abréger (en supposant comme auparavant $Su=x$, &c.):

$$A=xY\quad,\qquad A'=\quad yX,$$

$$B=X_{,}Y_{,},\qquad B'=-\frac{1}{k}xyX_{,,}Y_{,,},$$

$$C=X_{,,}Y_{,,},\qquad C'=-kxyX_{,}Y_{,}\,,$$

$$P=x^2-y^2,$$

$$Q=1-\frac{1}{k}x^2-\frac{1}{k}y^2+x^2y^2,$$

$$R=1-kx^2-ky^2+x^2y^2,$$

$$S=\;X\;Y\,,\qquad S'=-\frac{k'^2}{k}xy\,,$$

$$T=xX_{,,}Y_{,},\qquad T'=-yY_{,,}X_{,},$$

$$U=xX_{,}Y_{,,},\qquad U'=\;yY_{,}X_{,,},$$

$$K=1-x^2y^2.$$

Alors on aura

$$S(u+v)=\frac{A+A'}{K}=\frac{P}{A-A'}=\frac{U+U'}{B-B'}=\frac{T-T'}{C-C'},$$

$$C(u+v)=\frac{B+B'}{K}=\frac{U-U'}{A-A'}=\frac{Q}{B-B'}=\frac{S+S'}{C-C'},$$

$$G(u+v)=\frac{C+C'}{K}=\frac{T+T'}{A-A'}=\frac{S-S'}{B-B'}=\frac{R}{C-C'},$$

et les valeurs correspondantes de $S(u-v)$, $C(u-v)$, $G(u-v)$ se trouveront en échangeant les signes de A', B', C', S', T', U'.

[Reverting to the functions sin am, cos am, Δ am, or say sn, cn, dn, instead of S, C, D, and introducing Dr Glaisher's very convenient notation s_1, c_1, d_1 for the sn, cn, dn of u, and s_2, c_2, d_2 for those of v, the formulæ just obtained may be written

$$\operatorname{sn}(u+v) = \frac{s_1c_2d_2 + s_2c_1d_1}{1-k^2s_1^2s_2^2} = \frac{s_1^2 - s_2^2}{s_1c_2d_2 - s_2c_1d_1} = \frac{s_1c_1d_2 + s_2c_2d_1}{c_1c_2 + s_1d_1s_2d_2} = \frac{s_1d_1c_2 + s_2d_2c_1}{d_1d_2 + k^2s_1c_1s_2c_2},$$

$$\operatorname{cn}(u+v) = \frac{c_1c_2 - s_1d_1s_2d_2}{1-k^2s_1^2s_2^2} = \frac{s_1c_1d_2 - s_2c_2d_1}{s_1c_2d_2 - s_2c_1d_1} = \frac{1 - s_1^2 - s_2^2 + k^2s_1^2s_2^2}{c_1c_2 + s_1d_1s_2d_2} = \frac{c_1d_1c_2d_2 - k'^2s_1s_2}{d_1d_2 + k^2s_1c_1s_2c_2},$$

$$\operatorname{dn}(u+v) = \frac{d_1d_2 - k^2s_1c_1s_2c_2}{1-k^2s_1^2s_2^2} = \frac{s_1d_1c_2 - s_2d_2c_1}{s_1c_2d_2 - s_2c_1d_1} = \frac{c_1d_1c_2d_2 + k'^2s_1s_2}{c_1c_2 + s_1d_1s_2d_2} = \frac{1 - k^2s_1^2 - k^2s_2^2 + k^2s_1^2s_2^2}{d_1d_2 + k^2s_1c_1s_2c_2},$$

viz. we have thus a fourfold representation of the addition-equation for each of the three functions.]

De ces formules il peut être tiré un grand nombre d'équations identiques; par exemple celles-ci:

$$(A^2 - A'^2) = KP, \text{ \&c.,} \qquad S^2 - S'^2 = QR, \text{ \&c.,}$$

$$(B+B')(C-C') = K(S+S'), \text{ \&c.,} \qquad (B-B')(C+C') = K(S-S'), \text{ \&c.,}$$

$$BC - B'C' = KS, \text{ \&c.,} \qquad B'C - BC' = KS', \text{ \&c.,}$$

$$(S+S')(T+T')(U+U') = (S-S')(T-T')(U-U') = PQR,$$

$$S'T'U' + S'TU + ST'U + STU' = 0,$$

$$STU + ST'U' + S'TU' + S'T'U = PQR,$$

$$(A-A')(S+S') = (C-C')(U-U'), \text{ \&c.,}$$

$$(A+A')(S-S') = (C+C')(U+U'), \text{ \&c.,}$$

$$(A-A')(T-T') = P(C-C'), \text{ \&c.,}$$

$$(A+A')(T+T') = P(C+C'), \text{ \&c.,}$$

$$(A-A')(U+U') = P(B-B'), \text{ \&c.,}$$

$$(A+A')(U-U') = P(B+B'), \text{ \&c.,}$$

&c., &c.,

et ces équations donnent immédiatement et dans les formes les plus simples, les formules qui se rapportent aux sommes et aux produits des fonctions de $u+v$ et $u-v$, par exemple

$$S(u+v)\,S(u-v)=\frac{A^2-A'^2}{K^2},$$

savoir au moyen de la première de ces équations identiques:

$$S(u+v)\,S(u-v)=\frac{P}{K};$$

de manière que toutes ces formules peuvent être considérées comme comprises dans les équations fondamentales et dans ce système d'équations identiques.

95.

NOTE SUR QUELQUES THÉORÈMES DE LA GÉOMÉTRIE DE POSITION.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), pp. 66—72. Continued from t. XXXVIII. p. 104, **70.**]

§ VII.

En considérant les soixante droites auxquelles donne lieu le théorème de Pascal, et en appliquant ce théorème aux hexagones différents qui peuvent être formés par six points sur une même conique, M. Kirkman a trouvé que ces soixante droites se coupent trois à trois non seulement dans les vingt points de M. Steiner (points que M. Kirkman nomme les points g), mais aussi dans soixante points h. Il a trouvé aussi qu'il y a quatre-vingt-dix droites J, dont chacune contient deux des points h et un des quarante cinq points p, dans lesquels s'entrecoupent, deux à deux, les droites menées par deux quelconques des six points. Les recherches étendues que M. Kirkman a faites dans la géométrie de position, paraitront dans un numéro prochain du *Cambridge and Dublin Mathematical Journal*, [t. v. (1850), pp. 185—200]. En attendant, M. Kirkman a publié dans le *Manchester Courier* du 27ième Juin 1849, vingt cinq théorèmes qui contiennent les résultats de ses recherches.

Moi, j'ai depuis trouvé que les soixante points h sont situés trois à trois sur vingt droites X. Tous ces théorèmes peuvent être démontrés assez facilement quand on connait la manière suivant laquelle les points et les droites doivent être combinés en construisant les points et les droites h, J, &c. Cela se fait alors d'une manière très simple, au moyen d'une notation que je vais expliquer.

Représentons les six points sur la conique par 1, 2, 3, 4, 5, 6. En combinant ces points deux à deux par les droites 12, 13 &c., les systèmes tels que 12, 34, 56

peuvent être représentés par les combinaisons binaires des six symboles a, b, c, d, e, f, et au moyen de la table qui se trouve § III. de ce mémoire, savoir la table

$$(A)\quad \left\{\begin{array}{l|l|l}
12.34.56=ac & 13.45.62=ab & 14.56.23=bd \\
12.35.64=be & 13.46.25=cd & 14.52.36=ae \\
12.36.45=df & 13.42.56=ef & 14.53.62=cf \\
 & 15.62.34=de & 16.23.45=ce \\
 & 15.63.42=bc & 16.24.53=ad \\
 & 15.64.23=af & 16.25.34=bf:
\end{array}\right.$$

le symbole ac dénote ici l'ensemble des droites 12, 34, 56; et ainsi de suite.

On voit que pour obtenir les six côtés d'un quelconque des soixante hexagones, il n'y a qu'à combiner les droites correspondantes, par paires, telles que ab, ac, qui ont une lettre en commun. Cela posé, les hexagones, ou, si l'on veut, les droites derivées de ces hexagones au moyen du théorème de Pascal (droites que je nommerai *droites de Pascal*), peuvent être représentées par les symboles $ab.ac$, &c., conformément à la table que voici:

$$(B)\quad \left\{\begin{array}{l|l|l|l}
213456=ab.ac & 214356=cf.ca & 215346=ed.eb & 216345=fb.fd \\
564=ef.eb & 563=db.df & 463=fa.fd & 453=ec.eb \\
645=dc.df & 635=ea.eb & 634=cb.ca & 534=ad.ac \\
465=cd.ca & 365=ae.ac & 364=bc.be & 354=da.df \\
546=ba.be & 536=fc.fd & 436=de.df & 435=bf.be \\
654=fe.fd & 653=bd.be & 643=af.ac & 543=ce.ca \\
314256=ea.ef & 315246=cb.cd & 316245=ad.ab & 415236=af.ae \\
562=bd.ba & 462=af.ab & 452=ce.cd & 362=cb.cf \\
625=cf.cd & 624=ed.ef & 524=fb.fe & 623=de.db \\
265=fc.fe & 264=de.dc & 254=bf.ba & 263=ed.ea \\
526=ae.ab & 426=bc.ba & 425=da.dc & 326=fa.fc \\
652=db.dc & 642=fa.fe & 542=ec.ef & 632=bc.bd \\
 & 416235=ce.cf & 516234=ec.ed & \\
 & 352=ad.ae & 342=bf.bc & \\
 & 523=bf.bd & 423=ad.af & \\
 & 253=fb.fc & 243=da.de & \\
 & 325=ec.ea & 324=ce.cb & \\
 & 532=da.db & 432=fb.fa &
\end{array}\right.$$

Remarquons maintenant que les droites de Pascal qui passent par un point p, tel que 12.45, sont $ca.ce$, $ba.be$, $ac.ab$, $ec.eb$. Cela étant, le point 12.45 peut être représenté par la notation $cb.ae$, et de cette manière le système complet des points p est représenté par la table suivante:

$$(C)\quad \left\{\begin{array}{l|l|l}
12.34 = bd.ef & 14.35 = ab.de & 23.45 = ad.bf \\
12.35 = af.cd & 14.36 = bf.cd & 23.46 = bc.de \\
12.36 = ab.ce & 14.56 = af.ce & 23.56 = ae.cf \\
12.45 = ae.be & 15.23 = be.cd & 24.35 = bf.ce \\
12.46 = ad.cf & 15.24 = ae.df & 24.36 = af.de \\
12.56 = bf.de & 15.26 = ac.bf & 24.56 = ab.ed \\
13.24 = ac.bd & 15.34 = ab.cf & 25.34 = ad.ce \\
13.25 = af.be & 15.36 = ad.fe & 25.36 = bd.cf \\
13.26 = bd.ec & 15.46 = bd.ce & 25.46 = ab.ef \\
13.45 = cf.de & 16.23 = ab.df & 26.34 = af.bc \\
13.46 = ae.bf & 16.24 = cf.be & 26.35 = ae.bd \\
13.56 = ad.bc & 16.25 = ac.de & 26.45 = cd.ef \\
14.23 = ac.ef & 16.34 = ae.cd & 34.56 = be.df \\
14.25 = ce.df & 16.35 = bc.ef & 35.46 = ac.df \\
14.26 = ad.be & 16.45 = af.bd & 36.45 = ac.be.
\end{array}\right.$$

Enfin les droites 12, &c. peuvent être représentées par des symboles tels que $ac.be.df$, &c., et au moyen de la table suivante, qui est pour ainsi dire la réciproque de la table (A):

$$(D)\quad \left\{\begin{array}{l|l|l}
ac.be.df = 12 & ab.ed.fc = 62 & ae.df.cb = 36 \\
ac.bd.fe = 56 & ab.ef.cd = 13 & ae.dc.bf = 52 \\
ac.bf.ed = 34 & ab.ec.df = 45 & ae.db.fc = 14 \\
 & ad.fb.ce = 16 & af.bc.ed = 15 \\
 & ad.fc.eb = 53 & af.be.dc = 64 \\
 & ad.fe.be = 24 & af.bd.ce = 32.
\end{array}\right.$$

Il y a à remarquer qu'une droite de Pascal $ab.ac$ contient les points $bc.ad$, $bc.ae$, $bc.af$, et que par un point $ab.cd$ passent les droites (les côtés opposés d'un hexagone) $ac.bd.ef$, $ab.bc.ef$, et les droites de Pascal $ca.cb$, $da.db$, $ac.ad$, $bc.bd$. Cela posé, en combinant les propriétés déjà connues d'avec celles que j'ai énoncées au commencement de cette section, en particularisant en même temps les combinaisons qui donnent lieu aux points et droites g, h, J, &c. et en adoptant une notation convenable pour ces points et droites, on trouvera ce qui suit:

(α) Les droites $ab.bc$, $bc.ca$, $ca.ab$ se rencontrent dans un même point abc qui est un des vingt points g, et que j'ai dénoté par ce symbole, § III. de ce mémoire.

(β) Les points abc, abd, abe, abf sont situés sur une même droite ab qui est une des quinze droites de M. Steiner ou de M. Plücker, et que j'ai dénotée (§ III.). Je nommerai droites I ces droites.

(γ) Les droites $ab.ac$, $ac.ad$, $ad.ab$ se rencontrent dans un même point $a.ef$ qui est un des soixante points h de M. Kirkman.

(δ) Les points $b.cd$, $c.db$, $d.bc$ sont situés sur la même droite $\{bcd\}$ qui est une de mes vingt droites X.

(ϵ) Les points $ab.cd$, $e.ab$, $f.ab$ sont situés sur une même droite $(ab)cd$ qui est un des quatre-vingt-dix points J de M. Kirkman.

Quant aux théorèmes (α) et (β), je vais reproduire dans la notation de cette section les démonstrations de M. Plücker.

Voici le principe de la démonstration du théorème (α): principe qui s'applique aussi, comme nous le verrons, aux démonstrations des théorèmes (γ, δ, et ϵ).

Supposons qu'il s'agit de démontrer généralement que trois droites X, X', X'' se rencontrent dans un même point, et supposons que ces droites sont déterminées:

$$\begin{array}{llll} X & \text{au moyen des points} & A\,, & B\,, \quad C\,, \\ X' & \quad\text{„}\quad\text{„}\quad\text{„} & A'\,, & B'\,, \quad C'\,, \\ X'' & \quad\text{„}\quad\text{„}\quad\text{„} & A''\,, & B''\,, \quad C''; \end{array}$$

formons d'abord la table

$$(\odot)\quad \left\{\begin{array}{lll} A'A'', & B'B'', & C'C'', \\ A''A\,, & B''B\,, & C''C\,, \\ A\,A'\,, & B\,B'\,, & C\,C'\,, \end{array}\right.$$

où $A'A''$, &c. sont les droites qui passent par les points A' et A'', &c.; et puis la table

$$(\flat)\quad \left\{\begin{array}{llll} B'B''.C'C'', & C'C''.A'A'', & A'A''.B'B'', \\ B''B\,.C''C\,, & C''C\,.A''A\,, & A''A\,.B''B\,, \\ B\,B'.C\,C'\,, & C\,C'\,.A\,A'\,, & A\,A'\,.B\,B'\,, \end{array}\right.$$

où $B'B''.C'C''$, &c. sont les points d'intersection des droites $B'B''$ et $C'C''$, &c. On sait que si les points de l'une quelconque des colonnes verticales de cette dernière table sont situés sur la même droite, les droites X, X', X'' se couperont dans un même point; et réciproquement. Précisément de la même manière on démontrerait que trois points X, X', X'' sont situés sur une même droite; seulement A, B, &c. seraient des droites, $A'A''$, &c. des points; et ainsi de suite. Or les droites du théorème (α) sont déterminées,

$$\begin{array}{llll} ab.bc & \text{au moyen des points} & ac.be, & ac.bf, \quad ac.bd, \\ bc.ca & \quad\text{„}\quad\text{„}\quad\text{„} & ba.ce, & ba.cf, \quad ba.cd, \\ ca.ab & \quad\text{„}\quad\text{„}\quad\text{„} & cb.ae, & cb.af, \quad cb.ad\,; \end{array}$$

donc la table (⊙) se réduit à

$$\begin{array}{lll} ac.be.df, & ac.bf.de, & ac.bd.ef, \\ ba.ce.df, & ba.cf.de, & ba.cd.ef, \\ cb.ae.df, & cb.af.de, & cb.ad.ef, \end{array}$$

et la table (☽) à

$$\begin{array}{lll} be.df, & bf.de, & bd.ef, \\ ce.df, & cf.de, & cd.ef, \\ ae.df, & af.de, & ad.ef; \end{array}$$

et les points de la première colonne verticale de cette table sont situés sur la droite $ed.ef$, ceux de la deuxième colonne verticale sur la droite $fd.fe$, et ceux de la troisième colonne verticale sur la droite $de.df$: l'existence de l'une quelconque de ces droites fait voir la vérité du théorème dont il s'agit.

Pour démontrer le théorème (β), considérons à part un quelconque des points abc, abd, abe, abf; par exemple le point abf. On peut envisager ce point comme déterminé par les droites $ab.af$, $ab.bf$, et ces droites contiennent:

$$\begin{array}{llll} ab.af & \text{les points} & bf.ac, & bf.ad, & bf.ae, \\ ab.bf & \quad\text{"} \quad\text{"} & af.bc, & af.bd, & af.be. \end{array}$$

Or $bf.ac$ et $af.bc$ sont situés sur la droite $ab.de.cf$; $bf.ad$ et $af.bd$ sur la droite $ab.ec.df$; et $bf.ae$ et $af.be$ sur la droite $ab.cd.ef$. De plus, les points $bf.ac$, $bf.ad$, $bf.ae$ sont situés sur les droites $ab.bc$, $ab.bd$, $ab.be$ respectivement, et les points $af.bc$, $af.bd$, $af.be$ sont situés sur les droites $ab.ac$, $ab.ad$, $ab.ae$ respectivement. Donc on peut représenter les points de la droite $ab.af$ par les symboles

$$(ab.de.cf)(ab.bc), \quad (ab.ec.df)(ab.bd), \quad (ab.cd.ef)(ab.be),$$

et les points de la droite $ab.bf$ par les symboles

$$(ab.de.cf)(ab.ac), \quad (ab.ec.df)(ab.ad), \quad (ab.cd.ef)(ab.ae).$$

Maintenant

Les droites	de même que les droites	se rencontrent sur la droite
$ab.bd$, $ab.ae$	$ab.be$, $ab.ad$,	$ab.de.cf$,
$ab.be$, $ab.ac$	$ab.bc$, $ab.ae$,	$ab.ec.df$,
$ab.bc$, $ab.ad$	$ab.bd$, $ab.ac$,	$ab.cd.ef$,

c'est à dire, il existe un système de trois hexagones dont les côtés sont

$$\begin{array}{llllll} (ab.de.cf, & ab.ec.df, & ab.cd.ef, & ab.bc, & ab.bd, & ab.be), \\ (ab.de.cf, & ab.ec.df, & ab.cd.ef, & ab.ac, & ab.ad, & ab.ae), \\ (ab.bc\ , & ab.bd\ , & ab.be\ , & ab.ac, & ab.ad, & ab.ae). \end{array}$$

Ces hexagones ont pour angles les mêmes six points. Or l'existence de l'une ou de l'autre des droites $ab.af$, $ab.bf$ suffit pour faire voir que ces six points sont situés sur la même conique: donc les côtés opposés du troisième hexagone se rencontrent dans trois points situés sur la même droite. De plus, on voit aisément que les hexagones sont précisément tels, qu'en vertu du théorème (α), les trois droites, auxquelles donnent lieu ces hexagones, se rencontrent dans un même point; et ce point sera évidemment le point abf. Mais les côtés opposés du troisième hexagone, savoir les droites $ab.bc$ et $ab.ac$; $ab.bd$ et $ab.ad$; $ab.be$ et $ab.ae$, se rencontrent dans les points abc, abd, abf: donc les quatre points abc, abd, abe, abf sont situés sur la même droite: théorème dont il s'agissait.

Pour démontrer le théorème (γ), on n'a qu'à considérer les droites de ce théorème comme déterminées,

$$\begin{array}{llllll} ab.ac & \text{par les points} & bc.ad, & bc.ae, & bc.af, \\ ac.ad & \quad\text{"} \quad\text{"} & cd.ab, & cd.ae, & cd.af, \\ ad.ab & \quad\text{"} \quad\text{"} & bd.ac, & bd.ae, & bd.af. \end{array}$$

La table ($\odot$) se réduit alors à

$$\begin{array}{lll} bc.ad.ef, & da.de, & da.df, \\ cd.ab.ef, & ba.be, & ba.bf, \\ bd.ac.ef, & ca.ce, & ca.cf, \end{array}$$

et la table ($\mathfrak{D}$) à

$$\begin{array}{lll} (da.df)(da.de), & af.de, & ae.df, \\ (ba.bf)(ba.be), & af.be, & ae.bf, \\ (ca.cf)(ca.ce), & af.ce, & ae.cf. \end{array}$$

Or les points de la deuxième colonne verticale sont situés sur la droite $ea.ef$, et les points de la troisième colonne verticale sur la droite $fa.fe$. L'existence de l'une ou de l'autre de ces droites fait voir que les droites $ab.ac$, $ac.ad$, $ad.ab$ se rencontrent dans un même point $a.be$.

Pour démontrer le théorème (δ), il est évident que les points de la première colonne verticale de la table qui vient d'être présentée, sont situés sur la même droite. Mais ces points sont précisément les points $d.bc$, $b.cd$, $c.db$; le théorème est donc démontré.

[En remarquant que les trois droites sur lesquelles sont situés les neuf points de la table générale ($\mathfrak{D}$) se rencontrent dans un même point, cette démonstration de l'existence des droites X fait voir que la droite $\{bcd\}$ passe par le point d'intersection des droites $ae.af$, $ef.fa$, savoir par le point afe; ou bien que chacune des vingt droites X passe, non seulement par trois points h, mais aussi par un seul point g. Ce théorème est dû à M. Salmon, qui indépendamment de mes recherches à trouvé l'existence des vingt droites X. 8 août 1849. *Inserted here from* Crelle, t. XLI. p. 84.]

Enfin, pour démontrer le théorème (ϵ), nous pouvons considérer les points de ce théorème comme déterminés,

$$\begin{array}{llllll} ab.cd & \text{par les droites} & bc.bd, & ad.be.ef, & ac.bd.ef, \\ f.ab & \quad\text{"} \quad\quad\text{"} & fc.fd, & fc\ .fe & ,\ fd.fe & , \\ e.ab & \quad\text{"} \quad\quad\text{"} & ec.ed, & ec\ .fe & ,\ ed.fe & . \end{array}$$

La table ($\odot$) se réduit alors à

$$\begin{array}{lll} cd.ef, & f.ce, & f.de, \\ cd.eb, & cf.eb, & df.eb, \\ cd.bf, & ce.bf, & de.bf, \end{array}$$

et la table ($\mathfrak{D}$) à

$$\begin{array}{lll} fe\ , & ce.cf, & de.df, \\ fe.fb, & cb.ce, & db.de, \\ fe.eb, & cb.cf, & db.df. \end{array}$$

Or les droites de la première colonne verticale de cette table se rencontrent dans le point $bf.be$, celles de la deuxième colonne verticale dans le point $c.ad$, et celles de la troisième colonne verticale dans le point $d.ac$; le théorème dont il s'agit est donc démontré. Dans cette démonstration on aurait aussi pu échanger les lettres a, b.

Les théorèmes (α) et (γ) peuvent être énoncés par le seul théorème suivant:

"Étant donnés six points sur la même conique, et menant par ces points neuf droites, de manière que chaque droite passe par deux points et que par chaque point il passe trois droites: on formera avec ces neuf droites trois hexagones différents dont chacun a les six points pour angles. Les droites de Pascal, auxquelles donnent lieu ces trois hexagones, se rencontreront dans un même point."

En supposant que le système de neuf droites contient toujours un même hexagone, il est possible de compléter de quatre manières différentes le système des neuf droites; savoir, on peut ajouter aux côtés de l'hexagone 1 les trois diagonales de l'hexagone 2, 3 ou 4, en menant une quelconque de ces diagonales et deux droites, chacune par deux angles alternés de l'hexagone. Ces quatre systèmes donnent lieu au point g, et aux trois points h, qui se trouvent sur la droite de Pascal, correspondante à l'hexagone dont il s'agit; savoir le premier système donne lieu au point g, et les trois derniers systèmes au point h.

96.

MÉMOIRE SUR LES CONIQUES INSCRITES DANS UNE MÊME SURFACE DU SECOND ORDRE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), pp. 73—86.]

En considérant une surface quelconque du second ordre, le problème se presente: d'examiner les proprietés des coniques inscrites dans cette surface et des cônes circonscrits. La plupart de ces propriétés est peut-être connue[1]; cependant je crois qu'on ne les a pas encore développées systématiquement. Je me propose de donner ici l'analyse des propriétés les plus simples d'un tel système de coniques, et la solution du problème analogue au problème des tactions qui se présente ici, ainsi que quelques théorèmes relatifs au passage à un système de coniques situées dans un même plan et inscrites dans une même conique, en me réservant pour une seconde partie de ce mémoire les développements ultérieurs concernant ce passage, et la solution complète du problème analogue au problème de Malfatti, généralisé par M. Steiner.

Remarquons d'abord que les coniques inscrites et les cônes circonscrits, ainsi que les plans des coniques inscrites et les sommets des cônes circonscrits, sont des figures *réciproques* par rapport à la surface du second ordre. En considérant deux coniques inscrites quelconques, et les cônes circonscrits correspondants, on remarquera que les plans des coniques inscrites se rencontrent dans une droite. Je la nommerai *Droite de symptose.* Les sommets des cônes circonscrits seront situés dans une droite que je nommerai *Droite d'homologie.* Ces deux droites seront évidemment réciproques l'une à l'autre. Il se trouvera sur la droite d'homologie deux points dont chacun est le sommet d'un cône qui passe par les deux coniques inscrites. Ces deux points peuvent être nommés *Points d'homologie.* De même il passera par la droite de *symptose* deux plans,

[1] Voyez le mémoire de M. Steiner "Einige geometrische Betrachtungen" Journal t. I. [1826] pp. 161—184, et un mémoire de M. Olivier, Quetelet, Corresp. Math. t. V. [1829].

qui sont les plans des coniques dans lesquelles se coupent les deux cônes circonscrits. Ces deux plans peuvent être nommés *Plans de symptose.* Les plans de symptose et les points d'homologie ne sont pas seulement des figures réciproques: les deux plans de symptose passent aussi par les deux points d'homologie, chacun par le point réciproque de l'autre plan; c'est à dire: les plans de symptose sont des plans *conjugués* par rapport à la surface du second ordre, et les points d'homologie sont des points *conjugués* par rapport à cette même surface. Remarquons aussi qu'en considérant le système formé par les plans des coniques inscrites et les plans tangents à la surface menés par la droite de symptose, on trouvera que les plans de symptose sont les plans doubles (ou si l'on veut les plans *auto-conjugués*) de l'involution. De même, en considérant le système formé par les sommets des cônes circonscrits et par les points de leur intersection avec la surface de la droite d'homologie, on trouvera que les points d'homologie sont les points doubles (ou *auto-conjugués*) de l'involution. Les deux cônes circonscrits qui ont pour sommets les deux points d'homologie, peuvent être nommés *Cônes d'homologie;* de même, les deux coniques inscrites, situées dans les deux plans de symptose, peuvent être nommées *Coniques de symptose.* (En passant, nous remarquerons que ces coniques de symptose correspondent aux *Potenzkreise* de M. Steiner.) Il est évident que les cônes d'homologie et les coniques de symptose sont des figures réciproques.

En considérant trois coniques inscrites, et les cônes circonscrits correspondants, on verra que les plans des coniques inscrites se rencontrent dans un point que je nommerai *Point de symptose.* Les sommets des cônes circonscrits seront situés dans un plan que je nommerai *Plan d'homologie.* Ce point et le plan seront réciproques l'un à l'autre. En combinant deux à deux les coniques inscrites ou les cônes circonscrits, cela donne lieu à trois droites de symptose qui passent chacune par le point de symptose, et à trois droites d'homologie situées chacune dans le plan d'homologie. Il existe aussi six plans de symptose qui se coupent trois à trois dans quatre droites, arêtes d'une pyramide quadrilatère qui a pour axes les trois droites de symptose. Les quatre droites dont il s'agit, peuvent être nommées *Axes de symptose.* Il existe également six points d'homologie, situés trois à trois dans quatre droites, côtés d'un quadrilatère qui a pour axes les trois droites d'homologie. Les quatre droites dont il s'agit, peuvent être nommées *Axes d'homologie.* La pyramide et le quadrilatère sont des figures réciproques, et il convient de remarquer (quoique cela soit assez évident) qu'il y a ici trois points d'homologie, non-situés dans un des côtés du quadrilatère, mais contenus dans trois points de symptose qui se coupent dans une arête de la pyramide.

Par l'un quelconque des axes d'homologie il passe deux plans dont chacun touche les trois coniques inscrites; de même il se trouve sur l'un quelconque des axes de symptose deux points, dont chacun est un point d'intersection des trois cônes circonscrits. Cela constitue la solution du problème: Trouver la conique inscrite, ou le cône circonscrit, qui touche trois coniques inscrites, ou trois cônes circonscrits. Il y a *huit* solutions de ce problème.

Avant d'aller plus loin, je vais indiquer quelques-unes des formules analytiques correspondantes à la théorie qui vient d'être expliquée.

Écrivons, pour abréger:

$$U = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Fyz + 2Gxz + 2Hxy + 2Lxw + 2Myw + 2Nzw,$$
$$V = \alpha x + \beta y + \gamma z + \delta w,$$

et représentons par $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ les coefficients du système inverse de A, B, C, D, F, G, H, L, M, N. Soit de plus

$$X = Ax + Hy + Gz + Lw,$$
$$\vdots$$
$$p^2 = \mathfrak{A}\alpha^2 + \mathfrak{B}\beta^2 + \mathfrak{C}\gamma^2 + \mathfrak{D}\delta^2 + 2\mathfrak{F}\beta\gamma + 2\mathfrak{G}\gamma\alpha + 2\mathfrak{H}\alpha\beta + 2\mathfrak{L}\alpha\delta + 2\mathfrak{M}\beta\delta + 2\mathfrak{N}\gamma\delta.$$

Cela posé, en prenant $U = 0$ pour équation de la surface du second ordre, et $V_1 = 0$, $V_2 = 0$, $V_3 = 0$ pour les équations des plans des coniques inscrites, on obtient pour l'un des plans de symptose des coniques inscrites $(U = 0,\ V_1 = 0)$, $(U = 0,\ V_2 = 0)$ l'équation très simple $p_2 V_1 = p_1 V_2 = 0$. De là on tire pour les coordonnées du point d'homologie, qui est le réciproque de ce plan de symptose, les équations

$$X : Y : Z : W = p_2\alpha_1 - p_1\alpha_2 : p_2\beta_1 - p_1\beta_2 : p_2\gamma_1 - p_1\gamma_2 : p_2\delta_1 - p_1\delta_2.$$

En formant également les expressions des coordonnées d'un point d'homologie des deux autres paires de coniques inscrites, on obtient pour équation de l'un des axes d'homologie:

$$\begin{vmatrix} & X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ p_2, & \alpha_2, & \beta_2, & \gamma_2, & \delta_2 \\ p_3, & \alpha_3, & \beta_3, & \gamma_3, & \delta_3 \end{vmatrix} = 0;$$

savoir, en choisissant quatre colonnes verticales quelconques de cette formule, on trouve que les déterminants que l'on obtient, sont tous égaux à zéro.

Nous ajouterons que la droite qui, par rapport à la conique $(U = 0,\ V_1 = 0)$, est réciproque de cet axe d'homologie, est donnée par les équations

$$V_1 = 0, \quad V_2 : V_3 = p_1 p_2 - \mathfrak{A}\alpha_1\alpha_2 - \ldots : p_1 p_3 - \mathfrak{A}\alpha_1\alpha_3 - \ldots;$$

il est clair que cette droite rencontre la surface du second ordre en deux points situés dans les plans des coniques inscrites qui, au moyen de l'axe d'homologie dont l'équation vient être donnée, sont déterminés de manière à toucher les trois coniques inscrites données. Mais sans se servir des équations de cette droite, on peut déterminer l'équation des deux plans menés par l'axe d'homologie dont il s'agit, de manière à toucher la conique inscrite $(U = 0,\ V_1 = 0)$; et la symétrie du résultat fera voir que ces deux plans touchent aussi deux autres coniques inscrites. La recherche de cette équation étant un peu difficile, je la donnerai en détail, en supposant cependant connu le théorème suivant:

En écrivant $v = \lambda x + \mu y + \nu z + \rho w$, $v' = \lambda' x + \mu' y + \nu' z + \rho' w$: les plans menés par la droite $(v = 0,\ v' = 0)$ de manière qu'ils touchent la conique inscrite $(U = 0,\ V = 0)$, sont donnés par l'équation

$$p^2 [\mathfrak{A}(\lambda v' - \lambda' v)^2 + \ldots] - [\mathfrak{A}\alpha(\lambda v' - \lambda' v) + \ldots]^2 = 0.$$

Pour appliquer ce théorème au problème dont il s'agit, nous n'avons qu'à substituer V_1 au lieu de V, et qu'à écrire

$$v = \begin{vmatrix} a, & b, & c, & d \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}, \quad v' = \begin{vmatrix} a', & b', & c', & d' \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}$$

où les coefficients a, b, c, d; a', b', c', d' sont des quantités quelconques.

Réduisons d'abord l'expression $\mathfrak{A}\alpha_1(\lambda v' - \lambda' v) + \ldots$. Pour cela, mettons dans les valeurs de v, v', les expressions $\mathfrak{A}\alpha_1 + \ldots$, $\mathfrak{H}\alpha_1 + \ldots$, &c. à la place de $x, y, \ldots$: les quantités X, Y, Z, W deviennent alors $K\alpha_1, K\beta_1, K\gamma_1, K\delta_1$ (où comme à l'ordinaire K est le déterminant formé par les quantités $A, B, \ldots$), et l'on obtient ainsi, aux signes près:

$$\mathfrak{A}\alpha_1\lambda + \ldots = Kp_1 \begin{vmatrix} a, & b, & c, & d \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}, \quad \mathfrak{A}\alpha_1\lambda' + \ldots + = Kp_1 \begin{vmatrix} a', & b', & c', & d' \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}$$

et de là

$$\mathfrak{A}\alpha_1(\lambda v' - \lambda' v) + \ldots = Kp_1 \square,$$

où

$$\square = \begin{vmatrix} a, & b', & c, & d \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a', & b', & c', & d' \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} - \begin{vmatrix} a', & b', & c', & d' \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a, & b, & c, & d \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}:$$

formule qui au moyen des propriétés des déterminants se réduit à

$$\square = \begin{vmatrix} X, & Y, & Z, & W \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a, & b, & c, & d, \\ a', & b', & c', & d', \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1, \\ \vdots \end{vmatrix}$$

Passons à l'expression $\mathfrak{A}(\lambda v' - \lambda' v)^2 + \ldots$, que nous mettrons sous la forme

$$\frac{1}{K}\{A[\mathfrak{A}(\lambda v' - \lambda' v)^2 + \ldots]^2 + \ldots\}.$$

En prenant des quantités quelconques $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, on obtient par une analyse semblable:

$$\mathfrak{A}\mathfrak{a}(\lambda v' - \lambda' v) + \ldots = K\overline{\square},$$

où

$$\overline{\square} = \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ a, & b, & c, & d \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a', & b', & c', & d' \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} - \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ a', & b', & c', & d' \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a, & b, & c, & d \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} :$$

formule qui se réduit à

$$\overline{\square} = \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix} \begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}.$$

De là, en supprimant les facteurs constants de $\square$ et de $\overline{\square}$, on obtient

$$\mathfrak{a}\xi + \mathfrak{b}\eta + \mathfrak{c}\zeta + \mathfrak{d}\omega = \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix},$$

expression qui sert à définir les fonctions ξ, η, ζ, ω. L'équation qu'il s'agissait de trouver devient

$$A\xi^2 + \ldots - K \begin{vmatrix} X, & Y, & Z, & W \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots \end{vmatrix}^2 = 0,$$

où il faut avoir égard que l'on a $X = Ax + \ldots$, &c. Savoir, l'équation qu'on vient d'écrire se décompose nécessairement en facteurs linéaires qui, égalés à zéro, donnent les équations des plans des coniques inscrites qui touchent chacune les trois coniques inscrites données.

Nous avons obtenu ce résultat en traduisant en analyse une construction géométrique; mais on y peut aussi parvenir en considérant le problème d'une manière purement analytique. En effet: soient comme plus haut, $U = 0$ l'équation de la surface du second ordre, $V_1 = 0$, $V_2 = 0$, $V_3 = 0$ les équations des plans des trois coniques inscrites données, $V = 0$ l'équation du plan de la conique inscrite qui touche chacune de ces trois coniques. La condition pour que cette conique touche la conique inscrite située dans le plan $V_1 = 0$, est $\mathfrak{A}\alpha\alpha_1 + \ldots = pp_1$. On a donc les trois équations

$$\mathfrak{A}\alpha\alpha_1 + \ldots = pp_1,$$
$$\mathfrak{A}\alpha\alpha_2 + \ldots = pp_2,$$
$$\mathfrak{A}\alpha\alpha_3 + \ldots = pp_3.$$

Au lieu de tirer de ces équations les quantités $\alpha : \beta : \gamma : \delta$, nous ajouterons au système la nouvelle équation

$$\alpha x + \ldots = 0,$$

par laquelle il sera possible d'éliminer les quatre quantités α, β, γ, δ. En attribuant à $X, \ldots$ la même signification qu'auparavant, nous mettrons les quatre équations sous la forme

$$\begin{aligned}
(\mathfrak{A}\alpha + \ldots)\,\alpha_1 + \ldots &= pp_1,\\
(\mathfrak{A}\alpha + \ldots)\,\alpha_2 + \ldots &= pp_2,\\
(\mathfrak{A}\alpha + \ldots)\,\alpha_3 + \ldots &= pp_3,\\
(\mathfrak{A}\alpha + \ldots)\,X + \ldots &= 0\,.
\end{aligned}$$

Écrivons de plus

$$(\mathfrak{A}\alpha + \ldots)\,\mathfrak{a} + \ldots = p\Theta,$$

où les quantités $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ sont arbitraires. En éliminant de ces équations les fonctions $(\mathfrak{A}\alpha + \ldots)$, puis en mettant à la place de $p\Theta$ la quantité à gauche de l'équation, on obtient, à un facteur constant près,

$$(\mathfrak{A}\alpha + \ldots)\,\mathfrak{a} + \ldots = \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots & & & & \end{vmatrix};$$

cela donne d'abord

$$\mathfrak{A}\alpha\alpha_1 + \ldots = p_1 \begin{vmatrix} X, & Y, & Z, & W \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots & & & \end{vmatrix},$$

puis, en écrivant $p^2 = \mathfrak{A}\alpha^2 + \ldots = \frac{1}{K}[A\,(\mathfrak{A}\alpha + \ldots)^2 + \ldots]$, on a

$$p^2 = \frac{1}{K}(A\xi^2 + \ldots),$$

et

$$\mathfrak{a}\xi + \ldots = \begin{vmatrix} \mathfrak{a}, & \mathfrak{b}, & \mathfrak{c}, & \mathfrak{d} \\ X, & Y, & Z, & W \\ p_1, & \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots & & & & \end{vmatrix},$$

et de là enfin, en substituant dans l'équation $\mathfrak{A}\alpha\alpha_1 + \ldots = pp_1$, on obtient comme plus haut l'équation

$$A\xi^2 + \ldots - K \begin{vmatrix} X, & Y, & Z, & W \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \vdots & & & \end{vmatrix}^2 = 0.$$

Il est clair que cette analyse peut être appliquée à la solution d'un nombre quelconque d'équations de la forme $\mathfrak{A}\alpha\alpha_1 + \ldots = pp_1$.

En revenant à la théorie *géométrique*, considérons un point quelconque que nous prendrons pour point de projection : le cône qui passe par une conique inscrite quelconque aura (comme on sait) un contact double avec le cône qui a pour sommet le point de projection. Le plan de contact sera le plan mené par le point de projection et par la droite d'intersection du plan de la conique inscrite et du plan réciproque au point de projection. En considérant plusieurs coniques inscrites ayant une droite de symptose commune, tous les cônes auxquels donnent lieu ces coniques inscrites, auront pour arêtes communes les deux droites menées par le point de projection aux points dans lesquels la surface est rencontrée par la droite de symptose commune. Ajoutons que les plans de contact des cônes dont il s'agit, avec le cône circonscrit, rencontrent le plan des deux arêtes communes dans une droite fixe, savoir dans l'une ou l'autre des droites doubles (ou auto-conjuguées) de l'involution formée par les deux arêtes communes et par les droites dans lesquelles le plan de ces deux arêtes communes rencontre le cône circonscrit. De plus, en considérant les plans tangents menés par l'une ou par l'autre des deux arêtes communes, ces plans tangents forment un système homologue à celui des plans des coniques inscrites. En considérant en particulier l'une ou l'autre des coniques de symptose de deux coniques inscrites quelconques : le plan tangent du cône correspondant est le plan double (ou auto-conjugué) de l'involution formée par les plans tangents des cônes qui correspondent aux deux coniques inscrites (c'est à dire par les plans tangents qui passent par l'arête commune dont il s'agit), et par les plans tangents de la surface du second ordre menés par cette même arête commune. C'est là en effet la propriété qui conduit à la construction des coniques de symptose de deux coniques situées dans le même plan et considerées comme inscrites dans une conique donnée.

97.

NOTE SUR LA SOLUTION DE L'ÉQUATION $x^{257}-1=0$.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), pp. 81—83.]

SOIT p_m la $m^{\text{ième}}$ puissance d'une racine quelconque (l'unité exceptée) de l'équation $x^{257}-1=0$, et représentons par α une racine quelconque (l'unité exceptée) de l'équation $\alpha^{256}-1=0$. En posant l'équation

$$(p_0+\alpha p_1+\alpha^2 p_2\ldots+\alpha^{255}p_{255})^2=M(p_0+\alpha^2 p_1+\alpha^4 p_2\ldots+\alpha^{254}p_{255}),$$

on sait que la quantité M peut être exprimée en fonction rationnelle de α. Cette fonction une fois connue, donnera tout de suite la valeur de l'expression $(p_0+\alpha p_1+\alpha^2 p_2\ldots+\alpha^{255}p_{255})^{256}$ en fonction rationnelle de α, et cela suffit pour résoudre l'équation dont il s'agit.

Une solution du problème a été donnée depuis longtemps par M. Richelot qui commence par supposer que α soit une racine primitive de l'équation $\alpha^{128}-1=0$. Cette solution est comprise, comme cas particulier, dans celle que je vais donner. La question est d'ailleurs intéressante, à cause de son rapport avec la théorie des nombres. En effet, quoiqu'en tant que je sache l'on n'a pas encore trouvé la règle pour former *à priori* la valeur de M, il est clair que les recherches de MM. Jacobi et Kummer doivent conduire à cette règle. Le résultat ici bas pourra servir pour la vérifier.

Voici la valeur que j'obtiens pour la fonction M:

$$\begin{aligned}
M=&-2+2\alpha-2\alpha^4+2\alpha^5+2\alpha^7+2\alpha^9-2\alpha^{10}-2\alpha^{14}-2\alpha^{16}+2\alpha^{21}+2\alpha^{23}\\
&+2\alpha^{25}-2\alpha^{26}-2\alpha^{28}+2\alpha^{29}-2\alpha^{30}+2\alpha^{33}-2\alpha^{34}-2\alpha^{36}+2\alpha^{37}-2\alpha^{38}+2\alpha^{45}\\
&+2\alpha^{47}-2\alpha^{48}+2\alpha^{49}-2\alpha^{50}+2\alpha^{51}+2\alpha^{53}-2\alpha^{54}-2\alpha^{60}+2\alpha^{61}-2\alpha^{64}+2\alpha^{65}\\
&-2\alpha^{66}+2\alpha^{67}-2\alpha^{68}+2\alpha^{69}-2\alpha^{72}-2\alpha^{74}+2\alpha^{75}-2\alpha^{76}+2\alpha^{79}-2\alpha^{80}-2\alpha^{82}\\
&-2\alpha^{84}-2\alpha^{86}+2\alpha^{89}-2\alpha^{92}-2\alpha^{94}-2\alpha^{100}+2\alpha^{101}-2\alpha^{104}-2\alpha^{106}-2\alpha^{108}+2\alpha^{109}\\
&+2\alpha^{111}+2\alpha^{113}-2\alpha^{114}+2\alpha^{115}+2\alpha^{117}+2\alpha^{119}-2\alpha^{122}+2\alpha^{123}-2\alpha^{124}+2\alpha^{127}-2\alpha^{128}\\
&+2\alpha^{129}-2\alpha^{130}-2\alpha^{134}+2\alpha^{135}-2\alpha^{138}+2\alpha^{141}+2\alpha^{145}+2\alpha^{147}+2\alpha^{151}+2\alpha^{153}-2\alpha^{154}\\
&-2\alpha^{156}-\alpha^{160}+2\alpha^{161}+2\alpha^{163}-2\alpha^{164}+2\alpha^{165}-2\alpha^{168}+2\alpha^{171}+2\alpha^{177}+2\alpha^{181}-2\alpha^{182}\\
&+2\alpha^{183}-2\alpha^{186}+2\alpha^{187}+2\alpha^{189}-2\alpha^{190}-2\alpha^{192}+2\alpha^{195}-2\alpha^{196}-2\alpha^{198}+2\alpha^{199}-2\alpha^{206}\\
&-2\alpha^{212}+2\alpha^{213}-2\alpha^{214}+2\alpha^{215}-2\alpha^{216}+2\alpha^{217}-2\alpha^{219}+2\alpha^{221}+2\alpha^{223}+2\alpha^{225}-2\alpha^{228}\\
&+2\alpha^{227}-2\alpha^{228}+2\alpha^{229}+2\alpha^{233}-2\alpha^{234}+2\alpha^{235}-2\alpha^{236}+2\alpha^{237}-2\alpha^{238}+2\alpha^{239}-2\alpha^{240}\\
&+2\alpha^{243}-2\alpha^{244}-2\alpha^{246}+2\alpha^{247}-2\alpha^{248}+2\alpha^{249}-2\alpha^{252}-2\alpha^{254}.
\end{aligned}$$

Représentons par M' ce que devient M, en supposant $\alpha^{128}+1=0$, nous obtiendrons

$$\begin{aligned}
M' = {} & 2\alpha^2 - 2\alpha^4 + 2\alpha^5 + 2\alpha^6 + 2\alpha^9 - 2\alpha^{13} - 2\alpha^{14} - 2\alpha^{16} - 2\alpha^{17} - 2\alpha^{19} + 2\alpha^{21} \\
& + 2\alpha^{29} - 2\alpha^{30} + \alpha^{32} - 2\alpha^{34} - 2\alpha^{35} - 2\alpha^{43} + 2\alpha^{45} + 2\alpha^{47} - 2\alpha^{48} - 2\alpha^{50} + 2\alpha^{51} \\
& - 2\alpha^{55} + 2\alpha^{58} - 2\alpha^{59} - 2\alpha^{60} + 2\alpha^{62} + 2\alpha^{65} - 2\alpha^{66} + 2\alpha^{69} + 2\alpha^{70} - 2\alpha^{71} - 2\alpha^{72} \\
& - 2\alpha^{74} + 2\alpha^{75} - 2\alpha^{76} + 2\alpha^{78} + 2\alpha^{79} - 2\alpha^{80} - 2\alpha^{82} - 2\alpha^{85} - 2\alpha^{87} + 2\alpha^{88} - 2\alpha^{93} \\
& - 2\alpha^{94} - 2\alpha^{95} - 2\alpha^{97} + 2\alpha^{98} - 2\alpha^{99} - 2\alpha^{104} - 2\alpha^{105} - 2\alpha^{107} + 2\alpha^{110} + 2\alpha^{112} + 2\alpha^{113} \\
& - 2\alpha^{114} + 2\alpha^{116} + 2\alpha^{117} + 2\alpha^{118} + 2\alpha^{120} - 2\alpha^{121} - 2\alpha^{122} + 2\alpha^{123} + 2\alpha^{126} + 2\alpha^{127}.
\end{aligned}$$

Soient M_1, M_1' ce que devient M en supposant successivement $\alpha^{128}-1=0$, $\alpha^{64}+1=0$, et soient M_2, M_2' ce que deviennent M ou M_1 en supposant successivement $\alpha^{64}-1=0$, $\alpha^{32}+1=0$, et ainsi de suite, jusqu'à M_7, M_7' qui seront ce que deviennent M ou M_1, &c. en supposant successivement $\alpha^2-1=0$, $\alpha+1=0$; nous aurons:

$$\begin{aligned}
M_1 = {} & -4 + 4\alpha - 2\alpha^2 - 2\alpha^4 + 2\alpha^5 - 2\alpha^6 + 4\alpha^7 + 2\alpha^9 - 4\alpha^{10} + 2\alpha^{13} - 2\alpha^{14} \\
& - 2\alpha^{16} + 2\alpha^{17} + 2\alpha^{19} + 2\alpha^{21} + 4\alpha^{23} + 4\alpha^{25} - 4\alpha^{26} - 4\alpha^{28} + 2\alpha^{29} - 2\alpha^{30} - \alpha^{32} \\
& + 4\alpha^{33} - 2\alpha^{34} + 2\alpha^{35} - 4\alpha^{36} + 4\alpha^{37} - 4\alpha^{38} + 2\alpha^{43} + 2\alpha^{45} + 2\alpha^{47} - 2\alpha^{48} + 4\alpha^{49} \\
& - 2\alpha^{50} + 2\alpha^{51} + 4\alpha^{53} - 4\alpha^{54} + 2\alpha^{55} - 2\alpha^{58} + 2\alpha^{59} - 2\alpha^{60} + 4\alpha^{61} - 2\alpha^{62} - 4\alpha^{64} \\
& + 2\alpha^{65} - 2\alpha^{66} + 4\alpha^{67} - 4\alpha^{68} + 2\alpha^{69} - 2\alpha^{70} + 2\alpha^{71} - 2\alpha^{72} - 2\alpha^{74} + 2\alpha^{75} - 2\alpha^{76} \\
& - 2\alpha^{78} + 2\alpha^{79} - 2\alpha^{80} - 2\alpha^{82} - 4\alpha^{84} + 2\alpha^{85} - 4\alpha^{86} + 2\alpha^{87} - 2\alpha^{88} + 4\alpha^{89} - 4\alpha^{92} \\
& + 2\alpha^{93} - 2\alpha^{94} + 2\alpha^{95} + 2\alpha^{97} - 2\alpha^{98} + 2\alpha^{99} - 4\alpha^{100} + 4\alpha^{101} - 2\alpha^{104} + 2\alpha^{105} - 4\alpha^{106} \\
& + 2\alpha^{107} - 4\alpha^{108} + 4\alpha^{109} - 2\alpha^{110} + 4\alpha^{111} - 2\alpha^{112} + 2\alpha^{113} - 2\alpha^{114} + 4\alpha^{115} - 2\alpha^{116} + 2\alpha^{117} \\
& - 2\alpha^{118} + 4\alpha^{119} - 2\alpha^{120} + 2\alpha^{121} - 2\alpha^{122} + 2\alpha^{123} - 4\alpha^{124} - 2\alpha^{126} + 2\alpha^{127}.
\end{aligned}$$

$$\begin{aligned}
M_1' = {} & 2\alpha - 4\alpha^3 + 2\alpha^4 + 2\alpha^7 + 2\alpha^8 + 2\alpha^9 - 2\alpha^{10} - 2\alpha^{11} + 2\alpha^{12} + 2\alpha^{13} - 2\alpha^{15} + 2\alpha^{17} \\
& + 2\alpha^{18} + 2\alpha^{19} + 4\alpha^{20} + 4\alpha^{22} + 2\alpha^{23} + 2\alpha^{24} - 4\alpha^{26} - 2\alpha^{31} - \alpha^{32} + 2\alpha^{33} - 4\alpha^{38} + 2\alpha^{40} \\
& - 2\alpha^{41} + 4\alpha^{42} + 4\alpha^{44} - 2\alpha^{45} + 2\alpha^{46} - 2\alpha^{47} + 2\alpha^{49} - 2\alpha^{51} + 2\alpha^{52} + 2\alpha^{53} - 2\alpha^{54} - 2\alpha^{55} \\
& + 2\alpha^{56} - 2\alpha^{57} + 2\alpha^{60} + 4\alpha^{61} - 2\alpha^{63}.
\end{aligned}$$

$$\begin{aligned}
M_2 = {} & -8 + 6\alpha - 4\alpha^2 + 4\alpha^3 - 6\alpha^4 + 4\alpha^5 - 4\alpha^6 + 6\alpha^7 - 2\alpha^8 + 2\alpha^9 - 6\alpha^{10} + 2\alpha^{11} \\
& - 2\alpha^{12} + 2\alpha^{13} - 2\alpha^{14} + 2\alpha^{15} - 4\alpha^{16} + 2\alpha^{17} - 2\alpha^{18} + 2\alpha^{19} - 4\alpha^{20} + 4\alpha^{21} - 4\alpha^{22} + 6\alpha^{23} \\
& - 2\alpha^{24} + 8\alpha^{25} - 4\alpha^{26} - 8\alpha^{28} + 4\alpha^{29} - 4\alpha^{30} + 2\alpha^{31} - \alpha^{32} + 6\alpha^{33} - 4\alpha^{34} + 4\alpha^{35} - 8\alpha^{36} \\
& + 8\alpha^{37} - 4\alpha^{38} - 2\alpha^{40} + 2\alpha^{41} - 4\alpha^{42} + 4\alpha^{43} - 4\alpha^{44} + 6\alpha^{45} - 2\alpha^{46} + 6\alpha^{47} - 4\alpha^{48} + 6\alpha^{49} \\
& - 4\alpha^{50} + 6\alpha^{51} - 2\alpha^{52} + 6\alpha^{53} - 6\alpha^{54} + 6\alpha^{55} - 2\alpha^{56} + 2\alpha^{57} - 4\alpha^{58} + 4\alpha^{59} - 6\alpha^{60} + 4\alpha^{61} \\
& - 4\alpha^{62} + 2\alpha^{63}.
\end{aligned}$$

$$\begin{aligned}
M_2' = {} & -7 + 2\alpha^4 - 4\alpha^6 + 6\alpha^7 - 2\alpha^{10} - 2\alpha^{11} + 2\alpha^{13} - 4\alpha^{13} - 2\alpha^{14} - 4\alpha^{15} - 4\alpha^{17} + 2\alpha^{18} \\
& - 4\alpha^{19} - 2\alpha^{20} - 2\alpha^{21} + 2\alpha^{22} + 6\alpha^{25} - 4\alpha^{27} - 2\alpha^{28}.
\end{aligned}$$

$$\begin{aligned}M_3 = &-9 \quad +12\alpha \ -8\alpha^2 + \ 8\alpha^3 - 14\alpha^4 + 12\alpha^5 - 8\alpha^6 + \ 6\alpha^7 - 4\alpha^8\\ &+4\alpha^9 - 10\alpha^{10} + 6\alpha^{11} - \ 6\alpha^{12} + \ 8\alpha^{13} - \ 6\alpha^{14} + 8\alpha^{15} - \ 8\alpha^{16} + 8\alpha^{17}\\ &-6\alpha^{18} + \ 8\alpha^{19} - 6\alpha^{20} + 10\alpha^{21} - 10\alpha^{22} + 12\alpha^{23} - 4\alpha^{24} + 10\alpha^{25} - 8\alpha^{26}\\ &+4\alpha^{27} - 14\alpha^{28} + 8\alpha^{29} - \ 8\alpha^{30} + \ 4\alpha^{31}.\end{aligned}$$

$$\begin{aligned}M_3' = &-1 \quad +4\alpha \ -2\alpha^2 - 8\alpha^4 + 2\alpha^5 + 2\alpha^6 - 6\alpha^7 - 8\alpha^8 - 6\alpha^9 - 2\alpha^{10}\\ &+2\alpha^{11} + 8\alpha^{12} + 2\alpha^{14} + 4\alpha^{15}.\end{aligned}$$

$$\begin{aligned}M_4 = &-17 \quad +20\alpha \ -14\alpha^2 + 16\alpha^3 - 20\alpha^4 + 22\alpha^5 - 18\alpha^6 + 18\alpha^7 - 8\alpha^8\\ &+14\alpha^9 - 18\alpha^{10} + 10\alpha^{11} - 20\alpha^{12} + 16\alpha^{13} - 14\alpha^{14} + 12\alpha^{15}.\end{aligned}$$

$$M_4' = -9 + 6\alpha + 4\alpha^2 + 6\alpha^3 + 6\alpha^5 - 4\alpha^6 + 6\alpha^7.$$

$$M_5 = -25 + 34\alpha - 32\alpha^2 + 26\alpha^3 - 40\alpha^4 + 38\alpha^5 - 32\alpha^6 + 30\alpha^7.$$

$$M_5' = -15 - 4\alpha - 4\alpha^2.$$

$$M_6 = -65 + 72\alpha - 64\alpha^2 + 56\alpha^3.$$

$$M_6' = -1 + 16\alpha.$$

$$M_7 = -129 + 128\alpha.$$

$$M_7' = -257.$$

Ces différentes expressions étant trouvées, supposons que α soit une racine primitive, et représentons par F, F_1, F_2, ... F_7 ce que devient M', M_1', M_2' ... M_7' en substituant α, α^2, α^4, ... α^{128} au lieu de α ($F = M'$, $F_7 = -257$); nous aurons

$$(p_0 + \alpha p_1 + \alpha^2 p_2 \ldots + \alpha^{255} p_{255})^{256} = -F^{128} . F_1^{64} . F_2^{32} . F_3^{16} . F_4^{8} . F_5^{4} . F_6^{2} . F_7;$$

cette équation constitue la solution dont il s'agit.

Ajoutons encore les formules beaucoup plus simples qui correspondent à l'équation $x^{17}-1=0$. En supposant $\alpha^{16}-1=0$, nous aurons

$$\begin{aligned}M &= -2 - \alpha^4 + 2\alpha^5 + 2\alpha^7 - 2\alpha^8 + 2\alpha^9 - 2\alpha^{10} + 2\alpha^{13} - 2\alpha^{14}.\\ M' &= -2\alpha + 2\alpha^2 - \alpha^4 + 2\alpha^6 + 2\alpha^7.\\ M_1 &= -4 + 2\alpha - 2\alpha^3 - \alpha^4 + 4\alpha^5 - 2\alpha^6 + 2\alpha^7.\\ M_1' &= -3 - 2\alpha - 2\alpha^3.\\ M_2 &= -5 + 6\alpha - 4\alpha^2 + 2\alpha^3.\\ M_2' &= -1 + 4\alpha.\\ M_3 &= -9 + 8\alpha.\\ M_3' &= -17;\end{aligned}$$

et de là, en supposant que α soit une racine primitive:

$$\begin{aligned}&(p_0 + \alpha p_1 + \ldots + \alpha^{15} p_{15})^{16}\\ &\qquad = (-2\alpha + 2\alpha^2 - \alpha^4 + 2\alpha^6 + 2\alpha^7)^8 . (-3 - 2\alpha^2 - 2\alpha^6)^4 . (-1 + 4\alpha^4)^2 . 17.\end{aligned}$$

Pour $x^5 - 1 = 0$ on obtient sans peine

$$(p_0 + \alpha p_1 + \alpha^2 p_2 + \alpha^3 p_3)^4 = (-2 - \alpha^2 + 2\alpha^3)^2 . 5;$$

où α est une racine primitive de l'équation $\alpha^4 - 1 = 0$, savoir $\alpha = \pm i$.

98.

NOTE RELATIVE À LA SIXIÈME SECTION DU "MÉMOIRE SUR QUELQUES THÉORÈMES DE LA GÉOMÉTRIE DE POSITION."

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), p. 84.]

This Note which is thus entitled in Crelle, but which in fact refers to the seventh section of the Memoir, is inserted in its proper place, *ante* p. 555.

99.

NOTE SUR QUELQUES FORMULES QUI SE RAPPORTENT À LA MULTIPLICATION DES FONCTIONS ELLIPTIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLI. (1851), pp. 85—92.]

En revenant sur l'équation

$$\begin{aligned}&\{-l\lambda - m\mu + (l-m)^2\} P_{l,m}\\ &+ l(\lambda - 2l + 2m + 2)(\lambda - 2l + 2m + 1) P_{l-1,m}\\ &+ m(\mu + 2l - 2m + 2)(\mu + 2l - 2m + 1) P_{l,m-1}\\ &\quad - 16lm\{\lambda\mu - (2l + 2m - 4)(\lambda + \mu)\} P_{l-1,m-1} = 0,\end{aligned}$$

dans laquelle $P_{0,0} = 1$, les expressions que j'ai données pour $P_{l,0}$, $P_{l,1}$ [93, see p. 535] peuvent être écrites comme suit:

$$P_{l,0} = \lambda[\lambda - l - 1]^{l-1},$$

$$P_{l,1} = \mu\lambda[\lambda - l - 1]^{l-1} + l\lambda[\lambda - l]^{l-2}\left(18l - 16 + \frac{2(l-1)(l-2)}{\lambda - l}\right) + \frac{\lambda\mu}{\lambda + \mu}(-10l\lambda[\lambda - l]^{l-2}):$$

équations qui peuvent être représentées par

$$P_{l,0} = Q_{l,0},$$

$$P_{l,1} = Q_{l,1} + R_{l,1;1}\frac{\lambda\mu}{\lambda + \mu},$$

ce qui conduisent à la forme

$$P_{l,2} = Q_{l,2} + R_{l,2;1}\frac{\lambda\mu}{\lambda + \mu} + R_{l,2;2}\frac{\lambda^2\mu^2}{(\lambda + \mu)^2},$$

où les coefficients R ne contiennent que la seule quantité λ, et où $Q_{l,2}$ est une fonction intégrale du $l^{\text{ième}}$ ordre par rapport à λ, et du second ordre par rapport à μ.

Cela donne

$$\{-l\lambda-2\mu+(l-2)^2\}\left\{Q_{l,2}+R_{l,2;1}\frac{\lambda\mu}{\lambda+\mu}+R_{l,2;2}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2}\right\}$$

$$+l(\lambda-2l+6)(\lambda-2l+5)\left\{Q_{l-1,2}+R_{l-1,2;1}\frac{\lambda\mu}{\lambda+\mu}+R_{l-1,2;2}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2}\right\}$$

$$+2(\mu+2l-2)(\mu+2l-3)\left\{Q_{l,1}+R_{l,1;1}\frac{\lambda\mu}{\lambda+\mu}\right\}$$

$$-32l\{\lambda\mu-2l(\lambda+\mu)\}\left\{Q_{l-1,1}+R_{l-1,1;1}\frac{\lambda\mu}{\lambda+\mu}\right\}=0,$$

ce qui se réduit à

$$\{-l\lambda-2\mu+(l-2)^2\}Q_{l,2}-(l-2)(\lambda-l+2)\left\{R_{l,2;1}\frac{\lambda\mu}{\lambda+\mu}+R_{l,2;2}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2}\right\}$$

$$-2\lambda\mu R_{l,2;1}-2\lambda^2\mu R_{l,2;2}+2\lambda^2R_{l,2;2}\frac{\lambda\mu}{(\lambda+\mu)}$$

$$+l(\lambda-2l+6)(\lambda-2l+5)\left\{Q_{l-1,2}+R_{l-1,2;1}\frac{\lambda\mu}{\lambda+\mu}+R_{l-1,2;2}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2}\right\}$$

$$+2(\mu+2l-2)(\mu+2l-3)Q_{l,1}+2(\mu-\lambda+4l-5)\lambda\mu R_{l,1;1}$$

$$+2(\lambda-2l+2)(\lambda-2l+3)R_{l,1;1}\frac{\lambda\mu}{(\lambda+\mu)}$$

$$-32l\{\lambda\mu-2l(\lambda+\mu)\}Q_{l-1,1}-32l\lambda\mu(\lambda-2l)R_{l-1,1;1}+32l\lambda^2R_{l-1,1;1}\frac{\lambda\mu}{\lambda+\mu}=0.$$

En ne faisant attention d'abord qu'aux termes qui contiennent des puissances négatives de $\lambda+\mu$, nous obtenons

$$-(l-2)(\lambda-l+2)R_{l,2;2}+l(\lambda-2l+6)(\lambda-2l+5)R_{l-1,2;2}=0$$

et

$$-(l-2)(\lambda-l+2)R_{l,2;1}+l(\lambda-2l+6)(\lambda-2l+5)R_{l-1,2;1}$$

$$+2(\lambda-2l+2)(\lambda-2l+3)R_{l,1;1}+32l\lambda^2R_{l-1,1;1}+2\lambda^2R_{l,2;2}=0.$$

La première équation, en calculant la constante arbitraire au moyen de $R_{2,2,2}=200$, donne

$$R_{l,2;2}=100\,l(l-1)\lambda[\lambda-l+1]^{l-3};$$

l'expression de $R_{2,2,2}=200$ se trouve par celle de $P_{2,2}$ qui peut être écrite sous la forme

$$P_{2,2}=\lambda(\lambda-3)\mu(\mu-3)+152\lambda\mu+336-40\lambda^2\mu+(40\lambda^2-1156)\frac{\lambda\mu}{\lambda+\mu}+\frac{200\lambda^2\mu^2}{(\lambda+\mu)^2}.$$

En substituant la valeur de $R_{l,2;2}$ et celles de

$$R_{l,1;1}=-10l\lambda[\lambda-l]^{l-3},\qquad R_{l-1,1;1}=-10(l-1)\lambda[\lambda-l+1]^{l-3}$$

dans la seconde équation, on obtient

$$-(l-2)(\lambda-l+2)R_{l,2;1}+l(\lambda-2l+6)(\lambda-2l+5)R_{l-1,2;1}$$
$$-20l\lambda(\lambda-2l+3)[\lambda-l]^{l-1}-120l(l-1)\lambda^2[\lambda-l+1]^{l-2}=0,$$

c'est à dire

$$-(l-2)(\lambda-l+2)R_{l,2;1}+l(\lambda-2l+6)(\lambda-2l+5)R_{l-1,2;1}$$
$$-20l\lambda[\lambda-l]^{l-4}\{6(l-1)\lambda^2(\lambda-l+1)+(\lambda-2l+4)(\lambda-2l+3)^2(\lambda-2l+2)\}.$$

Mettons

$$R_{l,2;2}=l(l-1)\lambda[\lambda-l+1]^{l-3}\Psi_l;$$

cela donne

$$\Psi_l-\Psi_{l-1}=\frac{-20}{(l-1)(l-2)(\lambda-l+1)(\lambda-l+2)}\{6(l-1)\lambda^2(\lambda-l+1)$$
$$+(\lambda-2l+4)(\lambda-2l+3)^2(\lambda-2l+2)\},$$

ce qui devient, quelques réductions faites,

$$\Psi_l-\Psi_{l-1}=\frac{-20}{(l-1)(l-2)}\{(\lambda+1)(\lambda+2)+17(l-1)(l-2)\}$$
$$-\frac{20(l-2)(l-3)}{\lambda-l+1}+\frac{20(l-2)(l-4)}{\lambda-l+2},$$

et de là on tire

$$\Psi_l=C-340l+\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(l-2)(l-3)}{\lambda-l+1}.$$

Or $R_{2,2;1}=2\Psi_2=40\lambda^2-1156$; donc $\Psi_2=20\lambda^2-578$, et de là $C=62-60\lambda$; donc enfin, en restituant la valeur de $R_{l,2;1}$,

$$R_{l,2;1}=l(l-1)\lambda[\lambda-l+1]^{l-3}\left\{62-340l-60\lambda+\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(l-2)(l-3)}{\lambda-l+1}\right\}.$$

Passons à l'expression de $Q_{l,2}$. Elle donne

$$\{-l\lambda-2\mu+(l-2)^2\}Q_{l,2}+l(\lambda-2l+6)(\lambda-2l+5)Q_{l-1,2}$$
$$+2(\mu+2l-2)(\mu+2l-3)Q_{l,1}\qquad-32l\{\lambda\mu-2l(\lambda+\mu)\}Q_{l-1,1}$$
$$+2(\mu-\lambda+4l-5)\lambda\mu R_{l,1;1}$$
$$-2\lambda\mu R_{l,2;1}-2\lambda^2\mu R_{l,2;2}-32l\lambda\mu(\lambda-2l)R_{l-1,1;1}=0;$$

où la dernière ligne se réduit à

$$-2l(l-1)\mu\lambda^2[\lambda-l+1]^{l-3}\left\{62-20l-120\lambda-\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(l-1)(l-2)}{\lambda-l+1}\right\}.$$

Donc on a

$$\begin{aligned}
&\{l\lambda+2\mu-(l-2)^2\}\,Q_{l,2}-l(\lambda-2l+6)(\lambda-2l+5)\,Q_{l-1,2}\\
&=2(\mu+2l-2)(\mu+2l-3)\left\{\mu\lambda[\lambda-l-1]^{l-1}+l\lambda[\lambda-l]^{l-2}\left(18l-16+\frac{2(l-1)(l-2)}{\lambda-l}\right)\right\}\\
&-32l\{\lambda\mu-2l(\lambda+\mu)\}\left\{\mu\lambda[\lambda-l]^{l-2}+(l-1)\lambda[\lambda-l+1]^{l-3}\left(18l-34+\frac{2(l-2)(l-3)}{\lambda-l+1}\right)\right\}\\
&-20l(4l-5+\mu-\lambda)\lambda^2\mu[\lambda-l]^{l-2}\\
&-2l(l-1)\lambda^2\mu[\lambda-l+1]^{l-3}\left\{62-20l-120\lambda+\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(\lambda-1)(\lambda-2)}{\lambda-l+1}\right\},
\end{aligned}$$

équation qui peut être représentée par

$$\{l\lambda+2\mu-(l-2)^2\}\,Q_{l,2}-l(\lambda-2l+6)(\lambda-2l+5)\,Q_{l-1,2}=\mathfrak{A}\mu^3+\mathfrak{B}\mu^2+\mathfrak{C}\mu+\mathfrak{D},$$

où les valeurs de $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ sont données par les formules

$$\mathfrak{A}=2\lambda[\lambda-l-1]^{l-1},$$

$$\begin{aligned}
\mathfrak{B}&=2(4l-5)\lambda[\lambda-l-1]^{l-1}+2l\lambda[\lambda-l]^{l-2}\left(18l-16+\frac{2(l-1)(l-2)}{\lambda-l}\right)\\
&\quad-32l\lambda(\lambda-2l)[\lambda-l]^{l-2}-20l\lambda^2[\lambda-l]^{l-2},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}&=2(2l-2)(2l-3)\lambda[\lambda-l-1]^{l-1}\\
&\quad+2(4l-5)l\lambda[\lambda-l]^{l-2}\left\{18l-16+\frac{2(l-1)(l-2)}{\lambda-l}\right\}\\
&\quad+64l^2\lambda^2[\lambda-l]^{l-2}\\
&\quad-32l(l-1)(\lambda-2l)\lambda[\lambda-l+1]^{l-3}\left\{18l-34+\frac{2(l-2)(l-3)}{\lambda-l+1}\right\}\\
&\quad-20l(4l-5-\lambda)\lambda^2[\lambda-l]^{l-2}\\
&\quad-2l(l-1)\lambda^2[\lambda-l+1]^{l-3}\left\{62-20l-120\lambda+\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(l-1)(l-2)}{\lambda-l+1}\right\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{D}&=2(2l-2)(2l-3)l\lambda[\lambda-l]^{l-2}\left\{18l-16+\frac{2(l-1)(l-2)}{\lambda-l}\right\}\\
&\quad+64l^2(l-1)\lambda^2[\lambda-l+1]^{l-3}\left\{18l-34+\frac{2(l-2)(l-3)}{\lambda-l+1}\right\}.
\end{aligned}$$

Sans m'arrêter à réduire ces expressions aux formes les plus simples, j'écris

$$Q_{l,2}=\mu(\mu-3)\lambda[\lambda-l-1]^{l-1}+\mu I_l+J_l;$$

en substituant cette valeur, les termes qui contiennent μ^3 se détruisent, et la comparaison des autres termes donne

$$\begin{aligned}
2I_l-6\lambda[\lambda-l-1]^{l-1}+\{l\lambda-(l-2)^2\}\lambda[\lambda-l-1]^{l-1}&\\
-l(\lambda-2l+6)(\lambda-2l+5)\lambda[\lambda-l]^{l-2}-\mathfrak{B}&=0,
\end{aligned}$$

$$2J_l + \{l\lambda - (l-2)^2\}\{-3\lambda[\lambda - l - 1]^{l-1} + I_l\}$$
$$- l(\lambda - 2l + 6)(\lambda - 2l + 5)\{-3\lambda[\lambda - l]^{l-2} + I_{l-1}\} - \mathfrak{C} = 0,$$

$$\{l\lambda - (l-2)^2\} J_l - l(\lambda - 2l + 6)(\lambda - 2l + 5) J_{l-1} - \mathfrak{D} = 0.$$

Les valeurs de I_l et J_l peuvent être tirées sans intégration de la première et de la seconde de ces équations. La valeur ainsi trouvée de J_l satisfera à la troisième équation (ce qui cependant doit être vérifié a posteriori). Il m'a paru plus simple de tirer la fonction I_l de la première équation, et celle de J_l en intégrant la troisième équation; alors ce sera la seconde équation qu'il y a à vérifier. En effet on obtient

$$I_l = l\lambda[\lambda - l]^{l-2}\left\{36l + 4 - 20\lambda + \frac{4(l-1)(l-2)}{\lambda - l}\right\}.$$

L'équation qui sert à déterminer J_l devient, en substituant la valeur de $\mathfrak{D}$,

$$\{l\lambda - (l-2)^2\} J_l - l(\lambda - 2l + 6)(\lambda - 2l + 5) J_{l-1}$$
$$= 4l(l-1)(2l-3)\lambda[\lambda - l]^{l-2}\left\{18l - 16 + \frac{2(l-1)(l-2)}{\lambda - l}\right\}$$
$$+ 64l^2(l-1)\lambda^2[\lambda - l + 1]^{l-3}\left\{18l - 34 + \frac{2(l-2)(l-3)}{\lambda - l + 1}\right\}.$$

En écrivant

$$J_l = l(l-1)\lambda[\lambda - l + 1]^{l-3} V_l,$$

on trouve

$$\{l\lambda - (l-2)^2\} V_l - (l-2)(\lambda - l + 2) V_{l-1}$$
$$= 8(2l-3)\frac{(\lambda - 2l + 3)(\lambda - 2l + 4)}{\lambda - l + 1}\left\{9l - 8 + \frac{(l-1)(l-2)}{\lambda - l}\right\}$$
$$+ 128l\lambda\left\{9l - 17 + \frac{(l-2)(l-3)}{\lambda - l + 1}\right\}.$$

En faisant

$$V_l = M_l + \frac{A_l}{\lambda - l + 1} + \frac{B_l}{(\lambda - l + 1)(\lambda - l)},$$

cette équation se réduit à

$$\{l\lambda - (l-2)^2\} M_l - (l-2)(\lambda - l + 2) M_{l-1}$$
$$+ lA_l - (l-2) A_{l-1} + \frac{(3l-4) A_l + lB_l - (l-2) B_{l-1}}{\lambda - l + 1} + \frac{4(l-1) B_l}{(\lambda - l + 1)(\lambda - l)}$$
$$= 8(2l-3)(9l-8)\left\{(\lambda - 3l + 6) - \frac{(l-2)(l-3)}{\lambda - l + 1}\right\}$$
$$+ 8(2l-3)(l-1)(l-2)\left\{1 - \frac{2(l-3)}{\lambda - l + 1} + \frac{(l-3)(l-4)}{(\lambda - l + 1)(\lambda - l)}\right\}$$
$$+ 128l\lambda(9l - 17) + 128\frac{l(l-1)(l-2)(l-3)}{\lambda - l + 1},$$

et cela donne tout de suite la valeur de B_l, et après quelques réductions celle de A_l.

On obtient ainsi

$$A_l = 2(l-2)(l-3)(36l+7),$$
$$B_l = 2(l-2)(l-3)(l-4)(2l-3).$$

En substituant ces valeurs, on a, toute réduction faite:

$$\{l\lambda-(l-2)^2\}\,M_l-(l-2)(\lambda-l+2)\,M_{l-1}=8\lambda(162l^2-315l+24)-24(l-2)^2(27l-26),$$

c'est à dire

$$lM_l-(l-2)\,M_{l-1}=1296l^3-2520l+192,$$
$$M_l-M_{l-1} \quad = \quad 648l-624,$$

ou enfin

$$M_l \;=324l^2-300l-528,$$
$$M_{l-1}=324l^2-948l+192:$$

valeurs qui (comme cela doit être) se changent l'une dans l'autre en entrechangeant les quantités l, $l-1$.

De là on obtient

$$J_l=l(l-1)\lambda[\lambda-l+1]^{l-3}\left\{324l^2-300l-528+\frac{2(l-2)(l-3)(36l+7)}{\lambda-l+1}\right.$$
$$\left.+\frac{2(2l-3)(l-2)(l-3)(l-4)}{(\lambda-l+1)(\lambda-l)}\right\},$$

valeur qu'on trouverait aussi par l'autre procédé indiqué ci-dessus. Or nous avons

$$Q_{l,2}=\mu(\mu-3)\lambda[\lambda-l-1]^{l-1}+\mu I_l+J_l,$$
$$P_{l,2}=Q_{l,2}+R_{l,2;1}\frac{\lambda\mu}{\lambda+\mu}+R_{l,2;2}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2};$$

donc enfin, en réunissant les valeurs des différentes parties de $P_{l,2}$, on obtient

$$P_{l,2}=\mu(\mu-3)\lambda[\lambda-l-1]^{l-1}$$
$$+\mu l\lambda[\lambda-l]^{l-2}\left\{36l+4-20\lambda+\frac{4(l-1)(l-2)}{\lambda-l}\right\}$$
$$+l(l-1)\lambda[\lambda-l+1]^{l-3}\left\{324l^2-300l-528+\frac{2(l-2)(l-3)(36l+7)}{\lambda-l+1}\right.$$
$$\left.+\frac{2(2l-3)(l-2)(l-3)(l-4)}{(\lambda-l+1)(\lambda-l)}\right\}$$
$$+l(l-1)\lambda[\lambda-l+1]^{l-3}\left\{62-340l-60\lambda+\frac{20(\lambda+1)(\lambda+2)}{l-1}-\frac{20(l-2)(l-3)}{\lambda-l+1}\right\}\frac{\lambda\mu}{\lambda+\mu}$$
$$+100l(l-1)\lambda[\lambda-l+1]^{l-3}\frac{\lambda^2\mu^2}{(\lambda+\mu)^2},$$

équation qui fait suite aux équations

$$P_{l,1} = \mu\lambda\,[\lambda - l - 1]^{l-1} + l\lambda\,[\lambda - l]^{l-2}\left\{18l - 16 + \frac{2(l-1)(l-2)}{\lambda - l}\right\} - 10l\lambda\,[\lambda - l]^{l-2} \cdot \frac{\lambda\mu}{\lambda + \mu},$$

$$P_{l,0} = \lambda\,[\lambda - l - 1]^{l-1}.$$

Je vais essayer maintenant à chercher d'une manière plus systématique les termes de $P_{l,m}$ qui ne contiennent pas la quantité μ, ou bien à chercher la solution de l'équation

$$\begin{aligned} &\{-l\lambda + (l-m)^2\}\,P_{l,m} \\ &+ l\,(\lambda - 2l + 2m + 2)(\lambda - 2l + 2m + 1)\,P_{l-1,m} \\ &+ 2m\,(l - m + 1)(2l - 2m + 1)\,P_{l,m-1} \\ &+ 32lm\,(l + m - 2)\,P_{l-1,m-1} = 0. \end{aligned}$$

En supposant que

$$P_{l,m} = [l]^m \lambda\,[\lambda - l + m - 1]^{l-m-1} \Sigma \frac{A_{l,m,p}}{[\lambda - l + m - 1]^p}$$

(où la sommation se rapporte à p, nombre qui doit être étendu depuis $p = 0$ jusqu'à $p = m$), on obtiendra sans peine

$$\begin{aligned} &\{-l\lambda + (l-m)^2\}\,\Sigma \frac{A_{l,m,p}}{[\lambda - l + m - 1]^p} \\ &+ (l - m)\,\Sigma \frac{A_{l-1,m,p}}{[\lambda - l + m - 1]^{p-1}} \\ &+ 2m\,(2l - 2m + 1)[\lambda - 2l + 2m]^2\,\Sigma \frac{A_{l,m-1,p}}{[\lambda - l + m - 1]^{p+1}} \\ &+ 32lm\,(l + m - 2)\,\lambda\,\Sigma \frac{A_{l-1,m-1,p}}{[\lambda - l + m - 1]^p} = 0; \end{aligned}$$

où p s'étend seulement jusqu'à $m - 1$ dans la troisième et dans la quatrième ligne.

Pour réduire la première ligne, j'écris

$$-l\lambda + (l-m)^2 = -l\,(\lambda - l + m - p) + [m^2 - (m + p)\,l],$$

ce qui réduit le terme général à

$$\frac{-lA_{l,m,p}}{[\lambda - l + m - 1]^{p-1}} + \frac{[m^2 - (m + p)\,l]A_{l,m,p}}{[\lambda - l + m - 1]^p}:$$

expression qui, en écrivant $r+1$ au lieu de p dans le premier terme, et r dans le second terme, peut être remplacée par

$$(\alpha)\qquad \frac{-lA_{l,m,r+1}}{[\lambda-l+m-1]^r}+\frac{\{m^2-(m+r)\,l\}\,A_{l,m,r}}{[\lambda-l+m-1]^r}.$$

La seconde ligne donne tout de suite le terme général

$$(\beta)\qquad \frac{(l-m)\,A_{l-1,m,r+1}}{[\lambda-l+m-1]^r}.$$

Pour réduire la troisième ligne, je mets

$$[\lambda-2l+2m]^2=[\lambda-l+m-p]^2-2\,[l-m-p+1]^1\,[\lambda-l+m-p-1]^1+[l-m-p]^2,$$

ce qui réduit le terme général à

$$2m\,(2l-2m+1)\frac{A_{l,m-1,p}}{[\lambda-l+m-1]^{p-1}}-4m\,(2l-2m+1)\frac{[l-m-p+1]^1\,A_{l,m-1,p}}{[\lambda-l+m-1]^p}$$
$$+\frac{2m\,(2l-2m+1)\,[l-m-p]^2\,A_{l,m-1,p}}{[\lambda-l+m-1]^{p+1}}:$$

expression qui (en écrivant $r+1$ au lieu de p dans le premier terme, r dans le second terme et $r-1$ dans le troisième terme) peut être remplacée par

$$(\gamma)\quad 2m\,(2l-2m+1)\frac{A_{l,m-1,r+1}}{[\lambda-l+m-1]^r}-4m\,(2l-2m+1)\frac{[l-m-r+1]^2\,A_{l,m-1,r}}{[\lambda-l+m-1]^r}$$
$$+\frac{2m\,(2l-2m+1)\,[l-m-r+1]^2\,A_{l,m-1,r-1}}{[\lambda-l+m-1]^r}.$$

Pour réduire la quatrième ligne, je mets

$$\lambda=(\lambda-l+m-p)+(l-m+p),$$

ce qui réduit le terme général à

$$32lm\,(l+m-2)\frac{A_{l-1,m-1,p}}{[\lambda-l+m-1]^{p-1}}+32lm\,(l+m-2)\frac{(l-m+p)\,A_{l-1,m-1,p}}{[\lambda-l+m-1]^p},$$

expression qui (en écrivant $r+1$ au lieu de p dans le premier et r dans le second terme) peut être remplacée par

$$(\delta)\quad 32lm\,(l+m-2)\frac{A_{l-1,m-1,r+1}}{[\lambda-l+m-1]^r}+32lm\,(l+m-2)\frac{(l-m+p)\,A_{l-1,m-1,r}}{[\lambda-l+m-1]^r}.$$

Donc en réunissant les expressions (α), (β), (γ), (δ) et en faisant attention que le coefficient du terme qui contient $[\lambda - l + m - 1]^r$ au dénominateur, doit se réduire à zéro, on obtient, en arrangeant encore les termes d'une manière convenable:

$$\begin{aligned}
&2m(2l-2m+1)A_{l,m-1,r+1}\\
&+32lm(l+m-2)A_{l-1,m-1,r+1}\\
&-lA_{l,m,r+1}\\
&+(l-m)A_{l-1,m,r+1}\\
-&4m(2l-2m+1)(l-m-r+1)A_{l,m-1,r}\\
+&32lm(l+m-2)(l+m-r)A_{l-1,m-1,r}\\
&+(m^2-(m+r)l)A_{l,m,r}\\
+&2m(2l-2m+1)(l-m-r+1)(l-m-r)A_{l,m-1,r-1}=0,
\end{aligned}$$

où r s'étend depuis $r=0$ jusqu'à $r=m$, en réduisant à zéro les termes pour lesquels le troisième suffixe est négatif ou plus grand que le second suffixe. Par exemple dans le cas de $r=m$, on obtient l'équation très simple

$$\begin{aligned}
&(2l-m)A_{l,m,m}\\
-&2(2l-2m+1)(l-2m+1)(l-2m)A_{l,m-1,m-1}=0
\end{aligned}$$

qui (sous la condition $A_{l,0,0}=1$) donne sans peine la valeur générale de $A_{l,m,m}$, savoir:

$$A_{l,m,m}=[2l-m-1]^m[l-m-1]^m:$$

valeur qui peut être présentée sous d'autres formes en considérant à part les deux cas de m pair et de m impair. En supposant $m=1$, $m=2$, on obtient

$$A_{l,1,1}=2(l-1)(l-2),\quad A_{l,2,2}=2(2l-3)(l-2)(l-3)(l-4):$$

valeurs qui servent à vérifier des résultats déjà trouvés.

100.

NOTE SUR LA THÉORIE DES HYPERDÉTERMINANTS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLII. (1851), pp. 368—371.]

DANS la théorie dont il s'agit, je suis parvenu à un théorème qui pourra, à ce qu'il me paraît, conduire à des développements intéressants.

Je ne considère ici que le cas d'une fonction homogène à *deux variables*, et en me servant des nouveaux termes de M. Sylvester, je nomme *Covariant* d'une fonction donnée, toute fonction qui ne change pas de forme en faisant subir aux variables des transformations linéaires quelconques, et *Invariant* toute fonction des seuls coefficients qui a la propriété mentionnée.

Cela posé, soit U une fonction donnée quelconque, du degré n par rapport aux variables, et, comme à l'ordinaire, contenant des coefficients arbitraires a, b, c, &c. Soit Q un *covariant* quelconque (y compris le cas particulier où Q est un *invariant*) de la fonction U, s le degré de Q par rapport aux variables, r le degré de cette même fonction Q par rapport aux coefficients. En supposant que la fonction U ait un facteur θ^ν (où $\theta = lx + my$ est une fonction linéaire des variables), ou autrement dit, en supposant l'équation $U = \theta^\nu V$, je dis que le *covariant* Q contiendra ce même facteur θ élevé à la puissance $r\nu - \frac{1}{2}(rn - s)$.

En effet, en se rappelant la méthode dont je me suis servi dans la seconde partie de mon mémoire sur les Hyperdéterminants (t. XXX. de ce Journal, [16]) (je suppose que le lecteur ait ce mémoire sous les yeux), on verra que cette fonction Q, supposée, comme plus haut, du degré r par rapport aux coefficients, sera nécessairement de la forme

$$Q = \overline{12}^\alpha\, \overline{13}^\beta\, \overline{23}^\gamma \ldots U_1 U_2 \ldots U_r,$$

puisque les coefficients n'entrent dans Q que par les fonctions U_1, U_2, &c. Or Q étant du degré s par rapport aux variables, on obtient $s = rn - 2(\alpha + \beta + \gamma + \ldots)$, c'est à dire:

$$\alpha + \beta + \gamma \ldots = \tfrac{1}{2}(rn - s).$$

Cela posé, puisque $U = \theta^\nu V$, on aura de même $U_1 = \theta_1^\nu V_1$, $U_2 = \theta_2^\nu V_2$, &c. Les expressions $\overline{12}$, &c. qui entrent dans l'expression de Q contiennent ∂_{x_1}, ∂_{y_1}, &c.: symboles

qui doivent être remplacés par $\partial_{x_1}+l\partial_{\theta_1}$, $\partial_{y_1}+m\partial_{\theta_1}$, &c. en supposant (comme il est permis) que les nouveaux symboles ∂_{x_1}, ∂_{y_1}, &c. ne se rapportent plus à $\theta_1^\nu V_1$, &c., mais seulement à V_1, &c. Cela donne

$$\begin{aligned}\overline{12} &= (\partial_{x_1}+l\partial_{\theta_1})(\partial_{y_2}+m\partial_{\theta_2})-(\partial_{x_2}+l\partial_{\theta_2})(\partial_{y_1}+m\partial_{\theta_1})\\ &= \partial_{x_1}\partial_{y_2}-\partial_{x_2}\partial_{y_1}+(l\partial_{y_2}-m\partial_{x_2})\partial_{\theta_1}-(l\partial_{y_1}-m\partial_{x_1})\partial_{\theta_2};\end{aligned}$$

c'est à dire: $\overline{12}$ est une fonction linéaire par rapport à ∂_{θ_1}, ∂_{θ_2}, et il en sera de même pour les expressions analogues $\overline{13}$, $\overline{23}$, &c.: donc le nombre des différentiations par rapport aux quantités θ_1, θ_2, &c., prises ensemble, ne surpasse pas $\alpha+\beta+\gamma+\ldots$, ou $\frac{1}{2}(rn-s)$. Or l'expression à différentier contient le facteur $\theta_1^\nu\theta_2^\nu\ldots\theta_r^\nu$; donc, en remettant, après les différentiations, θ au lieu de θ_1, $\theta_2,\ldots\theta_r$, la fonction Q contiendra le facteur θ élévé à la puissance $r\nu-\frac{1}{2}(rn-s)$.

Tout cela suppose implicitement que l'on ait $r\nu-\frac{1}{2}(rn-s)\overline{\overline{<}}s$. Or le même raisonnement, modifié très peu, fait voir aussi que pour

$$r\nu-\tfrac{1}{2}(nr-s)>s,$$

ou plus simplement pour

$$r(\nu-\tfrac{1}{2}n)>\tfrac{1}{2}s,$$

la fonction Q doit s'évanouir d'elle-même, savoir en établissant entre les coefficients de U les relations qui expriment l'existence du facteur θ^r. De là on tire le théorème suivant:

Étant donnée une fonction U du degré n, tout *covariant* du degré r par rapport aux coefficients et du degré s par rapport aux variables, s'évanouit en supposant que la fonction U ait un facteur θ pour lequel $r(\nu-\frac{1}{2}n)>\frac{1}{2}s$; et en particulier:

Un *invariant* quelconque de la fonction U s'évanouit en supposant que la fonction U ait un facteur θ^ν, pour lequel $\nu>\frac{1}{2}n$.

En mettant $n=2m$ ou $2m+1$, l'*invariant* s'évanouit en supposant que U ait le facteur θ^{m+1}.

Les conditions pour que la fonction U ait un tel facteur, se trouvent en égalant à zéro les coefficients différentiels de U du $m^{\text{ième}}$ ordre par rapport aux variables x, y, et en éliminant ces variables.

Mais avant d'aller plus loin il convient d'entrer dans quelques détails de la théorie d'une telle élimination. Je prends l'exemple le plus simple, et je suppose que l'on ait à éliminer x, y des équations

$$\begin{aligned}ax+by&=0,\\ bx+cy&=0,\\ cx+dy&=0.\end{aligned}$$

On est habitué à dire que ce système équivaut à *deux équations* entre les seuls coefficients: mais cela n'est juste que dans un sens qui manque de precision. Le système équivaut plutôt à *deux relations* entre les coefficients, et ces deux relations sont exprimées par les *trois équations*, $bd-c^2=0$, $bc-ad=0$, $ac-b^2=0$. Il n'est pas

vrai que deux de ces équations embrassent *nécessairement* la troisième. En effet, la première et la seconde équations sont satisfaites en écrivant $c=0$, $d=0$, mais ces valeurs sont absolument étrangères à la question, et ne satisfont pas à la troisième équation, de manière que toutes les trois équations sont nécessaires pour exprimer les relations entre les coefficients. C'est pourquoi je dis que ces trois équations sont des résultats *distincts* de l'élimination. Et de même, pour un système quelconque d'équations, le nombre des résultats distincts de l'élimination n'est pas généralement à beaucoup près si faible que le nombre des relations entre les coefficients. Qu'on veuille consulter sur ce sujet mon mémoire "On the order of certain systems of algebraical equations," *Cambridge and Dublin Mathematical Journal*, t. IV. [1849] pp. 132—137 [77], et le mémoire de M. Salmon "On the Classification of curves of double curvature," t. V. [1850] pp. 23—46.

Je reviens à l'objet de cette note, et je suppose qu'en égalant à zéro les coefficients différentiels du $m^{\text{ième}}$ ordre de la fonction U, les équations $P=0$, $Q=0$, $R=0$, &c. forment le système entier des résultats distincts de l'élimination. Un *invariant* quelconque I s'évanouira en supposant $P=0$, $Q=0$, $R=0$, &c. Il doit donc exister une équation telle que $\lambda I=\alpha P+\beta Q+\gamma R+\ldots$, où λ, α, β, $\gamma\ldots$ sont des fonctions rationnelles et intégrales des coefficients. Mais de plus, la fonction λ doit être purement numérique, ou ce qui est le même, doit se réduire à *l'unité*, car autrement $I=0$ serait un résultat de l'élimination différent des résultats $P=0$, $Q=0$, $R=0$, &c., et ces équations ne seraient plus le système entier des résultats distincts. Donc enfin: un *invariant* quelconque I sera exprimé par une équation telle que

$$I=\alpha P+\beta Q+\gamma R+\ldots,$$

α, β, γ, ... étant des fonctions intégrales et rationnelles des coefficients.

Les résultats que je viens d'obtenir s'accordent parfaitement avec ceux dans ma "Note sur les hyperdéterminants," t. XXXIV. [1847] pp. 148—152 [54]. En effet, j'y ai fait voir qu'en supposant qu'une fonction $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$ ait un facteur $(\alpha x+\beta y)^2$, l'élimination des variables entre les équations

$$\begin{aligned} ax^2+2bxy+cy^2&=0,\\ bx^2+2cxy+dy^2&=0,\\ cx^2+2dxy+ey^2&=0, \end{aligned}$$

donne lieu aux équations $ae-4bd+3c^2=0$, $ace+2bcd-ad^2-b^2e-c^3=0$; et les fonctions égalées à zero sont en effet les seuls *invariants* de la fonction du quatrième ordre. J'ajoute que la théorie actuelle fait voir aussi que dans le cas dont il s'agit, la dérivée

$$(ax^2+2bxy+cy^2)(cx^2+2dxy+ey^2)-(bx^2+2cxy+dy^2)^2,$$

ou son développement

$$(ac-b^2)x^4+2(ad-bc)x^3y+(ae+2bd-3c^2)x^2y^2+2(be-cd)xy^3+(ce-d^2)y^4,$$

se réduit (à un coefficient constant près) à $(\alpha x+\beta y)^4$: et au cas où la fonction donnée du quatrième ordre est supposée être un carré, cette fonction et la dérivée qui vient d'être écrite sont égales à un facteur constant près; resultat dont je me suis servi ailleurs.

NOTES AND REFERENCES.

1. As to the history of Determinants, see Dr Muir's "List of Writings on Determinants," *Quart. Math. Jour.* vol. XVII. (1882), pp. 110—149; and the interesting analyses of the earlier papers in course of publication by him in the *R. S. E. Proceedings*, vol. XIII. (1885—86) *et seq.*

The (new?) theorem for the multiplication of two determinants was given by Binet in his "Mémoire sur un système de formules analytiques &c." *Jour. École Polyt.* t. X. (1815), pp. 29—112.

An expression for the relation between the distances of five points in space, but not by means of a determinant or in a developed form, is given by Lagrange in the Memoir "Solutions analytiques de quelques problèmes sur les pyramides triangulaires," *Mém. de Berlin*, 1773: the question was afterwards considered by Carnot in his work "Sur la relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace, suivi d'un essai sur la théorie des transversales," 4to Paris, 1806. Carnot projected four of the points on a spherical surface having for its centre the fifth point, and then, from the relation connecting the cosines of the sides and diagonals of the spherical quadrilateral, deduced the relation between the distances of the five points: this is given in a completely developed form, containing of course a large number of terms.

Connected with the question we have the theorem given by Staudt in the paper "Ueber die Inhalte der Polygone und Polyeder," *Crelle* t. XXIV. (1842), pp. 252—256; the product of the volumes of two polyhedra is expressible as a rational and integral function of the distances of the vertices of the one from those of the other polyhedron.

More general determinant-formulæ relating to the "powers" of circles and spheres have been subsequently obtained by Darboux, Clifford and Lachlan: see in particular Lachlan's Memoir, "On Systems of Circles and Spheres," *Phil. Trans.* vol. CLXXVII. (1886), pp. 481—625.

2 and 3. The investigation was suggested to me by a passage in the *Mécanique Analytique*, Ed. 2 (1811), t. I. p. 113 (Ed. 3, p. 106); after referring to a formula of Laplace, whereby it appeared that the attraction of an ellipsoid on an exterior point depends only on the quantities $B^2 - A^2$ and $C^2 - A^2$ which are the squares of

the eccentricities of the two principal sections through the major semiaxis A, Lagrange remarks that, starting from this result and making use of a theorem of his own in the Berlin Memoirs 1792—93, he was able to construct the series by means of the development of the radical $1 \div \sqrt{x^2+y^2+z^2-2by-2cz+b^2+c^2}$ in powers of b, c, preserving therein only the even powers of b and c, and transforming a term such as $Hb^{2m}c^{2n}$ into a determinate numerical multiple of $\frac{4}{3}\pi ABC \,.\, H(B^2-A^2)^m(C^2-A^2)^n$.

It occurred to me that Lagrange's series must needs be a series

$$\Sigma\, \zeta_p \left(A^2\frac{d^2}{da^2}+B^2\frac{d^2}{db^2}+C^2\frac{d^2}{dc^2}\right)^p \phi(a,\ b,\ c),$$

reducible to his form as a function of B^2-A^2, C^2-A^2, in virtue of the equation $\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}+\frac{d^2}{dc^2}\right)\phi(a,\ b,\ c)=0$ satisfied by the function ϕ (I wrote this out some time before the Senate House Examination 1842, in an examination paper for my tutor, Mr Hopkins): and I was thus led to consider how the series in question could be transformed so as to identify it with the known expression for the attraction as a single definite integral.

I remark that my formulæ relate to the case of n variables: as regards ellipsoids the number of variables is of course $=3$: in the earlier solutions of the problem of the attraction of ellipsoids there is no ready method of making the extension from 3 to n. The case of n variables had however been considered in a most able manner by Green in his Memoir "On the determination of the exterior and interior attractions of Ellipsoids of variable densities," *Camb. Phil. Trans.* vol. v. 1835, pp. 395—430 (and Mathematical Papers, 8vo London, 1871, pp. 187—222); and in the Memoir by Lejeune-Dirichlet, "Sur une nouvelle méthode pour la détermination des intégrales multiples," *Liouv.* t. IV. (1839), pp. 164—168, although the case actually treated is that of three variables, the method can be at once extended to the case of any number of variables: it is to be noticed also that the methods of Green and Lejeune-Dirichlet are each applicable to the case of an integral involving an integer or fractional negative power of the distance. This is far more general than my formulæ, for in them the negative exponent for the squared distance is $=\frac{1}{2}n$, and, by differentiation in regard to the coordinates $a, b, \ldots$ of the attracted point, we can only change this into $\frac{1}{2}n+p$, where p is a positive integer. But in 28, the radical contained in the multiple integral is $\frac{1}{\{(a_1-x_1t)^2+\ldots\}^{\frac{1}{2}n-s}}$, where s is integer or fractional, and by a like process of expansion and summation I obtain a result depending on a single integral $\int_0^1 \frac{(1-u)^\sigma u^{i+\kappa-1}du}{\{(\zeta+h_1^2u)\ldots\}^{\frac{1}{2}}}$. And in 29, retaining throughout the general function $\phi(a_1-x_1, \ldots)$ and making the analogous transformation of the multiple integral itself, I express the integral

$$V=\int dx_1\ldots dx_n\ x_1^{2a_1+1}\ldots x_{f+1}^{2a_f+1}\ldots \phi(a_1-x_1,\ldots)$$

in terms of an integral $\int_0^1 T^{n+1}(1-T^2)^{k+f}WdT$, where $W=\int dx_1\ldots dx_n\,\phi(a_1-x_1T,\ldots)$.

I recall the fundamental idea of Lejeune-Dirichlet's investigation; starting with an integral $\iiint U dx dy dz$ over a given volume he replaces this by $\iiint \rho U dx dy dz$ where ρ is a discontinuous function, $=1$ for points inside, and $=0$ for points outside, the given volume; such a function is expressible as a definite integral (depending on the form of the bounding surface) in regard to a new variable θ: the limits for x, y, z, may now be taken to be ∞, $-\infty$ for each of the variables x, y, z, and it is in many cases possible to effect these integrations and thus to express the original multiple integral as a single definite integral in regard to θ.

I have not ascertained how far the wholly different method in Lejeune-Dirichlet's Memoir "Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque homogène ou hétérogène," *Crelle*, t. XXXII. (1846), pp. 80—84, admits of extension in regard to the number of variables, or the exponent of the radical.

4. As noticed p. 22, the investigation was suggested to me by Mr Greathead's paper, "Analytical Solutions of some problems in Plane Astronomy," *Camb. Math. Jour.* vol. I. (1839), pp. 182—187, giving the expression of the true anomaly in multiple sines of the mean anomaly. I am not aware that this remarkable expression has been elsewhere at all noticed except in a paper by Donkin, "On an application of the Calculus of Operations in the transformation of trigonometrical series," *Quart. Math. Journal*, vol. III. (1860), pp. 1—15; see p. 9, *et seq.*

5. In a terminology which I have since made use of:

The Postulandum or Capacity ($\square$) of a curve of the order r is $=\frac{1}{2}r(r+3)$; and the Postulation (∇) of the condition that the curve shall pass through k given points is in general $=k$.

If however the k points are the mn intersections of two given curves of the orders m and n respectively, and if r is not less than m or n, and not greater than $m+n-3$, then the postulation for the passage through the mn points, instead of being $=mn$, is $=mn-\frac{1}{2}(m+n-r-1)(m+n-r-2)$.

Writing $\gamma=m+n-r$, and $\delta=\frac{1}{2}(\gamma-1)(\gamma-2)$, the theorem may be stated in the form, a curve of the order r passing through $mn-\delta$ of the mn points of intersection will pass through the remaining δ points. The method of proof is criticised by Bacharach in his paper, "Ueber den Cayley'schen Schnittpunktsatz," *Math. Ann.* t. 26 (1886), pp. 275—299, and he makes what he considers a correction, but which is at any rate an important addition to the theorem, viz. if the δ points lie in a curve of the order $\gamma-3$, then the curve of the order r through the $mn-\delta$ points does not of necessity nor in general pass through the δ points. See my paper "On the Intersection of Curves," *Math. Ann.* t. XXX. (1887), pp. 85—90.

6. The formulæ in Rodrigues' paper for the transformation of rectangular coordinates afterwards presented themselves to me in connexion with Quaternions, see 20; and again in connexion with the theory of skew determinants, see 52.

8. A correction to the theorem (18), p. 42, is made in my paper "Notes on Lagrange's theorem," *Camb. and Dubl. Math. Jour.* vol. VI. (1851), pp. 37—45.

10. This paper is connected with 5, but it is a particular investigation to which I attach little value. The like remark applies to 40.

12. The second part of this paper, pp. 75—80, relates to the functions obtained from n columns of symbolical numbers in such manner as a determinant is obtained from 2 columns, and which are consequently sums of determinants: they are the functions which have since been called Commutants; the term is due to Sylvester.

13. In modern language: Boole (in his paper "Exposition of a general theory of linear transformations," *Camb. Math. Jour.* vol. III. (1843), pp. 1—20 and 106—119) had previously shown that a discriminant was an invariant; and Hesse in the paper "Ueber die Wendepunkte der Curven dritter Ordnung," *Crelle*, t. XXVIII. (1844), pp. 68—96, had established certain covariantive properties of the ternary cubic function. I first proposed in this paper the general problem of invariants (that is, functions of the coefficients, invariantive for a linear transformation of the facients), treating it by what may be called the "tantipartite" theory: the idea is best seen from the example p. 89, viz. for the tripartite function

$$U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2,$$

we have a function of the coefficients which is simultaneously of the forms

$$H\begin{vmatrix} a, & b, & c, & d \\ e, & f, & g, & h \end{vmatrix}, \quad H\begin{vmatrix} a, & b, & e, & f \\ c, & d, & g, & h \end{vmatrix}, \quad H\begin{vmatrix} a, & c, & e, & g \\ b, & d, & f, & h \end{vmatrix},$$

and as such it is invariantive for linear transformations of the (x_1, x_2), (y_1, y_2), (z_1, z_2).

Passing from the tantipartite form to a binary form, I obtained for the binary quartic the quadrinvariant $(I =)\ ae - 4bd + 3c^2$: as noticed at the end of the paper, the remark that there is also the cubinvariant $(J =)\ ace - ad^2 - b^2e - c^3 + 2bcd$ was due to Boole. The two functions present themselves, but without reference to the invariantive property and not in an explicit form, in Cauchy's Memoir "Sur la détermination du nombre des racines réelles dans les équations algébriques," *Jour. École Polyt.* t. X. (1815), pp. 457—548.

In p. 92 it is assumed that the invariant called θu is the discriminant of the function $U = ax_1y_1z_1w_1 \ldots + px_2y_2z_2w_2$: but, as mentioned in [ ], the assumption was incorrect. This was shown by Schläfli in his Memoir, "Ueber die Resultante eines Systemes mehrerer algebraischen Gleichungen," *Wiener Denks.* t. IV. Abth. 2 (1852), pp. 1—74: see pp. 35 *et seq.* The discriminant is there found by actual calculation to be a function (not of the order 6 as is θu, but) of the order 24, not breaking up into factors; in the particular case where the coefficients $a, \ldots p$ are equal, 1, 4, 6, 4, 1 of them to a, b, c, d, e respectively, in such wise that changing only the variables the function becomes $= (a, b, c, d, e \between x, y)^4$, then the discriminant in question does break

up into factors, the value in this case being $J^6(I^3 - 27J^2)$ of the order 24 as in the general case, but containing the factor $I^3 - 27J^2$ which is the discriminant of the binary quartic.

14. In this paper I developed what (to give it a distinctive name) may be called the "hyperdeterminant" theory, viz. the expressions considered are of the form

$$\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots U_1U_2U_3\ldots,$$

where after the differentiations the variables (x_1, y_1), (x_2, y_2), ... are to be or may be put equal to each other: it is to be noticed that although in the examples I chiefly consider constant derivatives, or invariants, the memoir throughout relates as well to covariants as invariants. The theory is to be distinguished from Gordan's process of Ueberschiebung, or derivational theory, viz. this may be considered as dealing exclusively, or nearly so, with the single class of derivatives $(V, W)^{\alpha}, = \overline{12}^{\alpha}V_1W_2$: the theorem that all the covariants of a binary function can be obtained successively by operating in this manner on the function itself and a covariant of the next inferior degree was a very important one.

15. Eisenstein's theorem may be stated as follows: the function $a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2$ (which is the discriminant of the binary cubic $(a, b, c, d \between x, y)^3$) is automorphic, viz. it is converted into a power of itself when for a, b, c, d we substitute the differential coefficients $\frac{d\phi}{da}, \frac{d\phi}{db}, \frac{d\phi}{dc}, \frac{d\phi}{dd}$ of the function itself. It is remarkable, see 54, that the function is automorphic in a different manner, viz. the Hessian determinant formed with the second differential coefficients $\frac{d^2\phi}{da^2}$, &c., is also equal to a power of the function itself. The first part of the paper relates to the function $a^2h^2 + b^2g^2 + \ldots + 4bceh$ which had presented itself to me, 13, in the theory of linear transformations, and which is in like manner automorphic for the change $a, b, \ldots$ into $\frac{d\phi}{da}, \frac{d\phi}{db}$, &c. The function however occurs in connexion with the arithmetical theory of the composition of quadratic forms, Gauss, *Disquisitiones Arithmeticæ* (1801), and see 92. The second part gives for the binary quartic covariant an automorphic formula analogous to those previously obtained by Hesse for the ternary cubic, viz. the Hessian of any linear function of the quartic and its Hessian, is itself a linear function of the quartic and its Hessian, the coefficients depending on the invariants I, J of the quartic form.

16. This is a mere reproduction of 13 and 14, and requires no remark.

19 and 23. These papers contain a mere sketch of the application of the doubly infinite product expression of the elliptic function sn u to the problem of transformation. As noticed in 23, I purposely abstained from any consideration of the infinite limiting values of m and n.

20. The discovery of the formula $q(ix+jy+kz)q^{-1}=ix'+jy'+kz'$, as expressing a rotation, was made by Sir W. R. Hamilton some months previous to the date of this paper. As appears by the paper itself, I was led to it by Rodrigues' formulæ, see 6. For the further development of the theory, see 68.

21. The system of imaginaries $i_1, i_2, \ldots i_7$ had presented itself to J. T. Graves about Christmas 1843, see his paper "On a Connection &c." *Phil. Mag.* vol. XXVI. (1845), pp. 315—320. They are called by him Octads, or Octaves.

24 and 25. These papers precede the researches of Eisenstein on the same subject. *Crelle*, t. XXXV. (1847). It was I think right that the theory of the doubly infinite products should have been investigated as in these papers: but the investigation is in some measure superseded by the beautiful theory of Weierstrass, viz. he takes for the element of the product (not a mere linear function of u, but) a linear function multiplied by an exponential factor,

$$\sigma u = u\Pi\left\{\left(1+\frac{u}{w}\right)e^{-\frac{u}{w}-\frac{1}{2}\frac{u^2}{w^2}}\right\},$$

$w=2m\omega+2m'\omega'$ where the ratio $\omega:\omega'$ is imaginary, and the product extends to all positive or negative integer values of m, m' (the simultaneous values 0, 0 excluded): in consequence of the introduction of the exponential factor the form of the bounding curve becomes immaterial, and the only condition is that it shall be ultimately everywhere at an infinite distance from the origin.

The general theory of Weierstrass in regard to the exponential factor is given in the Memoir, "Zur Theorie der eindeutigen analytischen Functionen," *Berlin. Abh.* 1876 (reprinted, *Abhandlungen aus der Functionenlehre*, 8° Berlin, 1886): and the application to Elliptic Functions is made in his lectures, edited by Schwarz, *Formeln und Lehrsätze* u. s. w. 4° Gött. 1883. See also Halphen, *Théorie des Fonctions Elliptiques*, Paris, 1887.

26. The geometrical results in regard to corresponding points on a cubic curve are many of them due to Maclaurin. See his "De linearum geometricarum proprietatibus generalibus tractatus," published as an Appendix to his Treatise on Algebra, 5 Ed. Lond. 1788. (See also De Jonquières' "Mélanges de Géométrie pure," 8° Paris, 1856.) But the theorem in the "Addition" was probably new: the curve of the third class touched by the line PP' is the curve called by me the Pippian (as represented by the contravariant equation $PU=0$), but which has since been called the Cayleyan of the cubic curve.

27. The theory of the conics of involution was so far as I am aware new.

28 and 29. See 2, 3.

30. As noticed in the paper, the investigation is directly founded upon that of Plücker for the singularities of a plane curve. It is to be observed that the definition of a "line through two points" *ligne menée par deux points (non consecutifs en général) du système,* does not exclude actual double points, for the line through an actual double point is a line through two points, coincident indeed, but not consecutive; but

it would have been proper to notice the distinction between actual and apparent double points (first made by Dr Salmon, to whom the term apparent double point, adp, is due): and the like in regard to the lines in two planes. In the translation of this paper, 83, there are two footnotes signed G. S. (Dr Salmon), giving for plane curves the formulæ $\iota - \kappa = 3(\nu - \mu)$ and $2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9)$: and for curves of double curvature and developable surfaces the analogous formulæ $\alpha - \beta = 2(n - m)$, $x - y = (n - m)$ and $2(g - h) = (n - m)(n + m - 7)$. Also in the second set of six equations, p. 210, the last three equations are replaced by

$$x = \tfrac{1}{2}m(m-1)(m^2 - m - 4) - \tfrac{1}{2}(2h + 3\beta)(2h + 3\beta + 1) - m(m-1)(2h + 3\beta) + 3h + 4\beta,$$

$$\alpha = 2m(3m - 7) - 3(4h + 5\beta),$$

$$g = \tfrac{1}{2}m(3m-7)(3m^2 - 5m - 7) + \tfrac{1}{2}(6h + 8\beta)(6h + 8\beta + 1) - 3m(m-2)(6h + 8\beta) + 19h + 24\beta,$$

viz. these are the equations serving to express x, α, g in terms of m, h, β.

For the discussion of some singularities not considered in the present paper see Zeuthen, "Sur les singularités ordinaires des courbes géométriques à double courbure," *Comptes Rendus*, t. LXVII. (1868), pp. 225—233, and "Sur les singularités ordinaires d'une courbe gauche et d'une surface développable," *Annali di Matem.* t. III. (1869–70), pp. 175—217.

40. See 10.

41 and 44. For demonstrations of Sir W. Thomson's theorem for the value of the definite integral $\int \frac{dx\ldots}{\{(x-a)^2 + \ldots + u^2\}^{\frac{1}{2}}(x^2 + \ldots + y^2)^{i+1}}$ see his papers "Démonstration d'un theorème d'Analyse" and "Extrait d'une lettre à M. Liouville," *Liouv.* t. X. (1845), pp. 137—147 and 364—367, also the paper "On certain definite integrals suggested by problems in the theory of Electricity," *Camb. Math. Jour.* t. II. (1845), pp. 109—121.

45. I am not aware that the equation $\sqrt{k}\,\mathrm{sn}\,u = \mathrm{H}(u) \div \Theta(u)$ had been previously demonstrated otherwise than by the circuitous process employed in the *Fundamenta Nova.*

The series $z = 1 + C_1 \frac{x^2}{1.2} \ldots + C_r \frac{x^r}{1.2\ldots r} + \ldots$, which is the solution of the differential equation $x^2 z + \alpha x \frac{dz}{dx} + \frac{d^2 z}{dx^2} - 2(\alpha^2 - 4)\frac{dz}{dx} = 0$, and in which C_1, C_2, ... denote the coefficients of the highest powers of n in the expressions given p. 299, is in fact (as remarked by me in a later paper, *Liouv.* t. VII. (1862)) the Weierstrassian function $\mathrm{Al}(x)$.

47. The surface here considered, the Tetrahedroid, is the general homographic transformation of the wave surface. It is a special case of the 16-nodal quartic surface considered by Kummer in his Memoir, "Algebraische Strahlen-systeme," *Berl. Abh.* 1866, pp. 1—120, and in various papers in the *Berliner Monatsberichte.*

48. The expressions $f_2 x = \Sigma (a-b)^2 (x-c)(x-d) \ldots$ for the Sturmian functions in terms of the roots, or (to use Sylvester's term), say the endoscopic expressions of these functions, were obtained by him, *Phil. Mag.* vol. XV. (1839).

It was interesting to express these in terms of the sums of powers S_1, S_2, &c., that is in terms of symmetrical functions of the coefficients, but for the actual expression of the Sturmian functions in terms of the coefficients the process is a very circuitous one, and the proper course is to start directly from the exoscopic expressions as linear functions of fx, $f'x$ also due to Sylvester (see his Memoir "On a theory of the syzygetic relations of two rational integral functions &c.," *Phil. Trans.* t. CXLIII. (1853), pp. 407—548), which is what is done in the subsequent paper 65.

49. I attach some value to this paper as a contribution to the theory of the Gamma function.

50, 55, 70, 95, 98. The general theorem of § I. was given in a very different and less suggestive notation, by an anonymous writer, "Théorèmes appartenant à la géométrie de la règle," *Gergonne* t. IX. (1818–19), pp. 289—291; viz. the statement is in effect as follows: considering in a plane or in space any n points 1, 2, 3 ... n; then joining these in order, take 12 any point in the line 1, 2; 23 any point in the line 2, 3, and so on to $n-1 . n$. Take then 123 the intersection of the lines 1, 23 and 12, 3; 234 the intersection of the lines 2, 34 and 23, 4; and so on to $n-2 . n-1 . n$. Take then 1234 the intersection of the lines 1, 234; 12, 34; and 123, 4 (viz. these three lines will meet in a point): 2345 the intersection of the lines 2, 345; 23, 45; 234, 5 (viz. these three lines will meet in a point)... and so on to $n-3 . n-2 . n-1 . n$. And so on for 12345, &c. up to $123 \ldots n$; in the successive constructions we have four, five, ... and finally n lines which in each case meet in a point. A proof is given by Gergonne, t. XI. (1820–21).

A large part of this paper relates to the theory of the relations to each other of the 60 Pascalian lines derived from the hexagons which can be formed with the same six points upon a conic: the literature of the question is very extensive, and I hope to refer to it again in the Notes to another volume.

54. See for an addition which should have been printed with this paper, the last paragraph of 92.

63. Boole's theorem of integration which is here demonstrated is a very remarkable one, and it would be very interesting to investigate the general forms, or a larger number of particular forms, for the functions P, Q satisfying the condition mentioned in the theorem. It may be remarked that the demonstration is very closely connected with Lejeune-Dirichlet's method for the determination of certain definite integrals referred to in 2 and 3, we have a triple integral the real part of which is $= fP \div Q^{n+q}$ or 0, according as P is or is not comprised between the limits 0 and 1. It includes Boole's formula mentioned in 44 and in 64.

65. See 48.

66. The method here employed of establishing the theory of Laplace's coefficients in n-dimensional space by means of rectangular coordinates, has I think some advantage

over the ordinary one, in which angular coordinates are introduced. The method is not given in Heine's *Kugelfunctionen*, Berlin 1878–81.

67, 93, and 99. These papers relate to the equation of differences, see p. 540, derived from Jacobi's equation $n(n-1)x^2z+(n-1)(\alpha x-2x^3)\frac{dz}{dx}+(1-\alpha x^2+x^4)\frac{d^2z}{dx^2}-2n(\alpha^2-4)\frac{dz}{d\alpha}=0$: the investigation seems to show that the integration cannot be effected in any tolerably simple form.

68. See 20.

69. This paper relates to the theory of the functions considered by Jacobi, and to which the name "Pfaffian" has since been given. It is shown that the definition of a determinant may be so extended as to include within it the Pfaffian: and it is proved that a symmetrical skew determinant is the square of a Pfaffian.

71. I doubt whether the theorem is true except in the cases $n=3$ and $n=4$.

76. As mentioned at the conclusion of the Memoir the whole subject was developed in a correspondence with Dr Salmon. Steiner's researches upon Cubic Surfaces are of later date, viz. we have his Memoir, "Ueber die Flächen dritten Grades" (read to the Berlin Academy 31 January 1856), *Crelle*, t. LIII. (1857), pp. 133—141 and *Werke*, t. II. pp. 651—659.

77. Contains the general definition of the 'order' of a system of equations.

81. Contains the remark that the Cartesian has a *cusp* at each of the two circular points at infinity.

82. The Problem of the fifteen school girls was proposed by Kirkman, *Lady's and Gentleman's Diary*, 1850: it is a particular case of the Prize-question proposed by him in the *Diary* for 1844. A great deal has been written on the subject.

83. See 30.

92. See 15.

94. This paper contains my fourfold formulæ for the addition of the elliptic functions sn, cn, and dn.

100. I remark that the terms covariant, invariant, here referred to as introduced by Sylvester, were first employed (together with many other valuable new terms) in the first part of his paper "On the Principles of the Calculus of Forms," *Camb. and Dubl. Math. Jour.* vol. VII. (1852), pp. 52—97. I hope to give in the next volume, in connexion with my Introductory Memoir on Quantics, a review of the earlier history of the subject.

END OF VOL. I.

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY & SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. II.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1889

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains fifty-eight papers (numbered 101, 102,..., 158) originally published, all but two of them, in the years 1851 to 1860: they are here reproduced nearly but not exactly in chronological order. The two excepted papers are 142, Numerical Tables Supplementary to Second Memoir on Quantics, now first published (1889); and, 143, Tables of the Covariants M to W of the Binary Quintic: from the Second, Third, Fourth, Fifth, Eighth, Ninth and Tenth Memoirs on Quantics (arranged in the present form, 1889): the determination of the finite number, 23, of the covariants of the quintic was made by Gordan in the year 1869, and the calculation of them having been completed in my Ninth and Tenth Memoirs, it appeared to me convenient in the present republication to unite together the values of all the covariants: viz. those of A to L are given in the Second Memoir 141, and the remainder M to W in the paper 143. I have added to the Third Memoir 144, in the notation thereof, some formulæ which on account of a difference of notation were omitted from a former paper, 35.

I remark that the present volume comprises the first six of the ten Memoirs on Quantics, viz. these are 139, 141, 144, 155, 156 and 158. I have, in the Notes and References, inserted a discussion of some length in reference to the paper 121, Note on a Question in the Theory of Probabilities: and also some remarks in reference to the theory of Distance developed in the Sixth Memoir on Quantics, 158.

CONTENTS.

PAGE

101. *Notes on Lagrange's Theorem* 1
Camb. and Dubl. Math. Jour. t. VI. (1851), pp. 37—45

102. *On a Double Infinite Series* 8
Camb. and Dubl. Math. Jour. t. VI. (1851), pp. 45—47

103. *On Certain Definite Integrals* 11
Camb. and Dubl. Math. Jour. t. VI. (1851), pp. 136—140

104. *On the Theory of Permutants* 16
Camb. and Dubl. Math. Jour. t. VII. (1852), pp. 40—51

105. *Correction to the Postscript to the Paper on Permutants* . 27
Camb. and Dubl. Math. Jour. t. VII. (1852), pp. 97—98

106. *On the Singularities of Surfaces* 28
Camb. and Dubl. Math. Jour. t. VII. (1852), pp. 166—171

107. *On the Theory of Skew Surfaces* 33
Camb. and Dubl. Math. Jour. t. VII. (1852), pp. 171—173

108. *On certain Multiple Integrals connected with the Theory of Attractions* 35
Camb. and Dubl. Math. Jour. t. VII. (1852), pp. 174—178

109. *On the Rationalisation of certain Algebraical Equations* . 40
Camb. and Dubl. Math. Jour. t. VIII. (1853), pp. 97—101

110. *Note on the Transformation of a Trigonometrical Expression* . 45
Camb. and Dubl. Math. Jour. t. IX. (1854), pp. 61—62

111. *On a Theorem of M. Lejeune-Dirichlet's* 47
Camb. and Dubl. Math. Jour. t. IX. (1854), pp. 163—165

PAGE

112. *Demonstration of a Theorem relating to the Products of Sums of Squares* 49
Phil. Mag. t. IV. (1852), pp. 515—519

113. *On the Geometrical Representation of the Integral* $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$ 53
Phil. Mag. t. V. (1853), pp. 281—284

114. *Analytical Researches connected with Steiner's Extension of Malfatti's Problem* 57
Phil. Trans. t. CXLII. (for 1852), pp. 253—278

115. *Note on the Porism of the In-and-circumscribed Polygon* . 87
Phil. Mag. t. VI. (1853), pp. 99—102

116. *Correction of two Theorems relating to the In-and-circumscribed Polygon* 91
Phil. Mag. t. VI. (1853), pp. 376—377

117. *Note on the Integral* $\int dx \div \sqrt{(m-x)(x+a)(x+b)(x+c)}$ . . 93
Phil. Mag. t. VI. (1853), pp. 103—105

118. *On the Harmonic Relation of two Lines or two Points* . . 96
Phil. Mag. t. VI. (1853), pp. 105—107

119. *On a Theorem for the Development of a Factorial* . . . 98
Phil. Mag. t. VI. (1853), pp. 182—185

120. *Note on a Generalisation of the Binomial Theorem* . . 102
Phil. Mag. t. VI. (1853), p. 185

121. *Note on a Question in the Theory of Probabilities* . . . 103
Phil. Mag. t. VI. (1853), p. 259

122. *On the Homographic Transformation of a Surface of the Second Order into Itself* 105
Phil. Mag. t. VI. (1853), pp. 326—333

123. *On the Geometrical Representation of an Abelian Integral* . 113
Phil. Mag. t. VI. (1853), pp. 414—418

124. *On a Property of the Caustic by Refraction of the Circle* . 118
Phil. Mag. t. VI. (1853), pp. 427—431

PAGE

125. *On the Theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$ 123
Phil. Mag. t. VII. (1854), pp. 40—47

126. *On the theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$. *Second Part* 131
Phil. Mag. t. VII. (1854), pp. 408—409

127. *On the Homographic Transformation of a Surface of the Second Order into itself* 133
Phil. Mag. t. VII. (1854), pp. 208—212: continuation of 122

128. *Developments on the Porism of the In-and-circumscribed Polygon* 138
Phil. Mag. t. VII. (1854), pp. 339—345

129. *On the Porism of the In-and-circumscribed Triangle, and on an irrational Transformation of two Ternary Quadratic Forms each into itself*. 145
Phil. Mag. t. IX. (1855), pp. 513—517

130. *Deuxième Mémoire sur les Fonctions doublement Périodiques* . 150
Liouville, t. XIX. (1854), pp. 193—208: sequel to 25

131. *Nouvelles Recherches sur les Covariants* 164
Crelle, t. XLVII. (1854), pp. 109—125

132. *Réponse à une Question proposée par M. Steiner* . . . 179
Crelle, t. L. (1855), pp. 277—278

133. *Sur un Théorème de M. Schläfli* 181
Crelle, t. L. (1855), pp. 278—282

134. *Remarques sur la Notation des Fonctions Algébriques* . . 185
Crelle, t. L. (1855), pp. 282—285

135. *Note sur les Covariants d'une Fonction Quadratique, Cubique, ou Biquadratique à deux Indéterminées* 189
Crelle, t. L. (1855), pp. 285—287

136. *Sur la Transformation d'une Fonction Quadratique en elle-même par des Substitutions linéaires* 192
Crelle, t. L. (1855), pp. 288—289

137. *Recherches Ultérieures sur les Déterminants gauches* . . 202
Crelle, t. L. (1855), pp. 299—313: continuation of 52 and 69.

PAGE

138. *Recherches sur les Matrices dont les termes sont des fonctions linéaires d'une seule Indéterminée* 216
Crelle, t. L. (1855), pp. 313—317

139. *An Introductory Memoir on Quantics* 221
Phil. Trans. t. CXLIV. (for 1854), pp. 244—258

140. *Researches on the Partition of Numbers* 235
Phil. Trans. t. CXLV. (for 1855), pp. 127—140

141. *A Second Memoir on Quantics* 250
Phil. Trans. t. CXLVI. (for 1856), pp. 101—126

142. *Numerical Tables Supplementary to Second Memoir on Quantics* 276
Now first published (1889)

143. *Tables of the Covariants* M *to* W *of the Binary Quintic: from the Second, Third, Fifth, Eighth, Ninth and Tenth Memoirs on Quantics* 282
Arranged in the present form, 1889

144. *A Third Memoir on Quantics* 310
Phil. Trans. t. CXLVI. (for 1856), pp. 627—647

145. *A Memoir on Caustics* 336
Phil. Trans. t. CXLVII. (for 1857), pp. 273—312

146. *A Memoir on Curves of the Third Order* 381
Phil. Trans. t. CXLVII. (for 1857), pp. 415—446

147. *A Memoir on the Symmetric Functions of the Roots of an Equation* 417
Phil. Trans. t. CXLVII. (for 1857), pp. 489—496

148. *A Memoir on the Resultant of a System of two Equations* . 440
Phil. Trans. t. CXLVII. (for 1857), pp. 703—715

149. *On the Symmetric Functions of the Roots of certain Systems of two Equations* 454
Phil. Trans. t. CXLVII. (for 1857), pp. 717—726

150. *A Memoir on the Conditions for the Existence of given Systems of Equalities among the Roots of an Equation* . . . 465
Phil. Trans. t. CXLVII. (for 1857), pp. 727—731

PAGE

151. *Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees* 471
Phil. Trans. t. CXLVII. (for 1857), pp. 733—736

152. *A Memoir on the Theory of Matrices* 475
Phil. Trans. t. CXLVIII. (for 1858), pp. 17—37

153. *A Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function* 497
Phil. Trans. t. CXLVIII. (for 1858), pp. 39—46

154. *Supplementary Researches on the Partition of Numbers* . . 506
Phil. Trans. t. CXLVIII. (for 1858), pp. 47—52

155. *A Fourth Memoir on Quantics* 513
Phil. Trans. t. CXLVIII. (for 1858), pp. 415—427

156. *A Fifth Memoir on Quantics* 527
Phil. Trans. t. CXLVIII. (for 1858), pp. 429—460

157. *On the Tangential of a Cubic* 558
Phil. Trans. t. CXLVIII. (for 1858), pp. 461—463

158. *A Sixth Memoir on Quantics* 561
Phil. Trans. t. CXLIX. (for 1859), pp. 61—90

Notes and References 593

CLASSIFICATION.

GEOMETRY

Theory of Distance, 158
Surfaces, 106, 107
Transformation of Quadric Surfaces, 122, 127, 129, 136, 153

Steiner's extension of Malfatti's Problem, 114
In-and-circumscribed triangle and polygon, 115, 116, 128, 129
Harmonic relation of two lines or points, 118
Question proposed by Steiner, 132
Caustics, 124, 145
Cubic Curves, 146, 157

ANALYSIS

Skew Determinants, 137
Attractions and Multiple Integrals, 108
Definite Integrals, 103
Elliptic and Abelian Integrals, 110, 113, 117, 123, 130
Covariants, Quantics &c., 131, 134, 135, 139, 141, 142, 143, 144, 155, 156, 158
Matrices, 138, 152
Partition of Numbers, 140, 154
Symmetric Functions &c., 147, 148, 149, 150

Lagrange's Theorem, 101
Double Infinite Series, 102
Permutants, 104, 105
Rationalisation of Algebraic Expression, 109
Transformation of Trigonometrical Expression, 110
Theorem of Lejeune-Dirichlet's, 111
Products of Sums of Squares, 112
Factorials, 119
Generalisation of Binomial Theorem, 120
Question in Probabilities, 121
Groups, 125, 126
Theorem of Schläfli's, on Elimination, 133
Sturmian Functions, 151

101.

NOTES ON LAGRANGE'S THEOREM.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VI. (1851), pp. 37—45.]

I.

IF in the ordinary form of Lagrange's theorem we write $(x+a)$ for x, it becomes

$$x = hf(a+x),$$

$$F(a+x) = Fa + \frac{h}{1} F'afa + \&c. \quad \text{.....} (1)$$

It follows that the equation

$$F(a+x) = Fa + \frac{1}{1} \frac{x}{f(a+x)} (F'afa) + , \text{.....} (2)$$

must reduce itself to an identity when the two sides are expanded in powers of x; or writing for shortness F, f instead of Fa, fa, and δ for $\frac{d}{da}$, we must have

$$\frac{1}{[r]^r} \delta^r F = S \left\{ \frac{1}{[p]^p} \delta^{p-1} (\delta F \,.\, f^p) \frac{1}{[r-p]^{r-1}} \delta^{r-p} f^{-p} \right\}, \text{.....} (3)$$

(where p extends from 0 to r). Or what comes to the same,

$$\frac{1}{[r]^r} \delta^r F = S \left\{ \frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}\,[s-1]^{s-1}} \delta^{p-s} f^p \,.\, \delta^{r-p} f^{-p} \,.\, \delta^s F \right\}, \text{.....} (4)$$

where s extends from 0 to $(r-p)$. The terms on the two sides which involve $\delta^r F$ are immediately seen to be equal; the coefficients of the remaining terms $\delta^s F$ on the second side must vanish, or we must have

$$S \left\{ \frac{1}{p\,[p-s]^{p-s}\,[r-p]^{r-p}} (\delta^{p-s} f^p)(\delta^{r-p} f^{-p}) \right\} = 0, \text{.....} (5)$$

(s being less than r). Or in a somewhat more convenient form, writing p, q and k for $p-s$, $r-p$ and $r-s$,

$$S\left\{\frac{1}{(p+s)\,[p]^p\,[s]^s}(\delta^p f^{p+s})\,(\delta^q f^{r-p-s})\right\}=0, \quad\text{.....}(6)$$

where s is constant and p and q vary subject to $p+q=k$, k being a given constant different from zero (in the case where $k=0$, the series reduces itself to the single term $\frac{1}{s}$). The direct proof of this theorem will be given presently.

II.

The following symbolical form of Lagrange's theorem was given by me in the *Mathematical Journal*, vol. III. [1843], pp. 283—286, [8].

If
$$x=a+hfx, \quad\text{.....}(7)$$
then
$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}F'a\,e^{hfa}.$$

Suppose $fx=\phi\,(b+k\psi x)$, or $x=a+h\phi\,(b+k\psi x)$, then

$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}F'a\,e^{h\phi\,(b+k\psi a)}.$$

But

$$e^{h\phi\,(b+k\psi a)}=\left(\frac{d}{db}\right)^{k\frac{d}{dk}}e^{h\phi b+k\psi a}.$$

(In fact the two general terms

$$\{\phi\,(b+k\psi a)\}^m \text{ and } \left(\frac{d}{db}\right)^{k\frac{k}{dk}}e^{k\psi a}\,(\phi b)^m,$$

of which the former reduces itself to $e^{k\psi a\frac{d}{db}}(\phi b)^m$, are equal on account of the equivalence of the symbols

$$e^{k\psi a\frac{d}{db}} \text{ and } \left(\frac{d}{db}\right)^{k\frac{d}{dk}}e^{k\psi a}).$$

Hence
$$x=a+h\phi\,(b+k\psi x), \quad\text{.....}(8)$$

$$Fx=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}\left(\frac{d}{db}\right)^{k\frac{d}{dk}}F'ae^h\phi^{b+k\psi a};$$

and the coefficient of $h^m k^n$ is

$$\frac{1}{[m]^m\,[n]^n}\left(\frac{d}{da}\right)^{m-1}F'a\,(\psi a)^n.\left(\frac{d}{db}\right)^n(\phi b)^n.$$

A similar formula evidently applies to the case of any finite number of functions ϕ, ψ, &c.: in the case of an infinite number we have

$$F\left(a+h\phi\left(b+k\psi\left(c+l\chi\left(d+\ldots\right)\right.\right.\right.=\left(\frac{d}{da}\right)^{h\frac{d}{dh}-1}\left(\frac{d}{db}\right)^{k\frac{d}{dk}}\left(\frac{d}{dc}\right)^{l\frac{d}{dl}}\ldots F'a\, e^{h\phi b+k\psi c+l\chi d+\ldots};\ldots(9)$$

or the coefficient of $h^m k^n l^p \ldots$ is

$$\frac{1}{[m]^m[n]^n[p]^p\ldots}\left(\frac{d}{da}\right)^m Fa\,.\left(\frac{d}{db}\right)^n(\phi b)^m\,.\left(\frac{d}{dc}\right)^p(\psi c)^r\ldots$$

the last of the series m, n, p..... being always zero; e.g. in the coefficient of $h^m k^n$, account must be had of the factor $\left(\frac{d}{dc}\right)^p(\psi c)^n$ or $(\psi c)^n$. The above form is readily proved independently by Taylor's theorem, without the assistance of Lagrange's. If in it we write $h=k$, &c., $a=b=$ &c., and $\phi=\psi=$ &c. $=f$, we have $F(a+hf(a+hf(a+\ldots)=Fx$, where $x=a+hfx$. Hence, comparing the coefficient of h^s with that given by Lagrange's theorem,

$$\frac{1}{[s]^s}\delta^{s-1}(\delta F\,.f^s)=S\left\{\frac{1}{[m]^m[n]^n[p]^p\ldots}\delta^m F\,.\,\delta^n f^m\,.\,\delta^p f^r\ldots\right\},\ \ldots\ldots\ (10)$$

where $m+n+$ &c. $=s$, and as before Fa, fa, $\frac{d}{da}$ have been replaced by F, f, δ. By comparing the coefficients of $\delta^m F$,

$$\frac{1}{[t]^t}\,\frac{s-t}{s}\,\delta^t f^s=\Sigma\left\{\frac{1}{[n]^n[p]^p\ldots}(\delta^n f^{s-t})\,(\delta^p f^n)\ldots\right\},\ \ldots\ldots\ldots\ldots\ldots(11)$$

where $n+p+\ldots=t$, the last of the series n, $p\ldots$ always vanishing. The formula (10) deduced, as above mentioned, from Taylor's theorem, and the subsequent formula (11) with an independent demonstration of it, not I believe materially different from that which will presently be given, are to be found in a memoir by M. Collins (volume II. (1833) of the Memoirs of the Academy of St Petersburg), who appears to have made very extensive researches in the theory of developments as connected with the combinatorial analysis.

III.

To demonstrate the formula (6), consider, in the first place, the expression

$$S\,\frac{\phi p}{[p]^p\,[q]^q}\,\{(\delta^p f^{p+s})\,(\delta^q f^{-p-s-q})\},$$

where $p+q=k$. Since

$$\frac{1}{[p]^p\,[q]^q}=\frac{1}{k}\left(\frac{1}{[p-1]^{p-1}\,[q]^q}+\frac{1}{[p]^p\,[q-1]^{q-1}}\right),$$

this is immediately transformed into

$$\frac{1}{k} S\phi p \left\{\frac{p+s}{[p-1]^{p-1}[q]^q}(\delta^{p-1}f^{p+s-1}\delta f)(\delta^q f^{-p-s-\theta}) - \frac{(p+s+\theta)}{[p]^p[q-1]^{q-1}}(\delta^p f^{p+s})(\delta^{q-1}.f^{-p-s-\theta-1}\delta f)\right.$$

$$= \frac{1}{k} S \frac{1}{[p]^p[q]^q}\{\phi(p+1)(p+s+1)(\delta^p.f^{p+s}\delta f)(\delta^q f^{-p-s-\theta-1})$$

$$\left. - \phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^q.f^{-p-s-\theta-1}\delta f)\right\},$$

in which last expression $p+q=(\rho-1)$. Of this, after separating the factor δf, the general term is

$$\frac{1}{k}\frac{1}{[\alpha]^\alpha}\delta^{\alpha+1}f.S\left\{\frac{1}{[p-\alpha]^{p-\alpha}[q]^q}\phi(p+1)(p+s+1)(\delta^{p-\alpha}f^{p+s})(\delta^q f^{-p-s-\theta-1})\right.$$

$$\left. - \frac{1}{[p]^p[q-\alpha]^{q-\alpha}}\phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^{q-\alpha}f^{-p-s-\theta-1})\right\},$$

equivalent to

$$\frac{1}{k}\frac{1}{[\alpha]^\alpha}\delta^{\alpha+1}f.S\frac{1}{[p]^p[q]^q}\{\phi(p+\alpha+1)(p+s+\alpha+1)(\delta^p f^{p+s+\alpha})(\delta^q f^{-p-s-\alpha-\theta-1})$$

$$- \phi p(p+s+\theta)(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta-1})\},$$

in which last expression $p+q=k-\alpha-1$. By repeating the reduction j times, the general term becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots$$

$$\times S\frac{1}{[p^p][q]^q}\Sigma\{(-)^{j-j'}\phi(p+\alpha+\beta\ldots+j')[p+s+\alpha+\beta\ldots+j']^{j'}$$

$$\times[p+s+\theta+\alpha+\beta\ldots+j-1]^{j-j'}(\delta^p f^{p+s+\alpha+\beta\ldots})(\delta^q f^{-p-s-\theta-j-\alpha-\beta\ldots})\},$$

where the sums $\alpha+\beta\ldots$ contain j' terms, j' being less than j or equal to it, and Σ extends to all combinations of the quantities α, $\beta\ldots$ taken j' and j' together (so that the summation contains 2^j terms). Also $p+q=k-\alpha-\beta\ldots$ (j terms) $-j$, and the products $k(k-\alpha-1)(k-\alpha-\beta-2)\ldots$ and $[\alpha]^\alpha[\beta]^\beta,\ldots\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots$ contain each of them j terms. Suppose the reduction continued until $k-\alpha-\beta\ldots$ (j terms) $-j=0$, then the only values of p, q are $p=0$, $q=0$; and the general term of

$$S\frac{\phi p}{[p]^p[q]^q}\{(\delta^p f^{p+s})(\delta^q f^{-p-s-\theta})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\delta^{\alpha+1}f.\delta^{\beta+1}f\ldots f^{-j-\theta}$$

$$\times\Sigma\{(-)^{j-j}\phi(\alpha+\beta\ldots+j')[s+\alpha+\beta\ldots+j']^{j'}[s+\theta+\alpha+\beta\ldots+j-1]^{j-j'}\}.$$

If $\theta = 0$, the general term reduces itself to

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\,\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\,\delta^{\alpha+1}f.\,\delta^{\beta+1}f\ldots f^{-j}.$$

$$\Sigma\{(-)^{j-j'}(s+\alpha+\beta\ldots+j')\,\phi(\alpha+\beta\ldots+j')\,[s+\alpha+\beta\ldots+j-1]^{j-1}\};$$

whence finally, if $\phi p = \dfrac{1}{p+s}$, the general term of

$$S\frac{1}{(p+s)[p]^p[q]^q}\{(\delta^p f^{p+s})(\delta^q f^{-p-s})\}$$

becomes

$$\frac{1}{k(k-\alpha-1)(k-\alpha-\beta-2)\ldots}\cdot\frac{1}{[\alpha]^\alpha[\beta]^\beta\ldots}\,\delta^{\alpha+1}f.\,\delta^{\beta+1}f\ldots f^{-j}\,\Sigma\{(-)^{j-j'}[s+\alpha+\beta\ldots+j-1]^{j-1}\};$$

and it is readily shown that the sum contained in this formula vanishes, which proves the equation in question.

IV.

The demonstration of the equation (11) is much simpler. We have

$$\delta^{t-1}(f^{s-1}\delta f) = \Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}\,\delta^{n-1}(f^{s-t-1}\delta f).(\delta^{t-n}f^t)\right\},$$

that is,

$$\delta^t f^s = \frac{s}{s-t}\,\Sigma\left\{\frac{[t-1]^{n-1}}{[n-1]^{n-1}}\,(\delta^n f^{s-t})(\delta^{t-n}f^t)\right\},$$

where n extends from $n=1$ to $n=t$. Similarly

$$\delta^{t-n}f^t = \frac{t}{n}\,\Sigma\left\{\frac{[t-n-1]^{p-1}}{[p-1]^{p-1}}\,(\delta^p f^n)(\delta^{t-n-p}f^{t-n})\right\},$$

$$\delta^{t-n-p}f^{t-n} = \frac{t-n}{p}\,\Sigma\left\{\frac{[t-n-p-1]^{q-1}}{[q-1]^{q-1}}\,(\delta^q f^p)(\delta^{t-n-p-q}f^{t-n-p})\right\},$$

&c.

Hence, substituting successively, and putting $t-n-p-q=r$, &c.,

$$\delta^t f^s = \frac{s}{s-t}\,\Sigma\left\{\frac{[t]^{t-r-1}}{[n]^n[p]^p(q+r)[q-1]^{q-1}}\,(\delta^n f^{s-t})(\delta^q f^p)(\delta^r f^{q+r})\right\},$$

&c.; and the last of these corresponding to a zero value of the last of the quantities $n, p, q\ldots$ is evidently the required equation (11).

V.

The formula (18) in my paper on Lagrange's theorem (before referred to) is incorrect. I propose at present, after giving the proper form of the formula in question, to develope the result of the substitution indicated at the conclusion of the paper. It will be convenient to call to mind the general theorem, that when any number

of variables $x, y, z \ldots$ are connected with as many other variables $u, v, w \ldots$ by the same number of equations (so that the variables of each set may be considered as functions of those of the other set) the quotient of the expressions $dxdy \ldots$ and $dudv \ldots$ is equal to the quotient of two determinants formed with the functions which equated to zero express the relations between the two sets of variables; the former with the differential coefficients of these functions with respect to $u, v \ldots$, the latter with the differential coefficients with respect to $x, y \ldots$. Consequently the notation $\dfrac{dxdy \ldots}{dudv \ldots}$ may be considered as representing the quotient of these determinants. This being premised, if we write

$$x - u - h\theta\,(x,\ y \ldots) = 0,$$
$$y - v - k\phi\,(x,\ y \ldots) = 0,$$

then the formula in question is

$$F(x,\ y \ldots)\frac{dxdy \ldots}{dudv \ldots} = \delta_u^{h\delta_h}\delta_v^{k\delta_k} \ldots e^{h\theta + k\phi \ldots} F,$$

if for shortness the letters $\theta, \phi, \ldots, F$ denote what the corresponding functions become when $u, v, \ldots$ are substituted for $x, y, \ldots$. Let $\dfrac{1}{\Delta}$ denote the value which $\dfrac{dxdy \ldots}{dudv \ldots}$, considered as a function of $x, y \ldots$, assumes when these variables are changed into $u, v, \ldots$, we have

$$\nabla = \begin{vmatrix} 1 - h\delta_u\theta, & -h\delta_v\theta \ldots \\ -k\delta_u\phi, & 1 - k\delta_v\phi \ldots \\ \vdots & \end{vmatrix}.$$

By changing the function F, we obtain

$$F(x,\ y \ldots) = \delta_u^{h\delta_h}\delta_v^{kd_k} \ldots e^{h\theta + k\phi \ldots} F\,\nabla;$$

where, however, it must be remembered that the $h, k, \ldots$, in so far as they enter into the function ∇, are not affected by the symbols $h\delta_h, k\delta_k, \ldots$ In order that we may consider them to be so affected, it is necessary in the function ∇ to replace h, k, &c. by $\dfrac{h}{\delta_u}, \dfrac{k}{\delta_v}$, &c. Also, after this is done, observing that the symbols $h\delta_u\theta, h\delta_v\theta \ldots$ affect a function $e^{h\theta + k\phi + \ldots} F$, the symbols $h\delta_u\theta, h\delta_v\theta, \ldots$ may be replaced by $\delta_u^{\theta}, \delta_v^{\theta}, \ldots$, where the θ is not an index, but an affix denoting that the differentiation is only to be performed with respect to $u, v \ldots$ so far as these variables respectively enter into the function θ. Transforming the other lines of the determinant in the same manner, and taking out from $\delta_u^{h\delta_h}\delta_v^{k\delta_k} \ldots$ the factor $\delta_u\delta_v \ldots$ in order to multiply this last factor into the determinant, we obtain

$$F(x,\ y \ldots) = \delta_u^{h\delta_h - 1}\delta_v^{k\delta_k - 1} \ldots e^{h\theta + k\phi \ldots} F\,\square;$$

where

$$\square = \begin{vmatrix} \delta_u - \delta_u^{\theta}, & -\delta_u^{\phi}, \ldots \\ -\delta_v^{\theta}, & \delta_v - \delta_v^{\phi}, \\ \vdots & \end{vmatrix}$$

in which expression δ_u, $\delta_v \ldots$ are to be replaced by

$$\delta_u^{\chi} + \delta_u^{\theta} + \delta_u^{\phi} \ldots \delta_v^{\chi} + \delta_v^{\theta} + \delta_v^{\phi} \ldots.$$

The complete expansion is easily arrived at by induction, and the form is somewhat singular. In the case of a single variable u we have $\square = \delta_u^{\chi}$, in the case of two variables, $\square = \delta_u^{\chi}\delta_v^{\chi} + \delta_u^{\chi}\delta_v^{\theta} + \delta_u^{\phi}\delta_v^{\chi}$. Or writing down only the affixes, in the case of a single variable we have F; in the case of two variables FF, $F\theta$, ϕF; and in the case of three variables FFF, ϕFF, χFF, $F\chi F$, $F\theta F$, $FF\theta$, $FF\phi$, $F\theta\theta$, $F\theta\phi$, $F\chi\theta$, $\phi F\phi$, $\chi F\phi$, $\phi F\theta$, $\chi\chi F$, $\phi\chi F$, $\chi\theta F$; where it will be observed that θ never occurs in the first place, nor ϕ in the second place, nor θ, ϕ (in any order) in the first and second places, &c., nor θ, ϕ, χ (in any order) in the first, second, and third places. And the same property holds in the general case for each letter and binary, ternary, &c. combination, and for the entire system of letters, and the system of affixes contains every possible combination of letters not excluded by the rule just given. Thus in the case of two letters, forming the system of affixes FF, $F\theta$, ϕF, θF, $F\phi$, $\theta\phi$, $\phi\theta$, the last four are excluded, the first three of them by containing θ in the first place or ϕ in the second place, the last by containing ϕ, θ in the first and second places: and there remains only the terms FF, $F\theta$, ϕF forming the system given above. Substituting the expanded value of $\square$ in the expression for $F(x, y \ldots)$, the equation may either be permitted to remain in the form which it thus assumes, or we may, in order to obtain the finally reduced form, after expanding the powers of h, $k \ldots$, connect the symbols δ_u^{θ}, $\delta_u^{\phi} \ldots \delta_u^{\chi}$, &c. with the corresponding functions θ, $\phi \ldots F$, and then omit the affixes; thus, in particular, in the case of a single variable the general term of Fx is

$$\frac{h^p}{[p]^p}\,\delta_u^{p-1}\,(\theta^p \delta_u F),$$

(the ordinary form of Lagrange's theorem). In the case of two letters the general term of $F(x, y)$ is

$$\frac{h^p k^q}{[p]^p\,[q]^q}\,\delta_u^{p-1}\delta_v^{q-1}\,\{\theta^p\phi^q\delta_u\delta_v F + \phi^q\delta_v\theta^p\delta_u F + \theta^p\delta_u\phi^q\delta^v F\}$$

(see the *Mécanique Céleste*, [Ed. 1, 1798] t. I. p. 176). In the case of three variables, the general term is

$$\frac{h^p k^q l^r}{[p]^p\,[q]^q\,[r]^r}\,\delta_u^{p-1}\delta_v^{q-1}\delta_w^{r-1}\,\{\theta^p\phi^q\chi^r\delta_u\delta_v\delta_w F + \ldots\},$$

the sixteen terms within the { } being found by comparing the product $\delta_u\delta_v\delta_w$ with the system FFF, ϕFF, &c., given above, and then connecting each symbol of differentiation with the function corresponding to the affix. Thus in the first term the δ_u, δ_v, δ_w, each affect the F, in the second term the δ_u affects ϕ^q, and the δ_v and δ_w each affect the F, and so on for the remaining terms. The form is of course deducible from Laplace's general theorem, and the actual development of it is given in Laplace's Memoir in the *Hist. de l'Acad.* 1777. I quote from a memoir by Jacobi which I take this opportunity of referring to, "De resolutione equationum per series infinitas," *Crelle*, t. VI. [1830], pp. 257—286, founded on a preceding memoir, "Exercitatio algebraica circa discerptionem singularem fractionum quæ plures variabiles involvunt," t. V. [1830], pp. 344—364.

Stone Buildings, April 6, 1850.

102.

ON A DOUBLE INFINITE SERIES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VI. (1851), pp. 45—47.]

THE following completely paradoxical investigation of the properties of the function Γ (which I have been in possession of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series.

Let $\Sigma_r \phi r$ denote the sum of the values of ϕr for all integer values of r from $-\infty$ to ∞. Then writing

$$u = \Sigma_r [n-1]^r x^{n-1-r}, \quad \text{.....} \quad (1)$$

(where n is any number whatever), we have immediately

$$\frac{du}{dx} = \Sigma_r [n-1]^{r+1} x^{n-2-r} = \Sigma_r [n-1]^r x^{n-1-r} = u;$$

that is,

$$\frac{du}{dx} = u, \text{ or } u = C_n e^x,$$

(the constant of integration being of course in general a function of n). Hence

$$C_n e^x = \Sigma_r [n-1]^r x^{n-1-r}; \quad \text{.....} \quad (2)$$

or e^x is expanded in general in a *doubly infinite necessarily divergent series of fractional powers of* x, (which resolves itself however in the case of n a positive or negative integer, into the ordinary singly infinite series, the value of C_n in this case being immediately seen to be Γn).

The equation (2) in its general form is to be considered as a definition of the function C_n. We deduce from it

$$\Sigma_r [n-1]^r (ax)^{n-1-r} = C_n e^{ax},$$
$$\Sigma_{r'} [n'-1]^{r'} (ax')^{n-1-r'} = C_{n'} e^{ax'};$$
$$\vdots$$

and also

$$\Sigma_k [n+n' \ldots -1]^k \{a(x+x' \ldots)\}^{n+n' \ldots -1-k} = C_{n+n' \ldots} e^{a(x+x' \ldots)}.$$

Multiplying the first set of series, and comparing with this last,

$$C_{n+n' \ldots} \Sigma_{r, r' \ldots} [n-1]^r [n'-1]^{r'} \ldots x^{n-1-r} x'^{n'-1-r'} \ldots$$
$$= C_n C_{n'} \ldots [n+n' \ldots -1]^k (x+x' \ldots)^{n+n' \ldots -1-k}, \ldots\ldots\ldots\ldots (3)$$

(where r, r' denote any positive or negative integer numbers satisfying $r+r'+\ldots = k+1-p$, p being the number of terms in the series $n, n', \ldots$). This equation constitutes a multinomial theorem of a class analogous to that of the exponential theorem contained in the equation (2).

In particular

$$C_{n+n' \ldots} \Sigma_{r, r' \ldots} [n-1]^r [n'-1]^{r'} \ldots = C_n C_{n'} \ldots [n+n' \ldots -1]^k p^{n+n' \ldots -1-k}, \ldots\ldots\ldots (4)$$

and if $p=2$, writing also m, n for n, n', and $k-1-r$ for r',

$$C_{m+n} \Sigma_r [m-1]^r [n-1]^{k-1-r} = C_m C_n [m+n-1]^k 2^{m+n-1-k}, \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

or putting $k=0$ and dividing,

$$C_m C_n \div C_{m+n} = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}. \ldots\ldots\ldots\ldots\ldots (6)$$

Now the series on the second side of this equation is easily seen to be convergent (at least for positive values of m, n). To determine its value write

$$\mathsf{F}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx;$$

then

$$\mathsf{F}(m, n) = \int_0^{\frac{1}{2}} x^{m-1} (1-x)^{n-1} \, dx + \int_0^{\frac{1}{2}} x^{n-1} (1-x)^{m-1} \, dx;$$

and by successive integrations by parts, the first of these integrals is reducible to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from -1 to $-\infty$ inclusively, and the second to $\frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r}$, r extending from 0 to ∞; hence

$$\mathsf{F}(m, n) = \frac{1}{2^{m+n-1}} \Sigma_r [m-1]^r [n-1]^{-1-r},$$

or

$$C_m C_n \div C_{m+n} = \mathsf{F}(m, n), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which proves the identity of C_m with the function $\Gamma(m)$. {Substituting in two of the preceding equations, we have

$$\Gamma n \Gamma n' \ldots \div \Gamma(n+n'\ldots) = \frac{1}{[n+n'\ldots-1]^k p^{n+n'\ldots-1-k}} \Sigma_{r,r'\ldots} [n-1]^r [n'-1]^{r'} \ldots, \quad \ldots (8)$$

(where, as before, p denotes the number of terms in the series $n, n', \ldots$ and $r+r'+\ldots=k+1-p$), the first side of which equation is, it is well known, reducible to a multiple definite integral by means of a theorem of M. Dirichlet's. And

$$\mathsf{F}(m, n) = \frac{1}{[m+n-1]^k 2^{m+n-1-k}} \Sigma_r [m-1]^r [n-1]^{k-1-r}, \quad \ldots\ldots\ldots\ldots (9)$$

where r extends from $-\infty$ to $+\infty$, and k is arbitrary. By giving large negative values to this quantity, very convergent series may be obtained for the calculation of $\mathsf{F}(m, n)$}.

103.

ON CERTAIN DEFINITE INTEGRALS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VI. (1851), pp. 136—140.]

SUPPOSE that for any positive or negative integral value of r, we have $\psi(x+ra) = U_r \psi x$, U_r being in general a function of x, and consider the definite integral

$$I = \int_{-\infty}^{\infty} \psi x \Psi x \, dx;$$

Ψx being any other function of x. In case of either of the functions ψx, Ψx becoming infinite for any real value α of x, the principal value of the integral is to be taken, that is, I is to be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{\infty} + \int_{-\infty}^{\alpha-\epsilon}\right) \psi x \Psi x \, dx, \quad (\epsilon = 0),$$

and similarly, when one of the functions becomes infinite for several of such values of x.

We have

$$I = \left(\ldots \int_{ra}^{(r+1)a} + \ldots\right) \psi x \Psi x \, dx;$$

or changing the variables in the different integrals so as to make the limits of each a, 0, we have

$$I = \int_0^a [\Sigma \psi(x+ra) \Psi(x+ra)] \, dx,$$

Σ extending to all positive or negative integer values of r, that is,

$$I = \int_0^a \psi x \, [\Sigma U_r \Psi(x+ra)] \, dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

which is true, even when the quantity under the integral sign becomes infinite for particular values of x, provided the integral be replaced by its principal value, that is, provided it be considered as the limit of

$$\left(\int_{\alpha+\epsilon}^{\alpha}+\int_{0}^{\alpha-\epsilon}\right)\psi x\left[\Sigma U_r\Psi(x+ra)\right]dx,$$

or

$$\int_{\epsilon}^{\alpha-\epsilon}\psi x\left[\Sigma U_r\Psi(x+ra)\right]dx;$$

where α, or one of the limiting values a, 0, is the value of x, for which the quantity under the integral sign becomes infinite, and ϵ is ultimately evanescent.

In particular, taking for simplicity $\alpha=\pi$, suppose

$$\psi(x+\pi)=\pm\psi x, \quad \text{or} \quad \psi(x+r\pi)=(\pm)^r\psi x;$$

then observing the equation

$$\Sigma\frac{(\pm)^r 1}{x+r\pi}=\cot x, \quad \text{or} = \operatorname{cosec} x,$$

according as the upper or under sign is taken, and assuming $\Psi x=x^{-\mu}$, we have finally

$$\int_{-\infty}^{\infty}\frac{\psi x\,dx}{x^\mu}=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^\pi\psi x\left[\left(\frac{d}{dx}\right)^{\mu-1}\cot x\right]dx,$$

$$\int_{-\infty}^{\infty}\frac{\psi x\,dx}{x^\mu}=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^\pi\psi x\left[\left(\frac{d}{dx}\right)^{\mu-1}\operatorname{cosec} x\right]dx,$$

the former equation corresponding to the case of $\psi(x+\pi)=\psi x$, the latter to that of $\psi(x+\pi)=-\psi x$.

Suppose $\psi_{,}x=\psi gx$, g being a positive integer. Then

$$\int_{-\infty}^{\infty}\frac{\psi_{,}x\,dx}{x^\mu}=g^{\mu-1}\int_{-\infty}^{\infty}\frac{\psi x\,dx}{x^\mu};$$

also if $\psi(x+\pi)=\psi x$, then $\psi_{,}(x+\pi)=\psi_{,}x$; but if $\psi(x+\pi)=-\psi x$, then $\psi_{,}(x+\pi)=\pm\psi_{,}x$, the upper or under sign according as g is even or odd. Combining these equations, we have

$\psi(x+\pi)=\psi x$, g even or odd,

$$\int_{-\infty}^{\infty}\frac{\psi gx\,dx}{x^\mu}=\frac{(-)^{\mu-1}}{\Gamma(\mu)}g^{\mu-1}\int_0^\pi\psi x\left[\left(\frac{d}{dx}\right)^{\mu-1}\cot x\right]dx=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^\pi\psi gx\left[\left(\frac{d}{dx}\right)^{\mu-1}\cot x\right]dx;$$

$\psi(x+\pi)=-\psi x$, g even,

$$\int_{-\infty}^{\infty}\frac{\psi gx\,dx}{x^\mu}=\frac{(-)^{\mu-1}}{\Gamma\mu}g^{\mu-1}\int_0^\pi\psi x\left[\left(\frac{d}{dx}\right)^{\mu-1}\operatorname{cosec} x\right]dx=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^\pi\psi gx\left[\left(\frac{d}{dx}\right)^{\mu-1}\operatorname{cosec} x\right]dx;$$

$\psi(x+\pi)=-\psi x$, g odd,

$$\int_{-\infty}^{\infty}\frac{\psi gx\,dx}{x^\mu}=\frac{(-)^{\mu-1}}{\Gamma\mu}g^{\mu-1}\int_0^\pi\psi x\left[\left(\frac{d}{dx}\right)^{\mu-1}\operatorname{cosec} x\right]dx=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^\pi\psi gx\left[\left(\frac{d}{dx}\right)^{\mu-1}\operatorname{cosec} x\right]dx.$$

In particular

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x} = \pi,$$

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^{\pi} \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx,$$

$$\int_0^{\pi} \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \cot x\right] dx = \quad g^{\mu-1} \int_0^{\pi} \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx, \quad g \text{ even},$$

$$\int_0^{\pi} \sin gx \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx = \quad g^{\mu-1} \int_0^{\pi} \sin x \left[\left(\frac{d}{dx}\right)^{\mu-1} \operatorname{cosec} x\right] dx, \quad g \text{ odd},$$

$$\int_0^{\pi} \sin gx \cot x\,dx = \pi, \quad g \text{ even},$$

$$\int_0^{\pi} \sin gx \operatorname{cosec} x\,dx = \pi, \quad g \text{ odd},$$

$$\int_0^{\pi} \frac{\tan x\,dx}{x} = 0, \text{ \&c.},$$

the number of which might be indefinitely extended.

The same principle applies to multiple integrals of any order: thus for double integrals, if $\psi(x+ra,\ y+rb) = U_{r,s}\psi(x,\ y)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x,\ y)\,\Psi(x,\ y)\,dx\,dy = \int_0^a\int_0^b \psi(x,\ y)\,\Sigma U_{r,s}\,\Psi(x+ra,\ y+sb). \quad \ldots \text{ (B)}$$

In particular, writing w, v for a, b, and assuming $\psi(x+rw,\ y+sv) = (\pm)^r(\pm)^s\psi(x,\ y)$; also $\Psi(x,\ y) = (x+iy)^{-\mu}$, where as usual $i = \sqrt{-1}$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\psi(x,\ y)\,dx\,dy}{(x+iy)^{\mu}} = \frac{(-)^{\mu-1}}{\Gamma\mu} \int_0^w\int_0^v \psi(x,\ y) \left[\left(\frac{d}{dx}\right)^{\mu-1} \Theta(x+iy)\right] dx\,dy, \ldots \text{(B}')$$

where

$$\Theta(x+iy) = \Sigma \frac{(\pm)^r(\pm)^s\,1}{(x+iy+rw+svi)},$$

Σ extending to all positive or negative integer values of r and s. Employing the notation of a paper in the *Cambridge Mathematical Journal*, "On the Inverse Elliptic Functions," t. IV. [1845], pp. 257—277, [24], we have for the different combinations of the ambiguous sign,

$$1. \quad -, \quad -, \quad \Theta(x+iy) = \frac{\mathfrak{G}(x+iy)}{\gamma(x+iy)} = \frac{1}{\phi(x+iy)},$$

$$2. \quad -, \quad +, \quad \Theta(x+iy) = \frac{G(x+iy)}{\gamma(x+iy} = \frac{F(x+iy)}{\phi(x+iy)},$$

$$3.\quad +,\quad -,\quad \Theta(x+iy)=\frac{g(x+iy)}{\gamma(x+iy)}=\frac{f(x+iy)}{\phi(x+iy)},$$

$$4.\quad +,\quad +,\quad \Theta(x+iy)=\frac{\gamma'(x+iy)}{\gamma(x+iy)};$$

where ϕ, f, F are in fact the symbols of the inverse elliptic functions (Abel's notation) corresponding very nearly to $\sin am$, $\cos am$, Δam. It is remarkable that the last value of Θ cannot be thus expressed, but only by means of the more complicated transcendant γx, corresponding to the $H(x)$ of M. Jacobi. The four cases correspond obviously to

$$\begin{aligned}
&1.\quad \psi(x+rw,\ y+sv)=(-)^{r+s}\psi(x,\ y),\\
&2.\quad \psi(x+rw,\ y+sv)=(-)^{r}\ \psi(x,\ y),\\
&3.\quad \psi(x+rw,\ y+sv)=(-)^{s}\ \psi(x,\ y),\\
&4.\quad \psi(x+rw,\ y+sv)=\qquad \psi(x,\ y).
\end{aligned}$$

The above formulæ may be all of them modified, as in the case of single integrals, by means of the obvious equation

$$\iint\frac{\psi(gx,\ gy)\,dxdy}{(x+iy)^{\mu}}=g^{\mu-2}\iint\frac{\psi(x,\ y)\,dxdy}{(x+iy)^{\mu}},\ \text{[limits } \infty,\ -\infty\text{]}.$$

The most important particular case is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(x+iy)\,dxdy}{(x+iy)}=wv,$$

for in almost all the others, for example in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(x+iy)\,dxdy}{(x+iy)^{\mu}}=\frac{(-)^{\mu-1}}{\Gamma\mu}\int_0^w\int_0^v\phi(x+iy)\left[\left(\frac{d}{dx}\right)^{\mu-1}\frac{1}{\phi(x+iy)}\right]dxdy,$$

the second integration cannot be effected.

Suppose next $\psi(x,\ y)$ is one of the functions $\gamma(x+iy)$, $g(x+iy)$, $G(x+iy)$, $\mathfrak{G}(x+iy)$, so that

$$\psi(x+rw,\ y+sv)=(\pm)^{r}(\pm)^{s}\,U_{r,s}\psi(x,\ y),$$

where

$$U_{r,s}=(-)^{rs}\,e^{\beta x(rw-svi)}\,q_{,}^{-\frac{1}{2}r^2}\,q^{-\frac{1}{2}s^2},$$

(see memoir quoted). Then, retaining the same value as before of $\Psi(x,\ y)$, we have still the formula (B), in which

$$\Theta(x+iy)=\Sigma\frac{(\pm)^{r}(\pm)^{s}\,U_{r,s}}{x+iy+rw+svi}.$$

But this summation has not yet been effected; the difficulty consists in the variable factor $e^{\beta x(rw-svi)}$ in the numerator, nothing being known I believe of the decomposition of functions into series of this form.

On the subject of the preceding paper may be consulted the following memoirs by Raabe, "Ueber die Summation periodischer Reihen," *Crelle*, t. XV. [1836], pp. 355—364, and "Ueber die Summation harmonisch periodischer Reihen," t. XXIII. [1842], pp. 105—125, and t. XXV. [1843], pp. 160—168. The integrals he considers, are taken between the limits 0, ∞ (instead of $-\infty$, ∞). His results are consequently more general than those given above, but they might be obtained by an analogous method, instead of the much more complicated one adopted by him: thus if $\phi(x+2\pi)=\phi x$, the integral $\int_0^\infty \phi x \frac{dx}{x}$ reduces itself to

$$\Sigma_0^\infty \int_0^{2\pi} \phi x \frac{dx}{x+2r\pi} = \int_0^{2\pi} dx\, \phi x \left[\frac{1}{x} + \Sigma_1^\infty \left(\frac{1}{x+2r\pi} - \frac{1}{2r\pi}\right)\right],$$

provided $\int_0^{2\pi} dx\, \phi x = 0$. The summation in this formula may be effected by means of the function Γ and its differential coefficient, and we have

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^{2\pi} dx \frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} dx,$$

which is in effect Raabe's formula (10), *Crelle*, t. XXV. p. 166.

By dividing the integral on the right-hand side of the equation into two others whose limits are 0, π, and π, 2π respectively, and writing in the second of these $2\pi - x$ instead of x, then

$$\int_0^\infty \phi x \frac{dx}{x} = -\frac{1}{2\pi}\int_0^\pi \left(\phi x \frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} + \phi(2\pi - x)\frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)}\right) dx;$$

or reducing by

$$\frac{\Gamma'\left(\frac{x}{2\pi}\right)}{\Gamma\left(\frac{x}{2\pi}\right)} = \frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)} - \pi \cot \tfrac{1}{2}x,$$

we have

$$\int_0^\infty \phi x \frac{dx}{x} = \tfrac{1}{2}\int_0^\pi \phi x \cot \tfrac{1}{2}x\, dx - \frac{1}{2\pi}\int_0^\pi [\phi x + \phi(2\pi - x)] \frac{\Gamma'\left(1-\frac{x}{2\pi}\right)}{\Gamma\left(1-\frac{x}{2\pi}\right)} dx,$$

which corresponds to Raabe's formula (10′). If $\phi(-x) = -\phi x$, so that $\phi x + \phi(2\pi - x) = 0$, the last formula is simplified; but then the integral on the first side may be replaced by $\frac{1}{2}\int_{-\infty}^{\infty} \phi x \frac{dx}{x}$, so that this belongs to the preceding class of formulæ.

104.

ON THE THEORY OF PERMUTANTS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VII. (1852), pp. 40—51.]

A FORM may by considered as composed of blanks which are to be filled up by inserting in them specializing characters, and a form the blanks of which are so filled up becomes a symbol. We may for brevity speak of the blanks of a symbol in the sense of the blanks of the form from which such symbol is derived. Suppose the characters are 1, 2, 3, 4, ..., the symbol may always be represented in the first instance and without reference to the nature of the form, by $V_{1234}\ldots$ And it will be proper to consider the blanks as having an invariable order to which reference will implicitly be made; thus, in speaking of the characters 2, 1, 3, 4, ... instead of as before 1, 2, 4, ... the symbol will be $V_{2134}\ldots$ instead of $V_{1234}\ldots$. When the form is given we shall have an equation such as

$$V_{1234} = P_{12}\, Q_3\, R_4 \ldots \quad \text{or} \quad = P_{12}\, P_{34} \ldots \quad \&\text{c.},$$

according to the particular nature of the form.

Consider now the characters 1, 2, 3, 4, ..., and let the primitive arrangement and every arrangement derivable from it by means of an even number of inversions or interchanges of two characters be considered as positive, and the arrangements derived from the primitive arrangement by an odd number of inversions or interchanges of two characters be considered as negative; a rule which may be termed "the rule of signs." The aggregate of the symbols which correspond to every possible arrangement of the characters, giving to each symbol the sign of the arrangement, may be termed a "Permutant;" or, in distinction from the more general functions which will presently be considered, a simple permutant, and may be represented by enclosing the symbol in brackets, thus $(V_{1234}\ldots)$. And by using an expression still more elliptical than the blanks of a symbol, we may speak of the blanks of a permutant, or the characters of a permutant.

As an instance of a simple permutant, we may take

$$(V_{123}) = V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321};$$

and if in particular $V_{123} = a_1b_2c_3$, then

$$(V_{123}) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

It follows at once that a simple permutant remains unaltered, to the sign *près* according to the rule of signs, by any permutations of the characters entering into the permutant. For instance,

$$(V_{123}) = (V_{231}) = (V_{312}) = -(V_{132}) = -(V_{213}) = -(V_{321}).$$

Consequently also when two or more of the characters are identical, the permutant vanishes, thus

$$V_{113} = 0.$$

The form of the symbol may be such that the symbol remains unaltered, to the sign *près* according to the rule of signs, for any permutations of the characters in certain particular blanks. Such a system of blanks may be termed a quote. Thus, if the first and second blanks are a quote,

$$V_{123} = -V_{213}, \quad V_{132} = -V_{312}, \quad V_{231} = -V_{321},$$

and consequently

$$(V_{123}) = 2(V_{123} + V_{231} + V_{312});$$

and if the blanks constitute one single quote,

$$(V_{123}\ldots) = N V_{123}\ldots,$$

where $N = 1.2.3\ldots n$, n being the number of characters. An important case, which will be noticed in the sequel, is that in which the whole series of blanks divide themselves into quotes, each of them containing the same number of blanks. Thus, if the first and second blanks, and the third and fourth blanks, form quotes respectively,

$$\tfrac{1}{6}(V_{1234}) = V_{1234} + V_{1342} + V_{1423} + V_{3412} + V_{4213} + V_{2314}.$$

It is easy now to pass to the general definition of a "Permutant." We have only to consider the blanks as forming, not as heretofore a single set, but any number of distinct sets, and to consider the characters in each set of blanks as permutable *inter se* and not otherwise, giving to the symbol the sign compounded of the signs corresponding to the arrangements of the characters in the different sets of blanks. Thus, if the first and second blanks form a set, and the third and fourth blanks form a set,

$$((V_{1234})) = V_{1234} - V_{2134} - V_{1243} + V_{2143}.$$

The word 'set' will be used throughout in the above technical sense. The particular mode in which the blanks are divided into sets may be indicated either in words or by some superadded notation. It is clear that the theory of permutants depends ultimately on that of simple permutants; for if in a compound permutant we first write down all the terms which can be obtained, leaving unpermuted the characters

of a particular set, and replace each of the terms so obtained by a simple permutant having for its characters the characters of the previously unpermuted set, the result is obviously the original compound permutant. Thus, in the above-mentioned case, where the first and second blanks form a set and the third and fourth blanks form a set,

$$((V_{1234})) = (V_{1234}) - (V_{1243}),$$

or

$$((V_{1234})) = (V_{1234}) - (V_{2134}),$$

in the former of which equations the first and second blanks in each of the permutants on the second side form a set, and in the latter the third and fourth blanks in each of the permutants on the second side form a set, the remaining blanks being simply supernumerary and the characters in them unpermutable. It should be noted that the term quote, as previously defined, is only applicable to a system of blanks belonging to the same set, and it does not appear that anything would be gained by removing this restriction.

The following rule for the expansion of a simple permutant (and which may be at once extended to compound permutants) is obvious. Write down all the distinct terms that can be obtained, on the supposition that the blanks group themselves in any manner into quotes, and replace each of the terms so obtained by a compound permutant having for a distinct set the blanks of each assumed quote; the result is the original simple permutant. Thus in the simple permutant (V_{1234}), supposing for the moment that the first and second blanks form a quote, and that the third and fourth blanks form a quote, this leads to the equation

$$(V_{1234}) = +((V_{1234})) + ((V_{1342})) + ((V_{1423})) + ((V_{3412})) + ((V_{4213})) + ((V_{2314})),$$

where in each of the permutants on the second side the first and second blanks form a set, and also the third and fourth blanks.

The blanks of a simple or compound permutant may of course, without either gain or loss of generality, be considered as having any particular arrangement in space, for instance, in the form of a rectangle: thus $V_{\substack{12\\34}}$ is neither more nor less general than V_{1234}. The idea of some such arrangement naturally presents itself as affording a means of showing in what manner the blanks are grouped into sets. But, considering the blanks as so arranged in a rectangular form, or in lines and columns, suppose in the first instance that this arrangement is independent of the grouping of the blanks into sets, or that the blanks of each set or of any of them are distributed at random in the different lines and columns. Assume that the form is such that a symbol

$$V_{\substack{\alpha\,\beta\,\gamma\ldots\\ \alpha'\beta'\gamma'\ldots\\ \vdots}}$$

is a function of symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, &c. Or, passing over this general case, and the case (of intermediate generality) of the function being a symmetrical function, assume that

$$V_{\substack{\alpha\,\beta\,\gamma\ldots\\ \alpha'\beta'\gamma'\ldots\\ \vdots}}$$

is the product of symbols $V_{\alpha\beta\gamma\ldots}$, $V_{\alpha'\beta'\gamma'\ldots}$, &c. Upon this assumption it becomes important to distinguish the different ways in which the blanks of a set are distributed in the different lines and columns. The cases to be considered are: (*A*). The blanks of a single set or of single sets are situated in more than one column. (*B*). The blanks of each single set are situated in the same column. (*C*). The blanks of each single set form a separate column. The case (*B*) (which includes the case (*C*)) and the case (*C*) merit particular consideration. In fact the case (*B*) is that of the functions which I have, in my memoir on Linear Transformations in the *Journal*, [13, 14] called hyperdeterminants, and the case (*C*) is that of the particular class of hyperdeterminants previously treated of by me in the *Cambridge Philosophical Transactions*, [12] and also particularly noticed in the memoir on Linear Transformations. The functions of the case (*B*) I now propose to call "Intermutants," and those in the case (*C*) "Commutants." Commutants include as a particular case "Determinants," which term will be used in its ordinary signification. The case (*A*) I shall not at present discuss in its generality, but only with the further assumption that the blanks form a single set (this, if nothing further were added, would render the arrangement of the blanks into lines and columns valueless), and moreover that the blanks of each line form a quote: the permutants of this class (from their connexion with the researches of Pfaff on differential equations) I shall term "Pfaffians." And first of commutants, which, as before remarked, include determinants.

The general expression of a commutant is

$$(V_{\substack{11\ldots\\22\\\vdots\\nn}}); \quad \text{or} \quad \begin{pmatrix} 11\ldots \\ 22 \\ \vdots \\ nn \end{pmatrix}$$

and (stating again for this particular case the general rule for the formation of a permutant) if, permuting the characters in the same column in every possible way, considering these permutations as positive or negative according to the rule of signs, one system be represented by

$$\begin{matrix} r_1 s_1 \ldots \\ r_2 s_2 \\ \vdots \\ r_n s_n \end{matrix}$$

the commutant is the sum of all the different terms

$$\pm\pm\ldots V_{r_1 s_1}\ldots V_{r_2 s_2}\ldots V_{r_n s_n}\ldots$$

The different permutations may be formed as follows: first permute the characters in all the columns except a single column, and in each of the arrangements so obtained permute entire lines of characters. It is obvious that, considering any one of the arrangements obtained by permutations of the characters in all the columns but one, the permutations of entire lines and the addition of the proper sign will only reproduce

the same symbol—in the case of an even number of columns constantly with the positive sign, but in the case of an odd number of columns with the positive or negative sign, according to the rule of signs. For the inversion or interchange of two entire lines is equivalent to as many inversions or interchanges of two characters as there are characters in a line, that is, as there are columns, and consequently introduces a sign compounded of as many negative signs as there are columns. Hence

THEOREM. A commutant of an even number of columns may be calculated by considering the characters of any one column (no matter which) as supernumerary unpermutable characters, and multiplying the result by the number of permutations of as many things as there are lines in the commutant.

The mark † added to a commutant of an even number of columns will be employed to show that the numerical multiplier is to be omitted. The same mark placed over any one of the columns of the commutant will show that the characters of that particular column are considered as non-permutable.

A determinant is consequently represented indifferently by the notations

$$\begin{pmatrix} 11 \\ 22 \\ \vdots \\ nn \end{pmatrix}^{\dagger} \quad \begin{pmatrix} \overset{\dagger}{11} \\ 22 \\ \vdots \\ nn \end{pmatrix}, \quad \begin{pmatrix} \overset{\dagger}{11} \\ 22 \\ \vdots \\ nn \end{pmatrix};$$

and a commutant of an odd number of columns vanishes identically.

By considering, however, a commutant of an odd number of columns, having the characters of some one column non-permutable, we obtain what will in the sequel be spoken of as commutants of an odd number of columns. This non-permutability will be denoted, as before, by means of the mark † placed over the column in question, and it is to be noticed that it is not, as in the case of a commutant of an even number of columns, indifferent over which of the columns the mark in question is placed; and consequently there would be no meaning in simply adding the mark † to a commutant of an odd number of columns.

A commutant is said to be symmetrical when the symbols $V_{\alpha\beta\gamma\ldots}$ are such as to remain unaltered by any permutations *inter se* of the characters $\alpha, \beta, \gamma \ldots$ A commutant is said to be skew when each symbol $V_{\alpha\beta\gamma\ldots}$ is such as to be altered in sign only according to the rule of signs for any permutations *inter se* of the characters $\alpha, \beta, \gamma \ldots$, this of course implies that the symbol $V_{\alpha\beta\gamma\ldots}$ vanishes when any two of the characters $\alpha, \beta, \gamma \ldots$ are identical. The commutant is said to be demi-skew when $V_{\alpha,\beta,\gamma\ldots}$ is altered in sign only, according to the rule of signs for any permutation *inter se* of non-identical characters $\alpha, \beta, \gamma, \ldots$

An intermutant is represented by a notation similar to that of a commutant. The sets are to be distinguished, whenever it is possible to do so, by placing in contiguity the symbols of the same set, and separating them by a stroke or bar from the symbols

of the adjacent sets. If, however, the symbols of the same set cannot be placed contiguously, we may distinguish the symbols of a set by annexing to them some auxiliary character by way of suffix or otherwise, these auxiliary symbols being omitted in the final result. Thus

$$\begin{pmatrix} 1 & 1 & 1a \\ 2 & 2 & 2b \\ \overline{3} & 3 & 5a \\ 4 & \overline{3} & 6b \end{pmatrix}$$

would show that 1, 2 of the first column and the 3, 4 of the same column, the 1, 2 and the upper 3 of the second column, and the lower 3 of the same column, the 1, 5 of the third column, and the 2, 6 of the same column, form so many distinct sets,—the intermutant containing therefore

$$(2.2.6.1.2.2=)\ 96 \text{ terms.}$$

A commutant of an even number of columns may be considered as an intermutant such that the characters of some one (no matter which) of its columns form each of them by itself a distinct set, and in like manner a commutant of an odd number of columns may be considered as an intermutant such that the characters of some one determinate column form each of them by itself a distinct set.

The distinction of symmetrical, skew and demi-skew applies obviously as well to intermutants as to commutants. The theory of skew intermutants and skew commutants has a connexion with that of Pfaffians.

Suppose $V_{\alpha\beta\gamma\ldots} = V_{\alpha+\beta+\gamma\ldots}$ (which implies the symmetry of the intermutant or commutant) and write for shortness $V_0 = a$, $V_1 = b$, $V_2 = c$, &c. Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 2(ac - b^2),$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 2(ae - 4bd + 3c^2),$$

$$\begin{bmatrix} 0 & \overset{\dagger}{0} \\ 1 & 1 \end{bmatrix} = (ac - b^2), \text{ \&c.}$$

The functions on the second side are evidently hyperdeterminants such as are discussed in my memoir on Linear Transformations, and there is no difficulty in forming directly from the intermutant or commutant on the first side of the equation the symbol of derivation (in the sense of the memoir on Linear Transformations) from which the hyperdeterminant is obtained. Thus

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{ is } \overline{12}^2 . UU, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \overline{12}^4 . UU,$$

$$\begin{bmatrix} 0 & \overset{\dagger}{0} \\ 1 & 1 \end{bmatrix} \text{ is } \overline{12} U^{,0} U^{,1}, \qquad \begin{bmatrix} 0 & 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \overline{12}^3 U^{,0} U^{,1},$$

each permutable column $\begin{matrix} 0 \\ 1 \end{matrix}$ corresponding to a $\overline{12}$ [1] and a non-permutable column $\begin{matrix} \overset{\dagger}{0} \\ 1 \end{matrix}$ changing UU into $U^{\cdot 0}U^{\cdot 1}$. Similarly

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix} \text{ becomes } (\overline{12}\,.\,\overline{13}\,.\,\overline{23})^2\,.\,UUU,$$

$$\begin{pmatrix} 0 & \overset{\dagger}{0} \\ 1 & 1 \\ 2 & 2 \end{pmatrix} \text{ becomes } \overline{12}\,.\,\overline{13}\,.\,\overline{23}\;U^{\cdot 0}U^{\cdot 1}U^{\cdot 2},$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} \text{ becomes } (\overline{12}\,.\,\overline{13}\,.\,\overline{14}\,.\,\overline{23}\,.\,\overline{24}\,.\,\overline{34})^2\;UUUU, \text{ \&c.}$$

The analogy would be closer if in the memoir on Linear Transformations, just as $\overline{12}$ is used to signify $\begin{vmatrix} \xi_1, & \eta_1 \\ \xi_2, & \eta_2 \end{vmatrix}$, $\overline{123}$ had been used to signify $\begin{vmatrix} \xi_1^2, & \xi_1\eta_1, & \eta_1^2 \\ \xi_2^2, & \xi_2\eta_2, & \eta_2^2 \\ \xi_3^2, & \xi_3\eta_3, & \eta_3^2 \end{vmatrix}$ &c., for then $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$ would have corresponded to $\overline{123}^2\,.\,UUU$, $\begin{pmatrix} 0 & \overset{\dagger}{0} \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$ to $\overline{123}\;U^{\cdot 0}U^{\cdot 1}U^{\cdot 2}$; and this would not only have been an addition of some importance to the theory, but would in some instances have facilitated the calculation of hyperdeterminants. The preceding remarks show that the intermutant

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \overline{1} \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & \overline{1} \end{pmatrix}$$

(where the first and fourth blanks in the last column are to be considered as belonging to the same set) is in the hyperdeterminant notation $(12\,.\,34)^2\,.\,(14\,.\,23)\;UUUU$.

[1] Viz. $\begin{matrix} 0 \\ 1 \end{matrix}$ corresponds to $\overline{12}$ because 0 and 1 are the characters occupying the *first* and *second* blanks of a column. If 0 and 1 had been the characters occupying the *second* and *third* blanks in a column, the symbol would have been $\overline{23}$ and so on. It will be remembered, that the symbolic numbers 1, 2..... in the hyperdeterminant notation are merely introduced to distinguish from each other functions which are made identical after certain differentiations are performed.

It will, I think, illustrate the general theory to perform the development of the last-mentioned intermutant. We have

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \overline{1} \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & \overline{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 \\ \overline{0} & \overline{0} & 0 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 0 \\ \overline{0} & \overline{0} & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \overset{\dagger}{1} \\ 1 & 1 & 1 \\ \overline{0} & \overline{0} & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \overset{\dagger}{1} \\ 1 & 1 & 0 \\ \overline{0} & \overline{0} & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= 2 \left\{ \begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \overset{\dagger}{0} \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \overset{\dagger}{1} \\ 1 & 1 & 1 \end{bmatrix} \right\}$$

$$= 2 \{(ad - bc)^2 - 4 (ac - b^2)(bd - c^2)\},$$

$$= 2 (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd),$$

the different steps of which may be easily verified.

The following important theorem (which is, I believe, the same as a theorem of Mr Sylvester's, published in the *Philosophical Magazine*) is perhaps best exhibited by means of a simple example. Consider the intermutant

$$\begin{pmatrix} x & 1 \\ y & 4 \\ \overline{x} & 3 \\ y & 2 \end{pmatrix}$$

where in the first column the sets are distinguished as before by the horizontal bar, but in the second column the 1, 2 are to be considered as forming a set, and the 3, 4 as forming a second set. Then, partially expanding, the intermutant is

$$\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 4 \\ x & 3 \\ y & 2 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 4 \\ x & 3 \\ y & 2 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 4 \\ y & 3 \\ x & 2 \end{pmatrix} + \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 4 \\ y & 3 \\ x & 2 \end{pmatrix},$$

or, since entire horizontal lines may obviously be permuted,

$$\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 2 \\ x & 3 \\ y & 4 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{y} & 1 \\ y & 2 \\ x & 3 \\ x & 4 \end{pmatrix} - \begin{pmatrix} \overset{\dagger}{x} & 1 \\ x & 2 \\ y & 3 \\ y & 4 \end{pmatrix} + \begin{pmatrix} \overset{\dagger}{y} & 1 \\ x & 2 \\ y & 3 \\ x & 4 \end{pmatrix};$$

and, observing that the 1, 2 form a permutable system as do also the 3, 4, the second and third terms vanish, while the first and fourth terms are equivalent to each other; we may therefore write

$$2\begin{pmatrix} \overset{\dagger}{x} & 1 \\ y & 2 \\ \hline x & 3 \\ y & 4 \end{pmatrix} = \begin{pmatrix} x & 1 \\ y & 4 \\ \hline x & 3 \\ y & 2 \end{pmatrix}$$

where on the first side of the equation the bar has been introduced into the second column, in order to show that *throughout* the equation the 1, 2 and the 3, 4 are to be considered as forming distinct sets.

Consider in like manner the expression

$$\begin{pmatrix} x & 1 \\ y & 7 \\ z & 6 \\ \hline x & 8 \\ y & 2 \\ z & 9 \\ \hline x & 4 \\ y & 5 \\ z & 3 \end{pmatrix}$$

where in the first column the sets are distinguished by the horizontal bars and in the second column the characters 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as belonging to distinct sets. The same reasoning as in the former case will show that this is a multiple of

$$\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \\ \hline x & 4 \\ y & 5 \\ z & 6 \\ \hline x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}$$

and to find the numerical multiplier it is only necessary to inquire in how many ways, in the expression first written down, the characters of the first column can be

permuted so that x, y, z may go with 1, 2, 3 and with 4, 5, 6 and with 7, 8, 9. The order of the x, y, z in the second triad may be considered as arbitrary; but once assumed, it determines the place of one of the letters in the first triad; for instance, $x8$ and $z9$ determine $y7$. The first triad must therefore contain $x1$ and $z6$ or $x6$ and $z1$. Suppose the former, then the third triad must contain $z3$, but the remaining two combinations may be either $x4$, $y5$, or $x5$, $y4$. Similarly, if the first triad contained $x6$, $z1$, there would be two forms of the third triad, or a given form of the second triad gives four different forms. There are therefore in all 24 forms, or

$$24\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \\ \hline x & 4 \\ y & 5 \\ z & 6 \\ \hline x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}^{\dagger} = \begin{pmatrix} x & 1 \\ y & 7 \\ z & 6 \\ \hline x & 8 \\ y & 2 \\ z & 9 \\ \hline x & 4 \\ y & 5 \\ z & 3 \end{pmatrix}$$

where the bars in the second column on the first side show that *throughout* the equation 1, 2, 3 and 4, 5, 6 and 7, 8, 9 are to be considered as forming distinct sets. The above proof is in reality perfectly general, and it seems hardly necessary to render it so in terms.

To perceive the significance of the above equation it should be noticed that the first side is a product of determinants, viz.

$$24\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 5 \\ y & 6 \\ z & 7 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}^{\dagger};$$

and if the second side be partially expanded by permuting the characters of the second column, each of the terms so obtained is in like manner a product of determinants, so that

$$24\begin{pmatrix} x & 1 \\ y & 2 \\ z & 3 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 4 \\ y & 5 \\ z & 6 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 7 \\ y & 8 \\ z & 9 \end{pmatrix}^{\dagger} = \begin{pmatrix} x & 1 \\ y & 7 \\ z & 6 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 8 \\ y & 2 \\ z & 9 \end{pmatrix}^{\dagger} \begin{pmatrix} x & 4 \\ y & 5 \\ z & 3 \end{pmatrix}^{\dagger} \pm \&c.,$$

the permutations on the second side being the permutations *inter se* of 1, 2, 3, of 4, 5, 6, and of 7, 8, 9.

It is obvious that the preceding theorem is not confined to intermutants of two columns.

POSTSCRIPT.

I wish to explain as accurately as I am able, the extent of my obligations to Mr Sylvester in respect of the subject of the present memoir. The term permutant is due to him—intermutant and commutant are merely terms framed between us in analogy with permutant, and the names date from the present year (1851). The theory of commutants is given in my memoir in the *Cambridge Philosophical Transactions*, [12], and is presupposed in the memoir on Linear Transformations, [13, 14]. It will appear by the last-mentioned memoir that it was by representing the coefficients of a biquadratic function by $a=1111$, $b=1112=1121=$ &c., $c=1122=$ &c., $d=1222=$ &c., $e=2222$, and forming the commutant $\begin{pmatrix} 1111 \\ 2222 \end{pmatrix}$ that I was led to the function $ae-4bd+3c^2$. The function $ace+2bcd-ad^2-b^2e-c^3$ or $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$ is mentioned in the memoir on Linear Transformations, as brought into notice by Mr Boole. From the particular mode in which the coefficients a, b,... were represented by symbols such as 1111, &c., I did not perceive that the last-mentioned function could be expressed in the commutant notation. The notion of a permutant, in its most general sense, is explained by me in my memoir, "Sur les déterminants gauches," *Crelle*, t. XXXVII. pp. 93—96, [69]; see the paragraph (p. 94) commencing "On obtient ces fonctions, &c." and which should run as follows: "On obtient ces fonctions (dont je reprends ici la théorie) par les propriétés génerales d'un determinant défini comme suit. En exprimant &c.;" the sentence as printed being ".....défini. Car en exprimant &c.," which confuses the sense. [The paragraph is printed correctly 69, p. 411.] Some time in the present year (1851) Mr Sylvester, in conversation, made to me the very important remark, that as one of a class the above-mentioned function,

$$ace+2bcd-ad^2-b^2e-c^3,$$

could be expressed in the commutant notation $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$, viz. by considering $00=a$, $01=10=b$, $02=11=20=c$, $12=21=d$, $22=e$; and the subject being thereby recalled to my notice, I found shortly afterwards the expression for the function

$$a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd$$

(which cannot be expressed as a commutant) in the form of an intermutant, and I was thence led to see the identity, so to say, of the theory of hyperdeterminants, as given in the memoir on Linear Transformations, with the present theory of intermutants. It is understood between Mr Sylvester and myself, that the publication of the present memoir is not to affect Mr Sylvester's right to claim the origination, and to be considered as the first publisher of such part as may belong to him of the theory here sketched out.

105.

CORRECTION OF THE POSTSCRIPT TO THE PAPER ON PERMUTANTS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VII. (1852), pp. 97—98.]

MR SYLVESTER has represented to me that the paragraph relating to his communications conveys an erroneous idea of the nature, purport, and extent of such communications; I have, in fact, in the paragraph in question, singled out what immediately suggested to me the expression of the function $6abcd + 3b^2c^2 - 4ac^3 - 4b^3d - a^2d^2$ as a partial commutant or intermutant, but I agree that a fuller reference ought to have been made to Mr Sylvester's results, and that the paragraph in question would more properly have stood as follows:

"Under these circumstances Mr Sylvester communicated to me a series of formal statements, not only oral but in writing, to the effect that he had discovered a permutation method of obtaining as many invariants—viz. commutantive invariants—by direct inspection from a function of any degree of any number of letters as the index of the degree contains even factors; one of these commutantive invariants being in fact the function $ace + 2bcd - ae^2 - bd^2 - c^3$, expressible, according to Mr Sylvester's notation, by $\begin{pmatrix} a^2, & ab, & b^2 \\ a^2, & ab, & b^2 \end{pmatrix}$; and, according to the notation of my memoir in the *Camb. Phil. Trans.*, supposing $00 = a$, $01 = 10 = b$, $02 = 11 = 20 = c$, &c. by $\begin{vmatrix} 00 \\ 11 \\ 22 \end{vmatrix}$."

Mr Sylvester and I shall, I have no doubt, be able to agree to a joint statement of any further correction or explanation which may be required.

106.

ON THE SINGULARITIES OF SURFACES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VII. (1852), pp. 166—171.]

IN the following paper, for symmetry of nomenclature and in order to avoid ambiguities, I shall, with reference to plane curves and in various phrases and compound words, use the term "node" as synonymous with double point, and the term "spinode" as synonymous with cusp. I shall, besides, have occasion to consider the several singularities which I call the "flecnode," the "oscnode," the "fleflecnode," and the "tacnode:" the flecnode is a double point which is a point of inflexion on one of the branches through it; the oscnode is a double point which is a point of osculation on one of the branches through it; the fleflecnode is a double point which is a point of inflexion on each of the branches through it; and the tacnode is a double point where two branches touch. And it may be proper to remark here, that a tacnode may be considered as a point resulting from the coincidence and amalgamation of two double points (and therefore equivalent to twelve points of inflexion); or, in a different point of view, [?] as a point uniting the characters of a spinode and a flecnode. I wish to call to mind here the definition of conjugate tangent lines of a surface, viz. that a tangent to the curve of contact of the surface with any circumscribed developable and the corresponding generating line of the developable, are conjugate tangents of the surface.

Suppose, now, that an absolutely arbitrary surface of any order be intersected by a plane: the curve of intersection has not in general any singularities other than such as occur in a perfectly arbitrary curve of the same order; but as a plane can be made to satisfy one, two, or three conditions, the curve may be made to acquire singularities which do not occur in such absolutely arbitrary curve.

Let a single condition only be imposed on the plane. We may suppose that the curve of intersection has a node; the plane is then a tangent plane and the node is the point of contact—of course any point on the surface may be taken for

the node. We may if we please use the term "nodes of a surface," "node-planes of a surface," as synonymous with the points and tangent planes of a surface. And it will be convenient also to use the word node-tangents to denote the tangents to the curve of intersection at the node; it may be noticed here that the node-tangents are conjugate tangents of the surface.

Next let two conditions be imposed upon the plane: there are three distinct cases to be considered.

First, the curve of intersection may have a flecnode. The plane (which is of course still a tangent plane at the flecnode) may be termed a flecnode-plane; the flecnodes are singular points on the surface lying on a curve which may be termed the "flecnode-curve[1]," and the flecnode-planes give rise to a developable which may be termed the flecnode-develope. The "flecnode-tangents" are the tangents to the curve of intersection at the flecnode; the tangent to the inflected branch may be termed the "singular flecnode-tangent," and the tangent to the other branch the "ordinary flecnode-tangent."

Secondly, the curve of intersection may have a spinode. The plane (which is of course still a tangent plane at the spinode) may be termed a spinode-plane; the spinodes are singular points on the surface lying on a curve which may be termed the "spinode-curve[2]." And the spinode-planes give rise to a developable which may be termed the "spinode-develope." Also the "spinode-tangent" is the tangent to the curve of intersection at the spinode.

Thirdly, the curve of intersection may have two nodes, or what may be termed a "node-couple." The plane (which is a tangent plane at each of the nodes and therefore a double tangent plane) may be also termed a "node-couple-plane." The node-couples are pairs of singular points on the surface lying in a curve which may be termed the "node-couple-curve," and the node-couple-planes give rise to a developable which may be termed the "node-couple-develope." The tangents to the curve of intersection at the two nodes of a node-couple might, if the term were required, be termed the "node-couple-tangents." Also one of the nodes of a node-couple may be termed a "node-with-node," and the tangents to the curve of intersection at such point will be the "node-with-node-tangents."

[1] The flecnode-curve, defined as the locus of the points through which can be drawn a line meeting the surface in four consecutive points, was, so far as I am aware, first noticed in Mr Salmon's paper "On the Triple Tangent Planes of a Surface of the Third Order" (*Journal*, t. IV. [1849], pp. 252—260), where Mr Salmon, among other things, shows that the order of the surface being n, the curve in question is the intersection of the surface with a surface of the order $11n-24$.

[2] The notion of a spinode, considered as the point where the indicatrix is a parabola (on which account the spinode has been termed a parabolic point) may be found in Dupin's *Développements de Géométrie*: the most important step in the theory of these points is contained in Hesse's memoir "Ueber die Wendepuncte der Curven dritter Ordnung" (*Crelle*, t. XXVIII. [1848], pp. 97—107), where it is shown that the spinode-curve is the curve of intersection of the surface supposed as before of the order n, with a certain surface of the order $4(n-2)$. See also Mr Salmon's memoir "On the Condition that a Plane should touch a surface along a Curve Line" (*Journal*, t. III. [1848], pp. 44—46).

It is hardly necessary to remark that the flecnode-curve is *not* the edge of regression of the flecnode-develope, and the like remark applies *m.m.* to the spinode-curve and the node-couple curve.

Finally, let three conditions be imposed upon the plane: there are six distinct cases to be considered, in each of which we have no longer curves and developes, but only singular points and singular tangent planes determinate in number.

First, the curve of intersection may have an oscnode. The plane (which continues a tangent plane at the oscnode) is an "oscnode-plane." The "oscnode-tangents" are the tangents to the curve of intersection at the oscnode; the tangent to the osculating branch is the "singular oscnode-tangent;" and the tangent to the other branch the "ordinary oscnode-tangent."

Secondly, the curve of intersection may have a fleflecnode. The plane (which continues a tangent plane at the fleflecnode) is a "fleflecnode-plane." The "fleflecnode-tangents" are the tangents to the curve of intersection at the fleflecnode.

Thirdly, the curve of intersection may have a tacnode. The plane (which continues a tangent plane at the tacnode) is a "tacnode-plane." The "tacnode-tangent" is the tangent to the curve of intersection at the tacnode.

Fourthly, the curve of intersection may have a node and a flecnode, or what may be termed a node-and-flecnode. The plane (which is a tangent plane at the node and also at the flecnode, where it is obviously a flecnode-plane) is a "node-and-flecnode-plane." The "node-and-flecnode-tangents," if the term were required, would be the tangents to the curve of intersection at the node and at the flecnode of the node-and-flecnode. The node of the node-and-flecnode may be distinguished as the node-with-flecnode, and the flecnode as the flecnode-with-node, and we have thus the terms "node-with-flecnode-tangents," "flecnode-with-node-tangents," "singular flecnode-with-node-tangent," and "ordinary flecnode-with-node-tangent."

Fifthly, the curve of intersection may have a node and also a spinode, or what may be termed a "node-and-spinode." The plane (which is a tangent plane at the node, and is also a tangent plane at the spinode, where it is obviously a spinode-plane) is a "node-and-spinode-plane." The node-and-spinode-tangents, if the term were required, would be the tangents at the node and the tangent at the spinode of the node-and-spinode to the curve of intersection. The node of the node-and-spinode may be distinguished as the "node-with-spinode," and the spinode as the "spinode-with-node," and we have thus the terms "node-with-spinode-tangent," "spinode-with-node-tangent."

Sixthly, the curve of intersection may have three nodes, or what may be termed a "node-triplet." The plane (which is a triple tangent plane touching the surface at each of the nodes) is a "node-triplet-plane." The "node-triplet-tangents," if the term were required, would be the tangents to the curve of intersection at the nodes of the node-triplet. Each node of the node-triplet may be distinguished as a "node-

with-node-couple," and the tangents to the curve of intersection at such nodes are "node-with-node-couple-tangents." The terms "node-couple-with-node," "node-couple-with-node-tangent," might be made use of if necessary.

It should be remarked that the oscnodes lie on the flecnode-curve, as do also the fleflecnodes; these latter points are real double points of the flecnode-curve. The tacnodes are points of intersection and (what will appear in the sequel) points of contact of the flecnode-curve, the spinode-curve, and the node-couple-curve. The spinode-with-nodes are points of intersection of the spinode-curve and node-couple-curve, and the flecnode-with-nodes are points of intersection of the flecnode-curve and node-couple-curve; the node-with-node-couples are real double points (entering in triplets) of the node-couple-curve.

Consider for a moment an arbitrary curve on the surface; the locus of the node-tangents at the different points of this curve is in general a skew surface, which may however, in cases to be presently considered, degenerate in different ways.

Reverting now to the flecnode-curve, it may be shown that the singular flecnode-tangent coincides with the tangent of the flecnode-curve. For consider on a surface two consecutive points such that the line joining them meets the surface in two points consecutive to the first-mentioned two points. The line meets the surface in four consecutive points, it is therefore a singular flecnode-tangent; *each* of the first-mentioned two points must be on the flecnode-curve, or the singular flecnode-tangent touches the flecnode-curve. The two flecnode-tangents are by a preceding observation conjugate tangents. It follows that the skew surface, locus of the flecnode-tangents, breaks up into two surfaces, each of which is a developable, viz. the locus of the singular flecnode-tangents is the developable having the flecnode-curve for its edge of regression, and the locus of the ordinary flecnode-tangents is the flecnode-develope. Of course at the tacnode, the tacnode-tangent touches the flecnode-curve.

Passing next to the spinode-curve, the spinode-plane and the tangent-plane at a consecutive point along the spinode-tangent are identical[1], or their line of intersection is indeterminate. The spinode-tangent is therefore the conjugate tangent to *any* other tangent line at the spinode, and therefore to the tangent to the spinode-curve. It follows that the surface locus of the spinode-tangents degenerates into a developable surface twice repeated, viz. the spinode-develope. Consider the tacnode as two coincident nodes; each of these nodes, by virtue of its constituting, in conjunction with the other, a tacnode, is on the spinode-curve; or, in other words, the tacnode-tangent touches the spinode-curve, and the same reasoning proves that it touches the node-couple-curve. It has already been seen that the tacnode-tangent touches the flecnode-curve; consequently the tacnode is a point, not of simple intersection only, but of contact, of the flecnode-curve, the spinode-curve, and the node-couple-curve.

In virtue of the principle of the spinode-plane being identical with the tangent plane at a consecutive point along the spinode tangent, it appears that the tacnode-

[1] It must not be inferred that the tangent plane at such consecutive point is a spinode-plane; this is obviously not the case.

plane is a stationary plane, as well of the flecnode-develope as of the spinode-develope, and it would at first sight appear that it must be also a stationary tangent plane of the node-couple-develope. But this is not so; the node-with-node-planes envelope, not the node-couple-develope, but the node-couple-develope twice repeated: the tacnode-plane is in a sense a stationary plane on such duplicate developable, but not in any manner on the single developable. The tacnode-plane is an ordinary tangent plane of the node-couple-develope.

Consider now a spinode-with-node, which we have seen is a point of intersection of the spinode-curve and node-couple-curve. The tangent plane at a consecutive point along the spinode-with-node-tangent, is *identical* with the spinode-with-node-plane; the curve of intersection of the tangent plane at such consecutive point has therefore a node at the node-with-spinode, or the tangent plane in question is a node-couple-plane, and the point of contact is a point on the node-couple-curve. Consequently the spinode-with-node-tangent touches the node-couple-curve, and thence also the spinode-with-node-plane is a stationary tangent plane of the node-couple-develope.

It should be remarked that no circumscribed developable can have a stationary tangent plane except the tangent planes at the points where the curve of contact meets the spinode-curve, and any one of these planes is only a stationary plane when the curve of contact touches the spinode-tangent; and that the node-couple-curve and the flecnode-curve do not intersect the spinode-curve except in the points which have been discussed.

Recapitulating, the node-couple-curve and the spinode-curve touch at the tacnodes, and intersect at the spinode-with-nodes: moreover, the tacnode-planes are stationary planes of the spinode-develope, and the spinode-with-node-planes are stationary planes of the node-couple-develope. Besides this, the two curves are touched at the tacnodes by the flecnode-curve, and the tacnode-planes are stationary planes of the flecnode-develope.

107.

ON THE THEORY OF SKEW SURFACES.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VII. (1852), pp. 171—173.]

A SURFACE of the nth order is a surface which is met by an indeterminate line in n points. It follows immediately that a surface of the nth order is met by an indeterminate plane in a curve of the nth order.

Consider a skew surface or the surface generated by a singly infinite series of lines, and let the surface be of the nth order. Any plane through a generating line meets the surface in the line itself and in a curve of the $(n-1)^{\text{th}}$ order. The generating line meets this curve in $(n-1)$ points. Of these points one, viz. that adjacent to the intersection of the plane with the consecutive generating line, is a unique point; the other $(n-2)$ points form a system. Each of the $(n-1)$ points are *sub modo* points of contact of the plane with the surface, but the proper point of contact is the unique point adjacent to the intersection of the plane with the consecutive generating line. Thus every plane through a generating line is an ordinary tangent plane, the point of contact being a point on the generating line. It is not necessary for the present purpose, but I may stop for a moment to refer to the known theorems that the anharmonic ratio of any four tangent planes through the same generating line is equal to the anharmonic ratio of their points of contact, and that the locus of the normals to the surface along a generating line is a hyperbolic paraboloid. Returning to the $(n-2)$ points in which, together with the point of contact, a generating line meets the curve of intersection of the surface and a plane through the generating line, these are fixed points independent of the particular plane, and are the points in which the generating line is intersected by other generating lines. There is therefore on the surface a double curve intersected in $(n-2)$ points by each generating line of the surface—a property which, though insufficient to determine the order of this double curve, shows that the order cannot be less than $(n-2)$. (Thus for $n=4$, the above reasoning shows that the double-curve must be

at least of the second order: assuming for a moment that it is in any case precisely of this order, it obviously cannot be a plane curve, and must therefore be two non-intersecting lines. This suggests at any rate the existence of a class of skew surfaces of the fourth order generated by a line which always passes through two fixed lines and by some other condition not yet ascertained; and it would appear that surfaces of the second order constitute a degenerate species belonging to the class in question.)

In particular cases a generating line will be intersected by the consecutive generating line. Such a generating line touches the double curve.

Consider now a point not on the surface; the planes determined by this point and the generating lines of the surface are the tangent planes through the point; the intersections of consecutive tangent planes are the tangent lines through the point; and the cone generated by these tangent lines or enveloped by the tangent planes is the tangent cone corresponding to the point. This cone is of the nth class. For considering a line through the point, this line meets the surface in n points, i.e. it meets n generating lines of the surface; and the planes through the line and these n generating lines, are of course tangent planes to the cone: that is, n tangent planes can be drawn to the cone through a given line passing through the vertex. The cone has not in general any lines of inflexion, or, what is the same thing, stationary tangent planes. For a stationary tangent plane would imply the intersection of two consecutive generating lines of the surface. And since the number of generating lines intersected by a consecutive generating line, and therefore the number of planes through two consecutive generating lines, is finite, no such plane passes through an indeterminate point. The tangent cone will have in general a certain number of double tangent planes; let this number be x. We have therefore a cone of the class n, number of double tangent planes x, number of stationary tangent planes 0. Hence, if m be the order of the cone, α the number of its double lines, and β the number of its cuspidal or stationary lines,

$$m = n(n-1) - 2x,$$

$$\beta = 3n(n-2) - 6x,$$

$$\alpha = \tfrac{1}{2}n(n-2)(n^2-9) - 2x(n^2-n-6) + 2x(x-1).$$

This is the proper tangent cone, but the cone through the double curve is *sub modo* a tangent cone, and enters as a square factor into the equation of the general tangent cone of the order $n(n-1)$. Hence, if X be the order of the double curve, and therefore of the cone through this curve,

$$m + 2X = n(n-1), \quad \text{and therefore } X = x;$$

that is, the number of double tangent planes to the tangent cone is equal to the order of the double curve. It does not appear that there is anything to determine x; and if this is so, skew surfaces of the nth order may be considered as forming different families according to the order of the double curve upon them.

To complete the theory, it should be added that a plane intersects the surface in a curve of the nth order having x double points but no cusps.

108.

ON CERTAIN MULTIPLE INTEGRALS CONNECTED WITH THE THEORY OF ATTRACTIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VII. (1852), pp. 174—178.]

IT is easy to deduce from Mr Boole's formula, given in my paper "On a Multiple Integral connected with the theory of Attractions," *Journal*, t. II. [1847], pp. 219—223, [44], the equation

$$\int \frac{d\xi\, d\eta \ldots}{[(\xi-\alpha)^2+(\eta-\beta)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{fg\ldots\pi^{\frac{1}{2}n}}{\theta_1^{\,n}\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)}\int_\epsilon^\infty \frac{s^{q-1}(\theta_1^2-\sigma)^q\, ds}{\sqrt{\left\{\left(s+\frac{f^2}{\theta_1^2}\right)\left(s+\frac{g^2}{\theta_1^2}\right)\ldots\right\}}}$$

where n is the number of variables of the multiple integral, and the condition of the integration is

$$\frac{(\xi-\alpha_1)^2}{f^2}+\frac{(\eta-\beta_1)^2}{g^2}+\ldots \leqq 1;$$

also where

$$\sigma = \frac{(\alpha-\alpha_1)^2}{s+\frac{f^2}{\theta_1^2}}+\frac{(\beta-\beta_1)^2}{s+\frac{g^2}{\theta_1^2}}\ldots+\frac{v^2}{s},$$

and ϵ is the positive root of

$$\theta_1^2 = \frac{(\alpha-\alpha_1)^2}{\epsilon+\frac{f^2}{\theta_1^2}}+\frac{(\beta-\beta_1)^2}{\epsilon+\frac{g^2}{\theta_1^2}}\ldots+\frac{v^2}{\epsilon}.$$

Suppose $f=g\ldots=\theta_1$, and write $(\alpha-\alpha_1)^2+\ldots=k^2$, we obtain

$$\int \frac{d\xi\ldots}{[(\xi-\alpha)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)}\int_\epsilon^\infty \frac{s^{q-1}(\theta_1^2-\sigma)^q\, ds}{(1+s)^{\frac{1}{2}n}},$$

the limiting condition for the multiple integral being

$$(\xi-\alpha_1)^2+\ldots \gtreqless \theta_1^2,$$

and the function σ, and limit ϵ, being given by

$$\sigma=\frac{k^2}{1+s}+\frac{v^2}{s}, \quad \theta_1^2=\frac{k^2}{1+\epsilon}+\frac{v^2}{\epsilon},$$

ϵ denoting, as before, the positive root. Observing that the quantity under the integral sign on the second side vanishes for $s=\epsilon$, there is no difficulty in deducing, by a differentiation with respect to θ_1, the formula

$$\int\frac{d\Sigma}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{2\theta_1\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q)}\int_\epsilon^\infty\frac{s^{q-1}(\theta_1^2-\sigma)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n}},$$

where $d\Sigma$ is the element of the surface $(\xi-\alpha_1)^2+\ldots=\theta_1^2$, and the integration is extended over the entire surface.

A slight change of form is convenient. We have

$$\theta_1^2-\sigma=\theta_1^2-\frac{k^2}{1+s}-\frac{v^2}{s}=\frac{1}{s(1+s)}(\theta_1^2s^2+\chi s-v^2),$$

if we suppose

$$\chi=\theta_1^2-k^2-v^2.$$

The formulæ then become

$$\int\frac{d\xi\ldots}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)}\int_\epsilon^\infty\frac{(\theta_1^2s^2+\chi s-v^2)^q\,ds}{s(1+s)^{\frac{1}{2}n+q}},$$

$$\int\frac{d\Sigma}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{2\pi^{\frac{1}{2}n}\theta_1}{\Gamma(\frac{1}{2}n-q)\,\Gamma q}\int_\epsilon^\infty\frac{(\theta_1^2s^2+\chi s-v^2)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

in which ϵ is the positive root of the equation

$$\theta_1^2\epsilon^2+\chi\epsilon-v^2=0.$$

I propose to transform these formulæ by means of the theory of images; it will be convenient to investigate some preliminary formulæ. Suppose $\lambda^2=\alpha^2+\beta^2\ldots$, $\lambda_1^2=\alpha_1^2+\beta_1^2\ldots$; also consider the new constants $a, b, \ldots, a_1, b_1, \ldots, u, f_1$, determined by the equations

$$\frac{\delta^2\alpha}{\lambda^2+v^2}=a, \qquad \frac{\delta^2\alpha_1}{\lambda_1^2-\theta_1^2}=a_1,$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{\delta^2 v}{\lambda^2+v^2}=u, \qquad \frac{\delta^2\theta_1}{\lambda_1^2\sim\theta_1^2}=f_1,$$

where δ is arbitrary. Then, putting

$$l^2=a^2+b^2\ldots, \quad l_1^2=a_1^2+b_1^2\ldots,$$

it is easy to see that

$$(\lambda^2+\upsilon^2)(l^2+u^2)=\delta^4, \quad (\lambda_1^2-\theta_1^2)(l_1^2-f_1^2)=\delta^4,$$

and

$$\frac{\delta^2 a}{l^2+u^2}=\alpha, \qquad \frac{\delta^2 a_1}{l_1^2-f_1^2}=\alpha_1,$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{\delta^2 u}{l^2+u^2}=\upsilon, \qquad \frac{\delta^2 f_1}{l_1^2\sim f_1^2}=\theta_1.$$

Proceeding to express the single integrals in terms of the new constants, we have in the first place $k^2=\delta^4 k^2$, where

$$k^2=\left(\frac{a}{l^2+u^2}-\frac{a_1}{l_1^2-f_1^2}\right)^2+\dots;$$

or if we write

$$aa_1+bb_1\dots=ll_1\cos\omega,$$

we have

$$k^2=\frac{l^2}{(l^2+u^2)^2}+\frac{l_1^2}{(l_1^2-f_1^2)^2}-\frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)}.$$

Hence also $\chi=\delta^4 j$, where

$$j=\frac{f_1^2}{(l_1^2-f_1^2)^2}-k^2-\frac{u^2}{(l^2+u^2)^2},$$

whence

$$-j=\frac{1}{l^2+u^2}+\frac{1}{l_1^2-f_1^2}+\frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)},$$
$$=\frac{1}{(l^2+u^2)(l_1^2-f_1^2)}\{p^2+u^2-f_1^2\},$$

where $p^2=l^2+l_1^2-2ll_1\cos\omega$, that is

$$p^2=(a-a_1)^2+(b-b_1)^2+\dots;$$

consequently $\theta_1^2 s^2+\chi s-\upsilon^2=\delta^4\Pi$, where Π is given by

$$\Pi=\frac{f_1^2}{(l_1^2-f_1^2)^2}s^2-\frac{(p^2+u^2-f_1^2)}{(l^2+u^2)(l_1^2-f_1^2)}s-\frac{u^2}{(l^2+u^2)^2};$$

and it is clear that ϵ will be the positive root of

$$0=\frac{f_1^2}{(l_1^2-f_1^2)^2}\epsilon^2-\frac{(p^2+u^2-f_1^2)}{(l^2+u^2)(l_1^2-f_1^2)}\epsilon-\frac{u^2}{(l^2+u^2)^2}.$$

It may be noticed that, in the particular case of $u=0$, the roots of this equation are 0, and $\frac{(p^2-f_1^2)(l_1^2-f_1^2)}{l^2 f_1^2}$. Consequently if $p^2-f_1^2$ and $l_1^2-f_1^2$ are of opposite signs, we have $\epsilon=0$; but if $p^2-f_1^2$ and $l_1^2-f_1^2$ are of the same sign, $\epsilon=\frac{(p^2-f_1^2)(l_1^2-f_1^2)}{l^2 f_1^2}$.

In order to transform the double integrals, considering the new variables $x, y, \ldots$, I write $x^2+y^2\ldots=r^2$ and

$$\xi=\frac{\delta^2 x}{r^2}, \ldots$$

whence also, if $\xi^2+\eta^2+\ldots=\rho^2$ (which gives $r\rho=\delta^2$), we have

$$x=\frac{d^2\xi}{\rho^2}, \ldots;$$

also it is immediately seen that

$$(\xi-\alpha)^2+\ldots v^2=\frac{\delta^4}{(l^2+u^2)\,r^2}\{(x^2-a)^2+\ldots+u^2\},$$

$$(\xi-\alpha_1)^2\ldots-\theta_1^2=\frac{\delta^4}{(l_1^2-f_1^2)\,r^2}\{(x-a_1)^2+\ldots-f_1^2\};$$

and from the latter equation it follows that the limiting condition for the first integral is $(x-a_1)^2+\ldots\overline{\gtrless}f_1^2$ (there is no difficulty in seeing that the sign $<$ in the former limiting condition gives rise here to the sign $>$), and that the second integral has to be extended over the surface $(x-a_1)^2+\ldots=f_1^2$. Also if dS represent the element of this surface, we may obtain

$$d\xi\,d\eta\ldots=\frac{\delta^{2n}}{r^{2n}}dx\,dy\ldots,\quad d\Sigma=\frac{\delta^{2n-2}}{r^{2n-2}}dS;$$

and, combining the above formulæ, we obtain

$$\int\frac{dx\,dy\ldots}{(x^2+y^2\ldots)^{\frac{1}{2}n+q}\{(x-a)^2+(y-b)^2\ldots+u^2\}^{\frac{1}{2}n-q}}$$
$$=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)\,(l^2+u^2)^{\frac{1}{2}n-q}}\int_\epsilon^\infty\frac{\Pi^q ds}{s\,(1+s)^{\frac{1}{2}n+q}},$$

the limiting condition of the multiple integral being

$$(x-a_1)^2+(y-b_1)^2\ldots\overline{\gtrless}f_1^2;$$

and

$$\int\frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n+q-1}\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-q}}$$
$$=\frac{2\pi^{\frac{1}{2}n}f_1}{\Gamma(\frac{1}{2}n-q)\,\Gamma q\,(l^2+u^2)^{\frac{1}{2}n-q}\,(l_1^2\sim f_1^2)}\int_\epsilon^\infty\frac{\Pi^{q-1}ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

where dS is the element of the surface $(x-a_1)^2+(y-b_1)^2\ldots=f_1^2$, and the integration extends over the entire surface. In these formulæ, l, l_1, p, Π denote as follows:

$$l^2=a^2+b^2+\ldots,\quad l_1^2=a_1^2+b_1^2+\ldots,\quad p^2=(a-a_1)^2+(b-b_1)^2+\ldots,$$

$$\Pi=\frac{f_1^2}{(l_1^2-f_1^2)^2}s^2-\frac{(p^2+u^2-f_1^2)}{(l_1^2-f_1^2)\,(l^2+u^2)}s-\frac{u^2}{(l^2+u^2)^2};$$

and ϵ is the positive root of the equation $\Pi=0$.

The only obviously integrable case is that for which in the second formula $q=1$; this gives

$$\int \frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n}\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-1}} = \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n)(l^2+u^2)^{\frac{1}{2}n-1}(l_1^2-f_1^2)(1+\epsilon)^{\frac{1}{2}n-1}}.$$

In the case of $u=0$, we have, as before, when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of opposite signs, $\epsilon=0$, and therefore $1+\epsilon=1$; but when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of the same sign, the value before found for ϵ gives

$$1+\epsilon=\frac{1}{l^2 f_1^2}\{l^2 f_1^2+(p^2-f_1^2)(l_1^2-f_1^2)\}.$$

Consider the image of the origin with respect to the sphere $(x-a_1)^2+(y-b_1)^2\ldots=f_1^2$, the coordinates of this image are

$$\frac{a_1}{l_1^2}(l_1^2-f_1^2), \quad \frac{b_1}{l_1^2}(l_1^2-f_1^2), \ldots,$$

and consequently, if μ be the distance of this image from the point $(a, b\ldots)$, we have

$$\mu^2=\{a-\frac{a}{l_1^2}(l_1^2-f_1^2)\}^2+\ldots$$

$$=\frac{1}{l_1^2}\{l^2 f_1^2+(p^2-f_1^2)(l_1^2-f_1^2)\};$$

whence, by a simple reduction,

$$1+\epsilon=\frac{l_1^2\mu^2}{l^2 f_1^2},$$

or the values of the integral are

$$p^2-f_1^2 \text{ and } l_1^2-f_1^2 \text{ opposite signs, } I=\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\frac{f_1}{l^{n-2}(l_1^2-f_1^2)},$$

$$p^2-f_1^2 \text{ and } l_1^2-f_1^2 \text{ the same sign, } I=\frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\frac{f_1^{n-1}}{l_1^{n-2}\mu^{n-2}(l_1^2-f_1^2)},$$

where μ is the distance from the point $(a, b\ldots)$ of the image of the origin with respect to the sphere $(x-a_1)^2+\ldots-f_1^2=0$.

Stone Buildings, August 6, 1850.

109.

ON THE RATIONALISATION OF CERTAIN ALGEBRAICAL EQUATIONS.

[From the *Cambridge and Dublin Mathematical Journal*, vol. VIII. (1853), pp. 97—101.]

SUPPOSE

$$x+y=0, \quad x^2=a, \quad y^2=b;$$

then if we multiply the first equation by 1, xy, and reduce by the two others, we have

$$x+\ \ y=0,$$
$$bx+ay=0,$$

from which, eliminating x, y,

$$\begin{vmatrix} 1, & 1 \\ b, & a \end{vmatrix}=0;$$

which is the equation between a and b; or, considering x, y as quadratic radicals, the rational equation between x, y. So if the original equation be multiplied by x, y, we have

$$a+xy=0,$$
$$b+xy=0;$$

or, eliminating 1, xy,

$$\begin{vmatrix} a, & 1 \\ b, & 1 \end{vmatrix}=0,$$

which may be in like manner considered as the rational equation between x, y.

The preceding results are of course self-evident, but by applying the same process to the equations

$$x+y+z=0, \quad x^2=a, \quad y^2=b, \quad z^2=c,$$

we have results of some elegance. Multiply the equation first by 1, yz, zx, xy, reduce and eliminate the quantities x, y, z, xyz, we have the rational equation

$$\begin{vmatrix} & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ 1 & b & a & . \end{vmatrix} = 0;$$

and again, multiply the equation by x, y, z, xyz, reduce and eliminate the quantities 1, yz, zx, xy, the result is

$$\begin{vmatrix} & a & b & c \\ a & . & 1 & 1 \\ b & 1 & . & 1 \\ c & 1 & 1 & . \end{vmatrix} = 0,$$

which is of course equivalent to the preceding one (the two determinants are in fact identical in value), but the form is essentially different. The former of the two forms is that given in my paper "On a theorem in the Geometry of Position" (*Journal*, vol. II. [1841] p. 270 [1]): it was only very recently that I perceived that a similar process led to the latter of the two forms.

Similarly, if we have the equations

$$x + y + z + w = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c, \quad w^2 = d,$$

then multiplying by 1, yz, zx, xy, xw, yw, zw, $xyzw$, reducing and eliminating the quantities in the outside row,

we have the result

$$\begin{array}{c|ccc|c|ccc|c|} & x, & y, & z, & w, & yzw, & zwx, & wxy, & xyz \\ \hline 1 & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & 1 & . & . & 1 \\ c & . & a & . & . & 1 & . & 1 \\ b & a & . & . & . & . & 1 & 1 \\ \hline d & . & . & a & . & 1 & 1 & . \\ . & d & . & b & 1 & . & 1 & . \\ . & . & d & c & 1 & 1 & . & . \\ \hline . & . & . & . & a & b & c & d \\ \hline \end{array} = 0;$$

so if we multiply the equations by x, y, z, w, yzw, zwx, wxy, and xyz, reduce and eliminate the quantities in the outside row,

we have the result

$$\begin{array}{c|ccc|ccc|c} 1, & yz, & zx, & xy, & xw, & yw, & zw, & xyzw \\ \hline a & . & 1 & 1 & 1 & . & . & . \\ b & 1 & . & 1 & . & 1 & . & . \\ c & 1 & 1 & . & . & . & 1 & . \\ \hline d & . & . & . & 1 & 1 & 1 & . \\ \hline . & d & . & . & . & c & b & 1 \\ . & . & d & . & c & . & a & 1 \\ . & . & . & d & b & a & . & 1 \\ \hline . & a & b & c & . & . & . & 1 \end{array} = 0,$$

which however is not essentially distinct from the form before obtained, but may be derived from it by an interchange of lines and columns.

And in general for any *even* number of quadratic radicals the two forms are not essentially distinct, but may be derived from each other by interchanging lines and columns, while for an *odd* number of quadratic radicals the two forms cannot be so derived from each other, but are essentially distinct.

I was indebted to Mr Sylvester for the remark that the above process applies to radicals of a higher order than the second. To take the simplest case, suppose

$$x + y = 0, \quad x^3 = a, \quad y^3 = b;$$

and multiply first by 1, x^2y, xy^2; this gives

$$\begin{aligned} x + y \quad\quad . &= 0 \\ . \quad ay + x^2y^2 &= 0 \\ bx \quad . + x^2y^2 &= 0; \end{aligned}$$

or, eliminating,

$$\begin{vmatrix} 1 & 1 & . \\ . & a & 1 \\ b & . & 1 \end{vmatrix} = 0;$$

next multiply by x, y, x^2y^2; this gives

$$\begin{aligned} x^2 \quad\quad . + xy &= 0 \\ . \quad\quad y^2 + xy &= 0 \\ bx^2 + ay^2 \quad\quad . &= 0; \end{aligned}$$

or, eliminating,

$$\begin{vmatrix} 1 & . & 1 \\ . & 1 & 1 \\ b & a & . \end{vmatrix} = 0;$$

and lastly, multiply by x^2, y^2, xy; this gives

$$\begin{aligned} a + x^2y \quad\quad . &= 0 \\ b \quad . + xy^2 &= 0 \\ . \quad x^2y + xy^2 &= 0; \end{aligned}$$

or, eliminating,

$$\begin{vmatrix} a & 1 & . \\ b & . & 1 \\ . & 1 & 1 \end{vmatrix} = 0;$$

where it is to be remarked that the second and third forms are not essentially distinct, since the one may be derived from the other by the interchange of lines and columns.

Applying the preceding process to the system

$$x + y + z = 0, \quad x^3 = a, \quad y^3 = b, \quad z^3 = c;$$

multiply first by 1, xyz, $x^2y^2z^2$, x^2z, y^2x, z^2y, x^2y, y^2z, z^2x, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{c} \begin{array}{ccc|ccc|ccc} x, & y, & z, & y^2z^2, & x^2yz, & y^2zx, & z^2xy, & z^2x^2, & x^2y^2 \end{array} \\ \begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & . & . & . & . & . & . \\ \hline . & . & . & 1 & 1 & 1 & . & . & . \\ \hline . & . & . & . & . & . & a & b & c \\ \hline . & . & a & 1 & . & . & . & 1 & . \\ b & . & . & . & 1 & . & . & . & 1 \\ . & c & . & . & . & 1 & 1 & . & . \\ \hline . & a & . & 1 & . & . & . & . & 1 \\ . & . & b & . & 1 & . & 1 & . & . \\ c & . & . & . & . & 1 & . & 1 & . \end{array} \end{array} = 0;$$

next multiply by x, y, z, y^2z^2, z^2x^2, x^2y^2, x^2yz, y^2zx, z^2xy, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{c} \begin{array}{ccc|ccc|ccc} x^2, & y^2, & z^2, & yz, & zx, & xy, & xy^2z^2, & yz^2x^2, & zx^2y^2 \end{array} \\ \begin{array}{ccc|ccc|ccc} 1 & . & . & . & 1 & 1 & . & . & . \\ . & 1 & . & 1 & . & 1 & . & . & . \\ . & . & 1 & 1 & 1 & . & . & . & . \\ \hline . & c & b & . & . & . & 1 & . & . \\ c & . & a & . & . & . & . & 1 & . \\ b & a & . & . & . & . & . & . & 1 \\ \hline . & . & . & a & . & . & . & 1 & 1 \\ . & . & . & . & b & . & 1 & . & 1 \\ . & . & . & . & . & c & 1 & 1 & . \end{array} \end{array} = 0;$$

lastly, multiply by x^2, y^2, z^2, yz, zx, xy, xy^2z^2, yz^2x^2, xy^2z^2, reduce and eliminate the quantities in the outside row,

the result is

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & xyz, & x^2y^2z^2, & yz^2, & zx^2, & xy^2, & y^2z, & z^2x, & x^2y \end{array} \\ \left|\begin{array}{ccc|ccc|ccc} a & . & . & . & 1 & . & . & . & 1 \\ b & . & . & . & . & 1 & 1 & . & . \\ c & . & . & 1 & . & . & . & 1 & . \\ \hline . & 1 & . & 1 & . & . & 1 & . & . \\ . & 1 & . & . & 1 & . & . & 1 & . \\ . & 1 & . & . & . & 1 & . & . & 1 \\ \hline . & . & 1 & . & . & c & . & b & . \\ . & . & 1 & a & . & . & . & . & c \\ . & . & 1 & . & b & . & a & . & . \end{array}\right| \end{array} = 0;$$

where, as in the case of two cubic radicals, two forms, viz. the first and third forms of the rational equation, are not essentially distinct, but may be derived from each other by interchanging lines and columns.

And in general, whatever be the number of cubic radicals, two of the three forms are not essentially distinct, but may be derived from each other by interchanging lines and columns.

110.

NOTE ON THE TRANSFORMATION OF A TRIGONOMETRICAL EXPRESSION.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IX. (1854), pp. 61—62.]

THE differential equation

$$\frac{dx}{(a+x)\sqrt{(c+x)}}+\frac{dy}{(a+y)\sqrt{(c+y)}}+\frac{dz}{(a+z)\sqrt{(c+z)}}=0,$$

integrated so as to be satisfied when the variables are simultaneously infinite, gives by direct integration

$$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)}+\tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)}=0;$$

and, by Abel's theorem,

$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix}=0.$$

To show *à posteriori* the equivalence of these two equations, I represent the determinant by the symbol $\square$, and expressing it in the form

$$\square=\begin{vmatrix} 1, & a+x, & (a+x)\sqrt{(c+x)} \\ \vdots & & \end{vmatrix},$$

I write for the moment $\xi=\sqrt{\left(\frac{a-c}{c+x}\right)}$ &c.; this gives

$$\square = \begin{vmatrix} 1, & (a-c)\left(1+\dfrac{1}{\xi^2}\right), & (a-c)^{\frac{3}{2}}\left(\dfrac{1}{\xi}+\dfrac{1}{\xi^3}\right) \\ \vdots & & \end{vmatrix}$$

$$= \frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \begin{vmatrix} \xi^3, & \xi^3+\xi, & \xi^3+1 \\ \vdots & & \end{vmatrix}$$

$$= \frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \begin{vmatrix} \xi^3, & \xi, & \xi^2+1 \\ \vdots & & \end{vmatrix}$$

$$= -\frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \left\{ \begin{vmatrix} 1, & \xi, & \xi^3 \\ \vdots & & \end{vmatrix} - \xi\eta\zeta \begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix} \right\}$$

$$= -\frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} (\xi+\eta+\zeta-\xi\eta\zeta) \begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix};$$

or, replacing ξ, η, ζ by their values, we have identically

$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix} =$$

$$-\frac{(c+x)^{\frac{3}{2}}(c+y)^{\frac{3}{2}}(c+z)^{\frac{3}{2}}}{(a-c)^2} \left\{ \sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} \right\} \begin{vmatrix} 1, & \sqrt{\dfrac{a-c}{c+x}}, & \dfrac{a-c}{c+x} \\ 1, & \sqrt{\dfrac{a-c}{c+y}}, & \dfrac{a-c}{c+y} \\ 1, & \sqrt{\dfrac{a-c}{c+z}}, & \dfrac{a-c}{c+z} \end{vmatrix},$$

and the equation

$$\sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} = 0$$

is of course equivalent to the trigonometrical equation

$$\tan^{-1}\sqrt{\frac{a-c}{c+x}} + \tan^{-1}\sqrt{\frac{a-c}{c+y}} + \tan^{-1}\sqrt{\frac{a-c}{c+z}} = 0,$$

which shows the equivalence of the two equations in question.

111.

ON A THEOREM OF M. LEJEUNE-DIRICHLET'S.

[From the *Cambridge and Dublin Mathematical Journal*, vol. IX. (1854), pp. 163—165.]

THE following formula,

$$\Sigma q^{ax^2+2bxy+cy^2} + \Sigma q^{a'x'^2+2b'xy+c'y^2} + \ldots = 2\Sigma\delta^{\frac{1}{2}(n-1)}\,\epsilon^{\frac{1}{8}(n^2-1)}\left(\frac{n}{P}\right)q^{nn'}, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is given in Lejeune-Dirichlet's well-known memoir "Recherches sur diverses applications &c." (*Crelle*, t. XXI. [1840] p. 8). The notation is as follows:—On the left-hand side (a, b, c), (a', b', c'), ... are a system of properly primitive forms to the *negative* determinant D (i.e. a system of positive forms); x, y are positive or negative integers including zero, such that in the sum $\Sigma q^{ax^2+2bxy+cy^2}$, $ax^2+2bxy+cy^2$ is prime to $2D$, and similarly in the other sums; q is indeterminate and the summations extend to the values first mentioned, of x and y. On the right-hand side we have to consider the form of D, viz. we have $D=PS^2$ or else $D=2PS^2$, where S^2 is the greatest square factor in D and where P is odd: this obviously defines P, and the values of δ, ϵ, which are always ± 1 (or, as I prefer to express it, are always $\underline{\pm}$) are given as follows, viz.

$$\begin{aligned}
D=PS^2\,,\quad & P\equiv 1 \pmod 4,\quad \delta,\quad \epsilon=++,\\
D=PS^2\,,\quad & P\equiv 3 \pmod 4,\quad \delta,\quad \epsilon=-+,\\
D=2PS^2,\quad & P\equiv 1 \pmod 4,\quad \delta,\quad \epsilon=+-,\\
D=2PS^2,\quad & P\equiv 3 \pmod 4,\quad \delta,\quad \epsilon=--,
\end{aligned}$$

n, n' are any positive numbers prime to $2D$, $\left(\frac{n}{P}\right)$ is Legendre's symbol as generalized by Jacobi, viz. in general if p be a positive or negative prime not a factor of n,

then $\left(\frac{n}{p}\right) = +$ or $-$ according as n is or is not a quadratic residue of p (or, what is the same thing, p being positive, $\left(\frac{n}{\pm p}\right) \equiv n^{\frac{1}{2}(n-1)} \pmod{p}$), and for $P = pp'p'' \ldots$,

$$\left(\frac{n}{P}\right) = \left(\frac{n}{p}\right)\left(\frac{n}{p'}\right)\left(\frac{n}{p''}\right)\ldots,$$

and the summation extends to all the values of n, n' of the form above mentioned. In the particular case $D = -1$, it is necessary that the second side should be doubled. The method of reducing the equation is indicated in the memoir. The following are a few particular cases.

$D = -1$,
$$\Sigma q^{x^2+y^2} = 4\Sigma\,(-)^{\frac{1}{2}(n-1)}\, q^{nn'},$$

or
$$(1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$$
$$= \frac{q}{1-q^2} - \frac{q^3}{1-q^6} + \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \ldots$$

$D = -2$,
$$\Sigma q^{x^2+2y^2} = 2\Sigma\,(-)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)}\, q^{nn'},$$

or
$$(1 + 2q^2 + 2q^8 + 2q^{18} \ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$$
$$= \frac{q}{1-q^2} + \frac{q^3}{1-q^6} - \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \&c.$$

an example given in the memoir.

$D = -3$,
$$\Sigma q^{x^2+xy+y^2} = 2\Sigma\left(\frac{n}{3}\right) q^{nn'},$$

or
$$(q^1 + q^{25} + q^{49} + q^{121} + q^{169} \ldots)(1 + 2q^{12} + 2q^{48} + 2q^{108} \ldots)$$
$$+ 2(q^3 + q^{27} + q^{75} + q^{147} + \ldots)(q^4 + q^{16} + q^{64} + q^{100} \ldots)$$
$$= \frac{q + q^5}{1-q^6} - \frac{q^5 + q^{25}}{1-q^{30}} + \frac{q^7 + q^{35}}{1-q^{42}} - \frac{q^{11} + q^{55}}{1-q^{66}} + \ldots$$

I am not aware that the above theorem is quoted or referred to in any subsequent memoir on Elliptic Functions, or on the class of series to which it relates; and the theorem is so distinct in its origin and form from all other theorems relating to the same class of series, and, independently of the researches in which it originates, so remarkable as a result, that I have thought it desirable to give a detached statement of it in this paper.

112.

DEMONSTRATION OF A THEOREM RELATING TO THE PRODUCTS OF SUMS OF SQUARES.

[From the *Philosophical Magazine*, vol. IV. (1852), pp. 515—519.]

MR KIRKMAN, in his paper "On Pluquaternions and Homoid Products of Sums of n Squares" (*Phil. Mag.* vol. XXXIII. [1848] pp. 447—459 and 494—509), quotes from a note of mine the following passage:—"The complete test of the possibility of the product of 2^n squares by 2^n squares reducing itself to a sum of 2^n squares is the following: forming the complete systems of triplets for $(2^n - 1)$ things, if *eab, ecd, fac, fdb* be any four of them, we must have, paying attention to the signs alone,

$$(\pm\, eab)\,(\pm\, ecd) = (\pm\, fac)\,(\pm\, fdb)\,;$$

i.e. if the first two are of the same sign, the last two must be so also, and *vice versâ;* I believe that, for a system of seven, two conditions of this kind being satisfied would imply the satisfaction of all the others: it remains to be shown that the complete system of conditions cannot be satisfied for fifteen things." I propose to explain the meaning of the theorem, and to establish the truth of it, without in any way assuming the existence of imaginary units.

The identity to be established is

$$(w^2 + a^2 + b^2 + \ldots)\,(w_{,}^2 + a_{,}^2 + b_{,}^2 \ldots) = w_{,,}^2 + a_{,,}^2 + b_{,,}^2 + \ldots$$

where the 2^n quantities $w, a, b, c, \ldots$ and the 2^n quantities $w_{,}, a_{,}, b_{,}, c_{,}, \ldots$ are given quantities in terms of which the 2^n quantities $w_{,,}, a_{,,}, b_{,,}, c_{,,}, \ldots$ have to be determined.

Without attaching any meaning whatever to the symbols $a_{\circ}, b_{\circ}, c_{\circ} \ldots$ I write down the expressions

$$w + aa_{\circ} + bb_{\circ} + cc_{\circ} \ldots, \quad w_{,} + a_{,}a_{\circ} + b_{,}b_{\circ} + c_{,}c_{\circ} \ldots,$$

and I multiply as if a_0, b_0, $c_0 \ldots$ really existed, taking care to multiply without making any transposition in the order *inter se* of two symbols a_0, b_0 combined in the way of multiplication. This gives a quasi-product

$$\begin{aligned} & ww_{,} + (aw_{,} + a_{,}w)\, a_0 + (bw_{,} + b_{,}w)\, b_0 + \ldots \\ & \quad + aa_{,}a_0^2 \;\; + bb_{,}b_0^2 \;\; + \ldots \\ & \quad + ab_{,}a_0b_0 + a_{,}bb_0a_0 + \ldots . \end{aligned}$$

Suppose, now, that a quasi-equation, such as

$$a_0b_0c_0 = +,$$

means that in the expression of the quasi-product

$$b_0c_0, \quad c_0a_0, \quad a_0b_0, \quad c_0b_0, \quad a_0c_0, \quad b_0a_0$$

are to be replaced by

$$a_0, \quad b_0, \quad c_0, \quad -a_0, \quad -b_0, \quad -c_0;$$

and that a quasi-equation, such as $a_0b_0c_0 = -$, means that in the expression of the quasi-product

$$b_0c_0, \quad c_0a_0, \quad a_0b_0, \quad c_0b_0, \quad a_0c_0, \quad b_0a_0$$

are to be replaced by

$$-a_0, \quad -b_0, \quad -c_0, \quad a_0, \quad b_0, \quad c_0.$$

It is in the first place clear that the quasi-equation, $a_0b_0c_0 = +$, may be written in any one of the six forms

$$\begin{aligned} & a_0b_0c_0 = +, \quad b_0c_0a_0 = +, \quad c_0a_0b_0 = +, \\ & a_0c_0b_0 = -, \quad c_0b_0a_0 = -, \quad b_0a_0c_0 = -; \end{aligned}$$

and so for the quasi-equation $a_0b_0c_0 = -$. This being premised, if we form a system of quasi-equations, such as

$$a_0b_0c_0 = \pm, \quad a_0d_0e_0 = \pm, \text{ \&c.}$$

where the system of triplets contains each duad once, and once only, and the arbitrary signs are chosen at pleasure; if, moreover, in the expression of the quasi-product we replace a_0^2, $b_0^2, \ldots$ each by -1, it is clear that the quasi-product will assume the form

$$w_{,,} + a_{,,}a_0 + b_{,,}b_0 + c_{,,}c_0 + \ldots,$$

$w_{,,}$, $a_{,,}$, $b_{,,}$, $c_{,,} \ldots$ being determinate functions of $w, a, b, c, \ldots$; $w_{,}$, $a_{,}$, $b_{,}$, $c_{,} \ldots$, homogeneous of the first order in the quantities of each set; the value of $w_{,,}$ being obviously in every case

$$w_{,,} = ww_{,} - aa_{,} - bb_{,} - cc_{,} \ldots,$$

and $a_{,,}$, $b_{,,}$, $c_{,,} \ldots$ containing in every case the terms $aw_{,} + a_{,}w$, $bw_{,} + b_{,}w$, $cw_{,} + c_{,}w, \ldots$ but the form of the remaining terms depending as well on the triplets entering into the

system of quasi-equations as on the values given to the signs $\pm$; *the quasi-equations serving, in fact, to prescribe a rule for the formation of certain functions* $w_{,,}, a_{,,}, b_{,,}, c_{,,}, \ldots$, the properties of which functions may afterwards be investigated.

Suppose, now, that the system of quasi-equations is such that

$$e_\circ a_\circ b_\circ, \quad e_\circ c_\circ d_\circ$$

being any two of its triplets, with a common symbol $e_\circ$, there occur also in the system the triplets

$$f_\circ a_\circ c_\circ, \quad f_\circ d_\circ b_\circ, \quad g_\circ a_\circ d_\circ, \quad g_\circ b_\circ c_\circ;$$

and suppose that the corresponding portion of the system is

$$\begin{aligned} e_\circ a_\circ b_\circ &= \epsilon, \quad e_\circ c_\circ d_\circ = \epsilon', \\ f_\circ a_\circ c_\circ &= \zeta, \quad f_\circ d_\circ b_\circ = \zeta', \\ g_\circ a_\circ d_\circ &= \iota, \quad g_\circ b_\circ c_\circ = \iota', \end{aligned}$$

where $\epsilon, \zeta, \iota, \epsilon', \zeta', \iota'$ each of them denote one of the signs $+$ or $-$; then $e_{,,}, f_{,,}, g_{,,}$ will contain respectively the terms

$$\begin{aligned} &\epsilon(ab_{,} - a_{,}b) + \epsilon'(cd_{,} - c_{,}d), \\ &\zeta(ac_{,} - a_{,}c) + \zeta'(db_{,} - d_{,}b), \\ &\iota(ad_{,} - a_{,}d) + \iota'(bc_{,} - b_{,}c); \end{aligned}$$

and $e_{,,}^2 + f_{,,}^2 + g_{,,}^2$ contains the terms

$$\begin{aligned} &(a^2 + b^2 + c^2 + d^2)(a_{,}^2 + b_{,}^2 + c_{,}^2 + d_{,}^2) - a^2a_{,}^2 - b^2b_{,}^2 - c^2c_{,}^2 - d^2d_{,}^2 \\ &\quad + 2[\epsilon\epsilon'(ab_{,} - a_{,}b)(cd_{,} - c_{,}d) \\ &\quad\quad + \zeta\zeta'(ac_{,} - a_{,}c)(db_{,} - d_{,}b) \\ &\quad\quad + \iota\iota'(ad_{,} - a_{,}d)(bc_{,} - b_{,}c)]; \end{aligned}$$

and by taking account of the terms $ew_{,} + e_{,}w$, $fw_{,} + f_{,}w$, $gw_{,} + g_{,}w$ in $e_{,,}, f_{,,}, g_{,,}$ respectively, we should have had besides in $e_{,,}^2 + f_{,,}^2 + g_{,,}^2$ the terms

$$\begin{aligned} &(e^2 + f^2 + g^2)w_{,}^2 + (e_{,}^2 + f_{,}^2 + g_{,}^2)w^2 \\ &\quad + 2(ee_{,} + ff_{,} + gg_{,})ww_{,}. \end{aligned}$$

Also $w_{,,}^2$ contains the terms

$$\begin{aligned} &w^2w_{,}^2 + a^2a_{,}^2 + b^2b_{,}^2 + c^2c_{,}^2 + d^2d_{,}^2 \\ &\quad - 2(ee_{,} + ff_{,} + gg_{,})ww_{,}; \end{aligned}$$

whence it is easy to see that

$$\begin{aligned} &w_{,,}^2 + a_{,,}^2 + b_{,,}^2 + c_{,,}^2 + \ldots = \\ &(w^2 + a^2 + b^2 + c^2 + \ldots)(w_{,}^2 + a_{,}^2 + b_{,}^2 + c_{,}^2 + \ldots) \\ &\quad + 2\Sigma[\epsilon\epsilon'(ab_{,} - a_{,}b)(cd_{,} - c_{,}d) \\ &\quad\quad + \zeta\zeta'(ac_{,} - a_{,}c)(db_{,} - d_{,}b) \\ &\quad\quad + \iota\iota'(ad_{,} - a_{,}d)(bc_{,} - b_{,}c)]. \end{aligned}$$

where the summation extends to all the quadruplets formed each by the combination of two duads such as ab and cd, or ac and db, or ad and bc, i. e. two duads, which, combined with the same common letter (in the instances just mentioned e, or f, or g), enter as triplets into the system of quasi-equations—so that if $\nu = 2^n - 1$, the number of quadruplets is

$$\tfrac{1}{2}\{\tfrac{1}{2}(\nu-1).\tfrac{1}{2}(\nu-3)\}\,\nu\,.\,\tfrac{1}{3}, = \tfrac{1}{24}\,\nu\,(\nu-1)\,(\nu-3),$$

and the terms under the sign Σ will vanish identically if only

$$\epsilon\epsilon' = \zeta\zeta' = \iota\iota' ;$$

but the relation $\epsilon\epsilon' = \iota\iota'$ is of the same form as the equation $\epsilon\epsilon' = \zeta\zeta'$; hence if all the relations

$$\epsilon\epsilon' = \zeta\zeta'$$

are satisfied, the terms under the sign Σ vanish, and we have

$$(w_{,,}^2 + a_{,,}^2 + b_{,,}^2 + c_{,,}^2 + \ldots) = (w^2 + a^2 + b^2 + c^2 + \ldots)(w_{,}^2 + a_{,}^2 + b_{,}^2 + c_{,}^2 + \ldots)$$

which is thus shown to be true, upon the suppositions—

1. That the system of quasi-equations is such that

$$e_\circ a_\circ b_\circ, \quad e_\circ c_\circ d_\circ$$

being any two of its triplets with a common symbol $e_\circ$, there occur also in the system the triplets

$$f_\circ a_\circ c_\circ, \quad f_\circ d_\circ b_\circ,$$
$$g_\circ a_\circ d_\circ, \quad g_\circ b_\circ c_\circ.$$

2. That for any two pairs of triplets, such as

$$e_\circ a_\circ b_\circ, \quad e_\circ c_\circ d_\circ \quad \text{and} \quad f_\circ a_\circ c_\circ, \quad f_\circ d_\circ b_\circ,$$

the product of the signs of the triplets of the first pair is equal to the product of the signs of the triplets of the second pair.

In the case of fifteen things $a, b, c, \ldots$ the triplets may, as appears from Mr Kirkman's paper, be chosen so as to satisfy the first condition; but the second condition involves, as Mr Kirkman has shown, a contradiction; and therefore the product of two sums, each of them of sixteen squares, is *not* a sum of sixteen squares. It is proper to remark, that this demonstration, although I think rendered clearer by the introduction of the idea of the system of triplets furnishing the rule for the formation of the expressions $w_{,,}$, $a_{,,}$, $b_{,,}$, $c_{,,}$, &c., is not in principle different from that contained in Prof. Young's paper "On an Extension of a Theorem of Euler, &c.", *Irish Transactions*, vol. XXI. [1848 pp. 311—341].

113.

NOTE ON THE GEOMETRICAL REPRESENTATION OF THE INTEGRAL $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$.

[From the *Philosophical Magazine*, vol. v. (1853), pp. 281—284.]

THE equation of a conic passing through the points of intersection of the conics

$$x^2 + y^2 + z^2 = 0,$$
$$ax^2 + by^2 + cz^2 = 0,$$

is of the form

$$w(x^2 + y^2 + z^2) + ax^2 + by^2 + cz^2 = 0,$$

where w is an arbitrary parameter. Suppose that the conic touches a given line, we have for the determination of w a quadratic equation, the roots of which may be considered as parameters for determining the line in question. Let one of the values of w be considered as equal to a constant quantity k, the line is always a tangent to the conic

$$k(x^2 + y^2 + z^2) + ax^2 + by^2 + cz^2 = 0;$$

and taking $w = p$ for the other value of w, p is a parameter determining the particular tangent, or, what is the same thing, determining the point of contact of this tangent.

The equation of the tangent is easily seen to be

$$x\sqrt{b-c}\sqrt{a+k}\sqrt{a+p} + y\sqrt{c-a}\sqrt{b+k}\sqrt{b+p} + z\sqrt{a-b}\sqrt{c+k}\sqrt{c+p} = 0;$$

suppose that the tangent meets the conic $x^2 + y^2 + z^2 = 0$ (which is of course the conic corresponding to $w = \infty$) in the points P, P', and let θ, ∞ be the parameters of the point P, and θ', ∞ the parameters of the point P', i.e. (repeating the defini-

tion of the terms) let the tangent at P of the conic $x^2+y^2+z^2=0$ be also touched by the conic $\theta(x^2+y^2+z^2)+ax^2+by^2+cz^2=0$, and similarly for θ'. The coordinates of the point P are given by the equations

$$x:y:z=\sqrt{b-c}\sqrt{a+\theta}:\sqrt{c-a}\sqrt{b+\theta}:\sqrt{a-b}\sqrt{c+\theta};$$

and substituting these values in the equation of the line PP', we have

$$(b-c)\sqrt{a+k}\sqrt{a+p}\sqrt{a+\theta}+(c-a)\sqrt{b+k}\sqrt{b+p}\sqrt{b+\theta}+(a-b)\sqrt{c+k}\sqrt{c+p}\sqrt{c+\theta}$$
$$=0\ldots(*),$$

an equation connecting the quantities p, θ. To rationalize this equation, write

$$\sqrt{(a+k)(a+p)(a+\theta)}=\lambda+\mu a,$$
$$\sqrt{(b+k)(b+p)(b+\theta)}=\lambda+\mu b,$$
$$\sqrt{(c+k)(c+p)(c+\theta)}=\lambda+\mu c,$$

values which evidently satisfy the equation in question. Squaring these equations, we have equations from which λ^2, $\lambda\mu$, μ^2 may be linearly determined; and making the necessary reductions, we find

$$\lambda^2=abc+kp\theta,$$
$$-2\lambda\mu=bc+ca+ab-(p\theta+kp+k\theta),$$
$$\mu^2=a+b+c+k+p+\theta;$$

or, eliminating λ, μ,

$$\{bc+ca+ab-(p\theta+kp+k\theta)\}^2-4(a+b+c+k+p+\theta)(abc+kp\theta)=0,\ \ldots\ldots(*),$$

which is the rational form of the former equation marked (*). It is clear from the symmetry of the formula, that the same equation would have been obtained by the elimination of L, M from the equations

$$\sqrt{(k+a)(k+b)(k+c)}=L+Mk,$$
$$\sqrt{(p+a)(p+b)(p+c)}=L+Mp,$$
$$\sqrt{(\theta+a)(\theta+b)(\theta+c)}=L+M\theta;$$

and it follows from Abel's theorem (but the result may be verified by means of Euler's fundamental integral in the theory of elliptic functions), that if

$$\Pi x=\int_\infty\frac{dx}{\sqrt{(x+a)(x+b)(x+c)}},$$

then the algebraical equations (*) are equivalent to the transcendental equation

$$\pm\Pi k\pm\Pi p\pm\Pi\theta=0;$$

the arbitrary constant which should have formed the second side of the equation having been determined by observing that the algebraical equation gives for $p=\theta$, $k=\infty$, a system of values, which, when the signs are properly chosen, satisfy the transcendental equation. In fact, arranging the rational algebraical equation according to the powers of k, it becomes

$$k^2(p-\theta)^2 - 2k\{p\theta(p+\theta) + 2(a+b+c)p\theta + (bc+ca+ab)(p+\theta) + 2abc\}$$
$$+ p^2\theta^2 - 2(bc+ca+ab)p\theta - 4abc(p+\theta) + b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2b^2ca - 2c^2ab = 0\,;(*)$$

which proves the property in question, and is besides a very convenient form of the algebraical integral. The ambiguous signs in the transcendental integral are not of course arbitrary (indeed it has just been assumed that for $p=\theta$, Πp and $\Pi\theta$ are to be taken with opposite signs), but the discussion of the proper values to be given to the ambiguous signs would be at all events tedious, and must be passed over for the present.

It is proper to remark, that $\theta=p$ gives not only, as above supposed, $k=\infty$, but another value of k, which, however, corresponds to the transcendental equation

$$\pm\Pi k \pm 2\Pi p = 0\,;$$

the value in question is obviously

$$4k = \frac{p^4 - 2(bc+ca+ab)p^2 - 8abcp + b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2b^2ca - 2c^2ab}{(p+a)(p+b)(p+c)}.$$

Consider, in general, a cubic function $\mathrm{a}x^3 + 3\mathrm{b}x^2y + 3\mathrm{c}xy^2 + \mathrm{d}y^3$, or, as I now write it in the theory of invariants, $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})(x, y)^3$, the Hessian of this function is

$$\left(\mathrm{ac}-\mathrm{b}^2,\ \tfrac{1}{2}(\mathrm{ad}-\mathrm{bc}),\ \mathrm{bd}-\mathrm{c}^2\right)(x, y)^2,$$

and applying this formula to the function $(p+a)(p+b)(p+c)$, it is easy to write the equation last preceding in the form

$$4k = p - (a+b+c) - \frac{9\ \text{Hessian}\ \{(p+a)(p+b)(p+c)\}}{(p+a)(p+b)(p+c)},$$

which is a formula for the duplication of the transcendent Πx.

Reverting now to the general transcendental equation

$$\pm\Pi k \pm \Pi p \pm \Pi\theta = 0,$$

we have in like manner

$$\pm\Pi k \pm \Pi p \pm \Pi\theta' = 0;$$

and assuming a proper correspondence of the signs, the elimination of Πp gives

$$\Pi\theta' - \Pi\theta = 2\Pi k\,;$$

i.e. if the points P, P' upon the conic $x^2+y^2+z^2=0$ are such that their parameters θ, θ' satisfy this equation, the line PP' will be constantly a tangent to the conic

$$k(x^2+y^2+z^2)+(ax^2+by^2+cz^2)=0.$$

Hence also, if the parameters k, k', k'' of the conics

$$\begin{aligned} k\ (x^2+y^2+z^2)+ax^2+by^2+cz^2&=0,\\ k'\ (x^2+y^2+z^2)+ax^2+by^2+cz^2&=0,\\ k''(x^2+y^2+z^2)+ax^2+by^2+cz^2&=0,\end{aligned}$$

satisfy the equation

$$\Pi k+\Pi k'+\Pi k''=0,$$

there are an infinity of triangles inscribed in the conic $x^2+y^2+z^2=0$, and the sides of which touch the last-mentioned three conics respectively.

Suppose $2\Pi k=\Pi\kappa$ (an equation the algebraic form of which has already been discussed), then

$$\Pi\theta'-\Pi\theta=\Pi\kappa,$$

$\theta=\infty$ gives $\theta'=\kappa$; or, observing that $\theta=\infty$ corresponds to a point of intersection of the conics $x^2+y^2+z^2=0$, $ax^2+by^2+cz^2=0$, κ is the parameter of the point in which a tangent to the conic $k(x^2+y^2+z^2)+ax^2+by^2+cz^2=0$ at any one of its intersections with the conic $x^2+y^2+z^2=0$ meets the last-mentioned conic. Moreover, the algebraical relation between θ, θ' and κ (where, as before remarked, κ is a given function of k) is given by a preceding formula, and is simpler than that between θ, θ' and k.

The preceding investigations were, it is hardly necessary to remark, suggested by a well-known memoir of the late illustrious Jacobi, and contain, I think, the extension which he remarks it would be interesting to make of the principles in such memoir to a system of two conics. I propose reverting to the subject in a memoir to be entitled "Researches on the Porism of the in- and circumscribed triangle." [This was, I think, never written.]

114.

ANALYTICAL RESEARCHES CONNECTED WITH STEINER'S EXTENSION OF MALFATTI'S PROBLEM.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLII. *for the year* 1852, pp. 253—278: Received April 12,—Read May 27, 1852]

THE problem, in a triangle to describe three circles each of them touching the two others and also two sides of the triangle, has been termed after the Italian geometer by whom it was proposed and solved, Malfatti's problem. The problem which I refer to as Steiner's extension of Malfatti's problem is as follows:—"To determine three sections of a surface of the second order, each of them touching the two others, and also two of three given sections of the surface of the second order," a problem proposed in Steiner's memoir, "Einige geometrische Betrachtungen," *Crelle*, t. I. [1826 pp. 161—184]. The geometrical construction of the problem in question is readily deduced from that given in the memoir just mentioned for a somewhat less general problem, viz. that in which the surface of the second order is replaced by a sphere; it is for the sake of the analytical developments to which the problem gives rise, that I propose to resume here the discussion of the problem. The following is an analysis of the present memoir:—

§ 1. Contains a lemma which appears to me to constitute the foundation of the analytical theory of the sections of a surface of the second order.

§ 2. Contains a statement of the geometrical construction of Steiner's extension of Malfatti's problem.

§ 3. Is a verification, founded on a particular choice of coordinates, of the construction in question.

§ 4. In this section, referring the surface of the second order to absolutely general coordinates, and after an incidental solution of the problem to determine a section touching three given sections, I obtain the equations for the solution of Steiner's extension of Malfatti's problem.

§ 5. Contains a separate discussion of a system of equations, including as a particular case the equations obtained in the preceding section.

§§ 6 and 7. Contain the application of the formulæ for the general system to the equations in § 4, and the development and completion of the solution.

§ 8. Is an extension of some preceding formulæ to quadratic functions of any number of variables.

§ 1. *Lemma relating to the sections of a surface of the second order.*

If

$$ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0$$

be the equation of a surface of the second order, and

$$\mathfrak{A}x^2 + \mathfrak{B}y^2 + \mathfrak{C}z^2 + \mathfrak{D}w^2 + 2\mathfrak{F}yz + 2\mathfrak{G}zx + 2\mathfrak{H}xy + 2\mathfrak{L}xw + 2\mathfrak{M}yw + 2\mathfrak{N}zw = 0$$

the reciprocal equation, the condition that the two sections

$$\lambda x + \mu y + \nu z + \rho w = 0,$$
$$\lambda' x + \mu' y + \nu' z + \rho' w = 0,$$

may touch, is

$$\begin{aligned}
&(\mathfrak{A}\lambda^2 + \mathfrak{B}\mu^2 + \mathfrak{C}\nu^2 + \mathfrak{D}\rho^2 + 2\mathfrak{F}\mu\nu + 2\mathfrak{G}\nu\lambda + 2\mathfrak{H}\lambda\mu + 2\mathfrak{L}\lambda\rho + 2\mathfrak{M}\mu\rho + 2\mathfrak{N}\nu\rho)^{\frac{1}{2}} \\
&\times (\mathfrak{A}\lambda'^2 + \mathfrak{B}\mu'^2 + \mathfrak{C}\nu'^2 + \mathfrak{D}\rho'^2 + 2\mathfrak{F}\mu'\nu' + 2\mathfrak{G}\nu'\lambda' + 2\mathfrak{H}\lambda'\mu' + 2\mathfrak{L}\lambda'\rho' + 2\mathfrak{M}\mu'\rho' + 2\mathfrak{N}\nu'\rho')^{\frac{1}{2}} \\
&= (\mathfrak{A}\lambda\lambda' + \mathfrak{B}\mu\mu' + \mathfrak{C}\nu\nu' + \mathfrak{D}\rho\rho' + \mathfrak{F}(\mu\nu' + \mu'\nu) + \mathfrak{G}(\nu\lambda' + \nu'\lambda) + \mathfrak{H}(\lambda\mu' + \lambda'\mu) \\
&\qquad + \mathfrak{L}(\lambda\rho' + \lambda'\rho) + \mathfrak{M}(\mu\rho' + \mu'\rho) + \mathfrak{N}(\nu\rho' + \nu'\rho));
\end{aligned}$$

and in particular if the equation of the surface be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + pw^2 = 0,$$

the condition of contact is

$$\begin{aligned}
&\left(\mathfrak{A}\lambda^2 + \mathfrak{B}\mu^2 + \mathfrak{C}\nu^2 + 2\mathfrak{F}\mu\nu + 2\mathfrak{G}\nu\lambda + 2\mathfrak{H}\lambda\mu + \frac{K}{p}\rho^2\right)^{\frac{1}{2}} \\
&\times \left(\mathfrak{A}\lambda'^2 + \mathfrak{B}\mu'^2 + \mathfrak{C}\nu'^2 + 2\mathfrak{F}\mu'\nu' + 2\mathfrak{G}\nu'\lambda' + 2\mathfrak{H}\lambda'\mu' + \frac{K}{p}\rho'^2\right)^{\frac{1}{2}} \\
&= \left(\mathfrak{A}\lambda\lambda' + \mathfrak{B}\mu\mu' + \mathfrak{C}\nu\nu' + \mathfrak{F}(\mu\nu' + \mu'\nu) + \mathfrak{G}(\nu\lambda' + \nu'\lambda) + \mathfrak{H}(\lambda\mu' + \lambda'\mu) + \frac{K}{p}\rho\rho'\right),
\end{aligned}$$

in which last formula

$$\begin{aligned}
&\mathfrak{A} = bc - f^2, \quad \mathfrak{B} = ca - g^2, \quad \mathfrak{C} = ab - h^2, \\
&\mathfrak{F} = gh - af, \quad \mathfrak{G} = hf - bg, \quad \mathfrak{H} = fg - ch, \\
&K = abc - af^2 - bg^2 - ch^2 + 2fgh.
\end{aligned}$$

§ 2.

In order to state in the most simple form the geometrical construction for the solution of Steiner's extension of Malfatti's problem, let the given sections be called for conciseness the determinators[1]; any two of these sections lie in two different cones, the vertices of which determine with the line of intersection of the planes of the determinators, two planes which may be termed bisectors; the six bisectors pass three and three through four straight lines; and it will be convenient to use the term bisectors to denote, not the entire system, but any three bisectors passing through the same line. Consider three sections, which may be termed tactors, each of them touching a determinator and two bisectors, and three other sections (which may be termed separators) each of them passing through the point of contact of a determinator and tactor and touching the other two tactors; the separators will intersect in a line which passes through the point of intersection of the determinators. The three required sections, or as I shall term them the resultors, are determined by the conditions that each resultor touches two determinators and two separators, the possibility of the construction being implied as a theorem. The *à posteriori* verification may be obtained as follows:—

§ 3.

Let $x=0$, $y=0$, $z=0$ be the equations of the resultors, $w=0$ the equation of the polar of the point of intersection of the resultors. Since the resultors touch two and two, the equation of the surface is easily seen to be of the form

$$2yz+2zx+2xy+w^2=0.\,(^2)$$

The determinators are sections each of them touching two resultors, but otherwise arbitrary; their equations are

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0,$$

$$\frac{1}{2\beta}x-\beta y\quad+\frac{1}{2\beta}z+w=0,$$

$$\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z\quad+w=0.$$

The separators are sections each of them touching two resultors at their point of contact (or what is the same thing, passing through the line of intersection of two resultors), and all of them having a line in common. Their equations may be taken to be

$$cy-bz=0,\quad az-cx=0,\quad bx-ay=0,$$

[1] I use the words "determinators," &c. to denote indifferently the sections or the planes of the sections; the context is always sufficient to prevent ambiguity.

[2] The reciprocal form is, it should be noted,

$$x^2+y^2+z^2-2yz-2zx-2xy-2w^2=0.$$

the values of a, b, c remaining to be determined. Now before fixing the values of these quantities, we may find three sections each of them touching a determinator at a point of intersection with the section which corresponds to it of the sections $cy-bz=0$, $az-cx=0$, $bx-ay=0$, and touching the other two of the last-mentioned sections; and when a, b, c have their proper values the sections so found are the tactors. For, let $\lambda x+\mu y+\nu z+\rho w=0$ be the equation of a section touching the determinator $-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$, and the two sections $bx-ay=0$, $az-cx=0$; and suppose

$$\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\lambda-2\lambda\mu-2\rho^2;$$

the conditions of contact with the sections $bx-ay=0$, $az-cx=0$ are found to be

$$(b+a)\,\Delta=(b+a)\,\lambda-(b+a)\,\mu-(b-a)\,\nu,$$
$$(c+a)\,\Delta=(c+a)\,\lambda-(c-a)\,\mu-(c+a)\,\nu,$$

values, however, which suppose a correspondence in the signs of the radicals. Thence $(b+a)\,\mu=(c+a)\,\nu$; or since the ratios only of the quantities λ, μ, ν, ρ are material, $\mu=c+a$, $\nu=b+a$, and therefore

$$\Delta^2=\lambda^2-2\,(2a+b+c)\,\lambda+(b-c)^2-2\rho^2,=(\lambda-b-c)^2,$$

or
$$\rho^2=-2\,(a\lambda+bc).$$

Hence the equation to a section touching $bx-ay=0$, $az-cx=0$ is

$$\lambda x+(c+a)\,y+(b+a)\,z+\sqrt{-2\,(a\lambda+bc)}\,w=0;$$

and to express that this touches the determinator in question, we have

$$\pm\,\alpha\,(\lambda-b-c)=\left(\alpha+\frac{1}{\alpha}\right)\lambda-\alpha\,(2a+b+c)+2\sqrt{-2\,(a\lambda+bc)};$$

and selecting the upper sign,

$$\frac{1}{\alpha}\lambda-2a\alpha=-2\sqrt{-2\,(a\lambda+bc)};$$

whence

$$\lambda=-2\alpha\,(a\alpha-\sqrt{-2bc}),\quad \sqrt{-2\,(a\lambda+bc)}=(2a\alpha-\sqrt{-2bc});$$

or the section touching the determinator and the sections $bx-ay=0$, $az-cx=0$ is

$$-2\alpha\,(a\alpha-\sqrt{-2bc})\,x+(c+a)\,y+(b+a)\,z+(2a\alpha-\sqrt{-2bc})\,w=0;$$

and at the point of contact with the determinator

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0,$$

$$2yz+2zx+2xy+w^2=0.$$

Eliminating w between the first and second equations and between the second and third equations,

$$\sqrt{-2bc}\left(ax+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)+cy+bz=0,$$

$$\left(ax+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)^2+2yz \quad =0\,;$$

and from these equations $(cy-bz)^2=0$, or the point of contact lies in the section $cy-bz=0$. It follows that the equations of the tactors are

$$-2\alpha\,(a\alpha-\sqrt{-2bc})\,x+(c+a)\,y+(b+a)\,z+(2a\alpha-\sqrt{-2bc})\,w=0,$$

$$(c+b)\,x-2\beta\,(b\beta-\sqrt{-2ca})\,y+(a+b)\,z+(2b\beta-\sqrt{-2ca})\,w=0,$$

$$(b+c)\,x+(a+c)\,y-2\gamma\,(c\gamma-\sqrt{-2ab})\,z+(2c\gamma-\sqrt{-2ab})\,w=0,$$

where a, b, c still remain to be determined.

Now the separators pass through the point of intersection of the determinators; the equations of these give for the point in question,

$$\begin{aligned} x : y : z : w &= (2\beta\gamma+1)\,(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\ &: (2\gamma\alpha+1)\,(\ \ \alpha-\beta+\gamma+2\alpha\beta\gamma)\\ &: (2\alpha\beta+1)\,(\ \ \alpha+\beta-\gamma+2\alpha\beta\gamma)\\ &: \qquad 4\alpha^2\beta^2\gamma^2-1+\alpha^2+\beta^2+\gamma^2\,; \end{aligned}$$

and the values of a, b, c are therefore

$$\begin{aligned} a : b : c &= (2\beta\gamma+1)\,(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\ &: (2\gamma\alpha+1)\,(\ \ \alpha-\beta+\gamma+2\alpha\beta\gamma)\\ &: (2\alpha\beta+1)\,(\ \ \alpha+\beta-\gamma+2\alpha\beta\gamma), \end{aligned}$$

which are to be substituted for a, b, c in the equations of the separators and tactors respectively.

Now proceeding to find the bisectors, let $\lambda x+\mu y+\nu z+\rho w=0$ be the equation of a section touching the determinators,

$$\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w=0,\quad \frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w=0\,;$$

and suppose, as before, $\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\lambda-2\lambda\mu-2\rho^2$; the conditions of contact are

$$\pm\,\beta\Delta=\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\rho,$$

$$\mp\,\gamma\Delta=\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\rho,$$

where it is necessary, for the present purpose, to give opposite signs to the radicals. For if the radicals had the same sign, it would follow that

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\rho\right]-\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\rho\right]=0;$$

hence the section $\lambda x+\mu y+\nu z+\rho w=0$ would pass through the point

$$x:y:z:w=0:\frac{1}{\beta^2}:\frac{1}{\gamma^2}:-\frac{2}{\beta}+\frac{2}{\gamma};$$

or the section would be a tangent section of the two determinators of the same class with the resultor $x=0$, which ought not to be the case. The proper formula is

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\rho\right]+\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\rho\right]=0;$$

and this equation being satisfied, the section

$$\lambda x+\mu y+\nu z+\rho w=0$$

passes through a point

$$x:y:z:w=2:-\frac{1}{\beta^2}:-\frac{1}{\gamma^2}:-\frac{2}{\beta}-\frac{2}{\gamma}.$$

The bisector passes through this point and the line of intersection of the determinators; its equation is

$$\frac{1}{\beta}\left(\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w\right)-\frac{1}{\gamma}\left(\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w\right)=0;$$

or reducing and completing the system, the equations of the bisectors are

$$\left(\frac{1}{2\beta^2}-\frac{1}{2\gamma^2}\right)x-\left(1+\frac{1}{2\gamma^2}\right)\quad y+\left(1+\frac{1}{2\beta^2}\right)\quad z+\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)w=0,$$

$$\left(1+\frac{1}{2\gamma^2}\right)\quad x+\left(\frac{1}{2\gamma^2}-\frac{1}{2\alpha^2}\right)y-\left(1+\frac{1}{2\alpha^2}\right)\quad z+\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)w=0,$$

$$-\left(1+\frac{1}{2\beta^2}\right)\quad x+\left(1+\frac{1}{2\alpha^2}\right)\quad y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0.$$

In order to verify the geometrical construction, it only remains to show that each bisector touches two tactors. Consider the bisector and tactor

$$-\left(1+\frac{1}{2\beta^2}\right)x+\left(1+\frac{1}{2\alpha^2}\right)y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0,$$

$$-2a(a\alpha-\sqrt{-2bc})\,x+(c+a)\,y+(b+a)\,z+(2a\alpha-\sqrt{-2bc})\,w=0;$$

and represent these for a moment by

$$\lambda x+\mu y+\nu z+\rho w=0,\quad \lambda' x+\mu' y+\nu' z+\rho' w=0;$$

if Δ be the same as before, and Δ' the like function of λ', μ', ν', ρ', also if

$$\Phi = \lambda\lambda' + \mu\mu' + \nu\nu' - (\mu\nu' + \mu'\nu) - (\nu\lambda' + \nu'\lambda) - (\lambda\mu' + \lambda'\mu) - 2\rho\rho',$$

then

$$\Delta^2 = \left(2 + \frac{1}{\alpha\beta}\right)^2,$$

$$\Delta'^2 = (2a\alpha^2 - 2\alpha\sqrt{-2bc} + b + c)^2,$$

$$\Phi = a\alpha^2\left(2 + \frac{1}{\alpha\beta}\right)^2 - 2\alpha\sqrt{-2bc}\left(2 + \frac{1}{\alpha\beta}\right) + c\left(2 + \frac{1}{\beta^2}\right);$$

and the condition of contact $\Delta\Delta' = \Phi$ (taking the proper sign for the radicals) becomes

$$\left(2 + \frac{1}{\alpha\beta}\right)(2a\alpha^2 - 2\alpha\sqrt{-2bc} + b + c) = a\alpha^2\left(2 + \frac{1}{\alpha\beta}\right)^2 - 2\alpha\sqrt{-2bc}\left(2 + \frac{1}{\alpha\beta}\right) + c\left(2 + \frac{1}{\beta^2}\right);$$

or reducing,

$$a\alpha - b\beta + c\,\frac{\alpha - \beta}{2\alpha\beta + 1} = 0,$$

an equation which is evidently not altered by the interchange of a, α and b, β. The conditions, in order that each bisector may touch two tactors, reduce themselves to the three equations,

$$a\alpha - b\beta + c\,\frac{\alpha - \beta}{2\alpha\beta + 1} = 0,$$

$$a\,\frac{\beta - \gamma}{2\beta\gamma + 1} + b\beta - c\gamma = 0,$$

$$-a\alpha + b\,\frac{\gamma - \alpha}{2\alpha\gamma + 1} + c\gamma = 0,$$

which are satisfied by the values above found for the quantities a, b, c. The possibility and truth of the geometrical construction are thus demonstrated.

§ 4.

Let it be in the first instance proposed to find the equation of a section touching all or any of the sections $x = 0$, $y = 0$, $z = 0$ of the surface of the second order,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + pw^2 = 0.$$

Any section whatever of this surface may be written in the form

$$(a\lambda + h\mu + g\nu)\,x + (h\lambda + b\mu + f\nu)\,y + (g\lambda + f\mu + c\nu)\,z + \sqrt{-p}\,\nabla w = 0,$$

where

$$\nabla^2 = a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu - K,$$

and λ, μ, ν are indeterminate. And considering any other section represented by a like equation,

$$(a\lambda' + h\mu' + g\nu')\,x + (h\lambda' + b\mu' + f\nu')\,y + (g\lambda' + f\mu' + c\nu')\,z + \sqrt{-p}\,\nabla' w = 0,$$

where

$$\nabla'^2 = a\lambda'^2 + b\mu'^2 + c\nu'^2 + 2f\mu'\nu' + 2g\nu'\lambda' + 2h\lambda'\mu' - K,$$

it may be shown by means of the lemma previously given, that the condition of contact is

$$a\lambda\lambda' + b\mu\mu' + c\nu\nu' + f(\mu\nu' + \mu'\nu) + g(\nu\lambda' + \nu'\lambda) + h(\lambda\mu' + \lambda'\mu) \pm K = \nabla\nabla'.$$

Suppose that λ', μ', ν' satisfy the equations

$$\nabla' = 0,$$
$$h\lambda' + b\mu' + f\nu' = 0,$$
$$g\lambda' + f\mu' + c\nu' = 0,$$

so that the last-mentioned section becomes $x = 0$; and observing that the first of the above equations may be transformed into

$$a\lambda' + h\mu' + g\nu' = \frac{K}{\lambda'},$$

it is easy to obtain $\lambda' = \sqrt{\mathfrak{A}}$, $\mu' = \frac{\mathfrak{H}}{\sqrt{\mathfrak{A}}}$, $\nu' = \frac{\mathfrak{G}}{\sqrt{\mathfrak{A}}}$. The condition of contact thus becomes

$$\frac{K}{\sqrt{\mathfrak{A}}}\lambda \pm K = 0;$$

and taking the under sign, $\lambda = \sqrt{\mathfrak{A}}$, so that if in the above written equation we establish all or any of the equations $\lambda = \sqrt{\mathfrak{A}}$, $\mu = \sqrt{\mathfrak{B}}$, $\nu = \sqrt{\mathfrak{C}}$, we have the equation of a section touching all or the corresponding sections of the sections

$$x = 0, \quad y = 0, \quad z = 0.$$

In particular we have for a solution of the problem of tactions, the following equation of the section touching $x = 0$, $y = 0$, $z = 0$, viz.

$$(a\sqrt{\mathfrak{A}} + h\sqrt{\mathfrak{B}} + g\sqrt{\mathfrak{C}})\,x + (h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} + f\sqrt{\mathfrak{C}})\,y + (g\sqrt{\mathfrak{A}} + f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})\,z$$
$$+ \frac{\sqrt{-p}}{\sqrt{K}}\sqrt{2(\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F})(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})} = 0.$$

Anticipating the use of a notation the value of which will subsequently appear, or putting

$$\mathrm{f} = \sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F}}, \quad \mathrm{g} = \sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G}}, \quad \mathrm{h} = \sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H}}, \quad J = \sqrt{2}\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

values which give

$$K^2 = -f^4 - g^4 - h^4 + 2g^2h^2 + 2h^2f^2 + 2f^2g^2 - \frac{4f^2g^2h^2}{J^2},$$

the equation of the section in question is

$$\frac{f^2}{\sqrt{\mathfrak{A}}}(-f^2+g^2+h^2)\,x + \frac{g^2}{\sqrt{\mathfrak{B}}}(f^2-g^2+h^2)\,y + \frac{h^2}{\sqrt{\mathfrak{C}}}(f^2+g^2-h^2)\,z + \frac{fgh\sqrt{-p}\sqrt{K}}{J}\,w = 0.$$

I proceed to investigate a transformation of the equation for the section with an indeterminate parameter λ, which touches the two sections $y=0$, $z=0$. We have

$$a\nabla^2 = (a\lambda + h\mu + g\nu)^2 + (\mathfrak{C}\mu^2 + \mathfrak{B}\nu^2 - 2\mathfrak{F}\mu\nu) - \mathfrak{B}\mathfrak{C} + \mathfrak{F}^2;$$

or putting for μ and ν their values $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$ in the second term,

$$a\nabla^2 = (a\lambda + h\mu + g\nu)^2 + (\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F})^2;$$

and introducing instead of λ an indeterminate quantity X, such that

$$a\lambda + h\mu + g\nu = (\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F})\,X,$$

we have

$$a\nabla^2 = (\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F})\sqrt{1+X^2};$$

also introducing throughout X instead of λ, and completing the substitution of $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$, for μ, ν, the equation of the section touching $y=0$, $z=0$, becomes

$$(ax+hy+gz)\,X + y\sqrt{\mathfrak{C}} + z\sqrt{\mathfrak{B}} + w\sqrt{-ap}\sqrt{1+X^2} = 0.$$

It may be remarked here in passing, that this is a very convenient form for the demonstration of the theorem; "If two sections of a surface of the second order touch each other, and are also tangent sections (of the same class) to two fixed sections, then considering the planes through the axis of the fixed sections and the poles of the tangent sections, and also the tangent planes through this axis, the anharmonic ratio of the four planes is independent of the position of the moveable tangent sections;" where by the axis of the fixed sections is to be understood the line joining their poles.

The sections touching $z=0$, $x=0$, and $x=0$, $y=0$, are of course

$$x\sqrt{\mathfrak{C}} + (hx+by+fz)\,Y + z\sqrt{\mathfrak{A}} + w\sqrt{-bp}\sqrt{1+Y^2} = 0,$$

$$x\sqrt{\mathfrak{B}} + y\sqrt{\mathfrak{A}} + (gx+fy+cz)\,Z + w\sqrt{-cp}\sqrt{1+Z^2} = 0,$$

where

$$h\lambda' + b\mu' + f\nu' = (\sqrt{\mathfrak{C}\mathfrak{A}} - \mathfrak{G})\,Y, \quad \lambda' = \sqrt{\mathfrak{A}}, \quad \mu' = \mu', \quad \nu' = \sqrt{\mathfrak{C}},$$

$$g\lambda'' + f\mu'' + c\nu'' = (\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\,Z, \quad \lambda'' = \sqrt{\mathfrak{A}}, \quad \mu'' = \sqrt{\mathfrak{B}}, \quad \nu'' = \nu''.$$

The conditions of contact of the sections represented by the above written equations would be perhaps most simply obtained directly from the lemma, but it is proper to deduce it from the formula for contact used in the present memoir. If for shortness

$$\Phi(\pm) = a\lambda'\lambda'' + b\mu'\mu'' + c\nu'\nu'' + f(\mu'\nu'' + \mu''\nu') + g(\nu'\lambda'' + \nu''\lambda') + h(\lambda'\mu'' + \lambda''\mu') \pm K,$$

where the symbol $\Phi(\pm)$ is used in order to mark the essentially different character of the results corresponding to the different values of the ambiguous sign, then

$$\begin{aligned}
bc\Phi(-) = {} & f(h\lambda' + b\mu' + f\nu')(g\lambda'' + f\mu'' + c\nu''), \\
& + (\mathfrak{A}\nu' - \mathfrak{G}\lambda')(g\lambda'' + f\mu'' + c\nu''), \\
& + (\mathfrak{A}\mu'' - \mathfrak{H}\lambda'')(h\lambda' + b\mu' + f\nu'), \\
& + \nu'\mu''(-\mathfrak{A}f) + \nu'\lambda''f\mathfrak{H} + \lambda'\mu''f\mathfrak{G} + \lambda'\lambda''(K - f\mathfrak{F}) \\
& - \mathfrak{A}K - f^2K. \\
= {} & f(h\lambda' + b\mu' + f\nu')(g\lambda'' + f\mu'' + c\nu'') \\
& + \sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(g\lambda'' + f\mu'' + c\nu'') \\
& + \sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})(h\lambda' + b\mu' + f\nu') \\
& + f\left(-\mathfrak{A}\sqrt{\mathfrak{B}\mathfrak{C}} + \mathfrak{H}\sqrt{\mathfrak{C}\mathfrak{A}} + \mathfrak{G}\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{A}\mathfrak{F} - (\mathfrak{G}\mathfrak{H} - \mathfrak{A}\mathfrak{F})\right) \\
= {} & f(h\lambda' + b\mu' + f\nu')(g\lambda'' + f\mu'' + c\nu'') \\
& + \sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(g\lambda'' + f\mu'' + c\nu'') \\
& + \sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})(h\lambda' + b\mu' + f\nu') \\
& - f(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H}),
\end{aligned}$$

that is, $\quad bc\Phi(-) = (\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\{fYZ + \sqrt{\mathfrak{A}}(Y + Z) - f\}.$

What, however, is really required[1], is the value of $\Phi(+)$; to find this, we have

$$\begin{aligned}
bc\Phi(+) &= bc\Phi(-) + 2bcK \\
&= (\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\{fYZ + \sqrt{\mathfrak{A}}(Y + Z) + f\} \\
&\quad + 2bcK - 2f(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H}),
\end{aligned}$$

the second line of which is

$$\begin{aligned}
& 2(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\left\{\frac{bcK}{K^2bc}(\sqrt{\mathfrak{A}\mathfrak{C}} + \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} + \mathfrak{H}) - f\right\} \\
&= \frac{2(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})}{K}\{(\sqrt{\mathfrak{A}\mathfrak{C}} + \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} + \mathfrak{H}) - \mathfrak{G}\mathfrak{H} + \mathfrak{A}\mathfrak{F}\} \\
&= 2(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\sqrt{\mathfrak{A}}\theta,
\end{aligned}$$

[1] It may be shown without difficulty that the $(-)$ sign would imply that the sections touching $z=0$, $x=0$, and $x=0$, $y=0$, were sections touching $x=0$ at the same point. By taking the $(-)$ sign in each equation we should have the solution of the problem "to determine three sections of a surface of the second order, the two sections of each pair touching one of three given sections at the same point," which is not without interest; the solution may be completed without any difficulty.

where

$$\theta = \frac{1}{K}(\sqrt{\mathfrak{A}\mathfrak{B}\mathfrak{C}} + \mathfrak{F}\sqrt{\mathfrak{A}} + \mathfrak{G}\sqrt{\mathfrak{B}} + \mathfrak{H}\sqrt{\mathfrak{C}});$$

and consequently

$$bc\Phi(+) = (\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\{fYZ + \sqrt{\mathfrak{A}}(Y+Z) + f + 2\theta\sqrt{\mathfrak{A}}\},$$

a reduction, which on account of its peculiarity, I have thought right to work out in full.

The condition of contact is

$$\Phi(+) = \nabla'\nabla'' = \frac{1}{\sqrt{bc}}(\sqrt{\mathfrak{A}\mathfrak{C}} - \mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H})\sqrt{1+Y^2}\sqrt{1+Z^2}.$$

Hence finally, the condition in order that the sections

$$x\sqrt{\mathfrak{C}} + (hx + by + fz)Y + z\sqrt{\mathfrak{A}} + w\sqrt{-bp}\sqrt{1+Y^2} = 0,$$

$$x\sqrt{\mathfrak{B}} + y\sqrt{\mathfrak{A}} + (gx + fy + cz)Z + w\sqrt{-cp}\sqrt{1+Z^2} = 0,$$

(the former of which is a section touching $z=0$, $x=0$, and the latter a section touching $x=0$, $y=0$) may touch, is

$$fYZ + \sqrt{\mathfrak{A}}(Y+Z) + (f + 2\theta\sqrt{\mathfrak{A}}) - \sqrt{bc}\sqrt{1+Y^2}\sqrt{1+Z^2} = 0.$$

The preceding researches show that the solution of Steiner's extension of Malfatti's problem depends on a system of equations, such as the system mentioned at the commencement of the following section.

§ 5.

Consider the system of equations

$$\alpha + \beta(Y+Z) + \gamma YZ + \delta\sqrt{1+Y^2}\sqrt{1+Z^2} = 0,$$

$$\alpha' + \beta'(Z+X) + \gamma' ZX + \delta'\sqrt{1+Z^2}\sqrt{1+X^2} = 0,$$

$$\alpha'' + \beta''(X+Y) + \gamma'' XY + \delta''\sqrt{1+X^2}\sqrt{1+Y^2} = 0;$$

these equations may, it will be seen, be solved by quadratics only, when the coefficients satisfy the relations

$$\frac{\beta}{\gamma-\alpha} = \frac{\beta'}{\gamma'-\alpha'} = \frac{\beta''}{\gamma''-\alpha''},$$

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2} = \frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2} = \frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2}.$$

It may be remarked that these equations are satisfied by

$$\beta=0,\quad \beta'=0,\quad \beta''=0,\quad \gamma=\delta,\quad \gamma'=\delta',\quad \gamma''=\delta'',$$

or if we write

$$\frac{\alpha}{\gamma}=-l,\quad \frac{\alpha'}{\gamma'}=-m,\quad \frac{\alpha''}{\gamma''}=-n,$$

the equations become by a simple reduction,

$$Y^2+Z^2+2l\ YZ=l^2-1,$$
$$Z^2+X^2+2mZX=m^2-1,$$
$$X^2+Y^2+2n\ XY=n^2-1,$$

which are equivalent to the equations discussed in my paper "On a system of Equations connected with Malfatti's Problem and on another Algebraical System," *Cambridge and Dublin Mathematical Journal*, t. IV. [1849] pp. 270—275, [79]; the solution might have been effected by the direct method, which I shall here adopt, of eliminating any one of the variables between the two equations into which it enters, and combining the result with the third equation.

Writing the second and third equations under the form

$$A'+B'X+C'\sqrt{1+X^2}=0,$$
$$A''+B''X+C''\sqrt{1+X^2}=0,$$

the result of the elimination may be presented in the form

$$A'A''+B'B''-C'C''=\sqrt{A'^2+B'^2-C'^2}\sqrt{A''^2+B''^2-C''^2},$$

which is most easily obtained by writing $X=\tan\phi$ and operating with the symbol $\cos^{-1}$; but if the rationalized equations be represented by

$$\lambda'+2\mu'X+\nu'X^2=0 \text{ and } \lambda''+2\mu''X+\nu''X^2=0,$$

the form

$$4(\lambda'\nu'-\mu'^2)(\lambda''\nu''-\mu''^2)=(\lambda'\nu''+\lambda''\nu'-2\mu'\mu'')^2$$

leads easily to the result in question. The values which enter are

$$A'=\alpha'+\beta'Z,\qquad A''=\alpha''+\beta''Y,$$
$$B'=\beta'+\gamma'Z,\qquad B''=\beta''+\gamma''Y,$$
$$C'=\delta'\sqrt{1+Z^2},\quad C''=\delta''\sqrt{1+Y^2};$$

whence, in the first place, by the equation connecting Y, Z,

$$C'C''=-\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\delta YZ\}.$$

It is obviously convenient that $A'A''+B'B''$ should be symmetrical with respect to Y and Z, and this will be the case if

$$\alpha'\beta''+\beta'\gamma''=\alpha''\beta'+\beta''\gamma',$$

that is, if

$$\beta'(\gamma''-\alpha'')=\beta''(\gamma'-\alpha');$$

or assuming that the equations are symmetrically related to the system, we have the first set of relations between the coefficients, relations which are satisfied by

$$\alpha=\gamma+2\phi\beta,\quad \alpha'=\gamma'+2\phi\beta',\quad \alpha''=\gamma''+2\phi\beta'',$$

and the values of α, α', α'' will be considered henceforth as given by these conditions. We have

$$A'A''+B'B''-C'C''=\alpha'\alpha''+\beta'\beta''+(\gamma'\beta''+\gamma''\beta'+2\phi\beta'\beta'')(Y+Z)+(\beta'\beta''+\gamma'\gamma'')\,YZ$$
$$+\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\gamma YZ\}.$$

The quantities $A'^2+B'^2-C'^2$, $A''^2+B''^2-C''^2$ are quadratic functions of Z and Y respectively, and with proper relations between the coefficients, we may assume

$$(A'^2+B'^2-C'^2)(A''^2+B''^2-C''^2)=l^2s^2\{U^2+k[(\alpha+\beta(Y+Z)+\gamma YZ)^2-\delta^2(1+Y^2)(1+Z^2)]\},$$

in which U is a linear function of $Y+Z$ and YZ, and k and ls are constants. The first side must, in the first place, be symmetrical with respect to Y and Z, or

$$\alpha'^2+\beta'^2-\delta'^2,\quad (\alpha'+\gamma')\beta',\quad \beta'^2+\gamma'^2-\delta'^2$$

must be proportional to

$$\alpha''^2+\beta''^2-\delta''^2,\quad (\alpha''+\gamma'')\delta'',\quad \beta''^2+\gamma''^2-\delta''^2.$$

But since

$$(\alpha'+\gamma')\beta',\quad (\alpha''+\gamma'')\beta''$$

are proportional to

$$\gamma'^2-\alpha'^2,\quad \gamma''^2-\alpha''^2,$$

it is only necessary that

$$\beta'^2+\gamma'^2-\delta'^2,\quad \beta''^2+\gamma''^2-\delta''^2$$

should be proportional to

$$\gamma'^2-\alpha'^2,\qquad \gamma''^2-\alpha''^2;$$

or since the equations are supposed symmetrically related to the system, we must have the second set of relations between the coefficients. Suppose

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2}=\frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2}=\frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2}=-\frac{s}{\phi},$$

then since

$$\gamma^2-\alpha^2=-4(\gamma+\phi\beta)\phi\beta,\ \&c.,$$

we have
$$\delta^2 = \beta^2 + \gamma^2 - 4s(\gamma + \phi\beta)\beta$$
$$\delta'^2 = \beta'^2 + \gamma'^2 - 4s(\gamma' + \phi\beta')\beta'$$
$$\delta''^2 = \beta''^2 + \gamma''^2 - 4s(\gamma'' + \phi\beta'')\beta'',$$

and δ, δ', δ'' will be supposed henceforth to satisfy these equations.

We have next

$$A'^2 + B'^2 - C'^2 = 4(\gamma' + \phi\beta')\beta'(s + \phi + Z + sZ^2)$$
$$A''^2 + B''^2 - C''^2 = 4(\gamma'' + \phi\beta'')\beta''(s + \phi + Y + sY^2),$$

which may be simplified by writing

$$s = \frac{\mu - \phi}{1 + \mu^2}, \qquad \nu = \frac{1 + \mu\phi}{\mu - \phi},$$

where μ, ν are to be considered as given functions of s and ϕ. These values give

$$A'^2 + B'^2 - C'^2 = 4(\gamma' + \phi\beta')\beta' s(Z + \mu)(Z + \nu),$$
$$A''^2 + B''^2 - C''^2 = 4(\gamma'' + \phi\beta'')\beta'' s(Y + \mu)(Y + \nu).$$

Hence, putting for simplicity

$$l^2 = 4(\gamma' + \phi\beta')(\gamma'' + \phi\beta'')\beta'\beta'',$$

we have

$$4(Z + \mu)(Z + \nu)(Y + \mu)(Y + \nu) = U^2 + k[(\alpha + \beta(Y + Z) + \gamma YZ)^2 - \delta^2(1 + Y^2)(1 + Z^2)];$$

and the two sides have next to be expressed in terms of $Y + Z$ and YZ.

If for symmetry we write

$$\xi = 1, \quad \eta = Y + Z, \quad \zeta = YZ,$$

then

$$4(\mu^2\xi + \mu\eta + \zeta)(\nu^2\xi + \nu\eta + \zeta) + k\delta^2[(\xi - \zeta)^2 + \eta^2] = U^2 + k(\alpha\xi + \beta\eta + \gamma\zeta)^2;$$

and U is now to be considered a linear function of ξ, η, ζ.

The condition that the first side of the equation may divide into factors, gives an equation for determining k; since the condition is satisfied for $k = 0$ and $k = \infty$, the equation will be linear, and it is easily seen that the value is $k = \frac{1}{\delta^2}(\mu - \nu)^2$. In fact

$$4(\mu^2\xi + \mu\eta + \zeta)(\nu^2\xi + \nu\eta + \zeta)^2 + (\mu - \nu)^2[(\xi - \zeta)^2 + \eta^2]$$
$$= (2\mu\nu\xi + (\mu + \nu)\eta + 2\zeta)^2 + (\mu - \nu)^2(\xi + \zeta)^2;$$

hence

$$\{2\mu\nu\xi + (\mu + \nu)\eta + 2\zeta\}^2 - U^2 = \frac{(\mu - \nu)^2}{\delta^2}\{(\alpha\xi + \beta\eta + \gamma\zeta)^2 - \delta^2(\xi + \zeta)^2\},$$

and we may assume

$$2\mu\nu\xi + (\mu+\nu)\eta + 2\zeta + U = \frac{\mu-\nu}{\delta}\Lambda\{(\alpha\xi+\beta\eta+\gamma\zeta) - \delta(\xi+\zeta)\},$$

$$2\mu\nu\xi + (\mu+\nu)\eta + 2\zeta - U = \frac{\mu-\nu}{\delta}\frac{1}{\Lambda}\{(\alpha\xi+\beta\eta+\gamma\zeta) + \delta(\xi+\zeta)\},$$

subject to its being shown that

$$4\mu\nu\xi + 2(\mu+\nu)\eta + 4\zeta = \frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)(\alpha\xi+\beta\eta+\gamma\zeta) - \delta\left(\Lambda-\frac{1}{\Lambda}\right)(\xi+\zeta)\right\}$$

gives a constant value for Λ. The comparison of coefficients gives

$$4\mu\nu = \frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\alpha - \left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\},$$

$$2\mu + 2\nu = \frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)\beta,$$

$$4 = \frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\gamma - \left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\};$$

the first and third of these give

$$4(1-\mu\nu) = \frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)(\gamma-\alpha),$$

which will be identical with the second, if

$$\frac{2(1-\mu\nu)}{\mu+\nu} = \frac{\beta}{\gamma-a} = -2\phi,$$

which follows at once from the equation

$$\nu = \frac{1+\mu\phi}{\mu-\phi}.$$

Forming next the two equations

$$\Lambda + \frac{1}{\Lambda} = \frac{2}{(\mu-\nu)\beta}(\mu+\nu)\delta,$$

$$\Lambda - \frac{1}{\Lambda} = \frac{2}{(\mu-\nu)\beta}\{(\mu+\nu)\gamma - 2\beta\},$$

these will be equivalent to a single equation if

$$(\mu+\nu)^2\delta^2 = \{(\mu+\nu)\gamma - 2\beta\}^2 + (\mu-\nu)^2\beta^2,$$

that is, if

$$(\mu+\nu)^2\delta^2 = (\mu+\nu)^2(\beta^2+\gamma^2) - 4(\mu+\nu)\beta\gamma - 4(\mu\nu-1)\beta^2;$$

or finally, if

$$\delta^2 = \beta^2 + \gamma^2 - 4s\beta\left(\gamma + \frac{\mu\nu - 1}{\mu + \nu}\beta\right) = \beta^2 + \gamma^2 - 4s(\gamma + \phi\beta)\beta,$$

which is in fact the case.

Writing the equations for

$$\Lambda + \frac{1}{\Lambda}, \quad \Lambda - \frac{1}{\Lambda},$$

in the form

$$\Lambda + \frac{1}{\Lambda} = \frac{2\delta}{(\mu - \nu)\beta s}\Lambda - \frac{1}{\Lambda} = \frac{2}{(\mu - \nu)\beta s}(\gamma - 2\beta s),$$

and substituting in

$$U = \frac{\mu - \nu}{2\delta}\left\{\left(\Lambda - \frac{1}{\Lambda}\right)(\alpha\xi + \beta\eta + \gamma\zeta) - \left(\Lambda + \frac{1}{\Lambda}\right)\delta^2(\xi + \zeta)\right\},$$

we have

$$U = \frac{1}{s\beta\delta}\{(\gamma - 2\beta s)(\alpha\xi + \beta\eta + \gamma\zeta) - \delta^2(\xi + \zeta)\}$$

$$= \frac{1}{s\beta\delta}\{(-\beta + 2s\gamma + 2\phi\gamma)\xi + (\gamma - 2s\beta)\eta + (-\beta + 2s\gamma + 4s\phi\beta)\zeta\};$$

and consequently, multiplying by

$$ls = 2\sqrt{(\gamma' + \phi\beta')(\gamma'' + \phi\beta'')\beta'\beta''}$$

we have

$$\sqrt{A'^2 + B'^2 - C'^2}\sqrt{A''^2 + B''^2 - C''^2}$$

$$= \frac{2}{\delta}\sqrt{(\gamma' + \phi\beta')(\gamma'' + \phi\beta'')\beta'\beta''}\{(-\beta + 2s\gamma + 2\phi\gamma)\xi + (\gamma - 2s\beta)\eta + (-\beta + 2s\gamma + 4s\phi\beta)\zeta\},$$

or collecting the different terms which enter into the equation

$$A'A'' + B'B'' - C'C'' = \sqrt{A'^2 + B'^2 - C'^2}\sqrt{A''^2 + B''^2 - C''^2}$$

the result is

$$(\alpha'\alpha'' + \beta'\beta'')\xi + (\gamma'\beta'' + \gamma''\beta' + 2\phi\beta'\beta'')\eta + (\beta'\beta'' + \gamma'\gamma'')\zeta + \frac{\delta'\delta''}{\delta}(\alpha\xi + \beta\eta + \gamma\zeta)$$

$$-\frac{2}{\delta}\sqrt{(\gamma' + 2\phi\beta')(\gamma'' + 2\phi\beta'')\beta'\beta''}\{(-\beta + 2s\gamma + 2\phi\gamma)\xi + (\gamma - 2s\beta)\eta + (-\beta + 2s\gamma + 4s\phi\beta)\zeta\} = 0,$$

which, combined with the first equation written under the form

$$(\alpha\xi + \beta\eta + \gamma\zeta)^2 - \delta^2[(\xi - \zeta)^2 + \eta^2] = 0,$$

determines the ratios of ξ, η, ζ, that is, the values of $Y + Z$ and YZ.

§ 6.

The system of equations

$$(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}\,(Y+Z)+fYZ-\sqrt{bc}\,\sqrt{1+Y^2}\,\sqrt{1+Z^2}=0,$$

$$(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}\,(Z+X)+gZX-\sqrt{ca}\,\sqrt{1+Z^2}\,\sqrt{1+X^2}=0,$$

$$(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}\,(X+Y)+hXY-\sqrt{ab}\,\sqrt{1+X^2}\,\sqrt{1+Y^2}=0,$$

where

$$\theta=\frac{1}{K}(\sqrt{\mathfrak{ABC}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}}),$$

on which depends the solution of Steiner's extension of Malfatti's problem, is at once seen to belong to the class of equations treated of in the preceding section, and we have $\phi=\theta$, $s=0$. The equations at the conclusion of the preceding section become

$$\{\sqrt{\mathfrak{BC}}+gh+2\theta(g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}})+4\theta^2\sqrt{\mathfrak{BC}}\}\,\xi+\{g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{BC}}\}\,\eta+\{\sqrt{\mathfrak{BC}}+gh\}\,\zeta$$

$$-a[(f+2\theta\sqrt{\mathfrak{A}})\,\xi+\sqrt{\mathfrak{A}}\,\eta+f\zeta]-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}\,[(\sqrt{\mathfrak{A}}-2\theta f)\,\xi-f\eta+\sqrt{\mathfrak{A}}\,\zeta]=0,$$

$$\{(f+2\theta\sqrt{\mathfrak{A}})\,\xi+\sqrt{\mathfrak{A}}\,\eta+f\zeta\}^2-bc\,\{(\xi-\zeta)^2+\eta^2\}=0,$$

which may also be written

$$(\sqrt{\mathfrak{BC}}+\mathfrak{F})\,(\xi+\zeta)+(-a\sqrt{\mathfrak{A}}+g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{BC}})\,(\eta+2\theta\xi)$$

$$-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}\,\{(\sqrt{\mathfrak{A}}-2\theta f)\,\xi-f\eta+\sqrt{\mathfrak{A}}\,\zeta\}=0,$$

$$\{f(\xi+\zeta)+\sqrt{\mathfrak{A}}\,(\eta+2\theta\xi)\}^2-bc\,\{(\xi-\zeta)^2+\eta^2\}=0.$$

Hence observing that

$$g+\theta\sqrt{\mathfrak{B}}=\frac{1}{K}(\sqrt{\mathfrak{BC}}+\mathfrak{F})\,(\sqrt{\mathfrak{AB}}+\mathfrak{H});\quad h+\theta\sqrt{\mathfrak{C}}=\frac{1}{K}(\sqrt{\mathfrak{BC}}+\mathfrak{F})\,(\sqrt{\mathfrak{AC}}+\mathfrak{G});$$

$$-a\sqrt{\mathfrak{A}}+h\sqrt{\mathfrak{B}}+g\sqrt{\mathfrak{C}}+2\theta\sqrt{\mathfrak{BC}}=\theta(\sqrt{\mathfrak{BC}}+\mathfrak{F}),$$

and putting for a moment

$$\lambda=\frac{1}{K}\sqrt{(\sqrt{\mathfrak{AC}}+\mathfrak{G})\,(\sqrt{\mathfrak{AB}}+\mathfrak{H})\sqrt{\mathfrak{BC}}},$$

and therefore

$$\sqrt{(g+\theta\sqrt{\mathfrak{B}})\,(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}=(\sqrt{\mathfrak{BC}}+\mathfrak{F})\,\lambda,$$

the first equation divides by $(\sqrt{\mathfrak{BC}}+\mathfrak{F})$, and the result is

$$(\xi+\zeta)+\theta(\eta+2\theta\xi)-\frac{2\lambda}{\sqrt{bc}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0.$$

Also, by an easy transformation, the second equation becomes

$$-\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}^2+4bc\zeta\left((\xi+\zeta)+\theta(\eta+2\theta\xi)-(1+\theta^2)\xi\right)=0,$$

or putting

$$\xi+\zeta\quad+\theta(\eta+2\theta\xi)=\Theta,$$

$$\frac{1}{\sqrt{bc}}\left(\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\right)=\Phi,$$

$$\zeta\qquad\qquad\qquad\qquad=\Psi,$$

the equations become

$$\Theta-2\lambda\Phi=0,$$

$$-\Phi^2+4\Psi\{\Theta-(1+\theta^2)\Psi\}=0;$$

hence eliminating Φ,

$$\left(2\Psi-\frac{\Theta}{1+\theta^2}\right)^2=\frac{\Theta^2}{(1+\theta^2)^2}\left(1-\frac{1+\theta^2}{4\lambda^2}\right),$$

or observing that

$$1+\theta^2=\frac{1}{K^2}(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{CA}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H}),$$

and reducing, we obtain

$$\Psi=\frac{K^2\Theta}{(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{CA}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})}\left(1+\frac{\sqrt{(\sqrt{\mathfrak{BC}}-\mathfrak{F})\sqrt{\mathfrak{BC}}}}{\sqrt{2\mathfrak{BC}}}\right);$$

also $\Theta=2\lambda\Phi$ gives

$$\Phi=\frac{K\Theta}{2\sqrt{(\sqrt{\mathfrak{AC}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})\sqrt{\mathfrak{BC}}}}.$$

Suppose

$$\sqrt{\mathfrak{BC}}+\mathfrak{F}=\alpha,\quad\sqrt{\mathfrak{BC}}-\mathfrak{F}=\alpha_{,},\quad\therefore\ \alpha\alpha_{,}=Ka,$$

$$\sqrt{\mathfrak{CA}}+\mathfrak{G}=\beta,\quad\sqrt{\mathfrak{CA}}-\mathfrak{G}=\beta_{,},\qquad\beta\beta_{,}=Kb,$$

$$\sqrt{\mathfrak{AB}}+\mathfrak{H}=\gamma,\quad\sqrt{\mathfrak{AB}}-\mathfrak{H}=\gamma_{,},\qquad\gamma\gamma_{,}=Kc;$$

then substituting

$$\Theta-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{bc}\,\Phi=0,$$

$$\Theta-\frac{4bc}{\beta_{,}\gamma_{,}}(\alpha+\alpha_{,})\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)\Psi=0,$$

that is,

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0,$$

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)\zeta=0;$$

these may be written

$$L\xi+M\eta+N\zeta=0,$$

$$L'\xi+M'\eta+N'\zeta=0,$$

where

$$L=1+2\theta^2-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(\sqrt{\mathfrak{A}}-2\theta f),\quad M=\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}f,\quad N=1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}},$$

$$L'=1+2\theta^2-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right),\quad M'=\theta\qquad,\quad N'=1;$$

or since ξ, η, ζ are equal to 1, $Y+Z$, YZ respectively,

$$1:Y+Z:YZ=MN'-M'N:NL'-N'L:LM'-L'M$$

$$=\quad -\frac{\sqrt{2}(\alpha+\alpha_{,})}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}})$$

$$:\quad \frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}})2\theta$$

$$:-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}f\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}}).$$

Also

$$f+\theta\sqrt{\mathfrak{A}}=\frac{\beta\gamma}{K}=\frac{Kbc}{\beta_{,}\gamma_{,}},$$

whence

$$Y+Z=-\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{,}}\sqrt{\beta_{,}\gamma_{,}}}{K}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)-2\theta,$$

$$YZ=\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{,}}\sqrt{\beta_{,}\gamma_{,}}}{K}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}f}{\sqrt{\beta_{,}\gamma_{,}}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)-1;$$

and by forming the analogous expressions for $Z+X$ and ZX, $X+Y$ and XY, the values of X, Y, Z may be determined. But the equations in question simplify themselves in a remarkable manner by the notation before alluded to.

Suppose

$$\mathfrak{f} = \sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}} - \mathfrak{F}},\quad \mathfrak{g} = \sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{C}\mathfrak{A}} - \mathfrak{G}},\quad \mathfrak{h} = \sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}} - \mathfrak{H}},\quad J = \sqrt{2}\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

these values give

$$\frac{K\sqrt{\mathfrak{A}}}{\sqrt{\mathfrak{B}\mathfrak{C}}}\alpha = 2\mathfrak{f}^2\left(1 - \frac{\mathfrak{f}^2}{J^2}\right),$$

$$\frac{K\sqrt{bc}}{\sqrt{\mathfrak{A}}} = 2\mathfrak{g}\mathfrak{h}\sqrt{1 - \frac{\mathfrak{g}^2}{J^2}}\sqrt{1 - \frac{\mathfrak{h}^2}{J^2}},$$

$$\frac{Kf}{\sqrt{\mathfrak{A}}} = \mathfrak{f}^2 - \mathfrak{g}^2 - \mathfrak{h}^2 + \frac{2\mathfrak{g}^2\mathfrak{h}^2}{J^2},$$

$$\vdots$$

$$K\theta = -\mathfrak{f}^2 - \mathfrak{g}^2 - \mathfrak{h}^2 + 2J^2,$$

$$K^2 = -\mathfrak{f}^4 - \mathfrak{g}^4 - \mathfrak{h}^4 + 2\mathfrak{g}^2\mathfrak{h}^2 + 2\mathfrak{h}^2\mathfrak{f}^2 + 2\mathfrak{f}^2\mathfrak{g}^2 - \frac{4\mathfrak{f}^2\mathfrak{g}^2\mathfrak{h}^2}{J^2}.$$

Applying these results to the preceding formulæ and forming for that purpose the equations

$$2\sqrt{2}\sqrt{\alpha + \alpha_{,}}\sqrt{\beta_{,}\gamma_{,}} = 4\mathfrak{g}\mathfrak{h},\quad \frac{\sqrt{2}\sqrt{\alpha + \alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}} = \frac{J^2}{\sqrt{\mathfrak{A}}\mathfrak{g}\mathfrak{h}},\quad \frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha + \alpha_{,}}} = \frac{\mathfrak{f}}{J},$$

$$\mathfrak{g}\mathfrak{h}K\theta + \frac{J^2}{\sqrt{\mathfrak{A}}}K\mathfrak{f} = (J^2 - \mathfrak{g}\mathfrak{h})\left(\mathfrak{f}^2 - (\mathfrak{g} - \mathfrak{h})^2\right) - 2\mathfrak{g}\mathfrak{h}(\mathfrak{g} - \mathfrak{h})^2,$$

we have

$$K(Y + Z) + 2K\theta = 4(J^2 - \mathfrak{g}\mathfrak{h})\left(1 - \frac{\mathfrak{f}}{J}\right),$$

$$K^2YZ + K^2 = \{(J^2 - \mathfrak{g}\mathfrak{h})\left(\mathfrak{f}^2 - (\mathfrak{g} - \mathfrak{h})^2\right) - 2\mathfrak{g}\mathfrak{h}(\mathfrak{g} - \mathfrak{h})^2\}\left(1 - \frac{\mathfrak{f}}{J}\right);$$

the former of which, combined with the similar equations for $Z + X$ and $X + Y$, gives for X, Y, Z the values to be presently stated, and these values will of course verify the second equation and the corresponding equations for ZX and XY.

Recapitulating the preceding notation, if $x = 0$, $y = 0$, $z = 0$ are the equations of the given sections, $w = 0$ the equation of the polar plane of their point of intersection with respect to the surface,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + pw^2 = 0$$

the equation of the surface, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, K as usual, and

$$\theta = \frac{1}{K}\left(\sqrt{\mathfrak{A}\mathfrak{B}\mathfrak{C}} + \mathfrak{F}\sqrt{\mathfrak{A}} + \mathfrak{G}\sqrt{\mathfrak{B}} + \mathfrak{H}\sqrt{\mathfrak{C}}\right),$$

then the equations of the required sections are

$$(ax+hy+gz)\,X+y\sqrt{\mathfrak{C}}+z\sqrt{\mathfrak{B}}+w\sqrt{-ap}\,\sqrt{1+X^2}=0,$$

$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\,Y+z\sqrt{\mathfrak{A}}+w\sqrt{-bp}\,\sqrt{1+Y^2}=0,$$

$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\,Z+w\sqrt{-cp}\,\sqrt{1+Z^2}=0,$$

where X, Y, Z are to be determined by the following equations,

$$(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}\,(Y+Z)+f\,YZ-\sqrt{bc}\,\sqrt{1+Y^2}\,\sqrt{1+Z^2}=0,$$

$$(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}\,(Z+X)+g\,ZX-\sqrt{ca}\,\sqrt{1+Z^2}\,\sqrt{1+X^2}=0,$$

$$(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}\,(X+Y)+hXY-\sqrt{ab}\,\sqrt{1+X^2}\,\sqrt{1+Y^2}=0;$$

and the solution of which, putting

$$\mathrm{f}=\sqrt[4]{\mathfrak{A}}\,\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F}},\quad \mathrm{g}=\sqrt[4]{\mathfrak{B}}\,\sqrt{\sqrt{\mathfrak{C}\mathfrak{A}}-\mathfrak{G}},\quad \mathrm{h}=\sqrt[4]{\mathfrak{C}}\,\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}},\quad J=\sqrt{2}\,\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

is given by the equations

$$KX=\frac{2\mathrm{fgh}}{J}+(-\mathrm{f}+\mathrm{g}+\mathrm{h})^2-2\,(-\mathrm{f}+\mathrm{g}+\mathrm{h})\,J,$$

$$KY=\frac{2\mathrm{fgh}}{J}+(\mathrm{f}-\mathrm{g}+\mathrm{h})^2-2\,(\mathrm{f}-\mathrm{g}+\mathrm{h})\,J,$$

$$KZ=\frac{2\mathrm{fgh}}{J}+(\mathrm{f}+\mathrm{g}-\mathrm{h})^2-2\,(\mathrm{f}+\mathrm{g}-\mathrm{h})\,J.\ [1]$$

Instead of the direct but very tedious process by which these values of X, Y, Z have been obtained, we may substitute the following *à posteriori* verification.

We have

$$K^2(1+X^2)=4\,(-\mathrm{f}+\mathrm{g}+\mathrm{h})^2J^2\left(1+\frac{\mathrm{f}}{J}\right)\left(1-\frac{\mathrm{g}}{J}\right)\left(1-\frac{\mathrm{h}}{J}\right),$$

$$K^2\sqrt{1+Y^2}\,\sqrt{1+Z^2}=4\left(\mathrm{f}^2-(\mathrm{g}^2-\mathrm{h})\right)J^2\left(1-\frac{\mathrm{f}}{J}\right)\sqrt{1-\frac{\mathrm{g}^2}{J^2}}\sqrt{1-\frac{\mathrm{h}^2}{J^2}},$$

$$K^2(1+YZ)=4\left(1-\frac{\mathrm{f}}{J}\right)\{(J^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}\,(\mathrm{g}-\mathrm{h})^2\},$$

$$K\,(Y+Z)-2\mathrm{f}^2-2\mathrm{g}^2-2\mathrm{h}^2+4J^2=4\left(1-\frac{\mathrm{f}}{J}\right)(J^2-\mathrm{gh}).$$

Putting also

$$\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{J}=\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-\frac{2\mathrm{gh}\,(J^2-\mathrm{gh})}{J^2},$$

$$K^2=\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)\left((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2-\frac{4\mathrm{g}^2\mathrm{h}^2}{J^2}\right)-\frac{4\mathrm{g}^2\mathrm{h}^2\,(\mathrm{g}-\mathrm{h})^2}{J^2},$$

[1] It is perhaps worth noticing that the value of the quantity λ previously made use of,

$$\lambda=\frac{\mathrm{f}^2}{Ka\sqrt{\mathfrak{A}}}\left\{\frac{2\mathrm{fgh}}{J}-\mathrm{g}^2-\mathrm{h}^2+(J+\mathrm{f}-\mathrm{g}-\mathrm{h})^2\right\}.$$

we have

$$\left(f^2-g^2-h^2+\frac{2g^2h^2}{J^2}\right)K^2(1+YZ)$$

$$=4\left(1-\frac{f}{J}\right)\left\{\left(f^2-(g-h)^2\right)\left[(J^2-gh)\left(f^2-(g-h)^2\right)-2gh\,(g-h)^2-2gh\frac{(J^2-gh)^2}{J^2}\right]\right.$$

$$\left.+\frac{4g^2h^2(g-h)^2(J^2-gh)}{J^2}\right\},$$

$$K^2\{K(Y+Z)-2f^2-2g^2-2h^2+4J^2\}$$

$$=4\left(1-\frac{f}{J}\right)\left\{f^2-(g-h)^2\right)\left[(J^2-gh)\left((g+h)^2-f^2-\frac{4g^2h^2}{J^2}\right)\right]-\frac{4g^2h^2(g-h)^2(J^2-gh)}{J^2}\right\}.$$

Also, since

$$\left(f^2-(g-h)^2\right)+\left((g+h)^2-f^2-\frac{4g^2h^2}{J^2}\right)=4gh\,\frac{(J^2-gh)}{J^2},$$

we have

$$\left(f^2-g^2-h^2+\frac{2g^2h^2}{J^2}\right)K^2(1+YZ)+K^2\{K(Y+Z)-2f^2-2g^2-2h^2\}$$

$$=4\left(1-\frac{f}{J}\right)\left(f^2-(g-h)^2\right)2ghJ^2\left(1-\frac{g^2}{J^2}\right)\left(1-\frac{h^2}{J^2}\right),$$

and the values obtained above give also

$$2gh\sqrt{1-\frac{g^2}{J^2}}\sqrt{1-\frac{h^2}{J^2}}K^2\sqrt{1+Y^2}\sqrt{1+Z^2}$$

$$=4\left(1-\frac{f}{J}\right)\left(f^2-(g-h)^2\right)2ghJ^2\left(1-\frac{g^2}{J^2}\right)\left(1-\frac{h^2}{J^2}\right),$$

which shows that the relation between Y and Z is verified by the assumed values of these quantities, and the other two equations are of course also verified. The solution of the problem will be rendered more complete if the equations of the required sections and of the auxiliary sections made use of in the geometrical construction are expressed in terms of f, g, h, J.

§ 7.

First, to substitute in the equations of the required sections or resultors. Writing the first equation in the form

$$\frac{K^2}{2\sqrt{\mathfrak{B}\mathfrak{C}}}\left\{aXx+(hX+\sqrt{\mathfrak{C}})\,y+(gX+\sqrt{\mathfrak{B}})\,z+\sqrt{-ap}\,\sqrt{1+X^2}\,w\right\}=0,$$

the coefficient of x will be

$$\frac{f^2}{\sqrt{\mathfrak{A}}}\left(1-\frac{f^2}{J^2}\right)\left\{\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right\},$$

or, as it is convenient to write it,

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)\left\{\frac{2fgh}{J(-f+g+h)}-f+g+h-2J\right\}.$$

The coefficient of y is

$$\frac{1}{2\sqrt{\mathfrak{B}}}\left\{\left(-f^2-g^2+h^2+\frac{2g^2h^2}{J^2}\right)\left(\frac{2fgh}{J^2}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$
$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1-\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f-g+h+\frac{2J(f^2+g^2-h^2)}{2fg}\right\};$$

and similarly the coefficient of z is

$$\frac{1}{2\sqrt{\mathfrak{C}}}\left\{\left(-f^2+g^2-h^2+\frac{2h^2f^2}{J^2}\right)\left(\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$
$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f+g-h+\frac{2J(f^2-g^2+h^2)}{2fh}\right\}$$

and the coefficient of w is

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\,2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}.$$

Hence, forming the equation of the resultor in question, and by means of it those of the other resultors, the equations of the resultors are

$$\left(\frac{2fgh}{J(-f+g+h)}-f+g+h-2J\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$
$$+\left(\frac{-2fgh}{J(-f+g+h)}+f-g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$
$$+\left(\frac{-2fgh}{J(-f+g+h)}+f+g-h+2J\frac{f^2-g^2+h^2}{2fh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$
$$+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w=0,$$

$$\left(\frac{2fgh}{J(f-g+h)}-f+g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$
$$+\left(\frac{2fgh}{J(f-g+h)}+f-g+h-2J\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$
$$+\left(\frac{-2fgh}{J(f-g+h)}+f+g-h+2J\frac{-f^2+g^2+h^2}{2gh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$
$$+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w=0$$

$$\left(\frac{-2fgh}{J(f+g-h)}-f+g+h+2J\frac{f^2-g^2+h^2}{2fh}\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$
$$+\left(\frac{-2fgh}{J(f+g-h)}+f-g+h+2J\frac{-f^2+g^2+h^2}{2gh}\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$
$$+\left(\frac{2fgh}{J(f+g-h)}+f+g-h-2J\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$
$$+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w=0,$$

values which might be somewhat simplified by writing ξ, η, ζ, ω instead of

$$\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x,\quad \frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y,\quad \frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z,\quad 2\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}\,w;$$

and it may be also remarked, that the coefficients as well of these formulæ as of those which follow may be elegantly expressed in terms of the parts of a triangle having f, g, h for its sides.

The equations of the separators are found by taking the differences two and two of the equations of the resultors (this requires to be verified *à posteriori*); thus subtracting the third equation from the second the result contains a constant factor,

$$\frac{1}{J(f^2-(g-h)^2)gh}\{4f^2g^2h^2-J(f^2-(g-h)^2)((g+h)^2-f^2)\},$$

equivalent to

$$\frac{1}{J(f^2-(g-h)^2)gh}\left(4f^2g^2h^2-J^2\left(K^2+\frac{4f^2g^2h^2}{J^2}\right)\right)\quad\text{or}\quad\frac{-JK^2}{(f^2-(g-h)^2)gh}.$$

Rejecting the factor in question and forming the analogous two equations, the equations of the separators are

$$-\frac{g-h}{f}\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x+\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y-\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z=0,$$
$$-\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x-\frac{h-f}{g}\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y+\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z=0,$$
$$\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x-\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y-\frac{f-g}{h}\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z=0;$$

and from the mode of formation of these equations it is obvious that the separators have a line in common.

The equations of the determinators being $x=0$, $y=0$, $z=0$, the equations of the tactors are

$$\sqrt{\mathfrak{B}}z-\sqrt{\mathfrak{C}}y=0,\quad \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0,\quad \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0;$$

and if $\alpha x+\beta y+\gamma z+\delta w=0$ be the equation of the tactor touching

$$x=0,\ \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0 \text{ and } \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0,$$

the conditions of contact are

$$\mathfrak{A}\left(\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2\right)=(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)^2,$$

$$2\sqrt{\mathfrak{A}\mathfrak{B}}\,(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\left(\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(\alpha\sqrt{\mathfrak{A}}-\beta\sqrt{\mathfrak{B}})+\gamma(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{H}\sqrt{\mathfrak{C}})\right\}^2,$$

$$2\sqrt{\mathfrak{A}\mathfrak{C}}\,(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\left(\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\alpha\sqrt{\mathfrak{A}}-\gamma\sqrt{\mathfrak{C}})+\beta(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\right\}^2,$$

whence

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{A}\mathfrak{B}}\,(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}\,(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$

$$(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{A}}\alpha-(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{B}}\beta+(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{F}\sqrt{\mathfrak{A}})\gamma,$$

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{A}\mathfrak{C}}\,(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})}\,(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$

$$(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\sqrt{\mathfrak{A}}\alpha+(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\beta-(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\gamma,$$

$$c\beta^2+b\gamma^2-2f\beta\gamma+\frac{\mathfrak{A}}{p}\delta^2=0,$$

and putting for a moment

$$\mu=\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G}-\sqrt{2\sqrt{\mathfrak{A}\mathfrak{C}}\,(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})},$$

$$\nu=\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}-\sqrt{2\sqrt{\mathfrak{A}\mathfrak{B}}\,(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})};$$

after some reductions, and observing that the ratios only of the quantities α, β, γ, δ are material, we obtain

$$\alpha=\frac{K}{\sqrt{\mathfrak{A}}}(K+h\nu+g\mu),$$

$$\beta=\frac{K}{\sqrt{\mathfrak{A}}}(\quad b\nu+f\mu),$$

$$\gamma=\frac{K}{\sqrt{\mathfrak{A}}}(\quad f\nu+c\mu),$$

$$\delta=\frac{K}{\sqrt{\mathfrak{A}}}\sqrt{-p(b\nu^2+c\mu^2+2f\mu\nu)}:$$

and it is easily seen also that the coordinates of the point of contact are

$$x=0,\quad y=\nu,\quad z=\mu,\quad w=-\frac{\delta}{p}\frac{\sqrt{\mathfrak{A}}}{K};$$

also

$$\mu=-\frac{Jg}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right),\quad \nu=-\frac{Jh}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right).$$

Hence substituting and introducing throughout the quantities f, g, h, J, also forming the analogous equations, the equations of the tactors are

$$\begin{aligned}
&\left\{f^2(-f^2+g^2+h^2)+(g+h)J\left(f^2-(g-h)^2-\frac{2f^2gh}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{A}}}x\\
&\quad-\left\{f^2-(g-h)^2+\frac{2gh(g-h)}{J}\right\}J\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y\\
&\quad-\left\{f^2-(g-h)^2-\frac{2gh(g-h)}{J}\right\}J\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z\\
&\quad+2\sqrt{K}\sqrt{gh\left(1-\frac{g}{J}\right)\left(1-\frac{h}{J}\right)(f^2-(g-h)^2)}\sqrt{-p}w=0,\\
&\quad-\left\{g^2-(h-f)^2+\frac{2hf(h-f)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x\\
&\quad+\left\{g^2(f^2-g^2+h^2)+(h+f)J\left(g^2-(h-f)^2-\frac{2fg^2h}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{B}}}y\\
&\quad-\left\{g^2-(h-f)^2-\frac{2hf(h-f)}{J}\right\}J\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z\\
&\quad+2\sqrt{K}\sqrt{hf\left(1-\frac{h}{J}\right)\left(1-\frac{f}{J}\right)(g^2-(h-f)^2)}\sqrt{-p}w=0,\\
&\quad-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x\\
&\quad-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y\\
&\quad+\left\{h^2(f^2+g^2-h^2)+(f+g)J\left(h^2-(f-g)^2-\frac{2fgh^2}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{C}}}z\\
&\quad+2\sqrt{K}\sqrt{fg\left(1-\frac{f}{J}\right)\left(1-\frac{g}{J}\right)(h^2-(f-g)^2)}\sqrt{-p}w=0.
\end{aligned}$$

It is obvious, from the equations, that each separator passes through the point of contact of a tactor and determinator, it consequently only remains to be shown that each separator touches two tactors. Consider the tactor which has been represented by $\alpha x + \beta y + \gamma z + \delta w = 0$, the unreduced values of the coefficients give

$$\mathfrak{A}\alpha + \mathfrak{H}\beta + \mathfrak{G}\gamma = K^2\sqrt{\mathfrak{A}},$$

$$\mathfrak{H}\alpha + \mathfrak{B}\beta + \mathfrak{F}\gamma = \frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{H} + \nu),$$

$$\mathfrak{G}\alpha + \mathfrak{F}\beta + \mathfrak{C}\gamma = \frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{G} + \mu),$$

$$\sqrt{\mathfrak{A}\alpha^2 + \ldots \frac{K}{p}}\,\delta^2 = \frac{1}{\sqrt{\mathfrak{A}}}(\mathfrak{A}\alpha + \mathfrak{H}\beta + \mathfrak{G}\gamma) = K^2.$$

Represent for a moment the separator

$$\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1 - \frac{\mathrm{f}}{J}\right)x - \frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1 - \frac{\mathrm{g}}{J}\right)y - \frac{\mathrm{f} - \mathrm{g}}{\mathrm{h}}\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1 - \frac{\mathrm{h}}{J}\right)z = 0$$

by $lx + my + nz + sw = 0$. Then putting $\mathfrak{A}l^2 + \ldots \frac{K}{p}s^2 = \square^2$, since

$$\mathfrak{A}\alpha l + \ldots + \frac{K}{p}\delta s = K^2\left\{l\sqrt{\mathfrak{A}} + \frac{m}{\sqrt{\mathfrak{A}}}(\mathfrak{H} + \nu) + \frac{n}{\sqrt{\mathfrak{A}}}(\mathfrak{G} + \mu)\right\}$$

$$= K^2\left\{\mathrm{f}\left(1 - \frac{\mathrm{f}}{J}\right) - \mathrm{g}\left(1 - \frac{2\mathrm{h}}{J}\right)\left(1 - \frac{\mathrm{g}}{J}\right) - (\mathrm{f} - \mathrm{g})\left(1 - \frac{2\mathrm{g}}{J}\right)\left(1 - \frac{\mathrm{h}}{J}\right)\right\}$$

$$= \frac{K^2}{J}\left\{-(\mathrm{f} - \mathrm{g})^2 + \mathrm{h}(\mathrm{f} + \mathrm{g}) - \frac{2\mathrm{fgh}}{J}\right\},$$

the condition of contact becomes

$$\square = \frac{1}{J}\left\{-(\mathrm{f} - \mathrm{g})^2 + \mathrm{h}(\mathrm{f} + \mathrm{g}) - \frac{2\mathrm{fgh}}{J}\right\};$$

or, forming the value of $\square^2$ and substituting,

$$\mathrm{f}^2\left(1 - \frac{\mathrm{f}}{J}\right)^2 + \mathrm{g}^2\left(1 - \frac{\mathrm{g}}{J}\right)^2 + (\mathrm{f} - \mathrm{g})^2\left(1 - \frac{\mathrm{h}}{J}\right)^2 + 2\left(1 - \frac{2\mathrm{f}^2}{J^2}\right)(\mathrm{f} - \mathrm{g})\,\mathrm{g}\left(1 - \frac{\mathrm{g}}{J}\right)\left(1 - \frac{\mathrm{h}}{J}\right)$$

$$- 2\left(1 - \frac{2\mathrm{g}^2}{J^2}\right)(\mathrm{f} - \mathrm{g})\,\mathrm{f}\left(1 - \frac{\mathrm{h}}{J}\right)\left(1 - \frac{\mathrm{f}}{J}\right) - 2\left(1 - \frac{2\mathrm{h}^2}{J^2}\right)\mathrm{fg}\left(1 - \frac{\mathrm{f}}{J}\right)\left(1 - \frac{\mathrm{g}}{J}\right)$$

$$= \frac{1}{J^2}\left(-(\mathrm{f} - \mathrm{g})^2 + \mathrm{h}(\mathrm{f} + \mathrm{g}) - \frac{2\mathrm{fgh}}{J}\right)^2,$$

which may be verified without difficulty, and thus the construction for the resultors is shown to be true.

§ 8

Several of the formulæ of the preceding sections of this memoir apply to any number of variables. Consider the surface (i.e. hypersurface)

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \ldots + pt^2 = 0,$$

and the section (i.e. hypersection)

$$(a\lambda + h\mu + g\nu \ldots)\, x + (h\lambda + b\mu + f\nu \ldots)\, y + (g\lambda + f\mu + c\nu \ldots)\, z \ldots + \sqrt{-p}\, \nabla t = 0,$$

where

$$\nabla^2 = a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu \ldots - K,$$

the condition of contact with any other section represented by a similar equation is

$$a\lambda\lambda' + b\mu\mu' + c\nu\nu' + f(\mu\nu' + \mu'\nu) + g\,(\nu\lambda' + \nu'\lambda) + h\,(\lambda\mu' + \lambda'\mu) \ldots \pm K = \nabla\nabla',$$

where K is the determinant formed with the coefficients $a, b, c, f, g, h, \ldots$ And consequently, by establishing all or any of the equations $\lambda = \sqrt{\mathfrak{A}}$, $\mu = \sqrt{\mathfrak{B}}$, $\nu = \sqrt{\mathfrak{C}}$, ... we have the condition in order that the section in question may touch all or the corresponding sections of the sections $x = 0$, $y = 0$, $z = 0$, ...

Let $\mathfrak{n}$ be the number of the variables $x, y, z \ldots$, then $K^{\mathfrak{n}-1} = \begin{vmatrix} \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \ldots \\ \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & \end{vmatrix}$,

also $$K^{\mathfrak{n}-2}\{(a\lambda + h\mu + g\nu \ldots)\, x + (h\lambda + b\mu + f\nu \ldots)\, y + (g\lambda + f\mu + c\nu \ldots)\, z \ldots\}$$

$$= -\begin{vmatrix} & x & y & z \ldots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix},$$

whence also

$$K^{\mathfrak{n}-2}(\nabla^2 + K) = -\begin{vmatrix} & \lambda & \mu & \nu \ldots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} \text{ or } K^{\mathfrak{n}-2}\nabla^2 = -\begin{vmatrix} 1 & \lambda & \mu & \nu \ldots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix},$$

and the equation of the section in question becomes

$$-\begin{vmatrix} & x & y & z \dots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\begin{vmatrix} 1 & \lambda & \mu & \nu \dots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\, t = 0,$$

also the condition of contact with the corresponding section is

$$-\begin{vmatrix} \mp 1 & \lambda & \mu & \nu \dots \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} = \sqrt{-\begin{vmatrix} 1 & \lambda & \mu & \nu \dots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\sqrt{-\begin{vmatrix} 1 & \lambda' & \mu' & \nu' \dots \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}.$$

In particular the equation of the sections which touches all the sections $x=0$, $y=0$, $z=0, \dots$, is

$$-\begin{vmatrix} & x & y & z \dots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\begin{vmatrix} 1 & \sqrt{\mathfrak{A}} & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}} \dots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\, t = 0.$$

Again, the equations of the section touching $y=0$, $z=0, \dots$ and the sections touching $x=0$, $z=0, \dots$ are

$$-\begin{vmatrix} & x & y & z \dots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\begin{vmatrix} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}} \dots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\, t = 0,$$

$$-\begin{vmatrix} & x & y & z \dots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\begin{vmatrix} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}} \dots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\, t = 0,$$

and the condition of contact of these two sections is

$$-\begin{vmatrix} \mp 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\ldots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} = \sqrt{-\begin{vmatrix} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\ldots \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}\sqrt{-\begin{vmatrix} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}}\ldots \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}}.$$

It would seem from the appearance of these equations that there should be some simpler method of obtaining the solution than the method employed in the previous part of this memoir.

115.

NOTE ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 99—102.]

THE equation of a conic passing through the points of intersection of the conics

$$U = 0, \quad V = 0$$

is of the form

$$wU + V = 0,$$

where w is an arbitrary parameter. Suppose that the conic touches a given line, we have for the determination of w a quadratic equation, the roots of which may be considered as parameters for determining the line in question. Let one of the values of w be considered as equal to a given constant k, the line is always a tangent to the conic

$$kU + V = 0;$$

and taking $w = p$ for the other value of w, p is a parameter determining the particular tangent, or, what is the same thing, the point of contact of this tangent.

Suppose the tangent meets the conic $U = 0$ (which is of course the conic corresponding to $w = \infty$) in the points P, P', and let θ, ∞ be the parameters of the point P, and θ', ∞ the parameters of the point P'. It follows from my "Note on the Geometrical representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$," [113] (1) and from the theory of invariants, that if $\square\xi$ represent the "Discriminant" of $\xi U + V$

[1] I take the opportunity of correcting an obvious error in the note in question, viz. $a^2 + b^2 + c^2 - 2bc - 2ca - 2ab$ is throughout written instead of (what the expression should be) $b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2b^2ca - 2c^2ab$. [This correction is made, *ante* p. 55.]

(I now use the term discriminant in the same sense in which determinant is sometimes used, viz. the discriminant of a quadratic function $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ or $(a, b, c, f, g, h)(x, y, z)^2$, is the determinant $k=abc-af^2-bg^2-ch^2+2fgh$), and if

$$\Pi\xi=\int_{\infty}\frac{d\xi}{\sqrt{\square\xi}},$$

then the following theorem is true, viz.

"If (θ, ∞), (θ', ∞) are the parameters of the points P, P' in which the conic $U=0$ is intersected by the tangent, the parameter of which is p, of the conic $kU+V=0$, then the equations

$$\Pi\theta=\Pi p-\Pi k,$$

$$\Pi\theta'=\Pi p+\Pi k,$$

determine the parameters θ, θ' of the points in question." And again,—

"If the variable parameters θ, θ' are connected by the equation

$$\Pi\theta'-\Pi\theta=2\Pi k,$$

then the line PP' will be a tangent to the conic $kU+V=0$." Whence, also,—

"If the sides of a triangle inscribed in the conic $U=0$ touch the conics

$$k\ U+V=0,$$

$$k'\ U+V=0,$$

$$k''U+V=0,$$

then the equation

$$\Pi k+\Pi k'+\Pi k''=0$$

must hold good between the parameters k, k', k''."

And, conversely, when this equation holds good, there are an infinite number of triangles inscribed in the conic $U=0$, and the sides of which touch the three conics; and similarly for a polygon of any number of sides.

The algebraical equivalent of the transcendental equation last written down is

$$\begin{vmatrix} 1, & k\ , & \sqrt{\square k} \\ 1, & k', & \sqrt{\square k'} \\ 1, & k'', & \sqrt{\square k''} \end{vmatrix}=0;$$

let it be required to find what this becomes when $k=k'=k''=0$, we have

$$\sqrt{\square k}=A+Bk+Ck^2+\ldots,$$

and substituting these values, the determinant divides by

$$\begin{vmatrix} 1, & k, & k^2 \\ 1, & k', & k'^2 \\ 1, & k'', & k''^2 \end{vmatrix},$$

the quotient being composed of the constant term C, and terms multiplied by k, k', k''; writing, therefore, $k=k'=k''=0$, we have $C=0$ for the condition that there may be inscribed in the conic $U=0$ an infinity of triangles circumscribed about the conic $V=0$; C is of course the coefficient of ξ^2 in $\sqrt{\square\xi}$, i.e. in the square root of the discriminant of $\xi U+V$; and since precisely the same reasoning applies to a polygon of any number of sides,—

THEOREM. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$ vanishes. [This and the theorem p. 90 are erroneous, see *post*, 116].

It is perhaps worth noticing that $n=2$, i.e. the case where the polygon degenerates into two coincident chords, is a case of exception. This is easily explained.

In particular, the condition that there may be in the conic[1]

$$ax^2+by^2+cz^2=0$$

an infinity of n-gons circumscribed about the conic

$$x^2+y^2+z^2=0,$$

is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1+a\xi)(1+b\xi)(1+c\xi)}$$

vanishes; or, developing each factor, the coefficient of ξ^{n-1} in

$$(1+\tfrac{1}{2}a\xi-\tfrac{1}{8}a^2\xi^2+\tfrac{1}{16}a^3\xi^3-\tfrac{5}{64}a^4\xi^4+\&c.)(1+\tfrac{1}{2}b\xi-\&c.)(1+\tfrac{1}{2}c\xi-\&c.)$$

vanishes.

Thus, for a triangle this condition is

$$a^2+b^2+c^2-2bc-2ca-2ab=0;$$

for a quadrangle it is

$$a^3+b^3+c^3-bc^2-b^2c-ca^2-c^2a-ab^2-a^2b+2abc=0,$$

which may also be written

$$(b+c-a)(c+a-b)(a+b-c)=0;$$

and similarly for a pentagon, &c.

[1] I have, in order to present this result in the simplest form, purposely used a notation different from that of the note above referred to, the quantities $ax^2+by^2+cz^2$ and $x^2+y^2+z^2$ being, in fact, interchanged.

Suppose the conics reduce themselves to circles, or write

$$U = \quad x^2 + y^2 - R^2 = 0,$$
$$V = (x - a)^2 + y^2 - r^2 = 0;$$

R is of course the radius of the circumscribed circle, r the radius of the inscribed circle, and a the distance between the centres. Then

$$\xi U + V = (\xi + 1,\ \xi + 1,\ -\xi R^2 - r^2 + a^2,\ 0,\ -a,\ 0)(x,\ y,\ 1)^2,$$

and the discriminant is therefore

$$-(\xi + 1)^2 (\xi R^2 + r^2 - a^2) - (\xi + 1) a^2 = -(1 + \xi) \{r^2 + \xi (r^2 + R^2 - a^2) + \xi^2 R^2\}.$$

Hence,

THEOREM. The condition that there may be inscribed in the circle $x^2 + y^2 - R^2 = 0$ an infinity of n-gons circumscribed about the circle $(x - a)^2 + y^2 - r^2 = 0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1 + \xi) \{r^2 + \xi (r^2 + R^2 - a^2) + \xi^2 R^2\}}$$

may vanish.

Now

$$(A + B\xi + C\xi^2)^{\frac{1}{2}} = \sqrt{A} \left\{1 + \tfrac{1}{2} B \frac{\xi}{A} + (\tfrac{1}{2} AC - \tfrac{1}{8} B^2) \frac{\xi^2}{A^2} + \ldots\right\},$$

or the quantity to be considered is the coefficient of ξ^{n-1} in

$$(1 + \tfrac{1}{2}\xi - \tfrac{1}{8}\xi^2 \ldots) \left\{1 + \tfrac{1}{2} B \frac{\xi}{A} + (\tfrac{1}{2} AC - \tfrac{1}{8} B^2) \frac{\xi^2}{A^2} + \ldots\right\},$$

where, of course,

$$A = r^2, \quad B = r^2 + R^2 - a^2, \quad C = R^2.$$

In particular, in the case of a triangle we have, equating to zero the coefficient of ξ^2,

$$(A - B)^2 - 4AC = 0;$$

or substituting the values of A, B, C,

$$(a^2 - R^2)^2 - 4r^2 R^2 = 0,$$

that is

$$(a^2 - R^2 + 2Rr)(a^2 - R^2 - 2Rr) = 0;$$

the factor which corresponds to the proper geometrical solution of the question is

$$a^2 - R^2 + 2Rr = 0,$$

Euler's well-known relation between the radii of the circles inscribed and circumscribed in and about a triangle, and the distance between the centres. I shall not now discuss the meaning of the other factor, or attempt to verify the formulæ which have been given by Fuss, Steiner and Richelot, for the case of a polygon of 4, 5, 6, 7, 8, 9, 12, and 16 sides. See Steiner, *Crelle*, t. II. [1827] p. 289, Jacobi, t. III. [1828] p. 376; Richelot, t. V. [1830] p. 250; and t. XXXVIII. [1849] p. 353.

2 *Stone Buildings, July* 9, 1853.

116.

CORRECTION OF TWO THEOREMS RELATING TO THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 376—377.]

THE two theorems in my "Note on the Porism of the in-and-circumscribed Polygon" (see August Number), [115], are erroneous, the mistake arising from my having inadvertently assumed a wrong formulæ for the addition of elliptic integrals. The first of the two theorems (which, in fact, includes the other as a particular case) should be as follows:—

THEOREM. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, depends upon the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$; viz. if this square root be

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+F\xi^5+G\xi^6+H\xi^7+\ldots,$$

then for $n=3$, 5, 7, &c. respectively, the conditions are

$$|\,C\,|=0,\qquad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix}=0,\qquad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix}=0,\ \&c.;$$

and for $n=4$, 6, 8, &c. respectively, the conditions are

$$|\,D\,|=0,\qquad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0,\qquad \begin{vmatrix} D, & E, & F \\ D, & F, & G \\ F, & G, & H \end{vmatrix}=0,\ \&c.$$

The examples require no correction; since for the triangle and the quadrilateral respectively, the conditions are (as in the erroneous theorem) $C=0$, $D=0$.

The second theorem gives the condition in the case where the conics are replaced by the circles $x^2+y^2-R^2=0$ and $(x-a)^2+y^2-r^2=0$, the discriminant being in this case

$$-(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\}.$$

As a very simple example, suppose that the circles are concentric, or assume $a=0$; the square root of the discriminant is here

$$(1+\xi)\sqrt{r^2+R^2\xi};$$

and putting for shortness $\frac{R^2}{r^2}=\alpha$, we may write

$$A+B\xi+\ldots=(1+\xi)\sqrt{1+\alpha\xi},$$

that is, $A=1$, $B=\frac{1}{2}\alpha+1$, $C=-\frac{1}{8}\alpha^2+\frac{1}{2}\alpha^2$, $D=\frac{1}{16}\alpha^3-\frac{1}{8}\alpha^2$, $E=-\frac{5}{128}\alpha^4+\frac{1}{16}\alpha^3$, &c.; thus in the case of the pentagon,

$$CE-D^2=\tfrac{1}{1024}\alpha^4\{(\alpha-4)(5\alpha-8)-4(\alpha-2)^2\}$$

$$=\tfrac{1}{1024}\alpha^4(\alpha^2-12\alpha+16),$$

and the required condition therefore is

$$\alpha^2-12\alpha+16=0.$$

It is clear that, in the case in question,

$$\frac{r}{R}=\cos 36^\circ=\tfrac{1}{4}(\sqrt{5}+1),$$

that is, $$\frac{R}{r}=\sqrt{5}-1, \text{ or } (R+r)^2-5r^2=0,$$

viz. $$(\sqrt{\alpha}+1)^2-5=0, \text{ or } \alpha+2\sqrt{\alpha}-4=0,$$

the rational form of which is

$$\alpha^2-12\alpha+16=0,$$

and we have thus a verification of the theorem for this particular case.

2 *Stone Buildings, Oct.* 10, 1853.

117.

NOTE ON THE INTEGRAL $\int dx \div \sqrt{(m-x)(x+a)(x+b)(x+c)}$.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 103—105.]

IF in the formulæ of my "Note on the Porism of the in-and-circumscribed Polygon," [115], it is assumed that

$$U = x^2 + y^2 + z^2 + \frac{1}{m}(ax^2 + by^2 + cz^2)$$

$$V = \qquad\qquad ax^2 + by^2 + cz^2,$$

and if a new parameter ω connected with the parameter w by the equation

$$w = \frac{\omega m}{m - \omega}$$

be made use of instead of w, then

$$wU + V = \frac{m}{m-\omega}\{\omega(x^2 + y^2 + z^2) + ax^2 + by^2 + cz^2\};$$

and thus the equation $wU + V = 0$, viz. the equation

$$\omega(x^2 + y^2 + z^2) + ax^2 + by^2 + cz^2 = 0,$$

is precisely of the same form as that considered in my "Note on the Geometrical Representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$," [113.] Moreover, introducing instead of ξ a quantity η, such that

$$\xi = \frac{m\eta}{m-\eta},$$

then

$$\frac{d\xi}{\sqrt{\square\xi}} = \frac{\sqrt{m}\,d\eta}{\sqrt{(m-\eta)(a+\eta)(b+\eta)(c+\eta)}}.$$

Also $\xi = \infty$ gives $\eta = m$, the integral to be considered is therefore

$$\Pi_{,}\eta = \int_m \frac{\sqrt{m}\, d\eta}{\sqrt{(m-\eta)(a+\eta)(b+\eta)(c+\eta)}};$$

i.e. if in the paper last referred to the parameter ∞ had been throughout replaced by the parameter m, the integral

$$\Pi\eta = \int_\infty \frac{d\eta}{\sqrt{(a+\eta)(b+\eta)(c+\eta)}}$$

would have had to be replaced by the integral $\Pi_{,}\eta$. It is, I think, worth while to reproduce for this more general case a portion of the investigations of the paper in question, for the sake of exhibiting the rational and integral form of the algebraical equation corresponding to the transcendental equation $\pm \Pi_{,}k \pm \Pi_{,}p \pm \Pi_{,}\theta = 0$. Consider the point ξ, η, ζ on the conic $m(x^2+y^2+z^2)+ax^2+by^2+cz^2=0$, the equation of the tangent at this point is

$$(m+a)\,\xi x + (m+b)\,\eta y + (m+c)\,\zeta z = 0;$$

and if θ be the other parameter of this line, then the line touches

$$\theta(x^2+y^2+z^2)+ax^2+by^2+cz^2=0;$$

or we have

$$\frac{(m+a)^2\xi^2}{\theta+a} + \frac{(m+b)^2\eta^2}{\theta+b} + \frac{(m+c)^2\zeta^2}{\theta+c} = 0;$$

and combining this with

$$(m+a)\,\xi^2 + (m+b)\,\eta^2 + (m+c)\,\zeta^2 = 0,$$

we have

$$\begin{aligned}\xi : \eta : \zeta = &\sqrt{b-c}\,\sqrt{a+\theta}\,\sqrt{b+m}\,\sqrt{c+m}\\ &: \sqrt{(c-a)}\,\sqrt{b+\theta}\,\sqrt{c+m}\,\sqrt{a+m}\\ &: \sqrt{(a-b)}\,\sqrt{c+\theta}\,\sqrt{a+m}\,\sqrt{b+m}\end{aligned}$$

for the coordinates of the point P. Substituting these for x, y, z in the equation of the line PP' (the parameters of which are p, k), viz. in

$$x\sqrt{b-c}\,\sqrt{(a+k)(a+p)} + y\sqrt{c-a}\,\sqrt{(b+k)(b+p)} + z\sqrt{a-b}\,\sqrt{c+k}\,\sqrt{c+p} = 0,$$

we have

$$(b-c)\frac{\sqrt{(a+p)(a+k)(a+\theta)}}{\sqrt{a+m}} + (c-a)\frac{\sqrt{(b+p)(b+k)(b+\theta)}}{\sqrt{b+m}}$$

$$+ (a-b)\frac{\sqrt{(c+p)(c+k)(c+\theta)}}{\sqrt{c+m}} = 0,$$

which is to be replaced by

$$\frac{(a+p)(a+k)(a+\theta)}{a+m} = (\lambda+\mu a)^2,$$

$$\frac{(b+p)(b+k)(b+\theta)}{b+m} = (\lambda+\mu b)^2,$$

$$\frac{(c+p)(c+k)(c+\theta)}{c+m} = (\lambda+\mu c)^2.$$

These equations give, omitting the common factor $(a+m)(b+m)(c+m)$,

$$\begin{aligned}\lambda^2 = \; & m^2(abc+pk\theta) \\ & + m\{-abc(p+k+\theta)+pk\theta(a+b+c)\} \\ & + \{abc(k\theta+\theta p+kp)+pk\theta(bc+ca+ab)\},\end{aligned}$$

$$\begin{aligned}2\lambda\mu = \; & m^2\{-(bc+ca+ab)+(k\theta+\theta p+pk)\} \\ & + m\{-abc-pk\theta+(bc+ca+ab)(p+k+\theta)+(k\theta+\theta p+pk)(a+b+c)\} \\ & + \{abc(p+k+\theta)-pk\theta(a+b+c)\},\end{aligned}$$

$$\begin{aligned}\mu^2 = \; & m^2\{a+b+c+p+k+\theta\} \\ & + m\{(bc+ca+ab)-(k\theta+\theta p+pk)\} \\ & + abc+pk\theta;\end{aligned}$$

and substituting in $4\lambda^2 . \mu^2 - (2\lambda\mu)^2 = 0$, we have the relation required. To verify that the equation so obtained is in fact the algebraical equivalent of the transcendental equation, it is only necessary to remark, that the values of λ^2, μ^2 are unaltered, and that of $\lambda\mu$ only changes its sign when a, b, c, m and p, k, θ, $-m$ are interchanged; and so this change will not affect the equation obtained by substituting in the equation $4\lambda^2 . \mu^2 - (2\lambda\mu)^2 = 0$. Hence precisely the same equation would be obtained by eliminating L, M from

$$(k+a)(k+b)(k+c) = (L+Mk)^2(m-k),$$
$$(p+a)(p+b)(p+c) = (L+Mp)^2(m-p),$$
$$(\theta+a)(\theta+b)(\theta+c) = (L+M\theta)^2(\theta-p);$$

or, putting $(L+Mk)(m-k) = \alpha+\beta k+\gamma k^2$, by eliminating α, β, γ from

$$\begin{aligned}(m-k)(k+a)(k+b)(k+c) &= (\alpha+\beta k+\gamma k^2)^2, \\ (m-p)(p+a)(p+b)(p+c) &= (\alpha+\beta p+\gamma p^2)^2, \\ (m-\theta)(\theta+a)(\theta+b)(\theta+c) &= (\alpha+\beta\theta+\gamma\theta^2)^2, \\ 0 &= (\alpha+\beta m+\gamma m^2)^2,\end{aligned}$$

which by Abel's theorem show that p, k, θ are connected by the transcendental equation above mentioned.

2 *Stone Buildings, July* 9, 1853.

118.

ON THE HARMONIC RELATION OF TWO LINES OR TWO POINTS.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 105—107.]

THE "harmonic relation of a point and line with respect to a triangle" is well known and understood[1]; but the analogous relation between two lines with respect to a quadrilateral, or between two points with respect to a quadrangle, is not, I think, sufficiently singled out from the mass of geometrical theorems so as to be recognized when implicitly occurring in the course of an investigation. The relation in question, or some particular case of it, is of frequent occurrence in the *Traité des Propriétés Projectives*, [Paris, 1822], and is, in fact, there substantially demonstrated (see No. 163); and an explicit statement of the theorem is given by M. Steiner, Lehrsätze 24 and 25, *Crelle*, t. XIII. [1835] p. 212 (a demonstration is given, t. XIX. [1839] p. 227). The theorem containing the relation in question may be thus stated.

THEOREM *of the harmonic relation of two lines with respect to a quadrilateral.* "If on each of the three diagonals of a *quadrilateral* there be taken two points harmonically related with respect to the angles upon this diagonal, then if three of the points lie in a *line*, the other three points will also lie in a *line*,"—the two lines are said to be harmonically related with respect to the quadrilateral.

It may be as well to exhibit this relation in a somewhat different form. The three diagonals of the quadrilateral form a triangle, the sides of which contain the six angles of the quadrilateral; and considering three only of these six angles (one angle on each side), these three angles are points which either lie in a line, or else

[1] The relation to which I refer is contained in the theorem, "If on each side of a *triangle* there be taken two points harmonically related with respect to the angles on this side, then if three of these points lie in *a line*, the lines joining the other three points with the opposite angles of the triangle meet in a *point*,"—the line and point are said to be harmonically related with respect to the triangle.

are such that the lines joining them with the opposite angles of the triangle meet in a point. Each of these points is, with respect to the involution formed by the two angles of the triangle, and the two points harmonically related thereto, a double point; and we have thus the following theorem of the harmonic relation of two lines to a triangle and line, or else to a triangle and point.

THEOREM. "If on the sides of a *triangle* there be taken three points, which either lie in a *line*, or else are such that the lines joining them with the opposite angles of a triangle meet in a *point;* and if on each side of the triangle there be taken two points, forming with the two angles on the same side an involution having the first-mentioned point on the same side for a double point; then if three of the six points lie in a *line*, the other three of the six points will also lie in a *line*,"—the two lines are said to be harmonically related to the triangle and line, or (as the case may be) to the triangle and point.

The theorems with respect to the harmonic relation of two points are of course the reciprocals of those with respect to the harmonic relation of two lines, and do not need to be separately stated.

The preceding theorems are useful in (among other geometrical investigations) the porism of the in-and-circumscribed polygon.

2 *Stone Buildings, July* 9, 1853.

119.

ON A THEOREM FOR THE DEVELOPMENT OF A FACTORIAL.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 182—185.]

THE theorem to which I refer is remarkable for the extreme simplicity of its demonstration. Let it be required to expand the factorial $\overline{x-a}\,\overline{x-b}\,\overline{x-c}\ldots$ in the form

$$\overline{x-\alpha}\,\overline{x-\beta}\,\overline{x-\gamma}\ldots+B\overline{x-\alpha}\,\overline{x-\beta}\ldots+C\overline{x-\alpha}\ldots+D\ldots \text{ \&c.}$$

We have first

$$x-a=\overline{x-\alpha}+\overline{\alpha-a};$$

multiply the two sides of this by $\overline{x-b}$; but in multiplying by this factor the term $\overline{x-\alpha}$, write the factor in the form $\overline{x-\beta}+\overline{\beta-b}$; and in multiplying the term $\overline{\alpha-a}$, write the factor in the form $\overline{x-\alpha}+\overline{\alpha-b}$; the result is obviously

$$\overline{x-a}\,\overline{x-b}=\overline{x-\alpha}\,\overline{x-\beta}\ +(\overline{\alpha-a}+\overline{\beta-b})\,\overline{x-\alpha}\ +\overline{\alpha-a}\,\overline{\alpha-b};$$

multiply this by $x-c$, this factor being in multiplying the quantity on the right-hand side written successively under the forms $\overline{x-\gamma}+\overline{\gamma-c}$, $\overline{x-\beta}+\overline{\beta-c}$, $\overline{\alpha-\alpha}+\overline{\alpha-c}$; the result is

$$\begin{aligned}\overline{x-a}\,\overline{x-b}\,\overline{x-c}= \quad & \overline{x-\alpha}\,\overline{x-\beta}\,\overline{x-\gamma}\\ & +(\overline{\alpha-a}+\overline{\beta-b}+\overline{\gamma-c})\,\overline{x-\alpha}\,\overline{x-\beta}\\ & +(\overline{\alpha-a}\,\overline{\alpha-b}+\overline{\alpha-a}\,\overline{\beta-c}+\overline{\beta-b}\,\overline{\beta-c})\,\overline{x-\alpha}\\ & +\overline{\alpha-a}\,\overline{\alpha-b}\,\overline{\alpha-c},\end{aligned}$$

which may be thus written,

$$(x-a)(x-b)(x-c)=$$

$$(x-\alpha)(x-\beta)(x-\gamma)+\begin{bmatrix}\alpha, & \beta, & \gamma\\ a, & b, & c\end{bmatrix}_1 \overline{x-\alpha}\,\overline{x-\beta}+\begin{bmatrix}\alpha, & \beta, & \\ a, & b, & c\end{bmatrix}_2 \overline{x-\alpha}+\begin{bmatrix}\alpha & & \\ a, & b, & c\end{bmatrix}_3.$$

Consider, for instance,

$$\begin{bmatrix}\alpha, & \beta, & \\ a, & b, & c\end{bmatrix}_2=\overline{\alpha-a}\,\overline{\alpha-b}+\overline{\alpha-a}\,\overline{\beta-c}+\overline{\beta-b}\,\overline{\beta-c};$$

then, paying attention in the first instance to the Greek letters only, it is clear that the terms on the second side contain the combinations two and two, with repetitions, of the Greek letters α, β, and these letters appear in each term in the alphabetical order. Each such combination may therefore be considered as derived from the primitive combination α, α by a change of one or both of the α's into β; and if we take (instead of the mere combination α, α) the complete first term $\overline{\alpha-a}\,\overline{\alpha-b}$, and simultaneously with the change of the α of either of the factors into β make a similar change in the Latin letter of the factor, we derive from the first term the other terms of the expression on the right-hand side of the expression. It is proper also to remark, that, paying attention to the Latin letters only, the different terms contain all the combinations two and two, without repetitions, of the letters a, b, c. The same reasoning will show that

$$\begin{aligned}\overline{x-a}\,\overline{x-b}\,\overline{x-c}\,\overline{x-d}= \quad & \overline{x-\alpha}\,\overline{x-\beta}\,\overline{x-\gamma}\,\overline{x-\delta}\\ &+\begin{bmatrix}\alpha, & \beta, & \gamma, & \delta\\ a, & b, & c, & d\end{bmatrix}_1 \overline{x-\alpha}\,\overline{x-\beta}\,\overline{x-\gamma}\\ &+\begin{bmatrix}\alpha, & \beta, & \gamma, & \\ a, & b, & c, & d\end{bmatrix}_2 \overline{x-\alpha}\,\overline{x-\beta}\\ &+\begin{bmatrix}\alpha, & \beta & & \\ a, & b, & c, & d\end{bmatrix}_3 \overline{x-\alpha}\\ &+\begin{bmatrix}\alpha & & & \\ a, & b, & c, & d\end{bmatrix}_4;\end{aligned}$$

where, for instance,

$$\begin{aligned}\begin{bmatrix}\alpha, & \beta & & \\ a, & b, & c, & d\end{bmatrix}_2 = \quad & (\alpha-a)(\alpha-b)(\alpha-c)\\ &+(\alpha-a)(\alpha-b)(\beta-d)\\ &+(\alpha-a)(\beta-c)(\beta-d)\\ &+(\beta-b)(\beta-c)(\beta-d), \text{ \&c.}\end{aligned}$$

It is of course easy, by the use of subscript letters and signs of summation, to present the preceding theorem under a more condensed form; thus writing

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_r & & \\ \alpha_1 & \alpha_2 & \dots & \alpha_r & \dots & \alpha_{r+s} \end{bmatrix}_{s+1} = \sum \left\{ \overline{\alpha_{k_s} - \alpha_{k_s+s}} \cdot \overline{\alpha_{k_{s-1}} - \alpha_{k_{s-1}+s-1}} \dots \overline{\alpha_{k_0} - \alpha_{k_0}} \right\}$$

where $k_s, k_{s-1}, \dots k_0$ form a decreasing series (equality of successive terms *not excluded*) of numbers out of the system $r, \overline{r-1}, \dots 3, 2, 1$; the theorem may be written in the form

$$\overline{x-\alpha_1}\ \overline{x-\alpha_2} \dots \overline{x-\alpha_p} = S_{q^0}^{\ p} \begin{bmatrix} \alpha_1 & \alpha_2 \dots & \alpha_{p-q+1} \\ \alpha_1 & \alpha_2 \dots\dots & \alpha_p \end{bmatrix}_q \overline{x-\alpha_1}\ \overline{x-\alpha_2} \dots \overline{x-\alpha_{p-q}};$$

but I think that a more definite idea of the theorem is obtained through the notation first made use of. It is clear that the above theorem includes the binomial theorem for positive integers, the corresponding theorem for an ordinary factorial, and a variety of other theorems relating to combinations.

Thus, for instance, if $C_q(\alpha_1, \dots \alpha_p)$ denote the combinations of $\alpha_1, \dots \alpha_p$, q and q together without repetitions, and $H_q(\alpha_1, \dots \alpha_p)$ denote the combinations of $\alpha_1, \dots \alpha_p$, q and q together with repetitions, then making all the α's vanish,

$$\overline{x-\alpha_1} \dots \overline{x-\alpha_p} = S_{q^0}^{\ p} (-)^q\, C_q(\alpha_1, \dots \alpha_p)\, x^{p-q};$$

and therefore

$$(x-a)^p = S_{q^0}^{\ p} (-)^q\, C_q(a, a \dots \text{plures})\, x^{p-q} = S_{q^0}^{\ p} (-)^q \frac{[p]^q}{[q]^q} a^q x^{p-q},$$

the ordinary binomial theorem for a positive and integral index p.

So making all the a's vanish,

$$x^p = S_{q^0}^{\ p} H_q(\alpha_1 \dots \alpha_{p-q+1})\, \overline{x-\alpha_1}\ \overline{x-\alpha_2} \dots \overline{x-\alpha_{p-q}}.$$

If m be any integer less than p, the coefficient of x^m on the right-hand side must vanish, that is, we must have identically

$$0 = S_{q^0}^{\ p-m} (-)^q\, C_{p-q-m}(\alpha_1, \alpha_2, \dots \alpha_{p-q})\, H_q(\alpha_1, \alpha_2, \dots \alpha_{p-q+1}).$$

So also

$$C_{p-m}(\alpha_1, \alpha_2, \dots \alpha_p) = S_{q^0}^{\ p-m} (-)^q\, C_{p-q-m}(\alpha_1, \alpha_2, \dots \alpha_{p-q}) \begin{bmatrix} \alpha_1 \dots & \alpha_{p-q+1} \\ \alpha_1 \dots\dots & \alpha_p \end{bmatrix}_q.$$

Suppose

$$\alpha_1 = 0, \quad \alpha_2 = 1 \dots \alpha_p = p-1; \quad \alpha_1 = k, \quad \alpha_2 = k-1, \dots \alpha_p = k-p+1,$$

then

$$\begin{bmatrix} \alpha_1, \dots & \alpha_{p-q+1} \\ \alpha_1 \dots\dots & \alpha_p \end{bmatrix}_q = \begin{bmatrix} k & \overline{k-1} & \dots & k-p+q \\ 0 & 1 & \dots\dots\dots\dots & p-1 \end{bmatrix}_q = \frac{[p]^q}{[q]^q} [k]^q\, \overline{x-\alpha_1} \dots \overline{x-\alpha_{p-q}} = [x]^{p-q};$$

and hence

$$[x+k]^p = S_{q^0}^{\ p} \frac{[p]^q}{[q]^q} [k]^q [x]^{p-q},$$

the binomial theorem for factorials.

A preceding formula gives at once the theorem

$$H_q(0, 1, \ldots p-q) = \frac{1}{[p-q]^{p-q}} \Delta^{p-q} 0^p.$$

It may be as well to remark, with reference to a demonstration frequently given of the binomial theorem, that in whatever way the binomial theorem is demonstrated for integer positive indices, it follows from what has preceded that it is quite as easy to demonstrate the corresponding theorem for the factorial $[m]^p$. But the theorem being true for the factorial $[m]^p$, it is at once seen that the product of the series for $(1+x)^m$ and $(1+x)^n$ is identical with the series for $(1+x)^{m+n}$, and thus it becomes *unnecessary* to employ for the purpose of proving this identity the so-called principle of the permanence of equivalent forms; a principle which however, in the case in question, may legitimately be employed.

120.

NOTE ON A GENERALIZATION OF A BINOMIAL THEOREM.

[From the *Philosophical Magazine*, vol. VI. (1853), p. 185.]

THE formula (*Crelle*, t. I. [1826] p. 367) for the development of the binomial $(x+a)^n$, but which is there presented in a form which does not put in evidence the law of the coefficients, is substantially equivalent to the theorem given by me as one of the Senate House Problems in the year 1851, and which is as follows:—

"If $\{a+\beta+\gamma\ldots\}^p$ denote the expansion of $(\alpha+\beta+\gamma\ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\delta^d\ldots$ only in which $b+c+d\ldots$ is not greater than $p-1$, $c+d+\ldots$ is not greater than $p-2$, &c., then

$$\begin{aligned} x^n = \quad & 1 & & (x+a)^n \\ & -\frac{n}{1} & \{\alpha\}^1 & (x+\alpha+\beta)^{n-1} \\ & +\frac{n(n-1)}{1.2} & \{\alpha+\beta\}^2 & (x+\alpha+\beta+\gamma)^{n-2} \\ & -\frac{n(n-1)(n-2)}{1.2.3} & \{\alpha+\beta+\gamma\}^3 & (x+\alpha+\beta+\gamma+\delta)^{n-3}. \\ & + \text{ \&c.''} \end{aligned}$$

The theorem is, I think, one of some interest.

121.

NOTE ON A QUESTION IN THE THEORY OF PROBABILITIES.

[From the *Philosophical Magazine*, vol. VI. (1853), p. 259.]

THE following question was suggested to me, either by some of Prof. Boole's memoirs on the subject of probabilities, or in conversation with him, I forget which; it seems to me a good instance of the class of questions to which it belongs.

Given the probability α that a cause A will act, and the probability p that A acting the effect will happen; also the probability β that a cause B will act, and the probability q that B acting the effect will happen; required the total probability of the effect.

As an instance of the precise case contemplated, take the following: say a day is called *windy* if there is at least w of wind, and a day is called *rainy* if there is at least r of rain, and a day is called *stormy* if there is at least W of wind, *or* if there is at least R of rain. The day may therefore be stormy because of there being at least W of wind, or because of there being at least R of rain, or on both accounts; but if there is less than W of wind *and* less than R of rain, the day will not be stormy. Then α is the probability that a day chosen at random will be windy, p the probability that a windy day chosen at random will be stormy, β the probability that a day chosen at random will be rainy, q the probability that a rainy day chosen at random will be stormy. The quantities λ, μ introduced in the solution of the question mean in this particular instance, λ the probability that a windy day chosen at random will be stormy by reason of the quantity of wind, or in other words, that there will be at least W of wind; μ the probability that a rainy day chosen at random will be stormy by reason of the quantity of rain, or in other words, that there will be at least R of rain.

The sense of the terms being clearly understood, the problem presents of course no difficulty. Let λ be the probability that the cause A acting will act efficaciously; μ the probability that the cause B acting will act efficaciously; then

$$p = \lambda + (1-\lambda)\mu\beta,$$
$$q = \mu + (1-\mu)\alpha\lambda,$$

which determine λ, μ; and the total probability ρ of the effect is given by

$$\rho = \lambda\alpha + \mu\beta - \lambda\mu\alpha\beta;$$

suppose, for instance, $\alpha = 1$, then

$$p = \lambda + (1-\lambda)\mu\beta, \quad q = \mu + \lambda - \lambda\mu, \quad \rho = \lambda + \mu\beta - \lambda\mu\beta,$$

that is, $\rho = p$, for p is in this case the probability that (acting a cause which is certain to act) the effect will happen, or what is the same thing, p is the probability that the effect will happen.

Machynlleth, August 16, 1853.

122.

ON THE HOMOGRAPHIC TRANSFORMATION OF A SURFACE OF THE SECOND ORDER INTO ITSELF.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 326—333.]

THE following theorems in plane geometry, relating to polygons of any number (odd or even) of sides, are well known.

"If there be a polygon of $(m+1)$ sides inscribed in a conic, and m of the sides pass through given points, the $(m+1)$th side will envelope a conic having double contact with the given conic." And "If there be a polygon of $(m+1)$ sides inscribed in a conic, and m of the sides touch conics having double contact with the given conic, the $(m+1)$th side will envelope a conic having double contact with the given conic." The second theorem of course includes the first, but I state the two separately for the sake of comparison with what follows.

As regards the corresponding theory in geometry of three dimensions, Sir W. Hamilton has given a theorem relating to polygons of an odd number of sides, which may be thus stated: "If there be a polygon of $(2m+1)$ sides inscribed in a surface of the second order, and $2m$ of the sides pass through given points, the $(2m+1)$th side will constantly touch two surfaces of the second order, each of them intersecting the given surface of the second order in the same four lines[1]."

[1] See *Phil. Mag.* vol. xxxv. [1849] p. 200. The form in which the theorem is exhibited by Sir W. Hamilton is somewhat different; the surface containing the angles is considered as being an ellipsoid, and the two surfaces touched by the last or $(2m+1)$th side of the polygon are spoken of as being an ellipsoid, and a hyperboloid of two sheets, having respectively double contact with the given ellipsoid: the contact is, in fact, a quadruple contact at the *same* four points; *real* as regards two of them in the case of the ellipsoid, and as regards the other two in the case of the hyperboloid of two sheets; and a quadruple contact is the coincidence of four generating lines belonging two and two to the two series of generating lines, these generating lines being of course (in the case considered by Sir W. Hamilton) all of them imaginary.

The entire theory depends upon what may be termed the transformation of a surface of the second order into itself, or analytically, upon the transformation of a quadratic form of four indeterminates into itself. I use for shortness the term transformation simply; but this is to be understood as meaning a homographic transformation, or in analytic language, a transformation by means of linear substitutions. It will be convenient to remark at the outset, that if two points of a surface of the second order have the relation contemplated in the data of Sir W. Hamilton's theorem (viz. if the line joining the two points pass through a fixed point), the transformation is, using the language of the *Recherches Arithmétiques*, an *improper* one, but that the relation contemplated in the conclusion of the theorem (viz. that of two points of a surface of the second order, connected by a line touching two surfaces of the second order each of them intersecting the given surface of the second order in the same four lines) depends upon a *proper* transformation; and that the circumstance that an *even* number of improper transformations is required in order to make a proper transformation (that this circumstance, I say), is the reason why the theorem applies to polygons in which an even number of sides pass through fixed points, that is, to polygons of an *odd* number of sides.

Consider, in the first place, two points of a surface of the second order such that the line joining them passes through a given point. Let x, y, z, w be current coordinates[1], and let the equation of the surface be

$$(a, \ldots)(x, y, z, w)^2 = 0,$$

and take for the coordinates of the two points on the surface x_1, y_1, z_1, w_1 and x_2, y_2, z_2, w_2, and for the coordinates of the fixed point $\alpha, \beta, \gamma, \delta$. Write for shortness

$$(a, \ldots)(\alpha, \beta, \gamma, \delta)^2 = p,$$

$$(a, \ldots)(\alpha, \beta, \gamma, \delta)(x_1, y_1, z_1, w_1) = q_1,$$

then the coordinates x_2, y_2, z_2, w_2 are determined by the very simple formulæ

$$x_2 = x_1 - \frac{2\alpha}{p} q_1,$$

$$y_2 = y_1 - \frac{2\beta}{p} q_1,$$

$$z_2 = z_1 - \frac{2\gamma}{p} q_1,$$

$$w_2 = w_1 - \frac{2\delta}{p} q_1.$$

[1] Strictly speaking, it is the ratios of these quantities, e.g. $x : w$, $y : w$, $z : w$, which are the coordinates, and consequently, even when the point is given, the values x, y, z, w are essentially indeterminate to a factor *près*. So that in assuming that a point is given, we should write $x : y : z : w = \alpha : \beta : \gamma : \delta$; and that when a point is obtained as the result of an analytical process, the conclusion is necessarily of the form just mentioned: but when this is once understood, the language of the text may be properly employed. It may be proper to explain here a notation made use of in the text: taking for greater simplicity the case of forms of two variables, $(l, m)(x, y)$ means $lx + my$; $(a, b, c)(x, y)^2$ means $ax^2 + 2bxy + cy^2$; $(a, b, c)(\xi, \eta)(x, y)$ means $a\xi x + b(\xi y + \eta x) + c\eta y$. The system of coefficients may frequently be indicated by a single coefficient only: thus in the text $(a, \ldots)(x, y, z, w)^2$ stands for the most general quadratic function of four variables.

In fact, these values satisfy identically the equations

$$\left\| \begin{matrix} x_2, & y_2, & z_2, & w_2 \\ x_1, & y_1, & z_1, & w_1 \\ \alpha, & \beta, & \gamma, & \delta \end{matrix} \right\| = 0,$$

that is, the point (x_2, y_2, z_2, w_2) will be a point in the line joining (x_1, y_1, z_1, w_1) and $(\alpha, \beta, \gamma, \delta)$. Moreover,

$$\begin{aligned}(a, \ldots)(x_2, y_2, z_2, w_2)^2 = \quad & (a, \ldots)(x_1, y_1, z_1, w_1)^2 \\ & - \frac{4q_1}{p}(a, \ldots)(\alpha, \beta, \gamma, \delta)(x_1, y_1, z_1, w_1) \\ & + \frac{4q_1^2}{p^2}(a, \ldots)(\alpha, \beta, \gamma, \delta)^2 \\ = \quad & (a, \ldots)(x_1, y_1, z_1, w_1)^2 - \frac{4q_1}{p}q_1 + \frac{4q_1^2}{p^2}p,\end{aligned}$$

that is,

$$(a, \ldots)(x_2, y_2, z_2, w_2)^2 = \quad (a, \ldots)(x_1, y_1, z_1, w_1)^2;$$

so that x_1, y_1, z_1, w_1 being a point on the surface, x_2, y_2, z_2, w_2 will be so too. The equation just found may be considered as expressing that the linear equations are a transformation of the quadratic form $(a, \ldots)(x, y, z, w)^2$ into itself. If in the system of linear equations the coefficients on the right-hand side were arranged square-wise, and the determinant formed by these quantities calculated, it would be found that the value of this determinant is -1. The transformation is on this account said to be *improper*. If in a system of linear equations for the transformation of the form into itself the determinant (which is necessarily $+1$ or else -1) be $+1$, the transformation is in this case said to be *proper*.

We have next to investigate the theory of the proper transformations of a quadratic form of four indeterminates into itself. This might be done for the absolutely general form by means of the theory recently established by M. Hermite, but it will be sufficient for the present purpose to consider the system of equations for the transformation of the form $x^2 + y^2 + z^2 + w^2$ into itself given by me some years since. (*Crelle*, vol. XXXII. [1846] p. 119, [52] (1).

I proceed to establish (by M. Hermite's method) the formulæ for the particular case in question. The thing required is to find x_2, y_2, z_2, w_2 linear functions of x_1, y_1, z_1, w_1, such that

$$x_2^2 + y_2^2 + z_2^2 + w_2^2 = x_1^2 + y_1^2 + z_1^2 + w_1^2.$$

Write

$$x_1 + x_2 = 2\xi, \quad y_1 + y_2 = 2\eta, \quad z_1 + z_2 = 2\zeta, \quad w_1 + w_2 = 2\omega;$$

[1] It is a singular instance of the way in which different theories connect themselves together, that the formulæ in question were generalizations of Euler's formulæ for the rotation of a solid body, and also are formulæ which reappear in the theory of quaternions; the general formulæ cannot be established by any obvious generalization of the theory of quaternions.

then putting $x_2 = 2\xi - x_1$, &c., the proposed equation will be satisfied if only

$$\xi^2 + \eta^2 + \zeta^2 + \omega^2 = \xi x_1 + \eta y_1 + \zeta z_1 + \omega w_1,$$

which will obviously be the case if

$$\begin{aligned} x_1 &= \quad \xi + \nu\eta - \mu\zeta + a\omega, \\ y_1 &= -\nu\xi + \eta + \lambda\zeta + b\omega, \\ z_1 &= \mu\xi - \lambda\eta + \zeta + c\omega, \\ w_1 &= -a\xi - b\eta - c\zeta + \omega, \end{aligned}$$

where λ, μ, ν, a, b, c are arbitrary.

Write for shortness

$$a\lambda + b\mu + c\nu = \phi, \quad 1 + \lambda^2 + \mu^2 + \nu^2 + a^2 + b^2 + c^2 + \phi^2 = k,$$

then we have

$$\begin{aligned} k\xi &= (1 + \lambda^2 + b^2 + c^2)\, x_1 + (\lambda\mu - \nu - ab - c\phi)\, y_1 + (\nu\lambda + \mu - ca + b\phi)\, z_1 + (b\nu - c\mu - a - \lambda\phi) w_1, \\ k\eta &= (\lambda\mu + \nu - ab + c\phi)\, x_1 + (1 + \mu^2 + c^2 + a^2)\, y_1 + (\mu\nu - \lambda - bc + a\phi)\, z_1 + (c\lambda - a\nu - b - \mu\phi) w_1, \\ k\zeta &= (\nu\lambda - \mu - ca - b\phi)\, x_1 + (\mu\nu + \lambda - bc + a\phi) y_1 + (1 + \nu^2 + a^2 + b^2)\, z_1 + (a\mu - b\lambda - c - \nu\phi) w_1, \\ kw &= (b\nu - c\mu + a + \lambda\phi)\, x_1 + (c\lambda - a\nu + b + \mu\phi) y_1 + (a\mu - b\nu + c + \nu\phi)\, z_1 + (1 + \lambda^2 + \mu^2 + \nu^2)\, w_1; \end{aligned}$$

and from these we obtain

$$\begin{aligned} kx_2 &= (1 + \lambda^2 + b^2 + c^2 - \mu^2 - \nu^2 - a^2 - \phi^2)\, x_1 + 2\,(\lambda\mu - \nu - ab - c\phi)\, y_1 + 2\,(\nu\lambda + \mu - ca + b\phi)\, z_1 \\ &\qquad + 2\,(b\nu - c\mu - a - \lambda\phi)\, w_1, \\ ky_2 &= 2\,(\lambda\mu + \nu - ab + c\phi)\, x_1 + (1 + \mu^2 + c^2 + a^2 - \nu^2 - \lambda^2 - b^2 - \phi^2)\, y_1 + 2\,(\mu\nu - \lambda - bc - a\phi)\, z_1 \\ &\qquad + 2\,(c\lambda - a\nu - b - \mu\phi)\, w_1, \\ kz_2 &= 2\,(\nu\lambda - \mu - ca - b\phi)\, x_1 + 2\,(\mu\nu + \lambda - bc + a\phi)\, y_1 + (1 + \nu^2 + a^2 + b^2 - \lambda^2 - \mu^2 - c^2 - \phi^2)\, z_1 \\ &\qquad + 2\,(a\mu - b\lambda - c - \nu\phi)\, w_1, \\ kw_2 &= 2\,(b\nu - c\mu + a + \lambda\phi)\, x_1 + 2\,(c\lambda - a\nu + b + \mu\phi)\, y_1 + 2\,(a\mu - b\nu + c + \nu\phi)\, z_1 \\ &\qquad + (1 + \lambda^2 + \mu^2 + \nu^2 - a^2 - b^2 - c^2 - \phi^2)\, w_1, \end{aligned}$$

values which satisfy identically $x_2^2 + y_2^2 + z_2^2 + w_2^2 = x_1^2 + y_1^2 + z_1^2 + w_1^2$.

Dividing the linear equations by k, and forming with the coefficients on the right-hand side of the equation so obtained a determinant, the value of this determinant is $+1$; the transformation is consequently a *proper* one. And conversely, what is very important, *every* proper transformation may be exhibited under the preceding form[1].

[1] The nature of the reasoning by which this is to be established may be seen by considering the analogous relation for two variables. Suppose that x_1, y_1 are linear functions of x and y such that $x_1^2 + y_1^2 = x^2 + y^2$; then if $2\xi = x + x_1$, $2\eta = y + y_1$, ξ, η will be linear functions of x, y such that $\xi^2 + \eta^2 = \xi x + \eta y$, or $\xi(\xi - x) + \eta(\eta - y) = 0$; $\xi - x$ must be divisible either by η or else by $\eta - y$. On the former supposition, calling the quotient ν, we have $x = \xi - \nu\eta$, and thence $y = \nu\xi + \eta$, leading to a transformation such as is considered in the text, and which is a proper transformation; the latter supposition leads to an improper transformation. The given transformation, assumed to be proper, exists and cannot be obtained from the second supposition; it must therefore be obtainable from the first supposition, i.e. it is a transformation which may be exhibited under a form such as is considered in the text.

Next considering the equations connecting x, y, z, w with ξ, η, ζ, ω, we see that

$$\begin{aligned} x_1^2+y_1^2+z_1^2+w_1^2 = (&\ \xi+\nu\eta-\mu\zeta+a\omega)^2 \\ &+(-\nu\xi+\eta+\lambda\zeta+b\omega)^2 \\ &+(\mu\xi-\lambda\eta+\zeta+c\omega)^2 \\ &+(-a\xi-b\eta-c\zeta+\omega)^2. \end{aligned}$$

We are thus led to the discussion (in connexion with the question of the transformation into itself of the form $x^2+y^2+z^2+w^2$) of the new form

$$\begin{aligned} &(x+\nu y-\mu z+aw)^2 \\ &+(-\nu x+y+\lambda z+bw)^2 \\ &+(\mu x-\lambda y+z+cw)^2 \\ &+(-ax-by-cz+w)^2; \end{aligned}$$

or, as it may also be written,

$$(x^2+y^2+z^2+w^2)+(\nu y-\mu z+aw)^2+(\lambda z-\nu x+bw)^2+(\mu x-\lambda y+cw)^2+(ax+by+cz)^2.$$

Represent for a moment the forms in question by U, V, and consider the surfaces $U=0$, $V=0$. If we form from this the surface $V+qU=0$, and consider the discriminant of the function on the left-hand side, then putting for shortness

$$\kappa=\lambda^2+\mu^2+\nu^2+a^2+b^2+c^2,$$

this discriminant is

$$\{(q+1)^2+\kappa(q+1)+\phi^2\}^2,$$

which shows that the surfaces intersect in four lines. Suppose the discriminant vanishes; we have for the determination of q a quadratic equation, which may be written

$$q^2+(2+\kappa)q+K=0;$$

let the roots of this equation be $q_{,}$, $q_{,,}$; then each of the functions $q_{,}U+V$, $q_{,,}U+V$ will break up into linear factors, and we may write

$$\begin{aligned} q_{,}U+V &= R_{,}S_{,}, \\ q_{,,}U+V &= R_{,,}S_{,,}. \end{aligned}$$

(U and V are of course linear functions of $R_{,}S_{,}$ and $R_{,,}S_{,,}$) forms which put in evidence the fact of the two surfaces intersecting in four lines.

The equations

$$x_1+x_2=2\xi,\quad y_1+y_2=2\eta,\quad z_1+z_2=2\zeta,\quad w_1+w_2=2\omega,$$

show that the point $(\xi, \eta, \zeta, \omega)$ lies in the line joining the points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2); and to show that this line touches the surface $V=0$, it is only necessary to form the equation of the tangent plane at the point $(\xi, \eta, \zeta, \omega)$ of the surface in question; this is

$$(x+\nu y-\mu z+aw)(\xi+\nu\eta-\mu\zeta+a\omega)+\ldots=0;$$

or what is the same thing,

$$(x+\nu y-\mu z+aw)\,x_1+\ldots=0,$$

which is satisfied by writing (x_1, y_1, z_1, w_1) for (x, y, z, w), that is, the tangent plane of the surface contains the point (x_1, y_1, z_1, w_1). We see, therefore, that the line through (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) touches the surface $V=0$ at the point $(\xi, \eta, \zeta, \omega)$.

Write now

$$a'=\frac{-\lambda}{\phi},\quad b'=\frac{-\mu}{\phi},\quad c'=\frac{-\nu}{\phi},\quad \lambda'=\frac{-a}{\phi},\quad \mu'=\frac{-b}{\phi},\quad \nu'=\frac{-c}{\phi};$$

if we derive from the coordinates x_1, y_1, z_1, w_1, by means of these coefficients $a', b', c', \lambda', \mu', \nu'$, new coordinates in the same way as x_2, y_2, z_2, w_2 were derived by means of the coefficients $a, b, c, \lambda, \mu, \nu$, the coordinates so obtained are $-x_2, -y_2, -z_2, -w_2$, i.e. we obtain the *very same* point (x_2, y_2, z_2, w_2) by means of the coefficients $(a, b, c, \lambda, \mu, \nu)$, and by means of the coefficients $(a', b', c', \lambda', \mu', \nu')$. Call $\xi', \eta', \zeta', \omega'$ what ξ, η, ζ, ω become when the second system of coefficients is substituted for the first; the point $\xi', \eta', \zeta', \omega'$ will be a point on the surface $V'=0$, where

$$V'=\phi^2(x^2+y^2+z^2+w^2)$$
$$+(-cy+bz-\lambda w)^2+(-az+cx-\mu w)^2+(-bx+ay-\nu w)^2+(-\lambda x-\mu y-\nu z)^2;$$

and since

$$V+V'=\kappa(x^2+y^2+z^2+w^2),$$

and $V=0$ intersects the surface $x^2+y^2+z^2+w^2=0$ in four lines, the surface $V'=0$ will also intersect this surface in the same four lines. And it is, moreover, clear that the line joining the points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) touches the surface $V'=0$ in the point $(\xi', \eta', \zeta', \omega')$. We thus arrive at the theorem, that when two points of a surface of the second order are so connected that the coordinates of the one point are linear functions of the coordinates of the other point, and the transformation is a proper one, the line joining the two points touches two surfaces of the second order, each of them intersecting the given surface of the second order in the same four lines. Any two points so connected may be said to be corresponding points, or simply a pair. Suppose the four lines and also a single pair is given, it is not for the determination of the other pairs necessary to resort to the two auxiliary surfaces of the second order; it is only necessary to consider each point of the surface as determined by the two generating lines which pass through it; then considering first

one point of the given pair, and the point the corresponding point to which has to be determined, take through each of these points a generating line, and take also two generating lines out of the given system of four lines, the four generating lines in question being all of them of the same set, these four generating lines intersecting either of the other two generating lines of the given system of four lines in four points. Imagine the same thing done with the other point of the given pair and the required point, we should have another system of four points (two of them of course identical with two of the points of the first-mentioned system of four points); these two systems must have their anharmonic ratios the same, a condition which enables the determination of the generating line in question through the required point: the other generating line through the required point is of course determined in the same manner, and thus the required point (i.e. the point corresponding to any point of the surface taken at pleasure) is determined by means of the two generating lines through such required point.

It is of course to be understood that the points of each pair belong to two distinct systems, and that the point belonging to the one system is not to be confounded or interchanged with the point belonging to the other system. Consider, now, a point of the surface, and the line joining such point with its corresponding point, but let the corresponding point itself be altogether dropped out of view. There are two directions in which we may pass along the surface to a consecutive point, in such manner that the line belonging to the point in question may be intersected by the line belonging to the consecutive point. We have thus upon the surface two series of curves, such that a curve of each series passes through a point chosen at pleasure on the surface. The lines belonging to the curves of the one series generate a series of developables, the edges of regression of which lie on one of the surfaces intersecting the surface of the second order in the four given lines; the lines belonging to the curves of the other series generate a series of developables, the edges of regression of which lie on the other of the surfaces intersecting the surface in the four given lines; the general nature of the system may be understood by considering the system of normals of a surface of the second order. Consider, now, the surface of the second order as given, and also the two surfaces of the second order intersecting it in the same four lines; from any point of the surface we may draw to the auxiliary surfaces four different tangents; but selecting any one of these, and considering the other point in which it intersects the surface as the point corresponding to the first-mentioned point, we may, as above, construct the entire system of corresponding points, and then the line joining any two corresponding points will be a tangent to the two auxiliary surfaces; the system of tangents so obtained may be called a system of *congruent* tangents. Now if we take upon the surface three points such that the first and second are corresponding points, and that the second and third are corresponding points, then it is obvious that the third and first are corresponding points;—observe that the two auxiliary surfaces for expressing the correspondence between the first and second point, those for the second and third point, and those for the third and first point, meet the surface, the two auxiliary surfaces of each pair in the same four lines, but that these systems of four lines are different

for the different pairs of auxiliary surfaces. The same thing of course applies to any number of corresponding points. We have thus, finally, the theorem, if there be a polygon of $(m+1)$ sides inscribed in a surface of the second order, and the first side of the polygon constantly touches two surfaces of the second order, each of them intersecting the surface of the second order in the same four lines (and the side belong always to the same system of congruent tangents), and if the same property exists with respect to the second, third, &c.... and mth side of the polygon, then will the same property exist with respect to the $(m+1)$th side of the polygon.

We may add, that, instead of satisfying the conditions of the theorem, any two consecutive sides of the polygon, or the sides forming any number of pairs of consecutive sides, may pass each through a fixed point. This is of course only a particular case of the improper transformation of a surface of a second order into itself, a question which is not discussed in the present paper.

123.

ON THE GEOMETRICAL REPRESENTATION OF AN ABELIAN INTEGRAL.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 414—418.]

THE equation of a surface passing through the curve of intersection of the surfaces

$$x^2 + y^2 + z^2 + w^2 = 0,$$
$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

is of the form

$$\theta(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0,$$

where θ is an arbitrary parameter. Suppose that the surface touches a given plane, we have for the determination of θ a cubic equation the roots of which may be considered as parameters defining the plane in question. Let one of the values of θ be considered equal to a given quantity k, the plane touches the surface

$$k(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0,$$

and the other two values of θ may be considered as parameters defining the particular tangent plane, or what is the same thing, determining its point of contact with the surface.

Or more clearly, thus:—in order to determine the position of a point on the surface

$$k(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0;$$

the tangent plane at the point in question is touched by two other surfaces

$$p(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0,$$
$$q(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0;$$

and, this being so, p and q are the parameters by which the point in question is determined. We may for shortness speak of the surface

$$k(x^2+y^2+z^2+w^2)+ax^2+by^2+cz^2+dw^2=0$$

as the surface (k). It is clear that we shall then have to speak of

$$x^2+y^2+z^2+w^2=0$$

as the surface (∞).

I consider now a chord of the surface (∞) touching the two surfaces (k) and (k'); and I take θ, ϕ as the parameters of the one extremity of this chord; (p, q) as the parameters of the point of contact with the surface (k); p', q' as the parameters of the point of contact with the surface (k'); and θ', ϕ' as the parameters of the other extremity of the chord; the points in question may therefore be distinguished as the points $(\infty\,;\ \theta, \phi)$, $(k;\ p, q)$, $(k';\ p', q')$, and $(\infty;\ \theta', \phi')$. The coordinates of the point $(\infty\,;\ \theta, \phi)$ are given by

$$\begin{aligned} x : y : z : w = {} & \sqrt{(a+\theta)(a+\phi)} \div \sqrt{(a-b)(a-c)(a-d)} \\ & : \sqrt{(b+\theta)(b+\phi)} \div \sqrt{(b-c)(b-d)(b-a)} \\ & : \sqrt{(c+\theta)(c+\phi)} \div \sqrt{(c-d)(c-a)(c-b)} \\ & : \sqrt{(d+\theta)(d+\phi)} \div \sqrt{(d-a)(d-b)(d-c)}\,; \end{aligned}$$

those of the point $(k;\ p, q)$ by

$$\begin{aligned} x : y : z : w = {} & \sqrt{(a+p)(a+q)} \div \sqrt{(a-b)(a-c)(a-d)}\,\sqrt{a+k} \\ & : \sqrt{(b+p)(b+q)} \div \sqrt{(b-c)(b-d)(b-a)}\,\sqrt{b+k} \\ & : \sqrt{(c+p)(c+q)} \div \sqrt{(c-d)(c-a)(c-b)}\,\sqrt{c+k} \\ & : \sqrt{(d+p)(d+q)} \div \sqrt{(d-a)(d-b)(d-c)}\,\sqrt{d+k}\,; \end{aligned}$$

and similarly for the other two points.

Consider, in the first place, the chord in question as a tangent to the two surfaces (k) and (k'). It is clear that the tangent plane to the surface (k) at the point $(k;\ p, q)$ must contain the point $(k';\ p', q')$, and *vice versâ*. Take for a moment ξ, η, ζ, ω as the coordinates of the point $(k;\ p, q)$, the equation of the tangent plane to (k) at this point is

$$\Sigma\,(a+k)\,\xi x=0\,;$$

or substituting for $\xi, \ldots$ their values

$$\Sigma\left(x\sqrt{(a+p)(a+q)}\,\sqrt{a+k} \div \sqrt{(a-b)(a-c)(a-d)}\right)=0\,;$$

or taking for $x, \ldots$ the coordinates of the point (k', p', q'), we have for the conditions that this point may lie in the tangent plane in question,

$$\Sigma\left(\sqrt{(a+p)(a+q)}\sqrt{(a+p')(a+q')}\sqrt{(a+k)} \div \sqrt{(a+k')}(a-b)(a-c)(a-d)\right)=0;$$

or under a somewhat more convenient form we have

$$\Sigma\left((b-c)(c-d)(d-b)\sqrt{(a+p)(a+q)}\sqrt{(a+p')(a+q')}\frac{\sqrt{a+k}}{\sqrt{a+k'}}\right)=0,$$

for the condition in order that the point (k', p', q') may lie in the tangent plane at $(k;\ p,\ q)$ to the surface (k). Similarly, we have

$$\Sigma\left((b-c)(c-d)(d-b)\sqrt{(a+p)(a+q)}\sqrt{(a+p')(a+q')}\frac{\sqrt{a+k'}}{\sqrt{a+k}}\right)=0,$$

for the condition in order that the point $(k,\ p,\ q)$ may lie in the tangent plane at $(k';\ p',\ q')$ to the surface (k'). The former of these two equations is equivalent to the system of equations

$$\sqrt{(a+p)(a+q)(a+p')(a+q')}\quad\sqrt{\frac{a+k}{a+k'}}=\lambda+\mu a+\nu a^2,$$
$$\vdots$$

and the latter to the system of equations

$$\sqrt{(a+p)(a+q)(a+p')(a+q')}\sqrt{\frac{a+k'}{a+k}}=\lambda'+\mu' a+\nu' a^2;$$
$$\vdots$$

where in each system a is to be successively replaced by b, c, d, and where λ, μ, ν and λ', μ', ν' are indeterminate. Now dividing each equation of the one system by the corresponding equation in the other system, we see that the equation

$$\frac{x+k}{x+k'}=\frac{\lambda+\mu x+\nu x^2}{\lambda'+\mu' x+\nu' x^2}$$

is satisfied by the values a, b, c, d of x; and, therefore, since the equation in x is only of the third order, that the equation in question must be *identically* true. We may therefore write

$$\lambda+\mu x+\nu x^2=(\rho x+\sigma)(x+k),\quad \lambda'+\mu' x+\nu' x^2=(\rho x+\sigma)(x+k'),$$

and the two systems of equations become therefore equivalent to the single system,

$$\sqrt{(a+p)(a+q)(a+p')(a+q')}=(\rho a+\sigma)\sqrt{(a+k)(a+k')},$$
$$\sqrt{(b+p)(b+q)(b+p')(b+q')}=(\rho b+\sigma)\sqrt{(b+k)(b+k')},$$
$$\sqrt{(c+p)(c+q)(c+p')(c+q')}=(\rho c+\sigma)\sqrt{(c+k)(c+k')},$$
$$\sqrt{(d+p)(d+q)(d+p')(d+q')}=(\rho d+\sigma)\sqrt{(d+k)(d+k')},$$

a set of equations which may be represented by the single equation

$$\psi(x+p)(x+q)(x+p')(x+q')-(\rho x+\sigma)^2(x+k)(x+k')=\chi(x-a)(x-b)(x-c)(x-d),$$

where x is arbitrary; or what is the same thing, writing $-x$ instead of x,

$$\chi(x+a)(x+b)(x+c)(x+d)+(\rho x-\sigma)^2(x-k)(x-k')=\psi(x-p)(x-q)(x-p')(x-q').$$

Hence, putting

$$\Pi x=\int\frac{dx}{\sqrt{(x+a)(x+b)(x+c)(x+d)(x-k)(x-k')}},$$

$$\Pi_1 x=\int\frac{xdx}{\sqrt{(x+a)(x+b)(x+c)(x+d)(x-k)(x-k')}},$$

we see that the algebraical equations between p, q; p', q' are equivalent to the transcendental equations

$$\Pi p \pm \Pi q \pm \Pi p' \pm \Pi q' = \text{const.}$$

$$\Pi_1 p \pm \Pi_1 q \pm \Pi_1 p' \pm \Pi_1 q' = \text{const.}$$

The algebraical equations which connect θ, ϕ with p, q; p', q', may be exhibited under several different forms; thus, for instance, considering the point $(\infty\,;\ \theta,\ \phi)$ as a point in the line joining $(k;\ p,\ q)$ and $(k';\ p',\ q')$, we must have

$$\left\|\begin{matrix} \sqrt{(a+p)(a+q)}\div\sqrt{a+k}, & \sqrt{(b+p)(b+q)}\div\sqrt{b+k},\ldots \\ \sqrt{(a+p')(a+q')}\div\sqrt{a+k'}, & \sqrt{(b+p')(b+q')}\div\sqrt{b+k'} \\ \sqrt{(a+\theta)(a+\phi)}, & \sqrt{(b+\theta)(b+\phi)} \end{matrix}\right\|=0,$$

i.e. the determinants formed by selecting any three of the four columns must vanish; the equations so obtained are equivalent (as they should be) to two independent equations.

Or, again, by considering $(\infty\,;\ \theta,\ \phi)$ first as a point in the tangent plane at $(k;\ p,\ q)$ to the surface (k), and then as a point in the tangent plane at $(k';\ p',\ q')$ to the surface (k'), we obtain

$$\Sigma\left((b-c)(c-d)(d-b)\sqrt{(a+p)(a+q)}\sqrt{(a+k)}\sqrt{(a+\theta)(a+\phi)}\right)=0,$$

$$\Sigma\left((b-c)(c-d)(d-b)\sqrt{(a+p')(a+q')}\sqrt{(a+k')}\sqrt{(a+\theta)(a+\phi)}\right)=0.$$

Or, again, we may consider the line joining $(\infty\,;\ \theta,\ \phi)$ and $(k;\ p,\ q)$ or $(k';\ p',\ q')$, as touching the surfaces (k) and (k'); the formulæ for this purpose are readily obtained by means of the lemma,—

"The condition in order that the line joining the points $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$ may touch the surface

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}w^2 = 0$$

is

$$\Sigma \mathrm{ab}\,(\xi\eta' - \xi'\eta)^2 = 0,$$

the summation extending to the binary combinations of a, b, c, d."

But none of all these formulæ appear readily to conduct to the transcendental equations connecting θ, ϕ with p, q; p', q'. Reasoning from analogy, it would seem that there exist transcendental equations

$$\pm \Pi\theta \pm \Pi\phi \pm \Pi p \pm \Pi p' = \text{const.}$$
$$\pm \Pi_{,}\theta \pm \Pi_{,}\phi \pm \Pi_{,}p \pm \Pi_{,}p' = \text{const.},$$

or the similar equations containing q, q', instead of p, p', into which these are changed by means of the transcendental equations between p, q, p', q'. If in these equations we write θ', ϕ' instead of θ, ϕ, it would appear that the functions Πp, $\Pi p'$, $\Pi_{,}p$, $\Pi_{,}p'$ may be eliminated, and that we should obtain equations such as

$$\pm \Pi\theta \pm \Pi\phi \pm \Pi\theta' \pm \Pi\phi' = \text{const.}$$
$$\pm \Pi_{,}\theta \pm \Pi_{,}\phi \pm \Pi_{,}\theta' \pm \Pi_{,}\phi' = \text{const.}$$

to express the relations that must exist between the parameters θ, ϕ and θ', ϕ' of the extremities of a chord of the surface

$$x^2 + y^2 + z^2 + w^2 = 0,$$

in order that this chord may touch the two surfaces

$$k\,(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0,$$
$$k'\,(x^2 + y^2 + z^2 + w^2) + ax^2 + by^2 + cz^2 + dw^2 = 0.$$

The quantities k, k', it will be noticed, enter into the radical of the integrals Πx, $\Pi_{,}x$. This is a very striking difference between the present theory and the analogous theory relating to conics, and leads, I think, to the inference that the theory of the polygon inscribed in a conic, *and the sides of which touch conics intersecting the conic in the same four points*, cannot be extended to surfaces in such manner as one might be led to suppose from the extension to surfaces of the much simpler theory of the polygon inscribed in a conic, *and the sides of which touch conics having double contact with the conic.* (See my paper "On the Homographic Transformation of a surface of the second order into itself," [122]).

The preceding investigations are obviously very incomplete; but the connexion which they point out between the geometrical question and the Abelian integral involving the root of a function of the sixth order, may I think be of service in the theory of these integrals.

124.

ON A PROPERTY OF THE CAUSTIC BY REFRACTION OF THE CIRCLE.

[From the *Philosophical Magazine*, vol. VI. (1853), pp. 427—431.]

M. ST LAURENT has shown (*Gergonne*, vol. XVIII. [1827] p. 1), that in certain cases the caustic by refraction of a circle is identical with the caustic of reflexion of a circle (the reflecting circle and radiant point being, of course, properly chosen), and a very elegant demonstration of M. St Laurent's theorems is given by M. Gergonne in the same volume, p. 48. A similar method may be employed to demonstrate the more general theorem, that the same caustic by refraction of a circle may be considered as arising from *six* different systems of a radiant point, circle, and index of refraction. The demonstration is obtained by means of the secondary caustic, which is (as is well known) an oval of Descartes. Such oval has three foci, any one of which may be taken for the radiant point: whichever be selected, there can always be found two corresponding circles and indices of refraction. The demonstration is as follows:—

Let c be the radius of the refracting circle, μ the index of refraction; and taking the centre of the circle as origin, let ξ, η be the coordinates of the radiant point, the secondary caustic is the envelope of the circle

$$\mu^2(\overline{x-\alpha}^2+\overline{y-\beta}^2)-(\overline{\xi-\alpha}^2+\overline{\eta-\beta}^2)=0,$$

where α, β are parameters which vary subject to the condition

$$\alpha^2+\beta^2-c^2=0;$$

the equation of the variable circle may be written

$$\{\mu^2(x^2+y^2+c^2)-(\xi^2+\eta^2+c^2)\}-2(\mu^2x-\xi)\alpha-2(\mu^2y-\eta)\beta=0,$$

which is of the form

$$C + A\alpha + B\beta = 0;$$

the envelope is therefore

$$C^2 = c^2(A^2 + B^2).$$

Hence substituting, we have for the equation of the envelope, i.e. for the secondary caustic,

$$\{\mu^2(x^2+y^2+c^2) - (\xi^2+\eta^2+c^2)\}^2 = 4c^2\{(\mu^2x-\xi)^2 + (\mu^2y-\eta)^2\},$$

which may also be written

$$\{\mu^2(x^2+y^2-c^2) - (\xi^2+\eta^2-c^2)\}^2 = 4c^2\mu^2(\overline{x-\xi}^2 + \overline{y-\eta}^2);$$

and this may perhaps be considered as the standard form.

To show that this equation belongs to a Descartes' oval, suppose for greater convenience $\eta = 0$, and write

$$\mu^2(x^2+y^2-c^2) - \xi^2 + c^2 = 2c\mu\sqrt{(x-\xi)^2+y^2};$$

multiplying this equation by $1-\frac{1}{\mu^2}$, and adding to each side $c^2\left(\mu-\frac{1}{\mu}\right)^2 + (x-\xi)^2 + y^2$, we have

$$\left(1-\frac{1}{\mu^2}\right)\{\mu^2(x^2+y^2-c^2) - \xi^2 + c^2\} + (x-\xi)^2 + y^2 + c^2\left(\mu-\frac{1}{\mu}\right)^2$$
$$= (x-\xi)^2 + y^2 + 2c\left(\mu-\frac{1}{\mu}\right)\sqrt{(x-\xi)^2+y^2} + c^2\left(\mu-\frac{1}{\mu}\right)^2;$$

or reducing

$$\mu^2\left\{\left(x-\frac{\xi}{\mu^2}\right)^2 + y^2\right\} = \left\{\sqrt{(x-\xi)^2+y^2} + c\left(\mu-\frac{1}{\mu}\right)\right\}^2;$$

again, multiplying the same equation by $\frac{1}{\mu^2}\left(1-\frac{c^2}{\xi^2}\right)$, and adding to each side

$$\frac{\xi^2}{\mu^2}\left(1-\frac{c^2}{\xi^2}\right)^2 + \frac{c^2}{\xi^2}(\overline{x-\xi}^2 + y^2),$$

we have

$$\frac{1}{\mu^2}\left(1-\frac{c^2}{\xi^2}\right)\{\mu^2(x^2+y^2-c^2) - \xi^2 + c^2\} + \frac{c^2}{\xi^2}(\overline{x-\xi}^2 + y^2) + \frac{\xi^2}{\mu^2}\left(1-\frac{c^2}{\xi^2}\right)^2$$
$$= \frac{c^2}{\xi^2}((x-\xi)^2 + y^2) + \frac{2c}{\mu}\left(1-\frac{c^2}{\xi^2}\right)\sqrt{(x-\xi)^2+y^2} + \frac{\xi^2}{\mu^2}\left(1-\frac{c^2}{\xi^2}\right)^2;$$

or reducing,

$$\left(x-\frac{c^2}{\xi}\right)^2 + y^2 = \left\{\frac{c}{\xi}\sqrt{(x-\xi)^2+y^2} + \frac{\xi}{\mu}\left(1-\frac{c^2}{\xi^2}\right)\right\}^2.$$

Hence, extracting the square roots of each side of the equations thus found, we have the equation of the secondary caustic in either of the forms

$$\sqrt{\left(x-\frac{\xi}{\mu^2}\right)^2+y^2}=\frac{1}{\mu}\sqrt{(x-\xi)^2+y^2}+\frac{c}{\mu}\left(\mu-\frac{1}{\mu}\right),$$

$$\sqrt{\left(x-\frac{c^2}{\xi}\right)^2+y^2}=\frac{c}{\xi}\sqrt{(x-\xi)^2+y^2}+\frac{1}{\mu}\left(\xi-\frac{c^2}{\xi}\right);$$

to which are to be joined

$$\sqrt{\left(x-\frac{c^2}{\xi}\right)^2+y^2}=\frac{c\mu}{\xi}\sqrt{\left(x-\frac{\xi}{\mu^2}\right)^2+y^2}+\frac{\xi}{\mu}-\frac{c^2\mu}{\xi},$$

$$c\left(\mu-\frac{1}{\mu}\right)\sqrt{\left(x-\frac{c^2}{\xi}\right)^2+y^2}+\left(-\xi+\frac{c^2}{\xi}\right)\sqrt{\left(x-\frac{\xi}{\mu^2}\right)^2+y^2}+\left(\frac{\xi}{\mu}-\frac{c^2\mu}{\xi}\right)\sqrt{(x-\xi)^2+y^2}=0.$$

Write successively,

$$\xi'=\xi\ ,\quad c'=c\ ,\quad \mu'=\mu\ ,\qquad (1)$$

$$\xi'=\frac{c^2}{\xi}\ ,\quad c'=\frac{c}{\mu}\ ,\quad \mu'=\frac{c}{\xi}\ ,\qquad (\alpha)$$

$$\xi'=\frac{\xi}{\mu^2}\ ,\quad c'=\frac{c}{\mu}\ ,\quad \mu'=\frac{1}{\mu}\ ,\qquad (\beta)$$

$$\xi'=\xi\ ,\quad c'=\frac{\xi}{\mu}\ ,\quad \mu'=\frac{\xi}{c}\ ,\qquad (\gamma)$$

$$\xi'=\frac{c^2}{\xi}\ ,\quad c'=c\ ,\quad \mu'=\frac{c\mu}{\xi}\ ,\qquad (\delta)$$

$$\xi'=\frac{\xi}{\mu^2}\ ,\quad c'=\frac{\xi}{\mu}\ ,\quad \mu'=\frac{\xi}{c\mu}\ ;\qquad (\epsilon)$$

or, what is the same thing,

$$\xi=\xi'\ ,\quad c=c'\ ,\quad \mu=\mu'\ ,\qquad (1)$$

$$\xi=\frac{\xi'}{\mu'^2}\ ,\quad c=\frac{\xi'}{\mu'}\ ,\quad \mu=\frac{\xi'}{c'\mu'}\ ,\qquad (\alpha)$$

$$\xi=\frac{\xi'}{\mu'^2}\ ,\quad c=\frac{c'}{\mu'}\ ,\quad \mu=\frac{1}{\mu'}\ ,\qquad (\beta)$$

$$\xi=\xi'\ ,\quad c=\frac{\xi'}{\mu'}\ ,\quad \mu=\frac{\xi'}{c'}\ ,\qquad (\gamma)$$

$$\xi=\frac{c'^2}{\xi'}\ ,\quad c=c'\ ,\quad \mu=\frac{c'\mu'}{\xi'}\ ,\qquad (\delta)$$

$$\xi=\frac{c'^2}{\xi'}\ ,\quad c=\frac{c'}{\mu'}\ ,\quad \mu=\frac{c'}{\xi'}\ ;\qquad (\epsilon)$$

or, again,

$$\xi' = \xi\ , \quad \frac{c'^2}{\xi'} = \frac{c^2}{\xi}\ , \quad \frac{\xi'}{\mu'^2} = \frac{\xi}{\mu^2}\ , \qquad (1)$$

$$\xi' = \frac{c^2}{\xi}\ , \quad \frac{c'^2}{\xi'} = \xi\ , \quad \frac{\xi'}{\mu'^2} = \xi\ , \qquad (\alpha)$$

$$\xi' = \frac{\xi}{\mu^2}\ , \quad \frac{c'^2}{\xi'} = \frac{c^2}{\xi}\ , \quad \frac{\xi'}{\mu'^2} = \xi\ , \qquad (\beta)$$

$$\xi' = \xi\ , \quad \frac{c'^2}{\xi'} = \frac{\xi}{\mu^2}\ , \quad \frac{\xi'}{\mu'^2} = \frac{c^2}{\xi}\ , \qquad (\gamma)$$

$$\xi' = \frac{c^2}{\xi}\ , \quad \frac{c'^2}{\xi'} = \xi\ , \quad \frac{\xi'}{\mu^2} = \frac{\xi}{\mu^2}\ , \qquad (\delta)$$

$$\xi' = \frac{\xi}{\mu^2}\ , \quad \frac{c'^2}{\xi'} = \xi\ , \quad \frac{\xi'}{\mu'^2} = \frac{c^2}{\xi}\ ; \qquad (\epsilon)$$

then, whichever system of values of ξ', c', μ' be substituted for ξ, c, μ, we have in each case identically the same secondary caustic, the effect of the substitution being simply to interchange the different forms of the equation; and we have therefore identically the same caustic. By writing

$$\begin{aligned}(\xi', c', \mu') &= (\xi, c, \mu)\\ &= \alpha(\xi, c, \mu),\\ &\text{\&c.,}\end{aligned}$$

α, β, γ, δ, ϵ will be functional symbols, such as are treated of in my paper "On the Theory of Groups as depending on the symbolic equation $\theta^n = 1$," [125], and it is easy to verify the equations

$$\begin{aligned}1 &= \alpha\beta = \beta\alpha = \gamma^2 = \delta^2 = \epsilon^2,\\ \alpha &= \beta^2 = \delta\gamma = \epsilon\delta = \gamma\epsilon,\\ \beta &= \alpha^2 = \epsilon\gamma = \gamma\delta = \delta\epsilon,\\ \gamma &= \delta\alpha = \epsilon\beta = \beta\delta = \alpha\epsilon,\\ \delta &= \epsilon\alpha = \gamma\beta = \alpha\gamma = \beta\epsilon,\\ \epsilon &= \gamma\alpha = \delta\beta = \beta\gamma = \alpha\delta.\end{aligned}$$

Suppose, for example, $\xi = -c$, i.e. let the radiant point be in the circumference; then in the fourth system $\xi' = -c$, $c' = -\frac{c}{\mu}$, (or, since c' is the radius of a circle, this radius may be taken $\frac{c}{\mu}$), $\mu' = -1$, or the new system is a reflecting system. This is one of M. St Laurent's theorems, viz.

THEOREM. The caustic by refraction of a circle when the radiant point is on the circumference, is the caustic by reflexion for the same radiant point, and a concentric circle the radius of which is the radius of the first circle divided by the index of refraction.

Again, if $\xi = -c\mu$, the fifth system gives $\xi' = \frac{c^2}{\xi'}$, $c' = c$, $\mu' = -1$, or the new system is in this case also a reflecting system. This is the other of M. St Laurent's theorems, viz.:—

THEOREM. The caustic by refraction of a circle when the distance of the radiant point from the centre is equal to the radius of the circle multiplied by the index of refraction, is the caustic by reflexion of the same circle for a radiant point which is the image of the first radiant point.

Of course it is to be understood that the image of a point means a point whose distance from the centre = square of radius ÷ distance.

2 *Stone Buildings, Nov.* 2, 1853.

125.

ON THE THEORY OF GROUPS, AS DEPENDING ON THE SYMBOLIC EQUATION $\theta^n = 1$.

[From the *Philosophical Magazine*, vol. VII. (1854), pp. 40—47.]

LET θ be a symbol of operation, which may, if we please, have for its operand, not a single quantity x, but a system $(x, y, \ldots)$, so that

$$\theta(x, y, \ldots) = (x', y', \ldots),$$

where $x', y', \ldots$ are any functions whatever of $x, y, \ldots$, it is not even necessary that $x', y', \ldots$ should be the same in number with $x, y, \ldots$. In particular x', y', &c. may represent a permutation of x, y, &c., θ is in this case what is termed a substitution; and if, instead of a set $x, y, \ldots$, the operand is a single quantity x, so that $\theta x = x' = fx$, θ is an ordinary functional symbol. It is not necessary (even if this could be done) to attach any meaning to a symbol such as $\theta \pm \phi$, or to the symbol 0, nor consequently to an equation such as $\theta = 0$, or $\theta \pm \phi = 0$; but the symbol 1 will naturally denote an operation which (either generally or in regard to the particular operand) leaves the operand unaltered, and the equation $\theta = \phi$ will denote that the operation θ is (either generally or in regard to the particular operand) equivalent to ϕ, and of course $\theta = 1$ will in like manner denote the equivalence of the operation θ to the operation 1. A symbol $\theta\phi$ denotes the compound operation, the performance of which is equivalent to the performance, first of the operation ϕ, and then of the operation θ; $\theta\phi$ is of course in general different from $\phi\theta$. But the symbols $\theta, \phi, \ldots$ are in general such that $\theta . \phi\chi = \theta\phi . \chi$, &c., so that $\theta\phi\chi$, $\theta\phi\chi\omega$, &c. have a definite signification independent of the particular mode of compounding the symbols; this will be the case even if the functional operations involved in the symbols θ, ϕ, &c. contain parameters such as the quaternion imaginaries i, j, k; but not if these functional operations contain parameters such as the imaginaries which enter into the theory of octaves, &c., and for which, e.g. $\alpha . \beta\gamma$ is something different from $\alpha\beta . \gamma$,

a supposition which is altogether excluded from the present paper. The order of the factors of a product $\theta\phi\chi\ldots$ must of course be attended to, since even in the case of a product of two factors the order is material; it is very convenient to speak of the symbols θ, $\phi\ldots$ as the first or furthest, second, third, &c., and last or nearest factor. What precedes may be almost entirely summed up in the remark, that the distributive law has no application to the symbols $\theta\phi\ldots$; and that these symbols are not in general convertible, but are associative. It is easy to see that $\theta^0 = 1$, and that the index law $\theta^m . \theta^n = \theta^{m+n}$, holds for all positive or negative integer values, not excluding 0. It should be noticed also, that if $\theta = \phi$, then, whatever the symbols α, β may be, $\alpha\theta\beta = \alpha\phi\beta$, and conversely.

A set of symbols,

$$1,\ \alpha,\ \beta, \ldots$$

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*[1]. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table, thus:

Nearer factors \ Further factors	1	α	β	..
1	1	α	β	..
α	α	α^2	$\beta\alpha$	
β	β	$\alpha\beta$	β^2	
	:			

that as well each line as each column of the square will contain all the symbols $1, \alpha, \beta, \ldots$. It also follows that the product of any number of the symbols, with or without repetitions, and in any order whatever, is a symbol of the group. Suppose that the group

$$1,\ \alpha,\ \beta,\ \ldots$$

contains n symbols, it may be shown that each of these symbols satisfies the equation

$$\theta^n = 1;$$

so that a group may be considered as representing a system of roots of this symbolic binomial equation. It is, moreover, easy to show that if any symbol α of the group

[1] The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations.

satisfies the equation $\theta^r = 1$, where r is less that n, then that r must be a submultiple of n; it follows that when n is a prime number, the group is of necessity of the form

$$1,\ \alpha,\ \alpha^2, \ldots \alpha^{n-1},\ (\alpha^n = 1);$$

and the same may be (but is not necessarily) the case, when n is a composite number. But whether n be prime or composite, the group, *assumed to be of the form in question*, is in every respect analogous to the system of the roots of the ordinary binomial equation $x^n - 1 = 0$; thus, when n is prime, all the roots (except the root 1) are prime roots; but when n is composite, there are only as many prime roots as there are numbers less than n and prime to it, &c.

The distinction between the theory of the symbolic equation $\theta^n = 1$, and that of the ordinary equation $x^n - 1 = 0$, presents itself in the very simplest case, $n = 4$. For, consider the group

$$1,\ \alpha,\ \beta,\ \gamma,$$

which are a system of roots of the symbolic equation

$$\theta^4 = 1.$$

There is, it is clear, at least one root β, such that $\beta^2 = 1$; we may therefore represent the group thus,

$$1,\ \alpha,\ \beta,\ \alpha\beta,\ (\beta^2 = 1);$$

then multiplying each term by α as further factor, we have for the group $1,\ \alpha^2,\ \alpha\beta,\ \alpha^2\beta$, so that α^2 must be equal either to β or else to 1. In the former case the group is

$$1,\ \alpha,\ \alpha^2,\ \alpha^3,\ (\alpha^4 = 1),$$

which is analogous to the system of roots of the ordinary equation $x^4 - 1 = 0$. For the sake of comparison with what follows, I remark, that, representing the last-mentioned group by

$$1,\ \alpha,\ \beta,\ \gamma,$$

we have the table

	1,	α,	β,	γ
1	1	α	β	γ
α	α	β	γ	1
β	β	γ	1	α
γ	γ	1	α	β

If, on the other hand, $\alpha^2 = 1$, then it is easy by similar reasoning to show that we must have $\alpha\beta = \beta\alpha$, so that the group in the case is

$$1,\ \alpha,\ \beta,\ \alpha\beta,\ (\alpha^2 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha);$$

or if we represent the group by

$$1,\ \alpha,\ \beta,\ \gamma,$$

we have the table

	1	α	β	γ
1	1	α	β	γ
α	α	1	γ	β
β	β	γ	1	α
γ	γ	β	α	1

,

or, if we please, the symbols are such that

$$\begin{aligned} \alpha^2 = \beta^2 &= \gamma^2 = 1, \\ \alpha &= \beta\gamma = \gamma\beta, \\ \beta &= \gamma\alpha = \alpha\beta, \\ \gamma &= \alpha\beta = \beta\alpha; \end{aligned}$$

[and we have thus a group essentially distinct from that of the system of roots of the ordinary equation $x^4 - 1 = 0$].

Systems of this form are of frequent occurrence in analysis, and it is only on account of their extreme simplicity that they have not been expressly remarked. For instance, in the theory of elliptic functions, if n be the parameter, and

$$\alpha(n) = \frac{c^2}{n}\quad \beta(n) = -\frac{c^2+n}{1+n}\quad \gamma(n) = -\frac{c^2(1+n)}{c^2+n},$$

then α, β, γ form a group of the species in question. So in the theory of quadratic forms, if

$$\begin{aligned} \alpha(a,\ b,\ c) &= (c,\ \ b,\ a) \\ \beta(a,\ b,\ c) &= (a,\ -b,\ c) \\ \gamma(a,\ b,\ c) &= (c,\ -b,\ a); \end{aligned}$$

although, indeed, in this case (treating forms which are properly equivalent as identical) we have $\alpha = \beta$, and therefore $\gamma = 1$, in which point of view the group is simply a group of two symbols 1, α, $(\alpha^2 = 1)$.

Again, in the theory of matrices, if I denote the operation of inversion, and tr that of transposition, (I do not stop to explain the terms as the example may be passed over), we may write

$$\alpha = I, \quad \beta = \text{tr}, \quad \gamma = I \,.\, \text{tr} = \text{tr} \,.\, I.$$

I proceed to the case of a group of six symbols,

$$1, \ \alpha, \ \beta, \ \gamma, \ \delta, \ \epsilon,$$

which may be considered as representing a system of roots of the symbolic equation

$$\theta^6 = 1.$$

It is in the first place to be shown that there is at least one root which is a prime root of $\theta^6 = 1$, or (to use a simpler expression) a root having the index 3. It is clear that if there were a prime root, or root having the index 6, the square of this root would have the index 3, it is therefore only necessary to show that it is impossible that *all* the roots should have the index 2. This may be done by means of a theorem which I shall for the present assume, viz. that if among the roots of the symbolic equation $\theta^n = 1$, there are contained a system of roots of the symbolic equation $\theta^p = 1$ (or, in other words, if among the symbols forming a group of the order there are contained symbols forming a group of the order p), then p is a submultiple of n. In the particular case in question, a group of the order 4 cannot form part of the group of the order 6. Suppose, then, that γ, δ are two roots of $\theta^6 = 1$, having each of them the index 2; then if $\gamma\delta$ had also the index 2, we should have $\gamma\delta = \delta\gamma$; and 1, γ, δ, $\delta\gamma$, which is part of the group of the order 6, would be a group of the order 4. It is easy to see that $\gamma\delta$ must have the index 3, and that the group is, in fact, 1, $\gamma\delta$, $\delta\gamma$, γ, δ, $\gamma\delta\gamma$, which is, in fact, one of the groups to be presently obtained; I prefer commencing with the assumption of a root having the index 3. Suppose that α is such a root, the group must clearly be of the form

$$1, \ \alpha, \ \alpha^2, \ \gamma, \ \alpha\gamma, \ \alpha^2\gamma, \ (\alpha^3 = 1);$$

and multiplying the entire group by γ as nearer factor, it becomes γ, $\alpha\gamma$, $\alpha^2\gamma$, γ^2, $\alpha\gamma^2$, $\alpha^2\gamma^2$; we must therefore have $\gamma^2 = 1$, α, or α^2. But the supposition $\gamma^2 = \alpha^2$ gives $\gamma^4 = \alpha^4 = \alpha$, and the group is in this case 1, γ, γ^2, γ^3, γ^4, γ^5 $(\gamma^6 = 1)$; and the supposition $\gamma^2 = \alpha$ gives also this same group. It only remains, therefore, to assume $\gamma^2 = 1$; then we must have either $\gamma\alpha = \alpha\gamma$ or else $\gamma\alpha = \alpha^2\gamma$. The former assumption leads to the group

$$1, \ \alpha, \ \alpha^2, \ \gamma, \ \alpha\gamma, \ \alpha^2\gamma, \ (\alpha^3 = 1, \ \gamma^2 = 1, \ \gamma\alpha = \alpha\gamma),$$

which is, in fact, analogous to the system of roots of the ordinary equation $x^6 - 1 = 0$; and by putting $\alpha\gamma = \lambda$, might be exhibited in the form 1, λ, λ^2, λ^3, λ^4, λ^5, $(\lambda^6 = 1)$, under which this system has previously been considered. The latter assumption leads to the group

$$1, \ \alpha, \ \alpha^2, \ \gamma, \ \alpha\gamma, \ \alpha^2\gamma, \ (\alpha^3 = 1, \ \gamma^2 = 1, \ \gamma\alpha = \alpha^2\gamma),$$

and we have thus two, and only two, essentially distinct forms of a group of six.

If we represent the first of these two forms, viz. the group

$$1, \ \alpha, \ \alpha^2, \ \gamma, \ \alpha\gamma, \ \alpha^2\gamma, \ (\alpha^3 = 1, \ \gamma^2 = 1, \ \gamma\alpha = \alpha\gamma)$$

by the general symbols

$$1,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,$$

we have the table

	1,	α,	β,	γ,	δ,	ϵ
1	1	α	β	γ	δ	ϵ
α	α	β	γ	δ	ϵ	1
β	β	γ	δ	ϵ	1	α
γ	γ	δ	ϵ	1	α	β
δ	δ	ϵ	1	α	β	γ
ϵ	ϵ	1	α	β	γ	δ

;

while if we represent the second of these two forms, viz. the group

$$1,\ \alpha,\ \alpha^2,\ \gamma,\ \alpha\gamma,\ \alpha^2\gamma,\ (\alpha^3=1,\ \gamma^2=1,\ \gamma\alpha=\alpha^2\gamma),$$

by the same general symbols

$$1,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,$$

we have the table

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	β	1	ϵ	γ	δ
β	β	1	α	δ	ϵ	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	ϵ	γ	β	1	α
ϵ	ϵ	γ	δ	α	β	1

;

or, what is the same thing, the system of equations is

$$\begin{aligned}
1 = \beta\alpha = \alpha\beta &= \gamma^2 = \delta^2 = \epsilon^2,\\
\alpha = \beta^2 &= \delta\gamma = \epsilon\delta = \gamma\epsilon,\\
\beta = \alpha^2 &= \epsilon\gamma = \gamma\delta = \delta\epsilon,\\
\gamma = \delta\alpha &= \epsilon\beta = \beta\delta = \alpha\epsilon,\\
\delta = \epsilon\alpha &= \gamma\beta = \alpha\gamma = \beta\epsilon,\\
\epsilon = \gamma\alpha &= \delta\beta = \beta\gamma = \alpha\delta.
\end{aligned}$$

An instance of a group of this kind is given by the permutation of three letters; the group

$$1,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon$$

may represent a group of substitutions as follows:—

$$\begin{array}{cccccc}
abc, & cab, & bca, & acb, & cba, & bac\\
abc & abc & abc & abc & abc & abc.
\end{array}$$

Another singular instance is given by the optical theorem proved in my paper "On a property of the Caustic by refraction of a Circle, [124]."

It is, I think, worth noticing, that if, instead of considering α, β, &c. as symbols of operation, we consider them as quantities (or, to use a more abstract term, 'cogitables') such as the quaternion imaginaries; the equations expressing the existence of the group are, in fact, the equations defining the meaning of the product of two complex quantities of the form

$$w + a\alpha + b\beta + \ldots;$$

thus, in the system just considered,

$$(w + a\alpha + b\beta + c\gamma + d\delta + e\epsilon)(w' + a'\alpha + b'\beta + c'\gamma + d'\delta + e'\epsilon) = W + A\alpha + B\beta + C\gamma + D\delta + E\epsilon,$$

where

$$\begin{aligned}
W &= ww' + ab' + a'b + cc' + dd' + ee',\\
A &= wa' + w'a + bb' + dc' + ed' + ce',\\
B &= wb' + w'b + aa' + ec' + cd' + de',\\
C &= wc' + w'c + da' + eb' + bd' + ae',\\
D &= wd' + w'd + ea' + cb' + ac' + be',\\
E &= we' + w'e + ca' + db' + bc' + ad'.
\end{aligned}$$

It does not appear that there is in this system anything analogous to the modulus $w^2 + x^2 + y^2 + z^2$, so important in the theory of quaternions.

I hope shortly to resume the subject of the present paper, which is closely connected, not only with the theory of algebraical equations, but also with that of

the composition of quadratic forms, and the 'irregularity' in certain cases of the determinants of these forms. But I conclude for the present with the following two examples of groups of higher orders. The first of these is a group of eighteen, viz.

$$1,\ \alpha,\ \beta,\ \gamma,\ \alpha\beta,\ \beta\alpha,\ \alpha\gamma,\ \gamma\alpha,\ \beta\gamma,\ \gamma\beta,\ \alpha\beta\gamma,\ \beta\gamma\alpha,\ \gamma\alpha\beta,\ \alpha\beta\alpha,\ \beta\gamma\beta,\ \gamma\alpha\gamma,\ \alpha\beta\gamma\beta,\ \beta\gamma\beta\alpha,$$

where

$$\alpha^2=1,\ \beta^2=1,\ \gamma^2=1,\ (\beta\gamma)^2=1,\ (\gamma\alpha)^2=1,\ (\alpha\beta)^2=1,\ (\alpha\beta\gamma)^2=1,\ (\beta\gamma\alpha)^2=1,\ (\gamma\alpha\beta)^2=1;$$

and the other a group of twenty-seven, viz.

$$1,\ \alpha,\ \alpha^2,\ \gamma,\ \gamma^2,\ \gamma\alpha,\ \alpha\gamma,\ \gamma\alpha^2,\ \alpha^2\gamma,\ \gamma^2\alpha,\ \alpha\gamma^2,\ \alpha^2\gamma^2,$$

$$\alpha\gamma\alpha,\ \alpha\gamma^2\alpha,\ \alpha^2\gamma\alpha,\ \alpha^2\gamma^2\alpha,\ \alpha\gamma\alpha^2,\ \alpha\gamma^2\alpha^2,\ \alpha^2\gamma\alpha^2,\ \alpha^2\gamma^2\alpha^2,\ \gamma\alpha\gamma^2,\ \gamma\alpha^2\gamma^2,\ \gamma^2\alpha\gamma,\ \gamma^2\alpha^2\gamma,\ \gamma^2\alpha\gamma\alpha^2,\ \gamma\alpha\gamma^2\alpha^2,$$

where

$$\alpha^3=1,\ \gamma^3=1,\ (\gamma\alpha)^3=1,\ (\gamma^2\alpha)^3=1,\ (\gamma\alpha^2)^3=1,\ (\gamma^2\alpha^2)^3=1.$$

It is hardly necessary to remark, that each of these groups is in reality perfectly symmetric, the omitted terms being, in virtue of the equations defining the nature of the symbols, identical with some of the terms of the group: thus, in the group of 18, the equations $\alpha^2=1$, $\beta^2=1$, $\gamma^2=1$, $(\alpha\beta\gamma)^2=1$ give $\alpha\beta\gamma=\gamma\beta\alpha$, and similarly for all the other omitted terms. It is easy to see that in the group of 18 the index of each term is 2 or else 3, while in the group of 27 the index of each term is 3.

2 *Stone Buildings, Nov.* 2, 1853.

126.

ON THE THEORY OF GROUPS, AS DEPENDING ON THE SYMBOLIC EQUATION $\theta^n = 1$.—SECOND PART.

[From the *Philosophical Magazine*, vol. VII. (1854), pp. 408—409.]

IMAGINE the symbols

$$L,\ M,\ N, \ldots$$

such that (L being any symbol of the system),

$$L^{-1}L,\ L^{-1}M,\ L^{-1}N, \ldots$$

is the *group*

$$1, \quad \alpha, \quad \beta, \ldots;$$

then, in the first place, M being any other symbol of the system, $M^{-1}L$, $M^{-1}M$, $M^{-1}N, \ldots$ will be the same group $1, \alpha, \beta, \ldots$. In fact, the system $L, M, N, \ldots$ may be written $L, L\alpha, L\beta \ldots$; and if e.g. $M = L\alpha$, $N = L\beta$ then

$$M^{-1}N = (L\alpha)^{-1} L\beta = \alpha^{-1} L^{-1} L\beta = \alpha^{-1}\beta,$$

which belongs to the group $1, \alpha, \beta, \ldots$.

Next it may be shown that

$$LL^{-1},\ ML^{-1},\ NL^{-1}, \ldots$$

is a group, although not in general the same group as $1, \alpha, \beta, \ldots$. In fact, writing $M = L\alpha$, $N = L\beta$, &c., the symbols just written down are

$$LL^{-1},\ L\alpha L^{-1},\ L\beta L^{-1}, \ldots$$

and we have e.g. $L\alpha L^{-1} . L\beta L^{-1} = L\alpha\beta L^{-1} = L\gamma L^{-1}$, where γ belongs to the group $1, \alpha, \beta$.

The system $L, M, N, \ldots$ may be termed a group-holding system, or simply a holder; and with reference to the two groups to which it gives rise, may be said to hold on the nearer side the group $L^{-1}L, L^{-1}M, L^{-1}N, \ldots$, and to hold on the further side the group $LL^{-1}, LM^{-1}, LN^{-1}, \ldots$ Suppose that these groups are one and the same group $1, \alpha, \beta \ldots$, the system $L, M, N, \ldots$ is in this case termed a symmetrical holder, and in reference to the last-mentioned group is said to hold such group symmetrically. It is evident that the symmetrical holder $L, M, N, \ldots$ may be expressed indifferently and at pleasure in either of the two forms $L, L\alpha, L\beta, \ldots$ and $L, \alpha L, \beta L$; i.e. we may say that the group is convertible with any symbol L of the holder, and that the group operating upon, or operated upon by, a symbol L of the holder, produces the holder. We may also say that the holder operated upon by, or operating upon, a symbol α of the group reproduces the holder.

Suppose now that the group

$$1, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \ldots$$

can be divided into a series of symmetrical holders of the smaller group

$$1, \alpha, \beta, \ldots;$$

the former group is said to be a multiple of the latter group, and the latter group to be a submultiple of the former group. Thus considering the two different forms of a group of six, and first the form

$$1, \alpha, \alpha^2, \gamma, \gamma\alpha, \gamma\alpha^2, \ (\alpha^3 = 1, \ \gamma^2 = 1, \ \alpha\gamma = \gamma\alpha),$$

the group of six is a multiple of the group of three, $1, \alpha, \alpha^2$ (in fact, $1, \alpha, \alpha^2$ and $\gamma, \gamma\alpha, \gamma\alpha^2$ are each of them a symmetrical holder of the group $1, \alpha, \alpha^2$); and so in like manner the group of six is a multiple of the group of two, $1, \gamma$ (in fact, $1, \gamma$ and $\alpha, \alpha\gamma$, and $\alpha, \alpha^2\gamma$ are each a symmetrical holder of the group $1, \gamma$). There would not, in a case such as the one in question, be any harm in speaking of the group of six as the product of the two groups $1, \alpha, \alpha^2$ and $1, \gamma$, but upon the whole it is, I think, better to dispense with the expression.

Considering, secondly, the other form of a group of six, viz.

$$1, \alpha, \alpha^2, \gamma, \gamma\alpha, \gamma\alpha^2 \, (\alpha^3 = 1, \ \gamma^2 = 1, \ \alpha\gamma = \gamma\alpha^2);$$

here the group of six is a multiple of the group of three, $1, \alpha, \alpha^2$ (in fact, as before, $1, \alpha, \alpha^2$ and $\gamma, \gamma\alpha, \gamma\alpha^2$, are each a symmetrical holder of the group $1, \alpha, \alpha^2$, since, as regards $\gamma, \gamma\alpha, \gamma\alpha^2$, we have $(\gamma, \gamma\alpha, \gamma\alpha^2) = \gamma(1, \alpha, \alpha^2) = (1, \alpha^2, \alpha)\gamma$). But the group of six is not a multiple of any group of two whatever; in fact, besides the group $1, \gamma$ itself, there is not any symmetrical holder of this group $1, \gamma$; and so, in like manner, with respect to the other groups of two, $1, \gamma\alpha$, and $1, \gamma\alpha^2$. The group of three, $1, \alpha, \alpha^2$, is therefore, in the present case, the only submultiple of the group of six.

It may be remarked, that if there be any number of symmetrical holders of the same group, $1, \alpha, \beta, \ldots$ then any one of these holders bears to the aggregate of the holders a relation such as the submultiple of a group bears to such group; it is proper to notice that the aggregate of the holders is not of necessity itself a holder.

127.

ON THE HOMOGRAPHIC TRANSFORMATION OF A SURFACE OF THE SECOND ORDER INTO ITSELF.

[From the *Philosophical Magazine*, vol. VII. (1854), pp. 208—212: continuation of **122**.]

I PASS to the improper transformation. Sir W. R. Hamilton has given (in the note, p. 723 of his Lectures on Quaternions [Dublin, 1853)] the following theorem:—If there be a polygon of $2m$ sides inscribed in a surface of the second order, and $(2m-1)$ of the sides pass through given points, then will the $2m$-th side constantly touch two cones circumscribed about the surface of the second order. The relation between the extremities of the $2m$-th side is that of two points connected by the general improper transformation; in other words, if there be on a surface of the second order two points such that the line joining them touches two cones circumscribed about the surface of the second order, then the two points are as regards the transformation in question a pair of corresponding points, or simply a pair. But the relation between the two points of a pair may be expressed in a different and much more simple form. For greater clearness call the surface of the second order U, and the sections along which it is touched by the two cones, θ, ϕ; the cones themselves may, it is clear, be spoken of as the cones θ, ϕ. And let the two points be P, Q. The line PQ touches the two cones, it is therefore the line of intersection of the tangent plane through P to the cone θ, and the tangent plane through P to the cone ϕ. Let one of the generating lines through P meet the section θ in the point A, and the other of the generating lines through P meet the section ϕ in the point B. The tangent planes through P to the cones θ, ϕ respectively are nothing else than the tangent planes to the surface U at the points A, B respectively. We have therefore at these points two generating lines meeting in the point P; the other two

generating lines at the points A, B meet in like manner in the point Q. Thus P, Q are opposite angles of a skew quadrangle formed by four generating lines (or, what is the same thing, lying upon the surface of the second order), and having its other two angles, one of them on the section θ and the other on the section ϕ; and if we consider the side PA as belonging determinately to one or the other of the two systems of generating lines, then when P is given, the corresponding point Q is, it is clear, completely determined. What precedes may be recapitulated in the statement, that in the improper transformation of a surface of the second order into itself, we have, as corresponding points, the opposite angles of a skew quadrangle lying upon the surface, and having the other two opposite angles upon given plane sections of the surface. I may add, that attending only to the sections through the points of intersection of θ, ϕ, if the point P be situate anywhere in one of these sections, the point Q will be always situate in the other of these sections, i.e. the sections correspond to each other in pairs; in particular, the sections θ, ϕ are corresponding sections, so also are the sections Θ, Φ (each of them two generating lines) made by tangent planes of the surface. Any three pairs of sections form an involution; the two sections which are the sibiconjugates of the involution are of course such, that, if the point P be situate in either of these sections, the corresponding point Q will be situate in the same section. It may be noticed that when the two sections θ, ϕ coincide, the line joining the corresponding points passes through a fixed point, viz. the pole of the plane of the coincident sections; in fact the lines PQ and AB are in every case reciprocal polars, and in the present case the line AB lies in a fixed plane, viz. the plane of the coincident sections, the line PQ passes therefore through the pole of this plane. This agrees with the remarks made in the first part of the present paper.

The analytical investigation in the case where the surface of the second order is represented under the form $xy-zw=0$ is so simple, that it is, I think, worth while to reproduce it here, although for several reasons I prefer exhibiting the final result in relation to the form $x^2+y^2+z^2+w^2=0$ of the equation of the surface of the second order. I consider then the surface $xy-zw=0$, and I take $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ for the coordinates of the poles of the two sections θ, ϕ, and also (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) as the coordinates of the points P, Q. We have of course $x_1y_1-z_1w_1=0$, $x_2y_2-z_2w_2=0$. The generating lines through P are obtained by combining the equation $xy-zw=0$ of the surface with the equation $xy_1+yx_1-zw_1-wz_1=0$ of the tangent plane at P. Eliminating x from these equations, and replacing in the result x_1 by its value $\dfrac{z_1w_1}{y_1}$, we have the equation

$$(yz_1-zy_1)(yw_1-wy_1)=0.$$

We may if we please take $yz_1-zy_1=0$, $xy_2+yx_1-zw_1-wz_1=0$ as the equations of the line PA; this leads to

$$\left.\begin{aligned} yz_1-zy_1&=0,\\ xy_1+yx_1-zw_1-wz_1&=0,\end{aligned}\right\}\quad \left.\begin{aligned} yw_2-wy_2&=0,\\ xy_2+yx_2-zw_2-wz_2&=0,\end{aligned}\right\}$$

for the equations of the lines PA, QA respectively; and we have therefore the coordinates of the point A, coordinates which must satisfy the equation

$$\beta x + \alpha y - \delta z - \gamma w = 0$$

of the plane θ. This gives rise to the equation

$$y_2(\alpha y_1 - \delta z_1) - w_2(\gamma y_1 - \beta z_1) = 0.$$

We have in like manner

$$\left.\begin{aligned} yw_1 - y_1 w &= 0, \\ xy_1 + yx_1 - zw_1 - wz_1 &= 0, \end{aligned}\right\} \quad \left.\begin{aligned} yz_2 - zy_2 &= 0, \\ xy_2 + yx_2 - zw_2 - wz_2 &= 0, \end{aligned}\right\}$$

for the equations of the lines PB, QB respectively; and we may thence find the coordinates of the point B, coordinates which must satisfy the equation

$$\beta' x + \alpha' y - \delta' z - \gamma' w = 0$$

of the plane ϕ. This gives rise to the equation

$$y_2(\alpha' y_1 - \gamma' w_1) - z_2(\delta' y_1 - \beta' w_1).$$

It is easy, by means of these two equations and the equation $x_2 y_2 - z_2 w_2 = 0$, to form the system

$$\begin{aligned} x_2 &= (\alpha y_1 - \delta z_1)(\alpha' y_1 - \gamma' w_1), \\ y_2 &= (\gamma y_1 - \beta z_1)(\delta' y_1 - \beta' w_1), \\ z_2 &= (\gamma y_1 - \beta z_1)(\alpha' y_1 - \gamma' w_1), \\ w_2 &= (\alpha y_1 - \delta z_1)(\delta' y_1 - \beta' w_1); \end{aligned}$$

or, effecting the multiplications and replacing $z_1 w_1$ by $x_1 y_1$, the values of x_2, y_2, z_2, w_2 contain the common factor y_1, which may be rejected. Also introducing on the left-hand sides the common factor MM', where $M^2 = \alpha\beta - \gamma\delta$, $M'^2 = \alpha'\beta' - \gamma'\delta'$, the equations become

$$\begin{aligned} MM' x_2 &= \gamma'\delta x_1 + \alpha\alpha' y_1 - \alpha'\delta z_1 - \alpha\gamma' w_1, \\ MM' y_2 &= \beta\beta' x_1 + \gamma\delta' y_1 - \beta\delta' z_1 - \beta'\gamma w_1, \\ MM' z_2 &= \beta\gamma' x_1 + \gamma\alpha' y_1 - \beta\alpha' z_1 - \gamma\gamma' w_1, \\ MM' w_2 &= \beta'\delta x_1 + \alpha\delta' y_1 - \delta\delta' z_1 - \alpha\beta' w_1, \end{aligned}$$

values which give identically $x_2 y_2 - z_2 w_2 = x_1 y_1 - z_1 w_1$. Moreover, by forming the value of the determinant, it is easy to verify that the transformation is in fact an improper one. We have thus obtained the equations for the improper transformation of the surface $xy - zw = 0$ into itself. By writing $x_1 + iy_1$, $x_1 - iy_1$ for x_1, y_1, &c., we have the following system of equations, in which (a, b, c, d), (a', b', c', d') represent, as before, the coordinates of the poles of the plane sections, and $M^2 = a^2 + b^2 + c^2 + d^2$, $M'^2 = a'^2 + b'^2 + c'^2 + d'^2$, viz. the system[1]

[1] The system is very similar in form to, but is *essentially* different from, that which could be obtained from the theory of quaternions by writing

$$MM'(w_2 + ix_2 + jy_2 + kz_2) = (d + ia + jb + kc)(w + ix + jy + kz)(d' + ia' + jb' + kc');$$

the last-mentioned transformation is, in fact, *proper*, and not *improper*.

$$\begin{aligned}
MM'x_2 &= (aa' - bb' - cc' - dd')\,x_1 + (\ \ ab' + a'b + cd' - c'd)\,y_1 \\
&\qquad + (\ \ ac' + a'c + db' - d'b)\,z_1 + (\ ad' + a'd + bc' - b'c)\,w_1, \\
MM'y_2 &= (ab' + a'b - cd' + c'd)\,x_1 + (-aa' + bb' - cc' - dd')\,y_1 \\
&\qquad + (\ \ bc' + b'c - da' + d'a)\,z_1 + (\ bd' + b'd - ac' + a'c)\,w, \\
MM'z_2 &= (ac' + a'c - db' + d'b)\,x_1 + (\ \ bc' + b'c - ad' + a'd)\,y_1 \\
&\qquad + (-aa' - bb' + cc' - dd')\,z_1 + (\ cd' + c'd - ba' + b'a)\,w_1, \\
MM'w_2 &= (ad' + a'd - bc' + b'c)\,x_1 + (\ \ bd' + b'd - ca' + c'a)\,y_1 \\
&\qquad + (\ \ cd' + c'd - ab' + a'b)\,z_1 + (-aa' - bb' - cc' + dd')\,w_1,
\end{aligned}$$

values which of course satisfy identically $x_2^2 + y_2^2 + z_2^2 + w_2^2 = x_1^2 + y_1^2 + z_1^2 + w_1^2$, and which belong to an improper transformation. We have thus obtained the improper transformation of the surface of the second order $x^2 + y^2 + z^2 + w^2 = 0$ into itself.

Returning for a moment to the equations which belong to the surface $xy - zw = 0$, it is easy to see that we may without loss of generality write $\alpha = \beta = \alpha' = \beta' = 0$; the equations take then the very simple form

$$MM'x_2 = \gamma'\delta x_1,\quad MM'y_2 = \gamma\delta' y_1,\quad MM'z_2 = -\gamma\gamma' w_1,\quad MM'w_2 = -\delta\delta' z_1,$$

where $MM' = \sqrt{-\gamma\delta}\,\sqrt{-\gamma'\delta'}$; and it thus becomes very easy to verify the geometrical interpretation of the formulæ.

It is necessary to remark, that, whenever the coordinates of the points Q are connected with the coordinates of the points B by means of the equations which belong to an improper transformation, the points P, Q have to each other the geometrical relation above mentioned, viz. there exist two plane sections θ, ϕ such that P, Q are the opposite angles of a skew quadrangle upon the surface, and having the other two opposite angles in the sections θ, ϕ respectively. Hence combining the theory with that of the proper transformation, we see that if A and B, B and $C, \ldots$, M and N are points corresponding to each other properly or improperly, then will N and A be points corresponding to each other, viz. properly or improperly, according as the number of the improper pairs in the series A and B, B and $C, \ldots$, M and N is even or odd; i.e. if all the sides but one of a polygon satisfy the geometrical conditions in virtue of which their extremities are pairs of corresponding points, the remaining side will satisfy the geometrical condition in virtue of which its extremities will be a pair of corresponding points, the pair being proper or improper according to the rule just explained.

I conclude with the remark, that we may by means of two plane sections of a surface of the second order obtain a proper transformation. For, if the generating lines through P meet the sections θ, ϕ in the points A, B respectively, and the remaining generating lines through A, B respectively meet the sections ϕ, θ respectively in B', A', and the remaining generating lines through B', A' respectively meet in a point P'; then will P, P' be a pair of corresponding points in a proper trans-

formation. In fact, the generating lines through P meeting the sections θ, ϕ in the points A, B respectively, and the remaining generating lines through A, B respectively meeting as before in the point Q, then P and Q will correspond to each other improperly, and in like manner P' and Q will correspond to each other improperly; i.e. P and P' will correspond to each other properly. The relation between P, P' may be expressed by saying that these points are opposite angles of the skew hexagon $PAB'P'A'B$ lying upon the surface, and having the opposite angles A, A' in the section θ, and the opposite angles B, B' in the section ϕ. It is, however, clear from what precedes, that the points P, P' lie in a section passing through the points of intersection of θ, ϕ, and thus the proper transformation so obtained is not the general proper transformation.

2 *Stone Buildings, January* 11, 1854.

128.

DEVELOPMENTS ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Magazine*, vol. VII. (1854), pp. 339—345.]

I PROPOSE to develope some particular cases of the theorems given in my paper, "Correction of two Theorems relating to the Porism of the in-and-circumscribed Polygon" (*Phil. Mag.* vol. VI. (1853), [116]). The two theorems are as follows:

THEOREM. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, depends upon the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$; viz. if this square root be

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+F\xi^5+G\xi^6+H\xi^7+\ldots,$$

then for $n=3$, 5, 7, &c. respectively, the conditions are

$$|\,C\,|=0,\qquad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix}=0,\qquad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix}=0,\ \&c.;$$

and for $n=4$, 6, 8, &c. respectively, the conditions are

$$|\,D\,|=0,\qquad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0,\qquad \begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix}=0,\ \&c.$$

THEOREM. In the case where the conics are replaced by the two circles

$$x^2+y^2-R^2=0,\quad (x-a)^2+y^2-r^2=0,$$

then the discriminant, the square root of which gives the series

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.,$$

is

$$(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\}.$$

Write for a moment

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=\sqrt{(1+a\xi)(1+b\xi)(1+c\xi)},$$

then

$$\begin{aligned}
A&=1,\\
2B&=a+b+c,\\
-8C&=a^2+b^2+c^2-2bc-2ca-2ab,\\
16D&=a^3+b^3+c^3-a^2(b+c)-b^2(c+a)-c^2(a+b)+2abc,\\
-128E&=5a^4+5b^4+5c^4-4a^3(b+c)-4b^3(c+a)-4c^3(a+b)\\
&\qquad+4a^2bc+4b^2ca+4c^2ab-2b^2c^2-2c^2a^2-2a^2b^2,\\
&\qquad\&c.
\end{aligned}$$

To adapt these to the case of the two circles, we have to write

$$r^2(1+a\xi)(1+b\xi)(1+c\xi)=(1+\xi)\{r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\},$$

and therefore

$$\begin{aligned}
c&=1,\\
r^2(a+b)&=r^2+R^2-a^2,\\
r^2ab&=R^2;
\end{aligned}$$

values which after some reductions give

$$\begin{aligned}
A&=1,\\
r^2\,.\,2B&=2r^2+R^2-a^2,\\
-r^4\,.\,8C&=(R^2-a^2)^2-4R^2r^2,\\
r^6\,.\,16D&=(R^2-a^2)\{(R^2-a^2)^2-2r^2(R^2+a^2)\},\\
-r^8\,.\,128E&=5(R^2-a^2)^4-8(R^2-a^2)^2(R^2+2r^2)r^2+16a^4r^4.
\end{aligned}$$

Hence also

$$\begin{aligned}
r^{12}\,.\,1024(CE-D^2)=\{&5(R^2-a^2)^4-8(R^2-a^2)^2(R^2+2r^2)r^2+16a^4r^4\}\{(R^2-a^2)^2-4R^2r^2)\}\\
&-4\{(R^2-a^2)^3-2(R^2-a^2)(R^2+a^2)r^2\}^2,
\end{aligned}$$

which after all reductions is

$$\begin{aligned}&(R^2-a^2)^6\\&-12R^2(R^2-a^2)^4r^2\\&+16R^2(R^2+2a^2)(R^2-a^2)^2r^4\\&-64R^2a^4r^6.\end{aligned}$$

Hence the condition that there may be, inscribed in the circle $x^2+y^2-R^2=0$ and circumscribed about the circle $(x-a)^2+y^2-r^2=0$, an infinity of n-gons, is for $n=3$, 4, 5, i.e. in the case of a triangle, a quadrangle and a pentagon respectively, as follows.

I. For the triangle, the relation is

$$(R^2-a^2)^2-4R^2r^2=0,$$

which is the completely rationalized form (the simple power of a radius being of course analytically a radical) of the well-known equation

$$a^2=R^2-2Rr,$$

which expresses the relation between the radii R, r of the circumscribed and inscribed circles, and the distance a between their centres.

II. For the quadrangle, the relation is

$$(R^2-a^2)^2-2r^2(R^2+a^2)=0,$$

which may also be written

$$(R+r+a)(R+r-a)(R-r+a)(R-r-a)-r^4=0.$$

(Steiner, *Crelle*, t. II. [1827] p. 289.)

III. For the pentagon, the relation is

$$(R^2-a^2)^6-12R^2(R^2-a^2)^4r^2+16R^2(R^2+2A^2)(R^2-a^2)^2r^4-64R^2a^4r^6=0,$$

which may also be written

$$(R^2-a^2)^2\{(R^2-a^2)^2-4R^2r^2\}^2-4R^2r^2\{(R^2-a^2)^2-4a^2r^2\}^2=0.$$

The equation may therefore be considered as the completely rationalized form of

$$(R^2-a^2)^2+2R(R^2-a^2)^2r-4R^2(R^2-a^2)r^2-8Ra^2r^3=0.$$

This is, in fact, the form given by Fuss in his memoir "De polygonis symmetrice irregularibus circulo simul inscriptis et circumscriptis," *Nova Acta Petrop.* t. XIII. [1802] pp. 166—189 (I quote from Jacobi's memoir, to be presently referred to). Fuss puts $R+a=p$, $R-a=q$, and he finds the equation

$$\frac{p^2q^2-r^2(p^2+q^2)}{r^2q^2-p^2(r^2+q^2)}=\pm\sqrt{\frac{q-r}{q+p}},$$

which, he remarks, is satisfied by $r=-p$ and $r=\frac{pq}{p+q}$, and that consequently the rationalized equation will divide by $p+r$ and $pq-r(p+q)$; and he finds, after the division,

$$p^2q^3+p^2q^2(p+q)r-pq(p+q)^2r^2-(p+q)(p-q)^2r^3=0,$$

which, restoring for p, q their values $R+a$, $R-a$, is the very equation above found.

The form given by Steiner (*Crelle*, t. II. p. 289) is

$$r(R-a)=(R+a)\sqrt{(R-r+a)(R-r-a)}+(R+a)\sqrt{(R-r-a)\,2R},$$

which, putting p, q instead of $R+a$, $R-a$, is

$$qr=p\sqrt{(p-r)(q-r)}+p\sqrt{(q-r)(q+p)};$$

and Jacobi has shown in his memoir, "Anwendung der elliptischen Transcendenten u. s. w.," *Crelle*, t. III. [1828] p. 376, that the rationalized equation divides (like that of Fuss) by the factor $pq-(p+q)r$, and becomes by that means identical with the rational equation given by Fuss.

In the case of two concentric circles $a=0$, and putting for greater simplicity $\frac{R^2}{r^2}=M$, we have

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=(1+\xi)\sqrt{1+M\xi}.$$

This is, in fact, the very formula which corresponds to the general case of two conics having double contact. For suppose that the polygon is inscribed in the conic $U=0$, and circumscribed about the conic $U+P^2=0$, we have then to find the discriminant of $\xi U+U+P^2$, i.e. of $(1+\xi)U+P^2$. Let K be the discriminant of U, and let F be what the polar reciprocal of U becomes when the variables are replaced by the coefficients of P, or, what is the same thing, let $-F$ be the determinant obtained by bordering K (considered as a matrix) with the coefficients of P. The discriminant of $(1+\xi)U+P^2$ is $(1+\xi)^3K+(1+\xi)^2F$, i.e. it is

$$(1+\xi)^2\{K(1+\xi)+F\},\ =(K+F)(1+\xi)^2(1+M\xi),$$

where $M=\frac{K}{K+F}$; or, what is the same thing, M is the discriminant of U divided by the discriminant of $U+P^2$. And M having this meaning, the condition of there being inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $U+P^2=0$, is found by means of the series

$$A+B\xi+C\xi^2+D\xi^3+E\xi^4+\&c.=(1+\xi)\sqrt{1+M\xi}.$$

We have, therefore,

$$\begin{aligned} A &= 1, \\ 2B &= M+2, \\ -8C &= M^2-4M, \\ 16D &= M^3-2M^2, \\ -128E &= 5M^4-8M^3, \\ &\&c. \\ 1024(CE-D^2) &= M^4(M^2-12M+16), \\ &\&c. \end{aligned}$$

Hence for the triangle, quadrangle and pentagon, the conditions are—

I. For the triangle,

$$M+2=0.$$

II. For the quadrangle,

$$M-4=0.$$

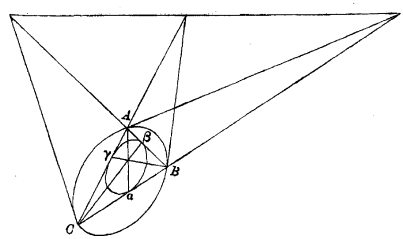
III. For the pentagon,

$$M^2-12M+16=0;$$

and so on.

It is worth noticing, that, in the case of two conics having a 4-point contact, we have $F=0$, and consequently $M=1$. The discriminant is therefore $(1+\xi)^3$, and as this does not contain any variable parameter, the conics cannot be determined so that there may be for a given value of n (nor, indeed, for any value whatever of n) an infinity of n-gons inscribed in the one conic, and circumscribed about the other conic.

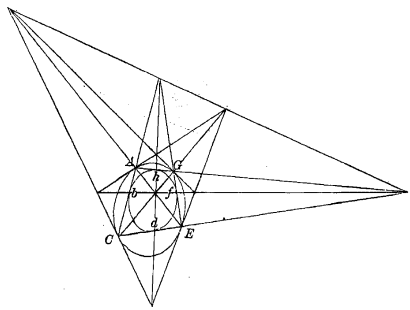
The geometrical properties of a triangle, &c. inscribed in a conic and circumscribed about another conic, these two conics having double contact with each other,



are at once obtained from those of the system in which the two conics are replaced

by concentric circles. Thus, in the case of a triangle, if ABC be the triangle, and α, β, γ be the points of contact of the sides with the inscribed conic, then the tangents to the circumscribed conic at A, B, C meet the opposite sides BC, CA, AB in points lying in the chord of contact, the lines $A\alpha$, $B\beta$, $C\gamma$ meet in the pole of contact, and so on.

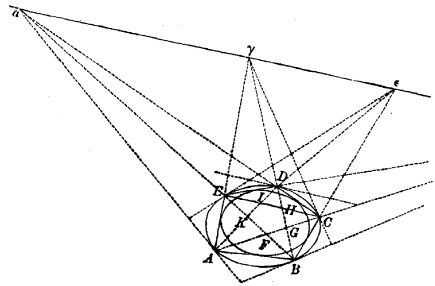
In the case of a quadrangle, if $ACEG$ be the quadrangle, and b, d, f, h the points of contact with the inscribed conic, then the tangents to the circumscribed



conic at the pair of opposite angles A, E and the corresponding diagonal CG, and in like manner the tangents at the pair of opposite angles C, G and the corresponding diagonal AE, meet in the chord of contact. Again, the pairs of opposite sides AC, EG, and the line dh joining the points of contact of the other two sides with the inscribed conic, and the pairs of opposite sides AG, CE, and the line bf joining the points of contact of the other two sides with the inscribed conic, meet in the chord of contact. The diagonals AE, CG, and the lines bf, dh through the points of contact of pairs of opposite sides with the inscribed conic, meet in the pole of contact, &c.

The beautiful systems of 'focal relations' for regular polygons (in particular for the pentagon and the hexagon), given in Sir W. R. Hamilton's Lectures on Quaternions, [Dublin, 1853] Nos. 379—393, belong, it is clear, to polygons which are inscribed in and circumscribed about two conics having double contact with each other. In fact, the focus of a conic is a point such that the lines joining such point with the circular points at infinity (i.e. the points in which a circle is intersected by the line infinity) are tangents to the conic. In the case of two concentric circles, these are to be considered as touching in the circular points at infinity; and consequently, when the concentric circles are replaced by two conics having double contact, the circular points at infinity are replaced by the points of contact of the two conics.

Thus, in the figure (which is simply Sir W. R. Hamilton's figure 81 put into



perspective), the system of relations

$$F,\ G\,(..)\,ABCI,$$
$$G,\ H\,(..)\,BCDK,$$
$$H,\ I\,(..)\,CDEF,$$
$$I,\ K\,(..)\,DEAG,$$
$$K,\ F\,(..)\,EABH,$$

will mean, $F,\ G\,(..)\,ABCI$, that there is a conic inscribed in the quadrilateral $ABCI$ such that the tangents to this conic through the points F and G pass two and two through the points of contact of the circumscribed and the inscribed conics, and similarly for the other relations of the system. As the figure is drawn, the tangents in question are of course (as the tangents through the foci in the case of the two concentric circles) imaginary.

2 *Stone Buildings, March* 7, 1854.

129.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE, AND ON AN IRRATIONAL TRANSFORMATION OF TWO TERNARY QUADRATIC FORMS EACH INTO ITSELF.

[From the *Philosophical Magazine*, vol. IX. (1855), pp. 513—517.]

THERE is an irrational transformation of two ternary quadratic forms each into itself, based upon the solution of the following geometrical problem,

Given that the line

$$lx + my + nz = 0$$

meets the conic

$$(a, b, c, f, g, h \between x, y, z)^2 = 0$$

in the point (x_1, y_1, z_1); to find the other point of intersection.

The solution is exceedingly simple. Take (x_2, y_2, z_2) for the coordinates of the other point of intersection, we must have identically with respect to x, y, z,

$$\begin{aligned}(a, \ldots \between x, y, z)^2 . (\mathfrak{A}, \ldots \between l, m, n)^2 - k\,(lx + my + nz)^2 \\ = (a, \ldots \between x_1, y_1, z_1 \between x, y, z) . (a, \ldots \between x_2, y_2, z_2 \between x, y, z)\end{aligned}$$

to a constant factor *près*.

Assume successively $x, y, z = \mathfrak{A}, \mathfrak{H}, \mathfrak{G}$; $\mathfrak{H}, \mathfrak{B}, \mathfrak{F}$; $\mathfrak{G}, \mathfrak{F}, \mathfrak{C}$; it follows that

$$\begin{aligned}x_2 : y_2 : z_2 = y_1 z_1 &\{\mathfrak{A}\,(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{A}l + \mathfrak{H}m + \mathfrak{G}n)^2\} \\ : z_1 x_1 &\{\mathfrak{B}\,(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{H}l + \mathfrak{B}m + \mathfrak{F}n)^2\} \\ : x_1 y_1 &\{\mathfrak{C}\,(\mathfrak{A}, \ldots \between l, m, n)^2 - (\mathfrak{G}l + \mathfrak{F}m + \mathfrak{C}n)^2\};\end{aligned}$$

or, what is the same thing,

$$\begin{aligned} x_2 : y_2 : z_2 = \; & y_1 z_1 (bn^2 + cm^2 - 2fmn) \\ : \; & z_1 x_1 (cl^2 + an^2 - 2gnl) \\ : \; & x_1 y_1 (am^2 + bn^2 - 2hlm). \end{aligned}$$

It is not necessary for the present purpose, but it may be as well to give the corresponding solution of the problem:

Given that one of the tangents through the point (ξ, η, ζ) to the conic

$$(a, b, c, f, g, h \between x, y, z)^2 = 0$$

is the line $l_1 x + m_1 y + n_1 z = 0$; to find the equation to the other tangent.

Let $l_2 x + m_2 y + n_2 z = 0$ be the other tangent, then

$$\begin{aligned} (a, \ldots \between \xi, \eta, \zeta)^2 . (a, \ldots \between x, y, z)^2 - \{(a \ldots \between \xi, \eta, \zeta \between x, y, z)\}^2 \\ = (l_1 x + m_1 y + n_1 z)(l_2 x + m_2 y + n_2 z) \end{aligned}$$

to a constant factor *près*. Assume successively $y = 0$, $z = 0$; $z = 0$, $x = 0$; $x = 0$, $y = 0$; then we have

$$\begin{aligned} l_2 : m_2 : n_2 = \; & m_1 n_1 \{a (a, \ldots \between \xi, \eta, \zeta)^2 - (a\xi + h\eta + g\zeta)^2\} \\ : \; & n_1 l_1 \{b (a, \ldots \between \xi, \eta, \zeta)^2 - (h\xi + b\eta + f\zeta)^2\} \\ : \; & l_1 m_1 \{c (a, \ldots \between \xi, \eta, \zeta)^2 - (g\xi + f\eta + c\zeta)^2\}; \end{aligned}$$

or, as they may be more simply written,

$$\begin{aligned} l_2 : m_2 : n_2 = \; & m_1 n_1 (\mathfrak{B}\zeta^2 + \mathfrak{C}\eta^2 + 2\mathfrak{F}\eta\zeta) \\ : \; & n_1 l_1 (\mathfrak{C}\xi^2 + \mathfrak{A}\zeta^2 - 2\mathfrak{G}\zeta\xi) \\ : \; & l_1 m_1 (\mathfrak{A}\eta^2 + \mathfrak{B}\xi^2 - 2\mathfrak{H}\xi\eta). \end{aligned}$$

Returning now to the solution of the first problem, I shall for the sake of simplicity consider the formulæ obtained by taking for the equation of the conic,

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0.$$

We see, therefore, that if this conic be intersected by the line $lx + my + nz = 0$ in the points (x_1, y_1, z_1) and (x_2, y_2, z_2), then

$$\begin{aligned} x_2 : y_2 : z_2 = \; & y_1 z_1 (\gamma m^2 + \alpha n^2) \\ : \; & z_1 x_1 (\alpha n^2 + \beta l^2) \\ : \; & x_1 y_1 (\beta l^2 + \alpha m^2). \end{aligned}$$

We have, in fact, *identically*

$$ly_1z_1(\beta n^2+\gamma m^2)+mz_1x_1(\gamma l^2+\alpha n^2)+nx_1y_1(\alpha m^2+\beta l^2)$$
$$=(\alpha mnx_1+\beta nly_1+\gamma lmz_1)(lx_1+my_1+nz_1)-lmn(\alpha x_1^2+\beta y_1^2+\gamma z_1^2),$$

$$\alpha y_1^2z_1^2(\beta n^2+\gamma m^2)^2+\beta z_1^2x_1^2(\gamma l^2+\alpha n^2)^2+\gamma x_1^2y_1^2(\alpha m^2+\beta l^2)^2$$
$$=\alpha\beta\gamma\{-l^2x_1^2-m^2y_1^2-n^2z_1^2$$
$$+(my_1+nz_1)l^2x_1^2+(nz_1+lx_1)m^2y_1^2+(lx_1+my_1)n^2y_1^2-2lmnx_1y_1z_1\}(lx_1+my_1+nz_1)$$
$$-(l^4\beta\gamma x_1^2+m^4\gamma\alpha y_1^2+n^4\alpha\beta z_1^2)(\alpha x_1^2+\beta y_1^2+\gamma z_1^2);$$

which show that if $lx_1+my_1+nz_1=0$ and $\alpha x_1^2+\beta y_1^2+\gamma_1 z^2=0$, then also $lx_2+my_2+nz_2=0$ and $\alpha x_2^2+\beta y_2^2+\gamma z_2^2=0$: this is, of course, as it should be.

I shall now consider l, m, n as *given functions* of x_1, y_1, z_1 satisfying identically the equations

$$lx_1+my_1+nz_1=0,$$
$$l^2bc+m^2ca+n^2ab=0,$$

equations which express that $lx+my+nz=0$ is the tangent from the point (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$. And I shall take for α, β, γ the following values, viz.

$$\alpha=ax_1^2+by_1^2+cz_1^2-a(x_1^2+y_1^2+z_1^2),$$
$$\beta=ax_1^2+by_1^2+cz_1^2-b(x_1^2+y_1^2+z_1^2),$$
$$\gamma=ax_1^2+by_1^2+cz_1^2-c(x_1^2+y_1^2+z_1^2);$$

so that x_1, y_1, z_1 continuing absolutely indeterminate, we have *identically* $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$. Also taking Θ as a function of x_1, y_1, z_1, the value of which will be subsequently given, I write

$$x_2=\Theta y_1z_1(\beta n^2+\gamma m^2),$$
$$y_2=\Theta z_1x_1(\gamma l^2+\alpha n^2),$$
$$z_2=\Theta x_1y_1(\alpha m^2+\beta l^2);$$

so that x_1, y_1, z_1 are arbitrary, and x_2, y_2, z_2 are taken to be determinate functions of x_1, y_1, z_1. The point (x_2, y_2, z_2) is geometrically connected with the point (x_1, y_1, z_1) as follows, viz. (x_2, y_2, z_2) is the point in which the tangent through (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$ meets the conic passing through the point (x_1, y_1, z_1) and the points of intersection of the conics $ax^2+by^2+cz^2=0$ and $x^2+y^2+z^2=0$. Consequently, in the particular case in which (x_1, y_1, z_1) is a point on the conic $x^2+y^2+z^2=0$, the point (x_2, y_2, z_2) is the point in which this conic is met by the tangent through (x_1, y_1, z_1) to the conic $ax^2+by^2+cz^2=0$.

It has already been seen that $lx_1+my_1+nz_1=0$ and $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$ identically; consequently we have identically $lx_2+my_2+nz_2=0$ and $\alpha x_2^2+\beta y_2^2+\gamma z_2^2=0$. The latter equation, written under the form

$$(ax_1^2+by_1^2+cz_1^2)(x_2^2+y_2^2+z_2^2)-(x_1^2+y_1^2+z_1^2)(ax_2^2+by_2^2+cz_2^2)=0,$$

shows that if x_2, y_2, z_2 are such that $x_2^2+y_2^2+z_2^2=x_1^2+y_1^2+z_1^2$, then that also $ax_2^2+by_2^2+cz_2^2=ax_1^2+by_1^2+cz_1^2$. I proceed to determine Θ so that we may have $x_2^2+y_2^2+z_2^2=x_1^2+y_1^2+z_1^2$. We obtain immediately

$$\frac{1}{\Theta^2}(x_2^2+y_2^2+z_2^2)=(l^4x_1^2+m^4y_1^2+n^4z_1^2)(\alpha^2x_1^2+\beta^2y_1^2+\gamma^2z_1^2)$$
$$-(\alpha^2l^4x_1^4+\beta^2m^4y_1^4+\gamma^2n^4z_1^4-2\beta\gamma m^2n^2y_1^2z_1^2-2\gamma\alpha n^2l^2z_1^2x_1^2-2\alpha\beta l^2m^2x_1^2y_1^2);$$

write for a moment

$$ax_1^2+by_1^2+cz_1^2=p,\ x_1^2+y_1^2+z_1^2=q,\ \text{so that}\ \alpha=p-aq,\ \beta=p-bq,\ \gamma=p-cq,$$

then

$$\alpha^2x_1^2+\beta^2y_1^2+\gamma^2z_1^2=qp^2-2p\,.\,pq+(a^2x_1^2+b^2y_1^2+c^2z_1^2)\,q^2,=q\,\{(a^2x_1^2+b^2y_1^2+c^2z_1^2)\,q-p^2\},$$
$$=q\,\{(b-c)^2\,y_1^2z_1^2+(c-a)^2\,z_1^2x_1^2+(a-b)^2\,x_1^2y_1^2\},$$

$$\alpha^2l^4x_1^4+\beta^2m^4y_1^4+\gamma^4n^4z_1^4-2\beta\gamma m^2n^2y_1^2z_1^2-2\gamma\alpha n^2l^2z_1^2x_1^2-2\alpha\beta l^2m^2x_1^2y_1^2$$
$$=\quad p^2\,\{l^4x_1^4+m^4y_1^4+n^4z_1^4-2m^2n^2y_1^2z_1^2-2n^2l^2z_1^2x_1^2-2l^2m^2x_1^2y_1^2\}$$
$$-2pq\,\{al^4x_1^4+bm^4y_1^4+cn^4z_1^4-(b+c)\,m^2n^2y_1^2z_1^2-(c+a)\,n^2l^2z_1^2x_1^2-(a+b)\,l^2m^2x_1^2y_1^2\}$$
$$+\quad q^2\,\{a^2l^4x_1^4+b^2m^4y_1^4+c^2n^4z_1^4-2bcm^2n^2y_1^2z_1^2-2can^2l^2z_1^2x_1^2-2abl^2m^2x_1^2y_1^2\},$$

the first line of which vanishes in virtue of the equation $lx_1+my_1+nz_1=0$; we have therefore

$$\frac{1}{\Theta^2}(x_2^2+y_2^2+z_2^2)\div(x_1^2+y_1^2+z_1^2)$$
$$=(l^4x_1^2+m^4y_1^2+n^4z_1^2)\,\{(b-c)^2\,y_1^2z_1^2+(c-a)^2\,z_1^2x_1^2+(a-b)^2\,x_1^2y_1^2\}$$
$$+2\,(ax_1^2+by_1^2+cz_1^2)\,\{al^4x_1^4+bm^4y_1^4+cn^4z_1^4-(b+c)\,m^2n^2y_1^2z_1^2-(c+a)\,n^2l^2z_1^2x_1^2-(a+b)\,l^2m^2x_1^2y_1^2\}$$
$$-(x_1^2+y_1^2+z_1^2)\,\{a^2l^4x_1^4+b^2m^4y_1^4+c^2n^4z_1^4-2bcm^2n^2y_1^2z_1^2-2can^2l^2z_1^2x_1^2-2abl^2m^2x_1^2y_1^2\}.$$

Hence reducing the function on the right-hand side, and putting

$$(x_2^2+y_2^2+z_2^2)\div(x_1^2+y_1^2+z_1^2)=1,$$

we have

$$\frac{1}{\Theta^2}=a^2l^4x_1^6+b^2m^4y_1^6+c^2n^4z_1^6$$
$$+(c^2m^4-2b^2m^2n^2)\,y_1^4z_1^2+(a^2n^4-2c^2n^2l^2)\,z_1^4x_1^2+(b^2l^4-2a^2l^2m^2)\,x_1^4y_1^2$$
$$+(b^2n^4-2c^2m^2n^2)\,y_1^2z_1^4+(c^2l^4-2a^2n^2l^2)\,z_1^2x_1^4+(a^2m^4-2b^2l^2m^2)\,x_1^2y_1^4$$
$$+\{l^4\,(b-c)^2+m^4\,(c-a)^2+n^4\,(a-b)^2$$
$$+2m^2n^2\,(bc-ca-ab)+2n^2l^2\,(-bc+ca-ab)+2l^2m^2\,(-bc-ca+ab)\}\,x_1^2y_1^2z_1^2.$$

The value of Θ might probably be expressed in a more simple form by means of the equations $lx_1+my_1+nz_1=0$ and $l^2bc+m^2ca+n^2ab=0$, even without solving these equations; but this I shall not at present inquire into.

Recapitulating, l, m, n are considered as functions of x_1, y_1, z_1 determined (to a common factor *près*) by the equations

$$lx_1 + my_1 + nz_1 = 0,$$
$$l^2bc + m^2ca + n^2ab = 0;$$

Θ is determined as above, and then writing

$$\alpha = ax_1^2 + by_1^2 + cz_1^2 - a(x_1^2 + y_1^2 + z_1^2),$$
$$\beta = ax_1^2 + by_1^2 + cz_1^2 - b(x_1^2 + y_1^2 + z_1^2),$$
$$\gamma = ax_1^2 + by_1^2 + cz_1^2 - c(x_1^2 + y_1^2 + z_1^2),$$

we have

$$x_2 = \Theta y_1 z_1 (\beta n^2 + \gamma m^2),$$
$$y_2 = \Theta z_1 x_1 (\gamma l^2 + \alpha n^2),$$
$$z_2 = \Theta x_1 y_1 (\alpha m^2 + \beta l^2);$$

and these values give

$$lx_2 + my_2 + nz_2 = 0,$$
$$x_2^2 + y_2^2 + z_2^2 = x_1^2 + y_1^2 + z_1^2,$$
$$ax_2^2 + by_2^2 + cz_2^2 = ax_1^2 + by_1^2 + cz_1^2.$$

In connexion with the subject I may add the following transformation, viz. if

$$3\sqrt{\alpha}\, x' = \sqrt{3\beta}\,(y - z) + \sqrt{(3\alpha - 2\beta)(x^2 + y^2 + z^2) + 2\beta(yz + zx + xy)},$$
$$\vdots$$

then reciprocally

$$3\sqrt{\beta}\, x = -\sqrt{3\alpha}\,(y' - z') + \sqrt{(3\beta - 2\alpha)(x'^2 + y'^2 + z'^2) + 2\alpha(y'z' + z'x' + x'y')}.$$
$$\vdots$$
$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$
$$\beta(x^2 + y^2 + z^2 - yz - zx - xy) = \alpha(x'^2 + y'^2 + z'^2 - y'z' - z'x' - x'y').$$

Suppose $1 + \rho + \rho^2 = 0$, then

$$x^2 + y^2 + z^2 - yz - zx - xy = (x + \rho y + \rho^2 z)(x + \rho^2 + \rho z);$$

and in fact

$$3\sqrt{\alpha}\,(x' + \rho y' + \rho^2 z') = -\sqrt{3\beta}\,(1 + 2\rho)(x + \rho y + \rho^2 z),$$
$$3\sqrt{\alpha}\,(x' + \rho^2 y' + \rho z) = \sqrt{3\beta}\,(1 + 2\rho)(x + \rho^2 y + \rho z).$$

The preceding investigations have been in my possession for about eighteen months.

2 *Stone Buildings, April* 18, 1855.

130.

DEUXIÈME MÉMOIRE SUR LES FONCTIONS DOUBLEMENT PÉRIODIQUES.

[From the *Journal de Mathématiques Pures et Appliquées* (Liouville), tom. XIX. (1854), pp. 193—208: Sequel to Memoir t. X. (1845), **25.**]

JE vais essayer de développer ici les propriétés qui se rapportent aux transformations linéaires des périodes des fonctions yx, gx, Gx, Zx, dont je me suis occupé dans le Mémoire sur les fonctions doublement périodiques que j'ai donné dans ce Recueil en 1845. Avant d'entrer en matière, je remarque que partant des expressions

$$\Omega = \omega + \omega' i,$$
$$\Upsilon = \upsilon + \upsilon' i,$$

des deux périodes, où $i = \sqrt{-1}$, on obtient, en écrivant

$$\Omega^* = \omega - \omega' i,$$
$$\Upsilon^* = \upsilon - \upsilon' i,$$

les équations

$$\Omega\,\Upsilon^* = \omega\upsilon + \omega'\upsilon' - i(\omega\upsilon' - \omega'\upsilon),$$
$$\Omega^*\Upsilon = \omega\upsilon + \omega'\upsilon' + i(\omega\upsilon' - \omega'\upsilon),$$

au moyen desquelles et des valeurs

$$\beta = \frac{\pi i(\omega\upsilon' - \omega'\upsilon)}{\Omega\Upsilon \bmod. (\omega\upsilon' - \omega'\upsilon)}, \qquad B = \frac{\pi(\omega\upsilon + \omega'\upsilon')}{\Omega\Upsilon \bmod. (\omega\upsilon' - \omega'\upsilon)}$$

des quantités β, B, on déduit les formules

$$B + \beta = \frac{\pi\Omega^*}{\Omega \bmod. (\omega\upsilon' - \omega'\upsilon)}, \qquad B - \beta = \frac{\pi\Upsilon^*}{\Upsilon \bmod. (\omega\upsilon' - \omega'\upsilon)}.$$

Je ne fais attention qu'aux transformations qui correspondent à des entiers impairs et premiers, et je suppose, de plus, que la transformation soit toujours propre et régulière; c'est-à-dire qu'en écrivant

$$(2k+1)\,\Omega_{,} = \lambda\Omega + \mu\Upsilon = (\lambda,\ \mu),$$
$$(2k+1)\,\Upsilon_{,} = \nu\Omega + \rho\Upsilon = (\nu,\ \rho),$$

où $2k+1$ est un entier positif, impair et premier, et où λ, μ, ν, ρ sont des entiers tels, qu'au signe près, $\lambda\rho - \mu\nu$ soit égal à $2k+1$, je suppose

$$\lambda\rho - \mu\nu = 2k+1,$$

(condition pour que la transformation soit propre), et, en outre,

$$\lambda \equiv 1, \quad \mu \equiv 0, \ (\text{mod. } 2)$$
$$\nu \equiv 0, \quad \rho \equiv 1,$$

(condition pour que la transformation soit régulière).

On trouve tout de suite

$$\Omega = \ \rho\Omega_{,} - \mu\Upsilon_{,} = (\rho,\ -\mu)_{,},$$
$$\Upsilon = -\nu\Omega_{,} + \lambda\Upsilon_{,} = (-\nu,\ \lambda)_{,};$$

j'écris aussi

$$\Omega_{,} = \omega_{,} + \omega_{,}'i, \quad \Omega_{,}^{*} = \omega_{,} - \omega_{,}'i,$$
$$\Upsilon_{,} = v_{,} + v_{,}'i, \quad \Upsilon_{,}^{*} = v_{,} - v_{,}'i,$$

et je suppose que $B_{,}$, $\beta_{,}$ soient des fonctions de $\omega_{,}$, $v_{,}$ telles que les fonctions B, β, de ω, v.

Cela étant, je forme d'abord l'équation

$$(2k+1)\,(\omega_{,}v_{,}' - \omega_{,}'v_{,}) = \omega v' - \omega' v,$$

au moyen de laquelle l'équation

$$(B+\beta)\,\Omega = \frac{\pi}{\text{mod.}\,(\omega v' - \omega' v)}\,\Omega^{*}$$

se transforme en

$$(2k+1)(B+\beta)\,\Omega = \frac{\pi}{\text{mod. }(\omega_{,}v_{,}' - \omega_{,}'v_{,})}\,\Omega^{*}.$$

De là

$$\{(2k+1)(B+\beta) - B_{,}\}\,\Omega = \frac{\pi}{\text{mod. }(\omega_{,}v_{,}' - \omega_{,}'v_{,})}\left\{\Omega^{*} - \Omega\,\frac{\omega_{,}v_{,} + \omega_{,}'v_{,}'}{\Omega_{,}\Upsilon_{,}}\right\},$$
$$= \frac{\pi}{\text{mod.}\,(\omega_{,}v_{,}' - \omega_{,}'v_{,})}\left\{\rho\Omega_{,}^{*} - \mu\Upsilon_{,}^{*} - \frac{\omega_{,}v_{,} + \omega_{,}'v_{,}'}{\Omega_{,}\Upsilon_{,}}\,(\rho\Omega_{,} - \mu\Upsilon_{,})\right\},$$
$$= \frac{\pi}{\text{mod.}\,(\omega_{,}v_{,}' - \omega_{,}'v_{,})}\left\{\rho\left(\Omega_{,}^{*} - \frac{\omega_{,}v_{,} + \omega_{,}'v_{,}'}{\Upsilon_{,}}\right) - \mu\left(\Upsilon_{,}^{*} - \frac{\omega_{,}v_{,} + \omega_{,}'v_{,}'}{\Omega_{,}}\right)\right\},$$
$$= \frac{\pi i\,(\omega_{,}v_{,}' - \omega_{,}'v_{,})}{\Omega_{,}\Upsilon_{,}\,\text{mod.}\,(\omega_{,}v_{,}' - \omega_{,}'v_{,})}\,(\rho\Omega_{,} + \mu\Upsilon_{,}),$$

ou enfin

$$\{\overline{2k+1}\,(B+\beta)-B_{,}\}\,\Omega = \quad \beta_{,}(\rho,\ \mu)_{,};$$

et de même

$$\{\overline{2k+1}\,(B-\beta)-B_{,}\}\,\Omega = -\beta_{,}(\nu,\ \lambda);$$

équations qui seront bientôt utiles.

Je suppose d'abord que $2k+1$ soit égal à l'unité, transformation que l'on peut nommer *triviale*. La fonction yx est définie par l'équation

$$yx = \epsilon^{-\frac{1}{2}Bx^2}\,x\Pi\left\{1+\frac{x}{(m,\ n)}\right\},\quad \text{mod.}\,(m,\ n) < T,\ T=\infty;$$

dans $(m,\ n) = m\Omega + n\Upsilon$, les entiers m, n doivent prendre toutes les valeurs positives ou négatives (le seul système $m=0$, $n=0$ excepté) qui satisfont à l'inégalité

$$\text{mod.}\,(m,\ n) < T,$$

dont le second membre T sera ensuite supposé infini. Soit $y_{,}x$ la fonction correspondante pour les périodes $\Omega_{,}$, $\Upsilon_{,}$; on aura

$$y_{,}\,x = \epsilon^{-\frac{1}{2}B_{,}x^2}\,x\Pi\left\{1+\frac{x}{(m,\ n)_{,}}\right\},\quad \text{mod.}\,(m,\ n)_{,} < T,\quad T=\infty.$$

Or

$$\begin{aligned}(m,\ n)_{,} &= m\Omega_{,} + n\Upsilon_{,},\\ &= m\,(\lambda\Omega+\mu\Upsilon)+n\,(\nu\Omega+\rho\Upsilon),\\ &= (\lambda m+\nu m)\,\Omega+(\mu m+\rho n)\,\Upsilon,\\ &= m_{,}\Omega+n_{,}\Upsilon,\\ &= (m_{,},\ n_{,}).\end{aligned}$$

En écrivant, comme nous venons de le faire,

$$m_{,} = \lambda m + \nu n,$$

$$n_{,} = \nu m + \rho n,$$

on voit tout de suite qu'à chaque système de valeurs entières de m, n, correspond un système, et un seul système, de valeurs entières de $m_{,}$, $n_{,}$; et que de même à chaque système de valeurs entières de $m_{,}$, $n_{,}$, correspond un système, et un seul système, de valeurs entières de m, n; de plus, les systèmes $m=0$, $n=0$ et $m_{,}=0$, $n_{,}=0$, correspondent l'un à l'autre. Il est donc permis d'écrire

$$x\Pi\left\{1+\frac{x}{(m,\ n)}\right\} = x\Pi\left\{1+\frac{x}{(m,\ n)_{,}}\right\},$$

les limites comme auparavant; car, à cause de

$$(m,\ n)_{,} = (m_{,},\ n_{,}),$$

la condition pour les limites, savoir :

$$\text{mod.}\,(m,\ n)_{,} < T,\ T = \infty,$$

devient

$$\text{mod.}\,(m,\ n) < T,\ T = \infty.$$

Cela donne enfin l'équation

$$\mathrm{y}_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-B)x^2}\,\mathrm{y}x;$$

et, au moyen de cette équation, on obtient une équation correspondante pour la transformation de l'une quelconque des fonctions $\mathrm{y}x$, $\mathrm{g}x$, Gx, Zx, définies par les équations

$$\mathrm{y}x = \epsilon^{-\frac{1}{2}Bx^2}.\,x\Pi\left\{1 + \frac{x}{(m,\ n)}\right\},\quad \text{mod.}\,(m,\ n) < T,$$

$$\mathrm{g}x = \epsilon^{-\frac{1}{2}Bx^2}.\ \Pi\left\{1 + \frac{x}{(\overline{m},\ n)}\right\},\quad \text{mod.}\,(\overline{m},\ n) < T,\quad T = \infty;$$

$$Gx = \epsilon^{-\frac{1}{2}Bx^2}.\ \Pi\left\{1 + \frac{x}{(m,\ \overline{n})}\right\},\quad \text{mod.}\,(m,\ \overline{n}) < T,$$

$$Zx = \epsilon^{-\frac{1}{2}Bx^2}.\ \Pi\left\{1 + \frac{x}{(\overline{m},\ \overline{n})}\right\},\quad \text{mod.}\,(\overline{m},\ \overline{n}) < T,$$

(équations dans lesquelles $\overline{m} = m + \frac{1}{2}$, $\overline{n} = n + \frac{1}{2}$). Je prends par exemple la fonction $\mathrm{g}x$, et j'écris dans l'équation entre $\mathrm{y}_{,}x$ et $\mathrm{y}x$, $x + \frac{1}{2}\Omega$ au lieu de x. Soit pour un moment $\rho = 2\rho' + 1$, $\mu = 2\mu'$; cela donne

$$x + \tfrac{1}{2}\Omega = x + \tfrac{1}{2}(\rho\Omega_{,} - \mu\Upsilon_{,}) = x + (\overline{\rho}',\ -\mu').$$

Donc

$$\mathrm{y}_{,}(x + \tfrac{1}{2}\Omega) = \epsilon^{B_{,}x(\overline{\rho}',\ \mu')}.\,M_{,}\mathrm{g}_{,}x,$$

c'est-à-dire

$$\mathrm{y}_{,}(x + \tfrac{1}{2}\Omega) = \epsilon^{\frac{1}{2}B_{,}x(\rho,\ \mu)}.\,M_{,}\mathrm{g}_{,}x;$$

de plus,

$$\mathrm{y}\,(x + \tfrac{1}{2}\Omega) = \epsilon^{\frac{1}{2}B\Omega x}\,M\mathrm{g}x.$$

Ces substitutions étant effectuées, les coefficients M, $M_{,}$ doivent être éliminés en écrivant $x = 0$; cela donne

$$\mathrm{g}_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-B)x^2}\,\epsilon^{\frac{1}{2}x[(B+B_{,})\Omega - \frac{1}{2}B_{,}(\rho,\ \mu)_{,}]}\,\mathrm{g}x,$$

ou enfin, au moyen d'une équation déjà trouvée,

$$\mathrm{g}_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-B)x^2}\,\mathrm{g}x,$$

et de même pour les fonctions Gx, Zx.

Donc enfin, en représentant par Jx l'une quelconque des fonctions yx, gx, Gx, Zx, on aura

$$J_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-B)x^2} Jx,$$

où $J_{,}x$ est ce que devient Jx au moyen d'une transformation triviale (propre et régulière) des périodes.

Je passe à présent à la transformation pour un nombre impair et premier $(2k+1)$ quelconque; mais pour cela on a besoin de connaître la valeur de la fonction

$$u' = \Pi\left\{1 + \frac{x}{(m,\ n)+y}\right\}, \quad \text{mod. } \{(m,\ n)+y\} < T, \quad T = \infty,$$

où $y = a + bi$ est une quantité réelle ou imaginaire quelconque.

Soit u ce que devient u' en prenant pour la condition par rapport aux limites

$$\text{mod. } (m,\ n) < T, \quad T = \infty;$$

on trouve sans peine

$$u = \epsilon^{\frac{1}{2}Rx^2 + Rxy} \frac{\mathrm{y}\,(x+y)}{\mathrm{y}\,(y)}.$$

Pour trouver u', je forme l'équation

$$u : u' = \Pi\left\{1 + \frac{x}{(m,\ n)}\right\},$$

la limite inférieure du produit infini double étant

$$\text{mod. } \{(m,\ n)+y\} > T,$$

et la limite supérieure

$$\text{mod. } (m,\ n) < T, \quad T = \infty;$$

cela donne

$$\begin{aligned}
\log u - \log u' &= x \sum \frac{1}{(m,\ n)+y} - \tfrac{1}{2}\, x^2 \sum \frac{1}{\{(m,\ n)+y\}^2} + \dots, \\
&= x \sum \frac{1}{(m,\ n)} - \tfrac{1}{2}\,(x^2 + 2yx) \sum \frac{1}{(m,\ n)^2} + \dots, \\
&= x \sum \frac{1}{(m,\ n)};
\end{aligned}$$

car on peut démontrer que

$$\sum \frac{1}{(m,\ n)^2} = 0, \quad \sum \frac{1}{(m,\ n)^3} = 0, \ \&c.$$

Pour cela, observons que m et n étant infinis puisque T l'est, la première des sommes dont il s'agit peut se remplacer par l'intégrale double

$$I = \iint \frac{dm\,dn}{(m,\ n)^2},$$

laquelle (en écrivant $m = r\cos\theta$, $n = r\sin\theta$, ce qui donne, comme on sait, $dm\,dn = rdr\,d\theta$) devient

$$I = \iint \frac{dr\,d\theta}{r(\Omega\cos\theta + r\sin\theta)^2},$$

d'où

$$I = \iint \frac{(\log r)\,d\theta}{(\Omega\cos\theta + r\sin\theta)^2},$$

en prenant $(\log r)$ entre les limites convenables. Pour trouver ces limites, j'écris

$$(m,\ n) + y = r(\Omega\cos\theta + \Upsilon\sin\theta) + y;$$

ce qui donne

$$\text{mod.}^2\,\{(m,\ n) + y\} = \{r(\Omega\cos\theta + \Upsilon\sin\theta) + y\}\,\{r(\Omega^*\cos\theta + \Upsilon^*\sin\theta) + y^*\},$$

savoir, à l'une des limites

$$r^2(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega^*\cos\theta + \Upsilon^*\sin\theta)$$
$$+ r\{y^*(\Omega\cos\theta + \Upsilon\sin\theta) + y(\Omega^*\cos\theta + \Upsilon^*\sin\theta)\} + T^2 = 0;$$

ou, en négligeant les puissances négatives de T,

$$r = \frac{T}{\sqrt{(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega^*\cos\theta + \Upsilon^*\sin\theta)}}$$
$$-\tfrac{1}{2}\left\{\frac{y}{\Omega\cos\theta + \Upsilon\sin\theta} + \frac{y^*}{\Omega^*\cos\theta + \Upsilon^*\sin\theta}\right\},$$

et à l'autre limite,

$$r = \frac{T}{\sqrt{(\Omega\cos\theta + \Upsilon\sin\theta)(\Omega^*\cos\theta + \Upsilon^*\sin\theta)}}.$$

Or, en représentant ces deux équations par

$$r = R - \phi, \quad r = R,$$

on trouve, pour la valeur de $(\log r)$ entre les deux limites,

$$\log R - \log(R - \phi) = -\log\left(1 - \frac{\phi}{R}\right) = 0,$$

à cause de la valeur infinie de R. Ainsi la somme cherchée est nulle; et il est tout clair que les sommes suivantes $\sum \frac{1}{(m,\ n)^3}$, &c., se réduisent de même à zéro.

Donc enfin,

$$\log u - \log u' = x \sum \frac{1}{(m,\ n)}.$$

Cela fait voir que

$$u' = \epsilon^{-kx} u,$$

le coefficient k étant donné au moyen de l'équation

$$k = \Sigma \frac{1}{(n, m)},$$

où la somme est prise, comme auparavant, entre les limites

$$\text{mod.}\{(m, n) + y\} > T, \quad \text{mod.}(m, n) < T, \; T = \infty.$$

Mais il n'est pas permis d'écrire

$$k = \iint \frac{dm\, dn}{(m, n)}.$$

En effet, cette intégrale n'est que le premier terme d'une suite dont il faudrait, pour obtenir un résultat exact, prendre deux termes; le second terme de la suite serait une intégrale prise le long d'un contour, et il serait, ce me semble, très-difficile d'en trouver la valeur. Pour trouver la valeur de k, je remarque que k sera fonction linéaire des quantités T, y, y^*, $\frac{y^2}{T}$, &c., qui entrent dans les valeurs de r; donc, puisqu'en dernière analyse $T = \infty$, k ne peut être que de la forme $Ly + My^*$. Cela étant, en substituant pour u' sa valeur, je forme l'équation

$$\frac{\mathrm{y}(x+y)}{\mathrm{y}(y)} = \epsilon^{-\frac{1}{2}Bx^2} \epsilon^{(-By+Ly+My^*)x} . \Pi \left\{1 + \frac{x}{(m, n) + y}\right\},$$

$$\text{mod.}\{(m, n) + y\} < T, \quad T = \infty,$$

et j'écris successivement

$$y = \tfrac{1}{2}\Omega, \quad y = \tfrac{1}{2}\Upsilon,$$

ce qui donne pour les valeurs correspondantes du produit infini double $\epsilon^{-\frac{1}{2}Bx^2} . gx$ et $\epsilon^{-\frac{1}{2}Bx^2} . Gx$; en comparant les valeurs ainsi obtenues avec les équations qui donnent les valeurs de $\mathrm{y}(x + \frac{1}{2}\Omega)$, $\mathrm{y}(x + \frac{1}{2}\Upsilon)$, on trouve

$$L = 0, \quad M = \frac{\pi}{\text{mod.}(\omega v' - \omega' v)},$$

ou enfin,

$$\frac{\mathrm{y}(x+y)}{\mathrm{y}(y)} = \epsilon^{-\frac{1}{2}Bx^2} \epsilon^{\left(-By + \frac{\pi}{\text{mod.}(\omega v' - \omega' v)} y^*\right)x} . \Pi \left\{1 + \frac{x}{(m, n) + y}\right\},$$

$$\text{mod.}\{(m, n) + y\} < T, \quad T = \infty,$$

laquelle est l'équation qu'il s'agissait d'établir. Il est à peine nécessaire de faire la remarque que pour $y = 0$, on doit considérer à part le facteur $1 + \frac{x}{y}$, lequel multiplié par $\mathrm{y}(y)$ devient tout simplement x; l'équation subsiste donc dans ce cas.

En revenant au problème des transformations linéaires, partant des équations

$$(2k+1)\,\Omega_{,} = \lambda\Omega + \mu\Upsilon,$$
$$(2k+1)\,\Upsilon_{,} = \nu\Omega + \rho\Upsilon,$$

je suppose d'abord que les coefficients λ, ν ne satisfassent pas à la fois aux deux conditions

$$\lambda \equiv 0, \quad \nu \equiv 0, \quad \text{mod.}\,(2k+1),$$

et je prends p, q des entiers quelconques tels, que $\lambda p + \nu q$ ne soit pas $\equiv 0$, mod. $(2k+1)$.

Cela étant, soient

$$\lambda p + \nu q = p_{,},$$
$$\mu p + \rho q = q_{,},$$
$$(2k+1)\,\psi = p_{,}\Omega + q_{,}\Upsilon,$$

et, par conséquent,

$$\psi = p\Omega_{,} + q\Upsilon_{,}.$$

Je forme l'équation

$$(m_{,},\ n_{,}) + s\psi = (m,\ n)_{,},$$

savoir

$$m_{,}\Omega + n_{,}\Upsilon + s\psi = m\Omega_{,} + n\Upsilon_{,},$$

c'est-à-dire

$$\lambda m + \nu n - sp_{,} = (2k+1)\,m_{,},$$
$$\mu m + \nu n - sq_{,} = (2k+1)\,n_{,},$$

ou, ce qui est la même chose,

$$m - sp = m_{,}\rho - n_{,}\nu,$$
$$n - sq = m_{,}\mu - n_{,}\lambda.$$

Or, $m_{,}$, $n_{,}$, s étant des entiers donnés, m, n seront aussi des entiers; de même, m, n étant des entiers donnés, on trouve de k à $-k$ un entier s qui donne $m_{,}$ un entier. Mais cela étant, $n_{,}$ sera aussi un entier; car autrement $n_{,}$ serait une fraction ayant pour dénominateur, lequel on voudrait, des nombres $2k+1$, λ, ν, ce qui est impossible à moins que

$$\lambda \equiv 0, \quad \nu \equiv 0, \quad \text{mod.}\,(2k+1).$$

Mais si ces équations avaient lieu, on trouverait d'abord s de manière à avoir $n_{,}$ entier, et alors, puisqu'on n'a pas aussi

$$\mu \equiv 0, \quad \rho \equiv 0, \quad \text{mod.}\,(2k+1)$$

(en effet, cela est impossible à cause de l'équation $\lambda\rho - \mu\nu = 2k+1$), on démontrerait, comme auparavant, pour $n_{,}$, que $m_{,}$ est entier. Donc, enfin, m, n étant des entiers donnés, on trouve pour $m_{,}$, $n_{,}$, s un système d'entiers tel que s soit compris de k à $-k$, et l'on voit sans peine qu'il n'y a qu'un seul système de cette espèce.

A présent, partant de l'équation

$$\frac{\mathrm{y}(x+y)}{\mathrm{y}(y)} = \epsilon^{-\frac{1}{2}Bx^2} \cdot e^{\left(-By+\frac{\pi}{\text{mod.}(\omega v'-\omega' v)}y^2\right)x} \cdot \Pi\left\{1+\frac{x}{(m_{,}, n_{,})+y}\right\}$$

(et faisant attention à la particularité que présente le cas de $y=0$), j'écris successivement

$$y=0, \quad y=\pm\psi, \ldots, \quad y=\pm k\psi,$$

et je forme le produit des équations ainsi trouvées. Cela donne, à cause de $(m_{,}, n_{,})+s\psi = (m, n)_{,}$,

$$\Pi\frac{\mathrm{y}(x+s\psi)}{\mathrm{y}(s\psi)} = \epsilon^{-\frac{1}{2}(2k+1)Bx^2} \cdot x\Pi\left\{1+\frac{x}{(m, n)_{,}}\right\},$$

la condition, par rapport aux limites, étant

$$\text{mod.}(m, n)_{,} < T, \quad T=\infty.$$

Or

$$\mathrm{y}_{,}x = \epsilon^{-\frac{1}{2}B_{,}x^2} \cdot x\Pi\left\{1+\frac{x}{(m, n)_{,}}\right\},$$

avec la même condition, par rapport aux limites; donc, enfin,

$$\mathrm{y}_{,}x = \epsilon^{-\frac{1}{2}[B_{,}-(2k+1)B]x^2} \cdot \Pi\left\{\frac{\mathrm{y}(x+s\psi)}{\mathrm{y}(s\psi)}\right\},$$

où, dans le numérateur, s doit avoir toutes les valeurs entières depuis $s=-k$ jusqu'à $s=+k$, y compris $s=0$, et dans le dénominateur ces mêmes valeurs, hormis la valeur $s=0$.

Il est, à présent, facile de faire voir que cette propriété subsiste pour l'une quelconque des fonctions $\mathrm{y}x$, $\mathrm{g}x$, Θx, Zx; en effet, pour la démontrer pour $\mathrm{g}x$, j'écris $x+\frac{1}{2}\Omega$ au lieu de x; en prenant, pour un moment, $\rho=2\rho'+1$, $\mu=2\mu'$, cela donne

$$x+\tfrac{1}{2}\Omega = +(\bar{\rho}', -\mu'),$$

$$\mathrm{y}_{,}(x+\tfrac{1}{2}\Omega) = \epsilon^{\beta_{,}x(\bar{\rho}', \mu')} \cdot M\mathrm{g}_{,}x = \epsilon^{\frac{1}{2}\beta_{,}x(\rho, \mu)_{,}} \cdot M_{,}\mathrm{g}x,$$

c'est-à-dire

$$\frac{\mathrm{y}_{,}(x+\frac{1}{2}\Omega)}{\mathrm{y}_{,}(\frac{1}{2}\Omega)} = \epsilon^{\frac{1}{2}\beta_{,}x(\rho, \mu)_{,}} \cdot \mathrm{g}_{,}x.$$

Or, on déduit de l'expression pour $\mathrm{y}_{,}x$,

$$\epsilon^{-\frac{1}{2}\beta_{,}x(\rho, \mu)_{,}} \cdot \frac{\mathrm{y}_{,}(x+\frac{1}{2}\Omega)}{\mathrm{y}_{,}(\frac{1}{2}\Omega)}$$

$$= \epsilon^{-\frac{1}{2}\beta_{,}x(\rho, \mu)_{,}} \epsilon^{-\frac{1}{2}(B_{,}-\overline{2k+1}B)(x^2+\Omega x)} \cdot \Pi\frac{\mathrm{y}(x+s\psi+\frac{1}{2}\Omega)}{\mathrm{y}(s\psi+\frac{1}{2}\Omega)},$$

$$= \epsilon^{-\frac{1}{2}\beta_{,}x(\rho, \mu)_{,}} \epsilon^{-\frac{1}{2}(B_{,}-\overline{2k+1}B)x^2} \epsilon^{\frac{1}{2}(2k+1)\beta\Omega x} \Pi\frac{\mathrm{g}(x+s\psi)}{\mathrm{g}(s\psi)};$$

c'est-à-dire

$$g_{,}x = e^{-\frac{1}{2}(B_{,}-\overline{2k+1}B)x^2}\, e^{\frac{1}{2}x\{-B_{,}\Omega+\overline{2k+1}(B+\beta)\Omega-\beta_{,}(\rho,\ \mu)_{,}\}} \prod \frac{g(x+s\psi)}{g(s\psi)},$$

ou enfin, à cause de l'équation

$$-B_{,}\Omega+\overline{2k+1}(B+\beta)\Omega-\beta_{,}(\rho,\ \mu)_{,}=0,$$

la valeur de $g_{,}x$ est

$$g_{,}x = e^{-\frac{1}{2}(B_{,}-\overline{2k+1}B)x^2}\,.\prod \frac{g(x+s\psi)}{g(s\psi)}:$$

et en représentant, comme auparavant, l'une quelconque des fonctions yx, gx, Gx, Zx par Jx, on a l'équation

$$J_{,}x = e^{-\frac{1}{2}(B_{,}-\overline{2k+1}B)x^2}\,.\prod \frac{J(x+s\psi)}{J(s\psi)},$$

équation dans laquelle s doit avoir, dans le numérateur, toutes les valeurs entières depuis $s=-k$ jusqu'à $s=k$, y compris $s=0$, et dans le dénominateur, ces mêmes valeurs, hormis la valeur $s=0$.

Je suppose que les valeurs de $p_{,}$, $q_{,}$ soient données (cela va sans dire que l'on ne doit pas avoir à la fois $p_{,}\equiv 0$, $q_{,}\equiv 0$, mod. $2k+1$), et je remarque que l'on a, pour déterminer λ, μ, ν, ρ, les conditions

$$\begin{aligned} \rho p_{,} - \nu q_{,} &\equiv 0, \qquad \text{mod. } (2k+1),\\ -\mu p_{,} + \lambda q_{,} &\equiv 0,\\ \lambda \equiv 1, \quad \mu &\equiv 0, \qquad \text{mod. } 2,\\ \nu \equiv 0, \quad \rho &\equiv 1,\\ \lambda\rho - \mu\nu &= 2k+1. \end{aligned}$$

Et cela étant, on aura ensuite, en rassemblant toutes les équations qui ont rapport à la transformation,

$$\begin{aligned} \rho p_{,} - \nu q_{,} &= (2k+1)p,\\ -\mu p_{,} + \lambda q_{,} &= (2k+1)q,\\ \Psi &= p_{,}\Omega + q_{,}\Upsilon,\\ (2k+1)\Omega_{,} &= \lambda\Omega + \mu\Upsilon,\\ (2k+1)\Upsilon_{,} &= \nu\Omega + \rho\Upsilon. \end{aligned}$$

Or, quoique les valeurs de λ, μ, ν, ρ ne soient pas complétement déterminées au moyen de ces conditions, cependant il est clair que la valeur de la fonction $J_{,}x$ ne dépend que des valeurs de $p_{,}$, $q_{,}$ (en effet, ces valeurs suffisent pour déterminer la quantité $\Psi = p_{,}\Omega + q_{,}\Upsilon$, de laquelle dépend la fonction $J_{,}x$). Les formes différentes de $J_{,}x$, pour les systèmes de valeurs de λ, μ, ν, ρ, qui correspondent à des valeurs données de $p_{,}$, $q_{,}$, doivent donc se dériver de l'une quelconque de ces formes, au moyen d'une transformation triviale des modules $\Omega_{,}$, $\Upsilon_{,}$. Il est, de plus, clair que les valeurs de $p_{,}$, $q_{,}$, qui sont égales à des multiples de $(2k+1)$ près, ne donnent qu'une seule valeur de $J_{,}x$. Je suppose d'abord que

$$p_{,} \equiv 0, \qquad \text{mod.}\,(2k+1),$$

on peut trouver un entier θ tel que

$$\theta p_{,} \equiv 1, \qquad \text{mod.}\,(2k+1);$$

en prenant alors

$$\theta q_{,} \equiv q_{,}, \qquad \text{mod.}\,(2k+1),$$

cela donne

$$\theta\,(p_{,}\Omega + q_{,}\Upsilon_{,}) \equiv \Omega + q_{,}\Upsilon, \quad \text{mod.}\,(2k+1),$$

savoir

$$\theta\Upsilon \equiv \Omega + q_{,}\Upsilon, \quad \text{mod.}\,(2k+1).$$

Mais en donnant à s des valeurs entières quelconques, depuis $-k$ jusqu'à k, le système des valeurs de $s\psi$ est équivalent au système des valeurs de $s\theta\psi$, mod. $(2k+1)$; il est donc permis d'écrire, sans perte de généralité,

$$\Psi = \Omega + q_{,}\Upsilon.$$

De même pour

$$p_{,} \equiv 0, \quad \text{mod.}\,(2k+1),$$

on démontre que l'on peut donner à $q_{,}$ une valeur quelconque, sans changer pour cela la valeur de $J_{,}x$; il convient d'avoir $p_{,}$ impair et $q_{,}$ pair. J'écris donc, pour le premier cas, $2q_{,}$ au lieu de $q_{,}$, et je suppose que, dans le deuxième cas, les valeurs de $p_{,}$, $q_{,}$ soient

$$p_{,} = 2k+1, \quad q_{,} = 2.$$

Cela donne :

Premier cas.

$$\Psi = \Omega + 2q_{,}\Upsilon,$$

$q_{,}$ un entier quelconque, y compris zéro, depuis $-k$ jusqu'à $+k$.

Deuxième cas.

$$\Psi = (2k+1)\,\Omega + 2\Upsilon;$$

le nombre des valeurs différentes de Ψ sera donc, en tout, $2k+2$.

On obtient tout de suite, pour le premier cas, le système d'équations

$$\begin{aligned} p_{,} &= 1, \quad q_{,} = 2\mathrm{q}_{,}, \\ \lambda &= 1, \quad \mu = 2\mathrm{q}_{,}, \\ \nu &= 0, \quad \rho = (2k+1), \\ p &= 1, \quad q = 0; \end{aligned}$$

$$\left\{\begin{aligned} \psi &= \frac{1}{2k+1}(\Omega + 2\mathrm{q}_{,}\Upsilon), \\ \Omega_{,} &= \frac{1}{2k+1}(\Omega + 2\mathrm{q}_{,}\Upsilon) = \psi, \\ \Upsilon_{,} &= \Upsilon. \end{aligned}\right.$$

Le cas particulier le plus simple est celui de $q_{,} = 0$; cela donne

$$\psi = \Omega_{,} = \frac{1}{2k+1}\Omega, \qquad \Upsilon_{,} = \Upsilon,$$

et, de là,

$$\frac{\Omega_{,}}{\Upsilon_{,}} = \frac{1}{2k+1}\frac{\Omega}{\Upsilon};$$

et même le cas général se réduit à celui-ci, car, au moyen d'une transformation *triviale*, on obtiendrait

$$\Omega' = \Omega + 2\mathrm{q}_{,}\Upsilon, \qquad \Upsilon' = \Upsilon,$$

et puis

$$\psi = \Omega_{,} = \frac{1}{2k+1}\Omega', \qquad \Upsilon_{,} = \Upsilon',$$

et, de là,

$$\frac{\Omega_{,}}{\Upsilon_{,}} = \frac{1}{2k+1}\frac{\Omega'}{\Upsilon'}.$$

Les équations correspondantes pour le deuxième cas sont:

$$\begin{aligned} p_{,} &= 2k+1, \quad q_{,} = 2, \\ \lambda &= 2k+1, \quad \mu = 0, \\ \nu &= 0, \quad \rho = 1, \\ p &= 1, \quad q = 2, \\ \psi &= \frac{1}{2k+1}\{(2k+1)\Omega + 2\Upsilon\} \\ \Omega_{,} &= \Omega, \\ \Upsilon_{,} &= \frac{1}{2k+1}\Upsilon; \end{aligned}$$

ce qui donne

$$\frac{\Omega_{,}}{\Upsilon_{,}} = (2k+1)\frac{\Omega}{\Upsilon}.$$

J'ajoute, sans m'arrêter pour les démontrer, quelques formules de transformation pour le nombre 2; je trouve d'abord

$$\Omega_{,} = \tfrac{1}{2}\Omega_{,}, \quad \Upsilon_{,} = \Upsilon,$$

$$\left\{\begin{aligned}
&\mathrm{y}_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\,\mathrm{y}x\,\mathrm{g}x,\\
&\mathrm{g}_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\,\frac{\mathrm{g}\,(x-\frac{1}{4}\Omega)\,\mathrm{g}\,(x+\frac{1}{4}\Omega)}{\mathrm{g}^2\,(\frac{1}{4}\Omega)},\\
&G_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\,Gx\,Zx,\\
&Z_{,}x = \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\,\frac{Z\,(x-\frac{1}{4}\Omega)\,Z\,(x+\frac{1}{4}\Omega)}{Z^2\,(\frac{1}{4}\Omega)}.
\end{aligned}\right.$$

Ces équations donnent, en introduisant les fonctions elliptiques, ϕx, fx, Fx données au moyen de

$$\phi x = \frac{\mathrm{y}x}{Zx}, \qquad fx = \frac{\mathrm{g}x}{Zx}, \qquad Fx = \frac{Gx}{Zx},$$

les équations

$$\frac{\phi_{,}x}{F_{,}x} = \phi x\,\frac{fx}{Fx},$$

$$f_{,}x = \frac{f(x-\frac{1}{4}\Omega)\,f(x+\frac{1}{4}\Omega)}{f^2\,(\frac{1}{4}\Omega)},$$

dont la seconde peut encore s'écrire sous la forme

$$f_{,}x = \frac{1 - c^2\,\dfrac{F^2\,(\frac{1}{4}\Omega)}{f^2\,(\frac{1}{4}\Omega)}\,\phi^2 x}{1 + e^2c^2\,\phi^2\,(\frac{1}{4}\Omega)\,\phi^2 x},$$

et les deux équations combinées ensemble conduisent sans peine à la valeur des modules $c_{,}$, $e_{,}$. On trouve en effet, en mettant comme à l'ordinaire $b^2 = c^2 + e^2$,

$$c_{,}^2 = 4bc,$$

$$e_{,}^2 = (b-c)^2,$$

et puis

$$\phi_{,}x = \frac{1 - c\,(c-b)\,\phi^2 x}{\phi x\,fx},$$

$$f_{,}x = \frac{1 - c\,(c+b)\,\phi^2 x}{1 - c\,(c-b)\,\phi^2 x},$$

$$F_{,}x = \frac{Fx}{1 - c\,(c-b)\,\phi^2 x},$$

formules qui correspondent à celles de la transformation de Lagrange. Les équations pour $\mathrm{y}_{,}x$, $Z_{,}x$ donnent encore une valeur de $\phi_{,}x$, laquelle, égalée à la valeur qui vient d'être trouvée, donne

$$\frac{\mathrm{y}x\,\mathrm{g}x\,Z^2\,(\frac{1}{4}\Omega)}{Z\,(x-\frac{1}{4}\Omega)\,Z\,(x+\frac{1}{4}\Omega)} = \frac{\phi x\,fx}{1 - c\,(c-b)\,\phi^2 x}.$$

On obtient tout de suite les formules pour la transformation analogue $\Omega_{,} = \Omega$, $\Upsilon_{,} = \frac{1}{2}\Upsilon$. Mais il faut de plus considérer le système

$$\Omega_{,} = \tfrac{1}{2}(\Omega - \Upsilon), \quad \Upsilon_{,} = \tfrac{1}{2}(\Omega + \Upsilon):$$

on aura alors

$$\left\{\begin{aligned}
\mathrm{y}_{,}x &= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, gx\, Zx,\\
\mathrm{g}_{,}x &= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, \frac{\mathrm{y}\,(x + \frac{1}{4}\overline{\Omega - \Upsilon})\,\mathrm{y}\,(x - \frac{1}{4}\overline{\Omega - \Upsilon})}{-\mathrm{y}^2(\frac{1}{4}\overline{\Omega - \Upsilon})}\\
&= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, \frac{Z\,(x + \frac{1}{4}\overline{\Omega - \Upsilon})\,Z\,(x + \frac{1}{4}\overline{\Omega - \Upsilon})}{Z^2(\frac{1}{4}\overline{\Omega - \Upsilon})},\\
G_{,}x &= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, \frac{\mathrm{y}\,(x + \frac{1}{4}\overline{\Omega + \Upsilon})\,\mathrm{y}\,(x - \frac{1}{4}\overline{\Omega + \Upsilon})}{-\mathrm{y}^2(\frac{1}{4}\overline{\Omega + \Upsilon})}\\
&= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, \frac{Z\,(x + \frac{1}{4}\overline{\Omega + \Upsilon})\,Z\,(x - \frac{1}{4}\overline{\Omega + \Upsilon})}{Z^2(\frac{1}{4}\overline{\Omega + \Upsilon})},\\
Z_{,}x &= \epsilon^{-\frac{1}{2}(B_{,}-2B)x^2}\, gx\, Gx\,;
\end{aligned}\right.$$

et puis, en écrivant

$$c_{,}^2 = (e - ic)^2, \quad -e_{,}^2 = (e + ic)^2,$$

on obtient

$$\phi_{,}x = \frac{\phi x}{fx\,Fx},$$

$$f_{,}x = \frac{1 + ice\,\phi^2 x}{fx\,Fx},$$

$$F_{,}x = \frac{1 - ice\,\phi^2 x}{fx\,Fx}\,;$$

$$\frac{1 + ice\,\phi^2 x}{fx\,Fx} = \frac{Z\,(x + \frac{1}{4}\overline{\Omega - \Upsilon})\,Z\,(x - \frac{1}{4}\overline{\Omega - \Upsilon})}{Z^2(\frac{1}{4}\overline{\Omega - \Upsilon})\,gx\,Gx},$$

$$\frac{1 - ice\,\phi^2 x}{fx\,Fx} = \frac{Z\,(x + \frac{1}{4}\overline{\Omega + \Upsilon})\,Z\,(x - \frac{1}{4}\overline{\Omega + \Upsilon})}{Z^2(\frac{1}{4}\overline{\Omega + \Upsilon})\,gx\,Gx},$$

$$\frac{1 + ice\,\phi^2 x}{1 - ice\,\phi^2 x} = \frac{Z\,(x + \frac{1}{4}\overline{\Omega - \Upsilon})\,Z\,(x - \frac{1}{4}\overline{\Omega - \Upsilon})\,Z^2(\frac{1}{4}\overline{\Omega + \Upsilon})}{Z\,(x + \frac{1}{4}\overline{\Omega + \Upsilon})\,Z\,(x - \frac{1}{4}\overline{\Omega + \Upsilon})\,Z^2(\frac{1}{4}\overline{\Omega - \Upsilon})}:$$

ou, au moins, ces formules seront exactes au signe de i près; car il serait peut-être difficile de déterminer quel est le signe qu'on doit donner à cette quantité.

131.

NOUVELLES RECHERCHES SUR LES COVARIANTS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. XLVII. (1854), pp. 109—125.]

JE me sers de la notation

$$(a_0,\ a_1, \dots a_n)(x,\ y)^n$$

pour représenter la fonction

$$a_0 x^n + \frac{n}{1} a_1 x^{n-1} y + \dots + a_n y^n:$$

en supposant que les coefficients a_0', a_1' &c. soient donnés par l'équation

$$(a_0,\ a_1, \dots a_n)(\lambda x + \mu y,\ \lambda' x + \mu' y)^n = (a_0',\ a_1', \dots a_n')(x,\ y)^n,$$

supposée identique par rapport à x, y, soit $\phi(a_0,\ a_1, \dots a_n;\ x,\ y)$ une fonction des coefficients et des variables, telle que

$$\phi(a_0',\ a_1', \dots a_n';\ \ x,\ y) = (\lambda\mu' - \lambda'\mu)^p\, \phi(a_0,\ a_1, \dots a_n;\ \ \lambda x + \mu y,\ \lambda' x + \mu' y);$$

cette fonction ϕ sera généralement un *Covariant*, et dans le cas particulier où ϕ est fonction des seuls coefficients, un *Invariant* de la fonction donnée.

Je suppose d'abord que les nouveaux coefficients soient donnés par l'équation

$$(a_0,\ a_1, \dots a_n)(x + \lambda y,\ y)^n = (a_0',\ a_1', \dots a_n')(x,\ y)^n;$$

cela donne les relations

$$\begin{aligned} a_0' &= a_0, \\ a_1' &= a_1 + \ \lambda a_0, \\ a_2' &= a_2 + 2\lambda a_1 + \lambda^2 a_0, \\ &\text{\&c.} \end{aligned}$$

Il faut donc que le *covariant* ϕ satisfasse à l'équation

$$\phi(a_0', a_1', \dots a_n'; \quad x, y) = \phi(a_0, a_1, \dots a_n; \quad x + \lambda y, y),$$

laquelle peut aussi être écrite comme suit:

$$\phi(a_0', a_1', \dots a_n'; \quad x - \lambda y, y) = \phi(a_0, a_1, \dots a_n; \quad x, y). \quad \dots\dots\dots\dots(X)$$

De même, en faisant

$$(a_0, a_1, \dots a_n)(x, \mu x + y)^n = (a_0^{\backprime}, a_1^{\backprime}, \dots a_n^{\backprime})(x, y)^n,$$

ce qui donne

$$\begin{aligned} a_n^{\backprime} &= a_n, \\ a_{n-1}^{\backprime} &= a_{n-1} + \mu a_n, \\ a_{n-2}^{\backprime} &= a_{n-2} + 2\mu a_{n-1} + a_n, \\ &\&c. \end{aligned}$$

le *covariant* ϕ doit satisfaire aussi à l'équation

$$\phi(a_0^{\backprime}, a_1^{\backprime}, \dots a_n^{\backprime}; \quad x, -\mu x + y) = \phi(a_0, a_1, \dots a_n; \quad x, y); \quad \dots\dots(Y)$$

et réciproquement, toute fonction ϕ homogène par rapport aux coefficients et aussi par rapport aux variables, qui satisfait à ces équations (X, Y), sera un *covariant* de la fonction donnée.

Examinons d'abord l'équation (X) que je représente par $\phi' = \phi$. Soit pour le moment, $a_1' - a_1 = \lambda\alpha_1$, $a_2' - a_2 = \lambda\alpha_2$, &c., alors on aura, comme à l'ordinaire, l'équation symbolique

$$\phi' = e^{\lambda(\alpha_1\partial_{a_1} + \alpha_2\partial_{a_2} \dots + \alpha_n\partial_{a_n} - y\partial_x)}\phi,$$

où les quantités a_1, a_2, &c., en tant qu'elles entrent dans α_1, α_2, &c., ne doivent pas être affectées par les symboles ∂_{a_1}, ∂_{a_2}, &c. de la différentiation. En substituant les valeurs de α_1, α_2, ..., et en ordonnant selon les puissances de λ, cette équation donne

$$\phi' = e^{\lambda\underset{\cdot}{\square} + \lambda^2\underset{\cdot}{\square}_1 \dots + \lambda^n\underset{\cdot}{\square}_{n-1} - \lambda y\partial_x}\phi,$$

où les symboles $\underset{\cdot}{\square}$, $\underset{\cdot}{\square}_1$, &c. sont donnés par

$$\begin{aligned} \underset{\cdot}{\square} &= a_0\partial_{a_1} + 2a_1\partial_{a_2} \dots + na_{n-1}\partial_{a_n}, \\ \underset{\cdot}{\square}_1 &= a_0\partial_{a_2} + 3a_1\partial_{a_3} \dots + \frac{n.n-1}{1.2}a_{n-2}\partial_{a_n}, \\ &\vdots \\ \underset{\cdot}{\square}_{n-1} &= a_0\partial_{a_n}, \end{aligned}$$

et les quantités a_1, a_2, &c., en tant qu'elles entrent dans les symboles $\underset{\cdot}{\square}$, $\underset{\cdot}{\square}_1$, &c., ne doivent pas être affectées par les symboles ∂_{a_1}, ∂_{a_2}, &c. de la différentiation. Il est assez remarquable que l'équation symbolique peut aussi être écrite sous la forme plus simple

$$\phi' = e^{\lambda(\underset{\cdot}{\square} - y\partial_x)}\phi,$$

où les quantités a_1, a_2, ..., en tant qu'elles entrent dans le symbole $\underset{\bullet}{\square}$, sont censées affectées des symboles ∂_{a_1}, ∂_{a_2}, &c. de la différentiation; de manière que dans le développement, $\underset{\bullet}{\square}^2 . \phi$ par exemple, signifie $\underset{\bullet}{\square} . \underset{\bullet}{\square}\phi$, et ainsi de suite. Je ne m'arrête pas sur ce point, parce que pour ce que je vais démontrer de plus important, il suffit de faire attention à la *première* puissance de λ. D'ailleurs l'intelligibilité des équations dont il s'agit, sera facilitée en faisant les développements et en comparant les puissances correspondantes de λ. Cela donne par exemple:

$$\underset{\bullet}{\square}^2 = \underset{\bullet}{\square}^2 + 2\underset{\bullet}{\square}_1, \quad \underset{\bullet}{\square}^3 = \underset{\bullet}{\square}^3 + 3\underset{\bullet}{\square}\underset{\bullet}{\square}_1 + 6\underset{\bullet}{\square}_2, \text{ \&c.}$$

où les symboles $\underset{\bullet}{\square}^2$, $\underset{\bullet}{\square}^3$ &c. à gauche de ces équations dénotent la double, triple, &c. répétition de l'opération $\underset{\bullet}{\square}$, tandis qu'à côté droit des équations, les quantités a_1, a_2, ... &c., en tant qu'elles entrent dans les symboles $\underset{\bullet}{\square}$, $\underset{\bullet}{\square}_1$, &c. sont censées ne pas être affectées des symboles ∂_{a_1}, ∂_{a_2}, &c. de la différentiation. Dans la suite, si le contraire n'est pas dit, je me servirai des expressions $\underset{\bullet}{\square}^2$, $\underset{\bullet}{\square}^3$, &c. pour dénoter les répétitions de l'opération, et de même pour les combinaisons de deux ou de plusieurs symboles.

Cela étant, l'équation $\phi' = e^{\lambda(\underset{\bullet}{\square} - y\partial_x)}\,\phi = \phi$ donne

$$\phi = \{1 + \lambda(\underset{\bullet}{\square} - y\partial_x) + \frac{\lambda^2}{1.2}(\underset{\bullet}{\square} - y\partial_x)^2 + \ldots\}\,\phi,$$

où $(\underset{\bullet}{\square} - y\partial_x)^2 . \phi$ (je le répète) équivaut à $(\underset{\bullet}{\square} - y\partial_x).(\underset{\bullet}{\square} - y\partial_x)\phi$; et ainsi de suite. Il faut d'abord que le coefficient de λ s'évanouisse, ce qui donne $(\underset{\bullet}{\square} - y\partial_x)\,\phi = 0$; et cette condition étant satisfaite, les coefficients des puissances supérieures s'évanouissent d'elles-mêmes; c'est-à-dire, l'équation (X) sera satisfaite en supposant que ϕ satisfait à l'équation aux différences partielles $(\underset{\bullet}{\square} - y\partial_x)\,\phi = 0$.

En posant

$$\begin{aligned}
\overset{\bullet}{\square} &= a_n\partial_{a_{n-1}} + 2a_{n-1}\partial_{a_{n-2}} \ldots + na_1\partial_{a_0},\\
\overset{\bullet}{\square}_1 &= a_n\partial_{a_{n-2}} + 3a_{n-1}\partial_{a_{n-3}} \ldots + \frac{n(n-1)}{1.2}\,a_2\partial_{a_0},\\
&\vdots\\
\overset{\bullet}{\square}_{n-1} &= a_n\partial_{a_0},
\end{aligned}$$

on fera un raisonnement analogue par rapport à l'équation (Y); et il sera ainsi démontré que ϕ doit satisfaire aussi à l'équation à différences partielles $(\overset{\bullet}{\square} - x\partial_y)\,\phi = 0$; donc enfin, on a le suivant

THÉORÈME. Tout *covariant* ϕ de la fonction

$$(a_0,\ a_1, \ldots a_n)(x,\ y)^n,$$

satisfait aux deux équations à différences partielles

$$(\underset{\bullet}{\square} - y\partial_x)\,\phi = 0, \quad (\overset{\bullet}{\square} - x\partial_y)\,\phi = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

où

$$\begin{aligned}
\underset{\bullet}{\square} &= a_0\partial_{a_1} + 2a_1\partial_{a_2} \ldots + na_{n-1}\partial_{a_n},\\
\overset{\bullet}{\square} &= na_1\partial_{a_0} + (n-1)a_2\partial_{a_1} \ldots + a_n\partial_{a_{n-1}};
\end{aligned}$$

et réciproquement toute fonction, homogène par rapport aux coefficients et par rapport aux variables, qui satisfait à ces équations, est un *covariant* de la fonction donnée.

Par exemple, l'*invariant* $\phi = ac - b^2$ de la fonction $ax^2 + 2bxy + cy^2$ satisfait aux équations

$$(a\partial_b + 2b\partial_c)\phi = 0, \quad (2b\partial_a + c\partial_b)\phi = 0,$$

et le *covariant* $\phi = (ac - b^2)x^2 + (a\partial - bc)xy + (b\partial - c^2)y^2$ de la fonction $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ satisfait aux équations

$$(a\partial_b + 2b\partial_c + 3c\partial_d - y\partial_x)\phi = 0, \quad (3d\partial_c + 2c\partial_b + b\partial_a - x\partial_y)\phi = 0.$$

Il est clair qu'en ne considérant que les fonctions qui restent les mêmes en prenant dans un ordre inverse les coefficients $a_0, a_1, \ldots a_n$ et les variables x, y, respectivement, les *covariants* seront définis par l'une ou l'autre des équations (A), et qu'il n'est plus nécessaire de considérer les deux équations. Cela posé, on trouve assez facilement les *covariants* par la méthode des coefficients indéterminés. Mais il y a à remarquer une circonstance de la plus grande importance dans cette théorie, savoir, que l'on obtient de cette manière un nombre d'équations plus grand qu'il n'en faut pour déterminer les coefficients dont il s'agit. Ces équations cependant, étant liées entre elles, se réduisent au nombre nécessaire d'équations indépendantes.

Cherchons par exemple pour la fonction $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ un *invariant* ϕ de la forme

$$\phi = Aa^2d^2 + Babcd + Cac^3 + Cb^3d + Db^2c^2,$$

contenant les quatre coefficients indéterminés A, B, C, D. En substituant dans l'équation $(a\partial_b + 2b\partial_c + 3c\partial_d)\phi = 0$, on obtient

$$(3C + 2B)ab^2d + (3B + 6C + 2D)abc^2 + (6A + B)ac^2d + (3C + 4D)b^3c = 0;$$

or les *quatre* équations données par cette condition, se réduisent à *trois* équations indépendantes, de sorte qu'en faisant par exemple $A = -1$, les autres coefficients seront déterminés, et l'on obtient le résultat connu:

$$\phi = -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2.$$

La circonstance mentionnée ci-dessus s'oppose à résoudre de la manière dont il s'agit, le problème de trouver le nombre des *invariants* d'un ordre donné: problème qui a toujours bravé mes efforts.

Avant d'entamer la solution des équations (A), je vais démontrer quelques propriétés générales des *covariants*, et des *invariants*. Pour abréger, je me servirai du mot *pesanteur*, en disant que les coefficients a_0, a_1, &c., ont respectivement les *pesanteurs* $0 - \frac{1}{2}n$, $1 - \frac{1}{2}n$, &c., que les variables x, y ont respectivement les pesanteurs $\frac{1}{2}$, $-\frac{1}{2}$, et que la *pesanteur*

d'un produit est égale à la somme des *pesanteurs* des facteurs. Cela posé, je dis que tout *covariant* est composé de termes dont chacun à la pesanteur *zéro*. Pour démontrer cela, j'écris:

$$(\underset{\bullet}{\square} - y\partial_x)(\overset{\bullet}{\square} - x\partial_y) = \underset{\bullet}{\square}\overset{\bullet}{\square} - y\partial_x\overset{\bullet}{\square} - x\partial_y\underset{\bullet}{\square} + xy\,\partial_x\partial_y + y\partial_y,$$

$$(\overset{\bullet}{\square} - x\partial_y)(\underset{\bullet}{\square} - y\partial_x) = \overset{\bullet}{\square}\underset{\bullet}{\square} - y\partial_x\overset{\bullet}{\square} - x\partial_y\underset{\bullet}{\square} + xy\,\partial_x\partial_y + x\partial_x;$$

cela donne

$$(\underset{\bullet}{\square} - y\partial_x)(\overset{\bullet}{\square} - x\partial_y) - (\overset{\bullet}{\square} - x\partial_y)(\underset{\bullet}{\square} - y\partial_x) = \underset{\bullet}{\square}\overset{\bullet}{\square} - \overset{\bullet}{\square}\underset{\bullet}{\square} + y\partial_y - x\partial_x;$$

or, en faisant attention aux valeurs de $\underset{\bullet}{\square}$, $\overset{\bullet}{\square}$, savoir

$$\underset{\bullet}{\square}\overset{\bullet}{\square} = (\underset{\bullet}{\square}\overset{\bullet}{\square}) + na_0\partial_{a_0} + 2(n-1)a_1\partial_{a_1} \ldots + n\,1\,a_{n-1}\partial_{a_{n-1}},$$

$$\overset{\bullet}{\square}\underset{\bullet}{\square} = (\overset{\bullet}{\square}\underset{\bullet}{\square}) + \qquad n\,.\,1\,.\,a_1\partial_{a_1} \ldots + 2(n-1)a_{n-1}\partial_{a_{n-1}} + 1\,.\,na_n\partial_{a_n},$$

où, en formant les produits $(\underset{\bullet}{\square}\overset{\bullet}{\square})$, $(\overset{\bullet}{\square}\underset{\bullet}{\square})$, les quantités a_0, a_1, ... a_n sont censées non affectées par les symboles ∂_{a_0}, ∂_{a_1}, ... ∂_{a_n} de la différentiation, on en tire

$$\begin{aligned}\underset{\bullet}{\square}\overset{\bullet}{\square} - \overset{\bullet}{\square}\underset{\bullet}{\square} &= n\,a_0\partial_{a_0} + (n-2)a_1\partial_{a_1} \ldots - na_n\partial_{a_n} \\ &= -2\,\{(0 - \tfrac{1}{2}n)a_0\partial_{a_0} + (1 - \tfrac{1}{2}n)a_1\partial_{a_1} \ldots + (n - \tfrac{1}{2}n)a_n\partial_{a_n}\} = -2\Theta,\end{aligned}$$

en représentant par Θ l'expression symbolique entre les crochets. De là enfin on obtient:

$$(\underset{\bullet}{\square} - y\partial_x)(\overset{\bullet}{\square} - x\partial_y) - (\overset{\bullet}{\square} - x\partial_y)(\underset{\bullet}{\square} - y\partial_x) = -2\,(\Theta + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}y\partial_y).$$

Or en supposant les deux parties de cette équation symbolique appliquées au *covariant* ϕ_1, la partie gauche de l'équation s'évanouit en vertu des équations (A) et l'équation se réduit à

$$(\Theta + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}y\partial_y)\phi = 0; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

ce qui est une nouvelle équation à différences partielles, à laquelle satisfait le *covariant* ϕ. Il est aisé de voir que cette équation exprime le théorème énoncé ci-dessus, savoir que tout *covariant* est composé de termes de la pesanteur *zéro*.

Il suit de là, en considérant un *covariant*

$$\phi = (A_0,\ A_1,\ \ldots\ A_s)(x,\ y)^s$$

qu'un coefficient quelconque A_i aura la pesanteur $i - \frac{1}{2}s$, ou bien que les *pesanteurs* forment une progression arithmétique aux différences 1, et dont les termes extrêmes sont $-\frac{1}{2}s$, $+\frac{1}{2}s$.

Substituons maintenant cette valeur de ϕ dans les équations (A). La première équation donne d'abord:

$$\underset{\bullet}{\square}A_0 = 0, \quad \underset{\bullet}{\square}A_1 = A_0, \quad \underset{\bullet}{\square}A_2 = 2A_1, \ldots \underset{\bullet}{\square}A_s = sA_{s-1}. \quad \ldots\ldots\ldots\ldots (\alpha)$$

Cela est un système qui équivaut aux deux équations

$$\underset{\bullet}{\square}^s\,.\,A_s = 0, \quad \phi = y^s\,.\,e^{\square_y^x}\,.\,A_s. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha')$$

De même, la seconde équation donne

$$\overset{\bullet}{\square} A_s = 0, \quad \overset{\bullet}{\square} A_{s-1} = A_s, \quad \overset{\bullet}{\square} A_{s-2} = 2A_{s-1}, \ldots \quad \overset{\bullet}{\square} A_0 = sA_1: \quad \ldots\ldots\ldots(\beta)$$

système qui équivaut aux deux équations

$$\overset{\bullet}{\square}{}^{s+1} A_0 = 0, \quad \phi = x^s e^{\overset{\bullet}{\square}\frac{y}{x}} . A_0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta')$$

On voit que A_0 satisfait aux deux équations

$$\underset{\bullet}{\square} A_0 = 0, \quad \overset{\bullet}{\square}{}^{s+1} A_0 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma)$$

et en supposant que cette quantité soit connue, on trouve les autres coefficients $A_1, A_2, \ldots, A_s$ par la seule différentiation, au moyen des équations (β). Or cela étant, je dis que les équations (α) seront satisfaites d'elles-mêmes. En effet: des équations $\underset{\bullet}{\square} A_0 = 0$, $\overset{\bullet}{\square} A_0 = sA_1$ on tire $\overset{\bullet}{\square}\underset{\bullet}{\square} A_0 = 0$, $\underset{\bullet}{\square}\overset{\bullet}{\square} A_0 = s\underset{\bullet}{\square} A_1$, et de là $(\overset{\bullet}{\square}\underset{\bullet}{\square} - \underset{\bullet}{\square}\overset{\bullet}{\square}) A_0 = -s\underset{\bullet}{\square} A_1$.

Or nous avons déjà vu que $\overset{\bullet}{\square}\underset{\bullet}{\square} - \underset{\bullet}{\square}\overset{\bullet}{\square} = 2\Theta$, et l'équation (B) donne $\Theta . A_0 + \frac{1}{2} s . A_0 = 0$: donc l'équation $(\overset{\bullet}{\square}\underset{\bullet}{\square} - \underset{\bullet}{\square}\overset{\bullet}{\square}) A_0 = -s\underset{\bullet}{\square} A_1$ se réduit à $A_0 = \underset{\bullet}{\square} A_1$: équation du système (α). De la même manière on obtient les autres équations de ce système. On peut dire que l'on aurait pu déterminer également le coefficient A_s au moyen des équations

$$\overset{\bullet}{\square} A_s = 0, \quad \underset{\bullet}{\square}{}^s . A_s = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\delta)$$

et de là les coefficients $A_{s-1}, \ldots A_0$ par les équations (α).

Prenons par exemple un *covariant* $(A_0, A_1, A_2)(x, y)^2$ de la fonction cubique $ax^3 + 3bx^2y + 3cxy^2 + dy^3$. A_0 doit satisfaire aux deux équations

$$(a\partial_b + 2b\partial_c + 3c\partial_d) A_0 = 0, \quad (3b\partial_a + 2c\partial_b + d\partial_c)^2 A_0 = 0.$$

Ces équations sont en effet satisfaites en mettant $A_0 = ac - b^2$. On a donc les équations

$$2A_1 = (3b\partial_a + 2c\partial_b + d\partial_c) A_0, \quad A_2 = (3b\partial_a + 2c\partial_b + d\partial_c) A_1,$$

pour déterminer A_1, A_2; ce qui donne $2A_1 = ad - bc$, $A_2 = bd - c^2$, et on est conduit ainsi au *covariant* mentionné ci-dessus, savoir à

$$(ac - b^2)x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2.$$

Soit maintenant

$$x^n \partial_{a_n} - x^{n-1} y\, \partial_{a_{n-1}} \ \ldots \pm y^n \partial_{a_0} = \Lambda,$$

on aura

$$\underset{\bullet}{\square}\Lambda = (\underset{\bullet}{\square}\Lambda), \quad \Lambda\underset{\bullet}{\square} = (\Lambda\underset{\bullet}{\square}) - y\frac{\partial\Lambda}{\partial x},$$

où dans $(\underset{\bullet}{\square}\Lambda)$, $(\Lambda\underset{\bullet}{\square})$ les quantités a_0, a_1, ... sont censées non affectées par les symboles ∂_{a_1}, ∂_{a_2}, &c. de la différentiation. Cela donne

$$\underset{\bullet}{\square}\Lambda - \Lambda\underset{\bullet}{\square} = y\,\frac{d\Lambda}{\partial x}.$$

Or $\partial_x\Lambda - \Lambda\partial_x = \dfrac{d\Lambda}{\partial x}$, donc:

$$(\underset{\bullet}{\square} - y\partial_x)\,\Lambda = \Lambda\,(\underset{\bullet}{\square} - y\partial_x),$$

et de même:

$$(\dot{\square} - x\partial_y)\,\Lambda = \Lambda\,(\dot{\square} - x\partial_y).$$

Appliquons ces deux équations symboliques à un *covariant* ϕ. Les termes à droite s'évanouissent à cause des équations (A), et l'on obtient les deux équations

$$(\underset{\bullet}{\square} - y\partial_x)\,\Lambda\phi = 0, \quad (\dot{\square} - x\partial_y)\,\Lambda\phi = 0,$$

c'est-à-dire: $\Lambda\phi$ sera aussi un *covariant* de la fonction donnée. Par exemple de *l'invariant*

$$\phi = -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2,$$

on tire le *covariant*

$$(x^3\partial_d - x^2y^2\partial_c + xy^2\partial_b - y^3\partial_a)\phi\,;$$

savoir:

$$\begin{aligned}
&(-a^2d + 3abc - 2b^3\,)x^3\\
-3\,(\;&abd - 2ac^2 + b^2c)x^2y\\
+3\,(\;&acd - 2b^2d + bc^2)xy^2\\
-\;&(-ad^2 + 3acd - 2c^3\,)y^3;
\end{aligned}$$

résultat déjà connu.

Essayons maintenant à intégrer les équations (A); savoir:

$$(\underset{\bullet}{\square} - y\partial_x)\phi = 0, \quad (\dot{\square} - x\partial_y)\phi = 0.$$

Pour intégrer la première, je reviens à une notation dont je me suis déjà servi dans ce mémoire et j'écris

$$\begin{aligned}
a_0' &= a_0,\\
a_1' &= a_1 + \lambda a_0,\\
a_2' &= a_2 + 2\lambda a_1 + \lambda^2 a_0,\\
&\vdots\\
a_n' &= a_n + n\lambda a_{n-1} \ldots + \lambda^n a_0.
\end{aligned}$$

En faisant $\lambda = -\frac{a_1}{a_0}$, ce qui donne $a_0' = 0$, on voit sans peine que l'on satisfera à l'équation, en mettant pour ϕ une fonction quelconque de quantités a_0', a_1', ... a_n', $x + \lambda y$, y; le nombre de ces quantités étant $n+2$. Et cela est la solution générale de l'équation.

Ce résultat doit être substituée dans la seconde équation, savoir dans $(\overset{\bullet}{\square} - x\partial_y)\phi = 0$. Pour cela, imaginons que les quantités a_0, a_1, ... a_n, x, y soient exprimées en fonction de a_0', a_2', ... a_n', x, y et a_1; puisque ϕ est fonction des seules quantités a_0', a_2' ... a_n', x, y, l'équation résultante doit être satisfaite, quelle que soit la valeur de a_1. Or on trouve que cette équation résultante a la forme $L + Ma_1 = 0$: donc il faut qu'on ait à la fois les deux équations $L = 0$, $M = 0$. (Je renvoie à une note les détails de la réduction.) En dernière analyse, et en remettant dans les équations $L = 0$, $M = 0$ les quantités a_0, a_2, ..., a_n au lieu de a_0', a_2', ..., a_n', je trouve les résultats suivants très simples, savoir, en écrivant

$$\begin{aligned}
\bar{\Theta} &= (0 - \tfrac{1}{2}n)a_0\partial_{a_0} + (2 - \tfrac{1}{2}n)a_2\partial_{a_2} + (3 - \tfrac{1}{2}n)a_3\partial_{a_3} \ldots + (n - \tfrac{1}{2}n)a_n\partial_{a_n}: \\
{}^{\bullet}\underset{\bullet}{\square} &= 3a_2\partial_{a_2} + 4a_3\partial_{a_4} \ldots + na_{n-1}\partial_{a_n}, \\
{}^{\bullet}\overset{\bullet}{\square} &= (n-2)a_2\partial_{a_2} + (n-3)a_4\partial_{a_3} \ldots + a_n\partial_{a_{n-1}}.
\end{aligned}$$

Les équations dont il s'agit sont

$$\{(n-1)a_2({}^{\bullet}\underset{\bullet}{\square} - y\partial_x) - a_0({}^{\bullet}\underset{\bullet}{\square} - x\partial_y)\}\bar{\phi} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(C)$$

$$(\bar{\Theta} + \tfrac{1}{2}x\partial_x - \tfrac{1}{2}y\partial_y)\bar{\phi} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(D)$$

et il y a à remarquer qu'on obtient l'équation (C) en éliminant entre les équations (A) le terme $\partial_{a_1}\phi$; et puis, en mettant $a_1 = 0$, on tire l'équation (D) de l'équation (B), en y mettant de même $a_1 = 0$. Il y a à remarquer aussi que la fonction $\bar{\phi}$ qui satisfait aux équations (C, D), est ce que devient un *covariant* quelconque ϕ, en y mettant $a_1 = 0$. On obtient d'abord la valeur générale en changeant a_0, a_2, ..., a_n en a_0', a_2', ..., a_n', et en mettant après pour ces quantités leurs valeurs en termes de a_0, a_1, a_2, ..., a_n. La solution du problème des *covariants* serait donc effectuée si l'on pourrait intégrer les équations (C, D).

Or la quantité a_0 entre dans l'équation (C) comme constante, et l'on voit sans peine que cette équation pourra être intégrée en mettant $a_0 = 1$; puis, en écrivant dans le résultat $\frac{a_2}{a_0}$, $\frac{a_3}{a_0}$, ... $\frac{a_n}{a_0}$ au lieu de a_2, a_3, ... a_n, et en multipliant par une puissance quelconque de a_0, le résultat ainsi obtenu, serait composé de termes de la même *pesanteur;* et en choisissant convenablement la puissance de a_0, on pourrait faire en sorte que ces termes fussent de la *pesanteur* zéro. Mais l'équation (D) ne fait qu'exprimer que la fonction $\bar{\phi}$ est composée de termes de la *pesanteur* zéro; le résultat obtenu de la manière dont il s'agit, satisfera donc par lui-même à l'équation (D), et il est permis de ne faire attention qu'à l'équation (C). Dans la pratique on intégrera cette

équation en ayant soin de faire en sorte que les solutions soient de la *pesanteur* zéro, ce qui peut être effectué en multipliant par une puissance convenablement choisie de a_0. Et puisqu'en faisant abstraction de cette quantité a_0, l'équation (C) contient $n+1$ quantités variables, savoir $a_2, a_3, \ldots, a_n, x, y$, la fonction $\overline{\phi}$ sera une fonction arbitraire de n quantités; et en supposant que cette fonction ne contienne pas les variables x, y (cas auquel $\overline{\phi}$ serait ce que deviendrait un *invariant* quelconque en y mettant $a_1 = 0$), $\overline{\phi}$ sera une fonction arbitraire de $n-2$ quantités.

La même chose sera évidemment vrai, si l'on rétablit la valeur générale de a_1: donc tout *invariant* sera une fonction d'un nombre $n-2$ d'*invariants*, que l'on pourra prendre pour primitifs; et tout *covariant* sera une fonction de ces *invariants* primitifs de la fonction donnée (laquelle est évidemment un de ses propres *covariants*), et d'un autre *covariant* que l'on peut prendre pour primitif. Cela ne prouve nullement (ce qui est néanmoins vrai pour les *invariants*, à ce que je crois) que tout invariant est une fonction rationnelle et intégrale de $n-2$ *invariants* convenablement choisis, et que tout *covariant* est une fonction rationnelle et intégrale (ce qui en effet n'est pas vrai) de ces *invariants*, de la fonction donnée, et d'un *covariant* convenablement choisi.

Le cas $n=2$ fait dans cette théorie une exception. On sait qu'il existe dans ce cas un *invariant*, savoir $ac-b^2$ qui, selon la théorie générale, ne doit pas exister, et il n'existe pas de *covariant*, hormis la fonction donnée elle-même. Or cette particularité peut être aisément expliquée.

Le cas $n=3$ rentre, comme cela doit être, dans la théorie générale. En effet, il existe dans ce cas un *invariant*, savoir la fonction $-a^2d^2+6abcd+4ac^3-4b^3d+3b^2c^2$ ci-dessus trouvée, et tout *covariant* de la fonction peut être exprimé par cet *invariant* de la fonction donnée elle-même, et par le *covariant* $(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2$ ci-dessus trouvé. Il en est ainsi par exemple pour le *covariant* de troisième ordre par rapport aux variables et aux coefficients; car en représentant par Φ le *covariant* dont il s'agit, par H le *covariant* du second ordre, par u la fonction donnée $ax^3+3bx^2y+3cxy^2+dy^3$ et par ∇ l'*invariant*, on obtient l'équation identique $\Phi^2+\square u^2=-4H^3$. Je fais mention de cette équation, parce que je crois qu'elle n'est pas généralement connue.

Je vais donner maintenant quelques exemples des équations $(C$ et $D)$. Soit d'abord $n=3$, et supposons que $\overline{\phi}$ ne contienne pas les variables x, y: $\overline{\phi}$ sera une fonction de a, c, d, et les équations reviendront à

$$(6c^2\partial_d - ad\partial_c)\overline{\phi}=0, \quad (-3a\partial_a + c\partial_c + 3d\partial_d)\overline{\phi}=0.$$

Les quantités ac^3, a^2d^3, dont chacune est de la pesanteur *zéro*, satisfont par là à la seconde équation, et en mettant $\overline{\phi}=Aa^2d^2+Cac^3$, on obtient $4A-C=0$, en vertu de la première équation; ou en faisant $A=-1$, cela donne $C=-4$; de là on tire $\overline{\phi}=-a^2d^2-4ac^3$, et la solution générale est $\overline{\phi}=F(-a^2d^2-4ac^3)$, F étant une fonction quelconque. La formule plus générale $\overline{\phi}=F(a, -a^2d^2-4ac^3)$ satisferait sans doute à la

première équation, mais pour que cette valeur satisfasse à la seconde équation, il faut que la quantité a, en tant qu'elle n'est pas contenue dans $-a^2d^2-4ac^3$, disparaisse. Ainsi la valeur donnée ci-dessus, savoir $\overline{\phi}=F(-a^2d^2-4ac^3)$, est la solution la plus générale des deux équations.

Écrivons a, $c-\frac{b^2}{a}$, $d-\frac{3bc}{a}+\frac{2b^3}{a^2}$ au lieu de a, c, d, et ϕ au lieu de $\overline{\phi}$, nous obtenons:

$$\phi=F(-a^3d^2+6abcd-4ac^3-4b^3d+3b^2c^2);$$

ce qui est l'expression la plus générale des *invariants* de la fonction $ax^3+3bx^2y+3cxy^2+y^3$, et l'on voit que tous ces *invariants* sont fonctions d'une seule quantité que nous avons prise ci-dessus pour l'*invariant* de la fonction de troisième ordre dont il s'agit.

Soit encore $n=4$, $\overline{\phi}$ sera une fonction de a, c, d, e qui satisfait aux équations

$$\{2ad\partial_c+(ae-9c^2)\,\partial_d-12cd\partial_e\}\,\overline{\phi}=0,$$

$$\{-2a\partial_a+d\partial_d+2e\partial_e\}\,\overline{\phi}=0,$$

dont la solution générale est $\overline{\phi}=F(ae+3c^2,\ ace-ad^2-c^3)$, F étant une fonction quelconque. On voit par là qu'il n'existe que les *invariants* indépendants $ae-4cd+3c^2$, $ace+2bcd-ad^2-b^2e-c^3$. Ce résultat est connu depuis longtemps.

Soit enfin $n=5$, $\overline{\phi}$ sera une fonction de a, c, d, e, f qui satisfait aux équations

$$\{3ad\partial_e+(2ae-12c^2)\,\partial_d+(af-16cd)\,\partial_e-20ce\partial_f\}\,\overline{\phi}=0,$$

$$\{-\tfrac{5}{2}a\partial_a-\tfrac{1}{2}c\partial_c+\tfrac{1}{2}d\partial_d+\tfrac{3}{2}e\partial_e+\tfrac{5}{2}f\partial_f\}\,\overline{\phi}=0.$$

On sait qu'il y en a une solution de quatrième ordre par rapport aux quantités a, c, d, e, f; et en prenant la fonction la plus générale dont les termes ont la *pesanteur* zéro, on aura:

$$\overline{\phi}=Aa^2f^2+Bacdf+Cace^3+Dad^2e+Ec^3e+Fc^2d^2:$$

fonction qui satisfait d'elle-même à la seconde équation. En substituant cette valeur dans la première équation, on trouvera que les coefficients A, B, &c. doivent satisfaire à ces *sept* équations:

$$2B+2C-40A=0,\quad 3B+D=0,\quad 3C+4D=0,\quad -12B+E=0,$$

$$9E-24D+4F-32C-20B=0,\quad 6F-16D=0,\quad -24F-16E=0,$$

qui se réduisent cependant (ce que l'on n'aurait pas facilement deviné par la forme des équations) à *cinq* équations indépendantes. En faisant donc $A=1$, on trouve aisément les autres coefficients B, C, &c. et on obtient ainsi:

$$\overline{\phi}=a^2f^2+4acdf+16ace^3-12ad^2e+48c^3e-32c^2d^2:$$

valeur qui peut être tirée d'une formule présentée dans mon mémoire sur les hyperdéterminants, [16], en y faisant $b=0$.

J'ai donné cet exemple pour faire voir qu'il serait impossible de déduire du nombre supposé connu des coefficients indéterminés qui correspondent à un ordre donné, le nombre des *invariants* de ce même ordre. Il est donc inutile de pousser plus loin cette discussion.

Note 1 *sur l'intégration des équations* (*A*).

En écrivant comme ci-dessus:

$$\underset{\bullet}{\square} = a_0\partial_{a_1} + 2a_1\partial_{a_2} \dots \quad + na_{n-1}\partial_{a_n},$$
$$\dot{\square} = na_1\partial_{a_0} + (n-1)a_2\partial_{a_1} \dots + a_n\partial_{a_{n-1}},$$

il s'agit de trouver une quantité ϕ, fonction de $a_0, a_1, \dots a_n, x$ et y qui satisfasse à la fois aux équations

$$(\underset{\bullet}{\square} - y\partial_x)\phi = 0,$$
$$(\dot{\square} - x\partial_y)\phi = 0.$$

Pour intégrer ces équations, j'écris, comme plus haut:

$$a_0' = a_0,$$
$$a_1' = a_1 + \lambda a_0,$$
$$a_2' = a_2 + 2\lambda a_1 + \lambda^2 a_0,$$
$$\vdots$$
$$a_n' = a_n + n\lambda a_{n-1} \dots \quad + \lambda^n a_0,$$

et aussi $x' = x - \lambda y$, $y' = y$. Cela posé, je fais remarquer d'abord que $\frac{da_1'}{d\lambda} = a_0'$, $\frac{da_2'}{d\lambda} = 2a_1'$, et ainsi de suite. En considérant λ comme fonction quelconque de $a_0, a_1, \dots a_n$, et en supposant que ϕ soit une fonction de $a_0', a_1', \dots a_n', x', y'$, on parvient assez facilement à l'équation identique $(\underset{\bullet}{\square} - y\partial_x)\phi = (1 + \underset{\bullet}{\square}\lambda)(\underset{\bullet}{\square}' - y'\partial_{x'})\phi$, où $\underset{\bullet}{\square}'$ est ce que devient $\underset{\bullet}{\square}$, en y écrivant $a_0', a_1', \dots a_n'$ au lieu de $a_0, a_1, \dots a_n$.

Nous pouvons donc satisfaire à la première équation, en déterminant λ au moyen de $1 + \underset{\bullet}{\square}\lambda = 0$: équation qui serait satisfaite en écrivant $\lambda = -\frac{a_1}{a_0}$, ou, si l'on veut, en déterminant λ par $a_1' = 0$. Donc, en supposant toujours que λ ait cette valeur, ϕ sera une fonction quelconque de $a_0', a_2', \dots a_n', x', y'$, c'est-à-dire d'un nombre $n + 2$ de quantités. Ce sera donc là (comme on aurait pu facilement prévoir), la solution générale de la première équation. Or en considérant ϕ comme fonction de $a_0', a_2', \dots a_n', x', y'$, ou, si l'on veut, de $a_0', a_1', a_2', \dots a_n', x', y'$ (où $a_1' = a_1 + \lambda a_0 = 0$), et en

substituant cette valeur dans l'équation $(\overset{\bullet}{\square} - x\partial_y)\phi = 0$, on voit d'abord que la variation de la quantité λ fournit au résultat le terme

$$\left(na_1 \frac{d\lambda}{da_0} + (n-1)a_2 \frac{d\lambda}{da_1}\right)(\underset{\bullet}{\square}' - y'\partial_{x'})\,\phi\,;$$

et puisque $na\dfrac{d\lambda}{da_0} + (n-1)a_2\dfrac{d\lambda}{da_1}$ se réduit à $n\dfrac{a_1^2}{a_0^2} - (n-1)\dfrac{a_0 a_2}{a_0^2}$, ou enfin à $\lambda^2 - \dfrac{(n-1)a_2'}{a_0}$, ce terme devient

$$\left(\lambda^2 - \frac{(n-1)\,a_2'}{a_0}\right)(\underset{\bullet}{\square}' - y'\partial_{x'}).$$

Le terme $-x'\partial_y \,.\, \phi$ se réduit à $-(x' + \lambda y')(-\lambda\partial_{x'} + \partial_{y'})\phi$, savoir à

$$(-x'\partial_{y'} + \lambda^2 y'\partial_{x'} + \lambda x'\partial_{x'} - \lambda y'\partial_{y'})\,\phi,$$

et en mettant pour un moment

$$\begin{aligned} M = \quad & na_1(\partial_{a_0'} + \lambda\partial_{a_1'} \ldots\ldots + \lambda^n\,\partial_{a_n'}) \\ + (n-1)\,a_2\,(& \qquad\quad \partial_{a_1'} \ldots\ldots + n\lambda^{n-1}\,\partial_{a_n'}) \\ & \vdots \\ + a_n\,(& \qquad\qquad\quad \partial_{a_{n-1}'} + n\lambda\,\partial_{a_n'}), \end{aligned}$$

nous obtenons

$$(\overset{\bullet}{\square} - x\partial_y)\,\phi = M\phi + \left(\lambda^2 - \frac{(n-1)a_2'}{a_0}\right)(\underset{\bullet}{\square}' - y'\partial_{x'})\phi + (-x'\partial_{y'} + \lambda^2 y'\partial_{x'} + \lambda x'\partial_{x'} - \lambda y'\partial_{y'})\,\phi,$$

c'est-à-dire

$$(\overset{\bullet}{\square} - x\partial_y)\phi = (M - x'\partial_{y'})\,\phi + \lambda^2\underset{\bullet}{\square}'\phi - \frac{(n-1)a_2'}{a_0'}(\underset{\bullet}{\square}' - y'\partial_{x'})\,\phi + \lambda(x'\partial_{x'} - y'\partial_{y'})\phi.$$

Or en supposant que $\overset{\bullet}{\square}'$ est ce que devient $\overset{\bullet}{\square}$ en y écrivant a_0', a_1', ... a_n' au lieu de a_0, a_1, ... a_n, et en posant

$$\Theta' = (0 - \tfrac{1}{2}n)a_0'\partial_{a_0'} + (1 - \tfrac{1}{2}n)\,a_1'\partial_{a_1'} + \ldots (n - \tfrac{1}{2}n)a_n'\partial_{a_n'},$$

on obtient, après avoir fait une réduction un peu pénible:

$$M\phi + \lambda^2\underset{\bullet}{\square}'\phi = \overset{\bullet}{\square}'\phi + 2\lambda\Theta'\phi,$$

(en effet les coefficients de $\partial_{a_0'}\phi$, $\partial_{a_1'}\phi$ &c. aux deux côtés de cette équation deviennent les mêmes après des réductions convenables.) Donc enfin on a

$$(\overset{\bullet}{\square} - x\partial_y)\phi = (\overset{\bullet}{\square}' - x'\partial_{y'})\phi - \frac{(n-1)a_2'}{a_0'}(\underset{\bullet}{\square}' - y'\partial_{x'})\,\phi + 2\lambda(\Theta' + \tfrac{1}{2}x'\partial_{x'} - \tfrac{1}{2}y'\partial_{y'})\phi = 0,$$

ou bien, puisque cette équation doit être satisfaite indépendamment de la quantité λ (qui seule contient a_1), elle se décompose dans les deux équations

$$\{a_0'(\overset{\bullet}{\square}' - x'\partial_{y'}) - (n-1)\,a_2'(\underset{\bullet}{\square}' - y'\partial_{x'})\}\,\phi = 0,$$

$$\{\Theta' + \tfrac{1}{2}x'\partial_{x'} - \tfrac{1}{2}y'\partial_{y'}\}\,\phi = 0,$$

lesquelles, en y mettant d'abord $a_1' = 0$, puis en remettant $a_0, a_2, \ldots, a_n, x, y$ au lieu de $a_0', a_2', \ldots, a_n, x', y'$, et en écrivant $\overline{\phi}$, $\overline{\Theta}$, $\cdot\underset{\bullet}{\square}$, $\cdot\overset{\bullet}{\square}$ au lieu de ϕ, Θ, $\underset{\bullet}{\square}$, $\overset{\bullet}{\square}$, donnent en effet les équations (C, D) dont je me suis servi dans le texte.

Note 2.

Je vais résumer dans cette note quelques formules qui feront voir la liaison qui existe entre les *invariants* d'une fonction de n-ième ordre et de la fonction de $(n-1)$ième ordre que l'on obtient en réduisant à zéro le coefficient de y^n, et en supprimant le facteur x.

Il convient pour cela de considérer une fonction telle que

$$(a_0, a_1, \ldots a_n)(x, y)_n = a_0 x^n + a_1 x^{n-1} y \ldots + a_n y^n,$$

dans laquelle n'entrent plus les coefficients numériques du binôme $(1+x)^n$.

Écrivons

$$(a_0, a_1, \ldots a_n)(x, y)_n = a_0 (x - \alpha_1 y)(x - \alpha_2 y) \ldots (x - \alpha_n y);$$

je tâche d'abord à représenter les *invariants* au moyen des racines $\alpha_1, \alpha_2, \ldots, \alpha_n$, et j'étends pour le moment le terme *invariant* à toute fonction, symétrique ou non, des racines qui ait la propriété caractéristique des *invariants:* fonctions qui jusqu'ici ont été considérées tacitement comme rationnelles par rapport aux coefficients.

Mettons d'abord

$$\nabla = a_0^{2n-2} (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 \ldots (\alpha_{n-1} - \alpha_n)^2;$$

cette quantité ∇ qui, égalée à zéro, exprime l'égalité de deux racines, et que je vais désormais nommer le *Discriminant* de la fonction, sera une fonction rationnelle des coefficients, et d'un *invariant* proprement dit. Mais de plus, toute fonction telle que $(\alpha_1 - \alpha_2)^n (\alpha_1 - \alpha_3)^n, \ldots,$ dans laquelle la somme des indices des facteurs qui contiennent α_1, celle des indices des facteurs qui contiennent α_2, &c. sont égales, sera un *invariant;* et en réunissant ces fonctions, pour trouver une somme en fonction symétrique des racines, on obtiendra des *invariants* proprement dits. Cela soit dit en passant. Pour le moment il suffit de prendre les invariants les plus simples, savoir ceux de la forme

$$\frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)},$$

lesquels en effet sont des rapports *anharmoniques* de quatre racines, prises à volonté. Soient $Q_1, Q_2, \ldots, Q_{n-3}$ la fonction qui vient d'être écrite et les fonctions que l'on en tire en mettant $\alpha_5, \alpha_6, \ldots, \alpha_n$ au lieu de α_4. Les fonctions $\nabla, Q_1, Q_2, \ldots, Q_{n-3}$ seront des *invariants* indépendants, et le nombre de ces *invariants* est $n-2$. Donc, tout autre

invariant sera une fonction des quantités ∇, Q_1, Q_2, ..., Q_{n-3}. Soit maintenant $a_n = 0$, et α_n la racine qui devient égale à zéro. Les quantités Q_1, Q_2, ..., Q_{n-4} seront toujours des rapports *anharmoniques* de quatre racines de l'équation du $(n-1)$ième ordre. Il n'y aura que la seule quantité Q_{n-3} qui change de forme, et elle ne sera pas un *invariant* de la fonction du $(n-1)$ième ordre. On voit aussi d'abord que le *discriminant* ∇ se réduit à $a^2_{n-1}\nabla_0$, en exprimant par ∇_0 le *discriminant* de la fonction du $(n-1)$ième ordre. (C'est je crois M. Joachimsthal qui a le premier remarqué cette circonstance.) Donc, en supposant $a_n = 0$, l'*invariant* de la fonction du n-ième ordre deviendra une fonction de $a^2_{n-1}\nabla_0$, Q_1, Q_2, ... Q_{n-4} et d'une quantité X qui n'est pas un *invariant* de la fonction du $(n-1)$ième ordre, mais qui sera toujours la même quel que soit l'invariant dont il s'agit. En considérant les *invariants* proprement dits de la fonction du $(n-1)$ième ordre, on peut former avec ces *invariants* des quotients I_1, I_2, ..., I_{n-4} du degré zéro par rapport aux coefficients. Nous pouvons remplacer par ces quotients les quantités Q_1, Q_2, ..., Q_{n-4}, et dire que l'*invariant* de la fonction du n-ième ordre, en mettant $a_n = 0$, deviendra une fonction des quantités $a^2_{n-1}\nabla_0$, I_1, I_2, ..., I_{n-4} et X.

Ces théorèmes auront, je crois, quelque utilité pour les recherches ultérieures: je les laisse à côté maintenant, et veux présenter une méthode assez simple pour calculer les *discriminants*.

Pour cela je remarque que les équations (A), en changeant, comme nous venons de le faire, les valeurs des coefficients, donnent pour les *invariants:*

$$(na_0\partial_{a_1} + (n-1)a_1\partial_{a_2} \ldots + a_{n-1}\partial_{a_n})\phi = 0,$$

$$(a_1\partial_{a_0} + 2a_2\partial_{a_1} \ldots + na_n\partial_{a_{n-1}})\phi = 0;$$

et ces équations seront satisfaites en mettant pour ϕ le *discriminant* ∇. Or, pour $a_n = 0$, la fonction ∇ devient $a^2_{n-1}\nabla_0$, ou, si l'on veut, $-a^2_{n-1}\nabla_0$; donc ∇ sera généralement de la forme

$$\nabla = -a^2_{n-1}\nabla_0 + Ba_n + Ca_n^2 + \ldots,$$

où a_n^{n-1} est la puissance la plus élevée de a_n. Donc, en supposant que ∇_0 soit connu, et en mettant la première des équations écrites ci-dessus sous la forme $(F + a_{n-1}\partial_{a_n})\nabla = 0$, où $F = na_0\partial_{a_1} + (n-1)a_1\partial_{a_2} \ldots + 2a_{n-2}\partial_{a_{n-1}}$, on obtiendra par la seule différentiation les coefficients B, C, &c. En effet, cette équation donne

$$a_{n-1}B = F(a^2_{n-1}\nabla_0), \quad 2a_{n-1}C = -F(B), \quad 3a_{n-1}D = -F(C);$$

et ainsi de suite.

En supposant par exemple $n = 3$, considérons la fonction du troisième ordre

$$\alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3:$$

le *discriminant* de $\alpha x^2 + \beta xy + \gamma y^2$ sera $4\alpha\gamma - \beta^2$. Nous avons alors

$$\nabla = -\gamma^2(4\alpha\gamma - \beta^2) + B\delta + C\delta^2,$$

et en mettant $F = 3\alpha\partial_\beta + 2\beta\partial_\gamma$, B, C seront donnés par

$$\gamma B = F(4\alpha\gamma^3 - \beta^2\gamma^2), \quad 2\gamma C = -F(B),$$

c'est-à-dire $B = 18\alpha\beta\gamma - 4\beta^3$, $C = -27\alpha^2$, et de là:

$$\nabla = -27\alpha^2\delta^2 + 18\alpha\beta\gamma\delta - 4\alpha\gamma^3 - 4\beta^3\delta + \beta^2\gamma^2:$$

valeur qui correspond en effet à la forme ordinaire

$$\nabla = -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2,$$

en changeant d'une manière convenable les coefficients.

Londres, Stone Buildings, 23 *Févr.* 1852.

132.

RÉPONSE À UNE QUESTION PROPOSÉE PAR M. STEINER (Aufgabe 4, Crelle t. XXXI. (1846) p. 90).

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 277—278.]

En partant des deux théorèmes:

I. Qu'il existe au moins une surface du second ordre qui touche neuf plans donnés quelconques;

II. Que le lieu d'intersection de trois plans rectangles qui touchent une surface du second ordre est une sphère concentrique avec la surface, tandis que pour le paraboloïde cette sphère se réduit à un plan,

M. Steiner suppose le cas d'un parallélepipède rectangle, ou même d'un cube P et d'un point quelconque D, par lequel passent trois plans rectangles. Les six plans du parallélepipède P et les trois plans qui passent par le point D seront touchés d'une surface F du second ordre (I.), et les huit angles E du parallélepipède P et le point D doivent donc se trouver tous les neuf sur la surface d'une sphère, ou dans un plan (II.). Les huit angles E sont en effet situés sur la surface d'une sphère, déterminée par eux; mais le point D étant arbitraire, ce point en général ne sera pas situé sur cette surface sphérique, de manière que les neuf points $8E$ et D ne seront situés, ni dans une surface sphérique, ni dans un plan; ce qui ne s'accorde pas avec le théorème II. Cela étant, M. Steiner dit, qu'il y a à prouver que la contradiction n'est qu'apparente, et que tout cela n'affaiblit pas la validité générale des deux théorèmes.

Il s'agit de savoir ce que devient dans le cas supposé par M. Steiner la surface du second ordre qui touche les six plans du parallélepipède P et les trois plans qui

passent par le point D. Cette surface sera en effet *la conique selon laquelle l'infini, considéré comme plan, est coupé par un cône déterminé, près la position du sommet.* En effet, menons par un point quelconque de l'espace trois plans parallèles aux plans du parallélepipède P, et par le point D trois autres plans parallèles à ces plans. Ces *six* plans seront touchés (en vertu d'un théorème connu) par un cône déterminé du second ordre, et on peut dire que ce cône, quelle que soit la position de son sommet, rencontre l'infini, considéré comme plan, dans une seule et même conique (cela n'est en effet autre chose que de dire que deux droites parallèles rencontrent l'infini, considéré comme plan, dans un seul et même point). Le cône dont il s'agit aura la propriété d'être touché par une infinité de systèmes de trois plans rectangles. En effet: le plan passant par le sommet, et perpendiculaire à la droite d'intersection de deux plans tangents quelconques sera un plan tangent du cône; les plans d'un tel système seront aussi des plans tangents de la conique mentionnée ci-dessus: donc le sommet du cône sera le point d'intersection de trois plans rectangles de la conique; et ce sommet étant un point entièrement indéterminé, le lieu de l'intersection des trois plans tangents rectangles de la conique, sera de même absolument indéterminé, ou si l'on veut, ce lieu sera l'espace entier près les points à une distance infinie. La contradiction apparente dont M. Steiner parle, a par conséquent son origine dans l'indétermination qui a lieu dans le cas dont il s'agit. Dans tout autre cas, le point d'intersection des trois plans rectangles de la surface du second ordre est parfaitement déterminé, et les théorèmes I. et II. sont tous deux légitimes.

133.

SUR UN THÉORÈME DE M. SCHLÄFLI.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 278—282.]

On lit dans (§ 13) d'un mémoire très intéressant de M. Schläfli intitulé "Über die Resultante eines Systems mehrerer algebraischer Gleichungen" (*Mém. de l'Acad. de Vienne*, t. IV. [1852]) un très beau théorème sur les *Résultants*.

Pour faire voir plus clairement en quoi consiste ce théorème, je prends un cas particulier. Soit

$$U = ax^3 + 3bx^2y + 3cxy^2 + dy^3 = (a,\ b,\ c,\ d)(x,\ y)^3,$$
$$V = \alpha x^2 + 2\beta xy + \gamma y^2 \qquad = (\alpha,\ \beta,\ \gamma)(x,\ y)^2.$$

Je fais $p = x^2$, $q = xy$, $r = y^2$, et je forme les opérateurs

$$\mathfrak{A} = \ \xi\partial_a + \tfrac{1}{3}\eta\partial_b + \tfrac{1}{3}\zeta\partial_c,$$
$$\mathfrak{B} = \tfrac{1}{3}\xi\partial_b + \tfrac{1}{3}\eta\partial_c + \ \zeta\partial_d,$$

lesquels, opérant sur U, donnent

$$x(p\xi + q\eta + r\zeta)\,;\quad y(p\xi + q\eta + r\zeta).$$

L'opérateur

$$\mathfrak{C} = \xi\partial_\alpha + \tfrac{1}{2}\eta\partial_\beta + \zeta\partial_\gamma,$$

opérant sur V, donne

$$p\xi + q\eta + r\zeta.$$

Cela étant, soit $\phi = 0$ le *résultant* des équations $U = 0$, $V = 0$, c'est-à-dire l'équation que l'on obtient en éliminant x, y entre les équations $U = 0$, $V = 0$, ou autrement dit, soit ϕ le résultant des fonctions U, V. Pour fixer les idées j'écris la valeur de ce résultant comme suit:

$$\phi = \begin{vmatrix} & a, & 3b, & 3c, & d \\ a, & 3b, & 3c, & d & \\ & & a, & 2\beta, & \gamma \\ & a, & 2\beta, & \gamma & \\ a, & 2\beta, & \gamma & & \end{vmatrix}.$$

Je suppose que les opérateurs $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ opèrent sur le résultant ϕ, ce qui donne les fonctions

$$\mathfrak{A}\phi, \quad \mathfrak{B}\phi, \quad \mathfrak{C}\phi,$$

ou en écrivant pour $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ leurs valeurs:

$$\begin{aligned} &(\xi\partial_a + \tfrac{1}{3}\eta\partial_b + \tfrac{1}{3}\zeta\partial_c)\,\phi, \\ &(\tfrac{1}{3}\xi\partial_b + \tfrac{1}{3}\eta\partial_c + \zeta\partial_d)\,\phi, \\ &(\xi\partial_\alpha + \tfrac{1}{2}\eta\partial_\beta + \zeta\partial_\gamma)\,\phi, \end{aligned}$$

et en considérant ces expressions comme des fonctions de ξ, η, ζ, j'en forme le résultant Φ, savoir

$$\Phi = \begin{vmatrix} \partial_a\phi, & \tfrac{1}{3}\partial_b\phi, & \tfrac{1}{3}\partial_c\phi \\ \tfrac{1}{3}\partial_b\phi, & \tfrac{1}{3}\partial_c\phi, & \partial_d\phi \\ \partial_\alpha\phi, & \tfrac{1}{2}\partial_\beta\phi, & \partial_\gamma\phi \end{vmatrix}.$$

Ce résultant Φ contiendra le carré de ϕ *comme facteur;* c'est ce qui donne, dans le cas particulier dont il s'agit, le théorème de M. Schläfli.

Généralement, en supposant que l'on ait autant de fonctions U, V, W, ... que d'indéterminées x, y, z, ..., on peut supposer que p, q, ... soient des monômes $x^l y^m z^n$, ... du même degré λ (il n'est pas nécessaire d'avoir la série entière de ces monômes), et on peut former des opérateurs $\mathfrak{A}$, $\mathfrak{B}$, &c. en même nombre que celui des monômes p, q, ... avec les indéterminées ξ, η, ..., tels que ces opérateurs $\mathfrak{A}$, $\mathfrak{B}$, ..., opérant sur les fonctions U, V, W, ... (chacun sur la fonction à laquelle il appartient), donnent $t(p\xi + q\eta \ldots)^\mu$, $t'(p\xi + q\eta \ldots)^{\mu'}$, &c.; t, t', &c. étant des monômes de la forme $x^f y^g z^h \ldots$.

Cela étant, soit ϕ le résultant des fonctions U, V, W, ...; en opérant sur ce résultant ϕ avec les opérateurs $\mathfrak{A}$, $\mathfrak{B}$, ... et en formant ainsi les fonctions $\mathfrak{A}\phi$, $\mathfrak{B}\phi$, ..., soit Φ le résultant de ces expressions considérées comme des fonctions de ξ, η, &c. Φ contiendra une puissance de ϕ *comme facteur*, et en supposant que μ ne soit plus petit qu'aucun autre des indices μ, μ', ...; $\pi = \mu\mu' \ldots$; et $\sigma = \frac{\pi}{\mu} + \frac{\pi}{\mu'} + \ldots$, l'indice de cette puissance sera au moins $\sigma - \frac{\pi}{\mu}$. Voilà le théorème général de M. Schläfli.

La démonstration donnée dans le mémoire cité est, on ne peut plus, simple et élégante. Elle repose d'abord sur un théorème connu (démontré au reste § 6) qui peut être énoncé ainsi; savoir, en supposant que les équations $U=0$, $V=0, \ldots$ soient satisfaites, on aura (près un facteur indépendant de ξ, $\eta, \ldots$):

$$\mathfrak{A}\phi = t\,(p\xi + q\eta \ldots)^{\mu}, \quad \mathfrak{B}\phi = t'\,(p\xi + q\eta \ldots)^{\mu'}, \text{ \&c.}$$

Puis, elle est fondée sur le théorème démontré (§ 12), savoir: le résultant des fonctions

$$\begin{aligned} &t\,(p\xi + q\eta \ldots)^{\mu} + f\,(\xi, \eta \ldots), \\ &t'\,(p\xi + q\eta \ldots)^{\mu'} + f'\,(\xi, \eta \ldots), \\ &\vdots \end{aligned}$$

(où f, $f', \ldots$ sont des polynômes de degrés μ, $\mu', \ldots$ en ξ, η, &c., et p, $q, \ldots$, t, $t', \ldots$ des constantes quelconques) sera, en supposant que μ ne soit plus petit qu'aucun autre des indices μ, $\mu', \ldots$, et en posant $\pi = \mu\mu' \ldots$, tout au plus du degré $\frac{\pi}{\mu}$ par rapport aux quantités t, t', &c. Voici cette démonstration, qui suppose aussi que le résultant ϕ soit *indécomposable*. Supposons que les coefficients de U, V, $W, \ldots$ soient assujettis à la seule condition d'être tels que le résultant ϕ soit un infiniment petit du premier ordre, il sera permis de supposer que tous ces coefficients des indéterminées x, $y, \ldots$ ne diffèrent des valeurs qui satisfont aux équations $U=0$, $V=0$, $W=0, \ldots$ que par des incréments infiniment petits du premier ordre; le résultant ϕ sera un infiniment petit du premier ordre, mais toute autre fonction des coefficients, à moins qu'elle ne contienne une puissance de ϕ comme facteur, aura une valeur finie, et toute fonction des coefficients infiniment petite de l'ordre k contiendra ϕ^k *comme facteur*. Dans cette supposition les équations $\mathfrak{A}\phi = 0$, $\mathfrak{B}\phi = 0$, &c. deviendront:

$$\begin{aligned} &t\,(p\xi + q\eta \ldots)^{\mu} + f\,(\xi, \eta \ldots) = 0, \\ &t'\,(p\xi + q\eta \ldots)^{\mu'} + f'\,(\xi, \eta \ldots) = 0, \\ &\vdots \end{aligned}$$

où f, $f', \ldots$ sont des polynômes de degrés μ, $\mu', \ldots$ dont les coefficients sont des infiniment petits du premier ordre. En supposant toujours que μ ne soit plus petit qu'aucun autre des indices $\mu, \mu', \ldots$ et en posant $\pi = \mu\mu' \ldots$, $\sigma = \frac{\pi}{\mu} + \frac{\pi}{\mu'} \ldots$, le resultant Φ du système sera tout au plus du degré $\frac{\pi}{\mu}$ par rapport aux quantités finies t, $t', \ldots$. Le degré par rapport à tous les coefficients est σ; le degré par rapport aux coefficients de f, $f', \ldots$ sera done au moins $\sigma - \frac{\pi}{\mu}$; c'est-à-dire, ce résultant sera un infiniment petit de l'ordre $\sigma - \frac{\pi}{\mu}$, ou enfin, Φ contiendra $\phi^{\sigma - \pi : \mu}$ *comme facteur*. Or les coefficients de U, V, $W, \ldots$ (assujettis à la seule condition ci-dessus mentionnée) étant d'ailleurs arbitraires, on voit sans peine qu'il est permis de faire abstraction de la *condition*, et que Φ contiendra en général cette même puissance $\phi^{\sigma - \pi : \mu}$ *comme facteur;* ce qu'il s'agissait de démontrer.

Rien n'empêche que Φ ne contienne une plus haute puissance que $\phi^{\sigma-\pi:\mu}$ comme facteur, ou que Φ ne s'évanouisse identiquement. On peut même assigner de plus près que l'a fait M. Schläfli, des cas où Φ s'évanouit identiquement. Soient $m, m', m'', \ldots$ les degrés de $U, V, W, \ldots$ par rapport à $x, y, z, \ldots$, $p = mm'm'' \ldots$, $s = \frac{p}{m} + \frac{p}{m'} + \frac{p}{m''} \ldots$, ϕ sera du degré $\frac{p}{m}$ par rapport aux coefficients de U. Soient aussi $\mu_{,}, \mu_{,,}, \ldots$ les degrés de celles des fonctions $\mathfrak{A}\phi, \mathfrak{B}\phi, \ldots$, pour lesquelles les opérateurs $\mathfrak{A}, \mathfrak{B}, \ldots$ contiennent des différentielles par rapport aux coefficients de U, $\rho = \frac{\pi}{\mu_{,}} + \frac{\pi}{\mu_{,,}} \ldots$: pour ces fonctions les coefficients seront du degré $\frac{p}{m} - 1$ par rapport aux coefficients de U; pour les autres ils seront du degré $\frac{p}{m}$. Φ sera donc du degré $\left(\frac{p}{m} - 1\right)\rho + \frac{p}{m}(\sigma - \rho), = \frac{p}{m}\sigma - \rho$, par rapport aux coefficients dé U, et $\Phi \div \phi^{\sigma-\pi:\mu}$ sera du degré $\frac{p}{m}\sigma - \rho - \frac{p}{m}\left(\sigma - \frac{\pi}{\mu}\right)$, c'est-à-dire du degré $\frac{p}{m} \cdot \frac{\pi}{\mu} - \rho$ par rapport aux coefficients de U. De même, en supposant que les lettres $m', \rho', \ldots$ aient rapport à V, &c., $\Phi \div \phi^{\sigma-\pi:\mu}$ sera du degré $\frac{p}{m'} \cdot \frac{\pi}{\mu} - \rho'$, &c. par rapport aux coefficients de V, &c. Si l'un quelconque des nombres $\frac{p}{m} \cdot \frac{\pi}{\mu} - \rho$, $\frac{p}{m'} \cdot \frac{\pi}{\mu} - \rho'$, &c. est *négatif*, et à plus forte raison, si leur somme $s \cdot \frac{\pi}{\mu} - \sigma$ est *negative*, Φ doit s'évanouir identiquement. En particulier, en supposant que le nombre des fonctions $\mathfrak{A}\phi, \mathfrak{B}\phi, \ldots$ (c'est-à-dire le nombre des indéterminés $\xi, \eta, \ldots$) soit ν, on aura $\sigma > \nu \frac{\pi}{\mu}$, et par cette raison Φ s'évanouira identiquement si $\frac{\pi}{\mu}(\sigma - \nu)$ est négatif, c'est-à-dire si $\nu > \sigma$. Je ne parlerai pas ici des cas examinés par M. Schläfli, où Φ contient comme facteur une plus haute puissance que $\phi^{\sigma-\pi:\mu}$.

134.

REMARQUES SUR LA NOTATION DES FONCTIONS ALGÉBRIQUES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 282—285.]

JE me sers de la notation

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \dots \\ \alpha', & \beta', & \gamma', & \dots \\ \alpha'', & \beta'', & \gamma'', & \dots \\ \vdots & & & \end{vmatrix}$$

pour représenter ce que j'appelle une *matrice;* savoir un *système* de quantités rangées en forme de *carré,* mais d'ailleurs tout à fait *indépendantes* (je ne parle pas ici des *matrices rectangulaires*). Cette notation me parait très commode pour la théorie des équations *linéaires;* j'écris par exemple

$$(\xi, \eta, \zeta, \dots) = \left(\begin{vmatrix} \alpha, & \beta, & \gamma, \dots \\ \alpha', & \beta', & \gamma', \dots \\ \alpha'', & \beta'', & \gamma'', \dots \\ \vdots & & \end{vmatrix}\right)(x, y, z, \dots)$$

pour représenter le système des équations

$$\begin{aligned} \xi &= \alpha\, x + \beta\, y + \gamma\, z \dots, \\ \eta &= \alpha' x + \beta' y + \gamma' z \dots, \\ \zeta &= \alpha'' x + \beta'' y + \gamma'' z \dots, \\ &\vdots \end{aligned}$$

On obtient par là l'équation:

$$(x, y, z, \ldots) = \left(\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \vdots & & & \end{vmatrix}^{-1} \right)(\xi, \eta, \zeta, \ldots),$$

qui représente le système d'équations qui donne $x, y, z, \ldots$ en termes de $\xi, \eta, \zeta, \ldots$, et on se trouve ainsi conduit à la notation

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \vdots & & & \end{vmatrix}^{-1}$$

de la matrice *inverse*. Les termes de cette matrice sont des fractions, ayant pour *dénominateur* commun le *déterminant* formé avec les termes de la matrice originale; les *numérateurs* sont les déterminants mineurs formés avec les termes de cette même matrice en supprimant l'une quelconque des lignes et l'une quelconque des colonnes.

Soit encore

$$(x, y, z, \ldots) = \left(\begin{vmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \\ a'', & b'', & c'', & \ldots \\ \vdots & & & \end{vmatrix} \right)(x, y, z, \ldots),$$

on peut écrire:

$$(\xi, \eta, \zeta, \ldots) = \left(\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \vdots & & & \end{vmatrix} \begin{vmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \\ a'', & b'', & c'', & \ldots \\ \vdots & & & \end{vmatrix} \right)(x, y, z, \ldots),$$

et l'on parvient ainsi à l'idée d'une *matrice composée*, par ex.

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \vdots & & & \end{vmatrix} \begin{vmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \\ a'', & b'', & c'', & \ldots \\ \vdots & & & \end{vmatrix}.$$

On voit d'abord que la valeur de cette matrice composée est

$$\begin{vmatrix} (\alpha, \beta, \gamma, \ldots)(a, a', a'', \ldots), & (\alpha, \beta, \gamma, \ldots)(b, b', b'', \ldots), & \ldots \\ (\alpha', \beta', \gamma', \ldots)(a, a', a'', \ldots), & (\alpha', \beta', \gamma', \ldots)(b, b', b'', \ldots), & \ldots \\ \vdots & & \end{vmatrix}$$

où $(\alpha, \beta, \gamma, \ldots)(a, a', a'', \ldots) = \alpha a + \beta a' + \gamma a'' + \ldots$. Il faut faire attention, dans la composition des matrices, de combiner les *lignes* de la matrice à gauche avec les *colonnes* de la matrice à droite, pour former les *lignes* de la matrice composée. Il y aurait bien des choses à dire sur cette théorie de matrices, laquelle doit, il me semble, précéder la théorie de *Déterminants.*

Une notation semblable peut être employée dans la théorie des fonctions *quadratiques.* En effet, on peut dénoter par

$$\left(\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \vdots & & & \end{vmatrix}\right)(\xi, \eta, \zeta)(x, y, z)$$

la fonction *lineo-linéaire*

$$\begin{aligned} &(\alpha\,\xi + \beta\,\eta + \gamma\,\zeta \ldots)\,x \\ &+ (\alpha'\,\xi + \beta'\,\eta + \gamma'\,\zeta \ldots)\,y \\ &+ (\alpha''\xi + \beta''\eta + \gamma''\zeta \ldots)\,z \\ &\quad\vdots \end{aligned}$$

et de là par

$$\left(\begin{vmatrix} a, & h, & g, & \ldots \\ h, & b, & f, & \ldots \\ g, & f, & c, & \ldots \\ \vdots & & & \end{vmatrix}\right)(x, y, z, \ldots)^2$$

la fonction *quadratique*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \ldots$$

que je représente aussi par

$$(a, b, c, \ldots f, g, h, \ldots)(x, y, z, \ldots)^2.$$

Je remarque qu'en général je représente une fonction rationnelle et intégrale, homogène et des *degrés* m, m', &c., par rapport aux indéterminées x, y, &c., x', y', &c., de la manière suivante:

$$(\,\Diamond\,)(x, y, \ldots)^m(x', y', \ldots)^{m'} \ldots.$$

Une fonction rationnelle et intégrale, homogène et du degré m par rapport aux deux indéterminées x, y sera donc représentée par

$$(\,\Diamond\,)(x, y)^m.$$

En introduisant dans cette notation les *coefficients*, j'écris par exemple

$$(a,\ b,\ c,\ d\,\mathfrak{X}\,x,\ y)^3,$$

pour représenter la fonction

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

tandis que je me sers de la notation

$$(a,\ b,\ c,\ d\,\tilde{\mathfrak{X}}\,x,\ y)^3,$$

pour représenter la fonction

$$ax^3 + bx^2y + cxy^2 + dy^3,$$

et de même pour les fonctions d'un degré quelconque. J'ai trouvé cette distinction très commode.

[In the foregoing Paper as here printed, except in the expression in the second line of this page,)(is used instead of)(: it appears by a remark (*Crelle*, t. LI, errata) that the manuscript had the interlaced parentheses $\mathfrak{X}$. Moreover in the manuscript $\begin{pmatrix} & \\ | & | \end{pmatrix}$ was used for a Matrix, which was thus distinguished from a Determinant, but in the absence of any real ambiguity, no alteration has been made in this respect. In the reprint of subsequent papers from Crelle, the arrowhead $\tilde{\mathfrak{X}}$ or $\tilde{\mathfrak{X}}$ is used instead of $(\smile)$.]

135.

NOTE SUR LES COVARIANTS D'UNE FONCTION QUADRATIQUE, CUBIQUE, OU BIQUADRATIQUE À DEUX INDÉTERMINÉES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 285—287.]

La théorie d'une fonction à deux indéterminées d'un degré quelconque, par exemple

$$(\,\diamond\,)(x,\ y)^m,$$

dépend du système des *covariants* de la fonction, lequel est censé contenir la fonction elle-même.

Pour une fonction *quadratique* le système de covariants est

$$(a,\ b,\ c)(x,\ y)^2,$$
$$ac-b^2.$$

Pour la fonction *cubique*, le système est

$$(a,\ b,\ c,\ d)(x,\ y)^3,$$
$$(ac-b^2,\ ad-bc,\ bd-c^2)(x,\ y)^2,$$
$$(-a^2d+3abc-2b^3,\ -abd+2ac^2-b^2c,\ acd-2b^2d+bc^2,\ ad^2-3bcd+2c^3)(x,\ y)^3,$$
$$-a^3d^2+6abcd-4ac^3-4b^3d+3b^2c^2,$$

fonctions lesquelles, en supposant qu'on les représente par U, H, Φ, $\square$, satisfont identiquement à l'équation

$$\Phi^2+\square U^2=-4H^3.$$

Pour la fonction *biquadratique*, le système est

$$(a,\ b,\ c,\ d,\ e)(x,\ y)^4,$$

$$ae - 4bd + 3c^2,$$

$$(ac - b^2,\ 2ad - 2bc,\ ae + 2bd - 3c^2,\ 2be - 2cd,\ ce - d^2)(x,\ y)^4,$$

$$ace + 2bcd - ad^2 - b^2e - c^3,$$

$$\left\{\begin{array}{l} -a^2d + 3abc - 2b^3, \\ -a^2e - 2abd + 9ac^2 - 6b^2c, \\ -5abe + 15acd - 10b^2d, \\ +10a^2d - 10b^2e, \\ +5ade + 10bd^2 - 15bce, \\ +ae^2 + 2bde - 9c^2e + 6cd^2, \\ +be^2 - 3cde + 2d^3, \end{array}\right\}(x,\ y)^6,$$

et ces fonctions, en supposant qu'on les représente par U, I, H, J, Φ, satisfont identiquement à l'équation

$$JU^3 - IU^2H + 4H^3 = -\Phi^2.$$

J'ajoute à ce système la fonction

$$\begin{aligned} I^3 - 27J^2 = {} & a^2e^2 - 12a^2bde^2 - 18a^2c^2e^2 + 54a^2cd^2e - 27a^2d \\ & + 54ab^2ce^2 - 6ab^2d^2e - 180abc^2de + 108abcd^3 + 81ac^4e \\ & - 54ac^3d^2 - 27b^4e^2 - 64b^3d^3 + 108b^3cde - 54b^3c^3e \\ & + 36b^2c^2d^2, \end{aligned}$$

qui est le *discriminant* de la fonction biquadratique.

Pour donner une application de ces formules, soit proposé de résoudre une équation quadratique, cubique ou biquadratique, ou autrement dit: de trouver un *facteur linéaire* de la fonction quadratique, cubique, ou biquadratique.

Il est assez singulier que pour la fonction quadratique la solution est en quelque sorte plus compliquée que pour les deux autres. En effet, il n'existe pas de solution symétrique, à moins qu'on n'introduise des quantités arbitraires et superflues; savoir, on trouve pour facteur linéaire de $(a,\ b,\ c)(x,\ y)^2$ l'expression

$$(a,\ b,\ c)(\alpha,\ \beta)(x,\ y) + \sqrt{\ } - \square\,.\,(\beta x - \alpha y),$$

où $(a,\ b,\ c)(\alpha,\ \beta)(x,\ y)$ dénote $a\alpha x + b(\alpha y + \beta x) + c\beta y$.

Pour la fonction *cubique*, l'équation $\Phi^2 + \square U^2 = -4H^3$ fait voir que les deux fonctions $\Phi + U\sqrt{\ } - \square$, $\Phi - U\sqrt{\ } - \square$ sont l'une et l'autre des cubes parfaits. L'expression

$$\sqrt[3]{\{\tfrac{1}{2}(\Phi + U\sqrt{\ } - \square)\}} - \sqrt[3]{\{\tfrac{1}{2}(\Phi - U\sqrt{\ } - \square)\}}$$

sera donc une fonction linéaire de x, y; et puisque cette fonction s'évanouit pour $U=0$, elle ne sera autre chose que l'un des facteurs linéaires de $(a, b, c, d)(x, y)^3$.

Pour la fonction *biquadratique*, en partant de l'équation

$$JU^3 - IU^2H + 4H^3 = -\Phi^2,$$

j'écris

$$M = \frac{I^3}{4J^2},$$

et je mets l'équation sous la forme

$$(1, 0, -M, M)(IH, JU)^3 = -\tfrac{1}{4} I^3 \Phi^2.$$

Donc, en supposant que ϖ_1, ϖ_2, ϖ_3 soient les racines de l'équation

$$(1, 0, -M, M)(\varpi, 1)^3 = 0,$$

ou plus simplement de l'équation

$$\varpi^3 - M(\varpi - 1) = 0,$$

ces expressions $IH - \varpi_1 JU$, $IH - \varpi_2 JU$, $IH - \varpi_3 JU$ seront toutes trois des carrés de fonctions quadratiques. L'expression

$$(\varpi_2 - \varpi_3)\sqrt{(IH - \varpi_1 JU)} + (\varpi_3 - \varpi_1)\sqrt{(IH - \varpi_2 JU)} + (\varpi_1 - \varpi_2)\sqrt{(IH - \varpi_3 JU)}$$

sera donc une fonction quadratique, et on voit sans peine qu'elle sera le carré d'une fonction *linéaire*. Or cette expression s'évanouit pour $U=0$; donc ce sera précisément le carré de l'un quelconque des facteurs linéaires de $(a, b, c, d, e)(x, y)^4$.

L'équation identique pour les covariants d'une fonction biquadratique donne lieu aussi (remarque que je dois à M. Hermite) à une transformation très élégante de *l'intégrale elliptique* $\int dx \div \sqrt{(a, b, c, d, e)(x, 1)^4}$.

136.

SUR LA TRANSFORMATION D'UNE FONCTION QUADRATIQUE EN ELLE-MÊME PAR DES SUBSTITUTIONS LINÉAIRES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 288—299.]

IL s'agit de trouver les transformations linéaires d'une fonction quadratique $(\diamond)(x, y, z, \ldots)^2$ *en elle-même*, c'est-à-dire de trouver pour $(x, y, z, \ldots)$ des fonctions linéaires de $x, y, z, \ldots$ telles que

$$(\diamond)(\mathrm{x}, \mathrm{y}, z, \ldots)^2 = (\diamond)(x, y, z, \ldots)^2.$$

En représentant la fonction quadratique par

$$(\diamond)(x, y, z, \ldots)^2 = \begin{vmatrix} a, & h, & g, \ldots \\ h, & b, & f, \ldots \\ g, & f, & c, \ldots \\ \vdots & & \end{vmatrix})(x, y, z, \ldots)^2,$$

la solution qu'a donnée M. Hermite de ce problème peut être résumée dans la seule équation

$$(\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots) =$$

$$\left(\begin{vmatrix} a, h, g, \ldots \\ h, b, f, \ldots \\ g, f, c, \ldots \\ \vdots \end{vmatrix}^{-1} \begin{vmatrix} a, & h-\nu, & g+\mu, \ldots \\ h+\nu, & b, & f-\lambda, \ldots \\ g-\mu, & f+\lambda, & c\ , \ldots \\ \vdots & & \end{vmatrix} \begin{vmatrix} a, & h+\nu, & g-\mu, \ldots \\ h-\nu, & b, & f+\lambda, \ldots \\ g+\mu, & f-\lambda, & c\ , \ldots \\ \vdots & & \end{vmatrix}^{-1} \begin{vmatrix} a, h, g, \ldots \\ h, b, f, \ldots \\ g, f, c, \ldots \\ \vdots \end{vmatrix} \right)(x, y, z, \ldots),$$

où $\lambda, \mu, \nu, \ldots$ sont des quantités quelconques.

En effet, pour démontrer que cela est une solution, on n'a qu'à reproduire dans un ordre inverse le procédé de M. Hermite. En introduisant les quantités auxiliaires $(\xi, \eta, \zeta, \ldots)$, on peut remplacer l'équation par les deux équations

$$\left(\begin{vmatrix} a, & h, & g, \ldots \\ h, & b, & f, \ldots \\ g, & f, & c, \ldots \\ \vdots & & \end{vmatrix}\right)(x, y, z, \ldots) = \left(\begin{vmatrix} a, & h+\nu, & g-\mu, \ldots \\ h-\nu, & b, & f+\lambda, \ldots \\ g+\mu, & f-\lambda, & c \quad, \ldots \\ \vdots & & \end{vmatrix}\right)(\xi, \eta, \zeta, \ldots)$$

$$\left(\begin{vmatrix} a, & h, & g, \ldots \\ h, & b, & f, \ldots \\ g, & f, & c, \ldots \\ \vdots & & \end{vmatrix}\right)(\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots) = \left(\begin{vmatrix} a, & h-\nu, & g+\mu, \ldots \\ h+\nu, & b, & f-\lambda, \ldots \\ g-\mu, & f+\lambda, & c \quad, \ldots \\ \vdots & & \end{vmatrix}\right)(\xi, \eta, \zeta, \ldots)$$

qui donnent tout de suite d'abord

$$(\Diamond)(x, y, z, \ldots)(\xi, \eta, \zeta, \ldots) = (\Diamond)(\xi, \eta, \zeta, \ldots)^2,$$

et puis

$$x + \mathrm{x} = 2\xi, \quad y + \mathrm{y} = 2\eta, \quad z + \mathrm{z} = 2\zeta, \quad \&c.$$

On obtient par là:

$$\begin{aligned}(\Diamond)(\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots)^2 &= (\Diamond)(2\xi - \mathrm{x}, 2\eta - \mathrm{y}, 2\zeta - \mathrm{z}, \ldots)^2, \\ &= 4(\Diamond)(\xi, \eta, \zeta, \ldots)^2 - 4(\Diamond)(\xi, \eta, \zeta, \ldots)(x, y, z, \ldots) \\ &\quad + (\Diamond)(x, y, z, \ldots)^2,\end{aligned}$$

c'est-à-dire l'équation

$$(\Diamond)(\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots)^2 = (\Diamond)(x, y, z, \ldots)^2,$$

qu'il s'agissait de vérifier.

Je remarque que la transformation est toujours *propre*. En effet, le déterminant de transformation est

$$\begin{vmatrix} a, h, g \ldots \\ h, b, f \ldots \\ g, f, c \ldots \\ \vdots \end{vmatrix}^{-1} \begin{vmatrix} a, & h-\nu, & g+\mu \ldots \\ h+\nu, & b, & f-\lambda \ldots \\ g-\mu, & f+\lambda, & c \quad \ldots \\ \vdots & & \end{vmatrix} \begin{vmatrix} a, & h+\nu, & g-\mu \ldots \\ h-\nu, & b, & f+\lambda \ldots \\ g+\mu, & f-\lambda, & c \quad \ldots \\ \vdots & & \end{vmatrix}^{-1} \begin{vmatrix} a, h, g \ldots \\ h, b, f \ldots \\ g, f, c \ldots \\ \vdots \end{vmatrix}.$$

Or les déterminants qui entrent dans les deux termes moyens, ne contiennent l'un ou l'autre que les puissances *paires* de $\lambda, \mu, \nu, \ldots$. Done ces deux déterminants sont égaux, et les quatre termes du produit sont *réciproques* deux à deux; le déterminant de transformation est donc $+1$, et la transformation est propre.

Pour obtenir une transformation *impropre*, il faut considérer une fonction quadratique qui contient outre les indéterminées $x, y, z, \ldots$ une indéterminée θ, et puis réduire à

zéro les coefficients de tous les termes dans lesquels entre cette indéterminée θ. Les valeurs de x, y, z, ... ne contiendront pas θ, et en représentant par ϑ l'indéterminée que l'on doit ajouter à la suite x, y, z, ..., la valeur de ϑ sera, comme on voit sans peine, $\vartheta = -\theta$; le déterminant de transformation pour la forme aux indéterminées $x, y, z, \ldots, \theta$ sera $+1$, et ce déterminant sera le produit du déterminant de transformation pour la forme aux indéterminées $x, y, z, \ldots$ multiplié par -1. Le déterminant de transformation pour la forme aux indéterminées $x, y, z, \ldots$ sera donc -1, c'est-à-dire, la transformation sera *impropre*.

Au lieu de la formule de transformation ci-dessus, on peut se servir des formules

$$(\xi, \eta, \zeta, \ldots) = \left(\begin{vmatrix} a, & h+\nu, & g-\mu, \ldots \\ h-\nu, & b, & f+\lambda, \ldots \\ g+\mu, & f-\lambda, & c, \ldots \\ \vdots & & \end{vmatrix}^{-1} \begin{vmatrix} a, & h, & g, \ldots \\ h, & b, & f, \ldots \\ g, & f, & c, \ldots \\ \vdots & & \end{vmatrix}\right)(x, y, z, \ldots),$$

$$\mathrm{x} = 2\xi - x, \quad \mathrm{y} = 2\eta - y, \quad \mathrm{z} = 2\zeta - z, \ldots.$$

Par exemple, en supposant que la forme à transformer soit

$$ax^2 + by^2 + cz^2 + \&c.,$$

on aura

$$(\xi, \eta, \zeta, \ldots) = \left(\begin{vmatrix} a, & \nu, & -\mu, \ldots \\ -\nu, & b, & \lambda, \ldots \\ \mu, & -\lambda, & c, \ldots \\ \vdots & & \end{vmatrix}^{-1}\right)(ax, by, cz, \ldots),$$

$$\mathrm{x} = 2\xi - x, \quad \mathrm{y} = 2\eta - y, \quad \mathrm{z} = 2\zeta - z, \ \&c.,$$

de manière qu'en posant

$$\begin{vmatrix} a, & \nu, & -\mu, \ldots \\ -\nu, & b, & \lambda, \ldots \\ \mu, & -\lambda, & c, \ldots \\ \vdots & & \end{vmatrix} = k,$$

$$\begin{vmatrix} a, & \nu, & -\mu, \ldots \\ -\nu, & b, & \lambda, \ldots \\ \mu, & -\lambda, & c, \ldots \\ \vdots & & \end{vmatrix}^{-1} = \frac{1}{k} \begin{vmatrix} A, & B, & C, \ldots \\ A', & B', & C', \ldots \\ A'', & B'', & C'', \ldots \\ \vdots & & \end{vmatrix},$$

on aura

$$(\mathrm{x}, \mathrm{y}, \mathrm{z} \ldots) = \frac{1}{k} \begin{vmatrix} 2A - \frac{k}{a}, & 2B, & 2C & \ldots \\ 2A', & 2B' - \frac{k}{b}, & 2C' & \ldots \\ 2A'', & 2B'', & 2C'' - \frac{k}{c} & \ldots \\ \vdots & & & \end{vmatrix}(ax, by, cz, \ldots),$$

ce qui est l'équation pour la transformation propre en elle-même, de la fonction $ax^2+by^2+cz^2+$ &c. On en déduira, comme dans le cas général, la formule pour la transformation impropre. On trouvera des observations sur cette formule dans le mémoire "Recherches ultérieures sur les déterminants gauches" [137].

Je reviens à l'équation générale

$$(\,\Diamond\,)(\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \ldots)^2=(\,\Diamond\,)(x,\ y,\ z,\ \ldots)^2,$$

et je suppose seulement que x, y, z, ... soient des fonctions linéaires de x, y, z, ... qui satisfont à cette équation sans supposer rien davantage par rapport à la forme de la solution. Cela étant, je forme les fonctions linéaires $\mathrm{x}-sx$, $\mathrm{y}-sy$, $\mathrm{z}-sz$, &c., où s est une quantité quelconque, et je considère la fonction

$$(\,\Diamond\,)(\mathrm{x}-sx,\quad \mathrm{y}-sy,\quad \mathrm{z}-sz\ \ldots)^2,$$

laquelle, en la développant, devient

$$(1+s^2)(\,\Diamond\,)(x,\ y,\ z\ \ldots)^2-2s(\,\Diamond\,)(x,\ y,\ z\ \ldots)(\xi,\ \eta,\ \zeta\ \ldots);$$

et en développant de la même manière la fonction quadratique

$$(\,\Diamond\,)\left(\mathrm{x}-\frac{1}{s}x,\quad \mathrm{y}-\frac{1}{s}y,\quad \mathrm{z}-\frac{1}{s}z,\ \ldots\right)^2,$$

on obtient l'équation identique

$$(\,\Diamond\,)(\mathrm{x}-sx,\ \mathrm{y}-sy,\ \mathrm{z}-sz,\ \ldots)^2=s^2\,.\,(\,\Diamond\,)\left(\mathrm{x}-\frac{1}{s}x,\quad \mathrm{y}-\frac{1}{s}y,\quad \mathrm{z}-\frac{1}{s}z,\ \ldots\right)^2.$$

Soit $\square$ le déterminant formé avec les coefficients de fonctions linéaires $\mathrm{x}-sx$, $\mathrm{y}-sy$, $\mathrm{z}-sz$, &c. En supposant que le nombre des indéterminées x, y, z, &c., est n, $\square$ sera évidemment une fonction rationnelle et intégrale du degré n par rapport à s. Soit de même $\square'$ le déterminant formé avec les coefficients de

$$\mathrm{x}-\frac{1}{s}x,\quad \mathrm{y}-\frac{1}{s}y,\quad \mathrm{z}-\frac{1}{s}z,\ \&c.;$$

l'équation qui vient d'être trouvée, donne $\square^2=s^{2n}\square'^2$, c'est-à-dire $\square=\pm s^n\square'$. Cela fait voir que les coefficients du premier et du dernier terme, du second et de l'avant-dernier terme, &c., sont égaux, aux signes près. De plus, le coefficient de la plus haute puissance s^n est toujours ± 1, et on voit sans peine qu'en supposant d'abord que n soit *impair*, on a pour la transformation *propre:*

$$\square=(1,\quad P,\ \ldots P,\quad 1)(-s,\ 1)^n$$

et pour la transformation *impropre*

$$\square=(1,\ -P,\ \ldots P,\ -1)(-s,\ 1)^n:$$

équation qui peut être changée en celle-ci: $\square=-(1,\ P,\ \ldots P,\ 1)(s,\ 1)^n$. Puis, en supposant que n soit *pair*, on a pour la transformation *propre:*

$$\square=(1,\quad P,\ \ldots P,\quad 1)(-s,\ 1)^n$$

et pour la transformation *impropre:*

$$\square=(1,\ -P,\ \ldots P,\ -1)(-s,\ 1)^n,$$

le coefficient moyen étant dans ce cas égal à zéro. Ces théorèmes pour la forme du déterminant des fonctions linéaires $x - sx$, $y - sy$, $z - sz$, ... sont dus à M. Hermite.

Il y a à remarquer que la forme $(\diamond)(x, y, z \ldots)^2$ est tout à fait indéterminée; c'est-à-dire, on suppose seulement une x, y, z, ... soient des fonctions linéaires de $x, y, z, \ldots$, telles qu'il y ait une forme quadratique $(\diamond)(x, y, z, \ldots)^2$ pour laquelle l'équation $(\diamond)(x, y, z \ldots)^2 = (\diamond)(x, y, z \ldots)^2$ est satisfaite.

Je regarde d'un autre point de vue ce problème de la transformation en elle-même, d'une fonction quadratique par des substitutions linéaires. Je suppose que x, y, z, &c. soient des fonctions linéaires données de $x, y, z, \ldots$, et je cherche une fonction linéaire de x, y, z, &c. qui, par la substitution de x, y, z, &c. au lieu de x, y, z, &c. se transforme en elle-même à un facteur près. Soit $(l, m, n, \ldots)(x, y, z, \ldots)$, cette fonction linéaire, il faut que $(l, m, n, \ldots)(x, y, z, \ldots)$ soit identiquement $= s.(l, m, n, \ldots)(x, y, z, \ldots)$, ou, ce qui est la même chose, que $(l, m, n, \ldots)(x - sx, y - sy, z - sz, \ldots)$ soit $= 0$; c'est-à-dire, les quantités $l, m, n, \ldots$ seront déterminées par autant d'équations linéaires dont les coefficients sont précisément ceux de $x - sx$, $y - sy$, $z - sz$, &c.; donc s sera déterminé si l'on rend égal à zéro le déterminant formé avec ces coefficients, et l, m, n, &c. se trouveront donnés rationnellement en termes de s. Cela étant, je suppose que les racines de l'équation en s soient $a, b, c, \ldots$, et ces différentes racines correspondront aux fonctions linéaires $x_a, x_b, x_c, \ldots$ qui ont la propriété dont il s'agit. Soit $(\diamond)(x, y, z, \ldots)^2$ une fonction quadratique qui se transforme en elle-même par la substitution de x, y, z, &c. au lieu de x, y, z, &c. Cette fonction peut être exprimée en fonction quadratique de x_a, x_b, x_c, &c.; quantités qui, en substituant x, y, z, &c. au lieu de $x, y, z, \ldots$ deviennent $ax_a, bx_b, cx_c, \ldots$.

Je prends les cas d'une fonction *binaire, ternaire*, &c., et d'abord le cas d'une fonction *binaire*.

En écrivant d'abord $(\diamond)(x, y)^2 = (A, B, C)(x_a, x_b)^2$, on doit obtenir identiquement $(A, B, C)(ax_a, bx_b)^2 = (A, B, C)(x_a, x_b)^2$, c'est-à-dire $A(a^2 - 1) = 0$, $B(ab - 1) = 0$, $C(b^2 - 1) = 0$. Or la solution $A = B = C = 0$ ne signifiant rien, on ne peut satisfaire à ces équations sans supposer des relations entre les quantités a, b; et pour obtenir une solution dans laquelle la fonction quadratique ne se réduit pas à un carré, il faut supposer, ou $ab - 1 = 0$, ou $a^2 - 1 = 0$ et $b^2 - 1 = 0$. Le premier cas est celui de la transformation *propre*. Il donne

$$ab = 1, \quad (\diamond)(x, y)^2 = l\, x_a x_b.$$

Le second cas est celui de la transformation *impropre*. Il donne

$$a = +1, \quad b = -1, \quad (\diamond)(x, y)^2 = l\, x_a^2 + m\, x_b^2.$$

En passant au cas d'une fonction *ternaire*, soit

$$(\diamond)(x, y, z)^2 = (A, B, C, F, G, H)(x_a, x_b, x_c)^2;$$

on doit avoir identiquement

$$(A, B, C, F, G, H)(ax_a, bx_b, cx_c)^2 = (A, B, C, F, G, H)(x_a, x_b, x_c)^2,$$

c'est-à-dire $A(a^2-1)=0$, $B(b^2-1)=0$, $C(c^2-1)=0$, $F(bc-1)=0$, $G(ca-1)=0$, $H(ab-1)=0$, et on voit que pour obtenir une solution dans laquelle la fonction quadratique ne se réduit pas à un carré, ou à une fonction de deux indéterminées, il faut supposer par exemple $a^2-1=0$, $bc-1=0$. On a donc dans le cas d'une fonction ternaire:

$$a^2=1, \quad bc=1, \quad (\diamond)(x, y, z)^2 = l\,\mathrm{x}_a^2 + m\,\mathrm{x}_b\mathrm{x}_c.$$

La transformation sera *propre*, ou *impropre*, selon que $a=+1$ ou $a=-1$.

Dans le cas d'une fonction *quaternaire*, on obtient pour la transformation *propre:*

$$ab=cd=1, \quad (\diamond)(x, y, z, w)^2 = l\,\mathrm{x}_a\mathrm{x}_b + m\,\mathrm{x}_c\mathrm{x}_d,$$

et pour la transformation *impropre:*

$$a=+1, \quad b=-1, \quad cd=1, \quad (\diamond)(x, y, z, w)^2 = l\,\mathrm{x}_a^2 + m\,\mathrm{x}_b^2 + n\,\mathrm{x}_c\mathrm{x}_d.$$

Dans le cas d'une fonction *quinaire* on obtient

$$a^2=1, \quad bc=de=1, \quad (\diamond)(x, y, z, w, t)^2 = l\,\mathrm{x}_a^2 + m\,\mathrm{x}_b\mathrm{x}_c + n\,\mathrm{x}_d\mathrm{x}_e$$

et la transformation est propre ou impropre, selon que $a=+1$ ou $a=-1$; et ainsi de suite.

Cette méthode a des difficultés dans le cas où l'équation en s a des racines *égales*. Je n'entre pas ici dans ce sujet.

Dans les formules qu'on vient de trouver, on peut considérer les coefficients l, m, &c. comme des quantités *arbitraires*. Mais en supposant que la fonction quadratique soit *donnée*, ces coefficients deviennent *déterminés*. On les trouvera par la formule suivante que je ne m'arrête pas à démontrer.

Soient α, β, γ, &c. les coefficients de la fonction linéaire x_a, α', β', γ', &c. les coefficients de la fonction linéaire x_b, et ainsi de suite; alors, dans les différentes formules qui viennent d'être données, le coefficient d'un terme x_a^2 à droite sera

$$\frac{-k}{(\dagger)(\alpha, \beta, \gamma, \ldots)},$$

et le coefficient d'un terme $\mathrm{x}_a\mathrm{x}_b$ à gauche sera

$$\frac{-k}{(\dagger)(\alpha, \beta, \gamma, \ldots)(\alpha', \beta', \gamma', \ldots)},$$

où k dénote le *discriminant* de la fonction quadratique à gauche, et où les coefficients des fonctions quadratiques des dénominateurs sont les coefficients inverses de cette même fonction quadratique à gauche[1].

[1] Je profite de cette occasion pour remarquer concernant ces recherches que les formules données dans la note sur les fonctions du second ordre (t. XXXVIII. [1848] p. 105) [71] pour les cas de trois et de quatre indéterminées, sont exactes, mais que je m'étais trompé dans la forme générale du théorème. [This correction is indicated vol. I. p. 589.]

L'application de la méthode à la forme binaire $(a, b, c)(x, y)^2$ donne lieu aux développements suivants.

J'écris $x = \alpha x + \beta y$, $y = \gamma x + \delta y$, et je représente par $(l, m)(x, y)$ une fonction linéaire qui par cette substitution est transformée en *elle-même*, au facteur s près. Nous aurons donc

$$(l, m)(\alpha x + \beta y, \gamma x + \delta y) = s\,(l, m)(x, y);$$

l'équation pour s sera

$$s^2 - s\,(\delta + \alpha) + \alpha\delta - \beta\gamma = 0;$$

laquelle peut aussi être écrite comme suit:

$$(1, -\delta - \alpha, \alpha\delta - \beta\gamma)(s, 1)^2 = 0.$$

Soient s', s'' les racines de cette équation. (Il est à peine nécessaire de remarquer que s', s'', et plus bas P, Q, sont ici ce que dans les formules générales j'ai représenté par a, b et x_a, x_b. De même les équations $\rho = s's''$, $\rho = s'^2$, $\rho = s''^2$, obtenues après, correspondent aux équations $ab = 1$, $a^2 = 1$, $b^2 = 1$.) On aura

$$s' + s'' = -\delta - \alpha, \qquad s's'' = \alpha\delta - \beta\gamma,$$

et les coefficients l, m seront déterminés rationnellement par s.

Mais on peut aussi déterminer ces coefficients par l'équation

$$l : m = l\alpha + m\gamma : l\beta + m\delta,$$

qui peut être écrite sous la forme

$$(\beta, \delta - \alpha, -\gamma)(l, m)^2 = 0,$$

et en éliminant entre cette équation et l'équation $lx + my = 0$ les quantités l, m, on voit que les fonctions linéaires $lx + my$ sont les facteurs de la fonction quadratique $(\beta, \delta - \alpha, -\gamma)(y, -x)^2$, ou, ce qui est la même chose, de la fonction quadratique

$$(\gamma, \delta - \alpha, -\beta)(x, y)^2;$$

je représente ces facteurs par P, Q et je remarque encore que l'équation en s aura des racines égales si

$$(\delta - \alpha)^2 + 4\beta\gamma = 0,$$

et que dans ce cas, et *exclusivement* dans ce cas, les fonctions P, Q ne forment qu'une seule et même fonction linéaire.

Je suppose maintenant que la fonction $(a, b, c)(x, y)^2$ se transforme en elle-même par la substitution $\alpha x + \beta y$, $\gamma x + \delta y$ au lieu de x, y, ou, ce qui est ici plus commode, je suppose que les deux fonctions sont égales à un facteur près, et j'écris

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 = \rho\,(a, b, c)(x, y)^2.$$

En développant cette équation, on obtient

$$\left.\begin{array}{lll} x^2(a,\ b,\ c)(\alpha^2-\rho, & 2\alpha\gamma, & \gamma^2\ ) \\ +\,2\,xy\,(a,\ b,\ c)(\alpha\beta, & \alpha\delta+\beta\gamma-\rho, & \gamma\delta\ ) \\ +\quad y^2(a,\ b,\ c)(\beta^2, & 2\beta\delta, & \delta^2-\rho) \end{array}\right\} = 0.$$

Voilà trois équations linéaires pour déterminer par les quantités α, β, γ, δ, considérées comme données, les coefficients $(a,\ b,\ c)$ de la fonction quadratique. Les coefficients de ces équations linéaires sont

$$\begin{array}{lll} \alpha^2-\rho, & 2\alpha\gamma, & \gamma^2, \\ \alpha\beta, & \alpha\delta+\beta\gamma-\rho, & \gamma\delta, \\ \beta^2, & 2\beta\delta, & \delta^2-\rho. \end{array}$$

Le système inverse par lequel on trouve les valeurs de a, b, c, est

$$\begin{array}{lll} \delta^2(\alpha\delta-\beta\gamma)-(\alpha\delta+\beta\gamma+\delta^2)\rho+\rho^2, & -\beta\delta(\alpha\delta-\beta\gamma)+\alpha\beta\rho, & \beta^2(\alpha\delta-\beta\gamma)-\beta^2\rho, \\ -2\gamma\delta(\alpha\delta-\beta\gamma)+2\alpha\gamma\rho, & \alpha^2\delta^2-\beta^2\gamma^2-(\delta^2+\alpha^2)\rho+\rho^2, & -2\alpha\beta(\alpha\delta-\beta\gamma)+2\beta\delta\rho, \\ \gamma^2(\alpha\delta-\beta\gamma)+\gamma^2\rho, & -\alpha\gamma(\alpha\delta-\beta\gamma)+\gamma\delta\rho, & \alpha^2(\alpha\delta-\beta\gamma)-(\alpha\delta+\beta\gamma+\alpha^2)\rho+\rho^2, \end{array}$$

et le déterminant, égalé à zéro, donne

$$(\alpha\delta-\beta\gamma-\rho)\,\{(\alpha\delta-\beta\gamma+\rho)^2-\rho\,(\alpha+\delta)^2\}=0:$$

équation dont les racines sont

$$\rho=\alpha\delta-\beta\gamma,\quad \rho=\{\tfrac{1}{2}(\alpha+\delta)\pm\tfrac{1}{2}\sqrt{(\alpha-\delta)^2+4\beta\gamma}\}^2.$$

En comparant ces valeurs avec celles de s', s'', on voit que les racines de l'équation en ρ sont

$$\rho=s's'',\quad \rho=s'^2,\quad \rho=s''^2,$$

et nous allons voir que ces valeurs de ρ donnent en général les valeurs PQ, P^2, Q^2, pour la fonction quadratique.

Soit d'abord $\rho=\alpha\delta-\beta\gamma\,(=s's'')$, et posons pour abréger $\alpha\delta-\beta\gamma-\rho=\phi$, le système inverse devient:

$$\begin{array}{lll} (\delta^2-\rho)\,\phi-\beta\rho\,.\,2\gamma, & -\beta\delta\phi-\beta\rho\,(\delta-\alpha), & \beta^2\phi+\beta\rho\,.\,2\beta, \\ -2\gamma\delta\phi-\rho\,(\delta-\alpha)\,2\gamma, & (\alpha\delta+\beta\gamma-\rho)\,\phi-\ \rho\,(\delta-\alpha)^2, & -2\alpha\beta\phi+\rho\,(\delta-\alpha)\,2\beta, \\ -\gamma^2\phi+\gamma\rho\,.\,2\gamma, & -\alpha\gamma\phi+\gamma\rho\,(\delta-\alpha), & (\alpha^2-\rho)\,\phi-\gamma\rho\,.\,2\beta, \end{array}$$

et en mettant $\phi=0$, les termes de chaque ligne (en omettant un facteur) deviennent γ, $\frac{1}{2}(\delta-\alpha)$, β. On obtient ainsi dans ce cas, pour la fonction quadratique $(a,\ b,\ c)(x,\ y)^2$ la valeur

$$(\gamma,\ \delta-\alpha,\ -\beta)(x,\ y)^2,$$

qui est en effet le produit PQ des fonctions linéaires.

Il y a à remarquer qu'en supposant $(\delta-\alpha)^2+4\beta\gamma=0$, ce qui est le cas pour lequel ρ sera une racine *triple*, il n'y aura pas de changement à faire dans ce résultat. La fonction quadratique est, comme auparavant, le produit PQ des fonctions linéaires;

seulement ces deux fonctions linéaires dans le cas actuel sont identiques, de manière que la fonction quadratique se réduit à P^2.

Soit ensuite

$$\rho = \{\tfrac{1}{2}(\alpha+\delta) \pm \tfrac{1}{2}\sqrt{(\alpha-\delta)^2+4\beta\gamma}\}^2 \,(= s'^2 \text{ ou } s''^2);$$

en écrivant $\rho = s^2$ et en mettant pour abréger $\alpha\delta - \beta\gamma - s(\delta+\alpha) + s^2 = \chi$, le système inverse devient

$$\begin{array}{ll}
(\delta^2+s^2)\chi + s(\delta-s)^2(\delta+\alpha), & -\beta\delta\chi - \beta s(\delta-s)(\delta+\alpha), \\
-2\gamma\delta\chi - 2\gamma s(\delta-s)(\delta+\alpha), & \{s^2+s(\delta+\alpha)+\alpha\delta+\beta\gamma\}\chi + 2\beta\gamma s(\delta+\alpha), \\
\gamma^2\chi + \gamma^2 s(\delta+\alpha), & -\alpha\gamma\chi - \gamma s(\alpha-s)(\delta+\alpha), \\
 & \beta^2\chi + \beta^2 s(\delta+\alpha), \\
 & -2\alpha\beta\chi - 2\beta s(\alpha-s)(\delta+\alpha), \\
 & (\alpha^2+s^2)\chi + s(\alpha-s)^2(\delta+\alpha).
\end{array}$$

Donc, en écrivant $\chi = 0$ et en omettant le facteur $s(\delta+\alpha)$, le système inverse devient

$$\begin{array}{lll}
(\delta-s)^2, & -\beta(\delta-s), & \beta^2, \\
\gamma(\delta-s), & \beta\gamma, & -\beta(\alpha-s), \\
\gamma^2, & -\gamma(\beta-s), & (\alpha-s)^2,
\end{array}$$

et les quantités dans chaque ligne sont dans le rapport $l^2 : lm : m^2$, de manière que la fonction quadratique est dans ce cas égale à P^2 ou Q^2. Cela suppose que $\delta+\alpha$ ne soit égal à zéro. En faisant pour le moment $\rho = 1$, on en tire la conclusion qu'à moins de supposer $\delta+\alpha = 0$, il n'existe pas de fonction quadratique binaire proprement dite (fonction non carrée) qui par la substitution *impropre* $\alpha x+\beta y$, $\gamma x+\delta y$ pour x, y, se transforme en elle-même. L'équation $\delta+\alpha = 0$ donne $\rho = \alpha\delta - \beta\gamma$, qui est une racine double de l'équation cubique. On remarquera en passant par rapport à la signification de l'équation $\delta+\alpha = 0$, que l'on a en général:

$$\begin{aligned}
&(\alpha, \beta)(\alpha x+\beta y, \gamma x+\delta y) : (\gamma, \delta)(\alpha x+\beta y, \gamma x+\delta y) \\
&= (\alpha^2+\beta\gamma)x + \beta(\delta+\alpha)y : \gamma(\delta+\alpha)x + (\delta^2+\beta\gamma)y,
\end{aligned}$$

et de là, qu'en supposant $\delta+\alpha = 0$, on a

$$(\alpha, \beta)(\alpha x+\beta y, \gamma x+\delta y) : (\gamma, \delta)(\alpha x+\beta y, \gamma x+\delta y) = x, y.$$

Cela revient à dire qu'en faisant deux fois la substitution $\alpha x+\beta y$, $\gamma x+\delta y$ au lieu de x, y, on retrouve les quantités x, y, ou que la substitution est *périodique* du second ordre. Il y a aussi à remarquer que dans le cas dont il s'agit, savoir pour $\delta+\alpha = 0$, on a $s'' = -s'$, et que les deux fonctions linéaires P, Q restent parfaitement déterminées.

Nous venons de voir qu'il n'existe pas de transformation impropre d'une fonction quadratique binaire proprement dite, à moins que $\delta+\alpha$ ne soit pas $= 0$. Mais en supposant $\delta+\alpha = 0$, on voit que les coefficients des équations pour a, b, c deviennent

$$\begin{array}{lll}
-\beta\gamma, & -\gamma(\delta-\alpha), & \gamma^2, \\
\alpha\beta, & \alpha(\delta-\alpha), & -\alpha\gamma, \\
\beta^2, & \beta(\delta-\alpha), & \beta\gamma,
\end{array}$$

c'est-à-dire: les coefficients de chaque équation sont dans le rapport de

$$\beta, \quad \delta - \alpha, \quad -\gamma,$$

de manière qu'en supposant que les coefficients a, b, c satisfont à la seule équation

$$(a, b, c)(\beta, \delta - \alpha, -\gamma) = 0,$$

où α, β, γ, δ sont des quantités quelconques, telles que $\delta + \alpha = 0$, on aura

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 = -(\alpha\delta - \beta\gamma)(a, b, c)(x, y)^2.$$

Ce n'est là qu'un cas particulier de l'équation identique

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 + (\alpha\delta - \beta\gamma) \,.\, (a, b, c)(x, y)^2 =$$
$$(\delta + \alpha) \,.\, \big(a\alpha + b\gamma, b(\delta + \alpha), b\beta + c\delta\big)(x, y)^2 \quad + (\beta, \delta - \alpha, -\gamma)(a, b, c) \,.\, (\beta, \delta - \alpha, -\gamma)(y, -x)^2.$$

Il faut remarquer qu'en supposant toujours l'équation

$$(a, b, c)(\beta, \delta - \alpha, -\gamma) = 0,$$

la fonction quadratique $(a, b, c)(x, y)^2$, *en supposant qu'elle se réduise à un carré*, est comme auparavant P^2 ou Q^2, c'est-à-dire le carré de l'une des fonctions linéaires. En effet: en écrivant $(a, b, c)(x, y)^2 = (lx + my)^2$, l'équation entre l, m serait évidemment $(\beta, \delta - \alpha, -\gamma)(l, m)^2 = 0$, de manière que l, m auraient les mêmes valeurs qu'auparavant. J'ajoute que tout ce qui précède par rapport à l'équation

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 = \rho\,(a, b, c)(x, y)^2$$

fait voir qu'à moins que la fonction quadratique ne soit un carré, on aura toujours $\rho = \pm(\alpha\delta - \beta\gamma)$; ce qu'on savait déjà dès le commencement, et ce qui peut être démontré comme à l'ordinaire, en égalant les discriminants $(ac - b^2)(\alpha\delta - \beta\gamma)^2$ et $(ac - b^2)\rho^2$ des deux côtés. Je fais enfin $\rho = 1$, ce qui donne l'équation

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 = (a, b, c)(x, y)^2,$$

et (en faisant abstraction du cas où la fonction quadratique est un carré) je tire de ce qui précède les résultats connus, savoir, que l'on a:

1. Pour la transformation propre:

$$\alpha\delta - \beta\gamma = 1,$$
$$a : 2b : c = \gamma : \delta - \alpha : -\beta.$$

2. Pour la transformation impropre:

$$\alpha\delta - \beta\gamma = -1, \quad \delta + \alpha = 0,$$
$$a\beta + b(\delta - \alpha) - c\gamma = 0.$$

Je crois que cette discussion a été utile pour compléter la théorie algébrique de la forme binaire $(a, b, c)(x, y)^2$.

137.

RECHERCHES ULTÉRIEURES SUR LES DÉTERMINANTS GAUCHES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 299—313: Continuation of the Memoir t. XXXII. (1846) and t. XXXVIII. (1849); **52** and **69.**]

J'AI déjà donné une formule pour le développement d'un *déterminant gauche.* En prenant, pour fixer les idées, un cas particulier, soit

$$\overline{12345 \mid 12345} = \begin{vmatrix} 11, & 12, & 13, & 14, & 15 \\ 21, & 22, & 23, & 24, & 25 \\ 31, & 32, & 33, & 34, & 35 \\ 41, & 42, & 43, & 44, & 45 \\ 51, & 52, & 53, & 54, & 55 \end{vmatrix},$$

(où $12 = -21$, &c., tandis que les quantités 11, 22, &c. ne s'évanouissent pas). Cette formule peut être écrite comme suit:

$$\begin{aligned} \overline{12345 \mid 12345} = \; & 11 \,.\, 22 \,.\, 33 \,.\, 44 \,.\, 55 \\ & + 11 \,.\, 22 \,.\, 33 \,.\, (45)^2 \\ & + 11 \,.\, 22 \,.\, 44 \,.\, (35)^2 \\ & + 11 \,.\, 22 \,.\, 55 \,.\, (34)^2 \\ & + 11 \,.\, 33 \,.\, 44 \,.\, (25)^2 \\ & + 11 \,.\, 33 \,.\, 55 \,.\, (24)^2 \\ & + 11 \,.\, 44 \,.\, 55 \,.\, (23)^2 \\ & + 22 \,.\, 33 \,.\, 44 \,.\, (15)^2 \\ & + 22 \,.\, 33 \,.\, 55 \,.\, (14)^2 \\ & + 22 \,.\, 44 \,.\, 55 \,.\, (13)^2 \\ & + 33 \,.\, 44 \,.\, 55 \,.\, (12)^2 \\ & + 11 \,.\, (2345)^2 \\ & + 22 \,.\, (1345)^2 \\ & + 33 \,.\, (1245)^2 \\ & + 44 \,.\, (1235)^2 \\ & + 55 \,.\, (1234)^2. \end{aligned}$$

Les expressions 12, 1234, &c. à droite sont ici des *Pfaffians*. On a

$$12 = 12,$$

$$1234 = 12.34 + 13.42 + 14.23$$

et en écrivant encore un terme, pour mieux présenter la loi:

$$\begin{aligned} 123456 = {} & 12.34.56 + 13.45.62 + 14.56.23 + 15.62.34 + 16.23.45 \\ & + 12.35.64 + 13.46.25 + 14.52.36 + 15.63.42 + 16.24.53 \\ & + 12.36.45 + 13.42.56 + 14.53.62 + 15.64.23 + 16.25.34. \end{aligned}$$

J'ai trouvé récemment une formule analogue pour le développement d'un *déterminant gauche* bordé, tel que

$$\overline{\alpha 1234 \mid \beta 1234} = \begin{vmatrix} \alpha\beta, & \alpha 1, & \alpha 2, & \alpha 3, & \alpha 4 \\ 1\beta, & 11, & 12, & 13, & 14 \\ 2\beta, & 21, & 22, & 23, & 24 \\ 3\beta, & 31, & 32, & 33, & 34 \\ 4\beta, & 41, & 42, & 43, & 44 \end{vmatrix};$$

cette formule est:

$$\begin{aligned} \overline{\alpha 1234 \mid \beta 1234} = {} & \alpha\beta \,.\, 11 \,.\, 22 \,.\, 33 \,.\, 44 \\ & + \alpha\beta \,.\, 12 \,.\, 12 \,.\, 33 \,.\, 44 \\ & + \alpha\beta \,.\, 13 \,.\, 13 \,.\, 22 \,.\, 44 \\ & + \alpha\beta \,.\, 14 \,.\, 14 \,.\, 22 \,.\, 33 \\ & + \alpha\beta \,.\, 23 \,.\, 23 \,.\, 11 \,.\, 44 \\ & + \alpha\beta \,.\, 24 \,.\, 24 \,.\, 11 \,.\, 33 \\ & + \alpha\beta \,.\, 34 \,.\, 34 \,.\, 11 \,.\, 22 \\ & + \alpha\beta \,.\, 1234 \,.\, 1234 \\ & + \alpha 1 \,.\, \beta 1 \,.\, 22 \,.\, 33 \,.\, 44 \\ & + \alpha 2 \,.\, \beta 2 \,.\, 11 \,.\, 33 \,.\, 44 \\ & + \alpha 3 \,.\, \beta 3 \,.\, 11 \,.\, 22 \,.\, 44 \\ & + \alpha 4 \,.\, \beta 4 \,.\, 11 \,.\, 22 \,.\, 33 \\ & + \alpha 123 \,.\, \beta 123 \,.\, 44 \\ & + \alpha 124 \,.\, \beta 124 \,.\, 33 \\ & + \alpha 134 \,.\, \beta 134 \,.\, 22 \\ & + \alpha 234 \,.\, \beta 234 \,.\, 11. \end{aligned}$$

Il est à peine nécessaire de remarquer que dans les *Pfaffians* à droite, où entrent des symboles tels que 1α, $\beta 1$, &c., qui ne se trouvent pas dans le déterminant dont il s'agit, il faut écrire $1\alpha = -\alpha 1$, $\beta 1 = -1\beta$, &c. Le symbole $\beta\alpha$ ne se trouve ni dans le déterminant, ni au côté droit. Cependant il convient de poser $\beta\alpha = -\alpha\beta$; car cela étant, il serait permis de transformer les *Pfaffians*, en écrivant par exemple $\alpha\beta 12 = -\beta\alpha 12$.

Je remarque en passant que, si avant de poser l'équation $\beta\alpha = -\alpha\beta$, on suppose que les deux symboles α, β deviennent identiques (si par exemple on écrit $\alpha = \beta = 5$), on aurait par exemple

$$\alpha\beta . 12 = \alpha\beta . 12 + \alpha 1 . 2\beta + \alpha 2 . \beta 1 = 55 . 12 + 51 . 25 + 52 . 51 = 55 . 12, \text{ \&c.},$$

et on retrouverait ainsi la formule pour le développement de $\overline{12345 \mid 12345}$.

La nouvelle formule peut être appliquée immédiatement au développement des *déterminants mineurs*. En effet, en se servant de la notation des *matrices*, on aura

$$\begin{vmatrix} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{vmatrix}^{-1} = \frac{1}{\overline{123 \mid 123}} \begin{vmatrix} +\overline{23 \mid 23}, & -\overline{13 \mid 23}, & -\overline{12 \mid 32} \\ -\overline{23 \mid 13}, & +\overline{13 \mid 13}, & -\overline{21 \mid 31} \\ -\overline{32 \mid 12}, & -\overline{31 \mid 21}, & +\overline{12 \mid 12} \end{vmatrix},$$

$$\begin{matrix} 11, & 12, & 13, & 14 \\ 21, & 22, & 23, & 24 \\ 31, & 32, & 33, & 34 \\ 41, & 42, & 43, & 44 \end{matrix}\Bigg|^{-1} = \frac{1}{\overline{1234 \mid 1234}} \begin{vmatrix} +\overline{234 \mid 234}, & -\overline{134 \mid 234}, & -\overline{124 \mid 324}, & -\overline{123 \mid 423} \\ -\overline{234 \mid 134}, & +\overline{134 \mid 134}, & -\overline{214 \mid 314}, & -\overline{213 \mid 413} \\ -\overline{324 \mid 124}, & -\overline{314 \mid 214}, & +\overline{214 \mid 214}, & -\overline{312 \mid 412} \\ -\overline{423 \mid 123}, & -\overline{413 \mid 213}, & -\overline{412 \mid 312}, & +\overline{123 \mid 123} \end{vmatrix}$$

et ainsi de suite. On suppose toujours que chaque terme de la matrice à droite soit divisé par le dénominateur commun. On voit que les déterminants *mineurs* qui correspondent à des termes tels que $\alpha\alpha$, sont des déterminants *gauches ordinaires*, avec le signe $+$, tandis que les déterminants mineurs qui correspondent à des termes tels que $\alpha\beta$, sont des déterminants *gauches bordés* tels que $\overline{\beta\ldots \mid \alpha\ldots}$, avec le signe $-$.

Pour donner des exemples de la vérification de ces formules, je remarque que l'on doit avoir

$$\begin{aligned} \overline{123 \mid 123} = \; & 11 \,.\, \overline{23 \mid 23} \\ & - 12 \,.\, \overline{23 \mid 13} \\ & - 13 \,.\, \overline{32 \mid 12}: \end{aligned}$$

équation qui peut aussi être écrite sous la forme

$$\begin{aligned} \overline{123 \mid 123} = \; & 11 \,.\, \overline{23 \mid 23} \\ & + 21 \,.\, \overline{23 \mid 13} \\ & + 31 \,.\, \overline{32 \mid 12}. \end{aligned}$$

En effet, en développant les deux côtés, on obtient:

$$\begin{aligned} 11.22.33 + 11.(23)^2 + 22.(13)^2 + 33.(12)^2 = {} & 11.(22.33 + (23)^2) \\ & + 21.(21.33 + 23.13\dagger) \\ & + 31.(31.22 + 32.12\dagger). \end{aligned}$$

On voit que les deux termes marqués par un † se détruisent et que l'équation est *identique*. On doit avoir de même,

$$\begin{aligned} \overline{1234 \mid 1234} = {} & 11 \,.\, \overline{234 \mid 234} \\ & - 12 \,.\, \overline{234 \mid 134} \\ & - 13 \,.\, \overline{324 \mid 124} \\ & - 14 \,.\, \overline{423 \mid 123}, \end{aligned}$$

ou, ce qui est la même chose:

$$\begin{aligned} \overline{1234 \mid 1234} = {} & 11 \,.\, \overline{234 \mid 234} \\ & + 21 \,.\, \overline{234 \mid 134} \\ & + 31 \,.\, \overline{324 \mid 124} \\ & + 41 \,.\, \overline{423 \mid 123}; \end{aligned}$$

c'est-à-dire, en développant des deux côtés:

$$\begin{aligned} 11.22.33.44 + 11.22.(34)^2 + 11.33.(24)^2 + 11.44.(23)^2 & \\ + 22.33.(14)^2 + 22.44.(13)^2 + 33.44.(12)^2 + (1234)^2 = & \\ 11\,[22.33.44 + 22\,(34)^2 + 33\,(42)^2 + 44\,(23)^2] & \\ + 21\,[21.33.44 + 2134.34^* + 23.13.44\dagger + 24.14.33\dagger] & \\ + 31\,[31.22.44 + 3124.24^* + 32.12.44\dagger + 34.14.22\dagger] & \\ + 41\,[41.22.33 + 4123.23^* + 42.12.33\dagger + 43.13.22\dagger]. & \end{aligned}$$

Cette expression est en effet identique, comme on le voit en observant que les six termes marqués par un † se détruisent deux à deux, et que les trois termes marqués par un (*) sont ensemble équivalents à $(1234)^2$.

Je remarque que le nombre des termes du développement du déterminant gauche est toujours une *puissance de* 2, et que de plus, ce nombre se réduit à la moitié, en réduisant à zéro un terme quelconque $\alpha\alpha$. Mais outre cela, le déterminant prend dans cette supposition la forme de déterminant d'un ordre inférieur de l'unité. Je considère par exemple le déterminant gauche $\overline{123 \mid 123}$. En y faisant $33 = 0$ et en accentuant, pour y mettre plus de clarté, tous les symboles, on trouve

$$\overline{123 \mid 123'} = 11' \,.\, (23')^2 + 22' \,.\, (13')^2.$$

De là, en écrivant

$$11 = 13'.11', \quad 12 = 11'.23',$$
$$22 = 13'.22',$$

on obtient

$$\overline{12 \mid 12} = 11\,.\,22 + (12)^2$$
$$= 11'.\{22'.(13')^2 + 11'.(23')^2\},$$

c'est-à-dire

$$\overline{12 \mid 12} = 11'\,.\,\overline{123 \mid 123}'.$$

On a de même

$$\overline{1234 \mid 1234}' = 11'\,.\,22'\,.\,(34')^2 + 11'\,.\,33'\,.\,(24')^2 + 22'33'(14')^2 + (1234')^2$$

et de là, en écrivant

$$11 = 14'.11', \quad 12 = 11'.24', \quad 23 = 1234',$$
$$22 = 14'.22', \quad 13 = 11'.34',$$
$$33 = 14'.33',$$

on obtient

$$\overline{123 \mid 123} = 11\,.\,22\,.\,33 + 11\,.\,(23)^2 + 22\,.\,(31)^2 + 33\,(12)^2$$
$$= 11'.\,14'\,\{22'\,.\,33'\,.\,(14')^2 + (1234')^2 + 11'\,.\,22'\,.\,(34')^2 + 11'\,.\,33'\,.\,(24')^2\},$$

c'est-à-dire,

$$\overline{123 \mid 123} = 11'\,.\,14'\,.\,\overline{1234 \mid 1234}'.$$

De même

$$\begin{aligned}\overline{12345 \mid 12345}' = {} & 11'\,.\,22'\,.\,33'\,.\,(45')^2 \\ & + 11'\,.\,22'\,.\,44'\,.\,(35')^2 \\ & + 11'\,.\,33'\,.\,44'\,.\,(25')^2 \\ & + 22'\,.\,33'\,.\,44'\,.\,(15')^2 \\ & + 11'\,.\,(2345')^2 \\ & + 22'\,.\,(1345')^2 \\ & + 33'\,.\,(1245')^2 \\ & + 44'\,.\,(1235')^2.\end{aligned}$$

Or il est permis d'écrire

$$11 = 15'.11', \quad 12 = 11'.25', \quad 23 = 1235', \quad 1234 = 2345'.11'.15',$$
$$22 = 15'.22', \quad 13 = 11'.35', \quad 24 = 1245',$$
$$33 = 15'.33', \quad 14 = 11'.45', \quad 34 = 1345',$$
$$44 = 15'.44'.$$

En effet, les quantités à gauche ne sont liées entre elles que par la seule équation $1234 = 12\,.\,34 + 13\,.\,42 + 14\,.\,23$ qui est satisfaite identiquement par les valeurs à substituer pour les quantités qui y entrent. Cela étant, on trouve d'abord:

$$\overline{1234 \mid 1234} = 11'(15')^2\,\overline{12345 \mid 12345}'.$$

Je prends encore un exemple. On a

$$\begin{aligned}\overline{123456 \mid 123456'} = {} & 11' \,.\, 22' \,.\, 33' \,.\, 44' \,.\, (56')^2 \\ & + 11' \,.\, 22' \,.\, 33' \,.\, 55' \,.\, (46')^2 \\ & + 11' \,.\, 22' \,.\, 44' \,.\, 55' \,.\, (36')^2 \\ & + 11' \,.\, 33' \,.\, 44' \,.\, 55' \,.\, (26')^2 \\ & + 22' \,.\, 33' \,.\, 44' \,.\, 55' \,.\, (16')^2 \\ & + 11' \,.\, 22' \,.\, (3456')^2 \\ & + 11' \,.\, 33' \,.\, (2456')^2 \\ & + 11' \,.\, 44' \,.\, (2356')^2 \\ & + 11' \,.\, 55' \,.\, (2346')^2 \\ & + 22' \,.\, 33' \,.\, (1456')^2 \\ & + 22' \,.\, 44' \,.\, (1356')^2 \\ & + 22' \,.\, 55' \,.\, (1346')^2 \\ & + 33' \,.\, 44' \,.\, (1256')^2 \\ & + 33' \,.\, 55' \,.\, (1246')^2 \\ & + 44' \,.\, 55' \,.\, (1236')^2 \\ & + (123456')^2.\end{aligned}$$

Ici, il est permis d'écrire:

$$\begin{aligned} & 11 = 16' \,.\, 11', \quad 12 = 11' \,.\, 26', \quad 23 = 1236', \quad 34 = 1346', \quad 45 = 1456', \\ & 22 = 16' \,.\, 22', \quad 13 = 11' \,.\, 36', \quad 24 = 1246', \quad 35 = 1356', \\ & 33 = 16' \,.\, 33', \quad 14 = 11' \,.\, 46', \quad 25 = 1256', \\ & 44 = 16' \,.\, 44', \quad 15 = 11' \,.\, 56', \\ & 55 = 16' \,.\, 55', \\ & 1234 = 2346' \,.\, 11' \,.\, 16', \quad 2345 = 123456' \,.\, 16', \\ & 1235 = 2356' \,.\, 11' \,.\, 16', \\ & 1245 = 2456' \,.\, 11' \,.\, 16', \\ & 1345 = 3456' \,.\, 11' \,.\, 16\,; \end{aligned}$$

car les valeurs des quantités à gauche satisfont identiquement aux équations qui ont lieu entre ces mêmes quantités. Par exemple l'équation $1234 = 12 \,.\, 34 + 13 \,.\, 42 + 14 \,.\, 23$ devient $2346' \,.\, 16' = 26' \,.\, 1346' + 63' \,.\, 1246' + 46' \,.\, 1236'$.

Or l'expression à droite devient, en développant:

$$\begin{aligned} & 26' (13' \,.\, 46' + 14' \,.\, 63' + 16' \,.\, 34') \\ + {} & 63' (12' \,.\, 46' + 14' \,.\, 62' + 16' \,.\, 24') \\ + {} & 46' (12' \,.\, 36' + 13' \,.\, 62' + 16' \,.\, 23'), \end{aligned}$$

et les termes qui contiennent le facteur $16'$, donnent ensemble $16'.2346'$, les autres termes se détruisent deux à deux. On obtient enfin, en effectuant la substitution :

$$\overline{12345 \mid 12345} = 11' . (16')^2\, \overline{123456 \mid 123456'};$$

et ainsi de suite.

Je fais les mêmes substitutions dans les *matrices inverses*, en supprimant cependant la dernière ligne et la dernière colonne de chaque matrice. On trouve ainsi, en y ajoutant les équations ci-dessus trouvées par rapport aux déterminants :

$$11' . \overline{123 \mid 123'} = \overline{12 \mid 12},$$

$$\frac{1}{13' . \overline{123 \mid 123'}} \begin{vmatrix} +\overline{23 \mid 23'}, & -\overline{13 \mid 23'} \\ -\overline{23 \mid 13'}, & +\overline{13 \mid 13'} \end{vmatrix} = \frac{1}{\overline{12 \mid 12}} \begin{vmatrix} -\overline{2 \mid 2} + \frac{\overline{12 \mid 12}}{11}, & -\overline{1 \mid 2} \\ +\overline{2 \mid 1}, & +\overline{1 \mid 1} \end{vmatrix},$$

$$11' . 14' . \overline{1234 \mid 1234'} = \overline{123 \mid 123},$$

$$\frac{1}{14' . \overline{1234 \mid 1234'}} \begin{vmatrix} +\overline{234 \mid 234}, & -\overline{134 \mid 234}, & -\overline{124 \mid 324} \\ -\overline{234 \mid 124}, & +\overline{134 \mid 134}, & -\overline{214 \mid 314} \\ -\overline{324 \mid 124}, & -\overline{314 \mid 214}, & +\overline{124 \mid 124} \end{vmatrix} =$$

$$\frac{1}{\overline{123 \mid 123}} \begin{vmatrix} -\overline{23 \mid 23} + \frac{\overline{123 \mid 123}}{11}, & -\overline{13 \mid 23}, & -\overline{12 \mid 32} \\ +\overline{23 \mid 13}, & +\overline{13 \mid 13}, & -\overline{21 \mid 31} \\ +\overline{32 \mid 12}, & -\overline{31 \mid 21}, & +\overline{12 \mid 12} \end{vmatrix},$$

$$11' . (15')^2 . \overline{12345 \mid 12345} = \overline{1234 \mid 1234},$$

$$\frac{1}{15' . \overline{12345 \mid 12345'}} \begin{vmatrix} +\overline{2345 \mid 2345'}, & -\overline{1345 \mid 2345'}, & -\overline{1245 \mid 3245'}, & -\overline{1235 \mid 4235'} \\ -\overline{2345 \mid 1345'}, & +\overline{1345 \mid 1345'}, & -\overline{2145 \mid 3145'}, & -\overline{2135 \mid 4135'} \\ -\overline{3245 \mid 1245'}, & -\overline{3145 \mid 2145'}, & +\overline{1245 \mid 1245'}, & -\overline{3125 \mid 4125'} \\ -\overline{4235 \mid 1235'}, & -\overline{4135 \mid 2135'}, & -\overline{4125 \mid 3125'}, & +\overline{1235 \mid 1235'} \end{vmatrix} =$$

$$\frac{1}{\overline{1234 \mid 1234}} \begin{vmatrix} -\overline{234 \mid 234} + \frac{\overline{1234 \mid 1234}}{11}, & -\overline{134 \mid 234}, & -\overline{124 \mid 324}, & -\overline{123 \mid 423} \\ +\overline{234 \mid 134}, & +\overline{134 \mid 134}, & -\overline{214 \mid 314}, & -\overline{213 \mid 413} \\ +\overline{324 \mid 124}, & -\overline{314 \mid 214}, & +\overline{124 \mid 124}, & -\overline{312 \mid 423} \\ +\overline{423 \mid 123}, & -\overline{413 \mid 213}, & -\overline{412 \mid 312}, & +\overline{123 \mid 123} \end{vmatrix},$$

et ainsi de suite.

Il est bon de changer un peu la forme de ces équations. On en déduit sans peine:

$$\frac{1}{13'\,.\,\overline{123\mid 123}'}\begin{vmatrix} 2\,.\,\overline{23\mid 23}' - \dfrac{\overline{123\mid 123}'}{11}, & -2\,.\,\overline{13\mid 23}' \\ -2\,.\,\overline{23\mid 13}', & +2\,.\,\overline{13\mid 13}' - \dfrac{\overline{123\mid 123}'}{22} \end{vmatrix}$$

$$= \frac{1}{\overline{12\mid 12}}\begin{vmatrix} -2\,.\,\overline{2\mid 2} + \dfrac{\overline{12\mid 12}}{11}, & -2\,.\,\overline{1\mid 2} \\ +2\,.\,\overline{2\mid 1}, & +2\,.\,\overline{1\mid 1} - \dfrac{\overline{12\mid 12}}{22} \end{vmatrix},$$

$$\frac{1}{14'\,.\,\overline{1234\mid 1234}'}\begin{vmatrix} 2\,.\,\overline{234\mid 234}' - \dfrac{\overline{1234\mid 1234}'}{11'}, & -2\,.\,\overline{134\mid 234}', & -2\,.\,\overline{124\mid 324}' \\ -2\,.\,\overline{234\mid 134}', & 2\,.\,\overline{134\mid 134}' - \dfrac{\overline{1234\mid 1234}'}{22'}, & -2\,.\,\overline{214\mid 314}' \\ -2\,.\,\overline{324\mid 124}', & -2\,.\,\overline{314\mid 214}', & 2\,.\,\overline{124\mid 124}' - \dfrac{\overline{1234\mid 12}}{33} \end{vmatrix}$$

$$= \frac{1}{\overline{123\mid 123}}\begin{vmatrix} -2\,.\,\overline{23\mid 23} - \dfrac{\overline{123\mid 123}}{11}, & -2\,.\,\overline{13\mid 23}, & -2\,.\,\overline{12\mid 32} \\ +2\,.\,\overline{23\mid 13}, & +2\,.\,\overline{13\mid 13} - \dfrac{\overline{123\mid 123}}{22}, & -2\,.\,\overline{21\mid 31} \\ +2\,.\,\overline{32\mid 12}, & -2\,.\,\overline{31\mid 21}, & +2\,.\,\overline{12\mid 12} - \dfrac{\overline{123\mid 123}}{33} \end{vmatrix},$$

et ainsi de suite.

Les formules que je viens de présenter, peuvent être appliquées aussitôt à la solution de la question suivante: "Trouver x_1, x_2, x_3, &c., fonctions linéaires de x_1, x_2, x_3, &c. telles que

$$11\,x_1^2 + 22\,x_2^2 + 33\,x_3^2 + \&c. = 11\,x_1^2 + 22\,x_2^2 + 33\,x_3^2 + \&c.,"$$

c'est-à-dire: transformer une fonction quadratique $11x_1^2 + 22x_2^2 + 33x_3^2 + \&c.$ *en elle-même* par des substitutions linéaires. Il suffira d'écrire la solution pour le cas de trois indéterminées: on satisfait identiquement à l'équation

$$11\,x_1^2 + 22\,x_2^2 + 33\,x_3^2 = 11\,x_1^2 + 22\,x_2^2 + 33\,x_3^2$$

en écrivant

$$(x_1,\ x_2,\ x_3) = \frac{1}{\overline{123\mid 123}} \times$$

$$\begin{vmatrix} +2\,.\,\overline{23\mid 23} - \dfrac{\overline{123\mid 123}}{11}, & -2\,.\,\overline{13\mid 23}, & -2\,.\,\overline{12\mid 32} \\ -2\,.\,\overline{23\mid 13}, & +2\,.\,\overline{13\mid 13} - \dfrac{\overline{123\mid 123}}{22}, & -2\,.\,\overline{21\mid 31} \\ -2\,.\,\overline{32\mid 12}, & -2\,.\,\overline{31\mid 21}, & +2\,.\,\overline{12\mid 12} - \dfrac{\overline{123\mid 123}}{33} \end{vmatrix} (11x_1,\ 22x_2,\ 33x_3).$$

Voilà la transformation *propre*. On en tire la transformation *impropre* de $11x_1^2 + 22x_2^2$ *en elle-même* en écrivant $33 = 0$; car, cela posé, les valeurs de x_1, x_2 ne contiennent pas x_3, et l'on n'a plus besoin de la valeur de x_3. On obtient ainsi la solution suivante; savoir, on satisfait identiquement à l'équation

$$11'\,x_1^2 + 22'\,x_2^2 = 11'\,x_1^2 + 22'\,x_2^2$$

en écrivant

$$(x_1,\ x_2) = \frac{1}{\overline{123 \mid 123'}} \begin{vmatrix} 2\,.\,\overline{23 \mid 23'} - \dfrac{\overline{123 \mid 123'}}{11'}, & -2\,.\,\overline{13 \mid 23'} \\ -2\,.\,\overline{23 \mid 13'}, & 2\,.\,\overline{13 \mid 13'} - \dfrac{\overline{123 \mid 123'}}{22} \end{vmatrix} (11'x_1,\ 22'x_2),$$

ce qui est une transformation *impropre*. Mais en y faisant la substitution $11 = 13'\,.\,11'$, $22 = 13'\,.\,22'$, $12 = 11'\,.\,23'$, on réduit la solution à celle-ci, savoir on satisfait identiquement à l'équation $11x_1^2 + 22x_2^2 = 11x_1^2 + 22x_2^2$ en écrivant

$$(x_1,\ x_2) = \frac{1}{\overline{12 \mid 12}} \begin{vmatrix} -2\,.\,\overline{2 \mid 2} + \dfrac{\overline{12 \mid 12}}{11}, & -2\,.\,\overline{1 \mid 2} \\ +2\,.\,\overline{2 \mid 1}, & +2\,.\,\overline{1 \mid 1} - \dfrac{\overline{12 \mid 12}}{22} \end{vmatrix} (11x_1,\ 22x_2),$$

ce qui est encore une transformation *impropre*, qui correspond de plus près à la formule pour la transformation *propre;* la seule différence est que les signes des termes de la première colonne de la matrice en sont changés.

En introduisant des lettres simples a, b, &c. à la place des symboles 11, 22, &c., je considère d'abord la transformation *propre*

$$ax^2 + by^2 = ax^2 + by^2.$$

Ici, en écrivant

$$\begin{vmatrix} 11, & 12 \\ 21, & 22 \end{vmatrix} = \begin{vmatrix} a, & \nu \\ -\nu, & b \end{vmatrix},$$

la formule donne

$$(x,\ y) = \frac{1}{ab + \nu^2} \begin{vmatrix} ab - \nu^2, & -2\nu b \\ 2\nu a\ \ , & ab - \nu^2 \end{vmatrix} (x,\ y).$$

La transformation *impropre*

$$ax^2 + by^2 = ax^2 + by^2$$

s'obtient au moyen de la formule donnée plus bas pour la transformation *propre* de la fonction $ax^2 + by^2 + cz^2$ *en elle-même*. En y écrivant $c = 0$, on obtient

$$(x,\ y) = \frac{1}{a\lambda^2 + b\mu^2} \begin{vmatrix} a\lambda^2 - b\mu^2, & 2\lambda\mu b \\ 2\lambda\mu a\ \ , & -a\lambda^2 + b\mu^2 \end{vmatrix} (x,\ y).$$

J'ai déjà fait voir de quelle manière cette formule se rattache à la formule pour la transformation *propre;* la différence entre les formes de ces transformations dans ce cas simple est assez frappante.

Pour obtenir la transformation propre

$$ax^2+by^2+cz^2=ax^2+by^2+cz^2,$$

j'écris

$$\begin{vmatrix} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{vmatrix} = \begin{vmatrix} a, & \nu, & -\mu \\ -\nu, & b, & \lambda \\ \mu, & -\lambda, & c \end{vmatrix};$$

cette formule donne

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = \frac{1}{abc+a\lambda^2+b\mu^2+c\nu^2}\times$$

$$\begin{vmatrix} abc+a\lambda^2-b\mu^2-c\nu^2, & 2(\lambda\mu-c\nu)b\ , & 2(\nu\lambda+b\mu)c \\ 2(\lambda\mu+c\nu)a\ , & abc-a\lambda^2+b\mu^2-c\nu^2, & 2(\mu\nu-a\lambda)c \\ 2(\nu\lambda-b\mu)a\ , & 2(\mu\nu+a\lambda)b\ , & abc-a\lambda^2-b\mu^2-c\nu^2 \end{vmatrix}(x,\ y,\ z).$$

La transformation *impropre en elle-même*

$$ax^2+by^2+cz^2=ax^2+by^2+cz^2$$

peut être tirée de la transformation *propre en elle-même* de la fonction donnée ci-après $ax^2+by^2+cz^2+dw^2$; en y écrivant $d=0$, on obtient

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = \frac{1}{bc\rho^2+ca\sigma^2+ab\tau^2+\phi^2}\times$$

$$\begin{vmatrix} -bc\rho^2+ca\sigma^2+ab\tau^2-\phi^2, & -2b\tau\phi-2bc\rho\sigma\ , & 2c\sigma\phi-2bc\tau\rho \\ 2a\tau\phi-2ac\rho\sigma\ , & bc\rho^2-ca\sigma^2+ab\tau^2-\phi^2, & -2c\rho\phi-2ca\sigma\tau \\ -2a\sigma\phi-2ab\rho\tau\ , & 2b\rho\phi-2ab\sigma\tau\ , & bc\rho^2+ca\sigma^2-ab\tau^2-\phi^2 \end{vmatrix}(x,\ y,\ z).$$

Pour vérifier que cette expression n'est en effet autre chose que la formule pour la transformation *propre,* en y changeant les signes de tous les termes, j'écris dans la formule pour la transformation propre, $a=b=c=\omega$. On a ainsi pour la transformation propre

$$\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2=x^2+y^2+z^2,$$

l'équation

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = \frac{1}{\omega^2+\lambda^2+\mu^2+\nu^2}\begin{vmatrix} \omega^2+\lambda^2-\mu^2-\nu^2, & 2\lambda\mu-2\nu\omega\ , & 2\nu\lambda+2\mu\omega \\ 2\lambda\mu+2\nu\omega\ , & \omega^2-\lambda^2+\mu^2-\nu^2, & 2\mu\nu-2\lambda\omega \\ 2\nu\lambda-2\mu\omega\ , & 2\mu\nu+2\lambda\omega\ , & \omega^2-\lambda^2+\mu^2-\nu^2 \end{vmatrix}(x,\ y,\ z),$$

et en écrivant dans la formule pour la transformation impropre, $a=b=c=1$, $d=0$, et λ, μ, ν, $-\omega$ au lieu de ρ, σ, τ, ϕ, on obtient pour la transformation impropre

$$\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2=x^2+y^2+z^2,$$

l'équation

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z})=\frac{1}{\omega^2+\lambda^2+\mu^2+\nu^2}\times$$

$$\begin{vmatrix} -\omega^2-\lambda^2+\mu^2+\nu^2, & -2\lambda\mu+2\nu\omega\ , & -2\nu\lambda-2\mu\omega \\ -2\lambda\mu-2\nu\omega\ , & -\omega^2+\lambda^2-\mu^2+\nu^2, & 2\mu\nu-2\lambda\omega \\ -2\nu\lambda+2\mu\omega\ , & -2\mu\nu-2\lambda\omega\ , & -\omega^2+\lambda^2-\mu^2-\nu^2 \end{vmatrix}(x,\ y,\ z).$$

Pour obtenir la transformation propre

$$a\mathrm{x}^2-b\mathrm{y}^2+c\mathrm{z}^2+d\mathrm{w}^2=ax^2+by^2+cz^2+dw^2,$$

j'écris

$$\begin{vmatrix} 11, & 12, & 13, & 14 \\ 21, & 22, & 23, & 24 \\ 31, & 32, & 33, & 34 \\ 41, & 42, & 43, & 44 \end{vmatrix}=\begin{vmatrix} a, & \nu, & -\mu, & \rho \\ -\nu, & b, & \lambda, & \sigma \\ \mu, & -\lambda, & c, & \tau \\ -\rho, & -\sigma, & -\tau, & d \end{vmatrix};$$

cela donne d'abord, en mettant pour abréger,

$$\phi=\lambda\rho+\mu\sigma+\nu\tau,$$

la valeur du déterminant

$$\overline{1234\mid 1234}=abcd+bc\rho^2+ca\sigma^2+ab\tau^2+ad\lambda^2+bd\mu^2+cd\nu^2+\phi^2$$

(ce que je représente par k).

J'ajoute aussi la valeur de la *matrice* inverse

$$\begin{vmatrix} 11, & 12, & 13, & 14 \\ 21, & 22, & 23, & 24 \\ 31, & 32, & 33, & 34 \\ 41, & 42, & 43, & 44 \end{vmatrix}^{-1},$$

savoir:

$$\frac{1}{k}\begin{vmatrix} bcd+b\tau^2+c\sigma^2+d\lambda^2, & -cd\nu-\tau\phi+d\lambda\mu-c\rho\sigma, & bd\mu+\sigma\phi+d\lambda\nu-b\rho\tau\ , & -bc\rho-\lambda\phi+c\nu\sigma-b\mu\tau \\ cd\nu+\tau\phi+d\lambda\mu-c\rho\sigma, & acd+b\tau^2+c\rho^2+d\mu^2, & -ad\lambda-\rho\phi+d\mu\nu-a\sigma\tau, & -ac\sigma-\mu\phi-c\nu\rho+a\lambda\tau \\ -bd\mu-\sigma\phi+d\lambda\nu-b\rho\tau, & ad\lambda+\rho\phi+d\mu\nu-a\sigma\tau, & abd+a\sigma^2+b\rho^2+d\nu^2\ , & -ab\tau-\nu\phi+b\mu\rho-a\lambda\sigma \\ bc\rho+\lambda\phi+c\nu\sigma-b\mu\tau, & ac\sigma+\mu\phi-c\nu\rho+a\lambda\tau, & ab\tau+\nu\phi+b\mu\rho-a\lambda\sigma, & abc+a\lambda^2+b\mu^2+c\nu^2 \end{vmatrix}.$$

On a pour la transformation, l'équation (x, y, z, w) =

$$\frac{1}{k}\left|\begin{array}{ll}
\begin{array}{l} abcd - bc\rho^2 + ca\sigma^2 + ab\tau^2 + ad\lambda^2 \\ - bd\mu^2 - cd\nu^2 - \phi^2 \end{array}, & 2b(-cd\nu - \tau\phi + d\lambda\mu - c\rho\sigma), \\
2a(cd\nu + \tau\phi + d\lambda\mu - c\rho\sigma), & \begin{array}{l} abcd + bc\rho^2 - ca\sigma^2 + ab\tau^2 - ad\lambda^2 \\ + bd\mu^2 - cd\nu^2 - \phi^2 \end{array}, \\
2a(-bd\mu - \sigma\phi + d\lambda\nu - b\rho\tau), & 2b(ad\lambda + \rho\phi + d\mu\nu - a\sigma\tau), \\
2a(bc\rho + \lambda\phi + c\nu\sigma - b\mu\tau), & 2b(ac\sigma + \mu\phi - c\nu\rho + a\lambda\tau),
\end{array}\right.$$

$$\left.\begin{array}{ll}
2c(bd\mu + \sigma\phi + d\lambda\nu - b\rho\tau), & 2d(-bc\rho - \lambda\phi + c\nu\sigma - b\mu\tau) \\
2c(-ad\lambda - \rho\phi + d\mu\nu - a\sigma\tau), & 2d(-ac\sigma - \mu\phi - c\nu\rho + a\lambda\tau) \\
\begin{array}{l} abcd + bc\rho^2 + ca\sigma^2 - ab\tau^2 - ad\lambda^2 \\ - bd\mu^2 + cd\nu^2 - \phi^2 \end{array}, & 2d(-ab\tau - \nu\phi + b\mu\rho - a\lambda\sigma) \\
2c(ab\tau + \nu\phi + b\mu\rho - a\lambda\sigma), & \begin{array}{l} abcd - bc\rho^2 - ca\sigma^2 - ab\tau^2 + ad\lambda^2 \\ + b\delta\mu^2 + cd\nu^2 - \phi^2 \end{array}
\end{array}\right| (x, y, z, w).$$

Je suppose que l'on ait $a = b = c = d = \omega$, et j'écris $\psi = -\frac{\phi}{\omega}$, c'est-à-dire $\psi = -\frac{\lambda\rho + \mu\sigma + \nu\tau}{\omega}$ ou $\lambda\rho + \mu\sigma + \nu\tau + \psi\omega = 0$. En faisant cette substitution, on trouve d'abord $k = \omega^2 R$, où

$$R = \lambda^2 + \mu^2 + \nu^2 + \psi^2 + \rho^2 + \sigma^2 + \tau^2 + \omega^2,$$

et puis pour la transformation propre

$$x^2 + y^2 + z^2 + w^2 = x^2 + y^2 + z^2 + w^2,$$

l'équation (x, y, z, w) =

$$\frac{1}{R}\left|\begin{array}{ll}
-\rho^2 + \sigma^2 + \tau^2 + \omega^2 + \lambda^2 - \mu^2 - \nu^2 - \psi^2, & -2\omega\nu + 2\tau\psi + 2\lambda\mu - 2\rho\sigma, \\
2\omega\nu - 2\tau\psi + 2\lambda\mu - 2\rho\sigma, & \rho^2 - \sigma^2 + \tau^2 + \omega^2 - \lambda^2 + \mu^2 - \nu^2 - \psi^2, \\
-2\omega\mu + 2\sigma\psi + 2\lambda\nu - 2\rho\tau, & 2\omega\lambda - 2\rho\psi + 2\mu\nu - 2\sigma\tau, \\
2\omega\rho - 2\lambda\psi + 2\nu\sigma - 2\mu\tau, & 2\omega\sigma - 2\mu\psi - 2\nu\rho + 2\lambda\tau,
\end{array}\right.$$

$$\left.\begin{array}{ll}
2\omega\mu - 2\sigma\psi + 2\lambda\nu - 2\rho\tau, & -2\omega\rho + 2\lambda\psi + 2\nu\sigma - 2\mu\tau \\
-2\omega\lambda + 2\rho\psi + 2\mu\nu - 2\sigma\tau, & -2\omega\sigma + 2\mu\psi - 2\nu\rho + 2\lambda\tau \\
\rho^2 + \sigma^2 - \tau^2 + \omega^2 - \lambda^2 - \mu^2 + \nu^2 - \psi^2, & -2\omega\tau + 2\nu\psi + 2\mu\rho - 2\lambda\sigma \\
2\omega\tau - 2\nu\psi + 2\mu\rho - 2\lambda\sigma, & -\rho^2 - \sigma^2 - \tau^2 + \omega^2 + \lambda^2 + \mu^2 + \nu^2 - \psi^2
\end{array}\right| (x, y, z, w).$$

On peut changer la forme de cette expression, en y écrivant

$$\lambda = \tfrac{1}{2}(\alpha - \alpha'), \quad \mu = \tfrac{1}{2}(\beta - \beta'), \quad \nu = \tfrac{1}{2}(\gamma - \gamma'), \quad \psi = \tfrac{1}{2}(\delta - \delta'),$$

$$\rho = \tfrac{1}{2}(\alpha + \alpha'), \quad \sigma = \tfrac{1}{2}(\beta + \beta'), \quad \tau = \tfrac{1}{2}(\gamma + \gamma'), \quad \omega = \tfrac{1}{2}(\delta + \delta');$$

cela donne

$$\alpha^2+\beta^2+\gamma^2+\delta^2-\alpha'^2-\beta'^2-\gamma'^2-\delta'^2=0,$$

$$R=\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2+\alpha'^2+\beta'^2+\gamma'^2+\delta'^2),$$

de manière qu'en écrivant

$$M=\alpha^2+\beta^2+\gamma^2+\delta^2,$$

$$M'=\alpha'^2+\beta'^2+\gamma'^2+\delta'^2,$$

on obtient

$$R=\tfrac{1}{2}(M+M')=\sqrt{(MM')},$$

et la formule pour la transformation devient

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \mathrm{w})=\frac{1}{\sqrt{(MM')}}\left|\begin{matrix} -\alpha\alpha'+\beta\beta'+\gamma\gamma'+\delta\delta', & -\alpha\beta'-\beta\alpha'-\gamma\delta'+\delta\gamma', & -\alpha\gamma'-\beta\delta'+\gamma\alpha'-\delta\beta', & \alpha\delta'+\beta\gamma'-\gamma\beta'+\delta\alpha' \\ -\alpha\beta'-\beta\alpha'+\gamma\delta'+\delta\gamma', & \alpha\alpha'-\beta\beta'+\gamma\gamma'+\delta\delta', & -\alpha\delta'-\beta\gamma'+\gamma\beta'+\delta\alpha', & -\alpha\gamma'+\beta\delta'+\gamma\alpha'+\delta\beta' \\ -\alpha\gamma'-\beta\delta'-\gamma\alpha'+\delta\beta', & \alpha\delta'-\beta\gamma'-\gamma\beta'-\delta\alpha', & \alpha\alpha'+\beta\beta'-\gamma\gamma'+\delta\delta', & \alpha\beta'-\beta\alpha'+\gamma\delta'+\delta\gamma' \\ -\alpha\delta'+\beta\gamma'-\gamma\beta'-\delta\alpha', & -\alpha\gamma'-\beta\delta'+\gamma\alpha'-\delta\beta', & \alpha\beta'-\beta\alpha'-\gamma\delta'-\delta\gamma', & -\alpha\alpha'-\beta\beta'-\gamma\gamma'+\delta\delta' \end{matrix}\right|(x,\ y,\ z,\ w).$$

On voit donc que même sans supposer l'équation $M=M'$, cette formule donne la transformation propre

$$\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2+\mathrm{w}^2=x^2+y^2+z^2+w^2.$$

Cette solution est à peu près de la même forme que la solution *impropre* donnée par Euler dans son mémoire "Problema algebraicum ob affectiones prorsus singulares memorabile" Nov. Comm. Petrop., t. XV. 1770, p. 75, et Comm. Arith. collectae, [4to. Petrop. 1849], t. I. p. 427. Je remarque aussi que cette même solution peut être déduite de la théorie des *Quaternions*. En effet, i, j, k étant des quantités imaginaires telles que $i^2=j^2=k^2=-1$, $jk=-kj=i$, $ki=-ik=j$, $ij=-ji=k$, on obtient, en effectuant la multiplication:

$$(\mathrm{x}i+\mathrm{y}j+\mathrm{z}k+\mathrm{w})=-\frac{1}{\sqrt{(MM')}}(\alpha i+\beta j+\gamma k+\delta)(xi+yj+zk+w)(\alpha' i+\beta' j+\gamma' k+\delta'),$$

x, y, z, w ayant les mêmes valeurs que dans la dernière formule de transformation.

En changeant les signes des termes de la quatrième colonne, on en tire pour la transformation impropre

$$\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2+\mathrm{w}^2=x^2+y^2+z^2+w^2,$$

la formule suivante plus symétrique :

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \mathrm{w}) = \frac{1}{\sqrt{(MM')}} \left| \begin{matrix} -\alpha\alpha' + \beta\beta' + \gamma\gamma' + \delta\delta', & -\alpha\beta' - \beta\alpha' - \gamma\delta' + \delta\gamma', \\ -\alpha\beta' - \beta\alpha' + \gamma\delta' - \delta\gamma', & \alpha\alpha' - \beta\beta' + \gamma\gamma' + \delta\delta', \\ -\alpha\gamma' - \beta\delta' - \gamma\alpha' + \delta\beta', & \alpha\delta' - \beta\gamma' - \gamma\beta' - \delta\alpha', \\ -\alpha\delta' + \beta\gamma' - \gamma\beta' - \delta\alpha', & -\alpha\gamma' - \beta\delta' + \gamma\alpha' - \delta\beta', \end{matrix} \right.$$

$$\left. \begin{matrix} -\alpha\gamma' + \beta\delta' - \gamma\alpha' - \delta\beta', & -\alpha\delta' - \beta\gamma' + \gamma\beta' - \delta\alpha' \\ -\alpha\delta' - \beta\gamma' - \gamma\beta' + \delta\alpha', & \alpha\gamma' - \beta\delta' - \gamma\alpha' - \delta\beta' \\ \alpha\alpha' + \beta\beta' - \gamma\gamma' + \delta\delta', & -\alpha\beta' + \beta\alpha' - \gamma\delta' - \delta\gamma' \\ \alpha\beta' - \beta\alpha' - \gamma\delta' - \delta\gamma', & \alpha\alpha' + \beta\beta' + \gamma\gamma' - \delta\delta' \end{matrix} \right| (x,\ y,\ z,\ w).$$

Ces formules pour la transformation, tant propre qu'impropre, de la fonction $x^2 + y^2 + z^2 + w^2$ *en elle-même,* sont utiles dans la théorie des polygones inscrits dans une surface du second ordre.

138.

RECHERCHES SUR LES MATRICES DONT LES TERMES SONT DES FONCTIONS LINÉAIRES D'UNE SEULE INDÉTERMINÉE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. L. (1855), pp. 313—317.]

Je pose la matrice

$$\begin{vmatrix} A\,, & B\,, & C\,, & \ldots \\ A'\,, & B'\,, & C'\,, & \ldots \\ A''\,, & B''\,, & C''\,, & \ldots \\ \vdots & & & \end{vmatrix}$$

dont les termes (n^2 en nombre) sont des fonctions linéaires d'une quantité s, et je considère le déterminant formé avec cette matrice, et les déterminants mineurs formés en supprimant un nombre quelconque des lignes et un nombre égal de colonnes de la matrice. En supprimant une *seule* ligne et une *seule* colonne, on obtient les *premiers mineurs;* en supprimant *deux* lignes et *deux* colonnes, on obtient les *seconds mineurs;* et ainsi de suite. Cela étant, je suppose que la quantité s a été trouvée en égalant à zéro le déterminant formé avec la matrice donnée; ce déterminant sera une fonction de s du n-ième degré qui généralement ne contiendra pas de facteurs multiples. On voit donc qu'un facteur simple du déterminant ne peut pas entrer comme facteur dans les premiers mineurs (c'est-à-dire dans *tous* les premiers mineurs); mais en supposant que le déterminant ait des facteurs multiples, un facteur multiple du déterminant peut entrer comme facteur (simple ou multiple) dans les premiers mineurs, ou dans les mineurs d'un ordre plus élevé. Il importe de trouver le degré selon lequel un facteur multiple du déterminant peut entrer comme facteur des premiers mineurs, ou des mineurs d'un ordre quelconque donné.

Cela se fait très facilement au moyen d'une propriété générale des déterminants; si les mineurs du $(r+1)$ième ordre contiennent le facteur $(s-a)^a$ (c'est-à-dire, si tous

les mineurs de cet ordre contiennent le facteur $(s-a)^{\alpha}$, mais non pas tous les facteurs $(s-a)^{\alpha+1}$); et si de même les mineurs du r-ième ordre contiennent le facteur $(s-a)^{\beta}$; alors les mineurs du $(r-1)$ième ordre contiendront *au moins* le facteur $(s-a)^{2\beta-\alpha}$. Autrement dit: les mineurs du $(r-1)$ième ordre contiendront le facteur $(s-a)^{\gamma}$ où $\gamma > 2\beta-\alpha$, ou, ce qui est la même chose, $\alpha-2\beta+\gamma \nless 0$; c'est-à-dire: en formant la suite des indices des puissances selon lesquelles le facteur $(s-a)$ entre dans les mineurs premiers, seconds, &c. (il va sans dire que cette suite sera une suite *décroissante*), les différences secondes seront *positives* [c'est-à-dire non *négatives*]. Je représente par $\alpha, \beta, \gamma, \ldots$ la suite dont il s'agit; je suppose, pour fixer les idées, que δ soit le dernier terme qui ne s'évanouisse pas, et j'écris

$$\begin{array}{lllllll}
\alpha, & \beta, & \gamma, & \delta, & 0, & 0, \ldots \\
\alpha-\beta, & \beta-\gamma, & \gamma-\delta, & \delta, & 0, \ldots & & \\
\alpha-2\beta+\gamma, & \beta-2\gamma+\delta, & \gamma-2\delta, & \delta, & 0, \ldots; & &
\end{array}$$

ici, quel que soit le nombre des termes, tous les nombres de la troisième ligne seront positifs, et en représentant ces nombres par $\int, \int', \int''$, &c., on obtient:

$$\begin{aligned}
\alpha &= \int + 2\int' + 3\int'' + 4\int''' + \ldots, \\
\beta &= \int' + 2\int'' + 3\int''' + \ldots, \\
\gamma &= \int'' + 2\int''' + \ldots, \\
\delta &= \int''' + \ldots, \\
&\vdots
\end{aligned}$$

Il y a ici à considérer que le nombre α, indice de la puissance selon laquelle le facteur $(s-a)$ entre dans le déterminant, est donné; il sera donc permis de prendre pour $\int, \int', \int'', \ldots$ des valeurs entières et positives quelconques (zéro y compris) qui satisfont à la première équation; les autres équations donnent alors les valeurs de β, γ, δ, &c. On forme de cette manière une table des particularités que peut présenter un facteur multiple $(s-a)^{\alpha}$ du déterminant; cette table est composée des symboles $\alpha, \beta, \gamma, \ldots$, et les nombres $\alpha, \beta, \ldots$ de chaque symbole font voir le degré selon lequel le facteur $(s-a)$ entre dans les déterminants, dans les mineurs premiers, seconds, &c. Or il est très facile de former, au moyen des tables pour $\alpha=1$, $\alpha=2, \ldots \alpha=k$, la table pour $\alpha=k+1$. On a par exemple pour $\alpha=1$, $\alpha=2$, $\alpha=3$, $\alpha=4$ les tables suivantes:

Pour	$\alpha=1$,	1.				
Pour	$\alpha=2$,	2,	21.			
Pour	$\alpha=3$,	3,	31,	321.		
Pour	$\alpha=4$,	4,	41,	42,	421,	4321.

De là on tire la table pour $\alpha=5$, savoir:

Pour	$\alpha=5$,	5,	51,	52,	521,	531,	5321,	54321.

En effet, le premier terme est 5, et on obtient les autres termes en mettant le nombre 5 devant les symboles des tables pour $\alpha=1$, $\alpha=2$, $\alpha=3$, $\alpha=4$, en ayant seulement soin

de supprimer les symboles 53, 54, 541, 542, 5421 pour lesquels le premier terme de la suite des différences secondes est négatif. On trouve de même pour $\alpha = 6$, la table suivante, savoir:

Pour $\alpha = 6$, 6, 61, 62, 621, 63, 631, 6321, 642, 6421, 64321, 654321;

et ainsi de suite. Les nombres des symboles pour $\alpha = 1, 2, 3, 4, 5, 6, 7, 8$, &c. sont 1, 2, 3, 5, 7, 11, 15, 22, [30, 42, 56], &c.; ce sont les coefficients des puissances x^1, x^2, x^3 &c. dans le développement de

$$(1-x)^{-1}\,(1-x^2)^{-1}\,(1-x^3)^{-1}\,(1-x^4)^{-1}\,(1-x^5)^{-1}\,\ldots \text{ \&c.}$$

fonctions qui se présentent, comme on sait, dans la théorie de la *partition des nombres*.

Maintenant, au lieu de considérer un seul facteur du déterminant, je considère *tous* les facteurs: par exemple pour $n = 4$, le déterminant peut avoir un facteur double $(s-a)^2$, et un autre facteur double $(s-b)^2$; il peut de plus arriver que le facteur $(s-a)$ soit facteur simple des premiers mineurs, mais que le facteur $(s-b)$ n'entre pas dans les premiers mineurs. Le symbole qui correspond au facteur $(s-a)$ sera 21, et le symbole qui correspond au facteur $(s-b)$ sera 2. En combinant ces deux symboles, on aura le symbole composé $\boxed{\begin{matrix}21\\2\end{matrix}}$, qui dénote que le déterminant a deux facteurs doubles de la classe dont il s'agit. Je forme de ces symboles composés des tables pour $n = 1$, $n = 2$, $n = 3$, $n = 4$, &c. On a:

Pour $n = 1$: $\boxed{1}$,

Pour $n = 2$: $\boxed{\begin{matrix}1\\1\end{matrix}}$, $\boxed{2}$, $\boxed{21}$,

Pour $n = 3$: $\boxed{\begin{matrix}1\\1\\1\end{matrix}}$, $\boxed{\begin{matrix}2\\1\end{matrix}}$, $\boxed{\begin{matrix}21\\1\end{matrix}}$, $\boxed{3}$, $\boxed{31}$, $\boxed{321}$,

Pour $n = 4$:

$\boxed{\begin{matrix}1\\1\\1\\1\end{matrix}}$, $\boxed{\begin{matrix}2\\1\\1\end{matrix}}$, $\boxed{\begin{matrix}21\\1\\1\end{matrix}}$, $\boxed{\begin{matrix}3\\1\end{matrix}}$, $\boxed{\begin{matrix}321\\1\end{matrix}}$, $\boxed{\begin{matrix}2\\2\end{matrix}}$, $\boxed{\begin{matrix}21\\2\end{matrix}}$, $\boxed{4}$, $\boxed{41}$, $\boxed{42}$, $\boxed{421}$, $\boxed{4321}$,

et ainsi de suite.

Pour donner encore un exemple du sens de ces symboles, le symbole $\boxed{\begin{matrix}321\\1\end{matrix}}$ dénote que le déterminant a un facteur $(s-a)$ qui entre comme facteur *triple* dans le déterminant, comme facteur *double* dans les premiers mineurs, et comme facteur *simple* dans les seconds mineurs; l'autre facteur du déterminant est un facteur simple $(s-b)$. Les nombres des symboles pour $n=1$, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. sont 1, 3, 6, 14, 27, 58, 111, 223, 424, 817, 1527, &c.; ces nombres sont les coefficients de x^1, x^2, x^3, &c. dans le développement de

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-5}(1-x^5)^{-7}(1-x^6)^{-11}(1-x^7)^{-15}(1-x^8)^{-22}(1-x^9)^{-30}\ldots \text{ \&c.}$$

où les indices 1, 2, 3, 5, 7, 11, &c. forment la suite qui se présente dans la théorie de la partition des nombres, dont j'ai parlé plus haut. Il est très facile de démontrer qu'il en est ainsi.

Les résultats que je viens de présenter sont en partie dus à M. Sylvester (voyez son mémoire "An enumeration of the contacts of lines and surfaces of the second order," *Philosophical Magazine*, [vol. I. (1851), pp. 18—20]). En effet, M. Sylvester commence par étendre à des fonctions d'un nombre quelconque d'indéterminées l'idée géométrique des contacts des courbes et des surfaces. En considérant les deux équations quadratiques $U=0$, $V=0$, il forme le discriminant de la fonction quadratique $U+sV$, et il cherche dans quel degré chaque facteur de ce discriminant peut entrer comme facteur dans les mineurs premiers, seconds, &c. Le discriminant de M. Sylvester est un déterminant symétrique; mais cela ne change rien à la question, et je n'ai fait que reproduire l'analyse de M. Sylvester, en donnant cependant l'algorithme pour la formation des symboles, et de plus la loi pour le nombre des symboles. M. Sylvester donne pour $n=2$, 3, 4, 5, 6, des nombres qui, en ajoutant à chacun le nombre 2, pour embrasser deux cas extrêmes qui ne sont pas comptés, seraient 3, 6, 14, 26, 58. Il se trouve dans le nombre 26 une erreur de calcul; ce nombre devrait être 27, et en suppléant le premier terme 1, on a la suite trouvée plus haut, savoir 1, 3, 6, 14, 27, 58, &c.; il y a de même une erreur de calcul dans les nombres donnés par M. Sylvester pour $n=7$ et $n=8$.

Mais tout cela s'applique à une autre théorie géométrique, savoir à la théorie des figures *homographes*. Pour fixer les idées, je ne considère que les figures dans le plan. En supposant que x, y, z soient les coordonnées d'un point, et en prenant pour (x, y, z) des fonctions linéaires de (x, y, z) on aura (x, y, z) comme coordonnées d'un point *homographe* au point (x, y, z). En cherchant les points qui sont homographes chacun à soi-même, on est conduit aux équations $\mathrm{x}-sx=0$, $\mathrm{y}-sy=0$, $\mathrm{z}-sz=0$. Les quantités à gauche $\mathrm{x}-sx$, $\mathrm{y}-sy$, $\mathrm{z}-sz$ sont des fonctions linéaires de x, y, z, ayant pour coefficients des fonctions linéaires de s. On a ainsi une matrice dont les termes sont des fonctions linéaires de s; la théorie entière se rattache aux propriétés de cette matrice. Pour le cas général de *l'homographie* ordinaire, on a le symbole

$\begin{array}{|c|}\hline 1\\ 1\\ 1\\ \hline\end{array}$, pour l'*homologie*, le symbole $\begin{array}{|c|}\hline 21\\ 1\\ \hline\end{array}$; les autres symboles $\begin{array}{|c|}\hline 2\\ 1\\ \hline\end{array}$, $\boxed{3}$, $\boxed{31}$ se rapportent à des cas moins généraux, et le symbole $\boxed{321}$ au cas de l'identité complète des deux figures; y compris ce cas-limite de l'identité complète, il existe pour le *plan* 6 espèces d'homographie; pour l'*espace ordinaire* il existe 14 espèces d'homographie. Je reviendrai à cette théorie à une autre occasion.

Londres, le 24 *Mai* 1854.

139.

AN INTRODUCTORY MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLIV. *for the year* 1854, pp. 244—258. Received April 20,—Read May 4, 1854.]

1. THE term Quantics is used to denote the entire subject of rational and integral functions, and of the equations and loci to which these give rise; the word "quantic" is an adjective, meaning *of such a degree*, but may be used substantively, the noun understood being (unless the contrary appear by the context) function; so used the word admits of the plural "quantics."

The quantities or symbols to which the expression "degree" refers, or (what is the same thing) in regard to which a function is considered as a quantic, will be spoken of as "facients." A quantic may always be considered as being, in regard to its facients, homogeneous, since to render it so, it is only necessary to introduce as a facient unity, or some symbol which is to be ultimately replaced by unity; and in the cases in which the facients are considered as forming two or more distinct sets, the quantic may, in like manner, be considered as homogeneous in regard to each set separately.

2. The expression "an equation," used without explanation, is to be understood as meaning the equation obtained by putting any quantic equal to zero. I make no absolute distinction between the words "degree" and "order" as applied to an equation or system of equations, but I shall in general speak of the order rather than the degree. The equations of a system may be independent, or there may exist relations of connexion between the different equations of the system; the subject of a system of equations so connected together is one of extreme complexity and difficulty. It will be sufficient to notice here, that in any system whatever of equations, assuming only that the equations are not more than sufficient to determine the ratios of the facients, and joining to the system so many linear equations between the facients as will render the ratios of the facients determinate, the order of the system is the same thing as the order of the equation which determines any one of these ratios; it is clear that for a single equation the order so determined is nothing else than the order of the equation.

3. An equation or system of equations represents, or is represented by a locus. This assumes that the facients depend upon quantities $x, y, \ldots$ the coordinates of a point in space; the entire series of points, the coordinates of which satisfy the equation or system of equations, constitutes the locus. To avoid complexity, it is proper to take the facients themselves as coordinates, or at all events to consider these facients as linear functions of the coordinates; this being the case, the order of the locus will be the order of the equation, or system of equations.

4. I have spoken of the *coordinates* of a *point* in *space*. I consider that there is an ideal space of any number of dimensions, but of course, in the ordinary acceptation of the word, space is of three dimensions; however, the plane (the space of ordinary plane geometry) is a space of two dimensions, and we may consider the line as a space of one dimension. I do not, it should be observed, say that the only idea which can be formed of a space of two dimensions is the plane, or the only idea which can be formed of space of one dimension is the line; this is not the case. To avoid complexity, I will take the case of plane geometry rather than geometry of three dimensions; it will be unnecessary to speak of space, or of the number of its dimensions, or of the plane, since we are only concerned with space of two dimensions, viz. the plane; I say, therefore, simply that x, y, z are the coordinates of a point (strictly speaking, it is the ratios of these quantities which are the coordinates, and the quantities x, y, z themselves are indeterminates, i.e. they are only determinate to a common factor *près*, so that in assuming that the coordinates of a point are α, β, γ, we mean only that $x : y : z = \alpha : \beta : \gamma$, and we never as a result obtain $x, y, z = \alpha, \beta, \gamma$, but only $x : y : z = \alpha : \beta : \gamma$; but this being once understood, there is no objection to speaking of x, y, z as coordinates). Now the notions of coordinates and of a point are merely relative; we may, if we please, consider $x : y : z$ as the parameters of a curve containing two variable parameters; such curve becomes of course determinate when we assume $x : y : z = \alpha : \beta : \gamma$, and this very curve is nothing else than the point whose coordinates are α, β, γ, or as we may for shortness call it, the point (α, β, γ). And if the coordinates (x, y, z) are connected by an equation, then giving to these coordinates the entire system of values which satisfy the equation, the locus of the points corresponding to these values is the locus representing or represented by the equation; this of course fixes the notion of a curve of any order, and in particular the notion of a line as the curve of the first order.

The theory includes, as a very particular case, the ordinary theory of reciprocity in plane geometry; we have only to say that the word "point" shall mean "line," and the word "line" shall mean "point," and that expressions properly or primarily applicable to a point and a line respectively shall be construed to apply to a line and a point respectively, and any theorem (assumed of course to be a purely descriptive one) relating to points and lines will become a corresponding theorem relating to lines and points; and similarly with regard to curves of a higher order, when the ideas of reciprocity applicable to these curves are properly developed.

5. A quantic of the degrees $m, m' \ldots$ in the sets $(x, y \ldots)$, $(x', y' \ldots)$ &c. will for the most part be represented by a notation such as

$$(*\between x, y \ldots \overset{m}{\between} x', y' \ldots \overset{m'}{)} \ldots),$$

where the mark * may be considered as indicative of the absolute generality of the quantic; any such quantic may of course be considered as the sum of a series of terms $x^\alpha y^\beta \ldots x'^{\alpha'} y'^{\beta'} \ldots$, &c. of the proper degrees in the different sets respectively, each term multiplied by a coefficient; these coefficients may be mere numerical multiples of single letters or elements such as $a, b, c, \ldots$, or else functions (in general rational and integral functions) of such elements; this explains the meaning of the expression "the elements of a quantic": in the case where the coefficients are mere numerical multiples of the elements, we may in general speak indifferently of the elements, or of the coefficients. I have said that the coefficients may be numerical multiples of single letters or elements such as $a, b, c, \ldots$; by the appropriate numerical coefficient of a term $x^\alpha y^\beta \ldots x'^{\alpha'} y'^{\beta'} \ldots$, I mean the coefficient of this term in the expansion of

$$(x + y \ldots)^m (x' + y' \ldots)^{m'} \ldots);$$

and I represent by the notation

$$(a, b, \ldots \between x, y, \ldots \between^m x', y', \ldots)^{m'} \ldots),$$

a quantic in which each term is multiplied as well by its appropriate numerical coefficient as by the literal coefficient or element which belongs to it in the set $(a, b, \ldots)$ of literal coefficients or elements. On the other hand, I represent by the notation

$$(a, b, \ldots \between x, y, \ldots \between^m x', y', \ldots)^{m'} \ldots),$$

a quantic in which each term is multiplied only by the literal coefficient or element which belongs to it in the set $(a, b, \ldots)$ of literal coefficients or elements. And a like distinction applies to the case where the coefficients are functions of the elements $(a, b, \ldots)$.

6. I consider now the quantic

$$(* \between x, y, \ldots \between^m x', y', \ldots)^{m'} \ldots),$$

and selecting any two facients of the same set, e.g. the facients x, y, I remark that there is always an operation upon the elements, tantamount as regards the quantic to the operation $x\partial_y$; viz. if we differentiate with respect to each element, multiply by proper functions of the elements and add, we obtain the same result as by differentiating with ∂_y and multiplying by x. The simplest example will show this as well as a formal proof; for instance, as regards $3ax^2 + bxy + 5cy^2$ (the numerical coefficients are taken haphazard), we have $\frac{1}{3}b\partial_a + 10c\partial_b$ tantamount to $x\partial_y$; as regards $a(x - \alpha y)(x - \beta y)$, we have $-a(\alpha + \beta)\partial_a + \alpha^2\partial_\alpha + \beta^2\partial_\beta$ tantamount to $x\partial_y$, and so in any other case. I represent by $\{x\partial_y\}$ the operation upon the elements tantamount to $x\partial_y$, and I write down the series of operations

$$\{x\partial_y\} - x\partial_y, \ldots \quad \{x'\partial_{y'}\} - x'\partial_{y'}, \ldots$$

where x, y are considered as being successively replaced by every permutation of two different facients of the set $(x, y, \ldots)$; x', y' as successively replaced by every permutation of two different facients of the set $(x', y', \ldots)$, and so on; this I call an entire system, and

I say that it is made up of partial systems corresponding to the different facient sets respectively; it is clear from the definition that the quantic is reduced to zero by each of the operations of the entire system. Now, besides the quantic itself, there are a variety of other functions which are reduced to zero by each of the operations of the entire system; any such function is said to be a covariant of the quantic, and in the particular case in which it contains only the elements, an invariant. (It would be allowable to define as a covariant *quoad any set or sets*, a function which is reduced to zero by each of the operations of the corresponding partial system or systems, but this is a point upon which it is not at present necessary to dwell.)

7. The definition of a covariant may however be generalized in two directions: we may instead of a single quantic consider two or more quantics; the operations $\{x\partial_y\}$, although represented by means of the same symbols x, y have, as regards the different quantics, different meanings, and we may form the sum $\Sigma\{x\partial_y\}$, where the summation refers to the different quantics: we have only to consider in place of the system before spoken of, the system

$$\Sigma\{x\partial_y\} - x\partial_y, \ldots; \ \Sigma\{x'\partial_{y'}\} - x'\partial_{y'}, \ldots \text{ \&c. \&c.},$$

and we obtain the definition of a covariant of two or more quantics.

Again, we may consider in connexion with each set of facients any number of new sets, the facients in any one of these new sets corresponding each to each with those of the original set; and we may admit these new sets into the covariant. This gives rise to a sum $S\{x\partial_y\}$, where the summation refers to the entire series of corresponding sets. We have in place of the system spoken of in the original definition, to consider the system

$$\{x\partial_y\} - S(x\partial_y), \ldots \ \{x'\partial_{y'}\} - S(x'\partial_{y'}), \ldots \text{ \&c. \&c.},$$

or if we are dealing with two or more quantics, then the system

$$\Sigma\{x\partial_y\} - S(x\partial_y), \ldots; \ \Sigma\{x'\partial_{y'}\} - S(x'\partial_{y'}), \ldots \text{ \&c. \&c.},$$

and we obtain the generalized definition of a covariant.

8. A covariant has been defined simply as a function reduced to zero by each of the operations of the entire system. But in dealing with given quantics, we may without loss of generality consider the covariant as a function of the like form with the quantic, i.e. as being a rational and integral function homogeneous in regard to the different sets separately, and as being also a rational and integral function of the elements. In particular in the case where the coefficients are mere numerical multiples of the elements, the covariant is to be considered as a rational and integral function homogeneous in regard to the different sets separately, and also homogeneous in regard to the coefficients or elements. And the term "covariant" includes, as already remarked, "invariant."

It is proper to remark, that if the same quantic be represented by means of different sets of elements, then the symbols $\{x\partial_y\}$ which correspond to these different forms

of the same quantic are mere transformations of each other, i.e. they become in virtue of the relations between the different sets of elements identical.

9. What precedes is a return to and generalization of the method employed in the first part of the memoir published in the *Camb. Math. Jour.*, t. IV. [1845], and *Camb. and Dubl. Math. Jour.*, t. I. [1846], under the title "On Linear Transformations," [13 and 14], and Crelle, t. XXX. [1846], under the title "Mémoire sur les Hyperdéterminants," [*16], and which I shall refer to as my original memoir. I there consider in fact the invariants of a quantic

$$(*\between x_1,\ x_2 \ldots x_m \between y_1,\ y_2 \ldots y_m) \ldots),$$

linear in regard to n sets each of them of m facients, and I represent the coefficients of a term $x_r y_s z_t \ldots$ by $rst \ldots$; there is no difficulty in seeing that α, β being any two different numbers out of the series $1, 2, \ldots m$, the operation $\{x_\beta \partial_{x_\alpha}\}$ is identical with the operation

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right),$$

where the summations refer to $s, t, \ldots$ which pass respectively from 1 to m, both inclusive; and the condition that a function, assumed to be an invariant, i.e. to contain only the coefficients, may be reduced to zero by the operation $\{x_\beta \partial_{x_\alpha}\} - x_\beta \partial_{x_\alpha}$, is of course simply the condition that such function may be reduced to zero by the operation $\{x_\beta \partial_{x_\alpha}\}$; the condition in question is therefore the same thing as the equation

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st}\right) u = 0$$

of my original memoir.

10. But the definition in the present memoir includes also the method made use of in the second part of my original memoir. This method is substantially as follows: consider for simplicity a quantic $U =$

$$(*\between x,\ y, \ldots)^m$$

containing only the single set $(x, y \ldots)$, and let U_1, $U_2 \ldots$ be what the quantic becomes when the set $(x, y \ldots)$ is successively replaced by the sets $(x_1, y_1, \ldots)$, $(x_2, y_2, \ldots), \ldots$ the number of these new sets being equal to or greater than the number of facients in the set. Suppose that $A, B, C, \ldots$ are any of the determinants

$$\left\| \begin{matrix} \partial_{x_1}, & \partial_{x_2}, & \partial_{x_3}, \ldots \\ \partial_{y_1}, & \partial_{y_2}, & \partial_{y_3}, \\ \vdots & & \end{matrix} \right\|,$$

then forming the derivative

$$A^p B^q C^r \ldots U_1 U_2 \ldots,$$

where $p, q, r \ldots$ are any positive integers, the function so obtained is a covariant involving the sets $(x_1, y_1, \ldots)$, $(x_2, y_2, \ldots)$ &c.; and if after the differentiations we replace

these sets by the original set $(x, y, \ldots)$, we have a covariant involving only the original set $(x, y, \ldots)$ and of course the coefficients of the quantic. It is in fact easy to show that any such derivative is a covariant according to the definition given in this Memoir. But to do this some preliminary explanations are necessary.

11. I consider any two operations P, Q, involving each or either of them differentiations in respect of variables contained in the other of them. It is required to investigate the effect of the operation $P \,.\, Q$, where the operation Q is to be in the first place performed upon some operand Ω, and the operation P is then to be performed on the operand $Q\Omega$. Suppose that P involves the differentiations ∂_a, $\partial_b, \ldots$ in respect of variables $a, b, \ldots$ contained in Q and Ω, we must as usual in the operation P replace ∂_a, $\partial_b, \ldots$ by $\partial_a + \partial'_a$, $\partial_b + \partial'_b, \ldots$ where the unaccentuated symbols operate only upon Ω, and the accentuated symbols operate only upon Q. Suppose that P is expanded in ascending powers of the symbols $\partial'_a, \partial'_b, \ldots$, viz. in the form $P + P_1 + P_2 + \&c.$, we have first to find the values of P_1Q, P_2Q, &c., by actually performing upon Q as operand the differentiations ∂'_a, $\partial'_b \ldots$. The symbols PQ, P_1Q, P_2Q, &c. will then contain only the differentiations ∂_a, $\partial_b, \ldots$ which operate upon Ω, and the meaning of the expression being once understood, we may write

$$P \,.\, Q = PQ + P_1Q + P_2Q + \&c.$$

In particular if P be a linear function of ∂_a, $\partial_b, \ldots$, we have to replace P by $P + P_1$, where P_1 is the same function of ∂'_a, $\partial'_b, \ldots$ that P is of ∂_a, $\partial_b, \ldots$, and it is therefore clear that we have in this case

$$P \,.\, Q = PQ + P(Q),$$

where on the right-hand side in the term PQ the differentiations ∂_a, $\partial_b, \ldots$ are considered as not in anywise affecting the symbol Q, while in the term $P(Q)$ these differentiations, or what is the same thing, the operation P, is considered to be performed upon Q as operand.

Again, if Q be a linear function of $a, b, c, \ldots$, then $P_2Q = 0$, $P_3Q = 0$, &c., and therefore $P \,.\, Q = PQ + P_1Q$; and I shall in this case also (and consequently whenever $P_2Q = 0$, $P_3Q = 0$, &c.) write

$$P \,.\, Q = PQ + P(Q),$$

where on the right-hand side in the term PQ the differentiations ∂_a, $\partial_b, \ldots$ are considered as not in anywise affecting the symbol Q, while the term $P(Q)$ is in each case what has been in the first instance represented by P_1Q.

We have in like manner, if Q be a linear function of ∂_a, ∂_b, $\partial_c, \ldots$, or if P be a linear function of $a, b, c, \ldots$,

$$Q \,.\, P = QP + Q(P);$$

and from the two equations (since obviously $PQ = QP$) we derive

$$P \,.\, Q - Q \,.\, P = P(Q) - Q(P),$$

which is the form in which the equations are most frequently useful.

12. I return to the expression

$$A^p B^q C^r \ldots U_1 U_2 \ldots,$$

and I suppose that after the differentiations the sets $(x_1, y_1, \ldots)$, $(x_2, y_2, \ldots)$, &c. are replaced by the original set $(x, y, \ldots)$. To show that the result is a covariant, we must prove that it is reduced to zero by an operation $\mathfrak{D} =$

$$\{x\partial_y\} - x\partial_y.$$

It is easy to see that the change of the sets $(x_1, y_1, \ldots)$, $(x_2, y_2, \ldots)$, &c. into the original set $(x, y, \ldots)$ may be deferred until after the operation $\mathfrak{D}$, provided that $x\partial_y$ is replaced by $x_1\partial_{y_1} + x_2\partial_{y_2} + \ldots$, or if we please by $Sx\partial_y$; we must therefore write $\mathfrak{D} = \{x\partial_y\} - Sx\partial_y$. Now in the equation

$$A \,.\, \mathfrak{D} - \mathfrak{D} \,.\, A = A(\mathfrak{D}) - \mathfrak{D}(A),$$

where, as before, $A(\mathfrak{D})$ denotes the result of the operation A performed upon $\mathfrak{D}$ as operand, and similarly $\mathfrak{D}(A)$ the result of the operation $\mathfrak{D}$ performed upon A as operand, we see first that $A(\mathfrak{D})$ is a determinant two of the lines of which are identical, it is therefore equal to zero; and next, since $\mathfrak{D}$ does not involve any differentiations affecting A, that $\mathfrak{D}(A)$ is also equal to zero. Hence $A \,.\, \mathfrak{D} - \mathfrak{D} \,.\, A = 0$ or A and $\mathfrak{D}$ are convertible. But in like manner $\mathfrak{D}$ is convertible with B, C, &c., and consequently $\mathfrak{D}$ is convertible with $A^p B^q C^r \ldots$. Now $\mathfrak{D} U_1 U_2 \ldots = 0$; hence

$$\mathfrak{D} \,.\, A^p B^q C^r \ldots U_1 U_2 \ldots = 0,$$

or $A^p B^q C^r \ldots U_1 U_2 \ldots$ is a covariant, the proposition which was to be proved.

13. I pass to a theorem which leads to another method of finding the covariants of a quantic. For this purpose I consider the quantic

$$(* \between x, y \ldots \overset{m}{\between} x', y' \ldots \overset{m'}{)} \ldots),$$

the coefficients of which are mere numerical multiples of the elements $(a, b, c, \ldots)$; and in connexion with this quantic I consider the linear functions $\xi x + \eta y \ldots$, $\xi' x' + \eta' y' \ldots$, which treating $(\xi, \eta, \ldots)$, $(\xi', \eta', \ldots)$, &c. as coefficients, may be represented in the form

$$(\xi, \eta, \ldots \between x, y, \ldots), \quad (\xi', \eta', \ldots \between x', y', \ldots), \ldots$$

we may from the quantic (which for convenience I call U) form an operative quantic

$$(* \between \xi, \eta, \ldots \overset{m}{\between} \xi', \eta', \ldots \overset{m'}{)} \ldots)$$

(I call this quantic Θ), the coefficients of which are mere numerical multiples of ∂_a, ∂_b, $\partial_c, \ldots$, and which is such that

$$\Theta U = \overline{(\xi, \eta, \ldots \between x, y, \ldots)}|^m \overline{(\xi', \eta', \ldots \between x', y', \ldots)}|^{m'} \ldots$$

i.e. a product of powers of the linear functions. And it is to be remarked that as regards the quantic Θ and its covariants or other derivatives, the symbols ∂_a, ∂_b, $\partial_c, \ldots$ are to be considered as elements with respect to which we may differentiate, &c.

The quantic Θ gives rise to the symbols $\{\xi\partial_\eta\}$, &c. analogous to the symbols $\{x\partial_y\}$, &c. formed from the quantic U. Suppose now that Φ is any quantic containing as well the coefficients as all or any of the sets of Θ. Then $\{x\partial_y\}$ being a linear function of $a, b, c, \ldots$ the variables to which the differentiations in Φ relate, we have

$$\Phi \,.\, \{x\partial_y\} = \Phi\,\{x\partial_y\} + \Phi\,(\{x\partial_y\})\,;$$

again, $\{\eta\partial_\xi\}$ being a linear function of the differentiations with respect to the variables $\partial_a, \partial_b, \partial_c, \ldots$ in Φ, we have

$$\{\eta\partial_\xi\} \,.\, \Phi = \{\eta\partial_\xi\}\,\Phi + \{\eta\partial_\xi\}\,(\Phi)\,;$$

these equations serve to show the meaning of the notations $\Phi\,(\{x\partial_y\})$ and $\{\eta\partial_\xi\}\,(\Phi)$, and there exists between these symbols the singular equation

$$\Phi\,(\{x\partial_y\}) = \{\eta\partial_\xi\}\,(\Phi).$$

14. The general demonstration of this equation presents no real difficulty, but to avoid the necessity of fixing upon a notation to distinguish the coefficients of the different terms and for the sake of simplicity, I shall merely exhibit by an example the principle of such general demonstration. Consider the quantic

$$U = ax^3 + 3bx^2y + 3cy^2 + dy^3,$$

this gives

$$\Theta = \xi^3\partial_a + \xi^2\eta\partial_b + \xi\eta^2\partial_c + \eta^3\partial_d\,;$$

or if, for greater clearness, $\partial_a, \partial_b, \partial_c, \partial_d$ are represented by $\alpha, \beta, \gamma, \delta$, then

$$\Theta = \alpha\xi^3 + \beta\xi^2\eta + \gamma\xi\eta^2 + \delta\eta^3,$$

and we have

$$\{x\partial_y\} = 3b\partial_a + 2c\partial_b + d\partial_c,$$

and

$$\{\eta\partial_\xi\} = 3\alpha\partial_\beta + 2\beta\partial_\gamma + \gamma\partial_\delta.$$

Now considering Φ as a function of $\partial_a, \partial_b, \partial_c, \partial_d$, or, what is the same thing, of $\alpha, \beta, \gamma, \delta$, we may write

$$\Phi\,(\{x\partial_y\}) = \Phi\,(3b\alpha + 2c\beta + d\gamma)\,;$$

and if in the expression of Φ we write $\alpha + \partial_a$, $\beta + \partial_b$, $\gamma + \partial_c$, $\delta + \partial_d$ for $\alpha, \beta, \gamma, \delta$ (where only the symbols $\partial_a, \partial_b, \partial_c, \partial_d$ are to be considered as affecting a, b, c, d as contained in the operand $3b\alpha + 2c\beta + d\gamma$), and reject the first term (or term independent of $\partial_a, \partial_b, \partial_c, \partial_d$ in the expansion) we have the required value of $\Phi\,(\{x\delta_y\})$. This value is

$$(\partial_\alpha\Phi\,\partial_a + \partial_\beta\Phi\,\partial_b + \partial_\gamma\Phi\,\partial_c)\,(3b\alpha + 2c\beta + d\gamma)\,;$$

performing the differentiations $\partial_a, \partial_b, \partial_c, \partial_d$, the value is

$$(3\alpha\partial_\beta + 2\beta\partial_\gamma + \gamma\partial_\delta)\,\Phi,$$

i.e. we have

$$\Phi\,(\{x\partial_y\}) = \{\eta\partial_\xi\}\,(\Phi).$$

15. Suppose now that Φ is a covariant of Θ, then the operation Φ performed upon any covariant of U gives rise to a covariant of the system

$$(*\between x, y, \ldots \overset{m}{\between} x', y', \ldots)^{m'} \ldots),$$

$$(\xi, \eta, \ldots \between x, y, \ldots), \qquad (\xi', \eta', \ldots \between x', y', \ldots), \text{ \&c.}$$

To prove this it is to be in the first instance noticed, that as regards $(\xi, \eta, \ldots \between x, y, \ldots)$, &c. we have $\{x\partial_y\} = \eta\partial_\xi$, &c. Hence considering $\{x\partial_y\}$, &c. as referring to the quantic U, the operation $\Sigma\{x\partial_y\} - x\partial_y$ will be equivalent to $\{x\partial_y\} + \eta\partial_\xi - x\partial_y$, and therefore every covariant of the system must be reduced to zero by each of the operations

$$\mathfrak{D} = \{x\partial_y\} + \eta\partial_\xi - x\partial_y.$$

This being the case, we have

$$\mathfrak{D} . \Phi = \mathfrak{D}\Phi + \mathfrak{D}(\Phi),$$

$$\Phi . \mathfrak{D} = \Phi\mathfrak{D} + \Phi(\mathfrak{D}),$$

equations which it is obvious may be replaced by

$$\mathfrak{D} . \Phi = \mathfrak{D}\Phi + \eta\partial_\xi(\Phi),$$

$$\Phi . \mathfrak{D} = \Phi\mathfrak{D} + \Phi(\{x\partial_y\}),$$

and consequently (in virtue of the theorem) by

$$\mathfrak{D} . \Phi = \mathfrak{D}\Phi + \eta\partial_\xi(\Phi),$$

$$\Phi . \mathfrak{D} = \Phi\mathfrak{D} + \{\eta\partial_\xi\}(\Phi);$$

and we have therefore

$$\mathfrak{D} . \Phi - \Phi . \mathfrak{D} = -(\{\eta\partial_\xi\} - \eta\partial_\xi)(\Phi);$$

or, since Φ is a covariant of Θ, we have $\mathfrak{D} . \Phi = \Phi . \mathfrak{D}$. And since every covariant of the system is reduced to zero by the operation $\mathfrak{D}$, and therefore by the operation $\Phi . \mathfrak{D}$, such covariant will also be reduced to zero by the operation $\mathfrak{D} . \Phi$, or what is the same thing, the covariant operated on by Φ, is reduced to zero by the operation $\mathfrak{D}$ and is therefore a covariant, i.e. Φ operating upon a covariant gives a covariant.

16. In the case of a quantic such as $U =$

$$(*\between x, y \overset{m}{\between} x', y')^{m'} \ldots),$$

we may instead of the new sets (ξ, η), (ξ', η')... employ the sets $(y, -x)$, $(y', -x')$, &c. The operative quantic Θ is in this case defined by the equation $\Theta U = 0$, and if Φ be, as before, any covariant of Θ, then Φ operating upon a covariant of U will give a covariant of U. The proof is nearly the same as in the preceding case; we have instead of the equation $\Phi(\{x\partial_y\}) = \{\eta\partial_\xi\}(\Phi)$ the analogous equation

$$\Phi(\{x\partial_y\}) = -\{x\partial_y\}(\Phi),$$

where on the left-hand side $\{x\partial_y\}$ refers to U, but on the right-hand side $\{x\partial_y\}$ refers to Θ, and instead of $\mathfrak{D} = \{x\partial_y\} + \eta\partial_\xi - x\partial_y$ we have simply $\mathfrak{D} = \{x\partial_y\} - x\partial_y$.

17. I pass next to the quantic

$$(*\between x, y)^m,$$

which I shall in general consider under the form

$$(a, b, \ldots b', a'\between x, y)^m,$$

but sometimes under the form

$$(a, b, \ldots b', a''\between x, y)^m,$$

the former notation denoting, it will be remembered,

$$ax^m + \frac{m}{1} bx^{m-1} y \ldots + \frac{m}{1} b'xy^{m-1} + a'y^m,$$

and the latter notation

$$ax^m + bx^{m-1} y \ldots + b'xy^{m-1} + a'y^m.$$

But in particular cases the coefficients will be represented all of them by unaccentuated letters, thus $(a, b, c, d\between x, y)^3$ will be used to denote $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, and $(a, b, c, d'\between x, y)^3$ will be used to denote $ax^3 + bx^2y + cxy^2 + dy^3$, and so in all similar cases.

Applying the general methods to the quantic

$$(a, b, \ldots b', a'\between x, y)^m,$$

we see that

$$\{y\partial_x\} = a\partial_b + \quad 2b\partial_c \ldots + mb'\partial_{a'},$$
$$\{x\partial_y\} = mb\partial_a + (m-1c\partial_b \ldots + \quad a'\partial_b;$$

in fact, with these meanings of the symbols the quantic is reduced to zero by each of the operations $\{y\partial_x\} - y\partial_x$, $\{x\partial_y\} - x\partial_y$; hence according to the definition any function which is reduced to zero by each of the last-mentioned operations is a covariant of the quantic. But in accordance with a preceding remark, the covariant may be considered as a rational and integral function, separately homogeneous in regard to the facients (x, y) and the coefficients $(a, b, \ldots b', a')$. If instead of the single set (x, y) the covariant contains the sets (x_1, y_1), (x_2, y_2), &c., then it must be reduced to zero by each of the operations $\{y\partial_x\} - Sy\partial_x$, $\{x\partial_y\} - Sx\partial_y$ (where $Sy\partial_x = y_1\partial_{x_1} + y_2\partial_{x_2} + \ldots$), but I shall principally attend to the case in which the covariant contains only the set (x, y).

Suppose, for shortness, that the quantic is represented by U, and let U_1, $U_2, \ldots$ be what U becomes when the set (x, y) is successively replaced by the sets (x_1, y_1), (x_2, y_2), &c. Suppose moreover that $\overline{12} = \partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}$, &c., then the function

$$\overline{12}^p\, \overline{13}^q\, \overline{23}^r \ldots U_1 U_2 U_3 \ldots,$$

in which, after the differentiations, the new sets (x_1, y_1), $(x_2, y_2), \ldots$ may be replaced by the original set (x, y), will be a covariant of the quantic U. And if the number

of differentiations be such as to make the facients disappear, i.e. if the sum of all the indices $p, q, \ldots$ of the terms $\overline{12}$, &c. which contain the symbolic number 1, the sum of all the indices $p, r, \ldots$ of the terms which contain the symbolic number 2, and so on, be severally equal to the degree of the quantic, we have an invariant. The operative quantic Θ becomes in the case under consideration

$$\Theta = (\partial_{a'}, -\partial_{b'}, \ldots \pm \partial_{a'} \between x, y)^m,$$

the signs being alternately positive and negative; in fact it is easy to verify that this expression gives identically $\Theta U = 0$, and any covariant of Θ operating on a covariant of U gives rise to a covariant of U.

18. But the quantic

$$(a, b, \ldots b', a' \between x, y)^m,$$

considered as decomposable into linear factors, i.e. as expressible in the form

$$a(x - \alpha y)(x - \beta y)\ldots,$$

gives rise to a fresh series of results. We have in this case

$$\begin{aligned} \{y\partial_x\} &= \qquad\qquad \partial_\alpha + \partial_\beta \ldots, \\ \{x\partial_y\} &= -(\alpha + \beta \ldots)\, a\partial_a + \alpha^2\partial_\alpha + \beta^2\partial_\beta + \ldots; \end{aligned}$$

in fact with these meanings of the symbols the quantic is reduced to zero by each of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$, and we have consequently the definition of the covariant of a quantic considered as expressed in the form $a(x - \alpha y)(x - \beta y)\ldots$. And it will be remembered that these and the former values of the symbols $\{x\partial_y\}$ and $\{y\partial_x\}$ are, when the same quantic is considered as represented under the two forms $(a, b, \ldots b', a' \between x, y)^m$ and $a(x - \alpha y)(x - \beta y)\ldots$, identical.

19. Consider now the expression

$$a^\theta (x - \alpha y)^j (x - \beta y)^k \ldots (\alpha - \beta)^p \ldots,$$

where the sum of the indices $j, p, \ldots$ of all the simple factors which contain α, the sum of the indices $k, p, \ldots$ of all the simple factors which contain β, &c. are respectively equal to the index θ of the coefficient a. The index θ and the indices p, &c. may be considered as arbitrary, nevertheless within such limits as will give positive values (0 inclusive) for the indices $j, k, \ldots$.

The expression in question is reduced to zero by each of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$; and this is of course also the case with the expressions obtained by interchanging in any manner the roots $\alpha, \beta, \gamma, \ldots$, and therefore with the expression

$$a^\theta \Sigma (x - \alpha y)^j (x - \beta y)^k \ldots (\alpha - \beta)^p \ldots,$$

where Σ denotes a summation with respect to all the different permutations of the roots $\alpha, \beta, \ldots$.

The function so obtained (which is of course a rational function of $(a, b, \ldots b', a')$) will be a covariant, and if we suppose $\mu = m\theta - 2Sp$, where Sp denotes the sum of all the indices p of the different terms $(\alpha - \beta)^p$, &c., then the covariant will be of the order μ (i.e. of the degree μ in the facients x, y), and of the degree θ in the coefficients.

20. In connexion with this covariant

$$a^\theta \Sigma (x - \alpha y)^p (x - \beta y)^k \ldots (\alpha - \beta)^p \ldots,$$

of the order μ and of the degree θ in the coefficients, of the quantic $U =$

$$a(x - \alpha y)(x - \beta y)\ldots,$$

consider the covariant

$$\Sigma (\overline{12}^p \ldots) V_1 V_2 \ldots V_m$$

of a quantic $V =$

$$(* \between x, y)^\phi,$$

in which, after the differentiations, the sets (x_1, y_1), (x_2, y_2), ... are replaced by the original set (x, y). The last-mentioned covariant will be of the order $m(\phi - \theta) + \mu$, and will be of the degree m in the coefficients; and in particular if $\phi = \theta$, i.e. if V be a quantic of the order θ, then the covariant will be of the order μ and of the degree m in the coefficients. Hence to a covariant of the degree θ in the coefficients, of a quantic of the order m, there corresponds a covariant of the degree m in the coefficients, of a quantic of the order θ; the two covariants in question being each of them of the same order μ. And it is proper to notice, that if we had commenced with the covariant of the quantic V, a reverse process would have led to the covariant of the quantic U. We may, therefore, say that the covariants of a given order and of the degree θ in the coefficients, of a quantic of the order m, correspond each to each with the covariants of the same order and of the degree m in the coefficients, of a quantic of the order θ; and in particular the invariants of the degree θ of a quantic of the order m, correspond each to each with the invariants of the degree m of a quantic of the order θ. This is the law of reciprocity demonstrated by M. Hermite, by a method which (I am inclined to think) is substantially identical with that here made use of, although presented in a very different form: the discovery of the law, considered as a law relating to the *number* of invariants, is due to Mr Sylvester. The precise meaning of the law, in the last-mentioned point of view, requires some explanation. Suppose that we know all the really independent invariants of a quantic of the order m, the law gives the number of invariants of the degree m of a quantic of the order θ (it is convenient to assume $\theta > m$), viz. of the invariants of the degree in question, which are linearly independent, or asyzygetic, i.e. such that there do not exist any merely numerical multiples of these invariants having the sum zero; but the invariants in question may and in general will be connected *inter se* and with the other invariants of the quantic to which they belong by non-linear equations: and in particular the system of invariants of the degree m will comprise all the invariants of that degree (if any) which are rational and integral

functions of the invariants of lower degrees. The like observations apply to the system of covariants of a given order and of the degree m in the coefficients, of a quantic of the order θ.

21. The number of the really independent covariants of a quantic $(*\between x, y)^m$ is precisely equal to the order m of the quantic, i.e. any covariant is a function (generally an irrational function only expressible as the root of an equation) of any m independent covariants, and in like manner the number of really independent invariants is $m-2$; we may, if we please, take $m-2$ really independent invariants as part of the system of the m independent covariants; the quantic itself may be taken as one of the other two covariants, and any other covariant as the other of the two covariants; we may therefore say that every covariant is a function (generally an irrational function only expressible as the root of an equation) of $m-2$ invariants, of the quantic itself and of a given covariant.

22. Consider any covariant of the quantic

$$(a, b, \ldots b', a'\between x, y)^m,$$

and let this be of the order μ, and of the degree θ in the coefficients. It is very easily shown that $m\theta-\mu$ is necessarily even. In particular in the case of an invariant (i.e. when $\mu=0$) $m\theta$ is necessarily even[1]: so that a quantic of an odd order admits only of invariants of an even degree. But there is an important distinction between the cases of $m\theta-\mu$ evenly even and oddly even. In the former case the covariant remains unaltered by the substitution of (y, x), $(a', b', \ldots b, a)$ for (x, y), $(a, b, \ldots b', a')$; in the latter case the effect of the substitution is to change the sign of the covariant. The covariant may in the former case be called a symmetric covariant, and in the latter case a skew covariant. It may be noticed in passing, that the simplest skew invariant is M. Hermite's invariant of the degree 18 of a quantic of the order 5.

23. There is another very simple condition which is satisfied by every covariant of the quantic

$$(a, b, \ldots b', a'\between x, y)^m,$$

viz. if we consider the facients (x, y) as being respectively of the weights $\frac{1}{2}, -\frac{1}{2}$, and the coefficients $(a, b, \ldots b', a')$ as being respectively of the weights $-\frac{1}{2}m, -\frac{1}{2}m+1, \ldots \frac{1}{2}m-1, \frac{1}{2}m$, then the weight of each term of the covariant will be zero. This is the most elegant statement of the law, but to avoid negative quantities, the statement may be modified as follows:—if the facients (x, y) are considered as being of the weights 1, 0 respectively, and the coefficients $(a, b \ldots b', a')$ as being of the weights $0, 1, \ldots, m-1, m$ respectively, then the weight of each term of the covariant will be $\frac{1}{2}(m\theta+\mu)$.

[1] I may remark that it was only M. Hermite's important discovery of an invariant of the degree 18 of a quantic of the order 5, which removed an erroneous impression which I had been under from the commencement of the subject, that $m\theta$ was of necessity *evenly even*.

24. The preceding laws as to the form of a covariant have been stated here by way of anticipation, principally for the sake of the remark, that they so far define the form of a covariant as to render it in very many cases practicable with a moderate amount of labour to complete the investigations by means of the operation $\{x\partial_y\} - x\partial_y$ and $\{y\partial_x\} - y\partial_x$. In fact, for finding the covariants of a given order, and of a given degree in the coefficients, we may form the most general function of the proper order and degree in the coefficients, satisfying the prescribed conditions as to symmetry and weight: such function, if reduced to zero by one of the operations in question, will, on account of the symmetry, be reduced to zero by the other of the operations in question; it is therefore only necessary to effect upon it, e.g. the operation $\{x\partial_y\} - x\partial_y$, and to determine if possible the indeterminate coefficients in such manner as to render the result identically zero: of course when this cannot be done there is not any covariant of the form in question. It is moreover proper to remark, as regards invariants, that if an invariant be expanded in a series of ascending powers of the first coefficient a, and the first term of the expansion is known, all the remaining terms can be at once deduced by mere differentiations. There is one very important case in which the value of such first term (i.e. the value of the invariant when a is put equal to 0) can be deduced from the corresponding invariant of a quantic of the next inferior order; the case in question is that of the discriminant (or function which equated to zero expresses the equality of a pair of roots); for by Joachimsthal's theorem, if in the discriminant of the quantic $(a, b, \ldots b', a' \between x, y)^m$ we write $a = 0$, the result contains b^2 as a factor, and divested of this factor is precisely the discriminant of the quantic of the order $m - 1$ obtained from the given quantic by writing $a = 0$ and throwing out the factor x: this is in practice a very convenient method for the calculation of the discriminants of quantics of successive orders. It is also to be noticed as regards covariants, that when the first or last coefficient of any covariant (i.e. the coefficient of the highest power of either of the facients) is known, all the other coefficients can be deduced by mere differentiations.

Postscript added October 7th, 1854.—I have, since the preceding memoir was written, found with respect to the covariants of a quantic $(* \between x, y)^m$, that a function of any order and degree in the coefficients satisfying the necessary condition as to weight, and such that it is reduced to zero by one of the operations $\{x\partial_y\} - x\partial_y$, $\{y\partial_x\} - y\partial_x$, will of necessity be reduced to zero by the other of the two operations, i.e. it will be a covariant; and I have been thereby led to the discovery of the law for the number of asyzygetic covariants of a given order and degree in the coefficients; from this law I deduce as a corollary, the law of reciprocity of MM. Sylvester and Hermite. I hope to return to the subject in a subsequent memoir.

140.

RESEARCHES ON THE PARTITION OF NUMBERS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLV *for the year* 1855, pp. 127—140. Received April 14,—Read May 24, 1855.]

I PROPOSE to discuss the following problem: "To find in how many ways a number q can be made up of the elements $a, b, c, \ldots$ each element being repeatable an indefinite number of times." The required number of partitions is represented by the notation

$$P(a, b, c, \ldots)q,$$

and we have, as is well known,

$$P(a, b, c, \ldots)q = \text{coefficient } x^q \text{ in } \frac{1}{(1-x^a)(1-x^b)(1-x^c)\ldots},$$

where the expansion is to be effected in ascending powers of x.

It may be as well to remark that each element is to be considered as a separate and distinct element, notwithstanding any equalities which may exist between the numbers $a, b, c, \ldots$; thus, although $a=b$, yet $a+a+a+$&c. and $a+a+b+$&c. are to be considered as two different partitions of the number q, and so in all similar cases.

The solution of the problem is thus seen to depend upon the theory, to which I now proceed, of the expansion of algebraical fractions.

Consider an algebraical fraction $\dfrac{\phi x}{fx}$,

where the denominator is the product of any number of factors (the same or different) of the form $1-x^m$. Suppose in general that $[1-x^m]$ denotes the irreducible factor of $1-x^m$, i.e. the factor which, equated to zero, gives the prime roots of the equation $1-x^m=0$. We have

$$1-x^m = \Pi[1-x^{m'}],$$

where m' denotes any divisor whatever of m (unity and the number m itself not excluded). Hence, if a represent a divisor of one or more of the indices m, and k be the number of the indices of which a is a divisor, we have

$$fx = \Pi[1 - x^a]^k.$$

Now considering apart from the others one of the multiple factors $[1 - x^a]^k$, we may write $fx = [1 - x^a]^k f_1 x$.

Suppose that the fraction $\frac{\phi x}{fx}$ is decomposed into simpler fractions, in the form

$$\begin{aligned}\frac{\phi x}{fx} = {} & I(x) \\ & + (x\partial_x)^{k-1}\frac{\theta x}{[1 - x^a]} + (x\partial_x)^{k-2}\frac{\theta_1 x}{[1 - x^a]} \cdots + \frac{\theta_{k-1} x}{[1 - x^a]} \\ & + \&c.,\end{aligned}$$

where $I(x)$ denotes the integral part, and the &c. refers to the fractional terms depending upon the other multiple factors such as $[1 - x^a]^k$. The functions θx are to be considered as functions with indeterminate coefficients, the degree of each such function being inferior by unity to that of the corresponding denominator; and it is proper to remark that the number of the indeterminate coefficients in all the functions θx together is equal to the degree of the denominator fx.

The term $(x\partial_x)^{k-1}\frac{\theta x}{[1 - x^a]}$ may be reduced to the form

$$\frac{gx}{[1 - x^a]^k} + \frac{g_1 x}{[1 - x^a]^{k-1}} + \&c.,$$

the functions gx being of the same degree as θx, and the coefficients of these functions being linearly connected with those of the function θx. The first of the foregoing terms is the only term on the right-hand side which contains the denominator $[1 - x^a]^k$; hence, multiplying by this denominator and then writing $[1 - x^a] = 0$, we find

$$\frac{\phi x}{f_1 x} = gx,$$

which is true when x is any root whatever of the equation $[1 - x^a] = 0$. Now by means of the equation $[1 - x^a] = 0$, $\frac{\phi x}{f_1 x}$ may be expressed in the form of a rational and integral function Gx, the degree of which is less by unity than that of $[1 - x^a]$. We have therefore $Gx = gx$, an equation which is satisfied by each root of $[1 - x^a] = 0$, and which is therefore an identical equation; gx is thus determined, and the coefficients of θx being linear functions of those of gx, the function θx may be considered as determined. And this being so, the function

$$\frac{\phi x}{fx} - (x\partial_x)^{k-1}\frac{\theta x}{[1 - x^a]}$$

will be a fraction the denominator of which does not contain any power of $[1-x^a]$ higher than $[1-x^a]^{k-1}$; and therefore $\theta_1 x$ can be found in the same way as θx, and similarly $\theta_2 x$, and so on. And the fractional parts being determined, the integral part may be found by subtracting from $\frac{\phi x}{fx}$ the sum of the fractional parts, so that the fraction $\frac{\phi x}{fx}$ can by a direct process be decomposed in the above-mentioned form.

Particular terms in the decomposition of certain fractions may be obtained with great facility. Thus m being a prime number, assume

$$\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^m)} = \&c. + \frac{\theta x}{[1-x^m]};$$

then observing that $(1-x^m)=(1-x)[1-x^m]$, we have for $[1-x^m]=0$,

$$\theta x = \frac{1}{(1-x)(1-x^2)\ldots(1-x^{m-1})}.$$

Now u being any quantity whatever and x being a root of $[1-x^m]=0$, we have identically

$$[1-u^m]=(u-x)(u-x^2)\ldots(u-x^{m-1});$$

and therefore putting $u=1$, we have $m=(1-x)(1-x^2)\ldots(1-x^{m-1})$, and therefore

$$\theta x = \frac{1}{m},$$

whence

$$\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^m)} = \&c. + \frac{1}{m}\frac{1}{[1-x^m]}.$$

Again, m being as before a prime number, assume

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} = \&c. + \frac{\theta x}{[1-x^m]},$$

we have in this case for $[1-x^m]=0$,

$$\theta x = \frac{1}{(1-x)^2(1-x^2)\ldots(1-x^{m-1})},$$

which is immediately reduced to $\theta x = \frac{1}{m}\frac{1}{1-x}$. Now

$$\frac{[1-u^m]}{u-x} = \frac{[1-u^m]-[1-x^m]}{u-x} = (1+u+\ldots+u^{m-2})+(1+u\ldots+u^{m-3})x\ldots+(1+u)x^{m-3}+x^{m-2};$$

or putting $u=1$,

$$\frac{m}{1-x} = (m-1)+(m-2)x\ldots+x^{m-2};$$

and substituting this in the value of θx, we find

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^m)} = \&c. + \frac{1}{m^2}\frac{(m-1)+(m-2)x\dots+x^{m-2}}{[1-x^m]}.$$

The preceding decomposition of the fraction $\frac{\phi x}{fx}$ gives very readily the expansion of the fraction in ascending powers of x. For, consider a fraction such as

$$\frac{\theta x}{[1-x^a]},$$

where the degree of the numerator is in general less by unity than that of the denominator; we have

$$1-x^a = [1-x^a]\,\Pi\,[1-x^{a'}],$$

where a' denotes any divisor of a (including unity, but not including the number a itself). The fraction may therefore be written under the form

$$\frac{\theta x \Pi\,[1-x^{a'}]}{1-x^a},$$

where the degree of the numerator is in general less by unity than that of the denominator, i.e. is equal to $a-1$. Suppose that b is any divisor of a (including unity, but not including the number a itself), then $1-x^b$ is a divisor of $\Pi\,[1-x^{a'}]$, and therefore of the numerator of the fraction. Hence representing this numerator by

$$A_0 + A_1 x \dots + A_{a-1}x^{a-1},$$

and putting $a = bc$, we have (corresponding to the case $b=1$)

$$A_0 + A_1 + A_2 \dots + A_{a-1} = 0,$$

and generally for the divisor b,

$$\begin{aligned} A_0 \quad &+ A_b \quad \dots + A_{(c-1)b} &= 0,\\ A_1 \quad &+ A_{b+1} \dots + A_{(c-1)b+1} &= 0,\\ &\vdots\\ A_{b-1} &+ A_{2b-1} \dots + A_{cb-1} &= 0. \end{aligned}$$

Suppose now that a_q denotes a circulating element to the period a, i.e. write

$$a_q = 1, \quad q = 0 \text{ (mod. } a),$$

$$a_q = 0 \text{ in every other case};$$

a function such as

$$A_0 a_q + A_1 a_{q-1} \dots + A_{a-1} a_{q-a+1}$$

will be a circulating function, or circulator to the period a, and may be represented by the notation

$$(A_0,\ A_1,\ \dots A_{a-1}) \text{ circlor } a_q.$$

In the case however where the coefficients A satisfy, for each divisor b of the number a, the above-mentioned equations, the circulating function is what I call a prime circulator, and I represent it by the notation

$$(A_0,\ A_1, \ldots A_{a-1}) \text{ per } a_q.$$

By means of this notation we have at once

$$\text{coefficient } x^q \text{ in } \quad \frac{\theta x}{[1-x^a]} = \quad (A_0,\ A_1 \ldots A_{a-1}) \text{ per } a_q,$$

and thence also

$$\text{coefficient } x^q \text{ in } (x\partial_x)^r \frac{\theta x}{[1-x^a]} = q^r (A_0,\ A_1 \ldots A_{a-1}) \text{ per } a_q.$$

Hence assuming that in the fraction $\frac{\phi x}{fx}$ the degree of the numerator is less than that of the denominator (so that there is not any integral part), we have

$$\text{coefficient } x^q \text{ in } \frac{\phi x}{fx} = \Sigma\, q^r (A_0,\ A_1,\ \ldots A_{a-1}) \text{ per } a_q;$$

or, if we wish to put in evidence the non-circulating part arising from the divisor $a=1$,

$$\text{coefficient } x^q \text{ in } \frac{\phi x}{fx} = Aq^{k-1} + Bq^{k-2} \ldots + Lq + M$$
$$+ \Sigma\, q^r (A_0,\ A_1 \ldots A_{a-1}) \text{ per } a_q;$$

where k denotes the number of the factors $1-x^m$ in the denominator fx, a is any divisor (unity excluded) of one or more of the indices m; and for each value of a r extends from $r=0$ to $r=k-1$, where k denotes the number of indices m of which a is a divisor. The particular results previously obtained show, that m being a prime number,

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x^2)(1-x^2)\ldots(1-x^m)} = \&c. + \frac{1}{m}(\quad 1, -1,\ \ 0,\ 0, \ldots) \text{ per } m_q,$$

and

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} = \&c. + \frac{1}{m^2}(m-1, -1, -1, \quad \ldots) \text{ per } m_q.$$

Suppose, as before, that the degree of ϕx is less than that of fx, and let the analytical expression above obtained for the coefficient of x^q in the expansion in ascending powers of x of the fraction $\frac{\phi x}{fx}$ be represented by Fq, it is very remarkable that if we expand $\frac{\phi x}{fx}$ in descending powers of x, then the coefficient of x^q in this new expansion (q is here of course negative, since the expansion contains only negative powers of x) is precisely equal to $-Fq$; this is in fact at once seen to be

the case with respect to each of the partial fractions into which $\frac{\phi x}{fx}$ has been decomposed, and it is consequently the case with respect to the fraction itself[1]. This gives rise to a result of some importance. Suppose that ϕx and fx are respectively of the degrees N and D; it is clear from the form of fx that we have $f\left(\frac{1}{x}\right)=(-)^D x^{-D} fx$; and I suppose that ϕx is also such that $\phi\left(\frac{1}{x}\right)=(\pm)^N x^{-N}\phi x$; then writing $D-N=h$, and supposing that $\frac{\phi x}{fx}$ is expanded in descending powers of x, so that the coefficient of x^q in the expansion is $-Fq$, it is in the first place clear that the expansion will commence with the term x^{-h}, and we must therefore have

$$Fq=0$$

for all values of q from $q=-1$ to $q=-(h-1)$.

Consider next the coefficient of a term x^{-h-q}, where q is 0 or positive; the coefficient in question, the value of which is $-F(-h-q)$, is obviously equal to the coefficient of x^{h+q} in the expansion in ascending powers of x of $\frac{\phi\left(\frac{1}{x}\right)}{f\left(\frac{1}{x}\right)}$, i.e. to

$$(\pm)^N(-)^D \text{ coefficient } x^{h+q} \text{ in } \frac{x^h\phi x}{fx},$$

or what is the same thing, to

$$(\pm)^N(-)^D \text{ coefficient } x^q \text{ in } \frac{\phi x}{fx};$$

and we have therefore, q being zero or positive,

$$F(-h-q)=-(\pm)^N(-)^D Fq.$$

In particular, when $\phi x=1$, $\qquad Fq=0$

for all values of q from $q=-1$ to $q=-(D-1)$; and q being 0 or positive,

$$F(-D-q)=(-)^{D-1}Fq.$$

The preceding investigations show the general form of the function $P(a, b, c, \ldots)q$, viz. that

$$P(a, b, c, \ldots)q=Aq^{k-1}+Bq^{k-2}\ldots+Lq+M+\Sigma q^r(A_0, A_1, \ldots A_{l-1}) \text{ per } l_q,$$

a formula in which k denotes the number of the elements $a, b, c, \ldots$ &c., and l is any divisor (unity excluded) of one or more of these elements; the summation in the case of each such divisor extends from $r=0$ to $r=k-1$, where k is the number of the elements $a, b, c, \ldots$ &c. of which l is a divisor; and the investigations indicate

[1] The property is a fundamental one in the general theory of developments.

how the values of the coefficients A of the prime circulators are to be obtained. It has been moreover in effect shown, that if $D = a + b + c + \ldots$, then, writing for shortness $P(q)$ instead of $P(a, b, c, \ldots)q$, we have

$$P(q) = 0$$

for all values of q from $q = -1$ to $q = -(D-1)$, and that q being 0 or positive,

$$P(-D-q) = (-)^{\nu-1} P(q);$$

these last theorems are however uninterpretable in the theory of partitions, and they apply only to the analytical expression for $P(q)$.

I have calculated the following particular results:—

$$P(1, 2)q = \frac{1}{4}\left\{2q + 3 + (1, -1) \text{ per } 2_q\right\}$$

$$P(1, 2, 3)q = \frac{1}{72}\left\{6q^2 + 36q + 47 + 9(1, -1) \text{ per } 2_q + 8(2, -1, -1) \text{ per } 3_q\right\}$$

$$P(1, 2, 3, 4)q = \frac{1}{288}\left\{2q^3 + 30q^2 + 135q + 175 + (9q + 45)(1, -1) \text{ per } 2_q + 32\ (1, 0, -1) \text{ per } 3_q + 36\ (1, 0, -1, 0) \text{ per } 4_q\right\}$$

$$P(1, 2, 3, 4, 5)q = \frac{1}{86400}\left\{30q^4 + 900q^3 + 9300q^2 + 38250q + 50651 + (1350q + 10125)\ (1, -1) \text{ per } 2_q + 3200\ (2, -1, -1) \text{ per } 3_q + 5400\ (1, 1, -1, -1) \text{ per } 4_q + 3456(4, -1, -1, -1, -1) \text{ per } 5_q\right\}$$

$$P(2)q = \frac{1}{2}\left\{1 + (1, -1) \text{ per } 2_q\right\}$$

$$P(2, 3)q = \frac{1}{12}\left\{2q + 5 + 3\ (1, -1) \text{ per } 2_q + 4(1, -1, 0) \text{ per } 3_q\right\}$$

$$P(2,\ 3,\ 4)\,q = \frac{1}{288}\Big\{6\,q^2+54\,q+107$$
$$+(18\,q+81)(1,\ -1)\ \text{per}\ 2_q$$
$$+32\quad(2,\ -1,\ -1)\ \text{per}\ 3_q$$
$$+36\ \ (1,\ -1,\ -1,\ 1)\ \text{per}\ 4_q\Big\}$$

$$P(2,\ 3,\ 4,\ 5)\,q = \frac{1}{1440}\Big\{2\,q^3+42\,q^2+267\,q+497$$
$$+(45\,q+315)(1,\ -1)\ \text{per}\ 2_q$$
$$+160\quad(1,\ -1,\ 0)\ \text{per}\ 3_q$$
$$+180\ \ (1,\ 0,\ -1,\ 0)\ \text{per}\ 4_q$$
$$+288\ (1,\ -1,\ 0,\ 0,\ 0)\ \text{per}\ 5_q\Big\}$$

$$P(2,\ 3,\ 4,\ 5,\ 6)\,q = \frac{1}{172800}\Big\{10\,q^4+400\,q^3+5550\,q^2+31000\,q+56877$$
$$+(450\,q^2+9000\,q+39075)(1,\ -1)\ \text{per}\ 2_q$$
$$+3200\,q\quad(1,\ -1,\ 0)\ \text{per}\ 3_q$$
$$+1600\quad(21,\ -19,\ -2)\ \text{per}\ 3_q$$
$$+10800\quad(1,\ 0,\ -1,\ 0)\ \text{per}\ 4_q$$
$$+6912\quad(4,\ -1,\ -1,\ -1,\ -1)\ \text{per}\ 5_q$$
$$+4800\quad(1,\ -1,\ -2,\ -1,\ 1,\ 2)\ \text{per}\ 6_q\Big\}$$

$$P(1,\ 2,\ 3,\ 5)\,q = \frac{1}{720}\Big\{4\,q^3+66\,q^2+324\,q+451$$
$$+45\quad(1,\ -1)\ \text{per}\ 2_q$$
$$+80\quad(1,\ -1,\ 0)\ \text{per}\ 3_q$$
$$+144\,(1,\ 0,\ 0,\ 0,\ -1)\ \text{per}\ 5_q\Big\}$$

$$P(1,\ 2,\ 2,\ 3,\ 4)\,q = \frac{1}{6912}\Big\{6\,q^4+144\,q^3+1194\,q^2+3960\,q+4267$$
$$+(54\,q^2+648\,q+1701)(1,\ -1)\ \text{per}\ 2_q$$
$$+256\quad(2,\ -1,\ -1)\ \text{per}\ 3_q$$
$$+432\quad(1,\ 0,\ -1,\ 0)\ \text{per}\ 4_q\Big\}$$

$$P(8)\,q = \frac{1}{8}\Big\{1$$
$$+1\quad(1,\ -1)\ \text{per}\ 2_q$$
$$+2\quad(1,\ 0,\ -1,\ 0)\ \text{per}\ 4_q$$
$$+8\,(1,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0)\ \text{per}\ 8_q\Big\}$$

$$
\begin{aligned}
P(7,\ 8)q \quad = \frac{1}{112}\Big\{ & 2q+43 \\
& +\ 7 \qquad\qquad\qquad (1,\ -1) \text{ per } 2_q \\
& +14 \qquad\qquad (1,\ -1,\ -1,\ 1) \text{ per } 4_q \\
& +16\ (3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3) \text{ per } 7_q \\
& +56\ (0,\ -1,\ -1,\ 0,\ 0,\ 1,\ 1,\ 0) \text{ per } 8_q \Big\},
\end{aligned}
$$

which are, I think, worth preserving.

Received April 14,—Read May 3 and 10, 1855.

I proceed to discuss the following problem: "To find in how many ways a number q can be made up as a sum of m terms with the elements $0,\ 1,\ 2, \ldots k$, each element being repeatable an indefinite number of times." The required number of partitions is represented by

$$P(0,\ 1,\ 2, \ldots k)^m q,$$

and the number of partitions of q less the number of partitions of $q-1$ is represented by

$$P'(0,\ 1,\ 2, \ldots k)^m q.$$

We have, as is well known,

$$P(0,\ 1,\ 2, \ldots k)^m q = \text{coefficient } x^q z^m \text{ in } \frac{1}{(1-z)(1-xz)\ldots(1-x^k z)},$$

where the expansion is to be effected in ascending powers of z. Now

$$\frac{1}{(1-z)(1-xz)\ldots(1-x^k z)} = 1 + \frac{1-x^{k+1}}{1-x} z + \frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)\ (1-x^2)} z^2 + \&c.,$$

the general term being

$$\frac{(1-x^{k+1})(1-x^{k+2})\ldots(1-x^{k+m})}{(1-x)(1-x^2)\ldots\ (1-x^m)} z^m,$$

or, what is the same thing,

$$\frac{(1-x^{m+1})(1-x^{m+2})\ldots(1-x^{m+k})}{(1-x)(1-x^2)\ldots\ (1-x^k)} z^m,$$

and consequently

$$P(0,\ 1,\ 2, \ldots k)^m q = \text{coefficient } x^q \text{ in } \frac{(1-x^{m+1})(1-x^{m+2})\ldots(1-x^{m+k})}{(1-x)(1-x^2)\ldots\ (1-x^k)};$$

to transform this expression I make use of the equation

$$(1+xz)(1+x^2 z)\ldots(1+z^k z) = 1 + \frac{x(1-x^k)}{1-x} z + \frac{x^3(1-x^k)(1-x^{k-1})}{(1-x)\ (1-x^2)} z^2 + \&c.,$$

where the general term is

$$x^{\frac{1}{2}s(s+1)}\frac{(1-x^k)(1-x^{k-1})\dots(1-x^{k-s+1})}{(1-x)\ (1-x^2)\ \dots(1-x^s)}z^s,$$

and the series is a finite one, the last term being that corresponding to $s=k$, viz. $x^{\frac{1}{2}k(k+1)}z^k$. Writing $-x^m$ for z, and substituting the resulting value of

$$(1-x^{m+1})(1-x^{m+2})\dots(1-x^{m+k})$$

in the formula for $P(0,\ 1,\ 2,\dots k)^m q$, we have

$$P(0,\ 1,\ 2,\dots k)^m q=\Sigma_s\left\{(-)^s \text{ coefficient } x^q \text{ in } \frac{x^{sm+\frac{1}{2}s(s+1)}}{(1-x)(1-x^2)\dots(1-x^s)(1-x)(1-x^2)\dots(1-x^{k+s})}\right\},$$

where the summation extends from $s=0$ to $s=k$; but if for any value of s between these limits $sm+\frac{1}{2}s(s+1)$ becomes greater than q, then it is clear that the summation need only be extended from $s=0$ to the last preceding value of s, or what is the same thing, from $s=0$ to the greatest value of s for which $q-sm-\frac{1}{2}s(s+1)$ is positive or zero.

It is obvious, that if $q>km$, then

$$P(0,\ 1,\ 2\dots k)^m q=0;$$

and moreover, that if $\theta \not> \frac{1}{2}km$, then

$$P(0,\ 1,\ 2,\dots k)^m\theta=P(0,\ 1,\ 2,\dots k)^m.\,km-\theta,$$

so that we may always suppose $q\not>\frac{1}{2}km$. I write therefore $q=\frac{1}{2}(km-\alpha)$ where α is zero or a positive integer not greater than km, and is even or odd according as km is even or odd. Substituting this value of q and making a slight change in the form of the result, we have

$$P(0,\ 1,\ 2\dots k)^m\tfrac{1}{2}(km-\alpha)=$$

$$\Sigma_s\left\{(-)^s \text{ coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+\frac{1}{2}s(s+1)}}{(1-x)(1-x^2)\dots(1-x^s)(1-x)(1-x^2)(1-x^{k-s})}\right\},$$

where the summation extends from $s=0$ to the greatest value of s for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero. But we may, if we please, consider the summation as extending, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when k is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$; the terms corresponding to values of s greater than the greatest value for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero being of course equal to zero. It may be noticed, that the fraction will be a proper one if $\alpha<(k-s)(k-s+1)$; or substituting for s its greatest value, the fraction will be a proper one for all values of s, if, when k is even, $\alpha<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha<\frac{1}{4}(k+1)(k+3)$.

We have in a similar manner,

$$P'(0,\ 1,\ 2\dots k)^m q=\text{coefficient } x^q z^m \text{ in } \frac{1-x}{(1-z)(1-xz)\dots(1-x^kz)},$$

which leads to

$$P'(0,\ 1,\ 2\ldots k)^m \tfrac{1}{2}(km-\alpha) =$$

$$\Sigma_s \left\{(-)^s \text{ coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+s(s+1)}}{(1-x^2)\ldots(1-x^s)(1-x)(1-x^2)\ldots(1-x^{k-s})}\right\},$$

where the summation extends, as in the former case, from $s=0$ to the greatest value of s, for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero, or, if we please, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when s is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$. The condition, in order that the fraction may be a proper one for all values of s, is, when k is even, $\alpha+1<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha+1<\frac{1}{4}(k+1)(k+3)$.

To transform the preceding expressions, I write when k is odd x^2 instead of x, and I put for shortness θ instead of $\frac{1}{2}k-s$ or $2(\frac{1}{2}k-s)$, and γ instead of $\frac{1}{2}\alpha+\frac{1}{2}s(s+1)$ or $\alpha+s(s+1)$; we have to consider an expression of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^\gamma}{Fx},$$

where Fx is the product of factors of the form $1-x^a$. Suppose that a' is the least common multiple of a and θ, then $(1-x^{a'})\div(1-x^a)$ is an integral function of x, equal χx suppose, and $1\div(1-x^a)=\chi x\div(1-x^{a'})$. Making this change in all the factors of Fx which require it (i.e. in all the factors except those in which a is a multiple of θ), the general term becomes

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^\gamma Hx}{Gx},$$

where Gx is a product of factors of the form $1-x^{a'}$, in which a' is a multiple of θ, i.e. Gx is a rational and integral function of x^θ. But in the numerator $x^\gamma Hx$ we may reject, as not contributing to the formation of the coefficient of $x^{\theta m}$, all the terms in which the indices are not multiples of θ; the numerator is thus reduced to a rational and integral function of x^θ, and the general term is therefore of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{\lambda(x^\theta)}{\kappa(x^\theta)},$$

or what is the same thing, of the form

$$\text{coefficient } x^m \text{ in } \frac{\lambda x}{\kappa x},$$

where κx is the product of factors of the form $1-x^a$, and λx is a rational and integral function of x. The particular value of the fraction depends on the value of s; and uniting the different terms, we have an expression

$$\text{coefficient } x^m \text{ in } S_s\ (-)^s \frac{\lambda x}{\kappa x},$$

which is equivalent to

$$\text{coefficient } x^m \text{ in } \frac{\phi x}{fx},$$

where fx is a product of factors of the form $1-x^a$, and ϕx is a rational and integral function of x. And it is clear that the fraction will be a proper one when each of the fractions in the original expression is a proper fraction, i.e. in the case of $P(0, 1, 2 \ldots k)^m \frac{1}{2}(km-\alpha)$, when for k even, $\alpha < \frac{1}{4}k(k+2)$, and for k odd, $\alpha < \frac{1}{4}(k+1)(k+3)$; and in the case of $P'(0, 1, 2 \ldots k)^m \frac{1}{2}(km-\alpha)$, when for k even, $\alpha+1 < \frac{1}{4}k(k+2)$, and for k odd, $\alpha+1 < \frac{1}{4}(k+1)(k+3)$.

We see, therefore, that

$$P(0, 1, 2 \ldots k)^m \tfrac{1}{2}(km-\alpha),$$

and

$$P'(0, 1, 2 \ldots k)^m \tfrac{1}{2}(km-\alpha),$$

are each of them of the form

$$\text{coefficient } x^m \text{ in } \frac{\phi x}{fx},$$

where fx is the product of factors of the form $1-x^a$, and up to certain limiting values of α the fraction is a proper fraction. When the fraction $\frac{\phi x}{fx}$ is known, we may therefore obtain by the method employed in the former part of this Memoir, analytical expressions (involving prime circulators) for the functions P and P'.

As an example, take

$$P(0, 1, 2, 3)^m \tfrac{3}{2}m,$$

which is equal to

$$\text{coefficient } x^{3m} \text{ in } \frac{1}{(1-x^2)(1-x^4)(1-x^6)}$$

$$-\text{coefficient } x^m \text{ in } \frac{1}{(1-x^2)(1-x^2)(1-x^4)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^6)(1-x^{12})}{(1-x^2)(1-x^4)},$$

which is equal to

$$1+x^2+2x^4+x^6+2x^8+x^{10}+x^{12}.$$

Hence, rejecting in the numerator the terms the indices of which are not divisible by 3, the first term becomes

$$\text{coefficient } x^{3m} \text{ in } \frac{1+x^6+x^{12}}{(1-x^2)(1-x^{12})(1-x^6)},$$

or what is the same thing, the first term is

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+x^4}{(1-x^2)^2(1-x^4)};$$

and, the second term being

$$-\text{coefficient } x^m \text{ in } \frac{x^2}{(1-x^2)^2(1-x^4)},$$

we have $$P(0,\ 1,\ 2,\ 3)^m\ \tfrac{3}{2}m = \text{coefficient } x^m \text{ in } \frac{1+x^4}{(1-x^2)^2(1-x^4)};$$

and similarly it may be shown, that

$$P(0,\ 1,\ 2,\ 3)^m\ \tfrac{1}{2}(3m-1) = \text{coefficient } x^m \text{ in } \frac{x+x^3}{(1-x^2)^2(1-x^4)}.$$

As another example, take

$$P'(0,\ 1,\ 2,\ 3,\ 4,\ 5)\ \tfrac{5}{2}m,$$

which is equal to

$$\text{coefficient } x^{5m} \text{ in } \frac{1}{(1-x^4)(1-x^6)(1-x^5)(1-x^{10})}$$

$$-\text{coefficient } x^{3m} \text{ in } \frac{x^2}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}$$

$$+\text{coefficient } x^m \text{ in } \frac{x^6}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^{20})(1-x^{30})(1-x^{40})}{(1-x^4)(1-x^6)(1-x^8)},$$

which is a function of x^2 of the order 36, the coefficients of which are

1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 4, 4, 6, 4, 6, 5, 7, 5, 7, 5, 7, 5, 6, 4, 6, 4, 4, 3, 4, 2, 3, 1, 2, 1, 1, 0, 1,

and the first part becomes therefore

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+4x^4+5x^6+7x^8+4x^{10}+3x^{12}}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}.$$

The multiplier for the second fraction is

$$\frac{(1-x^6)(1-x^{12})(1-x^{24})}{(1-x^2)(1-x^4)(1-x^8)},$$

which is a function of x^2 of the order 14, the coefficients of which are

1, 1, 2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 2, 1, 1;

and the second term becomes

$$-\text{coefficient } x^m \text{ in } \frac{2x^2+2x^4+3x^6+x^8+x^{10}}{(1-x^2)^2(1-x^4)(1-x^8)};$$

and the third term is

$$\text{coefficient } x^m \text{ in } \frac{x^6}{(1-x^2)(1-x^4)^2(1-x^6)}.$$

Now the fractions may be reduced to a common denominator

$$(1-x^2)(1-x^4)(1-x^6)(1-x^8)$$

by multiplying the terms of the second fraction by $\frac{1-x^6}{1-x^2}(=1+x^2+x^4)$, and the terms of the third fraction by $\frac{1-x^8}{1-x^4}(=1+x^4)$; performing the operations and adding, the numerator and denominator of the resulting fraction will each of them contain the factor $1-x^2$; and casting this out, we find

$$P(0,\ 1,\ 2,\ 3,\ 4,\ 5)^m\,\tfrac{5}{2}m = \text{coefficient } x^m \text{ in } \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)}.$$

I have calculated by this method several other particular cases, which are given in my "Second Memoir upon Quantics", [141], the present researches were in fact made for the sake of their application to that theory.

Received April 20,—Read May 3 and 10, 1855.

Since the preceding portions of the present Memoir were written, Mr Sylvester has communicated to me a remarkable theorem which has led me to the following additional investigations[1].

Let $\frac{\phi x}{fx}$ be a rational fraction, and let $(x-x_1)^k$ be a factor of the denominator fx, then if

$$\left\{\frac{\phi x}{fx}\right\}_{x_1}$$

denote the portion which is made up of the simple fractions having powers of $x-x_1$ for their denominators, we have by a known theorem

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{z} \text{ in } \frac{1}{x-x_1-z}\,\frac{\phi(x_1+z)}{f(x_1+z)}.$$

Now by a theorem of Jacobi's and Cauchy's,

$$\text{coefficient } \frac{1}{z} \text{ in } Fz = \text{coefficient } \frac{1}{t} \text{ in } F(\psi t)\,\psi' t;$$

whence, writing $x_1+z=x_1e^{-t}$, we have

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{t} \text{ in } \frac{x_1}{x_1-xe^t}\,\frac{\phi(x_1e^{-t})}{f(x_1e^{-t})}.$$

[1] Mr Sylvester's researches are published in the *Quarterly Mathematical Journal*, July 1855, [vol. I. pp. 141—152], and he has there given the general formula as well for the circulating as the non-circulating part of the expression for the number of partitions.—Added 23rd February, 1856.—A. C.

Now putting for a moment $x = x_1 e^\theta$, we have

$$\frac{1}{x_1 - xe^t} = \frac{1}{x_1(1 - e^{\theta+t})} = \frac{1}{x_1(1 - e^\theta)} + \partial_\theta \frac{1}{x_1(1 - e^\theta)} + \dots,$$

and $\partial_\theta = x\partial_x$, whence

$$\frac{1}{x_1 - xe^t} = \frac{1}{x_1 - x} + \frac{t}{1} x\partial_x \frac{1}{x_1 - x} + \frac{t^2}{1.2} (x\partial_x)^2 \frac{1}{x_1 - x} + \dots,$$

the general term of which is

$$\frac{t^{s-1}}{\Pi(s-1)} (x\partial_x)^{s-1} \frac{1}{x_1 - x}.$$

Hence representing the general term of

$$\frac{x_1 \phi(x_1 e^{-t})}{f(x_1 e^{-t})}$$

by $\chi x_1 t^{-s}$, so that

$$\chi x_1 = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1} \frac{x_1 \phi(x_1 e^{-t})}{f(x_1 e^{-t})},$$

we find, writing down only the general term,

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \dots + \frac{1}{\Pi(s-1)} (x\partial_x)^{s-1} \frac{\chi x_1}{x_1 - x} + \dots,$$

where the value of χx_1 depends upon that of s, and where s extends from $s = 1$ to $s = k$.

Suppose now that the denominator is made up of factors (the same or different) of the form $1 - x^m$. And let a be any divisor of one or more of the indices m, and let k be the number of the indices of which a is a divisor. The denominator contains the divisor $[1 - x^a]^k$, and consequently if ρ be any root of the equation $[1 - x^a] = 0$, the denominator contains the factor $(\rho - x)^k$. Hence writing ρ for x_1 and taking the sum with respect to all the roots of the equation $[1 - x^a] = 0$, we find

$$\left\{\frac{\phi x}{fx}\right\}_{[1-x^a]} = \dots + \frac{1}{\Pi(s-1)} (x\partial_x)^{s-1} S \frac{\chi\rho}{\rho - x} + \dots$$

$$= \dots + \frac{1}{\Pi(s-1)} (x\partial_x)^{s-1} \frac{\theta x}{[1 - x^a]} + \dots,$$

where

$$\chi\rho = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1} \frac{\rho\phi(\rho e^{-t})}{f(\rho e^{-t})},$$

and as before s extends from $s = 1$ to $s = k$. We have thus the actual value of the function θx made use of in the memoir.

A preceding formula gives

$$\left\{\frac{\phi x}{fx}\right\}_1 = \text{coefficient } \frac{1}{t} \text{ in } \frac{1}{1 - xe^t} \frac{\phi(e^{-t})}{f(e^{-t})},$$

which is a very simple expression for the non-circulating part of the fraction $\frac{\phi x}{fx}$.

This is, in fact, Mr Sylvester's theorem above referred to.

141.

A SECOND MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVI. *for the year*, 1856, pp. 101—126. Received April 14,—Read May 24, 1855.]

THE present memoir is intended as a continuation of my Introductory Memoir upon Quantics, t. CXLIV. (1854), p. 245, and must be read in connexion with it; the paragraphs of the two Memoirs are numbered continuously. The special subject of the present memoir is the theorem referred to in the Postscript to the Introductory Memoir, and the various developments arising thereout in relation to the number and form of the covariants of a binary quantic.

25. I have already spoken of asyzygetic covariants and invariants, and I shall have occasion to speak of irreducible covariants and invariants. Considering in general a function u determined like a covariant or invariant by means of a system of partial differential equations, it will be convenient to explain what is meant by an asyzygetic integral and by an irreducible integral. Attending for greater simplicity only to a single set $(a, b, c, \ldots)$, which in the case of the covariants or invariants of a single function will be as before the coefficients or elements of the function, it is assumed that the system admits of integrals of the form $u = P$, $u = Q$, &c., or as we may express it, of integrals P, Q, &c., where P, Q, &c. are rational and integral homogeneous functions of the set $(a, b, c, \ldots)$, and moreover that the system is such that P, Q, &c. being integrals, $\phi(P, Q, \ldots)$ is also an integral. Then considering only the integrals which are rational and integral homogeneous functions of the set $(a, b, c, \ldots)$, integrals $P, Q, R, \ldots$ not connected by any linear equation or syzygy (such as $\lambda P + \mu Q + \nu R \ldots 0$), [1] are said to be asyzygetic; but in speaking of the asyzygetic integrals of a particular degree, it is implied that the integrals are a system such that every other integral of

[1] It is hardly necessary to remark, that the multipliers $\lambda, \mu, \nu, \ldots$, and generally any coefficients or quantities not expressly stated to contain the set $(a, b, c, \ldots)$, are considered as independent of the set, or to use a convenient word, are considered as "trivials."

the same degree can be expressed as a linear function (such as $\lambda P + \mu Q + \nu R \ldots$) of these integrals; and any integral P not expressible as a rational and integral homogeneous function of integrals of inferior degrees is said to be an irreducible integral.

26. Suppose now that A_1, A_2, A_3, &c. denote the number of asyzygetic integrals of the degrees 1, 2, 3, &c. respectively, and let α_1, α_2, α_3, &c. be determined by the equations

$$A_1 = \alpha_1,$$
$$A_2 = \tfrac{1}{2}\alpha_1(\alpha_1+1) + \alpha_2,$$
$$A_3 = \tfrac{1}{6}\alpha_1(\alpha_1+1)(\alpha_1+2) + \alpha_1\alpha_2 + \alpha_3,$$
$$A_4 = \tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3) + \tfrac{1}{2}\alpha_1(\alpha_1+1)\alpha_2 + \alpha_1\alpha_3 + \tfrac{1}{2}\alpha_2(\alpha_2+1) + \alpha_4, \text{ \&c.},$$

or what is the same thing, suppose that

$$1 + A_1 x + A_2 x^2 + \text{\&c.} = (1-x)^{-\alpha_1}(1-x^2)^{-\alpha_2}(1-x^3)^{-\alpha_3}\ldots;$$

a little consideration will show that α_r represents the number of irreducible integrals of the degree r less the number of linear relations or syzygies between the composite or non-irreducible integrals of the same degree. In fact the asyzygetic integrals of the degree 1 are necessarily irreducible, i.e. $A_1 = \alpha_1$. Represent for a moment the irreducible integrals of the degree 1 by X, X', &c., then the composite integrals X^2, XX', &c., the number of which is $\frac{1}{2}\alpha_1(\alpha_1+1)$, must be included among the asyzygetic integrals of the degree 2; and if the composite integrals in question were asyzygetic, there would remain $A_2 - \frac{1}{2}\alpha_1(\alpha_1+1)$ for the number of irreducible integrals of the degree 2; but if there exist syzygies between the composite integrals in question, the number to be subtracted from A_2 will be $\frac{1}{2}\alpha_1(\alpha_1+1)$ less the number of these syzygies, and we shall have $A_2 - \frac{1}{2}\alpha_1(\alpha_1+1)$, i.e. α_2 equal to the number of the irreducible integrals of the degree 2 less the number of syzygies between the composite integrals of the same degree. Again, suppose that α_2 is negative $= -\beta_2$, we may for simplicity suppose that there are no irreducible integrals of the degree 2, but that the composite integrals of this degree, X^2, XX', &c., are connected by β_2 syzygies, such as $\lambda X^2 + \mu XX' + \text{\&c.} = 0$, $\lambda_1 X^2 + \mu_1 XX' + \text{\&c.} = 0$. The asyzygetic integrals of the degree 4 include X^4, X^3X', &c., the number of which is $\frac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)$; but these composite integrals are not asyzygetic, they are connected by syzygies which are augmentatives of the β_2 syzygies of the second degree, viz. by syzygies such as

$$(\lambda X^2 + \mu XX' \ldots) X^2 = 0, \quad (\lambda X^2 + \mu XX' \ldots) XX' = 0, \text{ \&c. } (\lambda_1 X^2 + \mu_1 XX' \ldots) X^2 = 0,$$
$$(\lambda_1 X^2 + \mu_1 XX' \ldots) XX' = 0, \text{ \&c.},$$

the number of which is $\frac{1}{2}\alpha_1(\alpha_1+1)\beta_2$. And these syzygies are themselves not asyzygetic, they are connected by secondary syzygies such as

$$\lambda_1(\lambda X^2 + \mu XX' \ldots) X^2 + \mu_1(\lambda X^2 + \mu XX' \ldots) XX' + \text{\&c.}$$
$$- \lambda(\lambda_1 X^2 + \mu_1 XX' \ldots) X^2 - \mu(\lambda_1 X^2 + \mu_1 XX' \ldots) XX' - \text{\&c.} = 0, \text{ \&c. \&c.},$$

the number of which is $\frac{1}{2}\beta_2(\beta_2-1)$. The real number of syzygies between the composite integrals X^4, X^2X', &c. (i.e. of the syzygies arising out of the β_2 syzygies between X^2, XX', &c.), is therefore $\frac{1}{2}\alpha_1(\alpha_1+1)\beta_2-\frac{1}{2}\beta_2(\beta_2-1)$, and the number of integrals of the degree 4, arising out of the integrals and syzygies of the degrees 1 and 2 respectively, is therefore

$$\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)-\tfrac{1}{2}\alpha_1(\alpha_1+1)\beta_2+\tfrac{1}{2}\beta_2(\beta_2-1);$$

or writing $-\alpha_2$ instead of β_2, the number in question is

$$\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)+\tfrac{1}{2}\alpha_1(\alpha_1+1)\alpha_2+\tfrac{1}{2}\alpha_2(\alpha_2+1).$$

The integrals of the degrees 1 and 3 give rise to $\alpha_1\alpha_3$ integrals of the degree 4; and if all the composite integrals obtained as above were asyzygetic, we should have

$$A_4-\tfrac{1}{24}\alpha_1(\alpha_1+1)(\alpha_1+2)(\alpha_1+3)-\tfrac{1}{2}\alpha_1(\alpha_1+1)\alpha_2-\tfrac{1}{2}\alpha_2(\alpha_2+1)-\alpha_1\alpha_3,$$

i.e. α_4 as the number of irreducible integrals of the degree 4; but if there exist any further syzygies between the composite integrals, then α_4 will be the number of the irreducible integrals of the degree 4 less the number of such further syzygies, and the like reasoning is in all cases applicable.

27. It may be remarked, that for any given partial differential equation, or system of such equations, there will be always a finite number ν such that given ν independent integrals every other integral is a function (in general an irrational function only expressible as the root of an equation) of the ν independent integrals; and if to these integrals we join a single other integral not a rational function of the ν integrals, it is easy to see that every other integral will be a rational function of the $\nu+1$ integrals; but every such other integral will not in general be a rational and integral function of the $\nu+1$ integrals; and [*incorrect*] there is not in general any finite number whatever of integrals, such that every other integral is a rational and integral function of these integrals, i.e. the number of irreducible integrals is in general infinite; and it would seem that this is in fact the case in the theory of covariants.

28. In the case of the covariants, or the invariants of a binary quantic, A_n is given (this will appear in the sequel) as the coefficient of x^n in the development, in ascending powers of x, of a rational fraction $\dfrac{\phi x}{fx}$, where fx is of the form

$$(1-x)^{\beta_1}(1-x^2)^{\beta_2}\ldots(1-x^k)^{\beta_k},$$

and the degree of ϕx is less than that of fx. We have therefore

$$1+A_1x+A_2x^2+\ldots=\frac{\phi x}{fx},$$

and consequently

$$\phi x=(1-x)^{\beta_1-\alpha_1}(1-x^2)^{\beta_2-\alpha_2}\ldots(1-x^k)^{\beta_k-\alpha_k}(1-x^{k+1})^{-\alpha_{k+1}}\ldots.$$

Now every rational factor of a binomial $1-x^m$ is the irreducible factor of $1-x^{m'}$, where m' is equal to or a submultiple of m. Hence in order that the series $\alpha_1, \alpha_2, \alpha_3, \ldots$ may terminate, ϕx must be made up of factors each of which is the irreducible factor of a binomial $1-x^m$, or if ϕx be itself irreducible, then ϕx must be the irreducible factor of a binomial $1-x^m$. Conversely, if ϕx be not of the form in question, the series $\alpha_1, \alpha_2, \alpha_3$, &c. will go on *ad infinitum*, and it is easy to see that there is no point in the series such that the terms beyond that point are all of them negative, i.e. there will be irreducible covariants or invariants of indefinitely high degrees; and the number of covariants or invariants will be infinite. The number of invariants is first infinite in the case of a quantic of the seventh order, or septimic; the number of covariants is first infinite in the case of a quantic of the fifth order, or quintic. [As is now well known, these conclusions are incorrect, the number of irreducible covariants or invariants is in every case finite.]

29. Resuming the theory of binary quantics, I consider the quantic

$$(a, b, \ldots b', a' \between x, y)^m.$$

Here writing

$$\{y\partial_x\} = a\partial_b + 2b\partial_c \ldots \qquad + mb'\partial_{a'}, = X,$$
$$\{x\partial_y\} = mb\partial_a + (m-1)\, c\partial_b \ldots + a'\partial_{b'}, = Y,$$

any function which is reduced to zero by each of the operations $X - y\partial_x$, $Y - x\partial_y$ is a covariant of the quantic. But a covariant will always be considered as a rational and integral function separately homogeneous in regard to the facients (x, y) and to the coefficients $(a, b, \ldots b', a')$. And the words order and degree will be taken to refer to the facients and to the coefficients respectively.

I commence by proving the theorem enunciated, No. 23. It follows at once from the definition, that the covariant is reduced to zero by the operation

$$\overline{X - y\partial_x} \,.\, \overline{Y - x\partial_y} - \overline{Y - x\partial_y} \,.\, \overline{X - y\partial_x},$$

which is equivalent to

$$X \,.\, Y - Y \,.\, X + y\partial_y - x\partial_x.$$

Now

$$X \,.\, Y = XY + X(Y)$$
$$Y \,.\, X = YX + Y(X),$$

where XY and YX are equivalent operations, and

$$X(Y) = 1ma\partial_a + 2(m-1)\, b\partial_b \ldots + m1b'\partial_{b'},$$
$$Y(X) = \qquad\qquad m1b\partial_b \ldots + 2(m-1)\, b'\partial_{b'} + 1ma'\partial_{a'},$$

whence

$$X(Y) - Y(X) = ma\partial_a + (m-2)\, b\partial_b \ldots - (m-2)\, b'\partial_b - ma'\partial_{a'}, = k \text{ suppose},$$

and the covariant is therefore reduced to zero by the operation

$$k + y\partial_y - x\partial_x.$$

Now as regards a term $a^\alpha b^\beta \ldots b'^{\beta'} a'^{\alpha'} \,.\, x^i y^j$, we have

$$k = m\alpha + (m-2)\,\beta \ldots, \; -(m-2)\,\beta' - m\alpha'$$
$$y\partial_y - x\partial_x = j - i;$$

and we see at once that for each term of the covariant we must have

$$m\alpha + (m-2)\beta \ldots - (m-2)\beta' - m\alpha' + j - i = 0,$$

i.e. if (x, y) are considered as being of the weights $\frac{1}{2}, -\frac{1}{2}$ respectively, and $(a, b, \ldots b', a')$ as being of the weights $-\frac{1}{2}m, -\frac{1}{2}m+1, \ldots \frac{1}{2}m-1, \frac{1}{2}m$ respectively, then the weight of each term of the covariant is zero.

But if (x, y) are considered as being of the weights 1, 0 respectively, and $(a, b, \ldots b', a')$ as being of the weights $0, 1, \ldots m-1, m$ respectively, then writing the equation under the form

$$m(\alpha + \beta \ldots + \beta' + \alpha') + j + i - 2(\beta + \ldots + \overline{m-1}\beta' + m\alpha' + i) = 0,$$

and supposing that the covariant is of the order μ and of the degree θ, each term of the covariant will be of the weight $\frac{1}{2}(m\theta + \mu)$.

I shall in the sequel consider the weight as reckoned in the last-mentioned manner. It is convenient to remark, that as regards any function of the coefficients of the degree θ and of the weight q, we have

$$X.Y - Y.X = m\theta - 2q.$$

30. Consider now a covariant

$$(A, B, \ldots B', A' \between x, y)^{\mu}$$

of the order μ and of the degree θ; the covariant is reduced to zero by each of the operations $X - y\partial_x$, $Y - x\partial_y$, and we are thus led to the systems of equations

$$\begin{aligned} XA &= 0, \\ XB &= \mu A, \\ XC &= (\mu - 1)B, \\ &\vdots \\ XB' &= 2C', \\ XA' &= B'; \end{aligned}$$

and

$$\begin{aligned} YA &= B, \\ YB &= 2C, \\ &\vdots \\ YC' &= (\mu - 1)B', \\ YB' &= \mu A', \\ YA' &= 0. \end{aligned}$$

Conversely if these equations are satisfied the function will be a covariant.

I assume that A is a function of the degree θ and of the weight $\frac{1}{2}(m\theta - \mu)$, satisfying the condition

$$XA = 0;$$

and I represent by YA, Y^2A, Y^3A, &c. the results obtained by successive operations with Y upon the function A. The function Y^sA will be of the degree θ and of the weight $\frac{1}{2}(m\theta-\mu)+s$. And it is clear that in the series of terms YA, Y^2A, Y^3A, &c., we must at last come to a term which is equal to zero. In fact, since m is the greatest weight of any coefficient, the weight of Y^s is at most equal to $m\theta$, and therefore if $\frac{1}{2}(m\theta-\mu)+s>m\theta$, or $s>\frac{1}{2}(m\theta+\mu)$, we must have $Y^sA=0$.

Now writing for greater simplicity XY instead of $X.Y$, and so in similar cases, we have, as regards Y^sA,

$$XY-YX=\mu-2s.$$

Hence

$$(XY-YX)A=\mu A,$$

and consequently

$$XYA=YXA+\mu A=\mu A.$$

Similarly

$$(XY-YX)YA=(\mu-2)YA,$$

and therefore

$$XY^2A=YXYA+(\mu-2)YA$$
$$=\mu YA+(\mu-2)YA=2(\mu-1)YA.$$

And again,

$$(XY-YX)Y^2A=(\mu-4)Y^2A,$$

and therefore

$$XY^3A=YXY^2A+(\mu-4)Y^2A$$
$$=2(\mu-1)Y^2A+(\mu-4)Y^2A=3(\mu-2)Y^2A,$$

or generally

$$XY^sA=s(\mu-s+1)Y^sA.$$

Hence putting $s=\mu+1$, $\mu+2$, &c., we have

$$XY^{\mu+1}A=0,$$
$$XY^{\mu+2}A=-(\mu+2)1.Y^{\mu+1}A,$$
$$XY^{\mu+3}A=-(\mu+3)2.Y^{\mu+2}A,$$
&c.,

equations which show that

$$Y^{\mu+1}A=0;$$

for unless this be so, i.e. if $Y^{\mu+1}A\neq 0$, then from the second equation $XY^{\mu+2}A\neq 0$, and therefore $Y^{\mu+2}A\neq 0$, from the third equation $XY^{\mu+3}\neq 0$, and therefore $X^{\mu+3}A\neq 0$, and so on *ad infinitum*, i.e. we must have $Y^{\mu+1}A=0$.

31. The suppositions which have been made as to the function A, give therefore the equations

$$\begin{aligned} XA &= 0, \\ XYA &= \mu A, \\ XY^2A &= 2(\mu-1)YA, \\ &\vdots \\ XY^\mu A &= \mu Y^{\mu+1}A, \\ Y^{\mu+1}A &= 0; \end{aligned}$$

and if we now assume

$$B = YA, \quad C = \tfrac{1}{2}YB, \ldots A^{\backprime} = \frac{1}{\mu}YB^{\backprime},$$

the system becomes

$$\begin{aligned} XA &= 0, \\ XB &= \mu A, \\ XC &= (\mu-1)B, \\ &\vdots \\ XA^{\backprime} &= B^{\backprime}, \\ YA' &= 0; \end{aligned}$$

so that the entire system of equations which express that $(A, B \ldots B^{\backprime}, A^{\backprime}\between x, y)^\mu$ is a covariant is satisfied; hence

THEOREM. Given a quantic $(a, b, \ldots b^{\backprime}, a^{\backprime}\between x, y)^m$; if A be a function of the coefficients of the degree θ and of the weight $\frac{1}{2}(m\theta-\mu)$ satisfying the condition $XA = 0$, and if $B, C, \ldots B^{\backprime}, A^{\backprime}$ are determined by the equations

$$B = YA, \quad C = \tfrac{1}{2}YB, \ldots A^{\backprime} = \frac{1}{\mu}YB^{\backprime},$$

then will

$$(A, B, \ldots B^{\backprime}, A^{\backprime}\between x, y)^\mu$$

be a covariant.

In particular, a function A of the degree θ and of the weight $\frac{1}{2}m\theta$, satisfying the condition $XA = 0$, will (also satisfy the equation $YA = 0$ and will) be an invariant.

32. I take now for A the most general function of the coefficients, of the degree θ and of the weight $\frac{1}{2}(m\theta-\mu)$; then XA is a function of the degree θ and of the weight $\frac{1}{2}(m\theta-\mu)-1$, and the arbitrary coefficients in the function A are to be determined so that $XA = 0$. The number of arbitrary coefficients is equal to the number of terms in A, and the number of the equations to be satisfied is equal to the number of terms in XA; hence the number of the arbitrary coefficients which remains indeterminate is equal to the number of terms in A less the number of terms in XA; and since the covariant is completely determined when the leading coefficient is known,

the difference in question is equal to the number of the asyzygetic covariants, i.e. the number of the asyzygetic covariants of the order μ and the degree θ is equal to the number of terms of the degree θ and weight $\frac{1}{2}(m\theta-\mu)$, less the number of terms of the degree θ and weight $\frac{1}{2}(m\theta-\mu)-1$.

33. I shall now give some instances of the calculation of covariants by the method just explained. It is very convenient for this purpose to commence by forming the literal parts by Arbogast's Method of Derivations: we thus form tables such as the following:—

a	b	c

a^2	ab	ac b^2	bc	b^2

a	b	c	d

a^2	ab	ac b^2	ad bc	bd c^2	cd	d^2

a^3	a^2b	a^2c ab^2	a^2d abc b^3	abd ac^2 b^2c	acd b^2d bc^2	ad^2 bcd c^3	bd^2 c^2d	cd^2	d^3

a^4	a^3b	a^3c a^2b^2	a^3d a^2bc ab^3	a^2bd a^2c^2 ab^2c b^4	a^2cd ab^2d abc^2 b^3c	a^2d^2 $abcd$ ac^3 b^3d b^2c^2	abd^2 ac^2d b^2cd bc^3	acd^2 b^3d^2 bc^2d c^4	ad^3 bcd^2 c^3d	bd^3 c^2d^2	cd^3	d^4

a	b	c	d	e

a^2	ab	ac b^2	ad bc	ae bd c^2	be cd	bd c^2	cd	d^2

a^3	a^2b	a^2c ab^2	a^2d abc b^3	a^2e abd ac^2 b^2c	abe acd b^2d bc^2	ace ad^2 b^2e bcd c^3	ade bce bd^2 c^2d	ae^2 bde c^2e cd^2	be^2 cde d^3	ce^2 d^2e	de^2	e^3

34. Thus in the case of a cubic $(a, b, c, d \between x, y)^3$, the tables show that there will be a single invariant of the degree 4. Represent this by

$$\begin{aligned} & A\,a^2d^2 \\ & + B\,abcd \\ & + C\,ac^3 \\ & + D\,b^3d \\ & + E\,b^2c^2, \end{aligned}$$

which is to be operated upon with $a\partial_b + 2b\partial_c + 3c\partial_d$. This gives

$+\ B$ $+\ 3D$ $+\ 2E$		$+\ 6A$	a^2cd ab^3d abc^2 b^3c
	$+\ 2B$ $+\ 6C$ $+\ 4E$	$+\ 3B$ $+\ 3D$	

i.e. $B + 6A = 0$, $3D + 2B = 0$, &c.; or putting $A = 1$, we find $B = -6$, $C = 4$, $D = 4$, $E = -3$, and the invariant is

$$\begin{aligned} & a^2d^2 \\ & - 6\,abcd \\ & + 4\,ac^3 \\ & + 4\,b^3d \\ & - 3\,b^2c^2. \end{aligned}$$

Again, there is a covariant of the order 3 and the degree 3. The coefficient of x^3 or leading coefficient is

$$\begin{aligned} & A\,a^2d \\ & + B\,abc \\ & + C\,b^3, \end{aligned}$$

which operated upon with $a\partial_b + 2b\partial_c + 3c\partial_d$, gives

$+\ B$ $+\ 3C$		$+\ 3A$	a^2c ab^2
	$+\ 2B$		

i.e. $B+3A=0$, $3C+2B=0$; or putting $A=1$, we have $B=-3$, $C=2$, and the leading coefficient is

$$\begin{array}{r} a^2d \\ -\,3\,abc \\ +\,2\,b^3. \end{array}$$

The coefficient of x^2y is found by operating upon this with $(3b\partial_a+2c\partial_b+d\partial_c)$, this gives

$+6$		-3	abd
	-6		ac^2
-9	$+12$		b^2c

i.e. the required coefficient of x^2y is

$$\begin{array}{r} 3\,abd \\ -\,6\,ac^2 \\ +\,3\,b^2c\,; \end{array}$$

and by operating upon this with $\frac{1}{2}(3b\partial_a+2c\partial_b+d\partial_c)$, we have for the coefficient of xy^2,

	$+3$	-6	acd
$+\frac{9}{2}$		$+\frac{3}{2}$	b^2d
-9	$+6$		bc^2

i.e. the coefficient of xy^2 is

$$\begin{array}{r} -\,3\,acd \\ +\,6\,b^2d \\ -\,3\,bc^2. \end{array}$$

Finally, operating upon this with $\frac{1}{3}(3b\partial_a+2c\partial_b+d\partial_c)$, we have for the coefficient of y^3,

		-1	ad^2
-3	$+8$	-2	bcd
	-2		c^3

i.e. the coefficient of y^3 is

$$\begin{array}{r} -\quad ad^2 \\ +\,3\,bcd \\ -\,2\,c^3, \end{array}$$

and the covariant is

a^2d+1	$abd+3$	$acd-3$	ad^2-1	
$abc-3$	ac^2-6	b^2d+6	$bcd+3$	$)(x,\ y)^2$
$b^3\ +2$	$b^2c\ +3$	$bc^2\ -3$	$c^3\ -2$	

[I now write the numerical coefficients after instead of before the literal terms.]

I have worked out the example in detail as a specimen of the most convenient method for the actual calculation of more complicated covariants[1].

35. The number of terms of the degree θ and of the weight q is obviously equal to the number of ways in which q can be made up as a sum of θ terms with the elements $(0, 1, 2, \ldots m)$, a number which is equal to the coefficient of $x^q z^\theta$ in the development of

$$\frac{1}{(1-z)(1-xz)(1-x^2z)\ldots(1-x^mz)};$$

and the number of the asyzygetic covariants of any particular degree for the quantic $(*\between x, y)^m$ can therefore be determined by means of this development. In the case of a cubic, for example, the function to be developed is

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)},$$

which is equal to

$$1 + z(1 + x + x^2 + x^3) + z^2(1 + x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + x^6) + \&c.,$$

where the coefficients are given by the following table; on account of the symmetry, the series of coefficients for each power of z is continued only to the middle term or middle of the series.

									1	(0)
							1	1		(1)
						1	1	2	2	(2)
					1	1	2	3	3	(3)
				1	1	2	3	4	4	5 (4)
			1	1	2	3	4	5	6	6 (5)
		1	1	2	3	4	5	7	7	8 8 (6)

[1] Note added Feb. 7, 1856.—The following method for the calculation of an invariant or of the leading coefficient of a covariant, is easily seen to be identical in principle with that given in the text. Write down all the terms of the weight next inferior to that of the invariant or leading coefficient, and operate on each of these separately with the symbol

$$\text{ind. } b \cdot \frac{b}{a} + 2 \text{ ind. } c \cdot \frac{c}{b} + \ldots (m-1) \text{ ind. } b' \cdot \frac{b'}{a},$$

where we are first to multiply by the fraction, rejecting negative powers, and then by the index of the proper letter in the term so obtained. Equating the results to zero, we obtain equations between the terms of the invariant or leading coefficient, and replacing in these equations each term by its numerical coefficient in the

and from this, by subtracting from each coefficient the coefficient which immediately precedes it, we form the table:

									1	(0)
							1	0		(1)
						1	0	1	0	(2)
					1	0	1	1	0	(3)
				1	0	1	1	1	0	1	(4)
			1	0	1	1	1	1	1	0	(5)
1	0	1	1	1	1	2	0	1	0	(6)

The successive lines fix the number and character of the covariants of the degrees 0, 1, 2, 3, &c. The line (0), if this were to be interpreted, would show that there is a single covariant of the degree 0; this covariant is of course merely the absolute constant unity, and may be excluded. The line (1) shows that there is a single covariant of the degree 1, viz. a covariant of the order 3; this is the cubic itself, which I represent by U. The line (2) shows that there are two asyzygetic covariants of the degree 2, viz. one of the order 6, this is merely U^2, and one of the order 2, this I represent by H. The line (3) shows that there are three asyzygetic covariants of the degree 3, viz. one of the order 9, this is U^3; one of the order 5, this is UH, and one of the order 3, this I represent by Φ. The line (4) shows that there are five asyzygetic covariants of the degree 4, viz. one of the order 12, this is U^4; one of the order 8, this is U^2H; one of the order 6, this is H^2; and one of the order 0, i.e. an invariant, this I represent by ∇. The line (5) shows that there are six asyzygetic covariants of the degree 5, viz. one of the order 15, this is U^5; one of the order 11, this is U^3H; one of the order 9, this is $U^3\Phi$; one of the order 7, this is UH^2; one of the order 5, this is $H\Phi$; and one of the order 3, this is ∇U. The line (6) shows that there are 8 asyzygetic covariants of the degree 6, viz. one of the order 18, this is U^6; one of the

invariant or leading coefficient, we have the equations of connexion of these numerical coefficients. Thus, for the discriminant of a cubic, the terms of the next inferior weight are a^2cd, ab^2d, abc^2, b^3c, and operating on each of these separately with the symbol

$$\text{ind. } b\,.\,\frac{b}{a}+2 \text{ ind. } c\,.\,\frac{c}{b}+3 \text{ ind. } d\,.\,\frac{d}{c},$$

we find

$abcd$		$+6\,a^2d^2$
$3\,b^3d$	$+2\,abcd$	
$2\,b^3c^2$	$+6\,ac^3$	$+3\,abcd$
	$+4\,b^2c^2$	$+3\,b^3d$

and equating the horizontal lines to zero, and assuming $a^2d^2=1$, we have $a^2d^2=1$, $abcd=-6$, $ac^3=4$, $b^3d=4$, $b^3c^2=-3$, or the value of the discriminant is that given in the text.

order 14, this is U^4H; one of the order 12, this is $U^3\Phi$; one of the order 10, this is U^2H^2; one of the order 8, this is $UH\Phi$; two of the order 6 (i.e. the three covariants H^3, Φ^2 and ∇U^2 are not asyzygetic, but are connected by a single linear equation or syzygy), and one of the order 2, this is ∇H. We are thus led to the irreducible covariants U, H, Φ, ∇ connected by a linear equation or syzygy between H^3, Φ^2 and ∇U^2, and this is in fact the complete system of irreducible covariants; ∇ is therefore the only invariant.

36. The asyzygetic covariants are of the form $U^pH^q\nabla^r$, or else of the form $U^pH^q\nabla^r\Phi$; and since U, H, ∇ are of the degrees 1, 2, 4 respectively, and Φ is of the degree 3, the number of asyzygetic covariants of the degree m of the first form is equal to the coefficient of x^m in $1 \div (1-x)(1-x^2)(1-x^4)$, and the number of the asyzygetic covariants of the degree m of the second form is equal to the coefficient of x^m in $x^3 \div (1-x)(1-x^2)(1-x^4)$. Hence the total number of asyzygetic covariants is equal to the coefficient of x^m in $(1+x^3) \div (1-x)(1-x^2)(1-x^4)$, or what is the same thing, in

$$\frac{1-x^6}{(1-x)(1-x^2)(1-x^3)(1-x^4)};$$

and conversely, if this expression for the number of the asyzygetic covariants of the degree m were established independently, it would follow that the irreducible invariants were four in number, and of the degrees 1, 2, 3, 4 respectively, but connected by an equation of the degree 6. As regards the invariants, every invariant is of the form ∇^p, i.e. the number of asyzygetic invariants of the degree m is equal to the coefficient of x^m in $\frac{1}{1-x^4}$, and conversely, from this expression it would follow that there was a single irreducible invariant of the degree 4.

37. In the case of a quartic, the function to be developed is:

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)},$$

and the coefficients are given by the table.

												1	(0)
										1	1	1	(1)
								1	1	2	2	3	(2)
						1	1	2	3	4	4	5	(3)
				1	1	2	3	5	5	7	7	8	(4)
		1	1	2	3	5	6	8	9	11	11	12	(5)
1	1	2	3	5	6	9	10	13	14	16	16	18	(6)

and subtracting from each coefficient the coefficient immediately preceding it, we have the table:

												1	(0)
									1	0	0		(1)
							1	0	1	0	1		(2)
						1	0	1	1	1	0	1	(3)
				1	0	1	1	2	0	2	0	1	(4)
		1	0	1	1	2	1	2	1	2	0	1	(5)
1	0	1	1	2	1	3	1	3	1	2	0	2	(6)

the examination of which will show that we have for the quartic the following irreducible covariants, viz. the quartic itself U; an invariant of the degree 2, which I represent by I; a covariant of the order 4 and of the degree 2, which I represent by H; an invariant of the degree 3, which I represent by J; and a covariant of the order 6 and the degree 3, which I represent by Φ; but that the irreducible covariants are connected by an equation of the degree 6, viz. there is a linear equation or syzygy between Φ^2, I^3H^3, I^2JH^2U, IJ^2HU^2 and J^3U^3; this is in fact the complete system of the irreducible covariants of the quartic: the only irreducible invariants are the invariants I, J.

38. The asyzygetic covariants are of the form $U^pI^qH^rJ^s$, or else of the form $U^pI^qH^rJ^s\Phi$, and the number of the asyzygetic covariants of the degree m is equal to the coefficient of x^m in $(1+x^3)\div(1-x)(1-x^2)^2(1-x^3)$, or what is the same thing, in

$$\frac{1-x^6}{(1-x)(1-x^2)^2(1-x^3)^2},$$

and the asyzygetic invariants are of the form I^pJ^q, and the number of the asyzygetic invariants of the degree m is equal to the coefficient of x^m in $1\div(1-x^2)(1-x^3)$. Conversely, if these formulæ were established, the preceding results as to the form of the system of the irreducible covariants or of the irreducible invariants, would follow.

39. In the case of a quintic, the function to be developed is

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)};$$

and the coefficients are given by the table:

												1	(0)
								1	1	1			(1)
						1	1	2	2	3	3		(2)
				1	1	2	3	4	5	6	6		(3)
		1	1	2	3	5	6	8	9	11	11	12	(4)
1	1	2	3	5	7	9	11	14	16	18	19	20	(5)
													⋮

and subtracting from each coefficient the one which immediately precedes it, we have the table:

												1	(0)
								1	0	0			(1)
						1	0	1	0	1	0		(2)
				1	0	1	1	1	1	1	0		(3)
		1	0	1	1	2	1	2	1	2	0	1	(4)
1	0	1	1	2	2	2	2	3	2	2	1	1	(5)
													⋮

We thus obtain the following irreducible covariants, viz.:

Of the degree 1; a single covariant of the order 5, this is the quintic itself.

Of the degree 2; two covariants, viz. one of the order 6, and one of the order 2.

Of the degree 3; three covariants, viz. one of the order 9, one of the order 5, and one of the order 3.

Of the degree 4; three covariants, viz. one of the order 6, one of the order 4, and one of the order 0 (an invariant).

Of the degree 5; three covariants, viz. one of the order 7, one of the order 3, and one of the order 1 (a linear covariant).

These covariants are connected by a single syzygy of the degree 5 and of the order 11; in fact, the table shows that there are only two asyzygetic covariants of this degree and order; but we may, with the above-mentioned irreducible covariants of the degrees, 1, 2, 3 and 4, form three covariants of the degree 5 and the order 11; there is therefore a syzygy of this degree and order.

40. I represent the number of ways in which q can be made up as a sum of m terms with the elements $0, 1, 2, \ldots m$, each element being repeatable an indefinite number of times by the notation

$$P(0, 1, 2, \ldots m)^\theta q,$$

and I write for shortness

$$P'(0, 1, 2, \ldots m)^\theta q = P(0, 1, 2 \ldots m)^\theta q - P(0, 1, 2 \ldots m)^\theta (q-1).$$

Then for a quantic of the order m, the number of asyzygetic covariants of the degree θ and of the order μ is

$$P'(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}(m\theta - \mu).$$

In particular, the number of asyzygetic invariants of the degree θ is

$$P'(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta.$$

To find the total number of the asyzygetic covariants of the degree θ, suppose first that $m\theta$ is even; then, giving to μ the successive values $0, 2, 4, \ldots m\theta$, the required number is

$$\begin{aligned}
& P(\tfrac{1}{2}m\theta) && - P(\tfrac{1}{2}m\theta - 1) \\
+ & P(\tfrac{1}{2}m\theta - 1) && - P(\tfrac{1}{2}m\theta - 2) \\
& \vdots \\
+ & P(2) && - P(1) \\
+ & P(1) \\
= & P(\tfrac{1}{2}m\theta),
\end{aligned}$$

i.e. when $m\theta$ is even, the number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta;$$

and similarly, when $m\theta$ is odd, the number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}(m\theta - 1).$$

But the two formulæ may be united into a single formula; for when $m\theta$ is odd $\tfrac{1}{2}m\theta$ is a fraction, and therefore $P(\tfrac{1}{2}m\theta)$ vanishes, and so when $m\theta$ is even $\tfrac{1}{2}(m\theta - 1)$ is a fraction, and $P\tfrac{1}{2}(m\theta - 1)$ vanishes; we have thus the theorem, that for a quantic of the order m:

The number of the asyzygetic covariants of the degree θ is

$$P(0, 1, 2 \ldots m)^\theta \tfrac{1}{2}m\theta + P(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}(m\theta - 1).$$

41. The functions $P(\tfrac{1}{2}m\theta)$, &c. may, by the method explained in my "Researches on the Partition of Numbers," [140], be determined as the coefficients of x^θ in certain functions of x; I have calculated the following particular cases:—

Putting, for shortness,

$$P'(0, 1, 2, \ldots m)^\theta \tfrac{1}{2}m\theta = \text{coefficient } x^\theta \text{ in } \phi m,$$

then $$\phi 2 = \frac{1}{1-x^2},$$

$$\phi 3 = \frac{1}{1-x^4},$$

$$\phi 4 = \frac{1}{(1-x^2)(1-x^3)},$$

$$\phi 5 = \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)},$$

$$\phi 6 = \frac{(1-x)(1+x-x^3-x^4-x^5+x^7+x^8)}{(1-x^2)^2(1-x^3)(1-x^4)(1-x^5)},$$

$$\phi 7 = \frac{1-x^6+2x^8-x^{10}+5x^{12}+2x^{14}+6x^{16}+2x^{18}+5x^{20}-x^{22}+2x^{24}-x^{26}+x^{32}}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})},$$

$$\phi 8 = \frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^6)(1-x^7)};$$

$$P(0,\ 1,\ 2, \ldots m)^\theta \tfrac{1}{2}m\theta = \text{coefficient of } x^\theta \text{ in } \psi m,$$

then $$\psi 2 = \frac{1}{(1-x)(1-x^2)},$$

$$\psi 3 = \frac{1+x^4}{(1-x^2)^2(1-x^4)},$$

$$\psi 4 = \frac{1-x+x^2}{(1-x)^2(1-x^3)(1-x^5)},$$

$$\psi 5 = \frac{1+x^2+6x^4+9x^6+12x^8+9x^{10}+6x^{12}+x^{14}+x^{16}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)};$$

$$P(0,\ 1,\ 2, \ldots m)^\theta \tfrac{1}{2}(m\theta - 1) = \text{coefficient of } x^\theta \text{ in } \psi_{,} m,$$

then $$\psi_{,} 3 = \frac{x+x^3}{(1-x^2)^2(1-x^4)},$$

$$\psi_{,} 5 = \frac{x+4x^3+8x^5+10x^7-10x^9+8x^{11}+4x^{13}+x^{15}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)}.$$

And from what has preceded, it appears that for a quantic of the order m, the number of asyzygetic covariants of the degree θ is for m even, coefficient x^θ in ψm, and for m odd, coefficient x^θ in $(\psi m + \psi_{,} m)$; and that the number of asyzygetic invariants of the degree θ is coefficient x^θ in ϕm. Attending first to the invariants:

42. For a quadric, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{1-x^2},$$

which leads to the conclusion that there is a single irreducible invariant of the degree 2.

43. For a cubic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{1-x^4},$$

i.e. there is a single irreducible invariant of the degree 4.

44. For a quartic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{(1-x^2)(1-x^3)},$$

i.e. there are two irreducible invariants of the degrees 2 and 3 respectively.

45. For a quintic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)}.$$

The numerator is the irreducible factor of $1-x^{36}$, i.e. it is equal to $(1-x^{36})(1-x^6) \div (1-x^{18})(1-x^{12})$; and substituting this value, the number becomes

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^{36}}{(1-x^4)(1-x^8)(1-x^{12})(1-x^{18})},$$

i.e. there are in all four irreducible invariants, which are of the degrees 4, 8, 12 and 18 respectively; but these are connected by an equation of the degree 36, i.e. the square of the invariant of the degree 18 is a rational and integral function of the other three invariants; a result, the discovery of which is due to M. Hermite.

46. For a sextic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1-x)(1+x-x^3-x^4-x^5+x^7+x^8)}{(1-x^2)^2(1-x^3)(1-x^4)(1-x^6)}:$$

the second factor of the numerator is the irreducible factor $1-x^{30}$, i.e. it is equal to $(1-x^{30})(1-x^5)(1-x^3)(1-x^2) \div (1-x^{15})(1-x^{10})(1-x^6)(1-x)$; and substituting this value, the number becomes

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^{30}}{(1-x^2)(1-x^4)(1-x^6)(1-x^{10})(1-x^{15})},$$

i.e. there are in all five irreducible invariants, which are of the degrees 2, 4, 6, 10 and 15 respectively; but these are connected by an equation of the degree 30, i.e. the square of the invariant of the degree 15 is a rational and integral function of the other four invariants.

47. For a septimic, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6+2x^8-x^{10}+5x^{12}+2x^{14}+6x^{16}+2x^{18}+5x^{20}-x^{22}+2x^{24}-x^{26}+x^{32}}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})},$$

the numerator is equal to

$$(1-x^8)(1-x^8)^{-2}(1-x^{10})(1-x^{12})^{-5}(1-x^{14})^{-4}\ldots,$$

where the series of factors does not terminate; hence [*incorrect*, see p. 253] the number of irreducible invariants is infinite; substituting the preceding value, the number of asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } (1-x^4)^{-1}(1-x^8)^{-3}(1-x^{12})^{-6}(1-x^{14})^{-4}\ldots$$

The first four indices give the number of irreducible invariants of the corresponding degrees, i.e. there are 1, 3, 6 and 4 irreducible invariants of the degrees 4, 8, 12 and 14 respectively, but there is no reason to believe that the same thing holds with respect to the indices of the subsequent terms. To verify this it is to be remarked, that there are 1, 4, 10 and 4 asyzygetic invariants of the degrees in question respectively; there is therefore one irreducible invariant of the degree 4; calling this X_4, there is only one composite invariant of the degree 8, viz. X_4^2; there are therefore three irreducible invariants of this degree, say X_8, X_8', X_8''. The composite invariants of the degree 12 are four in number, viz. X_4^3, X_4X_8, X_4X_8', X_4X_8'', *and these cannot be connected by any syzygy*, for if they were so, X_4^3, X_8, X_8', X_8'' would be connected by a syzygy, or there would be less than 3 irreducible invariants of the degree 8. Hence there are precisely 6 irreducible invariants of the degree 12. And since the irreducible invariants of the degrees 4, 8 and 12 do not give rise to any composite invariant of the degree 14, there are precisely 4 irreducible invariants of the degree 14.

48. For an octavic, the number of the asyzygetic invariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^6)(1-x^7)};$$

and the second factor of the numerator is

$$(1-x)^{-1}(1-x^2)(1-x^3)^{-1}(1-x^6)^{-1}(1-x^8)^{-1}(1-x^9)^{-1}(1-x^{10})^{-1}(1-x^{16})(1-x^{17})(1-x^{18})\ldots,$$

where the series of factors does not terminate, hence [*incorrect*] the number of irreducible invariants is infinite. Substituting the preceding value, the number of the asyzygetic invariants of the degree θ is

$$\text{coeff.} x^\theta \text{ in } (1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^6)^{-1}(1-x^7)^{-1}(1-x^8)^{-1}(1-x^9)^{-1}(1-x^{10})^{-1}(1-x^{16})(1-x^{17})(1-x^{18})\ldots$$

There is certainly one, and only one irreducible invariant for each of the degrees 2, 3, 4, 5 and 6 respectively; but the formula does not show the number of the irreducible invariants of the degrees 7, &c.; in fact, representing the irreducible invariants of the degrees 2, 3, 4, 5 and 6 by X_2, X_3, X_4, X_5, X_6, these give rise to 3 composite invariants of the degree 7, viz. $X_2X_2X_3$, X_2X_5, X_3X_4, which may or may not be connected by a syzygy; if they are not connected by a syzygy, there will be a single irreducible invariant of the degree 7; but if they are connected by a syzygy, there will be two irreducible invariants of the degree 7; it is useless at present to pursue the discussion further.

Considering next the covariants,—

49. For a quadric, the number of asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1}{(1-x)(1-x^2)},$$

i.e. there are two irreducible covariants of the degrees 1 and 2 respectively; these are of course the quadric itself and the invariant.

50. For a cubic, the number of the asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{(1+x)(1+x^2)}{(1-x^2)^2(1-x^4)}.$$

The first factor of the numerator is the irreducible factor of

$$1-x^3, = (1-x^2) \div (1-x),$$

and the second factor of the numerator is the irreducible factor of

$$1-x^4, = (1-x^4) \div (1-x^2);$$

substituting these values, the number is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6}{(1-x)(1-x^2)(1-x^2)(1-x^4)},$$

i.e. there are 4 irreducible covariants of the degrees 1, 2, 3, 4 respectively; but these are connected by an equation of the degree 6; the covariant of the degree 1 is the cubic itself U, the other covariants are the covariants already spoken of and represented by the letters H, Φ and ∇ respectively (H is of the degree 2 and the order 3, Φ of the degree 3 and the order 3, and ∇ is of the degree 4 and the order 0, i.e. it is an invariant).

51. For a quartic, the number of the asyzygetic covariants of the degree θ is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x+x^2}{(1-x)^2(1-x^2)(1-x^3)},$$

the numerator of which is the irreducible factor of $1-x^6$, i.e. it is equal to $(1-x^6)(1-x) \div (1-x^2)(1-x^3)$. Making this substitution, the number is

$$\text{coefficient } x^\theta \text{ in } \frac{1-x^6}{(1-x)(1-x^2)^2(1-x^3)^2},$$

i.e. there are five irreducible covariants, one of the degree 1, two of the degree 2, and two of the degree 3, but these are connected by an equation of the degree 6. The irreducible covariant of the degree 1 is of course the quartic itself U, the other irreducible covariants are those already spoken of and represented by I, H, J, Φ respectively (I is of the degree 2 and the order 0, and J is of the degree 3 and the order 0, i.e. I and J are invariants, H is of the degree 2 and the order 4, Φ is of the degree 3 and the order 6).

52. For a quintic, the number of irreducible covariants of the degree θ is

$$\text{coeff. } x^\theta \text{ in } \frac{1+x+x^2+4x^3+6x^4+8x^5+9x^6+10x^7+12x^8+10x^9+9x^{10}+8x^{11}+6x^{12}+4x^{13}+x^{14}+x^{15}+x^{16}}{(1-x^2)^2(1-x^4)(1-x^6)(1-x^8)},$$

the numerator of which is

$$(1+x)^2(1-x+2x^2+x^3+2x^4+3x^5+x^6+5x^7+x^8+3x^9+2x^{10}+x^{11}+2x^{12}-x^{13}+x^{14});$$

the first factor is $(1-x)^{-2}(1-x^2)^2$, the second factor is

$$(1-x)(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-2}(1-x^5)^{-2}(1-x^6)^5(1-x^7)^5(1-x^8)^7(1-x^9)^1(1-x^{10})^{-9}(1-x^{11})^{-19}\ldots,$$

which does not terminate; hence [*incorrect*] the number of irreducible covariants is infinite. Substituting the preceding values, the expression for the number of the asyzygetic covariants of the degree θ is

$$\text{coeff. } x^\theta \text{ in } (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-3}(1-x^5)^{-2}(1-x^6)^4(1-x^7)^5(1-x^8)^6(1-x^9)^1(1-x^{10})^{-9}(1-x^{11})^{-19}\ldots,$$

which agrees with a previous result: the numbers of irreducible covariants for the degrees 1, 2, 3, 4 are 1, 2, 3 and 3 respectively, and for the degree 5, the number of irreducible covariants is three, but there is one syzygy between the composite covariants of the degree in question; the difference $3-1=2$ is the index taken with its sign reversed of the factor $(1-x^2)^{-2}$.

53. I consider a system of the asyzygetic covariants of any particular degree and order of a given quantic, the system may of course be replaced by a system the terms of which are any linear functions of those of the original system, and it is necessary to inquire what covariants ought to be selected as most proper to represent the system of asyzygetic covariants; the following considerations seem to me to furnish a convenient rule of selection. Let the literal parts of the terms which enter into the coefficients of the highest power of x or leading coefficients be represented by M_α, M_β, M_γ,... these quantities being arranged in the natural or alphabetical order; the first in order of these quantities M, which enters into the leading coefficient of a particular covariant, may for shortness be called the leading term of such covariant, and a covariant the leading term of which is posterior in order to the leading term of another covariant, may be said to have a lower leading term.

It is clear, that by properly determining the multipliers of the linear functions we may form a covariant the leading term of which is lower than the leading term of any other covariant (the definition implies that there is but one such covariant); call this Θ. We may in like manner form a covariant such that its leading term is lower than the leading term of every other covariant except Θ_1; or rather we may form a system of such covariants, since if Φ_2 be a covariant having the property in question, $\Phi_2+k\Theta_1$ will have the same property, but k may be determined so that the covariant shall not contain the leading term of Θ_1, i.e. we may form a covariant Θ_2 such that its leading term is lower than the leading term of every other covariant excepting Θ_1, and that the leading term of Θ_1 does not enter into Θ_2; and there is but one such covariant, Θ_2. Again, we may form a covariant Θ_3 such that its leading term is lower than the leading term of every other covariant excepting Θ_1 and Θ_2, and that the

leading terms of Θ_1 and Θ_2 do not either of them enter into Θ_3; and there is but one such covariant, Θ_3. And so on, until we arrive at a covariant the leading term of which is higher than the leading terms of the other covariants, and which does not contain the leading terms of the other covariants. We have thus a series of covariants Θ_1, Θ_2, Θ_3, &c. containing the proper number of terms, and which covariants may be taken to represent the asyzygetic covariants of the degree and order in question.

In order to render the covariants Θ definite as well numerically as in regard to sign, we may suppose that the covariant is in its least terms (i.e. we may reject numerical factors common to all the terms), and we may make the leading term positive. The leading term with the proper numerical coefficient (if different from unity) and with the proper power of x (or the order of the function) annexed, will, when the covariants of a quantic are tabulated, be sufficient to indicate, without any ambiguity whatever, the particular covariant referred to. I subjoin a table of the covariants of a quadric, a cubic and a quartic, and of the covariants of the degrees 1, 2, 3, 4 and 5 respectively of a quintic, and also two other invariants of a quintic.

[Except for the quantic itself, the algebraical sum of the numerical coefficients in any column is $=0$, viz. the sum of the coefficients with the sign $+$ is equal to that of the coefficients with the sign $-$, and I have as a numerical verification inserted at the foot of each column this sum with the sign $\pm$].

Covariant Tables (Nos. 1 to 26).

No. 1.

$$\left(\begin{array}{|c|c|c|}\hline a+1 & b+2 & c+1 \\ \hline\end{array}\right\}(x,\ y)^2.$$

No. 2.

$$\begin{array}{|c|}\hline ac+1 \\ b^2-1 \\ \hline \pm 1 \end{array}$$

The tables Nos. 1 and 2 are the covariants of a binary quadric. No. 1 is the quadric itself; No. 2 is the quadrinvariant, which is also the discriminant.

No. 3.

$$\left(\begin{array}{|c|c|c|c|}\hline a+1 & b+3 & c+3 & d+1 \\ \hline\end{array}\right\}(x,\ y)^3.$$

No. 4.

$$\left(\begin{array}{|c|c|c|}\hline ac+1 & ad+1 & bd+1 \\ b^2-1 & bc-1 & c^2-1 \\ \hline \pm 1 & \pm 1 & \pm 1 \end{array}\right\}(x,\ y)^2$$

No. 5.

$$\left(\begin{array}{|c|c|c|c|}\hline a^2d+1 & abd+3 & acd-3 & ad^2-1 \\ abc-3 & ac^2-6 & b^2d+6 & bcd+3 \\ b^3+2 & b^2c+3 & bc^2-3 & c^3-2 \\ \hline \pm 3 & \pm 6 & \pm 6 & \pm 3 \end{array}\right\}(x,\ y)^3.$$

No. 6.

$$\begin{array}{|c|}\hline a^2d^2+1 \\ abcd-6 \\ ac^3+4 \\ b^2d+4 \\ b^2c^2-3 \\ \hline \pm 9 \end{array}$$

The tables Nos. 3, 4, 5 and 6 are the covariants of a binary cubic. No. 3 is the cubic itself; No. 4 is the quadricovariant, or Hessian; No. 5 is the cubicovariant; No. 6 is the invariant, or discriminant. And if we write No. $3 = U$, No. $4 = H$, No. $5 = \Phi$, No. $6 = \nabla$,

then identically,

$$\Phi^2 - \nabla U^2 + 4H^3 = 0.$$

No. 7.

$$\left(\begin{array}{|c|c|c|c|c|}\hline a+1 & b+4 & c+6 & d+4 & e+1 \\ \hline\end{array}\right)(x, y)^4$$

No. 8.

$$\begin{array}{|c|}\hline ae+1 \\ bd-4 \\ c^2+3 \\ \hline \end{array}$$

± 4

No. 9.

$$\left(\begin{array}{|c|c|c|c|c|}\hline ac+1 & ad+2 & ae+1 & be+2 & bd+1 \\ b^2-1 & bc-2 & bd+2 & cd-2 & c^2-1 \\ & & c^2-3 & & \\ \hline \end{array}\right)(x, y)^4$$

± 1 ± 2 ± 3 ± 2 ± 1

No. 10.

$$\begin{array}{|c|}\hline ace+1 \\ ad^2-1 \\ b^2e-1 \\ bcd+2 \\ c^3-1 \\ \hline \end{array}$$

± 3

No. 11.

$$\left(\begin{array}{|c|c|c|c|c|c|c|}\hline a^2d+1 & a^2e+1 & abe+5 & ace\ \ldots & ade-5 & ae^2-1 & be^2-1 \\ abc-3 & abd+2 & acd-15 & ad^2-10 & bce+15 & bde-2 & cde+3 \\ b^3+2 & ac^2-9 & b^2d+10 & b^2e+10 & bd^2-10 & c^2e+9 & d^3-2 \\ & b^2c+6 & bc^2\ \ldots & bcd\ \ldots & c^2d\ \ldots & cd^2-6 & \\ & & & c^3\ \ldots & & & \\ \hline \end{array}\right)(x, y)^6$$

± 3 ± 3 ± 9 ± 15 ± 10 ± 15 ± 9 ± 3

No. 12.

$$\begin{array}{|l r|}\hline a^3e^3 & +\ 1 \\ a^2bde^2 & -\ 12 \\ a^2c^2e^2 & -\ 18 \\ a^2cd^2e & +\ 54 \\ a^2d^4 & -\ 27 \\ ab^2ce^2 & +\ 54 \\ ab^2d^2e & -\ 6 \\ abc^2de & -\ 180 \\ abcd^3 & +\ 108 \\ ac^4e & +\ 81 \\ ac^3d^2 & -\ 54 \\ b^4e^2 & -\ 27 \\ b^3cde & +\ 108 \\ b^3d^3 & -\ 64 \\ b^2c^2e & -\ 54 \\ b^2c^2d^2 & +\ 36 \\ bc^4d & \ldots \\ c^6 & \ldots \\ \hline \end{array}$$

± 442

The tables Nos. 7, 8, 9, 10 and 11 are the irreducible covariants of a quartic. No. 7 is the quartic itself; No. 8 is the quadrinvariant; No. 9 is the quadricovariant, or Hessian; No. 10 is the cubinvariant; and No. 11 is the cubicovariant. The table No. 12 is the discriminant. And if we write No. $7 = U$, No. $8 = I$, No. $9 = H$, No. $10 = J$, No. $11 = \Phi$, No. $12 = \nabla$,

then the irreducible covariants are connected by the identical equation

$$JU^2 - IU^2H + 4H^3 + \Phi^2 = 0,$$

and we have

$$\nabla = I^3 - 27J^2.$$

[The Tables Nos. 13 to 24 which follow, and also Nos. 25 and 26 which are given in 143 relate to the binary quintic. I have inserted in the headings the capital letters A, B, ... L and also Q and Q′ by which I refer to these covariants of the quintic. A is the quintic itself, C is the Hessian, G is the quartinvariant, J a linear covariant: Q is the simplest octinvariant, and Q′ is the discriminant. As noticed in the original memoir we have $AI + BF - CE = 0$; and $Q' = G^2 - 128\,Q$, only the coefficient 128 was by mistake given as 1152.]

A. No. 13.

$$(\ a+1 \quad b+5 \quad c+10 \quad d+10 \quad e+5 \quad f+1\ \ \between x, y\between^5$$

B. No. 14.

$$(\ \begin{array}{ccc} ae+1 & af+1 & bf+1 \\ bd-4 & be-3 & ce-4 \\ c^2+3 & cd+2 & d^2+3 \\ \pm 4 & \pm 3 & \pm 4 \end{array}\ \between x, y\between^2$$

C. No. 15.

$$(\ \begin{array}{ccccccc} ac+1 & ad+3 & ae+3 & af+1 & bf+3 & cf+3 & df+1 \\ b^2-1 & bc-3 & bd+3 & be+7 & ce+3 & de-3 & e^2-1 \\ & & c^2-6 & cd-8 & d^2-6 & & \\ \pm 1 & \pm 3 & \pm 6 & \pm 8 & \pm 6 & \pm 3 & \pm 1 \end{array}\ \between x, y\between^6$$

D. No. 16.

$$(\ \begin{array}{cccc} abf\ \ldots & acf+1 & adf+1 & aef\ \ldots \\ ace+1 & ade-1 & ae^2-1 & bdf+1 \\ ad^2-1 & b^2f-1 & bcf-1 & be^2-1 \\ b^2e-1 & bee+1 & bde+1 & c^2f-1 \\ bcd+2 & bd^2+1 & c^2e+1 & cde+2 \\ c^3\ -1 & c^2d-1 & cd^2-1 & d^3\ -1 \\ \pm 3 & \pm 3 & \pm 3 & \pm 3 \end{array}\ \between x, y\between^3$$

E. No. 17.

$$(\ \begin{array}{cccccc} a^2f+1 & abf+5 & acf+2 & adf-2 & aef-5 & af^2-1 \\ abe-5 & ace-16 & ade-12 & ae^2-8 & bdf+16 & bef+5 \\ acd+2 & ad^2+6 & b^2f+8 & bcf+12 & be^2+9 & cdf-2 \\ b^2d+8 & b^2e-9 & bce-38 & bde+38 & c^2f-6 & ce^2-8 \\ bc^2-6 & bcd+38 & bd^2+72 & c^2e-72 & cde-38 & d^2e+6 \\ & c^3\ -24 & c^2d-32 & cd^2+32 & d^3\ +24 & \\ \pm 11 & \pm 49 & \pm 82 & \pm 82 & \pm 49 & \pm 11 \end{array}\ \between x, y\between^5$$

F. No. 18.

$$(\ \begin{array}{ccccccccc} a^2d+1 & a^2e+2 & a^2f+1 & abf+7 & acf+5 & adf-5 & aef-7 & af^2-1 & bf^2-2 & cf^2-1 \\ abc-3 & abd+1 & abe+11 & ace-8 & ade-40 & ae^2-16 & bdf+8 & bef-11 & cef-1 & def+3 \\ b^3\ +2 & ac^2-12 & acd-34 & ad^2-34 & b^2f+16 & bcf+40 & be^2-29 & cdf+34 & d^2f+12 & e^3\ -2 \\ & b^2c+9 & b^2d+16 & b^2e+29 & bce+47 & bde-47 & c^2f+34 & ce^2-16 & de^2-9 & \\ & & bc^2+6 & bcd-2 & bd^2-44 & c^2e+44 & cde+2 & d^2e-6 & & \\ & & & c^3\ +8 & c^2d+16 & cd^2-16 & d^3\ -8 & & & \\ \pm 3 & \pm 12 & \pm 34 & \pm 44 & \pm 84 & \pm 84 & \pm 44 & \pm 34 & \pm 12 & \pm 3 \end{array}\ \between x, y\between^9$$

G. No. 19.

$a^2f^2 + 1$
$abef - 10$
$acdf + 4$
$ace^2 + 16$
$ad^2e - 12$
$b^2df + 16$
$b^2e^2 + 9$
$bc^2f - 12$
$bcde - 76$
$bd^3 + 48$
$c^3e + 48$
$c^2d^2 - 32$
± 142

H. No. 20.

	$a^2df + 1$	$a^2ef + 2$	$a^2f^2 + 1$	$abf^2 + 2$	$acf^2 + 1$
	a^2e^2 ...	$abdf - 4$	$abef - 4$	$acef - 4$	$adef - 3$
	$abcf - 3$	$abe^2 - 10$	$acdf - 2$	$ad^2f - 2$	$ae^3 + 2$
	$abde - 5$	$ac^2f - 2$	$ace^2 + 4$	$ade^2 + 4$	b^2f^2 ...
	$ac^2e + 10$	$acde + 24$	ad^2e ...	$b^2ef - 10$	$bcef - 5$
(	$acd^2 - 4$	$ad^3 - 12$	$b^2df + 4$	$bcdf + 24$	$bd^2f + 10$) $(x, y)^4$
	$b^3f + 2$	$b^2cf + 4$	$b^2e^2 - 9$	$bce^2 + 16$	$bde^2 - 5$
	$b^3ce - 5$	$b^2de + 16$	bc^2f ...	$bd^2e - 22$	$c^3df - 4$
	$b^2d^2 + 14$	$bc^2e - 22$	$bcde + 50$	$c^3f - 12$	$c^2e^2 + 14$
	$bc^3d - 16$	$bcd^2 - 4$	$bd^3 - 36$	$c^2de - 4$	$cd^2e - 16$
	$c^4 + 6$	$c^3d + 8$	$c^2e - 36$	$cd^3 + 8$	$d^4 + 6$
			$c^2d^2 + 28$		
	± 33	± 54	± 87	± 54	± 33

I. No. 21.

	$a^2cf + 1$	$a^2df + 2$	a^2ef ...	a^2f^2 ...	abf^2 ...	$acf^2 - 2$	$adf^2 - 1$	
	$a^3de - 1$	$a^2e^2 - 2$	$abdf + 2$	$abef$ ...	$acef - 2$	$adef$ ...	$ae^2f + 1$	
	$ab^2f - 1$	$abcf - 10$	$abe^2 - 2$	$acdf$ ...	$ad^2f + 1$	$ae^3 + 2$	$bcf^2 + 1$	
	$abce - 2$	$abde + 10$	$ac^2f - 1$	$ace^2 - 20$	$ade^2 + 1$	$b^2f^2 + 2$	$bdef + 2$	
	$abd^2 + 4$	ac^3e ...	$acde - 2$	$ad^2e + 20$	$b^2ef + 2$	$bcef$ ...	$be^3 - 3$	
	$ac^2d - 1$	acd^2 ...	$ad^3 + 3$	$b^2df + 20$	$bcdf + 2$	$bd^2f + 10$	$c^2ef - 4$	
(	$b^3e + 3$	$b^3f - 2$	$b^2cf - 1$	b^2e^2 ...	$bce^2 - 5$	$bde^2 - 14$	$cd^2f + 1$	) $(x, y)^6$.
	$b^2cd - 6$	$b^2ce + 14$	$b^2de + 5$	$bc^2f - 20$	$bd^2e - 1$	$c^3df - 10$	$cde^2 + 6$	
	$be^3 + 3$	$b^2d^2 + 2$	$bc^3e + 1$	$bcde$ ...	$c^3f - 3$	$c^2e^2 - 2$	$d^3e - 3$	
		$bc^3d - 26$	$bcd^2 - 9$	$bd^3 - 20$	$c^3de + 9$	$cd^2e + 26$		
		$c^4 + 12$	$c^3d + 4$	$c^3e + 20$	$cd^3 - 4$	$d^4 - 12$		
				c^2d^2 ...				
	± 11	± 40	± 15	± 60	± 15	± 40	± 11	

J. No. 22.

	$a^2cf^2 + 1$	$a^2df^2 + 1$	
	$a^3def - 2$	$a^2e^2f - 1$	
	$a^2e^3 + 1$	$abcf^2 - 2$	
	$ab^2f^2 - 1$	$abdef - 4$	
	$abcef - 4$	$abe^3 + 6$	
	$abd^2f + 8$	$ac^2ef + 8$	
	$abde^2 - 2$	$acd^2f - 2$	
	$ac^3df - 2$	$acde^2 - 12$	
	$ac^2e^2 + 14$	$ad^2e + 6$	
	$acd^2e - 22$	$b^3f^2 + 1$	
(	$ad^4 + 9$	$b^2cef - 2$	) $(x, y)^1$
	$b^3ef + 6$	$b^2d^2f + 14$	
	$b^2cdf - 12$	$b^2de^2 - 15$	
	$b^2ce^2 - 15$	$bc^3df - 22$	
	$b^2d^2e + 10$	$bc^2e^2 + 10$	
	$bc^3f + 6$	$bcd^2e + 30$	
	$bc^2de + 30$	$bd^4 - 15$	
	$bcd^3 - 20$	$c^4f + 9$	
	$c^4e - 15$	$c^3de - 20$	
	$c^3d^2 + 10$	$c^2d^2 + 10$	
	± 95	± 95	

K. No. 23.

a^2bf^2 ...	a^2cf^2 + 1	a^2df^2 − 1	a^2ef^2 ...	
a^2cef + 1	a^2def − 5	a^2e^2f + 1	$abdf^2$ − 1	
a^2d^2f − 3	a^2e^3 + 4	$abcf^2$ + 5	abe^2f + 1	
a^2de^2 + 2	ab^2f^2 − 1	$abdef$ − 8	ac^2f^2 + 3	
ab^2ef − 1	$abcef$ + 8	abe^3 + 3	$acdef$ − 14	
$abcdf$ + 14	abd^2f + 11	ac^2ef − 11	ace^3 + 8	
$abce^2$ − 11	$abde^2$ − 17	acd^2f + 11	ad^2f + 9	
abd^2e − 1	ac^2df − 11	$acde^2$ + 6	ad^2e^2 − 6	
ac^2f − 9	ac^2e^2 − 16	ad^3e − 6	b^2cf^2 − 2	
ac^2de + 14	acd^2e + 44	b^3f^2 − 4	b^2def + 11	$\mathfrak{Q}(x, y)^3$
acd^3 − 6	ad^4 − 18	b^2cef + 17	b^2e^3 − 9	
b^3df − 8	b^2ef − 3	b^2d^2f + 16	bc^2ef + 1	
b^3e^2 + 9	b^2cdf − 6	b^2de^2 − 21	bcd^2f + 14	
b^2c^2f + 6	b^2ce^2 + 21	bc^3df − 44	$bcde^2$ + 16	
b^2cde − 16	b^2d^2e − 5	bc^2e^2 + 5	bd^2e − 3	
b^2d^3 + 8	bc^3f + 6	bcd^2e + 39	c^3df + 6	
bc^3e + 3	bc^2de − 39	bd^4 − 12	c^2e^2 − 8	
bc^2d^2 − 2	bcd^3 + 22	c^4f + 18	c^2d^2e + 2	
c^4d ...	c^4e + 12	c^3de − 22	cd^4 ...	
	c^2d^2 − 8	c^2d^3 + 8		
± 57	± 129	± 129	± 57	

L. No. 24.

a^3ef ...	a^3f^2 ...	a^3bf^2 ...	a^3cf^2 − 1	a^2df^2 + 1	a^2ef^2 ...	a^2f^3 ...	abf^3 ...	
a^3bdf ...	a^3bef ...	a^3cef − 3	a^2def + 7	a^2e^2f − 1	$abdf^2$ + 3	$abef^2$ ...	$acef^2$ ...	
a^3be^2 ...	a^2cdf + 7	a^2d^2f + 12	a^2e^3 − 6	$abcf^2$ − 7	abe^2f − 3	$acdf^2$ − 7	ad^2f^2 − 2	
a^2c^2f + 2	a^2ce^2 − 10	a^2de^2 − 9	ab^2f^2 + 1	$abdef$ + 26	ac^2f^2 − 12	ace^2f + 7	ade^2f + 4	
a^2cde − 5	a^2d^2e + 3	ab^2ef + 3	$abcef$ − 26	abe^3 − 19	$acdef$ + 18	ad^2ef + 7	ae^4 − 2	
a^2d^3 + 3	ab^2df − 7	$abcdf$ − 18	abd^2f + 32	ac^2ef − 32	ace^2 + 6	ade^3 − 7	b^2ef^2 ...	
ab^2cf − 4	ab^2e^2 + 10	$abce^2$ − 18	$abde^2$ − 8	acd^2f + 18	ad^2f + 3	b^2df^2 + 10	$bcdf^2$ + 5	
ab^2de + 5	abc^2f − 7	abd^2e + 30	ac^2df − 18	$acde^2$ + 53	ad^2e^2 − 15	b^2e^2f − 10	bce^2f − 5	
abc^2e + 5	$abcde$ − 8	ac^2f − 3	ac^2e^2 + 6	ad^3e − 39	b^2ef^2 + 9	bc^2f^2 − 3	bd^2ef − 5	
$abcd^2$ − 7	abd^3 + 9	ac^2de + 45	acd^2e + 52	b^3f^2 + 6	b^2def + 18	$bcdef$ + 8	bde^3 + 5	$\mathfrak{Q}(x, y)^7$
ac^3d + 1	ac^2e + 22	acd^3 − 39	ad^4 − 39	b^2cef + 8	b^2e^3 − 27	bce^3 − 2	c^2f^2 − 3	
b^4f + 2	ac^2d^2 − 19	b^3df − 6	b^3ef + 19	b^2d^2f − 6	bc^2ef − 30	bd^2f − 22	c^2def + 7	
b^3ce − 5	b^3ef + 7	b^3e^2 + 27	b^2cdf − 53	b^2de^2 − 20	bcd^2f − 45	bd^2e^2 + 19	c^2e^3 + 2	
b^3d^2 − 2	b^3de + 2	b^2c^2f + 15	b^2ce^2 + 20	bc^2df + 45	$bcde^2$ + 87	c^3ef − 9	cd^2f − 1	
b^2c^2d + 8	b^2c^2e − 19	b^2cde − 87	b^2d^2e − 25	bc^2e^2 + 25	bd^3e − 12	c^3d^2f + 19	cd^2e^2 − 8	
bc^4 − 3	b^2cd^2 − 11	b^2d^3 + 6	bc^3f + 39	bcd^2e − 52	c^3df + 39	c^3de^2 + 11	d^4e + 3	
	bc^3d + 33	bc^3e + 12	bc^2de − 45	bd^4 ...	c^3e^2 − 6	cd^3e − 33		
	c^5 − 12	bc^2d^2 + 57	bcd^3 + 65	c^4f + 39	c^2d^2e − 57	d^5 + 12		
		c^4d − 24	c^4e ...	c^3de − 65	cd^4 + 24			
			c^2d^2 − 20	c^2d^3 + 20				
± 26	± 93	± 207	± 241	± 241	± 207	± 93	± 26	

No. 25, $Q = +1\, a^3cdf^2 + \&c.$
No. 26, $Q' = +1\, a^4f^4 + \&c.$ } [see *post*, 143.]

142.

NUMERICAL TABLES SUPPLEMENTARY TO SECOND MEMOIR ON QUANTICS.

[Now first published (1889).]

In the present paper I arrange in a more compendious form and continue to a much greater extent the tables (first of each pair) given Nos. 35—39 of my Second Memoir on Quantics, 141, pp. 260—264, which relate to the cubic, the quartic and the quintic functions; and I give the like tables for the sextic, the septimic and the octavic functions respectively. The cubic table exhibits the coefficients of the several xz terms of the function $1 \div (1 - z \,.\, 1 - xz \,.\, 1 - x^2z \,.\, 1 - x^3z)$, or, what is the same thing, it gives the number of partitions of a given number into a given number of parts, the parts being 0, 1, 2, 3, (repetitions admissible): or again, regarding the letters a, b, c, d, as having the weights 0, 1, 2, 3 respectively, it shows the number of literal terms of a given degree and given weight. And similarly for the quartic, quintic, sextic, septimic and octavic tables respectively, the parts of course being 0, 1, ... up to 4, 5, 6, 7 or 8, and the letters being a, b, ... up to e, f, g, h or i. The extent of the tables is as follows:

cubic table	extends to	deg-weight		18—27
quartic	"	"	"	18—36
quintic	"	"	"	18—45
sextic	"	"	"	15—45
septimic	"	"	"	12—42
octavic	"	"	"	10—40

viz. for the quintic, the sextic and the octavic functions these are the deg-weights of the highest invariants respectively. I designate the Tables as the *ad*-, *ae*-, *af*-, *ag*-, *ah*- and *ai*-tables respectively.

It is to be noticed that in the several tables the lower part of each column is for shortness omitted; the column has to be completed by taking into it the series

of bottom terms of each of the preceding columns: thus in the *af*- or quintic table the complete column for degree 3 would be

D	3
W	8-7
-0	6
-1	6
-2	5
-3	4
-4	3
-5	2
-6	1
-7	1

where the concluding terms 2, 1, 1 are the bottom terms of the three preceding columns respectively. And the meaning is that for degree 3, and weight 8, or 7, the number of terms is $=6$; for weight $7-1$, $=6$, the number of terms is $=6$; and similarly for weights 5, 4, 3, 2, 1, 0 the numbers are 5, 4, 3, 2, 1, 1; the numbers are those of the terms

W.	0	1	2	3	4	5	6	7	8
	a^3	a^2b	a^2c	a^2d	a^2e	a^2f	abf	acf	adf
			ab^2	abc	abd	abe	ace	ade	ae^2
				b^3	ac^2	acd	ad^2	b^2f	bef
					b^2c	b^2d	b^2e	b^2ce	bde
						bc^2	bed	bd^2	c^2e
							c^3	c^2d	cd^2
No.	1	1	2	3	4	5	6	6	6

The like remarks and explanations apply to the other tables.

ad-TABLE.

D	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
W	0	2-1	3	5-4	6	8-7	9	11-10	12	14-13	15	17-16	18	20-19	21	23-22	24	26-25	27	
	1	1	2	3	5	6	8	10	13	15	18	21	25	28	32	36	41	45	50	-0
			2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	50	-1
					4	5	7	9	12	14	17	20	24	27	31	35	40	44	49	-2
							7	8	11	13	17	19	23	26	31	34	39	43	49	-3
									10	12	15	18	22	25	29	33	38	42	47	-4
											14	16	20	23	28	31	36	40	46	-5
													19	21	26	29	35	38	44	-6
															24	27	32	36	42	-7
																	30	33	39	-8
																			37	-9

ae-TABLE.

D	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
W	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	
	1	1	3	5	8	12	18	24	33	43	55	69	86	104	126	150	177	207	241	−0
		1	2	4	7	11	16	23	31	41	53	67	83	102	123	147	174	204	237	−1
			2	4	7	11	16	23	31	41	53	67	83	102	123	147	174	204	237	−2
				3	5	9	14	20	28	38	49	63	79	97	118	142	168	198	231	−3
					5	8	13	19	27	36	48	61	77	95	116	139	166	195	228	−4
						6	10	16	23	32	43	56	71	89	109	132	158	187	219	−5
							9	14	21	30	40	53	68	85	105	128	153	182	214	−6
								11	17	25	35	47	61	78	97	119	144	172	203	−7
									15	22	32	43	57	73	92	113	138	165	196	−8
										18	26	37	50	65	83	104	127	154	184	−9
											23	33	45	60	77	97	120	146	175	−10
												27	38	52	68	87	109	134	162	−11
													34	46	62	80	101	125	153	−12
														39	53	70	90	113	139	−13
															47	63	82	104	129	−14
																54	71	92	116	−15
																	64	83	106	−16
																		72	93	−17
																			84	−18

af-TABLE.

D	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
W	0	3-2	5	8-7	10	13-12	15	18-17	20	23-22	25	28-27	30	33-32	35	38-37	40	43-42	45	
	1	1	3	6	12	20	32	49	73	102	141	190	252	325	414	521	649	795	967	−0
		1	3	6	11	19	32	48	71	101	141	188	249	322	414	518	645	791	966	−1
			2	5	11	18	30	46	70	98	137	184	247	317	408	511	641	783	957	−2
			2	4	9	16	29	43	66	93	134	178	240	309	402	501	630	770	948	−3
				3	8	14	25	39	63	88	127	170	233	299	390	488	619	754	930	−4
					6	11	23	35	57	81	121	161	222	286	379	472	601	734	912	−5
					5	9	19	30	52	74	111	150	212	272	362	453	583	711	886	−6
						7	16	26	45	66	103	139	197	256	346	433	559	684	860	−7
							12	21	40	58	92	126	184	239	325	409	536	655	827	−8
							10	17	33	50	83	114	168	220	306	385	507	623	795	−9
								13	28	43	72	101	154	202	283	359	480	590	756	−10
									22	35	63	89	137	183	262	333	448	554	719	−11
									18	29	53	77	123	165	238	306	419	519	677	−12
										23	45	66	107	146	217	280	386	482	637	−13
											36	55	94	129	194	253	356	446	593	−14
											30	46	80	112	174	228	324	409	553	−15
												37	68	97	152	203	295	374	509	−16
													56	82	134	180	264	339	469	−17
													47	69	115	157	237	306	427	−18
														57	99	137	209	273	389	−19
															83	117	185	243	350	−20
															70	100	160	214	315	−21
																84	139	188	279	−22
																	118	162	248	−23
																	101	140	217	−24
																		119	190	−25
																			163	−26
																			141	−27

ag-TABLE.

D	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
W	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	
	1	1	4	8	18	32	58	94	151	227	338	480	676	920	1242	1636	−0
		1	3	8	16	32	55	94	147	227	332	480	668	920	1232	1635	−1
		1	3	7	16	30	55	90	146	221	330	471	664	907	1226	1617	−2
			2	7	14	29	51	88	139	217	319	464	648	896	1203	1601	−3
			2	5	13	25	48	81	134	205	310	446	634	870	1182	1565	−4
				4	10	23	42	76	123	196	293	431	608	847	1145	1533	−5
				3	9	19	39	68	116	182	280	408	587	813	1113	1483	−6
					6	16	32	61	103	169	258	387	553	780	1064	1435	−7
					5	12	28	52	94	152	241	359	525	737	1021	1373	−8
						10	22	46	81	139	218	335	488	699	965	1316	−9
						7	18	37	71	121	199	304	455	650	914	1244	−10
							13	31	59	107	175	278	415	607	852	1178	−11
							11	24	51	91	157	248	382	557	798	1102	−12
								19	40	78	134	222	341	512	733	1031	−13
								14	33	64	117	193	308	462	677	952	−14
									25	54	98	170	271	419	614	882	−15
									20	42	83	144	240	371	559	803	−16
										34	67	124	206	331	499	734	−17
										26	56	103	180	289	449	661	−18
											43	86	151	253	394	596	−19
											35	69	129	216	349	529	−20
												57	106	187	302	472	−21
												44	88	156	263	412	−22
													70	132	223	362	−23
													58	108	192	303	−24
														89	159	270	−25
														71	134	228	−26
															109	195	−27
															90	161	−28
																135	−29
																110	−30

ah-TABLE.

D	0	1	2	3	4	5	6	7	8	9	10	11	12	
W	0	4-3	7	11-10	14	18-17	21	25-24	28	32-31	35	39-38	42	
	1	1	4	10	24	49	94	169	289	468	734	1117	1656	−0
		1	4	10	23	48	94	166	285	464	734	1109	1646	−1
		1	3	9	23	46	90	162	282	454	722	1093	1634	−2
			3	8	20	43	88	155	272	441	709	1069	1605	−3
			2	7	19	39	81	146	263	424	686	1038	1572	−4
			2	5	16	35	76	136	247	403	663	1000	1524	−5
				4	14	30	68	125	233	379	629	957	1475	−6
				3	11	26	61	112	214	354	598	908	1410	−7
					9	21	52	100	197	325	558	856	1346	−8
					6	17	46	87	176	297	520	799	1271	−9
					5	13	37	75	158	268	477	742	1197	−10
						10	31	63	137	239	437	682	1114	−11
						7	24	53	120	210	392	623	1036	−12
							19	42	101	184	353	563	950	−13
							14	34	86	157	311	506	871	−14
							11	26	70	134	274	449	788	−15
								20	58	112	236	397	711	−16
								15	45	93	204	346	633	−17
									36	75	171	300	564	−18
									27	61	145	256	493	−19
									21	47	119	218	432	−20
										37	98	182	372	−21
										28	78	152	320	−22
											63	124	270	−23
											48	101	229	−24
											38	80	189	−25
												64	157	−26
												49	127	−27
													103	−28
													81	−29
													65	−30

ai-TABLE.

	0	1	2	3	4	5	6	7	8	9	10	
W	0	4	8	12	16	20	24	28	32	36	40	
	1	1	5	13	33	73	151	289	526	910	1514	−0
		1	4	12	31	71	147	285	519	902	1502	−1
		1	4	12	31	70	146	282	515	894	1492	−2
		1	3	11	28	66	139	272	499	873	1460	−3
			3	10	27	63	134	263	486	851	1430	−4
			2	8	23	57	123	247	461	816	1379	−5
			2	7	21	52	116	233	440	783	1331	−6
				5	17	45	103	214	409	738	1265	−7
				4	15	40	94	197	383	696	1214	−8
				3	11	33	81	176	348	645	1127	−9
					9	28	71	158	319	597	1057	−10
					6	22	59	137	284	543	974	−11
					5	18	51	120	255	495	900	−12
						13	40	101	221	441	816	−13
						10	33	86	194	394	742	−14
						7	25	70	164	345	662	−15
							20	58	141	302	593	−16
							14	45	116	258	519	−17
							11	36	97	222	457	−18
								27	77	185	393	−19
								21	63	156	340	−20
								15	48	127	286	−21
									38	104	243	−22
									28	82	200	−23
									22	66	167	−24
										50	134	−25
										39	109	−26
										29	85	−27
											68	−28
											51	−29
											40	−30

The numbers of each table are connected in several ways with those of the preceding tables. One of these connexions, which is of some importance, is best explained by an example: in the *af*-table, **8-20**, the number of terms of degree **8** and weight **20** is 73; and we have $73 = 1 + 6 + 16 + 23 + 27$, viz. (see p. 288) these are the numbers of the terms in a^4, a^3, a^2, a^1, a^0 respectively: the complementary factors, (for example) of a^3 are bef^3, &c. terms in b, c, d, e, f of the degree **5** and weight **20**, and (replacing therein each letter by that which immediately precedes it) these are in number equal to the terms in a, b, c, d, e of the degree **5** and weight **20 − 5**, **= 15**; thus the number 6 of the terms in question is that for the deg-weight **5-15** of the *ae*-table: and so 1, 6, 16, 23, 27 are the numbers in the *ae*-table for the deg-weights **4-16**, **5-15**, **6-14**, **7-13** and **8-12** respectively, or (making a change rendered necessary by the abbreviated form of the tables) say for the deg-weights **4-0**, **5-10**, **6-14**, **7-13** and **8-12**.

143.

TABLES OF THE COVARIANTS M TO W OF THE BINARY QUINTIC: FROM THE SECOND, THIRD, FIFTH, EIGHTH, NINTH AND TENTH MEMOIRS ON QUANTICS.

[Arranged in the present form, 1889.]

THE binary quintic has in all (including the quintic itself and the invariants) 23 covariants, which I have represented by the capital letters, A, B, C, ... W (alternative forms of two of these are denoted by Q′ and S′). The covariants A, ... L, and also Q, Q′ were given in my Second Memoir on Quantics, and except Q and Q′ are reproduced in the present reprint thereof, 141; in all these I gave not only the literal terms actually presenting themselves, but also the terms with zero coefficients; in the other covariants however, or in most of them, the terms with zero coefficients were omitted. It is very desirable to have in every case the complete series of literal terms, and in the covariants as here printed they are accordingly inserted: the number of terms is in each case known beforehand by the foregoing af-table, 142, and any omission is thus precluded; by means of this af-table we have the numbers of terms as shown in the following list.

I have throughout (as was done in the Ninth and Tenth Memoirs) expressed the literal terms in a slightly different form from that employed in the Second Memoir: this is done in order to show at a glance in each column the set of terms which contain a given power of a, and in each such set the terms which contain a given power of b.

The numerical verifications are also given not only for the entire column but for each set of terms containing the same power of a; viz. in most cases, but not always, the positive and negative coefficients of a set have equal sums, which are shown by

a number with the sign ± prefixed. The verification is in some cases given in regard to the subsets involving the same powers of a and b, here also the sums of the positive and negative coefficients are not in every case equal. The cases of inequality will be referred to at the end of this paper.

The whole series of covariants is as follows:

Mem.	No. of table.				deg-weight
2	13	A	=	$(1, 1, 1, 1, 1, 1 \between x, y)^5$	1 (0....5)
„	14	B	=	$(3, 3, 3 \between x, y)^2$	2 (4.6)
„	15	C	=	$(2, 2, 3, 3, 3, 2, 2 \between x, y)^5$	2 (2.....8)
„	16	D	=	$(6, 6, 6, 6 \between x, y)^3$	3 (6..9)
„	17	E	=	$(5, 6, 6, 6, 6, 5 \between x, y)^5$	3 (5....10)
„	18	F	=	$(3, 4, 5, 6, 6, 6, 6, 5, 4, 3 \between x, y)^9$	3 (3.....12)
„	19	G	=	$(12 \between x, y)^0$, Invt.	4-10
„	20	H	=	$(11, 11, 12, 11, 11 \between x, y)^4$	4 (8...12)
„	21	I	=	$(9, 11, 11, 12, 11, 11, 9 \between x, y)^6$	4 (7.....13)
„	22	J	=	$(20, 20 \between x, y)^1$	5 (12, 13)
„	23	K	=	$(19, 20, 20, 19 \between x, y)^3$	5 (11..14)
„	24	L	=	$(16, 18, 19, 20, 20, 19, 18, 16 \between x, y)^7$	5 (9.....16)
8	83	M	=	$(32, 32, 32 \between x, y)^2$	6 (14.16)
„	84	N	=	$(30, 32, 32, 32, 30 \between x, y)^4$	6 (13...17)
9	90	O	=	$(49, 49 \between x, y)^1$	7 (17, 18)
„	91	P	=	$(46, 48, 49, 49, 48, 46 \between x, y)^5$	7 (15....20)
2	Q 25 Q′ 26	Q, Q′	=	$(73 \between x, y)^0$, Invt.	8-20
9	92	R	=	$(71, 73, 71 \between x, y)^2$	8 (19.21)
9 10	S 93 S 93 *bis*	S, S′	=	$(101, 102, 102, 101 \between x, y)^3$	9 (21..24)
9	94	T	=	$(190, 190 \between x, y)^1$	11 (27, 28)
3	29	U	=	$(252 \between x, y)^0$, Invt.	12-30
9	95	V	=	$(325, 325 \between x, y)^1$	13 (32, 33)
5	29A	W	=	$(967 \between x, y)^0$, Invt.	18-45

M. No. 83.

$a^2\ b^0ef^2$		...	$a^3\ b^0f^2$		...	$a^2\ b^1f^3$		...
$a^2\ b^1df^2$		...	$a^2\ b^1ef^2$		...	b^0cef^2		...
e^2f		...	b^0cdf^2	−	1	d^2f^2	−	1
$b^0c^2f^2$	−	1	ce^2f	+	1	de^2f	+	2
$cdef$	+	5	d^2ef	+	1	e^4	−	1
ce^3	−	3	de^3	−	1	$a^1\ b^2ef^2$		...
d^2f	−	3	$a^1\ b^2df^2$	+	1	b^1cdf^2	+	5
d^2e^2	+	2	e^2f	−	1	ce^2f	−	5
$a^1\ b^2cf^2$	+	2	$b^1c^2f^2$	+	1	d^2ef	−	5
def	−	5	$cdef$	+	6	de^3	+	5
e^3	+	3	ce^3	−	8	$b^0c^3f^2$	−	3
b^1c^2ef	−	5	d^2f	−	10	c^2def	+	7
cd^2f	+	7	d^2e^2	+	11	c^2e^2	+	2
cde^2	−	1	b^0c^2ef	−	10	cd^2f	−	1
d^2e	−	1	c^2d^2f	+	11	cd^2e^2	−	8
b^0c^3df	−	1	c^2de^2	+	18	d^4e	+	3
c^3e^2	+	6	cd^3e	−	28	$a^0\ b^3df^2$	−	3
c^2d^2e	−	8	d^5	+	9	e^2f	+	3
cd^4	+	3	$a^0\ b^3cf^2$	−	1	$b^2c^2f^2$	+	2
$a^0\ b^4f^2$	−	1	def	−	8	$cdef$	−	1
b^2cef	+	5	e^3	+	9	ce^3	−	3
d^2f	+	2	b^2c^2ef	+	11	d^2f	+	6
de^2	−	3	cd^2f	+	18	d^2e^2	−	4
b^2c^2df	−	8	cde^2	−	37	b^1c^3ef	−	1
c^2e^2	−	4	d^3e	+	8	c^2d^2f	−	8
cd^2e	+	7	b^1c^3df	−	28	c^2de^2	+	7
d^4	−	1	c^2e^2	+	8	cd^3e	+	5
b^1e^4f	+	3	c^2d^2e	+	37	d^5	−	3
c^3de	+	5	cd^4	−	17	b^0c^4df	+	3
c^2d^3	−	4	b^0e^3f	+	9	c^4e^2	−	1
b^0c^5e	−	3	c^4de	−	17	c^3d^2e	−	4
c^4d^2	−	2	c^3d^3	+	8	c^2d^4	+	2
	±	7		±	2		±	2
		21			57			22
		24			108			28
	±	52		±	167		±	52

$\mathfrak{X}(x, y)^2$

N. No. 84.

$a^3\ b^0df^2$	$-\ 1$	$a^3\ b^0ef^2$	$\ldots$	$a^3\ b^0f^2$	$\ldots$	$a^2\ b^1f^2$	$\ldots$	$a^2\ b^0cf^3$	$+\ 1$
e^2f	$+\ 1$	$a^2\ b^1df^2$	$-\ 4$	$a^2\ b^1ef^2$	$\ldots$	b^0cef^2	$+\ 4$	def^2	$-\ 3$
$a^2\ b^1ef^2$	$+\ 3$	e^2f	$+\ 4$	b^0cdf^2	$\ldots$	d^2f^2	$-\ 4$	e^3f	$+\ 2$
def	$+\ 2$	$b^0c^2f^2$	$+\ 4$	ce^2f	$+\ 6$	de^2f	$-\ 4$	$a^1\ b^2f^3$	$-\ 1$
e^3	$-\ 5$	$cdef$	$-\ 8$	d^2ef	$-\ 12$	e^4	$+\ 4$	b^1cef^2	$-\ 2$
b^0c^2ef	$-\ 8$	ce^3	$+\ 4$	de^3	$+\ 6$	$a^1\ b^2ef^2$	$-\ 4$	d^2f^2	$+\ 8$
cd^2f	$+\ 2$	d^2f	$\ldots$	$a^1\ b^2df^2$	$-\ 6$	b^1cdf^2	$+\ 8$	de^2f	$-\ 2$
cde^2	$+\ 12$	d^2e^2	$\ldots$	e^2f	$\ldots$	ce^2f	$-\ 16$	e^4	$-\ 6$
d^3e	$-\ 6$	$a^1\ b^2cf^2$	$+\ 4$	$b^1c^2f^2$	$+\ 12$	d^2ef	$+\ 48$	$b^0c^2df^2$	$-\ 2$
$a^1\ b^2f^2$	$-\ 2$	def	$+\ 16$	$cdef$	$\ldots$	de^3	$-\ 32$	c^2e^3f	$+\ 6$
b^2cef	$-\ 2$	e^3	$-\ 24$	ce^3	$-\ 36$	$b^0c^3f^2$	$\ldots$	cd^2ef	$-\ 20$
d^2f	$-\ 6$	b^1c^2ef	$-\ 48$	d^3f	$+\ 48$	c^3def	$-\ 40$	cde^3	$+\ 12$
de^2	$+\ 13$	cd^2f	$+\ 40$	d^2e^2	$-\ 12$	c^2e^3	$+\ 56$	d^4f	$+\ 9$
b^1c^2df	$+\ 20$	cde^2	$+\ 40$	b^0c^2ef	$-\ 48$	cd^3f	$+\ 8$	d^3e^2	$-\ 6$
c^2e^2	$+\ 4$	d^3e	$-\ 24$	c^3d^2f	$\ldots$	cd^2e^2	$-\ 40$	$a^0\ b^2ef^2$	$+\ 5$
cd^2e	$-\ 52$	b^0c^3df	$-\ 8$	c^3de^2	$+\ 156$	d^4e	$+\ 12$	b^2cdf^2	$-\ 12$
d^4	$+\ 24$	c^3e^2	$+\ 56$	cd^3e	$-\ 168$	$a^0\ b^3df^2$	$-\ 4$	ce^2f	$-\ 13$
b^0c^4f	$-\ 9$	c^2d^2e	$-\ 88$	d^5	$+\ 54$	e^2f	$+\ 24$	d^2ef	$-\ 4$
c^3de	$+\ 20$	cd^4	$+\ 36$	$a^0\ b^3cf^2$	$-\ 6$	$b^2c^2f^2$	$\ldots$	de^3	$+\ 15$
c^2d^3	$-\ 10$	$a^0\ b^4f^2$	$-\ 4$	def	$+\ 36$	$cdef$	$-\ 40$	$b^1c^2f^2$	$+\ 6$
$a^0\ b^1ef$	$+\ 6$	b^3cef	$+\ 32$	e^3	$\ldots$	ce^3	$-\ 60$	c^2def	$+\ 52$
b^2cdf	$+\ 12$	d^2f	$-\ 56$	b^2c^2ef	$+\ 12$	d^3f	$-\ 56$	c^2e^3	$-\ 10$
ce^2	$-\ 15$	de^2	$+\ 60$	cd^2f	$-\ 156$	d^2e^2	$+\ 100$	cd^2f	$-\ 20$
d^2e	$+\ 10$	b^2c^2df	$+\ 40$	cde^2	$\ldots$	b^1c^2ef	$+\ 24$	cd^2e^2	$-\ 30$
b^2c^2f	$+\ 6$	c^2e^2	$-\ 100$	d^3e	$+\ 60$	c^2d^2f	$+\ 88$	d^4e	$+\ 15$
c^2de	$+\ 30$	cd^2e	$-\ 80$	b^1c^2df	$+\ 168$	c^2de^2	$+\ 80$	b^0c^4ef	$-\ 24$
cd^3	$-\ 20$	d^4	$+\ 60$	c^2e^2	$-\ 60$	cd^2e	$-\ 200$	c^3d^2f	$+\ 10$
b^1c^4e	$-\ 15$	b^1c^4f	$-\ 12$	c^2d^2e	$\ldots$	d^5	$+\ 60$	c^3de^2	$+\ 20$
c^3d^2	$+\ 10$	c^3de	$+\ 200$	cd^4	$-\ 30$	b^0c^4df	$-\ 36$	c^2d^3e	$-\ 10$
b^0c^5d	$\ldots$	c^2d^3	$-\ 120$	b^0c^3f	$-\ 54$	c^4e^2	$-\ 60$	cd^5	$\ldots$
		b^0c^5e	$-\ 60$	c^4de	$+\ 30$	c^3d^2e	$+\ 120$		
		c^4d^2	$+\ 40$	c^3d^3	$\ldots$	c^3d^4	$-\ 40$		
	$\pm\ 1$								
	19		$\pm\ 12$		$\pm\ 12$		$\pm\ 8$		$\pm\ 3$
	81		192		270		132		37
	62		432		306		496		123
	$\pm\ 163$		$\pm\ 636$		$\pm\ 588$		$\pm\ 636$		$\pm\ 163$

$\mathfrak{Q}(x, y)^4$

O. No. 90.

$a^3\ b^0cf^3$	+ 1	$a^3\ b^0df^3$	− 1		
def^2	− 4	e^2f^2	+ 1		
e^3f	+ 3	$a^2\ b^1cf^3$	+ 4		
$a^2\ b^2f^3$	− 1	def^2	+ 3		
b^1cef^2	− 3	e^3f	− 7		
d^2f^2	+ 16	$b^0c^2ef^2$	− 16		
de^2f	+ 4	cd^2f^2	+ 6		
e^4	− 15	cde^2f	+ 30		
$b^0c^2df^2$	− 6	ce^4	− 8		
c^2e^2f	+ 4	d^3ef	− 18		
cd^2ef	− 22	d^2e^3	+ 6		
cde^3	+ 26	$a^1\ b^3f^3$	− 3		
d^4f	+ 9	b^2cef^2	− 4		
d^3e^2	− 12	d^2f^2	− 4		
$a^1\ b^3ef^2$	+ 7	de^3f	− 1		
b^2cdf^2	− 30	e^4	+ 18		
ce^2f	+ 1	$b^1c^2df^2$	+ 22		
d^2ef	− 74	c^2e^2f	+ 74		
de^3	+ 84	cd^2ef	− 160		
$b^1c^3f^2$	+ 18	cde^3	− 32		
c^2def	+ 160	d^4f	+ 81		$\mathfrak{X}(x, y)^1$
c^2e^3	− 98	d^3e^2	+ 6		
cd^3f	− 20	$b^0c^4f^2$	− 9		
cd^2e^2	− 94	c^3def	+ 20		
d^4e	+ 51	c^3e^3	− 112		
b^0c^4ef	− 81	c^2d^3f	− 18		
c^3d^2f	+ 18	$c^2d^2e^2$	+ 284		
c^3de^2	+ 140	cd^4e	− 216		
c^2d^3e	− 100	d^5	+ 54		
cd^5	+ 18	$a^0\ b^4ef^2$	+ 15		
$a^0\ b^4df^2$	+ 8	b^3cdf^2	− 26		
e^2f	− 18	ce^2f	− 84		
$b^3c^2f^2$	− 6	d^2ef	+ 98		
$cdef$	+ 32	de^3	− 45		
ce^3	+ 45	$b^3c^2f^2$	+ 12		
d^3f	+ 112	c^3def	+ 94		
d^2e^2	− 150	c^2e^3	+ 150		
b^3c^3ef	− 6	cd^3f	− 140		
c^3d^2f	− 284	cd^2e^2	− 50		
c^3de^2	+ 50	d^4e	+ 15		
cd^3e	+ 320	b^1c^4ef	− 51		
d^5	− 120	c^3d^2f	+ 100		
b^1c^4df	+ 216	c^3de^2	− 320		
c^4e^2	− 15	c^2d^3e	+ 310		
c^3d^2e	− 310	cd^5	− 90		
c^2d^4	+ 130	b^0c^5df	− 18		
b^0c^5f	− 54	c^5e^2	+ 120		
c^5de	+ 90	c^4d^2e	− 130		
c^4d^3	− 40	c^3d^4	+ 40		
	± 4		± 1		
	59		49		
	497		559		
	1003		954		
	±1563		±1563		

P. No. 91.

$a^4\ b^0f^3$ ...	$a^3\ b^2f^2$ ...	$a^2\ b^0cf^3$ − 1	$a^3\ b^0df^3$ + 1	$a^2\ b^0ef^3$ ...	$a^3\ b^0f^4$ ...
$a^3\ b^1ef^2$ ...	b^0cef^2 − 2	def^2 + 6	e^2f^2 − 1	$a^2\ b^1df^3$ + 2	$a^2\ b^1ef^3$ ...
b^0cdf^2 + 1	d^2f^2 + 5	e^3f − 5	$a^2\ b^1cf^3$ − 6	e^2f^2 − 2	b^0cdf^3 − 1
ce^2f − 2	de^2f − 1	$a^2\ b^2f^3$ + 1	def^2 + 11	$b^0c^2f^3$ − 5	ce^2f^2 + 1
d^2ef + 2	e^4 − 2	b^1cef^2 − 11	e^3f − 5	$cdef^2$ + 17	d^2ef^2 + 3
de^3 − 1	$a^2\ b^2ef^2$ + 2	d^2f^2 − 4	$b^0c^2ef^2$ + 4	ce^2f − 7	de^3f − 5
$a^3\ b^2df^2$ − 1	b^1cdf^2 − 17	de^2f − 4	cd^2f^2 − 2	d^3f^2 − 4	e^5 + 2
e^3f + 2	ce^2f + 13	e^4 + 17	cde^2f + 4	d^2e^2f − 6	$a^1\ b^2df^3$ + 2
$b^1c^2f^2$ − 3	d^2ef − 32	$b^0c^3df^2$ + 2	ce^4 − 4	de^4 + 5	e^2f^2 − 2
$cdef$ − 6	de^3 + 32	c^2e^2f + 26	d^2ef − 10	$a^1\ b^3cf^3$ + 1	$b^1c^2f^3$ − 2
ce^3 + 13	$b^0c^3f^2$ + 4	cd^2ef − 2	d^2e^3 + 8	def^2 − 13	$cdef^2$ + 6
d^2f − 8	c^2def + 36	cde^2 − 40	$a^1\ b^3f^3$ + 5	e^3f + 12	ce^2f − 2
d^2e^2 + 2	c^2e^3 − 24	d^4f − 9	b^2cef^2 + 4	$b^1c^2ef^2$ + 32	d^3f^2 − 16
b^0c^3ef + 16	cd^2f − 10	d^3e^2 + 24	d^3f^2 − 26	cd^2f^2 − 36	d^2e^2f + 24
c^3d^2f − 2	cd^2e^2 − 16	$a^1\ b^3ef^2$ + 5	de^2f − 35	cde^2f − 42	de^4 − 10
c^3de^2 − 38	d^4e + 12	b^2cdf^2 − 4	e^4 + 42	ce^4 + 24	$b^0c^3ef^2$ + 8
cd^2e + 34	$a^1\ b^3df^2$ + 7	ce^2f + 35	$b^1c^3df^2$ + 2	d^2ef + 56	$c^3d^2f^2$ + 2
d^5 − 9	e^2f − 12	d^2ef − 26	c^2e^2f + 26	d^2e^3 − 34	c^3de^2f − 52
$a^1\ b^3cf^2$ + 5	$b^2c^2f^2$ + 6	de^3 − 22	cd^2ef + 72	$b^0c^3df^2$ + 10	c^2e^4 + 28
def + 2	$cdef$ + 42	$b^1c^3f^2$ + 10	cde^3 − 124	c^3e^2f − 54	cd^2ef + 52
e^3 − 12	ce^3 ...	c^2def − 72	d^4f + 13	c^3d^2ef + 64	cd^2e^3 − 32
b^2c^2ef − 24	d^3f + 54	c^2e^3 − 106	d^3e^2 + 26	c^2de^3 + 46	d^3f − 18
cd^2f + 52	d^2e^2 − 91	cd^3f + 76	$b^0c^4f^2$ + 9	cd^1f − 37	d^4e^2 + 12
cde^2 + 7	b^1c^3ef − 68	cd^2e^2 + 210	c^3def − 76	cd^2e^2 − 50	$a^0\ b^3cf^2$ + 1
d^3e − 22	c^3d^2f − 64	d^4e − 99	c^3e^3 − 56	d^3e + 21	def^2 − 13
b^1c^3df − 52	c^3de^2 + 14	b^0c^4ef − 13	c^2d^3f + 10	$a^0\ b^4f^3$ + 2	e^3f + 12
c^3e^2 + 34	cd^3e + 204	c^3d^2f − 10	$c^2d^2e^2$ + 296	b^3cef^2 − 32	$b^2c^2ef^2$ − 2
c^2d^2e + 8	d^5 − 93	c^3de^2 + 128	cd^1e − 260	d^1f^2 + 24	cd^4f^2 + 38
cd^4 − 1	b^0c^4df + 37	c^2d^3e − 184	d^6 + 72	de^2f ...	cde^2f − 7
b^0c^2f + 18	c^4e^2 + 86	cd^5 + 72	$a^0\ b^5ef^2$ − 17	e^4 ...	ce^4 − 30
c^4de − 25	c^3d^2e − 208	$a^0\ b^4df^2$ + 4	b^3cdf^2 + 40	$b^2c^3df^2$ + 16	d^3ef − 34
c^3d^3 + 10	c^3d^4 + 86	e^3f − 42	ce^2f + 22	c^2e^2f + 91	d^2e^3 + 35
$a^0\ b^3f^2$ − 2	$a^0\ b^4ef^2$ − 5	$b^2c^2f^2$ − 8	d^2ef + 106	cd^2ef − 14	$b^1c^3df^2$ − 34
b^4cef + 10	def − 12	$cdef$ + 124	de^3 − 105	cde^3 − 105	c^2e^2f + 22
d^2f − 28	e^3 ...	ce^3 + 105	$b^2c^3f^2$ − 24	d^4f − 86	c^3d^2ef − 8
de^2 + 30	b^3c^2ef + 34	d^3f + 56	c^3def − 210	d^3e^2 + 110	c^3de^2 + 50
b^3c^2df + 32	cd^2f − 46	d^2e^2 − 130	c^2e^3 + 130	$b^1c^4f^2$ − 12	cd^4f + 25
c^2e^2 − 35	cde^2 + 105	b^2c^3ef − 26	cd^3f − 128	c^3def − 204	cd^3e^2 − 70
cd^2e − 50	d^3e − 20	c^2d^2f − 296	cd^2e^2 + 170	c^3e^3 + 20	d^5e + 15
d^4 + 30	b^3c^3df + 50	c^2de^2 − 170	d^4e − 25	c^2d^3f + 208	$b^0c^4f^2$ + 9
b^3c^3f − 12	c^3e^2 − 110	cd^3e + 340	b^1c^4ef + 99	$c^2d^2e^2$ + 170	c^4def + 1
c^2de + 70	c^2d^2e − 170	d^5 − 60	c^3d^2f + 184	cd^3e − 250	c^4e^3 − 30
c^2d^3 − 40	cd^4 + 115	b^1c^4df + 260	c^3de^2 − 340	d^6 + 60	c^3d^3f − 10
b^1c^3e − 15	b^1c^3f − 21	c^4e^2 + 25	c^3d^2e + 150	b^0c^5ef + 93	$c^3d^2e^2$ + 40
c^4d^2 + 10	c^4de + 250	c^3d^2e − 150	cd^5 − 40	c^4d^2f − 86	c^2d^4e − 10
b^0c^3d ...	c^3d^3 − 150	c^2d^4 ...	b^0c^5df − 72	c^4de^2 − 115	cd^6 ...
	b^0c^6e − 60	b^0c^6f − 72	c^5e^2 + 60	c^3d^3e + 150	
	c^5d^2 + 40	c^5de + 40	c^4d^2e ...	c^3d^5 − 40	
		c^4d^3 ...	c^3d^4 ...		
± 3	± 5	± 6	± 1		
67	99	70	27	± 24	± 6
136	536	536	577	266	134
182	594	954	961	944	248
± 388	±1234	±1566	±1566	±1234	±388

$\mathfrak{Q}(x,\ y)^5$

	Q. No. 25.	Q′. No. 26.
$a^4\ b^0f^4$	...	+ 1
$a^3\ b^1ef^3$	...	− 20
b^0cdf^3	+ 1	− 120
ce^2f^2	− 1	+ 160
d^2ef^2	− 3	+ 360
de^3f	+ 5	− 640
e^5	− 2	+ 256
$a^2\ b^2df^3$	− 1	+ 160
e^2f^2	+ 1	− 10
$b^1c^2f^3$	− 3	+ 360
$cdef^2$	+ 11	− 1640
ce^3f	− 5	+ 320
d^3f^2	+ 12	− 1440
d^2e^2f	− 30	+ 4080
de^4	+ 15	− 1920
$b^0c^3ef^2$	+ 12	− 1440
$c^2d^2f^2$	− 21	+ 2640
c^2de^2f	− 34	+ 4480
c^2e^4	+ 22	− 2560
cd^3ef	+ 78	− 10080
cd^2e^3	− 48	+ 5760
d^5f	− 27	+ 3456
d^4e^2	+ 18	− 2160
$a^1\ b^3cf^3$	+ 5	− 640
def^2	− 5	+ 320
e^3f	...	− 180
$b^2c^3ef^2$	− 30	+ 4080
cd^3f^2	− 34	+ 4480
cde^2f	+ 133	− 14920
ce^4	− 54	+ 7200
d^3ef	− 18	+ 960
d^2e^3	+ 3	− 600
$b^1c^3df^2$	+ 78	− 10080
c^2e^2f	− 18	+ 960
c^2d^2ef	− 220	+ 28480
c^2de^3	+ 106	− 16000
cd^4f	+ 93	− 11520
cd^3e^2	− 30	+ 7200
d^5e	− 9	...
$b^0c^5f^2$	− 27	+ 3456
c^4def	+ 93	− 11520
c^4e^3	− 38	+ 6400
c^3d^2f	− 42	+ 5120
$c^3d^2e^2$	+ 8	− 3200
c^2d^3e	+ 6	...
cd^6	...	...
$a^0\ b^5f^3$	− 2	+ 256
b^4cef^2	+ 15	+ 1920
d^2f^2	+ 22	− 2560
de^2f	− 54	+ 7200

	Q. No. 25.	Q′. No. 26.
$a^0\ b^4e^4$	+ 27	− 3375
$b^3c^2df^2$	− 48	+ 5760
c^2e^2f	+ 3	− 600
cd^2ef	+ 106	− 16000
cde^3	− 81	+ 9000
d^4f	− 38	+ 6400
d^3e^2	+ 38	− 4000
$b^3c^4f^2$	+ 18	− 2160
c^3def	− 30	+ 7200
c^3e^3	+ 38	− 4000
c^2d^3f	+ 8	− 3200
$c^2d^2e^2$	+ 25	+ 2000
cd^4e	− 57	...
d^5	+ 18	...
b^1c^5ef	− 9	...
c^4d^2f	+ 6	...
c^4de^2	− 57	...
c^3d^3e	+ 74	...
c^2d^5	− 24	...
$b^0\ c^6df$	...	...
c^6e^2	+ 18	...
c^5d^2e	− 24	...
c^4d^4	+ 8	...

The sums for Q' are

1	= 1
776 −	780 = − 4
21256 −	21250 = + 6
68656 −	68660 = − 4
37816 −	37815 = + 1
128505 −	128505 = 0

± 6	± 128505
169	
525	
424	
± 1124	

R. No. 92.

$a^4\ b^0ef^3$	...	$a^0\ b^4ce^2f$	-15
$a^3\ b^1df^3$	...	d^2ef	-38
e^2f^2	...	de^3	$+45$
$b^0c^2f^3$	-1	$b^3c^3f^2$	$+3$
$cdef^2$	$+6$	c^3def	$+102$
ce^3f	-4	c^2e^3	-15
d^3f^2	-3	cd^2f	$+76$
d^2e^2f	$+1$	cd^2e^2	-175
de^4	$+1$	d^4e	$+35$
$a^2\ b^3cf^3$	$+2$	b^5c^4ef	-42
def^2	-6	c^3d^2f	-182
e^3f	$+4$	c^2de^2	$+120$
$b^1c^3ef^2$	-3	c^3d^3e	$+150$
cd^2f^2	$+3$	cd^5	-70
cde^2f	-18	b^1c^5df	$+126$
ce^4	$+17$	c^3e^2	-15
d^2ef	$+22$	c^4d^2e	-175
d^2e^3	-21	c^3d^4	$+75$
$b^0c^3df^2$	...	b^0c^7f	-27
c^2e^2f	$+13$	c^3de	$+45$
c^2d^2ef	-12	c^3d^3	-20
c^2de^3	-21		
cd^4f	-3		
cd^3e^2	$+32$		
d^5e	-9		
$a^1\ b^4f^3$	-1		
b^2cef^2	...		
d^2f^2	$+6$		
de^2f	$+16$		
e^4	-18		
$b^2c^3df^2$	-3		
c^2e^2f	$+3$		
cd^2ef	-18		
cde^3	$+14$		
d^4f	-41		
d^3e^2	$+39$		
$b^1c^4f^2$	...		
c^3def	-32		
c^3e^3	-2		
c^2d^3f	$+84$		
$c^2d^2e^2$	$+24$		
cd^4e	-106		
d^6	$+36$		
b^0c^5ef	$+18$		
c^4d^2f	-33		± 8
c^4de^2	-25		93
c^3d^3e	$+60$		800
c^2d^5	-21		780
$a^0\ b^5ef^2$	$+3$		± 1181
b^1cdf^2	-6		

$a^4\ b^0f^4$	...	$a^0\ b^4e^4$	...
$a^3\ b^1ef^3$	...	$b^3c^3df^2$	$+18$
b^0cdf^3	...	c^3e^2f	-66
ce^2f^2	...	cd^2ef	$+20$
d^2ef^2	$+2$	cde^3	...
de^3f	-4	d^4f	$+58$
e^5	$+2$	d^3e^2	-50
$a^2\ b^2df^2$	...	$b^2c^4f^2$	-6
e^2f^2	...	c^3def	$+72$
$b^1c^2f^3$	-2	c^3e^3	$+50$
$cdef^2$	...	c^2d^2f	-156
ce^3f	$+4$	$c^2d^2e^2$	...
d^3f^2	-14	cd^4e	$+90$
d^2e^2f	$+30$	d^6	-30
de^4	-18	b^1c^5ef	-24
$b^0c^2ef^2$	$+14$	c^4d^2f	$+94$
$c^2d^2f^2$	...	c^4de^2	-90
c^2de^2f	-66	c^3d^2e	...
c^2e^4	$+26$	c^2d^5	$+10$
cd^3ef	$+56$	b^1c^6df	-18
cd^2e^3	-18	c^6e^2	$+30$
d^3f	-18	c^5d^2e	-10
d^3e^2	$+6$	c^4d^4	...
$a^1\ b^3cf^2$	$+4$		
def^2	-4		
e^3f	...		
$b^0c^2ef^2$	-30		
cd^2f^2	$+66$		
cde^2f	...		
ce^4	-18		
d^3ef	-84		
d^2e^3	$+66$		
$b^1c^3df^2$	-56		
c^3e^2f	$+84$		
c^2d^2ef	...		
c^2de^3	-20		
cd^4f	$+40$		
cd^3e^2	-72		
d^5e	$+24$		
$b^0c^3f^2$	$+18$		
c^4def	-40		
c^4e^3	-58		
c^3d^2f	...		
$c^3d^2e^2$	$+156$		
c^3d^3e	-94		± 4
cd^6	$+18$		136
$a^0\ b^2f^3$	-2		476
b^4cef^2	$+18$		478
d^2f^2	-26		± 1094
de^2f	$+18$		

$a^3\ b^1f^4$	...	$a^0\ b^3d^2ef$	$+2$
b^0cef^3	...	d^2e^3	$+15$
d^2f^3	$+1$	$b^3c^3df^2$	-32
de^2f^2	-2	c^2e^2f	-39
e^4f	$+1$	c^2d^2ef	-24
$a^2\ b^2ef^3$	...	c^2de^3	$+175$
b^1cdf^3	-6	cd^4f	$+25$
ce^2f^2	$+6$	cd^3e^2	-120
d^2ef^2	$+3$	d^5e	$+15$
de^3f	...	$b^1c^5f^2$	$+9$
e^5	-3	c^4def	$+106$
$b^0c^2f^3$	$+3$	c^4e^3	-35
c^2def^2	-3	c^3d^2f	-60
c^2e^3f	-6	$c^3d^2e^2$	-150
cd^3f^2	...	c^2d^4e	$+175$
cd^2e^2f	$+3$	cd^6	-45
cde^4	$+6$	b^0c^6ef	-36
d^4ef	...	c^5d^2f	$+21$
d^3e^3	-3	c^5de^2	$+70$
$a^1\ b^3df^2$	$+4$	c^4d^3e	-75
e^2f^2	-4	c^3d^5	$+20$
$b^2c^2f^3$	-1		
$cdef^2$	$+18$		
ce^3f	-16		
d^3f^2	-13		
d^2e^2f	-3		
de^4	$+15$		
$b^1c^3ef^2$	-22		
$c^2d^2f^2$	$+12$		
c^2de^2f	$+18$		
c^2e^4	$+38$		
cd^3ef	$+32$		
cd^2e^3	-102		
d^5f	-18		
d^4e^2	$+42$		
$b^0c^4df^2$	$+3$		
c^4e^2f	$+41$		
c^3d^2ef	-84		
c^3de^3	-76		
c^2d^4f	$+33$		
$c^2d^3e^2$	$+182$		
cd^5e	-126		
d^7	$+27$		
$a^0\ b^4cf^3$	-1		
def^2	-17		± 2
e^3f	$+18$		21
$b^3c^2ef^2$	$+21$		465
cd^2f^2	$+21$		693
cde^2f	-14		± 1181
ce^4	-45		

$\Upsilon(x, y)^2$.

S. No. 93 *bis*; S′. No. 93. $(*\mathfrak{Q}x, y)^3$.

Coef. x^3	S	S′	Coef. x^3	S	S′	Coef. x^2y	S	S′	Coef. x^2y	S	S′
$a^4\ b^1f^4$	...	...	$a^1\ b^3d^3ef$	− 66	+ 528	$a^4\ b^0cf^4$	...	+ 9	$a^1\ b^3c^3e^4$	+ 66	+ 12960
b^0cef^3	...	+ 9	d^3e^3	+ 72	− 45	def^3	...	− 45	cd^5ef	+ 78	+ 18612
d^2f^2	...	+ 21	$b^2c^2df^2$	− 21	− 2592	e^3f^2	...	+ 36	cd^2e^3	− 186	− 18900
de^2f^2	...	− 78	c^3e^2f	− 96	− 9747	$a^3\ b^2f^4$	...	− 9	d^3f	+ 51	− 3888
c^4f	...	+ 48	c^2d^2ef	+ 36	− 8496	b^1cef^3	...	− 18	d^4e^2	− 9	+ 2970
$a^3\ b^2ef^3$	...	− 9	c^2de^3	+ 213	+ 26610	d^2f^3	...	+ 243	$b^1c^4df^2$	+ 111	+ 15228
b^1cdf^3	...	− 162	cd^4f	+ 120	+ 8544	de^2f^2	...	+ 9	c^4e^2f	− 78	− 4968
ce^2f^2	...	+ 99	cd^3e^2	− 303	− 16650	e^4f	...	− 216	c^3d^2ef	− 36	− 14544
d^2ef^2	...	+ 309	d^5e	+ 51	+ 720	$b^0c^2df^3$	− 3	− 351	c^3de^3	− 54	− 12960
de^3f	...	+ 12	$b^1c^3f^2$	+ 9	+ 972	$c^2e^2f^2$	+ 3	+ 144	c^3d^4f	− 96	+ 1296
e^5	...	− 240	c^4def	+ 174	+ 24624	cd^2ef^2	+ 24	+ 1836	$c^2d^3e^2$	+ 150	+ 22500
$b^0e^3f^3$	− 2	− 81	c^4e^3	− 36	− 5040	cde^3f	− 42	− 2592	cd^5e	+ 30	− 6480
c^2def^2	+ 15	+ 1026	c^3d^3f	− 204	− 15984	ce^5	+ 18	+ 1152	d^7	− 27	...
c^2e^3f	− 9	− 768	$c^3d^2e^2$	− 174	− 29340	d^4f^2	− 18	− 1458	$b^0c^5f^2$	− 27	− 3888
cd^2f^2	− 9	− 738	c^2d^4e	+ 330	+ 34320	d^3e^2f	+ 33	+ 2268	c^5def	+ 24	+ 5184
cd^2e^2f	− 6	− 564	cd^6	− 99	− 8640	d^2e^4	− 15	− 1008	c^5e^3	+ 54	+ 5760
cde^4	+ 9	+ 1056	b^0c^4ef	− 63	− 7776	$a^2\ b^3ef^3$	...	+ 63	c^4d^3f	+ 27	− 576
d^4ef	+ 9	+ 756	c^3d^2f	+ 66	+ 5184	b^3cdf^3	+ 6	− 234	$c^4d^2e^2$	− 93	− 9360
d^2e^3	− 7	− 696	c^3de^2	+ 99	+ 12960	ce^2f^2	− 6	− 18	c^3d^4e	+ 6	+ 2880
$a^2\ b^3df^3$	...	+ 120	c^4d^2e	− 147	− 14400	d^2ef^2	− 24	− 3231	c^3d^5	+ 9	...
e^3f^2	...	− 21	c^3d^5	+ 45	+ 3840	de^3f	+ 42	+ 4293	$a^0\ b^5ef^3$	+ 3	+ 288
$b^2c^2f^3$	+ 6	+ 486	$a^0\ b^6f^3$	+ 2	+ 192	e^5	− 18	− 972	def^2	− 30	− 3888
$cdef^2$	− 30	− 2160	b^5cef^2	− 15	− 1440	$b^1c^3f^3$	+ 3	+ 810	e^3f	+ 27	+ 3645
ce^3f	+ 18	+ 1023	d^2f^2	− 6	− 192	c^2def^2	− 78	− 3825	$b^4c^2ef^2$	...	+ 756
d^3f^2	+ 9	+ 120	de^2f	− 18	− 1080	c^2e^2f	+ 69	+ 4032	cd^2f^2	+ 51	+ 7488
d^2e^2f	+ 6	− 1053	e^4	+ 27	+ 2025	cd^2f^2	+ 93	+ 7938	cde^2f	− 39	− 4050
de^4	− 9	+ 1314	$b^4c^3df^2$	+ 24	+ 1728	cd^3e^2f	− 51	− 9360	ce^4	− 27	− 6072
$b^1c^3ef^2$	− 15	− 1863	c^2e^2f	+ 51	+ 4410	cde^4	− 33	− 864	d^3ef	+ 60	− 4320
$c^3d^2f^2$	+ 21	+ 2538	cd^2ef	+ 102	+ 5280	d^4ef	− 57	− 1296	d^2e^3	− 45	+ 6072
c^3de^2f	− 6	+ 2340	cde^3	− 171	− 13500	d^3e^3	+ 54	+ 2700	$b^3c^3df^2$	− 39	− 7128
c^2e^4	+ 18	+ 672	d^4f	+ 6	− 4800	$b^0c^4ef^2$	+ 24	− 324	c^3e^2f	+ 45	+ 2970
cd^3ef	+ 30	+ 2820	d^3e^2	+ 18	+ 7800	$c^3d^2f^2$	− 36	− 2484	c^2d^2ef	− 108	+ 3060
cd^2e^3	− 51	− 7812	$b^3c^4f^2$	− 9	− 648	c^3de^2f	− 9	+ 6624	c^2de^3	+ 96	+ 10128
d^3f	− 36	− 3024	c^3def	− 210	− 14040	c^3e^4	− 54	− 6912	cd^4f	− 111	+ 1440
d^4e^2	+ 39	+ 4572	c^3e^3	+ 43	+ 3075	c^2d^2ef	+ 24	− 4428	cd^3e^2	+ 147	− 13950
$b^0c^4df^2$	− 3	− 324	c^2d^3f	− 120	+ 9120	$c^2d^2e^3$	+ 129	+ 12672	d^5e	− 30	+ 3600
c^4e^2f	+ 45	+ 3888	$c^2d^2e^2$	+ 345	+ 16350	cd^3f	+ 9	+ 1944	$b^3c^3f^2$	+ 9	+ 194
c^3d^2ef	− 84	− 8748	cd^4e	− 87	− 19200	cd^4e^2	− 114	− 9072	c^4def	+ 6	− 162
c^3de^3	− 63	− 4800	d^6	− 2	+ 4800	d^5e	+ 27	+ 1944	c^4e^3	− 48	− 450
c^2d^4f	+ 45	+ 4248	b^2c^5ef	+ 72	+ 4860	$a^1\ b^4df^3$	− 3	+ 144	c^3d^3f	+ 234	− 36
$c^2d^2e^2$	+ 150	+ 14520	c^4d^2f	+ 240	− 3240	e^4f^2	+ 3	− 243	$c^3d^2e^2$	− 150	+ 630
cd^3e	− 117	− 11448	c^4de^2	− 192	− 8100	$b^3c^2f^3$	− 6	− 900	c^3d^4e	− 108	− 180
d^7	+ 27	+ 2592	c^3d^3e	− 186	+ 9000	$cdef^2$	+ 108	+ 10620	cd^6	+ 57	...
$a^1\ b^4ef^3$	− 6	− 576	c^2d^5	+ 96	− 2400	ce^2f	− 96	− 8586	b^1c^6ef	+ 9	...
def^2	+ 15	+ 672	b^1c^6df	− 144	...	d^3f^2	− 21	− 864	c^6d^2f	− 141	...
e^3f	− 9	− 459	c^6e^2	+ 18	...	d^2e^2f	− 48	− 1215	c^6de^2	+ 87	...
$b^2c^2ef^2$	+ 30	+ 3456	c^5d^2e	+ 201	...	de^4	+ 63	+ 1215	c^4d^3e	+ 96	...
cd^2f^2	− 15	− 864	c^4d^4	− 87	...	$b^2c^3ef^2$	− 24	− 1836	c^3d^5	− 51	...
cde^2f	+ 24	+ 2094	b^0c^5f	+ 27	...	$c^2d^2f^2$	− 123	− 16812	b^0c^7df	+ 27	...
ce^4	− 45	− 3915	c^7de	− 45	...	c^2de^2f	+ 147	+ 6651	c^7e^2	− 18	...
			c^6d^3	+ 20	...				c^6d^2e	− 21	...
									c^5d^4	+ 12	...
				± 78							±
				± 33	3258					± 78	56
				414	41253					480	430
				1284	124524					927	1060
				1292	68640					966	476
				± 3023	± 237753					± 2451	± 2024

For the Numerical Verifications for S see further pp. 304, 305.

S. No. 93 *bis*; S′. No. 93.

Coef. xy^2	S	S′	Coef. xy^2	S	S′	Coef. y^3	S	S′	Coef. y^3	S	S′
$t^4\ b^0df^4$	...	− 9	$a^1\ b^1c^2e^4$	− 60	+ 4320	$a^4\ b^0ef^4$	...	...	$a^1\ b^1d^2e^2$	− 72	− 4860
e^2f^3	...	+ 9	c^2d^3ef	+ 36	+ 14544	$a^3\ b^1df^4$	...	− 9	$b^0c^2ef^2$	+ 36	+ 3024
$t^3\ b^1cf^4$	..	+ 45	$c^2d^2e^3$	+ 108	− 3060	e^2f^3	...	+ 9	$c^4d^2f^2$	− 45	− 4248
def^3	...	+ 18	cd^5f	− 24	− 5184	$b^0c^2f^4$	...	− 21	c^4de^2f	− 120	− 8544
e^3f^2	...	− 63	cd^4e^2	− 6	+ 1620	$cdef^3$	...	+ 162	c^4e^4	− 6	+ 4800
$b^0c^2ef^3$	...	− 243	d^6e	− 9	...	ce^2f^2	...	− 120	c^3d^2ef	+ 204	+ 15984
cd^2f^2	+ 3	+ 351	$b^0c^3df^2$	− 9	− 1944	d^2f^3	+ 2	+ 81	$c^3d^2e^3$	+ 120	− 9120
cde^2f^2	− 6	+ 234	c^5e^2f	− 51	+ 3888	$d^2e^2f^2$	− 6	− 486	c^2d^3f	− 66	− 5184
ce^4f	+ 3	− 144	c^4d^2ef	+ 96	− 1296	de^4f	+ 6	+ 576	$c^2d^4e^2$	− 240	+ 3240
d^2ef^2	− 3	− 810	c^4de^3	+ 111	− 1440	e^6	− 2	− 192	cd^5e	+ 144	...
d^2e^3f	+ 6	+ 900	c^3d^4f	− 27	+ 576	$a^2\ b^2cf^4$	...	+ 78	d^8	− 27	...
de^5	− 3	− 288	$c^3d^2e^2$	− 234	+ 360	def^3	...	− 99	$a^0\ b^2ef^3$	...	+ 240
$^2\ b^2f^4$	...	− 36	c^2d^5e	+ 141	...	e^3f^2	...	+ 21	b^4cdf^3	− 9	− 1056
b^2cef^3	...	− 9	cd^7	− 27	...	$b^1c^2ef^3$	...	− 309	ce^3f^2	+ 9	− 1314
d^2f^3	− 3	− 144	$a^0\ b^3df^3$	− 18	− 1152	cd^2f^3	− 15	− 1026	d^2ef^2	− 18	− 672
de^2f^2	+ 6	+ 18	e^3f^2	+ 18	+ 972	cde^2f^2	+ 30	+ 2160	de^3f	+ 45	+ 3915
e^4f	− 3	+ 243	$b^4c^2f^3$	+ 15	+ 1008	ce^4f	− 15	− 672	e^5	− 27	− 2025
$b^1c^2df^3$	− 24	− 1836	$cdef^2$	+ 33	+ 864	d^2ef^2	+ 15	+ 1863	$b^3c^2f^3$	+ 7	+ 696
$c^2e^2f^2$	+ 24	+ 3231	ce^3f	− 63	− 1215	d^2e^3f	− 30	− 3456	c^2def^2	+ 51	+ 7812
cd^2ef^2	+ 78	+ 3825	d^3f^2	+ 54	+ 6912	de^5	+ 15	+ 1440	c^2e^3f	− 72	+ 45
cde^2f	− 108	− 10620	d^2e^2f	− 66	− 12960	$b^0c^3df^3$	+ 9	+ 738	cd^3f^2	+ 63	+ 4800
ce^3	+ 30	+ 3888	de^4	+ 27	+ 6075	$c^3e^2f^2$	− 9	− 120	cd^2e^2f	− 213	− 26610
d^4f^2	− 24	+ 324	$b^1c^3ef^2$	− 54	− 2700	$c^2d^2ef^2$	− 21	− 2538	cde^4	+ 171	+ 13500
d^3e^2f	+ 24	+ 1836	$c^2d^3f^2$	− 129	− 12672	c^2de^3f	+ 15	+ 864	d^4ef	+ 36	+ 5040
d^2e^4	...	− 756	c^2de^2f	+ 186	+ 18900	c^2e^5	+ 6	+ 192	d^3e^3	− 43	− 3075
$b^0c^4f^3$	+ 18	+ 1458	c^2e^4	+ 45	− 6075	cd^4f^2	+ 3	+ 324	$b^2c^4ef^2$	− 39	− 4572
c^3def^2	− 93	− 7938	cd^3ef	+ 54	+ 12960	cd^3e^2f	+ 21	+ 2592	$c^3d^2f^2$	− 150	− 14520
c^3e^3f	+ 21	+ 864	cd^2e^3	− 96	− 10125	cd^2e^4	− 24	− 1728	c^3de^2f	+ 303	+ 16650
$c^2d^3f^2$	+ 36	+ 2484	d^5f	− 54	− 5760	d^5ef	− 9	− 972	c^3e^4	− 18	− 7800
$c^2d^2e^2f$	+ 123	+ 16812	d^4e^2	+ 48	+ 4500	d^4e^3	+ 9	+ 648	c^2d^3ef	+ 174	+ 29340
c^2de^4	− 51	− 7488	$b^2c^4df^2$	+ 114	+ 9072	$a^1\ b^4f^4$	...	− 48	$c^2d^2e^3$	− 345	− 16350
cd^3ef	− 111	− 15228	c^4e^2f	+ 9	− 2970	b^2cef^2	...	− 12	cd^5f	− 99	− 12960
cd^2e^3	+ 39	+ 7128	c^3d^2ef	− 150	− 22500	d^2f^3	+ 9	+ 768	cd^4e^2	+ 192	+ 8100
d^6f	+ 27	+ 3888	c^3de^3	− 147	+ 13950	de^2f^2	− 18	− 1023	d^6e	− 18	...
d^5e^2	− 9	− 1944	c^3d^4f	+ 93	+ 9360	e^4f	+ 9	+ 459	$b^1c^6df^2$	+ 117	+ 11448
b^4ef^3	...	+ 216	$c^2d^3e^2$	+ 150	− 6300	$b^2c^2df^3$	+ 6	+ 564	c^6e^2f	− 51	− 720
b^3cdf^3	+ 42	+ 2592	cd^5e	− 87	...	$c^2e^2f^2$	− 6	+ 1053	c^4d^2ef	− 330	− 34320
ce^2f^2	− 42	− 4293	d^7	+ 18	...	cd^2ef^2	+ 6	− 2340	c^4de^3	+ 87	+ 19200
d^2ef^2	− 69	− 4032	$b^1c^4f^2$	− 27	− 1944	cde^2f	− 24	− 2094	c^3d^4f	+ 147	+ 14400
de^2f	+ 96	+ 8586	c^5def	− 30	+ 6480	ce^5	+ 18	+ 1080	$c^3d^2e^2$	+ 186	− 9000
e^5	− 27	− 3645	c^5e^3	+ 30	− 3600	d^3f^2	− 45	− 3888	c^2d^3e	− 201	...
$b^2c^3f^3$	− 33	− 2268	c^4d^2f	− 6	− 2880	d^2e^2f	+ 96	+ 9747	cd^7	+ 45	...
c^2def^2	+ 51	+ 9360	$c^4d^2e^2$	+ 108	+ 1800	d^2e^4	− 51	− 4410	$b^0c^7f^2$	− 27	− 2592
c^2e^3f	+ 48	+ 1215	c^3d^4e	− 96	...	$b^1c^4f^3$	− 9	− 756	c^6def	+ 99	+ 8640
cd^3f^2	+ 9	− 6624	c^2d^6	+ 21	...	c^5def^2	− 30	− 2820	c^6e^3	+ 2	− 4800
cd^2e^2f	− 147	− 6651	b^0c^7ef	+ 27	...	c^5e^2f	+ 66	− 528	c^5d^2f	− 45	− 3840
cde^4	+ 39	+ 4050	c^6d^2f	− 9	...	$c^2d^3f^2$	+ 84	+ 8748	$c^5d^2e^2$	− 96	+ 2400
d^4ef	+ 78	+ 4968	c^6de^2	− 57	...	$c^2d^2e^2f$	− 36	+ 8496	c^4d^3e	+ 87	...
d^3e^3	− 45	− 2970	c^5d^2e	+ 51	...	c^2de^4	− 102	− 5280	c^3d^5	− 20	...
$b^1c^4ef^2$	+ 57	+ 1296	c^4d^5	− 12	...	cd^4ef	− 174	− 24624			
$c^3d^2f^2$	− 24	+ 4428				cd^3e^3	+ 210	+ 14040			
c^3de^2f	− 78	− 18612				d^6f	+ 63	+ 7776			
					± 9						
				± 12	1548					± 8	± 828
				426	45999					123	10920
				912	62019					1071	79779
				1101	92853					1821	146226
				± 2451	± 202428					± 3023	± 237753

T. No. 94.

x coefficient.		x coefficient.			
$a^5\ b^0cf^5$	...	$a^2\ b^3d^2f^3$	− 20	$a^1\ b^3c^2d^2ef^2$	+ 153
def^4	...	$d^3e^2f^2$	+ 33	c^2de^3f	− 390
e^3f^3	...	de^4f	− 48	c^3e^5	− 234
$a^4\ b^2f^5$	...	e^6	+ 27	cd^4f^2	− 114
b^1cef^4	...	$b^3c^3ef^3$	+ 39	cd^3e^2f	− 308
d^2f^4	...	$c^3d^2f^3$	− 105	cd^2e^4	+ 735
de^2f^3	...	$c^3de^2f^2$	+ 18	d^3ef	+ 208
e^4f^2	...	c^3e^4f	− 6	d^4e^2	− 283
$b^0c^2df^4$	− 1	cd^2ef^2	+ 114	$b^3c^3f^3$	+ 27
$c^2e^2f^3$	+ 1	cd^2e^3f	− 57	c^4def^2	− 396
cd^2ef^3	+ 7	cde^5	+ 12	c^4e^3f	− 337
cde^3f^2	− 12	d^5f^2	− 6	$c^3d^2f^2$	+ 222
ce^5f	+ 5	d^4e^2f	+ 3	$c^3d^2e^2f$	+ 783
d^4f^3	− 6	d^3e^4	− 12	c^3de^4	+ 880
$d^3e^2f^2$	+ 12	$b^1c^4df^3$	+ 90	c^2d^4ef	+ 93
d^2e^4f	− 7	$c^4e^2f^2$	− 198	$c^2d^2e^3$	− 1986
de^6	+ 1	$c^3d^2ef^2$	− 9	cd^6f	− 240
$a^3\ b^3ef^4$	...	c^3de^2f	+ 238	cd^5e^2	+ 1098
b^2cdf^4	+ 2	c^3e^5	+ 116	d^7e	− 144
ce^2f^3	− 2	$c^2d^4f^2$	− 6	$b^1c^6ef^2$	+ 81
d^5ef^3	− 7	$c^2d^3e^2f$	+ 108	$c^5d^2f^2$	− 54
de^3f^2	+ 12	$c^2d^2e^4$	− 513	c^5de^2f	+ 570
e^5f	− 5	cd^5ef	− 294	c^5e^4	− 148
$b^1c^3f^4$	+ 3	cd^4e^2	+ 513	c^4d^3ef	− 1116
c^3def^3	− 30	d^7f	+ 108	$c^4d^2e^3$	− 527
$c^3e^3f^2$	+ 21	d^6e^2	− 153	c^3d^2f	+ 474
cd^2f^3	+ 44	$b^0c^6f^3$	− 27	$c^3d^3e^2$	+ 1662
$cd^2e^2f^2$	− 69	c^5def^2	+ 108	c^2d^6e	− 1185
cde^4f	+ 62	c^5e^2f	+ 194	cd^8	+ 243
ce^6	− 28	$c^4d^3f^2$	− 42	$b^0c^7df^2$	...
d^4ef^2	− 6	$c^4d^2e^2f$	− 663	c^7e^3f	− 216
d^3e^3f	− 8	c^4de^4	− 274	c^6d^2ef	+ 369
d^2e^5	+ 11	c^3d^4ef	+ 570	c^6de^3	+ 340
$b^0c^4ef^3$	− 6	$c^3d^3e^2$	+ 914	c^5d^4f	− 149
$c^3d^2f^3$	− 11	c^2d^6f	− 153	$c^5d^3e^2$	− 730
$c^3de^2f^2$	+ 96	$c^2d^5e^2$	− 1032	c^4d^5e	+ 488
c^3e^4f	− 64	cd^7e	+ 486	c^3d^7	− 102
$c^2d^3ef^2$	− 66	d^9	− 81	$a^0\ b^7f^4$	− 2
$c^2d^2e^3f$	− 29	$a^1\ b^5cf^4$	+ 7	b^6cef^3	+ 20
c^3de^5	+ 68	def^3	− 16	d^2f^3	− 24
cd^3f^2	+ 18	e^3f^2	+ 9	de^2f^2	+ 72
cd^4e^2f	+ 75	$b^4c^2ef^3$	− 53	e^4f	− 54
cd^2e^4	− 78	cd^2f^2	+ 104	$b^5c^2df^3$	+ 16
d^6ef	− 27	cde^2f^2	− 150	$c^2e^2f^2$	− 129
d^5e^2	+ 24	ce^4f	+ 117	cd^2ef^2	− 108
$a^2\ b^4df^4$	− 1	d^2ef^2	− 48	cde^3f	+ 72
e^2f^3	+ 1	d^2e^2f	+ 138	ce^5	+ 135
$b^3c^2f^4$	− 8	de^3	− 108	d^4f^2	+ 84
$cdef^3$	+ 46	$b^3c^3df^3$	− 82	d^3e^2f	− 112
ce^2f^2	− 30	$c^3e^2f^2$	+ 315	d^2e^4	...

$a^0\ b^4c^4f^3$	− 6
c^3def^2	+ 240
c^3e^2f	+ 179
$c^2d^3f^2$	− 144
$c^2d^2e^2f$	+ 306
c^2de^4	− 765
cd^4ef	+ 28
cd^3e^2	+ 280
d^5f	− 88
d^4e^2	+ 40
$b^3c^5ef^2$	− 63
$c^4d^2f^2$	+ 42
c^4de^2f	− 798
c^4e^4	+ 175
c^3d^3ef	− 224
$c^3d^2e^3$	+ 1365
c^3d^5f	+ 368
$c^3d^4e^2$	− 1025
cd^6e	+ 60
d^8	+ 30
$b^3c^6df^2$	...
c^6e^2f	+ 252
c^5d^2ef	+ 798
c^5de^3	− 700
c^4d^4f	− 578
$c^4d^3e^2$	− 370
c^3d^5e	+ 880
c^2d^7	− 240
$b^1c^8f^2$	...
c^7def	− 486
c^7e^3	+ 60
c^6d^3f	+ 312
$c^6d^2e^2$	+ 645
c^5d^4e	− 735
c^4d^6	+ 190
b^0c^9ef	+ 81
c^8d^2f	− 54
c^8de^2	− 135
c^7d^3e	+ 150
c^6d^5	− 40

± 26
436
3738
9116
6880

±20196

and see further p. 306.

T. No. 94.

	y coefficient.			y coefficient.			y coefficient.	
$a^5\ b^0df^5$	...	$a^2\ b^3cd^2e^2f^2$	$-\ 18$	$a^1\ b^2c^4df^3$	$-\ 75$	$a^0\ b^4cd^3e^2f$	$-\ 880$
e^3f^4	...	cde^4f	$+\ 150$	$c^4e^2f^2$	$-\ 3$	cd^5e^4	$+\ 765$
$a^4\ b^1cf^5$	...	ce^6	$-\ 72$	$c^3d^3ef^2$	$-\ 108$	d^5ef	$+\ 148$
def^4	...	d^4ef^2	$+\ 198$	c^3de^3f	$+\ 308$	d^4e^3	$-\ 175$
e^3f^3	...	d^3e^2f	$-\ 315$	c^3e^5	$+\ 112$	$b^3c^3f^3$	$-\ 24$
$b^0c^2ef^4$	...	d^2e^3	$+\ 129$	$c^2d^4f^2$	$+\ 663$	c^4def^2	$-\ 513$
cd^2f^4	$+\ 1$	$b^1c^4ef^3$	$+\ 6$	$c^2d^3e^2f$	$-\ 783$	c^4e^3f	$+\ 283$
cde^2f^3	$-\ 2$	$c^3d^2f^3$	$+\ 66$	$c^2d^2e^4$	$-\ 306$	$c^3d^3f^2$	$-\ 914$
ce^4f^2	$+\ 1$	$c^3de^2f^2$	$-\ 114$	cd^5ef	$-\ 570$	$c^3d^2e^2f$	$+\ 1986$
d^3ef^3	$-\ 3$	c^3e^4f	$+\ 48$	cd^4e^2	$+\ 798$	c^3de^4	$-\ 280$
$d^2e^2f^2$	$+\ 8$	$c^2d^3ef^2$	$+\ 9$	d^7f	$+\ 216$	c^2d^4ef	$+\ 527$
de^3f	$-\ 7$	$c^2d^2e^2f$	$-\ 153$	d^6e^2	$-\ 252$	$c^3d^2e^3$	$-\ 1365$
e^7	$+\ 2$	c^2de^5	$+\ 108$	$b^1c^5f^3$	$+\ 27$	cd^6f	$-\ 340$
$a^3\ b^2f^5$	...	cd^5f^2	$-\ 108$	c^5def^2	$+\ 294$	cd^5e^2	$+\ 700$
b^3cef^4	...	cd^4e^2f	$+\ 396$	c^5e^2f	$-\ 208$	d^7e	$-\ 60$
d^2f^4	$-\ 1$	cd^3e^4	$-\ 240$	$c^4d^2e^2f$	$-\ 93$	$b^2c^6ef^2$	$+\ 153$
de^2f^3	$+\ 2$	d^6ef	$-\ 81$	$c^4d^3f^2$	$-\ 570$	$c^5d^2f^2$	$+\ 1032$
e^4f^2	$-\ 1$	d^5e^3	$+\ 63$	c^4de^4	$-\ 28$	c^5de^2f	$-\ 1098$
$b^1c^2df^4$	$-\ 7$	$b^0c^5df^3$	$-\ 18$	c^3d^4ef	$+\ 1116$	c^2e^4	$-\ 40$
$c^2e^2f^3$	$+\ 7$	$c^5e^2f^2$	$+\ 6$	$c^3d^3e^3$	$+\ 224$	c^4d^3ef	$-\ 1662$
cd^2ef^3	$+\ 30$	$c^4d^2ef^2$	$+\ 6$	c^2d^6f	$-\ 369$	$c^4d^2e^3$	$+\ 1025$
cde^3f^2	$-\ 46$	c^4de^2f	$+\ 114$	$c^2d^5e^2$	$-\ 798$	c^3d^5f	$+\ 730$
ce^5f	$+\ 16$	c^4e^5	$-\ 84$	cd^7e	$+\ 486$	$c^3d^4e^2$	$+\ 370$
d^4f^3	$+\ 6$	$c^3d^4f^2$	$+\ 42$	d^9	$-\ 81$	c^2d^6e	$-\ 645$
$d^3e^2f^2$	$-\ 39$	$c^3d^3e^2f$	$-\ 222$	$b^0c^7ef^2$	$-\ 108$	cd^8	$+\ 135$
d^2e^4f	$+\ 53$	$c^3d^2e^4$	$+\ 144$	$c^6d^2f^2$	$+\ 153$	$b^1c^7df^2$	$-\ 486$
de^6	$-\ 20$	c^3d^5ef	$+\ 54$	c^6de^2f	$+\ 240$	c^7e^2f	$+\ 144$
$b^0c^4f^4$	$+\ 6$	$c^3d^4e^2$	$-\ 42$	c^6e^4	$+\ 88$	c^6d^2ef	$+\ 1185$
c^3def^3	$-\ 44$	cd^7f	...	c^5d^3ef	$-\ 474$	c^6de^3	$-\ 60$
$c^3e^2f^2$	$+\ 20$	cd^6e^2	...	$c^5d^2e^3$	$-\ 368$	c^5d^4f	$-\ 488$
$c^2d^3f^3$	$+\ 11$	d^8e	...	c^4d^5f	$+\ 149$	$c^5d^3e^2$	$-\ 880$
$c^2d^2e^2f^2$	$+\ 105$	$a^1\ b^5df^4$	$-\ 5$	$c^4d^4e^2$	$+\ 578$	c^4d^6e	$+\ 735$
c^2de^4f	$-\ 104$	e^3f^3	$+\ 5$	c^3d^6e	$-\ 312$	c^3d^7	$-\ 150$
c^2e^6	$+\ 24$	$b^4c^2f^4$	$+\ 7$	c^2d^8	$+\ 54$	$b^0c^8f^2$	$+\ 81$
cd^4ef^2	$-\ 90$	$cdef^3$	$-\ 62$	$a^0\ b^6cf^4$	$-\ 1$	c^8def	$-\ 243$
cd^3e^2f	$+\ 82$	ce^2f^2	$+\ 48$	def^3	$+\ 28$	c^8e^3	$-\ 30$
cd^2e^5	$-\ 16$	d^3f^3	$+\ 64$	e^3f^2	$-\ 27$	c^7d^3f	$+\ 102$
d^3f^2	$+\ 27$	$d^2e^2f^2$	$+\ 6$	$b^0c^2ef^3$	$-\ 11$	$c^7d^2e^2$	$+\ 240$
d^2e^2f	$-\ 27$	de^4f	$-\ 117$	cd^2f^3	$-\ 68$	c^6d^4e	$-\ 190$
d^4e^2	$+\ 6$	e^6	$+\ 54$	cde^2f^2	$-\ 12$	c^5d^6	$+\ 40$
$a^2\ b^4ef^4$	...	$b^3c^3ef^3$	$+\ 8$	c^4f^2	$+\ 108$		
b^3cdf^4	$+\ 12$	$c^3d^2f^3$	$+\ 29$	d^3ef^2	$-\ 116$		
ce^2f^3	$-\ 12$	$c^3de^2f^2$	$+\ 57$	d^2e^3f	$+\ 234$	$\pm\ 12$	
d^2ef^3	$-\ 21$	c^3e^4f	$-\ 138$	de^5	$-\ 135$	395	
de^3f^2	$+\ 30$	cd^3ef^2	$-\ 238$	$b^4c^3df^3$	$+\ 78$	1650	
e^5f	$-\ 9$	cd^2e^2f	$+\ 390$	$c^3e^2f^2$	$+\ 12$	6511	
$b^2c^3f^4$	$-\ 12$	cde^5	$-\ 72$	$c^2d^3ef^2$	$+\ 513$	11628	
c^2def^3	$+\ 69$	d^3f^2	$-\ 194$	c^2de^3f	$-\ 735$	± 20196	
$c^2e^2f^2$	$-\ 33$	d^2e^2f	$+\ 337$	c^2e^5	...	and see further p. 306.	
cd^3f^2	$-\ 96$	d^3e^4	$-\ 179$	cd^4f^2	$+\ 274$		

$\mathfrak{X}(x, y)^1$

U. No. 29.

a^5	b^9f^6	...	a^3	$b^1d^3e^5$	− 22	a^2	$b^1c^2d^2f^2$	− 108	a^1	$b^3c^3e^2f^2$	− 90
a^5	b^1ef^5	...		$b^9c^3f^4$	− 4		$c^3d^4e^2f$	− 42		$c^4d^2ef^2$	− 42
	b^9cdf^5	...		c^4def^3	+ 36		$c^3d^2e^4$	+ 298		c^4de^3f	+ 674
	c^2f^4	...		$c^4e^3f^2$	− 16		cd^3ef	+ 242		c^4e^5	− 4
	d^2ef^4	...		$c^3d^3f^3$	− 22		cd^3e^3	− 294		$c^3d^4f^2$	+ 394
	de^3f^2	...		$c^3d^2e^2f^2$	− 50		d^5f	− 72		$c^3d^3e^4$	− 652
	e^4f^2	...		c^3de^4f	+ 16		d^7e^2	+ 78		$c^3d^2e^2f$	− 714
a^4	b^3f^5	...		c^3e^6	+ 16		$b^9c^5df^3$	− 6		c^3d^3ef	− 498
	b^3df^5	...		$c^2d^4ef^2$	+ 54		$c^6e^3f^2$	+ 62		$c^3d^4e^3$	+ 1246
	e^2f^4	...		$c^2d^3e^3f$	+ 46		$c^5d^3ef^2$	− 108		cd^7f	+ 224
	$b^1c^2f^5$	...		$c^2d^2e^5$	− 60		c^5de^3f	− 164		cd^6e^2	− 516
	$cdef^4$	...		cd^5e^2f	− 6		c^5e^5	− 24		d^8e	+ 48
	d^3f^4	...		cd^5e^3f	− 70		$c^4d^4f^2$	+ 63		$b^1c^7f^3$	+ 18
	$d^2e^3f^3$	...		cd^4e^4	+ 56		$c^4d^3e^3f$	+ 394		c^6def^2	+ 242
	de^4f^2	...		d^7ef	+ 18		$c^4d^2e^4$	+ 194		c^6e^3f	− 128
	e^6f	...		d^6e^2	− 14		c^3d^3ef	− 324		$c^5d^3f^2$	− 324
	$b^9c^3ef^4$	...	a^3	b^5f^5	...		$c^3d^4e^3$	− 440		$c^5d^2e^2f$	− 498
	$c^3d^2f^4$	− 1		b^4cef^4	...		c^3d^7f	+ 78		c^5de^4	+ 136
	$e^3de^2f^3$	+ 2		d^3f^4	− 1		$c^3d^6e^2$	+ 428		c^4d^5ef	+ 1078
	$c^3e^4f^2$	− 1		de^2f^3	+ 2		cd^8e	− 180		$c^4d^3e^3$	+ 206
	cd^3ef^3	+ 6		e^4f^2	− 1		d^{10}	+ 27		c^3d^6f	− 342
	$cd^2e^2f^2$	− 16		$b^3c^2df^4$	− 16	a^1	b^6ef^4	...		$c^3d^5e^2$	− 804
	cde^3f	+ 14		$c^2e^3f^3$	+ 16		b^3cdf^4	+ 14		c^3d^7e	+ 506
	ce^7	− 4		cd^2ef^3	+ 82		ce^2f^3	− 14		cd^9	− 90
	d^5f^3	− 4		cde^3f^2	− 132		d^2ef^3	− 32		$b^9c^8ef^2$	− 72
	$d^4e^2f^2$	+ 11		ce^5f	+ 50		de^3f^2	+ 50		$c^7d^2f^2$	+ 78
	d^3e^4f	− 10		d^4f^2	− 16		e^3f	− 18		c^7de^2f	+ 224
	d^2e^6	+ 3		$d^3e^2f^2$	− 14		$b^4c^3f^4$	− 10		c^7e^4	+ 16
a^3	b^3ef^3	...		d^2e^4f	+ 60		c^3def^3	− 30		c^6d^3ef	− 342
	def^4	...		de^6	− 30		$c^2e^3f^2$	+ 60		$c^6d^2e^3$	− 220
	e^3f^3	...		$b^2c^4f^4$	+ 11		$cd^2e^3f^2$	− 48		c^6d^4f	+ 106
	$b^2c^3ef^4$	...		c^3def^3	− 30		cd^3f^3	+ 16		$c^6d^3e^2$	+ 392
	cd^2f^4	+ 2		$c^3e^3f^2$	− 14		cde^4f	+ 38		c^4d^6e	− 222
	cde^2f^3	− 4		$c^2d^3f^3$	− 50		ce^6	− 36		c^3d^8	+ 40
	ce^4f^2	+ 2		$c^2d^2e^2f^2$	+ 168	a^0	b^7df^4	− 4		c^5d^5f	+ 392
	d^3ef^3	− 6		c^2de^4f	− 48		e^2f^3	+ 4		$c^4d^4e^2$	+ 143
	$d^2e^2f^2$	+ 16		c^2e^5	− 4		$b^6c^2f^4$	+ 3		c^4d^6e	− 354
	de^3f	− 14		cd^4ef^2	− 48		$cdef^2$	+ 24		c^3d^8	+ 83
	e^7	+ 4		cd^3e^3f	− 2		ce^3f^2	− 30		$b^1c^5df^2$	− 180
	$b^1c^3df^4$	+ 6		cd^2e^5	+ 6		d^3f^3	+ 16		c^5e^2f	+ 48
	$c^3e^2f^3$	− 6		d^5f^2	+ 62		$d^2e^2f^2$	− 4		c^7d^2ef	+ 506
	$c^3d^2ef^3$	− 50		d^5e^2f	− 90		de^4f	− 36		c^7de^3	− 56
	$c^3de^3f^2$	+ 82		d^4e^4	+ 39		e^6	+ 27		c^6d^4f	− 222
	c^3e^3f	− 32		$b^1c^5ef^3$	− 28		$b^5cd^3ef^2$	− 104		c^6d^3e	+ 330
	cd^4f^3	+ 36		$c^4d^2f^3$	+ 54		cd^5f^2	− 164		c^8d^7	− 72
	$cd^3e^2f^2$	− 30		$c^4de^2f^2$	− 48		cd^4e^2f	+ 674		$b^9c^{10}f^2$	+ 27
	cd^2e^4f	− 30		c^4e^4f	+ 112		cd^3e^4	− 590		c^9def	− 90
	cde^6	+ 24		$c^3d^3ef^2$	+ 82		d^6ef	− 128		c^9e^3	− 4
	d^5ef^2	− 28		$c^3d^2e^2f$	− 170		d^5e^3	+ 138		c^8d^3f	+ 40
	d^4e^3f	+ 50		c^3de^5	− 104		$b^2c^5df^3$	− 70		$c^8d^3e^2$	+ 83
										c^7d^4e	− 72
										c^6d^6	+ 16

± 36, ± 464, ± 2608, ± 7278, ± 6878, together ± 17264: and see further p. 307.

V. No. 95. $(*\mathfrak{Q}x, y)^1$.

x coefficient.

a^6	b^0cf^6	...	a^3	b^4df^3	−	2	a^3	$b^0cd^3e^3$	+	876	a^2	$b^1c^3d^3e^4$	+	2800
	def^5	...		e^3f^4	+	2		d^6f	+	162		c^3d^6ef	+	6624
	e^6f^4	...		$b^3c^2f^5$	−	16		d^3e^2	−	162		$c^2d^5e^3$	+	2052
a^5	b^2f^6	...		$cdef^4$	+	32	a^2	b^5cf^5	+	14		cd^8f	−	918
	$b^1c\,ef^5$	...		ce^3f^3		...		def^4	−	6		cd^7e^2	−	2304
	d^2f^5	...		d^3f^4	−	8		e^3f^3	−	8		d^9e	+	486
	de^2f^4	...		$d^2e^2f^3$	+	80		$b^4c^2ef^4$	−	50		c^7df^2		...
	e^4f^3	...		de^4f^3	−	160		cd^2f^4	+	90		$b^0c^7e^3f^2$	+	504
	$b^0c^3df^5$	− 2		e^5f	+	72		cde^2f^3	−	120		$c^6d^2ef^2$	−	576
	$c^2e^3f^4$	+ 2		$b^3c^2ef^4$	+	84		ce^4f^2	+	60		c^6de^2f	−	2288
	cd^2ef^4	+ 10		$c^3d^2f^4$	−	104		d^3ef^3	−	280		c^6e^5	+	1172
	cde^3f^3	− 16		$c^3de^2f^3$	−	160		$d^2e^2f^2$	+	300		$c^5d^4f^2$	−	124
	ce^5f^2	+ 6		$c^2e^4f^2$	+	60		de^3f	+	216		$c^5d^3e^2f$	+	4336
	d^4f^4	− 6		cd^2ef^3	+	320		e^7	−	216		$c^5d^2e^4$	−	2540
	$d^3e^2f^3$	+ 12		$cd^2e^3f^2$	+	80		$b^3c^3df^4$	−	160		c^4d^3ef	−	1912
	$d^2e^4f^2$	− 10		cde^5f	−	496		$c^3e^2f^3$	−	80		$c^4d^4e^2$	+	2100
	de^6f	+ 6		ce^7	+	252		$c^3d^2ef^3$	+	1280		c^3d^7f	+	240
	e^8	− 2		d^5f^3	−	72		$c^3de^3f^2$		...		$c^3d^6e^2$	−	1560
a^4	b^3ef^5	...		$d^4e^2f^2$	−	420		c^2e^5f	−	312		c^2d^3e	+	810
	b^2cdf^5	+ 4		d^3e^4f	+	860		cd^4f^2	−	440		cd^{10}	−	162
	ce^2f^4	− 4		d^2e^6	−	404		$cd^3e^2f^2$	−	2160	a^1	b^7f^5	−	4
	d^2ef^4	− 10		$b^1c^4df^4$	+	96		cd^2e^4f	+	1740		b^6cef^4	−	22
	de^3f^3	+ 16		$c^4e^2f^3$	−	120		cde^6	−	216		d^2f^4	−	26
	e^5f^2	− 6		$c^3d^2ef^3$	−	560		d^5ef^2	+	2344		de^2f^3	+	76
	$b^1c^3f^3$	+ 6		$c^3de^3f^2$	+	160		d^4e^2f		3240		e^4f^2		...
	c^2def^4	− 26		c^2e^5f	+	304		d^3e^5	+	1244		$b^5c^3df^4$	+	124
	$c^2e^3f^3$	+ 8		$c^2d^4f^3$	+	280		$b^2c^3f^4$	+	72		$c^3e^3f^3$	+	368
	cd^3f^4	+ 32		$c^2d^3e^2f^2$	+	1440		c^4def^3	−	240		cd^2ef^3	−	688
	$cd^2e^2f^3$	− 116		$c^2d^2e^4f$	−	960		$c^4e^3f^2$	+	940		cde^3f^2	−	192
	cde^4f^2	+ 180		c^2de^6	−	376		$c^3d^3f^3$		...		ce^5f		...
	ce^6f	− 78		cd^5ef^2	−	1296		$c^3d^2e^2f^2$	−	1320		d^4f^3	+	400
	d^4ef^3	+ 24		cd^4e^2f	+	80		c^3de^4f	−	2640		$d^3e^2f^2$	+	984
	$d^3e^3f^2$	− 20		cd^3e^6	+	832		c^3e^6	+	908		d^2e^4f	−	2160
	d^2e^5f	− 44		d^7f^2	+	432		$c^2d^4ef^2$	+	600		de^6	+	1080
	de^7	+ 34		d^6e^2f	−	72		$c^2d^3e^2f$	+	3360		$b^4c^4f^4$	−	60
	$b^0c^4ef^4$	− 30		d^5e^4	−	240		$c^2d^2e^3$	−	168		c^3def^3	−	480
	$c^3d^2f^4$	+ 4		$b^0c^3f^4$	−	36		cd^3f^2	−	1656		$c^3e^3f^2$	−	1580
	$c^3de^2f^3$	+ 240		c^3def^3	+	288		cd^2e^2f	+	3408		$c^2d^2f^3$	+	40
	$c^3e^4f^2$	− 130		$c^3e^3f^2$	−	56		cd^4e^4	−	3480		$c^2d^2e^2f^2$	+	2040
	$c^2d^3ef^3$	− 160		$c^4d^2f^3$	−	140		d^7ef	−	1008		c^2de^4f	+	2910
	$c^2d^2e^3f^2$	− 280		$c^4d^2e^2f^2$	−	480		d^6e^3	+	1224		c^2e^6	−	810
	c^2de^5f	+ 332		c^4de^4f	+	420		$b^1c^6ef^2$	−	144		cd^4ef^2	−	3420
	c^2e^7	− 54		c^4e^6	−	276		$c^5d^2f^2$	+	108		cd^3e^2f	+	4800
	cd^5f^3	+ 24		$c^3d^4ef^2$	+	420		$c^5de^2f^2$	−	768		cd^3e^5	−	3510
	$cd^4e^2f^2$	+ 360		$c^3d^3e^2f$	−	1120		c^5e^4f	−	700		d^5f^2	−	1516
	cd^3e^4f	− 320		$c^3d^2e^5$	+	1112		$c^4d^3ef^2$	+	900		d^5e^2f	+	2156
	cd^2e^6	+ 38		$c^2d^5f^2$	−	144		$c^4d^2e^2f$	+	8160		d^4e^4	−	430
	d^6ef^2	− 108		$c^2d^5e^2f$	+	1620		c^4de^5	−	2148		$b^3c^5ef^2$	+	336
	d^5e^3f	+ 96		$c^2d^4e^4$	−	1620		$c^3d^3f^2$	+	912		$c^4d^2f^3$	−	40
	d^4e^5	− 12		cd^7ef	−	864		$c^3d^2e^2f$	−	15060		$c^4de^2f^2$	+	2640

For the Numerical Verifications see p. 308.

V. No. 95 (continued).

x coefficient.

a^1	$b^3c^4e^4f$	+ 1840	a^0	$b^7d^2ef^3$	+ 184	a^0	$b^2c^8ef^2$	− 594
	$c^3d^3ef^2$	− 1280		de^3f^2	− 108		$c^7d^2f^2$	− 10296
	$c^3d^2e^3f$	− 13360		e^5f	...		c^7de^3f	+ 10080
	c^3de^5	+ 3200		$b^6c^3f^4$	+ 18		c^7e^4	+ 900
	$c^2d^5f^2$	+ 7312		c^5def^3	+ 264		c^6d^3ef	+ 19440
	$c^2d^4e^2f$	− 2360		$c^2e^3f^2$	+ 756		$c^6d^2e^3$	− 8800
	$c^2d^3e^4$	+ 3840		cd^2f^3	− 368		c^6d^5f	− 9160
	cd^6ef	− 5344		$cd^2e^2f^2$	− 732		$c^5d^4e^2$	− 11900
	cd^5e^2	+ 2800		cde^4f	+ 540		c^4d^6e	+ 13900
	d^8f	+ 1956		ce^6	...		c^3d^8	− 3150
	d^7e^2	− 1680		d^4ef^2	− 1172		$b^1c^9df^2$	+ 3564
	$b^2c^6df^3$	− 36		d^3e^3f	+ 2520		c^9e^2f	− 1350
	$c^6e^2f^2$	− 1296		d^2e^5	− 1350		c^8d^2ef	− 9540
	$c^5d^2ef^2$	+ 1668		$b^5c^4ef^3$	− 144		c^8de^3	− 750
	c^5de^3f	− 1312		$c^3d^2f^2$	+ 376		c^7d^4f	+ 4260
	c^5e^5	− 2060		$c^3de^2f^2$	− 1440		$c^7d^3e^2$	+ 10800
	$c^4d^4f^2$	− 8020		c^3e^4f	− 1530		c^6d^5e	− 9100
	$c^4d^3e^2f$	+ 15220		$c^2d^3ef^2$	+ 6360		c^5d^7	+ 2000
	$c^4d^2e^4$	+ 1180		$c^2d^2e^3f$	− 6000		$b^0c^{11}f^2$	− 486
	c^3d^5ef	+ 3712		c^2de^5	+ 1350		$c^{10}def$	+ 1620
	$c^3d^4e^3$	− 8540		cd^5f^2	+ 2344		$c^{10}e^3$	+ 450
	c^3d^7f	− 2952		cd^4e^2f	− 9260		c^9d^3f	− 720
	$c^3d^6e^2$	...		cd^3e^4	+ 7200		$c^9d^2e^2$	− 2250
	cd^8e	+ 3330		d^6ef	+ 1720		c^8d^4e	+ 1800
	d^{10}	− 810		d^5e^3	− 1900		c^7d^6	− 400
	$b^1c^8f^3$	...		$b^4c^5df^3$	− 168			
	c^7def^2	− 576		$c^5e^2f^2$	+ 648			
	c^7e^3f	+ 1824		$c^4d^3ef^2$	− 6420			
	$c^6d^2f^2$	+ 3792		c^4de^3f	+ 9360			
	$c^6d^2e^2f$	− 5808		c^4e^5	+ 450			
	c^6de^4	+ 3240		$c^3d^4f^2$	− 10100			
	c^5d^4ef	− 4768		$c^3d^3e^2f$	+ 19920			
	$c^5d^3e^3$	− 6240		$c^3d^2e^4$	− 10300			
	c^4d^6f	+ 2608		c^3d^5ef	+ 4920			
	$c^4d^5e^2$	+ 12440		$c^3d^4e^3$	− 10100			
	c^3d^7e	− 8160		cd^7f	− 3440			
	c^2d^9	+ 1620		cd^5e^2	+ 7100			
	$b^0c^9ef^2$	+ 162		d^8e	− 750			
	$c^8d^2f^2$	− 702		$b^3c^7f^3$	+ 36			
	c^8de^2f	− 90		c^6def^2	+ 2988			
	c^8e^4	− 1290		c^6e^3f	− 2880			
	c^7d^3ef	+ 1920		$c^5d^3f^2$	+ 14688			
	$c^7d^2e^3$	+ 3640		$c^5d^2e^2f$	− 22740			
	c^6d^5f	− 796		c^5de^4	+ 600			
	$c^6d^4e^2$	− 5340		c^4d^4ef	− 16520			
	c^5d^6e	+ 3100		$c^4d^3e^3$	+ 23300			
	c^4d^8	− 600		c^3d^6f	+ 8760			
a^0	b^8ef^3	+ 18		$c^3d^5e^2$	− 5200			
	b^7cdf^4	− 36		c^2d^7e	− 5400			
	ce^2f^3	− 180		cd^9	+ 1500			

V. No. 95 (continued).

y coefficient.

a^3	b^0df^6		...	a^3	$b^0c^3def^4$	−	116	a^2	b^4e^6f		...
	c^2f^5		...		$c^2e^3f^3$	+	80		$b^2c^2ef^4$	−	20
a^5	b^1cf^6		...		cd^3f^4	+	240		$c^3d^2f^4$	−	280
	def^5		...		$cd^2e^2f^3$	−	160		$c^3de^3f^3$	+	80
	e^2f^4		...		cde^4f^2	−	120		$c^2e^4f^2$	+	300
	$b^0c^2ef^5$		...		ce^6f	+	76		cd^2ef^3	+	160
	cd^2f^5	−	2		d^4ef^3	−	120		$cd^2e^2f^2$		...
	cde^2f^4	+	4		$d^3e^3f^2$	−	80		cde^3f	−	192
	ce^4f^3	−	2		d^2e^5f	+	368		ce^7	−	108
	d^2ef^4	+	6		de^7	−	180		d^3f^3	−	56
	$d^2e^2f^3$	−	16		$b^1c^4ef^4$	+	24		$d^2e^2f^2$	+	940
	de^3f^2	+	14		$c^3d^2f^4$	−	160		d^2e^4f	−	1580
	e^7f	−	4		$c^3de^2f^3$	+	320		d^2e^6	+	756
a^4	b^2f^6		...		$c^2e^4f^2$	−	280		$b^2c^4df^4$	+	360
	b^2cef^5		...		$c^2d^3ef^3$	−	560		$c^4e^3f^3$	−	420
	d^3f^5	+	2		$c^2d^2e^2f^2$	+	1280		$c^3d^2ef^3$	+	1440
	de^2f^4	−	4		c^3de^3f	−	688		$c^3de^2f^2$	−	2160
	e^4f^3	+	2		c^2e^7	+	184		c^3e^3f	+	984
	$b^1c^2df^5$	+	10		cd^3f^3	+	288		$c^2d^4f^3$	−	480
	$c^2e^2f^4$	−	10		$cd^2e^2f^2$	−	240		$c^2d^2e^2f^2$	−	1320
	cd^2ef^4	−	26		cd^3e^4f	−	480		$c^2d^2e^4f$	+	2040
	cde^3f^3	+	32		cd^2e^6	+	264		c^2de^6	−	732
	ce^5f^2	−	6		d^5ef^2	−	144		cd^5ef^2	−	768
	d^4f^4	−	30		d^3e^2f	+	336		cd^4e^3f	+	2640
	$d^3e^2f^3$	+	84		d^4e^5	−	144		cd^3e^5	−	1440
	$d^2e^4f^2$	−	50		$b^0c^3df^4$	+	24		d^7f^2	+	504
	de^6f	−	22		$c^3e^3f^2$	−	72		d^3e^2f	−	1296
	e^8	+	18		$c^2d^2ef^3$	+	280		d^2e^4	+	648
	$b^0c^4f^5$	−	6		$c^2de^2f^2$	−	440		$b^1c^6f^4$	−	108
	c^3def^4	+	32		c^2e^4f	+	400		c^5def^3	−	1296
	$c^3e^2f^3$	−	8		$c^3d^4f^3$	−	140		$c^5e^2f^2$	+	2344
	$c^2d^3f^4$	+	4		$c^2d^3e^2f^2$		...		$c^4d^2f^3$	+	420
	$c^2d^2e^2f^3$	−	104		$c^2d^2e^4f$	+	40		$c^4d^2e^2f^2$	+	600
	$c^2de^4f^2$	+	90		c^2de^6	−	368		c^4de^4f	−	3420
	c^2e^6f	−	26		$c^2d^5ef^2$	+	108		c^4e^6	−	1172
	cd^4ef^3	+	96		$c^2d^4e^2f$	−	40		$c^3d^4ef^2$	+	900
	$cd^3e^3f^2$	−	160		$c^2d^3e^5$	+	376		$c^3d^3e^2f$	−	1280
	cd^2e^5f	+	124		cd^7f^2		...		$c^3d^2e^5$	+	6360
	cde^7	−	36		cd^6e^2f	−	36		$c^2d^6f^2$	−	576
	d^5f^3	−	36		cd^5e^4	−	168		$c^2d^5e^2f$	+	1668
	$d^4e^2f^2$	+	72		d^8ef		...		$c^2d^4e^4$	−	6420
	d^4e^4f	−	60		d^7e^3	+	36		cd^7ef	−	576
	d^3e^6	+	18	a^2	b^5df^5	+	6		cd^6e^3	+	2988
a^3	b^2ef^6		...		e^4f^4	−	6		d^9f	+	162
	b^3cdf^5	−	16		$b^4c^2f^5$	−	10		d^8e^2	−	594
	ce^2f^4	+	16		$cdef^4$	+	180		$b^0c^7ef^3$	+	432
	d^2ef^4	+	8		ce^3f^3	−	160		$c^6d^2f^3$	−	144
	de^3f^3		...		d^3f^4	−	130		$c^6de^2f^2$	−	1656
	e^5f^2	−	8		$d^2e^2f^3$	+	60		c^6e^4f	−	1516
	$b^2c^3f^5$	+	12		de^4f^2	+	60		$c^5d^3ef^2$	+	912

a^3	$b^0c^3d^2e^2f$	+	7312
	c^5de^5	+	2344
	$c^4d^3f^2$	−	124
	$c^4d^4e^2f$	−	8020
	$c^4d^2e^4$	−	10100
	c^3d^6ef	+	3792
	$c^3d^5e^3$	+	14648
	c^3d^3f	−	702
	$c^2d^7e^2$	−	10296
	$c\,d^9e$	+	3564
	d^{11}	−	486
a^1	b^6cf^5	+	6
	def^4	−	78
	e^3f^3	+	72
	$b^5c^2ef^4$	−	44
	cd^2f^4	+	332
	cde^3f^3	−	496
	ce^4f^2	+	216
	d^3ef^3	+	304
	$d^2e^3f^2$	−	312
	de^5f		...
	e^7		...
	$b^4c^3df^4$	−	320
	$c^3e^2f^3$	+	860
	$c^2d^2ef^3$	−	960
	$c^2de^3f^2$	+	1740
	c^2e^5f	−	2160
	cd^4f^3	+	420
	$cd^3e^2f^2$	−	2640
	cd^2e^4f	+	2910
	cde^6	+	540
	d^2ef^2	−	700
	d^2e^3f	+	1840
	d^2e^5	−	1530
	$b^3c^2f^4$	+	96
	c^3def^3	+	80
	$c^3e^3f^2$	−	3240
	$c^3d^3f^3$	−	1120
	$c^3d^2e^2f^2$	+	3360
	c^3de^4f	+	4800
	c^3e^6	+	2520
	$c^2d^4ef^2$	+	8160
	$c^2d^3e^3f$	−	13360
	$c^2d^2e^5$	−	6000
	cd^6f^2	−	2288
	cd^5e^3f	−	1312
	cd^4e^4	+	9360
	d^7ef	+	1824
	d^6e^3	−	2880
	$b^2c^6ef^3$	−	72

For the Numerical Verifications see p. 308.

V. No. 95 (concluded).

y coefficient.

$a^1\ b^2c^5d^2f^3$	+	1620
$c^5de^3f^2$	+	3408
c^5e^4f	+	2156
$c^4d^5ef^2$	−	15060
$c^4d^2e^3f$	−	2360
c^4de^5	−	9260
$c^3d^5f^2$	+	4336
$c^3d^4e^2f$	+	15220
$c^3d^3e^4$	+	19920
c^2d^8ef	−	5808
$c^2d^5e^3$	−	22740
cd^8f	−	90
cd^7e^2	+	10080
d^9e	−	1350
$b^4c^2df^3$	−	864
$c^7e^2f^2$	−	1008
$c^6d^2ef^2$	+	6624
c^6de^3f	−	5344
c^6e^5	+	1720
$c^5d^4f^2$	−	1912
$c^5d^3e^2f$	+	3712
$c^5d^2e^4$	+	4920
c^4d^5ef	−	4768
$c^4d^4e^3$	−	16520
c^3d^7f	+	1920
$c^3d^6e^2$	+	19440
c^2d^8e	−	9540
cd^{10}	+	1620
$b^6e^3f^3$	+	162
c^5def^2	−	918
c^5e^3f	+	1956
$c^7d^3f^2$	+	240
$c^7d^2e^2f$	−	2952
c^7de^4	−	3440
c^6d^4ef	+	2608
$c^6d^3e^3$	+	8760
c^5d^6f	−	796
$c^5d^5e^2$	−	9160
c^4d^7e	+	4260
c^4d^9	−	720
$a^0\ b^8f^5$	−	2
$b^7c\,ef^4$	+	34
d^9f^4	−	54
de^2f^3	+	252
e^4f^2	−	216
$b^5c^3df^4$	+	38
$c^2e^3f^2$	−	404
cd^2ef^3	−	376
cde^3f^2	−	216
ce^5f	+	1080

$a^0\ b^6d^4f^3$	−	276
$d^5e^2f^2$	+	908
d^2e^4f	−	810
de^6		...
$b^5c^4f^4$	−	12
c^3def^3	+	832
$c^3e^3f^2$	+	1244
$c^3d^2f^3$	+	1112
$c^3d^3e^3f^2$	−	168
c^3de^4f	−	3510
c^3e^5	−	1350
cd^4ef^2	−	2148
cd^3e^3f	+	3200
cd^2e^5	+	1350
d^8f^2	+	1172
d^5e^2f	−	2060
d^4e^4	+	450
$b^4c^5ef^3$	−	240
$c^4d^2f^3$	−	1620
$c^4de^2f^2$	−	3480
c^4e^4f	−	430
$c^3d^5ef^2$	+	2800
$c^3d^2e^3f$	+	3840
c^3de^5	+	7200
$c^2d^5f^2$	−	2540
$c^2d^4e^2f$	+	1180
$c^2d^3e^4$	−	10300
cd^8ef	+	3240
cd^7e^3	+	600
d^8f	−	1290
d^7e^2	+	900
$b^3c^6df^3$	+	876
$c^6e^3f^2$	+	1224
$c^5d^2ef^2$	+	2052
c^5de^3f	+	2800
c^5e^5	−	1900
$c^4d^4f^2$	+	2100
$c^4d^3e^2f$	−	8540
$c^4d^2e^4$	−	10100
c^3d^5ef	−	6240
$c^3d^4e^3$	+	23300
c^2d^7f	+	3640
$c^2d^6e^2$	−	8800
cd^8e	−	750
d^{10}	+	450
$b^2c^8f^3$	−	162
c^7def^2	−	2304
c^7e^3f	−	1680
$c^6d^3f^2$	−	1560
$c^6d^2e^2f$		...

$a^0\ b^3c^6de^4$	+	7100
c^5d^4ef	+	12440
$c^5d^3e^3$	−	5200
c^4d^8f	−	5340
$c^4d^5e^2$	−	11900
c^3d^7e	+	10800
c^2d^9	−	2250
$b^4c^9ef^2$	+	486
$c^8d^2f^2$	+	810
c^8de^2f	+	3330
c^8e^4	−	750
c^7d^3ef	−	8160
$c^7d^2e^3$	−	5400
c^6d^5f	+	3100
$c^6d^4e^2$	+	13900
c^5d^6e	−	9100
c^4d^8	+	1800
$b^6c^{10}df^2$	−	162
$c^{10}e^2f$	−	810
c^9d^2ef	+	1620
c^9de^3	+	1500
c^8d^4f	−	600
$c^8d^3e^2$	−	3150
c^7d^5e	+	2000
c^6d^7	−	400

W, 29 A.

a^9	b^0f^9	...	a^6	$b^1ce^2f^2$	− 15	a^5	$b^2c^2de^4f^4$	− 90	a^5	$b^0c^3d^2e^2f$	+ 1320
a^8	b^1ef^8	...		d^5ef^8	+ 10		$c^3e^6f^3$	+ 30		c^3de^9	− 260
	b^0cdf^8	...		$d^4e^2f^4$	− 35		cd^4ef^5	− 210		$c^2d^7f^4$	+ 60
	ce^3f^7	...		$d^3e^3f^3$	+ 40		$cd^3e^3f^4$	+ 120		$c^3d^6e^2f^3$	− 500
	d^2ef^7	...		$d^2e^2f^2$	− 10		$cd^2e^3f^3$	+ 360		$c^3d^5e^4f^2$	+ 2235
	de^3f^6	...		de^3f	− 10		cde^7f^2	− 420		$c^2d^4e^5f$	− 1995
	e^5f^5	...		e^{11}	+ 5		ce^9f	+ 130		$c^2d^3e^6$	+ 370
a^7	b^3df^8	...		$b^0c^5f^7$	− 1		d^8f^5	− 5		cd^5ef^2	+ 360
	e^2f^7	...		c^4def^8	+ 15		$d^5e^2f^4$	+ 195		$cd^7e^2f^2$	− 1320
	$b^1c^2f^8$	...		$c^4e^3f^5$	− 10		$d^4e^4f^3$	− 315		cd^6e^2f	+ 1110
	$cdef^7$	...		$c^3d^3f^6$	...		$d^3e^6f^2$	+ 40		cd^5e^3	− 210
	ce^2f^6	...		$c^3d^2e^2f^5$	− 90		d^2e^8f	+ 165		$d^{10}f^3$	− 81
	d^3f^7	...		$c^3de^4f^4$	+ 120		de^{10}	− 75		$d^9e^2f^2$	+ 270
	$d^2e^2f^6$	...		$c^3e^6f^3$	− 40		$b^1c^6ef^6$	− 10		d^8e^4f	− 225
	de^4f^6	...		$c^2d^4ef^5$	+ 60		$c^4d^2f^6$	− 60		d^7e^6	+ 45
	e^6f^4	...		$c^2d^3e^3f^4$	+ 30		$c^4de^3f^5$	+ 210	a^4	b^6ef^7	...
	$b^0c^3ef^7$	...		$c^2d^2e^4f^3$	− 180		$c^4e^4f^4$	− 110		b^5cdf^7	...
	$c^3d^2f^7$	...		$c^2de^7f^2$	+ 120		$c^3d^3ef^5$	...		ce^2f^6	...
	$c^2de^2f^6$	...		c^2e^9f	− 20		$c^3d^2e^2f^4$	+ 60		d^2ef^6	...
	$c^2e^4f^5$	...		cd^5f^5	− 15		$c^3de^3f^3$	− 360		de^4f^5	...
	cd^3ef^6	...		$cd^3e^2f^4$	− 110		$c^3e^7f^2$	+ 240		e^6f^4	...
	$cd^2e^3f^5$	...		$cd^4e^4f^3$	+ 265		$c^2d^5f^5$	+ 30		$b^4c^2f^7$	− 10
	cde^5f^4	...		$cd^3e^6f^2$	− 200		$c^2d^4e^2f^4$	− 210		c^2def^6	+ 90
	ce^7f^3	...		cd^2e^8f	+ 65		$c^2d^3e^4f^3$	− 180		$c^2e^3f^5$	− 60
	d^5f^6	+ 1		cde^{10}	− 10		$c^2d^2e^6f^2$	+ 1140		cd^3f^6	− 120
	$d^4e^2f^5$	− 5		d^7ef^4	+ 45		c^2de^8f	− 870		$cd^2e^2f^5$	+ 90
	$d^3e^4f^4$	+ 10		$d^6e^4f^3$	− 100		c^2e^{10}	+ 130		cde^4f^4	...
	$d^2e^6f^3$	− 10		$d^5e^2f^2$	+ 81		cd^5ef^4	+ 310		ce^6f^3	...
	de^8f^2	+ 5		d^4e^7f	− 30		$cd^5e^3f^3$	− 240		d^4ef^5	+ 110
	$e^{10}f$	− 1		d^3e^9	+ 5		$cd^4e^5f^2$	− 390		$d^3e^3f^4$	− 50
a^6	b^3ef^8	...	a^5	b^5f^8	...		cd^3e^7f	+ 280		$d^2e^5f^3$	− 240
	def^7	...		b^4cef^7	...		cd^2e^9	+ 30		de^7f^2	+ 280
	e^3f^6	...		d^2f^7	...		d^8f^4	− 180		e^9f	− 90
	$b^2c^2ef^7$	...		de^2f^6	...		$d^7e^2f^6$	+ 300		$b^3c^4ef^5$	+ 35
	cd^2f^7	...		e^4f^5	...		$d^6e^4f^3$	− 120		$c^3d^2f^6$	− 30
	cde^2f^6	...		$b^3c^3df^7$	...		d^5e^6f	+ 30		$c^3de^2f^5$	− 120
	ce^4f^5	...		$c^2e^2f^8$	...		d^4e^8	− 30		$c^3e^4f^4$	+ 50
	d^2ef^6	...		cd^2ef^6	...		$b^0c^6df^6$	+ 15		$c^2d^3ef^6$	− 60
	$d^2e^3f^5$	...		cde^3f^5	...		$c^6e^2f^5$	+ 5		$c^2d^2e^3f^4$	...
	de^3f^4	...		ce^5f^4	...		$c^5d^2ef^5$	− 30		$c^2de^6f^3$	+ 360
	e^7f^3	...		d^4f^6	+ 10		$c^5de^3f^4$	− 270		$c^2e^7f^2$	− 210
	$b^1c^3df^7$	...		$d^3e^2f^5$	− 40		$c^5e^5f^3$	+ 196		cd^5f^5	+ 270
	$c^3e^2f^6$	...		$d^2e^4f^4$	+ 60		$c^4d^4f^5$	...		$cd^4e^2f^4$	+ 575
	$c^2d^2ef^6$	...		de^6f^3	− 40		$c^4d^2e^2f^4$	+ 225		$cd^3e^4f^3$	− 1700
	$c^3de^2f^5$	...		e^8f^2	+ 10		$c^4d^2e^4f^3$	+ 615		$cd^2e^6f^2$	+ 480
	$c^2e^5f^4$	...		$b^3c^4f^7$	+ 5		$c^4de^6f^2$	− 660		cde^8f	+ 670
	cd^4f^6	− 15		c^3def^6	− 60		c^4e^8f	+ 45		ce^{10}	− 315
	$cd^3e^2f^5$	+ 60		$c^3e^2f^3$	+ 40		$c^3d^3ef^4$	− 120		d^3ef^4	− 685
	$cd^2e^4f^4$	− 90		$c^3d^3f^6$	+ 90		$c^3d^4e^3f^3$	− 220		$d^2e^3f^3$	+ 540
	cde^6f^3	+ 60		$c^2d^3e^2f^5$	...		$c^3d^3e^5f^2$	− 980		$d^4e^5f^2$	+ 1515

For the Numerical Verifications see p. 309.

W, 29 A (continued).

$a^4\ b^3d^3e^2f$	−	2080	$a^4\ b^1cd^6e^6$	−	615	$a^3\ b^6d^2f^5$	−	196	$a^3\ b^3d^6e^6$	−	3880
d^2e^9	+	705	$d^{10}ef^2$	−	945	$d^4e^3f^4$	−	660	$b^2c^7ef^3$	−	300
$b^2c^5df^6$	+	110	d^9e^3f	+	900	$d^3e^4f^3$	+	1840	$c^6d^2f^5$	+	500
$c^5e^2f^5$	−	195	d^8e^5	+	45	$d^2e^5f^2$	−	1040	$c^6de^2f^4$	+	3810
$c^4d^2ef^5$	+	210	$b^9c^6ef^5$	+	180	de^6f	−	180	$c^6e^4f^3$	−	3710
$c^4de^3f^4$	−	575	$c^7d^2f^5$	−	60	e^{10}	+	216	$c^5d^3ef^4$	−	14040
$c^4e^5f^3$	+	660	$c^7de^2f^4$	−	1420	$b^4c^4df^6$	−	265	$c^5d^2e^3f^3$	+	16120
$c^3d^4f^5$	−	225	$c^7e^4f^3$	+	25	$c^4e^2f^5$	+	315	$c^5de^2f^2$	−	540
$c^3d^2e^2f^4$	+	1350	$c^6d^3ef^4$	+	780	$c^3d^2ef^5$	+	180	c^5e^7f	+	600
$c^3d^2e^4f^3$		...	$c^6d^2e^3f^3$	+	5760	$c^3de^3f^4$	+	1700	$c^4d^3f^4$	+	7020
$c^3de^6f^2$	−	1440	$c^6de^5f^2$	−	2945	$e^3e^5f^3$	−	1840	$c^4d^4e^2f^3$		...
c^3e^8f	−	75	c^6e^7f	+	1390	$c^3d^4f^5$	−	615	$c^4d^3e^4f^2$	−	1950
$c^2d^5ef^4$	−	1965	$c^5d^5f^4$		...	$c^3d^3e^2f^4$	−	1350	$c^4d^2e^6f$	−	17670
$c^2d^4e^3f^3$	+	6000	$c^5d^4e^2f^3$	−	7020	$c^3d^2e^4f^3$		...	c^4de^8	+	4170
$c^2d^3e^5f^2$	−	7050	$c^5d^3e^4f^2$	−	180	$c^3d^6f^2$	+	1560	$c^3d^5ef^3$	+	480
$c^2d^2e^7f$	+	3000	$c^5d^2e^6f$	−	1275	c^3e^8f	+	135	$c^3d^3e^2f^2$	−	31040
c^2de^9	+	265	c^5de^8	−	1110	cd^7ef^4	+	2210	$c^3d^4e^3f$	+	45180
cd^7f^4	+	1420	$c^4d^6ef^3$	+	3120	$cd^6e^3f^2$	−	4100	$c^3d^3e^7$	−	3160
$cd^6e^2f^2$	−	3810	$c^4d^5e^2f^2$	+	3900	$cd^5e^5f^2$	+	6000	$c^2d^5f^3$	−	140
$cd^5e^4f^2$	+	2310	$c^4d^4e^3f$	+	1240	cd^5e^7f	−	4880	$c^2d^4e^2f^2$	+	18000
cd^4e^6f	+	1795	$c^4d^3e^7$	+	3155	cde^9	+	990	$c^2d^3e^4f$	−	12180
cd^3e^8	−	1800	$c^3d^8f^3$	−	515	d^8f^4	−	25	$c^2d^2e^6$	−	13430
d^8ef^3	+	240	$c^3d^7e^2f^2$	−	2920	$d^6e^2f^3$	+	3710	cd^9ef^2	−	7200
$d^7e^2f^2$	+	30	$c^3d^6e^4f$	−	940	$d^5e^4f^2$	−	10755	cd^8e^2f	−	120
d^6e^5f	−	870	$c^3d^5e^6$	−	4300	d^4e^6f	+	9875	cd^7e^6	+	9960
d^5e^7	+	615	$c^3d^9ef^2$	+	675	d^3e^8	−	2845	$d^{11}f^2$	+	1890
$b^1c^7f^6$	−	45	$c^3d^8e^3f$	+	510	$b^3c^6f^6$	+	100	$d^{10}e^2f$	−	540
c^6def^5	−	310	$c^3d^7e^5$	+	2940	c^6def^5	+	240	d^9e^4	−	1710
$c^6e^3f^4$	+	685	$cd^{11}f^2$		...	$c^5e^3f^4$	−	540	$b^1c^6df^5$	−	360
$c^5d^3f^5$	+	120	$cd^{10}e^2f$	−	135	$c^4d^3f^5$	+	220	$c^6e^2f^4$	−	240
$c^5d^2e^2f^4$	+	1965	cd^9e^4	−	990	$c^4d^2e^2f^4$	−	6000	$c^7d^2ef^4$	+	5840
$c^5de^4f^3$	−	2210	$d^{12}ef$		...	$c^4de^4f^3$	+	4100	$c^7de^3f^2$	−	6560
$c^5e^6f^2$	−	960	$d^{11}e^2$	+	135	$c^4e^6f^2$	+	1340	$c^7e^5f^2$	+	8460
$c^4d^4ef^4$		...	$a^3\ b^7df^7$		...	$c^3d^4ef^4$	+	11700	$c^6d^4f^4$	−	3120
$c^4d^3e^3f^3$	−	11700	e^2f^6		...	$c^3d^3e^2f^3$		...	$c^6d^3e^2f^3$	−	480
$c^4d^2e^5f^2$	+	15435	$b^8c^3f^7$	+	10	$c^3d^2e^3f^2$	−	15240	$c^6d^2e^4f^2$	−	25880
c^4de^7f	−	2760	$cdef^6$	−	60	c^3de^7f	+	6960	c^5de^6f	−	1820
c^4e^9	+	555	ce^3f^5	+	40	c^3e^9	−	1620	c^6e^8	−	3620
$c^3d^3f^4$	−	780	d^3f^6	+	40	$c^3d^6f^4$	−	5760	$c^5d^5ef^3$	+	49680
$c^3d^2e^2f^3$	+	14040	$d^2e^2f^5$	−	30	$c^3d^5e^2f^3$	−	16120	$c^5d^4e^3f^2$		...
$c^3d^4e^4f^2$	−	10625	de^4f^4		...	$c^3d^4e^4f^2$	+	26700	$c^5d^3e^5f$	+	17520
$c^3d^3e^6f$	−	3220	e^6f^3		...	$c^3d^3e^6f$	−	5240	$c^5d^2e^7$	+	13500
$c^3d^2e^8$	−	570	$b^5c^5ef^6$	−	40	$c^3d^2e^8$	−	1640	$c^4d^7f^3$	−	120
$c^3d^7ef^3$	−	5840	$c^5d^2f^6$	+	180	cd^7ef^3	+	6560	$c^4d^6e^2f^2$	−	32280
$c^3d^6e^3f^2$	−	540	$c^5de^2f^5$	−	360	$cd^6e^3f^2$	+	7240	$c^4d^5e^4f$	−	46880
$c^3d^5e^5f$	+	5550	$c^5e^4f^4$	+	240	cd^5e^5f	−	24240	$c^4d^4e^6$	−	30040
$c^3d^4e^7$	+	1285	cd^5ef^5	+	360	cd^4e^7	+	11420	$c^3d^8ef^2$	+	12860
cd^9f^3	+	990	$cd^2e^2f^4$	−	360	d^9f^3	−	980	$c^3d^7e^3f$	+	32000
$cd^8e^2f^2$	+	3150	cde^5f^3		...	$d^8e^2f^2$	−	3420	$c^3d^6e^5$	+	46160
cd^7e^4f	−	3600	ce^7f^2		...	d^7e^4f	+	8100	$c^2d^{10}f^2$	−	2700

W, 29 A (continued).

$a^3\ b^1c^2d^9e^2f$	$-\ 8820$	$a^2\ b^6cd^3e^2f^4$	$+\ 1440$	$a^2\ b^4cd^5e^6$	$-\ 18750$	$a^2\ b^3c^3d^7e^4$	$+\ 243000$
$c^2d^8e^4$	$-\ 34620$	$cd^7e^4f^3$	$-\ 1560$	d^9ef^2	$+\ 14115$	$c^2d^{10}ef$	$+\ 2340$
$cd^{11}ef$	$+\ 1080$	cde^6f^2	...	d^8e^2f	$-\ 23790$	$c^2d^9e^3$	$-\ 89550$
$cd^{10}e^2$	$+\ 12060$	ce^8f	...	d^7e^3	$+\ 8175$	$cd^{12}f$	$+\ 270$
$d^{13}f$	...	d^5ef^4	$+\ 960$	$b^5c^7df^5$	$+\ 1320$	$cd^{11}e^2$	$+\ 15120$
$d^{12}e^2$	$-\ 1620$	$d^4e^3f^3$	$-\ 1340$	$c^7e^2f^4$	$-\ 30$	$d^{13}e$	$-\ 810$
$b^9c^{10}f^5$	$+\ 81$	$d^3e^5f^2$	$-\ 2440$	$c^6d^2ef^4$	$+\ 540$	$b^1c^{10}ef^4$	$+\ 945$
c^9def^4	$-\ 990$	d^2e^7f	$+\ 4320$	$c^6de^3f^3$	$-\ 7240$	$c^9d^2f^4$	$-\ 675$
$c^8e^2f^3$	$+\ 980$	de^9	$-\ 1620$	$c^6e^5f^2$	$-\ 20390$	$c^8de^3f^3$	$+\ 7200$
$c^8d^3f^4$	$+\ 515$	$b^5c^5f^6$	$-\ 81$	$c^5d^4f^4$	$-\ 3900$	$c^8e^4f^2$	$-\ 14115$
$c^8d^2e^2f^3$	$+\ 140$	c^4def^3	$+\ 390$	$c^5d^3e^2f^3$	$+\ 31040$	$c^8d^2ef^3$	$-\ 12860$
$c^8de^4f^2$	$-\ 195$	$c^4e^3f^4$	$-\ 1515$	$c^5d^2e^4f^2$	$+\ 32370$	$c^8d^3e^2f^2$	$+\ 8220$
c^8e^6f	$-\ 5575$	$c^3d^2f^5$	$+\ 980$	c^5de^6f	$+\ 38820$	c^8de^5f	$+\ 150$
$c^7d^4ef^3$	$+\ 120$	$c^3d^3e^2f^4$	$+\ 7050$	c^5e^8	$+\ 9310$	c^8e^7	$+\ 6155$
$c^7d^3e^3f^2$	$-\ 800$	$c^3de^4f^3$	$-\ 6000$	$c^4d^3ef^3$	$-\ 49680$	$c^7d^5f^3$	$-\ 480$
$c^7d^2e^5f$	$+\ 22600$	$c^3e^6f^2$	$+\ 2440$	$c^4d^2e^2f^2$	...	$c^7d^4e^2f^2$	$+\ 26700$
c^7de^7	$+\ 7240$	$c^3d^4ef^4$	$-\ 15435$	$c^4d^2e^5f$	$-\ 91260$	$c^7d^3e^4f$	$+\ 63960$
$c^6d^6f^3$	...	$c^3d^3e^3f^3$	$+\ 15240$	$c^4d^2e^7$	$-\ 50550$	c^7de^6	$-\ 6660$
$c^6d^5e^2f^2$	$-\ 1260$	$c^3d^2e^5f^2$	...	$c^3d^7f^3$	$+\ 800$	$c^6d^6ef^2$	...
$c^6d^4e^4f$	$-\ 42330$	c^3de^7f	$-\ 6480$	$c^3d^6e^2f^2$	$+\ 81840$	$c^6d^5e^2f$	$-\ 180600$
$c^6d^3e^6$	$-\ 34340$	c^3e^9	$+\ 1215$	$c^3d^5e^4f$	$+\ 360$	$c^6d^4e^3$	$-\ 71610$
$c^5d^7ef^2$	$+\ 480$	cd^8f^4	$+\ 2945$	$c^3d^4e^6$	$+\ 101450$	$c^5d^8f^2$	$-\ 4755$
$c^5d^6e^2f$	$+\ 48360$	$cd^7e^2f^3$	$+\ 540$	$c^2d^8ef^2$	$-\ 8220$	$c^5d^7e^2f$	$+\ 141240$
$c^5d^5e^5$	$+\ 73828$	$cd^6e^4f^2$	$-\ 795$	$c^2d^7e^3f$	$-\ 58080$	$c^5d^6e^4$	$+\ 219730$
$c^4d^9f^2$	$+\ 105$	cd^5e^6f	$-\ 4180$	$c^2d^6e^5$	$-\ 34300$	c^4d^9ef	$-\ 45130$
$c^4d^8e^2f$	$-\ 30265$	cd^4e^8	$+\ 4185$	$cd^{10}f^3$	$-\ 7590$	$c^4d^8e^2$	$-\ 240975$
$c^4d^7e^4$	$-\ 92290$	d^7ef^3	$-\ 8460$	cd^9e^2f	$+\ 41640$	$c^3d^{11}f$	$+\ 5580$
$c^3d^{10}ef$	$+\ 9540$	$d^6e^3f^2$	$+\ 20390$	cd^8e^4	$-\ 4650$	$c^3d^{10}e^2$	$+\ 128490$
$c^3d^9e^3$	$+\ 69220$	d^5e^5f	$-\ 16194$	$d^{11}ef$	$-\ 5580$	$c^2d^{12}e$	$-\ 34155$
$c^3d^{12}f$	$-\ 1215$	d^4e^7	$+\ 3765$	$d^{10}e^3$	$+\ 1980$	cd^{14}	$+\ 3645$
$c^2d^{11}e^2$	$-\ 30510$	$b^4c^6ef^5$	$+\ 120$	$b^2c^2f^5$	$-\ 270$	$b^0c^{11}df^4$	...
$cd^{13}e$	$+\ 7290$	$c^5d^2f^5$	$-\ 2235$	c^5def^4	$-\ 3150$	$c^{11}e^2f^3$	$-\ 1890$
d^{15}	$-\ 729$	$c^3de^3f^4$	$-\ 2310$	$c^5e^3f^3$	$+\ 3420$	$c^{10}d^2ef^3$	$+\ 2700$
$a^2\ b^8cf^7$	$-\ 5$	$c^3e^4f^3$	$+\ 10755$	$c^4d^3f^4$	$+\ 2920$	$c^{10}de^3f^2$	$+\ 7590$
def^6	$+\ 15$	$c^4d^2ef^4$	$+\ 10625$	$c^4d^2e^2f^3$	$-\ 18000$	$c^{10}e^5f$	$+\ 8256$
e^3f^5	$-\ 10$	$c^4d^2e^3f^3$	$-\ 26700$	$c^4de^4f^2$	$+\ 43800$	$c^9d^4f^3$	$-\ 105$
$b^7c^2ef^6$	$+\ 10$	$c^4de^5f^2$	$+\ 795$	c^4e^6f	$+\ 5030$	$c^9d^3e^2f^2$	$-\ 14360$
cd^2f^6	$-\ 120$	c^4e^7f	$-\ 10070$	$c^3d^5ef^3$	$+\ 32280$	$c^9d^2e^4f$	$-\ 43605$
ede^2f^5	$+\ 420$	$c^3d^5f^4$	$+\ 180$	$c^3d^3e^3f^2$	$-\ 81840$	c^9de^6	$-\ 12310$
ce^4f^4	$-\ 280$	$c^3d^4e^2f^3$	$+\ 1950$	$c^3d^2e^5f$	$-\ 85800$	$c^8d^5ef^2$	$+\ 4755$
d^3ef^5	$-\ 240$	$c^3d^3e^4f^2$	...	c^3de^7	$-\ 28710$	$c^8d^4e^3f$	$+\ 77790$
$d^2e^3f^4$	$+\ 210$	$c^3d^2e^6f$	$+\ 36510$	$c^2d^6f^3$	$+\ 1260$	$c^8d^3e^5$	$+\ 59835$
de^5f^3	...	c^3de^8	...	$c^2d^5e^2f^2$	...	$c^7d^7f^2$	...
e^7f^2	...	$c^2d^6ef^3$	$+\ 25880$	$c^2d^4e^4f$	$+\ 181980$	$c^7d^6e^2f$	$-\ 57060$
$b^6c^3df^3$	$+\ 200$	$c^2d^5e^3f^2$	$-\ 32370$	$c^2d^3e^6$	$+\ 153480$	$c^7d^5e^4$	$-\ 114960$
$c^3e^2f^5$	$-\ 40$	$c^2d^4e^5f$	$-\ 12180$	$c^4d^7ef^2$	$-\ 26700$	c^6d^8ef	$+\ 19020$
$c^2d^2ef^5$	$-\ 1140$	$c^2d^3e^7$	$-\ 9850$	$c^4d^6e^2f$	$-\ 41360$	$c^6d^7e^3$	$+\ 109660$
$c^2de^4f^4$	$-\ 480$	cd^8f^3	$+\ 195$	$c^4d^5e^5$	$-\ 306900$	$c^5d^{10}f$	$-\ 2481$
$c^2e^6f^3$	$+\ 1040$	$cd^7e^2f^2$	$-\ 43800$	$c^3d^9f^2$	$+\ 14360$	$c^5d^9e^2$	$-\ 56110$
cd^4f^5	$+\ 660$	cd^6e^4f	$+\ 72755$	$c^3d^8e^3f$	$-\ 16170$	$c^4d^{11}e$	$+\ 14895$

W, 29 A (continued).

$a^2\ b^6c^3d^{13}$	−	1620	$a^1\ b^8d^8f^3$	+	5575	$a^1\ b^4cd^9e^3$	+	41250	$a^1\ b^4c^{11}e^3f^2$	+	5580
$a^1\ b^{10}f^7$	+	1	$d^7e^2f^2$	−	5030	$d^{12}f$	+	5445	$c^{10}d^3f^3$	−	9540
b^9cef^6	+	10	d^8e^4f	−	4255	$d^{11}e^2$	−	6525	$c^{10}d^2e^2f^2$	−	2340
d^2f^6	+	20	d^8e^6	+	2175	$b^3c^5ef^4$	−	900	$c^{10}de^4f$	+	20610
de^2f^5	−	130	$b^5c^6df^5$	−	1110	$c^8d^2f^4$	−	510	$c^{10}e^6$	−	4350
e^4f^4	+	90	$c^6e^2f^4$	+	870	$c^8de^3f^3$	+	120	$c^8d^4ef^2$	+	45130
$b^8c^2df^6$	−	65	$c^6d^2ef^4$	−	5550	$c^8e^4f^2$	+	23790	$c^8d^3e^2f$	−	92200
$c^2e^2f^5$	−	165	$c^6de^3f^3$	+	24240	$c^7d^3ef^3$	−	32000	$c^6d^2e^5$	−	25050
cd^2ef^6	+	870	$c^6e^5f^2$	+	16194	$c^7d^2e^2f^2$	+	58080	$c^8d^6f^2$	−	19020
cde^3f^4	−	670	$c^4d^4f^4$	−	1240	c^7de^3f	−	15440	$c^8d^5e^2f$	+	46050
ce^5f^3	+	180	$c^4d^3e^2f^3$	−	45180	c^7e^7	−	12500	$c^8d^4e^4$	+	138750
d^4f^5	−	45	$c^4d^2e^4f^2$	+	12180	$c^6d^2f^3$	−	48360	c^7d^7ef		...
$d^3e^2f^4$	+	75	c^4de^6f	−	66650	$c^6d^4e^2f^2$	+	41360	$c^7d^6e^3$	−	178200
$d^2e^4f^3$	−	135	c^4e^8	−	8550	$c^6d^3e^4f$	−	181600	c^6d^9f	−	1650
de^6f^2		...	$c^3d^5ef^3$	−	17520	$c^6d^2e^6$	−	18400	$c^6d^8e^2$	+	103950
e^8f		...	$c^3d^4e^2f^2$	+	91260	$c^5d^6ef^2$	+	180600	$c^5d^{10}e$	−	30250
$b^7c^4f^6$	+	30	$c^3d^3e^5f$		...	$c^5d^5e^3f$		...	c^4d^{12}	+	3600
c^3def^5	−	280	$c^3d^2e^7$	+	62100	$c^5d^4e^5$	+	289800	$b^9c^{13}ef^3$		...
$c^3e^3f^4$	+	2080	$c^2d^7f^3$	−	22600	$c^4d^8f^2$	−	77790	$c^{12}d^2f^3$	+	1215
$c^2d^3f^5$	−	1320	$c^2d^6e^2f^2$	+	85800	$c^4d^7e^2f$	−	87000	$c^{12}de^2f^2$	−	270
$c^3d^2ef^4$	−	3000	$c^2d^5e^4f$	−	148890	$c^4d^6e^4$	−	318500	$c^{12}e^4f$	−	5445
$c^2de^4f^3$	+	4880	$c^2d^4e^6$	+	1850	c^3d^8ef	+	92200	$c^{11}d^3ef^2$	−	5580
$c^3e^6f^2$	−	4320	cd^8ef^2	−	150	$c^3d^7e^2$	+	179500	$c^{11}d^2e^3f$	+	17520
cd^4ef^4	+	2760	cd^7e^3f	+	15440	$c^3d^{11}f$	−	17520	$c^{11}de^5$	+	8700
$cd^3e^2f^3$	−	6960	cd^6e^5	+	10350	$c^3d^{10}e^2$	−	69000	$c^{10}d^5f^2$	+	2481
$cd^2e^4f^2$	+	6480	$d^{10}f^2$	−	8256	$cd^{12}e$	+	15300	$c^{10}d^4e^2f$	−	10595
cde^7f		...	d^9e^2f	+	12210	d^{14}	−	1350	$c^{10}d^3e^4$	−	31150
ce^9		...	d^8e^4	−	7050	$b^2c^{10}df^4$	+	135	c^9d^6ef	+	1650
d^6f^4	−	1390	$b^4c^5f^5$	+	225	$c^{10}e^3f^3$	+	540	$c^9d^5e^2$	+	37950
$d^5e^2f^3$	−	600	c^7def^4	+	3600	$c^9d^2ef^3$	+	8820	c^8d^8f		...
$d^4e^4f^2$	+	10070	$c^7e^3f^3$	−	8100	$c^9de^3f^2$	−	41640	$c^8d^7e^2$	−	22275
d^3e^6f	−	12600	$c^6d^3f^4$	+	940	c^9e^5f	−	12210	c^7d^9e	+	6600
d^2e^8	+	4050	$c^6d^2e^2f^3$	+	12180	$c^8d^4f^3$	+	30265	c^6d^{11}	−	800
$b^6c^3ef^5$	−	30	$c^6de^4f^2$	−	72755	$c^8d^3e^2f^2$	+	16170	$a^0\ b^{11}ef^6$	−	5
$c^4d^2f^5$	+	1995	c^6e^6f	+	4255	$c^8d^2e^4f$	+	62025	$b^{10}cdf^6$	+	10
$c^4de^2f^4$	−	1795	$c^5d^4ef^3$	+	46880	c^8de^6	+	44225	ce^2f^5	+	75
$c^4e^4f^3$	−	9875	$c^5d^3e^2f^2$	−	360	$c^7d^5ef^2$	−	141240	d^2ef^5	−	130
$c^3d^3ef^4$	+	3220	$c^5d^2e^6f$	+	148890	$c^7d^4e^2f$	+	87000	de^3f^4	+	315
$c^3d^2e^2f^3$	+	5240	c^5de^7	+	38950	$c^7d^3e^5$	−	129000	e^5f^3	−	216
$c^3de^5f^2$	+	4180	$c^4d^6f^3$	+	42330	$c^6d^7f^2$	+	57060	$b^9c^3f^6$	−	5
c^3e^7f	+	12600	$c^4d^5e^2f^2$	−	181980	$c^6d^6e^2f$		...	c^3def^5	−	30
$c^2d^5f^4$	+	1275	$c^4d^4e^4f$		...	$c^6d^5e^4$	−	5250	$c^3e^3f^4$	−	705
$c^2d^4e^2f^3$	+	17670	$c^4d^3e^6$	−	220125	c^5d^8ef	−	46050	cd^2f^5	+	260
$c^2d^3e^4f^2$	−	36510	$c^3d^7ef^2$	−	63960	$c^5d^7e^3$	+	122800	$cd^2e^2f^4$	−	265
$c^2d^2e^6f$		...	$c^3d^6e^3f$	+	181600	$c^4d^{10}f$	+	10595	cde^4f^3	−	990
c^2de^8	−	6075	$c^3d^5e^5$	+	159000	$c^4d^9e^2$	−	88125	ce^4f^2	+	1620
cd^6ef^3	+	1820	$c^2d^9f^2$	+	43605	$c^3d^{11}e$	+	27300	d^4ef^4	−	555
$cd^5e^3f^2$	−	38820	$c^2d^8e^2f$	−	62025	c^2d^{13}	−	3375	$d^3e^3f^3$	+	1620
cd^4e^5f	+	66650	$c^2d^7e^4$	−	92500	$b^1c^{12}f^4$		...	$d^2e^5f^2$	−	1215
cd^3e^7	−	19800	$cd^{10}ef$	−	20610	$c^{11}def^3$	−	1080	de^7f		...

W, 29 A (concluded).

$a^0\ b^9e^9$		...	$a^0\ b^6c^3d^6f^3$	+	34340	$a^0\ b^4c^4d^8ef$	−	138750	$a^0\ b^1c^{10}d^4e^3$	−	17875
$b^5c^4ef^5$	+	30	$c^3d^7e^3f^2$	−	153480	$c^4d^7e^3$	−	1250	c^9d^7f	−	6600
$c^3d^2f^5$	−	370	$c^3d^4e^4f$	+	220125	$c^3d^{10}f$	+	31150	$c^8d^5e^2$	+	4125
$c^3de^2f^4$	+	1800	$c^3d^3e^6$		...	$c^3d^9e^2$	+	40000	c^8d^3e		...
$c^3e^4f^3$	+	2845	$c^2d^7ef^2$	+	6660	$c^2d^{11}e$	−	18750	c^7d^{10}		...
$c^3d^3ef^4$	+	570	$c^2d^6e^3f$	+	18400	cd^{13}	+	2250	$b^0c^{15}f^3$	+	729
$c^3d^2e^2f^3$	+	1640	$c^2d^5e^5$	−	73375	$b^5c^{11}f^4$	−	135	$c^{14}def^2$	−	3645
$c^2de^5f^2$	−	4185	cd^9f^2	+	12310	$c^{10}def^3$	−	12060	$c^{14}e^3f$	+	1350
c^2e^7f	−	4050	cd^8e^2f	−	44225	$c^{10}e^3f^2$	−	1980	$c^{13}d^3f^2$	+	1620
cd^3f^4	+	1110	cd^7e^4	+	42500	$c^9d^3f^3$	−	69220	$c^{13}d^2e^2f$	+	3375
$cd^4e^2f^3$	−	4170	$d^{10}ef$	+	4350	$c^9d^2e^2f^2$	+	89550	$c^{12}de^4$	−	2250
$cd^3e^3f^2$		...	d^9e^3	−	5125	c^9de^3f	−	41250	$c^{12}d^4ef$	−	3600
cd^2e^6f	+	6075	$b^6c^6ef^4$	−	45	c^9e^6	+	5125	$c^{12}d^3e^2$	+	2125
cde^8		...	$c^7d^2f^4$	−	2940	$c^8d^4ef^2$	+	240975	$c^{11}d^6f$	+	800
d^6ef^3	+	3620	$c^7de^2f^3$	−	9960	$c^8d^3e^3f$	−	179500	$c^{11}d^5e^2$	−	500
$d^5e^3f^2$	−	9310	$c^7e^4f^2$	−	8175	$c^8d^2e^5$	+	80125	$c^{10}d^7e$		...
d^4e^5f	+	8550	$c^6d^3ef^3$	−	46160	$c^7d^8f^2$	−	109660	c^9d^9		...
d^3e^7	−	3375	$c^6d^2e^3f^2$	+	34300	$c^7d^6e^2f$	−	122800			
$b^7c^5df^5$	+	210	c^6de^5f	−	10350	$c^7d^4e^4$	+	1250			
$c^5e^2f^4$	−	615	c^6e^7	+	7375	c^6d^7ef	+	178200			
$c^4d^2ef^4$	−	1285	$c^5d^3f^3$	−	73828	$c^6d^6e^3$		...			
$c^4de^2f^3$	−	11420	$c^5d^2e^2f^2$	+	306900	c^5d^9f	−	37950			
$c^4e^5f^2$	−	3765	$c^5d^2e^4f$	−	159000	$c^8d^8e^2$	−	37125			
$c^3d^4f^4$	−	3155	$c^5d^2e^6$	+	73375	$c^4d^{10}e$	+	17875			
$c^3d^3e^2f^3$	+	3160	$c^4d^6ef^2$	+	71610	c^3d^{12}	−	2125			
$c^3d^2e^4f^2$	+	9850	$c^4d^5e^3f$	−	289800	$b^2c^{12}ef^3$	+	1620			
c^3de^6f	+	19800	$c^4d^4e^5$		...	$c^{11}d^2f^3$	+	30510			
c^3e^8	+	3375	$c^3d^8f^2$	−	59835	$c^{11}de^2f^2$	−	15120			
$c^3d^5ef^3$	−	13500	$c^3d^7e^2f$	+	129000	$c^{11}e^4f$	+	6525			
$c^3d^4e^2f^2$	+	50550	$c^3d^6e^4$	+	80500	$c^{10}d^3ef^2$	−	128490			
$c^3d^3e^4f$	−	62100	c^2d^9ef	+	25050	$c^{10}d^2e^3f$	+	69000			
$c^3d^2e^6$		...	$c^2d^8e^3$	−	80125	$c^{10}de^5$	−	19875			
cd^7f^2	−	7240	$cd^{11}f$	−	8700	$c^9d^5f^2$	+	56110			
$cd^6e^2f^2$	+	28710	$cd^{10}e^2$	+	19875	$c^8d^4e^2f$	+	88125			
cd^5e^4f	−	38950	$d^{12}e$	−	1125	$c^8d^3e^4$	−	40000			
cd^4e^6	+	25875	$b^4c^9df^4$	+	990	c^8d^6ef	−	103950			
d^8ef^2	−	6155	$c^8e^3f^3$	+	1710	$c^8d^5e^3$	+	37125			
d^7e^3f	+	12500	$c^8d^2ef^3$	+	34620	c^7d^8f	+	22275			
d^6e^5	−	7375	$c^8de^3f^2$	+	4650	$c^7d^7e^2$		...			
$b^8c^7f^5$	−	45	c^8e^5f	+	7050	c^6d^9e	−	4125			
c^8def^4	+	615	$c^7d^4f^3$	+	92290	c^5d^{11}	+	500			
$c^8e^3f^3$	+	3880	$c^7d^3e^2f^2$	−	243000	$b^1c^{13}df^3$	−	7290			
$c^8d^3f^4$	+	4300	$c^7d^2e^4f$	+	92500	$c^{13}e^2f^2$	+	810			
$c^8d^2e^2f^3$	+	13430	c^7de^6	−	42500	$c^{12}d^2ef^2$	+	34155			
$c^8de^4f^2$	+	18750	$c^6d^5ef^2$	−	219730	$c^{12}de^3f$	−	15300			
c^8e^6f	−	2175	$c^6d^4e^3f$	+	318500	$c^{12}e^5$	+	1125			
$c^7d^4ef^3$	+	30040	$c^6d^3e^5$	−	80500	$c^{11}d^4f^2$	−	14895			
$c^7d^3e^2f^2$	−	101450	$c^5d^7f^2$	+	114960	$c^{11}d^3e^2f$	−	27300			
$c^7d^2e^5f$	−	1850	$c^5d^6e^3f$	+	5250	$c^{11}d^2e^4$	+	18750			
c^7de^7	−	25875	$c^5d^5e^4$		...	$c^{10}d^5ef$	+	30250			

For the lower covariants the numerical verifications are given for the entire coefficient, but for the higher ones where the number of terms in a coefficient is considerable they are given separately for the different powers of a; and it is also interesting to consider them for the separate combinations of a and b. I recall that the positive and negative numerical coefficients are summed separately, so that ($\pm$ a number) means that the sum of the positive numerical coefficients is equal to the sum of the negative numerical coefficients and thus that the whole sum is $=0$.

It is to be observed that for the lower covariants the sums of the numerical coefficients do not vanish for the separate powers of a: thus in the invariant G, **141**, the sums of the numerical coefficients for the terms in a^2, a^1, a^0 are $=1$, -2, 1 respectively.

As regards the invariants Q and Q'; for the first of these, Q, the sums of the numerical coefficients for the terms in a^4, a^3, a^2, a^1, a^0 are each of them $=0$, but this is not the case as regards Q'; in fact Q' is $=G^2+$ a multiple of Q; hence the sums for Q are the same as those for G^2, viz. they are $=1, -4, +6, -4, +1$ respectively. Like results present themselves in other cases, and they might probably be accounted for in a similar manner; we have a series of sums not each $=0$, but which are equal to a set of binomial coefficients taken with the signs $+$ and $-$ alternately and thus the sum of these sums is $=0$.

For R, S and S', I have given the sums for the different powers of a; and in regard to S I give here the following paragraphs from the Tenth Memoir on Quantics:—

I remark that I calculated the first two coefficients S_0, S_1, and deduced the other two, S_2 from S_1, and S_3 from S_0, by reversing the order of the letters (or which is the same thing, interchanging a and f, b and e, c and d) and reversing also the signs of the numerical coefficients. This process for S_2, S_3 is to a very great extent a verification of the values of S_0, S_1. For, as presently mentioned, the terms of S_0 form subdivisions such that in each subdivision the sum of the numerical coefficients is $=0$: in passing by the reversal process to the value of S_3, the terms are distributed into an entirely new set of subdivisions, and then in each of these subdivisions the sum of the numerical coefficients is found to be $=0$; and the like as regards S_1 and S_2.

If in the expressions for S_0, S_1, S_2, S_3 we first write $d=e=f=1$, thus in effect combining the numerical coefficients for the terms which contain the same powers in a, b, c, we find

$$\begin{aligned}
S_0 = \quad & a^3(-2c^3+6c^2-6c+2) \\
& + a^2\{b^2(6c^2-12c-6)+b(-15c^3+33c^2-21c+3) \\
& \qquad\qquad + b^0(42c^4-147c^3+195c^2-117c+27)\} \\
& + a\,\{b^4.\,0+b^3(30c^2-36c+6)+b^2(-117c^3+249c^2-183c+51) \\
& \qquad + b\,(9c^5+148c^4-378c^3+330c^2-99c)+b^0(-63c^6+165c^5-147c^4+45c^3)\}
\end{aligned}$$

$$+ a^0 . \{b^6 . 2 + b^5(-15c+3) + b^4(75c^2 - 69c + 24) + b^3(-9c^4 - 167c^3 + 225c^2 - 87c - 2)$$
$$+ b^2(72c^5 + 48c^4 - 186c^3 + 96c^2) + b(-126c^6 + 201c^5 - 87c^4)$$
$$+ b^0(27c^3 - 45c^7 + 20c^6)\}$$

which for $c=1$ becomes

$$= 2b^6 - 12b^5 + 30b^4 - 40b^3 + 30b^2 - 12b + 2, \text{ that is } 2(b-1)^6,$$

and for $b=1$, becomes $=0$.

$$S_2 = \quad a^3(0c^2 + 0c + 0)$$
$$+ a^2\{b^2(0c+0) + b(3c^3 - 9c^2 + 9c - 3) + b^0(24c^4 - 99c^3 + 153c^2 - 105c + 27)\}$$
$$+ a\ \{b^4 . 0 + b^3(-6c^3 + 12c - 6) + b^2(-24c^3 + 90c^2 - 108c + 42)$$
$$+ b(33c^4 - 90c^3 + 54c^2 + 30c - 27) + b^0(-27c^5 + 78c^3 - 66c^4 + 6c^3 + 9c^2)\}$$
$$+ a^0\{b^5(3c-3) + b^4(-15c+15) + b^3(6c^3 - 12c^2 + 36c - 30)$$
$$+ b^2(9c^5 - 42c^4 + 84c^3 - 108c^2 + 57c) + b(9c^6 - 54c^5 + 96c^4 - 51c^3)$$
$$+ b^0(9c^7 - 9c^6)\}$$

which for $c=1$ becomes $=0$.

$$S_3 = \quad a^3(0c+0)$$
$$+ a^2\{b^2 . 0 + b(0c^2 + 0c + 0) + b^0(18c^4 - 72c^3 + 108c^2 - 72c + 18)\}$$
$$+ a\ \{b^3(0c+0) + b^2(-33c^3 + 99c^2 - 99c + 33) + b(57c^4 - 162c^3 + 144c^2 - 30c - 9)$$
$$+ b^0(-60c^5 + 207c^4 - 261c^3 + 141c^2 - 27c)\}$$
$$+ a^0\{b^5 . 0 + b^4(15c^2 - 30c + 15) + b^3(-54c^3 + 102c^2 - 42c - 6)$$
$$+ b^2(123c^4 - 297c^3 + 243c^2 - 87c + 18) + b(-27c^5 + 102c^4 - 96c^3 + 21c^2)$$
$$+ b^0(27c^7 - 60c^6 + 51c^5 - 12c^4)\}$$

which for $c=1$ becomes $=0$.

$$S_4 = \quad a^3 . 0$$
$$+ a^2\{b(0c+0) + b^0(0c^3 + 0c^2 + 0c + 0)\}$$
$$+ a\ \{b^3 . 0 + b^2(0c^2 + 0c + 0) + b(-9c^4 + 36c^3 - 54c^2 + 36c - 9)$$
$$+ b^0(36c^5 - 171c^4 + 324c^3 - 306c^2 + 144c - 27)\}$$
$$+ a^0\{b^4(0c+0) + b^3(7c^3 - 21c^2 + 21c - 7) + b^2(-39c^4 + 135c^3 - 171c^2 + 93c - 18)$$
$$+ b(66c^5 - 243c^4 + 333c^3 - 201c^2 + 45c)$$
$$+ b^0(-27c^7 + 101c^6 - 141c^5 + 87c^4 - 20c^3)\}$$

which for $c=1$ becomes $=0$.

It follows that for $c=d=e=f=1$, the value of the covariant S is $=2(b-1)^6x^3$, which might be easily verified.

For T, U, V and W, I look at the sums for the different combinations of a and b.

Thus for T we have

	x coefficient.				y coefficient.		
$a^4 b^0$		26		$a^4 b^0$		± 12	
			± 26				± 12
$a^3 b^2$		± 14		$a^3 b^2$		± 2	
b^4		141		b^4		112	
b^0		281		b^0		281	
			± 436				± 395
$a^2 b^4$		± 1		$a^2 b^0$		± 42	
b^3		106		b^2		546	
b^2		186		b^1		696	
b^1		1173		b^0		366	
b^0		2272					
			± 3738				± 1650
$a^1 b^6$		± 16		$a^1 b^5$		± 5	
b^4		359		b^4		179	
b^3		1411		b^3		821	
b^2		3103		b^2		2097	
b^1		3030		b^1		2147	
b^0		1197		b^0		1262	
			± 9116				± 6511
$a^0 b^7$		− 2					
b^6	92	− 78		$a^0 b^0$		± 28	
b^5	307	− 349		b^5		342	
b^4	1073	− 1003		b^4		1790	
b^3	2040	− 2110		b^3		3496	
b^2	1930	− 1880		b^2		3445	
b^1	1207	− 1221		b^1		2064	
b^0	231	− 239		b^0		463	
			± 6880				± 11628
			± 20196				± 20196

Observe here that in the x-coefficient for the terms in a^0 the successive sums are -2, $+14$, -42, $+70$, -70, $+42$, $-14+2$, which are the coefficients of $-2(\theta-1)^7$.

For U we have

$a^4 b^0$	$\pm$ 36	
		$\pm$ 36
$a^3 b^2$	$\pm$ 24	
b^1	198	
b^0	242	
		$\pm$ 464
$a^2 b^4$	$\pm$ 2	
b^3	208	
b^2	286	
b^1	866	
b^0	1246	
		$\pm$ 2608
$a^1 b^5$	$\pm$ 64	
b^4	328	
b^3	1258	
b^2	2586	
b^1	2186	
b^0	856	
		$\pm$ 7278
$a^0 b^7$	$\pm$ 4	
b^0	70	
b^5	448	
b^4	1488	
b^3	2140	
b^2	1678	
b^1	884	
b^0	166	
		$\pm$ 6878
		$\pm$ 17264

For V we have

	x coefficient.		y coefficient.	
$a^5 b^0$	$\pm$ 36		$a^5 b^0$	$\pm$ 24
		$\pm$ 36		$\pm$ 24
$a^4 b^2$	$\pm$ 20		$a^4 b^2$	$\pm$ 4
b^1	284		b^1	144
b^0	1094		b^0	436
		$\pm$ 1398		$\pm$ 584
$a^3 b^4$	2		$a^3 b^0$	$\pm$ 24
b^3	184		b^2	776
b^2	1656		b^1	2696
b^1	3624		b^0	1264
b^0	4898			$\pm$ 4760
		$\pm$ 10364		
$a^2 b^5$	$\pm$ 14		$a^2 b^0$	$\pm$ 6
b^4	666		b^4	300
b^3	6608		b^3	2236
b^2	10512		b^2	8616
b^1	22042		b^1	15442
b^0	9162		b^0	33044
		$\pm$ 49004		$\pm$ 59644
$a^1 b^7$	$-$ 4		$a^1 b^6$	$\pm$ 78
b^3	76 $-$ 48		b^5	852
b^3	2956 $-$ 3040		b^4	8310
b^4	11946 $-$ 11806		b^3	30200
b^3	23924 $-$ 24064		b^2	56740
b^2	25110 $-$ 25026		b^1	39956
b^1	25524 $-$ 25552		b^3	17986
b^0	8822 $-$ 8812			$\pm$ 154122
		$\pm$ 98358		
$a^0 b^8$	18		$a^0 b^8$	$-$ 2
b^7	184 $-$ 324		b^7	286 $-$ 270
b^6	4098 $-$ 3622		b^6	2026 $-$ 2082
b^5	19350 $-$ 20274		b^5	9360 $-$ 9248
b^4	42398 $-$ 41278		b^4	19760 $-$ 19900
b^3	51872 $-$ 52740		b^3	36442 $-$ 36330
b^2	44320 $-$ 43900		b^2	30340 $-$ 30396
b^1	20624 $-$ 20740		b^1	23426 $-$ 23410
b^0	3870 $-$ 3856		b^0	5120 $-$ 5122
		$\pm$ 186734		$\pm$ 126760
		$\pm$ 345894		$\pm$ 345894

Here in the x-coefficient for a^1 the successive sums are -4, $+28$, -84, $+140$, -140, $+84$, -28, $+4$, which are the coefficients of $-4(\theta-1)^7$; and for a^0 the successive sums are 18, -140, $+476$, -924, $+1120$, -868, $+420$, -116, $+14$, which are the coefficients of $18(\theta-1)^8+4(\theta-1)^7$. In the y-coefficient the successive sums are -2, $+16$, -56, $+112$, -140, $+112$, -56, $+16$, -2, which are the coefficients of $-2(\theta-1)^8$.

Finally for W we have

$a^7 b^0$	$\pm$	16	
		$\pm$	16
$a^6 b$	$\pm$	175	
b^0		806	
		$\pm$	981
$a^5 b^6$	$\pm$	80	
b^2		1175	
b^1		2760	
b^0		6871	
		$\pm$	10886
$a^4 b^4$	$\pm$	570	
b^3		5200	
b^2		18005	
b^1		44720	
b^0		23810	
		$\pm$	92305
$a^3 b^6$	$\pm$	90	
b^5		2386	
b^4		26675	
b^3		84680	
b^2		107730	
b^1		199160	
b^0		240499	
		$\pm$	661220
$a^2 b^6$	$\pm$	15	
b^7		640	
b^6		8260	
b^5		59135	
b^4		182055	
b^3		341470	
b^2		699260	
b^1		612015	
b^0		304501	
		$\pm$	2207351

				$\pm$ 2972759
$a^1 b^{10}$	+	1		
b^9		120	− 130	
b^8		1125	− 1080	
b^7		30350	− 30470	
b^6		122400	− 122190	
b^5		332494	− 332746	
b^4		729150	− 728940	
b^3		880750	− 880870	
b^2		466935	− 466890	
b^1		363670	− 363680	
b^0		76116	− 76115	
				$\pm$ 3003111
$a^0 b^{11}$	+		− 5	
b^{10}		400	− 346	
b^9		3500	− 3765	
b^8		26240	− 25460	
b^7		154030	− 155560	
b^6		409700	− 407600	
b^5		747985	− 750043	
b^4		745920	− 744480	
b^3		613100	− 613805	
b^2		311790	− 311560	
b^1		89215	− 89260	
b^0		9999	− 9995	
				$\pm$ 3111879
				$\pm$ 9087749

Here for the terms in a^1 the successive sums are

$$1,\ -10,\ +45,\ -120,\ +210,\ -252,\ +210,\ -120,\ +45,\ -10,\ +1,$$

which are the coefficients of $(\theta-1)^{10}$; and for the terms in a^0 the successive sums are

$$-5,\ +54,\ -265,\ +780,\ -1530,\ +2100,\ -2058,\ +1440,\ -705,\ +230,\ -45,\ +4,$$

which are the coefficients of $-5(\theta-1)^{11}-(\theta-1)^{10}$.

144.

A THIRD MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVI. *for the year* 1856, pp. 627—647. Received March 13,—Read April 10, 1856.]

MY object in the present memoir is chiefly to collect together and put upon record various results useful in the theories of the particular quantics to which they relate. The tables at the commencement relate to binary quantics, and are a direct sequel to the tables in my Second Memoir upon Quantics, vol. CXLVI. (1856), [141]. The definitions and explanations in the next part of the present memoir are given here for the sake of convenience, the further development of the subjects to which they relate being reserved for another occasion. The remainder of the memoir consists of tables and explanations relating to ternary quadrics and cubics.

Covariant and other Tables, Nos. 27 to 50 (Nos. 1 to 50 binary quantics)[1].

Nos. 27 to 29 are a continuation of the tables relating to the quintic

$$(a, b, c, d, e, f \between x, y)^5.$$

No. 27 gives the values of the different determinants of the matrix

$$\left(\begin{array}{cccccc} & a, & 4b, & 6c, & 4d, & e \\ a, & 4b, & 6c, & 4d, & e & \\ & b, & 4c, & 6d, & 4e, & f \\ b, & 4c, & 6d, & 4e, & f & \end{array}\right)$$

determinants which are represented by 1234, 1235, &c., where the numbers refer to

[1] The Tables 49 and 50 were inserted October 6, 1856.—A. C.

the different columns of the matrix. No. 28 gives the values of certain linear functions of these determinants, viz.

$$L = 1256 + 2345 - 2\,.\,1346,$$
$$L' = 3\,.\,1256 - 1346,$$
$$8M = -1345 + 2\,.\,1246,$$
$$8M' = -2346 + 2\,.\,1356,$$
$$8N = -1245 + 3\,.\,1236,$$
$$8N' = -2356 + 3\,.\,1456,$$
$$80P = L' - 3L = 5\,.\,1346 - 3\,.\,2345,$$
$$16P' = -5L' - L = -16\,.\,1256 - 3\,.\,1346 - 2356.$$

At the end of the two tables there are given certain relations which exist between the terms of Tables 14, 16, 25, 26, 27 and 28.

No. 27.

1234.	1235.	1236.	1245.	1246.	1345.	1256.	2345.
a^2bf ...	$a^2cf - 4$	$a^2df + 6$	$a^2df - 6$	$a^2ef + 4$	a^2ef ...	$a^2f^2 + 1$	a^2f^2 ...
$a^2ce - 16$	$a^2de + 24$	a^2e^2 ...	$a^2e^2 + 16$	$abdf - 4$	$abdf - 24$	$abef - 2$	$abef$ ...
$a^2d^2 + 36$	$ab^2f + 4$	$abcf - 22$	$abcf + 6$	$abc^2 - 4$	$abe^2 + 64$	$acdf - 16$	$acdf + 20$
$ab^2e + 16$	$abce - 84$	$abde - 6$	$abde - 26$	$ac^2f - 24$	$ac^2f + 24$	$ace^2 + 16$	$ace^2 - 80$
$abcd - 152$	$abd^2 - 24$	$ac^2e + 16$	$ac^2e - 96$	$acde + 24$	$acde - 208$	$ad^2e + 16$	$ad^2e + 60$
$ac^3 + 96$	$ac^2d + 64$	acd^2 ...	$acd^2 + 96$	ad^3 ...	$ad^3 + 144$	$b^2df - 15$	$b^2df - 80$
$b^2d + 80$	$b^3e + 60$	$b^3f + 16$	b^3f ...	$b^2cf + 24$	b^2cf ...	b^2e^2 ...	$b^2e^2 + 240$
$b^2c^2 - 60$	$b^2cd - 40$	$b^2ce - 10$	$b^2ce + 90$	$b^2de - 20$	$b^2de - 40$	bc^2f ...	$bc^2f + 60$
	be^3 ...	b^2d^2 ...	$b^2d^2 - 80$	bc^2e ...	$bc^2e + 60$	$bcde$ ...	$bcde - 860$
		bc^2d ...	bc^2d ...	bcd^2 ...	$bcd^2 - 40$	bd^3 ...	$bd^3 + 960$
		c^4 ...	c^4 ...	c^3d ...	c^3d ...	c^3e ..	$c^3e + 960$
						c^2d^2 ...	$c^2d^2 - 320$

1346.	2346.	1356.	2356.	1456.	2456.	3456.
a^2f^2 ...	abf^2 ...	$abf^2 + 4$	$acf^2 - 6$	$acf^2 + 6$	$adf^2 - 4$	aef^2 ...
$abef + 16$	$acef - 24$	$acef - 4$	$adef + 6$	$adef - 22$	$ae^2f + 4$	$bdf^2 - 16$
$acdf - 36$	$ad^2f + 24$	$ad^2f - 24$	ae^3 ...	$ae^3 + 16$	$bcf^2 + 24$	$be^2f + 16$
$ace^2 - 16$	ade^2 ...	$ade^2 + 24$	$b^2f^2 + 16$	b^2f^2 ...	$bdef - 84$	$c^2f^2 + 36$
$ad^2e + 36$	$b^2ef + 64$	$b^2ef - 4$	$bcef - 26$	$bcef - 6$	$be^3 + 60$	$cdef - 152$
$b^2df - 16$	$bcdf - 208$	$bcdf + 24$	$b^2df - 96$	$bd^2f + 16$	$c^2ef - 24$	$ce^3 + 80$
b^2e^2 ...	$bce^2 - 40$	$bce^2 - 20$	$bde^2 + 90$	$bde^2 - 10$	$cd^2f + 64$	$d^3f + 96$
$bc^2f + 36$	$bd^2e + 60$	bd^2e ...	$c^3df + 96$	c^3df ...	$cde^2 - 40$	$d^2e^2 - 60$
$bcde - 20$	$c^3f + 144$	c^3f ...	$c^2e^2 - 80$	c^2e^2 ...	d^3e ...	
bd^3 ...	$c^2de - 40$	c^2de ...	cd^2e ...	cd^2e ...		
c^3e ...	cd^3 ...	cd^3 ...	d^4 ...	d^4 ...		
c^2d^2 ...						

No. 28.

N.	M.	L.	L′.	P.	P′.	M′.	N′.
$a^2df + 3$	$a^2ef + 1$	$a^2f^2 + 1$	$a^2f^2 + 3$	a^2f^2 ...	$a^2f^2 - 1$	$abf^2 + 1$	$acf^2 + 3$
$a^2e^2 - 2$	$abdf + 2$	$abef - 34$	$abef - 22$	$abef + 1$	$abef + 9$	$acef + 2$	$adef - 9$
$abcf - 9$	$abe^2 - 9$	$acdf + 76$	$acdf - 12$	$acdf - 3$	$acdf - 1$	$ad^2f - 9$	$ae^3 + 6$
$abde + 1$	$ac^2f - 9$	$ace^2 - 32$	$ace^2 + 64$	$ace^2 + 2$	$ace^2 - 18$	$ade^2 + 6$	$b^2f^2 - 2$
$ac^2e + 18$	$acde + 32$	$ad^2e - 12$	$ad^2e - 36$	ad^2e ...	$ad^2e + 12$	$b^2ef - 9$	$bcef + 1$
$acd^2 - 12$	$ad^3 - 18$	$b^2df - 32$	$b^2df + 64$	$b^2df + 2$	$b^2df - 18$	$bcdf + 32$	$bd^2f + 18$
$b^3f + 6$	$b^2cf + 6$	$b^2e^2 + 225$	$b^2e^2 - 45$	$b^2e^2 - 9$	b^2e^2 ...	bce^2 ...	$bde^2 - 15$
$b^2ce - 15$	b^2de ...	$bc^2f - 12$	$bc^2f - 36$	bc^2f ...	$bc^2f + 12$	$bd^2e - 15$	$c^2df - 12$
$b^2d^2 + 10$	$bc^2e - 15$	$bcde - 820$	$bcde + 20$	$bcde + 31$	$bcde + 45$	$c^3f - 18$	$c^2e^2 + 10$
bc^2d ...	$bcd^2 + 10$	$bd^3 + 480$	bd^3 ...	$bd^3 - 18$	$bd^3 - 30$	$c^2de + 10$	cd^2e ...
c^4 ...	c^3d ...	$c^3e + 480$	c^3e ...	$c^3e - 18$	$c^3e - 30$	cd^3 ...	d^4 ...
		$c^2d^2 - 320$	c^2d^2 ...	$c^2d^2 + 12$	$c^2d^2 + 20$		

If the coefficients of the table 14 are represented by $\frac{1}{2}A$, B, $\frac{1}{2}C$, viz. writing

$$A = 2\,(ae - 4bd + 3c^2),$$
$$B = \quad af - 3be + 2cd,$$
$$C = 2\,(bf - 4ce + 3d^2),$$

then we have the following relations between 1234, &c. and A, B, C, viz.

	$C\times$	$+B\times$	$+A\times$
$1234 =$	$+ 6\,a^2$	$- 12\,ab$	$+ 16\,ac - 10\,b^2$
$1235 =$	$+ 6\,ab$	$- 2\,ac - 10\,b^2$	$+ 6\,ad$
$1236 =$	$- 2\,ac + 8\,b^2$	$+ 6\,ad - 18\,bc$	$- 2\,df + 8\,e^2$
$1245 =$	$+ 18\,ac$	$- 6\,ad - 30\,bc$	$+ 8\,ae + 10\,bd$
$1246 =$	$+ 12\,bc$	$+ 4\,ae - 4\,bd - 24\,c^2$	$+ 4\,be + 8\,cd$
$1345 =$	$+ 24\,ad$	$- 8\,ae - 40\,bd$	$+ 4\,af + 20\,be$
$1256 =$	$- 1\,ae + 4\,bd + 3\,c^2$	$+ 1\,af + 5\,be - 18\,cd$	$- 1\,bf + 4\,ce + 3\,d^2$
$2345 =$	$+ 20\,ae + 40\,bd - 30\,c^2$	$- 80\,be + 20\,cd$	$+ 20\,bf + 40\,ce - 30\,d^2$
$1346 =$	$+ 4\,ae + 8\,bd + 6\,c^2$	$- 36\,cd$	$+ 4\,bf + 8\,ce + 6d^2$
$2346 =$	$+ 4\,af + 20\,be$	$- 8\,bf - 4\,ce$	$+ 24\,cf$
$1356 =$	$+ 4\,be + 8\,cd$	$+ 4\,bf - 4\,ce - 24\,d^2$	$+ 12\,de$
$2356 =$	$+ 8\,bf + 10\,ce$	$- 6\,cf - 30\,de$	$+ 18\,df$
$1456 =$	$+ 6\,ce$	$+ 6\,cf - 18\,de$	$- 2\,df + 8\,e^2$
$2456 =$	$+ 6\,cf$	$- 2\,df - 10\,e^2$	$+ 6\,ef$
$3456 =$	$+ 16\,df - 10\,e^2$	$- 12\,ef$	$+ 6\,f^2$

and the following relations between L, L', &c. and A, B, C, viz.

	$C\times$	$+B\times$	$+A\times$
$N =$	$- 3\,ac + 3\,b^2$	$+ 3\,ad - 3\,bc$	$- 1\,ae + 1\,bd$
$M =$	$- 3\,ad + 3\,bc$	$+ 3\,ae - 3\,c^2$	$- 1\,af + 1\,cd$
$L =$	$+ 11\,ae + 28\,bd - 39\,c^2$	$+ 1\,af - 75\,be + 74\,cd$	$+ 11\,bf + 28\,ce - 39\,d^2$
$L' =$	$- 7\,ae + 4\,bd + 3\,c^2$	$+ 3\,af + 15\,be - 18\,cd$	$- 7\,bf + 4\,ce + 3\,d^2$
$2P =$	$- 1\,ae - 2\,bd + 3\,c^2$	$+ 3\,be - 3\,cd$	$+ 1\,bf + 2\,ce - 3\,d^2$
$P' =$	$+ 3\,ae - 6\,bd + 3\,c^2$	$- 1\,af + 1\,cd$	$+ 3\,bf - 6\,ce + 3\,d^2$
$M' =$	$- 1\,af + 1\,cd$	$+ 3\,bf - 3\,d^2$	$- 3\,cf + 3\,de$
$N' =$	$- 1\,bf + 1\,ce$	$+ 3\,cf - 3\,de$	$- 3\,df + 3\,e^2$

We have also the following relations between L, L', &c. and a, b, c, d, e, f, viz.

$$\begin{aligned}
aP - bM + cN &= 0,\\
aM' + bP' - 2cM + 3dN &= 0,\\
aN' + 2bM' - cL' \quad . \quad + 3eN &= 0,\\
3bN' \quad . \quad - dL' + 2eM + fN &= 0,\\
3cN' - 2dM' + eP' + fM &= 0,\\
dN' - eM' + fP &= 0.
\end{aligned}$$

The quartinvariant No. 19 [G] is equal to

$$-AC + B^2,$$

i.e. it is in fact equal to -4 into the discriminant of the quintic No. 14, [A].

The octinvariant No. 25 [Q] is expressible in terms of the coefficients of Nos. 14 and 16, viz. A, B, C, as before, and $\frac{1}{3}\alpha$, β, γ, $\frac{1}{3}\delta$ the coefficients of No. 16, [D], i.e.

$$\begin{aligned}
\alpha &= 3(ace - ad^2 - b^2e + 2bcd - c^3),\\
\beta &= acf - ade - b^2f + bd^2 + bce - c^2d,\\
\gamma &= adf - ae^2 - bcf + bde + c^2e - cd^2,\\
\delta &= 3(bdf - be^2 + 2cde - c^2f - d^3),
\end{aligned}$$

then No. 25 is equal to

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}.$$

The value of the discriminant No. 26, [Q'], is

$$(\text{No. } 19)^2 - 128 \text{ No. } 25. \quad [\text{that is } Q' = G^2 - 128Q.]$$

We have also an expression for the discriminant in terms of L, L', &c., viz. three times the discriminant No. 26 is equal to

$$[\text{or say } 3Q' =]\ LL' + 64MM' - 64NN',$$

a remarkable formula, the discovery of which is due to Mr Salmon.

It may be noticed, that in the particular case in which the quintic has two square factors, if we write

$$(a, b, c, d, e, f \between x, y)^5 = 5\{(p, q, r \between x, y)^2\}^2 \,.\, (\lambda, \mu \between x, y),$$

then

$$a = 5\lambda p^2, \quad b = 4pq\lambda + p^2\mu, \quad c = (2q^2 + pr)\lambda + 2pq\mu,$$
$$f = 5r^2\mu, \quad e = r^2\lambda + 4qr\mu, \quad d = 2qr\lambda + (2q^2 + pr)\mu;$$

and these values give

$$P = K(6q^2 - pr), \qquad P' = K(10q^2 - 15pr),$$
$$M = K \,.\, 10pq, \qquad M' = K \,.\, 10qr,$$
$$N = K \,.\, 5p^2, \qquad N' = K \,.\, 5r^2,$$

where the value of K is

$$8(p\mu^2 - 2q\mu\lambda + r\lambda^2)^2(pr - q^2)^2.$$

The table No. 29 is the invariant of the twelfth degree of the quintic, given in its simplest form, i.e. in a form not containing any power higher than the fourth of the leading coefficient a: this invariant was first calculated by M. Faa de Bruno.

No. 29. [See U. No. 29, p. 294.]

The tables Nos. 30 to 35 relate to a sextic. No. 30 is the sextic itself; No. 31 the quadrinvariant; Nos. 32 and 33 the quadricovariants (the latter of them the Hessian); No. 34 is the quartinvariant or catalecticant; and No. 35 is the sextinvariant in its best form, i.e. a form not containing any power higher than the second of the leading coefficient a.

No. 30.

$$\left(\begin{array}{|c|c|c|c|c|c|c|}\hline a+1 & b+6 & c+15 & d+20 & e+15 & f+6 & g+1 \\ \hline\end{array}\right)(x, y)^6$$

No. 31.

$$\begin{array}{|c|}\hline ag + 1 \\ bf - 6 \\ ce + 15 \\ d^2 - 10 \\ \hline \end{array}$$
$$\pm 16$$

No. 32.

$$\left(\begin{array}{|c|c|c|c|c|}\hline ae + 1 & af + 2 & ag + 1 & bg + 2 & cg + 1 \\ bd - 4 & be - 6 & ce - 9 & cf - 6 & df - 4 \\ c^2 + 3 & cd + 4 & d^2 + 8 & de + 4 & e^2 + 3 \\ \hline \end{array}\right)(x, y)^4$$
$$\pm 4 \qquad \pm 6 \qquad \pm 9 \qquad \pm 6 \qquad \pm 4$$

No. 33.

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline ac + 1 & ad + 4 & ae + 6 & af + 4 & ag + 1 & bg + 4 & cg + 6 & dg + 4 & eg + 1 \\ b^2 - 1 & bc - 4 & bd + 4 & be + 16 & bf + 14 & cf + 16 & df + 4 & ef - 4 & f^2 - 1 \\ & & c^2 - 10 & cd - 20 & ce + 5 & de - 20 & e^2 - 10 & & \\ & & & & d^2 - 20 & & & & \\ \hline \end{array}\right)(x, y)^8$$
$$\pm 1 \qquad \pm 4 \qquad \pm 10 \qquad \pm 20 \qquad \pm 20 \qquad \pm 20 \qquad \pm 10 \qquad \pm 4 \qquad \pm 1$$

No. 34.

$aceg$	$+1$
acf^2	-1
ad^2g	-1
$adef$	$+2$
ae^3	-1
b^2eg	-1
b^2f^2	$+1$
$bcdg$	$+2$
$bcef$	-2
bd^2f	-2
bde^2	$+2$
c^3g	-1
c^2df	$+2$
c^2e^2	$+1$
cd^2e	-3
d^4	$+1$

± 12

No. 35.

$a^2d^2g^2$	$+1$	$acde^2f$ -42	bc^2df^2 $+60$
a^2defg	-6	ace^4 $+12$	bc^2e^2f -30
a^2df^3	$+4$	ad^4g -20	bcd^3g $+24$
a^2e^3g	$+4$	ad^3ef $+24$	bcd^2ef -84
$a^2e^2f^2$	-3	ad^2e^3 -8	$bcde^3$ $+66$
$abcdg^2$	-6	b^3dg^2 $+4$	bd^4f $+24$
$abcefg$	$+18$	b^3efg -12	bd^3e^2 -24
$abef^3$	-12	b^2f^3 $+8$	c^4eg $+12$
abd^2fg	$+12$	$b^2c^2g^2$ -3	c^4f^2 -27
$abde^2g$	-18	b^2ce^2g $+30$	c^3d^2g -8
abe^3f	$+6$	b^2cef^2 -24	c^3def $+66$
ac^3g^2	$+4$	b^2d^2eg -12	c^3e^3 -8
ac^2e^2g	-24	$b^2d^2f^2$ -24	c^2d^3f -24
ac^2dfg	-18	b^2de^2f $+60$	$c^2d^2e^2$ -39
ac^2ef^2	$+30$	b^2e^4 -27	cd^4e $+36$
acd^2eg	$+54$	bc^2fg $+6$	d^6 -8
acd^2f^2	-12	bc^2deg -42	

± 565

The sextinvariant may be thus represented by means of a determinant of the sixth order and of the quadrinvariant and quartinvariant.

$$5 \times \text{No. } 35 = \begin{vmatrix} & a, & 2b, & 3c, & 4d, & e \\ & b, & 2c, & 3d, & 4e, & f \\ & c, & 2d, & 3e, & 4f, & g \\ a, & 4b, & 3c, & 2d, & e & \\ b, & 4c, & 3d, & 2e, & f & \\ c, & 4d, & 3e, & 2f, & g & \end{vmatrix}$$

$$+4(ag-6bf+15ce-10d^2)\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

The tables Nos. 36 and 37 relate to a septimic. No. 36 is the septimic itself; No. 37 the quartinvariant.

No. 36.

$a+1$	$b+7$	$c+21$	$d+35$	$e+35$	$f+21$	$g+7$	$h+1$

$(\ \ )(x, y)^7$

No. 37.

a^2h^2	$-\ 1$	bd^2h	$-\ 40$
$abgh$	$+\ 14$	$bdeg$	$-\ 50$
$acfh$	$-\ 18$	bdf^2	$-\ 360$
acg^2	$-\ 24$	be^2f	$+\ 240$
$adeh$	$+\ 10$	c^2eg	$-\ 360$
$adfg$	$+\ 60$	c^3f^2	$-\ 81$
ae^2g	$-\ 40$	cd^2g	$+\ 240$
b^2fh	$-\ 24$	$cdef$	$+\ 990$
b^2g^2	$-\ 25$	ce^3	$-\ 600$
$bcfg$	$+\ 234$	d^3f	$-\ 600$
$bceh$	$+\ 60$	d^2e^2	$+\ 375$
			± 2223

The tables Nos. 38 to 45 relate to the octavic. No. 38 is the octavic itself; No. 39 the quadrinvariant; Nos. 40, 41 and 42 are the quadricovariants, the last of them being the Hessian; No. 43 is the cubinvariant; No. 44 the quartinvariant, and No. 45 the quintinvariant, which is also the catalecticant.

No. 38.

$$\left(\begin{array}{ccccccccc} a+1 & b+8 & c+28 & d+56 & e+70 & f+56 & g+28 & h+8 & i+1 \end{array}\right) \between(x, y)^8$$

No. 39.

$ai + 1$
$bh - 8$
$cg + 28$
$df - 56$
$e^2 + 35$
± 64

No. 40.

$$\left(\begin{array}{ccccc} ag + 1 & ah + 2 & ai + 1 & bi + 2 & ci + 1 \\ bf - 6 & bg - 10 & bh - 2 & ch - 10 & dh - 6 \\ ce + 15 & cf + 18 & cg - 8 & dg + 18 & eg + 15 \\ d^2 - 10 & de - 10 & df + 34 & ef - 10 & f^2 - 10 \\ & & e^2 - 25 & & \\ \pm 16 & \pm 20 & \pm 35 & \pm 20 & \pm 16 \end{array}\right) \between(x, y)^4.$$

No. 41.

$$\left(\begin{array}{ccccccccc} ae + 1 & af + 4 & ag + 6 & ah + 4 & ai + 1 & bi + 4 & ci + 6 & di + 4 & ei + 1 \\ bd - 4 & be - 12 & bf - 8 & bg + 8 & bh + 12 & ch + 8 & dh - 8 & eh - 12 & fh - 4 \\ c^2 + 3 & cd + 8 & ce - 22 & cf - 48 & cg - 22 & dg - 48 & eg - 22 & fg + 8 & g^2 + 3 \\ & & d^2 + 24 & de + 36 & df - 36 & ef + 36 & f^2 + 24 & & \\ & & & & e^2 + 45 & & & & \\ \pm 4 & \pm 12 & \pm 30 & \pm 48 & \pm 58 & \pm 48 & \pm 30 & \pm 12 & \pm 4 \end{array}\right) \between(x, y)^8$$

No. 42.

$$\left(\begin{array}{ccccccccccccc} ac + 1 & ad + 6 & ae + 15 & af + 20 & ag + 15 & ah + 6 & ai + 1 & bi + 6 & ci + 15 & di + 20 & ei + 15 & fi + 6 & gi + 1 \\ b^2 - 1 & bc - 6 & bd + 6 & be + 50 & bf + 90 & bg + 78 & bh + 34 & ch + 78 & dh + 90 & eh + 50 & fh + 6 & gh - 6 & h^2 - 1 \\ & & c^2 - 21 & cd - 70 & d^2 - 105 & cf + 126 & cg + 154 & dg + 126 & f^2 - 105 & fg - 70 & g^2 - 21 & & \\ & & & & & de - 210 & df - 14 & ef - 210 & & & & & \\ & & & & & & e^2 - 175 & & & & & & \\ \pm 1 & \pm 6 & \pm 21 & \pm 70 & \pm 105 & \pm 210 & \pm 189 & \pm 210 & \pm 105 & \pm 70 & \pm 21 & \pm 6 & \pm 1 \end{array}\right) \between(x, y$$

No. 43.

aei	$+1$
afh	-4
ag^2	$+3$
bdi	-4
beh	$+12$
bfg	-8
c^2i	$+3$
cdh	-8
ceg	-22
cf^2	$+24$
d^2g	$+24$
def	-36
e^3	$+15$
	± 82

No. 44.

$acgi$	-1	$bcgh$	$+3$	cd^2i	-2
ach^2	$+1$	$bdei$	$+1$	$cdeh$	-23
$adfi$	$+3$	$bdfh$	-10	$cdfg$	$+27$
$adgh$	-3	bdg^2	$+9$	ce^2g	$+19$
ae^2i	-2	be^2h	$+11$	cef^2	-21
$aefh$	$+1$	$befg$	-23	d^3h	$+12$
aeg^2	$+3$	bf^3	$+12$	d^2eg	-21
af^2g	-2	c^3ei	$+3$	d^2f^2	-13
b^2gi	$+1$	c^2fh	$+9$	de^2f	$+32$
b^2h^2	-1	c^2g^2	-12	e^4	-10
$bcfi$	-3				
					± 147

No. 45.

$acegi$	$+1$	af^4	$+1$	$bdeg^2$	-4	cd^2g^2	$+1$
$aceh^2$	-1	b^2egi	-1	bdf^2g	$+2$	$cdefg$	-2
acf^2i	-1	b^2eh^2	$+1$	be^3h	-2	edf^3	-2
$acfgh$	$+2$	b^2fgh	-2	be^2fg	$+4$	ce^3g	-3
acg^3	-1	b^2fi^2	$+1$	bef^3	-2	ce^2dh	$+4$
ad^2gi	-1	b^2g^3	$+1$	c^3gi	-1	ce^2f^2	$+3$
ad^2h^2	$+1$	$bcdgi$	$+2$	c^3h^2	$+1$	d^4i	$+1$
$adefi$	$+2$	$bcdh^2$	-2	c^2dfi	$+2$	d^3eh	-2
$adegh$	-2	$bcefi$	-2	c^2dgh	-2	d^3fg	-2
adf^2h	-2	$bcegh$	$+2$	c^2e^2i	$+1$	d^2e^2g	$+3$
$adfg^2$	$+2$	bcf^2h	$+2$	c^2efh	-4	d^2ef^2	$+3$
ae^3i	-1	$bcfg^2$	-2	c^2eg^2	$+2$	de^3f	-4
ae^2fh	$+2$	bd^2fi	-2	c^2f^2g	$+1$	e^5	$+1$
ae^2g^2	$+1$	bd^2gh	$+2$	cd^2ei	-3		
aef^2g	-3	bde^2i	$+2$	cd^2fh	$+2$		
							± 56

If we write

$$\text{No. } 39 = I,$$
$$\text{No. } 43 = J,$$
$$\text{No. } 44 = K,$$
$$\text{No. } 45 = L,$$

then the determinant called the lambdaic, viz.

$$\begin{vmatrix} a, & b, & c, & d, & e-12\lambda \\ b, & c, & d, & e+3\lambda, & f \\ c, & d, & e-2\lambda, & f, & g \\ d, & e+3\lambda, & f, & g, & h \\ e-12\lambda, & f, & g, & h, & i \end{vmatrix}$$

is equal to

$$L + 2\lambda K + 3\lambda^2 J + 18\lambda^3 I - 2592\lambda^5.$$

Nos. 46 to 48 relate to the nonic. No. 46 is the nonic itself; Nos. 47 and 48 are the two quartinvariants, each of them in its best form, viz. No. 48 does not contain a^2, and No. 47 does not contain aci^2, the leading term of No. 48. The nonic is the lowest quantic with two quartinvariants.

No. 46.

$a+1$	$b+9$	$c+36$	$d+84$	$e+126$	$f+126$	$g+84$	$h+36$	$i+9$	$j+1$	$\mathfrak{Q}(x, y)^9$.

No. 47.

a^2j^2	-1	bf^2h	...
$abij$	$+18$	bfg^2	-720
aci^2	...	c^2fj	$+432$
$achj$	-72	c^2gi	-1728
$adgj$	$+168$	c^3h^2	...
$adhi$	...	$cdej$	-720
$aefj$	-108	$cdfi$	$+2160$
$aegi$	-576	$cdgh$	$+4608$
aeh^2	$+432$	ce^2i	...
af^2i	$+540$	$cefh$	-2592
$afgh$	-720	ceg^2	-5760
ag^3	$+320$	cf^2g	$+4320$
b^2hj	...	d^2j	$+320$
b^2i^2	-81	d^2ei	-720
$begj$	...	d^2fh	-5760
$bchi$	$+648$	d^2g^2	-1536
$bdfj$	-576	$defg$	$+14688$
$bdgi$	$+792$	de^2h	$+4320$
bdh^2	-1728	df^3	-8640
$begh$	$+2160$	e^3g	-8640
be^2j	$+540$	e^2f^2	$+5184$
$befi$	-972		
			± 41650

No. 48.

a^2j^2	...	bf^2h	$+70$
$abij$	...	bfg^2	-45
aci^2	$+2$	c^2fj	$+27$
$achj$	-2	c^2gi	-52
$adgj$	$+7$	c^2h^2	$+25$
$adhi$	-7	$cdej$	-45
$aefj$	-5	$cdfi$	$+23$
$aegi$	-22	$cdgh$	$+22$
aeh^2	$+27$	ce^2i	$+70$
af^2i	$+25$	$cefh$	-127
$afgh$	-45	ceg^2	$+32$
ag^3	$+20$	cf^2g	$+25$
b^2hj	$+2$	d^2j	$+20$
b^2i^2	-2	d^2ei	-45
$begj$	-7	d^2fh	$+32$
$bchi$	$+7$	d^2g^2	$+47$
$bdfj$	-22	$defg$	$+85$
$bdgi$	$+74$	de^2h	$+25$
bdh^2	-52	df^3	-50
$begh$	$+23$	e^3g	-50
be^2j	$+25$	e^2f^2	$+30$
$befi$	-73		
			± 698

Nos 49, [49 A] and 50 relate to the dodecadic. No. 49 is the dodecadic itself; [No. 49 A, inserted in this place, but originally printed in the Fifth Memoir on Quantics, is the dodecadic quadricovariant], No. 50 is the cubinvariant. [The numerical coefficients in this last table as originally printed in the Third Memoir were altogether erroneous, and the table as here printed is in fact the table No. 50 *bis*, of the Fifth Memoir on Quantics.]

No. 49.

$a+1$	$b+12$	$c+66$	$d+220$	$e+495$	$f+792$	$g+924$	$h+792$	$i+495$	$j+220$	$k+66$	$l+12$	$m+1$	$\mathfrak{Q}(x, y)$

No. 49 A.

$ag+1$	$ah+6$	$ai+15$	$aj+20$	$ak+15$	$al+6$	$am+1$	
$bf-6$	$bg-30$	$bh-54$	$bi-30$	$bj+30$	$bk+54$	$bl+30$	
$ce+15$	$cf+54$	$cg+24$	$ch-150$	$ci-270$	$cj-150$	$ck+24$	
d^2-10	$de-30$	$df+150$	$dg+430$	$dh+270$	$di-270$	$dj-430$	
		e^2-135	$ef-270$	$eg+495$	$eh+1080$	$ei+495$	
				f^2-540	$fg-720$	$fh+720$	
						g^2-840	
± 16	± 60	± 189	± 450	± 810	± 1140	± 1270	

$bm+6$	$cm+15$	$dm+20$	$em+15$	$fm+6$	$gm+1$	
$cl+54$	$dl+30$	$el-30$	$fl-54$	$gl-30$	$hl-6$	
$dk-150$	$ek-270$	$fk-150$	$gk+24$	$hk+54$	$ik+15$	$\between(x, y)^{12}$.
$ej-270$	$fj+270$	$gj+430$	$hj+150$	$ij-30$	j^2-10	
$fi+1080$	$gi+495$	$hi-270$	i^2-135			
$gh-720$	h^2-540					
± 1140	± 810	± 450	± 189	± 60	± 16	

No. 50.

$agm+1$	$cfl-54$	$dhi+270$
$ahl-6$	$cgk+24$	e^2k-135
$aik+15$	$chj+150$	$efj+270$
aj^2-10	ci^2-135	$egi+495$
$bfm-6$	d^2m-10	eh^2-540
$bgl+30$	$del+30$	f^2i-540
$bhk-54$	$dfk+150$	$fgh+720$
$bij+30$	$dgj-430$	g^3-280
$cem+15$		
		± 2200

Resuming now the general subject,—

54. The simplest covariant of a system of quantics of the form

$$(*\between x, y, \ldots)^m$$

(where the number of quantics is equal to the number of the facients of each quantic) is the functional determinant or *Jacobian*, viz. the determinant formed with the differential coefficients or derived functions of the quantics with respect to the several facients.

55. In the particular case in which the quantics are the differential coefficients or derived functions of a single quantic, we have a corresponding covariant of the single quantic, which covariant is termed the *Hessian;* in other words, the Hessian is the determinant formed with the second differential coefficients or derived functions of the quantic with respect to the several facients.

56. The expression, an *adjoint linear form*, is used to denote a linear function $\xi x+\eta y+\ldots$, or in the notation of quantics $(\xi, \eta, \ldots \between x, y, \ldots)$, having the same facients as

the quantic or quantics to which it belongs, and with indeterminate coefficients $(\xi, \eta, \ldots)$. The invariants of a quantic or quantics, and of an adjoint linear form, may be considered as quantics having $(\xi, \eta, \ldots)$ for facients, and of which the coefficients are of course functions of the coefficients of the given quantic or quantics. An invariant of the class in question is termed a contravariant of the quantic or quantics. The idea of a contravariant is due to Mr Sylvester.

In the theory of binary quantics, it is hardly necessary to consider the contravariants; for any contravariant is at once turned into an invariant by writing $(y, -x)$ for (ξ, η).

57. If we imagine, as before, a system of quantics of the form

$$(*\between x, y, \ldots)^m,$$

where the number of quantics is equal to the number of the facients in each quantic, the function of the coefficients, which, equated to zero, expresses the result of the elimination of the facients from the equations obtained by putting each of the quantics equal to zero, is said to be the *Resultant* of the system of quantics. The resultant is an invariant of the system of quantics.

And in the particular case in which the quantics are the differential coefficients, or derived functions of a single quantic with respect to the several facients, the resultant in question is termed the *Discriminant* of the single quantic; the discriminant is of course an invariant of the single quantic.

58. Imagine two quantics, and form the equations which express that the differential coefficients, or derived functions of the one quantic with respect to the several facients, are proportional to those of the other quantic. Join to these the equations obtained by equating each of the quantics to zero; we have a system of equations, one of which is contained in the others, and from which therefore the facients may be eliminated. The function which, equated to zero, expresses the result of the elimination is an invariant which (from its geometrical signification) might be termed the *Tactinvariant* of the two quantics, but I do not at present propose to consider this invariant except in the particular case where the system consists of a given quantic and of an adjoint linear form. In this case the tactinvariant is a contravariant of the given quantic, viz. the contravariant termed the *Reciprocant.*

59. Consider now a quantic

$$(*\between x, y, \ldots)^m,$$

and let the facients $x, y, \ldots$ be replaced by $\lambda x + \mu X$, $\lambda y + \mu Y, \ldots$ the resulting function may, it is clear, be considered as a quantic with the facients (λ, μ) and of the form

$$\left\{\begin{array}{l} (*\between x, y, \ldots)^m \\ (*\between x, y, \ldots)^{m-1}(X, Y, \ldots) \\ \quad\vdots \\ (*\between X, Y, \ldots)^m \end{array}\right\}(\lambda, \mu)^m.$$

The coefficients of this quantic are termed *Emanants*, viz., excluding the first coefficient, which is the quantic itself (but which might be termed the 0-th emanant), the other coefficients are the first, second, and last or ultimate emanants. The ultimate emanant is, it is clear, nothing else than the quantic itself, with $(X, Y, \ldots)$ instead of $(x, y, \ldots)$ for facients: the penultimate emanant is, in like manner, obtained from the first emanant by interchanging $(x, y, \ldots)$ with $(X, Y, \ldots)$, and similarly for the other emanants. The facients $(X, Y, \ldots)$ may be termed the *facients of emanation*, or simply the *new facients*. The theory of emanation might be presented in a more general form by employing two or more sets of emanating facients; we might, for example, write $\lambda x + \mu X + \nu X'$, $\lambda y + \mu Y + \nu Y', \ldots$ for $x, y, \ldots$, but it is not necessary to dwell upon this at present.

The invariants, in respect to the new facients, of any emanant or emanants of a quantic (i.e. the invariants of the emanant or emanants, considered as a function or functions of the new facients), are, it is easy to see, covariants of the original quantic, and it is in many cases convenient to define a covariant in this manner; thus the Hessian is the discriminant of the second or quadric emanant of the quantic.

60. If we consider a quantic

$$(a, b, \ldots \between x, y, \ldots)^m,$$

and an adjoint linear form, the operative quantic

$$(\partial_a, \partial_b, \ldots \between \xi, \eta, \ldots)^m$$

(which is, so to speak, a contravariant operator) is termed the *Evector*. The properties of the evector have been considered in the introductory memoir, and it has been in effect shown that the evector operating upon an invariant, or more generally upon a contravariant, gives rise to a contravariant. Any such contravariant, or rather such contravariant considered as so generated, is termed an *Evectant*. In the case of a binary quantic,

$$(a, b, \ldots \between x, y)^m,$$

the covariant operator

$$(\partial_a, \partial_b, \ldots \between y, -x)^m$$

may, if not with perfect accuracy, yet without risk of ambiguity, be termed the *Evector*, and a covariant obtained by operating with it upon an invariant or covariant, or rather such covariant considered as so generated, may in like manner be termed an *Evectant*.

61. Imagine two or more quantics of the same order,

$$(a, b, \ldots \between x, y)^m,$$
$$(\alpha, \beta, \ldots \between x, y)^m,$$
$$\vdots$$

we may have covariants such that for the coefficients of each pair of quantics the covariant is reduced to zero by the operators

$$a\partial_\alpha + b\partial_\beta + \ldots,$$
$$\alpha\partial_a + \beta\partial_b + \ldots.$$

Such covariants are called *Combinants*, and they possess the property of being invariantive, quoad the system, i.e. the covariant remains unaltered to a factor *près*, when each quantic is replaced by a linear function of all the quantics. This extremely important theory is due to Mr Sylvester.

Proceeding now to the theory of ternary quadrics and cubics,—

First for a ternary quadric, we have the following tables:—

Covariant and other Tables, Nos. 51 to 56 (a ternary quadric).

No. 51.

The quadric is represented by

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2,$$

which means

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

No. 52.

The first derived functions (omitting the factor 2) are—

$$\begin{array}{llll} (a, & h, & g \between x, & y, \quad z), \\ (h, & b, & f \between x, & y, \quad z), \\ (g, & f, & c \between x, & y, \quad z). \end{array}$$

No. 53.

The operators which reduce a covariant to zero are

$$\begin{array}{llll} (\ h, & b, & 2f \between \partial_g, & \partial_f, \quad \partial_c) - z\partial_y, \\ (2g, & f, & c \between \partial_a, & \partial_h, \quad \partial_g) - x\partial_z, \\ (\ a, & 2h, & g \between \partial_h, & \partial_b, \quad \partial_f) - y\partial_x, \\ (\ g, & 2f, & c \between \partial_h, & \partial_b, \quad \partial_f) - y\partial_z, \\ (\ a, & h, & 2g \between \partial_g, & \partial_f, \quad \partial_c) - z\partial_x, \\ (2h, & b, & f \between \partial_a, & \partial_h, \quad \partial_g) - x\partial_y. \end{array}$$

No. 54.

The evector is

$$(\partial_a,\ \partial_b,\ \partial_c,\ \partial_f,\ \partial_g,\ \partial_h \between \xi,\ \eta,\ \zeta)^2.$$

No. 55.

The discriminant is

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

which is equal to

$$abc - af^2 - bg^2 - ch^2 + 2fgh.$$

No. 56.

The reciprocant is

$$-\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix}$$

which is equal to

$$(bc - f^2,\ \ ca - g^2,\ \ ab - h^2,\ \ gh - af,\ \ hf - bg,\ \ fg - ch \between \xi,\ \eta,\ \zeta)^2.$$

The discriminant is, it will be noticed, the same function as the Hessian. The reciprocant is the evectant of the discriminant. The covariants are the quadric itself and the discriminant; the reciprocant is the only contravariant.

Next, for a ternary cubic, we have the following Tables:

Covariant and other Tables, Nos. 57 to 70 (a ternary cubic).

No. 57.

The cubic is $U =$

$$(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l \between x,\ y,\ z)^3,$$

which means—

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2 + 6lxyz.$$

No. 58.

The first derived functions (omitting the factor 3) are

$$(a,\ k,\ g,\ l,\ j,\ h \between x,\ y,\ z)^2,$$
$$(h,\ b,\ i,\ f,\ l,\ k \between x,\ y,\ z)^2,$$
$$(j,\ f,\ c,\ i,\ g,\ l \between x,\ y,\ z)^2.$$

The second derived functions (omitting the factor 6) are

$$\begin{array}{l}
(a,\ h,\ j\between x,\ y,\ z),\\
(k,\ b,\ f\between x,\ y,\ z),\\
(g,\ i,\ c\between x,\ y,\ z),\\
(l,\ f,\ i\between x,\ y,\ z),\\
(j,\ l,\ g\between x,\ y,\ z),\\
(h,\ k,\ l\between x,\ y,\ z).
\end{array}$$

No. 59.

The operators which reduce a covariant to zero are

$$\begin{array}{l}
(\ j,\ 3f,\ c,\ 2i,\ g,\ 2l\between\partial_h,\ \partial_b,\ \partial_i,\ \partial_f,\ \partial_l,\ \partial_k)-y\partial_z,\\
(\ a,\ k,\ 3g,\ 2l,\ 2j,\ h\between\partial_j,\ \partial_f,\ \partial_c,\ \partial_i,\ \partial_g,\ \partial_l)-z\partial_z,\\
(3h,\ b,\ i,\ f,\ 2l,\ 2k\between\partial_a,\ \partial_k,\ \partial_g,\ \partial_l,\ \partial_j,\ \partial_h)-x\partial_y,\\
(\ h,\ b,\ 3i,\ 2f,\ 2l,\ k\between\partial_j,\ \partial_f,\ \partial_c,\ \partial_i,\ \partial_g,\ \partial_l)-z\partial_y,\\
(3j,\ f,\ c,\ i,\ 2g,\ 2l\between\partial_a,\ \partial_k,\ \partial_g,\ \partial_l,\ \partial_j,\ \partial_h)-x\partial_z,\\
(\ a,\ 3k,\ g,\ 2l,\ j,\ 2h\between\partial_h,\ \partial_b,\ \partial_i,\ \partial_f,\ \partial_l,\ \partial_k)-y\partial_x.
\end{array}$$

No. 60.

The evector is

$$(\partial_a,\ \partial_b,\ \partial_c,\ \partial_f,\ \partial_g,\ \partial_h,\ \partial_i,\ \partial_j,\ \partial_k,\ \partial_l\between\xi,\ \eta,\ \zeta)^3.$$

No. 61.

The Hessian is $HU=$

$$-\left|\begin{array}{lllll}
(a,\ h,\ j\between x,\ y,\ z), & (h,\ k,\ l\between x,\ y,\ z), & (j,\ l,\ g\between x,\ y,\ z)\\
(h,\ k,\ l\between x,\ y,\ z), & (k,\ b,\ f\between x,\ y,\ z), & (l,\ f,\ i\between x,\ y,\ z)\\
(j,\ l,\ g\between x,\ y,\ z), & (l,\ f,\ i\between x,\ y,\ z), & (g,\ i,\ c\between x,\ y,\ z)
\end{array}\right|$$

which is equal to

$$\left\{\begin{array}{|cccccccccc|}
agk-1 & bhi-1 & cij-1 & bch-1 & acf-1 & abg-1 & bcj-1 & ach-1 & abi-1 & abc-1\\
al^2+1 & bl^2+1 & cl^2+1 & bgl+2 & ai^2+1 & afl+2 & bg^2+1 & afg-1 & af^2+1 & afi+1\\
gh^2+1 & f^2h+1 & fg^2+1 & bij-1 & chl+2 & aik-1 & cfh-1 & ail+2 & bgh-1 & bgj+1\\
hjl-2 & fkl-2 & fkl-2 & ck^2+1 & cjk-1 & bj^2+1 & ckl+2 & ch^2+1 & bjl+2 & chk+1\\
j^2k+1 & ik^2+1 & i^2j+1 & f^2j+1 & fgj+1 & fhj-2 & fij+1 & fj^2+1 & fjk-2 & fgh-3\\
 & & & fgk-2 & g^2k+1 & ghk+1 & gik-2 & gjk+1 & gk^2+1 & fjl+2\\
 & & & fhi+1 & ghi-2 & h^2i+1 & hi^2+1 & hij-2 & hki+1 & gkl+2\\
 & & & fl^2-1 & gl^2-1 & hl^2-1 & il^2-1 & jl^2-1 & kl^2-1 & hil+2\\
 & & & & & & & & & ijk-3\\
 & & & & & & & & & j^3-2\\
\hline
\pm3 & \pm3 & \pm3 & \pm5 & \pm5 & \pm5 & \pm5 & \pm5 & \pm5 & \pm9
\end{array}\right\}(x,\ y,\ z)^2.$$

No. 62.

The quartinvariant is $S=$

$abcl$ -1	f^2j^2 $+1$
$abgi$ $+1$	$fghl$ $+3$
$acfk$ $+1$	$fgjk$ -1
af^2g -1	$fhij$ -1
$afil$ $+1$	fj^4 -2
ai^2k -1	g^2k^2 $+1$
$bchj$ $+1$	$ghki$ -1
bg^2h -1	gkl^2 -2
$bgjl$ $+1$	h^2i^2 $+1$
bij^2 -1	hil^2 -2
cf^2h -1	$ijkl$ $+3$
$chkl$ $+1$	l^4 $+1$
cjk^2 -1	

± 16

No. 63.

The sextinvariant is $T=$

$a^2b^3c^2$ $+1$	acf^2hl -24	$bcfhj^2$ -12	cfh^3i -12	$fgjkl^2$ -12
a^2bcfi -6	acf^2jk -12	$bcgh^2l$ -24	cfh^2l^2 $+12$	fh^2i^2j -12
a^2bi^3 $+4$	$acfgk^2$ -12	$bcghjk$ $+18$	$cfhjkl$ -60	$fhijl^2$ -12
a^2cf^3 $+4$	$acfhki$ $+18$	bch^2ij -12	cfj^2k^2 $+24$	fij^2kl $+36$
$a^2f^2i^2$ -3	$acfkl^2$ $+36$	$bchjl^2$ $+36$	$cghk^2l$ $+12$	fil^4 $+24$
ab^3cgj -6	$acik^2l$ -24	bcj^2kl -24	$cgjk^3$ -12	g^3k^2 $+8$
ab^2g^3 $+4$	af^2gj -12	$bfij^2hj$ $+6$	ch^2ikl $+12$	g^2hk^2i -12
abc^2hk -6	af^2g^2k $+24$	$bfjg^2l$ $+12$	$chijk^2$ $+6$	$g^2k^2l^2$ -24
$abcfgh$ $+6$	af^2ghi $+6$	$bfij^3$ -12	$chkl^2$ -12	gh^2ki^2 -12
$abcfjl$ $+12$	af^2gl^2 $+12$	bg^3hk -12	cjk^2l^2 $+12$	$ghkil^2$ -12
$abcgkl$ $+12$	af^2ijl $+12$	bg^2h^2i $+24$	f^3j^3 $+8$	$gijk^2l$ $+36$
$abchil$ $+12$	$afgkil$ -60	bg^2hl^2 $+12$	$f^2g^2k^2$ -27	gkl^4 $+24$
$abcijk$ $+6$	$afhi^2l$ $+12$	bg^2jkl $+12$	f^2gj^2k -12	h^3i^3 $+8$
$abcl^3$ -20	afi^2jk $+6$	$bghijl$ -60	f^2ghjl $+36$	$h^2i^2l^2$ -24
$abfgij$ $+18$	$afil^3$ -12	$bgij^2k$ $+6$	f^2hij^2 -12	hil^4 $+24$
$abfg^2l$ -24	agk^2i^2 $+24$	$bgjl^3$ -12	$f^2j^2l^2$ -24	hi^2jkl $+36$
abg^2ki -12	ahi^2k -12	bhi^2j^2 $+24$	fg^2hkl $+36$	$i^2j^2k^2$ -27
$abghi^2$ -12	ai^2kl^2 $+12$	bij^2l^2 $+12$	fg^2jk^2 -12	$ijkl^2$ -36
$abgil^2$ $+36$	b^2cj^3 $+4$	$c^2h^2k^2$ -3	fgh^2il $+36$	l^6 -8
abi^2jl -24	$b^2g^2j^2$ -3	cf^2h^2j $+24$	$fghijk$ -6	
ac^2k^3 $+4$	bc^3h^3 $+4$	$cfgh^2k$ $+6$	$fghl^3$ -36	

± 871

The discovery of the invariants S and T is due to Aronhold, the developed expressions were first obtained by Mr Salmon.

No. 64.

There is an octicovariant for which we may take

$$\Theta U = \begin{vmatrix} & \partial_x HU, & \partial_y HU, & \partial_z HU \\ \partial_x HU, & \frac{1}{6}\partial_x^2\ U, & \frac{1}{6}\partial_x\partial_y U, & \frac{1}{6}\partial_x\partial_z U \\ \partial_y HU, & \frac{1}{6}\partial_y\partial_x U, & \frac{1}{6}\partial_y^2\ U, & \frac{1}{6}\partial_y\partial_z U \\ \partial\ HU, & \frac{1}{6}\partial_z\partial_x U, & \frac{1}{6}\partial_y\partial_z U, & \frac{1}{6}\partial_z^2\ U \end{vmatrix} -,$$

or else

$$\Theta_{,}U = \begin{vmatrix} & \frac{1}{3}\partial_x \quad U, & \frac{1}{3}\partial_y \quad U, & \frac{1}{3}\partial_z \quad U \\ \frac{1}{3}\partial_x U, & \partial_x^2 \quad HU, & \partial_x\partial_y HU, & \partial_x\partial_z HU \\ \frac{1}{3}\partial_y U, & \partial_y\partial_x HU, & \partial_y^2 \quad HU, & \partial_y\partial_z HU \\ \frac{1}{3}\partial_z U, & \partial_z\partial_x HU, & \partial_z\partial_y HU, & \partial_z^2 \quad HU \end{vmatrix}$$

or else, what I believe is more simple, a function $\Theta_{,,}U$, which is a linear function of the last-mentioned two functions.

The relations between ΘU, $\Theta_{,}U$, $\Theta_{,,}U$ are

$$-\Theta_{,}U + 4\Theta U = T\,.\,U^2 - 24S\,.\,U\,.\,HU,$$
$$\Theta_{,,}U + 2\Theta U = T\,.\,U^2 - 10S\,.\,U\,.\,HU.$$

I have not worked out the developed expressions.

No. 65.

The cubicontravariant is $PU =$

$$\left\{\begin{array}{cccccccccc}
bcl\ -1 & acl\ -1 & abl\ -1 & ack\ +1 & abi\ +1 & bcj\ +1 & abg\ +1 & bch\ +1 & acf\ +1 & abc\ -1 \\
bgi\ +1 & agi\ +1 & afk\ +1 & afg\ -2 & af^2\ -1 & bg^2\ -1 & afl\ +1 & bgl\ +1 & ai^2\ -1 & afi\ +1 \\
cfk\ +1 & chj\ +1 & bhj\ +1 & ail\ +1 & bgh\ -2 & cfh\ -2 & aik\ -2 & bij\ -2 & chl\ +1 & bgj\ +1 \\
f^2g\ -1 & g^2h\ -1 & fk^2\ -1 & ch^2\ -1 & bjl\ +1 & ckl\ +1 & bj^2\ -1 & ck^2\ -1 & cjk\ -2 & chk\ +1 \\
fil\ +1 & gjl\ +1 & ckl\ +1 & fj^2\ +2 & fhl\ +3 & fgl\ +3 & fhj\ -1 & f^2j\ +2 & fgj\ -1 & fgh\ +3 \\
i^2k\ -1 & ij^2\ -1 & jk^2\ -1 & fkl\ +3 & fjk\ -1 & fij\ -1 & ghk\ -1 & fgk\ -1 & g^2k\ +2 & fjl\ -4 \\
 & & & gjk\ -1 & gk^2\ +2 & gki\ -1 & h^2i\ +2 & fhi\ -1 & ghi\ -1 & gkl\ -4 \\
 & & & hij\ -1 & hki\ -1 & hi^2\ +2 & hl^2\ -2 & fl^2\ -2 & gl^2\ -2 & hil\ -4 \\
 & & & jl^2\ -2 & kl^2\ -2 & il^2\ -2 & jkl\ +3 & ikl\ +3 & ijl\ +3 & ijk\ +3 \\
 & & & & & & & & & l^3\ +4 \\
\pm 3 & \pm 3 & \pm 3 & \pm 7 & \pm 7 & \pm 7 & \pm 7 & \pm 7 & \pm 7 & \pm 13
\end{array}\right\}(\xi, \eta, \zeta)^3.$$

No. 66.

The quintic contravariant is $QU =$

b^5c^3	+ 1	a^2bc^2	+ 1	a^2b^3c	+ 1	a^2bci	− 3	ab^2cj	− 3	abc^2k	− 3	a^2bcf	− 3	ab^2cg	− 3	abc^2h	− 3	$abcfj$	+ 6		
$bcfi$	− 6	a^2cfi	− 3	a^2b^2fi	− 3	a^2cf^2	+ 6	ab^2g^2	+ 6	$abcfg$	+ 3	a^2bi^2	+ 6	$abcfl$	+ 6	$abcgl$	+ 6	$abcgk$	+ 6		
bi^2	+ 4	a^2i^3	+ 2	a^2f^3	+ 2	a^2fi^2	− 3	$abcfh$	+ 3	$abcil$	+ 6	a^2f^2i	− 3	$abcik$	+ 3	$abcij$	+ 3	$abchi$	+ 6		
cf^2	+ 4	$abcgj$	− 6	ab^2gj	− 3	$abcgh$	+ 3	$abckl$	+ 6	$abgi^2$	− 6	$abchl$	+ 6	$abfgi$	+ 9	abg^2i	− 6	$abcl^2$	− 30		
f^2i^2	− 3	abg^3	+ 4	$abchk$	− 6	$abcjl$	+ 6	$abfij$	+ 9	acf^2l	− 12	$abcjk$	+ 3	abi^2l	− 12	ac^3k^2	+ 6	$abfg^2$	− 12		
$bcgj$	− 3	ac^3hk	− 3	$abfgh$	+ 3	$abfgl$	− 24	$abfgl$	− 24	$acfki$	+ 9	$abfgj$	+ 9	acf^2k	− 6	acf^2j	− 6	$abgil$	+ 36		
bg^3	+ 2	$acfgh$	+ 3	$abfjl$	+ 6	abg^2l	− 12	$abgki$	− 12	af^2gi	+ 3	abg^2k	− 6	af^2g	− 6	$acfgk$	− 12	abi^2j	− 12		
c^2hk	− 3	$acfjl$	+ 6	$abgkl$	+ 6	$acfhl$	− 24	$abhi^2$	− 6	af^2i^2l	+ 6	$abghi$	− 12	af^2il	+ 6	$acfhi$	+ 9	acf^2h	− 12		
$cfgh$	+ 3	$acgkl$	+ 6	$abhil$	+ 6	$acfjk$	− 12	$abil^2$	+ 18	ai^3k	− 6	$abgl^2$	+ 18	af^2i^2k	+ 3	$acfl^2$	+ 18	$acfkl$	+ 36		
$cfjl$	+ 6	$achil$	+ 6	$abijk$	+ 3	$acgk^2$	− 6	$acfk^2$	− 6	bc^2h^2	+ 6	$abijl$	− 24	b^2cj^2	+ 6	$acikl$	− 24	$acik^2$	− 12		
$cgkl$	+ 6	$acijk$	+ 3	abl^3	− 10	$achki$	+ 9	af^3j	− 6	$bcfj^2$	− 6	$acfhk$	+ 9	b^2gj	− 3	af^2g^2	+ 12	af^2gl	+ 12		
$chil$	+ 6	acl^3	− 10	ack^3	+ 4	$ackl^2$	+ 18	af^2gk	+ 24	$bcghl$	− 24	ack^2l	− 12	$bcfhj$	− 12	$afgil$	− 30	af^2ij	+ 6		
ijk	+ 3	$afgij$	+ 9	af^2hl	− 12	af^2gj	− 18	af^2hi	+ 3	$bcgjk$	+ 9	af^2gh	+ 3	$bcghk$	+ 9	af^2i^2j	+ 3	$afgki$	− 30		
gij	+ 9	afg^2l	− 12	af^2jk	− 6	afg^2k	+ 24	af^2l^2	+ 6	$bchij$	− 12	af^2jl	+ 6	$bchi^2$	− 6	$agki^2$	+ 24	$afhi^2$	+ 6		
g^2l	− 12	ag^2ki	− 6	$afgk^2$	− 6	$afghi$	+ 6	$afkil$	− 30	$bcjl^2$	+ 18	$afgkl$	− 30	$bchl^2$	+ 18	ahi^3	− 6	$afil^2$	− 18		
g^2ki	− 6	$aghi^2$	− 6	$afhki$	+ 9	$afgl^2$	+ 12	ak^2i^2	+ 12	bfg^2j	+ 3	$afhil$	+ 12	$bcjkl$	− 24	ai^2l^2	+ 6	ai^2kl	+ 12		
hi^2	− 6	$agil^2$	+ 18	$ajkl^2$	+ 18	$afijl$	+ 12	b^2gj^2	− 3	bg^3k	− 6	$afijk$	+ 6	bfg^2h	+ 3	$bcghj$	+ 9	$bcgh^2$	− 12		
ail^2	+ 18	aik^2l	− 12	aik^2l	− 12	$agkil$	− 30	bch^2l	− 12	bg^2hi	+ 24	afl^3	− 6	$bfgjl$	+ 12	bcj^2l	− 12	$bchjl$	+ 36		
cjl	− 12	bcj^3	+ 4	b^2j^3	+ 2	ahi^2l	+ 6	$bchjk$	+ 9	bg^2l^2	+ 6	agk^2i	+ 24	bf^2ij^2	− 18	bg^2h	− 6	bcj^2k	− 12		
k^3	+ 2	bg^2j^2	− 3	bch^3	+ 4	ai^2jk	+ 3	$bfghj$	+ 6	$bgijl$	− 30	ahi^2k	− 18	bg^2kl	+ 6	bg^2jl	+ 6	$bfgj^3$	+ 6		
c^2hl	− 12	c^2h^3	+ 2	$bfhj^2$	− 6	ail^3	− 6	bf^2l	+ 6	bi^2j^2	+ 12	aik^2l	+ 12	$bghil$	− 30	bg^2ij	+ 3	bg^2hl	+ 12		
cjk	− 6	$cfhj^2$	− 6	bgh^2l	− 12	$bchj^2$	− 6	bg^3hk	− 18	c^3hk^2	− 3	$bchi^2j$	− 6	$bgijk$	+ 6	c^2h^2k	− 3	bg^2jk	+ 6		
gk^2	− 6	cgh^2l	− 12	$bghjk$	+ 9	bg^2hj	+ 3	bgh^2i	+ 24	cf^2hj	+ 24	bfj^3	− 6	bgl^3	− 6	$cfgh^2$	+ 3	$bghij$	− 30		
hki	+ 9	bh^2ij	− 6	ch^2ij	− 6	bgj^2l	+ 6	$bghl^2$	+ 12	$cfghk$	+ 6	bg^2h^2	+ 12	bla^2j	+ 24	$cfhjl$	− 30	$bgjl^2$	− 18		
kl^2	+ 18	$chjl^2$	+ 18	$bhjl^2$	+ 18	bij^3	− 6	$bgjkl$	+ 12	$cfhl^2$	− 18	$bghjl$	− 30	$bijl^2$	+ 12	cfj^2k	+ 24	bij^2l	+ 12		
k^2l	− 12	cj^2kl	− 12	b^2kl	− 12	cf^2ij	+ 24	$bhijl$	− 30	$cfhl^2$	+ 12	bgj^2k	+ 3	cf^2h^2	+ 12	$cghkl$	+ 12	cfl^2l	+ 12	$(\xi, \eta\ \zeta)^n$	
gj	− 6	ch^2k^2	− 3	ch^2k^2	− 3	cgh^2k	+ 3	bij^2k	+ 3	$cfjkl$	− 30	$bhijl$	+ 24	$cfhkl$	− 30	$cgjk^2$	− 18	$cfhjk$	− 30		
g^2k	+ 12	fg^2hj	+ 3	f^2hj^2	+ 12	ch^2i	− 6	bjl^3	− 6	cgk^2l	+ 6	bj^2l^2	+ 6	$cfjk^2$	+ 24	ch^2il	+ 6	$cghk^2$	+ 6		
ghi	+ 3	fg^2l	+ 6	$fghk$	+ 3	ch^2l^2	+ 6	cfh^3k	+ 3	$chikl$	+ 12	cfk^3	− 6	cgk^3	− 6	$chijk$	+ 6	ch^2ik	+ 6		
gl^2	+ 6	f^2ij^2	− 6	fh^3i	− 6	$chjkl$	− 30	chk^2l	+ 6	$cijk^2$	+ 3	ch^2kl	+ 6	$chik^2$	+ 3	chl^3	− 6	$chkl^2$	− 18		
ijl	+ 6	g^3hk	− 6	fh^2l^2	+ 6	cj^2l^2	+ 12	cjk^3	− 6	ckl^3	− 6	$chjk^2$	+ 3	ck^2l^2	+ 6	$cjkl^2$	+ 12	cjk^2l	+ 12		
kil	− 30	g^2h^2i	+ 12	$fhjkl$	− 30	f^2j^3	+ 12	f^2gl^2	− 27	f^2g^2h	− 27	f^2hj^2	− 6	f^2j^2	+ 12	f^2gj^2	− 6	f^2ghj	+ 18		
i^2l	+ 6	g^2hl^2	+ 6	fg^2k^2	+ 12	fg^2h^2	− 27	f^2g^2k	− 6	f^3gjl	+ 18	fgh^2l	+ 18	f^2gjk	− 12	fg^2hl	+ 18	f^2g^2l	− 24		
ejk	+ 3	g^2jkl	+ 6	ghk^2l	+ 6	fgj^2k	− 12	f^2hjl	+ 18	f^2i^2j	− 6	$fghjk$	− 3	f^2ghl	+ 18	fg^2jk	− 12	fg^2hk	+ 18		
l^2	− 6	$ghijl$	− 30	gjk^3	− 6	$fghjl$	+ 36	$fghkl$	+ 36	fg^2kl	+ 18	fh^2ij	− 12	f^2hij	− 12	$fghij$	− 3	fgk^2i	+ 18		
a^2k^2	+ 12	gij^3	− 6	h^2ikl	+ 6	$fhij^2$	− 12	$fgjk^2$	− 12	$fghil$	+ 36	$fhjl^2$	− 6	f^2jl^2	− 24	$fgjl^2$	− 6	$fgjkl$	− 54		
ak	− 6	hi^2j^2	+ 12	hkl^3	− 6	fj^2l^2	− 24	fhl^2l	+ 18	fg^2ijk	− 3	fg^2kl	+ 18	fg^2k^2	− 6	f^2ij^2l	+ 18	$fgjkl$	− 12		
il^2	+ 6	ij^2l^2	+ 6	jk^2l^2	+ 6	g^2jk^2	− 6	fhl^3	− 18	fhi^2j	− 12	g^3hk^2	− 6	g^2hk	+ 12	g^2k^2	+ 12	$fhijl$	− 12		
						gh^2il	+ 18	$fjkl^2$	− 6	$fijl^2$	− 6	gh^2ki	− 12	fhi^2k	− 6	g^2hki	− 12	fij^2l	+ 18		
						$ghijk$	− 3	g^2k^2i	+ 12	g^2k^2i	− 6	$ghkl^2$	− 6	fhl^2i	− 6	fhi^2k	− 6	g^2kl^2	− 24		
						ghl^3	− 18	ghk^2i	− 12	$ghki^2$	− 12	gjk^2l	+ 18	$fhil^2$	− 6	g^2kl^2	− 24	gh^2ij	− 6		
						$gjki^2$	− 6	gk^2l^2	− 24	$gkil^2$	− 6	h^3i^2	+ 12	$fijkl$	+ 36	gh^2il	+ 18	$ghkil$	− 12		
						h^2i^2j	− 6	h^2ki^2	− 6	h^2i^3	+ 12	hl^4	+ 12	f^4	+ 12	gl^4	+ 12	$gijk^2$	+ 18		
						$hijl^2$	− 6	$hkil^2$	− 6	hi^2l^2	− 24	gik^2l	+ 18	$gijkl$	+ 36	hi^2kl	+ 18	gkl^3	+ 48		
						ij^2kl	+ 18	ijk^2l	+ 18	i^2jkl	+ 18	$hijkl$	+ 36	ij^2k^2	− 27	i^2jl^2	− 24	hk^2i^2l	− 24		
						jl^4	+ 12	kl^4	+ 12	il^4	+ 12	ij^2k^2	− 27	i^2j^2k	− 27	hil^2	+ 48				
												ikl^3	− 18	jkl^3	− 18	jk^3	− 18	hi^2jk	+ 18		
																		$ijkl^2$	− 54		
																		l^5	− 24		
± 145		± 145		± 145		± 282		± 282		± 282		± 282		± 282		± 282		± 486			

No. 67.

The reciprocant is $FU = (*\between \xi, \eta, \zeta)^6 =$

ξ^6.	η^6.	ζ^6.	$\eta^5\zeta$.	$\zeta^5\xi$.	$\xi^5\eta$.	$\eta\zeta^5$.	$\zeta\xi^5$.	$\xi\eta^5$.
$b^2c^2 + 1$	$a^2c^2 + 1$	$a^2b^2 + 1$	$a^2ci - 6$	$ab^2j - 6$	$bc^2k - 6$	$a^2bf - 6$	$b^2cg - 6$	$ac^2h - 6$
$bcfi - 6$	$acgj - 6$	$abhk - 6$	$acgh + 6$	$abfh + 6$	$bcfg + 6$	$abhl + 12$	$bcfl + 12$	$acgl + 12$
$bi^3 + 4$	$ag^3 + 4$	$ak^3 + 4$	$acjl + 12$	$abkl + 12$	$bcil + 12$	$abjk + 6$	$bcik + 6$	$acij + 6$
$cf^3 + 4$	$cj^3 + 4$	$bh^3 + 4$	$agij + 18$	$afk^2 - 12$	$bgi^2 - 12$	$afhk + 18$	$bfgi + 18$	$ag^2i - 12$
$f^2i^2 - 3$	$g^2j^2 - 3$	$h^2k^2 - 3$	$ag^2l - 24$	$bh^2l - 24$	$cf^2l - 24$	$ak^2l - 24$	$bi^2l - 24$	$cghj + 18$
			$chj^2 - 12$	$bhjk + 18$	$cfki + 18$	$bh^2j - 12$	$cf^2k - 12$	$cj^2l - 24$
			$g^2hj + 6$	$fh^2k + 6$	$f^2gi + 6$	$fh^3 - 12$	$f^3g - 12$	$g^3h - 12$
			$gj^2l + 12$	$hk^2l + 12$	$fi^2l + 12$	$h^2kl + 12$	$f^2il + 12$	$g^2jl + 12$
			$ij^3 - 12$	$jk^3 - 12$	$i^3k - 12$	$hjk^2 + 6$	$fi^2k + 6$	$gij^2 + 6$
± 9	± 9	± 9	± 54	± 54	± 54	± 54	± 54	± 54

$\eta^4\zeta^2$.	$\zeta^4\xi^2$.	$\xi^4\eta^2$.	$\eta^3\zeta^4$.	$\zeta^3\xi^4$.	$\xi^3\eta^4$.	$\eta^3\zeta^3$.	$\zeta^3\xi^3$.	$\xi^3\eta^3$.
$a^2cf + 6$	$ab^2g + 6$	$bc^2h + 6$	$a^2bi + 6$	$b^2cj + 6$	$ac^2k + 6$	$a^2bc - 2$	$ab^2c - 2$	$abc^2 - 2$
$a^2h^2 + 9$	$abfl - 12$	$bcgl - 12$	$a^2f^2 + 9$	$b^2g^2 + 9$	$acfg - 6$	$a^2fi - 18$	$abfi + 6$	$acfi + 6$
$achl - 12$	$abik - 6$	$bcij - 6$	$abgh - 6$	$bcfh - 6$	$acil - 12$	$abgj + 6$	$af^3 - 4$	$ai^3 - 4$
$acjk - 6$	$af^2k + 12$	$bg^2i + 12$	$abjl - 12$	$bckl - 12$	$agi^2 + 12$	$achk + 6$	$b^2gj - 18$	$bcgj + 6$
$afgj - 18$	$b^2j^2 + 9$	$c^2k^2 + 9$	$afhl - 36$	$bfgl - 36$	$c^2h^2 + 9$	$afgh + 18$	$bchk + 6$	$bg^3 - 4$
$ag^2k + 12$	$bfhj - 18$	$cf^2j + 12$	$afjk - 18$	$bfij - 18$	$cfj^2 + 12$	$afjl + 36$	$bfgh + 18$	$c^2hk - 18$
$aghi - 18$	$bghk - 18$	$cfgk - 18$	$agk^2 + 12$	$bgik - 18$	$cghl - 36$	$agkl - 48$	$bfjl + 36$	$cfgh + 18$
$agl^2 + 48$	$bh^2i + 12$	$cf^2hi - 18$	$aihk - 18$	$bhi^2 + 12$	$cgjk - 18$	$ahil + 36$	$bgkl + 36$	$cfjl - 48$
$aijl - 36$	$bhl^2 + 48$	$cfl^2 + 48$	$akl^2 + 48$	$bil^2 + 48$	$chij - 18$	$aijk + 18$	$bhil - 48$	$cgkl + 36$
$ch^2j + 12$	$bjkl - 36$	$cikl - 36$	$bhj^2 + 12$	$cfk^2 + 12$	$cjl^2 + 48$	$ai^3 - 32$	$bijk + 18$	$cijk + 18$
$fj^3 + 12$	$f^3h^2 - 3$	$f^2g^2 - 3$	$gh^2k - 6$	$f^3j + 12$	$fg^2j - 6$	$bj^3 - 4$	$bl^3 - 32$	$chil + 36$
$g^2h^2 - 3$	$fhkl - 24$	$fgil - 24$	$fhj^2 + 36$	$f^2gk + 36$	$g^3k + 12$	$ch^3 - 4$	$ck^3 - 4$	$cl^3 - 32$
$ghjl - 24$	$fjk^2 + 36$	$fi^2j - 6$	$h^3i + 12$	$f^2hi - 6$	$g^2hi + 36$	$fhj^2 - 36$	$f^3hl + 12$	$fg^2l + 12$
$gj^3k - 6$	$gk^3 + 12$	$gi^2k + 36$	$h^2l^2 - 12$	$f^2l^2 - 12$	$g^2l^2 - 12$	$gh^2l + 12$	$f^2jk - 36$	$fgij + 12$
$hij^2 + 36$	$hik^2 - 6$	$hi^3 + 12$	$hjkl - 24$	$fikl - 24$	$gijl - 24$	$ghjk + 12$	$fgk^2 - 36$	$g^2ki - 36$
$j^3l^2 - 12$	$k^3l^2 - 12$	$i^3l^2 - 12$	$j^2k^2 - 3$	$i^2k^2 - 3$	$i^2j^2 - 3$	$h^2ij - 36$	$fhki + 12$	$ghi^2 - 36$
						$hjl^2 + 24$	$fkl^2 + 24$	$gil^2 + 24$
						$j^2kl + 12$	$ik^2l + 12$	$i^2jl + 12$
± 135	± 135	± 135	± 135	± 135	± 135	± 180	± 180	± 180

$\xi^3\eta\zeta$.	$\eta^4\zeta\xi$.	$\zeta^4\xi\eta$.	$\xi^3\eta^2\zeta$.	$\eta^3\zeta^2\xi$.	$\zeta^3\xi^2\eta$.	$\xi^3\eta\zeta^2$.	$\eta^3\zeta\xi^2$.	$\zeta^3\xi^2\eta$.	$\xi^3\eta^3\zeta^2$.
$bcfj$ -12	$acfj$ -12	$abgk$ -12	$abci$ $+6$	$abcj$ $+6$	$abck$ $+6$	$abcf$ $+6$	$abcg$ $+6$	$abch$ $+6$	$abcl$ -24
$bcgk$ $+30$	$acgk$ -12	$abhi$ -12	acf^2 -12	abg^2 -12	$abfg$ -24	abi^2 -12	$acfl$ $+36$	$abgl$ $+36$	$abgi$ $+6$
$bchi$ -12	$achi$ $+30$	$abij$ $+30$	$af\,i^2$ $+6$	$acfh$ -24	$abil$ $+36$	af^2i $+6$	$acik$ -24	$abij$ -24	$acfk$ $+6$
bcl^2 -24	acl^2 -24	abl^2 -24	$bcgh$ -24	$ackl$ $+36$	af^2l $+12$	$bchl$ $+36$	$af\,gi$ -30	ack^2 -12	af^2g $+30$
bfg^2 -18	afg^2 $+24$	af^2h -18	$bcgl$ $+36$	$afgl$ -60	$af\,ik$ -30	$bcjk$ -24	ai^2l $+12$	af^2j -36	$af\,il$ -48
$bgil$ $+12$	$agil$ $+12$	$afkl$ $+12$	bg^2l $+12$	af^2j $+54$	bch^2 -12	$bfgj$ $+54$	bcj^2 -12	$afgk$ $+12$	ai^2k $+30$
bi^2j $+24$	ai^2j -18	aik^2 $+24$	$bgij$ -30	ahi^2 -36	bf^2j -36	bg^2k -36	bg^2j $+6$	$afhi$ $+54$	$bchj$ $+6$
cf^2h $+24$	cgh^2 -18	bgh^2 $+24$	$cfhl$ -60	$agik$ $+12$	$bghl$ -60	$bghi$ $+12$	cf^2hj $+12$	af^2l $+24$	bg^2h $+30$
$cfkl$ $+12$	$chjl$ $+12$	$bhjl$ $+12$	$cfjk$ $+12$	ail^2 $+24$	$bgjk$ $+54$	bgl^2 $+24$	$cghk$ $+54$	$aikl$ -60	$bgjl$ -48
cik^2 -18	cj^2k $+24$	bj^2k -18	cgk^2 -36	bgj^2 $+6$	$bhij$ $+12$	$bijl$ -60	ch^2i -36	$bghj$ -30	bij^2 $+30$
f^2gl $+60$	fgj^2 -12	fh^2l $+60$	$chik$ $+54$	ch^2l $+12$	bjl^2 $+24$	$cfhk$ -30	chl^2 $+24$	bj^2l $+12$	cfh^2 $+30$
f^2ij -12	g^2hl $+60$	$fhjk$ -66	ckl^2 $+24$	$chjk$ -30	chk^2 $+6$	ck^2l $+12$	$cjkl$ -60	ch^2k $+6$	$chkl$ -48
$fgik$ -66	g^2jk -12	ghk^2 -12	f^2gj -24	$fghj$ $+78$	f^2hj $+60$	f^2gh -66	fg^2h -66	fgk^2 -66	cjk^2 $+30$
fhi^2 -12	$ghij$ -66	h^2ki -12	fg^2k $+60$	fj^2l -48	$fghk$ $+78$	f^2jl -48	$fgjl$ $+72$	$fhjl$ -12	f^2j^2 $+24$
fil^2 -48	gjl^2 -48	hkl^2 -48	$fghi$ $+78$	g^2hk -24	fh^2i -24	$fgkl$ -12	fij^2 -24	fj^2k $+60$	$fghl$ $+108$
i^2kl $+60$	j^2il $+60$	jk^2l $+60$	fgl^2 -96	gh^2i $+60$	fhl^2 -96	$fhil$ $+72$	g^2kl -48	$ghkl$ $+72$	$fgjk$ -114
			$fijl$ $+72$	ghl^2 -96	$fjkl$ -12	$fijk$ $+78$	$ghil$ -12	gjk^2 -24	$fhij$ -114
			$gikl$ -12	$gjkl$ $+72$	gk^2l -48	fl^3 $+48$	$gijk$ $+78$	h^2il -48	fjl^2 $+24$
			hi^2l -48	$hijl$ -12	$hikl$ $+72$	gik^2 $+60$	gl^3 $+48$	$hijk$ $+78$	g^2k^2 $+24$
			i^2jk -66	ij^2k -66	ijk^2 -66	hi^2k -24	hi^2j $+60$	hl^3 $+48$	$ghki$ -114
			il^3 $+48$	jl^3 $+48$	kl^3 $+48$	ikl^2 -96	ijl^2 -96	jkl^2 -96	gkl^2 $+24$
									h^2i^2 $+24$
									hil^2 $+24$
									$ijkl$ $+108$
									l^4 -48
± 222	± 222	± 222	± 408	± 408	± 408	± 408	± 408	± 408	± 558

The preceding Tables contain the complete system [not so] of the covariants and contravariants of the ternary cubic, i.e. the covariants are the cubic itself U, the quartinvariant S, the sextinvariant T, the Hessian HU, and an octicovariant, say ΘU; the contravariants are the cubicontravariant PU, the quinticontravariant QU, and the reciprocant FU.

The contravariants are all of them evectants, viz. PU is the evectant of S, QU is the evectant of T, and the reciprocant FU is the evectant of QU, or what is the same thing, the second evectant of T.

The discriminant is a rational and integral function of the two invariants; representing it by R, we have $R = 64\,S^3 - T^2$.

If we combine U and HU by arbitrary multipliers, say α and 6β, so as to form the sum $\alpha U + 6\beta HU$, this is a cubic, and the question arises, to find the covariants and contravariants of this cubic: the results are given in the following Table:

No. 68.

$$\alpha U + 6\beta HU \quad = \alpha U + 6\beta HU.$$

$$H(\alpha U + 6\beta HU) = \quad (0,\ 2S,\quad T,\quad 8S^2\,\between\,\alpha,\ \beta)^3\,U$$
$$\qquad\qquad + (1,\quad 0,\ -12S,\ -2T\,\between\,\alpha,\ \beta)^3\,HU.$$

$$P(\alpha U+6\beta HU) = (1,\ 0,\ 12S,\ 4T\between\alpha,\ \beta)^3 PU$$
$$+(0,\ 1,\ 0,\ -4S\between\alpha,\ \beta)^3 QU.$$
$$Q(\alpha U+6\beta HU) = (0,\ 60S,\ 30T,\ 0,\ -120TS,\ -24T^2+576S^3\between\alpha,\ \beta)^5 PU$$
$$+(1,\ 0,\ 0,\ 10T,\ 240S^2,\ 24TS\between\alpha,\ \beta)^5 QU.$$
$$S(\alpha U+6\beta HU) = (S,\ T,\ 24S^2,\ 4TS,\ T^2-48S^3\between\alpha,\ \beta)^4.$$
$$T(\alpha U+6\beta HU) = (T,\ 96S^2,\ 60TS,\ 20T^2,\ 240TS^2,\ -48T^2S+4608S^4,\ -8T^3+576TS^3\between\alpha,\ \beta)^6.$$
$$R(\alpha U+6\beta HU) = [(1,\ 0,\ -24S,\ -8T,\ -48S^2\between\alpha,\ \beta)^4]^3 R.$$
$$F(\alpha U+6\beta HU) = (1,\ 0,\ -24S,\ -8T,\ -48S^2\between\alpha,\ \beta)^4 FU$$
$$+(0,\ 24,\ 0,\ 0,\ -48T\between\alpha,\ \beta)^4 (PU)^2$$
$$+(0,\ 0,\ 24,\ 0,\ 96S\between\alpha,\ \beta)^4 PU.QU$$
$$+(0,\ 0,\ 0,\ 8,\ 0\between\alpha,\ \beta)^4.(QU)^2.$$

We have, in like manner, for the covariants and contravariants of the cubic $6\alpha PU+\beta QU$, the following Table:

No. 69.

$$6\alpha PU+\beta QU = 6\alpha PU+\beta QU.$$
$$H(6\alpha PU+\beta QU) = (-2T,\ 48S^2,\ 18TS,\ T^2+16S^3\between\alpha,\ \beta)^3 PU$$
$$+(8S,\ T,\ -8S^2,\ -TS\between\alpha,\ \beta)^3 QU.$$
$$P(6\alpha PU+\beta QU) = (32S^2,\ 12TS,\ T^2+32S^3,\ 4TS^2\between\alpha,\ \beta)^3 U$$
$$+(4T,\ 96S^2,\ 12TS,\ T^2-32S^3\between\alpha,\ \beta)^3 HU.$$

$$Q(6\alpha PU+\beta QU) = \left\{\begin{array}{l} +384T\ S^2, \\ +120T^2S+7680\ S^4, \\ +\ 10T^3\ +3200TS^3, \\ +480T^2S^2, \\ +\ 30T^3S, \\ +\ 1T^4-\ 24T^2S^3+512S^6 \end{array}\right\}(\alpha,\ \beta)^5\ U$$

$$+\left\{\begin{array}{l} -\ 24T^2\ +4608\ S^3, \\ +1920TS^2, \\ +\ 480T^2S, \\ +\ 30T^3\ +1920TS^2, \\ +\ 120T^2S^2+7680\ S^5, \\ -\ 6T^3S+\ 768TS^4 \end{array}\right\}(\alpha,\ \beta)^5\ HU.$$

$$S(6\alpha PU+\beta QU)=\left\{\begin{array}{l} +\ 1T^2\ +192\ S^3, \\ +128T\,S^2, \\ +\ 18T^2S+384\ S^4, \\ +\ 1T^3\ +\ 64TS^3, \\ +\ 5T^2S^2-\ 64\ S^5 \end{array}\right\}(\alpha,\ \beta)^4.$$

$$T(6\alpha PU+\beta QU)=\left\{\begin{array}{l} -\ 8T^3\ +\ 4608TS^3, \\ +1920T^2S^2+73728\ S^5, \\ +\ 360T^3S+38400TS^4, \\ +\ 20T^4\ +\ 8960T^2S^3, \\ +\ 840T^3S^2+\ 7680TS^5, \\ +\ 36T^4S+\ 384T^2S^4+24576\ S^7, \\ +\ 1T^5\ -\ 40T^2S^2+\ 2560TS^6 \end{array}\right\}(\alpha,\ \beta)^6.$$

$$R(6\alpha PU+\beta QU)=[(48S,\ 8T,\ -96S^2,\ -24TS,\ -T^2-16S^3\,\between\alpha,\ \beta)^4]^3\,R^2.$$

$$\begin{aligned} F(6\alpha PU+\beta QU)=&(\ 192\,S,\ \ 32T\ \ ,-384\ \ S^2\ \ \ \ \ \ ,-\ \ 96T\,S\ \ \ \ \ \ \ ,-\ \ 4T^2\ \ -64S^3\between\alpha,\ \beta)^4\,\Theta U \\ &+(\ \ 0\ \ ,512\ \ S^3,\ \ 192T\,S^2\ \ \ \ \ ,\ \ 24T^2S\ \ \ \ \ \ ,\ \ T^3\ \ \ \ \ \ \ \ \between\alpha,\ \beta)^4.\,U^2 \\ &+(1344S^2,\ 352TS\ ,\ \ 24T^2-1152S^3,\ -288T\,S^2\ \ \ \ ,-\ \ 20T^2S+64\,S^3\between\alpha,\ \beta)^4\,U.\,HU \\ &+(\ \ 48\,T,\ \ 0\ \ \ ,\ \ 288T\,S\ \ \ \ \ \ \ ,\ \ 24T^2+1536S^3,\ \ 144T\,S^3\ \ \ \ \between\alpha,\ \beta)^4\,(HU)^2. \end{aligned}$$

The tables for the ternary cubic become much more simple if we suppose that the cubic is expressed in Hesse's canonical form; we have then the following table:

No. 70.

$$\begin{aligned} U\ &=x^3+y^3+z^3+6lxyz. \\ S\ &=-l+l^4. \\ T\ &=1-20l^3-8l^6. \\ R\ &=-(1+8l^3)^2. \\ HU\ &=l^2(x^3+y^3+z^3)-(1+2l^3)\,xyz. \\ \Theta U\ &=(1+8l^3)^2(y^3z^3+z^3x^3+x^3y^3) \\ &\quad+(-9l^6)(x^3+y^3+z^3)^2 \\ &\quad+(-2l-5l^4-20l^7)(x^3+y^3+z^3)\,xyz \\ &\quad+(-15l^2-78l^5+12l^8)\,x^3y^3z^3. \\ \Theta_1U\ &=4(1+8l^3)^3(y^3z^3+z^3x^3+x^3y^3) \\ &\quad+(-1-4l^3-4l^6)(x^3+y^3+z^3)^2 \\ &\quad+(4l+100l^4+112l^7)(x^3+y^3+z^3)\,xyz \\ &\quad+(48l^2+552l^5+48l^8)\,x^3y^3z^3. \end{aligned}$$

$$\begin{aligned}
\Theta_{\prime\prime}U = &-2(1+8l^3)^2(y^3z^3+z^3x^3+x^3y^3)\\
&+(1-10l^3)(x^3+y^3+z^3)^2\\
&+(6l-180l^4+96l^7)(x^3+y^3+z^3)\,xyz\\
&+(6l^2-624l^5-192l^8)\,x^2y^2z^2.\\
PU = &-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\,\xi\eta\zeta.\\
QU = &(1-10l^3)(\xi^3+\eta^3+\zeta^3)-6l^2(5+4l^3)\,\xi\eta\zeta.\\
FU = &-4(1+8l^3)(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\
&+(\xi^3+\eta^3+\zeta^3)^2\\
&-24l^2(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\
&-24l(1+2l^3)\,\xi^2\eta^2\zeta^2,
\end{aligned}$$

to which it is proper to join the following transformed expressions for ΘU, $\Theta_{\prime}U$, $\Theta_{\prime\prime}U$, viz.

$$\begin{aligned}
\Theta U = &(1+8l^3)^2(y^3z^3+z^3x^3+x^3y^3)\\
&+(-2l^3-l^6)\,U^2\\
&+(2l-5l^4)\,U.HU\\
&+(-3l^2\quad)(HU).\\
\Theta_{\prime}U = &\,4(1+8l^3)^2(y^3z^3+z^3x^2+x^3y^3)\\
&+\quad(-1+12l^3+4l^6)\,U^2\\
&+\quad(-16l+4l^4\qquad)\,U.HU\\
&+\quad(-12l^2\qquad\qquad)(HU)^2.\\
\Theta_{\prime\prime}U = &-2(1+8l^3)^2(y^3z^3+z^3x^3+x^3y^3)\\
&+\quad(1-16l^3-6l^6)\,U^2\\
&+\quad(6l\qquad\qquad)\,U.HU\\
&+\quad(6l^2\qquad\qquad)(HU)^2.
\end{aligned}$$

The last preceding table affords a complete solution of the problem to reduce a ternary cubic to its canonical form.

[I add to the present Memoir, in the notation hereof $(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3$ for the ternary cubic, some formulæ originally contained in the paper "On Homogeneous Functions of the third order with three variables," (1846), but which on account of the difference of notation were omitted from the reprint, 35, of that paper.

Representing the determinant

$$-\begin{vmatrix}
ax+hy+jz, & hx+ky+bz, & jx+ly+gz, & \xi\\
hx+ky+bz, & kx+by+fz, & lx+fy+iz, & \eta\\
jx+ly+gz, & lx+fy+iz, & gx+iy+cz, & \zeta\\
\xi\quad, & \eta\quad, & \zeta & \\
\end{vmatrix}$$

by $$(A, B, C, F, G, H \between x, y, z)^2$$

the values of A, B, C, F, G, H (equations (10) of 35) are

	A	B	C	F	G	H
ξ^2	$gk - l^2$	$bi - f^2$	$cf - i^2$	$bc - fi$	$fg + ck - 2il$	$ki + bg - 2fl$
η^2	$ag - j^2$	$hi - l^2$	$cj - g^2$	$ij + ch - 2gl$	$ca - gj$	$gh + ai - 2jl$
ζ^2	$ak - h^2$	$bh - k^2$	$fj - l^2$	$hf + bj - 2kl$	$jk + af - 2hl$	$ab - hk$
$\eta\zeta$	$2hj - 2al$	$2kl - 2hf$	$2gl - 2ij$	$l^2 + gk - hi - fj$	$gh - ai$	$jk - af$
$\zeta\xi$	$2hl - 2jk$	$2fk - 2bl$	$2il - 2fg$	$ki - bg$	$l^2 + hi - fj - gk$	$hf - bj$
$\xi\eta$	$2jl - 2gh$	$2fl - 2ki$	$2gi - 2cl$	$fg - ck$	$ij - ch$	$l^2 + fj - gk - hi$

Moreover writing

$$FU = \begin{vmatrix} a, & k, & g, & l, & j, & h, & \xi \\ h, & b, & i, & f, & l, & k, & \eta \\ j, & f, & c, & i, & g, & l, & \zeta \\ 2\xi & . & . & . & \zeta & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A, & B, & C, & F, & G, & H & . \end{vmatrix},$$

so that

$$FU = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h},$$

then the values of a, b, c, f, g, h (equations (13) of 35) are

	a	b	c	f	g	h
ξ^4	0	$2cf$ $-2i^2$	$2bi$ $-2f^2$	fi $-bc$	0	0
η^4	$2cj$ $-2g^2$		$2ag$ $-2j^2$	0	gj $-ca$	0
ζ^4	$2bh$ $-2k^2$	$2ak$ $-2h^2$		0	0	hk $-ab$
$\eta^3\zeta$	$8gl$ $-6ij$ $-2ch$	0	$4hj$ $-4al$	$2j^2$ $-2ag$	$3ai$ $-2jl$ $-gh$	ca $-gj$
$\zeta^3\xi$	$4fk$ $-4bl$	$8hl$ $-6jk$ $-2af$	0	ab $-hk$	$2k^2$ $-2bh$	$3bj$ $-2kl$ $-hf$
$\xi\eta^3$	0	$4gi$ $-4cl$	$8fl$ $-6ki$ $-2bg$	$3ck$ $-2il$ $-fg$	bc $-fi$	$2i^2$ $-2cf$
$\eta\zeta^3$	$8kl$ $-6hf$ $-2bj$	$4hj$ $-4al$	0	$2h^2$ $-2ak$	ab $-hk$	$3af$ $-2hl$ $-jk$
$\zeta\xi^3$	0	$8il$ $-6fg$ $-2ck$	$4fk$ $-4bl$	$3bg$ $-2fl$ $-ki$	$2f^2$ $-2bi$	bc $-fi$
$\xi\eta^3$	$4gi$ $-4cl$	0	$8jl$ $-6gh$ $-2ai$	ca $-gj$	$3ch$ $-2gl$ $-ij$	$2g^2$ $-2cj$
$\eta^2\zeta^2$	$6hi$ $+6fj$ $-4gk$ $-8l^2$	$2ag$ $-2j^2$	$2ak$ $-2h^2$	$4al$ $-4hj$	jk $+2hl$ $-3af$	gh $+2jl$ $-3ai$
$\zeta^2\xi^2$	$2bi$ $-2f^2$	$6fj$ $+6gk$ $-4hi$ $-8l^2$	$2bh$ $-2k^2$	hf $+2kl$ $-3bj$	$4bl$ $-4fk$	ki $+2fl$ $-3bg$
$\xi^2\eta^2$	$2cf$ $-2i^2$	$2cj$ $-2g^2$	$6gk$ $+6hi$ $-4fj$ $-8l^2$	ij $+2gl$ $-3ch$	fg $+2il$ $-3ck$	$4cl$ $-4gi$
$\xi^2\eta\zeta$	$2fi$ $-2bc$	$4ch$ $+4gl$ $-8ij$	$4bj$ $+4kl$ $-8hf$	$4l^2$ $+2hi$ $+2fj$ $-8gk$	$7ki$ $-6fl$ $-bg$	$7fg$ $-6il$ $-ck$
$\xi\eta^2\zeta$	$4ck$ $+4il$ $-8fg$	$2gj$ $-2ca$	$4af$ $+4hl$ $-8jk$	$7gh$ $-6jl$ $-ai$	$4l^2$ $+2fj$ $+2gk$ $-8hi$	$7ij$ $-6gl$ $-ch$
$\xi\eta\zeta^2$	$4bg$ $+4fl$ $-8ki$	$4ai$ $+4jl$ $+8gh$	$2hk$ $-2ab$	$7jk$ $-6hl$ $-af$	$7hf$ $-6kl$ $-bj$	$4l^2$ $+2gk$ $+2hi$ $-8fj$

Also if the discriminant be written

$$K(U) = \begin{vmatrix} a & k & g & l & j & h \\ h & b & i & f & l & k \\ j & i & c & i & g & l \\ \mathfrak{A} & \mathfrak{K} & \mathfrak{G} & \mathfrak{L} & \mathfrak{J} & \mathfrak{H} \\ \mathfrak{H} & \mathfrak{B} & \mathfrak{I} & \mathfrak{F} & \mathfrak{L} & \mathfrak{K} \\ \mathfrak{J} & \mathfrak{I} & \mathfrak{C} & \mathfrak{I} & \mathfrak{G} & \mathfrak{L} \end{vmatrix},$$

then the values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{I}$, $\mathfrak{J}$, $\mathfrak{K}$, $\mathfrak{L}$ (equations (20) of 35) are

$$\begin{aligned}
\mathfrak{A} &= akg + 2hjl - al^2 - gh^2 - j^2k, \\
\mathfrak{B} &= bih + 2fkl - bl^2 - hf^2 - k^2i, \\
\mathfrak{C} &= cjf + 2gil - cl^2 - fg^2 - i^2j, \\
3\mathfrak{F} &= bch + bij - ck^2 + 2gfk - 2bgl + fl^2 - f^2j - fih, \\
3\mathfrak{G} &= caf + cjk - ai^2 + 2hgi - 2chl + gl^2 - g^2k - gjf, \\
3\mathfrak{H} &= abg + aki - bj^2 + 2fhj - 2afl + hl^2 - h^2i - hkg, \\
3\mathfrak{I} &= bcj + cfh - bg^2 + 2kig - 2ckl + jl^2 - i^2h - fij, \\
3\mathfrak{J} &= cak + agf - ch^2 + 2ijh - 2ail + kl^2 - j^2f - gjk, \\
3\mathfrak{K} &= abi + bhg - af^2 + 2jkf - 2bjl + il^2 - k^2g - hki, \\
6\mathfrak{L} &= abc + 3fgh + 3ijk + 2l^3 - afi - bgj - chk - 2lgk - 2lhi - 2lfj.
\end{aligned}$$

The equation $K(U) = R = 64S^3 - T^2$ would however afford a perhaps easier formula for the calculation of the discriminant.]

145.

A MEMOIR UPON CAUSTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 273—312. Received May 1,—Read May 8, 1856.]

THE following memoir contains little or nothing that can be considered new in principle; the object of it is to collect together the principal results relating to caustics *in plano*, the reflecting or refracting curve being a right line or a circle, and to discuss, with more care than appears to have been hitherto bestowed upon the subject, some of the more remarkable cases. The memoir contains in particular researches relating to the caustic by refraction of a circle for parallel rays, the caustic by reflexion of a circle for rays proceeding from a point, and the caustic by refraction of a circle for rays proceeding from a point; the result in the last case is not worked out, but it is shown how the equation in rectangular coordinates is to be obtained by equating to zero the discriminant of a rational and integral function of the sixth degree. The memoir treats also of the secondary caustic, or orthogonal trajectory of the reflected or refracted rays, in the general case of a reflecting or refracting circle and rays proceeding from a point; the curve in question, or rather *a* secondary caustic, is, as is well known, the Oval of Descartes or 'Cartesian': the equation is discussed by a method which gives rise to some forms of the curve which appear to have escaped the notice of geometers. By considering the caustic as the evolute of the secondary caustic, it is shown that the caustic, in the general case of a reflecting or refracting circle and rays proceeding from a point, is a curve of the sixth class only. The concluding part of the memoir treats of the curve which, when the incident rays are parallel, must be taken for the secondary caustic in the place of the Cartesian, which, for the particular case in question, passes off to infinity. In the course of the memoir, I reproduce a theorem first given, I believe, by me in the Philosophical Magazine, viz. that there are six different systems of a radiant point

and refracting circle which give rise to identically the same caustic, [see *post*, XXVIII]. The memoir is divided into sections, each of which is to a considerable extent intelligible by itself, and the subject of each section is for the most part explained by the introductory paragraph or paragraphs.

I.

Consider a ray of light reflected or refracted at a curve, and suppose that ξ, η are the coordinates of a point Q on the incident ray, α, β the coordinates of the point G of incidence upon the reflecting or refracting curve, a, b the coordinates of a point N upon the normal at the point of incidence, x, y the coordinates of a point q on the reflected or refracted ray.

Write for shortness,

$$(b-\beta)(\xi-\alpha)-(a-\alpha)(\eta-\beta)=\nabla QGN,$$
$$(a-\alpha)(\xi-\alpha)+(b-\beta)(\eta-\beta)=\square QGN,$$

then ∇QGN is equal to twice the area of the triangle QGN, and if ξ, η instead of being the coordinates of a point Q on the incident ray were current coordinates, the equation $\nabla QGN=0$ would be the equation of the line through the points G and N, i.e. of the normal at the point of incidence; and in like manner the equation $\square QGN=0$ would be the equation of the line through G perpendicular to the line through the points G and N, i.e. of the tangent at the point of incidence.

We have

$$\overline{NG}^2=(a-\alpha)^2+(b-\beta)^2,$$
$$\overline{QG}^2=(\xi-\alpha)^2+(\eta-\beta)^2,$$

and therefore identically,

$$\overline{NG}^2\cdot\overline{QG}^2=\overline{\nabla QGN}^2+\overline{\square QGN}^2.$$

Suppose for a moment that ϕ is the angle of incidence and ϕ' the angle of reflexion or refraction; and let μ be the index of refraction (in the case of reflexion $\mu=-1$), then writing

$$(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta)=\nabla qGN,$$
$$(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta)=\square qGN,$$

and

$$\overline{qG}^2=(x-\alpha)^2+(y-\beta)^2,$$

we have

$$\sin\phi=\frac{\nabla QGN}{\overline{NG}\,.\,\overline{GQ}},\quad \sin\phi'=\frac{\nabla qGN}{\overline{NG}\,.\,\overline{Gq}};$$

and substituting these values in the equation

$$\sin^2\phi-\mu^2\sin^2\phi'=0,$$

we obtain

$$\overline{qG}^2\,\overline{\nabla QGN}^2 - \mu^2\,\overline{QG}^2\,\overline{\nabla qGN}^2 = 0,$$

an equation which is rational of the second order in x, y, the coordinates of a point q on the refracted ray; this equation must therefore contain, as a factor, the equation of the refracted ray; the other factor gives the equation of a line equally inclined to, but on the opposite side of the normal; this line (which of course has no physical existence) may be termed *the false refracted ray*. The caustic is *geometrically* the envelope of *the pair of rays*, and for finding the equation of the caustic it is obviously convenient to take the equation of the two rays conjointly in the form under which such equation has just been found, without attempting to break the equation up into its linear factors.

It is however interesting to see how the resolution of the equation may be effected; for this purpose multiply the equation by $\overline{NG}^3$, then reducing by means of a previous formula, the equation becomes

$$(\overline{\nabla qGN}^2 + \overline{\square qGN}^2)\,\overline{\nabla QGN}^2 - \mu^2(\overline{\nabla QGN}^2 + \overline{\square QGN}^2)\,\overline{\nabla qGN}^2 = 0,$$

which is equivalent to

$$\overline{\nabla qGN}^2\,(\mu^2\,\overline{\square QGN}^2 + (\mu^2-1)\,\overline{\nabla QGN}^2) - \overline{\square qGN}^2\,\overline{\nabla QGN}^2 = 0,$$

and the factors are

$$\nabla qGN\sqrt{\mu^2\,\overline{\square QGN}^2 + (\mu^2-1)\,\overline{\nabla QGN}^2} \mp \square qGN\,.\,\nabla QGN = 0;$$

it is in fact easy to see that these equations represent lines passing through the point G and inclined to GN at angles $\pm\phi'$, where ϕ' is given by the equations

$$\sin\phi = \mu\sin\phi',$$

$$\tan\phi = \frac{\nabla QGN}{\square QGN},$$

and there is no difficulty in distinguishing in any particular case between the refracted ray and the false refracted ray.

In the case of reflexion $\mu = -1$, and the equations become

$$\nabla qGN\,.\,\square QGN \mp \square qGN\,.\,\nabla QGN = 0;$$

the equation

$$\nabla qGN\,.\,\square QGN - \square qGN\,.\,\nabla QGN = 0$$

is obviously that of the incident ray, which is what the false refracted ray becomes in the case of reflexion; and the equation

$$\nabla qGN\,.\,\square QGN + \square qGN\,.\,\nabla QGN = 0$$

is that of the reflected ray.

II.

But instead of investigating the nature of the caustic itself, we may begin by finding the secondary caustic or orthogonal trajectory of the refracted rays, i.e. a curve having the caustic for its evolute; suppose that the incident rays are all of them normal to a certain curve, and let Q be a point upon this curve; and considering the ray through the point Q, let G be the point of incidence upon the refracting curve; then if the point G be made the centre of a circle the radius of which is $\mu^{-1} . GQ$, the envelope of the circles will be the secondary caustic. It should be noticed that, if the incident rays proceed from a point, the most simple course is to take such point for the point Q. The remark, however, does not apply to the case where the incident rays are parallel; the point Q must here be considered as the point in which the incident ray is intersected by some line at right angles to the rays, and there is not in general any one line which can be selected in preference to another. But if the refracting curve be a circle, then the line perpendicular to the incident rays may be taken to be a diameter of the circle. To translate the construction into analysis, let ξ, η be the coordinates of the point Q, and α, β the coordinates of the point G, then ξ, η, α, β are in effect functions of a single arbitrary parameter; and if we write

$$\overline{GQ}^2 = (\xi - \alpha)^2 + (\eta - \beta)^2,$$

$$\overline{Gq}^2 = (x - \alpha)^2 + (y - \beta)^2,$$

then the equation

$$\mu^2 \overline{Gq}^2 - \overline{GQ}^2 = 0,$$

where x, y are to be considered as current coordinates, and which involves of course the arbitrary parameter, is the equation of the circle, and the envelope is obtained in the usual manner. This is the well-known theory of Gergonne and Quetelet.

III.

There is however a simpler construction of the secondary caustic in the case of the reflexion of rays proceeding from a point. Suppose, as before, that Q is the radiant point, and let G be the point of incidence. On the tangent at G to the reflecting curve, let fall a perpendicular from Q, and produce it to an equal distance on the other side of the tangent; then if q be the extremity of the line so produced, it is clear that q is a point on the reflected ray Gq, and it is easy to see that the locus of q is the secondary caustic. Produce now QG to a point Q' such that $GQ' = QG$, it is clear that the locus of Q' will be a curve similar to and similarly situated with and twice the magnitude of the reflecting curve, and that the two curves have the point Q for a centre of similitude. And the tangent at Q' passes through the point q, i.e. q is the foot of the perpendicular let fall from Q upon the tangent at Q'; we have therefore the theorem due to Dandelin, viz.

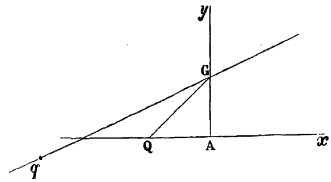
If rays proceeding from a point Q are reflected at a curve, then the secondary caustic is the locus of the feet of the perpendiculars let fall from the point Q upon the tangents of a curve similar to and similarly situated with and twice the magnitude of the reflecting curve, and such that the two curves have the point Q for a centre of similitude.

IV.

If rays proceeding from a point Q are reflected at a line, the reflected rays will proceed from a point q situate on the perpendicular let fall from Q, and at an equal distance on the other side of the reflecting line. The point q may be spoken of as the image of Q; it is clear that if Q be considered as a variable point, then the locus of the image q will be a curve equal and similar but oppositely situated to the curve, the locus of Q, and which may be spoken of as the image of such curve. Hence it at once follows, that if the incidental rays are tangent, or normal, or indeed in any other manner related to a curve, then the reflected rays will be tangent, or normal, or related in a corresponding manner to a curve the image of the first-mentioned curve. The theory of the combined reflexions and refractions of a pencil of rays transmitted through a plate or prism, is, by the property in question, rendered very simple. Suppose, for instance, that a pencil of rays is refracted at the first surface of a plate or prism, and after undergoing any number of internal reflexions, finally emerges after a second refraction at the first or second surface; in order to find the caustic enveloped by the rays after the second refraction, it is only necessary to form the successive images of the first caustic corresponding to the different reflexions, and finally to determine the caustic for refraction in the case where the incident rays are the tangents of the caustic which is the last of the series of images; the problem is not in effect different from that of finding the caustic for refraction in the case where the incident rays are the tangents to the caustic after the first refraction, but the line at which the second refraction takes place is arbitrarily situate with respect to the caustic. Thus e.g. suppose the incident rays proceed from a point, the caustic after the first refraction is, it will be shown in the sequel, the evolute of a conic; for the complete theory of the combined reflexions and refractions of the pencil by a plate or prism, it is only necessary to find the caustic by refraction, where the incident rays are the normals of a conic, and the refracting line is arbitrarily situate with respect to the conic.

V.

Suppose that rays proceeding from a point Q are refracted at a line; and take the refracting line for the axis of y, the axis of x passing through the radiant point Q, and take the distance QA for unity. Suppose that the index of refraction μ is put equal to $\frac{1}{k}$. Then if ϕ be the angle of incidence and ϕ' the angle of refraction,



we have $\sin\phi' = k\sin\phi$, and the equation $y - x\tan\phi' = \tan\phi$ of the refracted ray becomes, putting for ϕ' its value,

$$y - \frac{k\sin\phi}{\sqrt{1-k^2\sin^2\phi}}\,x - \tan\phi = 0.$$

Differentiating with respect to the variable parameter and combining the two equations, we obtain, after a simple reduction,

$$kx = -\frac{(1-k^2\sin^2\phi)^{\frac{3}{2}}}{\cos^3\phi},$$

$$k'y = -\frac{k'^3\sin^3\phi}{\cos^3\phi},$$

where $k' = \sqrt{1-k^2}$; hence eliminating,

$$(kx)^{\frac{2}{3}} - (k'y)^{\frac{2}{3}} = 1,$$

which is the equation of the caustic. When the refraction takes place into a denser medium k is less than 1, and k'^2 is positive, the caustic is therefore the evolute of a hyperbola (see fig. 1); but when the refraction takes place in a rarer medium k is greater than 1, and k'^2 is negative, the caustic is therefore the evolute of an ellipse (see fig. 2). These results appear to have been first obtained by Gergonne. The conic (hyperbola or ellipse) is the secondary caustic, and as such may be obtained as follows.

VI.

The equation of the variable circle is

$$x^2 + (y - \tan\phi)^2 - k^2\sec^2\phi = 0;$$

or reducing, the equation is

$$x^2 + y^2 - 2y\tan\phi + k'^2\tan^2\phi - k^2 = 0:$$

whence, considering $\tan\phi$ as the variable parameter, the equation of the envelope is

$$k'^2(x^2 + y^2 - k^2) - y^2 = 0,$$

that is,

$$k'^2x^2 - k^2y^2 - k^2k'^2 = 0,$$

or

$$\frac{x^2}{k^2}-\frac{y^2}{k'^2}=1$$

is the equation of the secondary caustic, or conic having the caustic for its evolute. The radiant point, it is clear, is a focus of the conic.

VII.

Let the equation of the refracted ray be represented by

$$Xx+Yy+Z=0,$$

we have

$$X : Y : Z=\frac{-k\sin\phi}{\sqrt{1-k^2\sin^2\phi}} : 1 : -\tan\phi,$$

from which we obtain

$$\frac{k^2}{X^2}-\frac{k'^2}{Y^2}=\frac{1}{Z^2}$$

for the tangential equation of the caustic; or if we represent the equation of the refracted ray by

$$Xx+Yy-k=0,$$

then we have

$$\frac{k^2}{X^2}-\frac{k'^2}{Y^2}=\frac{1}{k^2}$$

for the tangential equation of the caustic.

Fig. 1.

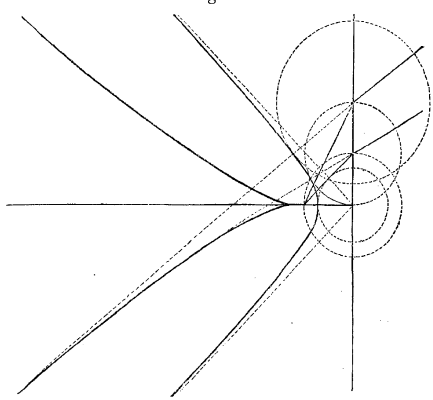
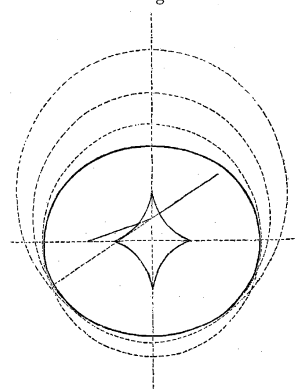


Fig. 2.



VIII.

If a ray be reflected at a circle; we may take a, b as the coordinates of the centre of the circle, and supposing as before that ξ, η are the coordinates of a point Q in the incident ray, α, β the coordinates of the point G of incidence, and x, y the coordinates of a point q in the reflected ray, the equation of the reflected ray, treating x, y as current coordinates, is

$$\{(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta)\}\{(a-\alpha)(\xi-\alpha)+(b-\beta)(\eta-\beta)\}$$
$$+\{(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta)\}\{(b-\beta)(\xi-\alpha)-(a-\alpha)(\eta-\beta)\}=0.$$

Write for shortness,

$$N_{q,G}=(b-\beta)(x-\alpha)-(a-\alpha)(y-\beta),$$
$$T_{q,G}=(a-\alpha)(x-\alpha)+(b-\beta)(y-\beta),$$

and similarly for $N_{Q,G}$, &c.; the equation of the reflected ray is

$$N_{q,G}T_{Q,G}+T_{q,G}N_{Q,G}=0.$$

Suppose that the reflected ray meets the circle again in G' and undergoes a second reflexion, and let x', y' be the coordinates of a point q' in the ray thus twice reflected. We see first (G' being a point in the first reflected ray) that

$$N_{G',G}T_{Q,G}+T_{G',G}N_{Q,G}=0.$$

Again, considering G as a point in the ray by the reflexion of which the second reflected ray arises, the equation of the second reflected ray is

$$N_{q',G'}T_{G,G'}+T_{q',G'}N_{G,G'}=0;$$

and from the form of the expressions $N_{q,G}$, $T_{q,G}$ it is clear that

$$N_{G,G'}=-N_{G',G},\ T_{G,G'}=+T_{G',G};$$

the equation for the second reflected ray may therefore be written under the form

$$N_{q',G'}T_{G',G}-T_{q',G'}N_{G',G}=0;$$

or reducing by a previous equation, we obtain finally for the equation of the second reflected ray,

$$N_{q',G'}T_{Q,G}+T_{q',G'}N_{Q,G}=0;$$

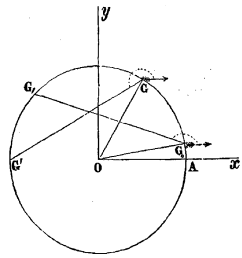
and in like manner the equation for the third reflected ray is

$$N_{q'',G''}T_{Q,G}+T_{q'',G''}N_{Q,G}=0,$$

and so on, the equation for the last reflected ray containing, it will be observed, the coordinates of the radiant point and of the first and last points of incidence (the coordinates of the last point of incidence can of course only be calculated from those of the radiant point and the first point of incidence, through the coordinates of the intermediate points of incidence), but not containing explicitly the coordinates of any of the intermediate points of incidence. The form is somewhat remarkable, but the result is really the same with that obtained by simple geometrical considerations, as follows.

IX.

Consider a ray reflected any number of times at a circle; and let $G_0G_{,}$ be the ray incident at $G_{,}$ and GG' the last reflected ray, the point at which the reflexion takes place or last point of incidence being G. Take the centre O of the circle for



the origin, and any two lines Ox, Oy through the centre and at right angles to each other for axes, and let Ox meet the circle in the point A. Write

$$\angle AOG_0 = \theta_0, \quad \angle xG_0G_{,} = \psi_0,$$
$$\angle AOG = \theta, \quad \angle xGG' = \psi,$$
$$\angle G_0G_{,}O = \phi;$$

then the radius of the circle being taken as the centre of the circle, the equation of the reflected ray is

$$y - \sin\theta = \tan\psi\,(x - \cos\theta);$$

and if there have been n reflexions, then

$$\theta = \theta_0 + n(\pi - 2\phi) = \theta_0 + n\pi - 2n\phi,$$
$$\psi = \psi_0 - 2n\phi,$$

and therefore the equation of the reflected ray is

$$y\cos(\psi_0 - 2n\phi) - x\sin(\psi_0 - 2n\phi) + (-)^n\sin(\psi_0 - \theta_0) = 0.$$

X.

If a pencil of parallel rays is reflected any number of times at a circle, then taking AO for the direction of the incident rays, we may write $\theta_0 = \phi$, $\psi_0 = \pi$, and the equation of a reflected ray is

$$x\sin 2n\phi + y\cos 2n\phi = (-)^n\sin\phi;$$

differentiating with respect to the variable parameter, we find

$$x\cos 2n\phi - y\sin 2n\phi = (-)^n \frac{1}{2n}\cos\phi;$$

and these equations give

$$x = \frac{(-)^n}{4n}\left\{(2n+1)\cos(2n-1)\phi - (2n-1)\cos(2n+1)\phi\right\},$$

$$y = \frac{(-)^n}{4n}\left\{-(2n+1)\sin(2n-1)\phi + (2n-1)\sin(2n+1)\phi\right\},$$

which may be taken for the equation of the caustic; the caustic is therefore an epicycloid: this is a well-known result.

XI.

If rays proceeding from a point upon the circumference are reflected any number of times at a circle, then taking the point A for the radiant point, we have $\alpha_0 = 0$, $\psi_0 = \pi - \phi$, and the equation of a reflected ray is

$$x\sin(2n+1)\phi + y\cos(2n+1)\phi = (-)^n\sin\phi;$$

differentiating with respect to the variable parameter, we find

$$x\cos(2n+1)\phi - y\sin(2n+1)\phi = (-)^n\frac{1}{2n+1}\sin\phi;$$

and these equations give

$$x = \frac{(-)^n}{2n+1}\left\{(n+1)\cos 2n\phi - n\cos(2n+2)\phi\right\},$$

$$y = \frac{(-)^n}{2n+1}\left\{-(n+1)\sin 2n\phi + n\cos(2n+2)\phi\right\},$$

which may be taken as the equation of the caustic; the caustic is therefore in this case also an epicycloid: this is a well-known result.

XII.

Consider a pencil of parallel rays refracted at a circle; take the radius of the circle as unity, and let the incident rays be parallel to the axis of x, then if ϕ, ϕ' be the angles of incidence and refraction, and μ or $\frac{1}{k}$ be the index of refraction, so that $\sin\phi' = k\sin\phi$, the coordinates of the point of incidence are $\cos\phi$, $\sin\phi$, and the equation of the refracted ray is

$$y - \sin\phi = \tan(\phi - \phi')(x - \cos\phi),$$

i.e.

$$\cos(\phi - \phi')(y - \sin\phi) = \sin(\phi - \phi')(x - \cos\phi),$$

or

$$y\cos(\phi-\phi')-x\sin(\phi-\phi')=\sin\phi',$$

which may also be written

$$(y\cos\phi-x\sin\phi)\cos\phi'+(y\sin\phi+x\cos\phi-1)\sin\phi'=0.$$

Writing $k\sin\phi$, $\sqrt{1-k^2\sin^2\phi}$ instead of $\sin\phi'$, $\cos\phi'$, and putting for shortness

$$y\cos\phi-x\sin\phi=Y,$$
$$y\sin\phi+x\cos\phi=X,$$
$$\frac{k\sin\phi}{\sqrt{1-k^2\sin^2\phi}}=\Phi,$$

the equation of the refracted ray becomes

$$Y+\Phi(X-1)=0;$$

and differentiating with respect to the variable parameter ϕ, observing that

$$\frac{dY}{d\phi}=\quad -X,\qquad \frac{dX}{d\phi}=Y,$$
$$\frac{d\Phi}{d\phi}=\frac{k\cos\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}=\frac{\cot\phi}{1-k^2\sin^2\phi}\Phi,$$

we have

$$-X+\Phi\left(Y+\frac{\cot\phi\,(X-1)}{1-k^2\sin^2\phi}\right)=0,$$

and the combination of the two equations gives

$$Y=-\frac{\Phi(1-k^2\sin^2\Phi)}{\Phi\cot\phi-1},$$
$$X=\frac{\Phi\cot\phi-k^2\sin^2\phi}{\Phi\cot\phi-1},$$

and we have therefore

$$y=Y\cos\phi+X\sin\phi=\frac{k^2\sin^3\phi\,(\Phi\cot\phi-1)}{\Phi\cot\phi-1}=k^2\sin^3\phi,$$

$$x=X\cos\phi-Y\sin\phi=\frac{\Phi\left(\frac{1}{\sin\phi}-k^2\sin^3\phi\right)-k^2\sin^2\phi\cos\phi}{\Phi\cot\phi-1},$$

i.e.

$$x=\frac{\Phi(1-k^2\sin^4\phi)-k^2\sin^3\phi\cos\phi}{\Phi\cos\phi-\sin\phi};$$

or multiplying the numerator and denominator by $(1-k^2\sin^2\phi)(\Phi\cos\phi+\sin\phi)$, the numerator becomes

$$\begin{aligned}(1-k^2\sin^2\phi)\{&\Phi^2\cos\phi\,(1-k^2\sin^4\phi)-k^2\sin^4\phi\cos\phi\\&+\Phi\left(\sin\phi\,(1-k^2\sin^2\phi)-k^2\sin^3\phi\cos\phi\right)\}\\&=k^2\sin^2\phi\cos\phi\,\{(1-k^2\sin^4\phi)-\sin^2\phi\,(1-k^2\sin^2\phi)\}\\&\quad+k\sin^2\phi\sqrt{1-k^2\sin^2\phi}\,(1-k^2\sin^2\phi)\\&=k^2\sin^2\phi\cos^3\phi+k\sin^2\phi\,(1-k^2\sin^2\phi)^{\frac{3}{2}},\end{aligned}$$

and the denominator becomes

$$\begin{aligned}&k^2\sin^2\phi\cos^2\phi-(1-k^2\sin^2\phi)\sin^2\phi\\&=-k'^2\sin^2\phi,\end{aligned}$$

if $k'^2=1-k^2$.

Hence we have for the coordinates of the point of the caustic,

$$\begin{cases}k'^2x=-k^2\cos^3\phi-k\,(1-k^2\sin^2\phi)^{\frac{3}{2}},\\ \quad\;\; y=\;\;k^2\sin^3\phi\,;\end{cases}$$

and eliminating ϕ, we obtain for the equation of the caustic,

$$k'^2x=-k^2\{1-k^{-\frac{4}{3}}y^{\frac{2}{3}}\}^{\frac{3}{2}}-k\{1-k^{\frac{2}{3}}y^{\frac{2}{3}}\}^{\frac{3}{2}};$$

or writing $\frac{1}{\mu}$ instead of k, we find

$$(1-\mu^2)\,x=(1-\mu^{\frac{4}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}+\mu\,(1-\mu^{-\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}}$$

for the equation of the caustic by refraction of the circle, for parallel rays. The equation was first obtained by St Laurent.

XIII.

The discussion of the preceding equation presents considerable interest. In the first place to obtain the rational form write

$$\alpha\;=(1-\mu^2)\,x,\;\;\beta=(1-\mu^{\frac{4}{3}}y^{\frac{2}{3}})^{\frac{3}{2}},\;\;\gamma=\mu\,(1-\mu^{-\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}};$$

this gives

$$\alpha^4\;-2\alpha^2(\beta^2+\gamma^2)+(\beta^2-\gamma^2)^2=0,$$

and we have

$$\beta^2=1-3\mu^{\frac{4}{3}}y^{\frac{2}{3}}+3\mu^{\frac{8}{3}}y^{\frac{4}{3}}-\mu^4y^2,$$

$$\gamma^2\;=\mu^2-3\mu^{\frac{4}{3}}y^{\frac{2}{3}}+3\mu^{\frac{8}{3}}y^{\frac{4}{3}}-y^2,$$

and consequently

$$\beta^2-\gamma^2=(1-\mu^2)\{1-3\mu^{\frac{2}{3}}y^{\frac{4}{3}}+(1+\mu^2)\,y^2\}.$$

Hence dividing out by the factor $(1-\mu^2)^2$, the equation becomes

$$(1-\mu^2)^2 x^4 - 2\left(1+\mu^2-6\mu^{\frac{4}{3}}y^{\frac{2}{3}}+3\mu^{\frac{2}{3}}(1+\mu^2)\,y^{\frac{4}{3}}-(1+\mu^4)\,y^2\right)2x^2+\left(1-3\mu^{\frac{2}{3}}y^{\frac{4}{3}}+(1+\mu^2)\,y^2\right)^2=0\,;$$

or reducing and arranging,

$$\begin{aligned}(1-\mu^2)^2 x^4 - 2\,(1+\mu^2)\,x^2 + 2\,(1+\mu^4)\,x^2y^2 + 1 + 2\,(1+\mu^2)\,y^2 + (1+\mu^2)^2\,y^4 \\ + (12\mu^{\frac{4}{3}}x^2 + 9\mu^{\frac{4}{3}}y^2)\,y^{\frac{2}{3}} - \left(6\mu^{\frac{2}{3}}(1+\mu^2)\,x^2 + 6\mu^{\frac{2}{3}} + 6\mu^{\frac{2}{3}}(1+\mu^2)\,y^2\right)y^{\frac{4}{3}} = 0,\end{aligned}$$

which is of the form

$$A + 3\mu^{\frac{4}{3}}By^{\frac{2}{3}} - 6\mu^{\frac{2}{3}}Cy^{\frac{4}{3}} = 0\,;$$

and the rationalized equation is

$$A^3 + 27\mu^4B^2y^2 - 216\mu^2C^3y^4 + 54\mu^2ABCy^2 = 0,$$

where the values of A, B, C may be written

$$\begin{aligned}A &= (x^2+y^2)\,[(1-\mu^2)^2x^2+(1+\mu^2)^2y^2] - 2\,(1+\mu^2)\,(x^2-y^2)+1,\\ B &= 4x^2+3y^2,\\ C &= (1+\mu^2)\,(x^2+y^2)+1\,;\end{aligned}$$

the caustic is therefore a curve of the 12th order.

To find where the axis of x meets the curve, we have

$$y=0,\ A_0^{\,3}=0,$$

where

$$\begin{aligned}A_0 &= (1-\mu^2)^2x^4 - 2\,(1+\mu^2)\,x^2+1\\ &= \{(1-\mu)^2x^2-1\}\,\{(1+\mu)^2x^2-1\},\end{aligned}$$

i.e.

$$\begin{cases} y=0,\\ x=\pm\dfrac{1}{1-\mu},\ x=\pm\dfrac{1}{1+\mu}\,,\end{cases}$$

or there are in all four points, each of them a point of triple intersection.

To find where the line ∞ meets the curve, we have

$$\infty\,,\ A'^3=0,$$

where

$$A' = (x^2+y^2)\,[(1-\mu^2)^2x^2+(1+\mu^2)^2y^2],$$

i.e.

$$\begin{cases}\infty\,,\\ x=\pm\,iy,\ x=\pm\dfrac{1+\mu^2}{1-\mu^2}\,iy,\end{cases}$$

or the curve meets the line ∞ in four points, each of them a point of triple intersection: two of these points are the circular points at ∞.

To find where the circle $x^2+y^2=1$ meets the curve, this gives $x^2=1-y^2$, and thence

$$\begin{aligned} A &= \mu^2(\mu^2-4)+4(1+2\mu^2)y^2, \\ B &= 4-y^2, \\ C &= \mu^2+2, \end{aligned}$$

and the equation becomes

$$\begin{aligned} &\{\mu^2(\mu^2-4)+4(1+2\mu^2)y^2\}^3+27\mu^4(4-y^2)^3y^2-216(\mu^2+2)^3y^4 \\ &\qquad +54\mu^2(\mu^2+2)y^2(4-y^2)\{\mu^2(\mu^2-4)+4(1+2\mu^2)y^2\}=0, \end{aligned}$$

which is only of the eighth order; it follows that each of the circular points at ∞ (which have been already shown to be points upon the curve) are quadruple points of intersection of the curve and circle. The equation of the eighth order reduces itself to

$$(y^2-\mu^2)^3\{27\mu^4y^2+(\mu^2-4)^3\}=0;$$

the values of x corresponding to the roots $y=\pm\mu$ are obtained without difficulty, and those corresponding to the other roots are at once found by means of the identical equation

$$(\mu^2-4)^3+27\mu^4+(1-\mu^2)(\mu^2+8)^2=0;$$

we thus obtain for the coordinates of the points of intersection of the curve with the circle $x^2+y^2=1$, the values

$$\begin{cases}\infty, \\ x=\pm iy,\end{cases} \quad \begin{cases} x=\pm\sqrt{1-\mu^2} \\ y=\pm\mu,\end{cases} \quad \begin{cases} x=\pm\dfrac{(\mu^2+8)\sqrt{1-\mu^2}}{3\sqrt{3}\mu^2}i, \\ y=\pm\dfrac{(\mu^2-4)^{\frac{3}{2}}}{3\sqrt{3}\mu^3}i,\end{cases}$$

each of the points of the first system being a quadruple point of intersection, each of the points of the second system a triple point of intersection, and each of the points of the third system a single point of intersection.

Next, to find where the circle $x^2+y^2=\dfrac{1}{\mu^3}$ meets the curve; writing $x^2=\dfrac{1}{\mu^2}-y^2$, we obtain for y an equation of the eighth order, which after all reductions is

$$\left(y^2-\frac{1}{\mu^4}\right)^3\{27\mu^4y^2+(1-4\mu^2)^3\}=0,$$

and we have for the coordinates of the points of intersection,

$$\begin{cases} \infty, \\ x = \pm iy, \end{cases} \quad \begin{cases} x = \pm \dfrac{1}{\mu}\sqrt{1-\dfrac{1}{\mu^2}}, \\ y = \pm \dfrac{1}{\mu^2}, \end{cases} \quad \begin{cases} x = \pm \dfrac{(1+8\mu^2)\sqrt{1-\dfrac{1}{\mu^2}}}{3\sqrt{3}\mu}\, i, \\ y = \pm \dfrac{(1-4\mu^2)^{\frac{3}{2}}}{3\sqrt{3}\mu}\, i, \end{cases}$$

each of the points of the first system being a quadruple point of intersection, each of the points of the second system a triple point of intersection, and each of the points of the third system a single point of intersection.

The points of intersection with the axes of x, and the points of triple intersection with the circles $x^2+y^2=1$ and $x^2+y^2=\dfrac{1}{\mu^2}$, are all of them cuspidal points; the two circular points at ∞ are, I think, triple points, and the other two points of intersection with the line ∞, cuspidal points, but I have not verified this: assuming that it is so, there will be a reduction 54 accounted for in the class of the curve, but the curve is, in fact, as will be shown in the sequel, of the class 6; there is consequently a reduction 72 to be accounted for by other singularities of the curve.

XIV.

It is obvious from the preceding formulæ that the caustic stands to the circle, radius $\dfrac{1}{\mu}$, in a relation similar to that in which it stands to the circle, radius 1, i.e. to the refracting circle. In fact, the very same caustic would have been obtained if the circle radius $\dfrac{1}{\mu}$ had been taken for the refracting circle, the index of refraction being $\dfrac{1}{\mu}$ instead of μ. This may be shown very simply by means of the irrational form of the equation as follows.

The equation of the caustic by refraction of the circle, radius 1, index of refraction μ, is as we have seen

$$(1-\mu^2)\,x = (1-\mu^{\frac{4}{3}}y^{\frac{2}{3}})^{\frac{3}{2}} + \mu\,(1-\mu^{-\frac{2}{3}}y^{\frac{2}{3}})^{\frac{3}{2}};$$

hence the equation of the caustic by refraction of the circle radius c', index of refraction μ', is

$$(1-\mu'^2)\,\frac{x}{c'} = \left\{1-\mu'^{\frac{4}{3}}\left(\frac{y}{c'}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}} + \mu'\left\{1-\mu'^{-\frac{2}{3}}\left(\frac{y}{c'}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

or, what is the same thing,

$$(1-\mu'^2)\frac{x}{c'\mu'}=\left\{1-\mu'^{-\frac{2}{3}}c'^{-\frac{2}{3}}y^{\frac{2}{3}}\right\}^{\frac{3}{2}}+\frac{1}{\mu'}\left\{1-\mu'^{\frac{4}{3}}c'^{-\frac{2}{3}}y^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

which becomes identical with the equation of the first-mentioned caustic if $\mu'=c'=\frac{1}{\mu}$.

Hence taking c instead of 1 as the radius of the first circle, we find,

THEOREM. The caustic by refraction for parallel rays of a circle, radius c, index of refraction μ, is the same curve as the caustic by refraction for parallel rays of a concentric circle, radius $\frac{c}{\mu}$, index of refraction $\frac{1}{\mu}$.

XV.

We may consequently in tracing the caustic confine our attention to the case in which the index of refraction is greater than unity. The circle, radius $\frac{c}{\mu}$, will in this case be within the refracting circle, and it is easy to see that if from the extremity of the diameter of the refracting circle perpendicular to the direction of the incident rays, tangents are drawn to the circle, radius $\frac{c}{\mu}$, the points of contact are the points of triple intersection of the caustic with the last-mentioned circle, and these points of intersection being, as already observed, cusps, the tangents in question are the tangents to the caustic at these cusps. The points of intersection with the axis of x are also cusps of the caustic, the tangents at these cusps coinciding with the axis of x: two of the last-mentioned cusps, viz. those whose distances from the centre are $\pm\frac{1}{\mu+1}$, lie within the circle, radius $\frac{c}{\mu}$, the other two of the same four cusps, viz. those whose distances from the centre are $\pm\frac{1}{\mu-1}$, lie without the circle, radius $\frac{c}{\mu}$; the last-mentioned two cusps lie without the refracting circle, when $\mu<2$, upon this circle, when $\mu=2$, and within it and therefore between the two circles, when $\mu>2$. The caustic is therefore of the forms in the annexed figures 3, 4, 5, in each

Fig. 3.

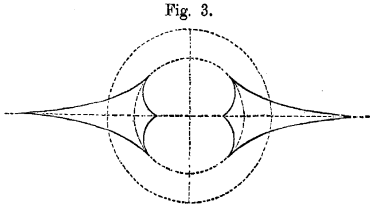


Fig. 4.

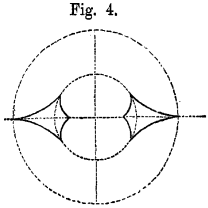
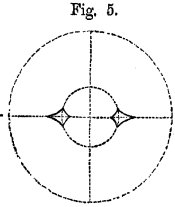


Fig. 5.



of which the outer circle is the refracting circle, and μ is > 1, but the three figures correspond respectively to the cases $\mu < 2$, $\mu = 2$ and $\mu > 2$. The same three figures will represent the different forms of the caustic when the inner circle is the refracting circle and μ is < 1, the three figures then respectively corresponding to the cases $\mu > \frac{1}{2}$, $\mu = \frac{1}{2}$, and $\mu < \frac{1}{2}$.

XVI.

To find the tangential equation, I retain k instead of its value $\frac{1}{\mu}$; the equation of the refracted ray then is

$$x(k\cos\phi - \sqrt{1-k^2\sin^2\phi}) + y(k\sin\phi + \cot\phi\sqrt{1-k^2\sin^2\phi}) - k = 0,$$

and representing this by

$$Xx + Yy - k = 0,$$

we have

$$X = k\cos\phi - \sqrt{1-k^2\sin^2\phi},$$

$$Y = k\sin\phi + \cot\phi\sqrt{1-k^2\sin^2\phi},$$

equations which give

$$X\cos\phi + Y\sin\phi = k,$$

$$X^2 + Y^2 \qquad = \frac{1}{\sin^2\phi},$$

and consequently

$$\sin\phi = \frac{1}{\sqrt{X^2+Y^2}}$$

$$\cos\phi = \frac{\sqrt{X^2+Y^2-1}}{\sqrt{X^2+Y^2}},$$

and we have

$$X\sqrt{X^2+Y^2-1} + Y - k\sqrt{X^2+Y^2} = 0,$$

which gives

$$(X^2+Y^2)(X^2-1-k^2) = -2kY\sqrt{X^2+Y^2};$$

or, dividing out by the factor $\sqrt{X^2+Y^2}$, the equation becomes

$$\sqrt{X^2+Y^2}(X^2-1-k^2) = -2kY,$$

from which

$$(X^2+Y^2)(X^2-1-k^2)^2 - 4k^2Y^2 = 0;$$

or reducing and arranging, we obtain

$$X^2(X^2-1-k^2)^2 + Y^2(X+1+k)(X+1-k)(X-1+k)(X-1-k) = 0$$

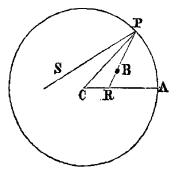
for the tangential equation of the caustic by refraction of a circle for parallel rays. The caustic is therefore of the class 6.

XVII.

Suppose next that rays proceeding from a point are reflected at a circle.

A very elegant solution of the problem is given by Lagrange in the *Mém. de Turin;* the investigation, as given by Mr P. Smith in a note in the *Cambridge and Dublin Mathematical Journal*, t. ii. [1847] p. 237, is as follows:

Let B be the radiant point, RBP an incident ray, and PS a reflected ray; CA a fixed radius; $ACP=\alpha$, $ACB=\epsilon$, reciprocal of $CB=c$, reciprocal of $CP=a$. The equations of the incident and reflected ray, where $u=\frac{1}{r}$, may be written



$$u=A\sin\theta+B\cos\theta;\ \text{incident ray},$$
$$u=A\sin(2\alpha-\theta)+B\cos(2\alpha-\theta);\ \text{reflected ray},$$

the conditions for determining A and B being

$$a=A\sin\alpha+B\cos\alpha,$$
$$c=A\sin\epsilon+B\cos\epsilon,$$

whence

$$A=\frac{a\cos\epsilon-c\cos\alpha}{\sin(\alpha-\epsilon)},\ B=\frac{c\sin\alpha-a\sin\epsilon}{\sin(\alpha-\epsilon)}.$$

Substituting these values, the equation of the reflected ray becomes

$$a\sin(2\alpha-\theta-\epsilon)=u\sin(\alpha-\epsilon)+c\sin(\alpha-\theta),$$

from which and its differential with respect to the arbitrary parameter α, the equation of the caustic, or envelope of the reflected rays, will be found by eliminating α.

In this, α being the only quantity treated as variable in the differentiation, let

$$2\alpha-\theta-\epsilon=2\phi,$$

therefore

$$\alpha=\phi+\tfrac{1}{2}(\theta+\epsilon),$$

and the equation becomes

$$a \sin 2\phi = u \sin \{\phi + \tfrac{1}{2}(\theta - \epsilon)\} + c \sin \{\phi - \tfrac{1}{2}(\theta - \epsilon)\}.$$

Make

$$P = \frac{(u+c)\cos \frac{1}{2}(\theta - \epsilon)}{2a},$$

$$Q = \frac{(u-c)\sin \frac{1}{2}(\theta - \epsilon)}{2a},$$

also

$$x = \frac{1}{\cos \phi},\ y = \frac{1}{\sin \phi},$$

then the equation becomes

$$Px + Qy = 1,$$

with the condition

$$x^{-2} + y^{-2} = 1.$$

Hence

$$P = \lambda x^{-3},\ Q = \lambda y^{-3};$$

multiplying by x and y, and adding, we find $\lambda = 1$; therefore

$$x^{-2} = P^{\frac{2}{3}},\ y^{-2} = Q^{\frac{2}{3}}.$$

Hence

$$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1;$$

or restoring the values of P and Q,

$$\{(u+c)\cos \tfrac{1}{2}(\theta - \epsilon)\}^{\frac{2}{3}} + \{(u-c)\sin \tfrac{1}{2}(\theta - \epsilon)\}^{\frac{2}{3}} = 1,$$

the equation of the caustic.

XVIII.

But the equation of the caustic for rays proceeding from a point and reflected at a circle may be obtained by a different method, as follows:

Take the centre of the circle for origin; let c be the radius of the circle, a, b the coordinates of the radiant point, α, β the coordinates of the point of incidence, x, y the coordinates of a point in the reflected ray. Then we have from the equation of the circle $\alpha^2 + \beta^2 = c^2$, and the equation of the reflected ray is by the general formula,

$$(b\alpha - a\beta)(\alpha x + \beta y - c^2) + (y\alpha - x\beta)(a\alpha + b\beta - c^2) = 0;$$

or arranging the terms in a different order,

$$(bx+ay)(\alpha^2-\beta^2)+2(by-ax)\alpha\beta-c^2(b+y)\alpha+c^2(a+x)\beta=0;$$

and writing herein $\alpha=c\cos\theta$, $\beta=c\sin\theta$, the equation becomes

$$(bx+ay)\cos 2\theta+(by-ax)\sin 2\theta-(b+y)c\cos\theta+(a+x)c\sin\theta=0,$$

where θ is a variable parameter.

Now in general to find the envelope of

$$A\cos 2\theta+B\sin 2\theta+C\cos\theta+D\sin\theta+E=0,$$

we may put $e^{i\theta}=z$, which gives the equation

$$(A-Bi)z^4+(C-Di)z^3+2Ez^2+(C+Di)z+(A+Bi)=0,$$

and equate the discriminant to zero: this gives

$$(4I)^3-27(-8J)^2=0,$$

where

$$4I=4(A^2+B^2)-(C^2+D^2)+\tfrac{4}{3}E^2,$$
$$-8J=A(C^2-D^2)+2BCD-\{8(A^2+B^2)+(C^2+D^2)\}\tfrac{1}{3}E+\tfrac{8}{27}E^3,$$

and consequently

$$\{4(A^2+B^2)-(C^2+D^2)+\tfrac{4}{3}E^2\}^3$$
$$-27\{A(C^2-D^2)+2BCD-\left(8(A^2+B^2)+(C^2+D^2)\right)\tfrac{1}{3}E+\tfrac{8}{27}E^3\}^2=0;$$

and substituting for A, B, C, D, E their values, we find

$$\{4(a^2+b^2)(x^2+y^2)-c^2\left((x+a)^2+(y+b)^2\right)\}^3-27(bx-ay)^2(x^2+y^2-a^2-b^2)^2=0,$$

for the equation of the caustic in the case of rays proceeding from a point and reflected at a circle: the equation was first obtained by St Laurent.

It will be convenient to consider the axis of x as passing through the radiant point; this gives $b=0$; and if we assume also $c=1$, the equation of the caustic becomes

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0.$$

XIX.

Reverting to the equation of the reflected ray, and putting, as before, $c=1$, $b=0$, this becomes

$$(-2a\cos\theta+1)x+\frac{a\cos 2\theta-\cos\theta}{\sin\theta}y+a=0;$$

differentiating with respect to θ, we have

$$(-2a\sin\theta)x+\frac{-a\cos\theta(1+2\sin^2\theta)+1}{\sin^2\theta}y=0;$$

and from these equations

$$x = \frac{a^2 \cos\theta (1 + 2\sin^2\theta) - a}{1 - 3a\cos 2\theta + 2a^2},$$

$$y = \frac{2a^2 \sin^3\theta}{1 - 3a\cos 2\theta + 2a^2},$$

which give the coordinates of a point of the caustic in terms of the angle θ which determines the position of the point of incidence. The values in question satisfy, as they should do, the equation

$$\{(4a^2 - 1)(x^2 + y^2) - 2ax - a^2\}^3 - 27a^2y^2(x^2 + y^2 - a^2)^2 = 0;$$

we have, in fact,

$$x^2 + y^2 \qquad - a^2 = \frac{4a^3(\cos\theta - a)^3}{(1 - 3a\cos 2\theta + 2a^2)^2},$$

$$(4a^2 - 1)(x^2 + y^2) - 2ax - a^2 = \frac{12a^4(\cos\theta - a)^2}{(1 - 3a\cos 2\theta + 2a^2)^2},$$

from which it is easy to derive the equation in question.

XX.

If we represent the equation of the reflected ray by

$$Xx + Yy + a = 0,$$

then we have

$$X = -2a\cos\theta + 1,$$

$$Y = \frac{a\cos 2\theta - \cos\theta}{\sin\theta},$$

and thence

$$(X - 1)^2 - 4a^2 = -4a^2\sin^2\theta,$$

$$X^2 + Y^2 \qquad = \frac{1}{\sin^2\theta}(1 - 2a\cos\theta + a^2),$$

$$X + a^2 \qquad = 1 - 2a\cos\theta + a^2,$$

and consequently

$$(X^2 + Y^2)\{(X - 1)^2 - 4a^2\} + 4a^2X + 4a^4 = 0,$$

or, what is the same thing,

$$\{X(X - 1) - 2a^2\}^2 + Y^2\{(X - 1)^2 - 4a^2\} = 0,$$

which may be considered as the tangential equation of the caustic by reflexion of a circle; or if we consider X, Y as the coordinates of a point, then the equation may be considered as that of the polar of the caustic. The polar is therefore a curve of the fourth order, having two double points defined by the equations $X(X - 1) - 2a^2 = 0$, $Y = 0$,

and a third double point at infinity on the axis of Y, i.e. three double points in all; the number of cusps is therefore 0, and there are consequently 4 double tangents and 6 inflections, and the curve is of the class 6. And as Y is given as an explicit function of X, there is of course no difficulty in tracing the curve. We thus see that the caustic by reflexion of a circle is a curve of the order 6, and has 4 double points and 6 cusps (the circular points at infinity are each of them a cusp, so that the number of cusps at a finite distance is 4): this coincides with the conclusions which will be presently obtained by considering the equation of the caustic.

XXI.

The equation of the caustic by reflexion of a circle is

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0.$$

Suppose first that $y=0$, we have

$$\{(4a^2-1)x^2-2ax-a^2\}^3=0,$$

i.e.

$$x=\frac{-a}{2a+1},\quad x=\frac{a}{2a-1},$$

or the curve meets the axis of x in two points, each of which is a triple point of intersection.

Write next $x^2+y^2=a^2$, this gives

$$\{(4a^2-1)a^2-2ax-a^2\}^3=0,$$

and consequently

$$x=-a(1-2a^2),$$
$$y=\pm 2a^2\sqrt{1-a^2},$$

or the curve meets the circle $x^2+y^2-a^2=0$ in two points, each of which is a triple point of intersection.

To find the nature of the infinite branches, we may write, retaining only the terms of the degrees six and five,

$$(4a^2-1)^3(x^2+y^2)^3-6(4a^2-1)^2a(x^2+y^2)^2x-27a^2y^2(x^2+y^2)^2=0;$$

and rejecting the factor $(x^2+y^2)^2$, this gives

$$(4a^2-1)^3x^2+\{(4a^2-1)^3-27a^2\}y^2-6(4a^2-1)^2ax=0;$$

or reducing,

$$(4a^2-1)^3x^2-(1-a^2)(8a^2+1)^2y^2-6(4a^2-1)^2ax=0;$$

and it follows that there are two asymptotes, the equations of which are

$$y=\frac{(4a^2-1)^{\frac{3}{2}}}{\sqrt{1-a^2}(8a^2+1)}\left\{x-\frac{3a}{4a^2-1}\right\}.$$

Represent for a moment the equation of one of the asymptotes by $y = A(x - \alpha)$, then the perpendicular from the origin or centre of the reflecting circle is $A\alpha \div \sqrt{1+A^2}$, and

$$A\alpha = \frac{3a\sqrt{4a^2-1}}{\sqrt{1-a^2}(1+8a^2)},$$

$$1 + A^2 = \frac{(1-a^2)(1+8a^2)^2 + (4a^2-1)^3}{(1-a^2)(1+8a^2)^2} = \frac{27a^2}{(1-a^2)(1+8a^2)^2},$$

$$\sqrt{1+A^2} = \frac{3\sqrt{3}\,a}{\sqrt{1-a^2}(1+8a^2)},$$

and the perpendicular is $\frac{1}{\sqrt{3}}\sqrt{4a^2-1}$, which is less than a if only $a^2 < 1$, i.e. in every case in which the asymptote is real.

The tangents parallel and perpendicular to the axis of x are most readily obtained from the equation of the reflected ray, viz.

$$(-2a\cos\theta + 1)x + \frac{a\cos 2\theta - \cos\theta}{\sin\theta}y + a = 0;$$

the coefficient of x (if the equation is first multiplied by $\sin\theta$) vanishes if $\sin\theta = 0$, which gives the axis of x, or if $\cos\theta = \frac{1}{2a}$, which gives $y = \pm\frac{\sqrt{4a^2-1}}{2a}$, for the tangents parallel to the axis of x.

The coefficient of y vanishes if $a\cos 2\theta - \cos\theta = 0$; this gives

$$\cos\theta = \frac{1 \pm \sqrt{8a^2+1}}{4a}, \qquad \sin\theta = \frac{1}{8a^2}(4a^2 - 1 \mp \sqrt{8a^2+1}),$$

and the tangents perpendicular to the axis of x are thus given by

$$x = \frac{-2a}{1 \mp \sqrt{8a^2+1}};$$

these tangents are in fact double tangents of the caustic. In order that the point of contact may be real, it is necessary that $\sin\theta$, $\cos\theta$ should be real; this will be the case for both values of the ambiguous sign if $a >$ or $= 1$, but only for the upper value if $a < 1$.

It has just been shown that for the tangents parallel to the axis of x, we have

$$y = \pm\frac{\sqrt{4a^2-1}}{2a},$$

the values of y being real for $a > \frac{1}{2}$: it may be noticed that the value $y = \frac{\sqrt{4a^2-1}}{2a}$

is greater, equal, or less than or to $y = 2a^2\sqrt{1-a^2}$, according as $a >$ = or $< \frac{1}{\sqrt{2}}$; this depends on the identity $(4a^2-1)-16a^6(1-a^2) = (2a^2-1)^2(2a^2+1)$.

To find the points of intersection with the reflecting circle, $x^2+y^2-1=0$, we have

$$(3a^2-1-2ax)^3-27a^2(1-x^2)(1-a^2)^2=0;$$

or, reducing,

$$8a^3x^3+(-27a^4+18a^2-15)\,a^2x^2+(54a^4-36a^2+6)\,ax+(-27a^4+18a^2+1)=0,$$

i.e.

$$(ax-1)^2(8ax-27a^4+18a^2+1)=0.$$

The factor $(ax-1)^2$ equated to zero shows that the caustic touches the circle in the points $x=\frac{1}{a}$, $y=\pm\sqrt{1-\frac{1}{a^2}}$, i.e. in the points in which the circle is met by the polar of the radiant point, and which are real or imaginary according as $a >$ or < 1. The other factor gives

$$x=\frac{27a^4-18a^2-1}{8a}.$$

Putting this value equal to ± 1, the resulting equation is $(a \mp 1)(27a^2+9a+1)=0$, and it follows that x will be in absolute magnitude greater or less than 1, i.e. the points in question will be imaginary or real, according as $a>1$ or $a<1$.

It is easy to see that the curve passes through the circular points at infinity, and that these points are cusps on the curve; the two points of intersection with the axis of x are cusps (the axis of x being the tangent), and the two points of intersection with the circle $x^2+y^2-a^2=0$ are also cusps, the tangent at each of the cusps coinciding with the tangent of the circle; there are consequently in all 6 cusps.

XXII.

To investigate the position of the double points we may proceed as follows: write for shortness $P=(4a^2-1)(x^2+y^2)-2ax-a^2$, $Q=ayS$, $S=x+y^2-a^2$; the equation of the caustic is

$$P^3-27Q^2=0;$$

hence, at a double point,

$$P^2\frac{dP}{dx}-18Q\frac{dQ}{dx}=0,$$

$$P^2\frac{dP}{dy}-18Q\frac{dQ}{dy}=0;$$

one of which equations may be replaced by

$$\frac{dP}{dx}\frac{dQ}{dy}-\frac{dP}{dy}\frac{dQ}{dx}=0.$$

Now

$$\frac{dP}{dx} = 2\{(4a^2-1)x-a\}, \quad \frac{dP}{dy} = 2(4a^2-1)y,$$

$$\frac{dQ}{dx} = 2axy, \qquad \frac{dQ}{dy} = a(x^2+3y^2-a^2) = a(S+2y^2);$$

substituting these values in the last preceding equation, we find

$$\frac{(4a^2-1)x-a}{(4a^2-1)y} = \frac{2xy}{S+2y^2},$$

or, reducing,

$$(4a^2-1)x-a = \frac{2ay^2}{S};$$

and using this to simplify the equation

$$P^2\frac{dP}{dx} - 18Q\frac{dQ}{dx} = 0,$$

we have

$$P^2\frac{4ay^2}{S} - 18ayS.\,2axy = 0,$$

i.e.

$$\frac{P^2}{S} - 9axS = 0,$$

and therefore

$$9x = \frac{P^2}{aS^2}.$$

Multiplying by P and writing for P^3 its value $27a^3y^3S^3$, we have

$$Px = 3ay^2,$$

and thence

$$P = \frac{3ay^2}{x}, \quad P^3 = \frac{27a^3y^6}{x^3} = 27a^2y^2S^3,$$

whence

$$\frac{ay^4}{x^3} = S^2, \quad \frac{a^{\frac{1}{2}}y^2}{x^{\frac{3}{2}}} = S, \quad \text{or} \quad \frac{y^2}{S} = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}};$$

and substituting in the equation

$$x = \frac{a}{4a^2-1}\left(1+\frac{2y^2}{S}\right),$$

we find

$$x = \frac{a}{4a^2-1}\left(1+\frac{2x}{a^{\frac{1}{2}}}\right),$$

or, rationalising,

$$4ax^3 - \{(4a^2-1)x-a\}^2 = 0,$$

or, what is the same thing,

$$(4ax-1)\left(x-a\left(a+i\sqrt{1-a^2}\right)^2\right)\left(x-a\left(a-i\sqrt{1-a^2}\right)^2\right)=0.$$

The factor $4ax-1$ equated to zero gives $x=\frac{1}{4a}$ from which y may be found, but the resulting point is not a double point; the other factors give each of them double points, and if we write

$$x=a\left(a+i\sqrt{1-a^2}\right)^2,$$

we find

$$y=\frac{2a^2i\left(a+i\sqrt{1-a^2}\right)^{\frac{3}{2}}}{\left(3a-i\sqrt{1-a^2}\right)^{\frac{3}{2}}},$$

values which, in fact, belong to one of the four double points. It is easy to see that the points in question are always imaginary.

It may be noticed, by way of verification, that the preceding values of x, y give

$$(4a^2-1)(x^2+y^2)-2ax-a^2=\frac{12a^4}{1+8a^2}\left(1-4a^2-4ai\sqrt{1-a^2}\right),$$

$$x^2+y^2-a^2=\frac{-4a^3}{1+8a^2}\left(3a+i\sqrt{1-a^2}\right),$$

$$y^2=\frac{4a^4}{1+8a^2}\left(-1+14a^2-16a^4+2a(3-8a^2)\,i\sqrt{1-a^2}\right);$$

and if the quantities within ( ) on the right-hand side are represented by A, B, C, then

$$\frac{A}{B}=-\left(a+i\sqrt{1-a^2}\right),$$

$$\frac{C}{B}=-\left(a+i\sqrt{1-a^2}\right)^3,$$

whence we have identically,

$$\left(\frac{A}{B}\right)^3=\frac{C}{B},\ \text{or}\ A^3=B^2C,$$

by means of which it appears that the values of x, y satisfy, as they should do, the equation of the caustic; and by forming the expressions for $(4a^2-1)x-a$ and $x^2+3y^2-a^2$, it might be shown, *à posteriori*, that the point in question was a double point.

XXIII.

The equation

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0$$

becomes when $a=1$ (i.e. when the radiant point is in the circumference),

$$\{3y^2+(x-1)(3x+1)\}^3-27y^2(y^2+x^2-1)^2=0;$$

it is easy to see that this divides by $(x-1)^2$; and throwing out this factor, we have for the caustic the equation of the fourth order,

$$27y^4+18y^2(3x^2-1)+(x-1)(3x+1)^3=0.$$

XXIV.

The equation

$$\{(4a^2-1)(x^2+y^2)-2ax-a^2\}^3-27a^2y^2(x^2+y^2-a^2)^2=0$$

becomes when $a=\infty$ (i.e. in the case of parallel rays),

$$(4x^2+4y^2-1)^3-27y^2=0,$$

which may also be written

$$64x^6+48x^4(4y^2-1)+12x^2(4y^2-1)^2+(8y^2+1)^2(y^2-1)=0.$$

XXV.

It is now easy to trace the curve. Beginning with the case $a=\infty$, the curve lies wholly within the reflecting circle, which it touches at two points; the line joining the points of contact, being in fact the axis of y, divides the curve into two equal portions; the curve has in the present, as in every other case (except one limiting case), two cusps on the axis of x (see fig. 6). Next, if a be positive and >1, the general form of the curve is the same as before, only the line joining the points of contact with the reflecting circle divides the curve into unequal portions, that in the

Fig. 6. $a=\infty$.

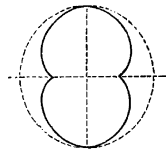
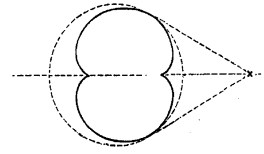


Fig. 7. $a>1$.



neighbourhood of the radiant point being the smaller of the two portions (see fig. 7). When $a=1$, the two points of contact with the reflecting circle unite together at the radiant point; the curve throws off, as it were, the two coincident lines $x=1$, and the order is reduced from 6 to 4. The curve has the form fig. 8, with only a single cusp on the axis of x. If a be further diminished, $a<1>\frac{1}{\sqrt{2}}$, the curve takes the form shown by fig. 9, with two infinite branches, one of them having simply a cusp on the axis of x, the other having a cusp on the axis of x, and a pair of cusps at its intersection with the circle through the radiant point; there are two asymptotes equally inclined to the axis of x. In the case $a=\frac{1}{\sqrt{2}}$, the form of the curve is nearly the same as before, only the cusps upon the circle through the radiant point lie on the axis of y (see fig. 10). The case $a<\frac{1}{\sqrt{2}}>\frac{1}{2}$ is shown, fig. 11. For $a=\frac{1}{2}$, the two

asymptotes coincide with the axis of x; one of the branches of the curve has wholly disappeared, and the form of the other is modified by the coincidence of the asymptotes

Fig. 8. $a = 1$.

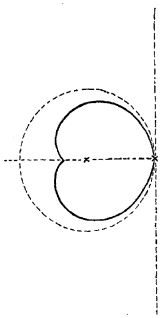


Fig. 9. $a < 1 > \frac{1}{\sqrt{2}}$.

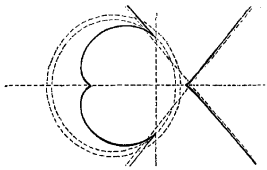


Fig. 13. $a = \frac{1}{3}$.

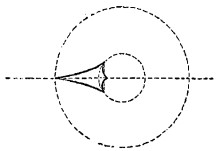


Fig. 11. $a < \frac{1}{\sqrt{2}} > \frac{1}{2}$.

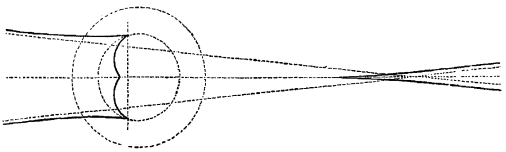


Fig. 10. $a = \frac{1}{\sqrt{2}}$.

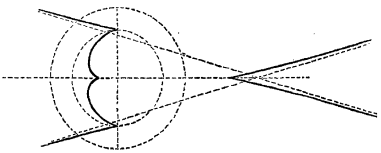
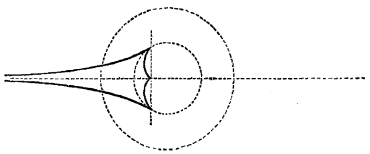


Fig. 12. $a = \frac{1}{2}$.



with the axis of x; it has in fact acquired a cusp at infinity on the axis of x (see fig. 12). When $a < \frac{1}{2}$, the curve consists of a single finite branch, with two cusps on the axis of x, and two cusps at the points of intersection with the circle through the radiant point; one of the last-mentioned cusps will be outside the reflecting circle as long as $a > \frac{1}{3}$; fig. 13 represents the case $a = \frac{1}{3}$, for which this cusp is upon the reflecting circle. For $a < \frac{1}{3}$, the curve lies wholly within the reflecting circle, one of the cusps upon the axis of x being always within, and the other always without the circle through the radiant point, and as a approaches O the curve becomes smaller and smaller, and ultimately disappears in a point. The case a negative is obviously included in the preceding one.

Several of the preceding results relating to the caustic by reflexion of a circle were obtained, and the curve is traced in a memoir by the Rev. Hamnet Holditch, *Quarterly Mathematical Journal*, t. I. [1857, pp. 93—111].

XXVI.

Suppose next that rays proceeding from a point are *refracted* at a circle. Take the centre of the circle as origin, let the radius be c, and take ξ, η as the coordinates of the radiant point, α, β the coordinates of the point of incidence, x, y the coordinates of a point in the refracted ray: then the general equation

$$-\overline{qG}^2\,\overline{\nabla QGN}^2+\mu^2\,\overline{QG}^2\,\overline{\nabla qGN}^2=0$$

becomes, taking the centre of the circle as the point N on the normal, or writing $a=b$, $b=0$,

$$-\{(x-\alpha)^2+(y-\beta)^2\}(\beta\xi-\alpha\eta)^2+\mu^2\{(\xi-\alpha)^2+(\eta-\beta)^2\}(\beta x-\alpha y)^2=0\,;$$

or putting $\alpha^2+\beta^2=c^2$, and expanding,

$$\begin{aligned}
&\alpha^3 && \{2(\eta^2x-\mu^2y^4\xi)\}\\
&+\alpha^2\beta && \{-4(\xi\eta x-\mu^2xy\xi)+2(\eta^2y-\mu^2y^2\eta)\}\\
&+\alpha\beta^2 && \{-4(\xi\eta y-\mu^2xy\eta)+2(\xi^2x-\mu^2x^2\xi)\}\\
&+\beta^3 && \{2(\xi^2y-\mu^2x^2\eta)\}\\
&-\alpha^2 && \{(x^2+y^2+c^2)\eta^2-\mu^2(\xi^2+\eta^2+c^2)y^2\}\\
&+2\alpha\beta && \{(x^2+y^2+c^2)\xi\eta-\mu^2(\xi^2+\eta^2+c^2)xy\}\\
&-\beta^2 && \{(x^2+y^2+c^2)\xi^2-\mu^2(\xi^2+\eta^2+c^2)x^2\}\\
&=0,
\end{aligned}$$

which may be represented by

$$A\alpha^3+B\alpha^2\beta+C\alpha\beta^2+D\beta^3+F\alpha^2+G\alpha\beta+H\beta^2=0.$$

Now $\alpha^2+\beta^2=c^2$, and we may write

$$\alpha=\tfrac{1}{2}c\left(z+\frac{1}{z}\right),\qquad \beta=-\tfrac{1}{2}ci\left(z-\frac{1}{z}\right).$$

The equation thus becomes

$$\begin{aligned}
&A\left(z+\frac{1}{z}\right)^3-Bi\left(z+\frac{1}{z}\right)^2\left(z-\frac{1}{z}\right)-C\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)^2-Di\left(z-\frac{1}{z}\right)^3\\
&\qquad+\frac{2}{c}F\left(z+\frac{1}{z}\right)^2-\frac{2}{c}Gi\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)-\frac{2}{c}H\left(z-\frac{1}{z}\right)^2=0
\end{aligned}$$

or expanding,

$$\left.\begin{aligned}
&(A-Bi-C-Di) && z^3\\
&+\frac{2}{c}(F-Gi-H) && z^2\\
&+\ (3A-Bi+C+3Di) && z\\
&+\frac{4}{c}(F+H)\\
&+\ (3A+Bi+C-3Di) && \frac{1}{z}\\
&+\frac{2}{c}(F+Gi-H) && \frac{1}{z^2}\\
&+\ (A+Bi-C+Di) && \frac{1}{z^3}
\end{aligned}\right\}=0,$$

in which z may be considered as the variable parameter; hence the equation of the caustic may be obtained by equating to zero the discriminant of the above function of z; but the discriminant of a sextic function has not yet been calculated. The equation would be of the order 20, and it appears from the result previously obtained for parallel rays, that the equation must be of the order 12 at the least; it is, I think, probable that there is not any reduction of order in the general case. It is however practicable, as will presently be seen, to obtain the tangential equation of the caustic by refraction, and the curve is thus shown to be only of the class 6.

XXVII.

Suppose that rays proceeding from a point are refracted at a circle, and let it be required to find the equation of the secondary caustic: take the centre of the circle as origin, let c be the radius, ξ, η the coordinates of the radiant point, α, β the coordinates of a point upon the circle, μ the index of refraction; the secondary caustic will be the envelope of the circle,

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-\{(\xi-\alpha)^2+(\eta-\beta)^2\}=0,$$

where α, β are variable parameters connected by the equation $\alpha^2+\beta^2-c^2=0$; the equation of the circle may be written in the form

$$\mu^2(x^2+y^2+c^2)-(\xi^2+\eta^2+c^2)-2(\mu^2x-\xi)\alpha-2(\mu^2y-\eta)\beta=0.$$

But in general the envelope of $A\alpha+B\beta+C=0$, where α, β are connected by the equation $\alpha^2+\beta^2-c^2=0$, is $c^2(A^2+B^2)-C^2=0$, and hence in the present case the equation of the envelope is

$$\{\mu^2(x^2+y^2+c^2)-(\xi^2+\eta^2+c^2)\}^2=4c^2\{(\mu^2x-\xi)^2+(\mu^2y-\eta)^2\},$$

which may also be written

$$\{\mu^2(x^2+y^2-c^2)-(\xi^2+\eta^2-c^2)\}^2=4c^2\mu^2\{(x-\xi)^2+(y-\eta)^2\}.$$

If the axis of x be taken through the radiant point, then $\eta=0$, and writing also $\xi=a$, the equation becomes

$$\{\mu^2(x^2+y^2-c^2)-a^2+c^2\}^2=4c^2\mu^2\{(x-a)^2+y^2\};$$

or taking the square root of each side,

$$\{\mu^2(x^2+y^2-c^2)-a^2+c^2\}=2c\mu\sqrt{(x-a)^2+y^2};$$

whence multiplying by $1-\dfrac{1}{\mu^2}$ and adding on each side $c^2\left(\mu-\dfrac{1}{\mu}\right)^2+(x-a)^2+y^2$, we have

$$\mu^2\left\{\left(x-\frac{a}{\mu^2}\right)^2+y^2\right\}=\left\{\sqrt{(x-a)^2+y^2}+c\left(\mu-\frac{1}{\mu}\right)\right\}^2,$$

or

$$\mu\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}=\sqrt{(x-a)^2+y^2}+c\left(\mu-\frac{1}{\mu}\right),$$

which shows that the secondary caustic is the Oval of Descartes, or as it will be convenient to call it, the Cartesian.

It is proper to remark, that the Cartesian consists in general of two ovals, one of which is the orthogonal trajectory of the refracted rays, the other the orthogonal trajectory of the false refracted rays. In the case of reflexion, the secondary caustic is a Cartesian having a double point; this may be either a conjugate point, or a real double point arising from the union and intersection of the two ovals; the same secondary caustic may arise also from refraction, as will be presently shown.

XXVIII.

Reverting to the original form of the equation of the secondary caustic, multiplying by $\frac{1}{\mu^2}\left(1-\frac{c^2}{a^2}\right)$ and adding on each side $\frac{a^2}{\mu^2}\left(1-\frac{c^2}{a^2}\right)^2+\frac{c^2}{a^2}\{(x-a)^2+y^2\}$, the equation becomes

$$\left\{\left(x-\frac{c^2}{a}\right)^2+y^2\right\}=\left\{\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{a}{\mu}\left(1-\frac{c^2}{a^2}\right)\right\}^2,$$

or extracting the square root,

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{a}{\mu}\left(1-\frac{c^2}{a^2}\right).$$

Combining this with the former result, we see that the equation may be expressed indifferently in any one of the four forms,

$$\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}=\frac{1}{\mu}\sqrt{(x-a)^2+y^2}+\frac{c}{\mu}\left(\mu-\frac{1}{\mu}\right),$$

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c}{a}\sqrt{(x-a)^2+y^2}+\frac{1}{\mu}\left(a-\frac{c^2}{a}\right),$$

$$\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}=\frac{c\mu}{a}\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}+\frac{a}{\mu}-\frac{c^2\mu}{a},$$

$$c\left(\mu-\frac{1}{\mu}\right)\sqrt{\left(x-\frac{c^2}{a}\right)^2+y^2}+\left(-a+\frac{c^2}{a}\right)\sqrt{\left(x-\frac{a}{\mu^2}\right)^2+y^2}+\left(\frac{a}{\mu}-\frac{c^2\mu}{a}\right)\sqrt{(x-a)^2+y^2}=0.$$

It follows, that if we write successively

$$a'=a,\quad c'=c,\quad \mu'=\mu \qquad (1)$$

$$a'=\frac{c^2}{a},\quad c'=\frac{c}{\mu},\quad \mu'=\frac{c}{a} \qquad (\alpha)$$

$$a'=\frac{a}{\mu^2},\quad c'=\frac{c}{\mu},\quad \mu'=\frac{1}{\mu} \qquad (\beta)$$

$$a'=a,\quad c'=\frac{a}{\mu},\quad \mu'=\frac{a}{c} \qquad (\gamma)$$

$$a'=\frac{c^2}{a},\quad c'=c,\quad \mu'=\frac{c\mu}{a} \qquad (\delta)$$

$$a'=\frac{a}{\mu^2},\quad c'=\frac{a}{\mu},\quad \mu'=\frac{a}{c\mu} \qquad (\epsilon),$$

or what is the same thing,

$$a = a', \quad c = c', \quad \mu = \mu' \qquad (1)$$

$$a = \frac{a'}{\mu'^2}, \quad c = \frac{a'}{\mu'}, \quad \mu = \frac{a'}{c'\mu'} \qquad (\alpha)$$

$$a = \frac{a'}{\mu'^2}, \quad c = \frac{c'}{\mu'}, \quad \mu = \frac{1}{\mu'} \qquad (\beta)$$

$$a = a', \quad c = \frac{a'}{\mu'}, \quad \mu = \frac{a'}{c'} \qquad (\gamma)$$

$$a = \frac{c'^2}{a'}, \quad c = c', \quad \mu = \frac{c'\mu'}{a'} \qquad (\delta)$$

$$a = \frac{c'^2}{a'}, \quad c = \frac{c'}{\mu'}, \quad \mu = \frac{c'}{a'} \qquad (\epsilon),$$

or what is again the same thing,

$$a' = a, \quad \frac{c'^2}{a'} = \frac{c^2}{a}, \quad \frac{a'}{\mu'^2} = \frac{a}{\mu^2} \qquad (1)$$

$$a' = \frac{c^2}{a}, \quad \frac{c'^2}{a'} = \frac{a}{\mu^2}, \quad \frac{a'}{\mu'^2} = a \qquad (\alpha)$$

$$a' = \frac{a}{\mu^2}, \quad \frac{c'^2}{a'} = \frac{c^2}{a}, \quad \frac{a'}{\mu'^2} = a \qquad (\beta)$$

$$a' = a, \quad \frac{c'^2}{a'} = \frac{a}{\mu^2}, \quad \frac{a'}{\mu'^2} = \frac{c^2}{a} \qquad (\gamma)$$

$$a' = \frac{c^2}{a}, \quad \frac{c'^2}{a'} = a, \quad \frac{a'}{\mu'^2} = \frac{a}{\mu^2} \qquad (\delta)$$

$$a' = \frac{a}{\mu^2}, \quad \frac{c'^2}{a'} = a, \quad \frac{a'}{\mu'^2} = \frac{c^2}{a} \qquad (\epsilon),$$

we have in each case identically the same secondary caustic, and therefore also identically the same caustic; in other words, the same caustic is produced by six different systems of a radiant point and refracting circle. It is proper to remark that if we represent the six systems of equations by $(a', c', \mu') = (a, c, \mu)$, $(a', c', \mu') = \alpha(a, c, \mu)$, &c., then, α, β, γ, δ, ϵ will be functional symbols satisfying the conditions

$$1 = \alpha\beta = \beta\alpha = \gamma^2 = \delta^2 = \epsilon^2,$$

$$\alpha = \beta^2 = \delta\gamma = \epsilon\delta = \gamma\epsilon,$$

$$\beta = \alpha^2 = \gamma\delta = \delta\epsilon = \epsilon\gamma,$$

$$\gamma = \delta\alpha = \alpha\epsilon = \epsilon\beta = \beta\delta,$$

$$\delta = \epsilon\alpha = \alpha\gamma = \gamma\beta = \beta\epsilon,$$

$$\epsilon = \gamma\alpha = \alpha\delta = \delta\beta = \beta\gamma.$$

XXIX.

The preceding formulæ, which were first given by me in the *Philosophical Magazine*, December 1853, [124] include as particular cases a preceding theorem with respect to the caustic by refraction of parallel rays, and also two theorems of St Laurent, *Gergonne*, t. XVIII., [1827, pp. 1—19] viz. if we suppose first that $a=c$, i.e. that the radiant point is in the circumference of the refracting circle, then the system (α) shows that the same caustic would be obtained by writing c, $\frac{c}{\mu}$, 1 (or what is the same thing -1) in the place of c, c, μ, and we have

THEOREM. The caustic by refraction for a circle when the radiant point is in the circumference is also the caustic by reflexion for the same radiant point, and for a reflecting circle concentric with the refracting circle, but having its radius equal to the quotient of the radius of the refracting circle by the index of refraction.

Next, if we write $a=c\mu$, then the refracted rays all of them pass through a point which is a double point of the secondary caustic, the entire curve being in this case the orthogonal trajectory, not of the refracted rays, but of the false refracted rays; the formula (δ) shows that the same caustic is obtained by writing $\frac{c^2}{a}$, c, 1 (or what is the same thing -1) in the place of a, c, $\mu\left(=\frac{a}{c}\right)$, and we have

THEOREM. The caustic by refraction for a circle when the distance of the radiant point from the centre is to the radius of the circle in the ratio of the index of refraction to unity, is also the caustic by reflexion for the same circle considered as a reflecting circle, and for a radiant point the image of the former radiant point.

XXX.

The curve is most easily traced by means of the preceding construction; thus if we take the radiant point outside the refracting circle, and consider μ as varying from a small to a large value (positive or negative values of μ give the same curve), we see that when μ is small the curve consists of two ovals, one of them within and the other without the refracting circle (see fig. 14). As μ increases the exterior oval continually increases, but undergoes modifications in its form; the interior oval in the first instance diminishes until we arrive at a curve, in which the interior oval is reduced to a conjugate point (see fig. 15); then as μ continues to increase the interior oval reappears (see fig. 16), and at last connects itself with the exterior oval, so as to form a curve with a double point (see fig. 17); and as μ increases still further the

curve again breaks up into an exterior and an interior oval (see fig. 18); and thenceforward as μ goes on increasing consists always of two ovals; the shape of the exterior oval is best perceived from the figures. An examination of the figures will also show how the same curves may originate from a different refracting circle and radiant point.

Fig. 14.

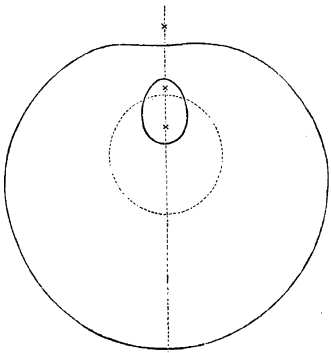


Fig. 17.

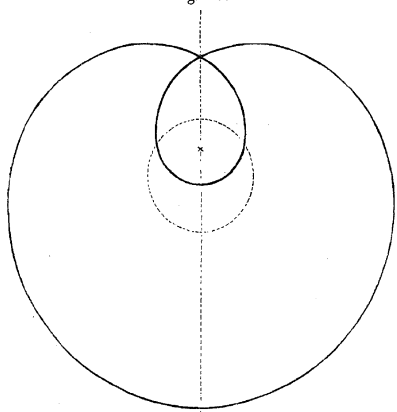


Fig. 15.

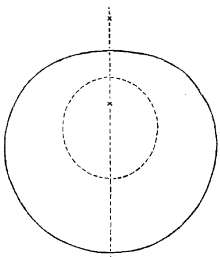


Fig. 16.

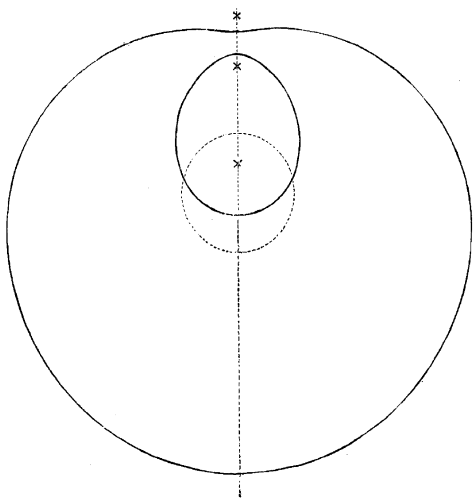
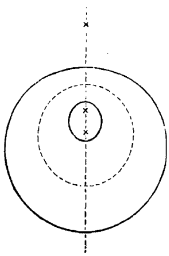


Fig. 18.



XXXI.

The theorem, "If a variable circle have its centre upon a circle S, and its radius proportional to the tangential distance of the centre from a circle C, the envelope is a Cartesian,"
is at once deducible from the theorem—

"If a variable circle have its centre upon a circle S and its radius proportional to the distance of the centre from a point C', the locus is a Cartesian,"

which last theorem was in effect given in discussing the theory of the secondary caustic. In fact, the locus of a point P such that its tangential distances from the circles C, C' are in a constant ratio, is a circle S. Conversely, if there be a circle C, and the locus of P be a circle S, then the circle C' may be found such that the tangential distances of P from the two circles are in a constant ratio, and the circle C' may be taken to be a point, i.e. if there be a circle C and the locus of P be a circle S, then a point C' may be found such that the tangential distance of P from the circle C is in a constant ratio to the distance from the point C'.

Hence treating P as the centre of the variable circle, it is clear that the variable circle is determined in the two cases by equivalent constructions, and the envelope is therefore the same in both cases.

XXXII.

The equation of the secondary caustic developed and reduced is

$$\mu^4(x^2+y^2)^2 - 2\mu^2\left(a^2+(\mu^2+1)c^2\right)(x^2+y^2) + 8c^2\mu^2ax + a^4 - 2a^2c^2(\mu^2+1) + (\mu^2-1)^2c^4 = 0,$$

or, what is the same thing,

$$\{\mu^2(x^2+y^2) - \left(a^2+(\mu^2+1)c^2\right)\}^2 + 8c^2\mu^2ax - 4c^2\left(c^2\mu^2+(\mu^2+1)a^2\right) = 0,$$

which may also be written

$$\left(x^2+y^2-\left(\frac{a^2}{\mu^2}+(1+\frac{1}{\mu^2})c^2\right)\right)^2 + \frac{8}{\mu^2}c^2ax - \frac{4c^2}{\mu^2}\left(c^2+(1+\frac{1}{\mu^2})a^2\right) = 0,$$

which is of the form

$$(x^2+y^2-\alpha)^2 + 16A(x-m) = 0;$$

and the values of the coefficients are

$$\alpha = \frac{1}{\mu^2}a^2 + (1+\frac{1}{\mu^2})c^2,$$

$$A = \frac{c^2a}{2\mu^2},$$

$$m = \frac{1}{2a}\left(c^2+(1+\frac{1}{\mu^2})a^2\right).$$

The equation just obtained should, I think, be taken as the standard form of the equation of the Cartesian, and the form of the equation shows that the Cartesian may be defined as the locus of a point, such that the fourth power of its tangential distance from a given circle is in a constant ratio to its distance from a given line.

XXXIII.

The Cartesian is a curve of the fourth order, symmetrical about a certain line which it intersects in four arbitrary points, and these points determine the curve. Taking the line in question (which may be called the axis) as the axis of x, and a line at right angles to it as the axis of y, let a, b, c, d be the values of x corresponding to the points of intersection with the axis, then the equation of the curve is

$$y^4 + y^2[2x^2 - (a+b+c+d)x - \tfrac{1}{4}(a^2+b^2+c^2+d^2-2ab-2ac-2ad-2bc-2bd-2cd)] \\ + (x-a)(x-b)(x-c)(x-d) = 0.$$

It is easy to see that the form of the equation is not altered by writing $x+\theta$ for x, and $a+\theta$, $b+\theta$, $c+\theta$, $d+\theta$ for a, b, c, d, we may therefore without loss of generality put $a+b+c+d=0$, and the equation of the curve then becomes

$$y^4 + y^2(2x^2 + ab + ac + ad + bc + bd + cd) + (x-a)(x-b)(x-c)(x-d) = 0,$$

where

$$a+b+c+d=0\,;$$

the curve is in this case said to be referred to the centre as origin.

The last-mentioned equation may be written

$$(x^2+y^2)^2 + (ab+ac+ad+bc+bd+cd)(x^2+y^2) - (abc+abd+acd+bcd)x + abcd = 0,$$

or

$$\begin{aligned} &\{x^2+y^2+\tfrac{1}{2}(ab+ac+ad+bc+bd+cd)\}^2 \\ &\quad - \quad (abc+abd+acd+bcd)x \\ &\quad - \tfrac{1}{4}\left\{\begin{array}{l} a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2 \\ +2a^2bc+2a^2bd+2a^2cd+2b^2ac+2b^2ad+2b^2cd \\ +2c^2ab+2c^2ad+2c^2bd+2d^2ab+2d^2ac+2d^2bc \\ +2abcd \end{array}\right\} = 0, \end{aligned}$$

or observing that

$$\begin{aligned} &a^2bc + a^2bd + a^2cd + b^2ac + b^2ad + b^2cd \\ &+ c^2ab + c^2ad + c^2bd + d^2ab + d^2ac + d^2bc \\ &\quad = abc(a+b+c) + abd(a+b+d) + acd(a+c+d) + bcd(b+c+d) \\ &\quad = -4abcd, \end{aligned}$$

the equation becomes

$$\begin{aligned}&\{x^2+y^2+\tfrac{1}{2}(ab+ac+ad+bc+bd+cd)\}^2\\&\quad-(abc+abd+acd+bcd)\,x\\&\quad-\tfrac{1}{4}(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2-6abcd)=0,\end{aligned}$$

which is of the form

$$(x^2+y^2-\alpha)^2+16A\,(x-m)=0,$$

and, as already remarked, signifies that the fourth power of the tangential distance of a point in the curve from a given circle, is proportional to the distance of the same point from a given line. The circle in question (which may be called the dirigent circle) has for its equation

$$x^2+y^2+\tfrac{1}{2}(ab+ac+ad+bc+bd+cd)=0\,;$$

the line in question, which may be called the directrix, has for its equation

$$x+\frac{a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2-6abcd}{4\,(abc+abd+acd+bcd)}=0\,;$$

the multiplier of the distance from the directrix is

$$abc+abd+acd+bcd.$$

It may be remarked that a, b, c, d being real, the dirigent circle is real; the equation may, in fact, be written

$$x^2+y^2=\tfrac{1}{8}\,[(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2].$$

XXXIV.

Considering the equation of the Cartesian under the form

$$(x^2+y^2-\alpha)^2+16A\,(x-m)=0,$$

the centre of the dirigent circle $x^2+y^2-\alpha=0$ must be considered as a real point, but α may be positive or negative, i.e. the radius may be either a real or a pure imaginary distance: the coefficients A, m must be real, the directrix is therefore a real line. The equation shows that for all points of the curve $x-m$ is always negative or always positive, according as A is positive or negative, i.e. that the curve lies wholly on one side of the directrix, viz. on the same side with the centre of the dirigent circle if A is positive, but on the contrary side if A is negative. In the former case the curve may be said to be an 'inside' curve, in the latter an 'outside' curve. If $m=0$, or the directrix passes through the centre of the dirigent circle, then the distinction between an inside curve and an outside curve no longer exists. It is clear that the curve touches the directrix in the points of intersection of this line and the dirigent circle, and that the points in question are the only points of intersection of the curve with the directrix or the dirigent circle; hence if the directrix and dirigent circle do not intersect, the curve does not meet either the directrix or the dirigent circle.

XXXV.

To discuss the equation

$$(x^2+y^2-\alpha)^2+16A(x-m)=0,$$

I write first $y=0$, which gives

$$x^4-2\alpha x^2+16Ax+\alpha^2-16Am=0$$

for the points of intersection with the axis of x. If this equation has equal roots, there will be a double point on the axis of x, and it is important to find the condition that this may be the case. The equation may be written in the form

$$(3,\ 0,\ -\alpha,\ 12A,\ 3\alpha^2-48Am \between x,\ 1)^4=0,$$

the condition for a part of equal roots is then at once seen to be

$$-(\alpha^2-12Am)^2+(\alpha^3-18Am\alpha+54A^2)^2=0;$$

or reducing and throwing out the factor A^2, this is

$$27A^2+2m(8m^2-9\alpha)A-\alpha^2(m^2-\alpha)=0.$$

This equation will give two equal values for A if

$$m^2(8m^2-9\alpha)^2+27\alpha^2(m^2-\alpha)=0,$$

an equation which reduces itself to

$$(4m^2-3\alpha)^3=0.$$

Hence, if $4m^2-3\alpha$ be negative, i.e. if $\alpha>\frac{4m^2}{3}$, the values of A will be imaginary, but if $4m^2-3\alpha$ be positive, or $\alpha<\frac{4m^2}{3}$, the values of A will be real. If $\alpha=\frac{4m^2}{3}$, then there will be two equal values of A, which in fact corresponds to a cusp upon the axis of x. Whenever the curve is real there will be at least two real points on the axis of x; and when $\alpha<\frac{4m^2}{3}$, but not otherwise, then for properly selected values of A there will be four real points on the axis of x.

Differentiating the equation of the curve, we have

$$\left((x^2+y^2-\alpha)x+4A\right)dx+(x^2+y^2-\alpha)ydy=0;$$

and if in this equation we put $dx=0$, we find $y=0$, or $x^2+y^2-\alpha=0$, i.e. that the points on the axis of x, and the points of intersection with the circle $x^2+y^2-\alpha=0$, are the only points at which the curve is perpendicular to the axis of x. To find the points at which the curve is parallel to the axis of x, we must write $dx=0$, this gives

$$(x^2+y^2-\alpha)x+4A=0,$$

and thence

$$x^2 + y^2 - \alpha = -\frac{4A}{x},$$

and

$$A + x^2(x - m) = 0:$$

this equation will have three real roots if $A < \frac{4m^3}{27}$, and only a single real root if $A > \frac{4m^3}{27}$; for $A = \frac{4m^3}{27}$, the equation in question will have a pair of equal roots. It is easy to see that there is always a single real root of the equation which gives rise to a real value of y, i.e. to a real point upon the curve; but, when the equation has three real roots, two of the roots may or may not give rise to real points upon the curve.

XXXVI.

It is now easy to trace the curve. First, when $m = 0$, or the directrix passes through the centre of the dirigent circle, the curve is here an oval bent in so as to have double contact with the directrix, and lying on the one or the other side of the directrix according to the sign of A. See fig. *a*.

Fig. *a*.

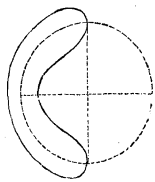


Fig. *b*.

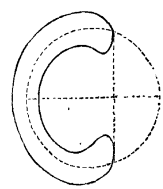


Fig. *c*.

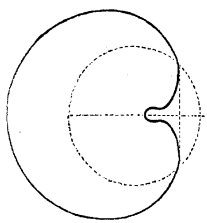
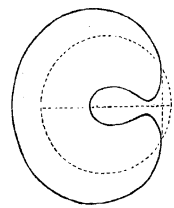


Fig. *d*.



Next, when the directrix does not pass through the centre of the dirigent circle, it will be convenient to suppose always that m is positive, and to consider A as passing first from 0 to ∞ and then from 0 to $-\infty$, i.e. to consider first the different inside curves, and then the different outside curves. Suppose $\alpha > \frac{4m^2}{3}$, the inside curve is at first an oval, as in fig. *b*, where (attending to one side only of the axis) it will be noticed that there are three tangents parallel to the axis, viz. one for the convexity of the oval, and two for the concavity. For $A = \frac{4m^3}{27}$ the two tangents for the concavity come together, and give rise to a stationary tangent (i.e. a tangent at an inflection) parallel to the axis, and for $A > \frac{4m^3}{27}$ the two tangents for the concavity disappear. The outside curve is an oval (of course on the opposite side of, and) bent in so as to have double contact with the directrix.

Next, if $\alpha = \frac{4m^2}{3}$, the inside curve is at first an oval, as in fig. *c*, and there are, as before, three tangents parallel to the axis: for $A = \frac{4m^3}{27}$, the tangents for the concavity of the oval come to coincide with the axis, and are tangents at a cusp, and for $A > \frac{4m^3}{27}$ the cusp disappears, and there are not for the concavity of the oval any tangents parallel to the axis. The outside curve is an oval as before, but smaller and more compressed.

Next, $\alpha < \frac{4m^2}{3} > m^2$, then the inside curve is at first an oval, as in fig. *d*, and there are, as before, three tangents parallel to the axis; when A attains a certain value which is less than $\frac{4m^3}{27}$, the curve acquires a double point; and as A further increases, the curve breaks up into two separate ovals, and there are then only two tangents parallel to the axis, viz. one for the exterior oval and one for the interior oval. As A continues to increase, the interior oval decreases; and when A attains a certain value which is less than $\frac{4m^3}{27}$, the interior oval reduces itself to a conjugate point, and it afterwards disappears altogether. The outside curve is an oval as before, but smaller and more compressed.

Next, if the directrix touch the dirigent circle, i.e. if $\alpha = m^2$. Then the inside curve is at first composed of an exterior oval which touches the dirigent circle, and of an interior oval which lies wholly within the dirigent circle. As A increases the interior oval decreases, reduces itself to a conjugate point, and then disappears. The outside curve is an oval which always touches the dirigent circle, at first very small (it may be considered as commencing from a conjugate point corresponding to $A = 0$), but increasing as A increases negatively.

Next, when the directrix does not meet the dirigent circle, i.e. if $\alpha < m^2$. The inside curve consists at first of two ovals, an exterior oval lying without the dirigent circle, and an interior oval lying within the dirigent circle. As A increases the interior oval decreases, reduces itself to a conjugate point and disappears. The outside curve is at first imaginary, but when A attains a sufficiently large negative value, it makes its appearance as a conjugate point, and afterwards becomes an oval which gradually increases.

Next, when the dirigent circle reduces itself to a point, i.e. if $\alpha = 0$. The inside curve makes its appearance as a conjugate point (corresponding to $A = 0$), and as A increases it becomes an oval and continually increases. The outside curve comports itself as in the last preceding case.

Finally, when the dirigent circle becomes imaginary, or has for its radius a pure imaginary distance, i.e. if α is negative. The inside curve is at first imaginary, but when A attains a certain value it makes its appearance as a conjugate point, and as A increases becomes an oval and continually increases. The outside curve, as in the preceding two cases, comports itself in a similar manner.

The discussion, in the present section, of the different forms of the curve is not a very full one, and a large number of figures would be necessary in order to show completely the transition from one form to another. The forms delineated in the four figures were selected as forms corresponding to imaginary values of the parameters by means of which the equation of the curve is usually represented, e.g. the equations in Section XXVIII.

XXXVII.

It has been shown that for rays proceeding from a point and refracted at a circle, the secondary caustic is the Cartesian; the caustic itself is therefore the evolute of the Cartesian; this affords a means of finding the tangential equation of the caustic. In fact, the equation of the Cartesian is

$$(x^2 + y^2 - \alpha)^2 + 16A\,(x - m) = 0\,;$$

and if we take for the equation of the normal

$$X\xi + Y\eta + Z = 0,$$

(where ξ, η are current coordinates), then

$$\begin{aligned} X : Y : Z &= -y\,(x^2 + y^2 - \alpha) \\ &: x\,(x^2 + y^2 - \alpha) + 4A \\ &: 4Ay, \end{aligned}$$

equations which give

$$\begin{aligned} Z^2Yx &= Y(mZ^2 - AX^2), \\ -Z^2Yy &= Z^3 + X(mZ^2 - AX^2), \\ Z^4Y^2(x^2+y^2-\alpha) &= 4AZ^3XY^2, \end{aligned}$$

whence eliminating, we have

$$\{Z^3 + X(mZ^2 - AX^2)\}^2 + Y^2(mZ^2 - AX^2)^2 - Z^3Y^2(\alpha Z + 4AX) = 0,$$

where if, as before, c denotes the radius of the refracting circle, a the distance of the radiant point from the centre, and μ the index of refraction, we have

$$\alpha = \frac{1}{\mu^2}a^2 + (1 + \frac{1}{\mu^2})c^2,$$

$$A = \frac{c^2 a}{2\mu^2},$$

$$m = \frac{1}{2a}\left(c^2 + (1 + \frac{1}{\mu^2})a^2\right).$$

The above equation is the condition in order that the line $Xx + Yy + Z = 0$ may be a normal to the secondary caustic $(x^2+y^2-\alpha)^2 + 16A(x-m) = 0$, or it is the tangential equation of the caustic, which is therefore a curve of the class 6 only. The equation may be written in the more convenient form

$$Z^6 + 2Z^3X(mZ^2 - AX^2) + (X^2+Y^2)(mZ^2 - AX^2)^2 - Z^3Y^2(\alpha Z + 4AX) = 0.$$

XXXVIII.

To compare the last result with that previously obtained for the caustic by reflexion, I write $\mu = -1$, and putting also $c = 1$ and $Z = a$ (for the equation of the reflected ray was assumed to be $Xx + Yy + a = 0$), we have

$$\alpha = a^2 + 2,\ A = \tfrac{1}{2}a,\ m = \frac{1}{2a}(1 + 2a^2),$$

and the equation becomes, after a slight reduction,

$$4a^4 + 4a^2X(2a^2 + 1 - X^2) + (X^2+Y^2)(2a^2 + 1 - X^2)^2 - 4a^2Y^2(a^2 + 2 + 2X) = 0,$$

which may be written

$$\left(2a^2 + X(2a^2 + 1 - X^2)\right)^2 + Y^2\left(-4a^2 + 1 - 8a^2X - 2(2a^2+1)X^2 + X^4\right) = 0;$$

this divides out by the factor $(X+1)^3$, and the equation then becomes,

$$(X^2 - X - 2a^2)^2 + Y^2\left((X-1)^2 - 4a^2\right) = 0,$$

which agrees with the result before obtained.

XXXIX.

Again, to compare the general equation with that previously obtained for parallel rays refracted at a circle, we must write $\mu = \frac{1}{k}$, $c = 1$, $a = \infty$, $Z = k$ (for the equation of the refracted ray was taken to be $Xx + Yy + k = 0$); we have then

$$\alpha = 1 + k^2 + k^3 a^2,\ A = \tfrac{1}{2}k^2 a^2,\ m = \frac{1}{2a}\left(1 + (1 + k^2)\, a^2\right),$$

and, after the substitution, $a = \infty$. The equation becomes in the first instance

$$k^6 + 2k^3 X \left\{\frac{1}{2a}\left(1 + (1 + k^2)\, a^2\right) k^2 - \tfrac{1}{2}k^3 a X^2\right\} + (X^2 + Y^2)\left\{\frac{1}{2a}\left(1 + (1 + k^2)\, a^2\right) k^2 - \tfrac{1}{2}k^3 a X^2\right\}^2$$
$$- k^2 Y^2 (1 + k^2 + k^3 a^2 + 2k^3 a X) = 0;$$

and then putting $a = \infty$, or, what is the same thing, attending only to the terms which involve a^2, and throwing out the constant factor k^4, we obtain

$$(X^2 + Y^2)(X^2 - 1 - k^2)^2 - 4k^2 Y^2 = 0,$$

or

$$X^2 (X^2 - 1 - k^2)^2 + Y^2 (X + 1 + k)(X - 1 - k)(X + 1 - k)(X - 1 - k) = 0,$$

which agrees with the former result.

XL.

It was remarked that the ordinary construction for the secondary caustic could not be applied to the case of parallel rays (the entire curve would in fact pass off to an infinite distance), and that the simplest course was to measure the distance GQ from a line through the centre of the refracting circle perpendicular to the direction of the rays. To find the equation of the resulting curve, take the centre of the circle as the origin and the direction of the incident rays for the axis of x; let the radius of the circle be taken equal to unity, and let μ denote, as before, the index of refraction. Then if α, β are the coordinates of the point of incidence of a ray, we have $\alpha^2 + \beta^2 = 1$, and considering α, β as variable parameters connected by this equation, the required curve is the envelope of the circle,

$$\mu^2 \{(x - \alpha)^2 + (y - \beta)^2\} - \alpha^2 = 0.$$

Write now $\alpha = \cos\theta$, $\beta = \sin\theta$, then multiplying the equation by -2, and writing $1 + \cos 2\theta$ instead of $2\cos^2\theta$, the equation becomes

$$1 + \cos 2\theta - 2\mu^2 (x^2 + y^2 - 2x\cos\theta - 2y\sin\theta + 1) = 0,$$

which is of the form

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

and the values of the coefficients are

$$\begin{aligned} A &= 1, \\ B &= 0, \\ C &= 4\mu^2 x, \\ D &= 4\mu^2 y, \\ E &= -2\mu^2(x^2+y^2) - 2\mu^2 + 1. \end{aligned}$$

Substituting these values in the equation

$$\begin{aligned} &\{12(A^2+B^2) - 3(C^2+D^2) + 4E^2\}^3 \\ &\qquad - \{27A(C^2-D^2) + 54BCD - \big(72(A^2+B^2) + 9(C^2+D^2)\big)E + 8E^3\}^2 = 0, \end{aligned}$$

the equation of the envelope is found to be

$$\begin{aligned} &16\{(1-\mu^2+\mu^4) - (\mu^2+\mu^4)(x^2+y^2) + \mu^4(x^2+y^2)^2\}^3 \\ &\quad - \left\{\begin{array}{l} 4 - 6\mu^2 - 6\mu^4 + 4\mu^6 \\ - (6\mu^2 + 3\mu^4 + 6\mu^6)(x^2+y^2) - 27\mu^4(x^2-y^2) \\ - (6\mu^4 + 6\mu^6)(x^2+y^2)^2 \\ + 4\mu^6(x^2+y^2)^3 \end{array}\right\}^2 = 0, \end{aligned}$$

which is readily seen to be only of the 8th order. But to simplify the result, write first $(x^2+y^2-1)+1$, and $2x^2-1-(x^2+y^2-1)$ in the place of x^2+y^2 and x^2-y^2 respectively, the equation becomes

$$\begin{aligned} &4\{(1-\mu^2)^2 - \mu^2(1-\mu^2)(x^2+y^2-1) + \mu^4(x^2+y^2-1)^2\}^3 \\ &\quad - \left\{\begin{array}{l} 2(1-\mu^2)^3 \\ - 3\mu^2(1-\mu^2)^2(x^2+y^2-1) - 27\mu^4x^2 \\ - 3\mu^4(1-\mu^2)(x^2+y^2-1)^2 \\ + 2\mu^6(x^2+y^2-1)^3 \end{array}\right\}^2 = 0. \end{aligned}$$

Write for a moment $1-\mu^2 = q$, $\mu^2(x^2+y^2-1) = \rho$, the equation becomes

$$4(q^2 - q\rho + \rho^2)^3 - (2q^3 - 3q^2\rho - 3\rho^2 + 2\rho^3 - 27\mu^4x^2)^2 = 0;$$

or developing,

$$\begin{aligned} &4(q^2 - q\rho + \rho^2)^3 - (2q^3 - 3q^2\rho - 3q\rho^2 + 2\rho^3)^2 \\ &\qquad + 54(2q^3 - 3q^2\rho - 3q\rho^2 + 2\rho^3)\mu^4x^2 - 729\mu^8x^4 = 0, \end{aligned}$$

and reducing and dividing out by 27, this gives

$$q^2\rho^2(\rho-q)^2 + 2(\rho+q)(2\rho-q)(\rho-2q)\mu^4x^2 - 27\mu^8x^4 = 0,$$

whence replacing q, ρ by their values, the required equation is

$$(1-\mu^2)^2(x^2+y^2-1)^2\left(\mu^2(x^2+y^2)-1\right)^2$$
$$+2\left(\mu^2(x^2+y^2)-2\mu^2+1\right)\left(2\mu^2(x^2+y^2)-\mu^2-1\right)\left(\mu^2(x^2+y^2)-2+\mu^2\right)x^2-27\mu^4x^4=0,$$

which is the equation of an orthogonal trajectory of the refracted rays.

In the case of reflexion, $\mu=-1$, and the equation becomes

$$4(x^2+y^2-1)^3-27x^2=0.$$

Comparing this with the equation of the caustic, it is easy to see,

THEOREM. In the case of parallel rays and a reflecting circle, there is a secondary caustic which is a curve similar to and double the magnitude of the caustic, the position of the two curves differing by a right angle.

XLI.

The entire system of the orthogonal trajectories of the refracted rays might in like manner be determined by finding the envelope of the circle (where, as before, α, β are variable parameters connected by the equation $\alpha^2+\beta^2=1$),

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-(\alpha+m)^2=0.$$

{The result, as far as I have worked it out, is as follows, viz.—

$$\left(3-12\left[m^2+2m\mu^2x+\mu^4(x^2+y^2)\right]+\left[1-2\mu^2+2m^2-2\mu^2(x^2+y^2)\right]^2\right)^3$$
$$-\left(\left[1-2\mu^2+2m^2-2\mu^2(x^2+y^2)\right]\left[9+18m^2+36m\mu^2x+18\mu^4(x^2+y^2)\right]\right.$$
$$\left.-54\left[m^2+2m\mu^2x+\mu^4(x^2-y^2)\right]-\left[1-2\mu^2+2m^2-2\mu^2(x^2+y^2)\right]^3\right)^2=0,$$

which, it is easy to see, is an equation of the order 8 only. Added Sept. 12.—A. C.}

146.

A MEMOIR ON CURVES OF THE THIRD ORDER.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 415—446. Received October 30,—Read December 11, 1856.]

A CURVE of the third order, or cubic curve, is the locus represented by an equation such as $U=(*\between x, y, z)^3=0$; and it appears by my "Third Memoir on Quantics," [144], that it is proper to consider, in connexion with the curve of the third order $U=0$, and its Hessian $HU=0$ (which is also a curve of the third order), two curves of the third class, viz. the curves represented by the equations $PU=0$ and $QU=0$. These equations, I say, represent curves of the third class; in fact, PU and QU are contravariants of U, and therefore, when the variables x, y, z of U are considered as point coordinates, the variables ξ, η, ζ of PU and QU must be considered as line coordinates, and the curves will be curves of the third class. I propose (in analogy with the form of the word Hessian) to call the two curves in question the Pippian and Quippian respectively. [The curve $PU=0$ is now usually called the Cayleyan.] A geometrical definition of the Pippian was readily found; the curve is in fact Steiner's curve R_0 mentioned in the memoir "Allgemeine Eigenschaften der algebraischen Curven," *Crelle*, t. XLVII. [1854] pp. 1—6, in the particular case of a basis-curve of the third order; and I also found that the Pippian might be considered as occurring implicitly in my "Mémoire sur les courbes du troisième ordre," *Liouville*, t. IX. [1844] pp. 285—293 [26] and "Nouvelles remarques sur les courbes du troisième ordre," *Liouville*, t. X. [1845] pp. 102—109 [27]. As regards the Quippian, I have not succeeded in obtaining a satisfactory geometrical definition; but the search after it led to a variety of theorems, relating chiefly to the first-mentioned curve, and the results of the investigation are contained in the present memoir. Some of these results are due to Mr Salmon, with whom I was in correspondence on the subject. The character of the results makes it difficult to develope them in a systematic order; but the results are given in such connexion one with another as I have been able to present them

in. Considering the object of the memoir to be the establishment of a distinct geometrical theory of the Pippian, the leading results will be found summed up in the nine different definitions or modes of generation of the Pippian, given in the concluding number. In the course of the memoir I give some further developments relating to the theory in the memoirs in *Liouville* above referred to, showing its relation to the Pippian, and the analogy with theorems of Hesse in relation to the Hessian.

Article No. 1.—*Definitions, &c.*

1. It may be convenient to premise as follows:—Considering, in connexion with a curve of the third order or cubic, *a point*, we have:

(*a*) The *first or conic polar* of the point.

(*b*) The *second or line polar* of the point.

The meaning of these terms is well known, and they require no explanation.

Next, considering, in connexion with the cubic, *a line*—

(*c*) The first or conic polars of each point of the line meet in four points, which are the *four poles* of the line.

(*d*) The second or line polars of each point of the line envelope a conic, which is the *lineo-polar envelope* of the line.

And reciprocally considering, in connexion with a curve of the third class, *a line*, we have:

(*e*) The *first or conic pole* of the line.

(*f*) The *second or point-pole* of the line.

And considering, in connexion with the curve of the third class, *a point*—

(*g*) The first or conic poles of each line through the point touch four lines, which are the *four polars* of the point.

(*h*) The second or point poles of each line through the point generate a conic which is the *point-pole locus* of the point.

But I shall not have occasion in the present memoir to speak of these reciprocal figures, except indeed the first or conic pole of the line.

The term *conjugate poles* of a cubic is used to denote two points, such that the first or conic polar of either of them, with respect to the cubic, is a pair of lines passing through the other of them. Reciprocally, the term *conjugate polars* of a curve of the third class denotes two lines, such that the first or conic pole of either of them, with respect to the curve of the third class, is a pair of points lying in the other of them.

The expression, a *syzygetic cubic*, used in reference to two cubics, denotes a curve of the third order passing through the points of intersection of the two cubics; but in the present memoir the expression is in general used in reference to a single cubic, to denote a curve of the third order passing through the points of intersection of the cubic and its Hessian. As regards curves of the third class, I use in the memoir the full expression, a curve of the third class syzygetically connected with two given curves of the third class.

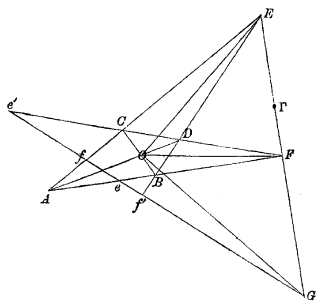
It is a well-known theorem, that if at the points of intersection of a given line with a given cubic tangents are drawn to the cubic, these tangents again meet the cubic in three points which lie in a line; such line is in the present memoir termed the *satellite line* of the given line, and the point of intersection of the two lines is termed the *satellite point* of the given line; the given line in reference to its satellite line or point is termed the *primary line*.

In particular, if the primary line be a tangent of the cubic, the satellite line coincides with the primary line, and the satellite point is the point of simple intersection of the primary line and the cubic.

Article No. 2.—*Group of Theorems relating to the Conjugate Poles of a Cubic.*

2. The theorems which I have first to mention relate to or originate out of the theory of the conjugate poles of a cubic, and may be conveniently connected together and explained by means of the accompanying figure.

The point E is a point of the Hessian; this being so, its first or conic polar, with respect to the cubic, will be a pair of lines passing through a point F of the



Hessian; and not only so, but the first or conic polar of the point F, with respect to the cubic will be a pair of lines passing through E. The pair of lines through

F are represented in the figure by FBA, FDC, and the pair of lines through E are represented by ECA, EDC, and the lines of the one pair meet the lines of the other pair in the points A, B, C, D. The point O, which is the intersection of the lines AD, BC, is a point of the Hessian, and joining EO, FO, these lines are tangents to the Hessian at the points E, F, that is, the points E, F are *corresponding points* of the Hessian, in the sense that the tangents to the Hessian at these points meet in a point of the Hessian. The two points E, F are, according to a preceding definition, conjugate poles of the cubic.

The line EF meets the Hessian in a third point G, and the points G, O are conjugate poles of the cubic. The first or conic polar of G, with respect to the cubic, is the pair of lines AOD, BOC meeting in O. The first or conic polar of O, with respect to the cubic, is the pair of lines GEF and $Gf'efe'$ meeting in G. The four poles of the line EO, with respect to the cubic, are the points of intersection of the first or conic polars of the two points E and O, that is, the four poles in question are the points F, F, e, e'. Similarly, the four poles of the line FO, with respect to the cubic, are the points E, E, f, f'.

The line EF, that is, any line joining two conjugate poles of the cubic, is a tangent to the Pippian, and the point of contact Γ is the harmonic with respect to the points E, F (which are points on the Hessian) of G, the third point of intersection with the Hessian. Conversely, any tangent of the Pippian meets the Hessian in three points, two of which are conjugate poles of the cubic, and the point of contact is the harmonic, with respect to these two points, of the third point of intersection with the Hessian.

The line GO in the figure is of course also a tangent of the Pippian, and moreover the lines FBA, FDC (that is, the pair of lines which are the first or conic polar of E) and the lines ECA, EDB (that is, the pair of lines which are the first or conic polar of F) are also tangents to the Pippian. The point E represents *any* point of the Hessian, and the three tangents through E to the Pippian are the line EFG and the lines ECA, EDB; the line EFG is the line joining E with the conjugate pole F, and the lines ECA, EDB are the first or conic polar of this conjugate pole F with respect to the cubic. The figure shows that the line EO (the tangent to the Hessian at the point E) and the before-mentioned three lines (the tangents through E to the Pippian), are harmonically related, viz. the line EO the tangent of the Hessian, and the line EF one of the tangents to the Pippian, are harmonics with respect to the other two tangents to the Pippian. It is obvious that the tangents to the Pippian through the point F are in like manner the line GFE, and the pair of lines FBA, FBC, and that these lines are harmonically related to FO the tangent at F of the Hessian. And similarly, the tangents to the Pippian through the point O are the line GO and the lines AOD, BOC, and the tangents to the Pippian through the point G are the line GO and the lines GFE and $Gf'efe'$. Thus all the lines of the figure are tangents to the Pippian except the lines EO, FO, which are tangents to the Hessian. It may be added, that the lineo-polar envelope of the line EF with respect to the cubic is the pair of lines OE, OF.

It will be presently seen that the analytical theory leads to the consideration of a line IJ (not represented in the figure): the line in question is the polar of E (or F) with respect to the conic which is the first or conic polar of F (or E) with respect to *any* syzygetic cubic. The line IJ is a tangent of the Pippian, and moreover the lines EF and IJ are conjugate polars of a curve of the third class syzygetically connected with the Pippian and Quippian, and which is moreover such that its Hessian is the Pippian.

Article Nos. 3 to 19.—*Analytical investigations, comprising the proof of the theorems, Article No.* 2.

3. The analytical theory possesses considerable interest. Take as the equation of the cubic,

$$U = x^3 + y^3 + z^3 + 6lxyz = 0;$$

then the equation of the Hessian is

$$HU = l^2(x^3 + y^3 + z^3) - (1 + 2l^3)\,xyz = 0;$$

and the equation of the Pippian in line coordinates (that is, the equation which expresses that $\xi x + \eta y + \zeta z = 0$ is a tangent of the curve) is

$$PU = -l(\xi^3 + \eta^3 + \zeta^3) + (-1 + 4l^3)\,\xi\eta\zeta = 0.$$

The equation of the Quippian in line coordinates is

$$QU = (1 - 10l^3)(\xi^3 + \eta^3 + \zeta^3) - 6l^2(5 + 4l^3)\,\xi\eta\zeta = 0;$$

and the values of the two invariants of the cubic form are

$$S = -l + l^4,$$
$$T = 1 - 20l^3 - 8l^6,$$

values which give identically,

$$T^2 - 64S^3 = (1 + 8l^3)^3;$$

the last-mentioned function being in fact the discriminant.

4. Suppose now that (X, Y, Z) are the coordinates of the point E, and (X', Y', Z') the coordinates of the point F; then the equations which express that these points are conjugate poles of the cubic, are

$$XX' + l(YZ' + Y'Z) = 0,$$
$$YY' + l(ZX' + Z'X) = 0,$$
$$ZZ' + l(XY' + X'Y) = 0;$$

and by eliminating from these equations, first (X', Y', Z'), and then (X, Y, Z), we find

$$l^2(X^3 + Y^3 + Z^3) - (1 + 2l^3)\,XYZ = 0,$$
$$l^2(X'^3 + Y'^3 + Z'^3) - (1 + 2l^3)\,X'Y'Z' = 0,$$

which shows that the points E, F are each of them points of the Hessian.

5. I may notice, in passing, that the preceding equations give rise to a somewhat singular *unsymmetrical* quadratic transformation of a cubic form. In fact, the second and third equations give $X' : Y' : Z' = YZ - l^2X^2 : l^2XY - lZ^2 : l^2ZX - lY^2$. And substituting these values for X', Y', Z' in the form

$$l^3(X'^3 + Y'^3 + Z'^3) - (1 + 2l^3)X'Y'Z',$$

the result must contain as a factor

$$l^2(X^3 + Y^3 + Z^3) - (1 + 2l^3)XYZ;$$

the other factor is easily found to be

$$-l^3\left(l^3(X^3 + Y^3 + Z^3) + 3lXYZ\right).$$

Several of the formulæ given in the sequel conduct in like manner to unsymmetrical transformations of a cubic form.

6. I remark also, that the last-mentioned system of equations gives, *symmetrically*,

$$\begin{aligned} X'^2 : Y'^2 : Z'^2 : Y'Z' : Z'X' : X'Y' \\ = YZ - l^2X^2 : ZX - l^2Y^2 : XY - l^2Z^2 : l^2YZ - lX^2 : l^2ZX - lY^2 : l^2XY - lZ^2; \end{aligned}$$

and it is, I think, worth showing how, by means of these relations, we pass from the equation between X', Y', Z' to that between X, Y, Z. In fact, representing, for shortness, the foregoing relations by

$$X'^2 : Y'^2 : Z'^2 : Y'Z' : Z'X' : X'Y' = A : B : C : F : G : H,$$

we may write

$$X' = AF = GH, \quad Y' = BG = HF, \quad Z' = CH = FG, \quad ABC = FGH;$$

and thence

$$X'^3 = AF \,.\, G^2H^2, \quad Y'^3 = BG \,.\, H^2F^2, \quad Z'^3 = CH \,.\, F^2G^2, \quad X'Y'Z' = F^3G^3H^3;$$

hence

$$l^3(X'^3 + Y'^3 + Z'^3) - (1 + 2l^3)X'Y'Z' = FGH\,\{l^3(AGH + BHF + CFG) - (1 + 2l^3)FGH\}.$$

But we have

$$\begin{aligned} l^3(AGH + BHF + CFG) &= -(2l^5 + l^3)(X^3 + Y^3 + Z^3)XYZ + (l^4 + 2l^7)(Y^3Z^3 + Z^3X^3 + X^3Y^3), \\ -(1 + 2l^3)FGH &= (l^5 + 2l^8)(X^3 + Y^3 + Z^3)XYZ + (l^4 + 2l^7)(Y^3Z^3 + Z^3X^3 + X^3Y^3) \\ &\qquad + l^3(1 - l^3)(1 + 2l^3)X^2Y^2Z^2; \end{aligned}$$

and thence

$$\begin{aligned} l^3(AGH + BHF + CFG) - (1 + 2l^3)FGH \\ = -l^3(1 - l^3)\{l^2(X^3 + Y^3 + Z^3)XYZ - (1 + 2l^3)X^2Y^2Z^2\}; \end{aligned}$$

and finally,

$$\begin{aligned} l^3(X'^3 + Y'^3 + Z'^3) - (1 + 2l^3)X'Y'Z' = l^3(-l + l^4)(lYZ - X^2)(lZX - Y^2)(lXY - Z^2)XYZ \\ \times\{l^2(X^3 + Y^3 + Z^3) - (1 + 2l^3)XYZ\}. \end{aligned}$$

We have also, identically,

$$ABC - FGH = \frac{1}{l}(-l + l^4)\,XYZ\,\{l^2(X^3 + Y^3 + Z^3) - (1 + 2l^3)\,XYZ\},$$

which agrees with the relation $ABC - FGH = 0$.

7. Before going further, it will be convenient to investigate certain relations which exist between the quantities (X, Y, Z), (X', Y', Z'), connected as before by the equations

$$\begin{aligned} XX' + l(YZ' + Y'Z) &= 0, \\ YY' + l(ZX' + Z'X) &= 0, \\ ZZ' + l(XY' + X'Y) &= 0, \end{aligned}$$

and the quantities

$$\begin{aligned} \xi &= YZ' - Y'Z, & \alpha &= XX' = -\frac{1}{l}(YZ' + Y'Z), \\ \eta &= ZX' - Z'X, & \beta &= YY' = -\frac{1}{l}(ZX' + Z'X), \\ \zeta &= XY' - X'Y, & \gamma &= ZZ' = -\frac{1}{l}(XY' + X'Y). \end{aligned}$$

We have identically,

$$2XX'(YZ' - Y'Z) + (XY' + X'Y)(ZX' - Z'X) + (ZX' + Z'X)(XY' - X'Y) = 0;$$

or expressing in terms of ξ, η, ζ, α, β, γ the quantities which enter into this equation, and forming the analogous equations, we have

$$\begin{aligned} 2l\alpha\xi - \gamma\eta - \beta\zeta &= 0, \\ -\gamma\xi + 2l\beta\eta - \alpha\zeta &= 0, \\ -\beta\xi - \alpha\eta + 2l\gamma\zeta &= 0. \end{aligned} \tag{A}$$

We have also

$$X^2Y'Z' - X'^2YZ = \tfrac{1}{2}\{-(XY' + X'Y)(ZX' - Z'X) + (ZX' + Z'X)(XY' - X'Y)\},$$

and thence in like manner,

$$\begin{aligned} X^2Y'Z' - X'^2YZ &= \frac{1}{2l}(\gamma\eta - \beta\zeta), \\ Y^2Z'X' - Y'^2ZX &= \frac{1}{2l}(\alpha\zeta - \gamma\xi), \\ Z^2X'Y' - X'^2YZ &= \frac{1}{2l}(\beta\xi - \alpha\eta). \end{aligned} \tag{B}$$

Again, we have

$$(YZ' - Y'Z)^2 = (YZ' + Y'Z)^2 - 4YY'ZZ',$$

$$(ZX' - Z'X)(XY' - X'Y) = -(ZX' + Z'X)(XY' + X'Y) + 2XX'(YZ' + Y'Z);$$

and thence

$$\begin{aligned}
\xi^2 &= \frac{1}{l^2}\alpha^2 - 4\beta\gamma, \\
\eta^2 &= \frac{1}{l^2}\beta^2 - 4\gamma\alpha, \\
\zeta^2 &= \frac{1}{l^2}\gamma^2 - 4\alpha\beta, \\
\eta\zeta &= -\frac{2}{l}\alpha^2 - \frac{1}{l^2}\beta\gamma, \\
\zeta\xi &= -\frac{2}{l}\beta^2 - \frac{1}{l^2}\gamma\alpha, \\
\xi\eta &= -\frac{2}{l}\gamma^2 - \frac{1}{l^2}\alpha\beta;
\end{aligned} \tag{C}$$

and conversely

$$\begin{aligned}
\frac{1}{l^2}(1+8l^3)\alpha^2 &= \xi^2 - 4l^2\eta\zeta, \\
\frac{1}{l^2}(1+8l^3)\beta^2 &= \eta^2 - 4l^2\zeta\xi, \\
\frac{1}{l^2}(1+8l^3)\gamma^2 &= \zeta^2 - 4l^2\xi\eta, \\
-\frac{1}{l^2}(1+8l^3)\beta\gamma &= 2l\xi^2 + \eta\zeta, \\
-\frac{1}{l^2}(1+8l^3)\gamma\alpha &= 2l\eta^2 + \zeta\xi, \\
-\frac{1}{l^2}(1+8l^3)\alpha\beta &= 2l\zeta^2 + \xi\eta.
\end{aligned} \tag{D}$$

8. It is obvious that

$$\xi x + \eta y + \zeta z = 0$$

is the equation of the line EF joining the two conjugate poles, and it may be shown that

$$\alpha x + \beta y + \gamma z = 0$$

is the equation of the line IJ, which is the polar of E with respect to a conic which is the first or conic polar of F with respect to *any* syzygetic cubic. In fact, the equation of a syzygetic cubic will be $x^3 + y^3 + z^3 + 6\lambda xyz = 0$, where λ is arbitrary, and the equation of the line in question is

$$(X\partial_x + Y\partial_y + Z\partial_z)(X'\partial_x + Y'\partial_y + Z'\partial_z)(x^3 + y^3 + z^3 + 6\lambda xyz) = 0;$$

or developing,

$$\begin{aligned}
&XX'x + YY'y + ZZ'z \\
&+ \lambda\{YZ' + Y'Z)x + (ZX' + Z'X)y + (XY' + X'Y)z\} = 0;
\end{aligned}$$

and the function on the left-hand side is

$$\left(1-\frac{\lambda}{l}\right)(\alpha x+\beta y+\gamma z),$$

which proves the theorem.

9. The equations (A) by the elimination of (ξ, η, ζ), give

$$-l(\alpha^2+\beta^2+\gamma^2)+(-1+4l^3)\,\alpha\beta\gamma=0,$$

which shows that the line IJ is a tangent of the Pippian: the proof of the theorem is given in this place because the relation just obtained between α, β, γ is required for the proof of some of the other theorems.

10. To find the coordinates of the point G in which the line EF joining two conjugate poles again meets the Hessian.

We may take for the coordinates of G,

$$uX+vX',\quad uY+vY',\quad uZ+vZ';$$

and, substituting in the equation of the Hessian, the terms containing u^3, v^3 disappear, and the ratio $u:v$ is determined by a simple equation. It thus appears that we may write

$$\begin{aligned} u &= -3l^2(XX'^2+YY'^2+ZZ'^2)+(1+2l^3)(Y'Z'X+Z'X'Y+X'Y'Z),\\ v &= 3l^2(X^2X'+Y^2Y'+Z^2Z')-(1+2l^3)(YZX'+ZXY'+XYZ'); \end{aligned}$$

hence introducing, as before, the quantities ξ, η, ζ, α, β, γ, we find

$$uX+vX'=3l^2(\gamma\eta-\beta\zeta)+(1+2l^3)(X^2Y'Z'-X'^2YZ);$$

but from the first of the equations (B),

$$X^2Y'Z'-X'^2YZ=\frac{1}{2l}(\gamma\eta-\beta\zeta),$$

and therefore the preceding value of $uX+vX'$ becomes

$$\left(3l^2-\frac{1+2l^3}{2l}\right)(\gamma\eta-\beta\zeta),$$

which is equal to

$$\frac{-1+4l^3}{2l}(\gamma\eta-\beta\zeta).$$

Hence throwing out the constant factor, we find, for the coordinates of the point G, the values

$$\gamma\eta-\beta\zeta,\quad \alpha\zeta-\gamma\xi,\quad \beta\xi-\alpha\eta.$$

11. To find the coordinates of the point O.

Consider O as the point of intersection of the tangents to the Hessian at the points E, F, then the coordinates of O are proportional to the terms of

$$\left\|\begin{matrix} 3l^2X^2-\overline{1+2l^3}YZ, & 3l^2Y^2-\overline{1+2l^3}ZX, & 3l^2Z^2-\overline{1+2l^3}XY \\ 3l^2X'^2-\overline{1+2l^3}Y'Z', & 3l^2Y'^2-\overline{1+2l^3}Z'X', & 3l^2Z'^2-\overline{1+2l^3}X'Y' \end{matrix}\right\|$$

Hence the x-coordinate is proportional to

$$(3l^2Y^2 - \overline{1+2l^3}ZX)(3l^2Z'^2 - \overline{1+2l^3}X'Y') - (3l^2Z^2 - \overline{1+2l^3}XY)\,3l^2Y'^2 - \overline{1+2l^3}Z'X'),$$

which is equal to

$$9l^4(Y^2Z'^2 - Y'^2Z^2) + 3l^2(1+2l^3)YY'(XY' - X'Y) + 3l^2(1+2l^3)ZZ'(ZX' - Z'X)$$
$$-(1+2l^3)^2XX'(YZ' - Y'Z);$$

or introducing, as before, the quantities ξ, η, ζ, α, β, γ, to

$$-9l^3\alpha\xi + 3l^2(1+2l^3)(\beta\zeta + \gamma\eta) - (1+2l^3)^2\alpha\xi,$$
$$= (-1 - 13l^3 - 4l^6)\alpha\xi + 2l^3(1+2l^3)(\beta\zeta + \gamma\eta).$$

But by the first of the equations (A) $\beta\zeta + \gamma\eta = 2l\alpha\xi$, and the preceding value thus becomes $(-1 - 7l^3 + 8l^6)\alpha\xi$. Hence throwing out the constant factor the coordinates of the point O are found to be

$$\alpha\xi, \quad \beta\eta, \quad \gamma\zeta.$$

12. The points G, O are conjugate poles of the cubic.

Take a, b, c for the coordinates of G, and a', b', c' for the coordinates of O, we have

$$a,\ b,\ c = \gamma\eta - \beta\zeta,\quad \alpha\zeta - \gamma\xi,\quad \beta\xi - \alpha\eta,$$
$$a',\ b',\ c' = \alpha\xi\quad,\quad \beta\eta\quad,\quad \gamma\zeta.$$

These values give $aa' + l(bc' + b'c)$

$$= \alpha\xi(\gamma\eta - \beta\zeta) + l\{\beta\eta(\beta\xi - \alpha\eta) + \gamma\zeta(\alpha\zeta - \gamma\xi)\}$$
$$= \xi\eta(\alpha\gamma + l\beta^2) + \eta^2(-l\alpha\beta) + \zeta^2(l\alpha\gamma) + \xi\zeta(-\alpha\beta - l\gamma^2);$$

or substituting for $\xi\eta$, η^2, ζ^2, $\xi\zeta$ their values in terms of α, β, γ, this is

$$\left(-\frac{2}{l}\gamma^2 - \frac{1}{l^2}\alpha\beta\right)(\alpha\gamma + l\beta^2)$$
$$+\left(\frac{\beta^2}{l^2} - 4\gamma\alpha\right)(\quad - l\alpha\beta)$$
$$+\left(\frac{\gamma^2}{l^2} - 4\alpha\beta\right)(\qquad l\alpha\gamma)$$
$$+\left(-\frac{2}{l}\beta^2 - \frac{1}{l}\alpha\gamma\right)(-\alpha\beta - l\gamma^2),$$

which is identically equal to zero. Hence, completing the system, we find

$$aa' + l(bc' + b'c) = 0,$$
$$bb' + l(ca' + c'a) = 0,$$
$$cc' + l(ab' + a'b) = 0,$$

equations which show that O (as well as G) is a point of the Hessian, and that the points G, O are corresponding poles of the cubic.

13. The line EF joining a pair of conjugate poles of the cubic is a tangent of the Pippian[1].

In fact, the equations (A), by the elimination of α, β, γ, give

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\,\xi\eta\zeta=0,$$

which proves the theorem.

14. To find the equation of the pair of lines through F, and to show that these lines are tangents of the Pippian.

The equation of the pair of lines considered as the first or conic polar of the conjugate pole E, is

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy)=0.$$

Let one of the lines be

$$\lambda x+\mu y+\nu z=0,$$

then the other is

$$\frac{X}{\lambda}x+\frac{Y}{\mu}y+\frac{Z}{\nu}z=0;$$

and we find

$$\begin{aligned} 2lX\mu\nu - \quad Y\nu^2 - \quad Z\mu^2 &= 0,\\ - \quad X\nu^2 + 2lY\nu\lambda - \quad Z\lambda^2 &= 0,\\ - \quad X\mu^2 - \quad Y\lambda^2 + 2lZ\mu\nu &= 0, \end{aligned}$$

any two of which determine the ratios λ, μ, ν.

The elimination of X, Y, Z gives

$$\begin{vmatrix} 2l\mu\nu, & -\nu^2, & -\mu^2 \\ -\nu^2, & 2l\nu\lambda, & -\lambda^2 \\ -\mu^2, & -\lambda^2, & 2l\lambda\mu \end{vmatrix}=0,$$

which is equivalent to

$$\lambda\mu\nu\{-l(\lambda^3+\mu^3+\nu^3)+(-1+4l^3)\,\lambda\mu\nu\}=0;$$

or, omitting a factor, to

$$-l(\lambda^3+\mu^3+\nu^3)+(-1+4l^3)\,\lambda\mu\nu=0,$$

which shows that the line in question is a tangent of the Pippian.

15. To find the equation of the pair of lines through O.

The equation of the pair of lines through E is in like manner

$$X'(x^2+2lyz)+Y'(y^2+2lzx)+Z'(z^2+2lxy)=0;$$

[1] Steiner's curve R_0, in the particular case of a cubic basis-curve, is according to definition the envelope of the line EF, that is, the curve R_0 in the particular case in question is the Pippian.

and combining this with the foregoing equation,

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy)=0$$

of the pair of lines through F, viz. multiplying the two equations by

$$X^2X'+Y^2Y'+Z^2Z', \quad -(XX'^2+YY'^2+ZZ'^2),$$

and adding, then if as before

$$a:b:c=\gamma\eta-\beta\zeta:\alpha\zeta-\gamma\xi:\beta\xi-\alpha\eta,$$

we find as the equation of a conic passing through the points A, B, C, D, the equation

$$a(x^2+2lyz)+b(y^2+2lzx)+c(z^2+2lxy)=0.$$

But putting, as before,

$$a':b':c'=\alpha\xi:\beta\eta:\gamma\zeta,$$

then a', b', c' are the coordinates of the point O, and the equations

$$aa'+l(bc'+b'c)=0,$$
$$bb'+l(ca'+c'a)=0,$$
$$cc'+l(ab'+a'b)=0,$$

show that the conic in question is in fact the pair of lines through the point O.

16. To find the coordinates of the point Γ, which is the harmonic of G with respect to the points E, F.

The coordinates of the point in question are

$$uX-vX', \quad uY-vY', \quad uZ-vZ',$$

where u, v have the values given in No. 10, viz.

$$u=-3l^2(XX'^2+YY'^2+ZZ'^2)+(1+2l^3)(Y'Z'X+Z'X'Y+X'Y'Z),$$
$$v=\quad 3l^2(X^2X'+Y^2Y'+Z^2Z')-(1+2l^3)(YZX'+ZXY'+XYZ');$$

these values give

$$uX-vX'=-3l^2\{2X^2X'^2+(XY'+X'Y)YY'+(XZ'+X'Z)ZZ'\}$$
$$+(1+2l^3)\{(XY'+X'Y)(XZ'+X'Z)+XX'(YZ'+Y'Z\};$$

and therefore

$$uX-vX'=-3l^2\left\{2\alpha^2-\frac{2}{l}\beta\gamma\right\}+(1+2l^3)\left\{\frac{1}{l^2}\beta\gamma-\frac{1}{l}\alpha^2\right\}$$
$$=\frac{1}{l^2}(1+8l^3)(-l\alpha^2+\beta\gamma);$$

and consequently, omitting the constant factor, the coordinates of Γ may be taken to be

$$-l\alpha^2+\beta\gamma, \quad -l\beta^2+\gamma\alpha, \quad -l\gamma^2+\alpha\beta.$$

17. The line through two consecutive positions of the point Γ is the line EF.

The coordinates of the point Γ are

$$-l\alpha^2+\beta\gamma,\quad -l\beta^2+\gamma\alpha,\quad -l\gamma^2+\alpha\beta\,;$$

and it has been shown that the quantities α, β, γ satisfy the equation

$$-l\,(\alpha^3+\beta^3+\gamma^3)+(-1+4l^3)\,\alpha\beta\gamma=0.$$

Hence, considering α, β, γ as variable parameters connected by this equation, the equation of the line through two consecutive positions of the point Γ is

$$\begin{vmatrix} & -3l\alpha^2+(-1+4l^3)\,\beta\gamma, & -3l\beta^2+(-1+4l^3)\,\gamma\alpha, & -3l\gamma^2+(-1+4l^3)\,\alpha\beta \\ x, & -2l\alpha & , \quad \gamma & , \quad \beta \\ y, & \gamma & , \quad -2l\beta & , \quad \alpha \\ z, & \beta & , \quad \alpha & , \quad -2l\gamma \end{vmatrix}=0\,;$$

and representing this equation by

$$Lx+My+Nz=0,$$

we find

$$\begin{aligned} L=\ & (4l^2\beta\gamma-\alpha^2)\left(-3l\alpha^2+(-1+4l^3)\beta\gamma\right) \\ & +(\alpha\beta+2l\gamma^2)\left(-3l\beta^2+(-1+4l^3)\,\gamma\alpha\right) \\ & +(\alpha\gamma+2l\beta^2)\left(-3l\gamma^2+(-1+4l^3)\,\alpha\beta\right); \end{aligned}$$

or, multiplying out and collecting,

$$L=3l\alpha^4+(-1-8l^3)\,\alpha^2\beta\gamma+(-5l+8l^4)\,(\alpha\beta^3+\alpha\gamma^3)+(-16l^2+16l^3)\,\beta^2\gamma^2\,;$$

but the equation

$$-l\,(\alpha^3+\beta^3+\gamma^3)+(-1+4l^3)\,\alpha\beta\gamma=0$$

gives

$$3l\alpha^4=-3l\,(\alpha\beta^3+\alpha\gamma^3)+(-3+12l^3)\,\alpha^2\beta\gamma,$$

and we have

$$\begin{aligned} L&=(-4+4l^3)\,\alpha^2\beta\gamma+(-8l+8l^4)\,(\alpha\beta^3+\alpha\gamma^3)+(-16l^2+16l^3)\,\beta^2\gamma^2 \\ &=(-4+4l^3)\left(\alpha^2\beta\gamma+2l\,(\alpha\beta^3+\alpha\gamma^3)+4l^2\beta^2\gamma^2\right) \\ &=(-4+4l^3)\,(\alpha\gamma+2l\beta^2)\,(\alpha\beta+2l\gamma^2)\,; \end{aligned}$$

or, in virtue of the equations (D),

$$L=(-4+4l^3)\,l^2\zeta\xi\,.\,l^2\xi\eta=(-4+4l^3)\,l^4\xi^2\eta\zeta=(-4+4l^3)\,l^4\xi\eta\zeta\,.\,\xi.$$

Hence, omitting the common factor, we find $L:M:N=\xi:\eta:\zeta$, and the equation $Lx+My+Nz=0$ becomes

$$\xi x+\eta y+\zeta z=0,$$

which is the equation of the line EF, that is, the line through two consecutive positions of Γ is the line EF; or what is the same thing, the line EF touches the Pippian in the point Γ which is the harmonic of G with respect to the points E, F.

18. The lineo-polar envelope of the line EF, with respect to the cubic, is the pair of lines OE, OF.

The equation of the pair of lines OE, OF, considered as the tangents to the Hessian at the points E, F, is

$$\left.\begin{array}{l}\{(3l^3X^2-\overline{1+2l^3}Y\,Z)\,x+(3l^3Y^2-\overline{1+2l^3}Z\,X)\,y+(3l^3Z^2-\overline{1+2l^3}X\,Y)\,z\}\\ \times\{(3l^3X'^2-\overline{1+2l^3}Y'Z')\,x+(3l^3Y'^2-\overline{1+2l^3}Z'X')\,y+(3l^3Z'^2-\overline{1+2l^3}X'Y')\,z\}\end{array}\right\}=0.$$

Here on the left-hand side the coefficient of x^2 is

$$9l^4X^2X'^2-3l^2(1+2l^3)(X^2Y'Z'+X'^2YZ)+(1+2l^3)^2\,YY'ZZ',$$

which is equal to

$$9l^4\alpha^2-3l^2(1+2l^3)(l^3\beta\gamma+\frac{1}{l}\alpha^2)+(1+2l^3)^2\,\beta\gamma,$$

that is

$$\frac{1}{l}(-l+l^4)\,\{3l\alpha^2+2(1+2l^3)\,\beta\gamma\}\,;$$

and the coefficient of yz is

$$9l^4(Y^2Z'^2+Y'^2Z^2)-3l^2(1+2l^3)\left(YY'(XY'+X'Y)+ZZ'(XZ'+X'Z)\right)$$
$$+(1+2l^3)^2\,XX'(YZ'+Y'Z),$$

which is equal to

$$9l^4\left(\frac{1}{l^3}\alpha^2-2\beta\gamma\right)-3l^2(1+2l^3)\left(-\frac{2}{l}\beta\gamma\right)+(1+2l^3)^2\,\alpha\left(-\frac{1}{l}\alpha\right),$$

that is

$$\frac{1}{l}(-l+l^4)\,\{(1-4l^3)\,\alpha^2-6l^3\beta\gamma\}.$$

Hence completing the system and throwing out the constant factor, the equation of the pair of lines is

$$\begin{array}{l}(3l\alpha^2+2(1+2l^3)\,\beta\gamma,\quad 3l\beta^2+2(1+2l^3)\,\gamma\alpha,\quad 3l\gamma^2+2(1+2l^3)\,\alpha\beta,\\ \qquad(1-4l^3)\,\alpha^2-6l^3\beta\gamma,\quad(1-4l^3)\,\beta^2-6l^3\gamma\alpha,\quad(1-4l^3)\,\gamma^2-6l^3\alpha\beta\between x,\ y,\ z)^2=0.\end{array}$$

But the equation of the line EF is $\xi x+\eta y+\zeta z=0$, and the equation of its lineo-polar envelope is

$$\begin{vmatrix} & \xi, & \eta, & \zeta\\ \xi, & x, & lz. & ly\\ \eta, & lz, & y, & lx\\ \zeta, & ly, & x, & z\end{vmatrix}=0\,;$$

or expanding,

$$\left(yz - l^2x^2,\quad zx - l^2y^2,\quad xy - l^2z^2,\quad l^2yz - lx^2,\quad l^2zx - ly^2,\quad l^2xy - lz^2\right)\!\!\!

We have

$$X : Y : Z = l\xi^2 - \eta\zeta : l\eta^2 - \zeta\xi : l\zeta^2 - \xi\eta,$$

and X, Y, Z are to be eliminated from these equations, and the equation

$$l^2(X^3 + Y^3 + Z^3) - (1 + 2l^3)\,XYZ = 0$$

of the Hessian. We have

$$\begin{aligned} X^3 + Y^3 + Z^3 = \;& l^3(\xi^3 + \eta^3 + \zeta^3)^2 \\ & - 3l^2(\xi^3 + \eta^3 + \zeta^3)\,\xi\eta\zeta \\ & + 9l\;\xi^2\eta^2\zeta^2 \\ & - (1 + 2l^3)(\eta^3\zeta^3 + \zeta^3\xi^3 + \xi^3\eta^3), \\ XYZ = \;& l(\xi^3 + \eta^3 + \zeta^3)\,\xi\eta\zeta \\ & + \;(-1 + l^3)\,\xi^2\eta^2\zeta^2 \\ & - l^2(\eta^3\zeta^3 + \zeta^3\xi^3 + \xi^3\eta^3), \end{aligned}$$

and thence

$$\begin{aligned} HU = \;& l^5\;(\xi^3 + \eta^3 + \zeta^3)^2 \\ & - (l + 5l^4)(\xi^3 + \eta^3 + \zeta^3)\,\xi\eta\zeta \\ & + (1 + 10l^3 - 2l^6)\,\xi^2\eta^2\zeta^2\,; \end{aligned}$$

and equating the right-hand side to zero, we have the equation in line coordinates of the curve in question, which is therefore a curve of the sixth class in quadratic syzygy with the Pippian and Quippian.

Article No. 21.—*Geometrical definition of the Quippian.*

21. I have not succeeded in obtaining any good geometrical definition of the Quippian, and the following is only given for want of something better.

The curve

$$T\,.\,PU\,\{P\,6H\,(\alpha U + 6\beta HU)\} - P\,(6HU)\,\{T\,(\alpha U + 6\beta HU)\,.\,P\,(\alpha U + 6\beta HU)\} = 0,$$

which is derived in what may be taken to be a known manner from the cubic, is in general a curve of the sixth class. But if the syzygetic cubic $\alpha U + 6\beta HU = 0$ be properly selected, viz. if this curve be such that its Hessian breaks up into three lines, then both the Pippian of the cubic $\alpha U + 6\beta HU = 0$, and the Pippian of its Hessian will break up into the same three points, which will be a portion of the curve of the sixth class, and discarding these three points the curve will sink down to one of the third class, and will in fact be the Quippian of the cubic.

To show this we may take

$$\alpha U + 6\beta HU = x^3 + y^3 + z^3, = 0$$

as the equation of the syzygetic cubic satisfying the prescribed condition, for this value in fact gives

$$H(\alpha U + 6\beta HU) = -xyz, = 0,$$

a system of three lines. We find, moreover,

$$P(\alpha U + 6\beta HU) = P(x^3 + y^3 + z^3), = -\xi\eta\zeta$$

and

$$P\{6H(\alpha U + 6\beta HU)\} = P(-6xyz), = -4\xi\eta\zeta,$$

the latter equation being obtained by first neglecting all but the highest power of l in the expression of PU, and then writing $l = -1$: we have also $T(\alpha U + 6\beta HU) = 1$. Substituting the above values, the curve of the sixth class is

$$\xi\eta\zeta\{-4T.PU + P(6HU)\} = 0;$$

or throwing out the factor $\xi\eta\zeta$, we have the curve of the third class,

$$-4T.PU + P(6HU) = 0.$$

Now the general expression in my Third Memoir, viz.

$$P(\alpha U + 6\beta HU) = (\alpha^3 + 12S\alpha\beta^2 + 4T\beta^3)PU + (\alpha^2\beta - 4S\beta^3)QU,$$

putting $\alpha = 0$, $\beta = 1$, gives

$$P(6HU) = 4T.PU - 4S.QU,$$

or what is the same thing,

$$-4T.PU + P(6HU) = -4S.QU;$$

and the curve of the third class is therefore the Quippian $QU = 0$. It may be remarked, that for a cubic $U = 0$ the Hessian of which breaks up into three lines, the above investigation shows that we have $PU = -\xi\eta\zeta$, $P(6HU) = -4\xi\eta\zeta$, and $T = 1$, and consequently that $-4T.PU + P(6HU)$ ought to vanish identically; this in fact happens in virtue of the factor S on the right-hand side, the invariant S of a cubic of the form in question being equal to zero; the appearance of the factor S on the right-hand side is thus accounted for *à priori*.

Article No. 22.—*Theorem relating to a line which meets three given conics in six points in involution.*

22. The envelope of a line which meets three given conics, the first or conic polars of any three points with respect to the cubic, in six points in involution, is the Pippian.

It is readily seen that if the theorem is true with respect to the three conics,

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0,$$

it is true with respect to any three conics whatever of the form

$$\lambda \frac{dU}{dx} + \mu \frac{dU}{dy} + \nu \frac{dU}{dz} = 0,$$

that is, with respect to any three conics, each of them the first or conic polar of some point (λ, μ, ν) with respect to the cubic. Considering then these three conics, take $\xi x + \eta y + \zeta z = 0$ as the equation of the line, and let (X, Y, Z) be the coordinates of a point of intersection with the first conic, we have

$$\begin{aligned} \xi X + \eta Y + \zeta Z &= 0, \\ X^2 + 2lYZ \quad &= 0; \end{aligned}$$

and combining with these a linear equation

$$\alpha X + \beta Y + \gamma Z = 0,$$

in which (α, β, γ) are arbitrary quantities, we have

$$X : Y : Z = \gamma\eta - \beta\zeta : \alpha\zeta - \gamma\xi : \beta\xi - \alpha\eta;$$

and hence

$$(\gamma\eta - \beta\zeta)^2 + 2l(\alpha\zeta - \gamma\xi)(\beta\xi - \alpha\eta) = 0,$$

an equation in (α, β, γ) which is in fact the equation in line coordinates of the two points of intersection with the first conic. Developing and forming the analogous equations, we find

$$\begin{aligned} &(-2l\eta\zeta, \quad \zeta^2, \quad \eta^2, \quad -\eta\zeta - l\xi^2, \quad l\xi\eta, \quad l\xi\zeta \between (\alpha, \beta, \gamma)^2 = 0, \\ &(\zeta^2, \quad -2l\zeta\xi, \quad \xi^2, \quad l\xi\eta, \quad -\zeta\xi - l\eta^2, \quad l\eta\zeta \between (\alpha, \beta, \gamma)^2 = 0, \\ &(\eta^2, \quad \xi^2, \quad -2l\xi\eta, \quad l\zeta\xi, \quad l\eta\zeta, \quad -\xi\eta - l\zeta^2 \between (\alpha, \beta, \gamma)^2 = 0, \end{aligned}$$

which are respectively the equations in line coordinates of the three pairs of intersections.

Now combining these equations with the equation $\gamma = 0$, we have the equations of the pairs of lines joining the points of intersection with the point $(x = 0, y = 0)$, and if the six points are in involution, the six lines must also be in involution, or the condition for the involution of the six points is

$$\begin{vmatrix} -2l\eta\zeta, & \zeta^2, & l\xi\zeta \\ \zeta^2, & -2l\zeta\eta, & l\eta\zeta, \\ \eta^2, & \xi^2, & -\xi\eta - l\zeta^2 \end{vmatrix} = 0,$$

that is,

$$4l^2\zeta^2\xi\eta(-\xi\eta - l\zeta^2) + l\eta^3\zeta^3 + l\xi^3\zeta^3 + 2l^2\xi^2\eta^2\zeta^2 + 2l^3\xi^2\eta^2\zeta^2 + \zeta^4(-\xi\eta - l\zeta^2) = 0;$$

or, reducing and throwing out the factor ζ^3, we find

$$-l(\xi^3 + \eta^3 + \zeta^3) + (-1 + 4l^3)\xi\eta\zeta = 0,$$

which shows that the line in question is a tangent of the Pippian.

It is to be remarked that any three conics whatever may be considered as the first or conic polars of three properly selected points with respect to a properly selected cubic curve. The theorem applies therefore to any three conics whatever, but in this case the cubic curve is not given, and the Pippian therefore stands merely for a curve of the third class, and the theorem is as follows, viz. the envelope of a line which meets any three conics in six points in involution, is a curve of the third class.

Article No. 23.—*Completion of the theory in Liouville, and comparison with analogous theorems of* HESSE.

In order to convert the foregoing theorem into its reciprocal, we must replace the cubic $U=0$ by a curve of the third class, that is we must consider the coordinates which enter into the equation as line coordinates; and it of course follows that the coordinates which enter into the equation $PU=0$ must be considered as point coordinates, that is we must consider the Pippian as a curve of the third order: we have thus the theorem; The locus of a point such that the tangents drawn from it to three given conics (the first or conic poles of any three lines with respect to a curve of the third class) form a pencil in involution, is the Pippian considered as a curve of the third order. This in fact completes the fundamental theorem in my memoirs in *Liouville* above referred to, and establishes the analogy with Hesse's results in relation to the Hessian; to show this I set out the two series of theorems as follows:

Hesse, in his memoirs On Curves of the Third Order and Curves of the Third Class, *Crelle*, tt. XXVIII. XXXVI. and XXXVIII. [1844, 1848, 1849], has shown as follows:

(α) The locus of a point such that its polars with respect to the three conics $X=0$, $Y=0$, $Z=0$ (or more generally its polars with respect to all the conics of the series $\lambda X+\mu Y+\nu Z=0$) meet in a point, is a curve of the third order $V=0$.

(β) Conversely, given a curve of the third order $V=0$, there exists a series of conics such that the polars with respect to all the conics of any point whatever of the curve $V=0$, meet in a point.

(γ) The equation of any one of the conics in question is

$$\lambda\frac{dU}{dx}+\mu\frac{dU}{dy}+\nu\frac{dU}{dz}=0,$$

that is, the conic is the first or conic polar of a point (λ, μ, ν) with respect to a certain curve of the third order $U=0$; and this curve is determined by the condition that its Hessian is the given curve $V=0$, that is, we have $V=HU$.

(δ) The equation $V=HU$ is solved by assuming $U=aV+bHV$, for we have then $H(aV+bHV)=AV+BHV$, where A, B are given cubic functions of a, b, and thence $V=HU=AV+BHV$, or $A=1$, $B=0$; the latter equation gives what is alone important, the ratio $a:b$; and it thus appears that there are three distinct series of conics,

each of them having the above-mentioned relation to the given curve of the third order $V=0$.

In the memoirs in *Liouville* above referred to, I have in effect shown that—

(α') The locus of a point such that the tangents from it to three conics, represented in line coordinates by the equations $X=0$, $Y=0$, $Z=0$ (or more generally with respect to any three conics of the series $\lambda X+\mu Y+\nu Z=0$) form a pencil in involution, is a curve of the third order $V=0$.

(β') Conversely, given a curve of the third order $V=0$, there exists a series of conics such that the tangents from any point whatever of the curve to any three of the conics, form a pencil in involution.

Now, considering the coordinates which enter into the equation of the Pippian as point coordinates, and consequently the Pippian as a curve of the third order, I am able to add as follows:

(γ') The equation in line coordinates of any one of the conics in question is

$$\lambda\frac{dU}{d\xi}+\mu\frac{dU}{d\eta}+\nu\frac{dU}{d\zeta}=0,$$

that is, the conic is the first or conic polar of a line (λ, μ, ν) with respect to a certain curve of the third class $U=0$; and this curve is determined by the condition that its Pippian is the given curve of the third order $V=0$, that is, we have $V=PU$.

(δ') The equation $V=PU$ is solved by assuming $U=aPV+bQV$, for we have then $P(aPV+bQV)=AV+BHV$, where A and B are given cubic functions of a, b; and thence $V=PU=AV+BHV$, or $A=1$, $B=0$; the latter equation gives what is alone important, the ratio $a:b$; and it thus appears that there are three distinct curves of the third class $U=0$, and therefore (what indeed is shown in the Memoirs in *Liouville*) three distinct series of conics having the above-mentioned relation to the given curve of the third order $V=0$.

It is hardly necessary to remark that the preceding theorems, although precisely analogous to those of Hesse, are entirely distinct theorems, that is the two series are not connected together by any relation of reciprocity.

Article Nos. 24 to 28.—*Various investigations and theorems.*

24. Reverting to the theorem (No. 18), that the lineo-polar envelope of the line EF is the pair of lines OE, OF; the line EF is any tangent of the Pippian, hence the theorem includes the following one:

The lineo-polar envelope with respect to the cubic, of any tangent of the Pippian, is a pair of lines.

And conversely,

The Pippian is the envelope of a line such that the lineo-polar envelope of the line with respect to the cubic is a pair of lines.

It is I think worth while to give an independent proof. It has been shown that the equation of the lineo-polar envelope with respect to the cubic, of the line $\xi x+\eta y+\zeta z=0$ (where ξ, η, ζ are arbitrary quantities), is

$$(-l^2\xi^2-2l\eta\zeta,\quad -l^2\eta^2-2l\zeta\xi,\quad -l^2\zeta^2-2l\xi\eta,\quad \tfrac{1}{2}\xi^2+l^2\eta\zeta,\quad \tfrac{1}{2}\eta^2+l^2\zeta\xi,\quad \tfrac{1}{2}\zeta^2+l^2\xi\eta \between x,\ y,\ x)^2=0;$$

and representing this equation by

$$\tfrac{1}{2}(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0,$$

we find

$$\begin{aligned}
bc-f^2&=\xi\,(-\xi^3+8l^3\eta^3+8l^3\zeta^3+12l^2\xi\eta\zeta),\\
ca-g^2&=\eta\,(8l^3\xi^3-\quad\eta^3+8l^3\zeta^3+12l^2\xi\eta\zeta),\\
ab-h^2&=\zeta\,(8l^3\xi^3+8l^3\eta^3-\quad\zeta^3+12l^2\xi\eta\zeta),
\end{aligned}$$

$$\begin{aligned}
gh-af&=\xi\left(2l^2(\xi^3+\eta^3+\zeta^3)+4l(1+2l^3)\,\xi\eta\zeta\right)+(1+8l^3)\,\eta^2\zeta^2,\\
hf-bg&=\eta\left(2l^2(\xi^3+\eta^3+\zeta^3)+4l(1+2l^3)\,\xi\eta\zeta\right)+(1+8l^3)\,\zeta^2\xi^2,\\
fg-ch&=\zeta\left(2l^2(\xi^3+\eta^3+\zeta^3)+4l(1+2l^3)\,\xi\eta\zeta\right)+(1+8l^3)\,\xi^2\eta^2;
\end{aligned}$$

and after all reductions,

$$\begin{aligned}
&abc-af^2-bg^2-ch^2+2fgh\\
&\qquad=[-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\,\xi\eta\zeta]^2=(PU)^2,
\end{aligned}$$

or the condition in order that the conic may break up into a pair of lines is $PU=0$.

25. The following formulæ are given in connexion with the foregoing investigation, but I have not particularly considered their geometrical signification. The lineo-polar envelope of an arbitrary line $\xi x+\eta y+\zeta z=0$, with respect to the cubic

$$x^3+y^3+z^3+6lxyz=0,$$

has been represented by

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0;$$

and if in like manner we represent the lineo-polar envelope of the same line, with respect to a syzygetic cubic

$$x^3+y^3+z^3+6l'xyz=0,$$

by

$$(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2=0,$$

then we have

$$\begin{aligned} a'(bc-f^2)+b'(ca-g^2)+c'(ab-h^2)+2f'(gh-af)+2g'(hf-bg)+2h'(fg-ch) \\ = \quad (l'^2+2l^2)(\xi^3+\eta^3+\zeta^3)^2 \\ +(2l'+4l-32l^3l'+8l^4)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ +(24ll'^2+48l^4l'^2-72l^3l'+24l^3+3)\xi^2\eta^2\zeta^2, \end{aligned}$$

which may be verified by writing $l'=l$, in which case the right-hand side becomes as it should do, $3(PU)^2$. If $l'=-\dfrac{1+2l^3}{6l^2}$, that is, if the syzygetic cubic be the Hessian, then the formula becomes

$$a'(bc-f^2)+\&c.=\frac{1}{36l^4}\left\{\begin{array}{c}(1+4l^3+76l^6)(\xi^3+\eta^3+\zeta^3)^2 \\ +12l^2(-1+26l^3+56l^6)(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ +12l\ (2+57l^3+168l^6+16l^{10})\xi^2\eta^2\zeta^2\end{array}\right\}$$

which is equal to

$$\frac{1}{36l^4}\left\{\overline{QU}^2-24S.\overline{PU}^2\right\}.$$

26. The equation

$$(bc'+b'c-2ff',\ldots gh'+g'h-af'-a'f,\ldots \between \xi,\ \eta,\ \zeta)^2=0$$

is the equation in line coordinates of a conic, the envelope of the line which cuts harmonically the conics

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0,$$
$$(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2=0;$$

and if a, b, &c., a', &c. have the values before given to them, then the coefficients of the equation are

$$\begin{aligned} bc'+b'c-2ff' &= \xi\{-\xi^3+4ll'(l+l')(\eta^3+\zeta^3)+(16ll'-2l^2-2l'^2)\xi\eta\zeta, \\ ca'+c'a-2gg' &= \eta\{-\eta^3+4ll'(l+l')(\zeta^3+\xi^3)+(16ll'-2l^2-2l'^2)\xi\eta\zeta, \\ ab'+a'b-2hh' &= \zeta\{-\zeta^3+4ll'(l+l')(\xi^3+\eta^3)+(16ll'-2l^2-2l'^2)\xi\eta\zeta, \end{aligned}$$

$$\begin{aligned} gh'+g'h-af'-a'f &= \xi\{(l^2+l'^2)(\xi^3+\eta^3+\zeta^3)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\eta^2\zeta^2, \\ hf'+h'f-bg'-b'g &= \eta\{(l^2+l'^2)(\xi^3+\eta^3+\zeta^3)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\zeta^2\xi^2, \\ fg'+f'g-ch'-c'h &= \zeta\{(l^2+l'^2)(\xi^3+\eta^3+\zeta^3)+(2l+2l'+8l^2l'^2)\xi\eta\zeta\}+\big(1+4ll'(l+l')\big)\xi^2\eta^2; \end{aligned}$$

and we thence obtain

$$\begin{aligned} (bc'+b'c-2ff',\ldots gh'+g'h-af'-a'f,\ldots \between \xi,\ \eta,\ \zeta)^2= \\ -(\xi^3+\eta^3+\zeta^3)^2 \\ +(l^2+l'^2+16ll')(\xi^3+\eta^3+\zeta^3)\xi\eta\zeta \\ +(6l+6l'+24l^2l'^2)\xi^2\eta^2\zeta^2 \\ +\big(4+16(l^2l'+ll'^2)\big)(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3),=0 \end{aligned}$$

as the condition which expresses that a line $\xi x + \eta y + \zeta z = 0$ cuts harmonically its lineo-polar envelopes with respect to the cubic and with respect to a syzygetic cubic.

27. To find the locus of a point such that its second or line polar with respect to the cubic may be a tangent of the Pippian. Let the coordinates of the point be (x, y, z); then if $\xi x + \eta y + \zeta z = 0$ be the equation of the polar, we have

$$\xi : \eta : \zeta = x^2 + 2lyz : y^2 + 2lzx : z^2 + 2lxy,$$

and the line in question being a tangent to the Pippian,

$$-l(\xi^3 + \eta^3 + \zeta^3) + (-1 + 4l^3)\,\xi\eta\zeta = 0.$$

But the preceding values give

$$\begin{aligned} \xi^3 + \eta^3 + \zeta^3 &= (x^3 + y^3 + z^3)^2 + 6l\,(x^3 + y^3 + z^3)\,xyz + 36l^2x^2y^2z^2 + (-2 + 8l^3)\,(y^3z^3 + z^3x^3 + x^3y^3) \\ \xi\eta\zeta &= 4l^2\,(x^3 + y^3 + z^3)\,xyz + (1 + 8l^3)\,x^2y^2z^2 + 2l\,(y^3z^3 + z^3x^3 + x^3y^3); \end{aligned}$$

and we have therefore

$$l\,(x^3 + y^3 + z^3)^2 + (10l^3 - 16l^6)\,(x^3 + y^3 + z^3)\,xyz + (1 + 40l^3 - 32l^6)\,x^2y^2z^2 = 0\,;$$

or introducing U, HU in place of $x^3 + y^3 + z^3$, xyz, the equation becomes

$$-S\,.\,U^2 + (HU)^2 = 0,$$

which is the equation of the locus in question.

28. The locus of a point such that its second or line polar with respect to the cubic is a tangent of the Quippian, is found in like manner by substituting the last-mentioned values of ξ, η, ζ in the equation

$$QU = (1 - 10l^3)\,(\xi^3 + \eta^3 + \zeta^3) - 6l^2\,(5 + 4l^3)\,\xi\eta\zeta.$$

We find as the equation of the locus,

$$\begin{aligned} (1 - 10l^3)\,(x^3 + y^3 + z^3)^2 + 6l\,(1 - 30l^3 - 16l^6)\,(x^3 + y^3 + z^3)\,xyz + 6l^2\,(1 - 104l^3 - 32l^6)\,x^2y^2z^2 \\ - 2\,(1 + 8l^3)^2\,(y^3z^3 + z^3x^3 + x^3y^3) = 0, \end{aligned}$$

where the function on the left-hand side is the octicovariant $\Theta_{\prime\prime}U$ of my Third Memoir, the covariant having been in fact defined so as to satisfy the condition in question. And I have given in the memoir the following expression for $\Theta_{\prime\prime}U$, viz.

$$\begin{aligned} \Theta_{\prime\prime}U = {} & (1 - 16l^3 - 6l^6)\,U^2 \\ & + (6l \qquad\quad)\,U\,.\,HU \\ & + (6l^2 \qquad\quad)\,(HU)^2 \\ & - \quad 2\,(1 + 8l^3)^2\,(y^3z^3 + z^3x^3 + x^3y^3). \end{aligned}$$

Article Nos. 29 to 31.—*Formulæ for the intersection of a cubic curve and a line.*

29. If the line $\xi x + \eta y + \zeta z = 0$ meet the cubic

$$x^3 + y^3 z^3 + 6lxyz = 0$$

in the points

$$(x_1,\ y_1,\ z_1),\quad (x_2,\ y_2,\ z_2),\quad (x_3,\ y_3,\ z_3),$$

then we have

$$x_1 x_2 x_3 : y_1 y_2 y_3 : z_1 z_2 z_3 = \eta^3 - \zeta^3 : \zeta^3 - \xi^3 : \xi^3 - \eta^3.$$

It will be convenient to represent the equation of the cubic by the abbreviated notation $(1,\ 1,\ 1,\ l\between x,\ y,\ z)^3 = 0$; we have the two equations

$$\begin{aligned} &(1,\ 1,\ 1,\ l\between x,\ y,\ z)^3 = 0, \\ &\xi x + \eta y + \zeta z \qquad\quad = 0; \end{aligned}$$

and if to these we join a linear equation with arbitrary coefficients,

$$\alpha x + \beta y + \gamma z = 0,$$

then the second and third equations give

$$x : y : z = \beta\zeta - \gamma\eta : \gamma\xi - \alpha\zeta : \alpha\eta - \beta\xi;$$

and substituting these values in the first equation, we obtain the resultant of the system. But this resultant will also be obtained by substituting, in the third equation, a system of simultaneous roots of the first and second equations, and equating to zero the product of the functions so obtained[1]. We must have therefore

$$(1,\ 1,\ 1,\ l\between \beta\zeta - \gamma\eta,\ \gamma\xi - \alpha\zeta,\ \alpha\eta - \beta\xi)^3 = (\alpha x_1 + \beta y_1 + \gamma z_1)(\alpha x_2 + \beta y_2 + \gamma z_2)(\alpha x_3 + \beta y_3 + \gamma z_3);$$

and equating the coefficients of α^3, β^3, γ^3, we obtain the above-mentioned relations.

30. If a tangent to the cubic

$$x^3 + y^3 + z^3 + 6lxyz = 0$$

at a point $(x_1,\ y_1,\ z_1)$ of the cubic meet the cubic in the point $(x_3,\ y_3,\ z_3)$, then

$$x_3 : y_3 : z_3 = x_1(y_1^3 - z_1^3) : y_1(z_1^3 - x_1^3) : z_1(x_1^3 - y_1^3).$$

For if the equation of the tangent is $\xi x + \eta y + \zeta z = 0$, then

$$x_1^2 x_3 : y_1^2 y_3 : z_1^2 z_3 = \eta^3 - \zeta^3 : \zeta^3 - \xi^3 : \xi^3 - \eta^3,$$

and

$$\xi : \eta : \zeta = x_1^2 + 2ly_1 z_1 : y_1^2 + 2lz_1 x_1 : z_1^2 + 2lx_1 y_1.$$

[1] This is in fact the general process of elimination given in Schläfli's Memoir, "Ueber die Resultante einer Systemes mehrerer algebraischer Gleichungen," Vienna Trans. 1852. [But the process was employed much earlier, by Poisson.]

These values give

$$\begin{aligned}\eta^3-\zeta^3 &= (y_1^3-z_1^3)(y_1^3+z_1^3+6lx_1y_1z_1-8l^3x_1^3)\\ &= (y_1^3-z_1^3)\times-(1+8l^3)\,x_1^3,\end{aligned}$$

since (x_1, y_1, z_1) is a point of the cubic; and forming in like manner the values of $\zeta^3-\xi^3$ and $\xi^3-\eta^3$, we obtain the theorem.

31. The preceding values of (x_3, y_3, z_3) ought to satisfy

$$(x_1^2+2ly_1z_1)\,x_3+(y_1^2+2lz_1x_1)\,y_3+(z_1^2+2lx_1y_1)\,z_3=0,$$
$$x_3^3+y_3^3+z_3^3+6lx_3y_3z_3=0;$$

in fact the first equation is satisfied identically, and for the second equation we obtain

$$\begin{aligned}x_3^2+y_3^2+z_3^2 &= x_1^3(y_1^3-z_1^3)^3+y_1^3(z_1^3-x_1^3)^3+z_1^3(x_1^3-y_1^3)^3\\ &= -x_1^9(y_1^3-z_1^3)-y_1^9(z_1^3-x_1^3)-z_1^9(x_1^3-y_1^3)\\ &= (x_1^3+y_1^3+z_1^3)(y_1^3-z_1^3)(z_1^3-x_1^3)(x_1^3-y_1^3),\\ x_3y_3z_3 &= x_1y_1z_1(y_1^3-z_1^3)(z_1^3-x_1^3)(x_1^3-y_1^3),\end{aligned}$$

and consequently

$$x_3^3+y_3^3+z_3^3+6lx_3y_3z_3=(x_1^3+y_1^3+z_1^3+6lx_1y_1z_1)(y_1^3-z_1^3)(z_1^3-x_1^3)(x_1^3-y_1^3)=0,$$

which verifies the theorem. It is proper to add (the remark was made to me by Professor Sylvester) that the foregoing values

$$x_3 : y_3 : z_3 = x_1(y_1^3-z_1^3) : y_1(z_1^3-x_1^3) : z_1(x_1^3-y_1^3)$$

satisfy *identically* the relation

$$\frac{x_3^3+y_3^3+z_3^3}{x_3y_3z_3}=\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1}.$$

Article Nos. 32 to 34.—*Formulæ for the Satellite line and point.*

32. The line $\xi x+\eta y+\zeta z=0$ meets the cubic

$$x^3+y^3+z^3+6lxyz=0$$

in three points, and the tangents to the cubic at these points meet the cubic in three points lying in a line, which has been called the Satellite line of the given line.

To find the equation of the satellite line; suppose that (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the coordinates of the point in which the given line meets the cubic; then we have, as before,

$$(1, 1, 1, l\{\beta\zeta-\gamma\eta,\ \gamma\xi-\alpha\zeta,\ \alpha\eta-\beta\xi\})^3=(\alpha x_1+\beta y_1+\gamma z_1)(\alpha x_2+\beta y_2+\gamma z_2)(\alpha x_3+\beta y_3+\gamma z_3).$$

The equation of the three tangents is

$$\left.\begin{aligned}\Pi = &[(x_1^2 + 2ly_1z_1)\,x + (y_1^2 + 2lz_1x_1)\,y + (z_1^2 + 2lx_1y_1)\,z]\\ &\times[(x_2^2 + 2ly_2z_2)\,x + (y_2^2 + 2lz_2x_2)\,y + (z_2^2 + 2lx_2y_2)\,z]\\ &\times[(x_3^2 + 2ly_3z_3)\,x + (y_3^2 + 2lz_3x_3)\,y + (z_3^2 + 2lx_3y_3)\,z]\end{aligned}\right\} = 0,$$

and if we put

$$F = (\xi^3 + \eta^3 + \zeta^3)^2 - 24l^2(\xi^3 + \eta^3 + \zeta^3)\,\xi\eta\zeta + (-24l - 48l^4)\,\xi^3\eta^3\zeta^3 + (-4 + 32l^3)\,(\eta^3\zeta^3 + \zeta^3\xi^3 + \xi^3\eta^3),$$

(F is the reciprocant FU of my Third Memoir), then we have identically

$$F\,.\,U - \Pi = (\xi x + \eta y + \zeta z)^2\,(\xi' x + \eta' y + \zeta' z),$$

and the equation of the satellite line is $\xi' x + \eta' y + \zeta' z = 0$. In fact the geometrical theory shows that we must have

$$F\,.\,U - N\Pi = (\xi x + \eta y + \zeta z)^2\,(\xi' x + \eta' y + \zeta' z),$$

and it is then clear that N is a mere number. To determine its value in the most simple manner, write $l = 0$, $y = 0$, $x = \zeta$, $z = -\xi$, we have then $F\,.\,U - N\Pi = 0$, where

$$F = \xi^6 + \eta^6 + \zeta^6 - 2\eta^3\zeta^3 - 2\zeta^3\xi^3 - 2\xi^3\eta^3,\quad U = \zeta^3 - \xi^3.$$

The value of Π is $\Pi = F\,.\,U$, and we thus obtain $N = 1$. For, substituting the above values,

$$\begin{aligned}\Pi = &(x_1^2\zeta - z_1^2\xi)\,(x_2^2\zeta - z_2^2\xi)\,(x_3^2\zeta - z_3^2\xi)\\ = &\quad \zeta^3 x_1^2x_2^2x_3^2\\ &- \zeta^2\xi\,(x_1^2x_2^2x_3^2 + \&c.)\\ &+ \zeta\xi^2\,(x_1z_2^2z_3^2 + \&c.)\\ &- \xi^3 z_1^2z_2^2z_3^2,\end{aligned}$$

and we have

$$\begin{aligned}x_1x_2x_3 &= \eta^3 - \zeta^3,\\ x_1x_2z_3 + \&c. &= 3\zeta^2\xi,\\ x_1z_2z_3 + \&c. &= -3\zeta\xi^2,\\ z_1z_2z_3 &= \xi^3 - \eta^3,\end{aligned}$$

and thence

$$\begin{aligned}x_1^2x_2^2x_3^2 + \&c. &= 9\zeta^4\xi^2 + 6\zeta\xi^2\,(\eta^3 - \zeta^3) = 3\zeta^4\xi^2 + 6\zeta\xi^2\eta^3,\\ x_1^2z_2^2z_3^2 + \&c. &= 9\zeta^2\xi^4 - 6\zeta^2\xi\,(\xi^3 - \eta^3) = 3\zeta^2\xi^4 + 6\zeta^2\xi\eta^3,\end{aligned}$$

and consequently

$$\begin{aligned}\Pi = &\quad \zeta^3\,(\eta^3 - \zeta^3)^2\\ &- \zeta^3\xi\,.\,3\zeta\xi^2\,(\zeta^3 + \eta^3)\\ &+ \zeta\xi^2\,.\,3\zeta^2\xi\,(\xi^3 + \eta^3)\\ &- \xi^3\,(\xi^3 - \eta^3)^2\\ = &(\zeta^3 - \xi^3)\,(\xi^3 + \eta^6 + \zeta^6 - 2\zeta^3\eta^3 - 2\zeta^3\xi^3 - 2\xi^3\eta^3).\end{aligned}$$

Now considering the equation

$$F\,.\,U-\Pi=(\xi x+\eta y+\zeta z)^3(\xi' x+\eta' y+\zeta' z),$$

in order to find ξ', η', ζ' it will be sufficient to find the coefficients of x^3, y^3, z^3 in the function on the left-hand side of the equation. The coefficient of x^3 in Π is

$$\begin{aligned}&(x_1^2+2ly_1z_1)(x_2^2+2lz_2x_2)(x_3^2+2lx_3y_3)\\ =\;&\quad x_1^2x_2^2x_3^2\\ &+2l\,(x_1^2x_2^2y_3z_3+\&c.)\\ &+4l^2\,(x_1^2y_2z_2y_3z_3+\&c.)\\ &+8l^3\quad y_1y_2y_3z_1z_2z_3;\end{aligned}$$

and it is easy to see that representing the function

$$(1,\,1,\,1,\,l\underset{\sim}{\times}\beta\zeta-\gamma\eta,\;\gamma\xi-\alpha\zeta,\;\alpha\eta-\beta\xi)^3$$

by

$$(\mathrm{a},\,\mathrm{b},\,\mathrm{c},\,\mathrm{f},\,\mathrm{g},\,\mathrm{h},\,\mathrm{i},\,\mathrm{j},\,\mathrm{k},\,l\underset{\sim}{\times}\alpha,\,\beta,\,\gamma)^3,$$

the symmetrical functions can be expressed in terms of the quantities a, b, &c., and that the preceding value of the coefficient of x^3 in Π is

$$\begin{aligned}&\mathrm{a}^3\\ &+2l\;(9\mathrm{hj}-6\mathrm{al})\\ &+4l^2\,(6\mathrm{gk}-3\mathrm{fj}-3\mathrm{hi}+3l^2)\\ &+8l^3\;\;\mathrm{bc};\end{aligned}$$

and substituting for a, &c. their values, this becomes

$$\begin{aligned}&(\eta^3-\zeta^3)^2\\ &+2l\;\{-9\,(\xi\eta^2+2l\eta\zeta^2)(\zeta^2\xi+2l\zeta\eta^2)\}\\ &+4l^2\{-6\,(\zeta\xi^2+2l\xi\eta^2)(\xi^2\eta+2l\xi\zeta^2)\\ &\qquad+3\,(\eta\zeta^2+2l\zeta\xi^2)(\zeta^2\xi+2l\zeta\eta^2)\\ &\qquad+3\,(\xi\eta^2+2l\eta\zeta^2)(\eta^2\zeta+2l\eta\xi^2)\}\\ &+8l^3\,(\xi^3-\eta^3)(\zeta^3-\xi^3),\end{aligned}$$

and reducing, we obtain for the coefficient of x^3 in Π the following expression,

$$\begin{aligned}&(\eta^3-\zeta^3)^2\\ &-18l\;\;\xi^2\eta^2\zeta^2\\ &-24l^3\,(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\ &-24l^3\,(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\ &+\;\;8l^3\,(\xi^3-\eta^3)(\zeta^3-\xi^3).\end{aligned}$$

Now the coefficient of x^3 in $F\,.\,U$ is simply F, which is equal to

$$\begin{aligned}
&\xi^6+\eta^6+\zeta^6-2\eta^3\zeta^3-2\zeta^3\xi^3-2\xi^3\eta^3\\
&-24l\quad\xi^2\eta^2\zeta^2\\
&-24l^2(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\
&-32l^3\,(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\
&-48l^4\;\xi^2\eta^2\zeta^2;
\end{aligned}$$

and subtracting, the coefficient of x^3 in $F\,.\,U-\Pi$ is

$$\begin{aligned}
&\xi^6-2\xi^3\eta^3-2\xi^3\zeta^3\\
&-6l\quad\xi^2\eta^2\zeta^2\\
&-8l^3\,(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\
&-8l^3\,(\xi^3-\eta^3)(\zeta^3-\xi^3)\\
&-48l^4\xi^2\eta^2\zeta^2,
\end{aligned}$$

which is equal to

$$(1+8l^3)\,\xi^2\,(\xi^4-2\xi\eta^3-2\xi\zeta^3-6l\eta^2\zeta^2).$$

The expression last written down is therefore the value of $\xi^2\xi'$, or dividing by ξ^2 we have ξ', and then the values of η', ζ' are of course known, and we obtain the identical equation

$$F\,.\,U-\Pi=$$

$$(1+8l^3)\,(\xi x+\eta y+\zeta z)^2\left\{\begin{aligned}&(\xi^4-2\xi\eta^3-2\xi\zeta^3-6l\eta^2\zeta^2)\,x\\&+(\eta^4-2\eta\zeta^3-2\eta\xi^3-6l\zeta^2\xi^2)\,y\\&+(\zeta^4-2\zeta\xi^3-2\zeta\eta^3-6l\xi^2\eta^2)\,z\end{aligned}\right\}$$

and the second factor equated to zero is the equation of the satellite line of $\xi x+\eta y+\zeta z=0$.

33. The point of intersection of the line $\xi x+\eta y+\zeta z=0$ with the satellite line $\xi' x+\eta' y+\zeta' z=0$ is the satellite point of the former line; and the coordinates of the satellite point are at once found to be

$$\begin{aligned}x:y:z=&\,(\eta^3-\zeta^3)(\eta\zeta+2l\xi^2)\\&:(\zeta^3-\xi^3)(\zeta\xi+2l\eta^2)\\&:(\xi^3-\eta^3)(\xi\eta+2l\zeta^2).\end{aligned}$$

34. If the primary line $\xi x+\eta y+\zeta z=0$ is a tangent to the cubic, then (x_1, y_1, z_1) being the coordinates of the point of contact, we have

$$\xi:\eta:\zeta=x_1^2+2ly_1z_1:y_1^2+2lz_1x_1:z_1^2+2lx_1y_1;$$

these values give as before

$$\eta^3 - \zeta^3 = -(1+8l^3)\,x_1^2\,(y_1^3 - z_1^3);$$

and they give also

$$\eta\zeta + 2l\xi^2 = (1+8l^3)\,y_1^2 z_1^2,$$

and consequently we obtain

$$x : y : z = x_1\,(y_1^3 - z_1^3) : y_1\,(z_1^3 - x_1^3) : z_1\,(x_1^3 - y_1^3),$$

that is, the satellite point of a tangent of the cubic is the point in which this tangent again meets the cubic.

Article Nos. 35 and 36.—*Theorems relating to the satellite point.*

35. If the line $\xi x + \eta y + \zeta z = 0$ be a tangent of the Pippian, then the locus of the satellite point is the Hessian.

Take (x, y, z) as the coordinates of the satellite point, then we have

$$\begin{aligned} x : y : z &= (\eta^3 - \zeta^3)(\eta\zeta + 2l\xi^2) \\ &: (\zeta^3 - \xi^3)(\zeta\xi + 2l\eta^2) \\ &: (\xi^3 - \eta^3)(\xi\eta + 2l\zeta^2); \end{aligned}$$

where the parameters ξ, η, ζ are connected by the equation

$$-l(\xi^3 + \eta^3 + \zeta^3) + (-1 + 4l^3)\,\xi\eta\zeta = 0.$$

We have

$$\begin{aligned} y^3 + z^3 = &(\zeta^3 - \xi^3)^3(\zeta\xi + 2l\eta^2)^3 \\ &+ (\xi^3 - \eta^3)^3(\xi\eta + 2l\zeta^2)^3, \end{aligned}$$

and it is easy to see that the function on the right-hand side must divide by $\eta^3 - \zeta^3$: hence $x^3 + y^3 + z^3$ will also divide by $\eta^3 - \zeta^3$, and consequently by $(\eta^3 - \zeta^3)(\zeta^3 - \xi^3)(\xi^3 - \eta^3)$. We have

$$\begin{aligned} (y^3 + z^3) \div (\eta^3 - \zeta^3) = \quad & \xi^3 \left\{ \begin{array}{l} -\zeta^9 - \zeta^6\eta^3 - \zeta^3\eta^6 - \eta^9 \\ +3\xi^3(\zeta^6 + \zeta^3\eta^3 + \eta^6) \\ -3\xi^6(\zeta^3 + \eta^3) \\ +\ \xi^9 \end{array} \right\} \\ & + 6l\xi^2\eta^2\zeta^2\,\{-\zeta^3 - \zeta^3\eta^3 - \eta^3 + 3\xi^3(\zeta^3 + \eta^3) - 3\xi^3\} \\ & + 12l^2\xi\eta\zeta\,\{-\eta^3\zeta^3(\eta^3 + \zeta^3) + 3\xi^3\eta^3\zeta^3 - \xi^9\} \\ & + 8l^3 \quad \{-\eta^6\zeta^6 + 3\eta^3\zeta^3\xi^3 - (\eta^3 + \zeta^3)\,\xi^3\} \end{aligned}$$

and

$$x^3 \div (\eta^3 - \zeta^3) = (\eta^3 - 2\eta^3\zeta^3 + \zeta^3)\,(\eta^3\zeta^3 + 6l\xi^2\eta^2\zeta^2 + 12l^2\xi^4\eta\zeta + 8l^3\xi^6).$$

Adding these values and completing the reduction, we find

$$\begin{aligned}(x^3+y^3+z^3)\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3) = &-\xi^6-\eta^6-\zeta^6+2\eta^3\zeta^3+2\zeta^3\xi^3+2\xi^3\eta^3\\ &+18l\ \ \xi^2\eta^2\zeta^2\\ &+12l^3(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\ &+\ \ 8l^3(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3);\end{aligned}$$

and we have also

$$\begin{aligned}xyz\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3) \quad = \quad &\xi^2\eta^2\zeta^2\\ &+2l\,(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\ &+4l^2(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3)\\ &+8l^3\,\xi^2\eta^2\zeta^2,\end{aligned}$$

and thence

$$\begin{aligned}\{A\,(x^3+y^3+z^3)+Bxyz\}\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)&\\ = \quad -A \qquad\qquad &(\xi^3+\eta^3+\zeta^3)^2\\ +(12l^2A+4lB) \qquad &(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\ +\big(18lA+(1+8l^3)B\big) \quad &\xi^2\eta^2\zeta^2\\ +\big((4l^3+8l^3)A+4l^2B\big)&(\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3).\end{aligned}$$

The coefficient of $\eta^3\zeta^3+\zeta^3\xi^3+\xi^3\eta^3$ on the right-hand side will vanish if $(1+2l^3)A+l^2B=0$, or, what is the same thing, if $A=l^2$, $B=-(1+2l^3)$; and substituting these values, we obtain

$$\begin{aligned}\{l^2(x^3+y^3+z^3)-(1+2l^3)\,\xi\eta\zeta\}\div(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)&\\ =-l^2 \qquad\qquad &(\xi^3+\eta^3+\zeta^3)\\ +(-4l+4l^4) \qquad &(\xi^3+\eta^3+\zeta^3)\,\xi\eta\zeta\\ +(-1+8l^9-16l^6)\,\xi^2\eta^2\zeta^2&,\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}l^2(x^3+y^3+z^3)-(1+2l^3)\,xyz = &-(\eta^3-\zeta^3)(\zeta^3-\xi^3)(\xi^3-\eta^3)\\ &\times\{-l\,(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\,\xi\eta\zeta\}^2.\end{aligned}$$

Hence the left-hand side vanishes in virtue of the relation between ξ, η, ζ, or we have

$$l^3(x^3+y^3+z^3)-(1+2l^3)\,xyz=0,$$

which proves the theorem.

36. Suppose that (X, Y, Z) are the coordinates of a point of the Hessian, and let (P, Q, R) be the coordinates of the point in which the tangent to the Hessian at the point (X, Y, Z) again meets the Hessian, or, what is the same thing, the

satellite point *in regard to the Hessian* of the tangent at (X, Y, Z). And consider the conic

$$X(x^2+2lyz)+Y(y^2+2lzx)+Z(x^2+2lxy),$$

which is the first or conic polar of the point (X, Y, Z) in respect of the cubic. The polar (in respect to this conic) of the point (P, Q, R) will be

$$\xi x+\eta y+\zeta z=0,$$

where

$$\begin{aligned}\xi &= PX+l(RY+QZ),\\ \eta &= QY+l(PZ+RX),\\ \zeta &= RZ+l(QX+PY);\end{aligned}$$

or putting for (P, Q, R) their values,

$$\begin{aligned}\xi &= (Y^3-Z^3)(X^2-lYZ),\\ \eta &= (Z^3-X^3)(Y^2-lZX),\\ \zeta &= (X^3-Y^3)(Z^2-lXY);\end{aligned}$$

and if from these equations and the equation of the Hessian we eliminate (X, Y, Z), we shall obtain the equation in line coordinates of the curve which is the envelope of the line $\xi x+\eta y+\zeta z=0$. We find, in fact,

$$\begin{aligned}\xi^3+\eta^3+\zeta^3 &= (Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)\\ &\quad\times\begin{cases} & l^3\ (X^3+Y^3+Z^3)^2\\ - & 3l\ (X^3+Y^3+Z^3)XYZ\\ + & 9l^2\ X^2Y^2Z^2\\ + & (1-4l^3)(Y^3Z^3+Z^3X^3+X^3Y^3),\end{cases}\\ \xi\eta\zeta &= (Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)\\ &\quad\times\begin{cases} & l^2(X^3+Y^3+Z^3)XYZ\\ + & (1-l^3)X^2Y^2Z^2\\ - & l\ (Y^3Z^3+Z^3X^3+X^3Y^3);\end{cases}\end{aligned}$$

and thence recollecting that

$$HU=l^2(X^3+Y^3+Z^3)-(1+2l^3)XYZ,$$

we find

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta=-(Y^3-Z^3)(Z^3-X^3)(X^3-Y^3)(HU)^2,$$

and the equation of the envelope is

$$-l(\xi^3+\eta^3+\zeta^3)+(-1+4l^3)\xi\eta\zeta=0,$$

which is therefore the Pippian. We have thus the theorem:

The envelope of the polar of the satellite point *in respect to the Hessian* of the tangent at any point of the Hessian, such polar being in respect of the conic which is the first or conic polar of the point of the Hessian in respect of the cubic, is the Pippian.

Article Nos. 37 to 40.—*Investigations and theorems relating to the first or conic polar of a point of the cubic.*

37. The investigations next following depend on the identical equations

$$\begin{aligned}
&\{\alpha(X^2+2lYZ)+\beta(Y^2+2lZX)+\gamma(Z^2+2lXY)\}\\
&\qquad\times\{-XYZ(x^3+y^3+z^3)+(X^3+Y^3+Z^3)xyz\}\\
&=\{X(x^2+2lyz)+Y(y^2+2lzx)+Z(z^2+2lxy)\}\\
&\qquad\times\{X(Y^3-Z^3)(\gamma y-\beta z)+Y(Z^3-X^3)(\alpha z-\gamma x)+Z(X^3-Y^3)(\beta x-\alpha y)\}\\
&+\{x(X^2+2lYZ)+y(Y^2+2lZX)+z(Z^2+2lXY)\}\\
&\quad\times\{-(\alpha YZ+\beta ZX+\gamma XY)(Xx^3+Yy^3+Zz^3)+(\alpha X^2+\beta Y^2+\gamma Z^2)(XYz+Yzx+Zxy)\},
\end{aligned}$$

which is easily verified.

I represent the equation in question by

$$K\Upsilon=WL+P\Theta;$$

then considering (x, y, z) as current coordinates, and (X, Y, Z) and (α, β, γ) as the coordinates of two given points Σ and Ω, we shall have $U=0$ the equation of the cubic, $W=0$ the equation of the first or conic polar of Σ with respect to the cubic, $P=0$ the equation of the second or line polar of Σ with respect to the cubic. The equation $\Upsilon=0$ is that of a syzygetic cubic passing through the point Σ: the coordinates of the satellite point in respect to this syzygetic cubic of its tangent at Σ are

$$X(Y^3-Z^3) : Y(Z^3-X^3) : Z(X^3-Y^3);$$

and calling the point in question Σ', then $L=0$ is the equation of a line through the points Σ', Ω. The equation $\Theta=0$ is that of a conic, viz. the first or conic polar of Σ with respect to a certain syzygetic cubic

$$-2(\alpha YZ+\beta ZX+\gamma XY)(x^3+y^3+z^3)+(\alpha X^2+\beta Y^2+\gamma Z^2)xyz=0,$$

depending on the points Σ, Ω, or, what is the same thing, the conic $\Theta=0$ is a properly selected conic passing through the points of intersection of the first or conic polars of Σ with respect to any two syzygetic cubics; and lastly, K is a constant coefficient. The equation expresses that the points of intersection of

$$(W=0,\ P=0),\ (W=0,\ \Theta=0),\ (L=0,\ P=0),\ (L=0,\ \Theta=0),$$

lie in the syzygetic cubic $\Upsilon=0$.

The left-hand side of the equation may be written

$$\begin{aligned} &-XYZ\{\alpha(X^2+2lYZ)+\beta(Y^2+2lZX)+\gamma(Z^2+2lXY)\}(x^3+y^3+z^3+6lxyz) \\ &+\quad xyz\{\alpha(X^2+2lYZ)+\beta(Y^2+2lZX)+\gamma(Z^2+2lXY)\}(X^3+Y^3+Z^3+6lXYZ); \end{aligned}$$

and it may be remarked also that we have

$$-3XYZ\{\alpha(X^2+2lYZ)+\beta(Y^2+2lZX)+\gamma(Z^2+2lXY)\}$$

equal identically to

$$\begin{aligned} &\{X(Y^3-Z^3)(\gamma Y-\beta Z)+Y(Z^3-X^3)(\alpha Z-\gamma X)+Z(X^3-Y^3)(\beta X-\alpha Y)\} \\ &\qquad\qquad -(\alpha YZ+\beta ZX+\gamma XY)(X^3+Y^3+Z^3+6lXYZ). \end{aligned}$$

Hence if we assume

$$X^3+Y^3+Z^3+6lXYZ=0,$$

the equation will take the form

$$KU=WL+P\Theta,$$

where the constant coefficient K may be expressed under the two different forms

$$\begin{aligned} K&=-XYZ\{\alpha(X^2+2lYZ)+\beta(Y^2+2lZX)+\gamma(Z^2+2lXY)\} \\ &=\tfrac{1}{3}\{X(Y^3-Z^3)(\gamma Y-\beta Z)+Y(Z^3-X^3)(\alpha Z-\gamma X)+Z(X^3-Y^3)(\beta X-\alpha Y)\}, \end{aligned}$$

and W, L, P, Θ have the same values as before. In the present case the point Σ is a point of the cubic: the equation $W=0$ represents the first or conic polar of the point in question, and the equation $P=0$ its second or line polar, which is also the tangent of the cubic. The line $L=0$ is a line joining the point Ω with the satellite point of the tangent at Σ, or dropping altogether the consideration of the point Ω, is an arbitrary line through the satellite point: the first or conic polar of Σ meets the cubic twice in the point Σ, and therefore also meets it in four other points; the conic $\Theta=0$ is a conic passing through these four points, and completely determined when the particular position of the line through the satellite point is given. And, as before remarked, $\Theta=0$ is a conic passing through the points of intersection of the first or conic polars of Σ with respect to any two syzygetic cubics. We have thus the theorem:

The first or conic polar of a point of the cubic touches the cubic at this point, and besides meets it in four other points; the four points in question are the points in which the first or conic polar of the given point in respect of the cubic is intersected by the first or conic polar of the same point in respect to any syzygetic cubic whatever.

38. The analytical result may be thus stated: putting

$$\kappa=\alpha YZ+\beta ZX+\gamma XY,\quad \lambda=\alpha X^2+\beta Y^2+\gamma Z^2,$$

or, if we please, considering κ, λ as arbitrary parameters, then the four points lie in the conic

$$(2\kappa X,\ 2\kappa Y,\ 2\kappa Z,\ -\lambda X,\ -\lambda Y,\ -\lambda Z \between x,\ y,\ z)^2 = 0,$$

or, what is the same thing, they are the points of intersection of the two conics

$$Xx^2 + Yy^2 + Zz^2 = 0,$$
$$Xyz + Yzx + Zxy = 0.$$

39. Considering the four points as the angles of a quadrangle, it may be shown that the three centres of the quadrangle lie on the cubic. To effect this, assume that the conic

$$(2\kappa X,\ 2\kappa Y,\ 2\kappa Z,\ -\lambda X,\ -\lambda Y,\ -\lambda Z \between x,\ y,\ z)^2 = 0$$

represents a pair of lines; these lines will intersect in a point, which is one of the three centres in question. And taking x, y, z as the coordinates of this point, we have

$$\begin{aligned} x^2 : y^2 : z^2 : yz : zx : xy &= 4\kappa^2\, YZ - \lambda^2 X^2 \\ &: 4\kappa^2\, ZX - \lambda^2 Y^2 \\ &: 4\kappa^2\, XY - \lambda^2 Z^2 \\ &: \lambda^2\, YZ + 2\kappa\lambda X^2 \\ &: \lambda^2\, ZX + 2\kappa\lambda Y^2 \\ &: \lambda^2\, XY + 2\kappa\lambda Z^2; \end{aligned}$$

and we may, if we please, use these equations to find the relation between κ, λ. Thus in the identical equation $x^2 . y^2 - (xy)^2 = 0$, substituting for x^2, xy, y^2 their values, and throwing out the factor Z, we find $(4\kappa^3 - \lambda^3)\, XYZ - \kappa\lambda^2 (X^3 + Y^3 + Z^3) = 0$, and thence, in virtue of the equation $X^3 + Y^3 + Z^3 + 6lXYZ = 0$, we obtain

$$4\kappa^3 + 6l\kappa\lambda^3 - \lambda^3 = 0.$$

But the preceding system gives conversely,

$$\begin{aligned} X^2 : Y^2 : Z^2 : YZ : ZX : XY &= 4\kappa^2 yz - \lambda^2 x^2 \\ &: 4\kappa^2 zx - \lambda^2 y^2 \\ &: 4\kappa^2 xy - \lambda^2 z^2 \\ &: \lambda^2 yz + 2\kappa\lambda x^2 \\ &: \lambda^2 zx + 2\kappa\lambda y^2 \\ &: \lambda^2 xy + 2\kappa\lambda z^2. \end{aligned}$$

Hence from the identical relation $X^2 . Y^2 - (XY)^2 = 0$, substituting for X^2, XY, Y^2 their values, and throwing out the factor z, we find $(4\kappa^3 - \lambda^3)\, xyz - \kappa\lambda^2 (x^3 + y^3 + z^3) = 0$, and thence, in virtue of the equation $4\kappa^3 - \lambda^3 = -6l\kappa\lambda^2$, we obtain

$$x^3 + y^3 + z^3 + 6lxyz = 0,$$

which shows that the point in question lies on the cubic. We have thus the theorem:

The first or conic polar of a point of the cubic touches the cubic at the point, and meets it besides in four points, which are the angles of a quadrangle the centres of which lie on the cubic. In other words, the quadrangle is an inscribed quadrangle.

40. To find the equations of the three *axes* of the quadrangle, that is of the lines through two centres.

We have

$$\begin{aligned}
&(4\kappa^2\, YZ - \lambda^2 X^2)\, x + (\lambda^2 XY + 2\kappa\lambda Z^2)\, y + (\lambda^2 ZX + 2\kappa\lambda Y^2)\, z = 0,\\
&(\lambda^2 XY + 2\kappa\lambda Z^2)\, x + (4\kappa^2 ZX - \lambda^2 Y^2)\, y + (\lambda^2 YZ + 2\kappa\lambda X^2)\, z = 0,\\
&(\lambda^2 ZX + 2\kappa\lambda Y^2)\, x + (\lambda^2 YZ + 2\kappa\lambda X^2)\, y + (4\kappa^2 XY - \lambda^2 Z^2)\, z = 0;
\end{aligned}$$

or arranging these equations in the proper form and eliminating κ^2, $\kappa\lambda$, λ^2, we find

$$\begin{vmatrix}
YZx, & Z^2y + Y^2z, & X(-Xx + Yy + Zz)\\
ZXy, & X^2z + Z^2x, & Y(\ Xx - Yy + Zz)\\
XYz, & Y^2x + X^2y, & Z(\ Xx + Yy + Zz)
\end{vmatrix} = 0;$$

or, multiplying out,

$$\begin{aligned}
XYZ\,\{&(Z^3 - Y^3)\,x^3 + (X^3 - Z^3)\,y^3 + (Y^3 - X^3)\,z^3\}\\
&+ x^2yZY^2(-2X^3 + Y^3 + Z^3) + zx^2YZ^2(2X^3 - Y^3 - Z^3)\\
&+ y^2zXZ^2(-2Y^3 + Z^3 + X^3) + xy^2ZX^2(2Y^3 - Z^3 - X^3)\\
&+ z^2xYX^2(-2Z^3 + X^3 + Y^3) + yz^2XY^2(2Z^3 - X^3 - Y^3) = 0.
\end{aligned}$$

We may simplify this result by means of the equation $X^3 + Y^3 + Z^3 + 6lXYZ = 0$, so as to make the left-hand side divide out by XYZ: we thus obtain

$$\begin{aligned}
&(Z^3 - Y^3)\,x^3 + (X^3 - Z^3)\,y^3 + (Y^3 - X^3)\,z^3\\
&+ (-3X^2Y - 6lY^2Z)\,x^2y + (-3Y^2Z - 6lZ^2X)\,y^2z + (-3Z^2X - 6lX^2Y)\,z^2x\\
&+ (\ 3XY^2 + 6lX^2Z)\,xy^2 + (\ 3YZ^2 + 6lY^2X)\,yz^2 + (\ 3ZX^2 + 6lZ^2Y)\,zx^2 = 0;
\end{aligned}$$

or in a different form,

$$\begin{aligned}
&(y^3 - z^3)\,X^3 + (z^3 - x^3)\,Y^3 + (x^3 - y^3)\,Z^3\\
&+ (-3x^2y - 6lz^2x)\,X^2Y + (-3y^2z - 6lx^2y)\,Y^2Z + (-3z^2x - 6ly^2z)\,Z^2X\\
&+ (\ 3xy^2 + 6lyz^2)\,XY^2 + (\ 3yz^2 + 6lzx^2)\,YZ^2 + (\ 3zx^2 + 6lxy^2)\,ZX^2 = 0,
\end{aligned}$$

as the equation of the three axes of the quadrangle.

Article No. 41. *Recapitulation of geometrical definitions of the Pippian.*

In conclusion, I will recapitulate the different modes of generation or geometrical definitions of the Pippian, obtained in the course of the present memoir. The curve in question is:

1. The envelope of the line joining a pair of conjugate poles of the cubic (see Nos. 2 and 13).

2. The envelope of each line of the pair forming the first or conic polar with respect to the cubic of a conjugate pole of the cubic (see Nos. 2 and 14).

3. The envelope of a line which is the polar of a conjugate pole of the cubic, with respect to the conic which is the first or conic polar of the other conjugate pole in respect to any syzygetic cubic (see Nos. 2 and 9).

4. The locus of the harmonic with respect to a pair of conjugate poles of the cubic of the third point of intersection with the Hessian of the line joining the two conjugate poles (see Nos. 2 and 17).

5. The envelope of a line such that its lineo-polar envelope with respect to the cubic breaks up into a pair of lines (see No. 24).

6. The envelope of a line which meets three conics, the first or conic polars of any three points in respect to the cubic, in six points in involution (see No. 22).

7. The envelope of the second or line polar with respect to the cubic, of a point the locus of which is a certain curve of the sixth order in quadratic syzygy with the cubic and Hessian, viz. the curve $-S\,.\,U^2+(HU)^2=0$ (see No. 27).

8. The envelope of a line having for its satellite point a point of the Hessian (see No. 35).

9. The envelope of the polar of the satellite point with respect to the Hessian of the tangent at a point of the Hessian, with respect to the first or conic polar of the point of the Hessian in respect to the cubic (see No. 36).

147.

A MEMOIR ON THE SYMMETRIC FUNCTIONS OF THE ROOTS OF AN EQUATION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 489—499. Received December 18, 1856,—Read January 8, 1857.]

THERE are contained in a work, which is not, I think, so generally known as it deserves to be, the "Algebra" of Meyer Hirsch [the work referred to is entitled Sammlung von Beispielen Formeln und Aufgaben aus der Buchstabenrechnung und Algebra, 8vo. Berlin, 1804 (8 ed. 1853), English translation by Ross, 8vo. London, 1827] some very useful tables of the symmetric functions up to the tenth degree of the roots of an equation of any order. It seems desirable to join to these a set of tables, giving reciprocally the expressions of the powers and products of the coefficients in terms of the symmetric functions of the roots. The present memoir contains the two sets of tables, viz. the new tables distinguished by the letter (a), and the tables of Meyer Hirsch distinguished by the letter (b); the memoir contains also some remarks as to the mode of calculation of the new tables, and also as to a peculiar symmetry of the numbers in the tables of each set, a symmetry which, so far as I am aware, has not hitherto been observed, and the existence of which appears to constitute an important theorem in the subject. The theorem in question might, I think, be deduced from a very elegant formula of M. Borchardt (referred to in the sequel), which gives the generating function of any symmetric function of the roots, and contains potentially a method for the calculation of the Tables (b), but which, from the example I have given, would not appear to be a very convenient one for actual calculation.

Suppose in general

$$(1,\ b,\ c\ldots \between 1,\ x)^{\infty} = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots,$$

so that

$$-b = \Sigma\alpha,\quad +c = \Sigma\alpha\beta,\quad -d = \Sigma\alpha\beta\gamma,\ \&c.,$$

and if in general

$$(pqr\ldots) = \Sigma\alpha^p\beta^q\gamma^r\ldots,$$

where as usual the summation extends only to the distinct terms, so that e.g. (p^2) contains only half as many terms as (pq), and so in all similar cases, then we have

$$-b=(1),\quad +c=(1^2),\quad -d=(1^3),\ \&c.;$$

and the two problems which arise are, first to express any combination $b^p c^q \ldots$ in terms of the symmetric functions $(l^x m^y \ldots)$, and secondly, or conversely, to express any symmetric function $(l^x m^y \ldots)$ in terms of the combinations $b^p c^q \ldots$.

It will conduce materially to brevity if $1^p 2^q \ldots$ be termed the partition belonging to the combination $b^p c^q \ldots$; and in like manner if $l^x m^y \ldots$ be termed the partition belonging to the symmetric function $(l^x m^y \ldots)$, and if the sum of the component numbers of the partition is termed the weight.

Consider now a line of combinations corresponding to a given weight, e.g. the weight 4, this will be

$$\begin{array}{lllll} e & bd & c^2 & b^2c & b^4 \quad \text{(line)} \\ 4 & 13 & 2^2 & 1^2 2 & 1^4, \end{array}$$

where I have written under each combination the partition which belongs to it, and in like manner a column of symmetric functions of the same weight, viz.

$$\begin{array}{l} (4) \quad \text{(column)} \\ (31) \\ (2^2) \\ (21^2) \\ (1^4), \end{array}$$

where, as the partitions are obtained by simply omitting the ( ), I have not separately written down the partitions.

It is at once obvious that the different combinations of the line will be made up of numerical multiples of the symmetric functions of the column; and conversely, that the symmetric functions of the column will be made up of numerical multiples of the combinations of the line; but this requires a further examination. There are certain restrictions as to the symmetric functions which enter into the expression of the combination, and conversely, as to the combinations which enter into the expression of the symmetric function. The nature of the first restriction is most clearly seen by the following Table:

Number of Parts.	Greatest Part.	Combinations with their several Partitions.		Contain Multiples of the Symmetric Functions.	Greatest Part does not exceed	Number of Parts not less than
1	4	e	4	(1^4),	1	4
2	3	bd	13	(1^4), (21^2),	2	3
2	2	c^2	2^2	(1^4), (21^2), (2^2),	2	2
3	2	b^2c	$1^2 2$	(1^4), (21^2), (2^2), (31),	3	2
4	1	b^4	1^4	(1^4), (21^2), (2^2), (31), (4)	4	1

Thus, for instance, the combination bd (the partition whereof is 13) contains multiples of the two symmetric functions (1^4), (21^2) only. The number of parts in the partition 13 is 2, and the greatest part is 3. And in the partitions (1^4), (21^2) the greatest part is 2, and the number of parts is not less than 3. The reason is obvious: each term of the developed expression of bd must contain at least as many roots as are contained in each term of d, that is 3 roots, and since the coefficients are linear functions in respect to each root, the combination bd cannot contain a power higher than 2 of any root. The reasoning is immediately applied to any other case, and we obtain

First Restriction.—A combination $b^p c^q \ldots$ contains only those symmetric functions $(l^z m^y \ldots)$, for which the greatest part does not exceed the number of parts in the partition $1^p 2^q \ldots$, and the number of parts is not less than the greatest part in the same partition.

Consider a partition such as $1^2 2$, then replacing each number by a line of units thus,

$$\begin{array}{c} 1 \\ 1 \\ 11, \end{array}$$

and summing the columns, we obtain a new partition 31, which may be called the conjugate[1] of $1^2 2$. It is easy to see that the expression for the combination $b^2 c$ (for which the partition is $1^2 2$) contains with the coefficient unity, the symmetric function (31), the partition whereof is the conjugate of $1^2 2$. In fact $b^2 c = (-\Sigma\alpha)^2 (\Sigma\alpha\beta)$, which obviously contains the term $+1\alpha^2\beta$, and therefore the symmetric function with its coefficient $+1\,(31)$; and the reasoning is general, or

THEOREM. A combination $b^q c^q \ldots$ contains the symmetric function (partition conjugate to $1^p 2^q \ldots$) with the coefficient unity, and sign + or − according as the weight is even or odd.

Imagine the partitions arranged as in the preceding column, viz. first the partition into one part, then the partitions into two parts, then the partitions into three parts, and so on; the partitions into the same number of parts being arranged according to the magnitude of the greatest part (the greatest magnitude first), and in case of equality according to the magnitudes of the next greatest part, and so on (for other examples, see the outside column of any one of the Tables). The order being thus completely defined, we may speak of a partition as being prior or posterior to another. We are now able to state a second restriction as follows.

Second Restriction.—The combination $b^p c^q \ldots$ contains only those symmetric functions which are of the form (partition not prior to the conjugate of $1^p 2^q \ldots$).

The terms excluded by the two restrictions are many of them the same, and it might at first sight appear as if the two restrictions were identical; but this is not

[1] The notion of Conjugate Partitions is, I believe, due to Professor Sylvester or Mr Ferrers. [It was due to Mr now Dr Ferrers.]

so: for instance, for the combination bd^2, see Table VII (a), the term (41^3) is excluded by the first restriction, but not by the second; and on the other hand, the term (3^21), which is not excluded by the first restriction, is excluded by the second restriction, as containing a partition 3^21 prior in order to 32^2, which is the partition conjugate to 13^2, the partition of bd^2. It is easy to see why bd^2 does not contain the symmetric function (3^21); in fact, a term of (3^21) is $(\alpha^3\beta^2\gamma)$, which is obviously not a term of $bd^3 = (-\Sigma\alpha)(\Sigma\alpha\beta\gamma)^3$; but I have not investigated the general proof.

I proceed to explain the construction of the Tables (a). The outside column contains the symmetric functions arranged in the order before explained; the outside or top line contains the combinations of the same weight arranged as follows, viz. the partitions taken in order from right to left are respectively conjugate to the partitions in the outside column, taken in order from top to bottom; in other words, each square of the sinister diagonal corresponds to two partitions which are conjugate to each other. It is to be noticed that the combinations taken in order, from left to right, are *not* in the order in which they would be obtained by Arbogast's Method of Derivations from an operand a^x, a being ultimately replaced by unity. The squares above the sinister diagonal are empty (i.e. the coefficients are zero), the greater part of them in virtue of both restrictions, and the remainder in virtue of the second restriction; the empty squares below the sinister diagonal are empty in virtue of the second restriction; but the property was not assumed in the calculation.

The greater part of the numbers in the Tables (a) were calculated, those of each table from the numbers in the next preceding table by the following method, depending on the derivation of the expression for $b^{p+1}c^q\ldots$ from the expression for $b^pc^q\ldots$ Suppose, for example, the column cd of Table V (a) is known, and we wish to calculate the column bcd of Table VI (a). The process is as follows:

Given

2^21	21^3	1^5
1	3	10

we obtain

321	2^3	31^3	2^21^2	21^4	1^6
1	3		2		
		3	6	12	
				10	60
1	3	3	8	22	60

where the numbers in the last line are the numbers in the column bcd of Table VI (a). The partition 2^21, considered as containing a part zero, gives, when the parts are successively increased by 1, the partitions 321, 2^3, 2^21^2, in which the indices of the increased part (i.e. the original part plus unity) are 1, 3, 2; these numbers are taken as multipliers of the coefficient 1 of the partition 2^21, and we thus have the new coefficients 1, 3, 2 of the partitions 321, 2^3, 2^21^2. In like manner the coefficient 3 of

the partition 21^3 gives the new coefficients 3, 6, 12 of the partitions 31^3, 2^21^2, 21^4, and the coefficient 10 of the partition 1^5 gives the new coefficients 10, 60 of the partitions 21^4 and 1^6, and finally, the last line is obtained by addition. The process in fact amounts to the multiplication separately of each term of $cd =$

$$1\,(2^21) + 3\,(21^3) + 10\,(1^5)$$

by $b = (1)$. It would perhaps have been proper to employ an analogous rule for the calculation of the combinations $c^2d^r\ldots$ not containing b, but instead of doing so I availed myself of the existing Tables (b). But the comparison of the last line of each Table (a) (which as corresponding to a combination b^p was always calculated independently of the Tables (b)) with such last line as calculated from the corresponding Table (b), seems to afford a complete verification of both the Tables; and my process has in fact enabled me to detect several numerical errors in the Tables (b), as given in the English translation of the work above referred to. It is not desirable, as regards facility of calculation and independently of the want of verification, to calculate either set of Tables wholly from the other; the rules for the independent calculation of the Tables (b) are fully and clearly explained in the work referred to, and I have nothing to add upon this subject.

The relation of symmetry, alluded to in the introductory paragraph of the present memoir, exists in each Table of either set, and is as follows: viz. the number in the Table corresponding to any two partitions in the outside column and the outside line respectively, is equal to the number corresponding to the same two partitions in the outside line and the outside column respectively. Or, calling the two partitions P, Q, and writing for shortness, combination (P) for the combination represented by the partition P, and for greater clearness, symmetric function (P) (instead of merely (P)) to denote the symmetric function represented by the partition P, we have the following two theorems, viz.

THEOREM. The coefficient in combination (P) of symmetric function (Q) is equal to the coefficient in combination (Q) of symmetric function (P);

and conversely,

THEOREM. The coefficient in symmetric function (P) of combination (Q) is equal to the coefficient in symmetric function (Q) of combination (P).

M. Borchardt's formula, before referred to, is given in the 'Monatsbericht' of the Berlin Academy (March 5, 1885)[1], and may be thus stated; viz. considering the case of n roots, write

$$(1,\ b,\ c, \ldots k\,\&\,1,\ x)^n = (1-\alpha x)(1-\beta x)\ldots(1-\kappa x) = fx,$$

then

$$\Sigma\left(\frac{1}{1-\alpha x}\,\frac{1}{1-\beta y}\cdots\frac{1}{1-\kappa u}\right) = \frac{1}{k}(-)^n\,\frac{fxfy\ldots fu}{\Pi(x, y, \ldots u)}\,\frac{d}{dx}\,\frac{d}{dy}\cdots\frac{d}{du}\,\frac{\Pi(x, y, \ldots u)}{fxfy\ldots fu},$$

[1] And in *Crelle*, t. liii. p. 195.—Note added 4th Dec. 1857, A. C.

where $\Pi\ (x, y, \ldots u)$ denotes the product of the differences of the quantities $x, y, \ldots u$, and on the left-hand side the summation extends to all the different permutations of $\alpha, \beta, \ldots \kappa$, or what is the same thing, of $x, y, \ldots u$.

Suppose for a moment that there are only two roots, so that

$$(1, b, c \between 1, x)^2 = (1-\alpha x)(1-\beta x),$$

then the left-hand side is

$$\frac{1}{(1-\alpha x)(1-\beta y)} + \frac{1}{(1-\alpha y)(1-\beta x)},$$

which is equal to

$$2 + (\alpha+\beta)(x+y) + (\alpha^2+\beta^2)(x^2+y^2) + 2\alpha\beta xy + (\alpha^3+\beta^3)(x^3+y^3) + (\alpha^2\beta+\alpha\beta^2)(x^2y+xy^2) + \&c.,$$

and the right-hand side is

$$\frac{1}{c} \cdot \frac{fxfy}{x-y} \frac{d}{dx} \frac{d}{dy} \frac{x-y}{fxfy},$$

which is equal to

$$\frac{1}{c} \frac{fxfy}{x-y} \left\{ \frac{f'xfy - f'yfx + (x-y) f'xf'y}{(fx)^2 (fy)^2} \right\},$$

and therefore to

$$\frac{1}{c} \cdot \frac{1}{fxfy} \left\{ \frac{f'xfy - f'yfx}{x-y} + f'xf'y \right\};$$

or substituting for fx, fy their values,

$$\frac{f'xfy - f'yfx}{x-y}$$

becomes equal to

$$2c - b^2 - bc(x+y) - 2c^2xy,$$

and $f'xf'y$ is equal to

$$b^2 + 2bc(x+y) + 4c^2xy.$$

The right-hand side is therefore equal to

$$\frac{2 + b(x+y) + 2cxy}{(1+bx+cx^2)(1+by+cy^2)};$$

and comparing with the value of the left-hand side, we see that this expression may be considered as the generating function of the symmetric functions of (α, β), viz. the expression in question is developable in a series of the symmetric functions of (x, y), the coefficients being of course functions of b and c, and these coefficients are (to given numerical factors *près*) the symmetric functions of the roots (α, β).

And in general it is easy to see that the left-hand side of M. Borchardt's formula is equal to

$$[0]+[1](1)(1)'+[2](2)(2)'+[1^2](1^2)(1^2)'+\&c.,$$

where (1), (2), (1^2), &c. are the symmetric functions of the roots $(\alpha, \beta, \ldots \kappa)$, $(1)'$, $(2)'$, $(1^2)'$, &c. are the corresponding symmetric functions of $(x, y, \ldots u)$, and [0], [1], [2], $[1^2]$, &c. are mere numerical coefficients; viz. [0] is equal to $1.2.3\ldots n$, and [1], [2], $[1^2]$, &c. are such that the product of one of these factors into the number of terms in the corresponding symmetric function (1), (2), (1^2), &c. may be equal to $1.2.3\ldots n$. The right-hand side of M. Borchardt's formula is therefore, as in the particular case, the generating function of the symmetric functions of the roots $(\alpha, \beta, \ldots \kappa)$, and if a convenient expression of such right-hand side could be obtained, we might by means of it express all the symmetric functions of the roots in terms of the coefficients.

Tables relating to the Symmetric Functions of the Roots of an Equation.

The outside line of letters contains the combinations (powers and products) of the coefficients, the coefficients being all with the positive sign, and the coefficient of the highest power being unity; thus in the case of a cubic equation the equation is

$$x^3+bx^2+cx+d=(x-\alpha)(x-\beta)(x-\gamma)=0.$$

The outside line of numbers is obtained from that of letters merely by writing 1, 2, 3... for b, c, d..., and may be considered simply as a different notation for the combinations. The outside column contains the different symmetric functions in the notation above explained, viz. (1) denotes $\Sigma\alpha$, (2) denotes $\Sigma\alpha^2$, (1^2) denotes $\Sigma\alpha\beta$, and so on. The Tables (a) are to be read according to the columns; thus Table II (a) means $b^2=1(2)+2(1)^2$, $c=(1^2)$. The Tables (b) are to be read according to the lines; thus Table II (b) means $(2)=-2c+1b^2$, $(1^2)=+1c$.

I (a).

	1
	b
(1)	-1

II (a).

	2	1^2
	c	b^2
(2)		$+1$
(1^2)	$+1$	$+2$

III (a).

	3	12	1^3
	d	bc	b^3
(3)			-1
(21)		-1	-3
(1^3)	-1	-3	-6

I (b).

	1
	b
(1)	-1

II (b).

	2	1^2
	c	b^2
(2)	-2	$+1$
(1^2)	$+1$	

III (b).

	3	12	1^3
	d	bc	b^3
(3)	-3	$+3$	-1
(21)	$+3$	-1	
(1^3)	-1		

IV (a).

‖	4 e	13 bd	2^2 c^2	$1^2 2$ b^2c	1^4 b^4
(4)					+ 1
(31)				+ 1	+ 4
(2^2)			+ 1	+ 2	+ 6
(21^2)		+ 1	+ 2	+ 5	+ 12
(1^4)	+ 1	+ 4	+ 6	+ 12	+ 24

IV (b).

=	4 e	13 bd	2^2 c^2	$1^2 2$ b^2c	1^4 b^4
(4)	− 4	+ 4	+ 2	− 4	+ 1
(31)	+ 4	− 1	− 2	+ 1	
(2^2)	+ 2	− 2	+ 1		
(21^2)	− 4	+ 1			
(1^4)	+ 1				

V (a).

‖	5 f	14 be	23 cd	$1^2 3$ b^2d	$1^2 2$ bc^2	$1^3 2$ b^3c	1^5 b^5
(5)							− 1
(41)						− 1	− 5
(32)					− 1	− 3	− 10
(31^2)				− 1	− 2	− 7	− 20
($2^2 1$)			− 1	− 2	− 5	− 12	− 30
(21^3)		− 1	− 3	− 7	− 12	− 27	− 60
(1^5)	− 1	− 5	− 10	− 20	− 30	− 60	− 120

V (b).

=	5 f	14 be	23 cd	$1^2 3$ b^2d	$1^2 2$ bc^2	$1^3 2$ b^3c	1^5 b^5
(5)	− 5	+ 5	+ 5	− 5	− 5	+ 5	− 1
(41)	+ 5	− 1	− 5	+ 1	+ 3	− 1	
(32)	+ 5	− 5	+ 1	+ 2	− 1		
(31^2)	− 5	+ 1	+ 2	− 1			
($2^2 1$)	− 5	+ 3	− 1				
(21^3)	+ 5	− 1					
(1^5)	− 1						

VI (a).

‖	6 g	15 bf	24 ce	$1^2 4$ b^2e	3^2 d^2	123 bcd	$1^3 3$ b^3d	2^3 c^3	$1^2 2^2$ b^2c^2	$1^4 2$ b^4c	1^6 b^6
(6)											+ 1
(51)										+ 1	+ 6
(42)									+ 1	+ 4	+ 15
(3^2)								+ 1	+ 2	+ 6	+ 20
(41^2)							+ 1	...	+ 2	+ 9	+ 30
(321)						+ 1	+ 3	+ 3	+ 8	+ 22	+ 60
(2^3)					+ 1	+ 3	+ 6	+ 6	+ 15	+ 36	+ 90
(31^3)				+ 1	...	+ 3	+ 10	+ 6	+ 18	+ 48	+ 120
($2^2 1^2$)			+ 1	+ 2	+ 2	+ 8	+ 18	+ 15	+ 34	+ 78	+ 180
(21^4)		+ 1	+ 4	+ 9	+ 6	+ 22	+ 48	+ 36	+ 78	+ 168	+ 360
(1^6)	+ 1	+ 6	+ 15	+ 30	+ 20	+ 60	+ 120	+ 90	+ 180	+ 360	+ 720

VI (b).

=	6 g	15 bf	24 ce	$1^2 4$ b^2c	3^2 d^2	123 bcd	$1^3 3$ b^3d	2^3 c^3	$1^2 2^2$ b^2c^2	$1^4 2$ b^4c	1^6 b^6
(6)	− 6	+ 6	+ 6	− 6	+ 3	− 12	+ 6	− 2	+ 9	− 6	+ 1
(51)	+ 6	− 1	− 6	+ 1	− 3	+ 7	− 1	+ 2	− 4	+ 1	
(42)	+ 6	− 6	+ 2	+ 2	− 3	+ 4	− 2	− 2	+ 1		
(3^2)	+ 3	− 3	− 3	+ 3	+ 3	− 3	...	+ 1			
(41^2)	− 6	+ 1	+ 2	− 1	+ 3	− 3	+ 1				
(321)	− 12	+ 7	+ 4	− 3	− 3	+ 1					
(2^3)	− 2	+ 2	− 2	...	+ 1						
(31^3)	+ 6	− 1	− 2	+ 1							
($2^2 1^2$)	+ 9	− 4	+ 1								
(21^4)	− 6	+ 1									
(1^6)	+ 1										

VII (*a*).

$\parallel$	7 h	16 bg	25 cf	1^25 b^2f	34 de	124 bce	1^34 b^3e	12^2 bd^2	2^23 c^2d	1^223 b^2cd	1^43 b^4d	12^3 bc^3	1^32^2 b^3c^2	1^52 b^5c	1^7 b^7
(7)															− 1
(61)														− 1	− 7
(52)													− 1	− 5	− 21
(43)												− 1	− 3	− 10	− 35
(51^2)											− 1	...	− 2	− 11	− 42
421)										− 1	− 4	− 3	− 11	− 35	− 105
(3^21)									− 1	− 2	− 6	− 7	− 18	− 50	− 140
(32^2)								− 1	− 2	− 5	− 12	− 12	− 31	− 80	− 210
(41^3)							− 1	...	...	− 3	− 13	− 6	− 24	− 75	− 210
(21^2)						− 1	− 3	− 2	− 5	− 13	− 34	− 27	− 68	− 170	− 420
(2^21)					− 1	− 3	− 6	− 7	− 12	− 27	− 60	− 51	− 117	− 270	− 630
$31^4)$				− 1	...	− 4	− 13	− 6	− 12	− 34	− 88	− 60	− 150	− 360	− 840
$2^21^3)$			− 1	− 2	− 3	− 11	− 24	− 18	− 31	− 68	− 150	− 117	− 258	− 570	− 1260
$21^5)$		− 1	− 5	− 11	− 10	− 35	− 75	− 50	− 80	− 170	− 360	− 270	− 570	− 1200	− 2520
(1^7)	− 1	− 7	− 21	− 42	− 35	− 105	− 210	− 140	− 210	− 420	− 840	− 630	− 1260	− 2520	− 5040

VII (*b*).

=	7 h	16 bg	25 cf	1^25 b^2f	34 de	124 bce	1^34 b^3e	12^2 bd^2	2^23 c^2d	1^223 b^2cd	1^43 b^4d	12^3 bc^3	1^32^2 b^3c^2	1^52 b^5e	1^7 b^7
(7)	− 7	+ 7	+ 7	− 7	+ 7	− 14	+ 7	− 7	− 7	+ 21	− 7	+ 7	− 14	+ 7	− 1
(61)	+ 7	− 1	− 7	+ 1	− 7	+ 8	− 1	+ 4	+ 7	− 9	+ 1	− 5	+ 5	− 1	
(52)	+ 7	− 7	+ 3	+ 2	− 7	+ 4	− 2	+ 7	− 3	− 6	+ 2	+ 3	− 1		
(43)	+ 7	− 7	− 7	+ 7	+ 5	+ 2	− 3	− 5	+ 1	+ 3	...	− 1			
(51^2)	− 7	+ 1	+ 2	− 1	+ 7	− 3	+ 1	− 4	− 2	+ 4	− 1				
(421)	− 14	+ 8	+ 4	− 3	+ 2	− 8	+ 3	+ 1	+ 2	− 1					
(3^21)	− 7	+ 4	+ 7	− 4	− 5	+ 1	...	+ 2	− 1						
(32^2)	− 7	+ 7	− 3	− 2	+ 1	+ 2	...	− 1							
(41^3)	+ 7	− 1	− 2	+ 1	− 3	+ 3	− 1								
(321^2)	+ 21	− 9	− 6	+ 4	+ 3	− 1									
(2^31)	+ 7	− 5	+ 3	...	− 1										
(31^4)	− 7	+ 1	+ 2	− 1											
(2^21^3)	− 14	+ 5	− 1												
(21^5)	+ 7	− 1													
(1^7)	− 1														

VIII (a). Runs on infrà.

‖	8 i	17 bh	26 cg	1^26 b^2g	35 df	125 bcf	1^25 b^2f	4^2 e^2	134 bde	2^24 c^2e	1^224 b^2ce	1^44 b^4e	23^2 cd^2
(8)													
(71)													
(62)													
(53)													
(4^2)													
(61^2)													
(521)													
(431)													
(42^2)													
(3^22)													+ 1
(51^3)												+ 1	...
(421^3)											+ 1	+ 4	...
(3^21^2)										+ 1	+ 2	+ 6	+ 2
(32^21)									+ 1	+ 2	+ 5	+ 12	+ 5
(2^4)								+ 1	+ 4	+ 6	+ 12	+ 24	+ 12
(41^4)							+ 1	...	...	...	+ 4	+ 17	...
(321^3)						+ 1	+ 3	...	+ 3	+ 7	+ 18	+ 46	+ 12
(2^21^2)					+ 1	+ 3	+ 6	+ 2	+ 11	+ 18	+ 39	+ 84	+ 31
(31^5)				+ 1	...	+ 5	+ 16	...	+ 10	+ 20	+ 55	+ 140	+ 30
(2^21^4)			+ 1	+ 2	+ 4	+ 14	+ 30	+ 6	+ 32	+ 53	+ 114	+ 246	+ 80
(21^6)		+ 1	+ 6	+ 13	+ 15	+ 51	+ 108	+ 20	+ 95	+ 150	+ 315	+ 660	+ 210
(1^8)	+ 1	+ 8	+ 28	+ 56	+ 56	+ 168	+ 336	+ 70	+ 280	+ 420	+ 840	+ 1680	+ 560

‖	1^23^2 b^2d^2	12^23 bc^2d	1^323 b^3cd	1^33 b^3d	2^4 c^4	1^22^3 b^2c^3	1^42^2 b^4c^2	1^62 b^6c	1^8 b^8
(8)									+ 1
(71)								+ 1	+ 8
(62)							+ 1	+ 6	+ 28
(53)						+ 1	+ 4	+ 15	+ 56
(4^2)					+ 1	+ 2	+ 6	+ 20	+ 70
(61^2)				+ 1	...	...	+ 2	+ 13	+ 56
(521)			+ 1	+ 5	...	+ 3	+ 14	+ 51	+ 168
(431)		+ 1	+ 3	+ 10	+ 4	+ 11	+ 32	+ 95	+ 280
(42^2)	+ 1	+ 2	+ 7	+ 20	+ 6	+ 18	+ 53	+ 150	+ 420
(3^22)	+ 2	+ 5	+ 12	+ 30	+ 12	+ 31	+ 80	+ 210	+ 560
(51^3)	...	...	+ 3	+ 16	...	+ 6	+ 30	+ 108	+ 336
(421^3)	+ 2	+ 5	+ 18	+ 55	+ 12	+ 39	+ 114	+ 315	+ 840
(3^21^2)	+ 4	+ 12	+ 30	+ 80	+ 28	+ 68	+ 172	+ 440	+ 1120
(32^21)	+ 12	+ 24	+ 58	+ 140	+ 48	+ 117	+ 284	+ 690	+ 1680
(2^4)	+ 28	+ 48	+ 108	+ 240	+ 90	+ 204	+ 468	+ 1080	+ 2520
(41^4)	+ 6	+ 12	+ 46	+ 140	+ 24	+ 84	+ 246	+ 660	+ 1680
(321^3)	+ 30	+ 58	+ 141	+ 340	+ 108	+ 258	+ 612	+ 1440	+ 3360
(2^21^2)	+ 68	+ 117	+ 258	+ 570	+ 204	+ 453	+ 1008	+ 2250	+ 5040
(31^5)	+ 80	+ 140	+ 340	+ 800	+ 240	+ 570	+ 1320	+ 3000	+ 6720
(2^21^4)	+ 172	+ 284	+ 612	+ 1320	+ 468	+ 1008	+ 2172	+ 4680	+ 10080
(21^6)	+ 440	+ 690	+ 1440	+ 3000	+ 1080	+ 2250	+ 4680	+ 9720	+ 20160
(1^8)	+ 1120	+ 1680	+ 3360	+ 6720	+ 2520	+ 5040	+ 10080	+ 20160	+ 40320

VIII (*b*). Runs on infrà.

=	8 i	17 bh	26 cg	1^26 b^2g	35 df	125 bcf	1^35 b^2f	4^2 e^2	134 bde	2^24 c^2e	1^224 b^2ce	1^14 b^4e	23^2 cd^2
(8)	− 8	+ 8	+ 8	− 8	+ 8	− 16	+ 8	+ 4	− 16	− 8	+ 24	− 8	− 8
(71)	+ 8	− 1	− 8	+ 1	− 8	+ 9	− 1	− 4	+ 9	+ 8	− 10	+ 1	+ 8
(62)	+ 8	− 8	+ 4	+ 2	− 8	+ 4	− 2	− 4	+ 16	− 4	− 6	+ 2	+ 2
(53)	+ 8	− 8	− 8	+ 8	+ 7	+ 1	− 3	− 4	+ 1	+ 8	− 9	+ 3	− 7
(4^2)	+ 4	− 4	− 4	+ 4	− 4	+ 8	− 4	+ 6	− 8	− 4	+ 4	...	+ 4
(61^2)	− 8	+ 1	+ 2	− 1	+ 8	− 3	+ 1	+ 4	− 9	− 2	+ 4	− 1	− 5
(521)	− 16	+ 9	+ 4	− 3	+ 1	− 8	+ 3	+ 8	− 10	− 4	+ 11	− 3	+ 5
(431)	− 16	+ 9	+ 16	− 9	+ 1	− 10	+ 4	− 8	+ 10	...	− 1	...	− 1
(42^2)	− 8	+ 8	− 4	− 2	+ 8	− 4	+ 2	− 4	...	+ 4	− 2	...	− 2
(3^22)	− 8	+ 8	+ 2	− 5	− 7	+ 5	...	+ 4	− 1	− 2	...	...	+ 1
(51^3)	+ 8	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+ 1	
(421^3)	+ 24	− 10	− 6	+ 4	− 9	+ 11	− 4	+ 4	− 1	− 2	+ 1		
(3^21^2)	+ 12	− 5	− 9	+ 5	+ 3	− 1	...	+ 2	− 2	+ 1			
(32^21)	+ 24	− 17	...	+ 5	+ 6	− 3	...	− 4	+ 1				
(2^4)	+ 2	− 2	+ 2	...	− 2	...	...	+ 1					
(41^4)	− 8	+ 1	+ 2	− 1	+ 3	− 3	+ 1						
(321^3)	− 32	+ 11	+ 8	− 5	− 3	+ 1							
(2^21^2)	− 16	+ 9	− 4	...	+ 1								
(31^5)	+ 8	− 1	− 2	+ 1									
(2^21^4)	+ 20	− 6	+ 1										
(21^6)	− 8	+ 1											
(1^8)	+ 1												

=	1^23^2 b^2d^2	12^23 bc^2d	1^323 b^3cd	1^53 b^5d	2^4 c^4	1^22^3 b^2c^3	1^42^2 b^4c^2	1^62 b^6c	1^8 b^8
(8)	+ 12	+ 24	− 32	+ 8	+ 2	− 16	+ 20	− 8	+ 1
(71)	− 5	− 17	+ 11	− 1	− 2	+ 9	− 6	+ 1	
(62)	− 9	...	+ 8	− 2	+ 2	− 4	+ 1		
(53)	+ 3	+ 6	− 3	...	− 2	+ 1			
(4^2)	+ 2	− 4	...	...	+ 1				
(51^3)	+ 5	+ 5	− 5	+ 1					
(521)	− 1	− 3	+ 1						
(431)	− 2	+ 1							
(42^2)	+ 1								

IX (a). Runs on to p. 430.

‖	9 j	18 bi	27 ch	1^27 b^3h	35 dg	125 bcg	1^36 b^3g	45 ef	135 bdf	2^25 c^2f	1^225 b^2cf	1^45 b^4f	14^2 be^2
(9)													
(81)													
(72)													
(63)													
(54)													
(71^2)													
(621)													
(531)													
(4^21)													
(52^2)													
(432)													
(3^3)													
(61^3)													
(521^2)													
(431^2)													
(42^21)													
(3^221)													
(32^3)													-1
(51^4)												-1	...
(421^3)											-1	-4	...
(3^21^3)										-1	-2	-6	...
(32^21^2)									-1	-2	-5	-12	-2
(2^41)								-1	-4	-6	-12	-24	-9
(41^5)							-1	...	...	...	-5	-21	...
(321^4)						-1	-3	...	-4	-9	-23	-58	-6
(2^31^3)					-1	-3	-6	-3	-15	-24	-51	-108	-24
(31^6)				-1	...	-6	-19	...	-15	-30	-81	-204	-20
(2^21^5)			-1	-2	-5	-17	-36	-10	-50	-81	-172	-366	-70
(21^7)		-1	-7	-15	-21	-70	-147	-35	-161	-252	-525	-1092	-210
(1^8)	-1	-9	-36	-72	-84	-252	-504	-126	-504	-756	-1512	-3024	-630

‖	234 cde	$1^2 34$ b^2de	$12^2 4$ bc^2e	$1^3 24$ b^3ce	$1^5 4$ b^5e	3^3 d^3	123^2 bcd^2	$1^2 3^2$ b^3d^2	$2^3 3$ c^3d	$1^2 2^2 3$ b^2c^2d
(9)										
(81)										
(72)										
(63)										
(54)										
(71^2)										
(621)										
(531)										-1
$(4^2 1)$									-1	-2
(52^2)								-1	...	-2
(432)							-1	-3	-3	-8
(3^3)						-1	-3	-6	-6	-15
(61^3)					-1	...	...	...	...	...
(521^2)				-1	-5	...	...	-2	...	-5
(431^2)			-1	-3	-10	...	-2	-6	-7	-19
$(42^2 1)$		-1	-2	-7	-20	...	-5	-17	-12	-36
$(3^2 21)$	-1	-2	-5	-12	-30	-3	-13	-30	-27	-65
(32^3)	-3	-7	-12	-27	-60	-6	-27	-64	-51	-120
(51^4)	...	...	...	-4	-21	...	...	-6	...	-12
(421^3)	...	-3	-7	-25	-75	...	-12	-42	-27	-85
$(3^2 1^3)$	-3	-6	-17	-42	-110	-6	-30	-72	-64	-152
$(32^2 1^2)$	-8	-19	-36	-85	-200	-15	-65	-152	-120	-281
$(2^4 1)$	-22	-48	-78	-168	-360	-36	-136	-300	-234	-516
(41^5)	...	-10	-20	-75	-225	...	-30	-110	-60	-200
(321^4)	-22	-54	-101	-241	-570	-36	-158	-372	-282	-656
$(2^3 1^3)$	-60	-129	-213	-459	-990	-93	-333	-720	-555	-1203
(31^6)	-60	-155	-270	-645	-1500	-90	-390	-920	-660	-1530
$(2^2 1^5)$	-165	-350	-565	-1200	-2550	-240	-820	-1740	-1320	-2800
(21^7)	-455	-945	-1470	-3045	-6300	-630	-2030	-4200	-3150	-6510
(1^9)	-1260	-2520	-3780	-7560	-15120	-1680	-5040	-10080	-7560	-15120

‖	$1^4 23$ b^4cd	$1^6 3$ b^6d	12^4 bc^4	$1^3 2^3$ b^3c^3	$1^5 2^2$ b^5c^2	$1^7 2$ b^7c	1^9 b^9
(9)							− 1
(81)						− 1	− 9
(72)					− 1	− 7	− 36
(63)				− 1	− 5	− 21	− 84
(54)			− 1	− 3	− 10	− 35	− 126
(71^2)		− 1	...	...	− 2	− 15	− 72
(621)	− 1	− 6	...	− 3	− 17	− 70	− 252
(531)	− 4	− 15	− 4	− 15	− 50	− 161	− 504
$(4^2 1)$	− 6	− 20	− 9	− 24	− 70	− 210	− 630
(52^2)	− 9	− 30	− 6	− 24	− 81	− 252	− 756
(432)	− 22	− 60	− 22	− 60	− 165	− 455	− 1260
(3^3)	− 36	− 90	− 36	− 93	− 240	− 630	− 1680
(61^3)	− 3	− 19	...	− 6	− 36	− 147	− 504
(521^2)	− 23	− 81	− 12	− 51	− 172	− 525	− 1512
(431^2)	− 54	− 155	− 48	− 129	− 350	− 945	− 2520
$(42^2 1)$	− 101	− 270	− 78	− 213	− 565	− 1470	− 3780
$(3^2 21)$	− 158	− 390	− 136	− 333	− 820	− 2030	− 5040
(32^3)	− 282	− 660	− 234	− 555	− 1320	− 3150	− 7560
(51^4)	− 58	− 204	− 24	− 108	− 366	− 1092	− 3024
(421^3)	− 241	− 645	− 168	− 459	− 1200	− 3045	− 7560
$(3^2 1^3)$	− 372	− 920	− 300	− 720	− 1740	− 4200	− 10080
$(32^2 1^2)$	− 656	− 1530	− 516	− 1203	− 2800	− 6510	− 15120
$(2^4 1)$	− 1140	− 2520	− 906	− 2016	− 4500	− 10080	− 22680
(41^5)	− 570	− 1500	− 360	− 990	− 2550	− 6300	− 15120
(321^4)	− 1516	− 3480	− 1140	− 2610	− 5940	− 13440	− 30240
$(2^3 1^3)$	− 2610	− 5670	− 2016	− 4383	− 9540	− 20790	− 45360
(31^6)	− 3480	− 7800	− 2520	− 5670	− 12600	− 27720	− 60480
$(2^2 1^5)$	− 5940	− 12600	− 4500	− 9540	− 20220	− 42840	− 90720
(21^7)	− 13440	− 27720	− 10080	− 20790	− 42840	− 88200	− 181440
(1^9)	− 30240	− 60480	− 22680	− 45360	− 90720	− 181440	− 362880

IX (*b*). Runs on to p. 432.

=	9 *j*	18 *bi*	27 *ch*	1^27 b^2h	35 *dg*	125 *bcg*	1^36 b^3g	45 *ef*	135 *bdf*	2^25 c^2f	1^225 b^2cf	1^45 b^4f	14^2 be^2	234 *cde*	1^234 b^2de
(9)	− 9	+ 9	+ 9	− 9	+ 9	− 18	+ 9	+ 9	− 18	− 9	+ 27	− 9	− 9	− 18	+ 27
(81)	+ 9	− 1	− 9	+ 1	− 9	+ 10	− 1	− 9	+ 10	+ 9	− 11	+ 1	+ 5	+ 18	− 11
(72)	+ 9	− 9	+ 5	+ 2	− 9	+ 4	− 2	− 9	+ 18	− 5	− 6	+ 2	+ 9	+ 4	− 20
(63)	+ 9	− 9	− 9	+ 9	+ 9	...	− 3	− 9	...	+ 9	− 9	+ 3	+ 9	...	− 9
(54)	+ 9	− 9	− 9	+ 9	− 9	+ 18	− 9	+ 11	− 2	− 1	− 7	+ 4	− 11	− 2	+ 13
(71^2)	− 9	+ 1	+ 2	− 1	+ 9	− 3	+ 1	+ 9	− 10	− 2	+ 4	− 1	− 5	− 11	+ 11
(621)	− 18	+ 10	+ 4	− 3	...	− 8	+ 3	+ 18	− 10	− 4	+ 11	− 3	− 14	− 4	+ 13
(531)	− 18	+ 10	+ 18	− 10	...	− 10	+ 4	− 2	− 5	− 8	+ 15	− 4	+ 6	+ 2	− 5
(4^21)	− 9	+ 5	+ 9	− 5	+ 9	− 14	+ 5	− 11	+ 6	+ 1	− 1	...	+ 5	+ 2	− 5
(52^2)	− 9	+ 9	− 5	− 2	+ 9	− 4	+ 2	− 1	− 8	...	+ 6	− 2	+ 1	+ 6	...
(432)	− 18	+ 18	+ 4	− 11	...	− 4	+ 5	− 2	+ 2	+ 6	− 5	...	+ 2	− 8	+ 1
(3^3)	− 3	+ 3	+ 3	− 3	− 6	+ 3	...	+ 3	+ 3	− 3	...	...	− 3	+ 3	...
(61^3)	+ 9	− 1	− 2	+ 1	− 3	+ 3	− 1	− 9	+ 4	+ 2	− 4	+ 1	+ 5	+ 5	− 5
(521^2)	+ 27	− 11	− 6	+ 4	− 9	+ 11	− 4	− 7	+ 15	+ 6	− 15	+ 4	− 1	− 5	+ 1
(431^2)	+ 27	− 11	− 20	+ 11	− 9	+ 13	− 5	+ 13	− 5	...	+ 1	...	− 5	+ 1	+ 2
(42^21)	+ 27	− 19	+ 1	+ 5	− 9	+ 12	− 5	+ 3	− 2	− 6	+ 3	...	+ 1	+ 2	− 1
(3^221)	+ 27	− 19	− 13	+ 12	+ 18	− 7	...	− 7	− 4	+ 3	...	...	+ 3	− 1	
(32^2)	+ 9	− 9	+ 5	+ 2	− 3	− 2	...	+ 1	+ 2	...	...	...	− 1		
(51^4)	− 9	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1			
(421^3)	− 36	+ 12	+ 8	− 5	+ 12	− 14	+ 5	− 4	+ 1	+ 2	− 1				
(3^21^3)	− 18	+ 6	+ 11	− 6	− 3	+ 1	...	− 2	+ 2	− 1					
(32^21^2)	− 54	+ 30	+ 5	− 9	− 9	+ 4	...	+ 4	− 1						
(2^41)	− 9	+ 7	− 5	...	+ 3	...	...	− 1							
(41^5)	+ 9	− 1	− 2	+ 1	− 3	+ 3	− 1								
(321^4)	+ 45	− 13	− 10	+ 6	+ 3	− 1									
(2^31^3)	+ 30	− 14	+ 5	...	− 1										
(31^6)	− 9	+ 1	+ 2	− 1											
(2^21^5)	− 27	+ 7	− 1												
(21^7)	+ 9	− 1													
(1^9)	− 1														

=	$1^2 2^4$ bc^5e	$1^3 2^4$ b^3ce	$1^5 4$ b^5e	3^3 d^3	$1 2^3 3^2$ bcd^2	$1^3 3^3$ b^3d^2	$2^3 3$ c^3d	$1^2 2^2 3$ b^2c^2d	$1^4 2 3$ b^4cd	$1^6 3$ b^6d	$1^2 4$ bc^4	$1^3 2^3$ b^3c^3	$1^5 2^2$ b^5c^2	$1^7 2$ b^7c	1^9 b^9
(9)	+ 27	− 36	+ 9	− 3	+ 27	− 18	+ 9	− 54	+ 45	− 9	− 9	+ 30	− 27	+ 9	− 1
(81)	− 19	+ 12	− 1	+ 3	− 19	+ 6	− 9	+ 30	− 13	+ 1	+ 7	− 14	+ 7	− 1	
(72)	+ 1	+ 8	− 2	+ 3	− 13	+ 11	+ 5	+ 5	− 10	+ 2	− 5	+ 5	− 1		
(63)	− 9	+ 12	− 3	− 6	+ 18	− 3	− 3	− 9	+ 3	...	+ 3	− 1			
(54)	+ 3	− 4	...	+ 3	− 7	− 2	+ 1	+ 4	...	...	− 1				
(71^2)	+ 5	− 5	+ 1	− 3	+ 12	− 6	+ 2	− 9	+ 6	− 1					
(621)	+ 12	− 14	+ 3	+ 3	− 7	+ 1	− 2	+ 4	− 1						
(531)	− 2	+ 1	...	+ 3	− 4	+ 2	+ 2	− 1							
$(4^2 1)$	+ 1	...	...	− 3	+ 3	...	− 1								
(52^2)	− 6	+ 2	...	− 3	+ 3	− 1									
(432)	+ 2	...	...	+ 3	− 1										
(3^3)	...	...	...	− 1											
(61^3)	− 5	+ 5	− 1												
(521^2)	+ 3	− 1													
(431^3)	− 1														

X (*a*). Runs on to p. 436.

‖	10 k	19 bj	28 ci	$1^2 8$ $b^2 i$	37 dh	127 bch	$1^2 7$ $b^3 h$	46 eg	136 bdg	$2^2 6$ $c^2 g$	$1^2 26$ $b^2 cg$	$1^4 6$ $b^4 g$	5^2 f^2
(10)													
(91)													
(82)													
(73)													
(64)													
(5^2)													
(81^2)													
(721)													
(631)													
(541)													
(62^2)													
(532)													
(4^22)													
(43^2)													
(71^3)													
(621^3)													
(531^2)													
(4^21^2)													
(52^21)													
(4321)													
(3^21)													
(42^3)													
(3^22^2)													
(61^4)													
(521^3)													
(431^3)													
(42^21^2)													
(3^21^3)													
(32^21)													
(2^5)													+ 1
(51^5)												+ 1	...
(421^4)											+ 1	+ 4	...
(3^21^4)										+ 1	+ 2	+ 6	...
(32^21^3)									+ 1	+ 2	+ 5	+ 12	...
(2^41^2)								+ 1	+ 4	+ 6	+ 12	+ 24	+ 2
(41^6)							+ 1	...	...	...	+ 6	+ 25	...
(321^5)						+ 1	+ 3	...	+ 5	+ 11	+ 28	+ 70	...
(2^31^4)					+ 1	+ 3	+ 6	+ 4	+ 19	+ 30	+ 63	+ 132	+ 6
(31^7)				+ 1	...	+ 7	+ 22	...	+ 21	+ 42	+ 112	+ 280	...
(2^21^6)			+ 1	+ 2	+ 6	+ 20	+ 42	+ 15	+ 72	+ 115	+ 242	+ 510	+ 20
(21^8)		+ 1	+ 8	+ 17	+ 28	+ 92	+ 192	+ 56	+ 252	+ 392	+ 812	+ 1680	+ 70
(1^{10})	+ 1	+ 10	+ 45	+ 90	+ 120	+ 360	+ 720	+ 210	+ 840	+ 1260	+ 2520	+ 5040	+ 252

$\parallel$	145 bef	235 cdf	$1^3 35$ b^3df	$12^2 5$ bc^2f	$1^3 25$ b^3cf	$1^5 5$ b^5f	24^2 ce^2	$1^2 4^2$ b^2e^2	$3^2 4$ d^3e	1234 $bcde$
(10)										
(91)										
(82)										
(73)										
(64)										
(5^2)										
(81^2)										
(721)										
(631)										
(541)										
(62^2)										
(532)										
$(4^2 2)$										
(43^2)										
(71^3)										
(621^2)										
(531^2)										
$(4^2 1^2)$										
$(52^2 1)$										
(4321)										+ 1
$(3^3 1)$									+ 1	+ 3
(42^3)								+ 1	...	+ 3
$(3^2 2^2)$							+ 1	+ 2	+ 2	+ 8
(61^4)						+ 1	...	...	...	...
(521^3)					+ 1	+ 5	...	...	...	...
(431^3)				+ 1	+ 3	+ 10	...	...	...	+ 3
$(42^2 1^2)$			+ 1	+ 2	+ 7	+ 20	...	+ 2	...	+ 8
$(3^2 21^2)$		+ 1	+ 2	+ 5	+ 12	+ 30	+ 2	+ 4	+ 5	+ 21
$(32^2 1)$	+ 1	+ 3	+ 7	+ 12	+ 27	+ 60	+ 7	+ 16	+ 12	+ 49
(2^5)	+ 5	+ 10	+ 20	+ 30	+ 60	+ 120	+ 20	+ 45	+ 30	+ 110
(51^5)	...	...	...	...	+ 5	+ 26	...	...	...	...
(421^4)	...	...	+ 4	+ 9	+ 32	+ 95	...	+ 6	...	+ 22
$(3^2 1^4)$	...	+ 4	+ 8	+ 22	+ 54	+ 140	+ 6	+ 12	+ 12	+ 56
$(32^2 1^3)$	+ 3	+ 11	+ 26	+ 48	+ 112	+ 260	+ 18	+ 42	+ 31	+ 128
$(2^4 1^2)$	+ 14	+ 32	+ 68	+ 108	+ 228	+ 480	+ 53	+ 114	+ 80	+ 284
(41^6)	...	...	+ 15	+ 30	+ 111	+ 330	...	+ 20	...	+ 60
(321^5)	+ 10	+ 35	+ 85	+ 156	+ 368	+ 860	+ 50	+ 120	+ 80	+ 335
$(2^3 1^4)$	+ 42	+ 99	+ 210	+ 339	+ 720	+ 1530	+ 144	+ 306	+ 213	+ 735
(31^7)	+ 35	+ 105	+ 266	+ 462	+ 1092	+ 2520	+ 140	+ 350	+ 210	+ 875
$(2^2 1^6)$	+ 130	+ 296	+ 622	+ 990	+ 2082	+ 4380	+ 400	+ 840	+ 570	+ 1900
(21^8)	+ 406	+ 868	+ 1792	+ 2772	+ 5712	+ 11760	+ 1120	+ 2310	+ 1540	+ 4900
(1^{10})	+ 1260	+ 2520	+ 5040	+ 7560	+ 15120	+ 30240	+ 3150	+ 6300	+ 4200	+ 12600

$1^3 34$ b^3de	$2^3 4$ c^3e	$1^2 2^2 4$ b^2c^2e	$1^4 24$ b^4ce	$1^6 4$ b^6e	13^3 bd^3	$2^2 3^2$ c^2d^2	$1^2 23^2$ b^2cd^2	$1^4 3^2$ b^4d^2	$12^3 3$ bc^3d
									+ 1
								+ 1	...
							+ 1	+ 4	+ 3
						+ 1	+ 2	+ 6	+ 7
					+ 1	+ 2	+ 5	+ 12	+ 12
				+ 1	...	...	...	...	...
			+ 1	+ 6	...	...	...	+ 2	...
		+ 1	+ 4	+ 15	...	...	+ 2	+ 8	+ 7
	+ 1	+ 2	+ 6	+ 20	...	+ 2	+ 4	+ 12	+ 16
+ 1	...	+ 2	+ 9	+ 30	...	...	+ 5	+ 22	+ 12
+ 3	+ 3	+ 8	+ 22	+ 60	+ 3	+ 8	+ 21	+ 56	+ 49
+ 6	+ 6	+ 15	+ 36	+ 90	+ 10	+ 18	+ 42	+ 96	+ 87
+ 10	+ 6	+ 18	+ 48	+ 120	+ 6	+ 15	+ 42	+ 115	+ 87
+ 18	+ 15	+ 34	+ 78	+ 180	+ 18	+ 34	+ 80	+ 188	+ 156
...	...	...	+ 4	+ 25	...	...	...	+ 6	...
+ 3	...	+ 7	+ 32	+ 111	...	...	+ 12	+ 54	+ 27
+ 9	+ 10	+ 27	+ 76	+ 215	+ 6	+ 18	+ 48	+ 132	+ 112
+ 27	+ 18	+ 54	+ 149	+ 390	+ 15	+ 34	+ 99	+ 270	+ 198
+ 48	+ 42	+ 99	+ 236	+ 570	+ 42	+ 80	+ 186	+ 436	+ 358
+ 112	+ 87	+ 198	+ 450	+ 1020	+ 87	+ 156	+ 358	+ 820	+ 645
+ 240	+ 180	+ 390	+ 840	+ 1800	+ 180	+ 310	+ 680	+ 1500	+ 1170
+ 10	...	+ 20	+ 95	+ 330	...	...	+ 30	+ 140	+ 60
+ 76	+ 48	+ 149	+ 416	+ 1095	+ 36	+ 78	+ 236	+ 650	+ 450
+ 132	+ 115	+ 270	+ 650	+ 1580	+ 96	+ 188	+ 436	+ 1032	+ 820
+ 294	+ 228	+ 523	+ 1196	+ 2730	+ 210	+ 370	+ 844	+ 1920	+ 1479
+ 612	+ 468	+ 1008	+ 2172	+ 4680	+ 444	+ 740	+ 1604	+ 3480	+ 2688
+ 215	+ 120	+ 390	+ 1095	+ 2850	+ 90	+ 180	+ 570	+ 1580	+ 1020
+ 775	+ 585	+ 1340	+ 3050	+ 6900	+ 510	+ 880	+ 2000	+ 4520	+ 3390
+ 1566	+ 1194	+ 2547	+ 5436	+ 11610	+ 1092	+ 1776	+ 3792	+ 8100	+ 6180
+ 2030	+ 1470	+ 3360	+ 7560	+ 16800	+ 1260	+ 2100	+ 4760	+ 10640	+ 7770
+ 3990	+ 3015	+ 6330	+ 13290	+ 27900	+ 2700	+ 4280	+ 8980	+ 18840	+ 14220
+ 10080	+ 7560	+ 15540	+ 31920	+ 65520	+ 6720	+ 10360	+ 21280	+ 43680	+ 32760
+ 25200	+ 18900	+ 37800	+ 75600	+ 151200	+ 16800	+ 25200	+ 50400	+ 100800	+ 75600

‖	$1^3 2^2 3$ b^3c^2d	$1^5 23$ b^5cd	$1^7 3$ b^7d	2^5 c^5	$2^2 3^2$ b^2c^4	$1^4 2^3$ b^4c^3	$1^6 2^2$ b^6c^2	$1^8 2$ b^8c	1^{10} b^{10}
(10)									+ 1
(91)								+ 1	+ 10
(82)							+ 1	+ 8	+ 45
(73)						+ 1	+ 6	+ 28	+ 120
(64)					+ 1	+ 4	+ 15	+ 56	+ 210
(5^2)				+ 1	+ 2	+ 6	+ 20	+ 70	+ 252
(81^2)			+ 1	...	...	...	+ 2	+ 17	+ 90
(721)		+ 1	+ 7	...	...	+ 3	+ 20	+ 92	+ 360
(631)	+ 1	+ 5	+ 21	...	+ 4	+ 19	+ 72	+ 252	+ 840
(541)	+ 3	+ 10	+ 35	+ 5	+ 14	+ 42	+ 130	+ 406	+ 1260
(62^2)	+ 2	+ 11	+ 42	...	+ 6	+ 30	+ 115	+ 392	+ 1260
(532)	+ 11	+ 35	+ 105	+ 10	+ 32	+ 99	+ 296	+ 868	+ 2520
(4^22)	+ 18	+ 50	+ 140	+ 20	+ 53	+ 144	+ 400	+ 1120	+ 3150
(43^3)	+ 31	+ 80	+ 210	+ 30	+ 80	+ 213	+ 570	+ 1540	+ 4200
(71^3)	...	+ 3	+ 22	...	...	+ 6	+ 42	+ 192	+ 720
(621^2)	+ 5	+ 28	+ 112	...	+ 12	+ 63	+ 242	+ 812	+ 2520
(531^2)	+ 26	+ 85	+ 266	+ 20	+ 68	+ 210	+ 622	+ 1792	+ 5040
(4^21^2)	+ 42	+ 120	+ 350	+ 45	+ 114	+ 306	+ 840	+ 2310	+ 6300
(52^21)	+ 48	+ 156	+ 462	+ 30	+ 108	+ 339	+ 990	+ 2772	+ 7560
(4321)	+ 128	+ 335	+ 875	+ 110	+ 284	+ 735	+ 1900	+ 4900	+ 12600
(3^21)	+ 210	+ 510	+ 1260	+ 180	+ 444	+ 1092	+ 2700	+ 6720	+ 16800
(42^3)	+ 228	+ 585	+ 1470	+ 180	+ 468	+ 1194	+ 3015	+ 7560	+ 18900
(3^22^2)	+ 370	+ 880	+ 2100	+ 310	+ 740	+ 1776	+ 4280	+ 10360	+ 25200
(61^4)	+ 12	+ 70	+ 280	...	+ 24	+ 132	+ 510	+ 1680	+ 5040
(521^3)	+ 112	+ 368	+ 1092	+ 60	+ 228	+ 720	+ 2082	+ 5712	+ 15120
(431^3)	+ 294	+ 775	+ 2030	+ 240	+ 612	+ 1566	+ 3990	+ 10080	+ 25200
(42^21^2)	+ 523	+ 1340	+ 3360	+ 390	+ 1008	+ 2547	+ 6330	+ 15540	+ 37800
(3^21^2)	+ 844	+ 2000	+ 4760	+ 680	+ 1604	+ 3792	+ 8980	+ 21280	+ 50400
(32^31)	+ 1479	+ 3390	+ 7770	+ 1170	+ 2688	+ 6180	+ 14220	+ 32760	+ 75600
(2^5)	+ 2580	+ 5700	+ 12600	+ 2040	+ 4530	+ 10080	+ 22500	+ 50400	+ 113400
(51^5)	+ 260	+ 860	+ 2520	+ 120	+ 480	+ 1530	+ 4380	+ 11760	+ 30240
(421^4)	+ 1196	+ 3050	+ 7560	+ 840	+ 2172	+ 5436	+ 13290	+ 31920	+ 75600
(3^21^4)	+ 1920	+ 4520	+ 10640	+ 1500	+ 3480	+ 8100	+ 18840	+ 43680	+ 100800
(32^21^3)	+ 3358	+ 7610	+ 17220	+ 2580	+ 5844	+ 13212	+ 29820	+ 67200	+ 151200
(2^41^2)	+ 5844	+ 12720	+ 27720	+ 4530	+ 9876	+ 21564	+ 47160	+ 103320	+ 226800
(41^6)	+ 2730	+ 6900	+ 16800	+ 1800	+ 4680	+ 11610	+ 27900	+ 65520	+ 151200
(321^5)	+ 7610	+ 17000	+ 37800	+ 5700	+ 12720	+ 28260	+ 62520	+ 137760	+ 302400
(2^31^4)	+ 13212	+ 28260	+ 60480	+ 10080	+ 21564	+ 46152	+ 98820	+ 211680	+ 453600
(31^7)	+ 17220	+ 37800	+ 82320	+ 12600	+ 27720	+ 60480	+ 131040	+ 282240	+ 604800
(2^21^6)	+ 29820	+ 62520	+ 131040	+ 22500	+ 47160	+ 98820	+ 207000	+ 433440	+ 907200
(21^8)	+ 67200	+ 137760	+ 282240	+ 50400	+ 103320	+ 211680	+ 433440	+ 887040	+ 1814400
(1^{10})	+ 151200	+ 302400	+ 604800	+ 113400	+ 226800	+ 453600	+ 907200	+ 1814400	+ 3628800

X (*b*). Runs on to p. 439.

=	10 k	19 bj	28 ci	$1^2 8$ b^2i	37 dh	127 bch	$1^3 7$ b^3h	46 eg	136 bdg	$2^2 6$ c^2g	$1^2 26$ b^2cg	$1^4 6$ b^4g	5^2 f^2	145 bef
(10)	− 10	+ 10	+ 10	− 10	+ 10	− 20	+ 10	+ 10	− 20	− 10	+ 30	− 10	+ 5	− 20
(91)	+ 10	− 1	− 10	+ 1	− 10	+ 11	− 1	− 10	+ 11	+ 10	− 12	+ 1	− 5	+ 11
(82)	+ 10	− 10	+ 6	+ 2	− 10	+ 4	− 2	− 10	+ 20	− 6	− 6	+ 2	− 5	+ 20
(73)	+ 10	− 10	− 10	+ 10	+ 11	− 1	− 3	− 10	− 1	+ 10	− 9	+ 3	− 5	+ 20
(64)	+ 10	− 10	− 10	+ 10	− 10	+ 20	− 10	+ 14	− 4	− 2	− 6	+ 4	− 5	− 4
(5^2)	+ 5	− 5	− 5	+ 5	− 5	+ 10	− 5	− 5	+ 10	+ 5	− 15	+ 5	+ 10	− 15
(81^2)	− 10	+ 1	+ 2	− 1	+ 10	− 3	+ 1	+ 10	− 11	− 2	+ 4	− 1	+ 5	− 11
(721)	− 20	+ 11	+ 4	− 3	− 1	− 8	+ 3	+ 20	− 10	− 4	+ 11	− 3	+ 10	− 31
(631)	− 20	+ 11	+ 20	− 11	− 1	− 10	+ 4	− 4	− 4	− 8	+ 15	− 4	+ 10	− 7
(541)	− 20	+ 11	+ 20	− 11	+ 20	− 31	+ 11	− 4	− 7	− 8	+ 18	− 5	− 15	+ 23
(62^2)	− 10	+ 10	− 6	− 2	+ 10	− 4	+ 2	− 2	− 8	...	+ 6	− 2	+ 5	− 8
(532)	− 20	+ 20	+ 4	− 12	− 1	− 3	+ 5	+ 20	− 19	− 4	+ 15	− 5	− 15	+ 10
($4^2 2$)	− 10	+ 10	+ 2	− 6	+ 10	− 12	+ 6	− 14	+ 4	+ 10	− 6	...	+ 5	+ 4
(43^2)	− 10	+ 10	+ 10	− 10	− 11	+ 1	+ 3	− 2	+ 13	− 4	− 3	...	+ 5	− 8
(71^3)	+ 10	− 1	− 2	+ 1	− 3	+ 3	− 1	− 10	+ 4	+ 2	− 4	+ 1	− 5	+ 11
(621^2)	+ 30	− 12	− 6	+ 4	− 9	+ 11	− 4	− 6	+ 15	+ 6	− 15	+ 4	− 15	+ 18
(531^2)	+ 30	− 12	− 22	+ 12	− 9	+ 13	− 5	− 6	+ 15	+ 10	− 19	+ 5	+ 10	− 12
($4^2 1^2$)	+ 15	− 6	− 11	+ 6	− 15	+ 17	− 6	+ 9	− 3	− 1	+ 1	...	+ 5	− 8
($52^2 1$)	+ 30	− 21	+ 2	+ 5	− 9	+ 12	− 5	− 18	+ 18	+ 4	− 17	+ 5	+ 10	− 1
(4321)	+ 60	− 42	− 28	+ 26	+ 3	+ 21	− 12	+ 12	− 15	− 8	+ 7	...	− 5	...
($3^3 1$)	+ 10	− 7	− 10	+ 7	+ 11	− 4	...	+ 2	− 7	+ 4	...	...	− 5	+ 5
(42^3)	+ 10	− 10	+ 6	+ 2	− 10	+ 4	− 2	+ 10	...	− 4	+ 2	...	− 5	...
($3^2 2^2$)	+ 15	− 15	+ 1	+ 7	+ 6	− 7	...	− 9	+ 3	+ 2	...	...	+ 5	− 1
(61^4)	− 10	+ 1	+ 2	− 1	+ 3	− 3	+ 1	+ 4	− 4	− 2	+ 4	− 1	+ 5	− 5
(521^3)	− 40	+ 13	+ 8	− 5	+ 12	− 14	+ 5	+ 16	− 19	− 8	+ 19	− 5	− 5	+ 1
(431^3)	− 40	+ 13	+ 24	− 13	+ 12	− 16	+ 6	− 8	+ 5	...	− 1	...	− 5	+ 5
($42^2 1^2$)	− 60	+ 33	+ 4	− 9	+ 18	− 23	+ 9	− 12	+ 3	+ 8	− 4	...	+ 5	− 1
($3^2 21^2$)	− 60	+ 33	+ 28	− 21	− 24	+ 9	...	...	+ 6	− 4	...	...	+ 5	− 3
($32^2 1$)	− 40	+ 31	− 8	− 7	− 2	+ 5	...	+ 8	− 3	...	...	...	− 5	+ 1
(2^5)	− 2	+ 2	− 2	...	+ 2	...	...	− 2	...	...	...	...	+ 1	
(51^5)	+ 10	− 1	− 2	+ 1	− 3	+ 3	− 1	− 4	+ 4	+ 2	− 4	+ 1		
(421^4)	+ 50	− 14	− 10	+ 6	− 15	+ 17	− 6	+ 4	− 1	− 2	+ 1			
($3^3 1^4$)	+ 25	− 7	− 13	+ 7	+ 3	− 1	...	+ 2	− 2	+ 1				
($32^2 1^3$)	+ 100	− 46	− 12	+ 14	+ 12	− 5	...	− 4	+ 1					
($2^4 1^2$)	+ 25	− 16	+ 9	...	− 4	...	...	+ 1						
(41^6)	− 10	+ 1	+ 2	− 1	+ 3	− 3	+ 1							
(321^5)	− 60	+ 15	+ 12	− 7	− 3	+ 1								
($2^3 1^4$)	− 50	+ 20	− 6	...	+ 1									
(31^7)	+ 10	− 1	− 2	+ 1										
($2^2 1^6$)	+ 35	− 8	+ 1											
(21^8)	− 10	+ 1												
(1^{10})	+ 1													

=	235	$1^2 35$	$12^2 5$	$1^3 25$	$1^5 5$	24^2	$1^2 4^2$	$3^2 4$	1234	$1^3 34$	$2^3 4$	$1^2 2^2 4$	$1^4 24$	$1^6 4$
	cdf	b^2df	bc^2f	b^3cf	b^5f	ce^2	b^2e^2	d^2e	$bcde$	b^3de	c^3e	b^2c^2e	b^4ce	b^6e
(10)	− 20	+ 30	+ 30	− 40	+ 10	− 10	+ 15	− 10	+ 60	− 40	+ 10	− 60	+ 50	− 10
(91)	+ 20	− 12	− 21	+ 13	− 1	+ 10	− 6	+ 10	− 42	+ 13	− 10	+ 33	− 14	+ 1
(82)	+ 4	− 22	+ 2	+ 8	− 2	+ 2	− 11	+ 10	− 28	+ 24	+ 6	+ 4	− 10	+ 2
(73)	− 1	− 9	− 9	+ 12	− 3	+ 10	− 15	− 11	+ 3	+ 12	− 10	+ 18	− 15	+ 3
(64)	+ 20	− 6	− 18	+ 16	− 4	− 14	+ 9	− 2	+ 12	− 8	+ 10	− 12	+ 4	...
(5^2)	− 15	+ 10	+ 10	− 5	...	+ 5	+ 5	+ 5	− 5	− 5	− 5	+ 5	...	...
(81^3)	− 12	+ 12	+ 5	− 5	+ 1	− 6	+ 6	− 10	+ 26	− 13	+ 2	− 9	+ 6	− 1
(721)	− 3	+ 13	+ 12	− 14	+ 3	− 12	+ 17	+ 1	+ 21	− 16	+ 4	− 23	+ 17	− 3
(631)	− 19	+ 15	+ 18	− 19	+ 4	+ 4	− 3	+ 13	− 15	+ 5	...	+ 3	− 1	...
(541)	+ 10	− 12	− 1	+ 1	...	+ 4	− 8	− 8	...	+ 5	...	− 1	...	...
(62^3)	− 4	+ 10	+ 4	− 8	+ 2	+ 10	− 1	− 4	− 8	...	− 4	+ 8	− 2	...
(532)	+ 17	− 4	− 13	+ 5	...	− 12	+ 1	+ 1	+ 5	− 1	+ 4	− 2	...	...
$(4^2 2)$	− 12	+ 2	+ 2	...	...	+ 2	− 3	+ 2	+ 4	...	− 2	...	...	...
(43^2)	+ 1	− 3	+ 3	...	...	+ 2	+ 3	− 1	− 3	...	...	...	...	...
(71^3)	+ 5	− 5	− 5	+ 5	− 1	+ 6	− 6	+ 3	− 12	+ 6	− 2	+ 9	− 6	+ 1
(621^2)	+ 15	− 19	− 17	+ 19	− 4	− 6	+ 1	− 3	+ 7	− 1	+ 2	− 4	+ 1	
(531^3)	− 4	+ 3	+ 2	− 1	...	+ 2	+ 2	− 3	+ 4	− 2	− 2	+ 1		
$(4^2 1^2)$	+ 1	+ 2	− 1	...	...	− 3	+ 3	+ 3	− 3	...	+ 1			
$(52^2 1)$	− 13	+ 2	+ 9	− 3	...	+ 2	− 1	+ 3	− 3	+ 1				
(4321)	+ 5	+ 4	− 3	...	...	+ 4	− 3	− 3	+ 1					
$(3^3 1)$	− 1	...	...	...	...	− 2	...	+ 1						
(42^3)	+ 4	− 2	...	...	...	− 2	+ 1							
$(3^3 2^2)$	− 2	...	...	...	..	+ 1								
(61^4)	− 5	+ 5	+ 5	− 5	+ 1									
(521^3)	+ 5	− 1	− 3	+ 1										
(431^3)	− 1	− 2	+ 1											
$(42^2 1^2)$	− 2	+ 1												
$(3^3 21^2)$	+ 1													

$1^3 3$	$2^2 3^2$	$1^2 2 3^3$	$1^4 3^2$	$1^2 3^3$	$1^3 2^2 3$	$1^5 2 3$	$1^7 3$	2^5	$2^2 3^3$	$1^4 2^3$	$1^6 2^2$	$1^8 2$	1^{10}
bd^3	c^2d^2	b^3cd^2	b^4d^2	bc^3d	b^3c^2d	b^5cd	b^7d	c^5	b^2c^4	b^4c^3	b^6c^2	b^8c	b^{10}
+ 10	+ 15	− 60	+ 25	− 40	+ 100	− 60	+ 10	− 2	+ 25	− 50	+ 35	− 10	+ 1
− 7	− 15	+ 33	− 7	+ 31	− 46	+ 15	− 1	+ 2	− 16	+ 20	− 8	+ 1	
− 10	+ 1	+ 28	− 13	− 8	− 12	+ 12	− 2	− 2	+ 9	− 6	+ 1		
+ 11	+ 6	− 24	+ 3	− 2	+ 12	− 3	...	+ 2	− 4	+ 1			
+ 2	− 9	...	+ 2	+ 8	− 4	...	...	− 2	+ 1				
− 5	+ 5	+ 5	...	− 5	...	...	...	+ 1					
+ 7	+ 7	− 21	+ 7	− 7	+ 14	− 7	+ 1						
− 4	− 7	+ 9	− 1	+ 5	− 5	+ 1							
− 7	+ 3	+ 6	− 2	− 3	+ 1								
+ 5	− 1	− 3	...	+ 1									
+ 4	+ 2	− 4	+ 1										
− 1	− 2	+ 1											
− 2	+ 1												
+ 1													

148.

MEMOIR ON THE RESULTANT OF A SYSTEM OF TWO EQUATIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 703—715. Received December 18, 1856,—Read January 8, 1857.]

THE Resultant of two equations such as

$$(a, b, \ldots \between x, y)^m = 0,$$
$$(p, q, \ldots \between x, y)^n = 0,$$

is, it is well known, a function homogeneous in regard to the coefficients of each equation separately, viz. of the degree n in regard to the coefficients $(a, b, \ldots)$ of the first equation, and of the degree m in regard to the coefficients $(p, q, \ldots)$ of the second equation; and it is natural to develope the resultant in the form $kAP + k'A'P' +$ &c., where A, A', &c. are the combinations (powers and products) of the degree n in the coefficients $(a, b, \ldots)$, P, P', &c. are the combinations of the degree m in the coefficients $(p, q, \ldots)$, and k, k', &c. are mere numerical coefficients. The object of the present memoir is to show how this may be conveniently effected, either by the method of symmetric functions, or from the known expression of the Resultant in the form of a determinant, and to exhibit the developed expressions for the resultant of two equations, the degrees of which do not exceed 4. With respect to the first method, the formula in its best form, or nearly so, is given in the *Algebra* of Meyer Hirsch, [for proper title see p. 417], and the application of it is very easy when the necessary tables are calculated: as to this, see my "Memoir on the Symmetric Functions of the Roots of an Equation"[1]. But when the expression for the Resultant of two equations is to be calculated without the assistance of such tables, it is I think by far the most simple process to develope the determinant according to the second of the two methods.

[1] *Philosophical Transactions*, 1857, pp. 489—497, [147].

Consider first the method of symmetric functions, and to fix the ideas, let the two equations be

$$(a,\ b,\ c,\ d\between x,\ y)^2 = 0,$$
$$(p,\ q,\ r\between x,\ y)^2 = 0.$$

Then writing

$$(a,\ b,\ c,\ d\between 1,\ z)^3 = a\,(1-\alpha z)\,(1-\beta z)\,(1-\gamma z),$$

so that

$$-\frac{b}{a} = \alpha+\beta+\gamma \qquad = (1),$$

$$+\frac{c}{a} = \alpha\beta+\alpha\gamma+\beta\gamma = (1^2),$$

$$-\frac{d}{a} = \alpha\beta\gamma \qquad = (1^3),$$

the Resultant is

$$(p,\ q,\ r\between \alpha,\ 1)^2\,.\,(p,\ q,\ r\between \beta,\ 1)^2\,.\,(p,\ q,\ r\between \gamma,\ 1)^2,$$

which is equal to

$$r^3 + qr^2(\alpha+\beta+\gamma) + pr^2(\alpha^2+\beta^2+\gamma^2) + pqr\,(\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma\alpha^2+\gamma^2\alpha) + \&c.;$$

or adopting the notation for symmetric functions used in the memoir above referred to, this is

$$\begin{array}{l}
\{\ \ r^3 \\
\{+qr^2\ (1) \\
\left\{\begin{array}{l}+pr^2\ (2)\\ +q^2r\ \ (1^2)\end{array}\right. \\
\left\{\begin{array}{l}+pqr\ (21)\\ +q^3\ \ \ (1^3)\end{array}\right. \\
\left\{\begin{array}{l}+p^2r\ \ (2^2)\\ +pq^2\ \ (21^2)\end{array}\right. \\
\{+p^2q\ \ (2^21) \\
\{+p^3\ \ \ (2^3)\ ,
\end{array}$$

the law of which is best seen by dividing by r^3 and then writing

$$\frac{q}{r} = [1], \qquad \frac{p}{r} = [2],$$

and similarly,

$$\frac{q^2}{r^2} = [1^2], \quad \frac{pq}{r^2} = [21], \ \&c\ ;$$

the expression would then become

$$1+[1](1)+[2](2)+[1^2](1^2)+[21](21)+[1^3](1^3)+[2^2](2^2)+[21^2](21^2)+[2^21](2^21)+[2^3](2^3),$$

where the terms within the [ ] and ( ) are simply all the partitions of the numbers 1, 2, 3, 4, 5, 6, the greatest part being 2, and the greatest number of parts being 3. And in like manner in the general case we have all the partitions of the numbers 1, 2, 3, ... mn, the greatest part being n, and the greatest number of parts being m.

The symmetric functions (1), (2), (1^2), &c. are given in the Tables (b) of the Memoir on Symmetric Functions, but it is necessary to remark that in the Tables the first coefficient a is put equal to unity, and consequently that there is a power of the coefficient a to be restored as a factor: this is at once effected by the condition of homogeneity. And it is not by any means necessary to write down (as for clearness of explanation has been done) the preceding expression for the Resultant; any portion of it may be taken out directly from one of the Tables (b). For instance, the bracketed portion

$$\begin{aligned} &+pqr\,(21),\\ &+q^3\quad(1^3), \end{aligned}$$

which corresponds to the partitions of the number 3, is to be taken out of the Table III (b). as follows: a portion of this Table (consisting as it happens of consecutive lines and columns, but this is not in general the case) is

$$\begin{array}{c|c|c|} = & d & bc \\ \hline (21) & +3 & -1 \\ \hline (1^3) & -1 & \\ \hline \end{array};$$

if in this we omit the sign $=$, and in the outside line write for homogeneity ad instead of d, and in the outside column, first substituting q, p for 1, 2, then write for homogeneity pqr instead of pq, we have

$$\begin{array}{c|c|c|} & ad & bc \\ \hline pqr & +3 & -1 \\ \hline q^3 & -1 & \\ \hline \end{array}$$

viz. $pqr\times(+3ad-1bc)+q^3(-1ad)$, for the value of the portion in question; this is equivalent to

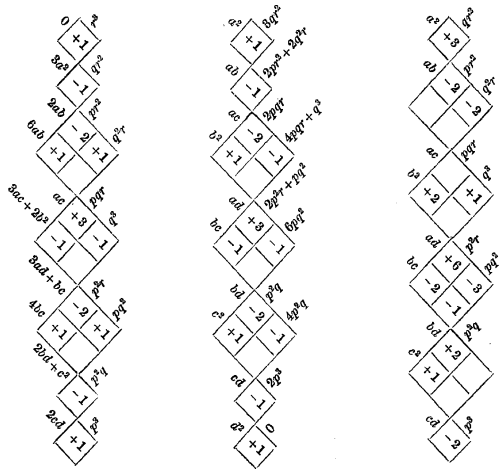
$$\begin{array}{c|c|c|} & pqr & q^3 \\ \hline ad & +3 & -1 \\ \hline bc & -1 & \\ \hline \end{array},$$ or as it may be more conveniently written, $$\begin{array}{ccc} & ad & pqr \\ bc & +3 & \\ & -1 & -1 \\ & & q^3 \end{array}$$

in which form it constitutes a part of the expression given in the sequel for the Resultant of the two functions in question; and similarly the remainder of the expression is at once derived from the Tables (b) I. to VI.

As a specimen of a mode of verification, it may be remarked that the Resultant quà invariant ought, when operated upon by the sum of the two operations,

$$3a\partial_b + 2b\partial_c + c\partial_d \text{ and } 2p\partial_q + q\partial_r,$$

to give a result zero. The results of the two operations are originally obtained in the forms in the first and second columns, and the first column, and the second column, with all the signs reversed, are respectively equal to the third column, and consequently the sum of the first and second columns vanishes, as it ought to do.



Next to explain the second method, viz. the calculation of the resultant from the expression in the form of a determinant.

Taking the same example as before, the resultant is

$$\begin{vmatrix} & a, & b, & c, & d \\ a, & b, & c, & d, & \\ & & p, & q, & r \\ & p, & q, & r & \\ p, & q, & r & & \end{vmatrix}$$

which may be developed in the form

$$\begin{array}{l}
+12\ .\ 345\} \\
-13\ .\ 245\} \\
+14\ .\ 235\} \\
+23\ .\ 145\} \\
-15\ .\ 234\} \\
-24\ .\ 135\} \\
+25\ .\ 134\} \\
+34\ .\ 125\} \\
-35\ .\ 124\} \\
+45\ .\ 123\}
\end{array}$$

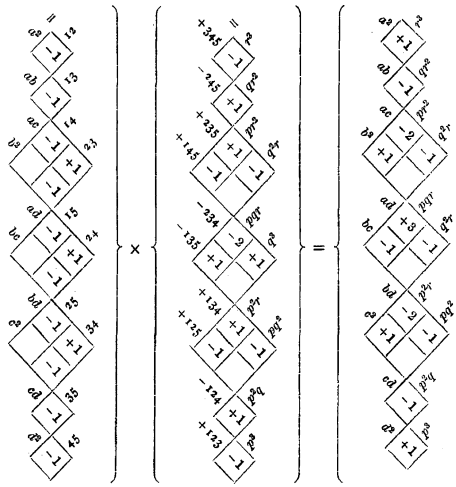
where 12, 13, &c. are the terms of

$$\left(\begin{array}{cccc} & a, & b, & c, & d \\ a, & b, & c, & d & \end{array}\right)$$

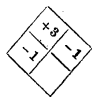
and 123, &c. are the terms of

$$\left(\begin{array}{ccc} & p, & q, & r \\ p, & q, & r & \end{array}\right)$$

viz. 12 is the determinant formed with the first and second columns of the upper matrix, 123 is the determinant formed with the first, second and third columns of the lower matrix, and in like manner for the analogous symbols. These determinants must be first calculated, and the remainder of the calculation may then be arranged as follows:—



where it is to be observed that the figures in the squares of the third column are obtained from those in the corresponding squares of the first and second columns by the ordinary rule for the multiplication of determinants,—taking care to multiply the dexter lines (i.e. lines in the direction $\backslash$) of the first square by the sinister lines (i.e. lines in the direction $/$) of the second square in order to obtain the sinister lines of the third square. Thus, for instance, the figures in the square



are obtained as follows, viz. the first sinister line $(+3, -1)$ by

$$(-1, +1)(-2, +1) = \quad 2+1 = +3,$$

$$(-1, +1)(+1, \quad 0) = -1+0 = -1,$$

and the second sinister line $(-1, 0)$ by

$$(0, -1)(-2, +1) = 0-1 = -1,$$

$$(0, -1)(+1, \quad 0) = 0+0 = \quad 0.$$

I have calculated the determinants required for the calculation, by the preceding process, of the Resultant of two quartic equations, and have indeed used them for the verification of the expression as found by the method of symmetric functions; as the determinants in question are useful for other purposes, I think the values are worth preserving.

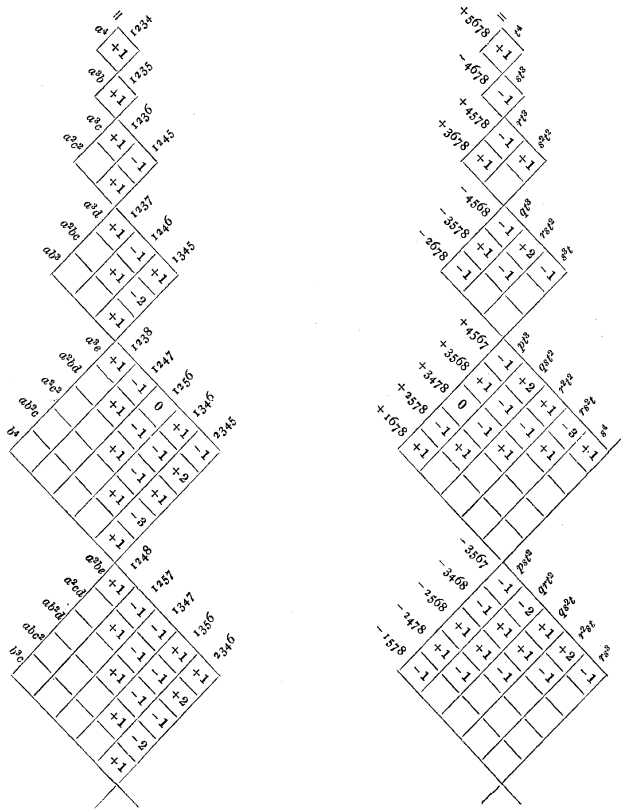
Table of the Determinants of the Matrices,

$$\left(\begin{array}{ccccccccc} & & & a, & b, & c, & d, & e & \\ & & a, & b, & c, & d, & e & & \\ & a, & b, & c, & d, & e & & & \\ a, & b, & c, & d, & e, & & & & \end{array}\right)$$

and

$$\left(\begin{array}{ccccccccc} & & & p, & q, & r, & s, & t & \\ & & p, & q, & r, & s, & t & & \\ & p, & q, & r, & s, & t & & & \\ p, & q, & r, & s, & t & & & & \end{array}\right)$$

arranged in the form adapted for the calculation of the Resultant of the two quartic equations $(a, b, c, d, e \between x, y)^4 = 0$, and $(p, q, r, s, t \between x, y)^4 = 0$, viz.



	a^3ce	ab^2c	a^2d^2	$abcd$	b^2d	ac^2	b^2c^2
1258	+1						
1348	−1	+1					
1267	−1	−1	+1				
1357	0	−1	−1	+1			
2347	0	−1	−1	+1	+1		
1456	+1	−1	−1	+1	+1	−2	
2356	−1	−1	+1	+1	+1	−2	+1
		+1	0	0	0	0	0
			−1	−1	−1	−1	−1
				+1	+1	+1	+1

	pr^2	q^2r	pqr	qrs	r^2s	qs^2	rs^2
3467	+1						
2567	0	+1					
3458	+1	−1	+1				
2468	+1	0	−1	+1			
1568	0	−1	−1	+1	+1		
2378	−1	−1	+1	+1	+1	−2	
1478	−1	0	−1	−1	−1	−1	+1
		+1	+1	+1	+1	+1	+1
			−1	−1	−1	−1	−1
				+1	+1	+1	+1

	a^2de	$abce$	b^2e	abd^2	ac^2d	b^2cd	bc^3
1268	+1						
1358	−1	+1					
2348	−1	+1	+1				
1367	+1	−1	−1	+1			
1457	−2	−1	−1	+1	+1		
2357	+1	0	0	−1	+1	−1	
2456	−2	+1	−1	−1	+1	0	−1
		+1	−1	−1	−1	+1	+1
			+1	−1	−1	+2	−1
				+1	−2	−1	−1
					+1	−2	+1

	pq^2	pqr	q^2s	qr^2	pr^2	rs^2
−3457	+1					
−2467	0	+1				
−1567	−1	−1	+1			
−3458	−2	−1	−1	+1		
−2368	+1	+2	+1	−1	+1	
−1468	+1	0	+1	+1	+2	−1
−1378	−1	+1	0	+1	+1	−1
		−1	−1	−1	−1	−1
			+1	+1	+1	+1
				−1	−1	−1
					+1	+1

	a^3e^2	ad^2e	ac^2e	b^2ce	acd^2	b^2d^2	bc^2d	c^4
1278	+1							
1368	−1	+1						
1458	−1	0	+1					
2358	+1	−1	−1	+1				
1467	+1	−1	−1	−1	+1			
2367	0	−1	−1	−1	−1	+1		
2457	−1	+1	+1	+1	+1	+1	−3	
3456	1	−2	+3	0	−2	+2	+1	+1
		+1	0	−1	−1	−1	−1	−3
			−1	−1	−1	−1	−1	+1
				+1	+1	+2	+1	+1
					−1	−2	−1	−1
						+1	+1	+1

	p^2q^2	$pqrs$	pr^2s	q^3r	pr^3s	q^2s^2	qr^2s^2	r^4s
3456	+1							
2457	−1	−2						
2367	0	+2	−2					
1467	+1	0	+2	+2				
2358	+1	−1	−1	−1	+2			
1458	0	−1	+1	−1	−1	+1		
1368	−1	−1	−1	−1	−1	−1	−3	
1278	+1	+1	0	+1	+1	+1	+1	+1
		+1	+1	+1	+1	+1	+1	−1
			−1	−1	−1	−1	−1	+1
				+1	+1	+1	+1	+1
					−1	−1	−1	−1
						+1	+1	+1

	1378	1468	2368	2458	1567	2467	3457
abc^2	$+1$	-1	-1	$+1$	$+1$	$+1$	-1
$acde$		$+1$	-1	-1	-2	$+1$	0
b^2de			$+1$	-1	0	-1	$+2$
bc^2e				$+1$	-1	-1	-1
ad^3					$+1$	-2	$+1$
bcd^2						$+1$	-1
c^3d							$+1$

	1478	2378	1568	2468	3458	2567	3467
ace^2	$+1$	-1	-1	$+1$	$+1$	$+1$	-1
b^2e^2		$+1$	0	-1	-2	$+1$	0
ad^2e			$+1$	-1	-1	-1	$+2$
$bcde$				$+1$	$+1$	-1	-1
c^2e					$+1$	-2	$+1$
bd^3						$+1$	-1
c^2d^2							$+1$

	1578	2478	2568	3468	3567
ade^2	$+1$	-1	-1	$+1$	-1
bce^2		$+1$	-1	-1	$+2$
bd^2e			$+1$	-1	-1
c^2de				$+1$	-2
cd^3					$+1$

	2456	2357	1457	1367	2348	1358	1268
p^3qr	$+1$	0	-1	-1	$+1$	$+1$	-1
$pqrt$		$+1$	-1	-1	$+1$	$+2$	-1
q^2t			$+1$	-1	-1	-1	-1
pqr^2				$+1$	0	-1	-1
p^2rs					$+1$	-2	$+1$
pq^2s						$+1$	$+1$
q^2rs							$+2$
qr^3							-1

	2356	1456	2347	1357	1267	1348	1258
p^2rt	$+1$	-1	-1	$+1$	$+1$	-1	$+1$
pq^2t		-1	-1	$+1$	-1	0	$+1$
p^2qs		0	$+1$	-1	-1	-1	$+1$
$pqrs$		0	-2	$+1$	-1	-1	$+1$
q^3s			-1	$+1$	$+1$	$+1$	$+1$
pr^2s			0	-1	-1	-1	$+1$
q^2r^2				$+1$	$+1$	$+1$	$+1$

	2346	1356	1347	1257	1248
p^2qt	-1	-1	$+1$	$+1$	-1
p^3rs		-2	$+1$	$+1$	-1
pq^3s			$+1$	-1	-1
pqr^2s				-1	-1
q^2r					-1

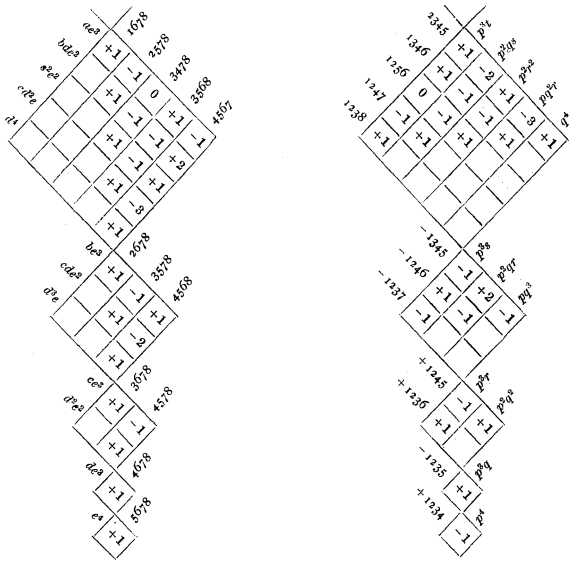

de²
bde²
e²e²
cde²
d⁴
+1
+1
+1

Table (2, 2).
Resultant of
$(a, b, c \between x, y)^2$,
$(p, q, r \between x, y)^2$.

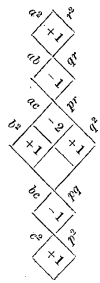


Table (3, 2).
Resultant of
$(a, b, c\ d \between x, y)^2$,
$(p, q, r\ \between x, y)^2$.

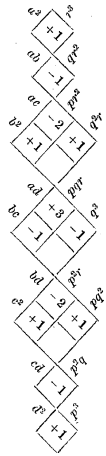


Table (4, 2).
Resultant of
$(a, b, c, d, e \between x, y)^4$,
$(p, q, r \between x, y)^2$.

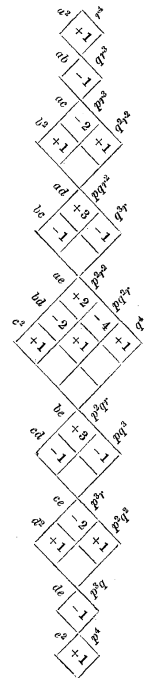


Table (3, 3).
Resultant of
$(a, b, c, d \between x, y)^3$,
$(p, q, r, s \between x, y)^3$.

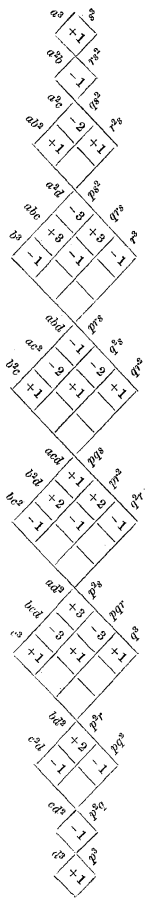
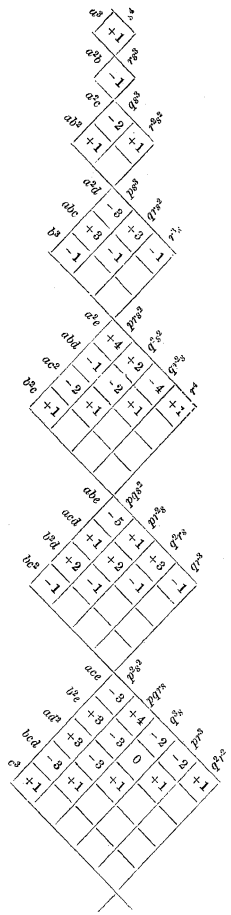


Table (4, 3).
Resultant of
$(a, b, c, d, e \between x, y)^4$,
$(p, q, r, s \between x, y)^3$.



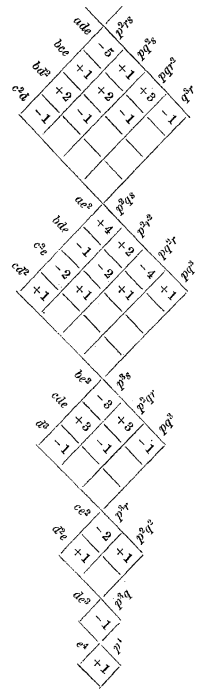


Table (4, 4).

Resultant of

$(a, b, c, d, e \between x, y)^4$,

$(p, q, r, s, t \between x, y)^4$.

| | | |

| | | |

149.

ON THE SYMMETRIC FUNCTIONS OF THE ROOTS OF CERTAIN SYSTEMS OF TWO EQUATIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 717—726. Received December 18, 1856,—Read January 8, 1857.]

SUPPOSE in general that $\phi = 0$, $\psi = 0$, &c. denote a system of $(n-1)$ equations between the n variables $(x, y, z, \ldots)$, where the functions ϕ, ψ, &c. are quantics (i.e. rational and integral homogeneous functions) of the variables. Any values $(x_1, y_1, z_1, \ldots)$ satisfying the equations, are said to constitute a set of roots of the system; the roots of the same set are, it is clear, only determinate to a common factor *près*, i.e. only the ratios *inter se* and not the absolute magnitudes of the roots of a set are determinate. The number of sets, or the degree of the system, is equal to the product of the degrees of the component equations. Imagine a function of the roots which remains unaltered when any two sets $(x_1, y_1, z_1, \ldots)$ and $(x_2, y_2, z_2, \ldots)$ are interchanged (that is, when x_1 and x_2, y_1 and y_2, &c. are simultaneously interchanged), and which is besides homogeneous of the same degree as regards each entire set of roots, although not of necessity homogeneous as regards the different roots of the same set; thus, for example, if the sets are (x_1, y_1), (x_2, y_2), then the functions x_1x_2, $x_1y_2 + x_2y_1$, y_1y_2 are each of them of the form in question; but the first and third of these functions, although homogeneous of the first degree in regard to each entire set, are not homogeneous as regards the two variables of each set. A function of the above-mentioned form may, for shortness, be termed a symmetric function of the roots; such function (disregarding an arbitrary factor depending on the common factors which enter implicitly into the different sets of roots) will be a rational and integral function of the coefficients of the equations, i.e. any symmetric function of the roots may be considered as a rational and integral function of the coefficients. The general process for the investigation of such expression for a symmetric function of the roots is indicated in Professor Schläfli's Memoir, "Ueber die Resultante eines Systemes mehrerer algebraischer

Gleichungen," *Vienna Transactions*, t. IV. (1852). The process is as follows:—Suppose that we know the resultant of a system of equations, one or more of them being linear; then if $\phi = 0$ be the linear equation or one of the linear equations of the system, the resultant will be of the form $\phi_1\phi_2\ldots$, where ϕ_1, ϕ_2, &c. are what the function ϕ becomes upon substituting therein the different sets $(x_1, y_1, z_1\ldots)$, $(x_2, y_2, z_2\ldots)$ of the remaining $(n-1)$ equations $\psi = 0$, $\chi = 0$, &c.; comparing such expression with the given value of the resultant, we have expressed in terms of the coefficients of the functions ψ, χ, &c., certain symmetric functions which may be called the fundamental symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c.; these are in fact the symmetric functions of the first degree in respect to each set of roots. By the aid of these fundamental symmetric functions, the other symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c. may be expressed in terms of the coefficients, and then combining with these equations a non-linear equation $\Phi = 0$, the resultant of the system $\Phi = 0$, $\psi = 0$, $\chi = 0$, &c. will be what the function $\Phi_1\Phi_2\ldots$ becomes, upon substituting therein for the different symmetric functions of the roots of the system $\psi = 0$, $\chi = 0$, &c. the expressions for these functions in terms of the coefficients. We thus pass from the resultant of a system $\phi = 0$, $\psi = 0$, $\chi = 0$, &c., to that of a system $\Phi = 0$, $\psi = 0$, $\chi = 0$, &c., in which the linear function ϕ is replaced by the non-linear function Φ. By what has preceded, the symmetric functions of the roots of a system of $(n-1)$ equations depend on the resultant of the system obtained by combining the $(n-1)$ equations with an arbitrary linear equation; and moreover, the resultant of any system of n equations depends ultimately upon the resultant of a system of the same number of equations, all except one being linear; but in this case the linear equations determine the ratios of the variables or (disregarding a common factor) the values of the variables, and by substituting these values in the remaining equation we have the resultant of the system. The process leads, therefore, to the expressions for the symmetric functions of the roots of any system of $(n-1)$ equations, and also to the expression for the resultant of any system of n equations. Professor Schläfli discusses in the general case the problem of showing how the expressions for the fundamental symmetric functions lead to those of the other symmetric functions, but it is not necessary to speak further of this portion of his investigations. The object of the present Memoir is to apply the process to two particular cases, viz. I propose to obtain thereby the expressions for the simplest symmetric functions (after the fundamental ones) of the following systems of two ternary equations; that is, first, a linear equation and a quadric equation; and secondly, a linear equation and a cubic equation.

First, consider the two equations

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

$$(\alpha, \beta, \gamma \between x, y, z) = 0,$$

and join to these the arbitrary linear equation

$$(\xi, \eta, \zeta \between x, y, z) = 0,$$

then the two linear equations give

$$x : y : z = \beta\zeta - \gamma\eta : \gamma\xi - \alpha\zeta : \alpha\eta - \beta\xi;$$

and substituting in the quadratic equation, we have for the resultant of the three equations,

$$(a, b, c, f, g, h \between \beta\zeta - \gamma\eta, \gamma\xi - \alpha\zeta, \alpha\eta - \beta\xi)^2 = 0,$$

which may be represented by

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between \xi, \eta, \zeta)^2 = 0,$$

where the coefficients are given by means of the Table.

	a	b	c	f	g	h	
a =		$+\gamma^2$	$+\beta^2$	$-2\beta\gamma$			(ξ^2)
b =	$+\gamma^2$		$+\alpha^2$		$-2\gamma\alpha$		(η^2)
c =	$+\beta^2$	$+\alpha^2$				$-2\alpha\beta$	(ζ^2)
f =	$-\beta\gamma$			$-\alpha^2$	$+\alpha\beta$	$+\gamma\alpha$	$2(\eta\zeta)$
g =		$-\gamma\alpha$		$+\alpha\beta$	$-\beta^2$	$+\beta\gamma$	$2(\zeta\xi)$
h =			$-\alpha\beta$	$+\gamma\alpha$	$+\beta\gamma$	$-\gamma^2$	$2(\xi\eta)$

viz. $\mathrm{a} = b\gamma^2 + c\beta^2 - 2f\beta\gamma$, &c.

But if the roots of the given system are

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2),$$

then the resultant of the three equations will be

$$(x_1, y_1, z_1 \between \xi, \eta, \zeta) \,.\, (x_2, y_2, z_2 \between \xi, \eta, \zeta) = 0;$$

and comparing the two expressions, we have

$$\begin{aligned}
\mathrm{a} &= x_1x_2, \\
\mathrm{b} &= y_1y_2, \\
\mathrm{c} &= z_1z_2, \\
2\mathrm{f} &= y_1z_2 + y_2z_1, \\
2\mathrm{g} &= z_1x_2 + z_2x_1, \\
2\mathrm{h} &= x_1y_2 + x_2y_1,
\end{aligned}$$

which are the expressions for the six fundamental symmetric functions, or symmetric functions of the first degree in each set, of the roots of the given system.

By forming the powers and products of the second order a^2, ab, &c., we obtain linear relations between the symmetric functions of the second degree in respect to each set of roots. The number of equations is precisely equal to that of the

symmetric functions of the form in question, and the solution of the linear equations gives—

$$\begin{aligned}
a^2 &= x_1^2x_2^2,\\
b^2 &= y_1^2y_2^2,\\
c^2 &= z_1^2z_2^2,\\
bc &= y_1z_1y_2z_2,\\
ca &= z_1x_1z_2x_2,\\
ab &= x_1y_1x_2y_2,\\
4f^2 - 2bc &= y_1^2z_2^2 + y_2^2z_1^2,\\
4g^2 - 2ca &= z_1^2x_2^2 + z_2^2x_1^2,\\
4h^2 - 2ab &= x_1^2y_2^2 + x_2^2y_1^2,\\
2af &= x_1y_1z_2x_2 + z_1x_1x_2y_2,\\
2bg &= y_1z_1x_2y_2 + x_1y_1y_2z_2,\\
2ch &= z_1x_1y_2z_2 + y_1z_1z_2x_2,\\
4gh - 2af &= x_1^2y_2z_2 + x_2^2y_1z_1,\\
4hf - 2bg &= y_1^2z_2x_2 + y_2^2z_1x_1,\\
4fg - 2ch &= z_1^2x_2y_2 + z_2^2x_1y_1,\\
2bf &= y_1^2y_2z_2 + y_2^2y_1z_1,\\
2cg &= z_1^2z_2x_2 + z_2^2z_1x_1,\\
2ah &= x_1^2x_2y_2 + x_2^2x_1y_1,\\
2cf &= z_1^2y_2z_2 + z_2^2y_1z_1,\\
2ag &= x_1^2z_2x_2 + x_2^2z_1x_1,\\
2bh &= y_1^2x_2y_2 + y_2^2x_1y_1.
\end{aligned}$$

Proceeding next to the powers and products of the third order a^3, a^2b, &c., the total number of linear relations between the symmetric functions of the third degree in respect to each set of roots exceeds by unity the number of the symmetric functions of the form in question; in fact the expressions for abc, af^2, bg^2, ch^2, fgh, contain, not five, but only four symmetric functions of the roots; for we have

$$\begin{aligned}
abc &= x_1y_1z_1 \,.\, x_2y_2z_2,\\
4af^2 &= (x_1y_1^2x_2z_2^2 + x_2y_2^2x_1z_1^2) + 2x_1y_1z_1x_2y_2z_2,\\
4bg^2 &= (y_1z_1^2y_2x_2^2 + y_2z_2^2y_1x_1^2) + 2x_1y_1z_1x_2y_2z_2,\\
4ch^2 &= (z_1x_1^2z_2y_2^2 + z_2x_2^2z_1y_1^2) + 2x_1y_1z_1x_2y_2z_2,\\
8fgh &= \left.\begin{aligned}&(x_1y_1^2x_2z_2^2 + x_2y_2^2x_1z_1^2)\\ &+ (y_1z_1^2y_2z_2^2 + y_2^2z_2^2y_1x_1^2)\\ &+ (z_1x_1^2z_2x_2^2 + z_2x_2^2z_1y_1^2)\end{aligned}\right\} + 2x_1y_1z_1x_2y_2z_2,
\end{aligned}$$

and consequently the quantities a, b, c, f, g, h, are not independent, but are connected by the equation

$$\mathrm{abc} - \mathrm{af}^2 - \mathrm{bg}^2 - \mathrm{ch}^2 + 2\mathrm{fgh} = 0,$$

an equation, which is in fact verified by the foregoing values of a, &c. in terms of the coefficients of the given system.

The expressions for the symmetric functions of the third degree considered as functions of a, b, c, f, g, h, are consequently not absolutely determinate, but they may be modified by the addition of the term $\lambda(\mathrm{abc} - \mathrm{af}^2 - \mathrm{bg}^2 - \mathrm{ch}^2 + 2\mathrm{fgh})$, where λ is an indeterminate numerical coefficient.

The simplest expressions are those obtained by disregarding the preceding equation for fgh, and the entire system then becomes:

$$\begin{aligned}
\mathrm{a}^3 &= x_1^3x_2^3,\\
\mathrm{b}^3 &= y_1^3y_2^3,\\
\mathrm{c}^3 &= z_1^3z_2^3,\\
\mathrm{b}^2\mathrm{c} &= y_1^2z_1y_2^2z_2,\\
\mathrm{c}^2\mathrm{a} &= z_1^2x_1z_2^2x_2,\\
\mathrm{a}^2\mathrm{b} &= x_1^2y_1x_2^2y_2,\\
\mathrm{bc}^2 &= y_1z_1^2y_2z_2^2,\\
\mathrm{ca}^2 &= z_1x_1^2z_2x_2^2,\\
\mathrm{ab}^2 &= x_1y_1^2x_2y_2^2,\\
\mathrm{abc} &= x_1y_1z_1x_2y_2z_2,\\
2\mathrm{a}^2\mathrm{f} &= x_1^2y_1z_2x_2^2 + x_2^2y_2z_1x_1^2,\\
2\mathrm{b}^2\mathrm{g} &= y_1^2z_1x_2y_2^2 + y_2^2z_2x_1y_1^2,\\
2\mathrm{c}^2\mathrm{h} &= z_1^2x_1y_2z_2^2 + z_2^2x_2y_1z_1^2,\\
2\mathrm{a}^2\mathrm{g} &= x_1^3z_2x_2^2 + x_2^3z_1x_1^2,\\
2\mathrm{b}^2\mathrm{h} &= y_1^3x_2y_2^2 + y_2^3x_1y_1^2,\\
2\mathrm{c}^2\mathrm{f} &= z_1^3y_2z_2^2 + z_2^3y_1z_1^2,\\
2\mathrm{a}^2\mathrm{h} &= x_1^3z_2^2x_2 + x_2^3z_1^2x_1,\\
2\mathrm{b}^2\mathrm{f} &= y_1^3x_2^2y_2 + y_2^3x_1^2y_1,\\
2\mathrm{c}^2\mathrm{g} &= z_1^3y_2^2z_2 + z_2^3y_1^2z_1,\\
2\mathrm{bcf} &= y_1^2z_1y_2z_2^2 + y_2^2z_2y_1z_1^2,\\
2\mathrm{cag} &= z_1^2x_1z_2x_2^2 + z_2^2x_2z_1x_1^2,\\
2\mathrm{abh} &= x_1^2y_1x_2y_2^2 + x_2^2y_2x_1y_1^2,\\
2\mathrm{bcg} &= y_1z_1^2x_2y_2z_2 + y_2z_2^2x_1y_1z_1,\\
2\mathrm{cah} &= z_1x_1^2x_2y_2z_2 + z_2x_2^2x_1y_1z_1,\\
2\mathrm{abf} &= x_1y_1^2x_2y_2z_2 + x_2y_2^2x_1y_1z_1,
\end{aligned}$$

$$\begin{aligned}
2\mathrm{bch} &= y_1^2 z_1 x_2 y_2 z_2 + y_2^2 z_2 x_1 y_1 z_1, \\
2\mathrm{caf} &= z_1^2 x_1 x_2 y_2 z_2 + z_2^2 x_2 x_1 y_1 z_1, \\
2\mathrm{abg} &= x_1^2 y_1 x_2 y_2 z_2 + x_2^2 y_2 x_1 y_1 z_1,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{af}^2 - 2\mathrm{abc} &= x_1 y_1^2 z_2^2 x_2 + x_2 y_2^2 z_1^2 x_1, \\
4\mathrm{bg}^2 - 2\mathrm{abc} &= y_1 z_1^2 x_2^2 y_2 + y^2 z_2^2 x_1^2 y_1, \\
4\mathrm{ch}^2 - 2\mathrm{abc} &= z_1 x_1^2 y_2^2 z_2 + z_2 x_2^2 y_1^2 z_1,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{bf}^2 - 2\mathrm{b}^2\mathrm{c} &= y_1^3 y_2 z_2^2 + y_2^3 y_1 z_1^2, \\
4\mathrm{cg}^2 - 2\mathrm{c}^2\mathrm{a} &= z_1^3 z_2 x_2^2 + z_2^3 z_1 x_1^2, \\
4\mathrm{ah}^2 - 2\mathrm{a}^2\mathrm{b} &= x_1^3 x_2 y_2^2 + x_2^3 x_1 y_1^2,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{cf}^2 - 2\mathrm{bc}^2 &= z_1^3 y_2^2 z_2 + z_2^3 y_1^2 z_1, \\
4\mathrm{ag}^2 - 2\mathrm{ca}^2 &= x_1^3 z_2^2 x_2 + x_2^3 z_1^2 x_1, \\
4\mathrm{bh}^2 - 2\mathrm{ab}^2 &= y_1^3 x_2^2 y_2 + y_2^3 x_1^2 y_1,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{agh} - 2\mathrm{a}^2\mathrm{f} &= x_1^3 x_2 y_2 z_2 + x_2^3 x_1 y_1 z_1, \\
4\mathrm{bhf} - 2\mathrm{b}^2\mathrm{g} &= y_1^3 x_2 y_2 z_2 + y_2^3 x_1 y_1 z_1, \\
4\mathrm{cfg} - 2\mathrm{c}^2\mathrm{h} &= z_1^3 x_2 y_2 z_2 + z_2^3 x_1 y_1 z_1,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{bgh} - 2\mathrm{abf} &= y_1^2 z_1 x_2^2 y_2 + y_2^2 z_2 x_1^2 y_1, \\
4\mathrm{chf} - 2\mathrm{bcg} &= z_1^2 x_1 y_2^2 z_2 + z_2^2 x_2 y_1^2 z_1, \\
4\mathrm{afg} - 2\mathrm{cah} &= x_1^2 y_1 z_2^2 x_2 + x_2^2 y_2 z_1^2 x_1,
\end{aligned}$$

$$\begin{aligned}
4\mathrm{cgh} - 2\mathrm{acf} &= y_1 z_1^2 x_2^2 z_2 + y_2 z_2^2 x_1^2 z_1, \\
4\mathrm{ahf} - 2\mathrm{bag} &= z_1 x_1^2 y_2^2 x_2 + z_2 x_2^2 y_1^2 x_1, \\
4\mathrm{bfg} - 2\mathrm{cbh} &= x_1 y_1^2 z_2^2 y_2 + x_2 y_2^2 z_1^2 y_1,
\end{aligned}$$

$$\begin{aligned}
8\mathrm{f}^2\mathrm{g} - 4\mathrm{chf} - 2\mathrm{bcg} &= z_1^3 x_2 y_2^2 + z_2^3 x_1 y_1^2, \\
8\mathrm{g}^2\mathrm{h} - 4\mathrm{afg} - 2\mathrm{cah} &= x_1^3 y_2 z_2^2 + x_2^3 y_1 z_1^2, \\
8\mathrm{h}^2\mathrm{f} - 4\mathrm{bgh} - 2\mathrm{abf} &= y_1^3 z_2 x_2^2 + y_2^3 z_1 x_1^2,
\end{aligned}$$

$$\begin{aligned}
8\mathrm{fg}^2 - 4\mathrm{chg} - 2\mathrm{acf} &= z_1^2 x_2^2 y_2 + z_2^2 x_1^2 y_1, \\
8\mathrm{gh}^2 - 4\mathrm{afh} - 2\mathrm{bag} &= x_1^3 y_2^2 z_2 + x_2^3 y_1^2 z_1, \\
8\mathrm{hf}^2 - 4\mathrm{bgf} - 2\mathrm{cbh} &= y_1^3 z_2^2 x_2 + y_2^3 z_1^2 x_1,
\end{aligned}$$

$$\begin{aligned}
8\mathrm{f}^3 - 6\mathrm{bcf} &= y_1^3 z_2^3 + y_2^3 z_1^3, \\
8\mathrm{g}^3 - 6\mathrm{cag} &= z_1^3 x_2^3 + z_2^3 x_1^3, \\
8\mathrm{h}^3 - 6\mathrm{abh} &= x_1^3 y_2^3 + x_2^3 y_1^3.
\end{aligned}$$

Secondly, consider the system of equations

$$\begin{aligned}
(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3 &= 0, \\
(\alpha, \beta, \gamma \between x, y, z) &= 0,
\end{aligned}$$

where the cubic function written at full length is

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3kxy^2 + 6lxyz.$$

Joining to the system the linear equation

$$(\xi,\ \eta,\ \zeta \between x,\ y,\ z) = 0,$$

the linear equations give

$$x : y : z = \beta\zeta - \gamma\eta : \gamma\xi - \alpha\zeta : \alpha\eta - \beta\xi,$$

and the resultant is

$$(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l \between \beta\zeta - \gamma\eta,\ \gamma\xi - \alpha\zeta,\ \alpha\eta - \beta\xi)^3 = 0,$$

which may be represented by

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h},\ \mathrm{i},\ \mathrm{j},\ \mathrm{k},\ \mathrm{l} \between \xi,\ \eta,\ \zeta)^3 = 0,$$

where the coefficients a, b, &c. are given by means of the Table :—

	a	b	c	f	g	h	i	j	k	l	
a =		$+\gamma^3$	$-\beta^3$	$-3\beta\gamma^2$			$+3\beta^2\gamma$				ξ^3
b =	$-\gamma^3$		$+\alpha^3$		$-3\gamma\alpha^2$			$+3\gamma^2\alpha$			η^3
c =	$+\beta^3$	$-\alpha^3$				$-3\alpha\beta^2$			$+3\alpha^2\beta$		ζ^3
f =	$+\beta\gamma^2$				$+\alpha^2\beta$	$-\gamma^2\alpha$	$-\alpha^3$	$-2\alpha\beta\gamma$		$+2\gamma\alpha^2$	$3\eta^2\zeta$
g =		$+\gamma\alpha^2$		$-\alpha^2\beta$		$+\beta^2\gamma$		$-\beta^3$	$-2\alpha\beta\gamma$	$+2\alpha\beta^2$	$3\zeta^2\xi$
h =			$+\alpha\beta^2$	$+\gamma^2\alpha$	$-\beta^2\gamma$		$-2\alpha\beta\gamma$		$-\gamma^3$	$+2\beta\gamma^2$	$3\xi^2\eta$
i =	$-\beta^2\gamma$			$+\alpha^3$		$+2\alpha\beta\gamma$		$+\alpha\beta^2$	$-\gamma\alpha^2$	$-2\alpha^2\beta$	$3\eta\zeta^2$
j =		$-\gamma^2\alpha$		$+2\alpha\beta\gamma$	$+\beta^3$		$-\alpha\beta^2$		$+\beta\gamma^2$	$-2\beta^2\gamma$	$3\zeta\xi^2$
k =			$-\alpha^2b$		$+2\alpha\beta\gamma$	$+\gamma^3$	$+\gamma\alpha^2$	$-\beta\gamma^2$		$-2\gamma^2\alpha$	$3\xi^2\eta$
l =				$-\gamma\alpha^2$	$-\alpha\beta^2$	$-\beta\gamma^2$	$+\alpha^2\beta$	$+\beta^2\gamma$	$+\gamma^2\alpha$		$6\xi\eta\zeta$

viz.

$$\mathrm{a} = b\gamma^3 - c\beta^3 - 3f\beta\gamma^2 + 3i\beta^2\gamma, \text{ \&c.}$$

But if the roots of the given system are

$$(x_1,\ y_1,\ z_1),\quad (x_2,\ y_2,\ z_2),\quad (x_3,\ y_3,\ z_3),$$

then the resultant of the three equations may also be represented by

$$(x_1,\ y_1,\ z_1 \between \xi,\ \eta,\ \zeta) \cdot (x_2,\ y_2,\ z_2 \between \xi,\ \eta,\ \zeta) \cdot (x_3,\ y_3,\ z_3 \between \xi,\ \eta,\ \zeta);$$

and comparing with the former expression, we find :

$$\mathrm{a} = x_1x_2x_3,$$
$$\mathrm{b} = y_1y_2y_3,$$
$$\mathrm{c} = z_1z_2z_3,$$

$$\begin{aligned}
3\mathrm{f} &= y_1y_2z_3 + y_2y_3z_1 + y_3y_1z_2,\\
3\mathrm{g} &= z_1z_2x_3 + z_2z_3x_1 + z_3z_1x_2,\\
3\mathrm{h} &= x_1x_2y_3 + x_2x_3y_1 + x_3x_1y_2,\\
3\mathrm{i} &= y_1z_2z_3 + y_2z_3z_1 + y_3z_1z_2,\\
3\mathrm{j} &= z_1x_2x_3 + z_2x_3x_1 + z_3x_1x_2,\\
3\mathrm{k} &= x_1y_2y_3 + x_2y_3y_1 + x_3y_1y_2,\\
6\mathrm{l} &= x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 + x_1y_3z_2 + x_2y_1z_3 + x_3y_2z_1.
\end{aligned}$$

But there is in the present case a relation independent of the quantities a, &c., viz. we have $(\alpha, \beta, \gamma \between x_1, y_1, z_1) = 0$, $(\alpha, \beta, \gamma \between x_2, y_2, z_2) = 0$, $(\alpha, \beta, \gamma \between x_3, y_3, z_3) = 0$, and thence eliminating the coefficients (α, β, γ), we find

$$\nabla = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1 = 0.$$

By forming the powers and products of the second degree a^2, ab, &c., we obtain 55 equations between the symmetric functions of the second degree in each set of roots. But we have $\nabla^2 = 0 =$ a symmetric function of the roots, and thus the entire number of linear relations is 56, and this is in fact the number of the symmetric functions of the second degree in each set. I use for shortness the sign S to denote the sum of the distinct terms obtained by permuting the different sets of roots, so that the equations for the fundamental symmetric functions are—

$$\begin{aligned}
\mathrm{a} &= x_1x_2x_3,\\
\mathrm{b} &= y_1y_2y_3,\\
\mathrm{c} &= z_1z_2z_3,\\
3\mathrm{f} &= S\, y_1y_2z_3,\\
3\mathrm{g} &= S\, z_1z_2x_3,\\
3\mathrm{h} &= S\, x_1x_2y_3,\\
3\mathrm{i} &= S\, y_1z_2z_3,\\
3\mathrm{j} &= S\, z_1x_2x_3,\\
3\mathrm{k} &= S\, x_1y_2y_3,\\
6\mathrm{l} &= S\, x_1y_2z_3;
\end{aligned}$$

then the complete system of expressions for the symmetric functions of the second order is as follows, viz.

$$\begin{aligned}
\mathrm{a}^2 &= x_1^2x_2^2x_3^2,\\
\mathrm{b}^2 &= y_1^2y_2^2y_3^2,\\
\mathrm{c}^2 &= z_1^2z_2^2z_3^2,\\
\mathrm{bc} &= y_1z_1y_2z_2y_3z_3,\\
\mathrm{ca} &= z_1x_1z_2x_2z_3x_3,\\
\mathrm{ab} &= x_1y_1x_2y_2x_3y_3,
\end{aligned}$$

$$3af = S\, x_1y_1x_2y_2z_3x_3,$$
$$3bg = S\, y_1z_1y_2z_2x_3y_3,$$
$$3ch = S\, z_1x_1z_2x_2y_3z_3,$$

$$3bf = S\, y_1^2y_2^2y_3z_3,$$
$$3cg = S\, z_1^2z_2^2z_3x_3,$$
$$3ah = S\, x_1^2x_2^2x_3y_3,$$

$$3cf = S\, y_1z_1y_2z_2z_3^2,$$
$$3ag = S\, z_1x_1z_2x_2x_3^2,$$
$$3bh = S\, x_1y_1x_2y_2y_3^2,$$

$$3ai = S\, x_1y_1z_2x_2z_3x_3,$$
$$3bj = S\, y_1z_1x_2y_2x_3y_3,$$
$$3ck = S\, z_1x_1y_2z_2y_3z_3,$$

$$3bi = S\, y_1^2y_2z_2y_3z_3,$$
$$3cj = S\, z_1^2z_2x_2z_3x_3,$$
$$3ak = S\, x_1^2x_2y_2x_3y_3,$$

$$3ci = S\, y_1z_1z_2^2z_3^2,$$
$$3aj = S\, z_1x_1x_2^2x_3^2,$$
$$3bk = S\, x_1y_1y_2^2y_3^2,$$

$$6al = S\, x_1^2x_2y_2z_3x_3,$$
$$6bl = S\, y_1^2y_2z_2x_3y_3,$$
$$6cl = S\, z_1^2z_2x_2y_3z_3,$$

$$9f^2 - 6bi = S\, y_1^2y_2^2z_3^2,$$
$$9g^2 - 6cj = S\, z_1^2z_2^2x_3^2,$$
$$9h^2 - 6ak = S\, x_1^2x_2^2y_3^2,$$

$$9i^2 - 6cf = S\, y_1^2z_2^2z_3^2,$$
$$9j^2 - 6ag = S\, z_1^2x_2^2x_3^2,$$
$$9k^2 - 6bh = S\, x_1^2y_2^2y_3^2,$$

$$9fg - 3ck = S\, x_1y_1y_2z_2z_3^2,$$
$$9gh - 3ai = S\, y_1z_1z_2x_2x_3^2,$$
$$9hf - 3bj = S\, z_1x_1x_2y_2y_3^2,$$

$$9\mathrm{jk} - 3\mathrm{af} = S\, x_1^2 x_2 y_2 y_3 z_3,$$
$$9\mathrm{ki} - 3\mathrm{bg} = S\, y_1^2 y_2 z_2 z_3 x_3,$$
$$9\mathrm{ij} - 3\mathrm{ch} = S\, z_1^2 z_2 x_2 x_3 y_3,$$

$$9\mathrm{fi} - 3\mathrm{bc} = S\, y_1^2 y_2 z_2 z_3^2,$$
$$9\mathrm{gj} - 3\mathrm{ca} = S\, z_1^2 z_2 x_2 x_3^2,$$
$$9\mathrm{hk} - 3\mathrm{ab} = S\, x_1^2 x_2 y_2 y_3^2,$$

$$3(\mathrm{fj} + \mathrm{gk} + \mathrm{hi} - \mathrm{l}^2) = S\, x_1 y_1 z_2 x_2 y_3 z_3,$$

$$3(2\mathrm{fj} - \mathrm{gk} - \mathrm{hi} + \mathrm{l}^2) = S\, x_1 y_1 x_2 y_2 z_3^2,$$
$$3(2\mathrm{gk} - \mathrm{hi} - \mathrm{fj} + \mathrm{l}^2) = S\, y_1 z_1 y_2 z_2 x_3^2,$$
$$3(2\mathrm{hi} - \mathrm{fj} - \mathrm{gk} + \mathrm{l}^2) = S\, z_1 x_1 z_2 x_2 y_3^2,$$

$$3(6\mathrm{fl} - 3\mathrm{ki} - \mathrm{bg}) = S\, x_1 y_1 y_2^2 z_3^2,$$
$$3(6\mathrm{gl} - 3\mathrm{ij} - \mathrm{ch}) = S\, y_1 z_1 z_2^2 x_3^2,$$
$$3(6\mathrm{hl} - 3\mathrm{jk} - \mathrm{af}) = S\, z_1 x_1 x_2^2 y_3^2,$$

$$3(6\mathrm{il} - 3\mathrm{fg} - \mathrm{ck}) = S\, z_1 x_1 y_2^2 z_3^2,$$
$$3(6\mathrm{jl} - 3\mathrm{gh} - \mathrm{ai}) = S\, x_1 y_1 z_2^2 x_3^2,$$
$$3(6\mathrm{kl} - 3\mathrm{hf} - \mathrm{bj}) = S\, y_1 z_1 x_2^2 y_3^2,$$

$$6(-\mathrm{fj} - \mathrm{gk} - \mathrm{hi} + 4\mathrm{l}^2) = S\, x_1^2 y_2^2 z_3^2.$$

As an instance of the application of the formulæ, let it be required to eliminate the variables from the three equations,

$$(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3 = 0,$$
$$(a', b', c', f', g', h' \qquad \between x, y, z)^2 = 0,$$
$$(\alpha, \beta, \gamma \qquad \between x, y, z) = 0.$$

This may be done in two different ways; first, representing the roots of the linear equation and the quadric equation by (x_1, y_1, z_1), (x_2, y_2, z_2), the resultant will be

$$(a, \ldots \between x_1, y_1, z_1)^3 . (a, \ldots \between x_2, y_2, z_2)^3,$$

which is equal to

$$a^2 x_1^3 x_2^3 + \&c.,$$

where the symmetric functions $x_1^3 x_2^3$, &c. are given by the formulæ $a'^3 = x_1^3 x_2^3$, &c., in which, since the coefficients of the quadratic equation are (a', b', c', f', g', h'), I have written a' instead of a. Next, if the roots of the linear equation and the cubic equation are represented by (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), then the resultant will be

$$(a', \ldots \between x_1, y_1, z_1)^2 . (a', \ldots \between x_2, y_2, z_2)^2 (a', \ldots \between x_3, y_3, z_3)^2,$$

which is equal to

$$a'^3\, x_1^2 x_2^2 x_3^2 + \&c.,$$

the symmetric functions $x_1^2 x_2^2 x_3^2$, &c. being given by the formulæ $a^2 = x_1^2 x_2^2 x_3^2$, &c. The expression for the Resultant is in each case of the right degree, viz. of the degrees 6, 3, 2, in the coefficients of the linear, the quadric, and the cubic equations respectively: the two expressions, therefore, can only differ by a numerical factor, which might be determined without difficulty. The third expression for the resultant, viz.

$$(\alpha, \beta, \gamma \between x_1, y_1, z_1).(\alpha, \beta, \gamma \between x_2, y_2, z_2)\ldots(\alpha, \beta, \gamma \between x_6, y_6, z_6),$$

(where $(x_1, y_1, z_1), \ldots (x_6, y_6, z_6)$ are the roots of the cubic and quadratic equations) compared with the foregoing value, leads to expressions for the fundamental symmetric functions of the cubic and quadratic equations, and thence to expressions for the other symmetric functions of these two equations; but it would be difficult to obtain the actually developed values even of the fundamental symmetric functions. I hope to return to the subject, and consider in a general point of view the question of the formation of the expressions for the other symmetric functions by means of the expressions for the fundamental symmetric functions.

150.

A MEMOIR ON THE CONDITIONS FOR THE EXISTENCE OF GIVEN SYSTEMS OF EQUALITIES AMONG THE ROOTS OF AN EQUATION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. *for the year* 1857, pp. 727—731. Received December 18, 1856,—Read January 8, 1857.]

IT is well known that there is a symmetric function of the roots of an equation, viz. the product of the squares of the differences of the roots, which vanishes when any two roots are put equal to each other, and that consequently such function expressed in terms of the coefficients and equated to zero, gives the condition for the existence of a pair of equal roots. And it was remarked long ago by Professor Sylvester, in some of his earlier papers in the *Philosophical Magazine*, that the like method could be applied to finding the conditions for the existence of other systems of equalities among the roots, viz. that it was possible to form symmetric functions, each of them a sum of terms containing the product of a certain number of the differences of the roots, and such that the entire function might vanish for the particular system of equalities in question; and that such functions expressed in terms of the coefficients and equated to zero would give the required conditions. The object of the present memoir is to extend this theory and render it exhaustive, by showing how to form a series of types of all the different functions which vanish for one or more systems of equalities among the roots; and in particular to obtain by the method distinctive conditions for all the different systems of equalities between the roots of a quartic or a quintic equation, viz. for each system conditions which are satisfied for the particular system, and are not satisfied for any other systems, except, of course, the more special systems included in the particular system. The question of finding the conditions for any particular system of equalities is essentially an indeterminate one, for given any set of functions which vanish, a function syzygetically connected with these will also vanish; the discussion of the nature of the

syzygetic relations between the different functions which vanish for any particular system of equalities, and of the order of the system composed of the several conditions for the particular system of equalities, does not enter into the plan of the present memoir. I have referred here to the indeterminateness of the question for the sake of the remark that I have availed myself thereof, to express by means of invariants or covariants the different systems of conditions obtained in the sequel of the memoir; the expressions of the different invariants and covariants referred to are given in my 'Second Memoir upon Quantics,' *Philosophical Transactions*, vol. CXLVI. (1856), [141].

1. Suppose, to fix the ideas, that the equation is one of the fifth order, and call the roots α, β, γ, δ, ϵ. Write $12=\Sigma\phi(\alpha-\beta)^l$, $12.13=\Sigma\phi(\alpha-\beta)^l(\alpha-\gamma)^m$, $12.34=\Sigma\phi(\alpha-\beta)^l(\gamma-\delta)^n$, &c., where ϕ is an arbitrary function and l, m, &c. are positive integers. It is hardly necessary to remark that similar types, such as 12, 13, 45, &c., or as 12.13 and 23.25, &c., denote identically the same sums. Two types, such as 12.13 and 14.15.23.24.25.34.35.45, may be said to be complementary to each other. A particular product $(\alpha-\beta)(\gamma-\delta)$ does or does not enter as a term (or factor of a term) in one of the above-mentioned sums, according as the type 12.34 of the product, or some similar type, does or does not form part of the type of the sum; for instance, the product $(\alpha-\beta)(\gamma-\delta)$ is a term (or factor of a term) of each of the sums 12.34, 13.45.24, &c., but not of the sums 12.13.14.15, &c.

2. If, now, we establish any equalities between the roots, e.g. $\alpha=\beta$, $\gamma=\delta$, the effect will be to reduce certain of the sums to zero, and it is easy to find in what cases this happens. The sum will vanish if each term contains one or both of the factors $\alpha-\beta$, $\gamma-\delta$, i.e. if there is no term the complementary of which contains the product $(\alpha-\beta)(\gamma-\delta)$, or what is the same thing, whenever the complementary type does not contain as part of it, a type such as 12.34. Thus for the sum 14.15.24.25.34.35.45, the complementary type is 12.13.23, which does not contain any type such as 12.34, i.e. the sum 14.15.24.25.34.35.45 vanishes for $\alpha=\beta$, $\gamma=\delta$. It is of course clear that it also vanishes for $\alpha=\beta=\epsilon$, $\gamma=\delta$ or $\alpha=\beta=\gamma=\delta$, &c., which are included in $\alpha=\beta$, $\gamma=\delta$. But the like reasoning shows, and it is important to notice, that the sum in question does not vanish for $\alpha=\beta=\gamma$: and of course it does not vanish for $\alpha=\beta$. Hence the vanishing of the sum 14.15.24.25.34.35.45 is characteristic of the system $\alpha=\beta$, $\gamma=\delta$. A system of roots α, β, γ, δ, ϵ may be denoted by 11111; but if $\alpha=\beta$, then the system may be denoted by 2111, or if $\alpha=\beta$, $\gamma=\delta$, by 221, and so on. We may then say that the sum 14.15.24.25.34.35.45 does not vanish for 2111, vanishes for 221, does not vanish for 311, vanishes for 32, 41, 5.

3. For the purpose of obtaining the entire system of results it is only necessary to form Tables, such as the annexed Tables, the meaning of which is sufficiently explained by what precedes: the mark (×) set against a type denotes that the sum represented by the complementary type vanishes, the mark (○) that the complementary type does not vanish, for the system of roots denoted by the symbol at the top or bottom of the column; the complementary type is given in the same horizontal line with the original type. It will be noticed that the right-hand columns do not extend to the foot of the Table; the reason of this of course is, to avoid a repetition of the same type. Some of

the types at the foot of the Tables are complementary to themselves, but I have, notwithstanding this, given the complementary type in the form under which it naturally presents itself.

4. The Tables are:

Table for the equal Roots of a Quartic.

211	22	31	4		211	22	31	4
×	×	×	×	12 . 13 . 14 . 23 . 24 . 34	o	o	o	o
o	×	×	×	12 . 13 . 14 . 23 . 24 . 34	o	o	o	×
o	×	×	×	12 . 13 . 14 . 23 . 24 . 34	o	o	o	×
o	o	×	×	12 . 34 . 13 . 14 . 23 . 24	o	o	×	×
o	×	×	×	12 . 13 . 14 . 23 . 24 . 34	211	22	31	4
o	o	×	×	12 . 13 . 24 . 14 . 23 . 34				
o	×	o	×	12 . 13 . 23 . 14 . 24 . 34				
211	22	31	4					

Table for the equal Roots of a Quintic.

2111	221	311	32	41	5		2111	221	311	32	41	5
×	×	×	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	o	o	o	o	o	o
o	×	×	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	o	o	o	o	o	×
o	×	×	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	o	o	o	o	o	×
o	o	×	×	×	×	12 . 34 . 13 . 14 . 15 . 23 . 24 . 25 . 35 . 45	o	o	o	o	×	×
o	×	×	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	o	o	o	o	o	×
o	o	×	×	×	×	12 . 13 . 45 . 14 . 15 . 23 . 24 . 25 . 34 . 35	o	o	o	o	×	×
o	o	×	×	×	×	12 . 13 . 24 . 14 . 15 . 23 . 25 . 34 . 35 . 45	o	o	o	o	×	×
o	×	o	×	×	×	12 . 13 . 23 . 14 . 15 . 24 . 25 . 34 . 35 . 45	o	o	o	×	×	×
o	×	×	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	o	o	o	×	o	×
o	o	o	×	×	×	12 . 13 . 14 . 23 . 15 . 24 . 25 . 34 . 35 . 45	o	o	o	×	×	×
o	o	×	×	×	×	12 . 13 . 14 . 25 . 15 . 23 . 24 . 34 . 35 . 45	o	o	o	×	×	×
o	o	×	×	×	×	12 . 13 . 24 . 34 . 14 . 15 . 23 . 25 . 35 . 45	o	o	o	o	×	×
o	o	×	×	×	×	12 . 14 . 23 . 35 . 13 . 15 . 24 . 25 . 34 . 45	o	o	o	o	×	×
o	o	o	o	×	×	12 . 13 . 23 . 45 . 14 . 15 . 24 . 25 . 34 . 35	o	o	×	×	×	×
o	o	o	×	×	×	12 . 13 . 14 . 15 . 23 . 24 . 25 . 34 . 35 . 45	2111	221	311	32	41	5
o	o	o	×	×	×	12 . 13 . 14 . 23 . 24 . 15 . 25 . 34 . 35 . 45						
o	o	o	×	×	×	12 . 13 . 14 . 23 . 25 . 15 . 24 . 34 . 35 . 45						
o	o	o	×	×	×	12 . 13 . 15 . 24 . 34 . 14 . 23 . 25 . 35 . 45						
o	o	o	o	×	×	12 . 13 . 14 . 23 . 45 . 15 . 24 . 25 . 34 . 35						
o	o	×	×	×	×	12 . 15 . 23 . 34 . 45 . 13 . 14 . 24 . 25 . 35						
2111	221	311	32	41	5							

The two Tables enable the discussion of the theory of the equal roots of a quartic or quintic equation: first for the quartic:

5. In order that a quartic may have a pair of equal roots, or what is the same thing, that the system of roots may be of the form 211, the type to be considered is

12.13.14.23.24.34;

this of course gives as the function to be equated to zero, the discriminant of the quartic.

6. In order that there may be two pairs of equal roots, or that the system may be of the form 22, the simplest type to be considered is

$$14.24.34;$$

this gives the function

$$\Sigma(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(x-\alpha y)^2(x-\beta y)^2(x-\gamma y)^2,$$

which being a covariant of the degree 3 in the coefficients and the degree 6 in the variables, can only be the cubicovariant of the quartic.

7. In order that the quartic may have three equal roots, or that the system of roots may be of the form 31, we may consider the type

$$13.14.23.24,$$

and we obtain thence the two functions

$$\Sigma(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta),$$
$$\Sigma(\alpha-\gamma)^2(\alpha-\delta)(\beta-\gamma)(\beta-\delta)^2,$$

which being respectively invariants of the degrees 2 and 3, are of course the quadrinvariant and the cubinvariant of the quartic. If we had considered the apparently more simple type

$$12.34,$$

this gives the function

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2,$$

which is the quadrivariant, but the cubinvariant is not included under the type in question.

8. Finally, if the roots are all equal, or the system of roots is of the form 4, then the simplest type is

$$12;$$

and this gives the function

$$\Sigma(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2,$$

a covariant of the degree 2 in the coefficients and the degree 4 in the variables; this is of course the Hessian of the quartic.

Considering next the case of the quintic:

9. In order that a quintic may have a pair of equal roots, or what is the same thing, that the system of roots may be of the form 2111, the type to be considered is

$$12.13.14.15.23.24.25.34.35.45;$$

this of course gives as the function to be equated to zero, the discriminant of the quintic.

10. In order that the quintic may have two pairs of equal roots, or that the system of roots may be 221, the simplest type to be considered is

$$14.15.24.25.34.35.45\,;$$

a type which gives the function

$$\Sigma\,(\alpha-\delta)(\alpha-\epsilon)(\beta-\delta)(\beta-\epsilon)(\gamma-\delta)(\gamma-\epsilon)(\delta-\epsilon)^2\,(x-\alpha y)^2\,(x-\beta y)^2\,(x-\gamma y)^2.$$

This is a covariant of the degree 5 in the coefficients and of the degree 9 in the variables; but it appears from the memoir above referred to, that there is not any irreducible covariant of the form in question; such covariant must be a sum of the products (No. 13)(No. 20), (No. 13)(No. 14)^2, (No. 15)(No. 16) (the numbers refer to the Covariant Tables given in the memoir), each multiplied by a merely numerical coefficient. These numerical coefficients may be determined by the consideration that there being two pairs of equal roots, we may by a linear transformation make these roots 0, 0, ∞, ∞, or what is the same thing, we may write $a=b=e=f=0$, the covariant must then vanish identically. The coefficients are thus found to be 1, -4, 50, and we have for a covariant vanishing in the case of two pairs of equal roots,

$$\begin{aligned} &1\ \text{(No. 13)(No. 20)}\\ -\ &4\ \text{(No. 13)(No. 14)}^2\\ +\ &50\ \text{(No. 15)(No. 16)}\end{aligned}$$

[or in the new notation $AH-4AB^2+50CD$].

In fact, writing $a=b=e=f=0$, and rejecting, where it occurs, a factor x^2y^2, the several covariants become functions of cx, dy; and putting, for shortness, x, y instead of cx, dy, the equation to be verified is

$$\begin{aligned} &1\,.\,10(x+y)(6x^4+8x^3y+28x^2y^2+8xy^3+6y^4)\\ -\ &4\,.\,10(x+y)(3x^2+2xy+3y^2)^2\\ +\ &50(6x^2+8xy+6y^2)(x^3+x^2y+xy^2+y^3)=0\,;\end{aligned}$$

and dividing out by $(x+y)$ and reducing, the equation is at once seen to be identically true.

11. In order that the quintic may have three equal roots, or that the system of roots may be of the form 311, the simplest type to be considered is

$$12.13.23.45\,;$$

this gives the function

$$\Sigma\,(\alpha-\beta)^2\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\delta-\epsilon)^4,$$

which being an invariant, and being of the fourth degree in the coefficients, must be the quartinvariant of the quintic [that is No. 19, $=G$]. The same type gives also the function

$$\Sigma\,(\alpha-\beta)^2(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\delta-\epsilon)^2\,(x-\delta y)^2\,(x-\epsilon y)^2,$$

which is a covariant of the degree 4 in the coefficients and the degree 4 in the variables; and it must vanish when $a=b=c=0$, this can only be the covariant

$$3\,(\text{No. } 20) - 2\,(\text{No. } 14)^2,\ [=3H-2B^2],$$

which it is clear vanishes as required.

12. In order that the quintic may have three equal roots and two equal roots, or that the system of roots may be of the form 32, the simplest type to be considered is

$$12.13.14.15,$$

which gives the function

$$\Sigma\,(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\,(\alpha-\epsilon)\,(x-\beta y)^3\,(x-\gamma y)^3\,(x-\delta y)^3\,(x-\epsilon y)^3,$$

a covariant of the degree 4 in the coefficients, and the degree 12 in the variables; and it must vanish when $a=b=c=0$, $e=f=0$; this can only be the covariant

$$3\,(\text{No. } 13)^2\,(\text{No. } 14) - 25\,(\text{No. } 15)^2,\ [=3A^2B-25C^2],$$

which it is clear vanishes as required.

13. In order that the quintic may have four equal roots, or that the system may be of the form 41, the simplest type to be considered is

$$12.34,$$

which gives the function

$$\Sigma\,(\alpha-\beta)^2\,(\gamma-\delta)^2\,(x-\epsilon y)^2,$$

a covariant of the degree 2 in the coefficients, and of the same degree in the variables; this can only be the covariant (No. 14), $[=B]$.

14. Finally, in order that all the roots may be equal, or that the system of roots may be of the form 5, the type to be considered is

$$12;$$

and this gives the function

$$\Sigma\,(\alpha-\beta)^2\,(x-\gamma y)^2\,(x-\delta y)^2\,(x-\epsilon y)^2,$$

a covariant of the degree 2 in the coefficients, and the degree 6 in the variables, and this can only be the Hessian (No. 15), $[=C]$.

It will be observed that all the preceding conditions are distinctive; for instance, the covariant which vanishes when the system of roots is of the form 311, does not vanish when the system is of the form 221, or of any other form not included in the form 311.

151.

TABLES OF THE STURMIAN FUNCTIONS FOR EQUATIONS OF THE SECOND, THIRD, FOURTH, AND FIFTH DEGREES.

[From the *Philosophical Transactions of the Royal Society of London,* vol. CXLVII. *for the year* 1857, pp. 733—736. Received December 18, 1856,—Read January 8, 1857.]

THE general expressions for the Sturmian functions in the form of determinants are at once deducible from the researches of Professor Sylvester in his early papers on the subject in the *Philosophical Magazine,* and in giving these expressions in the Memoir 'Nouvelles Recherches sur les Fonctions de M. Sturm,' *Liouville,* t. XIII. p. 269 (1848), [65], I was wrong in claiming for them any novelty. The expressions in the last-mentioned memoir admit of a modification by which their form is rendered somewhat more elegant; I propose on the present occasion merely to give this modified form of the general expression, and to give the developed expressions of the functions in question for equations of the degrees two, three, four, and five.

Consider in general the equation

$$U = (a,\ b,\ \ldots\ j,\ k \between x,\ 1)^n,$$

and write

$$P = (a,\ b,\ \ldots\ \ j \between x,\ 1)^{n-1},$$

$$Q = (b,\ \ \ldots\ j,\ k \between x,\ 1)^{n-1},$$

then supposing as usual that the first coefficient a is positive, and taking for shortness n_1, n_2, &c. to represent the binomial coefficients $\frac{n-1}{1}$, $\frac{n-1\,.\,n-2}{1\,.\,2}$, &c. corresponding to the index $(n-1)$, the Sturmian functions, each with its proper sign, are as follows, viz.

$$U,\quad P,\quad \begin{vmatrix} P, & Q \\ a, & b \end{vmatrix},\quad -\begin{vmatrix} xP, & P, & xQ, & Q \\ a, & ., & b, & . \\ n_1b, & a, & n_1c, & b \\ n_2c, & n_1b, & n_2d, & n_1c \end{vmatrix},$$

$$+\begin{vmatrix} x^2P, & xP, & P, & x^2Q, & xQ, & Q \\ a, & ., & ., & b, & ., & . \\ n_1b, & a, & ., & n_1c, & b, & . \\ n_2c, & n_1b, & a, & n_2d, & n_1c, & b \\ n_3d, & n_2c, & n_1b, & n_3e, & n_2d, & n_1c \\ n_4e, & n_3d, & n_2c, & n_4f, & n_3e, & n_2d \end{vmatrix}, \text{ \&c.}$$

where the terms containing the powers of x, which exceed the degrees of the several functions respectively, vanish identically (as is in fact obvious from the form of the expressions), but these terms may of course be omitted *ab initio*.

The following are the results which I have obtained; it is well known that the last or constant function is in each case equal to the discriminant, and as the expressions for the discriminant of equations of the fourth and fifth degrees are given, Tables No. 12 and No. 26 [Q', see 143] in my 'Second Memoir upon Quantics'(1), I have thought it sufficient to refer to these values without repeating them at length.

Table for the degree 2.

The Sturmian functions for the quadric $(a, b, c \between x, 1)^2$ are

$$\left(\boxed{a+1 \mid b+2 \mid c+1} \between x, 1\right)^2,$$

$$\left(\boxed{a+1 \mid b+1} \between x, 1\right),$$

$$\boxed{\begin{matrix} ac-1 \\ b^2+1 \end{matrix}}$$

Table for the degree 3.

The Sturmian functions for the cubic $(a, b, c, d \between x, 1)^3$ are

$$\left(\boxed{a+1 \mid b+3 \mid c+3 \mid d+1} \between x, 1\right)^3,$$

[1] *Philosophical Transactions*, t. CXLVI. p. 101 (1856), [141].

$$\left(\begin{array}{|c|c|c|}\hline a+1 & b+2 & c+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1)^2,$$

$$\left(\begin{array}{|c|c|}\hline ac-2 & ad-1 \\ b^2+2 & bc+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1),$$

$$\begin{array}{|l|}\hline a^2d^2\ +1 \\ abcd\ +6 \\ ac^3\ \ -4 \\ bd^3\ \ -4 \\ b^2c^2\ -3 \\ \hline\end{array}$$

Table for the degree 4.

The Sturmian functions for the quartic $(a,\ b,\ c,\ d,\ e\mathfrak{X}x,\ 1)_4$ are

$$\left(\begin{array}{|c|c|c|c|c|}\hline a+1 & b+4 & c+6 & d+4 & e+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1)^4,$$

$$\left(\begin{array}{|c|c|c|c|}\hline a+1 & b+3 & c+3 & d+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1)^3,$$

$$\left(\begin{array}{|c|c|c|}\hline ac-3 & ad-3 & ae-1 \\ b^2+3 & bc+3 & bd+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1)^2,$$

$$3\left(\begin{array}{|c|c|}\hline a^2ce\ -1 & a^2de\ +1 \\ a^2d^2\ +3 & abce\ -4 \\ ab^2e\ +1 & abd^2\ -1 \\ abcd-14 & ac^2d\ +3 \\ ac^3\ +9 & b^2e\ \ +3 \\ b^3d\ +8 & b^2cd\ -2 \\ b^2c^2\ -6 & \\ \hline\end{array}\right.\mathfrak{X}x,\ 1),$$

$$-\begin{array}{|l|}\hline a^3e^3+1 \\ \&c. \\ \text{Disct. Tab.} \\ \text{No. 12.} \\ \hline\end{array}$$

Table for the degree 5.

The Sturmian functions for the quintic $(a,\ b,\ c,\ d,\ e,\ f\mathfrak{X}x,\ 1)^5$ are

$$\left(\begin{array}{|c|c|c|c|c|c|}\hline a+1 & b+5 & c+10 & d+10 & e+5 & f+1 \\ \hline\end{array}\right.\mathfrak{X}x,\ 1)^5,$$

$$\left(\begin{array}{|c|c|c|c|c|}\hline a+1 & b+4 & c+6 & d+4 & e+1 \\ \hline\end{array}\right) \mathfrak{X}(x,\ 1)^4,$$

$$\left(\begin{array}{|c|c|c|c|}\hline \begin{array}{c} ac-4 \\ b^2+4 \end{array} & \begin{array}{c} ad-6 \\ bc+6 \end{array} & \begin{array}{c} ae-4 \\ bd+4 \end{array} & \begin{array}{c} af-1 \\ be+1 \end{array} \\ \hline\end{array}\right) \mathfrak{X}(x,\ 1)^3,$$

$$2\left(\begin{array}{|c|c|c|}\hline \begin{array}{l} a^3ce\ -\ 8 \\ a^3d^2\ +18 \\ ab^2e\ +\ 8 \\ abcd\ -76 \\ ac^3\ +48 \\ b^3d\ +40 \\ b^2c^2\ -30 \end{array} & \begin{array}{l} a^2cf\ -\ 2 \\ a^2de\ +12 \\ ab^2f\ +\ 2 \\ abce\ -42 \\ abd^2\ -12 \\ ac^2d\ +32 \\ b^3e\ +30 \\ b^2cd\ -20 \end{array} & \begin{array}{l} a^2df\ +\ 3 \\ abcf\ -11 \\ abde\ -\ 3 \\ ac^2e\ +\ 8 \\ b^3f\ +\ 8 \\ b^2ce\ -\ 5 \end{array} \\ \hline\end{array}\right) \mathfrak{X}(x,\ 1)^2,$$

$$2\left(\begin{array}{|c|c|}\hline \begin{array}{l} a^3cf^2\ -\ 2 \\ a^3def\ +\ 24 \\ a^3e^3\ -\ 32 \\ a^2b^2f^2\ +\ 2 \\ a^2bde^2\ +\ 264 \\ a^2bcef\ -\ 52 \\ a^2bd^2f\ -\ 96 \\ a^2c^3df\ +\ 64 \\ a^2c^2e^2\ +\ 352 \\ a^2cd^2e\ -\ 938 \\ a^2d^4\ +\ 432 \\ ab^3ef\ +\ 28 \\ ab^2ce^2\ -\ 970 \\ ab^2d^2e\ +\ 120 \\ abc^2de\ +\ 2480 \\ ab^2cdf\ +\ 264 \\ abcd^3\ -\ 1440 \\ abc^2f\ -\ 192 \\ ac^4e\ -\ 960 \\ ae^3d^2\ +\ 640 \\ b^4df\ -\ 160 \\ b^4e^2\ +\ 450 \\ b^3cde\ -\ 1400 \\ b^3d^3\ +\ 800 \\ b^3c^2f\ +\ 120 \\ b^3c^3e\ +\ 600 \\ b^3c^2d^2\ -\ 400 \end{array} & \begin{array}{l} a^3df^2\ +\ 3 \\ a^3e^2f\ -\ 8 \\ a^2bcf^2\ -\ 11 \\ a^2bdef\ +\ 58 \\ a^2be^3\ +\ 8 \\ a^2c^2ef\ +\ 104 \\ a^2cd^2f\ -\ 156 \\ a^2cde^2\ -\ 96 \\ a^2d^3e\ +\ 108 \\ ab^3f^2\ +\ 8 \\ ab^2cef\ -\ 266 \\ ab^2d^2f\ -\ 8 \\ ab^2de^2\ +\ 35 \\ abc^3df\ +\ 584 \\ abc^2e^2\ +\ 120 \\ abcd^2e\ -\ 360 \\ ac^4f\ -\ 288 \\ ac^3de\ +\ 160 \\ b^4ef\ +\ 120 \\ b^3cdf\ -\ 320 \\ b^3ce^2\ -\ 75 \\ b^3d^2e\ +\ 200 \\ b^2c^3f\ +\ 180 \\ b^2c^2de\ -\ 100 \end{array} \\ \hline\end{array}\right) \mathfrak{X}(x,\ 1),$$

a^4f^4+1
$+$ &c.
Disct. Tab.
No. 26, [Q'].

152.

A MEMOIR ON THE THEORY OF MATRICES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVIII. *for the year*, 1858, pp. 17—37. Received December 10, 1857,—Read January 14, 1858.]

THE term matrix might be used in a more general sense, but in the present memoir I consider only square and rectangular matrices, and the term matrix used without qualification is to be understood as meaning a square matrix; in this restricted sense, a set of quantities arranged in the form of a square, e.g.

$$\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)$$

is said to be a matrix. The notion of such a matrix arises naturally from an abbreviated notation for a set of linear equations, viz. the equations

$$\begin{aligned} X &= ax + by + cz, \\ Y &= a'x + b'y + c'z, \\ Z &= a''x + b''y + c''z, \end{aligned}$$

may be more simply represented by

$$(X, Y, Z) = \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x, y, z),$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices. It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities; they may be added,

multiplied or compounded together, &c.: the law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix, and thence to arrive at the notion of a rational and integral function, or generally of any algebraical function, of a matrix. I obtain the remarkable theorem that any matrix whatever satisfies an algebraical equation of its own order, the coefficient of the highest power being unity, and those of the other powers functions of the terms of the matrix, the last coefficient being in fact the determinant; the rule for the formation of this equation may be stated in the following condensed form, which will be intelligible after a perusal of the memoir, viz. the determinant, formed out of the matrix diminished by the matrix considered as a single quantity involving the matrix unity, will be equal to zero. The theorem shows that every rational and integral function (or indeed every rational function) of a matrix may be considered as a rational and integral function, the degree of which is at most equal to that of the matrix, less unity; it even shows that in a sense, the same is true with respect to any algebraical function whatever of a matrix. One of the applications of the theorem is the finding of the general expression of the matrices which are convertible with a given matrix. The theory of rectangular matrices appears much less important than that of square matrices, and I have not entered into it further than by showing how some of the notions applicable to these may be extended to rectangular matrices.

1. For conciseness, the matrices written down at full length will in general be of the order 3, but it is to be understood that the definitions, reasonings, and conclusions apply to matrices of any degree whatever. And when two or more matrices are spoken of in connexion with each other, it is always implied (unless the contrary is expressed) that the matrices are of the same order.

2. The notation

$$\left(\begin{array}{ccc|c} a\,, & b\,, & c & \between x, y, z) \\ a', & b', & c' & \\ a'', & b'', & c'' & \end{array}\right.$$

represents the set of linear functions

$$((a, b, c \between x, y, z),\ (a', b', c' \between x, y, z),\ (a'', b'', c'' \between x, y, z)),$$

so that calling these (X, Y, Z), we have

$$(X, Y, Z) = \left(\begin{array}{ccc|c} a\,, & b\,, & c & \between x, y, z) \\ a', & b', & c' & \\ a'', & b'', & c'' & \end{array}\right.$$

and, as remarked above, this formula leads to most of the fundamental notions in the theory.

3. The quantities (X, Y, Z) will be identically zero, if all the terms of the matrix are zero, and we may say that

$$\begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}$$

is the matrix zero.

Again, (X, Y, Z) will be identically equal to (x, y, z), if the matrix is

$$\begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}$$

and this is said to be the matrix unity. We may of course, when for distinctness it is required, say, the matrix zero, or (as the case may be) the matrix unity *of such an order*. The matrix zero may for the most part be represented simply by 0, and the matrix unity by 1.

4. The equations

$$(X, Y, Z) = \begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} (x, y, z), \quad (X', Y', Z') = \begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} (x, y, z)$$

give

$$(X+X', Y+Y', Z+Z') = \begin{pmatrix} a+\alpha, & b+\beta, & c+\gamma \\ a'+\alpha', & b'+\beta', & c'+\gamma' \\ a''+\alpha'', & b''+\beta'', & c''+\gamma'' \end{pmatrix} (x, y, z)$$

and this leads to

$$\begin{pmatrix} a+\alpha, & b+\beta, & c+\gamma \\ a'+\alpha', & b'+\beta', & c'+\gamma' \\ a''+\alpha'', & b''+\beta'', & c''+\gamma'' \end{pmatrix} = \begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} + \begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix}$$

as a rule for the addition of matrices; that for their subtraction is of course similar to it.

5. A matrix is not altered by the addition or subtraction of the matrix zero, that is, we have $M \pm 0 = M$.

The equation $L = M$, which expresses that the matrices L, M are equal, may also be written in the form $L - M = 0$, i.e. the difference of two equal matrices is the matrix zero.

6. The equation $L = -M$, written in the form $L + M = 0$, expresses that the sum of the matrices L, M is equal to the matrix zero, the matrices so related are said to be *opposite* to each other; in other words, a matrix the terms of which are equal but opposite in sign to the terms of a given matrix, is said to be opposite to the given matrix.

7. It is clear that we have $L+M=M+L$, that is, the operation of addition is commutative, and moreover that $(L+M)+N=L+(M+N)=L+M+N$, that is, the operation of addition is also associative.

8. The equation

$$(X,\ Y,\ Z)=\left(\begin{array}{ccc} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(mx,\ my,\ mz)$$

written under the forms

$$(X,\ Y,\ Z)=m\left(\begin{array}{ccc} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x,\ y,\ z)=\left(\begin{array}{ccc} ma\ , & mb\ , & mc \\ ma', & mb', & mc' \\ ma'', & mb'', & mc'' \end{array}\right)(x,\ y,\ z)$$

gives

$$m\left(\begin{array}{ccc} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)=\left(\begin{array}{ccc} ma\ , & mb\ , & mc \\ ma', & mb', & mc' \\ ma'', & mb'', & mc'' \end{array}\right)$$

as the rule for the multiplication of a matrix by a single quantity. The multiplier m may be written either before or after the matrix, and the operation is therefore commutative. We have it is clear $m(L+M)=mL+mM$, or the operation is distributive.

9. The matrices L and mL may be said to be similar to each other; in particular, if $m=1$, they are equal, and if $m=-1$, they are opposite.

10. We have, in particular,

$$m\left(\begin{array}{ccc} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{array}\right)=\left(\begin{array}{ccc} m, & 0, & 0 \\ 0, & m, & 0 \\ 0, & 0, & m \end{array}\right),$$

or replacing the matrix on the left-hand side by unity, we may write

$$m=\left(\begin{array}{ccc} m, & 0, & 0 \\ 0, & m, & 0 \\ 0, & 0, & m \end{array}\right);$$

the matrix on the right-hand side is said to be the single quantity m considered as *involving the matrix unity.*

11. The equations

$$(X,\ Y,\ Z)=\left(\begin{array}{ccc} a\ , & b\ , & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x,\ y,\ z),\quad (x,\ y,\ z)=\left(\begin{array}{ccc} \alpha\ , & \beta\ , & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)(\xi,\ \eta,\ \zeta),$$

give

$$(X, Y, Z) = \left(\begin{array}{ccc} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right) \between (\xi, \eta, \zeta) = \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) \between \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right) \between (\xi, \eta, \zeta),$$

and thence, substituting for the matrix

$$\left(\begin{array}{ccc} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right)$$

its value, we obtain

$$\left(\begin{array}{ccc} (a, b, c \between \alpha, \alpha', \alpha''), & (a, b, c \between \beta, \beta', \beta''), & (a, b, c \between \gamma, \gamma', \gamma'') \\ (a', b', c' \between \alpha, \alpha', \alpha''), & (a', b', c' \between \beta, \beta', \beta''), & (a', b', c' \between \gamma, \gamma', \gamma'') \\ (a'', b'', c'' \between \alpha, \alpha', \alpha''), & (a'', b'', c'' \between \beta, \beta', \beta''), & (a'', b'', c'' \between \gamma, \gamma', \gamma'') \end{array}\right) = \left(\begin{array}{ccc} a, b, c \\ a', b', c' \\ a'', b'', c'' \end{array}\right) \between \left(\begin{array}{ccc} \alpha, \beta, \gamma \\ \alpha', \beta', \gamma' \\ \alpha'', \beta'', \gamma'' \end{array}\right)$$

as the rule for the multiplication or composition of two matrices. It is to be observed, that the operation is not a commutative one; the component matrices may be distinguished as the first or further component matrix, and the second or nearer component matrix, and the rule of composition is as follows, viz. any *line* of the compound matrix is obtained by combining the corresponding *line* of the first or further component matrix successively with the several *columns* of the second or nearer compound matrix.

[We may conveniently write

$$\begin{array}{c|ccc} & (\alpha, \alpha', \alpha''), & (\beta, \beta', \beta''), & (\gamma, \gamma', \gamma'') \\ \hline (a, b, c) & \text{"} & \text{"} & \text{"} \\ (a', b', c') & \text{"} & \text{"} & \text{"} \\ (a'', b'', c'') & \text{"} & \text{"} & \text{"} \end{array}$$

to denote the left-hand side of the last preceding equation.]

12. A matrix compounded, either as first or second component matrix, with the matrix zero, gives the matrix zero. The case where any of the terms of the given matrix are infinite is of course excluded.

13. A matrix is not altered by its composition, either as first or second component matrix, with the matrix unity. It is compounded either as first or second component matrix, with the single quantity m considered as involving the matrix unity, by multiplication of all its terms by the quantity m: this is in fact the before-mentioned rule for the multiplication of a matrix by a single quantity, which rule is thus seen to be a particular case of that for the multiplication of two matrices.

14. We may in like manner multiply or compound together three or more matrices: the order of arrangement of the factors is of course material, and we may

distinguish them as the first or furthest, second, third, &c., and last or nearest component matrices: any two consecutive factors may be compounded together and replaced by a single matrix, and so on until all the matrices are compounded together, the result being independent of the particular mode in which the composition is effected; that is, we have $L.MN = LM.N = LMN$, $LM.NP = L.MN.P$, &c., or the operation of multiplication, although, as already remarked, not commutative, is associative.

15. We thus arrive at the notion of a positive and integer power L^p of a matrix L, and it is to be observed that the different powers of the same matrix are convertible. It is clear also that p and q being positive integers, we have $L^p.L^q = L^{p+q}$, which is the theorem of indices for positive integer powers of a matrix.

16. The last-mentioned equation, $L^p.L^q = L^{p+q}$, assumed to be true for all values whatever of the indices p and q, leads to the notion of the powers of a matrix for any form whatever of the index. In particular, $L^p.L^0 = L^p$ or $L^0 = 1$, that is, the 0th power of a matrix is the matrix unity. And then putting $p = 1$, $q = -1$, or $p = -1$, $q = 1$, we have $L.L^{-1} = L^{-1}.L = 1$; that is, L^{-1}, or as it may be termed the inverse or reciprocal matrix, is a matrix which, compounded either as first or second component matrix with the original matrix, gives the matrix unity.

17. We may arrive at the notion of the inverse or reciprocal matrix, directly from the equation

$$(X, Y, Z) = \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x, y, z),$$

in fact this equation gives

$$(x, y, z) = \left(\begin{array}{ccc} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{array}\right)(X, Y, Z) = \left(\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)^{-1}\right)(X, Y, Z),$$

and we have, for the determination of the coefficients of the inverse or reciprocal matrix, the equations

$$\left(\begin{array}{ccc} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{array}\right)\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) = \left(\begin{array}{ccc} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{array}\right),$$

$$\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)\left(\begin{array}{ccc} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{array}\right) = \left(\begin{array}{ccc} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{array}\right),$$

which are equivalent to each other, and either of them is by itself sufficient for the complete determination of the inverse or reciprocal matrix. It is well known that if ∇ denote the determinant, that is, if

$$\nabla = \begin{vmatrix} a\,, & b\,, & c \\ a'\,, & b'\,, & c' \\ a''\,, & b''\,, & c'' \end{vmatrix}$$

then the terms of the inverse or reciprocal matrix are given by the equations

$$A = \frac{1}{\nabla}\begin{vmatrix} 1, & 0\,, & 0 \\ 0, & b'\,, & c' \\ 0, & b''\,, & c'' \end{vmatrix}, \qquad B = \frac{1}{\nabla}\begin{vmatrix} 0\,, & 1, & 0 \\ a'\,, & 0, & c' \\ a''\,, & 0, & c'' \end{vmatrix}, \text{ \&c.}$$

or what is the same thing, the inverse or reciprocal matrix is given by the equation

$$\begin{pmatrix} a\,, & b\,, & c \\ a'\,, & b'\,, & c' \\ a''\,, & b''\,, & c'' \end{pmatrix}^{-1} = \frac{1}{\nabla}\begin{pmatrix} \partial_a\nabla, & \partial_{a'}\nabla, & \partial_{a''}\nabla \\ \partial_b\nabla\,, & \partial_{b'}\nabla, & \partial_{b''}\nabla \\ \partial_c\nabla\,, & \partial_{c'}\nabla\,, & \partial_{c''}\nabla \end{pmatrix}$$

where of course the differentiations must in every case be performed as if the terms a, b, &c. were all of them independent arbitrary quantities.

18. The formula shows, what is indeed clear *à priori*, that the notion of the inverse or reciprocal matrix fails altogether when the determinant vanishes: the matrix is in this case said to be indeterminate, and it must be understood that in the absence of express mention, the particular case in question is frequently excluded from consideration. It may be added that the matrix zero is indeterminate; and that the product of two matrices may be zero, without either of the factors being zero, if only the matrices are one or both of them indeterminate.

19. The notion of the inverse or reciprocal matrix once established, the other negative integer powers of the original matrix are positive integer powers of the inverse or reciprocal matrix, and the theory of such negative integer powers may be taken to be known. The theory of the fractional powers of a matrix will be further discussed in the sequel.

20. The positive integer power L^m of the matrix L may of course be multiplied by any matrix of the same degree: such multiplier, however, is not in general convertible with L; and to preserve as far as possible the analogy with ordinary algebraical functions, we may restrict the attention to the case where the multiplier is a single quantity, and such convertibility consequently exists. We have in this manner a matrix cL^m, and by the addition of any number of such terms we obtain a rational and integral function of the matrix L.

21. The general theorem before referred to will be best understood by a complete development of a particular case. Imagine a matrix

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

and form the determinant

$$\begin{vmatrix} a-M, & b \\ c & , d-M \end{vmatrix},$$

the developed expression of this determinant is

$$M^2 - (a+d)\,M^1 + (ad-bc)\,M^0;$$

the values of M^2, M^1, M^0 are

$$\begin{pmatrix} a^2+bc\ , & b(a+d) \\ c(a+d), & d^2+bc \end{pmatrix}, \quad \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

and substituting these values the determinant becomes equal to the matrix zero, viz. we have

$$\begin{vmatrix} a-M, & b \\ c & , d-M \end{vmatrix} = \begin{pmatrix} a^2+bc\ , & b(a+d) \\ c(a+d), & d^2+bc \end{pmatrix} - (a+d)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (a^2+bc)-(a+d)\,a+(ad-bc), & b(a+d)-(a+d)\,b \\ c(a+d)-(a+d)\,c & , d^2+bc-(a+d)\,d+ad-bc \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix};$$

that is

$$\begin{vmatrix} a-M, & b \\ c & , d-M \end{vmatrix} = 0,$$

where the matrix of the determinant is

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} - M\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

that is, it is the original matrix, diminished by the same matrix considered as a single quantity involving the matrix unity. And this is the general theorem, viz. the determinant, having for its matrix a given matrix less the same matrix considered as a single quantity involving the matrix unity, is equal to zero.

22. The following symbolical representation of the theorem is, I think, worth noticing: let the matrix M, considered as a single quantity, be represented by $\tilde{M}$, then writing 1 to denote the matrix unity, $\tilde{M}.1$ will represent the matrix M, considered as a single quantity involving the matrix unity. Upon the like principles of notation, $\tilde{1}.M$ will represent, or may be considered as representing, simply the matrix M, and the theorem is

$$\text{Det. } (\tilde{1}\,.\,M - \tilde{M}\,.\,1) = 0.$$

23. I have verified the theorem, in the next simplest case of a matrix of the order 3, viz. if M be such a matrix, suppose

$$M = \begin{pmatrix} a, & b, & c \\ d, & e, & f \\ g, & h, & i \end{pmatrix},$$

then the derived determinant vanishes, or we have

$$\begin{vmatrix} a-M, & b & , & c \\ d & , & e-M, & f \\ g & , & h & , & i-M \end{vmatrix} = 0,$$

or expanding

$$M^3 - (a+e+i)M^2 + (ei+ia+ae-fh-cg-bd)M - (aei+bfg+cdh-afh-bdi-ceg) = 0;$$

but I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree.

24. If we attend only to the general form of the result, we see that any matrix whatever satisfies an algebraical equation of its own order, which is in many cases the material part of the theorem.

25. It follows at once that every rational and integral function, or indeed every rational function of a matrix, can be expressed as a rational and integral function of an order at most equal to that of the matrix, less unity. But it is important to consider how far or in what sense the like theorem is true with respect to irrational functions of a matrix. If we had only the equation satisfied by the matrix itself, such extension could not be made; but we have besides the equation of the same order satisfied by the irrational function of the matrix, and by means of these two equations, and the equation by which the irrational function of the matrix is determined, we may express the irrational function as a rational and integral function of the matrix, of an order equal at most to that of the matrix, less unity; such expression will however involve *the coefficients of the equation satisfied by the irrational function*, which are functions (in number equal to the order of the matrix) of the terms, assumed to be unknown, of the irrational function itself. The transformation is nevertheless an important one, as reducing the number of unknown quantities from n^2 (if n be the order of the matrix) down to n. To complete the solution, it is necessary to compare the value obtained as above, with the assumed value of the irrational function, which will lead to equations for the determination of the n unknown quantities.

26. As an illustration, consider the given matrix

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

and let it be required to find the matrix $L=\sqrt{M}$. In this case M satisfies the equation

$$M^2-(a+d)M+ad-bc=0;$$

and in like manner if

$$L=\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$$

then L satisfies the equation

$$L^2-(\alpha+\delta)L+\alpha\delta-\beta\gamma=0;$$

and from these two equations, and the rationalized equation $L^2=M$, it should be possible to express L in the form of a linear function of M: in fact, putting in the last equation for L^2 its value $(=M)$, we find at once

$$L=\frac{1}{\alpha+\delta}[M+(\alpha\delta-\beta\gamma)],$$

which is the required expression, involving as it should do the coefficients $\alpha+\delta$, $\alpha\delta-\beta\gamma$ of the equation in L. There is no difficulty in completing the solution; write for shortness $\alpha+\delta=X$, $\alpha\delta-\beta\gamma=Y$, then we have

$$L=\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}=\begin{pmatrix} \dfrac{a+Y}{X}, & \dfrac{b}{X} \\ \dfrac{c}{X}, & \dfrac{d+Y}{X} \end{pmatrix},$$

and consequently forming the values of $\alpha+\delta$ and $\alpha\delta-\beta\gamma$,

$$X=\frac{a+d+2Y}{X},$$

$$Y=\frac{(a+Y)(d+Y)-bc}{X^2},$$

and putting also $a+d=P$, $ad-bc=Q$, we find without difficulty

$$X=\sqrt{P+2\sqrt{Q}},$$

$$Y=\sqrt{Q},$$

and the values of α, β, γ, δ are consequently known. The sign of $\sqrt{Q}$ is the same in both formulæ, and there are consequently in all four solutions, that is, the radical $\sqrt{M}$ has four values.

27. To illustrate this further, suppose that instead of M we have the matrix

$$M^2=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2=\begin{pmatrix} a^2+bc, & b(a+d) \\ c(a+d), & d^2+bc \end{pmatrix},$$

so that $L^2 = M^2$, we find

$$P = (a+d)^2 - 2(ad-bc),$$
$$Q = (ad-bc)^2,$$

and thence $\sqrt{Q} = \pm (ad-bc)$. Taking the positive sign, we have

$$Y = \quad ad - bc,$$
$$X = \pm (a+d),$$

and these values give simply

$$L = \pm \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} = \pm M.$$

But taking the negative sign,

$$Y = -ad + bc,$$
$$X = \pm \sqrt{(a-d)^2 + 4bc},$$

and retaining X to denote this radical, we find

$$L = \begin{pmatrix} \dfrac{a^2 - ad + 2bc}{X}, & \dfrac{b(a+d)}{X} \\ \dfrac{c(a+d)}{X}, & \dfrac{d^2 - ad + 2bc}{X} \end{pmatrix},$$

which may also be written

$$L = \frac{a+d}{X} \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} - \frac{2(ad-bc)}{X} \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

or, what is the same thing,

$$L = \frac{a+d}{X} M - \frac{2(ad-bc)}{X};$$

and it is easy to verify *à posteriori* that this value in fact gives $L^2 = M^2$. It may be remarked that if

$$M^2 = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}^2 = 1,$$

the last-mentioned formula fails, for we have $X = 0$; it will be seen presently that the equation $L^2 = 1$ admits of other solutions besides $L = \pm 1$. The example shows how the values of the fractional powers of a matrix are to be investigated.

28. There is an apparent difficulty connected with the equation satisfied by a matrix, which it is proper to explain. Suppose, as before,

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

so that M satisfies the equation

$$\begin{vmatrix} a-M, & b \\ c & , \; d-M \end{vmatrix} = 0,$$

or

$$M^2-(a+d)M+ad-bc=0,$$

and let $X_{,}$, $X_{,,}$ be the single quantities, roots of the equation

$$\begin{vmatrix} a-X, & b \\ c & , \; d-X_{,} \end{vmatrix} = 0$$

or

$$X^2-(a+d)X+ad-bc=0.$$

The equation satisfied by the matrix may be written

$$(M-X_{,})(M-X_{,,})=0,$$

in which $X_{,}$, $X_{,,}$ are to be considered as respectively involving the matrix unity, and it would at first sight seem that we ought to have one of the simple factors equal to zero; this is obviously not the case, for such equation would signify that the perfectly indeterminate matrix M was equal to a single quantity, considered as involving the matrix unity. The explanation is that each of the simple factors is an indeterminate matrix, in fact $M-X_{,}$ stands for the matrix

$$\begin{pmatrix} a-X_{,}, & b \\ c & , \; d-X_{,} \end{pmatrix},$$

and the determinant of this matrix is equal to zero. The product of the two factors is thus equal to zero without either of the factors being equal to zero.

29. A matrix satisfies, we have seen, an equation of its own order, involving the coefficients of the matrix; assume that the matrix is to be determined to satisfy some other equation, the coefficients of which are given single quantities. It would at first sight appear that we might eliminate the matrix between the two equations, and thus obtain an equation which would be the only condition to be satisfied by the terms of the matrix; this is obviously wrong, for more conditions must be requisite, and we see that if we were then to proceed to complete the solution by finding the value of the matrix common to the two equations, we should find the matrix equal in every case to a single quantity considered as involving the matrix unity, which it is clear ought not to be the case. The explanation is similar to that of the difficulty before adverted to; the equations may contain one, and only one, common factor, and may be both of them satisfied, and yet the common factor may not vanish. The necessary condition seems to be, that the one equation should be a factor of the other; in the case where the assumed equation is of an order equal or superior to the matrix, then if this equation contain as a factor the equation which is always satisfied by the matrix, the assumed equation will be satisfied identically, and the condition is sufficient as well as necessary: in the other case, where the assumed equation is of an order inferior to that of the matrix, the condition is necessary, but it is not sufficient.

30. The equation satisfied by the matrix may be of the form $M^n = 1$; the matrix is in this case said to be periodic of the nth order. The preceding considerations apply to the theory of periodic matrices; thus, for instance, suppose it is required to find a matrix of the order 2, which is periodic of the second order. Writing

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

we have

$$M^2 - (a+d)M + ad - bc = 0,$$

and the assumed equation is

$$M^2 - 1 = 0.$$

These equations will be identical if

$$a + d = 0, \quad ad - bc = -1,$$

that is, these conditions being satisfied, the equation $M^2 - 1 = 0$ required to be satisfied, will be identical with the equation which is always satisfied, and will therefore itself be satisfied. And in like manner the matrix M of the order 2 will satisfy the condition $M^3 - 1 = 0$, or will be periodic of the third order, if only $M^3 - 1$ contains as a factor

$$M^2 - (a+d)M + ad - bc,$$

and so on.

31. But suppose it is required to find a matrix of the order 3,

$$M = \begin{pmatrix} a, & b, & c \\ d, & e, & f \\ g, & h, & i \end{pmatrix}$$

which shall be periodic of the second order. Writing for shortness

$$\begin{vmatrix} a-M, & b & , & c \\ d & , & e-M, & f \\ g & , & h & , & i-M \end{vmatrix} = -(M^3 - AM^2 + BM - C),$$

the matrix here satisfies

$$M^3 - AM^2 + BM - C = 0,$$

and, as before, the assumed equation is $M^3 - 1 = 0$. Here, if we have $1 + B = 0$, $A + C = 0$, the left-hand side will contain the factor $(M^2 - 1)$, and the equation will take the form $(M^2 - 1)(M + C) = 0$, and we should have then $M^2 - 1 = 0$, provided $M + C$ were not an indeterminate matrix. But $M + C$ denotes the matrix

$$\begin{pmatrix} a+C, & b & , & c \\ d & , & e+C, & f \\ g & , & h & , & i+C \end{pmatrix}$$

the determinant of which is C^3+AC^2+BC+C, which is equal to zero in virtue of the equations $1+B=0$, $A+C=0$, and we cannot, therefore, from the equation $(M^2-1)(M+C)=0$, deduce the equation $M^2-1=0$. This is as it should be, for the two conditions are not sufficient, in fact the equation

$$M^2=\begin{pmatrix} a^2+bd+cg, & ab+be+ch, & ac+bf+ci \\ da+ed+fg, & db+e^2+fh, & dc+ef+fi \\ ga+hd+ig, & gb+he+ih, & gc+hf+i^2 \end{pmatrix}=1$$

gives nine equations, which are however satisfied by the following values, involving in reality four arbitrary coefficients; viz. the value of the matrix is

$$\begin{pmatrix} \dfrac{\alpha}{\alpha+\beta+\gamma}, & \dfrac{-(\beta+\gamma)\nu\mu^{-1}}{\alpha+\beta+\gamma}, & \dfrac{-(\beta+\gamma)\nu\mu^{-1}}{\alpha+\beta+\gamma} \\ \dfrac{-(\gamma+\alpha)\mu\nu^{-1}}{\alpha+\beta+\gamma}, & \dfrac{\beta}{\alpha+\beta+\gamma}, & \dfrac{-(\gamma+\alpha)\lambda\mu^{-1}}{\alpha+\beta+\gamma} \\ \dfrac{-(\alpha+\beta)\mu\nu^{-1}}{\alpha+\beta+\gamma}, & \dfrac{-(\alpha+\beta)\nu\lambda^{-1}}{\alpha+\beta+\gamma}, & \dfrac{\gamma}{\alpha+\beta+\gamma} \end{pmatrix}$$

so that there are in all five relations (and not only two) between the coefficients of the matrix.

32. Instead of the equation $M^n-1=0$, which belongs to a periodic matrix, it is in many cases more convenient, and it is much the same thing to consider an equation $M^n-k=0$, where k is a single quantity. The matrix may in this case be said to be periodic to a factor *près*.

33. Two matrices L, M are convertible when $LM=ML$. If the matrix M is given, this equality affords a set of linear equations between the coefficients of L equal in number to these coefficients, but these equations cannot be all independent, for it is clear that if L be any rational and integral function of M (the coefficients being single quantities), then L will be convertible with M; or what is apparently (but only apparently) more general, if L be any algebraical function whatever of M (the coefficients being always single quantities), then L will be convertible with M. But whatever the form of the function is, it may be reduced to a rational and integral function of an order equal to that of M, less unity, and we have thus the general expression for the matrices convertible with a given matrix, viz. any such matrix is a rational and integral function (the coefficients being single quantities) of the given matrix, the order being that of the given matrix, less unity. In particular, the general form of the matrix L convertible with a given matrix M of the order 2, is $L=\alpha M+\beta$, or what is the same thing, the matrices

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad \begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix}$$

will be convertible if $a'-d' : b' : c'=a-d : b : c$.

34. Two matrices L, M are skew convertible when $LM = -ML$; this is a relation much less important than ordinary convertibility, for it is to be noticed that we cannot in general find a matrix L skew convertible with a given matrix M. In fact, considering M as given, the equality affords a set of linear equations between the coefficients of L equal in number to these coefficients; and in this case the equations are independent, and we may eliminate all the coefficients of L, and we thus arrive at a relation which must be satisfied by the coefficients of the given matrix M. Thus, suppose the matrices

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad \begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix}$$

are skew convertible, we have

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix} = \begin{pmatrix} aa'+bc', & ab'+bd' \\ ca'+dc', & cb'+dd' \end{pmatrix},$$

$$\begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix}\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} = \begin{pmatrix} aa'+b'c, & a'b+b'd \\ c'a+d'c, & c'b+d'd \end{pmatrix},$$

and the conditions of skew convertibility are

$$\begin{aligned} 2aa'+bc'+b'c &= 0, \\ b'(a+d)+b(a'+d') &= 0, \\ c'(a+d)+c(a'+d') &= 0, \\ 2dd'+bc'+b'c &= 0. \end{aligned}$$

Eliminating a', b', c', d', the relation between a, b, c, d is

$$\begin{vmatrix} 2a, & c\ , & b\ , & . \\ b\ , & a+d, & . & b \\ c\ , & . & a+d, & c \\ . & c\ , & b\ , & 2d \end{vmatrix} = 0,$$

which is

$$(a+d)^2(ad-bc) = 0.$$

Excluding from consideration the case $ad-bc=0$, which would imply that the matrix was indeterminate, we have $a+d=0$. The resulting system of conditions then is

$$a+d=0, \quad a'+d'=0, \quad aa'+bc'+b'c+dd'=0,$$

the first two of which imply that the matrices are respectively periodic of the second order to a factor *près*.

35. It may be noticed that if the compound matrices LM and ML are similar, they are either equal or else opposite; that is, the matrices L, M are either convertible or skew convertible.

36. Two matrices such as

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad \begin{pmatrix} a, & c \\ b, & d \end{pmatrix},$$

are said to be formed one from the other by transposition, and this may be denoted by the symbol tr.; thus we may write

$$\begin{pmatrix} a, & c \\ b, & d \end{pmatrix} = \text{tr.} \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}.$$

The effect of two successive transpositions is of course to reproduce the original matrix.

37. It is easy to see that if M be any matrix, then

$$(\text{tr. } M)^p = \text{tr. } (M^p),$$

and in particular,

$$(\text{tr. } M)^{-1} = \text{tr. } (M^{-1}).$$

38. If L, M be any two matrices,

$$\text{tr.}(LM) = \text{tr. } M. \text{ tr. } L,$$

and similarly for three or more matrices, L, M, N, &c.,

$$\text{tr.}(LMN) = \text{tr. } N. \text{ tr. } M. \text{ tr. } L, \text{ \&c.}$$

40. A matrix such as

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}$$

which is not altered by transposition, is said to be symmetrical.

41. A matrix such as

$$\begin{pmatrix} 0, & \nu, & -\mu \\ -\nu, & 0, & \lambda \\ \mu, & -\lambda, & 0 \end{pmatrix}$$

which by transposition is changed into its opposite, is said to be skew symmetrical.

42. It is easy to see that any matrix whatever may be expressed as the sum of a symmetrical matrix, and a skew symmetrical matrix; thus the form

$$\begin{pmatrix} a, & h+\nu, & g-\mu \\ h-\nu, & b, & f+\lambda \\ g+\mu, & f-\lambda, & c \end{pmatrix}$$

which may obviously represent any matrix whatever of the order 3, is the sum of the two matrices last before mentioned.

43. The following formulæ, although little more than examples of the composition of transposed matrices, may be noticed, viz.

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \between \begin{pmatrix} a, & c \\ d, & b \end{pmatrix} = \begin{pmatrix} a^2+b^2, & ac+bd \\ ac+bd, & c^2+d^2 \end{pmatrix}$$

which shows that a matrix compounded with the transposed matrix gives rise to a symmetrical matrix. It does not however follow, nor is it the fact, that the matrix and transposed matrix are convertible. And also

$$\begin{pmatrix} a, & c \\ b, & d \end{pmatrix} \between \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \between \begin{pmatrix} a, & c \\ b, & d \end{pmatrix} = \begin{pmatrix} a^3+bcd+a(b^2+c^2), & c^3+abd+c(a^2+d^2) \\ b^3+acd+b(a^2+d^2), & d^3+abc+d(b^2+c^2) \end{pmatrix}$$

which is a remarkably symmetrical form. It is needless to proceed further, since it is clear that

$$\begin{pmatrix} a, & c \\ b, & d \end{pmatrix} \between \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \between \begin{pmatrix} a, & c \\ b, & d \end{pmatrix} \between \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} = \left(\begin{pmatrix} a, & c \\ b, & d \end{pmatrix} \between \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\right)^2.$$

44. In all that precedes, the matrix of the order 2 has frequently been considered, but chiefly by way of illustration of the general theory; but it is worth while to develope more particularly the theory of such matrix. I call to mind the fundamental properties which have been obtained, viz. it was shown that the matrix

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

satisfies the equation

$$M^2 - (a+d)M + ad - bc = 0,$$

and that the two matrices

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad \begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix},$$

will be convertible if

$$a'-d' : b' : c' = a-d : b : c,$$

and that they will be skew convertible if

$$a+d=0, \quad a'+d'=0, \quad aa'+bc'+b'c+dd'=0,$$

the first two of these equations being the conditions in order that the two matrices may be respectively periodic of the second order to a factor *près*.

45. It may be noticed in passing, that if L, M are skew convertible matrices of the order 2, and if these matrices are also such that $L^2=-1$, $M^2=-1$, then putting $N=LM=-ML$, we obtain

$$L^2=-1, \qquad M^2=-1, \qquad N^2=-1,$$
$$L=MN=-NM, \quad M=NL=-NL, \quad N=LM=-ML,$$

which is a system of relations precisely similar to that in the theory of quaternions.

46. The integer powers of the matrix

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

are obtained with great facility from the quadratic equation; thus we have, attending first to the positive powers,

$$\begin{aligned} M^2 &= (a+d)\,M - (ad-bc), \\ M^3 &= [(a+d)^2 - (ad-bc)]\,M - (a+d)(ad-bc), \\ &\&c., \end{aligned}$$

whence also the conditions in order that the matrix may be to a factor *près* periodic of the orders 2, 3, &c. are

$$\begin{aligned} a+d \qquad\qquad &= 0, \\ (a+d)^2 - (ad-bc) &= 0, \\ \&c.\,; \end{aligned}$$

and for the negative powers we have

$$(ad-bc)\,M^{-1} = -M + (a+d),$$

which is equivalent to the ordinary form

$$(ad-bc)\,M^{-1} = \begin{pmatrix} d, & -b \\ -c, & a \end{pmatrix};$$

and the other negative powers of M can then be obtained by successive multiplications with M^{-1}.

47. The expression for the nth power is however most readily obtained by means of a particular algorithm for matrices of the order 2.

Let h, b, c, J, q be any quantities, and write for shortness $R = -h^2 - 4bc$; suppose also that h', b', c', J', q' are any other quantities, such nevertheless that $h' : b' : c' = h : b : c$, and write in like manner $R' = -h'^2 - 4b'c'$. Then observing that $\frac{h}{\sqrt{R}}, \frac{b}{\sqrt{R}}, \frac{c}{\sqrt{R}}$ are respectively equal to $\frac{h'}{\sqrt{R'}}, \frac{b'}{\sqrt{R'}}, \frac{c'}{\sqrt{R'}}$, the matrix

$$\begin{pmatrix} J\left(\cot q - \frac{h}{\sqrt{R}}\right), & \frac{2bJ}{\sqrt{R}} \\ \frac{2cJ}{\sqrt{R}}, & J\left(\cot q + \frac{h}{\sqrt{R}}\right) \end{pmatrix}$$

contains only the quantities J, q, which are not the same in both systems; and we may therefore represent this matrix by (J, q), and the corresponding matrix with

h', b', c', J', q' by (J', q'). The two matrices are at once seen to be convertible (the assumed relations $h' : b' : c' = h : b : c$ correspond in fact to the conditions,

$$a' - d' : b' : c' = a - d : b : c,$$

of convertibility for the ordinary form), and the compound matrix is found to be

$$\left(\frac{\sin(q+q')}{\sin q \sin q'} JJ',\ q+q'\right);$$

and in like manner the several convertible matrices (J, q), (J', q'), (J'', q'') &c. give the compound matrix

$$\left(\frac{\sin(q+q'+q''\ldots)}{\sin q \sin q' \sin q''\ldots} JJ'J''\ldots,\ q+q'+q''\ldots\right).$$

48. The convertible matrices may be given in the first instance in the ordinary form, or we may take these matrices to be

$$\left(\begin{matrix} a, & b \\ c, & d \end{matrix}\right),\ \left(\begin{matrix} a', & b' \\ c', & d' \end{matrix}\right),\ \left(\begin{matrix} a'', & b'' \\ c'', & d'' \end{matrix}\right), \text{ \&c.}$$

where of course $d - a : b : c = d' - a' : b' : c' = d'' - a'' : b'' : c'' =$ &c. Here writing $h = d - a$, and consequently $R = -(d-a)^2 - 4bc$, and assuming also $J = \frac{1}{2}\sqrt{R}$ and $\cot q = \dfrac{d+a}{\sqrt{R}}$, and in like manner for the accented letters, the several matrices are respectively

$$(\tfrac{1}{2}\sqrt{R},\ q)\,(\tfrac{1}{2}\sqrt{R'},\ q'),\ (\tfrac{1}{2}\sqrt{R''},\ q''), \text{ \&c.},$$

and the compound matrix is

$$\left(\frac{\sin(q+q'+q''\ldots)}{\sin q \sin q' \sin q''\ldots}(\tfrac{1}{2}\sqrt{R})(\tfrac{1}{2}\sqrt{R'})(\tfrac{1}{2}\sqrt{R''})\ldots,\ q+q'+q''+\ldots\right).$$

49. When the several matrices are each of them equal to

$$\left(\begin{matrix} a, & b \\ c, & d \end{matrix}\right),$$

we have of course $q = q' = q''\ldots$, $R = R' = R''\ldots$, and we find

$$\left(\begin{matrix} a, & b \\ c, & d \end{matrix}\right)^n = \left(\frac{\sin nq}{\sin^n q}(\tfrac{1}{2}\sqrt{R})^n,\ nq\right);$$

or substituting for the right-hand side, the matrix represented by this notation, and putting for greater simplicity

$$\frac{\sin nq}{\sin^n q}(\tfrac{1}{2}\sqrt{R})^n = (\tfrac{1}{2}\sqrt{R})\,L, \text{ or } L = \frac{\sin nq}{\sin^n q}(\tfrac{1}{2}\sqrt{R})^{n-1},$$

we find

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2} L\left(\sqrt{R}\cot nq - (d-a)\right), & Lb \\ Lc & , \frac{1}{2} L\left(\sqrt{R}\cot nq + (d-a)\right) \end{pmatrix}$$

where it will be remembered that

$$R = -(d-a)^2 - 4bc \text{ and } \cot q = \frac{d+a}{\sqrt{R}},$$

the last of which equations may be replaced by

$$\cos q + \sqrt{-1}\sin q = \frac{d+a+\sqrt{-R}}{2\sqrt{ad-bc}}.$$

The formula in fact extends to negative or fractional values of the index n, and when n is a fraction, we must, as usual, in order to exhibit the formula in its proper generality, write $q + 2m\pi$ instead of q. In the particular case $n = \frac{1}{2}$, it would be easy to show the identity of the value of the square root of the matrix with that before obtained by a different process.

50. The matrix will be to a factor *près*, periodic of the nth order if only $\sin nq = 0$, that is, if $q = \frac{m\pi}{n}$ (m must be prime to n, for if it were not, the order of periodicity would be not n itself, but a submultiple of n); but $\cos q = \frac{d+a}{2\sqrt{ad-bc}}$, and the condition is therefore

$$(d+a)^2 - 4(ad-bc)\cos^2\frac{m\pi}{n} = 0,$$

or as this may also be written,

$$d^2 + a^2 - 2ad\cos\frac{2m\pi}{n} + 4bc\cos^2\frac{m\pi}{n} = 0,$$

a result which agrees with those before obtained for the particular values 2 and 3 of the index of periodicity.

51. I may remark that the last preceding investigations are intimately connected with the investigations of Babbage and others in relation to the function $\phi x = \frac{ax+b}{cx+d}$.

I conclude with some remarks upon rectangular matrices.

52. A matrix such as

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \end{pmatrix}$$

where the number of columns exceeds the number of lines, is said to be a broad matrix; a matrix such as

$$\begin{pmatrix} a, & b \\ a', & b' \\ a'', & b'' \end{pmatrix}$$

where the number of lines exceeds the number of columns, is said to be a deep matrix.

53. The matrix zero subsists in the present theory, but not the matrix unity. Matrices may be added or subtracted when the number of the lines and the number of the columns of the one matrix are respectively equal to the number of the lines and the number of the columns of the other matrix, and under the like condition any number of matrices may be added together. Two matrices may be equal or opposite, the one to the other. A matrix may be multiplied by a single quantity, giving rise to a matrix of the same form; two matrices so related are similar to each other.

54. The notion of composition applies to rectangular matrices, but it is necessary that the number of lines in the second or nearer component matrix should be equal to the number of columns in the first or further component matrix; the compound matrix will then have as many lines as the first or further component matrix, and as many columns as the second or nearer component matrix.

55. As examples of the composition of rectangular matrices, we have

$$\left(\begin{array}{c|c} a, b, c & \\ d, e, f & \end{array}\right) \between \left|\begin{array}{c} a', b', c', d' \\ e', f', g', h' \\ i', j', k', l' \end{array}\right| = \left(\begin{array}{c} (a, b, c \between a', e', i'), (a, b, c \between b', f', j') (a, b, c \between c', g', k'), (a, b, c \between d', h' l') \\ (d, e, f \between a', e', i'), (d, e, f \between b', f', j') (d, e, f \between c', g', k'), (d, e, f \between d', h' l') \end{array}\right),$$

and

$$\left(\begin{array}{c|c} a, d & \\ b, e & \\ c, f & \end{array}\right) \between \left|\begin{array}{c} a', b', c', d' \\ e', f', g', h' \end{array}\right| = \left(\begin{array}{c} (a, d \between a', e'), (a, d \between b', f'), (a, d \between c', g'), (a, d \between d', h') \\ (b, e \between a', e'), (b, e \between b', f'), (b, e \between c', g'), (b, e \between d', h') \\ (c, f \between a', e'), (c, f \between b', f'), (c, f \between c', g'), (c, f \between d', h') \end{array}\right).$$

56. In the particular case where the lines and columns of the one component matrix are respectively equal in number to the columns and lines of the other component matrix, the compound matrix is square, thus we have

$$\left(\begin{array}{c|c} a, b, c & \\ d, e, f & \end{array}\right) \between \left|\begin{array}{c} a', d' \\ b', e' \\ c', f' \end{array}\right| = \left(\begin{array}{c} (a, b, c \between a', b', c'), (a, b, c \between d', e', f') \\ (d, e, f \between a', b', c'), (d, e, f \between d', e', f') \end{array}\right)$$

and

$$\left(\begin{array}{c|c} a', d' & \\ b', e' & \\ c', f' & \end{array}\right) \between \left|\begin{array}{c} a, b, c \\ d, e, f \end{array}\right| = \left(\begin{array}{c} (a', d' \between a, d), (a', d' \between b, e), (a', d' \between c, f) \\ (b', e' \between a, d), (b', e' \between b, e), (b', e' \between c, f) \\ (c', f' \between a, d), (c', f' \between b, e), (c', f' \between c, f) \end{array}\right)$$

The two matrices in the case last considered admit of composition in the two different orders of arrangement, but as the resulting square matrices are not of the same order, the notion of the convertibility of two matrices does not apply even to the case in question.

57. Since a rectangular matrix cannot be compounded with itself, the notions of the inverse or reciprocal matrix and of the powers of the matrix and the whole resulting theory of the functions of a matrix, do not apply to rectangular matrices.

58. The notion of transposition and the symbol tr. apply to rectangular matrices, the effect of a transposition being to convert a broad matrix into a deep one and reciprocally. It may be noticed that the symbol tr. may be used for the purpose of expressing the law of composition of square or rectangular matrices. Thus treating (a, b, c) as a rectangular matrix, or representing it by $\begin{vmatrix} a, & b, & c \end{vmatrix}$, we have

$$\text{tr.} \begin{vmatrix} a', & b', & c' \end{vmatrix} = \begin{vmatrix} a' \\ b' \\ c' \end{vmatrix},$$

and thence

$$\begin{vmatrix} a, & b, & c \end{vmatrix} \text{tr.} \begin{vmatrix} a', & b', & c' \end{vmatrix} = \begin{vmatrix} a, & b, & c \end{vmatrix} \begin{vmatrix} a' \\ b' \\ c' \end{vmatrix} = (a, b, c \between a', b', c'),$$

so that the symbol

$$(a, b, c \between a', b', e')$$

would upon principle be replaced by

$$\begin{vmatrix} a, & b, & c \end{vmatrix} \text{tr.} \begin{vmatrix} a', & b', & c' \end{vmatrix}:$$

it is however more convenient to retain the symbol

$$(a, b, c \between a', b', c').$$

Hence introducing the symbol tr. only on the left-hand sides, we have

$$\begin{vmatrix} a, & b, & c \\ d, & e, & f \end{vmatrix} \text{tr.} \begin{vmatrix} a', & b', & c' \\ d', & e', & f' \end{vmatrix} = \begin{vmatrix} (a, b, c \between a', b', c'), & (a, b, c \between d', e', f') \\ (d, e, f \between a', b', c'), & (d, e, f \between d', e', f') \end{vmatrix},$$

or to take an example involving square matrices,

$$\begin{vmatrix} a, & b \\ d, & e \end{vmatrix} \text{tr.} \begin{vmatrix} a', & b' \\ d', & e' \end{vmatrix} = \begin{vmatrix} (a, b \between a', b'), & (a, b \between d', e') \\ (d, e \between a', b'), & (d, e \between d', e') \end{vmatrix};$$

it thus appears that in the composition of matrices (square or rectangular), when the second or nearer component matrix is expressed as a matrix preceded by the symbol tr., any *line* of the compound matrix is obtained by compounding the corresponding *line* of the first or further component matrix successively with the several *lines* of the matrix which preceded by tr. gives the second or nearer component matrix. It is clear that the terms 'symmetrical' and 'skew symmetrical' do not apply to rectangular matrices.

153.

A MEMOIR ON THE AUTOMORPHIC LINEAR TRANSFORMATION OF A BIPARTITE QUADRIC FUNCTION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVIII. *for the year* 1858, pp. 39—46. Received December 10, 1857,—Read January 14, 1858.]

THE question of the automorphic linear transformation of the function $x^2+y^2+z^2$, that is the transformation by linear substitutions, of this function into a function $x_,^2+y_,^2+z_,^2$ of the same form, is in effect solved by some formulæ of Euler's for the transformation of coordinates, and it was by these formulæ that I was led to the solution in the case of the sum of n squares, given in my paper "Sur quelques propriétés des déterminants gauches"(1). A solution grounded upon an *à priori* investigation and for the case of any quadric function of n variables, was first obtained by M. Hermite in the memoir "Remarques sur une Mémoire de M. Cayley relatif aux déterminants gauches"(2). This solution is in my Memoir "Sur la transformation d'une function quadratique en elle-même par des substitutions linéaires"(3), presented under a somewhat different form involving the notation of matrices. I have since found that there is a like transformation of a bipartite quadric function, that is a lineo-linear function of two distinct sets, each of the same number of variables, and the development of the transformation is the subject of the present memoir.

1. For convenience, the number of variables is in the analytical formulæ taken to be 3, but it will be at once obvious that the formulæ apply to any number of variables whatever. Consider the bipartite quadric

$$\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x, y, z)(\mathrm{x}, \mathrm{y}, \mathrm{z}),$$

1 *Crelle*, t. XXXII. (1846) pp. 119—123, [52].
2 *Cambridge and Dublin Mathematical Journal*, t. IX. (1854) pp. 63—67.
3 *Crelle*, t. L. (1855) pp. 288—299, [136].

which stands for

$$\begin{aligned}&(ax+by+cz)\,\mathrm{x}\\&+(a'x+b'y+c'z)\,\mathrm{y}\\&+(a''x+b''y+c''z)\,\mathrm{z},\end{aligned}$$

and in which (x, y, z) are said to be the nearer variables, and $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ the further variables of the bipartite.

2. It is clear that we have

$$\left(\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}\right)(x, y, z)(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \left(\begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix}\right)(\mathrm{x}, \mathrm{y}, \mathrm{z})(x, y, z)$$

and the new form on the right-hand side of the equation may also be written

$$\left(\mathrm{tr.}\left(\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}\right)\right)(\mathrm{x}, \mathrm{y}, \mathrm{z})(x, y, z),$$

that is, the two sets of variables may be interchanged, provided that the matrix is transposed.

3. Each set of variables may be linearly transformed: suppose that the substitutions are

$$(x, y, z) = \left(\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}\right)(x_{,}, y_{,}, z_{,})$$

and

$$(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \left(\begin{vmatrix} \mathrm{l}, & \mathrm{l}', & \mathrm{l}'' \\ \mathrm{m}, & \mathrm{m}', & \mathrm{m}'' \\ \mathrm{n}, & \mathrm{n}', & \mathrm{n}'' \end{vmatrix}\right)(\mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,}).$$

Then first substituting for (x, y, z) their values in terms of $(x_{,}, y_{,}, z_{,})$, the bipartite becomes

$$\left(\left(\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}\right)\left(\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}\right)\right)(x_{,}, y_{,}, z_{,})(\mathrm{x}, \mathrm{y}, \mathrm{z});$$

represent for a moment this expression by

$$\left(\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}\right)(x_{,}, y_{,}, z_{,})(\mathrm{x}, \mathrm{y}, \mathrm{z}),$$

then substituting for (x, y, z) their values in terms of $(\mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,})$, it is easy to see that the expression becomes

$$\left(\left(\begin{array}{ccc} \mathrm{l}, & \mathrm{m}, & \mathrm{n} \\ \mathrm{l}', & \mathrm{m}', & \mathrm{n}' \\ \mathrm{l}'', & \mathrm{m}'', & \mathrm{n}'' \end{array}\right) \between \left(\begin{array}{ccc} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{array}\right)\right) \between x_{,}, y_{,}, z_{,} \between \mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,}),$$

and re-establishing the value of the auxiliary matrix, we obtain, as the final result of the substitutions,

$$\left(\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}\right) = \left(\left(\begin{array}{ccc} \mathrm{l}, & \mathrm{m}, & \mathrm{n} \\ \mathrm{l}', & \mathrm{m}', & \mathrm{n}' \\ \mathrm{l}'', & \mathrm{m}'', & \mathrm{n}'' \end{array}\right) \between \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) \between \left(\begin{array}{ccc} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{array}\right)\right) \between x_{,}, y_{,}, z_{,} \between \mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,}),$$

that is, the matrix of the transformed bipartite is obtained by compounding in order, first or furthest the transposed matrix of substitution of the further variables, next the matrix of the bipartite, and last or nearest the matrix of substitution of the nearer variables.

4. Suppose now that it is required to find the automorphic linear transformation of the bipartite

$$\left(\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}\right),$$

or as it will henceforward for shortness be written,

$$(\Omega \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z});$$

this may be effected by a method precisely similar to that employed by M. Hermite for an ordinary quadric. For this purpose write

$$x + x_{,} = 2\xi, \quad y + y_{,} = 2\eta, \quad z + z_{,} = 2\zeta,$$
$$\mathrm{x} + \mathrm{x}_{,} = 2\Xi, \quad \mathrm{y} + \mathrm{y}_{,} = 2\mathrm{H}, \quad \mathrm{z} + \mathrm{z}_{,} = 2\mathrm{Z},$$

or, as these equations may be represented,

$$(x + x_{,}, \quad y + y_{,}, \quad z + z_{,}) = 2(\xi, \eta, \zeta),$$
$$(\mathrm{x} + \mathrm{x}_{,}, \quad \mathrm{y} + \mathrm{y}_{,}, \quad \mathrm{z} + \mathrm{z}_{,}) = 2(\Xi, \mathrm{H}, \mathrm{Z});$$

then we ought to have

$$(\Omega \between 2\xi - x, 2\eta - y, 2\zeta - z \between 2\Xi - \mathrm{x}, 2\mathrm{H} - \mathrm{y}, 2\mathrm{Z} - \mathrm{z}) = (\Omega \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}).$$

5. The left-hand side is

$$4(\Omega \between \xi, \eta, \zeta \between \Xi, \mathrm{H}, \mathrm{Z}) - 2(\Omega \between x, y, z \between \Xi, \mathrm{H}, \mathrm{Z}) - 2(\Omega \between \xi, \eta, \zeta \between \mathrm{x}, \mathrm{y}, \mathrm{z}) + \Omega \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}),$$

and the equation becomes

$$2(\Omega \between \xi, \eta, \zeta \between \Xi, \mathrm{H}, \mathrm{Z}) - (\Omega \between x, y, z \between \Xi, \mathrm{H}, \mathrm{Z}) - (\Omega \between \xi, \eta, \zeta \between \mathrm{x}, \mathrm{y}, \mathrm{z}) = 0,$$

or as it may be written,

$$\left.\begin{array}{r}(\Omega\between\xi,\ \eta,\ \zeta\between\Xi,\ \mathrm{H},\ \mathrm{Z})-(\Omega\between x,\ y,\ z\between\Xi,\ \mathrm{H},\ \mathrm{Z})\\ +(\Omega\between\xi,\ \eta,\ \zeta\between\Xi,\ \mathrm{H},\ \mathrm{Z})-(\Omega\between\xi,\ \eta,\ \zeta\between\mathrm{x},\ \mathrm{y},\ \mathrm{z})\end{array}\right\}=0;$$

or again,

$$\left.\begin{array}{r}(\Omega\between\xi-x,\ \eta-y,\ \zeta-z\between\Xi,\ \mathrm{H},\ \mathrm{Z})\\ +(\Omega\between\xi,\ \eta,\ \zeta\between\Xi-\mathrm{x},\ \mathrm{H}-\mathrm{y},\ \mathrm{Z}-\mathrm{z})\end{array}\right\}=0,$$

or what is the same thing,

$$\left.\begin{array}{r}(\Omega\between\xi-x,\ \eta-y,\ \zeta-z\between\Xi,\ \mathrm{H},\ \mathrm{Z})\\ +(\mathrm{tr.}\ \Omega\between\Xi-\mathrm{x},\ \mathrm{H}-\mathrm{y},\ \mathrm{Z}-\mathrm{z}\between\xi,\ \eta,\ \zeta)\end{array}\right\}=0,$$

and it is easy to see that the equation will be satisfied by writing

$$\begin{aligned}(\ \ \Omega\between\xi-x,\ \eta-y,\ \zeta-z)&=(\ \ \Upsilon\between\xi,\ \eta,\ \zeta),\\ (\mathrm{tr.}\ \Omega\between\Xi-\mathrm{x},\ \mathrm{H}-\mathrm{y},\ \mathrm{Z}-\mathrm{z})&=-(\mathrm{tr.}\ \Upsilon\between\Xi,\ \mathrm{H},\ \mathrm{Z}),\end{aligned}$$

where Υ is any arbitrary matrix. In fact we have then

$$\begin{aligned}(\ \ \Omega\between\xi-x,\ \eta-y,\ \zeta-z\between\Xi,\ \mathrm{H},\ \mathrm{Z})&=(\ \ \Upsilon\between\xi,\ \eta,\ \zeta\between\Xi,\ \mathrm{H},\ \mathrm{Z}),\\ (\mathrm{tr.}\ \Omega\between\Xi-\mathrm{x},\ \mathrm{H}-\mathrm{y},\ \mathrm{Z}-\mathrm{z}\between\xi,\ \eta,\ \zeta)&=-(\mathrm{tr.}\ \Upsilon\between\Xi,\ \mathrm{H},\ \mathrm{Z}\between\xi,\ \eta,\ \zeta)\\ &=-(\ \ \Upsilon\between\xi,\ \eta,\ \zeta\between\Xi,\ \mathrm{H},\ \mathrm{Z}),\end{aligned}$$

and the sum of the two terms consequently vanishes.

6. The equation

$$(\Omega\between\xi-x,\ \eta-y,\ \zeta-z)=(\Upsilon\between\xi,\ \eta,\ \zeta)$$

gives

$$(\Omega-\Upsilon\between\xi,\ \eta,\ \zeta)=(\Omega\between x,\ y,\ z),$$

and we then have

$$(\Omega+\Upsilon\between\xi,\ \eta,\ \zeta)=(\Omega\between x_{,},\ y_{,},\ z_{,}).$$

In fact the two equations give

$$(2\Omega\between\xi,\ \eta,\ \zeta)=(\Omega\between x+x_{,},\ y+y_{,},\ z+z_{,}),$$

or what is the same thing,

$$2(\xi,\ \eta,\ \zeta)=(x+x_{,},\ y+y_{,},\ z+z_{,}),$$

which is the equation assumed as the definition of $(\xi,\ \eta,\ \zeta)$; and conversely, this equation, combined with either of the two equations, gives the other of them.

7. We have consequently

$$\begin{aligned}(x,\ y,\ z)&=(\Omega^{-1}(\Omega-\Upsilon)\between\xi,\ \eta,\ \zeta),\\ (\xi,\ \eta,\ \zeta)&=((\Omega+\Upsilon)^{-1}\Omega\between x_{,},\ y_{,},\ z_{,}),\end{aligned}$$

and thence

$$(x,\ y,\ z)=(\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega\between x_{,},\ y_{,},\ z_{,}).$$

8. But in like manner the equation

$$(\text{tr. } \Omega \between \Xi - \mathrm{x},\ \mathrm{H} - \mathrm{y},\ \mathrm{Z} - \mathrm{z}) = -(\text{tr. } \Upsilon \between \Xi,\ \mathrm{H},\ \mathrm{Z})$$

gives

$$(\text{tr. } \overline{\Omega + \Upsilon} \between \Xi,\ \mathrm{H},\ \mathrm{Z}) = (\text{tr. } \Omega \between \mathrm{x},\ \mathrm{y},\ \mathrm{z}),$$

and we then obtain

$$(\text{tr. } \overline{\Omega - \Upsilon} \between \Xi,\ \mathrm{H},\ \mathrm{Z}) = (\text{tr. } \Omega \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}).$$

9. In fact these equations give

$$(\text{tr. } 2\Omega \between \Xi,\ \mathrm{H},\ \mathrm{Z}) = (\text{tr. } \Omega \between \mathrm{x} + \mathrm{x}_{,},\ \mathrm{y} + \mathrm{y}_{,},\ \mathrm{z} + \mathrm{z}_{,}),$$

or

$$2(\Xi,\ \mathrm{H},\ \mathrm{Z}) = (\mathrm{x} + \mathrm{x}_{,},\ \mathrm{y} + \mathrm{y}_{,},\ \mathrm{z} + \mathrm{z}_{,});$$

and conversely, this equation, combined with either of the two equations, gives the other of them. We have then

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = ((\text{tr. } \Omega)^{-1} (\text{tr. } \overline{\Omega + \Upsilon} \between \Xi,\ \mathrm{H},\ \mathrm{Z}),$$

$$(\Xi,\ \mathrm{H},\ \mathrm{Z}) = ((\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{ tr. } \Omega \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}),$$

and thence

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = ((\text{tr. } \Omega)^{-1} (\text{tr. } \overline{\Omega + \Upsilon})(\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{tr. } \Omega \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}).$$

10. Hence, recapitulating, we have the following theorem for the automorphic linear transformation of the bipartite

$$(\Omega \between x,\ y,\ z \between \mathrm{x},\ \mathrm{y},\ \mathrm{z}),$$

viz. Υ being an arbitrary matrix, if

$$(x,\ y,\ z) = (\Omega^{-1} (\Omega - \Upsilon)(\Omega + \Upsilon)^{-1} \Omega \between x_{,},\ y_{,},\ z_{,}),$$

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = ((\text{tr. } \Omega)^{-1} (\text{tr. } \overline{\Omega + \Upsilon})(\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{ tr. } \Omega \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}),$$

then

$$(\Omega \between x,\ y,\ z \between \mathrm{x},\ \mathrm{y},\ \mathrm{z}) = (\Omega \between x_{,},\ y_{,},\ z_{,} \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}),$$

which is the theorem in question.

11. I have thought it worth while to preserve the foregoing investigation, but the most simple demonstration is the verification *à posteriori* by the actual substitution of the transformed values of (x, y, z), $(\mathrm{x}, \mathrm{y}, \mathrm{z})$. To effect this, recollecting that in general $\text{tr. } (A^{-1}) = (\text{tr. } A)^{-1}$ and $\text{tr. } ABCD = \text{tr. } D.\ \text{tr. } C.\ \text{tr. } B.\ \text{tr. } A$, the transposed matrix of substitution for the further variables is

$$\Omega\, (\Omega - \Upsilon)^{-1} (\Omega + \Upsilon)\, \Omega^{-1};$$

and compounding this with the matrix Ω of the bipartite, and the matrix

$$\Omega^{-1} (\Omega - \Upsilon)(\Omega + \Upsilon)^{-1} \Omega$$

of substitution for the nearer variables, the theorem will be verified if the result is equal to the matrix Ω of the bipartite; that is, we ought to have

$$\Omega(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\Omega^{-1}\Omega\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega=\Omega,$$

or what is the same thing,

$$\Omega(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega=\Omega;$$

this is successively reducible to

$$\begin{aligned}(\Omega+\Upsilon)\Omega^{-1}(\Omega-\Upsilon) &= (\Omega-\Upsilon)\Omega^{-1}(\Omega+\Upsilon),\\ \Omega^{-1}(\Omega+\Upsilon)\Omega^{-1}(\Omega-\Upsilon) &= \Omega^{-1}(\Omega-\Upsilon)\Omega^{-1}(\Omega+\Upsilon),\\ (1+\Omega^{-1}\Upsilon)(1-\Omega^{-1}\Upsilon) &= (1-\Omega^{-1}\Upsilon)(1+\Omega^{-1}\Upsilon),\end{aligned}$$

which is a mere identity, and the theorem is thus shown to be true.

12. It is to be observed that, in the general theorem, the transformations or matrices of substitution for the two sets of variables respectively are not identical, but it may be required that this shall be so. Consider first the case where the matrix Ω is symmetrical, the necessary condition is that the matrix Υ shall be skew symmetrical; in fact we have then

$$\text{tr. }\Omega=\Omega,\ \text{ tr. }\Upsilon=-\Upsilon,$$

and the transformations become

$$\begin{aligned}(x,\ y,\ z) &= (\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega\between x_{,},\ y_{,},\ z_{,}),\\ (\mathrm{x},\ \mathrm{y},\ \mathrm{z}) &= (\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega\between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}),\end{aligned}$$

which are identical. We may in this case suppose that the two sets of variables become equal, and we have then the theorem for the automorphic linear transformation of the ordinary quadric

$$(\Omega\between x,\ y,\ z)^2,$$

viz. Υ being a skew symmetrical matrix, if

$$(x,\ y,\ z)=(\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega\between x_{,},\ y_{,},\ z_{,}),$$

then

$$(\Omega\between x,\ y,\ z)^2=(\Omega\between x_{,},\ y_{,},\ z_{,})^2.$$

13. Next, if the matrix Ω be skew symmetrical, the condition is that the matrix Υ shall be symmetrical; we have in this case tr. $\Omega=-\Omega$, tr. $\Upsilon=\Upsilon$, and the four factors in the matrix of substitution for (x, y, z) are respectively $-\Omega^{-1}$, $-(\Omega-\Upsilon)$, $-(\Omega+\Upsilon)^{-1}$ and $-\Omega$, and such matrix of substitution becomes therefore, as before, identical with that for $(x,\ y,\ z)$; we have therefore the following theorem for the automorphic linear transformation of a skew symmetrical bipartite

$$(\Omega\between x,\ y,\ z\between \mathrm{x},\ \mathrm{y},\ \mathrm{z}),$$

when the transformations for the two sets of variables are identical, viz. Υ being any symmetrical matrix, if

$$(x,\ y,\ z) = (\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega \between x_{,},\ y_{,},\ z_{,}),$$

$$(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = (\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}),$$

then

$$(\Omega \between x,\ y,\ z \between \mathrm{x},\ \mathrm{y},\ \mathrm{z}) = (\Omega \between x_{,},\ y_{,},\ z_{,} \between \mathrm{x}_{,},\ \mathrm{y}_{,},\ \mathrm{z}_{,}).$$

14. Lastly, in the general case where the matrix Ω is anything whatever, the condition is

$$\Omega^{-1}\Upsilon = -(\text{tr. } \Omega)^{-1} \text{ tr. } \Upsilon$$

for assuming this equation, then first

$$\Omega^{-1}(\Omega - \Upsilon) = (\text{tr. } \Omega)^{-1}(\text{tr. } \overline{\Omega + \Upsilon}),$$

and in like manner

$$\Omega^{-1}(\Omega + \Upsilon) = (\text{tr. } \Omega)^{-1}(\text{tr. } \overline{\Omega - \Upsilon}).$$

But we have

$$1 = (\text{tr. } \Omega)^{-1}(\text{tr. } \overline{\Omega - \Upsilon})(\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{ tr. } \Omega,$$

and therefore, secondly,

$$(\Omega + \Upsilon)^{-1}\Omega = (\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{ tr. } \Omega\,;$$

and thence

$$\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega = (\text{tr. } \Omega)^{-1}(\text{tr. } \overline{\Omega + \Upsilon})(\text{tr. } \overline{\Omega - \Upsilon})^{-1} \text{tr. } \Omega,$$

or the two transformations are identical.

15. To further develope this result, let Ω^{-1} be expressed as the sum of a symmetrical matrix Q_0 and a skew symmetrical matrix $Q_{,}$, and let Υ be expressed in like manner as the sum of a symmetrical matrix Υ_0 and a skew symmetrical matrix $\Upsilon_{,}$. We have then

$$\begin{aligned} \Omega^{-1} &= Q_0 + Q_{,}, \\ (\text{tr. } \Omega)^{-1} = \text{tr. } (\Omega^{-1}) &= Q_0 - Q_{,}, \\ \Upsilon &= \Upsilon_0 + \Upsilon_{,}, \\ \text{tr. } \Upsilon &= \Upsilon_0 - \Upsilon_{,}, \end{aligned}$$

and the condition, $\Omega^{-1}\Upsilon = -(\text{tr. } \Omega)^{-1} \text{ tr. } \Upsilon$, becomes

$$(Q_0 + Q_{,})(\Upsilon_0 + \Upsilon_{,}) = -(Q_0 - Q_{,})(\Upsilon_0 - \Upsilon_{,}),$$

that is,

$$Q_0\Upsilon_0 + Q_{,}\Upsilon_{,} = 0,$$

and we have

$$\Upsilon_0 = -Q_0^{-1}Q_{,}\Upsilon_{,},$$

or as we may write it,

$$\Upsilon_0 = -(\tfrac{1}{2}\{\Omega^{-1} + \text{tr. }\Omega^{-1}\})^{-1}(\tfrac{1}{2}\{\Omega^{-1} - \text{tr. }\Omega^{-1}\})\Upsilon_{,},$$

and thence

$$\Upsilon = -(\tfrac{1}{2}\{\Omega^{-1} + \text{tr. }\Omega^{-1}\})^{-1}(\tfrac{1}{2}\{\Omega^{-1} - \text{tr. }\Omega^{-1}\})\Upsilon_{,} + \Upsilon_{,},$$

where $\Upsilon_{,}$ is an arbitrary skew symmetrical matrix.

16. This includes the before-mentioned special cases; first, if Ω is symmetrical, then we have simply $\Upsilon = \Upsilon_{,}$, an arbitrary skew symmetrical matrix, which is right. Next, if Ω is skew symmetrical, then $\Upsilon = -0^{-1}\Omega^{-1}\Upsilon_{,} + \Upsilon_{,}$, which can only be finite for $\Upsilon_{,} = 0$, that is, we have $\Upsilon = -0^{-1}\Omega^{-1}0$, and (the first part of Υ being always symmetrical) this represents an arbitrary symmetrical matrix. The mode in which this happens will be best seen by an example. Suppose

$$\Omega^{-1} = \begin{pmatrix} A, & H+\nu \\ H-\nu, & B \end{pmatrix}, \quad \text{tr. }\Omega^{-1} = \begin{pmatrix} A, & H-\nu \\ H+\nu, & B \end{pmatrix},$$

and write

$$\Upsilon_{,} = \begin{pmatrix} 0, & \theta \\ -\theta, & 0 \end{pmatrix},$$

then we have

$$\Upsilon = -\begin{pmatrix} A, & H \\ H, & B \end{pmatrix}^{-1}\begin{pmatrix} 0, & \nu \\ -\nu, & 0 \end{pmatrix}\begin{pmatrix} 0, & \theta \\ -\theta, & 0 \end{pmatrix} + \begin{pmatrix} 0, & \theta \\ -\theta, & 0 \end{pmatrix}$$

$$= -\frac{\nu\theta}{AB-H^2}\begin{pmatrix} -B, & H \\ H, & -A \end{pmatrix} + \begin{pmatrix} 0, & \theta \\ -\theta, & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\nu B\theta}{AB-H^2}, & \dfrac{-\nu H\theta}{AB-H^2} + \theta \\ \dfrac{-\nu H\theta}{AB-H^2} - \theta, & \dfrac{\nu A\theta}{AB-H^2} \end{pmatrix} + \begin{pmatrix} 0, & \theta \\ -\theta, & 0 \end{pmatrix}.$$

When Ω is skew symmetrical, A, B, H vanish; but since their ratios remain arbitrary, we may write κA, κB, κH for A, B, H, and assume ultimately $\kappa = 0$. Writing $\kappa\theta$ in the place of θ, and then putting $\kappa = 0$, the matrix becomes

$$\begin{pmatrix} \dfrac{\nu B\theta}{AB-H^2}, & \dfrac{-\nu B\theta}{AB-H^2} \\ \dfrac{-\nu H\theta}{AB-H^2}, & \dfrac{\nu A\theta}{AB-H^2} \end{pmatrix}$$

which, inasmuch as $A:\theta$, $B:\theta$, and $C:\theta$ remain arbitrary, represents, as it should do, an arbitrary symmetrical matrix.

17. Hence, finally, we have the following Theorem for the automorphic linear transformation of the bipartite quadric,

$$(\Omega \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}),$$

when the two transformations are identical, viz. if $\Upsilon_{,}$ be a skew symmetrical matrix, and if

$$\Upsilon = -(\tfrac{1}{2}\{\Omega^{-1} + \mathrm{tr.}\,\Omega^{-1}\})(\tfrac{1}{2}\{\Omega^{-1} - \mathrm{tr.}\,\Omega^{-1}\})\Upsilon_{,} + \Upsilon_{,};$$

then if

$$(x, y, z) = (\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega \between x_{,}, y_{,}, z_{,}),$$

$$(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega \between \mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,});$$

we have

$$(\Omega \between x, y, z \between \mathrm{x}, \mathrm{y}, \mathrm{z}) = (\Omega \between x_{,}, y_{,}, z_{,} \between \mathrm{x}_{,}, \mathrm{y}_{,}, \mathrm{z}_{,});$$

and in particular,

If Ω is a symmetrical matrix, then Υ is an arbitrary skew symmetrical matrix;

If Ω is a skew symmetrical matrix, then Υ is an arbitrary symmetrical matrix.

154.

SUPPLEMENTARY RESEARCHES ON THE PARTITION OF NUMBERS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVIII. *for the year* 1858, pp. 47—52. Received March 19,—Read June 18, 1857.]

THE general formula given at the conclusion of my memoir, "Researches on the Partition of Numbers"(1), is somewhat different from the corresponding formula of Professor Sylvester[2], and leads more directly to the actual expression for the number of partitions, in the form made use of in my memoir; to complete my former researches, I propose to explain the mode of obtaining from the formula the expression for the number of partitions.

The formula referred to is as follows, viz. if $\frac{\phi x}{fx}$ be a rational fraction, the denominator of which is made up of factors (the same or different) of the form $1-x^m$, and if a is a divisor of one or more of the indices m, and k is the number of indices of which it is a divisor, then

$$\left\{\frac{\phi x}{fx}\right\}_{[1-x^a]} = \ldots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1} S \frac{\chi\rho}{\rho - x} \ldots$$

$$= \ldots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1} S \frac{\theta x}{[1-x^a]} \ldots$$

where

$$\chi\rho = \text{coeff.}\ \frac{1}{t} \text{ in } t^{s-1} \frac{\rho\phi(\rho e^{-t})}{f(\rho e^{-t})},$$

[1] *Philosophical Transactions*, tom. CXLVI. (1856) p. 127, [140].

[2] Professor Sylvester's researches are published in the *Quarterly Mathematical Journal*, tom. I. [1857, pp. 141—152]; there are some numerical errors in his value of P (1, 2, 3, 4, 5, 6) q.

in which formula $[1-x^a]$ denotes the irreducible factor of $1-x^a$, that is, the factor which equated to zero gives the prime roots, and ρ is a root of the equation $[1-x^a]=0$; the summation of course extends to all the roots of the equation. The index s extends from $s=1$ to $s=k$; and we have then the portion of the fraction depending on the denominator $[1-x^a]$. In the partition of numbers, we have $\phi x=1$, and the formula becomes therefore

$$\left\{\frac{1}{fx}\right\}_{[1-x^a]}=\ldots+\frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}S\frac{\chi\rho}{\rho-x}\ldots$$

$$=\ldots+\frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\frac{\theta x}{[1-x^a]},$$

where

$$\chi\rho=\text{coeff.}\frac{1}{t}\text{ in }t^{s-1}\frac{\rho}{f(\rho e^{-t})}.$$

We may write

$$fx=\Pi(1-x^m),$$

where m has a given series of values the same or different. The indices not divisible by a may be represented by m, the other indices by ap, we have then

$$fx=\Pi(1-x^n)\,\Pi(1-x^{ap}),$$

where the number of indices ap is equal to k. Hence

$$f(\rho e^{-t})=\Pi(1-\rho^n e^{-nt})\,\Pi(1-\rho^{ap}e^{-apt});$$

or since ρ is a root of $[1-x^a]=0$, and therefore $\rho^a=1$, we have

$$f(\rho e^{-t})=\Pi(1-\rho^n e^{-nt})\,\Pi(1-e^{-apt});$$

and it may be remarked that if $n\equiv\nu$ (mod. a), where $\nu<a$, then instead of ρ^n we may write ρ^ν, a change which may be made at once, or at the end of the process of development.

We have consequently to find

$$\chi\rho=\text{coeff.}\frac{1}{t}\text{ in }t^{s-1}\frac{\rho}{\Pi(1-\rho^n e^{-nt})\,\Pi(1-e^{-apt})}.$$

The development of a factor $\dfrac{1}{1-\rho^n e^{-nt}}$ is at once deduced from that of $\dfrac{1}{1-ce^{-t}}$, and is a series of positive powers of t. The development of a factor $\dfrac{1}{1-e^{-apt}}$ is deduced from that of $\dfrac{1}{1-e^{-t}}$, and contains a term involving $\dfrac{1}{t}$. Hence we have

$$\frac{1}{\Pi(1-\rho^n e^{-nt})\,\Pi(1-e^{-apt})}=A_{-k}\frac{1}{t^k}+A_{-(k-1)}\frac{1}{t^{k-1}}\ldots+A_{-1}\frac{1}{t}+A_0+\&c.,$$

and thence

$$\chi\rho=\rho A_{-s}.$$

The actual development, when k is small (for instance $k=1$ or $k=2$), is most readily obtained by developing each factor separately and taking the product. To do this we have

$$\frac{1}{1-ce^{-t}}=\frac{1}{1-c}-\frac{c}{(1-c)^2}t+\frac{c+c^2}{(1-c)^3}\tfrac{1}{2}t^2-\frac{c+4c^2+c^3}{(1-c)^4}\tfrac{1}{6}t^3+\&c.,$$

where by a general theorem for the expansion of any function of e^t, the coefficient of t^f is

$$=\frac{(-)^f}{\Pi f}\frac{1}{1-c(1+\Delta)}0^f$$

$$=\frac{(-)^f}{\Pi f}\left(\frac{1}{1-c}+\frac{c}{(1-c)^2}\Delta\ldots+\frac{c^f}{(1-c)^{f+1}}\right)0^f$$

(where as usual $\Delta 0^f=1^f-0^f$, $\Delta^2 0^f=2^f-2\,.\,1^f+0^f$, &c.) and

$$\frac{1}{1-e^{-t}}=\frac{1}{t}+\frac{1}{2}+\frac{1}{12}t-\frac{1}{720}t^3+\frac{1}{30240}t^5-\&c.,$$

where, except the constant term, the series contains odd powers only and the coefficient of t^{2f-1} is $\frac{(-)^{f+1}B_f}{\Pi 2f}$; B_1, B_2, $B_3\ldots$ denoting the series $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}\ldots$ of Bernoulli's numbers.

But when k is larger, it is convenient to obtain the development of the fraction from that of the logarithm, the logarithm of the fraction being equal to the sum of the logarithms of the simple factors, and these being found by means of the formulæ

$$\log\frac{1}{1-ce^{-t}}=\log\frac{1}{1-c}-\frac{c}{1-c}t+\frac{c}{(1-c)^2}\frac{t^2}{2}-\frac{c+c^2}{(1-c)^3}\frac{t^3}{6}+\frac{c+4c^2+c^3}{(1-c)^4}\frac{t^2}{24}+\&c.$$

$$\log\frac{1}{1-e^{-t}}=-\log t+\frac{1}{2}t-\frac{1}{24}t^2+\frac{1}{2880}t^4-\frac{1}{181440}t^6+\&c.$$

The fraction is thus expressed in the form

$$\frac{1}{\Pi(1-\rho^n)\,\Pi(ap)}\frac{1}{t^k}e^{k_1t+k_2t^2+\ldots};$$

and by developing the exponential we obtain, as before, the series commencing with $A_{-k}\frac{1}{t^k}$.

Resuming now the formula

$$\chi\rho=\rho A_{-s},$$

which gives $\chi\rho$ as a function of ρ, we have

$$\frac{\theta x}{[1-x^a]}=S\frac{\chi\rho}{\rho-x};$$

but this equation gives

$$\chi\rho = \theta\rho \left(\frac{\rho - x}{[1 - x^a]}\right)_{x=\rho},$$

and we have

$$[1 - x^a] = (x - \rho)(x - \rho^{a_2}) \dots (x - \rho^{a_\alpha}),$$

if 1, $a_2, \dots a_\alpha$ are the integers less than a and prime to it (α is of course the degree of $[1 - x^a]$). Hence

$$\chi\rho = \theta\rho \frac{-1}{\rho^{\alpha-1} \Pi (1 - \rho^{a_i - 1})},$$

and therefore

$$\theta\rho = -\rho^{\alpha-1} \Pi (1 - \rho^{a_i-1}) \chi\rho;$$

or putting for $\chi\rho$ its value

$$\theta\rho = -\rho^{\alpha} \Pi (1 - \rho^{a_i-1}) A_{-\beta},$$

where α is the degree of $[1 - x^a]$ and a_i denotes in succession the integers (exclusive of unity) less than a and prime to it. The function on the right hand, by means of the equation $[1 - \rho^a] = 0$, may be reduced to an integral function of ρ of the degree $\alpha - 1$, and then by simply changing ρ into x we have the required function θx. The fraction $\dfrac{\theta x}{[1 - x^a]}$ can then by multiplication of the terms by the proper factor be reduced to a fraction with the denominator $1 - x^a$, and the coefficients of the numerator of this fraction are the coefficients of the corresponding prime circulator ( ) per a_q.

Thus, let it be required to find the terms depending on the denominator $[1 - x^3]$ in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)};$$

these are

$$S\frac{\chi_1\rho}{\rho - x}, \quad x\partial_x S\frac{\chi_2\rho}{\rho - x},$$

where

$$\chi_1\rho = \text{coeff. } \frac{1}{t} \text{ in } \frac{\rho}{f(\rho e^{-t})},$$

$$\chi_2\rho = \text{coeff. } \frac{1}{t} \text{ in } t\frac{\rho}{f(\rho e^{-t})}$$

and

$$\frac{1}{f(\rho e^{-t})} = \frac{1}{(1-\rho e^{-t})(1-\rho^2 e^{-2t})(1-\rho^4 e^{-4t})(1-\rho^5 e^{-5t})(1-e^{-3t})(1-e^{-6t})}$$

$$= A_{-2}\frac{1}{t^2} + A_{-1}\frac{1}{t} + \&c.,$$

where it is easy to see that

$$A_{-2}=\frac{1}{18}\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)},$$

$$A_{-1}=\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}\left\{\frac{1}{4}-\frac{1}{18}\left(\frac{\rho}{1-\rho}+\frac{2\rho^2}{1-\rho^2}+\frac{4\rho^4}{1-\rho^4}+\frac{4\rho^5}{1-\rho^5}\right)\right\},$$

and we have

$$\theta_2\rho=-\rho^2(1-\rho)A_{-2},$$
$$\theta_1\rho=-\rho^2(1-\rho)A_{-1}.$$

But $[1-\rho^3]=1+\rho+\rho^2=0$. Hence $\rho^3=1$, and therefore

$$(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)=(1-\rho)^2(1-\rho^2)^2=9.$$

Hence

$$\theta_2\rho=-\frac{1}{162}\rho^2(1-\rho)=\frac{1}{162}(1-\rho^2)=\frac{1}{162}(2+\rho),$$

whence

$$\theta_2 x=\frac{1}{162}(2+x),$$

and the partial fraction is

$$\frac{1}{162}\frac{2+x}{1+x+x^2},$$

which is

$$=\frac{1}{162}\frac{2-x-x^2}{1-x^3},$$

and gives rise to the prime circulator $\frac{1}{162}(2,\ -1,\ -1)$ per 3_q.

The reduction $\theta_1\rho$ is somewhat less simple; we have

$$\begin{aligned}\theta_1\rho&=-\frac{1}{9}\rho^2(1-\rho)\left\{\frac{1}{4}-\frac{1}{18}\left(\frac{\rho}{1-\rho}+\frac{2\rho^2}{1-\rho^2}+\frac{4\rho}{1-\rho}+\frac{5\rho^2}{1-\rho^2}\right)\right\}\\&=-\frac{1}{9}\rho^2(1-\rho)\left\{\frac{1}{4}-\frac{5}{18}\frac{\rho}{1-\rho}-\frac{7}{18}\frac{\rho^2}{1-\rho^2}\right\}\\&=\frac{1}{9}(1-\rho^2)\left\{\frac{1}{4}-\frac{5}{54}\rho(1-\rho^2)-\frac{7}{54}\rho^2(1-\rho)\right\}\\&=\frac{1}{972}(1-\rho^2)(51-10\rho-14\rho^2)\\&=\frac{1}{972}(61+4\rho-65\rho^2);\end{aligned}$$

hence finally

$$\theta_1\rho = \frac{1}{324}(42+23\rho), \quad \theta_1 x = \frac{1}{324}(42+23x);$$

and the partial fraction is

$$\frac{1}{324}x\partial_x \frac{42+23x}{1+x+x^2},$$

which is

$$= \frac{1}{324}x\partial_x \frac{42-19x-23x^2}{1-x^3},$$

and gives rise to the prime circulator $\frac{1}{324}q(42, -19, -23)$ per 3_q.

The part depending on the denominator $1-x$ is

$$\frac{A_{-1}}{1-x} + \frac{1}{1}x\partial_x\frac{A_{-2}}{1-x} + \frac{1}{1.2}(x\partial_x)^2\frac{A_{-3}}{1-x}\cdots + \frac{1}{1.2.3.4.5}(x\partial_x)^5\frac{A_{-6}}{1-x},$$

where

$$\frac{1}{(1-e^{-t})(1-e^{-xt})(1-e^{-x^2t})(1-e^{-x^3t})(1-e^{-x^4t})(1-e^{-x^5t})}$$

$$= A_{-6}\frac{1}{t^6} + A_{-5}\frac{1}{t^5}\cdots + A_{-1}\frac{1}{t} + \&c.$$

We have here

$$\log\frac{1}{1-e^{-t}} = -\log t + \frac{1}{2}t - \frac{1}{24}t^2 + \frac{1}{2880}t^4 - \&c.,$$

and thence the fraction is

$$\frac{1}{720\,t^6}e^{\frac{21}{2}t - \frac{91}{24}t^2 + \frac{455}{576}t^4 - \&c.}$$

which is equal to

$$\frac{720\,t^6}{1}\Big(1 + \frac{21}{2}t + \frac{441}{8}t^2 + \frac{3087}{16}t^3 + \frac{64827}{128}t^4 + \frac{1361367}{1280}t^5 + \cdots$$
$$\times\Big(1 - \frac{91}{24}t^2 + \frac{8281}{1152}t^4 + \ldots\Big)$$
$$\times\Big(1 + \frac{455}{576}t^4 + \ldots\Big)$$
$$= \frac{1}{720}\frac{1}{t^6} + \frac{7}{480}\frac{1}{t^5} + \frac{77}{1080}\frac{1}{t^4} + \frac{245}{1152}\frac{1}{t^3} + \frac{43981}{103680}\frac{1}{t^2} + \frac{199577}{345600}\frac{1}{t} + \cdots$$

and consequently the partial fractions are

$$\frac{1}{86400}(x\partial_x)^5\frac{1}{1-x} + \frac{7}{11520}(x\partial_x)^4\frac{1}{1-x} + \frac{77}{6480}(x\partial_x)^3\frac{1}{1-x} + \frac{245}{2304}(x\partial_x)^2\frac{1}{1-x}$$
$$+ \frac{43981}{103680}(x\partial_x)\frac{1}{1-x} + \frac{199577}{345600}\frac{1}{1-x},$$

from which the non-circulating part is at once obtained.

The complete expression for the number of partitions is $P(1, 2, 3, 4, 5, 6)\,q =$

$$\frac{1}{1036800}(12q^5 + 630q^4 + 1230q^3 + 110250q^2 + 439810q + 598731)$$

$$+ \quad \frac{1}{4608}(6q^2 + 126q + 581)(1, 1) \text{ per } 2_q$$

$$+ \quad \frac{1}{162}q \ldots\ldots\ldots\ldots (2, -1, -1) \text{ per } 3_q$$

$$+ \quad \frac{1}{324} \ldots\ldots (42, -19, -23) \text{ per } 3_q$$

$$+ \quad \frac{1}{32} \ldots\ldots\ldots (1, 1, -1, -1) \text{ per } 4_q$$

$$+ \quad \frac{1}{25} \ldots\ldots (2, 1, 0, -1, -2) \text{ per } 5_q$$

$$+ \quad \frac{1}{36} \ldots (2, 1, -1, -2, -1, 1) \text{ per } 6_q.$$

155.

A FOURTH MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVIII. *for the year* 1858, pp. 415—427. Received February 11,—Read March 18, 1858.]

THE object of the present memoir is the further development of the theory of binary quantics; it should therefore have preceded so much of my third memoir, t. 147 (1857), p. 627, [144], as relates to ternary quadrics and cubics. The paragraphs are numbered continuously with those of the former memoirs. The first three paragraphs, Nos. 62 to 64, relate to quantics of the general form $(*\between x, y, \ldots)^m$, and they are intended to complete the series of definitions and explanations given in Nos. 54 to 61 of my third memoir; Nos. 68 to 71, although introduced in reference to binary quantics, relate or may be considered as relating to quantics of the like general form. But with these exceptions the memoir relates to binary quantics of any order whatever: viz. Nos. 65 to 80 relate to the covariants and invariants of the degrees 2, 3 and 4; Nos. 81 and 82 (which are introduced somewhat parenthetically) contain the explanation of a process for the calculation of the invariant called the Discriminant; Nos. 83 to 85 contain the definitions of the Catalecticant, the Lambdaic and the Canonisant, which are functions occurring in Professor Sylvester's theory of the reduction of a binary quantic to its canonical form; and Nos. 86 to 91 contain the definitions of certain covariants or other derivatives connected with Bezout's abbreviated method of elimination, due for the most part to Professor Sylvester, and which are called Bezoutiants, Cobezoutiants, &c. I have not in the present memoir in any wise considered the theories to which the catalecticant, &c. and the last-mentioned other covariants and derivatives relate; the design is to point out and precisely define the different covariants or other derivatives which have hitherto presented themselves in theories relating to binary quantics, and so to complete, as far as may be, the explanation of the terminology of this part of the subject.

62. If we consider a quantic

$$(a, b, \ldots \between x, y, \ldots)^m$$

and an adjoint linear form, the operative quantic

$$(a, b, \ldots \between \partial_\xi, \partial_\eta, \ldots)^m,$$

or more generally the operative quantic obtained by replacing in any covariant of the given quantic the facients $(x, y, \ldots)$ by the symbols of differentiation $(\partial_\xi. \partial_\eta, \ldots)$ (which operative quantic is, so to speak, a contravariant operator), may be termed the *Provector;* and the Provector operating upon any contravariant gives rise to a contravariant, which may of course be an invariant. Any such contravariant, or rather such contravariant considered as so generated, may be termed a *Provectant;* and in like manner the operative quantic obtained by replacing in any contravariant of the given quantic the facients $(\xi, \eta, \ldots)$ by the symbols of differentiation $(\partial_x, \partial_y, \ldots)$ (which operative quantic is a covariant operator), is termed the *Contraprovector;* and the contraprovector operating upon any covariant gives rise to a covariant, which may of course be an invariant. Any such covariant, or rather such covariant considered as so generated, may be termed a *Contraprovectant.*

In the case of a binary quantic,

$$(a, b, \ldots \between x, y)^m,$$

the two theorems coalesce together, and we may say that the operative quantic

$$(a, b, \ldots \between \partial_y, -\partial_x)^m,$$

or more generally the operative quantic obtained by replacing in any covariant of the given quantic the facients (x, y) by the symbols of differentiation $(\partial_y, -\partial_x)$ (which is in this case a covariant operator), may be termed the Provector. And the Provector operating on any covariant gives a covariant (which as before may be an invariant), and which considered as so generated may be termed the Provectant.

63. But there is another allied theory. If in the quantic itself or in any covariant we replace the facients $(x, y, \ldots)$ by the first derived functions $(\partial_\xi P, \partial_\eta P, \ldots)$ of any contravariant P of the quantic, we have a new function which will be a contravariant of the quantic. In particular, if in the quantic itself we replace the facients $(x, y, \ldots)$ by the first derived functions $(\partial_\xi P, \partial_\eta P, \ldots)$ of the Reciprocant, then the result will contain as a factor the Reciprocant, and the other factor will be also a contravariant. And similarly, if in any contravariant we replace the facients $(\xi, \eta, \ldots)$ by the first derived functions $(\partial_x W, \partial_y W, \ldots)$ of any covariant W (which may be the quantic itself) of the quantic U, we have a new function which will be a covariant of the quantic. And in particular if in the Reciprocant we replace the facients $(\xi, \eta, \ldots)$ by the first derived functions $(\partial_x U, \partial_y U, \ldots)$ of the quantic, the result will contain U as a factor, and the other factor will be also a covariant. In the case of a binary quantic $(a, b, \ldots \between x, y)^m$ the two theorems coalesce and we have the following theorem, viz. if in the quantic U or in any covariant the facients (x, y) are replaced by the first derived functions $(\partial_y W, -\partial_x W)$ of a covariant W, the result will be a covariant; and if in the quantic

U the facients (x, y) are replaced by the first derived functions $(\partial_y U, -\partial_x U)$ of the quantic, the result will contain U as a factor, and the other factor will be also a covariant.

Without defining more precisely, we may say that the function obtained by replacing as above the facients of a covariant or contravariant by the first derived functions of a contravariant or covariant is a *Transmutant* of the first-mentioned covariant or contravariant.

64. Imagine any two quantics of the same order, for instance, the two quantics

$$U = (a, b, \ldots \between x, y \ldots)^m,$$
$$V = (a', b', \ldots \between x, y \ldots)^m,$$

then any quantic such as $\lambda U + \mu V$ may be termed an *Intermediate* of the two quantics; and a covariant of $\lambda U + \mu V$, if in such covariant we treat λ, μ as facients, will be a quantic of the form

$$(A, B, \ldots B`, A` \between \lambda, \mu)^n,$$

where the coefficients $(A, B, \ldots B`, A`)$ will be covariants of the quantics U, V, viz. A will be a covariant of the quantic U alone; $A`$ will be the same covariant of the quantic $V$ alone, and the other coefficients (which in reference to $A$, $A`$ may be termed the *Connectives*) will be covariants of the two quantics; and any coefficient may be obtained from the one which precedes it by operating on such preceding coefficient with the combinantive operator

$$a'\partial_a + b'\partial_b + \ldots,$$

or from the one which succeeds it by operating on such succeeding coefficient with the combinantive operator

$$a\partial_{a'} + b\partial_{b'} + \ldots,$$

the result being divided by a numerical coefficient which is greater by unity than the index of μ or (as the case may be) λ in the term corresponding to the coefficient operated upon. It may be added, that any invariant in regard to the facients (λ, μ) of the quantic

$$(A, B, \ldots B`, A` \between \lambda, \mu)^n$$

is not only a covariant, but it is also a combinant of the two quantics U, V.

As an example, suppose the quantics U, V are the quadrics

$$(a, b, c \between x, y)^2 \text{ and } (a', b', c' \between x, y)^2,$$

then the quadrinvariant of

$$\lambda U + \mu V \text{ is } (\lambda a + \mu a')(\lambda c + \mu c') - (\lambda b + \mu b')^2,$$

which is equal to

$$(ac - b^2,\ ac' - 2bb' + ca',\ a'c' - b'^2 \between \lambda, \mu)^2,$$

and $ac' - 2bb' + ca'$ is the connective of the two discriminants $ac - b^2$ and $a'c' - b'^2$.

65. The law of reciprocity for the number of the invariants of a binary quantic[1], leads at once to the theorems in regard to the number of the quadrinvariants, cubinvariants and quartinvariants of a binary quantic of a given degree, first obtained by the method in the second part of my original memoir[2]. Thus a quadric has only a single invariant, which is of the degree 2; hence, by the law of reciprocity, the number of quadrinvariants of a quantic of the order m is equal to the number of ways in which m can be made up with the part 2, which is of course unity or zero, according as m is even or odd. And we conclude that

The quadrinvariant exists only for quantics of an even order, and for each such quantic there is one, and only one, quadrinvariant.

66. Again, a cubic has only one invariant, which is of the degree 4, and the number of cubinvariants of a quantic of the degree m is equal to the number of ways in which m can be made up with the part 4. Hence

A cubinvariant only exists for quantics of an evenly even order, and for each such quantic there is one, and only one, cubinvariant.

67. But a quartic has two invariants, which are of the degrees 2 and 3 respectively, and the number of quartinvariants of a quantic of the degree m is equal to the number of ways in which m can be made up with the parts 2 and 3. When m is even, there is of course a quartinvariant which is the square of the quadrinvariant, and which, if we attend only to the irreducible quartinvariants, must be excluded from consideration. The preceding number must therefore, when m is even, be diminished by unity. The result is easily found to be

Quartinvariants exist for a quantic of any order, even or odd, whatever, the quadric and the quartic alone excepted; and according as the order of the quantic is

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

the number of quartinvariants is

$$g\,,\quad g\,,\quad g\,,\quad g+1\,,\quad g\,,\quad g+1\,.$$

In particular, for the orders

$$2,\quad 3,\quad 4,\quad 5;\quad 6,\quad 7,\quad 8,\quad 9,\quad 10,\quad 11;\quad 12,\ \&c.,$$

the numbers are

$$0,\quad 1,\quad 0,\quad 1;\quad 1,\quad 1,\quad 1,\quad 2,\quad 1,\quad 2;\quad 2,\ \&c.$$

Thus the ninthic is the lowest quantic which has more than one quartinvariant.

68. But the whole theory of the invariants or covariants of the degrees 2, 3, 4 is most easily treated by the method above alluded to, contained in the second part of my original memoir; and indeed the method appears to be the appropriate one for the

[1] Introductory Memoir, [139], No. 20. [2] Ibid. Nos. 10—17.

treatment of the theory of the invariants or covariants of any given degree whatever, although the application of it becomes difficult when the degree exceeds 4. I remark, in regard to this method, that it leads naturally, and in the first instance, to a special class of the covariants of a system of quantics, viz. these covariants are linear functions of the derived functions of any quantic of the system. (It is hardly necessary to remark that the derived functions referred to are the derived functions of any order of the quantic with regard to the facients.) Such covariants may be termed *tantipartite* covariants; but when there are only two quantics, I use in general the term *lineo-linear*. The tantipartite covariants, while the system remains general, are a special class of covariants, but by particularizing the system we obtain all the covariants of the particularized system. The ordinary case is when all the quantics of the system reduce themselves to one and the same quantic, and the method then gives all the covariants of such single quantic. And while the order of the quantic remains indefinite, the method gives covariants (not invariants); but by particularizing the order of the quantic in such manner that the derived functions become simply the coefficients of the quantic, the covariants become invariants: the like applies of course to a system of two or more quantics.

69. To take the simplest example, in seeking for the covariants of a single quantic U, we in fact have to consider two quantics U, V. An expression such as $\overline{12}\,UV$ is a lineo-linear covariant of the two quantics; its developed expression is

$$\partial_x U \,.\, \partial_y V - \partial_y U \,.\, \partial_x V,$$

which is the Jacobian. In the particular case of two linear functions $(a, b \between x, y)$ and $(a', b' \between x, y)$, the lineo-linear covariant becomes the lineo-linear invariant $ab' - a'b$, which is the Jacobian of the two linear functions.

In the example we cannot descend from the two quantics U, V to the single quantic U (for putting $V = U$ the covariant vanishes); but this is merely accidental, as appears by considering a different lineo-linear covariant $\overline{12}^2 UV$, the developed expression of which is

$$\partial_x^2 U \,.\, \partial_y^2 V - 2\partial_x\partial_y U \,.\, \partial_x\partial_y V + \partial_y^2 U \,.\, \partial_x^2 V.$$

In the particular case of two quadrics $(a, b, c \between x, y)^2$, $(a', b', c' \between x, y)^2$, the lineo-linear covariant becomes the lineo-linear invariant

$$ac' - 2bb' + ca'.$$

If we have $V = U$, then the lineo-linear covariant gives the quadricovariant

$$\partial_x^2 U \,.\, \partial_y^2 U - (\partial_x\partial_y U)^2$$

of the single quantic U (such quadricovariant is in fact the Hessian); and if in the last-mentioned formula we put for U the quadric $(a, b, c \between x, y)^2$, or what is the same thing, if in the expression of the lineo-linear invariant $ac' - 2bb' + ca'$, we put the two quadrics equal to each other, we have the quadrinvariant

$$ac - b^2$$

of the single quadric.

70. The lineo-linear invariant $ab'-a'b$ of two linear functions may be considered as giving the lineo-linear covariant $\partial_x U \,.\, \partial_y V - \partial_y U \,.\, \partial_x V$ of the two quantics U and V, and in like manner the lineo-linear invariant $ac'-2bb'+ca'$ may be considered as giving the lineo-linear covariant $\partial_x^2 U \,.\, \partial_y^2 V - 2\partial_x\partial_y U \,.\, \partial_x\partial_y V + \partial_y^2 U \,.\, \partial_x^2 V$ of the quantics U, V. And generally, any invariant whatever of a quantic or quantics of a given order or orders leads to a covariant of a quantic or quantics of any higher order or orders: viz. the coefficients of the original quantic or quantics are to be replaced by the derived functions of the quantic or quantics of a higher order or orders.

71. The same thing may be seen by means of the theory of Emanants. In fact, consider any emanants whatever of a quantic or quantics; then, attending only to the facients of emanation, the emanants will constitute a system of quantics the coefficients of which are derived functions of the given quantic or quantics; the invariants of the system of emanants will be functions of the derived functions of the given quantic or quantics, and they will be covariants of such quantic or quantics; and we thus pass from the invariants of a quantic or quantics to the covariants of a quantic or quantics of a higher order or orders.

72. It may be observed also, that in the case where a tantipartite invariant, when the several quantics are put equal to each other, does not become equal to zero, we may pass back from the invariant of the single quantic to the tantipartite invariant of the system; thus the lineo-linear invariant $ac'-2bb'+ca'$ of two quadrics leads to the quadrinvariant $ac-b^2$ of a single quantic; and *conversely*, from the quadrinvariant $ac-b^2$ of a single quadric, we obtain by an obvious process of derivation the expression $ac'-2bb'+ca'$ of the lineo-linear invariant of two quadrics This is in fact included in the more general theory explained, No. 64.

73. Reverting now to binary quantics, two quantics of the same order, even or odd, have a lineo-linear invariant. Thus the two quadrics

$$(a,\ b,\ c \between x,\ y)^2,\ (a',\ b',\ c' \between x,\ y)^2$$

have (it has been seen) the lineo-linear invariant

$$ac'-2bb'+ca';$$

and in like manner the two cubics

$$(a,\ b,\ c,\ d \between x,\ y)^3,\ (a',\ b',\ c',\ d' \between x,\ y)^3$$

have the lineo-linear invariant

$$ad'-3bc'+3cb'-da',$$

which examples are sufficient to show the law.

74. The lineo-linear invariant of two quantics of the same odd order is a combinant, but this is not the case with the lineo-linear invariant of two quantics of the same even order. Thus the last-mentioned invariant is reduced to zero by each of the operations

$$a\partial_{a'}+b\partial_{b'}+c\partial_{c'}+d\partial_{d'}$$

and

$$a'\partial_a + b'\partial_b + c'\partial_c + d'\partial_d;$$

but the invariant

$$ac' \quad -2bb' + ca'$$

is by the operations

$$a\partial_{a'} + b\partial_{b'} + c\partial_{c'}$$

and

$$a'\partial_a + b'\partial_b + c'\partial_c$$

reduced respectively to

$$2(ac - b^2)$$

and

$$2(a'c' - b'^2).$$

75. For two quantics of the same odd order, when the quantics are put equal to each other, the lineo-linear invariant vanishes; but for two quantics of the same even order, when these are put equal to each other, we obtain the quadrinvariant of the single quantic. Thus the quadrinvariant of the quadric $(a, b, c \between x, y)^2$ is

$$ac - b^2;$$

and in like manner the quadrinvariant of the quartic $(a, b, c, d, e \between x, y)^4$ is

$$ae - 4bd + 3c^2.$$

76. When the two quantics are the first derived functions of the same quantic of any odd order, the lineo-linear invariant does not vanish, but it is not an invariant of the single quantic. Thus the lineo-linear invariant of

$$(a, b, c \between x, y)^2$$

and

$$(b, c, d \between x, y)^2$$

is

$$(ad - 2bc + cb =)\, ad - bc,$$

which is not an invariant of the cubic

$$(a, b, c, d \between x, y)^3.$$

But for two quantics which are the first derived functions of the same quantic of any even order, the lineo-linear invariant is the quadrinvariant of the single quantic. Thus the lineo-linear invariant of

$$(a, b, c, d \between x, y)^3$$

and

$$(b, c, d, e \between x, y)^3$$

is

$$(ae - 3bd + 3c^2 - db =)\, ae - 4bd + 3c^2,$$

which is the quadrinvariant of the quartic

$$(a, b, c, d, e \between x, y)^4.$$

77. I do not stop to consider the theory of the lineo-linear covariants of two quantics, but I derive the quadricovariants of a single quantic directly from the quadrinvariant. Imagine a quantic of any order even or odd. Its successive even emanants will be in regard to the facients of emanation quantics of an even order, and they will each of them have a quadrinvariant, which will be a quadricovariant of the given quantic. The emanants in question, beginning with the second emanant, are (in regard to the facients of the given quantic assumed to be of the order m) of the orders $m-2$, $m-4, \ldots$ down to 1 or 0, according as m is odd or even, or writing successively $2p+1$ and $2p$ in the place of m, and taking the emanants in a reverse order, the emanants for a quantic of any odd order $2p+1$ are of the orders $1, 3, 5 \ldots 2p-1$, and for a quantic of any even order $2p$, they are of the orders $0, 2, 4 \ldots 2p-2$. The quadricovariants of a quantic of an odd order $2p+1$, are consequently of the orders $2, 6, 10 \ldots 4p-2$, and the quadricovariants of a quantic of an even order $2p$, are of the orders $0, 4, 8 \ldots 4p-4$. We might in each case carry the series one step further, and consider a quadricovariant of the order $4p+2$, or (as the case may be) $4p$, which arises from the 0th emanant of the given quantic; such quadricovariant is, however, only the square of the given quantic.

78. In the case of a quantic of an evenly even order (but in no other case) we have a quadricovariant of the same order with the quantic itself. We may in this case form the lineo-linear invariant of the quantic and the quadricovariant of the same order: such lineo-linear invariant is an invariant of the given quantic, and it is of the degree 3 in the coefficients, that is, it is a cubinvariant. This agrees with the before-mentioned theorem for the number of cubinvariants.

79. In the case of the quartic $(a, b, c, d, e \between x, y)^4$, the cubinvariant is, by the preceding mode of generation, obtained in the form

$$e(ac-b^2) - 4d\tfrac{2}{4}(ad-bc) + 6c\tfrac{1}{6}(ae-4bd+3c^2) - 4b\tfrac{2}{4}(be-cd) + a(ce-d^2),$$

which is in fact equal to

$$3(ace - ad^2 - b^2e + 2bcd - c^3);$$

and omitting the numerical factor 3, we have the cubinvariant of the quartic.

80. In the case of a quantic of any order even or odd, the quadrinvariants of the quadricovariants are quartinvariants of the quantic. But these quartinvariants are not all of them independent, and there is no obvious method grounded on the preceding mode of generation for obtaining the number of the independent (asyzygetic) quartinvariants, and thence the number of the irreducible quartinvariants of a quantic of a given order.

81. I take the opportunity of giving some additional developments in relation to the discriminant of a quantic

$$(a, b, \ldots b', a' \between x, y)^m.$$

To render the signification perfectly definite, it should be remarked that the discriminant contains the term $a^{m-1}a'^{m-1}$, and that the coefficient of this term may be taken to be

$+1$. It was noticed in the Introductory Memoir, that, by Joachimsthal's theorem, the discriminant, on putting $a=0$, becomes divisible by b^2, and that throwing out this factor it is to a numerical factor *près* the discriminant of the quantic of the order $(m-1)$ obtained by putting $a=0$ and throwing out the factor x; and it was also remarked, that this theorem, combined with the general property of invariants, afforded a convenient method for the calculation of the discriminant of a quantic when that of the order immediately preceding is known. Thus let it be proposed to find the discriminant of the cubic

$$(a,\ b,\ c,\ d \between x,\ y)^3.$$

Imagine the discriminant expanded in powers of the leading coefficient a in the form

$$Aa^2+Ba+C,$$

then this function *quà* invariant must be reduced to zero by the operation $3b\partial_a+2c\partial_b+d\partial_c$; or putting for shortness $\nabla=2c\partial_b+d\partial_c$, the operation is $\nabla+3b\partial_a$, and we have

$$\left.\begin{array}{r} a^2\nabla A+a\nabla B+\nabla C \\ +a\,6bA+3bB \end{array}\right\}=0,$$

and consequently

$$B=-\frac{1}{3b}\nabla C,\quad A=-\frac{1}{6b}\nabla B,\quad \nabla A=0.$$

But C is equal to b^2 into the discriminant of $(3b,\ 3c,\ d \between x,\ y)^2$, that is, its value is $b^2(12bd-9c^2)$, or throwing out the factor 3, we may write

$$C=4b^3d-3b^3c^2;$$

this gives

$$B=-\frac{1}{3b}(-6b^2cd+24b^2cd-12bc^3),$$

or reducing

$$B=-6bcd+4c^3;$$

and thence

$$A=-\frac{1}{6b}(-6bd^2+12c^2d-12c^2d),$$

or reducing

$$A=d^3,$$

which verifies the equation $\nabla A=0$, and the discriminant is, as we know,

$$a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2.$$

82. If we consider the quantic $(a,\ b,\ldots a' \between x,\ 1)^m$ as expressed in terms of the roots in the form $a(x-\alpha y)(x-\beta y)\ldots$, then the discriminant $(=a^{m-1}a'^{m-1}+\&c.$ as above) is to a factor *près* equal to the product of the squares of the differences of the roots, and the factor may be determined as follows: viz. denoting by $\zeta(\alpha,\ \beta,\ \ldots)$ the product of the squares of the differences of the roots, we may write

$$a^{2m-2}\zeta(\alpha,\ \beta,\ \ldots)=N(a^{m-1}a'^{m-1}+\&c.),$$

where N is a number; and then considering the equation $x^m - 1 = 0$, we have to determine N the equation

$$\zeta(\alpha, \beta, \ldots) = (-)^{m-1} N.$$

But in general

$$\zeta(\alpha, \beta \ldots) = (-)^{\frac{1}{2}m(m-1)} (\alpha - \beta)(\alpha - \gamma) \ldots (\beta - \alpha)(\beta - \gamma) \ldots$$

and if

$$\phi x = (x - \alpha)(x - \beta) \ldots,$$

then

$$(\alpha - \beta)(\alpha - \gamma) \ldots = \phi' \alpha, \text{ \&c.},$$

or

$$\zeta(\alpha, \beta, \ldots) = (-)^{\frac{1}{2}m(m-1)} \phi' \alpha \phi' \beta \ldots;$$

here

$$\phi x = x^m - 1, \quad \phi' x = m x^{m-1},$$

and therefore

$$\phi' \alpha \phi' \beta \ldots = m^m (\alpha \beta \gamma \ldots)^{m-1},$$

but

$$(-)^m \alpha \beta \gamma \ldots = -1,$$

or

$$\alpha \beta \gamma \ldots = (-)^{m-1} 1,$$

and

$$\phi' \alpha \phi' \beta \ldots = (-)^{(m-1)2} m^m = (-)^{m+1} m^m;$$

whence

$$\zeta(\alpha, \beta \ldots) = (-)^{m-1 + \frac{1}{2}m(m-1)} m^m = (-)^{m-1} N,$$

or

$$N = (-)^{\frac{1}{2}m(m-1)} m^m,$$

and consequently

$$a^{m-2} \zeta(\alpha, \beta, \ldots) = (-)^{\frac{1}{2}m(m-1)} m^m (a^{m-1} a'^{m-1} + \text{\&c.}),$$

or what is the same thing, the value of the discriminant $\square$ $(= a^{m-1} a'^{m-1} + \text{\&c.})$ is

$$(-)^{\frac{1}{2}m(m-1)} m^{-m} a^{m-2} \zeta(\alpha, \beta, \ldots).$$

It would have been allowable to define the discriminant so as that the leading term should be

$$(-)^{\frac{1}{2}m(m-1)} a^{m-1} a'^{m-1},$$

in which case the discriminant would have constantly the same sign as the product of the squared differences; but I have upon the whole thought it better to make the leading term of the discriminant always positive.

83. A quantic of an even order $2p$ has an invariant of peculiar simplicity, viz. the determinant the terms of which are the coefficients of the pth differential coefficients, or derived functions of the quantic with respect to the facients; such invariant may also be considered as a tantipartite invariant of the pth emanants. Thus the sextic

$$(a, b, c, d, e, f, g \between x, y)^6$$

has for one of its invariants, the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

The invariant in question is termed by Professor Sylvester the *Catalecticant.*

84. Professor Sylvester also remarked, that we may from the catalecticant form a function containing an indeterminate quantity λ, such that the coefficients of the different powers of λ are invariants of the quantic; thus for the sextic, the function in question is

$$\begin{vmatrix} a & , & b & , & c & , & d-\lambda \\ b & , & c & , & d+\frac{1}{3}\lambda, & & e \\ c & , & d-\frac{1}{3}\lambda, & & e & , & f \\ d+\lambda, & & e & , & f & , & g \end{vmatrix}$$

where the law of formation is manifest; the terms in the sinister diagonal are modified by annexing to their numerical submultiples of λ with the signs $+$ and $-$ alternately, and in which the multipliers are the reciprocals of the binomial coefficients. The function so obtained is termed the *Lambdaic.*

85. If we consider a quantic of an odd order, and form the catalecticant of the penultimate emanant, we have the covariant termed the Canonisant. Thus in the case of the quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

the canonisant is

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}$$

which is equivalent to

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}$$

and a like transformation exists with respect to the canonisant of a quantic of any odd order whatever. The canonisant and the lambdaic (which includes of course the catalecticant) form the basis of Professor Sylvester's theory of the *Canonical Forms* of quantics of an odd and an even order respectively.

86. There is another family of covariants which remains to be noticed. Consider any two quantics of the same order,

$$(a,\ b,\ldots \between x,\ y)^m,$$
$$(a',\ b',\ldots \between x,\ y)^m,$$

and join to these a quantic of the next inferior order,

$$(u,\ v,\ldots \between y,\ -x)^{m-1},$$

where the coefficients $(u,\ v,\ldots)$ are considered as indeterminate, and which may be spoken of as the adjoint quantic.

Take the odd lineo-linear covariants (viz. those which arise from the odd emanants) of the two quantics; the term arising from the $(2i+1)$th emanants is of the form

$$(A,\ B,\ldots \between x,\ y)^{2(m-1-2i)},$$

where $(A,\ B,\ldots)$ are lineo-linear functions of the coefficients of the two quantics.

Take also the quadricovariants of the adjoint quantic; the term arising from the $(2i-m)$th emanant is of the form

$$(U,\ V,\ldots \between x,\ y)^{2(m-1-2i)},$$

where $(U,\ V,\ldots)$ are quadric functions of the indeterminate coefficients $(u,\ v,\ldots)$. We may then form the quadrinvariant of the two quantics of the order $2(m-1-2i)$: this will be an invariant of the two quantics and the adjoint quantic, lineo-linear in the coefficients of the two quantics and of the degree 2 in regard to the coefficients $(u,\ v,\ldots)$ of the adjoint quantic; or treating the last-mentioned coefficients as facients, the result is a lineo-linear m-ary quadric of the form

$$(\mathfrak{A},\ \mathfrak{B},\ldots \between u,\ v,\ldots)^2,$$

viz. in this expression the coefficients $\mathfrak{A},\ \mathfrak{B},\ldots$ are lineo-linear functions of the coefficients of the two quantics. And giving to i the different admissible values, viz. from $i=0$ to $i=\frac{1}{2}m-1$ or $\frac{1}{2}(m-1)-1$, according as m is even or odd, the number of the functions obtained by the preceding process is $\frac{1}{2}m$ or $\frac{1}{2}(m-1)$, according as m is even or odd. The functions in question, the theory of which is altogether due to Professor Sylvester, are termed by him *Cobezoutiants;* we may therefore say that a cobezoutiant is an invariant of two quantics of the same order m, and of an adjoint quantic of the next preceding order $m-1$, viz. treating the coefficients of the adjoint quantic as the facients of the cobezoutiant, the cobezoutiant is an m-ary quadric, the coefficients of which are lineo-linear functions of the coefficients of the two quantics, and the number of the cobezoutiants is $\frac{1}{2}m$ or $\frac{1}{2}(m-1)$, according as m is even or odd.

87. If the two quantics are the differential coefficients, or first derived functions (with respect to the facients) of a single quantic

$$(a,\ b,\ldots \between x,\ y)^m,$$

then we have what are termed the *Cobezoutoids* of the single quantic, viz. the cobezoutoid is an invariant of the single quantic of the order m, and of an adjoint quantic of the order $(m-2)$; and treating the coefficients of the adjoint quantic as facients, the cobezoutoid is an $(m-1)$ary quadric, the coefficients of which are quadric functions of the coefficients of the given quantic. The number of the cobezoutoids is $\frac{1}{2}(m-1)$ or $\frac{1}{2}(m-2)$, according as m is odd or even.

88. Consider any two quantics of the same order,

$$(a, \ldots \between x, y)^m, \quad (a', \ldots \between x, y)^m,$$

and introducing the new facients (X, Y), form the quotient of determinants,

$$\begin{vmatrix} (a, \ldots \between x, y)^m, & (a, \ldots \between x, y)^m \\ (a, \ldots \between X, Y)^m, & (a', \ldots \between X, Y)^m \end{vmatrix} \div \begin{vmatrix} x, y \\ X, Y \end{vmatrix}$$

which is obviously an integral function of the order $(m-1)$ in each set of facients separately, and lineo-linear in the coefficients of the two quantics; for instance, if the two quantics are

$$(a, b, c, d \between x, y)^3,$$
$$(a', b', c', d' \between x, y)^3,$$

the quotient in question may be written

$$\left(\begin{vmatrix} 3(ab'-a'b), & 3(ac'-a'c) & , & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & & 3(bd'-b'd) \\ ad'-a'd, & 3(bd'-b'd) & , & 3(cd'-c'd) \end{vmatrix} \between x, y)^2 (X, Y)^2.$$

The function so obtained may be termed the *Bezoutic Emanant* of the two quantics.

89. The notion of such function was in fact suggested to me by Bezout's abbreviated process of elimination, viz. the two quantics of the order m being put equal to zero, the process leads to $(m-1)$ equations each of the order $(m-1)$: these equations are nothing else than the equations obtained by equating to zero the coefficients of the different terms of the series $(X, Y)^{m-1}$ in the Bezoutic emanant, and the result of the elimination is consequently obtained by equating to zero the determinant formed with the matrix which enters into the expression of the Bezoutic emanant. In other words, this determinant is the Resultant of the two quantics. Thus the resultant of the last-mentioned two cubics is the determinant

$$\begin{vmatrix} 3(ab'-a'b), & 3(ac'-a'c) & , & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & & 3(bd'-b'd) \\ ad'-a'd, & 3bd'-b'd & , & 3(cd'-c'd) \end{vmatrix}$$

90. If the two quantics are the differential coefficients or first derived functions (with respect to the facients) of a single quantic of the order m, then we have in like manner the *Bezoutoidal Emanant* of the single quantic; this is a function of the order $(m-2)$ in each set of facients, and the coefficients whereof are quadric functions of the coefficients of the single quantic. Thus the Bezoutoidal emanant of the quartic

$$(a, b, c, d, e \between x, y)^4$$

is

$$\left(\begin{array}{ccc} 3(ac-b^2), & 3(ad-bc), & ae-bd \\ 3(ad-bc), & ae+8bd-9c^2, & 3(be-cd) \\ ae-bd, & 3(be-cd), & 3(ce-d^2) \end{array}\right) \between x, y)^2 (X, Y)^2$$

and of course the determinant formed with the matrix which enters into the expression of the Bezoutoidal Emanant, is the discriminant of the single quantic.

91. Professor Sylvester forms with the matrix of the Bezoutic emanant and a set of m facients $(u, v, \ldots)$ an m-ary quadric function, which he terms the *Bezoutiant*. Thus the Bezoutiant of the before-mentioned two cubics is

$$\left(\begin{array}{ccc} 3(ab'-a'b), & 3(ac'-a'c), & ad'-a'd \\ 3(ac'-a'c), & ad'-a'd+9(bc'-b'c), & 3(bd'-b'd) \\ ad'-a'd, & 3bd'-b'd, & 3(cd'-c'd) \end{array}\right) \between u, v, w)^2;$$

and in like manner with the Bezoutoidal emanant of the single quantic of the order m and a set of $(m-1)$ new facients $(u, v, \ldots)$, an $(m-1)$ary quadric function, which he terms the *Bezoutoid*. Thus the Bezoutoid of the before-mentioned quartic is

$$\left(\begin{array}{ccc} 3(ac-b^2), & 3(ad-bc), & ae-bd \\ 3(ad-bc), & ae+8bd-9c^2, & 3(be-cd) \\ ae-bd, & 3(be-cd), & 3(ce-d^2) \end{array}\right) \between u, v, w)^2.$$

To him also is due the important theorem, that the Bezoutiant is an invariant of the two quantics of the order m and of the adjoint quantic $(u, v, \ldots \between y, -x)^{m-1}$, being in fact a linear function with mere numerical coefficients of the invariants called Cobezoutiants, and in like manner that the Bezoutoid is an invariant of the single quantic of the order m and of the adjoint quantic $(u, v, \ldots \between y, -x)^{m-2}$, being a linear function with mere numerical coefficients of the invariants called Cobezoutoids.

The modes of generation of a covariant are infinite in number, and it is to be anticipated that, as new theories arise, there will be frequent occasion to consider new processes of derivation, and to single out and to define and give names to new covariants. But I have now, I think, established the greater part by far of the definitions which are for the present necessary.

156.

A FIFTH MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVIII. *for the year* 1858, pp. 429—460. Received February 11,—Read March 18, 1858.]

THE present memoir was originally intended to contain a development of the theories of the covariants of certain binary quantics, viz. the quadric, the cubic, and the quartic; but as regards the theories of the cubic and the quartic, it was found necessary to consider the case of two or more quadrics, and I have therefore comprised such systems of two or more quadrics, and the resulting theories of the harmonic relation and of involution, in the subject of the memoir; and although the theory of homography or of the anharmonic relation belongs rather to the subject of bipartite binary quadrics, yet from its connexion with the theories just referred to, it is also considered in the memoir. The paragraphs are numbered continuously with those of my former memoirs on the subject: Nos. 92 to 95 relate to a single quadric; Nos. 96 to 114 to two or more quadrics, and the theories above referred to; Nos. 115 to 127 to the cubic, and Nos. 128 to 145 to the quartic. The several quantics are considered as expressed not only in terms of the coefficients, but also in terms of the roots,—and I consider the question of the determination of their linear factors,—a question, in effect, identical with that of the solution of a quadric, cubic, or biquadratic equation. The expression for the linear factor of a quadric is deduced from a well-known formula; those for the linear factors of a cubic and a quartic were first given in my "Note sur les Covariants d'une fonction quadratique, cubique ou biquadratique à deux indéterminées," Crelle, vol. L. (1855), pp. 285—287, [135]. It is remarkable that they are in one point of view more simple than the expression for the linear factor of a quadric.

92. In the case of a quadric the expressions considered are

$$(a, b, c \between x, y)^2, \tag{1}$$

$$ac - b^2 \quad , \tag{2}$$

where (1) is the quadric, and (2) is the discriminant, which is also the quadrinvariant, catalecticant, and Hessian.

And where it is convenient to do so, I write

$$(1) = U,$$
$$(2) = \square.$$

93. We have

$$(\partial_c, -\partial_b, \partial_a \between x, y)^2 \square = U,$$

which expresses that the evectant of the discriminant is equal to the quadric;

$$(a, b, c \between \partial_y, -\partial_x)^2 U = 4\square,$$

which expresses that the provectant of the quadric is equal to the discriminant;

$$(a, b, c \between bx + cy, -ax - by)^2 = \square U,$$

which expresses that a transmutant of the quadric is equal to the product of the quadric and the discriminant.

94. When the quadric is expressed in terms of the roots, we have

$$a^{-1} U = (x - \alpha y)(x - \beta y),$$
$$a^{-2} \square = -\tfrac{1}{4}(\alpha - \beta)^2;$$

and in the case of a pair of equal roots,

$$a^{-1} U = (x - \alpha y)^2,$$
$$\square \quad = 0.$$

95. The problem of the solution of a quadratic equation is that of finding a linear factor of the quadric. To obtain such linear factor in a symmetrical form, it is necessary to introduce arbitrary quantities which do not really enter into the solution, and the form obtained is thus in some sort more complicated than in the like problem for a cubic or a quartic. The solution depends on the linear transformation of the quadric, viz. if we write

$$(a, b, c \between \lambda x + \mu y, \nu x + \rho y)^2 = (a', b', c' \between x, y)^2,$$

so that

$$a' = (a, b, c \between \lambda, \nu)^2,$$
$$b' = (a, b, c \between \lambda, \nu \between \mu, \rho),$$
$$c' = (a, b, c \between \mu, \rho)^2,$$

then

$$a'c' - b'^2 = (ac - b^2)(\lambda\rho - \mu\nu)^2,$$

an equation which in a different notation is

$$(a, b, c \between x, y)^2 . (a, b, c \between X, Y)^2 - \{(a, b, c \between x, y \between X, Y)\}^2 = \square (Yx - Xy)^2,$$

in which form it is a theorem relating to the quadric and its first and second emanants. The equation shows that

$$(a,\ b,\ c \between x,\ y \between X,\ Y) + \sqrt{-\square}\,(Yx - Xy),$$

where $(X,\ Y)$ are treated as supernumerary arbitrary constants, is a linear factor of $(a,\ b,\ c \between x,\ y)^2$, and this is the required solution.

96. In the case of two quadrics, the expressions considered are

$$(a,\ b,\ c \between x,\ y)^2, \tag{1}$$

$$(a',\ b',\ c' \between x,\ y)^2, \tag{2}$$

$$ac - b^2\ , \tag{3}$$

$$ac' - 2bb' + ca', \tag{4}$$

$$a'c' - b'^2\ , \tag{5}$$

$$\left.\begin{array}{ccccccc}
 & & & a^2 & & c^2 & \\
 & & & +1 & & & \\
 & & ab & & b'c' & & \\
 & & & -4 & & & \\
 & ac & & & a'c' & & \\
 & & & -2 & & & \\
b^2 & & & & & & b'^2 \\
 & & & +4 & & +4 & \\
 & & ba & & a'b' & & \\
 & & & -4 & & & \\
 & & a^2 & & a'^2 & & \\
 & & & +1 & & &
\end{array}\right\}, \tag{6}$$

$$(ab' - a'b,\quad ac' - a'c\quad ,\ bc' - b'c \between x,\ y)^2, \tag{7}$$

$$(\lambda a + \mu a',\quad \lambda b + \mu b'\quad ,\ \lambda c + \mu c' \between x,\ y)^2, \tag{8}$$

$$(ac - b^2\ ,\ ac' - 2bb' + ca',\ a'c' - b'^2 \between \lambda,\ \mu)^2, \tag{9}$$

(1) and (2) are the quadrics, (3) and (5) are the discriminants, and (4) is the lineo-linear invariant, or connective of the discriminants; (6) is the resultant of the two quadrics, (7) is the Jacobian, (8) is an intermediate, and (9) is the discriminant of the intermediate. And where it is convenient to do so, I write

$$\begin{aligned}
(1) &= U,\\
(2) &= U',\\
(3) &= \square,\\
(4) &= Q,\\
(5) &= \square',\\
(6) &= R,\\
(7) &= H,\\
(8) &= W,\\
(9) &= \Theta.
\end{aligned}$$

97. The Jacobian (7) may also be written in the form

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}.$$

The Resultant (6) may be written in the form

$$\begin{vmatrix} & a, & 2b, & c \\ a, & 2b, & c, & \\ & a', & 2b', & c' \\ a', & 2b', & c', & \end{vmatrix},$$

and also, taken negatively, in the form

$$4(ab'-a'b)(bc'-b'c)-(ac'-a'c)^2,$$

which is the discriminant of the Jacobian; and in the form

$$4(ac-b^2)(a'c'-b'^2)-(ac'-2bb'+ca')^2,$$

which is the discriminant of the Intermediate.

98. We have the following relations:

$$\begin{aligned}(a,\ b,\ c \between b'x+c'y,\ -a'x-b'y)^2 &= -(a'c'-b'^2)\quad (a,\ b,\ c \between x,\ y)^2 \\ &\quad +(ac'-2bb'+ca')\ (a',\ b',\ c' \between x,\ y)^2, \\ (a',\ b',\ c' \between bx+cy,\ -ax-by)^2 &= +(ac'-2bb'+ca')\ (a,\ b,\ c \between x,\ y)^2 \\ &\quad -(ac-b^2)\quad (a',\ b',\ c' \between x,\ y)^2,\end{aligned}$$

and moreover

$$\begin{aligned}&(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between U',\ -U)^2 \\ &\qquad = -\{(ab'-a'b,\ ac'-a'c,\ bc'-b'c \between x,\ y)^2\}^2,\end{aligned}$$

an equation, the interpretation of which will be considered in the sequel.

99. The most important relations which may exist between the two quadrics are:

First, when the connective vanishes, or

$$ac'-2bb'+ca'=0,$$

in which case the two quadrics are said to be *harmonically* related: the nature of this relation will be further considered.

Secondly, when $R=0$, the two quadrics have in this case a common root, which is given by any of the equations,

$$\begin{aligned} x^2 : 2xy : y^2 &= \partial_a R \quad : \partial_b R \quad : \partial_c R \\ &= \partial_{a'} R \quad : \partial_{b'} R \quad : \partial_{c'} R \\ &= bc' - b'c : ca' - c'a : ab' - a'b. \end{aligned}$$

The last set of values express that the Jacobian is a perfect square, and that the two roots are each equal to the common root of the two quadrics.

The preceding values of the ratios $x^2 : 2xy : y^2$ are consistent with each other in virtue of the assumed relation $R=0$, hence in general the functions

$$4\partial_a R \,.\, \partial_c R - (\partial_b R)^2,\ \partial_a R \,.\, \partial_{b'} R - \partial_b R \,.\, \partial_{a'} R,\ \&c.$$

all of them contain the Resultant R as a factor.

It is easy to see that the Jacobian is harmonically related to each of the quadrics; in fact we have identically

$$\begin{aligned} a\,(bc' - b'c) + b\,(ca' - c'a) + c\,(ab' - a'b) &= 0, \\ a'\,(bc' - b'c) + b'\,(ca' - c'a) + c'\,(ab' - a'b) &= 0, \end{aligned}$$

which contain the theorem in question.

100. When the quadrics are expressed in terms of the roots, we have

$$\begin{aligned} a^{-1}\,U &= (x - \alpha y)(x - \beta y), \\ a'^{-1}\,U' &= (x - \alpha' y)(x - \beta' y), \\ 4\,a^{-2}\,\square &= -(\alpha - \beta)^2, \\ 2\,(aa')^{-1} Q &= 2\alpha\beta + 2\alpha'\beta' - (\alpha + \beta)(\alpha' + \beta'), \\ 4\,a'^{-2}\,\square' &= -(\alpha' - \beta')^2, \\ (aa')^{-2}\,R &= (\alpha - \alpha')(\alpha - \beta')(\beta - \alpha')(\beta - \beta'), \\ (aa')^{-1}\,H &= \begin{vmatrix} y^2, & 2yx, & x^2 \\ 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \end{vmatrix}. \end{aligned}$$

101. The comparison of the last-mentioned value of R with the expression in terms of the roots obtained from the equation

$$-R = 4\square\,\square' - Q^2,$$

gives the identical equation

$$(\alpha - \beta)^2(\alpha' - \beta')^2 - \{2\alpha\beta + 2\alpha'\beta' - (\alpha + \beta)(\alpha' + \beta')\}^2 = -4\,(\alpha - \alpha')(\alpha - \beta')(\beta - \alpha')(\beta - \beta'),$$

which may be easily verified.

102. We have identically

$$\begin{aligned}
&2\alpha\beta + 2\alpha'\beta' - (\alpha+\beta)(\alpha'+\beta') \\
&\quad = 2(\alpha - \alpha')(\alpha - \beta') - (\alpha - \beta)(2\alpha - \alpha' - \beta') \\
&\quad = 2(\beta - \alpha')(\beta - \beta') - (\beta - \alpha)(2\beta - \alpha' - \beta') \\
&\quad = 2(\alpha' - \alpha)(\alpha' - \beta) - (\alpha' - \beta')(2\alpha' - \alpha - \beta) \\
&\quad = 2(\beta' - \alpha)(\beta' - \beta) - (\beta' - \alpha')(2\beta' - \alpha - \beta);
\end{aligned}$$

and the equation $Q = ac' - 2bb' + ca' = 0$ may consequently be written in the several forms

$$\frac{2}{\alpha-\beta} = \frac{1}{\alpha-\alpha'} + \frac{1}{\alpha-\beta'},$$

$$\frac{2}{\beta-\alpha} = \frac{1}{\beta-\alpha'} + \frac{1}{\beta'-\beta},$$

$$\frac{2}{\alpha'-\beta'} = \frac{1}{\alpha'-\alpha} + \frac{1}{\alpha'-\beta},$$

$$\frac{2}{\beta'-\alpha'} = \frac{1}{\beta'-\alpha} + \frac{1}{\beta'-\beta},$$

so that the roots (α, β), (α', β') are harmonically related to each other, and hence the notion of the harmonic relation of the two quadrics.

103. In the case where the two quadrics have a common root $\alpha = \alpha'$,

$$\begin{aligned}
a^{-1}\, U &= (x - \alpha y)(x - \beta y), \\
a'^{-1}\, U' &= (x - \alpha y)(x - \beta' y), \\
4\, a^{-2}\, \square &= -(\alpha - \beta)^2, \\
2\,(aa')^{-1} Q &= (\alpha - \beta)(\alpha - \beta'), \\
4a'^{-2}\, \square' &= -(\alpha - \beta')^2, \\
R &= 0, \\
(aa')^{-1} H &= (\beta' - \beta)(x - \alpha y)^2.
\end{aligned}$$

104. In the case of three quadrics, of the expressions which are or might be considered, it will be sufficient to mention

$$(a\,,\, b\,,\, c \between x,\, y)^2, \tag{1}$$

$$(a',\, b',\, c' \between x,\, y)^2, \tag{2}$$

$$(a'',\, b'',\, c'' \between x,\, y)^2, \tag{3}$$

$$\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}, \tag{4}$$

where (1), (2), (3) are the quadrics themselves, and (4) is an invariant, linear in the coefficients of each quadric. And where it is convenient to do so, I write

$$(1) = U,$$
$$(2) = U',$$
$$(3) = U'',$$
$$(4) = \Omega.$$

105. The equation $\Omega = 0$ is, it is clear, the condition to be satisfied by the coefficients of the three quadrics, in order that there may be a syzygetic relation $\lambda U + \mu U' + \nu U'' = 0$, or what is the same thing, in order that each quadric may be an intermediate of the other two quadrics; or again, in order that the three quadrics may be *in Involution*. Expressed in terms of the roots, the relation is

$$\begin{vmatrix} 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \\ 1, & \alpha'' + \beta'', & \alpha''\beta'' \end{vmatrix} = 0;$$

and when this equation is satisfied, the three pairs, or as it is usually expressed, the six quantities α, β; α', β'; α'', β'', are said to be in involution, or to form an involution. And the two perfectly arbitrary pairs α, β; α', β' considered as belonging to such a system, may be spoken of as an involution. If the two terms of a pair are equal, e.g. if $\alpha'' = \beta'' = \theta$, then the relation is

$$\begin{vmatrix} 1, & 2\theta, & \theta^2 \\ 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \end{vmatrix} = 0;$$

and such a system is sometimes spoken of as an involution of five terms. Considering the pairs (α, β), (α', β') as given, there are of course two values of θ which satisfy the preceding equation; and calling these $\theta_{,}$ and $\theta_{,,}$, then $\theta_{,}$ and $\theta_{,,}$ are said to be the sibiconjugates of the involution α, β; α', β'. It is easy to see that $\theta_{,}, \theta_{,,}$ are the roots of the equation $H = 0$, where H is the Jacobian of the two quadrics U and U' whose roots are (α, β), (α', β'). In fact, the quadric whose roots are $\theta_{,}, \theta_{,,}$ is

$$\begin{vmatrix} y^2, & 2yx, & x^2 \\ 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \end{vmatrix}$$

which has been shown to be the Jacobian in question. But this may be made clearer as follows:—If we imagine that λ, μ are determined in such manner that the intermediate $\lambda U + \mu U'$ may be a perfect square, then we shall have $\lambda U + \mu U' = a''(x - \theta y)^2$, where θ denotes one or other of the sibiconjugates $\theta_{,}, \theta_{,,}$ of the involution. But the condition in order that $\lambda U + \mu U'$ may be a square is

$$(ac - b^2,\ ac' - 2bb' + ca',\ a'c' - b'^2 \between \lambda, \mu)^2;$$

and observing that the equation $\lambda : \mu = U' : -U$ implies $\lambda U + \mu U' = 0 = a''(x-\theta y)^2$, it is obvious that the function

$$(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between U',\ -U)^2$$

must be to a factor *près* equal to $(x-\theta_{,}y)^2(x-\theta_{,,}y)^2$. But we have identically

$$(ac-b^2,\ ac'-2bb'+ca',\ a'c'-b'^2 \between U',\ -U)^2 = -\{(ab'-a'b,\ ac'-a'c,\ bc'-b'c \between x,\ y)^2\}^2,$$

and we thus see that $(x-\theta_{,}y)$, $(x-\theta_{,,}y)$ are the factors of the Jacobian.

106. It has been already remarked that the Jacobian is harmonically related to each of the quadrics U, U'; hence we see that the sibiconjugates $\theta_{,}$, $\theta_{,,}$ of the involution α, β, α', β' are a pair harmonically related to the pair α, β, and also harmonically related to the pair α', β', and this properly might be taken as the definition for the sibiconjugates $\theta_{,}$, $\theta_{,,}$ of an involution of four terms. And moreover, α, β; α', β' being given, and $\theta_{,}$, $\theta_{,,}$ being determined as the sibiconjugates of the involution, if α'', β'' be a pair harmonically related to $\theta_{,}$, $\theta_{,,}$, then the three pairs α, β; α', β'; α'', β'' will form an involution; or what is the same thing, any three pairs α, β; α', β'; α'', β'', each of them harmonically related to a pair $\theta_{,}$, $\theta_{,,}$, will be an involution, and $\theta_{,}$, $\theta_{,,}$ will be the sibiconjugates of the involution.

107. In particular, if α, β be harmonically related to $\theta_{,}$, $\theta_{,,}$, then it is easy to see that $\theta_{,}$, $\theta_{,}$ may be considered as harmonically related to $\theta_{,}$, $\theta_{,,}$, and in like manner $\theta_{,,}$, $\theta_{,,}$ will be harmonically related to $\theta_{,}$, $\theta_{,,}$; that is, the pairs $\theta_{,}$, $\theta_{,}$; $\theta_{,,}$, $\theta_{,,}$ and α, β will form an involution. This comes to saying that the equation

$$\begin{vmatrix} 1, & 2\theta_{,}, & \theta_{,}^2 \\ 1, & 2\theta_{,,}, & \theta_{,,}^2 \\ 1, & \alpha+\beta, & \alpha\beta \end{vmatrix} = 0$$

is equivalent to the harmonic relation of the pairs α, β; $\theta_{,}$, $\theta_{,,}$; and in fact the determinant is

$$(\theta_{,}-\theta_{,,})\left(2\alpha\beta + 2\theta_{,}\theta_{,,} - (\alpha+\beta)(\theta_{,}+\theta_{,,})\right),$$

which proves the theorem in question.

108. Before proceeding further, it is proper to consider the equation

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix} = 0,$$

which expresses that the sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ are homographic; for although the homographic equation may be considered as belonging to the theory of

the bipartite quadric $(x-\alpha y)(x-\alpha' y)$, yet the theory of involution cannot be completely discussed except in connexion with that of homography. If we write

$$A=(\beta-\gamma)(\alpha-\delta),\quad B=(\gamma-\alpha)(\beta-\delta),\quad C=(\alpha-\beta)(\gamma-\delta),$$
$$A'=(\beta'-\gamma')(\alpha'-\delta'),\quad B'=(\gamma'-\alpha')(\beta'-\delta'),\quad C'=(\alpha'-\beta')(\gamma'-\delta'),$$

then we have

$$A+B+C=0,$$
$$A'+B'+C'=0,$$

and thence

$$BC'-B'C=CA'-C'A=AB'-A'B;$$

and either of these expressions is in fact equal to the last-mentioned determinant, as may be easily verified. Hence, when the determinant vanishes, we have

$$A:B:C=A':B':C'.$$

Any one of the three ratios $A:B:C$, for instance the ratio $B:C$, $=$

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)},$$

is said to be the anharmonic ratio of the set $(\alpha, \beta, \gamma, \delta)$, and consequently the two sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ will be homographically related when the anharmonic ratios (that is, the corresponding anharmonic ratios) of the two sets are equal.

If any one of the anharmonic ratios be equal to unity, then the four terms of the set taken in a proper manner in pairs, will be harmonics; thus the equation $\frac{B}{C}=1$ gives

$$\frac{(\gamma-\alpha)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)}=1,$$

which is reducible to

$$2\alpha\delta+2\beta\gamma-(\alpha+\delta)(\beta+\gamma)=0,$$

which expresses that the pairs α, δ and β, γ are harmonics.

109. Now returning to the theory of involution (and for greater convenience taking α, α' &c. instead of α, β &c. to represent the terms of the same pair), the pairs α, α'; β, β'; γ, γ'; δ, δ'; &c. will be in involution if each of the determinants formed with any three lines of the matrix

$$\begin{matrix} 1, & \alpha+\alpha', & \alpha\alpha', \\ 1, & \beta+\beta', & \beta\beta', \\ 1, & \gamma+\gamma', & \gamma\gamma', \\ 1, & \delta+\delta', & \delta\delta', \\ \&c. & & \end{matrix}$$

vanishes: but this being so, the determinant

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}$$

which is equal to

$$\begin{vmatrix} \alpha, & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta, & 1, & \beta+\beta', & \beta\beta' \\ \gamma, & 1, & \gamma+\gamma', & \gamma\gamma' \\ \delta, & 1, & \delta+\delta', & \delta\delta' \end{vmatrix}$$

will vanish, or the two sets $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ will be homographic; that is, if any number of pairs are in involution, then, considering four pairs and selecting in any manner a term out of each pair, these four terms and the other terms of the same four pairs form respectively two sets, and the two sets so obtained will be homographic.

110. In particular, if we have only three pairs α, α'; β, β'; γ, γ', then the sets $\alpha, \beta, \gamma, \alpha'$ and $\alpha', \beta', \gamma', \alpha$ will be homographic; in fact, the condition of homography is

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \alpha', & \alpha, & \alpha\alpha' \end{vmatrix} = 0,$$

which may be written

$$\begin{vmatrix} \alpha, & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta, & 1, & \beta+\beta', & \beta\beta' \\ \gamma, & 1, & \gamma+\gamma', & \gamma\gamma' \\ \alpha', & 1, & \alpha+\alpha', & \alpha\alpha' \end{vmatrix} = 0,$$

or what is the same thing,

$$\begin{vmatrix} \alpha & , & 1, & \alpha+\alpha', & \alpha\alpha' \\ \beta & , & 1, & \beta+\beta', & \beta\beta' \\ \gamma & , & 1, & \gamma+\gamma', & \gamma\gamma' \\ \alpha'-\alpha, & & 0, & 0 & , 0 \end{vmatrix} = 0,$$

so that the first-mentioned relation is equivalent to

$$(\alpha'-\alpha)\begin{vmatrix} 1, & \alpha+\alpha', & \alpha\alpha' \\ 1, & \beta+\beta', & \beta\beta' \\ 1, & \gamma+\gamma', & \gamma\gamma' \end{vmatrix} = 0,$$

and the two sets give rise to an involution. The condition of homography as expressed by the equality of the anharmonic ratios may be written

$$\frac{\alpha-\beta\,.\,\gamma-\alpha'}{\alpha-\gamma\,.\,\alpha'-\beta}=\frac{\alpha'-\beta'\,.\,\gamma'-\alpha}{\alpha'-\gamma'\,.\,\alpha-\beta'};$$

or multiplying out,

$$(\alpha-\beta)(\alpha-\beta')(\alpha'-\gamma)(\alpha'-\gamma')-(\alpha'-\beta)(\alpha'-\beta')(\alpha-\gamma)(\alpha'-\gamma')=0,$$

which is a form for the equation of involution of the three pairs. But this and the other transformations of the equation of involution is best obtained by a different method, as will be presently seen.

111. Imagine now any number of pairs α, α'; β, β'; γ, γ'; δ, δ'; &c. in involution, and let x, y, z, w be the fourth harmonics of the same quantity λ with respect to the pairs α, α'; β, β'; γ, γ' and δ, δ' respectively; then the anharmonic ratios of the set (x, y, z, w) will be independent of λ, or what is the same thing, if x', y', z', w' are the fourth harmonics of any other quantity λ' with respect to the same four pairs, the sets (x, y, z, w) and (x', y', z', w') will be homographic, or we shall have

$$\begin{vmatrix} 1, & x, & x', & xx' \\ 1, & y, & y', & yy' \\ 1, & z, & z', & zz' \\ 1, & w, & w', & ww' \end{vmatrix}=0.$$

It will be sufficient to show this in the case where λ is anything whatever, but λ' has a determinate value, say $\lambda'=\infty$; and since if all the terms α, α', &c. are diminished by the same quantity λ the relations of involution and homography will not be affected, we may without loss of generality assume $\lambda=0$, but in this case

$$x=\frac{2\alpha\alpha'}{\alpha+\alpha'},\quad x'=\tfrac{1}{2}(\alpha+\alpha'),$$

and the equation to be proved is

$$\begin{vmatrix} 1, & \dfrac{\alpha\alpha'}{\alpha+\alpha'}, & \alpha+\alpha', & \alpha\alpha' \\ 1, & \dfrac{\beta\beta'}{\beta+\beta'}, & \beta+\beta', & \beta\beta' \\ 1, & \dfrac{\gamma\gamma'}{\gamma+\gamma'}, & \gamma+\gamma', & \gamma\gamma' \\ 1, & \dfrac{\delta\delta'}{\delta+\delta'}, & \delta+\delta', & \delta\delta' \end{vmatrix}=0,$$

which is obviously a consequence of the equations which express the involution of the four pairs.

A set homographic with $x, y, z\ w$, which are the fourth harmonics of any quantity whatever λ with respect to the pairs in involution, α, α'; β, β'; γ, γ'; δ, δ', is said to be homographic with the four pairs, and we have thus the notion of a set of single quantities homographic with a set of pairs in involution. This very important theory is due to M. Chasles.

112. Let r; s; t be the anharmonic ratios of a set $\alpha, \beta, \gamma, \delta$, and let $r_{,}$; $s_{,}$; $t_{,}$ be the anharmonic ratios (corresponding or not corresponding) of a set $\alpha_{,}, \beta_{,}, \gamma_{,}, \delta_{,}$. And suppose that r'; s'; t'; $r_{,}'$; $s_{,}'$; $t_{,}'$; r''; s''; t''; $r_{,}''$; $s_{,}''$; $t_{,}''$; r'''; s'''; t'''; $r_{,}'''$; $s_{,}'''$; $t_{,}'''$, are the analogous quantities for three other pairs of sets; then an equation such as

$$\begin{vmatrix} 1, & \frac{r}{s}, & \frac{r_{,}}{s_{,}}, & \frac{rr_{,}}{ss_{,}} \\ \vdots & & & \end{vmatrix} = 0,$$

or as it is more conveniently written,

$$\begin{vmatrix} ss_{,} & , & rs_{,} & , & r_{,}s & , & rr_{,} \\ s's_{,}' & , & r's_{,}' & , & r_{,}'s' & , & r'r_{,}' \\ s''s_{,}'' & , & r''s_{,}'' & , & r_{,}''s'' & , & r''r_{,}'' \\ s'''s_{,}''' & , & r'''s_{,}''' & , & r_{,}'''s''' & , & r'''r_{,}''' \end{vmatrix} = 0$$

is a relation independent of the particular ratios $r : s$ which have been chosen for the anharmonic ratios of the sets; this is easily shown by means of the equations

$$r + s + t = 0, \quad r_{,} + s_{,} + t_{,} = 0,$$

which connect the anharmonic ratios. The equation in fact expresses a certain relation between four sets $(\alpha, \beta, \gamma, \delta)$ and four other sets $(\alpha_{,}, \beta_{,}, \gamma_{,}, \delta_{,})$; a relation which may be termed the relation of the homography of the anharmonic ratios of four and four sets: the notion of this relation is also due to M. Chasles.

113. The general relation

$$\begin{vmatrix} 1, & \alpha + \beta, & \alpha\beta \\ 1, & \alpha' + \beta', & \alpha'\beta' \\ 1, & \alpha'' + \beta'', & \alpha''\beta'' \end{vmatrix} = 0$$

may be exhibited in a great variety of forms. In fact, if the determinant is denoted by Υ, then multiplying by this determinant the two sides of the identical equation

$$\begin{vmatrix} u^2, & -u, & 1 \\ v^2, & -v, & 1 \\ w^2, & -w, & 1 \end{vmatrix} = (u-v)(v-w)(w-u),$$

we obtain

$$\Upsilon(u-v)(v-w)(w-u) = \begin{vmatrix} (u-\alpha)(u-\beta), & (v-\alpha)(v-\beta), & (w-\alpha)(w-\beta) \\ (u-\alpha')(u-\beta'), & (v-\alpha')(v-\beta'), & (w-\alpha')(w-\beta') \\ (u-\alpha'')(u-\beta''), & (v-\alpha'')(v-\beta''), & (w-\alpha'')(w-\beta'') \end{vmatrix}.$$

If, for example, $u=\alpha$, $v=\beta$, then we have

$$\Upsilon(\alpha-\beta)=-(\alpha-\alpha')(\alpha-\beta')(\beta-\alpha'')(\beta-\beta'')+(\beta-\alpha')(\beta-\beta')(\alpha-\alpha'')(\alpha-\beta'');$$

and again, if $u=\alpha$, $v=\alpha'$, $w=\alpha''$, then we have

$$\Upsilon=-(\alpha-\beta'')(\alpha'-\beta)(\alpha''-\beta')+(\alpha-\beta')(\alpha'-\beta'')(\alpha''-\beta).$$

Putting $\Upsilon=0$, the two equations give respectively

$$\frac{(\alpha-\alpha')(\beta-\alpha'')}{(\alpha-\alpha'')(\alpha'-\beta)}=\frac{(\alpha-\beta'')(\beta-\beta')}{(\alpha-\beta')(\beta''-\beta)};$$

and

$$(\alpha-\beta'')(\alpha'-\beta)(\alpha''-\beta')=(\alpha-\beta')(\alpha'-\beta'')(\alpha''-\beta),$$

which are both of them well-known forms.

114. A corresponding transformation applies to the equation

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix}=0,$$

which expresses the homography of two pairs. In fact, calling the determinant Ψ and representing by V the similar determinant

$$\begin{vmatrix} ss', & -s', & -s, & 1 \\ tt', & -t', & -t, & 1 \\ uu', & -u', & -u, & 1 \\ vv', & -v', & -v, & 1 \end{vmatrix},$$

which, equated to zero, would express the homography of the sets (s, t, u, v) and (s', t', u', v'), we have

$$V\Psi=\begin{vmatrix} (s-\alpha)(s'-\alpha'), & (s-\beta)(s'-\beta'), & (s-\gamma)(s'-\gamma'), & (s-\delta)(s'-\delta') \\ (t-\alpha)(t'-\alpha'), & (t-\beta)(t'-\beta'), & (t-\gamma)(t'-\gamma'), & (t-\delta)(t'-\delta') \\ (u-\alpha)(u'-\alpha'), & (u-\beta)(u'-\beta'), & (u-\gamma)(u'-\gamma'), & (u-\delta)(u'-\delta') \\ (v-\alpha)(v'-\alpha'), & (v-\beta)(v'-\beta'), & (v-\gamma)(v'-\gamma'), & (v-\delta)(v'-\delta') \end{vmatrix},$$

which gives various forms of the equation of homography. In particular, if $s=\alpha$, $s'=\beta'$, $t=\beta$, $t'=\alpha'$, $u=\gamma$, $u'=\delta'$, $v=\delta$, $v'=\gamma$, then

$$V\Psi=\begin{vmatrix} . & . & (\alpha-\gamma)(\beta'-\gamma'), & (\alpha-\delta)(\beta'-\delta') \\ . & . & (\beta-\gamma)(\alpha'-\gamma'), & (\beta-\delta)(\alpha'-\delta') \\ (\gamma-\alpha)(\delta'-\alpha'), & (\gamma-\beta)(\delta'-\beta') & . & . \\ (\delta-\alpha)(\gamma'-\alpha'), & (\delta-\beta)(\gamma'-\beta') & . & . \end{vmatrix},$$

and the right-hand side breaks up into factors, which are equal to each other (whence also $V = \Psi$), and the equation $\Psi = 0$ takes the form

$$(\alpha-\gamma)(\beta-\delta)(\alpha'-\delta')(\beta'-\gamma')-(\alpha-\delta)(\beta-\gamma)(\alpha'-\gamma')(\beta'-\delta')=0,$$

which is, in fact, one of the equations which express the equality of the anharmonic ratios of $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$.

115. In the case of a cubic, the expressions considered are

$$(a, b, c, d \between x, y)^3, \tag{1}$$

$$(ac-b^2,\ ad-bc,\ bd-c^2 \between x, y)^2, \tag{2}$$

$$\left\{\begin{array}{l} -a^3d+3abc-2b^3 \\ -abd+2ac^2-b^2c \\ +acd-2b^2d+bc^2 \\ +ad^2-3bcd+2c^3 \end{array}\right\} \between x, y)^3, \tag{3}$$

$$a^3d^2-6abcd+4ac^3+4b^3d-3b^2c^2, \tag{4}$$

where (1) is the cubic, (2) is the quadricovariant or Hessian, (3) is the cubicovariant, and (4) is the quartinvariant or discriminant.

And where it is convenient to do so, I write

$$\begin{aligned}(1) &= U,\\ (2) &= H,\\ (3) &= \Phi,\\ (4) &= \square,\end{aligned}$$

so that we have

$$\Phi^2-\square U^2+4H^3=0.$$

116. The Hessian may be written under the form

$$(ax+by)(cx+dy)-(bx+cy)^2,$$

(which, indeed, is the form under which *quâ* Hessian it is originally given), and under the form

$$\begin{vmatrix} y^2, & -yx, & x^2 \\ a\,, & b\,, & c \\ b\,, & c\,, & d \end{vmatrix}.$$

The cubicovariant may be written under the form

$$\begin{aligned}&\{2(ac-b^2)x+\ \ (ad-bc)y\}(bx^2+2cxy+dy^2)\\ -&\{\ (ad-bc)x+2(bd-c^2)y\}(ax^2+2bxy+cy^2),\end{aligned}$$

that is, as the Jacobian of the cubic and Hessian; and under the form

$$\tfrac{1}{2}(\partial_a,\ \partial_b,\ \partial_c,\ \partial_d \between y,\ -x)^3\,\square,$$

that is, as the evectant of the discriminant.

The discriminant, taken negatively, may be written under the form

$$+4(ac-b^2)(bd-c^2)-(ad-bc)^2,$$

that is, as the discriminant of the Hessian.

117. We have

$$(a,\ b,\ c,\ d \between bx^2+2cxy+dy^2,\ -ax^2-2bxy-cy^2)^3 = U\Phi,$$

which expresses that a transmutant of the cubic is the product of the cubic and the cubicovariant. The equation

$$\{(\partial_a,\ \partial_b,\ \partial_c,\ \partial_d \between y,\ -x)^3\}^2\,\square = 2U^2$$

expresses that the second evectant of the discriminant is the square of the cubic.

The equation

$$\begin{vmatrix} d^2 & , & -3cd & , & -3bd+6c^2 & , & -3bc+2ad \\ -3cd & , & -3c^2+12bd, & & -3ad-6bc & , & -3ac+6b^2 \\ -3bd+6c^2 & , & -3ad-6bc & , & -3b^2+12ac, & & -3ab \\ -3bc-12ad, & & -3ac+6b^2 & , & 3ab & , & a^2 \end{vmatrix} = 27\,\square^2$$

expresses that the determinant formed with the second differential coefficients of the discriminant gives the square of the discriminant.

The covariants of the intermediate $\alpha U+\beta\Phi$ are as follows, viz.

118. For the Hessian, we have

$$\begin{aligned} \tilde{H}(\alpha U+\beta\Phi) &= (1,\ 0,\ -\square \between \alpha,\ \beta)^2 H \\ &= (\alpha^2-\beta^2\square)\,H; \end{aligned}$$

for the cubicovariant,

$$\begin{aligned} \tilde{\Phi}(\alpha U+\beta\Phi) &= (0,\ \square,\ 0,\ -\square^2 \between \alpha,\ \beta)^3 U \\ &\quad +(1,\ 0,\ -\square,\ 0 \between \alpha,\ \beta)^3\Phi \\ &= (\alpha^2-\beta^2\square)(\alpha\Phi+\beta\square U); \end{aligned}$$

and for the discriminant,

$$\begin{aligned} \tilde{\square}(\alpha U+\beta\Phi) &= (1,\ 0,\ -2\square,\ 0,\ \square^2 \between \alpha,\ \beta)^4\Phi \\ &= (\alpha^2-\beta^2\square)^2\,\square, \end{aligned}$$

where on the left-hand sides I have, for greater distinctness, written $\tilde{H}$, &c. to denote the functional operation of taking the Hessian, &c. of the operand $\alpha U+\beta\Phi$.

In particular, if $\alpha = 0$, $\beta = 1$,

$$\tilde{H}\Phi = -\square \,.\, H,$$
$$\tilde{\Phi}\Phi = -\square^2 \,.\, U,$$
$$\tilde{\square}\Phi = \quad \square^3 \,.$$

119. Solution of a cubic equation.

The question is to find a linear factor of the cubic

$$(a, b, c, d \between x, y)^3,$$

and this can be at once effected by means of the relation

$$\Phi^2 - \square U^2 = -4H^3$$

between the covariants. The equation in fact shows that each of the expressions

$$\tfrac{1}{2}(\Phi + U\sqrt{\square}), \quad \tfrac{1}{2}(\Phi - U\sqrt{\square})$$

is a perfect cube, and consequently that the cube root of each of these expressions is a linear function of (x, y). The expression

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\square})} + \sqrt[3]{\tfrac{1}{2}(\Phi - U\sqrt{\square})}$$

is consequently a linear function of x, y, and it vanishes when $U = 0$, that is, the expression is a linear factor of the cubic.

It may be noticed here that the cubic being $a(x - \alpha y)(x - \beta y)(x - \gamma y)$, then we may write

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\square})} - \sqrt[3]{\tfrac{1}{2}(\Phi - U\sqrt{\square})} = \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y),$$

where ω is an imaginary cube root of unity: this will appear from the expressions which will be presently given for the covariants in terms of the roots.

120. Canonical form of the cubic.

The expressions $\frac{1}{2}(\Phi + U\sqrt{\square})$, $\frac{1}{2}(\Phi - U\sqrt{\square})$ are perfect cubes; and if we write

$$\tfrac{1}{2}(\Phi + U\sqrt{\square}) = \quad \sqrt{\square}\,\mathrm{x}^3,$$
$$\tfrac{1}{2}(\Phi - U\sqrt{\square}) = -\sqrt{\square}\,\mathrm{y}^3,$$

then we have

$$U = \quad \mathrm{x}^3 + \mathrm{y}^3,$$
$$\Phi = \sqrt{\square}\,(\mathrm{x}^3 - \mathrm{y}^3),$$

and thence also

$$H = -\sqrt[3]{\square}\,\mathrm{xy}.$$

121. When the cubic is expressed in terms of the roots, we have

$$a^{-1}U = (x - \alpha y)(x - \beta y)(x - \gamma y);$$

and then putting for shortness

$$A = (\beta - \gamma)(x - \alpha y), \quad B = (\gamma - \alpha)(x - \beta y), \quad C = (\alpha - \beta)(x - \gamma y),$$

so that

$$A + B + C = 0,$$

we have

$$\begin{aligned} a^{-2}H &= -\tfrac{1}{18}(A^2 + B^2 + C^2) = \tfrac{1}{9}(BC + CA + AB), \\ a^{-3}\Phi &= -\tfrac{1}{27}(B - C)(C - A)(A - B), \\ a^{-4}\square &= -\tfrac{1}{27}(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2. \end{aligned}$$

122. The covariants H, Φ are most simply expressed as above, but it may be proper to add the equations

$$\begin{aligned} a^{-2}H &= -\tfrac{1}{18}\Sigma(\beta - \gamma)^2(x - \alpha y)^2 \\ &= -\tfrac{1}{9}\left\{\begin{matrix} \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta, \\ 6\alpha\beta\gamma - \beta\gamma^2 - \gamma\alpha^2 - \alpha\beta^2 - \beta^2\gamma - \gamma^2\alpha - \alpha^2\beta, \\ \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 - \alpha^2\beta\gamma - \beta^2\gamma\alpha - \gamma^2\alpha\beta \end{matrix}\right\}(x, y)^2 \\ &= -\tfrac{1}{9}\{(\alpha + \omega\beta + \omega^2\gamma)x + (\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta)y\}\{(\alpha + \omega^2\beta + \omega\gamma)x + (\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta)y\} \end{aligned}$$

(where ω is an imaginary cube root of unity),

$$\begin{aligned} a^{-3}\Phi &= \tfrac{1}{27}\Sigma(\alpha - \beta)(\alpha - \gamma)^2(x - \beta y)^2(x - \gamma y) \\ &= \left\{\begin{matrix} 2(\alpha^3 + \beta^3 + \gamma^3) - 3(\beta\gamma^2 + \gamma\alpha^2 + \alpha\beta^2 + \beta^2\gamma + \gamma^2\alpha + \alpha^2\beta) + 12\alpha\beta\gamma, \\ -2(\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta) + 4(\beta^2\gamma^3 + \gamma^2\alpha^2 + \alpha^2\beta^2) - (\beta\gamma^3 + \gamma\alpha^3 + \alpha\beta^3 + \beta^3\gamma + \gamma^3\alpha + \alpha^3\beta), \\ -2(\alpha\beta^2\gamma^2 + \beta\gamma^2\alpha^2 + \gamma\alpha^2\beta^2) + 4(\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta) - (\beta^2\gamma^3 + \gamma^2\alpha^3 + \alpha^2\beta^3 + \beta^3\gamma^2 + \gamma^3\alpha^2 + \alpha^3\beta^2), \\ +2(\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3) - 3(\alpha\beta^2\gamma^3 + \beta\gamma^2\alpha^3 + \gamma\alpha^2\beta^3 + \alpha\beta^3\gamma^2 + \beta\gamma^3\alpha^2 + \gamma\alpha^3\beta^2) + 12\alpha^2\beta^2\gamma^2 \end{matrix}\right\}(x, y)^3 \\ &= \{(2\alpha - \beta - \gamma)x + (2\beta\gamma - \gamma\alpha - \alpha\beta)y\}\{(2\beta - \gamma - \alpha)x + (2\gamma\alpha - \alpha\beta - \beta\gamma)y\}\{(2\gamma - \alpha - \beta)x + (2\alpha\beta - \beta\gamma - \gamma\alpha)y\}. \end{aligned}$$

123. It may be observed that we have $a^{-6}\square U^2 = -\frac{1}{27}A^2B^2C^2$, which, with the above values of H, Φ in terms of A, B, C and the equation $A + B + C = 0$, verifies the equation $\Phi^2 - \square U^2 + 4H^3 = 0$, which connects the covariants. In fact, we have identically,

$$\begin{aligned} &(B - C)^2(C - A)^2(A - B)^2 = \\ &-4(A + B + C)^3ABC + (A + B + C)^2(BC + CA + AB)^2 + 18(A + B + C)(BC + CA + AB)ABC \\ &\qquad - 4(BC + CA + AB)^3 - 27A^2B^2C^2, \end{aligned}$$

by means of which the verification can be at once effected.

124. If, as before, ω is an imaginary cube root of unity, then we may write

$$27a^{-3}\Phi \quad = -(B-C)(C-A)(A-B),$$

$$27a^{-3}U\sqrt{\square} = \quad 3(\omega-\omega^2)ABC,$$

and these values give

$$27a^{-3}\tfrac{1}{2}(\Phi+U\sqrt{\square} = \{(\alpha+\omega^2\beta+\omega\gamma)x+(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)y\}^3,$$

$$27a^{-3}\tfrac{1}{2}(\Phi-U\sqrt{\square} = \{(\alpha+\omega\beta+\omega^2\gamma)x+(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)y\}^3,$$

and we thence obtain

$$\sqrt[3]{\tfrac{1}{2}(\Phi+U\sqrt{\square})}-\sqrt[3]{\tfrac{1}{2}(\Phi-U\sqrt{\square})} = -\tfrac{1}{3}a(\omega-\omega^2)(\beta-\gamma)(x-\alpha y),$$

which agrees with a former result.

125. The preceding formulæ show without difficulty, that each factor of the cubicovariant is the harmonic of a factor of the cubic with respect to the other two factors of the cubic; and moreover, that the factors of the cubic and the cubicovariant form together an involution having for sibiconjugates the factors of the Hessian. In fact, the harmonic of $x-\alpha y$ with respect to $(x-\beta y)(x-\gamma y)$ is $(2\alpha-\beta-\gamma)x+(2\beta\gamma-\gamma\alpha-\alpha\beta)y$, which is a factor of the cubicovariant; the product of the pair of harmonic factors is

$$(2\alpha-\beta-\gamma)x^2+2(\beta\gamma-\alpha^2)xy+(-2\alpha\beta\gamma+\alpha^2\beta+\alpha^2\gamma)y^2;$$

and multiplying this by $\beta-\gamma$, and taking the sum of the analogous expressions, this sum vanishes, or the three pairs form an involution. That the Hessian gives the sibiconjugates of the involution is most readily shown as follows:—the last-mentioned quadric may be written

$$\big(-(\alpha+\beta+\gamma)+3\alpha\big)x^2+2\big(\alpha\beta+\alpha\gamma+\beta\gamma-\alpha(\alpha+\beta+\gamma)\big)xy+\big(-3\alpha\beta\gamma+\alpha(\alpha\beta+\alpha\gamma+\beta\gamma)\big)y^2,$$

which is equal to

$$\left(3\frac{b}{a}+3\alpha\right)x^2+2\left(3\frac{c}{a}-3\frac{b}{a}\alpha\right)xy+\left(3\frac{d}{a}+3\frac{c}{a}\alpha\right)y^2,$$

or, throwing out the factor $3a^{-1}$, to

$$(b+a\alpha,\quad 2c-2b\alpha,\quad d+c\alpha\between x,\ y)^2,$$

which is harmonically related to the Hessian

$$(ac-b^2,\quad ad-bc,\quad bd-c^2\between x,\ y)^2;$$

and in like manner the other two pairs of factors will be also harmonically related to the Hessian.

126. In the case of a pair of equal roots, we have

$$\begin{aligned} a^{-1}U &= \quad (x-\alpha y)^2(x-\gamma y), \\ a^{-2}H &= -\tfrac{1}{9}(\alpha-\gamma)^2(x-\alpha y)^2, \\ a^{-3}\Phi &= -\tfrac{2}{27}(\alpha-\gamma)^3(x-\alpha y)^3, \\ \square &= \quad 0. \end{aligned}$$

And in the case of all the roots equal, we have

$$a^{-1}U = (x-\alpha y)^3,$$

$$H=0, \quad \Phi=0, \quad \square=0.$$

127. In the solution of a biquadratic equation we have to consider the cubic equation $\varpi^3 - M(\varpi - 1) = 0$. The cubic here is $(1,\ 0,\ -M,\ M \between \varpi,\ 1)^3$, or what is the same thing,

$$(1,\ 0,\ -\tfrac{1}{3}M,\ M \between \varpi,\ 1)^3;$$

the Hessian is

$$M(-\tfrac{1}{3},\ 1,\ -\tfrac{1}{9}M \between \varpi,\ 1)^2;$$

the cubicovariant is

$$M(-1,\ \tfrac{2}{9}M,\ -\tfrac{1}{3}M,\ M+\tfrac{2}{27}M^2 \between \varpi,\ 1)^3;$$

and the discriminant is

$$M^2(1-\tfrac{4}{27}M).$$

128. In the case of a quartic, the expressions considered are

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4, \tag{1}$$

$$ae - 4bd + 3c^2, \tag{2}$$

$$(ac-b^2,\ 2(ad-bc),\ ae+2bd-3c^2,\ 2(be-cd),\ ce-d^2 \between x,\ y)^4, \tag{3}$$

$$ace + 2bcd - ad^2 - b^2e - c^3, \tag{4}$$

$$\left(\begin{array}{l} -\ a^2d + 3\,abc - 2\,b^3, \\ -\ a^2e - 2\,abd + 9\,ac^2 - 6\,b^2c, \\ -\ 5\,abe + 15\,acd - 10\,b^2d, \\ +10\,ad^2 - 10\,b^2e, \\ +\ 5\,ade + 10\,bd^2 - 15\,bee, \\ +\ ae^2 + 2\,bde - 9\,c^2e + 6\,cd^2, \\ +\ be^2 - 3\,cde + 2\,d^3 \end{array}\right) \between x,\ y)^6, \tag{5}$$

where (1) is the quartic, (2) is the quadrinvariant, (3) is the quadricovariant or Hessian, (4) is the cubinvariant, and (5) is the cubicovariant.

And where it is convenient to do so, I write

$$\begin{aligned}(1) &= U,\\ (2) &= I,\\ (3) &= H,\\ (4) &= J,\\ (5) &= \Phi.\end{aligned}$$

The preceding covariants are connected by the equation

$$JU^3 - IU^2H + 4H^3 = -\Phi^2.$$

The discriminant is not an irreducible invariant, its value is

$$\square = I^3 - 27J^2 = a^3e^3 + \&c.,$$

for which see Table No. 12, [p. 272].

129. It is for some purposes convenient to arrange the expanded expression of the discriminant in powers of the middle coefficient c. We thus have

$$\begin{aligned}\square = \; & a^3e^3 - 12\,a^2bde^2 - 27\,a^2d^4 - 6\,ab^2d^2e - 27\,b^4e^2 - 64\,b^3d^3\\ & + c\,(54\,a^2d^2e + 54\,ab^2e^2 + 108\,abd^3 + 108\,b^3de)\\ & + c^2\,(-18\,a^2e^2 - 180\,abde + 36\,b^2d^2)\\ & + c^3\,(-54\,ad^2 - 54\,b^2e)\\ & + c^4\,(81\,ae).\end{aligned}$$

130. Solution of a biquadratic equation.

We have to find a linear factor of the quartic

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4.$$

The equation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, putting for shortness

$$M = \frac{I^3}{4J^2},$$

may be written

$$(1,\ 0,\ -M,\ M \between IH,\ JU)^3 = -\tfrac{1}{4}I^3\Phi^2.$$

Hence, if ϖ_1, ϖ_2, ϖ_3 are the roots of

$$(1,\ 0,\ -M,\ M \between \varpi,\ 1)^3 = 0,$$

the expressions $IH - \varpi_1 JU$, $IH - \varpi_2 JU$, $IH - \varpi_3 JU$ are each of them squares; write

$$\begin{aligned}(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) &= X^2,\\ (\varpi_3 - \varpi_1)(IH - \varpi_2 JU) &= Y^2,\\ (\varpi_1 - \varpi_2)(IH - \varpi_3 JU) &= Z^2,\end{aligned}$$

so that, identically,

$$X^2 + Y^2 + Z^2 = 0;$$

and consequently $X + \iota Y$, $X - \iota Y$ are each of them squares. The expression

$$\alpha X + \beta Y + \gamma Z$$

will be a square if only

$$\alpha^2 + \beta^2 + \gamma^2 = 0,$$

as may be seen by writing it under the form

$$\tfrac{1}{2}(\alpha + \iota\beta)(X - \iota Y) + \tfrac{1}{2}(\alpha - \iota\beta)(X + \iota Y) - \gamma\iota\sqrt{X^2 + Y^2};$$

and in particular, writing $\sqrt{\varpi_2 - \varpi_3}$, $\sqrt{\varpi_3 - \varpi_1}$, $\sqrt{\varpi_1 - \varpi_2}$ for α, β, γ, the expression

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}$$

is a square; and since the product of the different values is a multiple of U^2 (this is most readily perceived by observing that the expression vanishes for $U = 0$), the expression is the square of a linear factor of the quartic.

131. To complete the solution: ϖ_1, ϖ_2, ϖ_3 are the roots of the cubic equation

$$(1,\ 0,\ -\tfrac{1}{3}M,\ M \between \varpi,\ 1)^3 = 0;$$

and hence, putting for shortness,

$$P^3 = \tfrac{1}{2}M\,\{(-1,\ \tfrac{2}{9}M,\ -\tfrac{1}{3}M,\ M + \tfrac{2}{27}M^2 \between IH,\ JU)^3 + \sqrt{1 - \tfrac{4}{27}M}\,(1,\ 0,\ -\tfrac{1}{3}M,\ M \between IH,\ JU)^3,$$

$$Q^3 = \tfrac{1}{2}M\,\{(-1,\ \tfrac{2}{9}M,\ -\tfrac{1}{3}M,\ M + \tfrac{2}{27}M^2 \between IH,\ JU)^3 - \sqrt{1 - \tfrac{4}{27}M}\,(1,\ 0,\ -\tfrac{1}{3}M,\ M \between IH,\ JU)^3,$$

we have (ω being an imaginary cube root of unity)

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) = P - Q;$$

and if

$$P_0^3 = \tfrac{1}{2}M\,\{-1 + \sqrt{1 - \tfrac{4}{27}M}\},$$

$$Q_0^3 = \tfrac{1}{2}M\,\{-1 - \sqrt{1 - \tfrac{4}{27}M}\},$$

then

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3) = P_0 - Q_0.$$

Hence, multiplying and observing that $(\omega - \omega^2)^2 = -3$, we find

$$-\frac{1}{(\omega - \omega^2)^2}(\varpi_2 - \varpi_3)^2(IH - \varpi_1 JU) = (P - Q)(P_0 - Q_0),$$

and consequently

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} = (\omega - \omega^2)\sqrt{-(P - Q)(P_0 - Q_0)}.$$

We have, in like manner,

$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_2 - \varpi_3)(IH - \varpi_1 JU) = P - Q,$$
$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_3 - \varpi_1)(IH - \varpi_2 JU) = \omega P - \omega^2 Q,$$
$$\tfrac{1}{3}(\omega - \omega^2)(\varpi_1 - \varpi_2)(IH - \varpi_3 JU) = \omega^2 P - \omega Q,$$

and
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_2-\varpi_3)= \quad P_0- \quad Q_0,$$
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_3-\varpi_1)=\omega\, P_0-\omega^2 Q_0,$$
$$\tfrac{1}{3}(\omega-\omega^2)(\varpi_1-\varpi_2)=\omega^2 P_0-\omega\, Q_0,$$

and therefore
$$(\varpi_2-\varpi_3)\sqrt{IH-\varpi_1 JU}=(\omega-\omega^2)\sqrt{-\quad(P-Q)\qquad(P_0-Q_0)},$$
$$(\varpi_3-\varpi_1)\sqrt{IH-\varpi_2 JU}=(\omega-\omega^2)\sqrt{-(\omega P-\omega^2 Q)(\omega P_0-\omega^2 Q)},$$
$$(\varpi_3-\varpi_1)\sqrt{IH-\varpi_3 JU}=(\omega-\omega^2)\sqrt{-(\omega^2 P-\omega Q)(\omega^2 P_0-\omega Q_0)}\,;$$

and hence disregarding the common factor $\omega-\omega^2$, the square of the linear factor of the quartic is
$$\sqrt{-(P-Q)(P_0-Q_0)}+\sqrt{-(\omega P-\omega^2 Q)(\omega P_0-\omega^2 Q_0)}+\sqrt{-(\omega^2 P-\omega Q)(\omega^2 P_0-\omega Q_0)},$$
which is the required solution.

It may be proper to add that
$$-\varpi_1= \quad P_0+ \quad Q_0,$$
$$-\varpi_2=\omega\, P_0+\omega^2 Q_0,$$
$$-\varpi_3=\omega^2 P_0+\omega\, Q_0.$$

132. The solution gives at once the canonical form of the quartic; in fact, writing
$$X+\iota Y=2\sqrt{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}\,\sqrt{J}\,\mathrm{x}^2,$$
$$X-\iota Y=2\sqrt{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}\,\sqrt{J}\,\mathrm{y}^2,$$
where X, Y have their former significations, we find, by a simple reduction,
$$IH-\varpi_1 JU= \quad(\varpi_3-\varpi_1)\,J\,(\mathrm{x}^2+\mathrm{y}^2)^2,$$
$$IH-\varpi_2 JU=-(\varpi_2-\varpi_3)\,J\,(\mathrm{x}^2-\mathrm{y}^2)^2,$$
$$IH-\varpi_3 JU=-\frac{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}{\varpi_1-\varpi_2}\,J\,.\,4\mathrm{x}^2\mathrm{y}^2,$$
and thence putting
$$\theta=-\frac{\varpi_2}{\varpi_1-\varpi_2}=\frac{\tfrac{1}{3}(\omega-\omega^2)(\omega^2 P_0+\omega Q_0)}{(\omega^2 P_0-\omega Q_0)},$$
we have
$$U=\mathrm{x}^4+\mathrm{y}^4+6\theta\mathrm{x}^2\mathrm{y}^2,$$
which is the form required.

133. The Hessian may be written under the form
$$(\partial_e,\ \ -\partial_d,\ \ \partial_c,\ \ -\partial_b,\ \ \partial_a\between x,\ y)^4 J,$$
that is, as the evectant of the cubinvariant.

The cubicovariant may be obtained by writing the quartic under the form

$$(ax+by,\ bx+cy,\ cx+dy,\ dx+ey \between x,\ y)^3,$$

and then, treating the linear functions as coefficients, or considering this as a cubic, the cubicovariant of the cubic gives the cubicovariant of the quartic.

If we represent the cubicovariant by

$$\Phi = (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f},\ \mathrm{g} \between x,\ y)^6,$$

then we have identically,

$$\mathrm{ag} - 9\mathrm{ce} + 8\mathrm{d}^2 = 0\,;$$

and moreover forming the quadrinavariant of the sextic, we find

$$\mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2 = \tfrac{1}{6}\square,$$

where $\square$ is the discriminant of the quartic. From these two equations we find

$$\mathrm{bf} - 4\mathrm{ce} + 3\mathrm{d}^2 = -\tfrac{1}{36}\square,$$

which is an expression given by Mr Salmon: it is the more remarkable as the left-hand side is the quadrinvariant of $(\mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f} \between x,\ y)^4$, which is *not* a covariant of the quartic. It may be noticed also that we have

$$\mathrm{af} - 3\mathrm{be} + 2\mathrm{cd} = 0,$$

$$\mathrm{bg} - 3\mathrm{cf} + 2\mathrm{de} = 0.$$

134. The covariants of the intermediate

$$\alpha U + 6\beta H$$

of the quartic and Hessian are as follows, viz.

The quadrinvariant is

$$\tilde{I}\,(\alpha U + 6\beta H) = \quad (I,\ 18J,\ 3I^2 \between \alpha,\ \beta)^2\,;$$

the cubinvariant is

$$\tilde{J}\,(\alpha U + 6\beta H) = \quad (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2 \between \alpha,\ \beta)^3;$$

the Hessian is

$$\begin{aligned}\tilde{H}\,(\alpha U + 6\beta H) = \quad & (1,\ 0,\ -3I \between \alpha,\ \beta)^3\, H \\ & + (0,\ I,\ \ 9J \between \alpha,\ \beta)^2\, U\,;\end{aligned}$$

and the cubicovariant is

$$\tilde{\Phi}\,(\alpha U + 6\beta H) = \quad (1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3\, \Phi\,;$$

to which may be added the discriminant, which is

$$\tilde{\square}\,(\alpha U + 6\beta H) = (1,\ 0,\ -18\,I,\ 108\,J,\ 81\,I^2,\ 972\,IJ,\ -2916\,J^2 \between \alpha,\ \beta)^6\,\square.$$

135. The expression for the lambdaic is

$$\begin{vmatrix} a & , & b & , & c-2\lambda \\ b & , & c+\lambda, & & d \\ c-2\lambda, & & d & , & e \end{vmatrix} = J+\lambda I-4\lambda^3.$$

If the determinant is represented by Λ, that is if

$$\Lambda = -4\lambda^3+\lambda I+J,$$

then if λ_1, λ_2, λ_3 are the roots of the equation $\Lambda=0$, and if the values of $\partial_a\Lambda$, &c. obtained by writing λ_1 in the place of λ are represented by $\partial_a\Lambda_1$, &c., then if x, y satisfy the equation

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4 = 0,$$

we have identically (X, Y being arbitrary),

$$\frac{(a,\ b,\ c,\ d,\ e \between X,\ Y \overset{3}{\between} x,\ y)}{Xy-Yx}$$

$$= \sqrt{-(\partial_e,\ -\partial_d,\ \partial_c,\ -\partial_b,\ \partial_a \between X,\ Y)^4\Lambda_1}$$

$$+ \sqrt{-(\partial_e,\ -\partial_d,\ \partial_c,\ -\partial_b,\ \partial_a \between X,\ Y)^4\Lambda_2}$$

$$+ \sqrt{-(\partial_e,\ -\partial_d,\ \partial_c,\ -\partial_b,\ \partial_a \between X,\ Y)^4\Lambda_3}$$

a theorem due to Aronhold. I have quoted this theorem in its original form as an application of the lambdaic, but I remark that

$$-(\partial_e,\ -\partial_d,\ \partial_c,\ -\partial_b,\ \partial_a \between X,\ Y)^4\Lambda = -\lambda(a, \ldots \between X,\ Y)^4 - (ac-b^2, \ldots \between X,\ Y)^4 = -\lambda U' - H'$$

if U', H' are what U, H become, substituting for $(x,\ y)$ the new facients $(X,\ Y)$. Moreover, we have

$$\lambda = -\frac{J\varpi}{I};$$

for substituting this value in the equation $\Lambda=0$, we obtain the before-mentioned equation $\varpi^3 - M(\varpi-1)=0$. We have, therefore,

$$-(\partial_e,\ -\partial_d,\ \partial_c,\ -\partial_b,\ \partial_a \between X,\ Y)^4\Lambda = \frac{J\varpi}{I}U' - H' = -\frac{1}{I}(IH' - J\varpi U'),$$

and the equation becomes

$$\frac{(a,\ b,\ c,\ d,\ e \between X,\ Y \overset{3}{\between} x,\ y)}{Xy-Yx}\sqrt{-I} = \sqrt{IH'-J\varpi_1 U'} + \sqrt{IH'-J\varpi_2 U'} + \sqrt{IH'-J\varpi_3 U'}.$$

Moreover, if $(x-\alpha y)$ be a factor of the quartic, then replacing in the formula y by the value αx, (x, y) will disappear altogether; and then changing (X, Y) into (x, y) where x, y are now arbitrary, we have

$$\frac{(a, b, c, d, e \between x, y)^2 (\alpha, 1)}{x-\alpha y}\sqrt{-I} = \sqrt{IH-\varpi_1 JU}+\sqrt{IH-\varpi_2 JU}+\sqrt{IH-\varpi_3 JU},$$

which is a form connected with the results in Nos. 130 and 131.

136. We have

$$\begin{vmatrix} y^4, & -4xy^3, & 6x^2y^2, & -4x^3y, & x^4 \\ & a\ , & 3b\ , & 3c\ , & d \\ a, & 3b\ , & 3c\ , & d\ , & \\ & b\ , & 3c\ , & 3d\ , & e \\ b, & 3c\ , & 3d\ , & e\ , & \end{vmatrix} = 6IH-9JU;$$

it will appear from the formulæ relating to the roots of the quartic, that the expression $6IH-9JU$ vanishes identically when there are two pairs of equal roots, or what is the same thing, when the quartic is a perfect square. The conditions in order that the expression may vanish are obviously

$$\begin{aligned} &6(ac-b^2) : 3(ad-bc) : ae+2bd-3c^2 : 3(be-cd) : 6(ce-d^2) : 9J \\ = \quad & a \quad : \quad b \quad : \quad c \quad : \quad d \quad : \quad e \quad : \quad I, \end{aligned}$$

conditions which imply that the several determinants

$$\left\| \begin{matrix} 6(ac-b^2), & 3(ad-bc), & ae+2bd-3c^2, & 3(be-cd), & 6(ce-d^2) \\ a\ , & b\ , & c\ , & d\ , & e \end{matrix} \right\|,$$

all of them vanish. If for a moment we write $6H=(a', b', c', d', e' \between x, y)^4$, then the determinants are

$$\left\| \begin{matrix} a', & b', & c', & d', & e' \\ a, & b, & c, & d, & e \end{matrix} \right\|;$$

we have identically

$$\begin{aligned} ad'-a'd &= 3(bc'-b'c), \\ eb'-e'b &= 3(dc'-d'c), \\ ae'-a'e &= 3(bd'-b'd), \end{aligned}$$

and the ten determinants thus reduce themselves to seven determinants only, these in fact being, to mere numerical factors *près*, the coefficients of the cubicovariant; this perfectly agrees with a subsequent result, viz. that the cubicovariant vanishes identically when the quartic is a perfect square.

137. It may be remarked that the equation $6IH-9JU=0$ will be satisfied identically if

$$a=\frac{b^2}{c-\phi},\quad e=\frac{d^2}{c-\phi},\quad bd=(c-\phi)(c+2\phi),$$

where ϕ is arbitrary; the quartic is in this case the square of

$$\left(\frac{b}{\sqrt{c-\phi}},\ \sqrt{c-\phi},\ \frac{d}{\sqrt{c-\phi}}\between x,\ y\right)^2.$$

If with the conditions in question we combine the equation $I=0$ (which in this case implies also $J=0$), we obtain $\phi=0$, and consequently

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e},$$

or the quartic will be a complete fourth power.

It is easy to express in terms of the coefficients a', b', c', d', e' of $6H$ the different determinants

$$\left\|\begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \end{matrix}\right\|,$$

we have in fact

$$\left\{\begin{aligned} ae-bd &= \tfrac{1}{2}\left(c'+\frac{1}{\sqrt{3}}\sqrt{a'e'+4b'd'-3c'^2}\right),\\ 3(bd-c^2) &= \tfrac{1}{2}\left(c'-\frac{1}{\sqrt{3}}\sqrt{a'e'+4b'd'-3c'^2}\right),\\ ac-b^2 &= \tfrac{1}{6}a',\\ ad-bc &= \tfrac{1}{3}b',\\ be-cd &= \tfrac{1}{3}c',\\ ce-d^2 &= \tfrac{1}{6}e', \end{aligned}\right.$$

whence all the above-mentioned determinants will vanish, or the quartic will be a perfect fourth power if only the Hessian vanishes identically.

138. Considering the quartic as expressed in terms of the roots, we have

$$a^{-1}U=(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y);$$

and if we write for shortness

$$A=(\beta-\gamma)(\alpha-\delta),$$
$$B=(\gamma-\alpha)(\beta-\delta),$$
$$C=(\alpha-\beta)(\gamma-\delta),$$

which are connected by

$$A+B+C=0,$$

then we have

$$a^{-2}I = \tfrac{1}{24}(A^2+B^2+C^2) = -\tfrac{1}{12}(BC+CA+AB),$$
$$a^{-3}J = \tfrac{1}{432}(B-C)(C-A)(A-B);$$

and for the discriminant we have

$$a^{-6}\square = \tfrac{1}{256}(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2,$$
$$= \tfrac{1}{256}A^2B^2C^2,$$

and it is easy by means of a preceding formula to verify the equation $\square = I^3 - 27J^2$.

139. The formulæ show a very remarkable analogy between the covariants of a cubic and the invariants of a quartic. In fact

For the cubic.	For the quartic.
$\begin{cases} A=(\beta-\gamma)(x-\alpha y), \\ B=(\gamma-\alpha)(x-\beta y), \\ C=(\alpha-\beta)(x-\gamma y), \end{cases}$	$\begin{cases} A=(\beta-\gamma)(\alpha-\delta), \\ B=(\gamma-\alpha)(\beta-\delta), \\ C=(\alpha-\beta)(\gamma-\delta); \end{cases}$

and then we have corresponding to each other:

For the cubic.	For the quartic.
The Hessian,	The quadrinvariant,
The cubicovariant,	The cubinvariant,
The cubic into the square root of the discriminant.	The discriminant.

140. For the two covariants, we have

$$a^{-2}H = -\tfrac{1}{48}\Sigma(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2,$$

and

$$a^{-3}\Phi = -\tfrac{1}{32}\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

if for shortness,

$$\mathfrak{A} = (\delta+\alpha-\beta-\gamma,\quad -\delta\alpha+\beta\gamma,\quad \delta\alpha(\beta+\gamma)-\beta\gamma(\delta+\alpha)\between x, y)^2,$$
$$\mathfrak{B} = (\delta+\beta-\gamma-\alpha,\quad -\delta\beta+\gamma\alpha,\quad \delta\beta(\gamma+\alpha)-\gamma\alpha(\delta+\beta)\between x, y)^2,$$
$$\mathfrak{C} = (\delta+\gamma-\alpha-\beta,\quad -\delta\gamma+\alpha\beta,\quad \delta\gamma(\alpha+\beta)-\alpha\beta(\delta+\gamma)\between x, y)^2.$$

141. We have

$$M = \tfrac{27}{8}\frac{(A^2+B^2+C^2)^3}{(B-C)^2(C-A)^2(A-B)^2};$$

or putting for shortness

$$\Lambda = \tfrac{3}{2}\frac{A^2+B^2+C^2}{(B-C)(C-A)(A-B)},$$

we have

$$M = \tfrac{3}{2}(A^2+B^2+C^2)\Lambda^2;$$

and it is then easy to deduce

$$\varpi_1 = \Lambda(B - C),$$
$$\varpi_2 = \Lambda(C - A),$$
$$\varpi_3 = \Lambda(A - B);$$

in fact, these values give

$$\begin{aligned} \varpi_1 + \varpi_2 + \varpi_3 &= 0, \\ \varpi_1\varpi_2 + \varpi_1\varpi_3 + \varpi_2\varpi_3 &= -M, \\ \varpi_1\varpi_2\varpi_3 &= M, \end{aligned}$$

and they are consequently the roots of the equation $\varpi^3 - M(\varpi - 1) = 0$.

142. The leading coefficient of $IH - \varpi_1 JU$ is then equal to a^4 into the following expression, viz.

$$\tfrac{1}{24}(A^2 + B^2 + C^2)\, a^{-2}(ac - b^2) - \tfrac{1}{288}(A^2 + B^2 + C^2)(B - C),$$

which is equal to

$$\tfrac{1}{1152}(A^2 + B^2 + C^2)\{48a^{-2}(ac - b^2) - 4(B - C)\},$$

and the term in { } is

$$8(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - 3(\alpha + \beta + \gamma + \delta)^2 - 4(\gamma - \alpha)(\beta - \delta) + 4(\alpha - \beta)(\gamma - \delta),$$

which is equal to

$$-3(\delta + \alpha - \beta - \gamma)^2.$$

But $IH - \varpi_1 JU$ is a square, and it is easy to complete the expression, and we have

$$a^{-4}(IH - \varpi_1 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \alpha - \beta - \gamma,\ -\delta\alpha + \beta\gamma,\ \delta\alpha(\beta + \gamma) - \beta\gamma(\delta + \alpha)\between x, y)^2\}^2$$
$$a^{-4}(IH - \varpi_2 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \beta - \gamma - \alpha,\ -\delta\beta + \gamma\alpha,\ \delta\beta(\gamma + \alpha) - \gamma\alpha(\delta + \beta)\between x, y)^2\}^2,$$
$$a^{-4}(IH - \varpi_3 JU) = -\tfrac{1}{384}(A^2 + B^2 + C^2)\{(\delta + \gamma - \alpha - \beta,\ -\delta\gamma + \alpha\beta,\ \delta\gamma(\alpha + \beta) - \alpha\beta(\gamma + \delta)\between x, y)^2\}^2.$$

We have, moveover,

$$\varpi_2 - \varpi_3 = -3\Lambda A,$$
$$\varpi_3 - \varpi_1 = -3\Lambda B,$$
$$\varpi_1 - \varpi_2 = -3\Lambda C,$$

and thence

$$a^{-2}(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} = \tfrac{1}{8}(\omega - \omega^2)\frac{A^2 + B^2 + C^2}{(B - C)(C - A)(A - B)}(\beta - \gamma)(\alpha - \delta)$$
$$\times(\delta + \alpha - \beta - \gamma,\quad -\delta\alpha + \beta\gamma,\quad \delta\alpha(\beta + \gamma) - \beta\gamma(\delta + \alpha)\between x, y)^2;$$

and taking the sum of the analogous expressions, we find

$$a^{-2}\{(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}\}$$
$$= -\tfrac{1}{4}(\omega - \omega^2)\frac{A^2 + B^2 + C^2}{(B - C)(C - A)(A - B)}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(x - \delta y)^2,$$

which agrees with a former result.

143. The equation $I=0$ gives

$$A : B : C = 1 : \omega : \omega^2,$$

where ω is an imaginary cube root of unity; the factors of the quartic may be said in this case to be *Symmetric Harmonics*.

The equation $J=0$ gives one of the three equations,

$$A=B, \quad B=C, \quad C=A;$$

in this case a pair of factors of the quartic are harmonics with respect to the other pair of factors. If we have simultaneously $I=0$, $J=0$, then

$$A=B=C=0,$$

and in this case three of the factors of the quartic are equal.

144. If any two of the linear factors of the quartic are considered as forming, with the other two linear factors, an involution, the sibiconjugates of the involution make up a quadratic factor of the cubicovariant; and considering the three pairs of sibiconjugates, or what is the same thing, the six linear factors of the cubicovariant, the factors of a pair are the sibiconjugates of the involution formed by the other two pairs of factors.

In fact, the sibiconjugates of the involution formed by the equations

$$(x-\alpha y)(x-\delta y)=0, \quad (x-\beta y)(x-\gamma y)=0$$

are found by means of the Jacobian of these two functions, viz. of the quadrics

$$(2, \; -\delta-\alpha, \; 2\delta\alpha \between x, y)^2,$$
$$(2, \; -\beta-\gamma, \; 2\beta\gamma \between x, y)^2,$$

which is

$$(\delta+\alpha-\beta-\gamma, \; -\delta\alpha+\beta\gamma, \; \delta\alpha(\beta+\gamma)-\beta\gamma(\delta+\alpha) \between x, y)^2,$$

viz. a quadratic factor of the cubicovariant; and forming the other two factors, there is no difficulty in seeing that any one of these is the Jacobian of the other two.

145. In the case of a pair of equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^2(x-\gamma y)(x-\delta y),\\
a^{-2}I &= \tfrac{1}{12}(\alpha-\gamma)^2(\alpha-\delta)^2,\\
a^{-3}J &= -\tfrac{1}{216}(\alpha-\gamma)^3(\alpha-\delta)^3,\\
\square &= 0,\\
a^{-3}H &= -\tfrac{1}{48}\{2(\alpha-\gamma)^2(x-\delta y)^2+2(\alpha-\delta)^2(x-\gamma y)^2+(\gamma-\delta)^2(x-\alpha y)^2\}(x-\alpha y)^2,\\
a^{-3}\Phi &= \tfrac{1}{32}(\gamma-\delta)^2(2\alpha-\gamma-\delta, \; \gamma\delta-\alpha^2, \; \gamma\alpha^2+\delta\alpha^2-2\gamma\alpha\delta \between x, y)^2(x-\alpha y)^4.
\end{aligned}$$

In the case of two pairs of equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^2(x-\gamma y)^2,\\
a^{-2}I &= \tfrac{1}{12}(\alpha-\gamma)^4,\\
a^{-3}J &= -\tfrac{1}{216}(\alpha-\gamma)^6,\\
\square &= 0,\\
a^{-1}H &= -\tfrac{1}{12}(\alpha-\gamma)^2(x-\alpha y)^2(x-\gamma y)^2,\\
\Phi &= 0;
\end{aligned}$$

these values give also

$$6IH-9JU=0.$$

146. In the case of three equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^3(x-\delta y),\\
I &= 0,\quad J=0,\quad \square=0,\\
a^{-2}H &= -\tfrac{1}{48}(\alpha-\delta)^2\{2(x-\delta y)^2+(x-\alpha y)^2\}(x-\alpha y)^2,\\
a^{-3}\Phi &= \tfrac{1}{32}(\alpha-\delta)^2(x-\alpha y)^6;
\end{aligned}$$

and in the case of four equal roots, we have

$$\begin{aligned}
a^{-1}U &= (x-\alpha y)^4,\\
I &= 0,\quad J=0,\quad \square=0,\\
H &= 0,\quad \Phi=0.
\end{aligned}$$

The preceding formulæ, for the case of equal roots, agree with the results obtained in my memoir on the conditions for the existence of given systems of equalities between the roots of an equation.

Addition, 7th October, 1858.

Covariant and other Tables (binary quadrics Nos. 25 bis, 29 A, 49 A, and 50 bis).

Mr Salmon has pointed out to me, that in the Table No. 25 of the simplest octinvariant of a binary quintic[1], the coefficients -210, -17, $+18$ and $+38$ are erroneous, and has communicated to me the corrected values, which I have since verified: the terms, with the corrected values of the coefficients, are [shown in the Table]

No. 25 bis.

[The terms with the erroneous coefficients were abc^3d^2ef, ac^5f^2, $b^4d^2f^2$, bc^3d^3e; the correct values -220, -27, $+22$, and $+74$ of the coefficients are given in the Table Q, No. 25, p. 288.]

[1] Second Memoir, *Philosophical Transactions*, t. CXLVI. (1856) p. 125.

Mr Salmon has also performed the laborious calculation of Hermites' 18-thic invariant of a binary quintic, and has kindly permitted me to publish the result, which is given in the following Table:

No. 29 A.

[This is the Table W No. 29 A given pp. 299—303, the form being slightly altered as appears p. 282.]

Mr Salmon has also remarked to me, that in the Table No. 50 of the cubinvariant of a binary dodecadic[1], the coefficients are altogether erroneous. There was, in fact, a fundamental error in the original calculation; instead of repeating it, I have, with a view to the deduction therefrom of the cubinvariant (see Fourth Memoir, No. 78), first calculated the dodecadic quadricovariant, the value of which is given in the following Table:

No. 49 A.

[For this Table see p. 319.]

It is now very easy to obtain the cubinvariant, which is

No. 50 bis.

[This is the Table No. 50, p. 319, the original No. 50 with coefficients which were altogether erroneous having been omitted.]

[1] Third Memoir, *Philosophical Transactions*, t. CXLVI. (1856) p. 635.

157.

ON THE TANGENTIAL OF A CUBIC.

[From the *Philosophical Transactions of the Royal Society of London*, vol. XLVIII. *for the year* 1858, pp. 461—463. Received February 11,—Read March 18, 1858.]

In my "Memoir on Curves of the Third Order" (1), I had occasion to consider a derivative which may be termed the "tangential" of a cubic, viz. the tangent at the point (x, y, z) of the cubic curve $(*\between x, y, z)^3 = 0$ meets the curve in a point (ξ, η, ζ), which is the tangential of the first-mentioned point; and I showed that when the cubic is represented in the canonical form $x^3 + y^3 + z^3 + 6lxyz = 0$, the coordinates of the tangential may be taken to be $x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3)$. The method given for obtaining the tangential may be applied to the general form $(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3$: it seems desirable, in reference to the theory of cubic forms, to give the expression of the tangential for the general form (2); and this is what I propose to do, merely indicating the steps of the calculation, which was performed for me by Mr Creedy.

The cubic form is

$$(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

which means

$$ax^3 + by^3 + cz^3 + 3fy^2z + 3gz^2x + 3hx^2y + 3iyz^2 + 3jzx^2 + 3hxy^2 + 6lxyz;$$

and the expression for ξ is obtained from the equation

$$\begin{aligned} x^2\xi = &\big(b, f, i, c\between(j, f, c, i, g, l\between x, y, z)^2, \quad -(h, b, i, f, l, k\between x, y, z)^2\big)^3 \\ &-(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3(\mathfrak{C}x + \mathfrak{D}), \end{aligned}$$

1 *Philosophical Transactions*, vol. CXLVII. (1857), [146].

2 At the time when the present paper was written, I was not aware of Mr Salmon's theorem (Higher Plane Curves, p. 156), that the tangential of a point of the cubic is the intersection of the tangent of the cubic with the first or line polar of the point with respect to the Hessian; a theorem, which at the same time that it affords the easiest mode of calculation, renders the actual calculation of the coordinates of the tangential less important. Added 7th October, 1858.—A. C.

where the second line is in fact equal to zero, on account of the first factor, which vanishes. And $\mathfrak{C}$, $\mathfrak{D}$ denote respectively quadric and cubic functions of (y, z), which are to be determined so as to make the right-hand side divisible by x^2; the resulting value of ξ may be modified by the adjunction of the evanescent term

$$(\mathrm{a}x+\mathrm{h}y+\mathrm{j}z)\,(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3,$$

where a, h, j are arbitrary coefficients; but as it is not obvious how these coefficients should be determined in order to present the result in the most simple form, I have given the result in the form in which it was obtained without the adjunction of any such term.

Write for shortness,

$$\begin{aligned}
P &= (k, l \quad \between y, z)\,,\\
Q &= (b, f, i \quad \between y, z)^2,\\
R &= (l, g, \quad \between y, z)\,,\\
S &= (f, i, c \quad \between y, z)^2,\\
B &= (h, j \quad \between y, z)\,,\\
C &= (k, l, g \quad \between y, z)^2,\\
D &= (b, f, i, c \between y, z)^3,
\end{aligned}$$

so that

$$\begin{aligned}
(h, b, i, f, l, k \quad \between x, y, z)^3 &= (h, P, Q \quad \between x, 1)^3,\\
(j, f, c, i, g, l \quad \between x, y, z)^3 &= (j, R, S \quad \between x, 1)^3,\\
(a, b, c, f, g, h, i, j, k, l \between x, y, z)^3 &= (a, B, C, D \between x, 1)^3.\\
\mathfrak{C}x+\mathfrak{D} \quad &= (\mathfrak{C}, \mathfrak{D} \quad \between x, 1),
\end{aligned}$$

and then for greater convenience writing $(h, 2P, Q \between x, 1)^3$, &c. for $(h, P, Q \between x, 1)^3$, &c., and omitting the $(x, 1)^2$, &c. and the arrow-heads, or representing the functions simply by $(h, 2P, Q)$, &c., we have

$$\begin{aligned}
x^2\xi = \;& b\,(j, 2R, S \quad)^3\\
& -3f\,(j, 2R, S \quad)^2 . (h\,, 2P, Q)\\
& +3i\,(j, 2R, S \quad) . (h\,, 2P, Q)^2\\
& -\;c \qquad\qquad\qquad . (h\,, 2P, Q)^3\\
& -\;(a, 3B, 3C, D) . (\mathfrak{C}, \mathfrak{D} \quad),
\end{aligned}$$

which can be developed in terms of the quantities which enter into it. The conditions, in order that the coefficients of x, x^0 may vanish, are thus seen to be

$$D\mathfrak{D} = bS^3 - 3fS^2Q + 3iSQ^2 - cQ^3,$$

$$D\mathfrak{C} - 3C\mathfrak{D} = b\,(6RS^2) - 3f\,(2S^2P + 4RSQ) + 3i\,(2RQ^2 + 4SPQ) - c\,(6PQ^2),$$

and from these we obtain

$$\mathfrak{C} = \left(\begin{array}{c|c|c}
bck - 3 & big + 6 & bcg + 3\\
bil + 6 & cfk - 6 & cfl - 6\\
fik + 3 & f^2g - 6 & fgi - 3\\
f^2l - 6 & i^2k + 6 & i^2l + 6
\end{array}\right) \between y, z)^2$$

$$\mathfrak{D} = \left(\begin{array}{|c|c|c|c|} b^2c - 1 & bcf - 3 & bci + 3 & bc^2 + 1 \\ bfi + 3 & bi^2 + 6 & cf^2 - 6 & cfi - 3 \\ f^3 - 2 & f^2i - 3 & fi^2 + 3 & i^3 + 2 \end{array}\right) \mathfrak{Y}(y, z)^3$$

and substituting these values, the right-hand side of the equation divides by x^3, and throwing out this factor we have the value of ξ; and the values of η, ζ may be thence deduced by a mere interchange of letters. The value for ξ is

x^4	x^3y	x^3z	x^2y^2	x^2yz	x^2z^2	xy^3	xy^2z
$bj^2 + 1$	$bj^2l + 6$	$bgj^2 + 6$	$abck + 3$	$abgi - 6$	$abcg - 3$	$ab^2c + 1$	$abcf + 3$
$ch^3 - 1$	$ch^2k - 6$	$ch^2l - 6$	$abil - 6$	$acfk + 6$	$acfl + 6$	$abfi - 3$	$abi^2 - 6$
$fhj^2 - 3$	$fhjl - 12$	$fghj - 12$	$af^2l + 6$	$af^2g + 6$	$afgi + 3$	$af^3 + 2$	$af^2i + 3$
$h^2ij + 3$	$fj^2k - 6$	$fj^2l - 6$	$afik - 3$	$ai^2k - 6$	$ai^2l - 6$	$bchk - 3$	$bchl - 12$
	$h^2il + 6$	$gh^2i + 6$	$bch^2 - 3$	$bgjl + 24$	$bgj^2 + 3$	$bhil - 6$	$bcjk + 9$
	$hijk + 12$	$hijl + 12$	$bhij + 6$	$bij^2 + 6$	$bgj^2j + 12$	$bijk + 12$	$bghi - 6$
			$bjl^2 + 12$	$cfh^2 - 6$	$cfhj - 6$	$bl^3 + 8$	$bgl^2 + 24$
			$chk^2 - 12$	$chkl - 24$	$fjl^2 - 24$	$ck^3 - 8$	$bijl + 18$
			$f^2hj - 6$	$f^2j^2 - 6$	$fg^2h - 12$	$f^2hl + 6$	$cfhk - 6$
			$flh^2i - 3$	$fghl - 24$	$fgjl - 24$	$f^2jk - 12$	$ck^2l - 24$
			$fhl^2 - 12$	$fggk - 24$	$fij^2 - 3$	$fhik + 3$	$f^2gh + 6$
			$fjkl - 24$	$fjl^2 - 24$	$ghil + 24$	$fkl^2 - 24$	$f^2jl - 18$
			$hikl + 24$	$ghik + 24$	$hi^2j + 6$	$ik^2l + 24$	$fghl - 48$
			$ijk^2 + 12$	$h^2i^2 + 6$	$ijl^2 + 12$		$fhil + 12$
				$hil^2 + 24$			$fijk - 9$
				$ijkl + 24$			$fl^3 - 24$
							$gik^2 + 24$
							$h^3k + 6$
							$ikl^2 + 48$

xyz^2	xz^3	y^4	y^3z	y^2z^2	yz^3	z^4
$abci - 3$	$abc^2 - 1$	$b^2ij + 3$	$b^2cj + 3$	$bcfj + 9$	$bc^2h - 3$	$bcg^2 + 3$
$acf^2 + 6$	$acfi + 3$	$bck^2 - 3$	$bcfh - 3$	$bchi - 9$	$bcgl + 6$	$c^2fh - 3$
$afi^2 - 3$	$ai^3 - 2$	$bf^2j - 3$	$bckl - 3$	$bgil + 18$	$bcij + 3$	$cfgl - 6$
$bcgh - 9$	$bcgj + 3$	$bfhi - 3$	$bfij + 3$	$cfkl - 18$	$bg^2i + 6$	$cfij + 3$
$bcjl + 12$	$bg^3 + 8$	$bikl + 6$	$bgik + 3$	$f^2gl - 18$	$cf^2j + 6$	$chi^2 + 3$
$bg^2l + 24$	$cfgh - 12$	$f^3h + 3$	$bhi^2 - 6$	$f^2ij - 9$	$cfgk - 6$	$fg^2i - 3$
$bgij + 6$	$cfjl + 3$	$f^2kl - 6$	$bil^2 + 12$	$fhi^2 + 9$	$cfhi - 3$	$gi^2l + 6$
$cfhl - 18$	$cl^3 - 8$	$fik^2 + 3$	$cfk^2 - 6$	$i^3kl + 18$	$cfl^2 - 12$	$i^2j - 3$
$cfjk + 6$	$fg^2l - 24$		$f^3j - 6$		$f^2g^2 - 6$	
$ckl^2 - 24$	$fgij - 3$		$f^2gk - 6$		$fgil - 6$	
$f^2gj - 6$	$ghi^2 + 12$		$f^2hi + 9$		$fi^2j - 9$	
$fg^2k - 24$	$gil^2 + 24$		$f^2l^2 - 12$		$gi^2k + 6$	
$fghi + 9$	$i^2jl - 6$		$fikl + 6$		$hi^3 + 6$	
$fgl^2 - 48$			$i^2k^2 + 6$		$i^2l^2 + 12$	
$fijl - 12$						
$gikl + 48$						
$hi^2l + 18$						
$i^2jk - 6$						
$il^2 + 24$						

;

and it is not necessary to write down the corresponding values for η, ζ.

158.

A SIXTH MEMOIR UPON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLIX. *for the year* 1859, pp. 61—90. Received November 18, 1858,—Read January 6, 1859.]

I PROPOSE in the present memoir to consider the geometrical theory: I have alluded to this part of the subject in the articles Nos. 3 and 4 of the Introductory Memoir, [139]. The present memoir relates to the geometry of one dimension and the geometry of two dimensions, corresponding respectively to the analytical theories of binary and ternary quantics. But the theory of binary quantics is considered for its own sake; the geometry of one dimension is so immediate an interpretation of the theory of binary quantics, that for its own sake there is no necessity to consider it at all; it is considered with a view to the geometry of two dimensions. A chief object of the present memoir is the establishment, upon purely descriptive principles, of the notion of distance. I had intended in this introductory paragraph to give an outline of the theory, but I find that in order to be intelligible it would be necessary for me to repeat a great part of the contents of the memoir in relation to this subject, and I therefore abstain from entering upon it. The paragraphs of the memoir are numbered consecutively with those of my former Memoirs on Quantics.

147. It will be seen that in the present memoir, the geometry of one dimension is treated of as a geometry of points in a line, and the geometry of two dimensions as a geometry of points and lines in a plane. It is, however, to be throughout borne in mind, that, in accordance with the remarks No. 4 of the Introductory Memoir, the terms employed are not (unless this is done expressly or by the context) restricted to their ordinary significations. In using the geometry of one dimension in reference to geometry of two dimensions considered as a geometry of points and lines in a plane, it is necessary to consider,—1°, that the word point may mean *point* and the word line mean *line*; 2°, that the word point may mean *line* and the

word line mean *point*. It is, I say, necessary to do this, for in such geometry of two dimensions we have systems of points in a line and of lines through a point, and each of these systems is in fact a system belonging to, and which can by such extended signification of the terms be included in, the geometry of one dimension. And precisely because we can by such extension comprise the correlative theorems under a common enunciation, it is not in the geometry of one dimension necessary to enunciate them separately; it may be and very frequently is necessary and proper in the geometry of two dimensions, where we are concerned with systems of each kind, to enunciate such correlative theorems separately. It may, by way of further illustration, be remarked, that in using the geometry of one dimension in reference to geometry of three dimensions considered as a geometry of points, lines, and planes in space, it would be necessary to consider,—1°, that the words point and line may mean respectively *point* and *line*; 2°, that the word line may mean *point in a plane*[1], and the word point mean *line*, viz. the expression points in a line mean *lines through a point and in a plane*; 3rd, that the word line may mean *line* and the word point mean *plane*, viz. the expression points in a line mean *planes through a line*. And so in using the geometry of two dimensions in reference to geometry of three dimensions considered as a geometry of points, lines, and planes in space, it would be necessary to consider,—1°, that the words point, line, and plane may mean respectively *point*, *line*, and *plane*; 2°, that the words point, line, and plane may mean respectively *plane*, *line*, and *point*. But I am not in the present memoir concerned with geometry of three dimensions. The thing to be attended to is, that in virtue of the extension of the signification of the terms, in treating the geometry of one dimension as a geometry of points in a line, and the geometry of two dimensions as a geometry of points and lines in a plane, we do in reality treat these geometries respectively in an absolutely general manner. In particular—and I notice the case because I shall have occasion again to refer to it—we do in the geometry of two dimensions include spherical geometry; the words plane, point, and line, meaning for this purpose, spherical surface, arc (of a great circle) and point (that is, pair of opposite points) of the spherical surface. And in like manner the geometry of one dimension includes the cases of points on an arc, and of arcs through a point.

148. I repeat also a remark which is in effect made in the same No. 4; the coordinates x, y of the geometry of one dimension, and the coordinates x, y, z and ξ, η, ζ of the geometry of two dimensions are only determinate to a common factor *près* (that is, it is the ratios only of the coordinates, and not their absolute magnitudes, which are determinate); hence in saying that the coordinates x, y are equal to a, b, or in writing $x, y = a, b$, we mean only that $x : y = a : b$, and we never as a result obtain $x, y = a, b$, but only $x : y = a : b$. And the like with respect to the coordinates x, y, z and ξ, η, ζ. (In the geometry of two dimensions, $x, y = a, b$, is for this reason considered and spoken of as a single equation.) But when this is once understood, there is no objection to treating the coordinates as if they were completely determinate.

[1] It would be more accurate to say that the word line may mean *point-in-and-with-a plane*, viz. the *locus in quo* of lines through the point and in the plane. Added, June 16, 1859.—A. C.

On Geometry of One Dimension, Nos. 149 to 168.

149. In geometry of one dimension we have the line as a space or *locus in quo*, which is considered as made up of points. The several points of the line are determined by the coordinates (x, y), viz. attributing to these any specific values, or writing $x, y = a, b$, we have a particular point of the line. And we may say also that the line is the *locus in quo* of the coordinates (x, y).

150. A linear equation,

$$(*\between x, y)^1 = 0,$$

is obviously equivalent to an equation of the before-mentioned form $x, y = a, b$, and represents therefore a point. An equation such as

$$(*\between x, y)^m = 0$$

breaks up into m linear equations, and represents therefore a system of m points, or point-system of the order m. The component points of the system, or the linear factors, or the values thereby given for the coordinates, are termed roots. When $m = 1$ we have of course a single point, when $m = 2$ we have a quadric or point-pair, when $m = 3$ a cubic or point-triplet, and so on. The point-system is the only figure or locus occurring in the geometry of one dimension. The quantic $(*\between x, y)^m$, when it is convenient to do so, may be represented by a single letter U, and we then have $U = 0$ for the equation of the point-system.

151. The equation

$$(*\between x, y)^m = 0$$

may have two or more of its roots equal to each other, or generally there may exist any systems of equalities between the roots of the equation, or what is the same thing, the system may comprise two or more coincident points, or any systems of coincident points. In particular, when the discriminant vanishes the equation will have a pair of equal roots, or the system will comprise a pair of coincident points; in the case of the quadric $(a, b, c\between x, y)^2 = 0$, the condition is $ac - b^2 = 0$, or as it may be written, $a, b = b, c$; in the case of the cubic

$$(a, b, c, d\between x, y)^3 = 0,$$

the condition is

$$a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2 = 0.$$

The preceding is the only special case for a quadric: for a cubic we have besides the special case where the three roots are equal, or the cubic reduces itself to three coincident points; the conditions for this are

$$ac - b^2 = 0, \quad ad - bc = 0, \quad bd - c^2 = 0,$$

equivalent to the two conditions

$$a : b = b : c = c : d.$$

For equations of a higher order the analytical question is considered, and as regards the quartic and the quintic respectively completely solved, in my "Memoir on the Conditions for the Existence of given Systems of Equalities between the Roots of an Equation" (1).

152. Any covariant of the equation

$$(*\between x, y)^m = 0,$$

equated to zero, gives rise to a point-system connected in a definite manner with the original point-system. And as regards the invariants, the evanescence of any invariant implies a certain relation between the points of the system; the identical evanescence of any covariant implies relations between the points of the system, such that the derived point-system obtained by equating the covariant to zero is absolutely indeterminate. The like remarks apply to the covariants or invariants of two or more equations, and the point-systems represented thereby.

153. In particular, for the two point-pairs represented by the quadric equations

$$(a, b, c\between x, y)^2 = 0,$$
$$(a', b', c'\between x, y)^2 = 0,$$

if the lineo-linear invariant vanishes, that is, if

$$ac' - 2bb' + ca' = 0,$$

we have the harmonic relation,—the two point-pairs are said to be harmonically related to each other, or the two points of the one pair are said to be harmonics with respect to the two points of the other pair. The analytical theory is fully developed in the "Fifth Memoir upon Quantics" (2). The chief results, stated under a geometrical form, are as follows:

1°. If either of the pairs and one point of the other pair are given, the remaining point of such other pair can be found.

2°. A point-pair can be found harmonically related to any two given point-pairs.

154. The last of the two theorems gives rise to the theory of involution. The two given point-pairs, viewed in relation to the harmonic pair, are said to be an involution of four points; and the points of the harmonic pair are said to be the (double or) sibiconjugate points of the involution. A system of three or more pairs, such that the third and every subsequent pair are each of them harmonically related to the sibiconjugate points of the first and second pairs, is said to be a system in involution. In particular, for three pairs we have what is termed an involution of six points; and it is clear that when two pairs and a point of the third pair are given, the remaining point of the third pair can be determined. And in like manner

[1] *Philosophical Transactions*, vol. CXLVII. (1857), pp. 727—731, [150].
[2] *Philosophical Transactions*, vol. CXLVIII. (1858), pp. 429—462, [156].

for a greater number of pairs, when two pairs and a point of each of the other pairs are given, the remaining point of each of the other pairs can be determined. Two points of the same pair are said to be conjugate to each other; or if we consider two pairs as given, then the points of the third or any subsequent pair are said to be conjugate to each other in respect to the given pairs. This explains the expression sibiconjugate points; in fact, the two pairs being given, either sibiconjugate point is, as the name imports, conjugate to itself. In other words, any two pairs and one of the sibiconjugate points considered as a pair of coincident points, form a system in involution, or involution of five points.

155. The three point-pairs, $U=0$, $U'=0$, $U''=0$, will be in involution when the quadrics U, U', U'' are connected by the linear relation or syzygy $\lambda U+\lambda' U'+\lambda'' U''=0$. This property, or the relation

$$\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0$$

to which it gives rise, might have been very properly adopted as the definition of the relation of involution, but I have on the whole preferred to deduce the theory of involution from the harmonic relation. The notion, however, of the linear relation or syzygy of three or more point-systems gives rise to a much more general theory of involution, but this is a subject that I do not now enter upon; it may, however, be noticed, that if $U=0$, $U'=0$ be any two point-systems of the same order, then we may find a point-system $U''=0$ of the same order, in involution with the given point-systems (that is, satisfying the condition $\lambda U+\lambda' U'+\lambda'' U''=0$), and such that the point-system $U''=0$ comprises a pair of coincident points; this is obviously an extension of the notion of the sibiconjugate points of an ordinary involution.

156. It was remarked in the Fifth Memoir, that the theories of the anharmonic ratio and of homography belong analytically to the subject of bipartite (lineo-linear) binary quantics; this may be further illustrated geometrically as follows: we may imagine two distinct spaces of one dimension, or lines, one of them the *locus in quo* of the coordinates (x, y), and the other the *locus in quo* of the coordinates (x, y), which are absolutely independent of, and are not in anywise related to, the coordinates of the first-mentioned system. There is no difficulty in the conception of this; for we may in a plane or in space of three dimensions imagine any two lines, and study the relations of analogy between the points of the one line *inter se*, and the points of the other line *inter se*, without in anywise adverting to the space of two or three dimensions which happens to be the common *locus in quo* of the two lines. It is proper to remark, that in speaking of the spaces of one dimension, which are the *loci in quibus* of the coordinates (x, y) and (x, y) respectively, as being each of them a line, we imply a restriction which is altogether unnecessary; the words line and point may, in regard to the two figures respectively, be used in different significations; for instance, one of the spaces may be a *line* and the points in it *points*; while the other of the spaces may be a *point* and the points in it *lines*, or it may be a *line* and the points in it *planes*.

157. A lineo-linear equation

$$(x - ay)(\mathrm{x} - \alpha\mathrm{y}) = 0$$

denotes then the two points $(x, y = a, 1)$ and $(\mathrm{x}, \mathrm{y} = \alpha, 1)$ existing irrespectively of each other in distinct spaces, and only by the equation itself brought into an ideal connexion; and any invariantive relation between the coefficients of any such bipartite function denotes geometrically a relation between a point-system in the space which is the *locus in quo* of the coordinates (x, y), and a point-system in the space which is the *locus in quo* of the coordinates (x, y); for instance, the equation

$$\begin{vmatrix} 1, & a, & \alpha, & a\alpha \\ 1, & b, & \beta, & b\beta \\ 1, & c, & \gamma, & c\gamma \\ 1, & d, & \delta, & d\delta \end{vmatrix} = 0$$

is the relation of homography between the four points $(a, 1)$, $(b, 1)$, $(c, 1)$, $(d, 1)$ in the first line, and the four points $(\alpha, 1)$, $(\beta, 1)$, $(\gamma, 1)$, $(\delta, 1)$ in the second line. The analytical theory is discussed in the Fifth Memoir; and, in particular, it is there shown, that writing

$$\begin{aligned} A &= (d-a)(b-c), & \mathfrak{A} &= (\delta-\alpha)(\beta-\gamma), \\ B &= (d-b)(c-a), & \mathfrak{B} &= (\delta-\beta)(\gamma-\alpha), \\ C &= (d-c)(a-b), & \mathfrak{C} &= (\delta-\gamma)(\alpha-\beta), \end{aligned}$$

then the condition may be expressed under any one of the forms

$$A : B : C = \mathfrak{A} : \mathfrak{B} : \mathfrak{C},$$

equations which denote the equality of the anharmonic ratios of the two point-systems.

158. The number of points in each system may be four, or any greater number; the homographic relation is then conveniently expressed under the form

$$\left\| \begin{matrix} 1, & 1, & 1, & 1, & 1, & \ldots \\ a, & b, & c, & d, & e, & \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \\ a\alpha, & b\beta, & c\gamma, & d\delta, & e\epsilon, & \end{matrix} \right\| = 0.$$

The relation is such that given three points of the one system and the corresponding three points of the other system, then to any fourth point whatever of the first system there can be found a corresponding fourth point of the second system. It is to be observed, however, that two systems of four points homographically related to each other, always correspond together in four different ways, viz. the two systems being (a, b, c, d) and $(\alpha, \beta, \gamma, \delta)$; then if the four points of the first system are (a, b, c, d), the corresponding four points of the second system may be taken in the four several orders, $(\alpha, \beta, \gamma, \delta)$, $(\beta, \alpha, \delta, \gamma)$, $(\gamma, \delta, \alpha, \beta)$, $(\delta, \gamma, \beta, \alpha)$.

159. What precedes is not to be understood as precluding the existence of a relation between the spaces which are the *loci in quibus* of the coordinates (x, y) and (x, y) respectively: not only may these be spaces of the same kind, but they may be one and the same space or line; and the points of the two systems may then be points of the same kind; and further, the coordinates (x, y) and (x, y) may belong to the same system of coordinates, that is, the equations $(x, y=a, 1)$ and $(\mathrm{x}, \mathrm{y}=a, 1)$ may denote one and the same point.

160. If the two point-systems are systems of the same kind, and are in one and the same line, then there are in general two points of the first system which coincide each of them with the corresponding point of the second system; such two points may be said to be the sibiconjugate points of the homography. In particular, the two sibiconjugate points of the homography may coincide together.

161. A system in involution affords an example of two homographic systems in the same line; in fact, taking arbitrarily a point out of each pair, the points so obtained form a system which is homographic with the system formed with the other points of the several pairs; and in this case the sibiconjugate points of the involution are also the sibiconjugate points of the homography. Thus if A and A', B and B', C and C', D and D' are pairs of the system in involution, then (A, B, C, D) and (A', B', C', D') will be homographic point-systems; and, as a particular case, (A, B, C, C') and (A', B', C', C) will be homographic point-systems. It is proper to notice that if F is a sibiconjugate point of the involution, then (A, B, F, F) and (A', B', F, F) are not (what at first sight they appear to be) homographic point-systems.

162. Imagine an involution of points; take on the line which is the *locus in quo* of the point-system a point O, and consider the point-system formed by the harmonics of O in respect to the several pairs of the involution; and in like manner take on the line any other point O', and consider the point-system formed by the harmonics of O' in respect to the several pairs of the involution; these two point-systems are homographically related to each other.—See Fifth Memoir, No. 111.

163. Two involutions may be homographically related to each other; in fact, take on the line which is the *locus in quo* of the first involution a point O, and consider the point-system formed by the harmonics of O in relation to the several pairs of the involution; take in like manner on the line which is the *locus in quo* of the second involution a point Q, and consider the point-system formed by the harmonics of Q with respect to the several pairs of the involution; then if the two point-systems are homographically related, the two involutions are said to be themselves homographically related: the last preceding article shows that the nature of the relation does not in anywise depend on the choice of the points O and Q. And it is not necessary that, as regards the two involutions respectively, the words line and point should have the same significations.—See Fifth Memoir, No. 111.

164. Four or more tetrads of points in a line may be homographically related to the same number of tetrads in another line. This is the case when the an-

harmonic ratios of the tetrads of the first system are homographically related to the anharmonic ratios of the tetrads of the second system. And it is not material which of the three anharmonic ratios of a tetrad of either system is selected, provided that the same selection is made for each of the other tetrads of the same system. The order of the points of a tetrad must be attended to, but there are in all four admissible permutations of the points of a tetrad, viz. if A, B, C, D are the points of a tetrad, then (A, B, C, D), (B, A, D, C), (C, D, A, B), (D, C, B, A) may be considered as one and the same tetrad. Any three tetrads whatever in the second system may correspond to any three tetrads of the first system; and then given a fourth tetrad of the first system, and three out of the four points of the corresponding tetrad of the second system, the remaining point of the tetrad may be determined. The words line and point need not, as regards the two systems of tetrads respectively, be understood in the same significations.—See Fifth Memoir, No. 112.

165. The foregoing theory of the harmonic relation shows that if we have a point-pair

$$(a, b, c \between x, y)^2 = 0,$$

the equation of any other point-pair whatever can be expressed, and that in two different ways, in the form

$$(a, b, c \between x, y)^2 + (lx + my)^2 = 0;$$

the points $(lx + my = 0)$ corresponding to the two admissible values of the linear function being in fact the harmonics of the point-pair in respect to the given point-pair $(a, b, c \between x, y)^2 = 0$, or what is the same thing, the sibiconjugate points of the involution formed by the two point-pairs (see Fifth Memoir, No. 105). The point-pair represented by the equation in question does not in itself stand in any peculiar relation to the given point-pair $(a, b, c \between x, y)^2 = 0$; but when thus represented it is said to be inscribed in the given point-pair, and the point $lx + my = 0$ is said to be the axis of inscription. And the harmonic of this point with respect to the given point-pair (that is, the other sibiconjugate point of the involution of the two point-pairs) is said to be the centre of inscription[1].

166. We may, if we please, (x', y') and θ being constants, exhibit the equation of the inscribed point-pair in the form

$$(a, b, c \between x, y)^2 (a, b, c \between x', y')^2 \sin^2\theta - (ac - b^2)(xy' - x'y)^2 = 0,$$

where we have for the axis of inscription and centre of inscription respectively, the equations

$$xy' - x'y = 0,$$
$$(a, b, c \between x, y \between x', y') = 0;$$

[1] The words inscribed, inscription, are used not in opposition to, but as identical with, the words circumscribed, circumscription; and in like manner (*post*, Nos. 203 et seq.) as regards conics.

or in the equivalent form,

$$(a,\ b,\ c\between x,\ y)^2(a,\ b,\ c\between x',\ y')^2\cos^2\theta - \{(a,\ b,\ c\between x,\ y\between x',\ y')\}^2 = 0,$$

where we have for the axis of inscription and the centre of inscription respectively, the equations

$$\begin{aligned}(a,\ b,\ c\between x,\ y\between x',\ y') &= 0,\\ xy' - x'y \qquad\qquad &= 0.\end{aligned}$$

167. The equivalence of the two forms depends on the identical equation

$$(a,\ b,\ c\between x,\ y)^2(a,\ b,\ c\between x',\ y')^2 - \{(a,\ b,\ c\between x,\ y\between x',\ y')\}^2 = (ac-b^2)(xy'-x'y)^2,$$

which is in fact the equation mentioned, Fifth Memoir, No. 95. If, for shortness, we write

$$\begin{aligned}&(a,\ b,\ c\between x,\ y)^2 \qquad\qquad = 00,\\ &(a,\ b,\ c\between x,\ y\between x',\ y') = 01 = 10,\\ &\&c.,\end{aligned}$$

then the equation may be represented in the form

$$\begin{vmatrix} 00, & 01 \\ 10, & 11 \end{vmatrix} = (ac-b^2)\begin{vmatrix} x, & y \\ x, & y' \end{vmatrix}^2.$$

168. There is a like equation for the three sets $(x,\ y)$, $(x',\ y')$, $(x'',\ y'')$; the right-hand side here vanishes, for there are not columns enough to form therewith a determinant, and the equation is

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix} = 0,$$

an equation which may also be written in the form

$$\cos^{-1}\frac{01}{\sqrt{00}\sqrt{11}} + \cos^{-1}\frac{12}{\sqrt{11}\sqrt{22}} = \cos^{-1}\frac{02}{\sqrt{00}\sqrt{22}},$$

as it is easy to verify by reducing this equation to an algebraical form. The various formulæ have been given in relation to the establishment of the notion of distance in the geometry of one dimension, but it will be convenient to defer the consideration of this theory so as to discuss it in connexion with geometry of two dimensions.

On Geometry of Two Dimensions, Nos. 169 to 208.

169. In geometry of two dimensions we have the plane as a space or *locus in quo,* which is considered under two distinct aspects, viz. as made up of points, and as made up of lines. The several points of the plane are determined by means of

the point-coordinates (x, y, z), viz. attributing to these any specific values, or writing $x, y, z = a, b, c$, we have a particular point of the plane; and in like manner the several lines of the plane are determined by the line-coordinates (ξ, η, ζ), viz. attributing to these any specific values, or writing $\xi, \eta, \zeta = \alpha, \beta, \gamma$, we have a particular line of the plane. And we may say that the plane is the *locus in quo* of the point-coordinates (x, y, z), and of the line-coordinates (ξ, η, ζ). It is not necessary to consider separately the analytical theories of point-coordinates and of line-coordinates; for the theory of the former in relation to points and lines respectively is identical with the theory of the latter in relation to lines and points respectively; but it is necessary to show how either system of coordinates, say the system of point-coordinates, is applicable to both points and lines, or in fact all loci whatever, and to explain the mutual relation of the two systems of coordinates.

170. Considering then point-coordinates, the equations

$$x, y, z = a, b, c,$$

determine, as already mentioned, a point.

A linear equation

$$(*\between x, y, z)^1 = 0$$

determines a line, viz. the line which is the locus of all the points, the coordinates of which satisfy this equation. And in like manner an equation

$$(*\between x, y, z)^m = 0$$

determines a curve of the mth order, viz. the curve which is the locus of all the points, the coordinates of which satisfy this equation. In particular, an equation of the second degree

$$(*\between x, y, z)^2 = 0$$

determines a conic.

171. If the quantic breaks up into rational factors, then the equation of the curve is satisfied by equating to zero any one of these factors, or the curve breaks up into curves of a lower order, and the order of the entire curve is equal to the sum of the orders of the component curves. In particular, a curve of any order may break up into a system of lines, the number of lines being of course equal to the order of the curve, and any two or more of these lines may coincide with each other. A curve not thus breaking up into curves of a lower order is said to be a proper curve.

172. Returning to the linear equation and expressing the coefficients, the equation is

$$(\xi, \eta, \zeta \between x, y, z) = 0,$$

or, what is the same thing,

$$\xi x + \eta y + \zeta z = 0;$$

and we say as a definition, that the coordinates (line-coordinates) of this line are (ξ, η, ζ).

173. But the same equation, considering (x, y, z) as constant coefficients, and (ξ, η, ζ) as line-coordinates, is the equation of a point, viz. the point which is the locus (envelope) of all those points the coordinates of which satisfy the equation in question; and such point is precisely the point, the coordinates (point-coordinates) of which are (x, y, z). In fact, if (ξ, η, ζ) are considered as variable parameters connected by the equation $\xi x + \eta y + \zeta z = 0$, then taking (X, Y, Z) as current point-coordinates, the equation $\xi X + \eta Y + \zeta Z = 0$ is satisfied by writing (x, y, z) for (X, Y, Z); or the several lines the coordinates whereof are (ξ, η, ζ), all pass through the point (x, y, z).

174. Hence recapitulating, the equation

$$(\xi, \eta, \zeta \between x, y, z) = 0,$$

or

$$\xi x + \eta y + \zeta z \quad = 0,$$

considering (x, y, z) as current point-coordinates, and (ξ, η, ζ) as constant coefficients, is the equation of a line the coordinates (line-coordinates) of which are (ξ, η, ζ); and the same equation, considering (ξ, η, ζ) as current line-coordinates, and (x, y, z) as constant coefficients, is the equation of a point the coordinates (point-coordinates) of which are (x, y, z).

175. The expression, the point (a, b, c), means the point whose point-coordinates are (a, b, c); and in like manner the expression, the line (α, β, γ), means the line whose line-coordinates are (α, β, γ). The last-mentioned expression may, without any impropriety or risk of ambiguity, be employed when we are dealing with point-coordinates; but it is of course always allowable, and it is frequently better, to substitute for the definition the thing signified, and say the line having for its equation $\alpha x + \beta y + \gamma z = 0$, or more briefly, the line $\alpha x + \beta y + \gamma z = 0$. It will be proper to do this in the following articles, Nos. 176 to 184, which contain some formulæ in point-coordinates relating to the theory of the point and the line.

176. The condition that the point (a, b, c) may lie in the line

$$\alpha x + \beta y + \gamma z = 0,$$

is of course

$$\alpha a + \beta b + \gamma c = 0.$$

177. The equation of the line passing through the points (a, b, c), (a', b', c'), is

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0;$$

and if in this equation (a', b', c') are considered as indeterminate, we have the equation of a line subjected to the single condition of passing through the point (a, b, c). The equation contains apparently two arbitrary parameters, but these in fact reduce themselves to a single one.

178. The coordinates of the point of intersection of the lines

$$\alpha x + \beta y + \gamma z = 0,$$
$$\alpha' x + \beta' y + \gamma' z = 0,$$

are given by the equations

$$x,\ y,\ z = \beta\gamma' - \beta'\gamma,\quad \gamma\alpha' - \gamma'\alpha,\quad \alpha\beta' - \alpha'\beta;$$

and if in these equations we consider α', β', γ' as indeterminate, we have the coordinates of a point subjected to the single condition of lying in the line $\alpha x + \beta y + \gamma z = 0$; the result, as in the last case, contains in appearance two arbitrary parameters, but these really reduce themselves to a single one.

179. The condition in order that the points (a, b, c), (a', b', c'), (a'', b'', c'') may lie in a line is

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0,$$

which may also be expressed by the equations

$$a'',\ b'',\ c'' = \lambda a + \mu a',\quad \lambda b + \mu b',\quad \lambda c + \mu c',$$

where λ, μ are arbitrary multipliers; these equations give therefore the coordinates of an indeterminate point in the line joining the points (a, b, c) and (a', b', c').

180. The condition that the lines

$$\alpha x + \beta y + \gamma z = 0,$$
$$\alpha' x + \beta' y + \gamma' z = 0,$$
$$\alpha'' x + \beta'' y + \gamma'' z = 0$$

may meet in a point is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = 0,$$

a relation which may also be expressed by the equations

$$\alpha'',\ \beta'',\ \gamma'' = l\alpha + m\alpha',\quad l\beta + m\beta',\quad l\gamma + m\gamma';$$

where l, m are arbitrary multipliers; and substituting these values in the equation $\alpha'' x + \beta'' y + \gamma'' z = 0$, we have for the equation of a line subjected to the single condition of passing through the point of intersection of the lines $\alpha x + \beta y + \gamma z = 0$, $\alpha' x + \beta' y + \gamma' z = 0$, the equation

$$l(\alpha x + \beta y + \gamma z) + m(\alpha' x + \beta' y + \gamma' z) = 0,$$

which is, in fact, at once obtained by the consideration that the values of (x, y, z) which satisfy simultaneously the equations $\alpha x+\beta y+\gamma z=0$ and $\alpha' x+\beta' y+\gamma' z=0$, satisfy also the equation in question.

181. The equation of the line passing through the point of intersection of the lines $\alpha x+\beta y+\gamma z=0$ and $\alpha' x+\beta' y+\gamma' z=0$, and also through the point (a, b, c), is obviously

$$\begin{vmatrix} \alpha x+\beta y+\gamma z, & \alpha' x+\beta' y+\gamma' z \\ \alpha a+\beta b+\gamma c, & \alpha' a+\beta' b+\gamma' c \end{vmatrix}=0,$$

which, or the equivalent form

$$\frac{\alpha x+\beta y+\gamma z}{\alpha a+\beta b+\gamma c}=\frac{\alpha' x+\beta' y+\gamma' z}{\alpha' a+\beta' b+\gamma' c},$$

is usually the most convenient one; but it is to be observed that the equation can also be written in the forms

$$\begin{vmatrix} x\ , & y\ , & z \\ a\ , & b\ , & c \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{vmatrix}=0,$$

and

$$\begin{vmatrix} bz-cy, & cx-az, & ay-bx \\ \alpha\ , & \beta\ , & \gamma \\ \alpha'\ , & \beta'\ , & \gamma' \end{vmatrix}=0,$$

or in the form

$$(\beta\gamma'-\gamma\beta')(bz-cy)+(\gamma\alpha'-\gamma'\alpha)(cx-az)+(\alpha\beta'-\beta\alpha')(ay-bx)=0,$$

which might also be represented by

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}\begin{vmatrix} x, & y, & z \\ a, & b, & c \end{vmatrix}=0.$$

182. To find the coordinates of the point of intersection of the line joining the points (a, b, c), (a', b', c'), with the line $\alpha x+\beta y+\gamma z=0$, we have

$$x,\ y,\ z=\lambda a+\mu a',\quad \lambda b+\mu b',\quad \lambda c+\mu c',$$

where λ, μ are given by

$$\lambda(\alpha a+\beta b+\gamma c)+\mu(\alpha a'+\beta b'+\gamma c')=0.$$

The preceding are elementary formulæ of almost constant occurrence; it may be proper to add to them the formulæ which follow.

183. To find the equation of the line passing through the point of intersection of the lines

$$\alpha_1 x+\beta_1 y+\gamma_1 z=0,\quad \alpha_2 x+\beta_2 y+\gamma_2 z=0,$$

and the point of intersection of the lines

$$\alpha_3 x + \beta_3 y + \gamma_3 z = 0, \quad \alpha_4 x + \beta_4 y + \gamma_4 z = 0.$$

Write for shortness $u_1 = \alpha_1 x + \beta_1 y + \gamma_1 z$, &c.; then we have identically

$$\begin{vmatrix} u_1, & u_2, & u_3, & u_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix} = 0,$$

and the two equations

$$\begin{vmatrix} u_1, & u_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} ., & ., & u_3, & u_4 \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix} = 0$$

are consequently equivalent to each other, and each of them represents the required line. It is easy to deduce the form

$$x \begin{vmatrix} \alpha_1, & \alpha_2, & ., & . \\ ., & ., & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix} + y \begin{vmatrix} \beta_1, & \beta_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ ., & ., & \beta_3, & \beta_4 \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4 \end{vmatrix} + z \begin{vmatrix} \gamma_1, & \gamma_2, & ., & . \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 \\ \beta_1, & \beta_2, & \beta_3, & \beta_4 \\ ., & ., & \gamma_3, & \gamma_4 \end{vmatrix} = 0.$$

184. The condition in order that the points of intersection of the lines $u_1 = 0$, $u_2 = 0$, of the lines $u_3 = 0$, $u_4 = 0$, and of the lines $u_5 = 0$, $u_6 = 0$ (where, as before, u_1 denotes $\alpha_1 x + \beta_1 y + \gamma_1 z$, &c.) may lie in the same line, is

$$\begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & ., & . \\ \beta_1, & \beta_2, & \beta_3, & \beta_4, & ., & . \\ \gamma_1, & \gamma_2, & \gamma_3, & \gamma_4, & ., & . \\ ., & ., & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6 \\ ., & ., & \beta_3, & \beta_4, & \beta_5, & \beta_6 \\ ., & ., & \gamma_3, & \gamma_4, & \gamma_5, & \gamma_6 \end{vmatrix} = 0,$$

which is of course really symmetrical with respect to the six sets. The last formula was given by me, *Cambridge Mathematical Journal*, t. IV. (1845), p. 18, [9].

185. Instead of the term point of a curve, it will be convenient to use the term 'ineunt' of the curve.

The line through two consecutive ineunts of the curve is the tangent at the ineunt. The point of intersection of two consecutive tangents is the ineunt on the tangent.

The equation of a curve in point-coordinates, or as it may be termed the point-equation of the curve, is the relation which exists between the point-coordinates of any ineunt of the curve.

The equation of a curve in line-coordinates, or line-equation of the curve, is the relation which exists between the line-coordinates of any tangent of the curve.

186. It has been mentioned, that the order of a curve is the degree of its point-equation: in like manner the class of a curve is the degree of its line-equation; and in the same way that a curve, as represented by a point-equation, may break up into curves having the order of the entire curve for the sum of their orders, so a curve as represented by a line-equation may break up into curves having the class of the entire curve for the sum of their classes. And, in particular, a curve may break up into a system of points, the number of points being equal to the class of the curve, and two or more of these points may coincide together.

187. A line is a curve of the order one and class zero; a point is a curve of the order zero and class one. A proper conic is a curve of the order two and class two; but when the conic breaks up into a pair of lines, the class sinks to zero; and when the conic breaks up into a pair of points, the order sinks to zero. It is to be noticed that a point, or system of points, cannot be represented by an equation in point-coordinates, nor a line or system of lines by an equation in line-coordinates. We may say, in general, that a curve is both a point-curve and a line-curve, but a point or system of points is a line-curve only, and a line or system of lines is a point-curve only.

188. The points of intersection (common ineunts) of two curves are the points the coordinates of which satisfy simultaneously the point-equations of the two curves. Hence the number of common ineunts is equal to the product of the orders of the two curves; and, in particular if one of the curves be a line, the number of points of intersection (common ineunts) is equal to the order of the curve. In like manner the common tangents of the two curves are the lines the coordinates of which satisfy simultaneously the line-equations of the two curves. Hence the number of common tangents is equal to the product of the classes of the two curves; and, in particular, if one of the curves be a point, the number of common tangents (tangents to the curve through the point) is equal to the class of the curve. Since the tangent is the line through two consecutive ineunts, it besides meets the curve, assumed to be of the order m, in $(m-2)$ points; and in like manner we may from any ineunt of a curve of the class n draw $(n-2)$ tangents to the curve.

189. The point-equation of a line passing through the points (x', y', z') and (x'', y'', z'') is, as already noticed,

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} = 0.$$

Suppose that (x, y, z) are the coordinates of a point (ineunt) of the curve $U=0$ the coordinates of the consecutive ineunt will be $(x+dx, y+dy, z+dz)$, and the line joining these two points will be the tangent to the curve at the point (x, y, z). Take (X, Y, Z) as current point-coordinates, the equation of the tangent is

$$\begin{vmatrix} X & , & Y & , & Z \\ x & , & y & , & z \\ x+dx, & & y+dy, & & z+dz \end{vmatrix} = 0,$$

or, what is the same thing,

$$X(ydz-zdy)+Y(zdx-xdz)+Z(xdy-ydx)=0.$$

But since U is a homogeneous function of (x, y, z), we have

$$x\partial_x U + y\partial_y U + z\partial_z U = mU = 0;$$

and since $(x+dx, y+dy, z+dz)$ is a point of the curve, we have

$$dx\partial_x U + dy\partial_y U + dz\partial_z U = 0;$$

and from these two equations

$$ydz-zdy : zdx-xdz : xdy-ydx = \partial_x U : \partial_y U : \partial_z U,$$

and the equation of the tangent consequently is

$$X\partial_x U + Y\partial_y U + Z\partial_z U = 0.$$

190. Take (ξ, η, ζ) as the line-coordinates of the tangent, then the equation of the tangent is

$$\xi X + \eta Y + \zeta Z = 0;$$

and comparing the two forms, we have

$$\xi : \eta : \zeta = \partial_x U : \partial_y U : \partial_z U;$$

and if from these equations and the equation $U=0$ (the point-equation of the curve) we eliminate (x, y, z), we obtain an equation between (ξ, η, ζ), which is the line-equation of the curve. We may, if we please, present the system under the form

$$\begin{aligned} \partial_x U + \lambda\xi &= 0, \\ \partial_y U + \lambda\eta &= 0, \\ \partial_z U + \lambda\zeta &= 0, \\ U \qquad &= 0, \end{aligned}$$

or, what is more simple, under the form

$$\begin{aligned} \partial_x U + \lambda\xi &= 0, \\ \partial_y U + \lambda\eta &= 0, \\ \partial_z U + \lambda\zeta &= 0, \\ \xi x + \eta y + \zeta z &= 0, \end{aligned}$$

and from either system eliminate x, y, z and λ.

191. If the point-equation of a conic be

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

then its line-equation is

$$-\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi & a, & h, & g \\ \eta & h, & b, & f \\ \zeta & g, & f, & c \end{vmatrix} = 0,$$

or writing

$$\begin{aligned} \mathfrak{A} &= bc - f^2, \\ \mathfrak{B} &= ca - g^2, \\ \mathfrak{C} &= ab - h^2, \\ \mathfrak{F} &= gh - af, \\ \mathfrak{G} &= hf - bg, \\ \mathfrak{H} &= fg - ch, \end{aligned}$$

and, to complete the system,

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

then the line-equation of the conic is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2 = 0.$$

192. The quantities $\mathfrak{A}$, &c. satisfy the relations

$$K^2 = \mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{F}^2 - \mathfrak{B}\mathfrak{G}^2 - \mathfrak{C}\mathfrak{H}^2 + 2\mathfrak{F}\mathfrak{G}\mathfrak{H},$$

$$\begin{aligned} &\mathfrak{A}a + \mathfrak{H}h + \mathfrak{G}g = K, && \mathfrak{H}a + \mathfrak{B}h + \mathfrak{F}g = 0, && \mathfrak{G}a + \mathfrak{F}h + \mathfrak{C}g = 0, \\ &\mathfrak{A}h + \mathfrak{H}b + \mathfrak{G}f = 0, && \mathfrak{H}h + \mathfrak{B}b + \mathfrak{F}f = K, && \mathfrak{G}h + \mathfrak{F}b + \mathfrak{C}f = 0, \\ &\mathfrak{A}g + \mathfrak{H}f + \mathfrak{G}c = 0, && \mathfrak{H}g + \mathfrak{B}f + \mathfrak{F}c = 0, && \mathfrak{G}g + \mathfrak{F}f + \mathfrak{C}c = K, \end{aligned}$$

and moreover

$$\begin{aligned} Ka &= \mathfrak{B}\mathfrak{C} - \mathfrak{F}^2, & Kf &= \mathfrak{G}\mathfrak{H} - \mathfrak{A}\mathfrak{F}, \\ Kb &= \mathfrak{C}\mathfrak{A} - \mathfrak{G}^2, & Kg &= \mathfrak{H}\mathfrak{F} - \mathfrak{B}\mathfrak{G}, \\ Kc &= \mathfrak{A}\mathfrak{B} - \mathfrak{H}^2, & Kh &= \mathfrak{F}\mathfrak{G} - \mathfrak{C}\mathfrak{H}. \end{aligned}$$

193. A system of points in a line is said to be a range, and a system of lines through a point is said to be a pencil. The theories of ranges and pencils, considered irrespectively of each other, are in fact a single theory, constituting the geometry of one dimension. It has been seen how in geometry of one dimension a range of points and a pencil of lines, although considered (as they must be considered) as existing in distinct spaces, may nevertheless stand in certain relations to each other. In geometry of two dimensions, the range and pencil may of course coexist in one and the same plane as their common *locus in quo*; and such coexistence occurs in fact very frequently: thus if we have a line and a point, and if lines are drawn

joining the point with the several points of the line, these lines constitute a pencil, and the points of the line constitute a range, and such pencil and range are homographically related.

194. The theory of homography in geometry of two dimensions may be made to depend upon the corresponding theory in geometry of one dimension, or what is the same thing, upon the homography of ranges or pencils. For consider two figures existing in distinct planes or spaces of two dimensions, any four points (not in a line) in the second figure may correspond to any four points (not in a line) in the first figure; and when this is so, we may, by the process about to be explained, given any other point of the first figure, construct the corresponding point of the second figure; and the two figures are then, by definition, homographically related. Suppose that the points A', B', C', D' of the second figure correspond respectively to the points A, B, C, D of the first figure, and let E be any other point of the first figure; suppose that E' is the corresponding point of the second figure; the pencils AB, AC, AD, AE and $A'B'$, $A'C'$, $A'D'$, $A'E'$ should be homographic to each other, that is, E' must lie on a given line through A'; and in like manner the pencils BA, BC, BD, BE and $B'A'$, $B'C'$, $B'D'$, $B'E'$ should be homographic to each other, that is, E' must lie on a given line through B'. And then, as a theorem, CA, CB, CD, CE and $C'A'$, $C'B'$, $C'D'$, $C'E'$, or DA, DB, DC, DE and $D'A'$, $D'B'$, $D'C'$, $D'E'$ will be homographic pencils, that is, the construction will be a determinate one whichever two of the four points are selected for the points A and B. The foregoing construction leads to an analytical relation, which I think constitutes a better foundation of the theory. Consider the first plane as the *locus in quo* of the coordinates (x, y, z), and the second plane as the *locus in quo* of the coordinates (X, Y, Z), these two coordinate systems being absolutely independent of each other. Consider any point of the first plane and a corresponding point of the second plane such that its coordinates (X, Y, Z) are given linear functions of the coordinates (x, y, z) of the point in the first plane. Any figure whatever in the first plane gives rise to a corresponding figure in the second plane, and the two figures are said to be homographic to each other. To a point of the first figure there corresponds in the second figure a point, to a line a line, to a range of points or pencil of lines, a homographic range of points or pencil of lines; the line or point which is the *locus in quo* of the range or pencil in the one figure corresponding with the line or point which is the *locus in quo* of the range or pencil in the other figure. And generally, to any curve of any order and class in the first figure, and to its ineunts and tangents, there correspond in the second figure a curve of the same order and class, and the ineunts and tangents of such curve.

195. It is to be remarked, that it is not by any means necessary that the word plane, or the words point and line, or consequently the words order and class, should have the same significations as regards the two figures respectively. The theory of homography, as above explained, in fact comprises what is commonly termed the theory of homography and also the theory of reciprocity.

196. Let the word plane have the ordinary signification as regards the two figures respectively; and Suppose, first, that the words point and line, and therefore order

and class, have also the ordinary significations as regards the two figures respectively: we have here the ordinary theory of homography, in which, to any range of points or pencil of lines in the first figure, there corresponds a homographic range of points or pencil of lines in the second figure, and to a curve of any order and class in the first figure there corresponds a curve of the same order and class in the second figure.

197. We may, as a specialization giving rise to further developments, assume that the two figures exist in one and the same plane. There is here in general a triangle, each of whose angles or sides, considered as a point or line in the first figure, corresponds to itself, considered as a point or line in the second figure: such triangle may be called the sibiconjugate triangle. Any one point of the plane, considered as belonging to the first figure, may correspond to any other point of the plane, considered as belonging to the second figure, and the second figure can be completely constructed by means of the sibiconjugate triangle and such pair of corresponding points. In certain special cases the sibiconjugate triangle becomes wholly or in part indeterminate; thus if the two figures are identical, each point of the plane, considered as belonging to the first figure, coincides with itself, considered as belonging to the second figure. But I reserve the further discussion of the theory of homography for another occasion.

198. Suppose, secondly, that in the foregoing general theory, as regards the first figure, the words point and line, and therefore order and class, signify *point* and *line*, *order* and *class*; while as regards the second figure, the words point and line signify *line* and *point* respectively, and therefore the words order and class, *class* and *order* respectively. We have in the present case the ordinary theory of reciprocity, viz. using all the words in the same significations as regards the two figures respectively; to a point in the first figure there corresponds in the second figure a line; to a line, a point; to a range of points or pencil of lines, a pencil of lines or range of points; to a curve of any order and class, and its ineunts and tangents, a curve of the same class and order, and the tangents and ineunts of such curve.

199. As a specialization giving rise to further developments, we may assume that the two figures exist in one and the same plane. In this case, the points which, considered indifferently as belonging to the first or the second figure, lie upon the corresponding lines in the second or first figure, generate a conic which may be termed the pole-conic; and the lines which, considered indifferently as belonging to the first or the second figure, pass through the corresponding points in the second or first figure, envelope a conic which may be termed the polar-conic, and these two conics have double contact with another. The further consideration of this subject is reserved for another occasion; but I remark that in the particular case where the two conics coincide, we have the ordinary theory of poles and polars in regard to a conic; a theory, which, in a different point of view, may be considered as arising out of the harmonic relation, and which must here be noticed.

200. Consider a conic and a point; any line through the point meets the conic in two points (ineunts of the conic), and the harmonic in relation to these two points of the given point has for its locus a line which is the polar of the given point.

The polar passes through the points of contact of the conic with the tangents through the given point.

In like manner considering the conic and a line; from any point of the line we may draw two tangents to the conic, and the harmonic of the given line with respect to the two tangents envelopes a point which is the pole of the given line. The pole is the point of intersection of the tangents of the conic at the points of intersection with the given line.

The polars of the several points of a line envelope a point which is the pole of the line; and the poles of the several lines through a point generate a line which is the polar of the point; and this proposition shows how the theory of poles and polars gives rise to a theory of reciprocity.

201. If the point-equation of a conic be

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

the point-equation of the polar with respect to this conic of the point (x', y', z') is

$$(a, b, c, f, g, h \between x, y, z \between x', y', z') = 0.$$

But it has been seen that the line-equation of the same conic is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2 = 0,$$

and the line-equation of the pole with respect to this conic of the line (ξ', η', ζ') (that is, the line whose point-equation is $\xi' x + \eta' y + \zeta' z = 0$) is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta \between \xi', \eta', \zeta') = 0,$$

in other words, the point-coordinates of the pole are

$$\mathfrak{A}\xi' + \mathfrak{H}\eta' + \mathfrak{G}\zeta', \quad \mathfrak{H}\xi' + \mathfrak{B}\eta' + \mathfrak{F}\zeta', \quad \mathfrak{G}\xi' + \mathfrak{F}\eta' + \mathfrak{C}\zeta'.$$

202. If $U = 0$, $V = 0$ be the point-equations of any two curves of the same order, then λ, μ being arbitrary coefficients,

$$\lambda U + \mu V = 0$$

is the equation of a curve of the same order passing through the points of intersection (common ineunts) of the two curves; such curve is said to be in involution with the given curves. The discussion of the general theory of involution is reserved for another occasion.

203. In particular, if $U = 0$ be the equation of a conic, and $P = 0$, $Q = 0$ the equations of two lines, then

$$U + \lambda PQ = 0$$

is the equation of a conic passing through the points of intersection of the conic with the two lines; and if the two lines coincide, then

$$U + \lambda P^2 = 0$$

is the equation of a conic having double contact with the conic $U=0$ at its points of intersection with the line $P=0$. Such conic is said to be inscribed in the conic $U=0$; the line $P=0$ is the axis of inscription; this line has the same pole with respect to each of the two conics, and the pole is termed the centre of inscription: the relation of the two conics is completely expressed by saying that the four common ineunts coincide in pairs upon the axis of inscription, and that the four common tangents coincide in pairs through the centre of inscription; it is consequently a similar relation in regard to ineunts and tangents respectively; and it is to be inferred *à priori*, that if $\Upsilon=0$ be the line-equation of the conic $U=0$, and $\Pi=0$ the line-equation of the centre of inscription, then the line-equation of the inscribed conic is $\Upsilon+\mu\Pi^2=0$.

204. To verify this, I remark that if the equation of the axis of inscription be

$$\xi' x+\eta' y+\zeta' z=0,$$

then (*ante*, No. 201) we have for the line-equation of the centre of inscription

$$\Pi=(\mathfrak{A},\ldots\between\xi,\ \eta,\ \zeta\between\xi',\ \eta',\ \zeta')=0.$$

The line-equation of the inscribed conic is in the first instance obtained in the form

$$(\mathfrak{A},\ldots\between\xi,\ \eta,\ \zeta)^2+\lambda\,(a,\ldots\between\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2=0;$$

but we have identically,

$$(\mathfrak{A},\ldots\between\xi,\ \eta,\ \zeta)^2(\mathfrak{A},\ldots\between\xi',\ \eta',\ \zeta')^2-\{(\mathfrak{A},\ldots\between\xi,\ \eta,\ \zeta\between\xi',\ \eta',\ \zeta')\}^2$$
$$=K\,(a,\ldots\between\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2,$$

and the equation thus becomes

$$[K+\lambda\,(\mathfrak{A},\ldots\between\xi',\ \eta',\ \zeta)^2](\mathfrak{A},\ldots\between\xi,\ \eta,\ \zeta)^2-\lambda\,\{(\mathfrak{A},\ldots\between\xi',\ \eta',\ \zeta'\between\xi,\ \eta,\ \zeta)\}^2=0,$$

which is of the form in question.

205. Take $(x',\ y',\ z')$ as the point-coordinates of the centre of inscription, the equation of the axis of inscription is

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z\between x',\ y',\ z')=0;$$

and we may, if we please, exhibit the equation of the inscribed conic in the form

$$(a,\ldots\between x,\ y,\ z)^2(a,\ldots\between x',\ y',\ z')^2\cos^2\theta-\{(a,\ldots\between x,\ y,\ z\between x',\ y',\ z')\}^2=0,$$

where θ is a constant. This equation may also be written

$$(a,\ldots\between x,\ y,\ z)^2(a,\ldots\between x',\ y',\ z')^2\sin^2\theta-(\mathfrak{A},\ldots\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2=0,$$

the two forms being equivalent in virtue of the identity,

$$(a,\ldots\between x,\ y,\ z)^2(a,\ldots\between x',\ y',\ z')^2-\{(a,\ldots\between x',\ y',\ z'\between x,\ y,\ z)\}^2$$
$$=(\mathfrak{A},\ldots\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2.$$

206. The line-coordinates (ξ', η', ζ') of the axis of inscription are

$$ax'+hy'+gz', \quad hx'+by'+fz', \quad gx'+fy'+cz',$$

and we thence deduce the relation

$$(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 = K(a, \ldots \between x', y', z')^2.$$

In order that the form

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 \cos^2\theta - \{(a, \ldots \between x', y', z' \between x, y, z)\}^2 = 0$$

may agree with the originally assumed form

$$(a, \ldots \between x, y, z)^2 + \lambda(\xi' x + \eta' y + \zeta' z)^2,$$

or what is the same thing,

$$(a, \ldots \between x, y, z)^2 + \lambda\{(a, \ldots \between x, y, z \between x', y', z')\}^2 = 0,$$

we must have

$$\lambda = \frac{-1}{(a, \ldots \between x', y', z')^2 \cos^2\theta},$$

which may also be written

$$\lambda = \frac{-K}{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \cos^2\theta},$$

or what is the same thing,

$$K + \lambda(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 - \lambda(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \sin^2\theta = 0;$$

and we thence, by a preceding formula, obtain the line-equation of the inscribed conic, viz.

207. The point-equation being

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 \cos^2\theta - \{(a, \ldots \between x, y, z \between x', y', z')\}^2 = 0,$$

or

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 \sin^2\theta - (\mathfrak{A}, \ldots \between yz'-y'z, zx'-z'x, xy'-x'y)^2 = 0,$$

equivalent in virtue of

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 - \{(a, \ldots \between x, y, z \between x', y', z')\}^2$$
$$= (\mathfrak{A}, \ldots \between yz'-y'z, zx'-z'x, xy'-x'y)^2;$$

then the corresponding forms of the line-equation are

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \sin^2\theta - \{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')\}^2 = 0,$$

and

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \cos^2\theta - K(a, \ldots \between \eta\zeta'-\eta'\zeta, \zeta\xi'-\zeta'\xi, \xi\eta'-\xi'\eta)^2 = 0,$$

equivalent to each other in virtue of the before mentioned identity

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 - \{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')\}^2$$
$$= K(a, \ldots \between \eta\zeta'-\eta'\zeta, \zeta\xi'-\zeta'\xi, \xi\eta'-\xi'\eta)^2.$$

208. Write for shortness

$$\begin{aligned}(a, \ldots \between x, y, z)^2 &= 00,\\ (a, \ldots \between x, y, z \between x', y', z) &= 01 = 10,\\ \&c.,\end{aligned}$$

then we have identically,

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix} = K \begin{vmatrix} x\,, & y\,, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}^2$$

and if the determinant on the right hand vanishes, that is if (x, y, z), (x', y', z'), (x'', y'', z'') are points in a line, then we have

$$\begin{vmatrix} 00, & 01, & 02 \\ 10, & 11, & 12 \\ 20, & 21, & 22 \end{vmatrix} = 0,$$

an equation, which, as already remarked, is equivalent to

$$\cos^{-1}\frac{01}{\sqrt{00}\sqrt{11}} + \cos^{-1}\frac{12}{\sqrt{11}\sqrt{22}} = \cos^{-1}\frac{02}{\sqrt{00}\sqrt{22}}.$$

The foregoing investigations in relation to the inscribed conic are given for the sake of the application thereof to the theory of distance, and it has been necessary to make use of analytical formulæ of some complexity which are introduced out of their natural place.

On the Theory of Distance, Nos. 209 to 229.

209. I return to the geometry of one dimension. Imagine in the line or *locus in quo* of the range of points, a point-pair, which I term the Absolute. Any point-pair whatever may be considered as inscribed in the Absolute, the centre and axis of inscription being the sibiconjugate points of the involution formed by the points of the given point-pair and the points of the Absolute; the centre and axis of inscription *quà* sibiconjugate points are harmonics with respect to the Absolute. A point-pair considered as thus inscribed in the Absolute is said to be a *point-pair circle*, or simply a *circle*; the centre of inscription and the axis of inscription are termed the centre and the axis. Either of the two sibiconjugate points may be considered as the centre, but the selection when made must be adhered to. It is proper to notice that, given the centre and one point of the circle, the other point of the circle is determined in a unique manner. In fact the axis is the harmonic of the centre in respect to the Absolute, and then the other point is the harmonic of the given point in respect to the centre and axis.

210. As a definition, we say that the two points of a circle are equidistant from the centre. Now imagine two points P, P'; and take the point P'' such that P, P'' are a circle having P' for its centre; take in like manner the point P''' such that

P', P''' are a circle having P'' for its centre; and so on: and again in the opposite direction, a point $P^{\backslash}$ such that P', $P^{\backslash}$ are a circle having P for its centre; a point $P^{\backslash\backslash}$ such that P, $P^{\backslash\backslash}$ are a circle having $P^{\backslash}$ for its centre, and so on. We have a series of points... $P^{\backslash\backslash}$, $P^{\backslash}$, P, P', P'', ... at equal intervals of distance: and if we take the points P, P' indefinitely near to each other, then the entire line will be divided into a series of equal infinitesimal elements; the number of these elements included between any two points measures the distance of the two points. It is clear that, according to the definition, if P, P', P'' be any three points taken in order, then

$$\text{Dist.}\,(P,\ P') + \text{Dist.}\,(P',\ P'') = \text{Dist.}\,(P,\ P''),$$

which agrees with the ordinary notion of distance.

211. To show how the foregoing definition leads to an analytical expression for the distance of two points in terms of their coordinates, take

$$(a,\ b,\ c \between x,\ y)^2 = 0$$

for the equation of the Absolute. The equation of a circle having the point $(x',\ y')$ for its centre is

$$(a,\ b,\ c \between x,\ y)^2 (a,\ b,\ c \between x',\ y')^2 \cos^2\theta - [(a,\ b,\ c \between x,\ y \between x',\ y')]^2 = 0;$$

and consequently if $(x,\ y)$, $(x'',\ y'')$ are the two points of the circle, then

$$\frac{(a,\ b,\ c \between x,\ y \between x',\ y')}{\sqrt{(a,\ b,\ c \between x,\ y)^2}\,\sqrt{(a,\ b,\ c \between x',\ y')^2}} = \frac{(a,\ b,\ c \between x',\ y' \between x'',\ y'')}{\sqrt{(a,\ b,\ c \between x',\ y')^2}\,\sqrt{(a,\ b,\ c \between x'',\ y'')^2}},$$

an equation which expresses that the points $(x'',\ y'')$ and $(x,\ y)$ are equidistant from the point $(x',\ y')$. It is clear that the distance of the points $(x,\ y)$ and $(x',\ y')$ must be a function of

$$\frac{(a,\ b,\ c \between x,\ y \between x',\ y')}{\sqrt{(a,\ b,\ c \between x,\ y)^2}\,\sqrt{(a,\ b,\ c \between x',\ y')^2}},$$

and the form of the function is determined from the before-mentioned property, viz. if P, P', P'' be any three points taken in order, then

$$\text{Dist.}\,(P,\ P') + \text{Dist.}\,(P',\ P'') = \text{Dist.}\,(P,\ P'').$$

This leads to the conclusion that the distance of the points $(x,\ y)$, $(x',\ y')$ is equal to a multiple of the arc having for its cosine the last-mentioned expression (see *ante*, No. 168); and we may in general assume that the distance is equal to the arc in question, viz. that the distance is

$$\cos^{-1}\frac{(a,\ b,\ c \between x,\ y \between x',\ y')}{\sqrt{(a,\ b,\ c \between x,\ y)^2}\,\sqrt{(a,\ b,\ c \between x',\ y')^2}},$$

or, what is the same thing,

$$\sin^{-1}\frac{(ac-b^2)\,(xy'-x'y)}{\sqrt{(a,\ b,\ c \between x,\ y)^2}\,\sqrt{(a,\ b,\ c \between x',\ y')^2}}.$$

It follows that the two forms

$$(a,\ b,\ c \between x,\ y)^2 (a,\ b,\ c \between x',\ y')^2 \cos^2\theta - \{(a,\ b,\ c \between x,\ y \between x',\ y')\}^2 = 0,$$
$$(a,\ b,\ c \between x,\ y)^2 (a,\ b,\ c \between x',\ y')^2 \sin^2\theta - (ac - b^2)(xy' - x'y) \quad = 0,$$

of the equation of a circle, each of them express that the distances of the two points from the centre are respectively equal to the arc θ; or, if we please, that θ is the radius of the circle.

212. When $\theta = 0$, we have

$$xy' - x'y = 0,$$

an equation which expresses that $(x,\ y)$ and $(x',\ y')$ are one and the same point. When $\theta = \frac{1}{2}\pi$, we have

$$(a,\ b,\ c \between x,\ y \between x',\ y') = 0,$$

an equation which expresses that the points $(x,\ y)$ and $(x',\ y')$ are harmonics with respect to the Absolute. The distance between any two points harmonics with respect to the Absolute is consequently a quadrant, and such points may be said to be quadrantal to each other. The quadrant is the unit of distance.

213. The foregoing is the general case, but it is necessary to consider the particular case where the Absolute is a pair of coincident points. The harmonic of any point whatever in respect to the Absolute is here a point coincident with the Absolute itself: the definition of a circle is consequently simplified; viz. any point-pair whatever may be considered as a circle having for its centre the harmonic of the Absolute with respect to the point-pair; we may, as before, divide the line into a series of equal infinitesimal elements, and the number of elements included between any two points measures the distance between the two points. As regards the analytical expression, in the case in question $ac - b^2$ vanishes, or the distance is given as the arc to an evanescent sine. Reducing the arc to its sine and omitting the evanescent factor, we have a finite expression for the distance. Suppose that the equation of the Absolute is

$$(qx - py)^2 = 0,$$

or what is the same thing, let the Absolute (treated as a single point) be the point $(p,\ q)$, then we find for the distance of the points $(x,\ y)$ and $(x',\ y')$ the expression

$$\frac{xy' - x'y}{(qx - py)(qx' - py')};$$

or, introducing an arbitrary multiplier,

$$\frac{(q\alpha - p\beta)(xy' - x'y)}{(qx - py)(qx' - py')},$$

which is equal to

$$\frac{\beta x - \alpha y}{qx - py} - \frac{\beta x' - \alpha y'}{qx' - py'}.$$

It is hardly necessary to remark, that in the present case the notion of the quadrantal relation of two points has altogether disappeared, and that the unit of distance is arbitrary.

214. Passing now to geometry of two dimensions, we have here to consider a certain conic, which I call the Absolute. Any line whatever determines with the Absolute (cuts it in) two points which are the Absolute in regard to such line considered as a space of one dimension, or *locus in quo* of a range of points, and in like manner any point whatever determines with the Absolute (has for tangents of the Absolute through the point) two lines which are the Absolute in regard to such point considered as a space of one dimension, or *locus in quo* of a pencil of lines. The foregoing theory for geometry of one dimension establishes the notion of distance as regards each of these ranges and pencils considered apart by itself; in order to bring the different ranges and pencils into relation with each other, it is necessary to assume that the quadrant which is the unit of distance for these several systems respectively, is one and the same distance for each system (of course, when, as in the analytical theory, we actually represent the quadrant by the ordinary symbol $\frac{1}{2}\pi$, the above assumption is tacitly made; but substituting the thing signified for the definition, and looking at the quadrant merely as the distance between two points, or as the case may be, lines, harmonically related to the point-pair, or as the case may be, line-pair, constituting the Absolute, the assumption is at once seen to be an assumption, and it needs to be made explicitly). But the assumption being made, the foregoing theory of distance in geometry of one dimension enables the comparison not only of the distances of points upon different lines, or of lines through different points, but of the distances of points on a line and of lines through a point. The pole of any line in relation to the Absolute may be termed simply the pole, and in like manner the polar of any line in relation to the Absolute may be termed simply the polar, and we have the theorem that the distance of two points or lines is equal to the distance of their polars or poles, or what is the same thing, that the distance of two poles and the distance of the two corresponding polars are equal. And we may, as a definition, establish the notion of the distance of a point from a line, viz. it is the complement of the distance of the polar of the point from the line, or what is the same thing, the complement of the distance of the point from the pole of the line. The distance of a pole and polar is therefore the complement of zero, that is, it is the quadrant.

215. It has, by means of the preceding assumption as to the quadrant, been possible to establish the notion of distance, without the assistance of the circle, but this figure must now be considered. A conic inscribed in the Absolute is termed a circle; the centre of inscription (or point of intersection of the common tangents) and the axis of inscription (or line of junction of the common ineunts) are the centre and axis of the circle. All the points of a circle are equidistant from the centre; all the tangents are equidistant from the axis, and this distance is the complement of the former distance.

216. These properties of the circle lead immediately to the analytical expressions for the distances of points or lines in terms of the coordinates. In fact, take

$$(a, b, c, f, g, h \between x, y, z)^2 = 0$$

for the point-equation of the Absolute; its line-equation will be

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2 = 0.$$

The point-equation of the circle having the point (x', y', z') for its centre, is

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 \cos^2\theta - \{(a, \ldots \between x, y, z \between x', y', z')\}^2 = 0,$$

or

$$(a, \ldots \between x, y, z)^2 (a, \ldots \between x', y', z')^2 \sin^2\theta - (\mathfrak{A}, \ldots \between yz' - y'z, zx' - z'x, xy' - x'y)^2 = 0,$$

from which (by the same reasoning as for the case of geometry of one dimension) it follows that the distance of the points (x, y, z), (x', y', z') is

$$\cos^{-1} \frac{(a, \ldots \between x, y, z \between x', y', z')}{\sqrt{(a, \ldots \between x, y, z)^2}\sqrt{(a, \ldots \between x', y', z')^2}},$$

or what is the same thing,

$$\sin^{-1} \frac{\sqrt{(\mathfrak{A}, \ldots \between yz' - y'z, zx' - z'x, xy' - x'y)^2}}{\sqrt{(a, \ldots \between x, y, z)^2}\sqrt{(a, \ldots \between x', y', z')^2}};$$

and it appears from the cosine formula (see *ante*, No. 208), that if P, P', P'' be points on the same line, then we have, as we ought to have,

$$\text{Dist. } (P, P') + \text{Dist. } (P', P'') = \text{Dist. } (P, P'').$$

217. In like manner, the line-equation of the same circle, the line-coordinates of the axis being (ξ', η', ζ'), is

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \sin^2\theta - \{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')\}^2 = 0,$$

or

$$(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2 (\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2 \cos^2\theta - K(a, \ldots \between \eta\zeta' - \eta'\zeta, \zeta\xi' - \zeta'\xi, \xi\eta' - \xi'\eta)^2 = 0,$$

from which it follows that the distance of the lines (ξ, η, ζ) and (ξ', η', ζ') is

$$\cos^{-1} \frac{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')}{\sqrt{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2}\sqrt{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2}},$$

or what is the same thing

$$\sin^{-1} \frac{\sqrt{K(a, \ldots \between \eta\zeta' - \eta'\zeta, \zeta\xi' - \zeta'\xi, \xi\eta' - \xi'\eta)^2}}{\sqrt{(\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2}\sqrt{(\mathfrak{A}, \ldots \between \xi, \eta', \zeta')^2}}.$$

218. And we may from the first formula of either set, deduce for the distance of the point (x, y, z) and the line (ξ', η', ζ'), the expression

$$\sin^{-1}\frac{\sqrt{K}\,(\xi' x + \eta' y + \zeta' z)}{\sqrt{(a, \ldots \between x, y, z)^2}\,\sqrt{(\mathfrak{A}, \ldots \between \xi', \eta', \zeta')^2}},$$

as may be easily seen by writing $\mathfrak{A}\xi' + \mathfrak{B}\eta' + \mathfrak{C}\zeta', \ldots$ for x', y', z', or $ax + hy + gz, \ldots$ for ξ, η, ζ, and putting $\sin^{-1}$ for $\cos^{-1}$.

219. It may be noticed that there are certain lines, viz. the tangents of the Absolute, in regard to which, considered as a space of one dimension, the Absolute is a pair of coincident points; and in like manner certain points, viz. the ineunts of the Absolute, in regard to which, considered as a space of one dimension, the Absolute is a pair of coincident lines.

220. We may, in particular, suppose that the Absolute, instead of being a proper conic, is a pair of points. The line through the two points may be called the Absolute line; such line is to be considered as a pair of coincident lines. Any point whatever determines with the Absolute, two lines, viz. the lines joining the point with the two points of the Absolute; this line-pair is the Absolute for the point considered as a space of one dimension or *locus in quo* of a pencil of lines, and the theory of the distances of lines through a point is therefore precisely the same as in the general case. But any line whatever determines with the Absolute (meets the Absolute line in) a pair of coincident points, which pair of coincident points is the Absolute in regard to such line considered as a space of one dimension or *locus in quo* of a range of points, and the theory of the distance of points on a line is therefore the theory before explained for this special case. But we cannot, in the same way as before, compare the distances of points upon different lines, since we have not in the present case the quadrant as a unit of distance. The comparison must be made by means of the circle, viz. in the present case any conic passing through the two points of the Absolute is termed a circle, and the point of intersection of the tangents to the circle at the two points of the Absolute (or what is the same thing, the pole of the Absolute line in respect to the circle) is the centre of the circle. The Absolute line itself may, if it is necessary to do so, be considered as the axis of the circle. It is assumed that the points of the circle are all of them equidistant from the centre, and by this assumption we are enabled to compare distances upon different lines. In fact we may, by a construction precisely similar to that of *Euclid*, Book I. Prop. II., from a given point A draw a finite line equal to a given finite line BC, and thence also upon a given line through A, determine the finite line AD equal to the given finite line BC. Since the unit of distance for points on a line is arbitrary, we cannot of course compare the distances of points with the distances of lines. The distance of a point from a line does, however, admit of comparison with the distance of two points; we have only to assume as a definition that the distance of a point from a line is the distance of the point from the point of intersection of the line with the quadrantal line through the point.

221. As regards the analytical theory, suppose that the point-coordinates of the two points of the Absolute are (p, q, r), (p_0, q_0, r_0), then the line-equation of the Absolute is

$$2(p\xi + q\eta + r\zeta)(p_0\xi + q_0\eta + r_0\zeta) = 0;$$

so that we have $\mathfrak{A} = 2pp_0$, $\mathfrak{B} = 2qq_0$, $\mathfrak{C} = 2rr_0$, $\mathfrak{F} = qr_0 + rq_0$, $\mathfrak{G} = rp_0 + pr_0$, $\mathfrak{H} = pq_0 + qp_0$, and thence $K = 0$; but

$$K(a, b, c, f, g, h \between x, y, z)^2 = \begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}^2$$

where obviously

$$\begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix} = 0$$

is the equation of the Absolute line.

222. The expression for the distance of the two points (x, y, z), (x', y', z') is given as the arc to an evanescent sine; but reducing the arc to its sine, and omitting the evanescent factor, the resulting expression is

$$\sqrt{2}\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ p, & q, & r \end{vmatrix}\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ p_0, & q_0, & r_0 \end{vmatrix} \div \begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}\begin{vmatrix} x', & y', & z' \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}$$

and the expression for the distance of the two lines (ξ, η, ζ), (ξ', η', ζ') is

$$\cos^{-1}\frac{(p\xi + q\eta + r\zeta)(p_0\xi' + q_0\eta' + r_0\zeta') + (p\xi' + q\eta' + r\zeta')(p_0\xi + q_0\eta + r_0\zeta)}{\sqrt{2(p\xi + q\eta + r\zeta)(p_0\xi + q_0\eta + r_0\zeta)}\,\sqrt{2(p\xi' + q\eta' + r\zeta')(p_0\xi' + q_0\eta' + r_0\zeta')}},$$

or, what is the same thing,

$$\sin^{-1}\frac{(qr_0 - rq_0)(\eta\zeta' - \eta'\zeta) + (rp_0 - pr_0)(\zeta\eta' - \zeta'\eta) + (pq_0 - qp_0)(\xi\eta' - \xi'\eta)}{\sqrt{2(p\xi + q\eta + r\zeta)(p_0\xi + q_0\eta + r_0\zeta)}\,\sqrt{2(p\xi' + q\eta' + r\zeta')(p_0\xi' + q_0\eta' + r_0\zeta')}};$$

and finally, the expression for the distance of the point x, y, z from the line (ξ', η', ζ'), reducing the arc to its sine and omitting the evanescent factor, is

$$(\xi'x + \eta'y + \zeta'z) \div \begin{vmatrix} x, & y, & z \\ p, & q, & r \\ p_0, & q_0, & r_0 \end{vmatrix}\sqrt{2(p\xi' + q\eta' + r\zeta')(p_0\xi' + q_0\eta' + r_0\zeta')}.$$

223. If in the above formula we put $(p, q, r) = (1, i, 0)$, $(p_0, q_0, r_0) = (1, -i, 0)$, where as usual $i = \sqrt{-1}$, then the line-equation of the Absolute is $\xi^2 + \eta^2 = 0$, or what is the same thing, the Absolute consists of the two points in which the line $z = 0$ intersects the line-pair $x^2 + y^2 = 0$; the last-mentioned line-pair, as passing through the Absolute, is by definition a circle; it is in fact the circle radius zero, or an evanescent circle. If we put also the coordinate z equal to unity, then the preceding assumption as to the coordinates of the points of the Absolute must be understood to mean only $x : y : 1 = 1 : i : 0$, or $1 : -i : 0$; that is, we must have x and y infinite, and, as before, $x^2 + y^2 = 0$, or in other words, the Absolute will consist of the points of intersection of the line infinity by the evanescent circle $x^2 + y^2 = 0$. With the values in question,

224. The expression for the distance of the points (x, y) and (x', y') is

$$\sqrt{(x - x')^2 + (y - y')^2};$$

that for the distance of the lines (ξ, η, ζ) and (ξ', η', ζ') is

$$\cos^{-1} \frac{\xi\xi' + \eta\eta'}{\sqrt{\xi^2 + \eta^2}\sqrt{\xi'^2 + \eta'^2}}$$

$$= \sin^{-1} \frac{\xi\eta' - \xi'\eta}{\sqrt{\xi^2 + \eta^2}\sqrt{\xi'^2 + \eta'^2}},$$

which may also be written

$$= \tan^{-1} \frac{\xi}{\eta} - \tan^{-1} \frac{\xi'}{\eta'};$$

and the expression for the distance of the point (x, y) from the line (ξ', η', ζ') is

$$\frac{\xi' x + \eta' y + \zeta'}{\sqrt{\xi'^2 + \eta'^2}},$$

which are obviously the formulæ of ordinary plane geometry, (x, y) being ordinary rectangular coordinates.

225. The general formulæ suffer no *essential* modification, but they are greatly simplified in form by taking for the point-equation of the Absolute

$$x^2 + y^2 + z^2 = 0,$$

or, what is the same, for the line-equation

$$\xi^2 + \eta^2 + \zeta^2 = 0.$$

In fact, we then have for the expression of the distance of the points (x, y, z), $(x', y' z')$,

$$\cos^{-1} \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}\sqrt{x'^2 + y'^2 + z'^2}};$$

for that of the lines (ξ, η, ζ), (ξ', η', ζ'),

$$\cos^{-1}\frac{\xi\xi'+\eta\eta'+\zeta\zeta'}{\sqrt{\xi^2+\eta^2+\zeta^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2}};$$

and for that of the point (x, y, z) and the line (ξ', η', ζ'),

$$\sin^{-1}\frac{\xi'x+\eta'y+\zeta'z}{\sqrt{x^2+y^2+z^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2}}.$$

226. Suppose (x, y, z) are ordinary rectangular coordinates in space satisfying the condition

$$x^2+y^2+z^2=1,$$

the point having (x, y, z) for its coordinates will be a point on the surface of the sphere, and (the last-mentioned equation always subsisting) the equation $\xi x+\eta y+\zeta z=0$ will be a great circle of the sphere; and since we are only concerned with the ratios of ξ, η, ζ, we may also assume $\xi^2+\eta^2+\zeta^2=1$. We may of course retain in the formulæ the expressions $x^2+y^2+z^2$ and $\xi^2+\eta^2+\zeta^2$, without substituting for these the values unity, and it is in fact convenient thus to preserve all the formulæ in their original forms. We have thus a system of spherical geometry; and it appears that the Absolute in such system is the (spherical) conic, which is the intersection of the sphere with the concentric cone or evanescent sphere $x^2+y^2+z^2=0$. The circumstance that the Absolute is a proper conic, and not a mere point-pair, is the real ground of the distinction between spherical geometry and ordinary plane geometry, and the cause of the complete duality of the theorems of spherical geometry.

227. I have, in all that has preceded, given the analytical theory of distance along with the geometrical theory, as well for the purpose of illustration, as because it is important to have the analytical expression of a distance in terms of the coordinates; but I consider the geometrical theory as perfectly complete in itself: the general result is as follows, viz. assuming in the plane (or space of geometry of two dimensions) a conic termed the Absolute, we may by means of this conic, by descriptive constructions, divide any line or range of points whatever, and any point or pencil of lines whatever, into an infinite series of infinitesimal elements, which are (as a definition of distance) assumed to be equal; the number of elements between two points of the range or two lines of the pencil, measures the distance between the two points or lines; and by means of the quadrant, as a distance which exists as well with respect to lines as points, we are enabled to compare the distance of two lines with that of two points; and the distance of a point and a line may be represented indifferently as the distance of two points, or as the distance of two lines.

228. In ordinary spherical geometry, the general theory undergoes no modification whatever; the Absolute is an actual conic, the intersection of the sphere with the concentric evanescent sphere.

229. In ordinary plane geometry, the Absolute degenerates into a pair of points, viz. the points of intersection of the line infinity with any evanescent circle, or what is the same thing, the Absolute is the two circular points at infinity. The general theory is consequently modified, viz. there is not, as regards points, a distance such as the quadrant, and the distance of two lines cannot be in any way compared with the distance of two points; the distance of a point from a line can be only represented as a distance of two points.

230. I remark in conclusion, that, *in my own point of view*, the more systematic course in the present introductory memoir on the geometrical part of the subject of quantics, would have been to ignore altogether the notions of distance and metrical geometry; for the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the Absolute. The original figure might comprise a conic; for instance, we might consider the properties of the figure formed by two or more conics, and we are then in the region of pure descriptive geometry: we pass out of it into metrical geometry by fixing upon a conic of the figure as a standard of reference and calling it the Absolute. Metrical geometry is thus a part of descriptive geometry, and descriptive geometry is *all* geometry, and reciprocally; and if this be admitted, there is no ground for the consideration, in an introductory memoir, of the special subject of metrical geometry; but as the notions of distance and of metrical geometry could not, without explanation, be thus ignored, it was necessary to refer to them in order to show that they are thus included in descriptive geometry.

NOTES AND REFERENCES.

101. No. V. of this paper gives a correction of a formula (18) in the paper 8, On Lagrange's Theorem.

102. I refer to this paper in my "Note on Riemann's paper 'Versuch einer allgemeinen Auffassung der Integration und Differentiation,' *Werke*, pp. 331—344." *Math. Ann.* t. XVI. (1880), pp. 81—82, for the sake of pointing out the connexion which it has with this paper of Riemann's (contained, as the Editors remark, in a MS. of his student time dated 14 Jan. 1847, and probably never intended for publication): the idea is in fact the same, Riemann considered a function of $x+h$ expanded in a doubly infinite, necessarily divergent, series of integer or fractional powers of h, according to an assigned law: and he thence deduces a theory of fractional differentiation.

114. This Memoir on Steiner's extension of Malfatti's problem is referred to by Clebsch in the paper "Anwendung der elliptischen Functionen auf ein Problem der Geometrie des Raumes," *Crelle*, t. LIII. (1857), pp. 292—308: it is there shown that my fundamental equations, p. 67, are the algebraical integrals of a system of equations

$$\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0,\quad \frac{dz}{\sqrt{Z'}}+\frac{dx}{\sqrt{X'}}=0,\quad \frac{dx}{\sqrt{X''}}+\frac{dy}{\sqrt{Y''}}=0,$$

the integrals of which become comparable when the quartic functions under the square roots differ only by constant factors; and expressing that this is so, he obtains the relations which I assumed to exist between the coefficients α, β, γ, δ, &c., under which the equations admit of solution by quadratics only. And he is thereby led to reduce the problem, not to the foregoing system of fundamental equations, but to other equations connecting themselves with the *usual* form of the Addition-theorem; and with a view thereto to develope a new solution of the Problem.

115, 116. The theory is further developed in my Memoir "On the Porism of the in-and-circumscribed Polygon," *Phil. Trans.* t. CLI., for 1861.

119. I attach some value to the process here explained: the most simple application is that referred to at the end of the paper, for the factorial binomial theorem; to multiply $m+n$ by $m+n-1$, we multiply the m by $(m-1)+n$, and the n by $m+(n-1)$, thus obtaining the result in the form $m(m-1)+2mn+n(n-1)$, and so in other cases.

121. The papers and works relating to the Question are

1. Boole. Proposed Question in the Theory of Probabilities, *Camb. and Dubl. Math. Jour.* t. VI. (1851), p. 286.

2. Cayley. 121, Note on a Question in the Theory of Probabilities, *Phil. Mag.* t. VI. (1853), p. 259.

3. Boole. Solution of a Question in the Theory of Probabilities, *Phil. Mag.* t. VII. (1854), pp. 29—32.

4. Boole. An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities, 8vo. London and Cambridge, 1854 (see in particular pp. 321—326).

5. Wilbraham. On the Theory of Chances developed in Prof. Boole's Laws of Thought, *Phil. Mag.* t. VII. (1854), pp. 465—476.

6. Dedekind. Bemerkungen zu einer Aufgabe der Wahrscheinlichkeitsrechnung, *Crelle*, t. L. (1855), pp. 268—271;

viz. Boole proposed the question in 1, I gave my solution in 2, Boole objected to it in 3, and gave without explanation or demonstration his solution, referring to his then forthcoming work 4, which contains (pp. 321—326) his investigation. Wilbraham in 5 defended my solution, and criticised Boole's: and finally Dedekind in 6 (which does not refer to 4 or 5) completed my solution, by determining the sign of a radical, and establishing between the data, as conditions of a possible experience, the relations $p-\beta q$ and $q-\alpha p$ neither of them negative.

I remark that although Boole in 1, 3, and 4 speaks throughout of "causes," yet it would seem that he rather means "concomitant events": I think that in his point of view the more accurate enunciation of the question would be—The probabilities of two events A and B are α and β respectively; the probability that if the event A present itself the event E will accompany it is p, and the probability that if the event B present itself the event E will accompany it is q; moreover it is assumed that the event E cannot appear in the absence of both the events A and B: required the probability of the event E.

He makes no assumption as to the independence *inter se* of A, and B: and moreover, in thus regarding A and B as events instead of causes, there is no room for regarding E as a consequence of one or the other of A and B, or of both of them.

In my solution I regard A and B as causes: I assume that they are independent causes; and further that either or both of them may act efficiently so as to

produce the event E, but that the event E cannot happen unless at least one of them act efficiently, viz. it cannot happen in consequence of the conjoint separately inefficient action of the two causes. On these assumptions it appears to me that my solution, as completed by Dedekind, is correct. This would not preclude the correctness of Boole's solution, if according to what precedes we consider it as the solution of a different question: but I am unable to understand it.

I resume my own solution, completing it according to Dedekind. I write with him u instead of ρ for the required probability of the event E; the equations of the text thus are

$$p=\lambda+(1-\lambda)\mu\beta,\quad q=\mu+(1-\mu)\lambda\alpha;\quad u=\lambda\alpha+\mu\beta-\lambda\mu\alpha\beta,$$

and we thence deduce

$$u-\beta q=(1-\beta)\lambda\alpha,\quad u-\alpha p=(1-\alpha)\mu\beta;$$

and then eliminating λ, μ, we find

$$u=\frac{u-\beta q}{1-\beta}+\frac{u-\alpha p}{1-\alpha}-\frac{(u-\beta q)(u-\alpha p)}{(1-\beta)(1-\alpha)},$$

or as this equation may be written

$$u^2-u(1-\alpha\beta+\alpha p+\beta q)+(1-\beta)\alpha p+(1-\alpha)\beta q+\alpha\beta pq=0;$$

say we have

$$u=\tfrac{1}{2}(1-\alpha\beta+\alpha p+\beta q-\rho),$$

where

$$\begin{aligned}\rho^2&=(1-\alpha\beta+\alpha p+\beta q)^2-4(1-\beta)\alpha p-4(1-\alpha)\beta q-4\alpha\beta pq,\\&=(1-2\alpha+\alpha\beta+\alpha p-\beta q)^2+4\alpha(1-\alpha)(1-\beta)(1-p),\\&=(1-2\beta+\alpha\beta-\alpha p+\beta q)^2+4\beta(1-\beta)(1-\alpha)(1-q),\\&=(1-\alpha\beta+\alpha p-\beta q)^2-4\alpha(1-\beta)(p-\beta q),\\&=(1-\alpha\beta-\alpha p+\beta q)^2-4\beta(1-\alpha)(q-\alpha p),\end{aligned}$$

and hence also

$$\lambda=\frac{\tfrac{1}{2}(1-\alpha\beta+\alpha p-\beta q-\rho)}{(1-\beta)\alpha},$$

$$\mu=\frac{\tfrac{1}{2}(1-\alpha\beta-\alpha p+\beta q-\rho)}{(1-\alpha)\beta}.$$

Here p, q, α, β, as probabilities, are none of them negative or greater than 1; p is the probability that, A acting, E will happen; and βq is the probability that B will act and E happen. But if A act, then even if B does not act, E may happen, or B may act and E happen, that is p is greater than or at least equal to βq, say $p-\beta q$ is not negative. And similarly $q-\alpha p$ is not negative. We thus have as conditions of a possible experience, $p-\beta q$ and $q-\alpha p$ neither of them negative.

The formulæ show that ρ^2 is real; and then further, taking for ρ its positive value, it at once appears that we have u, λ, μ no one of them negative or greater than 1, viz. the values are such as these quantities, as probabilities, ought each of them to have: and we have thus a real solution.

Boole in 1 after remarking that the quadratic equation in u may be written in the form

$$\frac{(1-\alpha p'-u)(1-\beta q'-u)}{1-u}=\alpha'\beta' \quad (p'=1-p, \text{ \&c.})$$

says that this is certainly erroneous; for in the particular case $p=1$, $q=0$ it gives $u=1$ or $u=\alpha(1-\beta)$, whereas the value should be $u=\alpha$. But observe that $p=1$, $q=0$, give $q-\alpha p, =-\alpha$, a negative value, so that the solution does not apply. If we further examine the meaning, A is a cause such that if it act then $(p=1)$ the event is sure to happen; and B is a cause (?) such that if it act then $(q=0)$ the event is sure not to happen; this is self-contradictory unless we make the new assumption that the causes A and B cannot both act. It is remarkable that even in this case my solution gives the plausible result $u=\alpha(1-\beta)$, viz. the probability of the event is the product of the probabilities of A acting, and B not acting.

In further illustration, and at the same time to examine Boole's solution, I write as follows:

	Wilbraham.	Boole.	Cayley.
ABE	ξ	$xyst$	$\alpha\beta(1-\lambda'\mu')$
ABE'	ξ'	$xys't'$	$\alpha\beta\lambda'\mu'$
$A'BE$	η	$x'yst$	$\alpha'\beta\mu$
$A'BE'$	η'	$x'ys't'$	$\alpha'\beta\mu'$
$AB'E$	ζ	$xy'st'$	$\alpha'\beta'\lambda$
$AB'E'$	ζ'	$xy's't'$	$\alpha\beta'\lambda'$
$A'B'E$	0	0	0
$A'B'E'$	σ'	$x'y's't'$	$\alpha'\beta'$

,

where in the first column the accent denotes negation: ABE means that the events A, B, E all happen, ABE' that A and B each happen, E' does not happen, and so for the other symbols. And in like manner in the third and fourth columns, where the unaccented letters denote probabilities, an accented letter is the probability of the contrary event, $x'=1-x$, &c.

By hypothesis E cannot happen unless either A or B happen, that is Prob. $A'B'E=0$, or writing $A'B'E$ for the probability (and so in other cases) say $A'B'E=0$. And I then (with Wilbraham) denote the probabilities of the other seven combinations of events by ξ, ξ', η, η', ζ, ζ' and σ'; and (as before) the required probability of the event E by u.

The data of the Problem are $1=1$, $A=\alpha$, $B=\beta$, $AE=\alpha p$, $BE=\beta q$, and we have thence to find $E=u$, where on the left-hand side of the first equation 1 means $ABE+ABE'+\text{\&c.}=\xi+\xi'+\eta+\eta'+\zeta+\zeta'+\sigma'$, and similarly A means

$$ABE+ABE'+AB'E+AB'E', =\xi+\xi'+\zeta+\zeta', \text{ \&c.};$$

we thus have

$$\begin{aligned}
&\xi+\xi'+\eta+\eta'+\zeta+\zeta'+\sigma'=1\ , \\
&\xi+\xi'+\zeta+\zeta' &&=\alpha\ , \\
&\xi+\xi'+\eta+\eta' &&=\beta\ , \\
&\xi+\zeta &&=\alpha p, \\
&\xi+\eta &&=\beta q, \\
&\xi+\eta+\zeta &&=\ \ u,
\end{aligned}$$

six equations for the determination of the eight quantities ξ, ξ', η, η', ζ, ζ', σ', and u.

For the determination of u, it is therefore necessary to find or assume two more equations: in my solution this is in effect done by giving to ξ, ξ', η, η', ζ, ζ', σ' the values in the fourth column, values which satisfy the six equations, and establish the two additional relations

$$\frac{\xi'}{\eta'}=\frac{\zeta'}{\sigma'}, \quad \frac{\xi+\xi'}{\eta+\eta'}=\frac{\zeta+\zeta'}{\sigma'},$$

or, as these may be written,

$$\frac{A\,BE'}{A'BE'}=\frac{A\,B'E'}{A'B'E'}, \quad \frac{A\,B}{A'B}=\frac{A\,B'}{A'B'};$$

these then are assumptions implicitly made in my solution; they amount to this, that the events A, B are treated as independent, *first* in the case where E does *not* happen; secondly in the case where it is not observed whether E does or does not happen.

Boole in his solution introduces what he calls logical probabilities (but what these mean, I cannot make out): viz. these are Prob. $A=x$, or say simply $A=x$; and similarly, $B=y$, $AE=s$, $BE=t$; then in the case ABE we have A, B, AE, BE, and the logical probability is taken to be $xyst$; and we obtain in like manner the other terms of the third column. And then taking ξ, ξ', η, η', ζ, ζ', σ' to be proportional to the terms of the third column, say $V\xi=xyst$, &c. and substituting in the six equations, we have six equations for the determination of x, y, s, t, V, u, and we thus arrive at the value of the required probability u.

But the assumed values of ξ, ξ', &c. give further

$$\frac{\xi}{\eta}=\frac{\zeta}{\sigma'}, \quad \frac{\xi'}{\eta'}=\frac{\zeta'}{\sigma'}, \text{ that is } \frac{A\,BE}{A'BE}=\frac{A\,B'E}{A'B'E'} \text{ and } \frac{A\,BE'}{A'BE'}=\frac{A\,B'E'}{A'B'E'},$$

which are assumptions made in Boole's solution. Wilbraham remarks that the second of these assumed equations, though perfectly arbitrary, is perhaps not unreasonable: it asserts that in those cases where E does not happen, the relation of independence exists between A and B, that is, provided E does not happen, A is as likely to happen whether B happens or does not happen. But that the first of these equations appears to him not only arbitrary, but eminently anomalous: no one (he thinks) can contend that it is either deduced from the data of the problem, or that the mind by the operation of any law of thought recognises it as a necessary or even a reasonable assumption.

To complete Boole's solution: the equations easily give

$$\frac{s't x' y}{u-\alpha p}=\frac{s t' x y'}{u-\beta q}=\frac{s't'}{1-u}=V;$$

and

$$\frac{s't'x'}{1-\alpha p'-u}=\frac{s't'y'}{1-\beta q'-u}=\frac{stxy}{\alpha p+\beta q-u}=V;$$

and multiplying together the first three values, and also the second three values, we have in each case the same numerator $ss'^2 tt'^2 xx'yy'$, and we thus obtain the equation

$$(u-\alpha p)(u-\beta q)(1-u)-(1-\alpha p'-u)(1-\beta p'-u)(\alpha p+\beta q-u)=0,$$

which, the term in u^2 disappearing, is a quadric equation; it is in fact

$$u^2(-1+\alpha p'+\beta q')+u\{1+\alpha(p-p')+\beta(q-q')-\alpha^2pp'-\beta^2qq'+\alpha\beta(-1+2p'q')\}$$
$$+\{-\alpha p-\beta q+\alpha^2pp'+\beta^2qq'+\alpha\beta(1-p'q')-(\alpha p+\beta q)\alpha\beta p'q'\}=0;$$

or, what is more simple, if we write with Boole $\alpha p=a$, $\beta q=b$, $1-\alpha p'=a'$, $1-\beta q'=b'$, $\alpha p+\beta y=c'$, then the equation is $(u-a)(u-b)(1-u)-(a'-u)(b'-u)(c'-u)=0$, that is

$$(1-a'-b')u^2-\{ab-a'b'+(1-a'-b')c'\}u+(ab-a'b'c')=0,$$

giving

$$u=\frac{ab-a'b'+(1-a'-b')c'+Q}{2(1-a'-b')},$$

where

$$Q^2=\{ab-a'b'+(1-a'-b')c'\}^2-4(1-a'-b')(ab-a'b'c').$$

We have as conditions which must be satisfied by the data, that each of the quantities a', b', c' is greater than each of the quantities a, b; or say, each of the quantities $1-\alpha p'$, $1-\beta q'$, $\alpha p+\beta q$ greater than each of the quantities αp, βq: Q^2 is then real, and taking Q positive, we have u equal to or greater than each of the three quantities and greater than each of the two quantities. The difficulties which I find in regard to this solution have been already referred to.

139. See volume I. Notes and References 13, 14, 15, 16 and 100. I have in the last of these noticed that the terms covariant and invariant were due to Sylvester: and I have referred to papers by Boole, Eisenstein, Hesse, Schläfli and Sylvester. Anterior to the present memoir 139 we have other papers by Boole and Sylvester, one by Hermite (with other papers not directly affecting the theory), a paper by Salmon, and a very important memoir by Aronhold: it will be convenient to give a list as follows:

Boole.

1. Researches on the theory of analytical transformations with a special application to the reduction of the general equation of the second order, *Camb. Math. Jour.* t. II. 1841, pp. 64—73.

2. Exposition of a general theory of linear transformations, Part I. *Camb. Math. Jour.* t. III. 1843, pp. 1—20.

Exposition of a general theory of linear transformations, Part II. *Camb. Math. Jour.* t. III. 1843, pp. 106—119.

3. Notes on linear transformations, *Camb. and Dubl. Math. Jour.* t. IV. 1845, pp. 166—171.

4. On the theory of linear transformations, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 87—106.

5. On the reduction of the general equation of the *n*th degree, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 106—113.

6. Letter to the Editor (reply to Prof. Sylvester), *Camb. and Dubl. Math. Jour.* t. VI. pp. 284, 285.

Sylvester.

1. On the intersections, contacts and other relations of two conics expressed by indeterminate coordinates, *Camb. and Dubl. Math. Jour.* t. V. 1850, pp. 262—282.

2. On a new class of theorems in elimination between quadratic functions, *Phil. Mag.* t. XXXVII. 1850, pp. 213—218.

3. On certain general properties of homogeneous functions, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 1—17.

4. On the intersections of two conics, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 18—20.

5. Reply to Prof. Boole's Observations contained in the November Number of the Journal, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 171—174.

6. Sketch of a memoir on Elimination, Transformation and Canonical forms, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 186—200.

7. On the general theory of Associated Algebraical forms, *Camb. and Dubl. Math. Jour.* t. VI. 1851, pp. 18—20.

8. On Canonical forms, 8vo. London, Bell, 1851.

9. On a remarkable discovery in the theory of Canonical forms and of hyper-determinants, *Phil. Mag.* t. II. 1851, pp. 391—410.

10. On the Principles of the Calculus of Forms. Part I. Generation of Forms. Sect 1. On Simple Concomitance. 2. On Complex Concomitance. 3. On Commutants. Notes in Appendix (1), (2), (3), (4), (5), (6), (7), (8), *Camb. and Dubl. Math. Jour.* t. VI. 1852, pp. 52—97.

11. On the Principles of the Calculus of Forms. Sect. 4. Reciprocity, also Properties and Analogies of Certain Invariants &c. 5. Applications and Extension of the theory of the Plexus. 6. On the partial differential equations to Concomitants, Orthogonal and Plagional Invariants, &c. Notes in Appendix (9), (10), (11). Postscript, *Camb. and Dubl. Math. Jour.* t. VI. 1852, pp. 179—217.

12. Note on the Calculus of Forms, *Camb. and Dubl. Math. Jour.* t. VIII. 1853 pp. 62—64.

13. On the Calculus of Forms otherwise the theory of Invariants. Sect. 7. On Combinants, *Camb. and Dubl. Math. Jour.* t. VIII. 1853, pp. 256—269.

14. On the Calculus of Forms otherwise the theory of Invariants. Sect. 7. Continued. 8. On the reduction of a sextic function of two variables to its canonical form, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 85, 103.

Salmon. Exercises in the Hyperdeterminant Calculus, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 19—33.

Hermite. Sur la théorie des fonctions homogènes à deux indéterminées, *Camb. and Dubl. Math. Jour.* t. IX. 1854, pp. 172—217.

Aronhold. Zur Theorie der homogenen Functionen von drei Variabeln, *Crelle* t. XXXIX. 1850, pp. 140—159.

In the present Memoir 139, dropping altogether the consideration of linear transformations, I start from the notion of certain operations upon the constants and facients of a quantic, viz. if to fix the ideas we consider the case of a binary quantic $(a, b, \ldots b', a')(x, y)^m$, then there is an operation $\{y\partial_x\}, = a\partial_b + 2b\partial_c \ldots + mb'\partial_{a'}$ which performed upon the quantic is tantamount to the operation $y\partial_x$: and similarly an operation $\{x\partial_y\}, = mb\partial_a + (m-1)\,c\partial_b \ldots + a'\partial_{b'}$ which performed upon the quantic is tantamount to the operation $x\partial_y$. Or, what is the same thing, there are two operations $\{y\partial_x\} - y\partial_x$, and $\{x\partial_y\} - x\partial_y$ each of which performed upon the quantic reduces it to zero: to use an expression subsequently introduced, say each of these is an *annihilator* of the quantic. The assumed definition is that any function of the coefficients and variables which is reduced to zero by each of these operators, is a Covariant: and in particular if the function contain the coefficients only (in which case obviously the operators may be reduced to $\{y\partial_x\}$ and $\{x\partial_y\}$ respectively) the function is an Invariant.

I believe I actually arrived at the notion by the simple remark, say that $a\partial_b + 2b\partial_c$ operating upon $ac - b^2$ reduced it to zero, and that the same operation performed upon $ax^2 + 2bxy + cy^2$ reduced it to $2axy + 2by^2$ which is $= y\partial_x\{ax^2 + 2bxy + cy^2\}$. But the earliest published mention of the notion is in the year 1852 in Note 7 of Sylvester's paper on the Principles of the Calculus of Forms (Sylvester 10). Here, connecting it with the theory of linear transformations, he writes "There is one principle of *paramount* importance which has not been touched upon in the preceding pages,... The principle now in question consists in introducing the idea of *continuous* or *infinitesimal* variation into the theory. To fix the ideas suppose C to be a function of the coefficients of $\phi(x, y, z)$ such that it remains unaltered when x, y, z become respectively fx, gy, hz, where $fgh = 1$. Next suppose that C does not alter when x becomes $x + ey + \epsilon z$, where e, ϵ are indefinitely small; it is easily and obviously demonstrable that if this be true for e, ϵ indefinitely small, it must be

true for all values of e, ϵ. Again suppose that C alters neither when x receives such infinitesimal increment, y and z remaining constant, nor when y and z separately receive corresponding increments z, x and x, y in the respective cases remaining constant. ...C will remain constant for any concurrent linear transformations of x, y, z when the modulus is unity. This all-important principle...also *instantaneously* gives the necessary and sufficient conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form." And in section 6 of the same paper (Sylvester 11) referring to the Note, he writes "This method may also be extended to concomitants generally. M. Aronhold as I collect from private information was the first to think of the application of this method to the subject: but it was Mr Cayley who communicated to me the equations which define the invariants of functions of two variables. The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr Cayley in a Memoir about to be published in *Crelle's Journal* [? 100]. I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears that the same idea and mode of treatment have been made use of. Mr Cayley's communication to me was made in the early part of December last [1851] and my method (the result of a remark made long before) of obtaining these and the more general equations and of demonstrating their sufficiency imparted a few weeks subsequently—I believe between January and February of the present year [1852]," and then applying the principle to the binary quadric, he proceeds to consider the theory of the operator $a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\ldots$, and the other operator with the coefficients in the reverse order, as applied to an invariant ϕ of the quantic. The theory of these operators was thus familiar to Sylvester in 1852, but it was in nowise made the foundation of the structure.

I notice as contained in the paper Boole (4), what is probably the first statement of the "provectant" process of forming an invariant; for example, from the quartic function $(a, b, c, d, e \between x, y)^4$ he derives

$$\tfrac{1}{48}(a, b, c, d, e \between \partial_y, -\partial_x)^4 . (a, b, c, d, e \between x, y)^4 = ae - 4bd + 3c^2, \text{ the quadrinvariant};$$

and similarly from the Hessian $(ac-b^2, 2(ad-bc), ae+2bd-3c^2, 2(be-cd), ce-d^2 \between x, y)^4$ is derived the cubinvariant $ace-ad^2-b^2e+2bcd-3c^2$. Mention is also made of the function $A(\beta\delta-\gamma^2)+B(\beta\gamma-\alpha\delta)+C(\alpha\gamma-\beta^2)$, ($A$, B, C given quadric functions, α, β, γ, δ given cubic functions of (a, b, c, d, e, f)), which is the octinvariant Q of the binary quintic.

The papers of Sylvester contain a great number of important results which will some of them be referred to in connexion with the later Memoirs on Quantics.

Hermite's discovery of the invariant of the degree 18 of the quintic, and the demonstration of his law of reciprocity are both given in the Memoir by him which is above referred to.

147. Upon looking at any one of the Tables, for instance VIII (*a*), it will be noticed (1) that the partition symbols in the outside top line and left-hand column respectively are differently arranged, (2) that the numbers of each pair of equal numbers (see the Memoir) are not symmetrically situate, and (3) that the table is what may be called a half-square; viz. the squares above (or, in the case of a (*b*) table, those below) the sinister diagonal are all vacant; the squares in the sinister diagonal itself are all occupied by units ($+1$ or -1 as the case may be). It is possible (and that in many ways) to give the same arrangement to the partition-symbols in the outside line and column respectively, and at the same time to retain the half-square form of the table: or (what is far more important) we may with Faà di Bruno, give the same arrangement to the partition-symbols, and at the same time make the table symmetrical, viz. cause the two numbers of each pair of equal numbers to be symmetrically situate in regard to the dexter diagonal of the square—but we cannot at the same time retain accurately the half-square form of the table. The general principle is that in the outside column (or line) the partition-symbols which are conjugate to each other have symmetrical positions, while the self-conjugate symbols are collected at the middle of the column (or line); there is then in regard to these self-conjugate symbols a sort of dislocation of the sinister diagonal, the units which belong to them being transferred to the dexter diagonal, and in the sinister diagonal replaced by zeros, for instance at the crossing of the two diagonals we may have

1	0
0	1

instead of

	1
1	

. A Table thus arranged may be called Symmetric.

Again as remarked by Fiedler, the two corresponding tables (*a*) and (*b*) may be united into a single table; the sinister diagonal is the same for each of them, and if we then insert into the (*b*) table below the sinister diagonal the numbers of the (*a*) table, we have a table which is to be read according to the lines for the numbers above and in the sinister diagonal; and according to the columns for the numbers in and below the same diagonal. This may be called a United table: it may be unsymmetric, or be rearranged so as to be made symmetric.

The tables have been rearranged as above, and extended to the order 14: I give the following references.

Fiedler. Elemente der Neueren Geometrie &c. (1862), pp. 73 et seq. (II. to X, (*a*) and (*b*) united, unsymmetric).

Faà di Bruno. Sur les Fonctions Symétriques, *Comptes Rendus*, t. 76 (1873), pp. 163—168 (II to VIII, (*b*), symmetric, there is some error in VIII, inasmuch as it is presented without the dislocation of the sinister diagonal).

——— Théorie des Fonctions Binaires, 8vo. Turin &c. 1876. II to XI (*b*) symmetric.

Durfee. Tables of the Symmetric Functions of the Twelfthic, *Amer. Math. Jour.* t. v. (1882), pp. 45—60. XII (*a*) and (*b*) unsymmetric.

Rehorovsky. Tafeln der symmetrischen Functionen der Wurzeln und der Coefficienten-Combinationen vom Gewichte eilf und zwölf. Wien, Denks. t. 26 (1883), pp.

53—60. XI (a) and (b), XII (a) and (b): unsymmetric, united. Is referred to in the next mentioned paper.

Durfee. The Tabulation of Symmetric Functions, *Amer. Math. Jour.* t. V. (1882), pp. 348, 349. XII (a) and (b); symmetric, united.

MacMahon. Symmetric Functions of the 13^{ic}, *Amer. Math. Jour.* t. VI. (1884), pp. 289—300. XIII (b); symmetric.

Cayley. Symmetric Functions of the roots for the degree 10 for the Form $1 + bx + \frac{cx^2}{1.2} + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$ *Amer. Math. Jour.* t. VII. (1885), pp. 47—56. II to X (b), unsymmetric. The calculation of the tables for this new form (MacMahon's) of the coefficients afforded a complete verification of the (b) tables, showing that there was not a single error in these tables as published in the *Philosophical Transactions.*

Durfee. Symmetric Functions of the 14^{ic}, *Amer. Math. Jour.* t. IX. (1887), pp. 278—292. XIV (b) symmetric, the arrangement is different from and seemingly better than that in the tables XII (b) and XIII (b).

MacMahon. Properties of a Complete Table of Symmetric Functions, *Amer. Math. Jour.* t. X. (1888), pp. 42—46.

——— Memoir on a New Theory of Symmetric Functions, *Amer. Math. Jour.* t. XI. (1889), pp. 1—36. (a) and (b) Tables for the weights 1 to 6 and their several partitions. To explain this, observe that the general idea is to ignore the coefficients altogether, regarding them as merely particular symmetric functions of the roots: thus the (b) table for the weight 4 (partition 1^4) is in fact the table IV (b) giving the symmetric functions (4), (31), (2^2), (21^2), (1^4) in terms of (1^4), $(1^3)(1)$, $(1^2)^2$, $(1^2)(1)^2$, $(1)^4$, that is in terms of the combinations e, bd, c^2, b^2c, b^4 of the coefficients, but that the other tables weight 4 to a different partition, give the values of symmetric functions (combinations of the foregoing) which are expressible in terms of other symmetric functions of the roots: for instance weight 4 (partition 21^2) gives (4), (31), (2^2), and (21^2) in terms of (21^2), $(21)(1)$, $(2)(1^2)$ and $(2)(1)^2$. A leading idea in this valuable memoir is that of the "Separations" of a Partition.

150. The theory is developed in an incomplete form. If to fix the ideas we consider a quintic equation $(a, b, c, d, e, f \between x, 1)^5 = 0$, then a single equality $\alpha = \beta$ between the roots implies a onefold relation between the coefficients (a, b, c, d, e, f): this is completely and precisely expressed by means of a single equation ($\nabla = 0$, where ∇ is the discriminant, $= a^4f^4 +$ &c.). Similarly a system of two equalities $\alpha = \beta = \gamma$, or $\alpha = \beta$, $\gamma = \delta$ as the case may be, implies a twofold relation between the coefficients (a, b, c, d, e, f) and the question arises, to determine the order of this twofold relation, and to find how it can be completely and precisely expressed, whether by two equations $A = 0$, $B = 0$, or if need be by a larger number of equations $A = 0$, $B = 0$, $C = 0$, &c. between the coefficients; this is not done in the memoir,

but what is done is only to find two or more equations satisfied in virtue of the system of the two equalities between the roots. And similarly in the case of a system of more than two equalities. See my paper 77, where this notion of the order of a system of equations was established.

152. The next later memoir on the theory of Matrices, so far as I am aware is that by Laguerre, "Sur le Cacul des Systèmes Lineaires," *Jour. Ec. Polyt.* t. XXV. (1867), pp. 215—264. A "système lineaire" is what I called a matrix, and the mode of treatment is throughout very similar to that of my memoir; in particular we have in it my theorem of the equation satisfied by a matrix of any order. The memoir contains a theorem relating to the integral functions of *two* matrices A, B of the same order, viz. this is expressible in the form $m + pA + qB + rAB$. For later developments see the papers by Sylvester in the *American Mathematical Journal.*

158. The notion of the "Absolute" was I believe first introduced in the present memoir. In reference to the theory of distance founded upon it and here developed, I refer to the papers

Klein, Ueber die sogenannte Nicht-Euklidische Geometrie, *Math. Ann.* t. IV. (1871), pp. 573—625.

Cayley, On the Non-Euclidian Geometry, *Math. Ann.* t. V. (1872), pp. 630—634.

Klein, Ueber die sogenannte Nicht-Euklidische Geometrie, *Math. Ann.* t. VI. (1873), pp. 112—145.

In his first paper Klein substitutes, for my $\cos^{-1}$ expression for the distance between two points, a logarithmic one; viz. in linear geometry if the two fixed points are A, B then the assumed definition for the distance of any two points P, Q is

$$\text{dist.}\,(PQ) = c \log \frac{AP\,.\,BQ}{AQ\,.\,BP};$$

this is a great improvement, for we at once see that the fundamental relation, $\text{dist.}\,(PQ) + \text{dist.}\,(QR) = \text{dist.}\,(PR)$, is satisfied: in fact we have

$$\text{dist.}\,(QR) = c \log \frac{AQ\,.\,BR}{AR\,.\,BQ},$$

and thence

$$\text{dist.}\,(PQ) + \text{dist.}\,(QR) = c \log \frac{AP\,.\,BR}{AR\,.\,BQ}, = \text{dist.}\,PR.$$

But in my Sixth Memoir, the question arises, what is meant by "coordinates": if in linear geometry (x, y) are the coordinates of a point P, does this mean that $x : y$ is the ratio of the distances in the ordinary sense of the word of the point P from two fixed points A, B: and if so, does the notion of distance in the new sense ultimately depend on that of distance in the ordinary sense? And similarly in Klein's definition, do AP, BQ, AQ, BP denote distances in the ordinary sense

of the word, and if so does the notion of distance in the new sense ultimately depend on that of distance in the ordinary sense?

As to my memoir, the point of view was that I regarded "coordinates" not as distances or ratios of distances, but as an assumed fundamental notion, not requiring or admitting of explanation. It recently occurred to me that they might be regarded as mere numerical values, attached arbitrarily to the point, in such wise that for any given point the ratio $x : y$ has a determinate numerical value, and that to any given numerical value of $x : y$ there corresponds a single point. And I was led to interpret Klein's formulæ in like manner; viz. considering A, B, P, Q as points arbitrarily connected with determinate numerical values a, b, p, q, then the logarithm of the formula would be that of $(a-p)(b-q) \div (a-q)(b-p)$. But Prof. Klein called my attention to a reference (p. 132 of his second paper) to the theory developed in Staudt's Geometrie der Lage, 1847 (more fully in the Beiträge zur Geometrie der Lage, Zweites Heft, 1857). The logarithm of the formula is $\log(A, B, P, Q)$, and, according to Staudt's theory (A, B, P, Q), the anharmonic ratio of any four points, has independently of any notion of distance the fundamental properties of a numerical magnitude, viz. any two such ratios have a sum and also a product, such sum and product being each of them a like ratio of four points determinable by purely descriptive constructions. The proof is easiest for the product: say the ratios are (A, B, P, Q) and (A', B', P', Q'): then considering these as given points we can construct R, such that $(A', B', P', Q') = (A, B, Q, R)$: the two ratios are thus (A, B, P, Q) and (A, B, Q, R), and we say that their product is (A, B, P, R) {observe as to this that introducing the notion of distance, the two factors are $\frac{AP.BQ}{AQ.BP}$ and $\frac{AQ.BR}{AR.BQ}$ and thus their product $= \frac{AP.BR}{AR.BP}$, which is (A, B, P, R), which is the foundation of the definition}. Next for the sum, we construct $Q_{,}$ such that $(A', B', P', Q') = (A, B, P, Q_{,})$; the sum then is $(A, B, P, Q) + (A, B, P, Q_{,})$; and if we then construct S such that (A, A), $(Q, Q_{,})$, (B, S) are an involution, we say that $(A, B, P, Q) + (A, B, P, Q_{,}) = (A, B, P, S)$. {Observe as to this that again introducing the notion of distance the last mentioned equation is $\frac{AP.BQ}{AQ.BP} + \frac{AP.BQ_{,}}{AQ_{1}.BP} = \frac{AP.BS}{AS.BP}$, that is $\frac{BQ}{AQ} + \frac{BQ_{,}}{AQ_{,}} = \frac{BS}{AS}$, which expresses that S is determined as above; in fact the equation $\frac{b-q}{a-q} + \frac{b-q_{,}}{a-q_{,}} = \frac{b-s}{a-s}$ is readily seen to be equivalent to

$$\begin{vmatrix} 1, & b+s\ , & bs \\ 1, & 2a\ \ , & a^2 \\ 1, & q+q_1, & qq_1 \end{vmatrix} = 0\}.$$

It must however be admitted that, in applying this theory of Staudt's to the theory of distance, there is at least the appearance of arguing in a circle, since the construction for the product of the two ratios, is in effect the assumption of the relation,

$$\text{dist. } PQ + \text{dist. } QR = \text{dist. } PR.$$

I may refer also to the Memoir, Sir R. S. Ball "On the theory of the Content," *Trans. R. Irish Acad.* vol. XXIX. (1889), pp. 123—182, where the same difficulty is discussed. The opening sentences are—"In that theory [Non-Euclidian Geometry] it seems as if we try to replace our ordinary notion of distance between two points by the logarithm of a certain anharmonic ratio. But this ratio itself involves the notion of distance measured in the ordinary way. How then can we supersede the old notion of distance by the Non-Euclidian notion, inasmuch as the very definition of the latter involves the former?"

An extensive list of papers is given, Halsted, Bibliography of Hyper-Space and of Non-Euclidean Geometry, *Amer. Math. Jour.* t. I. (1878), pp. 261—276 and 384—385, also t. II. (1879), pp. 65—70.

END OF VOL. II.

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Scientific Papers of the late Prof. J. Clerk Maxwell. Edited by W. D. NIVEN, M.A., late Fellow of Trinity College. In 2 vols. Royal 4to. [*Nearly ready.*

The Electrical Researches of the Hon. H. Cavendish, F.R.S. Written between 1771 and 1781. Edited from the original MSS. in the possession of the Duke of Devonshire, K.G., by the late J. CLERK MAXWELL, F.R.S. Demy 8vo. 18*s.*

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.*
Vol. II. By the same Editor. [*In the Press.*

A History of the Study of Mathematics at Cambridge. By W. W. ROUSE BALL, M.A., Fellow and Lecturer on Mathematics of Trinity College, Cambridge. Crown 8vo. 6*s.*

Mathematical and Physical Papers, by Sir W. THOMSON, LL.D., D.C.L., F.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Vol. I. Demy 8vo. 18*s.* Vol. II. 15*s.*
[Vol. III. *In the Press.*

Mathematical and Physical Papers, by G. G. STOKES, Sc.D., LL.D., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge. Reprinted from the Original Journals and Transactions, with Additional Notes by the Author. Vol. I. Demy 8vo. 15*s.* Vol. II. 15*s.*
[Vol. III. *In the Press.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Royal 4to. cloth (vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half-morocco 28*s.* (net). New Series for the years 1874—1883 in the Press.

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., Fellow of Sidney Sussex College. Crown 8vo. 7*s.* 6*d.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of St Peter's College, Cambridge. [*In the Press.*

A Treatise on Plane Trigonometry. By E. W. HOBSON, M.A., Fellow and Lecturer of Christ's College, Cambridge. Demy 8vo. [*In the Press.*

A Catalogue of the Portsmouth Collection of Books and Papers written by or belonging to Sir ISAAC NEWTON. Demy 8vo. 5*s.*

A Treatise on Natural Philosophy. By Sir W. THOMSON, LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Professors Sir W. THOMSON and P. G. TAIT. Demy 8vo. 9*s.*

An Elementary Treatise on Quaternions. By P. G. TAIT, M.A. *Third Edition.* Demy 8vo. 18*s.*

A Treatise on the Theory of Determinants and their applications in Analysis and Geometry, by R. F. SCOTT, M.A., Fellow of St John's College. Demy 8vo. 12*s.*

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A. Demy 8vo. 12*s.*

The Analytical Theory of Heat, by JOSEPH FOURIER. Translated, with Notes, by A. FREEMAN, M.A., formerly Fellow of St John's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on the General Principles of Chemistry, by M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A., and D. J. CARNEGIE, B.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

Lectures on the Physiology of Plants, by S. H. VINES, Sc.D., Professor of Botany in the University of Oxford. Demy 8vo. With Illustrations. 21*s.*

A Short History of Greek Mathematics. By J. GOW, Litt.D., Fellow of Trinity College. Demy 8vo. 10*s.* 6*d.*

Diophantos of Alexandria; a Study in the History of Greek Algebra. By T. L. HEATH, M.A., Fellow of Trinity College, Cambridge. Demy 8vo. 7*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Books I. & II. Extra Fcap. 8vo. 1*s.* 6*d.* (*Pitt Press Mathematical Series.*)

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

MATHEMATICAL PAPERS.

London: C. J. CLAY & SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. III.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1890

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 64 papers numbered 159 to 222 originally published in the years 1857 to 1862: the chronological order is slightly departed from for the sake of including a series of papers in the Memoirs and the Monthly Notices of the Royal Astronomical Society: there are thus some earlier papers which have been left over for the next volume.

As papers are referred to by their Numbers only it will be convenient to give in each volume a Table such as the following:

Vol.	I.	Numbers	1	to	100.
,,	II.	,,	101	,,	158.
,,	III.	,,	159	,,	222.

CONTENTS.

[An Asterisk denotes that the Paper is not printed in full.]

PAGE

159. *On some Integral Transformations* 1
Quart. Math. Jour. t. I. (1857), pp. 4—6

160. *On a Theorem relating to Reciprocal Triangles* 5
Quart. Math. Jour. t. I. (1857), pp. 7—10

161. *A problem in Permutations* 8
Quart. Math. Jour. t. I. (1857), p. 79

162. *Two letters on Cubic Forms* 9
Quart. Math. Jour. t. I. (1857), pp. 85—87 and 90—91

163. *On Hansen's Lunar Theory* 13
Quart. Math. Jour. t. I. (1857), pp. 112—125

164. *On Gauss' Theory for the Attraction of Ellipsoids* 25
Quart. Math. Jour. t. I. (1857), pp. 162—166

165. *On some Geometrical Theorems relating to a triangle circumscribed about a Conic* 29
Quart. Math. Jour. t. I. (1857), pp. 169—175

166. *Note on the Homology of Sets* 35
Quart. Math. Jour. t. I. (1857), p. 178

167. *Apropos of Partitions* 36
Quart. Math. Jour. t. I. (1857), pp. 183—184

PAGE

*168. *A demonstration of the Fundamental Property of Geodesic Lines* 38
Quart. Math. Jour. t. I. (1857), pp. 185—186

169. *Eisenstein's Geometrical Proof of the Fundamental Theorem for Quadratic Residues* (*Translated from the Original Memoir*, Crelle, t. XXVIII. (1844), *with an addition by A. Cayley*) . 39
Quart. Math. Jour. t. I. (1857), pp. 186—191

170. *On Schellbach's Solution of Malfatti's Problem* 44
Quart. Math. Jour. t. I. (1857), pp. 222—226

171. *Note on Mr Salmon's Equation of the Orthotomic Circle* . . 48
Quart. Math. Jour. t. I. (1857), pp. 242—244

172. *Note on the Logic of Characteristics* 51
Quart. Math. Jour. t. I. (1857), pp. 257—259

173. *On Laplace's Method for the Attraction of Ellipsoids* . . 53
Quart. Math. Jour. t. I. (1857), pp. 285—300

*174. *On the Oval of Descartes* 66
Quart. Math. Jour. t. I. (1857), pp. 320—328

175. *On the Porism of the In-and-circumscribed Triangle* . . 67
Quart. Math. Jour. t. I. (1857), pp. 344—354

176. *Note on Jacobi's Canonical Formulæ for Disturbed Motion in an Elliptic Orbit*. 76
Quart. Math. Jour. t. I. (1857), pp. 355—356

177. *Solution of a Mechanical Problem* 78
Quart. Math. Jour. t. I. (1857), pp. 405—406

178. *On the à posteriori Demonstration of the Porism of the In-and-circumscribed Triangle* 80
Quart. Math. Jour. t. II. (1858), pp. 31—38

179. *On certain Forms of the Equation of a Conic* 86
Quart. Math. Jour. t. II. (1858), pp. 44—48

PAGE

180. *Note on the Reduction of an Elliptic Orbit to a fixed plane* . 91
Quart. Math. Jour. t. II. (1858), pp. 49—54

181. *On Sir W. R. Hamilton's Method for the Problem of three or more Bodies* 97
Quart. Math. Jour. t. II. (1858), pp. 66—73

182. *On Lagrange's Solution of the Problem of two fixed Centres* . 104
Quart. Math. Jour. t. II. (1858), pp. 76—83

183. *Note on Certain Systems of Circles* 111
Quart. Math. Jour. t. II. (1858), pp. 83—88

184. *A Theorem relating to Surfaces of the Second Order* . . 115
Quart. Math. Jour. t. II. (1858), pp. 140—142

185. *Note on the 'Circular Relation' of Prof. Möbius* . . . 118
Quart. Math. Jour. t. II. (1858), p. 162

186. *On the determination of the value of a certain Determinant* . 120
Quart. Math. Jour. t. II. (1858), pp. 163—166

187. *On the Sums of Certain Series arising from the Equation* $x=u+tfx$ 124
Quart. Math. Jour. t. II. (1858), pp. 167—171

188. *On the Simultaneous Transformation of two Homogeneous Functions of the Second Order* 129
Quart. Math. Jour. t. II. (1858), pp. 192—195

189. *Note on a formula in finite Differences* 132
Quart. Math. Jour. t. II. (1858), pp. 198—201

190. *On the System of Conics which pass through the same four points* 136
Quart. Math. Jour. t. II. (1858), pp. 206—207

191. *Note on the Expansion of the true Anomaly* 139
Quart. Math. Jour. t. II. (1858), pp. 229—232

PAGE

192. *On the Area of the Conic Section represented by the General Trilinear Equation of the Second Degree* 143
Quart. Math. Jour. t. II. (1858), pp. 248—253

193. *On Rodrigues' Method for the Attraction of Ellipsoids* . . 149
Quart. Math. Jour. t. II. (1858), pp. 333—337

194. *Note on the Theory of Attraction* 154
Quart. Math. Jour. t. II. (1858), pp. 338, 339

195. *Note on the Recent Progress of Theoretical Dynamics* . . 156
Report of British Association, 1857, pp. 1—42

196. *Note sur un Problème d'Analyse Indéterminée*. 205
Nouvelles Annales de Math. t. XVI. (1857), pp. 161—165

197. *Note on the Theory of Logarithms* 208
Phil. Mag. t. XI. (1856), pp. 275—280

198. *Note on a Result of Elimination* 214
Phil. Mag. t. XI. (1856), pp. 378—379

199. *Note on the Theory of Elliptic Motion* 216
Phil. Mag. t. XI. (1856), pp. 425—428

200. *On the Cones which pass through a given Curve of the Third Order in Space* 219
Phil. Mag. t. XII. (1856), pp. 20—22

201. *Second Note on the Theory of Logarithms* 222
Phil. Mag. t. XII. (1856), pp. 354—360

202. *Supplementary Remarks on the Porism of the In-and-circumscribed Triangle* 229
Phil. Mag. t. XIII. (1857), pp. 19—30

203. *On the Theory of the Analytical Forms called Trees* . . 242
Phil. Mag. t. XIII. (1857), pp. 172—176

204. *On a Problem in the Partition of Numbers* 247
Phil. Mag. t. XIII. (1857), pp. 245—248

PAGE

205. *Note on the Summation of a Certain Factorial Expression* . 250
Phil. Mag. t. XIII. (1857), pp. 419—423

206. *Note on a Theorem relating to the Rectangular Hyperbola* . 254
Phil. Mag. t. XIII. (1857), p. 423

207. *Analytical Solution of the Problem of Tactions* . . . 255
Phil. Mag. t. XIII. (1857), pp. 507—509

208. *Note on the Equipotential Curve* $\frac{m}{r}+\frac{m'}{r'}=C$. 258
Phil. Mag. t. XIV. (1857), pp. 142—146

209. *A Demonstration of Sir W. R. Hamilton's Theorem of the Isochronism of the Circular Hodograph* 262
Phil. Mag. t. XIV. (1857), pp. 427—430

210. *On the Cubic Transformation of an Elliptic Function* . . 266
Phil. Mag. t. XV. (1858), pp. 363—364

211. *On a Theorem relating to Hypergeometric Series* . . . 268
Phil. Mag. t. XVI. (1858), pp. 356, 357

212. *A Memoir on the Problem of Disturbed Elliptic Motion* . 270
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 1—29

213. *On the Development of the Disturbing Function in the Lunar Theory* 293
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 69—95

214. *The First part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories* . 319
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 187—215

215. *A Supplementary Memoir on the Problem of Disturbed Elliptic Motion* 344
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 217—234

PAGE

216. *Tables of the Development of Functions in the Theory of Elliptic Motion* 360

Mem. R. Astron. Soc. t. XXIX. (1861), pp. 191—306

217. *A Memoir on the Problem of the Rotation of a solid Body* . 475

Mem. R. Astron. Soc. t. XXIX. (1861), pp. 307—342

218. *A Third Memoir on the Problem of Disturbed Elliptic Motion* 505

Mem. R. Astron. Soc. t. XXXI. (1863), pp. 43—56

219. *On some formulæ relating to the Variation of the Plane of a Planet's Orbit* 516

Monthly Notices R. Astron. Soc. t. XXI. (1861), pp. 43—46

220. *Note on a Theorem of Jacobi's in relation to the Problem of Three Bodies* 519

Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 76—78

221. *On the Secular Acceleration of the Moon's Mean Motion* . 522

Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 173—230

222. *On Lambert's Theorem for Elliptic Motion* 562

Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 238—242

Notes and References 567

CLASSIFICATION.

GEOMETRY

Circles, Conics and Quadric Surfaces, 170, 171, 179, 183, 184, 190, 192, 206, 207
Malfatti's Problem, 170
Problem of Tactions, 207
In-and-Circumscribed Triangle, 165, 175, 178, 202
Reciprocal Triangles, 160
Homology of Sets, 166
Geodesic Lines, 168
Cartesians, 174
Circular relation of Möbius, 185
Cones through Cubic Curve, 200
Equipotential Curve, 208

ANALYSIS

Attractions, 164, 173, 193, 194
Dynamics
Report on Progress of Theoretical Dynamics, 195
Hansen's Lunar Theory, 163
Jacobi's Canonical Formulæ, 176
Plane of Elliptic Orbit, 180, 219
Problem of two fixed Centres, 182
Hamilton's Method for three or more Bodies, 181
Elliptic Motion, 191, 192, 199, 222; Disturbed, 212, 215, 218; Tables, 216
Hamilton's Circular Hodograph, 209
Disturbing Function in Lunar and Planetary Theories, 213, 214
Theorem of Jacobi's, 220
Secular Acceleration of Moon, 221
Rotation of Solid Body, 217

ANALYSIS

Elliptic Functions, 210
Transformation of two Quadric Functions, 188
Integral Transformations, 159
Permutations, 161
Binary Cubic Forms, 162
Partitions, 167, 204; Trees, 203
Quadratic Residues, 169
Logic of Characteristics, 172
Mechanical Problem, 177
Special Determinant, 186
Sums of Certain Series, 187
Formula in Finite Differences, 189
Indeterminate Analysis, 196
Theory of Logarithms, 197, 201
Elimination, 198
Hypergeometric Series, 205, 211

159.

ON SOME INTEGRAL TRANSFORMATIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 4—6.]

SUPPOSE that x, a, b, c and x', a', b', c' have the same anharmonic ratios, or what is the same thing, let these quantities satisfy the equation

$$\begin{vmatrix} 1\,, & 1\,, & 1\,, & 1 \\ x\,, & a\,, & b\,, & c \\ x'\,, & a'\,, & b'\,, & c' \\ xx', & aa', & bb', & cc' \end{vmatrix} = 0;$$

this equation may be represented under a variety of different forms, which are obtained without difficulty; thus, if for shortness

$$K = a(b'-c')(x'-a') + b\,(c'-a')(x'-b') + c\,(a'-b')(x'-c'),$$

then

$$\begin{aligned} Kx &= -\{bc(b'-c')(x'-a') + ca(c'-a')(x'-b') + ab(a'-b')(x'-c')\}, \\ K(x-a) &= (c-a)(a-b)(b'-c')(x'-a'), \\ K(x-b) &= (a-b)(b-c)(c'-a')(x'-b'), \\ K(x-c) &= (b-c)(c-a)(a'-b')(x'-b'). \end{aligned}$$

Consider x, x' as variables; then

$$K^2\,dx = (b-c)\,(c-a)\,(a-b)\,(b'-c')(c'-a')(a'-b')\,dx';$$

let, d, d' be any corresponding values of x, x'; then

$$\begin{vmatrix} 1\,, & 1\,, & 1\,, & 1 \\ a\,, & b\,, & c\,, & d \\ a'\,, & b'\,, & c'\,, & d' \\ aa', & bb', & cc', & dd' \end{vmatrix} = 0$$

and we have

$$K(x-d) = D(x'-d');$$

where

$$D = (c'-a')(a'-b')(b'-c')\Lambda,$$

and

$$\Lambda = \frac{(a-d)(b-c)}{(a'-d')(b'-c')} = \frac{(b-d)(c-a)}{(b'-d')(c'-a')} = \frac{(c-d)(a-b)}{(c'-d')(a'-b')}.$$

Suppose $\alpha+\beta+\gamma+\delta=-2$; then

$$(x-a)^\alpha(x-b)^\beta(x-c)^\gamma(x-d)^\delta\,dx = J(x'-a')^\alpha(x'-b')^\beta(x'-c')^\gamma(x'-d')^\delta\,dx,$$

where

$$J = (b-c)^{\beta+\gamma+1}(c-a)^{\gamma+\alpha+1}(a-b)^{\alpha+\beta+1}(b'-c')^{\alpha+\delta+1}(c'-a')^{\beta+\delta+1}(a'-b')^{\gamma+\delta+1}D^\delta.$$

We may in particular take for a', b', c', d' the systems b, a, d, c : c, d, a, b and d, c, b, a respectively; this gives, writing successively y, z, w instead of x',

$$\begin{aligned}
&(x-a)^\alpha(x-b)^\beta(x-c)^\gamma(x-d)^\delta\,dx\\
&= M(y-a)^\beta(y-b)^\alpha(y-c)^\delta(y-d)^\gamma\,dy\\
&= N(z-a)^\gamma(z-b)^\delta(z-c)^\alpha(z-d)^\beta\,dz\\
&= P(w-a)^\delta(w-b)^\gamma(w-c)^\beta(w-d)^\alpha\,dw,
\end{aligned}$$

where

$$\begin{aligned}
M &= -(-)^{\gamma+\delta}(a-c)^{\alpha+\gamma+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(b-d)^{\beta+\delta+1},\\
N &= (-)^{\gamma+\delta}(a-b)^{\alpha+\beta+1}(a-d)^{\alpha+\delta+1}(b-c)^{\beta+\gamma+1}(c-d)^{\gamma+\delta+1},\\
P &= \phantom{-(-)^{\gamma+\delta}}(a-b)^{\alpha+\beta+1}(a-c)^{\alpha+\gamma+1}(b-d)^{\beta+\delta+1}(c-d)^{\gamma+\delta+1};
\end{aligned}$$

the relations between the variables x, y, z, w being

$$\begin{aligned}
x &= \frac{(c+d)\,ab-(a+b)\,cd-(ab-cd)\,y}{ab-cd-(a+b-c-d)\,y}\\
&= \frac{(b+d)\,ac-(a+c)\,bd-(ac-bd)\,z}{ac-bd-(a+c-b-d)\,z}\\
&= \frac{(b+c)\,ad-(a+d)\,bc-(ad-bc)\,w}{ad-bc-(a+d-b-c)\,w}:
\end{aligned}$$

these are, in fact, the formulæ in my note, "On an Integral Transformation," *Camb. and Dubl. Math. Jour.* t. III. (1848), p. 286 [62], which was suggested to me by Gudermann's transformation for elliptic functions, (*Crelle*, t. XXIII. (1846), p. 330).

Suppose now that the values of a', b', c', d' are 0, 1, ∞, ζ, we have in this case

$$x = \frac{a(b-c)+c(a-b)\,y}{(b-c)+(a-b)\,y},$$

and representing the denominator by K, then

$$\begin{aligned} K(x-a) &= -(a-b)(a-c)\,y, \\ K(x-b) &= (a-b)(b-c)(1-y), \\ K(x-c) &= (a-c)(b-c), \\ K(x-d) &= (a-b)(c-d)(y-\zeta), \end{aligned}$$

where

$$\zeta = \frac{(a-d)(b-c)}{(a-b)(c-d)},$$

and we have

$$K^2\,dx = -(a-b)(a-c)(b-c)\,dy,$$

whence

$$\begin{aligned} &(x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta\,dx = \\ &\qquad -(-)^\alpha (a-b)^{\alpha+\beta+\delta+1} (a-c)^{\alpha+\gamma+1} (b-c)^{\beta+\gamma+1} (c-d)^\delta\, y^\alpha (1-y)^\beta (y-\zeta)^\delta\,dy. \end{aligned}$$

It is easy, by means of this equation, to generalise a remarkable formula given by M. Serret in his memoir, "Sur la Représentation géométrique des Fonctions elliptiques et ultrà-elliptiques," *Liouville*, t. XI. and XII. [1846 and 1847], and *Recueil des Savans étrangers*, t. XI. [1851].(1) In fact, suppose that the indices α, β, γ, δ are integers, and that two of these indices, e.g. γ, δ, are negative, the remaining two indices being positive, then writing $-\gamma$, $-\delta$ instead of γ, δ the integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta\,dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma+\delta = \alpha+\beta+2$, depends on the integral

$$\int \frac{y^\alpha (1-y)^\beta\,dy}{(y-\zeta)^\delta}.$$

Suppose that the fraction under the integral sign is resolved into simple fractions, each of these fractions will be integrable algebraically, except the fraction having for its denominator the simple power $y-\zeta$, the integral of which is a logarithm. The coefficient of this fraction is at once found by writing in the numerator $\zeta+(y-\zeta)$ for y; and expanding in ascending powers of $y-\zeta$ and equating this coefficient to zero, we have

$$\left(\frac{d}{d\zeta}\right)^{\delta-1} \zeta^\alpha (1-\zeta)^\beta = 0,$$

which [observing that $(\gamma-1)+(\delta-1) = \alpha+\beta$] is easily seen to be equivalent to

$$\left(\frac{d}{d\zeta}\right)^{\gamma-1} \zeta^\beta (1-\zeta)^\alpha = 0.$$

[1] M. Serret has reproduced the theorem in his very interesting and instructive treatise, "Cours d'Algèbre supérieure," deuxième édition, Paris, 1854, [quatrième édition, Paris, 1877].

Hence if the function $\zeta = \dfrac{(a-d)(b-c)}{(a-b)(c-d)}$ satisfy this condition, the indefinite integral

$$\int \frac{(x-a)^\alpha (x-b)^\beta \, dx}{(x-c)^\gamma (x-d)^\delta},$$

where $\gamma + \delta = \alpha + \beta + 2$, will be expressible as a rational algebraical fraction.

It may be noticed, that in the general case, observing that $x = a$, $x = b$ give $y = 0$, $y = 1$, the integral

$$\int_a^b (x-a)^\alpha (x-b)^\beta (x-c)^\gamma (x-d)^\delta \, dx$$

depends on

$$\int_0^1 y^\alpha (1-y)^\beta (\zeta - y)^\delta \, dy,$$

or, putting $\zeta = \dfrac{1}{u}$, upon

$$\int_0^1 y^\alpha (1-y)^\beta (1-uy)^\delta \, dy,$$

which is expressible by means of a hypergeometric series having u for its argument or fourth element.

2, *Stone Buildings, Lincoln's Inn, Feb.* 1854.

160.

ON A THEOREM RELATING TO RECIPROCAL TRIANGLES.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 7—10.]

THE following theorem is, I assume, known; but the analytical demonstration of it depends upon a formula in determinants which is not without interest. The theorem referred to may be thus stated:

"A triangle and its reciprocal are in perspective;" where by the reciprocal of a triangle is meant the triangle the sides of which are the polars of the angles of the first-mentioned triangle with respect to a conic; and triangles are in perspective when the three lines forming the corresponding angles meet in a point, or what is the same thing, when the three points of intersection of the corresponding sides lie in a line.

Let the equation of the conic be

$$x^2 + y^2 + z^2 = 0,$$

and take (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ for the coordinates of the angles of the triangle, then if K be the determinant, and (A, B, C) (A', B', C') (A'', B'', C'') the inverse system, i.e. if

$$\begin{aligned}
KA &= (\beta'\gamma'' - \beta''\gamma'), & KB &= \gamma'\alpha'' - \gamma''\alpha', & KC &= \alpha'\beta'' - \alpha''\beta', \\
KA' &= (\beta''\gamma - \beta\gamma''), & KB' &= \gamma''\alpha - \gamma\alpha'', & KC' &= \alpha''\beta - \alpha\beta'', \\
KA'' &= (\beta\gamma' - \beta'\gamma), & KB'' &= \gamma\alpha' - \gamma'\alpha, & KC'' &= \alpha\beta' - \alpha'\beta,
\end{aligned}$$

equations which may be represented in the notation of matrices by the single equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma' \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}^{-1} = \begin{vmatrix} A, & A', & A'' \\ B, & B', & B'' \\ C, & C', & C'' \end{vmatrix},$$

then the equations of the sides of the triangle are

$$A\,x + B\,y + C\,z = 0,$$
$$A'x + B'y + C'z = 0,$$
$$A''x + B''y + C''z = 0,$$

and the coordinates of the angles of the reciprocal triangle may be taken to be (A, B, C) (A', B', C') (A'', B'', C''); the equations of the lines joining the corresponding angles of the two triangles are therefore

$$(B\gamma - C\beta)x + (C\alpha - A\gamma)y + (A\beta - B\alpha)z = 0,$$
$$(B'\gamma' - C'\beta')x + (C'\alpha' - A'\gamma')y + (A'\beta' - B'\alpha')z = 0,$$
$$(B''\gamma'' - C''\beta'')x + (C''\alpha'' - A''\gamma'')y + (A''\beta'' - B''\alpha'')z = 0;$$

the condition that these lines may meet in a point is therefore

$$\begin{vmatrix} B\gamma - C\beta, & C\alpha - A\gamma, & A\beta - B\alpha \\ B'\gamma' - C'\beta', & C'\alpha' - A'\gamma', & A'\beta' - B'\alpha' \\ B''\gamma'' - C''\beta'', & C''\alpha'' - A''\gamma'', & A''\beta'' - B''\alpha'' \end{vmatrix} = 0,$$

an equation which is satisfied identically when A, B, C; A', B', C'; A'', B'', C'' are replaced by their values. To prove this I transform the different quantities which enter into the determinant as follows: putting

$$F = \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'',$$
$$G = \alpha''\alpha + \beta''\beta + \gamma''\gamma,$$
$$H = \alpha\alpha' + \beta\beta' + \gamma\gamma';$$

we have

$$\begin{aligned} K(B\gamma - C\beta) &= \gamma\,(\gamma'\alpha'' - \gamma''\alpha') - \beta(\alpha\beta'' - \alpha''\beta') \\ &= \alpha''(\beta\beta' + \gamma\gamma') - \alpha'(\beta\beta'' + \gamma\gamma'') \\ &= \alpha''(\alpha\alpha' + \beta\beta' + \gamma\gamma') - \alpha'(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'') \\ &= \alpha''H - \alpha'G, \\ &\&c.; \end{aligned}$$

and the equation becomes

$$\begin{vmatrix} \alpha''H - \alpha'G, & \beta''H - \beta'G, & \gamma''H - \gamma'G \\ \alpha F - \alpha''H, & \beta F - \beta''H, & \gamma F - \gamma''H \\ \alpha'G - \alpha F, & \beta'G - \beta F, & \gamma'G - \gamma F \end{vmatrix} = 0.$$

Now the minor $(\beta F - \beta''H)(\gamma'G - \gamma F) - (\gamma F - \gamma''H)(\beta'G - \beta F)$ is equal to

$$GH(\beta'\gamma'' - \beta''\gamma') + HF(\beta''\gamma - \beta\gamma'') + FG(\beta\gamma' - \beta'\gamma),$$

i.e. to

$$K(GHA + HFA' + FGA'');$$

and expressing the other minors in a similar form, the equation to be proved is

$$\left.\begin{array}{l}(GHA+HFA'+FGA'')(B\gamma-C\beta)\\ +(GHB+HFB'+FGB'')(C\alpha-A\gamma)\\ +(GHC+HFC'+FGC'')(A\beta-B\alpha)\end{array}\right\}=0,$$

i.e.

$$HF\begin{vmatrix}A', & B', & C'\\ A, & B, & C\\ \alpha, & \beta, & \gamma\end{vmatrix}+FG\begin{vmatrix}A'', & B'', & C''\\ A, & B, & C\\ \alpha, & \beta, & \gamma\end{vmatrix}=0.$$

The first determinant is

$$-\{\alpha(BC'-B'C)+\beta(CA'-C'A)+\gamma(AB'-A'B)\}=-\frac{1}{K}(\alpha\alpha''+\beta\beta''+\gamma\gamma'')=-\frac{1}{K}G,$$

and the second determinant is

$$\{\alpha(B''C-BC'')+\beta(C''A-CA'')+\gamma(A''B-AB'')\}=\frac{1}{K}(\alpha\alpha'+\beta\beta'+\gamma\gamma')=\frac{1}{K}H,$$

and we have therefore identically

$$HF(-G)+FG(H)=0.$$

The corresponding theorem in geometry of three dimensions is that a tetrahedron and its reciprocal have to each other a certain relation, viz. the four lines joining the corresponding angles are generating lines of a hyperboloid, or, what is the same thing, the four lines of intersection of corresponding faces are generating lines of a hyperboloid. The demonstration would show how the theorem in determinants is to be generalised.

2, *Stone Buildings, Lincoln's Inn, February,* 1855.

161.

A PROBLEM IN PERMUTATIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), p. 79.]

THE game called Mousetrap gives rise to a singular problem in permutations. A set of cards, ace, two, three, &c., say up to thirteen, are arranged in a circle with their faces upwards—you begin at any card, and count one, two, three, &c., and if upon counting suppose the number five, you arrive at the card five, that card is thrown out; and beginning again with the next card, you count one, two, three, &c., throwing out if the case happen a new card as before, and so on until you have counted up to thirteen, without coming to a card which ought to be thrown out. It is easy to see that, whatever the number of the cards is, they may be so arranged as to be all thrown out in the order of their numbers; but that it is not possible in general to arrange the cards so that all the cards, or any specified cards, may be thrown out in a given order. Thus, if all the cards are to be thrown out in the order of their numbers, the arrangements in the case of a single card, two, three, &c. cards, are

1							
1	2						
1	3	2					
1	4	2	3				
1	3	2	5	4			
1	4	2	5	6	3		
1	5	2	7	4	3	6	
1	6	2	4	5	3	7	8

&c.

It is required to investigate the general theory.

162.

TWO LETTERS ON CUBIC FORMS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 85—87 and 90—91.]

CHER MONS. HERMITE,

Il y a longtemps que j'ai voulu vous écrire, mais j'en ai été empêché je ne sais comment; j'ai assez à vous dire par rapport aux covariants, mais à présent je vais vous parler des formes cubiques à deux indéterminées. Il me semble que l'on peut simplifier la théorie de Eisenstein, et l'étendre au cas d'un déterminant négatif quelconque, de la manière que voici.

Soit $(a, b, c, d \between x, y)^3$ une forme cubique, je représente par Hessn. $(a, b, c, d \between x, y)^3$ la forme quadratique dérivée $(ac-b^2, \frac{1}{2}(ad-bc), bd-c^2 \between x, y)^2$. Cela étant, soit (A, B, C) une forme représentative (réduite et proprement primitive) au déterminant $-D$; à moins que $(A, B, C)^2 = (A, -B, C)$, c'est-à-dire, à moins que (A, B, C) ne soit une forme laquelle par sa triplication produit la forme principale, il n'existe pas de forme cubique (a, b, c, d) telle que $-$ Hessn. $(a, b, c, d \between x, y)^3 = (A, B, C \between x, y)^2$, ou, si l'on veut, telle que $b^2-ac=A$, $bc-ad=2B$, $c^2-bd=C$; mais en supposant que l'on ait $(A, B, C)^2 = (A, -B, C)$ on peut trouver une seule forme cubique qui satisfait à l'équation dont il s'agit. J'écarte, cela va sans dire, l'une ou l'autre des deux formes (a, b, c, d) et $(-a, -b, -c, -d)$.

En effet on a identiquement

$$(b^2-ac, -\tfrac{1}{2}(bc-ad), c^2-bd \between bxx'+cxy'+dyy', axx'+bxy'+bx'y+cyy')^2$$
$$= (b^2-ac, \tfrac{1}{2}(bc-ad), c^2-bd \between x, y)^2 \times (b^2-ac, \tfrac{1}{2}(bc-ad), c^2-bd \between x', y')^2,$$

donc, en supposant que $b^2-ac=A$, $bc-ad=2B$, $c^2-bd=C$, il s'ensuit que $(A, B, C)^2 = (A, -B, C)$.

Je suppose donc $(A, B, C)^2 = (A, -B, C)$, et je dis qu'il ne peut pas y avoir deux formes cubiques, (a, b, c, d) et $(a_{,}, b_{,}, c_{,}, d_{,})$, qui aient la propriété dont il s'agit; car, en écrivant

$$\left.\begin{aligned}\xi &= bxx' + cxy' + cx'y + dyy',\\ \eta &= axx' + bxy' + bx'y + cyy',\end{aligned}\right\} \qquad \left.\begin{aligned}\xi_{,} &= b_{,}xx' + c_{,}xy' + c_{,}x'y + d_{,}yy',\\ \eta_{,} &= a_{,}xx' + b_{,}xy' + b_{,}x'y + c_{,}yy',\end{aligned}\right\}$$

on trouverait

$$(A, -B, C \between \xi, \eta)^2 = (A, -B, C \between \xi_{,}, \eta_{,})^2,$$

ce qui implique d'abord que $\xi_{,}, \eta_{,}$ soient des fonctions linéaires de ξ, η. Mais $(A, -B, C)$ étant une forme réduite et proprement primitive au déterminant $-D$, il n'existe pas de transformation de la forme quadratique en elle-même, hormis $\xi_{,} = \xi$, $\eta_{,} = \eta$. Le cas $D = 1$ doit se traiter à part; dans ce cas particulier il n'y a que la forme cubique $(0, 1, 0, 1)$. Donc &c. Enfin, si $(A, B, C)^2 = (A, -B, C)$, il existe une forme cubique (a, b, c, d) telle que $b^2 - ac = A$, &c.; car en cherchant par la méthode de Gauss les valeurs des coefficients p, p', p'', p''' et q, q', q'', q''' qui donnent cette transformation, on obtient d'abord $p' = p''$, $q' = q''$. On peut donc représenter ces coefficients par $b_{,}, c_{,}, c_{,}, d_{,}$; a, b, b, c; savoir, on peut trouver $a, b, c, b_{,}, c_{,}, d_{,}$ de manière que

$$\begin{aligned}(A, -B, C \between b_{,}xx' + c_{,}xy' + c_{,}x'y + d_{,}yy',\ & axx' + bxy' + bx'y + cyy')^2\\ &= (A, B, C \between x, y)^2 . (A, B, C \between x', y')^2.\end{aligned}$$

Cela étant, les équations de Gauss donnent

$$\begin{aligned} A &= bb_{,} - ac_{,}, & A &= b^2 - ac,\\ 2B &= cb_{,} - ad_{,}, & -2B &= cb_{,} + ad_{,} - 2bc_{,},\\ C &= cc_{,} - bd_{,}, & C &= c_{,}^2 - b_{,}d_{,},\end{aligned}$$

et de là on obtient

$$\begin{aligned} b\ (b - b_{,}) - a\ (c - c_{,}) &= 0,\\ c\ (b - b_{,}) - b\ (c - c') &= 0,\\ c_{,}(b - b_{,}) - b_{,}\ (c - c_{,}) &= 0,\\ d_{,}(b - b_{,}) - c_{,}\ (c - c_{,}) &= 0;\end{aligned}$$

c'est-à-dire, ou $\frac{a}{b} = \frac{b}{c} = \frac{b_{,}}{c_{,}} = \frac{c_{,}}{d_{,}}$, ce qui n'est pas vrai (car cela donnerait $A = 0$, $B = 0$, $C = 0$), ou $b - b_{,} = 0$, $c - c_{,} = 0$. Donc $b_{,} = b$, $c_{,} = c$; et en écrivant d au lieu de $d_{,}$, on voit que l'équation de transformation devient

$$\begin{aligned}(A, -B, C \between bxx' + cxy' + cx'y + dyy',\ & axx' + bxy' + bx'y + cyy')^2\\ &= (A, B, C \between x, y)^2 . (A, B, C \between x, y')^2,\end{aligned}$$

où

$$A = b^2 - ac, \quad 2B = bc - ad, \quad C = c^2 - bd. \quad \text{C. q. f. à d.}$$

Je ne sais pas si je vous ai mentionné que j'ai calculé les formes quadratiques pour les treize numéros -307, &c. aux déterminants irréguliers. Pour $D=-307$ ces formes sont:

Ordre P. P., 1 genre, 9 classes, c'est-à-dire:

Composition.				
1	δ	δ^2	$\frac{m}{307}$	$(1, 0, 307), (7, 1, 44), (7, -1, 44),$
$\delta_{,}$	$\delta\delta_{,}$	$\delta^2\delta_{,}$		$(4, 1, 77), (11, -1, 28), (17, 4, 19),$
$\delta_{,}^2$	$\delta\delta_{,}^2$	$\delta^2\delta_{,}^2$	$+$	$(4, -1, 77), (17, -4, 19), (11, 1, 28),$

où $\delta^3=1$, $\delta_{,}^3=1$.

Ordre I. P., 1 genre, 3 classes, c'est-à-dire:

$$\sigma,\quad \sigma\delta,\quad \sigma\delta^2 \quad | \quad + \quad | \, (2, 1, 154),\quad (14, 1, 22),\quad (14, -1, 22).$$

A chaque forme de l'ordre P. P. il corresponde done une forme et une seule forme cubique au déterminant -1228. Ces formes sont

$$\begin{aligned}
&(0, 1, 0, -307), (1, 1, -6, 8), (1, -1, -6, -8),\\
&(0, 2, 1, -38), (1, -3, -2, 8), (4, 1, -4, -3),\\
&(0, 2, -1, -38), (4, -1, -4, 3), (1, 3, -2, 8).
\end{aligned}$$

Je serai bien aise d'avoir de vos nouvelles, et je vous prie de me croire votre très-dévoué

A. CAYLEY.

CHER MONS. HERMITE,

On démontre sans peine la proposition avancée dans ma dernière lettre, savoir qu'en supposant

$$\begin{aligned}
\xi &= b\,xx' + c\,xy' + c\,x'y + d\,yy',\\
\eta &= a\,xx' + b\,xy' + b\,x'y + c\,yy',\\
\xi_{,} &= b_{,}xx' + c_{,}xy' + c_{,}x'y + d_{,}yy',\\
\eta_{,} &= a_{,}xx' + b_{,}xy' + c_{,}x'y + d_{,}yy',
\end{aligned}$$

$$(A, -B, C \between \xi, \eta)^2 = (A, -B, C \between \xi_{,}, \eta_{,})^2,$$

on doit avoir

$$\begin{aligned}
\xi_{,} &= \alpha\xi + \beta\eta,\\
\eta_{,} &= \gamma\xi + \delta\eta,
\end{aligned}$$

où α, β, γ, δ sont des entiers. En effet en trouve d'abord en éliminant par ex. xy' et $x'y$,

$$\xi_{,} = \alpha\xi + \beta\eta + \lambda xx' + \mu yy',$$
$$\eta_{,} = \gamma\xi + \delta\eta + \nu xx' + \rho yy',$$

où α, β, &c., λ, μ, &c., sont des quantités rationnelles. En substituant ces valeurs de ξ, η, on aura $(A, -B, C)(\lambda, \nu)^2 = 0$, $(A, -B, C)(\mu, \rho)^2 = 0$. Donc le déterminant n'étant pas un carré, $\lambda = 0$, $\nu = 0$, $\mu = 0$, $\rho = 0$ et

$$\xi_{,} = \alpha\xi + \beta\eta,$$
$$\eta_{,} = \gamma\xi + \delta\eta,$$

où α, β, γ, δ sont des quantités rationnelles. Donc

$$b_{,} = \alpha b + \beta a, \quad a_{,} = \gamma b + \delta a,$$
$$c_{,} = \alpha c + \beta b, \quad b_{,} = \gamma c + \delta b,$$
$$d_{,} = \alpha d + \beta c, \quad c_{,} = \gamma d + \delta c,$$

donc α, β seront des quantités rationnelles ayant pour dénominateur l'une quelconque à volonté des trois quantités $b^2 - ac$, $c^2 - bd$, $ad - bc$, c'est-à-dire, des quantités A, B, C; et, puisque (A, B, C) est une forme P. P., cela ne peut arriver à moins que α, β ne soient des entiers; de même, γ, δ seront des entiers. Je remarque de plus les équations

$$a\beta + b(\alpha - \delta) - c\gamma = 0,$$
$$b\beta + c(\alpha - \delta) - d\gamma = 0,$$

lesquelles donnent

$$\begin{aligned} \beta : \alpha - \delta : -\gamma = bd - c^2 &: bc - ad : ac - b^2 \\ = \quad C \quad &: -2B \quad : A, \end{aligned}$$

cela fait voir que la transformation en elle-même de la forme $(A, -B, C \between \xi, \eta)^2$ à moyen de $\xi_{,} = \alpha\xi + \beta\eta$, $\eta_{,} = \gamma\xi + \delta\eta$ est une transformation propre. J'aime cependant mieux la manière dont vous vous êtes servi pour déduire d'une forme cubique donnée toutes les autres formes cubiques qui correspondent à la même forme quadratique. Il serait facile de la même manière, étant donnée une transformation quelconque d'une forme quadratique dans le produit de deux autres formes quadratiques, d'en déduire toutes les autres transformations; car il y a pour la fonction $axx'x'' + \&c \ldots$ un covariant qui corresponde au cubicovariant de la fonction $ax^3 + \&c. \ldots$ Je suis votre très-dévoué

A. CAYLEY.

2, Stone Buildings, Lincoln's Inn, 6 Mars, 1855.

163.

ON HANSEN'S LUNAR THEORY.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 112—125.]

THE following paper was written in order to exhibit, in as clear a form as may be, the investigation of the remarkable equations for the motion of the moon established in Hansen's "Fundamenta Nova Investigationis Orbitæ veræ quam Luna perlustrat," &c., Gothæ, 1838. I have availed myself for this purpose of the remarks in Jacobi's two letters in answer to a letter of Hansen's, *Crelle*, t. XLII. [1851], p. 12; it may be convenient to remark that the quantity there represented by Λ, and which does not occur in Hansen's own investigation, is in this paper represented by Θ.

The position of the moon referred to the earth as centre is determined by

r, the radius vector,
L, the longitude,
Λ, the latitude.

Suppose, moreover, that the attractive force at distance unity, $=\kappa(M+E)$, is represented by n^2a^3, then the principal function will be $V=-\dfrac{n^2a^2}{r}$, and the disturbing function R may be represented by $n^2a^3\Omega$; the expression for the half of the vis viva is

$$T=\tfrac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2+r^2\cos^2\Lambda\left(\frac{dL}{dt}\right)^2+r^2\left(\frac{d\Lambda}{dt}\right)^2\right\},$$

and the equations of motion are therefore

$$\frac{d}{dt}\frac{dr}{dt}-r\cos^2\Lambda\left(\frac{dL}{dt}\right)^2-r\left(\frac{d\Lambda}{dt}\right)^2+\frac{n^2a^3}{r^2}=n^2a^3\frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2\cos^2\Lambda\frac{dL}{dt}\right)=n^2a^3\frac{d\Omega}{dL},$$

$$\frac{d}{dt}\left(r^2\frac{d\Lambda}{dt}\right)+r^2\cos\Lambda\sin\Lambda\left(\frac{dL}{dt}\right)^2=n^2a^3\frac{d\Omega}{d\Lambda},$$

where Ω is considered as a function of r, L, Λ.

Consider the orbit as an ellipse, then putting

a, the mean distance,

n, the mean motion $= \sqrt{\dfrac{\kappa(M+E)}{a^3}}$,

e, the excentricity,

c, the mean anomaly at epoch,

ω, the distance of perigee from node,

θ, the longitude of node,

i, the inclination,

Φ, the distance from node,

Ψ, the reduced distance from node, $= L - \theta$,

U, the excentric anomaly,

f, the true anomaly, $= \Phi - \omega$,

the elements of the orbit are a, e, c, ω, θ, i, and we have from the theory of elliptic motion,

$$nt + c = U - e \sin U,$$

$$f = \tan^{-1} \frac{\sqrt{(1-e^2)} \sin U}{\cos U - e},$$

$$r = a(1 - e \cos U) = \frac{a(1-e^2)}{1 + e \cos f}.$$

Moreover i is the angle at the base of a right-angled spherical triangle, the base, perpendicular, and hypothenuse of which are Ψ, Λ, Φ, hence

$$\tan \Psi = \cos i \tan \Phi,$$

$$\sin \Lambda = \sin i \sin \Phi,$$

$$\sin \Psi = \cot i \tan \Lambda,$$

$$\cos \Phi = \cos \Lambda \cos \Psi.$$

Considering the elements as constant, we have

$$\frac{dr}{dt} = \frac{nae \sin f}{\sqrt{(1-e^2)}},$$

$$\frac{df}{dt} = \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Phi}{dt} = \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Psi}{dt} = \frac{\cos i}{\cos^2 \Lambda} \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{dL}{dt} = \frac{\cos i}{\cos^2 \Lambda} \frac{na^2 \sqrt{(1-e^2)}}{r^2},$$

$$\frac{d\Lambda}{dt} = \sin i \cos \Psi \frac{na^2 \sqrt{(1-e^2)}}{r^2}.$$

Hence also

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{n^2a^3e\cos f}{r^3},$$

$$\frac{d}{dt}\left(r^2\cos^2\Lambda\frac{dL}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(r^2\frac{d\Lambda}{dt}\right) = -\frac{\cos^2 i\sin\Lambda}{\cos^3\Lambda}\,\frac{n^2a^4(1-e^2)}{r^2},$$

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = -\frac{2n^2a^3}{r^3}\,e\sin f.$$

The foregoing values show that the equations of motion, neglecting the terms which involve the disturbing functions, are satisfied by the elliptic values of r, L, Λ: and in order to satisfy the actual equations of motion, we have only to consider the elements as variable and to write

$$dr = 0,$$

$$dL = 0,$$

$$d\Lambda = 0,$$

$$d\frac{dr}{dt} = n^2a^3\frac{d\Omega}{dr}\,dt,$$

$$d\left(r^2\cos^2\Lambda\frac{dL}{dt}\right) = n^2a^3\frac{d\Omega}{dL}\,dt,$$

$$d\left(r^2\frac{d\Lambda}{dt}\right) = n^2a^3\frac{d\Omega}{d\Lambda}\,dt,$$

where the differentiations relate only to the elements, or, what is the same thing, to t in so far only as it enters through the variable elements: the system is at once transformed into

$$dr = 0,$$

$$dL = 0,$$

$$d\Lambda = 0,$$

$$d\frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\frac{d\Omega}{dr}\,dt,$$

$$d\,na^2\sqrt{(1-e^2)}\cos i = n^2a^3\frac{d\Omega}{dL}\,dt,$$

$$d\,na^2\sqrt{(1-e^2)}\sin i\cos\Psi = n^2a^3\frac{d\Omega}{d\Lambda}\,dt.$$

Now $\Psi = L - \theta$, or (supposing, as before, that the differentiations relate to t, only in so far as it enters through the variable elements) $d\Psi = -d\theta$, and thence $d\theta = \frac{\tan\Phi}{\sin i}\,di$; we have also $d\Phi = -\cos i\,d\theta$. The equations containing $\frac{d\Omega}{dL}$ and $\frac{d\Omega}{d\Lambda}$ give

$$\cos i\,d\,na^2\sqrt{(1-e^2)} - na^2\sqrt{(1-e^2)}\sin i\,di \qquad = n^2a^3\frac{d\Omega}{dL}\,dt,$$

$$\cos\Psi\sin i\,d\,na^2\sqrt{(1-e^2)} + na^2\sqrt{(1-e^2)}\,(\cos\Psi\cos i\,di + \sin\Psi\sin i\,d\theta) = n^2a^3\frac{d\Omega}{d\Lambda}\,dt;$$

or, expressing $d\theta$ by means of di and reducing, the second of these equations becomes

$$\sin i\,d\,na^2\sqrt{(1-e^2)} + na^2\sqrt{(1-e^2)}\frac{\cos i}{\cos^2\Lambda\cos^2\Psi}\,di \qquad = \frac{1}{\cos\Psi}n^2a^3\frac{d\Omega}{d\Lambda}\,dt,$$

and combining this with the first of the two equations, and observing that

$$\frac{\cos^2 i}{\cos^2\Lambda\cos^2\Psi} + \sin^2 i = \frac{1}{\cos^2\Psi},$$

we find

$$d\,na^2\sqrt{(1-e^2)} = \quad n^2a^3\left(\frac{\cos i}{\cos^2\Lambda}\qquad\frac{d\Omega}{dL} + \sin i\cos\Psi\frac{d\Omega}{d\Lambda}\right)dt,$$

$$di \qquad = \frac{na^2}{\sqrt{(1-e^2)}}\left(-\sin i\cos^2\Psi\frac{d\Omega}{dL} + \cos i\cos\Psi\frac{d\Omega}{d\Lambda}\right)dt.$$

Now, considering Ω as a function of r, θ, i, Φ, then Λ, L are given as functions of θ, i, Φ by the equations $\sin\Lambda = \sin i\sin\Phi$, $\tan\Psi = \cos i\tan\Phi$, $\Psi = L - \theta$, and after some simple reductions,

$$\frac{d\Omega}{dr} = \quad\frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\theta} = \quad\frac{d\Omega}{dL},$$

$$\frac{d\Omega}{di} = \tan\Phi\left(-\sin i\cos^2\Psi\frac{d\Omega}{dL} + \cos i\cos\Psi\frac{d\Omega}{d\Lambda}\right),$$

$$\frac{d\Omega}{d\Phi} = \qquad\left(\qquad\frac{\cos i}{\cos^2\Lambda}\frac{d\Omega}{dL} + \sin i\cos\Psi\frac{d\Omega}{d\Lambda}\right);$$

whence also

$$\frac{d\Omega}{d\theta} = \cos i\frac{d\Omega}{d\Phi} - \sin i\cot\Phi\frac{d\Omega}{di}.$$

We have therefore

$$
\begin{aligned}
d\, na^2\sqrt{(1-e^2)} &= n^2a^3\frac{d\Omega}{d\Phi}\,dt,\\
di &= \frac{na\cot\Phi}{\sqrt{(1-e^2)}}\frac{d\Omega}{di}\,dt,\\
d\theta &= \frac{na}{\sqrt{(1-e^2)}\sin i}\frac{d\Omega}{di}\,dt,\\
d\Phi &= \frac{-na\cot i}{\sqrt{(1-e^2)}}\frac{d\Omega}{di}\,dt,\\
dr &= 0,\\
d\frac{nae\sin f}{\sqrt{(1-e^2)}} &= n^2a^3\frac{d\Omega}{dr}\,dt.
\end{aligned}
$$

Suppose now that we have

ρ, a radius vector, τ for t,

ϕ, a true anomaly, do.,

v, an excentric anomaly, do.,

i.e. let ρ, ϕ, v, be what the radius vector, the true anomaly and the excentric anomaly become when the time t, in so far as it enters directly, and not through the variable elements, is replaced by a new variable τ. We have

$$
\begin{aligned}
n\tau + c &= v - e\sin v,\\
\phi &= \tan^{-1}\frac{\sqrt{(1-e^2)}\sin v}{\cos v - e},\\
\rho &= a(1-e\cos v) = \frac{a(1-e^2)}{1+e\cos\phi};
\end{aligned}
$$

and of course the differential coefficients of ρ, ϕ with respect to τ may be at once deduced from the corresponding expressions for the differential coefficients of r, f with respect to t, the elements being considered as constant. Now, using l to denote a logarithm, and supposing that the differentiations affect only the elements, we have

$$
\begin{aligned}
dl\rho &= \frac{da}{a} - \frac{2ede}{1-e^2} - \frac{\rho\cos\phi de}{a(1-e^2)} + \frac{\rho e\sin\phi d\phi}{a(1-e^2)},\\
dlr &= \frac{da}{a} - \frac{2ede}{1-e^2} - \frac{r\sin f de}{a(1-e^2)} + \frac{re\sin f df}{a(1-e^2)};
\end{aligned}
$$

and putting for shortness

$$
\begin{aligned}
X_{,} &= dl\rho - \frac{\rho e\sin\phi d\phi}{a(1-e^2)},\\
X &= dlr - \frac{re\sin f df}{a(1-e^2)},
\end{aligned}
$$

we find

$$
X_{,} - \frac{\rho\sin\phi}{r\sin f}X = \left(1 - \frac{\rho\sin\phi}{r\sin f}\right)\left(\frac{da}{a} - \frac{2ede}{1-e^2}\right) - \frac{\rho}{a(1-e^2)}\left(\cos\phi - \frac{\sin\phi\cos f}{\sin f}\right)de,
$$

or reducing

$$X_{,} - \frac{\rho \sin \phi}{r \sin f} X =$$

$$\frac{1}{\sin f (1 + e \cos \phi)} \left\{ \left(\sin f - \sin \phi + e \sin (f - \phi) \right) \left(\frac{da}{a} - \frac{2ede}{1 - e^2} \right) - \sin (f - \phi)\, de \right\}.$$

Write for a moment

$$P = na^2 \sqrt{(1 - e^2)}, \quad Q = \frac{nae \sin f}{\sqrt{(1 - e^2)}}, \quad R = \frac{a (1 - e^2)}{1 + e \cos f},$$

so that

$$a (1 - e^2) = \frac{P^2}{n^2 a^3},$$

$$e^2 \qquad = \frac{1}{n^4 a^6} P^2 Q^2 + \frac{1}{n^4 a^6} \frac{P^4}{R^2} - \frac{2}{n^2 a^3} \frac{P^2}{R} + 1.$$

We have therefore

$$\frac{da}{a} - \frac{2ede}{1 - e^2} = \frac{2dP}{P} = \frac{2}{na^2 (1 - e^2)} dP,$$

$$ede = \left(\frac{1}{n^4 a^6} PQ^2 - \frac{2}{n^4 a^6} \frac{P^3}{R^2} - \frac{2}{n^2 a^3} \frac{P}{R} \right) dP + \frac{1}{n^4 a^6} P^2 Q dQ + \left(- \frac{1}{n^4 a^6} \frac{P^4}{R^3} + \frac{2}{n^2 a^3} \frac{P^2}{R^2} \right) dR,$$

which, after reduction, becomes

$$de = \frac{1}{na^2 (1 - e^2)} \left(e \sin^2 f + 2 (\cos f + e \cos^2 f) \right) dP + \frac{1}{na} \sqrt{(1 - e^2)} \sin f\, dQ - \frac{(1 + e \cos f)^2 \cos f}{a (1 - e^2)} dR,$$

and substituting these values,

$$X_{,} - \frac{\rho \sin \phi}{r \sin f} X = \frac{1}{1 + e \cos \phi} \left\{ \left(2 - 2 \cos (f - \phi) + e \sin f \sin (f - \phi) \right) \frac{1}{na^2 \sqrt{(1 - e^2)}} dP \right.$$

$$- a (1 - e^2) \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}} dQ$$

$$\left. + \frac{(1 + e \cos f)^2 \cot f \sin (f - \phi)}{\sqrt{(1 - e^2)}} \frac{1}{na^2 \sqrt{(1 - e^2)}} dR \right\},$$

or substituting for X, $X_{,}$, P, Q, R, their values

$$dl\rho - \frac{\rho \sin \phi}{r \sin f} dlr + \frac{\rho e \sin \phi}{a (1 - e^2)} (df - d\phi) =$$

$$\frac{\rho}{a (1 - e^2)} \left(2 - 2 \cos (f - \phi) + e \sin f \sin (f - \phi) \right) \frac{1}{na^2 \sqrt{(1 - e^2)}} dna^2 \sqrt{(1 - e^2)}$$

$$- \rho \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}} d \frac{nae \sin f}{\sqrt{(1 - e^2)}}$$

$$+ \frac{\rho a \sqrt{(1 - e^2)}}{r^2} \cot f \sin (f - \phi) \frac{1}{na^2 \sqrt{(1 - e^2)}} dr.$$

Now

$$-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}$$
$$=\frac{\rho}{a(1-e^2)}\Big(2-2\cos(f-\phi)+e\sin f\sin(f-\phi)\Big);$$

therefore

$$dl\rho-\frac{\rho\sin\phi}{r\sin f}\,dlr+\frac{\rho e\sin\phi}{a(1-e^2)}(df-d\phi)=$$
$$-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}\frac{1}{na^2\sqrt{(1-e^2)}}\,dna^2\sqrt{(1-e^2)}$$
$$-\rho\sin(f-\phi)\frac{1}{na^2\sqrt{(1-e^2)}}\,d\,\frac{nae\sin f}{\sqrt{(1-e^2)}}$$
$$+\frac{\rho a\sqrt{(1-e^2)}}{r^2}\cot f\sin(f-\phi)\frac{1}{na^2\sqrt{(1-e^2)}}\,dr.$$

So far the variations of the elements have, in fact, been treated as independent; but if we substitute for $dna^2\sqrt{(1-e^2)}$, $d\,\frac{nae\sin f}{\sqrt{(1-e^2)}}$, dr, their values in the disturbed motion, the equation becomes

$$dl\rho+\frac{\rho e\sin\phi}{a(1-e^2)}(df-d\phi)=-\left\{\frac{\rho}{r}\cos(f-\phi)-1+\frac{\rho}{a(1-e^2)}\Big(\cos(f-\phi)-1\Big)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{d\Phi}dt$$
$$-\rho\sin(f-\phi)\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dr}dt.$$

Consider now the point in which the orbit is intersected by any orthogonal trajectory to the successive positions of the orbit, or to fix the ideas, the orthogonal trajectory passing through ♈, the point in question may, for want of a recognised name, be called the "departure point;" and the angular distances in the orbit measured from this point may be termed "departures;" the expression, "the departure," is to be understood as meaning the departure of the moon. Write now

χ, the departure of the perigee,

$v_{,}$, the departure, $=f+\chi$,

σ, the departure of the node, $=\chi-\omega$,

Θ, the longitude in orbit of departure point, $=\theta-\sigma$.

It should be remarked that χ is not properly an element, i.e. it is not a function of a, e, c, ω, θ, i without t, and in like manner σ and Θ (which depend upon χ) are not elements.

We have from the geometrical definition

$$d\chi=d\omega+\cos i\,d\theta;$$

and therefore

$$d\sigma = \cos i\, d\theta,$$
$$d\Theta = (1 - \cos i)\, d\theta.$$

Moreover $v_{,} = \Phi + \sigma$, which gives (assuming that the differentiations are performed with respect to t, only in so far as it enters through the variable elements) $dv_{,} = d\Phi + d\sigma = d\Phi + \cos i\, d\theta$, i.e. $dv_{,} = 0$, an equation which might have been assumed for the purpose of defining the departure point; the equation, in fact, expresses that the departure $v_{,}$ is measured from a point not actually fixed, but such that the increment of $v_{,}$ in the interval of time dt is the angular distance between two consecutive positions of the moon.

We have, as above noticed, $d\sigma = \cos i\, d\theta$, and thence and from what has preceded

$$di = \frac{na \cot \Phi}{\sqrt{(1-e^2)}} \frac{d\Omega}{di},$$

$$d\sigma = \frac{na \cot i}{\sqrt{(1-e^2)}} \frac{d\Omega}{di}.$$

Now the position of the moon can be determined by means of the quantities r, $v_{,}$, Θ, σ, i; hence Ω (which has been considered as a function of r, Φ, θ, i) may, if we please, be considered as a function of r, $v_{,}$, Θ, σ, i and from the differential relations

$$dr = dr,$$
$$dv_{,} = d\Phi + \cos i\, d\theta,$$
$$d\Theta = (1 - \cos i)\, d\theta,$$
$$d\sigma = d\omega + \cos i\, d\theta,$$
$$di = di,$$

we find

$$\frac{d\Omega}{dr} = \frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\Phi} = \frac{d\Omega}{dv_{,}},$$

$$\frac{d\Omega}{d\theta} = \cos i \left(\frac{d\Omega}{dv_{,}} + \frac{d\Omega}{d\sigma}\right) + \left(1 - \cos i\right) \frac{d\Omega}{d\Theta},$$

$$\frac{d\Omega}{di} = \frac{d\Omega}{di};$$

we have therefore

$$\frac{d\Omega}{d\sigma} + \frac{1 - \cos i}{\cos i} \frac{d\Omega}{d\Theta} = -\frac{d\Omega}{d\Phi} + \frac{1}{\cos i} \frac{d\Omega}{d\theta},$$

or in virtue of a preceding equation

$$\frac{d\Omega}{d\sigma} + \frac{1 - \cos i}{\cos i} \frac{d\Omega}{d\Theta} = -\tan i \cot \Phi \frac{d\Omega}{di};$$

and effecting the substitutions, and collecting the results,

$$d\ na^2\sqrt{(1-e^2)} = n^2a^3\frac{d\Omega}{dv_{,}}\,dt,$$

$$dr = 0,$$

$$d\,\frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\frac{d\Omega}{dr}\,dt,$$

$$di = -\frac{na\cot i}{\sqrt{(1-e^2)}}\left(\frac{d\Omega}{d\sigma}+\frac{1-\cos i}{\cos i}\,\frac{d\Omega}{d\Theta}\right)dt,$$

$$d\sigma = \frac{na\cot i}{\sqrt{(1-e^2)}}\,\frac{d\Omega}{di}\,dt,$$

where Ω is considered as a function of r, $v_{,}$, Θ, σ, i.

Instead of σ, i we may introduce the new quantities p, q defined by the equations

$$p = \sin i \sin\sigma,$$
$$q = \sin i\cos\sigma;$$

this gives $\sin^2 i = p^2+q^2$, $\sigma = \tan^{-1}\frac{p}{q}$ and retaining in the formulæ the sine and cosine of i, to avoid the introduction of irrational functions of p^2+q^2, we have

$$d\Theta = (1-\cos i)\,d\theta = \frac{1-\cos i}{\cos i}\,d\sigma = \frac{1-\cos i}{\cos i\sin^2 i}(qdp-pdq),$$

i.e.

$$d\Theta = \frac{qdp-pdq}{\cos i\,(1+\cos i)},$$

which determines Θ by means of p and q. We have moreover

$$dp = \sin i\cos\sigma\,d\sigma + \cos i\sin\sigma\,di,$$

$$dq = -\sin i\sin\sigma\,d\sigma + \cos i\cos\sigma\,di,$$

$$\frac{d\Omega}{dp} = \frac{\sin\sigma}{\cos i}\,\frac{d\Omega}{di} + \frac{\cos\sigma}{\sin i}\,\frac{d\Omega}{d\sigma},$$

$$\frac{d\Omega}{dq} = \frac{\cos\sigma}{\cos i}\,\frac{d\Omega}{di} - \frac{\sin\sigma}{\sin i}\,\frac{d\Omega}{d\sigma},$$

from which equations and the foregoing values of di and $d\sigma$ we find the values of dp and dq; the other equations of the system remain unaltered, and we have therefore

$$d\ na^2\sqrt{(1-e^2)} = n^2a^3\,\frac{d\Omega}{dv_{,}}\,dt,$$

$$dr = 0,$$

$$d\,\frac{nae\sin f}{\sqrt{(1-e^2)}} = n^2a^3\,\frac{d\Omega}{dr}\,dt,$$

$$dp = \frac{na\cos^2 i}{\sqrt{(1-e^2)}}\left(\frac{d\Omega}{dq} - \frac{p}{\cos i\,(1+\cos i)}\frac{d\Omega}{d\Theta}\right)dt,$$

$$dq = -\frac{na\cos^2 i}{\sqrt{(1-e^2)}}\left(\frac{d\Omega}{dp} + \frac{q}{\cos i\,(1+\cos i)}\frac{d\Omega}{d\Theta}\right)dt,$$

where Ω is considered as a function of r, $v_{,}$, Θ, p, q. The symbols $\frac{d\Omega}{dp}$, $\frac{d\Theta}{dq}$, as employed by Hansen, mean that the differentiations are to be performed as if Θ was a function of p, q, such that

$$d\Theta = \frac{d\Theta}{dp}dp + \frac{d\Theta}{dq}dq = \frac{qdp - pdq}{\cos i\,(1+\cos i)};$$

the last two equations being therefore nothing else than what Hansen represents by

$$dp = \frac{na\cos^2 i}{\sqrt{(1-e^2)}}\frac{d\Omega}{dq}dt,$$

$$dq = \frac{na\cos^2 i}{\sqrt{(1-e^2)}}\frac{d\Omega}{dp}dt.$$

Write now

λ, the departure, τ for t,

i.e. λ is what $v_{,}$ becomes when t, in so far as it enters explicitly, and not through the variable elements, is replaced by the new variable τ; so that, in fact $\lambda = \phi + \chi$. The values of r, $v_{,}$ could be at once found from those of ρ, λ by changing τ into t; and to determine the values of ρ, λ, Hansen proceeds as follows:

writing

$$\lambda = \Pi(\zeta, t),$$

$$l\rho = \Gamma(\zeta, t) + \beta,$$

where ζ and β are new variables functions of τ and t, and Π, Γ are arbitrary functional symbols; so that if z, w are what ζ, β become when τ is changed into t, we should have

$$v_{,} = \Pi(z, t),$$

$$lr = \Gamma(z, t) + w;$$

then the foregoing equations give

$$\frac{d\lambda}{d\tau} = \Pi'(\zeta, t)\frac{d\zeta}{d\tau},$$

$$\frac{d\lambda}{dt} = \Pi'(\zeta, t)\frac{d\zeta}{dt} + \Pi_{,}(\zeta, t),$$

$$\frac{dl\rho}{d\tau} = \Gamma'(\zeta, t)\frac{dl\rho}{d\tau} + \frac{d\beta}{d\tau},$$

$$\frac{dl\rho}{dt} = \Gamma'(\zeta, t)\frac{dl\rho}{dt} + \Gamma_{,}(\zeta, t) + \frac{d\beta}{dt};$$

[where the accents and strokes denote differentiation in regard to ζ, t respectively].

Hence eliminating $\Pi'(\zeta, t)$ and $\Gamma'(\zeta, t)$ we have

$$\frac{d\lambda}{d\tau}\frac{d\zeta}{dt}-\frac{d\lambda}{dt}\frac{d\zeta}{d\tau}=-\Pi_{,}(\zeta, t)\frac{d\zeta}{d\tau},$$

$$\frac{dl\rho}{d\tau}\frac{d\zeta}{dt}-\frac{dl\rho}{dt}\frac{d\zeta}{d\tau}=\frac{d\beta}{d\tau}\frac{d\zeta}{dt}-\frac{d\beta}{dt}\frac{d\zeta}{d\tau}-\Gamma_{,}(\zeta, t)\frac{d\zeta}{dt},$$

or, what is the same thing,

$$\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}=\frac{\frac{d\lambda}{dt}}{\frac{d\lambda}{d\tau}}-\Pi_{,}(\zeta, t)\frac{1}{\frac{d\lambda}{d\tau}},$$

$$\frac{d\beta}{dt}-\frac{d\beta}{d\tau}\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}}=\frac{dl\rho}{dt}-\frac{\frac{d\lambda}{dt}}{\frac{d\lambda}{d\tau}}\frac{dl\rho}{d\tau}+\Pi_{,}(\zeta, t)\frac{\frac{dl\rho}{d\tau}}{\frac{d\lambda}{d\tau}}-\Gamma_{,}(\zeta, t),$$

or writing

$$T=\frac{d}{d\tau}\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}},\quad R=\frac{d\beta}{dt}-\frac{d\beta}{d\tau}\frac{\frac{d\zeta}{dt}}{\frac{d\zeta}{d\tau}},$$

we have

$$T=\frac{1}{\frac{d\lambda}{d\tau}}\frac{d^2\lambda}{dtd\tau}-\frac{\frac{d^2\lambda}{d\tau^2}}{\left(\frac{d\lambda}{d\tau}\right)^2}\frac{d\lambda}{dt}+\Pi_{,}(\zeta, t)\frac{\frac{d^2\lambda}{d\tau^2}}{\left(\frac{d}{d\tau}\right)^2}-\Pi_{,}(\zeta, t)\frac{\frac{d\zeta}{d\tau}}{\frac{d\lambda}{d\tau}},$$

$$R=\frac{dl\rho}{dt}-\frac{\frac{dl\rho}{d\tau}}{\frac{d\lambda}{d\tau}}\frac{d\lambda}{d\tau}+\Pi_{,}(\zeta, t)\frac{\frac{dl\rho}{d\tau}}{\frac{d\lambda}{d\tau}}-\Gamma_{,}(\zeta, t).$$

Now from the equation $\lambda=\phi+\chi$ where χ is independent of τ, the differential coefficients of λ and ρ with respect to τ are at once deduced from those of f, r with respect to t, and we have,

$$\frac{d^2\lambda}{d\tau^2}=-\frac{2n^2a^2}{\rho^3}e\sin\phi,$$

$$\frac{d\lambda}{d\tau}=\frac{na^2\sqrt{(1-e^2)}}{\rho^2},$$

$$\frac{dl\rho}{d\tau}=\frac{nae\sin\phi}{\rho\sqrt{(1-e^2)}}.$$

Consequently

$$T=\frac{\rho^2}{na^2\sqrt{(1-e^2)}}\frac{d}{dt}\frac{na^2\sqrt{(1-e^2)}}{\rho^2}+\frac{2\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}$$

$$-\frac{2\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t)-\frac{\rho^2}{na^2\sqrt{(1-e^2)}}\Pi'_{,}(\zeta,\ t)\frac{d\zeta}{dt},$$

$$=-2\frac{dl\rho}{dt}+\frac{2\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}+\frac{1}{na^2\sqrt{(1-e^2)}}\frac{d}{dt}na^2\sqrt{(1-e^2)}$$

$$-\frac{2\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t)-\frac{\rho^2}{na^2\sqrt{(1-e^2)}}\Pi'_{,}(\zeta,\ t)\frac{d\zeta}{dt},$$

$$R=\frac{dl\rho}{dt}-\frac{\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}+\frac{\rho e\sin\phi}{a(1-e^2)}\Pi_{,}(\zeta,\ t)-\Gamma_{,}(\zeta,\ t),$$

and substituting in these equations the values of

$$\frac{dl\rho}{dt}-\frac{\rho e\sin\phi}{a(1-e^2)}\frac{d\lambda}{dt}\text{ and }\frac{d}{dt}na^2\sqrt{(1-e^2)}$$

we find

$$T=\left\{2\frac{\rho}{r}\cos(v_{,}-\lambda)-1+\frac{2\rho}{a(1-e^2)}\left(\cos(v_{,}-\lambda)-1\right)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dv_{,}}+2\frac{\rho}{r}\sin(v_{,}-\lambda)\frac{na}{\sqrt{(1-e^2)}}r\frac{d\Omega}{dr}$$

$$-2\Pi_{,}(\zeta,\ t)\frac{\rho e\sin\phi}{a(1-e^2)}-\Pi'_{,}(\zeta,\ t)\frac{\rho^2}{na^2\sqrt{(1-e^2)}}\frac{d\zeta}{dt}$$

$$R=-\left\{\frac{\rho}{r}\cos(v_{,}-\lambda)-1+\frac{\rho}{a(1-e^2)}\left(\cos(v_{,}-\lambda)-1\right)\right\}\frac{na}{\sqrt{(1-e^2)}}\frac{d\Omega}{dv_{,}}-\frac{\rho}{r}\sin(v_{,}-\lambda)\frac{na}{\sqrt{(1-e^2)}}r\frac{d\Omega}{dr}$$

$$+\Pi_{,}(\zeta,\ t)\frac{\rho e\sin\phi}{a(1-e^2)}-\Gamma_{,}(\zeta,\ t),$$

which are Hansen's values, except that $\frac{d\lambda}{d\tau}$ in the coefficient of $\Pi'_{,}(\zeta,\ t)$ has been replaced by its value.

2, *Stone Buildings*, 31*st March*, 1855.

164.

ON GAUSS' METHOD FOR THE ATTRACTION OF ELLIPSOIDS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 162—166.]

THE following is the method employed in Gauss' Memoir "Theoria Attractionis Corporum Sphæroidicorum ellipticorum homogeneorum methodo novo tractata," 1813. *Comm. Gott. recent.*, t. II. [and *Werke* t. VI. pp. 1—22]. I have somewhat developed the geometrical considerations upon which the method depends.

The attraction of the ellipsoid is found by means of the following theorems, which apply generally to the case of a homogeneous solid bounded by a closed surface:— M denotes the attracted point, P a point of the surface, PQ is the normal (lying outside the surface) at the point P, dS is the element of the surface at this point, MQ, QX, and MX denote angles at the point P, viz. MQ the $\angle MPQ$, and QX and MX the inclinations of QP and MP respectively to a line PX drawn in a direction assumed as that of the axis of X, $\overline{MP}$ denotes the distance between the points M and P. And X is the attraction in the direction opposite to that of the axis of x; the integrations extend over the entire surface.

THEOREM. The integral

$$\iint \frac{dS \cos MQ}{\overline{MP}^2}$$

has for its value

$$0, \quad -2\pi, \text{ or } -4\pi,$$

according as M is exterior to, upon, or interior to, the surface.

This is obviously a purely geometrical theorem.

THEOREM. The attraction is given by the formula

$$X = \iint \frac{dS \cos QX}{\overline{MP}}.$$

THEOREM. The attraction is also given by the formula

$$X = -\iint \frac{dS \cos MQ \cos MX}{\overline{MP}}.$$

Consider now an ellipsoid, the semi-axes of which are A, B, C, and putting a, b, c for the coordinates of the attracted point M, and x, y, z for the coordinates of P, assume

$$x = A\xi,$$
$$y = B\eta,$$
$$z = C\zeta,$$

so that ξ, η, ζ are the coordinates of a point P' on a sphere, radius unity, corresponding in a definite manner to the point P on the ellipsoid; and let $d\sigma$ be the corresponding element of the spherical surface, we have

$$dS = \frac{ABC}{p} d\sigma,$$

where p denotes the perpendicular let fall from the centre of the ellipsoid upon the tangent plane at P.

Moreover,

$$\cos QX = \frac{p\xi}{A}, \quad \cos MX = \frac{a-x}{\overline{MP}} = \frac{a - A\xi}{\overline{MP}};$$

and therefore

$$dS \cos QX = \frac{BC\xi d\sigma}{\overline{MP}}.$$

The second theorem gives therefore

$$A \frac{X}{ABC} = \iint \frac{\xi d\sigma}{\overline{MP}},$$

where the integration is extended over the surface of the sphere, and the third theorem gives

$$\frac{X}{ABC} = -\iint \frac{(a - A\xi) \cos MQ dS}{\overline{MP}^2},$$

where the integration is extended over the surface of the ellipsoid.

Suppose now a confocal ellipsoid, the semi-axes of which are $A + \delta A$, $B + \delta B$, $C + \delta C$, and let $P_{,}$ be the point on this new ellipsoid which *corresponds* to the point P on the original ellipsoid, i.e. let $P_{,}$ be the point whose coordinates are $(A + \delta A)\xi$, $(B + \delta B)\eta$, $(C + \delta C)\zeta$; the decrement of MP will be equal to the normal distance δN between the two ellipsoids at the point P, multiplied into the cosine of the angle MQ, and we have, by a property of confocal ellipsoids, $A\delta A = B\delta B = C\delta C = p\delta N$; we have therefore

$$\delta \overline{MP} = -\frac{A\delta A \cos MQ}{p},$$

which gives

$$d\sigma \,.\, \delta\overline{MP} = \frac{A\delta A}{ABC}\, dS \cos MQ.$$

Now from the equation

$$A\,\frac{X}{ABC} = \iint \frac{\xi d\sigma}{\overline{MP}},$$

we find

$$A\delta\,\frac{X}{ABC} + \frac{X}{ABC}\,\delta A = \iint \frac{\xi d\sigma \delta\overline{MP}}{\overline{MP}^2}$$

$$= \frac{\delta A}{ABC}\iint \frac{A\xi \cos MQ dS}{\overline{MP}^2}.$$

But

$$-\frac{X}{ABC}\,\delta A = \frac{\delta A}{ABC}\iint \frac{(a - A\xi)\cos MQ dS}{\overline{MP}^2},$$

and consequently

$$A\delta \,.\, \frac{X}{ABC} = \frac{a\delta A}{ABC}\iint \frac{\cos MQ dS}{\overline{MP}^2}.$$

Hence, by the first theorem:

In the case of an exterior point, we have

$$\delta \,.\, \frac{X}{ABC} = 0,$$

i.e. the attractions, in the directions of the axes, of confocal ellipsoids vary as the masses; which is Maclaurin's theorem for the attractions of ellipsoids upon an exterior point.

In the case of an interior point, we have

$$\delta \,.\, \frac{X}{ABC} = -4\pi\,\frac{a\delta A}{A^2BC};$$

or, taking α, β, γ as the semi-axes of an ellipsoid confocal with the ellipsoid (A, B, C), but exterior to it, and supposing that (X) refers to the ellipsoid (α, β, γ), we have

$$\delta\frac{(X)}{\alpha\beta\gamma} = -4\pi a\,\frac{\delta\alpha}{\alpha^2\beta\gamma}.$$

Now introducing instead of α the new variable θ, such that $\alpha^2 = A^2 + \theta$, we have $\frac{\delta\alpha}{\alpha^2} = \frac{\delta\theta}{(A^2+\theta)^{\frac{3}{2}}}$, $\beta = (B^2+\theta)^{\frac{1}{2}}$, $\gamma = (C^2+\theta)^{\frac{1}{2}}$, and consequently writing d for δ,

$$d\frac{(X)}{\alpha\beta\gamma} = -4\pi a\,\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

and thence, effecting the integration,

$$\frac{X_{,}}{(A^2+\theta_{,})^{\frac{1}{2}}(B^2+\theta_{,})^{\frac{1}{2}}(C^2+\theta_{,})^{\frac{1}{2}}}-\frac{X}{ABC}=-4\pi a\int_0^{\theta_{,}}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

where $X_{,}$ refers to the ellipsoid whose semi-axes are $(A^2+\theta_{,})^{\frac{1}{2}}$, $(B^2+\theta_{,})^{\frac{1}{2}}$, $(C^2+\theta_{,})^{\frac{1}{2}}$. In the case where $\theta_{,}=\infty$, we have

$$\frac{X_{,}}{(A^2+\theta_{,})^{\frac{1}{2}}(B^2+\theta_{,})^{\frac{1}{2}}(C^2+\theta_{,})^{\frac{1}{2}}}=0,$$

and consequently

$$X=4\pi a\,ABC\int_0^{\infty}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

which is the expression for the attraction, in the direction opposite to that of the axis of x, of the ellipsoid (A, B, C) upon an interior point, the coordinates of which are (a, b, c).

In the case of an exterior point, let $A_{,}$, $B_{,}$, $C_{,}$ be the semi-axes of the confocal ellipsoid passing through the attracted point; so that putting $A_{,}=\sqrt{(A^2+\eta)}$, $B_{,}=\sqrt{(B^2+\eta)}$, $C_{,}=\sqrt{(C^2+\eta)}$, we have

$$\frac{a^2}{A^2+\eta}+\frac{b^2}{B^2+\eta}+\frac{c^2}{C^2+\eta}=1,$$

the attraction is equal to $\dfrac{ABC}{A_{,}B_{,}C_{,}}\times$ attraction of the ellipsoid which passes through the point, i.e.

$$X=4\pi a\,ABC\int_0^{\infty}\frac{d\theta}{(A_{,}^2+\theta)^{\frac{3}{2}}(B_{,}^2+\theta)^{\frac{1}{2}}(C_{,}^2+\theta)^{\frac{1}{2}}};$$

or, putting $\theta-\eta$ instead of θ,

$$X=4\pi a\,ABC\int_{\eta}^{\infty}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

which is the expression for the attraction upon an exterior point. The formulæ coincide, as they ought to do, in the case of a point upon the surface.

2, *Stone Buildings*, *9th April*, 1855.

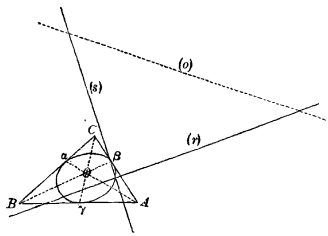
165.

ON SOME GEOMETRICAL THEOREMS RELATING TO A TRIANGLE CIRCUMSCRIBED ABOUT A CONIC.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 169—175.]

THE following investigations were suggested to me by Sir F. Pollock's interesting paper "On a Geometrical Theorem relating to an Equilateral Triangle circumscribed about a Circle," [*Quart. Math. Jour.* t. I. (1857), pp. 167—169].

If on the sides of a triangle ABC, there be taken points α, β, γ, such that $A\alpha$, $B\beta$, $C\gamma$ meet in a point O; and if on each side of the triangle there be taken two points forming with the two angles on the same side an involution having the first-mentioned point on the same side for a double point; then if three of the six points lie in a line, the two lines are said to be harmonically related with respect to the triangle ABC and point O. Call these the lines (r), (s).



The triangle ABC and point O give rise to a determinate conic; viz., the conic touching the sides at the points α, β, γ.

The harmonic of the point O with respect to the triangle is a line (o), which is also the polar of O with respect to the conic. The conic meets this line in two points (as the figure is drawn, imaginary ones), I, J.

Suppose now that the harmonic lines (r), (s) are harmonically related to the points I, J, (i.e. let the lines (r), (s) and the lines through their point of intersection and I, J, be a harmonic pencil, (or, what is the same thing, let I, J, and the points in which the line of junction meets the lines (r), (s) be a harmonic range), then, *Theorem*, the point of intersection of the lines (r), (s) lies on the conic.

In order to prove this, take

$$x=0,\ y=0,\ z=0, \text{ for the equations of } BC,\ CA,\ AB,$$

$$x=y=z, \text{ for the point } O,$$

the equations of the harmonic lines will be

$$ax+by+cz=0,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,$$

the equation of the conic is

$$x^2+y^2+z^2-2yz-2zx-2xy=0\,;$$

the equation of the line (o) is

$$x+y+z=0.$$

The coordinates of the points I, J, are

$$x:y:z=1:\omega:\omega^2,$$

and

$$x:y:z=1:\omega^2:\omega,$$

where ω is an imaginary cube root of unity.

The equations of the lines joining the point of intersection of (r), (s) with the points I, J, are

$$\frac{ax+by+cz}{a+b\omega+c\omega}=\frac{bcx+cay+abz}{bc+ca\omega+ab\omega^2},$$

and

$$\frac{ax+by+cz}{a+b\omega^2+c\omega}=\frac{bcx+cay+abz}{bc+ca\omega^2+ab\omega},$$

and these will form with the lines

$$ax+by+cz=0,$$

$$bcx+cay+abz=0,$$

a harmonic pencil, if

$$(a+b\omega+c\omega^2)(bc+ca\omega^2+ab\omega)$$
$$+(a+b\omega^2+c\omega)(bc+ca\omega+ab\omega^2)=0;$$

i.e., if

$$6abc-b^2c-bc^2-ca^2-c^2a-ab^2-a^2b=0;$$

which is, therefore, the condition in order that (r), (s), I, J may be harmonically related.

The coordinates of the point of intersection of (r), (s) are

$$x:y:z=a(b^2-c^2):b(c^2-a^2):c(a^2-b^2);$$

and substituting these values in

$$x^2+y^2+z^2-2yz-2zx-2xy,$$

we see, *à priori*, that the result contains the factor

$$2abc+bc^2+b^2c+ca^2+c^2a+ab^2+a^2b=(b+c)(c+a)(a+b);$$

in fact $b+c=0$ gives $x:y:z=0:b(b^2-a^2):b(b^2-a^2)$, i.e. $x=0$, $y-z=0$, which makes $x^2+y^2+z^2-2yz-2zx-2xy$ vanish; and so for the other factors.

Effecting the substitution, we find

$$x^2+y^2+z^2-2yz-2zx-xy=(2abc+bc^2+b^2c+ca^2+c^2a+ab^2+a^2b)$$
$$\times(-6abc+bc^2+b^2c+ca^2+c^2a+ab^2+a^2b),$$

which equals 0, on account of the second factor.

Hence the point of intersection of the lines (r), (s) lies upon the conic. Q.E.D.

Consider, next, a side of the triangle: then, *Theorem*,—the following four points on a side of the triangle, viz. the point of contact, the point of intersection with the line (o), and the points of intersection with the harmonic lines (r), (s), form a harmonic range.

In fact, for the side $x=0$ the points in question are given by means of the equations

$$y-z=0,\quad y+z=0,\quad by+cz+0,\quad \frac{y}{b}+\frac{z}{c}=0,$$

and forming the equations

$$y^2-z^2=0,$$

$$y^2+\left(\frac{b}{c}+\frac{c}{b}\right)yz+z^2=0;$$

the theorem is at once seen to be true. Q.E.D.

It is interesting to give the analytical expressions for some other of the points and lines of the figure. The harmonic lines (r), (s), meet in a point of the conic given by the equations

$$\xi : \eta : \zeta = a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2).$$

This point may be spoken of as the common point of intersection of the harmonic lines with the conic. The foregoing values give

$$\xi^2 - \eta^2 - \zeta^2 = \eta\zeta\left(\frac{b}{c} + \frac{c}{b}\right),$$

$$\eta^2 - \zeta^2 - \xi^2 = \zeta\xi\left(\frac{c}{a} + \frac{a}{c}\right),$$

$$\zeta^2 - \xi^2 - \eta^2 = \xi\eta\left(\frac{a}{b} + \frac{b}{a}\right)$$

which may also be written

$$\xi^2 - (\eta - \zeta)^2 = \eta\zeta\frac{1}{bc}(b + c)^2 = QR,$$

$$\eta^2 - (\zeta - \xi)^2 = \zeta\xi\frac{1}{ca}(c + a)^2 = RP,$$

$$\zeta^2 - (\xi - \eta)^2 = \xi\eta\frac{1}{ab}(a + b)^2 = PQ,$$

if for shortness

$$-\xi + \eta + \zeta = P,$$
$$\xi - \eta + \zeta = Q,$$
$$\xi + \eta - \zeta = R,$$

formulæ which will be presently useful.

Each harmonic line meets the conic in another point, which may be termed the simple point of intersection. For the line $ax + by + cz = 0$, the coordinates of the simple point of intersection are given by $x : y : z = \eta\zeta(b + c)^2 : \zeta\xi(c + a)^2 : \zeta\eta(a + b)^2$; or, what is the same thing, by

$$x : y : z = \frac{1}{Pa} : \frac{1}{Qb} : \frac{1}{Rc},$$

and, in like manner, the coordinates of the simple point of intersection of the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ are given by

$$x : y : z = \frac{a}{P} : \frac{b}{Q} : \frac{c}{R}.$$

Hence the equation of the line joining the two simple points of intersection is

$$\begin{vmatrix} x\,, & y\,, & z \\ \dfrac{1}{Pa}, & \dfrac{1}{Qb}, & \dfrac{1}{Rc} \\ \dfrac{a}{P}\,, & \dfrac{b}{Q}\,, & \dfrac{c}{R} \end{vmatrix} = 0,$$

or expanding and reducing

$$P\xi x + Q\eta y + R\zeta z = 0.$$

The equation of the tangent to the conic at the common point of intersection is evidently

$$Px + Qy + Rz = 0.$$

The last-mentioned lines, together with the harmonic lines (r), (s), viz. the lines

$$ax + by + cz = 0,$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0,$$

may be considered as the sides of an inscribed quadrilateral; the equation of the conic must therefore be expressible in a form in which this is put in evidence; to do this, I first form the equation

$$\left(ax + by + cz\right)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = x^2 + y^2 + z^2 - 2\left(1 - \frac{QR}{\eta\zeta}\right)yz - \left(1 - \frac{RP}{\zeta\xi}\right)zx - 2\left(1 - \frac{PQ}{\zeta\eta}\right)xy,$$

which may also be written

$$\left(ax + by + cz\right)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = x^2 + y^2 + z^2 - 2\lambda yz - 2\mu zx - 2\nu xy,$$

where

$$\lambda = \frac{\eta^2 + \zeta^2 - \xi^2}{2\eta\zeta},$$

$$\mu = \frac{\zeta^2 + \xi^2 - \eta^2}{2\zeta\xi},$$

$$\nu = \frac{\xi^2 + \eta^2 - \zeta^2}{2\xi\eta},$$

and then putting

$$\triangle = \xi^2 + \eta^2 + \zeta^2 - 2\eta\zeta - 2\zeta\xi - 2\xi\eta,$$

an equation which gives

$$\begin{aligned} \triangle &= P^2 - 4\eta\zeta \\ &= Q^2 - 4\zeta\xi \\ &= R^2 - 4\xi\eta, \end{aligned}$$

and

$$-PQR-8\xi\eta\zeta=(\xi+\eta+\zeta)\,\Delta,$$

it is easy by means of these relations to verify the identical equation

$$3\xi\eta\zeta(x^2+y^2+z^2-2yz-2zx-2xy)+\xi\eta\zeta(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)$$

$$-(P\xi x+Q\eta y+R\zeta z)(Px+Qy+Rz)=\Delta\,\{-\xi x^2-\eta y^2-\zeta z^2+(\xi+\eta+\zeta)\,(yz+zx+xy)\},$$

or, writing for Δ its value $\Delta=0$, the equation of the conic takes the form

$$\xi\eta\zeta(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)-(P\xi x+Q\eta y+R\zeta z)(Px+Qy+Rz)=0,$$

which is as it should be.

It may be added, that the common point of intersection and the points in which the harmonic lines (r), (s) meet a side of the triangle lie in a conic passing through the points I, J, such that with respect to this conic the point of contact is the pole of the line (o). Thus for the side $x=0$ the equation of the conic in question is

$$\theta x(x+y+z)+\left(1-\frac{b}{c}-\frac{c}{b}\right)x^2+(by+cz)\left(\frac{y}{b}+\frac{z}{c}\right)=0,$$

where

$$\theta=\frac{a\,(b-c)^2(b+c)}{bc\,(a-b)(c-a)}=\frac{(b+c)^2}{2bc},$$

and similarly for the other two conics.

In the case where the triangle is an equilateral triangle and the line (o) is the line at ∞, the conic becomes a circle, the points I, J, are the circular points at infinity, and lines harmonic in respect to the points I, J, become lines at right angles to each other: the foregoing results agree, therefore, with Sir F. Pollock's Theorem.

166.

NOTE ON THE HOMOLOGY OF SETS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), p. 178.]

LET L denote a set of any four elements a, b, c, d, and in like manner Λ, L_1 &c. sets of the four elements α, β, γ, δ; $a_{,}$, $b_{,}$, $c_{,}$, $d_{,}$, &c.; then we may establish a relation of homology between four sets L, L_1, L_2, L_3, and four other sets Λ, Λ_1, Λ_2, Λ_3; viz., considering the corresponding anharmonic ratios of the different sets, we may suppose a relation of homology between these ratios. Thus considering the set to L, write

$$x=(a-b)(c-d),$$
$$y=(a-c)(d-b),$$
$$z=(a-d)(b-c),$$

then $x+y+z=0$ and the anharmonic ratios of the set are $x:y:z$—we may, if we please, take $x:y$ as the anharmonic ratio of the set. And in like manner taking $\xi:\eta$ as the anharmonic ratio of the set α, β, γ, δ, &c., the assumed relation between the sets L, L_1, L_2, L_3 and the sets Λ, Λ_1, Λ_2, Λ_3 will be

$$\begin{vmatrix} x\xi, & x\eta, & y\xi, & y\eta \\ x_1\xi_1, & x_1\eta_1, & y_1\xi_1, & y_1\eta_1 \\ x_2\xi_2, & x_2\eta_2, & y_2\xi_2, & y_2\eta_2 \\ x_3\xi_3, & x_3\eta_3, & y_3\xi_3, & y_3\eta_3 \end{vmatrix}=0;$$

and it is to be observed, that this relation is independent of the particular ratio $x:y$ which has been chosen as the anharmonic ratio of the set; in fact, if we write $x=-y-z$, $\xi=-\eta-\zeta$, &c., then reducing the result by means of an elementary property of determinants, the equation will preserve its original form, but will contain the ratios $y:z$; $\eta:\zeta$, &c., instead of the ratios $x:y$; $\xi:\eta$, &c.

167.

APROPOS OF PARTITIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 183—184.]

LET $\Pi(1-x^a)=(1-x^a)\ (1-x^b)\ldots(\kappa$ factors) and assume that $\left[\frac{1}{\Pi(1-x^a)}\right]_{1-x}$ is the part of $\frac{1}{\Pi(1-x^a)}$ which involves negative powers of $1-x$, then

$$\text{Coefficient } x^q \text{ in } \left[\frac{1}{\Pi(1-x^a)}\right]_{1-x} = \text{coefficient } z^{\kappa-1} \text{ in } \left(\frac{1}{(1-z)^{q+1}}\,\Pi\,\frac{z}{1-(1-z)^a}\right),$$

which suggests the question of the expansion in powers of z, of the function

$$\frac{1}{(1-z)^{q+1}}\,\Pi\,\frac{z}{1-(1-z)^a}.$$

Now by the definition of Bernoulli's numbers

$$\frac{1}{e^t-1}=\frac{1}{t}-\frac{1}{2}+B_1\frac{t}{1.2}-B_2\frac{t^3}{1.2.3.4}+B_3\frac{t^5}{1.2.3.4.5.6}-\&c.$$

from which it is easy to deduce

$$\frac{t}{1-e^{-t}}=e^{\frac{1}{2}t-B_1\frac{t^2}{1.2^2}+B_2\frac{t^4}{1.2.3.4^2}-B_3\frac{t^6}{1.2.3.4.5.6^2}+\&c.};$$

and, writing in this formula $t=-a\log(1-z)$, we have

$$\frac{-a\log(1-z)}{1-(1-z)^a}=e^{-\frac{1}{2}\log(1-z)a-B_1\frac{\log^2(1-z)}{1.2^2}a^2+B_2\frac{\log^4(1-z)}{1.2.3.4^2}a^4+\&c.},$$

i.e.

$$\frac{z}{1-(1-z)^a}=\frac{1}{a}\left(\frac{-z}{\log(1-z)}\right)e^{-\frac{1}{2}\log(1-z)a-\&c.}=\frac{1}{a}e^{\log\left(\frac{-z}{\log(1-z)}\right)-\frac{1}{2}\log(1-z)a-\&c.},$$

and putting p_κ for $abc\ldots$ and S_1, $S_2\ldots$ for the sums of the powers, we have, taking the product

$$\Pi\frac{z}{1-(1-z)^a}=\frac{1}{p_\kappa}e^{\kappa\log\left(\frac{-z}{\log(1-z)}\right)-\frac{1}{2}\log(1-z)S_1-B_1\frac{\log^2(1-z)}{1.2^2}S_2+B_2\frac{\log_4(1-z)}{1.2.3.4^2}S_4-\&c.},$$

whence also

$$\frac{1}{(1-z)^{q+1}}\Pi\frac{z}{1-(1-z)^a}=\frac{1}{p_\kappa}e^{\kappa\log\left(\frac{-z}{\log(1-z)}\right)-(q+1+\frac{1}{2}S_1)\log(1-z)-B_1\frac{\log^2(1-z)}{1.2^2}S_2+B_2\frac{\log^4(1-z)}{1.2.3.4^2}S_4-\&c.}$$

from which the development may be found.

The index of e is

$$\begin{aligned}&(q+1-\tfrac{1}{2}\kappa+\tfrac{1}{2}S_1)z\\&+(\tfrac{1}{2}q+\tfrac{1}{2}-\tfrac{\pi}{24}\kappa+\tfrac{1}{4}S_1-\tfrac{1}{24}S_2)z^2\\&+(\tfrac{1}{3}q+\tfrac{1}{3}-\tfrac{1}{8}\kappa+\tfrac{1}{6}S_1-\tfrac{1}{24}S_3)z^3\\&+\&c.\end{aligned}$$

and developing the exponential,

1°. The coefficient of z is

$$q+\tfrac{1}{2}S_1-\tfrac{1}{2}(\kappa-2).$$

2°. The coefficient of z^2 is

$$\tfrac{1}{2}q^2+q\{\tfrac{1}{2}S_1-\tfrac{1}{2}(\kappa-3)\}+\tfrac{1}{8}S_1^2-\tfrac{1}{24}S_2-\tfrac{1}{4}(\kappa-3)S_1+(\kappa-3)(\tfrac{1}{6}\kappa-\tfrac{1}{3}),$$

and so on.

The peculiarity is the appearance of the factors $\kappa-2$, $\kappa-3$, &c. If we neglect these terms, and consider as well q as a, b, $c\ldots$ to be each of them of the dimension unity, the coefficients will be homogeneous.

168.

A DEMONSTRATION OF THE FUNDAMENTAL PROPERTY OF GEODESIC LINES.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 185—6.]

This is the translation of a paragraph in M. Bertrand's Memoir "Démonstration géométrique de quelques théorèmes relatifs à la théorie des Surfaces," Liouv. t. XIII. (1848), p. 73, in which he shows that the theorem that the geodesic lines on a surface have at each point their osculating plane normal to the surface is an immediate consequence of Meunier's theorem.

169.

EISENSTEIN'S GEOMETRICAL PROOF OF THE FUNDAMENTAL THEOREM FOR QUADRATIC RESIDUES. (*Translated from the Original Memoir,* Crelle, t. XXVIII. (1844), *with an Addition, by* A. CAYLEY.)

[From the *Quarterly Mathematical Journal,* vol. I. (1857), pp. 186—191.]

LET p be a positive odd prime, a the aggregate of all the even numbers $<p$ and >0, viz. $a=2,\ 4,\ 6,\ldots p-1$; and let q be any integer not divisible by the modulus p; then if r denote the general term of the residues of the multiples qa in respect to the modulus p, it is clear that the numbers of the series the general term of which is $(-)^r r$ will coincide to multiples of p près with the numbers of the series a; so that we shall have the two congruences

$$q^{\frac{1}{2}(p-1)}\,\Pi a \equiv \Pi r \,(\text{mod. } p), \text{ and } \Pi a \equiv (-)^{\Sigma r}\,\Pi r \,(\text{mod. } p),$$

from which it follows that

$$q^{\frac{1}{2}(p-1)} \equiv (-)^{\Sigma r}\,1 \,(\text{mod. } p), \text{ and therefore } \left(\frac{p}{q}\right) = (-)^{\Sigma r}\,1.$$

Let $E\left(\frac{qa}{p}\right)$ denote the greatest whole number contained in the fraction $\frac{qa}{p}$, then it is clear that $\Sigma qa = p\Sigma E\left(\frac{qa}{p}\right) + \Sigma r$; and since all the a's are even, and $p \equiv 1 \,(\text{mod. } 2)$ it follows that $\Sigma r \equiv \Sigma E\left(\frac{qa}{p}\right) (\text{mod. } 2)$; and we have, therefore,

$$\left(\frac{q}{p}\right) = (-)^{\Sigma E\left(\frac{qa}{p}\right)}.$$

When $q=2$, the formula gives at once the value of $\left(\frac{2}{p}\right)$; when, on the other hand, q is odd, and therefore $q-1$ even, we find, by a simple transformation,

$$\Sigma E\left(\frac{qa}{p}\right) \equiv -E\left(\frac{q}{p}\right)+E\left(\frac{2q}{p}\right)-E\left(\frac{3q}{p}\right)\ldots\pm E\left(\frac{\frac{1}{2}(p-1)q}{p}\right)$$

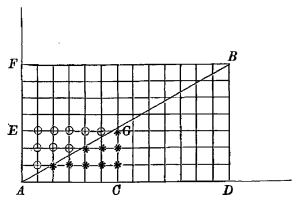
$$\equiv \quad E\left(\frac{q}{p}\right)+E\left(\frac{2q}{p}\right)+E\left(\frac{3q}{p}\right)\ldots+E\left(\frac{\frac{1}{2}(p-1)q}{p}\right), \text{ (mod. 2)};$$

and if the last sum be represented by μ, then also

$$\left(\frac{q}{p}\right)=(-)^{\mu}1.$$

Imagine now in a plane, a rectangular system of coordinates (x, y) and the whole plane divided by lines parallel to the axes at distances $=1$ from each other into squares of the dimension $=1$. And let the angles which do not lie on the axes of coordinates be called "lattice points."

Take, now, upon any vertical parallel a point corresponding to the ordinate y, then $E(y)$ will denote the number of lattice points which lie between this point and the horizontal axis; and, in like manner, taking upon any horizontal parallel a point



corresponding to the abscissa x, then $E(x)$ will denote the number of lattice points which lie between this point and the vertical axis. If, therefore, we draw in the plane a curve the equation of which is $y=\phi x$, then the sum

$$E\phi 1+E\phi 2+E\phi 3+E\phi 4+\&c.,$$

will denote the number of lattice points which lie between the curve and the axis of x, including any lattice points which lie upon the curve.

Suppose now, to return to the subject, that AB represents the straight line which has for its equation $y=\frac{q}{p}x$, where p and q are now assumed to be both of them positive

odd primes; and let $AD = FB$ be $= p$, $AF = DB = q$, $AC = EG = \frac{1}{2}(p-1)$, $AE = CG = \frac{1}{2}(q-1)$. Then if μ denote the number of the lattice points between AB and AD as far as the ordinate CG inclusively (these lattice points are distinguished in the figure by the mark [*]), by what precedes $\left(\frac{q}{p}\right) = (-)^{\mu} 1$. But since the equation of the straight line AB may also be written $x = \frac{p}{q} y$, we have in the same way, if ν denote the number of the lattice points which lie between AB and AF as far as the abscissa EG inclusively (these are distinguished in the figure by the mark [∘]), $\left(\frac{p}{q}\right) = (-)^{\nu} 1$. But the lattice points marked with [*] and [∘] taken together, i.e. all the lattice points to the right, and all the lattice points to the left of AB, make up the entire system of lattice points of the rectangle $AEGC$, the number of which is $\frac{1}{2}(p-1) . \frac{1}{2}(q-1)$; consequently, $\mu + \nu = \frac{1}{2}(p-1) . \frac{1}{2}(q-1)$, and therefore

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-)^{\mu+\nu} 1 = (-)^{\frac{1}{2}(p-1) \cdot \frac{1}{2}(q-1)} 1$$

which is the theorem in question.

It may be noticed that the foregoing transformation $\Sigma E\left(\frac{qa}{p}\right) \equiv \mu \pmod{2}$ may itself be proved by the simple geometrical consideration that $\Sigma E\left(\frac{qa}{p}\right)$ is nothing else than the number of lattice points which lie on the even ordinates (those corresponding to $x = 2, 4, 6, \ldots p-1$) between AB and AD as far as BD, and that each ordinate, between the axes AD and BF exclusively, contains $(q-1)$, i.e. an even number of lattice points; and besides, that the two triangles BAD and ABF are congruent, and that the latter of them stands in the same relation to BF and BD as the former does to AD and AF; the completion of which is left to the reader.

Remark. There are figures for which simple formulæ may be obtained for the number of the interior lattice points. Imagine, for example, a circle, the centre of which lies in the axis of coordinates, and the radius of which is $\sqrt{m}$, then the number S of the lattice points which lie within the circle, including those which lie on the axes, is given by the following formula,

$$S = 1 + 4\{E(m) - E(\tfrac{1}{3}m) + E(\tfrac{1}{5}m) - \&c.\}$$

continued until the series stops of itself. It is easy to see that the equation expresses a relation between the number of lattice points of the circle and the number of lattice points of a segment included between two rectangular hyperbolas. Writing in the formula

$$\frac{1}{m}S = \frac{1}{m} + 4\left\{\frac{1}{m}Em - \frac{1}{m}E(\tfrac{1}{3}m) + \frac{1}{m}E(\tfrac{1}{5}m) - \&c.\right\}$$

$m = \infty$, the left-hand side becomes equal to π, and the right-hand side becomes equal to $4(1 - \frac{1}{3} + \frac{1}{5} - \&c.)$, which gives the formula of Leibnitz. There are similar

formulæ for the number of lattice points of a system of sectors of ellipses or hyperbolas; and similar relations exist also in space, and in cases of more than three dimensions. We shall return to this important subject, which has the closest connection with the properties of the higher forms, upon another occasion.

(*Addition by the Translator.*) Eisenstein is now, alas! dead; too soon for the complete development of his various and profound researches in elliptic functions and the theory of numbers; and the promise at the conclusion of the foregoing memoir has not, I believe, been fulfilled. The formula in the Remark must, I think, have been established by geometrical considerations, and would have served to give the number of decompositions of a number into the sum of two squares; but, as I do not perceive how this is to be done, I shall follow a reverse course, and establish the theorem from considerations founded on the theory of numbers. I remark, first, that the number of lattice points in a quadrant of the circle, inclusive of those on the vertical axis, but exclusive of those on the horizontal axis and of the centre, is equal to $E\sqrt{m}+E\sqrt{(m-1)}+E\sqrt{(m-4)}+E\sqrt{(m-9)}+\&c.$; and that this sum, multiplied by four and increased by unity, gives precisely the number of lattice points of the circle, including those on the horizontal and vertical axes. The formula to be established is therefore

$$E\sqrt{m}+E\sqrt{(m-1)}+E\sqrt{(m-4)}+E\sqrt{(m-9)}+\&c.=E(m)-E(\tfrac{1}{3}m)+E(\tfrac{1}{5}m)-E(\tfrac{1}{7}m)+\&c.$$

where, as already noticed, the left-hand side denotes the number of lattice points of a quadrant of the circle, inclusive of those on the vertical axis, but exclusive of those on the horizontal axis and of the centre.

Let Xn be the number of ways in which the integer number n can be expressed as the sum of two squares. {If $n=\alpha^2+\beta^2$, then if α and β are unequal, and neither of them is zero, this counts as two decompositions, viz. $x=\alpha$, $y=\beta$ or $x=\beta$, $y=\alpha$; but if $\alpha=\beta$, this counts only as a single decomposition; or if either of the numbers α, β, e.g. α, is zero, then, since $y=0$ is excluded, this counts as a single decomposition, $x=0$, $y=\beta$.} Xn will denote the number of lattice points *on* the quadrant of the circle radius $\sqrt{n}$. Suppose also that $E'\left(\frac{1}{k}n\right)$ stands for unity or zero, according as $\frac{1}{k}n$ is or is not an integer. Then as m passes through the integer number n, i.e. from a value between $n-1$ and n to a value between n and $n+1$, the left-hand side of the equation is increased by Xn, and the right-hand side of the equation is increased by $E'(n)-E'(\frac{1}{3}n)+E'(\frac{1}{5}n)-\&c.$ We ought therefore to have

$$Xn=E'(n)-E'(\tfrac{1}{3}n)+E'(\tfrac{1}{5}n)-E'(\tfrac{1}{7}n)+\&c.,$$

and conversely from this equation, the original equation will at once follow. The right-hand side denotes, it should be observed, the number of factors of n of the form $\equiv 1$ (mod. 4), less the number of factors of the form $\equiv 3$ (mod. 4). Let $n=2^k f^\alpha f'^{\alpha'}\dots g^\beta g'^{\beta'}\dots$ where f, $f'\dots$ are odd primes $\equiv 1$ (mod. 4) and g, $g'\dots$ are odd primes $\equiv 3$ (mod. 4). Consider first the factors g, $g'\dots$, and forming the product

$$(1+g\dots+g^\beta)(1+g+\dots g'^{\beta'})\dots,$$

it is easy to see that if all or any one or more of the indices $\beta, \beta' \ldots$ are odd, then the number of terms of the product which are $\equiv 1$ (mod. 4) is equal to the number of terms of the product which are $\equiv 3$ (mod. 4); but if all the indices $\beta, \beta' \ldots$ are even, then the number of terms of the first form is greater by unity than the number of terms of the second form. Now the terms of the product $(1+f+\ldots f^{\alpha})(1+f'+\ldots f'^{\alpha'})\ldots$ are all $\equiv 1$ (mod. 4), and the number of terms is $(1+\alpha)(1+\alpha')\ldots$. Hence if all or any one or more of the indices $\beta, \beta' \ldots$ are odd, then the number of factors of n of the first form *less* the number of factors of the second form is zero; but if the indices $\beta, \beta' \ldots$ are all even, the number of factors of the first form *less* the number of factors of the second form is $(1+\alpha)(1+\alpha')\ldots$, i.e. we have

$$E'(n) - E'(\tfrac{1}{3}n) + E'(\tfrac{1}{5}n) - \&c. = 0, \text{ or } = (1+\alpha)(1+\alpha')\ldots,$$

according as $\beta, \beta' \ldots$ are all or any one or more of them odd, or according as they are all of them even. Now it is well known that the number $n = 2^k f^{\alpha} f'^{\alpha'} \ldots g^{\beta} g'^{\beta'} \ldots$ does not, in the case of all or any one or more of the indices $\beta, \beta' \ldots$ being odd, admit of decomposition into two squares, i.e. in this case $Xn = 0$; but if the indices $\beta, \beta' \ldots$ are all even, then the number n will admit of precisely as many decompositions into two squares as the number $n' = 2^k f^{\alpha} f'^{\alpha'} \ldots$ (in fact, the only decompositions of n are those obtained from the decompositions of n' by multiplying the roots into the common factor $g^{\frac{1}{2}\beta} g'^{\frac{1}{2}\beta'} \ldots$), and the number of decompositions of n' is moreover equal to the number of decompositions of $n'' = f^{\alpha} f'^{\alpha'} \ldots$, which last number is in fact the product of the numbers of decompositions of $f^{\alpha}, f'^{\alpha'} \ldots$; the number of decompositions of n into two squares (estimated according to the foregoing convention) is thus shown to be, in the case of $\beta, \beta' \ldots$, all of them even, equal to $(1+\alpha)(1+\alpha')\ldots$. And we have therefore in every case

$$Xn = E'(n) - E'(\tfrac{1}{3}n) + E'(\tfrac{1}{5}n) - \&c.,$$

and the principal theorem is thus shown to be true.

170.

ON SCHELLBACH'S SOLUTION OF MALFATTI'S PROBLEM.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 222—226.]

THE following elegant solution of Malfatti's Problem as applied to spherical triangles is given by Dr Schellbach (*Crelle*, t. XLV. (1853), p. 186); for the reason which will be mentioned I have made a change of notation.

In a spherical triangle ABC to describe three small circles, each of them touching the other two, and also two sides of the triangle.

Let the sides of the triangle be a, b, c, and let x, y, z, be the distances of the points of contact from the adjacent angles of the triangle. Then writing

$$a+b+c=2s,$$

$$a-\tfrac{1}{2}s=l,\ b-\tfrac{1}{2}s=m,\ c-\tfrac{1}{2}s=n,$$

whence

$$l+m+n=\tfrac{1}{2}s,$$

and putting also

$$\tfrac{1}{2}s-x=\xi,\ \tfrac{1}{2}s-y=\eta,\ \tfrac{1}{2}s-z=\zeta,$$

it is easy to obtain

$$\left\{\begin{aligned}
&\frac{\cos l\ \cos\eta\cos\zeta}{\cos\frac{1}{2}s}-\frac{\sin l\ \sin\eta\sin\zeta}{\sin\frac{1}{2}s}=1,\\
&\frac{\cos m\cos\zeta\cos\xi}{\cos\frac{1}{2}s}-\frac{\sin m\sin\zeta\sin\xi}{\sin\frac{1}{2}s}=1,\\
&\frac{\cos n\ \cos\xi\cos\eta}{\cos\frac{1}{2}s}-\frac{\sin n\sin\xi\sin\eta}{\sin\frac{1}{2}s}=1,
\end{aligned}\right.$$

from which equations the unknown quantities ξ, η, ζ, are to be determined. And the equations may be solved without assuming the existence of the relation $l+m+n=\frac{1}{2}s$.

To solve the equations, let the subsidiary angles λ, μ, ν, be determined by the conditions

$$\begin{cases} \dfrac{\cos\lambda\cos m\cos n}{\cos\frac{1}{2}s} + \dfrac{\sin\lambda\sin m\sin n}{\sin\frac{1}{2}s} = 1, \\ \dfrac{\cos\mu\cos n\cos l}{\cos\frac{1}{2}s} + \dfrac{\sin\mu\sin n\sin l}{\sin\frac{1}{2}s} = 1, \\ \dfrac{\cos\nu\cos l\cos m}{\cos\frac{1}{2}s} + \dfrac{\sin\nu\sin l\sin m}{\sin\frac{1}{2}s} = 1, \end{cases}$$

then it may be shown that

$$\begin{cases} \cos(\eta+\zeta) = \dfrac{\cos\frac{1}{2}(s+\lambda-\zeta)}{\cos\frac{1}{2}(\lambda+l)}, \ \cos(\eta-\zeta) = \dfrac{\cos\frac{1}{2}(s-\lambda+l)}{\cos\frac{1}{2}(\lambda+l)}, \\ \cos(\zeta+\xi) = \dfrac{\cos\frac{1}{2}(s+\mu-m)}{\cos\frac{1}{2}(\mu+m)}, \ \cos(\zeta-\xi) = \dfrac{\cos\frac{1}{2}(s-\mu+m)}{\cos\frac{1}{2}(\mu+m)}, \\ \cos(\xi+\eta) = \dfrac{\cos\frac{1}{2}(s+\nu-n)}{\cos\frac{1}{2}(\nu+n)}, \ \cos(\xi-\eta) = \dfrac{\cos\frac{1}{2}(s-\nu+n)}{\cos\frac{1}{2}(\nu+n)}. \end{cases}$$

If we write

$$\tan\phi = \tan m\tan n\cot\tfrac{1}{2}s,$$
$$\tan\chi = \tan n\tan l\cot\tfrac{1}{2}s,$$
$$\tan\psi = \tan l\tan m\cot\tfrac{1}{2}s,$$

then

$$\cos(\lambda-\phi) = \frac{\cos\frac{1}{2}s\cos\phi}{\cos m\cos n},$$

$$\cos(\mu-\chi) = \frac{\cos\frac{1}{2}s\cos\chi}{\cos n\cos l},$$

$$\cos(\nu-\psi) = \frac{\cos\frac{1}{2}s\cos\psi}{\cos l\cos m},$$

equations which give the values of λ, μ, ν, from which ξ, η, ζ are determined as above.

If we suppose that the sides become indefinitely small, we have the case of a plane triangle, and the equations then are

$$\eta^2+\zeta^2+\frac{4l}{s}\eta\zeta = \tfrac{1}{4}s^2-l^2,$$

$$\zeta^2+\xi^2+\frac{4m}{s}\zeta\xi = \tfrac{1}{4}s^2-m^2,$$

$$\xi^2+\eta^2+\frac{4n}{s}\xi\eta = \tfrac{1}{4}s^2-n^2.$$

We have here

$$(\eta+\zeta)^2 = \tfrac{1}{4}\{(s+\lambda-l)^2-(\lambda+l)^2\} = (\tfrac{1}{2}s+\lambda)(\tfrac{1}{2}s-l),$$
$$(\eta-\zeta)^2 = \tfrac{1}{4}\{(s-\lambda+l)^2-(\lambda+l)^2\} = (\tfrac{1}{2}s-\lambda)(\tfrac{1}{2}s+l),$$

and consequently

$$\eta^2+\zeta^2=\tfrac{1}{2}s^2-2\lambda l, \quad \eta\zeta=s(\lambda-l),$$

where if

$$\phi=\frac{2mn}{s}, \ (\lambda-\phi)^2=\tfrac{1}{4}s^2+\phi^2-m^2-n^2,$$

i.e.

$$\lambda=\frac{2mn}{s}+\sqrt{\tfrac{1}{4}s^2+\frac{4m^2n^2}{s^2}-m^2-n^2}$$

$$=\frac{2mn}{s}+\frac{1}{2s}\sqrt{(s^2-4m^2)(s^2-4n^2)}$$

Hence

$$\eta^2+\zeta^2=\tfrac{1}{2}s^2-\frac{4lmn}{s}-\frac{l}{s}\sqrt{s^2-4m^2}\sqrt{s^2-4n^2},$$

$$\eta\zeta=2mn-sl+\tfrac{1}{2}\sqrt{s^2-4m^2}\sqrt{s^2-4n^2},$$

$$\zeta^2+\xi^2=\tfrac{1}{2}s^2-\frac{4lmn}{s}-\frac{m}{s}\sqrt{s^2-4n^2}\sqrt{s^2-4l^2},$$

$$\zeta\xi=2nl-sm+\tfrac{1}{2}\sqrt{s^2-4n^2}\sqrt{s^2-4l^2},$$

$$\xi^2+\eta^2=\tfrac{1}{2}s^2-\frac{4lmn}{s}-\frac{n}{s}\sqrt{s^2-4l^2}\sqrt{s^2-4m^2},$$

$$\xi\eta=2lm-sn+\tfrac{1}{2}\sqrt{s^2-4l^2}\sqrt{s^2-4m^2},$$

which is in fact at once deducible from the formulæ in my paper "On a System of Equations connected with Malfatti's Problem and on another Algebraical System," (*Camb. and Dubl. Math. Journ.* t. IV. (1849), p. 270 [79]).

Write now for l, m, n, ξ, η, ζ, their values in terms of a, b, c, x, y, z. We have

$$(\tfrac{1}{2}s-y)^2+(\tfrac{1}{2}s-z)^2-\frac{4}{s}(\tfrac{1}{2}s-a)(\tfrac{1}{2}s-y)(\tfrac{1}{2}s-z)=\tfrac{1}{4}s^2-(\tfrac{1}{2}s-a)^2,$$

i.e.

$$y^2+z^2-\frac{4}{s}(\tfrac{1}{2}s-a)yz-2a(y+z)+a^2=0,$$

or reducing

$$(y+z-a)^2-4\left(1-\frac{a}{s}\right)yz=0,$$

and we have thus the system

$$y+z+2\sqrt{1-\frac{a}{s}}\sqrt{yz}=a,$$

$$z+x+2\sqrt{1-\frac{b}{s}}\sqrt{zx}=b,$$

$$x+y+2\sqrt{1-\frac{c}{s}}\sqrt{xy}=c,$$

which are given by Schellbach in the same volume, p. 29; and it was for the sake of facilitating the comparison that the notation has been altered in the case of the spherical triangle. To solve the system, Schellbach writes

$$a = s\sin^2\phi,\ b = s\sin^2\chi,\ c = s\sin^2\psi,$$

reducing the equations to

$$y + z + 2\sqrt{yz}\cos\phi = s\sin^2\phi,$$

$$z + x + 2\sqrt{zx}\cos\chi = s\sin^2\chi,$$

$$x + y + 2\sqrt{xy}\cos\psi = s\sin^2\psi,$$

whence, putting

$$\phi + \chi + \psi = 2\sigma,$$

the equations are satisfied by

$$x = s\sin^2(\sigma - \phi),\ y = s\sin^2(\sigma - \chi),\ z = s\sin^2(\sigma - \psi),$$

which leads to a simple geometrical construction. And if we substitute for ϕ, χ, ψ, σ, their values, it is easy to obtain

$$x = \tfrac{1}{2}s\left\{1 - \sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} + \sqrt{1-\frac{a}{s}}\sqrt{\frac{b}{s}}\sqrt{\frac{c}{s}} - \sqrt{1-\frac{b}{s}}\sqrt{\frac{c}{s}}\sqrt{\frac{a}{s}} - \sqrt{1-\frac{c}{s}}\sqrt{\frac{a}{s}}\sqrt{\frac{b}{s}}\right\},$$

$$y = \tfrac{1}{2}s\left\{1 - \sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} - \sqrt{\left(1-\frac{a}{s}\right)}\sqrt{\frac{b}{s}}\sqrt{\frac{c}{s}} + \sqrt{1-\frac{b}{s}}\sqrt{\frac{c}{s}}\sqrt{\frac{a}{s}} - \sqrt{1-\frac{c}{s}}\sqrt{\frac{a}{s}}\sqrt{\frac{b}{s}}\right\},$$

$$z = \tfrac{1}{2}s\left\{1 - \sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} - \sqrt{1-\frac{a}{s}}\sqrt{\frac{b}{s}}\sqrt{\frac{c}{s}} - \sqrt{1-\frac{b}{s}}\sqrt{\frac{c}{s}}\sqrt{\frac{a}{s}} + \sqrt{1-\frac{c}{s}}\sqrt{\frac{a}{s}}\sqrt{\frac{b}{s}}\right\},$$

$$yz = \tfrac{1}{2}s\left\{\sqrt{1-\frac{b}{s}}\sqrt{1-\frac{c}{s}} - \sqrt{1-\frac{a}{s}} + \sqrt{\frac{b}{s}}\sqrt{\frac{c}{s}}\right\},$$

$$zx = \tfrac{1}{2}s\left\{\sqrt{1-\frac{c}{s}}\sqrt{1-\frac{a}{s}} - \sqrt{1-\frac{b}{s}} + \sqrt{\frac{c}{s}}\sqrt{\frac{a}{s}}\right\},$$

$$xy = \tfrac{1}{2}s\left\{\sqrt{1-\frac{a}{s}}\sqrt{1-\frac{b}{s}} - \sqrt{1-\frac{c}{s}} + \sqrt{\frac{a}{s}}\sqrt{\frac{b}{s}}\right\},$$

values which are also at once obtained from the formula in my paper above referred to. It may be remarked that the above equations for the determination of x, y, z (the distances of the points of contact from the adjacent angles of the triangle) are very similar in form to those given in the same paper for the determination of X, Y, Z, the radii of the inscribed circles.

171.

NOTE ON MR SALMON'S EQUATION OF THE ORTHOTOMIC CIRCLE.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 242—244.]

LET $U_1=0$, $U_2=0$, $U_3=0$ be the equations of three circles, and let V be the functional determinant of U_1, U_2, U_3, the functions being in the first instance made homogeneous by the introduction of a variable z, which is ultimately replaced by unity; then the equation of the circle cutting at right angles the three given circles, or, as it may be called, the orthotomic circle, is $V=0$. This elegant theorem of Mr Salmon's is connected with the theory developed by Hesse in the memoir, "Ueber die Wendepuncte der Curven dritter Ordnung," *Crelle*, t. XXVIII. (1844), p. 97).

In fact, let $U_1=0$, $U_2=0$, $U_3=0$ be the equations of three conics, the locus of a point such that its polars with respect to each of these conics, or indeed with respect to any conic having for its equation $\lambda U_1+\mu U_2+\nu U_3=0$ (where λ, μ, ν are arbitrary), pass through the same point, is a curve of the third order $V=0$, where V is the functional determinant of U_1, U_2, U_3.

Conversely, if the curve of the third order $V=0$ be given, and U be a function of the third order, such that the functional determinant of $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$, or, what is the same thing, the "Hessian" of the function U is equal to V, a condition which may be written $V=H(U)$, then we may take for the conics any three conics the equations of which are of the form $\lambda\frac{dU}{dx}+\mu\frac{dU}{dy}+\nu\frac{dU}{dz}=0$. The equation $V=H(U)$ affords the means of determining U; in fact, we shall have $U=aV+bH(V)$, where a and b are constants to be determined. This gives $H(U)=H\left(aV+bH(V)\right)=AV+BH(V)$, where A and B are given functions of a, b (a practical method of determining these functions was first given in Aronhold's memoir, "Zur Theorie der homogenen Functionen dritten Grades von zwei Variabeln," *Crelle*, t. XXXIX. (1850), pp. 140—159); and we have therefore

$V = AV + BH(V)$, i.e. $A = 1$, $B = 0$: the latter equation determines, what is alone important, the ratio $b : a$; the equation is of the third order, so that there are in general three distinct solutions $U = aV + BH(V) = 0$.

In the particular case in which the curves of the third order $V = 0$ is made up of a line $P = 0$ and a conic $W = 0$, i.e. where $V = PW = 0$, the curve $H(V) = 0$ is made up of the same line $P = 0$ and of a conic having double contact with the conic $W = 0$ at the point of intersection with the line $P = 0$, i.e. $H(PW) = P(lW + mP^2)$. And $U = aPW + bH(PW)$ is consequently a function of the same form, i.e. the cubic $U = 0$ is made up of the line $P = 0$ and of a conic having double contact with the conic $W = 0$ at its points of intersection with the line $P = 0$. We may therefore write $U = P(fW + gP^2)$, and forming with this value the equation $PW = P(FW + GP^2)$, it may be noticed that, owing to the occurrence of a special factor which may be rejected, the resulting equation $G = 0$ gives only a single value for the ratio $f : g$. Forming from the value $U = P(fW + gP^2)$, the equation $\lambda \frac{dU}{dx} + \mu \frac{dU}{dy} + \nu \frac{dU}{dz} = 0$, the equation thus obtained will be of the form $W + PQ = 0$, which is the equation of a conic passing through the points of intersection of the line and conic $P = 0$, $W = 0$, and besides intersecting the conic $W = 0$ in two other points. And it may also be shown that the four points of intersection, (i.e. the points given by the equations $W = 0$, $W + PQ = 0$), the pole of the line $P = 0$ with respect to the conic $W = 0$, and the pole of this same line with respect to the conic $W + PQ = 0$, lie all six in the same conic. We see, therefore, that, given a curve of the third order, the aggregate of a line $P = 0$ and a conic $W = 0$, as the locus of the point such that its polars, with respect to three several conics (or a system depending on three conics), meet in a point, each conic of the system is a conic passing through the points of intersection of the line and conic $P = 0$, $W = 0$, and, moreover, such that the four points of intersection with the conic $W = 0$ and the poles of the line $P = 0$, with respect to the conic of the system, and with respect to the conic $W = 0$, lie all six in the same conic. In the particular case in which the line and conic $P = 0$, $W = 0$ are the line at ∞ and a circle, each conic of the system is a circle such that its points of intersection with the circle $W = 0$ and the centres of the two circles lie in a circle, i.e. the conics are circles cutting at right angles the circle $W = 0$, which agrees with Mr Salmon's theorem.

To verify the assumed theorems in the case of the curve of the third order $V = PW = 0$, we may take

$$P = \alpha x + \beta y + \gamma z = 0$$

for the equation of the line, and

$$W = x^2 + y^2 + z^2 = 0$$

for the equation of the conic. I write, for greater convenience, $U = P(\frac{1}{2} fW + \frac{1}{6} gP^2)$; the Hessian of this is

$$\begin{vmatrix} f(P + 2\alpha x) + g\alpha^2 P, & f(\beta x + \alpha y) + g\alpha\beta P, & f(\alpha z + \gamma x) + g\gamma\alpha P \\ f(\beta x + \alpha y) + g\alpha\beta P, & f(P + 2\beta y) + g\beta^2 P, & f(\gamma y + \beta z) + g\beta\gamma P \\ f(z z + \gamma x) + g\gamma\alpha P, & f(\gamma y + \beta z) + g\beta\gamma P, & f(P + 2\gamma z) + g\gamma^2 P \end{vmatrix},$$

which is equal to

$$f^3P\left(4P^2-(\alpha^2+\beta^2+\gamma^2)\,W\right)+f^2gP\,(\alpha^2+\beta^2+\gamma^2)\,P^2,$$

i.e. we must have $4f+g\,(\alpha^2+\beta^2+\gamma^2)=0$, or putting $g=-24$ and $\therefore f=6\,(\alpha^2+\beta^2+\gamma^2)$, we have

$$U=P\left(3\,(\alpha^2+\beta^2+\gamma^2)\,W-4P^2\right).$$

Forming the function $\lambda\dfrac{dU}{dx}+\mu\dfrac{dU}{dy}+\nu\dfrac{dU}{dz}$, and dividing by the constant factor $3\,(\alpha^2+\beta^2+\gamma^2)\,(\lambda\alpha+\mu\beta+\nu\gamma)$, we have for the equation of any one of the conics

$$W+P\left\{\frac{2\,(\lambda x+\mu y+\nu z)}{\lambda\alpha+\mu\beta+\nu\gamma}-\frac{4P}{\alpha^2+\beta^2+\gamma^2}\right\}=0,$$

which may be written under the form $W+2PQ=0$, where

$$Q=ax+by+cz=\frac{\lambda x+\mu y+\nu z}{\lambda\alpha+\mu\beta+\nu\gamma}-\frac{2\,(\alpha x+\beta y+\gamma z)}{\alpha^2+\beta^2+\gamma^2}.$$

We have therefore $a\alpha+b\beta+c\gamma=1-2=-1$, i.e. $a\alpha+b\beta+c\gamma+1=0$. And there is no difficulty in showing that, given the two conics

$$x^2+y^2+z^2=0,$$
$$x^2+y^2+z^2+2\,(\alpha x+\beta y+\gamma z)\,(ax+by+cz)=0,$$

the condition in order that the four points of intersection and the poles, with respect to each conic, of the line $\alpha x+\beta y+\gamma z=0$, may lie in a conic is precisely this equation $a\alpha+b\beta+c\gamma+1=0$.

172.

NOTE ON THE LOGIC OF CHARACTERISTICS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 257—259.]

THE conditions in order that an equation of the sixth degree

$$(a, b, c, d, e, f, g), (x, y)^6 = 0$$

may have five of its roots equal are

$$\begin{aligned} A &= ae - 4bd + 3c^2 = 0, \\ B &= af - 3be + 2cd = 0, \\ C &= bf - 4ce + 3d^2 = 0, \\ D &= ag - 9ce + 8d^2 = 0, \\ E &= bg - 3cf + 2de = 0, \\ F &= cg - 4df + 3e^2 = 0, \end{aligned}$$

equivalent of course to four relations between the coefficients: among the connections of these equations are

$$fA - eB - bF + cE = 0,$$

$$(3e^2 - 2df)A - 2deB + ecD - AF - 2cdE + (3c^2 - 2bd)F = 0.$$

The system is one of the tenth order. To verify this, I write first

$$(A, B, C, F) = (A, B, C, F, cE) = (A, B, C, F, c) + (A, B, C, F, E),$$

i.e. the equations $A=0$, $B=0$, $C=0$, $F=0$ imply (by the first of the connectives) the additional equation $cE=0$, viz. the system $A=0$, $B=0$, $C=0$, $F=0$, $cE=0$, or what is the same thing, one of the systems $A=0$, $B=0$, $C=0$, $F=0$, $c=0$ and $A=0$, $B=0$, $C=0$, $F=0$, $E=0$.

We have in like manner

$$(A, B, C, F, E) = (A, B, C, F, E, ceD) = (A, B, C, F, E, D)$$

since (A, B, C, F, E, c) and (A, B, C, F, E, e) respectively vanish as being each of them equivalent not to four but to five relations, and therefore as not adding to the order of the system.

Again,

$$\begin{aligned}(A, B, C, c) &= (A, B, C, c, bF)\\ &= (A, B, C, c, b) + (A, B, C, c, F)\\ &= (ae, af, d^2, b, c) + (A, B, C, c, F).\end{aligned}$$

But here

$$\begin{aligned}(ae, af, d^2, b, c) &= (a, af, d^2, b, c) + (e, af, d^2, b, c)\\ &= (a, af, d^2, b, c),\end{aligned}$$

(for (e, af, d^2, b, c) vanishes as being equivalent to five relations, and therefore as not adding to the order of the system),

$$\begin{aligned}&= (a, a, d^2, b, c) + (a, f, d^2, b, c)\\ &= (a, d^2, b, c),\end{aligned}$$

since, (a, f, d^2, b, c) vanishes for the above-mentioned reason.

Hence

$$(A, B, C, c) = (a, d^2, b, c) + (A, B, C, c, F).$$

We have consequently

$$\begin{aligned}(A, B, C, D, E, F) &= (A, B, C, E, F)\\ &= (A, B, C, F) - (A, B, C, F, c)\\ &= (A, B, C, F) - \{(A, B, C, c) - (a, b, c, d^2)\},\end{aligned}$$

which may be thus interpreted:—the system (A, B, C, c) contains the system (a, b, c, d^2), or what is the same thing, contains twice-over the system (a, b, c, d). Discarding this contained system, the remainder of the system (A, B, C, c) is contained in the system (A, B, C, F), and the residue of the last-mentioned system is the system (A, B, C, D, E, F), i.e. the system represented by the equations which express the equality of five roots of the given equation of the sixth degree.

It follows immediately that the order of the system (A, B, C, D, E, F) is $16-(8-2)=10$, i.e. that the system is, as above stated, one of the tenth order. The preceding is, I think, a good example of the kind of reasoning to be employed in what Mr Sylvester has most happily termed the Logic of Characteristics.

173.

ON LAPLACE'S METHOD FOR THE ATTRACTION OF ELLIPSOIDS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 285—300.]

[THE method referred to is that given in Book III. of the Mécanique Céleste, Ed. I. 1798.] Let the equation of the surface of the ellipsoid be

$$lx^2 + my^2 + nz^2 - k = 0,$$

and take a, b, c as the coordinates of the attracted point, which is supposed to be exterior to the surface. Imagine an indefinitely thin cone, having the attracted point for vertex, and intersecting the surface of the ellipsoid; let ξ, η, ζ be the direction-cosines of the cone, and dS its spherical angle. Then if for a moment r denote the radius vector corresponding to a point within the ellipsoid, the element of the mass within the cone is $r^2 dr dS$; and we may thence, by an integration with respect to r, find the attractions parallel to the axes (and tending towards the centre) and the potential of the mass within the cone, viz. if r', r'' be the values of r at the surface of the ellipsoid, the attractions are $(r''-r')\,\xi dS$, $(r''-r')\,\eta dS$, $(r''-r')\,\zeta dS$, and the potential is $\frac{1}{2}(r''^2 - r'^2)\,dS$. And putting for shortness

$$\begin{aligned} L &= l\xi^2 + m\eta^2 + n\zeta^2, \\ I &= la\xi + mb\eta + nc\zeta, \\ P &= la^2 + mb^2 + nc^2 - 1, \\ R &= I^2 - PL, \end{aligned}$$

then r', r'' will be the roots of the equation $Lr^2 - 2Ir + P = 0$, and we have consequently $r' = \dfrac{I - \sqrt{R}}{L}$, $r'' = \dfrac{I + \sqrt{R}}{L}$. We have, therefore, for the attractions parallel to the axes

(and tending towards the centre) and for the potential of the entire ellipsoid the well-known expressions

$$A = 2\int dS\,\frac{\xi\sqrt{R}}{L},$$

$$B = 2\int dS\,\frac{\eta\sqrt{R}}{L},$$

$$C = 2\int dS\,\frac{\zeta\sqrt{R}}{L},$$

and

$$V = 2\int \frac{I\sqrt{R}dS}{L^2},$$

where the integrations extend over the spherical angle of the circumscribed cone $R=0$.

We have moreover, by the general theory of attractions,

$$A = -\frac{dV}{da},\ B = -\frac{dV}{db},\ C = -\frac{dV}{dc}.$$

Since at the limits of the integration the quantity under the integral sign vanishes, it is easy to see that the first differentials of V, A, B, C with respect to any of the quantities a, b, c, l, m, n, k, may be found by simply differentiating under the integral sign without its being necessary to pay attention to the variation of the limits, so that, for instance, $\frac{dV}{da} = 2\int dS\,\frac{d}{da}\,\frac{I\sqrt{R}}{L}$. It is proper to remark that the expressions thus obtained for the attractions A, B, C, are of a different form from the foregoing expressions for the same quantities. Laplace writes

$$F = aA + bB + cC,$$

and he remarks that it may be shown by differentiation that the quantities, B, C, F, V, are connected by an equation which (writing k for k^2, and $\frac{m}{l}$, $\frac{n}{l}$ for m, n, the equation of the ellipsoid being with Laplace $x^2+my^2+nz^2=k^2$) becomes, in the notation of the present memoir,

$$\begin{aligned}
&\{l(a^2+b^2+c^2)-k\}\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+V-F\\
&+\left(1-\frac{l}{m}\right)b\left(\frac{dF}{db}-\tfrac{1}{2}\frac{dV}{db}-B\right)\\
&+\left(1-\frac{l}{n}\right)c\left(\frac{dF}{dc}-\tfrac{1}{2}\frac{dV}{dc}-C\right)\\
&-(m-l)\frac{dF}{dm}-(n-l)\frac{dF}{dn}=0.
\end{aligned}$$

I write this equation in the form

$$\begin{aligned}
&(la^2+mb^2+nc^2-k)\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+V-F\\
&\quad+(l\ \ -l)\left\{-a^2\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+\frac{a}{l}\left(\frac{dF}{da}-\tfrac{1}{2}\frac{dV}{da}-A\right)-\frac{dF}{dl}\right\}\\
&\quad+(m-l)\left\{-b^2\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+\frac{b}{m}\left(\frac{dF}{db}-\tfrac{1}{2}\frac{dV}{db}-B\right)-\frac{dF}{dm}\right\}\\
&\quad+(n\ \ -l)\left\{-c^2\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+\frac{c}{n}\left(\frac{dF}{dc}-\tfrac{1}{2}\frac{dV}{dc}-C\right)-\frac{dF}{dn}\right\}\\
&\quad=0;
\end{aligned}$$

and I remark that this equation may be broken up into two equations, each of which separately is satisfied; these equations are

$$\begin{aligned}
&-k\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+(V-F)-l\frac{dF}{dl}-m\frac{dF}{dm}-n\frac{dF}{dn}\\
&\qquad\qquad+a\left(\frac{dF}{da}-\tfrac{1}{2}\frac{dV}{da}-A\right)+b\left(\frac{dF}{db}-\tfrac{1}{2}\frac{dV}{db}-B\right)+c\left(\frac{dF}{dc}-\tfrac{1}{2}\frac{dV}{dc}-C\right)=0,
\end{aligned}$$

and

$$\begin{aligned}
&(a^2+b^2+c^2)\left(\frac{dV}{dk}-\frac{dF}{dk}\right)-\frac{dF}{dl}-\frac{dF}{dm}-\frac{dF}{dn}\\
&\qquad\qquad+\frac{a}{l}\left(\frac{dF}{da}-\tfrac{1}{2}\frac{dV}{da}-A\right)+\frac{b}{m}\left(\frac{dF}{db}-\tfrac{1}{2}\frac{dV}{db}-B\right)+\frac{c}{n}\left(\frac{dF}{dc}-\tfrac{1}{2}\frac{dV}{dc}-C\right)=0.
\end{aligned}$$

It may be added that the functions under the integral signs, and consequently the integrals, are all of them homogeneous of the degree zero in l, m, n, k. The thing to be verified is that the foregoing two equations are satisfied by the functions under the integral signs, independently of the integrations, in fact by the values

$$\left\{\begin{aligned}
&A=\frac{\xi\sqrt{R}}{L},\ B=\frac{\eta\sqrt{R}}{L},\ C=\frac{\zeta\sqrt{R}}{L},\\
&V=\frac{I\sqrt{R}}{L},\\
&F=\frac{(a\xi+b\eta+c\zeta)\sqrt{R}}{L}.
\end{aligned}\right.$$

We find by differentiating these values, and after a few obvious reductions,

$$\begin{aligned}
\frac{dV}{da}&=l\xi\left(\frac{\sqrt{R}}{L^2}+\frac{I^2}{L^2\sqrt{R}}\right)+la\left(\frac{-I}{L\sqrt{R}}\right),\\
\frac{dV}{dl}&=a\xi\left(\frac{\sqrt{R}}{L^2}+\frac{I^2}{L^2\sqrt{R}}\right)+a^2\left(\frac{-I}{2L\sqrt{R}}\right)+\xi^2\left(-\frac{3I\sqrt{R}}{2L^3}-\frac{I^3}{2L^3\sqrt{R}}\right),\\
\frac{dV}{dk}&=\qquad\qquad\qquad\qquad\frac{I}{2L\sqrt{R}},
\end{aligned}$$

$$\frac{dF}{da} - \frac{\xi\sqrt{R}}{L} = (a\xi + b\eta + c\zeta)\left\{l\zeta\left(\frac{I}{L\sqrt{R}}\right) + la\left(-\frac{1}{\sqrt{R}}\right)\right\},$$

$$\frac{dF}{dl} = (a\xi + b\eta + c\zeta)\left\{a\xi\left(\frac{I}{L\sqrt{R}}\right) + a^2\left(\frac{-1}{2\sqrt{R}}\right) + \xi^2\left(-\frac{\sqrt{R}}{2L^2} - \frac{I^2}{2L^2\sqrt{R}}\right)\right\},$$

$$\frac{dF}{dk} = (a\xi + b\eta + c\zeta)\left\{\frac{1}{2\sqrt{R}}\right\},$$

values which give, as they should do,

$$l\frac{dF}{dl} + m\frac{dF}{dm} + n\frac{dF}{dn} + k\frac{dF}{dk} = 0.$$

Hence forming the values of the different parts of the first equation,

$$-k\left(\frac{dV}{dk} - \frac{dF}{dk}\right) + V - F - l\frac{dF}{dl} - m\frac{dF}{dm} - n\frac{dF}{dn}$$

$$= \frac{I\sqrt{R}}{L^2} - \frac{kI}{2L\sqrt{R}} + (a\xi + b\eta + c\zeta)\left\{-\frac{\sqrt{R}}{L} + \frac{k}{\sqrt{R}}\right\},$$

$$a\left(\frac{dF}{da} - A\right) + b\left(\frac{dF}{db} - B\right) + c\left(\frac{dF}{dc} - C\right) = (a\xi + b\eta + c\zeta)\left\{\frac{\sqrt{R}}{L} - \frac{k}{\sqrt{R}}\right\},$$

$$-\tfrac{1}{2}\left(a\frac{dV}{da} + b\frac{dV}{db} + c\frac{dV}{dc}\right) = -\frac{I\sqrt{R}}{L^2} + \frac{kI}{2L\sqrt{R}},$$

which satisfy the first equation.

And in like manner for the second equation,

$$-(a^2 + b^2 + c^2)\left(\frac{dV}{dk} - \frac{dF}{dk}\right) = -(a^2 + b^2 + c^2)\frac{I}{2L\sqrt{R}} + (a^2 + b^2 + c^2)(a\xi + b\eta + c\zeta)\frac{1}{2\sqrt{R}},$$

$$\frac{a}{l}\left(\frac{dF}{da} - A\right) + \frac{b}{m}\left(\frac{dF}{db} - B\right) + \frac{c}{n}\left(\frac{dF}{dc} - C\right)$$

$$= (a\xi + b\eta + c\zeta)^2\frac{I}{L\sqrt{R}} - (a^2 + b^2 + c^2)(a\xi + b\eta + c\zeta)\frac{1}{\sqrt{R}},$$

$$-\tfrac{1}{2}\left(\frac{a}{l}\frac{dV}{da} + \frac{b}{m}\frac{dV}{db} + \frac{c}{n}\frac{dV}{dc}\right) = (a^2 + b^2 + c^2)\frac{I}{2L\sqrt{R}} - (a\xi + b\eta + c\zeta)\left(\frac{\sqrt{R}}{2L^2} + \frac{I^2}{2L^2\sqrt{R}}\right),$$

$$-\frac{dF}{dl} - \frac{dF}{dm} - \frac{dF}{dn} = -(a\xi + b\eta + c\zeta)^2\frac{I}{2L\sqrt{R}} + (a^2 + b^2 + c^2)(a\xi + b\eta + c\zeta)\frac{1}{2\sqrt{R}}$$

$$\div (a\xi + b\eta + c\zeta)\left(\frac{\sqrt{R}}{2L^2} + \frac{I^2}{2L^2\sqrt{R}}\right),$$

which satisfy the second equation

Now considering V, F, A, B, C as standing for the definite integrals, we may replace A, B, C by the differential coefficients of V, and retaining for shortness F to stand for

$$F = -a\frac{dV}{da} - b\frac{dV}{db} - c\frac{dV}{dc},$$

the first equation becomes

$$-k\left(\frac{dV}{dk} - \frac{dF}{dk}\right) + V - F - l\frac{dF}{dl} - m\frac{dF}{dm} - n\frac{dF}{dn}$$
$$+ a\left(\frac{dF}{da} + \tfrac{1}{2}\frac{dV}{da}\right) + b\left(\frac{dF}{db} + \tfrac{1}{2}\frac{dV}{db}\right) + c\left(\frac{dF}{dc} + \tfrac{1}{2}\frac{dV}{dc}\right) = 0;$$

and the second equation becomes

$$-(a^2+b^2+c^2)\left(\frac{dV}{dk} - \frac{dF}{dk}\right) - \frac{dF}{dl} - \frac{dF}{dm} - \frac{dF}{dn}$$
$$+ \frac{a}{l}\left(\frac{dF}{da} + \tfrac{1}{2}\frac{dV}{da}\right) + \frac{b}{m}\left(\frac{dF}{db} + \tfrac{1}{2}\frac{dV}{db}\right) + \frac{c}{n}\left(\frac{dF}{dc} + \tfrac{1}{2}\frac{dV}{dc}\right) = 0.$$

If we put as before $V = \frac{I\sqrt{R}}{L^2}$, the preceding values of the differential coefficients of V give $F = \frac{2I\sqrt{R}}{L^2} + \frac{kI}{L\sqrt{R}}$, or as we may write it $F = -2V + W$, where $W = \frac{kI}{L\sqrt{R}}$. I put then for the moment

$$\begin{cases} V = \dfrac{I\sqrt{R}}{L^2}, \quad W = \dfrac{kI}{L\sqrt{R}}, \\ F = -2V + W. \end{cases}$$

It should be remarked that there is nothing in what has preceded which tends to show that these values must satisfy the differential equations. The definite integrals must, of course, satisfy as before the equations, but it does not follow that the equations are satisfied by the elements separately. And in fact only the first equation is so satisfied; the second equation is not satisfied. To verify this I form the differential coefficients

$$\frac{dW}{da} = l\xi\left(\frac{k}{L\sqrt{R}} - \frac{kI^2}{LR\sqrt{R}}\right) + la\left(\frac{kI}{R\sqrt{R}}\right),$$

$$\frac{dW}{dk} = \frac{I}{L\sqrt{R}} - \frac{kI}{2R\sqrt{R}},$$

$$\frac{dW}{dl} = a\xi\left(\frac{k}{L\sqrt{R}} - \frac{kI^2}{LR\sqrt{R}}\right) + \xi^2\left(\frac{-3kI}{2L^2\sqrt{R}} + \frac{kI^3}{2L^2R\sqrt{R}}\right) + a^2\frac{kI}{2R\sqrt{R}}.$$

We have as before

$$l\frac{dF}{dl}+m\frac{dF}{dm}+n\frac{dF}{dn}+k\frac{dF}{dk}=0,$$

and forming the expressions for the different parts of the first equation,

$$-k\left(\frac{dV}{dk}-\frac{dF}{dk}\right)+V-F-l\frac{dF}{dl}-m\frac{dF}{dm}-n\frac{dF}{dn}$$

$$=-5k\frac{dV}{dk}+2k\frac{dW}{dk}+3V-W \qquad =\frac{3I\sqrt{R}}{L^2}-\frac{3kI}{2L\sqrt{R}}-\frac{k^2I}{R\sqrt{R}},$$

$$a\left(\frac{dF}{da}+\tfrac{1}{2}\frac{dV}{da}\right)+\&c.=$$

$$-\tfrac{3}{2}\left(a\frac{dV}{da}+b\frac{dV}{db}+c\frac{dV}{dc}\right)+\left(a\frac{dW}{da}+b\frac{dW}{db}+c\frac{dW}{dc}\right),$$

and

$$-\tfrac{3}{2}\left(a\frac{dV}{da}+b\frac{dV}{db}+c\frac{dV}{dc}\right) \qquad =-\frac{3I\sqrt{R}}{L^2}+\frac{3kI}{2L\sqrt{R}},$$

$$\left(a\frac{dW}{da}+b\frac{dW}{db}+c\frac{dW}{dc}\right) \qquad =\frac{k^2I}{R\sqrt{R}},$$

which show that the first equation is satisfied.

Proceeding in the same manner with the second equation,

$$-(a^2+b^2+c^2)\left(\frac{dV}{dk}-\frac{dF}{dk}\right)=$$

$$-(a^2+b^2+c^2)\left(3\frac{dV}{dk}-\frac{dW}{dk}\right)=(a^2+b^2+c^2)\left(-\frac{I}{L\sqrt{R}}-\frac{kI}{2R\sqrt{R}}\right)$$

$$\frac{a}{l}\left(\frac{dF}{da}+\tfrac{1}{2}\frac{dV}{da}\right)+\&c.=$$

$$-\tfrac{3}{2}\left(\frac{a}{l}\frac{dV}{da}+\frac{b}{m}\frac{dV}{db}+\frac{c}{n}\frac{dV}{dc}\right)+\left(\frac{a}{l}\frac{dW}{da}+\frac{b}{m}\frac{dW}{db}+\frac{c}{n}\frac{dW}{dc}\right);$$

and

$$-\tfrac{3}{2}\left(\frac{a}{l}\frac{dV}{da}+\frac{b}{m}\frac{dV}{db}+\frac{c}{n}\frac{dV}{dc}\right)=(a^2+b^2+c^2)\frac{3I}{2L\sqrt{R}}+(a\xi+b\eta+c\zeta)\left(-\frac{3\sqrt{R}}{2L^2}-\frac{3I^2}{2L^2\sqrt{R}}\right),$$

$$\left(\frac{a}{l}\frac{dW}{da}+\frac{b}{m}\frac{dW}{db}+\frac{c}{n}\frac{dW}{dc}\right)=(a^2+b^2+c^2)\frac{kI}{R\sqrt{R}}+(a\xi+b\eta+c\zeta)\left(\frac{k}{L\sqrt{R}}-\frac{kI^2}{LR\sqrt{R}}\right);$$

$$-\frac{dF}{dl}-\frac{dF}{dm}-\frac{dF}{dn}=2\left(\frac{dV}{dl}+\frac{dV}{dm}+\frac{dV}{dn}\right)-\left(\frac{dW}{dl}+\frac{dW}{dm}+\frac{dW}{dn}\right),$$

and

$$2\left(\frac{dV}{dl}+\frac{dV}{dm}+\frac{dV}{dn}\right)=(a^2+b^2+c^2)\left(\frac{-1}{L\sqrt{R}}\right)+(a\xi+b\eta+c\zeta)\left(\frac{2\sqrt{R}}{L^2}+\frac{2I^2}{L^2\sqrt{R}}\right)-\frac{3I\sqrt{R}}{L^3}-\frac{I^3}{L^3\sqrt{R}}$$

$$-\left(\frac{dW}{dl}+\frac{dW}{dm}+\frac{dW}{dn}\right)=(a^2+b^2+c^2)\left(\frac{-kI}{2R\sqrt{R}}\right)+(a\xi+b\eta+c\zeta)\left(\frac{-k}{L\sqrt{R}}+\frac{kI^2}{LR\sqrt{R}}\right)$$

$$+\frac{3kI}{2L^2\sqrt{R}}-\frac{kI^3}{2L^3R\sqrt{R}};$$

and the value of the left-hand side of the second equation is therefore,

$$(a\xi+b\eta+c\zeta)\left(\frac{\sqrt{R}}{2L^2}+\frac{3I^2}{2L^2\sqrt{R}}\right)-\frac{3I\sqrt{R}}{L^3}-\frac{I^3}{2L^3\sqrt{R}}+\frac{3kI}{2L^2\sqrt{R}}-\frac{kI^3}{2L^3R\sqrt{R}},$$

which is not equal to zero, or the second equation is not satisfied.

I consider again V, F as denoting the definite integrals, and I eliminate $\frac{dV}{dk}$, $\frac{dF}{dk}$ by means of the equations

$$l\frac{dV}{dl}+m\frac{dV}{dm}+n\frac{dV}{dn}+k\frac{dV}{dk}=0,$$

$$l\frac{dF}{dl}+m\frac{dF}{dm}+n\frac{dF}{dn}+k\frac{dF}{dk}=0.$$

The first equation thus becomes

$$l\frac{dV}{dl}+m\frac{dV}{dm}+n\frac{dV}{dn}-2\left(l\frac{dF}{dl}+m\frac{dF}{dm}+n\frac{dF}{dn}\right)+V-F$$

$$+\tfrac{1}{2}\left(a\frac{dV}{da}+b\frac{dV}{db}+b\frac{dV}{dc}\right)+\left(a\frac{dF}{da}+b\frac{dF}{db}+c\frac{dF}{dc}\right)=0,$$

and the second equation becomes

$$(a^2+b^2+c^2)\frac{1}{k}\left(l\frac{dV}{dl}+m\frac{dV}{dm}+n\frac{dV}{dn}\right)-(a^2+b^2+c^2)\frac{1}{k}\left(l\frac{dF}{dl}+m\frac{dF}{dm}+n\frac{dF}{dn}\right)$$

$$+\tfrac{1}{2}\left(\frac{a}{l}\frac{dV}{da}+\frac{bd}{m}\frac{V}{db}+\frac{c}{n}\frac{dV}{dc}\right)+\left(\frac{a}{l}\frac{dF}{da}+\frac{b}{m}\frac{dF}{db}+\frac{c}{n}\frac{dF}{dc}\right)-\left(\frac{dF}{dl}+\frac{dF}{dm}+\frac{dF}{dn}\right)=0;$$

and it will be remembered that in these equations

$$F=-a\frac{dV}{da}-b\frac{dV}{db}-c\frac{dV}{dc}.$$

The first equation (it is easy to perceive) shows merely that V is made up of terms separately homogeneous in a, b, c, and in l, m, n, and such that the degrees in the two sets respectively being κ, λ, then $\lambda-\frac{1}{2}(\kappa-2)$. In fact V being a function of the form in question, if we attend only to the term the degrees of which are κ, λ,

then, by the properties of homogeneous functions, $F=-V$, and the first equation is satisfied if only

$$\lambda+2\lambda\kappa+1+\kappa+\tfrac{1}{2}\kappa-\kappa^2=0,$$

i.e. if $\lambda(2\kappa+1)=\kappa^2-\frac{3}{2}\kappa-1=\frac{1}{2}(\kappa-2)(2\kappa+1)$ or $\lambda=\frac{1}{2}(\kappa-2)$. Or, what is the same thing, we may say that the first equation shows that V is made up of terms the degrees of which in a, b, c and in l, m, n are $-2i-1$, and $-i-\frac{3}{2}$ respectively. Attending henceforth only to the second equation, I write $\frac{l}{k}=\frac{1}{\alpha+\theta}$, $\frac{m}{k}=\frac{1}{\beta+\theta}$, $\frac{n}{k}=\frac{1}{\gamma+\theta}$; so that $\alpha+\theta$, $\beta+\theta$, $\gamma+\theta$ denote the squared semiaxes of the ellipsoid. We have

$$\frac{d}{dl}=-\frac{(\alpha+\theta)^2}{k}\frac{d}{da}, \text{ \&c.},$$

and the equation becomes

$$\begin{aligned}
&-(a^2+b^2+c^2)\left((\alpha+\theta)\frac{dV}{d\alpha}+(\beta+\theta)\frac{dV}{d\beta}+(\gamma+\theta)\frac{dV}{d\gamma}\right)\\
&+(a^2+b^2+c^2)\left((\alpha+\theta)\frac{dF}{d\alpha}+(\beta+\theta)\frac{dF}{d\beta}+(\gamma+\theta)\frac{dF}{d\gamma}\right)\\
&\qquad+\tfrac{1}{2}\left(a(\alpha+\theta)\frac{dV}{da}+b(\beta+\theta)\frac{dV}{db}+c(\gamma+\theta)\frac{dV}{dc}\right)\\
&\qquad+\left(a(\alpha+\theta)\frac{dF}{da}+b(\beta+\theta)\frac{dF}{db}+c(\gamma+\theta)\frac{dF}{dc}\right)\\
&\qquad+\left((\alpha+\theta)^2\frac{dF}{d\alpha}+(\beta+\theta)^2\frac{dF}{d\beta}+(\gamma+\theta)^2\frac{dF}{d\gamma}\right)=0.
\end{aligned}$$

Put for shortness $\Theta=(\alpha+\theta)(\beta+\theta)(\gamma+\theta)$, and write

$$V=v\sqrt{\Theta},\ F=f\sqrt{\Theta},$$

($\sqrt{\Theta}$ is to a constant factor *près* the mass of the ellipsoid) then v, f are connected by the equation

$$f=-a\frac{dv}{da}-b\frac{dv}{db}-c\frac{dv}{dc};$$

and observing that

$$(\alpha+\theta)\frac{d\sqrt{\Theta}}{d\alpha}+(\beta+\theta)\frac{d\sqrt{\Theta}}{d\beta}+(\gamma+\theta)\frac{d\sqrt{\Theta}}{d\gamma}=3\Theta\frac{1}{2\sqrt{\Theta}}=\tfrac{3}{2}\sqrt{\Theta},$$

$$\begin{aligned}
&(\alpha+\theta)^2\frac{d\sqrt{\Theta}}{d\alpha}+(\beta+\theta)^2\frac{d\sqrt{\Theta}}{d\beta}+(\gamma+\theta)^2\frac{d\sqrt{\Theta}}{d\gamma}\\
&\qquad=(\alpha+\beta+\gamma+3\theta)\Theta\frac{1}{2\sqrt{\Theta}}=\tfrac{1}{2}(\alpha+\beta+\gamma+3\theta)\sqrt{\theta},
\end{aligned}$$

the equation becomes

$$
\begin{aligned}
&-(a^2+b^2+c^2)\left((\alpha+\theta)\frac{dv}{d\alpha}+(\beta+\theta)\frac{dv}{d\beta}+(\gamma+\theta)\frac{dv}{d\gamma}\right)\\
&+(a^2+b^2+c^2)\left((\alpha+\theta)\frac{df}{d\alpha}+(\beta+\theta)\frac{df}{d\beta}+(\gamma+\theta)\frac{df}{d\gamma}\right)\\
&-\tfrac{3}{2}(a^2+b^2+c^2)\,v+\tfrac{3}{2}(a^2+b^2+c^2)f\\
&+\tfrac{1}{2}\left(a(\alpha+\theta)\frac{dv}{d\alpha}+b(\beta+\theta)\frac{dv}{db}+c(\gamma+\theta)\frac{dv}{dc}\right)\\
&+\left(a(\alpha+\theta)\frac{df}{da}+b(\beta+\theta)\frac{df}{db}+c(\gamma+\theta)\frac{df}{dc}\right)\\
&+\left((\alpha+\theta)^2\frac{df}{d\alpha}+(\beta+\theta)^2\frac{df}{d\beta}+(\gamma+\theta)^2\frac{df}{d\gamma}\right)\\
&+\tfrac{1}{2}(\alpha+\beta+\gamma+3\theta)f=0.
\end{aligned}
$$

Now $v\left(=\frac{1}{\sqrt{\Theta}}V\right)$ may be expanded in the form

$$v=U_0+U_1+U_2\ldots+U_i\ldots,$$

where U_i is of the degree $2i$ in the semiaxes $\sqrt{(\alpha+\theta)}$, $\sqrt{(\beta+\theta)}$, $\sqrt{(\gamma+\theta)}$, and of the degree $-2i-1$ in the coordinates a, b, c. And it is easy to see that the first term of the expansion is

$$U_0=\frac{4\pi}{3}\frac{1}{\sqrt{(a^2+b^2+c^2)}}.$$

The preceding value of v gives

$$f=U_0+3U_1+5U_2\ldots+(2i+1)U_i+\ldots,$$

and substituting the values of v, f in the differential equation, and attending only to the terms which are of the same dimensions as $(a^2+b^2+c^2)U_{i+1}$, we have

$$
\begin{aligned}
&\left\{-(i+1)+(i+1)(2i+3)-\tfrac{3}{2}+\tfrac{3}{2}(2i+3)\right\}(a^2+b^2+c^2)U_{i+1}\\
&+\left\{a\alpha\frac{dU_i}{d\alpha}+b\beta\frac{dU_i}{d\beta}+c\gamma\frac{dU_i}{d\gamma}-(2i+1)\theta U_i\right\}\left(\tfrac{1}{2}+2i+1\right)\\
&+\left\{\alpha^2\frac{dU_i}{d\alpha}+\beta^2\frac{dU_i}{d\beta}+\gamma^2\frac{dU_i}{d\gamma}+2i\theta U_i-\theta^2\left(\frac{dU_i}{d\alpha}+\frac{dU_i}{d\beta}+\frac{dU_i}{d\gamma}\right)\right\}(2i+1)\\
&+\tfrac{1}{2}(\alpha+\beta+\gamma+3\theta)(2i+1)U_i=0;
\end{aligned}
$$

or, reducing,

$$
\begin{aligned}
&(i+1)(2i+5)(a^2+b^2+c^2)\,U_{i+1}\\
&+(2i+\tfrac{3}{2})\left(a\alpha\frac{dU_i}{da}+b\beta\frac{dU_i}{db}+c\gamma\frac{dU_i}{dc}\right)\\
&+(2i+1)\left(\alpha^2\frac{dU_i}{d\alpha}+\beta^2\frac{dU_i}{d\beta}+\gamma^2\frac{dU_i}{d\gamma}+\tfrac{1}{2}(\alpha+\beta+\gamma)\,U_i\right)\\
&-(2i+1)\left(\frac{dU_i}{d\alpha}+\frac{dU_i}{d\beta}+\frac{dU_i}{d\gamma}\right)=0.
\end{aligned}
$$

Now U_i is a function of $\alpha+\theta$, $\beta+\theta$, $\gamma+\theta$; if, therefore, for any particular value of i, U_i is independent of θ, it is clear that U_i must be a function of the differences of these quantities, and we shall have $\frac{dU_i}{d\alpha}+\frac{dU_i}{d\beta}+\frac{dU_i}{d\gamma}=0$; and this being so, the equation of differences, and consequently U_{i+1}, will be independent of θ. But U_0 is independent of θ, hence U_1, U_2, &c. are all of them independent of θ, or v is independent of θ; i.e. for ellipsoids having the same foci for their principal sections, and acting on the same external point, the potentials, and therefore the attractions, are proportional to the masses, which is Maclaurin's theorem for the attractions of ellipsoids upon an external point.

The foregoing equation, omitting the term which vanishes, gives

$$
U_{i+1}=\frac{\begin{aligned}&-(2i+\tfrac{3}{2})\left(a\alpha\frac{dU_i}{da}+b\beta\frac{dU_i}{db}+c\gamma\frac{dU_i}{dc}\right)\\&-(2i+1)\left(\alpha^2\frac{dU_i}{d\alpha}+\beta^2\frac{dU_i}{d\beta}+\gamma^2\frac{dU_i}{d\gamma}+\tfrac{1}{2}(\alpha+\beta+\gamma)\,U_i\right)\end{aligned}}{(i+1)(2i+5)(a^2+b^2+c^2)}.
$$

It may be remarked that this equation, with the assistance of the equation

$$
\frac{dU_i}{d\alpha}+\frac{dU_i}{d\beta}+\frac{dU_i}{d\gamma}=0,
$$

gives

$$
\begin{aligned}
&(i+1)(2i+5)\left(\frac{dU_{i+1}}{d\alpha}+\frac{dU_{i+1}}{d\beta}+\frac{dU_{i+1}}{d\gamma}\right)=-(2i+\tfrac{3}{2})\left(a\frac{dU_i}{da}+b\frac{dU_i}{db}+c\frac{dU_i}{dc}\right)\\
&-(2i+1)\left(2\left(\alpha\frac{dU_i}{d\alpha}+\beta\frac{dU_i}{d\beta}+\gamma\frac{dU_i}{d\gamma}\right)+\tfrac{3}{2}U_i\right)=\left((2i+\tfrac{3}{2})(2i+1)-(2i+1)(2i+\tfrac{3}{2})\right)U_i=0,
\end{aligned}
$$

which is as it should be.

Write

$$
U_i=\frac{Q_i}{(a^2+b^2+c^2)^{i+\frac{1}{2}}},
$$

where U_i is of the degree i in α, β, γ, and of the degree $2i$ in a, b, c. We have

$$a\alpha\frac{dU_i}{da}+b\beta\frac{dU_i}{db}+c\gamma\frac{dU_i}{dc}=(a^2+b^2+c^2)^{-2i-\frac{1}{2}}\left(a\alpha\frac{dQ_i}{da}+b\beta\frac{dQ_i}{db}+c\gamma\frac{dQ_i}{dc}\right)$$
$$-(4i+1)(a^2+b^2+c^2)^{-2i-\frac{3}{2}}(a^2\alpha+b^2\beta+c^2\gamma)\,Q_i,$$

$$\alpha^2\frac{dU_i}{d\alpha}+\beta^2\frac{dU_i}{d\beta}+\gamma^2\frac{dU_i}{d\gamma}+\tfrac{1}{2}(\alpha+\beta+\gamma)\,U_i$$
$$=(a^2+b^2+c^2)^{-2i-\frac{1}{2}}\left(\alpha^2\frac{dQ_i}{d\alpha}+\beta^2\frac{dQ_i}{d\beta}+\gamma^2\frac{dQ_i}{d\gamma}\right)+\tfrac{1}{2}(\alpha+\beta+\gamma)\,Q_i.$$

Hence, putting in like manner

$$(i+1)(2i+5)\,Q_{i+1}=-(2i+\tfrac{3}{2})\left\{(a^2+b^2+c^2)\left(a\alpha\frac{dQ_i}{da}+b\beta\frac{dQ_i}{db}+c\gamma\frac{dQ_i}{dc}\right)\right.$$
$$\left.-(4i+1)(a^2\alpha+b^2\beta+c^2\gamma)\,Q_i\right\}$$
$$-(2i+1)(a^2+b^2+c^2)\left(\alpha^2\frac{dQ_i}{d\alpha}+\beta^2\frac{dQ_i}{d\beta}+\gamma^2\frac{dQ_i}{d\gamma}+\tfrac{1}{2}(\alpha+\beta+\gamma)\,Q_i\right),$$

i.e.

$$(i+1)(2i+5)\,Q_{i+1}=-(a^2+b^2+c^2)\left\{(2i+\tfrac{3}{2})\left(a\alpha\frac{dQ_i}{da}+d\beta\frac{dQ_i}{db}+c\gamma\frac{dQ_i}{dc}\right)\right.$$
$$\left.+(2i+1)\left(\alpha^2\frac{dQ_i}{d\alpha}+\beta^2\frac{dQ_i}{d\beta}+\gamma^2\frac{dQ_i}{d\gamma}+\tfrac{1}{2}(\alpha+\beta+\gamma)\,Q_i\right)\right\}+(2i+\tfrac{3}{2})(4i+1)(a^2\alpha+b^2\beta+c^2\gamma)\,Q_i,$$

from which the functions Q_i may be calculated successively.

We may, it is clear, write

$$U_i=\frac{4\pi}{3}\cdot\frac{1}{2^i\,1.2.3..i\,5.7..2i+3}\,K_i,$$

and we shall then have

$$(a^2+b^2+c^2)\,K_{i+1}+(4i+3)\left(a\alpha\frac{dK_i}{da}+b\beta\frac{dK_i}{db}+c\gamma\frac{dK_i}{dc}\right)$$
$$+(4i+2)\left(\alpha^2\frac{dK_i}{d\alpha}+\beta^2\frac{dK_i}{d\beta}+\gamma^2\frac{dK_i}{d\gamma}\right)$$
$$+(2i+1)\,(\alpha+\beta+\gamma)\,K_i=0,$$

where

$$K_0=\frac{1}{\sqrt{(a^2+b^2+c^2)}}.$$

I proceed to show that

$$K_i=\left(\alpha\frac{d^2}{da^2}+\beta\frac{d^2}{db^2}+\gamma\frac{d^2}{dc^2}\right)^i\frac{1}{\sqrt{(a^2+b^2+c^2)}}.$$

In fact, assuming this equation for any particular value of i, we find first

$$\left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_i=i\left(\alpha\frac{d^2}{da^2}+\beta\frac{d^2}{db^2}+\gamma\frac{d^2}{dc^2}\right)^{i-1}\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)\frac{1}{\sqrt{(a^2+b^2+c^2)}}$$

$$=i\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)\left(\alpha\frac{d^2}{da^2}+\beta\frac{d^2}{db^2}+\gamma\frac{d^2}{dc^2}\right)^{i-1}\frac{1}{\sqrt{(a^2+b^2+c^2)}}$$

$$=i\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)K_{i-1}.$$

Now putting for shortness

$$\Delta=\alpha\frac{d^2}{da^2}+\beta\frac{d^2}{db^2}+\gamma\frac{d^2}{dc^2},$$

we find, replacing ΔK_{i+1} and ΔK_i by their values K_{i+2} and K_{i+1},

$$\begin{aligned}\Delta(a^2+b^2+c^2)K_{i+1}=&\ (a^2+b^2+c^2)K_{i+2}\\&+4\left(a\alpha\frac{d}{da}+b\beta\frac{d}{db}+c\gamma\frac{d}{dc}\right)K_{i+1}\\&+2(\alpha+\beta+\gamma)K_{i+1},\end{aligned}$$

$$\begin{aligned}\Delta\left(a\alpha\frac{d}{da}+b\beta\frac{d}{db}+c\gamma\frac{d}{dc}\right)K_i=&\ \left(a\alpha\frac{d}{da}+b\beta\frac{d}{db}+c\gamma\frac{d}{dc}\right)K_{i+1}\\&+2\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)K_i,\end{aligned}$$

$$\begin{aligned}\Delta\left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_i=&\ \left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_{i+1}\\&-\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)K_i,\end{aligned}$$

$$\Delta(\alpha+\beta+\gamma)K_i=(\alpha+\beta+\gamma)K_{i+1};$$

hence operating on the equation of differences with the symbol Δ, we obtain

$$\begin{aligned}(a^2+b^2+c^2)K_{i+2}&+(4i+7)\left(a\alpha\frac{d}{da}+b\beta\frac{d}{db}+c\gamma\frac{d}{dc}\right)K_{i+1}\\&+(4i+2)\left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_{i+1}\\&+\{2(4i+3)-(4i+2)\}\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)K_i\\&+(2i+3)(\alpha+\beta+\gamma)K_{i+1}=0,\end{aligned}$$

the third line of which is

$$4(i+1)\left(\alpha^2\frac{d^2}{da^2}+\beta^2\frac{d^2}{db^2}+\gamma^2\frac{d^2}{dc^2}\right)K_i, = 4\left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_{i+1},$$

by a foregoing equation, and the assumed equation of difference thus leads to

$$\begin{aligned}(a^2+b^2+c^2)K_{i+2}&+(4i+7)\left(a\alpha\frac{d}{da}+b\beta\frac{d}{db}+c\gamma\frac{d}{dc}\right)K_{i+1}\\&+(4i+6)\left(\alpha^2\frac{d}{d\alpha}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}\right)K_{i+1}\\&+(2i+3)(\alpha\quad+\beta\quad+\gamma\quad)K_{i+1}=0,\end{aligned}$$

which is the assumed equation, writing $i+1$ instead of i. The equation, if true for i, is therefore true for $i+1$, and it is easily seen to be true for $i=0$; hence it is true generally, or the value

$$K_i=\left(\alpha\frac{d^2}{da^2}+\beta\frac{d^2}{db^2}+\gamma\frac{d^2}{dc^2}\right)^i\frac{1}{\sqrt{(a^2+b^2+c^2)}}$$

satisfies the equation obtained by Laplace's method, and gives, moreover, the proper value for K_0. We have thus the value of K_i; and remembering that

$$V=\sqrt{(\alpha+\theta)(\beta+\theta)(\gamma+\theta)}\,v,$$

and observing that the symbol Δ may be replaced by

$$\Delta=(\alpha+\theta)\frac{d^2}{da^2}+(\beta+\theta)\frac{d^2}{db^2}+(\gamma+\theta)\frac{d^2}{dc^2},$$

the value of V is

$$V=\frac{4\pi}{3}\sqrt{(\alpha+\theta)(\beta+\theta)(\gamma+\theta)}\,S_0^\infty\left(\frac{1}{2^i\,1.2\ldots i.5.7\ldots 2i+3}\Delta^i\frac{1}{\sqrt{(a^2+b^2+c^2)}}\right);$$

which is in fact the value which I have found by a much more simple method in the *Cambridge Mathematical Journal*, t. III. p. 69 [2].

174.

ON THE OVAL OF DESCARTES.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 320—328.]

THIS is an extract of the passage in the Geometry (La Géométrie de M. Descartes, 12° Paris, 1705, pp. 79—92) in which he applies his method of coordinates to this now well-known curve: the marginal reference is "Explication de quatre nouveaux genres d'Ovales qui servent à l'Optique." The paper was intended to be introductory to a discussion in some detail of the history and properties of the Curve, but no continuation was written.

175.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 344—354.]

THE porism of the in-and-circumscribed triangle in its most general form relates to a triangle the angles of which lie in fixed curves, and the sides of which touch fixed curves; but at present I consider only the case in which the angles lie in one and the same fixed curve, which for greater simplicity I assume to be a conic. We have therefore a triangle ABC, the angles of which lie in a fixed conic $\mathfrak{S}$, and the sides of which touch the fixed curves $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; the points of contact may be represented by α, β, γ. And if we consider the conic $\mathfrak{S}$ and the curves $\mathfrak{A}$, $\mathfrak{B}$ as given, the curve $\mathfrak{C}$ will be the envelope of the side AB; to construct this side we have only to take at pleasure a point C on the curve $\mathfrak{S}$ and to draw through this point tangents to the curve $\mathfrak{B}$, $\mathfrak{A}$ respectively meeting the conic $\mathfrak{S}$ in the points A and B; the line joining these points is the required side AB. I may notice that in the case supposed of the curve $\mathfrak{S}$ being a conic, the lines $A\alpha$, $B\beta$, $C\gamma$ meet in a point; which gives at once a construction for γ, the point of contact of AB with the curve C. For the sake however of exhibiting the reasoning in a form which may be modified so as to be applicable to a curve $\mathfrak{S}$ of any order, instead of the conic $\mathfrak{S}$, I shall dispense with the employment of the property just mentioned, which is peculiar to the case of the conic.

Suppose for a moment (figs. 1 and 1 *bis*) that the curves $\mathfrak{A}$, $\mathfrak{B}$ are points, and let the line through $\mathfrak{A}$, $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in the points M, N. If we take the point N for the angle C of the triangle, the points A, B will each of them coincide with M, and the side AB will be the tangent at M to the conic $\mathfrak{S}$: call this tangent T. Consider next a point C in the neighbourhood of N, we shall have two points A, B in the neighbourhood of M, and the point in which AB intersects T will be the point of contact γ of T with the curve $\mathfrak{C}$. To find this point, suppose that $M\mathfrak{A} = a$, $M\mathfrak{B} = b$, $N\mathfrak{A} = a'$, $N\mathfrak{B} = b'$ and let the distance of C from N be ds; the distances parallel to T of A, B from the line MN will be proportional to $\frac{b}{b'}$, ds, $\frac{a}{a'}ds$, and

the perpendicular distances of these points from T are consequently proportional to $\frac{b^2}{b'^2}ds^2$, $\frac{a^2}{a'^2}ds^2$. The inclination of AB to T is therefore proportional to

$$\left(\frac{b^2}{b'^2}-\frac{a^2}{a'^2}\right)ds^2 \div \left(\frac{b}{b}-\frac{a}{a'}\right)ds, \text{ i.e. to } \left(\frac{b}{b'}+\frac{a}{a'}\right)ds,$$

which is of the same order as ds, and it is at once seen that the point γ will

Fig. 1.

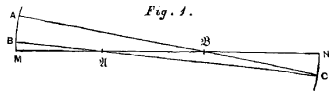
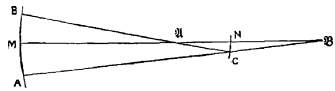


Fig. 1, bis.

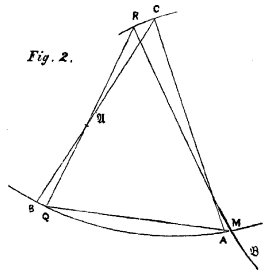


coincide with M, i.e. that the curve $\mathfrak{C}$ touches the conic $\mathfrak{S}$ at the point M. If, however, $\frac{a}{a'}+\frac{b}{b'}=0$, i.e. if the points M, N are harmonically related to $\mathfrak{A}$, $\mathfrak{B}$, then the inclination is in general of the order ds^2, and the point γ will be at a finite distance from M; moreover T is in this case a stationary tangent (i.e. a tangent at a point of inflexion) of the curve $\mathfrak{C}$. Now reverting to the general case of $\mathfrak{A}$, $\mathfrak{B}$ being any two curves, then if there be a common tangent touching these curves in the points α, β, and meeting the curve $\mathfrak{S}$ in the points M, N, the like reasonings apply to this case. Hence

First Lemma. If a common tangent to the curves $\mathfrak{A}$, $\mathfrak{B}$ touch these curves in α, β and meet the conic $\mathfrak{S}$ in the points M, N; the point N gives rise to a branch of the curve $\mathfrak{C}$ which (except in the case after mentioned) touches the conic $\mathfrak{S}$ at the point M. If however M, N are harmonically related with respect to α, β, then the branch of the curve $\mathfrak{C}$ does *not* pass through M, but it has for a *stationary* tangent the tangent M to the conic $\mathfrak{S}$.

Suppose again that the curve $\mathfrak{A}$ (fig. 2) is a point, and let the curve $\mathfrak{B}$ intersect the conic $\mathfrak{S}$ at the point M, and let the tangent to $\mathfrak{B}$ at M meet $\mathfrak{S}$ in a point R, and $R\mathfrak{A}$ meet $\mathfrak{S}$ in a point Q. Then taking the point R for the angle C of the triangle, we shall have A, B coinciding with M, Q respectively, and thence MQ a tangent to the curve $\mathfrak{C}$. To find the point of contact, take C in the neighbourhood of R at a distance from it ds. B will be a point in the neighbourhood of Q and distant from it by an infinitesimal of the order ds, A will be in the neighbourhood of M and distant from it by an infinitesimal of the order ds^2. Hence AB will intersect MQ at the point M, or the curve $\mathfrak{C}$ will pass through M. Reverting to the general case where $\mathfrak{A}$ is a curve, we have only to consider RQ as a tangent to the curve $\mathfrak{A}$ at a point α, and the like reasoning will apply to this case: hence

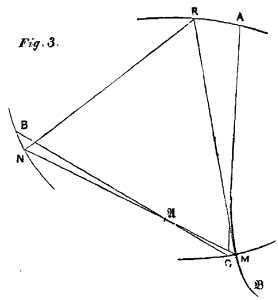
Fig. 2.



SECOND LEMMA. If the curve $\mathfrak{B}$ intersect the conic $\mathfrak{S}$ at a point M, and the tangent to $\mathfrak{B}$ at M intersect $\mathfrak{S}$ in R, if moreover a tangent through R to the curve $\mathfrak{A}$ intersect $\mathfrak{S}$ in Q; then to each of the points Q there corresponds a branch through M of the curve $\mathfrak{C}$, viz. a branch touching MQ at the point M.

Suppose as before (fig. 3) that $\mathfrak{A}$ is a point, and let the curve $\mathfrak{B}$ intersect the

Fig. 3.



conic $\mathfrak{S}$ at the point M, and the tangent to $\mathfrak{B}$ at M meet $\mathfrak{S}$ in the point R. And let $M\mathfrak{A}$ meet $\mathfrak{S}$ in the point N. Then taking the point M for the angle C of the triangle we shall have A, B coinciding with R, N respectively, and thence NR a tangent to the curve $\mathfrak{C}$. To find the point of contact, take A in the neighbourhood of R at a distance from it ds, then C will be in the neighbourhood of M and distant from it by an infinitesimal of the order ds^2, and B will be in the neighbourhood of N and distant from it by an infinitesimal of the same order ds^2, and consequently AB will intersect NR at the point N, or the curve $\mathfrak{C}$ passes through N. Reverting to the general case where $\mathfrak{A}$ is a curve, we have only to consider MN as a tangent to the curve $\mathfrak{A}$ at a point α and the like reasoning applies: hence,

THIRD LEMMA. If the curve $\mathfrak{B}$ intersect the conic $\mathfrak{A}$ at a point M, and the tangent to $\mathfrak{B}$ at M intersect $\mathfrak{S}$ in R, if moreover a tangent through M to the curve

$\mathfrak{A}$ meet $\mathfrak{S}$ in N; then to each of the points R there corresponds a branch through N of the curve $\mathfrak{C}$, viz. a branch touching NR at N.

Suppose again (fig. 4) that the curve $\mathfrak{A}$ is a point, and let the curve $\mathfrak{B}$ *touch* $\mathfrak{S}$ at the point M. And let $M\mathfrak{A}$ meet $\mathfrak{S}$ in the point N. Then taking the point

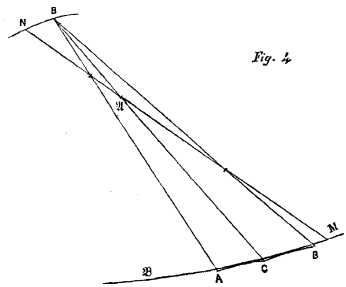


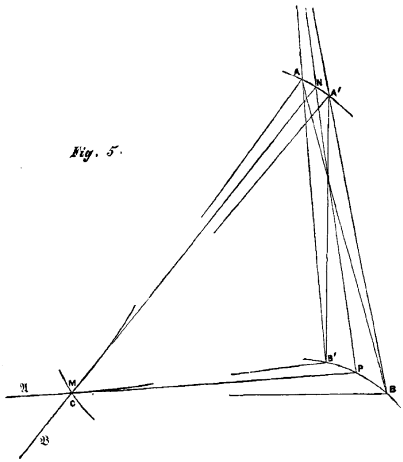
Fig. 4

M for the angle C of the triangle, the angle A will coincide with M and the angle B will coincide with N, and consequently we shall have MN a tangent to the curve $\mathfrak{C}$. And MN is in fact a double tangent, for proceeding to find the point of contact, take C in the neighbourhood of M at a distance ds; then from C we may draw to the curve $\mathfrak{B}$ two tangents each of them meeting $\mathfrak{S}$ in a point $\mathfrak{A}$ in the neighbourhood of M and distant from it by an infinitesimal of the order ds; again CA meets $\mathfrak{S}$ in a point B in the neighbourhood of N and distant from it by an infinitesimal of the same order ds; hence AB will meet MN at a point which will be in general at a finite distance from M, or rather (since there are two positions of the point A) the lines AB will meet MN in two points, each of them in general at a finite distance from M. Reverting to the case of $\mathfrak{A}$ being a curve, we must as before consider MN as a tangent to the curve $\mathfrak{A}$ at the point α, and the same reasoning applies: hence

FOURTH LEMMA. If the curve $\mathfrak{B}$ *touch* the conic $\mathfrak{S}$ at the point M, and if a tangent through M to the curve $\mathfrak{A}$ meet $\mathfrak{S}$ in N, then MN is a double tangent of the curve $\mathfrak{C}$, viz. the line MN has two distinct points of contact with the curve $\mathfrak{C}$.

Suppose (fig. 5) that the curves $\mathfrak{A}$ and $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in one and the same point M, and let the tangent at M to the curve $\mathfrak{A}$ meet $\mathfrak{S}$ in the point P, and the tangent at M to the curve $\mathfrak{B}$ meet $\mathfrak{S}$ in the point N; then if we take the point M for the angle C of the triangle, the angles A and B of the triangle will coincide with N, P respectively, and NP will be a tangent of the curve $\mathfrak{C}$. But NP will be a double tangent, for proceeding to find the point of contact, take C in the neighbourhood of M, and distant from it by an infinitesimal of the second order ds^2; then since from the point C there may be drawn two tangents touching

$\mathfrak{A}$ in the neighbourhood of M, and two tangents touching $\mathfrak{B}$ in the neighbourhood of M, there will be two points A in the neighbourhood of N and distant from it by infinitesimals of the order ds, and in like manner two points B in the neighbourhood of P and distant from it by infinitesimals of the order ds. Call these points A, A';



B, B'; then AB', $A'B'$ will meet NP in one and the same point, and AB', $A'B'$ will in like manner meet NP in one and the same point; these two points, which will be in general at finite distances from N, P, will be points of contact of NP with the curve $\mathfrak{C}$: hence,

FIFTH LEMMA. If the curves $\mathfrak{A}$, $\mathfrak{B}$ meet the conic $\mathfrak{S}$ in one and the same point M, and if the tangents at M to the curves $\mathfrak{B}$ and $\mathfrak{A}$ respectively meet $\mathfrak{S}$ in the points N and P, then, joining these points, the line NP will be a double tangent to the curve $\mathfrak{C}$, viz. there will be two distinct points of contact of the line NP with the curve $\mathfrak{C}$.

The double tangents of the fourth and fifth lemmas exist by virtue of the particular relations assumed between the curves $\mathfrak{A}$, $\mathfrak{B}$ and the conic $\mathfrak{S}$, viz. from one or both of the curves $\mathfrak{A}$, $\mathfrak{B}$ touching the conic $\mathfrak{S}$, or from the two curves having a common point of intersection or common points of intersection with $\mathfrak{S}$; there are besides double tangents which exist independently of any such relations, and the theory of which will be presently investigated, but it will be convenient in the first instance to find the class of $\mathfrak{C}$.

The class of $\mathfrak{C}$ is at once determinable from the classes (which I represent by m, n) of the curves $\mathfrak{A}$, $\mathfrak{B}$. In fact, take any point M on the conic $\mathfrak{S}$ for the angle A of the triangle. Through the point M we may draw N tangents to $\mathfrak{B}$, each of which will intersect the conic $\mathfrak{S}$ in a single point, or there will be N positions of the point C, and since from each of these we may draw M tangents to the curve $\mathfrak{A}$, each tangent intersecting $\mathfrak{S}$ in a single point; or we have mn positions of the angle B, i.e. this same number of tangents through M to the curve $\mathfrak{C}$. But if the same point M had been taken as the angle B of the triangle, there would have been in like manner mn positions of the angle A, i.e. this same number of tangents through M to the curve $\mathfrak{C}$. Hence there are in all $2mn$ tangents through M to the curve $\mathfrak{C}$, or the curve $\mathfrak{C}$ is of the class $2mn$.

Considering now the double tangents of the curve $\mathfrak{C}$; imagine a quadrilateral inscribed in the conic $\mathfrak{S}$ of which two opposite sides touch the curve $\mathfrak{A}$, and the other two opposite sides touch the curve $\mathfrak{B}$: suppose $MNPQ$, of which the sides MN and PQ touch $\mathfrak{A}$ and the sides NP and QM touch $\mathfrak{B}$. If we take M for the angle C of the triangle, then the angles A, B will coincide with Q, N, i.e. NQ is a tangent to the curve $\mathfrak{C}$, and the point of contact may be determined as before by considering a point C in the neighbourhood of M; but in like manner if we take P for the angle C of the triangle, then the angles A, B will coincide with N, Q, i.e. QN is again a tangent to the curve $\mathfrak{C}$, and the point of contact may be determined by considering a point C in the neighbourhood of P; QN is therefore a double tangent of the curve $\mathfrak{C}$. But in like manner MP is a double tangent of the curve $\mathfrak{C}$, i.e. the diagonals of the quadrilateral are each of them double tangents of the curve $\mathfrak{C}$, and the number of double tangents is consequently double the number of quadrilaterals. Imagine a pentilateral inscribed in the conic $\mathfrak{S}$, the first and third sides of which touch the curve $\mathfrak{A}$ and the second and fourth sides of which touch the curve $\mathfrak{B}$, the fifth or closing side will envelope a curve $\mathfrak{C}'$, and in the cases in which the curve $\mathfrak{C}'$ and the conic $\mathfrak{S}$ have a common tangent, the fifth side will vanish, and the pentilateral becomes a quadrilateral of the kind before referred to. Now as $\mathfrak{C}$ was shown to be of the class $2mn$, so it may be shown that $\mathfrak{C}'$ is of the class $2m^2n^2$, hence $\mathfrak{C}'$ and $\mathfrak{S}$ have $4m^2n^2$ common tangents. The quadrilateral may reduce itself to a common tangent of the curves $\mathfrak{A}$ and $\mathfrak{B}$: this gives rise to $2mn$ points of contact of $\mathfrak{C}'$ and $\mathfrak{S}$, and the common tangent at a point of contact reckons as two common tangents; the number of the remaining common tangents is therefore $4m^2n^2 - 4mn$. And these are in fact tangents at points of contact of $\mathfrak{C}'$ and $\mathfrak{S}$, i.e. $\mathfrak{C}'$ and $\mathfrak{S}$ touch in $2m^2n^2 - 2mn$ points. And since each angle of the quadrilateral may be taken as the first angle, the number of quadrilaterals is one fourth of this, or $\frac{1}{2}(m^2n^2 - mn)$, and the number of double tangents of the curve $\mathfrak{C}$ from the before-mentioned cause is therefore $m^2n^2 - mn$.

But there is another way in which double tangents arise; we may have a quadrilateral $MNPQ$ inscribed in the conic $\mathfrak{S}$, such that two adjacent sides MN, NP touch the curve $\mathfrak{A}$, and the other two adjacent sides PQ, QM touch the curve $\mathfrak{B}$. In fact in this case one of the diagonals, viz. NQ, is a double tangent of the curve $\mathfrak{C}$; the number of double tangents is therefore equal to the number of quadrilaterals.

Consider a pentilateral inscribed in the conic, and such that the first and second sides touch the curve $\mathfrak{A}$ and the third and fourth sides touch the curve $\mathfrak{B}$, the fifth or closing side will envelope a curve $\mathfrak{C}''$, and in the cases in which the curve $\mathfrak{C}''$ and the conic $\mathfrak{S}$ have a common tangent, the fifth side will vanish, and the pentilateral will become a quadrilateral of the kind last referred to. The curve $\mathfrak{C}''$ is of the class $2mn(m-1)(n-1)$, hence $\mathfrak{C}''$ and $\mathfrak{S}$ have $4mn(m-1)(n-1)$ common tangents; but these are tangents at points of contact, or the curves touch in $2mn(m-1)(n-1)$ points, and the number of quadrilaterals is $mn(m-1)(n-1)$. The number of double tangents of the curve $\mathfrak{C}$ from the cause last referred to is therefore $mn(m-1)(n-1)$, which is equal to $mn(mn-m-n+1)$. The total number of double tangents of the curve $\mathfrak{C}$ is consequently $mn(2mn-m-n)$. And the curve $\mathfrak{C}$ has not in general any stationary tangents or what is the same thing any inflexions. It has been shown that $\mathfrak{C}$ is of the class $2mn$, it is therefore of the order $2mn(2mn-1)-2mn(2mn-m-n)$, which is equal to $2mn(m+n-1)$. Hence

THEOREM. If a triangle ABC be inscribed in a conic $\mathfrak{S}$, and the sides BC, AC touch curves $\mathfrak{A}$, $\mathfrak{B}$ of the classes m, n respectively, the side AB will envelope a curve $\mathfrak{C}$ of the class $2mn$ with in general $mn(2mn-m-n)$ double tangents, but no stationary tangents, and therefore of the order $2mn(m+n-1)$. If the curve $\mathfrak{A}$ touch the conic $\mathfrak{S}$, each point of contact will give rise to n double tangents of the curve $\mathfrak{C}$, and so if the curve $\mathfrak{B}$ touch the conic $\mathfrak{S}$, each point of contact will give rise to m double tangents of the curve $\mathfrak{C}$. And if $\mathfrak{A}$ and $\mathfrak{B}$ intersect on the conic $\mathfrak{S}$, each such intersection will give rise to a double tangent of the curve $\mathfrak{C}$. The curve $\mathfrak{C}$ in general touches the conic $\mathfrak{S}$ in the points in which it is intersected by any common tangent of the curves $\mathfrak{A}$ and $\mathfrak{B}$; but if the points of contact be harmonically situated with respect to the conic $\mathfrak{S}$, then $\mathfrak{C}$ does *not* pass through the points of intersection, but the tangents to $\mathfrak{S}$ at the points of intersection are stationary tangents of $\mathfrak{C}$. There is of course in the above-mentioned special cases a corresponding reduction in the order of $\mathfrak{C}$.

It sometimes happens that the number of double tangents of $\mathfrak{C}$ becomes infinite, i.e. in fact that $\mathfrak{C}$ is made up of two coincident curves; instances of this will be presently mentioned.

Suppose that the curves $\mathfrak{A}$, $\mathfrak{B}$ are points; $\mathfrak{C}$ is in general a conic having double contact with the conic $\mathfrak{S}$ in the points in which it is intersected by the line joining $\mathfrak{A}$, $\mathfrak{B}$. But if the points $\mathfrak{A}$, $\mathfrak{B}$ are harmonically situated with respect to the conic $\mathfrak{S}$, then $\mathfrak{C}$ does *not* pass through the points of intersection, but the tangents to $\mathfrak{S}$ at the points of intersection are stationary tangents of $\mathfrak{C}$. This implies that the curve $\mathfrak{C}$ is made up of two coincident points at the point of intersection of the two tangents of $\mathfrak{S}$: call this the point $\mathfrak{C}$, then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are conjugate points of the conic $\mathfrak{S}$, and we have the well-known theorem that in a conic $\mathfrak{S}$ there may be inscribed an infinity of triangles the sides of which pass through three conjugate points of the conic. It should be remarked that for each position of the side AB, there are two positions of the angle $\mathfrak{C}$, i.e. each side AB is properly a double tangent of the curve (point) $\mathfrak{C}$.

Let $\mathfrak{A}$ be a point and $\mathfrak{B}$ be a conic, the curve $\mathfrak{C}$ is in general of the class 4, with two double tangents, and therefore of the order 8. It is proper to remark that the double tangents originate in a quadrilateral of the kind first considered, viz. a quadrilateral of which two opposite sides pass through the point $\mathfrak{A}$ and the other two opposite sides touch the conic $\mathfrak{B}$. It is easy in the present case to construct the quadrilateral: consider the point $\mathfrak{A}$ as a pole, and take its polar with respect to the conic $\mathfrak{S}$, take the pole of this polar with respect to the conic $\mathfrak{B}$. Join the two poles, and from the point in which the joining line meets the polar draw tangents to the conic $\mathfrak{B}$, these tangents will meet the conic $\mathfrak{S}$ in four points, lying two and two in lines passing through the point $\mathfrak{A}$: we have thus the quadrilateral, and the diagonals of the quadrilateral (or the other two lines passing through the four points) are the double tangents of the curve $\mathfrak{C}$.

There are two particular cases to be considered. First, where the conic $\mathfrak{B}$ has double contact with the conic $\mathfrak{S}$. Here the lines joining the point $\mathfrak{A}$ with the points of contact are double tangents of the curve $\mathfrak{C}$, which has therefore in all four double tangents, and being a curve of the fourth class it must break up into two curves of the second class, i.e. into two conics. And these are curves having double contact with the conic $\mathfrak{S}$; for the curve $\mathfrak{C}$ touches the conic $\mathfrak{S}$ in the four points in which $\mathfrak{S}$ is intersected by the tangents through $\mathfrak{A}$ to the conic $\mathfrak{B}$. Secondly, where $\mathfrak{A}$ is one of the conjugate points of the conics $\mathfrak{B}$, $\mathfrak{S}$. The general construction for the quadrilateral shows that if from any point of the common polar of $\mathfrak{A}$ with respect to $\mathfrak{B}$ and with respect to $\mathfrak{S}$, we draw tangents to $\mathfrak{B}$, these will meet $\mathfrak{S}$ in four points, lying two and two in lines passing through $\mathfrak{A}$, i.e. that the number of the double tangents of the curve $\mathfrak{C}$ is indefinite; $\mathfrak{C}$ is therefore made up of two coincident curves of the second class, i.e. of two coincident conics. Moreover, $\mathfrak{C}$ passes through the points of intersection of $\mathfrak{B}$, $\mathfrak{S}$; hence, disregarding one of the two coincident conics, we may say that the curve $\mathfrak{C}$ is a conic passing through the points of intersection of the conics $\mathfrak{B}$, $\mathfrak{S}$.

Next, let the curves $\mathfrak{A}$, $\mathfrak{B}$ be each of them a conic. The curve $\mathfrak{C}$ is of the class 8, with in general 16 double tangents, and therefore of the order 24. But there are two particular cases to be considered: first, where the conics $\mathfrak{A}$, $\mathfrak{B}$ have each of them double contact with the conic $\mathfrak{S}$. Here the tangents drawn from the points of contact of either of the conics $\mathfrak{A}$ or $\mathfrak{B}$ with $\mathfrak{S}$ to the other of the conics, $\mathfrak{B}$ or $\mathfrak{A}$, is a double tangent of the curve $\mathfrak{C}$, i.e. there are 8 new double tangents, or in all 24 double tangents of the curve $\mathfrak{C}$, which is therefore of the order 8; and being of the class 8 with 24 double tangents, it must break up into four curves of the second class, i.e. into four conics. And the curve $\mathfrak{C}$ touches the conic $\mathfrak{S}$ in the points in which $\mathfrak{S}$ is intersected by any one of the four common tangents of the conics $\mathfrak{A}$, $\mathfrak{B}$, viz. 8 points in all; hence each of the four conics has double contact with the conic $\mathfrak{S}$. Attending only to one of the four conics of which $\mathfrak{C}$ is made up, we have thus what (in a restricted sense of the expression, the porism of the in-and-circumscribed triangle) I call the porism (homographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides touch conics having double contact with the circumscribed conic, then will the third side touch a conic having double contact with the circumscribed conic.

Secondly, the conics $\mathfrak{A}$ and $\mathfrak{B}$ may cut the conic $\mathfrak{S}$ in the same four points. Here it may be seen that there are an infinity of inscribed quadrilaterals of the kind first considered, viz. of which two opposite sides touch the conic $\mathfrak{A}$, and the other two opposite sides touch the conic $\mathfrak{B}$. Hence, the curve $\mathfrak{C}$ is made up of two coincident curves of the class 4. But the curve of the class 4 has in fact 4 double tangents, viz. considering each of the points of intersection of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, and drawing tangents to $\mathfrak{A}$ and $\mathfrak{B}$ meeting $\mathfrak{S}$ in two new points, the line joining these points is a double tangent of the curve in question, which is therefore of the 4th order, and being of the class 4 with 4 double tangents, it must break up into two curves of the second class, i.e. into two conics. Each of these conics passes through the points of intersection of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{S}$, and touches the four lines last referred to, the conics would of course be completely determined by the condition of passing through the four points and touching one of the four lines. Attending only to one of the two conics, we have thus what I call the porism (allographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides touch conics meeting the circumscribed conic in the same four points, the remaining side will touch a conic meeting the circumscribed conic in the four points.

The *à posteriori* demonstration of these theorems will form the subject of another paper, [178].

176.

NOTE ON JACOBI'S CANONICAL FORMULÆ FOR DISTURBED MOTION IN AN ELLIPTIC ORBIT.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 355—356.]

CONSIDER a body (afterwards called the disturbed body) revolving about a central body under the influence of their mutual attraction and of any disturbing forces. Then referring the disturbed body to axes through the central body and parallel to fixed lines in space, write

x, y, z, the coordinates of the disturbed body,

r, , the radius vector, $= \sqrt{(x^2 + y^2 + z^2)}$,

M , the mass of the disturbed body,

M'' , the mass of the central body.

Write also

R, the disturbing function, taken negatively, i.e. the sign of R is taken as in the Mécanique Céleste.

The equations of motion then are

$$\frac{d^2x}{dt^2} = -\frac{(M+M'')\,x}{r^3} - \frac{dR}{dx},$$

$$\frac{d^2y}{dt^2} = -\frac{(M+M'')\,y}{r^3} - \frac{dR}{dy},$$

$$\frac{d^2z}{dt^2} = -\frac{(M+M'')\,z}{r^3} - \frac{dR}{dz},$$

and the motion may be represented by supposing that the body moves in an ellipse with variable elements, and such that the direction and velocity of motion are always the same as in the actual orbit.

We may, to fix the ideas, take the plane of xy to be the plane of the ecliptic and the axis of x to be the line through the first point of Aries, or origin of longitude.

Jacobi's canonical elements may be taken to be,

First, the constant of vis viva or invariable part of the half-square of the velocity, which is equal to the sum of the masses divided by twice the mean distance and with the sign minus.

Secondly, the constant of areas, which is equal to the square root of the sum of the masses into the half of the latus rectum.

Thirdly, the constant of the reduced area (i.e. of the area described on the plane of the ecliptic), which is equal to the square root of the sum of the masses into the half of the latus rectum into the cosine of the inclination.

Fourthly, the constant attached by addition to the time t, or what is the same thing, the epoch or time of pericentric passage, taken with the sign minus.

Fifthly, the angular distance from node (or argument of latitude) of the pericentre.

Sixthly, the longitude of the node.

(The first and fourth elements are taken by Jacobi with the contrary sign, but this difference is not material.)

Representing the preceding system of canonical elements by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, and observing that Jacobi's disturbing function Ω is the same as the disturbing function R of the Mécanique Céleste, except that the sign is reversed (i.e. $\Omega = -R$), the expressions for the variations of the canonical elements are

$$\frac{d\mathfrak{A}}{dt} = -\frac{dR}{d\mathfrak{F}}, \quad \frac{d\mathfrak{B}}{dt} = -\frac{dR}{d\mathfrak{G}}, \quad \frac{d\mathfrak{C}}{dt} = -\frac{dR}{d\mathfrak{H}},$$

$$\frac{d\mathfrak{F}}{dt} = +\frac{dR}{d\mathfrak{A}}, \quad \frac{d\mathfrak{G}}{dt} = +\frac{dR}{d\mathfrak{B}}, \quad \frac{d\mathfrak{H}}{dt} = +\frac{dR}{d\mathfrak{C}}.$$

In the ordinary case in which the disturbing force is the attraction of a third body, write

x', y', z', the coordinates of the disturbing body,

r' , the radius vector, $= \sqrt{(x'^2 + y'^2 + r'^2)}$,

M' , the mass of the disturbing body.

Then the expression for the disturbing function is

$$R = M' \left\{ \frac{xx' + yy' + zz'}{r'^3} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right\}.$$

The preceding formulæ form a convenient standard of reference for the various systems of elements which have been made use of by writers upon Physical Astronomy; any such system may be without difficulty derived from the canonical system by expressing the elements adopted in terms of the above-mentioned canonical elements.

177.

SOLUTION OF A MECHANICAL PROBLEM.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 405—406.]

A HEAVY plane is supported by parallel elastic strings of small extensibility; and the strings are of the same length and extensibility: required the position of equilibrium.

Imagine the plane horizontal, and let n be the number of strings, (a, b), (a', b'), &c. the coordinates of the points of attachment; ξ, η the coordinates of the centre of gravity of the plane; W the weight; let the equation of the horizontal line about which the plane turns be

$$x \cos \alpha + y \sin \alpha - p = 0;$$

and let $\delta\theta$ be the inclination of the plane in its position of equilibrium to the horizontal plane, and $\omega\delta l$ the force generated by an increase δl in the length of one of the strings.

We have for the conditions of equilibrium

$$\Sigma (a \cos \alpha + b \sin \alpha - p) \ \omega\delta\theta - W = 0,$$
$$\Sigma (a \cos \alpha + b \sin \alpha - p) \, a\omega\delta\theta - W\xi = 0,$$
$$\Sigma (a \cos \alpha + b \sin \alpha - p) \, b\omega\delta\theta - W\eta = 0;$$

or putting $\Sigma a = L$, $\Sigma b = M$, $\Sigma a^2 = A$, $\Sigma ab = H$, $\Sigma b^2 = B$, we have

$$L \cos \alpha + M \sin \alpha - np - \frac{W}{\omega\delta\theta} = 0,$$

$$A \cos \alpha + H \sin \alpha - Lp - \frac{W\xi}{\omega\delta\theta} = 0,$$

$$H \cos \alpha + B \sin \alpha - Mp - \frac{W\eta}{\omega\delta\theta} = 0.$$

Combining with these the equations

$$x \cos \alpha + \eta \sin \alpha - p = 0,$$

and eliminating linearly $\cos \alpha$, $\sin \alpha$, p and W, we have

$$\begin{vmatrix} \cdot & x, & y, & 1 \\ \xi, & A, & H, & L \\ \eta, & H, & B, & M \\ 1, & L, & M, & n \end{vmatrix} = 0$$

for the equation of the required line $x \cos \alpha + y \sin \alpha - p = 0$. Replacing L, M, A, H, B by their values, the equation is readily transformed into

$$\Sigma \left\{ \begin{vmatrix} x, & y, & 1 \\ a, & b, & 1 \\ a', & b', & 1 \end{vmatrix} \times \begin{vmatrix} \xi, & \eta, & 1 \\ a, & b, & 1 \\ a', & b', & 1 \end{vmatrix} \right\} = 0$$

where the summation extends to each pair of points (a, b) and (a', b'). This is, in fact, an extension of the harmonic relation of a point and line with respect to a triangle.

178.

ON THE A POSTERIORI DEMONSTRATION OF THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 31—38.]

IN my former paper "On the Porism of the In-and-circumscribed Triangle" (*Journal*, t. I. p. 344 [175]) the two porisms (the homographic and the allographic) were established *à priori*, i.e. by means of an investigation of the order of the curve enveloped by the third side of the triangle. I propose in the present paper to give the *à posteriori* demonstration of these two porisms; first according to Poncelet, and then in a form not involving (as do his demonstrations) the principle of projections. My objection to the employment of the principle may be stated as follows: viz. that in a systematic development of the subject, the theorems relating to a particular case and which are by the principle in question extended to the general case, are not in anywise more simple or easier to demonstrate than are the theorems for the general case; and, consequently that the circuity of the method can and ought to be avoided.

The porism (homographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

The following is Poncelet's demonstration, the Nos. are those of the *Traité des Propriétés Projectives* [Paris, 1822]:

No. 431. If a triangle be inscribed in a circle and two of the sides are parallel to given lines, then the third side envelopes a concentric circle.

This is evident, for, the angle in the segment subtended by the third side being constant, the length of the third side is constant; hence, the length of the perpendicular from the centre upon the third side is also constant, and the third side envelopes a concentric circle.

Hence, by the principle of projections:

If a triangle be inscribed in a conic and two of the sides pass through given points, the remaining side envelopes a conic having double contact with the circumscribed conic, the line through the two points being the chord of contact.

No. 434. Conversely, if there be a triangle inscribed in a conic and the first side envelope a conic having double contact with the circumscribed conic, and the second side pass through a fixed point in the chord of contact, then will the third side also pass through a fixed point in the chord of contact.

No. 437. In particular, if there be a triangle inscribed in a conic and two of the sides pass through fixed points, then will the third side pass through a fixed point, viz. the point forming with the other two points a conjugate system.

No. 439. It follows that:

If there be a triangle inscribed in a conic and the first side passes through a fixed point, and the second side envelopes a conic having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

For the chord of contact meets the polar of the fixed point with respect to the circumscribed conic in a point; the line joining this point with the third angle (i.e. the angle opposite the third side) of the triangle meets the conic in a variable point; and joining this variable point with the first and second angles of the triangle we have a new triangle; two of the sides of this new triangle (by Nos. 434 and 437) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

We have thus passed from the case of the two sides passing through fixed points to that of one of the two sides enveloping a conic having double contact with the given conic and the other of them passing through a fixed point; and, by a repetition of the reasoning, Poncelet passes to the general case, viz.

If there be a triangle inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

But it is somewhat more simple to omit the intermediate case of a conic and point, and pass directly, by the reasoning of No. 439, from the case of two points to that of two conics.

In fact, considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the conic in a variable point, and joining this variable point with the first and second angles of the triangle we have a new triangle: two of the sides of this new triangle (by No. 434) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, envelopes a conic having double contact with the circumscribed conic; and the general case is thus established.

The porism (allographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic and two of the sides envelope conics meeting the circumscribed conic in the same four points, then the third side will touch a conic meeting the circumscribed conic in the four points.

The following is Poncelet's demonstration:

No. 433. In the particular case of the homographic porism, viz.—that in which two of the sides of the triangle pass through fixed points and the remaining side envelopes a conic having double contact with the circumscribed conic—it is easy to see that the lines joining the angles of the triangle with the two fixed points and with the point of contact on the third side, meet in a point; this follows at once by the principle of projection from the case in No. 431, viz. the case of a triangle inscribed in a circle when two of the sides are parallel to given lines and the third side touches a concentric circle. Hence,

No. 531. If there be a triangle inscribed in a conic, and two of the sides envelope fixed curves, and the third side envelopes a certain curve; the lines joining the angles of the triangle with the points of contact meet in a point.

In fact, attending only to the infinitesimal variation of the position of the triangle, the curves enveloped by the first and second sides may be replaced by the points of contact on these sides, and the curve enveloped by the third side may be replaced by a conic having double contact with the circumscribed conic, and the general case thus follows at once from the particular one.

Nos. 162 and 163. LEMMA[1]. If, on the sides of a triangle ABC, there are taken any three points L, M, N in the same line, and the harmonics A', B', C' of these points (i.e. the harmonic of each point with respect to the two vertices on the same side of the triangle), then the lines AA', BB', CC' will meet in a point; and, moreover, if $A'L$, $B'M$, $C'N$ are bisected in F, G, H (or, what is the same thing, if $\overline{FA'}^2 = FB \,.\, FC$, $\overline{GB}^2 = GC \,.\, GA$, $\overline{HC'}^2 = HA \,.\, HB$), then will the three points F, G, H lie in a line. This is, in fact, the theorem No. 164,—In any complete quadrilateral the middle points of the three diagonals lie in a line.

It is now easy to prove a particular case of the allographic porism, viz.

No. 531. If there be a triangle inscribed in a circle, such that two of the sides envelope circles having a common secant (real or ideal) with the circumscribed circle; then will the third side envelope a circle having the same common secant with the circumscribed circle.

For if the triangle be ABC, and the points of contact of the sides CB, CA with the enveloped circles and the point of contact of the side AB with the enveloped curve, be A', B', C'; if moreover the points of intersection of the circumscribed circle and the two enveloped circles be M, N, and the common secant MN meet the sides

[1] I have not thought it necessary to give the figures; they can be supplied without difficulty.

of the triangle in F, G, H; then F, G, H and A', B', C' are points on the sides of the triangle ABC, such that F, G, H lie in a line, and AA', BB', CC' meet in a point. And by a property of the circle

$$\overline{FA'}^2 = FM \,.\, FN = FB \,.\, FC,$$

$$\overline{GB'}^2 = GM \,.\, GN = GC \,.\, GA\,;$$

whence by the lemma (or rather its converse) $\overline{HC'}^2 = HA \,.\, HB$ and by a property of the circle $HA \,.\, HB = HM \,.\, HN$; and therefore, $\overline{HC'}^2 = HM \,.\, HN$, a property which can only belong to a circle having, with the other circles, the common secant MN: the particular case is thus demonstrated. And the principle of projections leads at once to the general case of the allographic porism.

To exhibit the demonstrations in a form independent of the principle of projections, it will be convenient to enunciate the following three lemmas: the first of them being, in fact, the theorem contained in No. 434, as generalised by No. 531; the second of them a theorem connected with and including the properties of the circle assumed in Poncelet's demonstration of the allographic porism; and the third of them a theorem derivable by the principle of projections from the theorem in Nos. 162 and 163.

LEMMA I. If there be a triangle inscribed in a conic, such that two of the sides envelope given curves and the third side envelopes a curve; then the lines joining the angles of the triangle with the points of contact of the opposite sides meet in a point.

LEMMA II. If there be three conics meeting in the same four points, then any line meets the conic in six points forming a system in involution.

COROLL. 1. If the line be a tangent to one of the conics, then the point of contact is the double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics. And conversely if the curve enveloped by the line is not given, but the preceding property holds for all positions of the tangent line; then the curve enveloped by such line is a conic passing through the points of intersection of the two given conics.

COROLL. 2. If one of the conics be a pair of coincident lines, then the other two conics are conics having double contact, with the line in question for their chord of contact; any line meets the chord of contact in a point which is a double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics; and if the line be a tangent to one of the conics, then the point of contact and the point of intersection with the chord of contact are harmonics with respect to the points of intersection with the other conic. And conversely if every tangent of a curve intersect a line and conic in such manner that the point of contact and the point of intersection with the line are harmonics with respect to the points of intersection with the conic; then the curve is a conic having double contact with the given conic, and the line in question is the chord of contact.

The third lemma is a theorem (first explicitly stated, so far as I am aware, by Steiner, Lehrsätze 24 and 25, Crelle, t. III. [1828], p. 212, and demonstrated by Bauer, t. XIX. [1839], p. 227) which, in a note in the *Phil. Mag.*, Augt. 1853 [118], I have called the *Theorem of the harmonic relation of two lines with respect to a quadrilateral.*

LEMMA III. If on each of the diagonals of a quadrilateral there be taken two points harmonically related with respect to the angles upon this diagonal; then if three of the points lie in a line, the other three points will also lie in a line: the two lines are said to be harmonically related with respect to the quadrilateral.

The relation may be exhibited under a different form. The three diagonals of the quadrilateral form a triangle, the sides of which contain the six angles of the quadrilateral; and considering only three of the six angles (one angle on each diagonal) these three angles are points which either lie in a line, or else are such that the lines joining them with the opposite angles of the triangle meet in a point. Each of the three points is, with respect to the involution formed by the two angles of the triangle and the two points harmonically related thereto, a double or sibi-conjugate point, and we have thus a theorem of the harmonic relation of two lines to a triangle and line, or else to a triangle and point, viz. *Theorem,* If on the sides of a *triangle* there be taken three points which either lie in a *line* or else are such that the lines joining them with the opposite angles of the triangle meet in a *point;* and if on each side of the triangle there be taken two points forming with the two angles on the same side an involution having the first-mentioned point on the same side for a double or sibi-conjugate point; then if three of the six points lie in a *line,* the other three of the six points will also lie in a line; the two lines are said to be harmonically related to the triangle and line, or (as the case may be) to the triangle and point.

The proof of the two porisms is by the preceding lemmas rendered very simple.

Demonstration of the homographic porism.

First, the particular case, where two of the sides pass through fixed points. Lemma I. gives the construction of the point of contact on the third side, and the figure shows that the point of contact and the point in which the third side is intersected by the line through the two given points are harmonics with respect to the points of intersection of the third side with the circumscribed conic. Hence, (Lemma II. Coroll. 2) the curve touched by the third side is a conic having double contact with the circumscribed conic, and the chord of contact is the line joining the two given points; and conversely if one of the sides touch a conic having double contact with the circumscribed conic and another of the sides passes through a fixed point on the chord of contact, then the third side will also pass through a fixed point on the chord of contact. The general case is deduced from the particular one precisely as before, viz. where two of the sides touch conics having double contact with the circumscribed conic, then considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the circumscribed conic in a variable point, and joining this variable point with the first

and second angles of the triangle, we have a new triangle, two of the sides of which (by the converse of the particular case) pass through fixed points: hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

Demonstration of the allographic porism.

Let ABC be the triangle, A', B', C' the points of contact on the three sides, then by Lemma I. the lines AA', BB', CC' meet in a point. Take a pair of lines passing through the points of intersection of the circumscribed conic with the two given conics enveloped by the sides CA, CB, and let one of these lines meet the sides of the triangle in the points F, G, H, and the other of them meet the sides of the triangle in the points F', G', H'. Then considering the following three conics, viz. the last-mentioned pair of lines, the circumscribed conic, and the conic enveloped by the side CA; these are conics passing through the same four points, and the side CA is a tangent to one of them: hence by Lemma II. Coroll. 1, G, G', C, A will be an involution having the point B' for a double or sibi-conjugate point, and similarly F, F', G, B are an involution having the point A' for a double or sibi-conjugate point. It follows from Lemma III. that H, H', A, B will be an involution having C' for a double or sibi-conjugate point. Hence by Lemma II. Coroll. 1 (the two given conics being the before-mentioned pair of lines and the circumscribed conic) the curve enveloped by the side AB will be a conic passing through the points of intersection of the pair of lines and the circumscribed conic, or, what is the same thing, the points of intersection of the circumscribed conic and the conics enveloped by the other two sides.

2, *Stone Buildings, Oct.* 2, 1856.

179.

ON CERTAIN FORMS OF THE EQUATION OF A CONIC.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 44—48.]

To find the general equation of a conic which passes through two given points and touches a given line.

Let the coordinates of the given points be (α, β, γ), $(\alpha', \beta', \gamma')$, and the equation of the given line be $\lambda x + \mu y + \nu z = 0$. Then writing

$$u = \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}, \quad v = s \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad w = \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix},$$

where a, b, c, s are arbitrary coefficients, the general equation of a conic passing through the two given points will be

$$uw - v^2 = 0.$$

We have identically

$$s \begin{vmatrix} \lambda x + \mu y + \nu z, & x, & y, & z \\ \lambda\alpha + \mu\beta + \nu\gamma, & \alpha, & \beta, & \gamma \\ \lambda\alpha' + \mu\beta' + \nu\gamma', & \alpha', & \beta', & \gamma' \\ \lambda a + \mu b + \nu c, & a, & b, & c \end{vmatrix} = 0;$$

and hence putting

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix},$$

$$A = \quad (\lambda\alpha' + \mu\beta' + \nu\gamma')\, s,$$
$$B = -(\lambda a + \mu b + \nu c)\, s,$$
$$C = -(\lambda\alpha + \mu\beta + \nu\gamma)\, s,$$

we have

$$(\lambda x + \mu y + \nu z)\, s\nabla + Au + Bv + Cw = 0,$$

and consequently the equation $\lambda x + \mu y + \nu z = 0$ is equivalent to

$$Au + Bv + Cw = 0,$$

and we have only to express that the line represented by this equation touches the conic $uw - v^2 = 0$.

Combining the two equations, we find $Au + Cw + B\sqrt{(uw)} = 0$, that is

$$(Au + Cw)^2 - B^2uw = 0,$$

an equation which must have equal roots; and the condition for this is obviously $4AC - B^2 = 0$. Or putting the condition under the form $-B + 2\sqrt{(AC)} = 0$ and substituting for A, B, C their values, the condition becomes $\{i = \sqrt{(-1)}$ as usual$\}$

$$\lambda a + \mu b + \nu c + 2is\sqrt{\{(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')\}} = 0.$$

We have consequently

$$s^2 = -\frac{(\lambda a + \mu b + \nu c)^2}{4(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')},$$

and the equation of the conic is

$$4(\lambda\alpha+\mu\beta+\nu\gamma)(\lambda\alpha'+\mu\beta'+\nu\gamma')\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}\begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix} + (\lambda a+\mu b+\nu c)^2\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}^2 = 0.$$

But the equation of the conic may be obtained in a different form as follows: we may first write $B^2v^2 = 4ACuw$, and then substituting for $-Bv$ the value

$$(\lambda x + \mu y + \nu z)\, s\nabla + Au + Cw,$$

the equation becomes

$$\{Au + Cw + (\lambda x + \mu y + \nu z)\, s\nabla\}^2 = 4ACuw,$$

or, extracting the root of each side and transposing,

$$\{\sqrt{(Au)} + \sqrt{(Cw)}\}^2 + (\lambda x + \mu y + \nu z)\, s\nabla = 0,$$

and thence

$$\sqrt{(Au)}+\sqrt{(Cw)}+i\sqrt{(s\nabla)}\sqrt{(\lambda x+\mu y+\nu z)}=0,$$

or substituting the values of A, C, ∇, u, w, and omitting the common factor $\sqrt{(s)}$ the equation becomes

$$\sqrt{(\lambda\alpha'+\mu\beta+\nu\gamma')}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}\right)}+i\sqrt{(\lambda\alpha+\mu\beta+\nu\gamma)}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}\right)}$$

$$+i\sqrt{(\lambda x+\mu y+\nu z)}\sqrt{\left(\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}\right)}=0,$$

a form symmetrically related to the three lines

$$\lambda x+\mu y+\nu z=0,\quad \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}=0,\quad \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}=0.$$

Let it be required to find the conic passing through the two points (α, β, γ), $(\alpha', \beta', \gamma')$, and touching the three lines

$$\lambda_1 x+\mu_1 y+\nu_1 z=0,\quad \lambda_2 x+\mu_2 y+\nu_2 z=0,\quad \lambda_3 x+\mu_3 y+\nu_3 z=0.$$

The constants a, b, c have to be determined in such manner that the equations obtained from the preceding, by writing successively $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ for (λ, μ, ν) may represent one and the same equation; the three equations so obtained will therefore subsist simultaneously, and we may from the equations in question eliminate a, b, c; the resulting equation

$$\begin{vmatrix} \sqrt{(\lambda_1 x+\mu_1 y+\nu_1 z)}, & \sqrt{(\lambda_2 x+\mu_2 y+\nu_2 z)}, & \sqrt{(\lambda_3 x+\mu_3 y+\nu_3 z)} \\ \sqrt{(\lambda_1\alpha+\mu_1\beta+\nu_1\gamma)}, & \sqrt{(\lambda_2\alpha+\mu_2\beta+\nu_2\gamma)}, & \sqrt{(\lambda_3\alpha+\mu_3\beta+\nu_3\gamma)} \\ \sqrt{(\lambda_1\alpha'+\mu_1\beta'+\nu_1\gamma')}, & \sqrt{(\lambda_2\alpha'+\mu_2\beta'+\nu_2\gamma')}, & \sqrt{(\lambda_3\alpha'+\mu_3\beta'+\nu_3\gamma')} \end{vmatrix}=0$$

is the equation of the conic in question; this is in fact evident from other considerations.

To find the condition in order that a conic passing through the points (α, β, γ), $(\alpha', \beta', \gamma')$ may touch the four lines

$$\lambda_1 x+\mu_1 y+\nu_1 z=0,\quad \lambda_2 x+\mu_2 y+\nu_2 z=0,\quad \lambda_3 x+\mu_3 y+\nu_3 z=0,\quad \lambda_4 x+\mu_4 y+\nu_4 z=0.$$

The relation first obtained between a, b, c, s gives four equations from which these quantities may be eliminated, the resulting equation

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1, & \sqrt{\{(\lambda_1\alpha+\mu_1\beta+\nu_1\gamma)\ (\lambda_1\alpha'+\mu_1\beta'+\nu_1\gamma')\}} \\ \lambda_2, & \mu_2, & \nu_2, & \sqrt{\{(\lambda_2\alpha+\mu_2\beta+\nu_2\gamma)\ (\lambda_2\alpha'+\mu_2\beta'+\nu_2\gamma')\}} \\ \lambda_3, & \mu_3, & \nu_3, & \sqrt{\{(\lambda_3\alpha+\mu_3\beta+\nu_3\gamma)\ (\lambda_3\alpha'+\mu_3\beta'+\nu_3\gamma')\}} \\ \lambda_4, & \mu_4, & \nu_4, & \sqrt{\{(\lambda_4\alpha+\mu_4\beta+\nu_4\gamma)\ (\lambda_4\alpha'+\mu_4\beta'+\nu_4\gamma')\}} \end{vmatrix} = 0$$

is the required relation.

The preceding investigations apply directly to the circle, which is a conic passing through two given points. Thus the equation of a circle touching the three lines

$$\begin{aligned} Ax + By + C &= 0, \\ A'x + B'y + C' &= 0, \\ A''x + B''y + C'' &= 0, \end{aligned}$$

is

$$\begin{vmatrix} \sqrt{(Ax+By+C)}, & \sqrt{(A'x+B'y+C')}, & \sqrt{(A''x+B''y+C'')} \\ \sqrt{(A\ \ +Bi)}, & \sqrt{(A'\ \ +B'i)}, & \sqrt{(A''\ \ +B''i)} \\ \sqrt{(A\ \ -Bi)}, & \sqrt{(A'\ \ -B'i)}, & \sqrt{(A''\ \ +B''i)} \end{vmatrix} = 0.$$

Hence also the equation of a circle touching the three lines

$$\begin{aligned} x\cos\alpha + y\sin\alpha - p &= 0, \\ x\cos\beta + y\sin\beta - q &= 0, \\ x\cos\gamma + y\sin\gamma - r &= 0, \end{aligned}$$

is

$$\sin\tfrac{1}{2}(\beta-\gamma)\sqrt{(x\cos\alpha+y\sin\alpha-p)} + \sin\tfrac{1}{2}(\gamma-\alpha)\sqrt{(x\cos\beta+y\sin\beta-q)}$$
$$+\sin\tfrac{1}{2}(\alpha-\beta)\sqrt{(x\cos\gamma+y\sin\gamma-r)} = 0.$$

To rationalise the equation, I remark that an equation $\sqrt{(A)}+\sqrt{(B)}+\sqrt{(C)}=0$ gives in general

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1})\ (A,\ B,\ C)^2 = 0,$$

and that

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1} \between 2p\sin^2\tfrac{1}{2}(\beta-\gamma),\quad 2q\sin^2\tfrac{1}{2}(\gamma-\alpha),\ 2r\sin^2\tfrac{1}{2}(\alpha-\beta)^2)$$

or as it may also be written

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1} \between p\{1-\cos(\beta-\gamma)\},\ q\{1-\cos(\gamma-a)\},\ r\{1-\cos(\alpha-\beta)\})^2,$$

is identically equal to

$$\begin{aligned} &\{p(\sin\beta - \sin\gamma) + q(\sin\ \gamma - \sin\alpha) + r(\sin\ \alpha - \sin\beta)\}^2 \\ &+ \{p(\cos\beta - \cos\gamma) + q(\cos\gamma - \cos\alpha) + r(\cos\alpha - \cos\beta)\}^2 \\ &- \{p\sin(\beta - \gamma) + q\sin(\gamma - \alpha) + r\sin(\alpha - \beta)\}^2. \end{aligned}$$

Hence if we replace p, q, r by

$$x\cos\alpha + y\sin\alpha - p, \quad x\cos\beta + y\sin\beta - q, \quad x\cos\gamma + y\sin\beta - r,$$

the last-mentioned expression equated to zero will give the equation of the circle, and we obtain

$$\begin{aligned} &\{\nabla x + p(\sin\ \beta - \sin\gamma) + q(\sin\ \gamma - \sin\alpha) + r(\sin\ \alpha - \sin\beta)\}^2 \\ &+ \{\nabla y - p(\cos\ \beta - \cos\gamma) - q(\cos\ \gamma - \cos\alpha) - r(\cos\ \alpha - \cos\beta)\}^2 \\ &- \{\quad p\ \sin(\beta - \quad \gamma) + q\ \sin(\gamma - \quad \alpha) + r\ \sin(\alpha - \quad \beta)\}^2 = 0, \end{aligned}$$

where

$$\nabla = \sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta),$$

and we have thus the equation of the circle in the usual form with the coordinates of the centre and the radius put in evidence.

The condition that there may be a circle touching the four lines

$$\begin{aligned} Ax + By + C &= 0, \\ A'x + B'y + C' &= 0, \\ A''x + B''y + C'' &= 0, \\ A'''x + B'''y + C''' &= 0, \end{aligned}$$

is by the general formula shown to be

$$\begin{vmatrix} A, & B, & C, & \sqrt{(A^2 + B^2)} \\ A', & B', & C', & \sqrt{(A'^2 + B'^2)} \\ A'', & B'', & C'', & \sqrt{(A''^2 + B''^2)} \\ A''', & B''', & C''', & \sqrt{(A'''^2 + B'''^2)} \end{vmatrix} = 0,$$

which is in fact obvious from other considerations.

180.

NOTE ON THE REDUCTION OF AN ELLIPTIC ORBIT TO A FIXED PLANE.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 49—54.]

THE principal object of the present Note is to obtain an expression for the quantity ϵ_0 which I call the modified mean longitude at epoch, viz. taking as the elements the longitude of the node, inclination and any four elements which determine the motion in the plane of the orbit, then the longitude measured in the fixed plane (or reduced longitude) will be a function of the form

$$nt + \epsilon_0 + \text{periodic terms},$$

where ϵ_0 is a determinate function of the elements, and it is proposed to find the expression of this function. But as the corresponding formulæ relating to the eccentricity and longitude of the pericentre are not in general given as part of the theory of elliptic motion, but occur only, so far as I am aware, in works on the lunar theory, I have thought it desirable to include these formulæ and take as the subject of this Note the reduction of an elliptic orbit to a fixed plane. Write

a_1, the semiaxis-major,

e_1, the eccentricity $(= \sin \kappa_1)$,

ϖ_1, the longitude of pericentre in orbit,

ϵ_1, the mean longitude in orbit at epoch,

θ, the longitude of node,

ϕ, the inclination $(= \tan^{-1} \gamma)$,

and moreover

n_1, the mean motion $\left\{= \sqrt{\left(\frac{\sigma}{a_1^3}\right)}\right\}$,

where by longitude in orbit is to be understood as usual a longitude measured in the fixed plane as far as the node and from the node in the plane of the orbit: the meaning of ϵ_1 is perhaps more clearly fixed by saying that $\epsilon_1 - \varpi_1$ denotes the mean anomaly at epoch.

The elements most nearly corresponding to the above, in the orbit reduced to the fixed plane, are

a_0, the modified semiaxis-major,

e_0, the modified eccentricity,

ϖ_0, the modified longitude of pericentre,

ϵ_0, the modified mean longitude at epoch,

θ, the longitude of node,

ϕ, the inclination $(= \tan^{-1}\gamma)$,

and moreover

n, the mean motion $\left\{\text{not equal to } \sqrt{\left(\frac{\sigma}{a_0^3}\right)}\right\}$,

where θ, ϕ, n are the same as in the actual orbit, but a_0, e_0, ϖ_0, ϵ_0 are defined as follows: viz. e_0, ϖ_0 are functions of e_1, ϖ_1, θ, ϕ given by the equations

$$\tan(\varpi_0 - \theta) = \sec\phi \, \tan(\varpi_1 - \theta),$$

$$e_0 = \frac{e_1 \cos(\varpi_1 - \theta)}{\cos(\varpi_0 - \theta)},$$

a_0 is determined by the condition

$$a_0(1 - e_0^2) = a_1(1 - e_1^2),$$

and ϵ_0 is determined so that the reduced longitude may be equal to

$$nt + \epsilon_0 + \text{periodic terms.}$$

It is easy to see that considering the orbit and the fixed plane as great circles of the sphere, and projecting the pericentre upon the fixed plane by an arc *perpendicular to the orbit*, then ϖ_0 denotes the longitude of such projection of the pericentre; and e_0 is equal to e_1 into the secant of the projecting arc. In fact we have a right-angled spherical triangle, of which the projecting arc in question is the perpendicular, and the hypothenuse and base are $\varpi_0 - \theta$ and $\varpi_1 - \theta$ respectively, and the base angle is the inclination ϕ. It is to be remarked that ϖ_0 is *not* the reduced longitude of the pericentre, an expression that would signify the longitude of the projection of the pericentre by an arc perpendicular to the fixed plane; this is the reason why I have throughout used the word *modified* instead of what would at first sight have appeared the natural one, viz. the word *reduced*. The modified semiaxis-major is obviously a semiaxis-major calculated from the latus rectum of the orbit by means of the modified eccentricity e_0.

The relations between e_0, ϖ_0, e_1, ϖ_1 may be written

$$\tan(\varpi_0 - \theta) = \sec\phi \tan(\varpi_1 - \theta),$$
$$e_0 = e_1 \sec\phi \sqrt{\{1 - \sin^2\phi \sin^2(\varpi_1 - \theta)\}},$$

or again

$$\tan(\varpi_1 - \theta) = \cos\phi \tan(\varpi_0 - \theta),$$
$$e_1 = e_0 \sqrt{\{1 - \sin^2\phi \sin^2(\varpi_0 - \theta)\}}.$$

Write now

r_1, the radius vector,
v_1, the longitude in orbit,
λ, the latitude $(= \tan^{-1} s)$,

and in like manner

r_0, the reduced radius vector,
v_0, the reduced longitude,
λ, the latitude $(= \tan^{-1} s)$.

Then $v_1 - \theta$ and $v_0 - \theta$ are the hypothenuse and base of a right-angled spherical triangle, the perpendicular being λ and the angle at the base being ϕ. We have

$$\tan\lambda = \tan\phi \sin(v_0 - \theta),$$
$$\sin\lambda = \sin\phi \sin(v_1 - \theta),$$
$$\tan(v_0 - \theta) = \cos\phi \tan(v_1 - \theta),$$
$$\cos(v_0 - \theta) = \sec\lambda \cos(v_1 - \theta).$$

We have for the radius vector

$$\frac{1}{r_1} = \frac{1}{a_1(1 - e_1^2)}\{1 + e_1 \cos(v_1 - \varpi_1)\},$$

and the reduced radius vector is thence found as follows: viz. we have $r_0 = r_1 \cos\lambda$, that is

$$\frac{1}{r_0} = \frac{1}{a_1(1 - e_1^2)}\{\sec\lambda + e_1 \sec\lambda \cos(v_1 - \varpi_1)\};$$

but $e_1 \sec\lambda \cos(v_1 - \varpi_1)$

$$= e_1 \sec\lambda \cos\{(v_1 - \theta) - (\varpi_1 - \theta)\},$$
$$= e_1 \sec\lambda \cos(v_1 - \theta)\cos(\varpi_1 - \theta) + e_1 \sec\lambda \sin(v_1 - \theta)\sin(\varpi_1 - \theta),$$
$$= e_1 \sec\lambda \cos(v_1 - \theta)\cos(\varpi_1 - \theta)\{1 + \tan(v_1 - \theta)\tan(\varpi_1 - \theta)\},$$
$$= e_0 \cos(v_0 - \theta)\cos(\varpi_0 - \theta)\{1 + \tan(v_0 - \theta)\tan(\varpi_0 - \theta)\},$$
$$= e_0 \cos(v_0 - \theta)\cos(\varpi_0 - \theta) + e_0 \sin(\varpi_0 - \theta)\sin(\varpi_0 - \theta),$$
$$= e_0 \cos(v_0 - \varpi_0),$$

and by the definition of a_0 we have $a_0(1-e_0^2)=a_1(1-e_1^2)$. Hence

$$\frac{1}{r_0}=\frac{1}{a_0(1-e_0^2)}\{\sec\lambda+e_0\cos(v_0-\varpi_0)\},$$

which, combined with the equation

$$\tan\lambda=\tan\phi\sin(v_0-\theta),$$

determines the position of the body in terms of the modified elements and of the reduced longitude v_0. Introducing into the two equations $s\,(=\tan\lambda)$ and $\gamma\,(=\tan\phi)$ in the place of λ and ϕ, they become

$$\frac{1}{r_0}=\frac{1}{a_0(1-e_0^2)}\{\sqrt{(1+s^2)}+e_0\cos(v_0-\varpi_0)\},$$

$$s=\gamma\sin(v_0-\theta),$$

which is the form in which the equations occur in the lunar theory.

Proceeding now to the formulæ which involve the time, it is to be remarked that the true anomaly and the quotient of the radius vector by the semiaxis-major are given functions of the eccentricity and the mean anomaly, and calling for a moment the last-mentioned quantities e, ξ, I represent the functions in question by

$$\text{elta}\,(e,\ \xi),\ \text{elqr}\,(e,\ \xi),$$

or more simply when the mean anomaly only is attended to by

$$\text{elta}\ \xi,\ \text{elqr}\ \xi.$$

I have found this notation very convenient as a means of dispensing with the introduction of the eccentric anomaly.

The reduced longitude is found in terms of the time by means of the equations

$$\tan(v_0-\theta)=\sec\phi\tan(v_1-\theta),$$
$$v_1-\varpi_1=\text{elta}\,(nt+\epsilon_1-\varpi_1),$$

the former equation gives, as is well known,

$$v_0-\theta=v_1-\theta-\tan^2\tfrac{1}{2}\phi\sin(2v_1-2\theta)+\tfrac{1}{2}\tan^4\tfrac{1}{2}\sin(4v_1-4\theta)-\&c.,$$

(where the successive coefficients are the reciprocals of the natural numbers) we have therefore

$$v_0=v_1-\tan^2\tfrac{1}{2}\phi\sin\{(2v_1-2\varpi_1)+(2\varpi_1-2\theta)\}+\&c.,$$

or, as it may be written,

$$\begin{aligned}v_0=v_1-\ &\tan^2\tfrac{1}{2}\phi\{\sin(2v_1-2\varpi_1)\cos(2\varpi_1-2\theta)+\cos(2v_1-2\varpi_1)\sin(2\varpi_1-2\theta)\}\\&+\tfrac{1}{2}\tan^4\tfrac{1}{2}\phi\{\sin(4v_1-4\varpi_1)\cos(4\varpi_1-4\theta)+\cos(4v_1-4\varpi_1)\sin(4\varpi_1-4\theta)\}\\&-\&c.,\end{aligned}$$

and for the present purpose it is only necessary to attend to the non-periodic part of the function on the right-hand side. Now

$$v_1 - \varpi_1 = \text{elta}\,(nt + \epsilon_1 - \varpi_1),$$

the non-periodic part of which is $nt + \epsilon_1 - \varpi_1$. And the non-periodic part of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\,\mu\,(v_1 - \varpi_1)$ is given by the equation (62) of Hansen's Memoir "Entwickelung des Products u. s. w." *Abhand. der K. Sächs. Gesellschaft zu Leipzig*, t. II. (1853). In fact, Hansen's

$$\beta \text{ is } = \frac{e_1}{1 + \sqrt{(1 - e_1^2)}}, = \tan \tfrac{1}{2}\kappa_1$$

and the formula gives for the non-periodic parts

$$\cos \mu\,(v_1 - \varpi_1) = (-)^\mu \tan^\mu \tfrac{1}{2}\kappa_1\,(1 + \mu \cos \kappa_1),$$
$$\sin \mu\,(v_1 - \varpi_1) = 0.$$

Hence, substituting these values and putting for the non-periodic part of v_0 the assumed value $nt + \epsilon_0$, we find

$$\begin{aligned}\epsilon_0 = \epsilon_1 - \quad & \tan^2 \tfrac{1}{2}\phi \tan^2 \tfrac{1}{2}\kappa_1 (1 + 2 \cos \kappa_1) \sin (2\varpi_1 - 2\theta) \\ & + \tfrac{1}{2} \tan^4 \tfrac{1}{2}\phi \tan^4 \tfrac{1}{2}\kappa_1 (1 + 4 \cos \kappa_1) \sin (4\varpi_1 - 2\theta) \\ & - \&c.\end{aligned}$$

The series on the right-hand side may be summed without difficulty, and we obtain

$$\begin{aligned}\epsilon_0 = \epsilon_1 - \tan^{-1} & \left\{\frac{\tan^2 \frac{1}{2}\phi \tan^2 \frac{1}{2}\kappa_1 \sin (2\varpi_1 - 2\theta)}{1 + \tan^2 \frac{1}{2}\phi \tan^2 \frac{1}{2}\kappa_1 \cos (2\varpi_1 - 2\theta)}\right\} \\ & - 2 \cos \kappa_1 \frac{\tan^2 \frac{1}{2}\phi \tan^2 \frac{1}{2}\kappa_1 \sin (2\varpi_1 - 2\theta)}{1 + 2 \tan^2 \frac{1}{2}\phi \tan^2 \frac{1}{2}\kappa_1 \cos (2\varpi_1 - 2\theta) + \tan^4 \frac{1}{2}\phi \tan^4 \frac{1}{2}\kappa_1},\end{aligned}$$

in which formula the values of $\tan \frac{1}{2}\phi$, $\tan \frac{1}{2}\kappa_1$ (in terms of γ, e_1) are

$$\frac{\gamma}{1 + \sqrt{(1 + \gamma^2)}}, \quad \frac{e_1}{1 + \sqrt{(1 - e_1^2)}},$$

and that of $\cos \kappa_1$ is $\sqrt{(1 - e_1^2)}$. We have thus the required expression for the modified mean longitude at epoch, and all the modified elements are now expressed in terms of the original elements.

The following investigation leads to a theorem which it is, I think, worth while to notice. We have

$$r_0^2 \frac{dv_0}{dt} = \sqrt{\{\sigma a_0 (1 - e_0^2)\}} \cos \phi,$$

and thence

$$dt = \frac{a_0^{\frac{3}{2}}(1-e_0^2)^{\frac{3}{2}}\,dv_0}{\sqrt{(\sigma)}\cos\phi\,\{\sec\lambda + e_0\cos(v_0-\varpi_0)\}^2}$$

$$= \frac{a_1^{\frac{3}{2}}(1-e_1^2)^{\frac{3}{2}}\,dv_0}{\sqrt{(\sigma)}\sqrt{(1+\gamma^2)}\,[\sqrt{\{1+\gamma^2\sin^2(v_0-\varpi_0)\}}+e_0\cos(v_0-\varpi_0)]^2},$$

or as it may be written

$$\frac{dv_0}{[\sqrt{\{1+\gamma^2\sin^2(v_0-\varpi_0)\}}+e_0\cos(v_0-\varpi_0)]^2} = \sqrt{\left(\frac{\sigma}{a_1^2}\right)}(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}\,dt,$$

$$= \qquad n\,(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}\,dt.$$

But it is easy to see that if the mean longitude $nt+\epsilon_0$ is expanded in terms of v_0, the relation between these quantities must be of the form $nt+\epsilon_0 = v_0 +$ periodic terms. It follows that in the preceding equation the non-periodic part of the function which multiplies dv_0 (the expansion being in multiple cosines of v_0) must be equal to $(1-e_1^2)^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}$. Hence, putting for e_1 its value, we find that the non-periodic part of

$$\frac{1}{[\sqrt{\{1+\gamma^2\sin^2(v_0-\theta)\}}+e_0\cos(v_0-\varpi_0)]^2},$$

expanded in multiple cosines of v_0 is

$$\left[1-e_0^2\left\{1-\frac{\gamma^2}{1+\gamma^2}\sin^2(\varpi_0-\theta)\right\}\right]^{-\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}},$$

a theorem which might, it is probable, be verified without much difficulty.

2, *Stone Buildings, October*, 1856.

181.

ON SIR W. R. HAMILTON'S METHOD FOR THE PROBLEM OF THREE OR MORE BODIES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 66—73.]

THE problem of three or more bodies is considered by Sir W. R. Hamilton in his two well-known memoirs on a general method in Dynamics, *Phil. Trans.* 1834 and 1835, and the differential equations for the relative motion, with respect to the central body, of all the other bodies are obtained in a form containing a single disturbing function only. Several methods of integration are given or indicated, among others, one which is in fact the method of the variation of the elements as applied to the particular form of the equations of motion. But the investigation shows (and Sir W. R. Hamilton notices this as a defect in his theory, as compared with the ordinary theory of the variation of the elements), that in the method in question, the elements are not osculating elements, i.e. that the positions only, and not the velocities of the bodies, can be calculated as if the elements remained constant during an element of time. The peculiar advantage of the method is of course the having a single disturbing function only, and this seems so important, that if I may venture to express an opinion, I cannot but think that the method will ultimately be employed for the purposes of Physical Astronomy. But, however this may be, it has appeared to me that it may be useful to present the method in a separate and distinct form, disengaged from the general theory as an illustration of which it was given by the author; and this is what I propose now to do.

Consider a central body M, and two other bodies M_1, M_2, and let the coordinates of M referred to a fixed origin be x, y, z, and the coordinates of M_1, M_2 referred to the body M as origin be x_1, y_1, z_1 and x_2, y_2, z_2 respectively. Then the coordinates of M_1, M_2 referred to the fixed origin, are $x+x_1$, $y+y_1$, $z+z_1$ and $x+x_2$, $y+y_2$, $z+z_2$

respectively, and if as usual T denotes the Vis-viva or half sum of each mass into the square of its velocity, and U denote the force function, then we have

$$\begin{aligned} T = \quad & \tfrac{1}{2}M\,(x'^2 + y'^2 + z'^2), \\ & + \tfrac{1}{2}M_1\,\{(x' + x_1')^2 + (y' + y_1')^2 + (z' + z_1')^2\}, \\ & + \tfrac{1}{2}M_2\,\{(x' + x_2')^2 + (y' + y_2')^2 + (z' + z_2')^2\}, \\ U = \quad & \frac{MM_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \\ & + \frac{MM_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}} \\ & + \frac{M_1M_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}, \end{aligned}$$

and the equations of motion are as usual

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} = \frac{dU}{dx},$$

&c.

If we assume that the centre of gravity of the bodies is at rest, then we have

$$Mx' + M_1(x' + x_1') + M_2(x' + x_2') = 0, \text{ \&c.},$$

and consequently

$$x' = -\frac{M_1x_1' + M_2x_2'}{M + M_1 + M_2},\ y' = -\frac{M_1y_1' + M_2y_2'}{M + M_1 + M_2},\ z' = -\frac{M_1z_1' + M_2z_2'}{M + M_1 + M_2}.$$

Now the value of T is

$$\begin{aligned} T = \quad & \tfrac{1}{2}(M + M_1 + M_2)(x'^2 + y'^2 + z'^2) \\ & + x'(M_1x_1' + M_2x_2') + y'(M_1y_1' + M_2y_2') + z'(M_1z_1' + M_2z_2') \\ & + \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2), \end{aligned}$$

or, putting for x', y', z' their values,

$$\begin{aligned} T = \quad & \tfrac{1}{2}M_1(x_1'^2 + y_1'^2 + z_1'^2) \\ & + \tfrac{1}{2}M_2(x_2'^2 + y_2'^2 + z_2'^2) \\ & - \tfrac{1}{2}\frac{1}{M + M_1 + M_2}\{(M_1x_1' + M_2x_2')^2 + (M_1y_1' + M_2y_2')^2 + (M_1z_1' + M_2z_2')^2\}, \end{aligned}$$

and with this new value of T the equations of motion still are

$$\frac{d}{dt}\frac{dT}{dx_1'} - \frac{dT}{dx_1} = \frac{dU}{dx_1}, \text{ \&c.}$$

Suppose now that the differential coefficients of T, with respect to x_1', y_1', z_1'; x_2', y_2', z_2', are respectively P_1, Q_1, R_1; P_2, Q_2, R_2, i.e. write

$$\frac{dT}{dx_1'} = P_1, \text{ \&c.},$$

and imagine T expressed as a function of P_1, Q_1, R_1; P_2, Q_2, R_2, and when this is done put $H = T - U$ (so that H stands for a function of P_1, Q_1, R_1; P_2, Q_2, R_2; x_1, y_1, z_1; x_2, y_2, z_2), then the equations of motion in Sir W. R. Hamilton's form are

$$\frac{dx_1}{dt} = \frac{dH}{dP_1}, \quad \frac{dP_1}{dt} = -\frac{dH}{dx_1}, \quad \text{\&c.}$$

Now from the last given value of T

$$P_1 = M_1 x_1' - \frac{M_1}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

$$P_2 = M_2 x_2' - \frac{M_2}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

and thence

$$P_1 + P_2 = \frac{M}{M + M_1 + M_2}(M_1 x_1' + M_2 x_2'),$$

and consequently

$$M_1 x_1' = P_1 + \frac{M_1}{M}(P_1 + P_2),$$

$$M_2 x_2' = P_2 + \frac{M_2}{M}(P_1 + P_2),$$

and we have

$$\begin{aligned} T = {} & \frac{1}{2M_1}\left[\left\{P_1 + \frac{M_1}{M}(P_1 + P_2)\right\}^2 + \left\{Q_1 + \frac{M_1}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_1 + \frac{M_1}{M}(R_1 + R_2)\right\}^2\right] \\ & + \frac{1}{2M_2}\left[\left\{P_2 + \frac{M_2}{M}(P_1 + P_2)\right\}^2 + \left\{Q_2 + \frac{M_2}{M}(Q_1 + Q_2)\right\}^2 + \left\{R_2 + \frac{M_2}{M}(R_1 + R_2)\right\}^2\right] \\ & - \frac{1}{2M^2}(M + M_1 + M_2)\left[(P_1 + P_2)^2 + (Q_1 + Q_2)^2 + (R_1 + R_2)^2\right], \end{aligned}$$

or, reducing,

$$\begin{aligned} T = {} & \left(\frac{1}{2M_1} + \frac{1}{2M}\right)(P_1^2 + Q_1^2 + R_1^2) \\ & + \left(\frac{1}{2M_2} + \frac{1}{2M}\right)(P_2^2 + Q_2^2 + R_2^2) \\ & + \frac{1}{M}(P_1 P_2 + Q_1 Q_2 + R_1 R_2), \end{aligned}$$

and consequently

$$\begin{aligned} H = & \left(\frac{1}{2M_1}+\frac{1}{2M}\right)(P_1^2+Q_1^2+R_1^2) \\ & +\left(\frac{1}{2M_2}+\frac{1}{2M}\right)(P_2^2+Q_2^2+R_2^2) \\ & +\frac{1}{M}(P_1P_2+Q_1Q_2+R_1R_2) \\ & -\frac{MM_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}} \\ & -\frac{MM_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}} \\ & -\frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}}}, \end{aligned}$$

and H having this value, the equations of motion are as before mentioned

$$\frac{dx_1}{dt}=\frac{dH}{dP_1},\quad \frac{dP_1}{dt}=-\frac{dH}{dx_1},\ \&c.$$

Instead of H write $H+\Upsilon$ where

$$\begin{aligned} H = & \left(\frac{1}{2M_1}+\frac{1}{2M}\right)(P_1^2+Q_1^2+R_1^2) \\ & +\left(\frac{1}{2M_2}+\frac{1}{2M}\right)(P_2^2+Q_2^2+R_2^2) \\ & -\frac{MM_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}} \\ & -\frac{MM_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \end{aligned}$$

and

$$\begin{aligned} \Upsilon = & \frac{1}{M}(P_1P_2+Q_1Q_2+R_1R_2) \\ & -\frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}}}, \end{aligned}$$

and the function Υ is to be treated as a disturbing function. The equations of motion for the body M_1 become

$$\frac{dx_1}{dt}=\frac{M+M_1}{MM_1}P_1+\frac{d\Upsilon}{dP_1},\quad \frac{dP_1}{dt}=-\frac{MM_1x_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dx_1},$$

$$\frac{dy_1}{dt}=\frac{M+M_1}{MM_1}Q_1+\frac{d\Upsilon}{dQ_1},\quad \frac{dQ_1}{dt}=-\frac{MM_1y_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dy_1},$$

$$\frac{dz_1}{dt}=\frac{M+M_1}{MM_1}R_1+\frac{d\Upsilon}{dR_1},\quad \frac{dR_1}{dt}=-\frac{MM_1z_1}{(x_1^2+y_1^2+z_1^2)^{\frac{3}{2}}}-\frac{d\Upsilon}{dz_1},$$

and there is of course a precisely similar system of equations of motion for the body M_2.

If we neglect Υ, the left-hand equations show that P_1, Q_1, R_1 denote the velocities or differential coefficients $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$ multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and substituting these values in the right-hand equations, we obtain the ordinary equations for the elliptic motion of the body M_1; and similarly for the body M_2. We may, if we please, complete the solution by the method of the variation of the arbitrary constants. Suppose for this purpose that a_1, b_1, c_1, e_1, f_1, g_1 are the elements for the elliptic motion of the body M_1, then treating these elements as variable we must have

$$\frac{dx_1}{da_1}\frac{da_1}{dt}+\frac{dx_1}{db_1}\frac{db_1}{dt}\dots+\frac{dx_1}{dg_1}\frac{dg_1}{dt}=\frac{d\Upsilon}{dP_1}, \text{ \&c.},$$

$$\frac{dP_1}{da_1}\frac{da_1}{dt}+\frac{dP_1}{db_1}\frac{db_1}{dt}\dots+\frac{dP_1}{dg_1}\frac{dg_1}{dt}=-\frac{d\Upsilon}{dx_1}, \text{ \&c.},$$

and it appears from these equations that as already noticed the disturbed values of the velocities are not (as they are in the ordinary theory) identical with the undisturbed values.

The disturbing function Υ may be considered as a function of the elements of the two orbits and of the time, and it is easy to obtain, as in the ordinary theory, the values of the differential coefficients $\frac{da_1}{dt}$, &c. in the form

$$\frac{da_1}{dt}=(a_1,\ b_1)\frac{d\Upsilon}{db_1}+(a_1,\ c_1)\frac{d\Upsilon}{dc_1}\dots+(a_1,\ g_1)\frac{d\Upsilon}{dg_1},$$

where

$$(a_1,\ b_1)=\frac{\delta(a_1,\ b_1)}{\delta(x_1,\ P_1)}+\frac{\delta(a_1,\ b_1)}{\delta(y_1,\ Q_1)}+\frac{\delta(a_1,\ b_1)}{\delta(z_1,\ R_1)},$$

if for shortness

$$\frac{\delta(a_1,\ b_1)}{\delta(x_1,\ P_1)}=\frac{da_1}{dx_1}\frac{db_1}{dP_1}-\frac{da_1}{dP_1}\frac{db_1}{dx_1}.$$

It will be remembered that in the ordinary theory, if Ω denote Lagrange's disturbing function ($\Omega=-R$ if R is the disturbing function of the Mécanique Céleste) the corresponding formulæ are

$$\frac{da}{dt}=(a,\ b)\frac{d\Omega}{db}+(a,\ c)\frac{d\Omega}{dc}\dots+(a,\ g)\frac{d\Omega}{dg},$$

where

$$(a,\ b)=\frac{\delta(a,\ b)}{\delta(x',\ x)}+\frac{\delta(a,\ b)}{\delta(y',\ y)}+\frac{\delta(a,\ b)}{\delta(z',\ z)},$$

if for shortness

$$\frac{\delta(a,\ b)}{\delta(x',\ x)}=\frac{da}{dx'}\frac{db}{dx}-\frac{da}{dx}\frac{db}{dx'},$$

or, what is the same thing, where

$$(a, b) = -\frac{\delta(a, b)}{\delta(x, x')} - \frac{\delta(a, b)}{\delta(y, y')} - \frac{\delta(a, c)}{\delta(z, z')},$$

and

$$\frac{\delta(a, b)}{\delta(x, x')} = \frac{da}{dx}\frac{db}{dx'} - \frac{da}{dx'}\frac{db}{dx}.$$

Now the values of the coefficients (a_1, b_1), &c. depend merely on the form of the expressions for a_1, b_1, &c. in terms of P_1, Q_1, R_1, x_1, y_1, z_1 and t; hence comparing the two systems of formulæ and observing P_1, Q_1, R_1 (which in the formulæ for the present theory correspond with x_1', y_1', z_1' in the other system of formulæ) are respectively equal to x_1', y_1', z_1', each of them multiplied by the constant factor $\frac{MM_1}{M+M_1}$, it is easy to see that the formulæ for the variations of any given system of elements in the present theory are at once deduced from the formulæ for the variations of the same system of elements in the ordinary theory by writing $-\Upsilon$ in the place of Ω and multiplying the values of the variations by the constant factor $\frac{MM_1}{M+M_1}$.

Take then as elements Jacobi's canonical system [1], viz. if we put

a_1, the semiaxis major,

e_1, the eccentricity,

ϖ_1, the longitude in orbit of pericentre,

ϵ_1, the mean longitude in orbit at epoch,

θ_1, the longitude of node,

ϕ_1, the inclination,

and

n_1, the mean motion $\left\{=\sqrt{\left(\frac{M+M_1}{a_1^3}\right)}\right\}$,

then the canonical elements are

$$\begin{aligned}
\mathfrak{A}_1 &= -\tfrac{1}{2}n_1^2 a_1^2, \\
\mathfrak{B}_1 &= n_1 a_1 \sqrt{(1-e_1^2)}, \\
\mathfrak{C}_1 &= n_1 a_1 \sqrt{(1-e_1^2)} \cos\phi_1, \\
\mathfrak{F}_1 &= \frac{1}{n_1}(\epsilon_1 - \varpi_1), \\
\mathfrak{G}_1 &= \varpi_1 - \theta_1, \\
\mathfrak{H}_1 &= \theta_1,
\end{aligned}$$

[1] I have for uniformity adopted Jacobi's canonical system, see his paper "Neues Theorem der analytischen Mechanik," *Crelle*, t. xxx. pp. 117—120 (1846); but it is proper to remark that Sir W. R. Hamilton, in his Memoirs above referred to, employs a slightly different but equally elegant system of canonical elements, and that the discovery of such a system belongs to Sir W. R. Hamilton, and is part of his general theory.

(the signs of the two elements $\mathfrak{A}_1$, $\mathfrak{F}_1$ have been changed, but this makes no difference in the formulæ) then the equations for the variations of the elements are

$$\frac{d\mathfrak{A}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{F}_1},$$

$$\frac{d\mathfrak{B}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{G}_1},$$

$$\frac{d\mathfrak{C}_1}{dt} = -\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{H}_1},$$

$$\frac{d\mathfrak{F}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{A}_1},$$

$$\frac{d\mathfrak{G}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{B}_1},$$

$$\frac{d\mathfrak{H}_1}{dt} = +\frac{M+M_1}{MM_1}\frac{d\Upsilon}{d\mathfrak{C}_1},$$

and it is easy thence to deduce the formulæ for the variations of any system of elements which it may be thought proper to make use of, for instance the system a_1, e_1, ϖ_1, ϵ_1, θ_1, ϕ_1.

It will be recollected that in the preceding system of formulæ the value of the disturbing function Υ is

$$\Upsilon = \frac{1}{M}(P_1P_2 + Q_1Q_2 + R_1R_2)$$
$$- \frac{M_1M_2}{\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\}}},$$

and that as a first approximation P_1, Q_1, R_1 are respectively equal to the velocities x_1', y_1', z_1', each multiplied by the constant factor $\frac{MM_1}{M+M_1}$, and P_2, Q_2, R_2 are respectively equal to the velocities x_2', y_2', z_2', each multiplied by $\frac{MM_2}{M+M_2}$.

2, *Stone Buildings*, 18*th Oct.*, 1856.

182.

ON LAGRANGE'S SOLUTION OF THE PROBLEM OF TWO FIXED CENTRES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 76—83.]

THE following variation of Lagrange's Solution of the Problem of Two Fixed Centres[1], is, I think, interesting, as showing more distinctly the connection between the differential equations and the integrals. The problem referred to is as follows: viz. to determine the motion of a particle acted upon by forces tending to two fixed centres, such that r, q being the distances of the particle from the two centres respectively, and α, β, γ being constants, the forces are $\frac{\alpha}{r^2}+2\gamma r$ and $\frac{\beta}{q^2}+2\gamma q$.

Take the first centre as origin and the line joining the two centres as axis of x; and let h be the distance between the two centres, then writing for symmetry

$$x = x_1 = x_2 + h,$$

(so that x_1 is the coordinate corresponding to the first centre as origin, and x_2 the coordinate corresponding to the second centre as origin) the distances are given by the equations

$$r^2 = x_1^2 + y^2 + z^2, \qquad q^2 = x_2^2 + y^2 + z^2,$$

and the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{\alpha x_1}{r^3} - \frac{\beta x_2}{q^3} - 2\gamma(x_1 + x_2),$$

$$\frac{d^2y}{dt^2} = -\frac{\alpha y}{r^3} - \frac{\beta y}{q^3} - 4\gamma y,$$

$$\frac{d^2z}{dt^2} = -\frac{\alpha z}{r^3} - \frac{\gamma z}{q^3} - 4\gamma z,$$

[1] Lagrange's Solution was first published in the *Anciens Mémoires de Turin*, t. IV., [1766—69], and is reproduced in the *Mécanique Analytique*.

and we obtain at once the integral of Vis-viva, viz. multiplying the three equations by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, adding and integrating $\left(\text{observing that } \frac{dx}{dt} = \frac{dx_1}{dt} = \frac{dx_2}{dt}\right)$, we have

$$\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \frac{\alpha}{r} + \frac{\beta}{q} - \gamma\,(r^2 + q^2) + 2H, \qquad (1,\ a)$$

and with equal facility, the equation of areas round the line joining the two centres, viz. multiplying the second and third equations by $-z$, y, adding and integrating, we have

$$y\frac{dz}{dt} - z\frac{dy}{dt} = B. \qquad (2,\ a)$$

So far Lagrange: to obtain a third integral I form the equation

$$\left[-2\,(y^2+z^2)\frac{dx}{dt} + (x_1+x_2)\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)\right] \times \left\{\frac{d^2x}{dt^2} + \frac{\alpha x_1}{r^3} + \frac{\beta x_2}{q^3} + 2\gamma\,(x_1+x_2)\right\}$$

$$+\left[\quad (x_1+x_2)\,y\frac{dx}{dt} - \qquad 2x_1x_2\frac{dy}{dt}\qquad\right] \times \left\{\frac{d^2y}{dt^2} + \frac{\alpha y}{r^3} + \frac{\beta y}{q^3} + 4\gamma y\qquad\right\}$$

$$+\left[\quad (x_1+x_2)\,z\frac{dx}{dt} - \qquad\qquad 2x_1x_2\frac{dz}{dt}\right] \times \left\{\frac{d^2z}{dt^2} + \frac{\alpha z}{r^3} + \frac{\beta z}{q^3} + 4\gamma z\qquad\right\} = 0.$$

The terms independent of the forces are

$$-2\,(y^2+z^2)\frac{dx}{dt}\frac{d^2x}{dt^2} + (x_1+x_2)\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)\frac{d^2x}{dt^2}$$

$$+(x_1+x_2)\frac{dx}{dt}\left(y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right) - 2x_1x_2\left(\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right),$$

which are equal to

$$\frac{d}{dt}\left[-(y^2+z^2)\left(\frac{dx}{dt}\right)^2 + (x_1+x_2)\frac{dx}{dt}\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right) - x_1x_2\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}\right],$$

and the terms depending on the forces are readily reduced to the form

$$\frac{d}{dt}\left\{-\frac{h\alpha x_1}{r} + \frac{h\beta x_2}{q} + h^2\gamma\,(y^2+z^2)\right\},$$

in fact, considering first the terms multiplied by α, these are

$$\frac{x_1}{r^3}\left\{\quad -2\ (y^2+z^2)\frac{dx}{dt} + (x_1+x_2)\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)\right\}$$

$$+\frac{1}{r^3}\left\{(x_1+x_2)\,(y^2+z^2)\frac{dx}{dt} - \ 2x_1x_2\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)\right\},$$

which is equal to

$$\frac{1}{r^3}\left\{(x_2-x_1)(y^2+z^2)\frac{dx}{dt}+x_1(x_1-x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{1}{r^3}\left\{-h(y^2+z^2)\frac{dx}{dt}+hx_1\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-(x_1^2+y^2+z^2)\frac{dx_1}{dt}+x_1\left(x\frac{dx_1}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)\right\}$$

$$=\frac{h}{r^3}\left\{-r^2\frac{dx_1}{dt}+rx_1\frac{dr}{dt}\right\}$$

$$=-h\frac{d}{dt}\frac{x_1}{r},$$

and similarly the term multiplied by β is

$$=\quad h\frac{d}{dt}\frac{x_2}{q},$$

lastly, the term multiplied by γ is

$$-4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}+2(x_1+x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$+4(x_1+x_2)(y^2+z^2)\frac{dx}{dt}-\quad 8x_1x_2\quad\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad 2(x_1-x_2)^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad 2h^2\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\quad \frac{d}{dt}h^2(y^2+z^2).$$

The preceding combination of the differential equations gives therefore an equation integrable per se, and effecting the integration we have

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\frac{dx}{dt}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)-x_1x_2\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^3\gamma(y^2+z^2)=K, \qquad (3,\ a)$$

which is the third integral equation. It may be convenient to mention here (what appears by the comparison of the formulæ obtained in the sequel with the corresponding formulæ of Lagrange) that the value of Lagrange's constant of integration C is

$$C=K-2Hh^2-B^2+\tfrac{1}{4}\gamma h^4.$$

Making use of the ordinary transformation

$$(y^2+z^2)\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2+\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)^2,$$

the integral equations may be written under the forms

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{1}{2(y^2+z^2)}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2=\frac{\alpha}{r}+\frac{\beta}{q}-\gamma(r^2+q^2)+2H-\frac{B^2}{2(y^2+z^2)}, \qquad (1,\ b)$$

$$y\frac{dz}{dt}-z\frac{dy}{dt}=B, \qquad (2,\ b)$$

$$-(y^2+z^2)\left(\frac{dx}{dt}\right)^2+(x_1+x_2)\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)\frac{dx}{dt}-\frac{x_1x_2}{y^2+z^2}\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2$$

$$-\frac{h\alpha x_1}{r}+\frac{h\beta x_2}{q}+h^2\gamma(y^2+z^2)=K+\frac{B^2x_1x_2}{y^2+z^2}, \qquad (3,\ b)$$

and observing that y^2+z^2, x_1, x_2 are in fact functions of r, q, it is clear that the determination of r, q in terms of t depends upon the first and third equations alone. Moreover the form of the equations shows that we can at once eliminate dt and thus obtain a differential equation between r, q alone. It would be difficult to discover *à priori*, before actually obtaining the differential equation in question, that it would be possible to effect the separation of the variables, but we know that this can be done by taking instead of r, q the new variables $u=r+q$, $s=r-q$. In order to complete the solution the first step is to introduce the variables r, q into the first and third equations: for this purpose we have

$$x_1=\frac{h^2+r^2-q^2}{2h}, \quad -x_2=\frac{h^2-r^2+q^2}{2h}, \quad x_1+x_2=\frac{r^2-q^2}{h},$$

$$y^2+z^2=\frac{\nabla}{4h^2},$$

if for shortness

$$\nabla=2r^2q^2+2h^2r^2+2h^2q^2-h^4-r^4-q^4,$$

and consequently

$$\frac{dx}{dt}=\frac{1}{h}\left(r\frac{dr}{dt}-q\frac{dq}{dt}\right),$$

$$y\frac{dy}{dt}+z\frac{dz}{dt}=\frac{1}{2h^2}\left\{(h^2-r^2+q^2)\,r\frac{dr}{dt}+(h^2+r^2-q^2)\,q\frac{dq}{dt}\right\}.$$

Substituting these values in the two equations, we find for the first equation

$$\nabla(rdr-qdq)^2+\{(h^2-r^2+q^2)\,rdr+(h^2+r^2-q^2)\,qdq\}^2$$

$$=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\,\nabla+2H\nabla-2h^2B^2\right\}dt^2, \qquad (1,\ c)$$

and for the second equation

$$\begin{aligned}&-\nabla^2(rdr-qdq)^2\\&+2\nabla(rdr-qdq)(r^2-q^2)\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}\\&+\{h^4-(r^2-q^2)^2\}\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}^2\\=h^4\Big[&\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2+4K\nabla-4B^2\{h^4-(r^2-q^2)^2\}\Big]dt^2. \quad (2,\ c)\end{aligned}$$

The first equation is easily reduced to

$$4h^2\{r^2q^2(dr^2+dq^2)+(h^2-r^2-q^2)rq\,dr\,dq\}=2h^2\left\{\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma(r^2+q^2)\nabla+2H\nabla-2h^2B^2\right\}dt^2,$$

the second equation gives

$$\begin{aligned}&h^4\{(h^2-r^2+q^2)rdr+(h^2+r^2-q^2)qdq\}^2-h^4\{(h^2-r^2-3q^2)rdr-(h^2-3r^2+q^2)qdq\}^2\\&=h^4\Big[\frac{2\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{2\beta\nabla}{q}(h^2-r^2+q^2)-\gamma\nabla^2+4K\nabla-4B^2\{h^4-(r^2-q^2)^2\}\Big]dt^2,\end{aligned}$$

and the function on the left-hand side is

$$8h^4(h^2-r^2-q^2)q^2r^2(dr^2+dq^2)+4h^4\{(h^2-r^2-q^2)^2+4q^2r^2\}rq\,dr\,dq.$$

Hence putting for a moment

$$M=\frac{\alpha\nabla}{r}+\frac{\beta\nabla}{q}-\gamma\nabla(r^2+q^2)+2H\nabla-2h^2B^2,$$

$$N=\frac{\alpha\nabla}{r}(h^2+r^2-q^2)+\frac{\beta\nabla}{q}(h^2-r^2+q^2)-\tfrac{1}{2}\gamma\nabla^2+2K\nabla-2B^2\{h^4-(r^2-q^2)^2\},$$

we have

$$2r^2q^2(dr^2+dq^2)+\qquad(h^2-r^2-q^2)2rq\,dr\,dq=Mdt^2,$$

$$2(h^2-r^2-q^2)2r^2q^2(dr^2+dq^2)+\{(h^2-r^2-q^2)^2+4q^2r^2\}2rq\,dr\,dq=Ndt^2,$$

and thence recollecting that

$$-(h^2-r^2-q^2)^2+4q^2r^2=\nabla,$$

we find

$$\nabla\,.\,2rq\,dr\,dq=\{(h^2-r^2-q^2)2M-N\}dt^2,$$

$$\nabla\,.\,2r^2q^2(dr^2+dq^2)=[-\{(h^2-r^2-q^2)^2+4q^2r^2\}M+(h^2-r^2-q^2)N]dt^2,$$

and substituting for M, N their values, the functions on the right-hand side contain ∇ as a factor, and dividing by ∇, we obtain

$$\begin{aligned}2rq\,dr\,dq=\Big[&\frac{\alpha}{r}(3r^2+q^2-h^2)+\frac{\beta}{q}(r^2+3q^2-h^2)\\&-\tfrac{1}{2}\gamma(3r^4+3q^4+10q^2r^2-2h^2r^2-2h^2q^2-h^4)\\&+4H(q^2+r^2-h^2)+2K-2B^2\Big]dt^2, \qquad (1,\ d)\end{aligned}$$

$$\begin{aligned}2r^2q^2(dr^2+dq^2) = \; & [2\alpha r(r^2+3q^2-h^2)+2\beta q(3r^2+q^2-h^2) \\ & -\tfrac{1}{2}\gamma\{r^6+q^6+15q^2r^2(r^2+q^2)-h^2(r^4+q^4+6r^2q^2)-h^4(r^2+q^2)+h^6\} \\ & +2H\{r^4+q^4+6r^2q^2-2h^2(r^2+q^2)+h^4\} \\ & +2K(r^2+q^2-h^2)-2(r^2+q^2)B^2]\,dt^2, \qquad (2,\ d)\end{aligned}$$

and by comparing the first of these formulæ with the corresponding formulæ of Lagrange, we find, as already observed, that the relation between the constant K and Lagrange's constant C is $K = C + 2Hh^2 + B^2 - \frac{1}{4}\gamma h^4$. And substituting this value of K, the two equations become identical with those of Lagrange[1].

The equation $y\dfrac{dz}{dt} - z\dfrac{dy}{dt} = B$, (putting $y = \sqrt{(y^2+z^2)}\cos\phi$, $z = \sqrt{(y^2+z^2)}\sin\phi$) gives at once $(y^2+z^2)\,d\phi = B\,dt$, and substituting for y^2+z^2 its value $= \dfrac{\nabla}{4h^2}$, we find

$$d\phi = \frac{4h^2B}{4q^2r^2-(h^2-r^2-q^2)^2}\,dt, \qquad (3,\ d)$$

which is the third of Lagrange's equations.

To complete the solution, the combination of the first and second equations gives

$$\begin{aligned}r^2q^2(dr \pm dq)^2 = \; & [\alpha\ \{(r\pm q)^3-h^2(r\pm q)\} \pm \beta\{(r\pm q)^3-h^2(r\pm q)\} \\ & -\tfrac{1}{4}\gamma\ \{(r\pm q)^6-h^2(r\pm q)^4-h^4(r\pm q)^2+h^6\} \\ & +H\ \{(r\pm q)^4-2h^2(r\pm q)^2+h^4\} \\ & +2K\{(r\pm q)^2-h^2\}-2B^2(r\pm q)^2]\,dt^2,\end{aligned}$$

and thence putting $r+q=s$, $r-q=u$ and writing for shortness

$$\begin{aligned}S = \; & (\alpha+\beta)(s^3-s^2h) \\ & -\tfrac{1}{4}\gamma(s^6-h^2s^4-h^4s^2+h^6) \\ & +H(s^4-2h^2s^2+h^4) \\ & +2K(s^2-h^2)-2B^2s^2,\end{aligned}$$

and

$$\begin{aligned}U = \; & (\alpha-\beta)(u^3-u^2h) \\ & -\tfrac{1}{4}\gamma(u^6-h^2u^4-h^4s^2+h^6) \\ & +H(u^4-2h^2u^2-h^4) \\ & +2K(u^2-h^2)-2B^2u^2,\end{aligned}$$

[1] The formulæ referred to are the formulæ (*b*), (*c*), *Méc. Anal.* t. II. page 112 of the second edition and page 97 of the third edition, but there is an inaccuracy in the formulæ (*c*), B^2 ought to be changed into B^2h^2; the error is continued in the subsequent formulæ and moreover the constant term $-Ch^2$ is omitted on the right-hand side of the formulæ (*e*) and in the subsequent formulæ, i.e. in the functions of s, u, the term $-B^2$ should be $-B^2h^2-Ch^2$.

we have

$$\tfrac{1}{16}(s^2-u^2)^2\,ds^2 = S dt^2, \qquad (1,\ e)$$

$$\tfrac{1}{16}(s^2-u^2)^2\,du^2 = U dt^2, \qquad (2,\ e)$$

$$d\phi = -\frac{4h^2 B}{(s^2-h^2)(u^2-h^2)}\,dt, \qquad (3,\ e)$$

and thence finally

$$\frac{ds}{\sqrt{(S)}} = \frac{du}{\sqrt{(U)}}, \qquad (1,\ f)$$

$$dt = \tfrac{1}{4}\left\{\frac{s^2 ds}{\sqrt{(S)}} - \frac{u^2 du}{\sqrt{(U)}}\right\}, \qquad (2,\ f)$$

$$d\phi = \frac{Bh^2 ds}{(s^2-h^2)\sqrt{(S)}} - \frac{Bu^2 du}{(u^2-h^2)\sqrt{(U)}}, \qquad (3,\ f)$$

so that the problem is reduced to quadratures, the functions to be integrated involving the square roots of two rational and integral functions of the sixth degree.

2, *Stone Buildings*, 10*th Nov.*, 1856.

183.

NOTE ON CERTAIN SYSTEMS OF CIRCLES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 83—88.]

It will be convenient to remark at the outset that two concentric circles, the radii of which are in the ratio of $1 : i$ (i being as usual the imaginary unit), are orthotomic[1], and that the most convenient quasi representation of a circle, the centre of which is real and the radius a pure imaginary quantity, is by means of the concentric orthotomic circle. This being premised consider a circle and a point C. The points of contact of the tangents through C to the circle may be termed the taction points; the points where the chord through C perpendicular to the line joining C with the centre meets the circle, may be termed the section points. It is clear that, for an exterior point, the taction points are real and the section points imaginary, while, for an interior point, the section points are real and the taction points imaginary. A circle having C for its centre and passing through the taction points (in fact the orthotomic circle having C for its centre) is said to be the taction circle. A circle having C for its centre and passing through the section points is said to be the section circle. Of course for an exterior point the taction circle is real and the section circle imaginary; while for an interior point the taction circle is imaginary and the section circle is real. It is proper also to remark that the taction circle and the section circle are concentric orthotomic circles.

Passing now to the case of two systems of orthotomic circles, let MM', NN' be lines at right angles to each other intersecting in R, and let M, M' be real or pure imaginary points on the line MM', equidistant from R. Imagine a system of circles,

[1] Two concentric circles are, it is well known, conics having a double contact at infinity, and it appears at first sight difficult to reconcile with this, the idea of two particular concentric circles being orthotomic. The explanation is that any two lines through a circular point at infinity may be considered as being at right angles to each other, and therefore any line through a circular point at infinity may be considered as being at right angles to itself. The two concentric circles in question have, in fact, at each circular point at infinity a common tangent, but this common tangent must be considered as being at right angles to itself. The paradox disappears entirely upon a homographic deformation of the figure; two lines KL, KM are then defined to be at right angles when joining K with the fixed points I, J, the four lines KL, KM, KI, KJ are a harmonic pencil; but when K coincides with I, then KI is indeterminate and may be taken to be the fourth harmonic of the pencil, i.e. any two lines IL, IM through the point I may be considered as being at right angles.

each of them having its centre on the line NN' and passing through the points M, M' (so that MM' is the radical axis of these circles). There are always on the line NN' two pure imaginary or real points N, N' equidistant from R, such that the circles, each of them having its centre on MM' and passing through the points N, N' (NN' being therefore the radical axis of these circles), are orthotomic to the first-mentioned system of circles, [the four points M, M', N, N' being thus as I have since termed them, two pairs of antipoints]. Moreover if R be made the centre of a circle passing through M, M', then the concentric orthotomic circle passes through N, N'; this is in fact only a particular case of the general property.

Suppose now that M, M' being given as the points of intersection of two circles having their centres on NN', it is required to find a circle having for its centre a given point C on NN' and passing through the points M, M'. In the case of M, M' being real, the required circle is obviously given and is always real. But if M, M' are imaginary; then if about any point of MM' as centre a circle be described orthotomic to one of the circles, it will be orthotomic to the other circle, and will meet NN' in the real points N, N'. Now if ρ be the radius of the required circle (i.e. of the circle having C for its centre and passing through the points M, M'), then $\rho^2 = (RC)^2 + (RM)^2 = (RC)^2 - (RN)^2$. Hence if $RC > RN$ or if C lies outside the space NN', ρ^2 is positive or the required circle is real, and the radius is at once constructed from the preceding expression

$$\rho^2 = RC^2 - RN^2.$$

But if $RC < RN$ or C lies within the space NN', then the required circle is imaginary, but the concentric orthotomic circle is at once constructed from the formula

$$\rho'^2 = RN^2 - RC^2.$$

Suppose now the point C is a centre of similitude of the two circles. The circle having C for its centre and passing through the points M, M' is a taction circle of all the taction circles of the two circles, it may be termed the tactaction circle. The concentric orthotomic of the circle having C for its centre and passing through the points M, M' is a section circle of all the taction circles of the two circles, it may be termed the sectaction circle. Consider first the case where the circles intersect in a pair of real points; here the two centres of similitude are on opposite sides of R; the tactaction circles are both real, the sectaction circles both imaginary. Secondly, the case where the two circles are wholly exterior each to the other, the two centres of similitude lie on the same side of R, viz. the centre of inverse similitude between R and N, the centre of direct similitude beyond N. Hence the tactaction circle corresponding to the centre of direct similitude and the sectaction circle corresponding to the centre of inverse similitude are real, the other tactaction circle and sectaction circle are imaginary. Thirdly, the case where one of the circles is wholly interior to the other; here the two centres of similitude are still on the same side of R, but the centre of direct similitude lies between R and N, and the centre of inverse similitude lies beyond N. Hence the sectaction circle corresponding to the centre of direct similitude and the tactaction circle corresponding to the centre of inverse similitude are real, the other sectaction circle and tactaction circle are imaginary.

To obtain a distinct idea of the methods made use of in Gaultier's "Mémoire sur les moyens généraux de construire graphiquement un cercle déterminé par trois conditions," (*Jour. École Polyt.* t. IX. [1813], p. 124), and in Steiner's "Geometrische Betrachtungen," *Crelle*, t. I. [1826], p. 161; it should be remarked that both of these geometers, confining as they do their attention to real circles, do not consider the section circle of an exterior point, or the taction circle of an interior point. The taction circle of an exterior point, or the section circle of an interior point, is Gaultier's "Cercle radical," and Steiner's "Potenzkreis," and Steiner also speaks of the radius of this circle as the "Potenz" of its centre in relation to the given circle. The nature of the Cercle radical, or Potenzkreis, (i.e. whether it is a taction circle or a section circle) is of course determined as soon as it is known whether the centre is an exterior or an interior point, and Gaultier distinguishes the two cases as the "radical réciproque" and the "radical simple," and in like manner Steiner speaks of the Potenz as being "äuszerlich" or "innerlich." Again, for two circles and for a given centre of similitude Gaultier and Steiner employ the tactaction circle or the sectaction circle, whichever of them is real, Gaultier without giving any distinctive appellation to the circle in question, Steiner calling it the Potenzkreis of the two circles, and in particular the "äuszere Potenzkreis" or the "innere Potenzkreis," according as it has for centre the centre of direct similitude or the centre of inverse similitude.

The preceding properties of circles are of course at once extended to conics passing each of them through the same two points; it is I think worth while to notice what the analogue is of a pair of concentric orthotomic circles. If the fixed points are I, J and if the point corresponding to the centre is K, then the conics are of course conics touching the lines KI, KJ in the points I, J, and, one of the conics being given, the other is to be determined. It is easily seen that if an arbitrary line through I meets the conics in P, P' and the line KJ in M, then the points I, M, P, P' are a harmonic range, and this condition gives the construction of the second conic; it of course follows that an arbitrary line through J meets the conics in points Q, Q' and the line KI in a point N such that the points J, N, Q, Q' are also a harmonic range. The two conics in question may be termed "inscribed harmonics" each of the other.

Addition. The equation of the tactaction circle, corresponding to the centre of direct (or inverse) similitude, of two given circles, may be found as follows:

Let the equations of the given circles be

$$(x-\alpha)^2+(y-\beta)^2=c^2,$$
$$(x-\alpha')^2+(y-\beta')^2=c'^2,$$

then the coordinates of the centre of direct similitude are

$$\frac{\alpha c'-\alpha' c}{c'-c'},\ \frac{\beta c'-\beta' c}{c'-c},$$

which are therefore the coordinates of the centre of the tactaction circle; and the equation of this circle is of the form

$$\lambda\left[(x-\alpha)^2+(y-\beta)^2-c^2\right]+(1-\lambda)\left[(x-\alpha')^2+(y-\beta')^2-c'^2\right]=0,$$

or, expanding and reducing,

$$(x^2+y^2)-2[\alpha\lambda+\alpha'(1-\lambda)]x-2[\beta\lambda+\beta'(1-\lambda)]y+\lambda(\alpha^2+\beta^2-c^2)+(1-\lambda)(\alpha'^2+\beta'^2-c'^2)=0.$$

We must therefore have

$$\alpha\lambda+\alpha'(1-\lambda)=\frac{\alpha c'-\alpha' c}{c'-c},$$

$$\beta\lambda+\beta'(1-\lambda)=\frac{\beta c'-\beta' c}{c'-c},$$

which are consistent with each other and give

$$\lambda=\frac{c'}{c'-c},\quad 1-\lambda=\frac{-c}{c'-c};$$

we have then

$$\lambda(\alpha^2+\beta^2-c^2)+(1-\lambda)(\alpha'^2+\beta'^2-c'^2)=\frac{1}{c'-c}[c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)];$$

and the equation of the tactaction circle is

$$x^2+y^2-2\frac{\alpha c'-\alpha' c}{c'-c}x-2\frac{\beta c'-\beta' c}{c'-c}y=\frac{1}{c'-c}[c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)],$$

which may also be written

$$\left(x-\frac{\alpha c'-\alpha' c}{c'-c}\right)^2+\left(y-\frac{\beta c'-\beta' c}{c'-c}\right)^2=\frac{cc'}{(c'-c)^2}[(\alpha'-\alpha)^2+(\beta'-\beta)^2-(c'-c)^2].$$

We have thus the equation of the tactaction circle corresponding to the centre of direct similitude, and that of the tactaction circle corresponding to the centre of inverse similitude is at once obtained from it by changing the sign of one of the two radii c, c'.

Consider any three circles and combining them in pairs, by what has preceded the equations of the tactaction circles corresponding to the centres of direct similitude will be

$$\begin{aligned}(c''-c')(x^2+y^2)-2(\alpha'c''-\alpha''c')x-2(\beta'c''-\beta''c')y\\ +c''(\alpha'^2+\beta'^2)-c'(\alpha''^2+\beta''^2)+c'c''(c''-c')=0,\\ (c-c'')(x^2+y^2)-2(\alpha''c-\alpha c'')x-2(\beta''c-\beta c'')y\\ +c(\alpha''^2+\beta''^2)-c''(\alpha^2+\beta^2)+c''c(c-c'')=0,\\ (c'-c)(x^2+y^2)-2(\alpha c'-\alpha' c)x-2(\beta c'-\beta' c)y\\ +c'(\alpha^2+\beta^2)-c(\alpha'^2+\beta'^2)+cc'(c'-c)=0,\end{aligned}$$

and representing these equations by $U=0$, $U'=0$, $U''=0$, we have identically

$$cU+c'U'+c''U''=0,$$

hence the three tactaction circles pass through the same two points, or what is the same thing, have a common radical axis.

184.

A THEOREM RELATING TO SURFACES OF THE SECOND ORDER.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 140—142.]

GIVEN a surface of the second order

$$(a, b, c, d, f, g, h, l, m, n)\,(x, y, z, w)^2 = 0,$$

and a fixed plane

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

imagine a variable plane

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

subjected to the condition that it always touches a surface of the second order, or what is the same thing such that the parameters ξ, η, ζ, ω satisfy a condition

$$(\mathrm{a, b, c, d, f, g, h, l, m, n})\,(\xi, \eta, \zeta, \omega)^2 = 0.$$

The given surface of the second order, and the variable plane meet in a conic, and the fixed plane and the variable plane meet in a line, it is required to find the locus of the pole of the line with respect to the conic.

The pole in question is the point in which the variable plane is intersected by the polar of the line with respect to the surface of the second order: this polar is the line joining the pole of the fixed plane with respect to the surface of the second order, and the pole of the variable plane with respect to the surface of the second order. Let α_1, β_1, γ_1, δ_1, be given linear functions of α, β, γ, δ, and ξ, η, ζ, ω, be given linear functions of ξ, η, ζ, ω, viz., if

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}),$$

are the inverse system to $(a, b, c, d, f, g, h, l, m)$, then let

$$\begin{aligned}
\alpha_1 &= \mathfrak{A}\alpha + \mathfrak{H}\beta + \mathfrak{G}\gamma + \mathfrak{L}\delta,\\
\beta_1 &= \mathfrak{H}\alpha + \mathfrak{B}\beta + \mathfrak{F}\gamma + \mathfrak{M}\delta,\\
\gamma_1 &= \mathfrak{G}\alpha + \mathfrak{F}\beta + \mathfrak{C}\gamma + \mathfrak{N}\delta,\\
\delta_1 &= \mathfrak{L}\alpha + \mathfrak{M}\beta + \mathfrak{N}\gamma + \mathfrak{D}\delta,
\end{aligned}$$

and in like manner,

$$\begin{aligned}
\xi_1 &= \mathfrak{A}\xi + \mathfrak{H}\eta + \mathfrak{G}\zeta + \mathfrak{L}\omega,\\
\eta_1 &= \mathfrak{H}\xi + \mathfrak{B}\eta + \mathfrak{F}\zeta + \mathfrak{M}\omega,\\
\zeta_1 &= \mathfrak{G}\xi + \mathfrak{F}\eta + \mathfrak{C}\zeta + \mathfrak{N}\omega,\\
\omega_1 &= \mathfrak{L}\xi + \mathfrak{M}\eta + \mathfrak{N}\zeta + \mathfrak{D}\omega,
\end{aligned}$$

then the coordinates of the pole of the fixed plane are as

$$\alpha_1 : \beta_1 : \gamma_1 : \delta_1,$$

and the coordinates of the pole of the variable plane are as

$$\xi_1 : \eta_1 : \zeta_1 : \delta_1,$$

whence the equations of the polar are

$$\left\| \begin{matrix} x, & y, & z, & w \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \end{matrix} \right\| = 0,$$

a system of equations which may be thus represented

$$\begin{aligned}
\xi_1 &= \mathrm{K}\lambda x + \mu\alpha_1,\\
\eta_1 &= \mathrm{K}\lambda y + \mu\beta_1,\\
\zeta_1 &= \mathrm{K}\lambda z + \mu\gamma_1,\\
\omega_1 &= \mathrm{K}\lambda w + \mu\delta_1,
\end{aligned}$$

where K is the discriminant of the system

$$(a, b, c, d, e, f, g, h, l, m, n).$$

Write

$$\begin{aligned}
\mathrm{x} &= ax + hy + gz + lw,\\
\mathrm{y} &= hx + by + fz + mw,\\
\mathrm{z} &= gx + fy + cz + nw,\\
\mathrm{w} &= lx + my + nz + dw.
\end{aligned}$$

the last preceding system of equations may be written

$$\begin{aligned}\xi &= \lambda \mathrm{x} + \mu\alpha,\\ \eta &= \lambda \mathrm{y} + \mu\beta,\\ \zeta &= \lambda \mathrm{z} + \mu\gamma,\\ \omega &= \lambda \mathrm{w} + \mu\delta,\end{aligned}$$

equations in which λ, μ are indeterminate, and where x, y, z, w may be considered as current coordinates, and this system represents the polar above referred to. Combining the equations in question with the equation

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

of the variable plane, we have

$$\lambda(x\mathrm{x} + y\mathrm{y} + z\mathrm{z} + w\mathrm{w}) + \mu(\alpha x + \beta y + \gamma z + \delta w) = 0,$$

i.e.

$$\lambda(a,\ldots)\,(x,\ y,\ z,\ w)^2 + \mu(\alpha x + \beta y + \gamma z + \delta w) = 0,$$

or what is the same thing

$$\lambda : \mu = \alpha x + \beta y + \gamma z + \delta w : (a,\ldots)(x,\ y,\ z,\ w)^2,$$

and substituting these values in the expressions for ξ, η, ζ, ω we have ξ, η, ζ, ω in terms of the coordinates x, y, z, w of the pole above referred to, i.e., if for shortness,

$$\begin{aligned}U &= (a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)\ (x,\ y,\ z,\ w)^2,\\ P &= \alpha x + \beta y + \gamma z + \delta w,\end{aligned}$$

then

$$\begin{aligned}\xi &= \tfrac{1}{2}Pd_xU - \alpha U,\\ \eta &= \tfrac{1}{2}Pd_yU - \beta U,\\ \zeta &= \tfrac{1}{2}Pd_zU - \gamma U,\\ \omega &= \tfrac{1}{2}Pd_wU - \delta U,\end{aligned}$$

and combining with these equations the equation

$$(\alpha,\ldots)\ (\xi,\ \eta,\ \zeta,\ \omega)^2 = 0,$$

we have

$$(\mathrm{a}\ldots)\ (\tfrac{1}{2}Pd_xU - \alpha U,\ \tfrac{1}{2}Pd_yU - \beta U,\ \tfrac{1}{2}Pd_zU - \gamma U,\ \tfrac{1}{2}Pd_wU - \delta U)^2 = 0,$$

for the required locus of the pole of the line of intersection of the variable plane and the fixed plane, with the conic of intersection of the given surface of the second order and the variable plane. The locus in question is a surface of the fourth order; and it may be remarked that this surface touches the given surface of the second order along the conic of intersection with the fixed plane.

7th April, 1857.

185.

NOTE ON THE 'CIRCULAR RELATION' OF PROF. MÖBIUS.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), p. 162.]

THEOREM.

Given the points

$$A,\ B,\ C;\ P,$$

and the points

$$A',\ B',\ C';$$

describe the circles α, β, γ, ω as follows: viz.

α through $(B,\ C,\ P)$,
β „ $(C,\ A,\ P)$,
γ „ $(A,\ B,\ P)$,
ω „ $(A,\ B,\ C)$,

and the circles α', β', γ', ω' as follows: viz.,

α' through $(B',\ C')$ cutting ω' at the angle at which α cuts ω,
β' „ $(C',\ A')$ „ ω' „ β „ ω,
γ' „ $(A',\ B')$ „ ω' „ γ „ ω, and
ω' „ $(A',\ B',\ C')$

then will α', β', γ' meet in a point P', i.e. we shall have the points A', B', C'; P'

such that the circles α', β', γ', ω' pass

$$\begin{aligned} &\alpha' \text{ through } (B', C', P'), \\ &\beta' \quad \text{,,} \quad (C', A', P'), \\ &\gamma' \quad \text{,,} \quad (A', B', P'), \\ &\omega' \quad \text{,,} \quad (A', B', C'). \end{aligned}$$

We may construct in this manner two figures, such that to three points of the first figure there correspond in the second figure three points which may be taken at pleasure, but these once selected to every other point of the first figure there will correspond in the second figure a perfectly determinate point. And the two figures will be such that whenever in the first figure four or more points lie in a circle, then in the second figure the corresponding points will also lie in a circle. The relation in question is due to Prof. Möbius, who has termed it the *Kreis-verwandschaft* (circular relation) of two plane figures. See his paper *Crelle*, t. LII. [1856], pp. 218—228, extracted from the Berichte of the *Royal Saxon Society of Sciences* of the 5th Feb. 1853, [and *Werke*, t. II. pp. 207—217].

186.

ON THE DETERMINATION OF THE VALUE OF A CERTAIN DETERMINANT.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 163—166.]

CONSIDERING the determinant

$$\begin{vmatrix} \theta, & 1, & . & . & . & \dots \\ n, & \theta, & 2 & & & \\ . & n-1, & \theta, & 3 & & \\ & & n-2, & \theta, & 4 & \\ \vdots & & & & & \end{vmatrix}$$

let the successive diagonal minors be U_0, U_1, $U_2, \dots U_x \dots$, it is easy to find

$$U_0 = 1,$$
$$U_1 = \theta,$$
$$U_2 = (\theta^2 - 1) - (n-1),$$
$$U_3 = \theta(\theta^2 - 4) - 3(n-2)\theta,$$
$$U_4 = (\theta^2 - 1)(\theta^2 - 9) - 6(n-3)(\theta^2 - 1) + 3(n-3)(n-1),$$

which in fact suggests the law, viz.

$$\begin{aligned} U_x = {} & (\theta + x - 1)(\theta + x - 3)(\theta + x - 5)\dots(\theta - x + 5)(\theta - x + 3)(\theta - x + 1) \\ & - \frac{x(x-1)}{2}(n - x + 1)(\theta + x - 3)(\theta + x - 5)\dots(\theta - x + 5)(\theta - x + 3) \\ & + \frac{x(x-1)(x-2)(x-3)}{2.4}(n - x + 1)(n - x + 3)(\theta + x - 5)\dots(\theta - x + 5) \\ & - \&c. \\ & \vdots \\ & + (-)^s \frac{x(x-1)\dots(x-2s+1)}{2.4\dots 2s}(n - x + 1)(n - x + 3)\dots(n - x + 2s - 1) \\ & \qquad (\theta + x - 2s - 1)(\theta + x - 2s - 3)\dots(\theta - x + 2s + 1) \\ & \vdots \end{aligned}$$

to $s = \frac{1}{2}x$ or $\frac{1}{2}(x-1)$, as x is even or odd.

And of course if x denote the number of lines or columns of the determinant, then U_x is the value of the determinant. This theorem, or a particular case of it, is due to Prof. Sylvester: I have not been able to find an easier demonstration than the following one, which, it must be admitted, is somewhat complicated. I observe that U_x satisfies the equation

$$U_x - \theta U_{x-1} + (x-1)(n-x+2)\, U_{x-2} = 0.$$

Hence writing $x-1$ and $x-2$ for x, we have the system

$$\begin{aligned} U_x - \theta U_{x-1} + (x-1)(n-x+2)\, U_{x-2} \qquad\qquad\qquad\qquad &= 0, \\ U_{x-1} - \qquad\qquad \theta U_{x-2} + (x-2)(n-x+3)\, U_{x-3} \qquad &= 0, \\ U_{x-2} - \qquad\qquad\qquad\qquad \theta U_{x-3} + (x-3)(n-x+4)\, U_{x-4} &= 0, \end{aligned}$$

or, eliminating U_{x-1} and U_{x-3},

$$\begin{aligned} U_x + \{(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2\}\, U_{x-2} \\ + (x-2)(x-3)(n-x+3)(n-x+4)\, U_{x-4} = 0. \end{aligned}$$

Suppose, for shortness,

$$(\theta + x - 1)(\theta + x - 3)(\theta + x - 5)\ldots(\theta - x + 5)(\theta - x + 3)(\theta - x + 1) = H_x,$$

and assume

$$U_x = A_{x,0}\, H_x - A_{x,1}\, H_{x-2} \ldots + (-)^s A_{x,s}\, H_{x-2s} \ldots,$$

where $A_{x,s}$ is independent of θ, then

$$\begin{aligned} U_x \quad &\text{contains the term } (-)^s A_{x,s} \quad H_{x-2s}, \\ U_{x-2} &\text{ contains the term } (-)^s A_{x-2,s}\, H_{x-2s-2}, \end{aligned}$$

which is to be multiplied by

$$(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2.$$

This multiplier may be written under the form

$$\begin{aligned} (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2 - \{\theta^2 - (x-2s-1)^2\} \\ = M_{x,s} - \{\theta^2 - (x-2s-1)^2\}, \end{aligned}$$

if, for shortness,

$$M_{x,s} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2.$$

Now

$$M_{x,s} - \{\theta^2 - (x-2s-1)^2\}$$

multiplied into

$$(-)^s A_{x-2,s}\, H_{x-2s-2}$$

gives rise to the terms

$$(-)^s M_{x,s}\, A_{x-2,s}\, H_{x-2s-2} - (-)^s A_{x-2,s}\, H_{x-2s},$$

(since $\{\theta^2 - (x-2s-1)^2\}\ H_{x-2s-2} = H_{x-2s}$), or, what is the same thing,

$$\begin{aligned} &-(-)^s M_{x,s-1}\, A_{x-2,s-1}\, H_{x-2s} - (-)^s A_{x-2,s}\, H_{x-2s} \\ &= -(-)^s \{M_{x,s-1}\, A_{x-2,s-1} + A_{x-2,s}\}\, H_{x-2s}, \end{aligned}$$

and moreover

$$U_{x-4} \text{ contains the term } (-)^s A_{x-4,s}\, H_{x-2s-4},$$

or, what is the same thing, $(-)^s A_{x-4,s-2}\ H_{x-2s}$.

Hence we must have

$$A_{x,s} - (A_{x-2,s} + M_{x,s-1}\ A_{x-2,s-1}) + (x-2)(x-3)(n-x+3)(n-x+4)\, A_{x-4,s-2} = 0,$$

where

$$M_{x,s-1} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s+1)^2.$$

This may be satisfied by assuming

$$A_{x,s} \quad = B_{x,s}\,(n-x+1)(n-x+3)\ldots(n-x+2s-1);$$

for then

$$\begin{aligned} A_{x-2,s} &= B_{x-2,s}\ \ (n-x+3)\ldots(n-x+2s-1)(n-x+2s+1), \\ A_{x-2,s-1} &= B_{x-2,s-1}(n-x+3)\ldots(n-x+2s-1), \\ (n-x+3)(n-x+4)\, A_{x-4,x-2} &= B_{x-4,s-2}\,(n-x+4)(n-x+3)\ldots(n-x+2s-1), \end{aligned}$$

and consequently

$$\begin{aligned} &B_{x,s}\ \ (n-x+1) \\ &- B_{x-2,s}\,(n-x+2s+1) \\ &- B_{x-2,s-1}\, M_{s,x-1} \\ &+ B_{x-4,s-2}\,(x-2)(x-3)(n-x+4) = 0; \end{aligned}$$

and if this equation be satisfied independently of n, we must have

$$\begin{aligned} &B_{x,s} - \qquad B_{x-2,s} - \qquad (2x-3)\, B_{x-2,s-1} + \ (x-2)(x-3)\, B_{x-4,s-2} = 0, \\ &B_{x,s} - (2s+1)\, B_{x-2,s} - \{5x-8-(x-2s+1)^2\}\, B_{x-2,s-1} + 4(x-2)(x-3)\, B_{x-4,s-2} = 0, \end{aligned}$$

and these are both satisfied by

$$B_{x,s} = \frac{x\,.\,x-1\ldots x-2s+1}{2^s.1.2.3\ldots s},$$

in fact, substituting this value and omitting the factor

$$\frac{(x-2)(x-3)\ldots(x-2s+1)}{2^s.1.2.3\ldots s},$$

the first equation becomes

$$x(x-1)- \qquad (x-2s)(x-2s-1)- \qquad (2x-3)\,2s+\ 4s(s-1)=0,$$

and the second equation becomes

$$x(x-1)-(2s+1)(x-2s)(x-2s-1)-\{5x-8-(x-2s+1)^2\}\,2s+16s(s-1)=0,$$

which are each of them an identical equation, the first being

$$\begin{aligned} x^2- \qquad & x \\ -x^2+(4s+1)\,x-2s\,(2s+1) & \\ -4sx+6s & \\ +4s\,(s-1)=0, & \end{aligned}$$

and the second being

$$\begin{aligned} x^2- \qquad & x \\ -(2s+1)\,\{x^2-(4s+1)\,x+\ 2s\,(2s+1)\} & \\ +2s\,\{x^2-(4s+3)\,x+(2s-1)^2+8\} & \\ +16s\,(s-1)=0. & \end{aligned}$$

as may be easily verified.

Hence writing for $B_{x,s}$ its value and recapitulating, the equation

$$\begin{aligned} U_x+\{(x-1)(n-x+2)+(x-2)(n-x+3)-\theta^2\}\,U_{x-2} & \\ +(x-2)(x-3)(n-x+3)(n-x+4)\,U_{x-4}=0 & \end{aligned}$$

is satisfied by

$$U_x=A_{x,0}\,H_x-A_{x,1}\,H_{x-2}\ldots+(-)^s A_{x,s}\,H_{x-2s}\ldots \text{to } s=\tfrac{1}{2}x \text{ or } \tfrac{1}{2}(x-1), \text{ as } x \text{ is even or odd,}$$

where

$$H_x=(\theta+x-1)(\theta+x-3)(\theta+x-5)\ldots(\theta-x+5)(\theta-x+3)(\theta-x+1),$$

$$A_{x,s}=\frac{x(x-1)\ldots(x-2s+1)}{2^s.1.2.3\ldots s}(n-x+1)(n-x+3)\ldots(n-x+2s-1),$$

and since for $x=0$, 1, 2, 3 the values of the expression U_x coincide with those of the first four diagonal minors, the expression gives in general the value of the diagonal minor, or when x denotes the number of lines or columns of the determinant, then the value of the determinant.

2, *Stone Buildings*, 1*st April*, 1857.

187.

ON THE SUMS OF CERTAIN SERIES ARISING FROM THE EQUATION $x = u + tfx$.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 167—171.]

LAGRANGE has given the following formula for the sum of the inverse n^{th} powers of the roots of the equation $x = u + tfx$,

$$\Sigma(z^{-n}) = u^{-n} + (-nu^{-n-1}fu)\frac{t}{1} + (-nu^{-n-1}f^2u)'\frac{t^2}{1.2} + \&c. \qquad (1)$$

where n is a positive integer and the series on the second side of the equation is to be continued as long as the exponent of u remains negative (*Théorie des Équations Numériques*, p. 225). Applying this to the equation $x = 1 + tx^s$, we have

$$\Sigma(z^{-n}) = 1^{-n} - \frac{n}{1}t.1^{-n+s-1} + \frac{n(n-2s+1)}{1.2}t^2.1^{-n+2s-2}\ldots$$

$$+ (-)^q\frac{n(n-qs+q-1)\ldots(n-qs+1)}{1.2\ldots q}t^q.1^{-n+qs-q} - \&c. \qquad (2)$$

to be continued while the exponent of 1 remains negative.

Let $n = \mu s + \rho$, ρ being not greater than $s-1$, the series may always be continued up to $q = \mu$, and no further. In fact writing the above value for n and putting $q = \mu + \theta$, the general term is

$$(-)^{\mu+\theta}\frac{t^{\mu+\theta}}{1.2\ldots(\mu+\theta)}(\mu s+\rho)(\rho-\theta s+\mu+\theta-1)\ldots(\rho-\theta s+1)\,1^{-(\rho-\theta s+\mu+\theta)}.$$

Now if $\rho + \mu - \theta(s-1)$ is negative or zero, the term is to be rejected on account of the index of 1 not being negative, and if this quantity be positive, then since

$\rho-\theta s+1$ is necessarily negative for any value of θ greater than zero, the factorial $(\rho-\theta s+\mu+\theta-1)\ldots(\rho-\theta s+1)$ begins with a positive and ends with a negative factor, and since the successive factors diminish by unity, one of them is necessarily equal to zero, or the term vanishes; hence the series is always to be continued up to $q=\mu$.

Hence

$$\Sigma(z^{-\mu s-\rho})=1-\frac{\mu s+\rho}{1}t+\frac{(\mu s+\rho)\{(\mu-2)s+\rho+1\}}{1.2}t^2\ldots$$

$$+(-)^q\frac{(\mu s+\rho)\{(\mu-q)s+\rho+q-1\}\ldots\{(\mu-q)s+\rho+1\}}{1.2\ldots q}t^q$$

$$-\&c. \tag{3}$$

continued to $q=\mu$.

By taking the terms in a reverse order, it is easy to derive

$$(-)^\mu t^{-\mu}\Sigma(z^{-\mu s-\rho})=(\mu s+\rho)\left\{\begin{array}{l}\dfrac{(\mu+\rho-1)\ldots(\mu+1)}{2.3\ldots\rho}-\dfrac{(\mu+\rho+s-2)\ldots\mu}{2.3\ldots\rho+s}t^{-1}\\ +(-)^q\dfrac{(\mu+\rho+qs-q-1)\ldots(\mu+1-q)}{2.3\ldots(\rho+q)s}t^{-q}\\ -\&c.\end{array}\right. \tag{4}$$

continued to $q=\mu$.

Suppose in particular $s=2$, and $t=-\dfrac{\alpha+1}{\alpha^2}$, so that the equation in x becomes $\dfrac{x-1}{x^2}=-\dfrac{\alpha+1}{\alpha^2}$, whence $x=-\alpha$ or $x=\dfrac{\alpha}{\alpha+1}$, or substituting in (2), we find

$$\frac{(\alpha+1)^n}{\alpha^n}+\frac{(-)^n}{\alpha^n}=1+\frac{n}{1}\frac{\alpha+1}{\alpha^2}+\frac{n(n-3)}{2}\left(\frac{\alpha+1}{\alpha^2}\right)^2+\&c. \tag{5}$$

continued to the term involving $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}n}$ or $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}(n-1)}$.

Put $\alpha=-\dfrac{a+b}{a}$; and therefore

$$\alpha+1=-\frac{b}{a},\quad \frac{\alpha+1}{\alpha}=\frac{b}{a+b},\quad \frac{\alpha+1}{\alpha^2}=\frac{ab}{(a+b)^2};$$

we obtain

$$\frac{a^n+b^n}{(a+b)^n}=1-\frac{n}{1}\frac{ab}{(a+b)^2}+\frac{n(n-3)}{1.2}\frac{a^2b^2}{(a+b)^4}-\&c. \tag{6},$$

or

$$\frac{(a+b)^n-a^n-b^n}{nab(a+b)}=(a+b)^{n-3}-\frac{n-3}{2}(a+b)^{n-5}ab$$

$$+\frac{(n-4)(n-5)}{2.3}(a+b)^{n-7}a^2b^2-\&c. \tag{7},$$

to be continued as long as the exponent of $(a+b)$ on the second side is negative.

This formula, which is easily deducible from that for the expansion of $\cos n\theta$ in powers of $\cos\theta$, is employed by M. Stern, *Crelle*, t. XX. [1840], in proving the following theorem: If

$$S = 1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{2.3} - \&c. \tag{8}$$

continued to the first term that vanishes, then according as n is of the form $6k+3$, $6k \pm 1$, $6k$ or $6k \pm 2$,

$$S = \frac{3}{n}, \quad S = 0, \quad S = -\frac{1}{n}, \quad S = \frac{2}{n}, \tag{9}$$

which is in fact immediately deduced from it by writing $b = \omega a$, ω being one of the impossible cube roots of unity. Substituting the above values of x in the equation (4),

$$(1+\alpha)^{p+1} - (1+\alpha)^{-p} = (2p+1)\,\alpha\left\{1 + \frac{(p+1)p}{2.3}\frac{\alpha^2}{\alpha+1} + \frac{(p+2)(p+1)p(p-1)}{2.3.4.5}\frac{\alpha^4}{(\alpha+1)^2} + \ldots\right\}, \tag{10}$$

$$(1+\alpha)^{p} + (1+\alpha)^{-p} = 2p\left\{\frac{1}{p} + \frac{p}{2}\frac{\alpha^2}{\alpha+1} + \frac{(p+1)p(p-1)}{2.3.4}\frac{\alpha^4}{(\alpha+1)^2} + \ldots\right\}, \tag{11}$$

whence

$$(1+\alpha)^{p+1} + (1+\alpha)^{p} = (2p+1)\,\alpha\left\{1 + \frac{(p+1)p}{2.3}\frac{\alpha^2}{\alpha+1} + \ldots\right\}$$
$$+ 2p\left\{\frac{1}{p} + \frac{p}{2}\frac{\alpha^2}{\alpha+1} + \ldots\right\}, = U \text{ suppose,} \tag{12}$$

i.e

$$\Delta(-)^p(1+\alpha)^p = (-)^{p+1}U \text{ or } (1+\alpha)^p = (-)^p\,\Sigma\,(-)^{p+1}U,$$

where Δ and Σ refer to the variable p. The summation is readily effected by means of the formulæ

$$\Sigma(-)^{p+1}(2p+1)(p+s+1)\ldots(p-s) = (-)^p(p+s+1)\ldots(p-s-1).$$
$$\Sigma(-)^{p+1}(p+s)\ldots(p-s)\,2p = (-)^p(p+s)\ldots(p-s-1),$$

and we thence find

$$(1+\alpha)^p = \left\{1 + \frac{p(p-1)}{1.2}\frac{\alpha^2}{1+\alpha} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\frac{\alpha^4}{(1+\alpha)^2} + \ldots\right\}$$
$$+ \alpha\left\{\frac{p}{1} + \frac{(p+1)p(p-1)}{1.2.3}\frac{\alpha^2}{1+\alpha} + \ldots\right\}, \tag{13}$$

a formula of Euler's (*Pet. Trans.* 1811) demonstrated likewise by M. Catalan (*Liouville*, t. IX. [1844], pp. 161—174) by induction. It may be expressed also in the slightly different form

$$(1+\alpha)^p = \left\{1+\frac{(p+1)p}{1.2}\frac{\alpha^2}{1+\alpha}+\frac{(p+2)(p+1)p(p-1)}{1.2.3.4}\frac{\alpha^4}{(1+\alpha)^2}+\dots\right\}$$

$$+\frac{\alpha}{1+\alpha}\left\{\frac{p}{1}+\frac{(p+1)p(p-1)}{1.2.3}\frac{\alpha^2}{1+\alpha}+\dots\right\}. \tag{14}$$

The two series (13), (14) are each of them supposed to contain $p+1$ terms, p being an integer; but since the terms after these all of them vanish, the series may be continued indefinitely. Suppose the two sides expanded in powers of p, the coefficients will be separately equal, and thus the identity of the two sides will be independent of the particular values of p, or the equations (13), (14), and similarly, (10), (11), (12) are true for any values of p whatever. It is to be observed that the series for negative values of p do not differ essentially from those for the corresponding positive values; as may be seen immediately by writing $-p$ for p, and $\frac{-\alpha}{1+\alpha}$ for α.

Suppose next $s=3$, or that the equation in x is $x=1+tx^3$; to rationalise the roots of this, assume $t=\frac{4(\beta^2-1)^2}{(\beta^2+3)^3}$, then values of x are

$$x=\frac{\beta^2+3}{2(\beta+1)},\ x=-\frac{\beta^2+3}{2(\beta-1)},\ x=\frac{\beta^2+3}{\beta^2-1},$$

and hence

$$\frac{2^n\{(\beta+1)^n+(-)^n(\beta-1)^n\}+(\beta^2-1)^n}{(\beta^2+3)^r}=$$

$$1-\frac{n}{1}t+\frac{n(n-5)}{1.2}t^2-\frac{n(n-7)(n-8)}{1.2.3}t^3\dots+(-)^r\frac{n(n-2r-1)\dots(n-3r+1)}{1.2\dots r}t^r+\&c. \tag{15}$$

where $t=\frac{4(\beta^2-1)^3}{(\beta^2+3)^3}$, and the series is to be continued up to the term involving $t^{\frac{1}{2}n}$, $t^{\frac{1}{2}(n-1)}$ or $t^{\frac{1}{2}(n-2)}$.

Again, from the formula (4) we deduce the three following forms,

$$(-)^\mu\frac{2^3\{(\beta+1)^{3\mu}+(-)^\mu(\beta-1)^{3\mu}\}+(\beta^2-1)^{3\mu}}{2^{2\mu}(\beta^2-1)^{2\mu}}=$$

$$3\mu\left\{\frac{1}{\mu}-\frac{(\mu+1)\mu}{2.3}t^{-1}+\frac{(\mu+3)(\mu+2)(\mu+1)\mu(\mu-1)}{2.3.4.5.6}t^{-2}+\dots(-)^q\frac{(\mu+2q-1)\dots(\mu-q+1)}{2.3\dots3q}t^{-q}\dots\right\}, \tag{16}$$

$$(-)^{\mu}\frac{2^{2\mu+1}\{(\beta+1)^{2\mu+1}-(-)^{\mu}(\beta-1)^{2\mu+1}\}+(\beta^2-1)^{2\mu+1}}{2^{2\mu}(\beta^2-1)^{2\mu}(\beta^2+3)}=$$

$$(3\mu+1)\left\{1-\frac{(\mu+2)(\mu+1)\mu}{2.3.4}t^{-1}\ldots+(-)^{q}\frac{(\mu+2q)\ldots(\mu-q+1)}{2.3\ldots3q+1}t^{-q}\ldots\right\}, \quad (17)$$

$$(-)^{\mu}\frac{2^{2\mu+2}\{(+1)^{2\mu+2}+(-)^{\mu}(\beta-1)^{2\mu+2}\}+(\beta^2-1)^{2\mu+2}}{2^{2\mu}(\beta^2-1)^{2}(\beta^2+3)^{2}}=$$

$$(3\mu+2)\left\{\frac{\mu+1}{2}-\frac{(\mu+3)(\mu+2)(\mu+1)\mu}{2.3.4.5}t^{-1}\ldots+(-)^{q}\frac{(\mu+2q+1)\ldots(\mu-q+1)}{2.3\ldots(3q+2)}t^{-q}\ldots\right\}, \quad (18)$$

all of them continued up to $q=\mu$.

2, *Stone Buildings*, 1*st April*, 1857.

188.

ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 192—195.]

In a former paper with this title, *Cambridge and Dublin Math. Journal*, t. IV. [1849], pp. 47—50 [74], I gave (founded on the methods of Jacobi and Prof. Boole) a simple solution of the problem, but the solution may I think be presented in an improved form as follows, where as before I consider for greater convenience the case of three variables only.

Suppose that by the linear transformation[1]

$$(x, y, z) = \left(\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix} \between x_1, y_1, z_1 \right),$$

we have identically

$$(a, b, c, f, g, h \between x, y, z)^2 = (a_1, b_1, c_1, f_1, g_1, h_1 \between x_1, y_1, z_1)^2,$$
$$(A, B, C, F, G, H \between x, y, z)^2 = (A_1, B_1, C_1, F_1, G_1, H_1 \between x_1, y_1, z_1)^2;$$

and write also

$$(\xi_1, \eta_1, \zeta_1) = \left(\begin{matrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{matrix} \between \xi, \eta, \zeta \right).$$

[1] I represent in this manner the system of equations

$$x = \alpha x_1 + \beta y_1 + \gamma z_1, \text{ \&c.}$$

and so in all like cases.

Comparing these with the relations between (x, y, z) and (x_1, y_1, z_1), we see that

$$(\xi, \eta, \zeta \between x, y, z) = (\xi_1, \eta_1, \zeta_1 \between x_1, y_1, z_1),$$

and multiplying the first of the relations between two quadrics by an indeterminate quantity λ, and adding it to the second, we have

$$(\lambda a + A, \ldots \between x, y, z)^2 = (\lambda a_1 + A_1, \ldots \between x_1, y_1, z_1)^2.$$

We have thus a linear function and a quadric transformed into functions of the same form by means of the linear substitutions, and any invariant of the system will remain unaltered to a factor près, such factor being a power of the determinant of substitution. The invariants are, 1° the discriminant of the quadric; 2° the reciprocant, considered not as a contravariant of the quadric, but as an invariant of the system. And if we write

$$K = \text{Disc.}\ (\lambda a + A, \ldots \between x, y, z)^2,$$

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2 = \text{Recip.}\ (\lambda a + A, \ldots \between x, y, z)^2,$$

then K_1, &c. being the analogous expressions for the transformed functions, and the determinant of substitution being represented by Π, we have

$$K_1 = \Pi^2 K,$$

$$(\mathfrak{A}_1, \ldots \between \xi_1, \eta_1, \zeta_1)^2 = \Pi^2 (\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2,$$

and substituting for ξ_1, η_1, ζ_1 their values in terms of ξ, η, ζ, the last equation breaks up into six equations, and we have

$$\begin{aligned} K_1 \qquad\qquad\qquad\qquad &= \Pi^2 K, \\ (\mathfrak{A}_1, \ldots \between \alpha, \alpha', \alpha'')^2 \qquad &= \Pi^2 \mathfrak{A}, \\ &\vdots \\ (\mathfrak{A}_1, \ldots \between \beta, \beta', \beta'')\ (\gamma, \gamma', \gamma'') \ &= \Pi^2 \mathfrak{F}, \\ &\vdots \end{aligned}$$

which is the system obtained in a somewhat different manner in my former paper. Putting $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$, and writing also (which is no additional loss of generality) $a_1 = b_1 = c_1 = 1$, the formulæ become

$$(a, b, c, f, g, h \between x, y, z)^2 = (1, 1, 1 \between x_1^2, y_1^2, z_1^2),$$

$$(A, B, C, F, G, H \between x, y, z)^2 = (A_1, B_1, C_1 \between x_1^2, y_1^2, z_1^2),$$

viz. there are two given quadrics which are to be by the same linear substitution transformed, one of them into the form $x_1^2 + y_1^2 + z_1^2$ and the other into the form $A_1 x_1^2 + B_1 y_1^2 + C_1 z_1^2$, where A_1, B_1, C_1 have to be determined. The solution is contained in the following system of formulæ, viz.

$$(A_1 + \lambda)(B_1 + \lambda)(C_1 + \lambda) = \Pi^2\ \text{Disc.}\ (\lambda a + A, \ldots),$$

which gives A_1, B_1, C_1 as the roots of a cubic equation, and gives also

$$1 = \Pi^2 \operatorname{Disc.}(a, \ldots) = \Pi^2\kappa, \text{ or } \Pi^2 = \frac{1}{\kappa} \text{ suppose,}$$

and we have then, writing for shortness, $(*\between X, Y, Z)$ for

$$((B_1+\lambda)(C_1+\lambda),\ (C_1+\lambda)(A_1+\lambda),\ (A_1+\lambda)(B_1+\lambda)\between X,\ Y,\ Z),$$

$$(*\between\alpha^2,\quad \alpha'^2,\quad \alpha''^2) = \frac{1}{\kappa}\mathfrak{A},$$

$$(*\between\beta^2,\quad \beta'^2,\quad \beta''^2) = \frac{1}{\kappa}\mathfrak{B},$$

$$(*\between\gamma^2,\quad \gamma'^2,\quad \gamma''^2) = \frac{1}{\kappa}\mathfrak{C},$$

$$(*\between\beta\gamma,\ \beta'\gamma',\ \beta''\gamma'') = \frac{1}{\kappa}\mathfrak{F},$$

$$(*\between\gamma\alpha,\ \gamma'\alpha',\ \gamma''\alpha'') = \frac{1}{\kappa}\mathfrak{G},$$

$$(*\between\alpha\beta,\ \alpha'\beta',\ \alpha''\beta'') = \frac{1}{\kappa}\mathfrak{H},$$

where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the coefficients of the reciprocant of $(\lambda a + A, \ldots \between x, y, z)^2$. Writing $\lambda = -A_1$, $-B_1$, or $-C_1$ the quadric functions on the left-hand side become mere monomials, and we have the actual values of the squares and products α^2, $\beta\gamma$, &c. of the coefficients of the linear substitutions: thus α^2, β^2, γ^2, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ are respectively equal to $\mathfrak{A}_0$, $\mathfrak{B}_0$, $\mathfrak{C}_0$, $\mathfrak{F}_0$, $\mathfrak{G}_0$, $\mathfrak{H}_0$ each into the common factor

$$\frac{1}{\kappa}(B_1 - A_1)(C_1 - A_1),$$

the suffix denoting that we are to write in the expressions for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ the value $-A_1$ for λ; and similarly for the sets $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$.

2, *Stone Buildings*, 27*th March*, 1857.

189.

NOTE ON A FORMULA IN FINITE DIFFERENCES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 198—201.]

In Jacobi's Memoir "De usu legitimo formulæ summatoriæ Maclaurinianæ," *Crelle*, t. XII. [1834], pp. 263—273 (1834), expressions are given for the sums of the odd powers of the natural numbers 1, 2, 3...x in terms of the quantity.

$$u = x(x+1),$$

viz. putting for shortness

$$Sx^r = 1^r + 2^r + \ldots + x^r,$$

the expressions in question are

$$\begin{aligned}
Sx^3 &= \tfrac{1}{4}u^2,\\
Sx^5 &= \tfrac{1}{6}u^2\,(u - \tfrac{1}{2}),\\
Sx^7 &= \tfrac{1}{8}u^2\,(u^2 - \tfrac{4}{3}u + \tfrac{2}{3}),\\
Sx^9 &= \tfrac{1}{10}u^2\,(u^3 - \tfrac{5}{2}u^2 + 3u - \tfrac{3}{2}),\\
Sx^{11} &= \tfrac{1}{12}u^2\,(u^4 - 4u^3 + \tfrac{17}{2}u^2 - 10u + 5),\\
Sx^{13} &= \tfrac{1}{14}u^2\,(u^5 - \tfrac{35}{6}u^4 + \tfrac{287}{18}u^3 - \tfrac{118}{3}u^2 + \tfrac{691}{15}u - \tfrac{691}{30}),\\
&\text{\&c.},
\end{aligned}$$

which, especially as regards the lower powers, are more simple than the ordinary expressions in terms of x.

The expressions are continued by means of a recurring formula, viz. if

$$Sx^{2p-3} = \frac{1}{2p-2}\,\{u^{p-1} - a_1u^{p-2}\ldots + (-)^{p-1}\,a_{p-3}u^2\},$$

$$Sx^{2p-1} = \frac{1}{2p}\,\{u^p - b_1u^{p-2}\ldots + (-)^p\,b_{p-3}u^2\},$$

then

$$\begin{aligned}
2p(2p-1)a_1 &= (2p-2)(2p-3)b_1 - p(p-1),\\
2p(2p-1)a_2 &= (2p-4)(2p-5)b_2 - (p-1)(p-2)b_1,\\
2p(2p-1)a_3 &= (2p-6)(2p-7)b_3 - (p-2)(p-3)b_2,\\
&\vdots\\
2p(2p-1)a_{p-3} &= 5\,.\,6\;b_{p-3} - 3\,.\,4\;b_{p-4},\\
0 &= 3\,.\,4\;b_{p-2} - 2\,.\,3\;b_{p-3},
\end{aligned}$$

by means of which the coefficients b can be determined when the coefficients a are known.

Jacobi remarks also that the expressions for the sums of the even powers may be obtained from those for the odd powers by means of the formula

$$Sx^{2p} = \frac{1}{2p+1}\,\partial_x Sx^{2p+1},$$

which shows that any such sum will be of the form $(2x+1)u$ into a rational and integral function of u: thus in particular

$$Sx^2 = \tfrac{1}{6}(2x+1)u.$$

To show *à priori* that Sx^{2p+1} can be expressed as a rational and integral function of u, it may be remarked that $Sx^{2p+1} = \phi_1 x$ where $\phi_1 x$ denotes the summatory integral $\Sigma(x+1)^{2p+1}$, taken so as to vanish for $x=0$: $\phi_1 x$ is a rational and integral function of x of the degree $2p+2$, and which, as is well known, contains x^2 as a factor. Suppose that y is any positive or negative integer less than x, we have

$$\phi_1 x - \phi_1 y = (y+1)^{2p+1} + (y+2)^{2p+1} \ldots + x^{2p+1},$$

and in particular putting $y = -1-x$,

$$\phi_1 x - \phi_1(-1-x) = (-x)^{2p+1} + (1-x)^{2p+1} \ldots + x^{2p+1}, = 0,$$

since the terms destroy each other in pairs; we have therefore $\phi_1 x = \phi_1(-1-x)$. Now $u = x^2 + x$, or writing this equation under the form $x^2 = -x + u$, we see that any rational and integral function of x may be reduced to the form $Px + Q$, where P and Q are rational and integral functions of u. Write therefore $\phi_1 x = Px + Q$: the substitution of $-1-x$ in the place of x leaves u unaltered, and the equation $\phi_1 x = \phi_1(-1-x)$ thus shows that $P = 0$; we have therefore $\phi_1 x = Q$, a rational and integral function of u. Moreover $\phi_1 x$ as containing the factor x^2, must clearly contain the factor u^2, and the expressions for Sx^{2p+1} are thus shown to be of the form given by Jacobi.

We may obtain a finite expression for Sx^n in terms of the differences of 0^n as follows: we have

$$Sx^n = 1^n + 2^n \ldots + x^n = \{(1+\Delta) + (1+\Delta)^2 \ldots + (1+\Delta)^x\}\,0^n = \frac{1+\Delta}{\Delta}\{(1+\Delta)^x - 1\}0^n,$$

and putting $(1+\Delta)^x = e^{x\log(1+\Delta)}$ and observing that the term independent of x vanishes, and that the terms containing powers higher than x^{n+1} also vanish, we have

$$Sx^n = S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{x^k}{\Pi k},$$

where the summation with respect to k, extends from $k=1$ to $k=n+1$, or what is the same thing (since the term corresponding to $k=1$ in fact vanishes) from $k=2$ to $k=n+1$.

The equation $x^2 = -x+u$ gives

$$x^k = P_k x + Q_k,$$

and it is easy to see that writing for shortness

$$M_k = 1 + \frac{k-3}{1}u + \frac{k-4\,.\,k-5}{1\,.\,2}u^2 + \frac{k-5\,.\,k-6\,.\,k-7}{1\,.\,2\,.\,3}u^3 + \ldots,$$

where the series is to be continued to the term $u^{\frac{1}{2}(k-2)}$ or $u^{\frac{1}{2}(k-3)}$) according as k is even or odd, we have

$$P_k = (-)^{k+1}M_{k+1}, \quad Q_k = (-)^k u M_k,$$

we have consequently

$$\begin{aligned} Sx^n = \; & xS_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{(-)^{k+1}M_{k+1}}{\Pi k} \\ & + S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{(-)^k u M_k}{\Pi k}. \end{aligned}$$

If n is odd, $=2p+1$, then (by what precedes) the first term vanishes, or we have

$$S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p+1} \cdot \frac{(-)^{k+1}M_{k+1}}{\Pi k} = 0, \; (k=1 \text{ to } k=2p+2),$$

and the formula becomes

$$Sx^{2p+1} = S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p+1} \cdot \frac{(-)^k u M_k}{\Pi k}, \qquad (k=1 \text{ to } k=2p+2),$$

which it may be noticed puts in evidence the factor u but not the factor u^2.

If n is even, $=2p$, then (by what precedes) the coefficient of x is to the constant term in the ratio 2 : 1, or we have

$$S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p} \cdot \frac{(-)^{k+1}(M_{k+1}-2uM_k)}{\Pi k} = 0, \; (k=1 \text{ to } k=2p+1),$$

and the formula becomes

$$Sx^{2p} = (2x+1)S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p} \cdot \frac{(-)^k u M_k}{\Pi k}, \; (k=1 \text{ to } k=2p+1).$$

The values of the functions M are as follows:

$$\begin{aligned}
M_1 &= 0,\\
M_2 &= 1,\\
M_3 &= 1,\\
M_4 &= 1 + u,\\
M_5 &= 1 + 2u,\\
M_6 &= 1 + 3u + u^2,\\
M_7 &= 1 + 4u + 3u^2,\\
&\&c.
\end{aligned}$$

As a simple example of the formulæ, we have

$$\begin{aligned}
Sx^3 = &\left\{\frac{1+\Delta}{\Delta}\log^2(1+\Delta)\right\} 0^3 . \tfrac{1}{2}u\\
&+\left\{\frac{1+\Delta}{\Delta}\log^3(1+\Delta)\right\} 0^3 . -\tfrac{1}{6}u\\
&+\left\{\frac{1+\Delta}{\Delta}\log^4(1+\Delta)\right\} 0^3 . \tfrac{1}{24}(u+u^2),
\end{aligned}$$

and the coefficients are

$$\begin{aligned}
(\Delta - \tfrac{1}{12}\Delta^3)\, 0^3 &= 1 - \tfrac{1}{12}6 = \tfrac{1}{2},\\
(\Delta^2 - \tfrac{1}{2}\Delta^3)\, 0^3 &= 6 - \tfrac{1}{2}6 = 3,\\
\Delta^3\ 0^3 &= \qquad\qquad 6,
\end{aligned}$$

and therefore

$$Sx^3 = \tfrac{1}{4}u - \tfrac{1}{2}u + \tfrac{1}{4}(u + u^2) = \tfrac{1}{4}u^2,$$

which is right; the example shows however that the calculation for the higher powers would be effected more readily by means of Jacobi's recurring formula.

2, *Stone Buildings*, 27*th Oct.*, 1857.

190.

ON THE SYSTEM OF CONICS WHICH PASS THROUGH THE SAME FOUR POINTS.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 206—207.]

I CONSIDER the system of conics passing through the same four points; these points may be real or imaginary, but it is assumed that there is a real system of conics, this will in fact be the case if two conics of the system are real. The four points are therefore given as the points of intersection of two real conics, and it will be proper to assume in the first instance that the conics intersect in four separate and distinct points, none of them at infinity. The four points may be all real, or two real and two imaginary, or all imaginary.

First, if the points are all real, we have here two cases, viz. each of the points may lie outside of the triangle formed by the other three, or as this may be expressed, the points may form a convex quadrangle; or else one of the points may be inside the triangle formed by the other three, or as this may be expressed, the points may form a triangle and interior point. In each case the pairs of lines joining the points, two and two together, will be conics (degenerate hyperbolas) forming part of the system of conics. Consider the two cases separately.

A

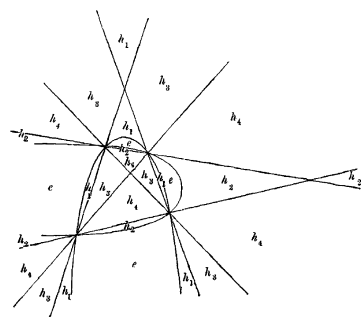
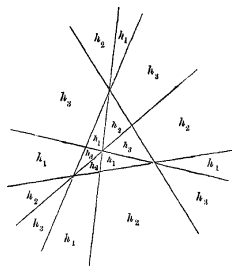


Fig. A. Four real points forming a convex quadrangle. The system contains two parabolas, and the pairs of lines and the parabolas divide the plane of the figure into five distinct regions, one of which contains only ellipses, and the other four contain each of them hyperbolas.

Fig. A′. Four real points forming a triangle and interior point. The system does not contain any parabolas, the three pairs of lines divide the plane of the figure into three distinct regions, each of which contains only hyperbolas.

A′



Next, if the points are two of them real and two of them imaginary. The line joining the two imaginary points will be real and this line may meet the line joining the two real points, in a point outside the two real points, or included between them, i.e. the real centre of the quadrangle may lie outside the real points, or may be included between them; I consider the two cases separately.

Fig. B. Two real and two imaginary points, the real centre of the quadrangle lying outside the real points. The system contains two parabolas, and these with the line joining the two real points and the line joining the two imaginary points divide the plane of the figure into three regions, one of which contains ellipses and the other two contain each of them hyperbolas.

B

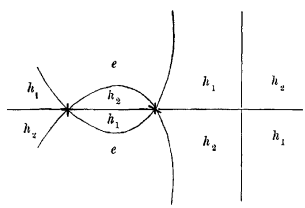
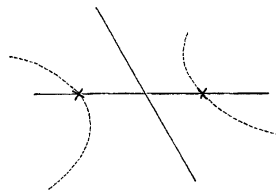


Fig. B′. Two real and two imaginary points, the real centre of the quadrangle lying between the real points. There are no parabolas, and the system contains only hyperbolas.

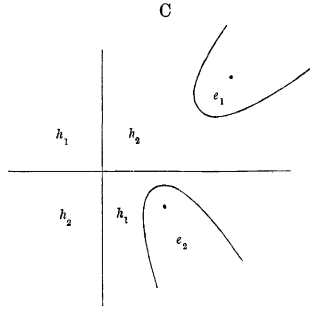
B′



Lastly, when the four points are imaginary. We have here only a single case.

Fig. C. Four imaginary points. The points lie on two real lines, there are (besides the point of intersection of these lines) two other real centres of the quadrangle, which lie harmonically with respect to the two lines. The system contains two parabolas and these and the two lines divide the plane of the figure into four regions, two of which contain each of them ellipses, and the other two contain each of them hyperbolas.

C



191.

NOTE ON THE EXPANSION OF THE TRUE ANOMALY.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 229—232.]

If the true anomaly and the mean anomaly are respectively denoted by u, m, and if e be the eccentricity, then as usual $u - e \sin u = m$; and if we write

$$\lambda = \frac{1 - \sqrt{(1 - e^2)}}{e}$$

and take c to denote the base of the hyperbolic system of logarithms, we have

$$u = m + 2\Sigma_1^\infty A_r \frac{\sin rm}{r},$$

and

$$A_r = \lambda^r c^{-\frac{1}{2}re(\lambda - \lambda^{-1})} + \lambda^{-r} c^{\frac{1}{2}re(\lambda - \lambda^{-1})},$$

where, after expanding the exponentials, the negative powers of λ are to be rejected and the term independent of λ is to be multiplied by $\frac{1}{2}$ (see *Camb. Math. Journal*, t. I. [1839] p. 228 and t. III. [1843] p. 165, [4]).

It is easily seen that e^r is the lowest power of e which enters into the value of A_r and the question arises to find the numerical coefficient of the term in question; this is readily obtained from the formula; in fact considering first a term of the form

$$\lambda^{-r} e^s (\lambda - \lambda^{-1})^s,$$

since λ is itself of the order e, when the negative powers of λ are rejected this is at least of the order e^s and it is consequently to be neglected if $s > r$. But if $s < r$ all the powers of λ are negative and the term is to be rejected. The only case to be

considered is therefore that of $s=r$, in which case there is a term containing e^r. We thus obtain from $\lambda^{-r} c^{\frac{1}{2}re(\lambda-\lambda^{-1})}$ the term

$$\frac{1}{2}\frac{r^r e^r}{2^r.1.2.3\ldots r}.$$

In the next place a term of the form $\lambda^r e^s(\lambda-\lambda^{-1})^s$ is at least of the order e^s if $s>r$, or the terms to be considered are those for which $s=$ or $<r$. But in such term the only part of the order e^r is

$$(-)^s \lambda^{r-s} e^s,$$

or, since neglecting higher powers of e we have $\lambda=\frac{1}{2}e$, this is

$$(-)^s 2^{-r+s} e^r,$$

and the set of terms arising from

$$\lambda^r c^{-\frac{1}{2}re(\lambda-\lambda^{-1})},$$

is

$$\frac{e^r}{2^r}\left\{1+\frac{r}{1}+\frac{r^2}{1.2}\ldots+\frac{r^{r-1}}{1.2\ldots(r-1)}+\frac{1}{2}\frac{r^r}{1.2\ldots r}\right\},$$

the last term being divided by 2 because arising from a term independent of λ. Hence the first term of A_r is

$$\frac{e^r}{2^r}\left\{1+\frac{r}{1}+\frac{r^2}{1.2}\ldots+\frac{r^r}{1.2\ldots r}\right\},$$

a result which it may be remarked is contained in the general formula given in Hansen's Memoir "Entwickelung des Products u. s. w.," *Leipzig Trans.*, t. II. p. 277 (1853).

The preceding expression is

$$=\frac{e^r c^r}{2^r}\frac{1}{\Gamma(r+1)}\int_r^\infty x^r c^{-x}\,dx,$$

and to find its value when r is large, we have

$$\int_r^\infty x^r c^{-x}\,dx=\int_0^\infty (y+r)^r e^{-y-r}\,dy=r^r c^{-r}\int_0^\infty\left(1+\frac{y}{r}\right)^r e^{-y}\,dy$$

$$=r^r c^{-r}\int_0^\infty c^{-y+r\log\left(1+\frac{y}{r}\right)}\,dy$$

$$=r^r c^{-r}\int_0^\infty c^{-\frac{y^2}{2r}+\frac{y^3}{3r^2}-\&c.}\,dy$$

$$=r^r c^{-r}\int_0^\infty\left(1+\frac{y^3}{3r^2}+\ldots\right)e^{-\frac{y^2}{2r}}\,dy$$

$$=r^r c^{-r}\sqrt{2r}\int_0^\infty\left(1+\frac{2\sqrt{2}}{3\sqrt{r}}z^3+\ldots\right)e^{-z^2}\,dz,$$

or neglecting all the terms except the first, this is

$$= r^r c^{-r} \sqrt{2r} \int_0^\infty e^{-z^2}\, dz$$

$$= \sqrt{2\pi r}\, r^r c^{-r}.$$

Hence multiplying by $\frac{1}{2^r} e^r c^r \frac{1}{\Gamma(r+1)}$ and observing that when r is large, we have, by a well-known formula,

$$\Gamma(r+1) = \sqrt{2\pi r}\, r^r c^{-r},$$

we obtain finally the result that when r is large the first term of A_r is approximately

$$= \left(\frac{ec}{2}\right)^r.$$

I take the opportunity of mentioning the following somewhat singular theorem, which seems to belong to a more general theory: viz. if $u - e \sin u = m$, then we have

$$\log(1 - e\cos u) = \frac{1}{\alpha} \log(1 - \alpha e \cos\phi),$$

where

$$\phi - \frac{1}{\alpha} \tan\phi = m,$$

provided that the negative powers of α are rejected, and α is then put equal to unity.

To show this, we have by Lagrange's theorem, observing that

$$\frac{d}{dm} F(1 - e\cos m) = e \sin m\, F'(1 - e\cos m),$$

$$F(1 - e\cos u) = F(1 - e\cos m) + \frac{e^2}{1} \sin^2 m\, F'(1 - e\cos m)$$

$$+ \frac{e^3}{1.2} \frac{d}{dm} \sin^3 m\, F'(1 - e\cos m) + \&c.,$$

and the coefficient of e^r in $F(1 - e\cos u)$ is

$$\frac{(-)^r}{1.2\ldots(r-1)} \left\{ \frac{1}{r} F_r \cos^r m + \frac{r-1}{1} F_{r-1} \cos^{r-2} m \sin^2 m \right.$$

$$\left. - \frac{(r-1)(r-2)}{1.2} F_{r-2} \frac{d}{dm} (\cos^{r-3} m \sin^3 m) + \&c. \right\},$$

where $F_r = F^r(1)$.

Hence in particular when $Fx = \log x$, $F_r = (-)^{r-1}\, 1 \,.\, 2 \ldots (r-1)$ and thence the coefficient of e^r in $\log(1 - e\cos u)$ is

$$-\left\{\frac{1}{r}\cos^r m - \frac{1}{1}\cos^{r-2} m \sin^2 m - \frac{1}{1\,.\,2}\frac{d}{dm}(\cos^{r-3} m \sin^2 m) - \&c.\right\},$$

continued as long as the exponent of $\cos m$ is not negative. Now in the expansion of $\frac{1}{\alpha}\log(1 - \alpha e \cos\phi)$, where $\phi - \frac{1}{\alpha}\tan\phi = m$, the coefficient of e^r is $-\frac{1}{r}\alpha^{r-1}\cos^r\phi$, and this (by Lagrange's theorem) is equal to

$$-\frac{1}{r}\alpha^{r-1}\left\{\cos^r m - \frac{1}{1\,.\,\alpha}\, r\cos^{r-1} m \sin m \tan m - \frac{1}{1\,.\,2\,.\,\alpha^2}\frac{d}{dm}(r\cos^{r-1} m \sin m \tan^2 m) - \&c.\right\}$$

$$= -\left\{\frac{1}{r}\alpha^{r-1}\cos^r m - \frac{1}{1}\alpha^{r-2}\cos^{r-2} m \sin^2 m - \frac{1}{1\,.\,2}\alpha^{r-3}\cos^{r-3} m \sin^3 m - \&c.\right\},$$

where the series is continued indefinitely; but if we reject the negative powers of α and then put α equal to unity this is precisely equal to the former expression for the coefficient of e^r, and the formula is thus shown to be true.

2, *Stone Buildings, W.C.*, 17*th Nov.*, 1857.

192.

ON THE AREA OF THE CONIC SECTION REPRESENTED BY THE GENERAL TRILINEAR EQUATION OF THE SECOND DEGREE.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 248—253.]

[THE original title was "Direct Investigation of the Question discussed in the Foregoing Paper," viz. a Paper with the present title by N. M. Ferrers (now Dr Ferrers), pp. 247—248. The area S of the conic section represented by the general equation $(A, B, C, A', B', C' \between x, y, z)^2 = 1$, where the coordinates are connected by the equation $x + y + z = 1$, was by considerations founded on the form of the function found to be

$$S = \frac{2\pi\,(AA'^2 + BB'^2 + CC'^2 - ABC - 2A'B'C')\,\Delta}{\{A'^2 - BC + B'^2 - CA + C'^2 - AB + 2\,(B'C' - AA') + 2\,(C'A' - BB') + 2\,(A'B' - CC')\}^{\frac{1}{2}}},$$

where Δ is the area of the fundamental triangle: and it was remarked that a similar method might be applied to determine the area of the conic section when it is defined by the distances of its several tangents from three given points.]

The position of a point P being determined as in the foregoing paper, let α, β, γ denote in like manner the coordinates of a point O, we have

$$\alpha + \beta + \gamma = 1,$$

and consequently if ξ, η, ζ are the relative coordinates $x - \alpha$, $y - \beta$, $z - \gamma$, we have

$$\xi + \eta + \zeta = 0.$$

The expression for the distance of the two points O, P is readily obtained in terms of the relative coordinates, viz. calling this distance r, we have

$$r^2 = L\xi^2 + M\eta^2 + N\zeta^2,$$

where, if l, m, n are the sides of the triangle ABC, we have

$$L = \tfrac{1}{2}(m^2 + n^2 + l^2),$$
$$M = \tfrac{1}{2}(n^2 + l^2 - m^2),$$
$$N = \tfrac{1}{2}(l^2 + m^2 - n^2);$$

and it is to be remarked that these values give

$$MN + NL + LM = \tfrac{1}{4}(2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4), = 4\Delta^2,$$

if Δ denote the area of the triangle ABC.

Consider now a conic

$$(a, b, c, f, g, h \between x, y, z)^2,$$

and suppose as usual that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ are the inverse coefficients and that K is the discriminant, suppose also for shortness

$$P = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between 1, 1, 1)^2.$$

The coordinates of the centre being α, β, γ, we have

$$\alpha = \frac{1}{P}(\mathfrak{A}, \mathfrak{H}, \mathfrak{G} \between 1, 1, 1),$$

$$\beta = \frac{1}{P}(\mathfrak{H}, \mathfrak{B}, \mathfrak{F} \between 1, 1, 1),$$

$$\gamma = \frac{1}{P}(\mathfrak{G}, \mathfrak{F}, \mathfrak{C} \between 1, 1, 1),$$

and writing as before ξ, η, ζ for $x-\alpha$, $y-\beta$, $z-\gamma$, so that ξ, η, ζ are the coordinates of a point P of the conic, in relation to the centre, we have x, y, z respectively equal to $\xi+\alpha$, $\eta+\beta$, $\zeta+\gamma$, and the equation of the conic gives

$$(a, \ldots \between \xi+\alpha, \eta+\beta, \zeta+\gamma)^2 = 0,$$

which may be written

$$\begin{aligned} &(a, \ldots \between \xi, \eta, \zeta)^2 \\ &+ 2(a, \ldots \between \alpha, \beta, \gamma)(\xi, \eta, \zeta) \\ &+ (a, \ldots \between \alpha, \beta, \gamma)^2 = 0. \end{aligned}$$

Now observing the equations

$$(a,\ h,\ g \between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

$$(h,\ b,\ f \between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

$$(g,\ f,\ c \between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

we have

$$(a,\ \ldots \between \alpha,\ \beta,\ \gamma)\ (\xi,\ \eta,\ \zeta) = \frac{K}{P}(\xi + \eta + \zeta) = 0,$$

$$(a,\ \ldots \between \alpha,\ \beta,\ \gamma)^2 \qquad = \frac{K}{P}(\alpha + \beta + \gamma) = \frac{K}{P};$$

and the equation of the conic gives therefore

$$(a,\ \ldots \between \xi,\ \eta,\ \zeta)^2 + \frac{K}{P} = 0,$$

and we have as before

$$\xi + \eta + \zeta = 0.$$

To find the axes we have only to make

$$r^2, = L\xi^2 + M\eta^2 + N\zeta^2,$$

a maximum or minimum, ξ, η, ζ varying subject to the preceding two conditions; this gives

$$(a,\ h,\ g \between \xi,\ \eta,\ \zeta) + \lambda L\xi + \mu = 0,$$

$$(h,\ b,\ f \between \xi,\ \eta,\ \zeta) + \lambda M\eta + \mu = 0,$$

$$(g,\ f,\ c \between \xi,\ \eta,\ \zeta) + \lambda N\zeta + \mu = 0,$$

and multiplying by ξ, η, ζ, adding and reducing, we have

$$-\frac{K}{P} + \lambda r^2 = 0,$$

which gives

$$\lambda = \frac{K}{Pr^2}.$$

Substituting this value, and joining to the resulting three equations the equation

$$\xi + \eta + \zeta = 0,$$

we may eliminate ξ, η, ζ, μ, and the result is

$$\begin{vmatrix} a+\frac{KL}{Pr^2}, & h, & g, & 1 \\ h, & b+\frac{KM}{Pr^2}, & f, & 1 \\ g, & f, & c+\frac{KN}{Pr^2}, & 1 \\ 1, & 1, & 1 & \end{vmatrix} = 0,$$

which may also be written

$$(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}' \between 1, 1, 1)^2 = 0,$$

where $(\mathfrak{A}', \ldots)$ are what $(\mathfrak{A}, \ldots)$ become when a, b, c are changed into

$$a+\frac{KL}{Pr^2}, \quad b+\frac{KM}{Pr^2}, \quad c+\frac{KN}{Pr^2};$$

we in fact have

$$\mathfrak{A}' = \mathfrak{A} + \frac{K}{Pr^2}(bN+cM) + \frac{K^2}{P^2r^4}MN,$$

$$\vdots$$

$$\mathfrak{F}' = \mathfrak{F} - \frac{K}{Pr^2}Lf,$$

$$\vdots$$

and (observing the value of P) the result consequently is

$$P + \frac{K}{Pr^2}\{(b+c-2f)L + (c+a-2g)M + (a+b-2h)N\} + \frac{K^2}{P^2r^4}(MN+NL+LM) = 0,$$

which may also be written

$$P^2r^4 + PKr^2\{(b+c-2f)L + (c+a-2g)M + (a+b-2h)N\} + 4\Delta^2K^2 = 0.$$

Hence if r_1, r_2 are the two semiaxes, we have

$$r_1^2r_2^2 = \frac{4\Delta^2K^2}{P^3},$$

and the area is $\pi r_1 r_2$ which is equal to

$$\frac{2\pi K\Delta}{\sqrt{(P^3)}},$$

which agrees with Mr Ferrers' result.

The formula $r^2 = L\xi^2 + M\eta^2 + N\zeta^2$ which is assumed in the preceding investigation may be proved as follows:

Writing a, b, c (instead of l, m, n) for the sides of the fundamental triangle and A, B, C for the angles, the equation in question is

$$r^2 = bc \cos A\, \xi^2 + ca \cos B\, \eta^2 + ab \cos C\, \zeta^2.$$

Now writing α, β, γ for the inclinations of the line r to the sides of the triangle, we have

$$A = \beta - \gamma,$$

$$B = \gamma - \alpha,$$

$$C = \pi + \alpha - \beta.$$

Moreover taking for a moment λ, μ, ν to denote the perpendiculars from the angles on the opposite sides, we have

$$\lambda = c \sin B = b \sin C,$$

$$\mu = a \sin C = c \sin A,$$

$$\nu = b \sin A = a \sin B,$$

and

$$\xi = \frac{r \sin \alpha}{\lambda}, \quad \eta = \frac{r \sin \beta}{\mu}, \quad \zeta = \frac{r \sin \gamma}{\nu};$$

the values of ξ^2, η^2, ζ^2 consequently are

$$\frac{r^2 \sin^2 \alpha}{bc \sin B \sin C}, \quad \frac{r^2 \sin^2 \beta}{ca \sin C \sin A}, \quad \frac{r^2 \sin^2 \gamma}{ab \sin A \sin B},$$

and the equation to be proved becomes

$$1 = \frac{\cos A \sin^2 \alpha}{\sin B \sin C} + \frac{\cos B \sin^2 \beta}{\sin C \sin A} + \frac{\cos C \sin^2 \gamma}{\sin A \sin B},$$

or, what is the same thing,

$$\sin A \sin B \sin C = \sin A \cos A \sin^2 \alpha + \sin B \cos B \sin^2 \beta + \sin C \cos C \sin^2 \gamma,$$

or again

$$4 \sin A \sin B \sin C = \sin 2A\,(1 - \cos 2\alpha) + \sin 2B\,(1 - \cos 2\beta) + \sin 2C\,(1 - \cos 2\gamma),$$

or putting for A, B, C their values in terms of α, β, γ this is

$$\begin{aligned} -4 \sin(\beta - \gamma) \sin(\gamma - \alpha) \sin(\alpha - \beta) = {} & \sin(2\beta - 2\gamma)(1 - \cos 2\alpha) \\ & + \sin(2\gamma - 2\alpha)(1 - \cos 2\beta) \\ & + \sin(2\alpha - 2\beta)(1 - \cos 2\gamma), \end{aligned}$$

which is an identical equation; it is most readily proved by writing x, y, z for $\tan\alpha$, $\tan\beta$, $\tan\gamma$; the equation thus becomes

$$\frac{-4}{(1+x^2)(1+y^2)(1+z^2)}(y-z)(z-x)(x-y)$$
$$= \Sigma \frac{1}{(1+y^2)(1+z^2)}\{2y(1-z^2)-2z(1-y^2)\}\frac{2x^2}{1+x^2},$$

or multiplying out

$$-(y-z)(z-x)(x-y) = \Sigma(y-z)(1+yz)x^2 = \Sigma x^2(y-z) + xyz\,\Sigma x(y-z),$$

that is

$$-(y-z)(z-x)(x-y) = \Sigma x^2(y-z) \qquad = x^2(y-z)+y^2(z-x)+z^2(x-y),$$

which is an identity.

[A different investigation of the formula $r^2 = L\xi^2 + M\eta^2 + N\zeta^2$, by Dr Ferrers, was appended to the original Paper.]

193.

ON RODRIGUES' METHOD FOR THE ATTRACTION OF ELLIPSOIDS.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 333—337.]

THE following is in substance the method given in the "Mémoire sur l'attraction des Spheroïdes," par M. Rodrigues, *Corresp. sur l'École Polyt.*, t. III. pp. 361—385 (1815). It will be seen that the method is very similar to that given two years before by Gauss, see my paper "On Gauss' Method for the Attraction of Ellipsoids," *Journal*, t. I. pp. 162—166 [164]: the solution in fact depends upon the geometrical theorem therein quoted, viz. if M be any point, P a point of a closed surface, PQ the normal (lying outside the surface) at the point P, dS the element of the surface at that point, and if MQ denotes the angle MPQ and $\overline{MP}$ the distance of the points M and P, then, theorem, the integral

$$\iint \frac{dS \cos MQ}{\overline{MP}^2}$$

has for its value

$$0,\ -2\pi \text{ or } -4\pi$$

according as M is exterior to, upon, or interior to the surface.

Suppose that M is the attracted point and taking A, B, C for the semiaxes of the surface of the attracting ellipsoid, or, if we please, for any semiaxes of an arbitrary ellipsoidal surface confocal with the surface of the attracting ellipsoid, let P be a point on the surface of the interior similar ellipsoid whose semiaxes are rA, rB, rC. The coordinates of M are taken to be a, b, c, and those of P are taken to be x, y, z, and the value of the potential is

$$V = \int \frac{dm}{\overline{MP}},$$

where dm is the element of mass.

We may write

$$x = rA\xi,$$
$$y = rB\eta,$$
$$z = rC\zeta,$$

and then ξ, η, ζ will be the coordinates of a point P' on the surface of a sphere, radius unity, corresponding in a definite manner to the point P on the surface of the internal similar ellipsoid. And if $d\sigma$ be the element of the spherical surface, then we have

$$dm = ABCr^2 dr\, d\sigma,$$

and therefore

$$\frac{V}{ABC} = \int \frac{r^2 dr\, d\sigma}{\overline{MP}};$$

where, in order to obtain the value of the potential V for the ellipsoid whose semi-axes are A, B, C, the integrations must be extended over the spherical surface and from $r = 0$ to $r = 1$.

Suppose that dS is the element of the internal similar surface at P, and let p be the perpendicular from the centre upon the tangent plane at P, we have

$$dS = \frac{r^3 ABC}{p} d\sigma.$$

Let P_0 be the point on the ellipsoid (A, B, C) similarly situated to the point P on the ellipsoid (rA, rB, rC); the coordinates of P_0 are $A\xi$, $B\eta$, $C\zeta$; and if p_0 be the perpendicular from the centre upon the tangent plane at P_0, then $p = rp_0$, and the preceding equation becomes

$$dS = \frac{r^2 ABC}{p_0} d\sigma.$$

Imagine now an ellipsoidal surface confocal with the surface (A, B, C) and having for its semiaxes

$$A + \delta A, \quad B + \delta B, \quad C + \delta C,$$

and let P_0' be the point on this surface which *corresponds* with the point P_0 on the surface (A, B, C); that is, let P_0' be the point whose coordinates are

$$(A + \delta A)\xi, \quad (B + \delta B)\eta, \quad (C + \delta C)\zeta;$$

and let P' be in like manner the point whose coordinates are

$$r(A + \delta A)\xi, \quad r(B + \delta B)\eta, \quad r(C + \delta C)\zeta;$$

the points P, P' will be in like manner *corresponding* points on the surface (rA, rB, rC) and on the confocal surface $\{r(A+\delta A), r(B+\delta B), r(C+\delta C)\}$; and if the normal distance at the point P_0 of the first two surfaces is δN, then the normal distance at the point P of the second two surfaces will be $r\delta N$. The decrement of $\overline{MP}$ will be equal to the normal distance $r\delta N$ of the two surfaces at the point P multiplied into the cosine of the angle MQ, and we have, by a property of confocal surfaces,

$$A\delta A = B\delta B = C\delta C = p_0\delta N = (\text{suppose})\ \tfrac{1}{2}\delta\theta,$$

we have therefore

$$d\overline{MP} = -\frac{\frac{1}{2}r\delta\theta}{p_0}\cos MQ.$$

Hence from the equation

$$\frac{V}{ABC} = \int \frac{r^2 dr\, d\sigma}{\overline{MP}}.$$

We deduce

$$\delta\frac{V}{ABC} = \int r^2 dr\, d\sigma\, \frac{\frac{1}{2}r\delta\theta}{P_0}\,\frac{\cos MQ}{\overline{MP}^2}.$$

But we have

$$\frac{r^2 d\sigma}{p_0} = \frac{dS}{ABC};$$

and the equation thus becomes

$$\delta\frac{V}{ABC} = \frac{\frac{1}{2}\delta\theta}{ABC}\int r dr\,\frac{dS\cos MQ}{\overline{MP}^2}.$$

It may be proper to remark here by way of recapitulation that the course of the investigation has been as follows: viz. that, with a view to obtaining the potential V of an attracting ellipsoid, we have found the increment of $\frac{V}{ABC}$ in passing from the ellipsoidal surface (A, B, C) to the ellipsoidal surface $(A+\delta A, B+\delta B, C+\delta C)$, each of them *confocal* with the surface of the attracting ellipsoid; and that for finding such increment we have had to consider the two surfaces (rA, rB, rC) and $\{r(A+\delta A)$ $r(B+\delta B), r(C+\delta C)\}$ confocal to each other and respectively *similar* to the first-mentioned two surfaces.

Resuming the formula just obtained, the integral with respect to dS is taken over the entire surface of the internal similar ellipsoid (rA, rB, rC), and if the attracted point M is external to the ellipsoid (A, B, C) it will be external to the interior similar ellipsoid (rA, rB, rC): hence in this case the double integral vanishes for all values of r, or we have

$$\delta\frac{V}{ABC} = 0;$$

that is the function $V \div ABC$, which represents the ratio of the potential to the mass, is not altered in passing from the ellipsoid (A, B, C) to the confocal ellipsoid

$$(A+\delta A,\quad B+\delta B,\quad C+\delta C),$$

or, what is the same thing, the potentials (and therefore the attractions) of confocal ellipsoids upon the same external point are proportional to their masses; this is in fact Maclaurin's theorem for the attraction of ellipsoids upon an external point.

But if the attracted point M is interior to the ellipsoid (A, B, C), then writing

$$\frac{a^2}{A^2}+\frac{b^2}{B^2}+\frac{c^2}{C^2}=r'^2,$$

where r' is less than unity, the double integral is $=0$ from $r=0$ to $r=r'$ and is $=-4\pi$ from $r=r'$ to $r=1$, and we have

$$\begin{aligned}\delta\frac{V}{ABC} &= -\frac{2\pi\delta\theta}{ABC}\int_{r'}^{1} r\,dr\\ &= -\frac{\pi\delta\theta}{ABC}(1-r'^2)\\ &= \frac{\pi\delta\theta}{ABC}\left(\frac{a^2}{A^2}+\frac{b^2}{B^2}+\frac{c^2}{C^2}-1\right);\end{aligned}$$

that is, the right-hand side of the equation is the increment (or taken with its sign reversed so as to be positive, it is the decrement) of the function $V \div ABC$ in passing from the ellipsoid (A, B, C) to the confocal ellipsoid

$$(A+\delta A,\quad B+\delta B,\quad C+\delta C),$$

where

$$\tfrac{1}{2}\delta\theta = A\delta A = B\delta B = C\delta C.$$

The preceding formula gives at once the potential for an interior point; in fact taking α, β, γ for the semiaxes of the ellipsoid and writing

$$A^2=\alpha^2+\theta,\quad B^2=\beta^2+\theta,\quad C^2=\gamma^2+\theta,$$

and using the ordinary symbol d instead of δ, we have

$$\frac{d}{d\theta}\frac{V}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}=\frac{\pi}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\},$$

and integrating from $\theta=0$ to $\theta=\infty$, we have

$$-\frac{V}{\alpha\beta\gamma}=\pi\int_0^\infty\frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\},$$

where V is now the potential for the ellipsoid whose semiaxes are α, β, γ; and we have therefore

$$V = -\pi\alpha\beta\gamma\int_0^\infty \frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\}.$$

To find the potential for an external point it is only necessary to remark that by the theorem above demonstrated $V \div \alpha\beta\gamma$ is equal to the corresponding function for the confocal ellipsoid through the attracted point, that is for the ellipsoid whose semiaxes are $\sqrt{(\alpha^2+\theta_1)}$, $\sqrt{(\beta^2+\theta_1)}$, $\sqrt{(\gamma^2+\theta_1)}$, where θ_1 is a positive quantity such that

$$\frac{a^2}{\alpha^2+\theta_1}+\frac{b^2}{\beta^2+\theta_1}+\frac{c^2}{\gamma^2+\theta_1}=1,$$

hence in the value of $V \div \alpha\beta\gamma$ we have only to write the above values in the place of α, β, γ; and if we then write $\theta-\theta_1$ in the place of θ the limits will be ∞, θ_1, and the expression for the potential is

$$V = -\pi\alpha\beta\gamma\int_{\theta_1}^\infty \frac{d\theta}{\sqrt{\{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)\}}}\left\{\frac{a^2}{\alpha^2+\theta}+\frac{b^2}{\beta^2+\theta}+\frac{c^2}{\gamma^2+\theta}-1\right\};$$

this completes the investigation.

194.

NOTE ON THE THEORY OF ATTRACTION.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 338—339.]

IMAGINE a closed surface, the equation of which contains the two parameters m, h. Call this the surface (m, h), and suppose also that for shortness the shell of uniform density included between the surfaces (m, h), (n, h) is called the shell (m, n, h). Suppose now that the surface is such:

1°. That the infinitesimal shell $(m, m+dm, h)$ exerts no attraction upon an internal point.

2°. That the equipotential surfaces of the shell in question for external points are the surfaces (m, k), where k is arbitrary.

Then, first, the attraction of the shell on a point of the equipotential surface (m, k) is proportional to the normal thickness at that point of the shell $(m, m+\delta m, k)$; or (more precisely) taking the density of the attracting shell as unity, the attraction is $= 4\pi \times$ mass of shell $(m, m+dm, h)$ into normal thickness of shell $(m, m+\delta m, k)$ divided by mass of the last-mentioned shell.

In fact the shell $(m, m+\delta m, k)$ exerts no attraction on an internal point, consequently if over the surface (m, k) we distribute the mass of the original shell $(m, m+dm, h)$ in such manner that the density at any point is proportional to the normal thickness of the shell $(m, m+\delta m, k)$ the distribution will be such that the attraction on an internal point may vanish; but in order that this may be the case, the density must be equal to $\frac{1}{4\pi}$ into the attraction upon that point of the shell $(m, m+\delta m, k)$. Hence the attraction is proportional to the normal thickness, and if the whole mass distributed over the surface (m, k) is precisely equal to the mass of the shell $(m, m+dm, h)$, then the density at any point must be equal to the mass

into normal thickness divided by mass of $(m, m+\delta m, k)$, and attraction $=4\pi$ into density, $=4\pi\times$ mass of shell $(m, m+dm, h)$ into normal thickness of shell $(m, m+\delta m, k)$ divided by mass of the last-mentioned shell.

And, secondly, the attractions of the solids bounded by the two surfaces (n, h), (n, h_1) respectively upon the same exterior point are proportional to their masses.

For the solid (n, h) may be divided into shells $(m, m+dm, h)$ and for this shell the equipotential surface is (m, k) and the attraction of the shell varies as mass of $(m, m+dm, h)$ into normal thickness of the shell $(m, m+\delta m, k)$. But in like manner the solid (n, h_1) may be divided into shells $(m, m+dm, h_1)$ and the attraction of the shell varies as mass of $(m, m+dm, h_1)$ into normal thickness of the shell $(m, m+\delta m, k)$ and the attractions are in each case in the direction normal to the shell (m, k), and therefore in the same direction; that is, the attraction of the shell $(m, m+dm, h)$ is in the same direction as that of the shell $(m, m+dm, h_1)$ and the two attractions are proportional to the masses. Hence integrating from $m=0$ (if for this value the included space is zero) to $m=n$, the attractions of the solids (n, h), (n, h_1) are composed of elements proportional and parallel, the elements of the attraction of (n, h) to the elements of the attraction of (n, h_1); and consequently the total attractions are in the same direction and proportional to the masses.

Thirdly, the attractions of the two surfaces (m, h), (n, h) upon the same interior point are equal.

A surface having the properties in question is of course the ellipsoidal surface

$$\frac{x^2}{m(a^2+h)}+\frac{y^2}{m(b^2+h)}+\frac{z^2}{m(c^2+h)}=1,$$

where if m varies (h being constant) the several surfaces are similar to each other, but if h varies (m being constant) the several surfaces are confocal to each other: for it is in fact well known that the infinitesimal shell bounded by similar ellipsoidal surfaces has the properties assumed with respect to the shell $(m, m+dm, h)$. The first theorem in effect reduces the problem of the determination of an ellipsoid upon an exterior point to a single integration, and constitutes the foundation of Poisson's method for the attraction of ellipsoids. The second theorem (Maclaurin's theorem for the attraction of ellipsoids on the same external point) shows that the attraction of an ellipsoid upon an external point can be found by means of the attraction of the confocal ellipsoid through the attracted point; and by the third theorem the attraction of an ellipsoid upon an interior point is equal to that of the similar ellipsoid through the attracted point; hence the second and third theorems reduce the determination of the attraction of an ellipsoid upon an external or internal point to that of an ellipsoid upon a point on the surface.

2, *Stone Buildings, W.C., 7th April,* 1858.

195.

REPORT ON THE RECENT PROGRESS OF THEORETICAL DYNAMICS.

[From the *Report of the British Association for the Advancement of Science*, 1857, pp. 1—42.]

THE object of the *Mécanique Analytique* of Lagrange is described by the author in the "Avertissement" to the first edition as follows:—"On a déjà plusieurs traités de mécanique, mais le plan de celui-ci est entièrement neuf. Je me suis proposé de réduire la théorie de cette science et l'art de résoudre tous les problêmes qui s'y rapportent à des formules générales dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problême." And the intention is carried out; the principle of virtual velocities furnishes the general formulæ for the solution of statical problems, and d'Alembert's principle then leads to the general formulæ for the solution of dynamical problems. The general theory of statics would seem to admit of less ulterior development; but as regards dynamics, the formulæ of the first edition of the *Mécanique Analytique* have been the foundation of a series of profound and interesting researches constituting the science of analytical dynamics. The present report is designed to give, so far as I am able, a survey of these researches; there will be found at the end a list, in chronological order, of the works and memoirs referred to, and I shall in the course of the report preserve as far as possible the like chronological order. It is proper to remark that I confine myself to the general theories of dynamics. There are various *special problems* of great generality, and susceptible of the most varied and extensive developments, such for instance as the problem of the motion of a single particle (which includes as particular cases the problem of central forces, that of two fixed centres, and that of the motion of a conical pendulum, either with or without regard to the motion of the earth round its axis), the problem of three bodies, and the problem of the rotation of a solid body about a fixed point. But a detailed account of the researches of geometers in relation to these special problems would properly form the subject of a separate report, and it is not my intention to enter upon them otherwise than incidentally, so far as it may appear desirable to do so. One problem, however, included in the first of the above-

mentioned special problems, I shall have frequent occasion to allude to: I mean the problem of the variation of the elements of a planet's orbit, which has a close historical connexion with the general theories which form the subject of this report. The so-called ideal coordinates of Hansen, and the principles of his method of integration in the planetary and lunar theories, have a bearing on the general subject, and might have been considered in the present report; but on the whole I have considered it better not to do so.

1. Lagrange, *Mécanique Analytique*, 1788.—The equations of motion are obtained, as before mentioned, by means of the principle of virtual velocities and d'Alembert's principle. In their original forms they involve the coordinates x, y, z of the different particles m or dm of the system, quantities which in general are not independent. But Lagrange introduces, in place of the coordinates x, y, z of the different particles, any variables or (using the term in a general sense) coordinates ξ, ψ, ϕ, ... whatever, determining the position of the system at the time t: these may be taken to be independent, and then if ξ', ψ', ϕ', ... denote as usual the differential coefficients of ξ, ψ, ϕ, ... with respect to the time, the equations of motion assume the form

$$\frac{d}{dt}\frac{dT}{d\xi'}-\frac{dT}{d\xi}+\Xi=0;$$
$$\vdots$$

or when Ξ, Ψ, Φ, ... are the partial differential coefficients with respect to ξ, ψ, ϕ, ... of one and the same function V, then the form

$$\frac{d}{dt}\frac{dT}{d\xi}-\frac{dT}{d\xi'}+\frac{dV}{d\xi}=0.$$
$$\vdots$$

In these equations, T, or the *vis viva* function, is the *vis viva* of the system, or sum of all the elements each into the half square of its velocity, expressed by means of the coordinates ξ, ψ, ϕ, ...; and (when such function exists) V, or the force function[1], is a function depending on the impressed forces and expressed in like manner by means of the coordinates ξ, ψ, ϕ, ...; the two functions T and V are given functions, by means of which the equations of motion for the particular problem in hand are completely expressed. In any dynamical problem whatever, the *vis viva* function T is a given function of the coordinates ξ, ψ, ϕ, ..., of their differential coefficients ξ', ψ', ϕ', ... and of the time t; and it is of the second order in regard to the differential coefficients ξ', ψ', ϕ', ...; and (when such function exists) the force function V is a given function of the coordinates ξ, ψ, ϕ, ... and of the time t. This is the most general form of the functions T, V, as they occur in dynamical problems, but in an extensive class of such problems the forms are less general, viz. T and V are each of them independent of the time, and T is a homogeneous function of the second order in regard to the differential coefficients ξ', ψ', ϕ', ...; the equations of motion

[1] The sign attributed to V is that of the *Mécanique Analytique*, but it would be better to write $V=-U$, and to call U (instead of V) the force function.

have in this case an integral $T+V=h$, which is the equation of *vis viva*, and the problems are distinguished as those in which the principle of *vis viva* holds good. It is to be noticed also that in this case since t does not enter into the differential equations, the integral equations will contain t in the form $t+c$, that is, in connexion with an arbitrary constant c attached to it by addition.

2. The above-mentioned form is *par excellence* the Lagrangian form of the equations of motion, and the one which has given rise to almost all the ulterior developments of the theory; but it is proper just to refer to the form in which the equations are in the first instance obtained, and which may be called the unreduced form, viz. the equations for the motion of a particle whose rectangular coordinates are x, y, z, are

$$m\frac{d^2x}{dt^2}=X+\lambda\frac{dL}{\delta x}+\mu\frac{dM}{\delta x}+\ldots$$

$$\vdots$$

where $L=0$, $M=0, \ldots$ are the equations of condition connecting the coordinates of the different points of the system, and λ, $\mu, \ldots$ are indeterminate multipliers.

3. The idea of a force function seems to have originated in the problems of physical astronomy. Lagrange, in a memoir "On the Secular Equation of the Moon," crowned by the French Academy of Sciences in the year 1774, expressed the attractive forces, decomposed in the directions of the axes of coordinates, by the partial differential coefficients of one and the same function with respect to these coordinates. And it was in these problems natural to distinguish the forces into principal and disturbing forces, and thence to separate the force function into two parts, a principal force function and a disturbing function. The problems of physical astronomy led also to the idea of the variation of the arbitrary constants of a mechanical problem. For as a fact of observation the planets move in ellipses the elements of which are slowly varying; the motion in a fixed ellipse was accounted for by the principal force, the attraction of the sun; the effect of the disturbing force is to produce a continual variation of the elements of such elliptic orbit. Euler, in a memoir published in 1749 in the *Memoirs of the Academy of Berlin* for that year, obtained differential equations of the first order for two of the elements, viz. the inclination and the longitude of the node, by making the arbitrary constants which express these elements in the fixed orbit to vary: this seems to be the first attempt at the method of the variation of the arbitrary constants. Euler afterwards treated the subject in a more complete manner, and the method is also made use of by Lagrange in his "Memoir on the Perturbations of the Planets" in the Berlin Memoirs for 1781, 1782, 1783, and by Laplace in the *Mécanique Céleste*, t. I. 1799. The method in its original form seeks for the expressions of the variations of the elements in terms of the differential coefficients of the disturbing function *with respect to the coordinates*. As regards one element, the longitude of the epoch, such expression (at least in a finite form) was first obtained by Poisson in his memoir of 1808, to be spoken of presently; but I am not able to refer to any place where such expressions in their best form are even now to be found; the question seems to have been unduly passed over in consequence of the new form immediately afterwards assumed by the method. It was very early

observed that the variation of one of the elements, viz. the mean distance, was expressible in a remarkable form by means of the differential coefficients of the disturbing function *taken with respect to the time t, in so far as it entered into the function through the coordinates of the disturbed planet.* I am not able to say at what time, or whether by Euler, Lagrange, or Laplace, it was observed that such differential coefficient with respect to the time was equivalent to the differential coefficient of the disturbing function with respect to one of the elements. But however this may be, the notion of the representation of the variations of the elements by means of the differential coefficients of the disturbing function *with respect to the elements* had presented itself *à posteriori*, and was made use of in an irregular manner prior to the year 1800, and therefore some eight years at any rate before the establishment by Lagrange of the general theory to which these forms belong.

4. Poisson's memoir of the 20th of June, 1808, "On the Secular Inequalities of the Mean Motion of the Planets," was presented by him to the Academy at the age of twenty-seven years. It contains, as already remarked, an expression in finite terms for the variation of the longitude of the epoch. But the memoir is to be considered rather as an application of known methods to an important problem of physical astronomy, than as a completion or extension of the theory of the variation of the planetary elements. The formulæ made use of are those involving the differential coefficients of the disturbing function with respect to the *coordinates;* and there is nothing which can be considered an anticipation of Lagrange's idea of the investigation, *à priori*, of expressions involving the differential coefficients with respect to the elements. But, as well for its own sake as historically, the memoir is a very important one. Lagrange, in his memoir of the 17th of August, 1808, speaks of it as having recalled his attention to a subject with which he had previously occupied himself, but which he had quite lost sight of; and Arago records that, on the death of Lagrange, a copy in his own handwriting of Poisson's memoir was found among his papers; and the memoir is referred to in, and was probably the occasion of, Laplace's memoir also of the 17th of August, 1808.

5. With respect to Laplace's memoir of the 17th of August, 1808, it will be sufficient to quote a sentence from the introduction to Lagrange's memoir:—"Ayant montré à M. Laplace mes formules et mon analyse, il me montra de son côté en même temps des formules analogues qui donnent les variations des élémens elliptiques par les différences partielles d'une même fonction, relatives à ces élémens. J'ignore comment il y est parvenu; mais je présume qu'il les a trouvées par une combinaison adroite des formules qu'il avait données dans la *Mécanique Céleste.*" This is, in fact, the character of Laplace's analysis for the demonstration of the formulæ.

6. In Lagrange's memoir of the 17th of August, 1808, "On the Theory of the Variations of the Elements of the Planets, and in particular on the Variations of the Major Axes of their Orbits," the question treated of appears from the title. The author obtains formulæ for the variations of the elements of the orbit of a planet in terms of the differential coefficients of the disturbing function with respect to the elements; but the method is a general one, quite independent of the particular form

of the integrals, and the memoir may be considered as the foundation of the general theory. The equations of motion are considered under the form,

$$\frac{d^2x}{dt^2} - \frac{1+m}{r^3}\,x = \frac{d\Omega}{dx},$$

$$\frac{d^2y}{dt^2} - \frac{1+m}{r^3}\,y = \frac{d\Omega}{dy},$$

$$\frac{d^2z}{dt^2} - \frac{1+m}{r^3}\,z = \frac{d\Omega}{dz},$$

and it is assumed that the terms in Ω being neglected, the problem is completely solved, viz., that the three coordinates, x, y, z, and their differential coefficients, x', y', z', are each of them given as functions of t, and of the constants of integration a, b, c, f, g, h; the disturbing function Ω is consequently also given as a function of t, and of the arbitrary constants. The velocities are assumed to be the same as in the undisturbed orbit. This gives the conditions

$$\delta x = 0, \qquad \delta y = 0, \qquad \delta z = 0;$$

and then the equations of motion give

$$\delta\frac{dx}{dt} = \frac{d\Omega}{dx}, \quad \delta\frac{dy}{dt} = \frac{d\Omega}{dy}, \quad \delta\frac{dz}{dt} = \frac{d\Omega}{dz},$$

equations in which δx, &c. denote the variations of x, &c., arising from the variations of the arbitrary constants, viz., $\delta x = \frac{dx}{da}\,\delta a + \frac{dx}{db}\,\delta b +$, &c. The differential coefficients $\frac{d\Omega}{\delta x}$, &c., can of course be expressed by means of $\frac{d\Omega}{da}$, &c.; and, by a simple combination of the several equations, Lagrange deduces expressions for $\frac{d\Omega}{da}$, &c., in terms of $\frac{da}{dt}$, &c.; viz.

$$\frac{d\Omega}{da} = (a,\ b)\frac{db}{dt} + (a,\ c)\frac{dc}{dt} + (a,\ f)\frac{df}{dt} + (a,\ g)\frac{dg}{dt} + (a,\ h)\frac{dh}{dt},$$

$$\vdots$$

where [1]

$$(a,\ b) = \frac{\partial(x,\ x')}{\partial(a,\ b)} + \frac{\partial(y,\ y')}{\partial(a,\ b)} + \frac{\partial(x,\ z')}{\partial(a,\ b)},$$

in which, for shortness,

$$\frac{\partial(x,\ x')}{\partial(a,\ b)} \text{ stands for } \frac{dx\,dx'}{da\,db} - \frac{dx'\,dx}{da\,db}.$$

[1] These are substantially the formulæ of Lagrange; but I have introduced here and elsewhere the very convenient abbreviation, due, I think, to Prof. Donkin, of the symbols $\frac{\partial(x,\ x')}{\partial(a,\ b)}$.

The form of the expressions shows at once that $(a, b) = -(b, a)$, so that the number of the symbols (a, b) is in fact fifteen.

Lagrange proceeds to show, that the differential coefficient with respect to t of the expression represented by the symbol (a, b) vanishes identically; and it follows, that the coefficients (a, b) are *functions of the elements only, without the time t.*

The general formulæ are applied to the problem in hand; and, in consequence of the vanishing of several of the coefficients (a, b), it is easy in the particular problem to pass from the expressions for $\frac{d\Omega}{da}$, &c. in terms of $\frac{da}{dt}$, &c. to those for $\frac{da}{dt}$, &c. in terms of $\frac{d\Omega}{da}$, &c. The author thus obtains an elegant system of formulæ for the variations of the elements of a planet's orbit, in terms of the differential coefficients of the disturbing function with respect to the elements; but it is not for the present purpose necessary to consider the form of the system, or the astronomical consequences deduced by means of it.

7. Lagrange's memoir of the 13th of March, 1809, "On the General Theory of the Variation of the Arbitrary Constants in all the Problems of Mechanics."—The method of the preceding memoir is here applied to the general problem; the equations of motion are considered under the form

$$\frac{d}{dt}\frac{dT}{dr'} - \frac{dT}{dr} + \frac{dV}{dr} = \frac{d\Omega}{dr},$$
$$\vdots$$

where T and V are of the degree of generality considered in the *Mécanique Analytique*, viz., T is a function of $r, s \ldots r', s', \ldots$ homogeneous of the second order as regards the differential coefficients $r', s', \ldots$, and V is a function of $r, s, \ldots$ only; or, rather, the equations are considered in a form obtained from the above, by writing $T - V = R$, viz., in the form

$$\frac{d}{dt}\frac{dR}{dr'} - \frac{dR}{dr} = \frac{d\Omega}{dr},$$

and, as in the preceding memoir, expressions are investigated for the differential coefficients $\frac{d\Omega}{dt}$, &c., in terms of $\frac{db}{dt}$, &c.: these are, as before, of the form

$$\frac{d\Omega}{da} = (a, b)\frac{db}{dt} +, \text{ \&c.}$$

where (a, b), &c., are in the body of the memoir obtained under a somewhat complicated form, and this complicates also the demonstration which is there given of the theorem that (a, b), &c. *are functions of the elements only, without the time t*; but in the Addition (published as part of the memoir, and without a separate date) and in the Supple-

ment the investigation is simplified, and the true form of the functions (a, b) obtained viz., writing $\frac{dT}{dr}=\rho, \ldots$ then

$$(a, b)=\frac{\partial(r, \rho)}{\partial(a, b)}+\frac{\partial(s, \sigma)}{\partial(a, b)}+\ldots$$

if, for shortness,

$$\frac{\partial(r, \rho)}{\partial(a, b)}=\frac{dr}{da}\frac{d\rho}{db}-\frac{d\rho}{da}\frac{dr}{db}, \text{ \&c.}$$

The representation of $\frac{dT}{dr'}$, $\frac{dT}{ds'}$, &c. by single letters *is* made by Lagrange in the Addition, No. 26 (Lagrange writes $\frac{dT}{dr'}=T'$, $\frac{dT}{ds'}=T''$, &c.), but quite incidentally in that number only, for the sake of the formula just stated: I have noticed this, as the step is an important one.

8. It is proper to remark that, in order to prove that the expressions (a, b), &c. are independent of the time, Lagrange, instead of considering the differential coefficients of each of these functions separately, establishes a general equation (see Nos. 25, 34, 35 of the Addition, and also the Supplement)

$$\frac{d}{dt}\left(\Delta r\delta\frac{dR}{dr'}-\delta r'\Delta\frac{dR}{dr'}+\ldots\right)=0,$$

where, if Δa, $\Delta b, \ldots$ denote any arbitrary increments whatever of the constants of integration a, $b \ldots$ then Δr, &c. are the corresponding increments of the coordinates r, &c.; this is, in fact, a grouping together of several distinct equations by means of arbitrary multipliers, and it is extremely elegant as a method of demonstration, and has been employed as well by Lagrange, here and elsewhere, as by others who have written on the subject; but I think the meaning of the formulæ is best seen when the component equations of the group are separately exhibited, and in the citation of formulæ I have therefore usually followed this course. Lagrange gives also an equation which is in fact a condensed form of the preceding expression for $\frac{d\Omega}{da}$, but which it is proper to mention, viz.:

$$\frac{d\Omega}{da}\,dt=\frac{dr}{da}\,\delta\frac{dR}{dr'}+\ldots-\delta r\frac{d}{da}\frac{dR}{dr'}-\ldots.$$

In fact, in the formula $\delta\frac{dR}{dr}$ stands for $\left(\frac{d}{da}\frac{dR}{dr'}\frac{da}{dt}+\frac{d}{db}\frac{dR}{dr}\frac{db}{dt}+\ldots\right)dt$, and δr for $\left(\frac{dr}{da}\frac{da}{dt}+\frac{dr}{dt}\frac{db}{dt}+\right)dt$; and, on substituting these values, the identity of the two expressions is seen without difficulty.

9. Lagrange remarks, that in the case where the condition of *vis viva* holds good, then if a be the constant of *vis viva* $(T+V=a)$, and c the constant attached by addition to the time, then $\frac{da}{dt}=\frac{d\Omega}{dc}$, which, he observes, is an equation remarkable as

well from its simplicity and generality as because it can be obtained *à priori*, independently of the variations of the other arbitrary constants: this is obviously the generalisation of the expression for the variation of the mean distance of a planet.

10. The consideration of Lagrange's function (a, b) originated, as appears from what has preceded, in the theory of the variation of the elements; but it is to be noticed, that the function (a, b) is altogether independent of the disturbing function, and the fundamental theorem that (a, b) is a function of the elements only, without the time, is a property of the undisturbed equations of motion. The like remark applies to Poisson's function (a, b), in the memoir next spoken of.

11. Poisson's memoir of the 16th of October, 1809.—The formulæ of this memoir are, so to speak, the reciprocals of those of Lagrange. The relations between the differential coefficients $\frac{d\Omega}{da}$, &c. of the disturbing function and the variations $\frac{da}{dt}$, &c. of the elements, depend with Lagrange, upon expressions for the coordinates and their differential coefficients in terms of the time and the elements; with Poisson, on expressions for the elements in terms of the time, and of the coordinates and their differential coefficients. The distinction is far more important than would at first sight appear, and the theory of Poisson gives rise to developments which seem to have nothing corresponding to them in the theory of Lagrange. The reason is as follows: when the system of differential equations is completely integrated, it is of course the same thing whether we have the integral equations in the form made use of by Lagrange, or in that by Poisson, the two systems are precisely equivalent the one to the other; but when the equations are not completely integrated, suppose, for instance, we have an expression for one of the coordinates in terms of the time and the elements, it is impossible to judge whether this is or is not one of the integral equations; the differential equations are not satisfied by means of this equation alone, but only by this equation with the assistance of the other integral equations. On the other hand, when we have an expression for one of the constants of integration in terms of the time and of the coordinates and their differential coefficients, it is possible, by mere substitution in the differential equations, and without the knowledge of any other integral equations, to see that the differential equations are satisfied, and that the assumed expression is, in fact, one of the system of integral equations. An expression of the form just referred to, viz., $c = \phi(t, x, y, \ldots x', y' \ldots)$, where the right-hand side does not contain any of the arbitrary constants, may, with great propriety, be termed an "integral," as distinguished from an integral equation, in which the constants and variables may enter in any conceivable manner; it is convenient also to speak of such equation simply as the integral c. [These locutions were introduced by Jacobi.]

12. Returning now to the consideration of Poisson's memoir, the equations of motion are considered under the same form as by Lagrange, viz., putting $T - V = R$, under the form

$$\frac{d}{dt}\frac{dR}{d\phi'} - \frac{dR}{d\phi} = \frac{d\Omega}{d\phi}:$$

$$\vdots$$

but Poisson writes

$$\frac{dR}{d\phi} = s, \ldots$$

thus, in effect, introducing a new set of variables, $s, \ldots$ equal in number to the coordinates $\phi, \ldots$, but he does not complete the transformation of the differential equations by the introduction therein of the new variables $s, \ldots$ in the place of the differential coefficients $\phi', \ldots$; this very important transformation was only effected a considerable time afterwards by Sir W. R. Hamilton. Poisson then assumes that the undisturbed equations are integrated in the form above adverted to, viz., that the several elements $a, b, \ldots$ are given as functions of the time t, and of the coordinates ϕ, &c. and their differential coefficients ϕ', &c. or, what is the form ultimately assumed, as functions of the time t, of the coordinates $\phi, \ldots$, and of the new variables s, &c.; and he then forms the functions

$$(a, b) = \frac{\partial (a, b)}{\partial (s, \phi)} + \ldots$$

where

$$\frac{\partial (a, b)}{\partial (s, \phi)} = \frac{da}{ds}\frac{db}{d\phi} - \frac{db}{ds}\frac{da}{d\phi},$$

(the notation is the abbreviated one before referred to), and he proves by differentiation that the differential coefficient of (a, b) with respect to the time vanishes: that is, that (a, b) which, by its definition is given as a function of t and of the coordinates $\phi, \ldots$, and of the new variables $s, \ldots$, is really a constant. Upon which Poisson remarks—"On conçoit que la constante...sera en général une fonction de a et b et des constantes arbitraires contenues dans les autres intégrales des équations du mouvement; quelquefois il pourra arriver que sa valeur ne renferme ni la constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais afin &c."

13. The importance of the remark seems to have been overlooked until the attention of geometers was called to it by Jacobi; it has since been developed by Bertrand and Bour.

It is clear from the definition that $(a, b) = -(b, a)$. It may be as well to remark that the denominator of the functional symbol is (s, ϕ) and not (ϕ, s), which would reverse the sign. [It may be noticed that throughout the Report, I speak of the Lagrange's Coefficients (a, b), and Poisson's Coefficients (a, b), distinguishing them in this manner, and not by any difference of notation.]

14. The equations for the variations of the elements are without difficulty shown to be

$$\frac{da}{dt} = (a, b)\frac{d\Omega}{db} + \ldots,$$

$$\vdots$$

which have the advantage over those of Lagrange of giving directly $\frac{da}{dt}$, &c. in terms of $\frac{d\Omega}{da}$, &c., instead of these expressions having to be determined from the value of $\frac{d\Omega}{da}$, &c. in terms of $\frac{da}{dt}$, &c.

15. Poisson applies his formulæ to the case of a body acted upon by a central force varying as any function of the distance, and also to the case of a solid body revolving round a fixed point. There is, as Poisson remarks, a complete similarity between the formulæ for these apparently very different problems, but this arises from the analogy which exists between the arbitrary constants chosen in the memoir for the two problems. The formulæ obtained form a very simple and elegant system, and one which, although not actually of the canonical form (the meaning of the term will be presently explained), might by a slight change be reduced to that form.

16. I may notice here a problem suggested by Poisson in a report to the Institute in the year 1830, on a manuscript work by Ostrogradsky on Celestial Mechanics, viz., in the case of a body acted upon by a central force, the effect of a disturbing function, *which is a function only of the distance from the centre,* is merely to alter the amount of the central force; and the expressions for the variations of the elements should therefore, in the case in question, admit of exact integration; the report is to be found in *Crelle,* t. VII. [1831], pp. 97—101.

17. The two memoirs of Lagrange and Poisson, which have been considered, establish the general theory of the variation of the arbitrary constants, and there is not, I think, very much added to them by Lagrange's memoir of 1810, the second edition of the *Mécanique Analytique,* 1811, or Poisson's memoir of 1816. The memoir by Maurice, in 1844, belongs to this part of the subject, and as its title imports, it is in fact a development of the theories of Lagrange and Poisson.

18. There is, however, one important point which requires to be adverted to. Lagrange, in the memoir of 1810, and the second edition of the *Mécanique Analytique,* remarks, that for a particular system of arbitrary constants, viz., if $\alpha, \ldots$ denote the initial values of the coordinates $\xi, \ldots$ and $\lambda, \ldots$ denote the initial values of $\frac{dT}{d\xi}, \ldots$ then the equations for the variations of the elements take the very simple form

$$\frac{d\alpha}{dt} = -\frac{d\Omega}{d\lambda}, \ldots, \quad \frac{d\lambda}{dt} = \frac{d\Omega}{d\alpha}, \ldots;$$

this is, in fact, the original idea and simplest example of a system of canonical elements; viz. of a system composed of pairs of elements, α, λ, the variations of which are given in the form just mentioned.

19. The "Avertissement" to the second edition of the *Mécanique Analytique,* contains the remark, that it is not necessary that the disturbing function Ω should actually exist; $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}, \frac{d\Omega}{dz}$ may be considered as mere conventional symbols standing for forces $X, Y, Z,$ not the differential coefficients of one and the same function, and then $\frac{d\Omega}{da}$ will be a conventional symbol standing for $\frac{d\Omega}{dx}\frac{dx}{da} + \frac{d\Omega}{dy}\frac{dy}{da} + \frac{d\Omega}{dz}\frac{dz}{da}$, and similarly for $\frac{d\Omega}{db}$, &c.; and this being so, all the formulæ will subsist as in the case of an actually existing disturbing function.

20. Cauchy, in a note in the *Bulletin de la Société Philomatique* for 1819 (reproduced in the "Mémoire sur l'Intégration des Equations aux Dérivées Partielles du Premier Ordre," *Exer. d'Anal. et de Physique Math.*, t. II. pp. 238—272 (1841)), showed that the integration of a partial differential equation of the first order could be reduced to that of a single system of ordinary differential equations. A particular case of this general theorem was afterwards obtained by Jacobi in the course of his investigations (founded on those of Sir W. R. Hamilton) on the equations of dynamics, and he was thence led to a slightly different form of the general theorem previously established by Cauchy, viz., Cauchy's method gives the *general*, Jacobi's the *complete* integral, of the partial differential equation. The investigations of the geometers who have written on the theory of dynamics are based upon those of Sir W. R. Hamilton and Jacobi, and it is therefore unnecessary, in the present report, to advert more particularly to Cauchy's very important discovery.

21. I come now to Sir W. R. Hamilton's memoirs of 1834 and 1835, which are the commencement of a second period in the history of the subject. The title of the first memoir shows the object which the author proposed to himself, viz., the discovery of a function by means of which the integral equations can be all of them actually represented. The method given for the determination of this function, or rather of each of the several functions which answer the purpose, presupposes the knowledge of the integral equations; it is therefore not a *method of integration*, but a theory of the representation of the integral equations assumed to be known. I venture to dissent from what appears to have been Jacobi's opinion, that the author missed the true application of his discovery; it seems to me, that Jacobi's investigations were rather a theory collateral to, and historically arising out of the Hamiltonian theory, than the course of development which was of necessity to be given to such theory. But the new form obtained in Sir W. R. Hamilton's memoirs for the equations of motion, is a result of not less importance than that which was the professed object of the memoirs.

22. Hamilton's principal function V.—The formulæ are given for the case of any number of free particles, but, for simplicity, I take the case of a single particle. The equations of motion are taken to be

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},$$

$$m\frac{d^2y}{dt^2}=\frac{dU}{dy},$$

$$m\frac{d^2y}{dt^2}=\frac{dU}{dz};$$

so that the *vis viva* function is

$$T=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2),$$

and the force function, taken with Lagrange's sign, would be $-U$. It is assumed that the condition of *vis viva* holds, that is, that U is a function of x, y, z only. The initial values of the coordinates are denoted by a, b, c, and those of the velocities

by a', b', c'. The equation of *vis viva* is $T = U + H$, and this gives rise to an equation $T_0 = U_0 + H$ of the same form for the initial values of the coordinates. The author then writes

$$V = \int_0^t 2T dt,$$

an equation, the form of which implies that T is expressed as a function of the time and of the constants of integration a, b, c, a', b', c'. The method of the calculus of variations leads to the equation

$$\delta V = m(x'\delta x + y'\delta y + z'\delta z) - m(a'\delta a + b'\delta b + c'\delta c) + t\delta H,$$

to understand which, it should be remarked that the coordinates x, y, z, and the velocities x', y', z', being functions of t and of a, b, c, a', b', c', then V is, in the first instance, given as a function of these quantities. But x, y, z being functions of a, b, c, a', b', c', t, we may conversely consider a', b', c' as functions of x, y, z, a, b, c, t, and thus V becomes a function of x, y, z, a, b, c, t. In like manner H is a function of x, y, z, a, b, c, t, and, eliminating t, we have V a function of x, y, z, a, b, c, H, which is the form in which in the last equation V is considered to be expressed. The equation then gives

$$\frac{dV}{dx} = mx', \quad \frac{dV}{dy} = my', \quad \frac{dV}{dz} = mz',$$

$$\frac{dV}{da} = -ma', \quad \frac{dV}{db} = -mb', \quad \frac{dV}{dc} = -mc',$$

$$\frac{dV}{dH} = t;$$

and, considering V as a known function of x, y, z, a, b, c, H, the elimination of H gives a set of equations which are in fact the integral equations of the problem, viz., the first three equations and the last equation give equations containing x, y, z, x', y', z', t and a, b, c, that is, the intermediate integrals; the second three equations and the last equation, give equations containing x, y, z, t, a, b, c, a', b', c', that is, the final integrals.

The function V satisfies the two partial differential equations

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} = U + H,$$

$$\frac{1}{2m}\left\{\left(\frac{dV}{da}\right)^2 + \left(\frac{dV}{db}\right)^2 + \left(\frac{dV}{dc}\right)^2\right\} = U_0 + H;$$

which, if they could be integrated, would give V as a function of x, y, z, a, b, c, H, and thus determine the motion of the system.

23. Hamilton's principal function S.—This is connected with the function V by the equation

$$V = tH + S;$$

or, what is the same thing, the new principal function S is defined by the equation

$$S = \int_0^t (T + U)\, dt\,;$$

but S is considered (not like V as a function of $x, y, z, a, b, c, H,$ but) as a function of x, y, z, a, b, c, t. The expression for the variation of S is

$$\delta S = -H\delta t + m\,(x'\delta x + y'\delta y + z'\delta z) - m\,(a'\delta a + b'\delta b + c'\delta c)$$

which is equivalent to the system

$$\frac{dS}{dx} = mx', \quad \frac{dS}{dy} = my', \quad \frac{dS}{dz} = mz',$$

$$\frac{dS}{da} = -ma', \quad \frac{dS}{db} = -mb', \quad \frac{dS}{dc} = -mc',$$

$$\frac{dS}{dt} = -H\,;$$

the first three and the second three of which give, respectively, the intermediate and the final integrals; the last equation leads only to the expression of the supernumerary constant H in terms of the initial coordinates $a, b, c,$ and it may be omitted from the system.

The function S satisfies the partial differential equations

$$\frac{dS}{dt} + \frac{1}{2m}\left\{\left(\frac{dS}{dx}\right)^2 + \left(\frac{dS}{dy}\right)^2 + \left(\frac{dS}{dz}\right)^2\right\} = U,$$

$$\frac{dS}{dt} + \frac{1}{2m}\left\{\left(\frac{dS}{da}\right)^2 + \left(\frac{dS}{db}\right)^2 + \left(\frac{dS}{dc}\right)^2\right\} = U_0\,;$$

which, if they could be integrated, would give S as a function of $x, y, z, a, b, c, t,$ and thus determine the motion of the system.

24. Hamilton's form of the equations of motion.—This is in fact the form obtained by carrying out the idea of introducing into the differential equations, in the place of the differential coefficients of the coordinates, the derived functions (with respect to these differential coefficients) of the *vis viva* function T. Taking η to denote any one of the series of coordinates, then the original system may be denoted by

$$\frac{d}{dt}\frac{dT}{d\eta'} - \frac{dT}{d\eta} = \frac{dU}{d\eta},$$

$$\vdots$$

(U is the force function taken with a contrary sign to that of Lagrange), and writing in like manner ϖ to denote any one of the new variables connected with the coordinates η by the equations

$$\frac{dT}{d\eta'} = \varpi,$$

$$\vdots$$

then T, in its original form, is a function of $\eta, \ldots \eta', \ldots$, homogeneous of the second order as regards the differential coefficients $\eta', \ldots$; and, consequently, these being linear functions (without constant terms) of the new variables ϖ, the *vis viva* function T can be expressed as a function of $\eta, \ldots \varpi, \ldots$, homogeneous of the second order as regards the variables $\varpi, \ldots$. And when T has been thus expressed, the equations of motion take the form

$$\frac{d\eta}{dt} = \frac{dH}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{dT}{d\eta} + \frac{dU}{d\eta},$$
$$\vdots \qquad\qquad \vdots$$

which is the required transformation. The force function U is independent of the differential coefficients $\eta', \ldots$ and, consequently, of the variables $\varpi, \ldots$, hence, writing $H = T - U$, the equations take the form

$$\frac{d\eta}{dt} = \frac{dH}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{dH}{d\eta},^{(1)}$$
$$\vdots \qquad\qquad \vdots$$

which correspond to the condensed form obtained by writing $T - V = R$ in Lagrange's equations. It is hardly necessary to remark that H is to be considered as a given function of $\eta, \ldots \varpi, \ldots$, viz., it is what $T - U$ becomes when the differential coefficients $\eta', \ldots$ are replaced by their values in terms of the new variables $\varpi, \ldots$.

25. I have, for greater simplicity, explained the theory of the functions V and S in reference to a very special form of the equations of motion; but the theory is, in fact, applicable to any form whatever of these equations; and, as regards the function V, is in the first memoir examined in detail with reference to Lagrange's general form of the equations of motion. The function S is considered at the end of the memoir, in reference only to the special form. The new form of the equations of motion is first established in the second memoir, and the theory of the functions V and S is there considered in reference to this form. The author considers also another function Q, which, when the matter is looked at from a somewhat more general point of view, is not really distinct from the function S.

26. The first memoir contains applications of the method to the problem of two bodies, and the problem of three or more bodies, and researches in reference to the approximate integration of the equations of motion by the separation of the function V into two parts, one of them depending on the principal forces, the other on the disturbing forces. The method, or one of the methods, given for this purpose, involves the consideration of the variation of the arbitrary constants, but it is not easy to single out any precise results, or explain their relation to the results of Lagrange and Poisson. The like remark applies to the investigations contained in Nos. 7 to 12 of the second memoir, but it is important to consider the theory described in the heading

[1] I find it stated in a note to M. Houel's "Thèse sur l'intégration des équations différentielles de la Mécanique," Paris, 1855, that this form of the equations of motion had been previously employed in an *unpublished* memoir by Cauchy, written in 1831. [Cauchy "Extrait du Mémoire présenté à l'Academie de Turin le 11 Oct. 1831" published in lithograph under the date Turin, 1832, with an Addition dated 6 Mar. 1833.]

of No. 13, as "giving formulæ for the variation of elements more analogous to those already known." The function H is considered as consisting of two parts, one of them being treated as a disturbing function; the equations of motion assume therefore the form

$$\frac{d\eta}{dt}=\frac{dH}{d\varpi}+\frac{d\Upsilon}{d\varpi}, \quad \frac{d\varpi}{dt}=-\frac{dH}{d\eta}-\frac{d\Upsilon}{d\eta},$$

(I have written H, Υ instead of the author's H_1, H_2). The terms involving Υ are in the first instance neglected, and it is assumed that the integrals of the resulting equations are presented in the form adopted by Poisson, viz., the constants of integration a, b, &c. are considered as given in terms of t, and of the two sets of variables $\eta, \ldots$ and $\varpi, \ldots$; the integrals are then extended to the complete equations by the method of the variation of the elements. The resulting expressions are the same in form as those of Poisson, viz.:

$$\frac{da}{dt}=(a,\ b)\frac{d\Upsilon}{db}+\ldots,$$

where

$$(a,\ b)=\frac{\partial(a,\ b)}{\partial(\eta,\ \varpi)}+\ldots,$$

if, for shortness,

$$\frac{\partial(a,\ b)}{\partial(\eta,\ \varpi)}=\frac{da}{d\eta}\frac{db}{d\varpi}-\frac{db}{d\eta}\frac{da}{d\varpi},$$

and conversely the values of $\frac{d\Upsilon}{da}$, &c. in terms of $\frac{da}{dt}$, &c. might have been exhibited in a form such as that of Lagrange. The expressions $(a,\ b)$, considered as functional symbols, have the same meanings as in the theories of Poisson and Lagrange; and, as in these theories, the differential coefficient of $(a,\ b)$ with respect to the time, vanishes, or $(a,\ b)$ is a function of the elements only.

27. It is to be observed that the disturbing function Υ is not necessarily in the same problem identical with the disturbing function Ω of Lagrange and Poisson (indeed, in any problem, the separation of the forces into principal forces and disturbing forces is an arbitrary one). Sir W. R. Hamilton, in the second memoir, gives a very beautiful application of his theory to the problem of three or more bodies, which has the peculiar advantage of making the motion of all the bodies depend upon one and the same disturbing function [1]. This disturbing function contains (as in the last-mentioned general formulæ) both sets of variables, and the consequence is that, as the author remarks, the varying elements employed by him are essentially different from those made use of in the theories of Lagrange and Poisson; the velocities cannot, in his theory, be obtained by differentiating the coordinates as if the elements were

[1] Lagrange has given formulæ for the determination of the motion of three or more bodies referred to their common centre of gravity by means of one and the same disturbing function. In Sir W. R. Hamilton's theory there is one central body to which all the others are referred. The method of Sir W. R. Hamilton is made use of in M. Houel's "Thèse d'Astronomie: Application de la Méthode de M. Hamilton au Calcul des Perturbations de Jupiter."—Paris, 1855.

constant. The investigation applies to the case where the attracting force is any function whatever of the distance, and the six elements ultimately adopted form a canonical system.

28. The precise relation of Sir W. R. Hamilton's form of the equations of motion to that of Lagrange, is best seen by considering Lagrange's equations, not as a system of differential equations of the second order between the coordinates and the time t, but as a system of twice as many differential equations of the first order between the coordinates, their differential coefficients treated as a new system of variables, and the time. It will be convenient to write $-U$, instead of Lagrange's force-function V, and (to conform to the usage of later writers who have treated the subject in the most general manner) to represent the coordinates by $q, \ldots$, their differential coefficients by $q', \ldots$, and the new variables which enter into the Hamiltonian form by $p, \ldots$; then the Lagrangian system will be

$$\frac{dq}{dt} = q', \quad \frac{d}{dt}\frac{dT}{dq'} - \frac{dT}{dq} = \frac{dU}{dq};$$
$$\vdots \qquad \vdots$$

or putting $T + U = Z$ (this is the same as Lagrange's substitution, $T - V = R$), the system becomes

$$\frac{dq}{dt} = q', \quad \frac{d}{dt}\frac{dZ}{dq'} = \frac{dZ}{dq},$$
$$\vdots \qquad \vdots$$

while the Hamiltonian system is

$$\frac{dq}{dt} = \frac{dT}{dp}, \quad \frac{dp}{dt} = -\frac{dT}{dq} + \frac{dU}{dq};$$
$$\vdots \qquad \vdots$$

or putting as before $T - U = H$, the system is

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq};$$
$$\vdots \qquad \vdots$$

where, in the Lagrangian systems, T and U, and consequently Z, are given functions of a certain form of $t, q, \ldots q', \ldots$, and in like manner, in the Hamiltonian system, T and U, and consequently H, are given functions of a certain form of $t, q, \ldots p \ldots$. The generalisation has since been made (it is not easy to say precisely when first made) of considering Z as standing for any function whatever of $t, q, \ldots q', \ldots$, and in like manner of considering H as standing for any function whatever of $t, q, \ldots p, \ldots$. It is to be noticed that in Sir W. R. Hamilton's memoir, the demonstration which is given of the transformation from Lagrange's equations to the new form depends essentially on the special form of the function T as a homogeneous function of the second order in regard to the differential coefficients of the coordinates; indeed the

transformation itself, as regards the actual value of the new function T ($=T$ expressed in terms of the new variables), which enters into the transformed equations, depends essentially upon the special form just referred to of the function T, although, as will be seen in the sequel, there is a like transformation applying to the most general form of the function T.

29. In the greater part of what has preceded, and especially in the above-mentioned substitutions $T+U=Z$ and $T-U=H$, it is of course assumed that the force function U exists; when there is no force function these substitutions cannot be made, but the forms corresponding to the untransformed forms in T and U are as follows, viz. the Lagrangian form is

$$\frac{dq}{dt}=q', \quad \frac{d}{dt}\frac{dT}{dq'}-\frac{dT}{dq}=Q,$$
$$\vdots \qquad\qquad \vdots$$

and the Hamiltonian form is

$$\frac{dq}{dt}=\frac{dT}{dp}, \quad \frac{dp}{dt}=-\frac{dT}{dq}+Q;$$
$$\vdots \qquad\qquad \vdots$$

that is, the only difference is, that the functions Q, instead of being the differential coefficients with respect to the variables $q\ldots$ of one and the same force function U, are so many separate and distinct functions of the variables $q, \ldots$, or more generally of the variables $q, \ldots p, \ldots$ of both sets.

30. Jacobi's letter of 1836.—This is a short note containing a mere statement of two results. The first is as follows, viz. the equations for the motion of a point *in plano* being taken to be

$$\frac{d^2x}{dt^2}=\frac{dU}{dx}, \quad \frac{d^2y}{dt^2}=\frac{dU}{dy},$$

where U is a function x, y without t; one integral is the equation of *vis viva* $\frac{1}{2}(x'^2+y'^2)=U+h$. Assume that another integral is $a=F(x, y, x', y')$, then x', y' will in general be functions of x, y, a, h, and considering them as thus expressed, it is stated that not only $x'dx+y'dy$ will be an exact differential, but its differential coefficients with respect to a, h will be so likewise, and the remaining integrals are

$$b=\int\left(\frac{dx}{da}dx+\frac{dy'}{da}dy\right),$$

$$t+T=\int\left(\frac{dx'}{dh}dx+\frac{dy'}{dh}dy\right),$$

a theorem, the relation of which to the general subject will presently appear.

The second result does not relate to the general subject, but I give it in a note for its own sake(1).

31. Poisson's memoir of 1837.—This contains investigations suggested by Sir W. R. Hamilton's memoir, and relating to the aid to be derived from a system of given integral equations (equal in number to the coordinates) in the determination of the principal function V. The equations $\frac{dV}{dx} = mx'$, &c. give $dV = m(x'dx + y'dy + z'dz)$, or in the case of a system of points, $dV = \Sigma m(x'dx + y'dy + z'dz)$. If the points, instead of being free, are connected together by any equations of condition, then, by means of these equations, the coordinates x, y, z of the different points and their differential coefficients, x', y', z', can be expressed as functions of a certain number of independent variables ϕ, ψ, θ, &c., and of their differential coefficients ϕ', ψ', θ', &c.; dV then takes the form $dV = Xd\phi + Yd\psi + Zd\theta + \ldots$, where X, Y, Z are functions of $\phi, \psi, \ldots$ $\phi', \psi', \ldots$. Imagine now a system of integrals (one of them the equation of *vis viva*) equal in number to the independent variables $\phi, \psi, \theta \ldots$; then, by the aid of these equations, $\phi', \psi', \theta' \ldots$, and, consequently, $X, Y, Z, \ldots$ can be expressed as functions (of the constants of integration and) of the variables $\phi, \psi, \theta, \ldots$. Hence, attending only to the variables, $dV = Xd\phi + Yd\psi + Zd\theta + \ldots$ is a differential expression involving only the variables $\phi, \psi, \theta \ldots$; but, as Poisson remarks, this expression is not in general a complete differential. In the cases in which it is so, V can of course be obtained directly by integrating the differential expression, viz. the function so obtained is in value, but not in form, Sir W. R. Hamilton's principal function V; for, with him, V is a function of the coordinates, and of a particular set of the constants of integration, viz. the constant of *vis viva* h, and the initial values of the coordinates. Poisson adds the very important remark, that V being determined by his process as above, then h being the constant of *vis viva*, and the constants of the other given integral equations being e, f, &c., the remaining integrals of the problem are(2)

$$\frac{dV}{dh} = t + \tau, \quad \frac{dV}{de} = l, \quad \frac{dV}{df} = m, \ldots$$

1 Jacobi imagines a point *without mass* revolving round the sun and disturbed by a planet moving in a circular orbit, which is taken for the plane of x, y; the coordinates of the point are x, y, z, those of the planet $a' \cos n't$, $a' \sin n't$, m' is the mass of the planet, M the mass of the sun; then we have accurately

$$\tfrac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} - n'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) =$$

$$\frac{M}{(x^2+y^2+z^2)^{\frac{1}{2}}} + m'\left\{\frac{1}{((x^2+y^2+z^2-2a'(x\cos n't + y\sin n't) + a'^2)^{\frac{1}{2}}} - \frac{x\cos n't + y\sin n't}{a'^2}\right\} + \text{const.}$$

which Jacobi suggests might be found useful in the lunar theory. The point being without mass, means only that it is considered as not disturbing the circular motion of the planet; the problem is properly a case of the problem of two centres, viz. one centre is fixed, and the other one revolves round it in a circle with a uniform velocity.

2 Poisson writes $\frac{dV}{dh} = -t + \epsilon$; there seems to be a mistake as to the sign of h running through the memoir. Correcting this, and putting $-\tau$ for ϵ, we have the formula $\frac{dV}{dh} = t + \tau$ given in the text.

where τ, l, m, ... are new arbitrary constants. But, as before remarked, the expression for dV is not always a complete differential. Poisson accordingly inquires into and determines (but not in a precise form) the conditions which must be satisfied, in order that the expression in question may be a complete differential. He gives, as an example, the case of the motion of a body in space under the action of a central force; and, secondly, the case considered in Jacobi's letter of 1836, which he refers to, viz., here $dV = x'dx + y'dy$, and when the two integral equations are one of them, the equations of *vis viva* $\frac{1}{2}(x'^2 + y'^2) = U + h$, and the other of them any integral equation $a = F(x, y, x', y')$ whatever (subject only to the restriction that a is not a function of x, y, $x'^2 + y'^2$, the necessity of which is obvious) the condition is satisfied *per se*, and, consequently, $x'dx + y'dy$ is a complete differential, and its integral gives (in value, although as before remarked not in form) the principal function V; and such value of V gives the two integral equations obtained in Jacobi's letter.

32. Jacobi's note of the 29th of November, 1836, "On the Calculus of Variations, and the Theory of Differential Equations."—The greater part of this note relates to the differential equations which occur in the calculus of variations, including, indeed, the differential equations of dynamics, but which belong to a different field of investigation. The latter part of the note relates more immediately to the differential equations of dynamics. The author remarks, that, in any dynamical problem of the motion of a single particle for which the principle of *vis viva* holds good, if, besides the integral of *vis viva*, there is given any other integral, the problem is reducible to the integration of an ordinary differential equation of two variables, and that it is always possible to integrate this equation, or at least *discover by a precise and general rule the factor which renders it integrable.* This would seem to refer to Jacobi's researches on the theory of the ultimate multiplier, but the author goes on to refer to a preceding communication to the Academy of Paris (the before-mentioned letter of 1836), which does not belong (or, at least, does not obviously belong) to this theory. He speaks also of a class of dynamical problems, viz. that of the motion of a system of bodies which mutually attract each other, and which may besides be acted upon by forces in parallel lines, or directed to fixed centres, or even to centres the motion of which is given; and, he remarks, in the solution of such a problem, the system of differential equations being in the first instance of the order $2n$ (that is, being a system admitting of $2n$ arbitrary constants), then if one integral is known, it is possible by a proper choice of the quantities selected for variables to reduce the system to the order $2n - 2$. If another integral is known, the equation may in like manner be reduced to a system of the order $2n - 4$, and so on until there are no more equations to be integrated; and thus the operations to be effected depend only upon quadratures. All this seems to refer to researches of Jacobi, which, so far as I am aware, have not hitherto been published. The results correspond with those recently obtained by Bour, *post*, Nos. 66 and 67.

33. Jacobi's memoir of 1837.—Jacobi refers to the memoirs of Sir W. R. Hamilton, and he reproduces, in a slightly different form, the investigation of the fundamental property of the principal function S. The case considered is that of a system of n

particles, the coordinates of which are connected together by any number of equations; but it will be sufficient here to attend to the case of a single free particle. The equations of motion are assumed to be

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz};$$

but U is considered as being a function of x, y, z and of the time t, that is, it is assumed that the condition of *vis viva* is not of necessity satisfied. The definition of the function S is

$$S+\int_0^t\left[U=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2)\right]dt,$$

which, when the equation of *vis viva* is satisfied, that is, when

$$T=\tfrac{1}{2}m\,(x'^2+y'^2+z'^2)=U+h,$$

agrees with Sir W. R. Hamilton's definition $S=2\int_0^t Udt+ht$. The function S is considered as being, by means of the integral equations assumed as known, expressed as a function of t, of the coordinates x, y, z, and of their initial values a, b, c. And then it is shown that S satisfies the equations

$$\frac{dS}{dx}=mx',\quad \frac{dS}{dy}=my',\quad \frac{dS}{dz}=mz',$$

$$\frac{dS}{da}=-ma',\quad \frac{dS}{db}=-mb',\quad \frac{dS}{dc}=-mc';$$

so that the intermediate and final integrals are expressed by means of the principal function S.

34. But Jacobi proceeds, "the definition assumes the integration of the differential equations of the problem. The results, therefore, are only interesting in so far as they have reduced the system of integral equations into a remarkable form. We may, however, define the function S in a quite different and *very much more general* manner." And then, attending only to the case of a system of free particles, he gives a definition, which, in the case of a single particle, is as follows:

Jacobi's principal function S.—The equations of motion being as before

$$\frac{d^2x}{dt^2}=m\frac{dU}{dx},\quad \frac{d^2y}{dt^2}=m\frac{dU}{dy},\quad \frac{d^2z}{dt^2}=m\frac{dU}{dz},$$

(where U is in general a function of x, y, z and t), then S is defined to be a *complete* solution of the partial differential equation

$$\frac{dS}{dt}+\frac{1}{2m}\left\{\left(\frac{dS}{dx}\right)^2+\left(\frac{dS}{dy}\right)^2+\left(\frac{dS}{dz}\right)^2\right\}=U.$$

A complete solution, it will be recollected, means a solution containing as many arbitrary constants as there are independent variables in the partial differential equation;

in the present case, therefore, four arbitrary constants. But one of these constants may be taken to be a constant attached to the function S by mere addition, and which disappears from the differential coefficients, and it is only necessary to attend to the other three arbitrary constants. S is consequently a function of t, x, y, z, and of the arbitrary constants α, β, γ, satisfying the partial differential equation. And this being so, it is shown that the integrals of the problem are

$$\frac{dS}{dx}=mx', \quad \frac{dS}{dy}=my', \quad \frac{dS}{dz}=mz',$$

$$\frac{dS}{d\alpha}=\lambda, \quad \frac{dS}{d\beta}=\mu, \quad \frac{dS}{d\gamma}=\nu;$$

where λ, μ, ν are any other arbitrary constants, viz., the first three equations give the intermediate integrals, and the last three equations give the final integrals of the problem.

Jacobi proceeds to give an analogous definition of the principal function V as follows:

35. Jacobi's principal function V.—First, when the condition of *vis viva* is satisfied. Here V is a complete solution of the partial differential equation

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}=U+h,$$

where h is the constant of *vis viva*. The partial differential equation contains only three independent variables; and since as before one of the constants of the complete solution may be taken to be a constant attached to V by mere addition, and which disappears from the differential coefficients, we may consider V as a function of t, x, y, z, and of the two constants of integration α and β. But V will of course also contain the constant h, which enters into the partial differential equation. The integrals of the problem are then shown to be

$$\frac{dV}{dx}=mx', \quad \frac{dV}{dy}=my', \quad \frac{dV}{dz}=mz',$$

$$\frac{dV}{dh}=t+\tau, \quad \frac{dV}{d\alpha}=\lambda, \quad \frac{dV}{d\beta}=\mu,$$

where τ, λ, μ are new arbitrary constants.

36. Jacobi's principal function V.—Secondly, when the equation of *vis viva* is not satisfied. Here U contains the time t, and we have no such equation as $T=U+h$, but along with the coordinates x, y, z there is introduced a new variable H, and V is defined to be a complete integral of the partial differential equation

$$\frac{1}{2m}\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)^2\right\}=U+H;$$

where, in the expression for U, it is assumed that t is replaced by $\frac{dV}{dH}$. There are, consequently, four independent variables, and a complete solution must contain, exclusively of the constant attached to V by mere addition, and which disappears from the differential coefficients, three arbitrary constants α, β, γ. The integral equations are shown to be

$$\frac{dV}{dx} = mx', \quad \frac{dV}{dy} = my', \quad \frac{dV}{dz} = mz',$$

$$\frac{dV}{d\alpha} = \lambda, \quad \frac{dV}{d\beta} = \mu, \quad \frac{dV}{d\gamma} = \nu,$$

$$\frac{dV}{dH} = t;$$

where λ, μ, ν are arbitrary constants, viz., eliminating H from the first three equations by the assistance of the last equation, we have the intermediate integrals; and eliminating H from the second three equations by the assistance of the last equation, we have the final integrals. The substitution of the above values $\frac{dV}{dx}$, &c. in the partial differential equation gives $T = U + H$, that is, $H (= T - U)$ is that function which, when the condition of *vis viva* is satisfied, becomes equal to h, the constant of *vis viva*.

Jacobi's extension of the theory to the case where the condition of *vis viva* is not satisfied, appears to have attracted very little attention; it is indeed true, as will be noticed in the sequel, that this general case can be reduced to the particular one in which the condition of *vis viva* is satisfied, but there is not it would seem any advantage in making this reduction; the formulæ for the general case are at least quite as elegant as those for the particular case.

37. Jacobi, after considering some particular dynamical applications, proceeds to apply the theory developed in the first part of the memoir to the general subject of partial differential equations; the differential equations of a dynamical problem lead to a partial differential equation, a complete solution of which gives the integral equations. Conversely, the integral equations give the complete solution of the partial differential equation, and applying similar considerations to any partial differential equation of the first order whatever, it is shown (what, but for Cauchy's memoir of 1819, which Jacobi was not acquainted with [1]), would have been a new theorem) that the solution of the partial differential equation depends on the integration of a single system of differential equations. The remainder of the memoir is devoted to the discussion of this theory and of the integration of the Pfaffian system of ordinary differential

[1] Jacobi refers to Lagrange's "Leçons sur la Théorie des Fonctions," and to a memoir by Pfaff in the Berlin Transactions for 1814, as containing, so far as he was aware, everything essential which was known in reference to the integration of partial differential equations of the first order; he refers also to his own memoir "Ueber die Pfaffsche Methode u. s. w." *Crelle*, t. II. pp. 347—358 (1827), as presenting the method in a more symmetrical and compendious form, but without adding to it anything essentially new.

equations, a system which is also treated of in Jacobi's memoir of 1844, "Theoria Novi Multiplicatoris &c." I take the opportunity of referring here to a short note by Brioschi, "Intorno ad una Proprietà delle Equazioni alle Derivate Parziali del Primo Ordine," *Tortolini*, t. VI. pp. 426—429 (1855), where the theory of the integration of a partial differential equation of the first order is presented under a singularly elegant form.

38. Jacobi's note of 1837, "On the Integration of the Differential Equations of Dynamics."—Jacobi remarks that it is possible to derive from Lagrange's form of the equations of motion an important profit for the integration of these equations, and he refers to his communication of the 29th of November 1839 to the Academy of Berlin, and to his former note to the Academy of Paris. He proceeds to say, that whenever the condition of *vis viva* holds good, he had found that it was possible in the integration of the equations of motion to follow a course such that each of the given integrals successively lowers by two unities the order of the system; and that the like theorem holds good when the condition of *vis viva* is not satisfied, that is, when the force function involves the time (this seems to be a restatement, in a more general form, of the theorems referred to in the note of the 29th of November 1836 to the Academy of Berlin); and he mentions that he had been, by his researches on the theory of numbers, led away from composing an extended memoir on the subject. The note then passes on to other subjects, and it concludes with two theorems, which are given without demonstration as extracts from the intended work he had before spoken of. These theorems are in effect as follows:

I. Let

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz},\ \&c.$$

be the $3n$ differential equations of the motion of a free system, and

$$\tfrac{1}{2}\Sigma m\,(x'^2+y'^2+z'^2)\,dt=U+h,$$

the equation of *vis viva*.

Let V be a complete solution of the partial differential equation

$$\tfrac{1}{2}\Sigma\frac{1}{m}\left\{\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2+\left(\frac{dV}{dz}\right)\right\}=U+h,$$

that is, a solution containing, besides the constants attached to V by mere addition, $3n-1$ constants $\alpha\,(\alpha_1,\ \alpha_2,\ldots\alpha_{3n-1})$, then first the integral equations are

$$\frac{dV}{d\alpha}=\beta,\ldots\quad\frac{dV}{dh}=t+\tau;$$

where $\beta\,(\beta_1,\ \beta_2,\ldots\beta_{3n-1})$ and τ are new arbitrary constants: this is in fact the theorem already quoted from Jacobi's memoir of 1837, and it is in the present place referred

to as an easy generalisation of Sir W. R. Hamilton's formulæ. But Jacobi proceeds (and this is given as entirely new) that the disturbed equations being

$$m\frac{d^2x}{dt^2}=\frac{dU}{dx}+\frac{d\Omega}{dx},\quad m\frac{d^2y}{dt^2}=\frac{dU}{dy}+\frac{d\Omega}{dy},\quad m\frac{d^2z}{dt^2}=\frac{dU}{dz}+\frac{d\Omega}{dz},$$

then the equations for the variations of the above system of arbitrary constants are

$$\frac{d\alpha}{dt}=\frac{d\Omega}{d\beta},\ldots\quad \frac{dh}{dt}=\frac{d\Omega}{d\tau},$$

$$\frac{d\beta}{dt}=-\frac{d\Omega}{d\alpha},\ldots\quad \frac{d\tau}{dt}=-\frac{d\Omega}{dh};$$

so that the constants form (I think the term is here first introduced) a canonical system.

Jacobi observes that, in the theory of elliptic motion, certain elements which he mentions form a system of canonical elements, and he remarks that, since one complete solution of a partial differential equation gives all the others, the theorem leads to the solution of another interesting problem, viz. "Given one system of canonical elements, to find all the other systems." This is effected by means of the second theorem, which is as follows:

II. Given the systems of differential equations between the variables $a\,(a_1,\ a_2\ldots a_m)$ and $b\,(b_1,\ b_2\ldots b_m)$

$$\frac{da}{dt}=-\frac{dH}{db},\ldots\quad \frac{db}{dt}=\frac{dH}{da},\ldots$$

where H is *any function* of the variables $a,\ldots$ and $b,\ldots$; let $\alpha\,(\alpha_1,\ \alpha_2,\ldots\alpha_m)$ and $\beta\,(\beta_1,\ \beta_2,\ldots\beta_m)$ be two new systems of variables connected with the preceding ones by the equations

$$\frac{d\psi}{d\alpha}=\beta,\ldots\quad \frac{d\psi}{da}=-b,\ldots$$

where ψ is a function of $\alpha,\ldots b,\ldots$ without t or the other variables, then expressing H as a function of t and the new variables $\alpha,\ldots$ and $\beta,\ldots$, these last variables are connected together by equations of the like form with the original system, viz.:

$$\frac{d\alpha}{dt}=-\frac{dH}{d\beta},\ldots\quad \frac{d\beta}{dt}=-\frac{dH}{d\alpha},\ldots$$

Jacobi concludes with the remark, that other theorems no less general may be deduced by putting $\psi+\lambda\psi_1+\mu\psi_2+\ldots$ instead of ψ, and eliminating the multipliers $\lambda,\ \mu,\ldots$ by means of the equations $\psi_1=0,\ \psi_2=0,\ldots$, and that the demonstrations of the theorems are obtained without difficulty.

39. Jacobi's note of the 21st of November, 1838.—Jacobi refers to a memoir by Encke in the Berlin *Ephemeris* for 1837, "über die speciellen Störungen," where expressions are given for the partial differential coefficients of the values in the theory

of elliptic motion of the coordinates x, y, z and the velocities x', y', z' with respect to the elements; and he remarks, that if Encke's elements are replaced by a system of elements α, β, γ, α', β', γ' which he mentions, connected with those of Encke by equations of a simple form, then considering first x, y, z, x', y', z' as given functions of t and the elements, and afterwards the elements α, β, γ, α', β', γ' as given functions of t and x, y, z, x', y', z', there exists the remarkable theorem that the thirty-six partial differential coefficients $\frac{d\alpha}{dx}$, $\frac{d\beta}{dx}$, &c., and the thirty-six partial differential co-coefficients $\frac{dx}{d\alpha}$, $\frac{dx}{d\beta}$, &c., are equal to each other, or differ only in their sign, viz.

$$\frac{dx}{d\alpha}=-\frac{d\alpha'}{dx'},\quad \frac{dx}{d\alpha'}=\frac{d\alpha}{dx'},\quad \frac{dx'}{d\alpha}=\frac{d\alpha'}{dx},\quad \frac{\partial x'}{\partial\alpha'}=-\frac{d\alpha}{dx};$$

thirty-six equations in all, viz. the pair α, α' of corresponding elements may be replaced by the pair β, β' or γ, γ': and then in each of the twelve equations y, y' or z, z' may be written instead of x, x'. The like applies to a system of constants which are the initial values of any system whatever of coordinates $p, \ldots$, and the initial values of the differential coefficients $q'=\frac{dT}{dp}$, &c. of the force function T with respect to $p, \ldots$; *and for every system of elements which possess the property first mentioned, the formulæ for the variations assume the simplest possible form, inasmuch as the variations of each element is equal to a single partial differential coefficient of the disturbing function with the coefficient* $+1$ *or* -1, *as is known to be the case with the last-mentioned system of elements;* in other words, if $a, \ldots$ and $b, \ldots$ be a system of elements corresponding to each other in pairs, such that

$$\frac{dp}{da}=-\frac{db}{dq},\quad \frac{dp}{db}=\frac{da}{dq},\quad \frac{dq}{da}=\frac{db}{dp},\quad \frac{dq}{db}=-\frac{da}{dp}$$
$$\vdots$$

(where a, b may be replaced by any other corresponding pair of elements, and p, q by any other corresponding pair of variables), then the elements $a, \ldots$ and $b, \ldots$ form a canonical system.

40. Jacobi's note of 1840 in the *Comptes Rendus*, calls attention to the theorem contained in the passage quoted above from Poisson's memoir of 1808, a theorem which Jacobi characterizes as "la plus profonde découverte de M. Poisson," and as the theorem "le plus important de la Mécanique et de cette partie du calcul intégral qui s'attache à l'intégration d'un système d'équations différentielles ordinaires"; and he proceeds, "le théorème dont il est question énoncé convenablement est le suivant—un nombre quelconque de points matériels étant tirés par des forces et soumis à des conditions telles que le principe des forces vives ait lieu, si l'on connaît outre que l'intégrale fournie par ce principe deux autres intégrales, on en peut déduire une troisième d'une manière directe et sans même employer des quadratures. En poursuivant le même procédé on pourra trouver une quatrième, une cinquième intégrale, et en général on parviendra à cette manière à déduire des deux intégrales données toutes

les intégrales, ou ce qui revient au même l'intégration complète du problême. *Dans des cas particuliers on retombera sur une combinaison des intégrales déjà trouvées avant qu'on soit parvenu à toutes les intégrales du problême, mais alors les deux intégrales données jouissent des propriétés particulières dont on peut tirer un autre profit pour l'intégration des équations dynamiques proposées.* C'est ce qu'on verra dans un ouvrage auquel je travaille depuis plusieurs années et dont peut-être je pourrai bientôt faire commencer l'impression."

41. Liouville's addition to Jacobi's letter of 1840.—This contains the demonstration of a theorem similar to that given in Jacobi's letter of 1836, and Poisson's memoir of 1837, but somewhat more general; the system considered is a system of four differential equations of the first order:

$$\frac{dx}{dt}=\lambda\frac{dU}{dx'},\quad \frac{dx'}{dt}=-\lambda\frac{dU}{dx},\quad \frac{dy}{dt}=\lambda\frac{dU}{dy'},\quad \frac{dy'}{dt}=-\lambda\frac{dU}{dy},$$

where U is a function of x, y, x', y', and λ is a function of x, y, x', y' and t. One integral is $U=a$, and if there be another integral $V=b$ where V is a function of x, y, x', y' only, then x', y' being by means of these two integrals expressed as a function of x, y, a, b, it is shown that $x'dx+y'dy$ is an exact differential, and putting $\int(x'dx+y'dy)=\theta$, then that $\frac{d\theta}{db}=\beta$ is a new integral of the given equations; and in the case where λ is a function of t only, the remaining integral is $\frac{d\theta}{da}=\int\lambda dt+\alpha$.

42. Binet's memoir of 1841 contains an exposition of the theory of the variation of the arbitrary constants as applied to the general system of equations

$$\frac{d}{dt}\frac{dF}{dx'}=\frac{dF}{dx'}+\lambda\frac{dL}{dx}+\mu\frac{dM}{dx}+\dots,$$
$$\vdots$$

where F is any function of t, and of the coordinates $x, y, z\dots$ of the different points of the system, and of their differential coefficients x', y', z', &c., and $L=0$, $M=0$, &c. are any equations of equation between the coordinates $x, y, z, \dots$ of the different points of the system; these equations may contain t, but they must not contain the differential coefficients $x', y', z', \dots$ The form is a more general one than that considered by Lagrange and Poisson. The memoir contains an elegant investigation of the variations of the elements of the orbit of a body acted upon by a central force, the expressions for the variations being obtained in a canonical form; and there is also a discussion of the problem suggested in Poisson's report of 1830 on the manuscript work of Ostrogradsky.

43. Jacobi's note of 1842, in the *Comptes Rendus*, announces the general principle (being a particular case of the theorem of the ultimate multiplier) stated and demonstrated in the memoir next referred to, and gives also the rule for the formation of the multiplier in the case to which the general principle applies.

44. Jacobi's memoir of 1842, "De Motu Puncti singularis": the author remarks, that the greater the difficulties in the general integration of the equations of dynamics, the greater the care which should be bestowed on the examination of the dynamical problems in which the integration can be reduced to quadratures; and the object of the memoir is stated to be the examination of the simplest case of all, viz. the problems relating to the motion of a single point. The first section, entitled, "De Extensione quadam Principii Virium vivarum," contains a remark which, though obvious enough, is of considerable importance: the forces X, Y, Z which act upon a particle, may be such that $Xdx+Ydy+Zdz$ is not an exact differential, so that if the particle were free, there would be no force function, and the equations of motion would not be expressible in the standard form. But if the point move on a surface or a curve, then in the former case $Xdx+Ydy+Zdz$ will be reducible to the form $Pdp+Qdq$, which will be an exact differential if a single condition (instead of the three conditions which are required in the case of a free particle) be satisfied, and in the latter case it will be reducible to the form Pdp, which is, *per se*, an exact differential. In the case of a surface, the requisite transformation is given by the Hamiltonian form of the equations of motion, which Jacobi demonstrates for the case in hand; and then in the third section, with a view to its application to the particular case, he enumerates the general proposition "quæ pro novo principio mechanico haberi potest," which is as follows:

"Consider the motion of a system of material points subjected to any conditions, and let the forces acting on the several points in the direction of the axes be functions of the coordinates alone: if the determination of the orbits of the several points is reduced to the integration of a single differential equation of the first order between two variables, for this equation there may be found, by a general rule, a multiplier which will render it integrable by quadratures only."

And for the particular case the theorem is thus stated:

"Given three differential equations of the first order between the four quantities q_1, q_2, p_1, p_2,

$$dq_1 : dq_2 : dp_1 : dp_2 = \frac{dT}{dp_1} : \frac{dT}{dp_2} : -\frac{dT}{dq_1}+Q_1 : -\frac{dT}{dq_2}+Q_2,$$

in which Q_1, Q_2 are functions of q_1, q_2 only; suppose that there are known two integrals α, β, and that by the aid of these p_1, p_2, $\frac{dT}{dp_1}$, $\frac{dT}{dp_2}$ are expressed by means of the quantities q_1, q_2 and the arbitrary constants α, β; there then remains to be integrated an equation of the first order, $\frac{dT}{dp_1}dq_2-\frac{dT}{dp_2}dq_1=0$ between the quantities q_1, q_2, by which is determined the orbit of the point on the given surface: I say that the left-hand side of the equation multiplied by the factor

$$\frac{dp_1}{d\alpha}\frac{dp_2}{d\beta}-\frac{dp_2}{d\alpha}\frac{dp_1}{d\beta},$$

will be a complete differential, or will be integrable by quadratures alone," and the demonstration of the theorem is given. The remainder of the memoir, sections 4 to 7, is occupied by a very interesting discussion of various important special problems.

45. There is an important memoir by Jacobi, which, as it relates to a special problem, I will merely refer to, viz. the memoir "Sur l'Elimination des Nœuds dans le problême des trois Corps," *Crelle*, t. XXVII. pp. 115—131 (1843). The solution is made to depend upon six differential equations, all of them of the first order except one, which is of the second order, and upon a quadrature.

46. Jacobi's memoir of 1844, "Theoria Novi Multiplicatoris &c."—This is an elaborate memoir establishing the definition and developing the properties of the "multiplier" of a system of ordinary differential equations, or of a linear partial differential equation of the first order, with applications to various systems of differential equations, and in particular to the differential equations of dynamics. The definition of the multiplier is as follows, viz. the multiplier of the system of differential equations

$$dx : dy : dz : dw \ldots = X : Y : Z : W \ldots$$

or of the linear partial differential equation of the first order

$$X\frac{df}{dx} + Y\frac{df}{dy} Z\frac{df}{dz} + W\frac{df}{dw} + \ldots = 0$$

is a function M, such that

$$\frac{dMX}{dx} + \frac{dMY}{dy} + \frac{dMZ}{dz} + \frac{dMW}{dw} + \ldots = 0.$$

One of the properties of the multiplier is that contained in the theorem of the ultimate multiplier, viz. that when all the integrals (except one) of the system of partial differential equations are known, and the system is thereby reduced to a single differential equation between two variables, then the multiplier (in the ordinary sense of the word) of this last equation is $M\nabla$, where M is the multiplier of the system, and ∇ is a given derivative of the known integrals, so that the multiplier of the system being known, the integration of the last differential equation is reduced to a mere quadrature. To explain the theorem more particularly, suppose that the system of given integrals, that is, all the integrals (except one) of the system are represented by $p = \alpha$, $q = \beta, \ldots$, and let u, v be any two functions whatever of the variables, so that $p, q, \ldots u, v$ are in number equal to the system $x, y, z, w, \ldots$ then if

$$X\frac{du}{dx} + Y\frac{du}{dy} + Z\frac{du}{dz} + W\frac{du}{dw} + \ldots = U,$$

$$X\frac{dv}{dx} + Y\frac{dv}{dy} + Z\frac{dv}{dz} + W\frac{dv}{dw} + \ldots = V,$$

the last differential equation takes the form

$$Udv - Vdu = 0,$$

where it is assumed that U and V are, by the assistance of the given integrals, expressed as functions of u, v and the constants of integration. The multiplier of the last-mentioned equation is $M\nabla$, where M is the multiplier of the system, and ∇ may be expressed in either of the two forms

$$\nabla = \frac{\partial(x, y, z, w, \ldots)}{\partial(\alpha, \beta, \ldots u, v)}$$

and

$$\nabla = \left\{\frac{\partial(p, q, \ldots u, v)}{\partial(x, y, z, w, \ldots)}\right\}^{-1};$$

where the symbols on the right-hand sides represent functional determinants; in the first form it is assumed that $x, y, z, w, \ldots$ are expressed as functions of $\alpha, \beta, \ldots u, v$, and in the second form that $p, q, \ldots u, v$ are expressed as functions of $x, y, z, w, \ldots$, but that ultimately $p, q, \ldots$ are replaced by their values in terms of the constants and u, v; the first of the two forms, from its not involving this transformation backwards, appears the more convenient.

47. I have thought it worth while to quote the theorem in its general form, but we may take for u, v any two of the original variables, and if, to fix the ideas, it is assumed that there are in all the four variables x, y, z, w, then the theorem will be stated more simply as follows:—given the system of differential equations

$$dx : dy : dz : dw = X : Y : Z : W,$$

and suppose that two of the integrals are $p = \alpha$, $q = \beta$, the last equation to be integrated will be

$$W dz - Z dW = 0,$$

where, by the assistance of the given integrals, W, Z are expressed as functions of z, w. And the multiplier of this equation is $M\nabla$, where M is the multiplier of the system, and ∇, attending only to the first of the two forms, is given by the equation

$$\nabla = \frac{\partial(x, y)}{\partial(\alpha, \beta)},$$

which supposes that x, y are expressed as functions of α, β, z, w.

48. Jacobi applies the theorem of the ultimate multiplier to the differential equations of dynamics, considered first in the unreduced Lagrangian form, where the coordinates are connected by any given system of equations of condition; secondly, in the reduced or ordinary Lagrangian form; and, thirdly, in the Hamiltonian form. The multiplier can be found for the first two forms, and the expressions obtained are simple and elegant; but, as regards the third form, there is a further simplification: the multiplier M of the system is equal to unity, and the multiplier of the last equation is therefore equal to ∇. The two cases are to be distinguished in which t does not, or does enter into the equations of motion; in the latter case the theorem furnished by the principle of the ultimate multiplier is the same as for the general

case of a system the multiplier of which is known, viz., the theorem is, given all the integrals except one, the remaining integral can be found by quadratures only. But in the former case, which is the ordinary one, including all the problems in which the condition of *vis viva* is satisfied, there is a further consequence deduced. In fact, the time t may be separated from the other variables, and the system of differential equations reduced to a system not involving the time, and containing a number of equations less by unity than the original system, and the theorem of the ultimate multiplier applies to this new system. But when the integrals of the new system have been obtained, the system may be completed by the addition of a single differential equation involving the time, and which is integrable by quadratures; the theorem consequently is, given all the integrals except two, these given integrals being independent of the time, the remaining integrals can be found by quadratures only. This is, in fact, the "Principium generale mechanicum" of the memoir of 1842.

The last of the published writings of Jacobi on the subject of dynamics is the "Auszug zweier Schreiben des Professors Jacobi an Herrn Director Hansen," *Crelle*, t. XLII. pp. 12—31 (1851): these relate chiefly to Hansen's theory of ideal coordinates.

49. The very interesting investigations contained in several memoirs by Liouville (*Liouville*, t. XI. XII. and XIV., and the additions to the "Connaissance des Temps" for 1849 and 1850) in relation to the cases in which the equations of motion of a particle or system of particles admit of integration, are based upon Jacobi's theory of the S function, that is, of the function which is the complete solution of a certain partial differential equation of the first order; the equation is given, in the first instance, in rectangular coordinates, and the author transforms it by means of elliptic coordinates or otherwise, and he then inquires in what cases, that is, for what forms of the force function, the equation is one which admits of solution. A more particular account of these memoirs does not come within the plan of the present report.

50. Desboves' memoir of 1848 contains a demonstration of the two theorems given in Jacobi's note of 1837, in the *Comptes Rendus;* and, as the title imports, there is an application of the theory to the problem of the planetary perturbations; the author refers to the above-mentioned memoirs of Liouville as containing a solution of the partial differential equation on which the problem depends, and also to a memoir of his own relating to the problem of two centres, where the solution is also given; and from this he deduces the solution just referred to, and which is employed in the present memoir. Jacobi's theorem gives at once the formulæ for the variation of the arbitrary constants contained in the solution. The material thing is to determine the signification of these constants, which can of course be done by a comparison of the formulæ with the known formulæ of elliptic motion; the author is thus led to a system of canonical elements similar to, but not identical with, those obtained by Jacobi.

51. Serret's two notes of 1848 in the *Comptes Rendus*.—These relate to the theory of Jacobi's S function, that is, of the function considered as the complete solution of a given partial differential equation of the first order. In the first of the two notes,

which relates to a single particle, the author gives a demonstration founded on a particular choice of variables, viz., those which determine orthotomic surfaces to the curve described by the moving point. The process seems somewhat artificial.

52. Sturm's note of 1848, in the *Comptes Rendus*, relates to the theory of Jacobi's S function, that is, of this function considered as the complete solution of a given partial differential equation of the first order. The force function is considered as involving the time t, which, however, is no more than had been previously done by Jacobi.

53. Ostrogradsky's note of 1848.—This contains an important step in the theory of the forms of the equations of motion, viz. it is shown how, in the case where the force function contains the time, the equations of motion may be transformed from the form of Lagrange to that of Sir W. R. Hamilton. If, as before, the force function (taken with the contrary sign to that of Lagrange) is represented by U, then putting, as before, $T+U=Z$ (the author writes V instead of Z), in the case under consideration Z will contain not only terms of the second order and terms of the order zero in the differential coefficients of the coordinates $q, \ldots$, but also terms of the first order, that is, Z will be of the form $Z=Z_2+Z_1+Z_0$, and putting $H=Z_2-Z_0$, this new function H being expressed as a function of the coordinates $q, \ldots$ and of the new variables $p, \ldots$, then the equations of motion take the Hamiltonian form, viz.—

$$\frac{dq}{dt}=\frac{dH}{dp}, \quad \frac{dp}{dt}=-\frac{dH}{dq};$$
$$\vdots$$

in the theory of the transformation, as originally given by Sir W. R. Hamilton, $Z_2=T$, $Z_1=0$, $Z_0=U$, and, consequently, $H=Z_2-Z_0=T-U$ as before.

[Ostrogradsky's memoir of 1850.—This among other important researches contains, and that *in the most general form*, the transformation of the equations of motion from the Lagrangian to the Hamiltonian form, and indeed the transformation of the general isoperimetric system (that is the system arising from any problem in the calculus of variations) to the Hamiltonian form.]

54. Brassinne's memoir of 1851.—The author reproduces for the Lagrangian equations of motion $\frac{d}{dt}\frac{dZ}{d\xi'}-\frac{dZ}{d\xi}=0$, &c. the demonstration of the theorem

$$\frac{d}{dt}\left(\delta\frac{dZ}{d\xi'}\Delta\xi-\Delta\frac{dZ}{d\xi'}\delta\xi+\ldots\right)=0;$$

and he shows that a similar theorem exists with regard to the system

$$-\frac{d^2}{dt^2}\frac{dZ}{d\xi''}+\frac{d}{dt}\frac{dz}{d\xi'}-\frac{dZ}{d\xi}=0;$$
$$\vdots$$

and with respect to the corresponding system of the mth order. The system in question, which is, in fact, the general form of the system of equations arising from

a problem in the calculus of variations, had previously been treated of by Jacobi, but the theorem is probably new. In conclusion, the author shows in a very elegant manner the interdependence of the theorem relating to Lagrange's coefficients (a, b), and of the corresponding theorem for the coefficients of Poisson.

55. Bertrand's memoir of 1851, "On the Integrals common to several Mechanical Problems," is one of great importance, but it is not very easy to explain its relation to other investigations. The author remarks that, given the integral of a mechanical problem, it is in general a question admitting of determinate solution to find the expression for the forces; in other words, to determine the problem which has given rise to the integral; at least, this is the case when it is assumed that the forces are functions of the coordinates, without the time or the velocities; and he points out how the solution of the question is to be obtained. But, in certain cases, the method fails, that is, it leads to expressions which are not sufficient for the determination of the forces; these are the only cases in which the given integral can belong to several different problems; and the method shows the conditions necessary in order that these cases may present themselves. It is to be remarked that the given integral must be understood to be one of an absolutely definite form, such for instance as the equations of the conservation of the motion of the centre of gravity or of areas, but not such as the equation of *vis viva*, which is a property common indeed to a variety of mechanical problems, but which involves the forces, and is therefore not the *same* equation for different problems. The author studies in particular the case where the system consists of a single particle; he shows, that when the motion is in a plane, the integrals capable of belonging to two or more different problems are two in number, each of them involving as a particular case the equation of areas. When the point moves on a surface, he arrives at the remarkable theorem—"In order that the equations of motion of a point moving on a surface may have an integral independent of the time, and common to two or more problems, it is necessary that the surface should be a surface of revolution, or one which is developable upon a surface of revolution." When the condition is satisfied, he gives the form of the integral, and the general expression of the forces in the problems for which such integral exists. He examines, lastly, the general case of a point moving freely in space. The number of integrals common to several problems is here infinite. After giving a general form which comprehends them all, the author shows how to obtain as many particular forms as may be desired: it is, in fact, only necessary to resolve any problem relative to motion in a plane, and to effect a certain simple transformation on the integrals; one thus obtains a new equation which is the integral of an infinite number of different problems relating to the motion in space.

As an instance of the analytical forms on which these remarkable results depend, I quote the following, which is one of the most simple:—"If an integral of the equations of motion of a point in a plane belongs to two different problems, it is of the form

$$\alpha = F(\phi', x, y, t),$$

where ϕ' is the derivative with respect to t, of a function of x, y, which equated to zero gives the equation of a system of right lines."

56. Bertrand's memoir of 1851, "On the Integration of the Differential Equations of Dynamics."—The author refers to Jacobi's note of 1840, in relation to Poisson's theorem; and after remarking that there are very few problems of which two integrals are not known, and which therefore might not be solved by the method if it never failed; he observes that unfortunately there are (as was known) cases of exception, and that, as his memoir shows, these cases are far more numerous than those to which the method applies; thus for example the equation of *vis viva*, combined with any other integral whatever, leads to an illusory result. The theorem of Poisson may lead to an illusory result in two ways; either the resulting integral may be an identity $0 = 0$, or it may be an integral contained in the integrals already known, and which consequently does not help the solution. It appears by the memoir that the two cases are substantially the same, and that it is sufficient to study the case in which the two integrals lead to the identity $0 = 0$. Suppose that one integral is given, the author shows that there always exist integrals which, combined with the given integral, lead to an illusory result, and he shows how the integrals which, combined with the given integral, leads to such illusory result, are to be obtained.

For instance, in the case of a body moving round a fixed centre [and attracted to it by a force which is a function of the distance], there are here two known integrals; first, the equation of *vis viva* (but this, as already remarked, combined with any other integral whatever, leads to an illusory result); secondly, the equation of areas. The question arises, what are the integrals which, combined with the equation of areas, lead to an illusory result? The integrals in question are, in fact, the other two integrals of the problem; so that the inquiry into the integrals which give an illusory result, leads here to the completion of the solution.

The like happens in two other cases which are considered, viz. first, the problem of motion *in plano* when the force function is a homogeneous function of the coordinates of the degree -2; secondly, the problem of two fixed centres. Indeed the case is the same for all problems whatever, where the coordinates of the points of the system can be expressed by means of two independent variables.

The next problem considered is that where two bodies attract each other, and are attracted to a fixed centre. Suppose, first, the motion is *in plano*, then as in the former case *all* the integrals will be found by seeking for the integrals which, combined with the equation of areas, give an illusory result. When the motion is in space, the principle of areas furnishes three integrals (the equation of *vis viva* is contained in these three equations); the integrals which, combined with the integrals in question, give illusory results, are eight in number, and, to complete the solution, there must be added to these one other integral, which alone does not put the method in default. The problem of three bodies is then shown to be reducible to the last-mentioned problem; and the same consequences therefore hold good with respect to the problem of three bodies, viz., there are eight integrals which, combined with the integrals furnished by the principle of areas, give illusory results. To complete the solution it would be necessary to add to these a ninth integral, which alone would not put the method into default.

57. The author remarks that it appears by the preceding enumeration that the method of integration, based on the theorem of Poisson, is far from having all the

importance attributed to it by Jacobi. The cases of exception are numerous; they constitute, in certain cases, the complete solutions of the problems, and embrace in other cases eleven integrals out of twelve. But it would be a misapprehension of his meaning to suppose that, according to him, the cases in which Poisson's theorem is usefully applicable ought to be considered as exceptions. The expression would not be correct even for the problems which are completely resolved in seeking for the integrals which put the method into default; there exists for these problems, it is true, a system of integrals which give illusory results; but these integrals, combined in a suitable manner, might furnish others to which the theorem could be usefully applied.

The author remarks, that, in seeking the cases of exception to Poisson's theorem, there is obtained a new method of integration, which may lead to useful results; and, after referring to Jacobi's memoir on the elimination of the nodes in the problem of three bodies, he remarks that, by his own new method, the problem is reduced to the integration of six equations, all of them of the first order; so that he effectuates one more integration than had been done by Jacobi [this is incorrect]; and he refers to a future memoir, (not, I believe, yet published) [? the Memoir of 1857] for the further development of his solution.

58. To give an idea of the analytical investigations, the equations of motion are considered under the Hamiltonian form

$$\frac{dq}{dt}=\frac{dH}{dp},\quad \frac{dp}{dt}=-\frac{dH}{dq},$$

where H is any function whatever of $q, \ldots p, \ldots$ without t, and then a given integral being

$$\alpha=\phi(q, \ldots p, \ldots),$$

the question is shown to resolve itself into the determination of an integral $\beta=\psi(q, \ldots p, \ldots)$ such that identically $(\alpha, \beta)=0$ or else $(\alpha, \beta)=1$, where (α, β) represents, as before, Poisson's symbol, viz.

$$(\alpha, \beta)=\frac{\partial(\alpha, \beta)}{\partial(q, p)}+\ldots,$$

if for shortness

$$\frac{\partial(\alpha, \beta)}{\partial(q, p)}=\frac{\partial\alpha}{dq}\frac{d\beta}{dp}-\frac{d\alpha}{dp}\frac{d\beta}{dq}.$$

The partial differential equations $(\alpha, \beta)=0$ or $(\alpha, \beta)=1$, satisfied by certain integrals β, are in certain cases, as Bertrand remarks, a precious method of integration leading to the classification of the integrals of a problem, so as to facilitate their ulterior determination: it is in fact by means of them that the several results before referred to are obtained in the memoir.

59. Bertrand's note of 1852 in the *Comptes Rendus*.—This contains the demonstration of a theorem analogous to Poisson's theorem $(\alpha, \beta)=\text{const.}$, but the function on the left-hand side is a function involving four of the arbitrary constants and binary combinations of pairs of corresponding variables, instead of two arbitrary constants and the series of pairs of corresponding variables.

60. Bertrand's notes, VI. and VII., to the third edition of the *Mécanique Analytique*, 1853, contain a concise and elegant exposition of various theorems which have been considered in the present report. The latter of the two notes relates to the above-mentioned theorem of Poisson, and places the theorem in a very clear light, in fact, establishing its connexion with the theory of canonical integrals. Bertrand in fact shows, that, given any integral α of the differential equations (in the last-mentioned form, the whole number of equations being $2k$), then the solution may be completed by joining to the integral α a system of integrals $\beta_1, \beta_2 \ldots \beta_{2k-1}$, which, combined with the integral α, give to Poisson's equation an identical form, viz. which are such that

$$(\alpha, \beta_1) = 1, \ (\alpha, \beta_2) = 0, \ldots (\alpha, \beta_{2k-1}) = 0.$$

This, he remarks, shows, that, given any integral α, the solution of the problem *may* be completed by integrals $\beta_1, \beta_2 \ldots \beta_{2k-1}$, which, combined with α, give all of them an identical form to the theorem of Poisson. But it is not to be supposed that all the integrals of the problem are in the same case. In fact, the most general integral is $\eta = \varpi(\alpha, \beta_1, \beta_2 \ldots \beta_{2k-1})$, and it is at once seen that $(\alpha, \eta) = (\alpha, \beta_1) \dfrac{d\eta}{d\beta_1} = \dfrac{d\eta}{d\beta_1}$, consequently the expression (α, η) will not be identically constant unless $\dfrac{d\eta}{d\beta_1}$ is so: but the integrals, in number infinite, which result from the combination of α, with $\beta_2, \beta_3 \ldots \beta_{2k-1}$ combined with the integral α, give identical results. Only the integrals which contain β_1 lead to results which are not identical. The integrals α and β_1, connected together in the above special manner, are termed by the author *conjugate integrals.*

61. Brioschi's two notes of 1853.—The memoir "Sulla Variazione &c." contains reflections and developments in relation to Bertrand's method of integration and to canonical systems of integrals, but I do not perceive that any new results are obtained.

The note, "Intorno ad un Teorema di Meccanica," contains a demonstration of the theorem in Bertrand's note of 1852 in the *Comptes Rendus*, and an extension of the theorem to the case of a combination of any even number of the arbitrary constants; the value of the symbol is shown by the theory of determinants to be a function of the Poissonian coefficients (α, β), and as these are constants, the value of the symbol considered is also constant.

62. Liouville's note of the 29th of June 1853 [1], contains the enunciation of a theorem which completes the investigations contained in Poisson's memoir of 1837. The equations considered are the Hamiltonian equations in their most general form, viz., H is any function whatever of t and the other variables: it is assumed that

[1] The date is that of the communication of the note to the Bureau of Longitudes, but the note is only published in Liouville's *Journal* in the May Number for 1855, which is subsequent to the date of the second part of Professor Donkin's memoir in the *Philosophical Transactions*, which contains the theorem in question. I have not had the opportunity of seeing a thesis by M. Adrien Lafon, Paris, 1854, where Liouville's theorem is quoted and demonstrated.

half of the integrals are known, and that the given integrals are such that for any two of them α, β, Poisson's coefficient (α, β) is equal to zero; this being so, the expression $pdq + \ldots - Hdt$, where, by means of the known integrals, the variables $p, \ldots$ are expressed in terms of $q, \ldots t$, is a complete differential in respect to $q, \ldots t$, viz. it will be the differential of Sir W. R. Hamilton's principal function V, which is thus determined by means of the known integrals, and the remaining integrals are then given at once by the general theory.

63. Professor Donkin's memoir of 1854 and 1855, Part I. (sections 1, 2, 3, articles 1 to 48).—The author refers to the researches of Lagrange, Poisson, Sir W. R. Hamilton, and Jacobi, and he remarks that his own investigations do not pretend to make any important step in advance. The investigations contained in section 1, articles 1 to 14, establish by an inverse process (that is, one setting out from the integral equations) the chief conclusions of the theories of Sir W. R. Hamilton and Jacobi, and in particular those relating to the canonical system of elements as given by Jacobi's theory. The theorem (3), article 1, which is a very general property of functional determinants, is referred to as probably new. The most important results of this portion of the memoir are recapitulated in section 4, in the form of seven theorems there given without demonstration; some of these will be presently again referred to. Articles 17 and 18 contain, I believe, the only demonstration which has been given of the equivalence of the generalised Lagrangian and Hamiltonian systems. The transformation is as follows: the generalised Lagrangian system is

$$\frac{d}{dt}\frac{dZ}{dq'} = \frac{dZ}{dq},$$
$$\vdots$$

where Z is any function of t and of $q, \ldots q', \ldots$; writing herein $\frac{dZ}{dq'} = p, \ldots$, and also $H = -Z + q'p + \ldots$, where, on the right-hand side, $q', \ldots$ are expressed in terms of $t, q, \ldots p, \ldots$, so that H is a function of $t, q, \ldots p, \ldots$; then the theorems in the preceding articles show that

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq},$$
$$\vdots \qquad \vdots$$

which is the generalised Hamiltonian system.

In section 2, articles 21 and 22, there is an elegant demonstration, by means of the Hamiltonian equations, of the theorem in relation to Poisson's coefficients (a, b), viz., that these coefficients are functions of the elements only. And there are contained various developments as to the consequences of this theorem; and as to systems of canonical, or, as the author calls them, *normal* elements. The latter part of the section and section 3, relate principally to the special problems of the motion of a body under the action of a central force, and of the motion of rotation of a solid body.

64. Part II. (sections 4, 5, 6 and 7, articles 49—93, appendices).—Section 4 contains the seven theorems before referred to. Although not given as new theorems, yet, to a considerable extent, and in form and point of view, they are new theorems.

Theorem 1 is a theorem standing apart from the others, and which is used in the demonstration of the transformation from the Lagrangian to the Hamiltonian system. It is as follows: viz., if X be a function of the n variables $x, \ldots$, and if $y, \ldots$ be n other variables connected with these by the n equations

$$\frac{dX}{dx} - y, \ldots$$

then will the values of $x, \ldots$, expressed by means of these equations in terms of $y, \ldots$, be of the form

$$x = \frac{dY}{dy}, \ldots$$

and if p be any other quantity explicitly contained in X, then also

$$\frac{dX}{dp} + \frac{dY}{dp} = 0,$$

the differentiation with respect to p being in each case performed only so far as p appears explicitly in the function.

The value of Y is given by the equation

$$Y = -X + xy + \ldots$$

where, on the right-hand side, $x, \ldots$ are expressed in terms of $y, \ldots$

Theorems 2, 3 and 4, and a supplemental theorem in article 50, relate to the deduction of the generalised Hamiltonian system of differential equations from the integral equations assumed to be known. In fact (writing V, $q, \ldots p, \ldots b, \ldots a, \ldots$, instead of the author's X, $x_1, \ldots x_n$, $y_1, \ldots y_n$, $a_1, \ldots a_n$, $b_1, \ldots b_n$), it is assumed that V is a given function of t, of the n variables $q, \ldots$, and of the n constants $b, \ldots$, and that the n variables $p, \ldots$, and the n constants $a, \ldots$, are determined by the conditions

$$\frac{dV}{dq} = p, \ldots \quad (1)$$

$$\frac{dV}{db} = a, \ldots \quad (2)$$

so that in fact by virtue of these $2n$ equations the $2n$ variables X, $q, \ldots p, \ldots$ may be considered as functions of t, and the $2n$ constants $b, \ldots a, \ldots$ (hypothesis 1), or conversely, the $2n$ constants $b, \ldots a, \ldots$, may be considered as functions of t and of the $2n$ variables $q, \ldots p, \ldots$ (hypothesis 2).

Theorem 2 is as follows: viz., if from the $2n$ equations (1, 2) and their total differential coefficients with respect to t, the $2n$ constants be eliminated, there will result the following $2n$ simultaneous differential equations of the first order, viz.:

$$\frac{dq}{dt} = \frac{dH}{dp}, \ldots$$

$$\frac{dp}{dt} = -\frac{dH}{dq}, \ldots$$

H is here a function of $q, \ldots p, \ldots$ (which will in general also contain t explicitly), given by the equation

$$H = -\frac{dV}{dt},$$

where, on the right-hand side, the differential coefficient $\frac{dV}{dt}$ is taken with respect to t, in so far as t appears explicitly in the original expression for V in terms of $q, \ldots b, \ldots$ and t, and after the differentiation, $b, \ldots$, are to be expressed in terms of the variables and t, by means of the equations (1).

Theorem 3 is, that there exists the following relations, viz.:

$$\frac{dq}{db} = -\frac{da}{bp}, \quad \frac{dq}{da} = \frac{db}{dp}, \ldots$$

$$\frac{dp}{db} = \frac{da}{dq}, \quad \frac{dp}{da} = -\frac{db}{dq}, \ldots$$

where (p, q) are *any* corresponding pair out of the systems $p, \ldots$ and $q, \ldots$, and (b, a) are any corresponding pair out of the systems $b, \ldots$ and $a, \ldots$, so that the total number of equations is $4n^2$: in each of the equations the left-hand side refers to hypothesis 1, and the right-hand side to hypothesis 2.

To these theorems should be added the supplemental theorem contained in article 50, viz., that there subsists also the system of equations

$$\frac{db}{dt} = \frac{dH}{da}, \ldots$$

$$\frac{da}{dt} = -\frac{dH}{db}, \ldots$$

where the left-hand sides refer to hypothesis 2, while the right-hand sides refer to hypothesis 1; as before $H = -\frac{dV}{dt}$, but here H is differentially expressed, being what the H of theorem 3 becomes when the variables are expressed according to hypothesis 1.

In theorem 4 the author's symbol (p, q) has a signification such as Poisson's (a, b), and if we write as before

$$(a, b) = \frac{\partial(a, b)}{\partial(p, q)} + \dots,$$

where

$$\frac{\partial(a, b)}{\partial(p\ q)} = \frac{da}{dp}\frac{db}{dq} - \frac{db}{dp}\frac{da}{dq}$$

(this refers of course to hypothesis 2), the theorem is, that the following equations subsist identically, viz., b, a being corresponding constants out of the two series $b, \dots$ and $a, \dots$, then

$$(b, a) = -(a, b) = 1,$$

but that for any other pairs b, a, or for any pairs whatever b, b or a, a, the corresponding symbol is $= 0$: in fact, that the constants $b, \dots$ and $a, \dots$ form a canonical system of elements.

Theorem 5 is a theorem including theorem 4, and relating to any two functions u, v either of the two $2n$ constants or else of the $2n$ variables, and which may besides contain t explicitly; it establishes, in fact, a relation between Poisson's coefficient (u, v) and the corresponding coefficient of Lagrange.

Theorem 6 is as follows: viz., if $q, \dots p, \dots$ are any $2n$ variables concerning which no supposition is made, except that they are connected by the n equations

$$b = \phi(q, \dots p, \dots),$$

which equations are only subject to the condition of being sufficient for the determination of $p, \dots$ in terms of $q, \dots$ and $a, \dots$, and they may contain explicitly any other quantities, for example, a variable t. Then, in order that the $\frac{1}{2}n(n-1)$ equations

$$\frac{dp_i}{dq_j} = \frac{dp_j}{dq_i}$$

may subsist identically, it is only necessary that each of the $\frac{1}{2}n(n-1)$ equations $(b_i, b_j) = O$ may be satisfied identically.

Theorem 7 is, in fact, the theorem previously established in its general form in Liouville's note of the 29th of June, 1853, viz., if, of the system of $2n$ differential equations

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq},$$

there be given n integrals involving the n arbitrary constants $b, \dots$, so that each of these constants can be expressed as a function of the variables $q, \dots, p, \dots$ (with or without t); then, *if the* $\frac{1}{2}n(n-1)$ *conditions* $(b_i, b_j) = O$ *subsist identically, the remaining* n *integrals can be found as follows:*—By means of the n integrals, let the n variables

$p, \ldots$ be expressed in terms of $x, \ldots$ $b, \ldots$ and t, and let H stand for what H, as originally given, becomes when $q, \ldots$ are thus expressed. Then the values of $p, \ldots$ and $-H$ are the differential coefficients of one and the same function of $p, \ldots$ and t; call this function V, then, since its differential coefficients are all given (by the equations $\frac{dV}{dq} = p, \ldots \frac{dV}{dt} = -H$), V may be found by integration; and it is therefore to be considered as a given function of $p, \ldots$ and t and of the constants $b, \ldots$. The remaining n integrals are given by the n equations

$$\frac{dV}{db} = a, \ldots$$

where the n quantities $a, \ldots$ are new arbitrary constants.

65. Section 5 of the memoir relates to the theory of the variation of the elements considered in relation to the following very general problem: viz., $Q, \ldots$ $P, \ldots$ being any functions whatever of the $2n$ variables $q, \ldots$ $p, \ldots$ and t; it is required to express the integrals of the system $2n$ differential equations

$$\frac{dq}{dt} = P, \quad \frac{dp}{dt} = Q,$$
$$\vdots \qquad \vdots$$

in the same form as the integrals (supposed given) of the standard system

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq},$$
$$\vdots \qquad \vdots$$

by substituting functions of t for the constant elements of the latter system. And section 6 contains some very general researches on the general problem of the transformation of variables, a problem of which, as the author remarks, the method of the variation of elements is a particular, and not the only useful case. In particular, the author considers what he terms a normal transformation of variables, and he obtains the theorem 8, which includes as a particular case the second of the two theorems in Jacobi's note of 1837, in the *Comptes Rendus*. This theorem is as follows: viz., if the original variables $q, \ldots$ $p, \ldots$ are given by the $2n$ equations

$$\frac{dq}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dq};$$
$$\vdots$$

and if the new variables $\eta, \ldots$ $\varpi, \ldots$ are connected with the original variables by the equations

$$\frac{dK}{dp} = q, \quad \frac{dK}{d\eta} = \varpi;$$
$$\vdots \qquad \vdots$$

where K is any function of $\eta, \ldots p, \ldots$ which may also contain t explicitly, then will the transformed equations be

$$\frac{d\eta}{dt} = \frac{d\Phi}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{d\Phi}{d\eta};$$

in which Φ is defined by the equation

$$\Phi = H - \frac{dK}{dt},$$

and is to be expressed in terms of the new variables, the substitution of the new variables in $\frac{dK}{dt}$ being made after the differentiation. In particular, if K does not contain t explicitly, then $\frac{dK}{dt} = 0$ and $\Phi = H$, so that, in this case, the transformation is effected merely by expressing H in terms of the new variables. There is also an important theorem relating to the *transformation of coordinates.* To explain this, it is necessary to go back to the generalised Lagrangian form

$$\frac{d}{dt}\frac{dZ}{dq'} = \frac{dZ}{dq};$$
$$\vdots$$

where the variables $q, \ldots$ correspond to the coordinates of a dynamical problem; if the new variables $\eta, \ldots$ are any given functions whatever of the original variables $q, \ldots$ and of t, this is what may be termed a transformation of coordinates. But the proposed system can be expressed, as shown in the former part of the memoir, in the generalised Hamiltonian form with the variables $q, \ldots$ and the derived variables $p, \ldots$ (the values of which are given by $\frac{dZ}{dq} = p, \ldots$): the problem is to transform the last-mentioned system by introducing, instead of the original coordinates $q, \ldots$, the new coordinates $\eta, \ldots$, and instead of the derived variables $p, \ldots$ the new derived variables $\varpi, \ldots$ defined by the analogous equations $\frac{dZ}{d\eta} = \varpi, \ldots$, in which Z is supposed to be expressed as a function of $\eta, \ldots$ and t. The method of transformation is given by the theorem 9, which states that the transformation is a normal transformation, and that the modulus of transformation (that is, the function corresponding to K in theorem 8) is

$$K = qp + \ldots$$

where $q, \ldots$ are to be expressed in terms of $\eta, \ldots$. The latter part of the same section contains researches relating to the case where the proposed equations are symbolically, but not actually, in the Hamiltonian form, viz., where the function H is considered as containing functions of $q, \ldots p, \ldots$ which are exempt from differentiation in forming the differential equations (the author calls this a pseudo-canonical system), and where, in like manner, the transformation of variables is a pseudo-normal transformation; the theorems 10 and 11 relate to this question, which is treated still more generally in Appendix C. The general methods are illustrated by applications to the problem of

three bodies and the problem of rotation; the former problem is specially discussed in section 7; but the results obtained (and which, as the author remarks, affords an example of the so-called "elimination of the nodes") do not come within the plan of the present report.

66. Bour's memoir of 1855, "On the Integration of the Differential Equations of Analytical Mechanics."—It has been already seen that the knowledge of half the entire system of the integrals of the differential equations (these known integrals satisfying certain conditions) leads by quadratures only to the knowledge of the remaining integrals; the researches contained in this most interesting and valuable memoir show that this theorem is, in fact, only the last of a series of theorems, here first established, relating to the successive reduction which results from the knowledge of each new integral. Speaking in general terms, it may be stated that the author operates on the linear partial differential equation of the first order, which is satisfied by the integrals of the differential equations; and that he effectuates upon this equation a reduction of two unities in the number of variables for every suitable new integral which is obtained [1]. The author shows also that an equal or greater reduction may sometimes be obtained by means of integrals which appear at first foreign to his method. Before going further, it may be convenient to remark that the author restricts himself to the case in which H is independent of the time, and where, consequently, the condition of *vis viva* is satisfied; it was, however, remarked by Liouville that the analysis, slightly modified, applies to the most general case where H is any function of t and the variables, and it is possible that when the entire memoir is published (it is given in Liouville's *Journal* as an extract), the theory will be exhibited under this more general form.

67. To give an idea of the analytical results, the equations are considered under the form

$$\frac{dp_i}{dt}=\frac{dH}{dq_i},\quad \frac{dq_i}{dt}=-\frac{dH}{dq_i}\,(i=1 \text{ to } i=n)$$

(where, as already remarked, H is independent of t). The integrals admit, therefore, of representation in the canonical form $\alpha, \beta, \alpha_1, \alpha_2, \ldots \alpha_{2n-2}$ where $\alpha(=H)$ is the equation of *vis viva*; $\beta\,(=G-t)$ is the integral conjugate to this, and the only integral involving the time; and the remaining integrals α_1 and α_2, α_3 and $\alpha_4 \ldots \alpha_{2n-3}$ and α_{2n-2} are conjugate pairs: we have

$$(\alpha_1, \alpha)\ (=(\alpha_1, H))=0,\quad (\alpha_1, \beta)\ (=(\alpha_1, G))=0,\quad (\alpha_1, \alpha_2)=1,\quad (\alpha_1, \alpha_3)=0, \ldots (\alpha_1, \alpha_{2n-2})=0.$$

The integrals $\alpha_1, \alpha_2, \ldots \alpha_{2n-2}$ verify the linear partial differential equation

$$\sum_{i=1}^{i=n}\left(\frac{dH}{dq_i}\frac{d\zeta}{dp_i}-\frac{dH}{dp_i}\frac{d\zeta}{dq_i}\right)=0 \text{ or, } (H, \zeta)=0 \qquad (1)$$

[1] I have borrowed this and the next sentence from Liouville's report. It would, I think, be more accurate to say, for every suitable new integral after the first one; in the case considered in the memoir, the condition of *vis viva* is satisfied, and there is always one integral, the equation of *vis viva*, which is known; but this alone, and in the general case the first known integral, will not cause a reduction of two unities.

which is also satisfied by $\zeta = H$, and of which the general solution is $\zeta = \phi(H, \alpha_1, \alpha_2 \ldots \alpha_{2n-2})$, while, on the contrary, the first member of the equation (1), becomes unity for $\zeta = G$, in other words $(H, G) = 1$. The equation (1) replaces the original differential equations; it is to the equation (1) that the theorems of Poisson and Bertrand may be supposed to be applied, and it is this equation (1) which is studied in the memoir, where it it shown how the order may be diminished when one or more integrals are known.

In the first place, the integral $\alpha = H$ which is known, may be made use of to eliminate one of the variables, suppose p_n; the result is found to be

$$\sum_{i=1}^{i=n-1} \left(\frac{dp_n}{dq_i} \frac{d\zeta}{dp_i} - \frac{dp_n}{dp_i} \frac{d\zeta}{dq_i} \right) + \frac{d\zeta}{dq_n} = 0, \tag{2}$$

which has the same integrals as the equation (1), except the integral of *vis viva* $\zeta = H$; it is this equation (2) which would have to be integrated if only the integral α were known.

Suppose now there is known a new integral α_1; this gives rise to the partial differential equation

$$\sum_{i=1}^{i=n} \left(\frac{d\alpha_1}{dq_i} \frac{d\zeta}{dp} - \frac{d\alpha_1}{dp_i} \frac{d\zeta}{dq_i} \right) = 0 \text{ or } (\alpha_1, \zeta) = 0, \tag{4}$$

which is satisfied by $\zeta = H, G, \alpha_1, \alpha_2, \alpha_4 \ldots \alpha_{2n-2}$, but not by α_2, which gives $(\alpha_1, \alpha_2) = 0$. The equation (4) is satisfied by $\zeta = H$, and it may be therefore transformed in the same manner as the equation (1) was, viz. p_n may be expressed in terms of the other variables and of α. The author remarks that it will happen, what causes the success of the method, that this operation (the object of which is to get rid of the solution $\zeta = H$) conducts to two different equations, according as $\zeta = G$ or $\zeta =$ any other integral of the equation (4); so that in the second form of the transformed equation the unknown integral $\zeta = G$ is also eliminated. This second form is found to be

$$\sum_{i=1}^{i=n-1} \left(\frac{d\alpha_1}{dq_i} \frac{d\zeta}{dp_i} - \frac{d\alpha_1}{dp_i} \frac{d\zeta}{dq_i} \right) = 0 \text{ or } (\alpha_1, \zeta) = 0, \tag{5}$$

which is precisely similar to the equation (1) (only the number of variables is diminished by two unities), and is possessed of the same properties. Its integrals are $\alpha_1, \alpha_3, \alpha_4 \ldots \alpha_{2n-2}$, which are all of them integrals of the problem, and give $(\alpha_1, \alpha_i) = 0$. And the theorems of Poisson and Bertrand apply equally to this equation; the only difference is, that the number of terms in the expressions (α, β) is less by two unities. A new integral (α_3) leads in like manner to an equation (8) similar to (5), but with the number of variables further diminished by two unities, and so on, until the half series of integrals $\alpha, \alpha_1, \alpha_3 \ldots \alpha_{2n-3}$ are known; the conjugate integrals $\beta, \alpha_2, \alpha_4 \ldots \alpha_{2n-2}$ are then obtained by quadratures only, in the method explained in the memoir, and which is in fact identical with that given by the theorem of Liouville and Donkin. The memoir contains other results, which have been already alluded to in a general manner; some of these are made use of by the author in his "Mémoire sur le problême des trois corps," *Journal École Polyt.*, t. XXI. pp. 35—58 (1856).

68. Liouville's note of July, 1855, on the occasion of Bour's memoir, mentions that the author of the memoir had recognized that, according to the remark made to him, his formulæ subsist with even increased elegance when H is considered as a function of t and the other variables. But (it is remarked) the general case can be always reduced to the particular one considered in the memoir, provided that the number of equations is augmented by two unities by the introduction of the new variables τ and u, the former of them, τ, equal to $t+$ constant, so that

$$\frac{dt}{d\tau}=1$$

the latter of them, u, defined by the equation

$$\frac{du}{d\tau}=-\frac{dH}{dt}.$$

Suppose in fact that

$$V=H+u,$$

then, since τ and u do not enter into H, which is a function only of t and the variables $q, \ldots p, \ldots$, we have

$$\frac{dV}{du}=1=\frac{dt}{d\tau};$$

and, moreover, the differential coefficients with respect to $t, q, \ldots p, \ldots$ of the functions H and V are equal. The system may be written

$$\frac{dt}{d\tau}=\frac{dV}{du}, \quad \frac{du}{d\tau}=-\frac{dV}{dt},$$

$$\frac{dp}{d\tau}=\frac{dV}{dq}, \quad \frac{dq}{d\tau}=-\frac{dV}{dp},$$

$$\vdots \qquad \vdots$$

which is a system containing two more variables, but in which V is independent of the variable τ, which stands in the place of t. The transformation is an elegant and valuable one, but it is not in anywise to be inferred that there is any advantage in considering the particular case (which is thus shown to be capable of including the general one), rather than the general one itself: such inference does not seem to be intended, and would, I think, be a wrong one.

69. Brioschi's note of 1855 contains an elegant demonstration (founded on the theory of skew determinants) of a property, which appears to be a new one, of the canonical integrals of a dynamical problem, viz. if q, p stand for a corresponding pair of the variables $q, \ldots p, \ldots$ then

$$\Sigma\frac{\partial(\alpha, \beta)}{\partial(q, p)}=1$$

where the summation refers to all the different pairs of conjugate integrals α, β of the canonical system, the pair q, p in the denominator being the same in each term;

but if the variables in the denominator are a non-corresponding pair out of the two series $q, \ldots$ and $p, \ldots$, or else a pair out of one series only (that is, both q's or both p's), then the expression on the left-hand side is equal to zero. This is in fact a sort of reciprocal theorem to the theorem which defines the canonical system of integrals. There are two or three memoirs of Brioschi in Crelle's *Journal* connected with this note and the note of 1853; but as they relate professedly to skew determinants and not to the equations of dynamics, it is not necessary here to refer to them more particularly.

70. Bertrand's memoir of 1857 forms a sequel to the memoir of 1851, on the integrals common to several problems of mechanics. The author calls to mind that he has shown in the first memoir, that, given an integral of a mechanical problem, and assuming only that the forces are functions of the coordinates, it is possible to determine the problem and find the forces which act upon each point; and (he proceeds) it is important to remark, that the solution leads often to contradictory results,—that, in fact, an equation assumed at hazard is not in general an integral of any problem whatever of the class under consideration: and he thereupon proposes to himself in the present memoir to develope some of the consequences of this remark, and to seek among the most simple forms, the equations which can present themselves as integrals, and the problems to which such integrals belong. The various special results obtained in the memoir are interesting and valuable.

71. In what precedes I have traced as well as I have been able the series of investigations of geometers in relation to the subject of analytical dynamics. The various theorems obtained have been in general stated with sufficient fulness to render them intelligible to mathematicians; the attempt to state them in a uniform notation and systematic order would be out of the province of the present report. The leading steps are,—first, the establishment of the Lagrangian form of the equations of motion; secondly, Lagrange's theory of the variation of the arbitrary constants, a theory perfectly complete in itself; and it would not have been easy to see *à priori* that it would be less fruitful in results than the theory of Poisson; thirdly, Poisson's theory of the variation of the arbitrary constants, and the method of integration thereby afforded; fourthly, Sir W. R. Hamilton's representation of the integral equations by means of a single characteristic function determinable *à posteriori* by means of the integral equations assumed to be known, or by the condition of its simultaneous satisfaction of two partial differential equations; fifthly, Sir W. R. Hamilton's form of the equations of motion; sixthly, Jacobi's reduction of the integration of the differential equations to the problem of finding a complete integral of a single partial differential equation, and the general theory of the connexion of the integration of a system of ordinary differential equations, and of a partial differential equation of the first order, a theory, however, of which Jacobi can only be considered as the second founder; seventhly, the notion (arising from the researches of Lagrange and Poisson) and ulterior development of the theory of a system of canonical integrals.

I remark in conclusion, that the differential equations of dynamics (including in the expression, as I have done throughout the report, the generalized Lagrangian and

Hamiltonian forms) are only one of the classes of differential equations which have occupied the attention of geometers. The greater part of what has been done with respect to the general theory of a system of differential equations is due to Jacobi, and he has also considered in particular, besides the differential equations of dynamics, the Pfaffian system of differential equations (including therein the system of differential equations which arise from any partial differential equation of the first order), and the so-called isoperimetric system of differential equations, that is, the system arising from any problem in the calculus of variations. In a systematic treatise it would be proper to commence with the general theory of a system of differential equations, and as a branch of this general theory, to consider the generalized Hamiltonian system, and in relation thereto to develope the various theorems which have a dynamical application. It would be shown that the generalized Lagrangian form could be transformed into the Hamiltonian form, but the first-mentioned form would, I think, properly be treated as a particular case of the isoperimetric system of differential equations.

List of Memoirs and Works above referred to.

Lagrange. Mécanique Analytique. 1st edition. 1788.

Laplace. Mécanique Céleste, t. I. 1799.

Poisson. Sur les inégalités séculaires des moyens mouvemens des planètes. Read to the Institute 20th June, 1808.—Jour. École Polyt., t. VIII. pp. 1—56. 1808.

Laplace. Mémoire.... Read to the Bureau of Longitudes 17th Aug., 1808.—Forms an Appendix to the 3rd volume of the Mécanique Céleste. 1808.

Lagrange. Mémoire sur la théorie des variations des éléments des planètes et en particulier des variations des grands axes de leurs orbites. Read to the Bureau of Longitudes the 17th Aug. and to the Institute the 22nd Aug., 1808.—Mém. de l'Instit., 1808, pp. 1—72. 1808.

Lagrange. Mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problêmes de la Mécanique. Read to the Institute 13th March, 1809.—Mém. de l'Instit., 1808, pp. 257—302 (includes an undated addition), and there is a Supplement also without date, pp. 363, 364. 1809.

Poisson. Mémoire sur la variation des constantes arbitraires dans les questions de Mécanique. Read to the Institute 16th Oct., 1809.—Jour. École Polyt., t. VIII. pp. 266—344. 1809.

Lagrange. Seconde mémoire sur la variation des constantes arbitraires dans les problêmes de Mécanique, dans lequel on simplifie l'application des formules générales à ces problêmes. Read to the Institute 19th Feb., 1810.—Mém. de l'Instit., 1809, pp. 343—352. 1810.

Lagrange. Mécanique Analytique. 2nd edition, t. I., 1811; t. II., 1813. 1811.

Poisson. Mémoire sur la variation des constantes arbitraires dans les questions de Mécanique. Read to the Academy, 2nd Sept., 1816.—Bulletins de la Soc. Philom., 1816, p. 109; and also, Mém. de l'Instit., t. I. pp. 1—70. 1816.

Hamilton, Sir W. R. On a general method in Dynamics, by which the study of all free systems of attracting or repelling points is reduced to the search and differentiation of one central relation or characteristic function.—Phil. Trans. for 1834, pp. 247—308. 1834.

Hamilton, Sir W. R. Second Essay on a general method in Dynamics.—Phil. Trans. for 1835, pp. 95—144. 1835.

Poisson. Remarques sur l'intégration des équations différentielles de la dynamique.—Liouville, t. II. pp. 317—337. 1837.

Jacobi. Lettre à l'Académie.—Comptes Rendus, t. III. pp. 59—61. 1836.

Jacobi. Zur Theorie der Variations-rechnung und der Differential-gleichungen. Extract from a letter of the 29th Nov. 1836, to Professor Encke, secretary of the mathematical class of the Academy of Berlin.—Crelle, t. XVII. pp. 61—82. 1836.

Jacobi. Ueber die Reduction der Integration der partiellen Differential-gleichungen erster Ordnung zwischen irgend einer Zahl Variabeln auf die Integration eines einzigen Systemes gewöhnlicher Differential-gleichungen.—Crelle, t. XXVII. pp. 97—162, and translated into French, Liouville, t. III. pp. 60—96, and 161—201. 1837.

Jacobi. Note sur l'intégration des équations différentielles de la dynamique.—Comptes Rendus, t. V. p. 61. 1837.

Jacobi. Neues Theorem der analytischen Mechanik.—Monatsbericht of the Academy of Berlin for 1838 (paper is dated 21st Nov., 1838); and Crelle, t. XXX. pp. 117—120. 1838.

Jacobi. Lettre adressée à M. le Président de l'Académie des Sciences à Paris.—Comptes Rendus, t. XI. p. 529; Liouville, t. V. pp. 350—351 (with an addition by Liouville, pp. 351—355). 1840.

Binet. Mémoire sur la variation des constantes arbitraires dans les formules générales de la dynamique et dans un système d'équations analogues plus étendues.—Jour. École Polyt., t. XVII. pp. 1—94. 1841.

Jacobi. Sur un nouveau principe général de la Mécanique Analytique.—Comptes Rendus, t. XV. pp. 202—205. 1842.

Jacobi. De motu puncti singularis.—Crelle, t. XXIV. pp. 5—27. 1842.

Maurice. Mémoire sur la variation des constantes arbitraires comme l'ont établie dans sa généralité les mémoires de Lagrange et celui de Poisson. Read to the Academy the 3rd June, 1844.—Mém. de l'Institut, t. XIX. pp. 553—638. 1844.

Jacobi. Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi.—Crelle, t. XXVII. pp. 199—268; and t. XXIX. pp. 213—279, and 333—376. 1844.

Desboves. Démonstration de deux théorèmes de M. Jacobi, application au problême des perturbations planétaires. Thesis presented to the Faculty of Sciences the 3rd April, 1848.—Liouville, t. XIII. pp. 397—411. 1848.

Serret. Sur l'intégration des équations différentielles du mouvement d'un point matériel.—Comptes Rendus, t. XXVI. pp. 605—610. 1848.

Serret. Sur l'intégration des équations différentielles de la dynamique.—Comptes Rendus, t. XXVI. pp. 639—643. 1848.

Sturm. Note sur l'intégration des équations générales de la dynamique.—Comptes Rendus, t. XXVI. pp. 658—673. 1848.

Ostrogradsky. Sur les intégrales des équations générales de la dynamique.—Mélanges de l'Acad. de St Pétersbourg, $\frac{6}{18}$th Oct., 1848.

[**Ostrogradsky.** Mémoire sur les équations différentielles relatives au problême des Isopérimètres.—Mém. de l'Acad. de St Pétersb., t. VI. pp. 385—517. 1850.]

Brassinne. Théorème relatif à une classe d'équations différentielles simultanées analogue à un théorème employé par Lagrange dans la théorie des perturbations.—Liouville, t. XVI. pp. 283—288. 1851.

Bertrand. Mémoire sur les intégrales communes à plusieurs problêmes de Mécanique. Presented to the Academy the 12th May, 1851.—Liouville, t. XVII. pp. 121—174. 1851.

Bertrand. Mémoire sur l'intégration des équations différentielles de la dynamique.—Liouville, t. XVII. pp. 393—436. 1852.

Bertrand. Sur un nouveau théorème de Mécanique Analytique.—Comptes Rendus, t. XXXV. pp. 698—699. 1852.

Bertrand's notes VI. and VII. to the third edition of the Mécanique Analytique, t. I. pp. 409—428, viz. note VI.—Sur les équations différentielles des problêmes de la Mécanique et la forme que l'on peut donner à leurs intégrales; and note VII.—Sur un théorème de Poisson. 1853.

Brioschi. Sulla variazione delle costanti arbitrarie nei problemi della Dinamica.—Tortolini, Annali, t. IV. pp. 298—311. 1853.

Brioschi. Intorno ad un teorema di Meccanica.—Tortolini, Annali, t. IV. pp. 395—400. 1853.

Liouville. Note sur l'intégration des équations différentielles de la dynamique, présentée au Bureau des Longitudes le 29 Juin, 1853.—Liouville, t. XX. pp. 137—138. 1853.

Donkin, Prof. On a Class of Differential Equations, including those which occur in Dynamical Problems. Part I.—Phil. Trans. 1854, pp. 71—113. Received Feb. 23rd, read Feb. 23rd, 1854. 1854.

Donkin, Prof. On a Class of Differential Equations, including those which occur in Dynamical Problems. Part II.—Phil. Trans. 1855, pp. 299—358. Received Feb. 17th, read March 22nd, 1855. 1855.

Liouville. Rapport sur un mémoire de M. Bour concernant l'intégration des équations différentielles de la Mécanique Analytique.—Comptes Rendus, t. XL. p. 661, séance du 26 Mars, 1855; Liouville, t. XX. pp. 135—136. 1855.

Bour. Sur l'intégration des équations différentielles de la Mécanique Analytique (extrait d'un mémoire présenté à l'Académie le 5 Mars, 1855).—Liouville, t. XX. pp. 185—200. 1855. [And Mém. Savants Étrang. t. XIV. pp. 35—58, 1856.]

Liouville. Note à l'occasion du mémoire précédent de M. Edmond Bour.—Liouville, t. XX. pp. 201—202. 1855.

Brioschi. Sopra una nuova proprietà degli integrali di un problema di dinamica.—Tortolini, t. VI. pp. 430—432. 1855.

Bertrand. Mémoire sur quelqu'une des formes les plus simples que puissent prendre les intégrales des équations différentielles du mouvement d'un point matériel.—Liouville, t. II. (2e série), pp. 113—140. 1857.

196.

NOTE SUR UN PROBLÈME D'ANALYSE INDÉTERMINÉE.

[From the *Nouvelles Annales de Mathématiques* (Terquem and Gerono), t. XVI. (1857), pp. 161—165.]

EULER a donné dans le Mémoire: *Regula facilis problemata Diophantea per numeros integros expedite solvendi* (*Comment. Arith. Coll.*, t. II., p. 263) la solution que voici de l'équation indéterminée

$$\alpha x^2 + \beta x + \gamma = \zeta y^2 + \eta y + \theta; \tag{1}$$

en supposant que l'on ait la solution $x = a$, $y = b$ de manière que

$$\alpha a^2 + \beta a + \gamma = \zeta b^2 + \eta b + \theta,$$

et en posant

$$s = \sqrt{\alpha r^2 + 1},$$

où r est une quantité quelconque, l'équation sera satisfaite par les valeurs

$$x = sa + \zeta rb + \frac{(s-1)\beta}{2\alpha} + \tfrac{1}{2} r\eta,$$

$$y = \alpha ra + sb + \frac{(s-1)\eta}{2\zeta} + \tfrac{1}{2} r\beta;$$

en effet, on voit sans peine que ces valeurs donnent identiquement

$$\alpha x^2 + \beta x + \gamma - (\zeta y^2 + \eta y + \theta) = \alpha a^2 + \beta a + \gamma - (\zeta b^2 + \eta b + \theta) = 0. \tag{2}$$

En supposant de plus que les coefficients α, β, γ, ζ, η, θ soient des nombres entiers tels que $\alpha\zeta$ soit un entier positif non carré, on peut toujours déterminer le nombre entier r de manière que s soit un nombre entier; cela étant, et en supposant que a, b soient des entiers, il est évident que x, y seront des nombres rationnels. Euler a de plus remarqué que l'on peut toujours faire en sorte que x, y soient des nombres entiers. En effet, si les formules donnent $x = a'$, $y = b'$ des valeurs non entières, en

substituant dans les formules au lieu de a, b les valeurs a', b', on obtiendra pour x, y des valeurs entières; cela se vérifie sans peine.

L'équation indéterminée (2) rentre dans celle-ci

$$(a, b, c, f, g, h \between x', y', z')^2 = (a, b, c, f, g, h \between x, y, z)^2; \tag{3}$$

en supposant que la forme ternaire

$$(a, b, c, f, g, h \between x, y, z)^2$$

se transforme en elle-même au moyen d'une substitution linéaire quelconque, on peut supposer que cette substitution soit telle que l'on ait $z' = z$; cela étant, en écrivant $z' = z = 1$ et en mettant de plus $h = 0$, l'équation (3) se réduit évidemment à une forme telle que l'équation (2). Or on peut trouver par la méthode générale de M. Hermite la solution convenable de l'équation (3). En supposant, comme à l'ordinaire,

$$\mathfrak{A} = bc - f^2, \ldots \qquad \mathfrak{F} = gh - af, \ldots$$

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

il faut pour cela écrire

$$x' = 2\xi - x, \quad y' = 2\eta - y, \quad z' = 2\zeta - z,$$

et

$$\begin{aligned} ax + hy + gz &= a\xi + h\eta + g\zeta \qquad - q\mathfrak{E}\eta + q\mathfrak{F}\zeta, \\ hx + bx + fz &= h\xi + b\eta + f\zeta + q\mathfrak{E}\xi \qquad - q\mathfrak{G}\zeta, \\ gx + fy + cz &= g\xi + f\eta + c\zeta - q\mathfrak{F}\xi + q\mathfrak{G}\eta \qquad , \end{aligned}$$

où q est une quantité arbitraire. En effet, en multipliant ces équations par ξ, η, ζ et en ajoutant, on obtient

$$(a, b, c, f, g, h \between \xi, \eta, \zeta \between x, y, z) = (a, b, c, f, g, h \between \xi, \eta, \zeta)^2$$

et, au moyen de cette équation et des valeurs

$$x' = 2\xi - x, \quad y' = 2\eta - y, \quad z' = 2\zeta - z,$$

on forme tout de suite l'équation (3). De plus, en multipliant les trois équations par $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ et en ajoutant, on obtient $Kz = K\zeta$, c'est-à-dire $z = \zeta$ et de là $z' = z$.

Cela étant, les deux équations donnent, en remplaçant ζ par z,

$$\begin{aligned} a\xi + (h - q\mathfrak{E})\,\eta &= ax + hy + (g - q\mathfrak{F})\,z, \\ (h + q\mathfrak{E})\,\xi \qquad + b\eta &= hx + by + (f + q\mathfrak{G})\,z, \end{aligned}$$

et de là, en remarquant que

$$ab - (h - q\mathfrak{E})\,(h + q\mathfrak{E}) = \mathfrak{C} + q^2\mathfrak{E}^2 = \mathfrak{E}\,(1 + q^2\mathfrak{E}),$$

on obtient très-facilement, éliminant successivement η et ξ,

$$(1+q^2\mathfrak{C})\,\xi=(1+qh)\,x+\qquad qby+(\;qf+q^2\mathfrak{G})\,z,$$
$$(1+q^2\mathfrak{C})\,\eta=-\qquad qax+(1-qh)\,y+(-qg+q^2\mathfrak{F})\,z$$

et ensuite

$$(1+q^2\mathfrak{C})\,x'=(1+2qh-q^2\mathfrak{G})\,x+\qquad 2qby+2\,(\;qf+q^2\mathfrak{G})\,z,$$
$$(1+q^2\mathfrak{C})\,y'=\qquad -\;2qax+(1-2qh-q^2\mathfrak{C})\,y+2\,(-qg+q^2\mathfrak{F})\,z,$$

c'est-à-dire, en écrivant $z=z'=1$, les valeurs

$$\left.\begin{aligned}(1+q^2\mathfrak{C})\,x'&=(1+2qh-q^2\mathfrak{G})\,x+\qquad 2qby+2\,(\;qf+q^2\mathfrak{G})\\(1+q^2\mathfrak{C})\,y'&=\qquad -\;2qax+(1-2qh-q^2\mathfrak{C})\,y+2\,(-qg+q^2\mathfrak{F})\end{aligned}\right\}\qquad(4),$$

satisfont identiquement à l'équation

$$(a,\ b,\ c,\ f,\ g,\ h\between x',\ y',\ 1)^2=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2\qquad(5).$$

En prenant $h=0$, on a

$$\mathfrak{C}=ab,\quad \mathfrak{F}=-af,\quad \mathfrak{G}=-bg,$$

et les formules deviennent

$$\left.\begin{aligned}(1+q^2ab)\,x'&=(1-q^2ab)\,x+\qquad 2qby+2q\,(f-qbg)\\(1+q^2ab)\,y'&=\qquad -2qax+(1-q^2ab)\,y-2q\,(g+qaf)\end{aligned}\right\}\qquad(6)$$

valeurs qui satisfont identiquement à l'équation

$$(ax'^2+2gx'+l)+(by'^2+2fy'+m)=(ax^2+2gx+l)+(by^2+2fy+m)\qquad(7),$$

[où pour c j'ai mis $l+m$] et en y écrivant

$$\frac{1-q^2ab}{1+q^2ab}=s=\sqrt{1-abr^2},$$

on obtient des formules qui correspondent précisément aux équations données par Euler pour x, y en termes de a, b.

Londres, 10 *Mars*, 1857.

197.

NOTE ON THE THEORY OF LOGARITHMS.

[From the *Philosophical Magazine*, vol. XI. (1856), pp. 275—280.]

AN imaginary quantity $x+yi$ may always be expressed in the form

$$x+yi=r(\cos\theta+i\sin\theta)=re^{\theta i},$$

where r is positive, and θ is included between the limits $-\pi$ and $+\pi$. We have in fact

$$r=\sqrt{x^2+y^2};$$

and when x is positive,

$$\theta=\tan^{-1}\frac{y}{x};$$

but when x is negative,

$$\theta=\tan^{-1}\frac{y}{x}\pm\pi;$$

where $\tan^{-1}$ denotes an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$, and where the upper or under sign is to be employed according as y is positive or negative. I use for convenience the mark $\equiv$ to denote identity of sign; we may then write

$$\theta=\tan^{-1}\frac{y}{x}+\epsilon\pi,$$

where

$$x\equiv+,\quad \epsilon=0,$$

$$x\equiv-,\quad \epsilon=\pm 1\equiv y.$$

It should be remarked that θ has a unique value except in the single case $x\equiv-$, $y=0$, where θ is indeterminately $\pm\pi$. We have, in fact, $\theta=+\pi$ or $\theta=-\pi$

according as x is considered as the limit of $x+yi$, $y \equiv +$, or of $x+yi$, $y \equiv -$. It is natural to write

$$\log(x+yi) = \log r + \theta i,$$

or what is the same thing,

$$\log(x+yi) = \log\sqrt{x^2+y^2} + \left(\tan^{-1}\frac{y}{x} + \epsilon\pi\right) i;$$

and I take this equation as the definition of the logarithm of an imaginary quantity. The question then arises, to find the value of the expression

$$\log(x+yi) + \log(x'+y'i) - \log(x+yi)(x'+y'i).$$

The preceding definition is, in fact, in the case of x positive, that given by M. Cauchy in the *Exercises de Mathématique*, vol. I. [1826]; and he has there shown that x, x', $xx'-yy'$ being all of them positive, the above-mentioned expression reduces itself to zero. The general definition is that given in my *Mémoire sur quelques Formules du Calcul Intégral*, Liouville, vol. XII. [1847], p. 231 [49]; but I was wrong in asserting that the expression always reduced itself to zero. We have, in fact, in general

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta},$$

when $1-\alpha\beta$ is positive; but when $1-\alpha\beta$ is negative (which implies that α, β have the same sign), then

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta} \pm \pi,$$

where the upper or under sign is to be employed according as α and β are positive or negative; or what is the same thing,

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta} + \epsilon\pi,$$

where

$$1-\alpha\beta \equiv +, \quad \epsilon = 0,$$
$$1-\alpha\beta \equiv -, \quad \epsilon = \pm 1 \equiv \alpha+\beta \equiv \alpha \equiv \beta.$$

This being premised, then writing

$$\log(x+yi) = \log\sqrt{x^2+y^2} + \left(\tan^{-1}\frac{y}{x} + \epsilon\pi\right) i,$$

$$\log(x'+y'i) = \log\sqrt{x'^2+y'^2} + \left(\tan^{-1}\frac{y'}{x'} + \epsilon'\pi\right) i,$$

$$\log(x+yi)(x'+y'i) = \log\left[(xx'-yy') + (xy'+yx')i\right]$$
$$= \log\sqrt{x^2+y^2}\sqrt{x'^2+y'^2} + \left(\tan^{-1}\frac{xy'+x'y}{xx'-yy'} + \epsilon''\pi\right) i,$$

$$\tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y'}{x'} = \tan^{-1}\frac{xy'+x'y}{xx'-yy'} + \epsilon'''\pi,$$

we find

$$\log(x+yi)+\log(x'+y'i)-\log(x+yi)(x'+y'i)=(\epsilon+\epsilon'-\epsilon''+\epsilon''')\pi i.$$

Hence, considering the different cases:

I.

$$\begin{aligned} x &\equiv +, \quad x' \equiv +, \quad xx'-yy' \equiv +, \\ \epsilon &= 0, \\ \epsilon' &= 0, \\ \epsilon'' &= 0, \\ \epsilon''' &= 0, \end{aligned}$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

II.

$$\begin{aligned} x &\equiv +, \quad x' \equiv +, \quad xx'-yy' \equiv -, \\ \epsilon &= 0, \\ \epsilon' &= 0, \\ \epsilon'' &= \pm 1 \equiv (xy'+x'y) \equiv \left(\frac{y}{x}+\frac{y'}{x'}\right), \\ \epsilon''' &= \pm 1 \qquad\qquad \equiv \left(\frac{y}{x}+\frac{y'}{x'}\right), \end{aligned}$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

III.

$$\begin{aligned} x &\equiv +, \quad x' \equiv -, \quad xx'-yy' \equiv +, \\ \epsilon &= 0, \\ \epsilon' &= \pm 1 \equiv y' \equiv -\frac{y'}{x'} \equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right), \\ \epsilon'' &= 0, \\ \epsilon''' &= \pm 1 \qquad\qquad \equiv \left(\frac{y}{x}+\frac{y'}{x'}\right), \end{aligned}$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

IV.

$$\begin{aligned} x &\equiv +, \quad x' \equiv -, \quad xx'-yy' \equiv -, \\ \epsilon &= 0, \\ \epsilon' &= \pm 1 \equiv y' \equiv -\frac{y'}{x'}, \\ \epsilon'' &= \pm 1 \equiv xy'+x'y \equiv -\left(\frac{y}{x}+\frac{y'}{x'}\right), \\ \epsilon''' &= 0, \end{aligned}$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0 \qquad \text{if } \frac{y'}{x'}\equiv \left(\frac{y}{x}+\frac{y'}{x'}\right),$$

but

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=\pm 2\equiv\left(\frac{y}{x}+\frac{y'}{x'}\right) \text{ if } \frac{y'}{x'}\equiv-\left(\frac{y}{x}+\frac{y'}{x'}\right).$$

V.

$$x \equiv -,\quad x'\equiv +,\quad xx'-yy'\equiv +,$$

$$\epsilon = \pm 1\equiv y\equiv -\frac{y}{x}\equiv-\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

$$\epsilon' = 0,$$

$$\epsilon'' = 0,$$

$$\epsilon''' = \pm 1 \qquad \equiv\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0.$$

VI.

$$x \equiv -,\quad x'\equiv +,\quad xx'-yy'\equiv -,$$

$$\epsilon = \pm 1\equiv y\equiv -\frac{y}{x},$$

$$\epsilon' = 0,$$

$$\epsilon'' = \pm 1\equiv (xy'+x'y)\equiv-\left(\frac{y}{x}+\frac{y'}{x'}\right),$$

$$\epsilon''' = 0,$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0 \qquad \text{if } \frac{y}{x}\equiv \left(\frac{y}{x}+\frac{y'}{x'}\right),$$

but

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=\pm 2\equiv\frac{y}{x}+\frac{y'}{x'} \text{ if } \frac{y}{x}\equiv-\left(\frac{y}{x}+\frac{y'}{x'}\right).$$

VII.

$$x \equiv -,\quad x'\equiv -,\quad xx'-yy'\equiv +,$$

$$\epsilon = \pm 1\equiv y\equiv -\frac{y}{x},$$

$$\epsilon' = \pm 1\equiv y'\equiv -\frac{y'}{x'},$$

$$\epsilon'' = 0,$$

$$\epsilon''' = 0;$$

and therefore

$$\epsilon+\epsilon'-\epsilon''+\epsilon'''=0 \qquad \text{if } \frac{y}{x}\equiv-\frac{y'}{x'},$$

but

$$\epsilon+\epsilon'-\epsilon''+\epsilon''=\pm 2\equiv-\left(\frac{y}{x}+\frac{y'}{x'}\right) \text{ if } \frac{y}{x}\equiv \frac{y'}{x'}.$$

VIII.

$$x \equiv -, \quad x' \equiv -, \quad xx' - yy' \equiv -,$$

$$\epsilon = \pm 1 \equiv y \equiv -\frac{y}{x} \quad \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

$$\epsilon' = \pm 1 \equiv y' \equiv -\frac{y'}{x'} \quad \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

$$\epsilon'' = \pm 1 \equiv (xy' + x'y) \equiv \quad \left(\frac{y}{x} + \frac{y'}{x'}\right),$$

$$\epsilon''' = \pm 1 \qquad \equiv \quad \left(\frac{y}{x} + \frac{y'}{x'}\right),$$

and therefore

$$\epsilon + \epsilon' - \epsilon'' + \epsilon''' = \pm 2 \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right).$$

Hence writing

$$\log(x+yi) + \log(x'+y'i) - \log(x+yi)(x'+y'i) = E\pi i,$$

we have $E = 0$, except in the following cases, viz.

1. (See IV.)

$$x \equiv +, \; x' = -, \; xx' - yy' \equiv -, \; \frac{y'}{x'} \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

where

$$E = \pm 2 \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right).$$

2. (See VI.)

$$x \equiv -, \; x' \equiv +, \; xx' - yy' \equiv -, \; \frac{y}{x} \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right),$$

where

$$E = \pm 2 \equiv \left(\frac{y}{x} + \frac{y'}{x'}\right).$$

3. (See VII.)

$$x \equiv -, \; x' \equiv -, \; xx' - yy' \equiv +, \; \frac{y}{x} \equiv \frac{y'}{x'},$$

where

$$E = \pm 2 \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right).$$

4. (See VIII.)

$$x \equiv -, \; x' \equiv -, \quad xx' - yy' \equiv -,$$

where

$$E = \pm 2 \equiv -\left(\frac{y}{x} + \frac{y'}{x'}\right).$$

It thus appears that when the real parts x, x', $xx'-yy'$ are all three of them positive, or any two of them positive and the third negative, E is equal to zero, or the logarithm of the product is equal to the sum of the logarithms of the factors; but that if the real parts are one of them positive and the other two of them negative, then if a certain relation between the real and imaginary parts is satisfied, but not otherwise, the property holds; and if the real parts are all three of them negative, the property does not hold in any case.

The preceding results do not apply to the case where any one of the arguments $x+yi$, $x'+y'i$, $(x+yi)(x'+y'i)$ is real and negative, for no definition applicable to such case has been given of a logarithm. If, however, we assume as a definition that the logarithm of a negative real quantity is equal to the logarithm of the corresponding positive quantity, then in the case, $x \equiv -$, $y=0$, we have

$$\log x + \log(x'+y'i) - \log x(x'+y'i) = \epsilon\pi i, \quad \epsilon = \pm 1 \equiv y';$$

an equation which is, in fact, equivalent to

$$\log(x'+y'i) - \log[-(x'+y'i)] = \epsilon\pi i, \quad \epsilon = \pm 1 \equiv y';$$

and in the case $xy'+x'y=0$, $xx'-yy' \equiv -$, which implies $y \equiv y'$, then

$$\log(x+yi) + \log(x'+y'i) - \log(x+yi)(x'+y'i) = \pi i, \quad \epsilon = \pm 1 \equiv y \text{ or } y';$$

an equation which is in fact equivalent to

$$\log(x+yi) + \log(-x+yi) - \log(x^2+y^2) = \epsilon\pi i, \ \epsilon = \pm 1 \equiv y.$$

The case where both of the arguments $x+yi$, $x'+y'i$ are real and negative, i.e. $x \equiv -$, $y=0$, $x' \equiv -$, $y'=0$ gives of course $\log x + \log x' - \log xx' = 0$, the logarithms of the negative real quantities x, x' being by the definition the same as the logarithms of the corresponding positive quantities. It should, however, be remarked that the definition, $(x \equiv -)$, $\log x = \log(-x)$ not only gives for $\log x$ a different value from that which would be obtained from the general definition of a logarithm, by considering $\log x$ as the limit of $\log(x+yi$, $y \equiv +$, or of $\log(x+yi)$, $y \equiv -$, but gives also a value, which, for the particular case in question, contradicts the fundamental equation $e^{\log x} = x$. It is therefore, I think, better not to establish any definition for the logarithm of a negative real quantity x, but to say that such logarithm is absolutely indeterminate and indeterminable, except in the case where, from the nature of the question, x is considered as the limit of $x+yi$, y positive, or of $x+yi$, y negative.

2, *Stone Buildings, March* 15, 1856.

198.

NOTE UPON A RESULT OF ELIMINATION.

[From the *Philosophical Magazine*, vol. XI. (1856), pp. 378—379.]

If the quadratic function

$$(a, b, c, f, g, h \between x, y, z)^2$$

break up into factors, then representing one of these factors by $\xi x + \eta y + \zeta z$, and taking any arbitrary quantities α, β, γ, the factor in question, and therefore the quadratic function is reduced to zero by substituting $\beta\zeta - \alpha\eta$, $\gamma\xi - \alpha\zeta$, $\alpha\eta - \beta\zeta$ in the place of x, y, z. Write

$$(a, b, c, f, g, h \between \beta\zeta - \alpha\eta, \gamma\xi - \alpha\zeta, \alpha\eta - \beta\zeta)^2 = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between \alpha, \beta, \gamma)^2;$$

the coefficients of the function on the right hand are

$$\begin{aligned}
\mathrm{a} &= \quad . \quad c\eta^2 + b\zeta^2 - 2f\eta\zeta \quad . \quad . \quad , \\
\mathrm{b} &= c\xi^2 + \quad . \quad + a\zeta^2 \quad . \quad - 2g\zeta\xi \quad . \quad , \\
\mathrm{c} &= b\xi^2 + a\eta^2 \quad . \quad . \quad . \quad - 2h\xi\eta, \\
\mathrm{f} &= -f\xi^2 \quad . \quad . \quad - a\eta\zeta + h\zeta\xi + g\xi\eta, \\
\mathrm{g} &= \quad . \quad - g\eta^2 \quad . \quad + h\eta\zeta - b\zeta\xi + f\xi\eta, \\
\mathrm{h} &= \quad . \quad . \quad - h\zeta^2 + g\eta\zeta + f\zeta\xi - c\xi\eta;
\end{aligned}$$

and it is to be remarked that we have identically

$$\begin{aligned}
\mathrm{a}\xi + \mathrm{h}\eta + \mathrm{g}\zeta &= 0, \\
\mathrm{h}\xi + \mathrm{b}\eta + \mathrm{f}\zeta &= 0, \\
\mathrm{g}\xi + \mathrm{f}\eta + \mathrm{c}\zeta &= 0.
\end{aligned}$$

Hence of the six equations, $\mathrm{a}=0$, $\mathrm{b}=0$, $\mathrm{c}=0$, $\mathrm{f}=0$, $\mathrm{g}=0$, $\mathrm{h}=0$, any three (except $\mathrm{a}=0$, $\mathrm{h}=0$, $\mathrm{g}=0$, or $\mathrm{h}=0$, $\mathrm{b}=0$, $\mathrm{f}=0$, or $\mathrm{g}=0$, $\mathrm{f}=0$, $\mathrm{c}=0$) imply the remaining three.

If from the six equations we eliminate ξ^2, η^2, &c., we obtain

$$\square = \begin{vmatrix} . & c, & b, & -2f, & . & . \\ c, & . & a, & . & -2g, & . \\ b, & a, & . & . & . & -2h \\ -f, & . & . & -a, & h, & g \\ . & -g, & . & f, & -b, & f \\ . & . & -h, & g, & c, & -c \end{vmatrix} = 0;$$

and the equation $\square = 0$ is therefore the result of the elimination of ξ, η, ζ from any three (other than the excepted combinations) of the six equations. But from what precedes, it appears that the equation $\square = 0$ must be satisfied when the quadratic function breaks up into factors, and consequently $\square$ must contain as a factor the discriminant

$$K = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

of the quadratic function. This agrees perfectly with the results obtained long ago by Prof. Sylvester in his paper, "Examples of the Dialytic Method of Elimination as applied to Ternary Systems of Equations," *Camb. Math. Journ.* vol. II. p. 232; but according to the assumption there made, the value of $\square$ would be (to a numerical factor *près*) $= abcK$. The correct value is by actual development shown to be $\square = -2K^2$. It would be interesting to show *à priori* that $\square$ contains K^2 as a factor.

2, *Stone Buildings, March* 28, 1856.

199.

NOTE ON THE THEORY OF ELLIPTIC MOTION.

[From the *Philosophical Magazine*, vol. XI. (1856), pp. 425—428.]

IF, as usual, r, θ denote the radius vector and longitude, and μ the central mass, then the Vis Viva and Force function are respectively

$$T = \tfrac{1}{2}(r'^2 + r^2\theta'^2),$$

$$U = \frac{\mu}{r};$$

and writing

$$\frac{dT}{dr'} = r' = p,$$

$$\frac{dT}{d\theta'} = r^2\theta' = q,$$

we have $r' = p$, $\theta' = \frac{q}{r^2}$, and $T = \frac{1}{2}\left(p^2 + \frac{q^2}{r^2}\right)$, whence, putting $H = T - U$, the value of H is

$$H = \tfrac{1}{2}\left(p^2 + \frac{q^2}{r^2}\right) - \frac{\mu}{r};$$

and by Sir W. R. Hamilton's theory, the equations of motion are

$$\frac{dr}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dr},$$

$$\frac{d\theta}{dt} = \frac{dH}{dq}, \quad \frac{dq}{dt} = -\frac{dH}{d\theta}:$$

or substituting for H its value, the equations of motion are

$$\frac{dr}{dt} = p,$$

$$\frac{d\theta}{dt} = \frac{q}{r^2},$$

$$\frac{dp}{dt} = \frac{q^2}{r^3} + \frac{\mu}{r^2},$$

$$\frac{dq}{dt} = 0.$$

Putting, as usual, $\mu = n^2a^3$, and introducing the eccentric anomaly u, which is given as a function of t by means of the equation

$$nt + c = u - e \sin u,$$

$\left(\text{so that } \frac{du}{dt} = \frac{n}{1 - e\cos u}\right)$, the integral equations are

$$q = na^2\sqrt{1-e^2},$$

$$p = \frac{nae \sin u}{1 - e\cos u},$$

$$r = a(1 - e\cos u),$$

$$\theta - \varpi = \tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u}{\cos u - e}\right);$$

where the constants of integration a, e, c, ϖ denote as usual the mean distance, the eccentricity, the mean anomaly at epoch, and the longitude of pericentre.

Suppose that q_0, p_0, r_0, θ_0, u_0 correspond to the time t_0 (q is constant, so that $q_0 = q$), and write

$$V = na^2(u - u_0 + e\sin u - e\sin u_0);$$

joining to this the equations

$$r = a(1 - e\cos u), \quad r_0 = a(1 - e\cos u_0),$$

$$\theta - \theta_0 = \tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u}{\cos u - e}\right) - \tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u_0}{\cos u - e}\right),$$

u, u_0, e will be functions of a, r, r_0, θ, θ_0, and consequently (n being throughout considered as a function of a) V will be a function of a, r, r_0, θ, θ_0. The function V so expressed as a function of a, r, r_0, θ, θ_0 is, in fact, the characteristic function of Sir W. R. Hamilton, and according to his theory we ought to have

$$dV = \tfrac{1}{2}n^2a(t - t_0)\,da + p\,dr + q\,d\theta - p_0\,dr_0 - q_0\,d\theta_0.$$

To verify this, I form the equation

$$\begin{aligned} dV = & \tfrac{1}{2}na\,(u - u_0 + e\sin u - e\sin u_0)\,da \\ & + na^2\,[(1 + e\cos u)\,du - (1 + e\cos u_0)\,du_0] \\ & + na^2\,(\sin u - \sin u_0)\,de \\ & + \frac{nae\sin u}{1 - e\cos u}\,\{dr - (1 - e\cos u)\,da - ae\sin u\,du + a\cos u\,de\} \\ & - \frac{nae\sin u_0}{1 - e\cos u_0}\,\{dr_0 - (1 - e\cos u_0)\,da - ae\sin u_0\,du_0 + a\cos u_0\,de\} \\ & + na^2\sqrt{1 - e^2}\left\{ d\theta - \frac{1}{\sqrt{1 - e^2}\,(1 - e\cos u)}[(1 - e^2)\,du + \sin u de] \right. \\ & \qquad \left. - d\theta_0 + \frac{1}{\sqrt{1 - e^2}\,(1 - e\cos u_0)}[(1 - e^2)\,du_0 + \sin u_0 de] \right\}; \end{aligned}$$

the coefficient of du on the right-hand side is

$$\begin{aligned} & na^2\,(1 + e\cos u) - \frac{na^2e^2\sin^2 u}{1 - e\cos u} - \frac{na^2\,(1 - e^2)}{1 - e\cos u} \\ & = na^2\left(1 + e\cos u - \frac{1 - e^2 + e^2\sin^2 u}{1 - e\cos u}\right), \end{aligned}$$

which vanishes, and similarly the coefficient of du_0 also vanishes: the coefficient of de is the difference of two parts, the first of which is

$$\begin{aligned} & na^2\sin u + \frac{na^2e\sin u\cos u}{1 - e\cos u} - \frac{na^2\sin u}{1 - e\cos u} \\ & = na^2\sin u\left(1 - \frac{1 - e\cos u}{1 - e\cos u}\right), \end{aligned}$$

which vanishes, and the second part in like manner also vanishes; the coefficient of da is the difference of two parts, the first of which is

$$\tfrac{1}{2}na\,(u + e\sin u) - nae\sin u = \tfrac{1}{2}na\,(u - e\sin u),$$

and the second is the like function of u_0; the entire coefficient therefore is

$$\tfrac{1}{2}na\,(u - u_0 - e\sin u + e\sin u_0).$$

We have therefore

$$\begin{aligned} dV = & \tfrac{1}{2}na\,(u - u_0 - e\sin u + e\sin u_0)\,da \\ & + \frac{nae\sin u}{1 - e\cos u}\,dr + na^2\sqrt{1 - e^2}\,d\theta \\ & - \frac{nae\sin u_0}{1 - e\cos u_0}\,dr_0 - na^2\sqrt{1 - e^2}\,d\theta_0; \end{aligned}$$

or what is the same thing,

$$dV = \tfrac{1}{2}n^2a\,(t - t_0)\,da + pdr + qd\theta - p_0dr_0 - q_0d\theta_0,$$

the equation which was to be verified.

2, *Stone Buildings, March* 28, 1856.

200.

ON THE CONES WHICH PASS THROUGH A GIVEN CURVE OF THE THIRD ORDER IN SPACE.

[From the *Philosophical Magazine*, vol. XII. (1856), pp. 20—22.]

THE following investigation is connected with the theory of the cubic

$$(a, b, c, d \between x, y)^3,$$

and in particular with a theorem that the determinant formed with the second differential coefficients of the discriminant gives the square of the discriminant.

Consider the coefficients a, b, c, d as linear functions of coordinates, the equations

$$ac - b^2 = 0, \quad bc - ad = 0, \quad bd - c^2 = 0$$

(equivalent, of course, to two equations) belong to a curve of the third order in space, the edge of regression of the developable surface obtained by putting the discriminant equal to zero, or which has for its equation

$$-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2 = 0\,;$$

and, moreover, the above forms are the general representations of any curve of the third order, and developable surface of the fourth order. The question arises, to find the equation of the cone of the third order having an arbitrary point for its vertex, and passing through the curve of the third order. This may be done by Joachimsthal's method: let $\alpha, \beta, \gamma, \delta$ be the values of a, b, c, d at the vertex of the cone, and in the equations of the curve, say in

$$ac - b^2 = 0, \quad bd - c^2 = 0,$$

for a, b, c, d write $ua + v\alpha$, $ub + v\beta$, $uc + v\gamma$, $ud + v\delta$; if from the equations so obtained u, v are eliminated, the resulting equation will be that of the cone of the third order.

The substitutions in question give

$$L u^2 + 2M\, uv + N v^2 = 0,$$
$$L' u^2 + 2M' uv + N' v^2 = 0,$$

where

$$L = 2(ac - b^2),\quad M = a\gamma + c\alpha - 2b\beta,\quad N = 2(\alpha\gamma - \beta^2),$$
$$L' = 2(bd - c^2),\quad M' = b\delta + d\beta - 2c\gamma,\quad N' = 2(\beta\delta - \gamma^2);$$

the result of the elimination is

$$4(LN - M^2)(L'N' - M'^2) - (LN' + L'N - 2MM')^2 = 0,$$

and we have

$$LN - M^2 = 4(b\gamma - c\beta)(a\beta - b\alpha) - (c\alpha - a\gamma)^2,$$
$$L'N' - M'^2 = 4(c\delta - d\gamma)(b\gamma - c\beta) - (b\delta - d\gamma)^2,$$
$$LN' + L'N - 2MM' = 4(b\gamma - c\beta)^2 + 4(a\beta - b\alpha)(c\delta - d\gamma) + 2(c\alpha - a\gamma)(b\delta - d\beta).$$

Write for shortness

$$b\gamma - c\beta = l,\quad c\alpha - a\gamma = m,\quad a\beta - b\alpha = n,$$
$$a\delta - d\alpha = f,\quad b\delta - d\beta = g,\quad c\delta - d\gamma = h,$$

values which give $lf + mg + nh = 0$. Then forming the expression

$$4(4ln - m^2)(4hl - g^2) - (4l^2 + 4nh + 2mg)^2,$$

this is equal to [1]

$$-16l(l^3 - nfh + lmg - 2lnh + ng^2 + m^2h),$$

and the equation of the cone is

$$l^3 - nfh + lmg - 2lnh + ng^2 + m^2h = 0;$$

or, substituting and expanding,

$$\begin{aligned}
&(\alpha\delta^2 - \gamma^3)\, b^3 - (\alpha^2\delta - \beta^3)\, c^3 - \delta(\beta\delta - \gamma^2)\, ab^2 + \alpha(\alpha\gamma - \beta^2)\, dc^2 \\
&+ \delta(\beta\delta - \gamma^2)\, a^2c - \alpha(\alpha\gamma - \beta^2)\, d^2b + 2\delta(\alpha\gamma - \beta^2)\, ac^2 - 2\alpha(\beta\delta - \gamma^2)\, db^2 \\
&- \gamma(\beta\delta - \gamma^2)\, a^3d + \beta(\alpha\gamma - \beta^2)\, ad^2 + 3\gamma(\beta\gamma - \alpha\delta)\, b^2c - 3\beta(\beta\gamma - \alpha\delta)\, bc^2 \\
&+ \delta(\beta\gamma - \alpha\delta)\, abc - \alpha(\beta\gamma - \alpha\delta)\, dbc + (\alpha\gamma\delta - 3\beta\gamma^2 + 2\beta^2\delta)\, abd \\
&- (\alpha\beta\delta - 3\beta^2\gamma + 2\alpha\gamma^2)\, acd = 0.
\end{aligned}$$

[1] With respect to the occurrence of the factor $l\,(=b\gamma - c\beta)$, it is worth noticing, that, putting $b = k\beta$, $c = k\gamma$, we have identically

$$L u^2 + 2M\, uv + N v^2 = 2[(\alpha\gamma - b\beta)\, u + (\alpha\gamma - \beta^2)\, v]\,(ku + v),$$
$$L' u^2 + 2M' uv + N' v^2 = 2[(\beta\delta - c\gamma)\, u + (\beta\delta - \gamma^2)\, v]\,(ku + v),$$

i.e. the two functions will contain a common factor if $b = k\beta$, $c = k\gamma$, or what is the same thing, if $b\gamma - c\beta = 0$. But if the functions contain a common factor, their resultant vanishes, i.e. the resultant will vanish in virtue of the relation $b\gamma - c\beta = 0$, or what is the same thing, $b\gamma - c\beta$ is a factor of the resultant.

Now putting

$$bd-c^2=p,\quad bc-ad=q,\quad ac-b^2=r,$$

and in like manner

$$\beta\delta-\gamma^2=P,\quad \beta\gamma-\alpha\delta=Q,\quad \alpha\gamma-\beta^2=R,$$

and reducing, the final result may be expressed in the form

$$\begin{aligned} P\,\{&\quad -2\alpha bp+(\gamma a-2\beta b)\,q+(\delta a-\gamma c)\,r\}\\ +Q\,\{&\quad -\alpha cp+(\gamma b-\beta c)\,q+\delta br\}\\ +R\,\{&-(\alpha d-\beta c)\,p-(\beta d-2\gamma c)\,q+2\delta cr\}=0, \end{aligned}$$

where a, b, c, d are current coordinates, and p, q, r are quadratic functions of a, b, c, d. The equation is (as it should be) satisfied by the equations ($p=0$, $q=0$, $r=0$) of the given curve; it is also satisfied *per se* when $P=0$, $Q=0$, $R=0$, i.e. when the vertex is a point on the curve; this indicates a change in form of the equation, and in fact the cone is in this case of the second order only. Suppose that the coordinates of the vertex are in this case given by $\frac{\alpha}{\beta}=\frac{\beta}{\gamma}=\frac{\gamma}{\delta}=\frac{1}{\sigma}$ (σ an arbitrary quantity), it may be easily shown that the equation of the cone is

$$p+\sigma q+\sigma^2 r=0;$$

or at full length

$$(bd-c^2)+\sigma(bc-ad)+\sigma^2(ac-b^2)=0;$$

in fact this equation is evidently that of a surface of the second order passing through the curve; and there is no difficulty in showing that it is a cone.

2, *Stone Buildings, May* 1, 1856.

201.

SECOND NOTE ON THE THEORY OF LOGARITHMS.

[From the *Philosophical Magazine*, vol. XII. (1856), pp. 354—360.]

THE theory of logarithms, as developed in my first note (*Phil. Mag.* April 1856), [197] may be exhibited in a clearer light by considering, instead of $\log a + \log b - \log ab = E\pi i$, the equation $\log \frac{b}{a} = \log b - \log a + E\pi i$, a form which more readily enables the accounting *à priori* for the discontinuity in the value of E. Writing then for b, a the complex values $x' + y'i$, $x + yi$, we have

$$\log \frac{x' + y'i}{x + yi} = \log (x' + y'i) - \log (x + yi) + E\pi i\,;$$

where, according to the assumed definition of a logarithm,

$$\log (x + yi) = \log \sqrt{x^2 + y^2} + i\left(\tan^{-1}\frac{y}{x} + \epsilon\pi\right),$$

in which $\log \sqrt{x^2 + y^2}$ is the real logarithm of $\sqrt{x^2 + y^2}$, and $\tan^{-1}\frac{y}{x}$ is an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. The coefficient ϵ is equal to zero when x is positive; but when x is negative, then $\epsilon = +1$ or -1, according as y is positive or negative, i.e. we have

$$x \equiv +, \quad \epsilon = 0,$$

$$x \equiv -, \quad \epsilon = \pm 1 \equiv y\,;$$

and of course the other logarithms in the equation have an analogous signification.

Hence, attending to the equation

$$\tan^{-1}\beta - \tan^{-1}\alpha = \tan^{-1}\frac{\beta-\alpha}{1+\alpha\beta} + \epsilon'''\pi,$$

where, when $1+\alpha\beta$ is positive, ϵ''' is equal to zero; but when $1+\alpha\beta$ is negative, ϵ''' is equal to $+1$ or -1, according as $\beta-\alpha$ is positive or negative: or what is the same thing (α, β being of opposite signs when $1+\alpha\beta$ is negative),

$$1+\alpha\beta \equiv +, \quad \epsilon''' = 0,$$
$$1+\alpha\beta \equiv -, \quad \epsilon''' = \pm 1 \equiv \beta-\alpha \equiv \beta \equiv -\alpha,$$

we find

$$E = \epsilon - \epsilon' + \epsilon'' - \epsilon''',$$

where ϵ, ϵ', ϵ'', ϵ''' are defined by the conditions

$$\begin{aligned}
&x \equiv +, && \epsilon = 0,\\
&x \equiv -, && \epsilon = \pm 1 \equiv y,\\
&x' \equiv +, && \epsilon' = 0,\\
&x' \equiv -, && \epsilon' = \pm 1 \equiv y',\\
&xx'+yy' \equiv +, && \epsilon'' = 0,\\
&xx'+yy' \equiv -, && \epsilon'' = \pm 1 \equiv xy'-x'y,\\
&1+\frac{y}{x}\frac{y'}{x'} \equiv +, && \epsilon''' = 0,\\
&1+\frac{y}{x}\frac{y'}{x'} \equiv -, && \epsilon''' = \pm 1 \equiv \frac{y'}{x'}-\frac{y}{x} \equiv \frac{y'}{x'} \equiv -\frac{y}{x}.
\end{aligned}$$

Suppose, to fix the ideas, that x, y are each of them positive, we have $\epsilon = 0$; and considering the several cases:

1. $x' \equiv +, \quad y' \equiv +.$

Here $xx'+yy' \equiv +$, $1+\frac{y}{x}\frac{y'}{x'} \equiv +$; and consequently not only $\epsilon' = 0$, but also $\epsilon'' = 0$, $\epsilon''' = 0$, and thence $E = 0$.

2. $x' \equiv +, \quad y' \equiv -.$

Here $\epsilon' = 0$. Moreover, xx' being positive, $xx'+yy'$ and $1+\frac{y}{x}\frac{y'}{x'}$ will have the same sign. If they are both positive, then $\epsilon'' = 0$, $\epsilon''' = 0$; but if they are both negative, then

$$\epsilon'' = \pm 1 \equiv xy'-x'y \equiv \frac{y'}{x'}-\frac{y}{x}$$

(since $xx' \equiv +$) and $\epsilon''' = \pm 1 \equiv \frac{y'}{x'}-\frac{y}{x}$, i.e. $\epsilon'' = \epsilon'''$. Hence in either case we have $E = 0$.

3. $x' \equiv -$, $y' \equiv +$.

Here $\epsilon' = \pm 1 \equiv y'$, i.e. $\epsilon' = 1$. Also xx' being negative, $xx' + yy'$ and $1 + \frac{y}{x}\frac{y'}{x'}$ will have opposite signs. Suppose first $xx' + yy'$ is positive, then $\epsilon'' = 0$. And $1 + \frac{y}{x}\frac{y'}{x'}$ being negative, we have $\epsilon''' = \pm 1 \equiv \frac{y'}{x'} - \frac{y}{x} \equiv \frac{y'}{x'}$, i.e. $\epsilon''' = -1$. But if $xx' + yy'$ is negative, then $\epsilon'' = \pm 1 \equiv xy' - x'y \equiv -\left(\frac{y'}{x'} - \frac{y}{x}\right)$ (since xx' is negative) $\equiv -\frac{y'}{x'}$, i.e. $\epsilon'' = +1$. And $1 + \frac{y}{x}\frac{y'}{x'}$ being positive, we have $\epsilon''' = 0$. Hence in each case $E = 0$.

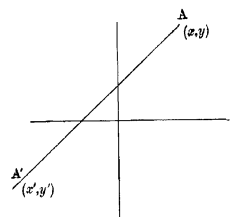
4. $x' \equiv -$, $y' \equiv -$.

Here $\epsilon' = \pm 1 \equiv y'$, i.e. $\epsilon' = -1$. Also xx' and yy' being each negative, $xx' + yy'$ will be negative, and therefore

$$\epsilon'' = \pm 1 \equiv xy' - x'y \equiv -\left(\frac{y'}{x'} - \frac{y}{x}\right);$$

i.e. if $\frac{y'}{x'} > \frac{y}{x}$, then $\epsilon'' = -1$; but if $\frac{y'}{x'} < \frac{y}{x}$, then $\epsilon'' = +1$. And $1 + \frac{y}{x}\frac{y'}{x'}$ being positive, $\epsilon''' = 0$. Hence if $\frac{y'}{x'} > \frac{y}{x}$, then $E = 1 - 1 = 0$; but if $\frac{y'}{x'} < \frac{y}{x}$, then $E = 1 + 1 = 2$.

Consider (x, y) (x', y') as the rectangular coordinates of two points A, A'. In the case which has been considered, the point A has been taken in the positive quadrant; and the preceding discussion shows that we have always $E = 0$, except in the case where the finite line AA' meets the negative portion of the axis of x, in which case we have $E = +2$. The same thing is true generally in whichever quadrant



A is situated, i.e. we have always $E = 0$, except in the cases in which the finite line AA' meets the negative portion of the axis of x. But when this happens, then if the line AA', considered as drawn from A to A', passes from above to below the axis of x, we have $E = +2$; but if the line AA', considered as drawn from A to A', passes from below to above the axis of x, then $E = -2$. So that treating the points

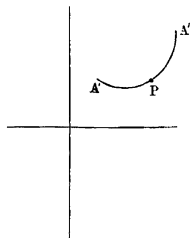
A, A' as the geometrical representations of the complex numbers $x+yi$, $x'+y'i$, we have in an exceedingly simple form the precise determination of the discontinuous number $E(=0$ or $\pm 2)$ in the formula

$$\log \frac{x'+y'i}{x+yi} = \log(x'+y'i) - \log(x+yi) + E\pi i.$$

Consider in general the definite integral

$$\int_z^{z'} \phi u du,$$

where z', z are complex numbers of the form $x+yi$, $x'+y'i$; and take A, A' as the geometrical representations of these limits, and the variable point P as the geometrical representation of the complex variable u. The value of the definite integral will depend to a certain extent on the series of values which we suppose u successively to assume in passing from z to z', or what is the same thing, on the path of the variable point P from A to A'. For (excluding altogether the case in which the path passes through a point for which ϕu becomes infinite) it is well known that the value of the definite integral is the same for any two paths which do not include between them a point for which ϕu becomes infinite; but when this condition is not satisfied, then the value of the definite integral is not in general the same for the



two paths[1]. In order therefore to give a precise signification to the notations, we must fix the path of the point P, and it is natural to assume that the path is a right line (of course there are an infinity of paths which give the same value to the definite integral, or as we may call them, paths equivalent to the right line; but the consideration of these would be a needless complication of the definition, and it is better to attend to the single path—the right line). The definition is at once converted into an analytical one; we have only to assume $u = z + r(z'-z)$, and to suppose that the new variable r passes from $r=0$ (which gives $u=z$) *through real*

[1] The theorem is, I believe, due to M. Cauchy. See the memoir of M. Puiseux, *Recherches sur les Fonctions Algébriques*, Liouville, vol. xv. [1850] pp. 365—480, where the subject is elaborately discussed.

values to $r=1$ (which gives $u=z'$), i.e. we have as the equivalent analytical definition of the definite integral between the complex limits z, z' the equation

$$\int_z^{z'} \phi u\, du = (z'-z)\int_0^1 \phi\,[z+r(z'-z)]\, dr,$$

where the new variable r is real. The only restriction is, that ϕu must not become infinite for any value of u along the path in question, i.e. $\phi\,[z+r(z'-z)]$ must not become infinite for any real value of r between the limits $r=0$, $r=1$.

Suppose next, the path being defined as above, or in any other manner, that $\phi_{,}u$ is a function of u such that $\phi_{,}'u=\phi u$. Then if $\phi_{,}u$ is continuous along the entire path, we have

$$\int_z^{z'} \phi u\, du = \phi_{,}z' - \phi_{,}z;$$

but if $\phi_{,}u$ is discontinuous at any points of the path, e.g. at the point $u=u_{,}$, and at no other point, then

$$\int_z^{z'} \phi u\, du = \phi_{,}z' - \phi_{,}(u_{,}+\alpha') + \phi_{,}(u_{,}-\alpha) - \phi_{,}z,$$

where $u_{,}-\alpha_{,}$, $u_{,}+\alpha'$ are values indefinitely near to $u_{,}$, the path being from z through $u_{,}-\alpha$ to $u_{,}+\alpha'$ and thence to z'. Or if we represent the break $\phi_{,}(u_{,}+\alpha')-\phi_{,}(u_{,}+\alpha)$ by the symbol ∇, then we have

$$\int_z^{z'} \phi u\, du = \phi_{,}z' - \phi_{,}z - \nabla.$$

Suppose now

$$\zeta = \tan^{-1}\frac{y}{x} + \epsilon\pi,$$

where, as before, $\tan^{-1}\dfrac{y}{x}$ denotes an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$, and

$$x \equiv +, \quad \epsilon = 0,$$
$$x \equiv -, \quad \epsilon = \pm 1 \equiv y;$$

and to fix the ideas, consider ζ as the z-coordinate of a surface, the other two coordinates being x and y. If x be negative and y be indefinitely small and positive, then $\epsilon=+1$, and we have $\zeta=\pi$; but if (x being still negative) y be indefinitely small and negative, then $\epsilon=-1$, and therefore $\zeta=-\pi$, i.e. there is a break or abrupt increment 2π of the coordinate ζ in passing across the negative part of the axis of x from a negative to a positive value of y, or, as we have before called it, from below to above; this is the only discontinuity in the surface, the form of the surface being, in fact, what is intended to be represented in the annexed figure.

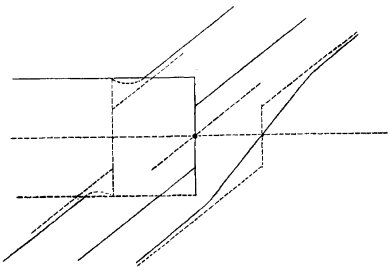
Suppose now $z'=x'+y'i$, $z=x+yi$, and consider the definite integral

$$\int_z^{z'} \frac{du}{u},$$

the path being, as before, a right line. We have, by the equivalent analytical definition,

$$\int_z^{z'} \frac{du}{u} = (z'-z)\int_0^1 \frac{dr}{z+r(z'-z)},$$

where the new variable r is real. And in like manner considering the integral $\int_1^{\frac{z'}{z}} \frac{du}{u}$,



the path being in this case also a right line, we have

$$\int_1^{\frac{z'}{z}} \frac{du}{u} = \left(\frac{z'}{z}-1\right)\int_0^1 \frac{dr}{1+r\left(\frac{z'}{z}-1\right)},$$

where the new variable r is real. The two integrals in r are identically the same, and consequently we have in every case

$$\int_1^{\frac{z'}{z}} \frac{du}{u} = \int_z^{z'} \frac{du}{u}.$$

Now $\log' u = \frac{1}{u}$; and in passing from $u=1$ to $u=\frac{z'}{z}$, there is no discontinuity in the value of $\log u$,

$$= \log \sqrt{p^2+q^2} + i\left(\tan^{-1}\frac{q}{p} + \epsilon\pi\right),$$

if for the moment $u = p+qi$; hence the value of the integral on the left-hand side is simply $\log\frac{z'}{z}$. The value of the integral on the right-hand side is in like manner $\log z' - \log z$, in the case in which the finite right line from $u=z$ to $u=z'$ does not meet the negative part of the axis of x; but when this happens, then there is a discontinuity in the value of the logarithm, and the integral on the right-hand side

will be $\log z' - \log z - 2\pi i$, or $\log z' - \log z + 2\pi i$, according as the right line considered as drawn from z to z' passes from below to above or from above to below the negative part of the axis of x. We have therefore in every case (E being defined as above) $\log z' - \log z + E\pi i$ for the value of the integral on the right-hand side, and the relation between the two integrals gives, as it ought to do, the equation

$$\log \frac{z'}{z} = \log z' - \log z + E\pi i,$$

or, in the form in which it was before written,

$$\log \frac{x' + y'i}{x + yi} = \log (x' + y'i) - \log (x + yi) + E\pi i.$$

The preceding discussion shows that the discontinuity in the value of E (-0 or ± 2) arises from, or is most intimately connected with, the geometrical discontinuity which necessarily exists in the surface $z = \tan^{-1} \frac{y}{x}$, whenever we define the symbol $\tan^{-1}$ in such manner as to give a *unique* value to the coordinate z.

2, *Stone Buildings, Sept.* 19, 1856.

202.

SUPPLEMENTARY REMARKS ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Philosophical Magazine*, vol. XIII. (1857), pp. 19—30.]

IN my former papers (see *Phil. Mag.* August and November, 1853, [115, 116]) I established (as part of a more general one) the following theorem, viz. the condition that there may be inscribed in the conic $U=0$ an infinity of triangles circumscribed about the conic $V=0$, is, that if we develope in ascending powers of k the square root of the discriminant of $kU+V$, the coefficient of k^2 in this development must vanish. Thus writing

$$\text{disct.}\,(kV+U)=(K,\ \Theta,\ \Theta',\ K'\between k,\ 1)^3,$$

the condition in question is found to be

$$3\Theta^2-4K\Theta'=0.$$

The following investigations, although relating only to particular cases of the theorem, are, I think, not without interest.

If the equation of the conic containing the angles is

$$U=2ayz+2bzx+2cxy=0,$$

and the equation of the conic touched by the sides is

$$V=x^2+y^2+z^2-2yz-2zx-2xy=0,$$

we have

$$\text{disct.}\,(k,\ k,\ k,\ a-k,\ b-k,\ c-k\between x,\ y,\ z)^2=(K,\ \Theta,\ \Theta',\ K'\between k,\ 1)^3,$$

that is,

$$\begin{aligned} K &= -4,\\ \Theta &= \tfrac{4}{3}(a+b+c),\\ \Theta' &= -\tfrac{1}{3}(a+b+c)^2,\\ K' &= 2abc. \end{aligned}$$

and the equation $3\Theta^2 - 4K\Theta' = 0$ becomes

$$\tfrac{16}{3}(a+b+c)^2 - \tfrac{16}{3}(a+b+c)^2 = 0,$$

which is satisfied identically. This is as it should be; for it is plain that there exists a triangle, viz. the triangle $(x=0,\ y=0,\ z=0)$, inscribed in the conic $U=0$, and circumscribed about the conic $V=0$.

Suppose that the equation of the conic containing the angles is

$$y^2 - 4zx = 0,$$

and the equation of the conic touched by the sides is

$$ax^2 + by^2 + cz^2 = 0,$$

then the tangential equation of the last-mentioned conic is

$$bc\xi^2 + ca\eta^2 + ab\zeta^2 = 0;$$

and if we take for the angles of the triangle $x : y : z = 1 : 2\lambda : \lambda^2$, or $1 : 2\mu : \mu^2$, or $1 : 2\nu : \nu^2$, then the equation of the line joining the angles (μ), (ν) is

$$2\mu\nu x - (\mu+\nu)\,y + z = 0,$$

which will touch the conic $ax^2 + by^2 + cz^2 = 0$ if

$$bc\,.\,4\mu^2\nu^2 + ca(\mu+\nu)^2 + ab\,.\,4 = 0;$$

and it is required to find under what circumstances the equations

$$bc\,.\,4\mu^2\nu^2 + ca(\mu+\nu)^2 + ab\,.\,4 = 0,$$
$$bc\,.\,4\nu^2\lambda^2 + ca(\nu+\lambda)^2 + ab\,.\,4 = 0,$$
$$bc\,.\,4\lambda^2\mu^2 + ca(\lambda+\mu)^2 + ab\,.\,4 = 0,$$

become equivalent to two equations only. The condition is of course included in the general formula; and putting

$$\text{disct.}\,(ka,\ kb+1,\ kc,\ 0,\ -2,\ 0 \between x,\ y,\ z)^2 = (K,\ \Theta,\ \Theta',\ K' \between k,\ 1)^3,$$

we must have

$$3\Theta^2 - 4K\Theta' = 0.$$

The discriminant in question is

$$k^3abc + k^2ac - k\,.\,4b - 4 = 0,$$

where $K=1$, $\Theta = \tfrac{1}{3}ac$, $\Theta' = -\tfrac{4}{3}b$, $K' = -4$; the required condition is therefore $ac + 16b^2 = 0$, or say

$$b = -\tfrac{1}{4}i\sqrt{ac}.$$

Substituting this value, the equations become

$$c\mu^2\nu^2 + i\sqrt{ca}\,(\mu+\nu)^2 + a = 0,$$
$$c\nu^2\lambda^2 + i\sqrt{ca}\,(\nu+\lambda)^2 + a = 0,$$
$$c\lambda^2\mu^2 + i\sqrt{ca}\,(\lambda+\mu)^2 + a = 0;$$

the first and second of these are

$$A + 2H\nu + B\nu^2 = 0$$
$$A' + 2H'\nu + B'\nu^2 = 0,$$

where

$$A = (-i\sqrt{a} + \mu^2\sqrt{c})\,i\sqrt{a}, \quad H = i\sqrt{ca}\mu, \quad B = (i\sqrt{a} + \mu^2\sqrt{c})\sqrt{c},$$
$$A' = (-i\sqrt{a} + \lambda^2\sqrt{c})\,i\sqrt{a}, \quad H' = i\sqrt{ca}\lambda, \quad B' = (i\sqrt{a} + \lambda^2\sqrt{c})\sqrt{c},$$

$$AB' + A'B - 2HH' = 2i\sqrt{ac}\,(a - i\sqrt{ac}\lambda\mu + c\lambda^2\mu^2),$$
$$AB - H^2 = i\sqrt{ac}\,(a - i\sqrt{ac}\mu^2 + c\mu^4),$$
$$A'B' - H'^2 = i\sqrt{ac}\,(a - i\sqrt{ac}\lambda^2 + c\lambda^4);$$

and the result of the elimination therefore is

$$(a - i\sqrt{ac}\lambda^2 + c\lambda^4)(a - i\sqrt{ac}\mu^2 + c\mu^4) - (a - i\sqrt{ac}\lambda\mu + c\lambda^2\mu^2)^2 = 0,$$

viz.

$$2\sqrt{ca}\,(\lambda-\mu)^2(c\lambda^2\mu^2 + i\sqrt{ca}\,(\lambda+\mu)^2 + a) = 0;$$

which agrees, as it should do, with the third equation.

To find the condition that it may be possible in the conic

$$x^2 + y^2 + z^2 = 0$$

to inscribe an infinity of triangles, each of them circumscribed about the conic

$$ax^2 + by^2 + cz^2 = 0:$$

let the equations of the sides be

$$l\sqrt{a}x + m\sqrt{b}y + n\sqrt{c}z = 0,$$
$$l'\sqrt{a}x + m'\sqrt{b}y + n'\sqrt{c}z = 0,$$
$$l''\sqrt{a}x + m''\sqrt{b}y + n''\sqrt{c}z = 0;$$

then the conditions of circumscription are

$$l^2 + m^2 + n^2 = 0,$$
$$l'^2 + m'^2 + n'^2 = 0,$$
$$l''^2 + m''^2 + n''^2 = 0;$$

and the conditions of inscription are

$$bc\,(m'n''-m''n')^2+ca\,(n'l''-n''l')^2+ab\,(l'm''-l''m')^2=0,$$
$$bc\,(m''n\ -mn'')^2+ca\,(n''l-nl'')^2+ab\,(l''m\ -l\,m'')^2=0,$$
$$bc\,(m\,n'\ -m'n\,)^2+ca\,(n\,l'-n'l\,)^2+ab\,(l\ m'\ -l'\,m\ )^2=0.$$

Now

$$\begin{aligned}(mn'-m'n)^2 &= (m^2+n^2)(m'^2+n'^2)-(mm'+nn')^2\\ &= l^2l'^2-(mm'+nn')^2\\ &= -(ll'+mm'+nn')(-ll'+mm'+nn');\end{aligned}$$

and making the like change in the analogous expressions, and putting for shortness

$$\begin{aligned}-bc+ca+ab&=\alpha,\\ bc-ca+ab&=\beta,\\ bc+ca-ab&=\gamma,\end{aligned}$$

the conditions in question become

$$(l'l''+m'm''+n'n'')(\alpha l'\ l''+\beta m'\,m''+\gamma n'\,n'')=0,$$
$$(l''l\ +m''m\ +n''n\ )(\alpha l''l\ +\beta m''m\ +\gamma n''n\ )=0,$$
$$(l\ l'+m\ m'+n\ n')(\alpha l\ l'\ +\beta m\ m'\ +\gamma n\ n')=0.$$

The proper solution is that given by the system of equations

$$\begin{aligned}l^2+\ m^2+\ n^2&=0,\\ l'^2+\ m'^2+\ n'^2&=0,\\ l''^2+\ m''^2+\ n''^2&=0,\end{aligned}$$

$$\begin{aligned}\alpha l'\,l''+\beta m'\,m''+\gamma n'\,n''&=0,\\ \alpha l''l\ +\beta m''m\ +\gamma n''n\ &=0,\\ \alpha l\ l'\ +\beta m\ m'\ +\gamma n\ n'\ &=0;\end{aligned}$$

and by writing $l=\frac{f}{\sqrt{\alpha}}$, $l'=\frac{f'}{\sqrt{\alpha}}$, $l''=\frac{f''}{\sqrt{\alpha}}$, $m=\frac{g}{\sqrt{\beta}}$, &c., $A=\frac{1}{\alpha}$, $B=\frac{1}{\beta}$, $C=\frac{1}{\gamma}$, these equations become

$$\begin{aligned}Af^2+Bg^2+Ch^2&=0,\\ Af'^2+Bg'^2+Ch'^2&=0,\\ Af''^2+Bg''^2+Ch''^2&=0,\end{aligned}$$

$$\begin{aligned}f'f''+g'g''+h'h''&=0,\\ f''f\ +g''g\ +h''h\ &=0,\\ f\,f'\ +g\,g'\ +h\,h'\ &=0.\end{aligned}$$

The first of which systems expresses that the points (f, g, h), (f', g', h'), (f'', g'', h'') are points in the conic

$$Ax^2 + By^2 + Cz^2 = 0\,;$$

and the second condition expresses that each of the points in question is the pole with respect to the conic

$$x^2 + y^2 + z^2 = 0$$

of the line joining the other two points, i.e. that the three points are a system of conjugate points with respect to the last-mentioned conic. The problem is thus reduced to the following one:

To find the condition in order that it may be possible in the conic

$$Ax^2 + By^2 + Cz^2 = 0$$

to inscribe an infinity of triangles such that the angles are a system of conjugate points with respect to the conic

$$x^2 + y^2 + z^2 = 0.$$

Before going further it is proper to remark that if, instead of assuming $\alpha l'l'' + \beta m'm'' + \gamma n'n'' = 0$, we had assumed

$$l'l'' + m'm'' + n'n'' = 0,$$

this, combined with the equations

$$l'^2 + m'^2 + n'^2 = 0, \quad l''^2 + m''^2 + n''^2 = 0,$$

would have given $l' : m' : n' = l'' : m'' : n''$, i.e. two of the angles of the triangle would have been coincident: this obviously does not give rise to any proper solution. Returning now to the system of equations in f, g, h, &c., since the equations give only the ratios $f : g : h$; $f' : g' : h'$; $f'' : g'' : h''$, we may if we please assume

$$\begin{aligned} f^2 + g^2 + h^2 &= 1, \\ f'^2 + g'^2 + h'^2 &= 1, \\ f''^2 + g''^2 + h''^2 &= 1, \end{aligned}$$

which, combined with the second system of equations, gives

$$\begin{aligned} f^2 + f'^2 + f''^2 &= 1, \\ g^2 + g'^2 + g''^2 &= 1, \\ h^2 + h'^2 + h''^2 &= 1. \end{aligned}$$

We have, consequently,

$$\begin{aligned} A + B + C &= A\,(f^2 + f'^2 + f''^2) + B\,(g^2 + g'^2 + g''^2) + C\,(h^2 + h'^2 + h''^2) \\ &= (Af^2 + Bg^2 + Ch^2) + (Af'^2 + Bg'^2 + Ch'^2) + (Af''^2 + Bg''^2 + Ch''^2), \end{aligned}$$

i.e.

$$A + B + C = 0,$$

for the condition that it may be possible in the conic

$$Ax^2 + By^2 + Cz^2 = 0$$

to describe an infinity of triangles the angles of which are conjugate points with respect to the conic $x^2 + y^2 + z^2 = 0$.

The equation of the conic $Ax^2 + By^2 + Cz^2 = 0$ may be written in the form

$$(b^2c^2 - c^2a^2 - a^2b^2 + 2a^2bc)\, x^2 + (c^2a^2 - a^2b^2 - b^2c^2 + 2ab^2c)\, y^2 + (a^2b^2 - b^2c^2 - c^2a^2 + 2abc^2)\, z^2 = 0,$$

which gives the values of A, B, C; or again in the form

$$\begin{aligned} &2(bc + ca + ab)(bcx^2 + cay^2 + abz^2) \\ &- (bc + ca + ab)^2(\ x^2 + \ y^2 + \ z^2) \\ &\qquad + 4abc(ax^2 + \ by^2 + \ cz^2) = 0; \end{aligned}$$

where it should be observed that $bcx^2 + cay^2 + abz^2 = 0$ is the equation of the conic which is the polar of $ax^2 + by^2 + cz^2 = 0$ with respect to $x^2 + y^2 + z^2 = 0$. It is very easy from the last form to deduce the equation of the auxiliary conic, when the conics $ax^2 + by^2 + cz^2 = 0$, $x^2 + y^2 + z^2 = 0$ are replaced by conics represented by perfectly general equations.

The condition $A + B + C = 0$ gives, substituting the values of A, B, C,

$$b^2c^2 + c^2a^2 + a^2b^2 - 2abc(a + b + c) = 0;$$

or in a more convenient form,

$$(bc + ca + ab)^2 - 4abc(a + b + c) = 0,$$

as the condition in order that it may be possible to inscribe in the conic $x^2 + y^2 + z^2 = 0$ an infinity of triangles, the sides of which touch the conic $ax^2 + by^2 + cz^2 = 0$: this agrees perfectly with the general theorem.

It is convenient to add (as a somewhat more general form of the equation $A + B + C = 0$), that the condition in order that it may be possible in the conic $Ax^2 + By^2 + Cz^2 = 0$ to inscribe an infinity of triangles the angles of which are conjugate points with respect to the conic $A_1x^2 + B_1y^2 + C_1z^2 = 0$, is

$$\frac{A}{A_1} + \frac{B}{B_1} + \frac{C}{C_1} = 0.$$

But the problem to find the condition in order that it may be possible in the conic $x^2 + y^2 + z^2 = 0$ to inscribe an infinity of triangles the sides of which touch the conic $ax^2 + by^2 + cz^2 = 0$, may, by the assistance of the geometrical theorem to be presently mentioned, be at once reduced to the problem:

To find the condition in order that it may be possible in the conic $x^2 + y^2 + z^2 = 0$ to inscribe an infinity of triangles the sides of which are conjugate points with respect to a conic

$$A_1x^2 + B_1y^2 + C_1z^2 = 0.$$

The theorem referred to is as follows:

THEOREM. If the chord PP' of a conic S envelope a conic σ, the points P, P' are harmonics with respect to a conic T which has with S, σ, a system of common conjugate points.

Take for the equation of S,

$$x^2 + y^2 + z^2 = 0,$$

and for the equation of σ,

$$ax^2 + by^2 + cz^2 = 0;$$

then if (x_1, y_1, z_1), (x_2, y_2, z_2) are the coordinates of the points P, P' respectively, we have

$$x_1^2 + y_1^2 + z_1^2 = 0,$$
$$x_2^2 + y_2^2 + z_2^2 = 0,$$

and the condition in order that the chord may touch the conic σ is

$$bc(y_1z_2 - y_2z_1)^2 + ca(z_1x_2 - z_2x_1)^2 + ab(x_1y_2 - x_2y_1)^2 = 0.$$

But we have

$$\begin{aligned}(y_1z_2 - y_2z_1)^2 &= (y_1^2 + z_1^2)(y_2^2 + z_2^2) - (y_1y_2 + z_1z_2)^2\\ &= x_1^2x_2^2 - (y_1y_2 + z_1z_2)^2\\ &= (x_1x_2 + y_1y_2 + z_1z_2)(x_1x_2 - y_1y_2 - z_1z_2),\end{aligned}$$

and making the like change in the analogous quantities, and putting for shortness

$$\begin{aligned}\alpha &= -bc + ca + ab,\\ \beta &= bc - ca + ab,\\ \gamma &= bc + ca - ab,\end{aligned}$$

the condition in question becomes

$$(x_1x_2 + y_1y_2 + z_1z_2)(\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2) = 0.$$

But the equation $x_1x_2 + y_1y_2 + z_1z_2 = 0$ must be rejected, as giving with the equations $x_1^2 + y_1^2 + z_1^2 = 0$, $x_2^2 + y_2^2 + z_2^2 = 0$ the relation $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$; we have therefore

$$\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 = 0,$$

which implies that the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are harmonics with respect to the conic

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 0,$$

which is a conic having with S, σ, a system of common conjugate points. The equation may also be written

$$(-bc + ca + ab)x^2 + (bc - ca + ab)y^2 + (bc + ca - ab)z^2 = 0;$$

or, as it may also be written,

$$(bc+ca+ab)(x^2+y^2+z^2)-2(bcx^2+caz^2+abx^2);$$

and, as before remarked, $bcx^2+cay^2+abz^2=0$ is the equation of the conic which is the polar of $ax^2+by^2+cz^2=0$ with respect to $x^2+y^2+z^2=0$.

The condition in order that there may be inscribed in the conic $x^2+y^2+z^2=0$ an infinity of triangles the angles of which are conjugate points with respect to the conic $\alpha x^2+\beta y^2+\gamma z^2=0$, is

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=0;$$

or writing this equation under the form $\beta\gamma+\gamma\alpha+\alpha\beta=0$, and substituting for α, β, γ their values, we have the equation already found, as the condition in order that it may be possible in the conic $x^2+y^2+z^2=0$ to inscribe an infinity of triangles the sides of which touch the conic $ax^2+by^2+cz^2=0$.

THEOREM. Let

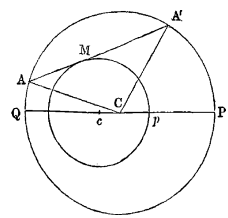
$$ax^2+by^2+cz^2=0$$

be the equation of a spherical conic, and let $(\xi:\eta:\zeta)$, a point on the conic, be the pole of a great circle cutting the conic in two points; the conic intersects upon the great circle an arc given by the equation

$$\cos\delta=\frac{(a+b+c)\sqrt{\xi^2+\eta^2+\zeta^2}}{\sqrt{(a+b+c)^2(\xi^2+\eta^2+\zeta^2)-4(bc\xi^2+ca\eta^2+ab\zeta^2)}};$$

hence if $a+b+c=0$, $\delta=90°$; or there may be inscribed in the conic an infinity of triangles having each of their sides equal to 90°.

It is worth while, in connexion with the subject, and for the sake of a remark to which they give rise, to reproduce in a short compass some results long ago obtained by Jacobi and Richelot. The following are Jacobi's formulæ for the chords of a circle subjected to the condition of touching another circle; viz. if in the figure we put



$$CP = R,$$
$$cp = r,$$
$$Cc = a,$$
$$\angle ACP = 2\phi,$$
$$\angle A'CP = 2\phi';$$

then it is clear from geometrical considerations that

$$\frac{d\phi}{\overline{MA}} = \frac{d\phi'}{\overline{MA'}}.$$

We have

$$\begin{aligned}\overline{MA}^2 = \overline{cA}^2 - \overline{cM}^2 &= a^2 + R^2 + 2aR\cos 2 - \phi r^2 \\ &= (a+R)^2 - r^2 - 4aR\sin^2\phi \\ &= \{(a+R)^2 - r^2\}(1 - k^2\sin 2\phi),\end{aligned}$$

or

$$\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{d\phi'}{\sqrt{1-k^2\sin^2\phi'}},$$

where

$$k^2 = \frac{4aR}{(a+R)^2 - r^2},$$

and therefore also

$$k'^2 = \frac{(R-a)^2 - r^2}{(R+a)^2 - r^2}.$$

It will be convenient for comparison with the formulæ of Richelot to write $\angle ACQ = 2\psi$; this gives

$$2\psi = \pi - 2\phi,$$

and the differential equation thus becomes

$$\frac{d\psi}{\sqrt{a^2 + R^2 - r^2 - 2aR\cos 2\psi}} = \frac{d\psi'}{\sqrt{a^2 + R^2 - r^2 - 2aR\cos 2\psi'}},$$

i.e.

$$\frac{d\psi}{\sqrt{m - n\cos 2\psi}} = \frac{d\psi'}{\sqrt{m - n\cos 2\psi'}};$$

or if

$$\tan\psi = g\tan\theta,$$

and therefore

$$\cos 2\psi = \frac{1 - g^2\tan^2\theta}{1 + g^2\tan^2\theta},$$

$$m - n\cos 2\psi = \frac{(m-n)\cos^2\theta + (m+n)g^2\sin^2\theta}{\cos^2\theta + g^2\sin^2\theta};$$

or if $g^2 = \dfrac{m-n}{m+n}$, then this is

$$= \frac{m-n}{\cos^2\theta + \dfrac{m-n}{m+n}\sin^2\theta},$$

$$= \frac{m-n}{1-\frac{2n}{m+n}\sin^2\theta}$$

and we have also

$$d\psi = \frac{\sqrt{\frac{m-n}{m+n}}\,d\theta}{1-\frac{2n}{m+n}\sin^2\theta},$$

and thence

$$\frac{d\psi}{\sqrt{m-n\cos 2\psi}} = \frac{d\theta}{\sqrt{m+n}\sqrt{1-\frac{2n}{m+n}\sin^2\theta}},$$

that is,

$$\frac{d\psi}{\sqrt{m-n\cos 2\psi}} = \frac{d\theta}{\sqrt{m+n}\sqrt{1-k^2\sin^2\theta}},$$

where $k^2 = \frac{2n}{m+n}$ has the same value as before. Hence the relation between θ, θ' is

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}},$$

which is identical with that between ϕ and ϕ'; and in fact the equation between θ, ϕ is

$$\tan\theta\tan\phi = \frac{1}{k'},$$

which, if $\phi = \text{am}\, u$, gives $\theta = \text{am}\,(K-u)$.

The differential equation contains only a single arbitrary parameter; hence the same differential equation might have been obtained from different values of a, R, the parameters which determine the circle enveloped by the moveable chord. The condition for this of course is $\frac{m'}{n'} = \frac{m}{n}$, that is

$$\frac{a^2+R^2-r^2}{2aR} = \frac{a'+R^2-r'^2}{2a'R},$$

or as the equation may be written

$$(aa'-R^2)(a-a') - R(a'r^2 - ar'^2) = 0;$$

this implies that the enveloped circles intersect the other circle in the same two points, or that all the circles have a common chord.

Suppose for $\psi = 0$, we have $\psi' = \beta$, then it is easy to see geometrically that

$$\tan^2 \beta = \frac{(R-a)^2 - r^2}{r^2};$$

let the corresponding value of θ' be $\theta' = \alpha$, i.e. suppose that for $\theta = 0$, we have $\theta' = \alpha$, then

$$\tan^2 \alpha = \frac{(R+a)^2 - r^2}{(R-a)^2 - r^2} \cdot \frac{(R-a)^2 - r^2}{r^2},$$

i.e.

$$\tan^2 \alpha = \frac{(R+a)^2 - r^2}{r^2};$$

or, what is the same thing,

$$\sec \alpha = \frac{R}{r} + \frac{a}{r};$$

and α having this value, we have for the finite relation between θ, θ',

$$F\theta' = F\theta + F\alpha.$$

Richelot has shown, by precisely similar reasoning, that for circles of the sphere we have

$$\frac{d\psi}{\sqrt{\cos^2 r - (\cos R \cos a + \sin R \sin a \cos 2\psi)^2}} = \frac{d\psi'}{\sqrt{\cos^2 r - (\cos R \cos a + \sin R \sin a \cos 2\psi')^2}}$$

which is of the form

$$\frac{d\psi}{\sqrt{1 - (\lambda + \mu \cos 2\psi)^2}} = \frac{d\psi'}{\sqrt{1 - (\lambda + \mu \cos 2\psi')^2}},$$

where

$$\lambda = \frac{\cos R \cos a}{\cos r},$$

$$\mu = \frac{\sin R \sin a}{\cos r}.$$

It is very important to remark, that this equation contains the two parameters λ, μ, so that the same equation cannot be obtained with any new values of the parameters a, r; or the formulæ *in plano* for three or more circles do not apply to circles of the sphere: the geometrical reason for this is as follows, viz. in the plane a circle is a conic passing through two fixed points (the circular points at ∞), and consequently any number of circles having a common chord are in fact to be considered as conics each of which passes through the same four points. But circles of the sphere are not spherical conics passing through two fixed points, but are merely spherical conics having a double contact with an imaginary spherical conic (viz. the curve of intersection of the sphere with a sphere radius zero); hence circles of the

sphere having a common spherical chord are not spherical conics passing through the same four points. I am not sure whether this remark as to the ground of the distinction between the theory of circles *in plano* and that of circles on the sphere has been explicitly made in any of the treatises on spherical geometry.

To reduce the equation, write

$$\tan\psi = \sqrt{\frac{1-(\lambda+\mu)}{1-(\lambda-\mu)}}\tan\theta;$$

then, after a simple reduction,

$$\frac{d\psi}{\sqrt{1-(\lambda+\mu\cos 2\psi)^2}} = \frac{d\theta}{\sqrt{(1+\mu)^2-\lambda^2}\sqrt{1-\frac{4\mu}{(1+\mu)^2-\lambda^2}\sin^2\theta}};$$

or the relation between the two values of θ is

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}},$$

where

$$k^2 = \frac{4\mu}{(1+\mu)^2-\lambda^2},$$

that is

$$k^2 = \frac{4\dfrac{\tan R}{\tan r}\cdot\dfrac{\sin a}{\cos R\sin r}}{\left(\dfrac{\tan R}{\tan r}+\dfrac{\sin a}{\cos R\sin r}\right)^2-1}.$$

Suppose that for $\psi=0$, we have $\psi'=\beta$, it is easy to see that

$$\tan^2\beta = \frac{\sin^2(R-a)-\sin^2 r}{\cos^2 R\sin^2 r};$$

let the corresponding value of θ' be $\theta'=\alpha$, i.e. suppose that for $\theta=0$, we have $\theta'=\alpha$, then

$$\begin{aligned}
\tan^2\alpha &= \frac{1-\dfrac{\cos(R+a)}{\cos r}}{1-\dfrac{\cos(R-a)}{\cos r}}\cdot\frac{\sin^2(R-a)-\sin^2 r}{\cos^2 R\sin^2 r}\\
&= \frac{\cos r-\cos(R+a)}{\cos r-\cos(R-a)}\cdot\frac{\cos^2 r-\cos^2(R-a)}{\cos^2 R\sin^2 r}\\
&= \frac{\big(\cos r-\cos(R+a)\big)\big(\cos r+\cos(R-a)\big)}{\cos^2 R\sin^2 r}\\
&= \frac{(\cos r+\sin R\sin a)^2-\cos^2 R\cos^2 a}{\cos^2 R\sin^2 r}\\
&= \frac{(\cos r\sin R+\sin a)^2-\cos^2 R\sin^2 r}{\cos^2 R\sin^2 r},
\end{aligned}$$

i.e.

$$\tan^2 \alpha = \left(\frac{\tan R}{\tan r} + \frac{\sin a}{\cos R \sin r}\right)^2 - 1,$$

whence

$$\sec \alpha = \left(\frac{\tan R}{\tan r} + \frac{\sin a}{\cos R \sin r}\right);$$

and α having this value, the finite relation between θ, θ' is

$$F\theta' = F\theta + F\alpha.$$

By comparing with the corresponding formula *in plano*, we arrive at Richelot's conclusion, that the formulæ for the sphere may be deduced from those *in plano* by writing in the place of $\frac{R}{r}$, $\frac{a}{r}$, the functions $\frac{\tan R}{\tan r}$, $\frac{\sin a}{\cos R \sin r}$, respectively.

2, *Stone Buildings, October* 1, 1856.

203.

ON THE THEORY OF THE ANALYTICAL FORMS CALLED TREES.

[*From the Philosophical Magazine*, vol. XIII. (1857), pp. 172—176.]

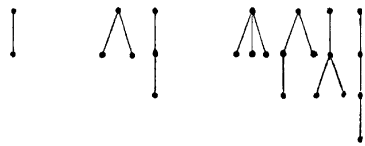
A SYMBOL such as $A\partial_x + B\partial_y + \ldots$, where A, B, &c. contain the variables x, y, &c. in respect to which the differentiations are to be performed, partakes of the natures of an operand and operator, and may be therefore called an Operandator. Let P, Q, R, ... be any operandators, and let U be a symbol of the same kind, or to fix the ideas, a mere operand; PU denotes the result of the operation P performed on U, and QPU denotes the result of the operation Q performed on PU; and generally in such combinations of symbols, each operation is considered as affecting the operand denoted by means of all the symbols on the right of the operation in question. Now considering the expression QPU, it is easy to see that we may write

$$QPU = (Q \times P)\, U + (QP)\, U,$$

where on the right-hand side $(Q \times P)$ and (QP) signify as follows: viz. $Q \times P$ denotes the mere algebraical product of Q and P, while QP (consistently with the general notation as before explained) denotes the result of the operation Q performed upon P as operand; and the two parts $(Q \times P)\, U$ and $(QP)\, U$ denote respectively the results of the operations $(Q \times P)$ and (QP) performed each of them upon U as operand. It is proper to remark that $(Q \times P)$ and $(P \times Q)$ have precisely the same meaning, and the symbol may be written in either form indifferently. But without a more convenient notation, it would be difficult to find the corresponding expressions for $RQPU$, &c. This, however, can be at once effected by means of the analytical forms called trees (see figs. 1, 2, 3), which contain all the trees which can be formed with one branch, two branches, and three branches respectively.

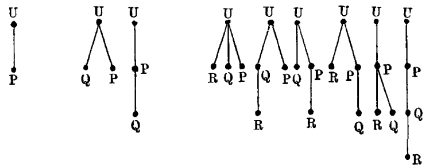
The inspection of these figures will at once show what is meant by the term in question, and by the terms *root, branches* (which may be either main branches, intermediate branches, or free branches), and *knots* (which may be either the root itself, or proper knots, or the extremities of the free branches). To apply this to the question in hand, PU consists of a single term represented by fig. 1 (*bis*); QPU consists, as above, of two terms represented by the two parts of fig. 2 (*bis*), viz. the

Fig. 1. Fig. 2. Fig. 3.



first part represents the term $(Q \times P)\, U$, and the second part represents the term $(QP)\, U$. And it is obvious that fig. 2 (*bis*) is at once formed from the figure 1 (*bis*) by adding on a branch terminated by Q at each of the knots of the single part of fig. 1 (*bis*). In like manner $RQPU$ consists of six terms represented by the six parts of fig. 3 (*bis*), and this figure is at once formed from fig. 2 (*bis*) by adding on a branch terminated by R at each knot of each part of fig. 2 (*bis*). It is

Fig. 1 (*bis*). Fig. 2 (*bis*). Fig. 3 (*bis*).



hardly necessary to remark that the first part of fig. 3 (*bis*) denotes what, in the notation first explained, would be denoted by $(R \times Q \times P)\, U$, the second term what would in like manner be denoted by $(RQ \times P)\, U$, and so on, the last part being the term which would be denoted by $((RQ)\, P)\, U$; viz. R operates upon Q, giving the operandator RQ, which operates upon P, giving the operandator $(RQ)\, P$, which finally operates upon U.

The figures 1 (*bis*), 2 (*bis*), &c. contain the same trees as are contained in the corresponding figures 1, 2, &c.; only, on account of the different modes of filling up, trees are considered as so many distinct trees in a figure of the second set which are considered as one and the same tree in the corresponding figure of the first set. A difference in the number of trees first occurs in the figures 3 and 3 (*bis*), the

first of which contains only four, while the latter contains six trees, viz. the first tree, the second, third and fourth trees, the fifth tree and the sixth tree of fig. 3 (*bis*) correspond respectively to the first tree, the second tree, the third tree, and the fourth tree of fig. 3. To derive fig. 3 (*bis*) from fig. 3, we must fill up the trees of fig. 3 with U at the root and R, Q, P at the other knots in every possible manner, subject only to the restriction, that, reckoning up from the extremity of a free branch to the root, there must not be any transposition in the order of the symbols RQP, and taking care to admit only distinct trees. Thus the first tree of fig. 3 might be filled up in six ways; but the trees so obtained are considered as one and the same tree, and we have only the first tree of fig. 3 (*bis*). Again, on account of the restriction, the fourth tree of fig. 3 can be filled up in one way only, and we have thus the sixth tree of fig. 3 (*bis*). And thus, in general, each figure of the second set can be formed at once from the corresponding figure of the first set; or when the first set of figures is given, the expression for $YX...QPU$ can be formed directly without the assistance of the expression for the preceding symbol $X...QPU$; the number of terms for the nth figure of the second set is obviously $1.2.3...n$, and consequently it is only necessary to count the terms in order to ascertain that no admissible mode of filling up has been omitted.

The number of parts in any one of the figures of the first set is much smaller than the number of parts in the corresponding figure of the second set; and the law for the number of parts, i.e. for the number A_n of the trees with n branches, is a very singular one. To obtain this law, we must consider how the trees with n branches can be formed by means of those of a smaller number of branches. A tree with n branches has either a single main branch, or else two main branches, three main branches, &c....to n main branches. If the tree has one main branch, it can only be formed by adding on to this main branch a tree with $(n-1)$ branches, i.e. A_n contains a part A_{n-1}. If the tree has two main branches, then $p+q$ being a partition of $n-2$, the tree can be formed by adding on to one main branch a tree of p branches, and to the other main branch a tree of q branches; the number of trees so obtained is A_pA_q: this, however, assumes that the parts p and q are unequal; if they are equal, it is easy to see that the number of trees is only $\frac{1}{2}A_p(A_p+1)$. Hence $p+q$ being any partition of $n-2$, A_n contains the part A_pA_q if p and q are unequal, and the part $\frac{1}{2}A_p(A_p+1)$ if p and q are equal. In like manner, considering the trees with three main branches, then if $p+q+r$ is any partition of $n-3$, A_n contains the part $A_pA_qA_r$ if p, q, r are unequal; but if two of these numbers, e.g. p and q, are equal, then the part $\frac{1}{2}A_p(A_p+1)A_r$; and if p, q, r are all equal, then the part $\frac{1}{6}A_p(A_p+1)(A_p+2)$; and so on, until lastly we have a single tree with n main branches, or A_n contains the part unity. A little consideration will show that the preceding rule for the formation of the number A_n is completely expressed by the equation

$$(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}(1-x^4)^{-A_3}... = 1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c.,$$

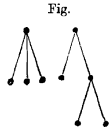
and consequently that we may, by means of this equation, calculate successively for the different values of n the number A_n of the trees with n branches. The calculation

may be effected very easily as follows: [the table as originally printed contained at the end of it some errors of calculation which were corrected, *B.A.* Report for 1875, p. 258].

$A_1 =$	1	(1)	1	1	1	1	1	1	1	1	1	$(1-x^2)^{-1}$
			1	1	1	1	1	1	1	1	1	
					1	1	1	1	1	1	1	
							1	1	1	1	1	
									1	1	1	
											1	
$A_2 =$	1	1	(2)	2	3	3	4	4	5	5	5	$(1-x^3)^{-2}$
				2	2	4	4	6	6	8	8	
							3	3	6	6	9	
										4	4	
$A_3 =$	1	1	2	(4)	5	7	11	13	17	23	27	$(1-x^4)^{-4}$
					4	4	8	16	20	28	44	
									10	10	20	
$A_4 =$	1	1	2	4	(9)	11	19	29	47	61	91	$(1-x^5)^{-9}$
						9	9	18	36	81	99	
											45	
$A_5 =$	1	1	2	4	9	(20)	28	47	83	142	235	$(1-x^6)^{-20}$
							20	20	40	80	180	
$A_6 =$	1	1	2	4	9	20	(48)	67	123	222	415	$(1-x^7)^{-48}$
								48	48	96	192	
$A_7 =$	1	1	2	4	9	20	48	(115)	171	318	607	$(1-x^8)^{-115}$
									115	115	230	
$A_8 =$	1	1	2	4	9	20	48	115	(286)	433	837	$(1-x^9)^{-306}$
										286	286	
$A_9 =$	1	1	2	4	9	20	48	115	286	(719)	1123	$(1-x^{10})^{-775}$
											719	
$A_r =$		1	2	4	9	20	48	115	286	719	1842	
for $r =$		1	2	3	4	5	6	7	8	9	10	

I have had occasion, for another purpose, to consider the question of finding the number of trees with a given number of free branches, bifurcations at least. Thus, when the number of free branches is three, the trees of the form in question are

Fig.



those in the annexed figure, and the number is therefore two. It is not difficult to see that we have in this case (B_r being the number of such trees with r free branches),

$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots = 1 + x + 2B_2x^2 + 2B_3x^3 + 2B_4x^4 + \&c.;$$

and a like process of development gives:

$B_r =$	1	2	5	12	33	90
for $r =$	2	3	4	5	6	7

I may mention, in conclusion, that I was led to the consideration of the foregoing theory of trees by Professor Sylvester's researches on the change of the independent variables in the differential calculus.

2, *Stone Buildings, January* 2, 1856.

204.

ON A PROBLEM IN THE PARTITION OF NUMBERS.

[From the *Philosophical Magazine*, vol. XIII. (1857), pp. 245—248.]

It is required to find the number of partitions into a given number of parts, such that the first part is unity, and that no part is greater than twice the preceding part.

Commencing to form the partitions in question, these are

1	1	1	1	1	1	1	1	1	&c.;
	1	2	1	1	2	2	2	2	
			1	2	1	2	3	4	

and if we were to proceed to the 4-partitions, each 3-partition ending in 1 would give rise to two such partitions; each 3-partition ending in 2 to four such partitions; each 3-partition ending in 3 to six partitions; and each 3-partition ending in 4 to eight such partitions. We form in this manner the Table:

	ending in																
Number of	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Totals
1-partitions	1																1
2-partitions	1	1															2
3-partitions	2	2	1	1													6
4-partitions	6	6	4	4	2	2	1	1									26
5-partitions	26	26	20	20	14	14	10	10	6	6	4	4	2	2	1	1	166

&c.;

and we are thus led to the series

$$\begin{array}{l} 1, \\ 1,\ 2, \\ 1,\ 2,\ 4,\ 6, \\ 1,\ 2,\ 4,\ 6,\ 10,\ 14,\ 20,\ 26, \\ \&c.; \end{array}$$

where, considering 0 as the first term of each series, the first differences of any series are the terms twice repeated of the next preceding series: thus the differences of the fourth series are 1, 1, 2, 2, 4, 4, 6, 6. It is moreover clear that the first half of each series is precisely the series which immediately precedes it; we need, in fact, only consider a single infinite series, 1, 2, 4, 6, &c. It is to be remarked, moreover, that in the column of totals, the total of any line is precisely the first number in the next succeeding line.

Consider in general a series A, B, C, D, E, &c., and a series A', B', C', D', E', &c. derived from it as follows:

$$\begin{aligned} A' &= 1A, \\ B' &= 2A, \\ C' &= 2A + B, \\ D' &= 2A + 2B, \\ E' &= 2A + 2B + C, \\ F' &= 2A + 2B + 2C, \\ &\&c.; \end{aligned}$$

viz. the first differences of the series 0, A', B', C', D', E', &c. are A, A, B, B, C, C, &c. Then multiplying by 1, x, x^2, &c. and adding, we have

$$\begin{aligned} A' + B'x + C'x^2 + \&c. &= (1 + 2x + 2x^2 + \ldots)(A + Bx^2 + Cx^4 + \&c.) \\ &= \frac{1+x}{1-x}(A + Bx^2 + Cx^4 + \&c.); \end{aligned}$$

and if we form in a similar manner A'', B'', C'', D'', &c. from A', B', C', D', &c. and so on, we have

$$\begin{aligned} A'' + B''x + C''x^2 + \&c. &= \frac{1+x}{1-x}(A' + B'x^2 + C'x^4 + \&c.) \\ &= \frac{1+x}{1-x}\,\frac{1+x^2}{1-x^2}(A + Bx^4 + Cx^8 + \&c.), \end{aligned}$$

and so on. Write $A = 1$, and suppose that the process is repeated an indefinite number of times, we have

$$1 + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \&c. = \frac{1+x\,.\,1+x^2\,.\,1+x^4\,.\,\&c.}{1-x\,.\,1-x^2\,.\,1-x^4\,.\,\&c.};$$

and the coefficients 1, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. are precisely those of the infinite series 1, 2, 4, 6, &c. We have more simply

$$1+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\&c. = \frac{1}{(1-x)^2(1-x^2)(1-x^4)(1-x^8)\,\&c.},$$

which gives rise to the following very simple algorithm for the calculation of the coefficients:

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16
0,	0;	1,	2,	4,	6,	9,	12,	16,	20,	25,	30,	36,	42,	49,	56
1,	2,	4,	6,	9,	12,	16,	20,	25,	30,	36,	42,	49,	56,	64,	72
0,	0,	0,	0;	1,	2,	4,	6,	10,	14,	20,	26,	35,	44,	56,	68
1,	2,	4,	6,	10,	14,	20,	26,	35,	44,	56,	68,	84,	100,	120,	140
0,	0,	0,	0,	0,	0,	0,	0;	1,	2,	4,	6,	10,	14,	20,	26
1 |	2 |	4,	6 |	10,	14,	20,	26 |	36,	46,	60,	74,	94,	114,	140,	166|

&c.

The last line is marked off into periods of (reckoning from the beginning) 1, 2, 4, 8, &c.; and by what has preceded, the series which gives the number of 1-partitions, 2-partitions, 3-partitions, &c. is found by summing to the end of each period and doubling the results; we thus, in fact, obtain (1), 2, 6, 26, 166, 1626, &c.: and the same series is also given by means of the last terms of the several periods.

The preceding expression for $1+\mathfrak{B}x+\mathfrak{C}x^2+\&c.$ shows that $\mathfrak{B}$, $\mathfrak{C}$, &c. are the number of partitions of 1, 2, 3, 4, 5, 6, &c. respectively into the parts 1, 1′, 2, 4, 8, &c.: and we are thus led to—

THEOREM. The number of x-partitions (first part unity, no part greater than twice the preceding one) is equal to the number of partitions of $2^{x-1}-1$ into the parts 1, 1′, 2, 4,... 2^{x-2}. Or, again, it is equal to twice the sum of the number of partitions of 0, 1, 2,... $2^{x-2}-1$ respectively into the parts 1, 1′, 2, 4,... 2^{x-3} (where the number of partitions of 0 counts for 1).

For example, the partitions of 0, 1, 2, 3, &c. with the parts 1, 1′, 2,... are

$(\cdot)$

$1,\ 1'$,

$1+1,\ 1+1',\ 1'+1',\ 2$,

$1+1+1,\ 1+1+1',\ 1+1'+1',\ 1'+1'+1',\ 2+1,\ 2+1'$,

the numbers of which are 1, 2, 4, 6. Hence, by the first part of the theorem, the number of 3-partitions is 6, and by the second part of the theorem, the number of 4-partitions is

$$2\,(1+2+4+6), = 26.$$

2, *Stone Buildings, March* 17, 1857.

205.

NOTE ON THE SUMMATION OF A CERTAIN FACTORIAL EXPRESSION.

[From the *Philosophical Magazine*, vol. XIII. (1857), pp. 419—423.]

MR KIRKMAN some months ago communicated to me a formula for the double summation of a factorial expression, to which formula he had been led by his researches on the partition of polygons. The formula in a slightly altered form is as follows: viz.

$$\Sigma_x \Sigma_y \frac{[x+y+2]^y}{[y+1]^{y+1}} \frac{[x]^y}{[y]^y} \frac{[r+k-x-y]^{k-1-y}}{[k-y]^{k-y}} \frac{[r-1-x]^{k-1-y}}{[k-1-y]^{k-1-y}} = \frac{2k}{r+3} \frac{[r+k+2]^k}{[k+1]^{k+1}} \frac{[r]^k}{[k]^k},$$

the summation extending from $x=0$ to $x=r-1$, and $y=0$ to $y=k-1$. In the particular case when $k=r$, then all the terms of the series except those in which $y=x$ vanish; and putting therefore $k=r$ and $y=x$, and making a slight change in the form of the right-hand side, the formula becomes

$$\Sigma \frac{[2x+2]^x}{[x+1]^{x+1}} \frac{[2r-2x]^{r-1-x}}{[r-x]^{r-x}} = 4 \frac{[2r+1]^{r-2}}{[r-1]^{r-1}},$$

the summation extending from $x=0$ to $x=r-1$.

We have, in the notation of Gauss, $[m]^m = m(m-1)\ldots 2.1 = \Pi m$, and a factorial $[m]^n$ is expressed in terms of the function Π by the formula $[m]^n = \Pi m \div \Pi(m-n)$. Write also

$$\Pi_1(m-\tfrac{1}{2}) = (m-\tfrac{1}{2})(m-\tfrac{3}{2})\ldots\tfrac{3}{2}\cdot\tfrac{1}{2},$$

we have

$$\Pi 2m \quad = 2^{2m} \quad \Pi m \Pi_1(m-\tfrac{1}{2}),$$

$$\Pi(2m+1) = 2^{2m+1} \Pi m \Pi_1(m+\tfrac{1}{2});$$

and transforming the factorials by these formulæ, the series becomes

$$\Sigma \frac{\Pi_1(x+\frac{1}{2})\,\Pi_1(r-x-\frac{1}{2})}{\Pi(x+2)\,\Pi(r-x+1)} = \frac{2r\Pi_1(r+\frac{1}{2})}{\Pi(r+3)},$$

the summation, as before, from $x=0$ to $x=r-1$. This may be written

$$\Sigma \frac{\Pi_1(x+\frac{1}{2})}{\Pi_1(\frac{1}{2})} \cdot \frac{\Pi_1(r-x-\frac{1}{2})}{\Pi_1(r-\frac{1}{2})} \frac{\Pi 2}{\Pi(x+2)} \frac{\Pi(r+1)}{\Pi(r-x+1)} = \frac{8r(r+\frac{1}{2})}{(r+2)(r+3)},$$

the summation from $x=0$ to $x=r-1$. The general term does not vanish for $x=r$ or $x=r+1$, but it vanishes for all greater values of x; hence if we add to the right-hand side the two terms corresponding to $x=r$ and $x=r+1$, the summation may be extended from $x=0$ to $x=r+1$, or what is the same thing, from $x=0$ indefinitely. The two terms in question are

$$\frac{4(r+\frac{1}{2})}{r+2} + \frac{-8(r+\frac{3}{2})(r+\frac{1}{2})}{(r+2)(r+3)}, = \frac{-4r(r+\frac{1}{2})}{(r+2)(r+3)},$$

and the resulting equation is

$$\Sigma \frac{\Pi_1(x+\frac{1}{2})}{\Pi_1(\frac{1}{2})} \frac{\Pi(r-x-\frac{1}{2})}{\Pi_1(r-\frac{1}{2})} \frac{\Pi 2}{\Pi(x+2)} \frac{\Pi(r+1)}{\Pi(r-x+1)} = \frac{4r(r+\frac{1}{2})}{(r+2)(r+3)},$$

the summation from $x=0$ indefinitely; or substituting for the functions Π and Π_1 their values, the formula is

$$1 + \frac{\frac{3}{2}.(r+1)}{3.(r-\frac{1}{2})} + \frac{\frac{3}{2}.\frac{5}{2}.(r+1)\,r}{3.4.(r-\frac{1}{2})(r-\frac{3}{2})} + \frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}.(r+1)\,r\,(r-1)}{3.4.5.(r-\frac{1}{2})(r-\frac{3}{2})(r-\frac{5}{2})} + \&c. = \frac{4r(r+\frac{1}{2})}{(r+2)(r+3)},$$

which is a formula obviously belonging to the theory of the hypergeometric series

$$\mathsf{F}(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha.\alpha+1.\beta.\beta+1}{1.2.\gamma.\gamma+1}x^2 + \&c.;$$

but the formula applicable to the case in hand has probably not been given. It may be proved as follows, premising that I disregard all difficulties arising from infinite values of the functions in the definite integrals, convergency, &c. We have

$$\int_0^1 \theta^{\alpha-1}(1-\theta)^{-\alpha-\gamma-1}(1-x\theta)^{\beta}\,d\theta$$

$$= \frac{\Pi(\alpha-1)\,\Pi(-\alpha-\gamma-1)}{\Pi(-\gamma-1)}\left(1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha.\alpha+1.\beta.\beta-1}{1.2.\gamma.\gamma-1}x^2 + \ldots\right).$$

Now we have

$$\int_0 dx \int_0 dx\,(1-x\theta)^{\beta} = \frac{1}{(\beta+1)(\beta+2)\,\theta^2}(1-x\theta)^{\beta+2} - \frac{1}{(\beta+1)(\beta+2)}\frac{1}{\theta^2} + \frac{x}{(\beta+1)\,\theta};$$

and hence multiplying by dx and integrating from $x=0$, and again multiplying by dx and integrating from $x=0$ to $x=1$, we find

$$\int_0^1 d\theta\,.\,\theta^{\alpha-3}(1-\theta)^{-\alpha-\gamma+\beta+1}-\int_0^1 d\theta\,.\,\theta^{\alpha-3}(1-\theta)^{-\alpha-\gamma-1}+(\beta+2)\int_0^1 d\theta\,.\,\theta^{\alpha-2}(1-\theta)^{-\alpha-\gamma-1}$$

$$=\frac{(\beta+1)(\beta+2)\,\Pi(\alpha-1)\,\Pi(-\alpha-\gamma-1)}{\Pi(-\gamma-1)}\,\tfrac{1}{2}S,$$

if for shortness

$$S=1+\frac{\alpha\,.\,\beta}{3\,.\,\gamma}+\frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta-1}{3\,.\,4\,.\,\gamma\,.\,\gamma-1}+\&c.$$

Substituting for the definite integrals their values,

$$\frac{\Pi(\alpha-3)\,\Pi(-\alpha-\gamma+\beta+1)}{\Pi(-\gamma+\beta-1)}-\frac{\Pi(\alpha-3)\,\Pi(-\alpha-\gamma-1)}{\Pi(-\gamma-3)}+(\beta+2)\frac{\Pi(\alpha-2)\,\Pi(-\alpha-\gamma-1)}{\Pi(-\gamma-2)}$$

$$=\frac{(\beta+1)(\beta+2)\,\Pi(\alpha-1)\,\Pi(-\alpha-\gamma-1)}{\Pi(-\gamma-1)}\,\tfrac{1}{2}S,$$

whence

$$\tfrac{1}{2}(\beta+1)(\beta+2)S=\frac{\Pi(\alpha-3)}{\Pi(\alpha-1)}\,\frac{\Pi(-\alpha-\gamma+\beta+1)}{\Pi(-\alpha-\gamma-1)}\,\frac{\Pi(-\gamma-1)}{\Pi(-\gamma+\beta-1)}$$

$$-\frac{\Pi(\alpha-3)}{\Pi(\alpha-1)}\,\frac{\Pi(-\gamma-1)}{\Pi(-\gamma-3)}+(\beta+2)\frac{\Pi(\alpha-2)}{\Pi(\alpha-1)}\cdot\frac{\Pi(-\gamma-1)}{\Pi(-\gamma-2)}.$$

The second and third terms are

$$-\frac{1}{(\alpha-1)(\alpha-2)}(\gamma+1)(\gamma+2)-(\beta+2)\frac{1}{\alpha-1}(\gamma+1),$$

which are

$$=-\frac{\gamma+1}{(\alpha-1)(\alpha-2)}(\alpha\beta+2\alpha-2\beta+\gamma-2).$$

For the reduction of the first term we have

$$\Pi(-\alpha-\gamma+\beta+1)=[\beta+1-\alpha-\gamma]^{\beta+2}\,\Pi(-\alpha-\gamma-1),$$

$$\Pi(\beta-\gamma-1)\qquad=[\beta-\gamma-1]^{\beta}\,\Pi(-\gamma-1);$$

and we thus find

$$\tfrac{1}{2}(\beta+1)(\beta+2)(\alpha-1)(\alpha-2)S=\frac{[\beta+1-\alpha-\gamma]^{\beta+2}}{[\beta-\gamma-1]^{\beta}}-(\gamma+1)(\alpha\beta+2\alpha-2\beta+\gamma-2),$$

where, as before,

$$S=1+\frac{\alpha\,.\,\beta}{3\,.\,\gamma}+\frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta-1}{3\,.\,4\,.\,\gamma\,.\,\gamma-1}+\&c.;$$

this is the formula in hypergeometric series required for the present purpose, and it is certainly true when the series is finite.

Write now

$$\alpha = \tfrac{3}{2}, \quad \beta = r + 1, \quad \gamma = r - \tfrac{1}{2};$$

then the first term is $[1]^{r+2} \div [\tfrac{1}{2}]^{r+1}$, which vanishes on account of the numerator, and the second term is $-\tfrac{1}{2}r(r+\tfrac{1}{2})$, and we have consequently

$$-\tfrac{1}{2}r(r+2)(r+3)\tfrac{1}{2}\,.\,\tfrac{1}{2}S = -\tfrac{1}{2}r(r+\tfrac{1}{2}),$$

which gives

$$S = \frac{4r(r+\frac{1}{2})}{(r+2)(r+3)},$$

S being here the series in r, the sum of which was required, and the particular case of Mr Kirkman's formula is thus verified. It is probable that the general case might be treated in an analogous manner by first grouping together the terms which correspond to a given difference $x \sim y$, and ultimately summing the sums of these partial series; but I have not examined this question.

2, *Stone Buildings, W.C., April* 18, 1857.

206.

NOTE ON A THEOREM RELATING TO THE RECTANGULAR HYPERBOLA.

[From the *Philosophical Magazine*, vol. XIII. (1857), p. 423.]

THE following theorem is given in a slightly different form by Brianchon and Poncelet, *Gergonne*, vol. XI. [1820], p. 205, viz. Any conic whatever which passes through the three angles of a triangle and the point of intersection of the perpendiculars let fall from the angles of the triangle upon the opposite sides is a rectangular hyperbola; and there is an elegant demonstration depending on the properties of the inscribed hexagon. The theorem is, however, a particular case of the following: viz. "Any conic whatever which passes through the four points of intersection of two rectangular hyperbolas is a rectangular hyperbola." And this, again, is a particular case of the following: viz. If there be a conic Ω and a line P, then considering any two conics U, V such that the points of intersection of P, U are harmonics in respect to the points of intersection of P, Ω, and the points of P, V are also harmonics in respect to the points of intersection of P, Ω, then any conic whatever W which passes through the four points of intersection of U, V will have the like property, viz. the points of intersection of P, W will be harmonics in respect of the points of intersection of P, Ω; a theorem which is an immediate consequence of the theorem that three conics which intersect in the same four points are intersected by any line whatever in six points which are in involution.

2, *Stone Buildings, W.C., April* 23, 1857.

207.

ANALYTICAL SOLUTION OF THE PROBLEM OF TACTIONS.

[From the *Philosophical Magazine*, vol. XIII. (1857), pp. 507—509.]

IT is well known that the eight circles, each of which touches three given circles, are determined as follows:—viz. considering any one in particular of the four axes of similitude of the given circles, and the perpendicular let fall from the radical centre (or centre of the orthotomic circle) of the given circles, there are two of the required tangent circles which have their centres upon the perpendicular, and pass through the points of intersection of the orthotomic circle and the axis of similitude, or in other words, the axis of similitude is a common chord (or radical axis) of the orthotomic circle and the two tangent circles. This suggests the choice of the radical centre for the origin of coordinates; and the resulting formulæ then take very simple forms, and the theorem is verified without difficulty.

Take then the centre of the orthotomic circle as the origin of coordinates, and let the radius of this circle be put equal to unity; then if (α, β), (α', β'), (α'', β'') are the coordinates of the centres of the given circles, the equations of these will be

$$x^2 + y^2 + 1 - 2\alpha\, x - 2\beta\, y = 0,$$
$$x^2 + y^2 + 1 - 2\alpha'\, x - 2\beta'\, y = 0,$$
$$x^2 + y^2 + 1 - 2\alpha'' x - 2\beta'' y = 0\,;$$

and the radii of the circles will be $\sqrt{\alpha^2+\beta^2-1}$, $\sqrt{\alpha'^2+\beta'^2-1}$, $\sqrt{\alpha''^2+\beta''^2-1}$. It will be convenient to write

$$\gamma = \pm\sqrt{\alpha^2 + \beta^2 - 1},$$
$$\gamma' = \pm\sqrt{\alpha'^2 + \beta'^2 - 1},$$
$$\gamma'' = \pm\sqrt{\alpha''^2 + \beta''^2 - 1},$$

where the three several signs $\pm$ are fixed once for all in a determinate manner. If, however, all the signs are reversed, the result is the same, so that the system is one of *four* (not of eight) different systems. The coordinates of a centre of similitude of the second and third circles are

$$\frac{\alpha'\gamma''-\alpha''\gamma'}{\gamma''-\gamma'},\quad \frac{\beta'\gamma''-\beta''\gamma'}{\gamma''-\gamma'},$$

and forming the corresponding expressions for the coordinates of the centres of similitude of the third and first circles, and of the first and second circles, the three centres of similitude lie on a line which will be an axis of similitude: to find the equation, write

$$\begin{aligned}
A &= \beta\gamma' - \beta'\gamma + \beta'\gamma'' - \beta''\gamma' + \beta''\gamma - \beta\gamma'',\\
B &= \gamma\alpha' - \gamma'\alpha + \gamma'\alpha'' - \gamma''\alpha' + \gamma''\alpha - \gamma\alpha'',\\
C &= \alpha\beta' - \alpha'\beta + \alpha'\beta'' - \alpha''\beta' + \alpha''\beta - \alpha\beta'',\\
\nabla &= \alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma,
\end{aligned}$$

values which, it will be observed, give

$$\begin{aligned}
A\alpha + B\beta + C\gamma &= \nabla,\\
A\alpha' + B\beta' + C\gamma' &= \nabla,\\
A\alpha'' + B\beta'' + C\gamma'' &= \nabla;
\end{aligned}$$

then the equation of the axis of similitude is found to be

$$Ax + By - \nabla = 0;$$

and hence the equation of the perpendicular let fall from the radical centre upon the axis of similitude is

$$Bx - Ay = 0.$$

It should therefore be possible to find two circles having their centres on the last-mentioned line and touching the three given circles Take $A\theta$, $B\theta$ as the coordinates of the centre of one of the two circles, and let r be its radius; the conditions of tangency are

$$\begin{aligned}
r &= \sqrt{(A\theta-\alpha)^2+(B\theta-\beta)^2} \pm \gamma,\\
&= \sqrt{(A\theta-\alpha')^2+(B\theta-\beta')^2} \pm \gamma',\\
&= \sqrt{(A\theta-\alpha'')^2+(B\theta-\beta'')^2} \pm \gamma'',
\end{aligned}$$

where the sign $\pm$ has the same value in each expression. We have consequently

$$(r \mp \gamma)^2 = (A\theta-\alpha)^2 + (B\theta-\beta)^2;$$

or, observing that $A\alpha + B\beta = \nabla - C\gamma$, and reducing,

$$r^2 - (A^2+B^2)\theta^2 + 2\nabla\theta - 1 = 2\gamma(\pm r + C\theta).$$

Forming the two analogous equations, the three equations will be satisfied if only

$$r^2 - (A^2 + B^2)\,\theta^2 + 2\nabla\theta - 1 = 0,$$
$$\pm\, r + C\theta \qquad\qquad = 0.$$

Eliminating r, we have

$$(A^2 + B^2 - C^2)\,\theta^2 - 2\nabla\theta + 1 = 0,$$

which gives for θ the two values

$$(A^2 + B^2 - C^2)\,\theta = \nabla \pm \sqrt{\nabla^2 - (A^2 + B^2 - C^2)}\,;$$

and then r is determined linearly by the equation

$$r = \mp\, C\theta.$$

The equation of the tangent circle is therefore

$$(x - A\theta)^2 + (y - B\theta)^2 = C^2\theta^2\,;$$

or reducing,

$$x^2 + y^2 - 1 - 2\theta\,(Ax + By - \nabla) = 0\,;$$

and recollecting that $Ax + By - \nabla = 0$ is the equation of the axis of similitude, the equation shows that the axis of similitude is a common chord or radical axis of the orthotomic circle and the two tangent circles.

2, *Stone Buildings, W.C., May* 15, 1857.

208.

NOTE ON THE EQUIPOTENTIAL CURVE $\frac{m}{r}+\frac{m'}{r'}=C$.

[From the *Philosophical Magazine*, vol. XIV. (1857), pp. 142—146.]

THE equation $\frac{m}{r}+\frac{m'}{r'}=C$, where m, m', C are constants, and r, r' are the distances of a point P of the locus from two given points M, M' respectively, expresses that the potential of the attracting or repelling masses m, m' has a constant value at all points of the locus. The locus is obviously a surface of revolution, having the line through the points M, M' for its axis; and instead of the surface, we may consider the section by a plane through the axis, or what is the same thing, we may consider r, r' as the distances *in plano* of a point P of the curve from the given points M, M': such curve may be termed the equipotential curve. I propose in the present Note to investigate in a general manner, and without entering into any analytical detail, the general form of the curve corresponding to different values of the quantity C.

It is proper to remark, that the curve is not altered by changing the signs of each or any of the quantities m, m', C (in fact, analytically the distances r, r' are essentially ambiguous in sign), so that we may without loss of generality consider m, m', C as all of them positive. The different branches of the complete analytical or geometrical curve have distinct mechanical significancies; thus, r, r' being positive, $\frac{m}{r}+\frac{m'}{r}=C$ is the curve for which the potential of the attracting masses m, m' is equal to C; but $\frac{m}{r}-\frac{m'}{r'}=C$ is the curve for which the attracting mass m, and the repulsive mass m', have the potential C; but this is a distinction to which I do not attend. I write for homogeneity $\frac{k}{a}$ instead of C, where a is the distance between the points M, M'; the equation thus becomes

$$\frac{m}{r}+\frac{m'}{r'}=\frac{k}{a},$$

where a is a positive distance, m, m', k may be considered as positive abstract numbers. The curve is obviously a curve of the eighth order. When k is large in comparison with m, m', then since r, r' cannot be both of them small in comparison of a (for if one be small, the other will be nearly equal to a), it is clear that one of these distances, for instance r, will be small, and the other, r', nearly equal to a. We in fact have $\left(\text{neglecting in the first instance } \frac{m'}{r'} \text{ in comparison with } \frac{m}{r}\right) \frac{m}{r}=\frac{k}{a}$, or more accurately, $\frac{m}{r}=\frac{k\pm m'}{a}$, i.e. $r=\frac{m}{k\pm m'}$, which shows that a part of the curve consists of two ovals, which are approximately concentric circles, radii $\frac{m'}{k\pm m'}a$, about the point M as centre. In like manner a part of the curve consists of two ovals, which are approximately concentric circles, radii $\frac{m'}{k\pm m}a$, about the point M' as centre. I denote by A, B, the two ovals about M, viz. A is the exterior, and B the interior oval; and in like manner by A', B' the two ovals about M', viz. A' is the exterior, and B' the interior oval. The distances *inter se* of the ovals A and B, or of the ovals A' and B', are small in comparison with the radii of these ovals respectively; and if, to fix the ideas, m' be greater than M, then the ovals A', B' are greater than the ovals A, B.

It is easy to see that the curve will have a node or double point on the axis if $k=(\sqrt{m'}\pm\sqrt{m})^2$; and we must first consider the case $k=(\sqrt{m'}+\sqrt{m})^2$. The node lies between the points M, M', and its distances from these points are respectively as $\sqrt{m}:\sqrt{m'}$, that is, it is nearest to M. The transition from the original form is very obvious; the exterior ovals A, A' have gradually expanded until they come in contact, and at the instant of doing so the two ovals change themselves into a figure of eight, AA'. The ovals B, B' also expand and change their form, but they preserve the general character of ovals enclosing the points M, M' respectively. The curve consists of a figure of eight AA', and (inside of the two divisions thereof respectively) of the ovals B, B' enclosing the points M, M'. The half of the curve nearest to M' is, as before, preponderant in magnitude.

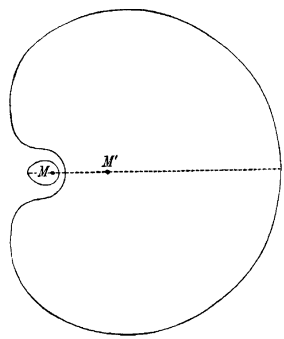
The next change when k continues to diminish is an obvious one: the figure of eight opens out into an hourglass-shaped oval AA', while the ovals B, B' continue increasing in magnitude and altering their form.

There will be again a node or double point when $k=(\sqrt{m'}-\sqrt{m})^2$; but to explain the transition to this special form, it is necessary to attend more particularly to the change of form in the oval B' as k approaches to the value in question, viz. this oval lengthens out and begins to twist itself round the oval B; and when k' becomes $=(\sqrt{m'}-\sqrt{m})^2$, then the oval B' has completely encircled B, the two extremities of B' meeting together at the double point, which is a point beyond M (i.e. on the other side to M'), such that its distances from M, M' are in the ratio of $\sqrt{m}:\sqrt{m'}$. And at the instant of contact there is, as in the former case, a modification of the

form of the portions which come into contact, so that the node is an ordinary double point. The oval B' has, in fact, become what may be termed a re-entrant figure of eight, , the small part of which encloses the oval B which encloses the point M, while the large part encloses the point M'. The curve consists of the exterior oval AA' (which has probably lost wholly or partially its hourglass form, and is more nearly an ordinary oval), of the re-entrant figure of eight, B', and of the enclosed oval B.

As k continues to diminish, the re-entrant figure of eight, B', breaks up into two detached ovals lB', mB', the larger of which, lB', encloses the other one and also the point M'; while the smaller one, mB', does not enclose M', but encloses the oval B which encloses M; the curve consists of the exterior oval AA', the ovals lB' and mB' which have arisen out of the oval B', and the oval B. As k further decreases, the ovals AA' and lB' continually increase in magnitude, and the ovals mB' and B approximate more and more nearly together; and at length, when k becomes $=0$, the ovals AA' and lB' disappear at infinity, while the ovals mB' and B unite themselves into a circle enclosing M, but not enclosing M': the equation of this circle is, in fact, $\frac{m}{r}+\frac{m'}{r'}=0$; or what is the same thing, $r^2=\frac{m^2}{m'^2}r'^2$, and the points M, M' have, in relation to this circle, the well-known relation that each is the image of the other.

The preceding description is, I think, intelligible without the assistance of a series of figures illustrating the different forms of the curve, but there is no difficulty in actually tracing the curve for any particular values of the constant parameters. Thus



(taking the distance MM' for unity) suppose that the equation of the curve is $\frac{1}{r}+\frac{4}{r'}=1{\cdot}2$; (the value 1·2 was selected as a value not far from that for which the oval B' becomes a re-entrant figure of eight, though the change of form is so rapid that this value shows

only the incipient tendency of the oval B' to take the form in question). The form of the portion of the curve consisting of the two ovals B, B' will be that shown by the figure, which was constructed by points on a double scale with some accuracy.

The case $m=m'$ is an exception, and must be considered separately: the curve is here in all its changes symmetrical about a perpendicular to the axis midway between the two centres M, M'. The curve in the first instance, i.e. when k is greater than $(\sqrt{m}+\sqrt{m})^2, = 4m$, consists of the two ovals B, A about M, and the two ovals B', A' about M'. As k decreases to $4m$, the two ovals A, A' gradually increase in magnitude, and at length come together, as before, into a figure of eight, AA'; and as k continues to diminish, the figure of eight opens out into an hourglass form AA', which continues increasing in magnitude, and degenerating into the form of an oval. The interior ovals B, B' approach more and more nearly together, lengthen out in the direction perpendicular to the axis, and present to each other a more and more flattened portion. The second value,

$$k=(\sqrt{m'}-\sqrt{m})^2,$$

which in the general case gives a node, in the present case only arises when $k=0$; and there is not then any node, but the curve degenerates in a similar manner to what happens for $k=0$ in the general case; viz. the oval AA' disappears at infinity, while the ovals B, B' coalesce together (their outer parts disappearing at infinity) into a pair of lines coincident with the perpendicular to the axis midway between the two centres.

2, *Stone Buildings, May* 31, 1857.

209.

A DEMONSTRATION OF SIR W. R. HAMILTON'S THEOREM OF THE ISOCHRONISM OF THE CIRCULAR HODOGRAPH.

[From the *Philosophical Magazine*, vol. XIV. (1857), pp. 427—430.]

IMAGINE a body moving *in plano* under the action of a central force, and let h denote, as usual, the double of the area described in a unit of time; let P be any point of the orbit, then measuring off, on the perpendicular let fall from the centre of force O on the tangent at P to the orbit, a distance OQ equal or proportional to h into the reciprocal of the perpendicular on the tangent, the locus of Q is the hodograph, and the points P, Q are corresponding points of the orbit and hodograph.

It is easy to see that the hodograph is the polar reciprocal of the orbit with respect to a circle having O for its centre, and having its radius equal or proportional to $\sqrt{h}$. And it follows at once that Q is the pole, with respect to this circle, of the tangent at P to the orbit.

In the particular case where the force varies inversely as the square of the distance, the hodograph is a circle. And if we consider two elliptic orbits described about the same centre, under the action of the same central force, and such that the major axes are equal, then (as will be presently seen) the common chord or radical axis of the two hodographs passes through the centre of force.

Imagine an orthotomic circle of the two hodographs (the centre of this circle is of course on the common chord or radical axis of the two hodographs), and consider the arcs intercepted on the two hodographs respectively by the orthotomic circle; then the theorem of the isochronism of the circular hodograph is as follows, viz. the times of hodographic description of the intercepted arcs are equal; in other words, the times of description in the orbits, of the arcs which correspond to the intercepted arcs of the hodographs, are equal. It was remarked by Sir W. R. Hamilton, that the theorem is in fact equivalent to Lambert's theorem, that the time depends only on the chord

of the described arc and the sum of the two radius vectors. And this remark suggests a mode of investigation of the theorem. Consider the intercepted arc of one of the hodographs: the tangents to the hodograph at the extremities of this arc are radii of the orthotomic circle; i.e. the corresponding arc of the orbit is the arc cut off by the polar (in respect to the directrix circle by which the hodograph is determined) of the centre of the orthotomic circle; the portion of this polar intercepted by the orbit is the elliptic chord, and this elliptic chord and the sum of the radius vectors at the two extremities of the elliptic chord determine the time of description of the arc; and the values of these quantities, viz. the elliptic chord and the sum of the radius vectors, must be the same in each orbit.

The analytical investigation is not difficult. I take as the equation of the first orbit,

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\varpi)};$$

then the polar of the orbit with respect to a directrix circle $r=c$ is

$$r^2 - \frac{c^2 e}{a(1-e^2)} r\cos(\theta-\varpi) - \frac{c^4}{a^2(1-e^2)} = 0,$$

and putting $c^2 = k\sqrt{k}\sqrt{a(1-e^2)}$ (where k is a constant quantity, i.e. it is the same in each orbit), the equation becomes

$$r^2 - \frac{ek\sqrt{k}}{\sqrt{a(1-e^2)}} r\cos(\theta-\varpi) - \frac{k^3}{a} = 0.$$

But since a is supposed to be the same in each orbit, we may for greater simplicity write $k^3 = m^2 a$; it will be convenient also to put $e = \sin\kappa$; we have then

$$r = \frac{a\cos^2\kappa}{1+\sin\kappa\cos(\theta-\varpi)}$$

for the equation of the orbit, $r^2 = ma\cos\kappa$ for the equation of the directrix circle, and

$$r^2 - r\,.\,m\tan\kappa\cos(\theta-\varpi) - m^2 = 0$$

for the equation of the hodograph.

We have in like manner

$$r = \frac{a\cos^2\kappa'}{1+\sin\kappa'\cos(\theta-\varpi')}$$

for the equation of the second orbit, $r^2 = ma\cos\kappa'$ for the equation of the corresponding directrix circle, and

$$r^2 - r\,.\,m\tan\kappa'\cos(\theta-\varpi') - m^2 = 0$$

for that of the hodograph.

The equations of the two hodographs give at once

$$\tan\kappa\cos(\theta-\varpi)-\tan\kappa'\cos(\theta-\varpi')=0$$

for the equation of the common chord or radical axis of the two hodographs,—an equation which shows that, as already noticed, the common chord passes through the origin or centre of force. This equation gives $\theta=\alpha$ if

$$\tan\kappa\cos(\alpha-\varpi)-\tan\kappa'\cos(\alpha-\varpi')=0\,;$$

i.e. α is a quantity such that the expressions $\tan\kappa\cos(\alpha-\varpi)$ and $\tan\kappa'\cos(\alpha-\varpi')$, which correspond to each other in the two orbits, are equal. We may take R, α as the polar coordinates of the centre of the orthotomic circle (where R is arbitrary); the equation of the polar of this point with respect to the directrix circle $r^2=ma\cos\kappa$, is then at once seen to be

$$r\cos(\theta-\alpha)=\frac{ma\cos\kappa}{R},$$

which is the equation of the line cutting off the arc of the elliptic orbit

$$r=\frac{a\cos^2\kappa}{1+\sin\kappa\sin(\theta-\varpi)}.$$

Writing $\theta-\varpi=\theta-\alpha+(\alpha-\varpi)$, the two equations give

$$\cos(\theta-\alpha)=\frac{A}{r},$$

$$\sin(\theta-\alpha)=\frac{B}{r}+C,$$

if for shortness

$$A=\frac{ma\cos\kappa}{R},$$

$$B=\frac{ma\cos\kappa}{R}\,\frac{\cos(\alpha-\varpi)}{\sin(\alpha-\varpi)}-\frac{a\cos^2\kappa}{\sin\kappa\sin(\alpha-\varpi)},$$

$$C=\frac{1}{\sin\kappa\sin(\alpha-\varpi)}\,;$$

we have therefore

$$\frac{A^2+B^2}{r^2}+\frac{2BC}{r}+C^2=1,$$

or, what is the same thing,

$$(1-C^2)\,r^2-2BCr-(A^2+B^2)=0\,;$$

and thence, if r', r'' are the two values of r,

$$r'+r''=\frac{2BC}{1-C^2},$$

$$r'r''=-\frac{A^2+B^2}{1-C^2}.$$

Let θ', θ'' be the corresponding values of θ, we have

$$\theta' - \theta'' = \theta' - \alpha - (\theta'' - \alpha),$$

and thence

$$\cos(\theta' - \theta'') = \frac{A}{r'} \cdot \frac{A}{r''} + \left(\frac{B}{r'} + C\right)\left(\frac{B}{r''} + C\right), \quad = \frac{A^2 + B^2}{r'r''} + BC\left(\frac{1}{r'} + \frac{1}{r''}\right) + C^2,$$

or adding unity to each side, multiplying by $r'r''$, and on the right-hand side substituting for $r' + r''$, $r'r''$ their values

$$r'r''\left(1 + \cos(\theta' - \theta'')\right) = -\frac{2C^2A^2}{1 - C^2};$$

the square of the chord is $r^2 + r'^2 - 2rr'\cos(\theta' - \theta'')$, or, what is the same thing, $(r' + r'')^2 - 2r'r''\left(1 + \cos(\theta' - \theta'')\right)$; hence to prove the theorem, it is only necessary to show that $r' + r''$ and $r'r''\left(1 + \cos(\theta' - \theta'')\right)$ have the same values in each orbit, that is, that $\frac{2BC}{1 - C^2}$ and $-\frac{2C^2A^2}{1 - C^2}$ have the same values in each orbit. But observing that

$$1 - \sin^2\kappa \sin^2(\alpha - \varpi) = \cos^2\kappa + \sin^2\kappa \cos^2(\alpha - \varpi) = \cos^2\kappa \left\{1 + \tan^2\kappa \cos^2(\alpha - \varpi)\right\},$$

the values of these expressions are respectively

$$-\frac{2a}{R} \frac{\left(m \tan\kappa \cos(\alpha - \varpi) - R\right)}{1 + \tan^2\kappa \cos^2(\alpha - \varpi)},$$

$$\frac{2a^2}{R^2} \frac{m^2}{1 + \tan^2\kappa \cos^2(\alpha - \varpi)},$$

which contain only the quantities m, a, R, $\tan\kappa \cos(\alpha - \varpi)$, which are the same for each orbit, and the theorem is therefore proved, viz. it is made to depend on Lambert's theorem. I may remark, that a geometrical demonstration which does not assume Lambert's theorem is given by Mr Droop in his paper "On the Isochronism of the Circular Hodograph," *Quarterly Mathematical Journal*, vol. I. [1857] pp. 374—378, where the dependence of the theorem on Lambert's theorem is also shown.

By what precedes, the theorem may be stated in a geometrical form as follows:—"Imagine two ellipses having a common focus, and their major axes equal; describe about the focus two directrix circles having their radii proportional to the square roots of the minor axes of the ellipses respectively; the polar reciprocal of each ellipse in respect to its own directrix circle will be a circle (the hodograph), and the common chord or radical axis of the two hodographs will pass through the focus. Consider any point on the common chord, and take the polar with respect to each directrix circle; such polar will cut off an arc of the corresponding ellipse; and then, *theorem*, the elliptic chord, and the sum of the radius vectors through the two extremities of the chord, will be respectively the same for each ellipse."

2, *Stone Buildings, W.C., June* 24, 1857.

210.

ON THE CUBIC TRANSFORMATION OF AN ELLIPTIC FUNCTION.

[From the *Philosophical Magazine*, vol. XV. (1858), pp. 363—364.]

LET

$$z = \frac{(a', b', c', d' \between x, 1)^3}{(a, b, c, d \between x, 1)^3}$$

be any cubic fraction whatever of x, then it is always possible to find quartic functions of z, x respectively, such that

$$\frac{dz}{\sqrt{(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e} \between z, 1)^4}} = \frac{dx}{\sqrt{(A, B, C, D, E \between x, 1)^4}}.$$

This depends upon the following theorem, viz. putting for shortness,

$$U = (a, b, c, d \between x, y)^3,$$
$$U' = (a', b', c', d' \between x, y)^3,$$

and representing by the notation

$$\text{disct. } (aU' - a'U, \quad bU' - b'U, \quad cU' - c'U, \quad dU' - d'U);$$

or more shortly by

$$\text{disct. } (aU' - a'U, \ldots),$$

the discriminant in regard to the facients (λ, μ) of the cubic function

$$(aU' - a'U, \quad bU' - b'U, \quad cU' - c'U, \quad dU' - d'U \between \lambda, \mu)^3;$$

or what is the same thing, the cubic function

$$(a, b, c, d \between \lambda, \mu)^3 . (a', b', c', d' \between x, y)^3$$
$$-(a', b', c', d' \between \lambda, \mu)^3 . (a, b, c, d \between x, y)^3;$$

and by $J(U, U')$ the functional determinant, or Jacobian, of the two cubics U, U'; the theorem is that the discriminant contains as a factor the square of the Jacobian, or that we have

$$\text{disct. } (aU' - a'U, \ldots) = \{J(U, U')\}^2 . (A, B, C, D, E \between x, y)^4.$$

For assuming this to be the case, then (disregarding a mere numerical factor) we have

$$UdU' - U'dU = J(U,\ U')(ydx - xdy),$$

and the two equations give

$$\frac{UdU' - U'dU}{\sqrt{\text{disct.}(aU' - a'U, \ldots)}} = \frac{ydx - xdy}{\sqrt{(A,\ B,\ C,\ D,\ E \between x,\ y)^4}},$$

whence writing z for $U' \div U$, and putting y equal to unity, we have

$$\frac{dz}{\sqrt{\text{disct.}(az - a', \ldots)}} = \frac{dx}{\sqrt{(A,\ B,\ C,\ D,\ E \between x,\ 1)^4}};$$

where disct. $(az - a', \ldots)$, or at full length,

$$\text{disct.}\ (az - a',\ bz - b',\ cz - c',\ dz - d'),$$

is a given quartic function of z,

$$= (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e} \between z,\ 1)^4$$

suppose; and this proves the theorem of transformation.

The assumed subsidiary theorem may be thus proved: suppose that the parameter θ is determined so that the cubic

$$U + \theta U'$$

may have a square factor, the cubic may be written

$$(a + \theta a',\ b + \theta b',\ c + \theta c',\ d + \theta d' \between x,\ y)^3,$$

and the requisite condition is

$$\text{disct.}\ (a + \theta a', \ldots) = 0;$$

there are consequently four roots; and calling these θ_1, θ_2, θ_3, θ_4, we have identically

$$\text{disct.}\ (a + \theta a', \ldots) = K(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4),$$

or what is the same thing,

$$\text{disct.}\ (aU' - a'U, \ldots) = K(U + \theta_1 U')(U + \theta_2 U')(U + \theta_3 U')(U + \theta_4 U').$$

Now any double factor of U or U' (that is the linear factor which enters twice into U or U') is a simple factor of $J(U,\ U')$, and we have $J(U,\ U') = J(U,\ U + \theta U')$, and consequently

$$J(U,\ U') = J(U,\ U + \theta_1 U') = \&\text{c.};$$

hence the double factors of each of the expressions $U + \theta_1 U'$, $U + \theta_2 U'$, $U + \theta_3 U'$, $U + \theta_4 U'$ are simple factors of $J(U,\ U')$, or what is the same thing, $J(U,\ U')$ is the product of four linear factors, which are respectively double factors of the product

$$(U + \theta_1 U')(U + \theta_2 U')(U + \theta_3 U')(U + \theta_4 U'),$$

or this product contains the factor $\{J(U,\ U')\}^2$, which proves the theorem.

2, *Stone Buildings, W.C., March* 5, 1858.

211.

ON A THEOREM RELATING TO HYPERGEOMETRIC SERIES.

[From the *Philosophical Magazine*, vol. XVI. (1858), pp. 356—357.]

In attempting to verify a formula of Hansen's relating to the development of the disturbing function in the planetary theory, I was led to a theorem in hypergeometric series: viz. writing, as usual,

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha . \alpha + 1 . \beta . \beta + 1}{1 . 2 . \gamma . \gamma + 1} x^2 + \ldots$$

then the product

$$F(\alpha, \beta, \gamma + \tfrac{1}{2}, x) F(\gamma - \alpha, \gamma - \beta, \gamma + \tfrac{1}{2}, x)$$

is connected with

$$(1 - x)^{-(\gamma - \alpha - \beta)} F(2\alpha, 2\beta, 2\gamma, x)$$

by a simple relation; for if the last-mentioned expression is put equal to

$$1 + Bx + Cx^2 + Dx^3 + \ldots$$

then the product in question is equal to

$$1 + \frac{\gamma}{\gamma + \frac{1}{2}} Bx + \frac{\gamma . \gamma + 1}{\gamma + \frac{1}{2} . \gamma + \frac{3}{2}} Cx^2 + \frac{\gamma . \gamma + 1 . \gamma + 2}{\gamma + \frac{1}{2} . \gamma + \frac{3}{2} . \gamma + \frac{5}{2}} Dx^3 + \&c.$$

The form of the identity thus arrived at will be best perceived by considering a particular case. Thus, comparing the coefficients of x^3, we have

$$\begin{aligned}
&\frac{\alpha.\alpha+1.\alpha+2.\beta.\beta+1.\beta+2}{1.2.3.\gamma+\frac{1}{2}.\gamma+\frac{3}{2}.\gamma+\frac{5}{2}}.1 \\
&+\frac{\alpha.\alpha+1.\beta.\beta+1}{1.2.\gamma+\frac{1}{2}.\gamma+\frac{3}{2}} \quad .\frac{\gamma-\alpha.\gamma-\beta}{1.\gamma+\frac{1}{2}} \\
&+\frac{\alpha.\beta}{1.2} \quad .\frac{\gamma-\alpha.\gamma-\alpha+1.\gamma-\beta.\gamma-\beta+1}{1.2.\gamma+\frac{1}{2}.\gamma+\frac{3}{2}} \\
&+1 \quad .\frac{\gamma-\alpha.\gamma-\alpha+1.\gamma-\alpha+2.\gamma-\beta.\gamma-\beta+1.\gamma-\beta+2}{1.2.3.\gamma+\frac{1}{2}.\gamma+\frac{3}{2}.\gamma+\frac{5}{2}}
\end{aligned}$$

$$\left.\begin{aligned}
=&\frac{2\alpha.2\alpha+1.2\alpha+2.2\beta.2\beta+1.2\beta+2}{1.2.3.2\gamma.2\gamma+1.2\gamma+2}.1 \\
&+\frac{2\alpha.2\alpha+1.2\beta.2\beta+1}{1.2.2\gamma.2\gamma+1}.\frac{\gamma-\alpha-\beta}{1} \\
&+\frac{2\alpha.2\beta}{1.2\gamma} \quad .\frac{\gamma-\alpha-\beta.\gamma-\alpha-\beta+1}{1.2} \\
&+1 \quad .\frac{\gamma-\alpha-\beta.\gamma-\alpha-\beta+1.\gamma-\alpha-\beta+2}{1.2.3}
\end{aligned}\right\}\frac{\gamma.\gamma+1.\gamma+2}{\gamma+\frac{1}{2}.\gamma+\frac{3}{2}.\gamma+\frac{5}{2}}.$$

It may be observed that the function on the right-hand side is, as regards α, a rational and integral function of the degree 3, and as such may be expanded in the form

$$\begin{aligned}
&A\alpha.\alpha+1.\alpha+2 \\
&+B\alpha.\alpha+1 \quad .\gamma-\alpha \\
&+C\alpha. \quad .\gamma-\alpha.\gamma-\alpha+1 \\
&+D \quad .\gamma-\alpha.\gamma-\alpha+1.\gamma-\alpha+2,
\end{aligned}$$

and that the last coefficient D can be obtained at once by writing $\alpha=0$; this in fact gives

$$D\gamma.\gamma+1.\gamma+2=\frac{\gamma-\beta.\gamma-\beta+1.\gamma-\beta+2}{1.2.3}\,\frac{\gamma.\gamma+1.\gamma+2}{\gamma+\frac{1}{2}.\gamma+\frac{3}{2}.\gamma+\frac{5}{2}},$$

and thence

$$D=\frac{\gamma-\beta.\gamma-\beta+1.\gamma-\beta+2}{1.2.3.\gamma+\frac{1}{2}.\gamma+\frac{3}{2}.\gamma+\frac{5}{2}},$$

which agrees with the left-hand side of the equation: and the value of the first coefficient A may be obtained in like manner with a little more difficulty; but I have not succeeded in obtaining a direct proof of the equation. The form of the equation shows that the left-hand side should vanish for $\gamma=-2$, which may be at once verified.

Grassmere, August 25, 1858.

212.

A MEMOIR ON THE PROBLEM OF DISTURBED ELLIPTIC MOTION.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXVII., 1859, pp. 1—29. Read March 9, 1858.]

I VENTURE to take up the problem of disturbed elliptic motion, for the sake of a further elaboration of the analytical theory. The points which present difficulty are the measurement of longitudes in the varying plane of the orbit, and (in the lunar theory) the determination of the position of the orbit by reference to the varying plane of the sun's orbit; it is, in memoirs and works on the lunar and planetary theories, often difficult to discover where or how (or whether at all) account is taken of these variations, and the analytical mode of treatment is for the most part very imperfect. I must except always Hansen's *Fundamenta Nova* [*investigationis orbitæ veræ quam Luna perlustrat*, Gotha 1838] where the points referred to are treated in a perfectly rigorous manner. There is, however, a want of clearness in the form under which his investigations are presented; and the comprehension of them is greatly facilitated by Jacobi's remarks, published under the title "Auszug zweier Schreiben des Prof. Jacobi an Herrn Director Hansen" (*Crelle*, t. XLII. pp. 12—31 (1851)). Jacobi observes that the integration of Hansen's system of differential equations introduces seven arbitrary constants, which, in the expressions for the coordinates referred to fixed axes, reduce themselves to six. The seventh constant, neglecting the disturbing forces, is in fact a constant which determines the position in the orbit of the arbitrary origin from which the longitudes in orbit are reckoned. I have, in my paper "On Hansen's Lunar Theory," *Quarterly Mathematical Journal*, vol. I. pp. 112—125 (1855), [163], termed this origin "the departure-point," and longitudes measured from it "departures." The seventh constant may be taken to be the departure of the node. I reproduce in the present memoir the explanation of what is meant by the departure when the plane of the orbit is variable. If the problem is treated by the method of the variation of the elements, the seventh constant becomes, like the other elements, variable; and we have

thus a seventh variable element, the departure of the node. The element just referred to (the departure of the node) forms, with the longitude of the node and the inclination, a group of three elements, which determine the position of the orbit and of the departure-point. The coordinates of the planet are in the first instance taken to be the radius vector, longitude, and latitude; but the before-mentioned three elements being considered as given, the position of the planet depends only on the radius vector and the departure. These may be then expressed in terms of the remaining four elements; as to the choice of these four elements, it is to be remarked that there is one element which only enters through the mean anomaly, and that there is great convenience in representing with Hansen the mean anomaly by a single letter; and that in the various formulæ we may use, in the place of the element implicitly involved in the mean anomaly, the mean anomaly itself, or treat the mean anomaly as an element; the four elements may be taken to be the semi-axis major, the eccentricity, the mean anomaly, and the departure of the pericentre. And joining to these the before-mentioned three elements, we have the system of elements represented in the memoir by a, e, g, ϖ, σ, θ, ϕ. It has been assumed so far that the three elements determine the position of the orbit and departure-point in reference to a fixed plane and origin of longitudes; but we may suppose more generally that, instead of the fixed plane and origin of longitudes, we have a variable plane or orbit of reference and a departure-point in this variable orbit of reference. The quantities which determine the orbit of reference and departure-point are naturally taken to be the departure of the node, longitude of the node, and inclination; these are assumed to be given functions of the time, and they are in the memoir represented by σ', θ', ϕ'. The three elements of the planet's orbit (viz. departure of node, longitude of node, and inclination) in relation to the orbit of reference and departure-point therein, are in the memoir represented by Σ, Θ, Φ, and the system of elements ultimately adopted is therefore a, e, g, ϖ, Σ, Θ, Φ. I obtain formulæ for the variations of these elements under two different modes of expression of the disturbing function: first, when the disturbing function is expressed in terms of the radius vector and departure and of the three elements Σ, Θ, Φ; secondly, when the disturbing function is expressed in terms of the seven elements a, e, g, ϖ, Σ, Θ, Φ. The establishment of the two sets of formulæ just referred to constitutes the chief object of the memoir; but the memoir contains some other investigations and formulæ in relation to the general subject.

The coordinates of the planet are

r, the radius vector,

v, the longitude,

y, the latitude.

The attractive force at distance unity is for convenience represented by n^2a^3, which denotes, therefore, an absolute constant; but the significations of n and a are not yet defined.

The disturbing function, as used by Lagrange, is denoted by Ω, that is $\Omega = -R$, if R be the disturbing function of the *Mécanique Céleste*.

The equations of motion are

$$\frac{d}{dt}\frac{dr}{dt} - r\cos^2 y\left(\frac{dv}{dt}\right)^2 - r\left(\frac{dy}{dt}\right)^2 + \frac{n^2a^2}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2\cos^2 y\frac{dv}{dt}\right) = \frac{d\Omega}{dv},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\cos y\sin y\left(\frac{dv}{dt}\right)^2 = \frac{d\Omega}{dy},$$

where Ω is regarded as a function of r, v, y, or (as this may be expressed) where $\Omega = \Omega(r, v, y)$.

If we neglect the disturbing forces, the planet moves in an ellipse; and taking a to represent the semi-axis major, the mean motion will be n. The mean anomaly, which I call g, will be a function of the form $nt+c$; but as c only enters through g, it will be convenient to use the mean anomaly g (considered as implicitly involving an arbitrary constant c) in the place of an element, and I write

a, the semi-axis major,

e, the eccentricity,

g, the mean anomaly,

θ, the longitude of node,

ϕ, the inclination,

$\mho$, the distance of pericentre from node.

I assume also

f, the true anomaly,

z, the distance of planet from node,

x, the reduced distance from node.

We have then r and f given functions of t and the elements, viz. we may write

$$r = a \text{ elqr } (e, g),$$
$$f = \text{ elta } (e, g),$$

(read elqr. elliptic quotient radius, and elta. elliptic anomaly). These values satisfy $r = \dfrac{a(1-e^2)}{1+e\cos f}$. Moreover z, x, y, are the hypothenuse, base, and perpendicular of a right-angled spherical triangle, the base angle whereof is ϕ; the equations which connect these quantities are therefore

$$\tan x = \tan z\cos\phi,$$
$$\sin y = \sin z\sin\phi,$$
$$\tan y = \sin x\tan\phi,$$
$$\cos z = \cos x\cos y,$$

equivalent, of course, to two equations. The first and second of them give in fact x, y, in terms of z and ϕ.

The value of z is

$$z = \mho + f,$$

so that x and y are given functions, and the longitude v is given in terms of x and θ by the equation

$$v = x + \theta,$$

and consequently the three coordinates r, v, y, are by the system of equations given in terms of t and the elements.

From the equations which connect z, x, y, ϕ, treating all these quantities as variable we deduce

$$\begin{aligned}
\sec^2 x\, dx &= \cos\phi \sec^2 z\, dz - \tan z \sin\phi\, d\phi,\\
\cos y\, dy &= \sin\phi \cos z\, dz + \sin z \cos\phi\, d\phi,\\
\sec^2 y\, dy &= \tan\phi \cos x\, dx + \sin x \sec\phi\, d\phi,\\
\sin z\, dz &= \cos y \sin x\, dx + \cos x \sin y\, dy,
\end{aligned}$$

equivalent of course to two equations; and the system is easily reduced to the more convenient form

$$\begin{aligned}
dx &= \cos\phi \sec^2 y\, dz - \tan z \cos^2 x \sin\phi\, d\phi,\\
dy &= \sin\phi \cos x\, dz + \cos x \tan z \cos\phi\, d\phi,\\
dx &= \cot\phi \sec x \sec^2 y\, dy - \tan z \operatorname{cosec}\phi\, d\phi,\\
dz &= \cos\phi\, dx + \cos x \sin\phi\, d\phi,
\end{aligned}$$

joining to these equations the

$$\begin{aligned}
dz &= d\mho + df,\\
dv &= dx + d\theta,
\end{aligned}$$

and considering at present the mere analytical forms, first if $d\phi = 0$, $d\mho = 0$, we have

$$\begin{aligned}
dx &= \cos\phi \sec^2 y\, dz,\\
dy &= \sin\phi \cos x\, dz,\\
dz &= df,\\
dv &= dx.
\end{aligned}$$

Next, if $dy = 0$, $dv = 0$, we have

$$\begin{aligned}
dx &= -\tan z \operatorname{cosec}\phi\, d\phi,\\
dz &= -\tan z \cot \phi\, d\phi,\\
dz &= \cos\phi\, dx,\\
dx &= -d\theta,\\
dz + \cos\phi\, d\theta &= 0.
\end{aligned}$$

I remark also that the equation, $\tan y = \sin x \tan\phi$, may be written in the form

$$\cos^2\phi \sec^2 y + \sin^2\phi \cos^2 x = 1.$$

The equations

$$r = a \operatorname{elqr}(e, g),$$
$$f = \operatorname{elta}(e, g),$$

treating all the quantities as variable, give

$$dr = \frac{ae \sin f}{\sqrt{1-e^2}}\, dg + \frac{1-e^2}{1+e\cos f}\, da - a\cos f\, de,$$

$$df = \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\, dg + \frac{\sin f\,(2+e\cos f)}{1-e^2}\, de,$$

to which is to be joined

$$dr = \frac{ae(1-e^2)\sin f}{(1+e\cos f)^2}\, df + \frac{1-e^2}{1+e\cos f}\, da + \frac{a(2e-\overline{1+e^2}\cos f)}{(1+e\cos f)^2}\, de,$$

all which formulæ will be useful.

If we treat the elements as constant, then in the foregoing expressions for dr and df, we must attend only to the part involving dg, and must put this equal to ndt; the values first obtained for dx, dy, dz, dv, correspond to this assumption, and we have

$$\frac{dr}{dt} = \frac{nae\sin f}{\sqrt{1-e^2}},$$

$$\frac{df}{dt} = \frac{na^2\sqrt{1-e^2}}{r^2},$$

$$\frac{dz}{dt} = \frac{na^2\sqrt{1-e^2}}{r^2},$$

$$\frac{dx}{dt} = \cos\phi \sec^2 y\, \frac{na^2\sqrt{1-e^2}}{r^2},$$

$$\frac{dv}{dt} = \cos\phi \sec^2 y\, \frac{na^2\sqrt{1-e^2}}{r^2},$$

$$\frac{dy}{dt} = \sin\phi \cos x\, \frac{na^2\sqrt{1-e^2}}{r^2},$$

and we then deduce

$$\frac{d}{dt}\frac{dr}{dt} = \frac{na^2 e\cos f}{r^2},$$

$$\frac{d}{dt}\left(r^2\cos^2 y\,\frac{dv}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) = -\cos^2\phi\sin y\sec^3 y\,\frac{n^2a^4(1-e^2)}{r^4},$$

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = -\frac{2n^2a^3e\cos f}{r^2},$$

values which satisfy the undisturbed equations.

The disturbed equations may be dealt with in the usual manner by the method of the variation of the elements, and attending only to the variations of the elements we have

$$dr = 0,$$
$$dv = 0,$$
$$dy = 0,$$
$$d\frac{dr}{dt} = \frac{d\Omega}{dr}\,dt,$$
$$d\left(r^2 \cos^2 y \frac{dv}{dt}\right) = \frac{d\Omega}{dv}\,dt,$$
$$d\left(r^2 \frac{dy}{dt}\right) = \frac{d\Omega}{dy}\,dt,$$

or, what is the same thing,

$$dr = 0,$$
$$dv = 0,$$
$$dy = 0,$$
$$d\frac{nae \sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr}\,dt,$$
$$d\,na^2\sqrt{1-e^2}\cos\phi = \frac{d\Omega}{dv}\,dt,$$
$$d\,na^2\sqrt{1-e^2}\sin\phi\cos x = \frac{d\Omega}{dy}\,dt,$$

where as before $\Omega = \Omega(r, v, y)$.

In virtue of the relations $dv = 0$, $dy = 0$, we have the above-mentioned equations,

$$dx = -\tan z \operatorname{cosec} \phi\, d\phi,$$
$$dz = -\tan z \cot \phi\, d\phi,$$
$$dz = \cos\phi\, dx,$$
$$dx = -d\theta,$$
$$dr + \cos\phi\, d\theta = 0,$$

we have

$$\begin{aligned} d\sin\phi\cos x &= -\sin\phi\sin x\,dx + \cos x\cos\phi\,d\phi, \\ &= \cos x\cos\phi\sec^2 z\,d\phi, \\ &= \sec x\cos\phi\sec^2 y\,d\phi\,; \end{aligned}$$

and the last two equations for the variations become

$$d\,na^2\sqrt{1-e^2}\cos\phi \qquad - na^2\sqrt{1-e^2}\sin\phi\,d\phi \qquad = \frac{d\Omega}{dv}\,dt,$$

$$d\,na^2\sqrt{1-e^2}\sin\phi\cos x + na^2\sqrt{1-e^2}\sec x\cos\phi\sec^2 y\,d\phi = \frac{d\Omega}{dy}\,dt,$$

and attending to the equations $\cos^2\phi \sec^2 y + \sin^2\phi \cos^2 x = 1$ we deduce at once

$$d\, na^2\sqrt{1-e^2} = \qquad \cos\phi \sec^2 y \frac{d\Omega}{dv}\, dt + \sin\phi \cos x \frac{d\Omega}{dy}\, dt,$$

$$d\phi \qquad = \frac{1}{na^2\sqrt{1-e^2}}\left(-\sin\phi \cos^2 x \frac{d\Omega}{dv}\, dt + \cos\phi \cos x \frac{d\Omega}{dy}\, dt\right).$$

Now the position of the planet may be determined by the quantities r, z, θ, ϕ, or we may consider Ω as a function of the last-mentioned quantities. And if on the right-hand side $\Omega = \Omega(r, v, y)$ as before, the formulæ of transformation are

$$\frac{d\Omega}{dr}\, dr + \frac{d\Omega}{dv}\, dv + \frac{d\Omega}{dy}\, dy = \frac{d\Omega}{dr}\, dr + \frac{d\Omega}{dz}\, dz + \frac{d\Omega}{d\theta}\, d\theta + \frac{d\Omega}{d\phi}\, d\phi$$

where

$$dv = \cos\phi \sec^2 y\, dz - \tan z \cos^2 x \sin\phi\, d\phi + d\theta,$$

$$dy = \sin\phi \cos x\, dz + \tan z \cos x \cos\phi\, d\phi,$$

and we have

$$\frac{d\Omega}{dr} = \frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\theta} = \frac{d\Omega}{dv},$$

$$\frac{d\Omega}{d\phi} = \tan z\left(-\sin\phi \cos^2 x \frac{d\Omega}{dv} + \cos\phi \cos x \frac{d\Omega}{dy}\right),$$

$$\frac{d\Omega}{dz} = \qquad \left(\cos\phi \sec^2 y \frac{d\Omega}{dv} + \sin\phi \cos x \frac{d\Omega}{dy}\right),$$

where on the left-hand side $\Omega = \Omega(r, z, \theta, \phi)$; and these equations give

$$\cot z \frac{d\Omega}{d\phi} = \cot\phi \frac{d\Omega}{dz} - \operatorname{cosec}\phi \frac{d\Omega}{d\theta},$$

an equation which is satisfied by $\Omega = \Omega(r, z, \theta, \phi)$. We have thus

$$dr = 0,$$

$$dv = 0,$$

$$dy = 0,$$

$$d\frac{nae \sin f}{\sqrt{1-e^2}} = \qquad \frac{d\Omega}{dr}\, dt,$$

$$d\, na^2\sqrt{1-e^2} = \qquad \frac{d\Omega}{dr}\, dt,$$

$$d\phi = \frac{\cot z}{na^2\sqrt{1-e^2}} \frac{d\Omega}{d\phi}\, dt,$$

which may be replaced by

$$dr = 0,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$dz = \frac{-\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$d\frac{nae\sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr}dt,$$

$$dna^2\sqrt{1-e^2} = \frac{d\Omega}{dz}dt,$$

$$d\phi = \frac{\cot z}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

where as before $\Omega = \Omega(r, z, \theta, \phi)$.

I remark that in the case of any central force whatever, we have an element h corresponding to $na^2\sqrt{1-e^2}$ in the elliptic theory, and the system for the variations is

$$dr = 0,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{h}\frac{d\Omega}{d\phi}dt,$$

$$dz = \frac{-\cot\phi}{h}\frac{d\Omega}{d\phi}dt,$$

$$d\frac{dr}{dt} = \frac{d\Omega}{dr}dt,$$

$$dh = \frac{d\Omega}{dz}dt,$$

$$d\phi = \frac{\cot z}{h}\frac{d\Omega}{d\phi}dt,$$

where $\Omega = \Omega(r, z, \theta, \phi)$.

Imagine a point in the orbit, which I call the departure-point, the angular distances from this point are termed departures. And I write

þ, the departure of planet,

ϖ, the departure of pericentre,

σ, the departure of node,

so that we have

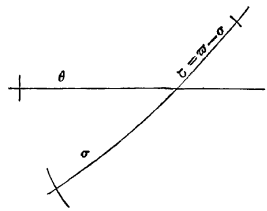
$$\text{þ} = \varpi + f,$$

$$z = \text{þ} - \sigma,$$

$$\text{ƀ} = \varpi - \sigma.$$

I write also

δ, the longitude in orbit of departure-point, or, as it may be termed, the adjustment;



that is

$$\delta = \theta - \sigma.$$

In the undisturbed motion the departure-point is simply a fixed point in the orbit, but when the orbit is variable, the departure-point is taken to be the point of intersection of the orbit with any orthogonal trajectory of the successive positions of the orbit, a definition which is expressed analytically by the equation,

$$d\sigma = \cos\phi\, d\theta.$$

The equation, $z = \flat - \sigma$, gives

$$dz = d\flat - d\sigma = d\flat - \cos\phi\, d\theta,$$

or, what is the same thing,

$$d\flat = dz + \cos\phi\, d\theta.$$

But we have $dz + \cos\phi\, d\theta = 0$, and consequently $d\flat = 0$, an equation which expresses that the increment of departure, in so far as such increment arises from the variation of the elements, is equal to zero. Or, what is the same thing, the total increment of departure is equal to the infinitesimal angle between two consecutive radius vectors of the planet.

I propose to consider the departure-point as a point which is constantly defined as above, viz., when the orbit is variable, the departure-point is the point of intersection of the orbit with any orthogonal trajectory of the successive positions of the orbit; and as a particular case of the definition, when the orbit is fixed, the departure-point is simply a fixed point on the orbit. The orbit here considered is that of the planet and the position of the planet is determined by the departure and radius vector (the latitude being zero), and this is assumed to be the case whenever the departure is spoken of, and it is such departure which is denoted by the letter $\flat$. But we might consider a departure-point (defined as above), upon any other orbit whatever, and use such departure-point as an origin of longitude (for instance, in the lunar theory we might consider a longitude measured along the variable plane of the

sun's orbit from a departure-point, defined as above, in that orbit), and the position of the planet would then be determined by means of the longitude, latitude, and radius vector. The term sidereal longitude is, I think, used in Physical Astronomy rather loosely to denote the longitude in the mean ecliptic from the mean equinox, less the precession; so defined it is not practically different from, and may I think in all cases be replaced by the longitude as measured from a departure-point in the mean ecliptic.

Returning from this digression, the assumed equation, $d\sigma = \cos\phi d\theta$, gives the expression for the variation $d\sigma$ of the departure of the node, and we now have in the place of the former six equations the seven equations

$$dr = 0,$$

$$d\mathfrak{p} = 0,$$

$$d\frac{nae\sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr}dt,$$

$$d\,na^2\sqrt{1-e^2} = \frac{d\Omega}{dz}dt,$$

$$d\phi = \frac{\cot z}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$d\sigma = \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

where as before $\Omega = \Omega\,(r,\ z,\ \theta,\ \phi)$.

But the value of z is $z = \mathfrak{p} - \sigma$, and Ω can be expressed, and that in a single way only, viz. by means of the substitution of $\mathfrak{p} - \sigma$ in the place of z, in the form $\Omega = \Omega\,(r,\ \mathfrak{p},\ \sigma,\ \theta,\ \phi)$, and if on the right-hand side $\Omega = \Omega\,(r,\ z,\ \theta,\ \phi)$ as before, then we have

$$\frac{d\Omega}{dr} = \frac{d\Omega}{dr},$$

$$\frac{d\Omega}{d\mathfrak{p}} = \frac{d\Omega}{dz},$$

$$-\frac{d\Omega}{d\sigma} = \frac{d\Omega}{dz},$$

$$\frac{d\Omega}{d\phi} = \frac{d\Omega}{d\phi},$$

$$\frac{d\Omega}{d\theta} = \frac{d\Omega}{d\theta},$$

where on the left-hand side $\Omega = \Omega\,(r,\ \text{ þ},\ \sigma,\ \theta,\ \phi)$. The function Ω so expressed satisfies, of course, the partial differential equation

$$\frac{d\Omega}{d\text{ þ}} + \frac{d\Omega}{d\sigma} = 0$$

(which conversely implies that þ, σ only enters through the function $\text{ þ} - \sigma$), and it also satisfies the partial differential equation obtained from the before-mentioned equation

$$\cot z \frac{d\Omega}{d\phi} = \cot\phi \frac{d\Omega}{dz} - \operatorname{cosec}\phi \frac{d\Omega}{d\theta}, \quad (\Omega = \Omega\,(r,\ z,\ \theta,\ \phi)),$$

by the introduction of the transformed expressions of the differential coefficients, and which may be written

$$\cot z \frac{d\Omega}{d\phi} = -\cot\phi \frac{d\Omega}{d\sigma} - \operatorname{cosec}\phi \frac{d\Omega}{d\theta},$$

where $\Omega = \Omega\,(r,\ \text{ þ},\ \sigma,\ \theta,\ \phi)$.

Using the last-mentioned equation to transform the value of $d\phi$, the expressions for the variations become

$$dr = 0,$$

$$d\text{ þ} = 0,$$

$$d\,\frac{nae\sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr}\,dt,$$

$$dna^2\sqrt{1-e^2} = \frac{d\Omega}{d\text{ þ}}\,dt,$$

$$d\phi = \frac{-\cot\phi}{na^2\sqrt{1-e^2}}\,\frac{d\Omega}{d\sigma}\,dt - \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\,\frac{d\Omega}{d\theta}\,dt,$$

$$d\sigma = \frac{\cot\phi}{na^2\sqrt{1-e^2}}\,\frac{d\Omega}{d\phi}\,dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\,\frac{d\Omega}{d\phi}\,dt,$$

where $\Omega = \Omega\,(r,\ \text{ þ},\ \sigma,\ \theta,\ \phi)$ as before.

I suppose now that the orbit of the planet, instead of being referred to a fixed plane, is referred to a moveable plane or orbit of reference. It is assumed that the longitudes in the orbit of reference are measured from a departure-point defined as above,—that is, from the point in which the orbit of reference is intersected by any orthogonal trajectory of the successive positions of the orbit of reference. And the

position in regard to the fixed plane, of the orbit of reference, and of the departure-point in this orbit, are determined by θ', σ', ϕ',—that is, we have for the orbit of reference,

θ', the longitude of node,

σ', the departure of node,

ϕ', the inclination.

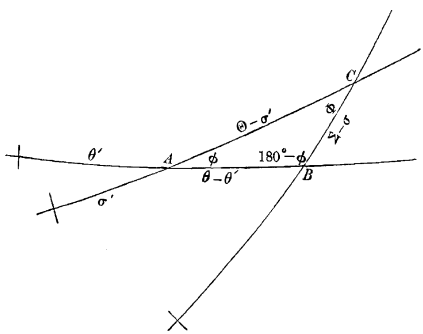
The position of the planet's orbit in relation to the moveable orbit of reference is determined in like manner by Θ, Σ, Φ,—that is, we have for the planet's orbit in relation to the orbit of reference,

Θ, the longitude of node,

Σ, the departure of node,

Φ, the inclination.

Hence if, as before, θ, σ, ϕ, belong to the orbit of the planet considered in relation to the fixed plane, $\Sigma-\sigma$, $\Theta-\sigma'$, $\theta-\theta'$, will be the sides of a spherical triangle, the opposite angles of which are ϕ', $180^\circ-\phi$ and Φ.



Putting for shortness $S=\Sigma-\sigma$, $S'=\Theta-\sigma'$, $G=\theta-\theta'$, so that these symbols denote

S, the distance of node, along planet's orbit, from fixed plane,

S', the distance of node, along orbit of reference, from fixed plane,

G, the distance in fixed plane of the nodes on fixed plane,

the sides of the spherical triangle are S, S', G, and the opposite angles are ϕ', $180^\circ-\phi$, Φ.

Calling the sides A, B, C, and the opposite angles a, b, c, the general formulæ for a spherical triangle give without difficulty,

$$dC = -\cos b\, dA - \cos a\, dB + \frac{\sin A \sin B \sin c}{\sin C}\, dc,$$

$$da = \frac{\sin b}{\sin C}\, dA + \frac{\cos C \sin a}{\sin C}\, dB + \frac{\sin A \cos b}{\sin C}\, dc,$$

$$db = \frac{\cos C \sin b}{\sin C}\, dA + \frac{\sin a}{\sin C}\, dB + \frac{\sin B \cos a}{\sin C}\, dc,$$

and conversely

$$dc = \cos B\, da + \cos A\, db + \frac{\sin a \sin b \sin C}{\sin C}\, dC,$$

$$dA = \frac{\sin B}{\sin c}\, da - \frac{\cos c \sin A}{\sin c}\, db - \frac{\sin a \cos B}{\sin c}\, dC,$$

$$dB = -\frac{\cos c \sin B}{\sin c}\, da + \frac{\sin A}{\sin c}\, db - \frac{\sin b \cos A}{\sin c}\, dC,$$

which, in the present case, become

$$d\Phi = \cos S\, d\phi - \cos S'\, d\phi' + \frac{\sin\phi \sin\phi' \sin G}{\sin\Phi}\, dG,$$

$$dS' = -\frac{\sin S}{\sin\Phi}\, d\phi + \frac{\cos\phi \sin S'}{\sin\Phi}\, d\phi' + \frac{\sin\phi \cos S}{\sin\Phi}\, dG,$$

$$dS = -\frac{\cos\phi \sin S}{\sin\phi}\, d\phi + \frac{\sin S'}{\sin\phi}\, d\phi' + \frac{\sin\phi' \cos S'}{\sin\phi}\, dG,$$

and

$$dG = \cos\phi'\, dS' - \cos\phi\, dS + \frac{\sin S \sin S' \sin\Phi}{\sin G}\, d\Phi,$$

$$d\phi = -\frac{\sin\phi'}{\sin G}\, dS' + \frac{\cos G \sin\phi}{\sin G}\, dS + \frac{\sin S \cos\phi'}{\sin G}\, d\Phi,$$

$$d\phi' = -\frac{\cos G \sin\phi'}{\sin G}\, dS' + \frac{\sin\phi}{\sin G}\, dS + \frac{\sin S \cos\phi}{\sin G}\, d\phi,$$

and we have also

$$dS = d\Sigma - d\sigma,$$
$$dS' = d\Theta - d\sigma',$$
$$dG = d\theta - d\theta'.$$

Hence, observing that $\sin S \sin\phi = \sin S' \sin\phi'$, the preceding equations may be written

$$d\Phi = \cos S\, d\phi - \cos S'\, d\phi' + \sin S \sin\phi\, d\theta - \sin S' \sin\phi'\, d\theta'$$

$$d\Theta = d\sigma' - \operatorname{cosec}\Phi \sin S\, d\phi + \cot\phi \sin S'\, d\phi' + \operatorname{cosec}\Phi \sin\phi' \cos S\,(d\theta - d\theta'),$$

$$d\Sigma = d\sigma - \cot\Phi \sin S\, d\phi + \operatorname{cosec}\phi \sin S'\, d\phi' + \operatorname{cosec}\Phi \sin\phi' \cos S'\,(d\theta - d\theta'),$$

and

$$\begin{aligned}
d\theta &= d\theta' + \cos\phi'(d\Theta - d\sigma') - \cos\phi(d\Sigma - d\sigma) + \sin S \sin\phi\, d\Phi,\\
d\phi &= -\operatorname{cosec} G \sin\phi'(d\Theta - d\sigma') + \cot G \sin\phi(d\Sigma - d\sigma) + \operatorname{cosec} G \sin S' \cos\phi'\, d\Phi,\\
d\phi' &= -\cot G \sin\phi'(d\Theta - d\sigma') + \operatorname{cosec} G \sin\phi(d\Sigma - d\sigma) + \operatorname{cosec} G \sin S \cos\phi\, d\Phi,
\end{aligned}$$

and it is proper to remark, that in obtaining these equations no use has been made of the equations $d\sigma = \cos\phi\, d\theta$, $d\sigma' = \cos\phi'\, d\theta'$.

The term in $d\Theta$ which contains $d\sigma'$, &c. may be written

$$(d\sigma' - \cos\phi'\, d\theta') + \cos\phi'\, d\theta' + \cot\Phi \sin S'\, d\phi' - \operatorname{cosec}\Phi \sin\phi \cos S\, d\theta',$$

which is equal to

$$(d\sigma' - \cos\phi'\, d\theta') + \cot\Phi\,(\sin S'\, d\phi' - \cos S' \sin\phi'\, d\theta');$$

and the term in $d\Sigma$ which contains $d\sigma$, &c. may be written

$$(d\sigma - \cos\phi\, d\theta) + \cos\phi\, d\theta - \cot\Phi \sin S\, d\phi + \operatorname{cosec}\Phi \sin\phi'\, d\theta,$$

which is equal to

$$(d\sigma - \cos\phi\, d\theta) - \cot\Phi\,(\sin S\, d\phi - \cos S \sin\phi\, d\theta);$$

reductions which depend on

$$\begin{aligned}
\cos\phi' - \operatorname{cosec}\Phi \sin\phi \cos S &= -\cot\Phi \sin\phi' \cos S',\\
\cos\phi + \operatorname{cosec}\Phi \sin\phi' \cos S' &= -\cot\Phi \sin\phi \cos S,
\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}
\cos\phi' \sin\Phi - \sin\phi' \cos\Phi \cos S' &= \sin\phi \cos S,\\
\cos\phi \sin\Phi - \sin\phi \cos\Phi \cos S &= -\sin\phi' \cos S',
\end{aligned}$$

which are relations between the sides and angles of the spherical triangle. And we then have

$$\begin{aligned}
d\Phi &= (\cos S\, d\phi + \sin S \sin\phi\, d\theta) - (\cos S'\, d\phi' + \sin S' \sin\phi'\, d\theta'),\\
d\Theta &= (d\sigma' - \cos\phi' d\theta') - \operatorname{cosec}\Phi\,(\sin S\, d\phi - \cos S \sin\phi\, d\theta) + \cot\Phi\,(\sin S'\, d\phi' - \cos S' \sin\phi'\, d\theta'),\\
d\Sigma &= (d\sigma - \cos\phi\, d\theta) - \cot\Phi\,(\sin S\, d\phi - \cos S \sin\phi\, d\theta) + \operatorname{cosec}\Phi\,(\sin S'\, d\phi' - \cos S' \sin\phi'\, d\theta'),
\end{aligned}$$

expressions which may be simplified by omitting the terms $(d\sigma - \cos\phi'\, d\theta')$ and $(d\sigma - \cos\phi\, d\theta)$.

Next substituting for $d\sigma$, $d\phi$, $d\theta$, their values, we obtain,

$$\begin{aligned}
dr &= 0,\\
dp &= 0,\\
d\frac{nae \sin f}{\sqrt{1-e^2}} &= \frac{d\Omega}{dr}\, dt,
\end{aligned}$$

$$d\,na^2\sqrt{1-e^2}=\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$d\Phi=\frac{1}{na^2\sqrt{1-e^2}}\left(\sin S\frac{d\Omega}{d\phi}-\cos S\left(\cot\phi\frac{d\Omega}{d\sigma}+\operatorname{cosec}\phi\frac{d\Omega}{d\theta}\right)\right)dt-\qquad(\sin S'd\phi'+\sin S'\sin\phi'd\theta'),$$

$$d\Sigma=\frac{\cot\Phi}{na^2\sqrt{1-e^2}}\left(\cos S\frac{d\Omega}{d\phi}+\sin S\left(\cot\phi\frac{d\Omega}{d\sigma}+\operatorname{cosec}\phi\frac{d\Omega}{d\theta}\right)\right)dt+\operatorname{cosec}\Phi(\sin S'd\phi'-\cos S'\sin\phi'd\theta'),$$

$$d\Theta=\frac{\operatorname{cosec}\Phi}{na^2\sqrt{1-e^2}}\left(\cos S\frac{d\Omega}{d\phi}+\sin S\left(\cot\phi\frac{d\Omega}{d\sigma}+\operatorname{cosec}\phi\frac{d\Omega}{d\theta}\right)\right)dt+\cot\Phi(\sin S'd\phi'-\cos S'\sin\phi'd\theta'),$$

where $\Omega=\Omega(r,\ \mathfrak{p},\ \sigma,\ \theta,\ \phi)$, as before.

But Ω may be expressed in the form $\Omega=\Omega(r,\ \mathfrak{p},\ \Sigma,\ \Theta,\ \Phi,\ \sigma',\ \theta',\ \phi')$, or disregarding σ', θ', ϕ', in the form $\Omega=\Omega(r,\ \mathfrak{p},\ \Sigma,\ \Theta,\ \Phi)$, and to effect the transformation of the differential coefficients we must write,

$$\begin{aligned}
d\Phi&=\cos S\,d\phi+\sin S\sin\phi\,d\theta,\\
d\Theta&=\qquad\qquad-\operatorname{cosec}\Phi(\sin S\,d\phi-\cos S\sin\phi\,d\theta),\\
d\Sigma&=(d\sigma-\cos\phi\,d\theta)-\cot\Phi(\sin S\,d\phi-\cos S\sin\phi\,d\theta),
\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}
d\phi&=\cos S\,d\phi-\sin S\sin\Phi\,d\Theta,\\
d\theta&=\qquad\qquad\operatorname{cosec}\phi(\sin S\,d\Phi+\cos S\sin\Phi\,d\Theta),\\
d\sigma&=d\Sigma-\cos\Phi\,d\Theta+\cot\phi(\sin S\,d\Phi+\cos S\sin\Phi\,d\Theta),
\end{aligned}$$

and substituting in

$$\begin{aligned}
&\frac{d\Omega}{dr}dr+\frac{d\Omega}{d\mathfrak{p}}d\mathfrak{p}+\frac{d\Omega}{d\sigma}d\sigma+\frac{d\Omega}{d\theta}d\theta+\frac{d\Omega}{d\phi}d\phi\\
&=\frac{d\Omega}{dr}dr+\frac{d\Omega}{d\mathfrak{p}}d\mathfrak{p}+\frac{d\Omega}{d\Sigma}d\Sigma+\frac{d\Omega}{d\Theta}d\Theta+\frac{d\Omega}{d\Phi}d\Phi,
\end{aligned}$$

if on the right-hand side $\Omega=\Omega(r,\ \mathfrak{p},\ \sigma,\ \theta,\ \phi)$ as before, then we have

$$\begin{aligned}
\frac{d\Omega}{dr}&=\frac{d\Omega}{dr},\\
\frac{d\Omega}{d\mathfrak{p}}&=\frac{d\Omega}{d\mathfrak{p}},\\
\frac{d\Omega}{d\Sigma}&=\frac{d\Omega}{d\sigma},\\
\frac{d\Omega}{d\Theta}&=(-\cos\Phi+\cot\phi\cos S\sin\Phi)\frac{d\Omega}{d\sigma}+\operatorname{cosec}\phi\cos S\sin\Phi\frac{d\Omega}{d\theta}-\sin S\sin\Phi\frac{d\Omega}{d\phi},\\
\frac{d\Omega}{d\Sigma}&=\qquad\qquad\cot\phi\sin S\frac{d\Omega}{d\sigma}+\qquad\operatorname{cosec}\phi\sin S\frac{d\Omega}{d\theta}+\qquad\cos S\frac{d\Omega}{d\phi},
\end{aligned}$$

where on the left-hand side $\Omega=\Omega(r,\ \mathfrak{p},\ \Sigma,\ \Theta,\ \Phi)$.

The last three equations give

$$\cos S \frac{d\Omega}{d\phi} + \sin S \left(\cot \phi \frac{d\Omega}{d\sigma} + \operatorname{cosec} \phi \frac{d\Omega}{d\theta}\right) = \frac{d\Omega}{d\Phi},$$

$$-\sin S \frac{d\Omega}{d\phi} + \cos S \left(\cot \phi \frac{d\Omega}{d\sigma} + \operatorname{cosec} \phi \frac{d\Omega}{d\theta}\right) = \cot \Phi \frac{d\Omega}{d\Sigma} + \operatorname{cosec} \Phi \frac{d\Omega}{d\Theta},$$

and the formulæ for the variations become

$$dr = 0,$$

$$d\mathfrak{p} = 0,$$

$$d \frac{nae \sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr} dt,$$

$$dna^2 \sqrt{1-e^2} = \frac{d\Omega}{d\mathfrak{p}} dt,$$

$$d\Phi = \frac{-\cot \Phi}{na^2 \sqrt{1-e^2}} \frac{d\Omega}{d\Sigma} dt - \frac{\operatorname{cosec} \Phi}{na^2 \sqrt{1-e^2}} \frac{d\Omega}{d\Theta} dt - (\cos S' d\phi' + \sin S' \sin \phi' d\theta'),$$

$$d\Sigma = \frac{\cot \Phi}{na^2 \sqrt{1-e^2}} \frac{d\Omega}{d\Phi} dt + \operatorname{cosec} \Phi (\sin S' d\phi' - \cos S' \sin \phi' d\theta'),$$

$$d\Theta = \frac{\operatorname{cosec} \Phi}{na^2 \sqrt{1-e^2}} \frac{d\Omega}{d\Phi} dt + \cot \Phi (\sin S' d\phi' - \cos S' \sin \phi' d\theta'),$$

where $\Omega = \Omega\ (r, \mathfrak{p}, \Sigma, \Theta, \Phi)$. It will be recollected that the value of S' is $= \Theta - \sigma'$. It may be noticed that

$$d\Sigma - \cos \Phi\, d\Theta = \sin \Phi (\sin S'\, d\phi' - \cos S' \sin \phi'\, d\theta').$$

The system just obtained is, except as regards the terms involving $d\phi'$ and $d\theta'$, precisely similar in its form to that in which the planet is referred to a fixed plane, or where $\Omega = \Omega\ (r, \mathfrak{p}, \sigma, \theta, \phi)$, and this is of course as it should be.

We have now

$$\mathfrak{p} = \varpi + f,$$

$$r = a \operatorname{elqr} (e, g),$$

$$f = \operatorname{elta} (e, g),$$

so that the position of the planet is determined by means of the elements $a, e, g, \varpi, \Sigma, \Theta, \Phi$. To find the variations of these elements, substituting for r its value in terms of f, the first, third, and fourth equations are

$$d \frac{a(1-e^2)}{1+e \cos f} = 0,$$

$$d \frac{nae \sin f}{\sqrt{1-e^2}} = \frac{d\Omega}{dr} dt,$$

$$d\, na^2 \sqrt{1-e^2} = \frac{d\Omega}{d\mathfrak{p}} dt,$$

which give

$$e\sin f\,df - \frac{2e+(1+e^2)\cos f}{1-e^2}\,de + \frac{1+e\cos f}{a}\,da = 0,$$

$$e\cos f\,df + \sin f\,de = \frac{\sqrt{1-e^2}}{na}\frac{d\Omega}{dr}\,dt + \frac{na^2\sqrt{1-e^2}}{e\sin f}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$-\,de + \frac{1-e^2}{2ae}\,da = \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

and we thence obtain

$$da = \frac{2e\sin f}{n\sqrt{1-e^2}}\frac{d\Omega}{dr}\,dt + \frac{2(1+e\cos f)^2}{na(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$de = \frac{\sqrt{1-e^2}\sin f}{na}\frac{d\Omega}{dr}\,dt + \frac{e+2\cos f+e\cos^2 f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$df = \frac{n\sqrt{1-e^2}\cos f}{nae}\frac{d\Omega}{dr}\,dt - \frac{(2+e\cos f)\sin f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

the last of which equations, combined with

$$df = \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\,dg + \frac{(2+e\cos f)\sin f}{1-e^2}\,de,$$

gives

$$dg = \frac{(1-e^2)(-2e+\cos f+e\cos^2 f)}{nae(1+e\cos f)}\frac{d\Omega}{dr}\,dt - \frac{(2+e\cos f)\sin f}{na^2e}\frac{d\Omega}{d\mathfrak{p}}\,dt.$$

The fourth equation of the formulæ for the variations, viz., $d\mathfrak{p}=0$, gives $0 = d\varpi + df$, and therefore $d\varpi = -df$, that is,

$$d\varpi = -\frac{\sqrt{1-e^2}\cos f}{nae}\frac{d\Omega}{dr}\,dt + \frac{(2+e\cos f)\sin f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

and the complete system becomes therefore

$$da = \frac{2e\sin f}{n\sqrt{1-e^2}}\frac{d\Omega}{dr}\,dt + \frac{2(1+e\cos f)^2}{na(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$de = \frac{\sqrt{1-e^2}\sin f}{na}\frac{d\Omega}{dr}\,dt + \frac{e+2\cos f+e\cos^2 f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$dg = \frac{(1-e^2)(-2e+\cos f+e\cos^2 f)}{nae(1+e\cos f)}\frac{d\Omega}{dr}\,dt - \frac{(2+e\cos f)\sin f}{na^2e}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$d\varpi = -\frac{\sqrt{1-e^2}\sin f}{nae}\frac{d\Omega}{dr}\,dt + \frac{(2+e\cos f)\sin f}{na^2e}\frac{d\Omega}{d\mathfrak{p}}\,dt,$$

$$d\Phi = \frac{-\cot\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Sigma}dt - \frac{\operatorname{cosec}\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Theta}dt - \qquad (\cos S'\,d\phi' + \sin S'\sin\phi'\,d\theta'),$$

$$d\Sigma = \frac{\cot\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Phi}dt \qquad + \operatorname{cosec}\Phi\,(\sin S'\,d\phi' - \cos S'\sin\phi'\,d\theta'),$$

$$d\Theta = \frac{\operatorname{cosec}\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Phi}dt \qquad + \cot\Phi\,(\sin S'\,d\phi' - \cos S'\sin\phi'\,d\theta'),$$

where, as before, $\Omega = \Omega\,(r, \mathfrak{p}, \Sigma, \Theta, \Phi)$. This is the first form of the expressions for the variations of the elements.

But we may in the disturbing function Ω replace r, $\mathfrak{p}$, by their values in terms of a, e, g, ϖ, and if on the right-hand side Ω has the last preceding value, we have

$$\frac{d\Omega}{da}da + \frac{d\Omega}{de}de + \frac{d\Omega}{dg}dg + \frac{d\Omega}{d\varpi}d\varpi + \frac{d\Omega}{d\Sigma}d\Sigma + \frac{d\Omega}{d\Theta}d\Theta + \frac{d\Omega}{d\Phi}d\Phi$$

$$= \frac{d\Omega}{dr}dr + \frac{d\Omega}{d\mathfrak{p}}d\mathfrak{p} + \frac{d\Omega}{d\Sigma}d\Sigma + \frac{d\Omega}{d\Theta}d\Theta + \frac{d\Omega}{d\Phi}d\Phi,$$

where on the left-hand side $\Omega = \Omega\,(a, e, g, \varpi, \Sigma, \Theta, \Phi)$; and the expressions for the differentials dr and $d\mathfrak{p}$, are

$$dr = \frac{1-e^2}{1+e\cos f}da - a\cos f\,de + \qquad \frac{ae\sin f}{\sqrt{1-e^2}}dg,$$

$$d\mathfrak{p} = \qquad \frac{(2+e\cos f)\sin f}{1-e^2}de + \frac{(1+e\cos f)^2}{(1-e)^{\frac{3}{2}}}dg + d\varpi,$$

and we have therefore

$$\frac{d\Omega}{da} = \frac{1-e^2}{1+e\cos f}\frac{d\Omega}{dr},$$

$$\frac{d\Omega}{de} = -a\cos f\frac{d\Omega}{dr} + \frac{(2+e\cos f)\sin f}{1-e^2}\frac{d\Omega}{d\mathfrak{p}},$$

$$\frac{d\Omega}{dg} = \frac{ae\sin f}{\sqrt{1-e^2}}\frac{d\Omega}{dr} + \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{d\mathfrak{p}},$$

$$\frac{d\Omega}{d\varpi} = \frac{d\Omega}{d\mathfrak{p}},$$

$$\frac{d\Omega}{d\Phi} = \frac{d\Omega}{d\Phi},$$

$$\frac{d\Omega}{d\Sigma} = \frac{d\Omega}{d\Sigma},$$

$$\frac{d\Omega}{d\Theta} = \frac{d\Omega}{d\Theta},$$

where on the left-hand side $\Omega = \Omega\,(a, e, g, \varpi, \Sigma, \Theta, \Phi)$,

and we thence obtain for the variations the new system of formulæ,

$$
\begin{aligned}
da &= \qquad\qquad + \frac{2}{na}\frac{d\Omega}{dg}dt,\\
de &= \qquad\qquad + \frac{1-e^2}{na^2e}\frac{d\Omega}{dg}dt - \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{d\varpi}dt,\\
dg &= -\frac{2}{na}\frac{d\Omega}{da}dt - \frac{1-e^2}{na^2e}\frac{d\Omega}{de}dt,\\
d\varpi &= \qquad + \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de}dt,\\
d\Phi &= \qquad\qquad \frac{-\cot\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Sigma}dt - \frac{\operatorname{cosec}\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Theta}dt\\
&\qquad\qquad - (\cos S'd\phi' + \sin S'\sin\phi' d\theta'),\\
d\Sigma &= \qquad + \frac{\cot\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Phi}dt\\
&\qquad\qquad + \operatorname{cosec}\Phi(\sin S'd\phi' - \cos S'\sin\phi' d\theta'),\\
d\Theta &= \qquad + \frac{\operatorname{cosec}\Phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\Phi}dt\\
&\qquad\qquad + \cot\Phi(\sin S'd\phi' - \cos S'\sin\phi' d\theta'),
\end{aligned}
$$

where, as before, $\Omega = \Omega(a, e, g, \varpi, \Sigma, \Theta, \Phi)$. This is the second form of the expressions for the variations of the elements. It is hardly necessary to remark, that if in either system of formulæ we omit the terms involving $d\theta'$ and $d\phi'$, and in the place of Σ, Θ, Φ, write σ, θ, ϕ, we have the formulæ for the variation of the elements when the orbit of the planet is referred to a fixed plane, and the disturbing function is given under the form $\Omega = \Omega(r, \mathfrak{p}, \sigma, \theta, \phi)$, or $\Omega = \Omega(a, e, g, \varpi, \sigma, \theta, \phi)$.

The demonstration of the two preceding systems forms, as before remarked, the object of the present Memoir. But it is proper to give also the systems for the variations of the elements in the form in which they would have been obtained, if the notion of the departure had not been introduced into the investigation. To do this I revert to a preceding system of equations, which may be written

$$
\begin{aligned}
dr &= 0,\\
d\theta &= \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,\\
dz &= \frac{-\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,\\
d\frac{nae\sin f}{\sqrt{1-e^2}} &= \frac{d\Omega}{dr}dt,
\end{aligned}
$$

$$d\,na^2\sqrt{1-e^2} = \frac{d\Omega}{dz}\,dt,$$

$$d\phi = \frac{-\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\theta}\,dt + \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{dz}\,dt\left(=\frac{\cot z}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}\,dt\right),$$

where $\Omega = \Omega\,(r,\ z,\ \theta,\ \phi)$.

Substituting in these equations for r, z, the values $\dfrac{a\,(1-e^2)}{1+e\cos f}$ and $\varpi + f$, we obtain

$$da = \frac{2e\sin f}{n\sqrt{1-e^2}}\frac{d\Omega}{dr}\,dt + \frac{2\,(1+e\cos f)^2}{na\,(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{dz}\,dt,$$

$$de = \frac{\sqrt{1-e^2}\sin f}{na}\frac{d\Omega}{dr}\,dt + \frac{e+2\cos f+e\cos^2 f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{dz}\,dt,$$

$$dg = \frac{(1-e^2)(-2e+\cos f+e\cos^2 f)}{nae\,(1+e\cos f)}\frac{d\Omega}{dr}\,dt - \frac{(2+e\cos f)\sin f}{na^2e}\frac{d\Omega}{dz}\,dt,$$

$$d\varpi = -\frac{\sqrt{1-e^2}\cos f}{nae}\frac{d\Omega}{dr}\,dt + \frac{(2+e\cos f)\sin f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{dz}\,dt - \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}\,dt,$$

$$d\phi = \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{dz}\,dt - \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\theta}\,dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}\,dt,$$

where $\Omega = \Omega\ (r,\ z,\ \theta,\ \phi)$, as before; and which is the first system for the variations of the six elements, a, e, g, ϖ, θ, ϕ.

But if in the disturbing function we replace r, z by their values, then if on the left-hand side $\Omega = \Omega\ (r,\ z,\ \theta,\ \phi)$, as before, we find

$$\frac{1-e^2}{1+e\cos f}\frac{d\Omega}{dr} = \frac{d\Omega}{dr},$$

$$-a\cos f\frac{d\Omega}{dr} + \frac{(2+e\cos f)\sin f}{1-e^2}\frac{d\Omega}{dz} = \frac{d\Omega}{de},$$

$$\frac{ae\sin f}{\sqrt{1-e^2}}\frac{d\Omega}{dr} + \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{dz} = \frac{d\Omega}{dg},$$

$$\frac{d\Omega}{d\phi} = \frac{d\Omega}{d\phi},$$

$$\frac{d\Omega}{dz} = \frac{d\Omega}{d\varpi},$$

$$\frac{d\Omega}{d\theta} = \frac{d\Omega}{d\theta},$$

where on the right-hand side $\Omega = \Omega\ (a,\ e,\ g,\ \varpi,\ \theta,\ \phi)$.

The formulæ for the variations thus become

$$
\begin{aligned}
da &= \frac{2}{na}\frac{d\Omega}{dg}dt, \\
de &= \frac{1-e^2}{na^2e}\frac{d\Omega}{dg}dt - \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{d\mho}dt, \\
dg &= -\frac{2}{na}\frac{d\Omega}{da}dt - \frac{1-e^2}{na^2e}\frac{d\Omega}{de}dt, \\
d\mho &= \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de}dt - \frac{\cot\phi}{na\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt, \\
d\phi &= \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mho}dt - \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\theta}dt, \\
d\theta &= +\frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,
\end{aligned}
$$

where, as before, $\Omega = \Omega\ (a,\ e,\ g,\ \mho,\ \theta,\ \phi)$. This is the second system of formulæ for the variations of the six elements $a,\ e,\ g,\ \mho,\ \theta,\ \phi$.

The last-mentioned system may be easily deduced from Jacobi's canonical system of formulæ, viz. putting

$\mathfrak{A}$, the constant of vis viva,

$\mathfrak{B}$, the constant of areas,

$\mathfrak{C}$, the constant of the reduced area,

$\mathfrak{F}$, the constant attached to the time,

$\mathfrak{G}$, the angular distance of pericentre from node,

$\mathfrak{H}$, the longitude of node;

then the canonical system is

$$
\begin{aligned}
d\mathfrak{A} &= \frac{d\Omega}{d\mathfrak{F}}dt, \\
d\mathfrak{B} &= \frac{d\Omega}{d\mathfrak{G}}dt, \\
d\mathfrak{C} &= \frac{d\Omega}{d\mathfrak{H}}dt, \\
d\mathfrak{F} &= -\frac{d\Omega}{d\mathfrak{A}}dt, \\
d\mathfrak{G} &= -\frac{d\Omega}{d\mathfrak{B}}dt, \\
d\mathfrak{H} &= -\frac{d\Omega}{d\mathfrak{C}}dt,
\end{aligned}
$$

and the expressions for $\mathfrak{A}$, &c. in terms of the elements a, e, g, $\mho$, θ, ϕ, are

$$\begin{aligned}
\mathfrak{A} &= -\tfrac{1}{2}n^2a^2,\\
\mathfrak{B} &= na^2\sqrt{1-e^2},\\
\mathfrak{C} &= na^2\sqrt{1-e^2}\cos\phi,\\
\mathfrak{F} &= \frac{1}{n}g-t,\\
\mathfrak{G} &= \mho,\\
\mathfrak{H} &= \theta,
\end{aligned}$$

and the transformation can be effected without the slightest difficulty.

I shall conclude with the demonstration of a formula which occurs implicitly in Hansen's Lunar Theory, and which may probably be useful for other purposes.

Write

ρ, the radius vector, τ for t,

ψ, the true anomaly, τ for t,

that is, let ρ, ψ, be what r, f, become when the time t, in so far as it enters explicitly in g, and not through the variable elements, is replaced by an arbitrary quantity, τ.

And suppose, in like manner,

γ, the mean anomaly, τ for t,

λ, the departure, τ for t,

and let $l\rho$ denote the logarithm of ρ; then we have, attending only to the variation of t, in so far as it enters through the variable elements,

$$\begin{aligned}
df &= \frac{(2+e\cos f)\sin f}{1-e^2}\,de + \frac{(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}}\,dg,\\
d\psi &= \frac{(2+e\cos\psi)\sin\psi}{1-e^2}\,de + \frac{(1+e\cos\psi)^2}{(1-e^2)^{\frac{3}{2}}}\,d\gamma,\\
dl\rho &= \frac{1}{a}\,da - \frac{(1+e\cos\psi)\cos\psi}{1-e^2}\,de + \frac{(1+e\cos\psi)\,e\sin\psi}{(1-e^2)^{\frac{3}{2}}}\,d\gamma.
\end{aligned}$$

We hence deduce,

$$dl\rho + \frac{\rho e\sin\psi}{a(1-e^2)}(df-d\psi)$$

$$= \frac{1}{a}\,da + \frac{e\sin\psi\sin f(2+e\cos f)-\cos\psi-2e-e^2\cos\psi}{(1-e^2)(1+e\cos\psi)}\,de + \frac{e\sin\psi(1+e\cos f)^2}{(1-e^2)^{\frac{3}{2}}(1+e\cos\psi)}\,dg,$$

the coefficient of $d\gamma$ being zero. And substituting for da, de, dg their values, viz.

$$da = \frac{2e\sin f}{n\sqrt{1-e^2}}\frac{d\Omega}{dr}dt + \frac{2(1+e\cos f)^2}{na(1-e^2)^{\frac{3}{2}}}\frac{d\Omega}{d\mathfrak{p}}dt,$$

$$de = \frac{\sqrt{1-e^2}\sin f}{na}\frac{d\Omega}{dr}dt + \frac{e+2\cos f+e\cos^2 f}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\mathfrak{p}}dt,$$

$$dg = \frac{(1-e^2)(-2e+\cos f+e\cos^2 f)}{nae(1+e\cos f)}\frac{d\Omega}{dr}dt + \frac{(2+e\cos f)\sin f}{na^2e}\frac{d\Omega}{d\mathfrak{p}}dt,$$

the equation becomes

$$dl\rho + \frac{\rho e\sin\psi}{a(1-e^2)}(df-d\psi)$$

$$= \frac{-\sqrt{1-e^2}}{na(1+e\cos\psi)}\sin(f-\psi)\frac{d\Omega}{dr}dt$$

$$+ \frac{1}{na^2\sqrt{1-e^2}(1+e\cos\psi)}\left\{2+e\cos\psi-\cos(f-\psi)(2+e\cos f)\right\}\frac{d\Omega}{d\mathfrak{p}}dt.$$

But we have $\mathfrak{p}=\varpi+f$, $\lambda=\varpi+\psi$, and, consequently, $f-\psi=\mathfrak{p}-\lambda$, and the equation becomes

$$dl\rho + \frac{\rho e\sin\psi}{a(1-e^2)}(d\mathfrak{p}-d\lambda)$$

$$= \frac{-\sqrt{1-e^2}}{na(1+e\cos\psi)}\sin(\mathfrak{p}-\lambda)\frac{d\Omega}{dr}dt$$

$$+ \frac{1}{na^2\sqrt{1-e^2}(1+e\cos\psi)}\left\{2+e\cos\psi-\cos(\mathfrak{p}-\lambda)(2+e\cos f)\right\}\frac{d\Omega}{d\mathfrak{p}}dt,$$

which is the equation referred to; the expression on the right-hand side, omitting the factor dt, is, in fact, the portion not involving the arbitrary functions Π, Γ, of Hansen's function R (*Fund.* p. 43), viz. it is in Hansen's notation,

$$-\left\{\frac{\rho}{r}\cos(v,-\lambda)-1+\frac{\rho}{a(1-e^2)}[\cos(v,-\lambda)-1]\right\}\frac{an}{\sqrt{1-e^2}}\frac{d\Omega}{dv}$$

$$-\frac{\rho}{r}\sin(v,-\lambda)\frac{an}{\sqrt{1-e^2}}r\frac{d\Omega}{dr},$$

where a, n, e, r, ρ, v, λ, Ω (Hansen), correspond to a, n, e, r, ρ, $\mathfrak{p}$, λ, $n^{-2}a^{-3}\Omega$, of the present Memoir.

2, *Stone Buildings, W. C.*, 3 *March*, 1858.

213.

ON THE DEVELOPMENT OF THE DISTURBING FUNCTION IN THE LUNAR THEORY.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXVII., (1859), pp. 69—95. Read November 12, 1858.]

THE development of the disturbing function for the lunar theory is effected in a very elegant manner in Hansen's *Fundamenta Nova*, and it requires only a single easy step to exhibit the result in a perfectly explicit form, and to compare it with those of other geometers. To do this is the immediate object of the present memoir, and the mode of development is a mere reproduction of that made use of by Hansen. But the memoir is written with a view to the development of and application to the lunar theory, of the theory contained in my "Memoir on the Problem of Disturbed Elliptic Motion," *ante* pp. 1—29, [212], and the notation adopted (differing from Hansen's very slightly) is consequently that of the memoir just referred to.

Taking, as usual, Ω to denote the disturbing function with the sign employed by Lagrange ($\Omega = -R$, if R be the disturbing function of the *Mécanique Céleste*), then

$$\Omega = m' \left\{ \frac{1}{(r^2 + r'^2 - 2rr' \cos H)^{\frac{1}{2}}} - \frac{r}{r'^2} \cos H \right\},$$

where we have

m', the mass of the sun,
r , the radius vector of the moon,
r' , the radius vector of the sun,
H, the angular distance of the sun and moon,

the earth being, of course, taken as the centre of motion; (Hansen's Ω is the above value divided by $M+m$, where M and m are the masses of the earth and moon respectively; that is, the disturbing function here represented by Ω is Hansen's Ω multiplied into $M+m$).

Write also

f , the true anomaly of the moon,

f' , the true anomaly of the sun,

$\mho$, the distance of moon's pericentre from ascending node of moon's orbit,

$\mho'$, the distance of sun's pericentre from same node,

Φ , the inclination of moon's orbit to that of the sun,

where the orbits referred to are the true or instantaneous orbits. Then the angular distance H is the third side of a spherical triangle, the other two sides whereof are $f+\mho$, $f'+\mho'$, the included angle between them being Φ, that is, we have

$$\cos H = \cos(f+\mho)\cos(f'+\mho') + \cos\Phi \sin(f+\mho)\sin(f'+\mho')$$

and in this equation, if we write

g , the mean anomaly of the moon,

a , the semi-axis major of the moon's orbit,

e , the eccentricity;

and in like manner,

g', the mean anomaly of the sun,

a', the semi-axis major of the sun's orbit,

e', the eccentricity;

then r, f, and r', f', are respectively given functions of a, e, g, and a', e', g', viz., we have

$$\begin{aligned} r &= a \text{ elqr } (e, g), \\ f &= \text{ elta } (e, g), \\ r' &= a' \text{ elqr } (e', g'), \\ f' &= \text{ elta } (e', g'), \end{aligned}$$

and introducing, instead of the inclination, the quantity

$$\eta\ (= \sin \tfrac{1}{2}\Phi), \text{ the sine of the semi-inclination};$$

the disturbing function Ω becomes a function of a, e, g, $\mho$, a', e', g', $\mho'$, η, and the required development is a development in multiple cosines of g, g', $\mho$, $\mho'$, the coefficients being of course functions of the remaining quantities a, e, a', e', η. The single symbols $\mho$, $\mho'$ (which denote the distances of the pericentres of the lunar and solar orbits from the mutual node) will be retained throughout the memoir; but if we write

ϖ, the departure of moon's pericentre,

Σ, the departure of moon's ascending node;

these departures being measured on the moon's orbit; and, in like manner, measured on the sun's orbit from a *departure point* on this orbit, but called for distinction "longitudes," instead of departures,

ϖ', the longitude of the sun's pericentre,

Θ, the longitude of the moon's ascending node;

then we have

$$\mho = \varpi - \Sigma,$$
$$\mho' = \varpi' - \Theta;$$

and the disturbing function Ω, so far as it depends on the position of the moon, is a function of the seven elements a, e, g, ϖ, η, Σ, Θ, and it contains also the quantities a', e', g', ϖ', which relate to the sun.

Proceeding now to develope Ω in ascending powers of $\frac{r}{r'}$, we have

$$\Omega = m'\left\{\frac{1}{r'} + \frac{r^2}{r'^3}\left[\tfrac{3}{2}\cos^2 H - \tfrac{1}{2}\right] + \frac{r^3}{r'^4}\left[\tfrac{5}{2}\cos^3 H - \tfrac{3}{2}\cos H\right] + \frac{r^4}{r'^5}\left[\tfrac{35}{8}\cos^4 H - \tfrac{15}{4}\cos^2 H + \tfrac{3}{8}\right] + \&c.\right\}$$

where, however, the last term is neglected in the sequel, and since we are only concerned with the differential coefficients of Ω in regard to the lunar elements, the first term $m'\frac{1}{r'}$ which depends only on the solar elements may also be neglected; we have

$$\begin{aligned}\cos H = &\ \cos^2\tfrac{1}{2}\Phi && \cos(f - f' + \mho - \mho')\\ &+ \sin^2\tfrac{1}{2}\Phi && \cos(f + f' + \mho + \mho'),\end{aligned}$$

and thence

$$\begin{aligned}\cos^2 H = &\ \cos^4\tfrac{1}{2}\Phi && \cos^2(f - f' + \mho - \mho')\\ &+ 2\cos^2\tfrac{1}{2}\Phi\sin^2\tfrac{1}{2}\Phi && \cos(f - f' + \mho - \mho')\ \cos(f + f' + \mho + \mho')\\ &+ \sin^4\tfrac{1}{2}\Phi && \cos^2(f + f' + \mho + \mho'),\end{aligned}$$

$$\begin{aligned}\cos^3 H = &\ \cos^6\tfrac{1}{2}\Phi && \cos^3(f - f' + \mho - \mho')\\ &+ 3\cos^4\tfrac{1}{2}\Phi\sin^2\tfrac{1}{2}\Phi && \cos^2(f - f' + \mho - \mho')\ \cos(f + f' + \mho + \mho')\\ &+ 3\cos^2\tfrac{1}{2}\Phi\sin^4\tfrac{1}{2}\Phi && \cos(f - f' + \mho - \mho')\ \cos^2(f + f' + \mho + \mho')\\ &+ \sin^6\tfrac{1}{2}\Phi && \cos^3(f + f' + \mho + \mho'),\end{aligned}$$

and converting the powers of the cosines of $f - f' + \mho - \mho'$, $f + f' + \mho + \mho'$ into multiple cosines, and expressing the coefficients in terms of $\eta\ (= \sin\frac{1}{2}\Phi)$ and neglecting η^6, we have

$$\cos H = \begin{array}{ll} -\eta^2 & \cos\ f - f' + \mho - \mho' \\ +\eta^2 & \cos\ f + f' + \mho + \mho', \end{array}$$

$$\cos^2 H = \begin{array}{lll} \frac{1}{2} - \eta^2 + \eta^4 & & \\ +\frac{1}{2} - \eta^2 + \frac{1}{2}\eta^4 & \cos & 2f - 2f' + 2\mho - 2\mho' \\ + \eta^2 - \eta^4 & \cos & 2f + 2\mho \\ + \eta^2 - \eta^4 & \cos & 2f' + 2\mho' \\ + \frac{1}{2}\eta^4 & \cos & 2f + 2f' + 2\mho + 2\mho', \end{array}$$

$$\cos^3 H = \begin{array}{lll} \frac{3}{4} - \frac{9}{4}\eta^2 + \frac{15}{4}\eta^4 & \cos & f - f' + \mho - \mho' \\ +\frac{1}{4} - \frac{3}{4}\eta^2 + \frac{3}{4}\eta^4 & \cos & 3f - 3f' + 3\mho - 3\mho' \\ + \frac{3}{2}\eta^2 - 3\eta^4 & \cos & f + f' + \mho + \mho' \\ + \frac{3}{4}\eta^2 - \frac{3}{2}\eta^4 & \cos & 3f - f' + 3\mho - \mho' \\ + \frac{3}{4}\eta^2 - \frac{3}{2}\eta^4 & \cos & f - 3f' + \mho - 3\mho' \\ + \frac{3}{4}\eta^4 & \cos & 3f + f' + 3\mho + \mho' \\ + \frac{3}{4}\eta^4 & \cos & f + 3f' + \mho + 3\mho'. \end{array}$$

Hence

$\Omega = m' \frac{r^2}{r'^3}$ multiplied into

$$\left\{ \begin{array}{lll} \frac{1}{4} - \frac{3}{2}\eta^2 + \frac{3}{2}\eta^4 & & \\ +\frac{3}{4} - \frac{3}{2}\eta^2 + \frac{3}{4}\eta^4 & \cos & 2f - 2f' + 2\mho - 2\mho' \\ + \frac{3}{2}\eta^2 - \frac{3}{2}\eta^4 & \cos & 2f + 2\mho \\ + \frac{3}{2}\eta^2 - \frac{3}{2}\eta^4 & \cos & 2f' + 2\mho' \\ + \frac{3}{4}\eta^4 & \cos & 2f + 2f' + 2\mho + 2\mho \end{array} \right.$$

$+ m' \frac{r^3}{r'^4}$ multiplied into

$$\left\{ \begin{array}{lll} \frac{3}{8} - \frac{33}{8}\eta^2 + \frac{75}{8}\eta^4 & \cos & f - f' + \mho - \mho' \\ +\frac{5}{8} - \frac{15}{8}\eta^2 + \frac{15}{8}\eta^4 & \cos & 3f - 3f' + 3\mho - 3\mho' \\ + \frac{9}{4}\eta^2 - \frac{15}{2}\eta^4 & \cos & f + f' + \mho + \mho' \\ + \frac{15}{8}\eta^2 - \frac{15}{4}\eta^4 & \cos & 3f - f' + 3\mho - \mho' \\ + \frac{15}{8}\eta^2 - \frac{15}{4}\eta^4 & \cos & f - 3f' + \mho - 3\mho' \\ + \frac{15}{8}\eta^4 & \cos & 3f + f' + 3\mho + \mho' \\ + \frac{15}{8}\eta^4 & \cos & f + 3f' + \mho + 3\mho', \end{array} \right.$$

but the last two terms of the part multiplied by $m'\frac{r^3}{r'^4}$ (which are besides of the fourth order in η) are neglected in the sequel.

Now i, i' denoting integer numbers, and writing down only the general terms (the summatory sign $\Sigma_{-\infty}^{\infty}$ being in each case understood), we may put

$$\frac{r^2}{a^2}\cos 2f = Q_c^i \cos ig, \quad \frac{r^2}{a^2}\sin 2f = Q_s^i \sin ig,$$

$$\frac{r^2}{a^2} = P^i \cos ig,$$

$$\frac{a'^3}{r'^3}\cos 2f' = G_c^{i'} \cos i'g', \quad \frac{a'^3}{r'^3}\sin 2f' = G_s^{i'} \sin i'g',$$

$$\frac{a'^3}{r'^3} = K^{i'} \cos i'g',$$

$$\frac{r^3}{a^3}\cos f = A_c^i \cos ig, \quad \frac{r^3}{a^3}\sin f = A_s^i \sin ig,$$

$$\frac{r^3}{a^3}\cos 3f = B_c^i \cos ig, \quad \frac{r^3}{a^3}\sin 3f = B_s^i \sin ig,$$

$$\frac{a'^4}{r'^4}\cos f' = C_c^{i'} \cos i'g', \quad \frac{a'^4}{r'^4}\sin f' = C_s^{i'} \sin i'g',$$

$$\frac{a'^4}{r'^4}\cos 3f' = D_c^{i'} \cos i'g', \quad \frac{a'^4}{r'^4}\sin 3f' = D_s^{i'} \sin i'g',$$

where

$$Q_c^i = Q_c^{-i}, \quad G_c^{i'} = G_c^{-i'}, \quad A_c^i = A_c^{-i}, \text{ \&c.}$$
$$Q_s^i = -Q_s^{-i}, \quad G_s^{i'} = -G_s^{-i'}, \quad A_s^i = -A_s^{-i}, \text{ \&c.}$$
$$P^i = P^{-i}, \quad K^{i'} = K^{-i'}.$$

Then by a known rule for the multiplication of doubly infinite sine or cosine series, and after an easy transformation of the original form of the coefficients, we obtain, for instance,

$$\frac{r^2}{r'^3}\cos(2f - 2f' + 2\varpi - 2\varpi') = \frac{a^2}{a'^3}(Q_c^i + Q_s^i)(G_c^{i'} - G_s^{i'})\cos(ig + i'g' + 2\varpi - 2\varpi')$$

where only the general term is written down, but the indices i, i', each of them extend from $-\infty$ to $+\infty$, zero included, or the summatory sign $\Sigma_{-\infty}^{\infty}\,\Sigma_{-\infty}^{\infty}$ is to be understood; and similarly for the other terms of Ω. And we may write

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 + \Omega_8 + \Omega_9 + \Omega_{10},$$

where

$$\left\{\begin{array}{llllll}
\Omega_1 = m'\dfrac{a^2}{a'^3} & \frac{1}{4} - \frac{3}{2}\eta^2 + \frac{3}{2}\eta^4 & P^i & K^{i'} & \cos & ig + i'g' \quad , \\[2ex]
\Omega_2 = m'\dfrac{a^2}{a'^3} & \frac{3}{4} - \frac{3}{2}\eta^2 + \frac{3}{4}\eta^4 & Q_c^i + Q_s^i & G_c^{i'} - G_s^{i'} & \cos & ig + i'g' + 2\mho - 2\mho', \\[2ex]
\Omega_3 = m'\dfrac{a^2}{a'^3} & \frac{3}{2}\eta^2 - \frac{3}{2}\eta^4 & Q_c^i + Q_s^i & K^{i'} & \cos & ig + i'g' + 2\mho \quad , \\[2ex]
\Omega_4 = m'\dfrac{a^2}{a'^3} & \frac{3}{2}\eta^2 - \frac{3}{2}\eta^4 & P^i & G_c^{i'} + G_s^{i'} & \cos & ig + i'g' \quad + 2\mho', \\[2ex]
\Omega_5 = m'\dfrac{a^2}{a'^3} & \frac{3}{4}\eta^4 & Q_c^i + Q_s^i & G_c^{i'} + G_s^{i'} & \cos & ig + i'g' + 2\mho + 2\mho',
\end{array}\right.$$

$$\left\{\begin{array}{llllll}
\Omega_6 = m'\dfrac{a^2}{a'^4} & \frac{3}{8} - \frac{33}{8}\eta^2 + \frac{75}{8}\eta^4 & A_c^i + A_s^i & C_c^{i'} - C_s^{i'} & \cos & ig + i'g' + \mho - \mho', \\[2ex]
\Omega_7 = m'\dfrac{a^2}{a'^4} & \frac{5}{8} - \frac{15}{8}\eta^2 + \frac{15}{8}\eta^4 & B_c^i + B_s^i & D_c^{i'} - D_s^{i'} & \cos & ig + i'g' + 3\mho - 3\mho', \\[2ex]
\Omega_8 = m'\dfrac{a^2}{a'^4} & \frac{9}{4}\eta^2 - \frac{15}{2}\eta^4 & A_c^i + A_s^i & C_c^{i'} + C_s^{i'} & \cos & ig + i'g' + \mho + \mho', \\[2ex]
\Omega_9 = m'\dfrac{a^2}{a'^4} & \frac{15}{8}\eta^2 - \frac{15}{4}\eta^4 & B_c^i + B_s^i & C_c^{i'} - C_s^{i'} & \cos & ig + i'g' + 3\mho - \mho', \\[2ex]
\Omega_{10} = m'\dfrac{a^2}{a'^4} & \frac{15}{8}\eta^2 - \frac{15}{4}\eta^4 & A_c^i + A_s^i & D_c^{i'} - D_s^{i'} & \cos & ig + i'g' + \mho - 3\mho'.
\end{array}\right.$$

The values of P^i, Q_c^i, Q_s^i, &c. are made to depend ultimately upon the development in multiple cosines of the square of the radius vector and the development in multiple sines of the true anomaly. And the actual values of P^i, Q_c^i, Q_s^i, &c., expanded in powers of e or e', are given pp. 174—179 of the work above referred to. I have verified all these values by a different process, and have discovered only a single inaccuracy, which, however, is rather an important one, as it affects the evection, viz., the value of Q_s^1 should be $-\frac{3}{2}e + \frac{23}{24}e^3 + \&c.$ instead of (Hansen) $-\frac{3}{2}e + \frac{33}{24}e^3 + \&c.$ The formation of the sums or differences $Q_c^i + Q_s^i$ is of course perfectly easy, and we thus obtain the actual developed expression of the disturbing function, which I represent under the following form, viz.:—

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 + \Omega_8 + \Omega_9 + \Omega_{10},$$

where

$\Omega_1 = m' \frac{a^2}{a'^3}\left(\frac{1}{4} - \frac{3}{2}\eta^2 + \frac{3}{2}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{1773}{256} e'^5$	cos	$-5g'$
	$\frac{77}{16} e'^4$		$-4g'$
	$\frac{53}{16} e'^3$		$-3g'$
	$\frac{9}{4} e'^2 + \frac{7}{4} e'^4$		$-2g'$
	$\frac{3}{2} e' + \frac{27}{16} e'^3$		$-g'$
	* $1 + \frac{3}{2} e'^2 + \frac{15}{8} e'^4 + \frac{35}{16} e'^6$		$0g'$
	$\frac{3}{2} e' + \frac{27}{16} e'^3$		$+g'$
	$\frac{9}{4} e'^2 + \frac{7}{4} e'^4$		$+2g'$
	$\frac{53}{16} e'^3$		$+3g'$
	$\frac{231}{48} e'^4$		$+4g'$
	$\frac{1773}{256} e'^5$		$+5g'$
	* Exact value is $(1 - e'^2)^{-\frac{3}{2}}$		
$-\frac{9}{160} e^6$	Solar part suprà.		$-6g$
$-\frac{25}{384} e^5$			$-5g$
$-\frac{1}{12} e^4 + \frac{1}{15} e^6$			$-4g$
$-\frac{1}{8} e^3 + \frac{9}{128} e^5$			$-3g$
$-\frac{1}{4} e^2 + \frac{1}{12} e^4 - \frac{1}{96} e^6$			$-2g$
$-e + \frac{1}{8} e^3 - \frac{1}{192} e^5$			$-g$
$1 + \frac{3}{2} e^2$ (exact)			$0g$
$-e + \frac{1}{8} e^3 - \frac{1}{192} e^5$			$+g$
$-\frac{1}{4} e^2 + \frac{1}{12} e^4 - \frac{1}{96} e^6$			$+2g$
$-\frac{1}{8} e^3 + \frac{9}{128} e^5$			$+3g$
$-\frac{1}{12} e^4 + \frac{1}{15} e^6$			$+4g$
$-\frac{25}{384} e^5$			$+5g$
$-\frac{9}{160} e^6$			$+6g$

* g' annual equation.

$\Omega_2 = m' \frac{a^2}{a'^2}\left(\frac{3}{4} - \frac{3}{2}\eta^2 + \frac{3}{4}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{228347}{3840} e'^5$	cos	$-7g'$	
	$\frac{533}{16} e'^4$		$-6g'$	
	$\frac{845}{48} e'^3$		$-5g'$	
	$\frac{17}{2} e'^2 - \frac{115}{6} e'^4$		$-4g'$	
	$\frac{7}{2} e' - \frac{123}{16} e'^3$		$-3g'$	$-2\varpi'$
	$1 - \frac{5}{2} e'^2 + \frac{13}{16} e'^4$		$-2g'$	
	$-\frac{1}{2} e' + \frac{1}{16} e'^3$		$-g'$	
	0 (exact)		$0g'$	
	$\frac{1}{48} e'^3$		$+g'$	
	$\frac{1}{24} e'^4$		$+2g'$	
$-\frac{11}{720} e^6$	Solar part suprà.		$-4g$	
$-\frac{17}{640} e^5$			$-3g$	
$-\frac{1}{16} e^4 - \frac{11}{480} e^6$			$-2g$	
$-\frac{7}{24} e^3 - \frac{47}{384} e^5$			$-g$	
$\frac{5}{2} e^2$ (exact)			$0g$	
$-3e + \frac{13}{8} e^3 + \frac{5}{192} e^5$			$+g$	
$1 - \frac{5}{2} e^2 + \frac{23}{16} e^4 - \frac{65}{288} e^6$			$+2g$	$+2\varpi$
$e - \frac{19}{8} e^3 + \frac{107}{64} e^5$			$+3g$	
$e^2 - \frac{5}{2} e^4 + \frac{101}{48} e^6$			$+4g$	
$\frac{25}{24} e^3 - \frac{1075}{384} e^5$			$+5g$	
$\frac{9}{8} e^4 - \frac{261}{80} e^6$			$+6g$	
$\frac{2401}{1920} e^5$			$+7g$	
$\frac{64}{45} e^6$			$+8g$	

$2g - 2g' + 2\varpi - 2\varpi'$ variation.

$g - 2g' + 2\varpi - 2\varpi'$ evection.

$\Omega_3 = m' \frac{a^2}{a'^3}\left(\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{1773}{256}e'^5$	cos	$-5g'$	
	$\frac{77}{16}e'^4$		$-4g'$	
	$\frac{53}{16}e'^3$		$-3g'$	
	$\frac{9}{4}e'^2 + \frac{7}{4}e'^4$		$-2g'$	
	$\frac{3}{2}e' + \frac{27}{16}e'^3$		$-g'$	
	* $1 + \frac{3}{2}e'^2 + \frac{15}{8}e'^4 + \frac{35}{16}e'^6$		$0g'$	
	$\frac{3}{2}e' + \frac{27}{16}e'^3$		$+g'$	
	$\frac{9}{4}e'^2 + \frac{7}{4}e'^4$		$+2g'$	
	$\frac{53}{16}e'^3$		$+3g'$	
	$\frac{231}{48}e'^4$		$+4g'$	
	$\frac{1773}{256}e'^5$		$+5g'$	
	*Exact value is $(1-e'^2)^{-\frac{3}{2}}$			
$-\frac{11}{720}e^6$	Solar part suprà.		$-4g$	
$-\frac{17}{640}e^5$			$-3g$	
$-\frac{1}{16}e^4 - \frac{11}{480}e^6$			$-2g$	
$-\frac{7}{24}e^3 - \frac{47}{384}e^5$			$-g$	
$\frac{5}{2}e^2$ (exact)			$0g$	
$-3e + \frac{13}{8}e^3 + \frac{5}{192}e^5$			$+g$	
$1 - \frac{5}{2}e^2 + \frac{23}{16}e^4 - \frac{65}{288}e^6$			$+2g$	$+2\varpi$
$e - \frac{19}{8}e^3 + \frac{107}{64}e^5$			$+3g$	
$e^2 - \frac{5}{2}e^4 + \frac{101}{48}e^6$			$+4g$	
$\frac{25}{24}e^3 - \frac{1075}{384}e^5$			$+5g$	
$\frac{9}{8}e^4 - \frac{261}{80}e^6$			$+6g$	
$\frac{2401}{1920}e^5$			$+7g$	
$\frac{64}{45}e^6$			$+8g$	

$\Omega_4 = m' \frac{a^2}{a'^3}\left(\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{1}{24}e'^4$	cos	$-2g'$	
	$\frac{1}{48}e'^3$		$-g'$	
	0 (exact)		$0\,g'$	
	$-\frac{1}{2}e' + \frac{1}{16}e'^3$		$+g'$	
	$1 - \frac{5}{2}e'^2 + \frac{13}{16}e'^4$		$+2g'$	
	$\frac{7}{2}e' - \frac{123}{16}e'^3$		$+3g'$	$+2\overline{\mathbf{C}}'$
	$\frac{17}{2}e'^2 - \frac{115}{6}e'^4$		$+4g'$	
	$\frac{845}{48}e'^3$		$+5g'$	
	$\frac{533}{16}e'^4$		$+6g$	
	$\frac{228347}{3840}e'^3$		$+7g'$	
$-\frac{9}{160}e^6$	Solar part suprà.		$-6g$	
$-\frac{25}{384}e^5$			$-5g$	
$-\frac{1}{12}e^4 + \frac{1}{15}e^6$			$-4g$	
$-\frac{1}{8}e^3 + \frac{9}{128}e^5$			$-3g$	
$-\frac{1}{4}e^2 + \frac{1}{12}e^4 - \frac{1}{96}e^6$			$-2g$	
$-e + \frac{1}{8}e^3 - \frac{1}{192}e^5$			$-g$	
$1 + \frac{3}{2}e^2$ (exact)			$0\,g$	
$-e + \frac{1}{8}e^3 - \frac{1}{192}e^5$			$+g$	
$-\frac{1}{4}e^2 + \frac{1}{12}e^4 - \frac{1}{96}e^6$			$+2g$	
$-\frac{1}{8}e^3 + \frac{9}{128}e^5$			$+3g$	
$-\frac{1}{12}e^4 + \frac{1}{15}e^6$			$+4g$	
$-\frac{25}{384}e^5$			$+5g$	
$-\frac{9}{160}e^6$			$+6g$	

$\Omega_5 = m' \frac{a^2}{a'^3}\left(\frac{3}{4}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{1}{24}e'^4$	cos	$-2g'$	
	$\frac{1}{48}e'^3$		$-g'$	
	0 (exact)		$0g'$	
	$-\frac{1}{2}e' + \frac{1}{16}e'^3$		$+g'$	
	$1 - \frac{5}{2}e'^2 + \frac{13}{16}e'^4$		$+2g'$	
	$\frac{7}{2}e' - \frac{123}{16}e'^3$		$+3g'$	$+2\varpi'$
	$\frac{17}{2}e'^2 - \frac{115}{6}e'^4$		$+4g'$	
	$\frac{845}{48}e'^3$		$+5g'$	
	$\frac{533}{16}e'^4$		$+6g'$	
	$\frac{2283347}{3840}e'^5$		$+7g'$	
$-\frac{11}{720}e^6$	Solar part suprà.		$-4g$	
$-\frac{17}{640}e^5$			$-3g$	
$-\frac{1}{16}e^4 - \frac{11}{480}e^6$			$-2g$	
$-\frac{7}{24}e^3 - \frac{47}{384}e^5$			$-g$	
$\frac{5}{2}e^2$ (exact)			$0g$	
$-3e + \frac{13}{8}e^3 + \frac{5}{192}e^5$			$+g$	
$1 - \frac{5}{2}e^2 + \frac{23}{16}e^4 - \frac{65}{288}e^6$			$+2g$	$+2\varpi$
$e - \frac{19}{8}e^3 + \frac{107}{64}e^5$			$+3g$	
$e^2 - \frac{5}{2}e^4 + \frac{101}{48}e^6$			$+4g$	
$\frac{25}{24}e^3 - \frac{1075}{384}e^5$			$+5g$	
$\frac{9}{8}e^4 - \frac{261}{80}e^6$			$+6g$	
$\frac{2401}{1920}e^5$			$+7g$	
$\frac{64}{45}e^6$			$+8g$	

$\dot{\Omega}_6 = m' \frac{a^3}{a'^4}\left(\frac{3}{8} - \frac{33}{8}\eta^2 + \frac{75}{8}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{77}{6}e'^3$	cos	$-4g'$	
	$\frac{53}{8}e'^2$		$-3g'$	
	$3e' + \frac{11}{4}e'^3$		$-2g'$	
	$1 + 2e'^2$		$-g'$	$-\mathfrak{T}'$
	* $e' + \frac{5}{2}e'^3$		$0g'$	
	$\frac{11}{8}e'^2$		$+g'$	
	$\frac{23}{12}e'^3$		$+2g'$	
	* Exact value is $e'(1-e'^2)^{-\frac{5}{2}}$			
$\frac{7}{128}e^4$	Solar part suprà.		$-3g$	
$\frac{1}{6}e^3$			$-2g$	
$\frac{11}{8}e^2 + \frac{7}{48}e^4$			$-g$	
$-\frac{5}{2}e - \frac{15}{8}e^3$ (exact)			$0g$	
$1 + 2e^2 - \frac{41}{64}e^4$			$+g$	$+\mathfrak{T}$
$-\frac{1}{2}e + e^3$			$+2g$	
$-\frac{3}{8}e^2 + \frac{11}{16}e^4$			$+3g$	
$-\frac{7}{24}e^3$			$+4g$	
$-\frac{95}{384}e^4$			$+5g$	

$g - g' + \mathfrak{T} - \mathfrak{T}'$ parallactic equation.

$\Omega_7 = m' \frac{a^2}{a'^4}\left(\frac{5}{8} - \frac{15}{8}\eta^2 + \frac{15}{8}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{163}{4}e'^3$	cos	$-6g'$	
	$\frac{127}{8}e'^2$		$-5g'$	
	$5e' - 22e'^3$		$-4g'$	
	$1 - 6e'^2$		$-3g'$	$-3\mathfrak{T}'$
	$-e' + \frac{5}{4}e'^3$		$-2g'$	
	$\frac{1}{8}e'^2$		$-g'$	
	0 (exact)		$0g'$	
$\frac{75}{128}e^4$	Solar part suprà.		$-g$	
$-\frac{35}{8}e^3$ (exact)			$0g$	
$\frac{57}{8}e^2 - \frac{65}{16}e^4$			$+g$	
$-\frac{9}{2}e + \frac{33}{4}e^3$			$+2g$	
$1 - 6e^2 + \frac{591}{64}e^4$			$+3g$	$+3\mathfrak{T}$
$\frac{3}{2}e - \frac{57}{8}e^3$			$+4g$	
$\frac{15}{8}e^2 - \frac{135}{16}e^4$			$+5g$	
$\frac{9}{4}e^3$			$+6g$	
$\frac{343}{128}e^4$			$+7g$	

$\Omega_8 = m' \frac{a^3}{a'^4}\left(\frac{9}{4}\eta^2 - \frac{15}{2}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{23}{12}e'^3$	cos	$-2g'$	
	$\frac{11}{8}e'^2$		$-g'$	
	$*\,e' + \frac{5}{2}e'^3$		$0g'$	
	$1 + 2e'^2$		$+g'$	$+\mho'$
	$3e' + \frac{11}{4}e'^3$		$+2g'$	
	$\frac{53}{8}e'^2$		$+3g'$	
	$\frac{77}{6}e'^3$		$+4g'$	
	* Exact value is $e'(1-e'^2)^{-\frac{3}{2}}$			
$-\frac{7}{128}e^4$	Solar part suprà.		$-3g$	
$\frac{1}{6}e^3$			$-2g$	
$\frac{11}{8}e^2 + \frac{7}{48}e^4$			$-g$	
$-\frac{5}{2}e - \frac{15}{8}e^3$ (exact)			$0g$	
$1 + 2e^2 - \frac{41}{64}e^4$			$+g$	$+\mho$
$-\frac{1}{2}e + e^3$			$+2g$	
$-\frac{3}{8}e^2 + \frac{11}{16}e^4$			$+3g$	
$-\frac{7}{24}e^3$			$+4g$	
$-\frac{95}{344}e^4$			$+5g$	

$\Omega_9 = m' \frac{a^2}{a'^4}\left(\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{77}{6}e'^3$	cos	$-4g'$	
	$\frac{53}{8}e'^2$		$-3g'$	
	$3e' + \frac{11}{4}e'^3$		$-2g'$	
	$1 + 2e'^2$		$-g'$	$-\varpi'$
	$*e' + \frac{5}{2}e'^3$		$0g'$	
	$\frac{11}{8}e'^2$		$+g'$	
	$\frac{23}{12}e'^3$		$+2g'$	
	*Exact value is $e'(1-e'^2)^{-\frac{5}{2}}$			
$\frac{75}{128}e^4$	Solar part suprà.		$-g$	
$-\frac{35}{8}e^3$ (exact)			$0g$	
$\frac{57}{8}e^2 - \frac{65}{16}e^4$			$+g$	
$-\frac{9}{2}e + \frac{33}{4}e^3$			$+2g$	
$1 - 6e^2 + \frac{591}{64}e^4$			$+3g$	$+3\varpi$
$\frac{3}{2}e - \frac{57}{8}e^3$			$+4g$	
$\frac{15}{8}e^2 - \frac{135}{16}e^4$			$+5g$	
$\frac{9}{4}e^3$			$+6g$	
$\frac{343}{128}e^4$			$+7g$	

$\Omega_{10} = m' \frac{a^2}{a'^4}\left(\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4\right)$ multiplied into

Lunar part infrà.	$\frac{163}{4}e'^3$	cos	$-6g'$	
	$\frac{127}{8}e'^2$		$-5g'$	
	$5e' - 22e'^3$		$-4g'$	
	$1 - 6e'^2$		$-3g'$	$-3\mho'$
	$-e' + \frac{5}{4}e'^3$		$-2g'$	
	$\frac{1}{8}e'^2$		$-g'$	
	0 (exact)		$0g'$	
$\frac{7}{128}e^4$	Solar part suprà.		$-3g$	
$\frac{1}{6}e^3$			$-2g$	
$\frac{11}{8}e^2 + \frac{7}{48}e^4$			$-g$	
$-\frac{5}{2}e - \frac{15}{8}e^3$ (exact)			$0g$	
$1 + 2e^2 - \frac{41}{64}e^4$			$+g$	$+\mho$
$-\frac{1}{2}e + e^3$			$+2g$	
$-\frac{3}{8}e^2 + \frac{11}{16}e^4$			$+3g$	
$-\frac{7}{24}e^3$			$+4g$	
$-\frac{95}{384}e^4$			$+5g$	

The arrangement of the tables hardly requires any explanation; each line of the upper half of a table is to be read in combination with each line of the lower half of the same table. Thus a term of Ω_2 is

$$m'\frac{a^2}{a'^3}(\tfrac{3}{4}-\tfrac{3}{2}\eta^2+\tfrac{3}{4}\eta^4)(-\tfrac{7}{24}e^3-\tfrac{47}{384}e^5)(1-\tfrac{5}{2}e'^2+\tfrac{13}{16}e'^4)\cos(-g-2g'+2\mho-2\mho').$$

It is to be observed that in the table for Ω_1 (where the arguments depend only on the mean anomalies), the several terms (other than the constant term) occur in pairs of equal terms, having respectively the arguments $ig+i'g'$ and $-ig-i'g'$; such equal terms are to be united together, or, what is the same thing, the coefficients must be multiplied by 2. Thus we have the two terms

$$m'\frac{a^2}{a'^3}(\tfrac{1}{4}-\tfrac{3}{2}\eta^2+\tfrac{3}{2}\eta^4)(-\tfrac{9}{160}e^6)(\tfrac{1773}{256}e'^5)\quad\cos(-6g-5g')$$

$$\text{Ditto}\qquad\qquad\qquad\qquad\cos(\ \ 6g+5g')$$

or, what is the same thing, the term is

$$m'\frac{a^2}{a'^3}(\tfrac{1}{4}-\tfrac{3}{2}\eta^2+\tfrac{3}{2}\eta^4)(-\tfrac{9}{160}e^6)(\tfrac{1773}{256}e'^5)\,2\cos(\ \ 6g+5g').$$

This is the case even when either i or i' vanishes; but, as already noticed, it is not the case for the constant term where i and i' both vanish. There are not any such equal pairs in the tables for Ω_2, &c., where all the arguments contain a part independent of the mean anomalies. The quantity $\mho'$ occurs in the upper or solar half of the table; but it is to be recollected that $\mho'(=\varpi'-\Theta)$ involves Θ, which is an element of the moon's orbit.

The peculiar form in which the coefficients are exhibited, viz. as the product of a term depending on the inclination, a term depending on the eccentricity of the moon's orbit, and a term depending on the eccentricity of the sun's orbit is not to be considered as a want of completion of the development; it is, on the contrary, an important advantage.

[The Errata in the Tables, noticed *Mem. R. Ast. Soc.* vol. XXVIII. p. 216, have been corrected. The values of P^i, Q_c^i, Q_s^i, &c. used in the construction of the Tables *ante* p. 298 may also be obtained from the Tables of the Developments of Functions in The Theory of Elliptic Motion, 216.]

Addition.

I deduce from the preceding an expression for the disturbing function in a form similar to and easily comparable with the forms given by Sir J. W. Lubbock in his work *On the Theory of the Moon* &c. (London, 1834), pp. 30—35, and by Pontécoulant in the *Théorie Analytique du Système du Monde*, t. IV. (Paris, 1846), pp. 58—61. The several notations are as follows, viz.:

Lubbock.	Pontécoulant.	Suprà.
ξ,	ϕ,	g
ξ',	ϕ',	g'
τ,	ξ,	$g-g'+\mho-\mho'$
η,	η,	$g+\mho$
γ,	γ,	$2\eta\sqrt{1-\eta^2}\div(1-2\eta^2)$

(γ, the tangent of the inclination, is employed by the above-named two authors instead of η, the sine of the semi-inclination, the relation between these two quantities gives to γ^4 or η^4,

$$\gamma^2 = 4\eta^2 + 12\eta^4,$$

$$\gamma^4 = 16\eta^4,$$

and conversely,

$$\eta^2 = \tfrac{1}{4}\gamma^2 - \tfrac{3}{16}\gamma^4,$$

$$\eta^4 = \tfrac{1}{16}\gamma^4,$$

which are useful for replacing one of these quantities by the other). The disturbing function is taken by Lubbock with the contrary sign, and he includes in the before-mentioned omitted term depending only on the sun's radius vector, that is, R (Lubbock) $= -m'\dfrac{1}{r'} - \Omega$; he uses also, in reference to the sun, subscript strokes instead of accents (so that, in referring to his notation, $\xi_{,}$ should properly be here used in the place of ξ', but this difference is obviously immaterial). Pontécoulant's disturbing function is taken with the same sign as in the present memoir, or we have R (Pontécoulant) $= \Omega$. Lubbock and Pontécoulant give in the principal terms the part involving $m'\dfrac{a^4}{a'^5}$, which depends on the square of the parallax, and which was disregarded in the preceding development. I have in the sequel inserted these terms from Lubbock's expression. We have, in fact:

$\Omega = m' \frac{a^2}{a'^3}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
$\frac{1}{4} + \frac{3}{8} e^2 + \frac{3}{8} e'^2 + \frac{9}{16} e^2 e'^2 + \frac{15}{32} e'^4 - \frac{3}{2} \eta^2 - \frac{9}{4} \eta^2 e^2 - \frac{9}{4} \eta^2 e'^2 + \frac{3}{2} \eta^4 + \frac{9}{64} \frac{a^2}{a'^2}$			0		0	
$+ \frac{3}{4} - \frac{15}{8} e^2 - \frac{15}{8} e'^2 + \frac{69}{64} e^4 + \frac{75}{16} e^2 e'^2 + \frac{39}{64} e'^4 - \frac{3}{2} \eta^2 + \frac{15}{4} \eta^2 e^2 + \frac{15}{4} \eta^2 e'^2 + \frac{3}{4} \eta^4 + \frac{5}{16} \frac{a^2}{a'^2}$	cos	2τ	1	2ξ	30	$2g - 2g' + 2\varpi - 2\varpi'$
$-\frac{1}{2} e + \frac{1}{16} e^3 - \frac{3}{4} e e'^2 + 3 \eta^2 e - \frac{9}{16} \frac{a^2}{a'^2} e$		ξ	2	ϕ	1	g
$-\frac{9}{4} e + \frac{39}{32} e^3 + \frac{45}{8} e e'^2 + \frac{9}{2} \eta^2 e - \frac{5}{4} \frac{a^2}{a'^2} e$		$2\tau - \xi$	3	$2\xi - \phi$	31	$g - 2g'$ } $+ 2\varpi - 2\varpi'$
$+ \frac{3}{4} e - \frac{57}{32} e^3 - \frac{15}{8} e e'^2 - \frac{3}{2} \eta^2 e$		$2\tau + \xi$	4	$2\xi + \phi$	32	$3g - 2g'$ }
$+ \frac{3}{4} e' + \frac{9}{8} e^2 e' + \frac{27}{32} e'^3 - \frac{9}{2} \eta^2 e' + \frac{35}{64} \frac{a^2}{a'^2} e'$		ξ'	5	ϕ'	6	g'
$+ \frac{21}{8} e' - \frac{105}{16} e^3 e' - \frac{369}{64} e'^3 - \frac{21}{4} \eta^2 e' + \frac{45}{32} \frac{a^2}{a'^2} e'$		$2\tau - \xi'$	6	$2\xi - \phi'$	33	$2g - 3g'$ } $+ 2\varpi - 2\varpi'$
$- \frac{3}{8} e' + \frac{15}{16} e^2 e' + \frac{3}{64} e'^3 + \frac{3}{4} \eta^2 e' + \frac{15}{32} \frac{a^2}{a'^2} e'$		$2\tau + \xi'$	7	$2\xi + \phi'$	34	$2g - g'$ }
$-\frac{1}{8} e^2 + \frac{1}{24} e^4 - \frac{3}{16} e^2 e'^2 + \frac{3}{4} \eta^2 e^2$		2ξ	8	2ϕ	2	$2g$
$+ \frac{15}{8} e^2 - \frac{75}{16} e^2 e'^2 - \frac{15}{4} \eta^2 e^2$		$2\tau - 2\xi$	9	$2\xi - 2\phi$	35	$-2g'$ } $+ 2\varpi - 2\varpi'$
$+ \frac{3}{4} e^2 - \frac{15}{8} e^4 - \frac{15}{8} e^2 e'^2 - \frac{3}{2} \eta^2 e^2$		$2\tau + 2\xi$	10	$2\xi + 2\phi$	36	$4g - 2g'$ }
$-\frac{3}{4} e e' + \frac{3}{32} e^3 e' - \frac{27}{32} e e'^3 + \frac{9}{2} \eta^2 e e'$		$\xi + \xi'$	11	$\phi + \phi'$	9	$g + g'$
$-\frac{63}{8} e e' + \frac{273}{64} e^3 e' + \frac{1107}{64} e e'^3 + \frac{63}{4} \eta^2 e e'$		$2\tau - \xi - \xi'$	12	$2\xi - \phi - \phi'$	37	$g - 3g'$ } $+ 2\varpi - 2\varpi'$
$-\frac{3}{8} e e' + \frac{57}{64} e^3 e' + \frac{3}{64} e e'^3 + \frac{3}{4} \eta^3 e e'$		$2\tau + \xi + \xi'$	13	$2\xi + \phi + \phi'$	40	$3g - g'$ }
$-\frac{3}{4} e e' + \frac{3}{32} e^3 e' - \frac{27}{32} e e'^3 + \frac{9}{2} \eta^2 e e'$		$\xi - \xi'$	14	$\phi - \phi'$	8	$g - g'$
$+ \frac{9}{8} e e' - \frac{39}{64} e^3 e' - \frac{9}{64} e e'^3 - \frac{9}{4} \eta^2 e e'$		$2\tau - \xi + \xi'$	15	$2\xi - \phi + \phi'$	38	$g - g'$ } $+ 2\varpi - 2\varpi'$
$+ \frac{21}{8} e e' - \frac{399}{64} e^3 e' - \frac{369}{64} e e'^3 - \frac{21}{4} \eta^2 e e'$		$2\tau + \xi - \xi'$	16	$2\xi + \phi - \phi'$	39	$3g - 3g'$ }
$+ \frac{9}{8} e'^2 + \frac{27}{16} e^2 e'^2 + \frac{7}{8} e'^4 - \frac{27}{4} \eta^2 e'^2$		$2\xi'$	17	$2\phi'$	7	$2g'$
$+ \frac{51}{8} e'^2 - \frac{255}{16} e^2 e'^2 - \frac{115}{8} e'^4 - \frac{51}{4} \eta^2 e'^2$		$2\tau - 2\xi'$	18	$2\xi - 2\phi'$	41	$2g - 4g' + 2\varpi - 2\varpi'$

* Variation. † Evection.

Ω (*continued*) $= m' \frac{a^2}{a'^3}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
$-\frac{1}{16} e^3$	cos	3ξ	20	3ϕ	3	$3g$
$-\frac{7}{32} e^3$		$2\tau - 3\xi$	21	$2\xi - 3\phi$	43	$-g - 2g' + 2\varpi - 2\varpi'$
$+\frac{25}{32} e^3$		$2\tau + 3\xi$	22	$2\xi + 3\phi$	44	$5g - 2g' + 2\varpi - 2\varpi'$
$-\frac{3}{16} e^2 e'$		$2\xi + \xi'$	23	$2\phi + \phi'$	11	$2g + g'$
$+\frac{105}{16} e^2 e'$		$2\tau - 2\xi - \xi'$	24	$2\xi - 2\phi - \phi'$	45	$-3g' + 2\varpi - 2\varpi'$
$-\frac{3}{8} e^2 e'$		$2\tau + 2\xi + \xi'$	25	$2\xi + 2\phi + \phi'$	48	$4g - g' + 2\varpi - 2\varpi'$
$-\frac{3}{16} e^2 e'$		$2\xi - \xi'$	26	$2\phi - \phi'$	10	$2g - g'$
$-\frac{15}{16} e^2 e'$		$2\tau - 2\xi + \xi'$	27	$2\xi - 2\phi + \phi'$	46	$-g' + 2\varpi - 2\varpi'$
$+\frac{21}{8} e^2 e'$		$2\tau + 2\xi - \xi'$	28	$2\xi + 2\phi - \phi'$	47	$4g - 3g' + 2\varpi - 2\varpi'$
$-\frac{9}{8} e e'^2$		$\xi + 2\xi'$	29	$\phi + 2\phi'$	13	$g + 2g'$
$-\frac{153}{8} e e'^2$		$2\tau - \xi - 2\xi'$	30	$2\xi - \phi - 2\phi'$	49	$g - 4g' + 2\varpi - 2\varpi'$
$-\frac{9}{8} e e'^2$		$\xi - 2\xi'$	32	$\phi - 2\phi'$	12	$g - 2g'$
$+\frac{51}{8} e e'^2$		$2\tau + \xi - 2\xi'$	34	$2\xi + \phi - 2\phi'$	51	$3g - 4g' + 2\varpi - 2\varpi'$
$+\frac{53}{32} e'^3$		$3\xi'$	35			$3g'$
$+\frac{845}{64} e'^3$		$2\tau - 3\xi'$	36			$2g - 5g' + 2\varpi - 2\varpi'$
$+\frac{1}{64} e'^3$		$2\tau + 3\xi'$	37			$2g + g' + 2\varpi - 2\varpi'$
$-\frac{1}{24} e^4$		4ξ	38			$4g$
$-\frac{3}{64} e^4$		$2\tau - 4\xi$	39	$2\xi - 4\phi$	52	$-2g - 2g' + 2\varpi - 2\varpi'$
$+\frac{27}{32} e^4$		$2\tau + 4\xi$	40			$6g - 2g' + 2\varpi - 2\varpi'$
$-\frac{3}{32} e^3 e'$		$3\xi + \xi'$	41	$3\phi + \phi'$	17	$3g + g'$
$-\frac{49}{64} e^3 e'$		$2\tau - 3\xi - \xi'$	42	$2\xi - 3\phi - \phi'$	53	$-g - 3g' + 2\varpi - 2\varpi'$

Ω (*continued*) $= m' \frac{a^2}{a'^2}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
$-\frac{25}{64} e^3 e'$	cos	$2\tau + 3\xi + \xi'$	43			$5g - g' + 2\mho - 2\mho'$
$-\frac{3}{32} e^3 e'$		$3\xi - \xi'$	44	$3\phi - \phi'$	16	$3g - g'$
$+\frac{7}{64} e^3 e'$		$2\tau - 3\xi + \xi'$	45	$2\xi - 3\phi + \phi'$	54	$-g - g'$ } $+ 2\mho - 2\mho'$
$+\frac{175}{64} e^3 e'$		$2\tau + 3\xi - \xi'$	46			$5g - 3g'$ }
$-\frac{9}{32} e^2 e'^2$		$2\xi + 2\xi'$	47	$2\phi + 2\phi'$	15	$2g + 2g'$
$+\frac{255}{16} e^2 e'^2$		$2\tau - 2\xi - 2\xi'$	48	$2\xi - 2\phi - 2\phi'$	55	$-4g' + 2\mho - 2\mho'$
$-\frac{9}{32} e^2 e'^2$		$2\xi - 2\xi'$	50	$2\phi - 2\phi'$	14	$2g - 2g'$
$+\frac{51}{8} e^2 e'^2$		$2\tau + 2\xi - 2\xi'$	52			$4g - 4g' + 2\mho - 2\mho'$
$-\frac{53}{32} e e'^3$		$\xi + 3\xi'$	53			$g + 3g'$
$-\frac{2535}{64} e e'^3$		$2\tau - \xi - 3\xi'$	54			$g - 5g'$ } $+ 2\mho - 2\mho'$
$+\frac{1}{64} e e'^3$		$2\tau + \xi + 3\xi'$	55			$3g + g'$ }
$-\frac{53}{32} e e'^3$		$\xi - 3\xi'$	56			$g - 3g'$
$-\frac{3}{64} e e'^3$		$2\tau - \xi + 3\xi'$	57			$g + g'$ } $+ 2\mho - 2\mho'$
$+\frac{845}{64} e e'^3$		$2\tau + \xi - 3\xi'$	58			$3g - 5g'$ }
$+\frac{77}{32} e'^4$		$4\xi'$	59			$4g'$
$+\frac{1599}{64} e'^4$		$2\tau - 4\xi'$	60			$2g - 6g'$ } $+ 2\mho - 2\mho'$
$+\frac{1}{32} e'^4$		$2\tau + 4\xi'$	61			$2g + 2g'$ }
$+\frac{3}{2}\eta^2 - \frac{15}{4}\eta^2 e^2 + \frac{9}{4}\eta^2 e'^2 - \frac{3}{2}\eta^4$		2η	62	2η	18	$2g + 2\mho$
$+\frac{3}{2}\eta^2 + \frac{9}{4}\eta^2 e^2 - \frac{15}{4}\eta^2 e'^2 - \frac{3}{2}\eta^4$		$2\tau - 2\eta$ rev.	63	$2\xi - 2\eta$ rev.	57	$2g' + 2\mho'$
$-\frac{9}{2}\eta^2 e$		$\xi - 2\eta$ rev.	65	$\phi - 2\eta$ rev.	19	g } $+ 2\mho$
$+\frac{3}{2}\eta^2 e$		$\xi + 2\eta$	66	$\phi + 2\eta$	20	$3g$ }

Ω (*continued*) $= m' \frac{a^2}{a'^3}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
$-\frac{3}{2}\eta^2 e$	cos	$2\tau - \xi - 2\eta$ rev.	67	$2\xi - \phi - 2\eta$ rev.	59	$g + 2g'$
$-\frac{3}{2}\eta^2 e$		$2\tau + \xi - 2\eta$ rev.	69	$2\xi + \phi - 2\eta$ rev.	61	$-g + 2g'$ $\Big\}$ $+2\varpi$
$+\frac{9}{4}\eta^2 e'$		$\xi' - 2\eta$ rev.	71	$\phi' - 2\eta$ rev.	21	$2g - g'$
$+\frac{9}{4}\eta^2 e'$		$\xi' + 2\eta$	72	$\phi' + 2\eta$	22	$2g + g'$ $\Big\}$ $+2\varpi$
$+\frac{21}{4}\eta^2 e'$		$2\tau - \xi' - 2\eta$ rev.	73	$2\xi - \phi' - 2\eta$ rev.	63	$3g'$
$-\frac{3}{4}\eta^2 e'$		$2\tau + \xi' - 2\eta$ rev.	75	$2\xi + \phi' - 2\eta$ rev.	65	g' $\Big\}$ $+2\varpi$
$+\frac{15}{4}\eta^2 e^2$		$2\xi - 2\eta$ rev.	77	$2\phi - 2\eta$ rev.	23	$0g$
$+\frac{3}{2}\eta^2 e^2$		$2\xi + 2\eta$	78	$2\phi + 2\eta$	24	$4g$ $\Big\}$ $+2\varpi$
$-\frac{3}{8}\eta^2 e^2$		$2\tau - 2\xi - 2\eta$ rev.	79	$2\xi - 2\phi - 2\eta$ rev.	67	$2g + 2g'$
$-\frac{3}{8}\eta^2 e^2$		$2\tau + 2\xi - 2\eta$ rev.	81	$2\xi + 2\phi - 2\eta$ rev.	69	$-2g + 2g'$ $\Big\}$ $+2\varpi$
$-\frac{27}{4}\eta^2 e e'$		$\xi + \xi' - 2\eta$ rev.	83	$\phi + \phi' - 2\eta$ rev.	27	$g - g'$
$+\frac{9}{4}\eta^2 e e'$		$\xi + \xi' + 2\eta$	84	$\phi + \phi' + 2\eta$	28	$3g + g'$ $\Big\}$ $+2\varpi$
$-\frac{21}{4}\eta^2 e e'$		$2\tau - \xi - \xi' - 2\eta$ rev.	85			$g + 3g'$
$+\frac{3}{4}\eta^2 e e'$		$2\tau + \xi + \xi' - 2\eta$ rev.	87			$-g + g'$ $\Big\}$ $+2\varpi$
$-\frac{27}{4}\eta^2 e e'$		$\xi - \xi' - 2\eta$ rev.	89	$\phi - \phi' - 2\eta$ rev.	25	$g + g'$
$+\frac{9}{4}\eta^2 e e'$		$\xi - \xi' + 2\eta$	90	$\phi - \phi' + 2\eta$	26	$3g - g'$ $\Big\}$ $+2\varpi$
$+\frac{3}{4}\eta^2 e e'$		$2\tau - \xi + \xi' - 2\eta$ rev.	91			$g + g'$
$-\frac{21}{4}\eta^2 e e'$		$2\tau + \xi - \xi' - 2\eta$ rev.	93			$-g + 3g'$ $\Big\}$ $+2\varpi$
$+\frac{27}{8}\eta^2 e'^2$		$2\xi' - 2\eta$ rev.	95			$2g - 2g'$
$+\frac{27}{8}\eta^2 e'^2$		$2\xi' + 2\eta$	96			$2g + 2g'$ $\Big\}$ $+2\varpi$
$+\frac{51}{4}\eta^2 e'^2$		$2\tau - 2\xi' - 2\eta$ rev.	97			$4g'$ $+2\varpi$

Ω (*continued*) $= m' \frac{a^2}{a'^4}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
* $\frac{3}{8} + \frac{3}{4}e^2 + \frac{3}{4}e'^2 - \frac{33}{8}\eta^2$	cos	τ	101	ξ	70	$g - g'$
$-\frac{15}{16}e$		$\tau - \xi$	102	$\xi - \phi$	71	$- g'$
$-\frac{3}{16}e$		$\tau + \xi$	103	$\xi + \phi$	72	$2g - g'$
$+\frac{9}{8}e'$		$\tau - \xi'$	104	$\xi - \phi'$	73	$g - 2g'$
$+\frac{3}{8}e'$		$\tau + \xi'$	105	$\xi + \phi'$	74	g
$+\frac{33}{64}e^2$		$\tau - 2\xi$	106	$\xi - 2\phi$	75	$- g - g'$
$-\frac{9}{64}e^2$		$\tau + 2\xi$	107	$\xi + 2\phi$	76	$3g - g'$ $+ \varpi - \varpi'$
$-\frac{45}{16}ee'$		$\tau - \xi - \xi'$	108	$\xi - \phi - \phi'$	77	$- 2g'$
$-\frac{3}{16}ee'$		$\tau + \xi + \xi'$	109	$\xi + \phi + \phi'$	79	$2g$
$-\frac{15}{16}ee'$		$\tau - \xi + \xi'$	110	$\xi - \phi + \phi'$	78	$0g + 0g'$
$-\frac{9}{16}ee'$		$\tau + \xi - \xi'$	111			$2g - 2g'$
$+\frac{159}{64}e'^2$		$\tau - 2\xi'$	112			$g - 3g'$
$+\frac{33}{64}e'^2$		$\tau + 2\xi'$	113			$g + g'$
$+\frac{9}{4}\eta^2$		$\tau - 2\eta$ rev.	114			$g + g' + \varpi + \varpi'$
$+\frac{15}{8}\eta^2$		$\tau + 2\eta$	115			$3g - 3g' + 3\varpi - \varpi'$
$+\frac{5}{8} - \frac{15}{4}e^2 - \frac{15}{4}e'^2 - \frac{15}{8}\eta^2$		3τ	116	3ξ	80	$3g - 3g'$
$-\frac{45}{16}e$		$3\tau - \xi$	117	$3\xi - \phi$	81	$2g - 3g'$
$+\frac{15}{16}e$		$3\tau + \xi$	118	$3\xi + \phi$	82	$4g - 3g'$
$+\frac{25}{8}e'$		$3\tau - \xi'$	119	$3\xi - \phi'$	83	$3g - 4g'$ $+ 3\varpi - 3\varpi'$
$-\frac{5}{8}e'$		$3\tau + \xi'$	120	$3\xi + \phi'$	84	$3g - 2g'$
$+\frac{285}{64}e^2$		$3\tau - 2\xi$	121	$3\xi - 2\phi$	85	$g - 3g'$

* Parallactic equation.

Ω (*continued*) $= m' \frac{a^3}{a'^4}$ multiplied into

		Lubbock.		Pontécoulant.		Arguments ut suprà.
		Arguments.	Nos.	Arguments.	Nos.	
$+\frac{75}{64}e^2$	cos	$3\tau + 2\xi$	122			$5g - 3g'$
$-\frac{225}{16}ee'$		$3\tau - \xi - \xi'$	123			$2g - 4g'$
$-\frac{15}{16}ee'$		$3\tau + \xi + \xi'$	124			$4g - 2g'$
$+\frac{45}{16}ee'$		$3\tau - \xi + \xi'$	125			$2g - 2g'$ $+ 3\varpi - 3\varpi'$
$+\frac{75}{16}ee'$		$3\tau + \xi - \xi'$	126			$4g - 4g'$
$+\frac{635}{64}e'^2$		$3\tau - 2\xi'$	127			$3g - 5g'$
$+\frac{5}{64}e'^2$		$3\tau + 2\xi'$	128			$3g - g'$
$+\frac{15}{8}\eta^2$		$3\tau - 2\eta$	129			$g - 3g' + \varpi - 3\varpi'$

where the abbreviation *rev.* denotes that the argument to which it is attached has its sign reversed in the third column of arguments: thus Arg. 63 (Lubbock), it is $-(2\tau - 2\eta)$ which is equal to $2g' + 2\varpi'$. As the formula contains only cosines, there is, of course, no change in the sign of the coefficient.

On comparing with Lubbock's value, I find some differences, which are as follows:— It will be recollected that R (Lubbock) $= -m' \frac{1}{r'} - \Omega$, so that the signs of the coefficients of Ω are to be reversed in order to deduce the corresponding coefficients of R. The Nos. refer to Lubbock's arguments, and the exterior factor $m' \frac{a^2}{a'^3}$ or $m' \frac{a^3}{a'^4}$ is disregarded[1].

Arg. 1. Lubbock's coefficient (viz. the coefficient in R) is

$$-\tfrac{3}{4}\left(1 - \tfrac{5}{2}e^2 - \tfrac{5}{2}e'^2 + \tfrac{23}{16}e^4 + \tfrac{25}{4}e^2e'^2 + \tfrac{13}{16}e'^4 + \tfrac{5}{12}\frac{a^2}{a'^2}\right)\cos^4\tfrac{1}{2}\iota$$

[1] I have been favoured with a note from Sir J. W. Lubbock, confirming my values of the coefficients for the arguments 8, 18, 58, 101 and 123.—Added 15th Feb. 1859, A. C.

which, substituting for $\cos^4 \frac{1}{2}\iota$ its value $1 - \frac{1}{2}\gamma^2 + \frac{7}{16}\gamma^4 -$ &c., and developing to the fourth order, gives

$$-\frac{3}{4}\left(1 - \frac{5}{2}e^2 - \frac{5}{2}e'^2 + \frac{23}{16}e^4 + \frac{25}{4}e^2e'^2 + \frac{13}{16}e'^4 - \frac{1}{2}\gamma^2 + \frac{5}{4}\gamma^2 e^2 + \frac{5}{4}\gamma^2 e'^2 + \frac{7}{16}\gamma^4 + \frac{5}{12}\frac{a^2}{a'^2}\right)$$

which, in fact, agrees with the value given suprà. I have only referred to this term in order to make the reduction.

Arg. 8, Lubbock's coefficient is

$$-\frac{1}{8}\left(1 - \frac{1}{3}e^2 + \frac{3}{2}e'^2 - \frac{3}{2}\gamma^2\right)e^2$$

the exterior sign should be + instead of −. The term is given with the correct sign, Pontécoulant, Arg. 2.

Arg. 18, Lubbock's coefficient is

$$-\frac{51}{8}\left(1 - \frac{5}{2}e^2 - \frac{115}{51}e'^2 - \frac{1}{2}\gamma^2\right)\cos^4 \frac{1}{2}\iota$$

which, developed to γ^2, would be

$$-\frac{51}{8}\left(1 - \frac{5}{2}e^2 - \frac{115}{51}e'^2 - \gamma^2\right).$$

I make it

$$-\frac{51}{8}\left(1 - \frac{5}{2}e^2 - \frac{115}{51}e'^2 - \frac{1}{2}\gamma^2\right).$$

The remaining differences are

No. of Arg. Lubbock.	Lubbock's coefficient.	Coefficient from Development suprà.
58	$+\frac{45}{64}e\,e'^3$	$-\frac{845}{64}e\,e'^3$
59	$+\frac{591}{64}e'^4$	$-\frac{77}{32}e'^4$
60	$-\frac{2453}{128}e'^4$	$-\frac{1599}{64}e'^4$
61	$+\frac{741}{128}e'^4$	$-\frac{1}{32}e'^4$
*62	$-\frac{3}{8}\left(1 - \frac{5}{2}e^2 + \frac{3}{2}e'^2\right)\gamma^2$	$-\frac{3}{8}\left(1 - \frac{5}{2}e^2 + \frac{3}{2}e'^2 - \gamma^2\right)\gamma^2$

No. of Arg. Lubbock.	Lubbock's coefficient.	Coefficient from Development suprà.
*63	$-\frac{3}{8}\left(1+\frac{3}{2}e^2-\frac{5}{2}e'^2+\frac{1}{8}\gamma^2\right)\gamma^2$	$-\frac{3}{8}\left(1+\frac{3}{2}e^2-\frac{5}{2}e'^2-\gamma^2\right)\gamma^2$
95	$-\frac{27}{16}\gamma^2 e'^2$	$-\frac{27}{32}\gamma^2 e'^2$
96	$-\frac{27}{16}\gamma^2 e'^2$	$-\frac{27}{32}\gamma^2 e'^2$
*101	$-\frac{3}{8}\left(1+3e^2+3e'^2-\frac{11}{4}\gamma^2\right)$	$-\frac{3}{8}\left(1+2e^2+2e'^2-\frac{11}{4}\gamma^2\right)$
115	$-\frac{15}{128}\gamma^2$	$-\frac{15}{32}\gamma^2$
123	$-\frac{225}{16}ee'$	$+\frac{225}{16}ee'$

The greater part of the discordant terms do not occur in Pontécoulant's development, which is not carried so far, and the only differences which I find in the coefficients of Pontécoulant's R $(=\Omega)$ are, as regards the arguments 18, 57, 70, corresponding respectively to Lubbock's arguments 62, 63, 101, included in the preceding table, and for which Pontécoulant's coefficients, correcting for the change of sign, correspond with those given by Lubbock. But I see no room for a mistake in the preceding investigation as regards the coefficients of these three terms; the terms in η of the coefficients of 62 and 63 are simply the quantity $(\frac{3}{2}\eta^2-\frac{3}{2}\eta^4)$, which forms the exterior factor of Ω_3 and Ω_4 respectively, and which, putting for η^2 and η^4, their values, is equal to $\frac{3}{8}(1-\gamma^2)\gamma^2$, and as regards the coefficient of 101, the portion $1+2e^2+2e'^2$ of this coefficient is obtained by the mere multiplication of the factors $1+2e^2-$&c. and $1+2e'^2$ set opposite to $g+\mho$ and $0g'-\mho'$ respectively in the table for Ω_6.

2, *Stone Buildings, W.C., 7th July,* 1858.

214.

THE FIRST PART OF A MEMOIR ON THE DEVELOPMENT OF THE DISTURBING FUNCTION IN THE LUNAR AND PLANETARY THEORIES.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXVIII. (1860), pp. 187—215. Read November 10, 1858.]

THE development, as is well known, depends upon that of the reciprocal of the distance of the two planets: and Hansen's Memoir "Entwickelung der negativen und ungeraden Potenzen der Quadratwurzel der Function $r^2 + r'^2 - 2rr'(\cos U \cos U' + \sin U \sin U' \cos J)$," *Abh. der K. Sächs. Ges. zu Leipzig*, t. II., pp. 286—376 (1854), contains a formula which is truly fundamental, viz. the expression of the coefficient of the general term

$$\frac{r^n}{r'^{n+1}} \cos(jU + j'U')$$

of the development of the reciprocal of the distance as expressed in the above-mentioned form, where r, r' are the radius vectors of the inferior and superior planets respectively, and U, U' are the angular distances from the mutual node. In the lunar theory, where the higher powers of $\frac{r}{r'}$ are neglected, we have in this manner a small number of terms each of which is to be separately developed in multiple cosines of the mean anomalies. This can be effected as in the *Fundamenta Nova*, and my "Memoir on the Development of the Disturbing Function in the Lunar Theory," *R. Ast. Soc. Mem.*, t. XXVII., 1859, [213], which is a mere completion of Hansen's process[1]. In fact if f, f' are the true anomalies, and $\mho$, $\mho'$ the distances

[1] I take the opportunity of mentioning the memoir of Hansen's which immediately precedes that above referred to, viz. "Entwickelung des Products einer Potenz des Radius Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie in Reihen die nach den Sinussen oder Cosinussen der Vielfachen der wahren der excentrischen oder mittleren Anomalie fortschreiten," t. II., pp. 183—281 (1853).

of the pericentres from the mutual node, then we have $U = f + \varpi$, $U' = f' + \varpi'$, and the general term is

$$\frac{r^n}{r'^{n+1}} \cos(jf + j'f' + j\varpi + j'\varpi')$$

where r, f are given functions of the mean anomaly g, and r', f' are the like functions of the mean anomaly g'. And the development depends upon those of

$$r^n \begin{matrix}\cos\\ \sin\end{matrix} jf, \qquad r'^{-n-1} \begin{matrix}\cos\\ \sin\end{matrix} j'f',$$

which (if we consider as well negative as positive values of the index n) are each of the form

$$r^n \begin{matrix}\cos\\ \sin\end{matrix} jf,$$

and when the developments of these expressions are known, we obtain at once by the mere addition and subtraction of the coefficients of the cosines and sines of the different multiples of g and g', the development of

$$\frac{r^n}{r'^{n+1}} \cos(jf + j'f' + j\varpi + j'\varpi')$$

in the tabular form employed in my memoir just referred to. In the planetary theory we must unite together the terms containing the different powers of $\frac{r}{r'}$ so as to form the entire coefficient $D(j, j')$ of $\cos(jU + j'U')$; if then we write

$$r = a(1 + x), \; r' = a'(1 + x'),$$

and develope the coefficient in powers of x, x' we have the general term

$$x^a x'^a \cos(jU + j'U')$$

which admits of development in multiple cosines of the mean anomalies, in the same manner precisely as the before-mentioned general term

$$\frac{r^n}{r'^{n+1}} \cos(jU + j'U')$$

in the lunar theory. It is proper to remark that this method is really identical with that commonly made use of in the planetary theory: the only difference is, that by Hansen's fundamental formula, we have the complete expression of the coefficient $D(j, j')$ developed in powers of sin or of $\tan \frac{1}{2}\phi$, instead of (as in the ordinary methods) the first two or three terms of this development.

The required development of

$$x^{\alpha} x'^{\alpha'} \cos(jU + j'U')$$

or, what is the same thing,

$$x^{\alpha} x'^{\alpha'} \cos(jf + j'f' + j\varpi + j'\varpi')$$

depends on the developments of

$$x^{\alpha} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf, \qquad x'^{\alpha'} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j'f'.$$

These are functions of the same form, and we may consider only

$$x^{\alpha} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf.$$

The value of x is $\left(\frac{r}{a} - 1\right)$ and we could of course calculate

$$\left(\frac{r}{a} - 1\right)^{\alpha} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf$$

by the methods of the *Fundamenta Nova* or the memoir of Hansen's referred to in the foot-note. But if we write $f = g + y$ (y is the equation of the centre), then the required expressions depend on

$$x^{\alpha} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jy$$

which are actually calculated as far as e^7 for $\alpha = 0, 1, 2 \ldots 7$, and j an undetermined symbol, by Leverrier in the *Annales de l'Observatoire de Paris*, t. I. pp. 346—348 (1855). Hence, by the mere substitution, in Leverrier's formula, of the numerical values of j and j', and by the addition and subtraction of the coefficients of the cosines or sines of the different multiples of g and g', we may obtain the development of

$$x^{\alpha} x'^{\alpha'} \cos(jf + j'f' + j\varpi + j'\varpi')$$

in a tabular form similar to that employed in my memoir already referred to.

I have thought it desirable to put together the various results above referred to, and to investigate by a different process the expression for Hansen's coefficient, which in his memoir is obtained by means of a long series of transformations which it is not very easy to follow, and is not exhibited in quite the most simple form. And this is what I have done in the present first part of a memoir on the development of the disturbing function. My object has been to exhibit, in as complete a form as possible, the preliminary development in multiple cosines of the true anomalies; and to indicate the process of the ulterior development in multiple cosines of the mean anomalies.

I.

Let $\mathfrak{M}$ be the inferior, $\mathfrak{M}'$ the superior of the two planets (in the lunar theory $\mathfrak{M}$ is the moon, $\mathfrak{M}'$ the sun) and let the quantities relating to the two bodies be, for $\mathfrak{M}$,

r, the radius-vector,
U, the distance from node,
$\mho$, the distance of pericentre from node,
f, the true anomaly,
g, the mean anomaly,
a, the mean distance,
e, the eccentricity,

and for $\mathfrak{M}'$ the accented letters, r', &c., in the like significations.

The node referred to is the ascending node of the orbit of $\mathfrak{M}$ upon that of $\mathfrak{M}'$, and I write also

Φ, the inclination of the orbit of $\mathfrak{M}$ to that of $\mathfrak{M}'$,
η, $=\sin\frac{1}{2}\Phi$.

The disturbing function is in the first instance given as a function of r, r', H, where

$$\cos H = \cos U \cos U' + \sin U \sin U' \cos \Phi,$$

that is, as a function of r, r', U, U', Φ, or, what is the same thing, of r, r', U, U', η. And the preliminary development is a development in multiple cosines of U, U'. We have then

$$U = f + \mho,$$
$$U' = f' + \mho',$$

and finally

$$r = a\,\mathrm{elqr}(e, g),$$
$$f = \mathrm{elta}(e, g),$$
$$r' = a'\,\mathrm{elqr}(e', g'),$$
$$f' = \mathrm{elta}(e', g'),$$

and the ulterior development is a development in multiple cosines of g, g', $\mho$, $\mho'$, the coefficients involving, as before, η, and also a, e, a', e'. But as usual it is not attempted to carry the development further, by introducing in the coefficients in place of the relative quantities $\mho$, $\mho'$, η, the remaining elements of the two orbits, which, if it were necessary to use them, would be

for $\mathfrak{M}$,

θ, the longitude of node,
σ, the departure of node,
ϕ, the inclination,
ϖ, the departure of pericentre,

and for $\mathfrak{M}'$, the like accented quantities, the orbit of reference being any fixed or moveable orbit whatever.

II.

The expression for the disturbing function on $\mathfrak{M}$ (that is, when the superior planet disturbs the inferior) is

$$\mathfrak{M}'\left\{-\frac{r\cos H}{r'^2}+\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}\right\}$$

and that for the disturbing function on $\mathfrak{M}'$ (that is, when the inferior planet disturbs the superior) is

$$\mathfrak{M}\left\{-\frac{r'\cos H}{r^2}+\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}\right\}$$

where the disturbing function is taken with Lagrange's sign ($=-R$, if R be the disturbing function of the *Mécanique Céleste*).

But we may in the first instance consider the development of the reciprocal of the distance of the two planets

$$=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}.$$

The preceding expression for $\cos H$ may be written

$$\cos H=\cos U\cos U'+\sin U\sin U'(1-2\eta^2),$$

or in either of the two forms

$$\cos H=\cos(U-U')-2\eta^2\sin U\sin U',$$

$$\cos H=(1-\eta^2)\cos(U-U')+\eta^2\cos(U+U').$$

Now imagine the function developed in ascending powers of $\frac{r}{r'}$, the coefficient of $\frac{r^n}{r'^{n+1}}$ will contain $\cos^n H$, $\cos^{n-2} H,\ldots$ to $\cos H$ or 1 according as n is even or odd; and if we then substitute for $\cos H$ the last given expression, and express the different powers of $\cos H$ in multiple cosines of $U-U'$ and $U+U'$, and make the final expression contain the cosines of opposite arguments each with the same coefficient, it is easy to see that the form of the general term is

$$\frac{r^n}{r'^{n+1}}C_n(j,j')\cos(jU+j'U'),$$

where j, j', each of them extend through the values n, $n-2,\ldots-n$, and where

$$C_n(-j,-j')=C_n(j,j').$$

Thus in particular for $n=0, 1, 2, 3$, the combinations (j, j') belonging to the several arguments are,

0, 0

,

1, 1	−1, 1
1, −1	−1, −1

,

2, 2	0, 2	−2, 2
2, 0	0, 0	−2, 0
2, −2	0, −2	−2, −2

,

3, 3	1, 3	−1, 3	−3, 3
3, 1	1, 1	−1, 1	−3, 1
3, −1	1, −1	−1, −1	−3, −1
3, −3	1, −3	−1, −3	−3, −3

.

But as the coefficients C satisfy the condition $C(-j, -j') = C(j, j')$, the two terms $C(j, j')\cos(jU+j'U')$ and $C(-j, -j')\cos(-jU-j'U')$ are equal to each other, and they may be combined together into a single term. The general term may consequently be written

$$\frac{r^n}{r'^{n+1}}\, 2C_n(j, j')\cos(jU+j'U'),$$

where j has only the values $n, n-2, \ldots 1$ or 0, and j' has as before the values $n, n-2, \ldots -n$; except (which occurs only when n is even) for $j=0$, when j' has only the values $n, n-2, \ldots 0$: and in the particular case $j=j'=0$, the last-mentioned expression for the general term must be multiplied by $\frac{1}{2}$, or, what is the same thing, the factor 2 must be omitted. In particular for $n=0, 1, 2, 3, 4$, the combinations (j, j') belonging to the several arguments are

0, 0

,

1, 1	
1, −1	

,

2, 2	0, 2	
2, 0	0, 0	
2, −2		

,

3, 3	1, 3		
3, 1	1, 1		
3, −1	1, −1		
3, −3	1, −3		

,

4, 4	2, 4	0, 4		
4, 2	2, 2	0, 2		
4, 0	2, 0	0, 0		
4, −2	2, −2			
4, −4	2, −4			

I remark that in a series $K(j, j')\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jU+j'U'$, where each argument occurs positively and negatively, and $K(-j, -j') = \pm K(j, j')$ according as the series is one of cosines

or of sines, we may say that the *discrete* general term is $K(j, j')\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(jU+j'U')$, or that $K(j, j')$ is the discrete coefficient of the cosine or sine, but if we unite together the terms with opposite arguments so as to form the general term $2K(j, j')\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(jU+j'U')$, then this may be called the *concrete* general term, and $2K(j, j')$ may be said to be the concrete coefficient of the cosine or sine. In a sine series the term corresponding to the argument zero vanishes; in a cosine series this is not in general the case, and the concrete term corresponding to the argument zero must be multiplied by $\frac{1}{2}$.

Returning to the question in hand, from the symmetry of the expression to be developed, we have

$$C_n(j, j') = C_n(j', j),$$

and it follows that the only coefficients which need be calculated are those for which j is not negative, and not less in absolute magnitude than j'; the remaining coefficients are respectively equal to coefficients which satisfy these conditions. Thus for $n = 0, 1, 2, 3, 4$, the combinations (j, j') corresponding to the coefficients in question are,

0, 0

,

1, 1	
1, −1	

,

2, 2		
2, 0	0, 0	
2, −2		

,

3, 3			
3, 1	1, 1		
3, −1	1, −1		
3, −3			

,

4, 4				
4, 2	2, 2			
4, 0	2, 0	0, 0		
4, −2	2, −2			
4, −4	2, −4			

;

and we have, for instance, $C(2, 4) = C(4, 2)$, $C(-2, -4) = C(2, 4) = C(4, 2)$, &c. Under the preceding restriction, viz. j not negative, and not less in absolute magnitude than j', the expression for $C_n(j, j')$ (deduced from the formulæ of Hansen's Memoir) is as follows; viz. putting as usual $\Pi x = 1.2.3\ldots x$, and also $\Pi_1(x-\frac{1}{2}) = \frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots(x-\frac{1}{2})$, and representing the hypergeometric series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha.\alpha+1.\beta.\beta+1}{1.\ 2.\ \gamma.\gamma+1}x^2 + \&c.$$

by $F(\alpha, \beta, \gamma, x)$, we have

$$C(j, j') = \frac{2^{j+j'}\,\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(n-j)\,\Pi\,\frac{1}{2}(n-j')\,\Pi\,(j+j')}$$
$$\times\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2)$$

which, it is to be noticed, is a rational and integral function of η, the highest power being η^{2n}, and the lowest power or order of the coefficient being $\eta^{j+j'}$. I reserve the demonstration of this formula for a separate paragraph.

Let the discrete general term involving $\cos(jU+j'U')$ be represented by

$$D(j, j')\cos(jU+j'U');$$

we have, in like manner as for the coefficients C, $D(-j, -j') = D(j, j')$, $D(j', j) = D(j, j')$, and consequently $D(j, j')$ will be known, if we know its value when the before-mentioned conditions are satisfied; viz., if j be not negative and not less in absolute magnitude than j'; and collecting together the different terms which involve the cosine in question, we find at once, the conditions being satisfied,

$$D(j, j') = \frac{\eta^j}{\eta'^{j+1}} C_j(j, j') + \frac{\eta^{j+2}}{\eta'^{j+3}} C_{j+2}(j, j') + \&c....$$

so that the value of $D(j, j')$ is known. A transformation of this expression will be given in the sequel.

III.

The *concrete* general term is

$$2D(j, j')\cos(jU+j'U');$$

in particular the concrete terms involving the arguments $U-U'$, $U+U'$, are

$$2D(1, -1)\cos(U-U'),$$
$$2D(1, \quad 1)\cos(U+U'),$$

or, substituting for the D coefficients their values, the terms are

$$2\left\{\frac{r}{r'^2} C_1(1, -1) + \frac{r^3}{r'^4} C_3(1, -1) + \ldots\right\}\cos(U-U'),$$

$$2\left\{\frac{r}{r'^2} C_1(1, \quad 1) + \frac{r^3}{r'^4} C_3(1, \quad 1) + \ldots\right\}\cos(U+U').$$

So far I have considered the reciprocal of the radius vector; but if we consider, instead, the disturbing functions, the only difference will be that we must add the term

$$-\frac{r}{r'^2}\cos H,$$

or

$$-\frac{r}{\eta^2}\cos H,$$

according as the superior planet disturbs the inferior one, or the inferior planet the superior one. The value of $\cos H$ is

$$= \quad (1-\eta^2)\cos(U-U') + \quad \eta^2 \quad \cos(U+U'),$$

$$= 2\,C_1(1,-1)\cos(U-U') + 2\,C_1(1,\ 1)\cos(U+U'),$$

observing that

$$C_1(1,-1) = \tfrac{1}{2}(1-\eta^2), \qquad C_1(1,\ 1) = \tfrac{1}{2}\eta^2;$$

and the terms to be added are consequently, when the superior planet disturbs the inferior,

$$-2\frac{r}{r'^2}C_1(1,-1)\cos(U-U')$$

$$-2\frac{r}{r'^2}C_1(1,\ \ 1)\cos(U+U'),$$

the effect of which is simply to destroy the same terms contained with the opposite sign in the reciprocal of the distance;

and when the inferior planet disturbs the superior one, the terms to be added are

$$-2\frac{r'}{r^2}C_1(1,-1)\cos(U-U')$$

$$-2\frac{r'}{r^2}C_1(1,\ \ 1)\cos(U+U'),$$

which are not equal to any terms in the reciprocal of the distance. We may write as follows:

Reciprocal of Distance is,

Disturbing function ÷ Mass of Disturbing Planet is,

		U	$+U'$
$D(0, 0)$	cos	0	0
$+2D(1, -1)$		1	−1
(1) $\left[-2\frac{r}{r'^2}C(1, -1)\right]$		1	−1
(2) $\left[-2\frac{r'}{r^2}C(1, -1)\right]$		1	−1
$+2D(1, 1)$		1	1
(1) $\left[-2\frac{r}{r'^2}C(1, 1)\right]$		1	1
(2) $\left[-2\frac{r'}{r^2}C(1, 1)\right]$		1	1
$+2D(2, -2)$		2	−2
$+2D(2, 0)$		2	0
$+2D(0, 2)$		0	2
$+2D(2, 2)$		2	2
$+2D(3, -3)$		3	−3
$+2D(3, -1)$		3	−1
$+2D(1, -3)$		1	−3
$+2D(3, 1)$		3	1
$+2D(1, 3)$		1	3
$+2D(3, 3)$		3	3
$+2D(4, -4)$		4	−4
$+2D(4, -2)$		4	−2
$+2D(2, -4)$		2	−4
$+2D(4, 0)$		4	0
$+2D(0, 4)$		0	4
$+2D(4, 2)$		4	2
$+2D(2, 4)$		2	4
$+2D(4, 4)$		4	4
+ &c., &c.			

[1] Only to be inserted for the disturbing function when the superior planet disturbs the inferior, and having the effect of destroying portions of the coefficients of the next preceding terms.

[2] Only to be inserted for the disturbing function when the inferior planet disturbs the superior one.

For the lunar theory, to the extent to which it is necessary to carry the development, and to which it is carried in Hansen's *Fundamenta Nova*, and my memoir before referred to, we might simply write,

Disturbing function ÷ Sun's Mass is

			U	$+U'$
	$\frac{1}{r'} C_0 (0, \quad 0)$	cos	0	0
(1)	$+ \frac{r^2}{r'^3} C_2 (0, \quad 0)$		0	0
(6)	$+ 2 \frac{r^3}{r'^4} C_3 (1, -1)$		1	-1
(8)	$+ 2 \frac{r^3}{r'^4} C_3 (1, \quad 1)$		1	1
(2)	$+ 2 \frac{r^2}{r'^3} C_2 (2, -2)$		2	-2
(3)	$+ 2 \frac{r^2}{r'^3} C_2 (2, \quad 0)$		2	0
(4)	$+ 2 \frac{r^2}{r'^3} C_2 (0, \quad 2)$		0	2
(5)	$+ 2 \frac{r^2}{r'^3} C_2 (2, \quad 2)$		2	2
(7)	$+ 2 \frac{r^3}{r'^4} C_3 (3, \quad 3)$		3	-3
(9)	$+ 2 \frac{r^3}{r'^4} C_3 (3, -1)$		3	-1
(10)	$+ 2 \frac{r^3}{r'^4} C_3 (1, -3)$		1	-3
	$+ 2 \frac{r^3}{r'^4} C_3 (3, \quad 1)$		3	1
	$+ 2 \frac{r^3}{r'^4} C_3 (1, \quad 3)$		1	3

where the prefixed numbers are those of Hansen's ten parts $\Omega_1, \Omega_2 \ldots \Omega_{10}$; or omitting the first term which depends only on the sun's radius vector, and the last two terms which are ultimately neglected, and substituting for the coefficients C their values we have,

Disturbing function ÷ Sun's Mass is

				U	$+U'$
(1)	$\frac{1}{4} - \frac{3}{2}\eta^2 + \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$	cos	0	0
(2)	$\frac{3}{4} - \frac{3}{2}\eta^2 + \frac{3}{4}\eta^4$	$\frac{r^2}{r'^3}$		2	− 2
(3)	$\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$		2	0
(4)	$\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$		0	2
(5)	$\frac{3}{4}\eta^4$	$\frac{r^3}{r'^4}$		2	2
(6)	$\frac{3}{8} - \frac{33}{8}\eta^2 + \frac{75}{8}\eta^4$	$\frac{r^3}{r'^4}$		1	− 1
(7)	$\frac{5}{8} - \frac{15}{8}\eta^2 + \frac{15}{8}\eta^4$	$\frac{r^3}{r'^4}$		3	− 3
(8)	$\frac{9}{4}\eta^2 - \frac{15}{2}\eta^4$	$\frac{r^3}{r'^4}$		1	1
(9)	$\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4$	$\frac{r^3}{r'^4}$		3	− 1
(10)	$\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4$	$\frac{r^3}{r'^4}$		1	− 3

If the same form be adopted for the planetary theory, the expressions for the leading coefficients will be,

$C_0(0, 0)$, $\quad D(0, 0) = \frac{1}{r'} \quad 1$

2 $\quad + \frac{r^2}{r'^3} \quad \frac{1}{4}(1 - 6\eta^2 + 6\eta^4)$

4 $\quad + \frac{r^4}{r'^5} \quad \frac{9}{64}(1 - 20\eta^2 + 90\eta^4 - 140\eta^6 + 70\eta^8)$

6 $\quad + \frac{r^6}{r'^7} \quad \frac{25}{256}(1 - 42\eta^2 + 420\eta^4 - 1680\eta^6 + 3150\eta^8 \ldots)$

8 $\quad + \frac{r^8}{r'^9} \quad \frac{1225}{16384}(1 - 72\eta^2 + 1260\eta^4 - 9240\eta^6 - 34650\eta^8 \ldots)$

⋮

$2p \quad + \frac{r^{2p}}{r'^{2p+1}} \frac{\Pi_1(p-\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi p\,\Pi p} F(-2p, 2p+1, 1, \eta^2);$

$C_1(1, -1)$, $\quad D(1, -1) = \dfrac{r}{r'^2} \quad \frac{1}{2}(1-\eta^2)$

3 $\quad + \dfrac{r^3}{r'^4} \quad \frac{3}{16}(1-\eta^2)(1-10\eta^2 + 15\eta^4)$

5 $\quad + \dfrac{r^5}{r'^6} \quad \frac{15}{128}(1-\eta^2)(1-28\eta^2 + 168\eta^4 - 330\eta^6 + 210\eta^8)$

7 $\quad + \dfrac{r^7}{r'^8} \quad \frac{175}{2048}(1-\eta^2)(1-54\eta^2 + 675\eta^4 - 3100\eta^6 + 7425\eta^8)$

$\vdots$

$2p+1 \quad + \dfrac{r^{2p+1}}{r'^{2p+2}} \dfrac{\Pi_1(p+\frac{1}{2})\Pi_1(p-\frac{1}{2})}{\Pi(p+1)\Pi p}(1-\eta^2)F(-2p, 2p+3, 1, \eta^2);$

$C_1(1, 1)$, $\quad D(1, 1) = \dfrac{r}{r'^2} \quad \frac{1}{2}\eta^2$

3 $\quad + \dfrac{r^3}{r'^4} \quad \frac{9}{8}\eta^2(1-\frac{10}{3}\eta^2 + \frac{5}{3}\eta^4)$

5 $\quad + \dfrac{r^5}{r'^6} \quad \frac{225}{128}\eta^2(1-\frac{28}{3}\eta^2 + 28\eta^4 - \frac{168}{5}\eta^6)$

7 $\quad + \dfrac{r^7}{r'^8} \quad \frac{1225}{512}\eta^2(1-18\eta^2 + \frac{225}{2}\eta^4 - 330\eta^6)$

$\vdots$

$2p+1 \quad + \dfrac{r^{2p+1}}{r'^{2p+2}} \; 2\dfrac{\Pi_1(p+\frac{1}{2})\Pi_1(p+\frac{1}{2})}{\Pi p\,\Pi p}\eta^2 F(-2p, 2p+3, 3, \eta^2);$

$C_2(2, -2)$, $\quad D(2, -2) = \dfrac{r^2}{r'^2} \quad \frac{3}{8}(1-\eta^2)^2$

4 $\quad + \dfrac{r^4}{r'^5} \quad \frac{5}{32}(1-\eta^2)^2(1-14\eta^2 + 28\eta^4)$

6 $\quad + \dfrac{r^6}{r'^7} \quad \frac{105}{1024}(1-\eta^2)^2(1-36\eta^2 + 270\eta^4 - 660\eta^6 + 495\eta^8)$

$\vdots$

$2p \quad + \dfrac{r^{2p}}{r'^{2p+1}} \dfrac{\Pi_1(p+\frac{1}{2})\Pi_1(p-\frac{3}{2})}{\Pi(p+1)\Pi(p-1)}(1-\eta^2)^2 F(-2p+2, 2p+3, 1, \eta^2);$

$C_2(2, 0)$, $\quad D(2, 0) = \dfrac{r^2}{r'^3} \quad \frac{3}{4}\eta^2(1-\eta^2)$

4 $\quad + \dfrac{r^4}{r'^5} \quad \frac{45}{32}\eta^2(1-\eta^2)(1-\frac{14}{3}\eta^2 + \frac{14}{3}\eta^4)$

6 $\quad + \dfrac{r^6}{r'^7} \quad \frac{525}{256}\eta^2(1-\eta^2)(1-12\eta^2 + 45\eta^4 - 66\eta^6 \ldots)$

$\vdots$

$2p \quad + \dfrac{r^{2p}}{r'^{2p+1}} \; 2\dfrac{\Pi_1(p+\frac{1}{2})\Pi_1(p-\frac{1}{2})}{\Pi p\,\Pi(p-1)}\eta^2(1-\eta^2)F(-2p+2, 2p+3, 3, \eta^2);$

$C_n(0,\ 2) = C_n(2,\ 0), \qquad D(0,\ 2) = D(2,\ 0);$

$C_2(2,\ 2), \qquad D(2,\ 2) = \dfrac{r^2}{r'^3}\ \tfrac{3}{8}\eta^4$

$4 \qquad + \dfrac{r^4}{r'^5}\ \tfrac{75}{32}\eta^4(1 - \tfrac{14}{5}\eta^2 + \tfrac{28}{15}\eta^4)$

$6 \qquad + \dfrac{r^5}{r'^6}\ \tfrac{3675}{512}\eta^4(1 - \tfrac{36}{5}\eta^2 + 18\eta^4)$

$\vdots$

$2p \qquad + \dfrac{r^{2p}}{r'^{2p+1}}\ \tfrac{2}{3}\ \dfrac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p+\frac{1}{2})}{\Pi(p-1)\,\Pi(p-1)}\eta^4 F(-2p+2,\ 2p+3,\ 5,\ \eta^2);$

$\vdots$

$C_3(3,\ -3), \qquad D(3,\ -3) = \dfrac{r^3}{r'^4}\ \tfrac{5}{16}(1-\eta^2)^3$

$5 \qquad + \dfrac{r^5}{r'^6}\ \tfrac{35}{256}(1-\eta^2)^3(1 - 18\eta^2 + 45\eta^4)$

$\vdots$

$2p+1 \qquad + \dfrac{r^{2p+1}}{r'^{2p+2}}\ \dfrac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p-\frac{3}{2})}{\Pi(p+2)\,\Pi(p-1)}(1-\eta^2)^3 F(-2p+2,\ 2p+5,\ 1,\ \eta^2);$

$C_3(3,\ -1), \qquad D(3,\ -1) = \dfrac{r^3}{r'^4}\ \tfrac{15}{16}\eta^2(1-\eta^2)^2$

$5 \qquad + \dfrac{r^5}{r'^6}\ \tfrac{105}{64}\eta^2(1-\eta^2)^2(1 - 6\eta^2 + \tfrac{15}{2}\eta^4)$

$\vdots$

$2p+1 \qquad + \dfrac{r^{2p+1}}{r'^{2p+2}}\ 2\,\dfrac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi(p+1)\,\Pi(p-1)}\eta^2(1-\eta^2)^2 F(-2p+2,\ 2p+5,\ 3,\ \eta^2);$

$C_n(1,\ -3) = C_n(3,\ -1), \qquad D(1,\ -3) = D(3,\ -1);$

$C_3(3,\ 1), \qquad D(3,\ 1) = \dfrac{r^3}{r'^4}\ \tfrac{15}{16}\eta^4(1-\eta^2)$

$5 \qquad + \dfrac{r^5}{r'^6}\ \tfrac{525}{128}\eta^4(1-\eta^2)(1 - \tfrac{18}{5}\eta^2 + 3\eta^4)$

$\vdots$

$2p+1 \qquad - \dfrac{r^{2p+1}}{r'^{2p+2}}\ \tfrac{2}{3}\ \dfrac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p+\frac{1}{2})}{\Pi p\ \Pi(p-1)}\eta^4(1-\eta^2) F(-2p+2,\ 2p+5,\ 5,\ \eta^2);$

$C_n(1,\ 3) = C_n(3,\ 1), \qquad D(1,\ 3) = D(3,\ 1);$

$C_3(3,\ 3), \qquad D(3,\ 3) = \dfrac{r^3}{r'^4}\ \tfrac{5}{16}\eta^6$

$5 \qquad + \dfrac{r^5}{r'^6}\ \tfrac{245}{64}\eta^6(1 + \tfrac{18}{7}\eta^2 \ldots)$

$\vdots$

$2p+1 \qquad + \dfrac{r^{2p+1}}{r'^{2p+2}}\ \tfrac{4}{45}\ \dfrac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p+\frac{3}{2})}{\Pi(p-1)\,\Pi(p-1)}\eta^6 F(-2p+2,\ 2p+5,\ 7,\ \eta^2);$ &c.

IV.

But for the planetary theory it is more suitable to arrange the expression of $D(j, j')$ according to the powers of η. We have, under the before-mentioned conditions, j not negative and not less in absolute magnitude than j',

$$D(j, j') = \Sigma \frac{r^{j+2\lambda}}{r'^{j+2\lambda+1}} C_{j+2\lambda}(j, j')$$

$$= \frac{r^j}{r'^{j+1}} \Sigma \left(\frac{r}{r'}\right)^{2\lambda} C_{j+2\lambda}(j, j')$$

where λ extends from 0 to ∞; and, writing in the expression of $C_n(j, j')$, $j+2\lambda$ for n, we find

$$C_{j+2\lambda}(j, j') = \frac{2^{j+j'} \Pi(j+\lambda-\frac{1}{2}) \Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})}{\Pi\lambda \Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+j')} F(-2\lambda, 2j+2\lambda+1, j+j'+1, \eta^2),$$

or, substituting for the hypergeometric series its value,

$$C_{j+2\lambda}(j, j') = \Sigma_\theta \frac{2^{j+j'} \Pi_1(j+\lambda-\frac{1}{2}) \Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})}{\Pi\lambda \Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+j')} \frac{(-)^\theta [2\lambda]^\theta [2j+2\lambda+\theta]^\theta}{[\theta]^\theta [j+j'+\theta]^\theta} \eta^{2\theta}$$

$$= \Sigma_\theta \frac{(-)^\theta 2^{j+j'} \Pi_1(j+\lambda-\frac{1}{2}) \Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2}) \Pi 2\lambda \Pi(2j+2\lambda+\theta)}{\Pi\lambda \Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+j'+\theta) \Pi\theta \Pi(2\lambda-\theta) \Pi(2j+2\lambda)} \eta^{2\theta}$$

from $\theta=0$ to $\theta=\infty$. Substituting in the numerator for $\Pi 2\lambda$, $2^{2\lambda} \Pi\lambda \Pi_1(\lambda-\frac{1}{2})$, and in the denominator for $\Pi(2j+2\lambda)$, $2^{2j+2\lambda} \Pi(j+\lambda) \Pi_1(j+\lambda-\frac{1}{2})$, and reducing, this becomes

$$C_{j+2\lambda}(j, j') = \Sigma_\theta \frac{(-)^\theta \eta^{2\theta}}{\Pi(j+j'+\theta) \Pi\theta} \frac{1}{2^{j-j'}} \frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2}) \Pi(2j+2\lambda+\theta) \Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+\lambda) \Pi(2\lambda-\theta)},$$

and substituting this expression in $D(j, j')$, we find

$$D(j, j') = \frac{1}{2^{j-j'}} \eta^{j+j'} (1-\eta^2)^{\frac{1}{2}(j-j')} \frac{r^j}{r'^{j+1}} \Sigma_\theta \frac{(-)^\theta \eta^{2\theta}}{\Pi(j+j'+\theta) \Pi\theta}$$

$$\Sigma_\lambda \left(\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2}) \Pi(2j+2\lambda+\theta) \Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+\lambda) \Pi(2\lambda-\theta)} \left(\frac{r}{r'}\right)^{2\lambda} \right)$$

from $\lambda=0$ to $\lambda=\infty$ and $\theta=0$ to $\theta=\infty$; which is the required expression. It is to be remarked that the series in $\left(\frac{r}{r'}\right)$ which multiplies $\eta^{2\theta}$ is not in general expressible as a hypergeometric series. If, however, we attend only to the leading term in η, or write $\theta=0$, we find for $D(j, j')$ the value

$$\frac{1}{2^{j-j'}} \eta^{j+j'} (1-\eta^2)^{\frac{1}{2}(j-j')} \frac{r^j}{r'^{j+1}} \frac{1}{\Pi(j+j')}$$

$$\Sigma_\lambda \left(\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2}) \Pi(2j+2\lambda) \Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda) \Pi(j+\lambda) \Pi 2\lambda} \left(\frac{r}{r'}\right)^{2\lambda} \right)$$

which may be simplified by putting in the numerator for $\Pi(2j+2\lambda)$,

$$2^{2j+2\lambda}\,\Pi(j+\lambda)\,\Pi_1(j+\lambda-\tfrac{1}{2}),$$

and in the denominator for $\Pi(j+j')$,

$$2^{j+j'}\,\Pi\,\tfrac{1}{2}(j+j')\,\Pi_1(\tfrac{1}{2}(j+j')-\tfrac{1}{2}),$$

and for $\Pi\,2\lambda$, $2^{2\lambda}\,\Pi\lambda\,\Pi_1(\lambda-\tfrac{1}{2})$. We thus obtain the value

$$\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{1}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,(\frac{1}{2}(j+j')-\frac{1}{2})}$$

$$\Sigma_\lambda\,\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi_1(j+\lambda-\frac{1}{2})}{\Pi\,(\frac{1}{2}(j-j')+\lambda)\,\Pi\lambda}\left(\frac{r}{r'}\right)^{2\lambda},$$

which is equal to

$$\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,\frac{1}{2}(j-j')}$$

$$\Sigma_\lambda\,\frac{[\frac{1}{2}(j+j')+\lambda-\frac{1}{2}]^\lambda\,[j+\lambda-\frac{1}{2}]^\lambda}{[\frac{1}{2}(j-j')+\lambda]^\lambda\,[\lambda]^\lambda}\left(\frac{r}{r'}\right)^{2\lambda},$$

or finally we have, as regards the leading term in η,

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,\frac{1}{2}(j-j')}$$

$$F\left(\tfrac{1}{2}(j+j')+\tfrac{1}{2},\ j+\tfrac{1}{2},\ \tfrac{1}{2}(j-j')+1,\ \frac{r^2}{r'^2}\right).$$

I remark that if in general

$$r^x r'^x(r^2+r'^2-2rr'\cos\vartheta)^{-x-\frac{1}{2}}=\Sigma R_x^{\,i}\cos i\vartheta,\quad R_x^{\,-i}=R_x^{\,i}$$

then, writing $\frac{1}{2}(j-j')$ for i and $\frac{1}{2}(j+j')$ for x, we have

$$R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')}=\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})\,\Pi\,\frac{1}{2}(j-j')}\,F\left(\tfrac{1}{2}(j+j')+\tfrac{1}{2},\ j+\tfrac{1}{2},\ \tfrac{1}{2}(j-j')+1,\ \frac{r^2}{r'^2}\right)$$

and the last-mentioned expression for $D(j,j')$, as regards the leading term in η, becomes therefore

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')}\,R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')},$$

and we have in general

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\left\{\frac{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')}\,R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')}+\text{terms in }\eta\right\}.$$

As a verification, it may be noticed that for $j+j'=0$ we have $D(j,-j')=(1-\eta^2)^j(R_0^{\,j}+\text{terms in }\eta)$, and for $\eta=0$, $D(j,-j)=R_0^{\,j}$, which is right, since the two sides each denote the coefficient of $\cos j\vartheta$ in $(r^2+r'^2-2rr'\cos\vartheta)^{-\frac{1}{2}}$. There is reason to believe that the expression for $D(j,j')$ might be further reduced so as to obtain in a convenient form the coefficients of the successive powers of η; but I have not yet accomplished this.

V.

Considering now $D(j, j')$ as a given function of r and r', we must in the planetary theory write $r = a(1+x)$, $r' = a'(1+x')$, and develope in powers of x and x'. The general term is, of course,

$$\frac{a^\alpha a'^{\alpha'}}{\Pi\alpha\,\Pi\alpha'} d_\alpha{}^\alpha d_{\alpha'}{}^{\alpha'} \overline{D(j, j')}\, x^\alpha x'^{\alpha'}$$

where $\overline{D(j, j')}$ is what $D(j, j')$ becomes when a, a' are substituted for r, r'; and, writing $f + \mho$, $f' + \mho'$, for U, U', we see that in the planetary theory the terms to be developed are of the form

$$x^\alpha x'^{\alpha'} \cos(jf + j'f' + j\mho + j'\mho'),$$

while, from what has preceded, the form for the lunar theory is

$$r^n r'^{n'} \cos(jf + j'f' + j\mho + j'\mho'),$$

$$\left(n' \text{ is always } = -n-1, \text{ except in the terms } \frac{r'}{r^2}\cos(U \pm U')\right)$$

the values which have to be substituted being

$$\begin{aligned} r &= a\ \mathrm{elqr}(e, g), & x &= \mathrm{elqr}(e, g) - 1, \\ f &= \ \ \mathrm{elta}(e, g), & & \\ r' &= a'\ \mathrm{elqr}(e', g'), & x' &= \mathrm{elqr}(e', g') - 1, \\ f' &= \ \ \mathrm{elta}(e', g'). & & \end{aligned}$$

Suppose that in the former case the developments of

$$x^\alpha \cos jf, \quad x^\alpha \sin jf, \quad x'^{\alpha'} \cos j'f', \quad x'^{\alpha'} \sin j'f'$$

and in the latter case the developments of

$$\left(\frac{r}{a}\right)^n \cos jf, \quad \left(\frac{r}{a}\right)^n \sin jf, \quad \left(\frac{r'}{a'}\right)^{n'} \cos j'f', \quad \left(\frac{r'}{a'}\right)^{n'} \sin j'f'$$

are

$$\Sigma\,[\cos]^i \cos ig, \quad \Sigma\,[\sin]^i \sin ig, \quad \Sigma\,[\cos]^{i'} \cos i'g', \quad \Sigma\,[\sin]^{i'} \sin i'g',$$

where the summations extend from i or $i' = -\infty$ to ∞, and where the coefficients $[\cos]^i$, $[\sin]^i$ satisfy

$$[\cos]^{-i} = [\cos]^i, \quad [\sin]^{-i} = -[\sin]^i,$$

and in like manner the coefficients $[\cos]^{i'}$, $[\sin]^{i'}$ satisfy

$$[\cos]^{-i'} = [\cos]^{i'}, \quad [\sin]^{-i'} = -[\sin]^{i'}.$$

It is to be observed that $[\cos]^i$, $[\sin]^i$ are functions of e, and $[\cos]^{i'}$, $[\sin]^{i'}$

functions of e'; the accents to the indices i and i' are sufficient to indicate this. Hence observing that

$$\begin{aligned}
\Sigma\,[\cos]^i \cos ig \,.\, \Sigma\,[\cos]^{i'} \cos i'g' &= \Sigma\Sigma\,[\cos]^i\,[\cos]^{i'} \cos (ig + i'g'),\\
\Sigma\,[\cos]^i \cos ig \,.\, \Sigma\,[\sin]^{i'} \sin i'g' &= \Sigma\Sigma\,[\cos]^i\,[\sin]^{i'} \sin (ig + i'g'),\\
\Sigma\,[\sin]^i \sin ig \,.\, \Sigma\,[\cos]^{i'} \cos i'g' &= \Sigma\Sigma\,[\sin]^i\,[\cos]^{i'} \sin (ig + i'g'),\\
\Sigma\,[\sin]^i \sin ig \,.\, \Sigma\,[\sin]^{i'} \sin i'g' &= -\Sigma\Sigma\,[\sin]^i\,[\sin]^{i'} \cos (ig + i'g'),
\end{aligned}$$

we have for the products of $x^a x'^{a'}$, or as the case may be $\left(\frac{r}{a}\right)^m \left(\frac{r'}{a'}\right)^n$ into

$$\begin{aligned}
&\cos (jf + j'f'), \text{ the values } \Sigma\Sigma\,([\cos]^i\,[\cos]^{i'} + [\sin]^i\,[\sin]^{i'}) \cos (ig + i'g'),\\
&\sin (jf + j'f'), \qquad \text{,,} \qquad \Sigma\Sigma\,([\sin]^i\,[\cos]^{i'} + [\cos]^i\,[\sin]^{i'}) \sin (ig + i'g'),
\end{aligned}$$

and thence observing that

$$\begin{aligned}
\cos (j\varpi + j'\varpi')\,\Sigma\Sigma\,P^{i,i'} \cos (ig + i'g') &= \Sigma\Sigma\,P^{i,i'} \cos (ig + i'g' + j\varpi + j'\varpi'),\\
-\sin (j\varpi + j'\varpi')\,\Sigma\Sigma\,Q^{i,i'} \sin (ig + i'g') &= \Sigma\Sigma\,Q^{i,i'} \cos (ig + i'g' + j\varpi + j'\varpi'),
\end{aligned}$$

provided only that $P^{-i,-i'} = P^{i,-i'}$, $Q^{-i,-i'} = -Q^{i,i'}$, we find for the product of $x^n x'^{n'}$ or as the case may be $\left(\frac{r}{a}\right)^n \left(\frac{r'}{a'}\right)^{n'}$ into $\cos (jf + j'f' + j\varpi + j'\varpi')$ the expression

$$\Sigma\Sigma\,([\cos]^i\,[\cos]^{i'} + [\sin]^i\,[\sin]^{i'} + [\sin]^i\,[\cos]^{i'} + [\cos]^i\,[\sin]^{i'}) \cos (ig + i'g' + j\varpi + j'\varpi')$$

or, finally, the expression

$$\Sigma\Sigma\,([\cos]^i + [\sin]^i)\,([\cos]^{i'} + [\sin]^{i'}) \cos (ig + i'g' + j\varpi + j'\varpi')$$

which is the required development of the general term in multiple cosines of the mean anomalies.

VI.

Investigation of the coefficient $C_n(j, j')$.

It is possible that there might be some advantage in developing in the first instance according to the powers of $\cos H$, a process which, as it has been seen, leads very readily to the form of the general term; but the mode which I have adopted is to develope in the first instance according to the powers of η. I put therefore

$$\cos H = \cos (U - U') - 2\eta^2 \sin U \sin U',$$

and we have then for the reciprocal of the distance,

$$\{r^2 + r'^2 - 2rr' \cos (U - U') - rr' (-4 \sin U \sin U')\,\eta^2\}^{-\frac{1}{2}}$$

which is to be developed in ascending powers of $\frac{r}{r'}$, and we have for the discrete general term of the development

$$\frac{r^n}{r'^{n+1}} C_n(j, j') \cos(jU + j'U');$$

which is the definition of $C_n(j, j')$, the coefficient the value of which is sought for. It has been seen that j, j', are each of them of the same parity with n (even or odd according as n is even or odd), and it has been seen also that it is sufficient to consider the case where j is not negative and not inferior in absolute magnitude to j'.

Expanding in powers of η, and putting as before

$$\Pi x = 1.2.3 \ldots x, \quad \Pi_1(x - \tfrac{1}{2}) = \tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2} \ldots (x - \tfrac{1}{2})$$

the general term is

$$\frac{\Pi_1(x - \frac{1}{2})}{\Pi x} r^x r'^x \{(r^2 + r'^2 - 2rr' \cos(U - U')\}^{-x-\frac{1}{2}} (-4 \sin U \sin U')^x \eta^{2x}$$

where x extends from 0 to ∞.

The factor $(-4 \sin U \sin U')^x$ consists of a series of multiple cosines, and as usual it is assumed that the cosines to opposite arguments are made to occur with equal coefficients. The form of the general term is $\cos(\lambda U + \lambda' U')$, where λ, λ' have each of them the values x, $x-2$, $x-4$, $\ldots -x$, that is, λ, λ' are each of them of the same parity with x. Hence j, j' being as before of the same parity with each other, and ϑ being even or odd according as j, j' and x are of the same parity or of opposite parities, the development of $(-4 \sin U \sin U')^x$ will contain a series of terms $\cos[(j + \vartheta) U + (j' - \vartheta) U']$, which (since the other factor contains only multiple cosines of $U - U'$) are the only terms which give rise to a term $\cos(jU + j'U')$. I represent the discrete term of $(-4 \sin U \sin U')^x$ which contains the before-mentioned argument by

$$M_x^{\vartheta} \cos[(j + \vartheta) U + (j - \vartheta) U'].$$

On the before-mentioned assumption, j not negative and not less in absolute magnitude than j', we have $j + j'$ and $j - j'$, each of them not negative. We must have $j + \vartheta \not> x$ and $j' - \vartheta \not> x$, that is $\vartheta \not> (x - j)$ and $\not< -(x - j')$, consequently $x - j \not< -(x - j')$ or $2x \not< j + j'$. And this relation being assumed, ϑ extends from the inferior limit $-(x - j')$ to the superior limit $(x - j)$ by steps of two units, the extreme terms being

$$\vartheta = -(x - j'), \quad M_x^{\vartheta} \cos[(j + j' - x) U + xU'],$$

$$\vartheta = (x - j), \quad M_x^{\vartheta} \cos[xU + (j + j' - x) U'],$$

the coefficient of U increasing from $j + j' - x$ to x, and that of U' diminishing from x to $j + j' - x$ by steps of two units.

Now expanding in ascending powers of $\frac{r}{r'}$, write

$$r^x r'^x \{(r^2 + r'^2 - 2rr' \cos(U - U')\}^{-x-\frac{1}{2}} = \Sigma R_x^i \cos i(U - U')$$

where i extends from $-\infty$ to $+\infty$ and $R_x^{-i} = R_x^i$; so that $R_x^i \cos i(U - U')$ is the discrete general term of the development. The term $R_x^{\mathfrak{s}} \cos \mathfrak{s}(U - U')$ in combination with the term $M_x^{\mathfrak{s}} \cos[(j + \mathfrak{s}) U + (j' - \mathfrak{s}) U']$ gives rise to $M_x^{\mathfrak{s}} R_x^{\mathfrak{s}} \cos(jU + j'U)$, and restoring the multiplier which has been omitted, and giving to $\mathfrak{s}$ and x the different admissible values, we find for the discrete general term containing $\cos(jU + j'U')$ the value

$$\Sigma \frac{\Pi_1(x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma (M_x^{\mathfrak{s}} R_x^{\mathfrak{s}}) \cos(jU + j'U')$$

where $\mathfrak{s}$ extends from the inferior limit $\mathfrak{s} = -(x - j')$ to the superior limit $\mathfrak{s} = x - j$, by steps of two units, and x extends from $x = \frac{1}{2}(j + j')$ to $x = \infty$. The portion of this containing $\frac{r^n}{r'^{n+1}} \cos(jU + j'U')$ is

$$\frac{r^n}{r'^{n+1}} C_n(j, j') \cos(jU + j'U')$$

and we have therefore

$$C_n(j, j') = \text{coeff. } \frac{r^n}{r'^{n+1}} \text{ in } \Sigma \frac{\Pi_1(x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma (M_x^{\mathfrak{s}} R_x^{\mathfrak{s}}).$$

There is some speciality in the case $j + j' = 0$, but the result just obtained subsists without variation. To find $M_x^{\mathfrak{s}}$ we have

$$M_x^{\mathfrak{s}} = \text{Discrete coeff. } \cos[(j + \mathfrak{s}) U + (j' - \mathfrak{s}) U'] \text{ in } (-4 \sin U \sin U')^x,$$

and putting $\sin U = \frac{1}{2i}\left(v - \frac{1}{v}\right)$, $\sin U' = \frac{1}{2i}\left(v' - \frac{1}{v'}\right)$ where $i = \sqrt{-1}$ we have

$$(-4 \sin U \sin U')^x = \left(v - \frac{1}{v}\right)^x \left(v' - \frac{1}{v'}\right)^x,$$

and the function on the right hand contains the term

$$(-)^{f+f'} \frac{\Pi x}{\Pi f \Pi(x - f)} \frac{\Pi x}{\Pi f' \Pi(x - f')} v^{x+2f} v'^{x-2f'},$$

or putting

$$x - 2f = j + \mathfrak{s},\ x - 2f' = j' - \mathfrak{s},$$

and therefore

$$f = \tfrac{1}{2}(x - j - \mathfrak{s}),\ f' = \tfrac{1}{2}(x - j' + \mathfrak{s}),$$

which give integer values for f, f', since $\mathfrak{s}$ is even or odd according as j, j', and x are of the same parity or of opposite parities, and replacing $v^{x-2f} v'^{x-2f'}$ by the *half* of its value $2\cos[(j + \mathfrak{s}) U + (j' - \mathfrak{s}) U']$, the term is

$$(-)^{x+\frac{1}{2}(j+j')} \frac{\Pi x}{\Pi \frac{1}{2}(x - j - \mathfrak{s}) \Pi \frac{1}{2}(x + j + \mathfrak{s})} \frac{\Pi x}{\Pi \frac{1}{2}(x - j' + \mathfrak{s}) \Pi \frac{1}{2}(x + j' - \mathfrak{s})}$$
$$\times \cos[(j + \mathfrak{s}) U + (j' - \mathfrak{s}) U'],$$

and consequently,

$$M_x^{\mathfrak{s}} = (-)^{x-\frac{1}{2}(j+j')} \frac{\Pi x}{\Pi\frac{1}{2}(x-j-\mathfrak{s})\,\Pi\frac{1}{2}(x+j+\mathfrak{s})} \frac{\Pi x}{\Pi\frac{1}{2}(x-j'+\mathfrak{s})\,\Pi\frac{1}{2}(x+j'-\mathfrak{s})},$$

which is the expression for $M_x^{\mathfrak{s}}$.

The expression for R_x^i is found in a similar manner, viz., by substituting for $\cos(U-U')$ its exponential expression, by which means the function

$$\left(r^2+r'^2-2rr'\cos(U-U')\right)^{-x-\frac{1}{2}}$$

breaks up into a pair of factors, each of which can be separately expanded; the result, i being positive, or zero, is

$$R_x^i = \frac{r^{x+i}}{r'^{x+i+1}} \Sigma \frac{\Pi_1(x+i+m-\frac{1}{2})\,\Pi_1(x+m-\frac{1}{2})}{\Pi(i+m)\,\Pi_1(x-\frac{1}{2})\,\Pi m\,\Pi_1(x-\frac{1}{2})} \left(\frac{r}{r'}\right)^{2m}$$

from $m=0$ to $m=\infty$: this may also be written

$$R_x^i = \frac{r^{x+i}}{r'^{x+i+1}} \frac{\Pi_1(x+i-\frac{1}{2})}{\Pi i\,\Pi_1(x-\frac{1}{2})} F\left(x+i+\tfrac{1}{2},\ x+\tfrac{1}{2},\ i+1,\ \frac{r^2}{r'^2}\right);$$

writing now $\mathfrak{s}$ for i, and $x+\mathfrak{s}+2m=n$, that is $m=\frac{1}{2}(n-x-\mathfrak{s})$ (n is of the same parity with j, j', and $\mathfrak{s}$ is even or odd according as j, j' and x are of the same parity or of opposite parities, m is therefore, as it should be, an integer), $R_x^{\mathfrak{s}}$ contains the term

$$\frac{r^n}{r'^{n+1}} \frac{\Pi_1(\frac{1}{2}(n+x+\mathfrak{s})-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+x-\mathfrak{s})-\frac{1}{2})}{\Pi\frac{1}{2}(n-x+\mathfrak{s})\,\Pi_1(x-\frac{1}{2})\,\Pi\frac{1}{2}(n-x-\mathfrak{s})\,\Pi_1(x-\frac{1}{2})},$$

$$= \frac{r^n}{r'^{n+1}} K_x^{\mathfrak{s}}$$

if for shortness

$$K_x^{\mathfrak{s}} = \frac{\Pi_1(\frac{1}{2}(n+x+\mathfrak{s})-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+x-\mathfrak{s})-\frac{1}{2})}{\Pi\frac{1}{2}(n-x+\mathfrak{s})\,\Pi_1(x-\frac{1}{2})\,\Pi_1\frac{1}{2}(n-x-\mathfrak{s})\,\Pi_1(x-\frac{1}{2})},$$

a formula which I assume to subsist as well for negative as for positive values of $\mathfrak{s}$, so that $K_x^{-\mathfrak{s}}=K_x^{\mathfrak{s}}$. It is to be noticed that if $x>n$, then either $n-x+\mathfrak{s}$ or $n-x-\mathfrak{s}$ vanishes, and we have therefore $K_x^{\mathfrak{s}}=0$.

Substituting for $R_x^{\mathfrak{s}}$ its value we have

$$C_n(j,j') = \Sigma \frac{\Pi_1(x-\frac{1}{2})}{\Pi x} \eta^{2x} \Sigma\,(M_x^{\mathfrak{s}} K_x^{\mathfrak{s}})$$

where $M_x^{\mathfrak{s}}$, $K_x^{\mathfrak{s}}$ have the values already obtained, and as before $\mathfrak{s}$ extends to $\mathfrak{s}=-(x-j')$ to $\mathfrak{s}=(x-j)$ by steps of two units, and x (since $K_x^{\mathfrak{s}}$ vanishes for $x<n$) extends from $x=\frac{1}{2}(j+j')$ to $x=n$.

To simplify; write $x=\frac{1}{2}(j+j')+u$; $\vartheta=-(x-j')+2s,=(x-j')-2t$, so that $s+t=u$, s and t being any integer values (zero not excluded) which satisfy this relation, and lastly u extends from $u=0$ to $u=n-\frac{1}{2}(j+j')$. We have

$$C_n(j,\,j')=\eta^{j+j'}\,\Sigma\eta^{2u}\,\frac{\Pi_1(\frac{1}{2}(j+j')+u-\frac{1}{2})}{\Pi(\frac{1}{2}(j+j)+u)}\,\Sigma\,\overline{M_u}^s\,\overline{K_u}^s,$$

where

$$\overline{M_u}^s=(-)^u\,\frac{\Pi(\frac{1}{2}(j+j')+u)}{\Pi t\,\Pi(\frac{1}{2}(j+j')+s)}\,\frac{\Pi(\frac{1}{2}(j+j')+u)}{\Pi s\,\Pi(\frac{1}{2}(j+j')+t)},$$

$$\overline{K_u}^s=\frac{\Pi_1(\frac{1}{2}(n+j')+s-\frac{1}{2})}{\Pi(\frac{1}{2}(n-j)-t)\,\Pi_1(\frac{1}{2}(j+j')+u-\frac{1}{2})}\,\frac{\Pi_1(\frac{1}{2}(n+j)+t-\frac{1}{2})}{\Pi(\frac{1}{2}(n-j')-s)\,\Pi_1(\frac{1}{2}(j+j')+u-\frac{1}{2})},$$

and substituting these values, and observing that the result contains a factor $\dfrac{1}{\Pi(\frac{1}{2}(j+j')+u)\,\Pi_1(\frac{1}{2}(j+j')+u-\frac{1}{2})}$ which may be replaced by $\dfrac{2^{j+j'+2u}}{\Pi(j+j'+2u)}$ the result is

$$C_n(j,\,j')=2^{j+j'}\,\eta^{j+j'}\,\Sigma_u\,\frac{(-)^u\,2^{2u}\,\eta^{2u}}{\Pi(j+j'+2u)}$$

$$\times\,\Sigma\left\{\frac{\Pi(\frac{1}{2}(j+j')+u)}{\Pi(\frac{1}{2}(j+j')+s)}\,\frac{\Pi(\frac{1}{2}(j+j')+u)}{\Pi(\frac{1}{2}(j+j')+t)}\,\frac{1}{\Pi s\Pi t}\,\frac{\Pi_1(\frac{1}{2}(n+j')+s-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j)+t-\frac{1}{2})}{\Pi(\frac{1}{2}(n-j')-s)\,\Pi(\frac{1}{2}(n-j)-t)}\right\}$$

which is easily transformed into

$$C_n(j,\,j')=j^{j+j'}\,\eta^{j+j'}\,\frac{\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j')-\frac{1}{2})}{\Pi\frac{1}{2}(n-j)\,\Pi\frac{1}{2}(n-j')\,\Pi(j+j')}\,S$$

if for shortness $S=$

$$\Sigma_u\,\frac{(-)^u\,2^{2u}\,\eta^{2u}}{[j+j'+2u]^{2u}}\,\Sigma\,\frac{[\frac{1}{2}(j+j')+u]^s[\frac{1}{2}(n+j')+s-\frac{1}{2}]^s[\frac{1}{2}(n-j')]^s[\frac{1}{2}(j+j')+u]^t[\frac{1}{2}(n+j)+t-\frac{1}{2}]^t[\frac{1}{2}(n-j)]}{[s]^s\,[t]^t}$$

$$s+t=u,\ u=0\ \text{to}\ u=n-\tfrac{1}{2}(j+j');$$

the value of the sum S is

$$S=(1-\eta^2)^{\frac{1}{2}(j-j')}\,F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

and we have consequently

$$C_n(j,\,j')=\frac{2^{j+j'}\,\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j')-\frac{1}{2})}{\Pi\frac{1}{2}(n-j)\,\Pi\frac{1}{2}(n-j')\,\Pi(j+j')}$$

$$\times\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

the required expression for $C_n(j,\,j')$. It only remains to prove the formula for S.

For this purpose, observing the equation $s+t=u$, I form the equations

$$\frac{2^{2u}}{[j+j'+2u]^{2u}} = \frac{1}{[\frac{1}{2}(j+j')+u]^u\,[\frac{1}{2}(j+j')+u-\frac{1}{2}]^u},$$

$$[\tfrac{1}{2}(j+j')+u]^s = \frac{[\frac{1}{2}(j+j')+u]^u}{[\frac{1}{2}(j+j')+t]^t},$$

$$[\tfrac{1}{2}(j+j')+u]^t = \frac{[\frac{1}{2}(j+j')+u]^u}{[\frac{1}{2}(j+j')+s]^s},$$

and thence

$$\frac{2^{2u}}{[j+j'+2u]^{2u}} = [\tfrac{1}{2}(j+j')+u]^s\,[\tfrac{1}{2}(j+j')+u]^t$$

$$= \frac{[\frac{1}{2}(j+j')+u]^u}{[\frac{1}{2}(j+j')+u-\frac{1}{2}]^u}\;\frac{1}{[\frac{1}{2}(j+j')+s]^s\,[\frac{1}{2}(j+j')+t]^t},$$

and we then have (putting also $(-)^u\eta^{2u} = (-)^{s+t}\eta^{2s+2t}$),

$$S = \Sigma_u \frac{[\frac{1}{2}(j+j')+u]^u}{[\frac{1}{2}(j+j')+u-\frac{1}{2}]^u}\,\Sigma\,\frac{(-)^s[\frac{1}{2}(n-j')]^s[\frac{1}{2}(n+j')+s-\frac{1}{2}]^s}{[s]^s\,[\frac{1}{2}(j+j')+s]^s}\,\eta^{2s}$$

$$\times\,\frac{(-)^t[\frac{1}{2}(\eta-j)]^t[\frac{1}{2}(n+j)+t-\frac{1}{2}]^t}{[t]^t\,[\frac{1}{2}(j+j')+t]^t}\,\eta^{2t},$$

and it is proper to remark that the summation as regards u may be continued indefinitely; for, if $u=s+t$ be $>n-\frac{1}{2}(j+j')$, then one at least of the relations $s>\frac{1}{2}(n-j)$, $t>\frac{1}{2}(n-j)$, must hold good, and at least one of the factorials $[\frac{1}{2}(n-j)]^s$, $[\frac{1}{2}(n-j)]^t$, will vanish. The two factors in the second sum are the general terms of two hypergeometric series. In fact, if we put

$$\tfrac{1}{2}(n+j)+\tfrac{1}{2} = \beta,$$
$$\tfrac{1}{2}(-n+j) = \alpha,$$
$$\tfrac{1}{2}(j+j')+\tfrac{1}{2} = \epsilon,$$

and therefore

$$\tfrac{1}{2}(n+j')+\tfrac{1}{2} = \epsilon-\alpha,$$
$$\tfrac{1}{2}(-n+j') = \epsilon-\beta,$$

and if, for shortness, we use $\{\alpha\}^s$ to denote the factorial $\alpha(\alpha+1)\ldots(\alpha+s-1)$ of the increment positive unity, then we have

$$S = \Sigma_u \frac{\{\epsilon+\frac{1}{2}\}^u}{\{\epsilon\}^u}\,\Sigma\,\frac{\{\alpha\}^s\{\beta\}^s}{\{1\}^s\{\epsilon+\frac{1}{2}\}^s}\,\eta^{2s}\,\frac{\{\epsilon-\alpha\}^s\{\epsilon-\beta\}^s}{\{1\}^s\{\epsilon+\frac{1}{2}\}^s}\,\eta^{2t}$$

the two hypergeometric series being consequently

$$F(\alpha,\ \beta,\ \epsilon+\tfrac{1}{2},\ \eta^2) = 1+\frac{\alpha\,.\,\beta}{1\,.\,\epsilon+\frac{1}{2}}\,\eta^2 + \frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,.\,2\,.\,\epsilon+\frac{1}{2}\,.\,\epsilon+\frac{3}{2}}\,\eta^4+\&c.,$$

$$F(\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon+\tfrac{1}{2},\ \eta^2) = 1+\frac{\epsilon-\alpha\,.\,\epsilon-\beta}{1\,.\,\epsilon+\frac{1}{2}}\,\eta^2+\frac{\epsilon-\alpha\,.\,\epsilon-\alpha+1\,.\,\epsilon-\beta\,.\,\epsilon-\beta+1}{1\,.\,2\,.\,\epsilon+\frac{1}{2}\,.\,\epsilon+\frac{3}{2}}\,\eta^4+\&c.$$

I assume the truth of the following remarkable theorem, [see 211] viz.:

"The series formed from the product of the two hypergeometric series,

$$F(\alpha,\ \beta,\ \epsilon+\tfrac{1}{2},\ \eta^2),\ F(\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon+\tfrac{1}{2},\ \eta^2),$$

by multiplying the successive terms of the product by 1, $\dfrac{\epsilon+\frac{1}{2}}{\epsilon}$, $\dfrac{\epsilon+\frac{1}{2}\,.\,\epsilon+\frac{3}{2}}{\epsilon\ .\,\epsilon+1}$, &c. respectively, is

$$=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)."$$

Hence, observing that the general term of S is formed precisely in the manner in question, we have

$$S=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)$$

or, substituting for α, β, ϵ their values

$$S=(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

which is the required value.

It is clear that α, β are interchangeable with $\epsilon-\alpha$, $\epsilon-\beta$; that is, we have

$$\begin{aligned} S&=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)\\ &=(1-\eta^2)^{\epsilon-\alpha-\beta}\ F(2\epsilon-2\alpha,\ 2\epsilon-2\beta,\ 2\epsilon,\ \eta^2)\end{aligned}$$

or

$$F(2\epsilon-2\alpha,\ 2\epsilon-2\beta,\ 2\epsilon,\ \eta^2)=(1-\eta^2)^{2\alpha+2\beta-2\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2),$$

which is a known property of hypergeometric series. The form

$$S=(1-\eta^2)^{-\frac{1}{2}(j-j')}F(-n+j',\ \ n+j'+1,\ \ j+j'+1,\ \ \eta^2)$$

is obviously less convenient than the one above mentioned, since the new expression is encumbered by a denominator $(1-\eta^2)^{\frac{1}{2}(j-j')}$, which really divides out, the finite hypergeometric series containing as a factor the square of such denominator. I have only noticed this for the verification which it affords by showing that j, j' may be interchanged.

VII.

The above expression of $C_n(j,\ j')$ is not, in its actual form, given in Hansen's memoir. The comparison with Hansen's formulæ is as follows:—The formula (36), p. 329, is

$$\begin{aligned} &C\left(n-2f,\ -(n-2f-2g)\right)\\ &\qquad=H\cos^{2n}\tfrac{1}{2}J\tan^{2g}\tfrac{1}{2}J\ F(-(2n-2f-2g),\ -2f,\ 2g+1,\ -\tan^2\tfrac{1}{2}J)\end{aligned}$$

where

$$H=\frac{\Pi(2n-2f)\,\Pi(2f+2g)}{2^{2n}\,\Pi(n-f)\,\Pi f\,\Pi(n-f-g)\,\Pi(f+g)\,\Pi\,2g};$$

and the formula (37), p. 330, is

$$F(-(2n-2f-2g),\ -2f,\ 2g+1,\ -\tan^2\tfrac{1}{2}J)$$
$$=\cos^{-2n+2f+2g}\tfrac{1}{2}J\ F(-(2n-2f-2g),\ 2f+2g+1,\ 2g+1,\ \sin^2\tfrac{1}{2}J).$$

Combining these, and putting $\sin\frac{1}{2}J=\eta$, we have

$$C\big(n-2f,\ -(n-2f-2g)\big)$$
$$=H\eta^{2g}(1-\eta^2)^{-n+2f+g}\ F(-(2n-2f-2g),\ 2f+2g+1,\ 2g+1,\ \eta^2),$$

and then, putting

$$n-2f\qquad=j',$$
$$-n+2f+2g=j,$$

and for $C(j', j), = C(j, j')$, writing $C_n(j, j')$, we have

$$C_n(j, j')=H\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2)$$

where

$$H=\frac{\Pi(n+j')\,\Pi(n+j)}{2^{2n}\,\Pi\frac{1}{2}(n+j')\,\Pi\frac{1}{2}(n-j')\,\Pi\frac{1}{2}(n+j)\,\Pi\frac{1}{2}(n-j)\,\Pi(j+j')},$$

or, finally, since

$$\Pi(n+j)=2^{n+j}\,\Pi\tfrac{1}{2}(n+j)\,\Pi_1(\tfrac{1}{2}(n+j)-\tfrac{1}{2}),$$
$$\Pi_1(n+j')=2^{n+j'}\,\Pi\tfrac{1}{2}(n+j')\,\Pi_1(\tfrac{1}{2}(n+j')-\tfrac{1}{2}),$$

we find

$$C_n(j, j')=\frac{2^{j+j'}\,\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})}{\Pi\frac{1}{2}(n-j)\,\Pi\frac{1}{2}(n-j')\,\Pi(j+j')}$$
$$\times\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

which is the formula of the present memoir.

215.

A SUPPLEMENTARY MEMOIR ON THE PROBLEM OF DISTURBED ELLIPTIC MOTION.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXVIII. pp. 217—234. Read January 12, 1860.]

THE present memoir contains an investigation upon fundamentally different principles of a system of formulæ given in my "Memoir on the Problem of Disturbed Elliptic Motion," *Ast. Soc. Mem.* t. XXVII. pp. 1—29 (1858), [212]; it is shown how the formulæ are deduced by means of the general equations of the *Mécanique Analytique*, from the expression for the *Vis viva* function T, in terms of the coordinates (r the radius vector, $\flat$ the departure) of the body in its instantaneous orbit, viz., the ultimate form of this function is $Tdt^2 = \frac{1}{2}(dr^2 + r^2 d\flat^2)$, but T contains in the first instance terms, not identically vanishing, but which are to be equated to zero, thus furnishing equations of the problem which could not be obtained from the foregoing ultimate form of T. The investigation throws, I think, a further light upon the system of formulæ, and completes the development which I was anxious to give of the dynamical problem. I have been a great deal indebted, in the composition of the memoir, to a correspondence some time ago with Professor Donkin on the general subject.

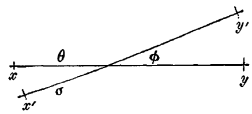
The word orbit is used to denote a great circle of the sphere, and it is assumed that in any orbit there is an origin of longitudes; the angular position of a body in reference to the orbit is determined by the longitude and latitude. It is ultimately assumed that the longitude is measured from an origin which is what I have called a departure-point; viz., in the general case of a variable orbit the departure-point is the point of intersection of the orbit by any orthogonal trajectory of the successive positions of the orbit: this includes the case of a fixed orbit, where the departure-point is simply a fixed point. As regards points in the orbit, the word departure may be used instead of longitude. In the present memoir the origin is in the first instance taken to be, not a departure-point, but an arbitrarily varying point of the orbit.

The mutual position of any two orbits whatever, say the position of an orbit $x'y'$ and of the origin x' therein, in reference to the orbit xy and the origin x therein, is determined by

θ, the longitude of node,

σ, the departure of node,

ϕ, the inclination,



where the expression departure of node is used by way of distinction from longitude of node, the departure referring to the orbit $x'y'$ and origin x', the positions of which are to be determined, and the longitude to the orbit of reference xy and origin x. And this distinction will be preserved throughout. It should be recollected that it is not as yet assumed that the origins are departure-points.

Consider now a fixed orbit x_0y_0 and fixed origin x_0 therein, and suppose that the orbit x_1y_1 and origin x_1 therein are determined in reference to the orbit x_0y_0 and origin x_0 therein, by

θ', the longitude of node,

σ', the departure of node,

ϕ', the inclination,

the orbit x_2y_2 and origin x_2 therein, in reference to the orbit x_1y_1 and origin x_1 therein, by

Θ, the longitude of node,

Σ, the departure of node,

Φ, the inclination,

and, finally, the position of the body in reference to the orbit x_2y_2 and origin x_2 therein, by

r, the radius vector,

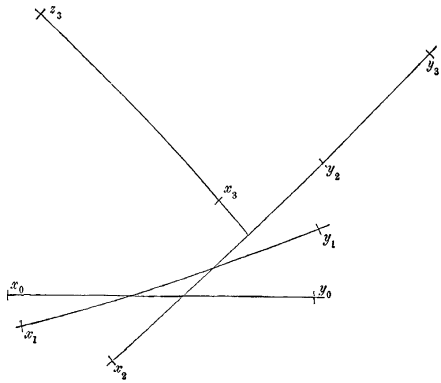
v, the longitude,

y, the latitude.

It is required to find the expression of T, the *Vis viva* function, or half square of the velocity. We may imagine the rectangular axes $x_0y_0z_0$, $x_1y_1z_1$, $x_2y_2z_2$, the positions of which are determined as above, and the rectangular axes $x_3y_3z_3$, that of x_3 passing through the body, that of y_3 lying in the orbit x_2y_2, 90° in advance of the body or

passing through the pole of the latitude circle, and that of z_3 in the latitude circle 90° above the body. Or considering x_3, y_3, z_3, as points of the sphere, their positions in reference to the orbit x_2y_2 and origin x_2 therein are determined as follows, viz.,

for x_3,	longitude is	v	, and latitude	y ,
y_3,	,,	$v+90^\circ$	,,	0 ,
z_3,	,,	v	,,	$y+90^\circ$.



and we may suppose that (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), are the coordinates of any point in reference to these sets of axes respectively, the coordinates of the body in reference to the axes $x_3y_3z_3$ being obviously $(r, 0, 0)$.

The displacements in the directions of the axes $x_3y_3z_3$ respectively of a point the coordinates of which are x_3, y_3, z_3, are as usual

$$dx_3 + (-Ry_3 + Qz_3)\,dt,$$
$$dy_3 + (-Pz_3 + Rx_3)\,dt,$$
$$dz_3 + (-Qx_3 + Py_3)\,dt,$$

where the expressions of the rotations P, Q, R are as follows, viz.,

$$\begin{aligned}
Pdt = \; & \sin y\, dv \\
& + \cos y\,[\sin(v-\Sigma)\sin\Phi\, d\Theta + \cos(v-\Sigma)\, d\Phi] \\
& + \sin y\,[\quad -d\Sigma + \cos\Phi\, d\Theta \qquad\qquad\qquad\qquad] \\
& + \{\cos y \cos(v-\Sigma) \qquad\qquad\qquad\

$$
\begin{aligned}
Qdt = &- dy \\
&+ [\cos(v-\Sigma)\sin\Phi\, d\Theta - \sin(v-\Sigma)\, d\Phi] \\
&\qquad - \sin(v-\Sigma)[\sin(\Theta-\sigma')\sin\phi'\, d\theta' + \cos(\Theta-\sigma')\, d\phi'] \\
&\qquad + \cos(v-\Sigma)\cos\Phi\,[\cos(\Theta-\sigma')\sin\phi'\, d\theta' - \sin(\Theta-\sigma')\, d\phi'] \\
&\qquad + \cos(v-\Sigma)\sin\Phi\,[\quad - d\sigma' + \cos\phi'\, d\theta' \qquad\qquad],
\end{aligned}
$$

$$
\begin{aligned}
Rdt = &\ \cos y\, dv \\
&- \sin y\,[\sin(v-\Sigma)\sin\Phi\, d\Theta + \cos(v-\Sigma)\, d\Phi] \\
&+ \cos y\,[\quad - d\Sigma + \cos\Phi\, d\Theta \qquad\qquad] \\
&+ \{-\sin y\cos(v-\Sigma) \qquad\qquad \}[\sin(\Theta-\sigma')\sin\phi'\, d\theta' + \cos(\Theta-\sigma')\, d\phi'] \\
&+ \{-\sin y\sin(v-\Sigma)\cos\Phi - \cos y\sin\Phi\}[\cos(\Theta-\sigma')\sin\phi'\, d\theta' - \sin(\Theta-\sigma')\, d\phi'] \\
&+ \{-\sin y\sin(v-\Sigma)\sin\Phi + \cos y\cos\Phi\}[\quad - d\sigma' + \cos\phi'\, d\theta' \qquad\qquad].
\end{aligned}
$$

For the body itself we must write r, 0, 0, in the place of x_3, y_3, z_3; the displacements are therefore

$$dr, \quad rRdt, \quad -rQdt;$$

and the expression for the *Vis viva* function is

$$Tdt^2 = \tfrac{1}{2}\{dr^2 + r^2(Q^2 + R^2)\, dt^2\}$$

where Q, R have the above-mentioned values.

Let Ω be a function of the coordinates x_0, y_0, z_0 of a point, Ω will be a function of x_3, y_3, z_3, and also of the quantities Θ, Σ, Φ, θ', σ', ϕ' which determine the positions of the axes x_3, y_3, z_3, and the complete differential of Ω will be

$$
\begin{aligned}
d\Omega = &\ \frac{d\Omega}{dx_3}\left\{dx_3 + (-Ry_3 + Qz_3)\, dt\right\} \\
&+ \frac{d\Omega}{dy_3}\left\{dy_3 + (-Pz_3 + Rx_3)\, dt\right\} \\
&+ \frac{d\Omega}{dz_3}\left\{dz_3 + (-Qx_3 + Py_3)\, dt\right\}.
\end{aligned}
$$

I assume for the present the truth of the following proposition, viz., that when Ω is a function of the coordinates of the body (in which case, as before, the values of x_3, y_3, z_3 are r, 0, 0), we have

$$
\begin{aligned}
\frac{d\Omega}{dx_3} &= \frac{d\Omega}{dr}, \\
\frac{d\Omega}{dy_3} &= \frac{1}{r}\sec y\,\frac{d\Omega}{dv}, \\
\frac{d\Omega}{dz_3} &= \frac{1}{r}\,\frac{d\Omega}{dy}.
\end{aligned}
$$

Ω is here a function of r, v, y, Θ, Σ, Φ, θ', σ', ϕ', or, as we may express it, $\Omega = \Omega(r, v, y, \Theta, \Sigma, \Phi, \theta', \sigma', \phi')$; and the expression for the complete differential, by what precedes, is

$$d\Omega = \frac{d\Omega}{dr}\,dr + Rdt\,.\sec y\,\frac{d\Omega}{dv} - Qdt\,\frac{d\Omega}{dy},$$

or, substituting for Q, R, their values, the expression is

$$d\Omega = \frac{d\Omega}{dr}\,dr$$

$$+\sec y\,\frac{d\Omega}{dv}\left\{\begin{array}{l} \cos y\,dv \\ -\sin y\,[\sin(v-\Sigma)\sin\Phi d\Theta + \cos(v-\Sigma)\,d\Phi] \\ +\cos y\,[\quad -d\Sigma + \cos\Phi d\Theta \qquad\qquad\qquad\qquad] \\ +\{-\sin y\cos(v-\Sigma) \qquad\qquad\qquad\qquad\}\,[\sin(\Theta-\sigma')\sin\phi' d\theta' + \cos(\Theta-\sigma')\,d\phi'] \\ +\{-\sin y\sin(v-\Sigma)\cos\Phi - \cos y\sin\Phi\}\,[\cos(\Theta-\sigma')\sin\phi' d\theta' - \sin(\Theta-\sigma')\,d\phi'] \\ +\{-\sin y\sin(v-\Sigma)\sin\Phi + \cos y\cos\Phi\}\,[\quad -d\sigma' + \cos\phi' d\theta' \qquad\qquad] \end{array}\right\}$$

$$+\quad\frac{d\Omega}{dy}\left\{\begin{array}{l} -dy \\ +[\cos(v-\Sigma)\sin\Phi d\Theta - \sin(v-\Sigma)\,d\Phi] \\ \qquad\qquad -\sin(v-\Sigma) \qquad [\sin(\Theta-\sigma')\sin\phi' d\theta' + \cos(\Theta-\sigma')\,d\phi'] \\ \qquad\qquad +\cos(v-\Sigma)\cos\Phi\,[\cos(\Theta-\sigma')\sin\phi' d\theta' - \sin(\Theta-\sigma')\,d\phi'] \\ \qquad\qquad +\cos(v-\Sigma)\sin\Phi\,[\quad -d\sigma' + \cos\phi' d\theta' \qquad\qquad] \end{array}\right\}$$

where, on the right-hand side, the terms involving the differentials of r, v, y, are, as they should be, $\frac{d\Omega}{dr}\,dr + \frac{d\Omega}{dv}\,dv + \frac{d\Omega}{dy}\,dy$.

I have given the complete expression, as it may be useful; but for the present purpose it will be sufficient to attend to the terms involving Θ, Σ, Φ. We have

$$\frac{d\Omega}{d\Theta}\,d\Theta + \frac{d\Omega}{d\Sigma}\,d\Sigma + \frac{d\Omega}{d\Phi}\,d\Phi$$

$$= \sec y\,\frac{d\Omega}{dv}\left\{\begin{array}{l} -\sin y\,[\sin(v-\Sigma)\sin\Phi\,d\Theta + \cos(v-\Sigma)\,d\Phi] \\ +\cos y\,[\quad -d\Sigma + \cos\Phi d\Theta \qquad\qquad\qquad\qquad] \end{array}\right\}$$

$$-\frac{d\Omega}{dy}\left\{\cos(v-\Sigma)\sin\Phi\,d\Theta - \sin(v-\Sigma)\,d\Phi \qquad\qquad\qquad\qquad\right\}.$$

and consequently

$$\frac{d\Omega}{d\Sigma} = -\frac{d\Omega}{dv},$$

$$\frac{d\Omega}{d\Theta} = \left\{\cos\Phi - \tan y \sin(v-\Sigma)\sin\Phi\right\}\frac{d\Omega}{dv} - \cos(v-\Sigma)\sin\Phi\frac{d\Omega}{dy},$$

$$\frac{d\Omega}{d\Phi} = -\tan y\cos(v-\Sigma)\frac{d\Omega}{dv} + \sin(v-\Sigma)\frac{d\Omega}{dy};$$

which, it is to be observed, are partial differential equations satisfied by Ω considered as a function of the form $\Omega = \Omega(r, v, y, \Theta, \Sigma, \Phi)$; Ω is, in fact, a function of three quantities only (say the coordinates x_0, y_0, z_0); and it is clear, *à priori*, that when thus expressed as a function of six quantities, it must satisfy three partial differential equations.

I write in the second equation $-\frac{d\Omega}{d\Sigma}$ for $\frac{d\Omega}{dv}$, and I then put $y=0$; we thus have

$$\cos(v-\Sigma)\frac{d\Omega}{dy} = -\cot\Phi\frac{d\Omega}{d\Sigma} - \operatorname{cosec}\Phi\frac{d\Omega}{d\Theta},$$

$$\sin(v-\Sigma)\frac{d\Omega}{dy} = \frac{d\Omega}{d\Phi}.$$

I propose, as already mentioned, to apply the formulæ to the demonstration of the results obtained in my Memoir on the Problem of Disturbed Elliptic Motion. It will be remembered that x_0y_0 and x_0 denote a fixed orbit and fixed origin therein, x_1y_1 and x_1 an arbitrarily varying orbit and origin therein. I assume, however, that x_1 is a departure-point, so that we have $-d\sigma' + \cos\phi' d\theta' = 0$. The orbit x_2y_2 will be taken to be the varying instantaneous orbit of the body itself, that is, we have constantly $y=0$; and it is assumed that x_2 is a departure-point in this orbit. To conform to my former notation, I write $\flat$ instead of v; the position of the body in the varying instantaneous orbit is consequently determined by

r, the radius vector,

$\flat$, the departure.

The entire displacement of the body arises from the displacements dr and $rd\flat$ in the direction of the radius vector, and perpendicular to this direction, in the instantaneous orbit; that is, we have $Q=0$, $R=d\flat$, and the expression for the *Vis viva* function is simply

$$Tdt^2 = \tfrac{1}{2}(dr^2 + r^2 d\flat^2),$$

and, putting in the foregoing expressions of Q and R, $y=0$, and $\flat$ in the place of v, the equations $Q=0$, $Rdt = d\flat$, give

$$\cos(\flat-\Sigma)\sin\phi d\Theta - \sin(\flat-\Sigma)d\Phi =$$
$$\sin(\flat-\Sigma)\quad[\sin(\Theta-\sigma')\sin\phi' d\theta' + \cos(\Theta-\sigma')d\phi']$$
$$-\cos(\flat-\Sigma)\cos\Phi\,[\cos(\Theta-\sigma')\sin\phi' d\theta' - \sin(\Theta-\sigma')d\phi'],$$

$$d\Sigma - \cos\Phi d\Theta =$$
$$-\sin\Phi\,[\cos(\Theta-\sigma')\sin\phi' d\theta' - \sin(\Theta-\sigma')d\phi'].$$

Now considering r, $\flat$, y as the coordinates (y being ultimately put equal to zero) the general equations of motion are

$$\frac{d}{dt}\left(\frac{dT}{d\dot{r}}\right)-\frac{dT}{dr}=\frac{dV}{dr},$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\flat}}\right)-\frac{dT}{d\flat}=\frac{dV}{d\flat},$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{y}}\right)-\frac{dT}{dy}=\frac{dV}{dy},$$

if for the moment $\dot{r}$, $\dot{\flat}$, $\dot{y}$ denote the differential coefficients of r, $\flat$, y: the expression of the force function V is $V=-\frac{n^2a^3}{r}+\Omega$ where n^2a^3 denotes an absolute constant, the sum of the masses, and the disturbing function Ω has a contrary sign to that of the *Mécanique Céleste*. We may in the first and second equations write at once $Tdt^2=\frac{1}{2}(dr^2+r^2d\flat^2)$, and these equations thus become

$$\frac{d}{dt}\frac{dr}{dt}-r\left(\frac{d\flat}{dt}\right)^2=\frac{n^2a^3}{r^2}+\frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2\frac{d\flat}{dt}\right)=\frac{d\Omega}{d\flat}.$$

If in the third equation we take the general value of T and after the differentiation with respect to $\dot{y}$, make the substitutions $y=0$, $Q=0$, $Rdt=d\flat$, we find

$$\frac{dT}{d\dot{y}}=0,$$

and the equation thus becomes

$$-\frac{dT}{dy}=\frac{d\Omega}{dy}.$$

But $Tdt^2=\frac{1}{2}\{dr^2+r^2(Q^2+R^2)\,dt^2\}$, consequently

$$-\frac{d\Omega}{dy}\,dt=\frac{dT}{dy}\,dt=r^2\left(Q\frac{dQ}{dy}+R\frac{dR}{dy}\right)dt=r^2d\flat\,\frac{dR}{dy},$$

and from the value of R, putting after the differentiation $y=0$, we find

$$\begin{aligned}\frac{dR}{dy}\,dt=-&\qquad[\sin(\flat-\Sigma)\sin\Phi\,d\Theta+\cos(\flat-\Sigma)\,d\Phi]\\ -\cos(\flat-\Sigma)&\qquad[\sin(\Theta-\sigma')\sin\phi'\,d\theta'+\cos(\Theta-\sigma')\,d\phi']\\ -\sin(\flat-\Sigma)\cos\Phi&[\cos(\Theta-\sigma')\sin\phi'\,d\theta-\sin(\Theta-\sigma')\,d\phi'].\end{aligned}$$

I write $r^2 \frac{d\text{p}}{dt} = h$. We thus have

$$\frac{d}{dt}\frac{dr}{dt} - r\left(\frac{d\text{p}}{dt}\right)^2 = \frac{n^2a^2}{r^2} + \frac{d\Omega}{dr},$$

$$r^2 \frac{d\text{p}}{dt} = h,$$

and

$$dh = \frac{d\Omega}{d\text{p}}\, dt;$$

the remaining equations are

$$d\Sigma - \cos\Phi\, d\Theta = \qquad - \sin\Phi\,[\cos(\Theta - \sigma')\sin\phi'\, d\theta' - \sin(\Theta - \sigma')\, d\phi'],$$

$$\cos(\text{p} - \Sigma)\sin\Phi\, d\Theta - \sin(\text{p} - \Sigma)\, d\Phi =$$

$$\sin(\text{p} - \Sigma) \qquad [\sin(\Theta - \sigma')\sin\phi'\, d\theta' + \cos(\Theta - \sigma')\, d\phi']$$

$$- \cos(\text{p} - \Sigma)\cos\Phi\,[\cos(\Theta - \sigma')\sin\phi'\, d\theta' - \sin(\Theta - \sigma')\, d\phi'],$$

$$\sin(\text{p} - \Sigma)\sin\Phi\, d\Theta + \cos(\text{p} - \Sigma)\, d\Phi =$$

$$\frac{1}{h}\frac{d\Omega}{dy}\, dt$$

$$- \cos(\text{p} - \Sigma) \qquad [\sin(\Theta - \sigma')\sin\phi'\, d\theta' + \cos(\Theta - \sigma')\, d\phi']$$

$$- \sin(\text{p} - \Sigma)\cos\Phi\,[\cos(\Theta - \sigma')\sin\phi'\, d\theta' - \sin(\Theta - \sigma')\, d\phi'].$$

Moreover we have

$$\cos(\text{p} - \Sigma)\frac{d\Omega}{dy} = -\cot\phi\,\frac{d\Omega}{d\Sigma} - \operatorname{cosec}\Phi\,\frac{d\Omega}{d\Theta},$$

$$\sin(\text{p} - \Sigma)\frac{d\Omega}{dy} = \qquad \frac{d\Omega}{d\Phi},$$

by means of which the preceding two equations are reduced to

$$d\Theta = \frac{\operatorname{cosec}\Phi}{h}\frac{d\Omega}{d\Phi}\, dt$$

$$- \cot\Phi\,[\cos(\Theta - \sigma')\sin\phi'\, d\theta' - \sin(\Theta - \sigma')\, d\phi'],$$

$$d\Phi = -\frac{\cot\Phi}{h}\frac{d\Omega}{d\Sigma}\, dt - \frac{\operatorname{cosec}\Phi}{h}\frac{d\Omega}{d\Theta}\, dt$$

$$- \qquad [\sin(\Theta - \sigma')\sin\phi'\, d\theta' + \cos(\Theta - \sigma')\, d\phi'];$$

and we then have

$$d\Sigma = \frac{\cot\Phi}{h}\frac{d\Omega}{d\Phi}\, dt$$

$$- \operatorname{cosec}\Phi\,[\cos(\Theta - \sigma')\sin\phi'\, d\theta' - \sin(\Theta - \sigma')\, d\phi'].$$

The entire system consequently is

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) - r\left(\frac{dp}{dt}\right)^2 = \frac{n^2a^3}{r^2} + \frac{d\Omega}{dr},$$

$$r^2\frac{dp}{dt} = h;$$

and

$$dh = \frac{d\Omega}{dp}\,dt,$$

$$d\Theta = \frac{\operatorname{cosec}\Phi}{h}\frac{d\Omega}{d\Phi}\,dt - \cot\Phi\,[\cos(\Theta-\sigma')\sin\phi'\,d\theta' - \sin(\Theta-\sigma')\,d\phi'],$$

$$d\Sigma = \frac{\cot\Phi}{h}\frac{d\Omega}{d\Phi}\,dt - \operatorname{cosec}\Phi\,[\cos(\Theta-\sigma')\sin\phi'\,d\theta' - \sin(\Theta-\sigma')\,d\phi'],$$

$$d\Phi = -\frac{\operatorname{cosec}\Phi}{h}\frac{d\Omega}{d\Theta}\,dt - \frac{\cot\Phi}{h}\frac{d\Omega}{d\Sigma}\,dt - [\sin(\Theta-\sigma')\sin\phi'\,d\theta' + \cos(\Theta-\sigma')\,d\phi'],$$

where $\Omega = \Omega(r, p, \Theta, \Sigma, \Phi)$. We may add the before-mentioned equation,

$$d\Sigma - \cos\Phi\,d\Theta = -\sin\Phi\,[\cos(\Theta-\sigma')\sin\phi'\,d\theta' - \sin(\Theta-\sigma')\,d\phi'].$$

To obtain the formulæ of my Memoir before referred to, it would only be necessary to write $h = na^2\sqrt{1-e^2}$, and complete the solution by applying to the first equation the method of the variation of the elements; but I prefer leaving the system in its present form, as it is one which may perhaps be useful.

The formulæ have especial reference to the lunar theory, and the orbit x_0y_0 denotes the fixed ecliptic, x_1y_1 the varying ecliptic, and x_2y_2 the instantaneous orbit of the moon. But if, instead of considering with Hansen the instantaneous orbit of the moon, the position of the moon is determined (as in the theories of Clairaut, Plana, and others) by

r, the radius vector,
v, the longitude,
y, the latitude,

where the longitude is measured on the variable ecliptic, then if, in the general expressions for P, Q, R, we first neglect the terms involving $d\theta'$, $d\sigma'$, $d\phi'$, and then write θ', σ', ϕ', in the place of Θ, Σ, Φ, we find

$$\begin{aligned} P\,dt = {} & \sin y\,dv \\ & + \cos y\,[\sin(v-\sigma')\sin\phi'\,d\theta' + \cos(v-\sigma')\,d\phi'] \\ & + \sin y\,[-d\sigma' + \cos\phi'\,d\theta'], \\ Q\,dt = {} & -dy \\ & + \quad [\cos(v-\sigma')\sin\phi'\,d\theta' - \sin(v-\sigma')\,d\phi'], \end{aligned}$$

$$\begin{aligned} R dt = & \cos y \, dv \\ & - \sin y \, [\sin (v - \sigma') \sin \phi' \, d\theta' + \cos (v - \sigma') \, d\phi'] \\ & + \cos y \, [- d\sigma' + \cos \phi' \, d\theta' \qquad\qquad], \end{aligned}$$

and with these values the expression for the *Vis viva* function is, as before,

$$T dt^2 = \tfrac{1}{2} \{dr^2 + r^2 (Q^2 + R^2) \, dt^2\}.$$

The longitude may be measured from a departure-point, and then the expressions for P, Q, R, may be simplified by omitting the terms which contain $-d\sigma' + \cos \phi' d\theta'$.

Addition.

I have in the course of the memoir used some properties connected with the theory of rotation, but it is proper to give the analytical investigation.

Suppose that the rectangular coordinates (X, Y, Z) are connected with the rectangular coordinates $(X_{,}, Y_{,}, Z_{,})$ by the equations

$$\begin{aligned} X &= \alpha \, X_{,} + \beta \, Y_{,} + \gamma \, Z_{,}, \\ Y &= \alpha' X_{,} + \beta' Y_{,} + \gamma' Z_{,}, \\ Z &= \alpha'' X_{,} + \beta'' Y_{,} + \gamma'' Z_{,}, \end{aligned}$$

then we have

$$\begin{aligned} dX &= \alpha \, dX_{,} + \beta \, dY_{,} + \gamma \, dZ_{,} + X_{,} d\alpha + Y_{,} d\beta + Z_{,} d\gamma, \\ dY &= \alpha' dX_{,} + \beta' dY_{,} + \gamma' dZ_{,} + X_{,} d\alpha' + Y_{,} d\beta' + Z_{,} d\gamma', \\ dZ &= \alpha'' dX_{,} + \beta'' dY_{,} + \gamma'' dZ_{,} + X_{,} d\alpha'' + Y_{,} d\beta'' + Z_{,} d\gamma'', \end{aligned}$$

and then, putting

$$\begin{aligned} \mathrm{p} dt &= \gamma d\beta + \gamma' d\beta' + \gamma'' d\beta'', \\ \mathrm{q} dt &= \alpha d\gamma + \alpha' d\gamma' + \alpha'' d\gamma'', \\ \mathrm{r} dt &= \beta d\alpha + \beta' d\alpha' + \beta'' d\alpha'', \end{aligned}$$

where p, q, r are the rotations round $X_{,}$, $Y_{,}$, $Z_{,}$ from $Y_{,}$ to $Z_{,}$, $Z_{,}$ to $X_{,}$, and $X_{,}$ to $Y_{,}$ respectively, we have

$$\begin{aligned} \alpha dX + \alpha' dY + \alpha'' dZ &= dX_{,} + (- \mathrm{r} Y_{,} + \mathrm{q} Z_{,}) \, dt, \\ \beta dX + \beta' dY + \beta'' dZ &= dY_{,} + (- \mathrm{p} Z_{,} + \mathrm{r} X_{,}) \, dt, \\ \gamma dX + \gamma' dY + \gamma'' dZ &= dZ_{,} + (- \mathrm{q} X_{,} + \mathrm{p} Y_{,}) \, dt, \end{aligned}$$

where, considering XYZ as fixed axes, the left-hand sides are the displacements in the directions of the moveable axes $X_{,}Y_{,}Z_{,}$ arising from the variations of the coordinates $X_{,}Y_{,}Z_{,}$ and the variation of the position of these axes; and these displace-

ments have therefore the values given by the right-hand sides. This is, of course, well known. It may be added that if the axes XYZ are moveable, and dX, dY, dZ denote the displacements in the directions of these axes, the formulæ are still true.

Suppose that Ω is a function of X, Y, Z; say $\Omega = \Omega(X, Y, Z)$, we have

$$d\Omega = \frac{d\Omega}{dX}dX + \frac{d\Omega}{dY}dY + \frac{d\Omega}{dZ}dZ,$$

which may be written

$$\begin{aligned} d\Omega = & \left(\alpha\frac{d\Omega}{dX} + \alpha'\frac{d\Omega}{dY} + \alpha''\frac{d\Omega}{dZ}\right)(\alpha dX + \alpha' dY + \alpha'' dZ) \\ & + \left(\beta\frac{d\Omega}{dX} + \beta'\frac{d\Omega}{dY} + \beta''\frac{d\Omega}{dZ}\right)(\beta dX + \beta' dY + \beta'' dZ) \\ & + \left(\gamma\frac{d\Omega}{dX} + \gamma'\frac{d\Omega}{dY} + \gamma''\frac{d\Omega}{dZ}\right)(\gamma dX + \gamma' dY + \gamma'' dZ), \end{aligned}$$

and the coefficients of transformation α, β, γ, &c., being independent of X, Y, Z and $X_{,}$, $Y_{,}$, $Z_{,}$, the first factors are $\frac{d\Omega}{dX_{,}}$, $\frac{d\Omega}{dY_{,}}$, $\frac{d\Omega}{dZ_{,}}$, and we have

$$\begin{aligned} d\Omega = & \frac{d\Omega}{dX_{,}}[dX_{,} + (-\mathrm{r}Y_{,} + \mathrm{q}Z_{,})\,dt] \\ & + \frac{d\Omega}{dY_{,}}[dY_{,} + (-\mathrm{p}Z + \mathrm{r}X_{,})\,dt] \\ & + \frac{d\Omega}{dZ}[dZ_{,} + (-\mathrm{q}X + \mathrm{p}Y_{,})\,dt], \end{aligned}$$

which is a theorem assumed in the memoir.

The expressions of P, Q, R in the memoir depend upon the composition of several rotations. If, in Lamé's very convenient notation, we write

	$X_{,}$	$Y_{,}$	$Z_{,}$
X	α	β	γ
Y	α'	β'	γ'
Z	α''	β''	γ''

to denote the before-mentioned relation between (X, Y, Z) and $(X_{,}, Y_{,}, Z_{,})$ then the tables

	$X_{,}$	$Y_{,}$	$Z_{,}$
X	α	β	γ
Y	α'	β'	γ'
Z	α''	β''	γ''

,

	X_2	Y_2	Z_2
$X_{,}$	A	B	C
$Y_{,}$	A'	B'	C'
$Z_{,}$	A''	B''	C''

lead to the table

	X_2	Y_2	Z_2
X	$(\alpha, \beta, \gamma)(A, A', A'')$	$(\alpha, \beta, \gamma)(B, B', B'')$	$(\alpha, \beta, \gamma)(C, C', C'')$
Y	$(\alpha', \beta', \gamma')(A, A', A'')$	$(\alpha', \beta', \gamma')(B, B', B'')$	$(\alpha', \beta', \gamma')(C, C', C'')$
Z	$(\alpha'', \beta'', \gamma'')(A, A', A'')$	$(\alpha'', \beta'', \gamma'')(B, B', B'')$	$(\alpha'', \beta'', \gamma'')(C, C', C'')$

where for shortness $(\alpha, \beta, \gamma)(A, A', A'')$, &c., stand for $\alpha A + \beta A' + \gamma A''$, &c. these are of course the ordinary formulæ for the composition of transformations from one set of rectangular axes to another. We may say that the coefficients of transformation from (X, Y, Z) to $(X_{,}, Y_{,}, Z_{,})$ are $(\alpha, \beta, \gamma$, &c.). And in dealing with several sets of coordinates, the coefficients of transformation may be distinguished by suffixes. Thus I assume that the coefficients of transformation

from (x_0, y_0, z_0) to (x_1, y_1, z_1) are $(\alpha, \beta, \gamma, \text{\&c.})_{01}$,
" (x_1, y_1, z_1) " (x_2, y_2, z_2) " $(\alpha, \beta, \gamma, \text{\&c.})_{12}$,
" (x_0, y_0, z_0) " (x_2, y_2, z_2) " $(\alpha, \beta, \gamma, \text{\&c.})_{02}$,

where $(\alpha, \beta, \gamma, \text{\&c.})_{02}$ can be obtained as above from $(\alpha, \beta, \gamma, \text{\&c.})_{01}$ and $(\alpha, \beta, \gamma, \text{\&c.})_{12}$. The rotations p, q, r may be distinguished by suffixes in like manner, viz., $(\mathrm{p}, \mathrm{q}, \mathrm{r})_{01}$ will denote the rotations from (x_0, y_0, z_0) to (x_1, y_1, z_1) and so on.

There is no difficulty in obtaining first

$$\mathrm{p}_{02} = \mathrm{p}_{12} + (\alpha, \alpha', \alpha'')_{12} (\mathrm{p}, \mathrm{q}, \mathrm{r})_{01},$$
$$\mathrm{q}_{02} = \mathrm{q}_{12} + (\beta, \beta', \beta'')_{12} (\mathrm{p}, \mathrm{q}, \mathrm{r})_{01},$$
$$\mathrm{r}_{02} = \mathrm{r}_{12} + (\gamma, \gamma', \gamma'')_{12} (\mathrm{p}, \mathrm{q}, \mathrm{r})_{01},$$

and then by a repetition of the like transformation

$$p_{03} = p_{23} + (\alpha, \alpha', \alpha'')_{23}\,(p, q, r)_{12} + (\alpha, \alpha', \alpha'')_{13}\,(p, q, r)_{01},$$
$$q_{03} = q_{23} + (\beta, \beta', \beta'')_{23}\,(p, q, r)_{12} + (\beta, \beta', \beta'')_{13}\,(p, q, r)_{01},$$
$$r_{03} = r_{23} + (\gamma, \gamma', \gamma'')_{23}\,(p, q, r)_{12} + (\gamma, \gamma', \gamma'')_{13}\,(p, q, r)_{01},$$

and in like manner for any number of systems, but for the present purpose this is enough, as p_{03}, q_{03}, r_{03} are in fact the P, Q, R of the memoir. And if θ', σ', ϕ', Θ, Σ, Φ, v, y signify as before, then the coefficients of transformation from (x_0, y_0, z_0) to (x_1, y_1, z_1) are given by the table

	x_1	y_1	z_1
x_0	$\cos\sigma'\cos\theta' + \sin\sigma'\sin\theta'\cos\phi'$	$\sin\sigma'\cos\theta' - \cos\sigma'\sin\theta'\cos\phi'$	$\sin\theta'\sin\phi'$
y_0	$\cos\sigma'\sin\theta' - \sin\sigma'\cos\theta'\cos\phi'$	$\sin\sigma'\sin\theta' + \cos\sigma'\cos\theta'\cos\phi'$	$-\cos\theta'\sin\phi'$
z_0	$-\sin\sigma'\sin\phi'$	$\cos\sigma'\sin\phi'$	$\cos\phi'$

and the corresponding rotation coefficients are

$$p_{01}\,dt = -\sin\sigma'\sin\phi'\,d\theta' + \cos\sigma'\,d\phi',$$
$$q_{01}\,dt = \cos\sigma'\sin\phi'\,d\theta' + \sin\sigma'\,d\phi',$$
$$r_{01}\,dt = -d\sigma' + \cos\phi'\,d\theta'.$$

The coefficients of transformation from (x_1, y_1, z_1) to (x_2, y_2, z_2) are given by the table

	x_2	y_2	z_2
x_1	$\cos\Sigma\cos\Theta + \sin\Sigma\sin\Theta\cos\Phi$	$\sin\Sigma\cos\Theta - \cos\Sigma\sin\Theta\cos\Phi$	$\sin\Theta\sin\Phi$
y_1	$\cos\Sigma\sin\Theta - \sin\Sigma\cos\Theta\cos\Phi$	$\sin\Sigma\sin\Theta + \cos\Sigma\cos\Theta\cos\Phi$	$-\cos\Theta\sin\Phi$
z_1	$-\sin\Sigma\sin\Phi$	$\cos\Sigma\sin\Phi$	$\cos\Phi$

and the corresponding rotation coefficients are

$$p_{12}\,dt = -\sin\Sigma\sin\Phi\,d\Theta + \cos\Sigma\,d\Phi,$$
$$q_{12}\,dt = +\cos\Sigma\sin\Phi\,d\Theta + \sin\Sigma\,d\Phi,$$
$$r_{12}\,dt = -d\Sigma + \cos\Phi\,d\Theta,$$

and the coefficients of transformation from (x_2, y_2, z_2) to (x_3, y_3, z_3) are given by the table

	x_3	y_3	z_3
x_2	$\cos v \cos y$	$-\sin v$	$-\cos v \sin y$
y_2	$\sin v \cos y$	$\cos v$	$-\sin v \sin y$
z_2	$\sin y$	0	$\cos y$

and the corresponding rotation coefficients are given by

$$\begin{aligned} \mathrm{p}_{23}\, dt &= -\sin y\, dv, \\ \mathrm{q}_{23}\, dt &= - \quad dy, \\ \mathrm{r}_{23}\, dt &= \cos y\, dv. \end{aligned}$$

From the last two tables it is easy to deduce the coefficients of transformation from (x_1, y_1, z_1) to (x_3, y_3, z_3), these are given by the table

	x_3	y_3	z_3
x_1	$\cos y \{\cos(v-\Sigma)\cos\Theta - \sin(v-\Sigma)\sin\Theta\cos\Phi\} + \sin y \sin\Theta \sin\Phi$	$-\sin(v-\Sigma)\cos\Theta - \cos(v-\Sigma)\sin\Theta\cos\Phi$	$-\sin y \{\cos(v-\Sigma)\cos\Theta - \sin(v-\Sigma)\sin\Theta\cos\Phi\} + \cos y \sin\Theta\sin\Phi$
y_1	$\cos y \{\cos(v-\Sigma)\sin\Theta + \sin(v-\Sigma)\cos\Theta\cos\Phi\} - \sin y \cos\Theta \sin\Phi$	$-\sin(v-\Sigma)\sin\Theta + \cos(v-\Sigma)\cos\Theta\cos\Phi$	$-\sin y \{\cos(v-\Sigma)\sin\Theta + \sin(v-\Sigma)\cos\Theta\cos\Phi\} - \cos y \cos\Theta \sin\Phi$
z_1	$\cos y \{\sin(v-\Sigma)\sin\Phi \quad\quad \} + \sin y \cos\Phi$	$\cos(v-\Sigma)\sin\Phi$	$-\sin y \{\sin(v-\Sigma)\sin\Phi \quad\quad \} + \cos y \cos\Phi$

It is now easy to form the values of P, Q, R. We have

$$\begin{aligned} P dt = {} & \sin y\, dv \\ & + [\cos y \{\cos(v-\Sigma)\cos\Theta - \sin(v-\Sigma)\sin\Theta\cos\Phi\} + \sin y \sin\Theta\sin\Phi](-\sin\Sigma\sin\Phi\, d\Theta + \cos\Sigma\, d\Phi) \\ & + [\cos y \{\cos(v-\Sigma)\sin\Theta + \sin(v-\Sigma)\cos\Theta\cos\Phi\} - \sin y \cos\Theta\sin\Sigma](\cos\Sigma\sin\Phi\, d\Theta + \sin\Sigma\, d\Phi) \\ & + [\cos y \{\sin(v-\Sigma)\sin\Phi \quad\quad \} + \sin y \cos\Phi \quad](-d\Sigma + \cos\Phi\, d\Theta \quad\quad) \end{aligned}$$

$$
\begin{aligned}
&+\cos v\cos y\,(-\sin\sigma'\sin\phi'\,d\theta'+\cos\sigma'\,d\phi')\\
&+\sin v\cos y\,(\ \ \cos\sigma'\sin\phi'\,d\theta'+\sin\sigma'\,d\phi')\\
&+\qquad\sin y\,(-d\sigma'+\cos\phi'\,d\theta'\qquad\qquad),
\end{aligned}
$$

$$
\begin{aligned}
Qdt=-dy\\
&+[-\sin(v-\Sigma)\cos\Theta-\cos(v-\Sigma)\sin\Theta\cos\Phi](-\sin\Sigma\sin\Phi\,d\Theta+\cos\Sigma\,d\Phi)\\
&+[-\sin(v-\Sigma)\sin\Theta+\cos(v-\Sigma)\cos\Theta\cos\Phi](\ \ \cos\Sigma\sin\Phi\,d\Theta+\sin\Sigma\,d\Phi)\\
&+[\ \ \cos(v-\Sigma)\sin\Phi\qquad\qquad\qquad\qquad](-d\Sigma+\cos\Phi\,d\Theta\qquad\qquad)\\
&\qquad-\sin v\,(-\sin\Sigma'\sin\phi'\,d\theta'+\cos\sigma'\,d\phi')\\
&\qquad+\cos v\,(\ \ \cos\Sigma'\sin\phi'\,d\theta'+\sin\sigma'\,d\phi')\\
&\qquad+\ \ 0\ \ (-d\Sigma'+\cos\phi'\,d\theta'\qquad\qquad),
\end{aligned}
$$

$$
\begin{aligned}
Rdt=\cos y\,dv\\
&+[-\sin y\,\{\cos(v-\Sigma)\cos\Theta-\sin(v-\Sigma)\sin\Theta\cos\Phi\}+\cos y\sin\Theta\sin\Phi](-\sin\Sigma\sin\Phi\,d\Theta+\cos\Sigma\,d\Phi)\\
&+[-\sin y\,\{\cos(v-\Sigma)\sin\Theta+\sin(v-\Sigma)\cos\Theta\cos\Phi\}-\cos y\cos\Theta\sin\Phi](\ \ \cos\Sigma\sin\Phi\,d\Theta+\sin\Sigma\,d\Phi)\\
&+[-\sin y\,\{\sin(v-\Sigma)\sin\Phi\qquad\qquad\qquad\}+\cos y\cos\Phi\qquad](-d\Sigma+\cos\Phi\,d\Theta\qquad\qquad)\\
&\qquad-\cos v\sin y\,(-\sin\sigma'\sin\phi'\,d\theta'+\cos\sigma'\,d\phi')\\
&\qquad-\sin v\sin y\,(\ \ \cos\sigma'\sin\phi'\,d\theta'+\sin\sigma'\,d\phi')\\
&\qquad+\cos y\qquad(-d\sigma'+\cos\phi'\,d\theta'\qquad\qquad),
\end{aligned}
$$

which are at once reducible into the before-mentioned form.

It only remains to prove the expressions for $\frac{d\Omega}{dx_3}$, $\frac{d\Omega}{dy_3}$, $\frac{d\Omega}{dz_3}$ assumed in the memoir. The relations between (x_3, y_3, z_3) and (x_2, y_2, z_2) give

$$
\begin{aligned}
\frac{d\Omega}{dx_3}&=\ \ \cos v\cos y\frac{d\Omega}{dx_2}+\sin v\cos y\frac{d\Omega}{dy_2}+\sin y\frac{d\Omega}{dz_2},\\
\frac{d\Omega}{dy_3}&=-\qquad\sin v\frac{d\Omega}{dx_2}+\cos v\qquad\frac{d\Omega}{dy_2}\qquad\qquad,\\
\frac{d\Omega}{dz_3}&=-\cos v\sin y\frac{d\Omega}{dx_2}-\sin v\sin y\frac{d\Omega}{dy_2}+\cos y\frac{d\Omega}{dz_2},
\end{aligned}
$$

these formulæ apply to any point whatever the coordinates of which to the one set of axes are (x_2, y_2, z_2) and to the other set (x_3, y_3, z_3) and in obtaining them the coefficients of transformation v, y are of course considered as independent of either set of coordinates. But the coordinates (x_2, y_2, z_2) of the body are

$$
\begin{aligned}
x_2&=r\cos y\cos v,\\
y_2&=r\cos y\sin v,\\
z_2&=r\sin y\,;
\end{aligned}
$$

and using the foregoing equations in order to introduce into the formulæ the differential coefficients $\frac{d\Omega}{dr}$, $\frac{d\Omega}{dv}$, $\frac{d\Omega}{dy}$ in the place of $\frac{d\Omega}{dx_2}$, $\frac{d\Omega}{dy_2}$, $\frac{d\Omega}{dz_2}$, we find

$$\frac{d\Omega}{dx_2} = \cos v \cos y \frac{d\Omega}{dr} - \frac{1}{r} \sin v \sec y \frac{d\Omega}{dv} - \frac{1}{r} \cos v \sin y \frac{d\Omega}{dy},$$

$$\frac{d\Omega}{dy_2} = \sin v \cos y \frac{d\Omega}{dr} + \frac{1}{r} \cos v \sec y \frac{d\Omega}{dv} - \frac{1}{r} \sin v \sin y \frac{d\Omega}{dy},$$

$$\frac{d\Omega}{dz_2} = \sin y \frac{d\Omega}{dr} + \frac{1}{r} \cos y \frac{d\Omega}{dy}.$$

Substituting these values we have the required formulæ

$$\frac{d\Omega}{dx_3} = \frac{d\Omega}{dr}, \quad \frac{d\Omega}{dy_3} = \frac{1}{r} \sec y \frac{d\Omega}{dv}, \quad \frac{d\Omega}{dz_3} = \frac{1}{r} \frac{d\Omega}{dy}$$

where on the right-hand side $\Omega = \Omega\,(r, v, y, \Theta, \Sigma, \Phi, \theta', \sigma', \phi')$.

2, *Stone Buildings, W.C.* 28*th Dec.*, 1858.

216.

TABLES OF THE DEVELOPMENTS OF FUNCTIONS IN THE THEORY OF ELLIPTIC MOTION.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXIX. (1861), pp. 191—306. Read January 12, 1859.]

THE notation made use of is as follows; viz., r, f, a, e, g denote

r, the radius vector;
f, the true anomaly;
a, the mean distance;
e, the eccentricity;
g, the mean anomaly;

so that

$$\frac{r}{a}, = \text{elqr}\ (e, g),$$

and

$$f, = \text{elta}\ (e, g),$$

are known functions of e, g. Moreover, x denotes the periodic part of the quotient radius $\frac{r}{a}$, and y the equation of the centre, or periodic part of the true anomaly f; so that

$$\frac{r}{a} = 1 + x,$$

$$f = g + y,$$

and (x, y) are therefore also known functions of e, g.

Formulæ for the development in multiple cosines or sines, up to the terms in e^r, of the functions

$$(x^0, x^1, \ldots x^r) \begin{matrix}\cos\\ \sin\end{matrix} jy,$$

where j is an indeterminate symbol, are given by Leverrier in the *Annales de l'Observatoire de Paris*, tom. I. (1855), pp. 346—348. It occurred to me that it would be desirable to form tables such as are contained in the present memoir, of various functions of the forms

$$x^m \begin{matrix}\cos\\ \sin\end{matrix} jf, \qquad \left(\frac{r}{a}\right)^{\pm m} \begin{matrix}\cos\\ \sin\end{matrix} jf,$$

obtainable from Leverrier's formulæ by assigning particular integer values to the symbol j. The calculations were performed under my superintendence by Messrs Creedy and Davis; and I have to acknowledge my obligation for a grant to defray the expenses, made to me out of the Donation Fund at the disposal of the Council of the Royal Society.

A cosine series is in general represented in the form

$$\Sigma\,[\cos]^i \cos ig,$$

where i extends from $-\infty$ to $+\infty$, and the coefficients $[\cos]^i$ satisfy the condition

$$[\cos]^{-i} = [\cos]^i;$$

and in like manner a sine series is represented in the form

$$\Sigma\,[\sin]^i \sin ig,$$

where i extends from $-\infty$ to $+\infty$, and the coefficients $[\sin]^i$ satisfy the condition

$$[\sin]^{-i} = -[\sin]^i, \text{ and in particular}$$
$$[\sin]^0 = 0.$$

Thus $[\cos]^i$ is a general symbol denoting the coefficient of $\cos ig$, in a cosine series considered as involving as well negative as positive multiples of the argument g, or, what is the same thing, $[\cos]^0$ is the constant term and $[\cos]^i$ is the half coefficient of $\cos ig$, when the series is considered as containing only positive multiples of the argument. And in like manner $[\sin]^i$ is a general symbol for the coefficient of $\sin ig$, in a sine series considered as involving as well negative as positive multiples of the argument g, or, what is the same thing, $[\sin]^i$ is the half coefficient of $\sin ig$, when the series is considered as containing only positive multiples of the argument.

In the case of a pair of corresponding functions $x^m \cos jf$ and $x^m \sin jf$, or $\left(\frac{r}{a}\right)^{\pm m} \cos jf$ and $\left(\frac{r}{a}\right)^{\pm m} \sin jf$, one of them expanded in the form

$$\Sigma\, [\cos]^i \cos ig,$$

and the other in the form

$$\Sigma\, [\sin]^i \sin ig,$$

the sums and differences of the corresponding coefficients $[\cos]^i$, $[\sin]^i$, are represented by the notation

$$[\cos \pm \sin]^i,$$

and it is clear that we have

$$[\cos + \sin]^{-i} = [\cos - \sin]^i, \quad [\cos \pm \sin]^0 = [\cos]^0.$$

These coefficients $[\cos \pm \sin]^i$ are for many purposes equally useful with the coefficients $[\cos]^i$, $[\sin]^i$, and they are here tabulated accordingly.

The functions tabulated are as follows:—

$$\left(x^1,\ x^2, \ldots . x^7\right) \qquad , [\cos],$$

$$\left(x^0,\ x^1,\ x^2, \ldots . x^7\right) \begin{matrix}\cos\\ \sin\end{matrix} jf,\ j=1 \text{ to } j=7,\ [\cos], [\sin], [\cos \pm \sin],$$

$$\left(\left(\frac{r}{a}\right)^{+4}, \ldots \left(\frac{r}{a}\right)^{+1}, \log \frac{r}{a}, \left(\frac{r}{a}\right)^{-1}, \ldots \left(\frac{r}{a}\right)^{-5}\right) \qquad , [\cos],$$

$$\left(\left(\frac{r}{a}\right)^{+4}, \ldots \left(\frac{r}{a}\right)^{+1}, \qquad \left(\frac{r}{a}\right)^{-1}, \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix} jf, j=1 \text{ to } j=5,\ [\cos], [\sin], [\cos \pm \sin],$$

all the developments being carried up to e^7, the limit of the formulæ from which they are deduced.

The calculations are based upon Leverrier's formulæ for

$$(x^0,\ x^1, \ldots\ldots x^7) \begin{matrix}\cos\\ \sin\end{matrix} jy,$$

which are made use of under the following forms, which I call the Datum Tables $(x^0 \ldots\ldots x^7) \begin{matrix}\cos\\ \sin\end{matrix} jy$.

Cosine Table. $\frac{1}{2}$ of	$[\ ]^0$				$[\ ]^1$			
	$\frac{e^0}{2}$	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e}{4}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
$x^0 \cos jy$	$+1$	$-8j^2$	$+96j^4$	$-1280j^6$	0	$-30j^2$	$+800j^4$	$-16800j^8$
			$-54j^2$	$+3920j^4$			$-120j^2$	$+30590j^2$
				$-3440j^2$				$-29470j^2$
$x \cos jy$		$+4$	$-48j^2$	$0j^4$	-1	$+12j^2$	$-160j^4$	$+2240j^6$
				$-360j^2$		$+9$	$-370j^2$	$-980j^4$
							-50	$+9695j^2$
								$+245$
$x^2 \cos jy$		$+4$	$-96j^2$	$+1920j^4$		-12	$-244j^2$	$+16800j^4$
				$-1320j^2$			$+80$	$-6300j^2$
								-420
$x^3 \cos jy$			$+144$	$-2880j^2$		-18	$+480j^2$	$-10080j^4$
							$+360$	$-22050j^2$
								-3150
$x^4 \cos jy$			$+144$	$-5760j^2$			-960	$-5040j^2$
								$+10080$
$x^5 \cos jy$				$+14400$			-1200	$+50400j^2$
								$+37800$
$x^6 \cos jy$				$+14400$				-151200
$x^7 \cos jy$				0				-176400

Datum Table $(x_0 \ldots x^7) \cos jy \ldots$

Cosine Table. $\frac{1}{2}$ of	$[\ ]^2$			$[\ ]^2$		
	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^6}{96}$	$\frac{e^8}{7680}$	$\frac{e^7}{1290240}$
$x^0 \cos jy$	$+4j^2$	$-64j^4$ $-256j^2$	$+960j^6$ $+11820j^4$ $+4710j^2$	$+30j^2$	$-1200j^4$ $-3240j^2$	$+30240j^6$ $+243810j^4$ $+173502j^2$
$x \cos jy$	-2	$+192j^2$ $+64$	$-4800j^4$ $-8190j^2$ -1440	$-12j^2$ -9	$+240j^4$ $+2745j^2$ $+675$	$-4032j^6$ $-122220j^4$ $-225099j^2$ -35721
$x^2 \cos jy$	$+2$	$0j^2$ -96	$-480j^4$ $+1590j^2$ $+2400$	$+12$	$-840j^2$ -1080	$+10080j^4$ $+102060j^2$ $+61236$
$x^3 \cos jy$		0	$+7920j^2$ $+1440$	-6	$-240j^2$ $+540$	$+10080j^4$ $+85050j^2$ -17010
$x^4 \cos jy$		$+96$	$-1440j^2$ -8640		$+480$	$-95760j^2$ -90720
$x^5 \cos jy$			$+3600$		-600	$-10080j^2$ $+113400$
$x^6 \cos jy$			$+10800$			$+30240$
$x^7 \cos jy$			0			-105840

... Datum Table $(x^0 \ldots x^7) \cos jy$ ...

Cosine Table. $\frac{1}{2}$ of	$[\ ]^4$		$[\ ]^6$		$[\ ]^6$	$[\ ]^7$
	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^6}{92160}$	$\frac{e^7}{1290240}$
$x^0 \cos jy$	$+16j^4$ $+283j^2$	$-384j^6$ $-22440j^4$ $-51528j^2$	$+400j^4$ $+3360j^2$	$-16800j^6$ $-467530j^4$ $-972790j^2$	$+64j^6$ $+8660j^4$ $+48538j^2$	$+3360j^6$ $+193130j^4$ $+828758j^2$
$x \cos jy$	$-168j^2$ -64	$+7680j^4$ $+46260j^2$ $+9216$	$-80j^4$ $-2375j^2$ -625	$+2240j^6$ $+205660j^4$ $+905135j^2$ $+153125$	$-2880j^4$ $-37890j^2$ -7776	$-448j^6$ $-82460j^4$ $-689731j^2$ -117649
$x^2 \cos jy$	$+48j^2$ $+96$	$-960j^4$ $-22380j^2$ -15360	$+1080j^2$ $+1000$	$-50400j^4$ $-540540j^2$ -262500	$+480j^4$ $+21450j^2$ $+12960$	$+23520j^4$ $+444780j^2$ $+201684$
$x^3 \cos jy$	-72	$+1440j^2$ $+11520$	$-240j^2$ -900	$+3360j^4$ $+147630j^2$ $+236250$	$-7920j^2$ -12960	$-3360j^4$ $-210630j^2$ -216090
$x^4 \cos jy$	$+24$	$+2880j^2$ $+0$	$+480$	$+35280j^2$ -84000	$+1440j^2$ $+8640$	$+65520j^2$ $+164640$
$x^5 \cos jy$		-7200	-120	$-30240j^2$ -63000	-3600	$-10080j^2$ -88200
$x^6 \cos jy$		$+4320$		$+90720$	$+720$	$+30240$
$x^7 \cos jy$		0		-35280	0	-5040

... Datum Table $(x^0 \ldots x^7) \cos jy$.

Sine Table. $\frac{1}{2}$ of	$[\ ]^0$				$[\ ]^1$			
	$\frac{e^0}{2}$	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e}{4}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
$x^0 \sin jy$					$+2j$	$-24j^3$	$+320j^5$	$-4480j^7$
						$-6j$	$+260j^3$	$-1400j^5$
							$+100j$	$-17290j^3$
								$+7490j$
$x \sin jy$						$+21j$	$-400j^3$	$+5040j^5$
							$-180j$	$+35j^3$
								$+6125j$
$x^2 \sin jy$						$+12j$	$-320j^3$	$+6720j^5$
							$+160j$	$-420j^3$
								$+3780j$
$x^3 \sin jy$							$+840j$	$-25200j^3$
								$-10080j$
$x^4 \sin jy$							$+480j$	$-20160j^3$
								$+25200j$
$x^5 \sin jy$								$+88200j$
$x^6 \sin jy$								$+50400j$
$x^7 \sin jy$								0

Datum Table $(x^8 \ldots x^7) \sin jy \ldots$

Sine Table. $\frac{1}{2}$ of	$[\ ]^2$			$[\ ]^3$		
	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
$x^0 \sin jy$	$+5j$	$-240j^3$ $-88j$	$+6000j^5$ $+8895j^3$ $+2040j$	$+8j^3$ $+26j$	$-160j^5$ $-3070j^3$ $-1290j$	$+2688j^7$ $+125160j^5$ $+257166j^3$ $+59850j$
$x \sin jy$	$-4j$	$+64j^3$ $+184j$	$-960j^5$ $-8340j^3$ $-5700j$	$-27j$	$+1400j^3$ $+2300j$	$-36624j^5$ $-207165j^7$ $-145971j$
$x^2 \sin jy$		$-72j$	$-1200j^3$ $+4260j$	$+12j$	$-160j^3$ $-1600j$	$+1344j^5$ $+39900j^3$ $+130788j$
$x^3 \sin jy$		$-96j$	$+2880j^3$ $+6120j$		$-180j$	$+55440j^3$ $+32760j$
$x^4 \sin jy$			$-8280j$		$+720j$	$-20160j^3$ $-156240j$
$x^5 \sin jy$			$-7200j$			$+37800j$
$x^6 \sin jy$			0			$+90720j$
$x^7 \sin jy$			0			0

... Datum Table $(x^0 \ldots x^7) \sin j - \ldots$

Sine Table. $\frac{1}{2}$ of	$[\ ]^4$		$[\ ]^5$		$[\ ]^6$	$[\ ]^7$
	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^6}{92160}$	$\frac{e^7}{1290240}$
$x^0 \sin j\,y$	$+120j^3$ $+206j$	$-4800j^5$ $-49200j^3$ $-21648j$	$+32j^3$ $+1790j^3$ $+2194j$	$-896j^7$ $-124600j^5$ $-939722j^3$ $-416990j$	$+1200j^5$ $+29835j^3$ $+29352j$	$+128j^7$ $+35560j^5$ $+563486j^3$ $+472730j$
$x \sin j\,y$	$-32j^3$ $-236j$	$+768j^5$ $+27840j^3$ $+35736j$	$-760j^3$ $-2640j$	$+34608j^5$ $+599935j^3$ $+657601j$	$-192j^5$ $-15780j^3$ $-36324j$	$-9744j^5$ $-339745j^3$ $-595231j$
$x^2 \sin j\,y$	$+156j$	$-7680j^3$ $-30120j$	$+160j^3$ $+2080j$	$-4032j^5$ $-237300j^3$ $-610764j$	$+5520j^3$ $+31620j$	$+1344j^5$ $+152460j^3$ $+552468j$
$x^3 \sin j\,y$	$-48j$	$0j^3$ $+8640j$	$-1020j$	$+35280j^3$ $+303240j$	$-960j^3$ $-19320j$	$-45360j^3$ $-384720j$
$x^4 \sin j\,y$		$+7200j$	$+240j$	$+6720j^3$ $+11760j$	$+7560j$	$+6720j^3$ $+193200j$
$x^5 \sin j\,y$		$-5760j$		$-113400j$	$-1440j$	$-63000j$
$x^6 \sin j\,y$		0		$+50400j$	0	$+10080j$
$x^7 \sin j\,y$		0		0	0	0

... Datum Table $(x^0 \ldots x^7) \sin j\,y$.

It may be remarked that with a view to the convenience of calculation, the Datum Tables give the half values of the several functions; thus, e.g., we have

$$\frac{1}{2}x^0\cos jy = \left\{+1\frac{e^0}{2} - 8j^2\frac{e^2}{16} + (96j^4 - 54j^2)\frac{e^4}{768} + \&c.\right\} + \left\{0\frac{e}{4} - 30j^3\frac{e^3}{96} + \&c.\right\} 2\cos g$$

$$+ \left\{4j^2\frac{e^2}{16} + \&c.\right\} 2\cos 2g + \&c.$$

and so for the other functions in the two tables.

The calculation, for a given value of j, of the functions $(x^0, x^1 \ldots\ldots x^7)\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf$ is performed as follows; the numerical value of j is in the first instance substituted throughout the two Datum tables, so as to obtain the numerical values of the separate terms $\pm Aj^m$; when this is done, the terms in each compartment are added together with the proper signs, and the results written down in a single table as follows, viz. if $j = 1$, and if we write down only the portion of the table which corresponds to $x^0\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$, (the portions corresponding to $x\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$, $x^2\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$ &c.... $x^7\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$, being of a precisely similar form, and coming into the table in order one under the other) this is

$\frac{1}{2}x^0\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$	$[\ ]^0$				$[\ ]^1$			
	$\frac{e^0}{2}$	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e}{4}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
[cos]	+ 1	− 8	+ 42	− 800	0	− 30	+ 680	− 15680
[sin]	0	0	0	0	2	− 30	+ 680	− 15680
[cos + sin]	+ 1	− 8	+ 42	− 800	+ 2	− 60	+ 1360	− 31360
[cos − sin]	Do.	"	"	"	− 2	0	0	0

$\frac{1}{2}x^0\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} y$	$[\ ]^2$			$[\ ]^3$		
	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
[cos]	+ 4	− 320	+ 17490	+ 30	− 4440	+ 447552
[sin]	+ 5	− 328	+ 16935	+ 34	− 4520	+ 444864
[cos + sin]	+ 9	− 648	+ 34425	+ 64	− 8960	+ 892416
[cos − sin]	− 1	+ 8	+ 555	− 4	+ 80	+ 2688

$\frac{1}{2}x^0 \begin{matrix}\cos\\ \sin\end{matrix} y$	$[\ ]^4$		$[\ ]^5$		$[\ ]^6$	$[\ ]^7$
	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^6}{92160}$	$\frac{e^7}{1290240}$
[cos]	+ 299	− 74352	+ 3760	− 1457120	+ 57262	+ 1025248
[sin]	+ 326	− 75648	+ 4016	− 1482208	+ 60387	+ 1071904
[cos + sin]	+ 625	− 150000	+ 7776	− 2939328	+ 117649	+ 2097152
[cos − sin]	− 27	+ 1296	− 256	+ 25088	− 3125	− 46656

where the lines [cos + sin], [cos − sin] are obtained by the addition and subtraction of the lines [cos], [sin].

The lines [cos + sin], [cos − sin], are then rearranged by reading them backwards and forwards from the terms which are in the [cos − sin] line, under $[\ ]^j$; viz., in the present case, where $j = 1$, from the terms − 2, 0, 0, 0 under $[\ ]^1$, as follows:—

$x^e \begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^0$				$[\ ]^1$			
	$\frac{e}{4}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^0}{2}$	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$
$\frac{1}{2}$ [cos + sin]	− 2	0	0	0	+ 1	− 8	+ 42	− 800
$\frac{1}{2}$ [cos − sin]	Do.	,,	,,	,,	0	− 1	+ 8	+ 555
[cos]	− 4	0	0	0	+ 1	− 9	+ 50	− 245
[sin]	0	0	0	0	+ 1	− 7	+ 34	− 1355

$x^0 \begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^2$				$[\ ]^3$			$[\ ]^4$		
	$\frac{e}{4}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^2}{16}$	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^3}{96}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$
$\frac{1}{2}$ [cos + sin]	+ 2	− 60	+ 1360	− 31360	+ 9	− 648	+ 34425	+ 64	− 8960	+ 892416
$\frac{1}{2}$ [cos − sin]	0	− 4	+ 80	+ 2688	0	− 27	+ 1296	0	− 256	+ 25088
[cos]	+ 2	− 64	+ 1440	− 28672	+ 9	− 675	+ 35721	+ 64	− 9216	+ 917504
[sin]	+ 2	− 56	+ 1280	− 24048	+ 9	− 621	+ 33129	+ 64	− 8704	+ 867328

$x^0 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	$[\ ]^8$
	$\frac{e^4}{768}$	$\frac{e^6}{92160}$	$\frac{e^5}{7680}$	$\frac{e^7}{1290240}$	$\frac{e^6}{92160}$	$\frac{e^7}{1290240}$
$\frac{1}{2}[\cos+\sin]$	+625	−150000	+7776	−2939328	+117649	+2097152
$\frac{1}{2}[\cos-\sin]$	0	− 3125	0	− 46656	0	0
$[\cos]$	+625	−153125	+7776	−2985984	+117649	+2097152
$[\sin]$	+625	−146875	+7776	−2892672	+117649	+2097152

where the lines [cos], [sin] are obtained by the addition and subtraction of the lines $\frac{1}{2}[\cos+\sin]$, $\frac{1}{2}[\cos-\sin]$; and it then only remains to reduce the fractions, at the same time multiplying by 2 the lines $\frac{1}{2}[\cos+\sin]$, $\frac{1}{2}[\cos-\sin]$, to obtain instead the lines $[\cos+\sin]$, $[\cos-\sin]$. We have thus,

$x^0 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^0$				$[\ ]^1$			
	e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$[\cos+\sin]$	−1	0	0	0	+1	−1	$+\frac{7}{64}$	$-\frac{5}{288}$
$[\cos-\sin]$	Do.	,,	,,	,,	0	$-\frac{1}{8}$	$+\frac{1}{48}$	$+\frac{37}{3072}$
$[\cos]$	−1	0	0	0	$+\frac{1}{2}$	$-\frac{9}{16}$	$+\frac{25}{384}$	$-\frac{49}{18432}$
$[\sin]$	0	0	0	0	$+\frac{1}{2}$	$-\frac{7}{16}$	$+\frac{17}{384}$	$-\frac{271}{18432}$

$x^0 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^2$				$[\ ]^3$			$[\ ]^4$		
	e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
$[\cos+\sin]$	+1	$-\frac{5}{4}$	$+\frac{17}{48}$	$-\frac{7}{144}$	$+\frac{9}{8}$	$-\frac{27}{16}$	$+\frac{765}{1024}$	$+\frac{4}{3}$	$-\frac{7}{3}$	$+\frac{83}{60}$
$[\cos-\sin]$	0	$-\frac{1}{12}$	$+\frac{1}{48}$	$+\frac{1}{240}$	0	$-\frac{9}{128}$	$+\frac{9}{320}$	0	$-\frac{1}{15}$	$+\frac{7}{180}$
$[\cos]$	$+\frac{1}{2}$	$-\frac{2}{3}$	$+\frac{3}{16}$	$-\frac{1}{45}$	$+\frac{9}{16}$	$-\frac{225}{256}$	$+\frac{3969}{10240}$	$+\frac{2}{3}$	$-\frac{6}{5}$	$+\frac{32}{45}$
$[\sin]$	$+\frac{1}{2}$	$-\frac{7}{12}$	$+\frac{1}{6}$	$-\frac{19}{720}$	$+\frac{9}{16}$	$-\frac{207}{256}$	$+\frac{3681}{10240}$	$+\frac{2}{3}$	$-\frac{17}{15}$	$+\frac{121}{180}$

$x^0 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	$[\ ]^8$
	e^4	e^6	e^5	e^7	e^6	e^7
$[\cos+\sin]$	$+\frac{625}{384}$	$-\frac{625}{192}$	$+\frac{81}{40}$	$-\frac{5103}{1120}$	$+\frac{117649}{46080}$	$+\frac{1024}{315}$
$[\cos-\sin]$	0	$-\frac{625}{9216}$	0	$-\frac{81}{1120}$	0	0
$[\cos]$	$+\frac{625}{768}$	$-\frac{30625}{18432}$	$+\frac{81}{80}$	$-\frac{81}{35}$	$+\frac{117649}{92160}$	$+\frac{512}{315}$
$[\sin]$	$+\frac{625}{768}$	$-\frac{29375}{18432}$	$+\frac{81}{80}$	$-\frac{2511}{1120}$	$+\frac{117649}{92160}$	$+\frac{512}{315}$

which is, in fact, part of the table $(x^0 \ldots x^7) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$. It is to be remarked, that when in any column the second term is zero, the third and fourth terms are each equal to one-half of the first term, and that I have in the tables represented them by the letters H, H.

The foregoing method depends, it is easy to see, on the following proposition; viz., writing

$$x^m \cos jf = \Sigma\, [\cos]_f{}^i \cos ig, \quad x^m \sin jf = \Sigma\, [\sin]_f{}^i \sin ig,$$
$$x^m \cos jy = \Sigma\, [\cos]_y{}^i \cos ig, \quad x^m \sin jy = \Sigma\, [\sin]_y{}^i \sin ig,$$

where, as before, $f = g + y$, then we have

$$[\cos+\sin]_f{}^i = [\cos+\sin]_y{}^{i-j},$$
$$[\cos-\sin]_f{}^i = [\cos-\sin]_y{}^{i+j},$$

the latter of which follows immediately from the former. To prove this, it is to be remarked that in general

$$\cos\alpha\, \Sigma\, [\cos]^i \cos ig = \Sigma\, [\cos]^i \cos(ig+\alpha),$$
$$\sin\alpha\, \Sigma\, [\sin]^i \sin ig = -\Sigma\, [\sin]^i \cos(ig+\alpha),$$
$$\sin\alpha\, \Sigma\, [\cos]^i \cos ig = \Sigma\, [\cos]^i \sin(ig+\alpha),$$
$$\cos\alpha\, \Sigma\, [\sin]^i \sin ig = \Sigma\, [\sin]^i \sin(ig+\alpha);$$

whence, putting $\alpha = jg$,

$$\cos jf = \cos(jy+jg) = \Sigma\, [\cos+\sin]_y{}^i \cos(i+j)\, g$$
$$= \Sigma\, \tfrac{1}{2}\, \{[\cos+\sin]_y{}^{i-j} + [\cos+\sin]_y{}^{-i-j}\} \cos ig,$$
$$\sin jf = \sin(jy+jg) = \Sigma\, [\cos+\sin]_y{}^i \sin(i+j)\, g$$
$$= \Sigma\, \tfrac{1}{2}\, \{[\cos+\sin]_y{}^{i-j} - [\cos+\sin]_y{}^{-i-j}\} \sin ig;$$

and therefore

$$[\cos]_f{}^i = \tfrac{1}{2}\{[\cos+\sin]_y{}^{i-j} + [\cos+\sin]_y{}^{-i-j}\},$$

$$[\sin]_f{}^i = \tfrac{1}{2}\{[\cos+\sin]_y{}^{i-j} - [\cos+\sin]_y{}^{-i-j}\};$$

and, adding and subtracting, we have the theorem. The coefficients contained in the Datum Tables $(x^0 \ldots x^7)\,{}^{\cos}_{\sin}\,jy$ were taken to be $\frac{1}{2}[\cos]_y{}^i$, $\frac{1}{2}[\sin]_y{}^i$, in order that the calculations might in the first instance give $\frac{1}{2}[\cos+\sin]_f{}^i$, $\frac{1}{2}[\cos+\sin]_f{}^j$, and thence by an addition and subtraction, without a division by 2, give $[\cos]_f{}^j$, $[\sin]_f{}^j$.

The resulting values admit of a very simple verification; in fact, writing down the two equations

$$x^m \cos jf = \Sigma\,[\cos]^i \cos ig, \qquad x^m \sin jf = \Sigma\,[\sin]^i \sin ig,$$

or, what is the same thing, the summation now extending only from $i=1$ to $i=\infty$,

$$\tfrac{1}{2}x^m \cos jf = \tfrac{1}{2}[\cos]^0 + \Sigma'\,[\cos]^i \cos ig, \qquad \tfrac{1}{2}x^m \sin jf = \Sigma'\,[\sin]^i \sin ig;$$

if in the first equation, and in the second equation differentiated with respect to g, we put $g=0$, then, observing that $g=0$ gives $f=0$, $x=-e$, $\dfrac{df}{dg} = \dfrac{(1+e)^2}{(1-e^2)^{\frac{3}{2}}}$, we have

$$\tfrac{1}{2}[\cos]^0 + \Sigma'\,[\cos]^i = \tfrac{1}{2}j(-e)^m,$$

$$\Sigma' i\,[\sin]^i = \tfrac{1}{2}j(-e)^m\left(1 + 2e + \tfrac{5}{2}e^2 + 3e^3 + \tfrac{27}{8}e^4 + \tfrac{15}{4}e^5 + \tfrac{65}{16}e^6 + \tfrac{35}{8}e^7 + \ldots\right).$$

The values of $(x^0 \ldots x^7)\,{}^{\cos}_{\sin}\,jf$ being known, those of $\left(\dfrac{r}{a}\right)^{\pm m}{}^{\cos}_{\sin}\,jf$ can, of course, be determined by merely substituting for $\dfrac{r}{a}$ its value $(1+x)$, expanding and reducing; and a convenient mode of arranging the calculations presents itself so obviously, that no explanation is required. The verification is similar to that for $x^m\,{}^{\cos}_{\sin}\,jf$. The only difference is, that for $g=0$ we have the value of r, viz. $1-e$, in the place of the value $-e$ of x. A similar verification is applicable to the Datum Tables.

To make the arrangement of the Tables clear, I have given at the end of them some examples to show how they are to be read off.

Table (x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7).

		$[\ ]^0$				$[\ ]^1$				$[\ ]^2$		
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^0		$+1$	0	0	0	0	0	0	0	0	0	0
x^1			$+\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+\frac{3}{16}$	$-\frac{5}{384}$	$+\frac{7}{18432}$	$-\frac{1}{4}$	$+\frac{1}{6}$	$-\frac{1}{32}$
x^2			$+\frac{1}{2}$	0	0		$-\frac{1}{4}$	$+\frac{1}{48}$	$-\frac{1}{1536}$	$+\frac{1}{4}$	$-\frac{1}{4}$	$+\frac{5}{96}$
x^3				$+\frac{3}{8}$	0		$-\frac{3}{8}$	$+\frac{3}{32}$	$-\frac{5}{1024}$		0	$+\frac{1}{32}$
	[cos]											
x^4				$+\frac{3}{8}$	0			$-\frac{1}{4}$	$+\frac{1}{64}$		$+\frac{1}{4}$	$-\frac{3}{16}$
x^5					$+\frac{5}{16}$			$-\frac{5}{16}$	$+\frac{15}{256}$			$+\frac{5}{64}$
x^6					$+\frac{5}{16}$				$-\frac{15}{64}$			$+\frac{15}{64}$
x^7									$-\frac{35}{128}$			

Table (x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7).

		$[\ ]^3$			$[\ ]^4$		$[\ ]^5$		$[\ ]^6$	$[\ ]^7$
		e^3	e^5	e^7	e^4	e^6	e^5	e^6	e^6	e^7
x^0		0	0	0	0	0	0	0	0	0
x^1		$-\frac{3}{16}$	$+\frac{45}{256}$	$-\frac{567}{10240}$	$-\frac{1}{6}$	$+\frac{1}{5}$	$-\frac{125}{768}$	$+\frac{4375}{18432}$	$-\frac{27}{160}$	$-\frac{16807}{91260}$
x^2		$+\frac{1}{4}$	$-\frac{9}{32}$	$+\frac{243}{2560}$	$+\frac{1}{4}$	$-\frac{1}{3}$	$+\frac{25}{96}$	$-\frac{625}{1536}$	$+\frac{9}{32}$	$+\frac{2401}{7680}$
x^3		$-\frac{1}{8}$	$+\frac{9}{64}$	$-\frac{27}{1024}$	$-\frac{3}{16}$	$+\frac{1}{4}$	$-\frac{15}{64}$	$+\frac{375}{1024}$	$-\frac{9}{32}$	$-\frac{343}{1024}$
	[cos]									
x^4			$+\frac{1}{8}$	$-\frac{9}{64}$	$+\frac{1}{16}$	0	$+\frac{1}{8}$	$-\frac{25}{192}$	$+\frac{3}{16}$	$+\frac{49}{192}$
x^5			$-\frac{5}{32}$	$+\frac{45}{256}$		$-\frac{5}{32}$	$-\frac{1}{32}$	$-\frac{25}{256}$	$-\frac{5}{64}$	$-\frac{35}{256}$
x^6				$+\frac{3}{64}$		$+\frac{3}{32}$		$+\frac{9}{64}$	$+\frac{1}{64}$	$+\frac{3}{64}$
x^7				$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^0$				$[\ ]^1$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^0	[cos + sin]	-1	0	0	0	$+1$	-1	$+\frac{7}{64}$	$-\frac{5}{288}$
	[cos − sin]	,,	,,	,,	,,	0	$-\frac{1}{8}$	$+\frac{1}{48}$	$+\frac{37}{3072}$
	[cos]	-1	0	0	0	H	$-\frac{9}{16}$	$+\frac{25}{384}$	$-\frac{49}{18432}$
	[sin]	0	0	0	0	H	$-\frac{7}{16}$	$+\frac{17}{384}$	$-\frac{271}{18432}$
x^1	[cos + sin]	$-\frac{1}{2}$	0	0	0		$+\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{128}$
	[cos − sin]	,,	,,	,,	,,		$+\frac{1}{4}$	$+\frac{1}{48}$	$+\frac{19}{1536}$
	[cos]	$-\frac{1}{2}$	0	0	0		$+\frac{3}{8}$	$-\frac{5}{96}$	$+\frac{7}{3072}$
	[sin]	0	0	0	0		$+\frac{1}{8}$	$-\frac{7}{96}$	$-\frac{31}{3072}$
x^2	[cos + sin]		$-\frac{1}{2}$	0	0		$+\frac{1}{2}$	$-\frac{1}{4}$	$+\frac{5}{384}$
	[cos − sin]		,,	,,	,,		$+\frac{1}{4}$	$-\frac{1}{16}$	$+\frac{5}{512}$
	[cos]		$-\frac{1}{2}$	0	0		$+\frac{3}{8}$	$-\frac{5}{32}$	$+\frac{35}{3072}$
	[sin]		0	0	0		$+\frac{1}{8}$	$-\frac{3}{32}$	$+\frac{5}{3072}$
x^3	[cos + sin]		$-\frac{3}{8}$	0	0			$+\frac{3}{8}$	$-\frac{1}{16}$
	[cos − sin]		,,	,,	,,			$+\frac{1}{4}$	$+\frac{1}{128}$
	[cos]		$-\frac{3}{8}$	0	0			$+\frac{5}{16}$	$-\frac{7}{256}$
	[sin]		0	0	0			$+\frac{1}{16}$	$-\frac{9}{256}$

N.B.—In all the Tables H, H, in the third and fourth lines of a set of four lines, denote one half of the number above them in the first line. The terms $[\cos - \sin]^0$, denoted by ,, ,, ... are the same as the terms $[\cos + \sin]^0$ immediately above them.

... *Table* $(x^0, x^1, x^2, x^3)\,{}^{\cos}_{\sin}f$ ...

	$\cos/\sin f$	$[\ ]^2$				$[\ ]^3$			$[\ ]^4$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
x^0	[cos + sin]	$+1$	$-\frac{5}{4}$	$+\frac{17}{48}$	$-\frac{7}{144}$	$+\frac{9}{8}$	$-\frac{27}{16}$	$+\frac{768}{1024}$	$+\frac{4}{3}$	$-\frac{7}{3}$	$+\frac{83}{60}$
	[cos − sin]	0	$-\frac{1}{12}$	$+\frac{1}{48}$	$+\frac{1}{240}$	0	$-\frac{9}{128}$	$+\frac{9}{320}$	0	$-\frac{1}{15}$	$+\frac{7}{180}$
	[cos]	H	$-\frac{2}{3}$	$+\frac{3}{16}$	$-\frac{1}{45}$	H	$-\frac{225}{256}$	$+\frac{3969}{10240}$	H	$-\frac{6}{5}$	$+\frac{32}{45}$
	[sin]	H	$-\frac{7}{12}$	$+\frac{1}{6}$	$-\frac{19}{720}$	H	$-\frac{207}{256}$	$+\frac{3681}{10240}$	H	$-\frac{17}{15}$	$+\frac{121}{180}$
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{7}{8}$	$-\frac{29}{96}$	$+\frac{5}{144}$	$-\frac{3}{4}$	$+\frac{21}{16}$	$-\frac{327}{512}$	-1	$+\frac{23}{12}$	$-\frac{289}{240}$
	[cos − sin]	0	$+\frac{1}{8}$	$-\frac{1}{96}$	$+\frac{1}{240}$	0	$+\frac{3}{32}$	$-\frac{33}{1280}$	0	$+\frac{1}{12}$	$-\frac{29}{720}$
	[cos]	H	$+\frac{1}{2}$	$-\frac{5}{32}$	$+\frac{7}{360}$	H	$+\frac{45}{64}$	$-\frac{1701}{5120}$	H	$+1$	$-\frac{28}{45}$
	[sin]	H	$+\frac{3}{8}$	$-\frac{7}{48}$	$+\frac{11}{720}$	H	$+\frac{39}{64}$	$-\frac{1569}{5120}$	H	$+\frac{11}{12}$	$-\frac{419}{720}$
x^2	[cos + sin]		0	$-\frac{1}{12}$	$+\frac{1}{32}$	$+\frac{1}{4}$	$-\frac{7}{16}$	$+\frac{73}{512}$	$+\frac{1}{2}$	$-\frac{23}{24}$	$+\frac{257}{480}$
	[cos − sin]		0	$-\frac{1}{24}$	$+\frac{1}{480}$	0	$-\frac{1}{32}$	$-\frac{5}{256}$	0	$-\frac{1}{24}$	$-\frac{1}{480}$
	[cos]		0	$-\frac{1}{16}$	$+\frac{1}{60}$	H	$-\frac{15}{64}$	$+\frac{63}{1024}$	H	$-\frac{1}{2}$	$+\frac{4}{15}$
	[sin]		0	$-\frac{1}{48}$	$+\frac{7}{480}$	H	$-\frac{13}{64}$	$+\frac{83}{1024}$	H	$-\frac{11}{24}$	$+\frac{43}{160}$
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{7}{16}$	$-\frac{7}{64}$		$-\frac{1}{4}$	$+\frac{51}{128}$	$-\frac{1}{8}$	$+\frac{1}{32}$	$+\frac{33}{128}$
	[cos − sin]		$-\frac{1}{8}$	$+\frac{1}{8}$	$-\frac{1}{64}$		$-\frac{1}{16}$	$+\frac{3}{32}$	0	$-\frac{1}{32}$	$+\frac{29}{384}$
	[cos]		$-\frac{1}{4}$	$+\frac{9}{32}$	$-\frac{1}{16}$		$-\frac{5}{32}$	$+\frac{63}{256}$	H	0	$+\frac{1}{6}$
	[sin]		$-\frac{1}{8}$	$+\frac{5}{32}$	$-\frac{3}{64}$		$-\frac{3}{32}$	$+\frac{39}{256}$	H	$+\frac{1}{32}$	$+\frac{35}{384}$

... *Table* (x^0, x^1, x^2, x^3) $\frac{\cos}{\sin} f$.

	$\frac{\cos}{\sin} f$	$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	$[\ ]^8$
		e^4	e^6	e^5	e^7	e^7	e^7
x^0	[cos + sin]	$+\frac{625}{384}$	$-\frac{625}{192}$	$+\frac{81}{40}$	$-\frac{729}{160}$	$+\frac{117649}{46080}$	$+\frac{1024}{315}$
	[cos − sin]	0	$-\frac{625}{9216}$	0	$-\frac{81}{1120}$	0	0
	[cos]	H	$-\frac{30625}{18432}$	H	$-\frac{81}{35}$	H	H
	[sin]	H	$-\frac{29375}{18432}$	H	$-\frac{2511}{1120}$	H	H
x^1	[cos + sin]	$-\frac{125}{96}$	$+\frac{2125}{768}$	$-\frac{27}{16}$	$+\frac{1269}{320}$	$-\frac{16807}{7680}$	$-\frac{128}{45}$
	[cos − sin]	0	$+\frac{125}{1536}$	0	$+\frac{27}{320}$	0	0
	[cos]	H	$+\frac{4375}{3072}$	H	$+\frac{81}{40}$	H	H
	[sin]	H	$+\frac{1375}{1024}$	H	$+\frac{621}{320}$	H	H
x^2	[cos + sin]	$+\frac{25}{32}$	$-\frac{425}{256}$	$+\frac{9}{8}$	$-\frac{423}{160}$	$+\frac{2401}{1536}$	$+\frac{32}{15}$
	[cos − sin]	0	$-\frac{25}{512}$	0	$-\frac{9}{160}$	0	0
	[cos]	H	$-\frac{875}{1024}$	H	$-\frac{27}{20}$	H	H
	[sin]	H	$-\frac{825}{1024}$	H	$-\frac{207}{160}$	H	H
x^3	[cos + sin]	$-\frac{5}{16}$	$+\frac{15}{32}$	$-\frac{9}{16}$	$+\frac{9}{8}$	$-\frac{343}{384}$	$-\frac{4}{3}$
	[cos − sin]	0	$-\frac{5}{384}$	0	0	0	0
	[cos]	H	$+\frac{175}{768}$	H	H	H	H
	[sin]	H	$+\frac{185}{768}$	H	H	H	H

Table (x^4, z^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^0$				$[\ ]^1$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^4	[cos + sin]			$-\frac{3}{8}$	0			$+\frac{3}{8}$	$-\frac{1}{8}$
	[cos − sin]			,,	,,			$+\frac{1}{4}$	$-\frac{5}{128}$
	[cos]			$-\frac{3}{8}$	0			$+\frac{5}{16}$	$-\frac{21}{256}$
	[sin]			0	0			$+\frac{1}{16}$	$-\frac{11}{256}$
x^5	[cos + sin]			$-\frac{5}{16}$	0				$+\frac{5}{16}$
	[cos − sin]			,,	,,				$+\frac{15}{64}$
	[cos]			$-\frac{5}{16}$	0				$+\frac{35}{128}$
	[sin]			0	0				$+\frac{5}{128}$
x^6	[cos + sin]				$-\frac{5}{16}$				$+\frac{5}{16}$
	[cos − sin]				,,				$+\frac{15}{64}$
	[cos]				$-\frac{5}{16}$				$+\frac{35}{128}$
	[sin]				0				$+\frac{5}{128}$
x^7	[cos + sin]				$-\frac{35}{128}$				
	[cos − sin]				,,				
	[cos]				$-\frac{35}{128}$				
	[sin]				0				

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^2$				$[\ ]^3$			$[\ ]^4$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
x^4	[cos + sin]			$-\frac{1}{8}$	$+\frac{1}{64}$		$+\frac{1}{4}$	$-\frac{51}{128}$		$+\frac{5}{16}$	$-\frac{9}{16}$
	[cos − sin]			$-\frac{1}{16}$	$-\frac{1}{64}$		$+\frac{1}{16}$	$-\frac{3}{32}$		$+\frac{1}{16}$	$-\frac{5}{48}$
	[cos]			$-\frac{3}{32}$	0		$+\frac{5}{32}$	$-\frac{63}{256}$		$+\frac{3}{16}$	$-\frac{1}{3}$
	[sin]			$-\frac{1}{32}$	$+\frac{1}{64}$		$+\frac{3}{32}$	$-\frac{39}{256}$		$+\frac{1}{8}$	$-\frac{11}{48}$
x^5	[cos + sin]			$-\frac{5}{16}$	$+\frac{35}{128}$			$-\frac{5}{64}$		$-\frac{5}{32}$	$+\frac{7}{32}$
	[cos − sin]			$-\frac{5}{32}$	$+\frac{13}{128}$			$-\frac{1}{32}$		$-\frac{1}{32}$	$+\frac{1}{32}$
	[cos]			$-\frac{15}{64}$	$+\frac{3}{16}$			$-\frac{7}{128}$		$-\frac{3}{32}$	$+\frac{1}{8}$
	[sin]			$-\frac{5}{64}$	$+\frac{11}{128}$			$-\frac{3}{128}$		$-\frac{1}{16}$	$+\frac{3}{32}$
x^6	[cos + sin]				$-\frac{5}{32}$			$+\frac{15}{64}$			$+\frac{3}{16}$
	[cos − sin]				$-\frac{3}{32}$			$+\frac{3}{32}$			$+\frac{1}{16}$
	[cos]				$-\frac{1}{8}$			$+\frac{21}{128}$			$+\frac{1}{8}$
	[sin]				$-\frac{1}{32}$			$+\frac{9}{128}$			$+\frac{1}{16}$
x^7	[cos + sin]				$-\frac{35}{128}$						$-\frac{21}{128}$
	[cos − sin]				$-\frac{21}{128}$						$-\frac{7}{128}$
	[cos]				$-\frac{7}{32}$						$-\frac{7}{64}$
	[sin]				$-\frac{7}{128}$						$-\frac{7}{128}$

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} f$	$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	$[\ ]^8$
		e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]	$+\frac{1}{16}$	$+\frac{7}{32}$	$+\frac{3}{16}$	$-\frac{3}{64}$	$+\frac{49}{128}$	$+\frac{2}{3}$
	[cos − sin]	0	$+\frac{7}{128}$	0	$+\frac{3}{64}$	0	0
	[cos]	H	$+\frac{35}{256}$	H	0	H	H
	[sin]	H	$+\frac{21}{256}$	H	$-\frac{3}{64}$	H	H
x^5	[cos + sin]		$-\frac{9}{32}$	$-\frac{1}{32}$	$-\frac{41}{128}$	$-\frac{7}{64}$	$-\frac{1}{4}$
	[cos − sin]		$-\frac{3}{64}$	0	$-\frac{7}{128}$	0	0
	[cos]		$-\frac{21}{128}$	H	$-\frac{3}{16}$	H	H
	[sin]		$-\frac{15}{128}$	H	$-\frac{17}{128}$	H	H
x^6	[cos + sin]		$+\frac{3}{32}$		$+\frac{7}{32}$	$+\frac{1}{64}$	$+\frac{1}{16}$
	[cos − sin]		$+\frac{1}{64}$		$+\frac{1}{32}$	0	0
	[cos]		$+\frac{7}{128}$		$+\frac{1}{8}$	H	H
	[sin]		$+\frac{5}{128}$		$+\frac{3}{32}$	H	H
x^7	[cos + sin]				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]				$-\frac{1}{128}$		0
	[cos]				$-\frac{1}{32}$		H
	[sin]				$-\frac{3}{128}$		H

Table (x^0, x^1, x^2, x^3) $\begin{matrix}\cos\\ \sin\end{matrix}$ $2f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}$ $2f$	$[\ ]^0$			$[\ ]^1$				$[\ ]^2$			
		e^2	e^4	e^6	e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^4	[cos + sin]	$+\frac{3}{4}$	$+\frac{1}{8}$	$+\frac{3}{64}$	-2	$+\frac{7}{4}$	$-\frac{5}{96}$	$+\frac{271}{4608}$	$+1$	-4	$+\frac{55}{16}$	$-\frac{103}{144}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{12}$	$+\frac{5}{192}$	$+\frac{49}{7680}$	0	0	$+\frac{1}{24}$	$+\frac{1}{60}$
	[cos]	$+\frac{3}{4}$	$+\frac{1}{8}$	$+\frac{3}{64}$	H	$+\frac{11}{12}$	$-\frac{5}{384}$	$+\frac{751}{23040}$	H	H	$+\frac{167}{96}$	$-\frac{503}{1440}$
	[sin]	0	0	0	H	$+\frac{5}{6}$	$-\frac{5}{128}$	$+\frac{151}{5760}$	H	H	$+\frac{163}{96}$	$-\frac{527}{1440}$
x	[cos + sin]	$+\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{3}{64}$	$-\frac{1}{2}$	$+\frac{5}{16}$	$-\frac{53}{384}$	$-\frac{203}{18432}$		$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{32}$
	[cos − sin]	,,	,,	,,	0	$-\frac{1}{16}$	$-\frac{61}{768}$	$-\frac{271}{6144}$		0	$-\frac{1}{48}$	$-\frac{17}{480}$
	[cos]	$+\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{3}{64}$	H	$+\frac{1}{8}$	$-\frac{167}{1536}$	$-\frac{127}{4608}$		H	$-\frac{25}{96}$	$-\frac{1}{30}$
	[sin]	0	0	0	H	$+\frac{3}{16}$	$-\frac{15}{512}$	$+\frac{305}{18432}$		H	$-\frac{23}{96}$	$+\frac{1}{480}$
x^2	[cos + sin]	$+\frac{1}{4}$	$+\frac{1}{8}$	$+\frac{3}{64}$		$-\frac{3}{4}$	$+\frac{17}{48}$	$+\frac{19}{512}$		$+\frac{1}{2}$	-1	$+\frac{53}{96}$
	[cos − sin]	,,	,,	,,		$-\frac{1}{4}$	$+\frac{1}{96}$	$+\frac{83}{7680}$		0	$-\frac{1}{16}$	$+\frac{1}{32}$
	[cos]	$+\frac{1}{4}$	$+\frac{1}{8}$	$+\frac{3}{64}$		$-\frac{1}{2}$	$+\frac{35}{192}$	$+\frac{23}{960}$		H	$-\frac{17}{32}$	$+\frac{7}{24}$
	[sin]	0	0	0		$-\frac{1}{4}$	$+\frac{11}{64}$	$+\frac{101}{7680}$		H	$-\frac{15}{32}$	$+\frac{25}{96}$
x^3	[cos + sin]		$+\frac{1}{2}$	$-\frac{3}{64}$		$-\frac{3}{8}$	$+\frac{5}{32}$	$-\frac{49}{1024}$			$+\frac{3}{8}$	$-\frac{1}{4}$
	[cos − sin]		,,	,,		$-\frac{1}{8}$	$-\frac{1}{64}$	$-\frac{39}{1024}$			$+\frac{1}{16}$	0
	[cos]		$+\frac{1}{2}$	$-\frac{3}{64}$		$-\frac{1}{4}$	$+\frac{9}{128}$	$-\frac{11}{256}$			$+\frac{7}{32}$	$-\frac{1}{8}$
	[sin]		0	0		$-\frac{1}{8}$	$+\frac{11}{128}$	$-\frac{5}{1024}$			$+\frac{5}{32}$	$-\frac{1}{8}$

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$	$[\ ]^3$				$[\ ]^4$			$[\ ]^5$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
x^0	[cos + sin]	$+2$	$-\frac{27}{4}$	$+\frac{207}{32}$	$-\frac{1147}{512}$	$+\frac{13}{4}$	$-\frac{259}{24}$	$+\frac{559}{48}$	$+\frac{59}{12}$	$-\frac{3221}{192}$	$+\frac{31087}{1536}$
	[cos − sin]	0	0	$+\frac{9}{320}$	$+\frac{27}{2560}$	0	0	$+\frac{1}{45}$	0	0	$+\frac{625}{32256}$
	[cos]	H	H	$+\frac{2079}{640}$	$-\frac{1427}{1280}$	H	H	$+\frac{8401}{1440}$	H	H	$+\frac{163363}{16128}$
	[sin]	H	H	$+\frac{2061}{640}$	$-\frac{2881}{2560}$	H	H	$+\frac{8369}{1440}$	H	H	$+\frac{326101}{32256}$
x	[cos + sin]	$-\frac{1}{2}$	$+\frac{33}{16}$	$-\frac{255}{128}$	$+\frac{1081}{2048}$	$-\frac{5}{4}$	$+\frac{107}{24}$	$-\frac{229}{48}$	$-\frac{37}{16}$	$+\frac{6259}{768}$	$-\frac{59723}{6144}$
	[cos − sin]	0	0	$-\frac{3}{256}$	$-\frac{231}{10240}$	0	0	$-\frac{1}{120}$	0	0	$-\frac{125}{18432}$
	[cos]	H	H	$-\frac{513}{512}$	$+\frac{2587}{10240}$	H	H	$-\frac{1147}{480}$	H	H	$-\frac{89647}{18432}$
	[sin]	H	H	$-\frac{507}{512}$	$+\frac{1409}{5120}$	H	H	$-\frac{381}{160}$	H	H	$-\frac{44761}{9216}$
x^2	[cos + sin]		$+\frac{1}{4}$	$-\frac{13}{16}$	$+\frac{367}{512}$	$+\frac{1}{4}$	$-\frac{5}{8}$	0	$+\frac{3}{4}$	$-\frac{223}{96}$	$+\frac{2987}{1536}$
	[cos − sin]		0	$-\frac{1}{32}$	$+\frac{71}{2560}$	0	0	$-\frac{1}{48}$	0	0	$-\frac{25}{1536}$
	[cos]		H	$-\frac{27}{64}$	$+\frac{953}{2560}$	H	H	$-\frac{1}{96}$	H	H	$+\frac{1481}{1536}$
	[sin]		H	$-\frac{25}{64}$	$+\frac{441}{1280}$	H	H	$+\frac{1}{96}$	H	H	$+\frac{251}{256}$
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{33}{32}$	$-\frac{753}{1024}$		$-\frac{1}{2}$	$+\frac{95}{64}$	$-\frac{1}{8}$	$-\frac{13}{64}$	$+\frac{1577}{1024}$
	[cos − sin]		0	$+\frac{3}{64}$	$-\frac{13}{1024}$		0	$+\frac{7}{192}$	0	0	$+\frac{95}{3072}$
	[cos]		H	$+\frac{69}{128}$	$-\frac{383}{1024}$		H	$+\frac{73}{96}$	H	H	$+\frac{2413}{3072}$
	[sin]		H	$+\frac{63}{128}$	$-\frac{185}{512}$		H	$+\frac{139}{192}$	H	H	$+\frac{1159}{1536}$

... *Table* (x^0, x^1, x^2, x^3) $\begin{matrix}\cos\\ \sin\end{matrix}$ $2f$.

	$\begin{matrix}\cos\\ \sin\end{matrix}$ $2f$	$[\ ]^6$ e^4	$[\ ]^6$ e^6	$[\ ]^7$ e^5	$[\ ]^7$ e^7	$[\ ]^8$ e^6	$[\ ]^9$ e^7
x^0	[cos + sin]	$+\frac{115}{16}$	$-\frac{2049}{80}$	$+\frac{9893}{960}$	$-\frac{889303}{23040}$	$+\frac{42037}{2880}$	$+\frac{367439}{17920}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x	[cos + sin]	$-\frac{61}{16}$	$+\frac{69}{5}$	$-\frac{4553}{768}$	$+\frac{2061389}{92160}$	$-\frac{8551}{960}$	$-\frac{26809}{2048}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{25}{16}$	$-\frac{21}{4}$	$+\frac{269}{96}$	$-\frac{77143}{7680}$	$+\frac{297}{64}$	$+\frac{18749}{2560}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{7}{16}$	$+\frac{3}{4}$	$-\frac{65}{64}$	$+\frac{8425}{3072}$	$-\frac{379}{192}$	$-\frac{3563}{1024}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{matrix}\cos\\ \sin\end{matrix}$ $2f\ldots$

$\begin{matrix}\cos\\ \sin\end{matrix}$ $2f$		$[\ ]^0$		$[\ ]^1$				$[\ ]^2$			
		e^2	e^4	e^6	e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^4	[cos + sin]		$+\frac{1}{4}$	$+\frac{3}{64}$			$-\frac{1}{2}$	$+\frac{5}{32}$			$+\frac{3}{8}$	$-\frac{1}{2}$
	[cos − sin]		,,	,,			$-\frac{1}{4}$	0			$+\frac{1}{16}$	$-\frac{1}{16}$
	[cos]		$+\frac{1}{4}$	$+\frac{3}{64}$			$-\frac{3}{8}$	$+\frac{5}{64}$			$+\frac{7}{32}$	$-\frac{9}{32}$
	[sin]		0	0			$-\frac{1}{8}$	$+\frac{5}{64}$			$+\frac{5}{32}$	$-\frac{7}{32}$
x^5	[cos + sin]			$+\frac{25}{64}$			$-\frac{5}{16}$	$+\frac{25}{256}$				$+\frac{5}{16}$
	[cos − sin]			,,			$-\frac{5}{32}$	$-\frac{1}{256}$				$+\frac{3}{32}$
	[cos]			$+\frac{25}{64}$			$-\frac{15}{64}$	$+\frac{3}{64}$				$+\frac{13}{64}$
	[sin]			0			$-\frac{5}{64}$	$+\frac{13}{256}$				$+\frac{7}{64}$
x^6	[cos + sin]			$+\frac{15}{64}$				$-\frac{25}{64}$				$+\frac{5}{16}$
	[cos − sin]			,,				$-\frac{15}{64}$				$+\frac{3}{32}$
	[cos]			$+\frac{15}{64}$				$-\frac{5}{16}$				$+\frac{13}{64}$
	[sin]			0				$-\frac{5}{64}$				$+\frac{7}{64}$
x^7	[cos + sin]							$-\frac{35}{128}$				
	[cos − sin]							$-\frac{21}{128}$				
	[cos]							$-\frac{7}{32}$				
	[sin]							$-\frac{7}{128}$				

... *Table* $(x^4, x^5, x^6, x^7)\,{}^{\cos}_{\sin}\,2f$ ...

	$\cos/\sin\ 2f$	$[\ ]^3$				$[\ ]^4$			$[\ ]^5$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
x^4	[cos + sin]			0	$-\frac{3}{16}$		$+\frac{1}{4}$	$-\frac{43}{64}$		$+\frac{1}{2}$	$-\frac{47}{32}$
	[cos − sin]			0	$-\frac{1}{32}$		0	$-\frac{1}{64}$		0	$-\frac{1}{48}$
	[cos]			0	$-\frac{7}{64}$		H	$-\frac{11}{32}$		H	$-\frac{143}{192}$
	[sin]			0	$-\frac{5}{64}$		H	$-\frac{21}{64}$		H	$-\frac{139}{192}$
x^5	[cos + sin]			$-\frac{5}{16}$	$+\frac{165}{256}$			$-\frac{15}{64}$		$-\frac{5}{32}$	$+\frac{59}{256}$
	[cos − sin]			$-\frac{1}{32}$	$+\frac{17}{256}$			$-\frac{1}{64}$		0	$-\frac{1}{256}$
	[cos]			$-\frac{11}{64}$	$+\frac{91}{256}$			$-\frac{1}{8}$		H	$+\frac{29}{256}$
	[sin]			$-\frac{9}{64}$	$+\frac{37}{128}$			$-\frac{7}{64}$		H	$+\frac{15}{128}$
x^6	[cos + sin]				$-\frac{5}{64}$			$+\frac{15}{64}$			$+\frac{21}{64}$
	[cos − sin]				$-\frac{1}{64}$			$+\frac{1}{64}$			$+\frac{1}{64}$
	[cos]				$-\frac{3}{64}$			$+\frac{1}{8}$			$+\frac{11}{64}$
	[sin]				$-\frac{1}{32}$			$+\frac{7}{64}$			$+\frac{5}{32}$
x^7	[cos + sin]				$-\frac{35}{128}$						$-\frac{21}{128}$
	[cos − sin]				$-\frac{7}{128}$						$-\frac{1}{128}$
	[cos]				$-\frac{21}{128}$						$-\frac{11}{128}$
	[sin]				$-\frac{7}{64}$						$-\frac{5}{64}$

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$	$[\ ]^6$		$[\ ]^7$		$[\ ]^8$	$[\ ]^9$
		e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]	$+\frac{1}{16}$	$+\frac{9}{16}$	$+\frac{1}{4}$	$+\frac{5}{24}$	$+\frac{41}{64}$	$+\frac{43}{32}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^5	[cos + sin]		$-\frac{13}{32}$	$-\frac{1}{32}$	$-\frac{163}{256}$	$-\frac{9}{64}$	$-\frac{101}{256}$
	[cos − sin]		0	0	0	0	0
	[cos]		H	H	H	H	H
	[sin]		H	H	H	H	H
x^6	[cos + sin]		$+\frac{3}{32}$		$+\frac{19}{64}$	$+\frac{1}{64}$	$+\frac{5}{64}$
	[cos − sin]		0		0	0	0
	[cos]		H		H	H	H
	[sin]		H		H	H	H
x^7	[cos + sin]				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]				0		0
	[cos]				H		H
	[sin]				H		H

Table (x^0, x^1, x^2, x^3) $\begin{matrix}\cos\\ \sin\end{matrix}\, 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 3f$	$[\ ]^0$			$[\ ]^1$			$[\ ]^2$			
		e^3	e^5	e^7	e^2	e^4	e^6	e	e^3	e^5	e^7
x^0	[cos + sin]	$-\frac{1}{2}$	$-\frac{3}{16}$	$-\frac{3}{32}$	$+\frac{21}{8}$	$-\frac{31}{16}$	$-\frac{103}{1024}$	-3	$+\frac{33}{4}$	$-\frac{89}{16}$	$+\frac{163}{192}$
	[cos − sin]	,,	,,	,,	0	$-\frac{5}{128}$	$-\frac{11}{320}$	0	0	$-\frac{1}{80}$	$-\frac{17}{960}$
	[cos]	$-\frac{1}{2}$	$-\frac{3}{16}$	$-\frac{3}{32}$	H	$-\frac{253}{256}$	$-\frac{691}{10240}$	H	H	$-\frac{223}{80}$	$+\frac{133}{320}$
	[sin]	0	0	0	H	$-\frac{243}{256}$	$-\frac{339}{10240}$	H	H	$-\frac{111}{40}$	$+\frac{13}{30}$
x^1	[cos + sin]	$-\frac{3}{4}$	$+\frac{1}{32}$	$+\frac{3}{64}$	$+\frac{5}{4}$	$-\frac{61}{48}$	$+\frac{129}{512}$	$-\frac{1}{2}$	$+\frac{9}{8}$	$-\frac{125}{96}$	$+\frac{709}{1152}$
	[cos − sin]	,,	,,	,,	0	$-\frac{1}{96}$	$+\frac{59}{1280}$	0	0	$-\frac{1}{96}$	$+\frac{89}{5760}$
	[cos]	$-\frac{3}{4}$	$+\frac{1}{32}$	$+\frac{3}{64}$	H	$-\frac{41}{64}$	$+\frac{763}{5120}$	H	H	$-\frac{21}{32}$	$+\frac{1817}{5760}$
	[sin]	0	0	0	H	$-\frac{121}{192}$	$+\frac{527}{5120}$	H	H	$-\frac{31}{48}$	$+\frac{3}{10}$
x^2	[cos + sin]	$-\frac{1}{2}$	$+\frac{1}{8}$	0	$+\frac{1}{4}$	$+\frac{5}{16}$	$-\frac{85}{1536}$		-1	$+\frac{19}{12}$	$-\frac{49}{96}$
	[cos − sin]	,,	,,	,,	0	$+\frac{5}{32}$	$+\frac{53}{768}$		0	$+\frac{1}{24}$	$+\frac{7}{480}$
	[cos]	$-\frac{1}{2}$	$+\frac{1}{8}$	0	H	$+\frac{15}{64}$	$+\frac{7}{1024}$		H	$+\frac{13}{16}$	$-\frac{119}{480}$
	[sin]	0	0	0	H	$+\frac{5}{64}$	$-\frac{191}{3072}$		H	$+\frac{37}{48}$	$-\frac{21}{80}$
x^3	[cos + sin]	$-\frac{1}{8}$	$-\frac{9}{32}$	$-\frac{3}{64}$		$+\frac{3}{4}$	$-\frac{65}{128}$		$-\frac{3}{8}$	$+\frac{9}{16}$	$-\frac{61}{128}$
	[cos − sin]	,,	,,	,,		$+\frac{3}{16}$	$-\frac{1}{32}$		0	0	$-\frac{5}{128}$
	[cos]	$-\frac{1}{8}$	$-\frac{9}{32}$	$-\frac{3}{64}$		$+\frac{15}{32}$	$-\frac{69}{256}$		H	H	$-\frac{33}{128}$
	[sin]	0	0	0		$+\frac{9}{32}$	$-\frac{61}{256}$		H	H	$-\frac{7}{32}$

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $3f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $3f$	$[\ ]^3$				$[\ ]^4$				$[\ ]^5$		
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^0	[cos + sin]	$+1$	-9	$+\frac{1215}{64}$	$-\frac{449}{32}$	$+3$	$-\frac{39}{2}$	$+\frac{155}{4}$	$-\frac{767}{24}$	$+\frac{51}{8}$	$-\frac{593}{16}$	$+\frac{75643}{1024}$
	[cos − sin]	0	0	0	$-\frac{27}{5120}$	0	0	0	$-\frac{1}{420}$	0	0	0
	[cos]	H	H	H	$-\frac{71867}{10240}$	H	H	H	$-\frac{8949}{560}$	H	H	H
	[sin]	H	H	H	$-\frac{71813}{10240}$	H	H	H	$-\frac{26843}{1680}$	H	H	H
x^1	[cos + sin]		$+\frac{1}{2}$	$-\frac{9}{8}$	$-\frac{9}{128}$	$-\frac{1}{2}$	$+\frac{15}{4}$	$-\frac{173}{24}$	$+\frac{161}{36}$	$-\frac{7}{4}$	$+\frac{509}{48}$	$-\frac{10439}{512}$
	[cos − sin]		0	0	$-\frac{21}{2560}$	0	0	0	$-\frac{1}{144}$	0	0	0
	[cos]		H	H	$-\frac{201}{5120}$	H	H	H	$+\frac{643}{288}$	H	H	H
	[sin]		H	H	$-\frac{159}{5120}$	H	H	H	$+\frac{215}{96}$	H	H	H
x^2	[cos + sin]		$+\frac{1}{2}$	$-\frac{9}{4}$	$+\frac{399}{128}$		$+\frac{1}{2}$	$-\frac{8}{3}$	$+\frac{437}{96}$	$+\frac{1}{4}$	$-\frac{13}{16}$	$-\frac{1393}{1536}$
	[cos − sin]		0	0	$+\frac{11}{512}$		0	0	$+\frac{7}{480}$	0	0	0
	[cos]		H	H	$+\frac{1607}{1024}$		H	H	$+\frac{137}{60}$	H	H	H
	[sin]		H	H	$+\frac{1585}{1024}$		H	H	$+\frac{363}{160}$	H	H	H
x^3	[cos + sin]			$+\frac{3}{8}$	$-\frac{9}{16}$		$-\frac{3}{8}$	$+\frac{15}{8}$	$-\frac{343}{128}$		$-\frac{3}{4}$	$+\frac{469}{128}$
	[cos − sin]			0	$-\frac{1}{128}$		0	0	$-\frac{1}{128}$		0	0
	[cos]			H	$-\frac{73}{256}$		H	H	$-\frac{43}{32}$		H	H
	[sin]			H	$-\frac{71}{256}$		H	H	$-\frac{171}{128}$		H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{matrix}\cos\\ \sin\end{matrix}$ $3f$.

	$\begin{matrix}\cos\\ \sin\end{matrix}$ $3f$	$[\ ]^6$			$[\ ]^7$		$[\ ]^8$		$[\ ]^9$	$[\ ]^{10}$
		e^3	e^5	e^7	e^4	e^6	e^5	e^7	e^6	e^7
x^0	[cos + sin]	$+\frac{47}{4}$	$-\frac{525}{8}$	$+\frac{43041}{320}$	$+\frac{2567}{128}$	$-\frac{35563}{320}$	$+\frac{2611}{80}$	$-\frac{87599}{480}$	$+\frac{263351}{5120}$	$+\frac{106469}{1344}$
	[cos − sin]	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{33}{8}$	$+\frac{373}{16}$	$-\frac{5919}{128}$	$-\frac{787}{96}$	$+\frac{58143}{1280}$	$-\frac{1423}{96}$	$+\frac{237373}{2880}$	$-\frac{64653}{2560}$	$-\frac{47603}{1152}$
	[cos − sin]	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H
x^2	[cos + sin]	$+1$	$-\frac{37}{8}$	$+\frac{891}{160}$	$+\frac{83}{32}$	$-\frac{9871}{768}$	$+\frac{133}{24}$	$-\frac{857}{30}$	$+\frac{5431}{512}$	$+\frac{1817}{96}$
	[cos − sin]	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{1}{8}$	$-\frac{9}{16}$	$+\frac{627}{128}$	$-\frac{9}{16}$	$+\frac{35}{32}$	$-\frac{51}{32}$	$+\frac{367}{64}$	$-\frac{467}{128}$	$-\frac{945}{128}$
	[cos − sin]	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 3f\ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 3f$	$[\ ]^0$			$[\ ]^1$			$[\ ]^2$			
		e^3	e^5	e^7	e^2	e^4	e^6	e	e^3	e^5	e^7
x^4	[cos + sin]		$-\frac{7}{16}$	$+\frac{3}{32}$		$+\frac{1}{4}$	$+\frac{9}{128}$			$-\frac{5}{8}$	$+\frac{43}{64}$
	[cos − sin]		,,	,,		$+\frac{1}{16}$	$+\frac{3}{32}$			$-\frac{1}{16}$	$+\frac{5}{192}$
	[cos]		$-\frac{7}{16}$	$+\frac{3}{32}$		$+\frac{5}{32}$	$+\frac{21}{256}$			$-\frac{11}{32}$	$+\frac{67}{192}$
	[sin]		0	0		$+\frac{3}{32}$	$-\frac{3}{256}$			$-\frac{9}{32}$	$+\frac{31}{96}$
x^5	[cos + sin]		$-\frac{5}{32}$	$-\frac{9}{64}$			$+\frac{35}{64}$			$-\frac{5}{16}$	$+\frac{45}{128}$
	[cos − sin]		,,	,,			$+\frac{7}{32}$			$-\frac{1}{32}$	$+\frac{1}{128}$
	[cos]		$-\frac{5}{32}$	$-\frac{9}{64}$			$+\frac{49}{128}$			$-\frac{11}{64}$	$+\frac{23}{128}$
	[sin]		0	0			$+\frac{21}{128}$			$-\frac{9}{64}$	$+\frac{11}{64}$
x^6	[cos + sin]			$-\frac{3}{8}$			$+\frac{15}{64}$				$-\frac{15}{32}$
	[cos − sin]			,,			$+\frac{3}{32}$				$-\frac{3}{32}$
	[cos]			$-\frac{3}{8}$			$+\frac{21}{128}$				$-\frac{9}{32}$
	[sin]			0			$+\frac{9}{128}$				$-\frac{3}{16}$
x^7	[cos + sin]			$-\frac{21}{128}$							$-\frac{35}{128}$
	[cos − sin]			,,							$-\frac{7}{128}$
	[cos]			$-\frac{21}{128}$							$-\frac{21}{128}$
	[sin]			0							$-\frac{7}{64}$

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 3f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 3f$	$[\ ]^3$				$[\ ]^4$				$[\ ]^5$		
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^4	[cos + sin]			$+\frac{3}{8}$	$-\frac{9}{8}$		$+\frac{1}{8}$	$-\frac{25}{32}$		$+\frac{1}{4}$	$-\frac{129}{128}$	
	[cos − sin]			0	$-\frac{3}{128}$		0	$-\frac{1}{96}$		0	0	
	[cos]			H	$-\frac{147}{256}$		H	$-\frac{19}{48}$		H	H	
	[sin]			H	$-\frac{141}{256}$		H	$-\frac{37}{96}$		H	H	
x^5	[cos + sin]				$+\frac{5}{16}$		$-\frac{5}{16}$	$+\frac{75}{64}$			$-\frac{25}{64}$	
	[cos − sin]				$+\frac{1}{64}$		0	$+\frac{1}{64}$			0	
	[cos]				$+\frac{21}{128}$		H	$+\frac{19}{32}$			H	
	[sin]				$+\frac{19}{128}$		H	$+\frac{37}{64}$			H	
x^6	[cos + sin]				$+\frac{5}{16}$			0			$+\frac{15}{64}$	
	[cos − sin]				$+\frac{1}{64}$			0			0	
	[cos]				$+\frac{21}{128}$			0			H	
	[sin]				$+\frac{19}{128}$			0			H	
x^7	[cos + sin]								$-\frac{35}{128}$			
	[cos − sin]								$-\frac{1}{128}$			
	[cos]								$-\frac{9}{64}$			
	[sin]								$-\frac{17}{128}$			

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $3f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $3f$	$[\ ]^8$			$[\ ]^7$		$[\ ]^8$		$[\ ]^9$	$[\ ]^{10}$
		e^3	e^5	e^7	e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]		$+\frac{11}{16}$	$-\frac{195}{64}$	$+\frac{1}{16}$	$+\frac{33}{32}$	$+\frac{5}{16}$	$+\frac{67}{96}$	$+\frac{123}{128}$	$+\frac{451}{192}$
	[cos − sin]		0	0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H	H
x^5	[cos + sin]		$-\frac{5}{32}$	$+\frac{27}{128}$		$-\frac{17}{32}$	$-\frac{1}{32}$	$-\frac{67}{64}$	$-\frac{11}{64}$	$-\frac{73}{128}$
	[cos − sin]		0	0		0	0	0	0	0
	[cos]		H	H		H	H	H	H	H
	[sin]		H	H		H	H	H	H	H
x^6	[cos + sin]			$+\frac{15}{32}$		$+\frac{3}{32}$		$+\frac{3}{8}$	$+\frac{1}{64}$	$+\frac{3}{32}$
	[cos − sin]			0		0		0	0	0
	[cos]			H		H		H	H	H
	[sin]			H		H		H	H	H
x^7	[cos + sin]			$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]			0				0		0
	[cos]			H				H		H
	[sin]			H				H		H

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$	$[\ ]^0$		$[\ ]^1$				$[\ ]^2$		
		e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^0	[cos + sin]	$+\frac{5}{16}$	$+\frac{3}{16}$	$-\frac{17}{6}$	$+\frac{161}{96}$	$+\frac{907}{3840}$		$+\frac{11}{2}$	$-\frac{149}{12}$	$+\frac{325}{48}$
	[cos − sin]	,,	,,	0	$+\frac{7}{480}$	$+\frac{277}{11520}$		0	0	$+\frac{1}{720}$
	[cos]	$+\frac{5}{16}$	$+\frac{3}{16}$	H	$+\frac{203}{240}$	$+\frac{1499}{11520}$		H	H	$+\frac{1219}{360}$
	[sin]	0	0	H	$+\frac{133}{160}$	$+\frac{611}{5760}$		H	H	$+\frac{2437}{720}$
x^1	[cos + sin]	$+\frac{5}{8}$	$+\frac{3}{32}$	$-\frac{31}{16}$	$+\frac{1447}{768}$	$-\frac{4537}{30720}$		$+\frac{7}{4}$	$-\frac{53}{12}$	$+\frac{743}{192}$
	[cos − sin]	,,	,,	0	$+\frac{19}{768}$	$-\frac{793}{92160}$		0	0	$+\frac{11}{960}$
	[cos]	$+\frac{5}{8}$	$+\frac{3}{32}$	H	$+\frac{733}{768}$	$-\frac{3601}{46080}$		H	H	$+\frac{621}{320}$
	[sin]	0	0	H	$+\frac{119}{128}$	$-\frac{6409}{92160}$		H	H	$+\frac{463}{240}$
x^2	[cos + sin]	$+\frac{5}{8}$	$-\frac{5}{32}$	$-\frac{3}{4}$	$+\frac{53}{96}$	$-\frac{2123}{7680}$		$+\frac{1}{4}$	$+\frac{1}{2}$	$-\frac{49}{64}$
	[cos − sin]	,,	,,	0	$-\frac{7}{96}$	$-\frac{649}{7680}$		0	0	$-\frac{1}{64}$
	[cos]	$+\frac{5}{8}$	$-\frac{5}{32}$	H	$+\frac{23}{96}$	$-\frac{231}{1280}$		H	H	$-\frac{25}{64}$
	[sin]	0	0	H	$+\frac{5}{16}$	$-\frac{737}{7680}$		H	H	$-\frac{3}{8}$
x^3	[cos + sin]	$+\frac{5}{16}$	0	$-\frac{1}{8}$	$-\frac{43}{64}$	$+\frac{389}{1024}$			$+1$	$-\frac{7}{4}$
	[cos − sin]	,,	,,	0	$-\frac{11}{64}$	$-\frac{59}{3072}$			0	$-\frac{1}{48}$
	[cos]	$+\frac{5}{16}$	0	H	$-\frac{27}{64}$	$+\frac{277}{1536}$			H	$-\frac{85}{96}$
	[sin]	0	0	H	$-\frac{1}{4}$	$+\frac{613}{3072}$			H	$-\frac{83}{96}$

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $4f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $4f$	$[\ ]^3$				$[\ ]^4$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^0	[cos + sin]	-4	$+\frac{45}{2}$	$-\frac{591}{16}$	$+\frac{5737}{256}$	$+1$	-16	$+\frac{247}{4}$	$-\frac{3355}{36}$
	[cos − sin]	0	0	0	$-\frac{9}{8960}$	0	0	0	0
	[cos]	H	H	H	$+\frac{100393}{8960}$	H	H	H	H
	[sin]	H	H	H	$+\frac{50201}{4480}$	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{39}{16}$	$-\frac{687}{128}$	$+\frac{12355}{2048}$		$+\frac{1}{2}$	-2	$-\frac{1}{8}$
	[cos − sin]	0	0	0	$+\frac{69}{10240}$		0	0	0
	[cos]	H	H	H	$+\frac{15461}{5120}$		H	H	H
	[sin]	H	H	H	$+\frac{30853}{10240}$		H	H	H
x^2	[cos + sin]		$-\frac{5}{4}$	$+\frac{67}{16}$	$-\frac{2119}{512}$		$+\frac{1}{2}$	-4	$+\frac{245}{24}$
	[cos − sin]		0	0	$-\frac{17}{2560}$		0	0	0
	[cos]		H	H	$-\frac{2653}{1280}$		H	H	H
	[sin]		H	H	$-\frac{5289}{2560}$		H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{39}{32}$	$-\frac{2037}{1024}$			$+\frac{3}{8}$	-1
	[cos − sin]		0	0	$-\frac{7}{1024}$			0	0
	[cos]		H	H	$-\frac{511}{512}$			H	H
	[sin]		H	H	$-\frac{1015}{1024}$			H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$	$[\ ]^5$				$[\ ]^6$			$[\ ]^7$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
x^4	[cos + sin]	$+4$	$-\frac{85}{2}$	$+\frac{6845}{48}$	$-\frac{490585}{2304}$	$+\frac{21}{2}$	$-\frac{377}{4}$	$+\frac{597}{2}$	$+\frac{137}{6}$	$-\frac{18113}{96}$	$+\frac{2249741}{3840}$
	[cos − sin]	0	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{95}{16}$	$-\frac{7325}{384}$	$+\frac{407635}{18432}$	$-\frac{9}{4}$	$+\frac{83}{4}$	$-\frac{4029}{64}$	$-\frac{103}{16}$	$+\frac{40967}{768}$	$-\frac{4894577}{30720}$
	[cos − sin]	0	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H	H
x^2	[cos + sin]		$+\frac{3}{4}$	$-\frac{295}{48}$	$+\frac{8785}{512}$	$+\frac{1}{4}$	-1	$-\frac{215}{64}$	$+\frac{5}{4}$	$-\frac{779}{96}$	$+\frac{103901}{7680}$
	[cos − sin]		0	0	0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{95}{32}$	$-\frac{7285}{1024}$		-1	$+\frac{117}{16}$	$-\frac{1}{8}$	$-\frac{67}{64}$	$+\frac{12069}{1024}$
	[cos − sin]		0	0	0		0	0	0	0	0
	[cos]		H	H	H		H	H	H	H	H
	[sin]		H	H	H		H	H	H	H	H

... *Table* $(x^0, x^1, x^2, x^3)\,{}^{\cos}_{\sin}\,4f$.

	$\cos/\sin\,4f$	$[\ ]^8$ e^4	$[\ ]^8$ e^6	$[\ ]^9$ e^5	$[\ ]^9$ e^7	$[\ ]^{10}$ e^6	$[\ ]^{11}$ e^7
x^0	[cos + sin]	$+\frac{2141}{48}$	$-\frac{84857}{240}$	$+\frac{13011}{160}$	$-\frac{809813}{1280}$	$+\frac{5087}{36}$	$+\frac{19116073}{80640}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{359}{24}$	$+\frac{56527}{480}$	$-\frac{7887}{256}$	$+\frac{2427367}{10240}$	$-\frac{11263}{192}$	$-\frac{9742939}{92160}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{31}{8}$	$-\frac{855}{32}$	$+\frac{307}{32}$	$-\frac{172907}{2560}$	$+\frac{3995}{192}$	$+\frac{317653}{7680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{11}{16}$	$+\frac{3}{2}$	$-\frac{147}{64}$	$+\frac{10999}{1024}$	$-\frac{145}{24}$	$-\frac{42325}{3072}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$	$[\ ]^0$		$[\ ]^1$			$[\ ]^2$			$[\ ]^3$			
		e^4	e^6	e^3	e^5	e^7	e^2	e^4	e^6	e^1	e^3	e^5	e^7
x^4	[cos + sin]	$+\frac{1}{16}$	$+\frac{3}{8}$		$-\frac{5}{8}$	$+\frac{29}{64}$		$+\frac{1}{4}$	$+\frac{1}{32}$			$-\frac{3}{4}$	$+\frac{111}{64}$
	[cos − sin]	,,	,,		$-\frac{1}{8}$	$+\frac{1}{192}$		0	$+\frac{1}{32}$			0	$+\frac{1}{64}$
	[cos]	$+\frac{1}{16}$	$+\frac{3}{8}$		$-\frac{3}{8}$	$+\frac{11}{48}$		H	$+\frac{1}{32}$			H	$+\frac{7}{8}$
	[sin]	0	0		$-\frac{1}{4}$	$+\frac{43}{192}$		H	0			H	$+\frac{55}{64}$
x^5	[cos + sin]		$+\frac{11}{32}$		$-\frac{5}{32}$	$-\frac{79}{256}$			$+\frac{45}{64}$			$-\frac{5}{16}$	$+\frac{195}{256}$
	[cos − sin]		,,		$-\frac{1}{32}$	$-\frac{37}{256}$			$+\frac{3}{64}$			0	$+\frac{1}{256}$
	[cos]		$+\frac{11}{32}$		$-\frac{3}{32}$	$-\frac{29}{128}$			$+\frac{3}{8}$			H	$+\frac{49}{128}$
	[sin]		0		$-\frac{1}{16}$	$-\frac{21}{256}$			$+\frac{21}{64}$			H	$+\frac{97}{256}$
x^6	[cos + sin]		$+\frac{3}{32}$			$-\frac{33}{64}$			$+\frac{15}{64}$				$-\frac{35}{64}$
	[cos − sin]		,,			$-\frac{11}{64}$			$+\frac{1}{64}$				$-\frac{1}{64}$
	[cos]		$+\frac{3}{32}$			$-\frac{11}{32}$			$+\frac{1}{8}$				$-\frac{9}{32}$
	[sin]		0			$-\frac{11}{64}$			$+\frac{7}{64}$				$-\frac{17}{64}$
x^7	[cos + sin]					$-\frac{21}{128}$							$-\frac{35}{128}$
	[cos − sin]					$-\frac{7}{128}$							$-\frac{1}{128}$
	[cos]					$-\frac{7}{64}$							$-\frac{9}{64}$
	[sin]					$-\frac{7}{128}$							$-\frac{17}{128}$

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 4f$	$[\ ]^4$				$[\ ]^5$				$[\ ]^6$		
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^4	[cos + sin]			$+\frac{3}{8}$	-2			$+\frac{1}{4}$	$-\frac{125}{64}$		$+\frac{1}{4}$	$-\frac{45}{32}$
	[cos − sin]			0	0			0	0		0	0
	[cos]			H	H			H	H		H	H
	[sin]			H	H			H	H		H	H
x^5	[cos + sin]				$-\frac{5}{16}$			$-\frac{5}{16}$	$+\frac{475}{256}$			$-\frac{35}{64}$
	[cos − sin]				0			0	0			0
	[cos]				H			H	H			H
	[sin]				H			H	H			H
x^6	[cos + sin]				$+\frac{5}{16}$				$+\frac{5}{64}$			$+\frac{15}{64}$
	[cos − sin]				0				0			0
	[cos]				H				H			H
	[sin]				H				H			H
x^7	[cos + sin]								$-\frac{35}{128}$			
	[cos − sin]								0			
	[cos]								H			
	[sin]								H			

... *Table* (x^4, x^5, x^6, x^7) $\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$.

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^7$		$[\ ]^8$		$[\ ]^9$		$[\ ]^{10}$	$[\ ]^{11}$
		e^3	e^5	e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]	$+\frac{7}{8}$	$-\frac{351}{64}$	$+\frac{1}{16}$	$+\frac{13}{8}$	$+\frac{3}{8}$	$+\frac{95}{64}$	$+\frac{43}{32}$	$+\frac{719}{192}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^5	[cos + sin]	$-\frac{5}{32}$	$+\frac{41}{256}$		$-\frac{21}{32}$	$-\frac{1}{32}$	$-\frac{397}{256}$	$-\frac{13}{64}$	$-\frac{199}{256}$
	[cos − sin]	0	0		0	0	0	0	0
	[cos]	H	H		H	H	H	H	H
	[sin]	H	H		H	H	H	H	H
x^6	[cos + sin]		$+\frac{39}{64}$		$+\frac{3}{32}$		$+\frac{29}{64}$	$+\frac{1}{64}$	$+\frac{7}{64}$
	[cos − sin]		0		0		0	0	0
	[cos]		H		H		H	H	H
	[sin]		H		H		H	H	H
x^7	[cos + sin]		$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]		0				0		0
	[cos]		H				H		H
	[sin]		H				H		H

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$ ...

$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$		$[\ ]^0$		$[\ ]^1$		$[\ ]^2$			$[\ ]^3$		
		e^5	e^7	e^4	e^6	e^3	e^5	e^7	e^2	e^4	e^6
x^0	[cos + sin]	$-\frac{3}{16}$	$-\frac{5}{32}$	$+\frac{1045}{384}$	$-\frac{229}{192}$	$-\frac{95}{12}$	$+\frac{185}{12}$	$-\frac{1295}{192}$	$+\frac{75}{8}$	$-\frac{665}{16}$	$+\frac{58515}{1024}$
	[cos − sin]	,,	,,	0	$-\frac{37}{9216}$	0	0	$+\frac{5}{4032}$	0	0	0
	[cos]	$-\frac{3}{16}$	$-\frac{5}{32}$	H	$-\frac{11029}{18432}$	H	H	$-\frac{13595}{4032}$	H	H	H
	[sin]	0	0	H	$-\frac{10955}{18432}$	H	H	$-\frac{425}{126}$	H	H	H
x^1	[cos + sin]	$-\frac{15}{32}$	$-\frac{11}{64}$	$+\frac{229}{96}$	$-\frac{7747}{3840}$	$-\frac{29}{8}$	$+\frac{205}{24}$	$-\frac{11831}{1920}$	$+\frac{9}{4}$	$-\frac{169}{16}$	$+\frac{9609}{512}$
	[cos − sin]	,,	,,	0	$-\frac{151}{7680}$	0	0	$-\frac{37}{5760}$	0	0	0
	[cos]	$-\frac{15}{32}$	$-\frac{11}{64}$	H	$-\frac{1043}{1024}$	H	H	$-\frac{3553}{1152}$	H	H	H
	[sin]	0	0	H	$-\frac{15343}{15360}$	H	H	$-\frac{277}{90}$	H	H	H
x^2	[cos + sin]	$-\frac{5}{8}$	$+\frac{1}{16}$	$+\frac{43}{32}$	$-\frac{357}{256}$	-1	$+\frac{37}{24}$	$-\frac{691}{480}$	$+\frac{1}{4}$	$+\frac{11}{16}$	$-\frac{1435}{512}$
	[cos − sin]	,,	,,	0	$+\frac{37}{1536}$	0	0	$+\frac{1}{480}$	0	0	0
	[cos]	$-\frac{5}{8}$	$+\frac{1}{16}$	H	$-\frac{2105}{3072}$	H	H	$-\frac{23}{32}$	H	H	H
	[sin]	0	0	H	$-\frac{2179}{3072}$	H	H	$-\frac{173}{240}$	H	H	H
x^3	[cos + sin]	$-\frac{15}{32}$	$+\frac{5}{32}$	$+\frac{7}{16}$	$+\frac{3}{32}$	$-\frac{1}{8}$	$-\frac{19}{16}$	$+\frac{261}{128}$		$+\frac{5}{4}$	$-\frac{531}{128}$
	[cos − sin]	,,	,,	0	$+\frac{47}{384}$	0	0	$+\frac{7}{384}$		0	0
	[cos]	$-\frac{15}{32}$	$+\frac{5}{32}$	H	$+\frac{83}{768}$	H	H	$+\frac{395}{384}$		H	H
	[sin]	0	0	H	$-\frac{11}{768}$	H	H	$+\frac{97}{96}$		H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$	$[\ ]^4$				$[\ ]^5$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^0	[cos + sin]	-5	$+\frac{95}{2}$	$-\frac{1675}{12}$	$+\frac{25085}{144}$	$+1$	-25	$+\frac{9775}{64}$	$-\frac{110225}{288}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{17}{4}$	$-\frac{365}{24}$	$+\frac{8413}{288}$		$+\frac{1}{2}$	$-\frac{25}{8}$	$-\frac{25}{128}$
	[cos − sin]	0	0	0	0		0	0	0
	[cos]	H	H	H	H		H	H	H
	[sin]	H	H	H	H		H	H	H
x^2	[cos + sin]		$-\frac{3}{2}$	$+\frac{26}{3}$	$-\frac{527}{32}$		$+\frac{1}{2}$	$-\frac{25}{4}$	$+\frac{9725}{384}$
	[cos − sin]		0	0	0		0	0	0
	[cos]		H	H	H		H	H	H
	[sin]		H	H	H		H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{17}{8}$	$-\frac{725}{128}$			$+\frac{3}{8}$	$-\frac{25}{16}$
	[cos − sin]		0	0	0			0	0
	[cos]		H	H	H			H	H
	[sin]		H	H	H			H	H

...Table $(x^0, x^1, x^2, x^3)\,{}^{\cos}_{\sin}\,5f$...

	$\cos/\sin\,5f$	$[\ ]^6$: e	e^3	e^5	e^7	$[\ ]^7$: e^2	e^4	e^6	$[\ ]^8$: e^3	e^5	e^7
x^0	[cos + sin]	$+5$	$-\frac{315}{4}$	$+\frac{6375}{16}$	$-\frac{59585}{64}$	$+\frac{125}{8}$	$-\frac{9605}{48}$	$+\frac{2825155}{3072}$	$+\frac{235}{6}$	$-\frac{21515}{48}$	$+\frac{187915}{96}$
	[cos − sin]	0	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{69}{8}$	$-\frac{1335}{32}$	$+\frac{10003}{128}$	$-\frac{11}{4}$	$+\frac{1723}{48}$	$-\frac{242573}{1536}$	$-\frac{37}{4}$	$+\frac{10145}{96}$	$-\frac{425783}{960}$
	[cos − sin]	0	0	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H	H	H
x^2	[cos + sin]		$+1$	$-\frac{47}{4}$	$+\frac{1553}{32}$	$+\frac{1}{4}$	$-\frac{19}{16}$	$-\frac{4295}{512}$	$+\frac{3}{2}$	$-\frac{313}{24}$	$+\frac{6977}{240}$
	[cos − sin]		0	0	0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{69}{16}$	$-\frac{1995}{128}$		$-\frac{5}{4}$	$+\frac{1639}{128}$	$-\frac{1}{8}$	$-\frac{53}{32}$	$+\frac{769}{32}$
	[cos − sin]		0	0	0		0	0	0	0	0
	[cos]		H	H	H		H	H	H	H	H
	[sin]		H	H	H		H	H	H	H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$	$[\ ]^9$		$[\ ]^{10}$		$[\ ]^{11}$	$[\ ]^{12}$
		e^4	e^6	e^5	e^7	e^6	e^7
x^0	[cos + sin]	$+\frac{11035}{128}$	$-\frac{59127}{64}$	$+\frac{8359}{48}$	$-\frac{1033885}{576}$	$+\frac{3050417}{9216}$	$+\frac{67335}{112}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{787}{32}$	$+\frac{334011}{1280}$	$-\frac{5455}{96}$	$+\frac{665623}{1152}$	$-\frac{918191}{7680}$	$-\frac{37647}{160}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{173}{32}$	$-\frac{12697}{256}$	$+\frac{365}{24}$	$-\frac{13481}{96}$	$+\frac{18859}{512}$	$+\frac{12907}{160}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{13}{16}$	$+\frac{63}{32}$	$-\frac{25}{8}$	$+\frac{7115}{384}$	$-\frac{3563}{384}$	$-\frac{3011}{128}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f\ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$	$[\ ]^0$		$[\ ]^1$		$[\ ]^2$			$[\ ]^3$		
		e^5	e^7	e^4	e^6	e^3	e^5	e^7	e^2	e^4	e^6
x^4	[cos + sin]	$-\frac{3}{16}$	$-\frac{5}{32}$	$+\frac{1}{16}$	$+\frac{25}{32}$		$-\frac{13}{16}$	$+\frac{81}{64}$		$+\frac{1}{4}$	$-\frac{9}{128}$
	[cos − sin]	,,	,,	0	$+\frac{19}{128}$		0	$-\frac{1}{192}$		0	0
	[cos]	$-\frac{3}{16}$	$-\frac{5}{32}$	H	$+\frac{119}{256}$		H	$+\frac{121}{192}$		H	H
	[sin]	0	0	H	$+\frac{81}{256}$		H	$+\frac{61}{96}$		H	H
x^5	[cos + sin]	$-\frac{1}{32}$	$-\frac{25}{64}$		$+\frac{15}{32}$		$-\frac{5}{32}$	$-\frac{65}{128}$			$+\frac{55}{64}$
	[cos − sin]	,,	,,		$+\frac{5}{64}$		0	$-\frac{1}{128}$			0
	[cos]	$-\frac{1}{32}$	$-\frac{25}{64}$		$+\frac{35}{128}$		H	$-\frac{35}{128}$			H
	[sin]	0	0		$+\frac{25}{128}$		H	$-\frac{15}{64}$			H
x^6	[cos + sin]		$-\frac{1}{4}$		$+\frac{3}{32}$			$-\frac{21}{32}$			$+\frac{15}{64}$
	[cos − sin]		,,		$+\frac{1}{64}$			$-\frac{1}{32}$			0
	[cos]		$-\frac{1}{4}$		$+\frac{7}{128}$			$-\frac{11}{32}$			H
	[sin]		0		$+\frac{5}{128}$			$-\frac{5}{16}$			H
x^7	[cos + sin]		$-\frac{7}{128}$					$-\frac{21}{128}$			
	[cos − sin]		,,					$-\frac{1}{128}$			
	[cos]		$-\frac{7}{128}$					$-\frac{11}{128}$			
	[sin]		0					$-\frac{5}{64}$			

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 5f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 5f$	$[\ ]^4$		$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	
		e e^3 e^5	e^7	e^0 e^2 e^4	e^6	e e^3 e^5	e^7	e^2 e^4	e^6
x^4	[cos + sin]	$-\frac{7}{8}$	$+\frac{113}{32}$	$+\frac{3}{8}$	$-\frac{25}{8}$	$+\frac{3}{8}$	$-\frac{249}{64}$	$+\frac{1}{4}$	$-\frac{239}{128}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^5	[cos + sin]	$-\frac{5}{16}$	$+\frac{85}{64}$		$+\frac{5}{16}$	$-\frac{5}{16}$	$+\frac{345}{128}$		$-\frac{45}{64}$
	[cos − sin]	0	0		0	0	0		0
	[cos]	H	H		H	H	H		H
	[sin]	H	H		H	H	H		H
x^6	[cos + sin]		$-\frac{5}{8}$		$+\frac{5}{16}$		$+\frac{5}{32}$		$+\frac{15}{64}$
	[cos − sin]		0		0		0		0
	[cos]		H		H		H		H
	[sin]		H		H		H		H
x^7	[cos + sin]		$-\frac{35}{128}$				$-\frac{35}{128}$		
	[cos − sin]		0				0		
	[cos]		H				H		
	[sin]		H				H		

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $5f$	$[\ ]^8$		$[\ ]^9$		$[\ ]^{10}$		$[\ ]^{11}$	$[\ ]^{12}$
		e^3	e^5	e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]		$+\frac{17}{16}$	$+\frac{1}{16}$	$+\frac{75}{32}$	$+\frac{7}{16}$	$+\frac{505}{192}$	$+\frac{229}{128}$	$+\frac{179}{32}$
	[cos − sin]		0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H
x^5	[cos + sin]		$-\frac{5}{32}$		$-\frac{25}{32}$	$-\frac{1}{32}$	$-\frac{275}{128}$	$-\frac{15}{64}$	$-\frac{65}{64}$
	[cos − sin]		0		0	0	0	0	0
	[cos]		H		H	H	H	H	H
	[sin]		H		H	H	H	H	H
x^6	[cos + sin]		$+\frac{3}{4}$		$+\frac{3}{32}$		$+\frac{17}{32}$	$+\frac{1}{64}$	$+\frac{1}{8}$
	[cos − sin]		0		0		0	0	0
	[cos]		H		H		H	H	H
	[sin]		H		H		H	H	H
x^7	[cos + sin]		$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]		0				0		0
	[cos]		H				H		H
	[sin]		H				H		H

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 6f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 6f$	$[\ ]^0$	$[\ ]^1$		$[\ ]^2$		$[\ ]^3$		
		e^6	e^5	e^7	e^4	e^6	e^3	e^5	e^7
x^0	[cos + sin]	$+\frac{7}{64}$	$-\frac{773}{320}$	$+\frac{5167}{7680}$	$+\frac{157}{16}$	$-\frac{1339}{80}$	$-\frac{67}{4}$	$+\frac{4053}{64}$	$-\frac{192531}{2560}$
	[cos − sin]	,,	0	$+\frac{17}{53760}$	0	0	0	0	0
	[cos]	$+\frac{7}{64}$	H	$+\frac{6031}{17920}$	H	H	H	H	H
	[sin]	0	H	$+\frac{4519}{13440}$	H	H	H	H	H
x^1	[cos + sin]	$+\frac{21}{64}$	$-\frac{1961}{768}$	$+\frac{162773}{92160}$	$+\frac{277}{48}$	$-\frac{993}{80}$	$-\frac{93}{16}$	$+\frac{6289}{256}$	$-\frac{384789}{10240}$
	[cos − sin]	,,	0	$+\frac{1091}{92160}$	0	0	0	0	0
	[cos]	$+\frac{21}{64}$	H	$+\frac{20483}{23040}$	H	H	H	H	H
	[sin]	0	H	$+\frac{26947}{30720}$	H	H	H	H	H
x^2	[cos + sin]	$+\frac{35}{64}$	$-\frac{179}{96}$	$+\frac{14689}{7680}$	$+\frac{37}{16}$	$-\frac{235}{48}$	$-\frac{5}{4}$	$+\frac{107}{32}$	$-\frac{12123}{2560}$
	[cos − sin]	,,	0	$-\frac{17}{7680}$	0	0	0	0	0
	[cos]	$+\frac{35}{64}$	H	$+\frac{917}{960}$	H	H	H	H	H
	[sin]	0	H	$+\frac{2451}{2560}$	H	H	H	H	H
x^3	[cos + sin]	$+\frac{35}{64}$	$-\frac{57}{64}$	$+\frac{739}{1024}$	$+\frac{9}{16}$	$+\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{117}{64}$	$+\frac{6249}{1024}$
	[cos − sin]	,,	0	$-\frac{75}{1024}$	0	0	0	0	0
	[cos]	$+\frac{35}{64}$	H	$+\frac{83}{256}$	H	H	H	H	H
	[sin]	0	H	$+\frac{407}{1024}$	H	H	H	H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$	$[\ ]^4$			$[\ ]^5$			
		e^2	e^4	e^6	e	e^3	e^5	e^7
x^0	[cos + sin]	$+\frac{57}{4}$	$-\frac{829}{8}$	$+\frac{8117}{32}$	-6	$+\frac{345}{4}$	$-\frac{12605}{32}$	$+\frac{1246315}{1536}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
x^1	[cos + sin]	$+\frac{11}{4}$	$-\frac{497}{24}$	$+\frac{1933}{32}$	$-\frac{1}{2}$	$+\frac{105}{16}$	$-\frac{13325}{384}$	$+\frac{1838665}{18432}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{1}{4}$	$+\frac{7}{8}$	$-\frac{685}{96}$		$-\frac{7}{4}$	$+\frac{745}{48}$	$-\frac{72955}{1536}$
	[cos − sin]	0	0	0		0	0	0
	[cos]	H	H	H		H	H	H
	[sin]	H	H	H		H	H	H
x^3	[cos + sin]		$+\frac{3}{2}$	$-\frac{517}{64}$		$-\frac{3}{8}$	$+\frac{105}{32}$	$-\frac{13265}{1024}$
	[cos − sin]		0	0		0	0	0
	[cos]		H	H		H	H	H
	[sin]		H	H		H	H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$	$[\ ]^6$				$[\ ]^7$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
x^0	[cos + sin]	$+1$	-36	$+\frac{5103}{16}$	$-\frac{19015}{16}$	$+6$	$-\frac{525}{4}$	$+\frac{29813}{32}$	$-\frac{4795063}{1536}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^1	[cos + sin]		$+\frac{1}{2}$	$-\frac{9}{2}$	$-\frac{9}{32}$	$-\frac{1}{2}$	$+\frac{189}{16}$	$-\frac{30821}{384}$	$+\frac{4080685}{18432}$
	[cos − sin]		0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H
x^2	[cos + sin]		$+\frac{1}{2}$	-9	$+\frac{1695}{32}$		$+\frac{5}{4}$	$-\frac{959}{48}$	$+\frac{175553}{1536}$
	[cos − sin]		0	0	0		0	0	0
	[cos]		H	H	H		H	H	H
	[sin]		H	H	H		H	H	H
x^3	[cos + sin]			$+\frac{3}{8}$	$-\frac{9}{4}$		$-\frac{3}{8}$	$+\frac{189}{32}$	$-\frac{30737}{1024}$
	[cos − sin]			0	0		0	0	0
	[cos]			H	H		H	H	H
	[sin]			H	H		H	H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$	$[\ ]^8$ e^2	e^4	e^6	$[\ ]^9$ e^3	e^5	e^7
x^0	[cos + sin]	$+\frac{87}{4}$	$-\frac{3011}{8}$	$+\frac{151205}{64}$	$+\frac{247}{4}$	$-\frac{59781}{64}$	$+\frac{13947303}{2560}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{13}{4}$	$+\frac{1369}{24}$	$-\frac{21969}{64}$	$-\frac{201}{16}$	$+\frac{48449}{256}$	$-\frac{10874049}{10240}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{1}{4}$	$-\frac{11}{8}$	$-\frac{3317}{192}$	$+\frac{7}{4}$	$-\frac{629}{32}$	$+\frac{145449}{2560}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]		$-\frac{3}{2}$	$+\frac{1313}{64}$	$-\frac{1}{8}$	$-\frac{153}{64}$	$+\frac{44889}{1024}$
	[cos − sin]		0	0	0	0	0
	[cos]		H	H	H	H	H
	[sin]		H	H	H	H	H

... *Table* $(x^0, x^1, x^2, x^3)\,{}^{\cos}_{\sin}\,6f$.

	$^{\cos}_{\sin}\,6f$	$[\ ]^{10}$ e^4	$[\ ]^{10}$ e^6	$[\ ]^{11}$ e^5	$[\ ]^{11}$ e^7	$[\ ]^{12}$ e^6	$[\ ]^{13}$ e^7
x^0	[cos + sin]	$+\frac{605}{4}$	$-\frac{16829}{8}$	$+\frac{107333}{320}$	$-\frac{33928027}{7680}$	$+\frac{110793}{160}$	$+\frac{72815971}{53760}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1805}{48}$	$+\frac{16547}{32}$	$-\frac{73961}{768}$	$+\frac{115203737}{92160}$	$-\frac{35499}{160}$	$-\frac{43634809}{92160}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$+\frac{115}{16}$	$-\frac{8135}{96}$	$+\frac{2173}{96}$	$-\frac{2039459}{7680}$	$+\frac{1937}{32}$	$+\frac{11111819}{7680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{15}{16}$	$+\frac{5}{2}$	$-\frac{261}{64}$	$+\frac{30707}{1024}$	$-\frac{863}{64}$	$-\frac{38507}{1024}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 6f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} 6f$	$[\ ]^0$	$[\ ]^1$		$[\ ]^2$		$[\ ]^3$		$[\ ]^4$		
		e^6	e^5	e^7	e^4	e^6	e^3	e^5	e^2	e^4	e^6
x^4	[cos + sin]	$+\frac{21}{64}$	$-\frac{1}{4}$	$-\frac{25}{48}$	$+\frac{1}{16}$	$+\frac{21}{16}$	-1	$+\frac{87}{32}$	$+\frac{1}{4}$	$-\frac{15}{64}$
	[cos − sin]	,,	0	$-\frac{13}{96}$	0	0	0	0	0	0
	[cos]	$+\frac{21}{64}$	H	$-\frac{21}{64}$	H	H	H	H	H	H
	[sin]	0	H	$-\frac{37}{192}$	H	H	H	H	H	H
x^5	[cos + sin]	$+\frac{7}{64}$	$-\frac{1}{32}$	$-\frac{187}{256}$		$+\frac{19}{32}$	$-\frac{5}{32}$	$-\frac{189}{256}$		$+\frac{65}{64}$
	[cos − sin]	,,	0	$-\frac{29}{256}$		0	0	0		0
	[cos]	$+\frac{7}{64}$	H	$-\frac{27}{64}$		H	H	H		H
	[sin]	0	H	$-\frac{79}{256}$		H	H	H		H
x^6	[cos + sin]	$+\frac{1}{64}$		$-\frac{21}{64}$		$+\frac{3}{32}$		$-\frac{51}{64}$		$+\frac{15}{64}$
	[cos − sin]	,,		$-\frac{3}{64}$		0		0		0
	[cos]	$+\frac{1}{64}$		$-\frac{3}{16}$		H		H		H
	[sin]	0		$-\frac{9}{64}$		H		H		H
x^7	[cos + sin]			$-\frac{7}{128}$				$-\frac{21}{128}$		
	[cos − sin]			$-\frac{1}{128}$				0		
	[cos]			$-\frac{1}{32}$				H		
	[sin]			$-\frac{3}{128}$				H		

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$	$[\ ]^5$ e^3	e^5	e^7	$[\ ]^6$ e^4	e^6	$[\ ]^7$ e^3	e^5	e^7	$[\ ]^8$ e^4	e^6
x^4	[cos + sin]	-1	$+\frac{25}{4}$		$+\frac{3}{8}$	$-\frac{9}{2}$	$+\frac{1}{2}$	$-\frac{217}{32}$		$+\frac{1}{4}$	$-\frac{153}{64}$
	[cos − sin]	0	0		0	0	0	0		0	0
	[cos]	H	H		H	H	H	H		H	H
	[sin]	H	H		H	H	H	H		H	H
x^5	[cos + sin]		$-\frac{5}{16}$	$+\frac{525}{256}$		$+\frac{5}{16}$		$-\frac{5}{16}$	$+\frac{945}{256}$		$-\frac{55}{64}$
	[cos − sin]		0	0		0		0	0		0
	[cos]		H	H		H		H	H		H
	[sin]		H	H		H		H	H		H
x^6	[cos + sin]			$-\frac{45}{64}$		$+\frac{5}{16}$			$+\frac{15}{64}$		$+\frac{15}{64}$
	[cos − sin]			0		0			0		0
	[cos]			H		H			H		H
	[sin]			H		H			H		H
x^7	[cos + sin]			$-\frac{35}{128}$					$-\frac{35}{128}$		
	[cos − sin]			0					0		
	[cos]			H					H		
	[sin]			H					H		

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $6f$	$[\ ]^9$			$[\ ]^{10}$		$[\ ]^{11}$		$[\ ]^{12}$	$[\ ]^{13}$
		e^3	e^5	e^7	e^4	e^6	e^5	e^7	e^6	e^7
x^4	[cos + sin]		$+\frac{5}{4}$	$-\frac{219}{16}$	$+\frac{1}{16}$	$+\frac{51}{16}$	$+\frac{1}{2}$	$+\frac{403}{96}$	$+\frac{147}{64}$	$+\frac{191}{24}$
	[cos − sin]		0	0	0	0	0	0	0	0
	[cos]		H	H	H	H	H	H	H	H
	[sin]		H	H	H	H	H	H	H	H
x^5	[cos + sin]		$-\frac{5}{32}$	$-\frac{9}{256}$		$-\frac{29}{32}$	$-\frac{1}{32}$	$-\frac{727}{256}$	$-\frac{17}{64}$	$-\frac{329}{256}$
	[cos − sin]		0	0		0	0	0	0	0
	[cos]		H	H		H	H	H	H	H
	[sin]		H	H		H	H	H	H	H
x^6	[cos + sin]			$+\frac{57}{64}$		$+\frac{3}{32}$		$+\frac{39}{64}$	$+\frac{1}{64}$	$+\frac{9}{64}$
	[cos − sin]			0		0		0	0	0
	[cos]			H		H		H	H	H
	[sin]			H		H		H	H	H
x^7	[cos + sin]			$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]			0				0		0
	[cos]			H				H		H
	[sin]			H				H		H

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^0$	$[\ ]^1$	$[\ ]^2$		$[\ ]^3$	
		e^7	e^6	e^5	e^7	e^4	e^6
x^0	[cos + sin]	$-\frac{1}{16}$	$+\frac{93289}{46080}$	$-\frac{329}{30}$	$+\frac{11809}{720}$	$+\frac{3227}{128}$	$-\frac{26943}{320}$
	[cos − sin]	,,	0	0	0	0	0
	[cos]	$-\frac{1}{16}$	H	H	H	H	H
	[sin]	0	H	H	H	H	H
x^1	[cos + sin]	$-\frac{7}{32}$	$+\frac{19081}{7680}$	$-\frac{187}{24}$	$+\frac{10927}{720}$	$+\frac{361}{32}$	$-\frac{55317}{1280}$
	[cos − sin]	,,	0	0	0	0	0
	[cos]	$-\frac{7}{32}$	H	H	H	H	H
	[sin]	0	H	H	H	H	H
x^2	[cos + sin]	$-\frac{7}{16}$	$+\frac{1131}{512}$	$-\frac{97}{24}$	$+\frac{4223}{480}$	$+\frac{113}{32}$	$-\frac{3177}{256}$
	[cos − sin]	,,	0	0	0	0	0
	[cos]	$-\frac{7}{16}$	H	H	H	H	H
	[sin]	0	H	H	H	H	H
x^3	[cos + sin]	$-\frac{35}{64}$	$+\frac{529}{384}$	$-\frac{23}{16}$	$+\frac{391}{192}$	$+\frac{11}{16}$	$+\frac{15}{32}$
	[cos − sin]	,,	0	0	0	0	0
	[cos]	$-\frac{35}{64}$	H	H	H	H	H
	[sin]	0	H	H	H	H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^4$ e^3	e^5	e^7	$[\ ]^5$ e^2	e^4	e^6
x^0	[cos + sin]	$-\frac{91}{3}$	$+\frac{4445}{24}$	$-\frac{47299}{120}$	$+\frac{161}{8}$	$-\frac{10409}{48}$	$+\frac{2509759}{3072}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{17}{2}$	$+\frac{2689}{48}$	$-\frac{3395}{24}$	$+\frac{13}{4}$	$-\frac{1721}{48}$	$+\frac{236923}{1536}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^2	[cos + sin]	$-\frac{3}{2}$	$+\frac{149}{24}$	$-\frac{5897}{480}$	$+\frac{1}{4}$	$+\frac{17}{16}$	$-\frac{7671}{512}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
x^3	[cos + sin]	$-\frac{1}{8}$	$-\frac{83}{32}$	$+\frac{1807}{128}$		$+\frac{7}{4}$	$-\frac{1781}{128}$
	[cos − sin]	0	0	0		0	0
	[cos]	H	H	H		H	H
	[sin]	H	H	H		H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^6$				$[\ ]^7$			
		e^1	e^3	e^5	e^7	e^0	e^2	e^4	e^6
x^0	[cos + sin]	-7	$+\frac{567}{4}$	$-\frac{14805}{16}$	$+\frac{89999}{32}$	$+1$	-49	$+\frac{37975}{64}$	$-\frac{883421}{288}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{75}{8}$	$-\frac{2199}{32}$	$+\frac{17549}{64}$		$+\frac{1}{2}$	$-\frac{49}{8}$	$-\frac{49}{128}$
	[cos − sin]	0	0	0	0		0	0	0
	[cos]	H	H	H	H		H	H	H
	[sin]	H	H	H	H		H	H	H
x^2	[cos + sin]		-2	$+\frac{101}{4}$	$-\frac{3611}{32}$		$+\frac{1}{2}$	$-\frac{49}{4}$	$+\frac{37877}{384}$
	[cos − sin]		0	0	0		0	0	0
	[cos]		H	H	H		H	H	H
	[sin]		H	H	H		H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{75}{16}$	$-\frac{411}{16}$			$+\frac{3}{8}$	$-\frac{49}{16}$
	[cos − sin]		0	0	0			0	0
	[cos]		H	H	H			H	H
	[sin]		H	H	H			H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 7f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 7f$	$[\ ]^8$				$[\ ]^9$		
		e	e^3	e^5	e^7	e^2	e^4	e^6
x^0	[cos + sin]	$+7$	-203	$+\frac{5768}{3}$	$-\frac{1255219}{144}$	$+\frac{231}{8}$	$-\frac{10381}{16}$	$+\frac{5454687}{1024}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
x^1	[cos + sin]	$-\frac{1}{2}$	$+\frac{31}{2}$	$-\frac{845}{6}$	$+\frac{154651}{288}$	$-\frac{15}{4}$	$+\frac{1363}{16}$	$-\frac{344031}{512}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
x^2	[cos + sin]		$+\frac{3}{2}$	$-\frac{94}{3}$	$+\frac{3791}{16}$	$+\frac{1}{4}$	$-\frac{25}{16}$	$-\frac{16155}{512}$
	[cos − sin]		0	0	0	0	0	0
	[cos]		H	H	H	H	H	H
	[sin]		H	H	H	H	H	H
x^3	[cos + sin]		$-\frac{3}{8}$	$+\frac{31}{4}$	$-\frac{3373}{64}$		$-\frac{7}{4}$	$+\frac{3945}{118}$
	[cos − sin]		0	0	0		0	0
	[cos]		H	H	H		H	H
	[sin]		H	H	H		H	H

... *Table* (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^{10}$ e^3	e^5	e^7	$[\ ]^{11}$ e^4	e^6
x^0	[cos + sin]	$+\frac{1099}{12}$	$-\frac{84889}{48}$	$+\frac{1273433}{96}$	$+\frac{94885}{384}$	$-\frac{4151693}{960}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
x^1	[cos + sin]	$-\frac{131}{8}$	$+\frac{30193}{96}$	$-\frac{436457}{192}$	$-\frac{5231}{96}$	$+\frac{3618557}{3840}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
x^2	[cos + sin]	$+2$	$-\frac{677}{24}$	$+\frac{9889}{96}$	$+\frac{295}{32}$	$-\frac{34789}{256}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
x^3	[cos + sin]	$-\frac{1}{8}$	$-\frac{13}{4}$	$+\frac{2361}{32}$	$-\frac{17}{16}$	$+\frac{99}{32}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H

Table (x^0, x^1, x^2, x^3) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f\ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^{12}$ e^5	$[\ ]^{12}$ e^7	$[\ ]^{13}$ e^6	$[\ ]^{14}$ e^7
x^0	[cos + sin]	$+\frac{23877}{40}$	$-\frac{390873}{40}$	$+\frac{61307827}{46080}$	$+\frac{1004057}{360}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
x^1	[cos + sin]	$-\frac{2451}{16}$	$+\frac{49413}{20}$	$-\frac{2945503}{7680}$	$-\frac{254051}{288}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
x^2	[cos + sin]	$+\frac{257}{8}$	$-\frac{74701}{160}$	$+\frac{144373}{1536}$	$+\frac{116977}{480}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
x^3	[cos + sin]	$-\frac{165}{32}$	$+\frac{5905}{128}$	$-\frac{7213}{384}$	$-\frac{10969}{192}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H

Table (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 7f \ldots$

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 7f$	$[\ ]^0$	$[\ ]^1$	$[\ ]^2$		$[\ ]^3$		$[\ ]^4$			$[\ ]^5$	
		e^7	e^6	e^5	e^7	e^4	e^6	e^3	e^5	e^7	e^4	e^6
x^4	[cos + sin]	$-\frac{7}{16}$	$+\frac{73}{128}$	$-\frac{5}{16}$	$-\frac{221}{192}$	$+\frac{1}{16}$	$+\frac{63}{32}$		$-\frac{19}{16}$	$+5$	$+\frac{1}{4}$	$-\frac{59}{128}$
	[cos − sin]	,,	0	0	0	0	0		0	0	0	0
	[cos]	$-\frac{7}{16}$	H	H	H	H	H		H	H	H	H
	[sin]	0	H	H	H	H	H		H	H	H	H
x^5	[cos + sin]	$-\frac{7}{32}$	$+\frac{9}{64}$	$-\frac{1}{32}$	$-\frac{149}{128}$		$+\frac{23}{32}$		$-\frac{5}{32}$	-1		$+\frac{75}{64}$
	[cos − sin]	,,	0	0	0		0		0	0		0
	[cos]	$-\frac{7}{32}$	H	H	H		H		H	H		H
	[sin]	0	H	H	H		H		H	H		H
x^6	[cos + sin]	$-\frac{1}{16}$	$+\frac{1}{64}$		$-\frac{13}{32}$		$+\frac{3}{32}$			$-\frac{15}{16}$		$+\frac{15}{64}$
	[cos − sin]	,,	0		0		0			0		0
	[cos]	$-\frac{1}{16}$	H		H		H			H		H
	[sin]	0	H		H		H			H		H
x^7	[cos + sin]	$-\frac{1}{128}$			$-\frac{7}{128}$					$-\frac{21}{128}$		
	[cos − sin]	,,			0					0		
	[cos]	$-\frac{1}{128}$			H					H		
	[sin]	0			H					H		

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^6$		$[\ ]^7$		$[\ ]^8$				$[\ ]^9$		
		e^5	e^7	e^4	e^6	e	e^3	e^5	e^7	e^2	e^4	e^6
x^4	[cos + sin]	$-\frac{9}{8}$	$+\frac{645}{64}$	$+\frac{3}{8}$	$-\frac{49}{8}$			$+\frac{5}{8}$	$-\frac{173}{16}$		$+\frac{1}{4}$	$-\frac{381}{128}$
	[cos − sin]	0	0	0	0			0	0		0	0
	[cos]	H	H	H	H			H	H		H	H
	[sin]	H	H	H	H			H	H		H	H
x^5	[cos + sin]	$-\frac{5}{16}$	$+\frac{375}{128}$		$+\frac{5}{16}$			$-\frac{5}{16}$	$+\frac{155}{32}$			$-\frac{65}{64}$
	[cos − sin]	0	0		0			0	0			0
	[cos]	H	H		H			H	H			H
	[sin]	H	H		H			H	H			H
x^6	[cos + sin]		$-\frac{25}{32}$		$+\frac{5}{16}$				$+\frac{5}{16}$			$+\frac{15}{64}$
	[cos − sin]		0		0				0			0
	[cos]		H		H				H			H
	[sin]		H		H				H			H
x^7	[cos + sin]		$-\frac{35}{128}$						$-\frac{35}{128}$			
	[cos − sin]		0						0			
	[cos]		H						H			
	[sin]		H						H			

... *Table* (x^4, x^5, x^6, x^7) $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$.

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $7f$	$[\ ]^{10}$ e^5	$[\ ]^{10}$ e^7	$[\ ]^{11}$ e^4	$[\ ]^{11}$ e^6	$[\ ]^{12}$ e^5	$[\ ]^{12}$ e^7	$[\ ]^{13}$ e^6	$[\ ]^{14}$ e^7
x^4	[cos + sin]	$+\frac{23}{16}$	$-\frac{3807}{192}$	$+\frac{1}{16}$	$+\frac{133}{32}$	$+\frac{9}{16}$	$+\frac{25}{4}$	$+\frac{367}{128}$	$+\frac{2093}{192}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
x^5	[cos + sin]	$-\frac{5}{32}$	$-\frac{23}{128}$		$-\frac{33}{32}$	$-\frac{1}{32}$	$-\frac{29}{8}$	$-\frac{19}{64}$	$-\frac{203}{128}$
	[cos − sin]	0	0		0	0	0	0	0
	[cos]	H	H		H	H	H	H	H
	[sin]	H	H		H	H	H	H	H
x^6	[cos + sin]		$+\frac{33}{32}$		$+\frac{3}{32}$		$+\frac{11}{16}$	$+\frac{1}{64}$	$+\frac{5}{32}$
	[cos − sin]		0		0		0	0	0
	[cos]		H		H		H	H	H
	[sin]		H		H		H	H	H
x^7	[cos + sin]		$-\frac{21}{128}$				$-\frac{7}{128}$		$-\frac{1}{128}$
	[cos − sin]		0				0		0
	[cos]		H				H		H
	[sin]		H				H		H

Table $\left(\left(\frac{r}{a}\right)^{-5}\ldots\left(\frac{r}{a}\right)^{-1},\ \log\frac{r}{a},\ \left(\frac{r}{a}\right)^{-1}\ldots\left(\frac{r}{a}\right)^{4}\right)\ldots$

		[ ]^0				[ ]^1			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-5}$		$+1$	$+5$	$+\frac{105}{8}$	$+\frac{105}{4}$	$+\frac{5}{2}$	$+\frac{135}{16}$	$+\frac{7285}{384}$	$+\frac{643015}{18432}$
$\left(\frac{r}{a}\right)^{-4}$		$+1$	$+3$	$+\frac{45}{8}$	$+\frac{35}{4}$	$+2$	$+\frac{17}{4}$	$+\frac{685}{96}$	$+\frac{48293}{4608}$
$\left(\frac{r}{a}\right)^{-3}$		$+1$	$+\frac{3}{2}$	$+\frac{15}{8}$	$+\frac{35}{16}$	$+\frac{3}{2}$	$+\frac{27}{16}$	$+\frac{261}{128}$	$+\frac{14309}{6144}$
$\left(\frac{r}{a}\right)^{-2}$		$+1$	$+\frac{1}{2}$	$+\frac{3}{8}$	$+\frac{5}{16}$	$+1$	$+\frac{3}{8}$	$+\frac{65}{192}$	$+\frac{2675}{9216}$
$\left(\frac{r}{a}\right)^{-1}$		$+1$	0	0	0	$+\frac{1}{2}$	$-\frac{1}{16}$	$+\frac{1}{384}$	$-\frac{1}{18432}$
	[cos]								
$\log\frac{r}{a}$		0	$+\frac{1}{4}$	$+\frac{1}{32}$	$+\frac{1}{96}$	$-\frac{1}{2}$	$+\frac{3}{16}$	$+\frac{1}{128}$	$+\frac{127}{18432}$
$\left(\frac{r}{a}\right)^{1}$		$+1$	$+\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+\frac{3}{16}$	$-\frac{5}{384}$	$+\frac{7}{18432}$
$\left(\frac{r}{a}\right)^{2}$		$+1$	$+\frac{3}{2}$	0	0	-1	$+\frac{1}{8}$	$-\frac{1}{192}$	$+\frac{1}{9216}$
$\left(\frac{r}{a}\right)^{3}$		$+1$	$+3$	$+\frac{3}{8}$	0	$-\frac{3}{2}$	$-\frac{9}{16}$	$+\frac{15}{128}$	$-\frac{35}{6144}$
$\left(\frac{r}{a}\right)^{4}$		$+1$	$+5$	$+\frac{15}{8}$	0	-2	$-\frac{9}{4}$	$+\frac{19}{96}$	$-\frac{29}{4608}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-5} \ldots \left(\frac{r}{a}\right)^{-1},\ \log\frac{r}{a},\ \left(\frac{r}{a}\right)^{1} \ldots \left(\frac{r}{a}\right)^{4}\right)$ ...

		$[\ ]^2$			$[\ ]^3$			$[\ ]^4$	
		e^2	e^4	e^6	e^3	e^5	e^7	e^4	e^6
$\left(\frac{r}{a}\right)^{-5}$		$+5$	$+\frac{155}{12}$	$+\frac{835}{32}$	$+\frac{145}{16}$	$+\frac{4715}{256}$	$+\frac{70893}{2048}$	$+\frac{745}{48}$	$+\frac{197}{8}$
$\left(\frac{r}{a}\right)^{-4}$		$+\frac{7}{2}$	$+\frac{67}{12}$	$+\frac{421}{48}$	$+\frac{23}{4}$	$+\frac{435}{64}$	$+\frac{27027}{2560}$	$+\frac{437}{48}$	$+\frac{899}{120}$
$\left(\frac{r}{a}\right)^{-3}$		$+\frac{9}{4}$	$+\frac{7}{4}$	$+\frac{141}{64}$	$+\frac{53}{16}$	$+\frac{393}{256}$	$+\frac{24753}{10240}$	$+\frac{77}{16}$	$+\frac{129}{160}$
$\left(\frac{r}{a}\right)^{-2}$		$+\frac{5}{4}$	$+\frac{1}{6}$	$+\frac{21}{64}$	$+\frac{13}{8}$	$-\frac{25}{128}$	$+\frac{393}{1024}$	$+\frac{103}{48}$	$-\frac{129}{160}$
$\left(\frac{r}{a}\right)^{-1}$		$+\frac{1}{2}$	$-\frac{1}{6}$	$+\frac{1}{48}$	$+\frac{9}{16}$	$-\frac{81}{256}$	$+\frac{729}{10240}$	$+\frac{2}{3}$	$-\frac{8}{15}$
	[cos]								
$\log\frac{r}{a}$		$-\frac{3}{8}$	$+\frac{11}{48}$	$-\frac{3}{128}$	$-\frac{17}{48}$	$+\frac{77}{256}$	$-\frac{743}{10240}$	$-\frac{71}{192}$	$+\frac{129}{320}$
$\left(\frac{r}{a}\right)^{1}$		$-\frac{1}{4}$	$+\frac{1}{6}$	$-\frac{1}{32}$	$-\frac{3}{16}$	$+\frac{45}{256}$	$-\frac{567}{10240}$	$-\frac{1}{6}$	$+\frac{1}{5}$
$\left(\frac{r}{a}\right)^{2}$		$-\frac{1}{4}$	$+\frac{1}{12}$	$-\frac{1}{96}$	$-\frac{1}{8}$	$+\frac{9}{128}$	$-\frac{81}{5120}$	$-\frac{1}{12}$	$+\frac{1}{15}$
$\left(\frac{r}{a}\right)^{3}$		0	$-\frac{1}{4}$	$+\frac{3}{32}$	$+\frac{1}{16}$	$-\frac{45}{256}$	$+\frac{189}{2048}$	$+\frac{1}{16}$	$-\frac{3}{20}$
$\left(\frac{r}{a}\right)^{4}$		$+\frac{1}{2}$	$-\frac{7}{12}$	$+\frac{1}{8}$	$+\frac{1}{4}$	$-\frac{19}{64}$	$+\frac{261}{2560}$	$+\frac{7}{48}$	$-\frac{1}{5}$

$$\ldots \textit{Table} \left(\left(\frac{r}{a}\right)^{-5} \ldots \left(\frac{r}{a}\right)^{-1},\ \log\frac{r}{a} \ldots \left(\frac{r}{a}\right)^{1} \ldots \left(\frac{r}{a}\right)^{4}\right).$$

		$[\ ]^5$		$[\ ]^6$	$[\ ]^7$
		e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^{-5}$		$+\frac{19669}{768}$	$+\frac{565135}{18432}$	$+\frac{1317}{32}$	$+\frac{1195093}{18432}$
$\left(\frac{r}{a}\right)^{-4}$		$+\frac{2701}{192}$	$+\frac{31997}{4608}$	$+\frac{1709}{80}$	$+\frac{737707}{23040}$
$\left(\frac{r}{a}\right)^{-3}$		$+\frac{1773}{256}$	$-\frac{4987}{6144}$	$+\frac{3167}{320}$	$+\frac{432091}{30720}$
$\left(\frac{r}{a}\right)^{-2}$		$+\frac{1097}{384}$	$-\frac{16621}{9216}$	$+\frac{1223}{320}$	$+\frac{47273}{9216}$
$\left(\frac{r}{a}\right)^{-1}$		$+\frac{625}{768}$	$-\frac{15625}{18432}$	$+\frac{81}{80}$	$+\frac{117649}{92160}$
	[cos]				
$\log\frac{r}{a}$		$-\frac{523}{1280}$	$+\frac{10039}{18432}$	$-\frac{899}{1920}$	$-\frac{355081}{645120}$
$\left(\frac{r}{a}\right)^{1}$		$-\frac{125}{768}$	$+\frac{4375}{18432}$	$-\frac{27}{160}$	$-\frac{16807}{92160}$
$\left(\frac{r}{a}\right)^{2}$		$-\frac{25}{384}$	$+\frac{615}{9216}$	$-\frac{9}{160}$	$-\frac{2401}{46080}$
$\left(\frac{r}{a}\right)^{3}$		$+\frac{15}{256}$	$-\frac{875}{6144}$	$+\frac{9}{160}$	$+\frac{343}{6144}$
$\left(\frac{r}{a}\right)^{4}$		$+\frac{19}{192}$	$-\frac{725}{4608}$	$+\frac{3}{40}$	$+\frac{1421}{23040}$

Table $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix} f \ldots$

$\begin{matrix}\cos\\ \sin\end{matrix} f$		$[\ ]^0$				$[\ ]^1$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$-\frac{3}{2}$	0	0	0	+1	$-\frac{1}{2}$	$-\frac{1}{64}$	$-\frac{29}{1152}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{1}{8}$	$+\frac{1}{24}$	$+\frac{25}{1024}$
	[cos]	$-\frac{3}{2}$	0	0	0	H	$-\frac{3}{16}$	$+\frac{5}{384}$	$-\frac{7}{18432}$
	[sin]	0	0	0	0	H	$-\frac{5}{16}$	$-\frac{11}{384}$	$-\frac{457}{18432}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	−2	$-\frac{1}{2}$	0	0	+1	$+\frac{1}{2}$	$-\frac{25}{64}$	$-\frac{23}{1152}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{5}{8}$	0	$+\frac{143}{3072}$
	[cos]	−2	$-\frac{1}{2}$	0	0	H	$+\frac{9}{16}$	$-\frac{25}{128}$	$+\frac{245}{18432}$
	[sin]	0	0	0	0	H	$-\frac{1}{16}$	$-\frac{25}{128}$	$-\frac{613}{18432}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{5}{2}$	$-\frac{15}{8}$	0	0	+1	+2	$-\frac{41}{64}$	$-\frac{37}{576}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{11}{8}$	$+\frac{7}{48}$	$+\frac{265}{3072}$
	[cos]	$-\frac{5}{2}$	$-\frac{15}{8}$	0	0	H	$+\frac{27}{16}$	$-\frac{95}{384}$	$+\frac{203}{18432}$
	[sin]	0	0	0	0	H	$+\frac{5}{16}$	$-\frac{151}{384}$	$-\frac{1387}{18432}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	−3	$-\frac{9}{2}$	$-\frac{3}{8}$	0	+1	+4	$-\frac{1}{64}$	$-\frac{199}{576}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{19}{8}$	$+\frac{47}{48}$	$+\frac{115}{1024}$
	[cos]	−3	$-\frac{9}{2}$	$-\frac{3}{8}$	0	H	$+\frac{51}{16}$	$+\frac{185}{384}$	$-\frac{2149}{18432}$
	[sin]	0	0	0	0	H	$+\frac{13}{16}$	$-\frac{191}{384}$	$-\frac{4219}{18432}$

...*Table* $\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix} f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^2$				$[\ ]^3$			$[\ ]^4$		
		e	e^3	e^5	e^7	e^2	e^4	e^6	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{1}{2}$	$-\frac{3}{8}$	$+\frac{5}{96}$	$-\frac{1}{72}$	$+\frac{3}{8}$	$-\frac{3}{8}$	$+\frac{111}{1024}$	$+\frac{1}{3}$	$-\frac{5}{12}$	$+\frac{43}{240}$
	[cos − sin]	0	$+\frac{1}{24}$	$+\frac{1}{96}$	$+\frac{1}{120}$	0	$+\frac{3}{128}$	$+\frac{3}{1280}$	0	$+\frac{1}{60}$	$-\frac{1}{720}$
	[cos]	H	$-\frac{1}{6}$	$+\frac{1}{32}$	$-\frac{1}{360}$	H	$-\frac{45}{256}$	$+\frac{567}{10240}$	H	$-\frac{1}{5}$	$+\frac{4}{45}$
	[sin]	H	$-\frac{5}{24}$	$+\frac{1}{48}$	$-\frac{1}{90}$	H	$-\frac{51}{256}$	$+\frac{543}{10240}$	H	$-\frac{13}{60}$	$+\frac{13}{144}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	0	$+\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{5}{96}$	$-\frac{1}{8}$	$+\frac{1}{2}$	$-\frac{397}{1024}$	$-\frac{1}{6}$	$+\frac{13}{24}$	$-\frac{47}{96}$
	[cos − sin]	0	$+\frac{1}{6}$	$-\frac{1}{24}$	$+\frac{7}{480}$	0	$+\frac{11}{128}$	$-\frac{11}{256}$	0	$+\frac{7}{120}$	$-\frac{7}{160}$
	[cos]	0	$+\frac{1}{3}$	$-\frac{3}{16}$	$+\frac{1}{30}$	H	$+\frac{75}{256}$	$-\frac{441}{2048}$	H	$+\frac{3}{10}$	$-\frac{4}{15}$
	[sin]	0	$+\frac{1}{6}$	$-\frac{7}{48}$	$+\frac{3}{160}$	H	$+\frac{53}{256}$	$-\frac{353}{2048}$	H	$+\frac{29}{120}$	$-\frac{107}{480}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{1}{2}$	$+1$	$-\frac{35}{96}$	$+\frac{23}{576}$	$-\frac{3}{8}$	$+\frac{11}{16}$	$-\frac{351}{1024}$	$-\frac{7}{24}$	$+\frac{55}{96}$	$-\frac{701}{1920}$
	[cos − sin]	0	$+\frac{1}{6}$	$-\frac{1}{96}$	$+\frac{7}{960}$	0	$+\frac{7}{128}$	$-\frac{9}{640}$	0	$+\frac{13}{480}$	$-\frac{73}{5760}$
	[cos]	H	$+\frac{7}{12}$	$-\frac{3}{16}$	$+\frac{17}{720}$	H	$+\frac{95}{256}$	$-\frac{1827}{10240}$	H	$+\frac{3}{10}$	$-\frac{17}{90}$
	[sin]	H	$+\frac{5}{12}$	$-\frac{17}{96}$	$+\frac{47}{2880}$	H	$+\frac{81}{256}$	$-\frac{1683}{10240}$	H	$+\frac{131}{480}$	$-\frac{203}{1152}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	-1	$+\frac{3}{4}$	$+\frac{13}{48}$	$-\frac{83}{576}$	$-\frac{3}{8}$	$+\frac{3}{16}$	$+\frac{249}{1024}$	$-\frac{1}{6}$	$+\frac{1}{48}$	$+\frac{119}{480}$
	[cos − sin]	0	$-\frac{1}{12}$	$+\frac{1}{6}$	$-\frac{43}{960}$	0	$-\frac{9}{128}$	$+\frac{57}{640}$	0	$-\frac{11}{240}$	$+\frac{91}{1440}$
	[cos]	H	$+\frac{1}{3}$	$+\frac{7}{32}$	$-\frac{17}{180}$	H	$+\frac{15}{256}$	$+\frac{1701}{10240}$	H	$-\frac{1}{80}$	$+\frac{7}{45}$
	[sin]	H	$+\frac{5}{12}$	$+\frac{5}{96}$	$-\frac{143}{2880}$	H	$+\frac{33}{256}$	$+\frac{789}{10240}$	H	$+\frac{1}{30}$	$+\frac{133}{1440}$

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix} f.$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^5$		$[\ ]^6$		$[\ ]^7$	$[\ ]^8$
		e^4	e^5	e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{125}{384}$	$-\frac{125}{256}$	$+\frac{27}{80}$	$-\frac{189}{320}$	$+\frac{16807}{46080}$	$+\frac{128}{315}$
	[cos − sin]	0	$+\frac{125}{9216}$	0	$+\frac{27}{2240}$	0	0
	[cos]	H	$-\frac{4375}{18432}$	H	$-\frac{81}{280}$	H	H
	[sin]	H	$-\frac{4625}{18432}$	H	$-\frac{135}{448}$	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$-\frac{25}{128}$	$+\frac{475}{768}$	$-\frac{9}{40}$	$+\frac{117}{160}$	$-\frac{2401}{9216}$	$-\frac{32}{105}$
	[cos − sin]	0	$+\frac{425}{9216}$	0	$+\frac{9}{224}$	0	0
	[cos]	H	$+\frac{6125}{18432}$	H	$+\frac{27}{70}$	H	H
	[sin]	H	$+\frac{5275}{18432}$	H	$+\frac{387}{1120}$	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{95}{384}$	$+\frac{205}{384}$	$-\frac{9}{40}$	$+\frac{171}{320}$	$-\frac{9947}{46080}$	$-\frac{68}{315}$
	[cos − sin]	0	$+\frac{155}{9216}$	0	$+\frac{27}{2240}$	0	0
	[cos]	H	$+\frac{5075}{18432}$	H	$+\frac{153}{560}$	H	H
	[sin]	H	$+\frac{4765}{18432}$	H	$+\frac{117}{448}$	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$-\frac{31}{384}$	$-\frac{7}{128}$	$-\frac{3}{80}$	$-\frac{33}{320}$	$-\frac{539}{46080}$	$+\frac{2}{315}$
	[cos − sin]	0	$-\frac{301}{9216}$	0	$-\frac{57}{2240}$	0	0
	[cos]	H	$-\frac{805}{18432}$	H	$-\frac{9}{140}$	H	H
	[sin]	H	$-\frac{203}{18432}$	H	$-\frac{87}{2240}$	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix} f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^0$				$[\ ]^1$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$	$+1$	-1	$-\frac{1}{64}$	$-\frac{17}{288}$
	[cos − sin]	,,	,,	,,	,,	0	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{115}{3072}$
	[cos]	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$	H	$-\frac{9}{16}$	$-\frac{5}{128}$	$-\frac{889}{18432}$
	[sin]	0	0	0	0	H	$-\frac{7}{16}$	$+\frac{3}{128}$	$-\frac{199}{18432}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	0	0	0	0	$+1$	$-\frac{1}{2}$	$-\frac{1}{64}$	$-\frac{29}{1152}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{1}{8}$	$+\frac{1}{24}$	$+\frac{25}{1024}$
	[cos]	0	0	0	0	H	$-\frac{3}{16}$	$+\frac{5}{384}$	$-\frac{7}{18432}$
	[sin]	0	0	0	0	H	$-\frac{5}{16}$	$-\frac{11}{384}$	$-\frac{457}{18432}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{1}{2}$	$+\frac{3}{4}$	$+\frac{15}{16}$	$+\frac{35}{32}$	$+1$	$+\frac{1}{2}$	$+\frac{55}{64}$	$+\frac{1177}{1152}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{5}{8}$	$+\frac{5}{6}$	$+\frac{3103}{3072}$
	[cos]	$+\frac{1}{2}$	$+\frac{3}{4}$	$+\frac{15}{16}$	$+\frac{35}{32}$	H	$+\frac{9}{16}$	$+\frac{325}{384}$	$+\frac{18725}{18432}$
	[sin]	0	0	0	0	H	$-\frac{1}{16}$	$+\frac{5}{384}$	$+\frac{107}{18432}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+1$	$+\frac{5}{2}$	$+\frac{35}{8}$	$+\frac{105}{16}$	$+1$	$+2$	$+\frac{239}{64}$	$+\frac{3323}{576}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{11}{8}$	$+\frac{49}{16}$	$+\frac{15665}{3072}$
	[cos]	$+1$	$+\frac{5}{2}$	$+\frac{35}{8}$	$+\frac{105}{16}$	H	$+\frac{27}{16}$	$+\frac{435}{128}$	$+\frac{100163}{18432}$
	[sin]	0	0	0	0	H	$+\frac{5}{16}$	$+\frac{43}{128}$	$+\frac{6173}{18432}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{3}{2}$	$+\frac{45}{8}$	$+\frac{105}{8}$	$+\frac{1575}{64}$	$+1$	$+4$	$+\frac{647}{64}$	$+\frac{11465}{576}$
	[cos − sin]	,,	,,	,,	,,	0	$+\frac{19}{8}$	$+\frac{371}{48}$	$+\frac{17179}{1024}$
	[cos]	$+\frac{3}{2}$	$+\frac{45}{8}$	$+\frac{105}{8}$	$+\frac{1575}{64}$	H	$+\frac{51}{16}$	$+\frac{3425}{384}$	$+\frac{338051}{18432}$
	[sin]	0	0	0	0	H	$+\frac{13}{16}$	$+\frac{457}{384}$	$+\frac{28829}{18432}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix} f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^{3/2}$				$[\ ]^{4}$		
		e	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{3}{2}$	$-\frac{7}{4}$	$+\frac{31}{96}$	$-\frac{1}{12}$	$+\frac{17}{8}$	$-\frac{47}{16}$	$+\frac{1069}{1024}$
	[cos − sin]	0	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{7}{240}$	0	$-\frac{9}{128}$	$-\frac{9}{320}$
	[cos]	H	$-\frac{11}{12}$	$+\frac{9}{64}$	$-\frac{9}{160}$	H	$-\frac{385}{256}$	$+\frac{5201}{10240}$
	[sin]	H	$-\frac{5}{6}$	$+\frac{35}{192}$	$-\frac{13}{480}$	H	$-\frac{367}{256}$	$+\frac{5489}{10240}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+2$	$-\frac{3}{2}$	$+\frac{5}{24}$	$-\frac{1}{18}$	$+\frac{27}{8}$	$-\frac{27}{8}$	$+\frac{999}{1024}$
	[cos − sin]	0	$+\frac{1}{6}$	$+\frac{1}{24}$	$+\frac{1}{30}$	0	$+\frac{27}{128}$	$+\frac{27}{1280}$
	[cos]	H	$-\frac{2}{3}$	$+\frac{1}{8}$	$-\frac{1}{90}$	H	$-\frac{405}{256}$	$+\frac{5103}{10240}$
	[sin]	H	$-\frac{5}{6}$	$+\frac{1}{12}$	$-\frac{2}{45}$	H	$-\frac{459}{256}$	$+\frac{4887}{10240}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{5}{2}$	$-\frac{1}{8}$	$+\frac{103}{96}$	$+\frac{1255}{1152}$	$+\frac{39}{8}$	-2	$+\frac{1803}{1024}$
	[cos − sin]	0	$+\frac{19}{24}$	$+\frac{43}{48}$	$+\frac{2063}{1920}$	0	$+\frac{131}{128}$	$+\frac{237}{256}$
	[cos]	H	$+\frac{1}{3}$	$+\frac{63}{64}$	$+\frac{779}{720}$	H	$-\frac{125}{256}$	$+\frac{2751}{2048}$
	[sin]	H	$-\frac{11}{24}$	$+\frac{17}{192}$	$+\frac{43}{5760}$	H	$-\frac{387}{256}$	$+\frac{855}{2048}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+3$	$+\frac{11}{4}$	$+\frac{245}{48}$	$+\frac{463}{64}$	$+\frac{53}{8}$	$+\frac{39}{16}$	$+\frac{7041}{1024}$
	[cos − sin]	0	$+\frac{23}{12}$	$+\frac{89}{24}$	$+\frac{5663}{960}$	0	$+\frac{343}{128}$	$+\frac{2819}{640}$
	[cos]	H	$+\frac{7}{3}$	$+\frac{141}{32}$	$+\frac{197}{30}$	H	$+\frac{655}{256}$	$+\frac{57757}{10240}$
	[sin]	H	$+\frac{5}{12}$	$+\frac{67}{96}$	$+\frac{641}{960}$	H	$-\frac{31}{256}$	$+\frac{12653}{10240}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{7}{2}$	$+\frac{15}{2}$	$+\frac{1529}{96}$	$+\frac{16207}{576}$	$+\frac{69}{8}$	$+\frac{183}{16}$	$+\frac{23865}{1024}$
	[cos − sin]	0	$+\frac{11}{3}$	$+\frac{997}{96}$	$+\frac{20279}{960}$	0	$+\frac{711}{128}$	$+\frac{8733}{640}$
	[cos]	H	$+\frac{67}{12}$	$+\frac{421}{32}$	$+\frac{8867}{360}$	H	$+\frac{2175}{256}$	$+\frac{189189}{10240}$
	[sin]	H	$+\frac{23}{12}$	$+\frac{133}{48}$	$+\frac{10099}{2880}$	H	$+\frac{753}{256}$	$+\frac{49461}{10240}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix} f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	$[\ ]^4$ e^3	e^5	e^7	$[\ ]^5$ e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{71}{24}$	$-\frac{229}{48}$	$+\frac{1169}{480}$	$+\frac{523}{128}$	$-\frac{1451}{192}$
	[cos − sin]	0	$-\frac{1}{15}$	$-\frac{1}{60}$	0	$-\frac{625}{9216}$
	[cos]	H	$-\frac{387}{160}$	$+\frac{387}{320}$	H	$-\frac{70273}{18432}$
	[sin]	H	$-\frac{1129}{480}$	$+\frac{1177}{960}$	H	$-\frac{69023}{18432}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{16}{3}$	$-\frac{20}{3}$	$+\frac{43}{15}$	$+\frac{3125}{384}$	$-\frac{3125}{256}$
	[cos − sin]	0	$+\frac{4}{15}$	$-\frac{1}{45}$	0	$+\frac{3125}{9216}$
	[cos]	H	$-\frac{16}{5}$	$+\frac{64}{45}$	H	$-\frac{109375}{18432}$
	[sin]	H	$-\frac{52}{15}$	$+\frac{13}{9}$	H	$-\frac{115625}{18432}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{103}{12}$	$-\frac{593}{96}$	$+\frac{721}{192}$	$+\frac{5485}{384}$	$-\frac{11053}{768}$
	[cos − sin]	0	$+\frac{643}{480}$	$+\frac{2569}{2880}$	0	$+\frac{16289}{9216}$
	[cos]	H	$-\frac{387}{160}$	$+\frac{1673}{720}$	H	$-\frac{116347}{18432}$
	[sin]	H	$-\frac{451}{120}$	$+\frac{4123}{2880}$	H	$-\frac{148925}{18432}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{77}{6}$	$-\frac{25}{48}$	$+\frac{4751}{480}$	$+\frac{2955}{128}$	$-\frac{3463}{384}$
	[cos − sin]	0	$+\frac{899}{240}$	$+\frac{2441}{480}$	0	$+\frac{48203}{9216}$
	[cos]	H	$+\frac{129}{80}$	$+\frac{899}{120}$	H	$-\frac{34909}{18432}$
	[sin]	H	$-\frac{32}{15}$	$+\frac{77}{32}$	H	$-\frac{131315}{18432}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{437}{24}$	$+\frac{1361}{96}$	$+\frac{63341}{1920}$	$+\frac{13505}{384}$	$+\frac{1541}{128}$
	[cos − sin]	0	$+\frac{3983}{480}$	$+\frac{100849}{5760}$	0	$+\frac{113027}{9216}$
	[cos]	H	$+\frac{899}{80}$	$+\frac{36359}{1440}$	H	$+\frac{223979}{18432}$
	[sin]	H	$+\frac{1411}{480}$	$+\frac{44587}{5760}$	H	$-\frac{2075}{18432}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix} f.$

	$\begin{matrix}\cos\\ \sin\end{matrix} f$	[ ]^6 e^5	[ ]^6 e^7	[ ]^7 e^8	[ ]^8 e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{899}{160}$	$-\frac{1879}{160}$	$+\frac{355081}{46080}$	$+\frac{47259}{4480}$
	[cos − sin]	0	$-\frac{81}{1120}$	0	0
	[cos]	H	$-\frac{6617}{1120}$	H	H
	[sin]	H	$-\frac{817}{140}$	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{243}{20}$	$-\frac{1701}{80}$	$+\frac{823543}{46080}$	$+\frac{8192}{315}$
	[cos − sin]	0	$+\frac{243}{560}$	0	0
	[cos]	H	$-\frac{729}{70}$	H	H
	[sin]	H	$-\frac{1215}{112}$	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{3669}{160}$	$-\frac{18847}{640}$	$+\frac{330911}{9216}$	$+\frac{556403}{10080}$
	[cos − sin]	0	$+\frac{2101}{896}$	0	0
	[cos]	H	$-\frac{7589}{560}$	H	H
	[sin]	H	$-\frac{71217}{4480}$	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{3167}{80}$	$-\frac{8999}{320}$	$+\frac{3024637}{46080}$	$+\frac{178331}{1680}$
	[cos − sin]	0	$+\frac{16337}{2240}$	0	0
	[cos]	H	$-\frac{729}{70}$	H	H
	[sin]	H	$-\frac{7933}{448}$	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{5127}{80}$	$-\frac{753}{320}$	$+\frac{5163949}{46080}$	$+\frac{3830591}{20160}$
	[cos − sin]	0	$+\frac{40263}{2240}$	0	0
	[cos]	H	$+\frac{2187}{280}$	H	H
	[sin]	H	$-\frac{22767}{2240}$	H	H

Table $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 2f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 2f$	$[\ ]^0$			$[\ ]^1$			
		e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{3}{2}$	0	0	$-\frac{5}{2}$	$+\frac{33}{16}$	$-\frac{73}{384}$	$+\frac{881}{18432}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{48}$	$-\frac{41}{768}$	$-\frac{1159}{30720}$
	[cos]	$+\frac{3}{2}$	0	0	H	$+\frac{25}{24}$	$-\frac{187}{1536}$	$+\frac{29}{5760}$
	[sin]	0	0	0	H	$+\frac{49}{48}$	$-\frac{35}{512}$	$+\frac{3941}{92160}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{5}{2}$	0	0	-3	$+\frac{13}{8}$	$+\frac{5}{192}$	$+\frac{227}{3072}$
	[cos − sin]	,,	,,	,,	0	$-\frac{7}{24}$	$-\frac{47}{384}$	$-\frac{1091}{15360}$
	[cos]	$+\frac{5}{2}$	0	0	H	$+\frac{2}{3}$	$-\frac{37}{768}$	$+\frac{11}{7680}$
	[sin]	0	0	0	H	$+\frac{23}{24}$	$+\frac{19}{256}$	$+\frac{371}{5120}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{15}{4}$	$+\frac{5}{8}$	0	$-\frac{7}{2}$	$+\frac{1}{16}$	$+\frac{289}{384}$	$+\frac{1645}{18432}$
	[cos − sin]	,,	,,	,,	,,	$-\frac{47}{48}$	$-\frac{151}{768}$	$-\frac{4043}{30720}$
	[cos]	$+\frac{15}{4}$	$+\frac{5}{8}$	0	H	$-\frac{11}{24}$	$+\frac{427}{1536}$	$-\frac{61}{2880}$
	[sin]	0	0	0	H	$+\frac{25}{48}$	$+\frac{243}{512}$	$+\frac{10177}{92160}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{21}{4}$	$+\frac{21}{8}$	0	-4	-3	$+\frac{79}{48}$	$+\frac{233}{1152}$
	[cos − sin]	,,	,,	,,	,,	$-\frac{13}{6}$	$-\frac{13}{24}$	$-\frac{989}{3840}$
	[cos]	$+\frac{21}{4}$	$+\frac{21}{8}$	0	H	$-\frac{31}{12}$	$+\frac{53}{96}$	$-\frac{637}{23040}$
	[sin]	0	0	0	H	$-\frac{5}{12}$	$+\frac{35}{32}$	$+\frac{5297}{23040}$

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right)$ $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$ ...

	$\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $2f$	$[\ ]^2$				$[\ ]^3$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+1$	$-\frac{7}{2}$	$+\frac{47}{16}$	$-\frac{215}{288}$	$+\frac{3}{2}$	$-\frac{75}{16}$	$+\frac{573}{128}$	$-\frac{3507}{2048}$
	[cos − sin]	0	0	$+\frac{1}{48}$	$-\frac{3}{160}$	0	0	$+\frac{21}{1280}$	$-\frac{123}{10240}$
	[cos]	H	H	$+\frac{71}{48}$	$-\frac{551}{1440}$	H	H	$+\frac{5751}{2560}$	$-\frac{8829}{10240}$
	[sin]	H	H	$+\frac{35}{24}$	$-\frac{131}{360}$	H	H	$+\frac{5709}{2560}$	$-\frac{4353}{5120}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+1$	$-\frac{5}{2}$	$+\frac{23}{16}$	$-\frac{65}{288}$	$+1$	$-\frac{19}{8}$	$+\frac{107}{64}$	$-\frac{479}{1024}$
	[cos − sin]	0	0	$-\frac{1}{16}$	$-\frac{11}{480}$	0	0	$-\frac{17}{640}$	$-\frac{7}{1024}$
	[cos]	H	H	$+\frac{11}{16}$	$-\frac{179}{1440}$	H	H	$+\frac{1053}{1280}$	$-\frac{243}{1024}$
	[sin]	H	H	$+\frac{3}{4}$	$-\frac{73}{720}$	H	H	$+\frac{1087}{1280}$	$-\frac{59}{256}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+1$	-1	$-\frac{11}{16}$	$+\frac{43}{72}$	$+\frac{1}{2}$	$-\frac{3}{16}$	$-\frac{117}{128}$	$+\frac{1553}{2048}$
	[cos − sin]	0	0	$-\frac{7}{48}$	$+\frac{1}{240}$	0	0	$-\frac{69}{1280}$	$+\frac{137}{10240}$
	[cos]	H	H	$-\frac{5}{12}$	$+\frac{433}{1440}$	H	H	$-\frac{1239}{2560}$	$+\frac{3951}{10240}$
	[sin]	H	H	$-\frac{13}{48}$	$+\frac{427}{1440}$	H	H	$-\frac{1101}{2560}$	$+\frac{1907}{5120}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+1$	$+1$	$-\frac{43}{16}$	$+\frac{35}{36}$		$+\frac{3}{2}$	$-\frac{9}{4}$	$+\frac{267}{256}$
	[cos − sin]	0	0	$-\frac{5}{48}$	0		0	$-\frac{3}{160}$	$+\frac{3}{640}$
	[cos]	H	H	$-\frac{67}{48}$	$+\frac{35}{72}$		H	$-\frac{363}{320}$	$+\frac{1341}{2560}$
	[sin]	H	H	$-\frac{31}{24}$	$+\frac{35}{72}$		H	$-\frac{357}{320}$	$+\frac{1329}{2560}$

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 2f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 2f$	$[\ ]^4$ e^2	e^4	e^6	$[\ ]^5$ e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+2$	$-\frac{19}{3}$	$+\frac{55}{8}$	$+\frac{125}{48}$	$-\frac{6625}{768}$	$+\frac{64625}{6144}$
	[cos − sin]	0	0	$+\frac{1}{72}$	0	0	$+\frac{1625}{129024}$
	[cos]	H	H	$+\frac{31}{9}$	H	H	$+\frac{679375}{129024}$
	[sin]	H	H	$+\frac{247}{72}$	H	H	$+\frac{3388875}{64512}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+1$	$-\frac{5}{2}$	$+\frac{101}{48}$	$+\frac{25}{24}$	$-\frac{1075}{384}$	$+\frac{8425}{3072}$
	[cos − sin]	0	0	$-\frac{11}{720}$	0	0	$-\frac{75}{7168}$
	[cos]	H	H	$+\frac{47}{45}$	H	H	$+\frac{29375}{21504}$
	[sin]	H	H	$+\frac{763}{720}$	H	H	$+\frac{925}{672}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{1}{4}$	$+\frac{5}{24}$	$-\frac{227}{192}$	$+\frac{5}{48}$	$+\frac{385}{768}$	$-\frac{9515}{6144}$
	[cos − sin]	0	0	$-\frac{83}{2880}$	0	0	$-\frac{2435}{129024}$
	[cos]	H	H	$-\frac{109}{180}$	H	H	$-\frac{101125}{129024}$
	[sin]	H	H	$-\frac{1661}{2880}$	H	H	$-\frac{49345}{64512}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$-\frac{1}{4}$	$+\frac{37}{24}$	$-\frac{139}{64}$	$-\frac{1}{3}$	$+\frac{151}{96}$	$-\frac{877}{384}$
	[cos − sin]	0	0	$-\frac{17}{2880}$	0	0	$-\frac{41}{16128}$
	[cos]	H	H	$-\frac{49}{45}$	H	H	$-\frac{36875}{32256}$
	[sin]	H	H	$-\frac{3119}{2880}$	H	H	$-\frac{36793}{32256}$

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 2f.$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 2f$	$[\ ]^6$		$[\ ]^7$		$[\ ]^8$	$[\ ]^9$
		e^4	e^5	e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{27}{8}$	$-\frac{189}{16}$	$+\frac{16807}{3840}$	$-\frac{1495823}{92160}$	$+\frac{256}{45}$	$+\frac{531441}{71680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{9}{8}$	$-\frac{261}{80}$	$+\frac{2401}{1920}$	$-\frac{12005}{3072}$	$+\frac{64}{45}$	$+\frac{59049}{35840}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	0	$+\frac{63}{80}$	$-\frac{343}{3840}$	$+\frac{102557}{92160}$	$-\frac{8}{45}$	$-\frac{19683}{71680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	0	H	H	H	H	H
	[sin]	0	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$-\frac{3}{8}$	$+\frac{33}{20}$	$-\frac{49}{120}$	$+\frac{20531}{11520}$	$-\frac{4}{9}$	$-\frac{2187}{4480}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 2f \ldots$

$\begin{matrix}\cos\\ \sin\end{matrix}\ 2f$		$[\ ]^0$			$[\ ]^1$			
		e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{1}{4}$	$+\frac{1}{8}$	$+\frac{5}{64}$	$-\frac{3}{2}$	$+\frac{17}{16}$	$+\frac{37}{384}$	$+\frac{197}{2048}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{48}$	$+\frac{29}{768}$	$+\frac{1013}{30720}$
	[cos]	$+\frac{1}{4}$	$+\frac{1}{8}$	$+\frac{5}{64}$	H	$+\frac{13}{24}$	$+\frac{103}{1536}$	$+\frac{31}{480}$
	[sin]	0	0	0	H	$+\frac{25}{48}$	$+\frac{15}{512}$	$+\frac{971}{30720}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	0	0	0	-1	$+\frac{3}{8}$	$+\frac{7}{192}$	$+\frac{295}{9216}$
	[cos − sin]	,,	,,	,,	0	$-\frac{1}{24}$	$-\frac{13}{384}$	$-\frac{389}{15360}$
	[cos]	0	0	0	H	$+\frac{1}{6}$	$+\frac{1}{768}$	$+\frac{77}{23040}$
	[sin]	0	0	0	H	$+\frac{5}{24}$	$+\frac{9}{256}$	$+\frac{1321}{46080}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	0	0	0	$-\frac{1}{2}$	$+\frac{1}{16}$	$-\frac{5}{384}$	$-\frac{143}{18432}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{48}$	$+\frac{11}{768}$	$+\frac{313}{30720}$
	[cos]	0	0	0	H	$+\frac{1}{24}$	$+\frac{1}{1536}$	$+\frac{7}{5760}$
	[sin]	0	0	0	H	$+\frac{1}{48}$	$-\frac{7}{512}$	$-\frac{827}{92160}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{1}{4}$	$+\frac{5}{8}$	$+\frac{35}{32}$	0	$+\frac{1}{2}$	$+\frac{11}{12}$	$+\frac{1099}{768}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{3}$	$+\frac{73}{96}$	$+\frac{2441}{1920}$
	[cos]	$+\frac{1}{4}$	$+\frac{5}{8}$	$+\frac{35}{32}$	0	$+\frac{5}{12}$	$+\frac{161}{192}$	$+\frac{3459}{2560}$
	[sin]	0	0	0	0	$+\frac{1}{12}$	$+\frac{5}{64}$	$+\frac{613}{7680}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{3}{4}$	$+\frac{21}{8}$	$+\frac{189}{32}$	$+\frac{1}{2}$	$+\frac{33}{16}$	$+\frac{1865}{384}$	$+\frac{169229}{18432}$
	[cos − sin]	,,	,,	,,	0	$+\frac{49}{48}$	$+\frac{2545}{768}$	$+\frac{219221}{30720}$
	[cos]	$+\frac{3}{4}$	$+\frac{21}{8}$	$+\frac{189}{32}$	H	$+\frac{37}{24}$	$+\frac{6275}{1536}$	$+\frac{23497}{2880}$
	[sin]	0	0	0	H	$+\frac{25}{48}$	$+\frac{395}{512}$	$+\frac{94241}{92160}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 2f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 2f$	$[\ ]^2$				$[\ ]^3$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+1$	-4	$+\frac{47}{16}$	$-\frac{55}{144}$	$+\frac{5}{2}$	$-\frac{131}{16}$	$+\frac{887}{128}$	$-\frac{3999}{2048}$
	[cos − sin]	0	0	0	$+\frac{1}{48}$	0	0	$-\frac{9}{1280}$	$+\frac{153}{10240}$
	[cos]	H	H	H	$-\frac{13}{72}$	H	H	$+\frac{8861}{2560}$	$-\frac{9921}{10240}$
	[sin]	H	H	H	$-\frac{29}{144}$	H	H	$+\frac{8879}{2560}$	$-\frac{5037}{5120}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+1$	$-\frac{7}{2}$	$+\frac{29}{16}$	$-\frac{53}{288}$	$+3$	$-\frac{69}{8}$	$+\frac{369}{64}$	$-\frac{1401}{1024}$
	[cos − sin]	0	0	$-\frac{1}{24}$	$-\frac{3}{80}$	0	0	$-\frac{27}{640}$	$-\frac{189}{5120}$
	[cos]	H	H	$+\frac{85}{96}$	$-\frac{319}{2880}$	H	H	$+\frac{3663}{1280}$	$-\frac{3597}{5120}$
	[sin]	H	H	$+\frac{89}{96}$	$-\frac{211}{2880}$	H	H	$+\frac{3717}{1280}$	$-\frac{213}{320}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+1$	$-\frac{5}{2}$	$+\frac{13}{16}$	$-\frac{35}{288}$	$+\frac{7}{2}$	$-\frac{123}{16}$	$+\frac{489}{128}$	$-\frac{1763}{2048}$
	[cos − sin]	0	0	$+\frac{1}{24}$	$+\frac{7}{240}$	0	0	$+\frac{81}{1280}$	$+\frac{81}{2048}$
	[cos]	H	H	$+\frac{41}{96}$	$-\frac{133}{2880}$	H	H	$+\frac{4971}{2560}$	$-\frac{841}{2048}$
	[sin]	H	H	$+\frac{37}{96}$	$-\frac{217}{2880}$	H	H	$+\frac{4809}{2560}$	$-\frac{461}{1024}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+1$	-1	$+\frac{17}{16}$	$+\frac{85}{72}$	$+4$	-5	$+\frac{51}{16}$	$+\frac{143}{128}$
	[cos − sin]	0	0	$+\frac{7}{16}$	$+\frac{109}{120}$	0	0	$+\frac{23}{40}$	$+\frac{1369}{1280}$
	[cos]	H	H	$+\frac{3}{4}$	$+\frac{47}{45}$	H	H	$+\frac{301}{160}$	$+\frac{2799}{2560}$
	[sin]	H	H	$+\frac{5}{16}$	$+\frac{49}{360}$	H	H	$+\frac{209}{160}$	$+\frac{61}{2560}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+1$	$+1$	$+\frac{65}{16}$	$+\frac{139}{18}$	$+\frac{9}{2}$	$-\frac{3}{16}$	$+\frac{963}{128}$	$+\frac{22737}{2048}$
	[cos − sin]	0	0	$+\frac{67}{48}$	$+\frac{333}{80}$	0	0	$+\frac{2451}{1280}$	$+\frac{53193}{10240}$
	[cos]	H	H	$+\frac{131}{48}$	$+\frac{8557}{1440}$	H	H	$+\frac{12081}{2560}$	$+\frac{83439}{10240}$
	[sin]	H	H	$+\frac{4}{3}$	$+\frac{2563}{1440}$	H	H	$+\frac{7179}{2560}$	$+\frac{15123}{5120}$

$$\ldots \textit{Table} \left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 2f \ldots$$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 2f$	$[\ ]^4$ e^2	e^4	e^6	$[\ ]^5$ e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{19}{4}$	$-\frac{121}{8}$	$+\frac{707}{48}$	$+\frac{389}{48}$	$-\frac{20267}{768}$	$+\frac{179141}{6144}$
	[cos − sin]	0	0	$-\frac{1}{90}$	0	0	$-\frac{625}{43008}$
	[cos]	H	H	$+\frac{10597}{1440}$	H	H	$+\frac{626681}{43008}$
	[sin]	H	H	$+\frac{10613}{1440}$	H	H	$+\frac{104551}{7168}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{13}{2}$	$-\frac{55}{3}$	$+\frac{239}{16}$	$+\frac{295}{24}$	$-\frac{13745}{384}$	$+\frac{105175}{3072}$
	[cos − sin]	0	0	$-\frac{2}{45}$	0	0	$-\frac{3125}{64512}$
	[cos]	H	H	$+\frac{10723}{1440}$	H	H	$+\frac{1102775}{64512}$
	[sin]	H	H	$+\frac{10787}{1440}$	H	H	$+\frac{276475}{16128}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{17}{2}$	$-\frac{115}{6}$	$+\frac{601}{48}$	$+\frac{845}{48}$	$-\frac{32525}{768}$	$+\frac{208225}{6144}$
	[cos − sin]	0	0	$+\frac{4}{45}$	0	0	$+\frac{15625}{129024}$
	[cos]	H	H	$+\frac{9079}{1440}$	H	H	$+\frac{2194175}{129024}$
	[sin]	H	H	$+\frac{8951}{1440}$	H	H	$+\frac{1089275}{64512}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{43}{4}$	$-\frac{129}{8}$	$+\frac{1985}{192}$	$+\frac{145}{6}$	$-\frac{1015}{24}$	$+\frac{23581}{768}$
	[cos − sin]	0	0	$+\frac{437}{576}$	0	0	$+\frac{2701}{2688}$
	[cos]	H	H	$+\frac{799}{144}$	H	H	$+\frac{56823}{3584}$
	[sin]	H	H	$+\frac{2759}{576}$	H	H	$+\frac{159665}{10752}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{53}{4}$	$-\frac{179}{24}$	$+\frac{977}{64}$	$+\frac{1541}{48}$	$-\frac{23479}{768}$	$+\frac{216917}{6144}$
	[cos − sin]	0	0	$+\frac{7579}{2880}$	0	0	$+\frac{467069}{129024}$
	[cos]	H	H	$+\frac{6443}{720}$	H	H	$+\frac{2511163}{129024}$
	[sin]	H	H	$+\frac{18193}{2880}$	H	H	$+\frac{1022047}{64512}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 2f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 2f$	$[\ ]^6$		$[\ ]^7$		$[\ ]^8$	$[\ ]^9$
		e^4	e^6	e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{209}{16}$	$-\frac{887}{20}$	$+\frac{78077}{3840}$	$-\frac{2228929}{30720}$	$+\frac{17807}{576}$	$+\frac{3313213}{71680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{345}{16}$	$-\frac{10569}{160}$	$+\frac{69251}{1920}$	$-\frac{5394109}{46080}$	$+\frac{42037}{720}$	$+\frac{3306951}{35840}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{533}{16}$	$-\frac{13827}{160}$	$+\frac{228347}{3840}$	$-\frac{3071075}{18432}$	$+\frac{73369}{720}$	$+\frac{12144273}{71680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+49$	-98	$+\frac{44569}{480}$	$-\frac{133707}{640}$	$+\frac{241517}{1440}$	$+\frac{2619177}{8960}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{555}{8}$	$-\frac{7149}{80}$	$+\frac{533617}{3840}$	$-\frac{20449667}{92160}$	$+\frac{379691}{1440}$	$+\frac{34431741}{71680}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table $\left(\frac{r}{a}\right) \ldots \left(\frac{r}{a}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix} 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} 3f$	$[\ ]^0$			$[\ ]^1$		
		e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$-\frac{5}{4}$	$-\frac{5}{32}$	$-\frac{3}{64}$	$+\frac{31}{8}$	$-\frac{77}{24}$	$+\frac{155}{1024}$
	[cos − sin]	,,	,,	,,	0	$-\frac{19}{384}$	$+\frac{3}{256}$
	[cos]	$-\frac{5}{4}$	$-\frac{5}{32}$	$-\frac{3}{64}$	H	$-\frac{417}{256}$	$+\frac{167}{2048}$
	[sin]	0	0	0	H	$-\frac{1213}{768}$	$+\frac{143}{2048}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$-\frac{5}{2}$	0	0	$+\frac{43}{8}$	$-\frac{25}{6}$	$+\frac{1069}{3072}$
	[cos − sin]	,,	,,	,,	0	$+\frac{37}{384}$	$+\frac{487}{3840}$
	[cos]	$-\frac{5}{2}$	0	0	H	$-\frac{521}{256}$	$+\frac{2431}{10240}$
	[sin]	0	0	0	H	$-\frac{1637}{768}$	$+\frac{3397}{30720}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{35}{8}$	0	0	$+\frac{57}{8}$	$-\frac{65}{16}$	$-\frac{19}{1024}$
	[cos − sin]	,,	,,	,,	0	$+\frac{75}{128}$	$+\frac{179}{640}$
	[cos]	$-\frac{35}{8}$	0	0	H	$-\frac{445}{256}$	$+\frac{1337}{10240}$
	[sin]	0	0	0	H	$-\frac{595}{256}$	$-\frac{1527}{10240}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	-7	$-\frac{7}{8}$	0	$+\frac{73}{8}$	$-\frac{91}{48}$	$-\frac{1419}{1024}$
	[cos − sin]	,,	,,	,,	0	$+\frac{641}{384}$	$+\frac{341}{640}$
	[cos]	-7	$-\frac{7}{8}$	0	H	$-\frac{29}{256}$	$-\frac{4367}{10240}$
	[sin]	0	0	0	H	$-\frac{1369}{768}$	$-\frac{9823}{10240}$

...*Table* $\left(\left(\frac{r}{a}\right) \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 3f$	$[\ ]^2$				$[\ ]^3$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$-\frac{7}{2}$	$+\frac{75}{8}$	$-\frac{659}{96}$	$+\frac{1687}{1152}$	$+1$	$-\frac{17}{2}$	$+\frac{1143}{64}$	$+\frac{1805}{128}$
	[cos − sin]	0	0	$-\frac{11}{480}$	$-\frac{13}{5760}$	0	0	0	$-\frac{69}{5120}$
	[cos]	H	H	$-\frac{551}{160}$	$+\frac{4211}{5760}$	H	H	H	$-\frac{72269}{10240}$
	[sin]	H	H	$-\frac{821}{240}$	$+\frac{11}{15}$	H	H	H	$-\frac{72131}{10240}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	-4	$+\frac{19}{2}$	$-\frac{79}{12}$	$+\frac{113}{72}$	$+1$	$-\frac{15}{2}$	$+\frac{927}{64}$	$-\frac{1415}{128}$
	[cos − sin]	0	0	$+\frac{1}{120}$	$+\frac{1}{36}$	0	0	0	$-\frac{1}{5120}$
	[cos]	H	H	$-\frac{263}{80}$	$+\frac{115}{144}$	H	H	H	$-\frac{56601}{10240}$
	[sin]	H	H	$-\frac{791}{240}$	$+\frac{37}{48}$	H	H	H	$-\frac{56599}{10240}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{9}{2}$	$+\frac{33}{4}$	$-\frac{133}{32}$	$+\frac{11}{16}$	$+1$	-6	$+\frac{591}{64}$	$-\frac{349}{64}$
	[cos − sin]	0	0	$+\frac{13}{160}$	$+\frac{1}{30}$	0	0	0	$+\frac{137}{5120}$
	[cos]	H	H	$-\frac{163}{80}$	$+\frac{173}{480}$	H	H	H	$-\frac{27783}{10240}$
	[sin]	H	H	$-\frac{339}{160}$	$+\frac{157}{480}$	H	H	H	$-\frac{28057}{10240}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	-5	$+\frac{21}{4}$	$+\frac{17}{48}$	$-\frac{71}{72}$	$+1$	-4	$+\frac{183}{64}$	$+\frac{65}{64}$
	[cos − sin]	0	0	$+\frac{2}{15}$	$+\frac{1}{720}$	0	0	0	$+\frac{37}{1024}$
	[cos]	H	H	$+\frac{30}{160}$	$-\frac{709}{1440}$	H	H	H	$+\frac{1077}{2048}$
	[sin]	H	H	$+\frac{53}{480}$	$-\frac{79}{160}$	H	H	H	$+\frac{1003}{2048}$

... *Table* $\left(\left(\frac{r}{a}\right) \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 3f$	$[\ ]^4$				$[\ ]^5$		
		e	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{5}{2}$	$-\frac{63}{4}$	$+\frac{757}{24}$	$-\frac{1979}{72}$	$+\frac{37}{8}$	$-\frac{635}{24}$	$+\frac{54765}{1024}$
	[cos − sin]	0	0	0	$-\frac{47}{5040}$	0	0	0
	[cos]	H	H	H	$-\frac{138577}{10080}$	H	H	H
	[sin]	H	H	H	$-\frac{15387}{1120}$	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+2$	$-\frac{23}{2}$	$+\frac{65}{3}$	$-\frac{5317}{288}$	$+\frac{25}{8}$	$-\frac{50}{3}$	$+\frac{98875}{3072}$
	[cos − sin]	0	0	0	$-\frac{17}{10080}$	0	0	0
	[cos]	H	H	H	$-\frac{2908}{315}$	H	H	H
	[sin]	H	H	H	$-\frac{31013}{3360}$	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{3}{2}$	$-\frac{57}{8}$	$+11$	$-\frac{2905}{384}$	$+\frac{15}{8}$	$-\frac{135}{16}$	$+\frac{13975}{1024}$
	[cos − sin]	0	0	0	$+\frac{57}{4480}$	0	0	0
	[cos]	H	H	H	$-\frac{793}{210}$	H	H	H
	[sin]	H	H	H	$-\frac{50923}{13440}$	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+1$	-3	$+\frac{37}{24}$	$+\frac{251}{144}$	$+\frac{7}{8}$	$-\frac{109}{48}$	$+\frac{535}{1024}$
	[cos − sin]	0	0	0	$+\frac{79}{5040}$	0	0	0
	[cos]	H	H	H	$+\frac{277}{315}$	H	H	H
	[sin]	H	H	H	$+\frac{1451}{1680}$	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right) \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 3f$	$[\ ]^6$			$[\ ]^7$	
		e^3	e^5	e^7	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{61}{8}$	$-\frac{677}{16}$	$+\frac{56487}{640}$	$+\frac{4553}{384}$	$-\frac{84109}{1280}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{9}{2}$	$-\frac{189}{8}$	$+\frac{3807}{80}$	$+\frac{2401}{384}$	$-\frac{127253}{3840}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{9}{4}$	$-\frac{81}{8}$	$+\frac{2781}{160}$	$+\frac{343}{128}$	$-\frac{7889}{640}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{3}{4}$	$-\frac{27}{16}$	$-\frac{81}{160}$	$+\frac{245}{384}$	$-\frac{147}{128}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right) \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 3f.$

$\begin{matrix}\cos\\ \sin\end{matrix}\ 3f$		$[\ ]^8$ e^5	$[\ ]^8$ e^7	$[\ ]^9$ e^6	$[\ ]^{10}$ e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{8551}{480}$	$-\frac{288221}{2880}$	$+\frac{26809}{1024}$	$+\frac{305593}{8064}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{128}{15}$	$-\frac{416}{9}$	$+\frac{59049}{5120}$	$+\frac{15625}{1008}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{16}{5}$	$-\frac{76}{5}$	$+\frac{19683}{5120}$	$+\frac{3125}{672}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{8}{15}$	$-\frac{26}{45}$	$+\frac{2187}{5120}$	$+\frac{625}{2016}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 3f$	$[\ ]^0$			$[\ ]^1$		
		e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$-\frac{1}{8}$	$-\frac{3}{32}$	$-\frac{9}{128}$	$+\frac{13}{8}$	$-\frac{41}{48}$	$-\frac{437}{3072}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{384}$	$-\frac{11}{960}$
	[cos]	$-\frac{1}{8}$	$-\frac{3}{32}$	$-\frac{9}{128}$	H	$-\frac{109}{256}$	$-\frac{787}{10240}$
	[sin]	0	0	0	H	$-\frac{329}{768}$	$-\frac{2009}{30720}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	0	0	0	$+\frac{7}{8}$	$-\frac{5}{24}$	$-\frac{29}{1024}$
	[cos − sin]	,,	,,	,,	0	$+\frac{5}{384}$	$+\frac{23}{1280}$
	[cos]	0	0	0	H	$-\frac{25}{256}$	$-\frac{53}{10240}$
	[sin]	0	0	0	H	$-\frac{85}{768}$	$-\frac{237}{10240}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	0	0	0	$+\frac{3}{8}$	0	$+\frac{23}{1024}$
	[cos − sin]	,,	,,	,,	0	$-\frac{1}{128}$	$-\frac{11}{1280}$
	[cos]	0	0	0	H	$-\frac{1}{256}$	$+\frac{71}{10240}$
	[sin]	0	0	0	H	$+\frac{1}{256}$	$+\frac{159}{10240}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	0	0	0	$+\frac{1}{8}$	$+\frac{1}{48}$	$+\frac{55}{3072}$
	[cos − sin]	,,	,,	,,	0	$+\frac{1}{384}$	$+\frac{1}{384}$
	[cos]	0	0	0	H	$+\frac{3}{256}$	$+\frac{21}{2048}$
	[sin]	0	0	0	H	$+\frac{7}{768}$	$+\frac{47}{6144}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{1}{8}$	$+\frac{7}{16}$	$+\frac{63}{64}$	$+\frac{1}{8}$	$+\frac{17}{48}$	$+\frac{837}{1024}$
	[cos − sin]	,,	,,	,,	0	$+\frac{65}{384}$	$+\frac{353}{640}$
	[cos]	$+\frac{1}{8}$	$+\frac{7}{16}$	$+\frac{63}{64}$	H	$+\frac{67}{256}$	$+\frac{7009}{10240}$
	[sin]	0	0	0	H	$+\frac{71}{768}$	$+\frac{1361}{10240}$

$$\ldots \textit{Table} \left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5} \right) \begin{matrix}\cos\\ \sin\end{matrix}\, 3f \ldots$$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 3f$	$[\ ]^2$				$[\ ]^3$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$-\frac{5}{2}$	$+\frac{13}{2}$	$-\frac{341}{96}$	$+\frac{187}{576}$	$+1$	-9	$+\frac{1143}{64}$	$-\frac{365}{32}$
	[cos − sin]	0	0	$+\frac{1}{120}$	$-\frac{1}{2880}$	0	0	0	$+\frac{9}{1024}$
	[cos]	H	H	$-\frac{567}{320}$	$+\frac{467}{2880}$	H	H	H	$-\frac{11671}{2048}$
	[sin]	H	H	$-\frac{1709}{960}$	$+\frac{13}{80}$	H	H	H	$-\frac{11689}{2048}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	-2	$+\frac{9}{2}$	$-\frac{41}{24}$	$+\frac{43}{288}$	$+1$	$-\frac{17}{2}$	$+\frac{951}{64}$	$-\frac{973}{128}$
	[cos − sin]	0	0	$+\frac{1}{120}$	$+\frac{23}{1440}$	0	0	0	$+\frac{27}{5120}$
	[cos]	H	H	$-\frac{17}{20}$	$+\frac{119}{1440}$	H	H	H	$-\frac{38893}{10240}$
	[sin]	H	H	$-\frac{103}{120}$	$+\frac{1}{15}$	H	H	H	$-\frac{38947}{10240}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$-\frac{3}{2}$	$+\frac{21}{8}$	$-\frac{19}{32}$	$+\frac{23}{192}$	$+1$	$-\frac{15}{2}$	$+\frac{687}{64}$	$-\frac{535}{128}$
	[cos − sin]	0	0	$-\frac{1}{80}$	$-\frac{1}{64}$	0	0	0	$-\frac{81}{5120}$
	[cos]	H	H	$-\frac{97}{320}$	$+\frac{5}{96}$	H	H	H	$-\frac{21481}{10240}$
	[sin]	H	H	$-\frac{93}{320}$	$+\frac{13}{192}$	H	H	H	$-\frac{21319}{10240}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	-1	$+\frac{5}{4}$	$-\frac{7}{48}$	$+\frac{23}{288}$	$+1$	-6	$+\frac{423}{64}$	$-\frac{125}{64}$
	[cos − sin]	0	0	$+\frac{1}{120}$	$+\frac{13}{1440}$	0	0	0	$+\frac{81}{5120}$
	[cos]	H	H	$-\frac{11}{160}$	$+\frac{2}{45}$	H	H	H	$-\frac{9919}{10240}$
	[sin]	H	H	$-\frac{37}{480}$	$+\frac{17}{480}$	H	H	H	$-\frac{10081}{10240}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$-\frac{1}{2}$	$+\frac{3}{4}$	$+\frac{61}{96}$	$+\frac{191}{144}$	$+1$	-4	$+\frac{255}{64}$	$+\frac{17}{64}$
	[cos − sin]	0	0	$+\frac{109}{480}$	$+\frac{31}{45}$	0	0	0	$+\frac{1553}{5120}$
	[cos]	H	H	$+\frac{69}{160}$	$+\frac{1451}{1440}$	H	H	H	$+\frac{2913}{10240}$
	[sin]	H	H	$+\frac{49}{240}$	$+\frac{51}{160}$	H	H	H	$-\frac{193}{10240}$

... *Table* $\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}$ $\begin{matrix}\cos\\ \sin\end{matrix}$ $3f$...

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 3f$	[ ]^4				[ ]^5		
		e	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{7}{2}$	$-\frac{179}{8}$	$+\frac{2009}{48}$	$-\frac{8893}{288}$	$+\frac{67}{8}$	$-\frac{2279}{48}$	$+\frac{274345}{3072}$
	[cos − sin]	0	0	0	$+\frac{11}{1260}$	0	0	0
	[cos]	H	H	H	$-\frac{311167}{20160}$	H	H	H
	[sin]	H	H	H	$-\frac{103781}{6720}$	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+4$	-24	$+\frac{241}{6}$	$-\frac{455}{18}$	$+\frac{85}{8}$	$-\frac{1355}{24}$	$+\frac{98525}{1024}$
	[cos − sin]	0	0	0	$+\frac{1}{315}$	0	0	0
	[cos]	H	H	H	$-\frac{15923}{1260}$	H	H	H
	[sin]	H	H	H	$-\frac{5309}{420}$	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{9}{2}$	-24	$+\frac{545}{16}$	$-\frac{71}{4}$	$+\frac{105}{8}$	$-\frac{125}{2}$	$+\frac{94825}{1024}$
	[cos − sin]	0	0	0	$-\frac{2}{105}$	0	0	0
	[cos]	H	H	H	$-\frac{7463}{840}$	H	H	H
	[sin]	H	H	H	$-\frac{7447}{840}$	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+5$	-22	$+\frac{607}{24}$	$-\frac{98}{9}$	$+\frac{127}{8}$	$-\frac{3065}{48}$	$+\frac{243805}{3072}$
	[cos − sin]	0	0	0	$+\frac{8}{315}$	0	0	0
	[cos]	H	H	H	$-\frac{1711}{315}$	H	H	H
	[sin]	H	H	H	$-\frac{191}{35}$	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{11}{2}$	$-\frac{141}{8}$	$+\frac{415}{24}$	$-\frac{5021}{1152}$	$+\frac{151}{8}$	$-\frac{2809}{48}$	$+\frac{63343}{1024}$
	[cos − sin]	0	0	0	$+\frac{16319}{40320}$	0	0	0
	[cos]	H	H	H	$-\frac{19927}{10080}$	H	H	H
	[sin]	H	H	H	$-\frac{32009}{13440}$	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 3f$	$[\ ]^6$			$[\ ]^7$	
		e^3	e^5	e^7	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+17$	$-\frac{2949}{32}$	$+\frac{57213}{320}$	$+\frac{12085}{384}$	$-\frac{32419}{192}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{47}{2}$	$-\frac{239}{2}$	$-\frac{33951}{160}$	$+\frac{17969}{384}$	$-\frac{301973}{1280}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{251}{8}$	$-\frac{4611}{32}$	$+\frac{36249}{160}$	$+\frac{8547}{128}$	$-\frac{391951}{1280}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{163}{4}$	$-\frac{2577}{16}$	$+\frac{1089}{5}$	$+\frac{35413}{384}$	$-\frac{709471}{1920}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{207}{4}$	$-\frac{2625}{16}$	$+\frac{30483}{160}$	$+\frac{47621}{384}$	$-\frac{262731}{640}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 3f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 3f$	$[\ ]^8$		$[\ ]^9$	$[\ ]^{10}$
		e^6	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{26371}{480}$	$-\frac{1710983}{5760}$	$+\frac{471527}{5120}$	$+\frac{604279}{4032}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{2611}{30}$	$-\frac{39893}{90}$	$+\frac{790053}{5120}$	$+\frac{532345}{2016}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{10531}{80}$	$-\frac{7363}{12}$	$+\frac{1258449}{5120}$	$+\frac{98715}{224}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{23029}{120}$	$-\frac{35614}{45}$	$+\frac{385095}{1024}$	$+\frac{44377}{63}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{65153}{240}$	$-\frac{2725199}{2880}$	$+\frac{2850411}{5120}$	$+\frac{2190757}{2016}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H

...*Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^0$		$[\ ]^1$			$[\ ]^2$		
		e^4	e^6	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{15}{16}$	$+\frac{9}{32}$	$-\frac{229}{48}$	$+\frac{2735}{768}$	$+\frac{2719}{30720}$	$+\frac{29}{4}$	$-\frac{101}{6}$	$+\frac{681}{64}$
	[cos − sin]	,,	,,	0	$+\frac{151}{3840}$	$+\frac{1423}{92160}$	0	0	$+\frac{37}{2880}$
	[cos]	$+\frac{15}{16}$	$+\frac{9}{32}$	H	$+\frac{6913}{3840}$	$+\frac{479}{9216}$	H	H	$+\frac{15341}{2880}$
	[sin]	0	0	H	$+\frac{1127}{640}$	$+\frac{3367}{92160}$	H	H	$+\frac{1913}{360}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{35}{16}$	$+\frac{7}{32}$	$-\frac{179}{24}$	$+\frac{2303}{384}$	$-\frac{1031}{3072}$	$+\frac{37}{4}$	$-\frac{83}{4}$	$+\frac{2639}{192}$
	[cos − sin]	,,	,,	0	$-\frac{17}{1920}$	$-\frac{1193}{15360}$	0	0	$+\frac{5}{576}$
	[cos]	$+\frac{35}{16}$	$+\frac{7}{32}$	H	$+\frac{5749}{1920}$	$-\frac{529}{2560}$	H	H	$+\frac{3961}{576}$
	[sin]	0	0	H	$+\frac{961}{320}$	$-\frac{1981}{15360}$	H	H	$+\frac{989}{144}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{35}{8}$	0	$-\frac{529}{48}$	$+\frac{6385}{768}$	$-\frac{20161}{30720}$	$+\frac{23}{2}$	$-\frac{139}{6}$	$+\frac{43}{3}$
	[cos − sin]	,,	,,	0	$-\frac{1159}{3840}$	$-\frac{25297}{92160}$	0	0	$-\frac{23}{720}$
	[cos]	$+\frac{35}{8}$	0	H	$+\frac{15383}{3840}$	$-\frac{4289}{9216}$	H	H	$+\frac{10297}{1440}$
	[sin]	0	0	H	$+\frac{2757}{640}$	$-\frac{17593}{92160}$	H	H	$+\frac{10343}{1440}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{63}{8}$	0	$-\frac{187}{12}$	$+\frac{1769}{192}$	$-\frac{311}{7680}$	$+14$	$-\frac{137}{6}$	$+\frac{171}{16}$
	[cos − sin]	,,	,,	0	$-\frac{1091}{960}$	$-\frac{13571}{23040}$	0	0	$-\frac{71}{720}$
	[cos]	$+\frac{63}{8}$	0	H	$+\frac{3877}{960}$	$-\frac{1813}{5760}$	H	H	$+\frac{953}{180}$
	[sin]	0	0	H	$+\frac{207}{40}$	$+\frac{6319}{23040}$	H	H	$+\frac{3883}{720}$

... *Table* $\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^3$				$[\ ]^4$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$-\frac{9}{2}$	$+\frac{399}{16}$	$-\frac{5415}{128}$	$+\frac{58251}{2048}$	$+1$	$-\frac{31}{2}$	$+\frac{239}{4}$	$-\frac{6719}{72}$
	[cos − sin]	0	0	0	$+\frac{411}{71680}$	0	0	0	0
	[cos]	H	H	H	$+\frac{509799}{35840}$	H	H	H	H
	[sin]	H	H	H	$+\frac{1019187}{71680}$	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	-5	$+\frac{209}{8}$	$-\frac{2783}{64}$	$+\frac{31065}{1024}$	$+1$	$-\frac{29}{2}$	$+\frac{215}{4}$	$-\frac{5993}{72}$
	[cos − sin]	0	0	0	$+\frac{209}{35840}$	0	0	0	0
	[cos]	H	H	H	$+\frac{271871}{17920}$	H	H	H	H
	[sin]	H	H	H	$+\frac{543533}{35840}$	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{11}{2}$	$+\frac{411}{16}$	$-\frac{5025}{128}$	$+\frac{53459}{2048}$	$+1$	-13	$+\frac{353}{8}$	$-\frac{1151}{18}$
	[cos − sin]	0	0	0	$-\frac{541}{71680}$	0	0	0	0
	[cos]	H	H	H	$+\frac{467631}{35840}$	H	H	H	H
	[sin]	H	H	H	$+\frac{935803}{71680}$	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	-6	$+\frac{93}{4}$	$-\frac{933}{32}$	$+\frac{7929}{512}$	$+1$	-11	$+\frac{253}{8}$	$-\frac{346}{9}$
	[cos − sin]	0	0	0	$-\frac{459}{17920}$	0	0	0	0
	[cos]	H	H	H	$+\frac{4329}{560}$	H	H	H	H
	[sin]	H	H	H	$+\frac{138987}{17920}$	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 4f$	$[\ ]^5$				$[\ ]^6$		
		e	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{7}{2}$	$-\frac{585}{16}$	$+\frac{47435}{384}$	$-\frac{3517045}{18432}$	$+\frac{33}{4}$	$-\frac{147}{2}$	$+\frac{15075}{64}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+3$	$-\frac{239}{8}$	$+\frac{18875}{192}$	$-\frac{155175}{1024}$	$+\frac{25}{4}$	$-\frac{215}{4}$	$+\frac{10831}{64}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{5}{2}$	$-\frac{365}{16}$	$+\frac{26845}{384}$	$-\frac{1884125}{18432}$	$+\frac{9}{2}$	-36	$+\frac{855}{8}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+2$	$-\frac{63}{4}$	$+\frac{3989}{96}$	$-\frac{239275}{4608}$	$+3$	-21	$+\frac{435}{8}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^7$			$[\ ]^8$	
		e^3	e^5	e^7	e^4	e^6
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{787}{48}$	$-\frac{103937}{768}$	$+\frac{13103351}{30720}$	$+\frac{1423}{48}$	$-\frac{37724}{160}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{269}{24}$	$-\frac{34601}{384}$	$+\frac{4312189}{15360}$	$+\frac{297}{16}$	$-\frac{13897}{96}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{343}{48}$	$-\frac{41503}{768}$	$+\frac{4923079}{30720}$	$+\frac{32}{3}$	$-\frac{1184}{15}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{49}{12}$	$-\frac{5243}{192}$	$+\frac{548261}{7680}$	$+\frac{16}{3}$	$-\frac{176}{5}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^9$ e^5	$[\ ]^9$ e^7	$[\ ]^{10}$ e^6	$[\ ]^{11}$ e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{64653}{1280}$	$-\frac{4051137}{10240}$	$+\frac{47603}{576}$	$+\frac{84728011}{645120}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{18749}{640}$	$-\frac{1157699}{5120}$	$+\frac{25799}{576}$	$+\frac{1449343}{21504}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{19683}{1280}$	$-\frac{1161297}{10240}$	$+\frac{3125}{144}$	$+\frac{19487171}{645120}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{2187}{320}$	$-\frac{115911}{2560}$	$+\frac{625}{72}$	$+\frac{1771561}{161280}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H

Table $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^0$		$[\ ]^1$			$[\ ]^2$		
		e^4	e^6	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{1}{16}$	$+\frac{1}{16}$	$-\frac{73}{48}$	$+\frac{421}{768}$	$+\frac{4231}{30720}$	$+4$	$-\frac{33}{4}$	$+\frac{331}{96}$
	[cos − sin]	,,	,,	0	$-\frac{19}{3840}$	$-\frac{1}{10240}$	0	0	$-\frac{7}{1440}$
	[cos]	$+\frac{1}{16}$	$+\frac{1}{16}$	H	$+\frac{1043}{3840}$	$+\frac{1057}{15360}$	H	H	$+\frac{2479}{1440}$
	[sin]	0	0	H	$+\frac{177}{640}$	$+\frac{2117}{30720}$	H	H	$+\frac{1243}{720}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	0	0	$-\frac{17}{24}$	$+\frac{25}{384}$	$+\frac{47}{15360}$	$+\frac{11}{4}$	$-\frac{29}{6}$	$+\frac{21}{16}$
	[cos − sin]	,,	,,	0	$-\frac{7}{1920}$	$-\frac{361}{46080}$	0	0	$-\frac{1}{1440}$
	[cos]	0	0	H	$+\frac{59}{1920}$	$-\frac{11}{4608}$	H	H	$+\frac{1889}{2880}$
	[sin]	0	0	H	$+\frac{11}{320}$	$+\frac{251}{46080}$	H	H	$+\frac{1891}{2880}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	0	0	$-\frac{13}{48}$	$-\frac{29}{768}$	$-\frac{197}{6144}$	$+\frac{7}{4}$	$-\frac{29}{12}$	$+\frac{1}{3}$
	[cos − sin]	,,	,,	0	$+\frac{11}{3840}$	$+\frac{407}{92160}$	0	0	$+\frac{1}{288}$
	[cos]	0	0	H	$-\frac{67}{3840}$	$-\frac{637}{46080}$	H	H	$+\frac{97}{576}$
	[sin]	0	0	H	$-\frac{13}{640}$	$-\frac{1681}{92160}$	H	H	$+\frac{95}{576}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	0	0	$-\frac{1}{12}$	$-\frac{5}{192}$	$-\frac{149}{7680}$	$+1$	-1	$+\frac{1}{24}$
	[cos − sin]	,,	,,	0	$-\frac{1}{960}$	$-\frac{11}{7680}$	0	0	$-\frac{1}{360}$
	[cos]	0	0	H	$-\frac{13}{960}$	$-\frac{1}{96}$	H	H	$+\frac{7}{360}$
	[sin]	0	0	H	$-\frac{1}{80}$	$-\frac{23}{2560}$	H	H	$+\frac{1}{45}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	0	0	$-\frac{1}{48}$	$-\frac{7}{768}$	$-\frac{209}{30720}$	$+\frac{1}{2}$	$-\frac{1}{3}$	0
	[cos − sin]	,,	,,	0	$+\frac{1}{3840}$	$+\frac{31}{92160}$	0	0	$+\frac{1}{720}$
	[cos]	0	0	H	$-\frac{17}{3840}$	$-\frac{149}{46080}$	H	H	$+\frac{1}{1440}$
	[sin]	0	0	H	$-\frac{3}{640}$	$-\frac{329}{92160}$	H	H	$-\frac{1}{1440}$

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^3$				$[\ ]^4$			
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$-\frac{7}{2}$	$+\frac{307}{16}$	$-\frac{3717}{128}$	$+\frac{30571}{2048}$	$+1$	-16	$+\frac{239}{4}$	$-\frac{3019}{36}$
	[cos − sin]	0	0	0	$-\frac{261}{71680}$	0	0	0	0
	[cos]	H	H	H	$+\frac{267431}{35840}$	H	H	H	H
	[sin]	H	H	H	$+\frac{535123}{71680}$	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	-3	$+\frac{123}{8}$	$-\frac{1305}{64}$	$+\frac{8547}{1024}$	$+1$	$-\frac{31}{2}$	$+\frac{433}{8}$	$-\frac{9793}{144}$
	[cos − sin]	0	0	0	$+\frac{27}{35840}$	0	0	0	0
	[cos]	H	H	H	$+\frac{74793}{17920}$	H	H	H	H
	[sin]	H	H	H	$+\frac{149559}{35840}$	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$-\frac{5}{2}$	$+\frac{183}{16}$	$-\frac{1611}{128}$	$+\frac{8035}{2048}$	$+1$	$-\frac{29}{2}$	$+\frac{365}{8}$	$-\frac{7111}{144}$
	[cos − sin]	0	0	0	$+\frac{243}{71680}$	0	0	0	0
	[cos]	H	H	H	$+\frac{70367}{35840}$	H	H	H	H
	[sin]	H	H	H	$+\frac{140491}{71680}$	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	-2	$+\frac{31}{4}$	$-\frac{215}{32}$	$+\frac{819}{512}$	$+1$	-13	$+\frac{283}{8}$	$-\frac{1147}{36}$
	[cos − sin]	0	0	0	$-\frac{81}{17920}$	0	0	0	0
	[cos]	H	H	H	$+\frac{7146}{8960}$	H	H	H	H
	[sin]	H	H	H	$+\frac{14373}{17920}$	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$-\frac{3}{2}$	$+\frac{75}{16}$	$-\frac{393}{128}$	$+\frac{1251}{2048}$	$+1$	-11	$+\frac{199}{8}$	$-\frac{655}{36}$
	[cos − sin]	0	0	0	$+\frac{243}{71680}$	0	0	0	0
	[cos]	H	H	H	$+\frac{11007}{35840}$	H	H	H	H
	[sin]	H	H	H	$+\frac{21771}{71680}$	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	[ ]^5				[ ]^6		
		e	e^3	e^5	e^7	e^2	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{9}{2}$	$-\frac{757}{16}$	$+\frac{58801}{384}$	$-\frac{1316215}{6144}$	$+13$	$-\frac{459}{4}$	$+\frac{11205}{32}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+5$	$-\frac{405}{8}$	$+\frac{29485}{192}$	$-\frac{1800725}{9216}$	$+\frac{63}{4}$	$-\frac{267}{2}$	$+\frac{12255}{32}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{11}{2}$	$-\frac{833}{16}$	$+\frac{55135}{384}$	$-\frac{2975165}{18432}$	$+\frac{75}{4}$	$-\frac{595}{4}$	$+\frac{12513}{32}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+6$	$-\frac{205}{4}$	$+\frac{11935}{96}$	$-\frac{184955}{1536}$	$+22$	$-\frac{317}{2}$	$+\frac{5945}{16}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{13}{2}$	$-\frac{765}{16}$	$+\frac{37925}{384}$	$-\frac{1493005}{18432}$	$+\frac{51}{2}$	$-\frac{321}{2}$	$+\frac{2613}{8}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^7$			$[\ ]^8$	
		e^2	e^5	e^7	e^4	e^6
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{1471}{48}$	$-\frac{190507}{768}$	$+\frac{22796399}{30720}$	$+\frac{1027}{16}$	$-\frac{119323}{240}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{959}{24}$	$-\frac{119119}{384}$	$+\frac{13442527}{15360}$	$+\frac{2141}{24}$	$-\frac{53003}{80}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{2443}{48}$	$-\frac{284557}{768}$	$+\frac{29629663}{30720}$	$+\frac{2893}{24}$	$-\frac{40387}{48}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{763}{12}$	$-\frac{81193}{192}$	$+\frac{7623959}{7680}$	$+\frac{1273}{8}$	$-\frac{61219}{60}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{3751}{48}$	$-\frac{352919}{768}$	$+\frac{29213303}{30720}$	$+\frac{4943}{24}$	$-\frac{23591}{20}$
	[cos − sin]	0	0	0	0	0
	[cos]	H	H	H	H	H
	[sin]	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 4f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 4f$	$[\ ]^9$ e^5	$[\ ]^9$ e^7	$[\ ]^{10}$ e^6	$[\ ]^{11}$ e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{159263}{1280}$	$-\frac{9671209}{10240}$	$+\frac{65773}{288}$	$+\frac{86564393}{215040}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{117099}{640}$	$-\frac{6819921}{5120}$	$+\frac{25435}{72}$	$+\frac{210276803}{322560}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{333513}{1280}$	$-\frac{18298713}{10240}$	$+\frac{75947}{144}$	$+\frac{131087143}{129024}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{115617}{320}$	$-\frac{5856453}{2560}$	$+\frac{110099}{144}$	$+\frac{27485931}{17920}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{627003}{1280}$	$-\frac{28615329}{10240}$	$+\frac{155681}{144}$	$+\frac{1454039059}{645120}$
	[cos − sin]	0	0	0	0
	[cos]	H	H	H	H
	[sin]	H	H	H	H

Table $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{4}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 5f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 5f$	$[\ ]^0$		$[\ ]^1$		$[\ ]^2$		
		e^6	e^7	e^4	e^6	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$-\frac{21}{32}$	$-\frac{21}{64}$	$+\frac{1961}{384}$	$-\frac{4109}{1280}$	$-\frac{277}{24}$	$+\frac{575}{24}$	$-\frac{24781}{1920}$
	[cos − sin]	,,	,,	0	$-\frac{1091}{46080}$	0	0	$-\frac{209}{40320}$
	[cos]	$-\frac{21}{32}$	$-\frac{21}{64}$	H	$-\frac{29803}{18432}$	H	H	$-\frac{52061}{8064}$
	[sin]	0	0	H	$-\frac{146833}{92160}$	H	H	$-\frac{2032}{315}$
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$-\frac{7}{4}$	$-\frac{7}{16}$	$+\frac{1131}{128}$	$-\frac{25429}{3840}$	$-\frac{97}{6}$	$+\frac{817}{24}$	$-\frac{2461}{120}$
	[cos − sin]	,,	,,	0	$-\frac{887}{46080}$	0	0	$-\frac{1}{105}$
	[cos]	$-\frac{7}{4}$	$-\frac{7}{16}$	H	$-\frac{61207}{18432}$	H	H	$-\frac{1149}{112}$
	[sin]	0	0	H	$-\frac{304261}{92160}$	H	H	$-\frac{17219}{1680}$
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$-\frac{63}{16}$	$-\frac{21}{64}$	$+\frac{5509}{384}$	$-\frac{21763}{1920}$	$-\frac{263}{12}$	$+\frac{2135}{48}$	$-\frac{2641}{96}$
	[cos − sin]	,,	,,	0	$+\frac{6067}{46080}$	0	0	$+\frac{13}{2016}$
	[cos]	$-\frac{63}{16}$	$-\frac{21}{64}$	H	$-\frac{103249}{18432}$	H	H	$-\frac{6931}{504}$
	[sin]	0	0	H	$-\frac{528379}{92160}$	H	H	$-\frac{27737}{2016}$
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$-\frac{63}{8}$	0	$+\frac{8501}{384}$	$-\frac{10543}{640}$	$-\frac{347}{12}$	$+\frac{2557}{48}$	$-\frac{3673}{120}$
	[cos − sin]	,,	,,	0	$+\frac{32251}{46080}$	0	0	$+\frac{281}{5040}$
	[cos]	$-\frac{63}{8}$	0	H	$-\frac{145369}{18432}$	H	H	$-\frac{30797}{2016}$
	[sin]	0	0	H	$-\frac{791347}{92160}$	H	H	$-\frac{154547}{10080}$

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right)$ $\begin{matrix}\cos\\ \sin\end{matrix}$ $5f$ ...

	$\begin{matrix}\cos\\ \sin\end{matrix}$ $5f$	$[\ ]^3$			$[\ ]^4$			
		e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{93}{8}$	$-\frac{417}{8}$	$+\frac{77733}{1024}$	$-\frac{11}{2}$	$+\frac{207}{4}$	$-\frac{3715}{24}$	$+\frac{58583}{288}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{113}{8}$	-62	$+\frac{94081}{1024}$	-6	$+\frac{109}{2}$	$-\frac{484}{3}$	$+\frac{6917}{32}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{135}{8}$	$-\frac{1119}{16}$	$+\frac{103311}{1024}$	$-\frac{13}{2}$	$+\frac{443}{8}$	$-\frac{1885}{12}$	$+\frac{238195}{1152}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{159}{8}$	$-\frac{1191}{16}$	$+\frac{101103}{1024}$	-7	$+54$	$-\frac{3379}{24}$	$+\frac{1558}{9}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 5f \ldots$

$\begin{matrix}\cos\\ \sin\end{matrix}\, 5f$		$[\ ]^5$				$[\ ]^6$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+1$	$-\frac{49}{2}$	$+\frac{9575}{64}$	$-\frac{441125}{1152}$	$+\frac{9}{2}$	$-\frac{561}{8}$	$+\frac{11415}{32}$	$-\frac{109167}{128}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+1$	$-\frac{47}{2}$	$+\frac{8975}{64}$	$-\frac{412175}{1152}$	$+4$	$-\frac{121}{2}$	$+\frac{1213}{4}$	$-\frac{11619}{16}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+1$	-22	$+\frac{7999}{64}$	$-\frac{177925}{576}$	$+\frac{7}{2}$	$-\frac{201}{4}$	$+\frac{7755}{32}$	$-\frac{9065}{16}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+1$	-20	$+\frac{6695}{64}$	$-\frac{138775}{576}$	$+3$	$-\frac{159}{4}$	$+\frac{2859}{16}$	$-\frac{12591}{32}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H

... *Table* $\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4 \begin{matrix}\cos\\ \sin\end{matrix} 5f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix} 5f$	$[\ ]^7$ e^2	e^4	e^6	$[\ ]^8$ e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{103}{8}$	$-\frac{3941}{24}$	$+\frac{780003}{1024}$		$+\frac{359}{12}$	$-\frac{32885}{96}$	$+\frac{1453367}{960}$
	[cos − sin]	0	0	0		0	0	0
	[cos]	H	H	H		H	H	H
	[sin]	H	H	H		H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{83}{8}$	$-\frac{259}{2}$	$+\frac{1829093}{3072}$		$+\frac{133}{6}$	$-\frac{2999}{12}$	$+\frac{263873}{240}$
	[cos − sin]	0	0	0		0	0	0
	[cos]	H	H	H		H	H	H
	[sin]	H	H	H		H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{65}{8}$	$-\frac{4667}{48}$	$+\frac{1331743}{3072}$		$+\frac{379}{24}$	$-\frac{8255}{48}$	$+\frac{141719}{192}$
	[cos − sin]	0	0	0		0	0	0
	[cos]	H	H	H		H	H	H
	[sin]	H	H	H		H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{49}{8}$	$-\frac{3283}{48}$	$+\frac{293853}{1024}$		$+\frac{32}{3}$	$-\frac{328}{3}$	$+\frac{6674}{15}$
	[cos − sin]	0	0	0		0	0	0
	[cos]	H	H	H		H	H	H
	[sin]	H	H	H		H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^1 \ldots \left(\frac{r}{a}\right)^4\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 5f.$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 5f$	$[\ ]^9$		$[\ ]^{10}$		$[\ ]^{11}$	$[\ ]^{12}$
		e^4	e^6	e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^1$	[cos + sin]	$+\frac{7887}{128}$	$-\frac{848529}{1280}$	$+\frac{11263}{96}$	$-\frac{1402147}{1152}$	$+\frac{9742939}{46080}$	$+\frac{409821}{1120}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^2$	[cos + sin]	$+\frac{5431}{128}$	$-\frac{578003}{1280}$	$+\frac{1817}{24}$	$-\frac{37429}{48}$	$+\frac{5931103}{46080}$	$+\frac{236641}{1120}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^3$	[cos + sin]	$+\frac{3563}{128}$	$-\frac{184221}{640}$	$+\frac{4433}{96}$	$-\frac{66859}{144}$	$+\frac{3389017}{46080}$	$+\frac{101971}{896}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^4$	[cos + sin]	$+\frac{2187}{128}$	$-\frac{107163}{640}$	$+\frac{625}{24}$	$-\frac{71875}{288}$	$+\frac{1771561}{46080}$	$+\frac{1944}{35}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

Table $\left(\left(\frac{r}{a}\right)^{-1} \dots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 5f \dots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 5f$	$[\ ]^0$		$[\ ]^1$		$[\ ]^2$		
		e^5	e^7	e^4	e^6	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$-\frac{1}{32}$	$-\frac{5}{128}$	$+\frac{167}{128}$	$-\frac{247}{960}$	$-\frac{31}{6}$	$+\frac{859}{96}$	$-\frac{18683}{6720}$
	[cos − sin]	,,	,,	0	$+\frac{151}{46080}$	0	0	$+\frac{13}{6720}$
	[cos]	$-\frac{1}{32}$	$-\frac{5}{128}$	H	$-\frac{2341}{18432}$	H	H	$-\frac{1867}{1344}$
	[sin]	0	0	H	$-\frac{12007}{92160}$	H	H	$-\frac{779}{560}$
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	0	0	$+\frac{209}{384}$	$+\frac{43}{1280}$	$-\frac{19}{6}$	$+\frac{55}{12}$	$-\frac{77}{96}$
	[cos − sin]	,,	,,	0	$+\frac{37}{46080}$	0	0	$-\frac{1}{2016}$
	[cos]	0	0	H	$+\frac{317}{18432}$	H	H	$-\frac{809}{2016}$
	[sin]	0	0	H	$+\frac{1511}{92160}$	H	H	$-\frac{101}{252}$
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	0	0	$+\frac{73}{384}$	$+\frac{211}{3840}$	$-\frac{43}{24}$	$+\frac{193}{96}$	$-\frac{103}{960}$
	[cos − sin]	,,	,,	0	$-\frac{47}{46080}$	0	0	$-\frac{17}{20160}$
	[cos]	0	0	H	$+\frac{497}{18432}$	H	H	$-\frac{109}{2016}$
	[sin]	0	0	H	$+\frac{2579}{92160}$	H	H	$-\frac{1073}{20160}$
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	0	0	$+\frac{7}{128}$	$+\frac{49}{1920}$	$-\frac{11}{12}$	$+\frac{35}{48}$	$+\frac{11}{480}$
	[cos − sin]	,,	,,	0	$+\frac{19}{46080}$	0	0	$+\frac{1}{1120}$
	[cos]	0	0	H	$+\frac{239}{18432}$	H	H	$+\frac{1}{84}$
	[sin]	0	0	H	$+\frac{1157}{92160}$	H	H	$+\frac{37}{3360}$
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	0	0	$+\frac{5}{384}$	$+\frac{1}{128}$	$-\frac{5}{12}$	$+\frac{5}{24}$	$+\frac{1}{96}$
	[cos − sin]	,,	,,	0	$-\frac{1}{9216}$	0	0	$-\frac{1}{2016}$
	[cos]	0	0	H	$+\frac{71}{18432}$	H	H	$+\frac{5}{1008}$
	[sin]	0	0	H	$+\frac{73}{18432}$	H	H	$+\frac{11}{2016}$

...*Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 5f$...

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 5f$	$[\ ]^3$			$[\ ]^4$			
		e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{59}{8}$	$-\frac{501}{16}$	$+\frac{39963}{1024}$	$-\frac{9}{2}$	$+\frac{337}{8}$	$-\frac{5683}{48}$	$+\frac{26119}{192}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{45}{8}$	$-\frac{177}{8}$	$+\frac{24501}{1024}$	-4	$+36$	$-\frac{565}{6}$	$+\frac{3475}{36}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{33}{8}$	$-\frac{29}{2}$	$+\frac{13281}{1024}$	$-\frac{7}{2}$	$+\frac{59}{2}$	$-\frac{3349}{48}$	$+\frac{17803}{288}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{23}{8}$	$-\frac{139}{16}$	$+\frac{6263}{1024}$	-3	$+23$	$-\frac{1145}{24}$	$+\frac{1703}{48}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{15}{8}$	$-\frac{75}{16}$	$+\frac{2535}{1024}$	$-\frac{5}{2}$	$+\frac{135}{8}$	$-\frac{715}{24}$	$+\frac{20945}{1152}$
	[cos − sin]	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\, 5f \ldots$

	$\begin{matrix}\cos\\ \sin\end{matrix}\, 5f$	$[\ ]^5$				$[\ ]^6$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+1$	-25	$+\frac{9575}{64}$	$-\frac{103325}{288}$	$+\frac{11}{2}$	-86	$+\frac{13593}{32}$	$-\frac{60877}{64}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+1$	$-\frac{49}{2}$	$+\frac{8999}{64}$	$-\frac{363365}{1152}$	$+6$	$-\frac{183}{2}$	$+\frac{3465}{8}$	$-\frac{29175}{32}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+1$	$-\frac{47}{2}$	$+\frac{8095}{64}$	$-\frac{298655}{1152}$	$+\frac{13}{2}$	$-\frac{759}{8}$	$+\frac{13509}{32}$	$-\frac{26213}{32}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+1$	-22	$+\frac{6935}{64}$	$-\frac{114085}{576}$	$+7$	$-\frac{383}{4}$	$+\frac{6275}{16}$	$-\frac{2751}{4}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+1$	-20	$+\frac{5615}{64}$	$-\frac{80455}{576}$	$+\frac{15}{2}$	$-\frac{375}{4}$	$+\frac{11055}{32}$	$-\frac{8595}{16}$
	[cos − sin]	0	0	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H	H	H

$$\ldots \textit{Table} \left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 5f \ldots$$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 5f$	$[\ ]^7$ e^2	e^4	e^6	$[\ ]^8$ e^3	e^5	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{149}{8}$	$-\frac{3771}{16}$	$+\frac{3242339}{3072}$	$+\frac{1201}{24}$	$-\frac{54151}{96}$	$+\frac{4603927}{1920}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{175}{8}$	$-\frac{6461}{24}$	$+\frac{1183371}{1024}$	$+\frac{188}{3}$	$-\frac{4115}{6}$	$+\frac{16781}{6}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{203}{8}$	$-\frac{896}{3}$	$+\frac{3712093}{3072}$	$+\frac{463}{6}$	$-\frac{38771}{48}$	$+\frac{46693}{15}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{233}{8}$	$-\frac{5149}{16}$	$+\frac{3701999}{3072}$	$+\frac{281}{3}$	$-\frac{22135}{24}$	$+\frac{99181}{30}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{265}{8}$	$-\frac{16135}{48}$	$+\frac{1169805}{1024}$	$+\frac{2695}{24}$	$-\frac{24485}{24}$	$+\frac{160513}{48}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

... *Table* $\left(\left(\frac{r}{a}\right)^{-1} \ldots \left(\frac{r}{a}\right)^{-8}\right) \begin{matrix}\cos\\ \sin\end{matrix}\ 5f.$

	$\begin{matrix}\cos\\ \sin\end{matrix}\ 5f$	$[\ ]^9$		$[\ ]^{10}$		$[\ ]^{11}$	$[\ ]^{12}$
		e^4	e^6	e^5	e^7	e^6	e^7
$\left(\frac{r}{a}\right)^{-1}$	[cos + sin]	$+\frac{14987}{128}$	$-\frac{394607}{320}$	$+\frac{11989}{48}$	$-\frac{161685}{64}$	$+\frac{22980061}{46080}$	$+\frac{2122251}{2240}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-2}$	[cos + sin]	$+\frac{19863}{128}$	$-\frac{2029257}{1280}$	$+\frac{8359}{24}$	$-\frac{983731}{288}$	$+\frac{33554587}{46080}$	$+\frac{40401}{28}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-3}$	[cos + sin]	$+\frac{25791}{128}$	$-\frac{2521323}{1280}$	$+\frac{1423}{3}$	$-\frac{1282973}{288}$	$+\frac{47722543}{46080}$	$+\frac{2391591}{1120}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-4}$	[cos + sin]	$+\frac{32907}{128}$	$-\frac{1516377}{640}$	$+\frac{7597}{12}$	$-\frac{67409}{12}$	$+\frac{66363649}{46080}$	$+\frac{1726833}{560}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H
$\left(\frac{r}{a}\right)^{-5}$	[cos + sin]	$+\frac{41355}{128}$	$-\frac{353187}{128}$	$+\frac{79711}{96}$	$-\frac{987035}{144}$	$+\frac{18100973}{9216}$	$+\frac{3906009}{896}$
	[cos − sin]	0	0	0	0	0	0
	[cos]	H	H	H	H	H	H
	[sin]	H	H	H	H	H	H

The foregoing tables are read as follows: for instance, in the Table $(x^0, x^1, x^2, x^3)\,{}^{\cos}_{\sin} f$, p. 376, *et seq.*, the third and fourth lines of the x^0-compartment show that

$$\begin{aligned}
\cos f = \quad & -e \\
& +(\tfrac{1}{2} \quad -\tfrac{9}{16}e^2 \quad +\tfrac{25}{384}e^4 \quad -\tfrac{49}{18432}e^6 \quad)\,2\cos g \\
& +(\ \tfrac{1}{2}e \quad -\tfrac{2}{3}e^3 \quad +\tfrac{3}{16}e^5 \quad -\tfrac{1}{45}e^7)\,2\cos 2g \\
& + \ \&c. \\
\sin f = \quad & (\tfrac{1}{2} \quad -\tfrac{7}{16}e^2 \quad +\tfrac{17}{384}e^4 \quad -\tfrac{271}{18432}e^6 \quad)\,2\sin g \\
& +(\ \tfrac{1}{2}e \quad -\tfrac{7}{12}e^3 \quad +\tfrac{1}{6}e^5 \quad -\tfrac{19}{720}e^7)\,2\sin 2g \\
& + \ \&c.
\end{aligned}$$

and the first and second lines give the sum and difference respectively of the corresponding coefficients of the cosine and sine series.

Addition, 28*th Dec.* 1860.—The tables have been verified by me, on the proof-sheets; the x-tables, the cosine lines of the $\left(\frac{r}{a}\right)$-tables, and a portion of the sine lines of the same tables, in the manner explained at the commencement of the memoir; but for the remainder of the sine lines it was found easier to employ the following mode of verification; viz. the equation $\frac{r}{a} = 1 + x$, gives

$$\left\{\left(\frac{r}{a}\right)^4 - 4\left(\frac{r}{a}\right)^3 + 6\left(\frac{r}{a}\right)^2 - 4\left(\frac{r}{a}\right)\right\}\sin jf = (-1 + x^4)\sin jf;$$

and as regards the terms up to e^7, the limit of the tables,

$$\left\{\left(\frac{r}{a}\right)^{-5} - 5\left(\frac{r}{a}\right)^{-4} + 10\left(\frac{r}{a}\right)^{-3} - 10\left(\frac{r}{a}\right)^{-2} + 5\left(\frac{r}{a}\right)^{-1}\right\}\sin jf = (1 - x^5 + 5x^6 - 15x^7)\sin jf,$$

which equations afford the verification referred to. In going over the earlier sheets I omitted to see that the fractions were in their least terms, and it may happen that, in some instances, they are not so. [A few reductions have been made, and I believe that the fractions are now all in their least terms.]

The expression for the true anomaly f itself has been repeatedly calculated to a much greater extent, in particular by Schubert (*Astronomie Théorique, Pet.* 1822), as far as e^{20}. The easiest way of obtaining it seems to be by means of the equation

$$\frac{df}{dg} = \left(\frac{r}{a}\right)^{-2}\sqrt{1 - e^2};$$

and, for the convenience of reference, I here give it as far as e^7, viz.,

$$f = g + \Sigma\,[\sin]^i \sin ig,$$

where, as in the other sine series, i is to be taken as well negatively as positively, and $[\sin]^{-i} = -[\sin]^i$. Or, what is the same thing,

$$f = g + \Sigma\,[\sin]^i\, 2 \sin ig,$$

where i has only positive values. And the coefficients are

	e	e^2	e^3	e^4	e^5	e^6	e^7
$[\sin]^1 =$	1		$-\frac{1}{8}$		$+\frac{5}{192}$		$+\frac{107}{9216}$
$[\sin]^2 =$		$+\frac{5}{8}$		$-\frac{11}{48}$		$+\frac{17}{384}$	
$[\sin]^3 =$			$+\frac{13}{24}$		$-\frac{43}{128}$		$+\frac{95}{1024}$
$[\sin]^4 =$				$+\frac{103}{192}$		$-\frac{451}{960}$	
$[\sin]^5 =$					$+\frac{1097}{1920}$		$-\frac{5957}{9216}$
$[\sin]^6 =$						$+\frac{1223}{1920}$	
$[\sin]^7 =$							$+\frac{47273}{64512}$

The expression for $\frac{r}{a}$, as far as e^{13}, is given in Schubert's work, above referred to; and that of $\log\left(\frac{r}{a}\right)$, as far as e^9, was calculated by Oriani, see the Introduction to Delambre's *Tables du Soleil*, Paris, 1806.

217.

A MEMOIR ON THE PROBLEM OF THE ROTATION OF A SOLID BODY.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXIX. (1861), pp. 307—342. Read May 11, 1860.]

THE present memoir was written for the sake of the further elaboration of the analytical theory of the Rotation of a Solid Body, upon principles similar to those of my "Memoir on the Problem of Disturbed Elliptic Motion," *Mem. R. Ast. Soc.* vol. XXVII. pp. 1—29 (1858) [212]; the like elements are adopted, and the course of the investigation corresponds precisely to that of the memoir just referred to. The formulæ for the variations of the elements in the two problems (the motion being in each case referred to a fixed plane of reference and origin of angles therein) are found to be (as it is known they should be) identical in form; an investigation, in the present memoir, of the transformation of the system to the case of a variable plane of reference and departure-point as an origin of angles in such plane, would have been a mere repetition of that contained in the former memoir, and it was therefore unnecessary to give it. A point in the present memoir to which attention may be called is the definition of the angle g (varying uniformly with the time, but used as an element) which corresponds to the mean anomaly in elliptic motion. Besides the ultimate system of formulæ for the variations of the elements in terms of the differential coefficients of the Disturbing Function with respect to the elements, it appears to me that the intermediate formulæ for the variations in terms of the differential coefficients with respect to the coordinates (which are in the ordinary investigation altogether passed over) are not without interest, and that it is possible that they might be employed with advantage in the integration of the equations of motion for the purposes of physical astronomy.

I.

In the theory of elliptic motion, where the elements are

a, the mean distance,

e, the eccentricity,

g, the mean anomaly,

ϖ, the departure of pericentre,

θ, the longitude of node,

σ, the departure of node,

ϕ, the inclination,

and if, besides, we have

n, the mean motion ($n^2a^3 =$ sum of the masses),

and Ω denote the disturbing function taken with Lagrange's sign ($\Omega = -R$, if R be the disturbing function of the *Mécanique Céleste*), then the formulæ for the variations of the elements are

$$da = \frac{2}{na}\frac{d\Omega}{dg}dt,$$

$$de = \frac{1-e^2}{na^2e}\frac{d\Omega}{dg}dt - \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{d\varpi}dt,$$

$$dg = -\frac{2}{na}\frac{d\Omega}{da}dt - \frac{1-e^2}{na^2e}\frac{d\Omega}{de}dt,$$

$$d\varpi = \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de}dt,$$

$$d\phi = -\frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\theta}dt,$$

$$d\sigma = \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

where $\Omega = \Omega(a, e, g, \varpi, \phi, \sigma, \theta)$.

And if in these equations we write

$$h = -\frac{1}{a}(\text{sum of the masses}) = -n^2a^2,$$

$$k = na^2\sqrt{1-e^2};$$

then attending to the equation $dn = -\frac{3}{2}\frac{n}{a}da$, we have

$$dh = n^2 a\,da,$$

$$dk = \tfrac{1}{2}na\sqrt{1-e^2}\,da - \frac{na^2 e}{\sqrt{1-e^2}},$$

and thence

$$\frac{d\Omega}{da} = n^2 a\frac{d\Omega}{dh} + \tfrac{1}{2}na\sqrt{1-e^2}\frac{d\Omega}{dk},$$

$$\frac{d\Omega}{de} = -\frac{n^2 a}{\sqrt{1-e^2}}\frac{d\Omega}{dk};$$

and the formulæ are very easily transformed into

$$dh = 2n\frac{d\Omega}{dg}dt,$$

$$dg = -2n\frac{d\Omega}{dh}dt,$$

$$dk = \frac{d\Omega}{d\varpi}dt,$$

$$d\varpi = \frac{d\Omega}{dk}dt,$$

$$d\phi = -\frac{\cot\phi}{k}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\theta}dt,$$

$$d\sigma = \frac{\cot\phi}{k}\frac{d\Omega}{d\phi}dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\phi}dt,$$

where $\Omega = \Omega\,(h, g, k, \varpi, \phi, \sigma, \theta)$.

This is the system of formulæ which will be obtained in the sequel for the variation of the elements in the problem of rotation, the new meanings of the symbols being explained *post*, Art. IV.

And if in either of the two problems, instead of the angles ϕ, σ, θ, which refer to a fixed plane of reference and origin of angles therein, we have the angles Φ, Σ, Θ, referring to a variable plane of reference and departure-point as an origin of angles in such plane, the position of these in respect to the fixed plane of reference and origin of angles therein being determined by

θ', the longitude of node,

σ', the departure of node,

ϕ', the inclination,

and if S' denote $\Theta - \sigma'$, then the system is

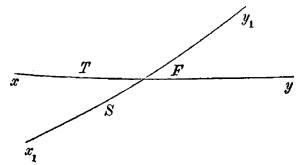
$$
\begin{aligned}
dh &= \quad 2n\frac{d\Omega}{dh}\,dt,\\
dg &= -\quad 2n\frac{d\Omega}{dh}\,dt,\\
dk &= \quad \frac{d\Omega}{d\varpi}\,dt,\\
d\varpi &= -\quad \frac{d\Omega}{dk}\,dt,\\
d\Phi &= -\frac{\cot\Phi}{k}\frac{d\Omega}{d\Sigma}\,dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\Theta}\,dt - (\cos S'd\phi' + \sin S'\sin\phi' d\theta'),\\
d\Sigma &= \frac{\cot\Phi}{k}\frac{d\Omega}{d\Phi}\,dt + \operatorname{cosec}\Phi\,(\sin S'd\phi' - \cos S'\sin\phi' d\theta'),\\
d\Theta &= \frac{\operatorname{cosec}\Phi}{k}\frac{d\Omega}{d\Phi}\,dt + \cot\Phi\,(\sin S'd\phi' - \cos S'\sin\phi' d\theta'),
\end{aligned}
$$

where $\Omega = \Omega\,(h,\ g,\ k,\ \varpi,\ \Phi,\ \Sigma,\ \Theta,\ \phi',\ \sigma',\ \theta')$, or what is the same thing, $\Omega = \Omega\,(h,\ g,\ k,\ \varpi,\ \Phi,\ \Sigma,\ \Theta)$. As already remarked, it is not necessary to repeat in the present memoir the transformation to this set of formulæ.

II.

Considering, now, the problem of rotation, let the axes xyz denote axes fixed in space, and the axes $x_{,}y_{,}z_{,}$ denote the principal axes of the body; and if, to fix the ideas, xy is called the ecliptic and $x_{,}y_{,}$ the equator (the ecliptic xy being considered as a fixed circle of the sphere and the origin of longitudes x as a fixed point therein), then we may write

T, the longitude of node,
S, the departure of node,
F, the inclination.



And if, as usual, p, q, r are the angular velocities round the principal axes, or axes of $x_{,}$, $y_{,}$, and $z_{,}$, from $y_{,}$ to $z_{,}$, $z_{,}$ to $x_{,}$, and $x_{,}$ to $y_{,}$ respectively, so that

$$
\begin{aligned}
pdt &= -\sin S\sin F\,dT + \cos S\,dF,\\
qdt &= \cos S\sin F\,dT + \sin S\,dF,\\
rdt &= \cos F\,dT - dS;
\end{aligned}
$$

and if A, B, C are the moments of inertia, then the *Vis Viva* function or sum of the elements of mass, each into the half square of its velocity, is

$$\mathbf{T} = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2);$$

and if Ω be the disturbing function (taken with Lagrange's sign), then the equations of motion

$$\frac{d}{dt}\frac{d\mathbf{T}}{d\dot{T}} - \frac{d\mathbf{T}}{dT} = \frac{d\Omega}{d\bar{T}}, \text{ \&c.}$$

give as usual

$$-\left[A\frac{dp}{dt}+(C-B)qr\right]\sin S\sin F+\left[B\frac{dq}{dt}+(A-C)rp\right]\cos S\sin F+\left[C\frac{dr}{dt}+(B-A)pq\right]\cos F=\frac{d\Omega}{d\bar{T}},$$

$$\left[A\frac{dp}{dt}+(C-B)qr\right]\cos S \quad +\left[B\frac{dq}{dt}+(A-C)rp\right]\sin S \qquad =\frac{d\Omega}{d\bar{F}},$$

$$-\left[C\frac{dr}{dt}+(B-A)pq\right] \qquad =\frac{d\Omega}{d\bar{S}},$$

or, as these equations may be written,

$$A\frac{dp}{dt}+(C-B)qr = -\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{d\bar{S}}\right)+\cos S\frac{d\Omega}{d\bar{F}},$$

$$B\frac{dq}{dt}+(A-C)rp = \cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{d\bar{S}}\right)-\sin S\frac{d\Omega}{d\bar{F}},$$

$$C\frac{dr}{dt}+(B-A)pq = -\frac{d\Omega}{d\bar{S}};$$

which, with the equations for p, q, r, determine the motion of the body.

III.

First, to integrate the equations of motion when the disturbing forces are neglected; we have, as usual, the integral equations

$$A\,p^2 + B\,q^2 + C\,r^2 = h,$$
$$A^2p^2 + B^2q^2 + C^2r^2 = k^2;$$

where h, k are constants of integration, viz., h is the constant of *Vis Viva*, and k is the constant of the principal area.

Moreover, if the coefficients α, β, &c. are those which belong to the transformation from xyz to $x_{,}y_{,}z_{,}$, viz., if in the table

	$x_,$	$y_,$	$z_,$
x	α	β	γ
y	α'	β'	γ'
z	α''	β''	γ''

the values of the coefficients are

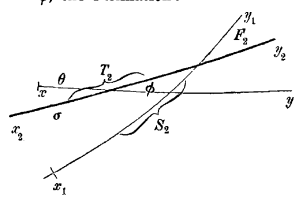
	$x_,$	$y_,$	$z_,$
x	$\cos S \cos T + \sin S \sin T \cos F$	$\sin S \cos T - \cos S \sin T \cos F$	$\sin T \sin F$
y	$\cos S \sin T - \sin S \cos T \cos F$	$\sin S \sin T + \cos S \cos T \cos F$	$-\cos T \sin F$
z	$-\sin S \sin F$	$\cos S \sin F$	$\cos F$

Then we have also, as usual, the integral equations

$$\begin{aligned} Ap\alpha + Bq\beta + Cr\gamma &= k \sin\theta \sin\phi, \\ Ap\alpha' + Bq\beta' + Cr\gamma' &= -k \cos\theta \sin\phi, \\ Ap\alpha'' + Bq\beta'' + Cr\gamma'' &= k \cos\phi; \end{aligned}$$

where θ, ϕ are constants of integration which determine the position of the principal plane (or invariable plane, in the undisturbed motion). Considering now a new system of axes $x_2y_2z_2$, where x_2 and z_2 are in the principal plane, and x_2 is in the first instance considered as an arbitrary fixed point therein (afterwards when the plane is treated as variable, x_2 is assumed to be a departure-point), let the position of the new set of axes in reference to the axes xyz be determined by

θ, the longitude of the node,
σ, the departure of the node,
ϕ, the inclination:



so that the relation between the coordinates xyz and $x_2y_2z_2$ is given by the table

	x_2	y_2	z_2
x	a	b	c
y	a'	b'	c'
z	a''	b''	c''

where the values of the coefficients are given by

	x_2	y_2	z_2
x	$\cos\sigma\cos\theta + \sin\sigma\sin\theta\cos\phi$	$\sin\sigma\cos\theta - \cos\sigma\sin\theta\cos\phi$	$\sin\theta\sin\phi$
y	$\cos\sigma\sin\theta - \sin\sigma\cos\theta\cos\phi$	$\sin\sigma\sin\theta + \cos\sigma\cos\theta\cos\phi$	$-\cos\theta\sin\phi$
z	$-\sin\sigma\sin\phi$	$\cos\sigma\sin\phi$	$\cos\phi$

and let the position of the axes $x_1y_1z_1$, in reference to the new axes $x_2y_2z_2$, be determined in a similar manner by

T_2, the longitude of node,

S_2, the departure of node,

F_2, the inclination;

so that we have the table

	x_1	y_1	z_1
x_2	α_2	β_2	γ_2
y_2	α_2'	β_2'	γ_2'
z_2	α_2''	β_2''	γ_2''

where the values of the coefficients are given by

	x_1	y_1	z_1
x_2	$\cos S_2 \cos T_2 + \sin S_2 \sin T_2 \cos F_2$	$\sin S_2 \cos T_2 - \cos S_2 \sin T_2 \cos F_2$	$\sin T_2 \sin F_2$
y_2	$\cos S_2 \sin T_2 - \sin S_2 \cos T_2 \cos F_2$	$\sin S_2 \sin T_2 + \cos S_2 \cos T_2 \cos F_2$	$-\cos T_2 \sin F_2$
z_2	$-\sin S_2 \sin F_2$	$\cos S_2 \sin F_2$	$\cos F_2$

The values of α_2, β_2, γ_2 are

$$\begin{aligned} &a\alpha + a'\alpha' + a''\alpha'', \\ &a\beta + a'\beta' + a''\beta'', \\ &a\gamma + a'\gamma' + a''\gamma'', \end{aligned}$$

with similar expressions for α_2', β_2', γ_2' and α_2'', β_2'', γ_2''; the last-mentioned three integrals are thus transformable into the form

$$\begin{aligned} Ap\alpha_2 + Bq\beta_2 + Cr\gamma_2 &= 0, \\ Ap\alpha_2' + Bq\beta_2' + Cr\gamma_2' &= 0, \\ Ap\alpha_2'' + Bq\beta_2'' + Cr\gamma_2'' &= k, \end{aligned}$$

which are, in fact, the equations which show that the plane x_2y_2 is the principal plane, or plane of maximum area.

The equations just obtained, attending to the values of α_2'', β_2'', γ_2'', give

$$\begin{aligned} Ap &= -k \sin S_2 \sin F_2, \\ Bq &= k \cos S_2 \sin F_2, \\ Cr &= k \cos F_2; \end{aligned}$$

and, since the expressions for p, q, r, in terms of T_2, S_2, F_2, must be similar to those in terms of T, S, F, we have

$$\begin{aligned} pdt &= -\sin S_2 \sin F_2 dT_2 + \cos S_2 dF_2, \\ qdt &= \cos S_2 \sin F_2 dT_2 + \sin S_2 dF_2, \\ rdt &= \cos F_2 dT_2 - dS_2; \end{aligned}$$

from which equations,

$$\sin F_2 dT_2 = (-p \sin S_2 + q \cos S_2)\, dt;$$

and substituting for $\sin S_2$, $\cos S_2$, the values $\dfrac{-Ap}{k \sin F_2}$, $\dfrac{Bq}{k \sin F_2}$, this becomes

$$k^2 \sin^2 F_2 dT_2 = k\,(Ap^2 + Bq^2)\, dt;$$

or, what is the same thing,

$$(k^2 - C^2r^2)\, dT_2 = k\,(h - Cr^2)\, dt.$$

The two equations $Ap^2+Bq^2+Cr^2=h$, $A^2p^2+B^2q^2+C^2r^2$ may be considered as determining p, q, in terms of r, and the equation $C\frac{dr}{dt}+(B-A)pq=0$, then gives

$$dt=\frac{-Cdr}{(B-A)pq},$$

whence also the equation for dT_2 becomes

$$dT_2=\frac{k(h-Cr^2)}{k^2-C^2r^2}\,\frac{-Cdr}{(B-A)pq}.$$

Instead of the time t, I consider a function $g=nt+\text{const.}$, n being for the present an arbitrary constant quantity which may be a function of the constants h and k; we have thus

$$dg=n\frac{-Cdr}{(B-A)pq},$$

and the integral equation is

$$g=n\int\frac{-Cdr}{(B-A)pq}.$$

The equation for dT_2 gives, in like manner,

$$T_2=\varpi+\int\frac{k(h-Cr^2)}{k^2-C^2r^2}\,\frac{-Cdr}{(B-A)pq};$$

where it is assumed that the integrals are each of them taken from $r=r_0$, r_0 being an arbitrary constant value, say a function of h and k. The quantity g is analogous to the mean anomaly in elliptic motion, or rather it will become so when the significations of n and r_0 are fixed; it is considered as implicitly involving a constant of integration, and, consequently, no constant of integration is added to the integral: as regards T_2, the constant of integration is ϖ, which denotes the initial value (corresponding to $r=r_0$) of the angle T_2.

IV.

Recapitulating the integral equations, we have

$$\begin{aligned}
&Ap^2+Bq^2+Cr^2=h,\\
&A^2p^2+B^2q^2+C^2r^2=k^2,\\
&Ap=-k\sin S_2\sin F_2,\\
&Bq=\ \ k\cos S_2\sin F_2,\\
&Cr=\ \ k\cos F_2,\\
&g\ =\ n\int\frac{-Cdr}{(B-A)pq},
\end{aligned}$$

which equations give r, p, q, and thence S_2 and F_2 in terms of g and the constants h and k.

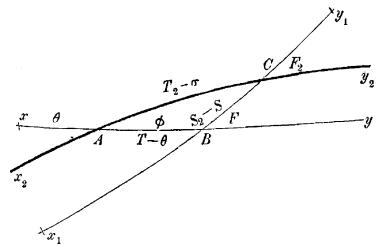
Moreover

$$T_2 = \varpi + \int \frac{k(h - Cr^2)}{k^2 - C^2r^2} \frac{-Cdr}{(B-A)pq},$$

which gives T_2 in terms of g and the constants h, k, ϖ; and then T, S, F are given in terms of T_2, S_2, F_2, and the constants θ, σ, ϕ, as follows, viz., we have a spherical triangle ABC, the sides whereof a, b, c, and the opposite angles A, B, C, are respectively,

Sides	$S_2 - S$,	$T_2 - \sigma$,	$T - \theta$,
Opposite angles	ϕ ,	$180^\circ - F$,	F_2 ,

as appears by the figure.



The above values of p, q, r, give

$$h - Cr^2 = \left(k^2 - C^2r^2\right)\left(\frac{\sin^2 S_2}{A} + \frac{\cos^2 S_2}{B}\right),$$

an equation which will be useful in the sequel.

The coefficient n, and the value r_0, which is the inferior limit of the integrals, are thus far considered as arbitrary functions of h and k. As already remarked, g (which is a variable quantity, $= nt + c$) is used as an element, and the elements are h, k, g, ϖ, σ, θ, ϕ.

As to the signification of the different elements, it is proper to remark that it is not for the purposes of the present memoir necessary to completely fix the significations of the quantities n and r_0; the only conditions imposed on these quantities in the sequel are that n shall be a function of h only, and that r_0 shall be a function of h and k, satisfying an equation of the form $r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right)$, where f denotes an arbitrary function. The values of n and r_0 might, in accordance with these conditions, be fixed more definitively by reference to the cone rolling and sliding on the principal plane, used in the theories of Poinsot and Jacobi for the representation of the undisturbed motion, but to do this would require a further discussion of the integral equations, and it is a point which is not here entered into.

This being premised, we have (corresponding to the orbit in the theory of elliptic motion) the principal plane, with a departure point therein, the positions whereof, in respect to the fixed plane of reference, are determined by θ, the longitude of node; ϕ, the inclination; and σ, the departure of node. The precise signification of ϖ depends upon that of r_0, and the signification of r_0 being assumed to be completely determined, that of ϖ will be so likewise; ϖ is to be considered as an angle measured in the principal plane from the departure point, and determining in that plane a line (or, treating the plane as an orbit, a point) which I call the rotation pericentre, or simply the pericentre; say ϖ is the departure of the pericentre; and then g is an angle varying uniformly with the time, used for expressing in terms of the time the angle T, which determines the position, in regard to the principal plane and departure point therein, of the node of the plane of x_1y_1, or *equator*, upon the principal plane,—such node corresponding with the moving body in the theory of elliptic motion. We may in fact say: T, the departure of the last-mentioned node, $=\varpi$, the departure of pericentre, $+(T-\varpi)$, the rotation true anomaly of such node; $T-\varpi$ being a function of g, the rotation mean anomaly of such node. The elements are then as follows, viz.:

h, the constant of *Vis Viva*,

k, the constant of areas,

g, the rotation mean anomaly,

ϖ, the departure of rotation pericentre,

θ, the longitude of node of principal plane,

σ, the departure of ditto,

ϕ, the inclination of principal plane;

where θ, σ, ϕ determine the position of the principal plane and departure point therein, in respect to a fixed plane of reference and departure point therein; but it has been already remarked that the position of the principal plane and departure point therein might be, by the analogous quantities Θ, Σ, Φ, determined in reference to an arbitrarily varying plane of reference and departure point therein, and that the expressions for the variations of the elements would then be of the form given for this case in Art. I.

In the problem of the Rotation of the Earth, the principal plane is sensibly coincident with the plane of the equator, and on this account there is no actual physical representation of various quantities occurring in the mathematical problem. But a complete discussion of the mathematical problem does not thereby become unnecessary for the purposes of physical astronomy.

V.

If, now, the Disturbing Function is taken into account, the equations are to be integrated by the method of the variation of the elements. I use dg to denote that part of the variation of $g\,(=tdn+dc$, if $g=nt+c)$, which depends on the variation of the constants, and in like manner for dp, dq, dr, &c. I assume, moreover, that x_2 is a departure point in the principal plane; this gives $d\sigma-\cos\phi\, d\theta=0$; and we see that the equations which lead to the expressions for the variations of the elements are

$$\begin{aligned}
A\,dp &= -\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos S\frac{d\Omega}{dF}\,dt,\\
B\,dq &= \cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\sin S\frac{d\Omega}{dF}\,dt,\\
C\,dr &= -\frac{d\Omega}{dS}\,dt,\\
dT &= 0,\\
dS &= 0,\\
dF &= 0,\\
d\sigma-\cos\phi\, d\theta &= 0,
\end{aligned}$$

where, of course, $\Omega=\Omega\,(T,\ S,\ F)$.

The variations dh and dk are obtained from the equations

$$\begin{aligned}
\tfrac{1}{2}dh &= A\,p\,dp+B\,q\,dq+C\,r\,dr,\\
k\,dk &= A^2p\,dp+B^2q\,dq+C^2r\,dr,
\end{aligned}$$

expressions which will presently be resumed and reduced; S_2 and F_2 may be considered as given functions of h, k, r, and the variations dS_2 and dF_2 can be thus obtained. The expressions for T, S, F, in terms of T_2, S_2, F_2, σ, θ, ϕ, and the equations $dT=0$, $dS=0$, $dF=0$, $d\sigma-\cos\phi\, d\theta=0$, then lead to the expressions for $d\theta$, $d\sigma$, $d\phi$, and to an equation $-dS_2+\cos F_2 dT_2=0$; dT_2 is thus also given in terms of dh, dk, dr. And this being so, the two equations

$$\begin{aligned}
g &= \int\frac{-C\,dr}{(B-A)\,pq},\\
T_2 &= \varpi+\int\frac{k\,(h-Cr^2)}{k^2-C^2r^2}\,\frac{-C\,dr}{(B-A)\,pq},
\end{aligned}$$

lead to the expressions of dg, $d\varpi$, in terms of dh, dk, dr; the form of these expressions is simplified by partially fixing, in the manner already referred to, the significations of the quantities n and r_0, considered as functions of h and k. In this manner (dh, dk, dr, being given linear functions of $\frac{d\Omega}{dT}\,dt$, $\frac{d\Omega}{dS}\,dt$, $\frac{d\Omega}{dF}\,dt$), we obtain

dh, dk, dg, $d\varpi$, $d\theta$, $d\sigma$, $d\phi$, all of them expressed in the same form; that is, in terms of the differential coefficients of the disturbing function Ω with respect to the coordinates T, S, F. We may then express the disturbing function Ω in terms of the elements, and transform the equations so as to obtain expressions for the variations of the elements in terms of the differential coefficients of Ω with respect to the elements.

VI.

Proceeding to carry out the foregoing plan, the equation

$$\tfrac{1}{2}dh = Apdp + Bqdq + Crdr,$$

putting for Adp, Bdq, Cdr, their values, gives

$$\tfrac{1}{2}dh = (-p\sin S + q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\,dt\right)$$
$$+ (p\cos S + q\sin S)\frac{d\Omega}{dF}\,dt - r\frac{d\Omega}{dS}\,dt,$$

which may also be written

$$\begin{aligned}\tfrac{1}{2}dh = \quad & \frac{h(h-Cr^2)}{k^2-C^2r^2}\left\{-\sin(S_2-S)\sin F_2\frac{d\Omega}{dF}\,dt\right.\\ & \left.+\cos(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt - \cos F_2\frac{d\Omega}{dS}\,dt\right\}\\ & +\frac{k(B-A)pq}{k^2-C^2r^2}\left\{\cos(S_2-S)\sin F_2\frac{d\Omega}{dF}\,dt\right.\\ & \left.+\sin(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt\right\}\\ & +\frac{r(Ch-k^2)}{k^2-C^2r^2}\frac{d\Omega}{dS}\,dt.\end{aligned}$$

In fact, if, in this expression, we consider first the terms involving $\frac{d\Omega}{dS}\,dt$, the coefficient is

$$\frac{-Cr(h-Cr^2)+r(Ch-k^2)}{k^2-C^2r^2} = \frac{-r(k^2-C^2r^2)}{k^2-C^2r^2} \text{ which is } = -r.$$

Next, the terms involving $\frac{d\Omega}{dF}\,dt$, the coefficient is

$$\left\{-(h-Cr^2)\sin(S_2-S)+(B-A)pq\cos(S_2-S)\right\}\frac{1}{k\sin F_2}$$
$$= \left[\left\{-(h-Cr^2)\sin S_2+(B-A)pq\cos S_2\right\}\cos S\right.$$
$$\left.+\left\{(h-Cr^2)\cos S_2+(B-A)pq\sin S_2\right\}\sin S\right]\frac{1}{k\sin F_2};$$

or, putting for $\sin S_2$, $\cos S_2$, their values $\dfrac{-Ap}{k\sin F_2}$, $\dfrac{Bq}{k\cos F_2}$, the coefficient is

$$\left[\left\{-(h-Cr^2)(-Ap)+(B-A)\,pq\,.\,Bq\right\}\cos S\right.$$
$$\left.+\left\{\ (h-Cr^2)\,Bq+(B-A)\,pq\,(-Ap)\right\}\sin S\right]\frac{1}{k^2\sin^2 F_2},$$

and the quantities in $\{\ \}$ being respectively $p\,(k^2-C^2r^2)$ and $q\,(k^2-C^2r^2)$, and since $k^2-C^2r^2=k^2\sin^2 F_2$, the coefficient is $=p\cos S+q\sin S$.

In like manner, for the terms involving $\left(\operatorname{cosec} F\dfrac{d\Omega}{dT}+\cot F\dfrac{d\Omega}{dS}\right)dt$, the coefficient is

$$\left\{(h-Cr^2)\cos(S_2-S)+(B-A)\,pq\sin(S_2-S)\right\}\frac{1}{k\sin F_2}$$
$$=\left[\left\{(h-Cr^2)\cos S_2+(B-A)\,pq\sin S_2\right\}\cos S\right.$$
$$\left.+\left\{(h-Cr^2)\sin S_2-(B-A)\,pq\cos S_2\right\}\sin S\right]\frac{1}{k\sin F_2},$$

which is $=-p\sin S+q\cos S$; and the foregoing transformed expression for $\frac{1}{2}dh$ is thus seen to be correct.

The equation

$$kdk=A^2pdp+B^2qdq+C^2rdr,$$

substituting for Adp, Bdq, Cdr, their values, becomes

$$\begin{aligned}kdk=\ & Ap\left\{-\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos S\frac{d\Omega}{dF}dt\right\}\\ &+Bq\left\{\ \cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\sin S\frac{d\Omega}{dF}dt\right\}\\ &+Cr\left\{-\frac{d\Omega}{ds}dt\qquad\qquad\qquad\qquad\qquad\qquad\right\}.\end{aligned}$$

But, from the equations $Ap=-k\sin S_2\sin F_2$, $Bq=k\cos S_2\sin F_2$, $Cr=k\cos F_2$, we deduce

$$\begin{aligned}-Ap\sin S+Bq\cos S&=\ \ k\cos(S_2-S)\sin F_2,\\ Ap\cos S+Bq\sin S&=-k\sin(S_2-S)\sin F_2,\\ Cr\qquad\qquad\qquad&=\ \ k\cos F_2,\end{aligned}$$

and we thence find

$$dk=-\sin(S_2-S)\sin F_2\frac{d\Omega}{dF}dt+\cos(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cos F_2\frac{d\Omega}{dS}dt;$$

so that dh and dk are now determined.

VII.

The expressions for dh and dk being thus obtained, and dr being also given by the equation $Cdr = -\frac{d\Omega}{dS}\,dt$, we may now express dF_2 and dS_2 in terms of dh, dk, dr. In fact, the equations

$$Cr = k\cos F_2,$$

$$h - Cr^2 = (k^2 - C^2r^2)\left(\frac{\sin^2 S_2}{A} + \frac{\cos^2 S_2}{B}\right),$$

give immediately

$$k\sin F_2 dF_2 = -Cdr + \cos F_2 dk,$$

$$\sin S_2 \cos S_2 \left(\frac{1}{A} - \frac{1}{B}\right) dS_2 = \frac{C(Ch - k^2)\,r}{(k^2 - C^2r^2)^2}\,dr + \frac{\frac{1}{2}dh}{k^2 - C^2r^2} - \frac{(h - Cr^2)\,k\,dk}{(k^2 - C^2r^2)^2};$$

or, multiplying by $k^2 - C^2r^2 (= k^2\sin^2 F_2)$, and replacing $\cos S_2 \sin F_2 \sin^2 F_2$ by $-ABpq$, and dividing, the last equation becomes

$$dS_2 = \frac{-C(Ch - k^2)\,r\,dr}{(k^2 - C^2r^2)(B - A)\,pq} - \frac{\frac{1}{2}dh}{(B - A)\,pq} + \frac{(h - Cr^2)\,k\,dk}{(k^2 - C^2r^2)(B - A)\,pq};$$

and thence also

$$\cos F_2\, dS_2 = \frac{-C^2(Ch - k^2)\,r^2 dr}{k\,(k^2 - C^2r^2)(B - A)\,pq} - \frac{\frac{1}{2}Cr\,dh}{k\,(B - A)\,pq} + \frac{C\,(h - Cr)\,r\,dk}{(k^2 - C^2r^2)(B - A)\,pq},$$

which is the value of dT_2, as given by the equation (not yet demonstrated)

$$dT_2 = \cos S_2\, dF_2.$$

The expression for dF_2, substituting for dr and dk, their values, gives

$$\begin{aligned} k\sin F_2\, dF_2 = \quad & \cos F_2 \sin F_2 \cos(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) dt \\ & - \cos F_2 \sin F_2 \sin(S_2 - S)\,\frac{d\Omega}{dF}\,dt \\ & - \cos^2 F_2\,\frac{d\Omega}{dS}\,dt + \frac{d\Omega}{dS}\,dt; \end{aligned}$$

or, combining the last two terms, and reducing,

$$dF_2 = \frac{1}{k}\left\{\cos F_2 \cos(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) dt - \cos F_2 \sin(S_2 - S)\frac{d\Omega}{dF}\,dt + \sin F_2\frac{d\Omega}{dS}\,dt\right\}.$$

We might, in like manner, transform the expression for dS_2, but it is somewhat more simple to obtain the new form by differentiation of the equation $\tan S_2 = -\frac{Ap}{Bq}$; this, in fact, gives

$$dS_2 = \frac{AB\,(pdq - qdp)}{A^2p^2 + B^2q^2} = \frac{1}{k^2\sin^2 F_2}(Ap\,.\,Bdq - Bq\,.\,Adp),$$

and the value of the numerator being

$$-k\sin S_2\sin F_2\left\{\quad\cos S\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\sin S\frac{d\Omega}{dF}\,dt\right\}$$

$$-k\cos S_2\sin F_2\left\{-\sin S\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos S\frac{d\Omega}{dF}\,dt\right\},$$

we find

$$dS_2=-\frac{1}{k\sin F_2}\left\{\sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos(S_2-S)\frac{d\Omega}{dF}\,dt\right\};$$

the last-mentioned expressions for dF_2 and dS_2 will be used for obtaining $d\theta$, $d\sigma$, $d\phi$.

VIII.

I form now the equations

$$\begin{aligned}-\sin S\sin F\,dT+\cos S\,dF=pdt-\sin S_2\sin F_2\,dT_2+\cos S_2\,dF_2\\ +\alpha_2\ (-\sin\sigma\sin\phi\,d\theta+\cos\sigma\,d\phi)\\ +\alpha_2'\ (\ \cos\sigma\sin\phi\,d\theta+\sin\sigma\,d\phi)\\ +\alpha_2''(-d\sigma+\cos\phi\,d\theta\qquad),\end{aligned}$$

$$\begin{aligned}\cos S\sin F\,dT+\sin S\,dF=qdt+\cos S_2\sin F_2dT_2+\sin S_2dF_2\\ +\beta_2\ (-\sin\sigma\sin\phi\,d\theta+\cos\sigma\,d\phi)\\ +\beta_2'\ (\ \cos\sigma\sin\phi\,d\theta+\sin\sigma\,d\phi)\\ +\beta_2''(-d\sigma+\cos\phi\,d\theta\qquad),\end{aligned}$$

$$\begin{aligned}-dS+\cos F\,dT\qquad=rdt-dS_2+\cos F_2dT_2\\ +\gamma_2\ (-\sin\sigma\sin\phi\,d\theta+\cos\sigma\,d\phi)\\ +\gamma_2'\ (\ \cos\sigma\sin\phi\,d\theta+\sin\sigma\,d\phi)\\ +\gamma_2''(-d\sigma+\cos\phi\,d\theta\qquad);\end{aligned}$$

which will presently be useful, but which require some explanation. As the equations stand, on the left hand, dT, dS, dF, denote the entire variations of T, S, F, treating not only the constants, but also the time, as variable; but on the right hand, dT_2, dS_2, dF_2, denote the variations of T_2, S_2, F_2, depending only on the variation of the constants; if, on the left hand and the right hand respectively, dT, dS, dF, and dT_2, dS_2, dF_2, were used to denote either the entire variations of T, S, F, and T_2, S_2, F_2, or else the variations of these quantities depending only on the variations of the constants, then the terms pdt, qdt, rdt, would have to be omitted, and the equations would still be true. And it is to be noticed that, omitting the terms pdt, qdt, rdt, the equations express the relations existing between dT, dS, dF, dT_2, dS_2, dF_2, $d\theta$, $d\sigma$, $d\phi$, in virtue of the integral relations implied by the existence of the spherical triangle, the sides and angles whereof are S_2-S, $T_2-\sigma$, $T-\theta$, and ϕ, $180^\circ-F$, F_2.

For the present purpose we are to omit the terms pdt, qdt, rdt, and to write $dT=0$, $dS=0$, $dF=0$. We have thus

$$\begin{aligned}
-(-\sin S_2 \sin F_2\, dT_2 + \cos S_2\, dF_2) = {} & \alpha_2\ (-\sin\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi) \\
& + \alpha_2'\ (\ \cos\sigma \sin\phi\, d\theta + \sin\sigma\, d\phi) \\
& + \alpha_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad), \\
-(\ \cos S_2 \sin F_2\, dT_2 + \sin S_2\, dF_2) = {} & \beta_2\ (-\sin\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi) \\
& + \beta_2'\ (\ \cos\sigma \sin\phi\, d\theta + \sin\sigma\, d\phi) \\
& + \beta_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad), \\
-(-dS_2 + \cos F_2\, dT_2) \qquad = {} & \gamma_2\ (-\sin\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi) \\
& + \gamma_2'\ (\ \cos\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi) \\
& + \gamma_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad).
\end{aligned}$$

Hence we have

$$\begin{aligned}
-\sin\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi = {} & -\alpha_2\,(-\sin S_2 \sin F_2\, dT_2 + \cos F_2\, dS_2) \\
& -\beta_2\,(\ \cos S_2 \sin F_2\, dT_2 + \sin F_2\, dS_2) \\
& -\gamma_2\,(\qquad\qquad \cos F_2\, dT_2 - dS_2\qquad),
\end{aligned}$$

which, attending to the values of α_2'', β_2'', γ_2'', may be written,

$$\begin{aligned}
-\sin\sigma \sin\phi\, d\theta + \cos\sigma\, d\phi = {} & -\alpha_2\,(\alpha_2''\, dT_2 + \cos S_2\, dF_2) \\
& -\beta_2\,(\beta_2''\, dT_2 + \sin S_2\, dF_2) \\
& -\gamma_2\,(\gamma_2''\, dT_2 - dS_2\qquad) \\
= {} & -(\alpha_2 \cos S_2 + \beta_2 \sin S_2)\, dF_2 + \gamma_2 dS_2,
\end{aligned}$$

since the coefficient of dT_2 vanishes; and, in like manner,

$$\begin{aligned}
\cos\sigma \sin\phi\, d\theta + \sin\sigma\, d\phi = {} & -\alpha_2'\,(\alpha_2''\, dT_2 + \cos S_2\, dF_2) \\
& -\beta_2'\,(\beta_2''\, dT_2 + \sin S_2\, dF_2) \\
& -\gamma_2'\,(\gamma_2''\, dT_2 - dS_2\qquad) \\
= {} & -(\alpha_2' \cos S_2 + \beta_2' \sin S_2)\, dF_2 + \gamma_2' dS_2;
\end{aligned}$$

and so also

$$\begin{aligned}
-d\sigma + \cos\phi\, d\theta \qquad = {} & -\alpha_2''\ (\alpha_2''\, dT_2 + \cos S_2\, dF_2) \\
& -\beta_2''\,(\beta_2''\, dT_2 + \sin S_2\, dF_2) \\
& -\gamma_2''\,(\gamma_2''\, dT_2 - dS_2\qquad) \\
= {} & -dT_2 - (\alpha_2'' \cos S_2 + \beta_2'' \sin S_2)\, dF_2 + \gamma_2'' dS_2.
\end{aligned}$$

But we have

$$\begin{aligned}
\alpha_2\ \cos S_2 + \beta_2\ \sin S_2 &= \cos T_2, & \gamma_2 &= \sin T_2 \sin F_2, \\
\alpha_2'\ \cos S_2 + \beta_2'\ \sin S_2 &= \sin T_2, & \gamma_2' &= -\cos T_2 \sin F_2, \\
\alpha_2''\cos S_2 + \beta_2''\sin S_2 &= \ \ 0\ \ , & \gamma_2'' &= \qquad \cos F_2,
\end{aligned}$$

and the foregoing equations thus become

$$\begin{aligned} -\sin\sigma\sin\phi\,d\theta+\cos\sigma\,d\phi &= -\cos T_2\,dF_2+\sin T_2\sin F_2\,dS_2,\\ \cos\sigma\sin\phi\,d\theta+\sin\sigma\,d\phi &= -\sin T_2\,dF_2-\cos T_2\sin F_2\,dS_2,\\ -d\sigma+\cos\phi\,d\theta &= -dT_2 \qquad +\cos F_2 dS_2. \end{aligned}$$

The last equation, making x_2 a departure point, or putting $-d\sigma+\cos\phi\,d\theta=0$, gives

$$dT_2=\cos F_2 dS_2,$$

an equation above referred to. The other two equations give

$$\begin{aligned} d\phi &= -\cos(T_2-\sigma)\,dF_2+\sin(T_2-\sigma)\sin F_2\,dS_2,\\ \sin\phi\,d\theta &= -\sin(T_2-\sigma)\,dF_2-\cos(T_2-\sigma)\sin F_2\,dS_2, \end{aligned}$$

which, with the equation,

$$d\sigma=\cos\phi\,d\theta,$$

give $d\theta$, $d\sigma$, $d\phi$, in terms of dS_2 and dF_2.

IX.

Proceeding to substitute for dS_2, dF_2, their values, we find

$$\begin{aligned} k\,d\phi= &-\cos(T_2-\sigma)\left\{\cos F_2\cos(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cos F_2\cos(S_2-S)\frac{d\Omega}{dF}dt+\sin F_2\frac{d\Omega}{dS}dt\right\}\\ &-\sin(T_2-\sigma)\left\{\sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos(S_2-S)\frac{d\Omega}{dF}dt\right\}, \end{aligned}$$

or reducing

$$\begin{aligned} k\,d\phi= &-\left[\sin(T_2-\sigma)\sin(S_2-S)+\cos(T_2-\sigma)\cos(S_2-S)\cos F_2\right]\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt\\ &-\left[\sin(T_2-\sigma)\cos(S_2-S)-\cos(T_2-\sigma)\sin(S_2-S)\cos F_2\right]\frac{d\Omega}{dF}dt\\ &-\left[\cos(T_2-\sigma)\sin F_2\right]\frac{d\Omega}{dF}dt; \end{aligned}$$

and, in like manner,

$$\begin{aligned} k\sin\phi\,d\theta= &-\sin(T_2-\sigma)\left\{\cos F_2\cos(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cot F_2\cos(S_2-S)\frac{d\Omega}{dF}dt+\sin F_2\frac{d\Omega}{dS}dt\right\}\\ &+\cos(T_2-\sigma)\left\{\sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos(S_2-S)\frac{d\Omega}{dF}dt\right\} \end{aligned}$$

or reducing

$$k\sin\phi\, d\theta = \left[\cos(T_2-\sigma)\sin(S_2-S)-\sin(T_2-\sigma)\cos(S_2-S)\cos F_2\right]\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt$$

$$+\left[\cos(T_2-\sigma)\cos(S_2-S)+\sin(T_2-\sigma)\sin(S_2-S)\cos F_2\right]\frac{d\Omega}{dF}\,dt$$

$$+\left[\qquad -\sin(T_2-\sigma)\sin F_2\qquad\right]\frac{d\Omega}{dS}\,dt.$$

These expressions for $d\phi$ and $d\theta$ are in a form which is convenient for some purposes, but they may be further reduced by means of the spherical triangle. In fact, in the expression for $d\phi$, the three coefficients in [ ] are respectively,

$$\begin{aligned}
\sin b\sin a+\cos b\cos a\cos C &= \sin B\sin A-\cos B\cos A\cos c\\
&= \sin F\sin\phi+\cos F\cos\phi\cos(T-\theta),\\
\sin b\cos a-\cos b\sin a\cos C &= \cos A\sin c\\
&= \cos\phi\sin(T-\theta),\\
\cos b\sin C &= \cos B\sin A+\sin B\cos A\cos c\\
&= -\cos F\sin\phi+\sin F\cos\phi\cos(T-\theta);
\end{aligned}$$

and we have thus

$$\begin{aligned}
k\,d\phi = -\Big(&\sin F\sin\phi+\cos F\cos\phi\cos(T-\theta)\Big)\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt\\
&-\cos\phi\sin(T-\theta)\,\frac{d\Omega}{dF}\,dt\\
-\Big(&-\cos F\sin\phi+\sin F\cos\phi\cos(T-\theta)\Big)\frac{d\Omega}{dS}\,dt;
\end{aligned}$$

and the right-hand side is

$$\begin{aligned}
-\Big(&\sin F\sin\phi+\cos F\cos\phi\cos(T-\theta)\Big)\operatorname{cosec} F\frac{d\Omega}{dT}\,dt\\
&-\cos\phi\sin(T-\theta)\,\frac{d\Omega}{dF}\,dt\\
&+\operatorname{cosec} F\cos\phi\cos(T-\theta)\,\frac{d\Omega}{dS}\,dt,
\end{aligned}$$

or we have finally

$$\begin{aligned}
k\,d\phi = &-\cos\phi\cos(T-\theta)\left(\cot F\frac{d\Omega}{dT}+\operatorname{cosec} F\frac{d\Omega}{dS}\right)dt\\
&-\sin\phi\,\frac{d\Omega}{dT}\,dt\\
&-\cos\phi\sin(T-\theta)\frac{d\Omega}{dF}\,dt.
\end{aligned}$$

In like manner, in the original expression for $k \sin \phi \, d\theta$, the three coefficients in [ ] are

$$\begin{aligned} \cos b \sin a - \sin b \cos a \cos C &= \cos B \sin c \\ &= -\cos F \sin (T-\theta), \\ \cos b \cos a + \sin b \sin a \cos C &= \cos c \\ &= \cos (T-\theta), \\ -\sin b \sin C \qquad\qquad &= -\sin B \sin c \\ &= -\sin T \sin (T-\theta), \end{aligned}$$

and thence

$$\begin{aligned} k \sin \phi \, d\theta = &-\cos F \sin (T-\theta) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS} \right) dt \\ &+ \qquad \cos (T-\theta) \frac{d\Omega}{dF} dt \\ &- \sin F \sin (T-\theta) \frac{d\Omega}{dS} dt, \end{aligned}$$

and the right-hand side is

$$\begin{aligned} &- \cot F \sin (T-\theta) \frac{d\Omega}{dT} dt \\ &+ \qquad \cos (T-\theta) \frac{d\Omega}{dF} dt \\ &- \operatorname{cosec} F \sin (T-\theta) \frac{d\Omega}{dS} dt, \end{aligned}$$

or we have finally

$$\begin{aligned} k \sin \phi \, d\theta = &- \sin (T-\theta) \left(\cot F \frac{d\Omega}{dT} + \operatorname{cosec} F \frac{d\Omega}{dS} \right) dt \\ &+ \cos (T-\theta) \frac{d\Omega}{dF} dt. \end{aligned}$$

The expression for $d\sigma$ can be obtained from either of those for $d\theta$, by the equation $d\sigma = \cos \phi \, d\theta$, and we have thus the values of $d\sigma$, $d\theta$, $d\phi$.

X.

It remains to find the expressions for dg and $d\varpi$. We have

$$g = \quad n \int \frac{-C dr}{(B-A) pq},$$

$$T_2 = \varpi + \int \frac{k (h - Cr^2)}{k^2 - C^2 r^2} \frac{-C dr}{(B-A) pq},$$

where p, q, are considered as functions of r, h, k, and where, besides, n and the inferior limit r_0 of the integrals are functions of h and k. We may write

$$dg = \frac{-nC}{(B-A)pq}\,dr + \mathfrak{A}\,dh + \mathfrak{B}\,dk,$$

$$dT_2 = d\varpi + \frac{k(h-Cr^2)}{k^2-C^2r^2}\,\frac{-C\,dr}{(B-A)pq}\,dr + \mathfrak{C}\,dh + \mathfrak{D}\,dk,$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, are functions which contain integrals with respect to r, and there is not any algebraical relation between them, except the equation $\mathfrak{B} = -2n\mathfrak{C}$, which will be obtained presently. I retain, therefore, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, in the formulæ. The first equation gives the value of dg: from the second equation we have

$$d\varpi = dT_2 + \frac{Ck(h-Cr^2)}{(k^2-C^2r^2)(B-A)pq}\,dr - \mathfrak{C}\,dh - \mathfrak{D}\,dk,$$

or substituting for dT_2 its value $\cos F_2 dS_2$, the expression for which has been obtained above [p. 489], we have

$$d\varpi = \frac{-C^2(Ch-k^2)r^2dr}{k(k^2-C^2r^2)(B-A)pq} - \frac{\frac{1}{2}Crdh}{k(B-A)pq} + \frac{C(h-Cr^2)rdk}{(k^2-C^2r^2)(B-A)pq}$$
$$+ \frac{Ck^2(h-Cr^2)dr}{k(k^2-C^2r^2)(B-A)pq} - \mathfrak{C}\,dh - \mathfrak{D}\,dk,$$

or reducing, this is

$$d\varpi = \frac{Chdr}{k(B-A)pq} + \left[\frac{-\frac{1}{2}Cr}{k(B-A)pq} - \mathfrak{C}\right]dh + \left[\frac{C(h-Cr^2)r}{(k^2-C^2r^2)(B-A)pq} - \mathfrak{D}\right]dk,$$

which might be retained in this form. I obtain, however, a different form as follows, viz., we have

$$d\varpi = \cos F_2\,dS_2 + \frac{Ck(h-Cr^2)dr}{(k^2-C^2r^2)(B-A)pq} - \mathfrak{C}\,dh - \mathfrak{D}\,dk;$$

or using the other form of dS_2 [p. 490], and substituting also for dr its value from the equation $Cdr = -\frac{d\Omega}{dS}dt$, we have

$$d\varpi = -\frac{1}{k}\cot F_2\left\{\sin(S_2-S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt + \cos(S_2-S)\frac{d\Omega}{dF}dt\right\}$$
$$-\frac{k(h-Cr^2)}{(k^2-C^2r^2)(B-A)pq}\frac{d\Omega}{dS}dt - \mathfrak{C}\,dh - \mathfrak{D}\,dk.$$

We have now to prove the before mentioned equation $BC = -2n\mathfrak{C}$. We have

$$\mathfrak{B} = \frac{d}{dk}n\int\frac{-Cdr}{(B-A)pq} = n\int dr\frac{d}{dk}\frac{-C}{(B-A)pq} + \frac{dn}{dk}\int\frac{-Cdr}{(B-A)pq} + \frac{nC}{(B-A)p_0q_0}\frac{dr_0}{dk},$$

$$\mathfrak{C} = \frac{d}{dh}\int\frac{k(h-Cr^2)}{k^2-C^2r^2}\frac{-Cdr}{(B-A)pq} = \int dr\frac{d}{dh}\frac{-Ck(h-Cr^2)}{(k^2-C^2r^2)(B-A)pq} + \frac{Ck(h-Cr_0^2)}{(k^2-C^2r_0^2)(B-A)p_0q_0}\frac{dr_0}{dh},$$

where p_0, q_0, are the values of p and q corresponding to $r = r_0$. If we assume that n is a function of h only, the term multiplied by $\frac{dn}{dk}$ will disappear, and by properly determining r_0 as a function of h and k, we can, as regards the terms which contain r_0, satisfy the equation $\mathfrak{B} = -2n\mathfrak{C}$; the condition for this is

$$\frac{dr_0}{dk} = \frac{-2k(h - Cr_0^2)}{k^2 - C^2r_0^2}\frac{dr_0}{dh},$$

which will be satisfied if r_0 is determined as a function of h and k by an equation of the form

$$r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right),$$

where f denotes an arbitrary function.

It remains to show that, as regards the terms involving integrals, we have the same relation $\mathfrak{B} = -2n\mathfrak{C}$, and this will be the case if

$$\frac{d}{dk}\frac{-C}{(B-A)pq} = -2\frac{d}{dh}\frac{-Ck(h - Cr^2)}{(k^2 - C^2r^2)(B-A)pq},$$

or, what is the same thing, if

$$\frac{d}{dk}\frac{1}{pq} = -2\frac{d}{dh}\frac{k(h - Cr^2)}{(k^2 - C^2r^2)pq} = \frac{-2k}{(k^2 - C^2r^2)}\frac{d}{dh}\frac{h - Cr^2}{pq},$$

in which equation p, q, are considered as functions of h, k, given by the equations

$$Ap^2 + Bq^2 + Cr^2 = h,$$
$$A^2p^2 + B^2q^2 + C^2r^2 = k^2.$$

We find without difficulty

$$\frac{d.pq}{dh} = \frac{-\frac{1}{2}(A^2p^2 - B^2q^2)}{AB(B-A)pq}, \qquad \frac{d.pq}{dk} = \frac{k(Ap^2 - Bq^2)}{AB(B-A)pq},$$

and thence

$$\frac{d}{dh}\frac{h - Cr^2}{pq} = \frac{1}{pq} + \frac{\frac{1}{2}(h - Cr^2)(A^2p^2 - B^2q^2)}{AB(B-A)p^3q^3}$$

$$= \frac{1}{AB(B-A)p^3q^3}\left[AB(B-A)p^2q^2 + \tfrac{1}{2}(Ap^2 + Bq^2)(A^2p^2 - B^2q^2)\right]$$

$$= \frac{1}{AB(B-A)p^3q^3}\tfrac{1}{2}(Ap^2 - Bq^2)(A^2p^2) + (B^2q^2)$$

$$= \frac{\frac{1}{2}(k^2 - C^2r^2)(Ap^2 - Bq^2)}{AB(B-A)p^3q^3},$$

and the right-hand side of the equation in question is therefore

$$= \frac{-k\,(Ap^2 - Bq^2)}{AB\,(B-A)\,p^2q^2},$$

which is obviously also the value of the left-hand side. Hence, under the assumed relations $\left(n \text{ a function of } h \text{ only, and } r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right)\right)$, we have the above-mentioned equation $\mathfrak{B} = -2n\mathfrak{C}$.

XI.

It will be convenient to recapitulate the various formulæ for the variations, as follows; we have

$$\tfrac{1}{2}dh = (-p\sin S + q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt$$

$$+\,(\;p\cos S + q\sin S)\frac{d\Omega}{dF}dt - r\frac{d\Omega}{dS}dt,$$

and

$$\tfrac{1}{2}dh = \frac{k\,(h - Cr^2)}{k^2 - C^2r^2}\left\{\cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt - \cos F_2\frac{d\Omega}{dS}dt\right\}$$

$$+\frac{k\,(B-A)\,pq}{k^2 - C^2r^2}\left\{\sin(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt + \cos(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt\right\}$$

$$+\frac{r\,(Ch - k^2)}{k^2 - C^2r^2}\frac{d\Omega}{dS}dt;$$

$$kdk = (-Ap\sin S + Bq\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt$$

$$+\,(\;Ap\cos S + Bq\sin S)\frac{d\Omega}{dF}dt - Cr\frac{d\Omega}{dS}dt,$$

and

$$dk = \cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt - \cos S_2\frac{d\Omega}{dS}dt;$$

$$kd\phi = -\cos\phi\cos(T - \theta)\left(\cot F\frac{d\Omega}{dT} + \operatorname{cosec} F\frac{d\Omega}{dS}\right)dt$$

$$-\sin\phi\frac{d\Omega}{dT}dt - \cos\phi\sin(T - \theta)\frac{d\Omega}{dF}dt,$$

and

$$kd\phi = -\left\{\sin(T_2-\sigma)\sin(S_2-S)+\cos(T_2-\sigma)\cos(S_2-S)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt$$

$$-\left\{\sin(T_2-\sigma)\cos(S_2-S)-\cos(T_2-\sigma)\sin(S_2-S)\cos F_2\right\}\frac{d\Omega}{dF}\,dt$$

$$-\left\{\qquad -\cos(T_2-\sigma)\sin F_2\qquad\right\}\frac{d\Omega}{dS}\,dt;$$

$$k\sin\phi d\theta = -\sin(T-\theta)\left(\cot F\frac{d\Omega}{dT}+\operatorname{cosec} F\frac{d\Omega}{dS}\right)dt$$

$$+\cos(T-\theta)\frac{d\Omega}{dF}\,dt,$$

and

$$k\sin\phi d\theta = \left\{\cos(T_2-\sigma)\sin(S_2-S)-\sin(T_2-\sigma)\cos(S_2-S)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt$$

$$+\left\{\cos(T_2-\sigma)\cos(S_2-S)+\sin(T_2-\sigma)\sin(S_2-S)\cos F_2\right\}\frac{d\Omega}{dF}\,dt$$

$$+\left\{\qquad -\sin(T_2-\sigma)\sin F_2\qquad\right\}\frac{d\Omega}{dS}\,dt;$$

$$d\sigma = \cos\phi d\theta;$$

$$dg = \frac{-nC}{(B-A)pq}\,dr \qquad +\mathfrak{A}dh+\mathfrak{B}dk,$$

and

$$dg = \frac{n}{(B-A)pq}\frac{d\Omega}{dS}\,dt+\mathfrak{A}dh+\mathfrak{B}dk;$$

$$d\varpi = \frac{Chdr}{k(B-A)pq}+\left\{\frac{-\frac{1}{2}Cr}{k(B-A)pq}-\mathfrak{C}\right\}dh+\left\{\frac{C(h-Cr^2)r}{(k^2-C^2r^2)(B-A)pq}-\mathfrak{D}\right\}dk,$$

and

$$d\varpi = -\frac{1}{k}\cot F_2\left\{\sin(S_2-S)\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos(S_2-S)\frac{d\Omega}{dF}\,dt\right\}$$

$$+\frac{B\sin^2 S_2+A\cos^2 S_2}{k(B-A)\sin^2 F_2\sin S_2\cos S_2}\frac{d\Omega}{dS}\,dt-\mathfrak{C}dh-\mathfrak{D}dk;$$

where it will be remembered that $\mathfrak{B}=-2n\mathfrak{C}$. The different forms for the variations of the same element are, or may be, each of them useful. The first expressions for dg and $d\varpi$ respectively are to be considered as giving these variations in terms of dr, dh, dk; and the second expressions are those obtained by substituting for these quantities their values, but in the terms multiplied by the integral expressions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, which, on

account of these multipliers, do not unite with the other terms, dh, dk are retained as standing for their values. The following equations may be added,

$$dT_2 = d\varpi - \frac{Ck(h - Cr^2)}{(k^2 - C^2r^2)(B - A)pq}\,dr + \mathfrak{C}dh + \mathfrak{D}dk,$$

$$dF_2 = \frac{-Ck\sin F_2}{k^2 - C^2r^2}\,dr + \frac{k\sin F_2\cos F_2}{k^2 - C^2r^2}\,dk,$$

$$dS_2 = \frac{-C(Ch - k^2)r}{(k^2 - C^2r^2)(B - A)pq}\,dr - \frac{\frac{1}{2}dh}{(B - A)pq} + \frac{(h - Cr^2)kdk}{(k^2 - C^2r^2)(B - A)pq},$$

which give dF_2, dS_2, in terms of dr, dh, dk; and I call to mind, also, the equations

$$Ap = -k\sin S_2\sin F_2, \qquad Bq = k\cos S_2\sin F_2, \qquad Cr = k\cos F_2.$$

XII.

To find the differential coefficients of Ω with respect to the elements, I proceed as follows; considering the function first under the form $\Omega = \Omega(T, S, F)$, the total differential is

$$\frac{d\Omega}{dT}dT + \frac{d\Omega}{dS}dS + \frac{d\Omega}{dF}dF,$$

which must be equal to the total differential of Ω considered under the form $\Omega = \Omega\ (h, k, g, \varpi, \sigma, \theta, \phi)$; that is, it must be

$$= \frac{d\Omega}{dh}dh + \frac{d\Omega}{dk}dk + \frac{d\Omega}{dg}(ndt + dg) + \frac{d\Omega}{d\varpi}d\varpi + \frac{d\Omega}{d\theta}d\theta + \frac{d\Omega}{d\sigma}d\sigma + \frac{d\Omega}{d\phi}d\phi;$$

where, as elsewhere dg denotes only the part of the variation of g, which depends on the variation of the constants; so that the total variation of g is represented by $ndt + dg$. The value of $d\Omega$,

$$= \frac{d\Omega}{dT}dT + \frac{d\Omega}{dS}dS + \frac{d\Omega}{dF}dF,$$

is to be obtained in a form comparable with the last-mentioned expression, by means of the formulæ *suprà*, Art. VIII., which, for shortness, I represent as follows:

$$\begin{aligned} -\sin S\sin F\,dT + \cos S\,dF &= dP, \\ \cos S\sin F\,dT + \sin S\,dF &= dQ, \\ -dS + \cos F\,dT \qquad\qquad &= dR; \end{aligned}$$

these equations give

$$\begin{aligned} dF &= \qquad\qquad \cos S\,dP + \sin S\,dQ\ , \\ dT &= \operatorname{cosec} F(-\sin S\,dP + \cos S\,dQ), \\ dS &= \quad \cot F(-\sin S\,dP + \cos S\,dQ) - dR; \end{aligned}$$

and we then have

$$\begin{aligned} d\Omega = & \quad (\quad \cos S\, dP + \sin S\, dQ) \frac{d\Omega}{dS} \\ & + (-\sin S\, dP + \cos S\, dQ) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) \\ & - dR \frac{d\Omega}{dS}; \end{aligned}$$

and substituting for dP, dQ, dR, their values, the resulting expression may, for shortness, be represented by $d\Omega = d_1\Omega + d_2\Omega + d_3\Omega$, where

$$\begin{aligned} d_1\Omega = & \quad (\quad p \cos S + q \sin S) \frac{d\Omega}{dF} dt \\ & + (-p \sin S + q \cos S) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) dt \\ & - r \frac{d\Omega}{dS} dt. \end{aligned}$$

$$\begin{aligned} d_2\Omega = & \left\{\cos (S_2 - S) \sin F_2 \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) - \sin (S_2 - S) \sin F_2 \frac{d\Omega}{dF} - \cos F_2 \frac{d\Omega}{dS}\right\} dT_2 \\ & + \left\{ \sin (S_2 - S) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) + \cos (S_2 - S) \frac{d\Omega}{dF} \right\} dF_2 \\ & + \frac{d\Omega}{dS} dS_2, \end{aligned}$$

which, for shortness, I represent by $d_2\Omega = X dT_2 + Y dF_2 + Z dS_2$; and

$$\begin{aligned} d_3\Omega = & \left\{ \begin{array}{l} (\quad \alpha_2 \cos S + \beta_2 \sin S) (-\sin \sigma \sin \phi\, d\theta + \cos \sigma\, d\phi) \\ + (\quad \alpha_2' \cos S + \beta_2' \sin S) (\quad \cos \sigma \sin \phi\, d\theta + \sin \sigma\, d\phi) \\ + (\quad \alpha_2'' \cos S + \beta_2'' \sin S) (\quad - d\sigma + \cos \phi\, d\theta \quad) \end{array} \right\} \frac{d\Omega}{dF} \\ & + \left\{ \begin{array}{l} (-\alpha_2 \sin S + \beta_2 \cos S) (-\sin \sigma \sin \phi\, d\theta + \cos \sigma\, d\phi) \\ + (-\alpha_2' \sin S + \beta_2' \cos S) (\quad \cos \sigma \sin \phi\, d\theta + \sin \sigma\, d\phi) \\ (-\alpha_2'' \sin S + \beta_2'' \cos S) (\quad - d\sigma + \cos \phi\, d\theta \quad) \end{array} \right\} \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) \\ & - \left\{ \begin{array}{r} \gamma_2 (-\sin \sigma \sin \phi\, d\theta + \cos \sigma\, d\phi) \\ + \gamma_2' (\quad \cos \sigma \sin \phi\, d\theta + \sin \sigma\, d\phi) \\ + \gamma_2'' (\quad - d\sigma + \cos \phi\, d\theta \quad) \end{array} \right\} \frac{d\Omega}{dS}, \end{aligned}$$

or substituting for α_2, &c., their values, and reducing, $d_2\Omega =$

$$\left\{\begin{array}{l}\left([\cos(S_2-S)\sin(T_2-\sigma)-\sin(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi-\sin(S_2-S)\sin F_2\cos\phi\right)d\theta\\ +\left(\cos(S_2-S)\cos(T_2-\sigma)+\sin(S_2-S)\sin(T_2-\sigma)\cos F_2\right)d\phi+\sin(S_2-S)\sin F_2 d\sigma\end{array}\right\}\frac{d\Omega}{dF}$$

$$+\left\{\begin{array}{l}\left([\sin(S_2-S)\sin(T_2-\sigma)+\cos(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi+\cos(S_2-S)\sin F_2\cos\phi\right)d\theta\\ +\left(\sin(S_2-S)\cos(T_2-\sigma)-\cos(S_2-S)\sin(T_2-\sigma)\cos F_2\right)d\phi-\cos(S_2-S)\cos F_2 d\sigma\end{array}\right\}\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$

$$-\left\{\left(-\cos(T_2-\sigma)\sin F_2\sin\phi+\cos F_2\cos\phi\right)d\theta+\sin(T_2-\sigma)\sin F_2 d\phi-\cos F_2 d\sigma\right\}\frac{d\Omega}{dS}.$$

Hence, comparing $d_1\Omega+d_2\Omega+d_3\Omega$ with the other expression for $d\Omega$, and observing that dT_2, dS_2, dF_2, do not involve $d\theta$, $d\sigma$, $d\phi$, we have

$$d_1\Omega=\frac{d\Omega}{dg}\,ndt,$$

$$d_2\Omega=\frac{d\Omega}{dh}\,dh+\frac{d\Omega}{dk}\,dk+\frac{d\Omega}{dg}\,dg+\frac{d\Omega}{d\varpi}\,d\varpi,$$

$$d_3\Omega=\frac{d\Omega}{d\theta}\,d\theta+\frac{d\Omega}{d\sigma}\,d\sigma+\frac{d\Omega}{d\phi}\,d\phi.$$

XIII.

The first equation gives

$$n\frac{d\Omega}{dg}=(-p\sin S+q\cos S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$
$$+(p\cos S+q\sin S)\frac{d\Omega}{dF}-r\frac{d\Omega}{dS},$$

and, comparing this with the first form of $\frac{1}{2}dh$, we have

$$dh=2n\frac{d\Omega}{dg}\,dt.$$

The third equation gives

$$\frac{d\Omega}{d\theta}=\left\{[\sin(S_2-S)\sin(T_2-\sigma)+\cos(S_2-S)\cos(T_2-S)\cos F_2]\sin\phi+\cos(S_2-S)\sin F_2\cos\phi\right\}\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$
$$+\left\{[\cos(S_2-S)\sin(T_2-\sigma)-\sin(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi-\sin(S_2-S)\sin F_2\cos\phi\right\}\frac{d\Omega}{dF}$$
$$+\left\{\cos(T_2-\sigma)\sin F_2\sin\phi-\cos F_2\cos\phi\right\}\frac{d\Omega}{dS}.$$

$$\frac{d\Omega}{d\sigma} = -\cos(S_2 - S)\cos F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} - \cot F\frac{d\Omega}{dS}\right)$$
$$+ \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} + \cos F_2\frac{d\Omega}{dS},$$

$$\frac{d\Omega}{d\phi} = \left\{\sin(S_2 - S)\cos(T_2 - \sigma) - \cos(S_2 - S)\sin(T_2 - \sigma)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$
$$+ \left\{\cos(S_2 - S)\cos(T_2 - \sigma) + \sin(S_2 - S)\sin(T_2 - \sigma)\cos F_2\right\}\frac{d\Omega}{dF}$$
$$- \sin(T_2 - \sigma)\sin F_2\frac{d\Omega}{dS},$$

and thence

$$\cot\phi\frac{d\Omega}{d\sigma} + \operatorname{cosec}\phi\frac{d\Omega}{d\theta} =$$
$$\left\{\sin(S_2 - S)\sin(T_2 - \sigma) + \cos(S_2 - S)\cos(T_2 - \sigma)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$
$$+ \left\{\cos(S_2 - S)\sin(T_2 - \sigma) - \sin(S_2 - S)\cos(T_2 - \sigma)\cos F_2\right\}\frac{d\Omega}{dF}$$
$$+ \cos(T_2 - \sigma)\sin F_2\frac{d\Omega}{dS};$$

and we have therefore

$$d\phi = -\frac{\cot\phi}{k}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\theta}dt,$$
$$d\sigma = \frac{\cot\phi}{k}\frac{d\Omega}{d\phi}dt,$$
$$d\theta = \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\phi}dt.$$

The second equation, viz.

$$\frac{d\Omega}{dh}dh + \frac{d\Omega}{dk}dk + \frac{d\Omega}{dg}dg + \frac{d\Omega}{d\varpi}d\varpi = d_2\Omega = XdT_2 + YdF_2 + ZdS_2,$$

if we substitute for dg, dT_2, dF_2, dS_2 their values, becomes

$$\frac{d\Omega}{dh}dh + \frac{d\Omega}{dk}dk + \frac{d\Omega}{dg}\left(-\frac{nC}{(B-A)pq}dr + \mathfrak{A}dh + \mathfrak{B}dk\right) + \frac{d\Omega}{d\varpi}d\varpi$$
$$= X\left\{d\varpi - \frac{Ck(h - Cr^2)}{(k^2 - C^2r^2)(B-A)pq}dr + \mathfrak{C}dh + \mathfrak{D}dk\right\}$$
$$+ Y\left\{-\frac{Ck\sin F_2}{k^2 - C^2r^2}dr + \frac{k\sin F_2\cos F_2}{k^2 - C^2r^2}dk\right\}$$
$$+ Z\left\{-\frac{C(Ch - k^2)r}{(k^2 - C^2r^2)(B-A)pq}dr - \frac{\frac{1}{2}dh}{(B-A)pq} + \frac{(n - Cr^2)kdk}{(k^2 - C^2r^2)(B-A)pq}\right\}.$$

The comparison of the terms involving $d\varpi$ gives at once $\frac{d\Omega}{d\varpi} = X$, or, substituting for X its value,

$$\frac{d\Omega}{d\varpi} = \cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$

$$- \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} - \cos F_2\frac{d\Omega}{dS};$$

and comparing with the expression for dk, we find

$$dk = \frac{d\Omega}{d\varpi}\,dt.$$

The terms involving dr give

$$n\frac{d\Omega}{dg} = \frac{k(h - Cr^2)}{k^2 - C^2r^2}X + \frac{k\sin F_2(B - A)pq}{k^2 - C^2r^2}Y + \frac{(Ch - k^2)r}{k^2 - C^2r^2}Z,$$

or, substituting for X, Y, Z, their values,

$$n\frac{d\Omega}{dg} = \frac{k(h - Cr^2)}{k^2 - C^2r^2}\left\{\cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} - \cos F_2\frac{d\Omega}{dS}\right\}$$

$$+ \frac{k(B - A)pq}{k^2 - C^2r^2}\left\{\sin(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) + \cos(S_2 - S)\sin F_2\frac{d\Omega}{dF}\right\}$$

$$+ \frac{r(Ch - k^2)}{k^2 - C^2r^2}\frac{d\Omega}{dS},$$

which agrees with the second form for $\frac{1}{2}dh$, and gives as before

$$dh = 2n\frac{d\Omega}{dg}\,dt;$$

the terms involving dh give

$$\frac{d\Omega}{dh} + \mathfrak{A}\frac{d\Omega}{dg} = \mathfrak{C}X - \frac{\frac{1}{2}}{(B - A)pq}Z,$$

and thence

$$-2n\frac{d\Omega}{dh} = 2n\mathfrak{A}\frac{d\Omega}{dg} - 2n\mathfrak{C}X + \frac{n}{(B - A)pq}Z;$$

or, substituting for $-2n\mathfrak{C}$ its value $\mathfrak{B}$, for X the value $\frac{d\Omega}{d\varpi}$, and for Z its value $\frac{d\Omega}{dS}$, we have

$$-2n\frac{d\Omega}{dh} = 2n\mathfrak{A}\frac{d\Omega}{dg} + \mathfrak{B}\frac{d\Omega}{d\varpi} + \frac{n}{(B - A)pq}\frac{d\Omega}{dS},$$

where, on the right hand, $\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ may be considered as standing for given functions of $\frac{d\Omega}{dT}$, $\frac{d\Omega}{dS}$, $\frac{d\Omega}{dF}$; for the present purpose, however, multiplying by dt, and substituting for $2n\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ the values dh and dk, we have

$$-2n\frac{d\Omega}{dh}dt = \mathfrak{A}dh + \mathfrak{B}dk + \frac{n}{(B-A)pq}\frac{d\Omega}{dS}dt,$$

which agrees with the foregoing value of dg, or we have

$$dg = -2n\frac{d\Omega}{dh}dt.$$

The comparison of the terms involving dk gives

$$\frac{d\Omega}{dk} + \mathfrak{B}\frac{d\Omega}{dg} = \mathfrak{D}X + \frac{k\sin F_2\cos F_2}{k^2 - C^2r^2}Y + \frac{(h - Cr^2)k}{(k^2 - C^2r^2)(B-A)pq}Z;$$

or, substituting and reducing,

$$-\frac{d\Omega}{dk} = -2n\mathfrak{C}\frac{d\Omega}{dg} - \mathfrak{D}\frac{d\Omega}{d\varpi} - \frac{1}{k}\cot F_2\left\{\cos(S_2 - S)\frac{d\Omega}{dF} + \sin(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)\right\}$$
$$-\frac{(h - Cr^2)k}{(k^2 - C^2r^2)(B-A)pq}\frac{d\Omega}{dS},$$

where, on the right-hand side, $\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ may be considered as standing for given functions of $\frac{d\Omega}{dT}$, $\frac{d\Omega}{dS}$, $\frac{d\Omega}{dF}$; for the present purpose, however, multiplying by dt and putting for $2n\frac{d\Omega}{dg}dt$ and $\frac{d\Omega}{d\varpi}dt$ the values dh and dk, we have

$$-\frac{d\Omega}{dk}dt = -\mathfrak{C}dh - \mathfrak{D}dk - \frac{1}{k}\cot F_2\left\{\cos(S_2 - S)\frac{d\Omega}{dF}dt + \sin(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt\right\}$$
$$-\frac{(h - Cr^2)k}{(k^2 - C^2r^2)(B-A)pq}\frac{d\Omega}{dS}dt,$$

which agrees with the foregoing value of $d\varpi$, or we have

$$d\varpi = -\frac{d\Omega}{dk}dt,$$

and we have thus the system of formulæ for the variation of the elements in the problem of rotation, given in Art. I.

218.

A THIRD MEMOIR ON THE PROBLEM OF DISTURBED ELLIPTIC MOTION.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXXI, pp. 43—56. Read January 10, 1863.]

THE object of the present Memoir is to obtain the differential equations for determining

r, the radius vector,

v, the longitude,

y, the latitude,

of the disturbed body, when the last two coordinates are measured in respect to an arbitrarily varying plane (which however, to fix the ideas, is called the variable ecliptic) and the departure point or origin of longitudes therein. This is very readily effected by means of an expression for the *Vis Viva* function given in my "Supplementary Memoir on the Problem of Disturbed Elliptic Motion," *Mem. Roy. Ast. Soc.*, t. XXVIII. pp. 217—234 (1859), [215]. Neglecting the squares of the variations of the variable ecliptic, and also the products of the variations by $\sin y$, or $\frac{dy}{dt}$, then (as might be expected) it is found that the equations for r and v are the same as for a fixed ecliptic, and the equation for y is found in a simple form, which is ultimately reduced to coincide with that obtained for the lunar theory by Laplace in the seventh book of the *Mécanique Céleste*, and which is used by him to show that the effect of the variation of the ecliptic on the latitude of the Moon (as measured from the variable ecliptic) is insensible. And it is shown conversely how the approximate formula of the Memoir may be obtained by a process similar to that made use of in the *Mécanique Céleste*.

I.

The position in respect to a fixed plane of reference and origin of longitudes therein, of the variable ecliptic and of the departure point or origin of longitudes therein, are determined by

θ', the longitude of node,

σ', the departure of node,

ϕ', the inclination,

where, by the definition of a departure point,

$$d\sigma' - \cos\phi'\, d\theta' = 0,$$

and then, in respect to the variable ecliptic and departure point or origin of longitudes therein, the position of the disturbed body is determined by

r, the radius vector,

v, the longitude,

y, the latitude;

and this being so, then (Supplementary Memoir, pp. 220, 227) the expression for the *Vis Viva* function is,

$$T = \tfrac{1}{2}\{\dot{r}^2 + r^2(Q^2 + R^2)\},$$

where

$$Q = \quad -\dot{y} \quad + [\cos(v-\sigma')\sin\phi' \,.\, \dot{\theta}' - \sin(v-\sigma')\,\dot{\phi}'],$$

$$R = \cos y \,.\, \dot{v} - \sin y\,[\sin(v-\sigma')\sin\phi' \,.\, \dot{\theta}' + \cos(v-\sigma')\,\dot{\phi}'],$$

the superscript dots being used to denote differentiation with respect to the time. The last-mentioned expressions may for shortness be denoted by

$$Q = \quad -\dot{y} + A,$$

$$R = \cos y \,.\, \dot{v} - B\sin y.$$

The equations of motion are of course,

$$\frac{d}{dt}\frac{dT}{d\dot{r}} - \frac{dT}{dr} = \frac{dV}{dr},$$

$$\frac{d}{dt}\frac{dT}{d\dot{v}} - \frac{dT}{dv} = \frac{dV}{dv},$$

$$\frac{d}{dt}\frac{dT}{d\dot{y}} - \frac{dT}{dy} = \frac{dV}{dy},$$

where $V = \dfrac{n^2a^3}{r} + \Omega$, if Ω is the disturbing function, taken with Lagrange's sign ($\Omega = -R$, if R is the disturbing function of the *Mécanique Céleste*).

To reduce these, we have in the first place

$$\frac{dT}{d\dot{r}} = \dot{r} \quad , \quad \frac{dT}{dr} = r(Q^2 + R^2),$$

$$\frac{dT}{d\dot{v}} = r^2 R \cos y, \quad \frac{dT}{dv} = r^2\left(Q\frac{dQ}{dv} + R\frac{dR}{dv}\right) = r^2 \quad (-QB - RA\sin y),$$

$$\frac{dT}{d\dot{y}} = -r^2 Q \quad , \quad \frac{dT}{dy} = r^2 R\frac{dR}{dy} \qquad = r^2 R(-\sin y \,.\, \dot{v} - B\cos y).$$

The equations are thus reduced to

$$\frac{d^2r}{dt^2} - r(Q^2 + R^2) + \frac{n^2a^3}{r^2} \qquad = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}(r^2\cos y \,.\, R) + r^2 \quad (QB + RA\sin y) \quad = \frac{d\Omega}{dv},$$

$$\frac{d}{dt}(-r^2 Q \quad) + r^2 R(\sin y\frac{dv}{dt} + B\cos y) = \frac{d\Omega}{dy},$$

and then substituting for Q, R their values, viz.

$$Q = \quad -\frac{dy}{dt} + A,$$

$$R = \cos y\frac{dv}{dt} - B\sin y,$$

we find

$$\frac{d^2r}{dt^2} - r\left\{\cos^2 y\left(\frac{dv}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + \frac{n^2a^3}{r^2} = \frac{d\Omega}{dr} + \mathfrak{A},$$

$$\frac{d}{dt}\left(r^2\cos^2 y\frac{dv}{dt}\right) \qquad = \frac{d\Omega}{dv} + \mathfrak{B},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\cos y\sin y\left(\frac{dv}{dt}\right)^2 \quad = \frac{d\Omega}{dy} + \mathfrak{C},$$

where

$$\mathfrak{A} = r\left(-2A\frac{dy}{dt} - 2B\sin y\cos y\frac{dv}{dt} + A^2 + B^2\sin^2 y\right),$$

$$\mathfrak{B} = \frac{d}{dt}(r^2 B\sin y\cos y) + r^2\left(B\frac{dy}{dt} - A\sin y\cos y\frac{dv}{dt} - AB\cos^2 y\right),$$

$$\mathfrak{C} = \frac{d}{dt}(r^2 A) \qquad + r^2\left(-(\cos^2 y - \sin^2 y)B\frac{dv}{dt} + B^2\sin y\cos y\right),$$

in which

$$A = \cos(v - \sigma')\sin\phi'\frac{d\theta'}{dt} + \sin(v - \sigma')\frac{d\phi'}{dt},$$

$$B = \sin(v - \sigma')\sin\phi'\frac{d\theta'}{dt} + \cos(v - \sigma')\frac{d\phi'}{dt};$$

θ', σ', ϕ' being given functions of t such that $d\sigma' - \cos\phi' \, d\theta' = 0$.

The foregoing equations of motion are rigorously accurate.

II.

Neglecting the terms which involve A^2, AB, B^2, we have

$$\mathfrak{A} = r\left(-2A\frac{dy}{dt} - 2B\sin y\cos y\frac{dv}{dt}\right),$$

$$\mathfrak{B} = \frac{d}{dt}(r^2 B\sin y\cos y) + r^2\left(B\frac{dy}{dt} - A\sin y\cos y\frac{dv}{dt}\right),$$

$$\mathfrak{C} = \frac{d}{dt}(r^2 A) \qquad - r^2(\cos^2 y - \sin^2 y)B\frac{dv}{dt};$$

and if we then neglect also the products of A and B by y and $\frac{dy}{dt}$, we have

$$\mathfrak{A} = 0,$$

$$\mathfrak{B} = 0,$$

$$\mathfrak{C} = \frac{d}{dt}(r^2 A) - r^2 B\frac{dv}{dt},$$

where it may be noticed that, in order to obtain this last value of $\mathfrak{C}$, the only neglected term is a term containing $B\sin^2 y$.

Now, attending to the values of A and B, we have

$$\frac{dA}{dt} = \frac{dA}{dv}\frac{dv}{dt} + \frac{d'A}{dt}$$

$$= -B\frac{dv}{dt} + \frac{d'A}{dt},$$

where here and in the sequel $\frac{d'}{dt}$ denotes differentiation in regard to t in so far only as it enters through the quantities σ', θ', ϕ', which determine the position of the variable ecliptic.

Hence

$$\mathfrak{C} = 2Ar\frac{dr}{dt} + r^2\left(-B\frac{dv}{dt} + \frac{d'A}{dt}\right) - r^2 B\frac{dv}{dt}$$

$$= 2Ar\frac{dr}{dt} + r^2\frac{d'A}{dt} - 2r^2 B\frac{dv}{dt};$$

and, as above, $\mathfrak{A} = 0$, $\mathfrak{B} = 0$.

Let r, v, y be the values obtained on the supposition that $\mathfrak{C} = 0$, and

$$r + \delta r,\ v + \delta v,\ y + \delta y,$$

the accurate values; the first and second equations show that, neglecting the products of y and $\frac{dy}{dt}$ into δy and $\frac{d\delta y}{dt}$, we have $\delta r = 0$, $\delta v = 0$; so that the values of r and v are not affected by the variation of the ecliptic. And then, substituting in the third equation $y + \delta y$ in the place of y, and for

$$\cos(y + \delta y)\ \sin\ (y + \delta y),\ = \cos y\sin y + (\cos^2 y - \sin^2 y)\,\delta y,$$

writing $\cos y \sin y + \delta y$, we have

$$\frac{d}{dt} \cdot r^2 \left(\frac{dy}{dt} + \frac{d\delta y}{dt}\right) + r^2 \left(\frac{dv}{dt}\right)^2 (\cos y \sin y + \delta y) = \frac{d\Omega}{dy} + \delta \frac{d\Omega}{dy} + \mathfrak{C};$$

or, since the terms independent of δy and $\mathfrak{C}$ must destroy each other, this is

$$\frac{d}{dt} \cdot r^2 \frac{d\delta y}{dt} + r^2 \left(\frac{dv}{dt}\right)^2 \delta y = \delta \frac{d\Omega}{dy} + \mathfrak{C};$$

or, as this may be written,

$$r^2 \frac{d^2\delta y}{dt^2} + 2r \frac{dr}{dt} \frac{d\delta y}{dt} + r^2 \left(\frac{dv}{dt}\right)^2 \delta y = \delta \frac{d\Omega}{dy} + 2Ar \frac{dr}{dt} + r^2 \frac{d'A}{dt} - 2r^2 B \frac{dv}{dt},$$

that is,

$$\frac{d^2\delta y}{dt^2} + \frac{2}{r} \frac{dr}{dt} \frac{d\delta y}{dt} + \left(\frac{dv}{dt}\right)^2 \delta y = \frac{1}{r^2} \delta \frac{d\Omega}{dy} + 2A \frac{1}{r} \frac{dr}{dt} + \frac{d'A}{dt} - 2B \frac{dv}{dt},$$

or, what is the same thing,

$$\begin{aligned} \frac{d^2\delta y}{dt^2} + n^2 (1 + \tfrac{3}{2} m^2) \delta y = \frac{1}{r^2} \delta \frac{d\Omega}{dy} &+ \left\{n^2 + \tfrac{3}{2} m^2 n^2 - \left(\frac{dv}{dt}\right)^2\right\} \delta y \\ &- \frac{2}{r} \frac{dr}{dt} \frac{d\delta y}{dt} + 2A \frac{1}{r} \frac{dr}{dt} + 2B \left(n - \frac{dv}{dt}\right) \\ &- 2Bn + \frac{d'A}{dt}, \end{aligned}$$

where as usual $m, = \frac{n'}{n}$, is the ratio of the mean motion of the Sun to that of the Moon; the term $\frac{3}{2} m^2 n^2 \delta y$ having been added on each side of the equation in order to destroy on the right-hand side the corresponding term arising from $\frac{1}{r^2} \delta \frac{d\Omega}{dy}$. In fact, to find the approximate expression for $\frac{d\Omega}{dy}$, we have

$$\Omega = \frac{m'r^2}{r'^3} (\tfrac{3}{2} \cos^2 H - \tfrac{1}{2}),$$

where H is the angular distance of the Sun and Moon; that is, $\cos H = \cos y \cos (v - v')$; here

$$\begin{aligned} \Omega &= \frac{m'r^2}{r'^3} \{\tfrac{3}{2} \cos^2 y \cos^2 (v - v') - \tfrac{1}{2}\}, \\ \frac{d\Omega}{dy} &= -\frac{m'r^2}{r'^3} \cdot 3 \sin y \cos y \cos^2 (v - v') \\ &= -\frac{m'r^2}{r'^3} \left(\tfrac{3}{2} + \tfrac{3}{2} \cos (2v - 2v')\right) \sin y \cos y; \end{aligned}$$

or, neglecting the periodic quantity $\cos(2v-2v')$, and writing y for $\sin y \cos y$; also putting as usual $m' = n'^2a'^3 = m^2n^2a'^3$, and $r' = a'$, we have

$$\frac{d\Omega}{dy} = -m^2n^2r^2 \cdot \tfrac{3}{2}\, y,$$

and thence

$$\delta \frac{d\Omega}{dy} = -m^2n^2r^2 \cdot \tfrac{3}{2}\, \delta y$$

or

$$\frac{1}{r^2}\, \delta \frac{d\Omega}{dy} = -\tfrac{3}{2}\, m^2n^2\, \delta y.$$

Substituting this value of $\frac{1}{r^2}\, \delta \frac{d\Omega}{dy}$, and putting also $r = a$, $\frac{dr}{dt} = 0$, $\frac{dv}{dt} = 0$, the equation for δy becomes

$$\frac{d^2\delta y}{dt^2} + n^2(1 + \tfrac{3}{2}\, m^2)\, \delta y = -2Bn + \frac{d'A}{dt}.$$

III.

To deduce the formula, seventh book of the *Mécanique Céleste*, I proceed as follows:

Putting

$$\frac{d\theta'}{dt} = \frac{1}{\cos\phi'} \frac{d\sigma'}{dt},$$

we have

$$A \cos\phi' = \frac{d\sigma'}{dt} \sin\phi' \cos(v-\sigma') - \frac{d\phi'}{dt} \cos\phi' \sin(v-\sigma'),$$

$$B \cos\phi' = \frac{d\sigma'}{dt} \sin\phi' \sin(v-\sigma') + \frac{d\phi'}{dt} \cos\phi' \cos(v-\sigma'),$$

which may be written

$$A \cos\phi' = -\sin v \frac{d}{dt}(\sin\phi' \cos\sigma') + \cos v \frac{d}{dt}(\sin\phi' \sin\sigma'),$$

$$B \cos\phi' = \quad \cos v \frac{d}{dt}(\sin\phi' \cos\sigma') + \sin v \frac{d}{dt}(\sin\phi' \sin\sigma').$$

Laplace in effect assumes that the variations of the ecliptic are given in the form

$$\sin\phi' \sin\sigma' = -\Sigma k \sin(int + \epsilon),$$
$$\sin\phi' \cos\sigma' = \quad \Sigma k \cos(int + \epsilon),$$

($it + \epsilon$ is there written for the argument, n being assumed $= 1$) where i, k, ϵ are absolute constants, the quantities i being all very small in comparison with m^2. Substituting these values, and putting $\cos\phi'$ equal to unity, we have

$$A = -\Sigma ik \cos(v + int + \epsilon),$$
$$B = -\Sigma ik \sin(v + int + \epsilon);$$

and thence also

$$\frac{d'A}{dt} = n\Sigma\, i^2 k \sin(v + int + \epsilon),$$

and

$$2nB - \frac{d'A}{dt} = -n\Sigma\,(2i + i^2)\, k \sin(v + int + \epsilon),$$

so that the equation for δy becomes

$$\frac{d^2\delta y}{dt^2} + n^2(1 + \tfrac{3}{2}m^2)\,\delta y + n\Sigma\,(2i + i^2)\, k \sin(v + int + \epsilon) = 0\,;$$

or, taking as the independent variable $v\,(= nt)$ in the place of t, this is

$$\frac{d^2\delta y}{dv^2} + (1 + \tfrac{3}{2}m^2)\,\delta y + \frac{1}{n}\Sigma\,(2i + i^2)\, k \sin(v + iv + \epsilon) = 0\,;$$

which is, in fact, Laplace's equation, n being retained instead of being put equal to unity, and δy being the part which depends on the variation of the ecliptic, of his $s_{,}$.

IV.

Conversely the equation

$$\frac{d^2\delta y}{dt^2} + n^2(1 + \tfrac{3}{2}m^2)\,\delta y = -2Bn + \frac{d'A}{dt}\,,$$

may be obtained by a process similar to Laplace's. Assuming that the Moon and Sun are each of them referred to a fixed plane of reference and origin of longitudes therein, by the coordinates

u, the reciprocal of the reduced radius vector,

v, the longitude,

s, the tangent of the latitude,

for the Moon, and by the corresponding coordinates u', v', s' for the Sun, then we have

$$\frac{d^2s}{dv^2} + s + \frac{\dfrac{ds}{dv}\dfrac{d\Omega}{dv} - su\dfrac{d\Omega}{du} - (1 + s^2)\dfrac{d\Omega}{ds}}{n^2u^2\left(1 + \dfrac{2}{n^2}\displaystyle\int\frac{d\Omega}{u^2dv}\,dv\right)} = 0.$$

Here, as before,

$$\Omega = \frac{m'r^2}{r'^3}\left(\tfrac{3}{2}\cos^2 H - \tfrac{1}{2}\right),$$

or, as it is now to be written,

$$\Omega = \frac{m'(1 + s^2)\,u'^3}{(1 + s'^2)^{\frac{3}{2}}\,u^2}\left\{\tfrac{3}{2}\left(\frac{\cos(v - v') + ss'}{\sqrt{1 + s^2}\,\sqrt{1 + s'^2}}\right)^2 - \tfrac{1}{2}\right\};$$

or, since the second term

$$\frac{m'u'^3}{(1+s'^2)^{\frac{3}{2}}} - \frac{1}{2}\frac{1+s^2}{u^2}$$

gives, as is immediately seen, no term in

$$\frac{ds}{dv}\frac{d\Omega}{dv} - su\frac{d\Omega}{du} - (1+s^2)\frac{d\Omega}{ds},$$

we may, in calculating this quantity, write

$$\Omega = \frac{m'(1+s^2)u'^3}{(1+s'^2)^{\frac{3}{2}}u^2}\cdot\frac{3}{2}\left(\frac{\cos(v-v')+ss'}{\sqrt{1+s^2}\sqrt{1+s'^2}}\right)^2$$

$$= \frac{3}{2}\frac{m'u'^3}{(1+s'^2)^{\frac{3}{2}}u^2}\left(\cos(v-v')+ss'\right)^2,$$

or, neglecting s'^2,

$$\Omega = \frac{3}{2}m'u'^3\left(\frac{\cos(v-v')+ss'}{u}\right)^2.$$

Hence, putting for a moment

$$\Theta = \frac{\cos(v-v')+ss'}{u},$$

we have

$$\frac{ds}{dv}\frac{d\Omega}{dv} - su\frac{d\Omega}{du} - (1+s^2)\frac{d\Omega}{ds} = 3m'u'^3.\,\Theta\left\{\frac{ds}{dv}\frac{d\Theta}{dv} - su\frac{d\Theta}{du} - (1+s^2)\frac{d\Theta}{ds}\right\},$$

where the factor in { } is

$$= \frac{1}{u}\left\{-\frac{ds}{dv}\sin(v-v') + s\left(\cos(v-v')+ss'\right) - (1+s^2)s'\right\}$$

$$= \frac{1}{u}\left\{s\cos(v-v') - \frac{ds}{dv}\sin(v-v') - s'\right\},$$

and the above-mentioned quantity is

$$= \frac{3m'u'^3}{u^3}\left\{\cos(v-v')+ss'\right\}\left\{s\cos(v-v') - \frac{ds}{dv}\sin(v-v') - s'\right\};$$

or, what is the same thing,

$$= \frac{3m'u'^3}{u^2}\cos(v-v')\left\{s\cos(v-v') - \frac{ds}{dv}\sin(v-v') - s'\right\}.$$

Considering now the Sun as moving in the variable ecliptic, its latitude is $= \sin\phi'\sin(v'-\sigma')$; that is, we have

$$s' = \sin\phi'\sin(v'-\sigma');$$

and if the Moon were in the variable ecliptic, its latitude would be $\sin\phi'\sin(v-\sigma')$: that is, the latitude, measured from the variable ecliptic, is $=s-\sin\phi'\sin(v-\sigma')$; or, putting

$s_{,}$, the Moon's latitude, measured from the variable ecliptic,

we have

$$s=\sin\phi'\sin(v-\sigma')+s_{,}.$$

Hence, disregarding the variations of ϕ', σ', we have

$$\frac{ds}{dv}=\sin\phi'\cos(v-\sigma')+\frac{ds_{,}}{dv};$$

and substituting these values of s, $\frac{ds}{dv}$, and s', we find

$$\begin{aligned}
&s\cos(v-v')-\frac{ds}{dv}\sin(v-v')-s'\\
&\quad= s_{,}\cos(v-v')-\frac{ds_{,}}{dv}\sin(v-v')\\
&\qquad+\sin\phi'\left\{\sin(v-\sigma')\cos(v-v')-\cos(v-\sigma')\sin(v-v')-\sin(v'-\sigma')\right\}\\
&\quad= s_{,}\cos(v-v')-\frac{ds_{,}}{dv}\sin(v-v');
\end{aligned}$$

or

$$\begin{aligned}
&\frac{ds}{dv}\frac{d\Omega}{dv}-su\frac{d\Omega}{du}(1+s^2)\frac{d\Omega}{ds}\\
&\qquad=\frac{3m'u'^3}{u^2}\cos(v-v')\left\{s_{,}\cos(v-v')-\frac{ds_{,}}{dv}\sin(v-v')\right\};
\end{aligned}$$

which, neglecting the periodic terms, is

$$=\tfrac{3}{2}\frac{m'u'^2}{u^2}s_{,};$$

and then

$$\frac{\dfrac{ds}{dv}\dfrac{d\Omega}{dv}-su\dfrac{d\Omega}{du}-(1+s^2)\dfrac{d\Omega}{ds}}{h^2u^2\left(1+\dfrac{2}{h^2}\displaystyle\int\frac{d\Omega}{u^3dv}\,dv\right)}=\frac{1}{h^2u^2}\cdot\tfrac{3}{2}\frac{m'u'^2}{u^2}s_{,}=\tfrac{3}{2}\frac{m'u'^2}{h^2u^4}s_{,};$$

which, putting $m'=n'^2a'^3=m^2n^2a'^3$, $u'=\frac{1}{a'}$, $h^2=n^2a^4$, $u=\frac{1}{a}$, becomes

$$=\tfrac{3}{2}m^2s_{,};$$

so that the differential equation is reduced to

$$\frac{d^2s}{dv^2}+s+\tfrac{3}{2}m^2s_{,}=0.$$

But

$$s = \sin\phi' \sin(v-\sigma') + s_{,},$$

$$\frac{ds}{dv} = \sin\phi' \cos(v-\sigma') + \frac{d'}{dv}\sin\phi' \sin(v-\sigma') + \frac{ds_{,}}{dv},$$

$$\frac{d^2s}{dv^2} = -\sin\phi' \sin(v-\sigma') + \frac{d'}{dv}\sin\phi' \cos(v-\sigma') + \frac{d}{dv}\frac{d'}{dv}\sin\phi' \sin(v-\sigma') + \frac{d^2s_{,}}{dv^2},$$

where $\frac{d'}{dv}$ denotes differentiation in respect to v, in so far only as it enters through ϕ' and σ' $\left(\text{these are functions of } t, \text{ which is } = \frac{1}{n}v\right)$. Hence

$$\frac{d^2s}{dv^2} + s = \frac{d^2s_{,}}{dv^2} + s_{,} + \frac{d'}{dv}\sin\phi' \cos(v-\sigma') + \frac{d}{dv}\frac{d'}{dv}\sin\phi' \sin(v-\sigma');$$

but

$$\frac{d'}{dv}\sin\phi' \sin(v-\sigma') = \frac{1}{n}\left(\sin v \frac{d}{dt}\sin\phi' \cos\sigma' - \cos v \frac{d}{dt}\sin\phi' \sin\sigma'\right)$$

$$= -\frac{1}{n}A\cos\phi'$$

$$= -\frac{1}{n}A;$$

and thence

$$\frac{d}{dv}\frac{d'}{dv}\sin\phi' \sin(v-\sigma') = -\frac{1}{n}\frac{dA}{dv} - \frac{1}{n}\frac{d'A}{dv} = \frac{1}{n}B - \frac{1}{n^2}\frac{d'A}{dt};$$

and similarly

$$\frac{d'}{dv}\sin\phi' \cos(v-\sigma') = \frac{1}{n}\left\{\cos v \frac{d}{dt}(\sin\phi' \cos\sigma') + \sin v \frac{d}{dt}\sin\phi' \sin\sigma'\right\}$$

$$= \frac{1}{n}B\cos\phi'$$

$$= \frac{1}{n}B.$$

Hence

$$\frac{d^2s}{dv^2} + s = \frac{d^2s_{,}}{dv^2} + s_{,} + \frac{2}{n}B - \frac{1}{n^2}\frac{d'A}{dt};$$

or, putting $v = nt$,

$$\frac{d^2s}{dv^2} + s = \frac{1}{n^2}\frac{d^2s_{,}}{dt^2} + s_{,} + \frac{2}{n}B - \frac{1}{n^2}\frac{d'A}{dt};$$

whence, substituting in the equation

$$\frac{d^2s}{dv^2} + s + \tfrac{3}{2}\, m^2 s_{,} = 0,$$

we have

$$\frac{d^2s_{,}}{dt^2} + n^2 (1 + \tfrac{3}{2}\, m^2)\, s_{,} + 2\, n\, B - \frac{d'A}{dt} = 0\,;$$

or putting $s_{,} = y + \delta y$, then

$$\frac{d^2y}{dt^2} + n^2 (1 + \tfrac{3}{2}\, m^2)\, y = 0\,;$$

which gives, in the approximation which is being considered, the principal term of the latitude, and then

$$\frac{d^2\delta y}{dt^2} + n^2 (1 + \tfrac{3}{2}\, m^2)\, \delta y + 2\, n\, B - \frac{d'A}{dt} = 0,$$

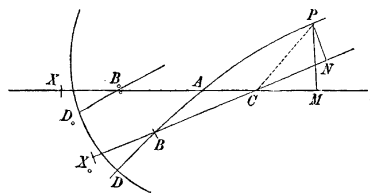
which is the approximate equation previously obtained by the method of the present memoir.

219.

ON SOME FORMULÆ RELATING TO THE VARIATION OF THE PLANE OF A PLANET'S ORBIT.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXI. (1861), pp. 43—47.]

IN Hansen's Memoir, "Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten," *Abhand. der K. Sächs. Gesell.* t. v. (1856), are contained, § 8, some very elegant formulæ for taking account of the variation of the plane of the orbit. These, in fact, depend upon the following geometrical theorem, viz., if (in the figure) ABC is a spherical triangle; P, a point on the side AB; and PM, PN, the perpendiculars let fall from P on the other two sides AC, CB; then we have



$$\cos PM \sin(BC + CM) = \cos PN \sin BN - \tan \tfrac{1}{2} C \cos BC (\sin PM + \sin PN),$$
$$\cos PM \cos(BC + CM) = \cos PN \cos BN + \tan \tfrac{1}{2} C \sin BC (\sin PM + \sin PN).$$

These equations, in fact, give

$$\cos PM \sin CM = \cos PN \sin CN - \tan \tfrac{1}{2} C (\sin PM + \sin PN),$$
$$\cos PM \cos CM = \cos PN \cos CN;$$

the latter of which is at once seen to be true, since joining the points C and P, the two sides are respectively equal to $\cos CP$. To verify the former one, write $\angle PCM = C_1$, $\angle PCN = C_2$, so that $C = C_1 - C_2$. Then, since $\cos CP = \cos PM \cos CM = \cos PN \cos CN$, $\sin PM = \sin CP \sin C_1$, $\sin PN = \sin CP \sin C_2$, the equation becomes $\cos CP (\tan CM - \tan CN) = -\tan \frac{1}{2} C \sin CP (\sin C_1 + \sin C_2)$, or since $\tan CM = \tan CP \cos C_1$, $\tan CN = \tan CP \cos C_2$, this is

$$\cos C_1 - \cos C_2 = -\tan \tfrac{1}{2} C (\sin C_1 + \sin C_2),$$

which is identically true, in virtue of the equation $C = C_1 - C_2$; and, conversely, we have the original two equations.

Suppose that XM is the ecliptic, X being the origin of longitudes, DP the instantaneous orbit, D the departure-point therein, and P the planet, DD_0 the orthogonal trajectory of the successive positions of the orbit; and writing

$\flat$, the departure of the planet,

v, the longitude of ditto,

y, the latitude of ditto,

θ, the longitude of node,

σ, the departure of ditto,

ϕ, the inclination;

then, in the figure, $DP = \flat$, $XM = v$, $PM = y$, $XA = \theta$, $DA = \sigma$, $\angle A = \phi$.

The quantities θ_0, σ_0, ϕ_0, might be considered as altogether arbitrary; but to fix the ideas it is better to assume at once that they denote

θ_0, the longitude of node,

σ_0, the departure,

ϕ_0, the inclination,

for the initial position of the orbit, viz., in the figure $XB_0 = \theta_0$, $D_0B_0 = \sigma_0$, $\angle B_0 = \phi_0$.

Take $DB = \sigma_0$, $\angle B = \phi_0$, $BX_0 = \theta_0$, this determines a travelling orbit of reference X_0N, and origin of longitudes X_0 therein; such that, with respect to this travelling orbit, the position of the planet's orbit is determined by

θ_0, the longitude of node,

σ_0, the departure of node,

ϕ_0, the inclination.

We have in the triangle ABC, $AB = \sigma - \sigma_0$, $\angle B = \phi_0$, $\angle A = 180^0 - \phi$; and if the other parts of the triangle are represented by

$$BC = \omega,$$
$$AC = \theta_0 - \theta + \omega + \Gamma,$$
$$\angle C = \Phi;$$

then ω, Γ, Φ, are given in terms of $\sigma - \sigma_0$, ϕ_0, ϕ; and we have, moreover, $XC = \theta + AC = \theta_0 + \omega + \Gamma$, $X_0C = \sigma_0 + \omega$; that is, the position of the travelling orbit X_0N, and origin of longitudes X_0 therein, are determined by

$$\begin{aligned} &\theta_0 + \omega + \Gamma, \text{ the longitude of node,} \\ &\sigma_0 + \omega \quad\quad\;\; \text{, the departure of node,} \\ &\Phi \quad\quad\quad\quad\;\; \text{, the inclination.} \end{aligned}$$

Suppose that in reference to this travelling orbit and origin of longitudes therein, we have

$$\begin{aligned} &v', \text{ the longitude of planet,} \\ &y', \text{ the latitude of ditto,} \end{aligned}$$

viz., in the figure $X_0N = v'$ (and therefore $BN = v' - \theta_0$), $PN = y'$.

Moreover, $BC + CM = BC + AM - AC = \omega + (v - \theta) - (\theta_0 - \theta + \omega + \Gamma) = v - \theta_0 - \Gamma$, hence the two equations are

$$\begin{aligned} \cos y \sin(v - \theta_0 - \Gamma) &= \cos y' \sin(v' - \theta_0) - \tan \tfrac{1}{2}\Phi \cos\omega (\sin y + \sin y'), \\ \cos y \cos(v - \theta_0 - \Gamma) &= \cos y' \cos(v' - \theta_0) + \tan \tfrac{1}{2}\Phi \sin\omega (\sin y + \sin y'), \end{aligned}$$

or, as they may also be written,

$$\begin{aligned} \cos y \sin(v - \theta_0 - \Gamma) &= \cos\phi_0 \sin(\text{þ} - \sigma_0) - \tan \tfrac{1}{2}\Phi \cos\omega (\sin y + \sin y'), \\ \cos y \cos(v - \theta_0 - \Gamma) &= \cos(\text{þ} - \sigma_0) + \tan \tfrac{1}{2}\Phi \sin\omega (\sin y + \sin y'), \end{aligned}$$

or, if we put $s = \sin y + \sin y'$, then observing that $\sin y = \sin\phi \sin(\text{þ} - \sigma)$, $\sin y' = \sin\phi_0 \sin(\text{þ} - \sigma_0)$, these become

$$\begin{aligned} \cos y \sin(v - \theta_0 - \Gamma) &= \cos\phi_0 \sin(\text{þ} - \sigma_0) - \tan \tfrac{1}{2}\Phi \,.\, s \cos\omega, \\ \cos y \cos(v - \theta_0 - \Gamma) &= \cos(\text{þ} - \sigma_0) + \tan \tfrac{1}{2}\Phi \,.\, s \sin\omega, \\ \sin y \{= \sin\phi \sin(\text{þ} - \sigma)\} &= -\sin\phi_0 \sin(\text{þ} - \sigma_0) + s, \end{aligned}$$

which are, in fact, Hansen's formulæ (16), p. 75, the letters corresponding as follows, viz.,

$$\begin{aligned} &v,\ \text{þ},\ y,\ \sigma,\ \sigma_0,\ \theta,\ \theta_0,\ \phi,\quad \phi_0\,,\ \Phi,\ \Gamma,\ \omega \text{ (suprà) to} \\ &l,\ v,\ b,\ \sigma,\ h\ ,\ \theta,\ h\ ,\ i\ ,\ -k\ ,\ 2\eta,\ \Gamma,\ \omega \text{ (Hansen).} \end{aligned}$$

where, of course, the correspondence ϕ_0 to $-k$, shows that these angles are measured in a contrary direction. I had from Hansen's equations expected that the above formulæ would have contained $\sin y - \sin y'$ in place of $\sin y + \sin y'$.

2, *Stone Buildings, W. C., 4th Dec,* 1860.

220.

NOTE ON A THEOREM OF JACOBI'S, IN RELATION TO THE PROBLEM OF THREE BODIES.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXII. (1861), pp. 76—78.]

THE following theorem of Jacobi's (*Comptes Rendus*, t. III., p. 61 (1836)) has not, I think, found its way in an explicit form into any treatise of physical astronomy. The theorem is as follows, viz. "Consider the movement of a point without mass round the Sun, disturbed by a planet the orbit of which is circular. Let xyz be the rectangular coordinates of the disturbed body, the orbit of the disturbing planet being taken as the plane of xy, and the Sun as the centre of coordinates; let a' be the distance of the disturbing planet, $n't$ its longitude, m' its mass, M the mass of the Sun: then we have, *rigorously*,

$$\tfrac{1}{2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]-n'\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

$$=\frac{M}{(x^2+y^2+z^2)^{\frac{1}{2}}}+m'\left\{\frac{1}{\{(x^2+y^2+z^2-2a'(x\cos n't+y\sin n't)+a'^2)^{\frac{1}{2}}}-\frac{x\cos n't+y\sin n't}{a'^2}\right\}+C.$$

This is therefore a new integral equation, which, in the problem of three bodies, subsists, as regards the terms independent of the eccentricity of the disturbing planet, and which is rigorous as regards all the powers of the mass of such planet. In the *Lunar Theory* the Earth must be substituted in the place of the Sun, and the Sun taken as the disturbing planet."

To prove the theorem, as expressed in polar coordinates, I take the equations of motion in the form in which I have employed them in my "Memoir on the Theory of Disturbed Elliptic Motion" (*Memoirs*, vol. XXVII. p. 1 (1859)), [212], viz.

$$\frac{d}{dt}\frac{dr}{dt} - r\cos^2 y\left(\frac{dv}{dt}\right)^2 - r\left(\frac{dy}{dt}\right)^2 + \frac{n^2a^3}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2\cos^2 y\frac{dv}{dt}\right) = \frac{d\Omega}{dv},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\cos y\sin y = \frac{d\Omega}{dy},$$

where

$$\Omega = m'\left\{\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}} - \frac{r\cos H}{r'^2}\right\};$$

or, since $\cos H = \cos y\cos(v-v')$, and the Sun is considered as moving in a circular orbit (i.e. $r'=a'$, $v'=n't$), we have

$$\Omega = m'\left\{\frac{1}{\sqrt{r^2+a'^2-2ra'\cos y\cos(v-n't)}} - \frac{r\cos y\cos(v-n't)}{a'^2}\right\};$$

so that Ω is a function of r, v, y and of t, which last quantity enters only in the combination $v-n't$. Hence the complete differential coefficient of Ω is

$$\frac{d(\Omega)}{dt} = \frac{d'\Omega}{dt} - n'\frac{d\Omega}{dv},$$

where $\frac{d'\Omega}{dt}$ denotes, as usual, the differential coefficient in regard to the time, in so far as it enters through the coordinates r, v, y of the disturbed body.

We have, as usual,

$$\frac{d}{dt}\frac{dr^2+r^2(\cos^2 y\,dv^2+dy^2)}{dt^2} = \frac{d'\Omega}{dt};$$

and, from the foregoing equation,

$$\frac{d'\Omega}{dt} = \frac{d(\Omega)}{dt} + n'\frac{d\Omega}{dv}$$

$$= \frac{d(\Omega)}{dt} + n'\frac{d}{dt}\left(r^2\cos^2 y\frac{dv}{dt}\right);$$

hence, substituting this value, transposing, and integrating, we have

$$\left(\frac{dr}{dt}\right)^2 + r^2\left\{\cos^2 y\left(\frac{dv}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} - n'r^2\cos^2 y\frac{dv}{dt} = \Omega + C,$$

which is Jacobi's equation expressed in terms of the coordinates r, v, y.

M. de Pontécoulant, in his *Lunar Theory* (1846), where the solar eccentricity is neglected, writes (p. 91),

$$\int d'R = R + m\int \frac{dR}{dv}\,dt,$$

($n' = mn = m$, since n is there put equal to unity); and combining this with the equation (p. 43),

$$\frac{r^2 dv}{(1+s^2)\,dt} = h + \int \frac{dR}{dv}\,dt,$$

we have

$$\int d'R = R - mh + \frac{mr^2\,dv}{(1+s^2)\,dt};$$

and substituting this value of $\int d'R$ in the integral of *Vis Viva* (p. 41),

$$\left(\frac{dr}{dt}\right)^2 + \frac{r^2 dv^2}{(1+s^2)\,dt^2} + \frac{r^2 ds^2}{(1+s^2)^2\,dt^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'R,$$

we have what is, in fact, Jacobi's equation.

221.

ON THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXII. (1862), pp. 171—231.]

THE present Memoir exhibits a new method of taking account, in the Lunar Theory, of the variation of the eccentricity of the Sun's orbit. The approximation is carried to the same extent as in Prof. Adams' Memoir "On the Secular Variation of the Moon's Mean Motion" (*Phil. Trans.*, vol. CXLIII. (1853), pp. 397—406); and I obtain results agreeing precisely with his, viz., besides his new periodic terms in the longitude and radius vector, I obtain in the longitude the secular term

$$\left(-\tfrac{3}{2}m^2+\tfrac{3771}{64}m^4\right)\int(e'^2-E'^2)\,ndt,$$

and in the quotient radius, or radius vector divided by the mean distance, the secular term

$$\left(\tfrac{9}{4}m^2-\tfrac{1273}{64}m^4\right)(e'^2-E'^2).$$

which is, in fact, as will be shown, included implicitly in the results given in Professor Adams' Memoir. In quoting the foregoing results, I have written $e'^2-E'^2$ in the place of $(e'+f't)^2-e'^2=2e'f't$, which in the notation of the present Memoir it should have been; and I purposely refrain from here explaining the precise signification of the symbols: this is carefully done in the sequel. The method appears to me a very simple one in principle; and it possesses the advantage that it is not incorporated step by step with a lunar theory in which the eccentricity of the Sun's orbit is treated as constant; but it is added on to such a lunar theory, giving in the Moon's coordinates the supplementary terms which arise from the variation of the solar eccentricity, and thus serving as a verification of any process employed for taking account of such variation.

I have given the details of the work in a series of Annexes, 1 to 23: this appears to me the best course for presenting the investigation in a readable form.

I.

The inclination and eccentricity of the Moon's orbit, and, *à fortiori*, the variation of the position of the Ecliptic, and the Sun's latitude, are neglected; and the longitudes are measured from a fixed point in the Ecliptic. I write

n, the actual mean motion of the Moon at a given epoch;

viz., it is assumed that the mean longitude at the time t is $\epsilon + nt + n_2 t^2 +$ &c. where ϵ, n, n_2, &c. are absolute constants; and, moreover,

a, the calculated mean distance of the Moon;

that is, n^2a^3 is the sum of the masses of the Earth and Moon; a is therefore an absolute constant; and, in like manner,

n', the actual mean motion of the Sun at the same epoch,

a', the calculated mean distance of the Sun;

that is, if it were necessary to pay attention to the secular variation of the mean motion of the Sun, the assumption would be that the mean longitude was $\epsilon' + n't + n_2' t^2 +$ &c., ϵ', n', n_2', &c. being absolute constants, and $n'^2a'^3$ the sum of the masses of the Sun and Earth; a' would thus also be an absolute constant. But for the purpose of the present investigation the secular variation of the mean longitude of the Sun is neglected, or it is assumed that the mean longitude of the Sun is $\epsilon' + n't$, ϵ', n' being absolute constants; and that $n'^2a'^3$ is the sum of the masses of the Sun and Earth, a' being thus also an absolute constant.

I put also

m, the ratio of the mean motions of the Sun and Moon;

that is,

$$m = \frac{n'}{n}, \text{ or } n' = mn;$$

m is also an absolute constant.

The Sun is considered as moving in an elliptic orbit, the eccentricity whereof is $e' + \delta e'$ or $e' + f't$, e' and f' being absolute constants; the longitude of the Sun's perigee may be taken to be $\varpi' + (1 - c')n't$; so that the mean anomaly g' is $= n't + \epsilon - [\varpi' - (1 - c')n't] = c'n't + \epsilon' - \varpi'$; c', ϖ', being absolute constants; but c' is in fact treated as being $= 1$. Hence, if r', v' are the radius vector and longitude of the Sun, we have

$$r' = a' \operatorname{elqr}(e' + \delta e', g'),$$

$$v' = \varpi' + (1 - c')n't + \operatorname{elta}(e' + \delta e', g')$$
$$= n't + \epsilon + [\operatorname{elta}(e' + \delta e', g') - g'],$$

where $$g' = c'n't + \epsilon' - \varpi'.$$

In the expression for the disturbing function the Sun's mass is taken to be $= n'^2a'^3$, or, what is the same thing, $= m^2n^2a'^3$.

Let r, v be the radius vector and longitude of the Moon; then, taking the usual approximate expression of the Disturbing Function, the equations of motion are

$$\frac{d}{dt}\frac{dr}{dt} - r\left(\frac{dv}{dt}\right)^2 + \frac{n^2a^3}{r^2} = m^2n^2a'^2\frac{r}{r'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right),$$

$$\frac{d}{dt}\frac{r^2dv}{dt} \qquad\qquad = m^2n^2a'^2\frac{r^2}{r'^3}\left(\quad -\tfrac{3}{2}\sin 2v - 2v'\right).$$

It will be convenient to assume

ρ, the quotient radius of the Moon's orbit,

ρ', the quotient radius of the Sun's orbit;

that is

$$r = \rho a, \quad r' = \rho' a'.$$

The equations of motion thus become

$$\frac{d}{dt}\frac{\delta\rho}{dt} - \rho\left(\frac{dv}{dt}\right)^2 + \frac{n^2}{\rho^3} = m^2n^2P,$$

$$\frac{d}{dt}\left(\rho^2\frac{dv}{dt}\right) \qquad\qquad = m^2n^2Q,$$

where for shortness

$$P = \frac{\rho}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right),$$

$$Q = \frac{\rho^2}{\rho'^3}\left(\quad -\tfrac{3}{2}\sin 2v - 2v'\right),$$

in which

$$\rho' = \text{elqr}\,(e' + \delta e',\ g'),$$

$$v' = n't + \epsilon' + [\text{elta}\,(e' + \delta e',\ g') - g'].$$

I now change the notation by writing $\rho' + \delta\rho'$, $v' + \delta v'$, in the place of ρ', v', respectively, using henceforward ρ', v' to denote

$$\rho' = \qquad\qquad \text{elqr}\,(e',\ g'),$$

$$v' = n't + \epsilon' + [\text{elta}\,(e',\ g') - g'];$$

and I write also $\rho + \delta\rho$, $v + \delta v$, in the place of ρ, v, using henceforward ρ, v to denote the solutions of the equations obtained from the equations of motion by writing therein ρ', v' instead of the complete values $\rho' + \delta\rho'$, $v' + \delta v'$.

Suppose, in like manner, that the complete values of P, Q are denoted by $P + \delta P$, $Q + \delta Q$, where

$$\delta P = \frac{dP}{d\rho}\delta\rho + \frac{dP}{dv}\delta v + \frac{dP}{d\rho'}\delta\rho' + \frac{dP}{dv'}\delta v',$$

with a like value for δQ, the first powers of $\delta\rho$, δv, $\delta\rho'$, $\delta v'$ being alone attended to. Then, observing that the equations of motion are satisfied when $\delta\rho$, δv, $\delta\rho'$, $\delta v'$ are neglected, we have, it is clear,

$$\frac{d}{dt}\frac{d\delta\rho}{dt} - \delta\rho\left(\frac{dv}{dt}\right)^2 - 2\rho\frac{dv}{dt}\frac{d\delta v}{dt} - \frac{2n^2}{\rho^3}\delta\rho = m^2n^2\delta P,$$

$$\frac{d}{dt}\left(\rho^2\frac{d\delta v}{dt} + 2\rho\delta\rho\frac{dv}{dt}\right) \qquad = m^2n^2\delta Q.$$

The second of these equations gives

$$\rho^2\frac{d\delta v}{dt} + 2\rho\delta\rho\frac{dv}{dt} = m^2n^2(C + \int dQdt),$$

where the constant of integration, C, is to be so determined that δv may not contain any term of the form kt (for any such term is taken to be included in the term nt of $v + \delta v$). Multiplying the equation just obtained by $\frac{2}{\rho}\frac{dv}{dt}$, and adding it to the first equation, we have

$$\frac{d^2\delta\rho}{dt^2} + \left\{3\left(\frac{dv}{dt}\right)^2 - \frac{2n^2}{\rho^3}\right\}\delta\rho = m^2n^2\left(\delta P + \frac{2}{\rho}\frac{dv}{dt}(C + \int dQdt)\right),$$

which, with the above-mentioned integral equation, are the equations for the solution of the problem; but it will be convenient to write them under the slightly different form

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = \left\{n^2 + \frac{2n^2}{\rho^3} - 3\left(\frac{dv}{dt}\right)^2\right\}\delta\rho + m^2n^2\left\{\delta P + \frac{2}{\rho}\frac{dv}{dt}\left(C + \int dQdt\right)\right\},$$

$$\frac{d\delta v}{dt} = \qquad -\frac{2}{\rho}\frac{dv}{dt}\delta\rho + \frac{m^2n^2}{\rho^2}(C + \int\delta Qdt).$$

In these equations C is determined, as above, by the condition that $\frac{d\delta v}{dt}$ may contain no constant term; the values of ρ', v', $\delta\rho'$, $\delta v'$ are of course given by the theory of elliptic motion, and those of ρ, v are given by the ordinary lunar theory, in which the eccentricity of the solar orbit is treated as a constant; and then, $\delta\rho$, δv being obtained by integrating the equations, the radius vector and longitude of the Moon are $a(\rho + \delta\rho)$ and $v + \delta v$ respectively.

We have

$$P = \frac{\rho}{\rho'^3}(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'),$$

$$Q = \frac{\rho}{\rho'^3}(\ -\tfrac{3}{2}\sin 2v - 2v').$$

Moreover, by the lunar theory, observing that Plana's a is, or may be considered, identical with the a of the present Memoir, and putting also

$$\tau = nt + \epsilon - (n't + \epsilon'),$$

we have

$$
\begin{aligned}
\frac{1}{\rho} = 1 + \tfrac{1}{6}\, m^2 - \tfrac{3}{4}\ & m^2 e'^2 \\
- \tfrac{3}{2}\ & m^2 e' && \cos\ g' \\
+ \quad & m^2 - \tfrac{5}{2}\, m^2 e'^2 && ,, \quad 2\tau \\
+ \tfrac{7}{2}\ & m^2 e' && ,, \quad 2\tau - g' \\
- \tfrac{1}{2}\ & m^2 e' && ,, \quad 2\tau + g' \\
- \tfrac{9}{4}\ & m^2 e'^2 && ,, \quad 2g' \\
+ \tfrac{17}{2}\ & m^2 e'^2 && ,, \quad 2\tau - 2g' \\
0\ & m^2 e'^2 && ,, \quad 2\tau + 2g',
\end{aligned}
$$

$$
\begin{aligned}
v = \quad nt + \epsilon & \\
- 3\ & me' + 0\ m^2 e' && \sin\ g' \\
+ \tfrac{11}{8}\ & m^2 - \tfrac{55}{16}\, m^2 e'^2 && ,, \quad 2\tau \\
+ \tfrac{77}{16}\ & m^2 e' && ,, \quad 2\tau - g' \\
- \tfrac{11}{16}\ & m^2 e' && ,, \quad 2\tau + g' \\
- \tfrac{9}{4}\ & me'^2 + 0\ m^2 e'^2 && ,, \quad 2g' \\
+ \tfrac{187}{16}\ & m^2 e'^2 && ,, \quad 2\tau - 2g' \\
0\ & m^2 e'^2 && ,, \quad 2\tau + 2g',
\end{aligned}
$$

where the series are carried as far as m^2 and e'^2; the terms in e'^2 are given, as I shall have occasion to refer to them, but they are not used in the investigation, and, omitting them, the values are

$$
\begin{aligned}
\frac{1}{\rho} = \quad 1 + \tfrac{1}{6}\, m^2 & \\
- \tfrac{3}{2}\, m^2 e' & \quad \cos\ g' \\
+ \quad m^2 & \quad ,, \quad 2\tau \\
+ \tfrac{7}{2}\, m^2 e' & \quad ,, \quad 2\tau - g' \\
- \tfrac{1}{2}\, m^2 e' & \quad ,, \quad 2\tau + g',
\end{aligned}
$$

$$
\begin{aligned}
v = \quad nt + \epsilon & \\
- 3\ me' & \quad \sin\ g' \\
+ \tfrac{11}{8}\, m^2 & \quad ,, \quad 2\tau \\
+ \tfrac{77}{16}\, m^2 e' & \quad ,, \quad 2\tau - g' \\
- \tfrac{11}{8}\, m^2 e' & \quad ,, \quad 2\tau + g'
\end{aligned}
$$

$$(g' = c'mnt + \text{const.}, \quad 2\tau = (2 - 2m)\, nt + \text{const.})$$

For the coordinates of the Sun we have

$$\begin{aligned}\frac{1}{\rho'} &= 1 \\ &\quad + e' \quad \cos g' \\ &\quad + e'^2 \quad \text{,,} \ 2g',\end{aligned}$$

$$\begin{aligned}v' &= n't + \epsilon' \\ &\quad + 2\,e' \quad \sin g' \\ &\quad + \tfrac{5}{4}\,e'^2 \quad \text{,,} \ 2g',\end{aligned}$$

the series being carried as far as e'^2; but the terms in e'^2 are only used for the formation of $\delta\rho'$, $\delta v'$; and, omitting them, we have

$$\begin{aligned}\frac{1}{\rho'} &= 1 \\ &\quad + e' \quad \cos g',\end{aligned}$$

$$\begin{aligned}v' &= n't + \epsilon' \\ &\quad + 2e' \quad \sin g'.\end{aligned}$$

If $e' + f't$ is written for e', then the value of $\delta e'$ is $= f't$; but as only the terms multiplied by the simple power f' are attended to, we may for convenience write $\delta e' = t$, the factor f' being restored in the final results: we thus have

$$\begin{aligned}\delta\frac{1}{\rho'} &= 1 \quad t\cos g' \\ &\quad + 2e' \quad \text{,,} \ 2g',\end{aligned}$$

$$\begin{aligned}\delta v' &= 2 \quad t\sin g' \\ &\quad + \tfrac{5}{2}\,e' \quad \text{,,} \ 2g',\end{aligned}$$

and we may add the equations

$$\begin{aligned}\frac{dv}{dt} &= \overbrace{1}^{n\,\times} \\ &\quad - 3\,m^2e' \quad \cos g' \\ &\quad + \tfrac{11}{4}\,m^2 \quad \text{,,} \ 2\tau \\ &\quad + \tfrac{77}{8}\,m^2e' \quad \text{,,} \ 2\tau - g' \\ &\quad - \tfrac{11}{8}\,m^2e' \quad \text{,,} \ 2\tau + g',\end{aligned}$$

$$\begin{aligned}\delta\frac{1}{\rho'^3} &= 3\,e' \quad t \\ &\quad + 3 \quad t\cos g' \\ &\quad + \tfrac{9}{2}\,e' \quad \text{,,} \ 2g',\end{aligned}$$

$$\begin{aligned}\frac{\delta v'}{\rho'^3} &= 2 \quad t\sin g' \\ &\quad + \tfrac{11}{2}\,e' \quad \text{,,} \ 2g',\end{aligned}$$

which will be found useful.

II.

Proceeding now to the development of the solution, we have

$$\begin{aligned}\delta P = & \frac{1}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\rho \\ & + \frac{\rho}{\rho'^3}\left(\quad - \tfrac{3}{2}\sin 2v - 2v'\right)\delta v \\ & + \rho\left[\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\frac{1}{\rho'^3} + (3\sin 2v - 2v')\frac{\delta v'}{\rho'^3}\right],\end{aligned}$$

where the terms containing $\delta\rho$ and δv are (see Annex 1)

$$\left.\begin{array}{l} \tfrac{1}{2} \\ + \tfrac{3}{2}\, e' \cos g' \\ + \tfrac{3}{2} \quad \text{,,} \quad 2\tau \\ + \tfrac{21}{4}\, e' \quad \text{,,} \quad 2\tau - g' \\ - \tfrac{3}{4}\, e' \quad \text{,,} \quad 2\tau + g' \end{array}\right| \delta\rho$$

$$\left.\begin{array}{l} - 3 \quad \sin 2\tau \\ - \tfrac{21}{2}\, e' \quad \text{,,} \quad 2\tau - g' \\ + \tfrac{3}{2}\, e' \quad \text{,,} \quad 2\tau + g' \end{array}\right| \delta v,$$

and also

$$\begin{aligned}\delta Q = & \frac{\rho}{\rho'^3}\left(-3\sin 2v - 2v'\right)\delta\rho \\ & + \frac{\rho^2}{\rho'^3}\left(-3\cos 2v - 2v'\right)\delta v \\ & + \rho^2\left[\left(-\tfrac{3}{2}\sin 2v - 2v'\right)\delta\frac{1}{\rho'^3} + (3\cos 2v - 2v')\frac{\delta v'}{\rho'^3}\right],\end{aligned}$$

where the terms containing $\delta\rho$ and δv are (see Annex 2)

$$\left.\begin{array}{l} - 3 \quad \sin 2\tau \\ - \tfrac{21}{2}\, e' \quad \text{,,} \quad 2\tau - g' \\ + \tfrac{3}{2}\, e' \quad \text{,,} \quad 2\tau + g' \end{array}\right| \delta\rho$$

$$\left.\begin{array}{ll} -3 & \cos 2\tau \\ -\frac{21}{2}e' & \text{,,}\ 2\tau - g' \\ +\frac{3}{2}e' & \text{,,}\ 2\tau + g' \end{array}\right| \delta v,$$

but the additional term $\frac{171}{4} m^2 e' \cos g'$ is ultimately added (see Annex 17) to the coefficient of δv.

Neglecting the terms in $\delta\rho$, δv, we have (see Annex 4)

$$\begin{array}{lll} \delta P = & \frac{3}{2}e' & t \\ & +\frac{3}{2} & t\cos g' \\ & -\frac{15}{2}e' & \text{,,}\ 2\tau \\ & +\frac{21}{4} & \text{,,}\ 2\tau - g' \\ & -\frac{3}{4} & \text{,,}\ 2\tau + g' \\ & +\frac{9}{2}e' & \text{,,}\ 2g' \\ & +\frac{51}{2}e' & \text{,,}\ 2\tau - 2g', \end{array}$$

and similarly (see Annex 5),

$$\begin{array}{lll} \delta Q = & \frac{15}{2}e' & t\sin 2\tau \\ & -\frac{21}{4} & \text{,,}\ 2\tau - g' \\ & +\frac{3}{4} & \text{,,}\ 2\tau + g' \\ & -\frac{51}{2}e' & \text{,,}\ 2\tau - 2g'. \end{array}$$

But in the foregoing expression of δP the terms belonging to the arguments g', $2g'$ give in $\delta\rho$, terms which rise by integration in δv; and in forming the expressions for δP, δQ, it is proper to take account of these terms. Attending only to the terms in question, we have

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = m^2 n^2 \delta P = \overbrace{\frac{3}{2}m^2 \quad t\cos g'}^{n^2\times}$$
$$+\frac{9}{2}m^2 e' \quad \text{,,}\ 2g'.$$

Now in general, if

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = n^2 \quad t\cos n\alpha t,$$

then

$$\delta\rho = \frac{2\alpha}{(1-\alpha^2)^2}\sin n\alpha t + \frac{1}{1-\alpha^2} \quad t\cos n\alpha t;$$

and hence the foregoing equation gives in $\delta\rho$ the terms

$$\begin{array}{lll} 3m^2 & \sin g' & +\frac{3}{2} \quad t\cos g' \\ +18m^2 e' & \text{,,}\ 2g' & +\frac{9}{2}m^2 e' \quad \text{,,}\ 2g'; \end{array}$$

or, neglecting the terms which contain m^3, the terms of $\delta\rho$ are

$$\begin{array}{ll} \tfrac{3}{2}m^2 & t\cos g' \\ +\tfrac{9}{2}m^2e' & \text{,,}\ 2g'. \end{array}$$

Substituting these terms in

$$\frac{d\delta v}{dt} = -2n\delta\rho,$$

we have

$$\frac{d\delta v}{dt} = \overbrace{\begin{array}{ll} -3\,m^2 & t\cos g' \\ -9\,m^2e' & \text{,,}\ 2g'; \end{array}}^{n\,\times}$$

and since, in general,

$$\int t\cos nat\,dt = \frac{1}{n^2a^2}\cos nat + \frac{1}{na}t\sin nat,$$

we obtain in δv the terms

$$\overbrace{\begin{array}{ll} -3 & \cos g' \\ -\tfrac{9}{4}e' & \text{,,}\ 2g' \end{array}}^{n^{-1}\,\times} \quad + \quad \overbrace{\begin{array}{ll} -3\,m & t\sin g' \\ -\tfrac{9}{2}me' & \text{,,}\ 2g', \end{array}}$$

or for the present purpose the terms

$$\overbrace{\begin{array}{ll} -3 & \cos g' \\ -\tfrac{9}{4}e' & \text{,,}\ 2g'; \end{array}}^{n^{-1}\,\times}$$

and these give in δP the additional terms (see Annex 6)

$$\overbrace{\begin{array}{ll} \tfrac{27}{2} & \sin 2\tau \\ +\tfrac{9}{2} & \text{,,}\ 2\tau - g' \\ +\tfrac{9}{2} & \text{,,}\ 2\tau + g' \\ +\tfrac{153}{8}e' & \text{,,}\ 2\tau - 2g' \\ +\tfrac{9}{8}e' & \text{,,}\ 2\tau + 2g', \end{array}}^{n^{-1}\,\times}$$

and in δQ the additional terms (see Annex 7)

$$\overbrace{\begin{array}{ll} \tfrac{27}{2} & \cos 2\tau \\ +\tfrac{9}{2} & \text{,,}\ 2\tau - g' \\ +\tfrac{9}{2} & \text{,,}\ 2\tau + g' \\ +\tfrac{153}{8}e' & \text{,,}\ 2\tau - 2g' \\ +\tfrac{9}{8}e' & \text{,,}\ 2\tau + 2g'. \end{array}}^{n^{-1}\,\times}$$

Combining the foregoing results, we have,

$$
\begin{array}{rlll}
\delta P = & \frac{3}{2}e' & t & + \overbrace{\qquad\qquad}^{n^{-1}\times} \\
& + \frac{3}{2} & t\cos g' & 0 \quad \sin g' \\
& -\frac{15}{2}e' & \text{,,}\ 2\tau & +\frac{27}{2}e' \quad \text{,,}\ 2\tau \\
& +\frac{21}{4} & \text{,,}\ 2\tau - g' & +\frac{9}{2} \quad \text{,,}\ 2\tau - g' \\
& -\frac{3}{4} & \text{,,}\ 2\tau + g' & +\frac{9}{2} \quad \text{,,}\ 2\tau + g' \\
& +\frac{9}{2}e' & \text{,,}\ 2g' & 0 \quad \text{,,}\ 2g' \\
& +\frac{51}{2}e' & \text{,,}\ 2\tau - 2g' & +\frac{153}{8}e' \quad \text{,,}\ 2\tau - 2g' \\
& 0 & \text{,,}\ 2\tau + 2g' & +\frac{9}{8}e' \quad \text{,,}\ 2\tau + 2g',
\end{array}
$$

and

$$
\begin{array}{rlll}
\delta Q = & 0 & t\sin g' & + \overbrace{0 \quad \cos g'}^{n^{-1}\times} \\
& +\frac{15}{2}e' & \text{,,}\ 2\tau & +\frac{27}{2}e' \quad \text{,,}\ 2\tau \\
& -\frac{21}{4} & \text{,,}\ 2\tau - g' & +\frac{9}{2} \quad \text{,,}\ 2\tau - g' \\
& +\frac{3}{4} & \text{,,}\ 2\tau + g' & +\frac{9}{2} \quad \text{,,}\ 2\tau + g' \\
& 0 & \text{,,}\ 2g' & 0 \quad \text{,,}\ 2g' \\
& -\frac{51}{2}e' & \text{,,}\ 2\tau - 2g' & +\frac{153}{8}e' \quad \text{,,}\ 2\tau - 2g' \\
& 0 & \text{,,}\ 2\tau + 2g' & +\frac{9}{8}e' \quad \text{,,}\ 2\tau + 2g';
\end{array}
$$

whence also (see Annex 8)

$$
\begin{array}{rlll}
n\int\delta Q dt = & 0 & t\cos g' & + \overbrace{0 \quad \sin g'}^{n^{-1}\times} \\
& -\frac{15}{4}e' & \text{,,}\ 2\tau & +\frac{69}{8}e' \quad \text{,,}\ 2\tau \\
& +\frac{21}{8} & \text{,,}\ 2\tau - g' & +\frac{15}{16} \quad \text{,,}\ 2\tau - g' \\
& -\frac{3}{8} & \text{,,}\ 2\tau + g' & +\frac{39}{16} \quad \text{,,}\ 2\tau + g' \\
& 0 & \text{,,}\ 2g' & 0 \quad \text{,,}\ 2g' \\
& +\frac{51}{4}e' & \text{,,}\ 2\tau - 2g' & +\frac{51}{16}e' \quad \text{,,}\ 2\tau - 2g' \\
& 0 & \text{,,}\ 2\tau + 2g' & +\frac{9}{16}e' \quad \text{,,}\ 2\tau + 2g'.
\end{array}
$$

The equation for $\delta\rho$ may be written,

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = m^2n^2\left(\delta P + 2n\int\delta Q dt\right),$$

and we have, (see Annex 9)

$$\delta P + 2n\int \delta Q dt = \begin{array}{lll} \frac{3}{2}\, e' & t & + \overbrace{\qquad\qquad}^{n^{-1}\times} \\ +\frac{3}{2} & t\cos g' & 0 \quad \sin g' \\ -15\, e' & \text{,,}\ 2\tau & +\frac{123}{4}\, e' \quad \text{,,}\ 2\tau \\ +\frac{21}{2} & \text{,,}\ 2\tau - g' & +\frac{51}{8} \quad \text{,,}\ 2\tau - g' \\ -\frac{3}{2} & \text{,,}\ 2\tau + g' & +\frac{75}{8} \quad \text{,,}\ 2\tau + g' \\ +\frac{9}{2}\, e' & \text{,,}\ 2g' & 0 \quad \text{,,}\ 2g' \\ +51\, e' & \text{,,}\ 2\tau - 2g' & +\frac{51}{2}\, e' \quad \text{,,}\ 2\tau - 2g' \\ 0 & \text{,,}\ 2\tau + 2g' & +\frac{9}{4}\, e' \quad \text{,,}\ 2\tau + 2g'. \end{array}$$

Hence observing that a term $n^2 t \cos nat$ in $\frac{d^2\delta\rho}{\delta t^2} + n^2\delta\rho$, gives in $\delta\rho$ the terms

$$\frac{1}{1-\alpha^2}\, t\cos nat \qquad + \frac{2\alpha}{(1-\alpha^2)^2}\,\frac{1}{n}\,\sin nat,$$

and a term $n \sin nat$ in $\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho$, gives in $\delta\rho$ the term

$$\frac{1}{1-\alpha^2}\,\frac{1}{n}\,\sin n\alpha t,$$

we have (see Annex 10, but restoring the factor f'),

$\delta\rho = f'$ into as follows, viz.

$$\begin{array}{lll} \frac{3}{2}\, m^2 e' & t & + \overbrace{\qquad\qquad}^{n^{-1}\times} \\ +\frac{3}{2}\, m^2 & t\cos g' & 3\, m^3 \quad \sin g' \\ +5\, m^2 e' & \text{,,}\ 2\tau & -\frac{203}{12}\, m^2 e' \quad \text{,,}\ 2\tau \\ -\frac{7}{2}\, m^2 & \text{,,}\ 2\tau - g' & +\frac{61}{24}\, m^2 \quad \text{,,}\ 2\tau - g' \\ +\frac{1}{2}\, m^2 & \text{,,}\ 2\tau + g' & -\frac{91}{24}\, m^2 \quad \text{,,}\ 2\tau + g' \\ +\frac{9}{2}\, m^2 e' & \text{,,}\ 2g' & -18\, m^3 e' \quad \text{,,}\ 2g' \\ -17\, m^2 e' & \text{,,}\ 2\tau - 2g' & +\frac{85}{6}\, m^2 e' \quad \text{,,}\ 2\tau - 2g' \\ 0 & \text{,,}\ 2\tau + 2g' & -\frac{3}{4}\, m^2 e' \quad \text{,,}\ 2\tau + 2g'. \end{array}$$

The first column, containing the term in t and the terms $t \cos$ arg., shows that the constant term of ρ, and the terms involving the cosines of the same arguments, as obtained without attending to the variation of the solar eccentricity, are correct as

regards the first power of t, when for the constant eccentricity e' we substitute $e'+f't$. In fact, the above-mentioned expression (correct to e'^2) of $\frac{1}{\rho}$ gives

$$\begin{array}{lll}
\rho = 1 - \frac{1}{6}m^2 + \frac{3}{4}m^2e'^2 & & \\
\qquad\qquad + \frac{3}{2}m^2e'^2 & & \cos g' \\
\qquad\qquad - \quad m^2 + \frac{5}{2}m^2e'^2 & & \text{,,}\ 2\tau \\
\qquad\qquad - \frac{7}{2}m^2e' & & \text{,,}\ 2\tau - g' \\
\qquad\qquad + \frac{1}{2}m^2e' & & \text{,,}\ 2\tau + g' \\
\qquad\qquad + \frac{9}{4}m^2e'^2 & & \text{,,}\ 2g' \\
\qquad\qquad - \frac{17}{2}m^2e'^2 & & \text{,,}\ 2\tau - 2g' \\
\qquad\qquad 0 & & \text{,,}\ 2\tau + 2g',
\end{array}$$

and putting therein $e'+f't$ in the place of e', we have the first column of the foregoing expression of $\delta\rho$.

The second column, involving sin arg., contains the new periodic terms considered in Prof. Adams' Memoir of 1853, and the coefficients for the arguments g', 2τ, $2\tau - g'$, $2\tau + g'$, agree with his values; observing that his terms belong to $\delta\frac{1}{\rho} = -\frac{\delta\rho}{\rho^2} = -\delta\rho$, so that the signs are reversed; those for the remaining arguments $2g'$, $2\tau - 2g'$, $2\tau + 2g'$, are not given by him.

The equation for δv may be written,

$$\frac{d\delta v}{dt} = -2n\delta\rho + m^2n^2\int\delta Q dt,$$

and we have (see Annex 11)

$$\begin{array}{lll}
\frac{d\delta v}{dt} = \overbrace{-\ 3\ m^2e'\ \ t}^{n\ \times} & & \\
\qquad - \ 3\ m^2 \quad t\cos g' & & -\ 6\ \ m^3 \quad \sin g \\
\qquad + \frac{55}{4}m^2e' & \text{,,}\ 2\tau & + \frac{1019}{24}m^2e' \quad \text{,,}\ 2\tau \\
\qquad + \frac{17}{8}m^2 & \text{,,}\ 2\tau - g' & - \frac{199}{48}m^2 \quad \text{,,}\ 2\tau - g' \\
\qquad - \frac{11}{8}m^2 & \text{,,}\ 2\tau + g' & + \frac{481}{48}m^2 \quad \text{,,}\ 2\tau + g' \\
\qquad - \ 9\ m^2e' & \text{,,}\ 2g' & -\ 36\ \ m^3e' \quad \text{,,}\ 2g' \\
\qquad + \frac{187}{8}m^2e' & \text{,,}\ 2\tau - 2g' & - \frac{1207}{48}m^2e' \quad \text{,,}\ 2\tau - 2g' \\
\qquad 0 & \text{,,}\ 2\tau + 2g' & + \frac{33}{16}m^2e' \quad \text{,,}\ 2\tau + 2g',
\end{array}$$

whence, integrating by the formulæ

$$\int t\cos nat\ \ dt = \frac{1}{n\alpha}t\sin nat + \frac{1}{n^2\alpha^2}\cos nat,$$

$$\int \sin nat\ \ dt = \qquad -\frac{1}{n\alpha}\cos nat,$$

we have (see Annex 12, but restoring the factor f'),

$\delta v = f'$ into as follows, viz.

		$-\frac{3}{2}\, m^2 n e' t^2$	
		$n^{-1} \times$	
$-3\, m$	$t \sin g'$	$+\overbrace{-3+6\,m^2}$	$\cos g'$
$-\frac{55}{8}\, m^3 e'$	„ 2τ	$-\frac{74}{3}\, m^2 e'$	„ 2τ
$+\frac{77}{16}\, m^2$	„ $2\tau - g'$	$+\frac{215}{48}\, m^2$	„ $2\tau - g'$
$-\frac{11}{16}\, m^2$	„ $2\tau + g'$	$-\frac{257}{48}\, m^2$	„ $2\tau + g'$
$-\frac{9}{2}\, m e'$	„ $2g'$	$(-\frac{9}{4} + 18\, m^2)\, e'$	„ $2g'$
$+\frac{187}{8}\, m^3 e'$	„ $2\tau - 2g'$	$+\frac{2239}{90}\, m^3 e'$	„ $2\tau - 2g'$
0	„ $2\tau + 2g'$	$-\frac{33}{32}\, m^3 e'$	„ $2\tau + 2g'$.

The first column, containing $t \sin$ arg., may be obtained from the before-mentioned expression (accurate to e'^2) of v, by substituting therein $e' + f't$ in the place of e'.

The term $-\frac{3}{2}\, m^2 n e' f' t^2$; or, as it may be written, $-\frac{3}{2}\, m^2 \int n\,[(e' + f't)^2 - e'^2]\, dt$, is the first term of the acceleration; the other terms of the second column are the new periodic terms in δv, considered by Prof. Adams; the coefficients for the arguments g', 2τ, $2\tau - g'$, $2\tau + g'$, agreeing with his values, but those for the remaining arguments $2g'$, $2\tau - 2g'$, $2\tau + 2g'$ not being given by him.

III.

Proceeding now to the calculation of the term in m^4 of the acceleration, we have,

$$\frac{d\delta v}{dt} = -\frac{2}{\rho}\frac{dv}{dt}\,\delta\rho + \frac{m^2 n^2}{\rho^2}\left(C + \int \delta Q dt\right),$$

where the non-periodic part of $\delta\rho$ is of the form,

$$\delta\rho = (\tfrac{3}{2}\, m^2 + \square m^4)\, e' f' t,$$

and it is in the first place necessary to find the value of the numerical coefficient $\square$, in fact to calculate the secular part of $\delta\rho$ as far as m^4. Reverting to the equation

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = \left(n^2 + \frac{2n^2}{\rho^3} - 3\left(\frac{dv}{dt}\right)^2\right)\delta\rho + m^2 n^2\left(\delta P + \frac{2}{\rho}\frac{dv}{dt}\left(C + \int \delta Q dt\right)\right),$$

and as before omitting in the process the factor f':

The part $\left(n^2 + \frac{2n^2}{\rho^3} - 3\left(\frac{dv}{dt}\right)^2\right)\delta\rho$ contains (see Annex 13), a term

$$= \tfrac{381}{8}\, m^4 n^2 e' t.$$

The part of $m^2n^2\delta P$, which involves $\delta\rho$, contains (see Annex 14) a term

$$= -\tfrac{495}{32}\, m^4n^2e't.$$

The part of $m^2n^2\delta P$, which depends on δv, contains a term

$$= -\tfrac{15}{4}\, m^4n^2e't.$$

The part of $m^2n^2\delta P$, depending on $\delta\rho'$ and $\delta v'$, is found (see Annex 18) to contain, besides the term $\frac{3}{2}\, m^2ne't$ in m^2 already obtained, a new term

$$= -\tfrac{647}{32}\, m^4n^2e't.$$

And finally the part $m^2n^2 \dfrac{2}{\rho}\dfrac{dv}{dt}\left(C+\int\delta Q dt\right)$ is found (see Annex 19) to contain a term

$$(2.\tfrac{-1455}{32} + \tfrac{675}{32} =) - \tfrac{2235}{32}\, m^4n^2e't,$$

where the component coefficient $-\frac{1455}{32}$, which arises from the new periodic terms of $\delta\rho$ and δv is separately calculated (see Annex 21).

Hence $\dfrac{d^2\delta\rho}{dt^2} + n^2\delta\rho$ contains the term

$$(\square = \tfrac{381}{8} - \tfrac{495}{32} - \tfrac{15}{4} - \tfrac{647}{32} - \tfrac{2235}{32} =) - \tfrac{1973}{32}\, m^4n^2e't,$$

and this gives in $\delta\rho$ the term

$$-\tfrac{1973}{32}\, m^4e't,$$

so that, restoring the term in m^2, and the common factor f', the complete secular term of $\delta\rho$ is

$$\delta\rho = (\tfrac{3}{2}\, m^2 - \tfrac{1973}{32}\, m^4)\, e'f't,$$

which, as will be shown, Art. IV., agrees with Prof. Adams' result.

Resuming now the equation,

$$\frac{d\delta v}{dt} = -\frac{2}{\rho}\frac{dv}{dt}\delta\rho + \frac{m^2n^2}{\rho^2}\left(C+\int\delta Q dt\right),$$

the part $-\dfrac{2}{\rho}\dfrac{dv}{dt}\delta\rho$ contains (see Annex 22) the term

$$(\tfrac{1973}{16} + \tfrac{275}{8} =)\, \tfrac{2523}{16}\, m^4ne't,$$

and the part $\dfrac{m^2n^2}{\rho}\left(C+\int\delta Q dt\right)$ contains (see Annex 23) the term

$$(\tfrac{45}{8} - \tfrac{1455}{32} =) - \tfrac{1275}{32}\, m^4ne't,$$

so that we have in $\dfrac{d\delta v}{dt}$ the term

$$(\tfrac{2523}{16} - \tfrac{1275}{32} =)\, \tfrac{3771}{32}\, m^4ne't,$$

giving in δv the term

$$\tfrac{3771}{64}\, m^4 n e' t^2,$$

or, restoring the term in m^2 and the common factor f', the complete secular term of δv is

$$\delta v = (-\tfrac{3}{2}\, m^2 + \tfrac{3771}{64}\, m^4)\, n e' f' t^2,$$

which agrees with the value obtained by Prof. Adams. It is right to remark that the m of Prof. Adams is different from that of the present Memoir; we have in fact,

$$m\ (\text{Adams}) = m\,\{1 + (\tfrac{3}{2}\, m^2 - \tfrac{3771}{64}\, m^4)\, 2e'f't\}$$

in the notation of the present Memoir; but as f'^2 is throughout neglected, we may in the foregoing expression of the secular part of δv substitute the m of Prof. Adams. And then if the term be written in the form

$$\delta v = (-\tfrac{3}{2}\, m^2 + \tfrac{3771}{64}\, m^2) \int [(e' + f't)^2 - e'^2]\, n dt,$$

the two results are seen to agree together. But as regards the before-mentioned secular term,

$$\delta \rho = (\tfrac{3}{2}\, m^2 - \tfrac{1972}{32}\, m^4)\, e' f' t,$$

the identification is less easy, and I shall consider it in the following article.

IV.

It will be convenient to write M, N, A, E', in place of the m, n, a, e', of the foregoing part of the present Memoir, and to now use m, n, e', in the significations in which they are employed by Prof. Adams; E' (the constant part of the solar eccentricity) is his E', and his e' is $E' + f't$. As to his symbols a, $a_{,}$, these, I think, ought to have been represented, and I shall here represent them by a, $\mathrm{a}_{,}$[1]. And I take a such that $n^2a^3 =$ Sum of the masses of the Earth and Moon; or, taking this to be unity, we have $N^2A^3 = 1$, $n^2a^3 = 1$.

The formulæ of Prof. Adams' memoir, which it will be necessary to make use of, may be written

$$\begin{aligned}
\frac{\mathrm{a}}{r} = 1 &- \tfrac{11}{8}\, m^2 - \tfrac{201}{16}\, m^4 e'^2 \\
&- \tfrac{3}{2}\, m^2 e' && \cos g' \\
&+ \quad m^2 - \tfrac{5}{2}\, m^2 e'^2 && ,, \ 2\tau \\
&+ \tfrac{7}{2}\, m^2 e' && ,, \ 2\tau - g' \\
&- \tfrac{1}{2}\, m^2 e' && ,, \ 2r + g'
\end{aligned}$$

[1] Plana, in his Lunar Theory, uses the three letters a, $\mathrm{a}_{,}$, a; his a and n being such that $n^2a^3 =$ Sum of the masses of the Earth and Moon. There is an obvious inconvenience in writing a, $a_{,}$, in the place of his a, $\mathrm{a}_{,}$.

the sine terms being disregarded,

$$n = N\,\{1 + (-\tfrac{3}{2}\,m^2 + \tfrac{3771}{64}\,m^4)\,2E''f''t\},$$

(whence also

$$m = M\,\{1 + (\ \ \tfrac{3}{2}\,m^2 - \tfrac{3771}{64}\,m^4)\,2E'f't\}$$

since $m = \dfrac{n'}{n}$, $M = \dfrac{N'}{N}$).

$$\frac{1}{n} = \mathrm{a}_{,}^{\frac{3}{2}}\,\{1 + \ \ m^2 - \tfrac{197}{64}\,m^4 + (\ \ \tfrac{3}{2}\,m^2 - \tfrac{3867}{64}\,m^4)\,e'^2\},$$

$$1 = \frac{\mathrm{a}}{\mathrm{a}_{,}}\,\{1 - \tfrac{1}{2}\,m^2 + \tfrac{13}{4}\,m^4 + (-\tfrac{3}{4}\,m^2 + \tfrac{3201}{64}\,m^4)\,e'^2\},$$

which formulæ $\left(\text{observing that } \dfrac{1}{\rho + \delta\rho} = \dfrac{A}{r}\right)$ lead to the value of $\delta\rho$, and we should obtain for the secular part the foregoing expression, which will now be

$$\delta\rho = (\tfrac{3}{2}\,M^2 - \tfrac{1973}{32}\,M^4)\,E'f't.$$

We in fact have

$$\frac{1}{n} = a^{\frac{3}{2}} = \mathrm{a}_{,}^{\frac{3}{2}}\,\{1 + \ \ m^2 - \tfrac{197}{64}\,m^4 + (\tfrac{3}{2}\,m^2 - \tfrac{3867}{64}\,m^4)\,e'^2\},$$

and thence

$$\begin{aligned} a &= \mathrm{a}_{,}\,\{1 + \tfrac{2}{3}\,m^2 - \tfrac{197}{96}\,m^4 + (\ \ m^2 - \tfrac{3867}{96}\,m^4)\,e'^2 \\ &\qquad\qquad - \tfrac{1}{9}\,(m^4 \qquad\qquad + 3\,m^4\,e'^2)\} \\ &= \mathrm{a}_{,}\,\{1 + \tfrac{2}{3}\,m^2 - \tfrac{623}{288}\,m^4 + (\ \ m^2 - \tfrac{3899}{96}\,m^4)\,e'^2\}\,; \end{aligned}$$

but

$$1 = \frac{\mathrm{a}}{\mathrm{a}_{,}}\,\{1 - \tfrac{1}{2}\,m^2 + \tfrac{13}{4}\,m^4 + (-\tfrac{3}{4}\,m^2 + \tfrac{3201}{64}\,m^4)\,e'^2\},$$

and therefore

$$\begin{aligned} a = \mathrm{a}\,\{1 &+ \tfrac{2}{3}\,m^2 - \tfrac{623}{288}\,m^4 + (\ \ m^2 - \tfrac{3899}{96}\,m^4)\,e'^2 \\ &- \tfrac{1}{2}\,m^2 + \tfrac{13}{4}\,m^4 \qquad\qquad - \tfrac{1}{2}\,m^4\,e'^2 \\ &\qquad - \tfrac{1}{3}\,m^4 + (-\tfrac{3}{4}\,m^2 + \tfrac{3201}{64}\,m^4)\,e'^2 \\ &\qquad\qquad\qquad\qquad - \tfrac{1}{2}\,m^4\,e'^2\} \\ = \mathrm{a}\,\{1 &+ \tfrac{1}{6}\,m^2 + \tfrac{217}{288}\,m^4 + (\ \ \tfrac{1}{4}\,m^2 + \tfrac{1613}{192}\,m^4)\,e'^2\}\,; \end{aligned}$$

and hence, since for the non-periodic part

$$\frac{\mathrm{a}}{r} = 1 - \tfrac{11}{8}\,m^2 - \tfrac{201}{16}\,m^4 e'^2,$$

we find

$$\begin{aligned}\frac{a}{r} &= 1 + \tfrac{1}{6} m^2 + \tfrac{217}{288} m^4 + (\tfrac{1}{4} m^2 + \tfrac{1613}{192} m^4) e'^2 \\ &\qquad - \tfrac{11}{8} m^4 \qquad\qquad - \tfrac{201}{16} m^4 e'^2 \\ &= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) e'^2 \\ &= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) 2E'f't.\end{aligned}$$

But

$$\frac{A}{a} = \left(\frac{n}{N}\right)^{\frac{2}{3}} = 1 + (-m^2 + \tfrac{3771}{96} m^4) 2E'f't;$$

and therefore

$$\begin{aligned}\frac{A}{r} &= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) 2E'f't \\ &\qquad + (- m^2 + \tfrac{3771}{96} m^4) 2E'f't \\ &\qquad - \tfrac{1}{6} m^4 . 2E'f't \\ &= 1 + \tfrac{1}{6} m^4 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (-\tfrac{3}{4} m^2 + \tfrac{2237}{64} m^4) 2E'f't.\end{aligned}$$

But we have

$$m = M \{1 + (\tfrac{3}{2} M^2 - \tfrac{3771}{64} M^4) 2E'f't\};$$

and thence in the foregoing expression

$$\begin{aligned} m^2 &= M^2 + 3M^4 . 2E'f't, \\ m^4 &= M^4; \end{aligned}$$

and therefore

$$\begin{aligned}\frac{A}{r} &= 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2 + (-\tfrac{3}{4} M^2 + \tfrac{2237}{64} M^4) 2E'f't \\ &\qquad + \tfrac{1}{2} M^4 . 2E'f't \\ &= 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2 + (-\tfrac{3}{4} M^2 + \tfrac{2269}{64} M^4) 2E'f't;\end{aligned}$$

and observing that in the periodic terms we may write A, in the place of a, and neglect the sine terms, we have

$$\begin{aligned}\frac{A}{r} = \frac{1}{\rho + \delta\rho} &= 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2 \\ &\qquad + (-\tfrac{3}{4} M^2 + \tfrac{2269}{64} M^4) 2E'f't \\ &\quad - \tfrac{3}{2} m^2 e' \qquad \cos g' \\ &\quad + m^2 (1 - \tfrac{5}{2} e'^2) \quad \text{,,} \quad 2\tau \\ &\quad + \tfrac{7}{2} m^2 e' \qquad \text{,,} \quad 2\tau - g' \\ &\quad - \tfrac{1}{2} m^2 e' \qquad \text{,,} \quad 2\tau + g',\end{aligned}$$

say $\frac{1}{\rho+\delta\rho}=1+X$; and thence $\rho+\delta\rho=\frac{1}{1+X}=1-X+X^2$, the non-periodic part whereof is

$$\begin{aligned}
1-\tfrac{1}{6}M^2+\tfrac{179}{128}\ M^4&+(-\tfrac{1}{4}M^2+\tfrac{179}{192}M^4)\,E'^2+(\tfrac{3}{4}M^2-\tfrac{2269}{64}M^4)\,2E'f't\\
+\tfrac{1}{36}\ \ M^4&\qquad\qquad+\tfrac{1}{12}M^4\ E'^2\qquad\qquad+\tfrac{1}{4}\ M^4.2E'f't\\
+\tfrac{1}{2}\cdot\tfrac{9}{4}\ m^4e'^2&\\
+\tfrac{1}{2}\ \ \ m^4(1-5e'^2)&\\
+\tfrac{1}{2}\cdot\tfrac{49}{4}m^4e'^2&\\
+\tfrac{1}{2}\cdot\tfrac{1}{4}\ m^4e'^2,&
\end{aligned}$$

where the terms in m^4 and $m^4e'^2$ are

$$=\tfrac{1}{2}m^4+(\tfrac{9}{8}-\tfrac{5}{2}+\tfrac{49}{8}+\tfrac{1}{8}=)\,\tfrac{39}{8}m^4e'^2,$$

which are

$$=\tfrac{1}{2}M^4\qquad\qquad+\tfrac{39}{8}M^4E'^2\qquad\qquad+\tfrac{39}{8}M^4.2E'f't;$$

so that the foregoing expression of the non-periodic part of $\rho+\delta\rho$ is

$$\begin{aligned}
=1\quad&-\tfrac{1}{6}M^2+(\ \tfrac{179}{128}+\tfrac{1}{36}+\tfrac{1}{2}=)\quad\tfrac{2219}{1152}M^4\\
&+(-\tfrac{1}{4}M^2+(\ \tfrac{179}{192}+\tfrac{1}{12}+\tfrac{39}{8}=)\quad\tfrac{1131}{192}M^4)\,E'^2\\
&+(\ \tfrac{3}{4}M^2+(-\tfrac{2269}{64}-\tfrac{1}{4}+\tfrac{39}{8}=)-\tfrac{1973}{64}M^4)\,2E'f't;
\end{aligned}$$

or the secular term of $\delta\rho$ is

$$=(\tfrac{3}{4}M^2-\tfrac{1973}{64}M^4)\,2E'f't,$$

which is the required formula.

V.

It is interesting to see how the coefficient $\frac{3771}{64}$ is made up. In Prof. Adams' Memoir we have

$$\begin{aligned}
\tfrac{3771}{64}=&\\
=&-\tfrac{3}{2}-\tfrac{3}{4}&&(=-\tfrac{9}{4}\ )\\
&+\tfrac{16}{4}+\tfrac{135}{64}-\tfrac{117}{8}+\tfrac{495}{128}-\tfrac{285}{16}-\tfrac{147}{16}-\tfrac{3}{16}-\tfrac{1893}{256}-\tfrac{27}{256}&&(=-\tfrac{2391}{64})\\
&+\tfrac{9}{2}-\tfrac{27}{4}-\tfrac{45}{2}-\tfrac{45}{2}-\tfrac{15}{2}-\tfrac{495}{32}+\tfrac{285}{4}-\tfrac{147}{4}-\tfrac{3}{4}+\tfrac{441}{8}+\tfrac{441}{8}+\tfrac{9}{8}+\tfrac{9}{8}&&(=+\tfrac{3153}{32}),
\end{aligned}$$

where it may be remarked that the terms

$$\tfrac{495}{128}-\tfrac{285}{16}=(-\quad\tfrac{1785}{128})$$

and

$$-\tfrac{495}{32}+\tfrac{285}{4}=(+4\,.\,\tfrac{1785}{128})$$

make together $3\,.\,\frac{1785}{128}=\frac{5355}{128}$, and that it is in fact by the addition of these terms that Plana's coefficient $\frac{2187}{128}$ is changed into $\frac{3771}{64}$.

But in the present Memoir the coefficient $\frac{3771}{64}$ is obtained by means of an entirely different set of component numbers, viz. we have

$$\tfrac{3771}{64} = -\tfrac{381}{8} + \tfrac{495}{32} + \tfrac{15}{4} + \tfrac{647}{32} + \tfrac{1455}{16} - \tfrac{675}{32} + \tfrac{275}{16} + \tfrac{45}{16} - \tfrac{1455}{64}.$$

I had imagined, from the way in which the numbers $\frac{1455}{16} - \frac{1455}{64}$ presented themselves, that, if they were omitted, Plana's value $\frac{2187}{128}$ would have been obtained; but the result shows that this is not so.

As just deduced from the formula of Prof. Adams, the number $\frac{1973}{64}$ is obtained as follows, viz.

$$\begin{aligned}\tfrac{1973}{64} &= (-\tfrac{3771}{96} - 1) - \tfrac{1}{3} - \tfrac{1}{2} + (\tfrac{3153}{64} + \tfrac{3}{4}) - \tfrac{1}{2} - \tfrac{201}{16} + \tfrac{3771}{96} - \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{39}{8} \\ &= -1 - \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{4} \qquad (= -1) \\ &\quad + \tfrac{3153}{64} - \tfrac{201}{16} - \tfrac{39}{8},\end{aligned}$$

where *ut suprà*

$$\tfrac{3153}{64} = \tfrac{1}{2}\left\{\tfrac{9}{2} - \tfrac{27}{4} - \tfrac{45}{2} - \tfrac{45}{2} - \tfrac{15}{2} - \tfrac{495}{32} + \tfrac{285}{4} - \tfrac{147}{4} - \tfrac{3}{4} + \tfrac{441}{8} + \tfrac{441}{8} + \tfrac{9}{8} + \tfrac{9}{8}\right\};$$

$-\frac{201}{16}$ is a number occurring in his Memoir, and which is in effect obtained irrespectively of the new periodic terms, and $-\frac{39}{8}$ is a number obtained as above, irrespectively of the new periodic terms. According to the method of the present Memoir, the number $\frac{1973}{64}$ was obtained in the form

$$\tfrac{1973}{64} = -\tfrac{1}{2}\left(\tfrac{381}{8} - \tfrac{495}{32} - \tfrac{15}{4} - \tfrac{647}{32} - \tfrac{1455}{16} + \tfrac{675}{32}\right).$$

VI.

If the investigation were pursued further, a question would arise as to the proper form to be given to the arguments; for in these, $nt + \epsilon$ seems to stand in the place of v, the value whereof is

$$v = nt + \epsilon - (\tfrac{3}{2} m^2 - \tfrac{3771}{64} m^4)\, ne'f't^2,$$

say $v = nt + \epsilon + kne'f't^2$, and it might be considered that in the arguments $nt + \epsilon$ should be changed into $nt + \epsilon + kne'f't^2$, or, what is the same thing, that τ should be changed into $\tau + kne'f't^2$, but that g' should remain unaltered (this assumes that there is not in the Sun's longitude any term corresponding to the acceleration). The arguments, instead of being of the simple form kt, would thus be of the form $kt + k_2 f't^2$. But this would not only increase the difficulty of integration, but would be inconsistent with the general plan of the solution; and it would seem to be the proper course to imagine the cosine or sine of such an argument to be developed $\left(\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} kt + k_2 f't^2 = \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} kt \mp k_2 f't^2 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} kt\right)$ in such manner as to bring the secular part of the argument outside the cos or sin; this is, in fact, the form which the solution takes when the arguments are left throughout in their original form, for the terms of the form $f't^2 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ arg. would present themselves in the subsequent approximations. But I shall not at present further examine the question.

Annexes containing the Details of the Calculation.

Annex 1.

Calculation of part of δP.

$$\delta P = \frac{1}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\rho$$

$$+ \frac{\rho}{\rho'^3}\left(-3\sin 2v - 2v'\right)\delta v.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$\frac{1}{2} + \frac{3}{2}\cos 2v - 2v' =$			$\frac{1}{\rho'^3} =$	
$\frac{1}{2}$			1	
$+\frac{3}{2}$	$\cos$	2τ	$+3\,e'$	$\cos g'$.
$+3\,e'$	„	$2\tau - g'$		
$-3\,e'$	„	$2\tau + g'$		

Product is =

$$\begin{array}{rlll}
 & \frac{1}{2} & & +\frac{3}{2}e' \quad \cos g' \\
 & +\frac{3}{2} & \cos\ 2\tau & +\frac{9}{2}e'\left(\frac{1}{2}\cos 2\tau - g' \quad +\frac{1}{2}\cos 2\tau + g'\right) \\
 & +3\,e' & „\ \ 2\tau - g' & \\
 & -3\,e' & „\ \ 2\tau + g' & \\
= & \frac{1}{2} & & \\
 & +\frac{3}{2}e' & \cos\ g' & \\
 & +\frac{3}{2} & „\ \ 2\tau & \\
(+3 + \frac{9}{4} =) & +\frac{21}{4}e' & „\ \ 2\tau - g' & \\
(-3 + \frac{9}{4} =) & -\frac{3}{4}e' & „\ \ 2\tau + g', &
\end{array}$$

which is the coefficient of $\delta\rho$.

And

$-3\sin 2v - 2v' =$			$\frac{1}{\rho'^3} =$	
-3	$\sin$	2τ	1	
$-6\,e'$	„	$2\tau - g'$	$+3\,e'$	$\cos g'$.
$+6\,e'$	„	$2\tau + g'$		

Product is =

$$
\begin{array}{lll}
-3 & \sin 2\tau & -9\,e'\,(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g') \\
-6\,e' & \text{,,}\ 2\tau - g' & \\
+6\,e' & \text{,,}\ 2\tau + g' & \\
= & & -3 \quad \sin 2\tau \\
 & (-6 - \tfrac{9}{2} =) & -\tfrac{21}{2}\,e' \quad \text{,,}\ 2\tau - g' \\
 & (+6 - \tfrac{9}{2} =) & +\tfrac{3}{2}\,e' \quad \text{,,}\ 2\tau + g';
\end{array}
$$

or, since $\rho = 1$, this is the coefficient of δv.

Annex 2.

Calculation of part of δQ.

$$
\delta Q = \frac{\rho}{\rho'^3}(-3\sin 2v - 2v')\,\delta\rho
$$
$$
+ \frac{\rho^2}{\rho'^3}(-3\cos 2v - 2v')\,\delta v.
$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$$
\begin{array}{lll|l}
-3\sin 2v - 2v' = & & & \dfrac{1}{\rho'^3} = \\
-3 & \sin 2\tau & & 1 \\
-6\,e' & \text{,,}\ 2\tau - g' & & +3\,e' \quad \cos g'. \\
-6\,e' & \text{,,}\ 2\tau + g' & &
\end{array}
$$

Product is =

$$
\begin{array}{lll}
-3 & \sin 2\tau & -9e'\,(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g') \\
-6\,e' & \text{,,}\ 2\tau - g' & \\
-6\,e' & \text{,,}\ 2\tau + g' & \\
= & & -3 \quad \sin 2\tau \\
 & (-6 - \tfrac{9}{2} =) & -\tfrac{21}{2}\,e' \quad \text{,,}\ 2\tau - g' \\
 & (+6 - \tfrac{9}{2} =) & +\tfrac{3}{2}\,e' \quad \text{,,}\ 2\tau + g'
\end{array}
$$

or, since $\rho = 1$, this is the coefficient of $\delta\rho$.

$$
\begin{array}{lll|l}
-3\cos 2v - 2v' = & & & \dfrac{1}{\rho'^3} = \\
-3 & \cos 2\tau & & 1 \\
-6\,e' & \text{,,}\ 2\tau - g & & +3\,e' \quad \cos g'. \\
+6\,e' & \text{,,}\ 2\tau + g & &
\end{array}
$$

Product is =

$$
\begin{array}{llll}
-\ 3 & \cos\ 2\tau & & -9e'\left(\frac{1}{2}\cos 2\tau - g' + \frac{1}{2}\cos 2\tau + g'\right) \\
-\ 6\,e' & \quad ,, \quad 2\tau - g' & & \\
+\ 6\,e' & \quad ,, \quad 2\tau + g' & & \\
= & & -\ 3 & \cos\ 2\tau \\
 & (-6 - \frac{9}{2} =) & -\ \frac{21}{2}\,e' & \quad ,, \quad 2\tau - g' \\
 & (+6 - \frac{9}{2} =) & +\ \frac{3}{2}\,e' & \quad ,, \quad 2\tau + g'\,;
\end{array}
$$

or, since $\rho^2 = 1$, this is the coefficient of δv.

Annex 3.

Calculation of $\begin{matrix}\cos\\ \sin\end{matrix}\, 2v - 2v'$.

$$
\begin{aligned}
v - \ v' &= \ \tau - 2e' \sin\ g' \\
2v - 2v' &= 2\tau - \ e' \sin\ g'
\end{aligned}
$$

$$
\begin{aligned}
\cos 2v - 2v' &= \qquad \cos\ 2\tau \\
&\qquad + \sin\ 2\tau\,.\,4e' \sin\ g' \\
&= \qquad \cos\ 2\tau \\
&\qquad + 2e' \quad ,, \quad 2\tau - g' \\
&\qquad - 2e' \quad ,, \quad 2\tau \quad g' \\
\sin 2v - 2v' &= \qquad \sin\ 2\tau \\
&\qquad - \quad \cos\ 2\tau\,.\,4e' \sin\ g' \\
&= \qquad \sin\ 2\tau \\
&\qquad + 2e' \quad ,, \quad 2\tau - g' \\
&\qquad - 2e' \quad ,, \quad 2\tau + g'.
\end{aligned}
$$

The expressions are calculated (*post*, Annex 16) as far as m^2.

Annex 4.

Calculation of a part of δP.

$$\delta P = \rho\left\{\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\frac{1}{\rho^3} + (3\sin 2v - 2v')\frac{\delta v'}{\rho^3}\right\}.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$$\frac{1}{2}+\frac{3}{2}\cos 2v-2v' = \qquad\qquad \delta\frac{1}{\rho'^3} =$$

$$\begin{array}{lll} & \frac{1}{2} & \\ + & \frac{3}{2} & \cos\ 2\tau \\ + & 3\,e' & \text{,,}\ \ 2\tau-g' \\ - & 3\,e' & \text{,,}\ \ 2\tau+g' \end{array} \qquad\qquad \begin{array}{lll} & 3\,e' & t \\ + & 3 & t\cos g' \\ + & 9\,e' & \text{,,}\ \ 2g'. \end{array}$$

Product is =

$$\begin{array}{lll} & \frac{3}{2}\,e' & t \\ + & \frac{9}{2}\,e' & t\cos 2\tau \\ + & \frac{3}{2} & \text{,,}\ \ g' \\ + & \frac{9}{2} & (\frac{1}{2}\,t\cos 2\tau-g' \quad + \frac{1}{2}\,t\cos 2\tau+g'\ ) \\ + & 9\,e' & (\frac{1}{2}\ \ \text{,,}\ \ 2\tau-2g' \quad + \frac{1}{2}\ \ \text{,,}\ \ 2\tau\ \ ) \\ - & 9\,e' & (\frac{1}{2}\ \ \text{,,}\ \ 2\tau \quad + \frac{1}{2}\ \ \text{,,}\ \ 2\tau+2g') \\ + & \frac{9}{2}\,e' & t\cos 2g' \\ + & \frac{27}{2}\,e' & (\frac{1}{2}\,t\cos 2\tau-2g' \quad + \frac{1}{2}\,t\cos 2\tau+2g'), \end{array}$$

which is =

$$\begin{array}{rlll} & & \frac{3}{2}\,e' & t \\ & + & \frac{3}{2} & t\cos g' \\ (\frac{9}{2}+\frac{9}{2}-\frac{9}{2}=) & + & \frac{9}{2}\,e' & \text{,,}\ \ 2\tau \\ & + & \frac{9}{4} & \text{,,}\ \ 2\tau-g' \\ & + & \frac{9}{4} & \text{,,}\ \ 2\tau+g' \\ & + & \frac{9}{2}\,e' & \text{,,}\ \ 2g' \\ (\ \ \frac{9}{2}+\frac{27}{4}=) & + & \frac{45}{4}\,e' & \text{,,}\ \ 2\tau-2g' \\ (-\frac{9}{2}+\frac{27}{4}=) & + & \frac{9}{4}\,e' & \text{,,}\ \ 2\tau+2g' \end{array}$$

$$3\sin 2v-2v' = \qquad\qquad \frac{\delta v'}{\rho'^3} =$$

$$\begin{array}{lll} & 3 & \sin\ 2\tau \\ + & 6\,e' & \text{,,}\ \ 2\tau-g' \\ - & 6\,e' & \text{,,}\ \ 2\tau+g' \end{array} \qquad\qquad \begin{array}{lll} & 2\ t & \sin g' \\ + & \frac{11}{2}\,e' & \text{,,}\ \ 2g'. \end{array}$$

Product is =

$$\begin{array}{lll} & 6 & (\frac{1}{2}\,t\cos 2\tau-g' \quad - \frac{1}{2}\,t\cos 2\tau+g'\ ) \\ + & 12\,e' & (\frac{1}{2}\ \ \text{,,}\ \ 2\tau-2g' \quad - \frac{1}{2}\ \ \text{,,}\ \ 2\tau\ \ ) \\ - & 12\,e' & (\frac{1}{2}\ \ \text{,,}\ \ 2\tau \quad - \frac{1}{2}\ \ \text{,,}\ \ 2\tau+2g') \\ + & \frac{33}{2}\,e' & (\frac{1}{2}\ \ \text{,,}\ \ 2\tau-2g' \quad - \frac{1}{2}\ \ \text{,,}\ \ 2\tau+2g'), \end{array}$$

which is =

$$\begin{array}{lll}
(-6-6=) & -12\,e' & t\cos 2\tau \\
 & +3 & \quad\text{,,}\quad 2\tau-g' \\
 & -3 & \quad\text{,,}\quad 2\tau+g' \\
(6+\frac{33}{4}=) & +\frac{57}{4}\,e' & \quad\text{,,}\quad 2\tau-2g' \\
(6-\frac{33}{4}=) & -\frac{9}{4}\,e' & \quad\text{,,}\quad 2\tau+2g',
\end{array}$$

whence, adding the two products, and observing that $\rho^2=1$, the required terms are

$$\begin{array}{lll}
= & \frac{3}{2}\,e' & t \\
 & +\frac{3}{2} & t\cos g' \\
(\frac{9}{2}-12=) & -\frac{15}{2}\,e' & \quad\text{,,}\quad 2\tau \\
(\frac{9}{4}+3=) & +\frac{21}{4} & \quad\text{,,}\quad 2\tau-g' \\
(\frac{9}{4}-3=) & -\frac{3}{4} & \quad\text{,,}\quad 2\tau+g' \\
 & +\frac{9}{2}\,e' & \quad\text{,,}\quad 2g' \\
(\frac{45}{4}+\frac{57}{4}=) & +\frac{51}{2}\,e' & \quad\text{,,}\quad 2\tau-2g' \\
(\frac{9}{4}-\frac{9}{4}=) & 0 & \quad\text{,,}\quad 2\tau+2g'.
\end{array}$$

Annex 5.

Calculation of a part of δQ, viz.

$$\delta Q=\rho^2\left[(-\tfrac{3}{2}\sin 2v-2v')\,\delta\frac{1}{\rho'^3}+(3\cos 2v-2v')\frac{\delta v'}{\rho'^3}\right],$$

$$\begin{array}{lll}
-\frac{3}{2}\sin 2v-2v'= & & \\
-\frac{3}{2} & \sin\ 2\tau & \\
-3\,e' & \quad\text{,,}\quad 2\tau-g' & \\
+3\,e' & \quad\text{,,}\quad 2\tau+g' &
\end{array}
\qquad
\begin{array}{lll}
\delta\frac{1}{\rho'^3}= & & \\
3\,e' & t & \\
+3 & t\cos g' & \\
+9\,e' & \quad\text{,,}\quad 2g'. &
\end{array}$$

Product is

$$\begin{array}{lll}
-\frac{3}{2}\,e' & t\sin 2\tau & \\
-\frac{3}{2} & (\frac{1}{2}\,t\sin 2\tau+g' & +\frac{1}{2}\,t\sin 2\tau-g'\ ) \\
-9\,e' & (\frac{1}{2}\quad\text{,,}\quad 2\tau & +\frac{1}{2}\quad\text{,,}\quad 2\tau-2g') \\
+9\,e' & (\frac{1}{2}\quad\text{,,}\quad 2\tau+2g' & +\frac{1}{2}\quad\text{,,}\quad 2\tau\quad\ ) \\
-\frac{27}{2}\,e' & (\frac{1}{2}\quad\text{,,}\quad 2\tau+2g' & +\frac{1}{2}\quad\text{,,}\quad 2\tau-2g'),
\end{array}$$

which is =

$$\begin{array}{rrl}
(-\tfrac{9}{2}-\tfrac{9}{2}+\tfrac{9}{2}=) & -\tfrac{9}{2}\,e' & t \sin 2\tau \\
 & -\tfrac{9}{4} & \quad ,, \quad 2\tau - g' \\
 & -\tfrac{9}{4} & \quad ,, \quad 2\tau + g' \\
(-\tfrac{9}{2}-\tfrac{27}{4}=) & -\tfrac{45}{4}\,e' & \quad ,, \quad 2\tau - 2g' \\
(+\tfrac{9}{2}-\tfrac{27}{4}=) & -\tfrac{9}{4}\,e' & \quad ,, \quad 2\tau + 2g';
\end{array}$$

and

$$\begin{array}{lll|lll}
\cos 2v - 2v' = & & & \dfrac{\delta v'}{\rho'^3} = & & \\
3 & \cos & 2\tau & 2 & t \sin & g' \\
+\,6\,e' & ,, & 2\tau - g' & & & \\
-\,6\,e' & ,, & 2\tau + g' & +\,\tfrac{11}{2}\,e' & ,, & 2g'.
\end{array}$$

Product is

$$\begin{array}{rrrrrr}
6 & (\tfrac{1}{2}\,t \sin & 2\tau + 2g' & -\tfrac{1}{2}\,t \sin & 2\tau - g' &) \\
+\,12\,e' & (\tfrac{1}{2} \quad ,, & 2\tau & -\tfrac{1}{2} \quad ,, & 2\tau - 2g' &) \\
-\,12\,e' & (\tfrac{1}{2} \quad ,, & 2\tau + 2g' & -\tfrac{1}{2} \quad ,, & 2\tau &) \\
+\,\tfrac{33}{2}\,e' & (\tfrac{1}{2} \quad ,, & 2\tau + 2g' & -\tfrac{1}{2} \quad ,, & 2\tau - 2g' &),
\end{array}$$

which is =

$$\begin{array}{rrl}
(6+6=) & 12\,e' & t \sin 2\tau \\
 & -\,3 & \quad ,, \quad 2\tau - g' \\
 & +\,3 & \quad ,, \quad 2\tau + g' \\
(-6-\tfrac{33}{4}=) & -\tfrac{57}{4}\,e' & \quad ,, \quad 2\tau - 2g' \\
(-6+\tfrac{33}{4}=) & +\tfrac{9}{4}\,e' & \quad ,, \quad 2\tau + 2g'.
\end{array}$$

Adding the two products together, and observing that $\rho^2 = 1$, the required terms are

$$\begin{array}{rrl}
(-\tfrac{9}{2} + 12 =) & \tfrac{15}{2}\,e' & t \sin 2\tau \\
(-\tfrac{9}{4} - 3 =) & -\tfrac{21}{4} & \quad ,, \quad 2\tau - g' \\
(-\tfrac{9}{4} + 3 =) & +\tfrac{3}{4} & \quad ,, \quad 2\tau + g' \\
(-\tfrac{45}{4} - \tfrac{57}{4} =) & -\tfrac{51}{2}\,e' & \quad ,, \quad 2\tau - 2g' \\
(-\tfrac{9}{4} + \tfrac{9}{4} =) & 0 & \quad ,, \quad 2\tau + 2g'.
\end{array}$$

Annex 6.

Calculation of terms in δP, viz.

$$\begin{array}{lll|lll}
 & & & & & n^{-1} \times \\
-\,3 & \sin & 2\tau & -\,3 & \cos & g' \\
-\tfrac{21}{2}\,e' & ,, & 2\tau - g' & -\tfrac{9}{4}\,e' & ,, & 2g'; \\
+\tfrac{3}{2}\,e' & ,, & 2\tau + g' & & &
\end{array}$$

the product of which is

$$\begin{array}{llll}
9 & (\tfrac{1}{2} \sin 2\tau + g' & + \tfrac{1}{2} \sin 2\tau - g' &) \\
+ \tfrac{63}{2} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau & + \tfrac{1}{2} \;\;,, \;\; 2\tau - 2g' &) \\
- \tfrac{9}{2} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau + 2g' & + \tfrac{1}{2} \;\;,, \;\; 2\tau &) \\
+ \tfrac{27}{4} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau + 2g' & + \tfrac{1}{2} \;\;,, \;\; 2\tau - 2g' &),
\end{array}$$

which is =

$$\begin{array}{lll}
 & \overbrace{\tfrac{27}{2} e' \quad \sin 2\tau}^{n^{-1}\times} & \\
(\tfrac{63}{4} - \tfrac{9}{4} =) & & \\
 & + \tfrac{9}{2} \quad ,, \quad 2\tau - g' & \\
 & + \tfrac{9}{2} \quad ,, \quad 2\tau + g' & \\
(\tfrac{63}{4} + \tfrac{27}{8} =) & + \tfrac{153}{8} e' \quad ,, \quad 2\tau - 2g' & \\
(-\tfrac{9}{4} + \tfrac{27}{8} =) & + \tfrac{9}{8} e' \quad ,, \quad 2\tau + 2g'. &
\end{array}$$

Annex 7.

Calculation of terms in δQ, viz.

$$\begin{array}{lll|lll}
 & & & & & \\
- 3 & \cos 2\tau & & & \overbrace{- 3 \quad \cos g'}^{n^{-1}\times} & \\
- \tfrac{21}{2} e' & ,, \quad 2\tau - g' & & & - \tfrac{9}{4} e' \quad ,, \quad 2g'; & \\
+ \tfrac{3}{2} e' & ,, \quad 2\tau + g' & & & &
\end{array}$$

the product of which is

$$\begin{array}{llll}
9 & (\tfrac{1}{2} \cos 2\tau - g' & + \tfrac{1}{2} \cos 2\tau + g' &) \\
+ \tfrac{63}{2} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau - 2g' & + \tfrac{1}{2} \;\;,, \;\; 2\tau &) \\
- \tfrac{9}{2} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau & + \tfrac{1}{2} \;\;,, \;\; 2\tau + 2g' &) \\
+ \tfrac{27}{4} e' & (\tfrac{1}{2} \;\;,, \;\; 2\tau - 2g' & + \tfrac{1}{2} \;\;,, \;\; 2\tau + 2g' &),
\end{array}$$

which is =

$$\begin{array}{lll}
 & \overbrace{\tfrac{27}{2} e' \quad \cos 2\tau}^{n^{-1}\times} & \\
(\tfrac{63}{4} - \tfrac{9}{4} =) & & \\
 & + \tfrac{9}{2} \quad ,, \quad 2\tau - g' & \\
 & + \tfrac{9}{2} \quad ,, \quad 2\tau + g' & \\
(\tfrac{63}{4} + \tfrac{27}{8} =) & + \tfrac{155}{8} e' \quad ,, \quad 2\tau - 2g' & \\
(\tfrac{9}{4} + \tfrac{27}{8} =) & + \tfrac{9}{8} e' \quad ,, \quad 2\tau + 2g'. &
\end{array}$$

Annex 8.

Calculation of $n\int \delta Q dt$.

We have

$$n\int \sin nat\, dt = -\frac{1}{a} t \cos nat + \frac{1}{na} \sin nat,$$

$$\int \cos nat\, dt = \qquad \frac{1}{na} \sin nat,$$

and in all the arguments a is taken $= 2$.

$n\int \delta Q dt =$	$-\frac{15}{4} e'$	$t \cos 2\tau$	$(+\frac{15}{8} + \frac{27}{4} =)$	$+\frac{69}{8} e'$	$\sin 2\tau$	($n^{-1}\times$)
	$+\frac{21}{8}$	„ $2\tau - g'$	$(-\frac{21}{16} + \frac{9}{4} =)$	$+\frac{15}{16}$	„ $2\tau - g'$	
	$-\frac{3}{8}$	„ $2\tau + g'$	$(+\frac{3}{16} + \frac{9}{4} =)$	$+\frac{39}{16}$	„ $2\tau + g'$	
	$+\frac{51}{4} e'$	„ $2\tau - 2g'$	$(-\frac{51}{8} + \frac{153}{16} =)$	$+\frac{51}{16} e'$	„ $2\tau - g'$	
	0	„ $2\tau + 2g'$	$(\ 0 + \frac{9}{16} =)$	$+\frac{9}{16}$	„ $2\tau + 2g'$.	

Annex 9.

Calculation of $\delta P + 2n\int \delta Q\, dt$; viz. this is

	$\frac{3}{2} e'$	t				
	$\frac{3}{2}$	$t \cos g'$				
$(-\frac{15}{2} - \frac{15}{2} =)$	$-15 e'$	„ 2τ	$(+\frac{27}{2} + \frac{69}{4} =)$	$+\frac{123}{4} e'$	$\sin 2\tau$	($n^{-1}\times$)
$(+\frac{21}{4} + \frac{21}{4} =)$	$+\frac{21}{2}$	„ $2\tau - g'$	$(+\frac{9}{2} + \frac{15}{8} =)$	$+\frac{51}{8}$	„ $2\tau - g'$	
$(-\frac{3}{4} - \frac{3}{4} =)$	$-\frac{3}{2}$	„ $2\tau + g'$	$(+\frac{9}{2} + \frac{39}{8} =)$	$+\frac{75}{8}$	„ $2\tau + g'$	
$(+\frac{9}{2} + 0 =)$	$+\frac{9}{2} e'$	„ $2g'$	$(\ 0 + 0 =)$	0	„ $2g'$	
$(+\frac{51}{2} + \frac{51}{2} =)$	$+51 e'$	„ $2\tau - 2g'$	$(+\frac{153}{8} + \frac{51}{8} =)$	$+\frac{51}{2} e'$	„ $2\tau - 2g'$	
$(\ 0 + 0 =)$	0	„ $2\tau + 2g'$	$(+\frac{9}{8} + \frac{9}{8} =)$	$+\frac{9}{4} e'$	„ $2\tau + 2g'$.	

Annex 10.

Calculation of $\delta\rho$ from the equation

$$\frac{d^2\delta\rho}{dt^2} + n^2 d\rho = m^2 n^2 \left(\delta P + 2n\int \delta Q\, dt\right).$$

In $\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho$, a term $n^2 t \cos nat$, gives in $\delta\rho$,

$$\frac{1}{1-\alpha^2} t \cos nat + \frac{2\alpha}{(1-\alpha^2)^2} \frac{1}{n} \sin nat;$$

and a term $n \sin nat$, gives in $\delta\rho$,

$$\frac{1}{1-\alpha^2} \frac{1}{n} \sin nat,$$

and $\alpha = 2$, and therefore $\frac{1}{1-\alpha^2} = -\frac{1}{3}$, for all the args. except g', $2g'$; for these, $\alpha = m$ or $2m$, and therefore $\frac{1}{1-\alpha^2} = 1$.

$\delta\rho =$	$\frac{3}{2}\, m^2 e'$	t						
	$+\frac{3}{2}\, m^2$	$t \cos g'$		$(+3 \quad =)$	$+3\, m^2$	$\sin g'$		
	$+5\, m^2 e'$	„ 2τ		$(-\frac{20}{3} - \frac{41}{4} =)$	$-\frac{203}{12}\, m^2 e'$	„ 2τ		
	$-\frac{7}{2}\, m^2$	„ $2\tau - g'$		$(+\frac{14}{3} - \frac{17}{8} =)$	$+\frac{61}{24}\, m^2$	„ $2\tau - g'$		
	$+\frac{1}{2}\, m^2$	„ $2\tau + g'$		$(-\frac{2}{3} - \frac{25}{8} =)$	$-\frac{91}{24}\, m^2$	„ $2\tau + g'$		
	$+\frac{9}{2}\, m^2 e'$	„ $2g'$		$(+18 + 0 =)$	$+18\, m^2 e'$	„ $2g'$		
	$-17\, m^2 e'$	„ $2\tau - 2g'$		$(+\frac{68}{3} + \frac{17}{2} =)$	$+\frac{85}{6}\, m^2 e'$	„ $2\tau - 2g'$		
	0	„ $2\tau + 2g'$		$(\ 0 - \frac{3}{4} =)$	$-\frac{3}{4}\, m^2 e'$	„ $2\tau + 2g'$.		

Annex 11.

Calculation of $\frac{d\delta v}{dt}$; viz. this is

$$= -2n\,\delta\rho + m^2 n^2 \int \delta Q\, dt.$$

$\frac{d\delta v}{dt} =$								
	$(-3 \quad =)$	$-3\, m^2 e'\, t$						
	$(-3 \quad =)$	$-3\, m^2\, t \cos g'$		$(-6 \quad =)$	$-6\, m^2$	$\sin g'$		
	$(-10 - \frac{15}{4} =)$	$-\frac{55}{4}\, m^2 e'$	„ 2τ	$(+\frac{203}{6} + \frac{69}{8} =)$	$+\frac{1019}{24}\, m^2 e'$	„ 2τ		
	$(+7 + \frac{21}{8} =)$	$+\frac{77}{8}\, m^2$	„ $2\tau - g'$	$(-\frac{61}{12} + \frac{15}{16} =)$	$-\frac{199}{48}\, m^2$	„ $2\tau - g'$		
	$(-1 - \frac{3}{8} =)$	$-\frac{11}{8}\, m^2$	„ $2\tau + g'$	$(+\frac{91}{12} + \frac{39}{16} =)$	$+\frac{481}{48}\, m^2$	„ $2\tau + g'$		
	$(-9 + 0 =)$	$-9\, m^2 e'$	„ $2g'$	$(-36 + 0 =)$	$-36\, m^2 e'$	„ $2g'$		
	$(+34 + \frac{51}{4} =)$	$+\frac{187}{4}\, m^2 e'$	„ $2\tau - 2g'$	$(-\frac{85}{3} + \frac{51}{16} =)$	$-\frac{1207}{48}\, m^2 e'$	„ $2\tau - 2g'$		
	$(\ 0 + 0\ )$	0	„ $2\tau + 2g'$	$(+\frac{3}{2} + \frac{9}{16} =)$	$+\frac{33}{16}\, m^2 e'$	„ $2\tau + 2g'$.		

Annex 12.

Calculation of δv from the foregoing value of $\frac{d\delta v}{dt}$.

We have

$$n\int \cos nat\,dt = \frac{1}{\alpha}\,t\sin nat + \frac{1}{n\alpha^2}\cos nat,$$

$$\int \sin nat\,dt = \qquad -\frac{1}{n\alpha}\cos nat\,;$$

$\alpha = 2$ for all the arguments, except only $\alpha = m$, $2m$, for the arguments g', $2g'$, respectively.

$\delta v =$

						$-$	$\frac{3}{2}\,m^2ne'$	t^2	
						$+$	$n^{-1}\times$		
$-\ 3\ m$	$t\sin g'$		$\left(-\frac{3}{m^2}+6=\right)$		$-\ 3$	$+6\,m^2$		$\cos g'$	
$-\ \frac{55}{8}\,m^2e'$	„ 2τ		$(-\frac{55}{16}$	$-\frac{1019}{48}=)$	$-\ \frac{74}{3}$	$m^2\,e'$		„ 2τ	
$+\ \frac{77}{16}\,m^2$	„ $2\tau-g'$		$(+\frac{77}{32}$	$+\frac{199}{96}=)$	$+\ \frac{215}{48}$	m^2		„ $2\tau-g'$	
$-\ \frac{11}{16}\,m^2$	„ $2\tau+g'$		$(-\frac{11}{32}$	$-\frac{481}{96}=)$	$-\ \frac{257}{48}$	m^2		„ $2\tau+g'$	
$-\ \frac{9}{2}\,me'$	„ $2g'$		$\left(-\frac{9}{4m^2}+18=\right)$		$(-\ \frac{9}{4}$	$+18\,m^2)\,e'$		„ $2g'$	
$+\frac{187}{8}\,m^2e'$	„ $2\tau-2g'$		$(+\frac{187}{16}$	$+\frac{1207}{96}=)$	$+\ \frac{2329}{96}$	$m^2\,e'$		„ $2\tau-2g'$	
0	„ $2\tau+2g'$		$(\ 0$	$-\frac{33}{32}=)$	$-\ \frac{33}{32}$	$m^2\,e'$		„ $2\tau+2g'$.	

The remaining Annexes relate to the determination of the non-periodic or secular terms of the order m^4, in $\delta\rho$ and δv respectively.

Annex 13.

Calculation of term of $\left(n^2+\frac{2n^2}{\rho^3}-3\left(\frac{dv}{dt}\right)^2\right)\delta\rho$.

We have

$n^2+\frac{2n^2}{\rho^3}-3\left(\frac{dv}{dt}\right)^2 =$

			$n^2\times$		
$(1+2+m^2-3=)$			m^2		
$(-\ 9+18=)$	$+\ 9$	m^2e'		$\cos g'$	
$(+\ 6-\frac{33}{2}=)$	$-\ \frac{21}{2}$	m^2		„ 2τ	
$(+21-\frac{231}{4}=)$	$-\ \frac{147}{4}$	m^2e'		„ $2\tau-g'$	
$(-\ 3+\frac{33}{4}=)$	$+\ \frac{21}{4}$	m^2e'		„ $2\tau+g'$	

$\delta\rho =$

	$\frac{3}{2}\,m^2e'$	t
$+$	$\frac{3}{2}\,m^2$	$t\cos g'$
$+$	$5\,m^2e'$	„ 2τ
$-$	$\frac{7}{2}\,m^2$	„ $2\tau-g'$
$+$	$\frac{1}{2}\,m^2$	„ $2\tau+g'$
$+$	&c.	

where, in the second factor, the arguments not occurring in the first factor are omitted, as not giving rise to any non-periodic term; and so in other similar cases. Hence term of product is

$$\begin{array}{lrcr} m^4n^2e't & 1 & . & \frac{3}{2} \\ & +\frac{1}{2} . & 9 & . \frac{3}{2} \\ & +\frac{1}{2} . & -\frac{21}{2} & . \ 5 \\ & +\frac{1}{2} . & -\frac{147}{4} & . -\frac{7}{2} \\ & +\frac{1}{2} . & \frac{21}{4} & . \ \frac{1}{2} \end{array}$$

$$=\left(\tfrac{3}{2}+\tfrac{27}{4}-\tfrac{105}{4}+\tfrac{1029}{16}+\tfrac{21}{16}=\right)+\tfrac{381}{8}\,m^4n^2e't.$$

Annex 14.

Calculation of term of $m^2n^2\delta P$.

$m^2n^2\delta P$, the part involving δv is

$$\begin{array}{ll} \multicolumn{2}{c}{\overbrace{m^2n^2 \times}} \\ -3 & \sin 2\tau \\ -\frac{21}{2}\,e' & \text{,,}\ 2\tau-g' \\ +\frac{3}{2}\,e' & \text{,,}\ 2\tau+g' \end{array}$$

$$\begin{array}{lll} \delta v = & & \\ & -3\,m & t \sin g' \\ & -\frac{55}{8}\,m^2e' & \text{,,}\ 2\tau \\ & +\frac{77}{16}\,m^2 & \text{,,}\ 2\tau-g' \\ & -\frac{11}{16}\,m^2 & \text{,,}\ 2\tau+g' \\ & +\ \&c. & \end{array}$$

and term of the product is

$$\begin{array}{lrcr} m^4n^2e't & \frac{1}{2} . & -3 & . -\frac{55}{8} \\ & +\frac{1}{2} . & -\frac{21}{2} & . \ \frac{77}{16} \\ & +\frac{1}{2} . & \frac{3}{2} & . -\frac{11}{16} \end{array}$$

$$=\left(\tfrac{165}{16}-\tfrac{1617}{64}-\tfrac{33}{64}=\right)-\tfrac{495}{32}\,m^4n^2e't.$$

Annex 15.

Calculation of term of $m^2n^2\delta P$.

$m^2n^2\delta P$, the term involving $\delta\rho$ is

$$\begin{array}{ll} \multicolumn{2}{c}{\overbrace{m^2n^2 \times}} \\ \frac{1}{2} & \\ +\frac{3}{2}\,e' & \cos g' \\ +\frac{3}{2} & \text{,,}\ 2\tau \\ +\frac{21}{4}\,e' & \text{,,}\ 2\tau-g' \\ -\frac{3}{4}\,e' & \text{,,}\ 2\tau+g' \end{array}$$

$$\begin{array}{lll} \delta\rho = & & \\ & \frac{3}{2}\,m^2e' & t \\ & +\frac{3}{2}\,m^2 & t \cos g' \\ & +5\,m^2e' & \text{,,}\ 2\tau \\ & -\frac{7}{2}\,m^2 & \text{,,}\ 2\tau-g \\ & +\frac{1}{2}\,m^2 & \text{,,}\ 2\tau+g' \\ & +\ \&c. & \end{array}$$

and term of the product is

$$
\begin{array}{lll}
m^4n^3e't & \frac{1}{2} \cdot & \frac{3}{2} \\
+\frac{1}{2} \cdot & \frac{3}{2} \cdot & \frac{3}{2} \\
+\frac{1}{2} \cdot & \frac{3}{2} \cdot & 5 \\
+\frac{1}{2} \cdot & \frac{21}{4} \cdot & -\frac{7}{2} \\
+\frac{1}{2} \cdot & -\frac{3}{4} \cdot & \frac{1}{2}
\end{array}
$$

$$= \left(\tfrac{3}{4} + \tfrac{9}{8} + \tfrac{15}{4} - \tfrac{147}{16} - \tfrac{3}{16} =\right) - \tfrac{15}{4} m^4n^3e't.$$

Annex 16.

Calculation of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 2v - 2v'$, as far as m^2.

$$
\begin{aligned}
\cos 2v - 2v' = \cos 2\tau + X = \quad & \cos 2\tau \\
& - X \sin 2\tau \\
& - \tfrac{1}{2} X^2 \cos 2\tau,
\end{aligned}
$$

$$
\begin{aligned}
\sin 2v - 2v' = \sin 2\tau + X = \quad & \sin 2\tau \\
& + X \cos 2\tau \\
& \tfrac{1}{2} X^2 \sin 2\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
X = & -(4 + 6m)\, e' && \sin g' \\
& + \tfrac{11}{4} m^2 && \text{,,}\ 2\tau \\
& + \tfrac{77}{8} m^2e' && \text{,,}\ 2\tau - g' \\
& - \tfrac{11}{8} m^2e' && \text{,,}\ 2\tau + g',
\end{aligned}
$$

and thence

$$
\begin{aligned}
X \sin 2\tau = & -(4 + 6\ m)\, e' && (\tfrac{1}{2} \cos 2\tau - g' && - \tfrac{1}{2} \cos 2\tau + g') \\
& + \tfrac{11}{4} m^2 && (\tfrac{1}{2} && - \tfrac{1}{2}\ \text{,,}\ 4\tau\ ) \\
& + \tfrac{77}{8} m^2e' && (\tfrac{1}{2} \cos g' && - \tfrac{1}{2}\ \text{,,}\ 4\tau - g') \\
& - \tfrac{11}{8} m^2e' && (\tfrac{1}{2}\ \text{,,}\ g' && - \tfrac{1}{2}\ \text{,,}\ 4\tau + g'),
\end{aligned}
$$

which is =

$$
\begin{aligned}
& \tfrac{11}{8} m^2 && \\
(\tfrac{77}{16} - \tfrac{11}{16} =) \quad & + \tfrac{33}{8} m^2 e' && \cos g' \\
& - (2 + 3\ m)\, e' && \text{,,}\ 2\tau - g' \\
& + (2 + 3\ m)\, e' && \text{,,}\ 2\tau + g' \\
& - \tfrac{11}{8} m^2 && \text{,,}\ 4\tau \\
& - \tfrac{77}{16} m^2e' && \text{,,}\ 4\tau - g' \\
& + \tfrac{11}{16} m^2e' && \text{,,}\ 4\tau + g';
\end{aligned}
$$

$$
\begin{aligned}
X\cos 2\tau = & -(4+6\,m)\,e' \quad (\tfrac{1}{2}\sin 2\tau + g' \;-\; \tfrac{1}{2}\sin 2\tau - g') \\
& + \tfrac{11}{4}\,m^2 \quad (\tfrac{1}{2}\;\;,,\;\; 4\tau \qquad\qquad\qquad\qquad) \\
& + \tfrac{77}{8}\,m^2 e' \quad (\tfrac{1}{2}\;\;,,\;\; 4\tau - g' \;-\; \tfrac{1}{2}\;\;,,\;\; g' \quad) \\
& - \tfrac{11}{8}\,m^2 e' \quad (\tfrac{1}{2}\;\;,,\;\; 4\tau + g' \;-\; \tfrac{1}{2}\;\;,,\;\; g' \quad),
\end{aligned}
$$

which is =

$$
\begin{aligned}
(-\tfrac{77}{16} - \tfrac{11}{16} =) \quad & \tfrac{11}{2}\,m^2 e' && \sin g' \\
& +(2+\;3\;m)\,e' && ,,\;\; 2\tau - g' \\
& -(2+\;3\;m)\,e' && ,,\;\; 2\tau - g' \\
& +\tfrac{11}{8}\,m^2 && ,,\;\; 4\tau \\
& +\tfrac{77}{16}\,m^2 e' && ,,\;\; 4\tau - g' \\
& -\tfrac{11}{16}\,m^2 e' && ,,\;\; 4\tau + g'\,;
\end{aligned}
$$

we have, moreover,

$$
\begin{aligned}
X^2 \quad & = -\,2\,(4+6m)\,e'\sin g' \,.\, \tfrac{11}{4}\,m^2 \sin 2\tau \\
& = -\,22\,m^2 e'\,(\tfrac{1}{2}\cos 2\tau - g' \;-\; \tfrac{1}{2}\cos 2\tau + g') \\
& = -\,11\,m^2 e' \quad \cos 2\tau - g' \\
& \quad +\,11\,m^2 e' \quad ,,\;\; 2\tau - g',
\end{aligned}
$$

and thence

$$
\begin{aligned}
X^2\cos 2\tau = & -\,11\,m^2 e'\,(\tfrac{1}{2}\cos 4\tau - g' + \tfrac{1}{2}\cos g') \\
& +\,11\,m^2 e'\,(\tfrac{1}{2}\;\;,,\;\; 4\tau + g' + \tfrac{1}{2}\;\;,,\;\; g') \\
= (-\tfrac{11}{2} + \tfrac{11}{2} =) \quad & 0 && \cos g' \\
& -\tfrac{11}{2}\,m^2 e' && ,,\;\; 4\tau - g' \\
& +\tfrac{11}{2}\,m^2 e' && ,,\;\; 4\tau + g'\,;
\end{aligned}
$$

and

$$
\begin{aligned}
X^2\sin 2\tau = & -\,11\;m^2 e'\,(\tfrac{1}{2}\sin 4\tau - g' + \tfrac{1}{2}\sin g') \\
& -\,11\;m^2 e'\,(\tfrac{1}{2}\;\;,,\;\; 4\tau + g' + \tfrac{1}{2}\;\;,,\;\; g') \\
= (-\tfrac{11}{2} - \tfrac{11}{2} =) & -\,11\;m^2 e' \quad \sin g' \\
& -\tfrac{11}{2}\;m^2 e' \quad ,,\;\; 4\tau - g' \\
& +\tfrac{11}{2}\;m^2 e' \quad ,,\;\; 4\tau + g'\,;
\end{aligned}
$$

and thence

$$
\begin{aligned}
\cos 2v - 2v' = \quad & -\tfrac{11}{8}\;m^2 \\
& +\;1 && \cos\;2\tau \\
& -\tfrac{33}{8}\;m^2 e' && ,,\;\; g' \\
& +(2+\;3\;m)\,e' && ,,\;\; 2\tau - g' \\
& -(2+\;3\;m)\,e' && ,,\;\; 2\tau + g' \\
& +\tfrac{11}{8}\;m^2 && ,,\;\; 4\tau \\
(+\tfrac{77}{16} + \tfrac{11}{4} =) \quad & +\tfrac{121}{16}\,m^2 e' && ,,\;\; 4\tau - g' \\
(-\tfrac{11}{16} - \tfrac{11}{4} =) \quad & -\tfrac{55}{16}\;m^2 e' && ,,\;\; 4\tau + g'\,;
\end{aligned}
$$

and

$$
\begin{array}{rrl}
\sin 2v - 2v' = & 1 & \sin\ 2\tau \\
(-\tfrac{11}{2} + \tfrac{11}{2} =) & 0\ m^2 e' & \text{,,}\ g' \\
& +(2 + 3\ m) e' & \text{,,}\ 2\tau - g' \\
& -(2 + 3\ m) e' & \text{,,}\ 2\tau + g' \\
& +\tfrac{11}{8} m^2 & \text{,,}\ 4\tau \\
(\tfrac{77}{16} + \tfrac{11}{4} =) & +\tfrac{121}{16} m^2 e' & \text{,,}\ 4\tau - g' \\
(-\tfrac{11}{16} - \tfrac{11}{4} =) & -\tfrac{55}{16} m^2 e' & \text{,,}\ 4\tau + g'.
\end{array}
$$

Annex 17.

Calculation of term in δQ.

The part of δQ containing δv is $\frac{\rho^2}{\rho'^3}(-3\cos 2v - 2v')\,\delta v$; and it is necessary to find in $\frac{\rho^2}{\rho'^3}(-3\cos 2v - 2v')$ the coefficient of $\cos g'$ as far as m^2; this is in fact required, *post* Annex 21, for the calculation of $m^2 n^2 \left(C + \int \delta Q\, dt\right)$.

$$
\begin{array}{rll}
\rho^2 = 1 - & \tfrac{1}{3}\ m^2 & \\
+ & 3\ m^2 e' & \cos\ g' \\
- & 2\ m^2 & \text{,,}\ 2\tau \\
- & 7\ m^2 e' & \text{,,}\ 2\tau - g' \\
+ & 1\ m^2 e' & \text{,,}\ 2\tau + g'
\end{array}
\qquad
\begin{array}{rl}
\frac{1}{\rho'^3} = & 1 \\
+ & 3\, e' \quad \cos\ g'
\end{array}
$$

and thence

$$
\begin{array}{rrll}
\frac{\rho^2}{\rho'^3} = & & & \\
& 1 - & \tfrac{1}{3}\ m^2 & \\
& +(3 + & 2\ m^2)\, e' & \cos\ g' \\
& - & 2\ m^2 & \text{,,}\ 2\tau \\
(-7 - \tfrac{1}{2}.2.3 =) & - & 10\ m^2 e' & \text{,,}\ 2\tau - g' \\
(+1 - \tfrac{1}{2}.2.3 =) & - & 2\ m^2 e' & \text{,,}\ 2\tau + g'
\end{array}
\qquad
\begin{array}{rrl}
-\ 3\ \cos 2v - 2v' = & & \\
& \tfrac{33}{8}\ m^2 & \\
& +\tfrac{99}{8}\ m^2\ e' & \cos\ g' \\
& -\ 3 & \text{,,}\ 2\tau \\
& -(6 + 9\ m)\, e' & \text{,,}\ 2\tau - g' \\
& +(6 + 9\ m)\, e' & \text{,,}\ 2\tau + g' \\
& \&c. &
\end{array}
$$

where, in the second column, the omitted terms have arguments containing 4τ, and consequently, do not, by combination with the first column, give rise to any term with the argument g'. The term arg. g' arises from the combinations

$$\begin{array}{lll}
\frac{33}{8} m^2 & . 3e' & \cos g' \\
+ \frac{99}{8} m^2 e' & . 1 & \text{,,} \; g' \\
- 3 & . - 10 m^2 e' & (\cos 2\tau \cos 2\tau - g' = \frac{1}{2} \cos 4\tau - g' + \frac{1}{2} \cos g') \\
- 3 & . - 2 m^2 e' & (\text{,,} \; 2\tau \;\; \text{,,} \; 2\tau + g' = \frac{1}{2} \;\; \text{,,} \; 4\tau + g' + \frac{1}{2} \;\; \text{,,} \; g') \\
- 6 e' & . - 2 m^2 & (\text{,,} \; 2\tau \;\; \text{,,} \; 2\tau - g' = \frac{1}{2} \;\; \text{,,} \; 4\tau - g' + \frac{1}{2} \;\; \text{,,} \; g') \\
+ 6 e' & . - 2 m^2 & (\text{,,} \; 2\tau \;\; \text{,,} \; 2\tau + g' = \frac{1}{2} \;\; \text{,,} \; 4\tau + g' + \frac{1}{2} \;\; \text{,,} \; g')
\end{array}$$

so that the required term is

$$(\tfrac{99}{8} + \tfrac{99}{8} + 15 + 3 + 6 - 6 =) + \tfrac{171}{4} m^2 e' \cos g',$$

and annexing this to the terms found Annex 2, the part of δQ which contains δv is

$$\left.\begin{array}{lll}
\frac{171}{4} m^2 e' & & \cos g' \\
- 3 & & \text{,,} \; 2\tau \\
- \frac{21}{2} & e' & \text{,,} \; 2\tau - g' \\
+ \frac{3}{2} & e' & \text{,,} \; 2\tau + g'
\end{array}\right| \delta v.$$

Annex 18.

Calculation of term of $m^2 n^2 \delta P$.

The part of δP involving $\delta v'$ and $\delta \rho'$ is

$$\delta P = \rho \left[(\tfrac{1}{2} + \tfrac{3}{2} \cos 2v - 2v') \, \delta \frac{1}{\rho'^3} + (3 \sin 2v - 2v') \frac{\delta v'}{\rho'^3} \right];$$

$$\begin{array}{|lll|}
\rho = & & \\
1 - \frac{1}{6} m^2 & & \\
+ \frac{3}{2} m^2 e' & \cos g' & \\
- \quad m^2 & \text{,,} \; 2\tau & \\
- \frac{7}{2} m^2 e' & \text{,,} \; 2\tau - g' & \\
+ \frac{1}{2} m^2 e' & \text{,,} \; 2\tau + g' &
\end{array}
\quad
\begin{array}{|lll|}
\frac{1}{2} + \frac{3}{2} \cos 2v - 2v' = & & \\
\frac{1}{2} - \frac{33}{16} m^2 & & \\
- \frac{99}{16} m^2 e' & \cos g' & \\
+ \frac{3}{2} & \text{,,} \; 2\tau & \\
+ (3 + \frac{9}{2} m) \, e' & \text{,,} \; 2\tau - g' & \\
- (3 + \frac{9}{2} m) \, e' & \text{,,} \; 2\tau + g' & \\
+ \&c. & &
\end{array}
\quad
\begin{array}{|ll|}
\delta \frac{1}{\rho'^3} = & \\
3 e' & t \\
+ 3 & t \cos 2g' \\
+ 9 e' & \text{,,} \; 2g'
\end{array}$$

where, in the second factor, the terms belonging to the arguments which contain 4τ (i. e. the arguments 4τ, $4\tau - g'$, $4\tau + g'$) are omitted. In fact, the terms in question would, in the product of the second and third factors, give rise to terms with arguments containing 4τ, and as there are no such terms in the first factor, there is no resulting secular term.

The product of the second and third factors is

$$\begin{array}{lll}
(\tfrac{3}{2} - \tfrac{99}{16} m^2) e' & t & \\
+ (\tfrac{3}{2} - \tfrac{99}{16} m^2) & t \cos g' & \\
+ \tfrac{9}{2} e' & \quad \text{,,} \; 2\tau & \\
\quad - \tfrac{297}{16} m^2 e' & (\tfrac{1}{2} t & + \tfrac{1}{2} t \cos 2g' \quad) \\
+ \tfrac{9}{2} & (\tfrac{1}{2} t \cos 2\tau - g' & + \tfrac{1}{2} \quad \text{,,} \; 2\tau + g' \;) \\
+ (9 + \tfrac{27}{2} m) e' & (\tfrac{1}{2} t \;\; \text{,,} \; 2\tau - 2g' & + \tfrac{1}{2} \quad \text{,,} \; 2\tau \quad) \\
- (9 + \tfrac{27}{2} m) e' & (\tfrac{1}{2} \quad \text{,,} \; 2\tau + 2g' & + \tfrac{1}{2} \quad \text{,,} \; 2\tau \quad) \\
+ (\tfrac{9}{2} - \tfrac{297}{16} m^2) e' & t \;\; \text{,,} \; 2g' & \\
+ \tfrac{27}{2} e' & (\tfrac{1}{2} t \;\; \text{,,} \; 2\tau - 2g' & + \tfrac{1}{2} t \;\; \text{,,} \; 2\tau + 2g'),
\end{array}$$

and we may in this product omit the terms with arguments containing $2g'$, since the first factor does not contain any such term. The product then is

$$\begin{array}{lll}
(-\tfrac{99}{16} - \tfrac{297}{32} = -\tfrac{495}{32}) & (\tfrac{3}{2} - \tfrac{495}{32} m^2) e' & t \\
 & + (\tfrac{3}{2} - \tfrac{99}{16} m^2) & t \cos g' \\
((\tfrac{9}{2} + \tfrac{9}{2} - \tfrac{9}{2}) e' + (\tfrac{27}{4} - \tfrac{27}{4}) m e' =) & + \tfrac{9}{2} e' & \text{,,} \; 2\tau \\
 & + \tfrac{9}{4} & \text{,,} \; 2\tau - g' \\
 & + \tfrac{9}{4} & \text{,,} \; 2\tau + g',
\end{array}$$

which is to be multiplied by the first factor, ρ, and the whole by the factor $m^2 n^2$.

The term in the product is

$$\begin{array}{ll}
m^2 n^2 e' t & (\tfrac{3}{2} - \tfrac{495}{32} m^2)(1 - \tfrac{1}{6} m^2) \\
 & + \tfrac{1}{2} \cdot \tfrac{3}{2} \quad . \quad \tfrac{3}{2} m^2 \\
 & + \tfrac{1}{2} \cdot \tfrac{9}{2} \quad . - \quad m^2 \\
 & + \tfrac{1}{2} \cdot \tfrac{9}{4} \quad . - \tfrac{7}{2} m^2 \\
 & + \tfrac{1}{2} \cdot \tfrac{9}{4} \quad . \quad \tfrac{1}{2} m^2,
\end{array}$$

giving the term $\tfrac{3}{2} m^2 n^2 e' t$, which was found above, Annex 9, and the new terms

$$(-\tfrac{495}{32} - \tfrac{1}{4} + \tfrac{9}{8} - \tfrac{9}{4} - \tfrac{63}{16} + \tfrac{9}{16} =) - \tfrac{647}{32} m^4 n^2 e' t.$$

The term of the part containing $\delta v'$ is found to be $=0$; in fact we have

$\rho =$	$3 \sin 2v - 2v' =$	$\dfrac{\delta v'}{\rho^3} =$
$1 - \frac{1}{6} m^2$	3 $\quad \sin 2\tau$	$2 \quad t \sin g'$
$+ \frac{3}{2} m^2 e' \quad \cos g'$	$+ (6 + 9\,m)\, e' \quad ,, \; 2\tau - g'$	$+ \frac{11}{2} e' \quad ,, \; 2g'$
$- \quad m^2 \quad ,, \; 2\tau$	$- (6 + 9\,m)\, e' \quad ,, \; 2\tau + g'$	
$- \frac{7}{2} m^2 e' \quad ,, \; 2\tau - g'$	&c.	
$+ \frac{1}{2} m^2 e' \quad ,, \; 2\tau + g'$		

where in the second factor the terms with arguments containing 4τ are for the before-mentioned reason omitted.

The product of the second and third factors is

$$\begin{aligned} & 6\,(\tfrac{1}{2}\, t \cos 2\tau - g' \quad - \tfrac{1}{2}\, t \cos 2\tau + g' \;) \\ + (12 + 18\,m)\, e' \,(\tfrac{1}{2} \quad ,, \; 2\tau - 2g' \quad & - \tfrac{1}{2} \quad ,, \; 2\tau \qquad) \\ - (12 + 18\,m)\, e' \,(\tfrac{1}{2} \quad ,, \; 2\tau \qquad & - \tfrac{1}{2} \quad ,, \; 2\tau + 2g') \\ + \tfrac{33}{2} \quad e' \,(\tfrac{1}{2} \quad ,, \; 2\tau - 2g' \quad & - \tfrac{1}{2} \quad ,, \; 2\tau + 2g') \end{aligned}$$

which, omitting the terms with arguments containing $2g'$, is

$$\begin{aligned} - (6 + 9m + 6 + 9m) = - (12 + 18m)\, e' \quad & t \cos 2\tau \\ + \; 3 \qquad & ,, \; 2\tau - g' \\ - \; 3 \qquad & ,, \; 2\tau + g' \end{aligned}$$

which is to be multiplied by the first factor, ρ, and the whole by $m^2 n^2$. The term is

$$\begin{aligned} m^4 n^2 e' t . \quad & \tfrac{1}{2} \,.\, - 12 \,.\, - 1 \\ + \tfrac{1}{2} \,. \quad & 3 \,.\, - \tfrac{7}{2} \\ + \tfrac{1}{2} \,.\, - 3 \,. \quad & \tfrac{1}{2} \end{aligned}$$

which is

$$(6 - \tfrac{21}{4} - \tfrac{3}{4} =) \; 0 \; m^4 n^2 e' t.$$

Hence the entire term in question is the before-mentioned value

$$- \tfrac{647}{32} \, m^4 n^2 e' t.$$

Annex 19.

Calculation of term in $m^2 n^2 \dfrac{2}{\rho} \dfrac{dv}{dt} \left(C + \int \delta Q \, dt \right)$.

We have

$$\frac{2}{\rho}\frac{dv}{dt} =$$

$$\begin{array}{lll} & \overbrace{2 + \tfrac{1}{3}\ m^2}^{n\,\times} & \\ & -\ 9\ m^2e' & \cos g' \\ & +\ \tfrac{15}{2}\ m^2 & \text{,,}\ 2\tau \\ & +\ \tfrac{105}{4}\ m^2e' & \text{,,}\ 2\tau - g' \\ & -\ \tfrac{15}{4}\ m^2e' & \text{,,}\ 2\tau + g' \end{array}$$

See *post* Annex 20.

$$m^2n^2\left(C + \int \delta Q\, dt\right) =$$

$$\begin{array}{lll} & \overbrace{-\ \tfrac{1455}{32}\ m^4e'\ t}^{n\,\times} & \\ & 0 & t \cos g' \\ & -\ \tfrac{15}{4}\ m^2 & \text{,,}\ 2\tau \\ & +\ \tfrac{21}{8}\ m^2 & \text{,,}\ 2\tau - g' \\ & -\ \tfrac{3}{8}\ m^2 & \text{,,}\ 2\tau - g' \end{array}$$

For the term $-\frac{1455}{32}\, m^4e'\, t$ see *post* Annex 21.

The term in the product therefore is

$$\begin{array}{lll} m^4n^2e'\,t\,. & 2\ .\ -\tfrac{1455}{32} & \\ & +\ \tfrac{1}{2}\,.\ \ \tfrac{15}{2}\ .\ -\ \tfrac{15}{4} & \\ & +\ \tfrac{1}{2}\,.\ \ \tfrac{105}{4}\ .\ -\ \tfrac{21}{8} & \\ & +\ \tfrac{1}{2}\,.\ -\ \tfrac{15}{4}\ .\ \ \tfrac{3}{8} & \end{array}$$

which is

$$= \left(2\ .\ -\tfrac{1455}{32} - \tfrac{225}{16} + \tfrac{2205}{64} + \tfrac{45}{64}\right) m^4n^2e'\,t$$

$$= \left(2\ .\ -\tfrac{1455}{32} + \tfrac{675}{32} =\right)\ \ -\tfrac{2335}{32}\, m^4n^2e'\,t.$$

Annex 20.

Calculation of $\frac{2}{\rho}\frac{dv}{dt}$.

We have

$$\frac{dv}{dt} =$$

$$\begin{array}{lll} & \overbrace{1}^{n\,\times} & \\ & -\ 3\ m^2e' & \cos g' \\ & +\ \tfrac{11}{4}\ m^2e' & \text{,,}\ 2\tau \\ & +\ \tfrac{77}{8}\ m^2e' & \text{,,}\ 2\tau - g' \\ & -\ \tfrac{11}{8}\ m^2e' & \text{,,}\ 2\tau + g' \end{array}$$

$$\frac{2}{\rho} =$$

$$\begin{array}{lll} & 2 + \tfrac{1}{3}\ m^2 & \\ & -\ 3\ m^2e' & \cos g' \\ & +\ 2\ m^2 & \text{,,}\ 2\tau \\ & +\ 7\ m^2e' & \text{,,}\ 2\tau - g' \\ & -\ 1\ m^2e' & \text{,,}\ 2\tau + g' \end{array}$$

so that the product is

$$\overbrace{2+\tfrac{1}{3}\ m^2}^{n\,\times}$$

$$\begin{array}{llll}
(-\ 6\ -3=) & -\ 9\ m^2e' & \cos\ g' \\
(+\tfrac{11}{2}+1=) & +\tfrac{15}{2}\ m^2 & \text{,,}\ \ 2\tau \\
(+\tfrac{77}{4}+7=) & +\tfrac{105}{4}\,m^2e' & \text{,,}\ \ 2\tau-g' \\
(-\tfrac{11}{4}\quad 1=) & -\tfrac{15}{8}\ m^2e' & \text{,,}\ \ 2\tau+g'.
\end{array}$$

Annex 21.

Calculation of a term in $m^2n^2\left(C+\int\delta Q\,dt\right)$.

The part $\frac{\rho}{\rho'^2}(-3\sin 2v-2v')\,\delta\rho$ of δQ gives

$$\frac{\rho}{\rho'^3}(-3\sin 2v-2v')=$$

$$\begin{array}{lll}
-\ 3 & \sin\ 2\tau & \\
\tfrac{21}{2}\ e' & \text{,,}\ \ 2\tau & g \\
+\ \tfrac{3}{2}\ e' & \text{,,}\ \ 2\tau+g' &
\end{array}$$

$$\delta\rho=$$

$$\overbrace{3\ m^3\qquad \sin\ g'}^{n^{-1}\,\times}$$

$$\begin{array}{ll}
-\ \tfrac{203}{12}\ m^2e' & \text{,,}\ \ 2\tau \\
+\ \tfrac{64}{24}\ m^2 & \text{,,}\ \ 2\tau-g' \\
-\ \tfrac{91}{24}\ m^2 & \text{,,}\ \ 2\tau+g'
\end{array}$$

and we have thence in $m^2n^2\,\delta Q$ the term

$$\begin{array}{lll}
m^4\,n\,e' & \tfrac{1}{2}\,.\,-\,3\,.\,-\tfrac{203}{12} \\
 & +\ \tfrac{1}{2}\,.\,-\tfrac{21}{2}\,.\quad \tfrac{61}{24} \\
 & +\ \tfrac{1}{2}\,.\quad \tfrac{3}{2}\,.\,-\tfrac{91}{24}\,;
\end{array}$$

that is

$$\left(\tfrac{203}{8}-\tfrac{427}{32}-\tfrac{91}{32}=\right)+\tfrac{147}{16}\ m^4\,n\,e'.$$

The part $\frac{\rho^2}{\rho'^3}(3\cos 2v-2v')\,\delta v$ of δQ gives

$$\frac{\rho^2}{\rho'^3}(-3\cos 2v-2v')=$$

$$\begin{array}{lll}
\tfrac{171}{4}\ m^2e' & \cos\ g' & \\
-\ 3 & \text{,,}\ \ 2\tau & \\
-\ \tfrac{21}{2}\quad e' & \text{,,}\ \ 2\tau-g' & \\
+\ \tfrac{3}{2}\quad e' & \text{,,}\ \ 2\tau+g' &
\end{array}$$

See *ante*, Annex 17.

$$\delta v=$$

$$\overbrace{-\ 3\qquad \cos\ g'}^{n^{-1}}$$

$$\begin{array}{ll}
-\ \tfrac{74}{3}\ m^2e' & \text{,,}\ \ 2\tau \\
+\ \tfrac{215}{48}\ m^2 & \text{,,}\ \ 2\tau-g' \\
-\ \tfrac{257}{48}\ m^2 & \text{,,}\ \ 2\tau+g'
\end{array}$$

and we have thence in $m^2n^2\,\delta Q$ the term

$$\begin{array}{lllll} m^4n\,e' & \tfrac{1}{2}\,. & \tfrac{171}{4}\,. & -\ 3 \\ & +\ \tfrac{1}{2}\,. & -\ 3\ \,. & -\ \tfrac{74}{3} \\ & +\ \tfrac{1}{2}\,. & -\ \tfrac{21}{2}\,. & \tfrac{215}{48} \\ & +\ \tfrac{1}{2}\,. & \tfrac{3}{2}\,. & -\ \tfrac{257}{48} \end{array}$$

that is

$$\left(-\tfrac{513}{8} + 37 - \tfrac{1505}{64} - \tfrac{257}{64} =\right) - \tfrac{1749}{32}\,m^4n\,e';$$

and, combining this with the other term just obtained, the two together are

$$\left(\tfrac{147}{16} - \tfrac{1749}{32} =\right) \tfrac{1455}{32}\,m^4n\,e';$$

and this term in $m^2n^2\,\delta Q$ gives in $m^2n^2\left(C + \int \delta Q\,dt\right)$ the term

$$-\ \tfrac{1455}{32}\,m^4n\,e'\,t.$$

Annex 22.

Calculation of term in $-\dfrac{2}{\rho}\dfrac{dv}{dt}\,\delta\rho$.

We have

$$-\frac{2}{\rho}\frac{dv}{dt}\ (\text{see Annex 19}) = \qquad\qquad \rho =$$

$$\begin{array}{lll} & n\times & \\ -\ 2 & -\ \tfrac{1}{3}\ m^2 & \\ & +\ 9\ m^2e' & \cos g' \\ & -\ \tfrac{15}{2}\ m^2 & \text{,,}\ \ 2\tau \\ & -\ \tfrac{105}{4}\ m^2e' & \text{,,}\ \ 2\tau - g' \\ & +\ \tfrac{15}{4}\ m^2e' & \text{,,}\ \ 2\tau + g' \end{array} \qquad \begin{array}{lll} (\tfrac{3}{2}\ m^2 - \tfrac{1973}{32}\ m^4)\ e' & & t \\ +\ \tfrac{8}{2}\ m^2 & & t\ \cos g' \\ +\ 5\ m^2e' & & \text{,,}\ \ 2\tau \\ -\ \tfrac{7}{2}\ m^2e' & & \text{,,}\ \ 2\tau - g' \\ +\ \tfrac{1}{2}\ m^2 & & \text{,,}\ \ 2\tau + g' \end{array}$$

giving, besides the term $\tfrac{3}{2}\,m^2n\,e'\,t$ already taken account of, the term

$$\begin{array}{lllll} m^4n\,e'\,t\,. & +\ \tfrac{1973}{16}\,. & -\ \tfrac{1}{3}\ . & \tfrac{3}{2} \\ & +\ \tfrac{1}{2}\ . & 9\ . & \tfrac{3}{2} \\ & +\ \tfrac{1}{2}\ . & -\ \tfrac{15}{2}\,. & 5 \\ & +\ \tfrac{1}{2}\ . & -\ \tfrac{105}{4}\,. & -\ \tfrac{7}{2} \\ & +\ \tfrac{1}{2}\ . & \tfrac{15}{4}\ . & \tfrac{1}{2} \end{array}$$

which is

$$= \left(\tfrac{1973}{16} - \tfrac{1}{2} + \tfrac{27}{4} - \tfrac{75}{4} + \tfrac{735}{16} - \tfrac{15}{16}\right) = \left(\tfrac{1973}{16} + \tfrac{275}{8} =\right) \tfrac{2523}{16}\,m^4n^2e'\,t.$$

Annex 23.

Calculation of term in $\frac{m^2n^2}{\rho^2}\left(C+\int \delta Q\, dt\right)$.

We have

$$\frac{1}{\rho^2} = \begin{array}{lll} 1 + \frac{1}{3} m^2 & & \\ \quad - 3\, m^2 e' & \cos g' & \\ \quad + 2\, m^2 & \quad \text{,,} \quad 2\tau & \\ \quad + 7\, m^2 e' & \quad \text{,,} \quad 2\tau - g' & \\ \quad - 1\, m^2 e' & \quad \text{,,} \quad 2\tau + g & \end{array}$$

$$m^2n^2\left(C+\int \delta Q\, dt\right) = \begin{array}{lll} \overbrace{-\frac{1455}{32} m^2 e' \quad t}^{n\,\times} & & \\ \quad - \frac{15}{4} m^2 e' & t \cos 2\tau & \\ \quad + \frac{21}{8} m^2 e' & \quad \text{,,} \quad 2\tau - g' & \\ \quad - \frac{3}{8} m^2 e' & \quad \text{,,} \quad 2\tau + g' & \end{array}$$

giving the term

$$\begin{array}{ll} m^4 n e' t & -\frac{1455}{32} \\ & +\frac{1}{2} \,.\; 2 \,.\, -\frac{15}{4} \\ & +\frac{1}{2} \,.\; 7 \,.\; \frac{21}{8} \\ & +\frac{1}{2} \,.\, -1 \,.\, -\frac{3}{8} \end{array}$$

which is

$$= \left(-\frac{1455}{32} - \frac{15}{4} + \frac{147}{16} \cdot \frac{3}{16}\right) = \left(-\frac{1455}{32} + \frac{45}{8} =\right) - \frac{1275}{32} m^4 n e' t,$$

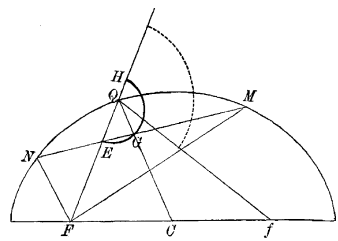
and this completes the series of calculations.

222.

ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXII. (1862), pp. 238—242.]

THE theorem referred to is that which gives the time of description of an elliptic arc in terms of the radius vectors and the chord. The demonstration given by the author in his "Insigniores Orbitæ Cometarum Proprietates," Augs. 1761, depends upon a series of geometrical propositions of great elegance, which may be thus stated.



Let FQ be a line given in magnitude and position, E a given point on this line, Qf a line given in magnitude only, the position thereof being determined by assigning a value to its variable inclination to the line FQ. With F, f, as foci describe an ellipse passing through the point Q (the axis major, $=FQ+Qf$, is of course a constant magnitude). Take C, the centre of the ellipse, and join CQ; through E draw a chord, MEN, conjugate to the diameter CQ and meeting it in G. Then treating the inclination as variable,

1°. The locus of G is a circle passing through E, and having its centre on the line FQ.

2°. The semichord GM or GN, and the sum $FM+FN$ of the radius vectors are respectively constant.

3°. The elliptic area NFM, divided by the square root of the latus rectum, is a constant.

It may also be mentioned, that taking 2θ to represent the external inclination (supplement of the angle FQf), and if, moreover, a is the semiaxis major, e the eccentricity, and u, u' the eccentric anomalies of the points M, N, then the square root of the latus rectum, or say $\sqrt{1-e^2}$, $\propto \sin\theta$, and moreover EM, EN, FM, FN, $e\cos u$, $e\cos u'$, $e\sin u$, $e\sin u'$, consists each of them of a constant part, *plus* a part which $\propto \cos\theta$; these expressions give as above $GM=GN=\frac{1}{2}(EM+EN)=\text{constant}$, $FM+FN=\text{constant}$; and they give moreover $e\cos u+e\cos u'=\text{constant}$; $e\sin u-e\sin u'=\text{constant}$; $u-u'=\text{const.}$ The expression for the area is

$$\tfrac{1}{2}a^2\sqrt{1-e^2}\,\{u-u'-(e\sin u-e\sin u')\},$$

and consequently the area divided by $\sqrt{1-e^2}$ is a constant; that is, the area is, as stated above, proportional to the square root of the latus rectum.

Hence, assuming the dynamical theorem that for a given central force at F, varying inversely as the square of the distance, the time of describing the elliptic arc is proportional to the area divided by the square root of the latus rectum, the time of describing the elliptic arc is constant. But in the extreme case, where the point f lies in the line FQ produced in the direction from F to Q, the ellipse reduces itself to a finite right line, length $FQ+Qf$, which is considered to be described by a body falling from the extremity with an initial velocity zero; and the arc MN is a portion thereof given in magnitude, and having for its centre the point H (where EH is the diameter of the before-mentioned circle, the locus of G). Hence the time of describing the elliptic arc is equal to the time of describing, under the action of the same central force, a given right line, and as such it is at once obtainable in the form

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left(\phi-\phi'-(\sin\phi-\sin\phi')\right)$$

where ϕ, ϕ' are functions of the major axis $FQ+Qf$, and of FM, FN, or, what is the same thing, of $FQ+Qf$, and of the chord MN and sum of the radius vectors FM, FN. The preceding is the geometrical mode of getting out the result, without the assistance of any expression for the elliptic area, and latus rectum, and assuming only that we know the formula for rectilineal motion; but, if the expressions for the elliptic area and latus rectum are obtained, then the expression for the time is known, and the problem is solved, without the necessity of passing from the ellipse to the right line.

Writing $FQ=\rho$, $QF=\sigma$, and as before the exterior angle of inclination $=2\theta$, the actual expressions for the various lines of the figure are easily found to be

$$\begin{aligned}
&\tfrac{1}{2}(\rho+\sigma) &&, =a,\\
CF=Cf&=\tfrac{1}{2}\sqrt{\rho^2+\sigma^2+2\rho\sigma\cos 2\theta}, &&=a,\\
CQ&=\tfrac{1}{2}\sqrt{\rho^2+\sigma^2-2\rho\sigma\cos 2\theta}, &&=a',\\
CR&=\sqrt{\rho\sigma} &&, =b',
\end{aligned}$$

where CR (not shown in the figure) denotes the semi-diameter conjugate to CQ.

$$1-e^2=\frac{4\rho\sigma}{(\rho+\sigma)^2}\sin^2\theta,$$

$$\cos F=\frac{\rho+\sigma\cos 2\theta}{2ae},\quad \cos Q=\frac{\rho-\sigma\cos 2\theta}{2a'},$$

$$\sin F=\frac{\sigma\sin 2\theta}{2ae},\quad \sin Q=\frac{\sigma\sin 2\theta}{2a'},\quad \sin C=\frac{\rho\sigma\sin 2\theta}{4a'ae},$$

where F, C, Q, denote the angles of the triangle FCQ, respectively,

$$EG=\frac{2k\sigma}{\rho+\sigma}\cos\theta,\quad \text{and therefore } EH=\frac{2k\sigma}{\rho+\sigma},$$

$$QG=\frac{k}{\rho+\sigma}2a',$$

and, if for shortness $\Lambda=\sqrt{k\rho\sigma(\rho+\sigma-k)}$, then

$$(EM,\ EN)=\frac{2}{\rho+\sigma}(\Lambda\pm k\sigma\cos\theta),$$

$$(FM,\ FN)=\frac{1}{\rho+\sigma}\{\rho(\sigma+\rho)+k(\sigma-\rho)\pm 2\Lambda\cos\theta\},$$

so that

$$GM=GN=\tfrac{1}{2}(EM+EN)=\frac{2\Lambda}{\rho+\sigma},$$

$$\tfrac{1}{2}(FM+FN)=\frac{1}{\rho+\sigma}\{\rho(\sigma+\rho)+k(\sigma-\rho)\},$$

and moreover

$$(e\cos u,\ e\cos u')=\frac{1}{(\rho+\sigma)^2}\{(\sigma-\rho)(\rho+\sigma-2k)\mp 4\Lambda\cos\theta\},$$

$$(e\sin u,\ e\sin u')=\frac{1}{(\rho+\sigma)^2}\left\{\pm\frac{2(\sigma-\rho)\Lambda}{\sqrt{\rho\sigma}}+2(\rho+\sigma-2k)\sqrt{\rho\sigma}\cos\theta\right\},$$

so that

$$e\cos u + e\cos u' = \frac{2}{(\rho+\sigma)^2}(\rho+\sigma-2k),$$

$$e\sin u - e\sin u' = \frac{4}{(\rho+\sigma)^2}\frac{(\sigma-\rho)\Lambda}{\sqrt{\rho\sigma}},$$

$$u-u' = 2\tan^{-1}\frac{2\Lambda}{\sqrt{\rho\sigma}(\rho+\sigma-2k)} = \sin^{-1}\frac{4\Lambda(\rho+\sigma-2k)}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

$$u-u'-(e\sin u - e\sin u') = \sin^{-1}\frac{4(\rho+\sigma-2k)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}} - \frac{4(\sigma-\rho)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}}$$

which is

$$= \phi - \phi' - (\sin\phi - \sin\phi'),$$

if

$$1-\cos\phi = \frac{1}{2a}(FM+FM+MN) = \frac{2}{(\rho+\sigma)^2}\{\rho(\sigma+\rho)+k(\sigma-\rho)+2\Lambda\},$$

$$1-\cos\phi' = \frac{1}{2a}(FM+FN-MN) = \frac{2}{(\rho+\sigma)^2}\{\rho(\sigma+\rho)+k(\sigma-\rho)-2\Lambda\}.$$

In fact we then have also

$$1+\cos\phi = \frac{2}{(\rho+\sigma)^2}\{\sigma(\sigma+\rho)-k(\sigma-\rho)-2\Lambda\},$$

$$1+\cos\phi' = \frac{2}{(\rho+\sigma)^2}\{\sigma(\sigma+\rho)-k(\sigma-\rho)+2\Lambda\},$$

and thence

$$\sin\tfrac{1}{2}\phi = \frac{1}{\rho+\sigma}\left(\sqrt{\rho(\rho+\sigma-k)}+\sqrt{k\sigma}\right),\quad \sin\tfrac{1}{2}\phi' = \left(\sqrt{\rho(\rho+\sigma-k)}-\sqrt{k\sigma}\right),$$

$$\cos\tfrac{1}{2}\phi = \frac{1}{\rho+\sigma}\left(\sqrt{\sigma(\rho+\sigma-k)}-\sqrt{k\rho}\right),\quad \cos\tfrac{1}{2}\phi' = \left(\sqrt{\sigma(\rho+\sigma-k)}+\sqrt{k\rho}\right),$$

whence

$$\sin\phi = \frac{2}{(\rho+\sigma)^2}\left\{\sqrt{\rho\sigma}(\rho+\sigma-2k)+\frac{(\sigma-\rho)\sqrt{\Lambda}}{\sqrt{\rho\sigma}}\right\},$$

$$\sin\phi' = \frac{2}{(\rho+\sigma)^2}\left\{\sqrt{\rho\sigma}(\rho+\sigma-2k)-\frac{(\sigma-\rho)\sqrt{\Lambda}}{\sqrt{\rho\sigma}}\right\},$$

and therefore $$\sin\phi - \sin\phi' = \frac{4(\sigma-\rho)\sqrt{\Lambda}}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

$$\sin\tfrac{1}{2}(\phi-\phi') = \frac{2k\Lambda}{(\rho+\sigma)\sqrt{\rho\sigma}},\quad \cos\tfrac{1}{2}(\phi-\phi') = \frac{\rho+\sigma-2k}{\rho+\sigma},$$

and therefore $$\sin(\phi-\phi') = \frac{4(\rho+\sigma-2k)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

which verifies the formula.

NOTES AND REFERENCES.

173. I ATTACH some value to this analysis and development of Laplace's Method, showing how it leads to the actual expression for the Potential of an Ellipsoid upon an exterior point in a series of terms of the form $\left(\alpha^2 \frac{d^2}{da^2} + \beta^2 \frac{d^2}{db^2} + \gamma^2 \frac{d^2}{dc^2}\right)^i \frac{1}{\sqrt{a^2+b^2+c^2}}$, being in fact the series deduced by me in the year 1842 from a result of Lagrange's; see vol. I., Notes and References 2 and 3.

191. The theorem obtained at the end of the paper is a very peculiar one; the only paper that I know of in anywise relating to the theory is Donkin, "On an application of the Calculus of Operations to the transformation of Trigonometric Series," *Quart. Math. Jour.* t. III. (1860), pp. 1—15, where (pp. 13—15) my theorem is referred to and a more general theorem involving two arbitrary functions ϕ, F, is arrived at.

194. In connexion herewith see the Memoir, Donkin, "On the Analytical Theory of the Attraction of Solids bounded by Surfaces of a hypothetical Class including the Ellipsoid," *Phil. Trans.* t. 150 (1860), pp 1—11. The author referring to my Note remarks that I there showed that if *two* of the principal theorems of attraction (in the case of the ellipsoid) be given the rest follow very simply and are common to all the surfaces of which these two can be predicated: but that the demonstration of the two assumed theorems constitute the most essential part of the analytical problem, and that it was his present object to show that they and the others connected with them are implied in the two partial differential equations

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 2\left(\frac{1}{a^2+h} + \frac{1}{b^2+h} + \frac{1}{c^2+h}\right)$$

and

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 + 4\frac{du}{dh} = 0,$$

satisfied by the function $\frac{x^2}{a+h} + \frac{y^2}{b+h} + \frac{z^2}{c+h}$: and he accordingly derives the whole theory, and in particular the theorems V. and VI. (equivalent to my assumed theorems) from these two partial differential equations.

221. It is well known that Plana, developing the explanation given by Laplace for the secular variation of the moon's mean motion, obtained in the expression of the true longitude the terms $-(\frac{3}{2}m^2 - \frac{2187}{128}m^4)\int(e'^2 - E'^2)\,ndt$, and that Prof. Adams in his memoir "On the Secular Variation of the Moon's Mean Motion," *Phil. Trans.* t. 143 (1853), pp. 397—406, corrected this into $-(\frac{3}{2}m^2 - \frac{3771}{64}m^4)\int(e'^2 - E'^2)\,ndt$. The validity of the correction was a good deal discussed, and it was interesting to establish the result by an entirely independent method.

END OF VOL. III.

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY & SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. IV.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1891

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 77 papers numbered 223 to 299 originally published all but one of them in the years 1856 to 1862, the excepted paper, No. 265, Addition to the Memoir on an Extension of Arbogast's Method of Derivations, being now first published.

The Table for the four volumes is

Vol.	I.	Numbers	1	to	100.
,,	II.	,,	101	,,	158.
,,	III.	,,	159	,,	222.
,,	IV.	,,	223	,,	299.

CONTENTS.

PAGE

223. *Note sur un théorème général par rapport à l'élimination* . . 1
Tortolini, t. VII. (1856), pp. 454—458

224. *Sur un théorème d'Abel. Note* 5
Tortolini, t. VIII. (1857), pp. 201—203

225. *On a Class of Dynamical Problems* 7
Proc. R. S. t. VIII. (1857), pp. 506—511

226. *On Professor Mac Cullagh's Theorem of the Polar Plane* . . 12
Proc. R. I. A. t. VI. (1858), pp. 481—491

227. *On the Theory of Reciprocal Surfaces* 21
Proc. R. I. A. t. VII. (1862), pp. 20—28

228. *Sur l'intégrale* $\int_0^1 \frac{t^{\mu+\frac{1}{2}}(1-t)^{\mu-\frac{1}{2}}dt}{(a+bt-ct^2)^{\mu+1}}$ 28
Liouville, t. II. (1857), pp. 49—51

229. *Note sur une formule pour la réversion des séries* . . . 30
Crelle, t. LII. (1856), pp. 276—284

230. *Note sur la méthode d'élimination de Bezout* 38
Crelle, t. LIII. (1857), pp. 366, 367

231. *Note sur l'équation* $x^2 - Dy^2 = \pm 4$, $D \equiv 5$ (mod. 8) . . . 40
Crelle, t. LIII. (1857), pp. 369—371

PAGE

232. *Mémoire sur la forme canonique des fonctions binaires* . . 43
Crelle, t. LIV. (1857), pp. 48—58

233. *Addition au Mémoire sur la forme canonique des fonctions binaires* 53
Crelle, t. LIV. (1857), p. 292

234. *Deuxième Note sur une formule pour la réversion des séries* . 54
Crelle, t. LIV. (1857), pp. 156—161

235. *Sur quelques formules pour la transformation des intégrales elliptiques* 60
Crelle, t. LV. (1858), pp. 15—24

236. *Note sur la composition du nombre* 47 *par rapport aux vingt-troisièmes racines de l'unité* 70
Crelle, t. LV. (1858), p. 192

237. *Théorème sur les déterminants gauches* 72
Crelle, t. LV. (1858), pp. 277, 278

238. *Note sur les normales d'une conique* 74
Crelle, t. LVI. (1859), pp. 182—185

239. *Addition à la Note sur la composition du nombre* 47 *par rapport aux vingt-troisièmes racines de l'unité* . . . 78
Crelle, t. LVI. (1859), pp. 186, 187

240. *Note on a Theorem in Spherical Trigonometry* 80
Phil. Mag. t. XVII. (1859), p. 151

241. *On Poinsot's four new Regular Solids* 81
Phil. Mag. t. XVII. (1859), pp. 123—128

242. *Second Note on Poinsot's four new Regular Polyhedra* . . 86
Phil. Mag. t. XVII. (1859), pp. 209, 210

243. *On the Theory of Groups as depending on the Symbolic Equation* $\theta^n = 1$. *Third Part* 88
Phil. Mag. t. XVIII. (1859), pp. 34—37. Sequel to 125 and 126

PAGE

244. *On an Analytical Theorem relating to the distribution of Electricity upon two Spherical Surfaces* 92
Phil. Mag. t. XVIII. (1859), pp. 119—127

245. *On an Analytical Theorem connected with the distribution of Electricity upon two Spherical Surfaces. Second Part* . 99
Phil. Mag. t. XVIII. (1859), pp. 193—202

246. *On Contour and Slope Lines* 108
Phil. Mag. t. XVIII. (1859), pp. 264—268

247. *On the Analytical Forms called Trees. Second Part* . . 112
Phil. Mag. t. XVIII. (1859), pp. 374—378. Continuation of **203**

248. *Sketch of a proof of the Theorem that every Algebraic Equation has a Root* 116
Phil. Mag. t. XVIII. (1859), pp. 436—439

249. *Note on Cones of the Third Order* 120
Phil. Mag. t. XVIII. (1859), pp. 439—442

250. *Sur la surface qui est l'enveloppe des plans conduits par les points d'un ellipsoïde perpendiculairement aux rayons menés par le centre* 123
Tortolini, t. II. (1859), pp. 168—179

251. *Sur quelques formules pour la différentiation* 135
Tortolini, t. II. (1859), pp. 214—230

252. *Note sur l'équation des différences pour une équation donnée de degré quelconque* 150
Tortolini, t. II. (1859), pp. 365, 366

253. *Sur la courbe parallèle à l'ellipse* 152
Tortolini, t. III. (1860), pp. 311—316

254. *Sur la surface parallèle à l'ellipsoïde* 158
Tortolini, t. III. (1860), pp. 345—352

255. *On a Problem of Double Partitions* 166
Phil. Mag. t. XX. (1860), pp. 337—341

PAGE

256. *On a System of Algebraic Equations* 171
Phil. Mag. t. xx. (1860), pp. 341, 342

257. *On the Cubic Centres of a line with respect to three lines and a line* 173
Phil. Mag. t. xx. (1860), pp. 418—423

258. *On a Relation between two Ternary Cubic Forms* . . . 179
Phil. Mag. t. xx. (1860), pp. 512—514

259. *The Problem of Polyhedra* 182
Phil. Trans. t. CXLVII. (for 1857), pp. 183—185

260. *On the Double Tangents of a Plane Curve* 186
Phil. Trans. t. CXLIX. (for 1859), pp. 193—212

261. *On the Conic of Five-pointic Contact at any Point of a Plane Curve* 207
Phil. Trans. t. CXLIX. (for 1859), pp. 371—400

262. *On the Equation of Differences for an Equation of any Order and in particular for the Equations of the Orders two, three, four, and five* 240
Phil. Trans. t. CL. (for 1860), pp. 93—112

263. *Demonstration of a Theorem in Finite Differences* . . . 262
Phil. Trans. t. CL. (for 1860), pp. 321—323

264. *On an Extension of Arbogast's Method of Derivations* . . 265
Phil. Trans. t. CLI. (for 1861), pp. 37—43

265. *Addition to the Memoir on an Extension of Arbogast's Method of Derivations* 272
Now first published, 1891

266. *On the Equation for the Product of the Differences of all but one of the Roots of a given Equation* 276
Phil. Trans. t. CLI. (for 1861), pp. 45—59

267. *On the Porism of the In-and-circumscribed Polygon* . . . 292
Phil. Trans. t. CLI. (for 1861), pp. 225—239

PAGE

268. *On a New Auxiliary Equation in the Theory of Equations of the Fifth Order* 309
Phil. Trans. t. CLI. (for 1861), pp. 263—276

269. *A Seventh Memoir on Quantics* 325
Phil. Trans. t. CLI. (for 1891), pp. 277—292

270. *On the Double Tangents of a Curve of the Fourth Order* . 342
Phil. Trans. t. CLI. (for 1861), pp. 357—362

271. *Sur l'invariant le plus simple d'une fonction quadratique bi-ternaire, et sur le résultant de trois fonctions quadratiques ternaires* 349
Crelle, t. LVII. (1860), pp. 139—148

272. *Démonstration d'un théorème de Jacobi par rapport au problème de Pfaff* 359
Crelle, t. LVII. (1860), pp. 273—277

273. *Note sur la transformation de Tschirnhausen* 364
Crelle, t. LVIII. (1861), pp. 259—262

274. *Deuxième Note sur la transformation de Tschirnhausen* . . 368
Crelle, t. LVIII. (1861), pp. 263—269

275. *On Tschirnhausen's Transformation* 375
Phil. Trans. t. CLI. (1861), pp. 561—578

276. *On the Analytical Theory of the Conic* 395
Phil. Trans. t. CLII. (1862), pp. 639—662

277. *On the Wave Surface* 420
Quart. Math. Jour. t. III. (1860), pp. 16—22

278. *Note on the Singular Solutions of Differential Equations* . . 426
Quart. Math. Jour. t. III. (1860), pp. 36, 37

279. *On a Theorem relating to Spherical Conics* 428
Quart. Math. Jour. t. III. (1860), p. 53

280. *On the Conics which touch four given lines* 429
Quart. Math. Jour. t. III. (1860), pp. 94—96

PAGE

281. *Note on the Wave Surface* 432
Quart. Math. Jour. t. III. (1860), pp. 142—144

282. *On a particular case of Castillon's Problem* 435
Quart. Math. Jour. t. III. (1860), pp. 157—164

283. *On a Theorem relating to Homographic Figures* 442
Quart. Math. Jour. t. III. (1860), pp. 177—180

284. *On a New Analytical Representation of Curves in Space* . 446
Quart. Math. Jour. t. III. (1860), pp. 225—236

285. *On the System of Conics having double contact with each other* 456
Quart. Math. Jour. t. III. (1860), pp. 246—250

286. *Note on the value of certain Determinants the terms of which are the squared distances of Points in a Plane or in Space* 460
Quart. Math. Jour. t. III. (1860), pp. 275—277

287. *Note on the Equation for the squared differences of the Roots of a Cubic Equation* 463
Quart. Math. Jour. t. III. (1860), pp. 307—309

288. *Note on the Curvature of a plane Curve at a double point, and on the Curvature of Surfaces* 466
Quart. Math. Jour. t. III. (1860), pp. 322—326

289. *On some Numerical Expansions* 470
Quart. Math. Jour. t. III. (1860), pp. 366—369

290. *A Discussion of the Sturmian Constants for cubic and quartic Equations* 473
Quart. Math. Jour. t. IV. (1861), pp. 7—12

291. *On the Demonstration of a Theorem relating to the Moments of Inertia of a Solid Body* 478
Quart. Math. Jour. t. IV. (1861), pp. 25—27

292. *A Theorem in Conics* 481
Quart. Math. Jour. t. IV. (1861), pp. 131—133

PAGE

293. *On a Certain System of Functional Symbols* 484
Quart. Math. Jour. t. IV. (1861), pp. 225—230

294. *On a New Analytical Representation of Curves in Space* . 490
Quart. Math. Jour. t. V. (1862), pp. 81—86

295. *On the Construction of the ninth point of Intersection of the Cubics which pass through eight given points* 495
Quart. Math. Jour. t. V. (1862), pp. 222—233

296. *On the Conics which pass through the four foci of a given Conic* 505
Quart. Math. Jour. t. V. (1862), pp. 275—280

297. *On some Formulæ relating to the Distances of a point from the vertices of a Triangle and to the Problem of Tactions* . 510
Quart. Math. Jour. t. V. (1862), pp. 381—384

298. *Report on the Progress of the Solution of certain Special Problems of Dynamics* 513
Report of the Brit. Assoc. for the Advancement of Science, 1862, pp. 184—252

299. *Mathematics, recent Terminology in* 594
English Cyclopædia, t. V. (1860), pp. 534—542

Notes and References 609

Plate . . . . . *to face p.* 134

CLASSIFICATION.

GEOMETRY

Reciprocal Surfaces, 227
Analytical Representation of Curves in Space, 284, 294
Contour and Slope Lines, 246
Polyhedra, 241, 242, 259
Conics, 238, 261, 276, 279, 280, 285, 292, 296
Cubic Curves, Cones, &c., 249, 257, 258, 295
Curvature of Curves and Surfaces, 288
Double Tangents, 260, 270
Envelopes, Parallel Curves and Surfaces, 250, 253, 254
Homographic Figures, 283
In-and-circumscribed Polygon, 267, 282
Mac Cullagh's Polar Plane, 226
Spherical Triangles, 240
Wave Surface, 277, 281

ANALYSIS

Mathematics, Recent Terminology in, 299
Dynamics. Report on the Progress of the solution of certain Special Problems, 298
—— Class of Problems in, 225, Moments of Inertia, 291
Quantics, binary and ternary, &c., 232, 233, 269, 271
Equation of Differences, Quintic Equation, &c., 252, 262, 266, 268, 287
Tschirnhausen's Transformation, 273, 274, 275
Theory of Numbers, 231, 236, 239
Determinants, 237, 286, 297
Derivations (Arbogast), 264, 265
Differentiation, 251, Differential Equations, 272, 278
Finite Differences; (Electricity), 244, 245; 263
Elimination, 223, 230
Elliptic Integrals, Modular Functions, 235, 293
Integral Calculus, Definite Integrals, 224, 228
Equations, 248, 256
Groups, 243
Numerical Expansions, 289
Partitions, 255
Reversion of Series, 229, 234
Sturmian Constants, 290
Trees, 247

223.

NOTE SUR UN THÉORÈME GÉNÉRAL PAR RAPPORT À L'ÉLIMINATION.

[From the *Annali di Scienze Mathematiche e Fisiche* (Tortolini), vol. VII. (1856), pp. 454—458.]

SOIENT

$$\phi = (a, b, c \ldots)(x, y)^n = 0,$$
$$\psi = (\alpha, \beta, \gamma \ldots)(x, y)^n = 0,$$

deux équations homogènes quelconques entre les variables x, y, et représentons par R la résultante des deux fonctions ϕ, ψ, de manière que $R=0$ sera la condition pour que les deux équations $\psi=0$, $\phi=0$ puissent avoir lieu. On sait depuis longtemps que les valeurs des variables x, y, qui satisfont à la fois aux deux équations $\psi=0$, $\phi=0$ sont données [1] par les conditions (équivalentes à une seule condition)

$$\frac{dR}{da} : \frac{dR}{db} : \frac{dR}{dc} : \text{etc.} = x^m : mx^{m-1}y : \frac{m(m-1)}{1.2} x^{m-2}y^2 : \text{etc.}$$
$$\frac{dR}{d\alpha} : \frac{dR}{d\beta} : \frac{dR}{d\gamma} : \text{etc.} = x^n : nx^{n-1}y : \frac{n(n-1)}{1.2} x^{n-2}y^2 : \text{etc.};$$

ce qui suppose cependant que les coefficients $a, b, c, \ldots \alpha, \beta, \gamma, \ldots$, sont des quantités absolument arbitraires: les conditions dont il s'agit peuvent aussi s'écrire sous la forme

$$\frac{dR}{da} : \frac{dR}{db} : \frac{dR}{dc} : \text{etc.} = \frac{d\phi}{da} : \frac{d\phi}{db} : \frac{d\phi}{dc} : \text{etc.}$$
$$\frac{dR}{d\alpha} : \frac{dR}{d\beta} : \frac{dR}{d\gamma} : \text{etc.} = \frac{d\psi}{d\alpha} : \frac{d\psi}{d\beta} : \frac{d\psi}{d\gamma} : \text{etc.}$$

[1] Il va sans dire que ce n'est que la valeur de $x : y$ laquelle est déterminée; mais dans la théorie des fonctions homogènes les valeurs absolues n'importent rien, et on peut dire que les valeurs x, y sont déterminées, quand $x : y$ est déterminée: on évite par cette locution des longueurs très-ennuyantes.

Or M. Schläfli dans son excellent mémoire "Ueber die Resultante eines Systemes mehrerer algebraïscher Gleichungen," *Trans. de l'Acad. de Vienne*, tom. IV. (1852), a généralisé ce théorème de la manière que voici. En considérant un nombre quelconque d'équations $\psi=0$, $\phi=0$, $\chi=0, \ldots$ entre le même nombre de variables $x=0$, $y=0$, $z=0, \ldots$ et en supposant que a, b, c, etc. soient des quantités qui entrent d'une manière quelconque dans la fonction ϕ, sans entrer dans les autres fonctions ψ, χ, etc. (il n'est nullement nécessaire que le nombre des quantités a, b, $c \ldots$ soit tel que la fonction ϕ reste absolument arbitraire, le nombre des quantités a, b, c, etc. peut même se réduire à 2) M. Schläfli fait voir que l'on a dans ce cas

$$\frac{dR}{da} : \frac{dR}{db} : \frac{dR}{dc} : \text{etc.} = \frac{d\phi}{da} : \frac{d\phi}{db} : \frac{d\phi}{dc} : \text{etc.}$$

Voici en effet le raisonnement fort simple dont se sert M. Schläfli pour établir la proposition dont il s'agit. Les équations $\psi=0$, $\phi=0$, $\chi=0$, etc. seront satisfaites par de certaines valeurs de $x : y : z$, etc. en supposant seulement que les quantités, a, b, c, etc. satisfont à la condition $R=0$. Donc les équations $\psi=0$ $\phi=0$, $\chi=0$, etc. seront encore satisfaites en donnant des variations infiniment petites quelconques δa, δb, δc, etc. aux quantités a, b, c, etc. en supposant seulement que ces variations soient telles que l'on ait

$$\delta R = \frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \frac{dR}{dc}\delta c + \text{etc.} = 0.$$

Or les équations $\psi=0$, $\chi=0$, etc. qui ne contiennent pas les quantités a, b, c, etc. suffisent seules (c'est-à-dire sans l'aide de l'équation $\phi=0$) à déterminer les valeurs de $x : y : z$, etc. qui satisfont au système $\phi=0$, $\psi=0$, $\chi=0$, (on suppose toujours l'équation $R=0$): donc les nouvelles valeurs de $x : y : z$, etc. seront les mêmes qu'auparavant, et l'on doit avoir

$$\delta\phi = \frac{d\phi}{da}\delta a + \frac{d\phi}{db}\delta b + \frac{d\phi}{dc}\delta c + \text{etc.} = 0;$$

savoir cette équation aura lieu en vertu de l'équation $\delta R=0$ qui est la seule condition à laquelle on a assujetti les variations δa, δb, δc, etc., ce qui donne évidemment les conditions

$$\frac{dR}{da} : \frac{dR}{db} : \frac{dR}{dc} : \text{etc.} = \frac{d\phi}{da} : \frac{d\phi}{db} : \frac{d\phi}{dc} : \text{etc.}$$

Cela étant, on peut encore généraliser le théorème de M. Schläfli: pour cela je suppose que les quantités a, b, $c, \ldots$ entrent d'une manière quelconque dans les fonctions ϕ, ψ, χ, etc. Les équations $\phi=0$, $\psi=0$, $\chi=0$, etc. impliquent l'équation $R=0$, et en donnant aux quantités a, b, c, etc. des variations infiniment petites quelconques δa, δb, δc, etc. qui satisfont à la condition $\delta R=0$, les équations $\phi=0$, $\psi=0$, $\chi=0$, etc. seront satisfaites à la fois, cependant par des nouvelles valeurs des variables;

on peut représenter par δx, δy, δz, etc. les variations qu'il faut attribuer aux variables x, y, z, etc. Les équations $\phi = 0$, $\psi = 0$, $\chi = 0$, etc. seront satisfaites en y variant à la fois les valeurs des variables x, y, z, etc. et des quantités a, b, c, etc.; les variations de ϕ, ψ, χ, etc. doivent donc s'évanouir: je représente de la manière que voici les conditions ainsi obtenues, savoir

$$\delta\phi + \frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y + \frac{d\phi}{dz}\,\delta z + \text{etc.} = 0,$$

$$\delta\psi + \frac{d\psi}{dx}\,\delta x + \frac{d\psi}{dy}\,\delta y + \frac{d\psi}{dz}\,\delta z + \text{etc.} = 0,$$

$$\delta\chi + \frac{d\chi}{dx}\,\delta x + \frac{d\chi}{dy}\,\delta y + \frac{d\chi}{dz}\,\delta z + \text{etc.} = 0.$$

etc.

En prenant L, M, N, etc. des fonctions absolument arbitraires, et en prenant aussi

$$\delta u = -\,L\delta x - M\delta y - N\delta z - \text{etc.}$$

on aura l'équation identique

$$\delta u \quad + L\delta x + M\delta y + N\delta z + \text{etc.} = 0,$$

et en éliminant les variations δx, δy, δz, etc. on obtient une équation $\square = 0$; la partie de $\square$ qui contient le terme δu sera évidemment

$$\delta u \left| \begin{matrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz}, \ldots \\ \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz}, \\ \frac{d\chi}{dx}, & \frac{d\chi}{dy}, & \frac{d\chi}{dz}, \end{matrix} \right|$$

et le déterminant, facteur de cette expression, s'évanouit en vertu des équations $\phi = 0$, $\psi = 0$, $\chi = 0$, etc. Cela est en effet un théorème de M. Hesse, lequel se démontre tout de suite en remarquant que l'on a

$$m\phi = x\frac{d\phi}{dx} + y\frac{d\phi}{dy} + z\frac{d\phi}{dz} + \text{etc.} = 0,$$

etc.

L'expression $\square$ ne contient donc pas de terme avec δu, et l'équation $\square = 0$, peut s'écrire comme suit:

$$\left| \begin{matrix} & L, & M, & N, \ldots \\ \delta\phi, & \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz}, \\ \delta\psi, & \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz}, \\ \delta\chi, & \frac{d\chi}{dx}, & \frac{d\chi}{dy}, & \frac{d\chi}{dz}, \end{matrix} \right| = 0$$

équation de la forme

$$X\delta\phi + Y\delta\psi + Z\delta\chi + \text{etc.} = 0$$

c'est-à-dire une équation entre les seules variations δa, δb, δc, etc. Or il ne peut pas y avoir entre ces variations d'autre équation que $\delta R = 0$, on doit donc avoir identiquement

$$X\delta\phi + Y\delta\psi + Z\delta\chi + \text{etc.} = k\delta R,$$

savoir cette équation sera satisfaite par les valeurs de $x : y : z$, etc. qui satisfont à $\phi = 0$, $\psi = 0$, $\chi = 0$, etc. C'est là le théorème qu'il s'agissait de démontrer; en supposant que les quantités a, b, c, etc. n'entrent que dans la fonction ϕ, on a $\delta\psi = 0$, $\delta\chi = 0$, etc., c'est-à-dire $X\delta\phi = k\delta R$, lequel est le théorème de M. Schläfli.

Je remarque que M. Schläfli a donné aussi un théorème par rapport au discriminant d'une fonction quelconque ϕ; savoir, en représentant par ∇ ce discriminant, et en supposant que les quantités a, b, c, etc. entrent d'une manière quelconque dans la fonction ϕ, les valeurs de x, y... qui satisfont aux équations

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0, \text{ etc.}$$

(lesquelles impliquent l'équation $\nabla = 0$) sont données par

$$\frac{d\nabla}{da} : \frac{d\nabla}{db} : \frac{d\nabla}{dc} : \text{etc.} = \frac{d\phi}{da} : \frac{d\phi}{db} : \frac{d\phi}{dc} : \text{etc.};$$

cela est déjà la forme la plus générale du théorème.

London, 2, *Stone Buildings*, 12 *Dec.* 1856.

224.

SUR UN THÉORÈME D'ABEL. NOTE.

[From the *Annali di Scienze Mathematiche e Fisiche* (Tortolini), vol. VIII. (1857), pp. 201—203.]

IL y a un petit mémoire d'Abel qui porte le titre "Ueber die Functionen welche der Gleichung $\phi(x)+\phi(y)=\psi\left(xf(y)+yf(x)\right)$ genugthun" (*Crelle*, tom. II. (1827), pp. 386—394). La solution du problème est contenue dans les équations que voici, savoir $f(x)$ est une fonction définie par l'équation

$$\alpha^{2n}=\left(f(x)-nx\right)^{n+\alpha'}\left(f(x)+nx\right)^{n-\alpha'},$$

et on a alors

$$\phi(x)=\frac{1}{n+\alpha'}\log C\left(f(x)+nx\right),$$

et (en réduisant un peu l'expression donné dans le mémoire)

$$\psi(x)=\frac{1}{n+\alpha'}\log C^2\alpha\left(f\left(\frac{x}{\alpha}\right)+\frac{nx}{\alpha}\right).$$

On a aussi pour $\phi(x)$ cette autre expression en forme d'intégrale indéfinie,

$$\phi(x)=\int\frac{dx}{f(x)+\alpha'x};$$

car le facteur $a\alpha$ par lequel dans le mémoire l'expression à côté droit est multiplié se réduit (comme on voit sans peine) à l'unité. En comparant les deux expressions de $\phi(x)$, on voit qu'il est permis de prendre l'intégrale depuis $x=0$, pourvu qu'on écrive $c=1$; cela donne

$$\int_0\frac{dx}{f(x)+\alpha'x}=\frac{1}{n+\alpha'}\log\left(f(x)+nx\right),$$

formule très simple pour l'intégration d'une expression algébrique laquelle ne peut pas s'exprimer à moyen de radicales.

On obtient une autre propriété de cette fonction $f(x)$ en substituant les valeurs des fonctions ϕ et ψ dans l'équation originale

$$\phi(x)+\phi(y)=\psi\left(xf(y)+yf(x)\right);$$

cela donne d'abord

$$\frac{1}{n+\alpha'}\log C^2\left(f(x)+nx\right)\left(f(x)-nx\right)=\frac{1}{n+\alpha'}\log C^2\alpha\left[f\left(\frac{xf(y)+yf(x)}{\alpha}\right)+\frac{n\left(xf(y)+yf(x)\right)}{\alpha}\right]$$

et de là en réduisant on obtient l'équation fonctionnelle très simple

$$f(x)f(y)+n^2xy=\alpha f\left(\frac{xf(y)-yf(x)}{\alpha}\right).$$

Je remarque que l'on peut sans perte de généralité écrire $\alpha=1$, et $n=1$: je mets β au lieu de α', et j'écris aussi pour plus de simplicité $f(x)=X$, $f(y)=Y$. On a alors pour déterminer la fonction $X\left(=f(x)\right)$, l'équation

$$(X-x)^{1+\beta}(X+x)^{1-\beta}=1$$

équation dans laquelle on pourrait remplacer les exposants $1+\beta$, $1-\beta$ par deux quantités quelconques.

La formule d'intégration devient

$$\int_0\frac{dx}{X+\beta x}=\frac{1}{1+\beta}\log(X+x)=-\frac{1}{1-\beta}\log(X-x)$$

formule que l'on peut vérifier sans peine à moyen de celle-ci,

$$(X+\beta x)X'=x+\beta X,$$

que l'on obtient en différentiant l'équation pour X. L'équation fonctionnelle sera

$$XY+xy=f(xY+yX),$$

c'est-à-dire en écrivant $xY+yX=z$, $XY+xy=Z$ on doit avoir

$$(Z-z)^{1+\beta}(Z+z)^{1-\beta}=1$$

et cela se vérifie tout de suite à moyen des équations

$$Z-z=(X-x)(Y-y),\quad Z+z=(X+x)(Y+y).$$

Je remarque aussi qu'en prenant le quotient des dérivées de cette équation par rapport à x et y on obtient

$$\frac{X'-Y'}{X'Y'-1}=\left(\frac{X}{x}-\frac{Y}{y}\right)\div\left(\frac{X}{x}\cdot\frac{Y}{y}-1\right),$$

laquelle est une propriété de la fonction X et sa dérivée X'.

Londres, 17 *Juillet*, 1857.

225.

ON A CLASS OF DYNAMICAL PROBLEMS.

[From the *Proceedings of the Royal Society of London*, vol. VIII. (1857), pp. 506—511.]

THERE are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to (which might be designated as continuous-impact problems) are those in which the system is continually taking into connexion with itself particles of infinitesimal mass (i.e. of a mass containing the increment of time dt as a factor), so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion. For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system, and in each element of time dt, the system takes into connexion with itself, and sets in motion with a finite velocity, an infinitesimal length ds of the chain; in fact, if v be the velocity of the part which hangs over, then the length vdt is set in motion with the finite velocity v. The general equation of dynamics applied to the case in hand will be

$$\Sigma\left\{\left(\frac{d^2x}{dt^2}-X\right)\delta x+\left(\frac{d^2y}{dt^2}-Y\right)\delta y+\left(\frac{d^2z}{dt^2}-Z\right)\delta z\right\}dm+\Sigma\left(\Delta u\delta\xi+\Delta v\delta\eta+\Delta w\delta\zeta\right)\frac{1}{dt}d\mu=0,$$

where the first term requires no explanation: in the second term ξ, η, ζ denote the coordinates at the time t of the particle $d\mu$ which then comes into connexion with the system; Δu, Δv, Δw are the finite increments of velocity (or, if the particle is originally at rest, then the finite velocities) of the particle $d\mu$ the instant that it has come into connexion with the system; $\delta\xi$, $\delta\eta$, $\delta\zeta$ are the virtual velocities of the same particle $d\mu$ considered as having come into connexion with and forming part of the system. The summation extends to the several particles or to the system of particles $d\mu$ which come into connexion with the system at the time t; of course, if there is only a single particle $d\mu$, the summatory sign Σ is to be omitted. The values of

Δu, Δv, Δw are $\frac{d\xi}{dt}-u$, $\frac{d\eta}{dt}-v$, $\frac{d\zeta}{dt}-w$, if by $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ we understand the velocities of $d\mu$ parallel to the axes, after it has come into connexion with the system; but it is to be observed, that considering ξ, η, ζ as the coordinates of the particle $d\mu$ which is continually coming into connexion with the system, then if the problem were solved and ξ, η, ζ given as functions of t (and, when there is more than one particle $d\mu$, of the constant parameters which determine the particular particle), $\frac{d\xi}{dt}$, &c., in the sense just explained, cannot be obtained by simple differentiation from such values of ξ, &c.: in fact, ξ, η, ζ so given as functions of t, belong at the time t to one particle, and at the time $t+dt$ to the next particle, but what is wanted is the increment in the interval dt of the coordinates ξ, η, ζ of one and the same particle.

Suppose as usual that x, y, z, and in like manner that ξ, η, ζ are functions of a certain number of independent variables θ, ϕ, &c., and of the constant parameters which determine the particular particle dm or $d\mu$, of which x, y, z, or ξ, η, ζ are the coordinates; parameters, that is, which vary from one particle to another, but which are constant during the motion for one and the same particle. The summations are in fact of the nature of definite integrations in regard to these constant parameters, which therefore disappear altogether from the final results. The first term,

$$\Sigma\left\{\left(\frac{d^2x}{dt^2}-X\right)\delta x+\left(\frac{d^2y}{dt^2}-Y\right)\delta y+\left(\frac{d^2z}{dt^2}-Z\right)\delta z\right\}dm,$$

may be reduced in the usual manner to the form

$$\Theta\delta\theta+\Phi\delta\phi+\ldots$$

where, writing as usual θ', ϕ', &c. for $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, &c., we have

$$\Theta=\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}+\frac{dV}{d\theta},$$

$$\Phi=\frac{d}{dt}\frac{dT}{d\phi'}-\frac{dT}{d\phi}+\frac{dV}{d\phi}, \text{ \&c.,}$$

(this supposes that $Xdx+Ydy+Zdz$ is an exact differential); only it is to be observed that in the problems in hand, the mass of the system is variable, or what is the same thing, the variables θ, ϕ, &c., are introduced into T and V through the limiting conditions of the summation or definite integration, besides entering directly into T and V in the ordinary manner. And in forming the differential coefficients $\frac{d}{dt}\frac{dT}{d\theta'}$, $\frac{dT}{d\theta}$, $\frac{dV}{d\theta}$, &c., it is necessary to consider the variables θ, ϕ, &c., in so far as they enter through the limiting conditions *as exempt from differentiation*, so that the expressions just given for Θ, Φ, &c., are, in the case in hand, rather conventional representations than actual analytical values; this will be made clearer in the sequel by the consideration of the before-mentioned particular problem.

Considering next the second term, or

$$\Sigma\left\{\left(\frac{d\xi}{dt}-u\right)\delta\xi+\left(\frac{d\eta}{dt}-v\right)\delta\eta+\left(\frac{d\zeta}{dt}-w\right)\delta\zeta\right\}\frac{1}{dt}d\mu,$$

we have here

$$\begin{aligned}\delta\xi&=a\ \delta\theta+b\ \delta\phi+\ldots,\\ \delta\eta&=a'\ \delta\theta+b'\ \delta\phi+\ldots,\\ \delta\zeta&=a''\ \delta\theta+b''\ \delta\phi+\ldots,\end{aligned}$$

where a, b, a', &c., are functions of the variables θ, ϕ, &c., and of the constant parameters which determine the particular particle $d\mu$. The virtual velocities or increments $\delta\theta$, $\delta\phi$, &c., are absolutely arbitrary, and if we replace them by $d\theta$, $d\phi$, &c., the actual increments of θ, ϕ, &c. in the interval dt during the motion, then $\delta\xi$, $\delta\eta$, $\delta\zeta$ will become $\frac{d\xi}{dt}dt$, $\frac{d\eta}{dt}dt$, $\frac{d\zeta}{dt}dt$, in the sense before attributed to $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$.

The particle $d\mu$ will contain dt as a factor, and the other factor will contain the differentials, or (as the case may be) products of differentials, of the constant parameters which determine the particular particle $d\mu$. We have thus the means of expressing the second line in the proper form; and if we write

$$\begin{aligned}\Sigma\,(a^2+a'^2+a''^2\ )\,d\mu&=\ \ Adt,\\ \Sigma\,(b^2+b'^2+b''^2\ )\,d\mu&=\ \ Bdt,\\ &\vdots\\ \Sigma\,(ab+a'b'+a''b'')\,d\mu&=\ \ Hdt,\\ &\vdots\\ \Sigma\,(au+a'v+a''w\,)\,d\mu&=-Pdt,\\ \Sigma\,(bu+b'v+b''w\,)\,d\mu&=-Qdt,\\ &\vdots\end{aligned}$$

then the required expression of the second line will be

$$(A\theta'+H\phi'\ldots+P)\,\delta\theta+(H\theta'+B\phi'\ldots+Q)\,\delta\phi+\ldots,$$

which, if we put

$$\begin{aligned}K&=\tfrac{1}{2}\,(A\theta'^2+B\phi'^2+\ldots+2H\theta'\phi'+\ldots+2P\theta'+2Q\phi'+\ldots),\\ &=\tfrac{1}{2}\,(A,\ B,\ldots H,\ldots P,\ Q,\ldots\between\theta',\ \phi',\ldots,\ 1)^2,\end{aligned}$$

may be more simply represented by

$$\frac{dK}{d\theta'}\,\delta\theta+\frac{dK}{d\phi'}\,\delta\phi+\ldots;$$

only it is to be remarked that A, $B,\ldots H,\ldots P$, $Q,\ldots$ will in general contain not only θ, $\phi,\ldots$, but also the differential coefficients θ', $\phi',\ldots$, and that in forming the differential coefficients $\frac{dK}{d\theta'}$, $\frac{dK}{d\phi''}$, &c., the quantities θ', $\phi',\ldots$, in so far as they enter

into K, not explicitly, but through the coefficients A, &c., must be considered *as exempt from differentiation*, so that the preceding expression for the second line by means of the function K is rather a conventional representation than an actual analytical value.

Uniting the two terms, and equating to zero the coefficients of $\delta\theta$, $\delta\phi$, &c., we obtain finally the equations of motion in the form

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} + \frac{dV}{d\theta} + \frac{dK}{d\theta'} = 0,$$

$$\frac{d}{dt}\frac{dT}{d\phi'} - \frac{dT}{d\phi} + \frac{dV}{d\phi} + \frac{dK}{d\phi'} = 0,$$

$$\vdots$$

where the several symbols are to be taken in the significations before explained.

In the particular problem, let z be measured vertically downwards from the plane of the table, then $Z = g$, and repeating for the particular case the investigation *ab initio*, the general equation of motion is

$$\Sigma\left(\frac{d^2z}{dt^2} - g\right)\delta z\, dm + \frac{d\zeta}{dt}\,\delta\zeta\,\frac{1}{dt}\,d\mu = 0.$$

Let s be the length in motion, or, what is the same thing, the z coordinate of the lower extremity; and suppose also that the mass of a unit of length is taken equal to unity, we have $\delta z = \delta s$, $\frac{d^2z}{dt^2} = \frac{d^2s}{dt^2}$, $dm = dz$, and the summation or integration with respect to z is from $z = 0$ to $z = s$, whence

$$\Sigma\left(\frac{d^2z}{dt^2} - g\right)\delta z\, dm = \left(\frac{d^2s}{dt^2} - g\right)\delta s\, \Sigma dz = \left(\frac{d^2s}{dt^2} - g\right) s\delta s\,;$$

which is of the form

$$\left(\frac{d}{dt}\frac{dT}{ds'} - \frac{dT}{ds} + \frac{dV}{ds}\right)\delta s,$$

if

$$T = \tfrac{1}{2}s'^2 . \bar{s}, \quad V = -gs\bar{s},$$

where the bar is used to denote exemption from differentiation, but ultimately $\bar{s}$ is to be replaced by s. Considering now the second term, here $\zeta = 0$, but $\delta\zeta = \delta s$, and thence $\frac{d\zeta}{dt} = s'$. Moreover, $d\mu = s'dt$, and thence finally the second term is s'^2, which is of the form $\frac{dK}{d\bar{s}'}$, if

$$K = \tfrac{1}{2}\bar{s}' . s'^2,$$

the bar having the same signification as before, but after the differentiation $\bar{s}' = s'$. The resulting equation is

$$\left(\frac{d^2s}{dt^2} - g\right)s + \left(\frac{ds}{dt}\right)^2 = 0,$$

which may be written in the form

$$s\frac{ds}{dt}d\left(s\frac{ds}{dt}\right)=gs^2ds,$$

and the first integral is therefore

$$\frac{sds}{\sqrt{s^3-a^3}}=\sqrt{\frac{2g}{3}}\,dt,$$

where a is the length hanging over at the commencement of the motion. If $a=0$, then the equation is

$$\frac{ds}{\sqrt{s}}=\sqrt{\frac{2g}{3}}\,dt,$$

and integrating from $t=0$, $2\sqrt{s}=\sqrt{\frac{2g}{3}}\,t$, or finally $s=\frac{1}{6}gt^2$, so that the motion is the same as that of a body falling under the influence of a constant force $\frac{1}{3}g$. It is perhaps worth noticing that the differential equation may be obtained as follows:— We have, in the first instance, a mass s moving with a velocity s', and after the particle $ds(=s'dt)$ has been set in motion, a mass $s+s'dt$ moving say with a velocity $s'+\delta s'$, whence neglecting for the moment the effect of gravity on the mass s, the momentum of the mass in motion will be constant, or we shall have

$$ss'=(s+s'dt)(s'+\delta s')=ss'+s'^2dt+s\delta s',$$

and therefore $s\delta s'=-s'^2dt$. Hence, adding on the right-hand side the term $gsdt$ arising from gravity, and substituting $\frac{d^2s}{dt^2}dt$ for $\delta s'$, we have the equation, $s\frac{d^2s}{dt^2}=gs-\left(\frac{ds}{dt}\right)^2$, as before.

226.

ON PROFESSOR MAC CULLAGH'S THEOREM OF THE POLAR PLANE.

[From the *Proceedings of the Royal Irish Academy*, vol. VI. (1858), pp. 481—491.]

A RAY of polarized light, incident on the surface of an extraordinary medium, may give rise to a reflected ray and a single refracted ray; but this will be the case only for a particular position, or positions, of the plane of polarization of the incident ray. According to Professor Mac Cullagh's theory, the plane of polarization, and the relative vibrations of the three rays, are deduced from two assumed principles, which may be referred to as

1°. The principle of equivalent vibrations.

2°. The principle of equivalent moments.

And from these principles are deduced

3°. The principle of vis viva.

4°. The theorem of the polar plane.

The directions of the vibrations are completely determined by means of 4°, the theorem of the polar plane; and the relative magnitudes are then given by 1°, the principle of equivalent vibrations. The other principles, viz.:—2°, the principle of equivalent moments, and 3°, the principle of vis viva, must therefore follow as mere geometrical consequences from the first-mentioned two principles, or theorems; and I have found that the deduction depends immediately upon the following two theorems in spherical trigonometry.

Suppose (Fig. 1) that $RR'R''$ is a spherical triangle, and let W be any point in the base RR'', and N be the central point of the base; then joining WR' and

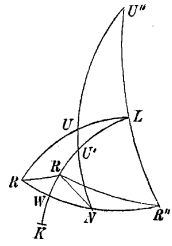


Fig. 1.

producing this arc (in the direction from W to R') to a point L, such that

$$\cot WL = \frac{\sin^2 NW}{\sin WR \sin WR'} \tan WR',$$

and joining NR', then

THEOREM I.

$$\frac{\sin^2 R''LR' - \sin^2 RLR'}{\sin^2 R''LR} = \frac{\sin NW \cos NR'}{\cos WR' \sin NR \cos NR'};$$

and if we suppose also, that an arc through N, perpendicular to the base RR'', cuts LR, LR', and LR'' produced in the points U, U', U'', then

THEOREM II.

$$\sin R''LR' \cos RU \sin NUR + \sin RLR' \cos R''U'' \sin NU''R''$$
$$= \sin R''LR \frac{\cos NW}{\cos WR' \sin NR} \cos R'U' \sin NU'R'.$$

The present memoir contains the proof of the two theorems, and the application of them to the optical theory.

To prove the first theorem, I write for shortness R, R'', W to denote the angles LRR'', $LR''R$, NWR', respectively; we have then,

$$\frac{\sin^2 R''LR' - \sin^2 RLR'}{\sin^2 R''LR} = \frac{\sin (R''LR' - RLR')}{\sin R''LR}$$
$$= \frac{1}{\sin R''LR} \{\sin R''LR' \cos RLR' - \sin RLR' \cos R''LR'\}$$
$$= \frac{1}{\sin R''LR} \left\{ \frac{\sin R''W \sin R''}{\sin LW} \cdot \frac{\cos RW - \cos LR \cos LW}{\sin LR \sin LW} \right.$$
$$\left. - \frac{\sin RW \sin R}{\sin LW} \cdot \frac{\cos R''W - \cos LR'' \cos LW}{\sin LR'' \sin LW} \right\}$$

$$= \frac{1}{\sin R''LR \sin^2 LW} \left\{ \frac{\sin R''}{\sin LR} \sin R''W (\cos RW - \cos LR \cos LW) \right.$$
$$\left. - \frac{\sin R}{\sin LR''} \sin RW (\cos R''W - \cos LR'' \cos LW) \right\}.$$

Observing that $\frac{\sin R''}{\sin LR}$, $\frac{\sin R}{\sin LR''}$, are each equal to $\frac{\sin R''LR}{\sin R''R}$, this becomes

$$= \frac{1}{\sin R''R \sin^2 LW} \{\sin R''W (\cos RW - \cos LR \cos LW)$$
$$- \sin RW (\cos R''W - \cos LR'' \cos LW)\};$$

and, substituting for $\cos LR$, $\cos LR''$, the values

$$\cos R\,W \cos LW - \sin R\,W \sin LW \cos W,$$
$$\cos R''W \cos LW + \sin R''W \sin LW \cos W,$$

the foregoing expression becomes

$$= \frac{1}{\sin R''R \sin^2 LW} \times \{\sin R''W (\cos R\,W \sin^2 LW + \sin R\,W \sin LW \cos LW \cos W)$$
$$- \sin RW (\cos R''W \sin^2 LW - \sin R''W \sin LW \cos LW \cos W)\},$$

$$= \frac{1}{\sin R''R} \{\sin R''W \cos RW - \sin RW \cos R''W + 2 \cot LW \sin RW \sin R''W \cos W\}$$

$$= \frac{1}{\sin R''R} \{\sin (R''W - RW) + 2 \cot LW \sin RW \sin R''W \cos W)\};$$

and, putting $R''W - RW = 2NW$, and substituting also for $\cot WL$ its value, which gives $\cot LW \sin RW \sin R''W = \sin^2 NW \tan WR'$, the expression becomes

$$= \frac{1}{\sin R''R} \{\sin 2NW + 2 \sin^2 NW \tan WR' \cos W\};$$

but we have

$$\cos W = \frac{\cos NR' - \cos NW \cos WR'}{\sin NW \sin WR};$$

and therefore

$$2 \sin^2 NW \tan WR' \cos W = 2 \frac{\sin NW}{\cos WR'} \cos NR' - \sin 2NW;$$

thus the expression becomes

$$= \frac{1}{\sin R''R} 2 \frac{\sin NW}{\cos WR'} \cos NR';$$

and $\sin R''R = \sin 2NR = 2 \sin NR \cos NR$, so that finally the expression becomes

$$= \frac{\sin NW \cos NR'}{\cos WR' \sin NR \cos NR},$$

which proves the theorem.

To prove the second theorem, take as before R, R'', W, to denote the angles LRR'', $LR''R$, NWR', respectively; and moreover, U, U', U'' to denote the angles NUR, $NU'R'$, $NU''R''$, respectively; then considering, first, the function on the left-hand side, viz.:

$$\sin R''LR' \cos RU \sin U + \sin RLR' \cos R''U'' \sin U'',$$

we have

$$\sin U = \frac{\sin NR}{\sin RU},$$

$$\cos RU \sin U = \sin NR \cot RU$$

$$= \sin NR \cos R \cot NR = \cos R \cos NR,$$

and, in like manner,

$$\sin U'' = \frac{\sin NR''}{\sin R''U''},$$

$$\cos R''U'' \sin U'' = \sin NR'' \cot R''U'' = \sin NR'' \cos R'' \cot NR''$$

$$= \cos R'' \cos NR'' = \cos R'' \cos NR;$$

the expression thus becomes

$$= \cos NR \{\sin R''LR' \cos R + \sin RLR' \cos R''\},$$

which is

$$= \cos NR \left\{ \frac{\sin R''W \sin W}{\sin R''L} \cdot \frac{\cos WL - \cos RW \cos RL}{\sin RW \sin RL} \right.$$

$$\left. - \frac{\sin RW \sin W}{\sin RL} \cdot \frac{\cos WL - \cos R''W \cos R''L}{\sin R''W \sin R''L} \right\},$$

or, substituting for $\cos RL$, $\cos R''L$ the values

$$\cos R\,W \cos WL - \sin R\,W \sin WL \cos W,$$

$$\cos R''W \cos WL + \sin R''W \sin WL \cos W,$$

the expression becomes

$$\frac{\cos NR \sin W}{\sin RL \sin R''L} \left\{ \frac{\sin R''W}{\sin RW} (\cos WL \sin^2 R\,W + \sin WL \sin R\,W \cos R\,W \cos W) \right.$$

$$\left. + \frac{\sin RW}{\sin R''W} (\cos WL \sin^2 R''W - \sin WL \sin R''W \cos R''W \cos W) \right\}$$

$$= \frac{\cos NR \sin W}{\sin RL \sin R''L} \{2 \cos WL \sin RW \sin R''W + \sin WL \sin (R''W - RW) \cos W\}$$

$$= \frac{\cos NR \sin W \sin WL}{\sin RL \sin R''L} \{2 \cot WL \sin RW \sin R''W + \sin (R''W - RW) \cos W\}.$$

Hence, putting for cot WL its value, which gives

$$\cot WL \sin RW \sin R''W = \sin^2 NW \tan WR',$$

and putting also

$$\sin(R''W - RW) = \sin 2NW = 2 \sin NW \cos NW,$$

the expression becomes

$$= \frac{2 \cos NR \sin W \sin WL \sin^2 NW}{\sin RL \sin R''L} (\tan WR' + \cot NW \cos W).$$

The right-hand side of the equation to be proved is

$$\sin R''LR \frac{\sin NW}{\cos WR' \sin NR} \cos R'U' \sin U',$$

and we have

$$\sin R''LR = \frac{\sin RR'' \sin R}{\sin R''L}, \qquad \sin R = \frac{\sin WL \sin W}{\sin RL},$$

and consequently

$$\sin R''LR = \frac{\sin RR'' \sin WL \sin W}{\sin RL \sin R''L} = \frac{2 \sin NR \cos NR \sin NL \sin W}{\sin RL \sin R''L},$$

or the expression is

$$= \frac{2 \cos NR \sin WL \sin W}{\sin RL \sin R''L} \cdot \frac{\sin NW}{\cos WR'} \cos R'U' \sin U'.$$

But we have

$$\sin U' = \frac{\sin NW}{\sin W'U'};$$

and therefore

$$\frac{\sin NW}{\cos WR'} \cos R'U' \sin U' = \sin^2 NW \frac{\cos R'U'}{\cos WR' \sin WU'}$$

$$= \sin^2 NW \frac{\cos(WU' - WR')}{\cos WR' \sin WU'}$$

$$= \sin^2 NW (\tan WR' + \cot WU').$$

Moreover we have $\cot WU' = \cot NW \cos W$, and the expression thus becomes

$$= \frac{2 \cos NR \sin W \sin WL \sin^2 NW}{\sin RL \sin R''L} (\tan WR' + \cot NW \cos W);$$

which is the expression previously found as the value of the left-hand side of the equation, and the theorem is therefore proved.

It is obvious that the point L might have been constructed by taking on $R'W$, produced in the direction from R' to W, a point K such that

$$\tan KW = \frac{\sin^2 NW}{\sin RW \sin R''W} \tan WR',$$

and then taking the arc KL in the reverse direction equal to 90°.

Passing now to the optical problem, it will be recollected that in Mac Cullagh's theory the direction of vibration in an extraordinary medium is perpendicular to the plane of the ray and wave normal, and that the polar plane of a refracted ray is by definition a plane through the point of incidence parallel to the direction of vibration, and also parallel to a line joining the extremity of the ray with the corresponding point on the *Index surface*,—the last-mentioned surface being the polar reciprocal of the refracted wave-surface, taken with respect to the reflected wave-surface, or wave-sphere, contemporaneously generated. We have to consider a ray of polarized light incident on the surface of an extraordinary medium, and giving rise to a reflected ray and a single refracted ray. Let the incident ray and the reflected ray be respectively produced within the medium, and let the three rays, viz., the incident ray produced, the refracted ray, and the reflected ray produced, be represented in direction (see Fig. 2) by AR, AR' and AR''; and take $AR = AR'' = 1$ as the radius of the wave-sphere and AR' as the radius of the wave-surface, corresponding at a given instant

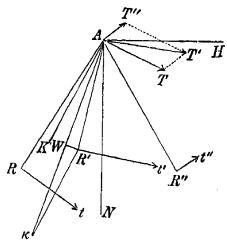


Fig. 2.

of time to the first or ordinary medium and the extraordinary medium respectively. Take also AW as the perpendicular on the tangent plane of the wave-surface at R', or 'wave-normal,' corresponding to the refracted ray AR'; and let AN represent the normal to the plane of separation of the two media, and AH the intersection of the last-mentioned plane with the plane of incidence. The lines AR, AR'', AW, AN, AH are of course all of them in the plane of incidence, the line AN bisects the angle made by the lines AR, AR'', and the lines AN, AH are at right angles to each other. The length of the wave-normal AW is given by the equation $AR \sin NAW = AW \sin NAR$, or putting, as above, $AR = 1$, and representing the two angles at A by NW, NR respectively, then, if p denote the length of the wave-normal, we have $\sin NW = p \sin NR$.

Take κ the pole of the tangent plane of the wave-surface at R' (or, what is the same thing, the image of the point W), in respect of the sphere radius AR, then κ will be the point on the index-surface corresponding to the point R' of the wave-surface; and let AK be drawn through the point A parallel to $R'\kappa$. Take AT' perpendicular to the plane WAR' (or, what is the same thing, the plane KAR') as the direction of the refracted vibration, the plane KAT' will be the polar plane; and by 4°, the theorem of the polar plane, the directions of the incident and reflected vibrations are given as the intersections of the polar plane with the wave-fronts or planes through A normal to the directions of the incident and reflected rays respectively; these intersections are represented in the figure by AT and AT''. The relative magnitudes of the vibrations are then determined by 2°, the principle of equivalent vibrations, viz., considering these vibrations as forces acting in the given directions AT', AT, AT'' respectively, the refracted vibration will be the resultant of the incident and reflected vibrations: the terminated lines AT', AT, AT'' in the figure are taken to represent in direction and magnitude the vibrations corresponding to the refracted ray and to the incident and reflected rays respectively, and the lines $R't'$, Rt, $R''t''$ are drawn through the extremities R', R, R'' of the three rays equal and parallel to AT', AT, and AT'' respectively. Let m', m, m'' denote the masses of ether set in motion by the three rays respectively, then, according to Mac Cullagh's hypothesis of equal densities, we have

$$m = m'' : m' :: AR \cos RN : \frac{AW \cos R'N}{\cos WR'},$$

(where RN, &c., denote the angles RAN, &c.); or writing as before, $AR = 1$, $AW = p$, where $\sin NW = p \sin RN$, we have

$$m = m'' : m' :: \quad \cos RN : \frac{p \cos R'N}{\cos WR'} \left(= \frac{\sin NW \cos R'N}{\cos WR' \sin RN} \right).$$

This being premised, then, 3°, the principle of vis viva is that

$$m\,(Rt)^2 = m'\,(R't')^2 + m''\,(R''t'')^2;$$

or, what is the same thing,

$$\frac{Rt^2 - R''t''^2}{R't'^2} = \frac{m'}{m} = \frac{\sin NW \cos R'N}{\cos WR' \sin RN \cos RN};$$

and 2°, the principle of equivalent moments, is that the moment of $R't'$ round the axis AH, is equal to the sum of the moments of Rt and $R''t''$ round the same axis. It only remains to show that these two properties are in fact contained in the Theorems I. and II.

The point κ is the image of W in a sphere, radius unity. Hence, $A\kappa = \frac{1}{p}\kappa W = \frac{1}{p} - p$, and therefore

$$\tan W\kappa R' = \frac{p^2 \tan WR'}{1 - p^2} = \tan KW,$$

but we have, as before, $\sin NW = p \sin RN$, and consequently,

$$\tan KW = \frac{\sin^2 NW \tan WR'}{\sin^2 RN - \sin^2 RW}$$

$$= \frac{\sin^2 NW}{\sin RW \sin R'W} \tan WR'.$$

Suppose now that the points R, R', R'', W, N, H, K, of Fig. 2, are all of them projected by radii through the centre A upon a sphere, radius unity (see Fig. 3, where the several points are represented by the same letters as in Fig. 2); and complete Fig. 3 by connecting the different points in question by arcs of great circles, and by producing KW (in the direction from K to W) to a point L, such that $KL = 90°$, and by joining LR, LR'', and drawing the arc $NU'UU''$ at right angles to $R''R$ (or, what is the same thing, with the pole H) meeting LR', LR, and LR'' produced, in the points U', U, U'' respectively. By what has preceded, the points K, L of Fig. 3

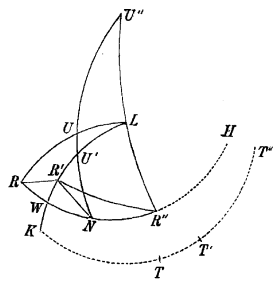


Fig. 3.

are constructed precisely in the same manner as the same points in Fig, 1, and in fact Fig. 3 is nothing else than Fig. 1 with some additional lines and points. The condition employed to determine the magnitude of the vibrations Rt, $R't'$, $R''t''$, gives that these vibrations are as

$$\sin T'T'' : \sin TT'' : \sin TT',$$

or, observing that LR, LR', LR'' are the great circles whose poles are T, T', T'' respectively, these vibrations are as

$$\sin R''LR' : \sin RLR'' : \sin RLR';$$

and, substituting these values, the equation given by the principle of vis viva becomes identical with that of Theorem I.

Proceeding to the condition given by the principle of equivalent moments, we have

$$\text{moment of } Rt \text{ round } AH$$
$$= Rt \times AR \times \cos[AR, \perp \text{dist.}(Rt, AH)] \times \sin(Rt, AH);$$

and in Fig. 3, observing that the radius through U is parallel to the perpendicular distance of (Rt, AH) (for LR has the pole T, and NU the pole H) then

$$\cos[AR, \perp \text{dist.}(Rt, AH)] = \cos RU,$$
$$\sin(Rt, AH) = \sin TH,$$

or, since T and H are the poles of LR and NW respectively, $TH = \angle NUR$, and, putting $AR = 1$, the moment is

$$= Rt \cos RU \sin NUR.$$

Similarly,

$$\text{moment of } R''t'' \text{ round } AH$$
$$= R''t'' \cos R''U'' \sin NU''R'';$$

and for the refracted ray,

$$\text{moment of } R't' \text{ round } AH$$
$$= AR' \times R't' \cos R'U' \sin NU'R'.$$

But we have

$$AR' = \frac{AW}{\cos WR'} = \frac{\sin NW}{\cos WR' \sin NR},$$

and therefore the moment is

$$= R't' \frac{\sin NW}{\cos WR' \sin NR} \cos R'U' \sin NU'R'.$$

Hence the vibrations Rt, $R''t''$, $R't'$, as before, are as

$$\sin R''LR' : \sin RLR' : \sin RLR'',$$

and thus the equation given by the principle of equivalent moments is precisely that of Theorem II.

227.

ON THE THEORY OF RECIPROCAL SURFACES.

[From the *Proceedings of the Royal Irish Academy*, vol. VII. (1862), pp. 20—28.]

THE present note is intended to be supplementary to Mr Salmon's memoir "On the Degree of a Surface reciprocal to a given one" (*Trans. R. Irish Acad.*, vol. XXI. pp. 461—488, 1857). I find that Mr Salmon's equations admit of a transformation which appears important in reference to the geometrical theory, and the object of the note is to present the system of equations under the new form.

Mr Salmon writes:

n, the order of the surface.

a, the order of the tangent cone drawn from any point to the surface.

δ, the number of the double edges of the cone.

κ, the number of its cuspidal edges.

b, the order of any double curve upon the surface.

k, the number of apparent double points of the double curve.

t, the number of triple points on the double curve.

c, the order of any cuspidal curve on the surface.

h, the number of apparent double points of the cuspidal curve.

β, the number of intersections of the double and cuspidal curves which are stationary points on the cuspidal curve.

γ, the number of intersections which are stationary points on the double curve.

i, the number of intersections which are not stationary points upon either curve.

ρ, the number of the points where the double curve is met by the curve of contact of the tangent cone.

σ, the number of the points where the cuspidal curve is met by the curve of contact.

And the accented letters denote the corresponding singularities of the reciprocal surface, or, if we choose that they should refer to the given surface, and its tangential or *class* singularities, then we have:

n', the class of the surface.

a', the class of the curve of intersection by any plane.

δ', the number of double tangents of the curve.

κ', the number of its cusps.

b', the class of the node-couple develope.

k', the number of apparent double planes of the node-couple develope.

t', the number of triple planes of the node-couple develope.

c', the class of the spinode develope.

h', the number of the apparent double planes of the spinode develope

β', the number of common tangent planes of the node-couple and spinode developes, stationary planes of the spinode develope.

γ', the number of common tangent planes, stationary planes of the node-couple develope.

ι', the number of the common tangent planes which are not stationary planes of either develope.

ρ', the number of the common tangent planes of the node-couple develope, and the tangent cone.

σ', the number of the common tangent planes of the spinode develope, and the tangent cone.

The terminology made use of is that of my paper "On the Singularities of Surfaces" (*Cambridge and Dublin Mathematical Journal*, vol. VII., 1852), [106]. To explain it, I need only remark that the term node is used as synonymous with double point, and the term spinode as synonymous with cusp; a spinode plane is a tangent plane meeting the surface in a curve, having a spinode at the point of contact; and a node-couple plane is a double tangent plane, or plane meeting the surface in a curve having two nodes; the term develope is used instead of developable surface.

To collect all the formulæ, it is proper to write also:

r, the class of the cuspidal curve.

q, the class of the double curve.

r', the order of the spinode develope.

q', the order of the node-couple develope,

where q' is what Mr Salmon, who only uses it incidentally in referring to a result of Professor Schläfli's, calls (after him) A.

Mr Salmon obtains, between the twenty-eight quantities,

$$n,\ a,\ \delta,\ \kappa,\ b,\ k,\ t,\ c,\ h,\ \beta,\ \gamma,\ i,\ \rho,\ \sigma,$$
$$n',\ a',\ \delta',\ \kappa',\ b',\ k',\ t',\ c',\ h',\ \beta',\ \gamma',\ i',\ \rho',\ \sigma',$$

the twenty-one equations,

$$a = a',$$
$$a' = n(n-1) - 2b - 3c,$$
$$\kappa' = 3n(n-2) - 6b - 8c,$$
$$\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{3}{2}c(c-1),$$

$$a(n-2) = \kappa + \rho + 2\sigma,$$
$$b(n-2) = \rho + 2\beta + 3\gamma + 3t,$$
$$c(n-2) = 2\sigma + 4\beta + \gamma,$$

$$a(n-2)(n-3) = 2\delta + 2ab + 3ac - 4\rho - 9\sigma,$$
$$b(n-2)(n-3) = 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho,$$
$$c(n-2)(n-3) = 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma,$$

$$n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 9c^2 + 8b + 15c - 8k - 18h + 18\beta + 12\gamma + 12i - 9t,$$

$$a = n'(n'-1) - 2b' - 3c',$$
$$\kappa = 3n'(n'-2) - 6b' - 8c',$$
$$*\delta = \tfrac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \tfrac{3}{2}c'(c'-1),$$

$$a'(n'-2) = \kappa' + \rho' + 2\sigma',$$
$$b'(n'-2) = \rho' + 2\beta' + 3\gamma' + 3t',$$
$$c'(n'-2) = 2\sigma' + 4\beta' + \gamma',$$

$$a'(n'-2)(n'-3) = 2\delta' + 2a'b' + 3a'c' - 4\rho' - 9\sigma',$$
$$b'(n'-2)(n'-3) = 4k' + a'b' + 3b'c' - 9\beta' - 6\gamma' - 3i' - 2\rho',$$
$$c'(n'-2)(n'-3) = 6h' + a'c' + 2b'c' - 6\beta' - 4\gamma' - 2i' - 3\sigma',$$

$$*n = n'(n'-1)^2 - n'(7b'+12c') + 4b'^2 + 9c'^2 + 8b' + 15c' - 8k' - 18h' + 18\beta' + 12\gamma' + 12i' - 9t';$$

to which may be joined

$$q = b^2 - b - 2k - 3\gamma - 6t,$$
$$r = c^2 - c - 2h - 3\beta,$$
$$q' = b'^2 - b' - 2k' - 3\gamma' - 6t',$$
$$r' = c'^2 - c' - 2h' - 3\beta',$$

Considering the twenty-one equations, and taking as data $n, b, c, \beta, \gamma, h, k$, then, by means of the several equations, other than the two equations marked (*), we may express in terms of the above data $a, \delta, \kappa, t, i, \rho, \sigma, n', a', \delta', \kappa', b', c', \rho', \sigma', 2\beta'+3\gamma'+3t', 4\beta'+\gamma', 4k'-3i', 6h'-2i'$; the quantities which enter into the first of the marked equations are then all given in terms of the above data, and it is clear that the equation must be satisfied identically: the quantities which enter into the second of the marked equations are given in terms of the data and of t', i', and it is not clear, *a priori*, but that the equation might lead to a relation between the data and t', i'; it will, however, appear in the sequel that the equation must be satisfied identically, independently of any particular values of t', i'. Thus, Mr Salmon's theory does not determine the values of these two quantities, nor, consequently, the values of β', γ', h', k'; it does, however, determine the values of the combinations $4\beta'+\gamma', 8k'-18h'$. But the twenty-one equations between the twenty-eight quantities may be replaced by seventeen equations between the twenty quantities

$$n,\ a,\ \delta,\ \kappa,\ b,\ c,\ \rho,\ \sigma,\ 4\beta+\gamma,\ 8k-18h,$$
$$n',\ a',\ \delta',\ \kappa',\ b',\ c',\ \rho',\ \sigma',\ 4\beta'+\gamma',\ 8k'-18h';$$

this will clearly be the case if it is only shown that the equation which gives n' can by the other equations be transformed into one of the form in question; for a similar transformation will, of course, apply to the equation for n, and then we have only to reject the equation containing t, and to replace the two equations which contain i, by the equation given by the elimination of this quantity, and in like manner to reject the equation containing t', and to replace the two equations containing i', by the equation given by the elimination of this quantity, and the system will be reduced to the required form.

The reduction of the equation which gives n' is effected as follows; we have

$$\begin{aligned}(2b+3c)(n-2)(n-3) &= 8k+18h+a(2b+3c)+12bc-36\beta-24\gamma-12i-4\rho-9\sigma,\\ 3b(n-2) &= 6\beta+9\gamma+9t+3\rho;\end{aligned}$$

and thence

$$\begin{aligned}&(2b+3c)(n-2)(n-3)+3b(n-2)\\ &\quad= a(2b+3c)+12bc+8k+18h-12i+9t-\rho-9\sigma-30\beta-15\gamma\\ &\quad= a(2b+3c)+12bc+8k+18h-12i+9t-18\beta-12\gamma-\rho-9\sigma-3(4\beta+\gamma);\end{aligned}$$

and consequently

$$\begin{aligned}&-8k-18h+18\beta+12\gamma+12i-9t\\ &\quad=\{a-(n-2)(n-3)\}(2b+3c)-3b(n-2)+12bc-\rho-9\sigma-3(4\beta+\gamma),\end{aligned}$$

which (observing that the left-hand side is precisely the combination of terms which enters into the equation for n') shows that the reduction is possible; to complete it, putting for a its value $n(n-1)-2b-3c$, we have

$$\begin{aligned}&-8k-18h+18\beta+12\gamma+12i-9t\\ &\quad=b(5n-6)+c(12n-18)-4b^2-9c^2-\rho-9\sigma-3(4\beta+\gamma);\end{aligned}$$

and substituting this value in the equation for n', we obtain

$$n' = n(n-1)^2 - b(2n-2) - 3c - \rho - 9\sigma - 3(4\beta + \gamma).$$

Some of the other equations admit of simplification: the equation

$$a(n-2)(n-3) = 2\delta + a(2b+3c) - 4\rho - 9\sigma,$$

if we put for a its value $n(n-1) - 2b - 3c$, becomes

$$(4n - 6 - 2b - 3c)(n-2)(n-3) = 2\delta - 4\rho - 9\sigma,$$

and the prescribed combination

$$(2b - 3c)(n-2)(n-3) = 8k - 18h + a(2b - 3c) - 4\rho + 9\sigma,$$

gives in like manner, putting for a its value,

$$(-n^2 + n + 4b)(n-2)(n-3) = (8k - 18h) - 4\rho + 9\sigma.$$

The system of seventeen equations then is:

$$\begin{aligned}
a &= a', \\
a' &= n(n-1) - 2b - 3c, \\
\kappa' &= 3n(n-2) - 6b - 8c, \\
\delta' &= \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{9}{2}c(c-1), \\
a(n-2) &= \kappa + \rho + 2\sigma, \\
c(n-2) &= 2\sigma + (4\beta + \gamma), \\
(4n - 6 - 2b - 3c)(n-2)(n-3) &= 2\delta - 4\rho - 9\sigma, \\
(-n^2 + n + 4b)(n-2)(n-3) &= (8k - 18h) - 4\rho + 9\sigma, \\
n' &= n(n-1)^2 - b(2n-2) - 3c - \rho - 9\sigma - 3(4\beta + \gamma), \\
a &= n'(n'-1) - 2b' - 3c', \\
\kappa &= 3n'(n'-2) - 6b' - 8c', \\
{}^*\delta &= \tfrac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \tfrac{9}{2}c'(c'-1), \\
a'(n'-2) &= \kappa' + \rho' + 2\sigma', \\
c'(n'-2) &= 2\sigma' + (4\beta' + \gamma'), \\
(4n' - 6 - 2b' - 3c')(n'-2)(n'-3) &= 2\delta' - 4\rho' - 9\sigma', \\
(-n'^2 + n' + 4b')(n'-2)(n'-3) &= (8k' - 18h') - 4\rho' + 9\sigma', \\
{}^*n &= n'(n'-1)^2 - b'(2n'-2) - 3c' - \rho' - 9\sigma' - 3(4\beta' + \gamma').
\end{aligned}$$

We may here take as data n, b, c, $4\beta + \gamma$, $8k - 18h$, the equations exclusively of the two marked (*), then give a, δ, κ, ρ, σ, n', a', δ', κ', b', c', ρ', σ', $4\beta' + \gamma'$, $8k' - 18h'$

and then, since all the quantities entering into the two excepted equations are expressed in terms of the data, these equations are satisfied identically, and it is easy to see that this proves what was before assumed, viz., that in the system of twenty-one equations, the second of the equations marked (*) is satisfied identically.

Several of the other quantities may be expressed without difficulty in terms of the data n, b, c, $4\beta+\gamma$, $8k-18h$: we in fact have (besides a, a', κ', δ', which are originally so expressed):

$$2\sigma = (n-2)\,c - (4\beta+\gamma),$$

$$8\rho = (16n-24)\,b - (15n-18)\,c - 2\,(4b^2-9c^2) + 2\,(8k-18h) - 9\,(4\beta+\gamma),$$

$$8\kappa = 8n\,(n-1)\,(n-2) - (32n-56)\,b - (17n-46)\,c + 2\,(4b^2-9c^2) - 2\,(8k-18h) + 17\,(4\beta+\gamma),$$

$$2\delta = n\,(n-1)\,(n-2)\,(n-3) - (4n^2-20n+24)\,b - (6n^2-15n+18)\,c + 12bc + 18c^2 + (8k-18h) - 9\,(4\beta+\gamma),$$

$$8n' = 8n\,(n-1)^2 - (32n-40)\,b - (21n-30)\,c + 2\,(4b^2-9c^2) - 2\,(8k-18h) + 21\,(4\beta+\gamma),$$

$$c' = 4n\,(n-1)\,(n-2) - (16n-28)\,b - (10n-26)\,c + (4b^2-9c^2) - (8k-18h) + 10\,(4\beta+\gamma),$$

but the expressions for the remaining quantities, viz., b', ρ', σ', $4\beta'+\gamma'$, $8k'-18h'$ would be very complicated. If we suppose that b, c, $4\beta+\gamma$, $8k-18h$, vanish, or, what is the same thing, attend only to the terms which contain n alone, we have:

$$2b' = n\,(n-1)\,(n-2)\,(n^3-n^2+n-12),$$

$$\rho' = n\,(n-2)\,(n^3-n^2+n-12),$$

$$\sigma' = 4n\,(n-2),$$

$$4\beta'+\gamma' = 4n^2\,(n-2)\,(n^3-3n^2+3n-3),$$

$$8k'-18h' = n\,(n-2)\,(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+403n^4 - 482n^3+348n^2-242n+60),$$

which agree with the values which Mr Salmon has obtained for β', γ', h', k' by means of the twenty-one equations, and the additional equations (peculiar to the case in question, of a surface of the degree n without singularities, and which are obtained by him from independent considerations), $i'=0$, and $\beta'=2n\,(n-2)\,(11n-24)$.

The system of seventeen equations completely accounts for the reduction of the order of the given surface considered as the reciprocal of the reciprocal surface, but the omitted equations are important for other purposes. We may by means of them express i, t in terms of the data for the system of twenty-one equations, viz., n, b, c, β, γ, h, k; and, effecting this, and annexing the corresponding values of i', t', we have the supplementary system:

$$4i = (5n-6)\,c - 6c^2 + 12h - 5\gamma,$$

$$24t = -(8n-8)\,b + (15n-18)\,c + 2\,(4b^2-9c^2) - 16k + 36h + 20\beta - 15\gamma,$$

$$4i' = (5n'-6)\,c' - 6c'^2 + 12h' - 5\gamma',$$

$$24t' = -(8n'-8)\,b' + (15n'-18c') + 2\,(4b'^2-9c'^2) - 16k' + 36h' + 20\beta' - 15\gamma,$$

to which I annex also, without transformation, the four equations for q, r, q', r', viz.:

$$q = b^2 - b - 2k - 3\gamma - 6t,$$
$$r = c^2 - c - 2h - 3\beta,$$
$$q' = b'^2 - b' - 2k' - 3\gamma' - 6t',$$
$$r' = b'^2 - c' - 2h' - 3\beta,$$

the last two of which, neglecting singularities, give

$$q' = 4n(n-2)(n-3)(n^2+2n-4),$$
$$r' = 2n(n-2)(3n-4),$$

which are the values given by Mr Salmon. I remark, in conclusion, that there is considerable difficulty in the geometrical conception of the points i and the planes i', and the subject appears to require further examination. In the case of a surface of the order n without multiple lines, we have not only $i=0$ (which is a matter of course), but also $i'=0$. In my paper before referred to, I showed, or attempted to show, by geometrical reasoning, that the common tangent planes of the spinode develope and the node-couple develope are stationary planes of the one or the other of the two developes, that is, $i'=0$, and the reasoning seems correct as far as it goes, but it was not shown how the demonstration would (as it ought to do) fail in the case of a surface having a double or cuspidal curve. I showed also that in the case where the common tangent plane is a stationary plane of the spinode develope (that is for the planes β'), the spinode curve and the node-couple curve touch instead of simply intersecting; it would seem that the tangent plane at such point is to be counted once, and not twice, in reckoning the number β' of such tangent planes: the like remark applies, of course, also to the points of intersection β of the double and cuspidal curves.

228.

SUR L'INTÉGRALE $\int_0^1 \frac{t^{\mu+\frac{1}{2}}(1-t)^{\mu-\frac{1}{2}}\,dt}{(a+bt-ct^2)^{\mu+1}}$.

Extrait d'une Lettre adressée à M. Liouville.

[From the *Journal de Mathématiques pures et appliquées* (Liouville), vol. II. (1857), pp. 49—51.]

Je suppose que le lecteur ait sous les yeux la Note de M. Liouville (tome I. [1856] page 421). Cela étant, en rétablissant les valeurs de g, h et en écrivant $i-1$ au lieu de μ, la formule de M. Liouville devient

$$\int_0^1 \frac{t^{i-\frac{1}{2}}(1-t)^{i-\frac{3}{2}}\,dt}{(a+bt-ct^2)^i} = \frac{\Gamma(\frac{1}{2})\,\Gamma(i-\frac{1}{2})}{\sqrt{a+b-c}\,\Gamma(i)\,[2a+b+2\sqrt{a(a+b-c)}]^{i-\frac{1}{2}}}, \tag{1}$$

laquelle, en y posant

$$t = \frac{x}{1+x},$$

se transforme en celle-ci:

$$\int_0^\infty \frac{x^{i-1}\,dx}{[(a+b-c)x^2+(2a+b)x+a]^i} = \frac{\Gamma(\frac{1}{2})\,\Gamma(i-\frac{1}{2})}{\sqrt{a+b-c}\,\Gamma(i)\,[2a+b+2\sqrt{a(a+b-c)}]^{i-\frac{1}{2}}}; \tag{2}$$

et en mettant le dénominateur de l'intégrale sous la forme

$$[(a+b-c)(x+\lambda)(x+\mu)]^i,$$

la formule devient

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}\,dx}{[(x+\lambda)(x+\mu)]^i} = \frac{\Gamma(\frac{1}{2})\,\Gamma(i-\frac{1}{2})}{\Gamma(i)} \cdot \frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}}: \tag{3}$$

formule qui se trouve dans mon Mémoire: "On certain formulæ for differentiation with applications to the evaluation of definite integrals," *Camb. and Dubl. Math. Journal,*

tom. II., page 122, (1847), [41]. La démonstration que j'y ai donnée n'est cependant rigoureuse (à moins que l'on n'admette la théorie de la différentiation à indices quelconques) que pour le cas de $i+\frac{1}{2}$ égal à un entier positif; en effet, en écrivant

$$U_{k,i}=[(x+\lambda)(x+\mu)]^{\frac{1}{2}k}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2i},$$

je suis parti de la formule

$$\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k(\lambda-\mu)^2U_{k-2,i-1}-(k+i)U_{k-1,i},$$

pour en déduire l'expression générale de $\left(\frac{d}{dx}\right)^s U_{0,i}$. Cette expression devient très-simple pour le cas $s=i+1$; on a alors

$$\frac{(-)^{i+1}}{i}\left(\frac{d}{dx}\right)^{i+1}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2i}=\frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i}\frac{1}{[(x+\lambda)(x+\mu)]^{i+\frac{1}{2}}}, \qquad (4)$$

formule de différentiation assez singulière. En y écrivant $i-\frac{1}{2}$, au lieu de i, et en intégrant $i+\frac{1}{2}$ fois par la formule

$$\int_0^\infty x^{i-\frac{1}{2}}fx\,dx=\frac{\Gamma(i+\frac{1}{2})}{(-)^{i+\frac{1}{2}}}\left(\int_\infty d\alpha\right)^{i+\frac{1}{2}}f\alpha, \qquad \alpha=0,$$

on obtient la formule intégrale ci-devant mentionnée. Il convient d'ajouter que cette formule, représentée sous la forme

$$\int_0^\infty\frac{x^{i-\frac{1}{2}}\,dx}{(ax^2+bx+c)^i}=\frac{\Gamma(\frac{1}{2})\,\Gamma(i-\frac{1}{2})}{\Gamma(i)}\frac{1}{(b+2\sqrt{ac})^{i-\frac{1}{2}}}, \qquad (5)$$

est comprise dans une formule générale donnée par M. Schlömilch: "Note sur la variation des constantes arbitraires d'une intégrale définie (*Crelle*, tom. XXXIII., page 268; 1847).

Je remarque aussi que M. Donkin, en comparant ses résultats concernant les fonctions de Laplace avec ceux de M. Boole sur le même sujet, a trouvé une identité, laquelle m'a conduit à cette autre formule de différentiation

$$\left\{\frac{[(x+\lambda)(x+\mu)]^2}{2x+\lambda+\mu}\frac{d}{dx}\right\}^n\frac{(2x+\lambda+\mu)^{n+s-1}}{[(x+\lambda)(x+\mu)]^{n-\frac{1}{2}}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2n-2s}}$$

$$=\frac{[(x+\lambda)(x+\mu)]^{n+1}}{(2x+\lambda+\mu)^{n-s+1}}\left(\frac{d}{dx}\right)^n\frac{(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2s}}{[(x+\lambda)(x+\mu)]^{\frac{1}{2}}}, \qquad (6)$$

formule que j'ai démontrée au moyen d'une analyse assez compliquée.

229.

NOTE SUR UNE FORMULE POUR LA RÉVERSION DES SÉRIES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LII. (1856), pp. 276—284.]

JE me propose de développer dans cette note, pour le cas de trois variables (ce qui suffit pour faire voir la loi dans le cas d'un nombre quelconque de variables) une formule qui se trouve dans le mémoire remarquable de Jacobi, "de resolutione aequationum per series infinitas," *Crelle*, t. VI. [1830] p. 257. Voici la formule dont il s'agit. Soit $f(x, y, z)$ une fonction rationnelle et entière des variables x, y, z, et mettons $u = X$, $v = Y$, $w = Z$, où X, Y, Z sont des fonctions rationnelles et entières des variables x, y, z, telles que $X - x$, $Y - y$, $Z - z$ ne contiennent que les puissances et les produits du deuxième ordre et des ordres supérieurs des variables. Cela étant, on aura

$$f(x, y, z) = \left[f(x, y, z) \frac{\partial(X, Y, Z)}{\partial(x, y, z)} \cdot \frac{1}{(X-u)(Y-v)(Z-w)} \right]_{x^{-1}y^{-1}z^{-1}},$$

où la notation $\dfrac{\partial(X, Y, Z)}{\partial(x, y, z)}$ dénote le déterminant fonctionnel ou "Jacobian" de X, Y, Z par rapport à x, y, z et la notation $[\ ]_{x^{-1}y^{-1}z^{-1}}$ signifie le coefficient de $x^{-1}y^{-1}z^{-1}$ dans le développement de la fonction en dedans des $[\ ]$; ce développement doit s'effectuer d'une manière déterminée, savoir il faut d'abord développer les facteurs $\dfrac{1}{X-u}$ selon les puissances descendantes de x, c'est-à-dire dans la forme $\dfrac{1}{X} + \dfrac{u}{X^2} + \dfrac{u^2}{X^3} + \text{etc.}$, et puis écrivant $X = x + P$ (où P est fonction des trois variables) il faut développer les puissances de X selon les puissances descendantes du monôme x, c'est-à-dire dans la forme $X^{-m} = x^{-m} - mx^{-m-1}P + \frac{1}{2}m(m+1)x^{-m-2}P^2 - \text{etc.}$, ce qui est en effet un développement selon les puissances *ascendantes* des variables.

La formule donne

$$\text{Coeff. } u^{\mathrm{a}}v^{\mathrm{b}}w^{\mathrm{c}} \text{ dans } f(x, y, z) = \left[f(x, y, z)\,\frac{\partial(X, Y, Z)}{\partial(x, y, z)} \cdot \frac{1}{X^{\mathrm{a}+1}\,Y^{\mathrm{b}+1}\,Z^{\mathrm{c}+1}}\right]_{x^{-1}y^{-1}z^{-1}},$$

ou ce qui est la même chose

$$\text{Coeff. } u^{\mathrm{a}}v^{\mathrm{b}}w^{\mathrm{c}} \text{ dans } f(x, y, z) = \left[f(x, y, z)\frac{\partial\left(-\frac{1}{\mathrm{a}}X^{-\mathrm{a}}, -\frac{1}{\mathrm{b}}Y^{-\mathrm{b}}, -\frac{1}{\mathrm{c}}Z^{-\mathrm{c}}\right)}{\partial(x, y, z)}\right]_{x^{-1}y^{-1}z^{-1}}.$$

Soit à présent

$$\begin{aligned} X &= x\,..+A_{f,g,h}\,x^f y^g z^h + \text{etc.},\\ Y &= y\,..+B_{i,j,k}\,x^i y^j z^k + \text{etc.},\\ Z &= z\,..+C_{l,m,n}\,x^l y^m z^n + \text{etc.},\\ f(x, y, z) &= \ldots\ \Theta_{P,Q,R}\,x^P y^Q z^R + \text{etc.}; \end{aligned}$$

dans ces expressions et partout dans ce qui suit, les etc.'s se rapportent à des termes que l'on obtient en affixant des accents en nombre quelconque aux symboles indéterminés.

En employant la notation de Gauss $\Pi\alpha = 1.2.3\ldots\alpha$, on obtient pour la terme général de $-\frac{1}{\mathrm{a}}X^{-\mathrm{a}}$,

$$(-)^{r-1}\frac{\Pi(\mathrm{a}+r-1)}{\Pi\mathrm{a}\,\Pi\alpha\text{ etc.}}A^{\mathrm{a}}{}_{f,g,h}\text{ etc. }x^{-\mathrm{a}-r+F}y^{G}z^{H},$$

où

$$\alpha + \text{etc.} = r,\quad f\alpha + \text{etc.} = F,\quad g\alpha + \text{etc.} = G,\quad h\alpha + \text{etc.} = H.$$

De même le terme général de $-\frac{1}{\mathrm{b}}Y^{-\mathrm{b}}$ est

$$(-)^{s-1}\frac{\Pi(\mathrm{b}+s-1)}{\Pi\mathrm{b}\,\Pi\beta\text{ etc.}}B^{\beta}{}_{i,j,k}\text{ etc. }x^{I}y^{-\mathrm{b}-s+J}z^{K},$$

où

$$\beta + \text{etc.} = s,\quad i\beta + \text{etc.} = I,\quad j\beta + \text{etc.} = J,\quad k\beta + \text{etc.} = K,$$

et le terme général de $-\frac{1}{\mathrm{c}}Z^{-\mathrm{c}}$ est

$$(-)^{t-1}\frac{\Pi(\mathrm{c}+t-1)}{\Pi\mathrm{c}\,\Pi\gamma\text{ etc.}}C^{\gamma}{}_{l,m,n,}\text{ etc. }x^{L}y^{M}z^{-\mathrm{c}-t+N},$$

où

$$\gamma + \text{etc.} = t,\quad l\gamma + \text{etc.} = L,\quad m\gamma + \text{etc.} = M,\quad n\gamma + \text{etc.} = N.$$

En formant de là le terme général du Jacobian et en multipliant par le terme général de $f(x, y, z)$ on obtient pour le terme général de l'expression en dedans des [ ], la valeur que voici,

$$(-)^{r+s+t}\frac{\Pi(\mathrm{a}+r-1)\,\Pi(\mathrm{b}+s-1)\,\Pi(\mathrm{c}+t-1)}{\Pi\mathrm{a}\,\Pi\mathrm{b}\,\Pi\mathrm{c}\,\Pi\alpha\text{ etc. }\Pi\beta\text{ etc. }\Pi\gamma\text{ etc.}}\times$$

$$A^{\mathrm{a}}{}_{f,g,h}\text{ etc. }B^{\beta}{}_{i,j,k}\text{ etc. }C^{\gamma}{}_{l,m,n}\text{ etc. }\Theta_{P,Q,R}\begin{vmatrix} \mathrm{a}+r-F, & -G, & -H\\ -I, & \mathrm{b}+s-J, & -K\\ -L, & -M, & \mathrm{c}+t-N \end{vmatrix}\times$$

$$x^{-\mathrm{a}-r+F+I+L+P-1}\,.\,y^{-\mathrm{b}-s+G+J+M+Q-1}\,.\,z^{-\mathrm{c}-t+H+K+N+R-1},$$

et pour trouver le terme qui contient $x^{-1}y^{-1}z^{-1}$ on n'a qu'à écrire dans cette expression

$$F+I+L+P=\mathrm{a}+r,\quad G+J+M+Q=\mathrm{b}+s,\quad H+K+N+R=\mathrm{c}+t,$$

le coefficient de $x^{-1}y^{-1}z^{-1}$ sera alors la valeur de l'expression $[\ ]_{x^{-1}y^{-1}z^{-1}}$ qu'il s'agissait de trouver. En effectuant cela et en recapitulant les formules on obtient le théorème suivant: en posant

$$\begin{aligned} X &= x\ldots+A_{f,g,h}\ x^f y^g z^h+\text{etc.}=u,\\ Y &= y\ldots+B_{i,j,k}\ x^i y^j z^k+\text{etc.}=v,\\ Z &= z\ldots+C_{l,m,n}\ x^l y^m z^n+\text{etc.}=w,\\ f(x,\ y,\ z) &= \ \ldots+\Theta_{P,Q,R}\ x^P y^Q z^R+\text{etc.} \end{aligned}$$

on aura pour le terme général du coeff. $u^{\mathrm{a}}v^{\mathrm{b}}w^{\mathrm{c}}$ dans $f(x,\ y,\ z)$ la valeur que voici,

$$(-)^{r+s+t}\frac{\Pi(\mathrm{a}+r-1)\,\Pi(\mathrm{b}+s-1)\,\Pi(\mathrm{c}+t-1)}{\Pi\mathrm{a}\,\Pi\mathrm{b}\,\Pi\mathrm{c}\,\Pi\alpha\text{ etc. }\Pi\beta\text{ etc. }\Pi\gamma\text{ etc.}}\times$$

$$A^{\alpha}_{f,g,h}\text{ etc. }B^{\beta}_{i,j,k}\text{ etc. }C^{\gamma}_{l,m,n}\text{ etc. }\Theta_{P,Q,R}\begin{vmatrix} P+I+L, & -G, & -H\\ -I, & Q+G+M, & -K\\ -L, & -M, & R+H+K \end{vmatrix}$$

dans laquelle

$$\begin{aligned} \alpha\ +\text{etc.}&=r, & \beta\ +\text{etc.}&=s, & \gamma\ +\text{etc.}&=t,\\ f\alpha+\text{etc.}&=F, & g\alpha+\text{etc.}&=G, & h\alpha+\text{etc.}&=H,\\ i\beta+\text{etc.}&=I, & j\beta+\text{etc.}&=J, & k\beta+\text{etc.}&=K,\\ l\gamma+\text{etc.}&=L, & m\gamma+\text{etc.}&=M, & n\gamma+\text{etc.}&=N,\\ P+F+I+L&=\mathrm{a}+r, & Q+G+J+M&=\mathrm{b}+s, & R+H+K+N&=\mathrm{c}+t. \end{aligned}$$

Les dernières équations peuvent s'écrire sous la forme

$$\begin{aligned} P+&(f-1)\,\alpha+\text{etc.}+ & i\beta+\text{etc.}+ & & l\gamma+\text{etc.}&=\mathrm{a},\\ Q+& & g\alpha+\text{etc.}+(j-1)\,\beta+\text{etc.}+ & & m\gamma+\text{etc.}&=\mathrm{b},\\ R+& & h\alpha+\text{etc.}+ & k\beta+\text{etc.}+(n-1)\,\gamma+\text{etc.}&=\mathrm{c}, \end{aligned}$$

qui sont les conditions auxquelles doivent satisfaire les valeurs de P, Q, R, α, etc., β, etc., γ, etc., f, g, h, etc., i, j, k, etc., l, m, n, etc.; en ajoutant ces équations on obtient

$$\begin{aligned} P+Q+R+(f+g+h-1)\,\alpha+\text{etc.}+(i+j+k-1)\,\beta+\text{etc.}+(l+m+n-1)\,\gamma+\text{etc.}\\ =\mathrm{a}+\mathrm{b}+\mathrm{c}, \end{aligned}$$

où les nombres $f+g+h-1$, etc. $i+j+k-1$, etc. $l+m+n-1$, etc. sont positifs, il n'y a donc qu'un nombre fini (comme cela doit être) de solutions des équations indéterminées.

Il y a une manière assez simple pour calculer le déterminant qui entre dans la formule, pour cela je représente les termes P, I, etc. par les notations symboliques $X\vartheta$, Xy, etc. de manière que le déterminant devient

$$\begin{vmatrix} X\vartheta + Xy + Xz, & -Yx, & -Zx \\ -Xy, & Y\vartheta + Yx + Yz, & -Zy \\ -Xz, & -Yz, & Z\vartheta + Zx + Zy \end{vmatrix},$$

or ce déterminant est ce que devient le produit

$$(X\vartheta + Xy + Xz)(Y\vartheta + Yz + Yx)(Z\vartheta + Zx + Zy),$$

en omettant du développement tous les termes qui contiennent un cycle tel que $Xy.Yx$, ou $Xy.Yz.Zx$. Cela donne pour le déterminant la somme des seize termes

$$\begin{aligned} X\vartheta.Y\vartheta.Z\vartheta + Y\vartheta.Z\vartheta(Xy + Xz) + Z\vartheta.X\vartheta(Yz + Yx) + X\vartheta.Y\vartheta(Zx + Zy) \\ + X\vartheta(Yx.Zx + Yx.Zy + Yz.Zx) + Y\vartheta(Zy.Xy + Zy.Xz + Zx.Xy) \\ + Z\vartheta(Xz.Yz + Xz.Yx + Xy.Yz); \end{aligned}$$

et la même chose est vraie quel que soit l'ordre du déterminant. C'est M. Sylvester qui m'a fait cette remarque.

Les formules de Jacobi s'appliquent aussi au cas où u, v, w, etc. sont données en termes de x, y, z, etc. au moyen d'équations et non pas explicitement comme auparavant; mais je ne chercherai pas à présent ce que deviennent les formules pour ce cas plus général.

On peut appliquer la formule au problème de la transformation des variables indépendantes dans le calcul différentiel. En effet, soient u, v, w des fonctions quelconques de x, y, z, et prenons ξ, η, ζ pour les incréments de x, y, z respectivement et υ, ν, ω pour les incréments de u, v, w respectivement; on aura

$$\xi = \left\{\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right) + \tfrac{1}{2}\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right)^2 + \text{etc.}\right\} x,$$

$$\eta = \left\{\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right) + \tfrac{1}{2}\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right)^2 + \text{etc.}\right\} y,$$

$$\zeta = \left\{\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right) + \tfrac{1}{2}\left(\upsilon\frac{d}{du} + \nu\frac{d}{dv} + \omega\frac{d}{dw}\right)^2 + \text{etc.}\right\} z.$$

Soit

$$\nabla = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

le Jacobian de x, y, z par rapport à u, v, w; et mettons

$$\left(\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}\right)x=p,$$

$$\left(\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}\right)y=q,$$

$$\left(\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}\right)z=r,$$

υ, ν, ω seront des fonctions linéaires de p, q, r, et, en formant avec ces valeurs l'expression de $\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}$, on trouve

$$\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}=p\bar{x}+q\bar{y}+r\bar{z},$$

où $\bar{x}$, $\bar{y}$, $\bar{z}$ sont des opérations de la forme $L\frac{d}{du}+M\frac{d}{dv}+N\frac{d}{dw}$, qui peuvent être représentées symboliquement de cette manière,

$$\bar{x}=\frac{1}{\nabla}\begin{vmatrix}\frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}\\ \frac{dy}{du}, & \frac{dy}{dv}, & \frac{dy}{dw}\\ \frac{dz}{du}, & \frac{dz}{dv}, & \frac{dz}{dw}\end{vmatrix}$$

et de même pour $\bar{y}$ et $\bar{z}$; il faut faire attention, qu'en opérant avec ces symboles, il faut traiter comme des constantes les fonctions de u, v, w qui entrent dans ces mêmes symboles.

Nous avons évidemment $\bar{x}x=1$, $\bar{y}x=0$, $\bar{z}x=0$, et de même $\bar{x}y=0$, etc.; on obtient de là

$$\xi=p+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2x+\text{etc.},$$
$$\eta=q+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2y+\text{etc.},$$
$$\zeta=r+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2z+\text{etc.}$$

Soit à présent ϑ une fonction quelconque de x, y, z, ou de u, v, w; en envisageant ϑ comme fonction de x, y, z, on trouve l'incrément de cette fonction en opérant sur ϑ avec le symbole

$$\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right)+\tfrac{1}{2}\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\zeta\frac{d}{dz}\right)^2+\text{etc.},$$

mais en envisageant ϑ comme fonction de u, v, w et en faisant attention à l'équation $\upsilon\frac{d}{du}+\nu\frac{d}{dv}+\omega\frac{d}{dw}=p\bar{x}+q\bar{y}+r\bar{z}$, on trouve ce même incrément en opérant sur ϑ avec le symbole

$$(p\bar{x}+q\bar{y}+r\bar{z})+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2+\text{etc.},$$

et les deux résultats deviendront identiques en substituant pour p, q, r les valeurs de ces quantités en termes de ξ, η, ζ, valeurs qui se trouvent par la réversion des séries qui donnent ξ, η, ζ en termes de p, q, r. C'est-à-dire nous aurons

$$\left(\frac{d}{dx}\right)^{\mathrm{a}}\left(\frac{d}{dy}\right)^{\mathrm{b}}\left(\frac{d}{dz}\right)^{\mathrm{c}} = \Pi\mathrm{a}\ \Pi\mathrm{b}\ \Pi\mathrm{c}.\ \mathrm{coeff.}\ \xi^{\mathrm{a}}\eta^{\mathrm{b}}\zeta^{\mathrm{c}}\ \text{dans}$$

$$\{(p\bar{x}+q\bar{y}+r\bar{z})+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2+\mathrm{etc.}\},$$

où

$$\xi = p+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2 x+\mathrm{etc.},$$
$$\eta = q+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2 y+\mathrm{etc.},$$
$$\zeta = r+\tfrac{1}{2}(p\bar{x}+q\bar{y}+r\bar{z})^2 z+\mathrm{etc.}$$

C'est là le problème de la réversion des séries qui vient d'être traité; et en substituant

$$\frac{1}{\Pi f\,\Pi g\,\Pi h}x^{-f}y^{-g}z^{-h}x,\quad \frac{1}{\Pi i\,\Pi j\,\Pi k}x^{-i}y^{-j}z^{-k}y,\quad \frac{1}{\Pi l\,\Pi m\,\Pi n}x^{-l}y^{-m}z^{-n}z,\quad \frac{1}{\Pi P\,\Pi Q\,\Pi R}x^{-P}y^{-Q}z^{-R}$$

au lieu de

$$A_{f,g,h},\qquad B_{i,j,k},\qquad C_{l,m,n},\qquad \Theta_{P,Q,R},$$

on trouve le théorème suivant:

THÉORÈME. Le terme général de $\left(\frac{d}{dx}\right)^{\mathrm{a}}\left(\frac{d}{dy}\right)^{\mathrm{b}}\left(\frac{d}{dz}\right)^{\mathrm{c}}$ est

$$K\Omega\,(x^{-f}y^{-g}z^{-h}x)^{\alpha}\ \mathrm{etc.}\ (x^{-i}y^{-j}z^{-k}y)^{\beta}\ \mathrm{etc.}\ (x^{-l}y^{-m}z^{-n}z)^{\gamma}\ \mathrm{etc.}\ x^{-P}y^{-Q}z^{-R},$$

expression dans laquelle

$$\begin{array}{lll}
\alpha+\mathrm{etc.}=r, & \beta+\mathrm{etc.}=s, & \gamma+\mathrm{etc.}=t,\\
f\alpha+\mathrm{etc.}=F, & g\alpha+\mathrm{etc.}=G, & h\alpha+\mathrm{etc.}=H,\\
i\beta+\mathrm{etc.}=I, & j\beta+\mathrm{etc.}=J, & k\beta+\mathrm{etc.}=K,\\
l\gamma+\mathrm{etc.}=L, & m\gamma+\mathrm{etc.}=M, & n\gamma+\mathrm{etc.}=N,\\
F+I+L+P=\mathrm{a}+r, & G+J+M+Q=\mathrm{b}+s, & H+K+N+R=\mathrm{c}+t;
\end{array}$$

$$K=(-)^{r+s+t}\times$$

$$\frac{\Pi(\mathrm{a}+r-1)\,\Pi(\mathrm{b}+s-1)\,\Pi(\mathrm{c}+t-1)}{\Pi\alpha\ \mathrm{etc.}\ \Pi\beta\ \mathrm{etc.}\ \Pi\gamma\ \mathrm{etc.}\ (\Pi f\,\Pi g\,\Pi h)^{\alpha}\ \mathrm{etc.}\ (\Pi i\,\Pi j\,\Pi k)^{\beta}\ \mathrm{etc.}\ (\Pi l\,\Pi m\,\Pi n)^{\gamma}\ \mathrm{etc.}\ \Pi P\,\Pi Q\,\Pi R};$$

$$\Omega=\begin{vmatrix} P+I+L, & -G, & -H\\ -I, & Q+G+M, & -K\\ -L, & -M, & R+H+K \end{vmatrix},$$

et où les nombres $f+g+h-1$, etc., $i+j+k-1$, etc., $l+m+n-1$, etc. sont tous positifs comme auparavant.

La formule contient les symboles $\bar{x}$, $\bar{y}$, $\bar{z}$ qui sont chacun une fonction linéaire de $\frac{d}{du}$, $\frac{d}{dv}$, $\frac{d}{dw}$, on pourrait donc se proposer la question de trouver le terme général en développant la formule de manière à ne contenir que des puissances et des produits de ces symboles $\frac{d}{du}$, $\frac{d}{dv}$, $\frac{d}{dw}$; c'est à quoi se rapportent les recherches très étendues que vient de faire M. Sylvester sur ce sujet, et qui embrassent aussi bien le cas où les nouvelles variables sont données explicitement que celui où les deux systèmes de variables sont liés par des équations données.

Il y a une autre manière de traiter cette question de la transformation des variables indépendantes, savoir en écrivant

$$R = \xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz},$$

$$\rho = \xi\ \bar{x} + \eta\ \bar{y} + \zeta\ \bar{z},$$

on peut exprimer les puissances de R au moyen de ρ et de cette autre quantite symbolique

$$\Lambda = x\ \bar{x} + y\ \bar{y} + z\ \bar{z}.$$

En effet, en mettant $\chi = p\bar{x} + q\bar{y} + r\bar{z}$, on a

$$R + \tfrac{1}{2} R^2 + \tfrac{1}{6} R^3 + \text{etc.} = \chi + \tfrac{1}{2} \chi^2 + \tfrac{1}{6} \chi^3 + \text{etc.},$$

où

$$\xi = p + \tfrac{1}{2} \chi^2 x + \tfrac{1}{6} \chi^3 x + \text{etc.},$$
$$\eta = q + \tfrac{1}{2} \chi^2 y + \tfrac{1}{6} \chi^3 y + \text{etc.},$$
$$\zeta = r + \tfrac{1}{2} \chi^2 z + \tfrac{1}{6} \chi^3 z + \text{etc.}$$

En se servant de la méthode des approximations successives, on trouve comme première approximation

$$p = \xi, \quad q = \eta, \quad r = \zeta,$$

et de là

$$\chi = \rho;$$

comme seconde approximation

$$p = \xi - \tfrac{1}{2}\rho^2 x, \quad q = \eta - \tfrac{1}{2}\rho^2 y, \quad r = \zeta - \tfrac{1}{2}\rho^2 z,$$

et de là

$$\chi = \rho - \tfrac{1}{2}\rho^3 \Lambda;$$

comme troisième approximation

$$p = \xi - \tfrac{1}{2}(\rho^2 - \rho^2 \Lambda\rho)\, x - \tfrac{1}{6}\rho^3 x,$$
$$q = \eta - \tfrac{1}{2}(\rho^2 - \rho^2 \Lambda\rho)\, y - \tfrac{1}{6}\rho^3 y,$$
$$r = \zeta - \tfrac{1}{2}(\rho^2 - \rho^2 \Lambda\rho)\, z - \tfrac{1}{6}\rho^3 z,$$

et de là

$$\chi = \rho - \tfrac{1}{2}\rho^2\Lambda + \tfrac{1}{2}\rho^2\Lambda\rho\Lambda - \tfrac{1}{6}\rho^3\Lambda, \quad \chi^2 = \rho^2 - \rho^2\Lambda\rho, \quad \chi^3 = \rho^3$$

et ainsi de suite; donc, en substituant,

$$\begin{aligned} R + \tfrac{1}{2}R^2 + \tfrac{1}{6}R^3 + \text{etc.} = \rho &- \tfrac{1}{2}\rho^2\Lambda + \tfrac{1}{2}\rho^2\Lambda\rho\Lambda - \tfrac{1}{6}\rho^3\Lambda \\ &+ \tfrac{1}{2}\rho^2 \quad - \tfrac{1}{2}\rho^2\Lambda\rho \\ &\qquad\qquad\qquad\qquad + \tfrac{1}{6}\rho^3 \quad + \text{etc. etc.} \end{aligned}$$

c'est-à-dire

$$\begin{aligned} R &= \rho, \\ R^2 &= \rho^2 - \rho^2\Lambda, \\ R^3 &= \rho^3 - 3\rho^2\Lambda\rho - (\rho^3 - 3\rho^2\Lambda\rho)\,\Lambda, \\ &\text{etc.} \end{aligned}$$

formules dans lesquelles un terme tel que $\rho^2\Lambda$ signifie $\rho^2 x \,.\, \bar{x} + \rho^2 y \,.\, \bar{y} + \rho^2 z \,.\, z$, c'est-à-dire l'opération du symbole ρ^2 s'arrête aux quantités x, y, z faisant partie du symbole Λ qui vient immédiatement après le ρ^2; la même chose a lieu dans tous les cas semblables. L'équation générale est

$$R^n = \Pi - \Pi\Lambda,$$

mais pour expliquer la forme de la fonction Π il faudrait faire des développements assez longs, dans lesquels je n'entrerai pas à cette occasion; la découverte de cette équation générale est due à M. Sylvester, qui l'a établie d'une autre manière.

Londres, 2, Stone Buildings, 16 *Avril,* 1855.

230.

NOTE SUR LA MÉTHODE D'ÉLIMINATION DE BEZOUT.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LIII. (1857), pp. 366—367.]

VOICI la forme la plus simple sous laquelle on peut présenter cette méthode. Pour éliminer les variables x, y entre deux équations du $n^{\text{ième}}$ degré

$$(a, \ldots \between x, y)^n = 0,$$
$$(a', \ldots \between x, y)^n = 0,$$

on n'a qu'à former l'équation identique

$$\frac{(a, \ldots \between x, y)^n (a', \ldots \between \lambda, \mu)^n - (a', \ldots \between x, y)^n (a, \ldots \between \lambda, \mu)^n}{\mu x - \lambda y} =$$

$$\left(\begin{vmatrix} a_{0,0}, & a_{1,0}, & \ldots & a_{n-1,0} \\ a_{0,1}, & a_{1,1}, & \ldots & a_{n-1,1} \\ \vdots & & & \\ a_{0,n-1}, & a_{1,n-1}, & \ldots & a_{n-1,n-1} \end{vmatrix} \between x, y)^{n-1} (\lambda, \mu)^{n-1}$$

où l'expression qui forme le second membre représente la fonction suivante,

$$\begin{aligned} &(a_{0,0} \quad x^{n-1} + a_{1,0} \quad x^{n-2} y \ldots + a_{n-1,0} \quad y^{n-1}) \lambda^{n-1} \\ &+ (a_{0,1} \quad x^{n-1} + a_{1,1} \quad x^{n-2} y \ldots + a_{n-1,1} \quad y^{n-1}) \lambda^{n-2} \mu \\ &\vdots \\ &+ (a_{0,n-1} x^{n-1} + a_{1,n-1} x^{n-2} y \ldots + a_{n-1,n-1} y^{n-1}) \mu^{n-1}; \end{aligned}$$

le résultat de l'élimination sera

$$\begin{vmatrix} a_{0,0} & , a_{1,0} & , \dots a_{n-1,0} \\ a_{0,1} & , a_{1,1} & , \dots a_{n-1,1} \\ \vdots & & \\ a_{0,n-1}, & a_{1,n-1}, & \dots a_{n-1,n-1} \end{vmatrix} = 0.$$

Par exemple on trouve

$$\frac{(a,\ b,\ c \between x,\ y)^2 (a',\ b',\ c' \between \lambda,\ \mu)^2 - (a',\ b',\ c' \between x,\ y)^2 (a,\ b,\ c \between \lambda,\ \mu)^2}{\mu x - \lambda y} =$$

$$\begin{pmatrix} 2(ab' - a'b), & ac' - a'c \between x,\ y \between \lambda,\ \mu); \\ ac' - a'c\ , & 2(bc' - b'c) \end{pmatrix}$$

$$\frac{(a,\ b,\ c,\ d \between x,\ y)^3 (a',\ b',\ c',\ d' \between \lambda,\ \mu)^3 - (a',\ b',\ c',\ d' \between x,\ y)^3 (a,\ b,\ c,\ d \between \lambda,\ \mu)^3}{\mu x - \lambda y} =$$

$$\begin{pmatrix} 3(ab' - a'b), & 3(ac' - a'c), & (ad' - a'd) \between x^2,\ xy,\ y^2 \between \lambda^2,\ \lambda\mu,\ \mu^2); \\ 3(ac' - a'c), & (ad' - a'd) + 9(bc' - b'c), & 3(bd' - b'd) \\ (ad' - a'd), & 3(bd' - b'd), & 3(cd' - c'd) \end{pmatrix}$$

d'où l'on tire immédiatement les résultats de l'élimination entre deux équations quadratiques ou cubiques.

Londres, 2 *Stone Buildings*, *Avril*, 1855.

231.

NOTE SUR L'ÉQUATION $x^2 - Dy^2 = \pm 4$, $D \equiv 5 \pmod{8}$.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LIII. (1857), pp. 369—371.]

On sait que pour un déterminant positif $D \equiv 5 \pmod{8}$ le nombre des formes quadratiques proprement primitives est égal au nombre des formes improprement primitives, quand il existe une solution en nombres impairs de l'équation $x^2 - Dy^2 = 4$; mais que le nombre des formes proprement primitives est égal à trois fois le nombre des formes improprement primitives, quand il n'existe pas de solution en nombres impairs de l'équation $x^2 - Dy^2 = 4$.

Les tables de Degen font voir immédiatement, pour les nombres $D \equiv 5 \pmod{8}$ qui ne sont pas plus grands que 997, s'il existe ou non une solution en nombres impairs de l'équation $x^2 - Dy^2 = 4$. À savoir si le nombre 4 se trouve dans la série des dénominateurs des quotients complets (la seconde ligne dans les tables), il existe une solution de l'équation; si non, il n'en existe pas. De plus, si la place où se trouve le nombre 4 est d'ordre pair il existe une solution tant de l'équation $x^2 - Dy^2 = +4$ que de l'équation $x^2 - Dy^2 = -4$, mais si cette place est d'ordre impair il existe seulement une solution de l'équation $x^2 - Dy^2 = 4$. On trouve la plus petite solution de ces équations au moyen de la série des quotients (la première ligne des tables), en s'arrêtant au nombre qui précède le nombre placé au-dessus du nombre 4, et en calculant la fraction continue déterminée par cette suite; par exemple, pour le nombre 61 on trouve dans les tables

	7, 1, 4, 3, 1 (2, 2)
61	1, 12, 3, 4, 9 (5, 5)
	226153980
	1766319049

Cela fait voir qu'il existe une solution de l'équation $x^2 - Dy^2 = -4$, solution que l'on obtient au moyen de la fraction continue $7 + \frac{1}{1 + \frac{1}{4}}, = \frac{39}{5}$ en faisant $x = 39$, $y = 5$. La plus petite solution de l'équation $x^2 - Dy^2 = 4$ se déduit très facilement de la plus petite solution de l'équation $\tau^2 - Dv^2 = -4$ au moyen de la formule $x + y\sqrt{D} = \frac{1}{2}(\tau + v\sqrt{D})^2$, ce qui donne $x = \tau^2 + 2$, $y = \tau v$.

On obtient pareillement la plus petite solution de l'équation $x^2 - Dy^2 = -1$ au moyen de la formule $(x + y\sqrt{D}) = \frac{1}{8}(\tau + v\sqrt{D})^3$, ce qui donne

$$x = \tfrac{1}{2}(\tau^3 + 3\tau), \quad y = \tfrac{1}{2}(\tau^2 + 1)\, v.$$

De même la plus petite solution de l'équation $x^2 - Dy^2 = 1$ se déduit de la plus petite solution de l'équation $T^2 - DU^2 = 4$ au moyen de la formule $x + y\sqrt{D} = \frac{1}{8}(T + U\sqrt{D})^3$, ce qui donne $x = \frac{1}{2}(T^3 - 3T)$, $y = \frac{1}{2}(T^2 - 1)\, U$.

Je fais observer à cette occasion que suivant une remarque de Göpel ("Notiz über A. Göpel," t. XXXV. [1847] p. 315 de ce Journal) si dans l'équation $\left(\frac{x + \sqrt{y}}{p}\right)^n = P + \sqrt{Q}$, où x, y, p, n, P, Q sont des entiers, le dénominateur p est plus grand que l'unité et x, y, p n'ont pas de dénominateur commun, on aura nécessairement $p = 2$, $n = 3$ ou égal à un multiple de 3, x impair, et y de la forme $8n + 5$.

J'ai calculé au moyen des tables de Degen la table suivante des plus petites solutions en nombres impairs de l'équation $x^2 - Dy^2 = -4$ ou, si cette équation n'en admet pas, de l'équation $x^2 - Dy^2 = 4$; en d'autres termes, une table des plus petites solutions impaires de l'équation $x^2 - Dy^2 = \pm 4$, $D \equiv 5$ (mod. 8).

Londres, 2 *Stone Buildings*, 10 Mars, 1855.

TABLE DES PLUS PETITES SOLUTIONS IMPAIRES DE L'ÉQUATION $x^2 - Dy^2 = \pm 4$, $D \equiv 5$ (MOD. 8).

D	$\pm$	x	y	D	$\pm$	x	y	D	$\pm$	x	y
5	−	1	1	341	+	277	15	677		imposs.	
13	−	3	1	349		imposs.		685	−	759	29
21	+	5	1	357	+	19	1	693	+	79	3
29	−	5	1	365	−	19	1	701		imposs.	
37		imposs.		373		imposs.		709		imposs.	
45	+	7	1	381		imposs.		717	+	241	9
53	−	7	1	389		imposs.		725	+	27	1
61	−	39	5	397	−	3447	173	733	−	27	1
69	+	25	3	405		imposs.		741	+	245	9
77	+	9	1	413	+	61	3	749	+	12945	473
85	−	9	1	421	−	444939	21685	757		imposs.	
93	+	29	3	429	+	145	7	765	+	83	3
101		imposs.		437	+	21	1	773	−	139	5
109	−	261	25	445	−	21	1	781		imposs.	
117	+	11	1	453	+	149	7	789	+	31825	1133
125	−	11	1	461	−	365	17	797	−	367	13
133	+	173	15	469	+	65	3	805	+	1447	51
141		imposs.		477	+	2599	119	813		imposs.	
149	−	61	5	485		imposs.		821	−	16189	565
157	−	213	17	493	−	111	5	829		imposs.	
165	+	13	1	501	+	28225	1261	837	+	29	1
173	−	13	1	509	−	925	41	845	−	29	1
181	−	1305	97	517	+	10573	465	853	−	27483	941
189		imposs.		525	+	23	1	861	+	1027	35
197		imposs.		533	−	23	1	869	+	49377	1675
205	+	43	3	541	−	1396425	60037	877		imposs.	
213	+	73	5	549	+	1523	65	885		imposs.	
221	+	15	1	557		imposs.		893	+	2301	77
229	−	15	1	565	−	309	13	901		imposs.	
237	+	77	5	573		imposs.		909		imposs.	
245	+	47	3	581	+	6725	279	917	+	1181	31
253	+	1177	74	589	+	4359377	179625	925		imposs.	
261	+	727	45	597	+	7949	399	933		imposs.	
269		imposs.		605	+	123	5	941	−	1135	37
277	−	2613	157	613	−	98763	3989	949	−	32685	1061
285	+	17	1	621	+	25	1	957	+	31	1
293	−	17	1	629	−	25	1	965	−	31	1
301	+	22745	1311	637	+	14159	561	973		imposs.	
309	+	5045	287	645	+	203	8	981	+	68123	2175
317	−	89	5	653	−	1661	65	989	+	103245	3283
325		imposs.		661	−	1789539	69605	997		imposs.	
333		imposs.		669	+	305285	11803				

232.

MÉMOIRE SUR LA FORME CANONIQUE DES FONCTIONS BINAIRES.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LIV. (1857), pp. 48—58.]

L'IDÉE de la forme canonique d'une fonction binaire de degré impair et la théorie de la réduction d'une pareille fonction à la forme canonique sont dues à M. Sylvester[1], qui a, en outre, étendu sa théorie aux fonctions binaires des degrés pairs 4 et 6. Mais il n'a établi nulle part l'idée générale de la forme canonique d'une fonction binaire d'un degré pair quelconque. Je me propose de reprendre cette théorie en considérant d'abord les fonctions binaires de degré impair et ensuite les fonctions binaires de degré pair.

Soit donc proposée la fonction de degré impair

$$(a, b, \ldots a' \between x, y)^{2m-1};$$

en désignant par le symbole Σ une somme de m termes, on voit qu'on pourra écrire

$$(a, b, \ldots a' \between x, y)^{2m-1} = \Sigma A (x + \alpha y)^{2m-1},$$

l'expression qui forme le second membre de cette équation est appelée la forme canonique de la fonction donnée, et il s'agit de trouver les valeurs des quantités A, α.

En supposant que le produit des facteurs linéaires $x + \alpha y$ soit égal, à un facteur constant près, à

$$(\mathrm{a}, \mathrm{b}, \ldots \mathrm{a}' \between x, y)^m$$

[1] Les recherches de M. Sylvester sont contenues dans le mémoire intitulé "Sketch of a Memoir on Elimination, Transformation, and Canonical Forms," *Camb. and Dublin Mathematical Journal*, t. VI. [1851] p. 186, dans un supplément publié conjointement avec ce mémoire et intitulé "Essay on Canonical Forms by J. J. Sylvester," London, Bell, 1851, et dans le mémoire intitulé, "On a remarkable discovery in the Theory of Canonical Forms and of Hyperdeterminants," *Phil. Mag.* Nov. 1851.

la question peut s'énoncer sous la forme suivante: trouver une fonction de degré m, telle qu'en représentant $x+\alpha y$ un facteur linéaire quelconque de cette fonction, la fonction donnée de degré $2m-1$ se réduise à la forme $\Sigma A(x+\alpha y)^{2m-1}$. La fonction de degré m contiendra un facteur constant dont la valeur peut être prise à volonté. Mais à ce facteur près la fonction de degré m sera complètement déterminée.

Pour fixer les idées je suppose que la fonction donnée soit la fonction cubique

$$(a,\ b,\ c,\ d\between x,\ y)^3,$$

dans ce cas il s'agira de trouver une fonction quadratique

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2$$

telle qu'en posant $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \mathrm{a}(x+\alpha y)(x+\beta y)$ on ait identiquement

$$(a,\ b,\ c,\ d\between x,\ y)^3 = A(x+\alpha y)^3 + B(x+\beta y)^3.$$

Cela se fait très-facilement par les méthodes ordinaires. En effet, on a

$$\begin{aligned} A \quad + B \quad &= a, \\ A\alpha + B\beta &= b, \\ A\alpha^2 + B\beta^2 &= c, \\ A\alpha^3 + B\beta^3 &= d, \end{aligned}$$

d'où il suit

$$\begin{aligned} a\xi\eta - b(\xi+\eta) + c &= A\ \ (\xi-\alpha)(\eta-\alpha) + B\ \ (\xi-\beta)(\eta-\beta), \\ b\xi\eta - c(\xi+\eta) + d &= A\alpha(\xi-\alpha)(\eta-\alpha) + B\beta(\xi-\beta)(\eta-\beta), \end{aligned}$$

ξ et η étant des quantités quelconques; donc, en prenant $\xi=\alpha$, $\eta=\beta$, on obtient

$$\begin{aligned} a\alpha\beta - b(\alpha+\beta) + c &= 0, \\ b\alpha\beta - c(\alpha+\beta) + d &= 0. \end{aligned}$$

Mais l'équation $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \mathrm{a}(x+\alpha y)(x+\beta y)$ donne

$$\alpha\beta : \alpha+\beta : 1 = \mathrm{c} : 2\mathrm{b} : \mathrm{a},$$

on a donc

$$\begin{aligned} a\mathrm{c} - 2b\mathrm{b} + c\mathrm{a} &= 0, \\ b\mathrm{c} - 2c\mathrm{b} + d\mathrm{a} &= 0, \end{aligned}$$

système qui donne les rapports $\mathrm{a} : \mathrm{b} : \mathrm{c}$. Mais pour compléter la solution de la manière la plus élégante, il faut ajouter à ces équations, l'équation $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = 0$, mise sous la forme $(\mathrm{c},\ \mathrm{b},\ \mathrm{a}\between y,\ x)^2 = 0$, ou ce qui est la même chose

$$y^2\mathrm{c} + 2yx\mathrm{b} + x^2\mathrm{a} = 0;$$

en éliminant a, b, c on obtient une équation de laquelle on déduit

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \begin{vmatrix} y^2, & -yx, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix}.$$

Mais on peut trouver ce résultat d'une manière encore plus facile. Considérant l'équation

$$(a,\ b,\ c,\ d\between x,\ y)^3 = A\,(x+\alpha y)^3 + B\,(x+\beta y)^3,$$

on n'a qu'à opérer sur cette équation en se servant du symbole

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between \partial_y,\ -\partial_x)^2;$$

le second membre se réduit à zéro à cause des équations $(\mathrm{a}, \mathrm{b}, \mathrm{c}\between \alpha, -1)^2 = 0$, $(\mathrm{a}, \mathrm{b}, \mathrm{c}\between \beta, -1)^2 = 0$ que donne l'équation $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \mathrm{a}\,(x+\alpha y\between x+\beta y)$. Par conséquent on a identiquement

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between \partial_y,\ -\partial_x)^2\,(a,\ b,\ c,\ d\between x,\ y)^3 = 0,$$

c'est-à-dire

$$x\,(ac - 2b\mathrm{b} + c\mathrm{a}) + y\,(b\mathrm{c} - 2c\mathrm{b} + d\mathrm{a}) = 0,$$

ou enfin

$$ac - 2b\mathrm{b} + c\mathrm{a} = 0,$$

$$b\mathrm{c} - 2c\mathrm{b} + d\mathrm{a} = 0,$$

et de là on tire comme auparavant la valeur de la fonction $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2$.

Les valeurs des coefficients A, B s'expriment alors en fonctions linéaires des coefficients de la fonction cubique, et des quantités α, β qui sont pour ainsi dire les racines de la fonction quadratique $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2$.

Il est évident que le procédé suivi dans ce qui précède est tout à fait général, et que l'on obtient toujours explicitement la fonction du degré m: par exemple pour la fonction du cinquième degré

$$(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5$$

la fonction dont il s'agit sera

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}$$

et de même dans tous les autres cas. Cette fonction du degré m (laquelle est un covariant de la fonction donnée) peut être appelée le *canonisant*. Pour la fonction du cinquième degré le canonisant peut être mis sous cette autre forme

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}$$

et on démontre facilement qu'il existe toujours une transformation semblable.

Considérons maintenant une fonction de degré pair

$$(a,\ b, \ldots a'\between x,\ y)^{2m},$$

et posons

$$(a,\ b, \ldots a'\between x,\ y)^{2m} = \Sigma A\,(x+\alpha y)^{2m} + \Lambda(\mathrm{a},\ \mathrm{b}, \ldots \mathrm{a}'\between x,\ y)^m\,(\mathrm{a}_1,\ \mathrm{b}_1, \ldots \mathrm{a}_1'\between x,\ y)^m$$

où l'on suppose $(\mathrm{a},\ \mathrm{b}, \ldots \mathrm{a}'\between x,\ y)^m = \mathrm{a}\,(x+\alpha y)\ldots$ et où $(\mathrm{a}_1,\ b_1, \ldots \mathrm{a}_1'\between x,\ y)^m$ représente un certain covariant parfaitement déterminé de $(\mathrm{a},\ \mathrm{b}, \ldots \mathrm{a}'\between x,\ y)^m$. L'expression qui forme le second membre de l'équation proposée s'appellera la forme canonique de la fonction de degré pair.

Pour fixer les idées je suppose que la fonction donnée soit la fonction du quatrième degré

$$(a,\ b,\ c,\ d,\ e\between x,\ y)^4;$$

dans ce cas, en prenant la fonction quadratique $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \mathrm{a}\,(x+\alpha y)\,(x+\beta y)$ et en représentant par $(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\between x,\ y)^2$ un covariant déterminé de $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2$ on a

$$(a,\ b,\ c,\ d,\ e\between x,\ y)^4 = A\,(x+\alpha y)^4 + B\,(x+\beta y)^4 + \Lambda\,(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2\,(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1)\,(x,\ y)^2.$$

Cela posé, la condition qui sert à déterminer le covariant $(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1)\,(x,\ y)^2$ est la suivante

$$\tfrac{1}{2}\,(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between \partial_y,\ -\partial_x)^2 . (\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2\,(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\between x,\ y)^2 = K\,(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2,$$

où K est un facteur constant dont la valeur est arbitraire.

Le premier membre sera

$$\left(\left(\begin{array}{ccc} 7\mathrm{ac}-6\mathrm{b}^2, & -2\mathrm{ab}\ , & \mathrm{a}^2\ , \\ 2\mathrm{bc}\ , & 12\mathrm{ac}-16\mathrm{b}^2, & 2\mathrm{ab}\ , \\ \mathrm{c}^2\ , & -2\mathrm{bc}\ , & 7\mathrm{ac}-6\mathrm{b}^2 \end{array}\right| \between \mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\right)(x,\ y)^2$$

en représentant par cette notation l'expression

$$\begin{array}{lrrr} & [(7\mathrm{ac}-6\mathrm{b}^2)\,\mathrm{a}_1 & -2\mathrm{abb}_1 & +\mathrm{a}^2\mathrm{c}_1]\,x^2 \\ +\,[& 2\mathrm{bca}_1 & +(12\mathrm{ac}-16\mathrm{b}^2)\,\mathrm{b}_1 & +2\mathrm{abc}_1]\,xy \\ +\,[& \mathrm{c}^2\mathrm{a}_1 & -2\mathrm{bcb}_1 & +(7\mathrm{ac}-6\mathrm{b}^2)\,\mathrm{c}_1]\,y^2. \end{array}$$

On a donc

$$\begin{array}{rrr} (7\mathrm{ac}-6\mathrm{b}^2)\,\mathrm{a}_1 & -2\mathrm{abb}_1 & +\mathrm{a}^2\mathrm{c}_1 = K\ \mathrm{a}, \\ 2\mathrm{bca}_1 & +(12\mathrm{ac}-16\mathrm{b}^2)\,\mathrm{b}_1 & +2\mathrm{abc}_1 = K2\mathrm{b}, \\ \mathrm{c}^2\mathrm{a}_1 & -2\mathrm{bcb}_1 & +(7\mathrm{ac}-6\mathrm{b}^2)\,\mathrm{c}_1 = K\ \mathrm{c}, \end{array}$$

et en ajoutant à ces équations la suivante

$$x^2\mathrm{a}_1 \qquad + 2xy\mathrm{b}_1 + y^2 \qquad \mathrm{c}_1 = \quad u,$$

on obtient, en éliminant a_1, b_1, c_1, l'équation

$$\begin{vmatrix} u\,, & x^2\,, & 2xy\,, & y^2 \\ Ka, & 7ac-6b^2, & -2ab\,, & a^2 \\ 2Kb, & 2bc\,, & 12ac-16b^2, & 2ab \\ Kc, & c^2\,, & -2bc\,, & 7ac-6b^2 \end{vmatrix} = 0,$$

laquelle peut s'écrire aussi comme il suit

$$u\begin{vmatrix} 7ac-6b^2, & -2ab\,, & a^2 \\ 2bc\,, & 12ac-16b^2, & 2ab \\ c^2\,, & -2bc\,, & 7ac-6b^2 \end{vmatrix} + K\begin{vmatrix} x^2\,, & 2xy\,, & y^2 \\ a, & 7ac-6b^2, & -2ab\,, & a^2 \\ 2b, & 2bc\,, & 12ac-16b^2, & 2ab \\ c, & c^2\,, & -2bc\,, & 7ac-6b^2 \end{vmatrix} = 0;$$

donc, en rétablissant la valeur de u, savoir $u = (a_1, b_1, c_1 \between x, y)^2$, et en prenant

$$K = \begin{vmatrix} 7ac-6b^2, & -2ab\,, & a^2 \\ 2bc\,, & 12ac-16b^2, & 2ab \\ c^2\,, & -2bc\,, & 7ac-6b^2 \end{vmatrix},$$

on obtient

$$(a_1, b_1, c_1 \between x, y)^2 = -\begin{vmatrix} x^2\,, & 2xy\,, & y^2 \\ a, & 7ac-6b^2, & -2ab\,, & a^2 \\ 2b, & 2bc\,, & 12ac-16b^2, & 2ab \\ c, & c^2\,, & -2bc\,, & 7ac-6b^2 \end{vmatrix}.$$

Ces valeurs se réduisent à des expressions très simples. En effet, on a $K = 576(ac-b^2)^3$, $(a_1, b_1, c_1 \between x, y)^2 = -72(ac-b^2)^2(a, b, c \between x, y)^2$, de sorte qu'en supprimant le facteur constant $-72(ac-b^2)^2$ on peut prendre

$$(a_1, b_1, c_1 \between x, y)^2 = (a, b, c \between x, y)^2.$$

C'est ce que l'on aurait pu prévoir dès le commencement; car le procédé par lequel on obtient $(a_1, b_1, c_1 \between x, y)^2$ fait voir que cette fonction est un covariant de $(a, b, c \between x, y)^2$, du second degré par rapport aux variables et du cinquième degré par rapport aux coefficients; donc cette fonction sera, à un facteur numérique près, identique avec $(ac-b^2)^2(a, b, c \between x, y)^2$. Une circonstance pareille se présente dans la réduction d'une fonction du sixième degré à la forme canonique. En effet le covariant $(a_1, b_1, c_1, d_1 \between x, y)^2$ sera une fonction du troisième degré par rapport aux variables et du septième degré par rapport aux coefficients, donc à un facteur numérique près, cette fonction sera identique avec le discriminant $(a^2d^2 + \text{etc.})$ multiplié par le cubicovariant $(a^2d - 3abc + 2b^3)x^3 + \text{etc.}$, ou, en supprimant le facteur constant, le covariant cherché sera tout simplement le cubicovariant de $(a, b, c, d \between x, y)^3$. De même pour la fonction du huitième degré on

trouve que le covariant $(a_1, b_1, c_1, d_1, e_1 \between x, y)^4$ sera tout simplement $(a, b, c, d, e \between x, y)^4$. Mais rien ne nous autorise à supposer qu'une réduction de cette espèce ait lieu dans le cas général, et il paraît qu'en général le covariant $(a_1, b_1, \ldots a_1' \between x, y)^m$ sera une fonction indécomposable du degré $2m+1$ par rapport aux coefficients; à savoir, en posant

$$(a, b, \ldots a' \between \partial_y, -\partial_x)^m (a, b, \ldots a' \between x, y)^m (a_1, b_1 \ldots a_1' \between x, y)^m = K (a, b, \ldots a' \between x, y)^m$$

le premier membre sera une fonction de la forme

$$\left(\left(\begin{array}{l} \mathfrak{A}, \quad \mathfrak{B} \ldots \between a_1, b_1, \ldots a_1' \\ \mathfrak{A}', \quad \mathfrak{B}' \ldots \\ \vdots \end{array} \right)\right) (x, y)^m$$

où $\mathfrak{A}, \mathfrak{B}, \ldots \mathfrak{A}', \mathfrak{B}', \ldots$ etc. sont des fonctions données du second degré par rapport aux coefficients $(a, b, \ldots a')$; cela étant, on aura

$$K = \left| \begin{array}{ll} \mathfrak{A}, & \mathfrak{B}, \ldots \\ \mathfrak{A}', & \mathfrak{B}', \ldots \\ \vdots & \end{array} \right|,$$

et

$$(a_1, b_1, \ldots a_1' \between x, y)^m = - \left| \begin{array}{llll} & x^m, & mx^{m-1}y & \\ a, & \mathfrak{A}, & \mathfrak{B} & \ldots \\ mb, & \mathfrak{A}', & \mathfrak{B}' & \\ \vdots & & & \end{array} \right|,$$

ce qui fait voir que le covariant dont il s'agit est une fonction du degré $2m+1$ par rapport aux coefficients.

Je reviens au cas particulier, le cas général pouvant être traité précisément de la même manière. On a

$$(a, b, c, d, e \between x, y)^4 = A (x + \alpha y)^4 + B (x + \beta y)^4 + \Lambda (a, b, c \between x, y)^2 (a_1, b_1, c_1 \between x, y)^2.$$

J'opère sur cette équation en me servant du symbole $(a, b, c \between \partial_y, -\partial_x)^2$ et j'obtiens

$$(a, b, c \between \partial_y, -\partial_x)^2 (a, b, c, d, e \between x, y)^4 = 2\Lambda K (a, b, c \between x, y)^2,$$

c'est-à-dire

$$\left. \begin{array}{r} a (c x^2 + 2dxy + e y^2) \\ - 2b (b x^2 + 2c xy + d y^2) \\ + \ c (a x^2 + 2bxy + c y^2) \end{array} \right\} = \tfrac{1}{6} \Lambda K (a x^2 + 2bxy + cy^2);$$

en posant, pour abréger, $\frac{1}{6} \Lambda K = \lambda$, on aura

$$\left. \begin{array}{lll} x^2 [ca & - 2bb & + a (c - \lambda)] \\ + 2xy [cb & - 2b (c + \tfrac{1}{2}\lambda) + ad &] \\ + \ y^2 [c (c - \lambda) - 2bd & + ae &] \end{array} \right\} = 0;$$

ce qui donne

$$\begin{aligned} \mathrm{c}\,a \quad & -2\mathrm{b}\,b \quad & +\mathrm{a}\,(c-\lambda) &= 0, \\ \mathrm{c}\,b \quad & -2\mathrm{b}\,(c+\tfrac{1}{2}\lambda) & +\mathrm{a}\,d \quad &= 0, \\ \mathrm{c}\,(c-\lambda) & -2\mathrm{b}\,d \quad & +\mathrm{a}\,e \quad &= 0, \end{aligned}$$

et de là on tire l'équation

$$\begin{vmatrix} a\,, & b\,, & c-\lambda \\ b\,, & c+\tfrac{1}{2}\lambda, & d \\ c-\lambda, & d\,, & e \end{vmatrix} = 0$$

qui servira à la détermination de la quantité λ. Les coefficients des différentes puissances de λ seront des invariants; en effet, si on développe le déterminant, l'équation devient $2(ace+2bcd-ad^2-b^2e+c^3)+\lambda(ae-4bd+3c^2)-\lambda^3=0$. La quantité λ étant connue, on peut au moyen de deux quelconques des trois équations trouver les rapports a : b : c, ou ce qui revient à la même chose, on peut se servir des trois équations en introduisant les quantités arbitraires L, M, N, et l'on trouve

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2 = \begin{vmatrix} & y^2\,, & -yx\,, & x^2 \\ L, & a\,, & b\,, & c-\lambda \\ M, & b\,, & c+\tfrac{1}{2}\lambda, & d \\ N, & c-\lambda, & d\,, & e \end{vmatrix}$$

expression qui, en réalité, ne contient rien d'indéterminé. La quantité K est une fonction connue de (a, b, c) donc $\Lambda = \dfrac{6\lambda}{K}$ sera pareillement connu, et les coefficients A, B s'expriment en fonctions linéaires des coefficients de la fonction du quatrième degré et des quantités α, β qui sont pour ainsi dire les racines de la fonction quadratique $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between x,\ y)^2$. Comme je l'ai déjà fait observer, tout ce qui précède s'applique également à une fonction de degré pair quelconque. Le déterminant qui contient λ peut être appelé le Lambdaïque. On voit aisément quelle est sa forme générale; par exemple pour la fonction $(a,\ b,\ c,\ d,\ e,\ f,\ g\between x,\ y)^6$ ce déterminant est

$$\begin{vmatrix} a\,, & b\,, & c\,, & d-\lambda \\ b\,, & c\,, & d+\tfrac{1}{3}\lambda, & e \\ c\,, & d-\tfrac{1}{3}\lambda, & e\,, & f \\ d+\lambda, & e\,, & f\,, & g \end{vmatrix}$$

expression dont le développement ne contient que les puissances paires de λ; cela arrive, comme M. Sylvester l'a remarqué depuis longtemps, toutes les fois que m est impair, c'est-à-dire toutes les fois que le degré de la fonction dont il s'agit est un nombre de la forme $4p+2$.

Revenons à la fonction du quatrième degré $(a, b, c, d, e \between x, y)^4$; en écrivant $(a_1, b_1, c_1 \between x, y)^2 = (a, b, c \between x, y)^2 = a(x+\alpha y)(x+\beta y)$ on obtient pour la forme canonique

$$(a, b, c, d, e \between x, y)^4 = A(x+\alpha y)^4 + B(x+\beta y)^4 + \Lambda(x+\alpha y)^2(x+\beta y)^2.$$

Relativement à la fonction du sixième degré $(a, b, c, d, e, f, g \between x, y)^6$, je fais observer que le cubicovariant de $(x+\alpha y)(x+\beta y)(x+\gamma y)$ est

$$\{(2\alpha-\beta-\gamma)x-(2\beta\gamma-\gamma\alpha-\alpha\beta)y\}\{(2\beta-\gamma-\alpha)x-(2\gamma\alpha-\alpha\beta-\beta\gamma)y\}$$
$$\{(2\gamma-\alpha-\beta)x-(2\alpha\beta-\beta\gamma-\gamma\alpha)y\};$$

donc en représentant cette fonction par Φ, on a pour la forme canonique

$$(a, b, c, d, e, f, g \between x, y)^6$$
$$= A(x+\alpha y)^6 + B(x+\beta y)^6 + C(x+\gamma y)^6 + \Lambda(x+\alpha y)(x+\beta y)(x+\gamma y)\Phi.$$

Pour la fonction du huitième degré le covariant $(a_1, b_1, c_1, d_1, e_1 \between x, y)^4$ est tout simplement égal à $(a, b, c, d, e \between x, y)^4$ ainsi que je l'ai déjà fait observer, et l'on a pour la forme canonique

$$(a, b, c, d, e, f, g, h, i \between x, y)^8$$
$$= A(x+\alpha y)^8 + B(x+\beta y)^8 + C(x+\gamma y)^8 + D(x+\delta y)^8 + \Lambda(x+\alpha y)^2(x+\beta y)^2(x+\gamma y)^2(x+\delta y)^2.$$

Je n'ai pas cherché à réduire, ou à présenter sous la forme d'un déterminant, le covariant $(a_1, b_1, c_1, d_1, e_1, f_1 \between x, y)^5$ qui entre dans la forme canonique de la fonction du dixième degré.

On a vu que le covariant $(a_1, b_1, \ldots a_1' \between x, y)^m$ s'obtient par le développement de la fonction

$$(a, b, \ldots a' \between \partial_y, -\partial_x)^m . (a, b, \ldots a_1' \between x, y)^m (a_1, b_1, \ldots a_1' \between x, y)^m$$

et j'ai donné ci-dessus un exemple de ce développement; en employant pour plus de commodité les lettres italiques au lieu des lettres romaines, on peut écrire la formule dont il s'agit de la manière suivante

$$(a, b, c \between \partial_y, -\partial_x)^2 (a, b, c \between x, y)^2 (a_1, b_1, c_1 \between x, y)^2 =$$
$$2\left(\left(\begin{array}{c|c|c} 7ac-6b^2 & -2ab & a^2 \\ 2bc & 12ac-16b^2 & 2ab \\ c^2 & -2bc & 7ac-6b^2 \end{array}\right) \between a_1, b_1, c_1\right)(x, y)^2.$$

Les équations analogues pour la fonction cubique etc. sont

$$(a, b, c, d \between \partial_y, -\partial_x)^3 (a, b, c, d \between x, y)^3 (a_1, b_1, c_1, d_1 \between x, y)^3 =$$

$$6 \times \left(\left(\begin{array}{c|c|c|c} -19ad+18bc & +39ac-36b^2 & -3ab & +a^2 \\ -39bd+36c^2 & -18ad+27bc & +72ac-81b^2 & +3ab \\ -3cd & -72bd+81c^2 & +18ad-27bc & +39ac-36b^2 \\ -d^2 & +3cd & -39bd+36c^2 & +19ad-18bc \end{array}\right) \between a_1, b_1, c_1, d_1\right)(x, y)^3$$

$$(a, b, c, d, e \between \partial_y, -\partial_x)^4 (a, b, c, d, e \between x, y)^4 (a_1, b_1, c_1, d_1, e_1 \between x, y)^4 =$$

$$24 \times \Big(\begin{vmatrix} +71\,ae - 160\,bd + 90\,c^2 & -124\,ad + 120\,bc & +126\,ac - 120\,b^2 & -4\,ab & +a^2 \\ +124\,be - 120\,cd & +160\,ae - 896\,bd + 720\,c^2 & -120\,ad + 144\,bc & +240\,ac - 256\,b^2 & +4\,ab \\ +126\,ce - 120\,d^2 & +120\,be - 144\,cd & +180\,ae - 1440\,bd + 1296\,c^2 & +120\,ad - 144\,bc & +126\,ac - 120\,b^2 \\ +4\,de & +240\,ce - 256\,d^2 & -120\,be + 144\,cd & +160\,ae - 896\,bd + 720\,c & +124\,ad - 120\,bc \\ +e^2 & -4\,de & +126\,ce - 120\,d^2 & -124\,be + 120\,cd & +71\,ae - 160\,bd + 90\,c^2 \end{vmatrix} \between (a_1, b_1, c_1, d_1, e_1) \Big) (x, y)^4.$$

$$(a, b, c, d, e, f \between \partial_y, -\partial_x)^5 (a, b, c, d, e, f \between x, y)^5 (a_1, b_1, c_1, d_1, e_1, f_1 \between x, y)^5 =$$

$$120 \times \Big(\begin{vmatrix} -251\,af + 600\,be - 350\,cd & +655\,ae - 1700\,bd + 1050\,c^2 & -460\,ad + 450\,bc & +310\,ac - 300\,b^2 & -5\,ab & +a^2 \\ -655\,bf + 1700\,ce - 1050\,d^2 & -600\,af + 2875\,be - 2250\,cd & +1700\,ae - 7750\,cd + 6000\,c^2 & -450\,ad + 500\,bc & +600\,ac - 625\,b^2 & +5\,ab \\ -460\,cf + 450\,de & -1700\,bf + 7750\,ce - 6000\,d^2 & -350\,af + 2250\,be - 2000\,cd & +2100\,ae - 12000\,bd + 10000\,c^2 & +450\,ad - 500\,bc & +310\,ac - 300\,b^2 \\ -310\,df + 300\,e^2 & -450\,cf + 500\,de & -2100\,bf + 12000\,ce - 10000\,d^2 & +350\,af - 2250\,be + 2000\,cd & +1700\,ae - 7750\,bd + 6000\,c^2 & +460\,ad - 450\,bc \\ -5\,ef & -600\,df + 625\,e^2 & +450\,cf - 500\,de & -1700\,bf + 7750\,ce - 6000\,d^2 & +600\,af - 2875\,be + 2250\,cd & +655\,ae - 1700\,bd + 1050\,c^2 \\ -e^2 & +5\,ef & -310\,df + 300\,e^2 & +460\,cf - 450\,de & -655\,bf + 1700\,ce - 1050\,d^2 & +251\,af - 600\,be + 350\,cd \end{vmatrix} \between (a_1, b_1, c_1, d_1, e_1, f_1) \Big) (x, y)^5.$$

Les résultats qu'on vient d'obtenir pour les cas particuliers les plus simples font voir que la théorie générale doit être susceptible de développements ultérieurs. Ainsi la

forme canonique de la fonction cubique $(a, b, c, d \between x, y)^3$ sera u^3+v^3 si l'on pose u, v au lieu de $\sqrt[3]{A}(x+\alpha y)$, $\sqrt[3]{B}(x+\beta y)$. La méthode générale ne donne que la valeur du produit u, v à un facteur constant près; mais on peut trouver les valeurs de u et v séparément. En effet en se rappelant les résultats obtenus dans la "Note sur les covariants d'une fonction quadratique, cubique, ou biquadratique à deux indéterminées" (t. L. p. 285 de ce Journal, [135]), on reconnaît que les deux fonctions $\Phi+U\sqrt{-\square}$, $\Phi-U\sqrt{-\square}$ sont l'une et l'autre des cubes parfaits et l'on a

$$2U\sqrt{-\square}=(\Phi+U\sqrt{-\square})-(\Phi-U\sqrt{-\square});$$

on a donc

$$u=\left\{\frac{\Phi+U\sqrt{-\square}}{2\sqrt{-\square}}\right\}^{\frac{1}{3}},\quad v=\left\{\frac{\Phi-U\sqrt{-\square}}{2\sqrt{-\square}}\right\}^{\frac{1}{3}}.$$

De même la forme canonique de la fonction du quatrième degré $(a, b, c, d, e \between x, y)^4$ sera $u^4+v^4+6\theta u^2v^2$, $=(u^2-v^2)^2+2(1+3\theta)u^2v^2$, si l'on pose u, v au lieu de $\sqrt[4]{A}(x+\alpha y)$, $\sqrt[4]{B}(x+\beta y)$; or $IH-\varpi_1 JU$, $IH-\varpi_2 JU$, $IH-\varpi_3 JU$ étant tous les trois des carrés de fonctions quadratiques, la fonction U peut s'exprimer à moyen de deux quelconques de ces trois quantités comme une somme de deux carrés et on déduit de là les valeurs de u et de v; on trouvera le développement de cette solution dans un mémoire "Sur quelques formules pour la transformation des intégrales elliptiques", [235], qui paraîtra prochainement dans ce Journal. Quant à la fonction du cinquième degré, on pourrait écrire u, v, w au lieu de $\sqrt[5]{A}(x+\alpha y)$, $\sqrt[5]{B}(v+\beta y)$, $\sqrt[5]{C}(x+\gamma y)$, ce qui donnerait pour forme canonique $u^5+v^5+w^5$, mais il vaut mieux, en multipliant ces valeurs par des facteurs convenables, écrire $u+v+w=0$ et prendre pour forme canonique $Au^5+Bv^5+Cw^5=0$, c'est ce qu'a fait M. Sylvester dans son mémoire ci-dessus cité. On trouve aussi des développements sur ce sujet dans le mémoire de M. Faà de Bruno, "Nota sulla teorica degli invarianti", *Tortolini, Annali di Scienze etc.* 1855.

Pour la fonction du sixième degré on peut supposer que la fonction cubique $(a, b, c, d \between x, y)^3$ soit réduite à la forme canonique u^3+v^3; cela étant, le cubicovariant sera u^3-v^3, et en représentant par ρ l'une des racines cubiques imaginaires de l'unité la forme canonique sera

$$A(u+v)^6+B(u+\rho v)^6+C(u+\rho^2 v)^6+\Lambda(u^6-v^6);$$

c'est encore ce qu'a fait voir M. Sylvester dans son mémoire.

Londres, 9 *Avril*, 1856.

233.

ADDITION AU MÉMOIRE SUR LA FORME CANONIQUE DES FONCTIONS BINAIRES.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LIV. (1857), p. 292.]

DANS le mémoire "Sur la forme canonique des fonctions binaires" (p. 48 de ce volume [232]) j'ai dit que M. Sylvester avait en outre étendu sa théorie aux fonctions binaires des degrés pairs 4 et 6; j'aurais dû dire, des degrés pairs 4, 6 et 8. J'ai aussi omis de faire observer que les termes *canonisant* et *lambdaïque* appartenaient à M. Sylvester. Enfin en citant dans la note les mémoires de M. Sylvester qui ont rapport à cette théorie j'ai omis de citer le mémoire "On the Calculus of Forms otherwise the Theory of Invariants, § VIII. (*Camb. and Dublin Math. Journal*, t. IX. [1854], p. 93) section qui porte le titre "On the Reduction of a Sextic Function of Two Variables to the Canonical Form."

Londres, 16 *Juillet*, 1857.

234.

DEUXIÈME NOTE SUR UNE FORMULE POUR LA RÉVERSION DES SÉRIES.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LIV. (1857), pp. 156—161: Sequel to Note t. LII. (1856), **229**.]

JE me propose de montrer, dans cette deuxième note, de quelle manière le théorème de Jacobi conduit à une formule donnée sans démonstration par M. Sylvester dans son mémoire intitulé "On the Change of Systems of Independent Variables," *Quarterly Math. Journal*, tom. I. [1857], p. 42 à 56 et 126 à 134. Pour fixer les idées je prends le cas de trois variables, et je suppose que $f(x, y, z)$ soit une fonction rationnelle et entière de x, y, z, et que ces variables soient données en fonction de u, v, w au moyen des équations $u = X$, $v = Y$, $w = Z$, où X, Y, Z sont des fonctions rationnelles et entières de x, y, z. Mais ces fonctions ne sont plus assujetties à la condition (admise dans ma première note) d'être telles que $X - x$, $Y - y$, $Z - z$ ne contiennent que les puissances et les produits du deuxième ordre et des ordres supérieurs des variables, et il s'agit de déterminer dans le cas général le développement de $f(x, y, z)$ en termes de u, v, w.

Pour résoudre ce problème j'écris

$$\begin{aligned} X &= A_{100}\, x + A_{010}\, y + A_{001}\, z + \ldots + A_{f,g,h}\, x^f y^g z^h + \text{etc.}, \\ Y &= B_{100}\, x + B_{010}\, y + B_{001}\, z + \ldots + B_{i,j,k}\, x^i y^j z^k + \text{etc.}, \\ Z &= C_{100}\, x + C_{010}\, y + C_{001}\, z + \ldots + C_{l,m,n}\, x^l y^m z^n + \text{etc.}, \\ f(x, y, z) &= \qquad\qquad\qquad\qquad \ldots + \Theta_{P,Q,R}\, x^P y^Q z^R + \text{etc.}; \end{aligned}$$

dans ces expressions et partout dans la suite les etc. représentent des termes qu'on obtient en donnant des accents en nombre quelconque aux symboles indéterminés. Je dois faire observer relativement au coefficient $A_{f,g,h}$ et aux coefficients semblables, que les termes qui correspondent à $f + g + h = 1$ sont écrits à part; on doit donc prendre pour les nombres f, g, h seulement les valeurs qui rendent $f + g + h > 1$.

Je pose

$$x' = A_{100}x + A_{010}y + A_{001}z,$$
$$y' = B_{100}x + B_{010}y + B_{001}z,$$
$$z' = C_{100}x + C_{010}y + C_{001}z,$$

et en représentant par

$$\begin{vmatrix} a_{100}, & b_{100}, & c_{100} \\ a_{010}, & b_{010}, & c_{010} \\ a_{001}, & b_{001}, & c_{001} \end{vmatrix}$$

la *matrice inverse* de

$$\begin{vmatrix} A_{100}, & A_{010}, & A_{001} \\ B_{100}, & B_{010}, & B_{001} \\ C_{100}, & C_{010}, & C_{001} \end{vmatrix},$$

on obtient les équations

$$x = a_{100}x' + b_{100}y' + c_{100}z',$$
$$y = a_{010}x' + b_{010}y' + c_{010}z',$$
$$z = a_{001}x' + b_{001}y' + c_{001}z'.$$

Il est presque superflu de faire observer qu'en représentant par ∇ le déterminant

$$\begin{vmatrix} A_{100}, & A_{010}, & A_{001} \\ B_{100}, & B_{010}, & B_{001} \\ C_{100}, & C_{010}, & C_{001} \end{vmatrix}$$

on a

$$a_{100} = \frac{1}{\nabla}\frac{d\nabla}{dA_{100}}, \qquad a_{010} = \frac{1}{\nabla}\frac{d\nabla}{dA_{010}}, \text{ etc.}$$

A présent, en supposant chacune des quantités u', v', w' égale à zéro, on a le système d'équations

$$u - X = 0, \qquad v - Y = 0, \qquad w - Z = 0$$
$$u' - X' = 0, \qquad v' - Y' = 0, \qquad w' - Z' = 0$$

où

$$X = x' \qquad \ldots + A_{f,g,h}\, x^f y^g z^h + \text{etc.},$$
$$Y = y' \qquad \ldots + B_{i,j,k}\, x^i y^j z^k + \text{etc.},$$
$$Z = z' \qquad \ldots + C_{l,m,n}\, x^l y^m z^n + \text{etc.},$$

$$X' = x - a_{100}x' - b_{100}y' - c_{100}z',$$
$$Y' = y - a_{010}x' - b_{010}y' - c_{010}z',$$
$$Z' = z - a_{001}x' - b_{001}y' - c_{001}z',$$

et comme auparavant

$$f(x, y, z) = \ldots + \Theta_{p,q,r}\, x^p y^q z^r + \text{etc.}$$

Or, en appliquant à ce nouveau système la formule de Jacobi, et en remarquant qu'il est permis de poser tout de suite $u'=0$, $v'=0$, $w'=0$, cette formule donne

$$f(x, y, z) =$$

$$\left[f(x, y, z)\frac{\partial(X, Y, Z, X', Y', Z')}{\partial(x, y, z, x', y', z')}\frac{1}{(X-u)(Y-v)(Z-w)X'Y'Z'}\right]_{x^{-1}y^{-1}z^{-1}x'^{-1}y'^{-1}z'^{-1}}$$

équation dans laquelle on doit d'abord développer le dernier facteur de l'expression renfermée entre crochets suivant les puissances ascendantes de u, v, w, et ensuite développer les puissances de X, Y, Z, X', Y', Z' suivant les puissances descendantes de x', y', z', x, y, z respectivement. On obtient ainsi

$$\text{coeff. de } u^a v^b w^c \text{ dans le développement de } f(x, y, z) =$$

$$\left[f(x, y, z)\frac{\partial(X, Y, Z, X', Y', Z')}{\partial(x, y, z, x', y', z')}\frac{1}{X^{a+1}Y^{b+1}Z^{c+1}X'Y'Z'}\right]_{x^{-1}y^{-1}z^{-1}x'^{-1}y'^{-1}z'^{-1}}$$

ou ce qui est la même chose

$$\text{coeff. de } u^a v^b w^c \text{ dans le développement de } f(x, y, z) =$$

$$\left[f(x, y, z)\frac{\partial\left(-\frac{1}{a}X^{-a}, -\frac{1}{b}Y^{-b}, -\frac{1}{c}Z^{-c}, \log X', \log Y', \log Z'\right)}{\partial(x, y, z, x', y', z')}\right]_{x^{-1}y^{-1}z^{-1}x'^{-1}y'^{-1}z'^{-1}}$$

Or, en posant $\Pi a = 1.2.3\ldots a$, etc., le terme général de $-\frac{1}{a}X^{-a}$ est

$$(-1)^{r-1}\frac{\Pi(a+r-1)}{\Pi a\,\Pi\alpha \text{ etc.}}A^{\alpha}_{f,g,h} \text{ etc. } x'^{-a-r}x^F y^G z^H,$$

où

$$\alpha + \text{etc.} = r, \quad f\alpha + \text{etc.} = F, \quad g\alpha + \text{etc.} = G, \quad h\alpha + \text{etc.} = H,$$

de même le terme général de $-\frac{1}{b}Y^{-b}$ est

$$(-1)^{s-1}\frac{\Pi(b+s-1)}{\Pi b\,\Pi\beta \text{ etc.}}B^{\beta}_{i,j,k} \text{ etc. } y'^{-b-s}x^I y^J z^K,$$

où

$$\beta + \text{etc.} = s, \quad i\beta + \text{etc.} = I, \quad j\beta + \text{etc.} = J, \quad k\beta + \text{etc.} = K$$

et le terme général de $-\frac{1}{c}Z^{-c}$ est

$$(-1)^{t-1}\frac{\Pi(c+t-1)}{\Pi c\,\Pi\gamma \text{ etc.}}C^{\gamma}_{l,m,n} \text{ etc. } z'^{-c-t}x^L y^M z^N,$$

où

$$\gamma + \text{etc.} = t, \quad l\gamma + \text{etc.} = L, \quad m\gamma + \text{etc.} = M, \quad n\gamma + \text{etc.} = N.$$

Le terme général du développement de $\log X'$ est

$$-\frac{\Pi(r'-1)}{\Pi\alpha'\,\Pi\delta'\,\Pi\iota'}\, a_{100}^{\alpha'}\, b_{100}^{\delta'}\, c_{100}^{\iota'}\, x^{-r'}\, x'^{\alpha'}\, y'^{\delta'}\, z'^{\iota'}$$

où

$$r' = \alpha' + \delta' + \iota',$$

et je fais observer que pour tenir compte du terme $\log x$ que contient $\log X'$, il suffit d'attribuer à α', δ', ι', r' les valeurs $\alpha' = \delta' = \iota' = r' = 0$. En effet on n'a besoin que des coefficients différentiels de $\log X'$, et, en gardant pour le moment r' au lieu de zéro, le terme dont il s'agit sera $-\Pi(r'-1)\,x^{-r'}$, ce qui différentié par rapport à x donne $\Pi r' \,.\, x^{-r'-1}$. En faisant $r' = 0$ cela devient $\frac{1}{x}$, ce qui est en effet la valeur du coefficient différentiel de $\log x$ par rapport à x.

De même le terme général de $\log Y'$ est

$$-\frac{\Pi(s'-1)}{\Pi\beta'\,\Pi\epsilon'\,\Pi\kappa'}\, a_{010}^{\beta'}\, b_{010}^{\epsilon'}\, c_{010}^{\kappa'}\, y^{-s'}\, x'^{\beta'}\, y'^{\epsilon'}\, z'^{\kappa'}$$

où

$$s' = \beta' + \epsilon' + \kappa',$$

et le terme général de $\log Z'$ est

$$-\frac{\Pi(t'-1)}{\Pi\gamma'\,\Pi\zeta'\,\Pi\lambda'}\, a_{001}^{\gamma'}\, b_{001}^{\zeta'}\, c_{001}^{\lambda'}\, z^{-t'}\, x'^{\gamma'}\, y'^{\zeta'}\, z'^{\lambda'}$$

où

$$t' = \gamma' + \zeta' + \lambda'.$$

Donc, en formant le terme général du Jacobien et en multipliant par le terme général de $f(x, y, z)$, on obtient pour le terme général de l'expression placée entre crochets la valeur suivante

$$\begin{aligned}
&(-1)^{r+s+t}\,\frac{\Pi(\mathrm{a}+r-1)\,\Pi(\mathrm{b}+s-1)\,\Pi(\mathrm{c}+t-1)}{\Pi\mathrm{a}\,\Pi\mathrm{b}\,\Pi\mathrm{c}\,\Pi\alpha\ \text{etc.}\ \Pi\beta\ \text{etc.}\ \Pi\gamma\ \text{etc.}}\,\frac{\Pi(r'-1)\,\Pi(s'-1)\,\Pi(t'-1)}{\Pi\alpha'\,\Pi\delta'\,\Pi\iota'\,\Pi\beta'\,\Pi\epsilon'\,\Pi\kappa'\,\Pi\gamma'\,\Pi\zeta'\,\Pi\lambda'}\\
&\times A^{a}_{f,g,h}\ \text{etc.}\ B^{\beta}_{i,j,k}\ \text{etc.}\ C\gamma_{l,m,n}\ \text{etc.}\ a_{100}^{\alpha'}\, b_{100}^{\delta'}\, c_{100}^{\iota'}\, a_{010}^{\beta'}\, b_{010}^{\epsilon'}\, c_{010}^{\kappa'}\, a_{001}^{\gamma'}\, b_{001}^{\zeta'}\, c_{001}^{\lambda'}\\
&\times \Omega\\
&\times x^{F+I+L+P-r'-1}\, y^{G+J+M+Q-s'-1}\, z^{H+K+N+R-t'-1}\, x'^{-\mathrm{a}-r+\alpha'+\beta'+\gamma'-1}\, y'^{-\mathrm{b}-s+\delta'+\epsilon'+\zeta'-1}\, z'^{-\mathrm{c}-t+\iota'+\kappa'+\lambda'-1}
\end{aligned}$$

où le facteur Ω représente le déterminant numérique suivant

$$\Omega = \begin{vmatrix}
F, & G, & H, & -\mathrm{a}-r, & 0, & 0\\
I, & J, & K, & 0, & -\mathrm{b}-s, & 0\\
L, & M, & N, & 0, & 0, & -\mathrm{c}-t\\
-r', & 0, & 0, & \alpha', & \delta', & \iota'\\
0, & -s', & 0, & \beta', & \epsilon', & \kappa'\\
0, & 0, & -t', & \gamma', & \zeta', & \lambda'
\end{vmatrix}.$$

Pour trouver le terme qui contient $x^{-1}y^{-1}z^{-1}x'^{-1}y'^{-1}z'^{-1}$ on n'a qu'à poser

$$F+I+L+P-r'=0, \qquad G+J+M+Q-s'=0, \qquad H+K+N+R-t'=0,$$
$$-\mathrm{a}-r+\alpha'+\beta'+\gamma'=0, \qquad -\mathrm{b}-s+\delta'+\epsilon'+\zeta'=0, \qquad -\mathrm{c}-t+\iota'+\kappa'+\lambda'=0.$$

En faisant cela, et en tenant compte des formules précédentes on obtient le théorème suivant: savoir en posant

$$\begin{aligned} X &= A_{100}\,x+A_{010}\,y+A_{001}\,z\ldots+A_{f,g,h}\quad x^f\,y^g\,z^h+\text{etc.}=u,\\ Y &= B_{100}\,x+B_{010}\,y+B_{001}\,z\ldots+B_{i,j,k}\quad x^i\,y^j\,z^k+\text{etc.}=v,\\ Z &= C_{100}\,x+C_{010}\,y+C_{001}\,z\ldots+C_{l,m,n}\quad x^l\,y^m\,z^n+\text{etc.}=w,\\ f(x,\,y,\,z) &= \qquad\qquad\qquad\qquad \ldots+\Theta^{P,\,Q,\,R}\quad x^P\,y^Q\,z^R+\text{etc.}\end{aligned}$$

on trouve pour le terme général du coeff. de $u^{\mathrm{a}}v^{\mathrm{b}}w^{\mathrm{c}}$ dans le développement de $f(x,\,y,\,z)$ l'expression

$$(-1)^{r+s+t}\,\frac{\Pi(\mathrm{a}+r-1)\,\Pi(\mathrm{b}+s-1)\,\Pi(\mathrm{c}+t-1)}{\Pi\mathrm{a}\,\Pi\mathrm{b}\,\Pi\mathrm{c}\;\Pi\alpha\,\text{etc.}\;\Pi\beta\,\text{etc.}\;\Pi\gamma\,\text{etc.}}\;\frac{\Pi(r'-1)}{\Pi\alpha'\,\Pi\delta'\,\Pi\iota'}\;\frac{\Pi(s'-1)}{\Pi\beta'\,\Pi\epsilon'\,\Pi\kappa'}\;\frac{\Pi(t'-1)}{\Pi\gamma'\,\Pi\zeta'\,\Pi\lambda'}$$
$$\times A^{\alpha}{}_{f,g,h}\ \text{etc.}\ B^{\beta}{}_{i,j,k}\ \text{etc.}\ C^{\gamma}{}_{l,m,n}\ \text{etc.}\ a_{100}^{\alpha'}\,b_{100}^{\delta'}\,c_{100}^{\iota'}\,a_{010}^{\beta'}\,b_{010}^{\epsilon'}\,c_{010}^{\kappa'}\,a_{001}^{\gamma'}\,b_{001}^{\zeta'}\,c_{001}^{\lambda'}$$
$$\times\,\Omega,$$

formule dans laquelle les a_{100} etc. sont les coefficients inverses des A_{100} etc. c'est-à-dire

$$\begin{vmatrix} a_{100}, & b_{100}, & c_{100}\\ a_{010}, & b_{010}, & c_{010}\\ a_{001}, & b_{001}, & c_{001}\end{vmatrix}=\begin{vmatrix} A_{100}, & A_{010}, & A_{001}\\ B_{100}, & B_{010}, & B_{001}\\ C_{100}, & C_{010}, & C_{001}\end{vmatrix}^{-1};$$

le déterminant numérique Ω a la valeur qui vient d'être donnée, en supposant seulement que les nombres α, β, etc. qui y entrent, satisfassent aux conditions suivantes

$$\begin{array}{lll} \alpha+\text{etc.}=r, & \beta+\text{etc.}=s, & \gamma+\text{etc.}=t,\\ f\alpha+\text{etc.}=F, & g\alpha+\text{etc.}=G, & h\alpha+\text{etc.}=H,\\ i\beta+\text{etc.}=I, & j\beta+\text{etc.}=J, & k\beta+\text{etc.}=K,\\ l\gamma+\text{etc.}=L, & m\gamma+\text{etc.}=M, & n\gamma+\text{etc.}=N,\\ \alpha'+\delta'+\iota'=r', & \beta'+\epsilon'+\kappa'=s', & \gamma'+\zeta'+\lambda'=t',\\ P+F+I+L=r', & Q+G+J+M=s', & R+H+K+N=t',\\ \alpha'+\beta'+\gamma'=\mathrm{a}+r, & \delta'+\epsilon'+\zeta'=\mathrm{b}+s, & \iota'+\kappa'+\lambda'=\mathrm{c}+t.\end{array}$$

De ces équations on tire

$$P+Q+R+F+G+H+I+J+K+L+M+N=\mathrm{a}+\mathrm{b}+\mathrm{c}+r+s+t,$$

ou, en substituant pour F, etc. leurs valeurs,

$$(f+g+h-1)\alpha+\text{etc.}+(i+j+k-1)\beta+\text{etc.}+(l+m+n-1)\gamma+\text{etc.}$$
$$=\mathrm{a}+\mathrm{b}+\mathrm{c}-P-Q-R,$$

les nombres $f+g+h-1$, $i+j+k-1$, $l+m+n-1$, etc. étant positifs. Il n'y a donc qu'un nombre fini de solutions des équations indéterminées, comme cela doit être.

On peut encore modifier un peu la forme du déterminant Ω; on voit d'abord que l'on peut changer les signes des quantités $-\mathrm{a}-r$, $-\mathrm{b}-s$, $-\mathrm{c}-t$, $-r'$, $-s'$, $-t'$; en faisant cela et en substituant pour ces quantités et aussi pour F, G, etc. les valeurs ci-dessus données, on obtient

$$\Omega=\begin{vmatrix} f\alpha+\text{etc.}, & g\alpha+\text{etc.}, & h\alpha+\text{etc.}, & \alpha'+\beta'+\gamma', & 0, & 0 \\ i\beta+\text{etc.}, & j\beta+\text{etc.}, & k\beta+\text{etc.}, & 0, & \delta'+\epsilon'+\zeta', & 0 \\ l\gamma+\text{etc.}, & m\gamma+\text{etc.}, & n\gamma+\text{etc.}, & 0, & 0, & \iota'+\kappa'+\lambda' \\ \alpha'+\delta'+\iota', & 0, & 0, & \alpha', & \delta', & \iota' \\ 0, & \beta'+\epsilon'+\kappa', & 0, & \beta', & \epsilon', & \kappa' \\ 0, & 0, & \gamma'+\zeta'+\lambda', & \gamma', & \zeta', & \lambda' \end{vmatrix}.$$

Ainsi se trouve démontré le théorème général de M. Sylvester relatif à la réversion des séries, car il est évident que les formules s'appliquent sans difficulté à un nombre quelconque de variables.

Londres, le 30 *Septembre*, 1856.

235.

SUR QUELQUES FORMULES POUR LA TRANSFORMATION DES INTÉGRALES ELLIPTIQUES.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LV. (1858), pp. 15—24.]

I.

En posant dans les formules "Fund. nova p. 15, Tab. III.," $\cos\phi = x$, on obtient

$$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{2}{\sqrt{(\alpha-\gamma)(\beta-\delta)}+\sqrt{(\alpha-\beta)(\gamma-\delta)}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

où

$$k^2 = \left(\frac{\sqrt{(\alpha-\gamma)(\beta-\delta)}-\sqrt{(\alpha-\beta)(\gamma-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)}+\sqrt{(\alpha-\beta)(\gamma-\delta)}}\right)^2$$

$$\frac{1-x}{1+x} = \frac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\gamma)(\gamma-\delta)}\,\frac{(y-\gamma)}{(\beta-y)}.$$

Maintenant soit

$$(y-\alpha)(y-\beta)(y-\gamma)(y-\delta) = (a,\ b,\ c,\ d,\ e \between y,\ 1)^4$$

et proposons-nous d'introduire dans les formules les coefficients (a, b, c, d, e) au lieu des racines $\alpha, \beta, \gamma, \delta$. En renvoyant d'ailleurs à la Note sur les covariants d'une fonction quadratique, cubique ou biquadratique insérée dans ce Journal, t. L. (1855), p. 285, [135], je pose pour abréger

$$(\alpha-\beta)(\gamma-\delta) = B,$$
$$(\alpha-\gamma)(\delta-\beta) = C,$$
$$(\alpha-\delta)(\beta-\gamma) = D,$$

de sorte que l'on a identiquement

$$B+C+D=0.$$

Les invariants I, J sont des fonctions rationnelles des quantités B, C, D; en effet on a

$$B^2+C^2+D^2=\ 24I,$$
$$(B-C)(C-D)(D-B)=432J,$$
$$B^2C^2D^2=256\,(I^3-27J^3).$$

Cela étant, nous avons

$$k^2=\left(\frac{i\sqrt{C}-\sqrt{B}}{i\sqrt{C}+\sqrt{B}}\right)^2,$$

ce qui donne d'abord

$$k^4+14k^2+1=16\frac{B^2+BC+C^2}{(i\sqrt{C}+\sqrt{B})^4},$$

ou, à cause de $B+C+D=0$, d'où $B^2+C^2+D^2=2\,(B^2+BC+C^2)$,

$$k^4+14k^2+1=\frac{8\,(B^2+C^2+D^2)}{(i\sqrt{C}+\sqrt{B})^4}.$$

On trouve ensuite

$$1-k^2=\frac{4i\sqrt{BC}}{(i\sqrt{C}+\sqrt{B})^4},$$

et de là

$$k^2(1-k^2)^4=\frac{(i\sqrt{C}-\sqrt{B})^2}{(i\sqrt{C}+\sqrt{B})^2}\cdot\frac{256B^2C^2}{(i\sqrt{C}+\sqrt{B})^8},$$

ou en multipliant le numérateur et le dénominateur par $(i\sqrt{C}+\sqrt{B})^2$, et posant D^2 au lieu de $(B+C)^2$, on obtient

$$k^2(1-k^2)^4=\frac{256B^2C^2D^2}{(i\sqrt{C}+\sqrt{B})^{12}}.$$

Ces équations donnent

$$\frac{k^2(1-k^2)^4}{(k^4+14k^2+1)^3}=\frac{B^2C^2D^2}{2\,(B^2+C^2+D^2)^3}$$

ou, en posant

$$N=\tfrac{27}{4}\frac{1}{1-\dfrac{27J^2}{I^3}},$$

$$\frac{k^2(1-k^2)^4}{(k^4+14k^2+1)^3}=\frac{1}{16N}.$$

On trouve aussi

$$k^4+14k^2+1=\frac{192I}{(i\sqrt{C}+\sqrt{B})^4},$$

ce qui donne

$$\frac{2}{i\sqrt{C}+\sqrt{B}}=\sqrt[4]{\frac{k^4+14k^2+1}{12\,I}},$$

et la formule de transformation devient ainsi

$$\frac{dy}{\sqrt{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4}}=\sqrt[4]{\frac{k^4+14k^2+1}{12\,I}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

le module k étant déterminé par l'équation

$$\frac{k^2(1-k^2)^4}{(k^4+14k^2+1)^3}=\frac{1}{16N},$$

où

$$N=\frac{27}{4}\,\frac{1}{1-\dfrac{27J^2}{I^3}}.$$

Cela revient à une formule que j'ai donnée dans le mémoire intitulé, "On the Reduction of $du\div\sqrt{U}$ when U is a Function of the Fourth Order," *Camb. and Dublin Math. Journal,* t. I. (1846), p. 70, [33], et de laquelle j'ai déduit des conséquences que je vais reproduire ici. En effet l'équation en k peut s'écrire sous la forme

$$(k^4+14k^2+1)^3-16Nk^2(k^2-1)^4=0,$$

c'est-à-dire

$$\left(k^2+\frac{1}{k^2}+14\right)^3-16N\left(k-\frac{1}{k}\right)^4=0\,;$$

en écrivant

$$k-\frac{1}{k}=\frac{4}{\sqrt{\vartheta-1}},$$

on obtient pour ϑ l'équation très-simple

$$\vartheta^3-N(\vartheta-1)=0,$$

et on a ensuite

$$k^2=\frac{7+\vartheta+4\sqrt{3+\vartheta}}{\vartheta-1},$$

ce qui donne aussi

$$k=\frac{2+\sqrt{3+\vartheta}}{\sqrt{\vartheta-1}}.$$

Soit $k=\beta^2$ l'une des valeurs de k, l'équation en k devient

$$\frac{(k^4+14k^2+1)^3}{k^2(k^2-1)^4}=\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4},$$

équation à laquelle on satisfait, en outre, par la valeur $k=\left(\frac{1-\beta}{1+\beta}\right)^2$. En effet cette valeur donne

$$k^4+14k^2+1=\frac{16(\beta^8+14\beta^4+1)^2}{(1+\beta)^8}, \qquad k^2-1=-\frac{8\beta(1+\beta^2)}{(1+\beta)^4},$$

expressions qui rendent l'équation identique. Cela fait voir que l'équation peut s'écrire sous la forme

$$(k^4+14k^2+1)^4-k^2(k^2-1)^4\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}$$
$$=(k^2-\beta^4)\left(k^2-\frac{1}{\beta^4}\right)\left(k^2-\left(\frac{1-\beta}{1+\beta}\right)^4\right)\left(k^2-\left(\frac{1+\beta}{1-\beta}\right)^4\right)\left(k^2-\left(\frac{1-\beta i}{1+\beta i}\right)^4\right)\left(k^2-\left(\frac{1+\beta i}{1-\beta i}\right)^4\right)$$

et les racines de l'équation en k^2 sont

$$\beta^4, \quad \frac{1}{\beta^4}, \quad \left(\frac{1-\beta}{1+\beta}\right)^4, \quad \left(\frac{1+\beta}{1-\beta}\right)^4, \quad \left(\frac{1-\beta i}{1+\beta i}\right)^4, \quad \left(\frac{1+\beta i}{1-\beta i}\right)^4,$$

ce qui s'accorde avec un résultat obtenu par Abel (voir les œuvres d'Abel t. I. p. 310 [Ed. 2, p. 459]).

Je fais observer à présent qu'en écrivant

$$\vartheta=\frac{3\varpi}{2\varpi-3},$$

on obtient pour ϖ l'équation $(27-4N)\varpi^3+27N(\varpi-1)=0$, qui, en posant $M=\frac{-27N}{27-4N}$, ou ce qui est la même chose

$$M=\frac{I^3}{4J^2},$$

devient

$$\varpi^3-M(\varpi-1)=0.$$

Cette équation est précisément la même que celle à laquelle je suis parvenu dans la Note sur les covariants etc. ci-dessus citée. La quantité ϖ est liée au module k par la relation

$$k=\frac{3\sqrt{\varpi-1}+2\sqrt{2\varpi-3}}{\sqrt{\varpi+3}}.$$

On peut introduire M au lieu de N dans l'équation en k, et en combinant les formules précédemment obtenues, on trouve

$$\frac{dy}{\sqrt{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4}}=\sqrt[4]{\frac{k^4+14k^2+1}{12I}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

où

$$\frac{27(1+14k^2+k^4)^3}{(1+k^2)^2(1-34k^2+k^4)^4}=4M, \quad M=\frac{I^3}{4J^2},$$

ou ce qui revient à la même chose,

$$k = \frac{3\sqrt{\varpi - 1} + 2\sqrt{2\varpi - 3}}{\sqrt{\varpi + 3}}, \qquad \varpi^3 - M(\varpi - 1) = 0.$$

De l'équation entre k et ϑ on déduit facilement la relation

$$\frac{k^4 + 14k^2 + 1}{k^4 - 34k^2 + 1} = \frac{-\vartheta}{2\vartheta - 3},$$

et en substituant dans cette équation pour ϑ sa valeur on obtient

$$\frac{1 - 34k^2 + k^4}{-3(1 + 14k^2 + k^4)} = \frac{1}{\varpi},$$

ce qui est encore une forme de la relation entre k et ϖ.

II

On peut obtenir les résultats qui viennent d'être déduits de la formule de Jacobi, en prenant pour point de départ la transformation d'une fonction du quatrième ordre dans sa forme canonique. Je suppose d'abord que l'on a identiquement

$$\begin{aligned}(a,\ b,\ c,\ d,\ e \between x,\ y)^4 &= (\lambda x + \mu y)^4 + (\lambda' x + \mu' y)^4 + 6\theta(\lambda x + \mu y)^2(\lambda' x + \mu' y)^2,\\ &= x_1^4 + y_1^4 + 6\theta x_1^2 y_1^2.\end{aligned}$$

Cela étant, je pose $\lambda\mu' - \lambda'\mu = \Lambda$, et je forme les covariants des deux expressions; on obtient par la propriété fondamentale de ces fonctions

$$\begin{aligned}I &= \Lambda^4(1 + 3\theta^2),\\ J &= \Lambda^6(\theta - \theta^3),\\ U &= x_1^4 + y_1^4 + 6\theta\, x_1^2 y_1^2,\\ H &= \Lambda^2\{\theta\, x_1^4 + \theta\, y_1^4 + (1 - 3\theta^2)\, x_1^2 y_1^2\},\\ \Phi &= \Lambda^3(9\theta^2 - 1)\, x_1\, y_1\,(x_1^4 - y_1^4).\end{aligned}$$

De ces relations on tire

$$\frac{J}{I} = \Lambda^2 \frac{\theta - \theta^3}{1 + 3\theta^2},$$

$$\frac{J^2}{I^3} = \frac{(\theta - \theta^3)^2}{(1 + 3\theta^2)^3},$$

de sorte qu'en posant

$$M = \frac{I^3}{4J^2},$$

on aura pour déterminer θ, l'équation

$$\frac{(1+3\theta^2)^2}{(\theta-\theta^3)^2}=4M,$$

et pour déterminer x_1, y_1 les équations

$$U=x_1^4+y_1^4+\qquad 6\theta x_1^2y_1^2,$$

$$\frac{I}{J}\,\frac{1-\theta^2}{1+3\theta^2}\,H=x_1^4+y_1^4+\frac{1-3\theta^2}{\theta}\,x_1^2y_1^2.$$

Je fais observer qu'en désignant par λ un coefficient tel que $U+6\lambda H$ soit un carré parfait, on obtient pour λ l'équation

$$1-9\lambda^2I-54\lambda^3J=0.$$

En effet le cubicovariant Φ d'une fonction qui est un carré parfait est identiquement égal à zéro, et le cubicovariant de la fonction $U+6\lambda H$ est $(1-9\lambda^2I-54\lambda^3J)\,\Phi$, on a donc pour λ l'équation qui vient d'être proposée; cela posé, en observant que

$$U-\frac{I}{J}\,\frac{1-\theta^2}{1+3\theta^2}\,H=\frac{9\theta^2-1}{\theta}\,x_1^2y_1^2,$$

on aura pour une des valeurs de λ,

$$\lambda=-\frac{I}{6J}\,\frac{1-\theta^2}{1+3\theta^2},$$

et en effet cette valeur donne

$$1-\frac{I^3}{J^2}\,\frac{(\theta-\theta^3)^2}{(1+3\theta^2)^3}=0,$$

ce qui est l'équation en θ. Cela étant, je pose

$$\lambda=-\frac{I}{6J}\,\frac{1}{\varpi_3};$$

alors ϖ_3 sera une racine de l'équation

$$\varpi_3-M(\varpi-1)=0,$$

on obtient

$$\varpi_3=\frac{1+3\theta^2}{1-\theta^2},$$

et de là

$$\theta^2=\frac{\varpi_3-1}{\varpi_3+3}.$$

Soient ϖ_1, ϖ_2 les deux autres racines de l'équation en ϖ, on aura

$$\varpi_1+\varpi_2=-\varpi_3,$$

$$\varpi\,\varpi_2=-\frac{\varpi_3^2}{\varpi_3-1},$$

ce qui donne

$$(\varpi_1-\varpi_2)^2=\frac{\varpi_3^2(\varpi_3+3)}{\varpi_3-1}=\frac{\varpi_3^2}{\theta^2}.$$

On pourra donc écrire

$$\theta=\frac{-\varpi_3}{\varpi_2-\varpi_1}$$

et au moyen des valeurs de U, H on obtient, par une très-simple réduction, les trois équations suivantes

$$\begin{aligned}
IH-\varpi_1 JU &= \qquad (\varpi_3-\varpi_1)\,J\,(x_1^2+y_1^2)^2,\\
IH-\varpi_2 JU &= -\qquad (\varpi_2-\varpi_3)\,J\,(x_1^2-y_1^2)^2,\\
IH-\varpi_3 JU &= -\frac{(\varpi_2-\varpi_3)(\varpi_3-\varpi_1)}{\varpi_1-\varpi_2}\,J\,.\,4x_1^2y_1^2,
\end{aligned}$$

dont deux quelconques donnent les valeurs de x_1, y_1; ainsi on a obtenu la solution complète du problème de la réduction de la fonction $(a, b, c, d, e\between x, y)^4$ à la forme canonique.

Je fais observer que ces équations montrent *a posteriori* que les expressions $IH-\varpi_1 JU$, $IH-\varpi_2 JU$, $IH-\varpi_3 JU$ sont toutes les trois des carrés de fonctions quadratiques. Je fais observer, en outre, que si l'on forme la valeur de l'expression

$$(\varpi_2-\varpi_3)\sqrt{IH-\varpi_1 JU}+(\varpi_3-\varpi_1)\sqrt{IH-\varpi_2 JU}+(\varpi_1-\varpi_2)\sqrt{IH-\varpi_3 JU},$$

en posant pour un moment $\varpi_2-\varpi_3=\alpha$, $\varpi_3-\varpi_1=\beta$, $\varpi_1-\varpi_2=\gamma$, l'expression dont il s'agit sera égale, à un facteur constant près, à

$$\sqrt{\alpha}\,(x_1^2+y_1^2)+i\sqrt{\beta}\,(x_1^2-y_1^2)+i\sqrt{\gamma}\,.\,2x_1y_1$$

ou, ce qui est la même chose, à

$$(\sqrt{\alpha}+i\sqrt{\beta},\ i\sqrt{\gamma},\ \sqrt{\alpha}-i\sqrt{\beta}\between x_1,\ y_1)^2.$$

Or, cette fonction doit être un carré parfait, ce qu'on reconnaît en effet, en se rappelant que

$$(\sqrt{\alpha}+i\sqrt{\beta})(\sqrt{\alpha}-i\sqrt{\beta})-(i\sqrt{\gamma})^2=\alpha+\beta+\gamma=0.$$

Sa racine carrée sera, à un facteur constant près, $(\sqrt{\alpha}+i\sqrt{\beta})\,x_1+i\sqrt{\gamma}y_1$, et cette dernière fonction sera un des facteurs linéaires de $x_1^4+y_1^4+6\theta x_1^2y_1^2$. Pour vérifier cela je pose $x_1^4+y_1^4+6\theta x_1^2y_1^2=0$, ce qui donne

$$x_1^2+(3\theta+\sqrt{9\theta^2-1})\,y_1^2=0,$$

et $\quad x_1-\left(\sqrt{\frac{3\theta+1}{2}}+\sqrt{\frac{3\theta-1}{2}}\right)y_1=0.$ Or $3\theta+1=\frac{-3\varpi_3}{\varpi_2-\varpi_1}+1=\frac{-3\varpi_3+\varpi_2-\varpi_1}{\varpi_2-\varpi_1}$

$=\frac{-2\varpi_3+2\varpi_2-(\varpi_1+\varpi_2+\varpi_3)}{\varpi_2-\varpi_1}=\frac{2\varpi_2-2\varpi_3}{\varpi_2-\varpi_3}=\frac{2\alpha}{-\gamma}$, c'est-à-dire $\sqrt{\frac{3\theta+1}{2}}=\frac{\sqrt{\alpha}}{i\sqrt{\gamma}}$, et de même $\sqrt{\frac{3\theta-1}{2}}=\frac{i\sqrt{\beta}}{i\sqrt{\gamma}}$, le facteur linéaire sera donc $i\sqrt{\gamma}\,.\,x_1-(\sqrt{\alpha}+i\sqrt{\beta})\,y_1$ ou, ce qui revient à la même chose, $(\sqrt{\alpha}+i\sqrt{\beta})\,x_1+i\sqrt{\gamma}y_1$, ce qu'il s'agissait de démontrer.

III.

Pour obtenir la formule qui sert à la transformation d'une intégrale elliptique, il faut présenter les résultats précédemment obtenus sous une forme un peu différente. J'écris

$$\begin{aligned}(a,\ b,\ c,\ d,\ e\between y,\ 1)^4 &= (\lambda+\mu y)^4-(1+k^2)(\lambda+\mu y)^2(\lambda'+\mu' y)^2+k^2(\lambda'+\mu' y)^4\\ &= (\lambda+\mu y)^4\,.\,(1-x^2)(1-k^2x^2),\end{aligned}$$

où

$$x=\frac{\lambda'+\mu' y}{\lambda+\mu y}.$$

Cela posé, en remplaçant comme auparavant $\lambda\mu'-\lambda'\mu$ par Λ, et en formant les covariants, on obtient

$$\begin{aligned}I &= \tfrac{1}{12}\Lambda^4(1+14k^2+k^4),\\ J &= \tfrac{1}{216}\Lambda^6(1+k^2)(1-34k^2+k^4),\\ U &= (\lambda+\mu y)^4\{1-(1+k^2)x^2+k^2x^4\},\\ H &= -\tfrac{1}{12}\Lambda^2(\lambda+\mu y)^4\{2(1+k^2)-(1-10k^2+k^4)x^2+2k^2(1+k^2)x^4\},\\ \Phi &= \tfrac{1}{4}\Lambda^3(\lambda+\mu y)^6(1-k^2)^3x(1-k^2x^4).\end{aligned}$$

On a d'abord

$$\frac{27\,(1+14k^2+k^4)^3}{(1+k^2)^2(1-34k^2+k^4)^2}=4M,$$

en posant comme auparavant $M=\dfrac{I^3}{4J^2}$; on obtient ensuite

$$\Lambda^4=\frac{12I}{1+14k^2+k^4},$$

et en remarquant que l'équation entre x et y donne

$$dx=\frac{(\lambda\mu'-\lambda'\mu)\,dy}{(\lambda+\mu y)^2}=\frac{\Lambda\,dy}{(\lambda+\mu y)^2},$$

on trouve

$$\frac{dy}{\sqrt{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4}}=\frac{1}{\Lambda}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

c'est-à-dire

$$\frac{dy}{\sqrt{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4}}=\sqrt[4]{\frac{k^4+14k^2+1}{12I}}\,\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

ce qui s'accorde avec la formule ci-dessus trouvée; pour compléter la solution, je pose

$$\frac{1-34k^2+k^4}{-3\,(1+14k^2+k^4)}=\frac{1}{\varpi_3},$$

ϖ_3 sera une des racines de l'équation

$$\varpi_3 - M(\varpi - 1) = 0,$$

en représentant par ϖ_1, ϖ_2 les deux autres racines, on obtient

$$2\varpi_1 = -\varpi_3 \frac{1 + 6k + k^2}{1 + k^2},$$

$$2\varpi_2 = -\varpi_3 \frac{1 - 6k + k^2}{1 + k^2},$$

et ensuite

$$IH - \varpi_1 JU = J(\lambda + \mu y)^4 (\varpi_3 - \varpi_1)(1 + kx^2)^2,$$
$$IH - \varpi_2 JU = J(\lambda + \mu y)^4 (\varpi_3 - \varpi_2)(1 - kx^2)^2,$$
$$IH - \varpi_3 JU = J\frac{(\varpi_3 - \varpi_1)(\varpi_3 - \varpi_2)}{\varpi_1 - \varpi_2}(\lambda + \mu y)^4 . 4kx^2.$$

Donc en posant $\varpi_2 - \varpi_3 = \alpha$, $\varpi_3 - \varpi_1 = \beta$, $\varpi_1 - \varpi_2 = \gamma$, on a

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}$$
$$= J(\lambda + \mu y)^4 \sqrt{\alpha\beta}\,(\sqrt{\alpha} + i\sqrt{\beta} + i\sqrt{\gamma}x\sqrt{k})^2,$$

de même

$$(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} - (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}$$
$$= J(\lambda + \mu y)^4 \sqrt{\alpha\beta}\,(\sqrt{\alpha} + i\sqrt{\beta} - i\sqrt{\gamma}x\sqrt{k})^2,$$

et de là enfin, en remplaçant α, β, γ par leurs valeurs,

$$\frac{(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} + (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}}{(\varpi_2 - \varpi_3)\sqrt{IH - \varpi_1 JU} + (\varpi_3 - \varpi_1)\sqrt{IH - \varpi_2 JU} - (\varpi_1 - \varpi_2)\sqrt{IH - \varpi_3 JU}}$$
$$= \left(\frac{\sqrt{\varpi_2 - \varpi_3} + i\sqrt{\varpi_3 - \varpi_1} + i\sqrt{\varpi_1 - \varpi_2}\,x\sqrt{k}}{\sqrt{\varpi_2 - \varpi_3} + i\sqrt{\varpi_3 - \varpi_1} - i\sqrt{\varpi_1 - \varpi_2}\,x\sqrt{k}}\right)^2,$$

équation dont le premier membre est le carré d'une fraction rationnelle de la forme

$$\frac{A + Bx}{C + Dx}.$$

IV.

Je terminerai ces recherches en démontrant le théorème de M. Hermite dont j'ai parlé dans la Note sur les covariants etc. ci-dessus citée. L'identité

$$JU^3 - IU^2H + 4H^3 = -\Phi^2$$

peut être mise sous la forme

$$-J+I.\frac{H}{U}-4\frac{H^3}{U^3}=\left(\frac{\Phi}{U^2}\right)^2.U.$$

Posons maintenant

$$z=\frac{H}{U},$$

formule dans laquelle je suppose qu'on ait fait $y=1$, de manière que U, H soient des fonctions rationnelles et entières de la seule variable x, savoir

$$U=(a,\ b,\ c,\ d,\ e \between x,\ 1)^4,$$
$$H=(ac-b^2,\ \tfrac{1}{2}(ad-bc),\ \tfrac{1}{6}(ae+2bd-3c^2),\ \tfrac{1}{2}(be-cd),\ ce-d^2 \between x,\ 1)^4,$$

alors on aura

$$\sqrt{-J+zI-4z^3}=\frac{\Phi}{U^2}\sqrt{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4},$$

et

$$dz=\frac{UdH-HdU}{U^2}.$$

En vertu de la théorie de Jacobi, $UdH-HdU$ est de la forme $M\Phi\,dx$, où M est un facteur constant; on trouve en effet très-facilement $UdH-HdU=2\Phi\,dx$, d'où l'on déduit la formule

$$\frac{dz}{\sqrt{-J+zI-4z^3}}=\frac{2dx}{\sqrt{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4}},$$

et je fais observer que l'intégrale du premier membre se ramène immédiatement à une forme qui ne contient que la seule constante M, $=\frac{I^3}{4J^2}$.

Londres, 9 *Avril*, 1856.

236.

NOTE SUR LA COMPOSITION DU NOMBRE 47 PAR RAPPORT AUX VINGT-TROISIÈMES RACINES DE L'UNITÉ.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LV. (1858), p. 192.]

M. KUMMER a trouvé (*Journal de Liouville*, t. XII. [1847] p. 208) que le nombre 47 peut être décomposé en onze facteurs qui se déduisent du suivant $\alpha^{10}+\alpha^{13}+\alpha^{8}+\alpha^{15}+\alpha^{7}+\alpha^{16}$, α désignant une racine $23^{\text{ème}}$ de l'unité, et on sait par la théorie générale qu'il doit y avoir une puissance 47^{3f} qui se décompose en vingt-deux facteurs. Le nombre 47^3 peut se décomposer en deux facteurs formés avec les demi-périodes des racines ; il était donc naturel d'essayer si le facteur $(\alpha^{10}+\alpha^{13}+\alpha^{8}+\alpha^{15}+\alpha^{7}+\alpha^{16})^3$ pourrait se décomposer de même en deux facteurs, ce qui donnerait la décomposition de 47^3 en vingt-deux facteurs. Mais on démontre très-facilement que cette décomposition n'est pas possible. En effet en posant $\alpha^\lambda=1$ (λ étant un nombre premier) et en faisant

$$A+B\alpha+\ldots K\alpha^{\lambda-1}=(a+b\alpha+\ldots k\alpha^{\lambda-1})(a+b\alpha^{\lambda-1}+\ldots k\alpha),$$

on aura $A=a^2+b^2+\ldots k^2$. Le nombre qui forme le premier membre peut se réduire au moyen de l'équation $1+\alpha+\ldots\alpha^{\lambda-1}=0$ à la forme $B'\alpha+C'\alpha^2\ldots+K'\alpha^{\lambda-1}$ et l'on aura

$$B'\alpha+C'\alpha^2\ldots+K'\alpha^{\lambda-1}=(a+b\alpha+\ldots k\alpha^{\lambda-1})(a+b\alpha^{\lambda-1}+\ldots k\alpha)-(a^2+b^2+\ldots k^2)(1+\alpha+\ldots\alpha^{\lambda-1}),$$

équation qui subsiste lorsqu'on y fait $\alpha=1$, ce qui donne

$$B'+C'+\ldots K'=(a+b+\ldots k)^2-\lambda(a^2+b^2+\ldots k^2);$$

or, la fonction qui forme le second membre, prise avec le signe négatif peut se mettre sous la forme $(a-b)^2+(a-c)^2+(b-c^2)+$ etc.; donc la décomposition n'existe pas à moins

que $B' + C' + \ldots K'$ ne soit négatif. Mais, en réduisant seulement au moyen de l'équation $\alpha^{22} - 1 = 0$, on trouve la suivante

$$(\alpha^{10} + \alpha^{13} + \alpha^{8} + \alpha^{15} + \alpha^{7} + \alpha^{16})^3 =$$

$$\left.\begin{array}{l} 6 + 7\alpha + 7\alpha^2 + 3\alpha^3 + 6\alpha^4 + 9\alpha^5 + 6\alpha^6 + 16\alpha^7 + 15\alpha^8 + 9\alpha^9 + 18\alpha^{10} + 9\alpha^{11} \\ + 7\alpha^{22} + 7\alpha^{21} + 3\alpha^{20} + 6\alpha^{19} + 9\alpha^{18} + 6\alpha^{17} + 16\alpha^{16} + 15\alpha^{15} + 9\alpha^{14} + 18\alpha^{13} + 9\alpha^{12} \end{array}\right\},$$

laquelle, en vertu de $1 + \alpha + \ldots \alpha^{22} = 0$, se réduit à

$$\left.\begin{array}{l} \alpha + \alpha^2 - 3\alpha^3 + 3\alpha^5 + 10\alpha^7 + 9\alpha^8 + 3\alpha^9 + 12\alpha^{10} + 3\alpha^{11} \\ + \alpha^{22} + \alpha^{21} - 3\alpha^{20} + 3\alpha^{18} + 10\alpha^{16} + 9\alpha^{15} + 3\alpha^{14} + 12\alpha^{13} + 3\alpha^{12} \end{array}\right\},$$

où la somme des coefficients est positive; donc la décomposition ne peut pas s'effectuer. On pourrait sans beaucoup de peine essayer de la même manière les nombres $f = 2$ ou $f = 3$, mais je ne sais pas si l'on a une idée quelconque de la grandeur du nombre f.

Londres, le 10 *Mai,* 1857.

237.

THÉORÈME SUR LES DÉTERMINANTS GAUCHES.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LV. (1858), pp. 277—278.]

DANS le mémoire intitulé "Recherches ultérieures sur les déterminants gauches," ce journal t. L. pp. 299—313 (1855), [137], j'ai donné une formule pour le développement d'un déterminant gauche bordé; mais j'ai omis de remarquer un cas particulier assez important. La formule générale se rapporte à un déterminant tel que:

$$\overline{\alpha 123 \mid \beta 123} = \begin{vmatrix} \alpha\beta, & \alpha 1, & \alpha 2, & \alpha 3 \\ 1\beta, & 11, & 12, & 13 \\ 2\beta, & 21, & 22, & 23 \\ 3\beta, & 31, & 32, & 33 \end{vmatrix},$$

que l'on obtient en bordant d'une manière quelconque la matrice gauche

$$\left(\begin{array}{ccc} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{array} \right).$$

On a par exemple:

$$\begin{aligned} \overline{\alpha 123 \mid \beta 123} = \; & \alpha\beta \,.\, 11 \,.\, 22 \,.\, 33 \\ & + \alpha\beta \,.\, 12 \,.\, 12 \,.\, 33 \\ & + \alpha\beta \,.\, 13 \,.\, 13 \,.\, 22 \\ & + \alpha\beta \,.\, 23 \,.\, 23 \,.\, 11 \\ & + \alpha 1 \,.\, \beta 1 \,.\, 22 \,.\, 33 \\ & + \alpha 2 \,.\, \beta 2 \,.\, 11 \,.\, 33 \\ & + \alpha 3 \,.\, \beta 3 \,.\, 11 \,.\, 22 \\ & + \alpha 123 \,.\, \beta 123, \end{aligned}$$

$$\begin{aligned}\overline{\alpha 1234 \mid \beta 1234} = \; & \alpha\beta \,.\, 11 \,.\, 22 \,.\, 33 \,.\, 44 \\ & + \alpha\beta \,.\, 12 \,.\, 12 \,.\, 33 \,.\, 44 \\ & + \;\ldots\ldots\ldots\ldots\ldots \\ & + \alpha\beta 1234 \,.\, 1234 \\ & + \alpha 1 \,.\, \beta 1 \,.\, 22 \,.\, 33 \,.\, 44 \\ & + \;\ldots\ldots\ldots\ldots\ldots \\ & + \alpha 123 \,.\, \beta 123 \,.\, 44 \\ & + \;\ldots\ldots\ldots\ldots\ldots \\ & + \alpha 234 \,.\, \beta 234 \,.\, 11\end{aligned}$$

où les expressions $\alpha\beta 12$, $\alpha\beta 13$ etc. sont des Pfaffiens. Cela étant, le déterminant formé en bordant d'une manière quelconque une matrice gauche et symétrique peut se nommer déterminant gauche et symétrique bordé, et la formule fait voir qu'un déterminant de cette espèce se réduit toujours au produit de deux Pfaffiens. En effet, en écrivant dans les exemples $11 = 22 = 33 = 44 = 0$, on obtient:

$$\overline{\alpha 123 \mid \beta 123} = \alpha 123 \,.\, \beta 123,$$

$$\overline{\alpha 1234 \mid \beta 1234} = \alpha\beta 1234 \,.\, 1234,$$

et de même pour un déterminant gauche et symétrique bordé quelconque, suivant que l'ordre du déterminant est pair ou impair. Je remarque à propos de cela, que dans le cas d'un déterminant d'ordre pair, le terme $\alpha\beta$ est multiplié par un mineur premier lequel (comme déterminant gauche et symétrique d'ordre impair) se réduit à zéro; le déterminant ne contient donc pas ce terme $\alpha\beta$, et sera par conséquent fonction linéo-linéaire des quantités $\alpha 1$, $\alpha 2$, etc. et 1β, 2β, etc.; de manière qu'on ne saurait être surpris de voir ce déterminant se présenter sous la forme d'un produit de deux facteurs dont l'un est fonction linéaire de $\alpha 1$, $\alpha 2$, etc. et l'autre fonction linéaire de 1β, 2β, etc. Mais pour un déterminant d'ordre impair, le coefficient du terme $\alpha\beta$ ne se réduit pas à zéro; en supposant donc que le déterminant puisse s'exprimer comme produit de deux facteurs, il est nécessaire que l'un de ces facteurs soit (comme le déterminant même) fonction linéaire de $\alpha\beta$ et linéo-linéaire de $\alpha 1$, $\alpha 2$, etc. et 1β, 2β, etc.; de cette manière on se rend compte de la différence de la forme des facteurs, qui a lieu dans les deux cas dont il s'agit.

En écrivant $\beta = \alpha$ (ce qui implique $\alpha\alpha = 0$, car on suppose toujours $\alpha\beta = -\beta\alpha$) le déterminant gauche et symétrique bordé se réduit à un déterminant gauche et symétrique ordinaire, de plus le Pfaffien $\alpha\beta 1234$ se réduit à zéro, et les équations deviennent:

$$\overline{\alpha 123 \mid \alpha 123} = (\alpha 123)^2,$$

$$\overline{\alpha 1234 \mid \alpha 1234} = 0;$$

savoir, quand l'ordre est pair, le déterminant se réduit au carré d'un Pfaffien, et quand l'ordre est impair, le déterminant s'évanouit; ce qui est en effet la propriété fondamentale des déterminants gauches et symétriques.

2 *Stone Buildings, Londres, le* 16 *Nov.* 1857.

238.

NOTE SUR LES NORMALES D'UNE CONIQUE.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LVI. (1859), pp. 182—185.]

ON connaît les recherches très élégantes de M. Joachimsthal sur les normales d'une conique [voir le Mémoire "Ueber die Normalen der Ellipse und des Ellipsoids," *Crelle*, t. XXVI. (1843) pp. 172—180]; en particulier l'auteur a obtenu le théorème suivant: en supposant que 1, 2, 3, 4 soient des points d'une conique tels que les quatre normales se rencontrent dans un même point, on prend le pôle de la droite (1, 2) par rapport à la conique et on mène par ce pôle des perpendiculaires aux diamètres de la conique; celà étant, en prenant sur chaque diamètre dans le sens opposé un point dont la distance du centre est égale à la distance du pied de la perpendiculaire sur ce même diamètre, la droite menée par ces deux points passe par les deux points 3 et 4. Mais cette propriété peut s'énoncer d'une manière beaucoup plus simple; la droite dont il s'agit est la polaire (ou autrement dit l'harmonicale)—par rapport au triangle formé par les deux diamètres et la droite située à l'infini—du pôle de la droite (1, 2) par rapport à la conique. Or on sait que l'idée de la perpendiculaire peut être généralisée. Savoir en prenant une conique quelconque que nous appelons *la conique absolue*, deux droites harmoniques par rapport à cette conique peuvent être appelées *perpendiculaires* (et de même deux points harmoniques par rapport à la conique absolue peuvent être appelés *perpendiculaires*). Cela posé, on peut parler dans un sens plus général des normales, etc. d'une courbe quelconque. En effet, que l'on s'imagine comme auparavant (outre la conique absolue) une conique donnée quelconque et quatre points 1, 2, 3 et 4 de cette conique tels que les normales se rencontrent dans un même point. Au lieu du triangle ci-dessus mentionné on a le triangle formé par les trois axes harmoniques (ou autrement dit, *conjugués*) communs aux deux coniques, et le théorème peut s'énoncer comme suit: En prenant le pôle de la droite (1, 2) par rapport à la conique donnée et puis la polaire (l'harmonicale) de ce pôle par rapport aux axes conjugués de la conique donnée et de la conique absolue, cette polaire passe par les deux points

3 et 4. Ou (ce qui revient à la même chose) on peut considérer les points 1, 2, 3, 4 comme les angles d'un quadrilatère inscrit dans la conique donnée, les quatre tangentes à cette conique aux points dont il s'agit seront les côtés d'un quadrilatère circonscrit à la conique donnée; cela étant, les six côtés du quadrilatère inscrit seront les polaires (les harmonicales)—par rapport aux trois axes conjugués de la conique donnée et de la conique absolue—des six sommets du quadrilatère circonscrit.

Pour démontrer cela, je fais observer qu'il est permis de rapporter la conique absolue et la conique donnée aux trois axes conjugués communs, c'est-à-dire de prendre

$$x^2 + y^2 + z^2 = 0,$$

pour équation de la conique absolue, et

$$ax^2 + by^2 + cz^2 = 0,$$

pour équation de la conique donnée: cela posé (et en observant que, d'après la définition, deux droites $Ax + By + Cz = 0$, $A'x + B'y + C'z = 0$ seront perpendiculaires si $AA' + BB' + CC' = 0$) on obtient sans peine

$$\frac{x(b-c)}{x_1} + \frac{y(c-a)}{y_1} + \frac{z(a-b)}{z_1} = 0,$$

pour équation de la normale au point 1, en désignant par (x_1, y_1, z_1) les coordonnées de ce point. On a de même, en désignant par (x_2, y_2, z_2) et (x_3, y_3, z_3) les coordonnées des points 2 et 3,

$$\frac{x(b-c)}{x_2} + \frac{y(c-a)}{y_2} + \frac{z(a-b)}{z_2} = 0,$$

$$\frac{x(b-c)}{x_3} + \frac{y(c-a)}{y_3} + \frac{z(a-b)}{z_3} = 0,$$

pour les équations des normales aux points 2 et 3; et la condition qui exprime que ces trois normales se rencontrent dans un même point sera évidemment

$$\begin{vmatrix} \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2} \\ \frac{1}{x_3}, & \frac{1}{y_3}, & \frac{1}{z_3} \end{vmatrix} = 0.$$

Mais les coordonnées (x_1, y_1, z_1) etc. satisfont à l'équation $ax^2 + by^2 + cz^2 = 0$, on a donc aussi

$$\begin{vmatrix} x_1^2, & y_1^2, & z_1^2 \\ x_2^2, & y_2^2, & z_2^2 \\ x_3^2, & y_3^2, & z_3^2 \end{vmatrix} = 0$$

et de ces deux équations on déduit la suivante

$$\left\{\begin{matrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{matrix}\right\} = 0$$

en désignant par le symbole qui forme le premier membre la fonction

$$x_1y_2z_3 + x_1y_3z_2 + x_2y_3z_1 + x_2y_1z_3 + x_3y_1z_2 + x_3y_2z_1.$$

C'est ce qui résulte de l'identité

$$\left\{\begin{matrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{matrix}\right\} \times \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \begin{vmatrix} x_1^2, & y_1^2, & z_1^2 \\ x_2^2, & y_2^2, & z_2^2 \\ x_3^2, & y_3^2, & z_3^2 \end{vmatrix} + 2x_1y_1z_1x_2y_2z_2x_3y_3z_3 \begin{vmatrix} \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2} \\ \frac{1}{x_3}, & \frac{1}{y_3}, & \frac{1}{z_3} \end{vmatrix};$$

car le déterminant qui forme le second facteur du premier membre de l'équation ne s'évanouissant pas, c'est l'autre facteur qui devra s'évanouir en vertu des deux relations données.

L'équation de la droite (1, 2) sera

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0:$$

les coordonnées du pôle de cette droite par rapport à la conique donnée $ax^2 + by^2 + cz^2 = 0$, seront

$$\frac{1}{a}(y_1z_2 - y_2z_1) : \frac{1}{b}(z_1x_2 - z_2x_1) : \frac{1}{c}(x_1y_2 - x_2y_1);$$

mais les deux équations $ax_1^2 + by_1^2 + cz_1^2 = 0$, $ax_2^2 + by_2^2 + cz_2^2 = 0$ donnent $a : b : c = y_1^2z_2^2 - y_2^2z_1^2 : z_1^2x_2^2 - z_2^2x_1^2 : x_1^2y_2^2 - x_2^2y_1^2$; par suite de cela les coordonneés du pôle deviennent

$$\frac{1}{y_1z_2 + y_2z_1} : \frac{1}{z_1x_2 + z_2x_1} : \frac{1}{x_1y_2 + x_2y_1};$$

donc l'équation de la polaire (l'harmonicale) de ce pôle par rapport aux trois droites $(x = 0,\ y = 0,\ z = 0)$ sera

$$x(y_1z_2 + y_2z_1) + y(z_1x_2 + z_2x_1) + z(x_1y_2 + x_2y_1) = 0,$$

laquelle peut être représentée comme suit

$$\left\{\begin{matrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right\} = 0,$$

et, en substituant (x_3, y_3, z_3) au lieu de (x, y, z), on voit que la droite dont il s'agit contient le point 3. De même cette droite contient le point 4, de sorte que le théorème se trouve démontré.

Observons encore que dans la géométrie de la sphère on peut prendre pour la conique absolue la conique imaginaire qui est l'intersection de la surface sphérique avec la surface conique imaginaire $x^2+y^2+z^2=0$. Le mot perpendiculaire aura alors la signification ordinaire, et on aura pour les coniques sphériques ce théorème très simple, savoir que les six côtés du quadrilatère inscrit seront les polaires (les harmonicales)—par rapport aux trois axes de la conique sphérique donnée—des six sommets du quadrilatère circonscrit. C'est ce que l'on reconnaît aussi par l'analyse que je viens de donner, laquelle en effet est précisément celle dont on se servirait naturellement pour les coniques sphériques.

Londres, le 10 *Mars*, 1857.

239.

ADDITION À LA NOTE SUR LA COMPOSITION DU NOMBRE 47 PAR RAPPORT AUX VINGT-TROISIÈMES RACINES DE L'UNITÉ.

[From the *Journal für die reine und angewandte Mathematik*, (Crelle), tom. LVI. (1859), pp. 186—187: Sequel to **236**.]

M. KUMMER a bien voulu m'écrire une lettre où il remarque que le cube du facteur complexe du nombre **47**, *multiplié par l'unité complexe convenable*, peut effectivement se décomposer en deux facteurs réciproques; c'est-à-dire qu'il existe une unité complexe $E(\alpha)$ telle que

$$E(\alpha)(\alpha^{10}+\alpha^{13}+\alpha^{8}+\alpha^{15}+\alpha^{7}+\alpha^{16})^3 = F(\alpha)\,F(\alpha^{-1}).$$

Pour $F(\alpha)$ M. Kummer a trouvé la valeur

$$F(\alpha) = \alpha^4+\alpha^5+\alpha^9+\alpha^{10}+\alpha^{16}-\alpha^{20}+\alpha^{22},$$

fonction qui par conséquent est l'un des **22** facteurs complexes de 47^3. Dans un postscriptum M. Kronecker m'a communiqué l'expression suivante de cette unité complexe

$$E(\alpha) = \frac{\alpha^2+\alpha^{21}}{(\alpha^8+\alpha^{15})(\alpha^9+\alpha^{14})(\alpha^{10}+\alpha^{13})^2(\alpha^{11}+\alpha^{12})}$$

équivalente à l'expression en fonction entière:

$$E(\alpha) = \begin{cases} -23\alpha + 2\alpha^2 - 20\alpha^3 - 21\alpha^5 - 3\alpha^6 - 17\alpha^7 - 4\alpha^8 - 14\alpha^9 - 8\alpha^{10} - 12\alpha^{11} \\ -23\alpha^{22} + 2\alpha^{21} - 20\alpha^{20} - 21\alpha^{18} - 3\alpha^{17} - 17\alpha^{16} - 4\alpha^{15} - 14\alpha^{14} - 8\alpha^{13} - 12\alpha^{12}. \end{cases}$$

En supposant que cette valeur de $E(\alpha)$ soit connue, on trouve sans peine une condition à laquelle $F(\alpha)$ doit satisfaire. La valeur que j'ai donnée pour $(\alpha^{10}+\alpha^{13}+\alpha^{8}+\alpha^{15}+\alpha^{7}+\alpha^{16})^3$

contient le terme constant $+6$; en y ajoutant la quantité évanouissante $-6(1+\alpha+\ldots+\alpha^{22})$ elle se réduit à

$$\begin{cases} \alpha + \alpha^2 - 3\alpha^3 + 3\alpha^5 + 10\alpha^7 + 9\alpha^8 + 3\alpha^9 + 12\alpha^{19} + 3\alpha^{11} \\ + \alpha^{22} + \alpha^{21} - 3\alpha^{20} + 3\alpha^{18} + 10\alpha^{16} + 9\alpha^{15} + 3\alpha^{14} + 12\alpha^{13} + 3\alpha^{12} \end{cases}$$

et en multipliant cette valeur par $E(\alpha)$, le terme constant sera $+808$; donc en y ajoutant la quantité évanouissante $-808(1+\alpha+\ldots+\alpha^{22})$ on obtient

$$E(\alpha)(\alpha^{16}+\alpha^{13}+\alpha^8+\alpha^{15}+\alpha^7+\alpha^{18})^3$$
$$= \begin{cases} -5\alpha - 8\alpha^2 - 7\alpha^3 - 7\alpha^4 - 4\alpha^5 - 3\alpha^6 - 7\alpha^7 - 8\alpha^8 - 7\alpha^9 - 7\alpha^{10} - 5\alpha^{11} \\ -5\alpha^{22} - 8\alpha^{21} - 7\alpha^{20} - 7\alpha^{19} - 4\alpha^{18} - 3\alpha^{17} - 7\alpha^{16} - 8\alpha^{15} - 7\alpha^{14} - 7\alpha^{13} - 5\alpha^{12}. \end{cases}$$

En représentant cette expression par $B'\alpha + C'\alpha^2 + \ldots + K'\alpha^{22}$, j'écris

$$B'\alpha + C'\alpha^2 + \ldots + K'\alpha^{22}$$
$$= (a + b\alpha + \ldots + k\alpha^{22})(a + b\alpha^{22} + \ldots + k\alpha) - (a^2 + b^2 + \ldots + k^2)(1 + \alpha + \ldots + \alpha^{22}),$$

équation qui subsiste pour $\alpha = 1$. On a donc

$$B' + C' + \ldots + K' = (a + b + \ldots + k)^2 - 23(a^2 + b^2 + \ldots + k^2),$$

ou, d'après les valeurs de B', C', ... K',

$$-136 = (a + b + \ldots + k)^2 - 23(a^2 + b^2 + \ldots + k^2),$$

ce qui donne

$$(a + b + \ldots + k)^2 \equiv -136 \pmod{23}.$$

On peut ajouter à $(a + b\alpha + \ldots + k\alpha^{22})$ un multiple quelconque de $(1 + \alpha + \ldots + \alpha^{22})$ et changer le signe; il est donc permis de prendre $(a + b + \ldots + k)$ positif et plus petit que $\frac{23}{2}$. Cela étant la congruence donne

$$a + b + \ldots + k = 5$$

et on obtient alors

$$a^2 + b^2 + \ldots + k^2 = 7.$$

Au moyen de cette valeur l'équation à laquelle il faut satisfaire devient

$$7 + 2\alpha - \alpha^2 + 3\alpha^5 + 4\alpha^6 - \alpha^8 + 2\alpha^{11}$$
$$+ 2\alpha^{22} - \alpha^{19} + 3\alpha^{18} + 4\alpha^{17} - \alpha^{15} + 2\alpha^{12}$$
$$= (a + b\alpha \ldots + k\alpha^{22})(a + b\alpha^{22} \ldots + k\alpha).$$

À cause des coefficients numériques négatifs, les coefficients cherchés a, b, ... k ne peuvent pas être tous à la fois positifs; et cela étant il n'y a qu'une seule manière de satisfaire aux deux conditions ci-dessus écrites, savoir il faut donner à sept des coefficients a, b, ... k les valeurs 1, 1, 1, 1, 1, 1, -1, et aux autres coefficients la valeur zéro; l'expression

$$F(\alpha) = \alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10} + \alpha^{16} - \alpha^{20} + \alpha^{22}$$

s'accorde en effet avec cette conclusion.

Londres, le 6 *Octobre,* 1858.

240.

NOTE ON A THEOREM IN SPHERICAL TRIGONOMETRY.

[From the *Philosophical Magazine*, vol. XVII. (1859), p. 151.]

I AM not aware that the following theorem has been noticed: viz., in any spherical triangle, if as usual a, b, c are the sides, and A, B, C the opposite angles, then

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a,$$
$$\sin c \sin a + \cos c \cos a \cos B = \sin C \sin A - \cos C \cos A \cos b,$$
$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c.$$

The demonstration is very simple; in fact we have

$$\begin{aligned}
&\sin b \sin c + \cos b \cos c \cos A \\
&\quad = \sin b \sin c (\sin^2 A + \cos^2 A) + \cos b \cos c \cos A \\
&\quad = \sin b \sin c \sin^2 A + \cos A (\cos b \cos c + \sin b \sin c \cos A) \\
&\quad = \sin B \sin C \sin^2 a + \cos A \cos a \\
&\quad = \sin B \sin C (1 - \cos^2 a) + \cos A \cos a \\
&\quad = \sin B \sin C + \cos a (\cos A - \sin B \sin C \cos a) \\
&\quad = \sin B \sin C - \cos B \cos C \cos a,
\end{aligned}$$

which proves the theorem.

2, *Stone Buildings, W.C., January* 5, 1859.

A geometrical proof and interpretation are given, G. B. Airy, "Remarks on Mr Cayley's Trigonometrical Theorem, etc." *Phil. Mag.* same volume, p. 176. I transfer to this place the concluding sentence of the subsequent paper 243. "I take the opportunity of noticing that the theorem in spherical trigonometry, which I gave in the February Number, is not new, but, as pointed out by Prof. Chauvenet in the Mathematical Monthly (Cambridge, U.S.), is to be found in Cagnoli's 'Trigonometry' (1808)."

241.

ON POINSOT'S FOUR NEW REGULAR SOLIDS.

[From the *Philosophical Magazine*, vol. XVII. (1859), pp. 123—128.]

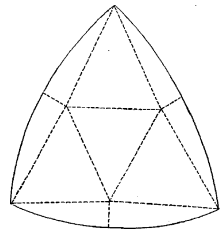
IT is shown by Poinsot, in the "Mémoire sur les Polygones et les Polyèdres," *Jour. École Polyt.* vol. IV. pp. 16 to 48 (1810), that, besides the regular polyhedrons of ordinary geometry, there are (of course in an extended signification of the term) four new regular polyhedrons, viz. an icosahedron, which I will call the great icosahedron (No. 33 of the Memoir), and three dodecahedrons, which I will call the great dodecahedron (No. 37), the great stellated dodecahedron (No. 38), and the small stellated dodecahedron (No. 39). The nature of Poinsot's generalization will be best understood by conceiving, as he does, that the polyhedron is projected on a concentric sphere, so that the faces become spherical polygons. Then for the ordinary polyhedrons of geometry, the sum of the angles at a vertex $=4$ right angles; but it may, according to the more general notion, be $=e$ times 4 right angles. In like manner for the ordinary polyhedrons, the sides of a face subtend at the centre angles the sum of which is $=4$ right angles; but according to the more general notion, this sum may be (viz. if the polygons are stellated) $=e'$ times four right angles. And finally, the sum of the spherical polygons is ordinarily equal to the entire spherical surface; but according to the more general notion, it may be $=D$ times the spherical surface. (e is Poinsot's e; e' does not occur in Poinsot; and, for a reason which will appear, I have written D for Poinsot's E.)

The new polyhedra are constructed as follows:

1. *The great Icosahedron.*—Each face is made up of seven faces, or rather four faces and six half faces of the ordinary icosahedron, in the manner shown by fig. 1. There are, as in the ordinary icosahedron, five angles at each vertex; but these make up together, not four, but eight right angles, or $e=2$; but, as in the ordinary poly-

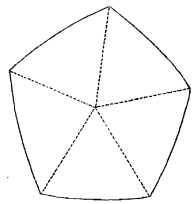
hedra, $e'=1$; and the sum of all the faces is obviously seven times the spherical surface, or $D=7$. (Also $E=7$.)

Fig. 1.



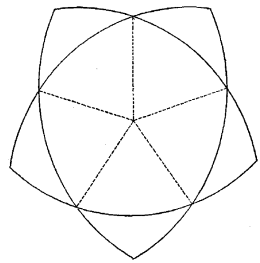
2. *The great Dodecahedron.*—Each face is made up of five faces of the ordinary icosahedron in the manner shown by the figure 2. There are five angles at each vertex, and these make up together eight right angles, or $e=2$; but, as in ordinary polyhedra, $e'=1$; and the sum of all the faces is obviously $12 \times \frac{5}{20}$, that is three times the spherical surface, or $D=3$. (Also $E=3$.)

Fig. 2.



3. *The great stellated Dodecahedron.*—Each face is formed by stellating a face of the great dodecahedron in the manner shown by fig. 3. There are, as in the ordinary

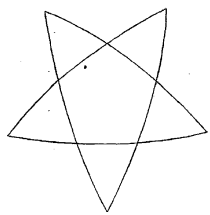
Fig. 3.



dodecahedron, three angles at each vertex, and the sum of these is simply four right angles, or $e=1$. On account of the stellation, $e'=2$. Each of the projecting parts of the face is equal $\frac{1}{3}$ of the face of the ordinary icosahedron; and if we reckon the area of the stellated pentagon to be that of the interior pentagon *plus* the projecting parts, the area of the face will be $5+\frac{5}{3}$, or $\frac{20}{3}$ of the face of the ordinary icosahedron; and the sum of the faces will be four times the spherical surface, and accordingly Poinsot writes $E=4$. If, however, what seems preferable, we reckon the area of the stellated pentagon as five times the triangle having for its vertex the centre of the face and standing upon a side (or what is the same thing, reckon the stellated pentagon as *twice* the interior pentagon *plus* the projecting parts), then the area of the face will be $10+\frac{5}{3}$ or $\frac{35}{3}$ of the face of the ordinary icosahedron, and the sum of the faces will be seven times the spherical surface, or $D=7$.

4. *The small stellated Dodecahedron.*—Each face is formed by stellating a face of the ordinary dodecahedron, as shown by fig. 4. There are five angles at each vertex; and the sum of these is four right angles, or $e=1$. On account of the stellation, $e'=2$. The area of each of the projecting parts is $\frac{1}{5}$ of the interior pentagon or face of the ordinary dodecahedron; and, according to the first mode of measurement, the

Fig. 4.



area of the stellated face is twice that of the face of the ordinary dodecahedron, and the sum of the faces is twice the spherical surface, and accordingly Poinsot writes $E=2$. But according to the second mode of measurement, the area of the stellated pentagon is three times that of the face of the ordinary dodecahedron, and the sum of the faces is three times the spherical surface, or we have $D=3$.

I form now the following Table, comprising as well the ordinary five figures as the new ones of Poinsot, and where we have

H, the number of faces.

S, the number of vertices.

A, the number of edges.

n, the number of sides to a face.

n', the number of sides (angles) at a vertex.

e, viz. the angles at a vertex make together e times four right angles.

e', viz. the angles which the sides of a face subtend at the centre of the face make together e' times four right angles.

E, viz. the faces make together E times the spherical surface, the area of a stellated face being reckoned (as by Poinsot), each portion being taken once only.

D, viz. the faces make together D times the spherical surface, the area of a stellated face being reckoned as the sum of the triangles having their vertices at the centre of the face and standing on the sides.

The Table is

Designation.	H.	S.	A.	n.	n'.	e.	e'.	D.		E.
Tetrahedron	4	4	6	3	3	1	1	1		1
Hexahedron	6	8	12	4	3	1	1	1		1
Octahedron	8	6	12	3	4	1	1	1		1
Dodecahedron	12	20	30	5	3	1	1	1		1
Icosahedron	20	12	30	3	5	1	1	1		1
Great stellated dodecahedron	12	20	30	5	3	1	2	7		4
Great icosahedron	20	12	30	3	5	2	1	7		7
Small stellated dodecahedron	12	12	30	5	5	1	2	3		2
Great dodecahedron	12	12	30	5	5	2	1	3		3

where the figures which are polar reciprocals of each other are written in pairs: viz. as is well known, the tetrahedron is its own reciprocal, the hexahedron and octahedron are reciprocals, and the dodecahedron and icosahedron are reciprocals; moreover the great stellated dodecahedron and the great icosahedron are reciprocals, and the small stellated dodecahedron and the great dodecahedron are reciprocals. The number which I have called D is reciprocal to itself; this is not the case for Poinsot's E; and I have not been able to define E in such a manner as to enable me to form the definition of a reciprocal number E': this may be possible, but in the mean time it seems better to discard E altogether, and use instead of it the number D.

Euler's well-known relation applying to ordinary polyhedra is

$$S + H = A + 2.$$

Poinsot in his memoir has (by an extension of Legendre's demonstration of Euler's theorem) obtained the more general relation,

$$eS + H = A + 2E,$$

which, however, does not apply to the two stellated figures where e' is different from unity; the general form is

$$eS + e'H = A + 2D,$$

which applies to all the nine figures. This applies to all polyhedra, regular or not, which are such that e has the same value for each vertex, and e' the same value for each face. To prove it, we have only to further extend Legendre's demonstration. If for any face, stellated or not, the sum of the angles is s, and the number of sides n, then, according to the foregoing mode of reckoning, the area of the face (measured in right angles) is

$$s + 4e' - 2n.$$

Now the sum of all the faces is D times the spherical surface, $= 8D$. But the sum of the term s is equal to the sum of the angles about each vertex, $= 4eS$; the sum of the term $4e'$ is $= 4e'H$, the sum of the term $2n$ is four times the number of edges, $= 4A$. Hence $4eS + 4e'H - 4A = 8D$, or $eS + e'H = 2D$.

I remark that the small stellated dodecahedron and the great dodecahedron are descriptively the same figures, and that, if we represent the vertices by $a, b, c, d, e, f, g, h, i, j, p, q$, and the faces by $A, B, C, D, E, F, G, H, I, J, P, Q$, then the relations of the vertices and faces is shown by either of the following Tables:

$a\ b\ c\ d\ e = P,$	$A\ C\ E\ B\ D = p,$
$p\ b\ i\ h\ e = A,$	$P\ I\ E\ B\ H = a,$
$p\ e\ j\ i\ a = B,$	$P\ J\ A\ C\ I = b,$
$p\ d\ f\ j\ b = C,$	$P\ F\ B\ D\ J = c,$
$p\ e\ g\ f\ c = D,$	$P\ G\ C\ E\ F = d,$
$p\ a\ h\ g\ d = E,$	$P\ H\ D\ A\ G = e,$
$j\ c\ d\ g\ q = F,$	$J\ D\ Q\ C\ G = f,$
$f\ d\ e\ h\ q = G,$	$F\ E\ Q\ D\ H = g,$
$g\ e\ a\ i\ q = H,$	$G\ A\ Q\ E\ I = h,$
$h\ a\ b\ j\ q = I,$	$H\ B\ Q\ A\ J = i,$
$i\ b\ c\ f\ q = J,$	$I\ C\ Q\ B\ F = j,$
$f\ g\ h\ i\ j = Q,$	$F\ H\ J\ G\ I = q,$

where it is to be noticed that in either Table each non-consecutive duad of any pentad occurs once, and only once, as a non-consecutive duad of another pentad. The restriction that a non-consecutive duad of any multiplet is *not* to occur as a duad, consecutive or non-consecutive, of any other multiplet (see my note appended to Mr Kirkman's paper "On Autopolar Polyhedra," *Phil. Trans.* 1857, p. 183 [259]), applies only to ordinary polyhedra, and not to the class here considered.

2, *Stone Buildings, W.C., January* 13, 1859.

242.

SECOND NOTE ON POINSOT'S FOUR NEW REGULAR POLYHEDRA.

[From the *Philosophical Magazine*, vol. XVII. (1859), pp. 209—210.]

THE Note on Poinsot's four new regular Polyhedra (February Number, p. 123), [241], was written without my being acquainted with Cauchy's first memoir, "Recherches sur les Polyèdres" (*Jour. Polyt.* vol. IX. pp. 68—86, 1813), the former part of which (pp. 68—76) relates to Poinsot's polyhedra. Cauchy considers the polyhedra, not as projected on the sphere, but *in solido;* and he shows, very elegantly, that all such polyhedra *must* be derived from the ordinary regular polyhedra by producing their sides or faces. The reciprocal method would be to produce the sides or join the vertices; and, adopting this reciprocal method, and projecting the figure on the sphere, we have the method employed by Poinsot, and explained and developed in my former Note. Cauchy does not at all consider Poinsot's generalized equation, $eS+H=A+2E$, nor of course my further generalization, $eS+e'H=A+2D$; but the latter part of the memoir relates to a generalization, in a different direction, of Euler's original formula, $S+H=A+2$: viz. Cauchy's theorem is—"If a polyhedron is partitioned into any number of polyhedra by taking at pleasure, in the interior of it, any number of new vertices, and if P be the total number of polyhedra thus formed, S the total number of vertices (including those of the original polyhedron), and A the total number of edges, then $S+H=A+P+1$; that is, the sum of the number of vertices and the number of faces exceeds by unity the sum of the number of edges and of the number of polyhedra."

For $P=1$, we have Euler's equation $S+H=A+2$; and for $P=0$, we have a theorem relating to the partition of a polygon; viz. if the polygon is divided into H polygons, and if S be the number of vertices, and A the number of sides, then $S+H=A+1$; from which it is easy to pass to Euler's equation, $S+H=A+2$, for

polyhedra. I remark that, in the equation $S+H=A+1$, H should, in analogy with Cauchy's notation for polyhedra, be replaced by P; so that we have for a single polygon,

$$A=S;$$

and for the partitions of a polygon,

$$A=S+P-1:$$

corresponding respectively to Euler's theorem for a single polyhedron, viz.

$$S+H=A+2;$$

and to Cauchy's theorem for the partitions of a polyhedron, viz.

$$S+H=A+2+(P-1).$$

Cauchy's second memoir (pp. 87—98) contains a very beautiful demonstration of the theorem implied in the ninth definition of the eleventh book of Euclid, viz. that two convex polyhedra are equal when they are bounded by the same number of faces equal each to each.

2, *Stone Buildings, W.C., February* 1, 1859.

243.

ON THE THEORY OF GROUPS AS DEPENDING ON THE SYMBOLIC EQUATION $\theta^n = 1$. THIRD PART.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 34—37: Sequel to **125** and **126**.]

THE following is, I believe, a complete enumeration of the groups of 8:

I. $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7 \quad (\alpha^8 = 1)$.

II. $1, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3 \ (\alpha^4 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha)$.

III. $1, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3 \ (\alpha^4 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha^3)$.

IV. $1, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3 \ (\alpha^4 = 1,\ \beta^2 = \alpha^2,\ \alpha\beta = \beta\alpha^3)$.

V. $1, \alpha, \beta, \beta\alpha, \gamma, \gamma\alpha, \gamma\beta, \gamma\beta\alpha \ (\alpha^2 = 1,\ \beta^2 = 1,\ \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \alpha\gamma = \gamma\alpha,\ \beta\gamma = \gamma\beta)$.

That the groups are really distinct is perhaps most readily seen by writing down the indices of the different terms of each group; these are

I.	1,	8,	4,	8,	2,	8,	4,	8.
II.	1,	4,	2,	4,	2,	4,	2,	4.
III.	1,	4,	2,	4,	2,	2,	2,	2.
IV.	1,	4,	2,	4,	4,	4,	4,	4.
V.	1,	2,	2,	2,	2,	2,	2,	2.

It will be presently seen why there is no group where the symbols α, β are such that $\alpha^4 = 1$, $\beta^2 = 1$, $\alpha\beta = \beta\alpha^2$. A group which presents itself for consideration is

$$1, \alpha, \alpha^2, \alpha^4, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3 (\alpha^4 = 1,\ \beta^2 = \alpha^2,\ \alpha\beta = \beta\alpha);$$

but the indices of the different terms of this group are

$$1,\ 4,\ 2,\ 4,\ 2,\ 4,\ 2,\ 4,$$

and if we write $\beta\alpha=\gamma$, then we find $\gamma^2=\beta\alpha\beta\alpha=\beta\beta\alpha\alpha=\alpha^4=1$, $\alpha\gamma=\alpha\beta\alpha=\beta\alpha\alpha=\gamma\alpha$; and the group is

$$1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \gamma,\ \gamma\alpha,\ \gamma\alpha^2,\ \gamma\alpha^3\ (\alpha^4=1,\ \gamma^2=1,\ \alpha\gamma=\gamma\alpha),$$

which is the group II.

The group IV is a remarkable one; it appears to arise from the circumstance that the factors 2 and 4 of the number 8 are not prime to each other; this can only happen when the number which denotes the order of the group contains a square factor. But the nature of the group in question will be better understood by presenting it under a different form. In fact, if we write $\beta\alpha^3=\gamma$, $\alpha^2=\beta^2=\vartheta$, then we find $\alpha^3=\vartheta\alpha$, $\beta\alpha^2=\vartheta\beta$, $\beta\alpha=\vartheta\gamma$, and the group will be

$$1,\ \alpha,\ \beta,\ \gamma,\ \vartheta,\ \vartheta\alpha,\ \vartheta\beta,\ \vartheta\gamma,$$

where the laws of combination are

$$\begin{array}{lll} \vartheta^2=1, & \alpha^2=\beta^2=\gamma^2=\vartheta, & \\ \beta\gamma=\alpha, & \gamma\alpha=\beta, & \alpha\beta=\gamma, \\ \gamma\beta=\alpha\vartheta=\vartheta\alpha, & \alpha\gamma=\beta\vartheta=\vartheta\beta, & \beta\alpha=\gamma\vartheta=\vartheta\gamma. \end{array}$$

Observe that ϑ is a symbol of operation such that $\vartheta^2=1$, and that ϑ is convertible with each of the other symbols α, β, γ. It will be not so much a restrictive assumption in regard to ϑ, as a definition of -1 considered as a symbol of operation if we write $\vartheta=-1$; the group thus becomes

$$1,\ \alpha,\ \beta,\ \gamma,\ -1,\ -\alpha,\ -\beta,\ -\gamma,$$

where

$$\begin{array}{l} \alpha^2=\beta^2=\gamma^2=-1, \\ \alpha=\beta\gamma=-\gamma\beta,\quad \beta=\gamma\alpha=\alpha\gamma,\quad \gamma=\alpha\beta=\beta\alpha. \end{array}$$

Hence α, β, γ combine according to the laws of the quaternion symbols i, j, k; and it is only the point of view from which the question is here considered which obliges us to consider the symbols as belonging to a group of 8, instead of (as in the theory of quaternions) a group of 4.

Suppose in general that the symbols α, β are such that

$$\alpha^m=1,\quad \beta^n=1,\quad \alpha\beta=\beta\alpha^s,$$

then we find

$$\alpha^u\beta^v=\beta^v\alpha^{us^v};$$

and therefore if $v=n$, $\alpha^u=\alpha^{us^n}$ or $\alpha^{u(s^n-1)}=1$, whence $u(s^n-1)\equiv 0\,(\text{mod. } m)$; or since u is arbitrary, $s^n-1\equiv 0\,(\text{mod. } m)$, an equation which, if m, n are given, determines the admissible values of s; thus, for example, if $n=2$, and m is a prime number, then $s=1$ or $s=m-1$. The equation $\alpha^u\beta^v=\beta^v\alpha^{us^v}$ shows that any combination whatever of the symbols α, β can be expressed in the form $\beta^q\alpha^p$ (or, if we please, in the form $\alpha^p\beta^q$). It is proper to show that the assumed law is consistent with the associative law, viz. that the expression

$$\beta^b\alpha^a\,.\,\beta^d\alpha^c\,.\,\beta^f\alpha^e$$

can be transformed in one way only into the form $\beta^q\alpha^p$. We in fact have

$$\beta^b\alpha^a\,.\,\beta^d\alpha^c=\beta^b\,.\,\alpha^a\beta^d\,.\,\alpha^c=\beta^b\,.\,\beta^d\alpha^{as^d}\,.\,\alpha^c=\beta^{b+d}\alpha^{as^d+c};$$

and multiplying this by the remaining factor $\beta^f\alpha^e$, we have

$$\beta^{b+d}\,.\,\alpha^{as^d+c}\beta^f\,.\,\alpha^e,$$

which is equal to

$$\beta^{b+d}\,.\,\beta^f\alpha^{as^{d+f}+cs^f}\,.\,\alpha^e,$$

or finally to

$$\beta^{b+d+f}\alpha^{as^{d+f}+cs^f+e};$$

and the result would have been precisely the same if, instead of thus combining together the first and second factors and the product with the third factor, we had combined the first factor with the product of the second and third factors, so that the associative law is satisfied.

It is now easy to see that if, as before,

$$\alpha^m=1,\quad \beta^n=1,\quad \alpha\beta=\beta\alpha^s,$$

conditions which it has been shown imply $s^n\equiv 1\,(\text{mod. } m)$, then the symbols $\beta^q\alpha^p$ (or, if we please, $\alpha^p\beta^q$), where p has the values $0, 1, 2, \ldots, m-1$, and q the values $0, 1, 2, \ldots, n-1$, form a group of mn terms. In particular, as already noticed, if $n=2$ and m is prime, then $s=1$ or $s=m-1$; the two groups so obtained are essentially distinct from each other. If $n=2$, but m is not prime, then s has in general more than two values: thus for $m=12$, $s^2\equiv 1\,(\text{mod. } 12)$, which is satisfied by $s=1, 5, 7$ and 11; the group corresponding to $s=1$ is distinct from that for any other value of s, but I have not ascertained whether the values other than unity do, or do not, give groups distinct from each other.

For the sake of an observation to which it gives rise, I write down an example of a group corresponding to $n=2$, $s=m-1$, say $m=5$, and therefore $s=4$, so that we have

$$\alpha^5=1,\quad \beta^2=1,\quad \alpha\beta=\beta\alpha^4,$$

and the group is

$$1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \beta,\ \beta\alpha,\ \beta\alpha^2,\ \beta\alpha^3,\ \beta\alpha^4,$$

the indices of the several terms being

$$1,\ 5,\ 5,\ 5,\ 5,\ 2,\ 2,\ 2,\ 2,\ 2.$$

The group is here expressed by means of the symbols α, β, having the indices 5 and 2 respectively, but it may be expressed by means of two symbols having each of them the index 2. Thus putting $\beta\alpha = \gamma$, we find $\beta^2 = 1$, $\gamma^2 = 1$, $(\beta\gamma)^5 = 1$, which is equivalent to $(\gamma\beta)^5 = 1$, and the group may be represented in the form

$$1,\ \beta,\ \gamma,\ \beta\gamma,\ \gamma\beta,\ \beta\gamma\beta,\ \gamma\beta\gamma,\ \beta\gamma\beta\gamma,\ \gamma\beta\gamma\beta,\ \beta\gamma\beta\gamma\beta = \gamma\beta\gamma\beta\gamma,$$

the equality of the last two symbols being an obvious consequence of the equation $(\beta\gamma)^5 = 1$. It is clear that for any even number $2p$ whatever, there is always a group which can be expressed in this form.

2, *Stone Buildings, W.C., June* 9, 1859.

244.

ON AN ANALYTICAL THEOREM RELATING TO THE DISTRIBUTION OF ELECTRICITY UPON SPHERICAL SURFACES.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 119—127.]

THERE is contained in Plana's "Mémoire sur la distribution de l'électricité à la surface de deux sphères conductrices complètement isolées" (*Mém. de Turin*, vol. VII. 1845), an identical relation which is remarkable, as well in itself as because by means of it the author corrects an error into which Poisson had fallen in his researches on the same subject. The development of a certain definite integral is obtained in the form (equation 165)

$$y = -2.3\frac{M_1 h}{b^2}\sin^2\tfrac{1}{2}\theta + \frac{3.4.5}{1.2}\frac{M_3 h}{b^2}\sin^4\tfrac{1}{2}\theta + \&c.$$

Poisson had in effect shown that $M_1 = 0$; and he thence inferred that, θ being small, the function in question $\propto \sin^4\frac{1}{2}\theta$, or what is the same thing, $\propto (1-\cos\theta)^2$. In the former part of the memoir, Plana shows that this is not the true form of the development; the foregoing development must therefore be illusory; and Plana in fact shows, by a laborious induction carried as far as M_7, that all the coefficients M vanish identically. The identical equation $M_i = 0$, where i is any positive integer whatever, constitutes the analytical theorem above referred to. Plana's expression for the function M_i is as follows:

B_1, B_3, B_5, &c. denote Bernoulli's numbers as given by the equation

$$\frac{t}{e^t-1} = 1 - \tfrac{1}{2}t + B_1\frac{t^2}{1.2} - B_3\frac{t^4}{1.2.3.4} + \&c.$$

($B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$, $B_7 = \frac{1}{30}$, &c. I have, in conformity with the usual practice written the equations so as to make these numbers all positive; with Plana they are

alternately positive and negative). And in the equation 162, writing for k its value $\frac{1}{1+b}$, we have, λ being any positive integer,

$$(1+b)^\lambda G_\lambda = \frac{1}{\lambda+1} - \tfrac{1}{2}(1+b) + \frac{\lambda.\lambda-1}{1.2} B_1 \frac{(1+b)^2}{\lambda-1}$$
$$- \frac{\lambda.\lambda-1.\lambda-2.\lambda-3}{1.2.3.4} B_3 \frac{(1+b)^4}{\lambda-3}$$
$$+ \frac{\lambda.\lambda-1.\lambda-2.\lambda-3.\lambda-4.\lambda-5}{1.2.3.4.5.6} B_5 \frac{(1+b)^6}{\lambda-5},$$
$$\pm \text{ \&c.},$$

where the series is continued for so long as the factor in the denominator is positive. It should be observed that this factor really divides out, and that the rule just mentioned amounts to this, viz. that when λ is odd, the finite series on the right-hand side is to be continued to its last term; but when λ is even, the series is to be continued only to the last term but one. And G_λ being thus defined, the expression for M_i (see equation 164, in which I have written for k its value $\frac{1}{1+b}$) is

$$M_i = (1+b)^{i-1}\left\{G_i + (i+2)\frac{1}{1}\frac{1+b}{b} G_{i+1} + (i+3)\frac{i}{1.2}\left(\frac{1+b}{b}\right)^2 G_{i+2}\right.$$
$$\left. + (i+4)\frac{i.i-1}{1.2.3}\left(\frac{1+b}{b}\right)^3 G_{i+3} + \text{\&c.}\right\},$$

where on the right-hand side the finite series is continued up to its last term, the value of which is obviously

$$(1+2i+1)\frac{1}{i+1}\left(\frac{1+b}{b}\right)^{i+1} G_{2i+1}, \text{ that is, } 2\left(\frac{1+b}{b}\right)^{i+1} G_{2i+1}.$$

But the form of this equation may be somewhat simplified. We in fact have

$$(1+b)M_i = 1\left\{\frac{1}{i+1} - \tfrac{1}{2}(1+b) + B_1\frac{i}{1.2}(1+b)^2 - B_3\frac{i.i-1.i-2}{1.2.3.4}(1+b)^4 + \text{\&c.}\right\}$$
$$+ \frac{i+2}{1}\frac{1}{b}\left\{\frac{1}{i+2} - \tfrac{1}{2}(1+b) + B_1\frac{i+1}{1.2}(1+b)^2 - B_3\frac{i+1.i.i-1}{1.2.3.4}(1+b)^4 + \text{\&c.}\right\}$$
$$+ \frac{i+3.i}{1.2}\frac{1}{b^2}\left\{\frac{1}{i+3} - \tfrac{1}{2}(1+b) + B_1\frac{i+2}{1.2}(1+b)^2 - B_3\frac{i+2.i+1.i}{1.2.3.4}(1+b)^4 + \text{\&c.}\right\},$$

which is at once changed into

$$(1+b)M_i =$$
$$\frac{1}{i+1}\left\{1 - \tfrac{1}{2}\frac{i+1}{1}(1+b) + B_1\frac{i+1.i}{1.2}(1+b)^2 - B_3\frac{i+1.i.i-1.i-2}{1.2.3.4}(1+b)^4 + \text{\&c.}\right\}$$

$$+\tfrac{1}{1}\frac{1}{b}\left\{1-\tfrac{1}{2}\frac{i+2}{1}(1+b)+B_1\frac{i+2.i+1}{1.2}(1+b)^2-B_3\frac{i+2.i+1.i.i-1}{1.2.3.4}(1+b)^4+\&c.\right\}$$

$$+\frac{i}{1.2}\frac{1}{b^2}\left\{1-\tfrac{1}{2}\frac{i+3}{1}(1+b)+B_1\frac{i+3.i+2}{1.2}(1+b)^2-B_3\frac{i+3.i+2.i+1.i}{1.2.3.4}(1+b)^4+\&c.\right\}$$

$$+\ \&c.;$$

or multiplying by $i+1$, and then putting $i-1$ in the place of i, we have

$$i(1+b)\,M_{i-1}=\Theta_i+\frac{i}{1}\frac{1}{b}\Theta_{i+1}+\frac{i.i-1}{1.2}\frac{1}{b^2}\Phi_{i+2}+\&c.=0,$$

where

$$\Theta_i=1-\tfrac{1}{2}\frac{i}{1}(1+b)+B_1\frac{i.i-1}{1.2}(1+b)^2-B_3\frac{i.i-1.i-2.i-3}{1.2.3.4}(1+b)^4\pm\&c.$$

In this last equation the finite series on the right-hand side is, when i is *even*, to be continued up to its last term, but when i is odd, then only up to the last term but one. The equation to be proved is

$$0=\Theta_i+\frac{i}{1}\frac{1}{b}\Theta_{i+1}+\frac{i.i-1}{1.2}\frac{1}{b^2}\Theta_{i+2}+\&c.,$$

where on the right-hand side the finite series is to be continued up to its last term: and the equation holds for any integer value of i which is >2. This is the simplest form of Plana's theorem.

We have

$$\Theta_i=i(1+b)^{i-1}G_{i-1};$$

or writing this equation under the form $\Theta_i=i(1+b)\dfrac{G_{i-1}}{b^{i-2}}$, and comparing with Plana's developed expressions for $\dfrac{G_{i-1}}{b^{i-2}}$ (which are continued by him as far as G_{17}), we find

$$\begin{aligned}
\Theta_2&=-\ b,\\
\Theta_3&=-\tfrac{1}{2}b+\tfrac{1}{2}b^2,\\
\Theta_4&=\qquad\qquad b^2,\\
\Theta_5&=\tfrac{1}{6}b+\tfrac{3}{8}b^2-\tfrac{3}{2}b^3-\tfrac{1}{6}b^4,\\
\Theta_6&=\qquad-\tfrac{1}{2}b^2-2b^3-\tfrac{1}{2}b^4,\\
\Theta_7&=-\tfrac{1}{6}b-\ b^2-\tfrac{4}{3}b^3+\tfrac{4}{3}b^4+\ b^5+\tfrac{1}{6}b^6,\\
\Theta_8&=\qquad\tfrac{3}{8}b^2+4b^3+\tfrac{23}{3}b^4+4b^5+\tfrac{3}{8}b^6,\\
\Theta_9&=\tfrac{3}{10}b+\tfrac{12}{5}b^2+\tfrac{32}{5}b^3+\tfrac{24}{5}b^4-\tfrac{24}{5}b^5-\tfrac{32}{5}b^6-\tfrac{12}{5}b^7-\tfrac{3}{10}b^8,\\
\Theta_{10}&=\qquad-\tfrac{3}{2}b^3-12b^4-37b^5-54b^6-37b^7-12b^8-\tfrac{3}{2}b^9,\\
&\&c.,
\end{aligned}$$

which are of course the results obtained by developing the foregoing expression for Θ_i, in powers of b, and collecting the terms. The formulæ put in evidence a remarkable symmetry which does not exist in the original expression in powers of $1+b$.

It would be now easy to verify, for moderately small values of the suffix, the equations

$$\Theta_2+\frac{2}{b}\Theta_3+\frac{1}{b^2}\Theta_4=0,$$

$$\Theta_3+\frac{3}{b}\Theta_4+\frac{3}{b^2}\Theta_5+\frac{1}{b^3}\Theta_6=0$$

&c.

This is, in fact, Plana's process, which, however, as the suffixes increase, becomes a very laborious one, and the law of the terms which destroy each other is not in anywise exhibited thereby.

I have succeeded in obtaining a complete demonstration, founded on Herschel's theorem for the development of a function of e^t, and the expression thereby given for Bernoulli's numbers. The theorem in question [See Herschel's *Collection of Examples in the Calculus of Finite Differences*, Cambridge, 1820, p. 70, where the theorem is given in the form $f\{(1+\Delta)^n\}\,0^x=n^x.f(1+\Delta)\,0^x$] is, that for any function of e^t which admits of development in positive integer powers of t,

$$f(e^t)=f(1+\Delta)e^{t\cdot 0},$$

where the right-hand side denotes the series the general term whereof is

$$\frac{t^n}{1.2.3\ldots n}f(1+\Delta)\,0^n;$$

and $f(1+\Delta)$ is of course to be developed in powers of Δ, and the different terms Δ, Δ^2, Δ^3 &c., applied to the symbol 0^n (viz. $\Delta 0^n=1^n-0^n$, $\Delta^2 0^n=2^n-2.1^n+0^n$, &c.). This gives

$$\frac{t}{e^t-1}=\frac{\log(1+\Delta)}{\Delta}e^{t\cdot 0},$$

and comparing the development of the right-hand side with the development

$$1-\tfrac{1}{2}t+B_1\frac{t^2}{1.2}-B_2\frac{t^4}{1.2.3.4}+\text{\&c.}$$

we find

$$\frac{\log(1+\Delta)}{\Delta}0^0=1,$$

$$\frac{\log(1+\Delta)}{\Delta}0^1=-\tfrac{1}{2},$$

$$\frac{\log(1+\Delta)}{\Delta}0^{2x}=(-)^{x+1}B_{2x-1},$$

$$\frac{\log(1+\Delta)}{\Delta}0^{2x-1}=0,\ (x>1).$$

It is now easy to obtain the equation

$$\Theta_i = \frac{\log(1+\Delta)}{\Delta}\{(1+0(1+b))^i - (-0(1+b))^i\}.$$

In fact, the first two terms of the development of the expression on the right-hand side agree with those of the foregoing expression for Θ_i. For any even power $2x$ (except, when i is even, the power $2x=i$) the term is

$$\frac{[i]^{2x}}{[2x]^{2x}}(1+b)^{2x}\frac{\log(1+\Delta)}{\Delta}0^{2x},$$

which agrees; and when i is even, then for the power $2x=i$ there are two equal and opposite terms which destroy each other, and the whole term in Θ_i is, as it ought to be, zero. For any odd power $2x-1$, $(x>1)$, (including, when i is odd, the power $2x-1=i$), the term vanishes as containing an evanescent factor. The expression for Θ_i is thus shown to be true.

I write for shortness,

$$\Theta_i = \frac{\log(1+\Delta)}{\Delta}\{X^i - Y^i\},$$

where

$$X = 1+0(1+b),$$
$$Y = \quad -0(1+b).$$

Forming the expression for $i(1+b)M_{i-1}$,

$$= \Theta_i + \frac{i}{1}\frac{1}{b}\Theta_{i+1} + \frac{i.i-1}{1.2}\frac{1}{b^2}\Theta_{i+2} + \&c.,$$

this is

$$i(1+b)M_{i-1} = \frac{\log(1+\Delta)}{\Delta}\left\{\left(X\left(1+\frac{X}{b}\right)\right)^i - \left(Y\left(1+\frac{Y}{b}\right)\right)^i\right\},$$

and we have

$$X \quad = (1+0)(1+b)-b,$$

and therefore

$$1+\frac{X}{b} \quad = (1+0)\frac{1+b}{b},$$

$$\left(X\left(1+\frac{X}{b}\right)\right)^i = \left(\frac{1+b}{b}\right)^i\{(1+0)\{(1+0)(1+b)-b\}\}^i,$$

$$Y \qquad = -0(1+b),$$

$$1+\frac{Y}{b} = 1-0\frac{1+b}{b} = -\frac{1}{b}\{0(1+b)-b\},$$

$$\left(Y\left(1+\frac{Y}{b}\right)\right)^i = \left(\frac{1+b}{b}\right)^i(0\{0(1+b)-b\})^i.$$

We see that the expression for $\left(X\left(1+\frac{X}{b}\right)\right)^i$ is deduced from that of $\left(Y\left(1+\frac{Y}{b}\right)\right)^i$ by writing therein $1+0$ in the place of 0; we have therefore

$$\left(X\left(1+\frac{X}{b}\right)\right)^i=(1+\Delta)\left(Y\left(1+\frac{Y}{b}\right)\right)^i,$$

and consequently

$$\left(X\left(1+\frac{X}{b}\right)\right)^i-\left(Y\left(1+\frac{Y}{b}\right)\right)^i=\Delta\left(Y\left(1+\frac{Y}{b}\right)\right)=\left(\frac{1+b}{b}\right)\Delta\left(0\left\{0\left(1+b\right)-b\right\}\right)^i;$$

whence also

$$i(1+b)\,M_{i-1}=\left(\frac{1+b}{b}\right)^i\log(1+\Delta)\left(0\left\{0\left(1+b\right)-b\right\}\right)^i.$$

We have by the general theorem,

$$t=\log e^t=\log(1+\Delta)\,e^{t\cdot 0};$$

and consequently whenever $n \nless 2$,

$$\log(1+\Delta)\,0^n=0.$$

But $i \nless 2$, and the function $\left(0\left\{0\left(1+b\right)-b\right\}\right)^i$ contains only 0^i and the superior powers; it is therefore reduced to zero by the operation $\log(1+\Delta)$, and we have

$$i(1+b)\,M_{i-1}=\Theta_i+\frac{i}{1}\frac{1}{b}\,\Theta_{i+1}+\frac{i\,.\,i-1}{1\,.\,2}\frac{1}{b^2}\,\Theta_{i+2}+\&c.=0;$$

and the theorem in question is thus proved. The foregoing expressions for Θ_2, Θ_3, &c. show that these functions all divide by b, and moreover that when i is even and greater than 2, then that Θ_i divides by b^2. The equation

$$\Theta_i=\frac{\log(1+\Delta)}{\Delta}\left\{\left(1+0\left(1+b\right)\right)^i-\left(-0\left(1+b\right)\right)^i\right\}$$

gives generally for the term in Θ_i involving b^α, the expression

$$\frac{[i]^\alpha}{[\alpha]^\alpha}\,b^\alpha\,\frac{\log(1+\Delta)}{\Delta}\left\{(1+0)^{i-\alpha}\,0^\alpha-(-0)^i\right\};$$

and it is to be shown, first, that the coefficient vanishes for $\alpha=0$; and next, that when i is even and >2, the coefficient also vanishes for $\alpha=1$. Putting $\alpha=0$, the coefficient is

$$\frac{\log(1+\Delta)}{\Delta}\left\{(1+0)^i-(-0)^i\right\},$$

which is equal to

$$\frac{\log(1+\Delta)}{\Delta}\left\{1+\Delta-(-)^i\,1\right\}0^i,$$

or to

$$\log(1+\Delta)\,0^i+\left\{1-(-)^i\,1\right\}\frac{\log(1+\Delta)}{\Delta}\,0^i,$$

where, since $i \nless 2$, the former term vanishes, as above remarked; and the latter term, when i is even, vanishes on account of the factor $1-(-)^i 1$; and when i is odd, on account of the other factor. Hence the coefficient vanishes for $\alpha = 0$.

Next, if i is even, and $\alpha = 1$, the coefficient becomes

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{i-1}0 - 0^i\},$$

which, writing $(1+0)-1$ for 0, becomes

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^i - (1+0)^{i-1} - 0^i\},$$

which, since $(1+0)^i - 0^i = \Delta 0^i$, $(1+0)^{i-1} = (1+\Delta)\,0^{i-1}$, is equal to

$$\log(1+\Delta)\,0^i - \frac{(1+\Delta)\log(1+\Delta)}{\Delta}\,0^{i-1},$$

or since the first term vanishes, to

$$-\frac{(1+\Delta)\log(1+\Delta)}{\Delta}\,0^{i-1}.$$

But this function is to a numerical factor *près* the coefficient of t^{i-1} in $\frac{e^t \log e^t}{e^t - 1}$, or (what is the same thing) in $\frac{-t}{1-e^{-t}}$; and if in the expression for $\frac{t}{e^t-1}$ we write $-t$ in the place of t, we find

$$-\frac{t}{1-e^{-t}} = 1 + \tfrac{1}{2}t + B_1\frac{t^2}{1.2} - B_2\frac{t^4}{1.2.3.4} + \&c.,$$

so that, i being even and greater than 2, the function in question vanishes. Hence in the case under consideration the coefficient vanishes for $\alpha = 1$.

Writing β for $i-\alpha$, or assuming $\alpha+\beta = i$, the symmetry of the foregoing expressions for Θ_2, Θ_3, &c. shows that we ought to have

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta - (-0)^{\alpha+\beta}\} = \pm\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\beta 0^\alpha - (-0)^{\alpha+\beta}\},$$

where the upper or under sign is to be taken according as $\alpha+\beta$ is even or odd. Or separating the two cases, we find

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta - (1+0)^\beta 0^\alpha\} \qquad = 0, \quad \alpha+\beta \text{ even},$$

and

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta + (1+0)^\beta 0^\alpha + 2\,.\,0^{\alpha+\beta}\} = 0, \quad \alpha+\beta \text{ odd}.$$

I have not attempted to verify *à posteriori* these elegant formulæ.

2, *Stone Buildings, W.C., June* 18, 1859.

245.

ON AN ANALYTICAL THEOREM CONNECTED WITH THE DISTRIBUTION OF ELECTRICITY ON SPHERICAL SURFACES. SECOND PART.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 193—202: continuation of **244**.]

THE theorem is certainly true; but its existence gives rise to a difficulty to which I shall advert in the sequel. I propose, in the first instance, to give a demonstration which starts from the expression for fx given by Plana's equation (115), instead of the deduced equation which was the basis of my former proof. It will be proper to explain the origin and meaning of the formulæ. We have two conducting spherical surfaces, radii 1 and b, in contact with each other (so that the distance between the centres is $1+b$): and then, if x is the distance from the centre of the sphere, radius 1, of an exterior point, and $\mu\,(=\cos\theta)$ the cosine of the inclination of this distance to the line from the centre to the centre of the other sphere, the potential $\phi(\mu, x)$ of the sphere, radius 1, at the point whose coordinates are (x, μ) is deduced from the potential fx of a point in the axis; that is, if

$$fx = A_0 \quad + A_1 x \quad + A_2 x^2 \quad + \&c.,$$

then

$$\phi(\mu, x) = A_0 P_0 + A_1 P_1 x + A_2 P_2 x^2 + \&c.,$$

where P_0, P_1, P_2, &c. are Legendre's functions, viz. the functions of μ which are the coefficients of the successive powers of x in the development of $(1-2\mu x+x^2)^{-\frac{1}{2}}$ in ascending powers of x. And the electrical thickness y at any point of the surface of the sphere, radius 1, is given by the formula

$$y = x\,\frac{d\phi(\mu, x)}{dx} + 2\phi(\mu, x)$$

where, after the differentiation, $x=1$.

The problem consequently depends on the determination of the potential fx for a point on the axis; and this is determined by the functional equation

$$fx - \frac{b}{1+2b-(1+b)x} f\left(\frac{1+b-x}{1+2b-(1+b)x}\right) = h - \frac{bh}{1+b-x}$$

(Plana's equation (G), in which I have written for β, γ, H their values, and substituted also for g its value $= h$). The solution of this equation is (equation (H), writing therein $g = h$)

$$fx = \frac{P}{1-x} + bh\Sigma_{n_0}^{\infty} \frac{1}{b+n(1+b)-n(1+b)x} - bh\Sigma_{n_0}^{\infty} \frac{1}{(n+1)(1+b)-\left(1+n(1+b)\right)x},$$

where P is an arbitrary constant *quoad* the functional equation, viz. it is any function whatever which has the property of remaining unaltered when x is changed into $\frac{1+b-x}{1+2b-(1+b)x}$. Poisson, and Plana after him, arrive at the conclusion that in the physical problem $P = 0$. It appears to me that there is ground for holding that this is only true *sub modo*, and that $\frac{P}{(1-x)^2}$ for $x = 1$ (which, if P were retained, would be a term occurring in the expression for the thickness at the point of contact) is not of necessity zero. But the term, if it exists, can be replaced at the conclusion; and I write therefore

$$fx = bh\Sigma_{n_0}^{\infty} \frac{1}{b+n(1+b)-n(1+b)x} - bh\Sigma_{n_0}^{\infty} \frac{1}{(n+1)(1+b)-\left(1+n(1+b)\right)x}.$$

According to the process by which the solution of the functional equation was obtained, this is the true form of the solution; for although the series are non-convergent, and the two sums are in fact each of them infinite, there is nothing to show a relation between the number of terms which must be taken in each series. However, nothing immediately turns upon this, as the expression is only used for obtaining an expression for fx in the form of a definite integral, viz., equation (36),

$$fx = bh \int_0^1 \frac{dt\, t^{b-1}(1-t^{1-x})}{1-t^{(1+b)(1-x)}};$$

or, equation (39),

$$fx = \frac{bh}{(1+b)(1-x)} \int_0^1 \frac{dt\,(t^{-\frac{1}{1+b}}-1)\, t^{\frac{bx}{(1+b)(1-x)}}}{1-t};$$

the latter of which gives (equation (115), in which I have written for a its value $\frac{b}{1+b}$)

$$fx = \frac{hb}{(1+b)(1-x)} \left\{ Z'\left(\frac{1+b-x}{(1+b)(1-x)}\right) - Z'\left(\frac{b}{1+b-x}\right) \right\},$$

where $Z'(p)$ is Legendre's function $\frac{d}{dp} \log \Gamma p$, which is developable in the form

$$Z'p = \log p - \frac{1}{2p} - \frac{B_1}{2p^2} + \frac{B_3}{4p^4} - \frac{B_5}{6p^6} + \&c.,$$

where B_1, B_3, &c. are Bernoulli's numbers.

This is the starting-point of the present investigation; and attending to the equations

$$\frac{\log(1+\Delta)}{\Delta}\,0^1 = -\tfrac{1}{2},$$

$$\frac{\log(1+\Delta)}{\Delta}\,0^{2x-1} = 0,\ (x>1),$$

$$\frac{\log(1+\Delta)}{\Delta}\,0^{2x} = B_{2x-1},$$

we see that the development of $Z'p$ becomes

$$Z'p = \log p + \frac{\log(1+\Delta)}{\Delta}\left(\frac{0}{p} - \frac{0^2}{2p^2} + \frac{0^3}{2p^3} - \&c.\right) = \log p + \frac{\log(1+\Delta)}{\Delta}\log\left(1+\frac{0}{p}\right),$$

which, observing that

$$\left(\frac{\log(1+\Delta)}{\Delta} - 1\right)\log p = 0,$$

can be expressed under the more simple form

$$Z'p = \frac{\log(1+\Delta)}{\Delta}\log(p+0).$$

We deduce hence

$$fx = \frac{hb}{(1-x)(1+b)}\,\frac{\log(1+\Delta)}{\Delta}\left\{\log\left(\frac{1+b-x}{(1-x)(1+b)}+0\right) - \log\left(\frac{b}{(1-x)(1+b)}+0\right)\right\};$$

or what is the same thing,

$$fx = \frac{hb}{(1-x)(1+b)}\,\frac{\log(1+\Delta)}{\Delta}\,\{\log\left(1+b-x+(1-x)(1+b)\,0\right) - \log\left(b+(1-x)(1+b)\,0\right)\},$$

which may be converted into

$$fx = \frac{hb}{1+b}\,\frac{\log(1+\Delta)}{\Delta}\int_0^1 \frac{dt}{b+t(1-x)}$$
$$+\frac{hb\log(1+\Delta)}{\Delta}\,0\left\{\int_0^1 \frac{dt}{1-x+b+(1-x)(1+b)t0} - \int_0^1 \frac{dt}{b+(1-x)(1+b)t0}\right\};$$

or what is the same thing,

$$fx = \frac{hb}{1+b}\,\frac{\log(1+\Delta)}{\Delta}\int_0^1 \frac{dt}{b+t-tx}$$
$$+\frac{hb\log(1+\Delta)}{\Delta}\,0\left\{\int_0^1 \frac{dt}{(1+b)(1+t0)-x\left(1+(1+b)t0\right)} - \int_0^1 \frac{dt}{b+(1+b)t0-x(1+b)t0}\right\},$$

the object of the transformation being to express fx so that x may only enter under the form $\frac{1}{A-Bx}$. The factor $\frac{\log(1+\Delta)}{\Delta}$ which multiplies the first of the three definite integrals, might be reduced to unity, but it is more convenient not to make this change.

Now if a fraction $\frac{1}{A-Bx}$ be operated upon by expanding in ascending powers of x, and multiplying the successive terms of the development by P_0, P_1, P_2, &c., it is converted into

$$\frac{1}{(A^2-2AB\mu x+B^2x^2)^{\frac{1}{2}}}.$$

Hence from the foregoing expression for fx we pass at once to the expression for $\phi(\mu, x)$; that is, we have

$$\phi(\mu, x)=\frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}0\int_0^1\frac{dt}{(A^2-2AB\mu x+B^2x^2)^{\frac{1}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{dt}{(A'^2-2A'B'\mu x+B'^2x^2)^{\frac{1}{2}}}-\int_0^1\frac{dt}{(A''^2-2A''B''\mu x+B''^2x^2)^{\frac{1}{2}}}\right\};$$

where for shortness,

$$A=b+t,\quad A'=(1+b)(1+t0),\quad A''=b+(1+b)t0,$$
$$B=\quad t,\quad B'=1+(1+b)t0,\quad B''=\quad(1+b)t0;$$

and it may be remarked that

$$A'=1+b+B'',\quad B'=1+B'',\quad A''=b+B''.$$

We thence obtain

$$x\frac{d\phi(\mu, x)}{dx}+2\phi(\mu, x)=\frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}\int_0^1\frac{(A^2-B^2x^2)\,dt}{(A^2-2AB\mu x+B^2x^2)^{\frac{3}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{(A'^2-B'^2x^2)\,dt}{(A'^2-2A'B'\mu x+B'^2x^2)^{\frac{3}{2}}}-\int_0^1\frac{(A''^2-B''^2x^2)\,dt}{(A''^2-2A''B''\mu x+B''^2x^2)^{\frac{3}{2}}}\right\},$$

and writing $x=1$,

$$y=\frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}\int_0^1\frac{(A^2-B^2)\,dt}{(A^2-2AB\mu+B^2)^{\frac{3}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{(A'^2-B'^2)\,dt}{(A'^2-2A'B'\mu+B'^2)^{\frac{3}{2}}}-\int_0^1\frac{(A''^2-B'')\,dt}{(A''^2-2A''B''\mu+B''^2)^{\frac{3}{2}}}\right\},$$

the integrals in the foregoing expression are of the form

$$\int_0^1\frac{(G+Ht)\,dt}{(L+2Mt+Nt^2)^{\frac{3}{2}}};$$

the value of the indefinite integral is

$$\frac{1}{LN-M^2}\frac{(NG-MH)\,t+MG-LH}{(L+2Nt+Mt^2)^{\frac{1}{2}}},$$

from which the value of the definite integral can be at once found. It is easy, by means of the values to be presently given, to verify that, in each of the three definite integrals, $NG-MH=0$; and the expression for the definite integral is therefore

$$\frac{MG-LH}{LN-M^2}\left\{\frac{1}{(L+2M+N)^{\frac{1}{2}}}-\frac{1}{L^{\frac{1}{2}}}\right\}.$$

In the first integral we have

$$\begin{aligned} G &= b^2, & L &= b^2,\\ H &= 2b, & M &= b(1-\mu),\\ & & N &= 2(1-\mu), \end{aligned}$$

whence

$$\begin{aligned} LN-M^2 &= b^2(1-\mu)(1+\mu), & MG-LH &= -b^3(1+\mu),\\ L+2M+N &= b^2+2(1-\mu)(1+b), & L &= b^2; \end{aligned}$$

and the integral is

$$\frac{-b}{1-\mu}\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)}}-\frac{1}{b}\right\}.$$

For the second integral we have

$$\begin{aligned} G &= b^2+2b, & L &= b^2+2\,(1-\mu)(1+b),\\ H &= 2b(1+b)\,0, & M &= (1-\mu)(2+b)(1+b)\,0,\\ & & N &= 2(1-\mu)(1+b)^2\,0^2; \end{aligned}$$

and thence

$$\begin{aligned} LN-M^2 &= (1-\mu)(1+\mu)\,b^2(1+b)^2\,0^2, & MG-LH &= -(1+\mu)\,b^3(1+b)\,0,\\ L+2M+N &= b^2+2(1-\mu)(1+b)\,\{(1+0)^2+b\,(0+0^2)\}; & L &= b^3+2(1-\mu)(1+b); \end{aligned}$$

and the value of the integral is

$$-\frac{b}{(1+b)(1-\mu)\,0}\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left((1+0)^2+b\,(0+0^2)\right)}}-\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)}}\right\}.$$

For the third integral,

$$\begin{aligned} G &= b^2, & L &= b^2,\\ H &= 2b(1+b)\,0, & M &= (1-\mu)\,b\,(1+b)\;0,\\ & & N &= 2(1-\mu)\;(1+b)^2\,0^2, \end{aligned}$$

and thence

$$\begin{aligned} LN-M^2 &= (1-\mu)(1+\mu)\,b^2(1+b)^2\,0^2, & MG-LH &= -(1+\mu)\,b^3(1+b)\,0,\\ L+2M+N &= b^2+2(1-\mu)(1+b)\;\left(0^2+b\,(0+0^2)\right), & L &= b^2; \end{aligned}$$

and the value of the integral is

$$\frac{-b}{(1+b)(1-\mu)\,0}\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left(0^2+b(0+0^2)\right)}}-\frac{1}{b}\right\}.$$

Hence the expression for y is

$$\begin{aligned} y = & \frac{-hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\left\{\frac{1}{\sqrt{b^2+2(1+\mu)(1+b)}}-\frac{1}{b}\right\}, \\ & -\frac{hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times \\ & \qquad\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left((1+0)^2+b(0+0^2)\right)}}-\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)}}\right\}, \\ & +\frac{hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times \\ & \qquad\left.\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left(0^2+b(0+0^2)\right)}}-\frac{1}{b}\right\}; \end{aligned}$$

the top line is destroyed by the second terms of the other two lines, and we have

$$\begin{aligned} y = & \frac{-hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times \\ & \left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left((1+0)^2+b(0+0^2)\right)}}-\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)\left(0^2+b(0+0^2)\right)}}\right\}. \end{aligned}$$

This expression admits of expansion in positive integer powers of $1-\mu$; and when so expanded the result ought, according to Plana's theorem, to be identically equal to zero. And I proceed to show that this is in fact the case. The coefficient of $(1-\mu)^{m-1}$ is to a factor *près* of the form

$$\frac{\log(1+\Delta)}{\Delta}\left\{\left((1+0)^2-b(0+0^2)\right)^m-\left(0^2+b(0+0^2)\right)^m\right\},$$

which is the sum of a series of terms each of the form

$$\frac{\log(1+\Delta)}{\Delta}\left\{(1+0)^{2m-2n}-0^{2m-2n}\right\}(0+0^2)^n;$$

this is equal to

$$\frac{\log(1+\Delta)}{\Delta}\left\{(1+0)^{2m-n}\,0^n-0^{2m-n}\,(1+0)^n\right\},$$

which is of the form

$$\frac{\log(1+\Delta)}{\Delta}\left\{(1+0)^\alpha\,0^\beta-(1+0)^\beta\,0^\alpha\right\},$$

where $\alpha+\beta=2m$ is even, or what is the same thing, $\alpha-\beta$ is even; and, as remarked in the first part of the present paper, such expression is in fact equal to zero. The demonstration, which is very simple, will be given in a note; but assuming for the moment the truth of the proposition, the coefficient of $(1-\mu)^{m-1}$ is the sum of a finite number of evanescent terms, and it is therefore identically equal to zero.

I consider this demonstration as identical *in principle* with that given by Plana; the same function is, by two processes, different indeed from each other, but which cannot but lead to the same result, developed in an infinite series of positive integer powers of $1-\mu$; and it is shown that the coefficient of each power of $1-\mu$ is equal to zero. But the difficulty I find is that the investigation *proves too much*, viz. it appears to prove that y is actually equal to zero. There are undoubtedly functions such as the function $e^{-\frac{1}{x^2}}$ (noticed by Cauchy and Sir W. R. Hamilton), which *in a sense* have the property in question, viz. that if we attempt to develope them in positive integer powers of x, the coefficients are found to be all of them zero; and it would appear that y is, in regard to $1-\mu$, a function of this nature. But it cannot be asserted *simpliciter* that $e^{-\frac{1}{x^2}}$ and its differential coefficients do in fact vanish for $x=0$; they only vanish for $x=0$ considered as the limit of an indefinitely small *real* positive or negative quantity. (This is quite consistent with a remarkable theorem of Cauchy's, by which it appears *à priori* that $e^{-\frac{1}{x^2}}$ cannot be expanded in positive integer powers of x, because it is discontinuous for the modulus zero.) And if, instead of a direct application of Maclaurin's theorem, we first expand $e^{-\frac{1}{x^2}}$, say in positive powers of $1-x$, and then develope the several terms in powers of x, we obtain for the coefficient of x^0, or any other power of x, an infinite series, which I apprehend is not convergent, and which can only be equal to zero in the same conventional sense in which $e^{-\frac{1}{x^2}}$ is equal to zero for $x=0$. This appears to be something very different from finding for the coefficient of x^0, or of any other power of x, an expression composed of a finite number of finite terms the sum whereof is identically equal to zero.

Plana has given for the calculation of y when μ is nearly equal to 1, an expression (equation (127)) which is deduced from the same development of $Z'p$ which is here made use of; but it appears to me that this expression is, for the following reason, open to objection. The expression referred to contains explicitly positive and integer powers of μ, and also powers of the radical $\sqrt{b^2+2(1-\mu)(1+b)}$: it would be, for anything that appears to the contrary, allowable to develope as well the positive and integer powers of μ as also the powers of the radical in question, in a series of positive and integer powers of $1-\mu$; but if this were done, we should obtain as a mere transformation of Plana's expression (127), an expression for y developed in a series of positive integer powers of $1-\mu$; and for consistency with the beforementioned result, the coefficients of the different powers of $1-\mu$ must be each equal to zero. But if this be so, it does not appear how the original expression (127) can be anything else than zero. The difficulty is, I think, a real one; and I do not see

how it is to be got over: it seems to render necessary a more careful study of the effect of the multiplication of the successive terms of the development of a function fx by Legendre's functions P_0, P_1, P_2, &c., so as to pass from fx to the function of two variables $\phi(\mu, x)$, as well generally as when this transformation is performed upon the as yet imperfectly studied transcendental function Z'.

I remark that the original expression for fx is of the form

$$fx = hb\Sigma_{n_0}^{\infty}\frac{1}{p-qx} - hb\Sigma_{n_0}^{\infty}\frac{1}{p'-q'x};$$

and this gives (Plana's equation (131))

$$y = hb\Sigma_{n_0}^{\infty}\frac{p^2-q^2}{(p^2-2pq\mu+q^2)^{\frac{3}{2}}} - hb\Sigma_{n_0}^{\infty}\frac{p'^2-q'^2}{(p'^2-2p'q'\mu+q'^2)^{\frac{3}{2}}},$$

the values of p, q, p', q' being

$$p = b + n(1+b), \quad p' = (n+1)(1+b),$$
$$q = \quad n(1+b), \quad q' = 1+n\,(1+b);$$

so that

$$p-q = b = p'-q', \text{ and } p'+q' = 2+b+2n(1+b) = p+q+2.$$

Hence, putting $\mu = 1$, we find

$$y = hb\Sigma_{n_0}^{\infty}\left(\frac{p+q}{b^2} - \frac{p'+q')}{b^2}\right) = -\frac{2h}{b^2}\Sigma_{n_0}^{\infty}1 = -\infty,$$

which is inconsistent with the expression $y=0$, deduced from the definite integral. If, however, it is assumed that fx contains the term $\frac{P}{1-x}$, then the corresponding term of y will be

$$\frac{P(1-x^2)}{(1-2\mu x+x^2)^{\frac{3}{2}}},$$

which, when $\mu = 1$, becomes $\frac{P(1+x)}{(1-x)^2}$; and if P be put equal to zero, then it is conceivable that, for $x=1$, $\frac{P}{1-x}$ may be equal to zero, but $\frac{P(1+x)}{(1-x)^2}$, or what will be the same thing, $\frac{2P}{(1-x)^2}$ may be finite or even infinite. This is perhaps the explanation of the apparent contradiction.

Note on the demonstration of the Theorem

$$\frac{\log(1+\Delta)}{\Delta}\{0^\alpha(1+0)^\beta - 0^\beta(1+0)^\alpha\} = 0, \quad \alpha-\beta \text{ even.}$$

Consider the function

$$\frac{e^t(t+z)}{e^{t+z}-1} = \phi(t, z),$$

which, it is clear, admits of expansion in positive integer powers of t and z. Changing the signs of t, z, we have

$$\frac{e^{-t}(-t-z)}{e^{-t-z}-1} = \phi(-t, -z),$$

or, what is the same thing,

$$\frac{e^z(t+z)}{e^{t+z}-1} = \phi(-t, -z),$$

and thence

$$\frac{(e^t-e^z)(t+z)}{e^{t+z}-1} = \phi(t, z) - \phi(-t, -z);$$

so that the development in positive integer powers of t, z, of the function on the right-hand side does not contain any term $t^\alpha z^\beta$ for which $\alpha-\beta$ is even. Writing the function under the form

$$\frac{e^t(t+z)}{e^{t+z}-1} - \frac{e^z(t+z)}{e^{t+z}-1},$$

and considering the two parts separately, then by Herschel's theorem extended to two variables, the coefficient of $t^\alpha z^\beta$ in the first term is

$$\frac{(1+\Delta_1)\log\{(1+\Delta_1)(1+\Delta_2)\}}{(1+\Delta_1)(1+\Delta_2)-1}\, 0_1^\alpha\, 0_2^\beta,$$

which is equal to

$$\frac{\log\{(1+\Delta_1)(1+\Delta_2)\}}{(1+\Delta_1)(1+\Delta_2)-1}(1+0_1)^\alpha\, 0_2^\beta,$$

or, what is the same thing,

$$\frac{\log(1+\Delta)}{\Delta}(1+0)^\alpha\, 0^\beta;$$

and forming in like manner the expression for the coefficient of $t^\alpha z^\beta$ in the second term, this is

$$\frac{\log(1+\Delta)}{\Delta}\, 0^\alpha(1+0)^\beta;$$

the difference of the two expressions therefore vanishes when $\alpha-\beta$ is even, which is the above-mentioned theorem. It would be easy to obtain a variety of similar theorems.

2, *Stone Buildings, W.C., June* 29, 1859.

246.

ON CONTOUR AND SLOPE LINES.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 264—268.]

It is, I think, interesting as a question of topography, to consider the general configuration of a system of contour lines and steepest or slope lines (*lignes de niveau* and *lignes de la plus grande pente*). Imagine, to fix the ideas, a mountainous island, the exterior or sea-level contour line being consequently a closed curve; the case where any contour line is a curve cutting itself is an important one, which will be considered; but disregarding it for the moment, and excluding (as I do throughout) a curve which cuts itself from the notion of a closed curve, the entire contour line corresponding to a given elevation will be either a single closed curve, or it will consist of two or more separate closed curves, in the latter case each of these may be considered as being by itself a contour line, and we may therefore say that the contour line is in general a closed curve. It may happen that the elevation of a given contour line is a maximum or minimum; in other words, that the consecutive curve without the given contour line and that within it are each of them higher or each of them lower than the given contour line; but this is a speciality which need not be particularly attended to; in general the consecutive curve without the given contour line will be lower, and that within it higher than the given contour line, in which case the tract bounded by the contour line is an elevation (hill, table-land, or mountain, as the case may be); or else the consecutive curve without the given contour line is higher, and that within it lower than the given contour line; in which case the tract bounded by the contour line is a depression. But there may be within the contour line bounding an elevation, spaces lower than the bounding line, and within the contour line bounding a depression, spaces higher than the bounding line. A depression usually contains water, and indeed is filled so as to overflow, in which case there is a lake with an outlet; if the depression is not filled to overflowing, the lake will have no outlet. The contour line bounding an elevation may become indefinitely small and ultimately reduce

itself to a point, which is a *summit;* the contour line bounding a depression may in like manner become indefinitely small, and ultimately reduce itself to a point, which is what I call an *immit.* A summit is a point of maximum elevation (though of course there may be summits, or even immits, which are higher); an immit is a point of minimum elevation. But there are besides, as at the heads of passes, points where the surface is horizontal, but where the elevation is neither a maximum nor a minimum; you descend backwards and forwards, but ascend right and left: I will for the present purpose call this kind of point a *knot.* And this leads to the consideration of a contour line which cuts itself: the point where this happens is in fact a knot, or geometrically the knot is a node or double point on the contour line. It may be assumed that the contour line through a knot does not pass through any other knot; for although there may be neighbouring passes of precisely the same elevation, yet the general configuration of the country will not be altered by giving a slight difference of elevation to such passes: the effect of this alteration is to distribute among contour lines of slightly different elevations (one to each line) the different knots which would otherwise occur upon one and the same contour line. The contour line through a knot cuts itself therefore at this point only: such contour line is either a figure of eight, or as I will term it, an *outloop* curve; or else it is the figure formed by the union (so as to give rise to a node or double point) of two closed curves, one of which lies within the other of them; this I call an *inloop* curve. An outloop curve consists of two loops; the spaces within these may also be spoken of as the loops. An inloop curve consists of an outer and an inner loop; the space within the inner loop may be spoken of as the inner loop, that between the two loops as the *lune.* It usually happens, and to fix the ideas I will assume, that for an outloop curve each of the loops is an elevation: this is the case of two mountain summits connected by a ridge or col, the lowest point whereof, or head of the pass, is the knot on the outloop contour line through this point. And in like manner, that for an inloop curve the lune is an elevation, the inner loop a depression; and that the outer loop, considered as a portion of the contour line, is higher than the consecutive exterior contour line. This is the case of a lake having an outlet; if the lake were dry, the passage up stream into the bed of it would be over a ridge, col, or barrier, the lowest point whereof, or point of outlet for the water of the lake, is the knot on the inloop contour line passing through this point, the shore of the lake being of course the inner loop of this contour line, and the waters being retained by means of the raised ground within the lune between the two loops of the contour line.

The slope lines cut at right angles the contour lines; and this property applies also to the projections of the two systems of lines; so that the two systems of lines delineated *in plano* intersect at right angles. Consider the contour lines which are closed curves surrounding a given summit or immit; the exterior contour line is intersected at each of its points by a slope line; and all these slope lines must, it is clear, intersect all the interior contour lines, and ultimately unite at the interior summit or immit. In order to see more distinctly the form of the system of slope lines, it is to be noticed that, if (as is in general the case) the indicatrix at the summit or immit be an ellipse, the contour lines in the immediate neighbourhood thereof will be

a system of similar and similarly situated concentric ellipses, the major and minor axes whereof correspond respectively with the directions of least and greatest curvature; the equation of any orthogonal trajectory of the ellipses, if a, b are the semi-axes, major and minor, of any one of them, is $y^{b^2} = Cx^{a^2}$; and unless $C = \infty$, the curve represented by this equation touches the axis of x, which is the direction of least curvature; if however $C = \infty$, then the equation becomes $x = 0$, and the curve touches the axis of y, which is the direction of greatest curvature. Hence in general at a summit or immit the slope curves all, except one (which is a limiting case) touch the line which is the direction of least curvature. The only exception is when the summit or immit is an umbilicus—the indicatrix is then a circle; the contour lines in the immediate neighbourhood of this point are concentric circles, and the slope lines pass in all directions through the summit or immit.

The indicatrix at a knot is in general a hyperbola, and consequently the contour lines in the neighbourhood of a knot are similar and similarly situated concentric hyperbolas; and if a, b are the semi-axes of one of these hyperbolas, the equation of an orthogonal trajectory is $x^{a^2}y^{b^2} = C$: and when this passes through the knot, $C = 0$; and therefore either $x = 0$ or else $y = 0$; there are consequently through the knot only two slope lines, which bisect the angles made by the two branches of the contour line and intersect each other at right angles. The slope lines through a knot may be termed ridge and course lines: and for one of these—the ridge line—the knot is a point of minimum elevation; for the other of them—the course line—the knot is a point of maximum elevation. But this requires some further development. To fix the ideas, consider the case where the contour line is an outloop curve, the loops being each of them elevations. The slope line through the knot, and which lies within the two loops, would be, according to the definition, a ridge line. Suppose that the contour lines within one of the loops are closed curves surrounding a summit, the ridge line will, it is clear, cut all these curves and ultimately arrive at the summit. But if the contour lines within the loop are not all of them closed curves; if, for instance, they are first closed curves, then an outloop curve, and within each of the loops of this, closed curves surrounding a summit, then it may happen that the above-mentioned ridge line will pass through the knot of the inner outloop curve: and with respect to this knot, it will be, not a ridge line, but a course line; so that the slope line in question cannot be spoken of *simpliciter* either as a ridge line or as a course line, but it is the one or the other *quoad* the knot in reference to which it is considered; and, considered by itself, it can only be spoken of as a ridge-or-course line. The case just referred to is, however, an exceptional one; in general the slope line in question would not pass through the knot of the inner outloop curve, but would cut one of the loops of this curve, and then cutting all the contour lines within such loop, arrive at last at the summit within such loop. And when the ridge line has once arrived at a summit, there is little meaning in continuing it further, and it may be considered as ending there; in fact there are through the summit an infinity of slope lines, all of them (except in the case where the summit is an umbilicus) coincident in direction with the ridge line, and consequently the ridge line may, without graphical discontinuity, be considered as proceeding along any one of these lines indifferently;

and although, when the surface is a geometrical one capable of being represented by an equation, there would be geometrically one of these slope lines which could be identified as the continuation of the ridge line, there would be no advantage in making this identification. Hence it may be considered that in general a ridge line passes from summit to summit, through a single intervening knot which is a point of minimum elevation on the ridge line; and in like manner, that in general a course line passes from immit to immit through a single intervening knot which is a point of maximum elevation on the course line; it need not be considered as an exception when, as is frequently the case, the course line arrives at the sea-level contour line without previously reaching an immit. It is to be noticed that a ridge line or a course line may commence and terminate at one and the same summit or immit, and thus form a closed curve.

The ridge lines, as above defined, determine the watershed. In the case of an isolated conical or dome-shaped mountain, and in general when the contour lines are all of them closed curves, there is no definable watershed; but in the case of a chain of mountain summits, the watershed runs from summit to summit through the heads of the passes over the connecting cols, i.e. it is made up of a series of ridge lines each extending from a summit to a summit through an intervening knot. And the course lines are, as nearly as may be, the beds of the streams which flow from the heads of the passes down the lateral valleys. The ridge line and the course line respectively are, I believe, the so-called *ligne de faîte* and *ligne de thalweg.*

2, *Stone Buildings, W.C., July* 20, 1859.

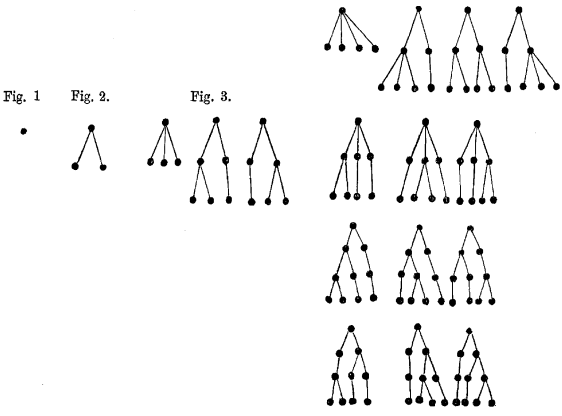
247.

ON THE ANALYTICAL FORMS CALLED TREES. SECOND PART.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 374—378. Continuation of **203**.]

THE following class of "trees" presented itself to me in some researches relating to functional symbols; viz., attending only to the terminal knots, the trees with one knot, two knots, three knots, and four knots respectively are shown in the figures 1, 2, 3 and 4:

Fig. 1 Fig. 2. Fig. 3. Fig. 4.



and similarly for any number of knots. The trees with four knots are formed first from those of one knot by attaching thereto in every possible way (one way only)

four knotted branches; secondly, from those with two knots by attaching thereto in every possible way (three different ways) four knotted branches; and thirdly, from those with three knots by attaching thereto in every possible way (three different ways) four knotted branches,—the original knots of the trees of one knot and two and three knots, being no longer terminal knots, are disregarded. The total numbers of trees with one knot and with two and three knots being respectively 1, 1, 3; the total number of trees with four knots is $1.1+3.1+3.3=13$. And in general, if the number of trees with m knots is ϕm, then it is easy to see that we have

$$\phi m = \phi 1 + \frac{m-1}{1}\phi 2 + \frac{m-1\,.\,m-2}{1\,.\,2}\phi 3 \ldots + \frac{m-1}{1}\phi(m-1);$$

or what is the same thing,

$$2\phi m = \phi 1 + \frac{m-1}{1}\phi 2 + \frac{m-1\,.\,m-2}{1\,.\,2}\phi 3 \ldots + \frac{m-1}{1}\phi(m-1) + \phi m.$$

Hence if

$$u = \phi 1 + \frac{x}{1}\phi 2 + \frac{x^2}{1\,.\,2}\phi 2 + \ldots,$$

we obtain

$$\begin{aligned} e^x\,.\,u = & \quad \phi 1 \\ & + \; x\;(\phi 1 + \phi 2) \\ & + \frac{x^2}{1\,.\,2}(\phi 1 + 2\phi 2 + \phi 3) \\ & + \quad \&c. \\ = & \quad 2\phi 1 - 1 \\ & + \; x^2\,.\,2\phi 2 \\ & + \frac{x^2}{1\,.\,2}\,.\,2\phi 3 \\ & + \quad \&c.; \end{aligned}$$

that is,

$$e^x u = 2u - 1,$$

and thence

$$u = \frac{1}{2-e^x},$$

which gives for ϕm the expression

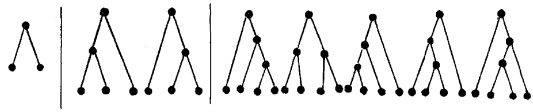
$$\phi m = 1\,.\,2\,.\,3 \ldots (m-1) \text{ coeff. } x^{m-1} \text{ in } \frac{1}{2-e^x};$$

and the value of ϕm might easily be obtained in an explicit form in terms of the differences of the powers of zero. The values of ϕm are, for

$m=$	1,	2,	3,	4,	5,	6,	7,	8, &c.
$\phi m=$	1,	1,	3,	13,	75,	541,	4683,	47293.

In the foregoing problem, the number of branches descending from a non-terminal knot is one, two, or more. But assume that the number of branches descending from a non-terminal knot is always two; so that attending, as before, only to the terminal knots, the trees with two knots, three knots, four knots respectively are shown in the figures, 5, 6, and 7.

Fig. 5. Fig. 6. Fig. 7.



This corresponds to the following problem in the theory of symbols; viz. if A, B, C, D, &c. are symbols capable of successive binary combinations, but do not satisfy the associative law, what is the number of the different significations of the ambiguous expressions ABC, $ABCD$, $ABCDE$, &c. respectively? For instance, AB has only one meaning; ABC may mean either $A.BC$ or $AB.C$. In like manner $ABCD$ may mean $A(B.CD)$, or $AB.CD$, or $(AB.C)D$, or $(A.BC)D$, or $A(BC.D)$; the numbers, 1, 2, 5 being those of the trees in the last three figures respectively; and similarly for any greater number of symbols.

Let ϕm be the required value corresponding to the number m; then we may in any manner whatever separate the number m into two parts m', m'', and then combining *inter se* the m' knots (or symbols) and the m'' knots (or symbols) respectively, ultimately combine the two combinations; hence a part of ϕm is $\phi m'.\phi m''$. The assumed definition of ϕm does not apply to the case $m=1$; but if we write $\phi 1 = 1$, then the foregoing consideration shows that we have

$$\begin{aligned}\phi m = \quad & \phi 1 \phi (m-1) \\ & + \phi 2 \phi (m-2) \\ & \quad \vdots \\ & + \phi (m-1) \phi 1 ;\end{aligned}$$

from which it is easy to calculate

$$\phi 1 = 1,\ \phi 2 = 1,\ \phi 3 = 2,\ \phi 4 = 5,\ \phi 5 = 14,\ \phi 6 = 42,\ \phi 7 = 132,\ \&c.$$

But to obtain the law, consider the generating function

$$u = \phi 1 + \quad x\phi 2 + \qquad x^3\phi 3 + \&c.;$$

we have

$$u^2 = \phi 1\ \phi 1 + x(\phi 1\ \phi 2 + \phi 2\ \phi 1) + x^2(\phi 1\ \phi 3 + \phi 2\ \phi 2 + \phi 3\ \phi 1) + \&c.,$$

which is

$$= \phi 2 + \quad x\phi 3 + \qquad x^2\phi 4 + \&c.;$$

and we have therefore

$$xu^2 = u - 1,$$

and consequently

$$u = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

But

$$\sqrt{1 - 4x} = 1 - \tfrac{1}{2}\,4x + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{1 \,.\, 2}(4x)^2 - \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{1 \,.\, 2 \,.\, 3}(4x)^3 + \&c.$$

$$= 1 - 2x - 2x^2 - 4x^3 - 10x^4 + \&c.,$$

and therefore

$$u = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + 1x + 2x^2 + 5x^3 + \&c.,$$

the series of coefficients 1, 1, 2, 5, &c. agreeing with the values already found. The expression for the general term is at once seen to be

$$\phi m = \frac{1 \,.\, 3 \,.\, 5 \ldots 2m - 3}{1 \,.\, 2 \,.\, 3 \ldots m} 2^{m-1},$$

which is a remarkably simple form.

2, *Stone Buildings, W.C., June* 9, 1859.

248.

SKETCH OF A PROOF OF THE THEOREM THAT EVERY ALGEBRAIC EQUATION HAS A ROOT.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 436—439.]

I HAVE referred to the theorem as usually stated; for it is an easy consequence of the existence of a single root of an equation of any order, that for an equation of the nth order there are n roots: the proof here proposed goes, however, to show directly the existence of the n roots: it is in form a geometrical one, and was suggested to me some months ago by a letter from Prof. De Morgan, containing the remark made in his memoir, "A Proof of the Existence of a Root in every Algebraic Equation," &c. (*Camb. Phil. Trans.* vol. X. 1858), viz. "that the curves $P=0$, $Q=0$, the intersections whereof determine the root-points, are such that two branches, one of each curve, cannot enclose a space." The proof which occurred to me was in character somewhat similar to that given by the Astronomer Royal in the paper, "Suggestion of a Proof of the Theorem that every Algebraic Equation has a Root" (*Camb. Phil. Trans.* vol. X. 1858), and which was suggested to him by Prof. De Morgan's memoir. I have since varied my proof by considering therein cones in the place of plane curves. It will be obvious, upon reading it, that the proof is closely connected with Cauchy's well-known theorem for the number of roots within a given circuit; the circuit being in this case infinity, and the number of roots included within it consequently equal to the order of the equation.

The curve represented by an equation of the nth degree between the coordinates (x, y) is by definition a curve of the nth order; and a cone standing on any such curve (taking the vertex for origin) is represented by a homogeneous equation of the nth degree between the coordinates (x, y, z), and is by definition a cone of the nth order. It is very easy to show that an equation of the nth degree cannot have more than n roots; and we have thence the geometrical theorems, that a curve of the nth

order is not intersected by a line in more than n points, and that a cone of the nth order is not intersected by a plane (I speak throughout of planes through the vertex) in more than n lines. I assume that an algebraic curve is always a continuous curve, viz. that it consists of a branch or branches, no one of which is a *courbe pointillée*, or a branch terminating abruptly in a point; an algebraic cone will be in the like sense a continuous surface. An algebraic curve cannot be an indefinite spiral, for in that case there would be lines meeting it in an infinity of points; and in like manner an algebraic cone cannot be an indefinite spiral surface: an algebraic cone consists, therefore, of a closed sheet or sheets. An algebraic curve may indeed have conjugate or isolated points, and an algebraic cone have conjugate or isolated lines: this is a circumstance which will be adverted to in the sequel. It will fix the ideas as to the general form of an algebraic cone, to remark that it may comprise twin-pair sheets, such as the sheet of a cone of the second order (this is properly spoken of as a twin-pair sheet, each of the two opposite portions of it being called, for distinction, a twin-sheet); and of single sheets, such as one at least of the sheets of a cone of the third or any other odd order (see the annexed "Note upon Cones of the Third Order," [249]). The advantage of the consideration of cones instead of plane curves, is that we have only closed sheets, and thus get rid of the distinction which exists for plane curves between infinite branches and the branches which are closed curves.

My proof depends on the following lemma, viz. "Consider two algebraic cones with the same vertex, each of them of the order n; then *if* there be some one plane meeting the first cone in n lines, and the second cone in n lines, such that the lines of each set occur alternately, the two cones intersect in at least n lines."

The truth of this lemma is, I conceive, a matter of intuition, depending only on the notion of the continuity of the sheets of the surface. Thus, if we have *in plano*, through a point O, the lines A, A' and B, B', such that, A, α being opposite points on the same line, and so for the other lines, the order round O is A, B, A', B', α, β, α', β', it is obvious that we cannot through the lines A, A' draw a cone, and through the lines B, B' draw a cone, without making these cones intersect in at least two lines: and in like manner for two sets, each of n lines. I have, in the enunciation of the lemma, said that the cones are each of them of the order n; this was necessary in order to exclude a case which might otherwise have happened, viz. a line of intersection of the plane with either of the cones might have been a conjugate or isolated line without any sheet through it; and if this were so, we could not infer the existence of the n lines of intersection of the two cones. But if a plane meet an algebraic cone of the nth order in n lines, no one of these can be a conjugate or isolated line; for such line is to be considered as two or more coincident lines, and there would be in all more than n lines of intersection of the plane and cone.

Consider now the equation

$$\phi u = 0,$$

where ϕu is a rational and integral function of u with (in general) imaginary coefficients, and write

$$\phi(x+y\sqrt{-1})=P+Q\sqrt{-1},$$

P, Q being real functions of (x, y), each of them of the degree n; if (x, y) are rectangular coordinates, then $P=0$, $Q=0$ are real curves each of the order n. And to each point of intersection of the two curves there corresponds a root of the equation. The two curves do not intersect in more than n points (for if they did, the equation $\phi u=0$ would have more than n roots); hence if it be shown that the two curves intersect in at least n points, they will intersect in precisely n points, and the equation will have n roots. Take any point as the common vertex of two cones standing upon the curves $P=0$, $Q=0$ respectively; each point of intersection of the two curves corresponds to a line of intersection of the two cones, and it is only necessary to show that the two cones intersect in at least n lines Take for the vertex a point in the perpendicular at the origin of (x, y) to the plane of the two curves, and at a distance unity from such origin, viz. a point such that, treating it as the origin of the coordinates (x, y, z), the coordinates in respect thereto of the origin (x, y) are $x=0$, $y=0$, $z=1$. The equations of the cones are at once deduced from those of the curves by writing therein $\left(\frac{x}{z}, \frac{y}{z}\right)$ in the place of (x, y) and, to render the equation integral, multiplying by z^n; or if $P'=0$, $Q'=0$ are the equations of the cones, we have

$$z^n\phi\left(\frac{x+y\sqrt{-1}}{z}\right)=P'+Q'\sqrt{-1}.$$

Consider the section by the plane through the vertex parallel to the plane of the two curves: the equation of this plane is $z=0$; and it is clear that, to obtain the intersections of this cone with the plane in question, we have only in $\phi\left(\frac{x+y\sqrt{-1}}{z}\right)$ to disregard all the terms after the first. Suppose that

$$\phi u=(a+b\sqrt{-1})\,u^n+\&c.;$$

then putting

$$(a+b\sqrt{-1})(x+y\sqrt{-1})^n=P_0'+Q_0'\sqrt{-1},$$

the equations $P_0'=0$, $Q_0'=0$ determine the intersections of the plane $z=0$ with the cones $P=0$, $Q=0$ respectively. But writing

$$a+b\sqrt{-1}=A(\cos\alpha+\sqrt{-1}\sin\alpha),$$

$$x+y\sqrt{-1}=r(\cos\theta+\sqrt{-1}\sin\theta),$$

we have

$$Ar^n\{\cos(n\theta+\alpha)+\sqrt{-1}\sin(n\theta+\alpha)\}=P_0'+Q_0'\sqrt{-1},$$

so that

$$P_0' = Ar^n \cos(n\theta + \alpha), \quad Q_0' = Ar^n \sin(n\theta + \alpha),$$

or the intersections with the cone $P = 0$ are the n lines given in direction by the equation

$$n\theta + \alpha = (m + \tfrac{1}{2})\pi,$$

and the intersections with the cone $Q = 0$ are the n lines given in direction by the equation

$$n\theta + \alpha = m\pi;$$

in each of which equations m is any integer number from 0 to $n-1$. Hence the plane $z = 0$ meets the cones in two sets of lines succeeding each other alternately, as required by the lemma, and the two cones intersect in at least n lines. And it is thus shown that the given equation of the nth degree has n roots.

2, *Stone Buildings, W.C., September* 26, 1859.

249.

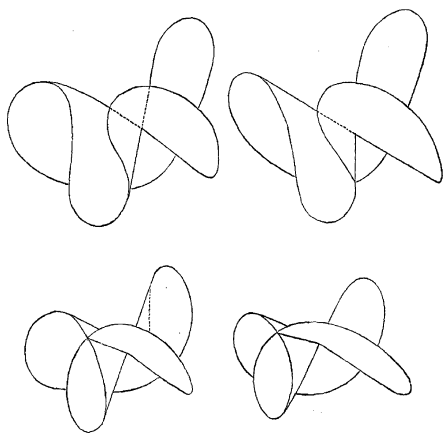
NOTE ON CONES OF THE THIRD ORDER.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 439—442.]

THE distinction adverted to in the preceding paper between the twin-pair sheets and single sheets of an algebraic cone is made (with respect to spherical curves, which is the same thing) by Möbius, in the interesting Memoir "Ueber die Grundformen der Linien der dritten Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, vol. I. (1849) [and *Werke*, vol. II. pp. 86—176]. Consider the generating line POP' of a cone, vertex O, and let pOp' be any position of this line, the points P, P', and in like manner the points p, p', being on opposite sides of the vertex; then if OP originally coincides with Op (and therefore OP' with Op'), and if, in the course of the generation of the surface, OP (without having first come to coincide with Op') comes to coincide with Op, at the same time OP' (without having first come to coincide with Op) will come to coincide with Op', and we have a *twin-pair sheet*, viz. one twin-sheet generated by OP, and the other twin-sheet generated by OP'. This is the ordinary case of a cone of the second order, and requires no further explanation. It is proper to remark that, for cones of superior orders, the conical angle of each twin-sheet is not (as for a cone of the second order) necessarily less than 360°. But suppose that OP, starting from the position Op, and before it again comes to coincide therewith, comes to coincide with Op', then at the same time OP' (without having first come again to coincide with Op') will come to coincide with Op; the generation is here complete, and we have a *single sheet*, which, if the motion were continued until OP came to coincide with Op, would only be generated over again. The conical angle of a single sheet is necessarily greater than 360°; for OP in coming to coincide with Op' must describe an angle greater than 180°, and OP' describing an equal angle, the entire angle is therefore greater than 360°; in the limiting case, where the entire angle is precisely 360°, the conical surface is a plane. It is easy to cut out in paper and join together two sectors of a circle so as to form therewith a sector the angle whereof exceeds 360°; such a sector can then, by joining

together the two radial edges, be converted into a cone of a single sheet; the generating lines being all finite lines equal in length, the curve formed by the circular edge is, it is clear, the spherical curve which is the intersection of the cone by a concentric sphere. It is shown by Möbius (stating his result with respect to cones instead of spherical curves) that a cone of an odd order must have at least one single sheet; a cone of the third order consists (1) of a single sheet, or else (2) of a single sheet and a twin-pair sheet. These are the two general forms of cones of the third order. But there are two special forms and one subspecial form, making in all five forms: viz., the two special forms are, (3) the cone has a nodal line; (4) the cone has an isolated line; and the subspecial form is, (5) the cone has a cuspidal line. The relation of the different forms may be explained as follows.

Starting from the form (1), as the constants of the equation change, the cone gathers itself up together so as to have a nodal line; this is the form (3). The loops of this form then detach themselves so as to form a twin-pair sheet, the remaining part of the surface reverting to a form similar to that of (1); we have thus a single sheet and twin-pair sheet, which is the form (2). The twin-pair sheet then dwindles away into an isolated line, giving the form (4); and lastly, the isolated line disappears and the cone resumes the form (1): these four forms constitute, there-



fore, a complete cycle. The constants may be such that the loop of the form (3) is evanescent, or, what is the same thing, that the forms (3) and (4) arise simultaneously; there is in this case a cuspidal line, or we have the form (5). It may be added that for the general forms (1) and (2) there are always three lines of inflexion. This is also the case with the form (4), where there is an isolated line; but in the form (3),

where there is a nodal line, there is but one line of inflexion; and in the form (5), where there is a cuspidal line, there is not any line of inflexion: the equivalent theorem for the spherical curves is given by Möbius [1]. It was remarked long ago by Sir I. Newton, that all curves of the third order could be generated as the shadows of the five cubical parabolas; these are, in fact, sections in a particular manner of the above-mentioned five forms of cones of the third order: the existence of five essentially distinct forms of cones of the third order is noticed by M. Chasles in the *Aperçu Historique*, 1837. The analytical distinction between the forms (1) and (2) depends on the sign of the function $1-\frac{64S^3}{T^2}$, where S, T are the quartinvariant and sextinvariant of the cubic form. I annex stereoscopic representations of the cones of the third order of the general form (1), and of the form with a nodal line (3). The generating lines are finite lines of equal length, and the curved contours shown in the figures are consequently the spherical curves which are the intersections of the cones by concentric spheres. The figures are intended to be looked at with the glasses of a Reeves's book stereoscope.

[1] It is hardly necessary to mention that, according to the general theory of cones of the third order, there are always nine lines of inflexion,—three real and six imaginary. Six of the lines of inflexion disappear when there is a double line, viz., in the case of a nodal line, two real and four imaginary lines of inflexion; but in the case of an isolated line, the six imaginary lines of inflexion. When there is a cuspidal line, eight lines of inflexion, viz. two real lines and the six imaginary lines, disappear.

2, *Stone Buildings, W.C., September* 26, 1859.

250.

SUR LA SURFACE QUI EST L'ENVELOPPE DES PLANS CONDUITS PAR LES POINTS D'UN ELLIPSOIDE PERPENDICULAIREMENT AUX RAYONS MENÉS PAR LE CENTRE.

[From the *Annali di Matematica pura ed applicata* (Tortolini), tom. II. (1859), pp. 3—14.]

La considération de la surface dont il s'agit me fut suggérée, il y a quelques années par le Prof. Stokes, mais il convient de remarquer que la courbe enveloppe des droites menées par les points d'une ellipse perpendiculaires aux rayons conduits par le centre, est mentionnée en passant dans le mémoire de M. Tortolini "Sulle relazioni ec." *Annali*, tom. VI. pp. 433—446, 1855 (voir p. 461), où il trouve que l'équation de la courbe est

$$\begin{aligned}&[4(a^4+b^4-a^2b^2)-3(a^2x^2+b^2y^2)]^2\\&\qquad=[9a^2(2b^2-a^2)x^2+9b^2(2a^2-b^2)y^2-4(a^2+b^2)(2a^2-b^2)(2b^2-a^2)]^2,\end{aligned}$$

équation qui était trouvée en égalant à zéro le discriminant d'une fonction quadratique. L'auteur remarque que cette équation fut trouvée par lui en 1846 dans la *Raccolta Scientifica di Roma*, et il remarque que la courbe est connue sous le nom de la courbe de Talbot.

Selon ma méthode l'équation de la courbe est trouvée en égalant à zéro le discriminant d'une fonction cubique, et l'équation de la surface est trouvée en égalant à zéro le discriminant d'une fonction biquadratique. Comme on a besoin de la courbe pour la discussion de la surface, je commence par la considération de la courbe.

On voit sans peine qu'en s'imaginant un cercle passant par le centre de l'ellipse et touchant l'ellipse à l'un quelconque de ses points, le point correspondant de la courbe est situé à l'extrémité du diamètre du cercle, lequel diamètre passe par le centre de

l'ellipse, ou ce qui est la même chose (prenant pour origine le centre de l'ellipse) les coordonnées du point de la courbe sont respectivement les doubles des coordonnées du centre du cercle. Prenez X, Y pour les coordonnées du point de l'ellipse, et x, y pour celles du point de la courbe, l'équation de l'ellipse sera

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

et, de la construction géométrique dont je viens de parler, on obtient sans peine

$$x=X\left(2-\frac{1}{a^2}(X^2+Y^2)\right),\qquad y=Y\left(2-\frac{1}{b^2}(X^2+Y^2)\right)$$

lesquelles sont les expressions de x, y en X, Y.

Ces expressions font voir que les points selon lesquels l'ellipse est coupée par le cercle $X^2+Y^2=2a^2$ (c'est-à-dire les points

$$(b^2-a^2)X^2=a^2(b^2-2a^2),\qquad (b^2-a^2)Y^2=a^2b^2,$$

qui sont des points imaginaires de l'ellipse) donnent pour la courbe des points doubles sur l'axe des Y (ces points sont toujours imaginaires) et de même les points selon lesquels l'ellipse est coupée par le cercle $X^2+Y^2=2b^2$ (c'est-à-dire les points

$$(a^2-b^2)X^2=a^2b^2,\qquad (a^2-b^2)Y^2=b^2(a^2-2b^2),$$

qui sont des points réels de l'ellipse en supposant $a^2>2b^2$) donnent pour la courbe des points doubles sur l'axe des X. Les coordonnées de ces points doubles sont données par $a^2x^2=4b^2(a^2-b^2)$, $y^2=0$, et elles sont ainsi réelles, soit pour $a^2<2b^2$, soit pour $a^2>2b^2$, mais dans le premier cas les points doubles sont des points conjugués.

En formant les équations

$$dx=\left(2-\frac{1}{a^2}(3X^2+Y^2)\right)dX-\frac{2XY}{a^2}dY=0,$$

$$dy=-\frac{2XY}{b^2}dX+\left(2-\frac{1}{b^2}(X^2+3Y^2)\right)dY=0,$$

et en éliminant dX, dY, on obtient l'équation

$$\left(2-\frac{1}{a^2}(3X^2+Y^2)\right)\left(2-\frac{1}{b^2}(X^2+3Y^2)\right)-\frac{4X^2Y^2}{a^2b^2}=0,$$

laquelle au moyen de l'équation de l'ellipse se réduit à

$$-2(X^2+Y^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+3(X^2+Y^2)^2\frac{1}{a^2b^2}=0.$$

On voit que les points de l'ellipse lesquels donnent lieu à des points de rebroussement de la courbe doivent être situés sur la courbe donnée par l'équation dernièrement mentionnée. Cette équation se divise en deux facteurs: l'équation $X^2+Y^2=0$ combinée avec l'équation de l'ellipse donne les points $X^2=-\dfrac{a^2b^2}{a^2-b^2}$, $Y^2=\dfrac{a^2b^2}{a^2-b^2}$, lesquels sont des points imaginaires de l'ellipse; mais on peut se convaincre sans peine que ces points *ne donnent pas lieu* à des points de rebroussement de la courbe. L'autre facteur, savoir $X^2=Y^2=\frac{2}{3}(a^2+b^2)$, combiné avec l'équation de l'ellipse donne les points

$$(a^2-b^2)\,X^2=\tfrac{1}{3}a^2(2a^2-b^2),\qquad (a^2-b^2)\,Y^2=\tfrac{1}{3}b^2(a^2-2b^2),$$

lesquels pour $a^2>2b^2$ sont des points réels de l'ellipse, et qui donnent lieu à des points de rebroussement de la courbe. On trouve alors pour les coordonnées des points de rebroussement

$$(a^2-b^2)\,a^2x^2=\tfrac{4}{27}(2a^2-b^2)^3,\qquad (b^2-a^2)\,b^2y^2=\tfrac{4}{27}(2b^2-a^2)^3$$

lesquels sont aussi des points réels pour $a^2>2b^2$.

On trouve sans peine les points dans lesquels la courbe coupe l'ellipse; en effet ces points se dérivent des points où l'ellipse est coupée par le cercle

$$X^2+Y^2=\frac{3a^2b^2}{a^2+b^2},$$

lesquels sont des points réels de l'ellipse si $a^2>2b^2$, et les coordonnées sont données par

$$(a^4-b^4)\,X^2=a^2b^2(2a^2-b^2),\qquad (b^4-a^4)\,Y^2=a^2b^2(2b^2-a^2),$$

de là on obtient, pour les coordonnées des points où l'ellipse est coupée par la courbe,

$$(a^2+b^2)^3(a^2-b^2)\,x^2=a^2b^2(2a^2-b^2)^3,\qquad (a^2+b^2)^3(b^2-a^2)\,y^2=a^2b^2(2b^2-a^2)^3,$$

lesquels sont de même des points réels si $a^2>2b^2$. A présent il est facile d'apercevoir la forme de la courbe; en effet on voit d'abord que la courbe est symétrique par rapport à chacune des deux axes et qu'elle touche l'ellipse aux extrémités des deux axes: pendant que $a^2<2b^2$ (c'est-à-dire pendant que l'ellipse n'est pas trop excentrique) la courbe est une ovale située entièrement en dedans de l'ellipse. Seulement il y a deux points conjugués sur l'axe de x. En supposant $a^2=2b^2$, on n'a plus les deux points conjugués, mais l'apparence de la courbe n'est pas autrement changée: cependant les points aux extrémités de l'axe majeur de l'ellipse sont ici des points singuliers d'une espèce particulière; car chacun de ces points est formé par l'union et l'amalgamation de deux points de rebroussement et d'un point double. Mais, pour $a^2>2b^2$, la forme de la courbe est entièrement changée: aux extrémités de l'axe majeur, la courbe est située en dehors de l'ellipse, avec sa convexité dans le sens opposé à celle de l'ellipse; la courbe va de chaque côté jusqu'à un point de rebroussement où elle change de direction, et puis elle coupe l'ellipse, coupe aussi l'axe majeur dans un point

en dedans de l'ellipse (ce point est l'intersection de deux branches de la courbe, et ainsi un point double), et enfin la courbe arrive à l'extrémité de l'axe mineur où elle se réunit avec la branche située de l'autre côté de l'axe mineur; la construction géométrique par points donne le moyen de tracer la courbe avec exactitude; et dans le cas $a^2 > 2b^2$ on pourrait aussi par les valeurs ci-devant données pour les coordonnées construire les points doubles, et les points de rebroussement. Pour trouver l'équation de la courbe il faut éliminer X, Y entre l'équation de l'ellipse, et les équations trouvées pour x, y; pour faire cela j'écris $X^2 + Y^2 = \theta$, on a

$$x = X\left(2 - \frac{\theta}{a^2}\right), \qquad y = Y\left(2 - \frac{\theta}{b^2}\right),$$

et l'équation $X^2 + Y^2 = \theta$ et l'équation de l'ellipse donnent alors

$$\theta = \frac{x^2}{\left(2 - \frac{\theta}{a^2}\right)^2} + \frac{y^2}{\left(2 - \frac{\theta}{b^2}\right)^2}, \qquad 1 = \frac{x^2}{a^2\left(2 - \frac{\theta}{a^2}\right)^2} + \frac{y^2}{b^2\left(2 - \frac{\theta}{b^2}\right)^2},$$

entre lesquelles il faut éliminer θ; en multipliant la première équation par 2, et la seconde équation par $-\theta$ et en ajoutant, on trouve

$$\theta = \frac{x^2}{2 - \frac{\theta}{a^2}} + \frac{y^2}{2 - \frac{\theta}{b^2}},$$

et la seconde équation est l'équation dérivée de celle-ci par rapport à θ; c'est-à-dire l'équation de la courbe sera trouvée en éliminant θ entre cette équation et sa dérivée par rapport à θ: en écartant les dénominateurs, l'équation devient

$$\theta^3 - \theta^2(2a^2 + 2b^2) + \theta(a^2x^2 + b^2y^2 + 4a^2b^2) - 2a^2b^2(x^2 + y^2) = 0,$$

et le discriminant de cette fonction cubique de θ, égalé à zéro donne l'équation de la courbe.

Je représente la fonction cubique par

$$(A, B, C, D \between \theta, 1)^3,$$

de manière que

$$\begin{aligned} A &= 1, \\ B &= -\tfrac{2}{3}(a^2 + b^2), \\ C &= \tfrac{1}{3}(a^2x^2 + b^2y^2 + 4a^2b^2), \\ D &= -2a^2b^2(x^2 + y^2). \end{aligned}$$

En représentant le discriminant par $\square$, dans le cas actuel où $A = 1$, il sera convenable de se servir de la forme

$$A^2\square = 4(AC - B^2)^3 - (3ABC - A^2D - 2B^3)^2 = 0;$$

l'on a

$$AC - B^2 = \tfrac{1}{9}(3a^2x^2 + 3b^2y^2 - 4a^4 + 4a^2b^2 - 4b^4),$$

$$AD - BC = \tfrac{2}{9}[(a^2 - 8b^2)a^2x^2 + (b^2 - 8a^2)b^2y^2 + 4(a^2 + b^2)a^2b^2],$$

$$BD - C^2 = -\tfrac{1}{9}[a^4x^4 + 2a^2b^2x^2y^2 + b^2y^4 - 4(a^2 + 3b^2)a^2b^2x^2 - 4(b^2 + 3a^2)a^2b^2y^2 + 16a^4b^4],$$

et de là

$$3ABC - A^2D - 2B^3 = 2B(AC - B^2) - A(AD - BC)$$
$$= -\tfrac{2}{27}[9(a^2 - 2b^2)a^2x^2 + 9(b^2 - 2a^2)b^2y^2 - 8a^6 + 12a^4b^2 + 12a^2b^4 - 8b^6],$$

et l'équation de la courbe est ainsi trouvée sous la forme

$$(3a^2x^2 + 3b^2y^2 - 4a^4 + 4a^2b^2 - 4b^4)^3$$
$$+ [9(a^2 - 2b^2)a^2x^2 + 9(b^2 - 2a^2)b^2y^2 - 8a^6 + 12a^4b^2 + 12a^2b^4 - 8b^6]^2 = 0$$

laquelle est en effet la forme ci-devant mentionnée; j'ajoute que l'on trouverait les expressions déjà données pour les coordonnées des points de rebroussement en égalant à zéro les deux fonctions quadratiques qui entrent dans l'équation.

Je considère à présent la surface qui est l'enveloppe des plans menés par les points d'un ellipsoïde perpendiculairement aux rayons conduits par le centre. En s'imaginant une sphère qui passe par le centre de l'ellipsoïde et touche l'ellipsoïde dans l'un quelconque de ses points, le point correspondant de la surface sera à l'extrémité d'un diamètre de la sphère, lequel diamètre passe par le centre de l'ellipsoïde: ou ce qui est la même chose (prenant le centre de l'ellipsoïde pour origine) les coordonnées du point de la surface sont respectivement les doubles des coordonnées du centre de la sphère. Prenez X, Y, Z pour les coordonnées du point de l'ellipsoïde, et x, y, z pour les coordonnées du point de la surface, l'équation de l'ellipsoïde sera

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1,$$

et la construction qui vient d'être mentionnée donne

$$x = X\left[2 - \frac{1}{a^2}(X^2 + Y^2 + Z^2)\right], \quad y = Y\left[2 - \frac{1}{b^2}(X^2 + Y^2 + Z^2)\right], \quad z = Z\left[2 - \frac{1}{c^2}(X^2 + Y^2 + Z^2)\right],$$

pour les expressions de x, y, z en fonction de X, Y, Z.

La discussion de la surface peut être effectuée au moyen de ces expressions. Il y a un assez grand nombre de différentes formes de la surface; mais le seul cas que je vais considérer (lequel apparemment est celui qui présente le plus grand nombre des singularités réelles) est le cas où $a^2 > 2b^2$, $b^2 > 2c^2$; en effet dans ce cas, les courbes du sixième ordre, ou sextiques, correspondantes aux sections principales de l'ellipsoïde ont chacune des points doubles et des points de rebroussement.

Les courbes selon lesquelles l'ellipsoïde est coupé par les sphères concentriques

$$X^2+Y^2+Z^2=2a^2, \qquad X^2+Y^2+Z^2=2b^2, \qquad X^2+Y^2+Z^2=c^2,$$

donnent lieu dans la surface à des courbes *nodales* (courbes doubles) dans les plans yz, zx, xy respectivement. La première de ces courbes d'intersection, et la courbe nodale dans le plan de zy qui y correspond, sont l'une et l'autre imaginaires: les deux autres courbes d'intersection, et les courbes nodales dans les plans de zx et xy qui y correspondent, sont réelles; la courbe selon laquelle l'ellipsoïde est coupé par la sphère

$$X^2+Y^2+Z^2=2b^2,$$

laquelle courbe est une espèce d'ovale autour de l'extrémité de l'axe le plus grand de l'ellipsoïde (il va sans dire que les ovales dont je parle sont des courbes à double courbure), cette courbe d'intersection, je dis, donne lieu à une courbe nodale hyperbolique sur le plan de ZX. Pour en trouver l'équation on a

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1, \qquad X^2+Y^2+Z^2=2b^2,$$

$$x=2X\left(1-\frac{b^2}{a^2}\right), \qquad y=0, \qquad z=2Z\left(1-\frac{b^2}{c^2}\right),$$

et de là, en éliminant X, Y, Z, on trouve

$$\frac{a^2x^2}{4b^2(a^2-b^2)}-\frac{c^2z^2}{4b^2(b^2-c^2)}=1$$

pour l'équation de la courbe nodale hyperbolique dans le plan de zx; on s'assure sans peine que cette courbe passe par les points doubles de la courbe sextique dans le plan de xy.

De même la courbe selon laquelle l'ellipsoïde est coupé par le sphère

$$X^2+Y^2+Z^2=2c^2,$$

laquelle courbe est une espèce d'ovale autour de l'extrémité de l'axe le plus petit, donne lieu à une courbe nodale elliptique dans le plan de xy, et l'on a d'une manière semblable

$$\frac{a^2x^2}{4c^2(a^2-c^2)}+\frac{b^2y^2}{4c^2(b^2-c^2)}=1$$

pour l'équation de cette courbe nodale elliptique dans le plan de xy: cette courbe passe par les points doubles des courbes sextiques dans les plans de yz et zy respectivement. La section de la surface par un plan principal de l'ellipsoïde est composée de la courbe sextique donnée par la section principale de l'ellipsoïde et de la courbe nodale (ellipse hyperbole, ou conique imaginaire) considérée comme deux coniques coïncidentes; les sections principales de la surface sont ainsi des courbes de l'ordre 10, et (ce qui sera montré plus complètement dans la suite) la surface elle-même est de l'ordre 10.

A présent je vais chercher la courbe sur l'ellipsoïde laquelle donne lieu à une courbe *cuspidale* (courbe de rebroussement) sur la surface. Pour cela je forme les équations:

$$dx = \left\{2 - \frac{1}{a^2}(3X^2 + Y^2 + Z^2)\right\} dX \qquad - \frac{2XY}{a^2} dY \qquad - \frac{2XZ}{a^2} dZ = 0,$$

$$dy = \qquad - \frac{2YX}{b^2} dX + \left\{2 - \frac{1}{b^2}(X^2 + 3Y^2 + Z^2)\right\} dY \qquad - \frac{2YZ}{b^2} dZ = 0,$$

$$dz = \qquad - \frac{2ZX}{c^2} dX \qquad - \frac{2ZY}{c^2} dY + \left\{2 - \frac{1}{c^2}(X^2 + Y^2 + Z^2)\right\} dZ = 0,$$

et en éliminant dX, dY, dZ et réduisant au moyen de l'équation de l'ellipsoïde on trouve

$$4\left(\frac{X^2}{a^4} + \frac{Y^2}{b^4} + \frac{Z^2}{c^4}\right) - 2(X^2 + Y^2 + Z^2)\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right) + 3(X^2 + Y^2 + Z^2)^2 \frac{1}{a^2b^2c^2} = 0,$$

laquelle est l'équation d'une surface du quatrième ordre qui coupe l'ellipsoïde suivant une courbe qui donne lieu à une courbe cuspidale sur la surface. La courbe sur l'ellipsoïde rencontre chacune des sections principales dans les points qui donnent lieu à la courbe sextique qui correspond à la section principale, et dans les points selon lesquels la section principale est coupée par la courbe qui donne lieu à une courbe nodale: ainsi les points d'intersection de la surface du quatrième ordre avec la section principale

$$\frac{X^2}{a^2} + \frac{Z^2}{c^2} = 1,\ Y = 0,$$

sont les points d'intersection de cette ellipse avec les courbes

$$X^2 + Z^2 = \tfrac{2}{3}(a^2 + c^2) \text{ et } X^2 + Z^2 = 2b^2,$$

lesquels points sont réels; les points d'intersection avec la section principale

$$\frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1,\ X = 0,$$

sont les points d'intersection de cette ellipse avec le cercle

$$Y^2 + Z^2 = \tfrac{2}{3}(b^2 + c^2),$$

lesquels points sont réels, et avec le cercle

$$Y^2 + Z^2 = 2a^2,$$

lesquels points sont imaginaires; les points d'intersection avec la section principale

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,\ Z = 0,$$

sont les points d'intersection de cette ellipse avec le cercle

$$X^2 + Y^2 = \tfrac{2}{3}(a^2 + b^2),$$

lesquels points sont réels, et avec le cercle

$$X^2 + Y^2 = 2c^2,$$

lesquels points sont imaginaires. La courbe sur l'ellipsoïde qui donne lieu à la courbe cuspidale rencontre les courbes sur l'ellipsoïde qui donnent lieu aux courbes nodales, seulement dans les points où ces dernières courbes sont rencontrées par les sections principales respectivement; et les points où cela arrive sont des points de contact des courbes dont il s'agit; les seuls points réels de contact sont les points où la section principale

$$\frac{X^2}{a^2} + \frac{Z^2}{c^2} = 1, \quad Y = 0,$$

est coupée par le cercle $X^2 + Z^2 = 2b^2$. Pour fixer les idées je suppose $a^2 + b^2 > 3b^2$, on a alors $2b^2 < \frac{2}{3}(a^2 + b^2)$ et les points dont je viens de parler sont situés plus près des extrémités de l'axe le plus petit que ne sont les points où cette même ellipse est coupée par le cercle

$$X^2 + Z^2 = \tfrac{2}{3}(a^2 + c^2).$$

La courbe sur l'ellipsoïde que donne lieu à la courbe cuspidale est composée de deux paires d'ovales: les ovales de l'une de ces paires sont situés autour des extrémités de l'axe le plus petit, et les ovales de l'autre paire autour des extrémités de l'axe le plus grand; et par la remarque déjà faite, les ovales de la première paire touchent la courbe sur l'ellipsoïde qui donnent lieu à la courbe hyperbolique nodale, les coordonnées des points de contact étant données par

$$\frac{X^2}{a^2} + \frac{Z^2}{c^2} = 1, \quad X^2 + Y^2 = 2b^2, \quad Y = 0,$$

les coordonnées des points correspondants sur la surface sont données par

$$x^2 = \frac{4(a^2 - b^2)(2b^2 - c^2)}{a^2(a^2 - c^2)}, \qquad y^2 = \frac{4(b^2 - c^2)(a^2 - 2b^2)}{c^2(a^2 - c^2)}.$$

Nous avons jusqu'ici parlé de la courbe nodale hyperbolique comme étant une hyperbole complète, mais il convient de remarquer que la courbe réelle entière sur l'ellipsoïde ne donne lieu qu'à des parties finies de cette hyperbole, savoir les parties finies comprises entre les points dont je viens de donner les coordonnées, et que pour ces parties de l'hyperbole la courbe nodale est une courbe nodale proprement dite, savoir il y a pour ces parties deux nappes réelles, mais pour les autres parties de l'hyperbole il n'y a aucune nappe réelle de la surface, ou autrement dit, la courbe nodale est une courbe conjuguée: les points dont il s'agit (lesquels pour plus de commodité je vais appeler les *points critiques*[1]) seront évidemment des points cuspidaux sur la courbe nodale (c'est-à-dire des points où les deux plans tangents deviennent identiques). Mais ces points sont ici des points d'une singularité plus élevée, savoir ils sont des

[1] Dans le mémoire anglais j'ai dit "points of cesser" ce qui était un terme plus expressif. A. C.

points de rebroussement sur la courbe cuspidale, et de plus la courbe nodale hyperbolique, la courbe cuspidale, et la courbe sextique dans le plan zx ont à chacun de ces points une tangente commune. Car, comme je l'ai auparavant mentionné, les courbes sur l'ellipsoïde qui donnent lieu à la courbe nodale hyperbolique et la courbe cuspidale respectivement, se touchent l'une l'autre aux points qui donnent lieu aux points critiques; de là il suit que la courbe nodale hyperbolique et la courbe cuspidale se touchent l'une l'autre aux points critiques, et par cela seulement que la tangente de la courbe cuspidale est, à l'un quelconque des points critiques, dans le plan de zx, (puisque la courbe cuspidale est symétrique par rapport à ce plan) on voit que le point critique sera un point de rebroussement sur la courbe cuspidale. Il ne reste qu'à montrer que la courbe nodale hyperbolique et la courbe sextique ont une tangente commune au point critique. La tangente de la courbe nodale hyperbolique est donnée par l'équation

$$\frac{a^2x\,dx}{a^2-b^2}-\frac{c^2z\,dz}{b^2-c^2}=0,$$

où

$$x=X\left[2-\frac{1}{a^2}(X^2+Y^2+Z^2)\right],\quad z=Z\left[2-\frac{1}{c^2}(X^2+Y^2+Z^2)\right],$$

et

$$\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1,\quad X^2+Z^2=2b^2,\quad Y=0,$$

c'est-à-dire que l'on a

$$x=\frac{2X}{a^2}(a^2-b^2),\qquad z=-\frac{2Z}{c^2}(b^2-c^2)$$

et l'équation devient ainsi $Xdx+Zdz=0$, ce qui fait voir que la tangente est perpendiculaire au rayon mené par le centre de l'ellipsoïde au point qui donne lieu au point critique, et ainsi la tangente de la courbe nodale hyperbolique coïncide avec celle de la courbe sextique dans le plan de zx. Et l'on peut à présent expliquer plus particulièrement la forme de la courbe cuspidale (le cas dont il s'agit est celui où l'on a non seulement $a^2>2b^2$, $b^2>2c^2$ mais aussi $a^2+c^2>3b^2$) en effet la courbe cuspidale est composée en premier lieu de deux ovales en forme d'œil situés symétriquement de chaque côté du plan de xy, et qui passent par les points critiques, et les points de rebroussement de la courbe sextique dans le plan de yz, et en second lieu de deux ovales ordinaires situés symétriquement de chaque côté du plan de yz, et qui passent par les points de rebroussement des courbes sextiques dans les plans de xy et zx respectivement: tous les ovales dont il s'agit étant, il va sans dire, des courbes à double courbure. Les relations des différentes courbes sur l'ellipsoïde seront mieux comprises au moyen de la figure 1ª, où les sections principales de l'ellipsoïde sont tracées, non pas en perspective, mais développées sur le plan de la figure; et les courbes qui donnent lieu aux courbes nodales et cuspidales respectivement sont seulement marquées par des courbes menées par les points dans les sections principales par lesquels points passent respectivement les courbes dont il s'agit. On aura une idée de la forme de la surface au moyen de la figure 2ª (laquelle est dessinée en perspective orthogonale au moyen de la première figure) et qui fait voir la forme des

sections principales de la surface (y compris les courbes nodales) et la forme de la courbe cuspidale. Les numéros et lettres dans les deux figures font voir quels points des différentes courbes sur l'ellipsoïde donnent lieu respectivement aux points des différentes courbes sur la surface. En particulier il convient de remarquer que dans la figure 1ª le point 4 qui donne lieu à un point critique est situé entre les points 3 et 5 qui donnent lieu respectivement au point double et au point de rebroussement de la courbe sextique dans le plan de zx, et cela fait voir très évidemment dans quelle partie de la courbe sextique est située le point critique, où cette courbe est touchée par la courbe nodale hyperbolique. Je n'ai pas cherché à dessiner d'autres sections de la surface, cela donnerait trop de complication à la figure, et il serait encore plus difficile d'apercevoir la forme de la surface, laquelle peut à peine être représentée sinon par un modèle ou au moins une stéréographie. On peut trouver l'équation de la surface de la même manière que celle de la courbe sextique, à savoir il faut éliminer X, Y, Z entre l'équation

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1,$$

et les équations

$$x=X\left[2-\frac{1}{a^2}(X^2+Y^2+Z^2)\right],\ y=Y\left[2-\frac{1}{b^2}(X^2+Y^2+Z^2)\right],\ z=Z\left[2-\frac{1}{c^2}(X^2+Y^2+Z^2)\right],$$

qui donnent les points de la surface.

En écrivant $\theta=X^2+Y^2+Z^2$ on trouve

$$\theta=\frac{x^2}{\left(2-\frac{\theta}{a^2}\right)^2}+\frac{y^2}{\left(2-\frac{\theta}{b^2}\right)^2}+\frac{z^2}{\left(2-\frac{\theta}{c^2}\right)^2},$$

$$1=\frac{x^2}{a^2\left(2-\frac{\theta}{a^2}\right)^2}+\frac{y^2}{b^2\left(2-\frac{\theta}{b^2}\right)^2}+\frac{z^2}{c^2\left(2-\frac{\theta}{c^2}\right)^2},$$

et en multipliant la seconde équation par 2, et la première équation par $-\theta$ et en ajoutant on trouve

$$\theta=\frac{x^2}{2-\frac{\theta}{a^2}}+\frac{y^2}{2-\frac{\theta}{b^2}}+\frac{z^2}{2-\frac{\theta}{c^2}}$$

laquelle a pour dérivée par rapport à θ la seconde équation. Cette équation peut s'écrire

$$\begin{aligned}
&\theta^4\\
&+\theta^3\left[-2(a^2+b^2+c^2)\right]\\
&+\theta^2\left[4(b^2c^2+c^2a^2+a^2b^2)+a^2x^2+b^2y^2+c^2z^2\right]\\
&+\theta\left[-8a^2b^2c^2-2(b^2+c^2)a^2x^2-2(c^2+a^2)b^2y^2-2(a^2+b^2)c^2z^2\right]\\
&+\quad 4a^2b^2c^2(x^2+y^2+z^2)=0,
\end{aligned}$$

et l'équation de la surface sera trouvée en égalant à zéro le discriminant de cette fonction de θ. En représentant l'équation par

$$\tfrac{1}{6}(A,\ B,\ C,\ D,\ E)\ (\theta,\ 1)^4 = 0,$$

on aura

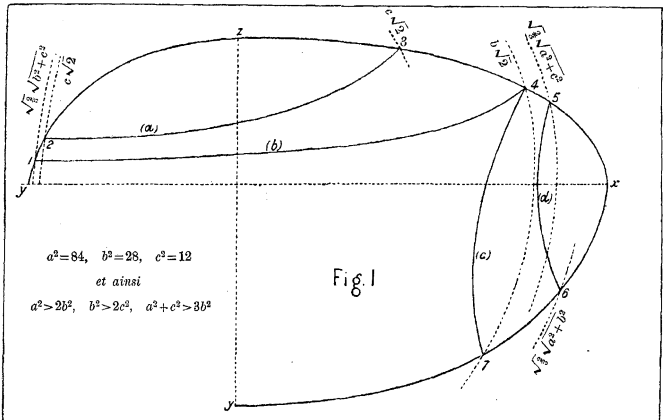
$$\begin{aligned}
A &= \quad 6,\\
B &= -\ 3\,(a^2+b^2+c^2),\\
C &= \quad 4\,(b^2c^2+c^2a^2+a^2b^2)+a^2x^2+b^2y^2+c^2z^2,\\
D &= -\ 12a^2b^2c^2-3\,(b^2+c^2)\,a^2x^2-3\,(c^2+a^2)\,b^2y^2-3\,(a^2+b^2)\,c^2z^2,\\
E &= \quad 24a^2b^2c^2\,(x^2+y^2+z^2),
\end{aligned}$$

et l'équation de la surface sera

$$(AE-4BD+3C^2)^3-27\,(ACE+2BCD-AD^2-B^2E-C^3)^2=0.$$

Il convient de remarquer que C^6, qui serait du douzième ordre en x, y, z n'entre pas dans l'équation, et que tous les autres termes sont au plus de l'ordre 10, la surface est donc (comme j'ai déjà remarqué) de l'ordre 10. Il ne m'a pas paru nécessaire de développer plus complètement l'équation[1].

[1] Mi sia lecito di richiamare qui per un' istante in seguito alla bella ed elegante Memoria dell' illustre Geometra, come più volte questa superficie di decimo ordine derivata dall' ellissoide sia stato fin da molti anni per me il soggetto di ricerche relativo al calcolo integrale per ciò che riguarda specialmente la sua quadratura dipendente da trascendenti ellittici di prima e seconda specie; così in un breve articolo inserito nella *Raccolta Scientifica* di Roma Anno 2° No. 9 in data del 2 Aprile 1846 ritrovai le coordinate della nuova superficie in funzione delle coordinate corrispondenti dell' ellissoide, e determinai ancora l' integrale definito duplicato di forma razionale per la quadratura della stessa superficie: più estesamente mi occupai dello stesso argomento in una Memoria inserita nel tom. 34° del Sig. *Crelle* in data 15 Aprile 1846: più completamente poi risolvei la stessa questione in altro articolo pubblicato nella citata *Raccolta Scientifica* dello stesso Anno 2° No. 21 con la data del 23 Ottobre 1846. Infine ripresi a risolvere lo stesso problema con altri metodi in una Memoria *Sulla riduzione di alcuni integrali definiti ai trascendenti ellittici* inserita pel mese di Agosto e Settembre 1848 nel giornale arcadico di Roma, e precisamente nel No. 18 e seguenti. Sovente ancora nei privati miei studii tentai di giungere all' equazione algebrica della superficie, il che dipendeva da un' eliminazione, ma proponendomi la questione sotto un' aspetto complicato, mi trovava arrestato dalla lunghezza dei calcoli. Il maraviglioso ripiego analitico ritrovato dal Sig. Cayley riduce quest' eliminazione ad una simplicità inaspettata, e rende inutile qualunque altra ricerca, che si volesse intraprendere sullo stesso soggetto. Termino questa breve nota col fare i miei distinti ringraziamenti al chiarissimo Autore, che si è compiaciuto dietro un mio suggerimento comunicatogli dal Sig. D. *Hirst* fare la traduzione della Memoria per essere inserita negli *Annali*.—*B. T.*



$a^2=84, \quad b^2=28, \quad c^2=12$

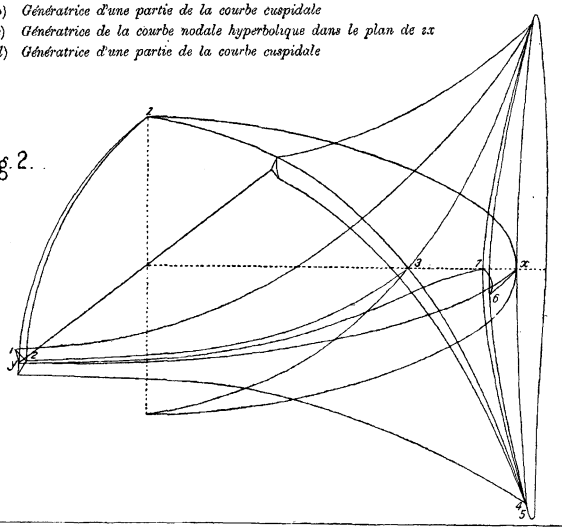
et ainsi

$a^2>2b^2, \quad b^2>2c^2, \quad a^2+c^2>3b^2$

Fig. 1

(a) *Génératrice de la courbe nodale elliptique dans le plan de xy*
(b) *Génératrice d'une partie de la courbe cuspidale*
(c) *Génératrice de la courbe nodale hyperbolique dans le plan de zx*
(d) *Génératrice d'une partie de la courbe cuspidale*

Fig. 2.



251.

SUR QUELQUES FORMULES POUR LA DIFFÉRENTIATION.

[From the *Annali di Matematica pura ed applicata* (Tortolini) tom. II. (1859), pp. 214—230. Traduction par l'auteur d'un mémoire présenté à la Société Royale de Londres le 26 Novembre, 1857.]

En cherchant une formule dans la théorie des intégrales définies multiples je fus conduit il y a plusieurs années à chercher les coefficients différentiels successifs de l'expression $(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2i}$, et les résultats que j'ai trouvés sont donnés dans le Mémoire "On certain formulæ for differentiation with applications to the evaluation of definite integrals," *Camb. and Dublin Mathematical Journal*, tom. II. pp. 122—128 (1847), [41]. J'ai depuis cherché les coefficients différentiels successifs de l'expression plus générale $[(x+\lambda)(x+\mu)]^{\frac{1}{2}k}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2i}$, mais l'investigation n'était pas achevée. Mon attention fut rappelée à ce sujet par deux identités remarquables trouvées par le Prof. Donkin dans son Mémoire "On the equation of Laplace's functions etc." *Phil. Trans.* tom. CXLVII. (1857), pp. 43—57, au moyen de la comparaison de ses résultats avec ceux du Prof. Boole; identités qui appartenaient, je l'aperçus, à la classe des formules ci-dessus mentionnées: la première de ces deux identités se déduit en effet assez facilement d'une formule exposée dans mon mémoire; la démonstration de la seconde identité est beaucoup plus difficile, et je n'ai réussi à l'établir qu'en la faisant dépendre de l'établissement de l'égalité des coefficients numériques de deux expressions de la même forme. Je suis depuis revenu aux investigations incomplètes dont j'ai parlé ci-dessus et les résultats que j'ai trouvés sont donnés dans le présent mémoire. Je remarque qu'en écrivant pour abréger

$$P=2x+\lambda+\mu,\quad Q=\sqrt{(x+\lambda)(x+\mu)},\quad R=(\sqrt{x+\lambda}-\sqrt{x+\mu})^2,$$

le sujet auquel appartiennent tous les résultats est la différentiation de l'expression $P^\alpha Q^\beta R^\gamma$: l'expression ci-dessus mentionnée $[(x+\lambda)(x+\mu)]^{\frac{1}{2}k}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2i}$ est de la

forme dont il s'agit, et la question qui s'y rapporte est celle d'obtenir le développement de $D_x^\mu P^\alpha Q^\beta R^\gamma$, où $\alpha = 0$.

La question suggérée par la seconde identité du Prof. Donkin est celle d'obtenir le développement de $(P^{-1}Q^\alpha D_x)^\mu P^\alpha Q^\beta R^\gamma$, où $\alpha = \gamma - \beta$. Comme la démonstration de ces identités est l'un des objets de ce Mémoire, j'ai donné dans la première section la réduction des identités à la forme sous laquelle je les ai depuis considérées. La seconde section se rapporte au développement de l'expression $D_x^\mu P^\alpha Q^\beta R^\gamma$, où $\alpha = 0$; la troisième section à celui de l'expression $(P^{-1}Q^\alpha D_x)^\mu P^\alpha Q^\beta R^\gamma$, où $\alpha = \gamma - \beta$: enfin la quatrième section contient l'application des deux identités, et quelques autres applications des formules.

§ I.

1. La première des deux identités du Prof. Donkin est

$$(\sin\theta D_\theta \sin\theta)^n (\tang \tfrac{1}{2}\theta)^n = 1 \,.\, 3 \,.\, 5 \ldots (2n-1)(\sin\theta)^{2n},$$

laquelle est l'équation (27) article No. 14, de son Mémoire[1].

En écrivant $\cot\theta = t$, l'équation devient

$$(-)^n D_t^n \frac{(\sqrt{1+t^2} - t)^n}{\sqrt{1+t^2}} = \frac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{(1+t^2)^{n+\frac{1}{2}}},$$

et en posant comme à l'ordinaire $i = \sqrt{-1}$ on obtient

$$\sqrt{1+t^2} - t = -\tfrac{1}{2}(\sqrt{t+i} - \sqrt{t-i})^2,$$

et en écrivant aussi

$$1 \,.\, 3 \,.\, 5 \ldots (2n-1) = 2^n \,.\, \tfrac{1}{2} \,.\, \tfrac{3}{2} \,.\, \tfrac{5}{2} \ldots (n - \tfrac{1}{2}) = 2^n [n - \tfrac{1}{2}]^n,$$

la formule devient

$$D_t^n \frac{(\sqrt{t+i} - \sqrt{t-i})^{2n}}{\sqrt{1+t^2}} = \frac{2^n [n - \frac{1}{2}]^n}{(1+t^2)^{n+\frac{1}{2}}}.$$

Ceci est un cas particulier de

$$D_x^n \frac{(\sqrt{x+\lambda} - \sqrt{x+\mu})^{2n}}{\sqrt{(x+\lambda)(x+\mu)}} = \frac{(-)^n (R-\mu)^{2n} [n - \frac{1}{2}]^n}{[(x+\lambda)(x+\mu)]^{n+\frac{1}{2}}},$$

ou, en écrivant comme auparavant

$$P = 2x + \lambda + \mu, \quad Q = \sqrt{(x+\lambda)(x+\mu)}, \quad R = [\sqrt{x+\lambda} - \sqrt{x+\mu}]^2,$$

la formule est

$$D_x^n Q^{-1} R^n = (-)^n (\lambda - \mu)^{2n} [n - \tfrac{1}{2}] Q^{-2n-1}.$$

[1] Vedi la nota in fine. [This Note by the Editor, Professor Tortolini, relating merely to the transformation $(\sin\theta D_\theta \sin\theta)^n = (\sin\theta)^{-1} (\sin^2\theta D_\theta)^n \sin\theta$, is not here reproduced.]

2. [1] La comparaison mentionnée article No. 5 du Mémoire du Prof. Donkin donne

$$(\sin\theta)^{-n}(\sin\theta\, D_\theta \sin\theta)^n \left\{ f\left[e^{\phi\sqrt{-1}} \operatorname{tang}\tfrac{1}{2}\theta + F\left(\frac{e^{\phi\sqrt{-1}}}{\operatorname{tang}\frac{1}{2}\theta}\right)\right]\right\}$$

$$= \mu^n \left(D_\mu \frac{1}{\mu}\right)^n \left\{ (\mu+\mu^2)^n f\left(\frac{\mu}{1+\mu}\,\frac{\sqrt{1-\mu'^2}}{\mu'}\, e^{\phi\sqrt{-1}}\right)\right.$$

$$\left. + (-)^n (\mu-\mu^2)^n F\left(\frac{\mu}{1-\mu}\,\frac{\sqrt{1-\mu'^2}}{\mu'}\, e^{\phi\sqrt{-1}}\right)\right\},$$

où $\mu = \cos\theta$, et après les différentiations $\mu' = \cos\theta$. Les parties qui contiennent les fonctions indéterminées f et F doivent être égales, chacune prise à part de l'autre. Les parties qui contiennent f seront égales, si cette égalité subsiste pour $fx = x^s$, où s est un indice quelconque; c'est-à-dire l'égalité subsistera si

$$(\sin\theta)^{-n}(\sin\theta\, D_\theta \sin\theta)^n (\operatorname{tang}\tfrac{1}{2}\theta)^s$$

$$= \mu^n \left(D_\mu \frac{1}{\mu}\right)^n (\mu+\mu^2)^n \frac{\mu^s}{(1+\mu)^s} \left(\frac{\sqrt{1-\mu'^2}}{\mu'}\right)^s,$$

$$= \mu^n \left(\frac{\sqrt{1-\mu'^2}}{\mu'}\right)^s \left(D_\mu \frac{1}{\mu}\right)^n (\mu+\mu^2)^n \frac{\mu^s}{(1+\mu)^s},$$

ou enfin en écrivant μ, au lieu de μ', si

$$(\sin\theta)^{-n}(\sin\theta\, D_\theta \sin\theta)^n (\operatorname{tang}\tfrac{1}{2}\theta)^s = \mu^{n-s}(1-\mu^2)^{\frac{1}{2}s}\left(D_\mu \frac{1}{\mu}\right)^n \mu^{n+s}(1+\mu)^{n-s},$$

où $\mu = \cos\theta$; c'est en effet la seconde des deux identités du Prof. Donkin. L'égalité des parties qui contiennent F dépend de la même manière de l'équation

$$(\sin\theta)^{-n}(\sin\theta\, D_\theta \sin\theta)^n (\operatorname{tang}\tfrac{1}{2}\theta)^s = (-)^n (1-\mu^2)^{\frac{1}{2}s}\left(D_\mu \frac{1}{\mu}\right)^2 \mu^{n+s}(1-\mu)^{n-s},$$

laquelle se déduit de l'autre équation en y écrivant $180° - \theta$ au lieu de θ.

3. La seconde identité (voir article No. 16) est

$$(\sin\theta)^{-n}(\sin\theta\, D_\theta \sin\theta)^n (\operatorname{tang}\tfrac{1}{2}\theta)^s = \mu^{n-s}(1-\mu^2)^{\frac{1}{2}s}\left(D_\mu \frac{1}{\mu}\right)^n \mu^{n+s}(1+\mu)^{n-s},$$

où comme auparavant $\mu = \cos\theta$. En écrivant $\cot\theta = t$, et en faisant attention que le côté gauche de l'équation peut s'écrire sous la forme

$$(\sin\theta)^{-n-1}(\sin^2\theta\, D_\theta)^n \sin\theta\, (\operatorname{tang}\tfrac{1}{2}\theta)^n,$$

et que l'on a

$$\sin\theta = \frac{1}{\sqrt{1+t^2}}, \quad \operatorname{tang}\tfrac{1}{2}\theta = \sqrt{1+t^2} - t, \quad \sin^2\theta\, D_\theta = -D_t,$$

[1] Le lecteur pourrait omettre cet article, qui ne fait que montrer qu'une certaine formule du Professeur Donkin se réduit à son identité seconde.

le côté gauche devient

$$(-)^n (1+t^2)^{\frac{1}{2}(n+1)} D_t^n \frac{(\sqrt{1+t^2}-t)^s}{\sqrt{1+t^2}}.$$

Le côté droit peut s'écrire sous la forme

$$\mu^{n-s+1}(1-\mu^2)^{\frac{1}{2}s}\left(\frac{1}{\mu}D_\mu\right)^n \mu^{n+s-1}(1+\mu)^{n+s},$$

et en observant que

$$\mu = \frac{t}{\sqrt{1+t^2}}, \quad \frac{1}{\mu}D_\mu = \frac{(1+t^2)^2}{t}D_t,$$

le côté droit devient

$$\frac{t^{n-s+1}}{(1+t^2)^{\frac{1}{2}(n+1)}}\left(\frac{(1+t^2)^2}{t}D_t\right)^2 \frac{t^{n+s-1}}{(1+t^2)^{n-\frac{1}{2}}(\sqrt{1+t^2}-t)^{n-s}};$$

en comparant les deux expressions on obtient

$$(-)^n \frac{(1+t^2)^{n+1}}{t^{n-s+1}} D_t^n \frac{(\sqrt{1+t^2}-t)^s}{\sqrt{1+t^2}} = \left(\frac{(1+t^2)^2}{t}D_t\right)^n \frac{t^{n+s-1}}{(1+t^2)^{n-\frac{1}{2}}(\sqrt{1+t^2}-t)^{n-s}},$$

et de là, en écrivant comme auparavant $i=\sqrt{-1}$, et

$$\sqrt{1+t^2}-t = -\tfrac{1}{2}(\sqrt{t+i}-\sqrt{t-i})^2$$

la formule devient

$$\left(\frac{(1+t^2)^2}{2t}D_t\right)^n \frac{(2t)^{n+s-1}}{(1+t^2)^{n-\frac{1}{2}}(\sqrt{1+i}-\sqrt{t-i})^{2n-2s}} = \frac{(1+t^2)^{n+1}}{(2t)^{n-s+1}} D_t^n \frac{(\sqrt{t+i}-\sqrt{t-i})^{2s}}{\sqrt{1+t^2}},$$

laquelle est un cas particulier de

$$\left[\frac{[(x+\lambda)(x+\mu)]^2}{2x+\lambda+\mu}D_x\right]^n \frac{(2x+\lambda+\mu)^{n+s-1}}{[(x+\lambda)(x+\mu)]^{n-\frac{1}{2}}(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2n-2s}}$$

$$= \frac{[(x+\lambda)(x+\mu)]^{n+1}}{(2x+\lambda+\mu)^{n-s+1}} D_x^n \frac{(\sqrt{x+\lambda}-\sqrt{x+\mu})^{2s}}{\sqrt{(x+\lambda)(x+\mu)}}$$

c'est-à-dire de

$$(P^{-1}Q^4 D_x)^n (P^{n+s-1}Q^{-2n+1}R^{-n+s}) = P^{-n+s-1}Q^{2n+2}D_x^n(Q^{-1}R^s).$$

§ II.

4. En écrivant comme auparavant

$$P = 2x+\lambda+\mu, \quad Q = \sqrt{(x+\lambda)(x+\mu)}, \quad R = (\sqrt{x+\lambda}-\sqrt{x+\mu})^2,$$

on a $R = P - 2Q$, et en écrivant pour abréger $(\lambda-\mu)^2 = \Lambda$,

on obtient aussi

$$\Lambda = P^2 - 4Q^2, \quad \frac{\Lambda}{R} = P + 2Q;$$

on trouve de plus

$$D_x P = 2, \quad D_x Q = \tfrac{1}{2}\frac{P}{Q}, \quad D_x R = -\frac{R}{Q}.$$

5. Les dernières formules donnent

$$D_x P^\alpha Q^\beta R^\gamma = \tfrac{1}{2}\beta P^{\alpha+1} Q^{\beta-2} R^\gamma - \gamma P^\alpha Q^{\beta-1} R^\gamma + 2\alpha P^{\alpha-1} Q^\beta R^\gamma,$$

formule dont on pourrait se servir pour chercher $D_x{}^r P^\alpha Q^\beta R^\gamma$; on a par exemple

$$\begin{aligned} D_x{}^2 P^\alpha Q^\beta R^\gamma = {} & \tfrac{1}{4}\beta(\beta-2) P^{\alpha+2} Q^{\beta-4} R^\gamma - \tfrac{1}{2}\gamma(2\beta-1) P^{\alpha+1} Q^{\beta-3} R^\gamma \\ & + [\beta(2\alpha+1) + \gamma^2] P^\alpha Q^{\beta-2} R^\gamma - 4\alpha\gamma P^{\alpha-1} Q^{\beta-1} R^\gamma + 4(\alpha-1)\gamma P^{\alpha-2} Q^\beta R^\gamma, \end{aligned}$$

et ainsi de suite. Et de même

$$P^{-1}Q^4 D_x P^\alpha Q^\beta R^\gamma = \tfrac{1}{2}\beta P^\alpha Q^{\beta+2} R^\gamma - \gamma P^{\alpha-1} Q^{\beta+3} R^\gamma + 2\alpha P^{\alpha-2} Q^{\beta+4} R^\gamma,$$

et de là

$$\begin{aligned} (P^{-1}Q^4 D_x)^2 P^\alpha Q^\beta R^\gamma = {} & \tfrac{1}{4}\beta(\beta+2) P^\alpha Q^{\beta+4} R^\gamma - \tfrac{1}{2}(2\beta+3)\gamma P^{\alpha-1} Q^{\beta+5} R^\gamma \\ & + [2\alpha(\beta+2) + \gamma^2] P^{\alpha-2} Q^{\beta+6} R^\gamma - 2(2\alpha-1)\gamma P^{\alpha-3} Q^{\beta+7} R^\gamma \\ & + 4\alpha(\alpha-2) P^{\alpha-4} Q^{\beta+8} R^\gamma, \end{aligned}$$

et ainsi de suite. Mais il serait difficile d'obtenir de cette manière l'expression de la r-ième répétition de l'opération D_x, ou $P^{-1}Q^4 D_x$, sur $P^\alpha Q^\beta R^\gamma$ et je ne poursuis pas la question.

6. Je vais à présent chercher le développement de $D_x{}^r P^\alpha Q^\beta R^\gamma\,(\alpha = 0)$ ou ce qui est la même chose $D_x{}^r Q^\beta R^\gamma$. On a

$$D_x Q^\beta R^\gamma = \tfrac{1}{2}\beta P Q^{\beta-2} R^\gamma - \gamma Q^{\beta-1} R^\gamma,$$

ou en substituant pour P la valeur

$$P = \frac{\Lambda}{R} - 2Q,$$

on a

$$D_x Q^\beta R^\gamma = -(\beta+\gamma) Q^{\beta-1} R^\gamma - \tfrac{1}{2}\beta \Lambda Q^{\beta-2} R^{\gamma-1}.$$

La répétition de l'opération D_x donne évidemment une expression de la forme

$$\begin{aligned} D_x{}^r Q^\beta R^\gamma = {} & (-)^r \quad L_r \quad \Lambda^0 Q^{\beta-r} \quad R^\gamma \\ & \vdots \\ & + (-)^{r-\theta} \frac{1}{2^\theta} L_{r,\theta} \quad \Lambda^\theta Q^{\beta-r-\theta} R^{\gamma-\theta} \\ & \vdots \\ & + \qquad \frac{1}{2^r} L_{r,\mu r} \,.\, \Lambda^r Q^{\beta-2r} \quad R^{\gamma-r}, \end{aligned}$$

et on obtient pour $L_{r,\theta}$ l'équation aux différences

$$L_{r+1,\theta+1}-(\beta+\gamma-r-2\theta-2)\,L_{r,\theta+1}-(\beta-r-\theta)\,L_{r,\theta}=0,$$

laquelle, avec les conditions particulières

$$L_{r,-1}=0,\quad L_{r,r+1}=0,\quad L_{0,0}=1,$$

suffit pour déterminer les coefficients $L_{r,\theta}$ de la formule.

7. Avant d'aller plus loin, je vais considérer le cas particulier $\beta=0$. L'équation aux différences est ici

$$L_{r+1,\theta+1}-(\gamma-r-2\theta-2)\,L_{r,\theta+1}+(r+\theta)\,L_{r,\theta}=0,$$

et l'on satisfait à cette équation en posant

$$L_{r,\theta}=\frac{(-)^\theta\,[r+\theta-1]^{2\theta}\,[\gamma-\theta-1]^{r-\theta-1}}{2^\theta\,[\theta]^\theta},$$

ou, ce qui est la même chose,

$$L_{r,\theta}=\frac{(-)^\theta\,2^\theta\gamma\,[\theta-\tfrac{1}{2}]^\theta\,[r+\theta-1]^{r-\theta-1}\,[\gamma-\theta-1]^{r-\theta-1}}{[r-\theta-1]^{r-\theta-1}}.$$

En effet on déduit de la première expression

$$\begin{aligned}L_{r+1,\theta+1}&-(\gamma-r-2\theta-2)\,L_{r,\theta+1}\\&=\frac{(-)^{\theta+1}\gamma}{2^{\theta+1}[\theta+1]^{\theta+1}}\,(r+\theta)\,[r+\theta-1]^{2\theta}\,[\gamma-\theta-2]^{r-\theta-2}\\&\qquad\times[(r+\theta+1)(\gamma-r)-(\gamma-r-2\theta-2)(r-\theta-1)],\end{aligned}$$

et le facteur en [ ] est $2(\theta+1)(\gamma-\theta-1)$, de manière que l'expression devient

$$=\frac{-(-)^\theta\gamma}{2^\theta[\theta]^\theta}\,(r+\theta)\,[r+\theta-1]^{2\theta}\,[\gamma-\theta-1]^{r-\theta-1},$$

ce qui est

$$=-(r+\theta)\,L_{r,\theta}.$$

L'équation aux différences est donc satisfaite ; les conditions pour les limites sont aussi satisfaites, et la valeur qui vient d'être donnée est donc celle de $L_{r,\theta}$ dans le cas particulier dont il s'agit où $\beta=0$.

8. Il sera convenable d'écrire

$$\frac{(-)^\theta}{2^\theta\gamma}\,L_{r,\theta}=\frac{[r+\theta-1]^{2\theta}\,[\gamma-\theta-1]^{r-\theta-1}}{2^{2\theta}\,[\theta]^\theta}=\frac{[\theta-\tfrac{1}{2}]^\theta\,[r+\theta-1]^{r-\theta-1}\,[\gamma-\theta-1]^{r-\theta-1}}{[r-\theta-1]^{r-\theta-1}}=L'_{r,\theta},$$

et alors en observant que $L_{r,\theta}$, ou $L'_{r,\theta}$ se réduit à zéro pour $\theta=r$, on obtient

$$\frac{(-)^r}{\gamma}\,D_x{}^rR^\gamma=L'_{r,0}\,Q^{-r}\,R^\gamma\ldots+L'_{r,\theta}\,\Delta^\theta\,Q^{-r-\theta}\,R^{\gamma-\theta}\ldots+L'_{r,r-1}\,\Delta^{r-1}\,Q^{-2r+1}\,R^{\gamma-r-1}.$$

9. Cela est en effet sous une forme un peu modifiée la formule fondamentale de mon mémoire dans le *Camb. and Dub. Math. Journal;* la valeur qu'on y trouve du coefficient $K_{r,\theta}$ (en mettant γ au lieu de i est)

$$K_{r,\theta} = \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(\gamma-r-\theta-1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2r-1-2\theta)\,\Gamma(\gamma-r+1)},$$

et le coefficient $L_{r,\theta}(\beta=0)$ du présent mémoire est lié avec $K_{r,\theta}$ par l'équation

$$\frac{(-)^\theta}{2^\theta\gamma} L_{r,\theta} = L'_{r,\theta} = K_{r,r-1-\theta}.$$

La valeur de $L'_{r,r-1}$ est $[r-\frac{3}{2}]^{r-1}$, et pour $r=\gamma+1$ le coefficient devient $[\gamma-\frac{1}{2}]^\gamma$, et tous les autres coefficients se réduisent à zéro: la formule devient donc tout simplement

$$\frac{(-)^{\gamma+1}}{\gamma} D_x^{\gamma+1} R^\gamma = [\gamma-\tfrac{1}{2}]^\gamma\, \Lambda^\gamma\, Q^{-2\gamma-1}.$$

10. Dans le cas $r>\gamma+1$ il convient de modifier la forme de l'équation. J'écris $r=\gamma+1+s$, puis en faisant attention que les coefficients $L'_{\gamma+1+s,\theta}$ se réduisent à zéro pour $\theta<\gamma$, on peut écrire $\gamma+\theta$ au lieu de θ, et dans la formule nouvelle étendre θ depuis $\theta=0$ jusqu'à $\theta=s$. On a ainsi

$$\begin{aligned}\frac{(-)^{\gamma+1+s}}{\gamma} D_x^{\gamma+1+s} R^\gamma = {} & L'_{\gamma+1+s,\gamma}\ \Lambda^\gamma\ Q^{-2\gamma-1-s}\ R^0 \\ & \vdots \\ & + L'_{\gamma+1+s,\gamma+\theta}\,\Lambda^{\gamma+\theta}\,Q^{-2\gamma-1-s-\theta}\,R^{-\theta} \\ & \vdots \\ & + L'_{\gamma+1+s,\gamma+s}\,\Lambda^{\gamma+s}\,Q^{-2\gamma-1-2s}\ R^{-s},\end{aligned}$$

et dans cette équation le côté gauche peut s'exprimer sous la forme

$$L'_{\gamma+1,\gamma}\,\Lambda^\gamma\,(-)^s\,D_x^s\,Q^{-2\gamma-1}.$$

On obtient ainsi

$$\begin{aligned}D_x^s\,Q^{-2\gamma-1} = {} & L''_{s,0}\ \ Q^{-2\gamma-1-s} \\ & \vdots \\ & + L''_{s,\theta}\,\Lambda^\theta\,Q^{-2\gamma-1-s-\theta}\,R^{-\theta} \\ & \vdots \\ & + L''_{s,s}\,\Lambda^s\,Q^{-2\gamma-1-2s}\ \ R^{-s},\end{aligned}$$

où $L''_{s,\theta} = (-)^s L'_{\gamma+1+s,\gamma+\theta} - L'_{\gamma+1,\gamma}$, et en substituant pour les coefficients L' leurs valeurs, et en faisant attention que $[-\theta-1]^{s-\theta} = (-)^{s-\theta}[s]^{s-\theta}$,

on trouve

$$L''_{s,\theta} = \frac{(-)^\theta\,[2\gamma+s+\theta]^{2\gamma+2\theta}\,[s]^{s-\theta}}{2^{2\theta}\,[2\gamma]^{2\theta}\,[\gamma+\theta]^\theta} = \frac{(-)^\theta\,[\gamma+\theta-\frac{1}{2}]^\theta\,[2\gamma+s+\theta]^{s-\theta}\,[s]^{s-\theta}}{[s-\theta]^{s-\theta}};$$

la formule qui vient d'être trouvée est donnée sous une forme différente dans mon mémoire déjà cité.

11. Je résume l'équation générale

$$L_{r+1,\,\theta+1}-(\beta+\gamma-r-2\theta-2)\,L_{r,\,\theta+1}-(\beta-r-\theta)\,L_{r,\,\theta}=0,$$

les conditions aux limites étant comme auparavant

$$L_{r,\,-1}=0,\quad L_{r,\,r+1}=0,\quad L_{0,\,0}=1.$$

On a en particulier

$$\begin{aligned}
&L_{r+1,\,0}-(\beta+\gamma-r\quad)\,L_{r,\,0}&&=0,\\
&L_{r+1,\,1}-(\beta+\gamma-r-2)\,L_{r,\,1}-(\beta-\ r\quad)\,L_{r,\,0}&&=0,\\
&L_{r+1,\,2}-(\beta+\gamma-r-4)\,L_{r,\,2}-(\beta-\ r-1)\,L_{r,\,1}&&=0,\\
&\vdots\\
&L_{r+1,\,r+1}\ldots\ldots\qquad\qquad-(\beta-2r\quad)\,L_{r,\,r}&&=0.
\end{aligned}$$

De là on voit tout de suite que les valeurs de $L_{r,\,0}$, $L_{r,\,r}$ sont

$$L_{r,\,0}=[\beta+\gamma]^r,\quad L_{r,\,r}=[\beta,\ -2]^r,$$

où dans la dernière équation la notation au côté droit dénote une factorielle à différence -2.

12. Les autres coefficients $L_{r,\,1}$, $L_{r,\,2}$, etc. peuvent s'obtenir successivement par une intégration directe des équations; et quoiqu'on ne les obtienne pas de cette manière sous la forme la plus commode, cependant il convient de donner l'investigation. Pour $L_{r,\,1}$ on peut écrire

$$L_{r,\,1}=[\beta+\gamma-2]^r\,M_{r,\,1},$$

l'équation pour $M_{r,\,1}$ sera

$$M_{r+1,\,1}-M_{r,\,1}=\frac{(\beta-r)\,[\beta+\gamma]^r}{[\beta+\gamma-2]^{r+1}},\ =\frac{(\beta-r)\,(\beta+\gamma)\,(\beta+\gamma-1)}{(\beta+\gamma-r)\,(\beta+\gamma-r-1)\,(\beta+\gamma-r-2)},$$

laquelle peut aussi s'écrire

$$M_{r+1,\,1}-M_{r,\,1}=1+\frac{-\frac{1}{2}(r+2)\,(r+1)\,(\gamma-2)}{\beta+\gamma-r-2}+\frac{(r+1)\,r\,(\gamma-1)}{\beta+\gamma-r-1}+\frac{-\frac{1}{2}\,r\,(r-1)\,\gamma}{\beta+\gamma-r}.$$

Cette équation a une solution

$$M_{r,\,1}=r+\frac{A_r}{\beta+\gamma-r-1}+\frac{B_r}{\beta+\gamma-r},$$

cela donne en effet

$$M_{r+1,\,1}-M_{r,\,1}=1+\frac{A_{r+1}}{\beta+\gamma-r-2}+\frac{B_{r+1}-A_r}{\beta+\gamma-r-1}+\frac{-B_r}{\beta+\gamma-r},$$

et l'on doit ainsi avoir

$$A_{r+1}=-\tfrac{1}{2}(r+2)\,(r+1)\,(\gamma-2),\quad B_{r+1}-A_r=(r+1)\,r\,(\gamma-1),\quad -B_r=-\tfrac{1}{2}r\,(r-1)\,\gamma,$$

équations qui sont satisfaites par

$$A_r=-\tfrac{1}{2}(r+1)\,r\,(\gamma-2),\quad B_r=\tfrac{1}{2}r\,(r-1)\,\gamma.$$

Les conditions aux limites sont aussi satisfaites, et en formant l'expression de L_{r+1} on obtient

$$L_{r,\,1}=[\beta+\gamma-2]^r\left(r+\frac{-\frac{1}{2}(r+1)\,r\,(\gamma-2)}{\beta+\gamma-r-1}+\frac{\frac{1}{2}r\,(r-1)\,\gamma}{\beta+\gamma-r}\right).$$

13. La valeur de $L_{r,2}$ peut s'obtenir par un procédé semblable, et l'on a

$$L_{r,2}=[\beta+\gamma-4]^r\left(\tfrac{1}{2}r(r-1)+\frac{A_r}{\beta+\gamma-r-3}+\frac{B_r}{\beta+\gamma-r-2}+\frac{C_r}{\beta+\gamma-r-1}+\frac{D_r}{\beta+\gamma-r}\right),$$

où

$$\begin{aligned}
A_r &= \tfrac{1}{48}(r+1)\,r\left((r^2+5r-6)\gamma^2+(-11r^2-31r+42)\gamma+(24r^2+48r-72)\right),\\
B_r &= -\tfrac{3}{48}\,r(r-1)\left((r^2+3r-2)\gamma^2+(-7r^2-13r+6)\gamma+8r^2+16r\right),\\
C_r &= \tfrac{3}{48}(r-1)(r-2)\left((r^2+r)\gamma^2+(-3r^2-3r)\gamma\right),\\
D_r &= -\tfrac{1}{48}(r-2)(r-3)\left((r^2-r)\gamma^2+(r^2-r)\gamma\right),
\end{aligned}$$

et l'on peut remarquer que les suites des coefficients numériques (1, 1, 1, 1), (5, 3, 1, −1), (−6, −2, 0, 0) ont respectivement les premières, secondes, et troisièmes différences égales à zéro, les suites (−11, −7, −3, 1), (−31, −13, −3, −1) ont respectivement les secondes, et troisièmes différences égales à zéro, et la suite (24, 8, 0, 0) a les troisièmes différences égales à zéro. Je remarque que pour $r=1$, $r=2$ les expressions de $L_{r,1}$, $L_{r,2}$ donnent

$$L_{1,1}=(\beta+\gamma-2)\left(1-\frac{(\gamma-2)}{\beta+\gamma-2}\right),\ =\beta,$$

$$L_{2,2}=(\beta+\gamma-4)(\beta+\gamma-5)\left(1+\frac{\gamma^2-8\gamma+15}{\beta+\gamma-5}+\frac{-\gamma^2+6\gamma-8}{\beta+\gamma-4}\right),\ =\beta(\beta-2),$$

lesquelles s'accordent avec les résultats donnés par l'expression générale de $L_{r,r}$.

14. L'expression de $L_{r,1}$ peut se transformer en

$$L_{r,1}=r\beta[\beta+\gamma-2]^{r-1}-\tfrac{1}{2}r(r-1)[\beta+\gamma-2]^{r-2},$$

et l'expression de $L_{r,2}$ peut de même, avec beaucoup plus de peine, se transformer en

$$L_{r,2}=\tfrac{1}{2}r(r-1)\left\{\begin{aligned}
&\beta(\beta-2)[\beta+\gamma-4]^{r-2}\\
&-\ (r-2)\beta(\beta-1)[\beta+\gamma-4]^{r-3}\\
&+\tfrac{1}{2}(r-2)(r-3)\tfrac{1}{2}(\beta+1)\beta[\beta+\gamma-4]^{r-4},
\end{aligned}\right.$$

et la forme de ces équations et d'autres considérations m'ont conduit à assumer en général

$$\begin{aligned}
L_{r,\theta}= &\quad \frac{1}{[\theta]^\theta} && A_{\theta,0}[r]^\theta\quad[\beta+\gamma-2\theta]^{r-\theta}\\
&\quad\vdots\\
&-\frac{1}{[\theta-1]^{\theta-1}[1]^1} && \tfrac{1}{2}A_{\theta,1}[r]^{\theta+1}\,[\beta+\gamma-2\theta]^{r-\theta-1}\\
&\quad\vdots\\
&+\frac{(-)^\sigma}{[\theta-\sigma]^{\theta-\sigma}[\sigma]^\sigma}\cdot\frac{1}{2^\sigma} && A_{\theta,\sigma}[r]^{\theta+\sigma}\,[\beta+\gamma-2\theta]^{r-\theta-\sigma}\\
&\quad\vdots\\
&\frac{(-)^\theta}{[\theta]^\theta} && A_{\theta,\theta}[r]^{2\theta}\quad[\beta+\gamma-2\theta]^{r-2\theta}.
\end{aligned}$$

15. En posant pour plus de commodité $\theta-1$ au lieu de θ dans l'équation aux différences, cette équation devient

$$L_{r+1,\,\theta}-(\beta+\gamma-r-2\theta)\,L_{r,\,\theta}=(\beta-r-\theta+1)\,L_{r,\,\theta-1},$$

et puis substituant dans cette équation la valeur assumée de $L_{r,\,\theta}$, et les valeurs correspondantes de $L_{r+1,\,\theta}$ et $L_{r,\,\theta-1}$, le terme général au côté gauche sera

$$\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-\sigma]^{\theta-\sigma}}\,\frac{1}{2^\sigma}\,[r]^{\theta+\sigma-1}\,[\beta+\gamma-2\theta]^{r-\theta-\sigma}\,A_{\theta,\,\sigma}$$
$$\times\,[(r+1)(\beta+\gamma-r-\theta-\sigma)-(\beta+\gamma-r-2\theta)(r-\theta-\sigma+1)];$$

l'expression en [ ] est $(\theta+\sigma)(\beta+\gamma-2\theta+1)$, et en substituant cette valeur le terme général au côté gauche devient

$$\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-\sigma]^{\theta-\sigma}}\cdot\frac{1}{2^\sigma}\,(\theta+\sigma)\,[r]^{\theta+\sigma-1}\,[\beta+\gamma-2\theta+1]^{r+1-\theta-\sigma}\,A_{\theta,\,\sigma}.$$

Le terme général au côté droit est

$$\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-1-\sigma]^{\theta-1-\sigma}}\,\frac{1}{2^\sigma}\,[r]^{\theta-1-\sigma}\,[\beta+\gamma-2\theta+1]^{r-\theta-\sigma}\,A_{\theta-1,\,\sigma}$$
$$\times\,(\beta-r-\theta+1)(\beta+\gamma-2\theta+2),$$

dont le dernier facteur est égal à

$$(\beta+\gamma-r-\theta+1+\sigma)(\beta-2\theta+2-\sigma)-(r-\theta+1-\sigma)(\gamma+\sigma);$$

et en substituant cette valeur le terme devient

$$\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-1-\sigma]^{\theta-1-\sigma}}\,\frac{1}{2^\sigma}\,[r]^{\theta-1+\sigma}\,[\beta+\gamma-2\theta-1]^{r-\theta-\sigma+1}\,(\beta-2\theta+2-\sigma)\,A_{\theta-1,\,\sigma}$$
$$-\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-1-\sigma]^{\theta-1-\sigma}}\cdot\frac{1}{2^\sigma}\,[r]^{\theta+\sigma}\,[\beta+\gamma-2\theta+1]^{r-\theta-\sigma}\,(\gamma+\sigma)A_{\theta-1,\,\sigma}.$$

En écrivant dans la seconde ligne (ce qui est permis) $\sigma-1$ au lieu de σ, cette ligne devient

$$-\frac{(-)^{\sigma-1}}{[\sigma-1]^{\sigma-1}\,[\theta-\sigma]^{\theta-\sigma}}\cdot\frac{1}{2^{\sigma-1}}\,[r]^{\theta+\sigma-1}\,[\beta+\gamma-2\theta+1]^{r-\theta-\sigma+1}\,(\gamma+\sigma-1)\,A_{\theta-1,\,\sigma-1},$$

et les deux lignes ensemble seront

$$\frac{(-)^\sigma}{[\sigma]^\sigma\,[\theta-\sigma]^{\theta-\sigma}}\,\frac{1}{2^\sigma}\,[r]^{\theta+\sigma-1}\,[\beta+\gamma-2\theta+1]^{r-\theta-\sigma+1}$$
$$\times\,[(\theta-\sigma)(\beta-2\theta+2-\sigma)\,A_{\theta-1,\,\sigma}+2\sigma\,(\gamma+\sigma-1)\,A_{\theta-1,\,\sigma-1}],$$

ce qui est le terme général au côté droit. En comparant cela avec l'expression ci-dessus trouvée pour le terme général au côté gauche on obtient

$$(\theta+\sigma)\,A_{\theta,\,\sigma}=(\theta-\sigma)\,(\beta-2\theta+2-\sigma)\,A_{\theta-1,\,\sigma}+2\sigma\,(\gamma+\sigma-1)\,A_{\theta-1,\,\sigma-1},$$

pour l'équation aux différences à laquelle doit satisfaire le coefficient $A_{\theta,\,\sigma}$; en y écrivant

$$A_{\theta,\,\sigma}=[\gamma+\sigma-1]^{\sigma}\,B_{\theta,\,\sigma},$$

on aura pour $B_{\theta,\,\sigma}$ l'équation

$$(\theta+\sigma)\,B_{\theta,\,\sigma}=(\theta-\sigma)\,(\beta-2\theta+2-\sigma)\,B_{\theta-1,\,\sigma}+2\sigma\,B_{\theta-1,\,\sigma-1},$$

et on voit sans peine que les conditions aux limites sont

$$B_{\theta,\,-1}=0,\quad B_{\theta,\,\theta+1}=0,\quad B_{0,\,0}=1.$$

16. On a en particulier

$$B_{\theta,\,0}=(\beta-2\theta+2)\,B_{\theta-1,\,0},\quad B_{\theta,\,\theta}=B_{\theta-1,\,\theta-1},$$

et de là

$$B_{\theta,\,0}=[\beta,\,-2]^{\theta},\quad B_{\theta,\,\theta}=1.$$

Les autres fonctions $B_{\theta,\,\sigma}$ sont des fonctions rationnelles et entières de β, du degré $\theta-\sigma$, données par l'équation aux différences; mais les coefficients numériques des différentes puissances de β n'admettent, à ce qu'il paraît, aucune expression simple; les fonctions $B_{\theta,\,\sigma}$ sont, pour ainsi dire, une espèce de transcendantes rationnelles propres pour exprimer la valeur de $L_{r,\,\theta}$, et il sera suffisant de tabuler les valeurs de $B_{\theta,\,\sigma}$ sans pousser plus loin la recherche de la loi de ces valeurs. On a en effet pour $B_{0,\,0}$, $B_{1,\,0}$, etc....$B_{1,\,1}$ les valeurs

$$\begin{array}{llll}
1,\ \beta, & \beta\,(\beta-2), & \beta\,(\beta-2)\,(\beta-4), & \beta\,(\beta-2)\,(\beta-4)\,(\beta-6),\ \text{etc.}\ \ldots \\
1, & \beta-1, & \beta^2-4\beta+\frac{5}{2}, & \beta^3-9\beta^2+\frac{43}{2}\beta-\frac{21}{2}, \\
 & 1\ , & \beta-2\ , & \beta^2-6\beta+7, \\
 & & 1\ , & \beta-3, \\
 & & & 1.
\end{array}$$

17. Puis en substituant pour $A_{\theta,\,\sigma}$ la valeur $[\gamma+\sigma-1]^{\sigma}\,B_{\theta,\,\sigma}$ on trouve

$$\begin{aligned}
L_{r,\theta}= & \frac{1}{[\theta]^{\theta}}\quad B_{\theta,\,0}\quad [r]^{\theta}\quad [\beta+\gamma-2\theta]^{r-\theta} \\
& +\frac{(-)}{[\theta-1]^{\theta-1}\,[1]^{1}}\ \tfrac{1}{2}\,B_{\theta,\,1}\ [\gamma]^{1}\,[r]^{\theta+1}\quad [\beta+\gamma-2\theta]^{r-\theta-1} \\
& \qquad\vdots \\
& +\frac{(-)^{\sigma}}{[\theta-\sigma]^{\theta-\sigma}\,[\sigma]^{\sigma}}\ \frac{1}{2^{\sigma}}\,B_{\theta,\,\sigma}\,[\gamma+\sigma-1]^{\sigma}\,[r]^{\theta+\sigma}\,[\beta+\gamma-2\theta]^{r-\theta-\sigma} \\
& \qquad\vdots \\
& +\quad\frac{(-)^{\theta}}{[\theta]^{\theta}}\quad \frac{1}{2^{\theta}}\,B_{\theta,\,\theta}\,[\gamma+\theta-1]^{\theta}\ \ [r]^{2\theta}\ \ [\beta+\gamma-2\theta]^{r-2\theta},
\end{aligned}$$

ce qui est l'expression de $L_{r,\theta}$ dans la formule

$$\begin{aligned} D_x^r Q^\beta B^\gamma = (-)^r & \quad L_{r,0} \quad Q^{\beta-\gamma} \quad R^\gamma \\ & \vdots \\ & + (-)^{r-\theta} \frac{1}{2^\theta} L_{r,\theta} \Lambda^\theta Q^{\beta-r-\theta} R^{\gamma-\theta} \\ & \vdots \\ & + \quad \frac{1}{2^r} L_{r,r} \Lambda^r Q^{\beta-2r} \quad R^{\gamma-r}. \end{aligned}$$

§ III.

18. Je passe à présent au développement de

$$(P^{-1} Q^4 D_x)^r P^\alpha \quad Q^\beta R^\gamma, \quad (\alpha = \gamma - \beta),$$

ou, ce qui est la même chose,

$$(P^{-1} Q^4 D_x)^r P^{\gamma-\beta} Q^\beta R^\gamma.$$

On a comme auparavant

$$P^{-1} Q^4 D_x P^\alpha Q^\beta R^\gamma = \tfrac{1}{2}\beta P^\alpha Q^{\beta+2} R^\gamma - \gamma P^{\alpha-1} Q^{\beta+3} R^\gamma + 2\alpha P^{\alpha-2} Q^{\beta+4} R^\gamma,$$

et de là

$$\begin{aligned} P^{-1} Q^4 D_x P^{\gamma-\beta} Q^\beta R^\gamma &= \tfrac{1}{2}\beta P^{\gamma-\beta} Q^{\beta+2} R^\gamma - \gamma P^{\gamma-\beta-1} Q^{\beta+3} R^\gamma + 2(\gamma-\beta) P^{\gamma-\beta-2} Q^{\beta+4} R^\gamma, \\ &= \tfrac{1}{2}\beta (P - 2Q) P^{\gamma-\beta-1} Q^{\beta+2} R^\gamma - (\gamma-\beta)(P-2Q) P^{\gamma-\beta-2} Q^{\beta+3} R^\gamma, \end{aligned}$$

c'est-à-dire

$$P^{-1} Q^4 D_x P^{\gamma-\beta} Q^\beta R^\gamma = \tfrac{1}{2}\beta P^{\gamma-\beta-1} Q^{\beta+2} R^{\gamma+1} - (\gamma-\beta) P^{\gamma-\beta-2} Q^{\beta+3} R^{\gamma+1}.$$

On peut donc écrire

$$\begin{aligned} (P^{-1} Q^4 D_x)^r P^{\gamma-\beta} Q^\beta R^\gamma = & \quad \frac{1}{2^r} N_{r,0} P^{\gamma-\beta-r} Q^{\beta+2r} R^{\gamma+r} \\ & \vdots \\ & + \frac{(-)^\theta}{2^{r-\theta}} N_{r,\theta} P^{\gamma-\beta-r-\theta} Q^{\beta+2r+\theta} R^{\beta+r} \\ & \vdots \\ & + (-)^r N_{r,r} P^{\gamma-\beta-2r} Q^{\beta+2r} R^{\gamma+r}, \end{aligned}$$

et on trouve pour $N_{r,\theta}$ l'équation aux différences

$$N_{r+1,\theta+1} - (\beta + 2r + \theta + 1) N_{r,\theta+1} - (\gamma - \beta - r - \theta) N_{r,\theta} = 0,$$

laquelle avec les conditions aux limites

$$N_{r,-1} = 0, \quad N_{r,r+1} = 0, \quad N_{0,0} = 1,$$

détermine le coefficient $N_{r,\theta}$.

On a, en particulier

$$\begin{aligned}
N_{r+1,0} &= (\beta+2r) \quad N_{r,0}, \\
N_{r+1,1} &= (\beta+2r+1) \quad N_{r,1}+(\gamma-\beta-r) \quad N_{r,0}, \\
N_{r+1,2} &= (\beta+2r+2) \quad N_{r,2}+(\gamma-\beta-r-1)\, N_{r,1}, \\
&\vdots \\
N_{r+1,r+1} &= (\gamma-\beta-2r)\, N_{r,r},
\end{aligned}$$

lesquelles donnent tout de suite

$$N_{r,0} = [\beta+2r-2,\ -2]^r, \quad N_{r,r} = [\gamma-\beta,\ -2]^r.$$

19. L'équation ressemble à celle pour $L_{r,\theta}$, et l'on pourrait croire que l'intégration s'accomplirait d'une manière semblable, mais cela n'est pas ainsi; car en considérant la seconde équation de la suite, c'est-à-dire

$$N_{r+1,1}-(\beta+2r+1)\, N_{r,1} = (\gamma-\beta-r)\,[\beta+2r-2,\ -2]^r,$$

en y mettant

$$N_{r,1} = [\beta+2r-1,\ -2]^r\, M_{r,1},$$

on obtient

$$[\beta+2r+1,\ -2]^{r+1}(M_{r+1,1}-M_{r,1}) = (\gamma-\beta-r)\,[\beta+2r-2,\ -2]^r,$$

et de là

$$M_{r+1,1}-M_{r,1} = \frac{(\gamma-\beta-r)\,[\beta+2r-2,\ -2]^r}{[\beta+2r+1,\ -2]^{r+1}};$$

mais les facteurs du numérateur, et du dénominateur ne sont pas ici (ce qui arrivait dans l'équation pour la quantité dénotée auparavant par le même symbole $M_{r,1}$) de la même forme, et il n'y a pas de simplification dans la forme de la fraction. En écrivant successivement $r-1$, $r-2, \ldots 2$, 1, 0 pour r on trouve

$$M_{r,1} = \gamma-\beta+\frac{(\gamma-\beta-1)\,\beta}{\beta+3}+\frac{(\gamma-\beta-2)(\beta+2)\,\beta}{(\beta+5)(\beta+3)}$$
$$\ldots+\frac{(\gamma-\beta-r+1)(\beta+2r-4)\ldots(\beta+2)\,\beta}{(\beta+2r-1)\ldots(\beta+5)(\beta+3)},$$

et de là

$$N_{r,1} = (\beta+2r-1)\ldots(\beta+5)(\beta+3)\left(\gamma-\beta+\frac{(\gamma-\beta-1)\,\beta}{\beta+3}+\frac{(\gamma-\beta-2)(\beta+2)\,\beta}{(\beta+5)(\beta+3)}\right.$$
$$\left.\ldots+\frac{(\gamma-\beta-r+1)(\beta+2r-4)\ldots(\beta+2)\,\beta}{(\beta+2r-1)\ldots(\beta+5)(\beta+3)}\right).$$

Il ne paraît pas que la suite en [ ] puisse être additionnée, et cela m'empêche de pousser plus loin la recherche de la forme des coefficients $N_{r,\theta}$; la solution, telle que je l'ai trouvée, est donc donnée par l'expression de $(P^{-1}Q^{\epsilon}D_x)^r\, P^{\gamma-\beta}\, Q^{\beta}\, R^{\gamma}$ en termes des coefficients numériques $N_{r,\theta}$, et par l'équation aux différences et conditions aux limites qui déterminent les coefficients $N_{r,\theta}$.

§ IV.

20. La première des identités du Prof. Donkin est

$$D_x^n Q^{-1} R^n = (-)^n \Lambda^n [n - \tfrac{1}{2}]^n Q^{-2n-1};$$

mais par une forme ci-dessus trouvée No. 9 en y écrivant n au lieu de γ on a

$$\frac{(-)^{n+1}}{n} D_x^{n+1} \quad R^n = \Lambda^n [n - \tfrac{1}{2}]^n Q^{-2n-1},$$

et, en remarquant que $D_x R^n = -nQ^{-1} R^n$, on voit que les deux formules sont identiques, et la vérité du théorème est ainsi démontrée.

21. La seconde des deux identités est

$$P^{-n+s-1} Q^{2n+2} D_x^n Q^{-1} R^s = (P^{-1} Q^i D_x)^n P^{n+s-1} Q^{-2n+1} R^{-n+s}.$$

On a par la formule (No. 10) en y mettant s au lieu de γ et $n+1$ au lieu de r,

$$\begin{aligned} \frac{(-)^{n+1}}{s} D_x^{n+1} R^s = \quad & L'_{n+1,0} \quad Q^{-n-1} \quad R^s \\ & \vdots \\ & + L'_{n+1,\theta} \Lambda^\theta \; Q^{-n-1-\theta} R^{s-\theta} \\ & \vdots \\ & + L'_{n+1,n} \Lambda^n Q^{-2n-1} \quad R^{s-n}, \end{aligned}$$

où

$$L'_{n+1,\theta} = \frac{[\theta - \frac{1}{2}]^\theta [n+\theta]^{n-\theta} [s-\theta-1]^{n-\theta}}{[n-\theta]^{n-\theta}}.$$

22. En renversant l'ordre des termes et mettant aussi $L'_{n+1,\theta} = V_{n-\theta}$, de manière que

$$V_\theta = \frac{[n - \frac{1}{2} - \theta]^{n-\theta} [2n-\theta]^\theta [s-n+\theta-1]^\theta}{[\theta]^\theta},$$

on obtient

$$\begin{aligned} \frac{(-)^{n+1}}{s} D_x^{n+1} R^s = \quad & V_0 \Lambda^n \quad Q^{-2n-1} \quad R^{s-n} \\ & \vdots \\ & + V^\theta \Lambda^{n-\theta} Q^{-2n-1+\theta} R^{s-n+\theta} \\ & \vdots \\ & + V_n \quad Q^{-n-1} \quad R^s, \end{aligned}$$

et en remarquant que $D_x R^s = -sQ^{-1} R^s$, et que l'on a aussi $\Lambda = P^2 - 4Q^2$ et $R = P - 2Q$, de manière que $\Lambda R^{-1} = (P + 2Q)$, l'équation peut s'écrire sous la forme

$$\begin{aligned} Q^{2n+2} P^{-n+s-1} D_x^n Q^{-1} R = (-)^n \times \; & V_0 (P+2Q)^n \quad P^{-n+s-1} Q R^s \\ & \vdots \\ & + V_\theta (P+2Q)^{n-\theta} P^{-n+s-1} Q^{1+\theta} R^s \\ & \vdots \\ & + V_n \qquad P^{-n+s-1} Q^{1+n} R^s, \end{aligned}$$

et en développant les binomiels on obtient pour $Q^{2n+2} P^{-n+s-1} D_x^n Q^{-1} R^s$ la valeur $(-)^n \times$

$$(V_0) P^{s-1} QR^s + \left(\frac{[n]^1}{[1]^1} 2^1 V_0 + V_1\right) P^{s-2} Q^2 R^s + \left(\frac{[n]^2}{[2]^2} 2^2 V_0 + \frac{[n-1]^1}{[1]^1} 2^1 V_1 + V_2\right) P^{s-3} Q^3 R^s$$

$$\ldots + (2^n V_0 + 2^{n-1} V_1 \ldots + 2^1 V_{n-1} + V_n) P^{s-n-1} Q^{n+1} R^s.$$

Cela devrait être égal à

$$(P^{-1} Q^4 D_x)^n P^{n+s-1} Q^{-2n+1} R^{-n+s}$$

et le développement de cette dernière expression se déduit de celui (No. 18) de

$$(P^{-1} Q^4 D_x)^r P^{\gamma-\beta} \quad Q^\gamma \qquad R^\beta$$

en y écrivant n, $-2n+1$, $-n+s$ au lieu de r, β, γ; la valeur sera ainsi

$$= \frac{1}{2^n} N_{n,0} P^{s-1} QR^s - \frac{1}{2^{n-1}} N_{n,1} P^{s-2} Q^2 R^s + \frac{1}{2^{n-2}} N_{n,2} P^{s-3} Q^3 R^s \ldots + (-)^n N_{n,n} P^{s-n-1} Q^{n+1} R^s,$$

laquelle a la même forme que le développement de la fonction au côté gauche de l'équation, et l'identité des deux expressions dépend des équations

$$\frac{1}{2^n} \quad N_{n,0} = (-)^n \qquad\qquad V_0,$$

$$\frac{1}{2^{n-1}} N_{n,1} = (-)^{n-1} \left(\frac{[n]^1}{[1]^1} 2^1 V_0 + V_1\right),$$

$$\frac{1}{2^{n-2}} N_{n,2} = (-)^{n-2} \left(\frac{[n]^2}{[2]^2} 2^2 V_0 + \frac{[n-1]^1}{[1]^1} 2^1 V_1 + V_2\right),$$

$$\vdots$$

$$N_{n,n} = (2^n V_0 + 2^{n-1} V_1 \ldots + 2 V_{n-1} + V_n);$$

mais comme on ne sait pas la forme générale des coefficients $N_{n,\theta}$ je n'ai pas pu vérifier complètement ces équations. On a cependant en mettant n, $-2n+1$, $-n+s$ pour r, β, γ

$$N_{n,0} = \quad [-1, +2]^n \ , \ = (-)^n \quad [2n-1, -2]^n \ , \ = (-1)^n 2^n \quad [n-\tfrac{1}{2}]^n,$$

$$N_{n,1} = s\,[-3, +2]^{n-1}, \ = (-)^{n-1} s\,[2n-1, -2]^{n-1}, \ = (-)^{n-1} 2^{n-1} s\,[n-\tfrac{1}{2}]^{n-1},$$

et les deux premières équations seront

$$[n-\tfrac{1}{2}]^n = [n-\tfrac{1}{2}]^n; \quad s\,[n-\tfrac{1}{2}]^{n-1} = \frac{n}{1} 2\,[n-\tfrac{1}{2}]^n + [n-\tfrac{3}{2}]^{n-1} (2n-1)(s-n),$$

où dans la seconde équation l'expression au côté droit est

$$[2n + 2(s-n)]\,[n-\tfrac{1}{2}]^n = 2s\,[n-\tfrac{1}{2}]^n = s\,[n-\tfrac{1}{2}]^{n-1},$$

comme cela devrait être. La dernière équation de la suite est

$$N_{n,n} = 2^n V_0 + 2^{n-1} V_1 \ldots + 2 V_{n-1} + V_n$$

et comme on a $N_{n,n} = [n+s-1, -2]^n$, cette dernière équation peut aussi se vérifier.

252.

NOTE SUR L'ÉQUATION DES DIFFÉRENCES POUR UNE ÉQUATION DONNÉE DE DEGRÉ QUELCONQUE.

[From the *Annali di Matematica pura ed applicata* (Tortolini), tom. II. (1859), pp. 365, 366.]

IL s'agit de trouver l'équation qui a pour racines les carrés des différences des racines d'une équation donnée

$$(*)\,(v,\ 1)^n = 0.$$

En représentant cette équation par $\phi v = 0$, soient x, y deux racines différentes quelconques, on a non seulement $\phi x = 0$, $\phi y = 0$; mais aussi

$$\phi x + \phi y = 0,\ \frac{\phi x - \phi y}{x - y} = 0,$$

et en écrivant dans ces équations

$$x + y = 2s,\quad (x-y)^2 = 4\theta,$$

(ou ce qui est la même chose $x = s + \sqrt{\theta}$, $y = s - \sqrt{\theta}$) on obtient deux équations rationnelles en s, et θ, et en éliminant s on obtient l'équation qui donne

$$\theta = \tfrac{1}{4}(x-y)^2.$$

Il convient de changer un peu la forme des équations; en effet la première équation est du degré n, la seconde du degré $n-1$ pour rapport à s, mais en écrivant les deux équations sous la forme

$$n(\phi x + \phi y) - (x+y)\frac{\phi x - \phi y}{x-y} = 0,\qquad \frac{\phi x - \phi y}{x-y} = 0,$$

l'une et l'autre équation sera du degré $n-1$ par rapport à s. La forme sous laquelle j'ai présenté la méthode de Bezout s'applique au problème; en représentant les deux équations par $Fs=0$, $Gs=0$ j'écris pour le moment

$$\phi(s+\sqrt{\theta})=A, \quad \phi(s-\sqrt{\theta})=B,$$

on a alors

$$Fs=n(A+B)-s\frac{A-B}{\sqrt{\theta}}, \quad Gs=\frac{A-B}{\sqrt{\theta}},$$

et en écrivant

$$\phi(s'+\sqrt{\theta})=A', \quad \phi(s'-\sqrt{\theta})=B',$$

on a de même

$$Fs'=n(A'+B')-\frac{s'(A'-B')}{\sqrt{\theta}}, \quad Gs'=\frac{A'-B'}{\sqrt{\theta}}.$$

On obtient de là

$$\Phi(s, s')=\frac{Fs\,Gs'-Fs'Gs}{s-s'},$$

$$=\left[\left(n(A+B)-\frac{s(A-B)}{\sqrt{\theta}}\right)\frac{A'-B'}{\sqrt{\theta}}-\left(n(A'+B')-\frac{s'(A'-B')}{\sqrt{\theta}}\right)\frac{(A-B)}{\sqrt{\theta}}\right]\frac{1}{s-s'},$$

ou, en réduisant,

$$-\Phi(s, s')=\frac{2n(AB'-A'B)}{(s-s')\sqrt{\theta}}+\frac{(A-B)(A'-B')}{\theta}.$$

Donc, en rétablissant les valeurs de A, B, A', B', on a la fonction

$$\frac{2n[\phi(s+\sqrt{\theta})\phi(s'-\sqrt{\theta})-\phi(s'+\sqrt{\theta})\phi(s-\sqrt{\theta})]}{(s-s')\sqrt{\theta}}$$

$$+\frac{[\phi(s+\sqrt{\theta})-\phi(s-\sqrt{\theta})][\phi(s'+\sqrt{\theta})-\phi(s'-\sqrt{\theta})]}{\theta}$$

laquelle sera de la forme

$$\left(\begin{array}{l} a_{0,0},\ a_{0,1},\ \ldots\ a_{0,n-2} \\ a_{1,0},\ a_{1,1},\ \ldots \\ \vdots \\ a_{n-2,0},\ \ldots \end{array}\right)(s, 1)^{n-2}(s', 1)^{n-2},$$

où les coefficients a sont des fonctions rationnelles de θ, et en égalant à zero le déterminant formé avec ces coefficients on a l'équation qu'il s'agissait de trouver. Quoique cette solution soit analytiquement la plus simple, j'ai une autre méthode nouvelle plus adaptée au calcul, laquelle j'ai appliquée à trouver l'équation des différences pour l'équation quintique

$$(a, b, c, d, e, f \between v, 1)^5=0.$$

Londres, 4 *Fév.* 1860.

253.

SUR LA COURBE PARALLÈLE À L'ELLIPSE.

[From the *Annali di Matematica pura ed applicata* (Tortolini), tom. III. (1860), pp. 311—316.]

Il fut remarqué par Cauchy (*Comptes Rendus*, tom. XIII. [1841], p. 1063) que l'équation de cette courbe pourrait se trouver en éliminant θ entre les équations

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}=1, \quad \frac{\theta^2x^2}{(\theta+a^2)^2}+\frac{\theta^2y^2}{(\theta+b^2)^2}=k^2,$$

et cette élimination fut effectuée, et le résultat trouvé sous la forme la plus simple par M. Catalan (*Terquem*, tom. III. 1844, p. 553); mais pour faire l'élimination de la manière la plus facile je remarque que ces deux équations donnent

$$\frac{x^2}{\theta+a^2}+\frac{y^2}{\theta+b^2}=1+\frac{k^2}{\theta}, \quad \frac{x^2}{(\theta+a^2)^2}+\frac{y^2}{(\theta+b^2)^2}=\frac{k^2}{\theta^2},$$

dont la seconde est la dérivée, par rapport à θ, de la première; or cette première équation est

$$(\theta+k^2)(\theta+a^2)(\theta+b^2)-x^2\theta(\theta+b^2)-y^2\theta(\theta+a^2)=0,$$

et, en égalant à zéro le discriminant de la fonction cubique, on aura l'équation de la courbe; en posant

$$A=x^2+y^2-k^2-a^2-b^2, \quad B=b^2x^2+a^2y^2-a^2k^2-b^2k^2-a^2b^2, \quad C=a^2b^2k^2,$$

la fonction cubique, multipliée par 3, sera

$$(3,-A,-B,3C\between\theta,1)^2;$$

on aura donc

$$4(A^2+3B)(B^2+3AC)-(AB-9C)^2=0,$$

ou ce qui est la même chose

$$A^2B^2 + 4B^3 + 4A^3C + 18ABC - 27C^2 = 0,$$

c'est-à-dire en substituant les valeurs de A, B, C, l'équation de la courbe sera

$$\begin{aligned}&(x^2+y^2-k^2-a^2-b^2)^2(b^2x^2+a^2y^2-a^2k^2-b^2k^2-a^2b^2)^2\\&+4(b^2x^2+a^2y^2-a^2k^2-b^2k^2-a^2b^2)^3\\&+4a^2b^2k^2(x^2+y^2-k^2-a^2-b^2)^3\\&+18a^2b^2k^2(x^2+y^2-k^2-a^2-b^2)(b^2x^2+a^2y^2-a^2k^2-b^2k^2-a^2b^2)\\&-27a^4b^4k^4=0,\end{aligned}$$

laquelle est en effet l'équation trouvée par M. Catalan.

En écrivant $k=0$ l'équation devient

$$0=(b^2x^2+a^2y^2-a^2b^2)\left[(x^2+y^2)^2-2(a^2-b^2)(x^2+y^2)+(a^2-b^2)^2\right],$$

ce qui équivaut à l'équation de l'ellipse

$$b^2x^2+a^2y^2-a^2b^2=0$$

deux fois répétée, et aux équations

$$(x\pm ae)^2+y^2=0,$$

(où comme à l'ordinaire $a^2e^2=a^2-b^2$) lesquelles appartiennent aux droites menées chacune par un foyer réel et un foyer imaginaire de l'ellipse; ces droites sont aussi les tangentes menées à l'ellipse par les quatre foyers. En effet en prenant sur l'ellipse le point imaginaire dont les coordonnées sont

$$x=\frac{a}{e},\quad y=ia\left(\frac{1}{e}-e\right),\ (i=\sqrt{-1})$$

l'équation du cercle, rayon 0, ayant pour centre le point dont il s'agit, sera

$$\left(x-\frac{a}{e}\right)^2+\left[y-ia\left(\frac{1}{e}-e\right)\right]^2=0;$$

ce cercle se réduit donc à deux droites, savoir la droite

$$x-\frac{a}{e}+i\left[y-ia\left(\frac{1}{e}-e\right)\right]=0,$$

ou, ce qui est la même chose,

$$x-ae+iy=0,$$

laquelle est tangente à l'ellipse, et la droite

$$\left(x-\frac{a}{e}\right)-i\left[y-ia\left(\frac{1}{e}-e\right)\right]=0,$$

ou, ce qui est la même chose,

$$x - a\left(\frac{2}{e} - e\right) - iy = 0,$$

laquelle est la droite menée par le point de contact à l'autre point circulaire à l'infini (celui qui n'est pas situé sur la tangente $x - ae + iy = 0$). Le cercle, comme courbe composée d'une droite tangente à l'ellipse et d'une autre droite menée par le point de contact, a même avec l'ellipse une intersection à trois points réunis. Ces considérations font voir pourquoi au cas $k = 0$ les quatre droites $(x \pm ae)^2 + y^2 = 0$ font partie de la courbe.

En supposant que k est quelconque, mais que l'on a $a = b$, l'équation devient

$$a^4 (x^2 + y^2)^2 [(x^2 + y^2 - a^2)^2 - 2k^2 (x^2 + y^2 + a^2) + k^4] = 0,$$

où, en écartant le facteur $(x^2 + y^2)^2$ et le facteur constant a^4, l'équation se réduit à

$$(x^2 + y^2 - a^2)^2 - 2k^2 (x^2 + y^2 + a^2) + k^4 = 0,$$

c'est-à-dire

$$[x^2 + y^2 - (a + k)^2] [x^2 + y^2 - (a - k)^2] = 0;$$

ainsi la courbe se réduit à l'ensemble des deux droites $(x^2 + y^2) = 0$ (chacune deux fois répétée, et des deux cercles

$$x^2 + y^2 - (a + k)^2 = 0, \quad x^2 + y^2 - (a - k)^2 = 0$$

comme évidemment cela doit être.

Par rapport aux singularités de la courbe, la forme de l'équation montre à premier coup d'œil qu'il y a quatre points doubles à l'infini; savoir les deux points circulaires à l'infini (points où l'infini est rencontré par les droites $x^2 + y^2 = 0$), et les deux points où l'infini est rencontré par les droites $b^2x^2 + a^2y^2 = 0$. Il y a de plus deux points doubles sur chacun des axes: en effet en écrivant dans l'équation de la courbe $y = 0$ l'équation résultante, toute réduction faite, devient

$$[(x - a)^2 - k^2] [(x + a)^2 + k^2] [b^2x^2 - (a^2 - b^2)(b^2 - k^2)]^2 = 0,$$

ce qui fait voir qu'il y a sur l'axe de x les deux points doubles dont les coordonnées sont données par l'équation

$$b^2x^2 - (a^2 - b^2)(b^2 - k^2) = 0.$$

Mais pour parvenir à cette conclusion il convient de considérer que l'axe de x rencontre la courbe dans des points tels que pour chacun la distance normale à l'ellipse est égale à k: il y a quatre distances normales k, mesurées le long de l'axe; et quatre distances normales k mesurées à des points situés symétriquement par rapport à l'axe, lesquels se rencontrent deux à deux sur l'axe aux points dont les coordonnées sont $b^2x^2 = (a^2 - b^2)(b^2 - k^2)$; c'est là l'origine des deux points doubles sur l'axe de x. On doit

aussi remarquer que les coordonnées des points sur l'ellipse qui correspondent à ces points doubles sont données par les équations

$$x^2 = \frac{a^4}{b^2}\frac{b^2 - k^2}{a^2 - b^2}, \quad y^2 = \frac{a^2k^2 - b^4}{a^2 - b^2};$$

on voit de là que les points sur l'ellipse qui donnent lieu aux deux points doubles sur l'axe de x, ne seront réels à moins que $k > \frac{b^2}{a^3}, < b$; les deux points doubles seront cependant réels pour toute valeur $k > 0 < b$. J'ajoute que pour les valeurs $k \overset{=}{>} 0, < \frac{b^2}{a}$, les deux points doubles seront des points conjugués ou isolés; pour $k = \frac{b^2}{a}$, chacun de ces points se réunit à la courbe, et à deux points de rebroussement, mais il n'y a pas de singularité visible dans la forme de la courbe; pour $k > \frac{b^2}{a} < b$ il y a deux points doubles avec des branches réelles; et pour $k = b$, ces deux points viennent se réunir au centre de l'ellipse et il y a deux branches ayant pour tangente commune à ce point l'axe de x; enfin pour $k > b$, les deux points doubles sur l'axe de x deviennent imaginaires.

Des considérations pareilles s'appliquent aux points doublés sur l'axe de y; pour $x = 0$ l'équation de la courbe se transforme en

$$[(y-b)^2 - k^2]\,[(y+b)^2 + k^2]\,[a^2y^2 - (a^2 - b^2)(k^2 - a^2)]^2 = 0;$$

il y a donc sur l'axe de y deux points doubles dont les coordonnées sont données par

$$a^2y^2 - (a^2 - b^2)(k^2 - a^2) = 0,$$

ces points correspondent à des points sur l'ellipse dont les coordonnées sont données par

$$x^2 = \frac{a^4 - b^2k^2}{a^2 - b^2}, \quad y^2 = \frac{b^4}{a^2}\frac{k^2 - a^2}{a^2 - b^2};$$

les points sur l'ellipse ne seront donc réels à moins que $k > \frac{a^2}{b}$; les deux points doubles seront cependant réels pour toute valeur $k > a$. Pour $k < a$ les deux points doubles seront imaginaires. Pour $k = a$ le centre est point de réunion de deux points doubles et il y a deux branches qui ont à ce point pour tangente commune l'axe de y: pour $k = \frac{a^2}{b}$, chacun des deux points doubles se réunit à deux points de rebroussement; mais il n'y a pas de singularité visible dans la forme de la courbe; enfin pour $k > \frac{a^2}{b}$ les deux points doubles se détachent de la courbe et deviennent des points conjugués ou isolés. En résumé, il y a huit points doubles, savoir quatre points doubles à l'infini, et quatre points doubles sur les deux axes.

Il y a de plus 12 points de rebroussement; ces points sont en effet les centres de courbure aux points de l'ellipse pour lesquels le rayon de courbure est égal à k;

les douze points seront tous imaginaires à moins que k n'ait une valeur entre les limites $\frac{b^2}{a}$, $\frac{a^2}{b}$: pour une telle valeur de k les douze points seront 4 réels, et 8 imaginaires. Pour $k=\frac{b^2}{a}$ les quatre points réels se réunissent deux à deux aux deux points doubles sur l'axe de x; pour $k=\frac{a^2}{b}$, ils se réunissent deux à deux aux deux points doubles sur l'axe de y; cela s'accorde avec ce qui est dit ci-dessus par rapport aux points doubles. On peut encore trouver que le nombre des points de rebroussement est 12 à moyen des équations

$$A^2+3B=0,\quad B^2+3AC=0,\quad AB-9C=0,$$

lesquelles appartiennent chacune à une courbe du quatrième ordre qui passe par les points de rebroussement; la forme des équations fait voir que ces courbes ont en commun 12 points, et seulement 12 points, d'intersection. C'est là en effet la manière la plus simple de trouver les coordonnées des points de rebroussement; car ces équations donnent tout de suite

$$A=-3C^{\frac{1}{3}},\qquad B=-3C^{\frac{2}{3}},$$

c'est-à-dire on a les équations

$$x^2+\ \ y^2=a^2\ \ +b^2+\ \ k^2-3\,(abk)^{\frac{2}{3}},$$
$$b^2x^2+a^2y^2=a^2b^2+(a^2+b^2)\,k^2-3\,(abk)^{\frac{4}{3}}$$

qui donnent les coordonnées des douze points de rebroussement.

Pour vérifier que ces points correspondent aux points de l'ellipse pour lesquels le rayon de courbure est égal à k; je prends $a\cos\theta$, $b\sin\theta$ pour les coordonnées d'un point sur l'ellipse, les coordonnées du centre de courbure seront données par

$$ax=(a^2-b^2)\cos^3\theta,\qquad by=-(a^2-b^2)\sin^3\theta,$$

et en supposant que le rayon de courbure soit égal à k, on a

$$k=\frac{1}{ab}\,(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}},$$

et de là

$$a^2\sin^2\theta+b^2\cos^2\theta=(abk)^{\frac{2}{3}}.$$

Cela donne

$$(a^2-b^2)\cos^2\theta=a^2-(abk)^{\frac{2}{3}},\qquad -(a^2-b^2)\sin^2\theta=b^2-(abk)^{\frac{2}{3}},$$

et de là on déduit

$$x^2+y^2=\frac{1}{a^2-b^2}\left[\frac{1}{a^2}\left(a^2-(abk)^{\frac{2}{3}}\right)^2-\frac{1}{b^2}\left(b^2-(abk)^{\frac{2}{3}}\right)^3\right]$$

$$b^2x^2+a^2y^2=\frac{1}{a^2-b^2}\left[\frac{b^2}{a^2}\left(a^2-(abk)^{\frac{2}{3}}\right)^3-\frac{a^2}{b}\left(b^2-(abk)^{\frac{2}{3}}\right)^3\right]$$

valeurs qui s'accordent en effet avec les valeurs ci-dessus trouvées.

La courbe parallèle à l'ellipse est de l'ordre 8 et il y a 8 points doubles, et 12 points de rebroussement; elle sera donc de la classe $56-16-36=4$. En effet pour mener d'un point donné une tangente à la courbe parallèle on peut décrire avec ce point comme centre un cercle rayon k, pour mener une tangente commune au cercle et à l'ellipse; la droite, menée par le point donné, parallèle à la tangente commune sera la tangente cherchée: on peut donc par un point donné mener 4 tangentes à la courbe, ou la courbe parallèle est de la classe 4. Au reste on peut trouver très facilement l'équation de la courbe parallèle en coordonnées tangentielles. Car en représentant par θ l'inclination d'une tangente quelconque de l'ellipse à l'axe de x, l'équation de cette tangente est

$$x\cos\theta+y\sin\theta \qquad -\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=0,$$

et l'équation de la tangente parallèle de la courbe sera ainsi

$$x\cos\theta+y\sin\theta-k-\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=0;$$

donc en représentant cette équation par $\xi x+\eta y+\zeta=0$ on aura

$$\xi : \eta : \zeta=\cos\theta : \sin\theta : k+\sqrt{a^2\cos^2\theta+b^2\sin^2\theta},$$

et de là

$$\zeta+k\sqrt{\xi^2+\eta^2}+\sqrt{a^2\xi^2+b^2\eta^2}=0,$$

laquelle est l'équation dont il s'agit; la forme rationnelle est

$$(a^2-k^2)^2\xi^4+(b^2-k^2)^2\eta^4+2(a^2-k^2)(b^2-k^2)\xi^2\eta^2-2(a^2+k^2)\zeta^2\xi^2-2(b^2+k^2)\zeta^2\eta^2+\zeta^4=0,$$

équation du quatrième ordre comme cela devait être.

J'ai cru qu'il n'était pas nécessaire de faire voir comment on pourrait à moyen de l'équation tracer la courbe; en effet on trouve les différentes formes assez facilement par des considérations géométriques, en considérant la courbe comme la développée de l'évolute de l'ellipse: on se rend compte à ce moyen de ce qui est déjà dit par rapport aux points doubles et aux points de rebroussement.

Je remarque enfin que l'équation d'une normale quelconque de l'ellipse est

$$ax\sin\theta-by\cos\theta=(a^2-b^2)\sin\theta\cos\theta$$

(où θ est un paramètre arbitraire): donc en considérant la trajectoire orthogonale de ce système de droites, on obtient

$$(xdx+ydy)\sqrt{b^2dx^2+a^2dy^2}+(a^2-b^2)\,dx\,dy=0$$

comme équation différentielle de la courbe parallèle à l'ellipse. L'intégrale de cette équation est donc l'équation du huitième ordre (contenant k comme constante arbitraire) qu'on a ci-dessus trouvée.

Londres, 2 *Stone Buildings*, 22 Oct. 1860.

254.

SUR LA SURFACE PARALLÈLE À L'ELLIPSOIDE.

[From the *Annali di Matematica pura ed applicata*, (Tortolini), tom. III. (1860), pp. 345—352.]

Il y a pour la surface parallèle à l'ellipsoïde une solution tout à fait semblable à celle pour la courbe parallèle à l'ellipse. En effet soit

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$$

l'équation de l'ellipsoïde; k la distance de la surface parallèle, en prenant cette distance sur la normale extérieure au point (X, Y, Z) et en écrivant pour abréger

$$\sqrt{\frac{X^2}{a^4}+\frac{Y^2}{b^4}+\frac{Z^2}{c^4}}=\frac{k}{\lambda},$$

on trouve pour les coordonnées (x, y, z) de l'extrémité de la normale

$$x=X\left(1+\frac{\lambda}{a^2}\right),\quad y=Y\left(1+\frac{\lambda}{b^2}\right),\quad z=Z\left(1+\frac{\lambda}{c^2}\right),$$

et, en substituant pour X, Y, Z les valeurs données par ces équations, on obtient

$$\frac{a^2x^2}{(a^2+\lambda)^2}+\frac{b^2y^2}{(b^2+\lambda)^2}+\frac{c^2z^2}{(c^2+\lambda)^2}=1,$$

$$\frac{\lambda^2x^2}{(a^2+\lambda)^2}+\frac{\lambda^2y^2}{(b^2+\lambda)^2}+\frac{\lambda^2z^2}{(c^2+\lambda)^2}=k^2.$$

Or ces équations donnent

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1+\frac{k^2}{\lambda},$$

$$\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}=\frac{k^2}{\lambda^2},$$

et la seconde équation est la dérivée par rapport à λ de la première. C'est-à-dire en obtient l'équation de la surface parallèle en égalant à zéro le discriminant par rapport à λ de l'équation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1+\frac{k^2}{\lambda},$$

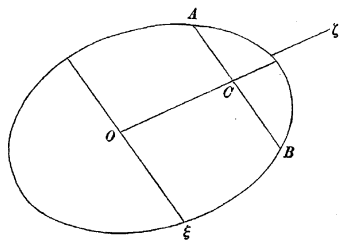
ou autrement dit, la surface parallèle est l'enveloppe par rapport à λ de cette équation. En multipliant par $\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$, on aura une équation du quatrième ordre, dont le discriminant égalé à zéro donne l'équation de la surface: on voit sans peine que cette équation sera de l'ordre 10.

J'avais à peine trouvé cette méthode, quand j'ai appris que le problème était déjà résolu par MM. Salmon et W. Roberts. M. Salmon considère la surface comme lieu des centres des sphères, rayon k, qui touchent à l'ellipsoïde $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$, ce qui lui donne tout de suite le théorème que voici; savoir on obtient l'équation de la surface parallèle, en égalant à zéro le discriminant par rapport à λ du discriminant par rapport à (X, Y, Z) de l'équation

$$\lambda\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}-1\right)+(X-x)^2+(Y-y)^2+(Z-z)^2=0;$$

cette solution est donnée en passant dans une Note publiée dans le No. de Septembre 1860 des Nouvelles Annales (Terquem et Gerono). En formant le discriminant par rapport à X, Y, Z on trouve l'équation en λ ci-dessus donnée.

La solution de M. W. Roberts n'avait pas été publiée: il a bien voulu me permettre d'en faire part aux géomètres. En considérant une section circulaire de l'ellipsoïde, l'enveloppe des sphères, rayon k, ayant leurs centres sur cette section, sera une surface annulaire, et l'enveloppe des surfaces annulaires qui correspondent aux différentes



sections circulaires de l'ellipsoïde sera la surface parallèle. Dans la figure (qui est censée située dans le plan des axes le plus grand et le plus petit) soit $O\xi$ la trace de la section circulaire centrale, $O\zeta$ le diamètre perpendiculaire, AB la trace d'une section quelconque parallèle à la section circulaire centrale: soit de plus $AB=2\delta$, et (α, γ)

les coordonnées du point C qui est le centre de la section AB. En prenant pour un moment ce point C pour origine, et l'axe de η parallèle à l'axe moyen, la surface annulaire sera l'enveloppe de la sphère

$$(\xi - \delta\cos\phi)^2 + (\eta - \delta\sin\phi)^2 + \zeta^2 = k^2,$$

(où ϕ est un paramètre arbitraire); donc l'équation de la surface annulaire sera

$$4\delta^2(\xi^2+\eta^2) = (\xi^2+\eta^2+\zeta^2+\delta^2-k^2)^2,$$

ou, en prenant le centre de l'ellipsoïde pour origine, cette équation sera

$$4\delta^2(\xi^2+\eta^2-2\alpha\xi+\alpha^2) = (\xi^2+\eta^2+\zeta^2+\delta^2-k^2-2\alpha\xi-2\gamma\zeta-\alpha^2-\gamma^2)^2.$$

Mais par les propriétés de l'ellipse on a $\gamma = l\alpha$ et $\delta^2 = m + n\alpha^2$, où l, m, n sont des quantités constantes, fonctions des axes a, b, c de l'ellipsoïde. L'équation de la surface annulaire devient ainsi

$$4(m+n\alpha^2)(\xi^2+\eta^2-2\alpha\xi+\alpha^2) = [\xi^2+\eta^2+\zeta^2+m-k^2-2\alpha(\xi+l\zeta)+\alpha^2(n+1+l^2)]^2,$$

équation qui contient le paramètre variable α au quatrième degré, et en égalant à zéro le discriminant par rapport à α de cette équation, on obtiendrait l'équation de la surface parallèle. À propos de cette solution je remarque qu'en considérant la surface annulaire, enveloppe des sphères, rayon k, ayant leurs centres sur l'ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1 = 0,$$

on obtient sans peine le système d'équations

$$\frac{a^2x^2}{(a^2+\lambda)^2} + \frac{b^2y^2}{(b^2+k)^2} = 1, \qquad \frac{\lambda^2x^2}{(a^2+\lambda)^2} + \frac{\lambda^2y^2}{(c^2+\lambda)^2} = k^2 - z^2,$$

et de là l'équation de la surface annulaire s'obtient en égalant à zéro le discriminant par rapport à λ de l'équation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 + \frac{k^2-z^2}{\lambda},$$

c'est-à-dire en égalant à zéro le discriminant d'une fonction *cubique* de λ. Mais en supposant que l'ellipse devient un cercle, l'on n'aura qu'une fonction quadratique de λ; c'est là pourquoi la surface annulaire considérée par M. Roberts s'exprime sous une forme assez simple pour qu'on puisse s'en servir pour trouver l'équation de la surface parallèle. Soit à présent l'ellipse, section de l'ellipsoïde

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1$$

par un plan quelconque

$$lX + mY + nZ = p;$$

il est clair que l'équation de la surface annulaire doit se trouver de même à moyen d'une fonction cubique de λ. Pour vérifier cela, je remarque que l'on a à trouver l'enveloppe de la sphère $(X-x)^2+(Y-y)^2+(Z-z)^2=k^2$, où les paramètres X, Y, Z sont liés par les deux équations qui viennent d'être mentionnées: cela donne tout de suite pour la surface annulaire le système des équations

$$x-X-\lambda\frac{X}{a^2}-\mu l=0,$$

$$y-Y-\lambda\frac{Y}{b^2}-\mu m=0,$$

$$z-Z-\lambda\frac{Z}{c^2}-\mu n=0,$$

où λ, μ sont des paramètres arbitraires: ces équations donnent

$$X=\frac{a^2(x-\mu l)}{a^2+\lambda},\quad Y=\frac{b^2(y-\mu m)}{b^2+\lambda},\quad Z=\frac{c^2(z-\mu n)}{c^2+\lambda},$$

et de là, en mettant pour abréger

$$\frac{la^2x}{a^2+\lambda}+\frac{mb^2y}{b^2+\lambda}+\frac{nc^2z}{c^2+\lambda}-p=P,$$

$$\frac{l^2a^2}{a^2+\lambda}+\frac{m^2b^2}{b^2+\lambda}+\frac{n^2c^2}{c^2+\lambda}=Q,$$

on trouve à moyen de l'équation linéaire

$$Q\mu-P=0,\ \text{où}\ \mu=\frac{P}{Q},$$

et les deux autres équations donnent alors

$$\frac{a^2(x-\mu l)^2}{(a^2+\lambda)^2}+\frac{b^2(y-\mu m)^2}{(b^2+\lambda)^2}+\frac{c^2(z-\mu n)^2}{(c^2+\lambda)^2}=1,$$

$$\frac{(\lambda x+a^2\mu l)^2}{(a^2+\lambda)^2}+\frac{(\lambda y+b^2\mu m)^2}{(b^2+\lambda)^2}+\frac{(\lambda z+c^2\mu n)^2}{(c^2+\lambda)^2}=k^2,$$

où μ est censé dénoter la valeur $\frac{P}{Q}$. Je déduis de là les équations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}+\frac{\mu^2}{\lambda}Q=1+\frac{k^2}{\lambda},$$

$$\begin{aligned}&\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\\&+2\left(\frac{a^2lx}{(a^2+\lambda)^2}+\frac{b^2my}{(b^2+\lambda)^2}+\frac{c^2nz}{(c^2+\lambda)^2}\right)\frac{\mu}{\lambda}\\&+\left(\frac{a^4l^2}{(a^2+\lambda)^2}+\frac{b^4m^2}{(b^2+\lambda)^2}+\frac{c^4n^2}{(c^2+\lambda)^2}\right)=\frac{k^2}{\lambda^2}.\end{aligned}$$

Soient P', Q' les dérivées de P, Q par rapport à λ, on a

$$-P' = \frac{la^2x}{(a^2+\lambda)^2} + \frac{mb^2y}{(b^2+\lambda)^2} + \frac{nc^2z}{(a^2+\lambda)^2},$$

$$-Q' = \frac{l^2a^2}{(a^2+\lambda)^2} + \frac{m^2b^2}{(b^2+\lambda)^2} + \frac{n^2c^2}{(c^2+\lambda)^2},$$

et de là aussi

$$Q+\lambda Q' = \frac{a^4l^2}{(a^2+\lambda)^2} + \frac{b^4m^2}{(b^2+\lambda)^2} + \frac{c^4n^2}{(c^2+\lambda)^2}.$$

A moyen de ces équations, et en substituant pour μ la valeur $\frac{Q}{P}$, les deux équations ci-dessus données deviennent

$$\left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 - \frac{k^2}{\lambda}\right) Q + \frac{P^2}{\lambda} = 0,$$

$$\left(\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2} - \frac{k^2}{\lambda^2}\right) Q - 2\frac{PP'}{\lambda} + (Q+\lambda Q')\frac{P^2}{Q\lambda^2} = 0;$$

en différentiant la première équation par rapport à λ on obtient

$$-\left(\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2} - \frac{k^2}{\lambda^2}\right) Q$$
$$+\left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 - \frac{k^2}{\lambda}\right) Q' + \frac{2PP'}{\lambda} - \frac{P^2}{\lambda^2} = 0,$$

et cette équation se réduit à une identité à moyen des deux équations: donc l'équation de la surface annulaire s'obtient en égalant à zéro le discriminant par rapport à λ de l'équation

$$\left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 - \frac{k^2}{\lambda}\right) Q + \frac{P^2}{\lambda} = 0,$$

c'est-à-dire, en substituant pour P, Q leurs valeurs, l'équation sera

$$\left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 - \frac{k^2}{\lambda}\right)\left(\frac{a^2l^2}{a^2+\lambda} + \frac{b^2m^2}{b^2+\lambda} + \frac{c^2n^2}{c^2+\lambda}\right)$$
$$+\frac{1}{\lambda}\left(\frac{a^2lx}{a^2+\lambda} + \frac{b^2my}{b^2+\lambda} + \frac{c^2nz}{c^2+\lambda} - p\right)^2 = 0.$$

Dans cette équation, en réunissant les termes

$$\frac{x^2}{a^2+\lambda}\,\frac{a^2l^2}{a^2+\lambda} + \frac{1}{\lambda}\,\frac{a^4l^2x^2}{(a^2+\lambda)^2},$$

on voit que ces termes se réduisent à

$$\frac{(a^2+\lambda)\,\dfrac{a^4l^2x^2}{\lambda}}{(a^2+\lambda)}, = \frac{a^4l^2x^2}{\lambda(a^2+\lambda)},$$

et de même pour les termes semblables en y, z. Donc en multipliant par

$$\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$$

les dénominateurs disparaissent, et l'équation est réellement de l'ordre 3 par rapport à λ.

La section sera circulaire en supposant

$$l=\sqrt{\frac{1}{b^2}-\frac{1}{a^2}},\quad m=0,\quad n=\sqrt{\frac{1}{c^2}-\frac{1}{b^2}};$$

cela donne

$$\frac{a^2l^2}{a^2+\lambda}+\frac{b^2m^2}{b^2+\lambda}+\frac{c^2n^2}{c^2+\lambda}=\frac{\frac{a^2}{b^2}-1}{a^2+\lambda}+\frac{1-\frac{c^2}{b^2}}{c^2+\lambda}=\frac{(a^2-c^2)(b^2+\lambda)}{b^2(a^2+\lambda)(c^2+\lambda)},$$

l'équation en λ devient ainsi

$$\left(\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1-\frac{k^2}{\lambda}\right)(a^2-c^2)(b^2+\lambda)$$
$$+\frac{b^2(a^2+\lambda)(c^2+\lambda)}{\lambda}\left(\frac{a^2x\sqrt{\frac{1}{b^2}-\frac{1}{a^2}}}{a^2+\lambda}+\frac{c^2z\sqrt{\frac{1}{c^2}-\frac{1}{b^2}}}{c^2+\lambda}-p\right)^2=0,$$

laquelle se réduit à

$$\begin{aligned}
&x^2[\lambda(a^2-c^2)+c^2(a^2-b^2)]+y^2\lambda(a^2-c^2)+z^2[\lambda(a^2-c^2)+a^2(b^2-c^2)]\\
&\quad+2acxz\sqrt{a^2-b^2}.\sqrt{b^2-c^2}\\
&\quad-2bp[ax(c^2+\lambda)\sqrt{a^2-b^2}+by(a^2+\lambda)\sqrt{b^2-c^2}]\\
&\quad-2(a^2-c^2)(b^2+\lambda)(c^2+\lambda)+b^2p^2(a^2+\lambda)(c^2+\lambda)=0,
\end{aligned}$$

équation de l'ordre 2 par rapport à (x, y, z), par rapport à λ, et par rapport à p. Donc en égalant à zéro le discriminant par rapport à λ, on obtiendrait une équation du quatrième ordre par rapport aux coordonnées (x, y, z), et par rapport à p; on aurait ainsi l'équation de la surface annulaire de M. Roberts, rapportée aux axes de l'ellipsoïde, et en termes des quantités a, b, c, k et du paramètre p qui donne la position du plan

$$X\sqrt{\frac{1}{b^2}-\frac{1}{a^2}}+Y\sqrt{\frac{1}{c^2}-\frac{1}{b^2}}-p=0$$

de la section circulaire.

M. Roberts a donné (*Comptes Rendus* Nov. 14, 1859) le théorème que voici qui se rapporte à une surface primitive quelconque:—"Soit D la surface semblable à la primitive en multipliant par 2 ses rayons vecteurs, et soit P la surface parallèle ou équidistante de D par la longueur constante k, l'équation qui résulte de la substitution de $\sqrt{x^2+y^2+z^2}$ au lieu de k dans l'équation de P, coïncide avec l'équation

de la première dérivée négative de la surface primitive"—ou je rappelle que dans la théorie des surfaces dérivées de M. Hirst la *première dérivée négative* est la surface enveloppe des plans conduits par les points de la surface primitive perpendiculairement aux rayons menés par un point quelconque. Pour démontrer le théorème, prenons $f(X, Y, Z)=0$ pour l'équation de la surface primitive: en supposant que (X, Y, Z) soient les coordonnées d'un point de cette surface, $(2X, 2Y, 2Z)$ seront les coordonnées d'un point de la surface D, et la surface parallèle à D sera l'enveloppe des sphères

$$(x-2X)^2+(y-2Y)^2+(z-2Z)^2=k^2,$$

avec la relation $f(X, Y, Z)=0$ entre les paramètres. Or on peut *avant d'effectuer l'élimination* écrire $\sqrt{x^2+y^2+z^2}$ au lieu de k, l'équation devient ainsi

$$X(x-X)+Y(y-Y)+Z(z-Z)=0,$$

avec cette même relation $f(X, Y, Z)=0$ entre les paramètres; or cette équation est celle d'un plan conduit par le point (X, Y, Z) perpendiculairement au rayon mené par l'origine des coordonnées: et ce dernier système donne ainsi l'équation de la surface dérivée.

L'équation de la surface parallèle de l'ellipsoïde est trouvée en égalant à zéro le discriminant par rapport à λ de l'équation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1+\frac{k^2}{\lambda};$$

donc, en écrivant dans cette équation $x^2+y^2+z^2$ au lieu de k, et $4a^2$, $4b^2$, $4c^2$ au lieu de a^2, b^2, c^2, on doit obtenir l'équation de la dérivée de l'ellipsoïde, les rayons étant menés par le centre. Pour me conformer à la notation de mon Mémoire "Sur la surface qui est l'enveloppe des plans conduits par les points d'un ellipsoïde perpendiculairement aux rayons menés par le centre" (*Journal*, tom. II. 1859 [250]) j'écris -2θ au lieu de λ: l'équation en λ devient ainsi

$$\frac{x^2}{4a^2-2\theta}+\frac{y^2}{4b^2-2\theta}+\frac{z^2}{4c^2-2\theta}=1-\frac{x^2+y^2+z^2}{2\theta},$$

laquelle se réduit tout de suite à

$$\frac{x^2}{2-\dfrac{\theta}{a^2}}+\frac{y^2}{2-\dfrac{\theta}{b^2}}+\frac{z^2}{2-\dfrac{\theta}{c^2}}=\theta,$$

et la surface dérivée est l'enveloppe de cette équation en θ, résultat trouvé dans le mémoire que je viens de mentionner.

Mais je dois à M. Roberts la remarque que réciproquement l'équation en λ pour la surface parallèle peut se déduire des formules de ce mémoire. En effet en prenant $a\cos\alpha$, $b\cos\beta$, $c\cos\gamma$ pour les coordonnées d'un point de l'ellipsoïde (on a comme à l'ordinaire $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$) l'équation du plan perpendiculaire au rayon par le centre sera

$$ax\cos\alpha+by\cos\beta+cz\cos\gamma=a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma,$$

et l'enveloppe de cette équation est l'enveloppe par rapport à θ de l'équation

$$\frac{x^2}{2-\frac{\theta}{a^2}}+\frac{y^2}{2-\frac{\theta}{b^2}}+\frac{z^2}{2-\frac{\theta}{c^2}}=\theta.$$

Donc *en général* l'enveloppe de

$$l\cos\alpha+m\cos\beta+n\cos\gamma=p\cos^2\alpha+q\cos^2\beta+r\cos^2\gamma,$$

(où comme auparavant $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$) sera l'enveloppe par rapport à θ de l'équation

$$\frac{l^2}{2p-\theta}+\frac{m^2}{2q-\theta}+\frac{n^2}{2r-\theta}=\theta.$$

Or la surface parallèle à l'ellipsoïde est l'enveloppe de

$$(X-x)^2+(Y-y)^2+(Z-z)^2=k^2,$$

ou, en écrivant $a\cos\alpha$, $b\cos\beta$, $c\cos\gamma$ au lieu de X, Y, Z, cette surface est l'enveloppe de

$$l\cos\alpha+m\cos\beta+n\cos\gamma=p\cos^2\alpha+q\cos^2\beta+r\cos^2\gamma,$$

en posant $\rho=x^2+y^2+z^2$, et

$$l=2ax,\ m=2by,\ n=2cz,$$
$$p=\rho+a^2-k^2,\ q=\rho+b^2-k^2,\ r=\rho+c^2-k^2.$$

Donc cette surface sera l'enveloppe par rapport à θ de l'équation

$$\frac{4a^2x^2}{2(\rho+a^2-k^2)-\theta}+\frac{4b^2y^2}{2(\rho+b^2-k^2)-\theta}+\frac{4c^2z^2}{2(\rho+c^2-k^2)-\theta}=\theta,$$

et en écrivant $(2\rho-k^2)-\theta=2\lambda$ cette équation devient

$$\frac{a^2x^2}{a^2+\lambda}+\frac{b^2y^2}{b^2+\lambda}+\frac{c^2z^2}{c^2+\lambda}=\rho-k^2-\lambda,$$

c'est-à-dire

$$x^2+y^2+z^2-\lambda\left(\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}\right)=\rho-k^2-\lambda,$$

ou enfin

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1+\frac{k^2}{\lambda},$$

ce qu'il s'agissait de faire voir. Je réserve à une autre occasion la discussion de la forme, et des singularités de la forme, et des singularités de la surface parallèle à l'ellipsoïde.

2, *Stone Buildings, W. C., Londres*, 7 *Nov.* 1860.

255.

ON A PROBLEM OF DOUBLE PARTITIONS.

[From the *Philosophical Magazine*, vol. XX. (1860), pp. 337—341.]

If $a+b+c+\ldots=m$, $\alpha+\beta+\gamma+\ldots=\mu$ (the quantities being all positive integer numbers, not excluding zero), then $(a, \alpha)+(b, \beta)+(c, \gamma)+\ldots$ is considered as a partition of (m, μ): and the partible quantity (m, μ), and parts (a, α), (b, β), &c. being each of them composed of two elements, such partition is said to belong to the theory of *Double Partitions*. The subject (so far as I am aware) has hardly been considered except by Professor Sylvester, and it is greatly to be regretted that only an outline of his valuable researches has been published: the present paper contains the demonstration of a theorem, due to him, by which (subject to certain restrictions) the question of Double Partitions is made to depend upon the ordinary theory of Single Partitions.

Let the question be proposed, "In how many ways can (m, μ) be made up of the given parts (a, α), (b, β), (c, γ), &c." under the following conditions (which are, it will be seen, necessary in the demonstration of the theorem constituting the solution), viz.

$$\frac{a}{\alpha}, \frac{b}{\beta}, \frac{c}{\gamma}, \text{ \&c. are unequal fractions, each in its least terms,}$$

and

$$\alpha, \beta, \gamma, \text{ \&c., are each less than } \mu+2.$$

The number of partitions is

$$= \text{coeff. } x^m y^\mu \text{ in } \frac{1}{(1-x^a y^\alpha)(1-x^b y^\beta)(1-x^c y^\gamma)\ldots}$$

the fraction being developed in ascending powers of x, y.

Considering the fraction as a function of y, it may be expressed as a sum of partial fractions in the form

$$\frac{A(x, y)}{1-x^a y^\alpha}+\frac{B(x, y)}{1-x^b y^\beta}+\frac{C(x, y)}{1-x^c y^\gamma}+\dots,$$

where

$A(x, y)$ is rational in x, rational and integral of degree $\alpha-1$ in y,
$B(x, y)$ " " " $\beta-1$ "
$C(x, y)$ " " " $\gamma-1$ "
&c.

To find $A(x, y)$ we have, when $y=x^{-\frac{a}{\alpha}}$,

$$A(x, y)=\frac{1}{(1-x^b y^\beta)(1-x^c y^\gamma)\dots};$$

or what is the same thing, we have

$$A(x, x^{-\frac{a}{\alpha}})=\frac{1}{(1-x^{b-\frac{a\beta}{\alpha}})(1-x^{c-\frac{a\gamma}{\alpha}})\dots}.$$

This in fact determines $A(x, y)$; for the right-hand side of the equation may be reduced to the form

$$\frac{A_0+A_1 x^{-\frac{a}{\alpha}}\dots+A_{\alpha-1}x^{-\frac{(\alpha-1)a}{\alpha}}}{(1-x^{ab-a\beta})(1-x^{ac-a\gamma})\dots},$$

where A_0, $A_1\dots A_{\alpha-1}$ are rational functions of x: to do this, it is only necessary (taking ω an imaginary α-th root of unity) to multiply the numerator and denominator by

$$\Pi(1-\omega x^{b-\frac{a\beta}{\alpha}})\Pi(1-\omega x^{c-\frac{a\gamma}{\alpha}})\dots,$$

where Π denotes the product of the factors corresponding to the $\alpha-1$ values of ω; the denominator is thus converted into

$$(1-x^{\alpha(b-\frac{a\beta}{\alpha})})(1-x^{\alpha(c-\frac{a\gamma}{\alpha})})\dots,$$

which is of the form in question; and the numerator becomes a rational function of x and $x^{-\frac{a}{\alpha}}$, integral as regards $x^{-\frac{a}{\alpha}}$, and therefore at once expressible in the form in question. And the equation, viz.

$$A(x, x^{-\frac{a}{\alpha}})=\frac{A_0+A_1 x^{-\frac{a}{\alpha}}\dots+A_{\alpha-1}x^{-\frac{(\alpha-1)a}{\alpha}}}{(1-x^{ab-a\beta})(1-x^{ac-a\gamma})\dots},$$

remains true if instead of $x^{-\frac{a}{\alpha}}$ we write $\omega x^{-\frac{a}{\alpha}}$; in fact, instead of writing in the first instance $y=x^{-\frac{a}{\alpha}}$, it would have been allowable to write $y=\omega x^{-\frac{a}{\alpha}}$, ω being any α-th

root, real or imaginary, of unity. Hence recollecting that $A(x, y)$ is a rational and integral function of the degree $\alpha-1$ in y, the equation

$$A(x, y)=\frac{A_0+A_1y\ldots+A_{\alpha-1}y^{\alpha-1}}{(1-x^{ab-a\beta})(1-x^{ac-a\gamma})\ldots},$$

which is true for the α values $\omega x^{-\frac{a}{\alpha}}$ of y, must be true identically; or this equation gives the value of $A(x, y)$. And the values of $B(x, y)$, $C(x, y)$, &c. are of course of the like form.

Now consider the term

$$\frac{A(x, y)}{1-x^a y^\alpha},$$

where $A(x, y)$ is a rational and integral function of the degree $\alpha-1$ in y, and $\frac{a}{\alpha}$ is by hypothesis a fraction in its least terms. The coefficient therein of $x^m y^\mu$ (the fraction being developed in ascending powers of x, y) is

$$= \text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1-x^{\frac{a}{\alpha}}y}$$

(the fraction being developed in ascending powers of x, y). In fact the two fractions only differ by a wholly *irrational* function of x, as is at once obvious by developing $\frac{1}{1-x^{\frac{a}{\alpha}}y}$ in ascending powers of y. We have, separating the integral part,

$$\frac{A(x, y)}{1-x^{\frac{a}{\alpha}}y}=U+\frac{A(x, x^{-\frac{a}{\alpha}})}{1-x^{\frac{a}{\alpha}}y},$$

where U is a rational and integral function of the degree $\alpha-2$ in y. But α being by hypothesis $<\mu+2$, or what is the same thing, $\alpha-2<\mu$, U does not contain any term of the form $x^m y^\mu$, and therefore

$$\text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1-x^{\frac{a}{\alpha}}y}$$

$$= \quad \text{do.} \quad \text{in } \frac{A(x, x^{-\frac{a}{\alpha}})}{1-x^{\frac{a}{\alpha}}y};$$

and this last is

$$= \text{coeff. } x^m \text{ in } x^{\frac{\mu a}{\alpha}} A(x, x^{-\frac{a}{\alpha}}),$$

$$= \text{coeff. } x^{m-\frac{\mu a}{\alpha}} \text{ in } A(x, x^{-\frac{a}{\alpha}}),$$

$$= \text{coeff. } x^{\alpha m-a\mu} \text{ in } A(x^\alpha, x^{-a});$$

and from the foregoing equation

$$A(x, x^{-\frac{\alpha}{a}}) = \frac{1}{(1 - x^{b - \frac{a\beta}{\alpha}})(1 - x^{c - \frac{a\gamma}{\alpha}})\ldots},$$

this is

$$= \text{coeff. } x^{\alpha m - a\mu} \text{ in } \frac{1}{(1 - x^{ab - a\beta})(1 - x^{ac - a\gamma})\ldots}.$$

The last-mentioned expression is thus the value of

$$\text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1 - x^\alpha y^\alpha};$$

and hence, *Theorem*,

$$\text{coeff. } x^m y^\mu \text{ in } \frac{1}{(1 - x^\alpha y^\alpha)(1 - x^b y^\beta)(1 - x^c y^\gamma)\ldots}$$

$$= \text{coeff. } x^{\alpha m - a\mu} \text{ in } \frac{1}{(1 - x^{ab - a\beta})(1 - x^{ac - a\gamma})\ldots}$$

$$+ \text{coeff. } x^{\beta m - b\mu} \text{ in } \frac{1}{(1 - x^{\beta a - b\alpha})(1 - x^{\beta c - b\gamma})\ldots}$$

$$+ \text{coeff. } x^{\gamma m - c\mu} \text{ in } \frac{1}{(1 - x^{\gamma a - c\alpha})(1 - x^{\gamma b - c\beta})\ldots}$$

$$+ \text{\&c.},$$

the fraction on the left-hand side being expanded in ascending powers of x, y, and those on the right-hand side being expanded in ascending powers of x, and the data satisfying the above-mentioned conditions. The number of partitions of (m, μ) is thus found to be equal to the expression on the right-hand side. It is to be noticed that on the right-hand side, when any of the indices $\alpha m - a\mu$, $\beta m - b\mu, \ldots$ is negative, the corresponding coefficient vanishes; and that when the index of the power of x in any factor of a denominator is negative, e.g. if $ab - a\beta = -p$, then (in order to develope in ascending powers of x) we must in the place of $\frac{1}{1 - x^{ab - a\beta}} = \frac{1}{1 - x^{-p}}$ write $\frac{x^p}{x^p - 1}$, $= -\frac{x^p}{1 - x^p}$, and develope in the form $-(x^p + x^{2p} + x^{3p} + \ldots)$. The right-hand side is thus seen to be the sum of a series of positive or negative numbers, each of which *taken positively* denotes the number of the single partitions of a given partible number into given parts.

If, using a term of Professor Sylvester's, we say that

$$\text{coeff. } x^m \text{ in } \frac{1}{(1 - x^a)(1 - x^b)\ldots}$$

(where $m, a, b, c \ldots$ are positive or negative integers, and the fraction is developed in ascending powers of x) is

$$= \text{Denumerant of } m \text{ in respect to the elements } (a, b, c, \ldots), \text{ say}$$

$$= \text{Denumerant } (m;\ a, b, c \ldots),$$

then when $m, a, b, c \ldots$ are positive, but not otherwise, Denumerant $(m;\ a, b, c \ldots)$ denotes the number of ways in which m can be made up of the parts $a, b, c \ldots$; and the foregoing result shows that the number of ways in which (m, μ) can be made up of the parts (a, α), (b, β), (c, γ), &c. is equal to the sum

$$\begin{aligned} &\text{Denumerant } (\alpha m - a\mu;\ \alpha b - a\beta,\ \alpha c - a\gamma, \ldots) \\ &+ \text{Denumerant } (\beta m - b\mu;\ \beta a - b\alpha,\ \beta c - b\gamma, \ldots) \\ &+ \text{Denumerant } (\gamma m - c\mu;\ \gamma a - c\alpha,\ \gamma b - c\beta, \ldots) \\ &+ \&c. \end{aligned}$$

But, as appears from what precedes, a denumerant may be equal to zero, or may denote a number of partitions taken negatively; and it is not allowable, in the place, e.g., of the first denumerant, to write *simpliciter*, number of partitions of $\alpha m - a\mu$ in respect of $\alpha b - a\beta$, $\alpha c - a\gamma$, &c. The notion of a Denumerant is, in fact, an important generalization of the notion of a number of partitions.

2, *Stone Buildings, W.C., October* 4, 1860.

256.

ON A SYSTEM OF ALGEBRAIC EQUATIONS.

[From the *Philosophical Magazine*, vol. XX. (1860), pp. 341, 342.]

THE determination of a, b, c from the system of equations

$$a^2 + bc = \lambda,$$
$$b^2 + ca = \mu,$$
$$c^2 + ab = \nu,$$

in the case where λ, μ, ν have the values 16, 17, and 18 respectively, is the problem known as Colonel Titus's Arithmetical Problem. See Masères' *Tracts on the Resolution of affected Algebraic Equations*, Lond. 1800. If for shortness we put

$$\sigma = \nu - c^2,$$

then the third equation gives $b = \frac{\sigma}{a}$; and substituting this value of b in the two other equations, we have

$$a^2 + \frac{\sigma c}{a} = \lambda,$$

$$\frac{\sigma^2}{a^2} + ca = \mu;$$

or what is the same thing,

$$a^3 - \lambda a + \sigma c = 0,$$
$$ca^3 - \mu a^2 + \sigma^2 = 0;$$

and from these equations, eliminating a, we have

$$\sigma^4 - 3c^2\sigma^3 + (3c^4 - 2\lambda\mu)\sigma^2 - c^2(c^4 - \lambda\mu)\sigma + c^4\lambda\mu - c^2(\lambda^2 + \mu^2) + \lambda^2\mu^2 = 0,$$

where $\sigma = \nu - c^2$. The equation in c^2 is thus of the fourth order; and in like manner, if instead of c^2 we take σ as the unknown quantity, and substitute therefore for c^2 its value $\nu - \sigma$, the equation in σ will be also of the fourth order: and effecting the reduction, this equation is

$$8\sigma^4 - 12\nu\sigma^3 + (6\nu^2 - 2\lambda\mu)\sigma^2 + (\lambda^3 + \mu^3 - \nu^3 - \lambda\mu\nu)\sigma + (\nu\lambda - \mu^2)(\nu\mu - \lambda^2) = 0.$$

It may be remarked that if $\sigma = 0$, then a or b vanishes; and therefore, from the original equations, $\nu\lambda - \mu^2 = 0$, or $\nu\mu - \lambda^2 = 0$, which agrees with the result afforded by the foregoing equation in σ. Again, if $\sigma = \nu$, then $c = 0$; and therefore, from the original equations, $\nu^2 - \lambda\mu = 0$. The left-hand side of the equation in σ, writing therein $\sigma = \nu$, should therefore contain the factor $\nu^2 - \lambda\mu$; its value in fact is $\nu^4 - 2\lambda\mu\nu^2 + \lambda^2\mu^2$, or $(\nu^2 - \lambda\mu)^2$.

If in the original equations we write $a = \frac{x}{z}$, $b = \frac{y}{z}$, the equations become

$$x^2 + cyz - \lambda z^2 = 0,$$

$$y^2 + czx - \mu z^2 = 0,$$

$$(c^2 - \nu)z^2 + xy = 0,$$

which are three homogeneous equations of the second order; from which, if the variables x, y, z are eliminated, we have the required equation in c. And it would not, I think, be difficult, from the known formula for the general case, to deduce the foregoing result corresponding to the very particular case which is here in question.

2, *Stone Buildings, W.C., September* 25, 1860.

257.

ON THE CUBIC CENTRES OF A LINE WITH RESPECT TO THREE LINES AND A LINE.

[From the *Philosophical Magazine*, vol. xx. (1860), pp. 418—423.]

CONSIDER a line L in relation to the three lines X, Y, Z and the line I: through the point of intersection of the lines X, L, draw any line meeting the lines I, Y, Z, and let the harmonic of the intersection with I, in relation to the intersections with Y, Z, be ξ; then the locus of the point ξ is a conic passing through the points YI, ZI, YZ.

If, in like manner, through the point of intersection of the lines Y, L, there is drawn any line meeting the lines I, Z, X, and the harmonic of the intersection with I, in relation to the intersections with Z, X, is called η, the locus of the point η is a conic passing through the points ZI, XI, ZX.

And so, if through the point of intersection of the lines Z, L there is drawn any line meeting the lines I, X, Y, and the harmonic of the intersection with I, in relation to the intersections with X, Y, is called ζ, then the locus of ζ is a conic passing through the points XI, YI, XY.

The pairs of conics, viz. the second and third, third and first, first and second conics, have obviously in common the points XI, YI, ZI respectively. They besides intersect all three of them in three points, which may be termed the *cubic centres* of the line L in relation to the lines X, Y, Z and the line I.

The line L may be such that two of the three cubic centres coincide; the locus of the coincident centres is in this case a conic which touches the lines X, Y, Z harmonically in regard to the line I; that is, it touches each of the three lines in the point which is the harmonic of its intersection with I in relation to its intersections with the other two lines.

Except that the line I is there taken to be infinity, the foregoing theorems occur in Plücker's *System der analytischen Geometrie* (Berlin, 1835), p. 177 *et seq.*; and they play an important part in his classification of curves of the third order (see p. 220 *et seq.*). It is, I think, an omission that he has not sought for the curve which is the envelope of the line L in the above-mentioned case of the two coincident centres: I find that the envelope is a curve of the fourth order, having four-pointic contact with the lines X, Y, Z harmonically in regard to the line I; viz., if the equations of the lines X, Y, Z are $x=0$, $y=0$, $z=0$ respectively, and the equation of the line I is $x+y+z=0$, then the equation of the envelope in question is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

a result which is also interesting as exhibiting a geometrical construction of the curve represented by this equation.

The investigation of the series of theorems is as follows; taking

$$\begin{array}{rcl} x=0 & \text{for the equation of} & X, \\ y=0 & \text{,,} & Y, \\ z=0 & \text{,,} & Z, \\ x+\ y+\ z=0 & \text{,,} & I, \\ \lambda x+\mu y+\nu z=0 & \text{,,} & L, \end{array}$$

then, first, in order to find the curve which is the locus of ξ, the coordinates of the point XL are given by $x : y : z = 0 : \nu : -\mu$; or if, as it is convenient to do, we take X, Y, Z (instead of x, y, z) for current coordinates, by $X : Y : Z = 0 : \nu : -\mu$. Hence taking x, y, z as the coordinates of ξ, the equation of the line through XL, ξ is

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ 0, & \nu, & -\mu \end{vmatrix} = 0,$$

viz.

$$X(\mu y+\nu z)-x(\mu Y+\nu Z)=0;$$

and at the point where this line meets the line I, the equation whereof is

$$X+Y+Z=0,$$

we have

$$(Y+Z)(\mu y+\nu z)+x(\mu Y+\nu Z)=0;$$

that is

$$Y(\mu x+\mu y+\nu z)+Z(\nu x+\mu y+\nu z)=0.$$

Hence this line, and the line

$$Yz-Zy=0,$$

with the lines

$$Y=0,\ Z=0,$$

are the lines which pass through the point YZ and the four harmonic points, and they form therefore a harmonic pencil; or we have

$$y(\mu x+\mu y+\nu z)-z(\nu x+\mu y+\nu z)=0,$$

or, what is the same thing,

$$(\mu y-\nu z)(x+y+z)+2yz(\nu-\mu)=0,$$

as the locus of the point ξ: the locus is therefore a conic passing through the points YI, ZI, YZ.

The equations of the conics which are the loci of X, Y, Z respectively, are therefore

$$\begin{aligned} U &= (\mu y-\nu z)(x+y+z)+2yz(\nu-\mu)=0,\\ V &= (\nu z-\lambda x)(x+y+z)+2zx(\lambda-\nu)=0,\\ W &= (\lambda x-\mu y)(x+y+z)+2xy(\mu-\lambda)=0; \end{aligned}$$

and the identical equation,

$$U\lambda x+V\mu y+W\nu z=0,$$

shows that these conics have three points of intersection in common. The three equations, and a fourth one to which they give rise, may be written

$$\begin{aligned} \frac{\mu}{z}-\frac{\nu}{y}+\frac{2(\nu-\mu)}{x+y+z} &= 0,\\ \frac{\nu}{x}-\frac{\lambda}{z}+\frac{2(\lambda-\nu)}{x+y+z} &= 0,\\ \frac{\lambda}{y}-\frac{\mu}{x}+\frac{2(\mu-\lambda)}{x+y+z} &= 0,\\ \frac{\nu-\mu}{x}+\frac{\lambda-\nu}{y}+\frac{\mu-\lambda}{z} &= 0; \end{aligned}$$

and each of these is the equation of a conic passing through the three cubic centres.

If two of the three centres coincide, then the conics all touch at the coincident centres. Consider the first and second conics: these intersect at the point $z=0$ $x+y+z=0$; and the line $x+y+z=2kz$, if k be properly determined, or what is the same thing, the line $x+y+z=\frac{2(\theta+\nu)}{\theta}z$, if θ is properly determined, will be a line passing through the last-mentioned point and one of the other points of intersection: k or θ will of course be determined by a cubic equation; and if this has a pair of equal

roots, the conics will touch. But the equation of the line, combined with those of the two conics, gives

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu};$$

and substituting these values in the equation of the line, we have

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

which is (as it should be) a cubic equation in θ.

If the equation in θ has equal roots, then

$$\frac{1}{(\theta+\lambda)^2}+\frac{1}{(\theta+\mu)^2}+\frac{1}{(\theta+\nu)^2}-\frac{2}{\theta^2}=0;$$

and putting in these two equations,

$$x=\frac{m}{\theta+\lambda},\quad y=\frac{m}{\theta+\mu},\quad z=\frac{m}{\theta+\nu},$$

we have

$$x+y+z-\frac{2m}{\theta}=0,$$

$$x^2+y^2+z^2-\frac{2m^2}{\theta^2}=0;$$

whence eliminating m,

$$(x+y+z)^2=2(x^2+y^2+z^2);$$

that is

$$x^2+y^2+z^2-2yz-2zx-2xy=0;$$

or, what is the same thing,

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

for the equation of the locus of the coincident centres: such locus is therefore a conic touching the lines $x=0$, $y=0$, $z=0$, in the points of intersection with the lines $y-z=0$, $z-x=0$, $x-y=0$ respectively; it is a conic touching the lines X, Y, Z harmonically in regard to the line I.

To find the envelope of the line L, the most convenient course is to take the equation in θ in the reduced form

$$\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-2\lambda\mu\nu=0;$$

this will have a pair of equal roots if

$$(\mu\nu+\nu\lambda+\lambda\mu)^3-27\lambda^2\mu^2\nu^2=0;$$

that is, if

$$\mu\nu + \nu\lambda + \lambda\mu - 3(\lambda\mu\nu)^{\frac{2}{3}} = 0;$$

or if

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} - 3\frac{1}{(\lambda\mu\nu)^{\frac{1}{3}}} = 0;$$

or finally if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

which is the relation between λ, μ, ν in order that the line

$$\lambda x + \mu y + \nu z = 0$$

may have two coincident centres; this gives at once for the equation of the envelope

$$\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0,$$

which is the equation of a curve of the fourth order having four-pointic contact with the lines $x=0$, $y=0$, $z=0$, at the points of intersection with the lines $y-z=0$, $z-x=0$, $x-y=0$ respectively, i.e. it has four-pointic contact with the lines X, Y, Z harmonically in regard to the line I.

It may be noticed that the rationalized form of the equation $\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0$ is

$$x^4 + y^4 + z^4 - 4(yz^3 + y^3z + zx^3 + z^3x + xy^3 + x^3y)$$
$$+ 6(y^2z^2 + z^2x^2 + x^2y^2) - 124(x^2yz + y^2zx + z^2xy) = 0.$$

If, to fix the ideas, the signs of the coordinates x, y, z are so determined that a point *within* the triangle $x=0$, $y=0$, $z=0$ has its coordinates positive (in which case the line $x+y+z=0$ will cut the three sides *produced*), the curve $\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0$ will lie wholly within the triangle, and will be of the form shown by the annexed

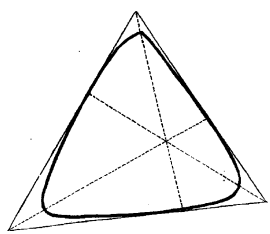


figure. This is, in fact, the form of the curve in the case considered by Plücker, where the line I is at infinity, the points of contact being the middle points of the sides. And his five groups of curves, α, β, γ, δ, ϵ, and two subdivisions of the group

β (see pp. 221—224), correspond to the following positions of the line in regard to the triangle and curve, viz.

α. The line cuts the three sides produced.

β. It passes through an angle, (a) cutting, or (b) not cutting the curve.

γ. It cuts two sides and a side produced, but does not cut or touch the curve.

δ. It cuts two sides and a side produced, and touches the curve.

ϵ. It cuts two sides and a side produced, and cuts the curve.

It is hardly necessary to remark that, in the general case, the tangential equation of the curve is

$$\xi^{-\frac{1}{3}} + \eta^{-\frac{1}{3}} + \zeta^{-\frac{1}{3}} = 0\,;$$

or what is the same thing,

$$(\eta\zeta + \zeta\xi + \xi\eta)^3 - 27\xi^2\eta^2\zeta^2 = 0\,;$$

and that the curve is therefore of the sixth class.

2, *Stone Buildings, W.C., October* 16, 1860.

258.

ON A RELATION BETWEEN TWO TERNARY CUBIC FORMS.

[From the *Philosophical Magazine*, vol. xx. 1860, pp. 512—514.]

THE cubic form

$$x^3 + y^3 + z^3 + 6lxyz$$

is in general linearly transformable into the form

$$(X + Y + Z)^3 + 27kXYZ;$$

in fact, writing

$$X = 2lx - y - z,$$
$$Y = 2ly - z - x,$$
$$Z = 2lz - x - y,$$

we have identically

$$(1 - 2l + 4l^2)(X + Y + Z)^3 + 24(l-1)^3 XYZ = 8(2l+1)^2(l-1)^3(x^3 + y^3 + z^3 + 6lxyz);$$

and the value of k consequently is

$$k = -\frac{8(l-1)^3}{9(1 - 2l + 4l^2)}.$$

If, however, $l = 1$ or $l = -\frac{1}{2}$, the transformation fails. In the former case, viz. for $l = 1$, the equations for the linear transformation become

$$X = 2x - y - z,$$
$$Y = 2y - z - x,$$
$$Z = 2z - x - y,$$

which give $X+Y+Z=0$, so that X, Y, Z are no longer independent; and the formula of transformation becomes

$$(X+Y+Z)^3=0.$$

It may be noticed that the invariant S of the form

$$x^3+y^3+z^3+6lxyz$$

is $S=-l+l^4$, so that $l=1$ is one of the values which make S vanish. And the above transformation is not applicable to the cubic form $x^3+y^3+z^3+6xyz$, which is a form for which S vanishes. The transformation, however, holds good for $l=0$, which is another value which makes S vanish; or it does apply to the form $x^3+y^3+z^3$, for which S vanishes. The transformation, in fact, is

$$(X+Y+Z)^3+24XYZ=-8(x^3+y^3+z^3),$$

with the linear equations

$$X=-y-z,$$
$$Y=-z-x,$$
$$Z=-x-y.$$

The above two forms for which S vanishes, viz.

$$x^3+y^3+z^3+6xyz,$$
$$x^3+y^3+z^3,$$

are, notwithstanding, equivalent to each other, as appears by the identical equation

$$(x+y+z)^3+(x+\omega y+\omega^2 z)^3+(x+\omega^2 y+\omega z)^3=3(x^3+y^3+z^3+6xyz),$$

where ω is an imaginary cube root of unity. In the latter of the two cases of failure, viz. for $l=-\frac{1}{2}$, the equations for the linear transformations are

$$X=Y=Z=-x-y-z;$$

so that X, Y, Z are not only not independent, but they are connected by *two* linear relations. And the formula of transformation becomes

$$(X+Y+Z)^3-27XYZ=0,$$

which is, in fact, true in virtue of the equations $X=Y=Z$.

The two forms of equation,

$$x^3+y^3+z^3+\ 6lxyz=0,$$
$$(x+y+z)^3+27kxyz=0,$$

represent each of them equally well a curve of the third order without a double point. In the first form the three real points of inflexion are given by

$$(x=0,\ y+z=0),\ (y=0,\ z+x=0),\ (z=0,\ x+y=0);$$

or what is the same thing, the points in question are the intersections of the lines $x=0$, $y=0$, $z=0$ with the line $x+y+z=0$; or we have $x+y+z=0$ for the equation of the line through the three points of inflexion; and the equations of the tangents at the points of inflexion are

$$2lx-y-z=0,\quad 2ly-z-x=0,\quad 2lz-x-y=0.$$

For the second form it is obvious that the points of inflexion are the intersections of the lines $x=0$, $y=0$, $z=0$ with the line $x+y+z=0$; and, moreover, that the lines $x=0$, $y=0$, $z=0$ are the tangents at the point of inflexion.

The first of the above-mentioned forms, however, cannot represent a curve with a double point. In fact the condition for its doing so would be $1+8l^3=0$; but when this condition is satisfied, the left-hand side breaks up into linear factors, and the equation represents, not a proper curve of the third order, but a system of three lines. The second form *can* represent a curve having a double point; viz. if $k=-1$, the curve will have a conjugate or isolated point at the point $x=y=z$. It is clear *à priori* that ($x=0$, $y=0$, $z=0$ being real lines) neither of the forms can represent a curve of the third order having a double point with two real branches through it, since in this case the curve has only one real point of inflexion.

I have elsewhere used the word "node" to denote a double point, and I take the opportunity of suggesting the employment of the words "crunode" (*crus*) and "acnode" (*acus*) to denote respectively a double point with two real branches through it, and a conjugate or isolated point.

2, *Stone Buildings*, *W.C.*, *October* 19, 1860.

259.

THE PROBLEM OF POLYHEDRA.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLVII. (for the year 1857), pp. 183—185: printed as a Note to Mr Kirkman's Memoir "On Autopolar Polyhedra," pp. 183—215.]

LET a, b, c, d, e, f, g, h, &c. represent the vertices of a polyhedron, then a face will be represented, e.g. by $abcde$, where the contiguous duads, viz. ab, bc, cd, de, ea are the edges of the face; and calling the face K, we may write

$$K = abcde. \tag{1}$$

It is to be noticed that the letters of a face-symbol may be taken forwards or backwards from any letter without altering the meaning of the symbol. Thus, $abcde$, $bcdea$, &c., $edcba$, &c. might any of them be taken to denote the face K. The diagonal of a face cannot be either an edge or a diagonal of any other face, i.e. a non-contiguous duad such as ac in a face-symbol K cannot be a duad, contiguous or non-contiguous, of any other face-symbol. But each edge of a face must be an edge of one and only one other face, i.e. each contiguous duad such as ab in the face-symbol K must be a contiguous duad of one and only one other face-symbol L. And moreover two faces cannot have more than a single edge in common, i.e. two face-symbols cannot contain more than a single contiguous duad, the same in each symbol.

The face K contains the edges ab, ac, i.e. the edge ab is contained in the face K; it will also be contained in one and only one other face, suppose L; this face will contain another edge through the vertex a, suppose the edge af, and so on, until we arrive at a face containing the edge ae; we have, for example,

$$K = eabcd,$$
$$L = baf\ldots,$$
$$M = fag\ldots,$$
$$N = gah\ldots,$$
$$P = hai\ldots,$$
$$Q = iae\ldots;$$

and we thence derive the vertex-symbol

$$a = KLMNPQ, \tag{2}$$

where the contiguous duads KL, LM, MN, NP, PQ, QK represent in order the edges through the vertex a. The remarks before made with respect to the face-symbols apply to the vertex-symbols. A non-contiguous duad such as KN of the vertex-symbol a cannot be a duad, contiguous or non-contiguous, of any other vertex-symbol; but each contiguous duad such as KL of the vertex-symbol a must be a contiguous duad of one and only one other vertex-symbol b. And the symbols of two vertices cannot contain more than one contiguous duad, the same in each symbol.

Any edge of the polyhedron admits of a double representation; it is the junction of two vertices, or the intersection of two faces. Thus ab and KL will represent the same edge, or we may write

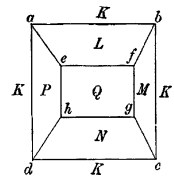
$$ab = KL. \tag{3}$$

It is to be remarked that in this system, to each equation $ab = KL$ there corresponds one and only one equation of the form $ae = KQ$, i.e. to an edge considered as drawn from a given vertex in a given face there corresponds one and only one other edge from the same vertex in the same face.

It has been shown how the system of face-symbols (1) leads to the system of vertex-symbols (2), and the system of edge symbols (3); and generally, any one of the three systems leads to the other two; and the three systems conjointly, or each system by itself, is a complete representation of the polyhedron. As an example, take the hexaedron: the three systems are:

$K = abcd,$	(1)	$a = LPK,$	(2)
$L = abfe,$		$b = LMK,$	
$M = bfge,$		$c = MNK,$	
$N = gcdh,$		$d = NPK,$	
$P = dhea,$		$e = LPQ,$	
$Q = hefg,$		$f = LMQ,$	
		$g = MNQ,$	
		$h = NPQ,$	

$$\begin{array}{lll} ab = KL, & ae = PL, & ef = LQ, \quad (3) \\ bc = KM, & bf = LM, & fg = MQ, \\ cd = KN, & cg = MN, & gh = NQ, \\ de = KP, & dh = NP, & he = PQ. \end{array}$$



Consider, now, two polyhedra having the same number of vertices and also the same number of faces. And let the vertices and faces of the first polyhedron taken in any order be represented by

$$abcde \ldots KLM \ldots,$$

and the vertices and faces of the second polyhedron taken in a certain order be represented by

$$a'b'c'd'e' \ldots K'L'M' \ldots;$$

then, forming the substitution symbol

$$a'b'c'd'e' \ldots K'L'M' \ldots abcde \ldots KLM \ldots,$$

which denotes that a' is to be written for a, b' for $b \ldots K'$ for K, &c., if operating with this upon the symbol system of the first polyhedron, we obtain the symbol system of the second polyhedron, the second polyhedron will be syntypic with the first. It should be noticed, that there may be several modes of arrangement of the vertices and faces of the second polyhedron, which will render it syntypic according to the foregoing definition with the first polyhedron, i.e. the second polyhedron may be syntypic in several different ways with the first polyhedron. This is, in fact, the same as saying that a polyhedron may be syntypic with itself in several different ways. Suppose, next, that the number of vertices of the second polyhedron is equal to the number of faces of the first polyhedron, and the number of faces of the second polyhedron is equal to the number of vertices of the first polyhedron; and let the vertices and faces of the first polyhedron in any order be represented by

$$abcde \ldots KLM \ldots,$$

and the faces and vertices of the second polyhedron in a certain order be represented by

$$A'B'C'D'E' \ldots k'l'm' \ldots.$$

Then, forming the substitution symbol

$$A'B'C'D'E' \ldots k'l'm' \ldots abcde \ldots KLM \ldots,$$

if, operating with this upon the symbol system of the first polyhedron, we obtain the symbol system of the second polyhedron, the second polyhedron is said to be polar-syntypic with the first; and, as in the case of syntypicism, this may happen in several different ways.

Lastly, if there be a polyhedron having the same number of vertices and faces, and if the vertices and faces in any order be represented by

$$abcd \ldots KLMN \ldots,$$

and the faces and vertices in a certain order be represented by

$$ABCD \ldots klmn \ldots;$$

then, forming the substitution symbol

$$ABCD \ldots klmn \ldots abcd \ldots KLMN \ldots,$$

if, operating with this upon the symbol system of the polyhedron, we reproduce such symbol system, i.e. in fact, if the polyhedron be polar-syntypic with itself, the polyhedron is said to be *autopolar;* and in accordance with a preceding remark, this may happen in several different ways. It is clear that the substitution symbol, operating on the symbol system of the vertices, must give the symbol system of the faces, and conversely; but operating on the symbol system of the edges, it must reproduce such symbol system of the edges: and this last condition will by itself suffice to make the polyhedron autopolar, i.e. the polyhedron will be autopolar if the substitution symbol, operating on the symbol system of the edges, reproduces such symbol system.

260.

ON THE DOUBLE TANGENTS OF A PLANE CURVE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLIX. *for the year* 1859, pp. 193—212. Received March 17,—Read April 14, 1859.]

It was first shown by Plücker on geometrical principles, that the number of the double tangents of a plane curve of the order m was $\frac{1}{2}m(m-2)(m^2-9)$: see the note, "Solution d'une question fondamentale concernant la théorie générale des Courbes," *Crelle*, t. XII. pp. 105—108 (1834), and the "Theorie der algebraischen Curven" (1839). The memoir by Hesse, "Ueber die Wendepuncte der Curven dritter Ordnung," *Crelle*, t. XXVIII. pp. 97—107 (1844), contains the analytical solution of the allied easier problem of the determination of the points of inflexion of a plane curve. In the memoir, "Recherches sur l'élimination et sur la théorie des Courbes," *Crelle*, t. XXXIV. (1847), pp. 30—45, [53], I showed how the problem of double tangents admitted of an analytical solution, viz. if $U=0$ is the equation of the curve, L, M, N the first derived functions of U, and

$$D=\alpha(M\partial_x-N\partial_y)+\beta(N\partial_y-L\partial_z)+\gamma(L\partial_z-M\partial_x)$$

(where α, β, γ are arbitrary), then the points of contact of the double tangents are given as the intersections of the curve $U=0$, with a curve the equation whereof is in the first instance obtained under the form $[Y]=0$; $[Y]$ being a given function of

$$D^2U,\ D^3U,\ \ldots\ D^mU,$$

of the degree m^2-m-6 in respect of (α, β, γ), the degree $m^3-2m^2-10m+12$ in respect of (x, y, z), and the degree m^2+m-12 in respect of the coefficients of U. It was necessary, in order that the points of intersection should be independent of the arbitrary quantities (α, β, γ), that we should have identically

$$[Y]=\Lambda\,.\,U+N\,.\,\Pi U,$$

N being of the degree m^2-m-6 in (α, β, γ), and consequently ΠU a function of (x, y, z) without (α, β, γ). Guided by Hesse's investigation for the points of inflexion, I asserted that it was probable that N was of the form $(\alpha x+\beta y+\gamma z)^{m^2-m-6}$; which being so, ΠU would be of the degree $(m-2)(m^2-9)$ in respect of (x, y, z), and the degree m^2+m-12 in respect of the coefficients, and I was thus led to the theorem, "On trouve les points de contact des tangentes doubles en combinant avec l'équation de la courbe une équation $\Pi U=0$, de l'ordre $(m-2)(m^2-9)$ par rapport aux variables et de l'ordre m^2+m-12 par rapport aux coefficients—c'est-à-dire, puisqu'il correspond deux points de contact à une tangente double, le nombre de ces tangentes est égal à $\frac{1}{2}m(m-2)(m^2-9)$: théorème démontré indirectement par M. Plücker."

Hesse, in the memoir "Ueber Curven dritter Ordnung u.s.w.," *Crelle*, t. XXXVI. pp. 143—176 (1848), showed how the components $D^2U, D^3U, \ldots D^mU$ of $[Y]$ could each of them be expressed in a simplified form, and he thus effected the actual reduction of $[Y]$ to the form $\Lambda \,.\, U+(\alpha x+\beta y+\gamma z)^{4(m-3)}R$, where R still contained the arbitrary quantities (α, β, γ) in the degree $(m-2)(m-3)$. In particular for a quartic curve, the equation $R=0$ was shown to be

$$3Q_2Q_4-Q_3^2=0,$$

where the left-hand side is of the degree 2 in (α, β, γ) and the degree 16 in (x, y, z); and which should therefore by means of the equation $U=0$ be reducible so as to contain the factor $(\alpha x+\beta y+\gamma z)^2$.

Jacobi's paper, "Beweis des Satzes, dass eine Curve n-ten Grades im allgemeinen $\frac{1}{2}n(n-2)(n^2-9)$ Doppeltangenten hat," *Crelle*, t. XL. pp. 237—260 (1850), did not, I think, materially advance the solution of the question. In a letter to Jacobi, dated the 30th December, 1849, published at the conclusion of the last-mentioned paper, Hesse gave the equation of the curve of the 14th order for the points of contact of the double tangents of a quartic, viz. in my notation,

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\between\partial_x H, \partial_y H, \partial_z H)^2-H(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\between\partial_x, \partial_y, \partial_z)^2 H=0,$$

and the demonstration is given in Hesse's paper, "Ueber die ganzen homogenen Functionen von der dritten und vierten Ordnung zwischen drei Variabeln," *Crelle*, t. XLI. pp. 285—292 (1851), and is reproduced in Mr Salmon's Treatise on the Higher Plane Curves (1852). Two very interesting memoirs by Hesse and Steiner, *Crelle*, t. XLIX. (1855), relate to the *geometrical* theory of the double tangents of a quartic, and it is not necessary to refer to them more particularly. It is to be observed that the curve which determines the points of contact of the double tangents is not absolutely determinate; for we may, it is clear, in the place of $\Pi U=0$, write $\Pi U+M \,.\, U=0$, where M is an arbitrary function of the proper degree: a very elegant transformation in the case of the quartic is given in Hesse's paper, "Transformation der Gleichung der Curven 14ten Grades, welche eine gegebene Curve 4ten Grades in den Berührungspuncten ihrer Doppeltangenten schneiden," *Crelle*, t. LII. pp. 97—103 (1856).

Mr Salmon's work above referred to, contains the fundamental theorem of the *tangential* of a cubic, viz. a tangent to a cubic meets the cubic in a third point

which lies on the second or line polar of the point of contact with respect to the Hessian. In my "Memoir on Curves of the Third Order," *Phil. Trans.* vol. CXLVII. (1857), pp. 415—446, art. No. 37, [146], I gave an identical equation relating to the tangential of a cubic, but which is not there exhibited in its proper form; this was afterwards effected by Mr Salmon, in the paper "On Curves of the Third Order," *Phil. Trans.* vol. CXLVIII. (1858), pp. 535—541. The equation, as given by Mr Salmon, is in the notation of the present memoir,

$$-\mathfrak{H}\,.\,U+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,DU-\tfrac{1}{3}DH\,.\,\mathfrak{D}\Upsilon+H\,.\,\Upsilon=0,$$

an equation which in fact puts in evidence the last-mentioned theorem for the tangential of a cubic.

The idea occurred to me of considering, in the case of the higher plane curves, the *tangentials* of a given point of the curve, viz. the points in which the tangent again meets the curve; for by expressing that two of these tangentials were coincident, we should have the condition that the given point is the point of contact of a double tangent. But I was not able to complete the solution.

Finally, Mr Salmon discovered the equation of a curve of the order $m-2$, which by its intersections with the tangent at the given point determines the tangentials, and by expressing that the curve in question is touched by the tangent, he was led to a complete solution of the Double-tangent problem. Mr Salmon's result is given in the note, "On the Double Tangents to Plane Curves," in the *Philosophical Magazine* for October 1858. The discovery just referred to led me to the investigations of the present memoir, in which it will be seen that I obtain, for a curve of any order whatever, the identical equation corresponding to the before-mentioned equation obtained by Mr Salmon in the case of a cubic; which identical equation puts in evidence the theorem as to the tangentials of the curve, and may thus be considered as containing in itself the solution of the Double-tangent problem: the identical equation is besides interesting for its own sake, as a part of the theory of ternary quantics.

1. Mr Salmon's solution of the problem of double tangents is based upon the following analytical determination of the tangentials of any point of the curve.

Let

$$\Upsilon=(*\between X,\ Y,\ Z)^n=0$$

be the equation of the given curve, $(X,\ Y,\ Z)$ being current coordinates; and let $(x,\ y,\ z)$ be the coordinates of a point on the curve, so that we have

$$U=(*\between x,\ y,\ z)^n=0,$$

a condition satisfied by the coordinates of the point in question.

Then the tangent

$$V=(X\partial_x+Y\partial_y+Z\partial_z)\,U=0$$

at the point $(x,\ y,\ z)$, meets the curve besides in $(n-2)$ points, which are the tangentials of the given point $(x,\ y,\ z)$, and which are determined as the intersections of the tangent $V=0$ with a certain curve,

$$\Omega=(\dagger\between x,\ y,\ z)^{n-2}=0.$$

2. To express the equation of this curve, let U_1, U_2, ... be the successive emanants of U, taken with the facients of emanation $(x_{,}, y_{,}, z_{,})$, viz.

$$U_1 = \frac{1}{n} \left[\text{should be } \frac{1}{n-2}\right] (x_{,}\partial_x + y_{,}\partial_y + z_{,}\partial_z)\, U,$$

$$U_2 = \frac{1}{n(n-1)} \left[\text{should be } \frac{1}{(n-2)(n-3)}\right] (x_{,}\partial_x + y_{,}\partial_y + z_{,}\partial_z)^2 U,$$

$$\vdots$$

where it should be noticed that the numerical determination is such, that putting (x, y, z) for $(x_{,}, y_{,}, z_{,})$, then U_1, U_2, ... become respectively equal to U. [The numerical determination should have been and in the latter part of the memoir is assumed to be such as to render H_1, H_2, &c. equal to H, on making the substitution in question: the correction was made in a later memoir "On the double tangents of a curve of the fourth order."] Suppose also that H, H_1, H_2, ... are the Hessians of U, U_1, U_2, ..., viz. H is the determinant formed with the second derived functions of U with respect to (x, y, z), H_1 the like determinant with the second derived functions of U_1 with respect to the same quantities (x, y, z); and so on. Moreover let $D^{n-2}H, = (X\partial_x + Y\partial_y + Z\partial_z)^{n-2}H$, denote the $(n-2)$thic emanant of H with respect to the current coordinates (X, Y, Z) as facients of emanation; and similarly let $D^{n-2}H_1$, $D^{n-2}H_2$, ... denote the $(n-2)$thic emanants of H_1, H_2, ... in respect to the same facient of emanation—it being understood that in all these functions, $(x_{,}, y_{,}, z_{,})$ are after the differentiations to be replaced by (x, y, z). It is to be observed that U_r is of the degree $(n-r)$ in (x, y, z), and consequently H_r of the degree $3(n-2-r)$; hence $D^{n-2}H_r$ is of the degree $3(n-2-r)-(n-2)$, $= 2(n-2)-3r$, which implies that $r \not> \frac{2}{3}(n-2)$, for otherwise $D^{n-2}H_r$ would be identically equal to zero. Upon replacing $(x_{,}, y_{,}, z_{,})$ by (x, y, z), $D^{n-2}H_r$ (r satisfying the above condition) becomes of the degree $2(n-2)$ in (x, y, z), and it is obviously of the degree 3 in the coefficients of U, and of the degree $(n-2)$ in the current coordinates (X, Y, Z).

3. This being premised, we have

$$\Omega = (\dagger\, \underset{\times}{\mathrm{O}}\, X, Y, Z)^{n-2},$$

$$= D^{n-2}H - \frac{n-1}{1} D^{n-2}H_1 + \&c. = 0,$$

for the equation of the curve of the order $(n-2)$, which by its intersection with the tangent gives the tangentials of the given point; the numerical coefficients are the binominal coefficients of the order $(n-1)$ taken with the signs + and − alternately, and the series is continued as long as the terms do not vanish, that is, if as before r denote the suffix of H, for so long as $r \not> \frac{2}{3}(n-2)$; but of course the value will not be altered by continuing the series to $r = n-1$. In particular, for the quartic we have

$$\Omega = D^2H - 3D^2H_1,$$

for the quintic

$$\Omega = D^3H - 4D^3H_1 + 6D^3H_2,$$

and so on. The function Ω, like the several component terms, is of course of the degree 3 in the coefficients of U, and of the degree $2(n-2)$ in (x, y, z).

4. It is to be remarked that the formula applies to a cubic; we have here simply $\Omega = DH$, which agrees with a result already mentioned. It may be noticed also that in the general case the formula gives at once the condition for the points of inflexion; in fact, if the point (x, y, z) be a point of inflexion, then one of the tangentials must coincide with this point, or the equation $\Omega = 0$ will be satisfied by writing therein (x, y, z) for (X, Y, Z); but when this is done $D^{n-2}H$, $D^{n-2}H_1$ &c. reduce themselves (to numerical factors *près*) to H, and the equation becomes simply $H = 0$, which is the well-known condition for the points of inflexion.

5. If two of the tangentials coincide, or what is the same thing, if the tangent $V = 0$ touches the curve $\Omega = 0$, then the point (x, y, z) will be the point of contact of a double tangent. The equation which expresses the condition in question, treating therein (x, y, z) as current coordinates, is consequently that of a curve, intersecting the given curve (now represented by $U = 0$) in the points of contact of the double tangents. The process leads to a determinate form $\Pi U = 0$, of the curve in question, but of course any curve whatever, $\Pi U + M \,.\, U = 0$, will intersect the curve $U = 0$ in the points of contact of the double tangents.

6. I write for the moment

$$\Omega = (A, \ldots)(X, Y, Z)^{n-2} = 0,$$
$$V = \xi X + \eta Y + \zeta Z \qquad = 0,$$

for the two equations; the coefficients $(A, \ldots)$, as already mentioned, are of the degree $2(n-2)$ in (x, y, z) and of the degree 3 in the coefficients of U; or as we may express it,

$$A, \ldots = (a, \ldots)^3 (x, y, z)^{2(n-2)}.$$

In like manner ξ, η, ζ are of the degree $(n-1)$ in (x, y, z), and the degree 1 in the coefficients of U, or we may write

$$\xi, \eta, \zeta = (a, \ldots)^1 (x, y, z)^{n-1}.$$

7. The equation which expresses that the line $V = 0$ touches the curve $\Omega = 0$, is $\mathsf{F}\Omega = 0$, where the facients of the Reciprocant $\mathsf{F}\Omega$ are the coefficients (ξ, η, ζ) of the linear function. This equation is of the form

$$(A, \ldots)^{2(n-3)} (\xi, \eta, \zeta)^{(n-2)(n-3)} = 0;$$

or attending to the forms of $(A, \ldots)$ and (ξ, η, ζ), it is of the form

$$(a, \ldots)^{6(n-2) + (n-2)(n-3)} (x, y, z)^{4(n-2)(n-3) + (n-1)(n-2)(n-3)} = 0,$$

or what is the same thing, the form

$$(a, \ldots)^{(n+4)(n-3)} \qquad (x, y, z)^{(n-2)(3n^2-9)} \qquad = 0,$$

viz. the curve through the points of contact of the double tangents is a curve of the order $(n-2)(n^2-9)$, and its equation contains the coefficients of the equation $U=0$ of the given curve in the degree $(n+4)(n-3)$. And since each double tangent corresponds to two points of contact, the number of double tangents is $\frac{1}{2}n(n-2)(n^2-9)$. This agrees with the before-mentioned results.

8. The whole problem is thus reduced to the demonstration of Mr Salmon's expression for the curve $\Omega=0$. To fix the ideas, consider the case of a quartic curve $\Upsilon=(*\between X, Y, Z)^4=0$, and let the function $U=(*\between x, y, z)^4$ (or as for shortness we may write it, $U=(x, y, z)^4$) and certain of its emanants be represented as follows, viz.—

$$\begin{aligned}
a &= (x, y, z)^4 \qquad\qquad , \\
b &= (x, y, z)^3(X, Y, Z), \\
c &= (x, y, z)^2(X, Y, Z)^2, \\
d &= (x, y, z)(X, Y, Z)^3, \\
e &= \ldots\ldots\ldots(X, Y, Z)^4, \\
a' &= (x, y, z)^3 \ldots\ldots\ldots (X', Y', Z'), \\
b' &= (x, y, z)^2(X, Y, Z)(X', Y', Z'), \\
c' &= (x, y, z)(X, Y, Z)^2(X', Y', Z'), \\
d' &= \ldots\ldots (X, Y, Z)^3(X', Y', Z'), \\
a'' &= (x, y, z)^2 \ldots\ldots\ldots (X', Y', Z')^2, \\
b'' &= (x, y, z)(X, Y, Z)(X', Y', Z')^2, \\
c'' &= \ldots\ldots\ldots(X, Y, Z)^2(X', Y', Z')^2,
\end{aligned}$$

where (X', Y', Z') are new arbitrary facients; but, as before, (X, Y, Z) are taken to be current coordinates, and (x, y, z) the coordinates of the given point on the curve:

$e=0$ is the equation of the curve;

$d=0$, the equation of the first or cubic polar of the point (x, y, z);

$b=0$, the equation of the last or line polar of the point (x, y, z), or what is the same thing (the point being on the curve), the tangent of the curve at this point;

$a=0$, the condition which expresses that the point is on the curve.

9. Imagine now an identical equation,

$$a\mathrm{I}+b\mathrm{II}+d\mathrm{III}+e\mathrm{IV}=0;$$

then, since $a=0$, we have

$$b\mathrm{II}+d\mathrm{III}+e\mathrm{IV}=0;$$

and if in this equation we write $b=0$, $e=0$, it becomes $\mathrm{III}d=0$, that is, the points of intersection of the curve $e=0$ and the tangent $b=0$ lie on one or other of the curves $d=0$, $\mathrm{III}=0$. But the points in question do not lie on the curve $d=0$, consequently they lie on the curve $\mathrm{III}=0$.

10. To explain the law of formation of the multipliers I, II, III, IV, I form the matrix

$$\left(\begin{array}{ccccccc} a, & b, & c, & d; & a', & b', & c' \\ b, & c, & d, & e; & b', & c', & d' \\ a', & b', & c', & d'; & a'', & b'', & c'' \end{array}\right),$$

and then we have

$$\mathrm{I} = -\begin{vmatrix} d, & c, & b' \\ e, & d, & c' \\ d', & c', & b'' \end{vmatrix} + \begin{vmatrix} d, & b, & c' \\ e, & c, & d' \\ d', & b', & d'' \end{vmatrix},$$

$$\mathrm{II} = -\begin{vmatrix} d, & c, & a' \\ e, & d, & b' \\ d', & c', & a'' \end{vmatrix} - \begin{vmatrix} d, & b, & b' \\ e, & c, & c' \\ d', & b', & b'' \end{vmatrix} - \begin{vmatrix} d, & a, & c' \\ e, & b, & d' \\ d', & a', & c'' \end{vmatrix},$$

$$\mathrm{III} = -\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} - \begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} - \begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix},$$

$$\mathrm{IV} = \begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix} + \begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix},$$

values which, as I proceed to show, satisfy the identical equation

$$a\mathrm{I} + b\mathrm{II} + d\mathrm{III} + e\mathrm{IV} = 0.$$

11. We have in fact

$$\begin{aligned} \mathrm{I} = \; & d\,(db'' - c'^2 + cc'' - b'd') \\ & + e\,(b'c' - b''c + b'c' - bc'') \\ & + d'(cc' - db' + bd' - cc'), \end{aligned}$$

where the last line is $= bd'^2 - db'd'$;

$$\begin{aligned} \mathrm{II} = \; & d\,(b'c' - a''d + b'c' - b''c + a'd' - bc'') \\ & + e\,(ca'' - a'c' + bb'' - b'^2 + ac'' - a'c') \\ & + d'(a'd - b'c + b'c - bc' + bc' - ad'), \end{aligned}$$

where the last line is $da'd' - ad'^2$;

$$\begin{aligned}\mathrm{III} = \; & a(-b'd' + cc'' + c'^2 - b''d + b'd' - ea'') \\ & + b(\; bc'' - b'c' + b''c - b'c' + a''d - a'd') \\ & + a'(\; cc' - bd' + b'd - cc' + a'e - b'd\;),\end{aligned}$$

where the last line is $-ba'd' + ea'^2$; and

$$\begin{aligned}\mathrm{IV} = \; & a(b''c - b'c' + da'' - b'c') \\ & + b(b'^2 - bb'' + a'c' - a''c) \\ & + a'(bc' - b'c + cb' - a'd),\end{aligned}$$

where the last line is $ba'c' - a'^2d$.

These values may be expressed as follows:

$$\begin{aligned}\mathrm{I} = \; & a(\; 0 \qquad\qquad\qquad\qquad\qquad\qquad) \\ & + b(\quad d'^2 \qquad\qquad\qquad\qquad\qquad) && (12) \\ & + d(\; b''d + c''c - b'd' - c'c' - d'b') && (13) \\ & + e(-b''c - c''b + b'c' + c'b' \qquad), && (14)\end{aligned}$$

$$\begin{aligned}\mathrm{II} = \; & a(-\; d'^2 \qquad\qquad\qquad\qquad\qquad) && (21) \\ & + b(\; 0 \qquad\qquad\qquad\qquad\qquad\qquad) && \ldots \\ & + d(-a''d - b''c - c''b + a'd' + b'c' + c'b' + d'a') && (23) \\ & + e(\; a''c + b''b + c''a - a'c' - b'b' - c'a' \qquad), && (24)\end{aligned}$$

$$\begin{aligned}\mathrm{III} = \; & a(-b''d - c''d + b'd' + c'c' + d'b' \qquad) && (31) \\ & + b(\; a''d + b''c + c''b - a'd' - b'c' - c'b' - d'a') && (32) \\ & + d(\; 0 \qquad\qquad\qquad\qquad\qquad\qquad) && \ldots \\ & + e(-a''a + a'a' \qquad\qquad\qquad), && (34)\end{aligned}$$

$$\begin{aligned}\mathrm{IV} = \; & a(\; b''c + c''b - b'c' - c'b' \qquad\qquad) && (41) \\ & + b(-a''c - b''b - c''a + a'c' + b'b' + c'a' \qquad) && (42) \\ & + d(\; a''a - a'a' \qquad\qquad\qquad\qquad) && (43) \\ & + e(\; 0 \qquad\qquad\qquad\qquad\qquad\qquad), && \ldots\end{aligned}$$

which are of the form

$$\begin{aligned}
\mathrm{I} &= a\ \ 0\ \ + b(12) + d(13) + e(14),\\
\mathrm{II} &= a(21) + b\ \ 0\ \ + d(23) + e(24),\\
\mathrm{III} &= a(31) + b(32) + d\ \ 0\ \ + e(34),\\
\mathrm{IV} &= a(41) + b(42) + d(43) + e\ \ 0\ \ ,
\end{aligned}$$

where $(12) = -(21)$, &c., and which therefore satisfy the equation

$$a\mathrm{I}\ \ + b\mathrm{II}\ \ + d\mathrm{III}\ \ + e\mathrm{IV} = 0.$$

12. The equation of the curve which by its intersection with the tangent gives the tangentials, is

$$\mathrm{III} = -\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} - \begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} - \begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix} = 0,$$

the degrees of which are

in the coefficients of U, 3,
in (x, y, z)..... 6,
in (X, Y, Z)..... 4,
in (X', Y', Z')..... 2;

and it only remains to divest this equation of a factor which it contains,

$$\begin{vmatrix} x, & y, & z \\ X, & Y, & Z \\ X', & Y', & Z' \end{vmatrix}^2,$$

which being thrown out, the equation will be independent of (X', Y', Z') and will be of the degrees

in the coefficients of U, 3,
in (x, y, z)..... 4,
in (X, Y, Z)..... 2,

and will in fact be the before-mentioned equation $\Omega = D^3H - 3D^3H_1 = 0$.

13. Write for shortness,

$$\begin{vmatrix} x, & y, & z \\ X, & Y, & Z \\ X', & Y', & Z' \end{vmatrix} = \Lambda,$$

it is to be shown that

$$\mathrm{III} = -\tfrac{1}{2}\Lambda^2(D^3H - 3D^3H_1).$$

14. To effect this I remark that we have identically

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix} = \Delta^2 H;$$

and I proceed to operate upon this equation with $D = X\partial_x + Y\partial_y + Z\partial_z$.

I notice that

$$a, \quad b, c, d, e; \quad a', \quad b', c', d'; \quad a'', \quad b'', c''$$

are in regard to (x, y, z) of the degrees

$$4, \quad 3, 2, 1, 0; \quad 3, \quad 2, 1, 0; \quad 2, \quad 1, 0;$$

or what is the same thing, since for the case in hand $n = 4$, of the degrees

$$n, n-1, \ldots; \quad n-1, n-2, \ldots; \quad n-2, n-3, \ldots$$

and we have

$$Da = nb, \; Db = (n-1)c, \ldots \quad Da' = (n-1)b', \; Db' = (n-2)c', \ldots \quad Da'' = (n-2)b'', \ldots$$

15. In the determinant

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix}, = \Delta^2 H,$$

the degrees of the terms (other than each top term, the degree of which is higher by unity) in the several columns are $n-1$, $n-2$, $n-2$; if then we operate on the determinant with D, and as regards the top terms we write

$$\begin{aligned} Da &= b + (n-1)b, \\ Db &= c + (n-2)c, \\ Da' &= b' + (n-2)b', \end{aligned}$$

we have in the first place a term

$$\begin{vmatrix} b, & c, & b' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix}$$

which vanishes, and next the terms

$$(n-1)\begin{vmatrix} b, & b, & a' \\ c, & c, & b' \\ b', & b', & a'' \end{vmatrix} + (n-2)\begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix} + (n-2)\begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix}$$

the first of which vanishes. On the right-hand side $D\Lambda = 0$ identically, and therefore $D.\Lambda^2 H = \Lambda^2 DH$, or we have

$$(n-2)\begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix} + (n-2)\begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix} = \Lambda^2 DH.$$

16. I repeat the operation D: we have

$$\left.\begin{array}{l} (n-2)(n-1)\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix} + (n-2)(n-1)\begin{vmatrix} b, & b, & b' \\ c, & c, & c' \\ b', & b', & b'' \end{vmatrix} \\ \\ +(n-2)(n-3)\begin{vmatrix} a, & d, & a \\ b, & c, & b \\ a', & d', & a'' \end{vmatrix} + (n-2)(n-2)\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} \\ \\ +(n-2)(n-2)\begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} + (n-2)(n-3)\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} \end{array}\right\} = \Lambda^2 D^2 H;$$

or, collecting the different terms,

$$(n-2)(n-3)\begin{vmatrix} a, b, c' \\ b, c, d' \\ a', b', c'' \end{vmatrix} + 2(n-2)^2\begin{vmatrix} a, c, b' \\ b, d, c' \\ a', c', b'' \end{vmatrix} + (n-2)(n-3)\begin{vmatrix} a, d, a' \\ b, e, b' \\ a', d', a'' \end{vmatrix} + (n-2)(n-1)\begin{vmatrix} b, c, a' \\ c, d, b' \\ b', c', a'' \end{vmatrix} = \Lambda^2 D^2 H.$$

17. A little consideration will show that in this equation we may write $n-1$ for n, and H_1 for H. In fact, putting for a moment $\delta = x_{,}\partial_x + y_{,}\partial_y + z_{,}\partial_z$, we have corresponding to the equation

$$\begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix} = \Lambda^2 H,$$

this other equation,

$$\begin{vmatrix} \delta a, & \delta b, & \delta a' \\ \delta b, & \delta c, & \delta b' \\ \delta a', & \delta b', & \delta a'' \end{vmatrix} = \Lambda^2 H_1,$$

where ultimately $(x_{,}, y_{,}, z_{,})$ are to be replaced by (x, y, z). We may operate upon this equation with D, D^2, ... as before, the only difference being that in the first

instance δa, δb, &c. are as regards (x, y, z) of degrees lower by unity than a, b, &c., that is $n-1$ must be substituted throughout in the place of n; and when at the end of the process $(x_{,}, y_{,}, z_{,})$ are replaced by (x, y, z), then δa, δb, &c. become equal to a, b, &c., from which the truth of the asserted proposition is manifest.

18. Hence writing $n=4$, we have

$$2\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} + 8\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix} + 2\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix} + 6\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix} = \Lambda^2 D^2 H,$$

$$2\begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a, & d', & a'' \end{vmatrix} + 2\begin{vmatrix} b, & c, & a' \\ c, & d, & b' \\ b', & c', & a'' \end{vmatrix} = \Lambda^2 D^2 H_1;$$

and hence

$$2\left\{\begin{vmatrix} a, & b, & c' \\ b, & c, & d' \\ a', & b', & c'' \end{vmatrix} + \begin{vmatrix} a, & c, & b' \\ b, & d, & c' \\ a', & c', & b'' \end{vmatrix} + \begin{vmatrix} a, & d, & a' \\ b, & e, & b' \\ a', & d', & a'' \end{vmatrix}\right\} = \Lambda^2(D^2H - 3D^2H_1),$$

which is the required equation

$$\mathrm{III} = -\tfrac{1}{2}\Lambda^2(D^2H - 3D^2H_1).$$

19. It is to be added, that the equation for $\Lambda^2 DH$ gives $\mathrm{IV} = \frac{1}{2}\Lambda^2 DH$; the values of II and I are at once obtained from those of III and IV by interchanging (x, y, z) and (X, Y, Z). Hence if we represent by $\mathfrak{H}$, $\mathfrak{D}$, &c. the values which H, D, &c. assume by this interchange, we may write

$$\begin{aligned} \mathrm{I} &= -\tfrac{1}{2}\Lambda^2 \mathfrak{D}\mathfrak{H}, \\ \mathrm{II} &= +\tfrac{1}{2}\Lambda^2(\mathfrak{D}^2\mathfrak{H} - 3\mathfrak{D}^2\mathfrak{H}_1), \\ \mathrm{III} &= -\tfrac{1}{2}\Lambda^2(D^2H - 3D^2H_1), \\ \mathrm{IV} &= +\tfrac{1}{2}\Lambda^2 DH; \end{aligned}$$

and the identical equation,

$$a\mathrm{I} + b\mathrm{II} + d\mathrm{III} + e\mathrm{IV} = 0,$$

gives therefore

$$-\mathfrak{D}\mathfrak{H}\,.\,U + \tfrac{1}{4}(\mathfrak{D}^2\mathfrak{H} - 3\mathfrak{D}^2\mathfrak{H}_1)DU - \tfrac{1}{4}(D^2H - 3D^2H_1)\mathfrak{D}\Upsilon + DH\,.\,\Upsilon = 0,$$

which is of itself sufficient to put in evidence the property that the curve $D^2H - 3D^2H_1 = 0$ gives by its intersections with the tangent $DU = 0$, the tangentials of the point (x, y, z). The last-mentioned equation is the equation for a quartic corresponding to Mr Salmon's equation

$$-\mathfrak{H}\,.\,U + \tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,DU - \tfrac{1}{3}DH\,.\,\mathfrak{D}\Upsilon + H\,.\,\Upsilon = 0$$

for the cubic $U = 0$.

20. It is worth while to give the investigation of the equation for the cubic; the matrix is

$$\left(\begin{array}{ccccc} a, & b, & c; & a', & b' \\ \left|\begin{array}{ccc} b, & c, & d; \\ a', & b', & c'; \end{array}\right. & & & \left.\begin{array}{cc} b', & c' \\ a'', & b'' \end{array}\right| & \end{array}\right)$$

and the identical equation is

$$a\mathrm{I} + b\mathrm{II} + c\mathrm{III} + d\mathrm{IV} = 0,$$

where

$$\mathrm{I} = \begin{vmatrix} c, & b, & b' \\ d, & c, & c' \\ c', & b', & b'' \end{vmatrix},$$

$$\mathrm{II} = -\begin{vmatrix} c, & b, & a' \\ d, & c, & b' \\ c', & b', & a'' \end{vmatrix} - \begin{vmatrix} c, & a, & b' \\ d, & b, & c' \\ c', & a', & b'' \end{vmatrix},$$

$$\mathrm{III} = -\begin{vmatrix} a, & b, & b' \\ b, & c, & c' \\ a', & b', & b'' \end{vmatrix} - \begin{vmatrix} a, & c, & a' \\ b, & d, & b' \\ a', & c', & a'' \end{vmatrix},$$

$$\mathrm{IV} = \begin{vmatrix} a, & b, & a' \\ b, & c, & b' \\ a', & b', & a'' \end{vmatrix},$$

or, as we may express them,

$$\begin{aligned} \mathrm{I} = {} & a(\ 0\qquad\qquad) && \\ & + b(\qquad c'^2\qquad) && (12) \\ & + c(b''c - b'c' - c'b') && (13) \\ & + d(-b''b + b'b'\ ), && (14) \end{aligned}$$

$$\begin{aligned} \mathrm{II} = {} & a(\qquad -c'^2\qquad\qquad) && (21) \\ & + b(\ 0\qquad\qquad) && \dots \\ & + c(-a''c - b''b + a'c' + b'b' + c'a') && (23) \\ & + d(\ a''b + b''a - a'b' - b'a'\ ), && (24) \end{aligned}$$

$$\begin{aligned} \mathrm{III} = {} & a(-b''c + b'c' + c'b'\qquad) && (31) \\ & + b(\ a''c + b''b - a'c' - b'b' - c'a') && (32) \\ & + c(\ 0\qquad\qquad) && \dots \\ & + d(-aa''\qquad + a'^2\qquad) , && (34) \end{aligned}$$

$$\begin{aligned} \mathrm{IV} = {} & a(b''b\ - b'b'\qquad) && (41) \\ & + b(-a''b - b''a + a'b' + b'a'\qquad) && (42) \\ & + c(\ a''a\qquad - a'a'\qquad) && (43) \\ & + d(\ 0\qquad\qquad) , && \dots \end{aligned}$$

which verify the identical equation. We have $\mathrm{III} = -\Lambda^2 DH$, $\mathrm{IV} = \Lambda^2 H$, and thence $\mathrm{II} = +\Lambda^2 \mathfrak{D}\mathfrak{H}$, $\mathrm{I} = -\Lambda^2 \mathfrak{H}$; hence the equation in question

$$-\mathfrak{H}\,.\,U + \tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,DU - \tfrac{1}{3}DH\,.\,\mathfrak{D}\Upsilon + H\,.\,\Upsilon = 0.$$

21. One other example will be sufficient to render manifest the law of the formation of the multipliers I, II, III, IV.

In the case of a sextic curve we have the matrix

$$\left(\begin{array}{cccccc} a, & b, & c, & d, & e, & f; \\ b, & c, & d, & e, & f, & g; \\ a', & b', & c', & d', & e', & f'; \end{array}\;\begin{array}{cccccc} a', & b', & c', & d', & e' \\ b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'' \end{array}\right),$$

the identical equation is

$$a\mathrm{I} + b\mathrm{II} + f\mathrm{III} + g\mathrm{IV} = 0;$$

and the expressions for the multipliers I, II, III, IV are:

$$\mathrm{I} = \begin{vmatrix} f, & e, & b' \\ g, & f, & c' \\ f', & e', & b'' \end{vmatrix} + \begin{vmatrix} f, & d, & c' \\ g, & e, & d' \\ f', & d', & c'' \end{vmatrix} + \begin{vmatrix} f, & c, & d \\ g, & d, & e \\ f', & c', & d'' \end{vmatrix} + \begin{vmatrix} f, & b, & e' \\ g, & c, & f' \\ f', & b', & e'' \end{vmatrix},$$

$$\mathrm{II} = -\begin{vmatrix} f, & e, & a' \\ g, & f, & b' \\ f', & e', & a'' \end{vmatrix} - \begin{vmatrix} f, & d, & b' \\ g, & e, & c' \\ f', & d', & b'' \end{vmatrix} - \begin{vmatrix} f, & c, & c' \\ g, & d, & d' \\ f', & c', & d'' \end{vmatrix} - \begin{vmatrix} f, & b, & d' \\ g, & c, & e' \\ f', & b', & d'' \end{vmatrix} - \begin{vmatrix} f, & a, & e' \\ g, & b, & f' \\ f', & a', & e'' \end{vmatrix},$$

$$\mathrm{III} = -\begin{vmatrix} a, & b, & e' \\ b, & c, & f' \\ a', & b', & e'' \end{vmatrix} - \begin{vmatrix} a, & c, & d' \\ b, & d, & e' \\ a', & c', & e'' \end{vmatrix} - \begin{vmatrix} a, & d, & c' \\ b, & e, & d' \\ a', & d', & e'' \end{vmatrix} - \begin{vmatrix} a, & e', & b' \\ b, & f', & c' \\ a', & e'', & b'' \end{vmatrix} - \begin{vmatrix} a, & f, & a' \\ b, & g, & b' \\ a', & f', & a'' \end{vmatrix},$$

$$\mathrm{IV} = \begin{vmatrix} a, & b, & d' \\ b, & c, & e' \\ a', & b', & d'' \end{vmatrix} + \begin{vmatrix} a, & c, & c' \\ b, & d, & d' \\ a', & d', & c'' \end{vmatrix} - \begin{vmatrix} a, & d, & b' \\ b, & e, & c' \\ a', & d', & b'' \end{vmatrix} - \begin{vmatrix} a, & e, & a' \\ b, & f, & b' \\ a', & e', & a'' \end{vmatrix}.$$

22. We have in fact

$$\begin{aligned}
\mathrm{I} = \; & a(0 && &&) \\
& +b(-ge'' && +f''f'' &&) && (12) \\
& +f(\;b''f + c''e + d''d + e''c && -b'f' - c'e' - d'd' - e'c' - f'b' &&) && (13) \\
& +g(-b''e - c''d - d''c && +b'e' + c'd' + d'c' + e'b' &&), && (14) \\
\mathrm{II} = \; & a(\;e''g && -f'f' &&) && (21) \\
& +b(0 && &&) && \ldots \\
& +f(-a''f - b''e - c''d - d''e - e''b + a'f' + b'e' + c'd' + d'c' + e'b' + f'a') && && && (23) \\
& +g(\;a''e + b''d + c''c + d''b && -a'e' - b'd' - c'c' - d'b' - e'a' &&), && (24)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{III} = \; & a(-b''f - c''e - d''d - e''c \quad + b'f' + c'e' + d'd' + e'c' + f'b' \quad) && (31)\\
& + b(+a''f + b''e + c''d + d''c + e''b - a'f'' - b'e' - c'd' - d'c' - e'b' - f'a') && (32)\\
& + f \;\; 0 && \cdots\\
& + g(-a''a \quad + a'^2 \quad), && (34)\\
\mathrm{IV} = \; & a(\;\; b''e + c''d + d''c \quad - b'e' - c'd' - d'c - e'b' \quad) && (41)\\
& + b(-a''e - b''d - c''c - d''b \quad + a'e' + b'd' + c'c' + d'b' + e'a' \quad) && (42)\\
& + f(\;\; a''a \quad - a'^2 \quad) && (43)\\
& + g(0 \quad), && \cdots
\end{aligned}
$$

which are of the form

$$
\begin{aligned}
\mathrm{I} &= a \;\; 0 \;\; + b(12) + f(13) + g(14),\\
\mathrm{II} &= a(21) + b \;\; 0 \;\; + f(23) + g(24),\\
\mathrm{III} &= a(31) + b(32) + f \;\; 0 \;\; + g(34),\\
\mathrm{IV} &= a(41) + b(42) + f(43) + g \;\; 0 \;\; ,
\end{aligned}
$$

where $(12) = -(21)$ &c., and the equation

$$a\mathrm{I} \;\; + b\mathrm{II} \;\; + f\mathrm{III} \;\; + g\mathrm{IV} = 0$$

is consequently satisfied.

23. The expression

$$
\mathrm{III} = -\begin{vmatrix} a, & b, & e' \\ b, & c, & f'' \\ a', & b', & e'' \end{vmatrix} - \begin{vmatrix} a, & c, & d' \\ b, & d, & e' \\ a', & c', & d'' \end{vmatrix} - \begin{vmatrix} a, & d, & c' \\ b, & e, & d' \\ a', & d', & c'' \end{vmatrix} - \begin{vmatrix} a, & f, & a' \\ b, & g, & b' \\ a', & f', & a'' \end{vmatrix}
$$

leads to

$$\mathrm{III} = -\Lambda^2(D^4H - 5D^4H_1 + 10D^4H_2),$$

and consequently the equation of the curve which by its intersections with the tangent determines the tangentials of a point of a sextic, is

$$D^4H - 5D^4H_1 + 10D^4H_2 = 0.$$

24. In the general case of a curve of the order n the matrix is

$$
\left(\begin{array}{llll|llll}
a_0, & a_1, & a_2 \ldots & a_{n-1}\,; & a_0', & a_1' & \ldots & a'_{n-2} \\
a_1, & a_2, & a_3 \ldots & a_n \;\; ; & a_1', & a_2' & \ldots & a'_{n-1} \\
a_0', & a_1', & a_2' \ldots & a'_{n-1}\,; & a_0'', & a_1'' & \ldots & a''_{n-2}
\end{array} \right)
$$

where, in analogy with what precedes,

$$
\begin{aligned}
a_0 \;\; &= (X, \; Y, \; Z)^n \qquad\qquad\qquad , \\
a_1 \;\; &= (X, \; Y, \; Z)^{n-1}(x, \; y, \; z) \;\; , \\
&\vdots \\
a_{n-1} &= (X, \; Y, \; Z) \;\; (x, \; y, \; z)^{n-1}, \\
a_n \;\; &= \qquad\qquad\qquad (x, \; y, \; z)^n \;\; ,
\end{aligned}
$$

and similarly for the accented letters, so that

$a_0 \quad = 0$ is the equation of the curve;

$a_1 \quad = 0$ is the equation of the first or $(n-1)$thic polar;

$a_{n-1} = 0$ is the equation of the last or line-polar, or what is the same thing, since (x, y, z) is a point on the curve, the tangent at this point;

$a_n \quad = 0$, the condition which expresses that (x, y, z) is a point of the curve;

and we have to form the identical equation

$$a_0\mathrm{I} + a_1\mathrm{II} + a_{n-1}\mathrm{III} + a_n\mathrm{IV} = 0.$$

25. If, for shortness, the columns of the last-mentioned matrix are represented by

$$1,\ 2,\ 3 \ldots n,\ (1),\ (2) \ldots (n-1),$$

and the determinants formed with these columns respectively by a corresponding notation $\{1, 2, (1)\}$, $\{1, 2, (2)\}$, &c., then the expressions for the multipliers I, II, III, IV are as follows, viz.

$$\begin{aligned}
\mathrm{I} &= \{n,\ n-1,\ (2)\} + \{n,\ n-2,\ (3)\} \ldots + \{n,\ 2,\ (n-1)\} \\
\mathrm{II} &= -\{n,\ n-1,\ (1)\} - \{n,\ n-2,\ (2)\} \ldots - \{n,\ 2,\ (n-2)\} - \{n,\ 1,\ (n-1)\}, \\
\mathrm{III} &= -\{1,\ 2,\ (n-1)\} - \{1,\ 3,\ (n-2)\} \ldots - \{1,\ n-1,\ (2)\} - \{1,\ n,\ (1)\ \}, \\
\mathrm{IV} &= \{1,\ 2,\ (n-2)\} + \{1,\ 3,\ (n-1)\} \ldots + \{1,\ n-1,\ (1)\};
\end{aligned}$$

the truth of the identical equation being shown, as in the foregoing special cases, by the transformation of the multipliers into the form

$$\begin{aligned}
\mathrm{I} &= a_0\ \ 0 + a_1(12) + a_{n-1}(13) + a_n(14), \\
\mathrm{II} &= a_0(21) + a_1\ \ 0 + a_{n-1}(23) + a_n(24), \\
\mathrm{III} &= a_0(31) + a_1(32) + a_{n-1}\ \ 0 + a_n(34), \\
\mathrm{IV} &= a_0(41) + a_1(42) + a_{n-1}(43) + a_n\ \ 0\ ,
\end{aligned}$$

where $(12) = -(21)$, &c.: the required expressions may be written down without difficulty.

26. Proceeding then to reduce the equation

$$\mathrm{III} = -\{1,\ 2,\ (n-1)\} - \{1,\ 3,\ (n-2)\} \ldots - \{1,\ n-1,\ (2)\} - \{1,\ n,\ (1)\},$$

we have the equation

$$\{1,\ 2,\ (1)\} = \Delta^2 H,$$

which is to be successively operated on with D. The degrees (less unity) of the columns

$$1, \quad 2, \quad \ldots n-1, \quad n, \quad (1) \quad (2), \quad \ldots n-1,$$

are

$$n-1,\ n-2, \ldots\ 1, \quad 0,\ n-2,\ n-3, \ \ldots\ \ 0;$$

and the rule is to operate on each column of the determinant, multiplying by the degree less unity, and increasing the symbolical number by unity. Thus

$$\begin{aligned}
D\{1,\ 2,\ (1)\} &= (n-1)\{2,\ 2,\ (1)\} + (n-2)\{1,\ 3,\ (1)\} + (n-2)\{1,\ 2,\ (2)\}, \\
&= \qquad\qquad\qquad\qquad (n-2)\{1,\ 3,\ (1)\} + (n-2)\{1,\ 2,\ (2)\},
\end{aligned}$$

since $\{2, 2, (1)\}$ vanishes identically. The following Table shows the mode of effecting the operations:

H	1	$= 1$	$[n-2]^0$ $[n-2]^0$		12(1)
DH	1	$= 1$	$[n-2]^0$	$[n-2]^1$	12(2)
	1	1	$[n-2]^1$	$[n-2]^0$	13(1)
D^2H	1	$= 1$	$[n-2]^0$	$[n-2]^2$	12(3)
	1 1	2	$[n-2]^1$	$[n-2]^1$	13(2)
	1	1	$[n-2]^2$	$[n-2]^0$	14(1)
	1	1	$[n-2]^1$	$[n-2]^0$ $[n-1]^1$	23(1)
D^3H	1	1	$[n-2]^0$	$[n-2]^3$	12(4)
	2 1	3	$[n-2]^1$	$[n-2]^2$	13(3)
	1 2	3	$[n-2]^2$	$[n-2]^1$	14(2)
	1	1	$[n-2]^3$	$[n-2]^0$	15(1)
	1 2	3	$[n-2]^1$	$[n-2]^1$	23(2)
	1 1	2	$[n-2]^2$	$[n-2]^0$ $[n-1]^1$	24(1)
D^4H	1	$= 1$	$[n-2]^0$	$[n-2]^4$	12(5)
	3 1	4	$[n-2]^1$	$[n-2]^3$	13(4)
	3 3	6	$[n-2]^2$	$[n-2]^2$	14(3)
	1 3	4	$[n-2]^3$	$[n-2]^1$	15(2)
	1	1	$[n-2]^4$	$[n-2]^0$	16(1)
	3 3	6	$[n-2]^1$	$[n-2]^2$	23(3)
	2 3 3	8	$[n-2]^2$	$[n-2]^1$ $[n-1]^1$	24(2)
	2 1	3	$[n-2]^3$	$[n-2]^0$	25(1)
	2	2	$[n-2]^2$	$[n-2]^0$ $[n-1]^2$	34(1)
D^5H	1	$= 1$	$[n-2]^0$	$[n-2]^5$	12(6)
	4 1	5	$[n-2]^1$	$[n-2]^4$	13(5)
	6 4	10	$[n-2]^2$	$[n-2]^3$	14(4)
	4 6	10	$[n-2]^3$	$[n-2]^2$	15(3)
	1 4	5	$[n-2]^4$	$[n-2]^1$	16(2)
	1	1	$[n-2]^5$	$[n-2]^0$	17(1)
	6 4	10	$[n-2]^1$	$[n-2]^4$	23(4)
	8 6 6	20	$[n-2]^2$	$[n-2]^3$ $[n-1]^1$	24(3)
	3 8 4	15	$[n-2]^3$	$[n-2]^2$	25(2)
	3 1	4	$[n-2]^4$	$[n-2]^1$	26(1)
	2 8	10	$[n-2]^2$	$[n-2]^1$	34(2)
	2 3	5	$[n-2]^3$	$[n-2]^0$ $[n-1]^2$	35(1)
D^6H	1	$= 1$	$[n-2]^0$	$[n-2]^6$	12(7)
	5 1	6	$[n-2]^1$	$[n-2]^5$	13(6)
	10 5	15	$[n-2]^2$	$[n-2]^4$	14(5)
	10 10	20	$[n-2]^3$	$[n-2]^3$	15(4)
	5 10	15	$[n-2]^4$	$[n-2]^2$	16(3)
	1 5	6	$[n-2]^5$	$[n-2]^1$	17(2)
	1	1	$[n-2]^6$	$[n-2]^0$	18(1)
	10 5	15	$[n-2]^1$	$[n-2]^4$	23(5)
	20 10 10	40	$[n-2]^2$	$[n-2]^3$	24(4)
	15 20 10	45	$[n-2]^3$	$[n-2]^2$ $[n-1]^1$	25(3)
	4 15 5	24	$[n-2]^4$	$[n-2]^1$	26(2)
	4 1	5	$[n-2]^5$	$[n-2]^0$	27(1)
	10 20	30	$[n-2]^2$	$[n-2]^2$	34(3)
	5 10 15	30	$[n-2]^3$	$[n-2]^1$ $[n-1]^2$	35(2)
	5 4	9	$[n-2]^4$	$[n-2]^0$	36(1)
	5	5	$[n-2]^5$	$[n-2]^0$ $[n-1]^3$	45(1)
D^7H	&c.				

where the first three columns show the numbers which give, by the addition of the numbers in the same horizontal line, the numerical coefficients of the factorials which multiply the different terms of H, DH, &c., and where in the last column 12(1), &c. are written for shortness in the place of $\{1, 2, (1)\}$, &c.

27. It is clear that we have in general

$$\begin{array}{rlll}
\Lambda^2 D^r H = & & 1[n-2]^0[n-2]^r & \{1,\ 2,\ (r+1)\} \\
& + \dfrac{r}{1} & [n-2]^1[n-2]^{r-1} & \{1,\ 3,\ (r)\ \} \\
& + \dfrac{r\,.\,r-1}{1\,.\,2} & [n-2]^2[n-2]^{r-2} & \{1,\ 4,\ (r-1)\} \\
& \vdots & & \\
& + \quad 1 & [n-2]^r[n-2]^0 & \{1,\ r+2,\ (1)\} \\
\hline
+[n-1]^1 & \left\{\begin{array}{l} + R_0' \\ + R_1' \\ \vdots \\ + R'_{r-2} \end{array}\right. & \begin{array}{ll} [n-2]^1 & [n-2]^{r-2} \\ [n-2]^2 & [n-2]^{r-3} \\ & \\ [n-2]^{r-1} & [n-2]^0 \end{array} & \begin{array}{l} \{2,\ 3,\ (r-1)\} \\ \{2,\ 4,\ (r-2)\} \\ \\ \{2,\ r+1,\ (1)\} \end{array} \\
\hline
+[n-1]^2 & \left\{\begin{array}{l} + R_0'' \\ \vdots \\ + R''_{r-4} \end{array}\right. & \begin{array}{ll} [n-2]^2 & [n-2]^{r-4} \\ & \\ [n-2]^{r-2} & [n-2]^0 \end{array} & \begin{array}{l} \{3,\ 4,\ (r-3)\} \\ \\ \{3,\ r,\ (1)\ \} \end{array} \\
\hline
& \vdots & & \\
+[n-1]^{\frac{1}{2}r} & \left\{\begin{array}{l} + R_0^{\frac{1}{2}r} \\ r \text{ even; or } r \text{ odd} \end{array}\right. & [n-2]^{\frac{1}{2}r}\ [n-2]^0 & \{\tfrac{1}{2}r+1,\ \tfrac{1}{2}r+2,\ (1)\} \\
+[n-1]^{\frac{1}{2}(r-1)} & \left\{\begin{array}{l} + R_0^{\frac{1}{2}(r-1)} \\ + R_0^{\frac{1}{2}(r+1)} \end{array}\right. & \begin{array}{ll} [n-2]^{\frac{1}{2}(r-1)} & [n-2]^1 \\ [n-2]^{\frac{1}{2}(r+1)} & [n-2]^0 \end{array} & \begin{array}{l} \{\tfrac{1}{2}(r+1),\ \tfrac{1}{2}(r+3),\ (2)\} \\ \{\tfrac{1}{2}(r+1),\ \tfrac{1}{2}(r+5),\ (1)\}; \end{array}
\end{array}$$

and the general term is

$$[n-1]^\delta\, R_s^\delta [n-2]^{\delta+s}[n-2]^{r-2\delta-s}\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\},$$

where s extends from $s=0$ to $s=r-2\delta$, and δ from $\delta=0$ to $\delta=\frac{1}{2}r$ or $\frac{1}{2}(r-1)$, according as r is even or odd. The expression for the coefficients R_s^δ is

$$R_s^\delta = \frac{[r]^\delta}{[s]^\delta},$$

and that of the other coefficients $R_{s\delta}\,(\delta = \text{or} < 1)$ is not required for the present purpose.

28. According to a remark already made, the expressions for $D^r H_1$, $D^r H_2$, &c. are at once obtained from that for $D^r H$ by merely writing $n-1$, $n-2$, &c. in the place of n: it is however to be noticed, that the quantity within the [ ] must not be negative, and that on its becoming so, the factorial is to be omitted.

29. I write now

$$+s=\alpha,\quad r-2\delta-s=\beta,$$

and I consider the expression

$$D^rH-\frac{n-1}{1}D^rH_1+\&c.;$$

the general term of which is

$$R_s^\delta\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\}$$
$$\times\left\{\begin{array}{l}\quad\quad [n-1]^\delta[n-2]^\alpha[n-2]^\beta\\ -\dfrac{n-1}{1}[n-2]^\delta[n-3]^\alpha[n-3]^\beta\\ +\quad \&c.\end{array}\right\},$$

or, as this may be written, putting $q=n-\delta-1$,

$$R_s^\delta\{\delta+1,\ \delta+2+s,\ (r-2\delta+1-s)\}$$
$$\times[n-1]^\delta\left\{\begin{array}{l}\quad\quad [n-2]^\alpha[n-2]^\beta\\ -\dfrac{q}{1}[n-3]^\alpha[n-3]^\beta\\ +\ \&c.\end{array}\right\}.$$

30. I assume $r\not>n-2$, we have then $\alpha+\beta=r-\delta\not>n-\delta-2$, and therefore $\alpha+\beta<q$. The general term of the series in $\{\}$ is

$$(-)^\vartheta\frac{[q]^\vartheta}{[\vartheta]^\vartheta}[n-2-\vartheta]^\alpha[n-2-\vartheta]^\beta,$$

where the terms for which $n-2-\vartheta$ is negative are to be excluded, or what is the same thing, the series is not to be continued beyond $\vartheta=n-2$. But observing that $[q]^\vartheta$ vanishes for $\vartheta>q$, that is, $\vartheta>n-\delta-1$, it is in fact the same thing whether the series is continued indefinitely or only to the term for which $\vartheta=n-\delta-1$, and we may consistently with the condition $\vartheta\not>n-2$, continue the series as far as $\vartheta=n-\delta-1$, except in the case $\delta=0$, when by doing so we include the term corresponding to $\vartheta=n-1$, which in virtue of the condition ought to be excluded. The expression for the term in question is $(-)^{n-1}[-1]^\alpha[-1]^\beta$; hence if the sum of the series continued to the proper point is S, the sum continued indefinitely (in the particular case $\delta=0$) is $S+(-)^{n-1}[-1]^\alpha[-1]^\beta$, but in every other case the sum continued indefinitely is simply S. And by a well-known theorem in finite differences, the sum continued indefinitely is in fact zero. That is, except in the case $\delta=0$, we have $S=0$, but in the excepted case

$$S+(-)^{n-1}[-1]^\alpha[-1]^\beta=0;$$

or observing that $\alpha+\beta(=r-\delta)$ is in this case $=r$, and transforming the factorials, we have

$$S=(-)^{n-r}[\alpha]^\alpha[\beta]^\beta,$$

or substituting for α and β their values,

$$S=(-)^{n-r}[s]^s[r-s]^{r-s}.$$

31. Hence the general term of

$$D^r H - \frac{n-1}{1} D^r H_1 + \&c.$$

vanishes except for $\delta = 0$, but when $\delta = 0$, its value is

$$R_s^0\{1,\ 2+s,\ (r+1-s)\} \times (-)^{n-r}[s]^s[r-s]^{r-s};$$

or observing that R_s^0 is equal to $[r]^r \div [s]^s[r-s]^{r-s}$, the value is simply

$$(-)^{n-r}[r]^r\{1,\ 2+s,\ (r+1-s)\},$$

that is, we have

$$D^r H - \frac{n-1}{1} D^r H_1 + \&c.$$

$$= (-)^{n-r}[r]^r S_s\{1,\ 2+s,\ (r+1-s)\},$$

the summation in respect to s extending from $s=0$ to $s=r$. In particular, giving to r the values $n-2$ and $n-1$, and attending to the expressions for III and IV, we find

$$\Lambda^2\left(D^{n-2}H - \frac{n-1}{1} D^{n-2}H_1 + \&c. \ldots\right) = -[n-2]^{n-2}\,\mathrm{III},$$

$$\Lambda^2\left(D^{n-3}H - \frac{n-1}{1} D^{n-3}H_1 + \&c. \ldots\right) = -[n-3]^{n-3}\,\mathrm{IV}.$$

32. The equation $\mathrm{III} = 0$ belongs to the curve which by its intersections with the tangent, gives the tangentials of a point of the curve $U=0$. Hence the equation of the curve in question is

$$D^{n-2}H - \frac{n-1}{1} D^{n-2}H_1 + \&c. = 0,$$

which is Mr Salmon's theorem, leading to the solution of the problem of double tangents.

33. The expressions for I and II are obtained from those of IV and III by interchanging (X, Y, Z) and (x, y, z), and reversing the sign. Hence if, as before, $\mathfrak{H}$, $\mathfrak{D}$, &c. denote the values which H, D, &c. assume by this interchange, we have

$$\Lambda^2\left(\mathfrak{D}^{n-2}\mathfrak{H} - \frac{n-1}{1} \mathfrak{D}^{n-2}H_1 + \&c. \ldots\right) = [n-2]^{n-2}\,\mathrm{II},$$

$$\Lambda^2\left(\mathfrak{D}^{n-2}\mathfrak{H} - \frac{n-1}{1} \mathfrak{D}^{n-3}H_1 + \&c. \ldots\right) = [n-3]^{n-3}\,\mathrm{I},$$

and the identical equation

$$a_0\mathrm{I} + a_1\mathrm{II} + a_{n-1}\mathrm{III} + a_n\mathrm{IV} = 0$$

becomes therefore

$$\left.\begin{aligned}&\left(\mathfrak{D}^{n-3}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-3}\mathfrak{H}_1+\&c.\right)U\\ +\frac{1}{n(n-2)}&\left(\mathfrak{D}^{n-2}\mathfrak{H}-\frac{n-1}{1}\mathfrak{D}^{n-2}\mathfrak{H}_1+\&c.\right)DU\\ -\frac{1}{n(n-2)}&\left(D^{n-2}H-\frac{n-1}{1}D^{n-2}H_1+\&c.\right)\mathfrak{D}\Upsilon\\ &\left(D^{n-3}H-\frac{n-1}{1}D^{n-3}H_1+\&c.\right)\Upsilon\end{aligned}\right\}=0,$$

which is the general identical equation referred to in the introduction to the present memoir.

34. It is to be noticed that for $n=3$, the equation is

$$(\mathfrak{H}-2\mathfrak{H}_1)U+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,DU-\tfrac{1}{3}DH\,.\,\mathfrak{D}\Upsilon-(H-2H_1)\Upsilon=0.$$

But we have $H_1=H$, and in like manner $\mathfrak{H}_1=\mathfrak{H}$, and the equation thus becomes

$$-\mathfrak{H}U+\tfrac{1}{3}\mathfrak{D}\mathfrak{H}\,.\,D\Upsilon-\tfrac{1}{3}DH\,.\,\mathfrak{D}\Upsilon+H\Upsilon=0.$$

And so also for $n=4$, the equation is

$$(\mathfrak{D}\mathfrak{H}-3\mathfrak{D}\mathfrak{H}_1)U+\tfrac{1}{8}(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}_1)DU-\tfrac{1}{8}(D^2H-3D^2H_1)\mathfrak{D}\Upsilon-(DH-3DH_1)\Upsilon=0.$$

But we have in general $DH_1=\dfrac{n-3}{n-2}DH$, and therefore in the present case $DH_1=\tfrac{1}{2}DH$, and consequently $\mathfrak{D}\mathfrak{H}_1=\tfrac{1}{2}\mathfrak{D}\mathfrak{H}$, and the equation thus becomes

$$-\mathfrak{D}\mathfrak{H}\,.\,U+\tfrac{1}{4}(\mathfrak{D}^2\mathfrak{H}-3\mathfrak{D}^2\mathfrak{H}_1)DU-\tfrac{1}{4}(D^2H-3D^2H_1)\mathfrak{D}\Upsilon+DH\,.\,\Upsilon=0,$$

which agree with the results previously obtained for the two particular cases.

261.

ON THE CONIC OF FIVE-POINTIC CONTACT AT ANY POINT OF A PLANE CURVE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CXLIX. (for the year 1859), pp. 371—400. Received March 1,—Read March 24, 1859.]

THE tangent is a line passing through two consecutive points of a plane curve, and we may in like manner consider the conic which passes through five consecutive points of a plane curve; and as there are certain singular points, viz. the points of inflexion where three consecutive points of the curve lie in a line, so there are singular points where six consecutive points of the curve lie in a conic. In the particular case where the given curve is a cubic, the last-mentioned species of singular points have been considered by Plücker and Steiner, and in the same particular case, the theory of the conic of five-pointic contact has recently been established by Mr Salmon. But the general case, where the curve is of any order whatever, has not so far as I am aware been hitherto considered;—the establishment of this theory is the object of the present memoir.

I. *Investigation of the Equation of the Conic of Five-pointic Contact.*

1. I take (X, Y, Z) as current coordinates, and I represent the equation of the given curve by

$$\Upsilon = (*\between X, Y, Z)^m = 0.$$

Let (x, y, z) be the coordinates of a given point on the curve, and let $U = (*\between x, y, z)^m$ be what Υ becomes when (x, y, z) are written in the place of (X, Y, Z); we have therefore $U = 0$ as a condition satisfied by the coordinates of the point in question.

2. Write for shortness

$$D\,U = (X\partial_x + Y\partial_y + Z\partial_z)\ U,$$
$$D^2U = (X\partial_x + Y\partial_y + Z\partial_z)^2\ U,$$

and let $\Pi = aX + bY + cZ = 0$ be the equation of a line. It is easy to see that

$$D^2U - \Pi\,.\,DU = 0$$

will be the equation of a conic having an ordinary (two-pointic) contact with the curve at the point (x, y, z). In fact the equation $DU=0$ is that of the tangent at the point in question, and the equation $D^2U=0$ is that of the penultimate polar (or polar conic) of the point, which conic is touched by the tangent; the assumed equation represents therefore a conic having an ordinary (two-pointic) contact with the polar conic, and therefore with the curve. It may be added that the two conics intersect besides in a pair of points, and that the line joining these, or common chord of the two conics, is the line represented by the equation $\Pi=0$; and this being so, the constants (a, b, c) of the line $\Pi=0$ can be so determined as to give rise to a five-pointic contact.

3. Consider the coordinates of a point of the curve as functions of a single variable parameter; then for the present purpose the coordinates of a point consecutive to (x, y, z) may be taken to be

$$\begin{aligned} &x+dx+\tfrac{1}{2}\,d^2x+\tfrac{1}{6}\,d^3x+\tfrac{1}{24}\,d^4x,\\ &y+dy+\tfrac{1}{2}\,d^2y+\tfrac{1}{6}\,d^3y+\tfrac{1}{24}\,d^4y,\\ &z+dz+\tfrac{1}{2}\,d^2z+\tfrac{1}{6}\,d^3z+\tfrac{1}{24}\,d^4z, \end{aligned}$$

values which, substituted for X, Y, Z, must satisfy the equations

$$\Upsilon=0,\quad D^2U-\Pi\,.\,DU=0.$$

4. I write for shortness

$$\begin{aligned} \partial_1&=d\,x\,\partial_x+d\,y\,\partial_y+d\,z\,\partial_z,\\ \partial_2&=d^2x\,\partial_x+d^2y\,\partial_y+d^2z\,\partial_z,\\ \partial_3&=d^3x\,\partial_x+d^3y\,\partial_y+d^3z\,\partial_z,\\ \partial_4&=d^4x\,\partial_x+d^4y\,\partial_y+d^4z\,\partial_z, \end{aligned}$$

then the consecutive value of Υ is

$$\exp.\,(\partial_1+\tfrac{1}{2}\,\partial_2+\tfrac{1}{6}\,\partial_3+\tfrac{1}{24}\,\partial_4)\,U$$

(Read exp. z, exponential of z, $=e^z$), which is

$$\begin{aligned} &= \left.\begin{array}{l} (1+\partial_1+\tfrac{1}{2}\partial_1^2+\tfrac{1}{6}\partial_1^3+\tfrac{1}{24}\partial_1^4)\\ \times(1\quad+\tfrac{1}{2}\partial_2\qquad\quad+\tfrac{1}{8}\,\partial_2^2)\\ \times(1\qquad\qquad+\tfrac{1}{6}\partial_3\qquad\quad)\\ \times(1\qquad\qquad\qquad\quad+\tfrac{1}{24}\partial_4) \end{array}\right\}U\\ &= \left.\begin{array}{l} 1+\partial_1+\tfrac{1}{2}\partial_1^2+\tfrac{1}{6}\partial_1^3\quad+\tfrac{1}{24}\partial_1^4\\ \qquad\quad+\tfrac{1}{2}\partial_2+\tfrac{1}{2}\partial_1\partial_2+\tfrac{1}{4}\partial_1^2\partial_2\\ \qquad\qquad\qquad\qquad\quad+\tfrac{1}{8}\partial_2^2\\ \qquad\qquad+\tfrac{1}{6}\partial_3\quad+\tfrac{1}{6}\partial_1\,\partial_3\\ \qquad\qquad\qquad\qquad\quad+\tfrac{1}{24}\partial_4 \end{array}\right\}U\\ &= \quad U\\ &\quad+\quad\partial_1 U\\ &\quad+\tfrac{1}{2}\,(\partial_1^2+\partial_2)\,U\\ &\quad+\tfrac{1}{6}\,(\partial_1^3+3\partial_1\partial_2+\partial_3)\,U\\ &\quad+\tfrac{1}{24}\,(\partial_1^4+6\partial_1^2\partial_2+4\partial_1\partial_3+3\partial_2^2+\partial_4)\,U, \end{aligned}$$

the several terms of which must respectively vanish, and we have therefore

$$\begin{aligned}
U &= 0,\\
\partial_1 U &= 0,\\
\partial_2 U &= -\ \partial_1^2 U,\\
\partial_3 U &= -(\partial_1^3 + 3\partial_1\partial_2)\ U,\\
\partial_4 U &= -(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\ U.
\end{aligned}$$

5. Next, preparing to substitute in the equation

$$D^2 U - \Pi\,.\,DU = 0,$$

the consecutive value of DU is

$$\begin{aligned}
&(x\ + dx + \tfrac{1}{2}d^2x + \tfrac{1}{6}d^3x + \tfrac{1}{24}d^4x)\,\partial_x U + \&c.\\
&= (\partial_0 + \partial_1\ + \tfrac{1}{2}\partial_2\ + \tfrac{1}{6}\partial_3\ + \tfrac{1}{24}\ \partial_4)\quad U,
\end{aligned}$$

where

$$\partial_0 U = (x\partial_x + y\partial_y + z\partial_z)\ U = mU.$$

Reducing by the above results, the consecutive value of DU is

$$= -\tfrac{1}{2}\partial_1^2 U - \tfrac{1}{6}(\partial_1^3 + 3\partial_1\partial_2)\ U - \tfrac{1}{24}(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\ U.$$

6. Hence also writing

$$\begin{aligned}
P &= ax\ \ + by\ \ + cz\ \ ,\\
\partial_1 P &= adx\ + bdy\ + cdz\,,\\
\partial_2 P &= ad^2x + bd^2y + cd^2z,
\end{aligned}$$

the consecutive value of $-\Pi DU$ is $-(P + \partial_1 P + \tfrac{1}{2}\partial_2 P)$ multiplied into the consecutive value of DU, and the product is

$$\begin{aligned}
&= \ P\,.\,\tfrac{1}{2}\ \partial_1^2 U\\
&\ + P\,.\,\tfrac{1}{6}\ (\partial_1^3 + 3\partial_1\partial_2)\ U \qquad\qquad\qquad + \partial_1 P\,.\,\tfrac{1}{2}\partial_1^2 U\\
&\ + P\,.\,\tfrac{1}{24}(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\ U + \partial_1 P\,.\,\tfrac{1}{6}(\partial_1^3 + 3\partial_1\partial_2)\ U + \tfrac{1}{2}\partial_2 P\,.\,\tfrac{1}{2}\partial_1^2 U.
\end{aligned}$$

7. The consecutive value of $D^2 U$ is

$$\begin{aligned}
&= \ (x + dx + \tfrac{1}{2}d^2x + \tfrac{1}{6}d^3x + \tfrac{1}{24}d^4x)^2\,\partial_x^2 U + \&c.\\
&= \left.\begin{aligned}
&\qquad x^2\\
&+\ 2x\ dx\\
&+\ \ x^2\ dx + (dx)^2\\
&+ \tfrac{1}{3}\ x\ d^3x + \ dx\ d^2x\\
&+ \tfrac{1}{12}x\ d^4x + \tfrac{1}{3}dx\ d^3x + \tfrac{1}{4}\ (d^2x)^2
\end{aligned}\right\}\partial_x^2 U + \&c.,
\end{aligned}$$

which is

$$\left.\begin{aligned} = \quad & \partial_0^2 \\ + \; & 2\partial_0\partial_1 \\ + \; & \partial_0\partial_2 + \partial_1^2 \\ + \; & \tfrac{1}{3}\partial_0\partial_3 + \partial_1\partial_2 \\ + \; & \tfrac{1}{12}\partial_0\partial_4 + \tfrac{1}{3}\partial_1\partial_3 + \tfrac{1}{4}\partial_2^2 \end{aligned}\right\} U;$$

and observing that

$$\begin{aligned} \partial_0^2\, U &= m(m-1)\; U, \\ \partial_0\partial_1 U &= \quad (m-1)\,\partial_1 U, \\ \partial_0\partial_2 U &= \quad (m-1)\,\partial_2 U, \\ \partial_0\partial_3 U &= \quad (m-1)\,\partial_3 U, \\ \partial_0\partial_4 U &= \quad (m-1)\,\partial_4 U, \end{aligned}$$

and reducing as before, the consecutive value of D^2U is

$$\begin{aligned} = - \quad & (m-2)\,\partial_1^2 U \\ & - \tfrac{1}{3}\,[(m-1)\,\partial_1^3 + 3\,(m-2)\,\partial_1\partial_2]\; U \\ & - \tfrac{1}{12}\,[(m-1)\,(\partial_1^4 + 6\partial_1^2\partial_2) + (m-2)\,(4\partial_1\partial_3 + 3\partial_2^2)]\; U. \end{aligned}$$

8. Substituting in the equation $D^2U - \Pi\,.\,DU = 0$, we obtain as the conditions of a five-pointic contact

$$\begin{aligned} & -(m-2)\,\partial_1^2 U + P\,.\,\tfrac{1}{2}\partial_1^2 U = 0, \\ & -\tfrac{1}{3}\,[(m-1)\;\partial_1^3 + 3\,(m-2)\,\partial_1\partial_2]\; U + P\,.\,\tfrac{1}{6}\,(\partial_1^3 + 3\partial_1\partial_2)\; U + \partial_1 P\,.\,\tfrac{1}{2}\partial_1^2 U = 0, \\ & -\tfrac{1}{12}\,[(m-1)\,(\partial_1^4 + 6\partial_1^2\partial_2) + (m-2)\,(4\partial_1\partial_3 + 3\partial_2^2)]\; U \\ & + P\,.\,\tfrac{1}{24}\,[\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2]\; U + \partial_1 P\,.\,\tfrac{1}{6}\,[\partial_1^3 + 3\partial_1\partial_2]\; U + \tfrac{1}{2}\partial_2 P\,.\,\tfrac{1}{2}\partial_1^2 U = 0\,; \end{aligned}$$

or reducing

$$P = 2\,(m-2),$$

$$\partial_1 P = \tfrac{2}{3}\,\frac{\partial_1^3 U}{\partial_1^2 U},$$

$$\partial_2 P = \tfrac{1}{2}\,\frac{[\partial_1^4 + 6\partial_1^2\partial_2]\; U}{\partial_1^2 U} - \tfrac{4}{9}\,\frac{\partial_1^3 U}{\partial_1^2 U}\,\frac{[\partial_1^3 + 3\partial_1\partial_2]\; U}{\partial_1^2 U},$$

which are the conditions of a five-pointic contact: it is to be remarked that if only the first and second conditions are satisfied, we have a four-pointic contact, and if only the first condition is satisfied, a three-pointic contact.

9. We have to reduce the last-mentioned equations; suppose that A, B, C are the first derived functions of U, then the equation $\partial_1 U = 0$ may be written

$$A\,dx + B\,dy + C\,dz = 0,$$

and this will be satisfied identically if

$$dx = B\nu - C\mu,$$
$$dy = C\lambda - A\nu,$$
$$dz = A\mu - B\lambda,$$

where λ, μ, ν are arbitrary multipliers, which may be taken to be constants. We have therefore $\partial_1 = \mathbf{D}$, where

$$\mathbf{D} = (B\nu - C\mu)\partial_x + (C\lambda - A\nu)\partial_y + (A\mu - B\lambda)\partial_z.$$

10. The resulting expressions for $\partial_1^2 U$, $\partial_1^3 U$, $\partial_1^4 U$ may be exhibited in the reduced forms given by Hesse, viz. if $\vartheta = \lambda x + \mu y + \nu z$, we have

$$\partial_1^2 U = P_2 U - Q_2 \vartheta^2,$$
$$\partial_1^3 U = P_3 U - Q_3 \vartheta^2,$$
$$\partial_1^4 U = P_4 U - Q_4 \vartheta^2,$$

where the values of P_2, P_3, P_4; Q_2, Q_3, Q_4 are as follows, viz. if (a, b, c, f, g, h) are the second derived functions of U, and if

$$H = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

be the Hessian; if, moreover,

$$\Phi = -\begin{vmatrix} & \lambda, & \mu, & \nu \\ \lambda & a, & h, & g \\ \mu & h, & b, & f \\ \nu & g, & f, & c \end{vmatrix}$$

be the bordered Hessian (we may also write $\Phi = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \lambda, \mu, \nu)^2$, where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the inverse coefficients of (a, b, c, f, g, h), viz. $\mathfrak{A} = (bc - f^2)$, &c.); and finally, if for shortness we write

$$\Pi = \partial_x\Phi \,.\, \partial_\lambda\Phi + \partial_y\Phi \,.\, \partial_\mu\Phi + \partial_z\Phi \,.\, \partial_\nu\Phi,$$
$$\square = \partial_x H \,.\, \partial_\lambda\Phi + \partial_y H \,.\, \partial_\mu\Phi + \partial_z H \,.\, \partial_\nu\Phi,$$

then we have

$$P_2 = \frac{m}{m-1}\,\Phi, \qquad Q_2 = \frac{1}{(m-1)^2}\,H,$$

$$P_3 = \frac{m}{m-1}\,\mathbf{D}\Phi, \qquad Q_3 = \frac{1}{(m-1)^2}\,\mathbf{D}\,H,$$

$$P_4 = \frac{m}{m-1}\,\mathbf{D}^2\Phi - \frac{m\vartheta}{(m-1)^2}\,\Pi, \qquad Q_4 = \frac{1}{(m-1)^2}\,\mathbf{D}^2 H - \frac{\vartheta}{(m-1)^3}\,\square + \frac{3(m-2)}{(m-1)^3}\,H\Phi.$$

In the present case $U=0$, and we have

$$\partial_1^2 U = -Q_2 \vartheta^2,$$
$$\partial_1^3 U = -Q_3 \vartheta^2,$$
$$\partial_1^4 U = -Q_4 \vartheta^2.$$

11. Hence, substituting for Q_2 and Q_3 their values, the first and second of the equations for the five-pointic contact give

$$P = 2(m-2),$$

$$\partial_1 P = \tfrac{2}{3}\frac{\mathbf{D}H}{H},$$

and observing that Π is a linear function of (X, Y, Z), and consequently that P, $\partial_1 P$ denote simply the values which Π assumes when (x, y, z), (dx, dy, dz) are respectively substituted for (X, Y, Z), we see at once that these two conditions will be satisfied if we put

$$\Pi = \tfrac{2}{3}\frac{1}{H} DH + \Lambda \,.\, DU,$$

where Λ is an arbitrary constant, or, what is the same thing, an arbitrary function of (x, y, z). We have thus the general equation of a conic of four-pointic contact.

12. The above value of Π gives

$$\partial_2 P = \tfrac{2}{3}\frac{1}{H}\partial_2 H + \Lambda\, \partial_2 U,$$

and the third equation of the system of conditions for a five-pointic contact is therefore

$$\tfrac{2}{3}\frac{1}{H}\partial_2 H + \Lambda\, \partial_2 U = \tfrac{1}{3}\frac{[\partial_1^4 + 6\partial_1^2\partial_2]\, U}{\partial_1^2 U} - \tfrac{4}{3}\frac{\partial_1^2 U}{\partial_1^2 U}\frac{[\partial_1^3 + 3\partial_1\partial_2]\, U}{\partial_1^2 U},$$

which leads to the value of Λ.

13. We have in general

$$\partial_1^2 U = \frac{m}{m-1}\Phi\, U - \frac{1}{(m-1)^2} H\, \vartheta^2,$$

$$\partial_1^3 U = \frac{m}{m-1}\mathbf{D}\Phi \,.\, U - \frac{1}{(m-1)^2}\mathbf{D}H \,.\, \vartheta^2,$$

$$(\partial_1^3 + 3\partial_1\partial_2)\, U = \frac{m}{m-1}\mathbf{D}\Phi \,.\, U - \frac{1}{(m-1)^2}\mathbf{D}H \,.\, \vartheta^2,$$

$$(\partial_1^4 + 6\partial_1^2\partial_2)\, U = \frac{m}{m-1}\left(2\partial_2\Phi + \mathbf{D}^2\Phi + \frac{\vartheta}{m-1}\Pi\right) U$$
$$- \frac{1}{(m-1)^2}\left(2\partial_2 H + \mathbf{D}^2 H + \frac{\vartheta}{m-1}\square - \frac{3(m-2)}{m-1} H\Phi\right)\vartheta^2,$$

the last two of which have not yet been demonstrated. The value of $\partial_2 H$ (which, however, is not required for the present purpose) is

$$\partial_2 H = \frac{-3(m-3)}{m-1} H\Phi + \frac{\frac{1}{2}\square}{m-1}\vartheta,$$

which also is not yet demonstrated.

14. Putting $U=0$, we have

$$\begin{aligned}
\partial_1^2 U &= -\frac{1}{(m-1)^2} \quad H\ \vartheta^2,\\
\partial_1^3 U &= -\frac{1}{(m-1)^2} \quad \mathbf{D}H\,.\,\vartheta^2,\\
(\partial_1^3 + 3\partial_1\partial_2) U &= -\frac{1}{(m-1)^2} \quad \mathbf{D}H\,.\,\vartheta^2,\\
(\partial_1^4 + 6\partial_1^2\partial_2) U &= -\frac{1}{(m-1)^2}\left(2\partial_2 H + \mathbf{D}^2 H + \frac{\vartheta}{m-1}\square - \frac{3(m-2)}{m-1} H\Phi\right)\vartheta^2,
\end{aligned}$$

and substituting,

$$\begin{aligned}
&\tfrac{2}{3}\frac{1}{H}\partial_2 H + \frac{1}{(m-1)^2} H\vartheta^2\Lambda\\
&\qquad = \tfrac{1}{3}\frac{1}{H}\left(2\partial_2 H + \mathbf{D}^2 H + \frac{\vartheta}{m-1}\square - \frac{3(m-2)}{m-1} H\Phi\right) - \tfrac{4}{9}\left(\frac{\mathbf{D}H}{H}\right)^2,
\end{aligned}$$

where the term involving $\partial_2 H$ disappears; the equation may be written

$$\frac{9H^3\vartheta^2}{(m-1)^2}\Lambda = 3H\left(\mathbf{D}^2 H + \frac{\vartheta}{m-1}\square - \frac{3(m-2)}{m-1} H\Phi\right) - 4(\mathbf{D}H)^2,$$

which I will represent by

$$\frac{9H^3\vartheta^2}{(m-1)^2}\Lambda = 3R_2R_4 - 4R_3^2,$$

the values of R_2, R_3, R_4 being

$$\begin{cases}
R_2 = H,\\
R_3 = \mathbf{D}H,\\
R_4 = \mathbf{D}^2 H + \dfrac{\vartheta}{m-1}\square - \dfrac{3(m-2)}{m-1} H\Phi.
\end{cases}$$

15. We have $R_2 = H$, and it will be shown that

$$\begin{aligned}
(m-1)^2 R_3^2 &= -\ 9(m-2)^2 H^2\Phi\\
&\quad +\ 3(m-2) H\square\vartheta\\
&\quad -\ \Psi\vartheta^2,\\
(m-1)^2 R_4 &= -12(m-2)^2 H\Phi\\
&\quad +\ 4(m-2)\square\vartheta\\
&\quad -\ \Omega\vartheta^2,
\end{aligned}$$

where for shortness

$$\Psi = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x H, \partial_y H, \partial_z H)^2,$$

$$\Omega = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x \;, \partial_y \;, \partial_z \;)^2 H,$$

and hence, writing

$$9H^3 \vartheta^2 \Lambda = 3R_2 (m-1)^2 R_4 - 4(m-1)^2 R_3^2,$$

we have

$$9H^3 \Lambda = -3\Omega H + 4\Psi;$$

or replacing Ω, Ψ by their values,

$$\Lambda = \frac{1}{9H^3} \left\{ \begin{array}{l} -3(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x \;, \partial_y \;, \partial_z \;)^2 H.H \\ +4(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \partial_x H, \partial_y H, \partial_z H)^2 \end{array} \right\},$$

and Λ having this value, the equation of the five-pointic conic is

$$D^2 U - \left(\tfrac{2}{3} \frac{1}{H} DH + \Lambda DU\right) DU = 0,$$

where it will be recollected that the current coordinates are (X, Y, Z), and that D denotes $X\partial_x + Y\partial_y + Z\partial_z$.

16. I remark, in passing, that the problem of finding the circle of curvature at a given point of a plane curve, is in fact that of determining the conic having with the curve at the given point a three-pointic contact, and besides passing through two given points. The equation of a conic having an ordinary contact, is

$$D^2 U - \Pi DU = 0,$$

where

$$\Pi = aX + bY + cZ,$$

and the condition of a three-pointic contact is

$$ax + by + cz = 2(m-2).$$

Let the coordinates of the two given points be

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2),$$

and let $(D^2U)_1$ &c. be the corresponding values of D^2U, &c., then we have

$$aX + bY + cZ = \frac{D^2 U}{DU},$$

$$ax_1 + by_1 + cz_1 = \left(\frac{D^2 U}{DU}\right)_1,$$

$$ax_2 + by_2 + cz_2 = \left(\frac{D^2 U}{DU}\right)_2;$$

and if from the four equations we eliminate a, b, c, we find

$$\begin{vmatrix} XDU, & YDU, & ZDU, & D^2U \\ x\ , & y\ , & z\ , & 2(m-2) \\ x_1\ , & y_1\ , & z_1\ , & \left(\dfrac{D^2U}{DU}\right)_1 \\ x_2\ , & y_2\ , & z_2\ , & \left(\dfrac{D^2U}{DU}\right)_2 \end{vmatrix} = 0$$

for the equation of the conic in question; x, y, z being the coordinates of the point of contact, and X, Y, Z current coordinates.

II. *Demonstration of Identities assumed in the preceding section.*

Proof of the expressions for $(\partial_1^3 + 3\partial_1\partial_2)\, U$ and $(\partial_1^4 + 6\partial_1^2\partial_2)\, U$:

17. It will be remembered that ∂_1, ∂_2 stand originally for

$$d\,x\,\partial_x + d\,y\,\partial_y + d\,z\,\partial_z,$$
$$d^2x\,\partial_x + d^2y\,\partial_y + d^2z\,\partial_z,$$

and that A, B, C being the first derived functions of U, dx, dy, dz are changed into

$$B\nu - C\mu, \quad C\lambda - A\nu, \quad A\mu - B\lambda,$$

and that the resulting value of ∂_1, viz.

$$(B\nu - C\mu)\,\partial_x + (C\lambda - A\nu)\,\partial_y + (A\mu - B\lambda)\,\partial_z,$$

is also represented by $\mathbf{D}$, so that $\partial_1 = \mathbf{D}$. The corresponding values of d^2x, d^2y, d^2z are

$$d^2x = \nu dB - \mu dC,$$
$$d^2y = \lambda dC - \nu dA,$$
$$d^2z = \mu dA - \lambda dB,$$

where we have

$$dA = adx + hdy + gdz, \text{ \&c.},$$

in which dx, dy, dz are to be replaced by the values

$$B\nu - C\mu, \quad C\lambda - A\nu, \quad A\mu - B\lambda;$$

and ∂_2 really denotes what

$$d^2x\,\partial_x + d^2y\,\partial_y + d^2z\,\partial_z$$

becomes when the above values are substituted for d^2x, d^2y, d^2z. But in the expressions $\partial_1^2 U$, $\partial_1\partial_2 U$ &c., the symbols ∂_x, ∂_y, ∂_z contained in ∂_1 and ∂_2 operate only on U, and not on the variable quantities A, B, C, &c. contained in ∂_1 and ∂_2.

18. If now we treat ∂_1 as an operand, that is, perform the differentiations on the variable quantities A, B, C which enter into ∂_1, we obtain

$$\partial_1 . \partial_1 = \partial_2,$$

or, what is the same thing, operating on $\partial_1 U$ with ∂_1, the result is

$$\partial_1 . \partial_1 U = \partial_1^2 U + \partial_2 U = (\partial_1^2 + \partial_2)\, U,$$

and in like manner

$$\partial_1 . \partial_1^2 U = (\partial_1^3 + 2\partial_1\, \partial_2)\, U,$$

$$\partial_1 . \partial_1^3 U = (\partial_1^4 + 3\partial_1^2 \partial_2)\, U.$$

It is, in fact, upon these principles that Hesse's values of $\partial_1^2 U$, $\partial_1^3 U$, &c. were obtained, and we may by means of them obtain the other expressions assumed in the preceding section.

19. In fact, starting from Hesse's equation,

$$\partial_1^2 U = \frac{m}{m-1}\, \Phi U - \frac{1}{(m-1)^2}\, H \vartheta^2,$$

we have

$$(\partial_1^3 + 2\partial_1 \partial_2)\, U = \frac{m}{m-1}\, (U \mathbf{D}\Phi + \Phi \mathbf{D} U) - \frac{1}{(m-1)^2} (\mathbf{D}H . \vartheta^2 + H . 2\vartheta \mathbf{D}\vartheta).$$

But we have identically $\mathbf{D}U = 0$, $\mathbf{D}\vartheta = 0$, and this equation becomes therefore

$$(\partial_1^3 + 2\partial_1 \partial_2)\, U = \frac{m}{m-1}\, U \mathbf{D}\Phi - \frac{1}{(m-1)^2}\, \mathbf{D}H . \vartheta^2;$$

this is precisely Hesse's value of $\partial_1^3 U$, and thus we have $\partial_1 \partial_2 U = 0$, and therefore

$$(\partial_1^3 + 3\partial_1 \partial_2)\, U = \partial_1^3\, U = \frac{m}{m-1}\, U \mathbf{D}\Phi - \frac{1}{(m-1)^2}\, \mathbf{D}H . \vartheta^2.$$

20. In like manner, starting from the expression of $\partial_1^3 U$, we have

$$(\partial_1^4 + 3\partial_1^2 \partial_2)\, U = \frac{m}{m-1}\, (\mathbf{D}U . \mathbf{D}\Phi + U \mathbf{D} . \mathbf{D}\Phi) - \frac{1}{(m-1)^2} (\mathbf{D}H . 2\vartheta \mathbf{D}\vartheta + \vartheta^2 \mathbf{D} . \mathbf{D}H);$$

or since $\mathbf{D}U$ and $\mathbf{D}\vartheta$ vanish identically, and the values of $\mathbf{D} . \mathbf{D}\Phi$, $\mathbf{D} . \mathbf{D}H$ are $\partial_2 \Phi + \mathbf{D}^2 \Phi$, $\partial_2 H + \mathbf{D}^2 H$, we have

$$(\partial_1^4 + 3\partial_1^2 \partial_2)\, U = \frac{m}{m-1}\, U (\partial_2 \Phi + \mathbf{D}^2 \Phi) - \frac{1}{(m-1)^2}\, \vartheta^2 (\partial_2 H + \mathbf{D}^2 H);$$

and if from the double of this equation we subtract Hesse's equation,

$$\partial_1^4 U = \frac{m}{m-1}\, U \left(\mathbf{D}^2 \Phi - \frac{\vartheta}{m-1}\, \Pi\right) - \frac{1}{(m-1)^2}\, \vartheta^2 \left(\mathbf{D}^2 H - \frac{\vartheta \square}{m-1} + \frac{3\,(m-2)}{m-1}\, H\Phi\right),$$

we find the required relation,

$$(\partial_1^4 + 6\partial_1^2\partial_2)\, U = \frac{m}{m-1}\left(2\partial_2\Phi + \mathbf{D}^2\Phi + \frac{\vartheta}{m-1}\,\Pi\right) U$$
$$- \frac{1}{(m-1)^2}\left(2\partial_2 H + \mathbf{D}^2 H + \frac{\vartheta\square}{m-1} - \frac{3\,(m-2)}{m-1}\, H\Phi\right)\vartheta^2.$$

Proof of the expression for $\partial_2 H$:

21. We have

$$\partial_2 H = \partial_x H\,(\nu dB - \mu dC) + \partial_y H\,(\lambda dC - \nu dB) + \partial_z H\,(\mu dA - \lambda dB),$$

where

$$dA = adx + hdy + gdz,$$
$$dB = hdx + bdy + fdz,$$
$$dC = gdx + fdy + cdz,$$

in which dx, dy, dz are to be replaced by their values: we have therefore

$$\partial_2 H = \partial_x H\,\{\quad \nu\,[\,(bC - fB)\,\lambda + (fA - hC)\,\mu + (hB - bA)\,\nu]$$
$$- \mu\,[(fC - cB)\,\lambda + (c\,A - gC)\,\mu + (gB - fA)\,\nu]\},$$
$$+ \&c.,$$

where the coefficient of $\partial_x H$ is

$$0\lambda^2 + (gC - cA)\,\mu^2 + (hB - bA)\,\nu^2 + (2fA - gB - hC)\,\mu\nu + (bC - fB)\,\nu\lambda + (cB - fC)\,\lambda\mu\,;$$

or, since we have

$$(m-1)\,A = ax + hy + gz,$$
$$(m-1)\,B = hx + by + fz,$$
$$(m-1)\,C = gx + fy + cz,$$

the coefficient, omitting the factor $\dfrac{1}{m-1}$, which will be afterwards restored, is

$$\begin{aligned}
& 0\lambda^2 \\
& + [\quad g\,(gx + fy + cz) - c\,(ax + hy + gz)]\,\mu^2 \\
& + [\quad h\,(hx + by + fz) - b\,(ax + hy + gz)]\,\nu^2 \\
& + [2f\,(ax + hy + gz) - g\,(hx + by + fz) - h\,(gx + fy + cz)]\,\mu\nu \\
& + [\quad b\,(gx + fy + cz) - f\,(hx + by + fz)]\,\nu\lambda \\
& + [\quad c\,(hx + fy + cz) - f\,(gx + fy + cz)]\,\lambda\mu \\
= \;& 0\lambda^2 \\
& + (-\ \mathfrak{B}x + \mathfrak{H}y)\,\mu^2 \\
& + (-\ \mathfrak{C}x + \mathfrak{G}z)\,\nu^2 \\
& + (-\ 2\mathfrak{F}x + \mathfrak{G}y + \mathfrak{H}z)\,\mu\nu \\
& + (-\ \mathfrak{G}x + \qquad \mathfrak{A}z)\,\nu\lambda \\
& + (-\ \mathfrak{H}x + \mathfrak{A}y \qquad)\,\lambda\mu,
\end{aligned}$$

which is equal to

$$-(\mathfrak{A}\lambda^2+\mathfrak{B}\mu^2+\mathfrak{C}\nu^2+2\mathfrak{F}\mu\nu+2\mathfrak{G}\nu\lambda+2\mathfrak{H}\lambda\mu)\,x$$
$$+(\mathfrak{A}\lambda+\mathfrak{H}\mu+\mathfrak{G}\nu)(\lambda x+\mu y+\nu z).$$

The coefficients of $\partial_y H$, ∂H_z have a similar form; and uniting the three terms, observing that $x\partial_x H+y\partial_y H+z\partial_z H$ is equal to $3(m-2)H$, and attending to the values of Φ, $\square$, ϑ, we have, restoring the omitted factor $\frac{1}{m-1}$,

$$\partial_2 H=\frac{-3(m-2)}{m-1}H\Phi+\frac{\frac{1}{2}\square}{m-1}\vartheta.$$

Proof of the expressions for $(\mathbf{D}H)^2$ and $\mathbf{D}^2H$:

22. These are obtained (for the particular case $m=4$, which makes but little difference) in Mr Salmon's *Higher Plane Curves*, pp. 88 and 89, and I merely reproduce his investigation; we have

$$(\mathbf{D}H)^2=\{(B\nu-C\mu)\,\partial_x H+(C\lambda-A\nu)\,\partial_y H+(A\mu-B\lambda)\,\partial_z H\}^2,$$

or, what is the same thing,

$$(\mathbf{D}H)^2=\{\lambda(C\partial_y-B\partial_z)H+\mu(A\partial_z-C\partial_x)H+\nu(B\partial_x-A\partial_z)H\}^2;$$

and if we consider first the term which contains λ^2, the coefficient is

$$\{(C\partial_y-B\partial_z)H\}^2.$$

Now making use of the equations

$$(m-1)A=ax+hy+gz,$$
$$(m-1)B=hx+by+fz,$$
$$(m-1)C=gx+fy+cz,$$

and

$$m(m-1)U=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

we have

$$\begin{aligned}(m-1)^2\ C^2&=(gx+fy+cz)^2-c(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)\\&=-\mathfrak{B}x^2+2\mathfrak{H}xy-\mathfrak{A}y^2,\\(m-1)^2BC&=(hx+by+fz)(gx+fy+cz)-f(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)\\&=\mathfrak{F}x^2-\mathfrak{G}xy-\mathfrak{H}xz+\mathfrak{A}yz,\\(m-1)^2B^2&=(hx+by+fz)^2-b(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)\\&=-\mathfrak{C}x^2+2\mathfrak{G}xz-\mathfrak{A}z^2;\end{aligned}$$

and hence

$$\begin{aligned}(m-1)^2((C\partial_y-B\partial_z)H)^2=&\ (-\mathfrak{B}x^2+2\mathfrak{H}xy\qquad-\mathfrak{A}y^2)(\partial_y H)^2\\&+2(-\mathfrak{F}x^2+\mathfrak{G}xy+\mathfrak{H}xz-\mathfrak{A}yz)\,\partial_y H\,.\,\partial_z H\\&+\ (-\mathfrak{C}x^2+\qquad2\mathfrak{G}xz-\mathfrak{A}z^2)(\partial_z H)^2\\=&-\ x^2(\mathfrak{B},\ \mathfrak{F},\ \mathfrak{C})(\partial_y H,\ \partial_z H)^2\\&+2\,x\,(y\partial_y H+z\partial_z H)(\mathfrak{H}\partial_y H+\mathfrak{G}\partial_z H)\\&-\ \mathfrak{A}\,(y\partial_y H+z\partial_z H)^2,\end{aligned}$$

and

$$y\partial_y H + z\partial_z H = 3(m-2)H - z\partial_z H,$$

so that the term is

$$\begin{aligned} &- x^2(\mathfrak{B}, \mathfrak{F}, \mathfrak{C}\mathfrak{X}\partial_y H, \partial_x H)^2 \\ &+ 2x\left(3(m-2)H - z\partial_z H\right)(\mathfrak{H}\partial_y H + \mathfrak{G}\partial_z H) \\ &- \mathfrak{A}\left(3(m-2)H - z\partial_z H\right)^2; \end{aligned}$$

or, reducing, we have

$$\begin{aligned} (m-1)^2\{(C\partial_y - B\partial_z)H\}^2 \\ = - \quad x^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x H, \partial_y H, \partial_z H)^2 \\ + 6(m-2)Hx(\mathfrak{A}\partial_x H + \mathfrak{H}\partial_y H + \mathfrak{G}\partial_z H) \\ - 9(m-2)^2\mathfrak{A}H^2. \end{aligned}$$

23. The other terms may be obtained in a similar manner; and it is easy to see that, collecting all the terms, the sum will be

$$\begin{aligned} &-(\lambda x + \mu y + \nu z)^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x H, \partial_y H, \partial_z H)^2 \\ &+ 6(m-2)H(\lambda x + \mu y + \nu z)\{(\mathfrak{A}\lambda + \mathfrak{H}\mu + \mathfrak{G}\nu)\partial_x H + (\mathfrak{H}\lambda + \mathfrak{B}\mu + \mathfrak{F}\nu)\partial_y H + (\mathfrak{G}\lambda + \mathfrak{F}\mu + \mathfrak{C}\nu)\partial_z H\} \\ &- 9(m-2)^2 H^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\lambda, \mu, \nu)^2; \end{aligned}$$

or, attending to the signification of the symbols $\mathfrak{S}$, Φ, $\square$, Ψ, we have

$$\begin{aligned} (m-1)^2(\mathbf{D}H)^2 = &-9(m-2)^2 H^2\Phi \\ &+ 3(m-2)H\square\mathfrak{S} \\ &- \Psi\mathfrak{S}^2. \end{aligned}$$

24. Next

$$\mathbf{D}^2 H = \left((B\nu - C\mu)\partial_x + (C\lambda - A\nu)\partial_y + (A\mu - B\lambda)\partial_z\right)^2 H,$$

or, what is the same thing,

$$\mathbf{D}^2 H = \{\lambda(C\partial_y - B\partial_z) + \mu(A\partial_z - C\partial_x) + \nu(B\partial_x - A\partial_z)\}^2 H;$$

and if we attend first to the term which contains λ^2, the coefficient is

$$(C\partial_y - B\partial_z)^2 H.$$

Now substituting for C^2, CB, B^2 as before, we have

$$\begin{aligned} (m-1)^2(C\partial_y - B\partial_z)^2 H = &\quad(-\mathfrak{B}x^2 + 2\mathfrak{H}xy \qquad - \mathfrak{A}y^2)\partial_y^2 H \\ &+ 2(-\mathfrak{F}x^2 + \mathfrak{G}xy + \mathfrak{H}xz - \mathfrak{A}yz)\partial_y\partial_z H \\ &+ \quad(-\mathfrak{C}x^2 + 2\mathfrak{G}xz \qquad - \mathfrak{A}z^2)\partial_z^2 H \\ = &-x^2(\mathfrak{B}, \mathfrak{F}, \mathfrak{C}\mathfrak{X}\partial_y, \partial_z)^2 H \\ &+ 2x(y\partial_y + z\partial_z)(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)H \\ &- \mathfrak{A}(y\partial_y + z\partial_z)^2 H, \end{aligned}$$

where it is to be observed that the symbols of differentiation affect H only. We have

$$\begin{aligned}(y\partial_y + z\partial_z)^2 H = \quad & (x\partial_x + y\partial_y + z\partial_z)^2 H \\ & - 2(x\partial_x + y\partial_y + z\partial_z)\, x\partial_x H \\ & + \ (x\partial_x)^2 H,\end{aligned}$$

where H is a homogeneous function of the degree $3m-6$; $x\partial_x H$, since the x is not affected by the differentiation, must be treated as of the degree $3m-7$, and $(x\partial_x)^2 H$, for the like reason, stands for $x^2\partial_x^2 H$; we have therefore

$$\begin{aligned}(y\partial_y + z\partial_z)^2 H = \quad & (3m-6)(3m-7)H \\ & - 2(3m-7)\, x\partial_x H \\ & + x^2\partial_x^2 H.\end{aligned}$$

In like manner,

$$\begin{aligned}(y\partial_y + z\partial_z)(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H = \quad & (x\partial_x + y\partial_y + z\partial_z)(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H - x\partial_x(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H \\ = & (3m-7)(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H - x\partial_x(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H;\end{aligned}$$

and hence

$$\begin{aligned}(m-1)^2(C\partial_y - B\partial_z)^2 H = & - x^2(\mathfrak{B}, \mathfrak{F}, \mathfrak{C}\mathfrak{X}\partial_y, \partial_z)^2 H \\ & + 2x[(3m-7)(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H - x\partial_x(\mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H] \\ & - \mathfrak{A}\ [(3m-6)(3m-7)H - 2(3m-7)\, x\partial_x H + x^2\partial_x^2 H] \\ = & - x^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x, \partial_y, \partial_z)^2 H \\ & + 2(3m-7)\, x\,(\mathfrak{A}\partial_x + \mathfrak{H}\partial_y + \mathfrak{G}\partial_z)\, H \\ & - (3m-6)(3m-7)\,\mathfrak{A} H.\end{aligned}$$

25. The other terms are formed in a similar manner; and collecting all the terms, we have

$$\begin{aligned}(m-1)^2\mathbf{D}^2 H = & -(3m-6)(3m-7)(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\ \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\lambda, \ \mu, \ \nu)^2 H \\ & + 2(3m-7)(\lambda x + \mu y + \nu z)\big((\mathfrak{A}\lambda + \mathfrak{H}\mu + \mathfrak{G}\nu)\,\partial_x H + (\mathfrak{H}\lambda + \mathfrak{B}\mu + \mathfrak{F}\nu)\,\partial_y H + (\mathfrak{G}\lambda + \mathfrak{F}\mu + \mathfrak{C}\nu)\,\partial_z H\big) \\ & - (\lambda x + \mu y + \nu z)^2(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x, \partial_y, \partial_z)^2 H,\end{aligned}$$

or, attending to the signification of the symbols $\mathfrak{I}$, Φ, $\square$, Ω, this is

$$\begin{aligned}(m-1)^2\mathbf{D}^2 H = & -(3m-6)(3m-7)\, H\Phi \\ & + (3m-7)\,\mathfrak{I}\square \\ & - \Omega\mathfrak{I}^2.\end{aligned}$$

Proof of the expressions for $(m-1)^2R_3^2$, $(m-1)^2R_4$:

26. We have

$$\begin{aligned}(m-1)^2R_3^2 &= (m-1)^2(\mathbf{D}H)^2\\ &= -9(m-2)^2H^2\Phi\\ &\quad +3(m-2)H\square\mathfrak{H}\\ &\quad -\Psi\mathfrak{H}^2,\end{aligned}$$

$$\begin{aligned}(m-1)^2R_4 &= (m-1)^2\mathbf{D}^2H+(m-1)\mathfrak{H}\square-(3m-6)(m-1)\Phi H\\ &= -12(m-2)^2H\Phi\\ &\quad +4(m-2)\square\mathfrak{H}\\ &\quad -\Omega\mathfrak{H}^2,\end{aligned}$$

which are the expressions required.

Proof of corresponding expressions for $(m-1)^6Q_3^2$, $(m-1)^4Q_4$:

27. We have

$$\begin{aligned}(m-1)^6Q_3^2 &= (m-1)^2(\mathbf{D}H)^2\\ &= -9(m-2)^2H^2\Phi\\ &\quad +3(m-2)H\square\mathfrak{H}\\ &\quad -\Psi\mathfrak{H},\end{aligned}$$

$$\begin{aligned}(m-1)^4Q_4 &= (m-1)^2\mathbf{D}^2H-(m-1)\mathfrak{H}\square+(3m-6)(m-1)H\Phi\\ &= -6(m-2)(m-3)H\Phi\\ &\quad +2(m-3)\square\mathfrak{H}\\ &\quad -\Omega\mathfrak{H}^2;\end{aligned}$$

to which I join

$$(m-1)^2Q_2 = H.$$

28. We have consequently

$$\begin{aligned}(m-1)^6(3Q_2Q_4-Q_3^2) &= -9(m-2)(m-4)H^2\Phi\\ &\quad +3(m-4)H\square\mathfrak{H}\\ &\quad +(-3\Omega H+\Psi)\mathfrak{H}^2,\end{aligned}$$

and

$$(m-1)^6\left(3(m-2)Q_2Q_4-2(m-3)Q_3^2\right)=\left(-3(m-2)\Omega H+2(m-3)\Psi\right)\mathfrak{H}^2,$$

which for $m=4$ becomes

$$729(3Q_2Q_4-Q_3^2)=(-3\Omega H+\Psi)\mathfrak{H}^2.$$

29. In the case $m=4$, we have Hesse's theorem, that the equation $3Q_2Q_4-Q_3=0$ gives a curve of the 14th order, which passes through the points of contact of the double tangents, viz. substituting for Ω, Ψ their values, the equation of this curve is

$$\begin{aligned}&-3H(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x\ , \partial_y\ , \partial_z)^2H\\ &+\quad(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_xH, \partial_yH, \partial_zH)^2=0.\end{aligned}$$

I have added these remarks for the sake of pointing out the striking resemblance of the expressions which occur in the double-tangent problem for $m=4$, and in the present theory of the five-pointic conic for any value whatever of m. It has not hitherto been shown what the expressions $3(m-2)Q_5Q_4-2(m-3)Q_3^2$ and $-3(m-2)\Omega H+2(m-3)\Psi$ respectively denote, except in the particular case $m=4$.

III. *Application of the Formulæ to the Cubic.*

30. I shall apply the formula for the five-pointic conic to a cubic; to avoid confusion as to a numerical factor, I write U', H' in the place of U, H, so that we have

$$\Lambda=\frac{1}{9H'^3}\left\{\begin{array}{l}-3(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_x\ ,\ \partial_y\ ,\ \partial_z)^2H'.H'\\ +4(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_xH',\ \partial_yH',\ \partial_zH')^2\end{array}\right\},$$

and then the equation of the five-pointic conic is

$$D^2U'-\tfrac{2}{3}\left(\frac{1}{H'}DH'+\Lambda DU'\right)DU'=0.$$

I take as the equation of the cubic,

$$U=x^3+y^3+z^3+6lxyz=0\,;$$

the formulæ Table No. 70 of my Third Memoir on Quantics, *Phil. Trans.*, vol. CXLVI. (1856), pp. 627—647, [144], putting H for HU, give

$$H=l^2(x^3+y^3+z^3)-(1+2l^3)xyz.$$

Hence writing

$$U'=\tfrac{1}{6}(x^3+y^3+z^3+6lxyz),$$

the first derived functions are

$$\tfrac{1}{2}(x^2+2lyz),\quad \tfrac{1}{2}(y^2+2lzx),\quad \tfrac{1}{2}(z^2+2lxy)\,;$$

the second derived functions, or (a, b, c, f, g, h), are (x, y, z, lx, ly, lz), whence $H'=-H$; the inverse coefficients $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are

$$(yz-l^2x^2,\ zx-l^2y^2,\ xy-l^2z^2,\ l^2yz-lx^2,\ l^2zx-ly^2,\ l^2xy-lz^2)\,;$$

and putting $U'=\frac{1}{6}U$ and $H'=-H$, we have

$$\Lambda=-\frac{1}{9H^2}\left\{\begin{array}{l}-3(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_x\ ,\ \partial_y\ ,\ \partial_z)^2H.H\\ +4(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_xH,\ \partial_yH,\ \partial_zH)^2\end{array}\right\};$$

and the equation of the five-pointic conic is

$$D^2U-\left(\tfrac{2}{3}\frac{1}{H}DH+\tfrac{1}{6}\Lambda DU\right)DU=0.$$

31. We have

$$
\begin{aligned}
&(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x, \partial_y, \partial_z)^2 H \\
&= \quad (yz - l^2x^2)\,.\,6l^2x \\
&\quad + \ (zx - l^2y^2)\,.\,6l^2y \\
&\quad + \ (xy - l^2z^2)\,.\,6l^2z \\
&\quad + 2(l^2yz - lx^2)\,.\,-(1+2l^3)\,x \\
&\quad + 2(l^2zx - ly^2)\,.\,-(1+2l^3)\,y \\
&\quad + 2(l^2xy - lz^2)\,.\,-(1+2l^3)\,z,
\end{aligned}
$$

which is

$$
\begin{aligned}
&= \ 18l^2xyz - 6l^4(x^3+y^3+z^3) \\
&\quad - 6l^2(1+2l^3)xyz + 2l(1+2l^3)(x^3+y^3+z^3), \\
&= \ (12l^2 - 12l^5)xyz + (2l - 2l^4)(x^3+y^3+z^3), \\
&= \ 2(l - l^4)(x^3+y^3+z^3+6lxyz),
\end{aligned}
$$

or we have

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x, \partial_y, \partial_z)^2 H = -2S\,.\,U,$$

where S is the quartinvariant (see the Table No. 70). For the present purpose $U=0$, and consequently

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x, \partial_y, \partial_z)^2 H = 0.$$

32. Next,

$$
\begin{aligned}
&(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{X}\partial_x H, \partial_y H, \partial_z H) \\
&= \quad (yz - l^2x^2)\,[3l^2x^2 - (1+2l^3)\,yz]^2 \\
&\quad + \ (zx - l^2y^2)\,[3l^2y^2 - (1+2l^3)\,zx]^2 \\
&\quad + \ (xy - l^2z^2)\,[3l^2z^2 - (1+2l^3)\,xy]^2 \\
&\quad + 2(l^2yz - lx^2)\,[3l^2y^2 - (1+2l^3)\,zx]\,[3l^2z^2 - (1+2l^3)\,xy] \\
&\quad + 2(l^2zx - ly^2)\,[3l^2z^2 - (1+2l^3)\,xy]\,[3l^2x^2 - (1+2l^3)\,yz] \\
&\quad + 2(l^2xy - lz^2)\,[3l^2x^2 - (1+2l^3)\,yz]\,[3l^2y^2 - (1+2l^3)\,zx],
\end{aligned}
$$

the first three lines of which are

$$
\begin{aligned}
&9l^4\,xyz\,(x^3+y^3+z^3) - 18l^2\,(1+2l^3)\,x^2y^2z^2 + (1+2l^3)^2(y^3z^3+z^3x^3+x^3y^3) \\
&- 9l^6\,(x^6+y^6+z^6) + 6l^4\,(1+2l^3)\,xyz\,(x^3+y^3+z^3) - 3l^2\,(1+2l^3)^2x^2y^2z^2;
\end{aligned}
$$

or collecting and reducing,

$$
\begin{aligned}
&(-9l^6 \qquad\qquad\quad)\,(x^6 + y^6 + z^6) \\
&+ (1 + 4l^3 + 4l^6 \qquad)\,(y^3z^3 + z^3x^3 + x^3y^3) \\
&+ (\ \ 15l^4 + 12l^7 \qquad)\,(x^3 + y^3 + z^3)\,xyz \\
&+ (-21l^3 - 48l^5 - 12l^8)\,x^2y^2z^2;
\end{aligned}
$$

the second three lines are

$$18l^6(y^2z^2+z^2x^2+x^2y^2)-12l^4(1+2l^3)\,xyz\,(x^3+y^3+z^3)+6l^2(1+2l^3)^2x^2y^2z^2$$
$$-54l^5x^2y^2z^2+12l^3(1+2l^3)(y^2z^2+z^2x^2+x^2y^2)-2l(1+2l^3)^2xyz\,(x^3+y^3+z^3);$$

or collecting and reducing,

$$\begin{aligned}&(12l^3+42l^6\qquad)(y^2z^2+z^2x^2+x^2y^2)\\&+(-2l-20l^4-32l^7)(x^3\ +y^3\ +z^3\ )\,xyz\\&+(6l^2-30l^5+24l^8)\,x^2y^2z^2.\end{aligned}$$

Hence in the first part replacing the top line by

$$-9l^6(x^3+y^3+z^3)^2+18l^6(y^2z^2+z^2x^2+x^2y^2),$$

and uniting the two parts, we find

$$\begin{aligned}&(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_xH,\ \partial_yH,\ \partial_zH)^2\\&=\qquad(1+8l^3)^2(y^2z^2+z^2x^2+x^2y^2)\\&\quad+(-9l^6\qquad)(x^3\ +y^3\ +z^3\ )^2\\&\quad+(-2l-5l^4-20l^7\quad)(x^3\ +y^3\ +z^3\ )\,xyz\\&\quad+(-15l^2-78l^5+12l^8)\,x^2y^2z^2;\end{aligned}$$

and referring to the Table No. 70, and writing Θ for ΘU, we have

$$(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_xH,\ \partial_yH,\ \partial_zH)^2=\Theta,$$

where Θ is the first of the three functions which may be chosen to represent the octicovariant of the cubic.

33. We have thus

$$\Lambda=-\tfrac{4}{9}\frac{\Theta}{H^3},$$

and thence

$$D^2U-\left(\tfrac{2}{3}\frac{1}{H}DH-\tfrac{2}{27}\frac{\Theta}{H^3}DU\right)DU=0$$

as the equation of the five-pointic conic: the investigation has been conducted by means of the canonical form of the equation of the cubic, but the form of the result shows that it applies to the equation of the cubic in any form whatever.

34. If, however, we continue to represent the cubic by the canonical equation

$$x^3+y^3+z^3+6lxyz=0,$$

the result may be further reduced. We have, putting $U=0$, or writing $x^3+y^3+z^3=-6lxyz$,

$$H=-(1+8l^3)\,xyz;$$

moreover, putting $U=0$, the Table No. 70 gives

$$\Theta=(1+8l^3)^2(y^2z^3+z^2x^3+x^3y^3)-3l^2H^2;$$

or substituting for H the last-mentioned value, and putting for shortness

$$Q=y^3z^3+z^3x^3+x^3y^3-3l^2x^2y^2z^2,$$

we have

$$\Theta=(1+8l^3)^2\,Q;$$

and with these values of H and Q, the equation of the five-pointic conic is

$$D^2U+\left(\tfrac{2}{3}\frac{1}{(1+8l^3)\,xyz}\,DH-\tfrac{2}{27}\frac{Q}{(1+8l^3)\,x^2y^2z^2}\,DU\right)DU=0,$$

where

$$D\,U=3\,\{(x^2+2lyz)\,X+(y^2+2lzx)\,Y+(z^2+2lxy)\,Z\}$$
$$D^2U=6\,\{(X^2+2lYZ)\,x+(Y^2+2lZX)\,y+(Z^2+2lXY)\,z\},$$

or, as it will be convenient to write it,

$$=6\,(x,\ y,\ z,\ lx,\ ly,\ lz\between X,\ Y,\ Z)^2,$$

$$DH=\left(3l^2x^2-(1+2l^3)\,yz\right)X+\left(3l^2y^2-(1+2l^3)\,zx\right)Y+\left(3l^2z^2-(1+2l^3)\,xy\right)Z;$$

whence, finally, the equation of the five-pointic conic of the cubic

$$X^3+Y^3+Z^3+6lXYZ=0$$

at the point $(x,\ y,\ z)$ is

$$\begin{aligned}&9\,(1+8l^3)\,x^2y^2z^2\,(x,\ y,\ z,\ lx,\ ly,\ lz\between X,\ Y,\ Z)^2\\&\qquad+\{(x^2+2lyz)\,X+(y^2+2lzx)\,Y+(z^2+2lxy)\,Z\}\times\\&\left\{\begin{array}{l}3x^2y^2z^2\,[\left(3l^2x^2-(1+2l^3)\,yz\right)X+\left(3l^2y^2-(1+2l^3)\,zx\right)Y+\left(3l^2z^2-(1+2l^3)\,xy\right)Z]\\-Q\,[(\ \ x^2+\ \ \ \ 2l\ \ yz)\,X+(\ \ y^2+\ \ \ \ 2l\ \ zx)\,Y+(\ \ z^2+\ \ \ \ 2l\ \ xy)\,Z]\end{array}\right\},\end{aligned}$$

a result which I had previously obtained by a special method.

35. But the expression may be exhibited in a different form by a transformation suggested by a geometrical theorem of Mr Salmon's. In fact the tangent at the point $(x,\ y,\ z)$ meets the cubic in the tangential of this point, and the coordinates of the tangential are $x\,(y^3-z^3)$, $y\,(z^3-x^3)$, $z\,(x^3-y^3)$. Calling these ξ, η, ζ, the equation of the tangent to the cubic at the tangential is

$$(\xi^2+2l\eta\zeta)\,X+(\eta^2+2l\zeta\xi)\,Y+(\zeta^2+2l\xi\eta)\,Z=0.$$

Now we have identically,

$$\begin{aligned}&3x^2y^2z^2\,\{\left(3l^2x^2-(1+2l^3)\,yz\right)X+\left(3l^2y^2-(1+2l^3)\,yz\right)Y+\left(3l^2z^2-(1+2l^3)\,zx\right)Z\}\\&\quad-Q\,\{(x^2+2lyz)\,X+(y^2+2lzx)\,Y+(z^2+2lxy)\,Z\}\\&=\quad\{(\xi^2+2l\eta\zeta)\,X+(\eta^2+2l\zeta\xi)\,Y+(\zeta^2+2l\xi\eta)\,Z\}\\&\quad+U\,\{x^2(-y^3-z^3+2lyz)\,X+y^2(-z^3-x^3+2lzx)\,Y+z^2(-x^3-y^3+2lxy)\,Z\}.\end{aligned}$$

In fact this relation will be true if only

$$3x^2y^2z^2\left(3l^2x^2-(1+2l^3)\,yz\right)-Q\,(x^2+2lyz)-(\xi^2+2l\eta\zeta)=Ux^2\,(-y^3-z^3+2lyz)\,;$$

and substituting for ξ, η, ζ and Q their values, the left-hand side is

$$\begin{aligned}&3x^2y^2z^2\left(3l^2x^2-(1+2l^3)\,yz\right)\\ &-(x^2+2lyz)\,(y^3z^3+z^3x^3+x^3y^3-3l^2x^2y^2z^2)\\ &-\ x^2\,(y^3-z^3)^2\\ &-\ 2lyz\,(z^3-x^3)\,(x^3-y^3)\,;\end{aligned}$$

and expanding and reducing, the result is

$$x^2\left(-(y^3+z^3)^2-x^3\,(y^3+z^3)\right)+2lyz\left(-2x^3\,(y^3+z^3)+x^6\right)+12l^2x^4y^2z^2\,;$$

whence, dividing by x^2, the equation becomes

$$\begin{aligned}-(y^3+z^3)\,(x^3+y^3+z^3)+2lxyz\left(x^3-2\,(y^3+z^3)\right)+12l^2x^2y^2z^2\\ =(-y^3-z^3+2lyz)\,(x^3+y^3+z^3+6lxyz),\end{aligned}$$

which is identically true.

36. Hence in the identical equation putting $U=0$, we see that the equation of the five-pointic conic may be written

$$\begin{aligned}&9\,(1+8l^3)\,x^3y^3z^3\,(x,\ y,\ z,\ lx,\ ly,\ lz\between X,\ Y,\ Z)^2\\ &+\{(x^2+2lyz)\,X+(y^2+2lzx)\,Y+(z^2+2lxy)\,Z\}\times\\ &\quad\{(\xi^2+2l\eta\zeta)\,X+(\eta^2+2l\zeta\xi)\,Y+(\zeta^2+2l\xi\eta)\,Z\}\\ &=0,\end{aligned}$$

where ξ, η, ζ stand for $x\,(y^3-z^3)$, $y\,(z^3-x^3)$, $z\,(x^3-y^3)$, the coordinates of the tangential of the given point, and which puts in evidence the geometrical theorem above referred to, viz.

THEOREM.—The common chord of the five-pointic conic and the polar conic is the tangent to the cubic at the tangential of the given point.

37. The five-pointic conic meets the cubic in the point of contact, considered as five coincident points, and in a remaining sixth point or point of simple intersection. The process by which I originally obtained the equation of the five-pointic conic, led also to the equation of the line joining the point of contact with the point of simple intersection: the equation of this line is

$$\begin{aligned}&9x^3y^3z^3\left\{\begin{array}{l}Xx\left((1+8l^3)\,x^4+(4l+41l^4)\,x^2yz+(-2l^2+2l^5)\,y^2z^2\right)\\ +\ Yy\left((1+8l^3)\,y^4+(4l+41l^4)\,y^2zx+(-2l^2+2l^5)\,z^2x^2\right)\\ +\ Zz\left((1+8l^3)\,z^4+(4l+41l^4)\,z^2xy+(-2l^2+2l^5)\,x^2y^2\right)\end{array}\right\}\\ &-6Qx^2y^2z^2\,\{X\left(3l^2x^2-(1+2l^3)\,yz\right)+Y\left(3l^2y^2-(1+2l^3)\,zx\right)+Z\left(3l^2z^2-(1+2l^3)\,xy\right)\}\\ &+\ Q^2\quad\{X\,(x^2+2lyz)+Y\,(y^2+2lzx)+Z\,(z^2+2lxy)\}=0.\end{aligned}$$

38. If the conic meet the cubic in six coincident points, that is, if the point of contact be a singular point of the kind already spoken of, or, as we may term it, a sextactic point, then the last-mentioned line must coincide with the tangent at the point. Represent for a moment the equation of the line by

$$AX + BY + CZ = 0,$$

then this line is to coincide with the line

$$(x^2 + 2lyz)\,X + (y^2 + 2lzx)\,Y + (z^2 + 2lxy)\,Z = 0,$$

or we must have

$$\begin{aligned} B\,(z^2 + 2lxy) - C\,(y^2 + 2lzx) &= 0, \\ C\,(x^2 + 2lyz) - A\,(z^2 + 2lxy) &= 0, \\ A\,(y^2 + 2lzx) - B\,(x^2 + 2lyz) &= 0, \end{aligned}$$

which must be equivalent to a single condition. The terms A, B, C, which contain

$$x^2 + 2lyz, \quad y^2 + 2lzx, \quad z^2 + 2lxy$$

respectively, may, it is clear, be omitted, and omitting also a factor $3x^2y^2z^2$, we may write

$$\begin{aligned} A &= 3x^2yz\left((1 + 8l^3)\,x^4 + (4l + 41l^4)\,x^2yz + (-2l^2 + 2l^5)\,y^2z^2\right) - 2Q\left(3l^2x^2 - (1 + 2l^3)\,yz\right), \\ B &= 3xy^2z\left((1 + 8l^3)\,y^4 + (4l + 41l^4)\,y^2zx + (-2l^2 + 2l^5)\,z^2x^2\right) - 2Q\left(3l^2y^2 - (1 + 2l^3)\,zx\right), \end{aligned}$$

and the like value for C. The last of the three equations is

$$\begin{aligned} &3xyz\left\{\begin{array}{l} (y^2 + 2lzx)\left((1 + 8l^3)\,x^5 + (4l + 41l^4)\,x^3yz + (-2l^2 + 2l^5)\,xy^2z^2\right) \\ -(x^2 + 2lyz)\left((1 + 8l^3)\,y^5 + (4l + 41l^4)\,y^3zx + (-2l^2 + 2l^5)\,yz^2x^2\right) \end{array}\right\} \\ &-2Q\left\{\quad (y^2 + 2lzx)\left(3l^2x^2 - (1 + 2l^3)\,yz\right) - (x^2 + 2lyz)\left(3l^2y^2 - (1 + 2l^3)\,zx\right)\right\} \\ &= 0, \end{aligned}$$

where the function on the left hand is

$$\begin{aligned} &= 3xyz\left\{\begin{array}{l} \quad (1 + 8l^3)\left(x^5y^3 - x^3y^5 + 2l\,(x^6z - y^6z)\right) \\ + (4l + 41l^4)\,2l\,(x^4yz^2 \quad - xy^4z^2) \\ + (-2l^2 + 2l^5)\,(xy^4z^2 \quad - x^4yz^2) \end{array}\right\} \\ &\quad -2Q\,\{(1 + 2l^3)\,(y^3z - x^3z) + 6l^2\,(x^3z - y^3z)\}, \end{aligned}$$

or, what is the same thing, throwing out the factor z, it is

$$\begin{aligned} &= \{3\,(1 + 8l^3)\,x^3y^3 + 6l\,(1 + 8l^3)\,xyz\,(x^3 + y^3) + 30l^2\,(1 + 8l^3)\,x^2y^2z^2\}\,(x^3 - y^3) \\ &\qquad - 2Q\,(1 + 8l^3)\,(x^3 - y^3)\,; \end{aligned}$$

or throwing out the factor $(1 + 8l^3)$ and substituting for Q its value, it is

$$= \{3x^3y^3 + 6lxyz\,(x^3 + y^3) + 30l^2x^2y^2z^2 - 2\,(y^3z^3 + z^3x^3 + x^3y^3 - 3l^2x^2y^2z^2)\}\,(x^3 - y^3).$$

The first factor, reducing by the equation $x^3+y^3+z^3+6lxyz=0$, is

$$\begin{aligned}
&= 3x^2y^3-(x^3+y^3)(x^3+y^3+z^3)+(x^3+y^3+z^3)^2-2(y^3z^3+z^3x^3+x^3y^3),\\
&= 3x^2y^3+z^3(x^3+y^3+z^3)-2(y^3z^3+z^3x^3+x^3y^3),\\
&= z^6-z^3(x^3+y^3)+x^3y^3,\\
&= (z^3-x^3)(z^3-y^3).
\end{aligned}$$

39. Hence putting for the moment

$$M=(y^3-z^3)(z^3-x^3)(x^3-y^3),$$

it appears that the last of the three equations is $Mz=0$; the first and second are of course $Mx=0$ and $My=0$, and the required condition is $M=0$, that is,

$$(y^3-z^3)(z^3-x^3)(x^3-y^3)=0,$$

the equation which, combined with the equation of the curve

$$x^3+y^3+z^3+6lxyz=0,$$

gives the sextactic points. There are consequently twenty-seven such points, and it is at once seen that these are the points of contact of the tangents to the cubic from the points of inflexion, or, what is the same thing, that the twenty-seven sextactic points form nine groups of three each, such that the three points of a group have for their common tangential one of the nine points of inflexion. In fact, let ω be a cube root (real or imaginary) of unity, the three sextactic points of one of the groups will be given by

$$\begin{cases} x-\omega y \quad\quad =0,\\ x^3+y^3+z^3+6lxyz=0.\end{cases}$$

Now consider the tangential of any one of these points, its coordinates are

$$\begin{aligned}
x_1&=x(y^3-z^3),\\
y_1&=y(z^3-x^3),\\
z_1&=z(x^3-y^3);
\end{aligned}$$

or, reducing by the equation $x-\omega y=0$, $x_1=\omega y(x^3-z^3)$, $y_1=-y(x^3-z^3)$, $z_1=0$, or, what is the same thing, $x_1+\omega y_1=0$, $z_1=0$; that is, the point (x_1, y_1, z_1) is one of the points of inflexion. This is the construction of the sextactic points obtained by Plücker and Steiner.

40. Reverting to the equation of the line joining the point of contact of the five-pointic conic with the point of simple intersection, this meets the cubic in a third point, and Mr Salmon has shown that this third point is in fact the second tangential (tangential of the tangential) of the point of contact, or, what is the same thing:

THEOREM. The point of simple intersection of the cubic and the five-pointic conic is the third point of intersection with the cubic, of the line joining the point of contact with the second tangential of this point.

41. I have not sought to verify this theorem by my formulæ. I remark, that combining it with the before-mentioned theorem, the five-pointic conic is completely determined as follows; viz.

THEOREM. The five-pointic conic touches the conic at the point of contact (two conditions); it passes through the two points in which the polar conic is intersected by the tangent to the cubic at the tangential of the point of contact (two conditions); and it passes through the point which is the third point of intersection with the cubic of the line joining the point of contact with its second tangential.

42. The construction for the point of simple intersection leads at once to that for the sextactic points; in fact, consider a point having for its tangential a point of inflexion: a point of inflexion is its own tangential, and the second tangential of the first-mentioned point is therefore the point of inflexion: the line joining the point with the second tangent is therefore the tangent at the point, and the point of simple intersection coincides with the point itself, that is, the point in question is a sextactic point.

43. I represent the equation of the five-pointic conic by

$$(a, b, c, f, g, h \between X, Y, Z)^2 = 0;$$

the value of a is

$$\begin{aligned} &= 9(1+8l^3)x^4y^3z^3 \\ &\quad + 3x^2y^2z^2\left(3l^2x^2 - (1+2l^3)yz\right)(x^2+2lyz) \\ &\quad - Q(x^2+2lyz)^2, \end{aligned}$$

in which equation

$$Q = y^3z^3 + z^3x^3 + x^3y^3 - 3l^2x^2y^2z^2;$$

or, reducing by the equation $x^3+y^3+z^3+6lxyz=0$,

$$Q = y^3z^3 + x^3(-x^3-6lxyz) - 3l^2x^2y^2z^2,$$

that is,

$$-Q = x^6 + 6lx^4yz + 3l^2x^2y^2z^2 - y^3z^3,$$

and we have

$$\begin{aligned} a &= 9(1+8l^3)x^4y^3z^3 \\ &\quad + 3x^2y^2z^2\left(3l^2x^4 + (-1+4l^3)x^2yz - 2l(1+2l^3)y^2z^2\right) \\ &\quad + (x^4+6lx^4yz+3l^2x^2y^2z^2-y^3z^3)(x^4+4xl^2yz+4l^2y^2z^2). \end{aligned}$$

We have in like manner

$$\begin{aligned} 2f &= 18(1+8l^3)lx^4y^2z^2 \\ &\quad + 3x^2y^2z^2\left\{\left(3l^2y^2-(1+2l^3)zx\right)(z^2+2lxy) + \left(3l^2z^2-(1+2l^3)xy\right)(y^2+2lzx)\right\} \\ &\quad - 2Q(y^2+2lzx)(z^2+2lxy); \end{aligned}$$

the coefficient of $3x^2y^2z^2$ in the second line is

$$= 6l^2y^2z^2 + (-1+4l^3)x(y^2+z^2) - 4l(1+2l^3)x^2yz;$$

or reducing by the equation of the curve,

$$= (1-4l^3)x^4 + (2l-32l^4)x^2yz + 6l^2y^2z^2;$$

and the coefficient of $-2Q$ is

$$= y^2z^2 + 2lx(y^3 + z^3) + 4l^2x^2yz;$$

or reducing by the equation of the curve,

$$= -2lx^4 - 8l^2x^2yz + y^2z^2.$$

We thus have

$$\begin{aligned} 2f = {} & 18l(1 + 8l^3)x^4y^3z^3 \\ & + 3x^2y^2z^2\left((1 - 4l^3)x^4 + (2l - 32l^4)x^2yz + 6l^2y^2z^2\right) \\ & + 2(x^6 + 6lx^4yz + 3l^2x^2y^2z^2 - y^3z^3)(-2lx^4 - 8l^2x^2yz + y^2z^2). \end{aligned}$$

Reducing the expressions of a and $2f$, we find for the coefficients (a, b, c, f, g, h),

$$\left\{\begin{aligned} a = {} & x^{10} + 10lx^8yz + 40l^2x^6y^2z^2 + (5 + 120l^3)x^4y^3z^3 - 10lx^3y^4z^4 - 4l^2y^5z^5, \\ & \vdots \\ 2f = {} & -4lx^{10} - 40l^2x^8yz + (5 - 120l^3)x^6y^2z^2 + 40lx^4y^3z^3 + 8l^2x^2y^4z^4 - 2y^5z^5, \\ & \vdots \end{aligned}\right.$$

which gives the completely developed form of the equation of the five-pointic conic.

44. I investigate the coordinates of the point of simple intersection of the cubic and the five-pointic conic as follows: the equations of the two curves are

$$\begin{aligned} & X^3 + Y^3 + Z^3 + 6lXYZ \qquad = 0, \\ & (a, b, c, f, g, h \between X, Y, Z)^2 = 0; \end{aligned}$$

or if we write

$$\begin{aligned} \alpha &= 1, & A &= c, \\ \gamma &= 6lXY, & B &= 2(gX + fY), \\ \delta &= X^3 + Y^3, & C &= aX^2 + 2hXY + bY^2, \end{aligned}$$

then the two equations are

$$\begin{aligned} \alpha Z^3 + \gamma Z + \delta &= 0, \\ AZ^2 + BZ + C &= 0, \end{aligned}$$

and the result of the elimination of Z will be

$$(\alpha Z_1^3 + \gamma Z_1 + \delta)(\alpha Z_2^3 + \gamma Z_2 + \delta) = 0,$$

where Z_1, Z_2 are the roots of the equation $AZ^2 + BZ + C = 0$; that is, we have

$$\left.\begin{aligned} & \alpha^2 \quad C^3 \\ & + \alpha\gamma \quad C(B^2 - 2AC) \\ & + \alpha\delta \quad (-B^3 + 2ABC) \\ & + \gamma^2 \quad CA^2 \\ & - \gamma\delta \quad BA^2 \\ & + \delta^2 \quad A^3 \end{aligned}\right\} = 0;$$

and substituting for A, B, C their values, but attending only to the terms which involve X^6 and Y^6, the result is

$$(a^3+c^3-8g^3+6acg)\,X^6+\ldots+(b^3+c^3-8f^3+6bcf)\,Y^6=0.$$

45. But the result of the elimination must obviously be

$$(Xy-Yx)^5\,(Xy_1-Yx_1)=0,$$

if $(x_1,\ y_1,\ z_1)$ are the coordinates of the point of simple intersection. Comparing the two results, and forming the analogous third equation, we may write

$$\begin{aligned} x^5x_1 &= b^3+c^3-8f^3+6bcf,\\ y^5y_1 &= c^3+a^3-8g^3+6cag,\\ z^5z_1 &= a^3+b^3-8h^3+6abh, \end{aligned}$$

where the value of x^5x_1 may also be written $(b+c-2f)\,(b+c\omega-2\omega^2f)\,(b+c\omega^2-2\omega f)$, ω being an imaginary cube root of unity, and so for the other two terms. The factors of x^5x_1 might be calculated from the identical equation

$$\begin{aligned} bY^2+2fYZ+cZ^2= \quad & 9\,(1+8l^3)\,x^3y^3z^3\,(yY^2+zZ^2+2lxYZ)\\ & +\{(y^3+2lzx)\,Y+(z^3+2lxy)\,Z\}\times\\ & \left\{\begin{array}{l} 3x^2y^2z^2\,[(3l^2y^2-(1+2l^3)\,zx)\,Y+(3l^2z^2-(1+2l^3)\,xy)\,Z]\\ \quad -Q\,[(y^3+2lzx)\,Y+(z^3+2lxy)\,Z] \end{array}\right\}. \end{aligned}$$

I remark, that putting $x=0$, we have

$$bY^2+2fYZ+cZ^2=-y^3z^3\,(y^3Y+z^3Z)^2,$$

and hence writing 1 for Y, and -1, $-\omega$, $-\omega^2$ for Z, we have

$$b+c-2f=-y^3z^3\,(y^3-z^3)^2,\ b+c\omega-2\omega^2f=-y^3z^3\,(y^3-\omega^4z^3)^2,\ b+c\omega^2-2\omega f=-y^3z^3\,(y^3-\omega^2z^3)^2;$$

and hence the product of the three factors is $-y^9z^9\,(y^3-z^3)^2\,(y^3-\omega^2z^3)^2\,(y^3-\omega^4z^3)^2$, which is equal to $-y^9z^9\,(y^3-z^3)^2\,(y^3+z^3)^2$, which vanishes in virtue of the assumed question $x=0$. This shows that the function $b^3+c^3-8f^3+6bcf$ contains the factor x. I have not verified *à posteriori*, but I assume it to be true, that it contains in fact the factor x^6, and consequently that the expressions for x_1, y_1, z_1 are rational and integral functions of $(x,\ y,\ z)$ of the degree 25, and containing respectively the factors x, y, z.

46. In the theory of the cubic, a point which depends linearly upon a given point may be termed a derivative of such point. According to a very beautiful theorem of Professor Sylvester's, the coordinates of a derivative point are necessarily rational and integral functions *of a square degree* of the coordinates $(x,\ y,\ z)$ of the given point; and moreover, there is but one derivative point having its coordinates of any given square degree m^2, or, as we may express it, only one derivative point of the degree m^2. The successive tangentials are derivative points of the degrees 4, 16, 64, &c.; the third point of intersection with the cubic, of the line joining two

derivative points of the degrees m^2 and n^2 respectively, is a derivative point of the degree $(m \pm n)^2$. Thus the third point of intersection with the cubic, of the line joining the given point with its second tangential, is a derivative point of the degree $(4 \pm 1)^2$, and it is easy to see that the degree is not 9; it is therefore 25. The point of simple intersection of the five-pointic conic is a derivative point of the degree 25; it is therefore, according to Professor Sylvester's general theory, identical with the point given by the former construction; this agrees with the before-mentioned theorem of Mr Salmon.

IV. *Independent investigation for the Cubic.*

47. The following is, in substance, the method by which I first obtained the equation of the five-pointic conic, for the cubic

$$X^3 + Y^3 + Z^3 + 6lXYZ = 0.$$

Write for shortness

$$\begin{aligned}
U &= x^3 + y^3 + z^3 + 6lxyz, \\
V &= (x^2 + 2lyz)\,X + (y^2 + 2lzx)\,Y + (z^2 + 2lxy)\,Z, \\
W &= (X^2 + 2lYZ)\,x + (Y^2 + 2lXZ)\,y + (Z^2 + 2lXY)\,z, \\
\Upsilon &= X^3 + Y^3 + Z^3 + 6lXYZ, \\
\mathrm{P} &= ax + by + cz, \\
\Pi &= aX + bY + cZ;
\end{aligned}$$

then, X, Y, Z being current coordinates, and x, y, z the coordinates of a point of the cubic (so that $U = 0$), the equation of the cubic will be

$$\Upsilon = 0,$$

and the equation of a conic having with it an ordinary (two-pointic) contact at the point (x, y, z), will be [1]

$$2W - \Pi V = 0.$$

48. Now imagine from the point of contact lines drawn to the other four intersections of the two curves; in the case of the five-pointic conic, three of these lines will coincide with the tangent $V = 0$, and the remaining line will be the line joining the point of contact with the point of simple intersection. The equations of the lines in question can be found by Joachimsthal's theorem, viz. if (x, y, z) be the coordinates of a given point, and (X, Y, Z) current coordinates, then if in the equations of any two curves we substitute for the coordinates, $\lambda x + \mu X$, $\lambda y + \mu Y$, $\lambda z + \mu Z$, and between the equations so obtained eliminate λ, μ, the resulting equation will be that

[1] I have introduced the factor 2, to make this correspond with the form $D^2U - \Pi DU = 0$, in the case in question, $m = 3$.

of the lines drawn from the point (x, y, z) to the points of intersection of the two curves. The point (x, y, z) is any point whatever, and it may therefore be a point of intersection, or, as in the present instance, a point of contact of the two curves; the only difference is, that in either case the degree of each equation as regards (λ, μ) is reduced by unity, and the degree of the resulting equation in X, Y, Z is also reduced by unity: in the case of a point of simple intersection this is the only reduction; but in the case of a point of contact, the resulting equation contains the equation of the tangent as a factor, and rejecting this factor, the reduction in degree is two units.

49. Applying the method to the two equations, $\Upsilon = 0$, $2W - \Pi V = 0$, and substituting therein for the original current coordinates X, Y, Z the values $\lambda x + \mu X$, $\lambda y + \mu Y$, $\lambda z + \mu Z$, the equations become

$$\lambda^3 U + 3\lambda^2\mu V + 3\lambda\mu^2 W + \mu^3\Upsilon = 0,$$
$$2(\lambda^2 U + 2\lambda\mu V + \mu^2 W) - (\lambda P + \mu\Pi)(\lambda U + \mu V) = 0;$$

or writing $U = 0$, and omitting from each equation the factor μ, the equations become

$$\lambda^2 . 3V + \lambda\mu . 3W + \mu^2\Upsilon \quad = 0,$$
$$\lambda(4 - P) V + \mu(2W - \Pi V) = 0;$$

and putting in the first equation $\lambda = 2W - \Pi V$, $\mu = -(4 - P) V$, the result of the elimination contains the factor V, rejecting which it becomes

$$3(2W - \Pi V)^2 - 3(4 - P)(2W - \Pi V) W + (4 - P)^2 V\Upsilon = 0,$$

which is of the fourth degree in (X, Y, Z), as it should be, and represents therefore the lines drawn from the point of contact to the other four points of intersection of the conic and cubic.

50. The equation may be written

$$-3(2W - \Pi V)\big((2 - P) W + \Pi V\big) + (4 - P)^2 V\Upsilon = 0,$$

and we obtain at once the condition that this may contain the factor V, viz. this condition is

$$P = 2;$$

and if this be satisfied the conic will have a three-pointic conic, and there will be three other points of intersection. And writing $P = 2$, and throwing out the factor V, we find

$$3\Pi^2 V - 6\Pi W + 4\Upsilon = 0$$

for the equation of the lines from the point of contact to the three points of intersection. And we have now to determine Π so that the function on the left hand may divide by V^2.

51. I simplify my original method by the use of a theorem of Mr Salmon's, viz. writing

$$\begin{array}{ll|ll}
\Upsilon = X^3 + Y^3 + Z^3 + 6lXYZ, & & \mathfrak{H} = l^2(X^3+Y^3+Z^3) - (1+2l^3)\,XYZ & , \\
W = (X^2 + 2lYZ)\,x + \ldots & , & \mathfrak{H}_1 = (3l^2X^2 - (1+2l^3)\,YZ \quad)\,x + \ldots & , \\
V = (x^2 + 2lyz \quad)\,X + \ldots & , & H_1 = (3l^2x^2 - (1+2l^3)\,yz \quad)\,X + \ldots, & \\
U = x^3 + y^3 + z^3 + 6lxyz & , & H = l^2(x^3+y^3+z^3) - (1+2l^3)\,xyz & ,
\end{array}$$

we have identically

$$\Upsilon H - U\mathfrak{H} = WH_1 - V\mathfrak{H}_1,$$

and in the present case, since $U = 0$,

$$\Upsilon H = WH_1 - V\mathfrak{H}_1.$$

Hence, multiplying by H and substituting this value of ΥH, the equation becomes

$$3H\Pi^2V - 6H\Pi\,W + 4WH_1 - 4V\mathfrak{H}_1 = 0,$$

or, as we may write it,

$$V(3H\Pi^2 - 4\mathfrak{H}_1) + 2W(2H_1 - 3H\Pi) = 0;$$

and we can at once make the equation divide by V, viz. by assuming

$$\Pi = \tfrac{2}{3}\frac{H_1}{H} + \tfrac{1}{2}\Lambda V,$$

where Λ is arbitrary: we have thus a four-pointic contact. And substituting for Π, and throwing out the factor V, the equation becomes

$$3H\left(\tfrac{2}{3}\frac{H_1}{H} + \tfrac{1}{2}\Lambda V\right)^2 - 4\mathfrak{H}_1 + 2W(\tfrac{3}{2}H\Lambda) = 0;$$

or reducing,

$$16(H_1^2 - 3H\mathfrak{H}_1) - 36H^2\Lambda W + 24HH_1\Lambda V + 9H^2\Lambda^2V^2 = 0,$$

which is the equation of the lines drawn from the point of contact to the remaining two points of intersection.

52. I write for greater convenience

$$\Lambda = -\tfrac{4}{9}\frac{\Theta}{H^3},$$

Θ being as yet indeterminate; the equation is thus reduced to

$$9H^3\{H(H_1^2 - 3H\mathfrak{H}_1) + \Theta W\} - 6H^2H_1\Theta V + \Theta^2V^2 = 0,$$

and we have then to determine Θ so that the left-hand side may divide by V; or, what is the same thing, we must determine Θ so that

$$H(H_1^2 - 3H\mathfrak{H}_1) + \Theta W$$

may divide by V. This implies the existence of an identical equation,

$$H(H_1^2 - 3H\mathfrak{H}_1) + \Theta W = MU + NV,$$

which for $U=0$ would give the decomposition in question; but I have not investigated the values of M and N. I assume at the outset $U=0$, and putting as before

$$Q = y^3z^3 + z^3x^3 + x^3y^3 - 3l^2x^2y^2z^2,$$

and writing also

$$\begin{aligned} S = \quad & Xx[(1+8l^3)x^4 + (4l+41l^4)x^2yz + (-2l^2+2l^5)y^2z^2] \\ & + Yy[(1+8l^3)y^4 + (4l+41l^4)y^2zx + (-2l^2+2l^5)z^2x^2] \\ & + Zz\,[(1+8l^3)z^4 + (4l+41l^4)z^2xy + (-2l^2+2l^5)x^2y^2], \end{aligned}$$

I remark that for $U=0$ we have

$$-xyzH_1^2 + VS + (1+8l^3)(QW - 3x^2y^2z^2\mathfrak{H}_1) = 0,$$

an equation which, observing that

$$H = -(1+8l^3)xyz,$$

and assuming also

$$\Theta = \quad (1+8l^3)^2 Q,$$

may be written

$$H(H_1^2 - 3H\mathfrak{H}_1) + \Theta W = -(1+8l^3)VS,$$

which gives the required decomposition, viz., Θ having this value, the conic will have a five-pointic contact. Reducing by the last equation, and throwing out the factor V, we find

$$-9H^3(1+8l^3)V - 6H^2\Theta H_1 + \Theta^2 V = 0$$

for the equation of the line joining the point of contact with the point of simple intersection. And if in this equation we write $H = -(1+8l^3)xyz$, and $\Theta = (1+8l^3)^2 Q$, we obtain, finally, for the equation of the line in question,

$$9x^3y^3z^3S - 6x^2y^2z^2QH_1 + Q^2V = 0,$$

which is the before-mentioned result.

53. It only remains to verify the assumed equation

$$-xyzH_1^2 + VS + (1+8l^3)(QW - 3x^2y^2z^2\mathfrak{H}_1) = 0.$$

We may write

$$-Q = x^6 + 6lx^4yz + 3l^2x^2y^2z^2 - y^3z^3,$$

and then observing that

$$\begin{aligned} W &= \quad xX^2 + \ldots + \qquad 2lxYZ + \ldots, \\ \mathfrak{H}_1 &= 3l^2xX^2 + \ldots - (1+2l^5)\,xYZ - \ldots, \end{aligned}$$

we find at once

$$(1+8l^3)(QW-3x^2y^2z^2\mathfrak{H}_1)=$$

$$-(1+8l^3)\left\{\begin{array}{l}(x^7+6lx^5yz+12l^2x^3yz-xy^3z^3)\,X^2\\ \quad\vdots\\ +(2lx^7+12l^2x^5yz-3x^3y^2z^2-2lxy^3z^3)\,YZ\\ \quad\vdots\end{array}\right.$$

Next writing

$$H_1=(3l^2x^2-(1+2l^3)\,yz)\,X+\ldots,$$
$$V=(x^2+2lyz)\,X+\ldots,$$
$$S=x\,\{(1+8l^3)\,x^4+(4l+41l^4)\,x^2yz+(-2l^2+2l^5)\,y^2z^2\}\,X+\ldots,$$

and forming the expression for

$$-xyzH_1^2+VS,$$

the coefficient of X^2 is

$$-xyz\qquad\{3l^2x^2-(1+2l^3)\,yz\}^2$$
$$+(x^2+2lyz)\,x\,\{(1+8l^3)\,x^4+(4l+41l^4)\,x^2yz+(-2l^2+2l^5)\,y^2z^2\},$$

which is

$$=(1+8l^3)\,(x^7+6lx^5yz+12l^2x^3y^2z^2-xy^3z^3);$$

the coefficient of YZ is

$$-2xyz\,(3l^2y^2-(1+2l^3)\,zx)\,(3l^2z^2-(1+2l^3)\,xy)$$
$$+(y^2+2lzx)\,z\,\{(1+8l^3)\,z^4+(4l+41l^4)\,z^2xy+(-2l^2+2l^5)\,x^2y^2\}$$
$$+(z^2+2lxy)\,y\,\{(1+8l^3)\,y^4+(4l+41l^4)\,y^2zx+(-2l^2+2l^5)\,x^2y^2\},$$

which is

$$=(1+8l^3)\,\{y^2z^2(y^3+z^3)+8lxy^3z^3+12l^2x^2yz\,(y^3+z^3)+2lx\,(y^6+z^6)-2x^2y^2z^2\};$$

or substituting for y^3+z^3 and y^6+z^6 the values $-x^3-6lxyz$ and $(x^3+6lxyz)^2-2y^3z^3$ respectively, this is

$$=(1+8l^3)\,(2lx^7+12l^2x^5yz-3x^2y^2z^2-2lxy^3z^3).$$

We have thus

$$-xyzH_1^2+VS$$

$$=(1+8l^3)\left\{\begin{array}{l}(x^7+6lx^5yz+12l^2x^3yz-xy^3z^3)\,X^2\\ \quad\vdots\\ +(2lx^7+12l^2x^5yz-3x^2y^2z^2-2lxy^3z^3)\,YZ\\ \quad\vdots\end{array}\right.$$

and the equation

$$-xyzH_1^2+VS+(1+8l^3)\,(QW-3x^2y^2z^2\mathfrak{H}_1)=0$$

is thus verified.

ADDITION.—The foregoing memoir was communicated to Mr Salmon, and I am indebted to him for two notes, containing the extension to a curve of any order, of the preceding investigation for the case of a cubic; I reproduce this extension in the following section.

V. *Extension of the last preceding method to a curve of any order.*

54. Consider the curve of the m-th order $\Upsilon = 0$, and in the place of the coordinates write $\lambda x + \mu X$, $\lambda y + \mu Y$, $\lambda z + \mu Z$, where, as before, (x, y, z) are the coordinates of the point of the curve, and X, Y, Z current coordinates; the term involving λ^m vanishes, and dividing out the factor μ the equation becomes

$$\lambda^{m-1} DU + \tfrac{1}{2}\lambda^{m-2}\mu D^2U + \tfrac{1}{6}\lambda^{m-3}\mu^2 D^3U + \tfrac{1}{24}\lambda^{m-4}\mu^3 D^4U + \&c. = 0.$$

Making the like substitution in $D^2U - \Pi DU = 0$, the assumed equation of the five-pointic conic, the factor μ divides out and the equation becomes

$$2\lambda(m-1)\,DU + \mu D^2U - (\lambda P + \mu\Pi)\,D^2U = 0,$$

or, what is the same thing,

$$\lambda DU\big(2(m-1) - P\big) + \mu\,(D^2U - \Pi DU) = 0;$$

and if from the two equations we eliminate λ, μ, the result, throwing out the factor DU, is

$$(D^2U - \Pi DU)^{m-1} - \tfrac{1}{2}\big(2(m-1) - P\big)\,D^2U\,(D^2U - \Pi DU)^{m-2} + \&c. = 0,$$

where all the terms after the second contain the factor DU; the condition in order that the equation may divide by DU, is consequently $2(m-1) - P = 2$, or $P = 2(m-2)$, the condition of a three-pointic contact. Substituting this value, and dividing by DU, the equation becomes

$$-\Pi\,(D^2U - \Pi DU)^{m-2} + \tfrac{2}{3}D^3U\,(D^2U - \Pi DU)^{m-3} - \tfrac{1}{3}D^4U\,.\,DU\,(D^2U - \Pi DU)^{m-4} + \&c. = 0,$$

which will be divisible by DU if $-\Pi D^2U + \frac{2}{3}D^3U$ is divisible by DU, and the condition for this is found to be, as before,

$$\Pi = \tfrac{2}{3}\frac{1}{H}DH + \Lambda DU,$$

where Λ is arbitrary; we have thus the conditions of a four-pointic contact.

55. Substituting this value of Π, we see that $D^3U - \frac{1}{H}DH\,.\,D^2U$ divides by DU, viz. there exists an identical equation,

$$D^3U - \frac{1}{H}DH\,.\,D^2U = IU + J\,.\,DU;$$

and hence if $U = 0$,

$$\frac{1}{DU}\left(D^3U - \frac{1}{H}DH\,.\,D^2U\right) = J,$$

where J is a quadric function of (X, Y, Z). I do not know the general form of this function, but Mr Salmon has obtained a result which may be generalized as

follows, viz. writing for X, Y, Z the values $B\nu - C\mu$, $C\lambda - A\nu$, $A\mu - B\lambda$ (where, as before, A, B, C are the first derived functions of U and λ, μ, ν are arbitrary), the expression for J is

$$J = \frac{1}{DU}\left(D^3U - \frac{1}{H}DH\,.\,D^2U\right) = \frac{3(m-2)}{(m-1)}\Phi - \frac{1}{(m-1)H}\square\vartheta;$$

a formula which will be presently useful.

56. The foregoing equation may be written

$$(D^2U)^{m-3}(-\Pi D^2U + \tfrac{2}{3}D^2U)$$
$$+ (D^2U)^{m-4}DU\{(m-2)\Pi^2D^2U - \tfrac{2}{3}(m-3)\Pi D^3U - \tfrac{1}{3}D^4U\} + \&c.\ (DU)^2 \ldots = 0;$$

and the term $-\Pi D^2U + \frac{2}{3}D^2U$ is equal to

$$\tfrac{2}{3}\left(D^2U - \frac{1}{H}DH\,.\,D^2U\right) - \Lambda DU\,.\,D^2U, = \tfrac{2}{3}JDU - \Lambda DU\,.\,D^2U.$$

Substituting this value the equation divides by DU, and throwing out this factor it becomes

$$(D^2U)^{m-3}(\tfrac{2}{3}J - \Lambda D^2U)$$
$$+ (D^2U)^{m-4}\{(m-2)\Pi^2D^2U - \tfrac{2}{3}(m-3)\Pi D^3U - \tfrac{1}{3}D^4U\} + \&c.\ DU = 0;$$

or observing that $\frac{2}{3}\Pi D^3U = \Pi^2D^2U +$ term containing DU, this may be written

$$(D^2U)^{m-3}(\tfrac{2}{3}J - \Lambda D^2U) + (D^2U)^{m-4}(\Pi^2D^2U - \tfrac{1}{3}D^4U) + \&c.\ DU = 0.$$

57. If the equation divides by DU we shall have a five-pointic contact; the condition for this is that

$$-\Lambda(D^2U)^2 + \tfrac{2}{3}JD^2U + \Pi^2D^2U - \tfrac{1}{3}D^4U$$

may divide by DU, or more simply that

$$-\Lambda(D^2U)^2 + \tfrac{2}{3}JD^2U + \tfrac{4}{9}\left(\frac{1}{H}DH\right)^2D^2U - \tfrac{1}{3}D^4U$$

may divide by DU, or, what is the same thing, the function in question must vanish in virtue of the substitution of the values $B\nu - C\mu$, $C\lambda - A\nu$, $A\mu - B\lambda$ in the place of X, Y, Z. The expression for J has just been given; we have besides

$$\left(\frac{1}{H}DH\right)^2 = \frac{(m-1)^4}{H^2}Q_3^2,\quad D^4U = -Q_4\vartheta^2,$$

where the values of Q_3^2, Q_4 are given (*ante*, No. 27); we have thus

58. $$\frac{1}{DU}\left(D^3U - \frac{1}{H}DH\,.\,D^2U\right) = \frac{3(m-2)}{m-1}\Phi - \frac{1}{(m-1)H}\square\vartheta,$$

$$\tfrac{4}{9}\left(\frac{1}{H}DH\right)^2 = -\frac{4(m-2)^2}{(m-1)^2}\Phi + \tfrac{4}{3}\frac{(m-2)^2}{(m-1)^2}\frac{\square}{H}\vartheta - \tfrac{4}{9}\frac{1}{(m-1)^2}\frac{1}{H^2}\Psi\vartheta^2,$$

$$-\tfrac{1}{3}D^4U = \vartheta^2\left\{-\frac{2(m-2)(m-3)}{(m-1)^4}\Phi H + \tfrac{4}{3}\frac{(m-3)}{(m-1)^4}\square\vartheta - \tfrac{1}{3}\frac{1}{(m-1)^4}\Omega\vartheta^2\right\},$$

and

$$D^2U = -\frac{1}{(m-1)^2} H\vartheta^2,$$

whence

$$\Lambda \frac{1}{(m-1)^4} H^2\vartheta^4 =$$

$$-\tfrac{2}{3}\frac{1}{(m-1)^2} H\vartheta^2 \left\{ \frac{3(m-2)}{m-1}\Phi - \frac{1}{(m-1)H}\square\vartheta \right\}$$

$$-\frac{1}{(m-1)^2} H\vartheta^2 \left\{ \frac{-4(m-2)^2}{(m-1)^2}\Phi + \tfrac{4}{3}\frac{(m-2)}{(m-1)^2 H}\square\vartheta - \tfrac{4}{9}\frac{1}{(m-1)^2}\frac{1}{H^2}\Psi\vartheta^2 \right\}$$

$$+\vartheta^2 \left\{ -\frac{2(m-2)(m-3)}{(m-1)^4}\Phi H + \tfrac{2}{3}\frac{(m-3)}{(m-1)^4}\square\vartheta - \tfrac{1}{3}\frac{1}{(m-1)^4}\Omega\vartheta^2 \right\},$$

$$= \frac{1}{9(m-1)^4 H}(4\Psi - 3\Omega H),$$

and consequently

$$\Lambda = \frac{1}{9H^3}(4\Psi - 3\Omega H),$$

which agrees with the result before obtained, and thus the present method gives the complete solution of the problem.

262.

ON THE EQUATION OF DIFFERENCES FOR AN EQUATION OF ANY ORDER, AND IN PARTICULAR FOR THE EQUATIONS OF THE ORDERS TWO, THREE, FOUR, AND FIVE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CL. (for the year 1860), pp. 93—112. Received March 2,—Read March 29, 1860.]

THE term, *equation of differences*, denotes the equation for the squared differences of the roots of a given equation; the equation of differences afforded a means of determining the number of real roots, and also limits for the real roots, of a given numerical equation, and was upon this account long ago sought for by geometers. In the *Philosophical Transactions* for 1763, Waring gives, but without demonstration or indication of the mode of obtaining it, the equation of differences for an equation of the fifth order wanting the second term: the result was probably obtained by the method of symmetric functions. This method is employed in the *Meditationes Algebraicæ* (1782), where the equation of differences is given for the equations of the third and fourth orders wanting the second terms; and in p. 85 the before-mentioned result for the equation of the fifth order wanting the second term, is reproduced. The formulæ for obtaining by this method the equation of differences, are fully developed by Lagrange in the *Traité des Equations Numériques* (1808); and he finds by means of them the equation of differences for the equations of the orders two and three, and for the equation of the fourth order wanting the second term; and in Note III. he gives, after Waring, the result for the equation of the fifth order wanting the second term. It occurred to me that the equation of differences could be most easily calculated by the following method. The coefficients of the equation of differences, *quà* functions of the differences of the roots of the given equation, are leading

coefficients of covariants, or (to use a shorter expression) they are "Seminvariants[1]," that is, each of them is a function of the coefficients which is reduced to zero by one of the two operators which reduce an invariant to zero. In virtue of this property they can be calculated, when their values are known for the particular case in which one of the coefficients of the given equation is zero. To fix the ideas, let the given equation be $(*\between v, 1)^n = 0$; then, when the last coefficient or constant term vanishes, the equation breaks up into $v = 0$ and into an equation of the degree $n-1$, which I call the reduced equation; the equation of differences will break up into two equations, one of which is the equation of differences for the reduced equation, the other is the equation for the squares of the roots of the same reduced equation. This hardly requires a proof; let the roots of the given equation be α, β, γ, δ, &c., those of the equation of differences are $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, $(\alpha-\delta)^2$, &c., $(\beta-\gamma)^2$, $(\beta-\delta)^2$, $(\gamma-\delta)^2$, &c.; but in putting the constant term equal to zero, we in effect put one of the roots, say α, equal to zero; the roots of the equation of differences thus become β^2, γ^2, δ^2 &c., $(\beta-\gamma)^2$, $(\beta-\delta)^2$, $(\gamma-\delta)^2$, &c. The equation for the squares of the roots can be found without the slightest difficulty; hence if the equation of differences for the reduced equation of the order $n-1$ is known, we can, by combining it with the equation for the squares of the roots, form the equation of differences for the given equation with the constant term put equal to zero, and thence by the above-mentioned property of the Seminvariancy of the coefficients, find the equation of differences for the given equation. The present memoir shows the application of the process to equations of the orders two, three, four, and five: part of the calculation for the equation of the fifth order was kindly performed for me by the Rev. R. Harley. It is to be noticed that the best course is to apply the method in the first instance to the forms $(a, b, \ldots \between v, 1)^n = 0$, without numerical coefficients (or, as they may be termed, the *denumerate forms*), and to pass from the results so obtained to those which belong to the forms $(a, b, \ldots \between v, 1)^n = 0$, or *standard forms*. The equation of differences, for $(\alpha-\beta)^2$, &c., the coefficients of which are seminvariants, naturally leads to the consideration of a more general equation having for its roots $(\alpha-\beta)^2(x-\gamma y)^2(x-\delta y)^2\ldots$, &c., the coefficients of which are covariants; and in fact, when, as for equations of the orders two, three, and four, all the covariants are known, such covariant equation can be at once formed from the equation of differences; for equations of the fifth order, however, where the covariants are not calculated beyond a certain degree, [they are now all calculated, see 141 and 143], only a few of the coefficients of the covariant equation can be thus at once formed. At the conclusion of the memoir, I show how the equation of differences for an equation of the order n can be obtained by the elimination of a single quantity from two equations each of the order $n-1$; and by applying to these two equations the simplification which I have made in Bezout's abridged method of elimination, I exhibit the equation of differences for the given equation of the order n, in a compendious form by means of a determinant; the first-mentioned method is, however, that which is best adapted for the actual development of the equation of differences for the equation of a given order.

[1] The term "Seminvariant" seems to me preferable to M. Brioschi's term "Peninvariant."

The equations successively considered are

$$\begin{array}{ll} (a,\ b,\ c & \mathfrak{X}v,\ 1)^2 = 0, \\ (a,\ b,\ c,\ d & \mathfrak{X}v,\ 1)^3 = 0, \\ (a,\ b,\ c,\ d,\ e & \mathfrak{X}v,\ 1)^4 = 0, \\ (a,\ b,\ c,\ d,\ e,\ f & \mathfrak{X}v,\ 1)^5 = 0. \end{array}$$

The equation of differences for the quadric, and that for the squares of the roots, are considered to be known, and the other results are derived from them: it will be convenient to write down in the first instance the results for the quadric, the cubic, and the quartic equations, and then explain the process of obtaining them.

For the quadric equation,

Equation of differences is, $0 =$

$$\left(\begin{array}{c|c} \overbrace{a^2 \times}^{} & \\ \overbrace{+1}^{} & \overbrace{\begin{array}{c} ac + 4 \\ b^2 - 1 \end{array}}^{} \end{array}\right. \mathfrak{X}\theta,\ 1).$$

Equation for the squares of the roots is, $0 =$

$$\left(\begin{array}{c|c|c} \overbrace{a^2 \times}^{} & & \\ \overbrace{+1}^{} & \overbrace{\begin{array}{c} ac + 2 \\ b^2 - 1 \end{array}}^{} & \overbrace{c^2 + 1}^{} \end{array}\right. \mathfrak{X}\theta,\ 1)^2.$$

For the cubic equation,

Equation of differences is, $0 =$

$$\left(\begin{array}{c|c|c|c} \overbrace{a^4 \times}^{} & \overbrace{a^2 \times}^{} & & \\ \overbrace{+1}^{} & \overbrace{\begin{array}{c} ac + 6 \\ b^2 - 2 \end{array}}^{} & \overbrace{\begin{array}{c} a^2c^2 + 9 \\ ab^2c - 6 \\ b^4 + 1 \end{array}}^{} & \overbrace{\begin{array}{c} a^2d^2 + 27 \\ abcd - 18 \\ ac^3 + 4 \\ b^3d + 4 \\ b^2c^2 - 1 \end{array}}^{} \end{array}\right. \mathfrak{X}\theta,\ 1)^3.$$

Equation for the squares of the roots is, $0 =$

$$\left(\begin{array}{c|c|c|c} \overbrace{a^2 \times}^{} & & & \\ \overbrace{+1}^{} & \overbrace{\begin{array}{c} ac + 2 \\ b^2 - 1 \end{array}}^{} & \overbrace{\begin{array}{c} c^2 + 1 \\ bd - 2 \end{array}}^{} & \overbrace{d^2 + 1}^{} \end{array}\right. \mathfrak{X}\theta,\ 1)^3.$$

For the quartic equation,

Equation of differences is, $0 =$

$a^6 \times$	$a^4 \times$	$a^2 \times$				
$+1$	$ac + 8$ $b^2 - 3$	$a^3e + 8$ $a^2bd - 2$ $a^2c^2 + 22$ $ab^2c - 16$ $b^4 + 3$	$a^4ce + 16$ $a^4d^2 + 26$ $a^3b^2e - 6$ $a^3bcd - 30$ $a^3c^3 + 28$ $a^2b^3d + 8$ $a^2b^2c^2 - 24$ $ab^4c + 8$ $b^6 - 1$	$a^4e^2 - 112$ $a^3bde + 56$ $a^3c^3e + 24$ $a^3cd^2 + 48$ $a^2b^2ce - 32$ $a^2b^3d^2 - 25$ $a^2bc^2d - 54$ $a^2c^4 + 17$ $ab^4e + 6$ $ab^3cd + 38$ $ab^2c^3 - 12$ $b^5d - 6$ $b^4c^2 + 2$	$a^3ce^2 - 192$ $a^3d^2e + 216$ $a^2b^2e^2 + 72$ $a^2bcde - 120$ $a^2bd^3 - 54$ $a^2c^3e + 32$ $a^2c^2d^2 + 18$ $ab^3de + 18$ $ab^2c^2e - 6$ $ab^2cd^2 + 42$ $abc^3d - 26$ $ac^5 + 4$ $b^4d^2 - 9$ $b^3c^3d + 6$ $b^2c^4 - 1$	$a^2e^3 + 256$ $a^2bde^2 - 192$ $a^2c^3e^2 - 128$ $a^2cd^2e + 144$ $a^2d^4 - 27$ $ab^2ce^2 + 144$ $ab^2d^2e - 6$ $abc^2de - 80$ $abcd^3 + 18$ $ac^4e + 16$ $ac^3d^2 - 4$ $b^4e^2 - 27$ $b^3cde + 18$ $b^3d^3 - 4$ $b^3c^2e - 4$ $b^3c^2d^2 + 1$

$(\quad \mathfrak{X}\theta,\ 1)^6.$

Equation for squares of the roots is, $0 =$

$a^2 \times$				
$+1$	$ac + 2$ $b^2 - 1$	$ae + 2$ $bd - 2$ $c^2 + 1$	$ce + 2$ $d^2 - 1$	$e^2 + 1$

$(\quad \mathfrak{X}\theta,\ 1)^4.$

The multiplication of the equation of differences and the equation of the squares of the roots of the quadric equation, gives the equation, $0 =$

$a^4 \times$	$a^2 \times$		
$+1$	$ac + 6$ $b^2 - 2$	$a^2c^2 + 9$ $ab^2c - 6$ $b^4 + 1$	$ac^3 + 4$ $b^2c^2 - 1$

$(\quad \mathfrak{X}\theta,\ 1)^3,$

where all the coefficients except the last are reduced to zero by the operator

$$3a\partial_b + 2b\partial_c + c\partial_d,$$

and they are consequently (without any alteration) coefficients of the equation of differences of the cubic equation: the last coefficient is not reduced to zero by the operator, and requires therefore to be completed by the adjunction of the terms in d (the series, here and in every other case, is of course a finite one, the number of terms might easily be calculated *à priori*). Let the value be $L_0 + L_1 d + L_2 d^2 + \&c.$, we have $L_0 = +4ac^3 - 1b^2c^2$; and putting for shortness $\nabla' = 3a\partial_b + 2b\partial_c$, the operator which reduces this to zero is $\nabla' + c\partial_d$; we ought therefore to have

$$\begin{array}{rl|l|l} 0 = & \nabla' L_0 + d & \nabla' L_1 + d^2 & \nabla' L_2 + \ldots \\ + & cL_1 & 2cL_2 & 3cL_3 \end{array}$$

and consequently

$$L_1 = -\frac{1}{c}\nabla' L_0,\ L_2 = -\frac{1}{2c}\nabla' L_1,\ L_3 = -\frac{1}{3c}\nabla' L_2,\ \&c.,$$

giving

$$L_1 = -\frac{1}{c}\{(-6+24=)\,18abc^2 - 4b^2c\} = -18abc + 4b^3,$$

$$L_2 = -\frac{1}{2c}\{-54a^2c + (36-36=)\,0\,ab^2\} = +27a^2,$$

$$L_3 = 0,\ \&c.,$$

and consequently for the last coefficient the value above written down; it will be presently seen how in more complicated cases the calculations should be arranged.

Again, multiplying together the equation of differences and the equation for the squares of the roots of the cubic equation, we obtain an equation which it is not necessary to write down, as it can be at once formed by putting $e = 0$ in the equation of differences for the quartic equation. And from the equation so obtained, by the adjunction of the terms in e, we find the equation of differences for the quartic equation, viz. each coefficient is of the form $L_0 + L_1 e + L_2 e^2 + \&c.$, where L_0 is known, and such coefficient is reduced to zero by the operator

$$4a\partial_b + 3b\partial_c + 2c\partial_d + d\partial_e;$$

or putting for shortness $\nabla' = 4a\partial_b + 3b\partial_c + 2c\partial_d$, the operator $\nabla' + d\partial_e$. We have therefore

$$L_1 = -\frac{1}{d}\nabla' L_0,\quad L_2 = -\frac{1}{2d}\nabla' L_1,\quad L_3 = -\frac{1}{3d}\nabla' L_2,\quad \&c.$$

It is to be observed that the last coefficient of the equation of differences is the discriminant, and that the above method of calculating the coefficients of the equation of differences, as applied to the last coefficient, is nothing else than the method of calculating the discriminant given in my Fourth Memoir on Quantics, [155].

The multiplication of the equation of differences, and the equation for the squares of the roots of the quartic equation, gives, in like manner, the equation of differences for the quintic equation, except as to the terms involving f; and these are obtained as above, viz. each coefficient is of the form $L_0 + L_1 f + L_2 f^2 + \&c.$, where L_0 is known; such coefficient is reduced to zero by the operator

$$5a\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e + e\partial_f;$$

or putting for shortness $\nabla' = 5a\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e$, the operator $\nabla' + e\partial_f$. We have therefore

$$L_1 = -\frac{1}{e}\nabla' L_0, \quad L_2 = -\frac{1}{2e}\nabla' L_1, \quad L_3 = -\frac{1}{3e}\nabla' L_2, \ \&c.,$$

which give L_1, L_2, &c. by means of L_0. For the calculation of $\nabla' L$ (where L is any one of the coefficients L_0, L_1, &c.), it is proper to separate the terms involving the different powers of e, and write $L = M_0 + M_1 e + M_2 e^2 + \&c.$, $\nabla' = \nabla'' + 2d\partial_e$, where $\nabla'' = 5a\partial_b + 4b\partial_c + 3c\partial_d$. We have then

$$\begin{array}{rl|l|l} \nabla L = & \nabla'' M_0 + e & \nabla'' M_1 + e^2 & \nabla'' M_2 + \&c., \\ & + 2dM_1 & + 4dM_2 & + 6dM_3, \end{array}$$

or, what is the same thing,

$$\begin{array}{rl|l} \frac{1}{e}\nabla L = & \nabla'' M_1 + e & \nabla'' M_2 + \ldots \\ & + 4dM_2 & + 4dM_3; \end{array}$$

and as an equation which should be satisfied identically, and which would therefore serve as a verification,

$$\nabla'' M_0 + 2dM_1 = 0;$$

but, since a verification was obtained by other means, the equations of this kind were not used for the purpose. It may be interesting to give the actual calculation of one of the coefficients, say of coefficient θ^2 (which, with coefficient θ, was calculated by Mr Harley).

Calculation of coefficient θ^2 in equation of differences for the quintic equation

$$(a,\ b,\ c,\ d,\ e,\ f\ \between v,\ 1)^5 = 0.$$

Calculation of L_0.

	$2ae$	$-2bd$	$+c^2$	$2ce$	$-d^2$	e^2	
	Coeff. θ^0.			Coeff. θ^1.		Coeff. θ^2.	In eq. of diff. for quartic.
a^4e^4	+512					−112 =	+400
a^3bde^3	−384	−512				+56	−840
$a^3c^2e^3$	−256		+256	−384		+24	−360
$a^3cd^2e^2$	+288			+432	+192	+48	+960
a^3d^3e	−54				−216		−270
$a^2b^2ce^3$	+288			+144		−32	+400
$a^2b^2d^2e^2$	−12	+384			−72	−25	+275
$a^2bc^2de^2$	−160	+256	−192	−240		−54	−390
a^2bcd^3e	+36	−288		−108	+120		−240
abd^5		+54			+54		+108
$a^2c^4e^2$	+32		−128	+64		+17	−15
$a^2c^3d^2e$	−8		+144	+36	−32		+140
$a^2c^2d^4$			−27		−18		−45
ab^4e^3	−54					+6	−48
ab^3cde^2	+36	−288		+36		+38	−178
ab^3d^2e	−8	+12			−18		−14
$ab^2c^3e^2$	−8		+144	−12		−12	+112
$ab^2c^2d^2e$	+2	+160	−6	+84	+6		+246
ab^2cd^4		−36			−42		−78
abc^4de		−32	−80	−52			−164
abc^2d^3		+8	+18		+26		+52
ac^6e			+16	+8			+24
ac^5d^2			−4		−4		−8
b^5de^2		+54				−6	+48
$b^4c^2e^2$			−27			+2	−25
b^4cd^2e		−36		−18			−54
b^4d^4		+8			+9		+17
b^3c^3de		+8	+18	+12			+38
$b^3c^2d^3$		−2	−4		−6		−12
b^2c^5e			−4	−2			−6
$b^2c^4d^2$			+1		+1		+2

$L_0 = A + Be + Ce^2 + De^3 + Ee^4$, suppose, then

$-B =$	$-C =$	$-D =$	$-E =$
a^2d^3 +270	a^3cd^2 −960	a^3bd +840	a^4 −400
a^2bcd +240	$a^2b^2d^2$ −275	a^3c^2 +360	
$a^2c^3d^2$ −140	a^2bc^2d +390	a^2b^2c −400	
ab^3d^3 +14	a^2c^4 +15	ab^4 +48	
$ab^2c^2d^2$ −246	ab^3cd +178		
abc^4d +164	ab^2c^3 −112		
ac^6 −24	b^5d −48		
b^4cd^2 +54	b^4c^2 +25		
b^3c^3d −38			
b^2c^5 +6			

Calculation of L_1.

Terms of $L_1 f$ not involving e.

$-f(\nabla''B+4dC)$ viz.	$5a\partial_b$	$+4b\partial_c$	$+3c\partial_d$	$4d$		
		$-B$		$-C$		
a^3cd^3f	$+1200$		$+3240$	-3840	$=$	$+600$
$a^2b^2d^3f$	$+210$	$+960$		-1100		$+70$
$a^2bc^2d^2f$	-2460	-1680	$+2160$	$+1560$		-420
a^2c^3df	$+820$		-840	$+60$		$+40$
ab^3cd^2f	$+1080$	-1968	$+126$	$+712$		-50
ab^2c^2df	-570	$+2624$	-1476	-448		$+130$
abc^3f	$+60$	-576	$+492$			-24
b^3d^2f		$+216$		-192		$+24$
b^2c^2df		-456	$+324$	$+100$		-32
b^2c^4f		$+120$	-114			$+6$

Terms of $L_1 f$ involving e^1.

$-ef(\nabla''C+6dD)$ viz.	$5a\partial_b$	$+4b\partial_c$	$+3c\partial_d$	$6d$		
		$-C$		$-D$		
a^2bd^2ef	-2750	-3840		$+5040$	$=$	-1550
a^2c^2def	$+1950$		-5760	$+2160$		-1650
a^2b^2cdef	$+2670$	$+3120$	-1650	-2400		$+1740$
a^2bc^2ef	-1120	$+240$	$+1170$			$+290$
ab^2def	-1200	$+712$		$+288$		-200
ab^2c^2ef	$+500$	-1344	$+534$			-310
b^2cef		$+200$	-144			$+56$

Terms of $L_1 f$ involving e^2.

$-e^2f(\nabla''D+8dE)$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$	$8d$		
		$-D$		$-E$		
a^2de^2f	$+4200$			-3200	$=$	$+1000$
a^2bce^2f	-4000	$+2880$	$+2520$			$+1400$
$a^2b^2e^2f$	$+960$	-1600				-640

Calculation of L_2.

$L_1 = A + Be + Ce^2$, suppose, where

$-\frac{1}{2}B =$		$-\frac{1}{2}C =$	
a^3bd^2	$+ 775$	a^4d	$- 500$
a^3c^2d	$+ 825$	a^3bc	$- 700$
a^2b^2cd	$- 870$	a^2b^3	$+ 320$
a^2bc^3	$- 145$		
ab^4d	$+ 100$		
ab^3c^2	$+ 155$		
b^5c	$- 28$		

Terms of $L_2 f^2$ not involving e.

$-f^2(\nabla''\frac{1}{2}B + 4d\frac{1}{2}C)$ viz.	$5a\partial_b$	$4b\partial_c$	$3c\partial_d$	$4d$		
		$-\frac{1}{2}B$		$-\frac{1}{2}C$		
$a^4d^2f^2$	$+ 3875$			$- 2000$	$=$	$+ 1875$
a^3bcdf^2	$- 8700$	$+ 6600$	$+ 4650$	$- 2800$		$- 250$
$a^3c^3f^2$	$- 725$		$+ 2475$			$+ 1750$
$a^2b^3df^2$	$+ 2000$	$- 3480$		$+ 1280$		$- 200$
$a^2b^2c^2f^2$	$+ 2325$	$- 1740$	$- 2610$			$- 2025$
ab^4cf^2	$- 700$	$+ 1240$	$+ 300$			$+ 840$
ab^6f^2		$- 112$				$- 112$

Terms of $L_2 f^2$ involving e^1.

$-ef^2\Delta''\frac{1}{2}C$	viz.	$5a\partial_b$	$4b_{dc}$	$3c\partial_d$		
			$-\frac{1}{2}C$			
a^4cef^2		$- 3500$		$- 1500$	$=$	$- 5000$
$a^3b^2df^2$		$+ 4800$	$- 2800$			$+ 2000$

Calculation of L_3 (gives $L_3 = 0$).

$$L_2 = A + Be,$$

$-\frac{1}{3}B =$	
a^4c	$+ 1666\frac{2}{3}$
a^3b^2	$- 666\frac{2}{3}$

Terms of $L_3 f^3$.

$-f^3\nabla''\frac{1}{3}B$	viz. $\underbrace{5a\partial_b \quad 4b\partial_c \quad 3c\partial_d}_{-\frac{1}{3}B}$		
a^4bf^3	$- 6666\frac{2}{3}$	$+ 6666\frac{2}{3}$ =	0

and the required coefficient of θ^2 is

$$L_0 + L_1 f + L_2 f^2.$$

All the coefficients were calculated in this manner, except the last coefficient, which was deduced from the known value of the discriminant for the standard form. And we have thus the complete expression of the equation of differences for the general quintic equation under the denumerate form $(a, b, c, d, e, f \between v, 1)^5 = 0$, viz.

Equation of differences for $(a, b, c, d, e\ f \between v, 1)^5 = 0$ is

θ^{10}	θ^9	θ^8	θ^7	θ^6	θ^5	θ^4
$a^8 \times$	$a^6 \times$	$a^4 \times$	$a^2 \times$			
$+1$	$ac + 10$	$a^3e + 10$	$a^4ce + 50$	$a^6df + 200$	$a^6f^2 + 625$	$a^5cf^2 + 1750$
	$b^2 - 4$	$a^2bd - 4$	$a^4d^2 + 25$	$a^6e^2 - 95$	$a^5bef - 250$	$a^5def - 950$
		$a^2c^2 + 39$	$a^3b^2e - 20$	$a^5bcf - 120$	$a^5cdf + 400$	$a^5e^3 + 40$
		$ab^2c - 30$	$a^3bcd - 50$	$a^5bde + 36$	$a^5ce^2 - 360$	$a^4b^2f^2 - 700$
		$b^4 + 6$	$a^3c^3 + 80$	$a^5c^2e + 124$	$a^5d^2e + 260$	$a^4bcef - 130$
			$a^2b^3d + 16$	$a^5cd^2 + 92$	$a^4b^2df - 110$	$a^4bd^2f + 380$
			$a^2b^2c^2 - 81$	$a^4b^3f + 32$	$a^4b^2e^2 + 169$	$a^4bde^2 + 142$
			$ab^4c + 30$	$a^4b^2ce - 98$	$a^4bc^2f - 240$	$a^4c^2df + 240$
			$b^6 - 4$	$a^4b^2d^2 - 44$	$a^4bcde - 104$	$a^4c^2e^2 - 522$
				$a^4bc^2d - 160$	$a^4bd^3 - 104$	$a^4cd^2e + 708$
				$a^4c^4 + 95$	$a^4c^3e + 196$	$a^4d^4 - 53$
				$a^3b^4e + 18$	$a^4c^2d^2 + 118$	$a^3b^3ef + 128$
				$a^3b^3cd + 116$	$a^3b^3cf + 150$	$a^3b^2ce^2 + 388$
				$a^3b^2e^3 - 104$	$a^3b^3de - 10$	$a^3b^2cdf - 394$
				$a^2b^5d - 20$	$a^3b^2c^2e - 180$	$a^3b^2d^2e - 378$
				$a^2b^4c^2 + 45$	$a^3b^2cd^2 + 20$	$a^3bc^3f - 144$
				$ab^6c - 10$	$a^3bc^3d - 220$	$a^3bc^2de - 480$
				$b^8 + 1$	$a^3c^5 + 66$	$a^3bcd^3 - 156$
					$a^2b^5f - 24$	$a^3c^4e + 194$
					$a^2b^4ce + 66$	$a^3c^3d^2 + 52$
					$a^2b^4d^2 - 3$	$a^2b^4df + 66$
					$a^2b^3c^2d + 192$	$a^2b^4e^2 - 84$
					$a^2b^3e^2 - 66$	$a^2b^3c^2f + 194$
					$ab^6e - 8$	$a^2b^3cde + 330$
					$ab^5cd - 66$	$a^2b^3d^2 + 92$
					$ab^4c^3 + 24$	$a^2b^2c^3e - 152$
					$b^7d + 8$	$a^2b^2c^2d^2 + 174$
					$b^6c^2 - 3$	$a^2bc^4d - 140$
						$a^2c^6 + 25$
						$ab^5cf - 70$
						$ab^5de - 42$
						$ab^4c^2e + 32$
						$ab^4cd^2 - 144$
						$ab^3c^3d + 100$
						$ab^2c^5 - 18$
						$b^7f + 8$
						$b^6ce - 2$
						$b^6d^2 + 22$
						$b^5c^2d - 16$
						$b^4c^4 + 3$

$(\mathfrak{X}\theta,\ 1)^{10} = 0$, viz. the function in θ is

θ^3		θ^2		θ		θ^0	
a^5ef^2	-3750	a^4cef^2	-5000	a^4df^3	-6250	a^4f^4	$+3125$
a^4bdf^2	$+1500$	$a^4d^2f^2$	$+1875$	$a^4e^2f^2$	$+5000$	a^2bef^3	-2500
a^4be^2f	$+1500$	a^4de^2f	$+1000$	a^3bcf^3	$+3750$	a^3cdf^3	-3750
$a^4c^2f^2$	$+2500$	a^4e^4	$+400$	a^3bdef^2	-250	$a^3ce^2f^2$	$+2000$
a^4cdef	-2150	$a^3b^2ef^2$	$+2000$	a^3be^2f	-2000	$a^3d^2ef^2$	$+2250$
a^4ce^3	-80	a^3bcdf^2	-250	$a^3c^2ef^2$	-3750	a^3de^3f	-1600
a^4d^3f	$+700$	a^3bce^2f	$+1400$	$a^3cd^2f^2$	$+3000$	a^3e^5	$+256$
$a^4d^2e^2$	$+570$	a^3bd^2ef	-1550	a^3cde^2f	$+200$	$a^2b^2df^3$	$+2000$
$a^3b^4cf^2$	-2450	a^3bde^3	-840	a^3ce^4	$+320$	$a^2b^2e^2f^2$	-50
a^3b^2def	-40	$a^3c^3f^2$	$+1750$	a^3d^3ef	-450	$a^2bc^2f^2$	$+2250$
$a^3b^2e^3$	-118	a^3c^2def	-1650	$a^3d^2e^3$	-40	a^2bcdef^2	-2050
a^3bc^3ef	$+290$	$a^3c^2e^3$	-360	$a^2b^2f^3$	-1000	a^2bce^2f	$+160$
a^3bcd^2f	-400	a^3cd^4f	$+600$	$a^2b^2cef^2$	$+1950$	$a^2bd^2f^2$	-900
a^3bcde^2	-158	$a^3cd^2e^2$	$+960$	$a^2b^2d^2f^2$	-1150	$a^2bd^2e^2f$	$+1020$
a^3bd^3e	-596	a^3d^4e	-270	$a^2b^2de^2f$	$+1170$	a^2bde^4	-192
a^3c^3df	$+80$	$a^3b^2df^2$	-200	$a^2b^2e^4$	$+72$	$a^2c^3ef^2$	-900
$a^3c^3e^2$	-308	$a^3b^2e^2f$	-640	$a^2bc^3df^2$	-2100	$a^2c^2d^2f^2$	$+825$
$a^3c^2d^2e$	$+612$	$a^3b^2c^2f^2$	-2025	$a^2bc^3e^2f$	$+1380$	$a^2c^2de^2f$	$+560$
a^3cd^4	-102	a^3b^2cdef	$+1740$	a^2bcd^2ef	-550	$a^2c^2e^4$	-128
$a^3b^4f^2$	$+490$	$a^3b^2ce^3$	$+400$	a^2bcde^3	-504	a^2cd^3ef	-630
a^3b^2cef	$+180$	$a^3b^2d^3f$	$+70$	a^2bd^4f	$+180$	$a^2cd^2e^2$	$+144$
$a^3b^2d^2f$	$+112$	$a^3b^2d^2e^2$	$+275$	$a^2bd^3e^2$	$+138$	a^2d^3f	$+108$
$a^3b^2de^2$	$+92$	a^3bc^3ef	$+290$	$a^2c^4f^2$	$+675$	$a^2d^4e^2$	-27
$a^3b^2c^2df$	$+86$	$a^3bc^3d^2f$	-420	a^2c^3def	-330	ab^3cf^3	-1600
$a^3b^2c^2e^2$	$+388$	$a^3bc^3de^2$	-390	$a^2c^3e^3$	-224	ab^3def^2	$+160$
$a^3b^2cd^2e$	$+150$	a^3bcd^2e	-240	$a^2c^2d^2f$	$+60$	ab^3e^2f	-36
$a^3b^3d^4$	$+160$	a^3bd^5	$+108$	$a^2c^2d^2e^2$	$+434$	$ab^2c^2ef^2$	$+1020$
a^3bc^4f	-48	a^3c^4df	$+40$	a^2cd^4e	-198	$ab^2cd^2f^2$	$+560$
a^3bc^3de	-504	$a^3c^4e^2$	-15	a^2d^6	$+27$	ab^2cde^3f	-746
a^3c^3e	$+106$	$a^3c^3d^2e$	$+140$	ab^4ef^2	-240	ab^2ce^4	$+144$
$a^3c^4d^2$	-7	$a^3c^2d^4$	-45	ab^3cdf^2	$+1320$	ab^2d^3ef	$+24$
ab^5ef	-68	ab^4cf^2	$+840$	ab^3ce^2f	-1230	$ab^2d^2e^3$	-6
ab^4cdf	-86	ab^4def	-200	ab^3d^2ef	-12	abc^3df^2	-630
ab^4ce^2	-178	ab^4e^3	-48	ab^3de^3	$+18$	abc^3e^2f	$+24$
ab^4d^2e	-54	ab^3c^2ef	-310	$ab^3c^3f^2$	-450	abc^2d^2ef	$+356$
ab^3c^3de	$+234$	ab^3cd^2f	-50	ab^3c^3def	$+594$	abc^2de^3	-80
ab^3c^3f	$+34$	ab^3cde^2	-178	$ab^3c^2e^3$	$+282$	$abcd^4f$	-72
ab^3cd^3	-148	ab^3d^3e	-14	ab^3cd^2f	-154	$abcd^3e^2$	$+18$
ab^3c^4e	-56	ab^2c^3df	$+130$	$ab^3cd^2e^2$	-114	ac^5f^2	$+108$
$ab^3c^3d^2$	$+112$	$ab^2c^3e^2$	$+112$	ab^3d^4e	$+6$	ac^4def	-72
abc^5d	-34	$ab^2c^3d^2e$	$+246$	abc^4ef	-72	ac^4e^2	$+16$
ac^7	$+4$	ab^3cd^4	-78	abc^3d^2f	$+24$	ac^3d^3f	$+16$
b^6df	$+16$	abc^5f	-24	abc^3de^2	-186	$ac^3d^2e^2$	-4
b^6e^2	$+26$	abc^4de	-164	abc^2d^2e	$+116$	b^5f^3	$+256$
b^5c^2f	-6	abc^3d^3	$+52$	$abcd^5$	-18	b^4cef^2	-192
b^5cde	-30	ac^6e	$+24$	ac^5e^2	$+36$	$b^4d^2f^2$	-128
b^5d^3	$+28$	ac^5d^2	-8	ac^4d^2e	-24	b^4de^2f	$+144$
b^4c^3e	$+8$	b^6f^2	-112	ac^3d^4	$+4$	b^4e^4	-27
$b^4c^2d^2$	-24	b^5cef	$+56$	b^5df^2	-192	$b^3c^3df^2$	$+144$
b^3c^4d	$+8$	b^5d^2f	$+24$	b^5e^2f	$+216$	$b^3c^2e^2f$	-6
b^3c^6	-1	b^5de^2	$+48$	$b^4ce^2f^2$	$+72$	b^3cd^2ef	-80
		b^4c^2df	-32	b^4cdef	-120	b^3cde^3	$+18$
		$b^4c^2e^2$	-25	b^4ce^3	-54	b^3d^4f	$+16$
		b^4cd^2e	-54	b^4d^3f	$+32$	$b^3d^3e^2$	-4
		b^4d^4	$+17$	$b^4d^2e^2$	$+18$	$b^3c^4f^2$	-27
		b^3c^4f	$+6$	b^3c^3ef	$+18$	b^3c^3def	$+18$
		b^3c^3de	$+38$	$b^3c^2d^2f$	-6	$b^3c^3e^3$	-4
		$b^3c^2d^3$	-12	$b^3c^2de^2$	$+42$	$b^3c^2d^3f$	-4
		b^3c^6e	-6	b^3cd^3e	-26	$b^2c^3d^2e^2$	$+1$
		$b^2c^4d^2$	$+2$	b^3d^5	$+4$		
				$b^2c^4e^2$	-9		
				$b^2c^3d^2e$	$+6$		
				$b^2c^2d^4$	-1		

$\mathfrak{X}\theta,\ 1)^{10}$

It may be remarked, that if ω is an imaginary cube root of unity, then the roots of the equation $(1, 1, 1, 1, 1, 1 \between v, 1)^6 = 0$ are $-1, \omega, \omega^2, -\omega, -\omega^2$; the differences of the roots are

$$\begin{array}{l} -1-\omega,\ -1-\omega^2,\ -1+\omega,\ -1+\omega^2,\ \omega-\omega^2,\ 2\omega,\ \omega+\omega^2,\ \omega^2+\omega,\ 2\omega^2,\ -\omega+\omega^2 \\ = \quad \omega^2\ ,\ \ \omega\ ,\ -1+\omega,\ -1+\omega^2,\ \omega-\omega^2,\ 2\omega,\ -1\ ,\ -1\ ,\ 2\omega^2,\ -\omega+\omega^2, \end{array}$$

and the squares of the differences are

$$\omega\ ,\ \omega^2\ ,\ -3\omega\ ,\ -3\omega^2\ ,\ -3\ ,\ 4\omega^2,\ 1\ ,\ 1\ ,\ 4\omega\ ,\ -3\ ,$$

from which the equation of differences is found to be

$$(\theta^2+\theta+1)(\theta^2-3\theta+9)(\theta^2+4\theta+16)(\theta^2-2\theta+1)(\theta^2+6\theta+9)=0;$$

or multiplying out, it is

$$(1, 6, 21, 46, 108, 546, 493, -1410, -567, -540, +1296 \between \theta, 1)^{10} = 0;$$

which is what the preceding expression of the equation of differences becomes upon writing therein $a=b=c=d=e=f=1$. Moreover, upon passing (as will presently be done) to the standard form, and then writing $a=b=c=d=e=f=1$, all the coefficients (except the first coefficient, which is equal to unity) should become equal to zero; these two tests afford a complete verification of the result.

The following corrections have to be made in Waring's result, as given by himself and Lagrange (Waring, *Phil. Trans.* 1762).

Waring, *Meditationes Algebraicæ*, p. 85:

for $+\ 169\, q^3 s$ *read* $+196\, q^3 s$ (in coefficient w^6).

Lagrange, *Equations Numériques*, p. 108:

for $+1200\, CE$ *read* $+200\, CE$ (in d)
for $-\ 169\, B^3 D$ *read* $-196\, B^3 D$ (in e)
for $-\ 25\, B^6$ *read* $+\ 25\, B^6$ (in f)
for $+\ 27\, C^4 D^2$ *read* $-\ 27\, C^4 D^2$ (in k).

It may be noticed, that if in the coefficients of the several powers of θ (as they are written down in the columns, without regarding the power of a which multiplies the entire column), we attend only to the terms independent of a, we have the series

$$\begin{array}{llllllll} 1, & b^2-4, & b^4+6, & b^6-4, & b^8+1, & b^7d+8, & b^7f\ +\ 8, & \&c. \\ & & & & & b^6c^2-3, & b^6ce\ -\ 2 & \\ & & & & & & b^6d^2\ +22 & \\ & & & & & & b^5c^2d-16 & \\ & & & & & & b^4c^4\ +\ 3, & \end{array}$$

the law of the first terms of which, up to the term $+1b^8$, is obvious; but the term $+1b^8$, which is the last term of this initial series, is also the first term of a terminal series, the terms of which are deduced from the coefficients in the equation of differences for the quartic equation $(a, b, c, d\ e \between v, 1)^4 = 0$, viz. these coefficients are

$a^6\times$	$a^4\times$	$a^2\times$	&c.
$+1$	$ac+8$	a^3e+8	
	b^2-3	a^2bd-2	
		a^2c^2+22	
		ab^2c-16	
		$b^4+3;$	

and by writing b, c, d, e, f in the place of a, b, c, d, e respectively, and multiplying by b^2, we have the above-mentioned series,

$$b^8+1,\quad b^7d+8,\ \&c.$$
$$b^6c^2-3.$$

It is easy to see, *à priori*, in the case of an equation of any order, that this property holds good.

Passing now to the standard forms:

For the quadric $(a, b, c\between v, 1)^2=0$, the equation of differences is, $0=$

$a^2\times$	$4\times$
$+1$	$ac+1$ b^2-1

$(\between\theta, 1)$.

For the cubic equation $(a, b, c, d\between v, 1)^3=0$, the equation of differences is, $0=$

$a^4\times$	$18a^2\times$	$81\times$	$27\times$
$+1$	$ac+1$ b^2-1	a^2c^2+1 ab^2c-2 b^4+1	a^2d^2+1 $abcd-6$ ac^3+4 b^3d+4 b^2c^2-3

$(\between\theta, 1)^2$.

For the quartic equation $(a, b, c, d, e\between v, 1)^4=0$, the equation of differences is, $0=$

$a^6\times$	$48a^4\times$	$8a^2\times$	$32\times$	$16\times$	$1152\times$	$256\times$
$+1$	$ac+1$ b^2-1	a^3e+1 a^2bd-4 a^2c^2+99 ab^2c-192 b^4+96	a^4ce+3 a^4d^2+13 a^3b^2e-3 $a^3bcd-90$ a^3c^3+189 a^2b^3d+64 $a^2b^2c^2-432$ ab^4c+384 b^6-128	a^4e^2-7 $a^3bde+56$ a^3c^2e+54 a^3cd^2+288 $a^2b^2ce-192$ $a^2b^2d^2-400$ $a^2bc^2d-1944$ a^2c^4+1377 ab^4e+96 $ab^3cd+3648$ ab^2c^3-2592 $b^5d-1536$ b^4c^2+1152	a^3ce^2-1 a^3d^2e+3 $a^2b^2e^2+1$ $a^2bcde-10$ a^2bd^3-12 a^2c^3e+6 $a^2c^2d^2+9$ ab^3de+4 ab^2c^2e-3 ab^2cd^2+56 abc^3d-78 ac^5+27 b^4d^2-32 b^3c^2d+48 b^2c^2-18	a^2e^3+1 a^2bde^2-12 $a^2c^2e^2-18$ a^2cd^2e+54 a^2d^4-27 ab^2ce^2+54 ab^2d^2e-6 $abc^3de-180$ $abcd^3+108$ ac^4e+81 ac^3d^2-54 b^4e^2-27 $b^3cde+108$ b^3d^3-64 b^2c^2e-54 $b^2c^2d^2+36$

$(\between\theta, 1)^6$.

For the quintic equation $(a, b, c, d, e, f \between v, 1)^5 = 0$, the equation of differences is,

θ^{10}	θ^{9}	θ^{8}	θ^{7}	θ^{6}	θ^{5}	θ^{4}
$a^8 \times$	$100\,a^6 \times$	$50\,a^4 \times$	$2500\,a^2 \times$	$125 \times$	$625 \times$	$2500 \times$
$+1$	$ac + 1$	$a^3e + 1$	$a^4ce + 1$	$a^8df + 16$	$a^6f^2 + 1$	$a^5cf^2 + 7$
	$b^2 - 1$	$a^2bd - 4$	$a^4d^2 + 1$	$a^8e^2 - 19$	$a^5bef - 10$	$a^5def - 19$
		$a^2c^2 + 78$	$a^3b^2e - 1$	$a^5bcf - 48$	$a^5cdf + 64$	$a^5e^3 + 2$
		$ab^2c - 150$	$a^3bcd - 10$	$a^5bde + 72$	$a^5ce^2 - 144$	$a^5b^2f^2 - 7$
		$b^4 + 75$	$a^3c^3 + 32$	$a^5e^2e + 496$	$a^5d^2e + 208$	$a^4bcef - 13$
			$a^2b^3d + 8$	$a^5cd^2 + 736$	$a^4b^2df - 44$	$a^4bd^2f + 76$
			$a^2b^2c^2 - 81$	$a^4b^4f + 32$	$a^4b^2e^2 + 169$	$a^4bde^2 + 71$
			$ab^4c + 75$	$a^4b^2ce - 980$	$a^4bc^2f - 192$	$a^4c^2df + 96$
			$b^6 - 25$	$a^4b^2d^2 - 880$	$a^4bcde - 416$	$a^4c^2e^2 - 522$
				$a^4bc^2d - 6400$	$a^4bd^3 - 832$	$a^4cd^2e + 1416$
				$a^4c^4 + 7600$	$a^4c^3e + 1568$	$a^4d^4 - 212$
				$a^3b^4e + 450$	$a^4c^2d^2 + 1888$	$a^3b^2ef + 32$
				$a^3b^3cd + 11600$	$a^3b^3cf + 300$	$a^3b^2ce^2 + 970$
				$a^3b^2c^3 - 20800$	$a^2b^3de - 100$	$a^3b^3cdf - 394$
				$a^3b^3d - 5000$	$a^2b^2c^2e - 3600$	$a^3b^2d^2e - 1890$
				$a^2b^4c^2 + 22500$	$a^3b^2cd^2 + 800$	$a^3bc^3f - 288$
				$ab^6c - 12500$	$a^3bc^3d - 17600$	$a^3bc^2de - 4800$
				$b^8 + 3125$	$a^3c^5 + 10560$	$a^3bcd^3 - 3120$
					$a^2b^5f - 120$	$a^3c^4e + 3880$
					$a^2b^4ce + 3300$	$a^3c^3d^2 + 2080$
					$a^2b^4d^2 - 300$	$a^2b^4df + 165$
					$a^2b^3c^2d + 38400$	$a^2b^4e^2 - 525$
					$a^2b^2c^4 - 26400$	$a^2b^3c^2f + 970$
					$ab^6e - 1000$	$a^2b^3cde + 8250$
					$ab^5cd - 33000$	$a^2b^3d^3 + 4600$
					$ab^4c^3 + 24000$	$a^2b^3c^2e - 7600$
					$b^7d + 10000$	$a^2b^2c^2d^2 + 17400$
					$b^6c^2 - 7500$	$a^2bc^4d - 28000$
						$a^2c^6 + 10000$
						$ab^5cf - 875$
						$ab^5de - 2625$
						$ab^4c^2e + 4000$
						$ab^4cd^2 - 36000$
						$ab^3c^3d + 50000$
						$ab^2c^5 - 18000$
						$b^7f + 250$
						$b^6ce - 625$
						$b^6d^2 + 13750$
						$b^5c^2d - 20000$
						$b^4c^4 + 7500$
$+1$	± 1	± 154	± 117	± 46627	± 91258	± 125515

$0 = (\mathfrak{X}\theta,\ 1)^{10} = 0$, viz. the function in θ is

θ^3	θ^2	θ	θ^0
6250 ×	62500 ×	62500 ×	3125 ×
a^5ef^2 − 3	a^4cef^2 − 4	a^4df^3 − 1	a^4f^4 + 1
a^4bdf^2 + 12	$a^4d^2f^2$ + 3	$a^4e^2f^2$ + 2	a^3bef^3 − 20
a^4be^2f + 30	a^4de^2f + 4	a^3bcf^3 + 3	a^3cdf^3 − 120
$a^4c^2f^2$ + 40	a^4e^4 + 4	a^2bdef^2 − 1	$a^3ce^2f^2$ + 160
a^4cdef − 172	$a^3b^2ef^2$ + 4	a^3be^3f − 20	$a^3d^2ef^2$ + 360
a^4ce^3 − 16	a^3bcdf^2 − 2	$a^3c^2ef^2$ − 30	a^3de^3f − 640
a^4d^3f + 112	a^3bce^2f + 28	$a^3cd^2f^2$ + 48	a^3e^5 + 256
$a^4d^2e^2$ + 228	a^3bd^2ef − 62	a^3cde^2f + 8	$a^2b^3df^3$ + 160
$a^3b^2cf^2$ − 98	a^3bde^2 − 84	a^3ce^4 + 32	$a^2b^2e^2f^2$ − 10
a^3b^2def − 8	$a^2c^3f^2$ + 28	a^3d^3ef − 36	$a^2bc^2f^3$ + 360
$a^3b^2e^3$ − 59	a^3c^2def − 132	$a^3d^2e^2$ − 8	a^2bcdef^2 − 1640
a^3bc^2ef + 116	$a^3c^2e^3$ − 72	$a^2d^3f^3$ − 2	a^2bce^2f + 320
a^3bcd^2f − 320	a^3cd^3f + 96	$a^2b^2cef^2$ + 39	$a^2bd^2f^2$ − 1440
a^3bcde^2 − 316	$a^3cd^2e^2$ + 384	$a^2b^2d^2f^2$ − 46	$a^2bd^2e^2f$ + 4080
a^3bd^3e − 2384	a^3d^4e − 216	$a^2b^2de^2f$ + 117	a^2bde^4 − 1920
a^2c^3df + 128	$a^2b^3df^2$ − 4	$a^2b^2e^4$ + 18	$a^2c^2ef^2$ − 1440
$a^2c^3e^2$ − 1232	$a^2b^3e^2f$ − 32	$a^2bc^2df^2$ − 168	$a^2c^2d^2f^2$ + 2640
$a^2c^2d^2e$ + 4896	$a^2b^2c^2f^2$ − 81	$a^2bc^2e^2f$ + 276	$a^2c^2de^2f$ + 4480
a^2cd^4 − 1632	a^2b^2cdef + 348	a^2bcd^2ef − 220	$a^2c^2e^4$ − 2560
$a^2b^4f^2$ + 49	$a^2b^2ce^2$ + 200	a^2bcde^2 − 504	a^2cd^3ef − 10080
a^2b^2cef + 180	$a^2b^3d^2f$ + 28	a^2bd^4f + 144	$a^2cd^2e^3$ + 5760
$a^2b^3d^2f$ + 224	$a^2b^2d^2e^2$ + 275	$a^2bd^3e^2$ + 276	a^2d^5f + 3456
$a^2b^2de^2$ + 460	a^2bc^3ef + 116	$a^2c^4f^2$ + 108	$a^2d^4e^2$ − 2160
$a^2b^2c^2df$ + 344	$a^2bc^2d^2f$ − 336	a^2c^3def − 264	ab^2cf^3 − 640
$a^2b^2c^2e^2$ + 3880	$a^2bc^2de^2$ − 780	$a^2c^3e^2$ − 448	ab^3def^2 + 320
$a^2b^2cd^2e$ + 3000	a^2bcd^3e − 960	$a^2c^2d^2f$ + 96	ab^3e^3f − 180
$a^2b^2d^4$ + 6400	a^2bd^5 + 864	$a^2c^2d^2e^2$ + 1736	$ab^2c^2ef^2$ + 4080
a^2bc^4f − 384	a^3c^4df + 64	a^2cd^4e − 1584	$ab^2cd^2f^2$ + 4480
a^2bc^3de − 20160	$a^2c^4e^2$ − 60	a^2d^6 + 432	ab^2cde^2f − 14920
a^2c^5e + 8480	$a^2c^3d^2e$ + 1120	ab^4ef^2 − 12	ab^2ce^4 + 7200
$a^2c^4d^2$ − 1120	$a^2c^2d^4$ − 720	ab^3cdf^2 + 264	ab^2d^2ef + 960
ab^5ef − 170	ab^4cf^2 + 84	ab^3ce^2f − 615	$ab^2d^2e^3$ − 600
ab^4cdf − 860	ab^4def − 100	ab^3d^2ef − 12	abc^3df^2 − 10080
ab^4ce^2 − 4450	ab^4e^3 − 60	ab^3de^2 + 45	abc^3e^2f + 960
ab^4d^2e − 2700	ab^3c^2ef − 310	$ab^2c^2f^2$ − 180	abc^2d^2ef + 28480
ab^3c^2de + 23400	ab^3cd^2f − 100	ab^2c^2def + 1188	abc^2de^3 − 16000
ab^3c^3f + 680	ab^3cde^2 − 890	$ab^2c^2e^3$ + 1410	$abcd^4f$ − 11520
ab^3cd^3 − 29600	ab^3d^2e − 140	ab^2cd^3f − 616	$abcd^3e^2$ + 7200
ab^2c^4e − 11200	ab^2c^2df + 520	$ab^2cd^2e^2$ − 1140	ac^3f^2 + 3456
$ab^2c^2d^2$ + 44800	$ab^2c^2e^2$ + 1120	ab^2d^4e + 120	ac^4def − 11520
abc^3d − 27200	$ab^2c^2d^2e$ + 4920	abc^4ef − 288	ac^4e^3 + 6400
ac^7 + 6400	ab^2cd^4 − 3120	abc^3d^2f + 192	ac^3d^3f + 5120
b^6df + 400	abc^5f − 192	abc^3de^2 − 3720	$ac^3d^2e^2$ − 3200
b^6e^2 + 1625	abc^4de − 6560	abc^2d^3e + 4640	b^3f^3 + 256
b^5c^2f − 300	abc^3d^3 + 4160	$abcd^5$ − 1440	b^4cef^2 − 1920
b^5cde − 7500	ac^6e + 1920	ac^5e^2 + 1440	$b^4d^2f^2$ − 2560
b^5d^3 + 14000	ac^5d^2 − 1280	ac^4d^2e − 1920	b^4de^2f + 7200
b^4c^3e + 4000	b^4f^2 − 28	ac^3d^4 + 640	b^4e^4 − 3375
$b^4c^2d^2$ − 24000	b^3cef + 140	b^3df^2 − 96	$b^3c^2df^2$ + 5760
b^3c^4d + 16000	b^3d^2f + 120	b^3c^2f + 270	$b^3c^2e^2f$ − 600
b^2c^6 − 4000	b^3de^2 + 600	$b^3c^2f^2$ + 72	b^3cd^2ef − 16000
	b^3c^3df − 320	b^3cdef − 600	b^3cde^3 + 9000
	$b^3c^2e^2$ − 625	b^3ce^3 − 675	b^3d^4f + 6400
	b^3cd^2e − 2700	b^3d^3f + 320	$b^3d^3e^2$ − 4000
	b^3d^4 + 1700	$b^3d^2e^2$ + 450	$b^2c^3f^2$ − 2160
	b^2c^4f + 120	b^2c^2ef + 180	b^2c^2def + 7200
	b^2c^3de + 3800	$b^2c^2d^2f$ − 120	$b^2c^2e^3$ − 4000
	$b^2c^2d^3$ − 2400	$b^2c^2de^2$ + 2100	$b^2c^2d^3f$ − 3200
	b^2c^3e − 1200	b^2cd^2e − 2600	$b^2c^2d^2e^2$ + 2000
	$b^2c^4d^2$ + 800	b^2d^5 + 800	
		$b^2c^4e^2$ − 900	
		$b^2c^3d^2e$ + 1200	
		$b^2c^2d^4$ − 400	
± 130884	± 23570	± 18666	± 128505

$\mathfrak{X}\theta,\ 1)^{10}$

[where in each column after the first the sum of the positive coefficients is equal to that of the negative coefficients, and I have inserted at the foot this sum with the sign +. A single coefficient -7500 in place of -3750 has been corrected.]

The coefficients in the preceding equations of differences are functions of the seminvariants of the quantics to which they belong; for instance, in the case of the quartic, the coefficient of θ^4 is

$$8a^2\{a^2(ae-4bd+3c^2)+96(ac-b^2)^2\},$$

that of θ^3 is

$$32\{-13a^3(ace-ad^2-b^2e+2bcd-c^3)+16a^2(ac-b^2)(ae-4bd+3c^2)+128(ac-b^2)^3\},$$

and so for the other coefficients; and by replacing each seminvariant by the covariant to which it belongs, we pass from the solution of the original problem of finding the equation for $\theta=(\alpha-\beta)^2$, to that of the problem of finding the equation for

$$\theta=\frac{(\alpha-\beta)^2}{(x-\alpha y)^2(x-\beta y)^2}.$$

The results are as follows:

For the quadric $(a, b, c \between x, y)^2$, the equation in θ is, $0=$

$$(U^2, 4\square \between \theta, 1),$$

where U is the quadric, $\square$ the discriminant.

For the cubic $(a, b, c, d \between x, y)^3$, the equation in θ is, $0=$

$$(U^4, 18U^2H, 81H^2, 27\square \between \theta, 1)^2,$$

where U is the cubic, H the Hessian, $\square$ the discriminant.

For the quartic $(a, b, c, d, e \between x, y)^4$, the equation in θ is, $0=$

$$\left\{\begin{array}{l} U^6, \\ 48U^4H, \\ 8U^2(U^2I+96H^2), \\ 32(-13U^3J+16U^3HI+128H^3), \\ 16(-\ 7U^2I^2-288UHJ+384H^2I), \\ 1152(-\ 3UIJ+2HI^3), \\ 256(\ \ I^3\ -27J^2), \end{array}\right\} \between \theta, 1)^6,$$

where U is the quartic, H the Hessian, I and J the quadrinvariant and the cubinvariant respectively.

For the quintic $(a, b, c, d, e, f \between x, y)^5$, the equation in θ, as far as it can be expressed in terms of known covariants, is, $0 =$

$$\left\{\begin{array}{ll} & U^8, \\ 100 & U^6 \text{ (Tab. No. 15)}, \\ 50 & U^4 [U^2 \text{ (Tab. No. 14)} + 75 \text{ (Tab. No. 15)}^2], \\ & \vdots \\ 3125 & \text{Discriminant } (= \text{Tab. No. 26}) \end{array}\right\} \between \theta, 1)^{10},$$

where the Tables referred to are those in my Second Memoir on Quantics, [141; for the completed equation see *post* p. 261].

The form of the preceding results may be modified by writing $\theta = \vartheta \div U^2$; we have thus the equations for

$$\vartheta = a^2 (\alpha - \beta)^2 (x - \gamma y)^2 (x - \delta y)^2 \ldots;$$

thus for example, in the case of the cubic $(a, b, c, d \between x, y)^3$, the equation for $\vartheta\, [= a^2 (\alpha - \beta)^2 (x - \gamma y)^2]$ is

$$0 = (1, 18H, 81H^2, 27\square U^2 \between \vartheta, 1)^2;$$

this equation may be written

$$(\vartheta + 9H)^2 \vartheta + 27\square U^2 = 0;$$

or putting $v = \sqrt{\vartheta}$, we have

$$v^3 + 9Hv + U\sqrt{-27\square} = 0,$$

an equation the roots of which are

$$a(\alpha - \beta)(x - \gamma y), \qquad a(\beta - \gamma)(x - \alpha y), \qquad a(\gamma - \alpha)(x - \beta y),$$

and which leads to the formula, given in my fifth Memoir on Quantics, [156], for the solution of a cubic equation. But this decomposition of the equation in ϑ is peculiar to the cubic.

The equation of differences for an equation of any order may be found by the following entirely distinct method. Let the proposed equation $(* \between v, 1)^n = 0$, be for shortness represented by $\phi v = 0$, and let x, y be any two distinct roots; we have not only $\phi x = 0$, $\phi y = 0$, but also $\phi x + \phi y = 0$, $\dfrac{\phi x - \phi y}{x - y} = 0$. Writing $\theta = (x - y)^2$, $s = x + y$, we have

$$x = \tfrac{1}{2}(s + \sqrt{\theta}), \qquad y = \tfrac{1}{2}(s - \sqrt{\theta}),$$

values which are to be substituted for x, y in the equations

$$\phi x + \phi y = 0, \qquad \frac{\phi x - \phi y}{x - y} = 0.$$

We have thus two equations rational in s and θ, and the elimination between them of the quantity s leads to the required equation in θ. But it is proper to modify the form of the system; in fact the two equations are, as regards s, the first of them of the degree n, the second of the degree $n-1$; but if we write

$$n(\phi x+\phi y)-\frac{(x+y)(\phi x-\phi y)}{x-y}=0, \qquad \frac{\phi x-\phi y}{x-y}=0,$$

then each of the equations will be of the same degree $n-1$ in s.

For instance, let $\phi v=(a,\ b,\ c,\ d \between v,\ 1)^3$, then $x=\frac{1}{2}(s+\sqrt{\theta})$, $y=\frac{1}{2}(s+\sqrt{\theta})$; the equations $\phi x+\phi y=0$, $\dfrac{\phi x-\phi y}{x-y}=0$, are

$$\begin{aligned} s^3.a+3s^2.2b+3s(4c+a\theta)+8d+6b\theta&=0,\\ 3s^2.a+3s\ .4b+\quad 12c+a\theta\qquad\qquad&=0;\end{aligned}$$

and multiplying the first equation by 3 and the second by $-s$, adding and dividing by 2, we have an equation

$$s^2.3b+s(12c+4a\theta)+12d+9b\theta=0.$$

The second equation and this equation may be written

$$\begin{aligned} &(3a,\ 12b\qquad\quad 12c+a\theta \between s,\ 1)^2=0,\\ &(3b,\ 12c+4a\theta,\ \ 12d+9b\theta \between s,\ 1)^2=0,\end{aligned}$$

and the elimination of s from these equations gives the required equation in θ. The result may be obtained under either of the two forms,

$$\{a^2\theta^2+(15ac-27b^2)\theta-36(bd-c^2)\}\{a^2\theta+\ 3(ac-b^2)\}+3\{2ab\theta+3(ad-bc)\}^2=0$$

and

$$\{4a^2\theta^2+(24ac-27b^2)\theta-36(bd-c^2)\}\{a^2\theta+12(ac-b^2)\}+3\{\ ab\theta+6(ad-bc)\}^2=0,$$

the expansions of which respectively coincide with the before-mentioned result.

In the case of the quartic equation $\phi v=(a,\ b,\ c,\ d \between v,\ 1)^4=0$, we have

$$\begin{aligned} s^4.a+4s^3.2b+6s^2(\ 4c+a\theta)+4s(8d+6b\theta)+16e+24c\theta+a\theta^2&=0,\\ 4s^3.a+6s^2.4b+4s\ (12c+a\theta)+\quad 32d+8b\theta\qquad\qquad&=0,\end{aligned}$$

from which we derive another cubic equation; the two cubic equations thus are

$$\begin{aligned} &(4a,\ \ 24b\qquad,\ \ 48c+\ 4a\theta,\ \ 32d+\ 8b\theta\qquad \between s,\ 1)^3=0,\\ &(4b,\ \ 24c+10a\theta,\ \ 48d+44b\theta,\ \ 32e+48c\theta+2a\theta^2 \between s,\ 1)^3=0,\end{aligned}$$

from which, if s be eliminated, we have the equation in θ.

Similarly, for the quintic equation $\phi v = (a, b, c, d, e, f \between v, 1)^5 = 0$, we have

$$s^5.a + 5s^4.2b + 10s^3(4c + a\theta) + 10s^2(8d + 6b\theta) + 5s(16e + 24c\theta + a\theta^2) + 32f + 80d\theta + 10b\theta^2 = 0,$$
$$5s^4.a + 10s^3.4b + 10s^2(12c + a\theta) + 5s(32d + 8b\theta) + (80e + 40c\theta + a\theta^2) = 0,$$

from which we derive another quartic equation; the two equations are

$$(5a, \quad 40b, \quad 120c + 10a\theta, \quad 160d + 40b\theta, \quad 80e + 40c\theta + a\theta^2 \between s, 1)^4 = 0,$$
$$(5b, \quad 40c + 20a\theta, \quad 120d + 130b\theta, \quad 160e + 280c\theta + 12a\theta^2, \quad 80f + 200d\theta + 25b\theta^2 \between s, 1)^4 = 0,$$

from which, if s be eliminated, we have the equation in θ.

But we may apply Bezout's method to the two equations each of the order $n-1$, which result from the equation of the nth order $\phi v = (* \between v, 1)^n = 0$. The process is as follows: Suppose, in general, that s is to be eliminated from the two equations

$$Fs = 0, \quad Gs = 0,$$

each of the order $n-1$; it is only necessary to form the expression

$$\frac{FsGs' - Fs'Gs}{s - s'},$$

which will be a function of s, s' of the form

$$\left(\begin{array}{l|l} a_{0,0}, & a_{0,1}, \ldots a_{0,n-2} \between s, 1)^{n-2}(s', 1)^{n-2} \\ a_{1,0}, & \\ \vdots & \\ a_{n-2,0}, & \end{array} \right.$$

where the coefficients are such that $a_{l,m} = a_{m,l}$; and by equating to zero the determinant formed with these coefficients, we have the result of the elimination.

In the present case, writing for a moment $\phi\frac{1}{2}(s + \sqrt{\theta}) = A$, $\phi\frac{1}{2}(s - \sqrt{\theta}) = B$, and in like manner $\phi\frac{1}{2}(s' + \sqrt{\theta}) = A'$, $\phi\frac{1}{2}(s' - \sqrt{\theta}) = B'$, we have

$$Fs = n(A + B) - \frac{s(A - B)}{\sqrt{\theta}}, \qquad Gs = \frac{A - B}{\sqrt{\theta}},$$

$$Fs' = n(A' + B') - \frac{s'(A' - B')}{\sqrt{\theta}}, \qquad Gs' = \frac{A' - B'}{\sqrt{\theta}},$$

and therefore

$$FsGs' - Fs'Gs = n\frac{(A + B)(A' - B') - (A - B)(A' + B')}{\sqrt{\theta}} - \frac{(s - s')(A - B)(A' - B')}{\theta};$$

or reducing and dividing by $s - s'$,

$$-\frac{FsGs' - Fs'Gs}{s - s'} = 2n\frac{AB' - A'B}{\sqrt{\theta}(s - s')} + \frac{(A - B)(A' - B')}{\theta}.$$

Hence, substituting for A, B, A', B' these values, we have the expression

$$2n\,\frac{\phi\tfrac{1}{2}(s+\sqrt{\theta})\,\phi\tfrac{1}{2}(s'-\sqrt{\theta})-\phi\tfrac{1}{2}(s-\sqrt{\theta})\,\phi\tfrac{1}{2}(s'+\sqrt{\theta})}{(s-s')\sqrt{\theta}}$$

$$+\frac{\{\phi\tfrac{1}{2}(s+\sqrt{\theta})-\phi\tfrac{1}{2}(s-\sqrt{\theta})\}\{\phi\tfrac{1}{2}(s'+\sqrt{\theta})-\phi\tfrac{1}{2}(s'-\sqrt{\theta})\}}{\theta},$$

which is of the form

$$\left(\begin{array}{l} a_{0,0}\ ,\quad a_{0,1},\,..\,a_{0,n-2}\between s,\ 1)^{n-2}(s,\ 1)^{n-2} \\ a_{1,0}\ , \\ \vdots \\ a_{n-2,0}, \end{array}\right|$$

and equating to zero the determinant formed with the coefficients, we have an equation in θ which is the equation of differences of the given equation $\phi v=0$. For instance, if the given equation is $\phi v=(a,\ b,\ c,\ d\between v,\ 1)^3=0$, then we have

$$8\phi\tfrac{1}{2}(s+\sqrt{\theta})=(a,\ 2b+a\sqrt{\theta},\ 4c+4b\sqrt{\theta}+a\theta,\ 8d+12c\sqrt{\theta}+6b\theta+a\theta\sqrt{\theta}\between s,\ 1)^3$$
$$=(A,\ B,\ C,\ D\between s,\ 1)^3,$$

$$8\phi\tfrac{1}{2}(s-\sqrt{\theta})=(a,\ 2b-a\sqrt{\theta},\ 4c-4b\sqrt{\theta}+a\theta,\ 8d-12c\sqrt{\theta}+6b\theta-a\theta\sqrt{\theta}\between s,\ 1)^3$$
$$=(A',\ B',\ C',\ D'\between s,\ 1)^3;$$

and the function in s, s' is

$$\frac{6}{\sqrt{\theta}}\left\{\begin{array}{l} \quad 3\,(AB'-A'B)\,s^2s'^2 \\ +3\,(AC'-A'C)\,ss'\,(s+s') \\ +\ \ (AD'-A'D)\,(s^2+ss'+s'^2) \\ +9\,(BC'-B'C)\,ss' \\ +3\,(BD'-B'D)\,(s+s') \\ +3\,(CD'-C'D) \end{array}\right\}$$

$$+\frac{1}{\theta}(A-A',\ B-B',\ C-C',\ D-D'\between s,\ 1)^3\,(A-A',\ B-B',\ C-C',\ D-D'\between s',\ 1)^3,$$

which is equal to

$$12\left\{\begin{array}{l} -3a^2s^2s'^2 \\ -12ab\,ss'\,(s+s') \\ +(-12ac-a^2\theta)\,(s^2+ss'+s'^2) \\ +(36ac-72b^2+9a^2\theta)\,ss' \\ +(24ad-72bc+12ab\theta)\,(s+s') \\ +96bd-144c^2+(-48ac+72b^2)\,\theta-3a^2\theta^2 \end{array}\right\}$$

$$+4\,(3as^2+12bs+12c+a\theta)\,(3as'^2+12bs'+12c+a\theta)\,;$$

or reducing and dividing by 32, this is

$$\{9(ac-b^2)+3a^2\theta\}\,ss'+\{9(ad-bc)+6ab\theta\}(s+s')+36(bd-c^2)+(-15ac+27b^2)\theta-a^2\theta^2,$$

the terms in s^2, s'^2 disappearing, as they should do. Writing this under the form

$$\left(\begin{array}{ll} 9(ac-b^2)+3a^2\theta, & 9(ad-bc)+6ab\theta \\ 9(ad-bc)+6ab\theta, & 36(bd-c^2)+(-15ac+27b^2)\theta-a^2\theta^2 \end{array}\right)\between(s,\,1)(s',\,1)$$

and equating the determinant to zero, we have the required equation in θ: the form is that obtained by the ordinary process of applying Bezout's method to the two equations $(3a,\ 12b,\ 12c+a\theta\between s,\ 1)^2=0$, $(3b,\ 12c+4a\theta,\ 12d+8b\theta\between s,\ 1)^2=0$, being in fact the before-mentioned equation

$$\left(a^2\theta^2+(15ac-27b^2)\theta-36(bd-c^2)\right)\left(a^2\theta+3(ac-b^2)\right)+3\left(2ab\theta+3(ad-bc)\right)^2=0.$$

But, as already remarked, this elimination process is less convenient for the complete development of the result, than the method first explained in the present memoir.

[The equation p. 257, changing the notation and inserting the omitted coefficients, becomes $0=$

θ^{10}	$\theta^9 \,.\, 100\times$	$\theta^8 \,.\, 50\times$	$\theta^7 \,.\, 2500\times$	$\theta^6 \,.\, 125\times$	$\theta^5 \,.\, 625\times$	$\theta^4 \,.\, \frac{1}{5}2500\times$
A^8+1	A^6C+1	A^2B+1 A^4C^2+75	A^2BC+1 A^2F^2+1 A^2C^3+29	A^4H+16 A^4B^2-19 A^3BC^2+450 CF^2+800 C^4+6325	A^4G+1 A^2B^2C-160 $AFI+60$ $AC^2D-1140$ BF^2+280 BC^3+3620	A^3J-212 $A^2BH-1545$ A^2B^3+230 $AEI+275$ $BEF+1225$ $B^2C^2+11650$ $C^2H+1125$

$\theta^3 \,.\, \frac{1}{5}6250\times$	$\theta^2 \,.\, 62500\times$	$\theta \,.\, 62500\times$	$\theta^0 \,.\, 3125\times$
A^2M-336 A^2BG-27 $AB^3D-2400$ $ACJ-1520$ $B^3C+3200$ $BCH-2800$ $C^2G+1625$	$ABJ-20$ $ADG-3$ B^4+24 $BCG+19$ B^2H-36 $CM-28$	$BM-16$ B^2G+2 $GH-1$ $DJ-48$	$Q'+1$

where the capital letters denote the covariants of the quantic as explained 141 and 143.]

263.

DEMONSTRATION OF A THEOREM IN FINITE DIFFERENCES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CL. (for the year 1860), pp. 321—323: printed as a note to Sir J. W. F. Herschel's Memoir "On the Formulæ investigated by Dr BRINKLEY for the general Term in the Development of LAGRANGE'S Expression for the Summation of Series and for Successive Integrations, pp. 319—321.]

THE formula (B) [of Sir J. W. F. Herschel's Memoir], substituting therein for A_x the value $\frac{1}{[x]}\nabla^{-n}0^x$, becomes

$$\nabla^{-n}0^x = -\frac{1}{[n-1]}\left\{\frac{[x+n]}{[x-1]}\frac{1}{[n+1]}2.3..n\nabla - \frac{[x+n]}{[x-2]}\frac{1}{[n+2]}3.4\ldots\overline{n+1}\,\nabla^2 - \&c.\right\}0^x;$$

or, as this may be written,

$$\nabla^{-n}0^x = \frac{[x+n]}{[n-1]}\left\{\qquad -\frac{1}{[1][x-1](n+1)}\nabla + \frac{1}{[2][x-2](n+2)}\nabla^2 - \&c.\right\}0^x;$$

or, inserting a first term which vanishes except in the case $x=0$, and which is required in order that the formula may hold good for this particular value,

$$\nabla^{-n}0^x = \frac{[x+n]}{[n-1]}\left\{\frac{1}{[0][x]n}\nabla^0 - \frac{1}{[1][x-1](n+1)}\nabla + \frac{1}{[2][x-2](n+2)}\nabla^2 - \&c.\right\}0^x;$$

where the series on the right-hand side need only be continued up to the term containing $\nabla^x 0^x$, since the subsequent terms vanish. [In these formulæ and throughout the present paper $[x]$ is written to denote the factorial $[x]^x$ or $\Pi(x)$, and so in other cases.]

Now $\nabla^{-n}0^x$, or $\left(\frac{\Delta}{\log(1+\Delta)}\right)^{-n}0^x$, is equal to $[x]\times$ coef. t^x in $\left(\frac{e^t-1}{t}\right)^{-n}$, and so $\nabla^q 0^x$, or $\left(\frac{\Delta}{\log(1+\Delta)}\right)^{q}0^x$, is equal to $[x]\times$ coef. t^x in $\left(\frac{e^t-1}{t}\right)^{q}$.

Hence, putting $R=\frac{e^t-1}{t}$, the last-mentioned formula will be true if, as regards the term which contains t^x, we have

$$R^{-n}=\frac{[x+n]}{[n-1]}\left\{\frac{1}{[0]\,[x]\,n}R^0-\frac{1}{[1]\,[x-1]\,(n+1)}R^1+\frac{1}{[2]\,[x-2]\,(n+2)}R^2-\&c.\right\},$$

the series on the right-hand side being continued up to the term in R^x. This formula is, in fact, true if R, instead of being restricted to denote $\frac{e^t-1}{t}$, denotes any function whatever of the form $1+bt+ct^2+\&c.$, and it is true not only for the term in t^x, but for all the powers of t not higher than t^x. And, moreover, R^{-n} may denote any positive or negative integral or fractional power of R. In fact, the formula (assuming for a moment the truth of it) shows that the expansion of any power whatever of a series of the form in question, can be obtained by means of the expansions of the successive positive integer powers of the same series: the existence of such a formula (at least for negative powers) was indicated by Eisenstein, *Crelle*, t. XXXIX. p. 181 (1850), and the formula itself, in a slightly different form, was obtained in a very simple manner by Professor Sylvester in his paper, "Development of an idea of Eisenstein," *Quart. Math. Journ.* t. I. p. 199 (1855); the demonstration was in fact as follows, viz. writing

$$R^n=(1+\overline{R-1})^n=1+\frac{n}{1}(R-1)+\frac{n\,.\,n-1}{1\,.\,2}(R-1)^2+\&c.:$$

if we attend only to the terms involving powers of t not higher than t^x, the series on the right-hand side needs only to be continued up to the term involving $(R-1)^x$, and the right side being thus converted into a rational and integral function of R, it may be developed in a series of powers of R (the highest power being of course R^x), and the coefficients of the several powers are finite series which admit of summation; this gives the required formula. But there is an easier method; the process shows that the series on the right-hand side, continued as above up to the term involving t^x, is, as regards n, a rational and integral function of the degree x; and by Lagrange's interpolation formula, any rational and integral function of n of the degree x, can be at once expressed in terms of the values corresponding to $x+1$ particular values of n. The investigation will be as follows:—Let R denote a series of the form $1+bt+ct^2+\&c.$, and let R^n denote the development of the nth power of R, continued up to the term containing t^x, the terms involving higher powers of t being rejected: R^0, R^1, R^2, &c., and generally R^s, will in like manner denote the developments of these powers up to the term involving t^x, or what is the same thing, they will be the values of R^n, corresponding to $n=0, 1, 2, \ldots s$. By what precedes R^n is a rational and integral function of n of the degree x, and it can therefore be expressed in terms of the values $R^0, R^1, R^2, \ldots R^x$, which correspond to $n=0, 1, 2, \ldots x$. Let s have any one of the last-mentioned values, then the expression

$$\frac{n\,.\,n-1\,.\,n-2\ldots n-x}{n-s}\;\frac{1}{s\,.\,s-1\ldots 2\,.\,1\,.\,-1\,.\,-2\ldots-(x-s)},$$

which, as regards n, is a rational and integral function of the degree x (the factor

$n-s$, which occurs in the numerator and in the denominator being of course omitted), vanishes for each of the values $n=0, 1, 2, \ldots x$, except only for the value $n=s$, in which case it becomes equal to unity. The required formula is thus seen to be

$$R^n = \Sigma \left\{ \frac{n \,.\, n-1 \,.\, n-2 \ldots n-x}{n-s} \; \frac{1}{s \,.\, s-1 \ldots 2 \,.\, 1 \,.\, -2 \ldots -(x-s)} R^s \right\},$$

where the summation extends to the several values $s = 0, 1, 2, \ldots x$; or, what is the same thing, it is

$$R^n = \Sigma \left\{ \frac{n \,.\, n-1 \,.\, n-2 \,..\, n-x}{n-s} \; \frac{(-)^{x-s}}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots (x-s)} R^s \right\};$$

or changing the sign of n, it is

$$R^{-n} = \Sigma \left\{ \frac{n \,.\, n+1 \,.\, n+2 \,..\, n+x}{n+s} \; \frac{(-)^s}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots x-s} R^s \right\},$$

or, as this may be written,

$$R^{-n} = \frac{[x+n]}{[n-1]} \Sigma \left\{ \frac{(-)^s}{[s]\,[x-s]\,(n+s)} R^s \right\}$$

or substituting for s the values $0, 1, 2, \ldots x$, the formula is

$$R^{-n} = \frac{[x+n]}{[n-1]} \left\{ \frac{1}{[0]\,[x]\,n} R^0 - \frac{1}{[1]\,[x-1]\,(n+1)} R^1 + \frac{1}{[2]\,[x-2]\,(n+2)} R^2 - \ldots \right\}$$

continued up to the term involving R^x, which is the theorem in question.

264.

ON AN EXTENSION OF ARBOGAST'S METHOD OF DERIVATIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861, pp. 37—43, Received October 18,—Read December 13, 1860.]

ARBOGAST'S Method of Derivations was devised by him with a view to the development of a function $\phi(a+bx+cx^2+\ldots)$, but it is at least as useful for the formation of only the literal parts of the coefficients, or, what is the same thing, the combinations of a given degree and weight in the letters $(a, b, c, d, \ldots)$, the weights of the successive letters being 0, 1, 2, 3, &c. Thus instead of applying the method to finding the coefficients

$$a^4,\ 4a^3b,\ 4a^3c+6a^2b^2,\ \&c.,$$

we may apply it merely to finding the sets of terms

$$\begin{array}{lll} a^4, & a^3b, & a^3c, \ \&c. \\ & & a^2b^2. \end{array}$$

To derive any column from the one which immediately precedes it, we operate on a letter by changing it into its immediate successor in the alphabet, and we must in each term operate on the last letter, and also, when the last but one letter in the term is the immediate antecessor in the alphabet of the last letter (but in this case only), operate on the last but one letter. Thus a^3c gives a^3d, but a^2b^2 gives a^2bc and ab^3, and the next succeeding column is therefore

$$\begin{array}{l} a^3d \\ a^2bc \\ ab^3. \end{array}$$

If the series of letters is finite, and the last letter of the term is also the last letter of the series, then it is impossible to operate on the last letter of the term, but the last but one letter (when the foregoing rule applies to it) is still to be operated on; and if the rule does not apply, then the term does not give rise to a term in the succeeding column; the operations will at length terminate, and a

complete series of columns be obtained. Thus, if the letters are (a, b, c, d), and the operations are (as before) performed upon a^4, the entire series of columns is

a^4	a^3b	a^3c a^2b^2	a^3d a^2bc ab^3	a^2bd a^2c^2 ab^2c b^4	a^2cd ab^2d abc^2 b^3c	a^2d^2 $abcd$ ac^3 b^3d b^2c^2	abd^2 ac^2d b^2cd bc^3	acd^2 b^2d^2 bc^2d c^4	ad^3 bcd^2 c^2d	bd^3 c^2d^2	cd^3	c^4

Nothing can be more convenient than the process when the entire series of columns is required; but it is very desirable to have a process for the formation of any column apart from the others; and the object of the present memoir is to investigate a rule for the purpose. But as Arbogast's rule, applied as above, depends upon very similar principles, I will commence by showing how this rule is to be demonstrated. If we take any combination b^3d and operate backwards on the last letter (viz. by changing it into its immediate antecessor in the alphabet), we obtain b^3c, which is a term in the next preceding column; b^3d is therefore obtainable from a term in the next preceding column, viz. b^3c, and the process is to operate on the last letter. If, instead of b^3d, the term is abc^2 (the last letter here entering as a power), the operation backwards on the last letter gives ab^2c, which is also a term of the preceding column; and it is to be noticed that the last but one letter b is here the immediate antecessor of the last letter c (and would have been so even if b had not entered into the given term abc^2, thus ac^2 operated on backwards would have given abc). Hence abc^2 is obtained by operating on a term in the next preceding column, viz. the term ab^2c, but in this case the operation is performed on the last but one letter. *Every* term is thus obtained from the next preceding column, viz. the terms are obtained by operating on the last letter, and (when the last but one letter is the immediate antecessor in the alphabet of the last letter) then also on the last but one letter, of each term of the next preceding column, and the correctness of the rule is thus demonstrated. It is to be observed that the terms are operated upon in order, the operation on the last but one letter (when it is operated on) being made immediately after that upon the last letter of the same term, and that the terms of a column are thus obtained in the proper alphabetical order.

I pass now to the above mentioned question of the formation of a single column by itself; it will be convenient, by way of illustration, to write down the columns

a^2e^3	,	ade^3
$abde^2$		bce^3
ac^2e^2		bd^2e^2
acd^2e		c^2de^2
ad^4		cd^3e
b^2ce^2		d^5
b^2d^2e		
bc^2de		
bcd^3		
c^4e		
c^3d^2		

which belong to the set (a, b, c, d, e), and are of the degree 5 and the weights 12 and 15 respectively.

Some definitions and explanations are required. I speak of the first and last letters *of the set*, simply as the first and the last letter: there is frequent occasion to speak of the last letter; and to avoid confusion, the last letter *of a term* will be spoken of as the ultimate letter; it is necessary also to consider the penultimate, antepenultimate, and pro-antepenultimate letters of the term. It will be convenient to distinguish between the ultimate letter and the ultimate, which may be either the ultimate letter or a power of such letter; and similarly for the penultimate, &c. Thus in the term bcd^3, the ultimate letter is d and the ultimate is d^3, the penultimate and the penultimate letter are each of them c; of course the ultimate, penultimate, &c. letters are always distinct from each other. We have also to consider the pairs of letters contained in a term; cd^3e contains the pairs (c, d), (c, e), (d, d), (d, e), and so in other cases; the letters of a pair are taken in the natural order. A pair not containing either the first letter or the last letter is *expansible;* thus the set being as before (a, b, c, d, e), the pairs (b, c), (d, d) are expansible: they are *expanded* by retreating the prior and advancing the posterior letter each one step; thus the just mentioned pairs (b, c), (d, d), are expanded into (a, d) and (c, e) respectively.

A pair composed of two distinct non-contiguous letters is *contractible;* it is *contracted* by advancing the prior and retreating the posterior letter each one step: thus (a, d), (c, e) are contractible pairs, and they give by contraction the pairs (b, c), (d, d) respectively: the processes of expansion and contraction are obviously converse to each other.

The expression the *last expansible pair* of a term hardly requires explanation; (d, d) is the last expansible pair of the term bcd^3, or of the term $b^2d^3e^2$, (c, d) the last expansible pair of the term c^2de (the set being always (a, b, c, d, e)), and so in all other cases. The expression the *last expansion*, in regard to any term, means the expansion of the last expansible pair of such term.

The expansion or contraction of any pair of a term leaves unaltered as well the weight as the degree, the resulting term belongs therefore to the same column as the original one. But the effect of an expansion is to diminish, and that of a contraction to increase the alphabetical rank, or rank in the column, the ranks being reckoned as the first or lowest, second, third, and so on, up to the last or highest rank. In particular, by performing upon any term the last expansion, we diminish the rank in the column, and by a succession of last expansions we bring the term up to the head of the column. Such term is not susceptible of any further expansion; it may therefore contain the first and last letters or either of them, and it may also contain a single intermediate letter in the first power only; thus the first and last letters being a, e, the head term of a column is of the form $a^m e^q$ or $a^m c e^q$, c being some intermediate letter, and the powers a^m, e^q being both or either of them omitable. In like manner, by a succession of contractions of any term we obtain the term at the foot of the column; such term is not susceptible of any further contraction, and it must therefore be composed either of a single letter or of two contiguous

letters, that is it must be of the form c^m, or of the form $c^m d^q$, where c, d are any two contiguous letters, not excluding the case where the single letter or each or either of the contiguous letters is a first or a last letter.

It is to be observed that the passage up from any term to the term at the head of the column (or top term) by means of a succession of last expansions, is a perfectly unique one; but as no selection has been made of a like unique process of contraction, this is not the case with respect to the passage down from any term to the term at the foot of the column (or bottom term) by a succession of contractions.

Every term gives by the last expansion a term above it; it can therefore be obtained from such term above it by means of a contraction. But the contraction of the upper term is by hypothesis such that, performing upon the contracted term the last expansion, we obtain the upper term; a contraction, which, performed on any term, gives a lower term which by performing upon it the last expansion reproduces the first mentioned or upper term, may be called *a reversible contraction*. And it is clear that if we perform on the top term all the reversible contractions, and on each of the resulting terms all the reversible contractions, and so on as long as the process is possible, we obtain without repetition all the terms of the column. The column is in fact similar to a genealogical tree in the male line, each lower term issuing from a single upper term, while each upper term generates a lower term or terms, or does not generate any such term, and the top term being the common origin of the entire series. It may be added that when the order of the reversible contractions of the same term is duly fixed, the alphabetical arrangement of the terms in the column agrees with the order as of primogeniture (an elder son and his issue male preceding all the younger sons and their respective issue male) in the genealogical tree.

It only remains then to inquire under what conditions a contraction is reversible. Now as regards any term, in order that a contraction performed on it may be reversible, it is necessary and sufficient that the pair produced by the contraction should be the last expansible pair of the contracted term. There are several cases to be considered. First, if the contraction affects the ultimate and penultimate letters of the term: this implies that the ultimate and penultimate letters are not contiguous. Let the term terminate in $e^m h$, the contracted term will terminate in $e^{m-1} fg$, and (f, g), the pair produced by the contraction, is the last expansible pair of the contracted term; the contraction is in this case reversible. If, however, the term terminate in $e^m h^p$ $(p > 1)$, the contracted term will terminate in $e^{m-1} fgh^{p-1}$, and the last expansible pair is not as before (f, g), but it is (according as $p = 2$ or $p > 2$) (g, h) or (h, h): unless indeed h is the last letter, in which case (f, g) remains the last expansible pair of the contracted term. In the example e^m has been written, but the case $m = 1$ is not excluded; moreover, the penultimate letter e is removed three steps from the ultimate letter h, but the result would have been the same if instead of e we had had any preceding letter, or had had the letter f; by hypothesis it is not g, the letter contiguous to the ultimate letter h. The conclusion is that a contraction on the ultimate and penultimate letters (these being non-contiguous) is reversible if the ultimate is a simple letter, or if, being a power, it is a power of the last letter.

Next let the contraction affect the ultimate and antepenultimate letters. The two letters are here separated by the penultimate letter, and are therefore not contiguous. Suppose that the termination is $f^l g^m k$ (f and g contiguous), the contracted term terminates in $f^{l-1} g g^m j$, and (g, j) the pair arising from the contraction is the last expansible pair of the contracted term; the contraction is therefore reversible. In the example f^l has been written, but the case $l=1$ is not excluded; moreover the ultimate letter k is taken non-contiguous to the penultimate letter g; but if the ultimate letter had been the contiguous letter h, the only difference is that the pair would be (g, g), and the conclusion is not altered. But suppose the termination of the term is $e^l g^m k$ (e, g, non-contiguous), then the contracted term terminates in $e^{l-1} f g^m j$, where the pair arising from the contraction is (f, j), but the last expansible pair is (g, j); the contraction therefore is not reversible. The case $l=1$ is not excluded; nor is it necessary that the ultimate letter should be non-contiguous to the penultimate; if the ultimate letter had been h, the pair arising from the contraction would have been (f, g), but the last expansible pair (g, g), and the contraction is still not reversible. In each of the cases considered the ultimate has been a simple letter; if in the first case the ultimate had been k^p $(p > 1)$, then the contracted pair would terminate in jk^{p-1}, and (according as $p=2$ or $p<2$) the last expansible pair would be (j, k) or (k, k), which is not the pair (g, j) produced by the contraction; the contraction is therefore not reversible, unless indeed k is the last letter, in which case it continues reversible. In the second case the contraction, which is not reversible when the ultimate is the simple letter k, remains not reversible when the ultimate is a power of such letter. The conclusion is that a contraction on the ultimate and antepenultimate letters is reversible, if the penultimate and antepenultimate letters are contiguous, and the ultimate is a simple letter, or if, being a power, it is a power of the last letter.

A contraction on the ultimate and pro-antepenultimate letters, or on the ultimate letter and any letter preceding the pro-antepenultimate letter, is never reversible. To show this, it will be sufficient to consider the case where the term terminates in $efgh$, the contracted term terminates in $ffgg$, the pair arising from the contraction being (f, g), and the last expansible pair being (g, g); and *à fortiori*, if the letters or any of them occur as powers, or are non-contiguous.

Consider, next, a contraction on the penultimate and antepenultimate letters; this assumes that these letters are non-contiguous. Such a contraction may be reversible if only the ultimate is the last letter or a power thereof; and the condition is then similar to that in the case of the ultimate and penultimate letters; only as the penultimate cannot be a power of the last letter, it must be a simple letter. And the conditions in order that the contraction may be reversible then are that the ultimate is the last letter or a power thereof, and the penultimate a simple letter.

The next case is that of a contraction on the penultimate and pro-antepenultimate letters. Such contraction may be reversible if the ultimate is the last letter or a power thereof; and the condition is then similar to that for the case of the ultimate and antepenultimate letters; only as the penultimate cannot be a power of the last

letter, it must be a simple letter. The conditions, in order that the contraction may be reversible, are that the ultimate may be the last letter or a power thereof, the penultimate a simple letter, and the antepenultimate and pro-antepenultimate letters contiguous.

A contraction on the penultimate letter and on any letter preceding the pro-antepenultimate letter, or upon any two letters preceding the penultimate letter, is never reversible. If the ultimate, penultimate, antepenultimate and pro-antepenultimate letters are denoted by U, P, A, P' respectively, then by what precedes, the following contractions, viz. UP, UA, PA, PP', may be reversible, and they will be so under the conditions shown in the annexed Table. It is to be noticed that the conditions for UA, PA are mutually exclusive, and consequently that the number of reversible contractions to be performed upon any term is at most 3. The Table is

	P	A	P'
U	U, P, non-contiguous letters. The ultimate a simple letter, or a power of the last letter.	P, A, contiguous letters. The ultimate a simple letter, or a power of the last letter.	
P		P, A, non-contiguous letters. Penultimate a simple letter. Ultimate the last letter, or a power thereof.	A, P', contiguous letters. Penultimate a simple letter. Ultimate the last letter, or a power thereof.

The contractions are to be applied in the order UP, UA, PA, PP', but all the contracted terms originating in a prior contraction of a given term are to be exhausted before proceeding to a posterior contraction of the same term. As an example of the process, the set being (a, b, c, d, e), I will take the column

$$\begin{array}{l} abe^3 \\ acde^2 \\ ad^3e \\ b^3de^2 \\ bc^3e^2 \\ bcd^2e \\ bd^4 \\ c^3de \\ c^2d^3 \end{array}$$

The top term abe^3 is given. The contractions applicable to it are UP, UA. And UP gives $acde^2$. The only contraction applicable to this is UA, giving ad^3e. And

there is not any contraction applicable to ad^3e. We revert therefore to UA on abe^3, this gives b^2de^2. The only applicable contraction is PA, giving bc^2e^2. The contractions applicable to this are UP, UA. And UP gives bcd^3e. The only contraction applicable to this is UA, giving bd^4, and there is not any contraction applicable to bd^4. We revert therefore to UA on bc^2e^2, this gives c^3de, and the only contraction applicable to this is UA, giving c^2d^2. There is not any contraction applicable to this, and the process is therefore complete. It may be remarked that the example presents no instance of the contraction PP'; in fact the only terms having a pro-antepenultimate are $acde^3$, in which A, P' are not contiguous letters, and bcd^3e, in which the penultimate is not a simple letter, so that the contraction PP' is in each case excluded. It must be confessed that a considerable amount of practice would be required before the process could be readily made use of.

265.

ADDITION TO THE MEMOIR ON AN EXTENSION OF ARBOGAST'S METHOD OF DERIVATIONS.

[Now first published, 1891.]

THE process explained in the foregoing Memoir is not easily workable, and I have in fact never used it. I have devised a new process theoretically less complete, and of a somewhat mixed character consisting, as it does, in the analytical reduction of the problem to the same problem for a smaller number of letters. I have however found it very convenient for obtaining the literal terms in the theory of the sextic, viz. where we have the seven letters (a, b, c, d, e, f, g); and I propose to explain the process by applying it to the determination of the terms of the discriminant, degree $= 6$, weight $= 18$.

Here writing a, b, c, d, e, f, g to denote the indices of these letters respectively in a term such as $\alpha^a\beta^b\gamma^c\ldots$ (that is, writing for convenience $a, b, c, \ldots$ instead of $\alpha, \beta, \gamma, \ldots$) we have

$$a + b + c + d + e + f + g = 6,$$
$$b + 2c + 3d + 4e + 5f + 6g = 18,$$

and thence

$$6a + 5b + 4c + 3d + 2e + f \quad = 18.$$

I separate off from the others the first three letters a, b, c. The equation gives $a = 3$ at most. And then

if $a = 3$, then $5b + 4c + 3d + 2e + f = 18$, $b = 0$, $c = 0$;

if $a = 2$, $5b + 4c + 3d + 2e + f = 6$, $b = 1$ at most;

if $b = 1$, $4c + 3d + 2e + f = 1$, $c = 0$,

„ $b = 0$, $4c + 3d + 2e + f = 6$, $c = 1$ at most,

if	$a = 1$,	$5b + 4c + 3d + 2e + f = 12$,	$b = 2$ at most;
	if $b = 2$,	$4c + 3d + 2e + f = 2$,	$c = 0$,
	„ $b = 1$,	$4c + 3d + 2e + f = 7$,	$c = 1$ at most,
	„ $b = 0$,	$4c + 3d + 2e + f = 12$,	$c = 3$ „ „
if	$a = 0$,	$5b + 4c + 3d + 2e + f = 18$,	$b = 3$ at most;
	if $b = 3$,	$4c + 3d + 2e + f = 3$,	$c = 0$,
	„ $b = 2$,	$4c + 3d + 2e + f = 8$,	$c = 2$ at most,
	„ $b = 1$,	$4c + 3d + 2e + f = 13$,	$c = 3$ „ „
	„ $b = 0$,	$4c + 3d + 2e + f = 18$,	$c = 4$ „ „

Hence considering the several terms as arranged in alphabetical order and writing down only the factors in a, b, c, we obtain col. 1 of the following diagram;

Col. 1.	Col. 2.	Col. 3.	
a^3	$(d^3)^9$	1	
a^2b	$(d^3)^8$	1	1
a^2b^0c	$(d^3)^7$	2	1
c^0	$(d^4)^6$	5	
$a\,b^2c^0$	$(d^3)^7$	2	7
$b\,c^1$	$(d^3)^6$	3	
c^0	$(d^4)^5$	4	
$a\,b^0c^3$	$(d^2)^6$	1	9
c^2	$(d^3)^5$	3	
c^1	$(d^4)^4$	4	
c^0	$(d^5)^3$	3	
$a^0b^3c^0$	$(d^3)^6$	3	11
b^2c^2	$(d^2)^6$	1	3
c^1	$(d^3)^5$	3	
c^0	$(d^4)^4$	4	
b^1c^3	$(d^2)^5$	1	8
c^2	$(d^3)^4$	3	
c^1	$(d^4)^3$	3	
c^0	$(d^5)^2$	2	
b^0c^4	$(d^2)^4$	2	9
c^3	$(d^3)^3$	3	
c^2	$(d^4)^2$	2	
c^1	$(d^5)^1$	1	
c^0	$(d^6)^0$	1	9
			58

We then form col. 2 by annexing to each term a term $(d^\theta)^\phi$ which denotes the derivative ϕ of d^θ; the value of θ being such that the whole term may be of the proper degree 6, and the value of ϕ being such that the whole term may be of the proper weight 18. Thus

$$a^3(d^3)^9, \text{ degree is } 3+3, = 6; \text{ weight is } 0+9+9 \quad, = 18,$$
$$ab^2c^0(d^3)^7, \quad \text{„} \quad 1+2+3, = 6; \quad \text{„} \quad \text{„} \quad 0+2+9+7, = 18,$$

viz. d^3 being of weight 9, then $(d^3)^9$ is of weight $9+9$, and $(d^3)^7$ of weight $9+7$.

The derivatives of any power of d are given by those of the indefinite power d^m, which, omitting therein the several powers of d, may be tabulated thus. I remark that the table is carried up to the column 21 in order that it may be applicable to the calculation of the literal terms of the degree 15 and weight 45 which belong to the highest invariant of the sextic.

SUBSIDIARY TABLE OF THE DERIVATIVES OF d^m.

	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	e	f	g	eg	fg	g^2	eg^2	fg^2	g^3	eg^3	fg^3	g^4	eg^4	fg^4	g^5	eg^5	fg^5	g^6	eg^6	fg^6	g^7
2			e^2	ef	f^2	e^2g	efg	f^2g	e^2fg	efg^2	f^2g^2	e^2g^2	efg^3	f^2g^3	e^2fg^3	efg^4	f^2g^4	e^2fg^4	efg^5	f^2g^5	e^2fg^5	efg^6
3				e^3	e^2f	ef^2	f^3	e^2fg	ef^2g	f^3g	e^2fg^2	ef^2g^2	f^3g^2	e^2fg^2	ef^2g^3	f^3g^3	e^2fg^3	ef^2g^4	f^3g^4	e^2fg^4	ef^2g^5	f^3g^5
4					e^4	e^3f	e^2g	ef^3	e^2g^2	efg^2	f^2g^2	e^2fg^2	ef^2g^3	f^2g^3	e^2fg^3	ef^2g^4	f^2g^4	e^2fg^4	ef^2g^5	f^2g^5	e^2fg^5	e^2g^6
5						e^5	e^2fg	e^2g	e^2fg^2	ef^2g	f^3g	e^2fg^2	ef^2g^2	f^2g^2	e^2fg^2	ef^2g^3	f^2g^3	e^2fg^3	ef^2g^4	f^2g^4	e^2fg^4	ef^2g^5
6							e^4f	e^2f^2	e^2g	e^2f^2g	ef^3g	e^2f^2g	ef^2g^2	f^2g^2	e^2fg^2	ef^2g^3	f^2g^3	e^2fg^3	ef^2g^4	f^2g^4	e^2fg^4	ef^2g^5
7							e^6	e^5f	e^2f	e^4fg	e^2f^3g	ef^5	f^6	e^4fg^2	e^2f^2g	f^6g	e^4fg^3	$e^2f^2g^2$	f^7g	e^4fg^4	$e^2f^2g^3$	f^7g^2
8								e^7	e^3f^2	e^3f^3	e^2f^4	e^2f^3g	e^2f^5	$e^2f^3g^2$	e^2f^4g	$e^2f^2g^2$	$e^2f^3g^3$	$e^2f^4g^2$	$e^2f^3g^4$	$e^2f^4g^3$	$e^2f^5g^2$	e^2f^6g
9									e^4f	e^3fg	e^2fg	e^2f^2g	e^2f^3g	e^2f^4g	$e^2f^3g^2$	e^2f^3g	$e^2f^4g^2$	$e^2f^3g^3$	$e^2f^4g^4$	$e^2f^5g^3$	$e^2f^6g^2$	e^2f^7g
10									e^5	e^3g	e^2g	e^2f^3	e^2f^4	e^2f^3g	e^2f^4g	$e^2f^3g^2$	$e^2f^4g^3$	$e^2f^5g^2$	e^2f^6g	$e^2f^7g^2$	e^2f^8g	e^2f^9
11										e^4f^2	e^3fg	e^2fg	e^2f^3g	e^2f^4g	e^2f^5g	$e^2f^4g^2$	$e^2f^5g^3$	$e^2f^6g^2$	$e^2f^7g^3$	$e^2f^8g^2$	e^2f^9g	e^2f^{10}
12										e^3f	e^3g	e^2f^2	e^2f^3	e^2f^4	e^2f^5	e^2f^6	$e^2f^7g^2$	$e^2f^8g^2$	$e^2f^9g^3$	$e^2f^{10}g^2$	$e^2f^{11}g$	e^2f^{12}
13											e^4f	e^3fg	e^3f^2	e^3fg	e^3f^3	e^3g^3	$e^3f^4g^2$	$e^3f^5g^2$	$e^3f^6g^3$	$e^3f^7g^3$	$e^3f^8g^2$	e^3f^9g
14											e^5	e^3g	e^4f^2	e^3f^2	e^3fg	$e^3f^2g^2$	$e^3f^3g^2$	$e^3f^4g^3$	$e^3f^5g^3$	$e^3f^6g^2$	$e^3f^7g^2$	e^3f^8g
15												e^4fg	e^4fg	e^4f^2	e^4f^3	e^4f^4g	e^3g^4	$e^3f^5g^2$	$e^3f^6g^3$	$e^3f^7g^2$	$e^3f^8g^3$	$e^3f^9g^2$
16												e^5f	e^5f^2	e^5fg	e^5f^3	e^5f^2g	$e^4f^4g^2$	$e^4f^5g^3$	$e^4f^6g^2$	$e^4f^7g^3$	$e^4f^8g^2$	e^4f^9g
17												e^6	e^6g	e^6f^2	e^6fg	e^6f^3	e^5fg^3	$e^5f^2g^2$	$e^5f^3g^3$	$e^5f^4g^2$	$e^5f^5g^2$	e^5f^6g
18													e^7f	e^7g	e^7f^2	e^7fg	e^6g^3	e^6fg^2	$e^6f^2g^3$	$e^6f^3g^2$	$e^6f^4g^2$	e^6f^5g
19													e^8	e^8f	e^8g	e^8f^2	e^7g^2	e^7fg^2	$e^7f^2g^3$	$e^7f^3g^2$	$e^7f^4g^2$	e^7f^5g
20														e^9	e^9f	e^9g	e^8fg^2	$e^8f^2g^2$	$e^8f^3g^3$	$e^8f^4g^2$	$e^8f^5g^2$	e^8f^6g
21														e^{10}	$e^{10}g$	$e^{10}f$	e^9g^2	e^9fg^2	$e^9f^2g^3$	$e^9f^3g^2$	$e^9f^4g^2$	e^9f^5g
22														e^{11}	$e^{11}g$	$e^{11}f$	$e^{10}fg$	$e^{10}g^2$	$e^{10}f^2g^2$	$e^{10}f^3g^2$	$e^{10}f^4g$	$e^{10}f^5$
23														e^{12}	$e^{12}f$	$e^{12}g$	$e^{11}g^2$	$e^{11}fg^2$	$e^{11}f^2g^2$	$e^{11}f^3g^2$	$e^{11}f^4g$	$e^{11}f^5$
24														e^{13}	$e^{13}g$	$e^{13}f$	$e^{12}g^2$	$e^{12}fg^2$	$e^{12}f^2g^2$	$e^{12}f^3g^2$	$e^{12}f^4g$	$e^{12}f^5$
25															e^{14}	$e^{14}f$	$e^{13}g^2$	$e^{13}fg^2$	$e^{13}f^2g^2$	$e^{13}f^3g^2$	$e^{13}f^4g$	$e^{13}f^5$
26															e^{15}	$e^{15}g$	$e^{14}g^2$	$e^{14}fg^2$	$e^{14}f^2g^2$	$e^{14}f^3g^2$	$e^{14}f^4g$	$e^{14}f^5$
27																e^{16}	$e^{15}g^2$	$e^{15}fg^2$	$e^{15}f^2g^2$	$e^{15}f^3g^2$	$e^{15}f^4g$	$e^{15}f^5$
28																	$e^{16}g^2$	$e^{16}fg^2$	$e^{16}f^2g^2$	$e^{16}f^3g^2$	$e^{16}f^4g$	$e^{16}f^5$
29																	$e^{17}g^2$	$e^{17}fg^2$	$e^{17}f^2g^2$	$e^{17}f^3g^2$	$e^{17}f^4g$	$e^{17}f^5$
30																	$e^{18}g^2$	$e^{18}fg^2$	$e^{18}f^2g^2$	$e^{18}f^3g^2$	$e^{18}f^4g$	$e^{18}f^5$
31																	$e^{19}g^2$	$e^{19}fg^2$	$e^{19}f^2g^2$	$e^{19}f^3g^2$	$e^{19}f^4g$	$e^{19}f^5$
32																		$e^{20}g^2$	$e^{20}fg^2$	$e^{20}f^2g^2$	$e^{20}f^3g^2$	$e^{20}f^4g$
33																		$e^{21}g^2$	$e^{21}fg^2$	$e^{21}f^2g^2$	$e^{21}f^3g^2$	$e^{21}f^4g$
34																			$e^{22}g^2$	$e^{22}fg^2$	$e^{22}f^2g^2$	$e^{22}f^3g^2$
35																			$e^{23}g^2$	$e^{23}fg^2$	$e^{23}f^2g^2$	$e^{23}f^3g$
36																			$e^{24}g^2$	$e^{24}fg^2$	$e^{24}f^2g^2$	$e^{24}f^3$
37																			$e^{25}g^2$	$e^{25}fg^2$	$e^{25}f^2g^2$	$e^{25}f^3$
38																			$e^{26}g^2$	$e^{26}fg^2$	$e^{26}f^2g^2$	$e^{26}f^3$
39																			$e^{27}g^2$	$e^{27}fg^2$	$e^{27}f^2g^2$	$e^{27}f^3$
40																			$e^{28}g^2$	$e^{28}fg^2$	$e^{28}f^2g^2$	$e^{28}f^3$
41																			$e^{29}g^2$	$e^{29}fg^2$	$e^{29}f^2g^2$	$e^{29}f^3$
42																			$e^{30}g^2$	$e^{30}fg^2$	$e^{30}f^2g^2$	$e^{30}f^3$
43																			$e^{31}g^2$	$e^{31}fg^2$	$e^{31}f^2g^2$	$e^{31}f^3$
44																			$e^{32}g^2$	$e^{32}fg^2$	$e^{32}f^2g^2$	$e^{32}f^3$
45																			$e^{33}g^2$	$e^{33}fg^2$	$e^{33}f^2g^2$	$e^{33}f^3$
46																			$e^{34}g^2$	$e^{34}fg^2$	$e^{34}f^2g^2$	$e^{34}f^3$
																			$e^{35}g^2$	$e^{35}fg^2$	$e^{35}f^2g^2$	$e^{35}f^3$
																			$e^{36}g^2$	$e^{36}fg^2$	$e^{36}f^2g^2$	$e^{36}f^3$
																			$e^{37}g^2$	$e^{37}fg^2$	$e^{37}f^2g^2$	$e^{37}f^3$
																			$e^{38}g^2$	$e^{38}fg^2$	$e^{38}f^2g^2$	$e^{38}f^3$
																			$e^{39}g^2$	$e^{39}fg^2$	$e^{39}f^2g^2$	$e^{39}f^3$
																			$e^{40}g^2$	$e^{40}fg^2$	$e^{40}f^2g^2$	$e^{40}f^3$
																			$e^{41}g^2$	$e^{41}fg^2$	$e^{41}f^2g^2$	$e^{41}f^3$
																			$e^{42}g^2$	$e^{42}fg^2$	$e^{42}f^2g^2$	$e^{42}f^3$
																			$e^{43}g^2$	$e^{43}fg^2$	$e^{43}f^2g^2$	$e^{43}f^3$
																			$e^{44}g^2$	$e^{44}fg^2$	$e^{44}f^2g^2$	$e^{44}f^3$
																			$e^{45}g^2$	$e^{45}fg^2$	$e^{45}f^2g^2$	$e^{45}f^3$
																			$e^{46}g^2$	$e^{46}fg^2$	$e^{46}f^2g^2$	$e^{46}f^3$
																				$e^{47}fg^2$	$e^{47}f^2g^2$	$e^{47}f^3$
																				$e^{48}fg^2$	$e^{48}f^2g^2$	$e^{48}f^3$
																				$e^{49}fg^2$	$e^{49}f^2g^2$	$e^{49}f^3$
																				$e^{50}fg^2$	$e^{50}f^2g^2$	$e^{50}f^3$
																				$e^{51}fg^2$	$e^{51}f^2g^2$	$e^{51}f^3$
																				$e^{52}fg^2$	$e^{52}f^2g^2$	$e^{52}f^3$
																				$e^{53}fg^2$	$e^{53}f^2g^2$	$e^{53}f^3$
																				$e^{54}fg^2$	$e^{54}f^2g^2$	$e^{54}f^3$
																				$e^{55}fg^2$	$e^{55}f^2g^2$	$e^{55}f^3$
																				$e^{56}fg^2$	$e^{56}f^2g^2$	$e^{56}f^3$
																				$e^{57}fg^2$	$e^{57}f^2g^2$	$e^{57}f^3$
																				$e^{58}fg^2$	$e^{58}f^2g^2$	$e^{58}f^3$
																				$e^{59}fg^2$	$e^{59}f^2g^2$	$e^{59}f^3$
																				$e^{60}fg^2$	$e^{60}f^2g^2$	$e^{60}f^3$
																				$e^{61}fg^2$	$e^{61}f^2g^2$	$e^{61}f^3$
																				$e^{62}fg^2$	$e^{62}f^2g^2$	$e^{62}f^3$
																				$e^{63}fg^2$	$e^{63}f^2g^2$	$e^{63}f^3$
																				$e^{64}fg^2$	$e^{64}f^2g^2$	$e^{64}f^3$
																				$e^{65}fg^2$	$e^{65}f^2g^2$	$e^{65}f^3$
																				$e^{66}fg^2$	$e^{66}f^2g^2$	$e^{66}f^3$
																				$e^{67}fg^2$	$e^{67}f^2g^2$	$e^{67}f^3$
																				$e^{68}fg^2$	$e^{68}f^2g^2$	$e^{68}f^3$
																				$e^{69}fg^2$	$e^{69}f^2g^2$	$e^{69}f^3$
																				$e^{70}fg^2$	$e^{70}f^2g^2$	$e^{70}f^3$
																				$e^{71}fg^2$	$e^{71}f^2g^2$	$e^{71}f^3$
																				$e^{72}fg^2$	$e^{72}f^2g^2$	$e^{72}f^3$
																				$e^{73}fg^2$	$e^{73}f^2g^2$	$e^{73}f^3$
																				$e^{74}fg^2$	$e^{74}f^2g^2$	$e^{74}f^3$
																				$e^{75}fg^2$	$e^{75}f^2g^2$	$e^{75}f^3$
																				$e^{76}fg^2$	$e^{76}f^2g^2$	$e^{76}f^3$
																				$e^{77}fg^2$	$e^{77}f^2g^2$	$e^{77}f^3$
																				$e^{78}fg^2$	$e^{78}f^2g^2$	$e^{78}f^3$
																				$e^{79}fg^2$	$e^{79}f^2g^2$	$e^{79}f^3$
																				$e^{80}fg^2$	$e^{80}f^2g^2$	$e^{80}f^3$
																				$e^{81}fg^2$	$e^{81}f^2g^2$	$e^{81}f^3$
																				$e^{82}fg^2$	$e^{82}f^2g^2$	$e^{82}f^3$
																				$e^{83}fg^2$	$e^{83}f^2g^2$	$e^{83}f^3$
																				$e^{84}fg^2$	$e^{84}f^2g^2$	$e^{84}f^3$
																				$e^{85}fg^2$	$e^{85}f^2g^2$	$e^{85}f^3$
																				$e^{86}fg^2$	$e^{86}f^2g^2$	$e^{86}f^3$
																				$e^{87}fg^2$	$e^{87}f^2g^2$	$e^{87}f^3$
																				$e^{88}fg^2$	$e^{88}f^2g^2$	$e^{88}f^3$
																				$e^{89}fg^2$	$e^{89}f^2g^2$	$e^{89}f^3$
																				$e^{90}fg^2$	$e^{90}f^2g^2$	$e^{90}f^3$

This means for instance that the derivatives of d^2 are

0	1	2	3	4	5	6
d^2	de	df e^2	dg ef	eg f^2	fg	g^2

those of d^3 are

0	1	2	3	4	5	6	7	8	9
d^3	d^2e	d^2f de^2	d^2g def e^3	deg df^2 e^2f	dfg e^2g efg	dg^2 efg f^3	eg^2 f^2g	fg^2	g^3

and so on; viz. to form any column we affix to the terms of the corresponding column of the subsidiary table the proper power of d, so that the degree in all the letters may be 2, 3, &c. as the case may be, using from each column of the subsidiary table only the terms which are not of too high a degree; for instance for col. 4 of the derivatives of d^3, only the terms eg, f^2, e^2f of the complete column eg, f^2, e^2f, e^3.

It is to be observed that these tables of the derivatives of d^2, d^3, &c. do not need to be actually formed; any column which is wanted can be written down at once, *currente calamo*, from the column of the subsidiary table. And if we require only the number of terms, then these numbers are at once taken out from the Subsidiary Table. Thus for the numerical column of the diagram, the Subsidiary Table, col. 9, contains but 1 term g^3 of degree 3, and thus the number set opposite to $(d^3)^9$ is 1; so col. 8 contains but 1 term fg^2 of the degree 3, and so the number opposite to $(d^2)^8$ is 1; col. 7 contains 2 terms eg^2, f^2g of degree 3, and so the number opposite to $(d^3)^7$ is 2; and so on: we have in this way the numbers 1, 1, 2, 5, 2, 3, 4, &c. the partial sums of which are 1, 1, 7, 9, 11, 3, 8, 9, 9 giving the total sum 58: viz. this is the number of the terms degree 6, weight 18, which can be formed with the seven letters (a, b, c, d, e, f, g).

When the literal terms are required, it is proper in the first instance as a safeguard against accidental omissions in copying, to take the numbers in this manner; and we can then take out the terms themselves, viz. these are

$$\begin{array}{ll} a^3 & g^3 \\ a^2b & fg^2 \\ a^2c & eg^2 \\ & f^2g \\ a^2 & d^2g^2 \\ & defg \\ & df^3 \\ & e^3g \\ & e^2f^2, \end{array}$$

and so on, the whole number being $= 58$ as just mentioned.

266.

ON THE EQUATION FOR THE PRODUCT OF THE DIFFERENCES OF ALL BUT ONE OF THE ROOTS OF A GIVEN EQUATION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. for the year 1861, pp. 45—59. Received November 30, 1860,—Read January 10, 1861.]

It is easy to see that for an equation of the order n, the product of the differences of all but one of the roots will be determined by an equation of the order n, the coefficients of which are alternately rational functions of the coefficients of the original equation, and rational functions multiplied by the square root of the discriminant. In fact, if the equation be $\phi v = (a, \ldots \between v, 1)^n = a(v-\alpha)(v-\beta)\ldots$, then putting for the moment $a=1$, and disregarding numerical factors, $\sqrt{\square}$, the square root of the discriminant, is equal to the product of the differences of the roots, and $\phi'\alpha$ is equal to $(\alpha-\beta)(\alpha-\gamma)\ldots$, consequently the product of the differences of the roots, all but α, is equal to $\sqrt{\square} \div \phi'\alpha$, and the expression $\dfrac{1}{\phi'\alpha}$ is the root of an equation of the order n, the coefficients of which are rational functions of the coefficients of the original equation. I propose in the present memoir to determine the equation in question for equations of the orders three, four, and five: the process employed is similar to that in my memoir "On the Equation of Differences for an Equation of any Order, and in particular for Equations of the Orders Two, Three, Four, and Five," *Phil. Trans.*, vol. CL. (1860), [262], viz. the last coefficient of the given equation is put equal to zero, so that the given equation breaks up into $v=0$ and into an equation of the order $n-1$ called the reduced equation; and this being so, the required equation breaks up into an equation of the order $n-1$ (which however is not, as for the equation of differences, that which corresponds to the reduced equation) and into a linear equation; the equation of the order $n-1$ is calculated by the method

of symmetric functions; and combining it with the linear equation, which is known, we have the required equation, except as regards the terms involving the last coefficient, which terms are found by the consideration that the coefficients of the required equation are seminvariants. The solution leads immediately to that of a more general question; for if the product of the differences of all the roots except α, of the given equation

$$\phi v = (*\between v,\ 1)^n = a\,(v-\alpha)\,(v-\beta)\ldots = 0$$

(which product is a function of the degree $n-2$ in regard to each of the roots $\beta,\ \gamma,\ \delta\,..$), is multiplied by $(x-\alpha y)^{n-2}$, the function so obtained will be the root of an equation of the order n, the coefficients of which are covariants of the quantic $(*\between x,\ y)^n$, and these coefficients can be at once obtained by writing, in the place of the seminvariants of the former result, the covariants to which they respectively belong. In the case of the quintic equation, one of these covariants is, in regard to the coefficients, of the degree 6, which exceeds the limit of the tabulated covariants, [the covariants are all tabulated, 141 and 143], the covariant in question has therefore to be now first calculated. The covariant equations for the cubic and the quartic might be deduced from the formulæ Nos. 119 and 142 of my Fifth memoir on Quantics *Phil. Trans.*, vol. CXLVIII. (1858), [156]; they are in fact the bases of the methods which are there given for the solution of the cubic and the quartic equations respectively; and it was in this way that I was led to consider the problem which is here treated of.

1. The notation $\zeta(\alpha,\ \beta,\ \gamma,\,..)$ is used (after Professor Sylvester) to denote the product of the squared differences of $(\alpha,\ \beta,\ \gamma,\,..)$, and the notation $\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\,..)$ to denote the product of the differences taken in a determinate order, viz.

$$\begin{aligned}\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta,\,..) = (\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\,..\\ (\beta-\gamma)\,(\beta-\delta)\,..\\ (\gamma-\delta)\,..\\ ..\end{aligned}$$

2. The product of the differences of the roots of an equation depends, as already noticed, on the square root of the discriminant; and in order to fix the numerical factors and signs, it will be convenient, in regard to the equations

$$(a,\ b,\ c\between v,\ 1)^2 = 0,$$

$$(a,\ b,\ c,\ d\between v,\ 1)^3 = 0,$$

$$(a,\ b,\ c,\ d,\ e\between v,\ 1)^4 = 0,$$

$$(a,\ b,\ c,\ d,\ e,\ f\between v,\ 1)^5 = 0,$$

to write as follows:

$$\zeta^{\frac{1}{2}}(\alpha, \beta) = \frac{1}{a}\sqrt{-(4ac-b^2)} = \frac{1}{a}\sqrt{-\square},$$

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma) = \frac{1}{a^2}\sqrt{-(27a^2d^2+4ac^3+\ldots)} = \frac{1}{a^2}\sqrt{-\square},$$

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) = -\frac{1}{a^3}\sqrt{256a^3e^3-27a^2d^4+\ldots} = -\frac{1}{a^3}\sqrt{\square},$$

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta, \epsilon) = -\frac{1}{a^4}\sqrt{3125a^4f^4+256a^3e^5+\ldots} = -\frac{1}{a^4}\sqrt{\square},$$

where it is to be observed, for example, that writing in the last equation $\epsilon=0$, and therefore $f=0$, we have $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta, 0) = -\frac{e}{a^4}\sqrt{256a^3e^3+\ldots}$, which agrees with the equation $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta, 0) = \alpha\beta\gamma\delta\ \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) = \frac{e}{a}\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$, if for $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$ we substitute the value given by the last equation but one.

For the cubic equation $(a, b, c, d \between v, 1)^3 = 0$;

3. We have to find the equation for $\theta = \zeta^{\frac{1}{2}}(\alpha, \beta) = \alpha - \beta$; the roots are

$$\theta_1 = \beta - \gamma, \quad \theta_2 = \gamma - \alpha, \quad \theta_3 = \alpha - \beta.$$

To apply the method above explained, write $\gamma = 0$, and therefore also $d = 0$; the roots thus become

$$\theta_1 = \beta, \quad \theta_2 = -\alpha, \quad \theta_3 = \alpha - \beta,$$

and we have the quadric and linear equations

$$(\theta + \alpha)(\theta - \beta) = 0, \quad \theta - (\alpha - \beta) = 0,$$

where (α, β) are the roots of the equation

$$(a, b, c \between v, 1)^2 = 0.$$

Hence, writing

$$Z = 4ac - b^2,$$

we have

$$\alpha - \beta = \frac{1}{a}\sqrt{-Z},$$

and the two equations become

$$\theta^2 a + \theta\sqrt{-Z} - c = 0, \quad \theta a - \sqrt{-Z} = 0;$$

or multiplying the two equations together,

$$\theta^3 a^2 + \theta^2 0 + \theta(3ac - b^2) + c\sqrt{-Z} = 0,$$

which is what the required equation becomes, on putting therein $d=0$; the coefficients of the complete equation are seminvariants, and the terms in d are to be inserted by means of this property. The coefficient $3ac-b^2$ is reduced to zero by the operator

$$3a\partial_b+2b\partial_c+c\partial_d,$$

it is therefore a seminvariant, and remains unaltered. The coefficient $c\sqrt{-Z}$ is what $\sqrt{-\square}$ becomes ($\square$ being the discriminant of the cubic equation) on putting therein $d=0$, it is therefore to be changed into $\sqrt{-\square}$. Hence

4. For the cubic equation $(a, b, c, d \between v, 1)^3$ the equation for $\theta\left(=\zeta^{\frac{1}{2}}(\alpha, \beta)\right)$ is $0=$

$$\left(\begin{array}{c|c|c|c} a^2\times & & & \sqrt{-\square}\times \\ \hline +1 & 0 & \begin{array}{c} ac+3 \\ b^2-1 \end{array} & +1 \end{array}\right) \between \theta, 1)^3.$$

5. For the quartic equation $(a, b, c, d, e \between v, 1)^4=0$;

$$\theta=-\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)=-(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma),$$

the roots are

$$\begin{aligned} \theta_1 &= \zeta^{\frac{1}{2}}(\beta, \gamma, \delta), \\ \theta_2 &= -\zeta^{\frac{1}{2}}(\gamma, \delta, \alpha), \\ \theta_3 &= \zeta^{\frac{1}{2}}(\delta, \alpha, \beta), \\ \theta_4 &= -\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma), \end{aligned}$$

the signs being in this case (and indeed for an equation of any even order) alternately positive and negative; in fact, if the equation is represented by $\phi v=0$, then the roots divided by $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$ should respectively be $\phi'\alpha$, $\phi'\beta$, $\phi'\gamma$, $\phi'\delta$, and this will be the case if the signs are taken as above.

6. Putting now $\delta=0$ (and therefore $e=0$) the roots become

$$\begin{aligned} \theta_1 &= \beta\gamma(\beta-\gamma), \\ \theta_2 &= \gamma\alpha(\gamma-\alpha), \\ \theta_3 &= \alpha\beta(\alpha-\beta), \\ \theta_4 &= -\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma), \end{aligned}$$

where (α, β, γ) are the roots of $(a, b, c, d \between v, 1)^3=0$. Let Z denote the discriminant of the cubic function, then $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)=\frac{1}{a^2}\sqrt{-Z}$, and we have thus the linear equation; the cubic equation is

$$\Pi_3\{\theta-\beta\gamma(\beta-\gamma)\}=0,$$

the coefficients of which can be calculated by the method of symmetric functions (see Annex No. 1).

7. The cubic equation being thus obtained, we have the two equations

$$\begin{array}{l|l} \theta^3 \,.\, a^4 & \theta \,.\, a^2 \\ + \theta^2 \,.\, - a^2 \sqrt{-Z} & + \sqrt{-Z} \\ + \theta \,.\, - 9a^2d^2 + 4abcd - b^3d & \\ + \qquad d^2 \sqrt{-Z} & \\ = 0 & = 0\,; \end{array}$$

and multiplying these together, the resultant equation is

$$\begin{aligned} & \theta^4 \,.\, a^6 \\ & + \theta^3 \,.\, 0 \\ & + \theta^2 \,.\, a^2 \,(- 9a^2d^2 + 4abcd - b^3d + Z) \\ & + \theta \,.\, \quad (- 8a^2d + 4abc \;\; - b^3)\, d \sqrt{-Z} \\ & \qquad\qquad - d^2Z = 0, \end{aligned}$$

where the coefficients have to be completed by adding the terms which contain e. We have $\sqrt{\square}$ in the place of $d\sqrt{-Z}$, and $\square$ in the place of $-d^2Z$. The coefficient $-8a^2d + 4abc - b^3$ is a seminvariant, and requires no alteration. The coefficient

$$- \;\; 9a^2d^2 + \;\; 4abcd - \;\; b^3d + Z$$

is

$$\begin{aligned} & - \;\; 9a^2d^2 + \;\; 4abcd \qquad\qquad\qquad - b^3d \\ & + 27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^3c^2\,; \end{aligned}$$

that is,

$$\begin{array}{ll} a^2d^2 & +18 \\ abcd & -14 \\ ac^3 & +\;\;4 \\ b^3d & +\;\;3 \\ b^2c^2, & -\;\;1 \end{array}$$

and the terms in e to be added to this, in order to make it a seminvariant, are easily found to be

$$\begin{array}{ll} a^2ce & -16 \\ ab^2e & +\;\;6 \end{array}$$

8. Hence, for the quartic equation $(a, b, c, d \between v, 1)^4$, the equation for $\theta\left(= \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)\right)$ is $0 =$

$a^4 \times$	$a^2\sqrt{\square} \times$	$a^2 \times$	$\sqrt{\square} \times$	$\square \times$
$a^2 + 1$	$a\,0$	$a^2ce - 16$ $a^2d^2 + 18$ $ab^2c + 6$ $abcd - 14$ $ac^3 + 4$ $b^3d + 3$ $b^2c^2 - 1$	$a^2d - 8$ $abc + 4$ $b^3 - 1$	$+ 1$

$\between (\theta, 1)^4.$

For the quintic equation $(a, b, c, d, e, f \between v, 1)^5 = 0\,;$

9. We have $\theta = \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)$, the roots being

$$\begin{array}{lll}
\theta_1 = \zeta^{\frac{1}{2}}(\beta, \gamma, \delta, \epsilon), & \text{which for } \epsilon = 0 \text{ becomes} & \beta\gamma\delta\zeta^{\frac{1}{2}}(\beta, \gamma, \delta), \\
\theta_2 = \zeta^{\frac{1}{2}}(\gamma, \delta, \epsilon, \alpha), & \quad ,, \qquad ,, & -\gamma\delta\alpha\zeta^{\frac{1}{2}}(\gamma, \delta, \alpha), \\
\theta_3 = \zeta^{\frac{1}{2}}(\delta, \epsilon, \alpha, \beta), & \quad ,, \qquad ,, & \delta\alpha\beta\zeta^{\frac{1}{2}}(\delta, \alpha, \beta), \\
\theta_4 = \zeta^{\frac{1}{2}}(\epsilon, \alpha, \beta, \gamma), & \quad ,, \qquad ,, & -\alpha\beta\gamma\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma), \\
\theta_5 = \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta), & \quad ,, \qquad ,, & \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta).
\end{array}$$

10. The linear equation is $\theta a^3 + \sqrt{Z} = 0$; the quartic equation may be written $\Pi_4(\theta - \theta_1) = 0$, for the determination of which see Annex No. 2. The two equations are

$$\begin{array}{l|l}
\theta^4 .\ a^9 & \theta . a^3 \\
+\theta^3 - a^6\sqrt{Z} & +\sqrt{Z} \\
+\theta^2 .\ a^3 Me & \\
+\theta .\ Ne^2\sqrt{Z} & \\
+\quad e^3 Z & \\
=0 & =0;
\end{array}$$

and multiplying these together, the resulting equation is

$$\begin{aligned}
&\theta^5 . a^{12} \\
&+\theta^4 . 0 \\
&+\theta^3 . a^6 (Me - Z) \\
&+\theta^2 . a^3 (Ne + M) e\sqrt{Z} \\
&+\theta\ .\quad (N + a^3 e) e^2 Z \\
&+\qquad e^3 Z\sqrt{Z} = 0,
\end{aligned}$$

where the coefficients have to be completed by the addition of the terms in f. We have $\sqrt{\square}$ in the place of $e\sqrt{Z}$, and therefore $\square$ in the place of $e^2 Z$.

11. The value of $Me - Z$ is

a^3e^3	$+96$	$-256 =$	-160
a^2bde^2	-60	$+192 =$	$+132$
$a^2c^2e^2$	-40	$+128 =$	$+88$
a^2cd^2e	$+27$	$-144 =$	-177
a^2d^4		$+27 =$	$+27$
ab^2ce^2	$+47$	$-144 =$	-97
ab^2d^2e		$+6 =$	$+6$
abc^2de	-18	$+80 =$	$+62$
$abcd^3$		$-18 =$	-18
ac^4e	$+4$	$-16 =$	-12
ac^2d^2		$+4 =$	$+4$
b^4e^2	-9	$+27 =$	$+18$
b^3cde	$+4$	$-18 =$	-14
b^3d^3		$+4 =$	$+4$
b^2c^3e	-1	$+4 =$	$+3$
$b^2c^2d^2$		$-1 =$	-1

and the terms in f are found to be

a^3def	$+300$
a^2bcef	-130
a^2bd^2f	-120
a^2c^2df	$+40$
ab^2ef	$+28$
ab^2cdf	$+66$
abc^3f	-24
b^4df	-16
b^3c^2f	$+6$
a^2cf^2	-125
$a^2b^2f^2$	$+50$

12. The value of $Ne+M$ is

a^3e^2	-16	$+96 =$	$+80$
a^2bde	$+6$	$-60 =$	-54
a^2c^2e	$+4$	$-40 =$	-36
a^2cd^2		$+27 =$	$+27$
ab^2ce	-5	$+47 =$	$+42$
abc^2d		$-18 =$	-18
ac^4		$+4 =$	$+4$
b^4e	$+1$	$-9 =$	-8
b^3cd		$+4 =$	$+4$
b^2c^3		$-1 =$	-1

and the terms in f to be added thereto are found to be

a^3df	-50
a^2bcf	$+30$
ab^2f	-8

13. The value of $N+a^3e$ is

a^3e	-15
a^2bd	$+6$
a^2c^2	$+4$
ab^2c	-5
b^4	$+1$

which is a seminvariant, and requires no addition.

14. Hence, for the quintic equation

$$(a,\ b,\ c,\ d,\ e,\ f\between v,\ 1)^6 = 0,$$

the equation for $\theta\left(=\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta)\right)$ is $0 =$

$a^{12}\times$		$a^6\times$	$a^3\sqrt{\square}\times$	$\square\times$	$\square\sqrt{\square}\times$
$+1$	0	a^2cf^2 -125	a^2df -50	a^2e -15	$+1$
		a^2def $+300$	a^2e^2 $+80$	a^2bd $+6$	
		a^2e^2 -160	a^2bcf $+30$	a^2c^2 $+4$	
		$a^2b^2f^2$ $+50$	a^2bde -54	ab^2c -5	
		a^2bde^2 $+132$	a^2c^2e -36	b^4 $+1$	
		a^2bcef -130	a^2cd^2 $+27$		
		a^2bd^2f -120	ab^2f -8		
		a^2c^2df $+40$	ab^2ce $+42$		
		$a^2c^2e^2$ $+88$	abc^2d -18		
		a^2cd^2e -117	ac^4 $+4$		
		a^2d^4 $+27$	b^4e -8		
(		ab^2ef $+28$	b^3cd $+4$		$\between\theta,\ 1)^5.$
		ab^2ce^2 -97	b^2c^3 -1		
		ab^2d^2e $+6$			
		abc^2de $+62$			
		ab^2cdf $+66$			
		$abcd^3$ -18			
		abc^2f -24			
		ac^4e -12			
		ac^3d^2 $+4$			
		b^4df -16			
		b^4e^2 $+18$			
		b^3cde -14			
		b^3d^3 $+4$			
		b^3c^2f $+6$			
		b^2c^3e $+3$			
		$b^2c^2d^2$ -1			

15. As a verification of this result, I remark that, taking for the quintic equation $v^5+v^4+v^3+v^2+v+1=0$, the roots of this equation are $-1,\ \omega,\ \omega^2,\ -\omega,\ -\omega^2$, where ω is an imaginary cube root of unity ($\omega^2+\omega+1=0$). We ought to have $\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon) = -\sqrt{\square} = -36$; and this will be the case if, for instance, $\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon$ are respectively $-1,\ \omega,\ \omega^2,\ -\omega^2,\ -\omega$. We have then

$$\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta) = -1-\omega\,.\,-1-\omega^2\,.\,-1+\omega^2\,.\,\omega-\omega^2\,.\,\omega+\omega^2\,.\,2\omega^2 \qquad = \quad 6,$$

$$\zeta^{\frac{1}{2}}(\beta,\ \gamma,\ \delta,\ \epsilon) = \quad \omega-\omega^2\,.\,\omega+\omega^2\,.\,2\omega\,.\,2\omega^2\,.\,\omega^2+\omega\,.\,-\omega^2+\omega \qquad = -12,$$

$$\zeta^{\frac{1}{2}}(\gamma,\ \delta,\ \epsilon,\ \alpha) = \quad 2\omega^2\,.\,\omega^2+\omega\,.\,\omega^2+1\,.\,-\omega^2+\omega\,.\,-\omega^2+1\,.\,-\omega+1 \qquad = +6\,(\omega-\omega^2),$$

$$\zeta^{\frac{1}{2}}(\delta,\ \epsilon,\ \alpha,\ \beta) = -\omega^2+\omega\,.\,-\omega^2+1\,.\,-\omega^2-\omega\,.\,-\omega+1\,.\,-2\omega\,.\,-1-\omega = -6\,(\omega-\omega^2),$$

$$\zeta^{\frac{1}{2}}(\epsilon,\ \alpha,\ \beta,\ \gamma) = -\omega+1\,.\,-2\omega\,.\,-\omega-\omega^2\,.\,-1-\omega\,.\,-1-\omega^2\,.\,\omega-\omega^2 \qquad = \quad 6.$$

The equation in θ is thus $(\theta-6)^2(\theta+12)(\theta^2+108)=0$, or multiplying out it is

$$(1,\ 0,\ 0,\ +432,\ -11664,\ +46656\between\theta,\ 1)^4=0,$$

which in fact (observing that $\sqrt{\square}=36$) is what the preceding formula becomes for the equation $(1,\ 1,\ 1,\ 1,\ 1,\ 1\between v,\ 1)^5=0$.

The analogous verifications for the cubic and the quartic equations are as follows:

16. For the cubic, if the assumed equation is $v^3+v^2+v+1=0$, the roots whereof are $-1,\ i,\ -i\,(i^2=-1)$, then we should have $\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma)=\sqrt{-\square}=4i$, which will be the case if $\alpha,\ \beta,\ \gamma=-1,\ i,\ -i$, respectively, and the roots $\beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta$ of the equation in θ then are $2i,\ -i+1,\ -i-1$, so that the equation in θ is $(\theta^2+2i\theta-2)(\theta-2i)=0$, or

$$(1,\ 0,\ 2,\ 4i\between\theta,\ 1)^3=0,$$

which (observing that $\sqrt{\square}=4i$) is what the formula for the equation in θ becomes for the equation $(1,\ 1,\ 1,\ 1\between v,\ 1)^3=0$.

17. For the quartic equation, taking this to be $v^4+v^3+v^2+v+1=0$, the roots are $\omega,\ \omega^2,\ \omega^3,\ \omega^4$, where ω is an imaginary fifth root of unity $(\omega^4+\omega^3+\omega^2+\omega+1=0)$, and putting $\alpha,\ \beta,\ \gamma,\ \delta$ equal to $\omega,\ \omega^2,\ \omega^3,\ \omega^4$ respectively, we have

$$-\sqrt{\square}=\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta)=-5(\omega+\omega^4-\omega^2-\omega^3),$$

giving, as it should do, $\square=125$. The equation in θ is therefore by the formula

$$\big(1,\ 0,\ 0,\ -25(\omega+\omega^4-\omega^2-\omega^3),\ 125\between\theta,\ 1\big)^4=0.$$

But the roots are

$$\begin{aligned}
\theta_1 &= \ \ \zeta^{\frac{1}{2}}(\beta,\ \gamma,\ \delta)=-.\,\omega^2-\omega^3.\,\omega^2-\omega^4.\,\omega^3-\omega^4= \ \ 2-\ \omega+\ \omega^2-2\omega^3= \ \ 2-X,\\
\theta_2 &= -\zeta^{\frac{1}{2}}(\gamma,\ \delta,\ \alpha)=-.\,\omega^3-\omega^4.\,\omega^3-\omega\ .\,\omega^4-\omega\ =-1+3\omega+2\omega^2+\ \ \omega^3=-1+Y,\\
\theta_3 &= \ \ \zeta^{\frac{1}{2}}(\delta,\ \alpha,\ \beta)= \ \ .\,\omega^4-\omega\ .\,\omega^4-\omega^2.\,\omega\ -\omega^2=-4-3\omega-2\omega^2-\ \ \omega^3=-4-Y,\\
\theta_4 &= -\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma)=-.\,\omega\ -\omega^2.\,\omega\ -\omega^3.\,\omega^2-\omega^3= \ \ 3+\ \omega-\ \omega^2+2\omega^3= \ \ 3+X,
\end{aligned}$$

if, for shortness,

$$X=\omega-\omega^2+2\omega^3,\quad Y=3\omega+2\omega^2+\omega^3.$$

The equation in θ is therefore

$$(\theta-2+X)(\theta+1-Y)(\theta+4+Y)(\theta-3-X)=0,$$

where the left-hand side is the product of the factors

$$(\theta-2+X)(\theta-3-X)=\theta^2-5\theta+6-\ X-X^2=\theta^2-5\theta+10-5(\omega+\omega^4)$$

and

$$(\theta+1-Y)(\theta+4+Y)=\theta^2+5\theta+4-3Y-Y^2=\theta^2+5\theta+10-5(\omega^2+\omega^3)$$

and the equation in θ is, therefore, as it should be,

$$\big(1,\ 0,\ 0,\ -25(\omega+\omega^4-\omega^2-\omega^3),\ 125\between\theta,\ 1\big)^4=0.$$

Passing from the denumerate to the standard forms:

18. For the cubic equation $(a, b, c, d \between v, 1)^3 = 0$, the equation for $\theta\left(= \zeta^{\frac{1}{2}}(\alpha, \beta)\right)$ is $0 =$

$a^2 \times$		$9 \times$	$\sqrt{-27\square} \times$
$+1$	0	$ac + 1$ $b^2 - 1$	$+1$

$\between \theta, 1)^3.$

19. For the quartic equation $(a, b, c, d, e \between v, 1)^4 = 0$, the equation for $\theta\left(= \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)\right)$ is $0 =$

$a^6 \times$		$96a^2 \times$	$512\sqrt{\square} \times$	$256\,\square \times$
$+1$	0	$a^2ce - 1$ $a^2d^2 + 3$ $ab^2e + 1$ $abcd - 14$ $ac^3 + 9$ $b^3d + 8$ $b^2c^2 - 6$	$a^2d - 1$ $abc + 3$ $b^3 - 2$	$+1$

$\between \theta, 1)^4.$

20. For the quintic equation $(a, b, c, d, e, f \between v, 1)^5 = 0$, the equation for $\theta\left(= \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)\right)$ is $0 =$

$a^{12} \times$		$625\,a^6 \times$	$12500\sqrt{\square}a^3 \times$	$15625\,\square \times$	$76125\,\square\sqrt{\square}$
$+1$	0	$a^3cf^2 - 2$ $a^3def + 24$ $a^3e^3 - 32$ $a^2b^2f^2 + 2$ $a^2bde^2 + 264$ $a^2bcef - 52$ $a^2bd^2f - 96$ $a^2c^2df + 64$ $a^2c^2e^2 + 352$ $a^2cd^2e - 936$ $a^2d^4 + 432$ $ab^2ef + 28$ $ab^2ce^2 - 970$ $ab^2d^2e + 120$ $ab^2cdf + 264$ $abc^2de + 2480$ $abcd^3 - 1440$ $abc^3f - 192$ $ac^4e - 960$ $ac^3d^2 + 640$ $b^4df - 160$ $b^4e^2 + 450$ b^3cde 1400 $b^3d^3 + 800$ $b^3c^2f + 120$ $b^2c^3e + 600$ $b^2c^2d^2 - 400$	$a^3df - 1$ $a^3e^2 + 4$ $a^2bcf + 3$ $a^2bde - 27$ $a^2c^2e - 36$ $a^2cd^2 + 54$ $ab^3f - 2$ $ab^2ce + 105$ $abc^2d - 180$ $ac^4 + 80$ $b^4e - 50$ $b^3cd + 100$ $b^3c^3 - 50$	$a^3e - 3$ $a^2bd + 12$ $a^2c^2 + 16$ $ab^2c - 50$ $b^4 + 25$	$+1$

$\between \theta, 1)^5.$

21. I remark, with respect to the equation in θ, for the cubic, that it leads at once to the equation of differences. In fact we have

$$a^2\theta^3 + 9(ac - b^2)\theta + \sqrt{-27\square} = \Pi_3\{\theta - (\alpha - \beta)\},$$

whence changing the sign of θ,

$$a^2\theta^3 + 9(ac - b^2)\theta - \sqrt{-27\square} = \Pi_3\{\theta + (\alpha - \beta)\};$$

or multiplying the two equations and putting u for θ^2,

$$u\{a^2u + 9(ac - b^2)\}^2 + 27\square = \Pi_3\{u - (\alpha - \beta)^2\},$$

that is, the equation of differences is

$$a^4u^3 + 18(ac - b^2)a^2u^2 + 81(ac - b^2)^2u + 27\square = 0;$$

but this mode of composition is peculiar to the case of the cubic.

If in the several equations in θ we substitute for the seminvariants the covariants to which they respectively belong, we obtain as follows:

22. For the cubic equation $(a, b, c, d \between v, 1)^3 = 0$, the equation for $\left(\vartheta = (\beta - \gamma)(x - \alpha y)\right)$ is

$$\left(\begin{array}{|c|c|c|c|}\hline U^2\times & & 9\times & \sqrt{-27}\,\square\times \\ \hline +1 & 0 & H & +1 \\ \hline\end{array}\right. \between \vartheta, 1)^3 = 0.$$

23. For the quartic equation $(a, b, c, d, e \between v, 1)^4 = 0$, the equation for

$$\vartheta\left(= (\beta - \gamma)(\gamma - \delta)(\delta - \beta)(x - \alpha y)^2\right) \text{ is}$$

$$\left(\begin{array}{|c|c|c|c|c|}\hline U^6\times & & 96U^3\times & 512\sqrt{\square}\times & 256\square\times \\ \hline +1 & & \begin{array}{c}-3JU\\+2IH\end{array} & -\Phi & +1 \\ \hline\end{array}\right. \between \vartheta, 1)^4 = 0.$$

24. And for the quintic equation $(a, b, c, d, e, f \between v, 1)^5 = 0$, [denoting the covariants of the quintic as in 141, A the quintic itself, &c.; and completing the expression for the coefficient of ϑ^2] the equation for

$$\vartheta\left(= (\beta - \gamma)(\beta - \delta)(\beta - \epsilon)(\gamma - \delta)(\gamma - \epsilon)(\delta - \epsilon)(x - \alpha y)^3\right) \text{ is}$$

$$\left\{\begin{array}{l} A^{12}, \\ 0\ , \\ 625A^6(48AJ - 80B^3 + 120BH - 50CG, \\ 12500\sqrt{\square}\,A^3\{4AB^2 - AH - 50CD\}, \\ 15625\ \square\ \{3A^2B + 25C^2\}, \\ 76125\ \square\sqrt{\square} \end{array}\right\} \between \vartheta, 1)^5 = 0,$$

where the covariant which enters into the coefficient of ϑ^2 being of the sixth degree in the coefficients, is not given in the Tables.

Its value (completed for me, from the first term, by Mr Davis) is

a^3cf^2 − 2	a^3df^2 + 18	a^2ef^2 − 30	a^3f^3 − 10	a^3bf^2 − 30	a^2cf^3 + 18	a^2df^3 − 2	
a^3def + 24	a^3e^2f − 48	a^2bdf^2 + 210	a^2bef^2 + 30	a^2cef^2 + 210	a^2def^2 − 66	$a^2e^2f^3$ + 2	
a^3e^3 − 32	a^2bcf^2 − 66	a^2be^2f + 60	a^2cdf^2 + 680	$a^2d^2f^2$ + 180	a^2e^3f + 48	$abcf^3$ + 24	
$a^2b^2f^2$ + 2	a^2bdef + 348	$a^2c^2f^2$ + 180	a^2ce^2f − 320	a^2de^2f − 840	ab^2f^3 − 48	$abdef^2$ − 52	
a^2bde^2 + 264	a^2be^3 + 48	a^2cdef + 120	a^2d^2ef − 1320	a^3e^4 + 480	$abcef^2$ + 348	abe^2f + 28	
a^2bcef − 52	a^2c^2ef + 624	a^2ce^3 − 480	a^2de^3 + 960	ab^2ef^2 + 60	abd^2f^2 + 624	ac^2ef^2 − 96	
a^2bd^2f − 96	a^2cd^2f − 936	a^2d^3f − 1080	ab^2df^2 − 320	$abcdf^2$ + 120	abe^4 + 720	acd^2f^2 + 64	
a^2c^2df + 64	a^2cde^2 − 576	$a^2d^2e^2$ + 1080	ab^2e^2f + 290	$abce^2f$ − 540	$abde^2f$ − 1596	$acde^2f$ + 264	
$a^2c^2e^2$ + 352	a^2d^3e + 648	ab^2cf^2 − 840	abc^2f^2 − 1320	abd^2ef + 960	ac^2df^2 − 936	ace^4 − 160	
a^2cd^2e − 936	ab^3f^2 + 48	ab^2def − 540	$abcdef$ + 4160	$abde^3$ − 600	ac^2e^2f − 48	ad^3ef − 192	
a^2d^4 + 432	ab^2cef − 1596	ab^2e^3 + 450	$abce^3$ − 3200	ac^3f^2 − 1080	acd^2ef + 3504	ad^2e^3 + 120	
ab^3ef + 28	ab^2de^2 + 210	abc^2ef + 960	abd^3f + 960	ac^2def + 4560	ad^4f − 1728	b^2f^3 − 32	
ab^2ce^2 − 970	ab^2d^2f − 48	$abcd^2f$ + 4560	abd^2e^2 − 600	ac^2e^3 − 2400	$acde^3$ − 1920	b^2cef^2 + 264	
ab^2d^2e + 120	abc^2df + 3504	$abcde^2$ − 4200	ac^3ef + 960	acd^3f − 2880	ad^2e^2 + 1080	$b^2d^2f^2$ + 352	$\mathfrak{J}(x, y)^6$.
ab^2cdf + 264	abc^2e^2 + 720	ac^4df − 2880	ac^3d^2f − 2560	acd^2e^2 + 1800	b^3ef^2 + 48	b^2de^2f − 970	
abc^2de + 2480	$abcd^2e$ − 2160	ac^3e^2 + 2400	ac^3de^2 + 1600	b^2df^2 − 480	b^2e^3f + 450	b^2e^4 + 450	
abc^3f − 192	ac^4f − 1728	b^4f^2 + 480	bc^3f^2 + 960	b^2e^2f + 450	b^2cdf − 576	b^2cef + 210	bc^3df^2 − 936
$abcd^3$ − 1440	ac^3de + 960	b^3cef − 600	b^3def − 3200	$b^2c^2f^2$ + 1080	b^2ce^2f + 210	bc^2df + 720	bcd^2ef + 2480
ac^4e − 960	b^4ef + 720	b^3d^2f − 2400	b^3e^3 + 2250	b^2cdef − 4200	b^2d^2ef + 720	b^2de^2 − 450	bc^2e^2f + 120
ac^3d^2 + 640	b^3cdf − 1920	b^3de^2 + 2250	b^2c^2ef − 600	b^2ce^3 + 2250	bc^3f^2 + 648	$bcde^2$ − 1400	
b^4df − 160	b^3ce^2 − 450	b^2c^2df + 1800	b^2cd^2f + 1600	b^2d^3f + 2400	bc^2def − 2160	bd^4f − 960	
b^4e^2 + 450	b^2d^2e + 1200	$b^2c^2e^2$ − 1500	b^2cde^2 − 1000	$b^2d^2e^2$ − 1500	bc^2e^3 + 1200	bd^3e^2 + 600	
b^3cde − 1400	b^2c^3f + 1080				bcd^2f + 960	c^4f^2 + 432	
b^3c^2f + 120	b^2c^2de − 600				bcd^2e^2 − 600	c^3def − 1440	
b^3d^3 + 800						c^3e^3 + 800	
b^2c^3e + 600						c^2d^2f + 640	
$b^2c^2d^2$ − 400						$c^2d^2e^2$ − 400	

[viz. this is $= 48AJ - 80B^3 + 120BH - 50CG$ as above].

In the following two Annexes, the notation of the symmetric functions is the same as in my "Memoir on the Symmetric Functions of the Roots of an Equation," *Phil. Trans.* vol. CXLVII. (1857), [147] and the values of the symmetric functions are taken from that memoir, the powers of a being restored by the principle of homogeneity. The suffixes of the Σ indicate the number of terms in the sum; thus in the first Annex

$$\Sigma_3\gamma(\beta-\gamma)(\gamma-\alpha) = \Sigma_3(\beta\gamma^2 - \alpha\beta\gamma - \gamma^3 + \alpha\gamma^2);$$

the terms $\Sigma_3(\beta\gamma^2 + \alpha\gamma^2)$ are equal to $\Sigma_6\alpha^2\beta$, the complete symmetric function; the correct result will be obtained (though of course neither of these equations is true) by writing $\Sigma_3\beta\gamma^2 = \frac{1}{2}\Sigma_6\alpha^2\beta$, $\Sigma_3\alpha\gamma^2 = \frac{1}{2}\Sigma_6\alpha^2\beta$, and so in similar cases; the insertion of the suffix to the Σ very much facilitates the calculation, and is a check on its accuracy.

Annex No. 1, containing the calculation of the equation $\Pi_3(\theta - \theta_1) = 0$, where

$$\theta_1 = \beta\gamma(\beta-\gamma),\ \theta_2 = \gamma\alpha(\gamma-\alpha),\ \theta_3 = \alpha\beta(\alpha-\beta),$$

α, β, γ being the roots of the cubic equation $(a, b, c, d\between y, 1)^3 = 0$.

We have

$$\Sigma_3\theta_1 = \Sigma\beta\gamma(\beta-\gamma) = -(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = \zeta^{\frac{1}{2}}(\alpha, \beta, \gamma) = a^{-2}\sqrt{-Z},$$

where $Z = 27a^2d^2 + \&c.$ is the discriminant of the cubic.

$$\Sigma_3\theta_1\theta_2 = \Sigma_3\beta\gamma\,(\beta-\gamma)\,\gamma\alpha\,(\gamma-\alpha) = \alpha\beta\gamma\Sigma_3\gamma\,(\beta-\gamma)\,(\gamma-\alpha),$$

where $\alpha\beta\gamma = -a^{-1}d$ and

$$\Sigma_3\gamma\,(\beta-\gamma)\,(\gamma-\alpha) = \Sigma_3\,(\beta\gamma^2 - \alpha\beta\gamma - \gamma^3 + \alpha\gamma^3)$$

$$\begin{array}{rl|l} = -\Sigma_3\alpha^3 & = -\ (\ 3) = a^{-2} & 3a^2d - 3abc + 1b^3 \\ +\Sigma_6\alpha^2\beta & +\ (21) & +3a^2d - 1abc \\ -3\alpha\beta\gamma & -3\,(13) & +3a^2d \end{array}$$

$$= a^{-2}\,(9a^2d - 4abc + 1b^3),$$

and therefore

$$\Sigma_3\theta_1\theta_2 = -a^{-3}\,(9a^3d^3 - 4abcd + 1b^3d).$$

And lastly,

$$\Sigma_1\theta_1\theta_2\theta_3,\ \text{or}\ \theta_1\theta_2\theta_3 = \alpha^3\beta^3\gamma^3\,(\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)$$

$$= -\alpha^3\beta^3\gamma^3\zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma)$$

$$= -a^{-4}d^3\sqrt{-Z}$$

so that the equation is the one given above, No. 7.

Annex No. 2, containing the calculation of the equation $\Pi_4(\theta - \theta_1) = 0$, where

$\theta_1 = \beta\gamma\delta\zeta^{\frac{1}{2}}\,(\beta,\ \gamma,\ \delta)$, $\theta_2 = -\gamma\delta\alpha\zeta^{\frac{1}{2}}\,(\gamma,\ \delta,\ \alpha)$, $\theta_3 = \delta\alpha\beta\zeta^{\frac{1}{2}}\,(\delta,\ \alpha,\ \beta)$, and $\theta_4 = -\alpha\beta\gamma\zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma)$,

$\alpha,\ \beta,\ \gamma,\ \delta$ being the roots of the quartic equation $(a,\ b,\ c,\ d,\ e\between v,\ 1)^4 = 0$.

$$\Sigma_4\theta_1 = (^1)\,\Sigma_4\beta\gamma\delta\zeta^{\frac{1}{2}}\,(\beta,\ \gamma,\ \delta) = -(\alpha-\beta)\,(\alpha-\gamma)\,(\alpha-\delta)\,(\beta-\gamma)\,(\beta-\delta)\,(\gamma-\delta)$$

$$= -\zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma,\ \delta)$$

$$= -a^{-3}$$

where $Z = 256a^3e^3 + \&c.$ is the discriminant of the quartic.

$$\Sigma_6\theta_1\theta_2 = \Sigma_6\theta_3\theta_4 = \Sigma_6\delta\alpha\beta\ \zeta^{\frac{1}{2}}\,(\delta,\ \alpha,\ \beta) \times -\alpha\beta\gamma\ \zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma)$$

$$= \Sigma_6\delta\alpha\beta\,(\delta-\alpha)\,(\delta-\beta)\,(\alpha-\beta) \times -\alpha\beta\gamma\,(\alpha-\beta)\,(\alpha-\gamma)\,(\beta-\gamma)$$

$$= -\alpha\beta\gamma\delta\ \Sigma_6\alpha\beta\,(\alpha-\beta)^2\,(\alpha-\gamma)\,(\alpha-\delta)\,(\beta-\gamma)\,(\beta-\delta),$$

where $\alpha\beta\gamma\delta = -a^{-1}e$, and

[1] The signs of θ_1, θ_2, θ_3, θ_4 are taken account of implicitly.

$$\Sigma_6\alpha\beta(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$$
$$=\Sigma_6\alpha\beta(\alpha-\beta)^2\{\alpha^2\beta^2-\alpha\beta(\alpha+\beta)(\gamma+\delta)+(\alpha^2+\beta^2)\gamma\delta+\alpha\beta(\gamma+\delta)^2-\gamma\delta(\alpha+\beta)(\gamma+\delta)+\gamma^2\delta^2\}$$

$$\left.\begin{array}{ll}
=\Sigma_6\alpha^3\beta^3(\alpha-\beta)^2, \text{ viz.} & \Sigma_{12}\alpha^5\beta^3 \\
 & -2\Sigma_6\alpha^4\beta^4 \\
-\Sigma_6\alpha^3\beta^2(\alpha-\beta)^2(\alpha+\beta)(\gamma+\delta), \text{ viz.} & -\Sigma_{24}\alpha^5\beta^2\gamma \\
 & +\Sigma_{24}\alpha^4\beta^3\gamma \\
+\alpha\beta\gamma\delta\Sigma_6(\alpha-\beta)^2(\alpha^2+\beta^2), \text{ viz} & +\frac{e}{a}3\Sigma_4\alpha^4 \\
 & -2\frac{e}{a}\Sigma_{12}\alpha^3\beta \\
 & +2\frac{e}{a}\Sigma_6\alpha^2\beta^2 \\
+\Sigma_6\alpha^2\beta^2(\alpha-\beta)^2(\gamma+\delta)^2, \text{ viz.} & +2\Sigma_{12}\alpha^4\beta^2\gamma^2 \\
 & -2\Sigma_{12}\alpha^3\beta^3\gamma^2 \\
 & +2\frac{e}{a}\Sigma_{12}\alpha^3\beta \\
 & -4\frac{e}{a}\Sigma_6\alpha^2\beta^2 \\
-\alpha\beta\gamma\delta\Sigma_6(\alpha-\beta)^2(\alpha+\beta)(\gamma+\delta), \text{ viz.} & -2\frac{e}{a}\Sigma_{12}\alpha^3\beta \\
 & +2\frac{e}{a}\Sigma_{12}\alpha^3\beta\gamma \\
+\alpha\beta\gamma\delta\Sigma_6(\alpha-\beta)^2\gamma\delta, \text{ viz.} & +\frac{e}{a}\Sigma_{12}\alpha^2\beta\gamma \\
 & -12\frac{e^2}{a^2}
\end{array}\right\}=\left.\begin{array}{ll}
 & \Sigma_{12}\alpha^5\beta^3 \\
 & -2\Sigma_6\alpha^4\beta^4 \\
 & -\Sigma_{24}\alpha^5\beta^2\gamma \\
 & +\Sigma_{24}\alpha^4\beta^3\gamma \\
 & +2\Sigma_{12}\alpha^4\beta^2\gamma^2 \\
 & -2\Sigma_{12}\alpha^3\beta^3\gamma^2 \\
 & +\frac{e}{a}3\Sigma_4\alpha^4 \\
 & -2\frac{e}{a}\Sigma_{12}\alpha^3\beta \\
 & -2\frac{e}{a}\Sigma_6\alpha^2\beta^2 \\
 & +3\frac{e}{a}\Sigma_{12}\alpha^2\beta\gamma \\
 & -12\frac{e^2}{a^2}
\end{array}\right\}=\begin{array}{l}
(53) \\
-2\,(4^2) \\
-1\,(521) \\
+1\,(431) \\
+2\,(42^2) \\
-2\,(3^22) \\
+3e\,(4) \\
-2e\,(31) \\
-2e\,(2^2) \\
+3e\,(21^2) \\
-12e^2,
\end{array}$$

where for a moment a is put equal to unity.

The value of the last-mentioned expression is then calculated as follows:

e^2	−4	−12	−8	−8	−8	−8	−12	−8	−4	−12	−12	=	−96
bde	+1	+16	+10	+10		+2	+12	+2	+4	+3		=	+60
c^2e	+8	+8	+4		+8	+4	+6	+4	−2			=	+40
b^2ce	−9	−8	−11	−1	−4		−12	−2				=	−47
b^4e	+3		+3				+3					=	+9
cd^2	−7	−8	−5	−1	−4	−2						=	−27
b^2d^2	+3	−4	+1	−2	+2							=	0
bc^2d	+6	+8	+3	+1								=	+18
b^3cd	−3		−1									=	−4
c^4	−2	−2										=	−4
b^2c^2	+1											=	+1

and restoring the powers of a by the principle of homogeneity, and putting

$$M = \boxed{\begin{array}{ll} a^3e^2 & +96 \\ a^2bde & -60 \\ a^2c^2e & -40 \\ a^2cd^2 & +27 \\ ab^2ce & +47 \\ abc^2d & -18 \\ ac^4 & +4 \\ b^4e & -9 \\ b^3ed & +4 \\ b^2c^3 & -1 \end{array}}$$

we have

$$\Sigma\theta_1\theta_2 = a^{-6}Me.$$

Next,

$$\begin{aligned}
\Sigma_4\theta_1\theta_2\theta_3 &= \Sigma_4\beta\gamma\delta\,\zeta^{\frac{1}{2}}(\beta,\ \gamma,\ \delta)\times-\gamma\delta\alpha\,\zeta^{\frac{1}{2}}(\gamma,\ \delta,\ \alpha)\times\delta\alpha\beta\,\zeta^{\frac{1}{2}}(\delta,\ \alpha,\ \beta)\\
&= \Sigma_4\beta\gamma\delta\,(\beta-\gamma)(\beta-\delta)(\gamma-\delta)\times-\gamma\delta\alpha\,(\gamma-\delta)(\gamma-\alpha)(\delta-\alpha)\times\delta\alpha\beta\,(\delta-\alpha)(\delta-\beta)(\alpha-\beta)\\
&= \alpha^2\beta^2\gamma^2\delta^2\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)\,\Sigma_4\alpha\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\\
&= \alpha^2\beta^2\gamma^2\delta^2\,\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma,\ \delta)\,\Sigma_4\alpha\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\\
&= -a^{-5}e^2\sqrt{Z}\,\Sigma_4 a\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta),
\end{aligned}$$

and observing that

$$(a,\ b,\ c,\ d,\ e\between v,\ 1)^4 = a\,(v-\alpha)(v-\beta)(v-\gamma)(v-\delta),$$

and therefore

$$4av^3 + bv^2 + 2cv + d = a\,(v-\beta)(v-\gamma)(v-\delta) + \&c.,$$

which, putting $v=\alpha$, gives

$$4a\alpha^3 + 3b\alpha^2 + 2c\alpha + d = a\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta),$$

we have

$$\begin{aligned}
\Sigma_4\alpha\,(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta) &= \frac{1}{a}\,(4a\Sigma\alpha^4 + 3b\Sigma\alpha^3 + 2c\Sigma\alpha^2 + d\Sigma\alpha)\\
&= 4\,(4) + 3b\,(3) + 2c\,(2) + d\,(1),
\end{aligned}$$

where for a moment a is put equal to 1. This is calculated by

e	-16				$=$	-16
bd	$+16$	-9		-1	$=$	$+6$
c^2	$+8$		-4		$=$	$+4$
b^2c	-16	$+9$	$+2$		$=$	-5
b^4	$+4$	-3			$=$	$+1$

or restoring the powers of a, and putting

$$N = \begin{vmatrix} a^3e & -16 \\ a^2bd & +6 \\ a^2c^2 & +4 \\ ab^2c & -5 \\ b^4 & +1 \end{vmatrix}$$

we have

$$\Sigma_4\theta_1\theta_2\theta_3 = a^{-9}Ne^2\sqrt{Z}.$$

Lastly,

$$\begin{aligned}\Sigma_4\theta_1\theta_2\theta_3\theta_4 \text{ or } \theta_1\theta_2\theta_3\theta_4 &= \alpha^2\beta^2\gamma^2\delta^2\,\zeta^{\frac{1}{2}}(\beta,\ \gamma,\ \delta)\,\zeta^{\frac{1}{2}}(\gamma,\ \delta,\ \alpha)\,\zeta^{\frac{1}{2}}(\delta,\ \alpha,\ \beta)\,\zeta^{\frac{1}{2}}(\alpha,\ \beta,\ \gamma)\\ &= \alpha^2\beta^2\gamma^2\delta^2\,\zeta(\alpha,\ \beta,\ \gamma,\ \delta)\\ &= a^{-9}e^2Z,\end{aligned}$$

and the equation $\Pi_4(\theta-\theta_1)=0$ is thus found to be the one given above, No. 10.

267.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861), pp. 225—239. Received February 20,—Read March 7, 1861.]

THE Porism referred to is as follows, viz. two conics may be so related to each other, that a polygon may be inscribed in the one, and circumscribed about the other conic, in such manner that any point whatever of the circumscribing conic may be taken for a vertex of the polygon. I gave in the year 1853, in the Philosophical Magazine(1), a general formula for the relation between the two conics, viz. if $U=0$ is the equation of the inscribed conic, $V=0$ that of the circumscribed conic, and if disct. $(U+\xi V)$, where ξ is an arbitrary multiplier, denotes the discriminant of $U+\xi V$ in regard to the coordinates (x, y, z) (such discriminant being of course a cubic function in regard to ξ, and also in regard to the coefficients of the two conics, U, V, jointly), then if we write

$$\sqrt{\text{disct. }(U+\xi V)} = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4 + F\xi^5 + G\xi^6 + \text{ \&c.},$$

the relations for the cases of the triangle, pentagon, heptagon, &c. are

$$C=0, \quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix} = 0, \quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} = 0, \text{ \&c.}$$

[1] See the papers—"On the Geometrical Representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$," *Phil. Mag.*, April 1853, [113].

"Note on the Porism of the in-and-circumscribed Polygon," *Phil. Mag.* August 1853, [115].

"Correction of two Theorems relating to the Porism of the in-and-circumscribed Polygon," *Phil. Mag.* November 1853, [116].

"Developments of the Porism of the in-and-circumscribed Polygon," *Phil. Mag.* May 1854, [128].

respectively, while those in the cases of the quadrangle, hexagon, octagon, &c. are

$$D=0,\quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix}=0,\quad \begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix}=0,\ \&c.$$

respectively. The demonstration of this fundamental theorem is for greater completeness here reproduced; but the chief object of the memoir is to direct attention to a curious analytical theorem which is an easy *à priori* consequence of the Porism, and to obtain the relations for the several polygons up to the enneagon, in a new and simple form which puts in evidence *à posteriori* for these cases the analytical theorem just referred to. The analytical theorem rests upon the following considerations:—the relation for a hexagon ought to include that for a triangle; in fact a triangle with its sides in order twice over is a form of hexagon; the condition for an octagon should in like manner include that for a quadrangle; and so in other cases. Let the cubic function, disct. $(U+\xi V)$, be represented by $1+\beta\xi+\gamma\xi^2+\delta\xi^3$, the coefficients A, B, C, D, E, &c. are functions of β, γ, δ. Write

$$C \qquad =(3),$$

$$D \qquad =(4),$$

$$\begin{vmatrix} C, & D \\ D, & E \end{vmatrix} =(5),$$

$$\begin{vmatrix} D, & E \\ E, & F \end{vmatrix} =(6),$$

$$\begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} =(7),$$

&c.

then (3), (4), (5) are respectively prime functions of β, γ, δ; that is they cannot be decomposed into factors, rational functions of these quantities; and it is convenient to denote this by writing $(3)=[3]$, $(4)=[4]$, $(5)=[5]$. But by what precedes, (6) contains the factor (3), that is [3]; and if the other factor, which is prime, is denoted by [6], then we have $(6)=[6][3]$. The next term (7) is prime, that is we have $(7)=[7]$; but the term (8) gives $(8)=[8][4]$; the term (9) gives $(9)=[9][3]$, and so on. Thus we have $(12)=[12][6][4][3]$, the numbers in [ ] being all the factors, the number itself included, and as well composite as prime, of the number in ( ), the factors 2 and 1 being however excluded. To make this clearer, it may be remarked that the last-mentioned equation has the geometrical signification that the relation for a dodecagon is the aggregate of the relations for a proper dodecagon, a proper hexagon, a quadrangle, and a triangle; that is, the relation for a dodecagon implies one or other of the last-mentioned relations. The relations for the several polygons up to the

enneagon are in the memoir obtained in a form which puts in evidence the property in question, that is, the series of equations

$$(3) = [3],$$

$$(4) = [4],$$

$$(5) = [5],$$

$$(6) = [6]\,[3],$$

$$(7) = [7],$$

$$(8) = [8]\,[4],$$

$$(9) = [9]\,[3].$$

To do this, the discriminant is represented, not as above in terms of the constants β, γ, δ, but in a somewhat different form, by means of the constants $b\,[=\beta]$, c, d, the last two whereof are such that $c=0$ is the relation for the triangle, $d=0$ the relation for the quadrangle; thus $[3]=c$, $[4]=d$, and for the particular cases considered, the analytical theorem consists herein, that c is a factor of (6), and of (9), and that d is a factor of (8). I have, for the sake of homogeneity, introduced into the formulæ the quantity $a\,(=1)$, but this is a matter of form only.

The functions [3], [4], &c. have been spoken of as *prime;* they are so, in fact, so far as they are calculated; and that they are so in general rests on the assumption that for a polygon of a given number of sides, there is but one form of relation: if, for instance, in the equation $[12]=0$, which is the condition for a proper dodecagon, the function [12] could be decomposed into rational factors, then equating each of these factors to zero, we should have so many distinct forms of relation for a proper dodecagon. I believe that the assumption and reasoning are valid; but without entering further into this, I take it for granted that in the general case the functions [3], [4], &c. are in fact prime. But the coefficients β, γ, δ, or b, c, d instead of being so many independent arbitrary quantities, may be given as rational functions of other quantities (if, for instance, the two conics are circles, radii R, r, and distance between the centres a, then β, γ, δ will be functions of R, r, a): and it is in a case of this kind quite conceivable that the functions [3], [4], &c., considered as functions of these new elements, should cease to be prime functions. In fact, in the case just referred to of the two circles (the original case of the Porism as considered by Füss), the functions [4], [6], &c., which correspond to a polygon of an *even* number of sides, appear to be each of them decomposable into two factors: the memoir contains some remarks tending to show *à priori* that in the case in question this decomposition takes place. I was led to examine the point by the elegant formulæ obtained in an essentially different manner by M. Mention, *Bull. de l'Acad. de St Pét.* t. I. pp. 15, 30, and 507 (1860), in reference to the case of the two circles (it thereby appears that the decomposition takes place for the quadrangle and the hexagon); and these formulae are reproduced in the memoir.

I.

Demonstration of the general Formula of the Relation between the two Conics.

The equation of a conic passing through the points of intersection of the conics

$$U=0, \quad V=0,$$

is of the form

$$U+mV=0,$$

where m is an arbitrary parameter. Suppose that the conic touches a given line, we have for the determination of m a quadratic equation; and conversely, if the roots of this quadratic equation are given, the line is also given; that is, the two roots may be considered as parameters which determine the particular line.

Let k be a given value of m; the parameters of any tangent of the conic $U+kV=0$ are k, p, but as k is common to all the tangents, we may consider the particular tangent as determined by the single parameter p. And a point of the conic $U+kV=0$ may be considered as determined by the same parameter p which determines the tangent at that point.

As regards the conic $V=0$, the common parameter for all its tangents is ∞, and we may consider any other tangent of this conic as determined by the parameter θ, and a point of the conic as determined by the same parameter θ.

Suppose, in the first instance, that the two conics are

$$U=ax^2+by^2+cz^2=0,$$
$$V=\ \ x^2+\ \ y^2+\ \ z^2=0;$$

the equation of the tangent of $U+kV=0$, the parameter whereof is p (in fact a common tangent of the conics $U+kV=0$, $U+pV=0$), is easily found to be

$$x\sqrt{b-c}\sqrt{a+k}\sqrt{a+p}+y\sqrt{c-a}\sqrt{b+k}\sqrt{b+p}+z\sqrt{a-b}\sqrt{c+k}\sqrt{c+p}=0;$$

and if this meet the conic $V=0$ in the points P, P', the parameters whereof are ∞, θ, and ∞, θ', or say θ and θ' respectively, then the coordinates of the point P are given by

$$x:y:z=\sqrt{b-c}\sqrt{a+\theta}:\sqrt{c-a}\sqrt{b+\theta}:\sqrt{a-b}\sqrt{c+\theta};$$

and substituting these values in the foregoing equation, we have

$$(b-c)\sqrt{a+k}\sqrt{a+p}\sqrt{a+\theta}+(c-a)\sqrt{b+k}\sqrt{b+p}\sqrt{b+\theta}+(a-b)\sqrt{c+k}\sqrt{c+p}\sqrt{c+\theta}=0$$

as an equation connecting the parameters p and θ. This equation may be replaced by

$$\sqrt{(a+k)(a+p)(a+\theta)}=\lambda+\mu a,$$
$$\sqrt{(b+k)(b+p)(b+\theta)}=\lambda+\mu b,$$
$$\sqrt{(c+k)(c+p)(c+\theta)}=\lambda+\mu c,$$

from which λ, μ are to be eliminated; and squaring and reducing, we have

$$\lambda^2 = abc + kp\theta,$$
$$-2\lambda\mu = bc + ca + ab - (p\theta + kp - k\theta),$$
$$\mu^2 = a + b + c + k + p + \theta,$$

and thence

$$(bc + ca + ab - p\theta - kp - k\theta)^2 - 4(a + b + c + k + p + \theta)(abc + kp\theta) = 0$$

as the rational form of the original equation. But the same rational equation would, it is clear, be obtained from the system

$$\sqrt{(k+a)(k+b)(k+c)} = L + Mk,$$
$$\sqrt{(p+a)(p+b)(p+c)} = L + Mp,$$
$$\sqrt{(\theta+a)(\theta+b)(\theta+c)} = L + M\theta,$$

by the elimination of L and M. And it follows from Abel's theorem (but the result might be verified by means of Euler's fundamental theorem for the addition of elliptic functions), that if

$$\Pi\xi = \int_{\infty} \frac{d\xi}{\sqrt{(\xi+a)(\xi+b)(\xi+c)}},$$

then the last-mentioned system is equivalent to the transcendental equation

$$\Pi\theta = \Pi p \pm \Pi k,$$

in which the arbitrary constant which should have been inserted, and the sign of $\Pi\theta$, are determined by the consideration that for $k = \infty$ (which gives $\Pi k = 0$) we ought to have $\theta = p$, and therefore $\Pi\theta = \Pi p$.

There is of course a similar equation in θ', and the terms with Πk must be taken with opposite signs, and we have thence the theorem:

"If θ, θ' are the parameters of the points P, P' in which the conic $V = 0$ is intersected by the tangent, the parameter whereof is p, of the conic $U + kV = 0$, then the equations

$$\Pi\theta = \Pi p - \Pi k,$$
$$\Pi\theta' = \Pi p + \Pi k,$$

determine the parameters θ, θ' of the points in question." And again:

"If the two variable parameters θ, θ' are connected by the equation

$$\Pi\theta' - \Pi\theta = 2\Pi k,$$

then the line PP' will be a tangent of the conic $U + kV = 0$."

The foregoing demonstration relates to the particular forms $U = ax^2 + by^2 + cz^2$, $V = x^2 + y^2 + z^2$; but observing that the function $\sqrt{(\xi+a)(\xi+b)(\xi+c)}$, which enters under the integral sign in the transcendental function $\Pi\xi$, is the square root of the discriminant of $U + \xi V$, the theory of covariants shows at once that the conclusions apply to any forms whatever of U, V, the expression for the transcendental function being

$$\Pi\xi = \int_{\infty} \frac{d\xi}{\sqrt{\text{disct.}\,(U+\xi V)}}.$$

Consider now a triangle inscribed in the conic $V = 0$, and with its sides touching the conics

$$U + k\ V = 0,$$
$$U + k'\ V = 0,$$
$$U + k''V = 0;$$

then if θ, θ', θ'' are the parameters of the angles, we have

$$\Pi\theta'' - \Pi\theta' = 2\Pi k\ ,$$
$$\Pi\theta\ - \Pi\theta'' = 2\Pi k'\ ,$$
$$\Pi\theta' - \Pi\theta\ = 2\Pi k'',$$

and thence

$$\Pi k + \Pi k' + \Pi k'' = 0$$

as the relation which must subsist between the parameters k, k', k'', of the conics touched by the sides; and similarly for a polygon of n sides, the relation between the parameters is

$$\Pi k_1 + \Pi k_2 + \ldots + \Pi k_n = 0.$$

But by Abel's theorem, this transcendental equation is equivalent to an algebraical one.

In fact, calling the radical $\sqrt{\square x}$, then if ϕx, χx are rational and integral functions of x with arbitrary coefficients, and if

$$\phi^2 x - \chi^2 x\,\square x = A\,(x-k_1)(x-k_2)\ldots(x-k_n),$$

(this implies that $\phi^2 x$ is of a degree not exceeding n and $\chi^2 x$ of a degree not exceeding $n-3$; that is, for n even the degrees of ϕx, χx are $\frac{1}{2}n$, $\frac{1}{2}(n-4)$; and for n odd they are $\frac{1}{2}(n-1)$, $\frac{1}{2}(n-3)$), then the algebraical equation is that obtained by the elimination of the arbitrary coefficients from the system of equations

$$\phi k_1 + \chi k_1 \square k_1 = 0,$$
$$\phi k_2 + \chi k_2 \square k_2 = 0,$$
$$\vdots$$
$$\phi k_n + \chi k_n \square k_n = 0;$$

or, what is the same thing, for n odd, $=2p-1$, it is

$$\{1,\ \theta, \ldots \theta^{p-1},\ \sqrt{\square\theta},\ \theta\sqrt{\square\theta}, \ldots \theta^{p-2}\sqrt{\square\theta}\} = 0,$$

and for n even, $=2p$, it is

$$\{1,\ \theta, \ldots \theta^{p},\ \sqrt{\square\theta},\ \theta\sqrt{\square\theta}, \ldots \theta^{p-2}\sqrt{\square\theta}\} = 0,$$

where the expressions in { } denote respectively the determinants, of $2p-1$ lines, and $2p$ lines, formed by substituting for θ the values $k_1, k_2, \ldots k_n$ respectively. Thus for $n=3$, the equation is

$$\begin{vmatrix} 1, & k_1, & \sqrt{\square k_1} \\ 1, & k_2, & \sqrt{\square k_2} \\ 1, & k_3, & \sqrt{\square k_3} \end{vmatrix} = 0;$$

and for $n=4$, it is

$$\begin{vmatrix} 1, & k_1, & k_1^2, & \sqrt{\square k_1} \\ 1, & k_2, & k_2^2, & \sqrt{\square k_2} \\ 1, & k_3, & k_3^2, & \sqrt{\square k_3} \\ 1, & k_4, & k_4^2, & \sqrt{\square k_4} \end{vmatrix} = 0,$$

and so on.

Suppose

$$\sqrt{\square\xi} = A + B\xi + C\xi^2 + D\xi^3 + E\xi^4 + \ldots;$$

then substituting the corresponding expressions for $\sqrt{\square k_1}$, $\sqrt{\square k_2}$, &c., the determinant will divide by $\{1,\ \theta,\ \theta^2, \ldots \theta^{n-1}\}$, and it may be seen without difficulty that the resulting equation on putting therein $k_1 = k_2 \ldots k_n = 0$, will, according as $n = 3, 4, 5, 6$, &c., be

$$C=0,\quad D=0,\quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix} = 0,\quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix} = 0,\quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} = 0, \text{ \&c.},$$

which is the theorem above referred to.

II.

Application to the several Polygons up to the Enneagon.

If the equation of the inscribed conic is $U=0$, and that of the circumscribed conic is $V=0$, and if the discriminant $(U+\xi V)$ is in the first instance represented by $1+4\beta\xi+4\gamma\xi^2+4\delta\xi^3$, then the square root of the discriminant is

$$1+2\beta\xi+2(\gamma-\beta^2)\xi^2+2(\delta-2\beta\gamma+2\beta^3)\xi^3+\text{\&c.},$$

so that the condition for the triangle is

$$\gamma - \beta^2 = 0,$$

and that for the quadrangle is

$$\delta - 2\beta\gamma + 2\beta^3 = 0.$$

It is obviously convenient to introduce into the formulæ, in the place of γ and δ, the quantities

$$c = \gamma - \beta^2,$$
$$d = \delta - 2\beta\gamma + 2\beta^3;$$

and writing also, for symmetry of notation, b in the place of β, we have

$$\beta = b,$$
$$\gamma = c + b^2,$$
$$\delta = d + 2bc,$$

so that the discriminant will be

$$= 1 + 4b\xi + 4(c + b^2)\xi^2 + 4(d + 2bc)\xi^3,$$

which is

$$= (1 + 2b\xi + 2c\xi^2)^2 + 4(d\xi^3 - c^2\xi^4).$$

But for homogeneity I introduce the quantity $a = 1$, and put the discriminant

$$= (1 + 2b\xi + 2ac\xi^2)^2 + 4a^2(d\xi^3 - c^2\xi^4).$$

The square root, divided by a^2, is

$$= a^{-2}(1 + 2b\xi + 2ac\xi^2)\sqrt{1 + \{4a^2(d\xi^3 - c^2\xi^4) \div (1 + 2b\xi + 2ac\xi^2)^2\}};$$

or developing, this is

$$\begin{aligned}
& a^{-2}(1 + 2b\xi + 2ac\xi^2) \\
+\ & 2\ (d\xi^3 - c^2\xi^4) \div (1 + 2b\xi + 2ac\xi^2) \\
-\ & 2a^2(d\xi^3 - c^2\xi^4)^2 \div (1 + 2b\xi + 2ac\xi^2)^3 \\
+\ & 4a^4(d\xi^3 - c^2\xi^4)^3 \div (1 + 2b\xi + 2ac\xi^2)^5 \\
-\ & 10a^6(d\xi^3 - c^2\xi^4)^4 \div (1 + 2b\xi + 2ac\xi^2)^7 \\
+\ & \&c.;
\end{aligned}$$

or representing this by $1 + 2B\xi + 2C\xi^2 + 2D\xi^3 + \&c.$, we have

$$C\xi^2 + D\xi^3 + E\xi^4 + F\xi^5 + G\xi^6 + H\xi^7 + I\xi^8 + J\xi^9 + K\xi^{10} + L\xi^{11} + M\xi^{12} + \ldots$$

$$= a^{-1}c\xi^2$$

$$+ (d\xi^3 - c^2\xi^4)\left\{\begin{array}{l|l|l|

which may be represented by 12, 13, &c., viz.

$$12 = \begin{vmatrix} C, & D \\ D, & E \end{vmatrix}, \quad 13 = \begin{vmatrix} C, & E \\ D, & F \end{vmatrix}, \text{ \&c.,}$$

we have, up to 45, which is all that is required,

$12 = a^{-1} \times$	$13 = a^{-1} \times$	$14 = a^{-1} \times$	$15 = a^{-1} \times$	$23 =$	$24 =$	$25 =$	$34 =$	$35 =$	$45 =$
$ad^2 - 1$	$abd^2 + 2$	$a^2cd^2 + 1$	$a^3d^3 + 1$	$acd^3 - 2$	$a^2d^3 - 1$	$a^2bd^3 + 4$	$a^2bd^3 + 2$	$a^3cd^3 - 2$	$a^4d^4 - 1$
$bcd - 2$	$ac^2d - 1$	$ab^2d^2 - 4$	$a^2bcd^2 - 2$	$b^2c^2d - 2$	$abcd^2 + 4$	$a^2c^2d^2 + 5$	$a^2c^2d^2 - 3$	$a^2b^2d^3 - 8$	$a^4bcd^3 + 4$
$c^3 - 1$	$b^2cd + 4$	$abc^3d + 6$	$a^2c^3d + 4$	$c^4 - 1$	$b^2c^2d + 4$	$ab^2cd^2 - 8$	$abc^3d - 4$	$a^2c^4d - 2$	$a^3c^3d^2 - 8$
	$bc^3 + 2$	$ac^4 + 2$	$ab^2d^2 + 8$		$bc^4 + 2$	$abc^3d + 4$	$ac^5 - 2$	$ab^2c^3d + 8$	$a^3b^2d^3 + 8$
		$b^3cd - 8$	$ab^2c^2d - 20$			$ac^5 + 2$		$abc^5 + 4$	$a^2b^2c^2d^2 + 12$
		$b^2c^3 - 4$	$abc^4 - 8$			$b^3c^2d - 8$			$a^2bc^4d - 4$
			$b^4cd + 16$			$b^2c^4 - 4$			$a^2c^5 - 4$
			$b^3c^3 + 8$						
-4	-7	$+7$	-7	-5	$+9$	-5	-7	0	$+7$

Forming in like manner the determinants of the matrix

$$\begin{pmatrix} C, & D, & E, & F, \ldots \\ D, & E, & F, & G, \ldots \\ E, & F, & G, & H, \end{pmatrix},$$

and representing these by 123, 124, &c., viz.

$$123 = \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix}, \text{ \&c.,}$$

we have, up to 234,

$123 = a^{-1} \times$	$124 = a^{-1} \times$	$134 = a^{-1} \times$	$234 =$
$a^3d^4 + 1$	$a^3bd^4 - 4$	$a^4cd^4 + 1$	$a^4d^5 - 1$
$a^2bcd^3 + 2$	$a^3c^2d^3 - 3$	$a^3b^2d^4 + 4$	$a^3bcd^4 - 4$
$a^2c^3d^2 - 1$	$a^2b^2cd^3 - 8$	$a^2bc^2d^3 + 8$	$a^2c^2d^3 - 4$
$abc^4d - 2$	$a^2bc^3d^2 - 4$	$a^2c^4d^2 - 3$	$a^2b^2c^2d^3 - 4$
$ac^6 - 1$	$a^2c^5d - 2$	$a^2b^3cd^3 + 8$	$a^2bc^4d^2 - 6$
	$ab^2c^4d + 4$	$a^2b^2c^3d^2 + 12$	$a^2c^6d - 2$
	$abc^6 + 2$	$a^3c^7 - 2$	
-1	-15	$+28$	-21

and further, the determinants of the matrix

$$\left(\begin{array}{l} C,\ D,\ E,\ F,\ \ldots \\ \left|\begin{array}{llll} D, & E, & F, & G \\ E, & F, & G, & H \\ F, & G, & H, & I \end{array}\right| \end{array}\right),$$

in the present case, the single determinant

$$1234 = \left|\begin{array}{llll} C, & D, & E, & F \\ D, & E, & F, & G \\ E, & F, & G, & H \\ F, & G, & H, & I \end{array}\right|,$$

we have

$$\boxed{1234 = a^{-1} \times \left\{\begin{array}{ll} a^6cd^8 & -\ 3 \\ a^5bc^2d^5 & -\ 12 \\ a^5c^4d^4 & -\ 4 \\ a^4b^2c^3d^4 & -\ 16 \\ a^4bc^5d^3 & -\ 14 \\ a^4c^7d^2 & -\ 3 \\ a^3b^3c^4d^3 & -\ 8 \\ a^3b^2c^6d^2 & -\ 12 \\ a^3bc^8d & -\ 6 \\ a^3c^{10} & -\ 1 \end{array}\right\}}$$

$$-79$$

The conditions for the triangle and the quadrangle are $c=0$, $d=0$; those for the pentagon and hexagon are $12=0$, $23=0$; for the heptagon and octagon, $123=0$, $234=0$; and that for the enneagon is $1234=0$. The foregoing values show that 23 and 1234 (which belong to the hexagon and the enneagon) divide by c (which belongs to the triangle), and that 234 (which belongs to the octagon) divides by d (which belongs to the quadrangle). But I was not prepared for the destruction which will be observed in the several determinants, of the terms involving the lower powers of a (that is, the terms of the highest orders in b, c, d), and which renders these expressions so much more simple than they would otherwise have been.

Representing the reduced equations for the several polygons, as before, by

$$\begin{array}{c} [3] = 0, \\ [4] = 0, \\ [5] = 0, \\ [6] = 0, \\ [7] = 0, \\ [8] = 0, \\ [9] = 0, \\ \&c.; \end{array}$$

then retaining the quantity $a(=1)$ for homogeneity, but rejecting the powers of a which divide out, and reversing in some cases the signs, the values of the functions [3], [4], &c. are

[3] =	[4] =	[5] =	[6] =	[7] =	[8] =	[9] =
$c+1$	$d+1$	ad^2+1	ad^2+2	a^2d^4+1	a^2d^4+1	a^3d^6+3
		$bcd+2$	$bcd+2$	$abcd^3+2$	$abcd^3+4$	a^2bcd^5+12
		c^3+1	c^3+1	ac^3d^2-1	ac^3d^3+4	$a^2c^3d^4+4$
				bc^4d-2	$b^2c^2d^2+4$	$ab^2c^2d^3+16$
				c^6-1	bc^4d+6	abc^4d^3-14
					c^6+2	ac^6d^2+3
						$b^3c^3d^3+8$
						$b^2c^3d^2+12$
						bc^7d+6
						c^9+1
$+1$	$+1$	$+4$	$+5$	-1	$+21$	$+79$

The similarity of form for the relations corresponding to the pentagon and the hexagon, and for those corresponding to the heptagon and the octagon, is, I am inclined to think, accidental; the functions are homogeneous as regards degree and weight; and the degrees and weights of the two consecutive functions being identical, the literal parts must be similarly constituted.

III.

M. MENTION'S *Formulæ for the Case of two Circles.*

In the case of two circles, if, as usual, the radii of the inscribed and circumscribed circles are put equal to r and R respectively, and the distance of their centres to a, then the equations of the inscribed and circumscribed circles respectively may be taken to be

$$x^2+y^2-r^2=0,$$
$$(x-a)^2+y^2-R^2=0;$$

and if, in the notation of M. Mention, we put

$$\frac{R^2-r^2}{a^2}=i,$$

$$\frac{1}{r^4}(r^4+R^4+a^4-2r^2R^2-2r^2a^2-2R^2a^2)=-\nu,$$

then the quadratic radical is

$$\sqrt{(1+4\xi)\{[1+(2+2i)\xi]^2+4\nu\xi^2\}};$$

and comparing this with

$$\sqrt{1+4b\xi+4(c+b^2)\xi^2+4(d+2bc)\xi^3},$$

we have

$$\begin{aligned} b &= i+2, \\ c+b^2 &= i^2+6i+\nu+5, \\ d+2bc &= 4(i^2+2i+\nu+1), \end{aligned}$$

and thence

$$\begin{aligned} b &= i+2, \\ c &= 2i+\nu+1, \\ d &= -2i\nu-2i; \end{aligned}$$

and by means of these values, or by effecting the development in a different manner, we find

$$B=\left\{\begin{array}{l} i \\ +2, \end{array}\right.$$

$$C=\left\{\begin{array}{l} i.2 \\ +\nu+1, \end{array}\right.$$

$$D=2\{i(-\nu-1)\},$$

$$E=\left\{\begin{array}{l} i^2.4\nu \\ +i\ .4(\nu+1) \\ \quad -(\nu+1)^2, \end{array}\right.$$

$$F=2\left\{\begin{array}{l} i^3.-4\nu \\ +i^2.-8\nu \\ +i.(\nu+1)(3\nu-5) \\ +2(\nu+1)^2, \end{array}\right.$$

$$G=2\left\{\begin{array}{l} i^4.8\nu \\ +i^3.24\nu \\ +i^2.4\nu(-3\nu+5) \\ +i\ .(\nu+1)(-18\nu+14) \\ +\quad(\nu+1)^2(\nu-7), \end{array}\right.$$

$$H=4\left\{\begin{array}{l} i^5.-8\nu \\ +i^4.-32\nu \\ +i^3.\quad 4\nu(5\nu-11) \\ +i^2.\quad 16\nu(3\nu-1) \\ +i\ .\quad(\nu+1)(\nu-7)(-5\nu+3) \\ \quad -4(\nu+1)^2(\nu-3). \end{array}\right.$$

These values give

$$C = \begin{cases} 2i \\ +\nu+1, \end{cases} \qquad = [3],$$

$$D = 2\{\, i(-\nu-1), \qquad = [4],$$

$$CE - D^2 = \begin{cases} i^3 . 8\nu \\ +i^2 . 4(\nu+1) \\ +i\ . 2(\nu+1)^2 \\ \quad -(\nu+1)^3, \end{cases} \qquad = [5],$$

$$DF - E^2 = \begin{pmatrix} 2i \\ +\nu+1 \end{pmatrix} \begin{pmatrix} 2i \\ -\nu-1 \end{pmatrix} \begin{Bmatrix} i^2 . 4\nu \\ +(\nu+1)^2 \end{Bmatrix} \qquad = [3]\ [6],$$

$$\begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix} = \begin{cases} i^3 .\ \ 64\nu \\ +i^3 . -32\nu(\nu+1)(\nu-1) \\ +i^4 . -16\nu(\nu+1)^2 \\ +i^3 . -8(3\nu-1)(\nu+1)^3 \\ +i^2 .\ \ 4(\nu+1)^4 \\ +i\ . -4(\nu+1)^5 \\ \quad -\ (\nu+1)^6 \end{cases} \qquad = [7],$$

$$\begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix} = i.(-\nu-1) \begin{Bmatrix} 16 i^4\nu \\ +(\nu+1)^4. \end{Bmatrix} \qquad = [4]\ [8].$$

It will be remarked that $[4], = i(-\nu-1)$, breaks up into the factors i and $\nu+1$; and so $[6], = (2i-\nu-1)\{4\nu i^3+(\nu+1)^2\}$, breaks up into the factors $2i-\nu-1$ and $4\nu i^3+(\nu+1)^3$.

It may be added that the developed expression of [6] is

$$= \begin{cases} i^3 . 8\nu \\ +i^2 . -4\nu(\nu+1) \\ +i\ . 2(\nu+1)^2 \\ \quad -(\nu+1)^3, \end{cases}$$

so that the difference between this and [5] is $i^2 . 4(\nu+1)^2$, which is $= d^2$; this agrees with a former result.

M. Mention has also given, but not in a developed form, the formulæ for the enneagon and the endecagon, and the following formula for the decagon, viz.

$$[16 i^4\nu + (\nu+1)^4]^2 + 16 i^2 \nu (\nu+1)^2 . \{2 i^2 (1-\nu) - (\nu+1)^2\}^2 = 0.$$

IV.

Considerations as to the form of relation, in the case of two circles, for Polygons of an odd and even number of sides respectively.

The relation between the two conics, or condition for the existence of the polygon, is the same whatever point of the circumscribed conic is taken as an angle of the polygon. Take for an angle, a point of intersection of the two conics. Consider first the case of the triangle; if a point of intersection is taken as a vertex A of the triangle, then the sides AB, AC coincide in direction with the tangent at A to the inscribed conic U, hence B and C coincide together at the point where this tangent meets the circumscribed conic V, BC is therefore a tangent of V, but it is by hypothesis a tangent of U; hence for the triangle the relation between the inscribed conic U and the circumscribed conic V is as follows: viz. a tangent to U at a point of intersection with V meets V at a point of contact of a common tangent of U and V.

In like manner for the quadrangle, if A be taken at a point of intersection, the sides AB and AD will coincide in direction with the tangent to U at this point, consequently B and D must coincide at the point where this tangent meets V; hence also CB, CD, the two tangents to U from the point C, must coincide, or C must be a point of intersection of the conics U, V. In other words, the pole, with respect to the inscribed conic U, of a common chord AC of the two conics must lie on the circumscribed conic V; this is therefore the condition for the quadrilateral.

In the ordinary mode of drawing the figures, with two conics which do not intersect, the points and lines employed in the foregoing constructions are imaginary, but the conics may be so drawn that these points and lines are all real.

In general, for a polygon of an odd number, $2n+1$, of sides, then starting from a point of intersection, the sides will coincide in pairs, viz. the first and last, second and last but one, and so on, the middle or $(n+1)$th side being a common tangent of the two conics. But for a figure of an even number, $2n$, of sides, then starting from a point of intersection of the two conics, the nth side will terminate at a second point of intersection, and then the same series of sides will be repeated in the reverse order, so that the sides will coincide in pairs, first and last, second and last but one, nth and $(n+1)$th. For a figure of an odd number of sides, the relation involves only a single point of intersection, but for a figure of an even number of sides, it involves two points of intersection.

Now in the case of two circles, for a polygon of an odd number of sides, the same relation is obtained, whether we take as the point of intersection one of the actual points of intersection, or a circular point at infinity, and the relation $[2n+1]=0$ does not break up into factors. And so for a polygon of an even number of sides, then taking for the two points of intersection, the two actual points of intersection, or the two circular points at infinity, we have one form of result; but taking for them an actual point of intersection and a circular point at infinity, we have a different form of result; and the equation $[2n]=0$ does break up into factors.

This is verified very simply in the case of the quadrangle. Taking for the two points of intersection the circular points at infinity, the line joining them is the line infinity, and its pole (with respect to the inscribed circle) is the centre of this circle; the relation therefore is that the centre of the inscribed circle lies on the circumscribed circle. But when this is the case, it is easy to see that the pole (with respect to the inscribed circle) of the radical axis, lies also on the circumscribed circle; this pole and the centre of the inscribed circle are in fact the extremities of a diameter of the circumscribed circle. The condition thus obtained is $R^2 - a^2 = 0$ (which is M. Mention's condition $i = 0$). We have next to find the analytical relation when the pole (with respect to the inscribed circle), of the line joining one of the actual points of intersection with a circular point at infinity is a point on the circumscribed circle. This I effect as follows:—taking $z = 0$ as the equation of line infinity, if the origin be taken on the middle point of the radical axis, and if $x = 0$ be the radical axis, then the equations of the two circles may be taken to be

$$\text{Inscribed circle,} \quad x^2 + y^2 - 2l\,xz - \nabla z^2 = 0,$$

$$\text{Circumscribed circle,} \quad x^2 + y^2 - 2Lxz - \nabla z^2 = 0,$$

a circular point at infinity is

$$x : y : z = 1 : \quad i : 0, \quad (i = \sqrt{-1}),$$

an actual point of intersection is

$$x : y : z = 0 : \sqrt{\nabla} : 1.$$

The line joining these is

$$xi - y + z\sqrt{\nabla} = 0,$$

its pole, with respect to the inscribed circle, is

$$x : y : z = -i\sqrt{\nabla} : \sqrt{\nabla} - il : 1;$$

and if this be a point of the circumscribed circle

$$-\nabla + (\nabla - 2il - l^2) + 2Li\sqrt{\nabla} - \nabla = 0,$$

that is

$$2(L - l)\,i\sqrt{\nabla} = l^2 + \nabla,$$

or

$$(l^2 + \nabla)^2 + 4(L - l)^2\nabla = 0,$$

which is the required relation: but to express it in terms of the ordinary data R, r, a, the equations of the circles, putting therein $z = 1$, become

$$(x - l)^2 + y^2 = \nabla + l^2,$$

$$(x - L)^2 + y^2 = \nabla + L^2,$$

and therefore

$$\begin{aligned} a &= L - l, \\ r^2 &= \nabla + l^2, \\ R^2 &= \nabla + L^2; \end{aligned}$$

these equations give

$$\frac{R^2 - r^2}{a} = L + l,$$

$$a = L - l;$$

and thence

$$L = \frac{R^2 - r^2 + a^2}{2a},$$

and

$$\nabla = R^2 - \left(\frac{R^2 - r^2 - a^2}{2a}\right)^2 = \frac{1}{4a^2}(2a^2R^2 + 2a^2r^2 + 2r^2R^2 - R^4 - r^4 - a^4)$$

$$= \frac{r^4}{4a^2}\nu,$$

if with M. Mention we write

$$\frac{1}{r^4}(r^4 + R^4 + a^4 - 2r^2R^2 - 2r^2a^2 - 2R^2a^2) = -\nu.$$

The equation

$$(l^2 + \nabla)^2 + 4(L - l)^2\nabla = 0$$

thus becomes

$$r^4 + 4a^2 \cdot \frac{r^4}{4a^2}\nu = 0,$$

that is, it becomes $\nu + 1 = 0$, which is the other factor of the complete condition $i(\nu + 1) = 0$.

268.

ON A NEW AUXILIARY EQUATION IN THE THEORY OF EQUATIONS OF THE FIFTH ORDER.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861), pp. 263—276. Received February 20,—Read March 7, 1861.]

CONSIDERING the equation of the fifth order, or quintic equation,

$$(*\between v,\ 1)^5 = (v - x_1)(v - x_2)(v - x_3)(v - x_4)(v - x_5) = 0,$$

and putting as usual

$$f\omega = x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5,$$

where ω is an imaginary fifth root of unity, then, according to Lagrange's general theory for the solution of equations, $f\omega$ is the root of an equation of the order 24, called the Resolvent Equation, but the solution whereof depends ultimately on an equation of the sixth order, viz.

$$(f\omega)^5,\ (f\omega^2)^5,\ (f\omega^3)^5,\ (f\omega^4)^5$$

are the roots of an equation of the fourth order, each coefficient whereof is determined by an equation of the sixth order; and moreover the other coefficients can be all of them rationally expressed in terms of any one coefficient assumed to be known; the solution thus depends on a single equation of the sixth order. In particular the last coefficient, or

$$(f\omega \,.\, f\omega^2 \,.\, f\omega^3 \,.\, f\omega^4)^5,$$

is determined by an equation of the sixth order; and not only so, but its fifth root, or

$$f\omega \,.\, f\omega^2 \,.\, f\omega^3 \,.\, f\omega^4,$$

(which is a rational function of the roots, and is the function called by Mr Cockle the Resolvent Product), is also determined by an equation of the sixth order: this

equation may be called the Resolvent-Product Equation. But the recent researches of Mr Cockle and Mr Harley[1] show that the solution of an equation of the fifth order may be made to depend on an equation of the sixth order, originating indeed in, and closely connected with, the resolvent-product equation, but of a far more simple form; this is the auxiliary equation referred to in the title of the present memoir. The connexion of the two equations, and the considerations which led to the new one, will be pointed out in the sequel; but I will here state synthetically the construction of the auxiliary equation. Representing for shortness the roots $(x_1, x_2, x_3, x_4, x_5)$ of the given quintic equation by 1, 2, 3, 4, 5, and putting moreover

$$12345 = 12 + 23 + 34 + 45 + 51, \text{ \&c.}$$

(where on the right-hand side 12, 23, &c. stand for x_1x_2, x_2x_3, &c.), then the auxiliary equation, say

$$(*\between\phi,\ 1)^6 = 0,$$

has for its roots

$$\phi_1 = 12345 - 24135, \quad \phi_4 = 21435 - 13245,$$
$$\phi_2 = 13425 - 32145, \quad \phi_5 = 31245 - 14325,$$
$$\phi_3 = 14235 - 43125, \quad \phi_6 = 41325 - 12435,$$

and, it follows therefrom, is of the form

$$(1,\ 0,\ C,\ 0,\ E,\ F,\ G\between\phi,\ 1)^6 = 0,$$

where C, E, G are rational and integral functions of the coefficients of the given equation, being in fact seminvariants, and F is a mere numerical multiple of the square root of the discriminant.

The roots of the given quintic equation are each of them rational functions of the roots of the auxiliary equation, so that the theory of the solution of an equation of the fifth order appears to be now carried to its extreme limit. We have in fact

$$\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5 = (*\between x_1,\ 1)^4,$$
$$\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6 = (*\between x_2,\ 1)^4,$$
$$\phi_1\phi_3 + \phi_2\phi_5 + \phi_4\phi_6 = (*\between x_3,\ 1)^4,$$
$$\phi_1\phi_5 + \phi_3\phi_6 + \phi_2\phi_4 = (*\between x_4,\ 1)^4,$$
$$\phi_1\phi_4 + \phi_2\phi_6 + \phi_3\phi_5 = (*\between x_5,\ 1)^4,$$

where $(*\between x_1,\ 1)^4$, &c. are the values, corresponding to the roots x_1, &c. of the given equation, of a given quartic function. And combining these equations respectively with the quintic equations satisfied by the roots x_1, &c. respectively, it follows that, conversely, the roots x_1, x_2, &c. are rational functions of the combinations $\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5$, $\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6$, &c. respectively, of the roots of the auxiliary equation.

[1] Cockle, "Researches in the Higher Algebra," *Manchester Memoirs*, t. xv. pp. 131—142 (1858).

Harley, "On the Method of Symmetric Products, and its Application to the Finite Algebraic Solution of Equations," *Manchester Memoirs*, t. xv. pp. 172—219 (1859).

Harley, "On the Theory of Quintics," *Quart. Math. Journ.* t. III. pp. 343—359 (1859).

It is proper to notice that, combining together in every possible manner the six roots of the auxiliary equation, there are in all fifteen combinations of the form $\phi_1\phi_2+\phi_3\phi_4+\phi_5\phi_6$. But the combinations occurring in the above-mentioned equations are a completely determinate set of five combinations: the equation of the order 15, whereon depend the combinations $\phi_1\phi_2+\phi_3\phi_4+\phi_5\phi_6$, is not rationally decomposable into three quintic equations, but only into a quintic equation having for its roots the above-mentioned five combinations, and into an equation of the tenth order, having for its roots the other ten combinations, and being an irreducible equation. Suppose that the auxiliary equation and its roots are known; the method of ascertaining what combinations of roots correspond to the roots of the quintic equation would be to find the rational quintic factor of the equation of the fifth order, and observe what combinations of the roots of the auxiliary equation are also roots of this quintic factor. The direct calculation of the auxiliary equation by the method of symmetric functions would, I imagine, be very laborious. But the coefficients are seminvariants, and the process explained in my memoir on the Equation of Differences, [262], was therefore applicable, and by means of it, the equation, it will be seen, is readily obtained. The auxiliary equation gives rise to a corresponding covariant equation, which is given at the conclusion of the memoir.

1. I will commence by referring to some of the results obtained by Mr Cockle and Mr Harley.

In the paper "Researches on the Higher Algebra," Mr Cockle, dealing with the quintic equation

$$v^5-5Qv+E=0,$$

obtains for the Resolvent Product $\theta\,(=f\omega f\omega^2 f\omega^3 f\omega^4)$ the equation

$$\theta^6+2QE\,5^5\,\theta^4+2Q^4\,5^7\,\theta^3+Q^2E^2\,5^{10}\,\theta^2-(58Q^5-E^3)\,E\theta+5^{14}\,Q^8=0\,;$$

and he remarks that this equation may be written

$$(\theta^3+5^5\,QE\theta+5^7\,Q^4)^2=5^{10}\,(108\,Q^5E-E^4)\,\theta,$$

so that $\sqrt{-\theta}$ is determined by an equation of the sixth order, involving the quadratic radical $\sqrt{E\,(E^3-108Q^5)}$, which is in fact the square root of the discriminant of the quintic equation.

2. Mr Harley, in his paper "On the Symmetric Product &c.," makes use of the functions

$$\tau\;=x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1\;(=12345),$$

$$\tau'=x_1x_3+x_3x_5+x_5x_2+x_2x_4+x_4x_1\;(=24135),$$

and he obtains for the form $v^5-5Qv^2+E=0$, the relation $\theta=5\tau\tau'$, which, since here $\tau+\tau'=0$, gives $\theta=-5\tau^2$.

Hence $\tau\,(=\sqrt{-\frac{1}{5}\theta})$ is the root of an equation of the sixth order involving the radical $\sqrt{E(E^3-108Q^5)}$, and which is in fact $(t=\tau\div\sqrt{5}=\frac{1}{5}\sqrt{-\theta})$, the equation

$$t^5+5QE\,t^2+\sqrt{E(E^3-108Q^5)}\,t-5Q^4=0,$$

given in Mr Harley's paper "On the Theory of Quintics."

3. And in the same paper there is given a system of equations

$$t_1t_3+t_2t_5+t_4t_6=x_1\,(3^2Q-x_1^3),\ \&c.,$$

connecting the five roots of the given quintic equation with the combinations

$$t_1t_3+t_2t_5+t_4t_6,\ \&c.$$

of the roots of the equation in t.

4. I quote also, with a slight change of notation, the following results from the paper "On the Symmetric Product &c.," viz. considering the quintic equation under the form

$$(a,\ b,\ c,\ d,\ e,\ f\between y,\ 1)^5=0,$$

we have

$$f\omega f\omega^4=\Sigma x^2+\tau\,(\omega\ +\omega^4)+\tau'\,(\omega^2+\omega^3),$$

$$f\omega^2f\omega^3=\Sigma x^2+\tau\,(\omega^2+\omega^3)+\tau'\,(\omega\ +\omega^4),$$

where

$$\Sigma x^2=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=\frac{1}{a^2}(b^2-2ac),$$

and thence, observing also that $\tau+\tau'=\dfrac{c}{a}$,

$$a^4\theta\,(=a^4f\omega f\omega^2f\omega^3f\omega^4)=5a^2c^2-5ab^2c+b^4+5a^4\tau\tau',$$

or, as this equation may also be written,

$$4a^4\theta=(5ac-2b^2)^2-5a^4(\tau-\tau')^2;$$

and hence the Resolvent Product $\theta\,(=f\omega f\omega^2f\omega^3f\omega^4)$ being determined by an equation, of the sixth order, this is also the case with the function $(\tau-\tau')^2$.

5. But the twelve functions $\pm(\tau-\tau')$ can be divided into two sets of six functions each, so that each set is determined by an equation of the sixth order involving a single quadratic radical. This was in fact suggested to me by Mr Harley's equation in t; for in the case considered $t+t'$ was $=0$, or $2t=t-t'$, and the equation in t was presumably the particular form of the equation for $\frac{1}{2}(t-t')$ in the general case. But it will presently appear in what manner the conclusion should have been arrived at *à priori*.

6. The preceding remarks show the connexion between the function $\phi\,(=\tau-\tau')$ to which belongs the new auxiliary equation, and the Resolvent Product $\theta\,(=f\omega f\omega^2 f\omega^3 f\omega^4)$. The relation was given for the denumerate form of the quintic; but taking, instead, the standard form $(a,\ b,\ c,\ d,\ e,\ f\between v,\ 1)^5=0$, it becomes

$$4a^4\theta=2500\,(ac-b^2)^2-5a^4\phi^2.$$

7. The foregoing equation shows that ϕ is a seminvariantive function of the roots. In fact

$$f\omega,\ =x_1-x_5+\omega\,(x_2-x_5)+\omega^2\,(x_3-x_5)+\omega^3\,(x_4-x_5),$$

is seminvariantive, and $f\omega^2$, $f\omega^3$, $f\omega^4$, being in like manner seminvariantive, the product $\theta\,(=f\omega f\omega^2 f\omega^3 f\omega^4)$ is also seminvariantive; $ac-b^2$ and a are seminvariants, and therefore ϕ is a seminvariantive function.

8. But it is easy to show this directly. For representing, as before, the roots by 1, 2, 3, 4, 5, we have

$$(1-5)(2-5)+(2-5)(3-5)+(3-5)(4-5)=12+23+34-5\,(1+22+23+4)+35^2,$$
$$(2-5)(4-5)+(4-5)(1-5)+(1-5)(3-5)=24+41+13-5\,(2+24+21+3)+35^2;$$

and the difference of the right-hand sides is

$$\begin{aligned}&12+23+34-5\,(2+3)\\&-24-41-13+5\,(4+1),\end{aligned}$$

which is $=12345-24135$. So that ϕ,

$$=(1-5)(2-5)+(2-5)(3-5)+(3-5)(4-5)-[(2-5)(4-5)+(4-5)(1-5)+(1-5)(3-5)],$$

is a function of the differences of the roots, that is, it is a seminvariantive function.

9. To account for the division of the twelve values of $\pm(\tau-\tau')$ into two sets as above, and to explain the formation of a set, consider the symbols 1, 2, 3, 4, 5 as belonging to five points. We may with these five points form in all $(\frac{1}{2}\,.\,1\,.\,2\,.\,3\,.\,4=)\,12$ pentagons, and the symbol 12345 of any pentagon may of course be read backwards or forwards from any point ($12345=23451=$ &c. $=15432=$ &c.) without alteration of its meaning. Now attaching to each arrangement of the five numbers a sign, + or −, according to the ordinary rule of signs, 12345 being as usual positive, the arrangements 12345, 23451, &c. . . 15432, &c., which belong to the same pentagon, have all of them the same sign; and we may consequently connect with each pentagon the sign + or −; there are, in fact, six pentagons with the sign + and six with the sign −; and to each positive pentagon there corresponds a negative pentagon, which is derived from it by *stellation*, viz. to the positive pentagon 12345 there corresponds the negative one 24135, and so for the other positive pentagons. The above-mentioned system of equations

$$\phi_1=12345-24135,\quad \phi_4=21435-13245,$$
$$\phi_2=13425-32145,\quad \phi_5=31245-14325,$$
$$\phi_3=14235-43125,\quad \phi_6=41325-12435,$$

in fact exhibits the six positive pentagons, each accompanied by its stellated negative pentagon, and the formation of the system of equations is thus completely explained; the order of arrangement of the pairs *inter se* (or, what comes to the same thing, the order of arrangement of the suffixes of the ϕ's) is wholly immaterial.

10. The six pairs of pentagons, or, what is the same thing, the ϕ's, correspond to each other in pairs in a fivefold manner, *quoad* the numbers 1, 2, 3, 4, 5 respectively; thus, *quoad* 5, the pairs are ϕ_1 and ϕ_4, ϕ_2 and ϕ_5, ϕ_3 and ϕ_6, or say 1 and 4, 2 and 5, 3 and 6. The relation is best seen by means of the positive pentagons; thus, *quoad* 5, in the pentagons 12345 and 21435, the points adjacent to 5 in the one of them are the points 2, 3, and in the other of them the complementary points 1, 4; and so in the other cases. The fivefold correspondence is shown by the symbolical equations

$$\begin{aligned} 1 &= 16,\ 24,\ 35, \\ 2 &= 12,\ 34,\ 56, \\ 3 &= 15,\ 23,\ 46, \\ 4 &= 13,\ 26,\ 35, \\ 5 &= 14,\ 25,\ 36, \end{aligned}$$

which, in fact, indicate the combinations of the ϕ's which correspond to the several roots of the quintic.

11. It is proper to notice that the right-hand sides of the last-mentioned equations contain all the duads formed with the six numbers 1, 2, 3, 4, 5, 6, each duad once, and once only. There are in all six such synthemes of duads, viz.

12 . 34 . 56	12 . 35 . 46	12 . 36 . 45
13 . 25 . 46	13 . 24 . 56	13 . 25 . 46
14 . 26 . 35	14 . 25 . 36	14 . 23 . 56
15 . 24 . 36	15 . 26 . 34	15 . 26 . 34
16 . 23 . 45	16 . 23 . 45	16 . 24 . 35
12 . 34 . 56	12 . 35 . 46	12 . 36 . 45
13 . 26 . 45	13 . 26 . 45	13 . 24 . 56
14 . 25 . 36	14 . 23 . 56	14 . 26 . 35
15 . 23 . 46	15 . 24 . 36	15 . 23 . 46
16 . 24 . 35	16 . 25 . 34	16 . 25 . 34

which is in fact the theorem whereon depends the existence, for six letters, of a 6-valued function not symmetrical in respect of five letters. There is not any peculiarity in the syntheme of duads which above presented itself; the occurrence of this particular syntheme, instead of any other, arises merely from the arbitrary selection of the suffixes of the ϕ's.

12. It is hardly necessary to remark that if the pentagon 12345 had been assumed negative instead of positive, the only difference would be that the ϕ's would have their signs reversed.

13. I proceed now to the calculation of the Auxiliary Equation. As the working is rather easier for that form, I shall in the first instance take for the given quintic the denumerate form

$$(a,\ b,\ c,\ d,\ e,\ f \between v,\ 1)^5 = 0.$$

Representing, as before, the roots x_1, x_2, x_3, x_4, x_5 of this equation by 1, 2, 3, 4, 5, and writing

$$12345 = 12 + 23 + 34 + 45 + 51, \text{ \&c.}$$

(where on the right-hand side 12, 23, &c. stand for x_1x_2, x_2x_3, &c.), we have to find the equation

$$(* \between \phi,\ 1)^6 = 0,$$

the roots whereof are

$$\phi_1 = 12345 - 24135, \quad \phi_4 = 21435 - 13245,$$

$$\phi_2 = 13425 - 32145, \quad \phi_5 = 31245 - 14325,$$

$$\phi_3 = 14235 - 43125, \quad \phi_6 = 41325 - 12435.$$

As already remarked, the coefficients are seminvariants, and if the equation is in the first instance calculated for the particular case $f = 0$, the terms in f can be separately determined. But putting $f = 0$, one of the roots, say 5, becomes $= 0$, and the remaining roots 1, 2, 3, 4 are the roots of the quartic equation $(a,\ b,\ c,\ d,\ e \between v,\ 1)^4 = 0$.

14. Writing for shortness

$$1234 = 12 + 23 + 34, \text{ \&c.}$$

and putting also

$$A = 12 + 34,$$

$$B = 13 + 42,$$

$$C = 14 + 23,$$

then we have

$$\phi_1 = 1234 - 2413 = 12 + 23 + 34 - 24 - 41 - 13 = A - B + 23 - 14,$$

$$\phi_2 = 1342 - 3214 = 13 + 34 + 42 - 32 - 21 - 14 = B - C + 34 - 12,$$

$$\phi_3 = 1423 - 4312 = 14 + 42 + 23 - 43 - 31 - 12 = C - A + 42 - 13,$$

$$\phi_4 = 2143 - 1324 = 21 + 14 + 43 - 13 - 32 - 24 = A - B - 23 + 14,$$

$$\phi_5 = 3124 - 1423 = 31 + 12 + 24 - 14 - 42 - 23 = B - C - 34 + 12,$$

$$\phi_6 = 4132 - 1243 = 41 + 13 + 32 - 12 - 24 - 43 = C - A - 42 + 13.$$

15. We have then

$$(\phi - \phi_1)(\phi - \phi_4) = (\phi - A + B)^2 - (14 - 23)^2$$

$$= (\phi - A + B)^2 - C^2 + 4 \,.\, 1234,$$

where 1234 denotes the product of the four roots; the functions A, B, C, and the product 1234, are each of the degree zero in the coefficients (a, b, c, d, e); and if we put

$$\begin{aligned} \mathrm{b} &= -\ c, \\ \mathrm{c} &= -4ae\ + bd, \\ \mathrm{d} &= \ 4ace - ad^2 - b^2e, \end{aligned}$$

then we in fact have

$$\begin{aligned} a\ \Sigma A\ &= -\mathrm{b}, \\ a^2\,\Sigma AB &= \ \mathrm{c}, \\ a^3\,ABC &= -\,d, \\ a\,.\,1234 &= \ e. \end{aligned}$$

But on the understanding that ϕ is ultimately to be changed into $a\phi$, it is allowable, and it will be convenient to write

$$\begin{aligned} \Sigma A\ &= -\,\mathrm{b}, \\ \Sigma AB &= \ \mathrm{c}, \\ ABC &= -\,\mathrm{d}, \\ 1234 &= \ ae. \end{aligned}$$

16. I assume also

$$\begin{aligned} B + C - A &= \alpha, \\ C + A - B &= \beta, \\ A + B - C &= \gamma; \end{aligned}$$

we have thus

$$\begin{aligned} (\phi - \phi_1)(\phi - \phi_4) &= (\phi + \alpha)(\phi - \beta) + 4ae, \text{ and therefore also} \\ (\phi - \phi_2)(\phi - \phi_5) &= (\phi + \beta)(\phi - \gamma) + 4ae, \\ (\phi - \phi_3)(\phi - \phi_6) &= (\phi + \gamma)(\phi - \alpha) + 4ae, \end{aligned}$$

so that the equation in ϕ is

$$[(\phi + \beta)(\phi - \gamma) + 4ae][(\phi + \gamma)(\phi - \alpha) + 4ae][(\phi + \alpha)(\phi - \beta) + 4ae] = 0.$$

17. To obtain the symmetrical functions of α, β, γ it is only necessary to remark that if in the identical equation

$$(1,\ \mathrm{b},\ \mathrm{c},\ \mathrm{d}\between\theta,\ 1)^3 = (\theta - A)(\theta - B)(\theta - C),$$

we put $\frac{1}{2}(\chi + A + B + C)$, $= \frac{1}{2}(\chi - \mathrm{b})$, in the place of θ, the equation becomes

$$(1,\ \mathrm{b},\ \mathrm{c},\ \mathrm{d}\between\chi - \mathrm{b},\ 2)^3 = (\chi + \alpha)(\chi + \beta)(\chi + \gamma),$$

so that we have

$$\begin{aligned} \Sigma\alpha\ &= -\,\mathrm{b} &&= \ \ c, \\ \Sigma\alpha\beta &= -\,\mathrm{b}^2 + 4\mathrm{c} &&= -16ae\ + 4bd\ - c^2, \\ \alpha\beta\gamma &= \ \mathrm{b}^3 - 4\mathrm{bc} + 8\mathrm{d} = &&\ 16ace - 8ad^2 - 8b^2e + 4bcd - c^3. \end{aligned}$$

18. The developed expression for the equation in ϕ is easily found to be

$$\left.\begin{array}{l} \phi^6 \\ +\phi^4 . -\Sigma\alpha^2 \quad +12\,ae \\ +\phi^2 . \quad \Sigma\alpha^2\beta^2 - \ 4\,ae\,(\Sigma\alpha^2+\Sigma\alpha\beta)+48\,a^2e^2 \\ +\phi \ . -4\,ae\,(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) \\ + \quad -\alpha^2\beta^2\gamma^2+4\,ae\,\alpha\beta\gamma\,\Sigma\alpha-16\,a^2e^2\,\Sigma\alpha\beta+64\,a^3e^3 \end{array}\right\}=0.$$

19. In this equation the coefficient of ϕ is

$$\begin{aligned} &-4 \ \ ae.8\,(B-A)(C-B)(A-C) \\ &= \ \ 32\,ae \quad (A-B)(B-C)(C-A); \end{aligned}$$

or, neglecting the multiplier a, it is

$$-32\,.\,1\,.\,2\,.\,3\,.\,4\,(1-2)(1-3)(1-4)(2-3)(2-4)(3-4),$$

which is the value for $5=0$, of

$$-32\,(1-2)(1-3)(1-4)(1-5)(2-3)(2-4)(2-5)(3-4)(3-5)(4-5),$$

i.e. the coefficient in question is

$$-32\,.\,25\sqrt{5}\sqrt{a^4f^4+\&c.}=-800\sqrt{5}\sqrt{a^4f^4+\&c.},$$

where $a^4f^4+\&c.$ denotes the discriminant of the denumerate form

$$(a,\ b,\ c,\ d,\ e,\ f\between v,\ 1)^5.$$

20. The remaining coefficients are rational functions of a, b, c, d, e, which have to be completed by the introduction of the terms in f. We have

Coeff. ϕ^4

$$\begin{array}{ll} =-\ \ (\Sigma\alpha)^2 & =-c^2 \\ \ \ +\ 2\,\Sigma\alpha\beta & \ \ +2\,(-16\,ae+4\,bd-c^2) \\ \ \ +12\,ae & \ \ +12\,ae. \end{array}$$

Coeff. ϕ^3

$$\begin{array}{ll} = \ \ (\Sigma\alpha\beta)^2 & = \ \ (-16\,ae+4\,bd-c^2)^2 \\ \ \ -\ 2\,\alpha\beta\gamma\Sigma\alpha & \ \ -2c\,(16\,ace-8\,ad^2-8\,b^2e+4\,bcd-c^3) \\ \ \ -\ 4\,ae\,(\Sigma\alpha)^2 & \ \ -4\,ac^2e \\ \ \ +\ 4\,ae\,(\Sigma\alpha\beta) & \ \ +4\,ae\,(-16\,ae+4\,bd-c^2) \\ \ \ +48\,a^2e^2 & \ \ +48\,a^2e^2. \end{array}$$

Coeff. ϕ^0

$$\begin{array}{ll} =-\ \ \alpha^2\beta^2\gamma^2 & =-(16\,ace-8\,ad^2-8\,b^2e+4\,bcd-c^3)^2 \\ \ \ +\ 4\,ae\,\alpha\beta\gamma\Sigma\alpha & \ \ +4\,ace\,(16\,ace-8\,ad^2-8\,b^2e+4\,bcd-c^3) \\ \ \ -16\,a^2e^2\,\Sigma\alpha\beta & \ \ -16\,a^2e^2\,(-16\,ae+4\,bd-c^2) \\ \ \ +64\,a^3e^3. & \ \ +64\,a^3e^3. \end{array}$$

21. Effecting the developments, these are

$ae - 20$	$a^2e^2 + 240$	$a^2e^3 + 320$
$bd + 8$	$abde - 112$	$a^2bde^2 - 64$
$c^2 - 3$	$ac^2e - 8$	$a^2c^2e^2 - 176$
	$acd^2 + 16$	$a^2cd^2e + 224$
	$b^2ce + 16$	$a^2d^4 - 64$
	$b^2d^2 + 16$	$ab^2ce^2 + 224$
	$bc^2d - 16$	$ab^2d^2e - 128$
	$c^4 + 3$	$abc^2de - 112$
		$abcd^3 + 64$
		$ac^4e + 28$
		$ac^3d^2 - 16$
		$b^4e^2 - 64$
		$b^3cde + 64$
		$b^2c^3e - 16$
		$b^2c^2d^2 - 16$
		$bc^4d + 8$
		$c^6 - 1$

the first of which is in fact complete; the others being completed, we obtain the equation in ϕ, viz.:

22. For the denumerate form $(a, b, c, d, e, f \between v, 1)^6 = 0$, the equation in ϕ is

$a^6 \times$		$a^4 \times$		$a^2 \times$	$-800\sqrt{5}a^2\sqrt{\square} \times$	
$+1$	0	$ae - 20$	0	$a^2df - 400$	$+1$	$a^3cf^2 + 4000$
		$bd + 8$		$a^2e^2 + 240$		$a^2def - 1600$
		$c^2 - 3$		$abcf + 240$		$a^2e^3 + 320$
				$abde - 112$		$a^2b^2f^2 - 1600$
				$ac^2e - 8$		$a^2bcef - 640$
				$acd^2 + 16$		$a^2bd^2f + 640$
				$b^3f - 64$		$a^2bde^2 - 64$
				$b^2ce + 16$		$a^2c^3df - 80$
				$b^2d^2 + 16$		$a^2c^2e^2 - 176$
				$bc^2d - 16$		$a^2cd^2e + 224$
				$c^4 + 3$		$a^2d^4 - 64$
(						$ab^2ef + 384$
						$ab^2cdf - 192$
						$ab^2ce^2 + 224$
						$ab^2d^2e - 128$
						$abc^2f + 48$
						$abc^2de - 112$
						$abcd^3 + 64$
						$ac^4e + 28$
						$ac^3d^2 - 16$
						$b^4e^2 - 64$
						$b^3cde + 64$
						$b^3c^3e - 16$
						$b^3c^2d^2 - 16$
						$bc^4d + 8$
						$c^6 - 1$

$\between(\phi, 1)^6 = 0,$

where $\square$, $= a^4f^4 + \&c.$, denotes the discriminant for the denumerate form.

I proceed now to form the expression for $\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6$.

23. Writing for convenience x in the place of the root 5, we have

$$\phi_1 = A - B + \{23 - 14 + x(1 + 4 - 2 - 3)\}$$
$$\phi_4 = A - B - \{23 - 14 + x(1 + 4 - 2 - 3)\},$$

or

$$\phi_1\phi_4 = (A - B)^2 - \{23 - 14 + x(1 + 4 - 2 - 3)\}^2.$$

The terms without x are, as before, $(A - B)^2 - C^2 + 4 \,.\, 1234$, or $-\alpha\beta + 4 \,.\, 1234$, and we have

$$\begin{aligned}\phi_1\phi_4 = &-\alpha\beta + 4 \,.\, 1234\\ &+ 2x(1 + 4 - 2 - 3)(14 - 23)\\ &- x^2(1 + 4 - 2 - 3)^2;\end{aligned}$$

and in like manner

$$\begin{aligned}\phi_2\phi_5 = &-\beta\gamma + 4 \,.\, 1234\\ &+ 2x(1 + 2 - 3 - 4)(12 - 34)\\ &- x^2(1 + 2 - 3 - 4)^2,\end{aligned}$$

and

$$\begin{aligned}\phi_3\phi_6 = &-\gamma\alpha + 4 \,.\, 1234\\ &+ 2x(1 + 3 - 4 - 2)(13 - 42)\\ &- x^2(1 + 3 - 4 - 2)^2.\end{aligned}$$

24. The roots 1, 2, 3, 4, contained in these expressions explicitly, and in α, β, γ, are the roots of the equation $\frac{1}{v - x}(a, b, c, d, e, f \between v, 1)^5 = 0$, or, what is the same thing,

$$(a', b', c', d', e' \between v, 1)^4 = 0,$$

where

$$\begin{aligned}a' &= a,\\ b' &= ax + b,\\ c' &= ax^2 + bx + c,\\ d' &= ax^3 + bx^2 + cx + d,\\ e' &= ax^4 + bx^3 + cx^2 + dx + e.\end{aligned}$$

Omitting, as before, a power of a, which is ultimately restored, we have

$$\begin{aligned}\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6 = &-\Sigma\alpha\beta + 12a'e'\\ &+ 2x\Sigma(1 + 4 - 2 - 3)(14 - 23)\\ &- x^2\Sigma(1 + 4 - 2 - 3)^2,\end{aligned}$$

where the Σ's in the second and third lines denote each of them the sum of the three terms obtained by the cyclical permutations of 2, 3, 4.

The first line is

$$(16\,a'e' - 4\,b'd' + c'^2) + 12\,a'e'$$
$$= 28\,a'e' - 4\,b'd' + 1\,c'^2.$$

The second line is $2x$ into $\Sigma\,1^2 2 - 3\,\Sigma\,123$,

$$= (-\,b'c' + 3\,a'd') + 3\,a'd'\,;$$

or it is

$$= 2x\,(6\,a'd' - 1\,b'c')\,;$$

and the third line is $-x^2$ into $3\,\Sigma\,1^2 - 2\,\Sigma\,12$,

$$= 3\,(b'^2 - 2\,a'c') - 2\,a'c'\,;$$

or it is

$$= x^2\,(8\,a'c' - 3\,b'^2).$$

Hence, combining the three terms,

$$\begin{aligned}\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6 = {} & 28\,a'e' - 4\,b'd' + 1\,c'^2 \\ & + x\ (12\,a'd' - 2\,b'c') \\ & + x^2\,(8\,a'c' - 3\,b'^2),\end{aligned}$$

or substituting for $(a',\ b',\ c',\ d',\ e')$ their values, the right-hand side is

$$= (40\,a^2,\ 32\,ab,\ 28\,ac - 8\,b^2,\ 44\,ad - 8\,bc,\ 28\,ae - 4\,bd + 1\,c^2\between x,\ 1)^4,$$

where x stands for x_3, and on the left-hand side the factor a^2 is to be restored.

25. Writing for shortness

$$(*\between x,\ 1)^4 = (40\,a^2,\ 32\,ab,\ 28\,ac - 8\,b^2,\ 44\,ad - 8\,bc,\ 28\,ae - 4\,bd + 1\,c^2\between x,\ 1)^4,$$

the equation is

$$a^2\,(\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6) = (*\between x_3,\ 1)^4;$$

and the system of equations to which this belongs is

$$\begin{aligned}a^2\,(\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5) &= (*\between x_1,\ 1)^4,\\ a^2\,(\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6) &= (*\between x_2,\ 1)^4,\\ a^2\,(\phi_1\phi_5 + \phi_2\phi_3 + \phi_4\phi_6) &= (*\between x_3,\ 1)^4,\\ a^2\,(\phi_1\phi_3 + \phi_2\phi_6 + \phi_3\phi_5) &= (*\between x_4,\ 1)^4,\\ a^2\,(\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6) &= (*\between x_5,\ 1)^4;\end{aligned}$$

so that the roots $x_1,\ x_2,\ x_3,\ x_4,\ x_5$ will be rational functions of the combinations $\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5$, &c. respectively, of the equation in ϕ.

26. Passing now to the standard form $(a, b, c, d, e, f \between v, 1)^5 = 0$, the equation in ϕ is

$$
\left(\begin{array}{cccccccc}
a^6 \times & & -100\,a^4 \times & & 2000\,a^2 \times & -800\,a^2\sqrt{5}\sqrt{\square} \times & 40000 \times & \\
+1 & 0 & \begin{array}{l} ae + 1 \\ bd - 4 \\ c^2 + 3 \end{array} & 0 & \begin{array}{l} a^2df - 2 \\ a^2e^2 + 3 \\ abef + 6 \\ abde - 14 \\ ac^2e - 2 \\ acd^2 + 8 \\ b^3f - 4 \\ b^2ce + 10 \\ b^2d^2 + 20 \\ bc^2d - 40 \\ c^4 + 15 \end{array} & +1 & \begin{array}{l} a^3cf^2 + 1 \\ a^3def - 2 \\ a^2e^3 + 1 \\ a^2b^2f^2 - 1 \\ a^2bcef - 4 \\ a^2bd^2f + 8 \\ a^2bde^2 - 2 \\ a^2c^2df - 2 \\ a^2c^2e^2 - 11 \\ a^2cd^2e + 28 \\ a^2d^4 - 16 \\ ab^3ef + 6 \\ ab^2cdf - 12 \\ ab^2ce^2 + 35 \\ ab^2d^2e - 40 \\ abc^3f + 6 \\ abc^2de - 70 \\ abcd^2 + 80 \\ ac^4e + 35 \\ ac^3d^2 - 40 \\ b^4e^2 - 25 \\ b^3cde + 100 \\ b^2c^2e - 50 \\ b^2c^2d^2 - 100 \\ bc^4d + 100 \\ c^6 - 25 \end{array} & \between \phi, 1)^6 = 0,
\end{array}\right.
$$

where $\square$, $= a^4f^4 + \&c.$, denotes the Discriminant for the Standard form.

27. And if we put

$$(*\between x, 1)^4 = 20\,(2\,a^2,\ 8\,ab,\ 22\,ac - 10\,b^3,\ 18\,ad - 10\,bc,\ 7\,ae - 10\,bd + 5\,c^2 \between x, 1)^4,$$

then we have

$$a^2(\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5) = (*\between x_1, 1)^4,$$
$$a^2(\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6) = (*\between x_2, 1)^4,$$
$$a^2(\phi_1\phi_5 + \phi_2\phi_3 + \phi_4\phi_6) = (*\between x_3, 1)^4,$$
$$a^2(\phi_1\phi_3 + \phi_2\phi_6 + \phi_3\phi_5) = (*\between x_4, 1)^4,$$
$$a^2(\phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6) = (*\between x_5, 1)^4,$$

which lead to rational expressions for the roots x_1, x_2, x_3, x_4, x_5 in terms of the combinations $\phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5$, &c. respectively.

28. Consider now the quintic function

$$U=(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5=a\,(x-\alpha y)\,(x-\beta y)\,(x-\gamma y)\,(x-\delta y)\,(x-\epsilon y)\,;$$

and treating the numbers 1, 2, 3, 4, 5 as corresponding to α, β, γ, δ, ϵ respectively, write

$$\Phi=\overbrace{12345}-\overbrace{24135},$$

where

$$\overbrace{12345}=\overbrace{12}+\overbrace{23}+\overbrace{34}+\overbrace{45}+\overbrace{51},\ \&c.,$$

in which $\overbrace{12}$, &c. denote respectively

$$\frac{1}{y^2}\,\frac{1}{x-\alpha y}\cdot\frac{1}{x-\beta y},\ \&c.$$

Then we have

$$\Phi=\frac{a}{U}[\phi x-\chi y],$$

where

$$\phi=12345-24135,$$

and

$$12345=12+23+34+45+51=\alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+\epsilon\alpha,$$

$$24135=24+41+13+35+52=\beta\delta+\delta\alpha+\alpha\gamma+\gamma\epsilon+\epsilon\beta,$$

and where

$$\chi=(12345)-(24135),$$

and

$$(12345)=123+234+345+451+512=\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\epsilon+\delta\epsilon\alpha+\epsilon\alpha\beta,$$

$$(24135)=241+413+135+352+524=\beta\delta\alpha+\delta\alpha\gamma+\alpha\gamma\epsilon+\gamma\epsilon\beta+\epsilon\beta\delta.$$

29. In fact,

$$\Phi-\frac{a}{U}\left\{\overbrace{(12345)}-\overbrace{(24135)}\right\},$$

where

$$\overbrace{(12345)}=\widetilde{123}+\widetilde{234}+\widetilde{345}+\widetilde{451}+\widetilde{512},$$

$$\overbrace{(24135)}=\widetilde{241}+\widetilde{413}+\widetilde{135}+\widetilde{352}+\widetilde{524},$$

where $\widetilde{123}$, &c. denote respectively

$$\frac{1}{y^2}(x-\alpha y)\,(x-\beta y)\,(x-\gamma y),\ \&c.,$$

and $\overbrace{(12345)}-\overbrace{(24135)}$ thus presents itself as a cubic function divided by y^2. But in this cubic function the coefficients of x^3, x^2y vanish. For the coefficient of any power of x will be

$$123+234+345+451+512-241-413-135-352-524,$$

where, first, for x^3, 123, &c. denote respectively unity; the coefficient of x^3 therefore vanishes. Next, for x^2y, 123, &c. denote respectively $-(1+2+3)$, &c. $(=\alpha+\beta+\gamma)$, and the coefficient of x^2y also vanishes. But for xy^2, 123, &c. denote respectively $12+23+31$ $(=\alpha\beta+\beta\gamma+\gamma\alpha)$, &c. respectively; the positive terms are

$$(12+23+31)+(23+34+42)+(34+45+53)+(45+51+14)+(51+12+25),$$

which are

$$\begin{aligned}&=2\,(12+23+34+45+51)+(24+41+13+35+52)\\&=2\,.\,12345+24153;\end{aligned}$$

and the negative terms, taken positively, are

$$(24+41+12)+(41+13+34)+(13+35+51)+(35+52+23)+(52+24+45),$$

which are

$$\begin{aligned}&=(12+23+34+45+51)+2\,(24+41+13+35+52)\\&=12345+2\,.\,24135;\end{aligned}$$

so that the difference, or coefficient of xy^2, is

$$=12345-24135,$$

which is $=\phi$.

And for y^3, 123, &c. denote respectively $-123\,(=-\alpha\beta\gamma)$, &c., so that the coefficient of y^3 is $=\chi$.

30. The cubic function is therefore $=\phi xy^2-\chi y^3$; and dividing by y^2, we have

$$\Phi=\frac{a}{U}(\phi x-\chi y).$$

Φ is thus a fractional covariantive function, the leading coefficient whereof is ϕ, and the equation for the determination of Φ is consequently that deduced from the equation for ϕ, by replacing therein the seminvariants by the corresponding covariants. The equation [denoting the covariants as in 141 and 143, A the quintic itself, &c.] is

$$\left\{\begin{array}{l} A^6, \\ 0, \\ -100A^4B, \\ 0, \\ +2000\,A^2\,(6B^2-4H) \\ -800\,A^2\sqrt{5}\,\sqrt{\text{disct.}}, =Q, \\ AJ-25D^2, \end{array}\right\}(\Phi,\ 1)^6=0,$$

where the coefficients are in regard to $(x,\ y)$ of the orders 30, —, 22, —, 14, 10, 6

respectively. The last coefficient [now given in the form $AJ-25D^2$], being of the degree 6 in the coefficients (a, b, c, d, e, f), is not given in the Tables; it is therefore merely indicated by $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}, \mathfrak{G}\between x, y)^6$, the leading coefficient $\mathfrak{A}$ being of course the last coefficient in the equation for ϕ, to the standard form.

I refrain from at present entering into the consideration of the values of the expressions $\Phi_1\Phi + \Phi_2\Phi_5 + \Phi_3\Phi_6$, &c.

ADDITION, Nov. 10, 1862. [Originally printed in a later memoir "On Tschirnhausen's Transformation" *post*, 275.]

I take the opportunity of remarking, with reference to [the foregoing] memoir, that I recently discovered that the auxiliary equation there considered is in fact due to Jacobi, who, in his paper, "Observatiunculæ ad theoriam æquationum pertinentes," *Crelle*, t. XIII. (1835), pp. 340—352, under the heading "Observatio de æquatione sexti gradus ad quam æquationes quinti gradus revocari possunt," gives the theory, and observes that the equation is of the form

$$\phi^6 + a_2\phi^4 + a_4\phi^2 + a_6 = 32\sqrt{\square}\,\phi,$$

and mentions that the value of a_2 is easily found to be (I adapt his notation to the denumerate form $(a, b, c, d, e, f\between v, 1)^6 = 0$)

$$= 40ae - 16bd + 6c^2$$

(this ought, however, to be divided by -2), but that the values of a_4, a_6 "paullo ampliores calculos poscunt."

The value of the coefficient in question is correctly obtained (page 270 of my memoir [as printed in the *Philosophical Transactions*]) in the form

$$\begin{aligned} &-c^2 \\ &+2(-16ae + 4bd - c^2) \\ &+12\,ae; \end{aligned}$$

but the reduced value is given in two places (page 271) as equal to

$$\begin{aligned} &-32\,ae, \text{ this should be } -20\,ae, \\ &+\ 8\,db, \qquad\quad ,, \qquad\quad +\ 8\,bd, \\ &-\ 3\,c^2, \qquad\quad ,, \qquad\quad -\ 3\,c^2. \end{aligned}$$

The last-mentioned correct value was used in obtaining the coefficient for the standard form, which coefficient is given correctly, page 274. [The correction here indicated -20 in place of -32, is made *ante* p. 318.]

269.

A SEVENTH MEMOIR ON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861), pp. 277—292. Received February 28,—Read March 14, 1861.]

THE present memoir relates chiefly to the theory of ternary cubics. Since the date of my Third Memoir on Quantics, [144], M. Aronhold has published the continuation of his researches on ternary cubics, in the memoir "Theorie der homogenen Functionen dritten Grades von drei Veränderlichen," *Crelle*, t. LV. pp. 97—191 (1858). He there considers two derived contravariants, linear functions of the fundamental ones, and which occupy therein the position which the fundamental contravariants PU, QU do in my Third Memoir; in the notation of the present memoir these derived contravariants are

$$\begin{aligned} YU &= 3T.PU - 4S.QU, \\ ZU &= -48S^2.PU + T.QU; \end{aligned}$$

and for the canonical form $x^3+y^3+z^3+6lxyz$, they acquire respectively the factor $(1+8l^3)^2$, viz. in this case

$$\begin{aligned} YU &= (1+8l^3)^2\{l\ (\xi^3+\eta^3+\zeta^3) - 3\ \xi\eta\zeta\}, \\ ZU &= (1+8l^3)^2\{(1+2l^3)(\xi^3+\eta^3+\zeta^3) + 18\,l^2\xi\eta\zeta\}. \end{aligned}$$

The derived contravariants have with the covariants U, HU, even a more intimate connexion than have the contravariants PU, QU; and the advantage of the employment of YU, ZU fully appears by M. Aronhold's memoir.

But the conclusion is, not that the contravariants PU, QU are to be rejected, but that the system is to be completed by the addition thereto of two derived covariants, linear functions of U, HU; these derived covariants, suggested to me by M. Aronhold's memoir, are in the present memoir called CU, DU; their values are

$$\begin{aligned} CU &= -T.U + 24S.HU, \\ DU &= 8S^2.U - 3T.HU: \end{aligned}$$

and for the canonical form $x^3+y^3+z^3+6lxyz$, they acquire respectively, not indeed $(1+8l^3)^2$, but the simple power $(1+8l^3)$, as a factor, viz. in this case

$$CU=(1+8l^3)\{\ (-1+4l^3)(x^3+y^3+z^3)+\qquad 18lxyz\},$$
$$DU=(1+8l^3)\{l^2(\ 5+4l^3)(x^3+y^3+z^3)+3(1-10l^3)xyz\};$$

it was in fact by means of this condition as to the factor $(1+8l^3)$, that the foregoing expressions for CU, DU were obtained [1].

The formulæ of my Third Memoir and those of M. Aronhold are by this means brought into harmony and made parts of a whole; instead of the two intermediates

$$\alpha U+6\beta HU,\quad 6\alpha PU+\beta QU,$$

in Tables 68 and 69 of my Third Memoir, or of the intermediates

$$\alpha U+6\beta HU,\quad -2\alpha YU+2\beta ZU,$$

of M. Aronhold's theory, we have the four intermediates

$$\alpha U+6\beta HU,\quad -2\alpha YU+2\beta ZU,\quad 2\alpha CU-2\beta DU,\quad 6\alpha PU+\beta QU,$$

in Tables 74, 75, 76, and 77 of the present memoir. These four Tables embrace the former results, and the new ones which relate to the covariants CU, DU; and they are what is most important in the present memoir. I have, however, excluded from the Tables, and I do not in the memoir consider (otherwise than incidentally) the covariant of the sixth order ΘU, or the contravariant (reciprocant) FU.

I have given in the memoir a comparison of my notation with that of M. Aronhold. A short part of the memoir relates to the binary cubic and the binary quartic, viz. each of these quantics has a covariant of its own order, forming with it an intermediate $\alpha U+\beta W$, the covariants whereof contain quantics in (α, β), the coefficients of which are invariants of the original quantic. The formulæ which relate to these cases are in fact given in my Fifth Memoir, [156], but they are reproduced here in order to show the relations between the quantics in (α, β) contained in the formulæ. As regards the binary quartic, these results are required for the discussion of the like question in regard to the ternary cubic, viz. that of finding the relations between the different quantics in (α, β) contained in the formulæ relating to the ternary cubic. Some of these relations have been obtained by M. Hermite in the memoir "Sur les formes cubiques à trois indéterminées," (*Liouville*, t. III. pp. 37—40 (1858), and in that "Sur la Résolution des équations du quatrième degré," *Comptes Rendus*, XLVI. p. 715 (1858), and by M. Aronhold in his memoir already referred to; and in particular I reproduce and demonstrate some of the results in the last-mentioned memoir of M. Hermite. But the relations in question are in the present memoir exhibited in a more complete and systematic form.

[1] M. Aronhold, in a letter dated Berlin, 17 June 1861, has pointed out to me that the covariants CU, DU are in his notation $P_{\xi\xi}$, $P_{\eta\eta}$, and that they belong to the forms called Conjugate Forms, § 27 of his memoir. But the explicit development of the properties of these covariants is not on this account the less interesting. Added 20 Sept. 1861.—A.C.

The paragraphs and Tables of the present memoir are numbered consecutively with those of my former memoirs on Quantics.

231. For the binary cubic $(a, b, c, d \between x, y)^3$, if U be the cubic itself, HU the Hessian, ΦU the cubicovariant, and $\square$ the discriminant (see Fifth Memoir, Nos. 115, 118), then

Covariant and other Tables, No. 71.

$$\begin{aligned} H(\alpha U+\beta\Phi U) &= (\alpha^2-\beta^2\square)\,HU, \\ \Phi(\alpha U+\beta\Phi U) &= -\tfrac{1}{2}\partial_\beta(\alpha^2-\beta^2\square)\,.\,U \\ &\quad +\tfrac{1}{2}\partial_\alpha(\alpha^2-\beta^2\square)\,.\,U, \\ \square(\alpha U+\beta\Phi U) &= (\alpha^2-\beta^2\square)^2\,\square, \end{aligned}$$

so that the quantics in (α, β) all of them depend on $\alpha^2-\beta^2\square$.

232. For the binary quartic $(a, b, c, d, e \between x, y)^4$, if U be the quartic itself, HU the Hessian, ΦU the cubicovariant, I, J, the quadrinvariant and the cubinvariant, and $\square\,(=I^3-27J^2)$ the discriminant (see Fifth Memoir, Nos. 128, 134), then

Table No. 72.

$$\begin{aligned} \Phi\,(\alpha U+6\beta HU) &= (1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3\,\Phi U, \\ H\,(\alpha U+6\beta HU) &= -\tfrac{1}{18}\,\partial_\beta\,(1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3\,.\,U \\ &\quad +\tfrac{1}{3}\,\partial_\alpha\,(1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3\,.\,HU, \\ I\,(\alpha U+6\beta HU) &= (I,\ 18J,\ 3I^2 \between \alpha,\ \beta)^2, \\ J\,(\alpha U+6\beta HU) &= (J,\ I^2,\ 9IJ,\ -I^3+54J^2 \between \alpha,\ \beta)^3, \\ \square\,(\alpha U+6\beta HU) &= ([1])\,(1,\ 0,\ -18I,\ 108J,\ 81I^2,\ 972IJ,\ 2916J^2 \between \alpha,\ \beta)^6\,\square \\ &= [(1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3]^2\,\square. \end{aligned}$$

233. Writing for the moment

$$G=(1,\ 0,\ -9I,\ -54J \between \alpha,\ \beta)^3,$$

then the Hessian, cubicovariant, and discriminant of this cubic function of (α, β) are respectively

$$\begin{aligned} HG &= -\ 3\,(I,\ 18J,\ 3I^2 \between \alpha,\ \beta)^2, \\ \Phi G &= \ 54\,(J,\ I^2,\ 9IJ,\ -I^3+54J^2 \between \alpha,\ \beta)^3, \\ \square G &= \ -108\square\,; \end{aligned}$$

so that the covariants of the intermediate $\alpha U+6\beta HU$ are all of them expressible by means of the cubic function G.

[1] The coefficient $2916J^2$ is in the Fifth Memoir erroneously given as $-2916J^2$. [This correction should have been made, vol. II. p. 549.]

It may be noticed that G is what the left-hand side of the equation

$$4(HU)^2 - 4I\,.\,HU\,.\,U^2 + JU^3 = -(\Phi U)^2$$

(see Fifth Memoir, No. 128) becomes on writing therein α, -6β, for U, HU respectively, and throwing out the factor 4.

234. I take the opportunity of remarking with respect to a binary quartic $U = (a, b, c, d \between x, y)^4$, that the Hessian of the cubicovariant, to fix the numerical factor, say $-\frac{1}{3}\{\partial_x^2 \Phi U\,.\,\partial_y^2 \Phi U - (\partial_x \partial_y \Phi U)^2\}$, is

$$= I^2 U^2 - 36 J\,.\,U\,.\,HU + 12 I (HU)^2,$$

which is

$$= \left(IU - \frac{18J}{I} HU\right)^2 + \frac{12}{I^2}(I^3 - 27J^2)(HU)^2;$$

or if $I^3 - 27J^2 = 0$, that is if the quartic has a pair of equal factors, the Hessian of the cubicovariant is a perfect square.

235. Coming now to the ternary cubic $U = (a, b, c, f, g, h, i, j, k, l \between x, y, z)^3$, I give in the first place the following comparison of my notation with that of M. Aronhold.

ARONHOLD.	CAYLEY.
f	U
Δf	$-6HU$
S	$4S$
T	$-T$
R	$-R$
S_f	$6PU$
T_f	$-2QU$
J	$64S^3 \div T^2$
P_f	$-2YU$
Q_f	$2ZU$
...	CU
...	DU
Θ	...
H	...
F	$-FU$
ψ	$2(\Theta_{\prime\prime} U - TU^2 + 4SU\,.\,HU)$,

where the notations YU, ZU (to correspond to M. Aronhold's P_f, Q_f) and the notations CU, DU are first employed in the present memoir. I remark in regard to $P_f (= -2YU)$, where, as already mentioned,

$$YU = (1 + 8l^3)^2 \{l(\xi^3 + \eta^3 + \zeta^3) - 3\xi\eta\zeta\},$$

that in my Memoir on Curves of the Third Order (*Phil. Trans.* t. CXLVII. (1857), see p. 427), [146], I was led incidentally to the curve

$$l(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta=0,$$

and that I there gave the equation

$$3T.PU-4S.QU=(1+8l^3)^3\{l(\xi^3+\eta^3+\zeta^3)-3\xi\eta\zeta\}.$$

But the curve

$$(1+2l^3)(\xi^3+\eta^3+\zeta^3)+18l^2\xi\eta\zeta=0,$$

which corresponds to $(Q_f=2ZU)$, does not occur in that memoir.

236. I remark, further, in regard to M. Aronhold's Θ, H, that these are what he calls "Zwischenformen," viz. they are covariants of the cubic and of the adjoint linear form $\xi x+\eta y+\zeta z$, or as they might be termed *Contracovariants*. For the canonical form $U=x^3+y^3+z^3+6lxyz$, the value of $\frac{1}{2}\Theta$ is

$$(yz-l^2x^2,\ zx-l^2y^2,\ xy-l^2z^2,\ l^2yz-lx^2,\ l^2zx-ly^2,\ l^2xy-lz^2\between\xi,\ \eta,\ \zeta)^2,$$

which is a form which occurs incidentally in my memoir last referred to (see p. 427). The value of H in the same case is

$$(-2l(1+2l^3)x^2-6lyz,\ldots,-(1+4l^3)x^2+2l(1+2l^3)yz,\ldots\between\xi,\ \eta,\ \zeta)^2,$$

which does not occur in that memoir. In my Third Memoir on Quantics I purposely abstained from the consideration of any such forms.

237. My covariants ΘU and $\Theta_{,}U$ involved unsymmetrically the cubic and its Hessian, and it did not occur to me how a similar covariant, such as M. Aronhold's ψ, which involves the two functions symmetrically, was to be formed. Let (A, B, C) be the first derived functions, (a, b, c, f, g, h) the second derived functions of the cubic U, and (A', B', C') the first derived functions, (a', b', c', f', g', h') the second derived functions of the Hessian HU, then disregarding numerical factors, we have

$$\Theta U=(bc-f^2,\ldots,gh-af,\ldots\between A',\ B',\ C')^2,$$

$$\Theta_{,}U=(b'c'-f'^2,\ldots,g'h'-a'f',\ldots\between A,\ B,\ C)^2,$$

and

$$\psi=(bc'+b'c-2ff',\ldots,gh'+g'h-af'-a'f,\ldots\between A,\ B,\ C\between A',\ B',\ C');$$

and considering $U=0$ as the equation of a curve of the third order, the equations $\Theta U=0$, $\Theta_{,}U=0$, $\psi=0$ have the following significations, viz. $\Theta U=0$ is the locus of a point, such that its second or line polar with respect to the Hessian touches its first or conic polar with respect to the cubic: $\Theta_{,}U$ is the locus of a point such that its second or line polar with respect to the cubic touches its first or conic polar with respect to the Hessian: and $\psi=0$ is the locus of a point such that its second or line polar with respect to the cubic, and its second or line polar with respect to the Hessian are reciprocals (that is, each passes through the pole of the other of them) with respect to the conic which is the envelope of a line cutting the first or conic polar of the point with respect to the cubic, and the first or conic polar of the point with respect to the Hessian, in two pairs of points which are harmonically related to each other: such being in fact the immediate interpretation of the analytical formula. But this in passing.

238. The formulæ (Tables 68 and 69 of my Third Memoir) for the discriminants of the intermediates $\alpha U + 6\beta HU$ and $6\alpha PU + \beta QU$ respectively are

$$R(\alpha U + 6\beta HU) = [(1, \quad 0, -24S, -8T, -48S^2 \quad \between \alpha, \beta)^4]^3 R,$$
$$R(6\alpha PU + \beta QU) = [(48S, 8T, -96S^2, -24TS, -T^2 - 16S^3 \between \alpha, \beta)^4]^3 R.$$

In M. Hermite's paper in the *Comptes Rendus*, already referred to, there are given between these quantics in (α, β) certain relations which (although less simple than the relations that will afterwards be obtained) I now proceed to investigate. Putting in the first formula $\alpha \div \beta = p$, and in the second formula $\alpha \div \beta = \theta$, we have

$$R(pU + 6HU) = 0, \text{ if } (1, \quad 0, -24S, -8T, -48S^2 \quad \between p, 1)^4 = 0,$$
$$R(6\theta PU + QU) = 0, \text{ if } (48S, 8T, -96S^2, -24TS, -T^2 - 16S^3 \between \theta, 1)^4 = 0,$$

which equations in p, θ, are about to be considered in place of the quantics from which they respectively arise.

239. It is convenient to write [1]

$$A = 4S,$$
$$B = \sqrt[3]{T^2 - 64S^3}$$

(so that $T^2 = A^3 + B^3$ and, for the canonical form,

$$A = -4l + 4l^3, \quad B = 1 + 8l^3).$$

Making this change, and joining to the equation in p that derived from it by writing q for p, and interchanging A, B, we have the three equations

$$(1, \quad 0, -6A, -8T, -3A^2 \quad \between p, 1)^4 = 0,$$
$$(1, \quad 0, -6B, -8T, -3B^2 \quad \between q, 1)^4 = 0,$$
$$(12A, 8T, -6A^2, -6TA, -T^2 - \tfrac{1}{4}A^3 \between \theta, 1)^4 = 0.$$

240. The signification of the equation in q is as follows, viz. if the quantic

$$U = (* \between x, y, z)^3$$

is transformed into the canonical form

$$X^3 + Y^3 + Z^3 + 6lXYZ$$

by means of the linear equations

$$(x, y, z) = (\Lambda \between X, Y, Z),$$

[1] A is (Aronhold's and) Hermite's S, B is Hermite's S_1, and p, q, θ, Λ are Hermite's δ, δ_1, Δ, d: there is a slight inaccuracy in three of his formulæ, which should be

$$\Delta = -\tfrac{1}{2}\frac{1}{S}\left(T + \frac{S_1^2}{\delta_1}\right), \quad \delta_1 = \frac{24S^2}{f'\delta}, \quad \delta = \frac{24S^2}{f_1'\delta},$$

corresponding to formulæ in the present memoir.

where Λ is a matrix, then using the same letter Λ to represent the determinant formed out of this matrix, or determinant of substitution, we have

$$q = \frac{3}{\Lambda^2},$$

so that the equation in q is one that presents itself in the question of the reduction of the cubic to its canonical form.

In fact the linear transformation gives

$$S\Lambda^4 = -l + l^4,$$
$$T\Lambda^6 = 1 - 20l^3 - 8l^6,$$

and thence

$$(T^2 - 64S^3)\,\Lambda'^2 = (1 + 8l^3)^3,$$

which, writing B^3 in the place of $T^2 - 64S^3$, becomes

$$B^3\Lambda'^2 = (1 + 8l^3)^3, \text{ or}$$
$$B\,\Lambda^4 = 1 + 8l^3\ , \text{ or } 8l^3 = B\Lambda^4 - 1,$$

whence also

$$\begin{aligned} 8T\Lambda^6 &= 8 - 20\,(B\Lambda^4 - 1) - (B\Lambda^4 - 1)^2 \\ &= 27 - 18\ B\Lambda^4 \qquad - B^2\Lambda^8, \end{aligned}$$

or, as this may be written,

$$\frac{81}{\Lambda^8} - \frac{54B}{\Lambda^4} - \frac{24T}{\Lambda^2} - 3B^3 = 0,$$

which, putting therein $q = \dfrac{3}{\Lambda^2}$, becomes

$$(1,\ 0,\ -6B,\ -8T,\ -3B^3 \between q,\ 1)^4 = 0,$$

the above-mentioned equation in q.

241. The relation between θ and q is

$$\theta = -\tfrac{1}{2}\frac{T}{A} + \frac{B^2}{2Aq},$$

as may be verified without difficulty. That between θ and p is

$$\theta = \tfrac{1}{4}\left(p + \frac{A}{p}\right),$$

as appears by the identical equation

$$(12A,\ 8T,\ -6A^2,\ -6TA,\ -T^2 - \tfrac{1}{4}A^3 \between \tfrac{1}{4}(p + \frac{A}{p}),\ 1)^4$$

$$= \frac{1}{64p^4}(3A,\ 8T,\ -12A^2,\ -72TA,\ -46A^3 - 64T^2,\ -72TA^2,\ -12A^4,\ 8TA^2,\ 3A^3 \between p,\ 1)^3$$

$$= \frac{1}{64p^4}(1,\ 0,\ -6A,\ -8T,\ -3A^2 \between p,\ 1)^4 \,.\, (3A,\ 8T,\ 6A^2,\ 0,\ -A^3 \between p,\ 1)^4,$$

where the second factor of the product on the right-hand side is

$$-\frac{p^4}{A}(1,\ 0,\ -6A,\ -8T,\ -3A^2\between\frac{A}{p},\ 1)^4.$$

The relation between p and q is then at once found to be

$$q=\frac{\dfrac{2B^2}{A}}{p+\dfrac{A}{p}+\dfrac{2T}{A}},$$

or (since p, q and A, B may be simultaneously interchanged)

$$p=\frac{\dfrac{2A^2}{B}}{q+\dfrac{B}{q}+\dfrac{2T}{B}}.$$

242. Let the equations in p, q be represented by $\phi p=0$, $\psi q=0$ respectively; then we have

$$\phi p=p^4-6Ap^2-8Tp-3A^2,$$

and therefore

$$\tfrac{1}{4}\phi' p=p^3-3Ap\ -2T,$$

whence

$$\begin{aligned}\tfrac{1}{4}p\phi' p&=p^4-3Ap^2-2Tp\\&=\quad 3Ap^2+6Tp+3A^2,\end{aligned}$$

and therefore

$$q=\frac{2B^2p}{\frac{1}{12}p\phi' p}=\frac{24B^2}{\phi' p},$$

with a like formula for p, that is we have

$$q=\frac{24B^2}{\phi' p},\ p=\frac{24A^2}{\psi' q},$$

which with the equation

$$\theta=\tfrac{1}{4}\left(p+\frac{A}{p}\right),$$

are the system of equations connecting θ, p, q.

243. As already remarked, we have to consider the two derived covariants

$$\begin{aligned}CU&=-T\,.\,U+24S\,.\,HU,\\DU&=8S^2\,.\,U-\ 3T\,.\,HU,\end{aligned}$$

and the two derived contravariants

$$\begin{aligned}YU&=\quad 3T\,.\,PU-4S\,.\,QU,\\ZU&=-48S^2\,.\,PU+\ T\,.\,QU,\end{aligned}$$

which for the canonical form $x^3+y^3+z^3+6lxyz$ are as follows:

Table No. 70 (addition to).

$$\begin{aligned}
CU &= (1+8l^3)\,[(-1+4l^3)(x^3+y^3+z^3)+ \qquad 18l\ xyz],\\
DU &= (1+8l^3)\,[l^2(5+4l^3)(x^3+y^3+z^3)+3(1-10l^3)\,xyz],\\
YU &= (1+8l^3)^2\,[\qquad l(\xi^3+\eta^3+\zeta^3)\ -3\xi\eta\zeta],\\
ZU &= (1+8l^3)^2\,[(1+2l^3)(\xi^3+\eta^3+\zeta^3)+18l^2\xi\eta\zeta].
\end{aligned}$$

244. We have conversely

$$\begin{aligned}
3R.\ U &= 3T\,.\,CU+24S\,.\,DU,\\
3R\,.\,HU &= 8S^2\,.\,CU+\ T\,.\,DU,
\end{aligned}$$

and

$$\begin{aligned}
-3R\,.\,PU &= \ T\,.\,YU+4S\,.\,QU,\\
-3R\,.\,QU &= 48S^2\,.\,YU+3T\,.\,PU,
\end{aligned}$$

and also the following formulæ, viz. if

$$2\alpha CU-2\beta DU=\alpha' U+6\beta' HU;$$

then

$$\begin{aligned}
\alpha' &= \qquad -2T\alpha-16S^2\beta,\\
\beta' &= \qquad 8S\alpha+\ T\beta,
\end{aligned}$$

which give, conversely,

$$\alpha=\frac{1}{2R}(\ T\alpha'+16S^2\beta'),$$

$$\beta=\frac{1}{2R}(-8S\alpha'-\ 2T\beta');$$

and moreover, if

$$-2\alpha YU+2\beta ZU=6\alpha' PU+\beta' QU,$$

then

$$\begin{aligned}
\alpha' &= \qquad -(\ T\alpha+16S^2\beta),\\
\beta' &= \qquad -(-8S\alpha-\ 2T\beta),
\end{aligned}$$

which give, conversely,

$$\alpha=-\frac{1}{2R}(-2T\alpha'-16S^2\beta'),$$

$$\beta=-\frac{1}{2R}(\ 8S\alpha'+\ T\beta');$$

so that the relation between (α, β) and (α', β') in the present case is similar to that between (α', β') and (α, β) in the former case. It may be noticed that in all these systems of linear equations, the determinant of transformation is a multiple of $64S^3-T^2\,(=R)$.

245. It will be convenient, before giving the Tables for the covariants of

$$\alpha U + 6\beta HU,\quad 2\alpha CU - 2\beta DU,\quad 6\alpha PU + \beta QU,\quad 2\alpha YU - 2\beta ZU,$$

which replace Tables 68 and 69 of my Third Memoir, to give the following separate Table of the quantics in (α, β) which enter into the expressions of the invariants in Tables 68 and 69, and in these new Tables.

Table No. 73.

$$(1,\ 0,\ -24S,\ -8T,\quad -48S^2\between\alpha,\ \beta)^4,$$
$$(S,\ T,\quad 24S^2,\ 4TS,\ T^2 - 48S^3\between\alpha,\ \beta)^4,$$
$$(T,\ 96S^2,\ 60TS,\ 20T^2,\ 240TS^2,\ -48T^3S + 4608S^4,\ -8T^3 + 576TS^3\between\alpha,\ \beta)^6.$$

$$(48S,\ 8T,\ -96S^2,\ -24TS,\ -T^2 - 16S^3\between\alpha,\ \beta)^4.$$

$$\left\{\begin{array}{l} T^2 \quad + 192\,S^3\ , \\ 128\,TS^2\ , \\ 18\,T^2S \ + 384\,S^4\ , \\ T^3 \quad + \ 64\,TS^3, \\ 5\,T^2S^2 \ - \ 64\,S^5\ , \end{array}\right\}\between\alpha,\ \beta)^4,$$

$$\left\{\begin{array}{l} -8\,T^3 \ + \ 4608\,TS^3\ , \\ 1920\,T^2S^2 + 73728\,S^5\ , \\ 360\,T^3S + 38400\,TS^4\ , \\ 20\,T^4 \ + \ 8960\,T^2S^3, \\ 840\,T^3S^2 + \ 7680\,TS^5\ , \\ 36\,T^3S \ + \ 384\,T^2S^4 + 24576\,S^7, \\ 1\,T^5 \ - \ 40\,T^3S^3 + \ 2560\,TS^6, \end{array}\right\}\between\alpha,\ \beta)^6,$$

where the first part of the Table contains the quantics in (α, β) which relate to the forms $\alpha U + 6\beta HU$ and $2\alpha YU - 2\beta ZU$, and the second part of the Table contains the quantics in (α, β) which relate to the forms $6\alpha PU + \beta QU$ and $-2\alpha CU + 2\beta DU$.

The quantics in (α, β) contained in the foregoing Table are in the sequel indicated by means of their leading coefficients; as thus,

$$(1,\ 0,\ -24S, \ldots\between\alpha,\ \beta)^4,\quad (S,\ T, \ldots\between\alpha,\ \beta)^4,\quad (T^2 + 192S^3, \ldots\between\alpha,\ \beta)^4,\ \&c.$$

246. It is easy to see what transformations must be performed on the results in Tables 68 and 69, in order to obtain the new Tables. Thus, in the formation of Table 74, Table 68 gives $\alpha U + 6\beta HU$ and $H(\alpha U + 6\beta HU)$, and from these $C(\alpha U + 6\beta HU)$,

$D(\alpha U+6\beta HU)$ have to be found: the same Table gives also $P(\alpha U+6\beta HU)$, $Q(\alpha U+6\beta HU)$, but the expressions of these quantities YU, ZU have to be introduced in the place of PU, QU; and from the expressions so transformed are deduced also the expressions for $Y(\alpha U+6\beta HU)$, $Z(\alpha U+6\beta HU)$. Table 75 is to be deduced from Table 69 by writing therein (α', β'), for (α, β), and then putting $6\alpha' PU+\beta' QU = 2\alpha YU-2\beta ZU$, which, as is seen above, gives α', β' as functions of α, β and of the invariants S and T; but in some of the formulæ YU, ZU, have to be introduced in the place of PU, QU. And so for the Tables 76 and 77. The actual effectuation of the transformations would, it is almost needless to remark, be very laborious, but the forms of the results are easily foreseen, and the results can then be verified by means of one or two coefficients only. The new Tables are

Table No. 74.

$$R(\alpha U+6\beta HU) = R\times[(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4]^3,$$
$$S(\alpha U+6\beta HU) = \quad (S,\ T\quad, \ldots \between \alpha,\ \beta)^4,$$
$$T(\alpha U+6\beta HU) = \quad [(T,\ 96S^2\ , \ldots \between \alpha,\ \beta)]^6.$$

$$(\alpha U+6\beta HU) = \qquad \alpha U+6\beta HU,$$
$$H(\alpha U+6\beta HU) = -\tfrac{1}{24}\times \begin{cases} \partial_\beta(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . U \\ -6\partial_\alpha(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . HU, \end{cases}$$
$$C(\alpha U+6\beta HU) = (1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 \times \begin{cases} \partial_\beta(S,\ T, \ldots \between \alpha,\ \beta)^4 . U \\ -6\partial_\alpha(S,\ T, \ldots \between \alpha,\ \beta)^4 . HU, \end{cases}$$
$$D(\alpha U+6\beta HU) = \tfrac{1}{12}(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 \times \begin{cases} \partial_\beta(T,\ 96S^2, \ldots \between \alpha,\ \beta)^6 . U \\ -6\partial_\alpha(T,\ 96S^2, \ldots \between \alpha,\ \beta)^6 . HU, \end{cases}$$
$$P(\alpha U+6\beta HU) = -\frac{1}{3R}\times \begin{cases} \partial_\beta(S,\ T, \ldots \between \alpha,\ \beta)^4 . YU \\ +\partial_\alpha(S,\ T, \ldots \between \alpha,\ \beta)^4 . ZU, \end{cases}$$
$$Q(\alpha U+6\beta HU) = -\frac{1}{6R}\times \begin{cases} \partial_\beta(T,\ 96S^2, \ldots \between \alpha,\ \beta)^6 . YU \\ +\partial_\alpha(T,\ 96S^2, \ldots \between \alpha,\ \beta)^6 . ZU, \end{cases}$$
$$Y(\alpha U+6\beta HU) = \qquad [(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4]^2 \times (-2\alpha YU+2\beta ZU),$$
$$Z(\alpha U+6\beta HU) = \qquad \tfrac{1}{4}[(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4]^2 \times \begin{cases} \partial_\beta(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . YU \\ +\partial_\alpha(1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . ZU. \end{cases}$$

No. 75.

$$R(-2\alpha YU+2\beta ZU)=-4096R^8\times[(S,\ T,\ldots\between\alpha,\ \beta)^4]^3,$$

$$S(-2\alpha YU+2\beta ZU)=\quad -R^3\times(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4,$$

$$T(-2\alpha YU+2\beta ZU)=\quad -8R^4\times(T,\ 96S^2,\ldots\between\alpha,\ \beta)^6.$$

$$-2\alpha YU+2\beta ZU\ =\qquad -2\alpha YU+2\beta ZU,$$

$$H(-2\alpha YU+2\beta ZU)=-\tfrac{2}{3}R\times\begin{cases}\partial_\beta(S,\ T,\ldots\between\alpha,\ \beta)^4.\,YU\\ +\ \partial_\alpha(S,\ T,\ldots\between\alpha,\ \beta)^4.\,ZU,\end{cases}$$

$$C(-2\alpha YU+2\beta ZU)=-16R^4(S,\ T,\ldots\between\alpha,\ \beta)^4\times$$

$$\begin{cases}\partial_\beta(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4.\,YU\\ +\ \partial_\alpha(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4.\,ZU\end{cases}$$

$$D(-2\alpha YU+2\beta ZU)=-\tfrac{32}{3}R^5(S,\ T,\ldots\between\alpha,\ \beta)^4\times$$

$$\begin{cases}\partial_\beta(T,\ 96S^2,\ldots\between\alpha,\ \beta)^6.\,YU\\ +\ \partial_\alpha(T,\ 96S^2,\ldots\between\alpha,\ \beta)^6.\,ZU,\end{cases}$$

$$P(-2\alpha YU+2\beta ZU)=\quad\tfrac{1}{6}R^2\times\begin{cases}\partial_\beta(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4.\quad U\\ -6\partial_\alpha(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4.\,HU,\end{cases}$$

$$Q(-2\alpha YU+2\beta ZU)=-\tfrac{2}{3}R^3\times\begin{cases}\partial_\beta(T,\ 96S^2,\ldots\between\alpha,\ \beta)^6.\quad U\\ -6\partial_\alpha(T,\ 96S^2,\ldots\between\alpha,\ \beta)^6.\,HU,\end{cases}$$

$$Y(-2\alpha YU+2\beta ZU)=\quad 256R^6[(S,\ T,\ldots\between\alpha,\ \beta)^4]^2\times$$

$$(\alpha U+6\beta HU),$$

$$Z(-2\alpha YU+2\beta ZU)=-512R^7[(S,\ T,\ldots\between\alpha,\ \beta)^4]^2\times$$

$$\begin{cases}\partial_\beta(S,\ T,\ldots\between\alpha,\ \beta)^4.\quad U\\ -6\partial_\alpha(S,\ T,\ldots\between\alpha,\ \beta)^4.\,HU.\end{cases}$$

No. 76.

$$R\,(2\alpha CU-2\beta DU) = -4096R^4 \times [(T^2+192S^3 \quad ,\ldots \between \alpha,\ \beta)^4]^3,$$

$$S\,(2\alpha CU-2\beta DU) = \quad -\ R^2 \times (48S,\ 8T \quad ,\ldots \between \alpha,\ \beta)^4,$$

$$T\,(2\alpha CU-2\beta DU) = \quad -8R^2 \times (-8T^3+4608TS^3,\ldots \between \alpha,\ \beta)^6.$$

$$2\alpha CU \quad -2\beta DU = \qquad 2\alpha CU-2\beta DU,$$

$$H\,(2\alpha CU-2\beta DU) = \quad \tfrac{2}{3} \times \begin{cases} \partial_\beta\,(T^2+192S^3,\ldots \between \alpha,\ \beta)^4 . CU \\ +\partial_\alpha\,(T^2+192S^3,\ldots \between \alpha,\ \beta)^4 . DU, \end{cases}$$

$$C\,(2\alpha CU-2\beta DU) = \quad -16R^2(T^2+192S^3,\ldots \between \alpha,\ \beta)^4 \times \begin{cases} \partial_\beta\,(48S,\ 8T \quad ,\ldots \between \alpha,\ \beta)^4 . CU \\ +\partial_\alpha\,(48S,\ 8T \quad ,\ldots \between \alpha,\ \beta)^4 . DU, \end{cases}$$

$$D\,(2\alpha CU-2\beta DU) = \quad \tfrac{32}{3} R^2(T^3+192S^3 \quad ,\ldots \between \alpha,\ \beta)^4 \times \begin{cases} \partial_\beta\,(-8T^3+4608TS^3,\ldots \between \alpha,\ \beta)^6 . CU \\ +\partial_\alpha\,(-8T^3+4608TS^3,\ldots \between \alpha,\ \beta)^6 . DU, \end{cases}$$

$$P\,(2\alpha CU-2\beta DU) = \quad \tfrac{1}{6}R \quad \times \begin{cases} 6\partial_\beta\,(48S,\ 8T,\ldots \between \alpha,\ \beta)^4 . PU \\ -\ \partial_\alpha\,(48S,\ 8T,\ldots \between \alpha,\ \beta)^4 . QU, \end{cases}$$

$$Q\,(2\alpha CU-2\beta DU) = \quad \tfrac{2}{3}R \quad \times \begin{cases} 6\partial_\beta\,(-8T^3+4608TS^3,\ldots \between \alpha,\ \beta)^6 . PU \\ -\ \partial_\alpha\,(-8T^3+4608TS^3,\ldots \between \alpha,\ \beta)^6 . QU, \end{cases}$$

$$Y\,(2\alpha CU-2\beta DU) = \quad 216R^3 \quad [(T^2+192S^3,\ldots \between \alpha,\ \beta)^4]^2 \times (6\alpha PU+\beta QU),$$

$$Z\,(2\alpha CU-2\beta DU) = -512R^3\,[(T^3+192S^3,\ldots \between \alpha,\ \beta)^4]^2 \times \begin{cases} 6\partial_\beta\,(T^2+192S^3,\ldots \between \alpha,\ \beta)^4 . PU \\ -\ \partial_\alpha\,(T^2+192S^3,\ldots \between \alpha,\ \beta)^4 . QU. \end{cases}$$

No. 77.

$$R\,(6\alpha PU+\beta QU)=R^2\times[(48S,\ 8T\quad\ ,\,.\,.\,\between\alpha,\ \beta)^4]^3,$$

$$S\,(6\alpha PU+\beta QU)=\quad(T^2+192S^3\quad\ ,\,.\,.\,\between\alpha,\ \beta)^4,$$

$$T\,(6\alpha PU+\beta QU)=\quad(-8T^3+4608TS^3,\,.\,.\,\between\alpha,\ \beta)^6.$$

$$6\alpha PU+\beta QU)=\qquad 6\alpha PU+\beta QU,$$

$$H\,(6\alpha PU+\beta QU)=-\tfrac{1}{24}\times\begin{cases}6\partial_\beta\,(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\,.\,PU\\ -\ \partial_\alpha\,(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\,.\,QU,\end{cases}$$

$$C\,(6\alpha PU+\beta QU)=\qquad -(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\times$$
$$\begin{cases}6\partial_\beta\,(T^2+192S^3,\,.\,.\,\between\alpha,\ \beta)^4\,.\,PU\\ -\ \partial_\alpha\,(T^2+192S^3,\,.\,.\,\between\alpha,\ \beta)^4\,.\,QU,\end{cases}$$

$$D\,(6\alpha PU+\beta QU)=\qquad \tfrac{1}{12}(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\times$$
$$\begin{cases}6\partial_\beta\,(-8T^3+4608TS^3,\,.\,.\,\between\alpha,\ \beta)^6\,.\,PU\\ -\ \partial_\alpha\,(-8T^3+4608TS^3,\,.\,.\,\between\alpha,\ \beta)^6\,.\,QU,\end{cases}$$

$$P\,(6\alpha PU+\beta QU)=\ \frac{1}{3R}\times\begin{cases}\partial_\beta\,(T^2+192S^3,\,.\,.\,\between\alpha,\ \beta)^4\,.\,CU\\ +\partial_\alpha\,(T^2+192S^3,\,.\,.\,\between\alpha,\ \beta)^4\,.\,DU,\end{cases}$$

$$Q\,(6\alpha PU+\beta QU)=\ \frac{1}{6R}\times\begin{cases}\partial_\beta\,(-8T^3+4608TS^3,\,.\,.\,\between\alpha,\ \beta)^6\,.\,CU\\ +\partial_\alpha\,(-8T^3+4608TS^3,\,.\,.\,\between\alpha,\ \beta)^6\,.\,DU,\end{cases}$$

$$Y\,(6\alpha PU+\beta QU)=\ -\tfrac{1}{2}R\,.\,[(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4]^2\times$$
$$(-2\alpha CU+2\beta DU),$$

$$Z\,(6\alpha PU+\beta QU)=\ -\tfrac{1}{4}R\,[(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4]^2\times$$
$$\begin{cases}\partial_\beta\,(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\,.\,CU\\ +\partial_\alpha\,(48S,\ 8T,\,.\,.\,\between\alpha,\ \beta)^4\,.\,DU.\end{cases}$$

247. It will be noticed how Tables 74 and 75 form a system involving only the quantics in (α, β) contained in the first part of Table 73, and how, in like manner, Tables 76 and 77 form a system involving only the quantics in (α, β) contained in the second part of Table 73; and, moreover, how in each pair of Tables the covariants, &c. correspond to each other as follows, viz.

$$R,\ S,\ T,\ 1,\ H,\ C,\ D,\ P,\ Q,\ Y,\ Z \text{ to}$$
$$S,\ R,\ T,\ Y,\ P,\ Z,\ Q,\ H,\ D,\ 1,\ C.$$

Thus in Table 74,—the formula for $H(\alpha U \quad + 6\beta HU)$,

and in Table 75,—the formula for $P(2\alpha YU - 2\beta ZU)$,

each of them involve the same factor

$$\begin{cases} \partial_\beta (1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . U \\ -6\partial_\alpha (1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4 . HU, \end{cases}$$

and so in all the other cases.

248. The quantics in (α, β) in each part of the foregoing Table 73 are covariantively connected together. In fact, considering the function $(1, 0, -24S, \ldots \between \alpha, \beta)^4$, which for shortness I call G, we have

$$\begin{aligned} G &= (1,\ 0,\ -24S, \ldots \between \alpha,\ \beta)^4, \\ IG &= 0, \\ JG &= 4(64S^3 - T^2) = 4R, \\ \square G &= (IG)^3 \quad -27(JG)^2 = -432R^2, \\ HG &= -4(S,\ T, \ldots \qquad \between \alpha,\ \beta)^4, \\ \Phi G &= 2(T,\ 96S^2, \ldots \qquad \between \alpha,\ \beta)^6. \end{aligned}$$

The last-mentioned formulæ, by the aid of Table 72, give rise to the following more general system in which they are themselves included.

Table No. 78.

$$\begin{aligned} \lambda G + 6\mu HG &= \lambda G + 6\mu HG, \\ H(\lambda G + 6\mu HG) &= 36\mu^2 G + \lambda^2 HG, \\ \Phi(\lambda G + 6\mu HG) &= (\lambda^3 - 216R\mu^3)\, 2(T,\ 96S^2, \ldots \between \alpha,\ \beta)^6, \\ I(\lambda G + 6\mu HG) &= \qquad 72R\lambda\mu, \\ J(\lambda G + 6\mu HG) &= 4R(\lambda^3 + 216R\mu^3), \\ \square(\lambda G + 6\mu HG) &= -432R^2(\lambda^3 - 216R\mu^3)^2. \end{aligned}$$

The expression for $H(\lambda G+6\mu HG)$, putting therein $\lambda=0$, shows that, to a numerical factor *près*, $H.HG$ is equal to G, and hence, disregarding numerical factors, we may say that each of the quartics $(1, 0, 24S, \ldots \between \alpha, \beta)^4$, $(S, T, \ldots \between \alpha, \beta)^4$, is the Hessian of the other of them, and that the sextic $(T, 96S^2, \ldots \between \alpha, \beta)^6$ is the cubicovariant of each of them.

249. Similarly, if the function $(48S, 8T, \ldots \between \alpha, \beta)^4$ is for shortness called G, then we have

$$\begin{aligned}
G &= (48S, 8T, \ldots \between \alpha, \beta)^4,\\
IG &= 0,\\
JG &= 4(64S^3-T^2)^2 = 4R^2,\\
\square G &= (IG)^3 - 27(JG)^2 = -432R^4,\\
HG &= -4(T^2+192S^3, \ldots \between \alpha, \beta)^4,\\
\Phi G &= -2(-8T^3+4608TS^2, \ldots \between \alpha, \beta)^4.
\end{aligned}$$

The last-mentioned formulæ, by the aid of the same Table 72, give rise to the more general system in which they are themselves included.

Table No. 79.

$$\begin{aligned}
\lambda G+6\mu HG &= \lambda G+6\mu HG,\\
H(\lambda G+6\mu HG) &= 36R^2\mu^2 G+\lambda^2 HG,\\
\Phi(\lambda G+6\mu HG) &= (\lambda^3-216R^2\mu^3)\times -2(-8T^3+4608TS^2, \ldots \between \alpha, \beta)^6,\\
I(\lambda G+6\mu HG) &= 72R^2\lambda\mu,\\
J(\lambda G+6\mu HG) &= 4R^2(\lambda^3+216R^2\mu^3),\\
\square(\lambda G+6\mu HG) &= -432R^4(\lambda^3-216R^2\mu^3)^2.
\end{aligned}$$

The expression for $H(\lambda G+6\mu HG)$, putting therein $\lambda=0$, shows that, to a numerical factor *près*, $H.HG$ is equal to G; so that, disregarding numerical factors, we may say that each of the quartics $(48S, T, \ldots \between \alpha, \beta)^4$, $(T^2+192S^3, \ldots \between \alpha, \beta)^4$, is the Hessian of the other of them, and that the sextic $(-8T^3+4608TS^2, \ldots \between \alpha, \beta)^6$ is the cubicovariant of each of them.

250. But besides this, the quantics in (α, β) in the two parts of the Table 73 are linearly connected together: the linear relations in question are in fact the equations whereon depend the expressions for the invariants in Tables 76 and 77 as deduced from those in Tables 74 and 75; and in the order of proof, they precede the formulæ in these four Tables. The linear relations are

Table No. 80.

$(1,\ 0,\ -24S,\ldots$	$\between -2T\alpha-16S^2\beta,\ \ 8S\alpha+\ T\beta)^4=-16R$	$(T^2+192S^2,\ldots$	$\between\alpha,\ \beta)^4,$
$(S,\ T,\ldots$	$\between -2T\alpha-16S^2\beta,\ \ 8S\alpha+\ T\beta)^4=$	$R^2(48S,\ 8T,\ldots$	$\between\alpha,\ \beta)^4,$
$(T,\ 96S^2,\ldots$	$\between -2T\alpha-16S^2\beta,\ \ 8S\alpha+\ T\beta)^6=-\ 8R^2$	$(-8T^3+4608TS^3,\ldots$	$\between\alpha,\ \beta)^6.$

$(48S,\ 8T,\ldots$	$\between\ T\alpha+16S^2\beta,\ -8S\alpha-2T\beta)^4=-16R^2$	$(S,\ T,\ldots$	$\between\alpha,\ \beta)^4,$
$(T^2+192S^2,\ldots$	$\between\ T\alpha+16S^2\beta,\ -8S\alpha-2T\beta)^4=-\ R^2$	$(1,\ 0,\ -24S,\ldots$	$\between\alpha,\ \beta)^4,$
$(-8T^3+4608TS^3,\ldots$	$\between\ T\alpha+16S^2\beta,\ -8S\alpha-2T\beta)^6=-\ 8R^4$	$(T,\ 96S^2,\ldots$	$\between\alpha,\ \beta)^6.$

Hence, attending to the remarks on the Tables 78 and 79, we may say that the quartics

$$(1,\ 0,\ -24S,\ldots\between\alpha,\ \beta)^4,$$
$$(48S,\ T,\ldots\between\alpha,\ \beta)^4,$$

which belong to the two parts respectively of Table 73, and which are related, the first of them to the discriminants of $\alpha U+6\beta HU$ and $2\alpha YU-2\beta ZU$, and the second to the discriminants of $6\alpha PU+\beta QU$, $-2\alpha CU+2\beta DU$, have these relations to each other, viz. each is a linear transformation of the Hessian of the other of them, and the cubicovariant of each is a linear transformation of the cubicovariant of the other.

270.

ON THE DOUBLE TANGENTS OF A CURVE OF THE FOURTH ORDER.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861), pp. 357—362. Received May 30,—Read June 20, 1861.]

THE present memoir is intended to be supplementary to that "On the Double Tangents of a Plane Curve (*Phil. Trans.*, vol. CXLIX. (1859), pp. 193—212) [260]." I take the opportunity of correcting an error which I have there fallen into, and which is rather a misleading one, viz. the emanants U_1, U_2, .. were numerically determined in such manner as to become equal to U on putting (x_1, y_1, z_1) equal to (x, y, z); the numerical determination should have been (and in the latter part of the memoir is assumed to be) such as to render H_1, H_2, &c. equal to H, on making the substitution in question; that is, in the place of the formulæ

$$U_1 = \frac{1}{n}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\,U,$$

$$U_2 = \frac{1}{n(n-1)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)^2\,U, \text{ \&c.,}$$

$$\vdots$$

there ought to have been

$$U_1 = \frac{1}{(n-2)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\,U,$$

$$U_2 = \frac{1}{(n-2)(n-3)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)^2\,U, \text{ \&c.}$$

$$\vdots$$

[this error is corrected *ante* p. 189].

The points of contact of the double tangents of the curve of the fourth order or quartic $U=0$, are given as the intersections of the curve with a curve of the fourteenth order $\Pi=0$; the last-mentioned curve is not absolutely determinate, since instead of $\Pi=0$, we may, it is clear, write $\Pi+MU=0$, where M is an arbitrary

function of the tenth order. I have in the memoir spoken of Hesse's original form (say $\Pi_1 = 0$) of the curve of the fourteenth order obtained by him in 1850, and of his transformed form (say $\Pi_2 = 0$) obtained in 1856. The method in the memoir itself (Mr Salmon's method) gives, in the case in question of a quartic curve, a third form, say $\Pi_3 = 0$. It appears by his paper "On the Determination of the Points of Contact of Double Tangents to an Algebraic Curve (*Quart. Math. Journ.* vol. III. p. 317 (1859))," that Mr Salmon has verified by algebraic transformations the equivalence of the last-mentioned form with those of Hesse; but the process is not given. The object of the present memoir is to demonstrate the equivalence in question, viz. that of the equation $\Pi_3 = 0$ with the one or other of the equations $\Pi_1 = 0$, $\Pi_2 = 0$, in virtue of the equation $U = 0$. The transformation depends, 1st, on a theorem used by Hesse for the deduction of his second form $\Pi_2 = 0$ from the original form $\Pi_1 = 0$, which theorem is given in his paper "Transformation der Gleichung der Curven 14ten Grades welche eine gegebene Curve 4ten Grades in den Berührungspuncten ihrer Doppeltangenten schneiden," *Crelle*, t. LII. pp. 97—103 (1856), containing the transformation in question; I prove this theorem in a different and (as it appears to me) more simple manner; 2nd, on a theorem relating to a cubic curve proved incidentally in my memoir "On the Conic of Five-pointic Contact at any point of a Plane Curve (*Phil. Trans.*, vol. CXLIX. (1859), see p. 385 [261])," the cubic curve being in the present case any first emanent of the given quartic curve: the demonstration occupies only a single paragraph, and it is here reproduced; and I reproduce also Hesse's demonstration of the equivalence of the two forms $\Pi_1 = 0$ and $\Pi_2 = 0$.

Let $U = (*\between x, y, z)^4$ be a quartic function of (x, y, z); (a, b, c, f, g, h) its second differential coefficients; $(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H})$ the reciprocal system

$$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch);$$

and let H be the Hessian of U, or determinant $abc - af^2 - bg^2 - ch^2 + 2fgh$ (H is of course a sextic function of x, y, z); (a', b', c', f', g', h') the second differential coefficients of H; $(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}')$ the reciprocal system

$$(b'c' - f'^2,\ c'a' - g'^2,\ a'b' - h'^2,\ g'h' - a'f',\ h'f' - b'g',\ f'g' - c'h').$$

Then $U = 0$ being the equation of a quartic curve, the equation of the curve of the fourteenth order which by its intersections determines the points of contact of the double tangents of the quartic curve, may be taken to be (Hesse's original form)

$$\Pi_1 = (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x H, \partial_y H, \partial_z H)^2 - 3H(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x, \partial_y, \partial_z)^2 H = 0\ (^1).$$

Or it may be taken to be (Hesse's transformed form)

$$\Pi_2 = 5(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x H, \partial_y H, \partial_z H)^2 - 3(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}'\between \partial_x U, \partial_y U, \partial_z U)^2 = 0.$$

And moreover, if $U_1 = \frac{1}{2}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\, U$, and if H_1 be the Hessian of U_1, and $(a'', b'', c'', f'', g'', h'')$ the second differential coefficients of $H - 3H_1$, where in the differentiations (x_1, y_1, z_1) are treated as constants but after the differentiations are

[1] In quoting this formula in my former memoir, the numerical factor 3 is by mistake omitted. [This correction should have been made *ante* p. 187.]

effected they are replaced by (x, y, z), and if $(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'')$ be the reciprocal system

$$(b''c''-f''^2,\ c''a''-g''^2,\ a''b''-h''^2,\ g''h''-a''f'',\ h''f''-b''g'',\ f''g''-c''h''),$$

then the equations of the curve of the fourteenth order may be taken to be (Salmon's form)

$$\Pi_3 = (\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 = 0.$$

I have preferred to write the three equations in the foregoing forms; but it is clear that the terms

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H \qquad ; \ (\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between \partial_x, \partial_y, \partial_z)^2 U$$

might also have been written

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between a', b', c', 2f', 2g', 2h');\ (\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between a, b, c, 2f, 2g, 2h).$$

As already noticed, it has been shown by Hesse (and his demonstration is to be here reproduced) that the two forms $\Pi_1 = 0$ and $\Pi_2 = 0$ are equivalent to each other. And the object of the memoir is to show that the third form $\Pi_3 = 0$ is equivalent to the other two. The equivalences in question subsist in virtue of the equation $U = 0$, that is, the functions Π_1, Π_2, Π_3 are not identical, but differ from each other by multiples of U.

Demonstration of Hesse's Theorem.

Let (a, b, c, f, g, h), (a', b', c', f', g', h') be any systems of coefficients of a ternary quadratic function; $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})$, $(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}')$ the reciprocal systems as above, (x, y, z) arbitrary quantities. Consider the function

$$\begin{aligned}\square = {} & (a, b, c, f, g, h \between x, y, z)^2 . (\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between a, b, c, 2f, 2g, 2h) \\ & - (\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between ax+hy+gz,\ hx+by+fz,\ gx+fy+cz)^2.\end{aligned}$$

The term involving A' is

$$a(a, b, c, f, g, h \between x, y, z)^2 - (ax+hy+gz)^2,$$

which is

$$\begin{aligned} &= (ab-h^2)\,y^2 + (ac-g^2)\,z^2 + 2\,(af-gh)\,yz, \\ &= Cy^2 + Bz^2 - 2Fyz; \end{aligned}$$

and the term involving $2F'$ is

$$f(a, b, c, f, g, h \between x, y, z)^2 - (hx+by+fz)(gx+fy+cz),$$

which is

$$\begin{aligned} &= (af-gh)\,x^2 + (f^2-bc)\,yz + (fg-ch)\,zx + (hf-bg)\,xy, \\ &= -Fx^2 - Ayz + Hzx + Gxy; \end{aligned}$$

and the entire expression for $\square$ is thus

$$\begin{aligned} & A'(Cy^2+Bz^2-2Fyz) \\ + \; & B'(Az^2+Cx^2-2Gzx) \\ + \; & C'(Bx^2+Ay^2-2Hxy) \\ + \; & 2F'(-Fx^2-Ayz+Hzx+Gxy) \\ + \; & 2G'(-Gy^2-Bzx+Fxy+Hyz) \\ + \; & 2H'(-Hz^2-Cxy+Gyz+Fzx); \end{aligned}$$

or, what is the same thing,

$$\square = (BC'+B'C-2FF',\ CA'+C'A-2GG',\ AB'+A'B-2HH',$$
$$GH'+G'H-AF'-A'F,\ HF'+H'F-BG'-B'G,\ FG'+F'G-CH'-C'H \between x,\ y,\ z)^2,$$

which is really the fundamental theorem. It is however used as follows; viz. the right-hand side being symmetrical in regard to the two systems

$$(a,\ b,\ c,\ f,\ g,\ h),\quad (a',\ b',\ c',\ f',\ g',\ h'),$$

the left-hand side, which is not in form symmetrical as regards the two systems, must be so in reality; or if $\square'$ is what $\square$ becomes by interchanging the two systems, then $\square' = \square$; or substituting for $\square$ and $\square'$ their values, we have

$$\begin{aligned} & (a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 . (\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between a,\ b,\ c,\ 2f,\ 2g,\ 2h) \\ & \qquad - (\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between ax+hy+gz,\ hx+by+fz,\ gx+fy+cz)^2 \\ = \; & (a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2 . (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between a',\ b',\ c',\ 2f',\ 2g',\ 2h') \\ & \qquad - (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between a'x+h'y+g'z,\ h'x+b'y+f'z,\ g'x+f'y+c'z)^2, \end{aligned}$$

which is Hesse's theorem.

If in particular $(a,\ b,\ c,\ f,\ g,\ h)$ are the second differential coefficients of a function $u = (* \between x,\ y,\ z)^p$, and $(a',\ b',\ c',\ f',\ g',\ h')$ the second differential coefficients of a function $u' = (* \between x,\ y,\ z)^{p'}$, then the equation becomes

$$\begin{aligned} & p(p-1)\,u . (\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between \partial_x,\ \partial_y,\ \partial_z)^2\, u - (p-1)^2 (\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between \partial_x u,\ \partial_y u,\ \partial_z u)^2 \\ = \; & p'(p'-1)\,u' . (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x,\ \partial_y,\ \partial_z)^2\, u' - (p'-1)^2 (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x u',\ \partial_y u',\ \partial_z u')^2; \end{aligned}$$

and if for $u,\ u'$ we take the quartic function U and the sextic function H, its Hessian, we have

$$\begin{aligned} & 12U . (\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between \partial_x,\ \partial_y,\ \partial_z)^2\, U - 9(\mathrm{A}',\ \mathrm{B}',\ \mathrm{C}',\ \mathrm{F}',\ \mathrm{G}',\ \mathrm{H}' \between \partial_x U,\ \partial_y U,\ \partial_z U)^2 \\ = \; & 30H . (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x,\ \partial_y,\ \partial_z)^2\, H - 25(\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x H,\ \partial_y H,\ \partial_z H)^2; \end{aligned}$$

and if in this identical equation we write $U=0$, then from the resulting equation and the equation

$$\Pi_1 = -3H(\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x,\ \partial_y,\ \partial_z)^2\, H + (\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{F},\ \mathrm{G},\ \mathrm{H} \between \partial_x H,\ \partial_y H,\ \partial_z H)^2$$

we may eliminate any one of the three terms

$$(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between \partial_x U, \partial_y U, \partial_z U)^2,$$

$$H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H,$$

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x H, \partial_y H, \partial_z H)^2;$$

and in particular if the second term be eliminated, we obtain the equation

$$\Pi_2 = 5(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x H, \partial_y H, \partial_z H)^2 - 3(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}' \between \partial_x U, \partial_y U, \partial_z U)^2,$$

and the equivalence of the two forms $\Pi_1 = 0$ and $\Pi_2 = 0$ is thus established.

But Hesse's theorem leads also to the demonstration of the equivalence of the third form $\Pi_3 = 0$. To use it for this purpose, I remark that if $(a'', b'', c'', f'', g'', h'')$ are the second differential coefficients of $H - 3H_1$, where after the differentiations (x_1, y_1, z_1) are to be replaced by (x, y, z), then the theorem gives

$$\begin{aligned} 12U.(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x, \partial_y, \partial_z)^2 U - 9(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = (a'', b'', c'', f'', g'', h'' \between x, y, z)^2.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 (H - 3H_1) \\ - (\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between a''x + h''y + g''z, h''x + b''y + f''z, g''x + f''y + c''z)^2. \end{aligned}$$

But on putting (x, y, z) for (x_1, y_1, z_1) we have (since H is a homogeneous function of the order 6, and H_1 before the change is a homogeneous function of the order 3 in (x, y, z)) $a''x + h''y + g''z = 5\partial_x H - 3.2\partial_x H_1 = 5\partial_x H - 3\partial_x H$ (since, on making the substitution, $H_1 = H$, but $\partial_x H_1 = \frac{1}{2}\partial_x H) = 2\partial_x H$; and thus

$$(a''x + h''y + g''z, h''x + b''y + f''z, g''x + f''y + c''z) = (2\partial_x H, 2\partial_y H, 2\partial_x H);$$

and similarly, on making the substitution,

$$(a'', b'', c'', f'', g'', h'' \between x, y, z)^2 = 6.5H - 3.3.2H_1 = (30 - 18)H = 12H.$$

Hence writing therein $U = 0$, the foregoing equation becomes

$$\begin{aligned} -9(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = 12H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 (H - 3H_1) \\ - 4(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x U, \partial_y U, \partial_z U)^2, \end{aligned}$$

which may also be written

$$\begin{aligned} -9(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = 12H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H \\ - 36H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H_1 \\ - 4(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x U, \partial_y U, \partial_z U)^2, \end{aligned}$$

where (x_1, y_1, z_1) are ultimately to be replaced by (x, y, z). The second line in fact vanishes, which I show as follows:

Demonstration of my Theorem for a Cubic Curve.

Let $U = (*\between x, y, z)^3$ be a cubic function; it may by a linear transformation of the coordinates be reduced to the canonical form $x^3 + y^3 + z^3 + 6lxyz$, and we then have

$$
\begin{aligned}
(\mathrm{A, B, C, F, G, H}\between \partial_x, \partial_y, \partial_z)^2\, H \div 6^5 \\
= \quad & (yz - l^2x^2) \,.\, -6l^2x \\
+ \quad & (zx - l^2y^2) \,.\, -6l^2y \\
+ \quad & (xy - l^2z^2) \,.\, -6l^2z \\
& + 2\,(l^2yz - lx^2) \,.\, (1 + 2l^3)\, x \\
& + 2\,(l^2zx - ly^2) \,.\, (1 + 2l^3)\, y \\
& + 2\,(l^2xy - lz^2) \,.\, (1 + 2l^3)\, z \\
= \quad & -18l^2\, xyz \qquad + 6l^4\ (x^3 + y^3 + z^3) \\
& + 6l^2\,(1 + 2l^3)\, xyz - 2l\,(1 + 2l^3)\,(x^3 + y^3 + z^3) \\
= \quad & (-12l^3 + 12l^6)\, xyz + (-2l + 2l^4)\,(x^3 + y^3 + z^3) \\
= \quad & 2\,(-l + l^4)\,(x^3 + y^3 + z^3 + 6lxyz);
\end{aligned}
$$

or since $-l + l^4$ is equal to the quartinvariant S, and the equation is an invariantive one, we have for any cubic function whatever

$$(\mathrm{A, B, C, F, G, H}\between \partial_x, \partial_y, \partial_z)^2\, H \div 6^5 = 2S \,.\, U,$$

which is the theorem in question. There is a difference of notation, and consequently a different numerical factor, in the theorem as stated in the memoir on the conic of five-pointic contact, referred to above.

If, as above, U is a quartic function $(*\between x, y, z)^4$, and $U_1 = \frac{1}{2}\,(x_1\partial_x + y_1\partial_y + z_1\partial_z)\, U$, then U_1 is a cubic function, and we have

$$(\mathrm{A_1, B_1, C_1, F_1, G_1, H_1}\between \partial_x, \partial_y, \partial_z)^2\, H_1 \div 6^5 = 2S_1 \,.\, U_1,$$

where it is to be noticed that S_1 denotes a quartic function in the coefficients of U_1, and consequently a quartic function in (x_1, y_1, z_1), the coefficients being quartic functions of the coefficients of U. On writing (x, y, z) in the place of (x_1, y_1, z_1), S_1 becomes a quartic function of (x, y, z), which is in fact a quarticovariant quartic of U.

If in the foregoing equation we write (x, y, z) in the place of (x_1, y_1, z_1), then U_1 becomes equal to $2U$; and consequently, if $U = 0$, the right-hand side of the equation vanishes. Moreover $(a_1, b_1, c_1, f_1, g_1, h_1)$ (the second differential coefficients of U_1) become

equal to (a, b, c, f, g, h), and consequently the coefficients $(\mathrm{A}_1, \mathrm{B}_1, \mathrm{C}_1, \mathrm{F}_1, \mathrm{G}_1, \mathrm{H}_1)$ become equal to $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})$. Hence, assuming always that $U=0$, the equation becomes

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H_1 = 0,$$

where after the differentiations (x_1, y_1, z_1) are replaced by (x, y, z). This is the form which is required for the present purpose.

Returning to the foregoing expression of $-9(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2$, this now becomes

$$\begin{aligned} -9\Pi_3 &= -9(\mathrm{A}'', \mathrm{B}'', \mathrm{C}'', \mathrm{F}'', \mathrm{G}'', \mathrm{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ &= 4\{3H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H - (\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x U, \partial_y U, \partial_z U)^2\}, \end{aligned}$$

so that the equation $\Pi_3 = 0$ gives

$$\Pi_1 = (\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x U, \partial_y U, \partial_z U)^2 - 3H.(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \partial_x, \partial_y, \partial_z)^2 H = 0,$$

and the equivalence of the equations $\Pi_1 = 0$ and $\Pi_3 = 0$ is thus established.

271.

SUR L'INVARIANT LE PLUS SIMPLE D'UNE FONCTION QUADRATIQUE BI-TERNAIRE, ET SUR LE RÉSULTANT DE TROIS FONCTIONS QUADRATIQUES TERNAIRES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LVII. (1860), pp. 139—148.]

M. SYLVESTER a trouvé depuis longtemps, pour le résultant des trois fonctions quadratiques ternaires

$$(a\,,\ b\,,\ c\,,\ f\,,\ g\,,\ h\,)\,(x,\ y,\ z)^2,$$
$$(a',\ b',\ c',\ f',\ g',\ h')\,(x,\ y,\ z)^2,$$
$$(a'',\ b'',\ c'',\ f'',\ g'',\ h'')\,(x,\ y,\ z)^2,$$

une expression remarquable

$$12R = 16C_{12} - C_6^2,$$

où C_{12}, C_6 représentent des fonctions données des coefficients qui sont non seulement des invariants mais aussi des *combinants* des trois fonctions quadratiques (Voir le mémoire de M. Sylvester, "On the Calculus of forms otherwise the theory of Invariants," § VII. *Camb. et Dub. Math. Journ.*, t. VIII. pp. 256—269 année 1853, équation (A.), p. 267). Pour expliquer la formation de la fonction C_6, il convient de considérer la fonction quadratique bi-ternaire

$$U = \left(\begin{array}{cccccc} a\,, & b\,, & c\,, & f\,, & g\,, & h \\ a', & b', & c', & f', & g', & h' \\ a'', & b'', & c'', & f'', & g'', & h'' \\ A\,, & B\,, & C\,, & F\,, & G\,, & H \\ A', & B', & C', & F', & G', & H' \\ A'', & B'', & C'', & F'', & G'', & H'' \end{array}\right)(x,\ y,\ z)^2(\xi,\ \eta,\ \zeta)^2$$

cette notation représentant la fonction

$$\begin{aligned}
&(a\ ,\ b\ ,\ c\ ,\ f\ ,\ g\ ,\ h\ )(x,\ y,\ z)^2 . \xi^2 \\
+&(a'\ ,\ b'\ ,\ c'\ ,\ f'\ ,\ g'\ ,\ h'\ )(x,\ y,\ z)^2 . \eta^2 \\
+&(a''\ ,\ b''\ ,\ c''\ ,\ f''\ ,\ g''\ ,\ h''\ )(x,\ y,\ z)^2 . \zeta^2 \\
+&(A\ ,\ B\ ,\ C\ ,\ F\ ,\ G\ ,\ H\ )(x,\ y,\ z)^2 . 2\eta\zeta \\
+&(A',\ B',\ C',\ F',\ G',\ H')(x,\ y,\ z)^2 . 2\zeta\xi \\
+&(A'',\ B'',\ C'',\ F'',\ G'',\ H'')(x,\ y,\ z)^2 . 2\xi\eta,
\end{aligned}$$

ou, ce qui est la même chose,

$$\begin{aligned}
&(a\ \ x^2+b\ \ y^2+c\ \ z^2+2f\ \ yz+2g\ \ zx+2h\ \ xy)\ \xi^2 \\
+&(a'\ \ x^2+b'\ \ y^2+c'\ \ z^2+2f'\ yz+2g'\ zx+2h'\ xy)\ \eta^2 \\
+&(a''\ x^2+b''\ y^2+c''\ z^2+2f''yz+2g''zx+2h''xy)\ \zeta^2 \\
+&(A\ x^2+B\ y^2+C\ z^2+2F\ yz+2G\ zx+2H\ xy)\ 2\eta\zeta \\
+&(A'x^2+B'y^2+C'z^2+2F'\ yz+2G'zx+2H'xy)\ 2\zeta\xi \\
+&(A''x^2+B''y^2+C''z^2+2F''yz\ +2G''zx+2H''xy)\ 2\xi\eta.
\end{aligned}$$

Soit $\overline{123}$ le déterminant

$$\begin{vmatrix} \partial_{x_1}, & \partial_{y_1}, & \partial_{z_1} \\ \partial_{x_2}, & \partial_{y_2}, & \partial_{z_2} \\ \partial_{x_3}, & \partial_{y_3}, & \partial_{z_3} \end{vmatrix},$$

et $\overline{1'2'3'}$ le déterminant

$$\begin{vmatrix} \partial_{\xi_1}, & \partial_{\eta_1}, & \partial_{\zeta_1} \\ \partial_{\xi_2}, & \partial_{\eta_2}, & \partial_{\zeta_2} \\ \partial_{\xi_3}, & \partial_{\eta_3}, & \partial_{\zeta_3} \end{vmatrix},$$

et soient U_1, U_2, U_3 ce que devient U lorsqu'on y écrit x_1, y_1, z_1, ξ_1, η_1, ζ_1; x_2, y_2, z_2, ξ_2, η_2, ζ_2; x_3, y_3, z_3, ξ_3, η_3, ζ_3 au lieu de x, y, z, ξ, η, ζ; on a alors cet invariant (analogue à l'invariant quadratique d'une fonction binaire d'ordre pair), savoir:

$$\frac{1}{384}\overline{123}^2 . \overline{1'2'3'}^2\ U_1U_2U_3$$

où comme à l'ordinaire les valeurs x, y, z, ξ, η, ζ sont rétablies après les différentiations au lieu de x_1, y_1, z_1, ξ_1, η_1, ζ_1, etc.

Je représente cet invariant par

$$\left\{\begin{matrix} a\ , & b\ , & c\ , & f\ , & g\ , & h \\ a'\ , & b'\ , & c'\ , & f'\ , & g'\ , & h' \\ a''\ , & b''\ , & c''\ , & f''\ , & g''\ , & h'' \\ A\ , & B\ , & C\ , & F\ , & G\ , & H \\ A'\ , & B'\ , & C'\ , & F'\ , & G'\ , & H' \\ A''\ , & B''\ , & C''\ , & F''\ , & G''\ , & H'' \end{matrix}\right\};$$

on voit sans peine que l'invariant peut être développé, et qu'alors il prend la forme

$$\left\{\begin{matrix} a, & b, & c, & f, & g, & h \\ a', & b', & c', & f', & g', & h' \\ a'', & b'', & c'', & f'', & g'', & h'' \end{matrix}\right\}$$

$$-\left\{\begin{matrix} a, & b, & c, & f, & g, & h \\ A, & B, & C, & F, & G, & H \\ A, & B, & C, & F, & G, & H \end{matrix}\right\}$$

$$-\left\{\begin{matrix} a', & b', & c', & f', & g', & h' \\ A', & B', & C', & F', & G', & H' \\ A', & B', & C', & F', & G', & H' \end{matrix}\right\}$$

$$-\left\{\begin{matrix} a'', & b'', & c'', & f'', & g'', & h'' \\ A'', & B'', & C'', & F'', & G'', & H'' \\ A'', & B'', & C'', & F'', & G'', & H'' \end{matrix}\right\}$$

$$+2\left\{\begin{matrix} A, & B, & C, & F, & G, & H \\ A', & B', & C', & F', & G', & H' \\ A'', & B'', & C'', & F'', & G'', & H'' \end{matrix}\right\}$$

où le premier terme

$$\left\{\begin{matrix} a, & b, & c, & f, & g, & h \\ a', & b', & c', & f', & g', & h' \\ a'', & b'', & c'', & f'', & g'', & h'' \end{matrix}\right\}$$

dénote la fonction

$$\begin{aligned} & ab'c'' + ab''c' + a'b''c + a'bc'' + a''bc' + a''b'c \\ & -2af'f'' - 2a'f''f - 2a''ff' \\ & -2bg'g'' - 2b'g''g - 2b''gg' \\ & -2ch'h'' - 2c'h''h - 2c''hh' \\ & +2fg'h'' + 2fg''h' + 2f'g''h + 2f'gh'' + 2f''gh' + 2f''g'h \end{aligned}$$

et de même pour les autres termes. L'invariant contient les termes

$$\begin{aligned} & ab'c'' + 2AB'C'' + 2fg'h'' + 4FG'H'' \\ & -2af'f'' - 2aBC + 2aF^2 \\ & -4AF'F'' + 4fAF - 4fGH, \end{aligned}$$

et on déduit de là son expression complète en y écrivant

$$ab'c'' + ab''c' + a'b''c + a'bc'' + a''bc' + a''b'c$$

au lieu de $ab'c''$;

$$af'f'' + a'f''f + a''ff' + bg'g'' + b'g''g + c''gg' + ch'h'' + c'h''h + c''hh'$$

au lieu de $af'f''$, et ainsi de suite; la somme des coefficients est $6+2.6+2.6+4.6+2.9+2.9+2.9+4.9+4.9+4.9=216$, ce qui est exacte, car il n'y a pas de termes qui se détruisent, et dans l'expression $\frac{1}{384}\overline{123}^2\,\overline{1'2'3'}^2\,U_1U_2U_3$, le nombre des termes qui composent le produit des deux déterminants carrés est $36\times 36=1296$, chacun de ces termes est multiplié par le facteur $8\times 8=64$ introduit par la différentiation, et l'on a

$$\frac{1}{384}\times 64\times 1296=216.$$

L'invariant qui vient d'être donné est lié avec l'invariant T de M. Aronhold (lequel se rapporte à une fonction ternaire cubique). C'est pour cela que je le désigne par la même lettre T, et pour fixer sa valeur j'écris

$$2T = ab'c'' + \text{etc.}$$

Cela posé, je forme les deux combinants C_6 et C_{12} de la manière suivante.

Combinant du sixième ordre C_6. Soient U, V, W les trois fonctions quadratiques ternaires; considérez la fonction syzygétique $\lambda U+\mu V+\nu W$, où λ, μ, ν sont des quantités arbitraires; formez avec les variables ξ, η, ζ le reciprocant [1] (multiplié par 2) de cette fonction, le résultat sera de l'ordre 2 par rapport aux coefficients de U, V, W, de l'ordre 2 par rapport à λ, μ, ν, et de l'ordre 2 par rapport à ξ, η, ζ; c. à d. ce résultat sera une fonction quadratique bi-ternaire de λ, μ, ν et ξ, η, ζ dont l'invariant $2T$ (lequel sera par conséquent de l'ordre 6 par rapport aux coefficients de U, V, W) est le combinant C_6 qu'il s'agissait de trouver.

Combinant du douzième ordre C_{12}. Considérez comme auparavant la fonction syzygétique $\lambda U+\mu V+\nu W$, formez le discriminant (multiplié par 6) de cette fonction; le résultat sera de l'ordre 3 par rapport aux coefficients de U, V, W, et de l'ordre 3 par rapport à λ, μ, ν; c. à d. il sera une fonction cubique ternaire de λ, μ, ν, dont

[1] Le réciprocant de $(a, b, c, f, g, h)(x, y, z)^2$ est

$$= -\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix}$$

$$= -(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\xi, \eta, \zeta)^2.$$

l'invariant S de M. Aronhold [1] (lequel sera de l'ordre 12 par rapport aux coefficients de U, V, W) est le combinant C_{12} qu'il s'agissait de trouver.

On peut exprimer ces résultats d'une manière abrégée en disant que C_6 *est l'invariant* $2T$ *de la fonction bi-ternaire* $2R$ et que C_{12} *est l'invariant* S *de la fonction cubique ternaire* $6D$, en désignant par R et D le réciprocant et le discriminant de la fonction $\lambda U + \mu V + \nu W$.

Pour démontrer l'équation

$$12R = 16C_{12} - C_6^2$$

il suffit de faire voir que cette équation est vérifiée lorsqu'au lieu des trois fonctions données U, V, W on substitue trois fonctions de la forme $lU + mV + nW$, les coefficients l, m, n ayant été choisis de manière à simplifier autant que possible les expressions des trois fonctions. Il est facile de voir qu'il est permis de prendre pour les trois fonctions les formes dont s'est servi M. Sylvester dans son mémoire, savoir

$$\rho\,(x^2 - y^2),$$
$$\sigma\,(y^2 - z^2),$$
$$y^2 + 2fyz + 2gzx + 2hxy.$$

Mais il suit des recherches de M. Hesse, et M. Sylvester aussi a depuis reconnu, qu'on peut prendre pour les trois fonctions les formes encore plus simples

$$x^2 + 2lyz,$$
$$y^2 + 2lzx,$$
$$z^2 + 2lxy,$$

qui se réduisent aux fonctions dérivées de la fonction cubique $x^3 + y^3 + z^3 + 6lxyz$. Or, en prenant ces valeurs de U, V, W, on obtient

$$\lambda U + \mu V + \nu W = (\lambda,\ \mu,\ \nu,\ \lambda l,\ \mu l,\ \nu l)\,(x,\ y,\ z)^2.$$

[1] Pour la fonction cubique ternaire

$$(a,\ b,\ c,\ i,\ j,\ k,\ i_1,\ j_1,\ k_1,\ l)\,(x,\ y,\ z)^3$$

l'invariant S est

$$\begin{aligned}
= \quad & l^6 \\
& - 2l^2\,(jk_1 + ki_1 + ij_1) \\
& + 3l\ (ijk + i_1j_1k_1) \\
& + \ l\ (aii_1 + bjj_1 + ckk_1) \\
& - \ l\ abc \\
& - \ (ii_1j_1k + jj_1k_1i + kk_1i_1j) \\
& + \ (j^2k_1^2 + k^2i_1^2 + i^2j_1^2) \\
& + \ (bcj_1k + cak_1i + abi_1j) \\
& - \ (ai^2j + bj^2k + ck^2i + ai_1^2k_1 + bj_1^2i_1 + ck_1^2j_1).
\end{aligned}$$

Le réciprocant (multiplié par 2) est

$$= -2 \begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & \lambda, & \nu l, & \mu l \\ \eta, & \nu l, & \mu, & \lambda l \\ \zeta, & \mu l, & \lambda l, & \nu \end{vmatrix},$$

lequel, en forme développée, donne le résultat

$$\begin{aligned}
&(-2l^2\lambda^2 \quad . \quad . \quad +2\mu\nu \quad . \quad . \quad)\,\xi^2 \\
+&(\ . \quad -2l^2\mu^2 \quad . \quad . \quad +2\nu\lambda \quad . \quad)\,\eta^2 \\
+&(\ . \quad . \quad -2l^2\nu^2 \quad . \quad . \quad +2\lambda\mu \quad)\,\zeta^2 \\
+&(-2l\lambda^2 \quad . \quad . \quad +2l^2\mu\nu \quad . \quad . \quad)\,2\zeta\eta \\
+&(\ . \quad -2l\mu^2 \quad . \quad . \quad +2l^2\nu\lambda \quad . \quad)\,2\xi\zeta \\
+&(\ . \quad . \quad -2l\nu^2 \quad . \quad . \quad +2l^2\lambda\mu)\,2\eta\xi,
\end{aligned}$$

que l'on peut aussi exprimer comme fonction quadratique bi-ternaire, savoir:

$$\left(\begin{vmatrix} -2l^2 & . & . & 1 & . & . \\ . & -2l^2 & . & . & 1 & . \\ . & . & -2l^2 & . & . & 1 \\ -2l & . & . & l^2 & . & . \\ . & -2l & . & . & l^2 & . \\ . & . & -2l & . & . & l^2 \end{vmatrix}\right)(\lambda, \mu, \nu)^2(\xi, \eta, \zeta)^2.$$

En représentant l'invariant $2T$ par le développement incomplet donné ci-dessus, on obtient dans le cas dont il s'agit

$$\begin{aligned}
2T = &\left\{\begin{matrix} -2l^2 & . & . & 1 & . & . \\ . & -2l^2 & . & . & 1 & . \\ . & . & -2l^2 & . & . & 1 \end{matrix}\right\} \\
&-\left\{\begin{matrix} -2l^2 & . & . & 1 & . & . \\ -2l & . & . & l^2 & . & . \\ -2l & . & . & l^2 & . & . \end{matrix}\right\} - \text{etc. etc.} \\
&+2\left\{\begin{matrix} -2l & . & . & l^2 & . & . \\ . & -2l & . & . & l^2 & . \\ . & . & -2l & . & . & l^2 \end{matrix}\right\},
\end{aligned}$$

ce qui se réduit à la valeur finale

$$\begin{aligned} & -8l^6+2 \\ & -(4l^6+4l^3+4l^3) \\ & -(4l^6+4l^3+4l^3) \\ & -(4l^6+4l^3+4l^3) \\ & +2(-8l^3+2l^6) \\ = \; & 2(1-20l^3-8l^6). \end{aligned}$$

Les invariants S et T de la fonction cubique $x^3+y^3+z^3+6lxyz$ sont

$$S=-l+l^4,$$
$$T=1-20l^3-8l^6;$$

l'invariant $2T$ de la fonction quadratique bi-ternaire est donc précisément égal à $2T$, où T est l'invariant qui vient d'être donné, et on a

$$C_6=2T.$$

Le discriminant (multiplié par 6) de la fonction $\lambda U+\mu V+\nu W$ est

$$=6\begin{vmatrix} \lambda, & l\nu, & l\mu \\ l\lambda, & \mu, & l\lambda \\ l\mu, & l\nu, & \nu \end{vmatrix}$$

$$=6\{-l^2\lambda^3-l^2\mu^3-l^2\nu^3+(1+2l^3)\lambda\mu\nu\},$$

où la fonction en parenthèses est ce que devient le Hessien de la fonction cubique $x^3+y^3+z^3+6lxyz$, lorsqu'on y substitue λ, μ, ν au lieu de x, y, z. Son invariant S est

$$\begin{aligned} &=(1+2l^3)\,216l^6+(1+2l^3)^4 \\ &=1+8l^3+240l^6+464l^9+16l^4. \end{aligned}$$

Cette quantité est en même temps un invariant de la fonction cubique

$$x^3+y^3+z^3+6lxyz,$$

et comme tel elle doit s'exprimer en fonction des deux invariants S et T de cette dernière: en effet elle se réduit à

$$(1-20l^3-8l^6)^2-48(-l+l^4)^3=T^2-48S^3.$$

Nous avons donc

$$C_{12}=T^2-48S^3,$$

et comme nous venons de trouver

$$C_6=2T,$$

l'équation $12R=16C_{12}-C_6^2$ se réduit à

$$\begin{aligned} 12R &= 16(T^2-48S^3)-4T^2 \\ &= 12T^2-768S^3, \end{aligned}$$

ou enfin à

$$R = T^2 - 64S^3,$$

équation qui fait voir que le résultant R dont il s'agit est effectivement égal à $R = (1+8l^3)^2$, c. à d. au discriminant de la fonction cubique

$$x^3 + y^3 + z^3 + 6lxyz,$$

ce qui donne la vérification du théorème général. Je m'étais servi d'abord d'une analyse moins simple, en considérant le cas dans lequel on prend pour les trois fonctions les formes

$$\begin{aligned} &x^2 - y^2, \\ &z^2 - y^2, \\ &x^2 + y^2 + z^2 + 2lyz + 2mzx + 2nxy. \end{aligned}$$

Il vaut la peine, je crois, de donner les expressions correspondantes de C_6, C_{12}, R. On a ici à considérer la fonction

$$\lambda(x^2 - y^2) + \mu(x^2 + y^2 + z^2 + 2lyz + 2mzx + 2nxy) + \nu(z^2 - y^2),$$

laquelle peut s'écrire comme suit,

$$(\mu + \lambda,\ \mu - \lambda - \nu,\ \mu + \nu,\ l\mu,\ m\mu,\ n\mu)(x,\ y,\ z)^2.$$

Le réciprocant (multiplié par 2) est

$$= -2 \begin{vmatrix} \xi & , \eta & , \zeta \\ \xi, & \mu + \lambda, \ n\mu & , m\mu \\ \eta, & n\mu \ , \ \mu - \lambda - \nu, & l\mu \\ \zeta, & m\mu \ , \ l\mu & , \mu + \nu \end{vmatrix}$$

et en écrivant a, b, c, f, g, h au lieu de $1 - l^2$, $1 - m^2$, $1 - n^2$, $mn - l$, $nl - m$, $lm - n$, cette fonction est

$$\begin{aligned} = \quad & (2a\mu^2 - 2\lambda\mu - 2\nu\lambda - 2\nu^2)\,\xi^2 \\ & + (2b\mu^2 + 2\lambda\mu + 2\mu\nu + 2\nu\lambda)\,\eta^2 \\ & + (2c\mu^2 - 2\nu\mu - 2\nu\lambda - 2\lambda^2)\,\zeta^2 \\ & + (2f\mu^2 - 2l\lambda\mu \qquad\qquad)\,2\eta\zeta \\ & + (2g\mu^2 + 2m\lambda\mu + 2m\mu\nu \quad)\,2\zeta\xi \\ & + (2h\mu^2 - 2n\nu\mu \qquad\qquad)\,2\xi\eta, \end{aligned}$$

ou, écrite en forme de fonction quadratique bi-ternaire,

$$= \begin{pmatrix} 0, & 2a, & -2, & 0, & 1, & -1 \\ 0, & 2b, & 0, & 1, & 1, & 1 \\ -2, & 2c, & 0, & -1, & -1, & 0 \\ 0, & 2f, & 0, & 0, & 0, & -l \\ 0, & 2g, & 0, & m, & 0, & m \\ 0, & 2h, & 0, & -n, & 0, & 0 \end{pmatrix} (\lambda,\ \mu,\ \nu)^2 (\xi,\ \eta,\ \zeta)^2.$$

En exprimant comme auparavant l'invariant $2T$ par une somme de cinq termes, on trouve sans peine que ces termes sont respectivement $18-4l^2-4m^2-4n^2$, $-4l^2$, $-4m^2$, $-4n^2$, 0; l'invariant sera donc $-8(l^2+m^2+n^2)+18$; et l'on a pour le combinant du sixième ordre

$$C_6 = -8(l^2+m^2+n^2)+18.$$

Le discriminant (multiplié par 6) de la fonction $\lambda U+\mu V+\nu W$ est

$$= 6\begin{vmatrix} \mu+\lambda, & n\mu & , & m\mu \\ n\mu & , & \mu-\lambda-\nu, & l\mu \\ m\mu & , & l\mu & , & \mu+\nu \end{vmatrix}$$

$$= 0\lambda^3 + 6(1-l^2-m^2-n^2+2lmn)\mu^3 + 0\nu^3$$
$$+ 6(m^2-n^2)\mu^2\nu - 6\nu^2\lambda - 6\lambda^2\mu - 6\mu\nu^2 - 6\nu\lambda^2 + 6(m^2-l^2)\lambda\mu^2 - 6\lambda\mu\nu$$
$$= \big(0,\ 6(1-l^2-m^2-n^2+2lmn),\ 0,\ 2(m^2-n^2),\ -2,\ -2,\ -2,\ -2,\ 2(m^2-l^2),\ -1\big)(\lambda,\ \mu,\ \nu)^3$$
$$= (a,\ b,\ c,\ i,\ j,\ k,\ i_1,\ j_1,\ k_1,\ l)(\lambda,\ \mu,\ \nu)^3,$$

où l'on a posé $a=0$, $b=6(1-l^2-m^2-n^2+2lmn)$, etc. L'invariant S de cette dernière fonction est

$$\begin{aligned} = \quad & 1 \\ & -2\big(-4(m^2-l^2)+4-4(m^2-n^2)\big) \\ & -3\big(\ 8(m^2-n^2) \qquad +8(m^2-l^2)\big) \\ & -4b \\ & +16(m^2-n^2)-16(m^2-n^2)(m^2-l^2)+16(m^2-l^2) \\ & +16(m^2-n^2)^2+16+16(m^2-l^2)^2 \\ & +16b \\ = \ & 9+12b+16(l^4+m^4+n^4-m^2n^2-n^2l^2-l^2m^2), \end{aligned}$$

ou, si l'on substitue la valeur de b,

$$\begin{aligned} C_{12} = 16&(l^4+m^4+n^4-m^2n^2-n^2l^2-l^2m^2) \\ &+144lmn-72(l^2+m^2+n^2)+81. \end{aligned}$$

On vient de trouver

$$C_6 = -8(l^2+m^2+n^2)+18,$$

équation qui donne

$$\begin{aligned} C_6^2 = 64&(l^4+m^4+n^4+2m^2n^2+2n^2l^2+2l^2m^2) \\ &-288(l^2+m^2+n^2)+324. \end{aligned}$$

De ces résultats combinés on conclue

$$16C_{12} - C_6^2 = 192\,(l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2) + 2304lmn - 864\,(l^2 + m^2 + n^2) + 972.$$

Évidemment l'expression du résultant est, à un facteur numérique près,

$$R = (3 + 2l + 2m + 2n)(3 + 2l - 2m - 2n)(3 - 2l + 2m - 2n)(3 - 2l - 2m + 2n),$$

ou en réduisant

$$R = 16\,(l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2) + 192lmn - 72\,(l^2 + m^2 + n^2) + 81.$$

On retrouve donc

$$12R = 16C_{12} - C_6^2,$$

équation qu'il s'agissait de vérifier.

En terminant je remarque que la plus grande partie de ce mémoire est tirée d'un manuscrit daté "Machynlleth, 13 Août 1853."

Londres, 22 *Mars* 1859.

272.

DÉMONSTRATION D'UN THÉORÈME DE JACOBI PAR RAPPORT AU PROBLÈME DE PFAFF.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LVII. (1860), pp. 273—277.]

DANS le mémoire de Jacobi "Theoria novi multiplicatoris etc." (t. XXIX. de ce Journal, 1845) on trouve p. 253 le passage que voici: "Methodum ad solvendum problema Pfaffianum ab ipso auctore adhibitam, data occasione observo per plures et altiores procedere integrationes quam methodus vera et genuina poscat. Quam novam methodum pro exemplo simplice explicabo. Ad aequationem differentialem

$$X_1dx_1 + X_2dx_2 + X_3dx_3 + X_4dx_4 = 0$$

per duas aequationes integrandam poscit Pfaffiana methodus...integrationem completam systematis trium aequationum differentialium primi ordinis inter quatuor variabiles ac deinde unius aequationis differentialis primi ordinis inter duas variabiles. Illius igitur systematis integrali uno invento, secundum illam methodum restat integratio completa duarum aequationum differentialium primi ordinis inter tres variabiles sive unius aequationis differentialis secundi ordinis inter duas variabiles ac deinde aequationis differentialis primi ordinis inter duas variabiles. At observo si integrali illo invento exprimatur x_4 per x_1, x_2, x_3, aequationem differentialem propositam abire in aliam linearem primi ordinis inter tres variabiles *conditioni integrabilitatis satisfacientem;* cujus integrationem vidimus (voir p. 246) absolvi posse per integrationes separatas duarum aequationum differentialium primi ordinis inter duas variabiles. Unde in locum aequationis differentialis secundi ordinis, tantum integrandae sunt duae aequationes differentiales separatae primi ordinis, quae est reductio maxime insignis; integrationi autem aequationis differentialis primi ordinis postremo praestandae omnino supersedetur. Tractatio hujus rei gravissimae completa ac generalis alii commentationi reservanda

est." Le théorème dont il s'agit est énoncé dans les mots "conditioni integrabilitatis satisfacientem," mais j'ai cité le passage en entier pour montrer l'importance qu'à bon droit Jacobi attachait à la théorie dont ce théorème fait partie. On sait que le problème de la solution d'une équation à différences partielles du premier ordre n'est qu'un cas particulier du problème de Pfaff et que la solution des équations différentielles de la Dynamique et celle d'autres systèmes est intimement liée avec la question de la solution de ce cas particulier: le problème de Pfaff est ainsi de la plus haute importance dans l'analyse. Jacobi a parlé autre part d'un mémoire sur la mécanique analytique dont il s'est beaucoup occupé, lequel aurait naturellement des rapports avec celui-là; malheureusement ni l'un ni l'autre de ces mémoires n'ont paru.

Pour démontrer le théorème, considerons l'expression différentielle

$$\nabla = X_1\,dx_1 + X_2\,dx_2 + X_3\,dx_3 + X_4\,dx_4.$$

En supposant que $a = \phi(x_1, x_2, x_3, x_4)$ soit une intégrale des équations différentielles de Pfaff, si au moyen de cette équation on exprime x_4 en fonction de (x_1, x_2, x_3, a) l'expression ∇ prendra la forme

$$\nabla_1 = Y_1\,dx_1 + Y_2\,dx_2 + Y_3\,dx_3 + A\,da$$

et il s'agit de faire voir que l'équation

$$Y_1\,dx_1 + Y_2\,dx_2 + Y_3\,dx_3 = 0$$

à laquelle $\nabla_1 = 0$ se réduit, lorsqu'on fait $a =$ constante, sera intégrable au moyen d'un facteur, ou autrement dit, que l'expression $Y_1dx_1 + Y_2dx_2 + Y_3dx_3$ sera de la forme Udu, c. à d. qu'il existe une quantité u, fonction de x_1, x_2, x_3, a, telle qu'en traitant a comme une constante on a

$$Y_1\,dx_1 + Y_2\,dx_2 + Y_3\,dx_3 = U\,du.$$

Ainsi dans cette formule du dénote $\frac{du}{dx_1}dx_1 + \frac{du}{dx_2}dx_2 + \frac{du}{dx_3}dx_3$, et si l'on veut regarder a comme variable il faudrait y écrire $du - \frac{du}{da}da$ au lieu de du; on aurait ainsi $\nabla_1 = U\left(du - \frac{du}{da}da\right) + A\,da$, ou en changeant la signification de A,

$$\nabla = X_1\,dx_1 + X_2\,dx_2 + X_3\,dx_3 + X_4\,dx_4 = U\,du + A\,da,$$

où u, a, et de même U, A, seront des fonctions données de x_1, x_2, x_3, x_4; c'est en effet la forme à laquelle dans le problème de Pfaff il s'agit de réduire l'expression différentielle ∇.

Or les équations différentielles de Pfaff sont le système que voici, savoir en écrivant

$$12 = \frac{dX_1}{dx_2} - \frac{dX_2}{dx_1}, \text{ etc.,}$$

ce qui implique $12 = -21$, $11 = 0$, etc., et en introduisant la variable auxiliaire t, qui doit être éliminée, le système est

$$\begin{aligned}
0 &= X_1dx_1 + X_2dx_2 + X_3dx_3 + X_4dx_4,\\
X_1dt &= \quad * \qquad 12dx_2 + 13dx_3 + 14dx_4,\\
X_2dt &= 21dx_1 \qquad * \quad + 23dx_3 + 24dx_4,\\
X_3dt &= 31dx_1 + 32dx_2 \qquad * \quad + 34dx_4,\\
X_4dt &= 41dx_1 + 42dx_2 + 43dx_3 \qquad * \quad ,
\end{aligned}$$

lequel ne contient que quatre équations indépendantes. On donne à ce système une forme plus symétrique en écrivant x_0 au lieu de t, et en y mettant de plus

$$X_1 = 01 = -10, \text{ etc.}$$

les équations deviennent par ce moyen

$$\begin{aligned}
0 &= \quad * \qquad 01dx_1 + 02dx_2 + 03dx_3 + 04dx_4,\\
0 &= 10dx_0 \qquad * \quad + 12dx_2 + 13dx_3 + 14dx_4,\\
0 &= 20dx_0 + 21dx_1 \qquad * \quad + 23dx_3 + 24dx_4,\\
0 &= 30dx_0 + 31dx_1 + 32dx_2 \qquad * \quad + 34dx_4,\\
0 &= 40dx_0 + 41dx_1 + 42dx_2 + 43dx \qquad * \quad ,
\end{aligned}$$

et on déduit delà

$$dx_0 : dx_1 : dx_2 : dx_3 : dx_4 = 1234 : 2340 : 3401 : 4012 : 0123,$$

où 1234, etc. sont les fonctions de Jacobi que j'ai nommées "Pfaffiens," savoir on a

$$\begin{aligned}
1234 &= 12 \,.\, 34 + 13 \,.\, 42 + 14 \,.\, 23,\\
2340 &= 23 \,.\, 40 + 24 \,.\, 03 + 20 \,.\, 34, = \quad * \quad - X_2 34 - X_3 42 - X_4 23,\\
3401 &= 34 \,.\, 01 + 30 \,.\, 14 + 31 \,.\, 40, = \quad X_1 34 \quad * \quad + X_3 41 + X_4 13,\\
4012 &= 40 \,.\, 12 + 41 \,.\, 20 + 42 \,.\, 01, = -X_1 24 - X_2 41 \quad * \quad - X_4 12,\\
0123 &= 01 \,.\, 23 + 02 \,.\, 31 + 03 \,.\, 12, = \quad X_1 23 + X_2 31 + X_3 12 \quad *
\end{aligned}$$

Cette transformation se trouve en effet dans le mémoire de Jacobi "Ueber die Pfaffsche Methode etc." (t. II. de ce Journal p. 347 et suivantes, 1827), où cependant Jacobi ne s'est pas servi de l'algorithme $X_1 = 01 = -10$ etc. au moyen duquel la forme de la solution devient si simple.

Faisons attention à présent à ce que l'équation

$$a = \phi\,(x_1,\ x_2,\ x_3,\ x_4)$$

est supposée être une intégrale des équations différentielles. On en déduit d'abord $x_4 = \text{fonc.}\,(x_1,\ x_2,\ x_3,\ a)$, et de là en traitant a comme constante

$$dx_4 = \frac{dx_4}{dx_1}dx_1 + \frac{dx_4}{dx_2}dx_2 + \frac{dx_4}{dx_3}dx_3,$$

équation qui doit être satisfaite par les valeurs qu'on vient de trouver pour $dx_1 : dx_2 : dx_3 : dx_4$, et qui par conséquent donne :

$$0 = 0123 - \frac{dx_4}{dx_1}\,2340 - \frac{dx_4}{dx_2}\,3401 - \frac{dx_4}{dx_3}\,4012.$$

En y substituant pour 0123, etc. leurs valeurs, modifiées en sorte que tous les termes deviennent positifs (ce qui se fait à l'aide des relations $12 = -21$ etc.), on trouve

$$\left.\begin{array}{l} \frac{dx_4}{dx_1}(\ \ * \qquad X_2 34 + X_3 42 + X_4 23) \\ + \frac{dx_4}{dx_2}(X_1 43 \qquad * \quad + X_3 14 + X_4 31) \\ + \frac{dx_4}{dx_3}(X_1 24 + X_2 41 \qquad * \quad + X_4 12) \\ + \qquad (X_1 32 + X_2 13 + X_3 21 \qquad * \ \ ) \end{array}\right\} = 0,$$

équation aux différences partielles à laquelle satisfait la variable x_4, lorsqu'au moyen de l'équation $a = \phi(x_1, x_2, x_3, x_4)$ (qui est une intégrale des équations différentielles de Pfaff) elle est donnée en fonction de x_1, x_2, x_3. Nous allons voir que c'est cette dernière équation, dans laquelle est contenu le théorème dont la démonstration fait l'objet de cette note. En effet l'expression différentielle

$$\nabla = X_1\,dx_1 + X_2\,dx_2 + X_3\,dx_3 + X_4\,dx_4$$

ayant été transformée en

$$\nabla_1 = Y_1\,dx_1 + Y_2\,dx_2 + Y_3\,dx_3 + A\,da$$

par la substitution de a au lieu de x_4, il s'en suit qu'on a les relations :

$$Y_1 = X_1 + X_4\frac{dx_4}{dx_1},$$

$$Y_2 = X_2 + X_4\frac{dx_4}{dx_2},$$

$$Y_3 = X_3 + X_4\frac{dx_4}{dx_3},$$

$$A = \qquad X_4\frac{dx_4}{da};$$

la variable x_4 contenue dans les quantités X_1, X_2, X_3, X_4 doit être remplacée par son expression en fonction de x_1, x_2, x_3, a.

Cela posé, la condition pour que l'équation

$$Y_1\,dx_1 + Y_2\,dx_2 + Y_3\,dx_3 = 0$$

soit intégrable par un facteur étant désignée par

$$Z = 0,$$

on sait que

$$Z = Y_1\left(\frac{dY_2}{dx_3} - \frac{dY_3}{dx_2}\right) + Y_2\left(\frac{dY_3}{dx_1} - \frac{dY_1}{dx_3}\right) + Y_3\left(\frac{dY_3}{dx_1} - \frac{dY_1}{dx_2}\right).$$

Mais d'après les relations que l'on vient d'établir on a

$$\frac{dY_2}{dx_3} = \frac{dX_2}{dx_3} + \frac{dX_2}{dx_4}\frac{dx_4}{dx_3} + \left(\frac{dX_4}{dx_3} + \frac{dX_4}{dx_4}\frac{dx_4}{dx_3}\right)\frac{dx_4}{dx_2} + X_4\frac{d^2x_4}{dx_2dx_3},$$

$$\frac{dY_3}{dx_2} = \frac{dX_3}{dx_2} + \frac{dX_3}{dx_4}\frac{dx_4}{dx_2} + \left(\frac{dX_4}{dx_2} + \frac{dX_4}{dx_4}\frac{dx_4}{dx_2}\right)\frac{dx_4}{dx_3} + X_4\frac{d^2x_4}{dx_2dx_3};$$

d'où l'on tire

$$\frac{dY_2}{dx_3} - \frac{dY_3}{dx_2} = 23 + 24\frac{dx_4}{dx_3} + 43\frac{dx_4}{dx_2},$$

et on trouve de même

$$\frac{dY_3}{dx_1} - \frac{dY_1}{dx_3} = 31 + 34\frac{dx_4}{dx_1} + 41\frac{dx_4}{dx_3},$$

$$\frac{dY_1}{dx_2} - \frac{dY_2}{dx_1} = 12 + 14\frac{dx_4}{dx_2} + 42\frac{dx_4}{dx_1};$$

donc en ajoutant ces équations après les avoir multipliées par $X_1 + X_4\frac{dx_4}{dx_1}$, $X_2 + X_4\frac{dx_4}{dx_2}$, $X_3 + X_4\frac{dx_4}{dx_3}$, les termes qui contiennent les produits $\frac{dx_4}{dx_1}\frac{dx_4}{dx_2}$, etc. se détruisent, et l'on obtient l'équation finale

$$Z = \left\{\begin{array}{l} \frac{dx_4}{dx_1}(\quad * \qquad X_2 34 + X_3 42 + X_4 23) \\ + \frac{dx_4}{dx_2}(X_1 43 \qquad * \quad + X_3 14 + X_4 31) \\ + \frac{dx_4}{dx_3}(X_1 24 + X_2 41 \qquad * \quad + X_4 12) \\ + \qquad (X_1 32 + X_2 13 + X_3 21 \qquad * \quad) \end{array}\right\},$$

dont la seconde partie s'évanouit comme il a été démontré ci-dessus. Par conséquent la condition d'intégrabilité $Z = 0$ se trouve remplie, ce qu'il s'agissait de prouver.

Londres, 3 *Septembre* 1859

273.

NOTE SUR LA TRANSFORMATION DE TSCHIRNHAUSEN.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LVIII. (1861), pp. 259—262.]

ON trouve dans le mémoire de M. Hermite "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré" (*Comptes Rendus*, t. XLVI. p. 961, 1858) un théorème très-important relatif à la transformation de Tschirnhausen, à l'aide de laquelle une équation $f(x)=0$ est transformée en une autre du même degré en y par la substitution $y=\phi x$, où ϕx désigne une fonction rationnelle. En considérant pour fixer les idées une équation du quatrième degré

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$

M. Hermite donne à l'équation $y=\phi x$ la forme que voici,

$$y = aT + (ax+4b)\,T_0 + (ax^2+4bx+6c)\,T_1 + (ax^3+4bx^2+6cx+4d)\,T_2,$$

et il fait voir que les coefficients de la transformée en y ont la propriété suivante: toute fonction de ces coefficients, laquelle exprimée en fonction de $a, b, c, d, e, T, T_0, T_1, T_2$ ne contient pas T, est un *invariant*, c. à d. un invariant des deux fonctions

$$(a,\ b,\ c,\ d,\ e \between \xi,\ \eta)^4,\quad (T_0,\ T_1,\ T_2 \between \eta,\ -\xi)^2.$$

Cela revient à dire qu'en supposant la valeur de T déterminée de manière à faire évanouir dans l'équation en y le coefficient du second terme (de y^3), les autres coefficients seront des invariants, de sorte que, si dans le polynome en y qui est égalé à zéro on considère y comme une constante absolue, le polynome tout entier sera un invariant des deux fonctions ci-dessus mentionnées. On trouve sans peine la valeur que doit avoir T, elle est donnée par l'équation

$$0 = aT + 3bT_0 + 3cT_1 + dT_2,$$

ce qui conduit pour y à la valeur

$$y = (ax + b)\, T_0 + (ax^2 + 4bx + 3c)\, T_1 + (ax^3 + 4bx^2 + 6cx + 3d)\, T_2,$$

et en même temps la transformée en y aura la propriété dont il s'agit.

Par rapport à la forme de l'expression que l'on vient de trouver pour y il est bon de remarquer que les coefficients numériques qu'on y rencontre, hormis ceux du terme en x^0, ou de $bT_0 + 3cT_1 + 3dT_2$, sont les coefficients de la puissance $(1, 1)^4$, tandis que les coefficients qui ont été désignés comme faisant exception sont ceux de la puissance $(1, 1)^3$. Une remarque pareille s'applique au cas général. Pour démontrer le théorème énoncé, je représente l'équation qui vient d'être écrite par $y = V$, la transformée en y sera donc

$$(y - V_1)(y - V_2)(y - V_3)(y - V_4) = 0,$$

où V_1, V_2, V_3, V_4 sont ce que devient V lorsqu'on y substitue successivement pour x les quatre racines de l'équation $(a, b, c, d, e \between x, 1)^4 = 0$. Or, en considérant y comme une constante, pour que l'expression qui forme la première partie de l'équation que l'on vient d'écrire soit un invariant, les conditions à remplir consistent en ce que cette expression se réduise à zéro au moyen de l'un et l'autre des opérateurs

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e - (\ T_2\partial_{T_1} + 2T_1\partial_{T_0}),$$
$$4b\partial_a + 3c\partial_b + 2d\partial_c + \ e\partial_d - (2T_1\partial_{T_2} + \ T_0\partial_{T_1}).$$

Ces conditions seront satisfaites si chacun des facteurs $y - V_1$, etc. est doué de cette même propriété, c. à d. si $y - V$, ou plus simplement si V, en y faisant x égal à l'une des racines de l'équation en x, se réduit à zéro au moyen de l'un et l'autre des deux opérateurs ci-dessus écrits. Je considère le premier des deux opérateurs et pour abréger je le désigne par

$$\Delta - (T_2\partial_{T_1} + 2T_1\partial_{T_0}).$$

Pour avoir ΔV, il faut d'abord former la valeur de Δx. On l'obtient en opérant avec Δ sur l'équation $(a, b, c, d, e \between x, 1)^4 = 0$, ce qui donne

$$(a, b, c, d \between x, 1)^3\, \Delta x + (a, b, c, d \between x, 1)^3 = 0, \text{ ou } \Delta x = -1.$$

La partie de ΔV qui dépend de la variation de x est par conséquent

$$-aT_0 + (-2ax - 4b)\, T_1 + (-3ax^2 - 8bx - 6c)\, T_2.$$

Pour l'autre partie de ΔV on trouve aisément

$$aT_0 + (4ax + 6b)\, T_1 + (4ax^2 + 12bx + 9c)\, T_2,$$

et de là en ajoutant

$$\Delta V = 2\,(ax + b)\, T_1 + (ax^2 + 4bx + 3c)\, T_2$$

ce qui est précisément égal à

$$(T_2\partial_{T_1} + 2T_1\partial_{T_0})\, V.$$

Donc V se réduit à zéro par l'opérateur

$$\Delta - (T_2 \partial_{T_1} + 2T_1 \partial_{T_0}).$$

De même en représentant le second opérateur par

$$\nabla - (2T_1 \partial_{T_2} + T_0 \partial_{T_1})$$

on trouve d'abord

$$(a,\ b,\ c,\ d \between x,\ 1)^3 \nabla x + x \,.\, (b,\ c,\ d,\ e \between x,\ 1)^3 = 0,$$

mais en ayant égard à l'équation $(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0$ la valeur de ∇x se réduit à $\Delta x = x^2$. La partie de ∇V due à la variation de x est par conséquent

$$ax^2 T_0 + (2ax^3 + 4bx^2)\, T_1 + (3ax^4 + 8bx^3 + 6cx^2)\, T_2.$$

L'autre partie de ∇V est

$$(4bx + 3c)\, T_0 + (4bx^2 + 12cx + 6d)\, T_1 + (4bx^3 + 12cx^2 + 12dx + 3c)\, T_2.$$

En les ajoutant, le coefficient de T_2 s'évanouit en vertu de l'équation en x, et l'on trouve

$$\nabla V = (ax^2 + 4bx + 3c)\, T_0 + 2\,(ax^3 + 4bx^2 + 6cx + 3d)\, T_1,$$

ce qui est précisément égal à

$$(2T_1 \partial_{T_2} + T_0 \partial_{T_1})\, V.$$

V se réduit donc à zéro au moyen de l'opérateur

$$\nabla - (2T_1 \partial_{T_2} + T_0 \partial_{T_1}),$$

ce qui achève la démonstration dont il s'agissait. Il va sans dire que la démonstration serait conduite d'une manière semblable pour une équation de degré quelconque. Pour avoir l'exemple le plus simple, je prends les équations

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$
$$y = (ax + b)\, T_0 + (ax^2 + 3bx + 2c)\, T_1,$$

et pour effectuer l'élimination j'écris

$$yx = (ax^2 + bx)\, T_0 + (-cx - d)\, T_1,$$
$$yx^2 = (-2bx^2 - 3cx - d)\, T_0 + (-cx^2 - dx)\, T_1.$$

Maintenant on a les trois équations

$$\begin{aligned} 0 &= bT_0 + 2cT_1 - y + x\,(aT_0 + 3bT_1) \qquad + x^2 \,.\, aT_1, \\ 0 &= \quad - dT_1 \quad + x\,(bT_0 - \ cT_1 - y) + x^2 \,.\, aT_0, \\ 0 &= \quad - dT_0 \quad + x\,(-3cT_0 - dT_1) \quad + x^2 (-2bT_0 - cT_1 - y). \end{aligned}$$

donc l'équation en y est

$$\begin{vmatrix} y - bT_0 - 2cT_1, & -aT_0 - 3bT_1, & -aT_1 \\ dT_1 \quad , & y - bT_0 + \ cT_1, & -aT_0 \\ dT_0 \quad , & 3cT_0 + \ dT_1, & y + 2bT_0 + cT_1 \end{vmatrix} = 0.$$

En ordonnant ce déterminant suivant les puissances de y, les coefficients de y^3, y^2, y, y^0 deviennent des formes binaires en T_0, T_1, des ordres 0, 1, 2, 3. En calculant les valeurs de ces quatre coefficients on les trouve respectivement

$$\begin{aligned}
&= 1,\\
&= 0,\\
&= 3\,(ac - b^2,\ ad - bc,\ bd - c^2 \between T_0,\ T_1)^2,\\
&= (a^2d - 3abc + 2b^3,\ 3abd - 6ac^2 + 3b^2c,\ -3acd + 6b^2d - 3bc^2,\ -ad^2 + 3bcd - 2c^3 \between T_0,\ T_1)^3,
\end{aligned}$$

c'est-à-dire que l'équation en y est celle-ci:

$$y^3 + 3y\,(ac - b^2,\ ad - bc,\ bd - c^2 \between T_0,\ T_1)^2$$
$$+ (a^2d - 3abc + 2b^3,\ 3abd - 6ac^2 + 3b^2c,\ -3acd + 6b^2d - 3bc^2,\ -ad^2 + 3bcd - 2c^3 \between T_0,\ T_1)^3 = 0,$$

équation qui remplit en effet la condition ci-dessus mentionnée, d'avoir pour coefficients des invariants des deux formes:

$$(a,\ b,\ c,\ d \between \xi,\ \eta)^3,\quad (T_0,\ T_1 \between \eta,\ -\xi).$$

Dans le cas particulier dont il s'agit, la fonction $(T_0,\ T_1 \between \eta,\ -\xi)$ est linéaire, et l'on peut même dire que les coefficients sont des covariants de la seule fonction $(a,\ b,\ c,\ d \between T_0,\ T_1)^3$ en y considérant T_0, T_1 comme les variables.

J'ai cru qu'il y avait de l'intérêt de donner cette vérification. D'ailleurs je remarque qu'au moyen du théorème même on aurait pu trouver tout de suite l'équation en y, en écrivant d'abord $T_1 = 0$, ce qui donne le système

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$
$$y = (ax + b)\,T_0,$$

et de là

$$\frac{1}{a}(a,\ b,\ c,\ d \between y - bT_0,\ aT_0)^3 = 0$$

ou enfin

$$y^3 + 3y\,(b^2 - ac)\,T_0^2 + (a^2d - 3abc + 2b^3)\,T_1^3 = 0.$$

Les valeurs des coefficients peuvent être complétées, eu égard à ce qu'ils doivent être des invariants de $(a,\ b,\ c,\ d \between \xi,\ \eta)^3$, $(T_0,\ T_1 \between \eta,\ -\xi)$ (ou, comme on vient de le dire, des covariants de $(a,\ b,\ c,\ d \between T_0,\ T_1)^3$). Mais ce n'est que dans le cas particulier, où les coefficients T_0, T_1 sont au nombre de deux, que l'on peut réduire la seconde équation à une équation linéaire.

Londres, 18 *Avril* 1860.

274.

DEUXIÈME NOTE SUR LA TRANSFORMATION DE TSCHIRNHAUSEN.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LVIII. (1861), pp. 263—269.]

À LA fin de ma première note sur ce sujet j'ai appliqué la transformation de Tschirnhausen à l'équation du troisième degré mise sous la forme

$$ax^3 + 3bx^2 + 3cx + d = (a,\ b,\ c,\ d \between x,\ 1)^3 = 0.$$

En y substituant $\frac{1}{3}b$, $\frac{1}{3}c$ au lieu de b, c, cette équation se change en

$$ax^3 + bx^2 + cx + d = (a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

et en même temps le résultat obtenu dans ma première note s'énonce de la manière suivante:

En calculant pour l'équation

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

la transformée en

$$y = (ax + \tfrac{1}{3}b)\, T_0 + (ax^2 + bx + \tfrac{2}{3}c)\, T_1,$$

on obtient

$$\left.\begin{array}{l} y^3 \\ + \frac{1}{3}y \left\{\begin{array}{ll} T_0^2 & (3ac - b^2) \\ + T_0T_1 & (9ad - bc) \\ + T_1^2 & (3bd - c^2) \end{array}\right\} \\ + \frac{1}{27} \left\{\begin{array}{ll} T_0^3 & (\ \ 27a^2d - 9abc + 2b^3) \\ + T_0^2T_1 & (\ \ 27abd - 18ac^2 + 3b^2c) \\ + T_0T_1^2 & (-27acd + 18b^2d - 3bc^2) \\ + T_1^3 & (-27ad^2 + 9bcd - 2c^3) \end{array}\right\} \end{array}\right\} = 0.$$

Je vais me servir de cette formule, pour en déduire l'équation qui d'une manière analogue est la transformée de l'équation du quatrième ordre

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0.$$

J'écris d'abord

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$

$$y = (ax + \tfrac{1}{4}b)\,T_0 + (ax^2 + bx + \tfrac{1}{2}c)\,T_1 + (ax^3 + bx^2 + cx + \tfrac{3}{4}d)\,T_2,$$

et je remarque qu'en faisant $e = 0$, le système proposé se partage en deux, dont le premier est:

$$x = 0,$$

$$y = \tfrac{1}{4}(bT_0 + 2cT_1 + 3dT_2),$$

et le second:

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

$$y = (ax + \tfrac{1}{4}b)\,T_0 + (ax^2 + bx + \tfrac{1}{2}c)\,T_1 - \tfrac{1}{4}dT_2;$$

ou, ce qui est la même chose,

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

$$y + \tfrac{1}{12}(bT_0 + 2cT_1 + 3dT_2) = (ax + \tfrac{1}{3}b)\,T_0 + (ax^2 + bx + \tfrac{2}{3}c)\,T_1.$$

Une circonstance analogue a lieu dans l'équation en y, résultat de l'élimination du système proposé. Pour $e = 0$ son premier membre se résout de même en deux facteurs qui égalés à zéro sont les résultats de l'élimination du premier et du second système ci-dessus écrits. Le premier de ces deux facteurs est donc

$$y - \tfrac{1}{4}(bT_0 + 2cT_1 + 3dT_2);$$

et le second (en vertu de la formule donnée antérieurement)

$$\begin{aligned} &\{y + \tfrac{1}{12}(bT_0 + 2cT_1 + 3dT_2)\}^3 \\ &+ \tfrac{1}{3}\,[(3ac - b^2)\,T_0^2 + \text{etc.}]\ \{y + \tfrac{1}{12}(bT_0 + 2cT_1 + 3dT_2)\} \\ &+ \tfrac{1}{27}\,[(27a^2d - 9abc + 2b^3)\,T_0^3 + \text{etc.}]\,; \end{aligned}$$

donc en multipliant les deux facteurs, et en égalant à zéro leur produit, on a la transformée en y de la forme $(a,\ b,\ c,\ d,\ 0 \between x,\ 1)^4$. Or dans le cas général, où e est différent de zéro, les coefficients de la transformée en y sont des invariants des deux formes

$$(a,\ b,\ c,\ d,\ e \between \xi,\ \eta)^4,\quad (T_0,\ T_1,\ T_2 \between \eta,\ -\xi)^2.$$

Cette propriété permet de déduire leurs valeurs générales des valeurs particulières qu'ils ont pour $e = 0$. Je formerai de cette manière la transformée en y pour la forme $(a,\ b,\ c,\ d,\ 0 \between x,\ 1)^4$, je passerai de là à la forme $(a,\ b,\ c,\ d,\ 0 \between x,\ 1)^4$ (ce qui se fait en écrivant $4b$, $6c$, $4d$ au lieu de b, c, d), et enfin je compléterai les valeurs des coefficients en y introduisant e au moyen de la propriété que doivent posséder les coefficients d'être des invariants des deux formes

$$(a,\ b,\ c,\ d,\ e \between \xi,\ \eta)^4,\quad (T_0,\ T_1,\ T_2 \between \eta,\ -\xi)^2.$$

On obtient d'abord l'équation en y sous la forme

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4 = 0,$$

où

$$
\begin{aligned}
8\mathfrak{C} = \quad & T_0^2 (\quad 8ac - 3b^2) \\
& + T_0T_1 (\quad 24ad - 4bc) \\
& + \quad T_1^2 (\quad 8bd - 4c^2) \\
& + T_0T_2 (- \quad 2bd) \\
& + T_1T_2 (- \quad 4cd) \\
& + \quad T_2^2 (- \quad 3d^2),
\end{aligned}
$$

$$
\begin{aligned}
8\mathfrak{D} = \quad & T_0^3 \quad (\quad 8a^2d - 4abc + 1b^3) \\
& + T_0^2T_1 \quad (\quad 4abd - 8ac^2 + 2b^2c) \\
& + T_0T_1^2 \quad (- 16acd + 4b^2d) \\
& + \quad T_1^3 \quad (- \quad 8ad^2) \\
& + T_0^2T_2 \quad (- \quad 4acd + 1b^2d) \\
& + T_0T_1T_2 (- 12ad^2) \\
& + T_1^2T_2 \quad (- \quad 4bd^2) \\
& + T_0T_2^2 \quad (- \quad 1bd^2) \\
& + T_1T_2^2 \quad (- \quad 2cd^2) \\
& + \quad T_2^3 \quad (- \quad 1d^3),
\end{aligned}
$$

$$
\begin{aligned}
256\mathfrak{E} = \quad & T_0^4 \quad (\quad 64a^2bd + 16ab^2c - 3b^4) \\
& + T_0^3T_1 \quad (- 128a^2cd - 80ab^2d + 64abc^2 - 8b^3c) \\
& + T_0^2T_1^2 \quad (- 128abcd + 64ac^3 - 48b^3d + 8b^2c^2) \\
& + T_0T_1^3 \quad (\quad 64abd^2 + 64ac^2d - 128b^2cd + 32bc^3) \\
& + \quad T_1^4 \quad (\quad 128acd^2 - 64bc^2d + 16c^4) \\
& + T_0^3T_2 \quad (- 192a^2d^2 + 32abcd - 4b^3d) \\
& + T_0^2T_1T_2 (- 288abd^2 + 64ac^2d + 8b^2cd) \\
& + T_0T_1^2T_2 (- 160b^2d^2 + 48bc^2d) \\
& + T_1^3T_2 \quad (+ 192ad^3 - 128bcd^2 + 32c^3d) \\
& + T_0^2T_2^2 \quad (- \quad 48acd^2 + 14b^2d^2) \\
& + T_0T_1T_2^2 (- 144ad^3 + 8bcd^2) \\
& + T_1^2T_2^2 \quad (- \quad 48bd^3 + 8c^2d^2) \\
& + T_0T_2^3 \quad (- \quad 4bd^3) \\
& + T_1T_2^3 \quad (- \quad 1cd^3) \\
& + \quad T_2^4 \quad (- \quad 8d^4).
\end{aligned}
$$

Ce calcul achevé et substituant la forme $(a, b, c, d, 0\between\xi, \eta)^4$ au lieu de $(a, b, c, d, 0\between\xi, \eta)^4$ (ou $4b$, $6c$, $4d$ au lieu de b, c, d) on obtient tous les termes de l'équation cherchée, hormis ceux qui contiennent e: et ces derniers s'obtiennent au moyen de ce que les coefficients des différentes puissances de y se réduisent à zéro par l'opérateur

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e - T_2\partial_{T_1} - 2T_1\partial_{T_0}.$$

Cela ne présente pas de difficulté, je supprime donc les calculs intermédiaires et je donne le résultat final que voici: les équations

$$(a,\ b,\ c,\ d,\ e\between x,\ 1)^4 = 0,$$

$$y = (ax+b)\,T_0 + (ax^2+4bx+3c)\,T_1 + (ax^3+4bx^2+6cx+3d)\,T_2,$$

conduisent à la transformée:

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4 = 0,$$

où l'on a

$\mathfrak{C} = 2$

T_0^2	T_0T_1	T_0T_2	T_1^2	T_1T_2	T_2^2
$ac+3$ b^2-3	$ad+6$ $bc-6$	$ae+2$ $bd-2$	$ae+1$ $bd+8$ c^2-9	$be+6$ $cd-6$	$ce+3$ d^2-3

$\mathfrak{D} = 4$

T_0^3	$T_0^2T_1$	$T_0^2T_2$	$T_0T_1^2$	$T_0T_1T_2$	T_1^3	$T_0T_2^2$	$T_1^2T_2$	$T_1T_2^2$	T_2^3
a^2d+1 $abc-3$ b^3+2	a^2e+1 $abd+2$ ac^2-9 b^2c+6	$abe+1$ $acd-3$ b^2d+2	$abe+4$ $acd-12$ b^2d+8	ad^2-6 b^2e+6	ad^2-4 b^2e+4	$ade-1$ $bce+3$ bd^2-2	$ade-4$ $bce+12$ bd^2-8	ae^2-1 $bde-2$ c^2e+9 cd^2-6	be^2-1 $cde+3$ d^3-2

$\mathfrak{E} =$

T_0^4	$T_0^3T_1$	$T_0^3T_2$	$T_0^2T_1^2$	$T_0^2T_1T_2$	$T_0T_1^3$	$T_0^2T_2^2$	$T_0T_1^2T_2$
a^2e+1 a^2bd-4 ab^2c+6 b^4-3	a^2be+8 a^2cd-12 ab^2d-20 abc^2+36 b^3c-12	a^2ce+12 a^2d^2-12 ab^2e-8 $abcd+12$ b^3d+4	a^2ce-6 ab^2e+30 $abcd-48$ ac^3+54 b^3d-48 b^2c^2+18	$abce+60$ abd^2-72 ac^2d+36 b^3e-36 b^2cd+12	a^2de-4 $abce-12$ abd^2+16 ac^2d+36 b^3e+48 $b^2cd-192$ bc^3+108	a^2e^2+2 $abde-16$ ac^2e+36 acd^2-18 b^2ce-18 b^2d^2+14	a^2e^2-4 $abde+20$ ac^2e+36 b^2d^2-160 bc^2d+108

T_1^4	$T_0T_1T_2^2$	$T_1^2T_2$	$T_0T_2^3$	$T_1^2T_2^2$	$T_1T_2^3$	T_2^4
a^2e^2+1 $abde-16$ ac^2e-18 acd^2+48 b^2ce+48 bc^2d-144 c^4+81	$acde+60$ ad^3-36 b^2de-72 bc^2e+36 bcd^2+12	abe^2-4 $acde-12$ ad^3+48 b^2de+16 bc^2e+36 bcd^2-192 c^3d+108	ace^2+12 ad^2e-8 b^2e^2-12 $bcde+12$ bd^3-4	ace^2-6 ad^2e+30 $bcde-48$ bd^3-48 c^3e+54 c^2d^2+18	ade^2+8 bce^2-12 bd^2e-20 c^2de+36 cd^3-12	ae^3+1 bde^2-4 cd^2e+6 d^4-3

J'écris

$$U' = aT_0^2 + 4bT_0T_1 + c\,(2T_0T_2 + 4T_1^2) + 4dT_1T_2 + eT_2^2,$$

$$H' = (ac-b^2)\,T_0^3 + 2\,(ad-bc)\,T_0T_1 + (ae-2bd+c^2)\,T_0T_2 + 4\,(bd-c^2)\,T_1^2$$
$$+ 2\,(be-cd)\,T_1T_2 + (ce-d^2)\,T_2^2,$$

et je représente par $4\Phi'$ la valeur qui vient d'être trouvée pour $\mathfrak{D}$. Ces expressions U', H', Φ' sont des invariants des deux formes $(a, b, c, d, e \between \xi, \eta)^4$, $(T_0, T_1, T_2 \between \eta, -\xi)^2$, on a de plus les invariants

$$ae - 4bd + 3c^2, \quad ace - ad^2 - b^2e + 2bcd - c^3,$$

que je représente comme à l'ordinaire par I, J, et l'invariant $T_0T_2 - T_1^2$ que je représente par Θ'. Cela posé on a

$$\mathfrak{C} = 6H' - 2I\Theta',$$

$$\mathfrak{D} = 4\Phi',$$

$$\mathfrak{E} = IU'^2 - 3H'^2 + I^2\Theta'^2 + 12J\Theta'U' + 2I\Theta'H'.$$

La dernière de ces équations peut être vérifiée aisément, pour cela on a seulement besoin de remarquer qu'en posant $a = e = 1$, $b = d = 0$, $c = \theta$, elle devient

$$\begin{aligned}
&(1+3\theta^2)\left(T_0^2 + \theta(2T_0T_2 + 4T_1^2) + T_2^2\right)^2 \\
&- 3\left(\theta T_0^2 + (1+\theta^2)\,T_0T_2 - 4\theta^2T_1^2 + \theta T_2^2\right)^2 \\
&+ (1+3\theta^2)^2(T_0T_2 - T_1^2)^2 \\
&+ 12(\theta - \theta^3)(T_0T_2 - T_1^2)\left(T_0^2 + \theta(2T_0T_2 + 4T_1^2) + T_2^2\right) \\
&+ 2(1+3\theta^2)(T_0T_2 - T_1^2)\left(\theta T_0^2 + (1+\theta^2)\,T_0T_2 - 4\theta^2T_1^2 + \theta T_2^2\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & T_0^4 \\
&+ T_0^3T_2(\ \ 12\theta) \\
&+ T_0^2T_1^2(-6\theta + 54\theta^3) \\
&+ T_0^2T_2^2(2 + 36\theta^2) \\
&+ T_0T_1^2T_2(-4 + 36\theta^2) \\
&+ T_1^4(1 - 18\theta^2 + 81\theta^4) \\
&+ T_1^2T_2^2(-6\theta + 54\theta^3) \\
&+ T_0T_2^3(\ \ 12\theta) \\
&+ T_2^4,
\end{aligned}$$

équation qui est identique. L'expression de l'invariant I (quadrinvariant) de la fonction $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$ est $\mathfrak{E} + 3\,(\frac{1}{6}\mathfrak{C})^2$, ou $\mathfrak{E} + 3\,(H' - \frac{1}{3}I\Theta')^2$, c'est-à-dire

$$\begin{aligned}
IU'^2 - 3H'^2 + \ & I^2\Theta'^2 + 12J\Theta'U' + 2I\Theta'H' \\
+ 3H'^2 + \tfrac{1}{3}I^2\Theta'^2 \ & \qquad\qquad - 2I\Theta'H',
\end{aligned}$$

ou enfin

$$IU'^2 + \tfrac{4}{3}I^2\Theta'^2 + 12J\Theta'U',$$

ce qui est égal à

$$\frac{1}{I}\left[(IU' + 6J\Theta')^2 + \tfrac{4}{3}(I^3 - 27J^2)\Theta'^2\right].$$

La condition à remplir pour que cet invariant se réduise à zéro peut donc être présentée sous la forme

$$IU' + \left[6J \pm 2\sqrt{-\tfrac{1}{3}(I^3 - 27J^2)}\right]\Theta' = 0,$$

ce qui s'accorde avec un résultat trouvé par M. Hermite.

Il doit y avoir, ce me semble, une équation identique de la forme

$$JU'^2 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2$$

qui servirait à exprimer le carré de Φ' au moyen des autres invariants U', H', Θ', I, J, mais en supposant que cette équation existe, la forme du facteur M, que je n'ai pas encore cherchée, reste à déterminer; l'invariant J (cubinvariant) de la forme $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\mathfrak{Y}y, 1)^4$ contient Φ'^2, et il faudrait employer l'identité dont je viens de parler pour réduire à sa forme la plus simple cet invariant; dans l'état actuel de la question je ne m'occupe donc pas de l'expression du cubinvariant de $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\mathfrak{Y}y, 1)^4$.

Pour passer au cas d'une équation du cinquième ordre, on devra faire usage de la formule qui se rapporte à la forme $(a, b, c, d, e\mathfrak{Y}x, 1)^4$. En faisant la substitution nécessaire on arrive à ce résultat que pour l'équation

$$(a, b, c, d, e\mathfrak{Y}x, 1)^4 = 0$$

la transformée en

$$y = (ax + \tfrac{1}{4}b)\,T_0 + (ax^2 + bx + \tfrac{1}{2}c)\,T_1 + (ax^3 + bx^2 + cx + \tfrac{3}{4}d)\,T_2$$

est la suivante

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\mathfrak{Y}y, 1)^4 = 0,$$

où

$\mathfrak{C} = \frac{1}{8}$

T_0^2	T_0T_1	T_0T_2	T_1^2	T_1T_2	T_2^2
$ac + 8$ $b^2 - 3$	$ad + 24$ $bc - 4$	$ae + 32$ $bd - 2$	$ae + 16$ $bd + 8$ $c^2 - 4$	$be + 24$ $cd - 4$	$ce + 8$ $d^2 - 3$

$\mathfrak{D} = \frac{1}{8}$

T_0^3	$T_0^2T_1$	$T_0^2T_2$	$T_0T_1^2$	$T_0T_1T_2$	T_1^3	$T_0T_2^2$	$T_1^2T_2$	$T_1T_2^2$	T_2^3
$a^2d + 8$ $abc - 4$ $b^3 + 1$	$a^2e + 32$ $abd + 4$ $ac^2 - 8$ $b^2c + 2$	$abe + 8$ $acd - 4$ $b^2d + 1$	$abe + 32$ $acd - 16$ $b^2d + 4$	$ad^2 - 12$ $b^2e + 12$	$ad^2 - 8$ $b^2e + 8$	$ade - 8$ $bce + 4$ $bd^2 - 1$	$ade - 32$ $bce + 16$ $bd^2 - 4$	$ae^2 - 32$ $bde - 4$ $c^2e + 8$ $cd^2 - 2$	$be^2 - 8$ $cde + 4$ $d^3 - 1$

$\mathfrak{E} = \frac{1}{256}$

T_0^4	$T_0^3T_1$	$T_0^3T_2$	$T_0^2T_1^2$	$T_0^2T_1T_2$	$T_0T_1^3$	$T_0^2T_2^2$	$T_0T_1^2T_2$
$a^2e + 256$	$a^2be + 512$	$a^2ce + 512$	$a^2ce - 256$	$abce + 640$	$a^2de - 256$	$a^2e^2 + 512$	$a^2e^2 - 1024$
$a^2bd - 64$	$a^2cd - 128$	$a^2d^2 - 192$	$ab^2e + 480$	$abd^2 - 288$	$abce - 128$	$abde - 256$	$abde + 320$
$ab^2c + 16$	$ab^2d - 80$	$ab^2e - 128$	$abcd - 128$	$ac^2d + 64$	$abd^2 + 64$	$ac^2e + 256$	$ac^2e + 256$
$b^4 - 3$	$abc^2 + 64$	$abcd + 32$	$ac^3 + 64$	$b^2e - 144$	$ac^2d + 64$	$acd^2 - 48$	$b^2d^2 - 160$
	$b^3c - 8$	$b^3d - 4$	$b^3d - 48$	$b^2cd + 8$	$b^2e + 192$	$b^2ce - 48$	$bc^2d + 48$
			$b^2c^2 + 8$		$b^2cd - 128$	$b^2d^2 + 14$	
					$bc^3 + 32$		

T_1^4	$T_0T_1T_2^2$	$T_1^2T_2$	$T_0T_2^3$	$T_1^2T_2^2$	$T_1T_2^3$	T_2^4
$a^2e^2 + 256$	$acde + 640$	$abe^2 - 256$	$ace^2 + 512$	$ace^2 - 256$	$ade^2 + 512$	$ae^3 + 256$
$abde - 256$	$ad^3 - 144$	$acde - 128$	$ad^2e - 128$	$ad^2e + 480$	$bce^2 - 128$	$bde^2 - 64$
$ac^2e - 128$	$b^2de - 288$	$ad^2 + 192$	$b^2e^2 - 192$	$bcde - 128$	$bd^2e - 80$	$cd^2e + 16$
$acd^2 + 128$	$bc^2e + 64$	$b^2de + 64$	$bcde + 32$	$bd^3 - 48$	$c^2de + 64$	$d^4 - 3$
$b^2ce + 128$	$bcd^2 + 8$	$bc^2e + 64$	$bd^3 - 4$	$c^3e + 64$	$cd^3 - 8$	
$bc^2d - 64$		$bcd^2 - 128$		$c^2d^2 + 8$		
$c^4 + 16$		$c^3d + 32$				

En m'appuyant sur ce résultat j'espère être à même de trouver la formule pour l'équation du cinquième ordre.

Londres, 11 *Mai* 1860.

275.

ON TSCHIRNHAUSEN'S TRANSFORMATION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLI. (for the year 1861), pp. 561—578. Received November 7,—Read December 5, 1861.]

THE memoir of M. Hermite, "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré," *Comptes Rendus*, t. XLVI. p. 961 (1858), contains a very important theorem in relation to Tschirnhausen's Transformation of an equation $f(x)=0$ into another of the same degree in y, by means of the substitution $y=\phi x$, where ϕx is a rational and integral function of x. In fact, considering for greater simplicity a quartic equation,

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$

M. Hermite gives to the equation $y=\phi x$ the following form,

$$y = aT + (ax+4b)\,B + (ax^2+4bx+6c)\,C + (ax^3+4bx^2+6cx+4d)\,D,$$

(I write B, C, D in the place of his T_0, T_1, T_2), and he shows that the transformed equation in y has the following property: viz., every function of the coefficients which, expressed as a function of a, b, c, d, e, T, B, C, D, does not contain T, is an *invariant*, that is, an invariant of the two quantics

$$(a,\ b,\ c,\ d,\ e \between X,\ Y)^4,\ (B,\ C,\ D \between Y,\ -X)^2.$$

This comes to saying that if T be so determined that in the equation for y the coefficient of the second term (y^3) shall vanish, the other coefficients will be invariants; or if, in the function of y which is equated to zero, we consider y as an absolute constant, the function of y will be an invariant of the two quantics. It is easy to find the value of T; this is in fact given by the equation

$$0 = aT + 3bB + 3cC + dD\,;$$

and we have thence for the value of y,

$$y = (ax + b)\,B + (ax^2 + 4bx + 3c)\,C + (ax^3 + 4bx^2 + 6cx + 3d)\,D\,;$$

so that for this value of y the function of y which equated to zero gives the transformed equation will be an invariant of the two quantics. It is proper to notice that in the last-mentioned expression for y, all the coefficients except those of the term in x^0, or $bB + 3cC + 3dD$, are those of the binomial $(1,\ 1)^4$, whereas the excepted coefficients are those of the binomial $(1,\ 1)^3$; this suffices to show what the expression for y is in the general case.

I have in the two papers, "Note sur la Transformation de Tschirnhausen," and "Deuxième Note sur la Transformation de Tschirnhausen," *Crelle*, t. LVIII. pp. 259 and 263 (1861), [273 and 274], obtained the transformed equations for the cubic and quartic equations; and by means of a grant from the Government Grant Fund, I have been enabled to procure the calculation, by Messrs Davis and Otter, under my superintendence, of the transformed equation for the quintic equation. The several results are given in the present memoir; and for greater completeness, I reproduce the demonstration which I have given in the former of the above-mentioned two Notes, of the general property, that the function of y is an invariant. At the end of the memoir I consider the problem of the reduction of the general quintic equation to Mr Jerrard's form $x^5 + ax + b = 0$.

Considering for simplicity the foregoing two equations

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$
$$y = (ax + b)\,B + (ax^2 + 4bx + 3c)\,C + (ax^3 + 4bx^2 + 6cx + 3d)\,D\,;$$

let the second of these be represented by $y = V$, the transformed equation in y is

$$(y - V_1)(y - V_2)(y - V_3)(y - V_4) = 0,$$

where V_1, V_2, V_3, V_4 are what V becomes upon substituting therein for x the roots x_1, x_2, x_3, x_4 of the quartic equation respectively. Considering y as a constant, the conditions to be satisfied in order that the function in y may be an invariant are that this function shall be reduced to zero by each of the two operators

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e - (\ D\partial_c + 2C\partial_B),$$
$$4b\partial_a + 3c\partial_b + 2d\partial_c + \ e\partial_d - (2C\partial_D + \ B\partial_c).$$

These conditions will be satisfied if each of the factors $y - V_1$, &c. has the property in question; that is, if $y - V$, or (what is the same thing) if V, supposing that x denotes therein a root of the quartic equation, is reduced to zero by each of the two operators. Considering the first operator, which for shortness I represent by

$$\Delta - (D\partial_C + 2C\partial_B),$$

in order to obtain ΔV we require the value of Δx. To find it, operating with Δ on the quartic equation, we have

$$(a,\ b,\ c,\ d \between x,\ 1)^3\,\Delta x + (a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

or $\Delta x = -1$. In ΔV, the part which depends on the variation of Δx then is

$$-aB + (-2ax - 4b)\,C + (-3ax^2 - 8bx - 6c)\,D,$$

and the other part of ΔV is at once found to be

$$+aB + (4ax + 6b)\,C + (4ax^2 + 12bx + 9c)\,D;$$

whence, adding,

$$\Delta V = 2\,(ax + b)\,C + (ax^2 + 4bx + 3c)\,D,$$

and this is precisely equal to

$$(D\partial_C + 2C\partial_B)\,V;$$

so that V is reduced to zero by the operator $\Delta - (D\partial_C + 2C\partial_B)$.

Similarly, if the second operator is represented by

$$\nabla - (2C\partial_D + B\partial_C),$$

then we have

$$(a,\ b,\ c,\ d \between x,\ 1)^3\,\nabla x + x\,(b,\ c,\ d,\ e \between x,\ 1)^3 = 0,$$

which by means of the equation

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0$$

is reduced to $\nabla x = x^2$. Hence in ∇V the part depending on the variation of x is

$$(ax^2)\,B + (2ax^3 + 4bx^2)\,C + (3ax^4 + 8bx^3 + 6cx^2)\,D,$$

and the other part of ∇V is at once found to be

$$(4bx + 3c)\,B + (4bx^2 + 12cx + 6d)\,C + (4bx^3 + 12cx^2 + 12dx + 3e)\,D;$$

and, adding, the coefficient of D vanishes on account of the quartic equation, and we have

$$\nabla V = (ax^2 + 4bx + 3c)\,B + 2\,(ax^3 + 4bx^2 + 6cx + 3d)\,C,$$

which is precisely equal to

$$(2C\partial_D + B\partial_C)\,V,$$

so that V is reduced to zero by the operator

$$\nabla - (2C\partial_D + B\partial_C),$$

which completes the demonstration; and the demonstration in the general case is precisely similar.

In the case of the cubic equation we have

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

$$y = (ax + b)\,B + (ax^2 + 3bx + 2c)\,C;$$

and writing the second equation in the form

$$(y-bB-2cC)+x(\ -aB-3bC)+x^2(\qquad -aC)=0,$$

multiplying by x and reducing by the cubic equation, we have

$$dC \qquad +x(y-bB+\ cC)+x^2(\ -\ aB) \qquad =0,$$

and repeating the process,

$$dB \qquad +x(\ 3cB+\ dC)+x^2(y+2bB+cC)=0;$$

or, what is the same thing, we have the system of equations

$$(\ y-bB-2cC,\quad -\ aB-3bC,\qquad -cC\ \between 1,\ x,\ x^2)=0,$$
$$\left|\begin{array}{cccc} & dC, & y-\ bB+\ cC, & -\ aB \\ dB & , & 3cB+\ dC, & y+2bB+cC \end{array}\right|$$

and the resulting equation in y is of course that formed by equating to zero the determinant formed out of the matrix in this equation. The developed expression is

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D}\between y,\ 1)^3=0,$$

where

$\frac{1}{3}\mathfrak{C}=$

	B^2		BC		C^2
ac b^2	$+1$ -1	ad bc	$+1$ -1	bd c^2	$+1$ -1
	± 1		± 1		± 1

$\mathfrak{D}=$

	B^3		B^2C		BC^2		C^3
a^2d abc b^3	$+1$ -3 $+2$	abd ac^2 b^2c	$+3$ -6 $+3$	acd b^2d bc^2	-3 $+6$ -3	ad^2 bcd c^3	-1 $+3$ -2
	± 3		± 6		± 6		± 3

The sum of the coefficients in each column should here and elsewhere in the present memoir be equal to zero, and I have by way of verification annexed to each column the sums ($\pm$ a number) of the positive and negative coefficients. The coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore the function in y, are invariants of the two forms,

$$(a,\ b,\ c,\ d\between X,\ Y)^3,\quad (B,\ C\between Y,\ -X);$$

or in the present case, where there are only two coefficients B, C, the coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore also the function in y, are covariants of the single form $(a,\ b,\ c,\ d\between B,\ C)^3$, considering therein $(B,\ C)$ as the facients.

It may be remarked that in the present case, assuming the invariance of the function in y, we may obtain the transformed equation in a very simple manner by writing in the first instance $C=0$, this gives

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

$$y = (ax+b)\,B,$$

and thence

$$\frac{1}{a}(a,\ b,\ c,\ d \between y - bB,\ aB)^3 = 0,$$

or developing,

$$y^3 + 3y\,(ac-b^2)\,B^2 + (a^2d - 3abc + 2b^3)\,B^3 = 0,$$

where the expressions for the coefficients are to be completed by the consideration that these coefficients are covariants of the form $(a,\ b,\ c,\ d \between B,\ C)^3$. But it is only in the case in hand of a cubic equation that the transformed equation can be obtained in this manner.

In the case of a quartic equation, we have

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$

$$y = (ax+b)\,B + (ax^2+4bx+3c)\,C + (ax^3+4bx^2+6cx+3d)\,D,$$

and these give the system of equations

$$\left(y - bB - 3cC - 3dD,\ -aB - 4bC - 6cD,\ -aC - 4bD,\ -aD \between 1,\ x,\ x^2,\ x^3\right) = 0,$$

$$\begin{vmatrix} eD, & y - bB - 3cC + dD, & -aB - 4bC, & -aC \\ eC, & 4dC + eD, & y - bB + 3cC + dD, & -aB \\ eB, & 4dB + eC, & 6cB + 4dC + eD, & y + 3bB + 3cC + dD \end{vmatrix}$$

and the transformed equation is therefore found by equating to zero the determinant formed out of the matrix contained in this equation.

The developed result, which was obtained by a different process in the 'Deuxième Note' above referred to, is

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4 = 0,$$

where

$\frac{1}{2}\mathfrak{C} =$

	B^2		BC		BD^2	C^2		CD		D^2
ac b^2	$+3$ -3	ad be	$+6$ -6	ae bd c^2	$+2$ -2	$+1$ $+8$ -9	be cd	$+6$ -6	ce d^2	$+3$ -3
	± 3		± 6		± 2	± 9		± 6		± 3

$\frac{1}{4}\mathfrak{D} =$

	B^3		B^2C		B^2D	BC^2		BCD	C^3		BD^2	C^2D		CD^2		D^3
a^2d	$+1$	a^2e	$+1$	abe	$+1$	$+4$	ac^2									
abc	-3	abd	$+2$	acd	-3	-12	ad^2	-6	-4	ade	-1	-4	ae^2	-1	be^2	-1
b^3	$+2$	ac^2	-9	b^2d	$+2$	$+8$	b^2e	$+6$	$+4$	bed	$+3$	$+12$	bde	-2	cde	$+3$
		b^2c	$+6$	bc^2			bcd			bd^2	-2	-8	c^2e	$+9$	d^3	-2
							c^3			c^2d			cd^2	-6		
	± 3		± 9		± 3	± 12		± 6	± 4		± 3	± 12		± 9		± 3

and

$\mathfrak{E} =$

	B^4		B^3C		B^3D	B^2C^2		B^2CD	BC^3		B^2D^2	BC^2D	C^4
a^3e	$+1$	a^2be	$+8$	a^2ce	$+12$	-6	a^2de	...	-4	a^2e^2	$+2$	-4	$+1$
a^2bd	-4	a^2cd	-12	a^2d^2	-12	...	$abce$	$+60$	-12	$abde$	-16	$+20$	-16
a^2c^2	...	ab^2d	-20	ab^2e	-8	$+30$	abd^2	-72	$+16$	ac^2e	$+36$	$+36$	-18
ab^3c	$+6$	abc^2	$+36$	$abed$	$+12$	-48	ac^2d	$+36$	$+36$	acd^2	-18	...	$+48$
b^4	-3	b^3c	-12	ac^3	...	$+54$	b^3e	-36	$+48$	b^2ce	-18	...	$+48$
				b^3d	-4	-48	b^3cd	$+12$	-192	b^3d^2	$+14$	-160	...
				b^3c^2		$+18$	bc^3		$+108$	bc^2d		$+108$	-144
										c^4			$+81$
	± 7		± 44		± 24	± 102		± 108	± 208		± 52	± 164	± 178

	BCD^2	C^3D		BD^3	C^2D^2		CD^3		D^4
abe^2	...	-4	ace^2	$+12$	-6	ade^2	$+8$	ae^3	$+1$
$acde$	$+60$	-12	ad^2e	-8	$+30$	bce^2	-12	bde^2	-4
ad^3	-36	$+48$	b^2e^2	-12	...	bd^2e	-20	c^2e^2	...
b^2de	-72	$+16$	$bcde$	$+12$	-48	c^2de	$+36$	cd^2e	$+6$
bc^2e	$+36$	$+36$	bd^2	-4	-48	cd^3	-12	d^4	-3
bed^2	$+12$	-192	c^3e		$+54$				
c^3d		$+108$	c^2d^2		$+18$				
	± 108	± 208		± 24	± 102		± 44		± 7

I write

$$U' = aB^2 + 4bBC + c(2BD + 4C^2) + 4dCD + eD^2,$$

$$H' = (ac - b^2)B^2 + 2(ad - bc)BC + (ae - 2bd + c^2)BD + 4(bd - c^2)C^2 + 2(be - cd)CD + (ce - d^2)D^2;$$

and I represent by Φ' the expression which has just been found for $\frac{1}{4}\mathfrak{D}$. These functions, U', H', Φ', are invariants of the two forms

$$(a, b, c, d, e \between X, Y)^4, \quad (B, C, D \between Y, -X)^2;$$

we have, moreover, the invariants

$$ae - 4bd + 3c^2, \; ace - ad^2 - b^2e + 2bcd - c^3,$$

which I represent as usual by I, J, and the invariant $BD-C^2$, which I represent by Θ'. This being so, we have

$$\mathfrak{C} = 6H' - 2I\Theta',$$
$$\mathfrak{D} = 4\Phi',$$
$$\mathfrak{E} = IU'^2 - 3H'^2 + I^2\Theta'^2 + 12J\Theta'U' + 2I\Theta'H',$$

the last of which may be verified as follows:—viz. writing $a=e=1$, $b=d=0$, $c=\theta$, it becomes

$$\begin{aligned}
&(1+3\theta^2)\{B^2+\theta(2BD+4C^2)+D^2\}^2\\
&-3\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}^2\\
&+(1+3\theta^2)^2(BD-C^2)^2\\
&+12(\theta-\theta^3)(BD-C^2)\{B^2+\theta(2BD+4C^2)+D^2\}\\
&+2(1+3\theta^2)(BD-C^2)\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}\\
=\ & B^4\\
&+B^3D\ (12\theta)\\
&+B^2C^2\ (-6\theta+54\theta^3)\\
&+B^2D^2\ (2+36\theta^2)\\
&+BC^2D\ (-4+36\theta^2)\\
&+C^4\ (1-18\theta^2+81\theta^4)\\
&+C^2D^2\ (-6\theta+54\theta^3)\\
&+BD^3\ (12\theta)\\
&+D^4,
\end{aligned}$$

which is an identical equation. [This is in effect the identity of the "Deuxième Note," see *ante* p. 372, but the left-hand side was in the original printed incorrectly, with four lines only instead of five.]

The expression for the invariant I (quadrinvariant) of the function $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\between y, 1)^4$ is $\mathfrak{E}+3(\frac{1}{3}\mathfrak{C})^2$, or $\mathfrak{E}+3(H'-\frac{1}{3}I\Theta')^2$, viz. it is

$$\begin{aligned}
&IU'^2-3H'^2+I^2\Theta'^2+12J\Theta'U'+2I\Theta'H'\\
&\quad+3H'^2+\tfrac{1}{3}I^2\Theta'^2 \qquad\qquad -2I\Theta'H',
\end{aligned}$$

or, finally, it is

$$IU'^2+\tfrac{4}{3}I^2\Theta'^2+12J\Theta'U',$$

which is equal to

$$\frac{1}{I}[(IU'+6J'\Theta')^2+\tfrac{4}{3}(I^3-27J^2)\Theta'^2];$$

so that the condition in order that this invariant may be equal to zero is

$$IU'+[6J\pm 2\sqrt{-\tfrac{1}{3}(I^3-27J^2)}]\,\Theta'=0,$$

which agrees with a result of M. Hermite's.

There should, I think, be an identical equation of the form

$$JU'^2 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2,$$

which would serve to express the square of the invariant Φ' in terms of the other invariants U', H', Θ', I, J; but assuming that such an equation exists, the form of the factor M remains to be ascertained. The invariant J (cubinvariant) of the form $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$ contains Φ'^2, and it would be necessary to make use of the identical equation just referred to in order to reduce it to its simplest form; and (this being so) I have not sought for the expression of the cubinvariant of $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$.

For the quintic we have the equations

$$(a, b, c, d, e, f \between x, 1)^5 = 0,$$

$$\begin{aligned} y = \ & (ax + b)\,B \\ & + (ax^2 + 5bx + 4c)\,C \\ & + (ax^3 + 5bx^2 + 10cx + 6d)\,D \\ & + (ax^4 + 5bx^3 + 10cx^2 + 10dx + 4e)\,E, \end{aligned}$$

and this leads to the system of equations

$$\left(\begin{array}{lllll} y - bB - 4cC - 6dD - 4eE, & -aB - 5bC - 10cD - 10dE, & -aC - 5bD - 10cE, & -aD - 5bE, & -aE \between 1, x, x^2, x^3, x^4) = 0, \\ fE, & y - bB - 4cC - 6cD + eE, & -aB - 5bC - 10cD, & -aC - 5bD, & -aD \\ fD, & 5eD + fE, & y - bB - 4cC + 4dD + eE, & -aB - 5bC, & -aC \\ fC, & 5eC + fD, & 10dC + 5eD + fE, & y - bB + 6cC + 4dD + eE, & -aB \\ fB, & 5eB + fC, & 10dB + 5eC + fD, & 10cB + 10dC + 5eD + fE, & y + 4bB + 6cC + 4dD + eE \end{array}\right.$$

and the transformed equation is obtained by equating to zero the determinant formed out of the matrix contained in this equation.

The determinant in question was calculated by the formula

	Div.
$\square = -12.345$	1
$+13.245$	2
-14.235	3
$+15.234$	4
-23.145	5
$+24.135$	6
-25.134	7
-34.125	8
$+35.124$	9
-45.123	10,

where the duadic symbols refer to the first and fifth columns, viz. 12 is the determinant formed out of the lines 1 and 2 of these columns, and so for the other like symbols; and the triadic symbols refer to the second, third, and fourth columns, viz. 345 is the determinant formed out of the lines 3, 4, 5 of these columns, and so for the other like symbols.

The ten divisions were separately calculated. It is to be noticed that these divisions other than 4 and 6 correspond to each other in pairs, while each of the divisions 4 and 6 corresponds to itself, as thus:

Div. 1,	-10
2,	-9
3,	-7
5,	-8
4,	-4
6,	-6,

viz. if in the place of

$$y;\ a,\ b,\ c,\ d,\ e,\ f;\ B,\ C,\ D,\ E,$$

we write

$$-y;\ f,\ e,\ d,\ c,\ b,\ a;\ E,\ D,\ C,\ B,$$

then division 1 becomes division 10 with its sign reversed, and so for divisions 2 and 9, 3 and 7, 5 and 8; while each of the divisions 4 and 6 is unaltered, except that the sign is reversed. But the corresponding divisions were each of them calculated, and the property in question was used as a verification. Another very convenient verification, which was employed for the several divisions, was obtained by putting

$$a = b = c = d = e = f = B = C = D = E = 1,$$

upon which supposition the determinant becomes

$$\begin{vmatrix} y-15, & -26, & -16, & -6, & -1 \\ 1, & y-10, & -16, & -6, & -1 \\ 1, & 6, & y, & -6, & -1 \\ 1, & 6, & 16, & y+10, & -1 \\ 1, & 6, & 16, & 16, & y+15 \end{vmatrix}$$

and the values of the ten divisions respectively are

y^5,	y^4,	y^3,	y^2,	y,	1	
		6,	−288,	+4608,	−24576	1
		16,	−576,	+6144,	−16384	2
		26,	−544,	+3584,	−24576	3
1,	0,	−96,	0,	−28672,	0	4
				0,	0	5
				0,	0	6
		26,	+544,	+3584,	+24576	7
				0,	0	8
		16,	+576,	+6144,	+16384	9
		6,	+288,	+4608,	+24576	10
1,	0,	0,	0,	0,	0	

A verification similar to this was in fact employed at each step of the calculation of a division: viz. in forming a product such as $(\lambda X+\mu Y+\&c.)(\lambda' X+\mu' Y+\&c.)$, where λ, μ, &c., λ', μ', .. &c. are numerical coefficients, and X, Y, &c. are monomial products of a, b, c, d, e, f and B, C, D, E, the sum of the numerical coefficients of the product is $(\lambda+\mu+\&c.)(\lambda'+\mu'+\&c.)$.

It was of course necessary to employ such verifications, as a test of the correctness of the several divisions, before proceeding to collect them together, but the collection itself affords an exceedingly good ultimate verification. The following is an exemplification: the terms in y which involve the product $BCDE$ are obtained by the collection of the corresponding terms in the ten divisions, as follows:

		1	2	3	4	5	6	7	8	9	10	
y	$BCDE$. $-5\,a^2f^2$	+1	−2	−2	+1	+1	−2	−2	+1	−2	+1	
	$+30\,abef$	+49	−16	−20	+4	−25	+50	−20	−25	−16	+49	
	$+980\,acdf$	+80	+200	+184	−148	+100		+184	+100	+200	+80	
	$-280\,ace^2$	+80		+80	−440							
	$-180\,ad^2e$		−60	−60	−60							
	$-280\,b^2df$				−440			+80			+80	
	$-825\,b^2e^2$				−825							
	$-180\,bc^2f$				−60			−60		−60		
	$+740\,bcde$				+740							
	bd^3											
	c^3e											
	c^2d^2											
	$\pm 1750 = 0$	+210	+122	+182	−1228	+76	+48	+182	+76	+122	+210	$=\pm 1228$

where it may be remarked that the greater part, but not all, of the component coefficients are divisible by 5. I soon observed in the process of summing the ten divisions that all the resulting coefficients should be divisible by 5 (the only exception is as to the terms in y^0 which contain B^5, C^5, D^5, and E^5 respectively), and the circumstance that they are so in each particular instance is as far as it goes a verification, which, however, only applies to those of the component coefficients which are not themselves divisible by 5. But it was known *à priori* (I will presently show how this is so) that the sum of the resulting coefficients should be equal to zero, and that they are so in fact is a verification as to *all* the coefficients. The foregoing specimen term $BCDE$ is one which remains unaltered when B, C, D, E are changed into E, D, C, B; and on making the further change a, b, c, d, e, f into f, e, d, c, b, a, the coefficient of $BCDE$ remains, as it should do, unaltered; this is a verification of the coefficients of the terms ace^2, b^2df; ad^2e, bc^2f, which are respectively interchanged by the substitution in question, but not of the other terms a^2f^2, $abef$, $acdf$, b^2e^2, $bcdf$, which are respectively unaltered by the substitution. I did *not* employ what would have been another convenient verification of the several divisions, viz. the comparison of their values on putting therein $a=b=c=d=e=f=1$, with the corresponding values as calculated independently from the determinant

$$\left|\begin{array}{lllllll}
y-B-4C-6D-E, & -B-5C-10D-10E, & -C-5D-10E, & D-5E, & -E \\
E, & y-B-4C-6D+E, & -B-5C-10D, & -C-5D, & -D \\
D, & 5D+E, & y-B-4C+4D+E, & -B-5C, & -C \\
C, & 5C+D, & 10C+5D+E, & y-B+6C+4D+E, & -B \\
B, & 5B+C, & 10B+5C+D, & 10B+10C+5D+E, & y+4B+6C+4D+E
\end{array}\right|;$$

the calculation of the ten divisions of this determinant would however itself have been a work of some labour.

The last-mentioned determinant is $=y^5$; in fact, equating it to zero, we have the transformed equation corresponding to the system of equations

$$(1, 1, 1, 1, 1, \between x, 1)^5=0,$$

$$y=(x+1)B+(x^2+5x+4)C+(x^3+5x^2+10x+6)D+(x^4+5x^3+10x^2+10x+4)E.$$

But the first of these equations is $(x+1)^5=0$, and the second is

$$y=(x+1)\{B+(x+4)C+(x^2+4x+6)D+(x^3+4x^2+6x+4)E\},$$

so that for each of the five equal roots $x=-1$, we have $y=0$, or the transformed equation in y is $y^5=0$.

And since upon writing $a=b=c=d=e=f=1$ the transformed equation becomes $y^5=0$, it is clear that in the coefficient of any monomial product of B, C, D, E, the sum of the numerical coefficients of the several monomial products of a, b, c, d, e, f must be $=0$, which is the property above referred to as affording a verification of the calculated expression of the transformed equation.

The final result is that the equations

$$\begin{aligned}(a,\ b,\ c,\ d,\ e,\ f\between x,\ 1)^5 &= 0,\\ y = \ & (ax+b)B\\ & +(ax^2+5bx+4c)C\\ & +(ax^3+5bx^2+10cx+6d)D\\ & +(ax^4+5bx^2+10cx^2+10dx+4e)E\end{aligned}$$

give for the transformed equation in y

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E},\ \mathfrak{F}\between y,\ 1)^5=0,$$

where

$\frac{1}{5}\mathfrak{C}=$

	B^2		BC		BD	C^2		BE	CD		CE	D^2		DE		E^2
ac	$+2$	ab	$+6$	ae	$+4$	$+2$	af	$+1$	$+1$	bf	$+4$	$+2$	cf	$+6$	df	$+2$
b^2	-2	bc	-6	bd	-4	$+10$	be	-1	$+15$	ce	-4	$+10$	de	-6	e^2	-2
				c^2		-12	cd		-16	d^2		-12				
	± 2		± 6		± 4	± 12		± 1	± 16		± 4	± 12		± 6		± 2

$\frac{1}{5}\mathfrak{D}=$

	B^3		B^2C		B^2D	BC^2		B^2E	BCD	C^3		BCE	BD^2	C^2D
a^2d	$+2$	a^2e	$+4$	a^2f	$+1$	$+1$	abf	$+1$	$+10$	$+3$	acf	$\dots$	$+6$	$+4$
abc	-6	abd	$+2$	abe	$+3$	$+19$	ace	-4	-8	-4	ade	-12	-22	-46
b^3	$+4$	ac^2	-24	acd	$\dots$	-52	ad^2	$\dots$	-48	-20	b^2f	$+8$	$+4$	$+20$
		b^2c	$+18$	b^2d	-4	$+20$	b^2e	$+3$	$+30$	$+25$	bce	$+4$	$+20$	$+70$
				bc^2		$+12$	bcd		$+16$	-20	bd^2		-8	-80
							c^3			$+16$	c^2d			$+32$
	± 6		± 24		± 4	± 52		± 4	± 56	± 44		± 12	± 30	± 126

	BDE	C^2E	CD^2		BE^2	CDE	D^3		CE^2	D^2E		DE^2		E^3
adf	$\dots$	-6	-4	aef	-1	-10	-3	af^2	-1	-1	bf^2	-4	cf^2	-2
ae^2	-8	-4	-20	bdf	$+4$	$+8$	$+4$	bef	-3	-19	cef	-2	def	$+6$
bcf	$+12$	$+22$	$+46$	be^2	-3	-30	-25	cdf	$\dots$	$+52$	d^2f	$+24$	e^3	-4
bde	-4	-20	-70	c^2f		$+48$	$+20$	ce^2	$+4$	-20	de^2	-18		
c^2e		$+8$	$+80$	cde		-16	$+20$	d^2e		-12				
cd^2			-32	d^3			-16							
	± 12	± 30	± 126		± 4	± 56	± 44		± 4	± 52		± 24		± 6

$\mathfrak{G} =$

	B^4		B^2C		B^2D	B^2C^2		B^3E	B^2CD	BC^3		B^2CE	B^2D^2	BC^2D	C^4
a^3e	+ 1	a^2f	+ 1	a^2bf	+ 2	+ 9	a^2cf	+ 6	+ 10	− 2	a^2df	+ 12	− 6	...	− 4
a^2bd	− 4	a^2be	+ 7	a^2ce	+ 20	− 22	a^2de	− 6	− 18	− 10	a^2e^2	− 12	+ 10	− 20	+ 5
a^2c^2	...	a^2cd	− 16	a^2d^2	− 24	...	ab^2f	− 5	+ 17	+ 29	$abcf$	+ 28	+ 38	+ 48	+ 6
ab^2c	+ 6	ab^2d	− 22	ab^2e	− 18	+ 31	$abce$	+ 8	+ 96	− 92	$abde$	− 36	− 86	− 8	− 50
b^4	− 3	abc^2	+ 48	$abcd$	+ 32	− 56	abd^2	...	− 144	+ 40	ac^2e	+ 32	+ 100	+ 72	− 64
		b^3c	− 18	ac^3	...	+ 96	ac^2d	...	+ 128	+ 128	acd^2	...	− 24	+ 144	+ 160
				b^3d	− 12	− 70	b^3e	− 3	− 105	+ 75	b^3f	− 28	− 14	+ 60	+ 25
				b^3c^2		+ 12	b^2cd		+ 16	− 360	b^2ce	+ 4	− 70	− 240	+ 50
							bc^3			+ 192	b^2d^2		+ 52	− 440	...
											bc^2d			+ 384	− 320
											c^4				+ 192
	± 7		± 56		±54	± 148		± 14	± 267	± 464		± 76	± 200	± 708	± 438

	B^2DE	BC^2E	BCD^2	C^3D		B^2E^2	$BCDE$	BD^3	C^3E	C^2D^2		BCE^2	BD^2E	C^2DE	CD^3
a^2ef	+ 5	− 5	− 7	+ 1	a^2f^2	+ 1	− 1	− 1	− 1	+ 1	abf^2	+ 5	− 5	− 7	+ 1
$abdf$	− 8	+ 92	− 8	− 36	$abef$	− 5	+ 6	− 4	− 4	− 25	$acef$	− 8	+ 92	− 8	− 36
abe^2	− 29	− 67	+ 15	− 25	$acdf$	+ 20	+ 196	+ 28	+ 28	− 80	ad^2f	+ 60	+ 12	+ 72	− 20
ac^2f	+ 60	+ 12	+ 72	− 20	ace^2	− 14	− 56	+ 100	− 28	+ 10	ade^2	− 42	− 54	+ 78	+ 190
$acde$	− 12	+ 36	+ 196	+ 28	ad^2e	...	− 36	− 68	+ 60	+ 336	b^2ef	− 29	− 67	+ 15	− 25
ad^3	...	...	− 72	+ 240	b^2df	− 14	− 56	− 28	+ 100	+ 10	$bcdf$	− 12	+ 36	+ 196	+ 28
b^2cf	− 42	− 54	+ 78	+ 190	b^2e^2	+ 12	− 165	− 75	− 75	...	bce^2	+ 26	− 110	− 510	− 50
b^2de	+ 26	− 110	− 510	− 50	bc^2f		− 36	+ 60	− 68	+ 336	bd^3e		+ 96	− 60	− 280
bc^3e		+ 96	− 60	− 280	$bcde$		+ 148	− 140	− 140	− 940	c^3f			− 72	+ 240
bcd^2			+ 296	− 560	bd^3			+ 128	...	− 120	c^2de			+ 296	− 560
c^3d				+ 512	c^3e				+ 128	− 120	cd^3				+ 512
					c^2d^2					+ 592					
	± 91	± 236	± 657	± 971		± 33	± 350	± 316	± 316	± 1285		± 91	± 236	± 657	± 971

	BDE^2	C^2E^2	CD^2E	D^4		BE^3	CDE^2	D^3E		CE^3	D^2E^2		DE^3		E^4
aef^2	+ 12	− 6	...	− 4	adf^2	+ 6	+ 10	− 2	aef^2	+ 2	+ 9	af^3	+ 1	bf^3	+ 1
$adef$	+ 28	+ 38	+ 48	+ 6	ae^2f	− 5	+ 17	+ 29	bdf^2	+ 20	− 22	bef^2	+ 7	cef^2	− 4
ae^3	− 28	− 14	+ 60	+ 25	bef^2	− 6	− 18	− 10	be^2f	− 18	+ 31	cdf^2	− 16	d^2f^2	...
b^2f^2	− 12	+ 10	− 20	+ 5	$bdef$	+ 8	+ 96	− 92	c^2f^2	− 24	...	ce^2f	− 22	de^2f	+ 6
$bcef$	− 36	− 86	− 8	− 50	be^3	− 3	− 105	+ 75	$cdef$	+ 32	− 56	d^2ef	+ 48	e^4	− 3
bd^2f	+ 32	+ 100	+ 72	− 64	c^2ef		− 144	+ 40	ce^3	− 12	− 70	de^3	− 18		
bde^2	+ 4	− 70	− 240	+ 50	cd^2f		+ 128	+ 128	d^2f		+ 96				
c^3df		− 24	+ 144	+ 160	cde^2		+ 16	− 360	d^2e^2		+ 12				
c^2e^2		+ 52	− 440	...	d^3e			+ 192							
cd^2e			+ 384	− 320											
d^4				+ 192											
	± 76	± 200	± 708	± 438		± 14	± 267	± 464		± 54	± 148		± 56		± 7

$\mathfrak{F} =$

	B^5		B^4C		B^4D	B^3C^2		B^4E	B^3CD	B^2C^3
a^4f	+ 1	a^3bf	+ 15	a^3cf	+ 30	− 10	a^3df	+ 20	...	− 10
a^3be	− 5	a^3ce	− 20	a^3de	− 30	...	a^3e^2	− 20	...	...
a^3cd	...	a^3d^2	...	a^2b^2f	− 15	+ 100	a^2bcf	− 30	+ 320	− 60
a^2b^2d	+ 10	a^2b^2e	− 55	a^2bce	− 100	− 190	a^2bde	+ 30	− 360	+ 50
a^2bc^2	...	a^2bcd	+ 80	a^2bd^2	+ 120	...	a^2c^2e	...	− 400	+ 120
ab^3c	− 10	a^2c^3	...	a^2c^2d	...	+ 160	a^2cd^2	...	+ 480	...
b^5	+ 4	ab^3d	+ 70	ab^3e	+ 55	− 260	ab^3f	+ 15	− 140	+ 340
		ab^2c^2	− 120	ab^3cd	− 80	+ 540	ab^2ce	− 20	− 440	− 1020
		b^4c	+ 30	abc^3	...	− 480	ab^2d^2	...	+ 960	− 100
				b^4d	+ 20	+ 200	abc^2d	...	− 640	+ 960
				b^3c^2		− 60	ac^4	...	...	− 640
							b^4e	+ 5	+ 300	− 500
							b^3cd		− 80	+ 1900
							b^3c^3			− 1040
	± 15		± 195		± 225	± 1000		± 70	± 2060	± 3370

	B^3CE	B^3D^2	B^2C^2D	BC^4		B^3DE	B^2C^2E	B^2CD^2	BC^3D	C^5
a^3ef	...	+ 10	− 20	+ 5	a^3f^2	+ 5	− 5	− 5	+ 5	− 1
a^2bdf	+ 240	− 40	+ 20	− 80	a^2bef	− 30	+ 30	+ 60	− 130	+ 25
a^2be^2	− 240	− 50	+ 100	− 25	a^2cdf	+ 220	− 40	+ 380	− 380	+ 80
a^2c^2f	− 120	+ 300	+ 20	− 40	a^2ce^2	− 400	+ 40	− 200	+ 400	− 100
a^2cde	+ 120	− 600	+ 60	+ 200	a^2d^2e	+ 180	...	− 180	+ 300	...
a^2d^3	...	+ 360	...	...	ab^2df	− 40	+ 1070	− 640	+ 380	− 250
ab^2cf	− 140	− 220	+ 1190	+ 20	ab^2e^2	+ 185	− 1145	+ 25	+ 125	...
ab^2de	+ 240	+ 540	− 1980	+ 250	abc^2f	− 300	− 940	+ 1640	+ 940	− 360
abc^2e	− 160	− 500	− 3160	+ 320	$abcde$	+ 60	+ 1020	− 4380	− 1140	+ 1000
$abcd^2$	...	+ 120	+ 4080	− 800	abd^3	...	...	+ 3960	− 1200	...
ac^3d	...	...	− 1280	− 320	ac^3e	...	− 320	− 2000	− 2080	+ 960
b^4f	+ 80	+ 40	− 400	+ 500	ac^2d^2	...	...	+ 480	+ 960	− 1600
b^3ce	− 20	+ 200	+ 850	− 2750	b^2cf	+ 120	+ 160	− 620	+ 800	+ 750
b^3d^2		− 160	+ 2600	...	b^2de	− 80	+ 650	+ 2900	− 3500	...
b^2c^2d			− 2080	+ 5600	b^2c^2e		− 520	− 100	− 2600	− 300
bc^4				− 2800	b^2cd^2			− 1320	+ 14800	...
					bc^3d				− 7680	+ 4800
					c^5					− 2304
	± 680	± 1570	± 8920	± 6895		± 810	± 2970	± 9445	± 18710	± 7615

	B^2E^2	B^2CDE	B^2D^3	BC^3E	BC^2D	C^4D		B^2CE^2	B^2D^2E	BC^2DE	BCD^3	C^4E	C^3D^2
a^2bf^2	+ 5	+ 15	− 10	− 25	...	+ 5	a^2cf^2	− 50	+ 70	+ 10	− 50	− 10	+ 20
a^2cef	− 70	− 60	+ 200	+ 80	− 210	+ 30	a^2def	+ 110	+ 80	− 340	+ 160	+ 130	− 30
a^2d^2f	+ 100	+ 240	+ 100	− 200	− 180	+ 120	a^2e^3	− 80	− 200	+ 400	...	− 100	...
a^3de^2	− 40	− 240	− 300	+ 200	+ 600	− 150	ab^2f^2	+ 80	− 70	− 40	+ 20	− 25	...
ab^2ef	+ 30	− 30	− 90	+ 70	...	− 125	$abcef$	− 520	− 460	+ 620	+ 380	− 20	− 500
$abcdf$	− 100	+ 1380	− 640	+ 260	+ 1920	− 1180	abd^2f	+ 700	+ 460	+ 1240	− 760	− 1000	+ 540
$abce^2$	+ 70	− 2820	− 500	− 360	+ 450	+ 500	$abde^2$	− 190	+ 620	− 2140	− 200	+ 1000	+ 750
abd^3e	...	+ 1980	+ 1540	− 300	− 3180	+ 1500	ac^2df	− 400	+ 200	− 1720	+ 2840	− 80	− 880
ac^3f	...	− 1200	+ 1000	− 560	+ 1080	− 240	ac^2e^2	+ 280	−2000	− 2240	− 2000	+ 880	+ 1400
ac^2de	...	+ 240	− 3000	+ 240	− 8160	+ 3120	acd^2e	...	+1080	+ 2520	− 6440	− 1200	− 120
acd^3	...	...	+ 1440	...	+ 5040	− 4800	ad^4	...	...	...	+ 4320	...	− 3600
b^3df	+ 40	+ 240	+ 120	+ 2000	− 1800	+ 500	b^2ef	+ 160	+ 430	− 200	− 500	+ 125	...
b^2e^2	− 35	+ 975	+ 500	− 2125	...	...	b^2cdf	− 20	− 440	+ 5320	− 2240	+ 3000	− 700
b^2c^2f		− 60	− 400	− 1060	+ 1620	+ 2100	b^2ce^2	− 70	+ 650	− 2950	+ 2750	− 3500	...
b^2cde		− 660	+ 600	+ 3700	+ 2700	− 6500	b^2d^2e		− 420	+ 4800	+ 7400	...	− 3000
b^2d^3			− 560	...	+ 9600	...	bc^3f			− 3240	+ 1800	− 1680	+ 3960
bc^3e				− 1920	− 5400	− 4800	bc^2de			− 2040	−10200	+ 4400	− 5600
bc^2d^2					− 4080	+17600	bcd^3				+ 2720	...	+20400
c^4d						− 7680	c^4e					− 1920	− 7200
							c^3d^2						− 5440
	± 245	± 5070	± 5500	± 6550	±23010	± 25475		±1330	± 3590	± 14910	±22390	± 9535	±27070

	B^2DE^2	BC^2E^2	BCD^2E	BD^4	C^3DE	C^2D^3		B^2E^3	$BCDE^2$	BD^3E	C^3E^2	C^2D^2E	CD^4
a^2df^2	+ 50	− 70	− 10	+ 10	+ 50	− 20	a^2ef^2	− 5	− 15	+ 25	+ 10	...	− 5
a^2e^2f	− 80	+ 70	+ 40	+ 25	− 20	...	$abdf^2$	+ 70	+ 60	− 80	− 200	+ 210	− 30
$abcf^2$	− 110	− 80	+ 340	− 130	− 160	+ 30	abe^2f	− 30	+ 30	− 70	+ 90	...	+ 125
$abdef$	+ 520	+ 460	− 620	+ 20	− 380	+ 500	ac^2f^2	− 100	− 240	+ 200	− 100	+ 180	− 120
abe^3	− 160	− 430	+ 200	− 125	+ 500	...	$acdef$	+ 100	− 1380	− 260	+ 640	− 1920	+ 1180
ac^2ef	− 700	− 460	− 1240	+ 1000	+ 760	− 540	ace^3	− 40	− 240	− 2000	− 120	+ 1800	− 500
acd^2f	+ 400	− 200	+ 1720	+ 80	− 2840	+ 880	ad^2f	...	+ 1200	+ 560	− 1000	− 1080	+ 240
$acde^2$	+ 20	+ 440	− 5320	− 3000	+ 2240	+ 700	ad^2e^2	...	+ 60	+ 1060	+ 400	− 1620	− 2100
ad^3e	...	...	+ 3240	+ 1680	− 1800	− 3960	b^2cf^2	+ 40	+ 240	− 200	+ 300	− 600	− 150
b^2f^2	+ 80	+ 200	− 400	+ 100	...	...	b^2def	− 70	+ 2820	+ 360	+ 500	− 450	− 500
b^2cef	+ 190	− 620	+ 2140	− 1000	+ 200	− 750	b^2e^3	+ 35	− 975	+ 2125	− 500	...	...
b^2d^2f	− 280	+2000	+ 2240	− 880	+ 2000	− 1400	bc^2ef		− 1980	+ 300	− 1540	+ 3180	+ 1500
b^2de^2	+ 70	− 650	+ 2950	+ 3500	− 2750	...	bcd^2f		− 240	− 240	+ 3000	+ 8160	− 3120
bc^2df		−1080	− 2520	+ 1200	+ 6440	+ 120	$bcde^2$		+ 660	− 3700	− 600	− 2700	+ 6500
bc^2e^2		+ 420	− 4800	...	− 7400	+ 3000	bd^3e			+ 1920	...	+ 5400	+ 4800
bcd^3e			+ 2040	− 4400	+10200	+ 5600	c^3df				− 1440	− 5040	+ 4800
bd^4				+ 1920	...	+ 7200	c^3e^2				+ 560	− 9600	...
c^4f					− 4320	+ 3600	c^2d^2e					+ 4080	−17600
c^3de					− 2720	−20400	cd^4						+ 7680
c^2d^3						+ 5440							
	±1330	± 3590	±14910	± 9535	±22390	±27070		± 245	± 5070	± 6550	± 5500	±23010	±25475

	BCE^3	BD^2E^2	CD^2E^2	CD^3E	D^5		BDE^3	C^2E^3	CD^2E^2	D^4E
a^2f^3	− 5	+ 5	+ 5	− 5	+ 1	abf^3	...	− 10	+ 20	− 5
$abef^2$	+ 30	− 30	− 60	+ 130	− 25	$acef^2$	− 240	+ 40	− 20	+ 80
$acdf^2$	− 220	+ 40	− 380	+ 380	− 80	ad^2f^2	+ 120	− 300	− 20	+ 40
ace^2f	− 40	− 1070	+ 640	− 380	+ 250	ade^2f	+ 140	+ 220	− 1190	− 20
ad^2ef	+ 300	+ 940	− 1640	− 940	+ 360	ae^4	− 80	− 40	+ 400	− 500
ade^3	− 120	− 160	+ 620	− 800	− 750	b^2ef^2	+ 240	+ 50	− 100	+ 25
b^2df^2	+ 400	− 40	+ 200	− 400	+ 100	$bcdf^2$	− 120	+ 600	− 60	− 200
b^2e^2f	− 185	+ 1145	− 25	− 125	...	bce^2f	− 240	− 540	+ 1980	− 250
bc^2f^2	− 180	...	+ 180	− 300	...	bd^2ef	+ 160	+ 500	+ 3160	− 320
$bcdef$	− 60	− 1020	+ 4380	+ 1140	− 1000	bde^3	+ 20	− 200	− 850	+ 2750
bce^3	+ 80	− 650	− 2900	+ 3500	...	c^3f^2		− 360	...	...
bd^3f		+ 320	+ 2000	+ 2080	− 960	c^2def		− 120	− 4080	+ 800
bd^2e^2		+ 520	+ 100	+ 2600	+ 3000	c^2e^3		+ 160	− 2600	...
c^3ef			− 3960	+ 1200	...	cd^3f			+ 1280	+ 320
c^2d^2f			− 480	− 960	+ 1600	cd^2e^2			+ 2080	− 5600
c^2de^2			+ 1320	− 14800	...	d^4e				+ 2880
cd^3e				+ 7680	− 4800					
d^5					+ 2304					
	± 810	± 2970	± 9445	± 18710	± 7615		± 680	± 1570	± 8920	± 6895

	BE^4	CDE^3	D^3E^2		CE^4	D^2E^3		DE^4		E^5
acf^3	− 20	...	+ 10	adf^3	− 30	+ 10	aef^2	− 15	af^4	− 1
$adef^2$	+ 30	− 320	+ 60	ae^2f^2	+ 15	− 100	bdf^2	+ 20	bef^3	+ 5
ae^2f	− 15	+ 140	− 340	bcf^3	+ 30	...	be^2f^2	+ 55	cdf^3	...
b^2f^3	+ 20	...	...	$bdef^2$	+ 100	+ 190	c^2f^3	...	ce^2f^2	− 10
$bcef^2$	− 30	+ 360	− 50	be^3f	− 55	+ 260	$cdef^2$	− 80	d^2ef^2	...
bd^2f^2	...	+ 400	− 120	c^2ef^2	− 120	...	ce^2f	− 70	de^3f	+ 10
bde^2f	+ 20	+ 440	+ 1020	cd^2f^2	...	− 160	d^3f^2	...	e^5	− 4
be^4	− 5	− 300	+ 500	cde^2f	+ 80	− 540	d^2e^2f	+ 120		
c^2df^2		− 480	...	ce^4	− 20	− 200	de^4	− 30		
c^2e^2f		− 960	+ 100	d^3ef		+ 480				
cd^2ef		+ 640	− 960	d^2e^3		+ 60				
cde^3		+ 80	− 1900							
d^4f			+ 640							
d^3e^2			+ 1040							
	± 70	± 2060	± 3370		± 225	± 1000		± 195		± 15

Upon writing $(B, C, D, E) = (x^3, xy^2, xy^2, y^3)$, the foregoing values of $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ become covariants of the quintic $(a, b, c, d, e, f \between x, y)^5$. In fact [using for the covariants of the quintic the notation of 141 and 143, A the quintic itself, &c.], we have

$$\tfrac{1}{5}\mathfrak{C} = 2C,$$

$$\tfrac{1}{5}\mathfrak{D} = 2F,$$

$$\tfrac{1}{5}\mathfrak{E} = A^2B - 3C^2,$$

$$\mathfrak{F} = A^2E - 2CF.$$

This [the circumstance that the values thus become covariants of the quintic] is a further verification of the foregoing result.

I will conclude by showing how the formula may be applied to the reduction of the general quintic equation to Mr Jerrard's form $x^5+ax+b=0$. It was long ago remarked by Professor Sylvester that Tschirnhausen's Transformation, in its original form, gave the means of effecting this reduction. In fact, if the transforming equation be

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4,$$

then the equation in y is of the form

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0,$$

where $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are, in regard to α, β, γ, δ, ϵ, of the degrees 1, 2, 3, 4, 5 respectively. And by assuming, say α a linear function of β, γ, δ, ϵ, we may make $B=0$, and we have then $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ functions of the degrees 2, 3, 4, 5 respectively of the quantities β, γ, δ, ϵ: and these can be determined by means of a quadric equation and a cubic equation in such manner that $\mathfrak{C}=0$, $\mathfrak{D}=0$, in which case the equation in y will be of the required form. For considering β, γ, δ, ϵ as the coordinates of a point in space, the equations $\mathfrak{C}=0$, $\mathfrak{D}=0$ will be the equations of a quadric surface and a cubic surface respectively, and if β, γ, δ, ϵ be the coordinates of a point on the curve of intersection, the required conditions will be satisfied. And by combining with the equation of the quadric surface, the equation of any tangent plane thereto (or by the different process which is made use of in the sequel), we may, by means of a quadric equation, find a generating line of the quadric surface, and then, by means of a cubic equation, a point of intersection of this line with the cubic surface, i.e. a point the coordinates whereof give the required values β, γ, δ, ϵ. And similarly for the new form of Tschirnhausen's Transformation; the only difference being that, starting with an equation in y which contains the four arbitrary quantities B, C, D, E, we obtain in the first instance an equation

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0$$

wanting the second term. And then B, C, D, E are to be so determined that $\mathfrak{C}=0$, $\mathfrak{D}=0$.

To proceed with the reduction, I write the foregoing value of $\mathfrak{C}$ in the form

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{cccc} 4ac-4b^2, & 6ad-6bc, & 4ae-4bd, & af-be \\ 6ad-6bc, & 4ae+20bd-24c^2, & af+15be-16cd, & 4bf-4ce \\ 4ae-4bd, & af+15be-16cd, & 4bf+20ce-24d^2, & 6cf-6de \\ af-be, & 4bf-4ce, & 6cf-6de, & 4df-4e^2 \end{array}\right) \between (B, C, D, E)^2,$$

which for shortness may be represented by

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{cccc} \mathrm{a}, & \mathrm{h}, & \mathrm{g}, & \mathrm{l} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f}, & \mathrm{m} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c}, & \mathrm{n} \\ \mathrm{l}, & \mathrm{m}, & \mathrm{n}, & \mathrm{p} \end{array}\right) \between (B, C, D, E)^2,$$

or say

$$\tfrac{2}{5}\mathfrak{C} = (\Omega \between B,\ C,\ D,\ E)^2.$$

Now, by a formula in my memoir "On the Automorphic Linear Transformation of a Bipartite Quadric Function," *Phil. Trans.*, vol. CXLVIII. (1858), see p. 44, [153], if Υ denote any skew symmetric matrix of the order 4, then if

$$(B,\ C,\ D,\ E) = (\Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\,\Omega \between B',\ C',\ D',\ E'),$$

in which formula Ω^{-1}, $\Omega - \Upsilon$, $(\Omega + \Upsilon)^{-1}$, Ω are all matrices which are to be compounded together into a single matrix, we have identically

$$(\Omega \between B,\ C,\ D,\ E)^2 = (\Omega \between B',\ C',\ D',\ E')^2.$$

Let Q denote the determinant $|\,\Omega + \Upsilon\,|$, then the terms of the matrix $(\Omega + \Upsilon)^{-1}$ are respectively divided by Q, and we may write

$$(\Omega + \Upsilon)^{-1} = \frac{1}{Q} \cdot Q(\Omega + \Upsilon)^{-1},$$

where $Q(\Omega + \Upsilon)^{-1}$ is the matrix obtained from the matrix $(\Omega + \Upsilon)^{-1}$ by multiplying each term by Q, the terms of $Q(\Omega + \Upsilon)^{-1}$ being thus rational and integral functions of the terms of the matrix $(\Omega + \Upsilon)$. Hence if, instead of the before-mentioned relation between $(B,\ C,\ D,\ E)$ and $(B',\ C',\ D',\ E')$, we assume

$$(B,\ C,\ D,\ E) = (\Omega^{-1}(\Omega - \Upsilon)\,Q\,(\Omega + \Upsilon)^{-1}\,\Omega \between B',\ C',\ D',\ E'),$$

we find

$$(\Omega \between B,\ C,\ D,\ E)^2 = Q^2\,(\Omega \between B',\ C',\ D',\ E')^2.$$

If here the matrix Υ is such that we have $Q = 0$, i.e. Det. $(\Omega + \Upsilon) = 0$ (which is a quadric relation between the terms of the skew matrix, that is, each term is contained therein in the first and second powers only), then the equation becomes

$$(\Omega \between B,\ C,\ D,\ E)^2 = 0.$$

It is clear that this can only be the case in consequence of the coefficients of transformation in the equation

$$(B,\ C,\ D,\ E) = (\Omega^{-1}(\Omega - \Upsilon)\,Q\,(\Omega + \Upsilon)^{-1}\,\Omega \between B',\ C',\ D',\ E')$$

being such that there shall exist at least two linear relations between the quantities $(B,\ C,\ D,\ E)$, and I assume (without stopping to prove it) that they are such that the number of such linear relations is in fact two. That is, the last-mentioned equation establishes between the quantities $(B,\ C,\ D,\ E)$ two linear relations, in virtue whereof $\mathfrak{C} = 0$. And this being so, we may, without loss of generality, write $D' = 0$, $E' = 0$, or put

$$(B,\ C,\ D,\ E) = (\Omega^{-1}(\Omega - \Upsilon)\,Q\,(\Omega + \Upsilon)^{-1}\,\Omega \between B',\ C',\ 0,\ 0);$$

so that B, C, D, E are linear functions of B', C', such that $\mathfrak{C}=0$. And then substituting these values for (B, C, D, E), we find $\mathfrak{D}$ a cubic function of B', C'; so that, putting $\mathfrak{D}=0$, we have a cubic equation to determine the ratio $B' : C'$.

The foregoing reasoning presents no real difficulty, but it is expressed by means of a very condensed notation, and it may be proper to illustrate it by the case of the quadric function $x^2+y^2+z^2$. Considering the equations

$$\begin{aligned} x &= (1+\lambda^2-\mu^2-\nu^2)\,x' + 2(\lambda\mu-\nu)\,y' + 2(\lambda\nu+\mu)\,z', \\ y &= 2(\lambda\mu+\nu)\,x' + (1-\lambda^2+\mu^2-\nu^2)\,y' + 2(\mu\nu-\lambda)\,z', \\ z &= 2(\nu\lambda-\mu)\,x' + 2(\mu\nu+\lambda)\,y' + (1-\lambda^2-\mu^2+\nu^2)\,z', \end{aligned}$$

these equations, if the expressions for x, y, z had been divided by $1+\lambda^2+\mu^2+\nu^2$, would have given

$$x^2+y^2+z^2 = x'^2+y'^2+z'^2.$$

Hence they actually do give

$$x^2+y^2+z^2 = (1+\lambda^2+\mu^2+\nu^2)^2\,(x'^2+y'^2+z'^2);$$

or if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

they give

$$x^2+y^2+z^2=0.$$

But if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

then

$$\begin{aligned} &1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu-\nu) : 2(\lambda\nu+\mu) \\ &= 2(\lambda\mu+\nu) : 1-\lambda^2+\mu^2-\nu^2 : 2(\mu\nu-\lambda) \\ &= 2(\nu\lambda-\mu) : 2(\mu\nu+\lambda) : 1-\lambda^2-\mu^2+\nu^2; \end{aligned}$$

so that we have

$$x : y : z = 1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu+\nu) : 2(\nu\lambda-\mu),$$

which is the same result as would have been found by writing $y'=z'=0$, and which comes to saying that x, y, z are not independent, but are connected by two linear relations.

The equation Det. $(\Omega+\Upsilon)=0$, written at length, will be

$$\begin{vmatrix} \mathrm{a}, & \mathrm{h}-\tau, & \mathrm{g}+\sigma, & \mathrm{l}+\lambda \\ \mathrm{h}+\tau, & \mathrm{b}, & \mathrm{f}-\rho, & \mathrm{m}+\mu \\ \mathrm{g}-\sigma, & \mathrm{f}+\rho, & \mathrm{c}, & \mathrm{n}+\nu \\ \mathrm{l}-\lambda, & \mathrm{m}-\mu, & \mathrm{n}-\nu, & \mathrm{p} \end{vmatrix} = 0,$$

where $\lambda, \mu, \nu, \rho, \sigma, \tau$ are the arbitrary constituents of the skew matrix; or developing, this is

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & p \end{vmatrix}$$

$$+\left(\begin{array}{cccccc} bc-f^2, & fg-ch, & hf-bg, & mg-nh, & fm-bn, & -fn+cm \\ fg-ch, & ca-g^2, & gh-af, & -gl+an, & nh-lf, & gn-cl \\ hf-bg, & gh-af, & ab-h^2, & hl-am, & -hm+bl, & lf-mg \\ mg-nh, & fm-bn, & -fn+cm, & ap-l^2, & ph-lm, & pg-ln \\ -gl+an, & nh-lf, & gn-cl, & ph-lm, & bp-m^2, & pf-mn \\ hl-am, & -hm+bl, & lf-mg, & pg-ln, & pf-mn, & cp-n^2 \end{array}\right)(\lambda, \mu, \nu, \rho, \sigma, \tau)^2$$

$$+(\lambda\rho+\mu\sigma+\nu\tau)^2=0,$$

the first term whereof, substituting for (a, b, c, f, g, h, l, m, n, p) their values, is in fact equal to the discriminant a^4f^4+ &c. of the quintic $(a, b, c, d, e, f \between X, Y)^5$. There is no loss of generality in putting all but two of the quantities $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ equal to zero; in fact this leaves in the formulæ a single arbitrary quantity, which is the right number, since the ratios $B : C : D : E$ have to satisfy only the two conditions $\mathfrak{C}=0$, $\mathfrak{D}=0$.

[An Addition of a half page, dated Nov. 10, 1862, and referring to the Memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order", 268, is printed at the end of that paper, *ante* p. 324.]

276.

ON THE ANALYTICAL THEORY OF THE CONIC.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLII. (for the year 1862), pp. 639—662. Received May 8,—Read May 15, 1862.]

THE decomposition into its linear factors of a decomposable quadric function cannot be effected in a symmetrical manner otherwise than by formulæ containing supernumerary arbitrary quantities; thus, for a binary quadric (which of course is always decomposable) we have

$$(a,\ b,\ c \between x,\ y)^2 = \frac{1}{(a,\ b,\ c \between x',\ y')^2} \text{ Prod. } \{(a,\ b,\ c \between x,\ y \between x',\ y') \pm \sqrt{ac-b^2}\,(xy'-x'y)\}\,;$$

or the expression for a linear factor is

$$\frac{1}{\sqrt{(a,\ b,\ c \between x',\ y')^2}}\{(a,\ b,\ c \between x,\ y \between x',\ y') \pm \sqrt{ac-b^2}\,(xy'-x'y)\},$$

which involves the arbitrary quantities (x', y'). And this appears to be the reason why, in the analytical theory of the conic, the questions which involve the decomposition of a decomposable ternary quadric have been little or scarcely at all considered: thus, for instance, the expressions for the coordinates of the points of intersection of a conic by a line (or say the line-equations of the two ineunts), and the equations for the tangents (separate each from the other) drawn from a given point not on the conic, do not appear to have been obtained. These questions depend on the decomposition of a decomposable ternary quadric, which decomposition itself depends on that for the simplest case, when the quadric is a perfect square. Or we may say that in the first instance they depend on the transformation of a given quadric function $U = (* \between x,\ y,\ z)^2$ into the form $W^2 + V$, where W is a linear function, given save as to a constant factor (that is, $W = 0$ is the equation of a given line), and V is a decomposable quadric function, which is ultimately decomposed into its linear factors, $= QR$, so that we have $U = W^2 + QR$.

The formula for this purpose, which is exhibited in the eight different forms I, II, III, IV, I (bis), II (bis), III (bis), IV (bis), is the analytical basis of the whole theory; and the greater part of the memoir relates to the establishment of these forms.

The solution of the geometrical questions above referred to is (as shown in the memoir) involved in and given immediately by these forms. It is also shown that the formulæ are greatly simplified in the case e.g. of tangents drawn to a conic from a point in a conic having double contact with the first-mentioned conic, and that in this case they lead to the linear Automorphic Transformation of the ternary quadric. The memoir concludes with some formulæ relating to the case of two conics, which however is treated of in only a cursory manner.

Article Nos. 1—17, *relating to a single conic.*

1. The point-equation of the conic is

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0,$$

which expresses that the point $(x,\ y,\ z)$ is an ineunt of the conic.

The line-equation of the same conic is

$$\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix} = 0,$$

or putting

$$(A,\ B,\ C,\ F,\ G,\ H) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

the line-equation is

$$(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2 = 0,$$

which expresses that the line $(\xi,\ \eta,\ \zeta)$ (that is, the line the point-equation whereof is $\xi x + \eta y + \zeta z = 0$) is a tangent of the conic. We are thus in the analytical theory of the conic concerned with the quadrics $(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2$ and $(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2$, which are the characteristics or *nilfactums* of these equations respectively.

2. I put also

$$K = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

or, what is the same thing,

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh.$$

3. It may be convenient to notice that when $(a,\ldots \between x, y, z)^2$ breaks up into factors, the conic the equation whereof is $(a,\ldots \between x, y, z)^2 = 0$, becomes a pair of lines; and that when $(a,\ldots \between x, y, z)^2$ is a perfect square, the conic becomes a pair of coincident lines, or say a *twofold* line. But a pair of lines, distinct or coincident, cannot be represented by a line-equation. The analytical formulæ presently given show that in the former case $(A,\ldots \between \xi, \eta, \zeta)^2$ is the square of a linear function, which equated to zero gives the line-equation of the point of intersection of the two lines, or node of the conic; and the equation $(A,\ldots \between \xi, \eta, \zeta)^2 = 0$ accordingly represents such point considered as a pair of coincident points, or say a *twofold* point. But in the latter case, where the conic is a twofold line, $(A,\ldots \between \xi, \eta, \zeta)^2$ is identically equal to zero, and the line-equation $(A,\ldots \between \xi, \eta, \zeta)^2 = 0$ is a mere identity $0 = 0$, thus ceasing to have any signification at all. And the like remarks apply to the conic as represented by the line-equation $(A,\ldots \between \xi, \eta, \zeta)^2 = 0$, the conic here breaking up into a pair of distinct or coincident points, &c.

4. It is proper to remark also that

$$(a, \ldots \between x', y', z' \between x, y, z) = 0$$

is the equation of the polar of the point (x', y', z') in regard to the conic, and that

$$(A,\ldots \between \xi', \eta', \zeta' \between \xi, \eta, \zeta) = 0$$

is the line-equation of the pole of the line (ξ', η', ζ'); or, what is the same thing, the point-coordinates of the pole are

$$A\xi' + H\eta' + G\zeta' : H\xi' + B\eta' + F\zeta' : G\xi' + F\eta' + C\zeta'.$$

5. The inverse matrix is

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right)^{-1}, = \frac{1}{K}\left(\begin{array}{ccc} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{array}\right);$$

but it is convenient to disregard the factor $\frac{1}{K}$, and speak of (A, B, C, F, G, H) as the inverse or reciprocal coefficients. The equation just written down implies the relations $Aa + Hh + Gg = K$, $Ah + Hb + Gf = 0$, &c., which may be arranged in two different ways as a system of nine equations.

6. We have also

$$(BC - F^2, CA - G^2, AB - H^2, GH - AF, HF - BG, FG - CH) = K(a, b, c, f, g, h),$$

and

$$ABC - AF^2 - BG^2 - CH^2 + 2FGH = K^2,$$

which are well-known theorems.

7. I notice also the theorem

$$(a,..\between x,\ y,\ z)^2.(a,..\between x',\ y',\ z')^2-[(a,..\between x,\ y,\ z\between x',\ y',\ z')]^2$$
$$=(A,..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2,$$

which is much used in the sequel: it may be mentioned, in passing, that this is included in the more general theorem

$$\begin{vmatrix} (a,..\between x,\ y,\ z\between l,\ m,\ n), & (a,..\between x',\ y',\ z'\between l,\ m,\ n) \\ (a,..\between x,\ y,\ z\between l',\ m',\ n'), & (a,..\between x',\ y',\ z'\between l,\ m,\ n) \end{vmatrix}$$
$$=(A,..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y\between mn'-m'n,\ nl'-n'l,\ lm'-l'm),$$

which is at once deducible from

$$\begin{vmatrix} Ll+Mm+Nn, & L'l+M'm+N'n \\ Ll'+Mm'+Nn', & L'l'+M'm'+N'n' \end{vmatrix}$$
$$=(MN'-M'N)(mn'-m'n)+(NL'-N'L)(nl'-n'l)+(LM'-L'M)(lm'-l'm),$$

by writing therein

$$(L,\ M,\ N)=(ax+hy+gz,\ hx+by+fz,\ gx+fy+cz),$$
$$(L',\ M',\ N')=(ax'+hy'+gz',\ hx'+by'+fz',\ gx'+fy'+cz').$$

8. Suppose now that

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2$$

breaks up into factors, or say that we have

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=2\,(ax+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z),$$

the values of the coefficients $(a,..)$ then are

$$(a,\ b,\ c,\ f,\ g,\ h)=(2\alpha\alpha',\ 2\beta\beta',\ 2\gamma\gamma',\ \beta\gamma'+\beta'\gamma,\ \gamma\alpha'+\gamma'\alpha,\ \alpha\beta'+\alpha'\beta),$$

and forming from these the inverse coefficients $(A,..)$ and the discriminant K, we find

$$(A,\ B,\ C,\ F,\ G,\ H)=-(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)^2.$$
$$K=0.$$

9. The last-mentioned equation, $K=0$, is the condition in order that $(a,..\between x,\ y,\ z)^2$ may break up into factors; and when it does so, we have

$$(A,..\between \xi,\ \eta,\ \zeta)^2=-[(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta\between \xi,\ \eta,\ \zeta)]^2,$$

that is, $(a,..\between x,\ y,\ z)^2$ breaking up into factors, $(A,..\between \xi,\ \eta,\ \zeta)^2$ is a perfect square; and equating it to zero, we have

$$[(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta\between \xi,\ \eta,\ \zeta)]^2=0;$$

which, $(\xi,\ \eta,\ \zeta)$ being line-coordinates, gives (as a two-fold point) the point of intersection of the lines $(\alpha,\ \beta,\ \gamma)$, $(\alpha',\ \beta',\ \gamma')$, that is, the lines $\alpha x+\beta y+\gamma z=0$, $\alpha' x+\beta' y+\gamma' z=0$.

10. If $(a,..\between x, y, z)^2$ is a perfect square, then $\alpha' : \beta' : \gamma' = \alpha : \beta : \gamma$; whence not only, as before, $K=0$, but the coefficients (A, B, C, F, G, H) all vanish (this implies the first-mentioned condition, $K=0$); and the line-equation $(A,..\between \xi, \eta, \zeta)^2=0$ becomes the mere identity $0=0$.

11. Conversely if $K=0$, then $(a,..\between x, y, z)^2$ breaks up into factors; and if (A, B, C, F, G, H) all vanish, then $(a,..\between x, y, z)^2$ is a perfect square. The conclusions stated *ante*, No. 3, are thus sustained.

12. I assume, first, that $(a,..\between x, y, z)^2$ is a perfect square (No. 13); and secondly, that it breaks up into factors (No. 14); and I proceed to inquire how in the one case the root, and in the other case the factors, can be determined in a symmetrical form.

13. Considering the before-mentioned identical equation

$$(a,..\between x, y, z)^2.(a,..\between x', y', z')^2-[(a,..\between x,y,z\between x', y', z')]^2=(A,..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2,$$

if $(a,..\between x, y, z)^2$ is a perfect square, then by what precedes, the right-hand side of the equation vanishes, and we have

$$(a,..\between x, y, z)^2=\frac{[(a,..\between x, y, z\between x', y', z')]^2}{(a,..\between x', y', z')^2};$$

and the root of $(a,..\between x, y, z)^2$ is thus seen to be

$$=\pm\frac{(a,..\between x, y, z\between x', y', z')}{\sqrt{(a,..\between x', y', z')^2}},$$

an expression which involves the quantities (x', y', z'), the values whereof may be assumed at pleasure without altering the value of the expression. For instance, assuming for (x', y', z') the values $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ successively, the different values of the expression are

$$\frac{ax+hy+gz}{\sqrt{a}},\quad \frac{hx+by+fz}{\sqrt{b}},\quad \frac{gx+fy+cz}{\sqrt{c}}.$$

But if, as assumed, $(a,..\between x, y, z)^2$ be a perfect square $=(\alpha x+\beta y+\gamma z)^2$, then

$$(a, b, c, f, g, h)=(\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta),$$

and each of the foregoing values becomes equal to the root $\alpha x+\beta y+\gamma z$. It is somewhat singular that it is not possible to obtain symmetrical formulæ without employing in this manner supernumerary arbitrary quantities such as (x', y', z').

14. Next, if $(a, ...\between x, y, z)^2$, instead of being a perfect square, only breaks up into factors, then in the foregoing identical equation the right-hand side is a perfect square, and by the formula just obtained its value is

$$\frac{[(A,..\between X, Y, Z\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)]^2}{(A,..\between X, Y, Z)^2},$$

where (X, Y, Z) are supernumerary arbitrary quantities. The identical equation then gives

$$(a,..\between x,y,z)^2=\frac{1}{(a,...\between x',y',z')^2}\left\{[(a,..\between x,y,z\between x',y',z')]^2+\frac{[(A,..\between X,Y,Z\between yz'-y'z,zx'-z'x,xy'-x'y)]^2}{(A,..\between X,\ Y,\ Z)^2}\right\},$$

and consequently

$$(a,..\between x,\ y,\ z)^2=\frac{1}{(a,..\between x',\ y',\ z')^2}\text{ Product of}$$

$$\left\{(a,..\between x,\ y,\ z\between x',\ y',\ z')\pm\frac{(A,..\between X,\ Y,\ Z\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)}{\sqrt{-(A,..\between X,\ Y,\ Z)^2}}\right\},$$

a formula which exhibits the decomposition of $(a,..\between x,\ y,\ z)^2$ assumed to be a function which breaks up into factors; the formula contains the two sets of supernumerary arbitrary quantities $(x',\ y',\ z')$ and $(X,\ Y,\ Z)$. It will be remembered that $(A,..)$ denotes the system of inverse or reciprocal coefficients $(bc-f^2,..)$.

15. Consider the formula

$$(a,\ b,\ c,\ f,\ g,\ h\between \eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2=(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}\between \xi',\ \eta',\ \zeta')^2,$$

which gives

$$\begin{aligned}
\mathrm{a} &= c\eta^2+b\zeta^2-2f\eta\zeta,\\
\mathrm{b} &= a\zeta^2+c\xi^2-2g\zeta\xi,\\
\mathrm{c} &= b\xi^2+a\eta^2-2h\xi\eta,\\
\mathrm{f} &= -a\eta\zeta-f\xi^2+g\xi\eta+h\xi\zeta,\\
\mathrm{g} &= -b\zeta\xi+f\xi\eta-g\eta^2+h\eta\zeta,\\
\mathrm{h} &= -c\xi\eta+f\xi\zeta+g\eta\zeta-h\zeta^2;
\end{aligned}$$

and from these we deduce

$$\left(\begin{vmatrix}\mathrm{a}, & \mathrm{h}, & \mathrm{g}\\ \mathrm{h}, & \mathrm{b}, & \mathrm{f}\\ \mathrm{g}, & \mathrm{f}, & \mathrm{c}\end{vmatrix}\between \xi,\ \eta,\ \zeta\right)=(0,\ 0,\ 0),$$

viz. $\mathrm{a}\xi+\mathrm{h}\eta+\mathrm{g}\zeta=0$, &c.

Also

$$(\mathrm{bc}-\mathrm{f}^2,\ \mathrm{ca}-\mathrm{g}^2,\ \mathrm{ab}-\mathrm{h}^2,\ \mathrm{gh}-\mathrm{af},\ \mathrm{hf}-\mathrm{bg},\ \mathrm{fg}-\mathrm{ch})=(\xi,\ \eta,\ \zeta)^2.(A,\ B,\ C,\ F,\ G,\ H\between \xi,\ \eta,\ \zeta)^2,$$

that is

$$\mathrm{bc}-\mathrm{f}^2=\xi^2(A,\ B,\ C,\ F,\ G,\ H\between \xi,\ \eta,\ \zeta)^2,\ \&c.$$

Hence also

$$(\mathrm{bc}-\mathrm{f}^2,..\between l,\ m,\ n)^2 \qquad =(l\xi+m\eta+n\zeta)^2(A,..\between \xi,\ \eta,\ \zeta)^2,$$

and

$$(\mathrm{bc}-\mathrm{f}^2,..\between l,\ m,\ n\between l',\ m',\ n')=(l\xi+m\eta+n\zeta)(l'\xi+m'\eta+n'\zeta)(A,..\between \xi,\ \eta,\ \zeta)^2;$$

and moreover

$$\mathrm{abc}-\mathrm{af}^2-\mathrm{bg}^2-\mathrm{ch}^2+2\mathrm{fgh}=0.$$

16. The last equation shows that $(a, \ldots \between \eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^2$, considered as a function of (ξ', η', ζ'), breaks up into factors. Or since the expression is not altered by interchanging (ξ', η', ζ') and (ξ, η, ζ), the same expression, considered as a function of (ξ, η, ζ), breaks up into factors. It is in fact easy to see that any quantic whatever, $(* \between \eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^m$, considered as a function of (ξ, η, ζ), breaks up into linear factors; for in virtue of the equation $\xi'(\eta\zeta' - \eta'\zeta) + \eta'(\zeta\xi' - \zeta'\xi) + \zeta'(\xi\eta' - \xi'\eta) = 0$, any one of the quantities $\eta\zeta' - \eta'\zeta$, $\zeta\xi' - \zeta'\xi$, $\xi\eta' - \xi'\eta$ can be expressed as a linear function of the other two; so that the quantic can be expressed as a linear function of any two of the three quantities; and *quà* homogeneous function of two quantities, it of course breaks up into factors, linear functions of these two quantities.

We may in all the formulæ interchange (x', y', z') and (x, y, z), writing (a′, b′, c′, f′, g′, h′) in the place of (a, b, c, f, g, h).

17. Putting, in like manner,

$$(A, B, C, F, G, H \between yz' - y'z,\ zx' - z'x,\ xy' - x'y)^2$$
$$= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between x', y', z')^2,$$

so that

$$\begin{aligned}
\mathfrak{A} &= Cy^2 + Bz^2 - 2Fyz,\\
\mathfrak{B} &= Az^2 + Cx^2 - 2Gzx,\\
\mathfrak{C} &= Bx^2 + Ay^2 - 2Hxy,\\
\mathfrak{F} &= -Ayz - Fx^2 + Gxy + Hzx,\\
\mathfrak{G} &= -Bzx + Fxy - Gy^2 + Hyz,\\
\mathfrak{H} &= -Cxy + Fzx + Gyz - Hz^2,
\end{aligned}$$

we obtain

$$\left(\begin{array}{ccc|} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C} \end{array} \between x, y, z \right) = (0, 0, 0),$$

viz.

$$\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z = 0, \text{ \&c.}$$

Also

$$(\mathfrak{B}\mathfrak{C} - \mathfrak{F}^2,\ \mathfrak{C}\mathfrak{A} - \mathfrak{G}^2,\ \mathfrak{A}\mathfrak{B} - \mathfrak{H}^2,\ \mathfrak{G}\mathfrak{H} - \mathfrak{A}\mathfrak{F},\ \mathfrak{H}\mathfrak{F} - \mathfrak{B}\mathfrak{G},\ \mathfrak{H}\mathfrak{G} - \mathfrak{C}\mathfrak{G})$$
$$= (x, y, z)^2 . K(a, b, c, f, g, h \between x, y, z)^2;$$

that is

$$\mathfrak{B}\mathfrak{C} - \mathfrak{F}^2 = x^2 K(a, b, c, f, g, h \between x, y, z)^2, \text{ \&c.};$$

whence also

$$(\mathfrak{B}\mathfrak{C} - \mathfrak{F}^2, \ldots \between \lambda, \mu, \nu)^2 \qquad = (\lambda x + \mu y + \nu z)^2 . K(a, \ldots \between x, y, z)^2,$$
$$(\mathfrak{B}\mathfrak{C} - \mathfrak{F}^2, \ldots \between \lambda, \mu, \nu \between \lambda', \mu', \nu') = (\lambda x + \mu y + \nu z)(\lambda' x + \mu' y + \nu' z) . K(a, \ldots \between x, y, z)^2;$$

and moreover

$$\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{F}^2 - \mathfrak{B}\mathfrak{G}^2 - \mathfrak{C}\mathfrak{H}^2 + 2\mathfrak{F}\mathfrak{G}\mathfrak{H} = 0.$$

The last equation shows that $(A, \ldots \between yz' - y'z,\ zx' - z'x,\ xy' - x'y)^2$, considered as a function of (x', y', z'), breaks up into factors, or (what is the same thing) this expression, considered as a function of (x, y, z), breaks up into factors; we may in all the formulæ interchange (x, y, z) and (x', y', z'), writing $(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}')$ in the place of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$.

Article Nos. 18 *to* 28, *relating to a single conic in connexion with a point or line.*

18. I apply the decomposition formula to the function $(A, \ldots \between yz' - y'z, \ldots)^2$, which, considered as a function of (x, y, z), breaks up into factors. We have

$$(A, \ldots \between yz' - y'z, \ldots)^2 = (\mathfrak{A}', \ldots \between x, y, z)^2$$

$$= \frac{1}{(\mathfrak{A}', \ldots \between l, m, n)^2} \text{ Product of}$$

$$(\mathfrak{A}', \ldots \between l, m, n \between x, y, z) \pm \frac{(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between mz - ny, \ldots \between \lambda, \mu, \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between \lambda, \mu, \nu)^2}}.$$

But we have

$$(\mathfrak{A}', \ldots \between l, m, n)^2 = (A, \ldots \between mz' - ny', \ldots)^2,$$

$$(\mathfrak{A}', \ldots \between l, m, n \between x, y, z) = (A, \ldots \between mz' - ny', \ldots \between yz' - y'z, \ldots),$$

$$(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between mz - ny, \ldots \between \lambda, \mu, \nu)$$

$$= [x'(mz - ny) + y'(nx - lz) + z'(ly - mx)](\lambda x' + \mu y' + \nu z')\ K(a, \ldots \between x', y', z')^2,$$

$$(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between \lambda, \mu, \nu)^2 = (\lambda x' + \mu y' + \nu z')^2 K(a, \ldots \between x', y', z')^2,$$

and thence

$$\frac{(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between mz - ny, \ldots \between \lambda, \mu, \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}' - \mathfrak{F}'^2, \ldots \between \lambda, \mu, \nu)^2}} = \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} \sqrt{-K(a, \ldots \between x', y', z')^2},$$

whence we have

$$(A, \ldots \between yz' - y'z, \ldots)^2 = \frac{1}{(A, \ldots \between mz' - ny')^2} \text{ Product of}$$

$$(A, \ldots \between mz' - ny', \ldots \between yz' - y'z, \ldots) \pm \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} \sqrt{-K(a, \ldots \between x', y', z')^2};$$

and the identical equation

$$(a, \ldots \between x, y, z)^2 . (a, \ldots \between x', y', z')^2 - [(a, \ldots \between x, y, z \between x', y', z')]^2 = (A, \ldots \between yz' - y'z, \ldots)^2$$

now gives

$$(a,..\between x,\ y,\ z)^2 = \text{Quotient by } (a,..\between x',\ y',\ z')^2 \text{ of}$$

$$\text{I.}\quad \left\{\begin{array}{l} [(a,..\between x,\ y,\ z\between x',\ y',\ z')]^2 \\ \quad + \text{Quotient by } (A,..\between mz'-ny',..)^2 \text{ of Product} \\ \{(A,..\between mz'-ny',..\between yz'-y'z,..) \pm \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} \sqrt{-K(a,..\between x',\ y',\ z')^2}\}, \end{array}\right.$$

where the Product part may also be written

$$\begin{array}{l} (a,...\between l\ ,\ m,\ n\between x',\ y',\ z')\,.\,(a,..\between x,\ y,\ z\between x',\ y',\ z') \\ -(a,...\between x',\ y',\ z')^2 \qquad\qquad .\,(a,..\between x,\ y,\ z\between l\ ,\ m,\ n) \\ \qquad\qquad \pm\sqrt{-K(a,..\between x',\ y',\ z')^2}\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}. \end{array}$$

19. Writing in the formula I.

$$(\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \between x',\ y',\ z') = (\xi',\ \eta',\ \zeta'),$$

we have

$$(x',\ y',\ z') = \frac{1}{K}(\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} \between \xi',\ \eta',\ \zeta'),$$

and thence

$$K(a,..\between x',\ y',\ z') \qquad\qquad = (A,..\between \xi',\ \eta',\ \zeta')^2$$

$$(a,..\between x\ ,\ y\ ,\ z\between x',\ y',\ z') = \xi' x + \eta' y + \zeta' z.$$

Assume

$$(l,\ m,\ n) = (\nu\eta' - \mu\zeta',\ \lambda\zeta' - \nu\xi',\ \mu\xi' - \lambda\eta'),$$

then from the foregoing values of $(x',\ y',\ z')$

$$\begin{aligned} mz' - ny' &= \frac{1}{K}\left\{(\lambda\zeta' - \nu\xi')(G\xi' + F\eta' + C\zeta') - (\mu\xi' - \lambda\eta')(H\xi' + B\eta' + F\zeta')\right\} \\ &= \frac{1}{K}\left\{\lambda\,[\xi'(A\xi' + H\eta' + G\zeta') + \eta'(H\xi' + B\eta' + F\zeta') + \zeta'(G\xi' + F\eta' + C\zeta')]\right. \\ &\qquad \left. - \lambda\xi'(A\xi' + H\eta' + G\zeta') - \mu\xi'(H\xi' + B\eta' + F\zeta') - \nu\xi'(G\xi' + F\eta' + C\zeta')\right\}, \end{aligned}$$

that is

$$mz' - ny' = \frac{1}{K}\left\{\lambda(A,..\between\xi', \eta', \zeta')^2 - \xi'(A,..\between\xi', \eta', \zeta'\between\lambda, \mu, \nu)\right\},$$

and similarly

$$nx' - lz' = \frac{1}{K}\left\{\mu(A,..\between\xi', \eta', \zeta')^2 - \eta'(A,..\between\xi', \eta', \zeta'\between\lambda, \mu, \nu)\right\},$$

$$ly' - mx' = \frac{1}{K}\left\{\nu(A,..\between\xi', \eta', \zeta')^2 - \zeta'(A,..\between\xi', \eta', \zeta'\between\lambda, \mu, \nu)\right\};$$

and thence

$$\begin{aligned}(A, H, G\between mz' - ny',..)\\ = \frac{1}{K}\Big\{ &(A, H, G\between\lambda, \mu, \nu).(A,..\between\xi', \eta', \zeta')^2\\ &-(A, H, G\between\xi', \eta', \zeta').(A,..\between\xi', \eta', \zeta'\between\lambda, \mu, \nu)\Big\}\end{aligned}$$

with the like equations, writing H, B, F and G, F, C in the place of A, H, G successively: and we then have

$$\begin{aligned}(A,..\between mz' - ny',..)^2\\ = \frac{1}{K}\Big\{ &(A,..\between\lambda, \mu, \nu\between mz' - ny',..).(A,..\between\xi', \eta', \zeta')^2\\ &-(A,..\between\xi', \eta', \zeta'\between mz' - ny,..).(A,..\between\xi', \eta', \zeta'\between\lambda, \mu, \nu)\Big\}.\end{aligned}$$

But the foregoing values of $mz' - ny'$, $nx' - lz'$, $ly' - mx'$ give also

$$\begin{aligned}(A,..\between\lambda, \mu, \nu\between mz' - ny',..)\\ = \frac{1}{K}\left\{(A,..\between\lambda, \mu, \nu)^2.(A,..\between\xi', \eta', \zeta')^2 - [(A,..\between\lambda, \mu, \nu\between\xi', \eta', \zeta')]^2\right\},\end{aligned}$$

$$\begin{aligned}(A,..\between\xi', \eta', \zeta,\between mz' - ny',..)\\ = \frac{1}{K}\left\{(A,..\between\lambda, \mu, \nu\between\xi', \eta', \zeta').(A,..\between\xi', \eta', \zeta')^2 - (A,..\between\xi', \eta', \zeta')^2.(A,..\between\lambda, \mu, \nu\between\xi', \eta', \zeta')\right\} = 0;\end{aligned}$$

so that

$$\begin{aligned}(A,..\between mz' - ny',..)^2\\ &= \frac{1}{K^2}(A,..\between\xi', \eta', \zeta')^2.\{(A,..\between\lambda, \mu, \nu)^2.(A,..\between\xi', \eta', \zeta')^2 - [(A,..\between\lambda, \mu, \nu\between\xi', \eta', \zeta')]^2\}\\ &= \frac{1}{K}(A,..\between\xi', \eta', \zeta')^2.(a,..\between\nu\eta' - \mu\zeta',..)^2.\end{aligned}$$

Similarly,

$$(A,..\between mz'-ny',..\between yz'-y'z,..)$$
$$=\frac{1}{K}\Big\{\ (A,..\between\lambda,\ \mu,\ \nu\between yz'-y'z,..).(A,..\between\xi',\ \eta',\ \zeta')^2$$
$$-(A,..\between\xi',\ \eta',\ \zeta'\between yz'-y'z,..).(A,..\between\xi',\ \eta',\ \zeta'\between\lambda,\ \mu,\ \nu)\Big\}.$$

But

$$\begin{aligned}(A,..\between\lambda,\ \mu,\ \nu\between yz'-y'z,..)\\ &= (A\lambda+H\mu+G\nu)(yz'-y'z)\\ &\quad+(H\lambda+B\mu+F\nu)(zx'-z'x)\\ &\quad+(G\lambda+F\mu+C\nu)(xy'-x'y),\\ &= x[y'(G\lambda+F\mu+C\nu)-z'(H\lambda+B\mu+F\nu)]\\ &\quad+y[z'(A\lambda+H\mu+G\nu)-x'(G\lambda+F\mu+C\nu)]\\ &\quad+z[x'(H\lambda+B\mu+F\nu)-y'(A\lambda+H\mu+G\nu)],\end{aligned}$$

which, substituting for x', y', z' their values

$$(x',\ y',\ z')=\frac{1}{K}\left(\begin{matrix}A, & H, & G\\ H, & B, & F\\ G, & F, & C\end{matrix}\right)\between\xi',\ \eta',\ \zeta'),$$

becomes

$$=\frac{1}{K}\Big\{\ x[(BC-F^2)(\nu\eta'-\mu\zeta')+(FG-CH)(\lambda\zeta'-\nu\xi')+(HF-BG)(\mu\xi'-\lambda\eta')]$$
$$+y[(FG-CH)(\nu\eta'-\mu\zeta')+(CA-G^2)(\lambda\zeta'-\nu\xi')+(GH-AF)(\mu\xi'-\lambda\eta')]$$
$$+z[(HF-BG)(\nu\eta'-\mu\zeta')+(GH-AF)(\lambda\zeta'-\nu\xi')+(AB-H^2)(\mu\xi'-\lambda\eta')]\Big\},$$

which is

$$=(a,..\between x,\ y,\ z\between\nu\eta'-\mu\zeta',..);$$

and by merely writing $(\xi',\ \eta',\ \zeta')$ in the place of $(\lambda,\ \mu,\ \nu)$, we have

$$(A,..\between\xi',\ \eta',\ \zeta'\between yz'-y'z,..)=0;$$

so that we find

$$(A,..\between mz'-ny',..\between yz'-y'z,..)$$
$$=\frac{1}{K}(A,..\between\xi',\ \eta',\ \zeta')^2.(a,..\between x,\ y,\ z\between\nu\eta'-\mu\zeta',\ \lambda\zeta'-\nu\xi',\ \mu\xi'-\lambda\eta').$$

Now, writing the formula I. in the form

$$(a,..\between x,\ y,\ z)^2 = \text{Quotient by } K(a,..\between x',\ y',\ z')^2 \text{ of } \left)\right.$$

$$\left\{\begin{array}{l} K\,[(a,..\between x,\ y,\ z\between x',\ y',\ z')]^2 \\ +\,\text{Quotient by } K(A,..\between mz'-ny',..)^2 \text{ of} \\ K^2\{[(A,..\between mz'-ny',..\between yz'-y'z,..)]^2 + \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}^2 K(a,..\between x,\ y,\ z)^2\}, \end{array}\right.$$

the right-hand side is

$$= \text{Quotient by } (A,..\between \xi',\ \eta',\ \zeta')^2 \text{ of } \left)\right.$$

$$\left\{\begin{array}{l} K(\xi'x+\eta'y+\zeta'z)^2 \\ +\,\text{Quotient by } (a,..\between \nu\eta'-\mu\zeta',..)^2(A,..\between \xi',\ \eta',\ \zeta')^2 \text{ of} \\ \{[(a,..\between \nu\eta'-\mu\zeta',..\between x,\ y,\ z)\,.\,(A,..\between \xi',\ \eta',\ \zeta')^2]^2 + \Pi^2(A,..\between \xi',\ \eta',\ \zeta')^2\}, \end{array}\right.$$

where

$$\Pi = K \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m & n \end{vmatrix},$$

or, what is the same thing,

$$\Pi = \begin{vmatrix} x\ , & y\ , & z \\ Kx', & Ky', & Kz' \\ l\ , & m\ , & n \end{vmatrix} = \begin{vmatrix} x & y & z \\ A\xi'+H\eta'+G\zeta', & H\xi'+B\eta'+F\zeta', & G\xi'+F\eta'+C\zeta' \\ \nu\eta'-\mu\zeta'\ , & \lambda\zeta'-\nu\xi'\ , & \mu\xi'-\lambda\eta'. \end{vmatrix}.$$

More simply, the right-hand side is

$$= \text{Quotient by } (A,..\between \xi',\ \eta',\ \zeta')^2 \text{ of } \left)\right.$$

$$\left\{\begin{array}{l} K(\xi'x+\eta'y+\zeta'z)^2 \\ +\,\text{Quotient by } (a,..\between \nu\eta'-\mu\zeta',..)^2 \text{ of} \\ \{[(a,..\between \nu\eta'-\mu\zeta',..\between x,\ y,\ z)]^2\,.\,(A,..\between \xi',\ \eta',\ \zeta')^2+\Pi^2\}\,; \end{array}\right.$$

or restoring the left-hand side, and resolving into its linear factors the function in { }, we have

$$(a,..\between x,\ y,\ z)^2 = \text{Quotient by } (A,..\between \xi',\ \eta',\ \zeta')^2 \text{ of } \left)\right.$$

$$\text{II.}\quad \left\{\begin{array}{l} K(\xi'x+\eta'y+\zeta'z)^2 \\ +\,\text{Quotient by } (a,..\between \nu\eta'-\mu\zeta',..)^2 \text{ of Product} \\ \Pi \pm \sqrt{-(A,..\between \xi',\ \eta',\ \zeta')^2}\,.\,(a,..\between \nu\eta'-\mu\zeta',..\between x,\ y,\ z), \end{array}\right.$$

where Π has the value given above, which may also be written

$$\begin{aligned}\Pi = \ & (A,..\between\xi',\ \eta',\ \zeta'\between\lambda,\ \mu,\ \nu)(\xi'x+\eta'y+\zeta'z)\\ & -(A,..\between\xi',\ \eta',\ \zeta')^2(\lambda x+\mu y+\nu z).\end{aligned}$$

20. We deduce at once the inverse or reciprocal formulæ

$$\text{III.}\quad \left\{\begin{array}{l}(A,..\between\xi,\ \eta,\ \zeta)^2 = \text{Quotient by } (A,..\between\xi',\ \eta',\ \zeta')^2 \text{ of}\\ \quad [(A,..\between\xi,\ \eta,\ \zeta\between\xi',\ \eta',\ \zeta')]^2\\ \quad + \text{Quotient by } (a,..\between\nu\eta'-\mu\zeta',..)^2 \text{ of } K \text{ into Product}\\ \quad (a,..\between\nu\eta'-\mu\zeta',..\between\eta\zeta'-\eta'\zeta,..) \pm \sqrt{-(A,..\between\xi',\ \eta',\ \zeta')^2}\begin{vmatrix}\xi, & \eta, & \zeta\\ \xi', & \eta', & \zeta'\\ \lambda, & \mu, & \nu\end{vmatrix},\end{array}\right.$$

where the Product part may also be written

$$\begin{aligned}\text{Product}\Bigg[\frac{1}{K}&(A,..\between\xi',\ \eta',\ \zeta'\between\lambda,\ \mu,\ \nu).(A,..\between\xi',\ \eta',\ \zeta'\between\xi,\ \eta,\ \zeta)\\ -\frac{1}{K}&(A,.\ \between\xi',\ \eta',\ \zeta')^2 \qquad .(A,..\between\lambda,\ \mu,\ \nu\between\xi,\ \eta,\ \zeta)\\ &\pm\sqrt{-(A,..\between\xi',\ \eta',\ \zeta')^2}\begin{vmatrix}\xi, & \eta, & \zeta\\ \xi', & \eta', & \zeta'\\ \lambda, & \mu, & \nu\end{vmatrix}\Bigg].\end{aligned}$$

21. And also

$$\text{IV.}\quad \left\{\begin{array}{l}(A,..\between\xi,\ \eta,\ \zeta)^2 = \text{Quotient by } (a,..\between x',\ y',\ z')^2 \text{ of}\\ \quad K(\xi x'+\eta y'+\zeta z')^2\\ \quad + \text{Quotient by } K(A,..\between ny'-mz',..)^2 \text{ of Product}\\ \quad K\Phi \pm \sqrt{-K(a,..\between x',\ y',\ z')^2}(A,..\between ny'-mz',..\between\xi,\ \eta,\ \zeta),\end{array}\right.$$

where

$$\Phi = \begin{vmatrix}\xi & \eta & \zeta\\ ax'+hy'+gz', & hx'+by'+fz', & gx'+fy'+cz'\\ ny'-mz' \ , & lz'-nx' \ , & mx'-ly'\end{vmatrix},$$

which may also be written

$$\begin{aligned}= \ & (a,..\between x',\ y',\ z'\between l,\ m,\ n)(x'\xi+y'\eta+z'\zeta)\\ & -(a,..\between x',\ y',\ z')^2 \quad . \quad (l\xi+m\eta+n\zeta).\end{aligned}$$

22. The geometrical signification is obvious. The formulæ I. and II. each of them show that the equation

$$(a,..\between x,\ y,\ z)^2 = 0$$

of the conic may be written in the form

$$W^2 + \frac{1}{M} QR = 0,$$

where $Q=0$, $R=0$ are any two tangents of the conic, and $W=0$ is the line joining the points of contact, or chord of contact corresponding to the two tangents; viz., in the formula I. we have

$$W = (a, \ldots \between x', y', z' \between x, y, z),$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = (A, \ldots \between mz' - ny', \ldots \between yz' - y'z, \ldots) \pm \sqrt{-K(a, \ldots \between x, y, z)^2} \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix},$$

(or for a different form of Q, R see the formula). The quantities (x', y', z') are the coordinates of the point of intersection of the two tangents, or pole of the chord of contact: (l, m, n) are supernumerary arbitrary quantities, the values whereof do not affect the result [1]. And in the formula II. we have

$$W = \xi' x + \eta' y + \zeta' z,$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = \Pi \pm \sqrt{-(A, \ldots \between \xi', \eta', \zeta')^2}\, (a, \ldots \between \nu\eta' - \mu\zeta', \ldots \between x, y, z),$$

(for the value of Π see the formula). The quantities (ξ', η', ζ') are the line-coordinates of the chord of contact (viz. the point-equation of this line is $\xi' x + \eta' y + \zeta' z = 0$); (λ, μ, ν) are supernumerary arbitrary quantities.

23. In the like manner the formulæ III. and IV. each of them show that the line-equation

$$(A, \ldots \between \xi, \eta, \zeta)^2 = 0$$

of the conic may be written in the form

$$W^2 + \frac{1}{M} QR = 0,$$

where $Q=0$, $R=0$ are any two ineunts of the conic, and $W=0$ is the point of intersection of the corresponding tangents; viz. in the formula III. we have

$$W = (A, \ldots \between \xi', \eta', \zeta' \between \xi, \eta, \zeta),$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = (a, \ldots \between \nu\eta' - \mu\zeta', \ldots) \pm \sqrt{-(A, \ldots \between \xi', \eta', \zeta')^2} \begin{vmatrix} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \lambda, & \mu, & \nu \end{vmatrix},$$

(for another form of Q, R see the formula).

[1] In a different point of view, viz. if we consider the formula I. as a transformation of the function $(a, \ldots \between x, y, z)^2$, then (x', y', z') and (l, m, n) would be each of them supernumerary arbitrary quantities: and so in the other like cases.

The quantities ξ', η', ζ' are the line-coordinates of the line through the two ineunts, or chord of contact; (λ, μ, ν) are supernumerary arbitrary quantities; and so in the formula IV. we have

$$W = x'\xi + y'\eta + z'\zeta,$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = K\Phi \pm \sqrt{-K(a,..\between x', y', z')^2}\,(A,..\between ny' - mz,..\between \xi, \eta, \zeta)$$

(for the value of Φ see the formula), where x', y', z' are the point-coordinates of the intersection of tangents at the two ineunts, or pole of the chord of contact; (l, m, n) are supernumerary arbitrary quantities.

24. We may, instead of the supernumerary arbitrary quantities (l, m, n) of the formula I., introduce the quantities (λ, μ, ν), where

$$(l, m, n) = \frac{1}{K}\left(\begin{matrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{matrix}\right)\between \lambda, \mu, \nu).$$

This gives

$$\begin{aligned}
&(A, H, G\between mz' - ny',..) \\
&\quad = A(mz' - ny') + H(nx' - lz') + G(ly' - mx') \\
&\quad = x'(Hn - Gm) + y'(Gl - An) + z'(Am - Hl) \\
&\quad = \frac{1}{K}\cdot\ x'[H(G\lambda + F\mu + C\nu) - G(H\lambda + B\mu + F\nu)] \\
&\qquad\quad + y'[G(A\lambda + H\mu + G\nu) - A(G\lambda + F\mu + C\nu)] \\
&\qquad\quad + z'[A(H\lambda + B\mu + F\nu) - H(H\lambda + B\mu + F\nu)] \\
&\quad = \qquad x'(g\mu - h\nu) + y'(f\mu - b\nu) + z'(c\mu - f\nu) \\
&\quad = \qquad \mu(gx' + fy' + cz') - \nu(hx' + by' + fz');
\end{aligned}$$

we have thus the system

$$\begin{aligned}
(A, H, G\between mz' - ny',..) &= \mu(gx' + fy' + cz') - \nu(hx' + by' + fz'), \\
(H, B, F\between mz' - ny',..) &= \nu(ax' + hy' + gz') - \lambda(gx' + fy' + cz'), \\
(G, F, C\between mz' - ny',..) &= \lambda(hx' + by' + fz') - \mu(ax' + hy' + gz'),
\end{aligned}$$

and thence

$$\begin{aligned}
&(A,..\between mz' - ny',..\between yz' - y'z,..) \\
&\quad = -\left|\begin{matrix} yz' - y'z, & zx' - z'x, & xy' - x'y \\ ax' + hy' + gz', & hx' + by' + fz', & gx' + fy' + cz' \\ \lambda, & \mu, & \nu \end{matrix}\right|;
\end{aligned}$$

or observing that the term in λ is

$$-(zx' - z'x)(gx' + fy' + cz') + (xy' - x'y)(hx' + by' + fz'),$$

which is

$$\begin{aligned}
&= x\left(x'(ax'+hy'+gz')+y'(hx'+fy'+fz')+z'(gx'+fy'+cz')\right)\\
&\quad -x\,.\,x'(ax'+hy'+gz')\\
&\quad -y\,.\,x'(hx'+by'+fz')\\
&\quad -z\,.\,x'(gx'+fy'+cz')\\
&= -x'(a,..\between x,\ y,\ z\between x',\ y',\ z')+x(a,..\between x',\ y',\ z')^2,
\end{aligned}$$

with similar expressions for the terms in μ, ν, we have

$$\begin{aligned}
&(A,..\between mz'-ny',..\between yz'-y'z,..)\\
&\qquad = -(\lambda x'+\mu y'+\nu z').(a,..\between x,\ y,\ z\between x',\ y',\ z')+(\lambda x+\mu y+\nu z).(a,..\between x',\ y',\ z')^2;
\end{aligned}$$

and so also

$$\begin{aligned}
&(A,..\between mz'-ny',..)^2\\
&\qquad = -(\lambda x'+\mu y'+\nu z').(a,..\between l,\ m,\ n\between x',\ y',\ z')+(\lambda l+\mu m+\nu n).(a,..\between x',\ y',\ z')^2,
\end{aligned}$$

where

$$\begin{aligned}
(a,..\between l,\ m,\ n\between x',\ y',\ z') &= \lambda x'+\mu y'+\nu z',\\
\lambda l+\mu m+\nu n &= \frac{1}{K}(A,..\between \lambda,\ \mu,\ \nu)^2,
\end{aligned}$$

so that

$$(A,..\between mz'-ny',..)^2 = -(\lambda x'+\mu y'+\nu z')^2+\frac{1}{K}(A,..\between \lambda,\ \mu,\ \nu)^2.(a,..\between x',\ y',\ z')^2.$$

Moreover,

$$\begin{vmatrix} x, & y, & z\\ x', & y', & z'\\ l, & m, & n \end{vmatrix} = \frac{1}{K}\left\{(A\lambda+H\mu+G\nu)(yz'-y'z)+(H\lambda+B\mu+F\nu)(zx'-z'x)+(G\lambda+F\mu+C\nu)(xy'-x'y)\right\},$$

which is

$$= \frac{1}{K}(A,..\between \lambda,\ \mu,\ \nu\between yz'-y'z,..);$$

and hence instead of the formula I. we have

$$\text{I. (bis)}\left\{\begin{array}{l}
(a,..\between x,\ y,\ z)^2 = \text{Quotient by } (a,..\between x',\ y',\ z')^2 \text{ of}\\
\quad [(a,..\between x,\ y,\ z\between x',\ y',\ z')]^2\\
\quad + \text{Quotient by } (A,..\between \lambda,\mu,\nu)^2(a,..\between x',y',z')^2 - K(\lambda x'+\mu y'+\nu z')^2 \text{ of } K \text{ into Product}\\
\quad \left\{\begin{array}{l}
(\lambda x'+\mu y'+\nu z').(a,..\between x,\ y,\ z\between x',\ y',\ z')-(\lambda x+\mu y+\nu z).(a,..\between x',\ y',\ z')^2\\
\pm\dfrac{1}{K}\sqrt{-K(a,..\between x',\ y',\ z')^2}\,(A,..\between \lambda,\ \mu,\ \nu\between yz'-y'z,..).
\end{array}\right\}.
\end{array}\right.$$

25. If, in like manner, in the formula II. we introduce, instead of $(\lambda,\ \mu,\ \nu)$, the new quantities $(l,\ m,\ n)$, where

$$(\lambda,\ \mu,\ \nu) = \left(\begin{array}{ccc} a, & h, & g\\ h, & b, & f\\ g, & f, & c \end{array}\right)\between (l,\ m,\ n),$$

or, what is the same thing,

$$(l,\ m,\ n)=\frac{1}{K}\left(\begin{array}{ccc} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{array}\right)\between(\lambda,\ \mu,\ \nu),$$

then we have

$$\begin{aligned}
(a,\ h,\ g\between\nu\eta'-\mu\zeta',..) &= n\ (H\xi'+B\eta'+F\zeta')-m\,(G\xi'+F\eta'+C\zeta'),\\
(h,\ b,\ f\between\nu\eta'-\mu\zeta',..) &= l\ (G\xi'+F\eta'+C\zeta')-n\ (A\xi'+H\eta'+G\zeta'),\\
(g,\ f,\ c\between\nu\eta'-\mu\zeta',..) &= m\,(A\xi'+H\eta'+G\zeta')-l\ (H\xi'+B\eta'+F\zeta');
\end{aligned}$$

and thence

$$\begin{aligned}
(a,..\between\nu\eta'-\mu\zeta',..\between x,\ y,\ z) &= \left|\begin{array}{ccc} x & , & y & , & z \\ A\xi'+H\eta'+G\zeta', & & H\xi'+B\eta'+F\zeta', & & G\xi'+F\eta'+C\zeta' \\ l & , & m & , & n \end{array}\right| \\
&= (A,..\between mz-ny,..\between\xi',\ \eta',\ \zeta'),
\end{aligned}$$

$$\begin{aligned}
(a,..\between\nu\eta'-\mu\zeta',..)^2 &= \frac{1}{K}\left\{(A,..\between\lambda,\ \mu,\ \nu)^2.(A,..\between\xi',\ \eta',\ \zeta')^2-[(A,..\between\lambda,\ \mu,\ \nu\between\xi',\ \eta',\ \zeta')]^2\right\}\\
&= (a,..\between l,\ m,\ n)^2.(A,..\between\xi',\ \eta',\ \zeta')^2-K\,(l\xi'+m\eta'+n\zeta')^2;
\end{aligned}$$

$$\begin{aligned}
V &= \left|\begin{array}{ccc} x & , & y & , & z \\ A\xi'+H\eta'+G\zeta', & & H\xi'+B\eta'+F\zeta', & & G\xi'+F\eta'+C\zeta' \\ \nu\eta'-\mu\zeta' & , & \lambda\zeta'-\nu\xi' & , & \mu\xi'-\lambda\eta' \end{array}\right| \\
&= (\nu\eta'-\mu\zeta').\,y\,(G\xi'+F\eta'+C\zeta')-z\,(H\xi'+B\eta'+F\zeta')\\
&\quad (\lambda\zeta'-\nu\xi').\,z\,(A\xi'+H\eta'+G\zeta')-x\,(G\xi'+F\eta'+C\zeta')\\
&\quad (\mu\xi'-\lambda\eta').\,x\,(H\xi'+B\eta'+F\zeta')-y\,(A\xi'+H\eta'+G\zeta')\\
&= \lambda\,\{(\xi'x+\eta'y+\zeta'z)(A\xi'+H\eta'+G\zeta')-x(A,..\between\xi',\ \eta',\ \zeta')^2\}\\
&\quad +\mu\,\{(\xi'x+\eta'y+\zeta'z)(H\xi'+B\eta'+F\zeta')-y(A,..\between\xi',\ \eta',\ \zeta')^2\}\\
&\quad +\nu\,\{(\xi'x+\eta'y+\zeta'z)(G\xi'+F\eta'+C\zeta')-z(A,..\between\xi',\ \eta',\ \zeta')^2\}\\
&= (\xi'x+\eta'y+\zeta'z).(A,..\between\lambda,\ \mu,\ \nu\between\xi',\ \eta',\ \zeta')-(\lambda x+\mu y+\nu z).(A,..\between\xi',\ \eta',\ \zeta')^2\\
&= K(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,..\between l,\ m,\ n\between x,\ y,\ z).(A,..\between\xi',\ \eta'\ \zeta')^2;
\end{aligned}$$

and the formula II. thus becomes

$$\text{II. (bis)}\quad \left\{\begin{array}{l}
(a,..\between x,\ y,\ z)^2=\text{Quotient by }(A,..\between\xi',\ \eta',\ \zeta')^2\text{ of}\\
K(\xi'x+\eta'y+\zeta'z)^2\\
+\text{Quotient by }(a,..\between l,\ m,\ n)^2.(A,..\between\xi',\ \eta',\ \zeta')^2-K(l\xi'+m\eta'+n\zeta')^2\text{ of Product}\\
\left\{\begin{array}{l}
K\,(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,..\between l,\ m,\ n\between x,\ y,\ z).(A,..\between\xi',\ \eta',\ \zeta')^2\\
\pm\sqrt{-(A,..\between\xi',\ \eta',\ \zeta')^2}\,(A,..\between mz-ny,..\between\xi',\ \eta',\ \zeta').
\end{array}\right\}
\end{array}\right.$$

26. And from these we at once deduce the inverse or reciprocal formulæ

$$\text{III. (bis)}\quad \left\{\begin{array}{l} (A,..\between\xi,\ \eta,\ \zeta)^2 = \text{Quotient by } (A,..\between\xi',\ \eta',\ \zeta')^2 \text{ of} \\ [(A,..\between\xi,\ \eta,\ \zeta\between\xi',\ \eta',\ \zeta')]^2 \\ + \text{Quotient by}(a,..\between l, m, n)^2 . (A,..\between\xi', \eta', \zeta')^2 - K(l\xi' + m\eta' + n\zeta')^2 \text{ of } K \text{ into Product} \\ \left\{\begin{array}{l} (l\xi' + m\eta' + n\zeta')(A,..\between\xi,\ \eta,\ \zeta\between\xi',\ \eta',\ \zeta') - (l\xi + m\eta + n\zeta) . (A,..\between\xi',\ \eta',\ \zeta')^2 \\ \pm\sqrt{-(A,..\between\xi',\ \eta',\ \zeta')^2}\,(a,..\between l,\ m,\ n\between\eta\zeta' - \eta'\zeta,..), \end{array}\right\}. \end{array}\right.$$

27. And

$$\text{IV. (bis)}\quad \left\{\begin{array}{l} (A,..\between\xi,\ \eta,\ \zeta)^2 = \text{Quotient by } (a,..\between x',\ y',\ z')^2 \text{ of} \\ K(x'\xi + y'\eta + z'\zeta)^2 \\ + \text{Quotient by } (A,..\between\lambda, \mu, \nu)^2 . (a,..\between x', y', z')^2 - K(\lambda x' + \mu y' + \nu z')^2 \text{ of Product} \\ \left\{\begin{array}{l} K(\lambda x' + \mu y' + \nu z') . (x'\xi + y'\eta + z'\zeta) - (A,..\between\xi,\ \eta,\ \zeta\between l, m, n) . (a,..\between x', y', z')^2 \\ \pm\sqrt{-K(a,..\between x',\ y',\ z')^2}\,(a,..\between x',\ y',\ z'\between\mu\zeta - \nu\eta,..), \end{array}\right\}. \end{array}\right.$$

These four formulæ have the same geometrical significations with the original four formulæ to which they correspond respectively.

28. The eight formulæ become all of them the same or very similar for the quadric form $(a,..\between x,\ y,\ z)^2 = x^2 + y^2 + z^2$, which of course implies $(A,..\between\xi,\ \eta,\ \zeta)^2 = \xi^2 + \eta^2 + \zeta^2$. Thus selecting any one of them at pleasure, e.g. the formula II. (bis), this becomes

$$\begin{aligned} & \{(x^2 + y^2 + z^2)(\xi'^2 + \eta'^2 + \zeta'^2) - (\xi' x + \eta' y + \zeta' z)^2\} \\ \times & \{(l^2 + m^2 + n^2)(\xi'^2 + \eta'^2 + \zeta'^2) - (l\xi' + m\eta' + n\zeta')^2\} \\ = & \{(l\xi' + m\eta' + n\zeta')(x\xi' + m\eta' + n\zeta') - (lx + my + nz)(\xi'^2 + \eta'^2 + \zeta'^2)\}^2 \\ & + (\xi'^2 + \eta'^2 + \zeta'^2) \begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2, \end{aligned}$$

where the terms independent of $\xi'^2 + \eta'^2 + \zeta'^2$ destroy each other. Omitting these terms, and dividing by $\xi'^2 + \eta'^2 + \zeta'^2$, the resulting equation is found to be

$$\begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2 = \begin{vmatrix} \xi'^2 + \eta'^2 + \zeta'^2, & \xi' x + \eta' y + \zeta' z, & \xi' l + \eta' m + \zeta' n \\ x\xi' + y\eta' + z\zeta', & x^2 + y^2 + z^2, & xl + ym + zn \\ l\xi' + m\eta' + n\zeta', & lx + my + nz, & l^2 + m^2 + n^2, \end{vmatrix}$$

which is a well-known identical equation.

Article Nos. 29 *to* 33, *relating to a single conic in connexion with an ineunt or a tangent of a conic of double contact.*

29. The formulæ assume a very simple form when the point of intersection of the two tangents, or the line of junction of the two ineunts of the conic, is an ineunt or a tangent of a conic having double contact with the first-mentioned conic. Thus, if to the conic

$$(a,..\between x,\ y,\ z)^2=0$$

tangents are drawn from a point $(x',\ y',\ z')$ of the conic

$$(a,..\between x,\ y,\ z)^2+(\xi'x+\eta'y+\zeta'z)^2=0,$$

then we have

$$(a,..\between x',\ y',\ z')^2=-(\xi'x'+\eta'y'+\zeta'z')^2;$$

and using the form I. (bis), and putting therein $(\xi',\ \eta',\ \zeta')$ in the place of the arbitrary quantities $(\lambda,\ \mu,\ \nu)$, the equation of the tangent divides out by $\xi'x'+\eta'y'+\zeta'z'$, and omitting this factor it becomes

$$(a,..\between x',\ y',\ z'\between x,\ y,\ z)+(\xi'x'+\eta'y'+\zeta'z')(\xi'x+\eta'y+\zeta'z)$$

$$\pm\frac{1}{\sqrt{K}}(A,..\between \xi',\ \eta',\ \zeta'\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)=0,$$

which is of the form

$$\left(\begin{matrix}\alpha, & \beta, & \gamma\\ \alpha', & \beta', & \gamma'\\ \alpha'', & \beta'', & \gamma''\end{matrix}\right)\between x',\ y',\ z'\between x,\ y,\ z)=0,$$

where the matrix

$$\left(\begin{matrix}\alpha, & \beta, & \gamma\\ \alpha', & \beta', & \gamma'\\ \alpha'', & \beta'', & \gamma''\end{matrix}\right) \text{ is} =$$

$$\begin{matrix} a+\xi'^2 & , h+\xi'\eta'+\frac{1}{\sqrt{K}}(G\xi'+F\eta'+C\zeta'), & g+\xi'\zeta'-\frac{1}{\sqrt{K}}(H\xi'+B\eta'+F\zeta') \\ h+\xi'\eta'-\frac{1}{\sqrt{K}}(G\xi'+F\eta'+C\zeta'), & b+\eta'^2 & , f+\eta'\zeta'+\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta') \\ g+\xi'\zeta'+\frac{1}{\sqrt{K}}(H\xi'+B\eta'+F\zeta'), & f+\eta'\zeta'-\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta'), & c+\zeta'^2. \end{matrix}$$

30. But instead of further developing these formulæ, I prefer to consider the formulæ which give the points of contact of the tangents in question, viz. the ineunts of the conic $(a,..\between x,\ y,\ z)^2=0$, or the tangents through the point $(x',\ y',\ z')$ of the conic $(a,..\between x,\ y,\ z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$.

We have as before

$$(a,..\between x',\ y',\ z')^2 = -(\xi' x' + \eta' y' + \zeta' z')^2,$$

and using the formula IV. (bis) and writing therein $(\xi',\ \eta',\ \zeta')$ in the place of the arbitrary quantities $(\lambda,\ \mu,\ \nu)$, the equation contains the factor $\xi' x' + \eta' y' + \zeta' z'$, and dividing by this factor, and by K, the line-equation of the ineunt is

$$x'\xi + y'\eta + z'\zeta + \frac{1}{K}(x'\xi' + y'\eta' + z'\zeta') \cdot (A,..\between \xi',\ \eta',\ \zeta' \between \xi,\ \eta,\ \zeta)$$

$$\pm \frac{1}{\sqrt{K}}(a,..\between x',\ y',\ z' \between \eta\zeta' - \eta'\zeta,..) = 0.$$

Selecting the positive sign, the coordinates of the corresponding ineunt are

$$x' + \frac{1}{K}(x'\xi' + y'\eta' + z'\zeta')(A\xi' + H\eta' + G\zeta') + \frac{1}{\sqrt{K}}\left\{\eta'(gx' + fy' + cz') - \zeta'(hx' + by' + fz')\right\},$$

$$y' + \frac{1}{K}(x'\xi' + y'\eta' + z'\zeta')(H\xi' + B\eta' + F\zeta') + \frac{1}{\sqrt{K}}\left\{\zeta'(ax' + hy' + gz') - \xi'(gx' + fy' + cz')\right\},$$

$$z' + \frac{1}{K}(x'\xi' + y'\eta' + z'\zeta')(G\xi' + F\eta' + C\zeta') + \frac{1}{\sqrt{K}}\left\{\xi'(hx' + by' + fz') - \eta'(ax' + hy' + gz')\right\};$$

and taking $(X,\ Y,\ Z)$ for the coordinates of the ineunt in question, and putting for shortness

$$\alpha = 1 - \frac{1}{\sqrt{K}}(g\eta' - h\zeta'),\quad \beta = -\frac{1}{\sqrt{K}}(f\eta' - b\zeta'),\quad \gamma = -\frac{1}{\sqrt{K}}(c\eta' - f\zeta'),$$

$$\alpha' = -\frac{1}{\sqrt{K}}(a\zeta' - g\xi'),\quad \beta' = 1 - \frac{1}{\sqrt{K}}(h\zeta' - f\xi'),\quad \gamma' = -\frac{1}{\sqrt{K}}(g\zeta' - c\xi'),$$

$$\alpha'' = -\frac{1}{\sqrt{K}}(h\xi' - a\eta'),\quad \beta'' = -\frac{1}{\sqrt{K}}(b\xi' - h\eta'),\quad \gamma'' = 1 - \frac{1}{\sqrt{K}}(f\xi' - g\eta'),$$

we may write

$$(1+P)X = (2-\alpha)x' - \beta y' - \gamma z' + \frac{1}{K}(A\xi' + H\eta' + G\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$(1+P)Y = -\alpha' x' + (2-\beta')y' - \gamma' z' + \frac{1}{K}(H\xi' + B\eta' + F\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$(1+P)Z = \alpha'' x'' - \beta'' y' + (2-\gamma'')z' + \frac{1}{K}(G\xi' + F\eta' + C\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

where P, which is arbitrary, may be put

$$= \frac{1}{K}(A,..\between \xi',\ \eta',\ \zeta')^2.$$

31. These equations then give

$$(x', y', z') = \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array} \right) (X, Y, Z),$$

which can be verified without difficulty by reversing the process; and we have thus the coordinates (X, Y, Z) in terms of (x', y', z'), and reciprocally.

32. If $(X_{,}, Y_{,}, Z_{,})$ are the coordinates of the other ineunt, we have, it is clear,

$$(x', y', z') = \left(\begin{array}{ccc} 2-\alpha, & -\beta, & -\gamma \\ -\alpha', & 2-\beta', & -\gamma' \\ -\alpha'', & -\beta'', & 2-\gamma'' \end{array} \right) (X_{,}, Y_{,}, Z_{,});$$

or substituting for (x', y', z') their values in terms of (X, Y, Z),

$$(2X_{,}, 2Y_{,}, 2Z_{,}) = \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array} \right) (X+X_{,}, Y+Y_{,}, Z+Z_{,}),$$

so that $(X+X_{,}, Y+Y_{,}, Z+Z_{,})$ are the same linear functions of $2X_{,}, 2Y_{,}, 2Z_{,}$ that (X, Y, Z) are of (x', y', z'); that is, we have

$$\tfrac{1}{2}(1+P)(X+X_{,}) = (2-\alpha)X_{,} - \beta Y_{,} - \gamma Z_{,} + \frac{1}{K}(A\xi' + H\eta' + G\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$\tfrac{1}{2}(1+P)(X+X_{,}) = -\alpha' X_{,} + (2-\beta')Y_{,} - \gamma' Z_{,} + \frac{1}{K}(H\xi' + B\eta' + F\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$\tfrac{1}{2}(1+P)(X+X_{,}) = -\alpha'' X_{,} - \beta'' Y_{,} + (2-\gamma'')Z_{,} + \frac{1}{K}(G\xi' + F\eta' + C\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

which equations may be written

$$(1+P)(X, Y, Z) = \left(\begin{array}{ccc} \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}' \\ \mathrm{a}'', & \mathrm{b}'', & \mathrm{c}'' \end{array} \right) (X_{,}, Y_{,}, Z_{,}),$$

where the values of the coefficients are

$$\begin{array}{lll}
-h\zeta') + \frac{1}{K}(A\xi'^2 - B\eta'^2 - 2F\eta'\zeta' - C\zeta'^2), & -\frac{2}{\sqrt{K}}(f\eta' - b\zeta') + \frac{2}{K}\eta'(A\xi' + H\eta' + G\zeta'), & -\frac{2}{\sqrt{K}}(c\eta' - f\zeta') + \frac{2}{K}\zeta'(A\xi' + H\eta' + G\zeta') \\
-g\xi') + \frac{2}{K}\xi'(H\xi' + B\eta' + F\zeta'), & 1 - \frac{2}{\sqrt{K}}(h\zeta' - f\eta') + \frac{1}{K}(B\eta'^2 - C\zeta'^2 - 2G\zeta'\xi' - A\xi'^2), & -\frac{2}{\sqrt{K}}(g\zeta' - c\xi') + \frac{2}{K}\zeta'(H\xi' + B\eta' + F\zeta'), \\
-a\eta') + \frac{2}{K}\xi'(G\xi' + F\eta' + C\zeta'), & -\frac{2}{\sqrt{K}}(b\xi' - h\eta') + \frac{2}{K}\eta'(G\xi' + F\eta' + C\zeta'), & 1 - \frac{2}{\sqrt{K}}(f\xi' - g\eta') + \frac{1}{K}(C\zeta'^2 - A\xi'^2 - 2H\xi'\eta' - B\eta'^2),
\end{array}$$

and considering (X, Y, Z) and $(X_{,}, Y_{,}, Z_{,})$ as quantities connected by the foregoing linear relations, we have identically

$$(a,..\between X, Y, Z)^2 = (a,..\between X_{,}, Y_{,}, Z_{,})^2.$$

The investigation leads thus to the automorphic transformation of the quadric function, a transformation first effected by M. Hermite[1].

33. It is to be remarked that the foregoing formulæ show that (x', y', z') being the coordinates of a point on the conic $(a,..\between x, y, z)^2 + (\xi' x + \eta' y + \zeta' z)^2 = 0$, from which point tangents are drawn to the conic $(a,..\between x, y, z)^2 = 0$, then the coordinates (x', y', z') enter *linearly* into the equations of the tangents, the ineunts (or points of contact), and the polar. And it may be added that the equation of the conic enveloped by the polar (that is, the polar conic of $(a,..\between x, y, z)^2 + (\xi' x + \eta' y + \zeta' z)^2 = 0$) has for its equation

$$\{K + (A,..\between \xi', \eta', \zeta')^2\}(a,..\between x, y, z)^2 - K(\xi' x + \eta' y + \zeta' z)^2 = 0,$$

and that the coordinates of the point of contact of the polar with this conic are

$$x' + \frac{1}{K}(A\xi' + H\eta' + G\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$y' + \frac{1}{K}(H\xi' + B\eta' + F\zeta')(\xi' x' + \eta' y' + \zeta' z'),$$

$$z' + \frac{1}{K}(G\xi' + F\eta' + C\zeta')(\xi' x' + \eta' y' + \zeta' z');$$

so that (x', y', z') also enter linearly into the expressions for the coordinates of the last-mentioned point.

Article Nos. 34 *to* 37, *relating to two conics.*

34. Considering now the two conics

$$U = (a, b, c, f, g, h\between x, y, z)^2 = 0,$$
$$U' = (a', b', c', f', g', h'\between x, y, z)^2 = 0;$$

suppose that the conic

$$\theta U + \theta' U' = (\theta a + \theta' a',.. \qquad \between x, y, z)^2 = 0$$

represents a pair of lines.

The condition for this is

$$\text{Disct. } (\theta a + \theta' a',..\between x, y, z)^2 = 0,$$

which is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\between \theta, \theta')^3 = 0,$$

where

$$\mathfrak{A} = K,$$
$$\mathfrak{B} = Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$
$$\mathfrak{C} = A'a + B'b + C'c + 2F'f + 2G'g + 2H'h,$$
$$\mathfrak{D} = K'$$

[1] See my "Memoir on the Automorphic Transformation of a Bipartite Quadric Function," *Phil. Trans.* vol. CXLVIII. (1858), pp. 39—46, [153].

(the significations of K', A', B', C', F', G', H' being of course analogous to those of K, A, B, C, F, G, H). The three roots $\theta : \theta'$ correspond, it is clear, to the three pairs of lines which can be drawn through the intersections of the two conics.

35. The equation

$$\text{Disct.}\,(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\between\theta, \theta')^3 = 0,$$

which is of the fourth order in $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, and of the sixth order as regards (a, b, c, f, g, h) and (a', b', c', f', g', h') respectively, is the condition in order that the two conics may touch each other. Assuming that it is satisfied, the cubic equation in $\theta : \theta'$ has a pair of equal roots; or say there is a twofold root and a onefold root; the twofold root gives the pair of lines drawn from the point of contact to the other two points of intersection, the onefold root gives the pair made up of the common tangent and the line joining the other two points of intersection.

36. In particular, suppose that the two conics are

$$2(\rho x + \sigma y + \tau z)(\rho' x + \sigma' y + \tau' z) = 0,$$
$$2(\lambda x + \mu y + \nu z)(\lambda' x + \mu' y + \nu' z) = 0;$$

so that

$$\begin{aligned}
(a\,, b\,, c\,, f\,, g\,, h\,) &= (2\rho\rho',\ 2\sigma\sigma',\ 2\tau\tau',\ \sigma\tau' + \sigma'\tau,\ \tau\rho' + \tau'\rho,\ \rho\sigma' + \rho'\sigma),\\
(a', b', c', f', g', h') &= (2\lambda\lambda',\ 2\mu\mu',\ 2\nu\nu',\ \mu\nu' + \mu'\nu,\ \nu\lambda' + \nu'\lambda,\ \lambda\mu' + \lambda'\mu),\\
(A\,, B\,, C\,, F\,, G\,, H\,) &= -(\sigma\tau' - \sigma'\tau,\ \tau\rho' - \tau'\rho,\ \rho\sigma' - \rho'\sigma)^2,\\
(A', B', C', F', G', H') &= -(\mu\nu' - \mu'\nu,\ \nu\lambda' - \nu'\lambda,\ \lambda\mu' - \lambda'\mu)^2;
\end{aligned}$$

and thence also

$$\begin{aligned}
\mathfrak{A} &= K \qquad\qquad = 0,\\
\mathfrak{B} &= Aa' + \&c. = -2 \begin{vmatrix} \lambda, & \mu, & \nu \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix} \begin{vmatrix} \lambda', & \mu', & \nu' \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix},\\
\mathfrak{C} &= A'a + \&c. = -2 \begin{vmatrix} \rho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} \begin{vmatrix} \rho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu, & \nu' \end{vmatrix},\\
\mathfrak{D} &= K' \qquad\qquad = 0;
\end{aligned}$$

and the equation in (θ, θ') is

$$\mathfrak{B}\theta + \mathfrak{C}\theta' = 0;$$

hence writing $\theta = \mathfrak{C}$, $\theta' = -\mathfrak{B}$, the equation of the pair of lines is

$$\begin{vmatrix} \rho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} \begin{vmatrix} \rho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} (\rho x + \sigma y + \tau z)(\rho' x + \sigma' y + \tau' z) - \begin{vmatrix} \lambda, & \mu, & \nu \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix} \begin{vmatrix} \lambda', & \mu', & \nu' \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix} (\lambda x + \mu y + \nu z)(\lambda' x + \mu' y + \nu' z) = 0;$$

and it is easy to see that the left-hand side does in fact break up into factors, and that the equation is

$$\begin{vmatrix} x\ , & y\ , & z \\ \mu\tau'-\nu\sigma', & \nu\rho'-\lambda\tau', & \lambda\sigma'-\mu\rho' \\ \sigma\nu'-\tau\mu', & \tau\lambda'-\rho\nu', & \rho\mu'-\sigma\lambda' \end{vmatrix} \begin{vmatrix} x\ , & y\ , & z \\ \mu\tau-\nu\sigma\ , & \nu\rho-\lambda\tau\ , & \lambda\sigma-\mu\rho \\ \sigma'\nu'-\tau'\mu', & \tau'\lambda'-\rho'\nu', & \rho'\mu'-\sigma'\lambda' \end{vmatrix} = 0,$$

which of course might have been obtained at once by means of the four points which are the intersection of each component line of the first conic by each component line of the second conic.

37. Suppose that the first conic is

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0,$$

while the second conic is the pair of lines

$$2(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z) = 0;$$

then, putting as before,

$$(\theta a+\theta'.2\lambda\lambda',..\between x,\ y,\ z)^2 = 0,$$

we have

$$(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D} \between \theta,\ \theta')^2 = 0,$$

where

$$\begin{aligned} \mathfrak{A} &= K, \\ \mathfrak{B} &= 2(A,\ B,\ C,\ F,\ G,\ H \between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu'), \\ \mathfrak{C} &= -(a,\ b,\ c,\ f,\ g,\ h \between \mu\nu'-\mu'\nu,\ \nu\lambda'-\nu'\lambda,\ \lambda\mu'-\lambda'\mu)^2, \\ \mathfrak{D} &= 0; \end{aligned}$$

and the equation in $(\theta,\ \theta')$ is

$$K\theta^2+2(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu')\,\theta\theta'-(a,..\between \mu\nu-\mu'\nu,..)^2\,\theta'^2 = 0,$$

which may be written

$$\begin{aligned} \{K\theta+(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu')\,\theta'\}^2 &= \{[(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu')]^2+K(a,..\between \mu\nu'-\mu'\nu,..)^2\}\theta'^2, \\ &= (A,..\between \lambda,\ \mu,\ \nu)^2.(A,..\between \lambda',\ \mu',\ \nu')^2.\theta'^2, \end{aligned}$$

that is

$$K\theta = [\pm\sqrt{(A,..\between \lambda,\ \mu,\ \nu)^2}\sqrt{(A,..\between \lambda',\ \mu',\ \nu')^2}-(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu')]\theta';$$

we may assume

$$\theta = \pm\sqrt{(A,..\between \lambda,\ \mu,\ \nu)^2}\sqrt{(A,..\between \lambda',\ \mu',\ \nu')^2}-(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu'),\quad \theta' = K,$$

so that the conic

$$\begin{aligned} \{\pm\sqrt{(A,..\between \lambda,\ \mu,\ \nu)^2}\sqrt{(A,..\between \lambda',\ \mu',\ \nu')^2}-(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu')\}\,(a,...\between x,\ y,\ z)^2 \\ +2K(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z) = 0 \end{aligned}$$

breaks up into a pair of lines.

Putting for shortness

$$\pm\sqrt{(A,..\between \lambda,\ \mu,\ \nu)^2}\sqrt{(A,..\between \lambda',\ \mu',\ \nu')^2}-(A,..\between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu') = \Omega$$

the coefficients on the left-hand side of the equation are

$$(\Omega a + 2K\lambda\lambda', \ldots \Omega f + K(\mu\nu' + \mu'\nu), \ldots),$$

whence, after all reductions, the inverse function is

$$\{(A, \ldots \between \lambda, \mu, \nu \between \xi, \eta, \zeta)\sqrt{(A, \ldots \between \lambda', \mu', \nu')^2} \mp (A, \ldots \between \lambda', \mu', \nu' \between \xi, \eta, \zeta)\sqrt{(A, \ldots \between \lambda, \mu, \nu)^2}\}^2,$$

and the remainder of the process of decomposition is effected without difficulty.

Addition, 18 December, 1862.

The formulæ II. and II. (bis) each of them give the tangents of the conic $(a, \ldots \between x, y, z)^2 = 0$ at the ineunts of intersection with the line $\xi' x + \eta' y + \zeta' z = 0$. A very elegant formula for these ineunts themselves was communicated to me by Mr Spottiswoode, and I have since found that the same or an equivalent formula is made use of by M. Aronhold in his recent valuable memoir, "Ueber eine neue algebraische Behandlungsweise der Integrale irrationaler Differentiale, &c.," *Crelle*, t. LXII. pp. 95—145 (1862). The formula is as follows, viz. for the conic and line,

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$
$$\xi' x + \eta' y + \zeta' z = 0,$$

then

$$x : y : z =$$

$$(l\xi' + m\eta' + n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\xi'} + \eta'(gl + fm + c\eta) - \zeta'(hl + bm + f\eta) + l\sqrt{\phi},$$

$$: (l\xi' + m\eta' + n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\eta'} + \zeta'(al + hm + g\eta) - \xi'(gl + fm + c\eta) + m\sqrt{\phi},$$

$$: (l\xi' + m\eta' + n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\zeta'} + \xi'(hl + bm + f\eta) - \eta'(al + hm + g\eta) + n\sqrt{\phi};$$

where

$$\phi = \begin{vmatrix} & \xi', & \eta', & \zeta' \\ \xi', & a, & h, & g \\ \eta', & h, & b, & f \\ \zeta', & g, & f, & c \end{vmatrix} = -(A, B, C, F, G, H \between \xi', \eta', \zeta')^2,$$

so that $\frac{1}{2}\frac{d\phi}{d\xi'}, \frac{1}{2}\frac{d\phi}{d\eta'}, \frac{1}{2}\frac{d\phi}{d\zeta'}$ are respectively

$$= -(A\xi' + H\eta' + G\zeta'), -(H\xi' + B\eta' + F\zeta'), -(G\xi' + F\eta' + C\zeta'),$$

and where l, m, n are supernumerary arbitrary quantities.

277.

ON THE WAVE SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 16—22.]

SOME very beautiful results in relation to the Wave Surface have been recently obtained by Herr Zech, in the Memoirs "Die Eigenschaften der Wellenflächen der zweiaxigen Krystalle mittels der höhern Geometrie abgeleitet," *Crelle*, t. LII. pp. 243—254 (1856), and "Die Krümmungslinien der Wellenfläche zweiaxiger Krystalle," *Crelle*, t. LIV. pp. 72—77 (1857). According to the former of Fresnel's two modes of generation, the wave surface is the envelope of a plane whose perpendicular distance v is a certain given function of the direction cosines l, m, n. For the same system of direction cosines, there are in fact two values of the perpendicular distance: call these v and w, and let the corresponding planes (parallel of course to each other) be called P and Q. Then the entire system of the planes P and Q envelope the wave surface, viz. the planes P may be considered as enveloping one sheet and the planes Q the other sheet of the surface. But if instead of considering the entire system of planes we consider only the planes P and the parallel planes Q, for which the perpendicular on the plane P has a given constant value v, then the planes P will envelope a developable F, and the planes Q will envelope a developable G, these two developables being, it is to be observed, distinct surfaces, not sheets of one and the same surface. Or, what is the same thing, the planes P for which the perpendicular distance has a given constant value v will envelope a developable F, and the planes P for which the perpendicular distance of the parallel planes Q has a given constant value w, will envelope a developable G. And it is obvious that the developables F and G touch the wave surface along curves. The equation of the developable F contains of course the arbitrary parameter v, and the equation of the developable G contains in like manner the arbitrary parameter w, so that we in fact have two series of developables

F and G respectively touching the wave surface along two series of curves. And it is shown in the second of the Memoirs above referred to that these curves are the *curves of curvature* of the wave surface.

The developable F is obtained as the envelope of the plane P, whose equation is

$$lx + my + nz = v,$$

where v has a given constant value and l, m, n are parameters which vary, subject to the two conditions

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0.$$

And in like manner the developable G is obtained as the envelope of the plane Q, whose equation is

$$lx + my + nz = \frac{abc}{w}\sqrt{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)},$$

where w has a given constant value and l, m, n are parameters which vary, subject to the two conditions

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{a^2 - w^2} + \frac{m^2}{b^2 - w^2} + \frac{n^2}{c^2 - w^2} = 0.$$

{It is hardly necessary to remark that if in the last-mentioned system of equations, the parameter w is also treated as variable, we obtain the wave surface: in fact v^2, w^2 being the roots of the equation

$$\frac{l^2}{a^2 - \theta} + \frac{m^2}{b^2 - \theta} + \frac{n^2}{c^2 - \theta} = 0,$$

we have, attending to the condition $l^2 + m^2 + n^2 = 1$, the identical equation

$$l^2(b^2 - \theta)(c^2 - \theta) + m^2(c^2 - \theta)(a^2 - \theta) + n^2(a^2 - \theta)(b^2 - \theta) = (v^2 - \theta)(w^2 - \theta),$$

and thence

$$vw = abc\sqrt{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)},$$

which shows that the system of equations for the plane Q (w being treated as variable) is in fact identical with the system of equations for the plane P.}

The form of the equations of the planes P and Q respectively shows that each of these planes is parallel to a tangent plane of the cone

$$\frac{x^2}{a^2 - v^2} + \frac{y^2}{b^2 - v^2} + \frac{z^2}{c^2 - v^2} = 0,$$

or in other words, that the planes P and Q are respectively tangents to the conic or infinitely thin surface of the second order, which is the second of the last-mentioned cone by the plane at infinity. Moreover it is obvious from the same equations that the plane P is a tangent plane of the sphere whose equation is

$$x^2 + y^2 + z^2 = v^2,$$

and that the plane Q is a tangent plane of the ellipsoid whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = \frac{a^2b^2c^2}{v^2}.$$

Hence each of the developables F and G is the envelope of a plane which is the common tangent plane of two surfaces of the second order; such developables are in general of the eighth order, see my paper "On the Developable Surfaces which arise from Two Surfaces of the Second Order," *Camb. and Dubl. Math. Journ.*, t. v. pp. 46—57 (1850), [84], and it will be presently seen that this is in fact the order of the developables F and G respectively.

The before-mentioned cone

$$\frac{x^2}{a^2 - v^2} + \frac{y^2}{b^2 - v^2} + \frac{z^2}{c^2 - v^2} = 0,$$

(Zech's cone K) is a cone having for its focal lines the optic axes (or normals to the circular sections) of the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ (Zech's ellipsoid E) which is used in the theory of the Wave Surface in the place of Fresnel's Surface of Elasticity. The complementary cone

$$(a^2 - v^2)\, x^2 + (b^2 - v^2)\, y^2 + (c^2 - v^2)\, z^2 = 0,$$

(Zech's cone C) meets the last-mentioned ellipsoid in a curve lying on the sphere whose equation is $x^2 + y^2 + z^2 = \frac{1}{v^2}$, a property which may be considered as affording the geometrical construction of the magnitude v, by means of the cone K. The ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = \frac{a^2b^2c^2}{v^2}$ is obviously an ellipsoid similar to the ellipsoid E, and the value of v being determined as above, the sphere $x^2+y^2+z^2=v^2$ and the ellipsoid $a^2x^2+b^2y^2+c^2z^2=\frac{a^2b^2c^2}{v^2}$ (which are used in the preceding geometrical construction of the developables F and G respectively) may be considered as given by construction.

To find the equation of the developable F, we have, from the equations

$$\begin{aligned} lx + my + nz &= v, \\ l^2 + m^2 + n^2 &= 1, \\ \frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} &= 0, \end{aligned}$$

treating them by the method of arbitrary multipliers in the usual manner, we obtain

$$x+\left(\rho+\frac{\theta}{a^2-v^2}\right)\ l=0,$$

$$y+\left(\rho+\frac{\theta}{b^2-v^2}\right)m=0,$$

$$z+\left(\rho+\frac{\theta}{c^2-v^2}\right)n=0,$$

equations which give in the first instance $v+\rho=0$ or $\rho=-v$, and then substituting this value for ρ,

$$l=\frac{x(v^2-a^2)}{v(v^2-a^2)+\theta},\qquad m=\frac{y(v^2-b^2)}{v(v^2-b^2)+\theta},\qquad z=\frac{z(v^2-c^2)}{v(v^2-c^2)+\theta},$$

and thence

$$\frac{x^2(v^2-a^2)}{v(v^2-a^2)+\theta}+\frac{y^2(v^2-b^2)}{v(v^2-b^2)+\theta}+\frac{z^2(v^2-c^2)}{v(v^2-c^2)+\theta}=v,$$

$$\frac{x^2(v^2-a^2)}{[v(v^2-a^2)+\theta]^2}+\frac{y^2(v^2-b^2)}{[v(v-b^2)+\theta]^2}+\frac{z^2(v^2-c^2)}{[v(v^2-c^2)+\theta]^2}=0,$$

the latter of which is the derived equation of the former with respect to the parameter θ, hence writing the former equation under the form

$$\begin{aligned}&\{\theta+v(v^2-a^2)\}\{\theta+v(v^2-b^2)\}\{\theta+v(v^2-c^2)\}\\&\quad-\frac{a^2}{v^2}v(v^2-a^2)\ \{\theta+v(v^2-b^2)\}\{\theta+v(v^2-c^2)\}\\&\quad-\&c.=0,\end{aligned}$$

or what is the same thing

$$(A,\ B,\ C,\ D)(\theta,\ 1)^3=0,$$

where

$A=3v^2,$

$B=(v^2-x^2)(v^2-a^2)+(v^2-y^2)(v^2-b^2)+(v^2-z^2)(v^2-c^2),$

$C=(v^2-y^2-z^2)(v^2-b^2)(v^2-c^2)+(v^2-z^2-x^2)(v^2-c^2)(v^2-a^2)+(v^2-x^2-y^2)(v^2-a^2)(v^2-b^2),$

$D=3(v^2-x^2-y^2-z^2)(v^2-a^2)(v^2-b^2)(v^2-c^2),$

the equation of the developable F is

$$(AD-BC)^2-4(AC-B^2)(BD-C^2)=0.$$

The investigation for the developable G is very similar to the preceding. Write for shortness

$$\Lambda^2=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2},$$

then the system of equations is

$$\begin{aligned}lx\ +\ my\ +\ nz-\frac{abc}{w}\Lambda&=0,\\ l^2\ +\ m^2\ +\ n^2\qquad&=1,\\ \frac{l^2}{a^2-w^2}+\frac{m^2}{b^2-w^2}+\frac{n^2}{c^2-w^2}\qquad&=0,\end{aligned}$$

and we then have

$$x - \frac{abc}{wa^2\Lambda}\, l + \left(\rho + \frac{\theta}{a^2 - w^2}\right) l = 0,$$

$$y - \frac{abc}{wb^2\Lambda}\, m + \left(\rho + \frac{\theta}{b^2 - w^2}\right) m = 0,$$

$$z - \frac{abc}{wc^2\Lambda}\, n + \left(\rho + \frac{\theta}{c^2 - w^2}\right) n = 0,$$

equations which give $\rho = 0$, and substituting this value, we obtain

$$l = \frac{x\,(w^2 - a^2)}{\dfrac{abc}{wa^2\Lambda}\,(w^2 - a^2) + \theta},$$

$$m = \frac{y\,(w^2 - b^2)}{\dfrac{abc}{wb^2\Lambda}\,(w^2 - b^2) + \theta},$$

$$n = \frac{z\,(w^2 - c^2)}{\dfrac{abc}{wc^2\Lambda}\,(w^2 - c^2) + \theta}.$$

Substituting these values in the equations

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = \Lambda^2,$$

$$\frac{l^2}{w^2 - a^2} + \frac{m^2}{w^2 - b^2} + \frac{n^2}{w^2 - c^2} = 0,$$

we find

$$\frac{\dfrac{x^2}{a^2}(w^2 - a^2)^2}{\left\{\dfrac{abc}{wa^2}(w^2 - a^2) + \Lambda\theta\right\}^2} + \frac{\dfrac{y^2}{b^2}(w^2 - b^2)^2}{\left\{\dfrac{abc}{wb^2}(w^2 - b^2) + \Lambda\theta\right\}^2} + \frac{\dfrac{z^2}{c^2}(w^2 - c^2)^2}{\left\{\dfrac{abc}{wc^2}(w^2 - c^2) + \Lambda\theta\right\}^2} = 1,$$

$$\frac{x^2\,(w^2 - a^2)}{\left\{\dfrac{abc}{wa^2}(w^2 - a^2) + \Lambda\theta\right\}^2} + \frac{y^2\,(w^2 - b^2)}{\left\{\dfrac{abc}{wb^2}(w^2 - b^2) + \Lambda\theta\right\}^2} + \frac{z^2\,(w^2 - c^2)}{\left\{\dfrac{abc}{wc^2}(w^2 - c^2) + \Lambda\theta\right\}^2} = 0,$$

and multiplying the first equation by $\dfrac{abc}{w}$ and the second equation by $\Lambda\theta$ and adding, we obtain

$$\frac{x^2\,(w^2 - a^2)}{\dfrac{abc}{wa^2}(w^2 - a^2) + \Lambda\theta} + \frac{y^2\,(w^2 - b^2)}{\dfrac{abc}{wb^2}(w^2 - b^2) + \Lambda\theta} + \frac{z^2\,(w^2 - c^2)}{\dfrac{abc}{wc^2}(w^2 - c^2) + \Lambda\theta} = \frac{abc}{w},$$

of which equation the latter of the foregoing two equations is the derived equation with respect to the parameter $\Lambda\theta$.

Comparing this with the foregoing equation

$$\frac{x^2(v^2-a^2)}{v(v^2-a^2)+\theta}+\frac{y^2(v^2-b^2)}{v(v^2-c^2)+\theta}+\frac{z^2(v^2-c^2)}{v(v^2-c^2)+\theta}=v,$$

we see that it is deduced from it by writing

$$a^2x^2,\ b^2y^2,\ c^2z^2,\ \frac{w^2}{a^2}-1,\ \frac{w^2}{b^2}-1,\ \frac{w^2}{c^2}-1,\ \frac{abc}{w^2},\ \Lambda\theta$$

in the place of

$$x^2,\quad y^2,\quad z^2,\ v^2-a^2,\ v^2-b^2,\ v^2-c^2,\ v\ ,\quad \theta$$

respectively, and the equation of the developable G is therefore

$$(A'D'-B'C')^2-4(A'C'-B'^2)(B'D'-C'^2)=0,$$

where A', B', C', D' are what A, B, C, D become when

$$x^2,\quad y^2,\quad z^2,\ v^2-a^2,\ v^2-b^2,\ v^2-c^2,\ v\ ,$$

are replaced by

$$a^2x^2,\ b^2y^2,\ c^2z^2,\ \frac{w^2}{a^2}-1,\ \frac{w^2}{b^2}-1,\ \frac{w^2}{c^2}-1,\ \frac{abc}{w}.$$

The equations of the developables F and G, although radically distinct from each other, are consequently similar in form, and each is at once deducible from the other.

2, *Stone Buildings, W.C., 9th March,* 1858.

278.

NOTE ON THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 36, 37.]

THE following investigation (which has been in my possession for a good many years) affords I think a simple explanation of the theory of the singular solutions of differential equations.

Let the primitive equation be

$$c^n + Pc^{n-1} + Qc^{n-2} + \ldots = 0,$$

where c is the arbitrary constant and P, $Q \ldots$ are any functions of x, y; then the differential equation is obtained by eliminating c from the foregoing equation and the derived equation

$$P'c^{n-1} + Q'c^{n-2} + \ldots = 0,$$

and the result may be represented by

$$F(P, Q, \ldots, P', Q', \ldots) = 0.$$

Assume now

$$c^n + Pc^{n-1} + Qc^{n-2} + \ldots = (c+X)(c+Y)(c+Z)\ldots,$$

then we have

$$\begin{aligned} P &= X + Y + Z + \&c., \\ Q &= XY + XZ + YZ + \&c., \\ &\&c., \end{aligned}$$

and consequently

$$\begin{aligned} P' &= X' + Y' + Z' + \&c., \\ Q' &= (Y + Z + \&c.)\, X' + \&c., \\ &\&c., \end{aligned}$$

and substituting these values in the function $F(P, Q, \ldots P', Q', \ldots)$ it is clear that we shall have $F(P, Q, \ldots, P', Q', \ldots) = UX'Y'Z'\ldots$ where U is a symmetrical function of X, Y, Z, &c., and therefore a function of $P, Q, \ldots$; and this equation will be identically true whatever values we attribute to $X', Y', Z', \ldots$, hence putting these quantities respectively equal to unity, we have

$$P' = n,$$
$$Q' = (n-1)P,$$
$$R' = (n-2)Q,$$
&c.,

and with these values

$$U = F(P, Q, \ldots, P', Q', \ldots),$$

that is $U = 0$ is the result obtained by eliminating c from the primitive equation and the equation

$$nc^{n-1} + (n-1)Pc^{n-2} + \ldots = 0,$$

which is the equation obtained by differentiating the primitive equation with respect to the arbitrary constant c: that is, $U = 0$ being the singular solution, the differential equation is

$$UX'Y'Z'\ldots = 0.$$

It is to be remarked that (P, Q, &c. being rational and integral functions) then if the roots X, Y, Z, &c. are also rational and integral functions, the differential equation contains U as a separable rational and integral factor, but if the roots are irrational then the differential equation does not really contain the rational and integral factor U, but $X'Y'Z'\ldots$ is here a rational fraction containing U in the denominator and $UX'Y'Z'\ldots$ is an indecomposable rational and integral function. This is easily verified *à posteriori* for a quadratic equation.

2, *Stone Buildings, W.C., 28th January*, 1858.

279.

ON A THEOREM RELATING TO SPHERICAL CONICS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), p. 53.]

THE following theorem was given by Prof. Maccullagh: "If three lines at right angles to each other pass through a fixed point O so that two of them are confined to given planes: the third line traces out a cone of the second order whose sections parallel to the given planes are circles, and the plane containing the other two lines envelopes a cone of the second order whose sections by planes parallel to the given planes are parabolas."

Referring the figure to the sphere we have a trirectangular triangle XYZ, of which two angles X, Y lie on fixed arcs A, B. The angle Z generates a spherical conic U' having A, B for its cyclic arcs. The side XY envelopes a spherical conic U touched by the arcs A, B. The conic U' is evidently the supplementary conic of U, hence the poles of A, B are the foci of U. We may drop altogether the consideration of the triangle XYZ and consider only the side XY, we have then the theorem:

If a quadrantal arc XY slides between the two fixed arcs A, B, the envelope of XY is a spherical conic U touched by the fixed arcs A, B, and which has for its foci the poles of these same arcs A, B.

It is worth while to notice the great reduction of order which takes place in consequence of the arc XY being a quadrant. If XY had been an arc of a given magnitude θ, the envelope would have been a spherical curve of an order certainly higher than 6. For considering the corresponding problem in plano, the envelope in the particular case where the fixed lines A, B are at right angles to each other is a curve of the sixth order, and in the general case where the two fixed lines are not at right angles the order is higher: the problem in plano corresponds of course, not to the general problem on the sphere, but to that in which θ is indefinitely small.

280.

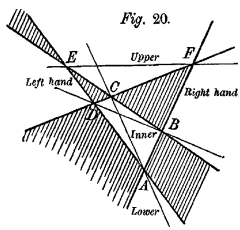
ON THE CONICS WHICH TOUCH FOUR GIVEN LINES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 94—96.

THERE are considerable practical difficulties in drawing a figure of the system of conics which touch four given lines, but a notion of the figure of the system may be obtained as follows:

Figure 20 may be taken to represent any quadrilateral whatever, having all its sides real; and if we attend only to the unshaded spaces, it will be seen that there are five regions which are called the inner, upper, lower, right-hand, and left-hand regions respectively. The inner diagonals are AC the vertical diagonal and BD the horizontal diagonal; the former of these traverses the inner, upper, and lower regions;

Fig. 20.



the latter the inner, right-hand, and left-hand regions; the outer diagonal is EF which traverses the upper region and the right-hand and left-hand regions. The inner or vertical and horizontal diagonals meet in the inner region, the vertical and outer diagonals meet in the upper region, the horizontal and outer diagonals meet in

the left-hand region. No conic touching the four lines lies wholly or in part in the shaded regions, and every conic touching the four lines lies wholly in the inner region or wholly in the upper region, or partly in the upper and partly in the lower region, or partly in the right-hand and partly in the left-hand region. It will be convenient to consider, 1°. the conics which lie in the inner region; 2°. the conics which lie in the upper and lower regions, or in the upper region only; 3°. the conics which lie in the right-hand and left-hand regions.

1°. The conics in the inner region are obviously ellipses; an extreme term is the finite right line BD, considered as an indefinitely thin ellipse; this gradually broadens out and there is (as a mean term) an ellipse which touches the four lines in the points in which they are intersected two and two by the lines joining the points E, F with the point of intersection of the diagonals AC and BD, the ellipse then narrows in the transverse direction and at length reduces itself to the finite line AC considered as an indefinitely thin ellipse.

2°. Considering first the conics which lie in the upper and lower regions, these are of course hyperbolas, an extreme term is the infinite portions $A\infty$ and $C\infty$ of the line AC, considered as an indefinitely thin hyperbola: we have then hyperbolas such that for each of them, one branch lies in the lower region and touches the two lines through A, while the other branch lies in the upper region and touches the two lines through C; the points of contact of the lower branch with the lines through A gradually recede from A, but the point of contact on the line AD, which adjoins the left-hand region, recedes with the greater rapidity, and it at last becomes infinite while the point of contact with the line AB which adjoins the right-hand region remains finite: we have thus a hyperbola having the line AD for its asymptote; the point at infinity of this line belongs of course indifferently to the upper and lower regions; and we may therefore consider one branch as lying in the lower region and touching AB and (at infinity) AD; the other branch as lying in the upper region and touching the two lines through C, and besides (at infinity) the line AD. We have next a series of hyperbolas such that for each of them, one branch lies in the lower region and touches only the line AB, the other branch lies in the upper region and touches the two lines through C and besides the line AD. We arrive again at a limiting case when the point of contact on the line AB passes off to infinity, or AB becomes an asymptote; we have here in the lower region a branch touching (at infinity) AB and in the upper region a branch touching the two lines through C, the line AD, and besides (at infinity) the line AB. To this succeeds a series of hyperbolas such that for each of them, one branch lies in the lower region but does not touch either of the lines through A, the other branch lies in the upper region and touches as well the two lines through C as also the two lines through A. Finally the branch in the lower region passes off to infinity, or the conic becomes a parabola lying wholly in the upper region.

We have thus arrived at the conics which lie wholly in the upper region, the extreme term being a parabola: this passes into an ellipse touching of course the four lines, and gradually reducing itself to the finite line EF considered as an indefinitely thin ellipse.

3°. We come now to the conics which lie in the right-hand and left-hand regions; an extreme term is the conic composed of the infinite portions $E\infty$ and $F\infty$ of the line EF, considered as an indefinitely thin hyperbola; we have then a series of hyperbolas such that for each of them the branch in the right-hand region touches the two lines through E, and the branch in the left-hand region the two lines through F; we then arrive at a limiting case where the branch in the right-hand region touches the line EC the upper boundary of the right-hand region at ∞, or where EC is an asymptote; we may say that the branch in the right-hand region touches the line EB and (at infinity) the line EC, while the branch in the left-hand region touches not only the two lines through F but also (at infinity) the line EC; we have next a series of hyperbolas such that for each of them the branch in the right-hand region touches only the line EB while the branch in the left-hand region touches as well the two lines through F as the line EC; we have then again a limiting case, viz. the branch in the right-hand region touches the line BC at infinity, or BC is an asymptote; we may here say that the branch in the right-hand region touches the line BE and (at infinity) the line BC, while the branch in the left-hand region touches the lines through D and (at infinity) the line BC; and to this succeeds a series of hyperbolas such that for each of them the branch in the right-hand region touches the two lines through B while the branch in the left-hand region touches the two lines through D; the hyperbola finally becomes the infinite portions $B\infty$ and $D\infty$ of the line BD, considered as an indefinitely thin hyperbola.

It is to be noticed that the entire system commences with the finite line BD considered as an indefinitely thin ellipse and passes on continuously to the infinite portions $B\infty$ and $D\infty$ of the same line, considered as an indefinitely thin hyperbola: it is therefore a cyclical series, and any term whatever might have been taken for the commencement of the series.

281.

NOTE ON THE WAVE SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 142—144.]

IN the paper "On the Wave Surface" in the last Number of the *Journal*, [277], I stated that it was shown in the second of Zech's Memoirs that the curves of contact of the developables F and G with the wave surface were *the curves of curvature* of the wave surface. I had not examined the demonstration of this theorem, and I had overlooked the author's note "Die Krümmungslinien der Wellenfläche zweiaxiger Krystalle, &c.," *Crelle*, t. LIV. p. 94 (Feb. 1858), where he points out that in a phrase which he quotes, he had assumed without demonstration a theorem which was in fact erroneous, and that he retracted all that he had said in No. 11 of his Memoir (the portion which contains the theorem as to the curves of curvature of the wave surface). M. Bertrand in a note in the *Comptes Rendus*, t. XLVII. pp. 817—819 (Nov. 1858), after referring to my paper, remarks that the theorem as to the curves of curvature of the wave surface appeared to him so remarkable that he hastened to investigate a proof of it, but that he very soon discovered that the theorem was unfortunately erroneous; and he proceeds to show why the theorem cannot be true. M. Bertrand's demonstration is as follows:

THEOREM I. If from a point O, we let fall perpendiculars on the tangent planes of a surface, the locus of their feet is a new surface. Let P be a point of this surface corresponding to the point M of the first surface, the normal at P passes through the middle point of OM.

THEOREM II. If the curve of curvature of a surface is such that the tangent planes at the several points of the curve are equidistant from a point O, the curve of curvature is situate on a sphere having the point O for its centre.

For suppose that we let fall from the point O perpendiculars on the tangent planes to the surface at the several points of the curve of curvature in question. The

locus of the feet will be a spherical curve, a normal at any point P of the curve will it is clear pass through the point O; besides, in virtue of the first theorem, another normal will pass through the middle point of the radius OM drawn to the corresponding point M of the surface; the tangent to the curve which is the locus of the points P is therefore perpendicular to the plane MOP, and consequently to the line MP. But when a developable surface is circumscribed about a sphere, the perpendiculars let fall from the centre of the sphere on the tangent planes of the developable surface have their feet on the generating lines; and consequently the curve which is the locus of the points P is situate on the developable surface (viz., the developable surface enveloped by the tangent planes at the several points of the curve of curvature of the given surface) and cuts the generating lines at right angles. But the curve, the locus of the point M, being by hypothesis a curve of curvature of the given surface, also cuts at right angles the generating lines of the developable surface, the two curves are therefore equidistant curves on the developable surface, viz. MP is constant, and since by hypothesis OP is constant, OM is also constant, which proves the second theorem.

This being premised, it is to be recollected that the wave surface may be generated in two different ways; 1°. it is the locus of the extremities of the central perpendiculars to the diametral sections of an ellipsoid E, equal to the axes of these sections; 2°. it is the envelope of planes parallel to the diametral sections of a second ellipsoid E' at distances inversely proportional to the axes of the sections. To obtain all the tangent planes of the wave surface situate at a distance h from the centre, it is necessary to find in the ellipsoid the diametral sections which have an axis equal to $\frac{1}{h}$; for this, we may cut the ellipsoid by a concentric sphere of the radius $\frac{1}{h}$, and draw tangent planes to the cone having for its vertex the centre of the ellipsoid and passing through the curve of intersection with the sphere. The tangent planes of the wave surface respectively parallel to the tangent planes of the cone will be at the distance h from the centre, and, if they touched the wave surface along a curve of curvature, it would follow from the foregoing theorem II. that their points of contact would be all at the same distance from the centre; but the distance of the centre of the wave surface from the point of contact of a tangent plane is inversely proportional to the perpendicular let fall from the centre of the ellipsoid E' on the tangent plane at the corresponding point of this ellipsoid: it follows that the curve of intersection of an ellipsoid with a concentric sphere is not such that the tangent planes of the ellipsoid at the several points of the curve are at the same distance from the centre, and consequently the tangent planes which were under consideration do *not* determine by their points of contact a curve of curvature of the wave surface.

I remark in relation to M. Bertrand's theorem I. that it is immediately connected with a well-known theorem which occurs in optics—viz., if rays proceeding from a point are reflected at a surface, and if from the radiant point to the surface and thence along the reflected ray, we measure off a constant distance, the surface which is the locus of the point so obtained (the secondary focal surface of Dandelin and Quetelet)

is an orthogonal trajectory of the reflected rays. In fact, if O be the radiant point, and OM' the incident ray, and if from M' we measure off on the reflected ray a distance $= -OM'$, that is, on the reflected ray produced backwards, a distance $M'P = OM'$, then the whole distance from O is $OM' - OM' = 0$, and the surface which is the locus of the point P is consequently an orthogonal trajectory to the reflected rays. But the point P may, it is clear, be constructed as follows: viz., on the tangent plane at M' let fall the perpendicular OP', and produce it to a point P such that $OP' = P'P$. And if we produce OM' to M so that $OM' = M'M$, then it is clear that the locus of M is a surface similar and similarly situated with the original surface, but of double the magnitude, and that OP is the perpendicular from O upon the tangent plane at M of the last-mentioned surface. And, by what precedes, the line PM' from P to the middle point of OM is a normal of the surface which is the locus of P: the corresponding theorem *in plano* is in fact actually given by Dandelin.

31*st Dec.*, 1858.

282.

ON A PARTICULAR CASE OF CASTILLON'S PROBLEM.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 157—164.

THE problem referred to as Castillon's problem, being itself a particular case of the problem of the in-and-circumscribed triangle, is as follows: viz. "In a circle to inscribe a triangle the sides of which pass through three given points," and the reciprocal problem of course presents itself "about a circle to circumscribe a triangle, the angles of which lie in given lines." If in Castillon's problem the three given points are the angles of a triangle circumscribed about the circle, or if in the reciprocal problem the three given lines are the sides of a triangle circumscribed about the circle, we have a circle and an inscribed and circumscribed triangle, such that the sides of the inscribed triangle pass through the angles of the circumscribed triangle, and the problem arises "given one of the triangles to determine the other triangle." The problem, so far as I am aware, was first proposed by Clausen, who has given, *Crelle*, t. IV. (1829), p. 391, a very elegant solution, which I propose to reproduce here.

Let the angle of a point be defined as the inclination to a fixed radius, of the line from the centre through the given point.

Let α, β, γ, be the angles of the points of contact of the sides of the circumscribed triangle.

α', β', γ', the angles of the angular points of the circumscribed triangle.

a, b, c, the angles of the angular points of the inscribed triangle.

a', b', c', the angles of the intersections of the perpendiculars from the centre on the sides of the inscribed triangle.

Therefore

$$\begin{aligned} 2a' &= b + c, & 2\alpha' &= \beta + \gamma, \\ 2b' &= c + a, & 2\beta' &= \gamma + \alpha, \\ 2c' &= a + b, & 2\gamma' &= \alpha + \beta. \end{aligned}$$

Hence observing that the distance of one of the angles of the circumscribed triangle is $\sec\frac{1}{2}(\beta-\gamma)$, and that the projection of this line perpendicular to the corresponding side of the inscribed triangle is equal to $\sec\frac{1}{2}(\beta-\gamma)\cos(\alpha'-a')$, which is equal to the projection of the radius perpendicular to the same side, or to $\cos\frac{1}{2}(b-c)$, we have

$$\cos(\alpha'-a') = \cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(b-c),$$
$$\cos(\beta'-b') = \cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(c-a),$$
$$\cos(\gamma'-c') = \cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(a-b).$$

Write

$$b-c=4f,\quad b+c-2\alpha=4x,$$
$$c-a=4g,\quad c+a-2\beta=4y,$$
$$a-b=4h,\quad a+b-2\gamma=4z;$$

therefore

$$\cos(y+z+g-h)=\cos 2f\cos(f+y-z),$$
$$\cos(z+x+h-f)=\cos 2g\cos(g+z-x),$$
$$\cos(x+y+f-g)=\cos 2h\cos(h+x-y),$$

or, since $f+g+h=0$,

$$\cos\{(y-h)+(z+g)\}=\cos 2f\cos\{(y-h)-(z+g)\},$$
$$\cos\{(z-f)+(x+h)\}=\cos 2g\cos\{(z-f)-(x+h)\},$$
$$\cos\{(x-g)+(y+f)\}=\cos 2h\cos\{(x-g)-(y+f)\},$$

or

$$\tan^2 f=\tan(y-h)\tan(z+g),$$
$$\tan^2 g=\tan(z-f)\tan(x+h),$$
$$\tan^2 h=\tan(x-g)\tan(y+f).$$

Put

$$x+h=\eta,\quad x-g=\zeta_1,$$
$$y+f=\zeta,\quad y-h=\xi_1,$$
$$z+g=\xi,\quad z-f=\eta_1,$$

$$\tan f=l,\quad \tan\xi=\mathrm{x},\quad \tan\xi_1=\mathrm{x}_1,$$
$$\tan g=m,\quad \tan\eta=\mathrm{y},\quad \tan\eta_1=\mathrm{y}_1,$$
$$\tan h=n,\quad \tan\zeta=\mathrm{z},\quad \tan\zeta_1=\mathrm{z}_1.$$

We have then

$$\tan^2 f=\tan\xi\tan\xi_1,$$
$$\tan^2 g=\tan\eta\tan\eta_1,$$
$$\tan^2 h=\tan\zeta\tan\zeta_1,$$

or, what is the same thing,

$$l^2=\mathrm{x}\mathrm{x}_1,\quad m^2=\mathrm{y}\mathrm{y}_1,\quad n^2=\mathrm{z}\mathrm{z}_1.$$

But to obtain equations involving only one of the sets (x, y, z), (x_1, y_1, z_1), it is proper to write

$$\tan^2 f = \tan(\zeta + g)\tan\xi = \tan\xi_1 \tan(\zeta_1 - h),$$
$$\tan^2 g = \tan(\xi + h)\tan\eta = \tan\eta_1 \tan(\xi_1 - f),$$
$$\tan^2 h = \tan(\eta + f)\tan\zeta = \tan\zeta_1 \tan(\eta_1 - g);$$

taking the first set of equations, we have

$$l^2 = \frac{x(z+m)}{1-mz},$$

$$m^2 = \frac{y(x+n)}{1-nx},$$

$$n^2 = \frac{z(y+l)}{1-ly};$$

therefore

$$z = \frac{l^2 - mx}{x + l^2 m}, \qquad y = \frac{m^2(1-nx)}{x+n};$$

therefore

$$y + l : 1 - ly = m^2 + ln + (l - m^2 n)x : n - lm^2 + (1 + lm^2 n)x;$$

therefore

$$n^2\{n - lm^2 + (1 + lm^2 n)x\}(l^2 m + x) = (l^2 - mx)\{m^2 + ln + (l - m^2 n)x\};$$

therefore

$$\begin{aligned}&(l^2m^2 + l^3n - l^2mn^3 + l^3m^3n^2)\\ &+ x\,(-n^3 + lm^2n^2 - l^2mn^2 - l^3m^3n^3 - m^3 - lmn + l^3 - l^2m^2n)\\ &+ x^2(-lm + m^3n - n^2 - lm^2n^3) = 0.\end{aligned}$$

Now $l + m + n = lmn$, and by means of this relation we find

$$\text{Coef. } x^\circ = l\{lm^2 + l^2n - n^2(l+m+n) + m(l+m+n)^2\},$$

$$\begin{aligned}\text{Coef. } x &= -n^3 + mn(l+m+n) - ln(l+m+n) - (l+m+n)^3 - m^3\\ &\qquad - lmn + l^3 - lm(l+m+n),\end{aligned}$$

$$\text{Coef. } x^2 = \frac{1}{l}\{-l^2m + m^2(l+m+n) - ln^2 - n(l+m+n)^2\};$$

or, reducing,

$$\begin{aligned}\frac{1}{l}\text{ coef. } x^\circ &= (m^3 + 2m^2n - n^3) + l(3m^2 + 2mn - n^2) + l^3(m+n)\\ &= (m+n)\{(m^2 + mn - n^2) + l(3m - n) + l^2\}\\ &= (m+n)\{(l+m)^2 + (l+m)(l+n) - (l+n)^2\},\end{aligned}$$

$$\begin{aligned}-\tfrac{1}{2}\text{ coef. } x &= m^3 + m^2n + mn^2 + n^3 + 2l(m^2 + 2mn + n^2) + 2l^3(m+n)\\ &= (m+n)\{m^2 + n^2 + 2l(m+n) + 2l^2\}\\ &= (m+n)\{(l+m)^2 + (l+n)^2\},\end{aligned}$$

$$\begin{aligned}l\text{ coef. } x^2 &= (m^3 - 2mn^2 - n^3) + l(m^2 - 2mn - 3n^2) - l^2(m+n)\\ &= (m+n)\{(m^2 - mn - n^2) + l(m - 3n) - l^2\}\\ &= (m+n)\{(l+m)^2 - (l+m)(l+n) - (l+n)^2\};\end{aligned}$$

and the equation becomes

$$\begin{aligned} & l^2\,\{(l+m)^2+(l+m)(l+n)-(l+n)^2\} \\ -\,& 2l\mathrm{x}\,\{(l+m)^2+(l+n)^2\} \\ +\,& \mathrm{x}^2\,\{(l+m)^2-(l+m)(l+n)-(l+n)^2\}=0, \end{aligned}$$

which may be written

$$\left[\frac{\mathrm{x}}{l}\,\{(l+m)^2-(l+m)(l+n)-(l+n)^2\}-\{(l+m)^2+(l+n)^2\}\right]^2=5\,(l+m)^2(l+n)^2,$$

and we have therefore

$$\frac{\mathrm{x}}{l}=\frac{(l+m)^2+\sqrt{(5)}\,(l+m)(l+n)+(l+n)^2}{(l+m)^2-(l+m)(l+n)-(l+n)^2},$$

or, reducing and observing also that $l^2=\mathrm{x}\mathrm{x}_1$, we have

$$\frac{\mathrm{x}}{l}=\frac{l}{\mathrm{x}_1}=\frac{l+m\ +\frac{1}{2}\{1+\sqrt{(5)}\}(l+n)}{(l+m)-\frac{1}{2}\{1+\sqrt{(5)}\}(l+n)}.$$

The values of $(l,\ m,\ n)$ are

$$l=\tan\tfrac{1}{4}(b-c),\quad m=\tan\tfrac{1}{4}(c-a),\quad n=\tan\tfrac{1}{4}(a-b),$$

and those of $(\mathrm{x},\ \mathrm{y},\ \mathrm{z})$, $(\mathrm{x}_1,\ \mathrm{y}_1,\ \mathrm{z}_1)$ are

$$\begin{aligned} \mathrm{x}\ &=\tan\tfrac{1}{4}(b+c-2\gamma), & \mathrm{y}\ &=\tan\tfrac{1}{4}(c+a-2\alpha), & \mathrm{z}\ &=\tan\tfrac{1}{4}(a+b-2\beta), \\ \mathrm{x}_1&=\tan\tfrac{1}{4}(b+c-2\beta), & \mathrm{y}_1&=\tan\tfrac{1}{4}(c+a-2\beta), & \mathrm{z}_1&=\tan\tfrac{1}{4}(a+b-2\alpha), \end{aligned}$$

and the foregoing result shows that the values of

$$\frac{\mathrm{x}}{l}\ \text{or}\ \frac{l}{\mathrm{x}_1};\quad \frac{\mathrm{y}}{m}\ \text{or}\ \frac{m}{\mathrm{y}_1};\quad \frac{\mathrm{z}}{n}\ \text{or}\ \frac{n}{\mathrm{z}_1}$$

are

$$\frac{\sin\frac{1}{2}(a-b)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(c-a)}{\sin\frac{1}{2}(a-b)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(c-a)},$$

$$\frac{\sin\frac{1}{2}(b-c)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(b-c)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(a-b)},$$

$$\frac{\sin\frac{1}{2}(c-a)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(c-a)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(b-c)}.$$

Write for a moment $\dfrac{\mathrm{x}}{l}=\dfrac{l}{\mathrm{x}_1}=k$; therefore

$$\frac{\mathrm{x}-\mathrm{x}_1}{1+\mathrm{x}\mathrm{x}_1}=\frac{lk-\dfrac{l}{k}}{1+l^2}=\frac{l\left(k-\dfrac{1}{k}\right)}{1+l^2}=\frac{2l}{1+l^2}\div\frac{2k}{k^2-1}=\sin\tfrac{1}{2}(b-c)\,.\,\frac{k^2-1}{2k}$$

$$=\sin\tfrac{1}{2}(b-c)\,.\,\frac{P^2-Q^2}{2PQ}\ \text{ if } k=\frac{P}{Q};$$

and taking P, Q for the numerator and the denominator respectively of the foregoing fractional expression for k, we find

$$P^2 - Q^2 = 2\{1+\sqrt{(5)}\}\sin\tfrac{1}{2}(a-b)\sin\tfrac{1}{2}(c-a),$$

$$\begin{aligned} 2PQ &= 2\left[\sin^2\tfrac{1}{2}(a-b) - \tfrac{1}{4}\{1+\sqrt{(5)}\}^2\sin^2\tfrac{1}{2}(c-a)\right] \\ &= -\{1+\sqrt{(5)}\}\left[\tfrac{1}{2}\{1-\sqrt{(5)}\}\sin^2\tfrac{1}{2}(a-b) + \tfrac{1}{2}\{1+\sqrt{(5)}\}\sin^2\tfrac{1}{2}(c-a)\right]; \end{aligned}$$

also

$$\frac{x-x_1}{1+xx_1} = \tan\tfrac{1}{2}(\beta-\gamma),$$

we have therefore

$$\tan\tfrac{1}{2}(\beta-\gamma) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\sqrt{(5)}\}\sin^2\frac{1}{2}(a-b)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin^2\frac{1}{2}(c-a)},$$

$$\tan\tfrac{1}{2}(\gamma-\alpha) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\sqrt{(5)}\}\sin^2\frac{1}{2}(b-c)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin^2\frac{1}{2}(a-b)},$$

$$\tan\tfrac{1}{2}(\alpha-\beta) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\sqrt{(5)}\}\sin^2\frac{1}{2}(c-a)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin^2\frac{1}{2}(b-c)},$$

equations which determine the circumscribed triangle when the inscribed triangle is given.

The more general problem, for a conic and an inscribed and circumscribed triangle such that the sides of the inscribed triangle pass through the angles of the circumscribed triangle, "given one of the triangles to determine the other" is solved by Möbius, *Crelle*, t. v. (1830), p. 103, by means of his Barycentric Calculus, which is in fact the method of trilinear coordinates. The solution is in effect as follows:

Let $\xi=0$, $\eta=0$, $\zeta=0$ be the equations of the sides of the inscribed triangle, ξ, η, ζ may be considered as containing each of them an implicit constant factor, and the equation of the conic may be taken to be

$$\eta\zeta+\zeta\xi+\xi\eta=0,$$

moreover, if $x=0$, $y=0$, $z=0$ be the equations of the sides of the circumscribed triangle, then considering x, y, z as also containing each of them an implicit constant factor, the equation of the conic may be taken to be

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

Suppose now the sides of the inscribed triangle pass through the angles of the circumscribed triangle, we have equations such as

$$\xi=b'y+cz,\quad \eta=c'z+ax,\quad \zeta=a'x+by;$$

substituting these values we must have identically

$$\eta\zeta+\zeta\xi+\xi\eta+m(x^2+y^2+z^2-2yz-2zx-2xy)=0,$$

or

$$\begin{aligned} aa'+m&=0, & bc+bc'+b'c&=2m, \\ bb'+m&=0, & ca+ca'+c'a&=2m, \\ cc'+m&=0, & ab+ab'+a'b&=2m, \end{aligned}$$

and, substituting for a', b', c' their values,

$$(m-bc)^2 = mb^2, \quad (m-ca)^2 = mc^2, \quad (m-ab)^2 = ma^2,$$

or, if $m = n^2$,

$$n^2 - bc = nb, \quad n^2 - ca = nc, \quad n^2 - ab = na,$$

{the signs on the second side must be all + or all − and the − sign may be omitted without loss of generality}.

Hence

$$a = b = c = -\tfrac{1}{2}\{1+\sqrt{(5)}\}\, n, \quad \sqrt{(5)} \text{ being written for } \pm\sqrt{(5)},$$

$$a' = b' = c' = -\frac{n^2}{a} = -\tfrac{1}{2}\{1-\sqrt{(5)}\}\, n = -\nu a,$$

if for shortness

$$-\nu = \frac{1-\sqrt{(5)}}{1+\sqrt{(5)}}, \text{ i.e. } \nu = \frac{\sqrt{(5)}-1}{\sqrt{(5)}+1}, \text{ or } \nu^2 - 3\nu + 1 = 0,$$

and ν having this value, the equations give

$$\xi = \nu y - z, \quad \eta = \nu z - x, \quad \zeta = \nu x - y,$$

whence also

$$\begin{aligned} 4(2\nu-1)\,x &= \nu\xi + \eta + (3\nu-1)\,\zeta, \\ 4(2\nu-1)\,y &= (3\nu-1)\,\xi + \nu\eta + \zeta, \\ 4(2\nu-1)\,z &= \xi + (3\nu-1)\,\eta + \nu\zeta. \end{aligned}$$

Each side of the circumscribed triangle has on it four points, viz. two angles of the circumscribed triangle, a point of contact with the conic, and an intersection with the corresponding side of the inscribed triangle. Thus for the side $x=0$, the four points are given as the intersections of $x=0$, with

$$y=0, \quad z=0, \quad y-z=0, \quad \nu y + z = 0,$$

and the anharmonic ratio of the four points is therefore a given quantity.(*)

Again, each side of the inscribed triangle has on it four points, viz. two angles of the inscribed triangle, a point of intersection with the tangent at the opposite angle of the inscribed conic, and a point of intersection with the corresponding side of the circumscribed triangle.

Thus for the side $\xi=0$, the four points are given as the points of intersection of $\xi=0$ with the lines

$$\eta=0, \quad \zeta=0, \quad \eta+\zeta=0, \quad \eta+(3\nu-1)\,\zeta=0,$$

and the anharmonic ratio of the four points is therefore a given quantity.(*)

If we draw tangents at the angles of the inscribed triangle, we have a new triangle, the sides of which are $\eta+\zeta=0$, $\zeta+\xi=0$, $\eta+\xi=0$, and joining the angles of this triangle with the points of contact of the opposite sides (i.e. the angles of the

inscribed triangle), we have three lines $\eta-\zeta=0$, $\zeta-\xi=0$, $\xi-\eta=0$ meeting in a point $\xi=\eta=\zeta$, which is obviously the same as the point $x=y=z$, which is the point of intersection of the lines joining the angles of the circumscribed triangle with the points of contact of the opposite sides.(*)

The coordinates of the points of contact of the sides of the circumscribed triangle are given by $(x=0,\ y-z=0)$, $(y=0,\ z-x=0)$, $(z=0,\ x-y=0)$, these points form therefore an inscribed triangle the sides of which are

$$y+z-x=0,\quad z+x-y=0,\quad x+y-z=0.$$

Again, the tangents at the angles of the inscribed triangle form a circumscribed triangle the sides of which are

$$\eta+\zeta=0,\quad \zeta+\xi=0,\quad \xi+\eta=0;$$

therefore

$$(\zeta+\xi)-\nu(\xi+\eta)=(1-\nu)\xi-\nu\eta+\zeta=2\nu x(1-\nu+\nu^2)y-(1-\nu+\nu^2)z=-2\nu(y+z-x),$$

and we thus have

$$\zeta+\xi-\nu(\xi+\eta)=-2\nu(y+z-x),$$
$$\xi+\eta-\nu(\eta+\zeta)=-2\nu(z+x-y),$$
$$\eta+\zeta-\nu(\zeta+\xi)=-2\nu(x+y-z),$$

equations which show that the sides of the second inscribed triangle pass through the angles of the second circumscribed triangle, and that the two systems are consequently reciprocal.(*)

The four theorems marked (*) are all of them contained in the paper by Möbius.

283.

ON A THEOREM RELATING TO HOMOGRAPHIC FIGURES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 177—180.]

THE following theorem is not new, but I do not remember where to find it:

Given any two homographic figures; there exists in the first figure a point S, which is the focus of an (ellipse or hyperbola, say an) ellipse Σ; and in the second figure a point S', which is the focus of a (hyperbola or ellipse, say a) hyperbola Σ', such that the points S, S' and the conics Σ, Σ' correspond to each other, and the conics Σ, Σ' are so related that the foci and vertices of the one may be superimposed upon the vertices and foci of the other. Moreover the perpendiculars from S, S' upon corresponding tangents of Σ, Σ' will be equal.

Write

$$x : y : 1 = \alpha\xi + \beta\eta + \gamma : \alpha'\xi + \beta'\eta + \gamma' : \alpha''\xi + \beta''\eta + \gamma'',$$

then l, m, λ, μ may be so determined that $(x-l)+i(y-m)$ shall vanish with $(\xi-\lambda)+i(\eta-\mu)$, for if we write

$$\begin{aligned} J &= \alpha - l\alpha'', & J' &= \alpha' - m\alpha'', \\ K &= \beta - l\beta'', & K' &= \beta' - m\beta'', \\ L &= \gamma - l\gamma'', & L' &= \gamma' - m\gamma'', \end{aligned}$$

we have

$$(x-l)+i(y-m) = \frac{(J+iJ')\{(\xi-\lambda)+i(\eta-\mu)\}}{\alpha''\xi+\beta''\eta+\gamma''};$$

and therefore

$$\begin{aligned} K + iK' &= i(J+iJ'), \\ L + iL' &= (J+iJ')(\lambda+i\mu), \end{aligned}$$

that is

$$K = -J, \quad K' = J,$$

or

$$l\beta'' + m\alpha'' = \alpha' + \beta,$$
$$-l\alpha'' + m\beta'' = -\alpha + \beta',$$

whence also

$$l(\alpha''^2 + \beta''^2) = \alpha'\beta'' - \alpha''\beta' + \alpha\alpha'' + \beta\beta'',$$
$$m(\alpha''^2 + \beta''^2) = \alpha''\beta - \alpha\beta'' + \alpha\alpha' + \beta\beta',$$

which determine l, m; and then λ, μ are given by

$$\lambda + i\mu = \frac{L + iL'}{J + iJ'},$$

but more simply as follows, viz. remarking that if $c = \beta'\gamma'' - \beta''\gamma'$, &c., so that

$$\xi : \eta : 1 = ax + a'y + a'' : bx + b'y + b'' : cx + c'y + c'',$$

then we have

$$\lambda(c^2 + c'^2) = bc' - b'c + ac + a'c',$$
$$\mu(c^2 + c'^2) = ca' - c'a + bc + b'c';$$

the equations $x = l$, $y = m$ give the point S and $\xi = \lambda$, $\eta = \mu$ the point S'. The values of J, J' are

$$J = \frac{-c\alpha'' - c'\beta''}{\alpha''^2 + \beta''^2}, \quad J' = \frac{-c'\alpha'' + c\beta''}{\alpha''^2 + \beta''^2};$$

and therefore

$$J^2 + J'^2 = \frac{c^2 + c'^2}{\alpha''^2 + \beta''^2} = \frac{k^2}{K^2},$$

if

$$k = \sqrt{(c^2 + c'^2)}, \quad K = \sqrt{(\alpha''^2 + \beta''^2)}:$$

we have therefore

$$\sqrt{\{(x-l)^2 + (y-m)^2\}} = \frac{k}{K} \frac{\sqrt{\{(\xi-\lambda)^2 + (\eta-\mu)^2\}}}{\alpha''\xi + \beta''\eta + \gamma''}. \qquad (*)$$

Consider now the expression

$$(x-l)\cos\vartheta + (y-m)\sin\vartheta - \varpi,$$

which made equal to 0 would be the equation of a line the perpendicular distance of which from S is ϖ, and which distance is inclined at an angle ϑ to the axis of x; then

$$(x-l)\cos\vartheta + (y-m)\sin\vartheta = \tfrac{1}{2}[(\cos\vartheta - i\sin\vartheta)\{(x-l) + i(y-m)\}$$
$$+ (\cos\vartheta + i\sin\vartheta)\{(x-l) - i(y-m)\}],$$

$$= \frac{1}{2(\alpha''\xi + \beta''\eta + \gamma'')}[(\cos\vartheta - i\sin\vartheta)\{(\xi-\lambda) + i(\eta-\mu)\}(J + iJ')$$
$$+ (\cos\vartheta + i\sin\vartheta)\{(\xi-\lambda) - i(\eta-\mu)\}(J - iJ')]$$

$$= \frac{k}{2(\alpha''\xi+\beta''\eta+\gamma'')K}\left[\{\cos(\vartheta-\vartheta_0)-i\sin(\vartheta-\vartheta_0)\}\{(\xi-\lambda)+i(\eta-\mu)\}\right.$$
$$\left.+\ \{\cos(\vartheta-\vartheta_0)+i\sin(\vartheta-\vartheta_0)\}\{(\xi-\lambda)-i(\eta-\mu)\}\right]$$
$$= \frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\ \{(\xi-\lambda)\cos(\vartheta-\vartheta_0)+(\eta-\mu)\sin(\vartheta-\vartheta_0)\},$$

where J, J' have been replaced by

$$J=\frac{k}{K}\cos\vartheta_0,\quad J'=\frac{k}{K}\sin\vartheta_0.$$

And putting besides

$$\vartheta-\vartheta_0=\theta,$$

we have more simply

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\{(\xi-\lambda)\cos\theta+(\eta-\mu)\sin\theta\},$$

whence

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi$$
$$=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\left\{(\xi-\lambda)\cos\theta+(\eta-\mu)\sin\theta-\frac{\varpi K}{k}(\alpha''\xi+\beta''\eta+\gamma'')\right\}$$
$$=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\left\{(\xi-\lambda)\left(\cos\theta-\frac{\varpi K\alpha''}{k}\right)+(\eta-\mu)\left(\sin\theta-\frac{\varpi K\beta''}{k}\right)-\frac{\varpi K}{k}(\alpha''\lambda+\beta''\mu+\gamma'')\right\}.$$

Write now

$$\alpha''=K\cos j,\ \beta''=K\sin j;\ \mathrm{a}=\frac{k}{K^2},\ \mathrm{b}=\frac{1}{K}(\alpha''\lambda+\beta''\mu+\gamma'');$$

we have

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\left\{\frac{\mathrm{a}}{\mathrm{b}}\left(\cos\theta-\frac{\varpi\cos j}{\mathrm{a}}\right)(\xi-\lambda)+\frac{\mathrm{a}}{\mathrm{b}}\left(\sin\theta-\frac{\varpi\sin j}{\mathrm{a}}\right)(\eta-\mu)-\varpi\right\}$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\{(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi\},$$

that is

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi\qquad(*)$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\{(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi\},$$

where it will be noticed that

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta-\varpi=0,$$

and

$$(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi=0,$$

are lines in the first figure and in the second figure, corresponding to each other and at the same distance ϖ from the points S, S' respectively. Call these lines T and T'.

But the relations between θ, ϕ, ϖ are given by

$$\varpi \cos j = \mathrm{a} \cos \theta - \mathrm{b} \cos \phi,$$
$$\varpi \sin j = \mathrm{a} \sin \theta - \mathrm{b} \sin \phi,$$

or, by changing the fixed axes from which θ, ϕ are respectively measured,

$$\varpi = \mathrm{a} \cos \theta - \mathrm{b} \cos \phi,$$
$$0 = \mathrm{a} \sin \theta - \mathrm{b} \sin \phi.$$

Now in these equations ϖ is the perpendicular distance from S upon the line $(x-l)\cos\vartheta + (y-m)\sin\vartheta - \varpi = 0$ and θ (which only differs from ϑ by a constant angle) is the inclination of this perpendicular to a certain fixed line; ϖ is also the perpendicular distance of the line $(\xi-\lambda)\cos\phi + (\eta-\mu)\sin\phi - \varpi = 0$ from the point S', and ϕ is the inclination of this perpendicular to a fixed line. Eliminating successively ϕ and θ, we have

$$\varpi = \mathrm{a} \cos \theta - \sqrt{(\mathrm{b}^2 - \mathrm{a}^2 \sin^2 \theta)},$$
$$\varpi = -\mathrm{b} \cos \phi + \sqrt{(\mathrm{a}^2 - \mathrm{b}^2 \sin^2 \phi)},$$

or, as these equations may also be written,

$$\varpi^2 - 2\varpi \mathrm{a} \cos \theta + \mathrm{a}^2 - \mathrm{b}^2 = 0,$$
$$\varpi^2 + 2\varpi \mathrm{b} \cos \phi + \mathrm{b}^2 - \mathrm{a}^2 = 0.$$

Suppose $\mathrm{a} > \mathrm{b}$, the former equation shows that the line T is a tangent to a certain hyperbola, and the latter equation shows that the line T' is a tangent to a certain ellipse, and it is easily seen that, taking for the transverse axes the lines from which the angles θ and ϕ are respectively measured, the equation of the hyperbola is

$$\frac{x^2}{\mathrm{b}^2} - \frac{y^2}{\mathrm{a}^2 - \mathrm{b}^2} = 1,$$

and that of the ellipse is

$$\frac{\xi^2}{\mathrm{a}^2} + \frac{y^2}{\mathrm{a}^2 - \mathrm{b}^2} = 1,$$

which are the conics Σ, Σ' referred to in the enunciation. And if the second conic is superimposed upon the first in such manner that the coordinates ξ, η may belong to the same axes with x, y; then the two conics will have the assumed relation, viz. the foci of either conic will coincide with the vertices of the other conic.

284.

ON A NEW ANALYTICAL REPRESENTATION OF CURVES IN SPACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 225—236.]

THE ordinary analytical representation of a curve in space by the equations of two surfaces passing through the curve is, even in the case where the curve is the complete intersection of the two surfaces, inappropriate as involving the consideration of surfaces which are extraneous to the curve; and the objection becomes more serious when the curve is not the complete intersection of any two surfaces; for in this case the curve can only be represented in conjunction with a curve or curves extraneous to itself. A curve in space can be in a mode represented by means of the equation of the developable surface having the curve for its edge of regression; but this corresponds to the representation of a plane curve by means of its equation in line coordinates; a representation which is very useful in addition to, but which is not to be substituted for, the equation in point coordinates. It occurred to me some years ago that it might be advantageous to represent a curve in space by means of the cone passing through the curve and having for its vertex an arbitrary point[1]; and although I have not advanced beyond the first steps of the theory, the results which I have obtained may, I think, be interesting to geometers. The conclusion is that a curve in space may be represented by a homogeneous equation $V=0$ between six coordinates (p, q, r, s, t, u) such that $ps+qt+ru=0$; the function V is moreover such that in virtue of this relation between the coordinates, and of the equation $V=0$ itself, we have

$$d_p V \,.\, d_s V + d_q V \,.\, d_t V + d_r V \,.\, d_u V = 0,$$

[1] See my paper "On the cones which pass through a curve of the third order in space," *Phil. Mag.*, t. XII. (1856), p. 20, [200], where a curve of the third order in space is in effect so represented.

or what is the same thing we have identically

$$d_p V . d_s V + d_q V . d_t V + d_r V . d_u V = LV + M(ps + qt + ru),$$

where L and M are functions of p, q, r, s, t, u. But the converse proposition, viz. any equation whatever $V=0$, where V satisfies the condition just referred to represents a curve in space, is not true; it would obviously be an important point in the theory to ascertain what further conditions must be satisfied by the function V.

The establishment of the foregoing results is very easy; in fact if x, y, z, w are current coordinates of the ordinary kind (point-coordinates) and $\alpha, \beta, \gamma, \delta$ the coordinates of an arbitrary point; the equation of any cone whatever having for its vertex the point $(\alpha, \beta, \gamma, \delta)$ may be represented by a homogeneous equation between the six determinants of the matrix

$$\begin{pmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{pmatrix}$$

or if we write

$$\begin{aligned} p &= \gamma y - \beta z, & s &= \delta x - \alpha w, \\ q &= \alpha z - \gamma x, & t &= \delta y - \beta w, \\ r &= \beta x - \alpha y, & u &= \delta z - \gamma w, \end{aligned}$$

values which it is well known give identically

$$ps + qt + ru = 0,$$

then the cone will be represented by a homogeneous equation

$$V = 0$$

between the six coordinates (p, q, r, s, t, u). It remains to find the conditions in order that all the cones so represented, viz. the cones obtained by giving any values whatever to the arbitrary quantities $\alpha, \beta, \gamma, \delta$ which enter implicitly into the coordinates p, q, r, s, t, u, pass through one and the same curve; for when this is the case, the equation $V=0$ may be properly considered as the equation of the curve.

Assume then that all the cones pass through the same curve; if we give to one of the arbitrary quantities $\alpha, \beta, \gamma, \delta$, say α, the infinitesimal variation $d\alpha$, then the function V becomes $V + d_\alpha V . d\alpha$, and each of the equations $V=0$, $V + d_\alpha V . d\alpha = 0$ belongs to a cone passing through the curve; the equation $d_\alpha V = 0$ is therefore the equation of a surface passing through the curve; and in like manner the four equations

$$d_\alpha V = 0, \quad d_\beta V = 0, \quad d_\gamma V = 0, \quad d_\delta V = 0$$

are each of them the equation of a surface passing through the curve, or these equations must be simultaneously satisfied for all the points of the curve, they must consequently reduce themselves to two independent relations. But V is given as a

function of p, q, r, s, t, u, through which quantities it is a function of $\alpha, \beta, \gamma, \delta$; the last-mentioned four equations become therefore

$$\begin{aligned}
0 &= \quad . \quad - d_r V . y + d_q V . z - d_s V . w, \\
0 &= d_r V . x \quad . \quad - d_p V . z - d_t V . w, \\
0 &= - d_q V . x + d_p V . y \quad . \quad - d_u V . w, \\
0 &= d_s V . x + d_t V . y + d_u V . z \quad . \quad ,
\end{aligned}$$

and from the first, second, and third equation, or any other combination of three equations, we obtain at once the condition

$$d_p V . d_s V + d_q V . d_t V + d_r V . d_u V = 0,$$

so that if this condition is satisfied, the four equations do in fact reduce themselves to two independent equations; the condition in question is thus shown to be necessary. But the condition only implies that all the cones having their vertices in the neighbourhood of the point $(\alpha, \beta, \gamma, \delta)$ pass through one and the same curve; this will be the case for instance for a series of cones all of them circumscribed about one and the same surface, those having their vertices in the neighbourhood of the point $(\alpha, \beta, \gamma, \delta)$ will all pass through the curve of contact with the surface of the cone having for its vertex the point in question. But the curve of contact is not a fixed curve for all positions of the vertex, and the condition before referred to is consequently insufficient.

It may be noticed that the systems p, q, r and s, t, u are not similar to each other and that the six coordinates cannot be in any way divided into two systems which are similar to each other: the symmetry of the coordinates is in fact that of the vertices (or sides) of a complete quadrilateral (or quadrangle); thus we may divide the coordinates into two sets in a fourfold manner as follows:

$$\begin{aligned}
&u, t, p; \quad r, q, s, \\
&s, q, u; \quad p, t, r, \\
&r, t, s; \quad u, q, p, \\
&p, q, r; \quad s, t, u,
\end{aligned}$$

where each left-hand set corresponds to three vertices forming a triangle and each right-hand set to the remaining three vertices *in lineo*. It may be noticed also that if in the equation $V = 0$ of any curve in space we substitute for p, q, r, s, t, u, their values, and equate to zero the coefficients of the different powers and products of $\alpha, \beta, \gamma, \delta$, each of the equations so obtained will belong to a surface passing through the curve, and the entire system of these equations will be equivalent to two relations only between the coordinates x, y, z, w. But any two of these surfaces will not in general intersect only in the curve, i.e. the curve will not be the complete intersection of any two of the surfaces. It may be added that the equation of any other surface whatever through the curve will be obtained by equating to zero a syzygetic function of the functions which equated to zero give the surfaces first referred to.

As an example of the theory, the equation, in the new coordinates, of a line in space will be

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where the constants are such that

$$AF + BG + CH = 0.$$

This may be verified as follows: suppose that the line in question is given as the line of junction of the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$; then the cone through the line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is nothing else than the plane through the three points; its equation is therefore

$$\begin{vmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ \alpha'', & \beta'', & \gamma'', & \delta'' \end{vmatrix} = 0,$$

which may be written

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

the values of A, &c. being

$$\begin{aligned} A &= \alpha'\delta'' - \alpha''\delta', & F &= \beta'\gamma'' - \beta''\gamma', \\ B &= \beta'\delta'' - \beta''\delta', & G &= \gamma'\alpha'' - \gamma''\alpha', \\ C &= \gamma'\delta'' - \gamma''\delta', & H &= \alpha'\beta'' - \alpha''\beta', \end{aligned}$$

which in fact satisfy the relation

$$AF + BG + CH = 0.$$

So again if the line is given as the intersection of the planes

$$\begin{aligned} ax + by + cz + dw &= 0, \\ a'x + b'y + c'z + d'w &= 0, \end{aligned}$$

then the equation of the cone (plane) through this line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is

$$(ax + by + cz + dw)(a'\alpha + b'\beta + c'\gamma + d'\delta) - (a'x + b'y + c'z + d'w)(a\alpha + b\beta + c\gamma + d\delta) = 0,$$

which is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

the values of A, &c. being

$$\begin{aligned} A &= bc' - b'c, & F &= ad' - a'd, \\ B &= ca' - c'a, & G &= bd' - b'd, \\ C &= ab' - a'b, & H &= cd' - c'd, \end{aligned}$$

which also satisfy the relation

$$AF + BG + CH = 0.$$

I annex the following further investigation: let p', q', r', s', t', u' and $p'', q'', r'', s'', t'', u''$ be what p, q, r, s, t, u become when $\alpha, \beta, \gamma, \delta$ are changed into $\alpha', \beta', \gamma', \delta'$ and $\alpha'', \beta'', \gamma'', \delta''$ respectively: the line represented by the equation

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

is also represented by the equations

$$Ap' + Bq' + Cr' + Fs' + Gt' + Hu' = 0,$$
$$Ap'' + Bq'' + Cr'' + Fs'' + Gt'' + Hu'' = 0,$$

which, reverting to the coordinates x, y, z, w, may be written

$$\mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'w = 0,$$
$$\mathfrak{A}''x + \mathfrak{B}''y + \mathfrak{C}''z + \mathfrak{D}''w = 0,$$

where

$$\begin{array}{l|l}
\mathfrak{A}' = \quad . \quad CB' - B\gamma' + F\delta', & \mathfrak{A}'' = \quad . \quad C\beta'' - B\gamma'' + F\delta'', \\
\mathfrak{B}' = -C\alpha' \quad . \quad + A\gamma' + G\delta', & \mathfrak{B}'' = -C\alpha'' \quad . \quad + A\gamma'' + G\delta'', \\
\mathfrak{C}' = \quad B\alpha' - A\beta' \quad . \quad + H\delta', & \mathfrak{C}'' = \quad B\alpha'' - A\beta'' \quad . \quad + H\delta'', \\
\mathfrak{D}' = -F\alpha' - G\beta' - H\gamma' \quad . \quad , & \mathfrak{D}'' = -F\alpha'' - G\beta'' - H\gamma'' \quad . \quad ,
\end{array}$$

in which form the equations represent two planes each of them through the given line: the equation of the cone (plane) through the given line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is

$$(\mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'w)(\mathfrak{A}''\alpha + \mathfrak{B}''\beta + \mathfrak{C}''\gamma + \mathfrak{D}''\delta)$$
$$- (\mathfrak{A}'\alpha + \mathfrak{B}'\beta + \mathfrak{C}'\gamma + \mathfrak{D}'\delta)(\mathfrak{A}''x + \mathfrak{B}''y + \mathfrak{C}''z + \mathfrak{D}''w) = 0,$$

or, developing,

$$(\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}')p + (\mathfrak{C}'\mathfrak{A}'' - \mathfrak{C}''\mathfrak{A}')q + (\mathfrak{A}'\mathfrak{B}'' - \mathfrak{A}''\mathfrak{B}')r$$
$$+ (\mathfrak{A}'\mathfrak{D}'' - \mathfrak{A}''\mathfrak{D}')s + (\mathfrak{B}'\mathfrak{D}'' - \mathfrak{B}''\mathfrak{D}')t + (\mathfrak{C}'\mathfrak{D}'' - \mathfrak{C}''\mathfrak{D}')u = 0.$$

Now

$$\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}' = (-C\alpha' + A\gamma' + G\delta')(B\alpha'' - A\beta'' + H\delta'')$$
$$- (-C\alpha'' + A\gamma'' + G\delta'')(B\alpha' - A\beta' + H\delta'),$$

which putting

$$\beta'\gamma'' - \beta''\gamma' = p, \quad \alpha'\delta'' - \alpha''\delta' = s,$$
$$\gamma'\alpha'' - \gamma''\alpha' = q, \quad \beta'\delta'' - \beta''\delta' = t,$$
$$\alpha'\beta'' - \alpha''\beta' = r, \quad \gamma'\delta'' - \gamma''\delta' = u,$$

become

$$A^2p_1 + ABq_1 + ACr_1 - (BG + CH)s_1 + AGt_1 + AHu_1,$$

or as it may be written

$$A(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1) - (AF + BG + CH)s_1.$$

Instead of writing at once $AF + BG + CH = 0$, I denote it by ∇ and I write also

$$Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1 = \Omega.$$

The series of coefficients $\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}''$, &c. is

$$\begin{array}{lll} A\Omega - \nabla s_1, & B\Omega - \nabla t_1, & C\Omega - \nabla u_1, \\ F\Omega - \nabla p_1, & G\Omega - \nabla q_1, & H\Omega - \nabla r_1, \end{array}$$

and the required equation, restoring for Ω, ∇ their values, is

$$\begin{aligned} &(Ap + Bq + Cr + Fs + Gt + Hu)(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + H_1) \\ &\qquad - (AF + BG + CH)(ps_1 + p_1s + qt_1 + q_1t + ru_1 + r_1u) = 0, \end{aligned}$$

which in virtue of $AF + BG + CH = 0$ becomes

$$(Ap + Bq + Cr + Fs + Gt + Hu)(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1) = 0.$$

The equation

$$Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1 = 0$$

would imply that the two points $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$ are in the same plane with the line

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

(or, what is the same thing, that this line is intersected by the line through the two points); in this exceptional case, the planes determined by the given line and the two points respectively are one and the same plane, and they do not by their intersection determine the given line. But in every other case the factor $Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1$ is not equal to zero, and the foregoing equation becomes

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

which is as it should be.

In what precedes it has been shown that the equation of the line through the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$ is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where

$$\begin{array}{ll} A = \alpha'\delta'' - \alpha''\delta', & F = \beta'\gamma'' - \beta''\gamma', \\ B = \beta'\delta'' - \beta''\delta', & G = \gamma'\alpha'' - \gamma''\alpha', \\ C = \gamma'\delta'' - \gamma''\delta', & H = \alpha'\beta'' - \alpha''\beta', \end{array}$$

and that the equation of the line of intersection of the planes $ax + by + cz + dw = 0$, $a'x + b'y + c'z + d'w = 0$ is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where

$$\begin{array}{ll} A = bc' - b'c, & F = ad' - a'd, \\ B = ca' - c'a, & G = bd' - b'd, \\ C = ab' - a'b, & H = cd' - c'd. \end{array}$$

It may be added that the condition of intersection of the two lines

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$
$$A'p + B'q + C'r + F's + G't + H'u = 0,$$

is

$$AF' + A'F + BG' + B'G + CH' + C'H = 0,$$

and any other problems in relation to the line, for instance to find the condition that a line may pass through a given point or lie in a given plane, &c. may also be solved by means of the new coordinates.

A curve of the second order in space is either a pair of lines or a plane conic, each of which species may degenerate into a pair of intersecting lines. If we write

$$W = Ap + Bq + Cr + Fs + Gt + Hu,$$
$$W' = A'p + B'q + C'r + F's + G't + H'u,$$

then the equation of the pair of lines will be $V = WW' = 0$, and it is worth while to show that this value of V satisfies the fundamental equation $d_pV.d_sV + d_qV.d_tV + d_rV.d_uV = 0$; we have in fact $d_pV = AW' + A'W$, &c. and thence the left-hand side is

$$W'^2(AF + BG + CH) + WW'(AF' + A'F + BG' + B'G + CH' + C'H) + W^2(A'F' + B'G' + G'H'),$$

which in fact vanishes in virtue of the relations

$$AF + BG + CH = 0, \quad A'F' + B'G' + C'H' = 0, \quad WW' = 0.$$

The like proof applies to any curve made up of two or more curves.

Consider in the next place the plane conic given by the equations

$$ax + by + cz + dw = 0,$$
$$x^2 + y^2 + z^2 + w^2 = 0.$$

To find the equation of the cone having for its vertex the point $(\alpha, \beta, \gamma, \delta)$ and passing through the conic we must, according to Joachimsthal's method, substitute in the two equations for x, y, z, w the values $\lambda x + \mu\alpha$, $\lambda y + \mu\beta$, $\lambda z + \mu\gamma$, $\lambda w + \mu\delta$, and from the resulting equations eliminate λ, μ. The two equations become

$$\lambda^2(x^2 + y^2 + z^2 + w^2) + 2\lambda\mu(\alpha x + \beta y + \gamma z + \delta w) + \mu^2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 0,$$
$$\lambda\ (ax + by + cz + dw) + \mu\ (\alpha x + \beta y + \gamma z + \delta w) = 0,$$

and thence eliminating λ, μ we find

$$\begin{aligned}&(a\alpha + b\beta + c\gamma + d\delta)^2(x^2 + y^2 + z^2 + w^2)\\ &- 2(a\alpha + b\beta + c\gamma + d\delta)(ax + by + cz + dw)(\alpha x + \beta y + \gamma z + \delta w)\\ &+ (ax + by + cz + dw)^2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 0,\end{aligned}$$

where the left-hand side is as it should be a function of p, q, r, s, t, u; the equation may in fact be written

$$\left.\begin{array}{l} \quad a^2(\,.\quad q^2+r^2+s^2) \\ +\ b^2(p^2\ \ .+r^2+t^2) \\ +\ c^2(p^2+q^2\ \ .+u^2) \\ +\ d^2(s^2+t^2+u^2\ \ .) \\ +2bc\,(-qr+tu) \\ +2ca\,(-rp+us) \\ +2ab\,(-pq+st) \\ +2ad\,(\ \ qu-rt) \\ +2bd\,(\ \ rs-pu) \\ +2cd\,(\ \ pt-qs) \end{array}\right\}=0,$$

which treated as a quadric in (a, b, c, d) may be written

$$\left(\begin{array}{|c|c|c|c|} \hline q^2+r^2+s^2 & -pq+st & -rp+us & qu-rt \\ \hline -pq+st & p^2+r^2+t^2 & -qr+tu & rs-pu \\ \hline -rp+us & -qr+tu & p^2+q^2+u^2 & pt-qs \\ \hline qu-rt & rs-pu & pt-qs & s^2+t^2+u^2 \\ \hline \end{array}\right)(a, b, c, d)^2=0,$$

or treated as a quadric in (p, q, r, s, t, u), in the form

$$\left(\begin{array}{|c|c|c|c|c|c|} \hline b^2+c^2 & -ab & -ac & . & cd & -bd \\ \hline -ba & c^2+a^2 & -bc & -cd & . & ad \\ \hline -ca & -cb & a^2+b^2 & bd & -ad & . \\ \hline . & -cd & bd & a^2+d^2 & ab & ac \\ \hline cd & . & -ad & ba & b^2+d^2 & bc \\ \hline -bd & ad & . & ca & cb & c^2+d^2 \\ \hline \end{array}\right)(p, q, r, s, t, u)^2=0.$$

Or again, in a form which is one of a system of four forms,

$$(b^2+c^2,\ c^2+a^2,\ a^2+b^2,\ -bc,\ -ca,\ -ab)(p,\ q,\ r)^2+(a^2+d^2,\ b^2+d^2,\ c^2+d^2,\ bc,\ ca,\ ab)(s,\ t,\ u)^2$$

$$+2\left\{\begin{array}{|c|c|c|c|} \hline & s & t & u \\ \hline p & . & cd & -bd \\ \hline q & -cd & . & ad \\ \hline r & bd & -ad & . \\ \hline \end{array}\right\}=0.$$

Representing the equation by $V=0$, the function V should verify the fundamental equation $d_pV\,.\,d_sV+d_qV\,.\,d_tV+d_rV\,.\,d_sV=0$, and we in fact have

$$d_pV=(b^2+c^2)\,p-abq-acr+\qquad\qquad cdt-bdu,$$
$$d_sV=\qquad\qquad -cdq+bdr+(a^2+d^2)\,s+abt+acu,$$

and thence forming the product $d_pV\,.\,d_sV$, and from it the two analogous products, we find

	$d_pV\,.\,d_sV=$	$d_qV\,.\,d_tV=$	$d_rV\,.\,d_uV=$
p^2	0	$-abcd$	$+abcd$
q^2	$+abcd$	0	$-abcd$
r^2	$-abcd$	$+abcd$	0
s^2	0	$-abcd$	$+abcd$
t^2	$+abcd$	0	$-abcd$
u^2	$-abcd$	$+abcd$	0
pq	$-cd\,(b^2+c^2)$	$+cd\,(c^2+a^2)$	$-cd\,(a^2-b^2)$
qr	$-ad\,(b^2-c^2)$	$-ad\,(c^2+a^2)$	$+ad\,(a^2+b^2)$
rp	$+bd\,(b^2+c^2)$	$-bd\,(c^2-a^2)$	$-bd\,(a^2+b^2)$
su	$-bd\,(a^2+d^2)$	$+bd\,(c^2+d^2)$	$-bd\,(c^2-a^2)$
ut	$-ad\,(b^2-c^2)$	$-ad\,(c^2+d^2)$	$+ad\,(b^2+d^2)$
ts	$+cd\,(a^2+d^2)$	$-cd\,(a^2-d^2)$	$-cd\,(b^2+d^2)$
ps	$+(a^2+d^2)(b^2+c^2)$	$-a^2b^2-c^2d^2$	$-a^2c^2-b^2d^2$
pt	$+ab\,(b^2+c^2)$	$-ab\,(b^2+d^2)$	$-ab\,(c^2-d^2)$
pu	$+ac\,(b^2+c^2)$	$-ac\,(b^2-d^2)$	$-ac\,(c^2+d^2)$
qs	$-ab\,(a^2+d^2)$	$+ab\,(c^2+a^2)$	$-ab\,(c^2-d^2)$
qt	$-a^2b^2-c^2d^2$	$+(b^2+d^2)(c^2+a^2)$	$-b^2c^2-a^2d^2$
qu	$-bc\,(a^2-d^2)$	$+bc\,(c^2+a^2)$	$-bc\,(c^2+d^2)$
rs	$-ac\,(a^2+d^2)$	$-ac\,(b^2-d^2)$	$+ac\,(a^2+b^2)$
rt	$-bc\,(a^2-d^2)$	$-bc\,(b^2+d^2)$	$+bc\,(a^2+b^2)$
ru	$-a^2c^2-b^2d^2$	$-b^2c^2-a^2d^2$	$+(c^2+d^2)(a^2+b^2)$

by which the equation is verified.

I conclude with some remarks relating to a different part of the subject. It is shown by M. E. de Jonquières in his "Essai sur la generation des courbes geometriques, &c." *Mem. Prés. à l'Acad. de Paris*, t. XVI. (1858), that the equation $U=0$ of any geometrical plane curve can be presented in a variety of different ways in the form

$$\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix}=0,$$

this being in fact the fundamental theorem of his very beautiful investigations. I am not aware that it has ever been considered whether the equation $U=0$ of a plane curve can in general be represented in the form

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ P'', & Q'', & R'' \end{vmatrix}=0,$$

or in the analogous forms where the left-hand side is a determinant of a higher order. As regards surfaces, the equation $U=0$ of a geometrical surface cannot in general be presented in the form

$$\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix} = 0,$$

nor in the similar forms where the left-hand side is a determinant of a higher order. We may consequently classify surfaces of a given order according to the forms of this nature by which they can be represented, or as I propose to term it, according to their "frangibility." It is obvious that this question is immediately connected with that of the representation of curves in space by means of the ordinary coordinates of analytical geometry; for instance if we have a surface $U=0$, which can be represented in the form

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ P'', & Q'', & R'' \end{vmatrix} = 0,$$

then we can describe upon the surface curves such as

$$\begin{Vmatrix} P', & Q', & R' \\ P'', & Q'', & R'' \end{Vmatrix} = 0,$$

viz. the curve so represented is the curve which in conjunction with the curve $(R'=0,\ R''=0)$ makes up the complete intersection of the two surfaces $P'R''-P''R'=0$, $Q'R''-Q''R'=0$: the curve in question is not in general the complete intersection of any two surfaces. If the surface $U=0$ can be represented in the form $\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix} = 0$, then we can describe upon the surface, curves such as the curve $(P=0,\ Q=0)$ which, although it is the complete intersection of two surfaces, is not the complete intersection of the given surface $U=0$ by any other surface. But if the surface $U=0$ cannot be represented in any such form, or as we may express it, if the surface is infrangible, then it would appear that the only curves which can be described upon the surface are those which are the complete intersection of the given surface by some other surface. The question is, I think, an interesting one in the theory of surfaces, but I doubt whether much will be done in this manner as regards the theory of curves in space, and it appears to me that there is more to be hoped for from the theory previously explained in the present paper.

2, *Stone Buildings, W.C., June* 2, 1859.

285.

ON THE SYSTEM OF CONICS HAVING DOUBLE CONTACT WITH EACH OTHER.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 246—250.]

CONSIDER the conics which pass through four points coinciding two and two together; the two points of each pair of coincident points are to be regarded as lying in a line which will be a tangent to the conic, and the system is thus a system of conics touching the same two lines at the same two points: or if we replace the two lines by a conic of the system, it is the system of conics having double contact with a given conic at the same two points. The lines may be spoken of as the tangents, and the points as the ineunts of the system; the line joining the two points is the axis, and the point of intersection of the two lines, the pole. It is assumed that the system contains real conics; the pole and axis must consequently be real, but the tangents and ineunts may be either all of them real or else all of them imaginary—we may for shortness say that in the former case the double contact is real, and that in the latter case it is imaginary. Consider first the case of a real double contact, the system of conics commences with an ellipse differing only slightly from the finite portion of the axis included between the ineunts, this ellipse being the first of a series of ellipses the last of which only slightly differs from a parabola, we have then a parabola, then a hyperbola differing only slightly from the parabola, and which is the first of a series of hyperbolas of which the last differs only slightly from the two tangents: all the before-mentioned conics are included within the angles at the pole or vertex of the triangle, viz. the ellipses and parabola within the angle towards the axis or base, and the hyperbolas, one branch of each of them within this angle and the other branch within the opposite angle. And it will be convenient to term these conics the "Multiform Series." Passing over the intermediate case, we come to a hyperbola differing only slightly from the two tangents, the first of a series of hyperbolas of which the

last differs only slightly from the infinite portions, not included between the ineunts, of the axis; all these hyperbolas lying without the angles at the pole or vertex of the triangle. The last-mentioned hyperbolas may be termed the "Uniform Series." Passing over another intermediate case we return to the multiform series, the entire system of conics forming in fact a complete cycle. The intermediate case forming the transition from the multiform to the uniform series is a conic which considered as generated by a point is the pair of tangents, but which considered as enveloped by a line is a pair of points coincident with the pole. The intermediate case forming the transition from the uniform series to the multiform series is a conic which considered as generated by a point is a pair of lines coincident with the axis, but considered as enveloped by a line is the pair of ineunts.

When the double contact is imaginary, the conics of the system may still be considered as forming a multiform and a uniform series, but the uniform series consists entirely of imaginary conics. The multiform series commences with an ellipse differing only slightly from the pole; this is the first of a series of ellipses of which the last differs only slightly from a parabola, we have next the parabola, and then a series of hyperbolas, the first of which differs only slightly from the parabola, and the last of which differs only slightly from the axis. The ellipses and parabolas lie all of them on the same side of the axis; the hyperbolas lie, one branch of each on the one side of the axis, and the other branch on the other side. The uniform series (being imaginary) of course does not admit of description. The intermediate or transition cases are the same as for a real double contact, with only the variation occasioned by the tangents and ineunts being imaginary. The conics of the entire system are considered (as in the case of a real double contact) to form a cycle, but here part of the cycle, viz. the uniform series, is imaginary.

Suppose that M and U stand for the multiform and uniform series, MU for the transition form from the multiform to the uniform series, and UM for the transition form from the uniform to the multiform series. The cycle may be represented as in the figure.

$$\begin{tikzcd}
 & MU \arrow[dr] & \\
M \arrow[ur] & & U \arrow[dl] \\
 & UM \arrow[ul] &
\end{tikzcd}$$

Call either of the conics MU, UM the centre of the cycle, and say that the cycle is arranged line-wise when MU is considered as the centre, and point-wise when UM is considered as the centre. We may pass from one conic of the cycle to another incentrically or excentrically, i.e. by passing towards the centre or away from the centre. The intermediates of two conics are the conics passed through in going from the first to the second incentrically, the extramediates are those passed through in going from the first to the second excentrically. In speaking of the extramediates or the intermediates of a single conic, such conic is considered as the first conic and

the second conic is understood to be the centre. The nearer extramediates are those passed through previous to reaching an extremity of the cycle, the further extramediates are those passed through subsequent to reaching an extremity of the cycle. It is clear that the intermediates and the nearer extramediates make up the series multiform or uniform to which the conic belongs, and that the further extramediates are the other series; and moreover that if the cycle be arranged line-wise and point-wise successively, the intermediates of the one arrangement are the nearer extramediates of the other arrangement and *vice versâ*; the further extramediates of each arrangement being the same.

Two points of a conic may be said to be conjunctive with respect to a given line when it is possible to pass from the one to the other along the curve without crossing the line and disjunctive in the contrary case—it being understood that both the parabola and the hyperbola are to be treated as closed curves, viz. the points at infinity of the parabola are to be considered as one and the same point, and so the points at each extremity of either asymptote of the hyperbola are to be considered as one and the same point—and in like manner two tangents of a conic are said to be conjunctive with respect to a given point, when it is possible by the revolution of the one tangent to arrive at the other tangent without sweeping through the point in question, and disjunctive in the contrary case.

Consider now any conic of the series, and call this simply the *conic*. Take upon the conic any two points a, a' and joining these with a variable point in the axis, let the lines so obtained meet the conic in the points b', b (so that ab' and $a'b$ meet on the axis). We have thus on the conic a series of points $a, b, c, \ldots$ and a second series $a', b', c', \ldots$ which are homographically related to each other, and which possess the property that taking any two points b, c of the first series and the corresponding points b', c' of the second series, the lines bc', $b'c$ meet on the axis. It is immaterial how the point a is chosen, but a being chosen at pleasure, the system will obviously depend on the way in which the point a' is chosen; so that the points of a conic may be considered as homographically related to each other in an infinity of different ways. The reciprocal construction of course applies to the tangents of a conic, so that the tangents of a conic may be considered as homographically related to each other in an infinity of different ways: and not only this, but it is clear that the tangents at points homographically related to each other are also homographically related to each other, and *vice versâ*.

The line joining corresponding points envelopes a conic, one of the conics of the system, and which may for shortness be spoken of simply as the *envelope*. The point of intersection of corresponding tangents generates a conic, one of the conics of a system, and which may in like manner be spoken of simply as the *locus*.

We may now enunciate the following theorem: Let the system be arranged line-wise, the envelope is an extramediate of the conic; viz. if a pair of corresponding points of the conic (all pairs have in this respect the same property) be conjunctive with respect to the axis, a nearer extramediate, but if disjunctive, then a further extramediate.

And again:

Let the system be arranged point-wise, the locus is an extramediate of the conic; viz. if a pair of corresponding tangents of the conic (all pairs have in this respect the same property) be conjunctive with respect to the pole, a nearer extramediate, but if disjunctive, then a further extramediate.

It is however proper to remark that when the double contact is imaginary, the pair of corresponding points or tangents (in fact any pair of points or tangents) is necessarily conjunctive, so that in this case the second alternatives in the two theorems have no application, and the envelope or locus is always a nearer extramediate. When the double contact is real, the pair of corresponding points or tangents may be either conjunctive or disjunctive, and the character (nearer or further) of the extramediate is determined accordingly.

The double contact being either real or imaginary, we may as a limiting case assume that the corresponding points or tangents coincide, the envelope or locus is in this case the conic itself. Suppose that the double contact is real, we have here two other limiting cases, viz. first, one of a pair (and therefore of each pair) of corresponding points or tangents may be situate in the axis or pass through the pole; this is the limit between the two cases of the pair of corresponding points or tangents being conjunctive and disjunctive, and therefore the envelope or locus must be the conic which is the limit between the nearer extramediates and the further extramediates, i.e. the envelope is the pair of ineunts and the locus is the pair of tangents.

Secondly, the line through a pair of corresponding points may pass through the pole, or the point of intersection of a pair of corresponding tangents may lie in the axis; the envelope or locus is here the furthest of the extramediates, viz. the envelope is the pair of points coincident with the pole, the locus is the pair of lines coincident with the axis. If considering one of the pair of corresponding points or lines as fixed, the other point or line passes through the last-mentioned limiting position, the envelope or locus returns back through the series of further extramediates.

In the statement of the preceding theorems, the envelope and locus have been considered separately, but we may if we please consider the locus as generated by the point of intersection of the tangents at the corresponding points: the locus and envelope are in this case reciprocal polars with respect to the conic; it should be noticed that the pair of corresponding points and the pair of corresponding tangents are either both conjunctive, or else both disjunctive.

2, *Stone Buildings, W.C., 3rd June,* 1859.

286.

NOTE ON THE VALUE OF CERTAIN DETERMINANTS, THE TERMS OF WHICH ARE THE SQUARED DISTANCES OF POINTS IN A PLANE OR IN SPACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 275—277.]

THE values of the several determinants mentioned in my paper "On a Certain Theorem in the Geometry of Position," *Cambridge Mathematical Journal*, Old Series, t. II. (1842), p. 267, [1], are as follows:

(1)
$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{21}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2,$$

where the Σ, Σ contain 3 and 6 terms respectively.

(2)
$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2 - \Sigma\, \overline{12}^2\, \overline{34}^2\, \overline{43}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2,$$

where the Σ, Σ, Σ contain 24, 12 and 8 terms respectively.

(3)
$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = \begin{matrix} -\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{45}^2 \\ -\Sigma\, \overline{12}^2\, \overline{21}^2\, \overline{34}^2\, \overline{43}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{45}^2\, \overline{54}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{41}^2 \\ +\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2\, \overline{45}^2 \end{matrix}$$

where the Σ, Σ, Σ, Σ, Σ contain 120, 15, 60, 30, and 40 terms respectively.

(4)
$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = \Sigma\, \overline{12}^2\, \overline{21}^2\, \overline{34}^2\, \overline{43}^2 - \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{41}^2,$$

where the Σ, Σ contain 3 and 6 terms respectively.

(5)
$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = \begin{matrix} \Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{34}^2\, \overline{45}^2\, \overline{51}^2 \\ -\Sigma\, \overline{12}^2\, \overline{23}^2\, \overline{31}^2\, \overline{45}^2\, \overline{54}^2 \end{matrix}$$

where the Σ, Σ contain 24 and 20 terms respectively.

And it is proper to remark that it is not in the preceding formulæ (as in the memoir above referred to in which $\overline{12}$ denotes a distance between two points 1 and 2) assumed that $\overline{12}$ and $\overline{21}$ are equal.

The formulæ (1) gives, if a, b, c represent the sides, the well-known expression for the area of a triangle

$$(\text{area})^2 = \tfrac{1}{16}(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4).$$

Similarly the formula (2), if a, b, c, f, g, h represent the edges, viz. $\overline{23} = a$, $\overline{31} = b$, $\overline{12} = c$, $\overline{14} = f$, $\overline{24} = g$, $\overline{34} = h$, gives for the volume of a tetrahedron

$$\begin{aligned}(\text{volume})^2 = \tfrac{1}{144}\{ & b^2c^2(g^2+h^2) + c^2a^2(h^2+f^2) + a^2b^2(f^2+g^2) \\ & + g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ & - a^2f^2(a^2+f^2) - b^2g^2(b^2+g^2) - c^2h^2(c^2+h^2) \\ & - a^2g^2h^2 \quad - b^2h^2f^2 \quad - c^2f^2g^2 \quad - a^2b^2c^2\}, \\ = \tfrac{1}{144} & W \text{ suppose.}\end{aligned}$$

Now

$$
\begin{aligned}
4 \times \text{surface} \\
= \quad & \sqrt{2h^2g^2 + 2g^2a^2 + 2a^2h^2 - a^4 - h^4 - g^4} \\
& + \sqrt{2f^2h^2 + 2h^2b^2 + 2b^2f^2 - h^4 - b^4 - f^4} \\
& + \sqrt{2g^2f^2 + 2f^2c^2 + 2c^2g^2 - g^4 - f^4 - c^4} \\
& + \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}, \\
& = x + y + z + w \text{ suppose,}
\end{aligned}
$$

and the norm of this is

$$(x^4 + y^4 + z^4 + w^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2 - 2x^2w^2 - 2y^2w^2 - 2z^2w^2)^2 - 64x^2y^2z^2w^2,$$

which is of the sixteenth order, and must be of the form WQ where Q is a function of a, b, c, f, g, h of the tenth order. The expression of this function is given by Prof. Sylvester in his paper "On the Relation between the Volume of a Tetrahedron &c.," *Camb. and Dubl. Math. Jour.*, t. VIII. (1853), pp. 171—178, viz. the value is

$$
\begin{aligned}
Q = \quad & a^2b^2c^2\{f^4 + g^4 + h^4 + g^2h^2 + h^2f^2 + f^2g^2 + b^2c^2 + c^2a^2 + a^2b^2 - (f^2 + g^2 + h^2)(a^2 + b^2 + c^2)\} \\
& + a^2g^2h^2\{f^4 + b^4 + c^4 + b^2c^2 + c^2f^2 + f^2b^2 + g^2h^2 + h^2a^2 + a^2g^2 - (f^2 + b^2 + c^2)(a^2 + g^2 + h^2)\} \\
& + b^2h^2f^2\{g^4 + c^4 + a^4 + c^2a^2 + a^2g^2 + g^2c^2 + h^2f^2 + f^2b^2 + b^2h^2 - (g^2 + c^2 + a^2)(b^2 + h^2 + f^2)\} \\
& + c^2f^2g^2\{h^4 + a^4 + b^4 + a^2b^2 + b^2h^2 + h^2a^2 + f^2g^2 + g^2c^2 + c^2f^2 - (h^2 + a^2 + b^2)(c^2 + f^2 + g^2)\},
\end{aligned}
$$

and, as there remarked, the equation $Q = 0$ expresses the condition that the radius of the inscribed sphere may be infinite.

2, *Stone Buildings, W.C., June* 10*th*, 1859.

287.

NOTE ON THE EQUATION FOR THE SQUARED DIFFERENCES OF THE ROOTS OF A CUBIC EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 307—309.]

THE question of finding the equation for the squared differences of the roots, presents, in the case of a cubic equation, a peculiarity which does not occur for equations of a higher order, viz. we may in the first instance form the equation for the differences of the roots taken in a given cyclical order, and thence deduce the equation for the squared differences of the roots. Let the cubic equation be

$$U = (a,\ b,\ c,\ d \between x,\ 1)^3 = a\,(x-\alpha)\,(x-\beta)\,(x-\gamma) = 0,$$

the function

$$\Pi\,\{\theta - (\beta - \gamma)\},$$

which equated to zero gives for θ the values $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, which are the differences of the roots taken in a given circular order, has for any interchanges whatever of the roots, two values only, viz. that just written down, and the value $\Pi\,\{\theta-(\gamma-\beta)\}$, which may be deduced therefrom by changing first the sign of θ and then the sign of the entire expression (or what is the same thing, by changing the signs of the terms containing the even powers of θ); we may consequently write

$$\Pi\,\{\theta - (\beta - \gamma)\} = P - Q\,\zeta^{\frac{1}{2}}\,(\alpha,\ \beta,\ \gamma),$$

which P, Q are symmetrical functions of the roots, and $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)$ or $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ is a function the square of which is a symmetrical function of the roots, and such

symmetrical functions of the roots can of course be expressed as functions of the coefficients. We have in fact

$$P = \theta^3 + \theta(\Sigma\beta\gamma - \Sigma\alpha^2) = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta\},$$
$$Q = 1,$$

and

$$\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma) = a^{-2}\sqrt{-27\square},$$

where $\square$ is the discriminant of the cubic function,

$$= a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2.$$

Consequently

$$\Pi\{\theta - (\beta - \gamma)\} = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta - \sqrt{-27\square}\},$$

and forming the similar equation

$$\Pi\{\theta + (\beta - \gamma)\} = a^{-2}\{a^2\theta^3 + 9(ac - b^2)\theta + \sqrt{-27\square}\},$$

multiplying the two equations together and writing u in the place of θ^2, we find

$$\Pi\{u - (\beta - \gamma)^2\} = a^{-4}\{[a^2u + 9(ac - b^2)]^2 u + 27\square\},$$

and the equation for the squared differences of the roots is thus seen to be

$$[a^2u + 9(ac - b^2)]^2 u + 27\square = 0,$$

or what is the same thing

$$a^4u^3 + 18a^2(ac - b^2)u^2 + 81(ac - b^2)^2 u + 27\square = 0.$$

I remark that if ω is an imaginary cube root of unity (so that $(\omega - \omega^2)^2 = -3$, $\omega - \omega^2$ being thus only another form of $\sqrt{-3}$) then if in the expression for $\Pi\{\theta - (\beta - \gamma)\}$ we write $\dfrac{3\theta}{(\omega - \omega^2)a}$ in the place of θ, the equation assumes the more simple form

$$\Pi\{\theta - \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)\} = \theta^3 - 3(ac - b^2)\theta - a\sqrt{\square},$$

which if U be the cubic function, H the Hessian $= (ac - b^2,\ ad - bc,\ bd - c^2 \between x, y)^2$, and $\square$ the discriminant as before, is a particular case (obtained by writing $x = 1$, $y = 0$) of the equation

$$\Pi\{\theta - \tfrac{1}{3}a(\omega - \omega^2)(x - \alpha y)\} = \theta^3 - 3H\theta - U\sqrt{\square},$$

which equation can be at once obtained from the equation (where Φ is the cubicovariant of the cubic function)

$$\sqrt[3]{\tfrac{1}{2}(\Phi + U\sqrt{\square})} - \sqrt[3]{\tfrac{1}{2}(\phi - U\sqrt{\square})} = \tfrac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y),$$

given in my Fifth Memoir on Quantics, *Phil. Trans.*, t. CXLVIII. (1858), [156]. For writing for a moment

$$\theta = \sqrt[3]{X} - \sqrt[3]{Y},$$

we find

$$\theta^3 = X - Y - 3\sqrt[3]{XY}\theta,$$

or

$$\theta^3 + 3\sqrt[3]{XY}\theta - (X - Y) = 0,$$

where $\sqrt[3]{XY} = \sqrt[3]{\frac{1}{4}(\Phi^2 - U^2\square)}$, which by the equation

$$\phi^2 - U^2\square = -4H^3$$

(given in the Memoir) is $= -H$, and $(X - Y)$ is $= U\sqrt{\square}$, so that the equation in θ is, as above, $\theta^3 - 3H\theta - U\sqrt{\square} = 0$, an equation which is satisfied by $\theta = \frac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y)$; and the other two roots being of course of the like form, the cubic function in θ is equal to $\Pi\{\theta - \frac{1}{3}a(\omega - \omega^2)(\beta - \gamma)(x - \alpha y)\}$ which proves the theorem.

2, *Stone Buildings, W.C., Nov. 3rd*, 1859.

288.

NOTE ON THE CURVATURE OF A PLANE CURVE AT A DOUBLE POINT, AND ON THE CURVATURE OF SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 322—326.]

THE radius of curvature of a plane curve at a double point is most readily determined as follows: viz. if x, y be the coordinates of the double point, θ the inclination to the axis of x of the tangent to the branch under consideration, and u, v the coordinates measured from the double point as origin, parallel and perpendicular to the tangent, of the consecutive point on such branch, then the radius of curvature for the branch in question is

$$= \frac{u^2}{2v};$$

the coordinates of the consecutive point are

$$x + u\cos\theta - v\sin\theta,$$
$$y + u\sin\theta + v\cos\theta,$$

and the value of $u^2 \div 2v$ is to be found by substituting these expressions for x, y in the equation of the curve. Let $U = 0$ be the equation of the curve; then at the double point, U and the differential coefficients of the first order vanish, let those of the second order be (a, b, c) and those of the third order $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$, then substituting we have

$$\tfrac{1}{2}(a, b, c)(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^2$$
$$+ \tfrac{1}{6}(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^3 = 0.$$

The coefficient of u^2 is

$$\tfrac{1}{2}(a, b, c)(\cos\theta, \sin\theta)^2,$$

which vanishes, since it is by putting this function equal to zero that we find the direction of the tangents at the double point. For the present purpose we are concerned with one of the two branches only, and in all that follows the ratio $\cos\theta : \sin\theta$ will denote a determinate root of the quadratic equation; viz. the root which corresponds to the branch in question.

The equation takes therefore the form

$$Buv + \tfrac{1}{2}Cv^2 + \tfrac{1}{6}Du^3 + \&c. = 0,$$

which might be satisfied by assuming $Buv + \frac{1}{2}Cv^2 = 0$, but the values so obtained belong to the branch which does not touch the tangent; the proper solution is

$$Buv + \tfrac{1}{6}Du^3 = 0,$$

and thence

$$\frac{u^2}{2v} = -\frac{3B}{D},$$

or substituting for B, D their values, the radius of curvature is

$$= \frac{3\,(a,\ b,\ c)\,(\cos\theta,\ \sin\theta)\,(-\sin\theta,\ \cos\theta)}{(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d})\,(\cos\theta,\ \sin\theta)^3},$$

where the expression in the numerator is

$$= (-a + c)\sin\theta\cos\theta + b\,(\cos^2\theta - \sin^2\theta),$$
$$= \tfrac{1}{2}\{(c - a)\sin 2\theta + 2b\cos 2\theta\}.$$

The curvature, at the point of contact, of the curve in which any surface is intersected by a tangent plane cannot be found by the ordinary theory of the curvature of surfaces, being (by reason that the point of contact is a double point on the curve) a case of exception from that theory. The foregoing method may be applied to the solution of the question as follows: Let $U = 0$ be the equation of the surface, and suppose that (x, y, z) refer to the point of contact of the tangent plane. Let L, M, N be the first differential coefficients of U at this point, (a, b, c, f, g, h) the second differential coefficients, (a, b, c, f, g, h, i, j, k, l) the third differential coefficients. Let λ, μ, ν be proportional to the cosines of the inclinations to the axis of the tangent to one of the two branches through the double point; the ratios $\lambda : \mu : \nu$ are determined by

$$(L,\ M,\ N)\,(\lambda,\ \mu,\ \nu) = 0,$$
$$(a,\ b,\ c,\ f,\ g,\ h)\,(\lambda,\ \mu,\ \nu)^2 = 0.$$

Let u, v be proportional to the coordinates of a consecutive point measured, from the point of contact as origin, in the direction of the tangent and in the perpendicular direction in the tangent plane. The cosines of the inclinations of u to the axes are as $\lambda : \mu : \nu$; those for the inclinations of the normal to the axes are as $L : M : N$; hence for the coordinate v which is perpendicular to the plane of the last-mentioned

two lines, the cosines of the inclinations to the axes are as $N\mu - M\nu : L\nu - N\lambda : M\lambda - L\mu$, and the coordinates of the consecutive point may be taken to be

$$\begin{aligned} &x + \lambda u + (N\mu - M\nu)\, v, \\ &y + \mu u + (L\nu - N\lambda)\, v, \\ &z + \nu u + (M\lambda - L\mu)\, v. \end{aligned}$$

Substituting these values in the equation of the surface, the terms involving the first powers of u, v vanish, and the term involving u^2 also vanishes in virtue of the relation $(a, b, c, f, g, h)(\lambda, \mu, \nu)^2 = 0$. The equation consequently becomes

$$Buv + \tfrac{1}{2}Cv^2 + \tfrac{1}{6}Du^3 + \ldots = 0,$$

and we have as before for the branch in question,

$$\frac{u^2}{2v} = -\frac{3B}{D}.$$

In the present case u, v have been taken, not as before equal, but only proportional, to the coordinates of the consecutive point measured from the point of contact parallel and perpendicular to the tangent, the values of the coordinates are in fact

$$\sqrt{\lambda^2 + \mu^2 + \nu^2}\, u, \quad \sqrt{L^2 + M^2 + N^2}\, \sqrt{\lambda^2 + \mu^2 + \nu^2}\, v,$$

and the expression for the radius of curvature is

$$= \frac{\sqrt{\lambda^2 + \mu^2 + \nu^2}}{\sqrt{L^2 + M^2 + N^2}}\, \frac{u^2}{2v},$$

or substituting for $\dfrac{u^2}{2v}$ the value $-\dfrac{3B}{D}$ and for B, D their values, the radius of curvature is

$$= -\frac{3\sqrt{\lambda^2 + \mu^2 + \nu^2}}{\sqrt{L^2 + M^2 + N^2}} \times \frac{(a, b, c, f, g, h)(\lambda, \mu, \nu)(N\mu - M\nu,\ L\nu - N\lambda,\ M\lambda - L\mu)}{(\mathrm{a, b, c, f, g, h, i, j, k, l})(\lambda, \mu, \nu)^3},$$

where, as already noticed, the ratios $\lambda : \mu : \nu$ are determined by the equations

$$\begin{aligned} (L, M, N)(\lambda, \mu, \nu) &= 0, \\ (a, b, c, f, g, h)(\lambda, \mu, \nu)^2 &= 0, \end{aligned}$$

the system of roots selected being that which corresponds to the branch under consideration. It may be noticed that these two equations give

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(L, M, N)^2 . (a, b, c, f, g, h)(\lambda, \mu, \nu)^2 - K\,[(L, M, N)(\lambda, \mu, \nu)]^2 = 0,$$

where as usual

$$\begin{aligned} (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}) &= (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch), \\ K &= abc - af^2 - bg^2 - ch^2 + 2fgh, \end{aligned}$$

and the expression on the left-hand side considered as a function of (λ, μ, ν) is decomposable into a pair of factors. Selecting the proper factor and equating it to zero, we have in addition to the linear equation

$$(L, M, N)(\lambda, \mu, \nu) = 0,$$

a new linear equation; these two equations determine the ratios $\lambda : \mu : \nu$.

P.S.—I remark that the process adopted for finding the radius of curvature at a double point of a plane curve is the most simple one for the case of an ordinary point on the curve. In fact let $U = 0$ be the curve, (x, y) the coordinates of the point in question, L, M the corresponding values of the first differential coefficients, (a, b, c) those of the second differential coefficients. The coordinates of the consecutive point are

$$x + u\cos\theta - v\sin\theta,$$
$$y + u\sin\theta + v\cos\theta.$$

Substituting these in the equation of the curve, we have

$$L(u\cos\theta - v\sin\theta) + M(u\sin\theta + v\cos\theta) + \tfrac{1}{2}(a, b, c)(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)^2 = 0,$$

the coefficient of u must be zero, or we have

$$L\cos\theta + M\sin\theta = 0,$$

giving

$$\sin\theta = -\frac{L}{\sqrt{L^2+M^2}},\quad \cos\theta = \frac{M}{\sqrt{L^2+M^2}},$$

and the equation may be reduced to

$$\sqrt{L^2+M^2}\,v + \tfrac{1}{2}(a, b, c)(\cos\theta, \sin\theta)^2 u^2 = 0,$$

or what is the same thing

$$(L^2+M^2)^{\frac{3}{2}} v + \tfrac{1}{2}(a, b, c)(M, -L)^2 u^2 = 0,$$

whence the radius of curvature

$$= \frac{u^2}{2v} = -\frac{(a, b, c)(M, -L)^2}{(L^2+M^2)^{\frac{3}{2}}},$$

which is the ordinary form for the radius of curvature at any point of a curve represented by the equation $U = 0$.

2, *Stone Buildings, W.C., October 2nd,* 1859.

289.

ON SOME NUMERICAL EXPANSIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 366—369.]

THERE are in the theory of series and the calculus of finite differences and the differential calculus, various sets of numbers, such for instance as Bernoulli's Numbers, or the successive differences of 0^m, which from their more or less frequent occurrence and the complexity of their law it is desirable to tabulate. To these belong, I think, the coefficients of the successive powers of x in the expansion of a given power of $\frac{\log(1+x)}{x}$, or when the index of the power of the function is an indeterminate quantity r, then the coefficients of the several terms of the rational and integral functions of r which form the coefficients of the successive powers of x in the expansion in question. We have

$$\log\frac{\log(1+x)}{x} = -\tfrac{1}{2}x + \tfrac{5}{24}x^2 - \tfrac{1}{8}x^3 + \tfrac{251}{2880}x^4 - \tfrac{19}{288}x^5 + \tfrac{19087}{362880}x^6 - \&c.,$$

which is most easily verified by differentiating and multiplying by $\log(1+x)$, which gives on the left-hand side $\frac{1}{1+x}$, and on the right-hand side the expansion $1 - x + x^2 - \&c.$ And from the above, multiplying each side by r, and taking the exponential, we find

$$\begin{aligned}\left\{\frac{(\log 1+x)}{x}\right\}^r = 1 & \\ & -(\tfrac{1}{2}r).x \\ & +(\tfrac{1}{8}r^2 + \tfrac{5}{24}r)\,x^2 \\ & -(\tfrac{1}{48}r^3 + \tfrac{5}{48}r^2 + \tfrac{1}{8}r)\,x^3 \\ & +(\tfrac{1}{384}r^4 + \tfrac{5}{192}r^3 + \tfrac{97}{1152}r^2 + \tfrac{251}{2880}r)\,x^4 \\ & -(\tfrac{1}{3840}r^5 + \tfrac{5}{1152}r^5 + \tfrac{61}{2304}r^3 + \tfrac{401}{5760}r^2 + \tfrac{19}{288}r)\,x^5 \\ & +(\tfrac{1}{46080}r^6 + \tfrac{5}{9216}r^5 + \tfrac{49}{9216}r^4 + \tfrac{10543}{414720}r^3 + \tfrac{4075}{69120}r^2 + \tfrac{19087}{362880}r)\,x^6 \\ & \mp \&c.\end{aligned}$$

which may be verified by writing $r = 1$.

As an instance of the use of the preceding table, let it be required to find expressions for the combinations without repetitions of the series of natural numbers $1, 2, 3, \ldots (n-1)$, or what is the same thing for the coefficients of the powers of k in $k(k-1)(k-2)\ldots(k-n+1)$. We have

$$\begin{aligned} & k(k-1)(k-2)\ldots(k-n+1) \\ & = k^n - A_1{}^n k^{n-1} + A_2{}^n k^{n-2} - \&c. \\ & = \Pi n \text{ coefficient } x^n \text{ in } (1+x)^k; \end{aligned}$$

whence

$$\begin{aligned} (-)^r A_r{}^n &= \Pi n \text{ coefficient } x^n k^{n-r} \text{ in } (1+x)^k \\ &= \Pi n \text{ coefficient } x^n k^{n-r} \text{ in } e^{k \log(1+x)} \\ &= \frac{\Pi n}{\Pi(n-r)} \text{ coefficient } x^n \text{ in } \{\log(1+x)\}^{n-r} \\ &= \frac{\Pi n}{\Pi(n-r)} \text{ coefficient } x^r \text{ in } \left\{\frac{\log(1+x)}{x}\right\}^{n-r}. \end{aligned}$$

Thus

$$\begin{aligned} A_1{}^n &= n\{\tfrac{1}{2}(n-1)\}, \\ A_2{}^n &= n(n-1)\{\tfrac{1}{8}(n-2)^2 + \tfrac{5}{24}(n-2)\}, \\ A_3{}^n &= n(n-1)(n-2)\{\tfrac{1}{48}(n-3)^3 + \tfrac{5}{48}(n-3)^2 + \tfrac{1}{8}(n-3)\}, \\ &\vdots \end{aligned}$$

and so on, as far as the expansion has been effected. It may be remarked that the general expression for the algebraical transcendent $A_r{}^n$ is given in Dr Schläfli's paper, "Sur les coefficients du développement du product $(1+x)(1+2x)\ldots\{1+(n-1)x\}$ suivant les puissances ascendantes de x." *Crelle*, t. XLIII. [1852], pp. 1—22, but the law is a very complicated one.

We have

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \tfrac{1}{7}x^7 - \&c.$$

and dividing by x, and taking the logarithm, we have as before

$$\log\left(\frac{1}{x}\log(1+x)\right) = -\tfrac{1}{2}x + \tfrac{5}{24}x^2 - \tfrac{1}{8}x^3 + \tfrac{251}{2880}x^4 - \tfrac{19}{288}x^5 + \tfrac{19087}{362880}x^6 - \&c.$$

which may be considered as the first of a series of logarithmic derivatives, viz., dividing by $-\tfrac{1}{2}x$ and taking the logarithm we have

$$\log\left(-\frac{2}{x}\log\left(\frac{1}{x}\log(1+x)\right)\right) = -\tfrac{5}{12}x + \tfrac{47}{288}x^2 - \tfrac{2443}{25920}x^3 + \tfrac{5303}{32944}x^4 - \tfrac{19631}{580608}x^5 + \&c.,$$

and by the like process

$$\log\left(-\frac{12}{5x}\log\left(-\frac{2}{x}\log\left(\frac{1}{x}\log(1+x)\right)\right)\right) =$$

$$-\tfrac{47}{120}x + \tfrac{11317}{86400}x^2 - \tfrac{439989}{5184000}x^3 + \tfrac{9390960318}{156764160000}x^4 - \&c.,$$

and so on.

Suppose in general that

$$\phi x = x + B_1 x^2 + C_1 x^3 + \ldots,$$

and let it be required to find the rth function $\phi^r x$. It is easy to see that the successive coefficients are rational and integral functions of r of the degrees 1, 2, 3, &c. respectively; we have in fact

$$\begin{aligned}
\phi^r x = & \qquad\qquad \phi^0 x \\
& + \frac{r}{1} \qquad (\phi^1 x - \phi^0 x) \\
& + \frac{r\,.\,r-1}{1\,.\,2} \qquad (\phi^2 x - 2\phi^1 x + \phi^0 x) \\
& + \frac{r\,.\,r-1\,.\,r-2}{1\,.\,2\,.\,3} (\phi^3 x - 3\phi^2 x + 3\phi^1 x - \phi^0 x), \\
& \qquad \&c.,
\end{aligned}$$

and by successive substitutions,

$$\begin{aligned}
\phi^0 x &= x, \\
\phi^1 x &= x + \ B_1 x^2 + C_1 x^3 + D_1 x^4 + \ldots, \\
\phi^2 x &= x + 2B_1 x^2 + (2B_1^2 + 2C_1)\, x^3 + (\ B_1^3 + \ 5B_1 C_1 + 2D_1)\, x^4 + \ldots, \\
\phi^3 x &= x + 3B_1 x^3 + (6B_1^2 + 3C_1)\, x^3 + (9B_1^3 + 15B_1 C_1 + 3D_1)\, x^4 + \ldots. \\
&\ \vdots
\end{aligned}$$

Whence

$$\begin{aligned}
\phi^r x = x & \\
& + rB_1\, x^2 \\
& + \{(r^2 - r)\, B_1^2 + rC_1\}\, x^3 \\
& + \{(r^3 - \tfrac{5}{2} r^2 + \tfrac{3}{2} r)\, B_1^3 + (\tfrac{5}{2} r^2 - \tfrac{5}{2} r)\, B_1 C_1 + rD_1\}\, x^4 \\
& + \&c.
\end{aligned}$$

It would, I think, be worth while to continue the expansion some steps further.

2, *Stone Buildings, W.C., Oct. 2nd,* 1859.

290.

A DISCUSSION OF THE STURMIAN CONSTANTS FOR CUBIC AND QUARTIC EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IV. (1861), pp. 7—12.]

FOR the cubic equation

$$(a,\ b,\ c,\ d)\,(x,\ 1)^3 = 0,$$

the Sturmian Constants (or leading coefficients of the Sturmian functions) are

$$a,\ a,\ b^2 - ac,\ -a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2.$$

If the signs of the constants, that is, of the functions for $+\infty$, are	then the signs of the functions for $-\infty$ are	
$+\ +\ +\ +$	$-\ +\ -\ +$	three real roots.
$+\ +\ -\ +$	$-\ +\ +\ +$	case cannot occur.
$+\ +\ +\ -$	$-\ +\ -\ -$	one real root.
$+\ +\ -\ -$	$-\ +\ +\ -$	one real root.

The second case would give a loss of variations of sign in passing from ∞ to $-\infty$, which is inconsistent with Sturm's theorem. To show *à posteriori* that the case cannot occur, we may form the identical equation

$$(a^2d - 3abc + 2b^3)^2 = -a^2(-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2) + 4(b^2 - ac)^3,$$

and, this being so, then in the case in question, the right-hand side would consist of two terms, each of them negative, while the left-hand side is essentially positive.

In the particular case where the third constant vanishes, or

$$b^2 - ac = 0,$$

we have

$$\begin{aligned} &-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2 \\ &= -(ad-bc)^2 + 4(b^2-ac)(c^2-bd) \\ &= -(ad-bc)^2, \text{ is negative;} \end{aligned}$$

hence, regarding the evanescent term as being at pleasure positive or negative, we have in each case a combination of signs corresponding to one real root.

The general result (which is well known) is, that there are three real roots or one real root according as

$$-a^2d^2 + 6abcd - 4ac^3 - 4b^3d + 3b^2c^2$$

is positive or negative.

For the quartic equation

$$(a,\ b,\ c,\ d,\ e)(x,\ 1)^4 = 0,$$

the Sturmian constants are

$$a,\ a,\ b^2 - ac,\ 3aJ + 2(b^2-ac)I,\ I^3 - 27J^2,$$

if, as usual,

$$I = ae - 4bd + 3c^2,$$

$$J = ace - ad^2 - b^2e + 2bcd - c^3.$$

If the signs of the constants, that is, of the functions for $+\infty$, are		then the signs of the functions for $-\infty$ are		
+ + + + +	0	+ − + − +	4	4 real roots.
+ + − + +	2	+ − − − +	2	no real root.
+ + + − +	2	+ − + + +	2	
+ + − − +	2	+ − − + +	2	
+ + + + −	1	+ − + − −	3	2 real roots.
+ + − + −	3	+ − − − −	1	−2, cannot occur.
+ + + − −	1	+ − + + −	3	2 real roots.
+ + − − −	1	+ − − + −	3	

The non-existing combination of signs is

$$\begin{aligned} I^3 - 27J^2 &= -, \\ 3aJ + 2(b^2-ac)I &= +, \\ b^2 - ac &= -. \end{aligned}$$

To show *à posteriori* that this case cannot occur, write

$$\begin{aligned} \vartheta &= a^2d - 3abc + 2b^3, \\ X &= 3aJ + 2(b^2-ac)I, \end{aligned}$$

then we have identically

$$9\,(3a^2J^2+X^2)\,\vartheta^2 = -4a^2X^3+36\,(b^2-ac)^2\,X^2-4a^2\,(b^2-ac)^2\,(I^3-27J^2),$$

which is impossible under the given combination of signs, since the left-hand side would be positive, and the right-hand side negative.

To prove the above identity—the relation $JU^3-IU^2H+4H^3+\Phi^2=0$, between the covariants of the quartic, gives

$$a^3J+a^2\,(b^2-ac)\,I-4\,(b^2-ac)^3+\vartheta^2=0,$$

or, what is the same thing,

$$\vartheta^2=-a^3J-a^2\,(b^2-ac)\,I+4\,(b^2-ac)^3.$$

But

$$X=3aJ+2\,(b^2-ac)\,I,$$

and thence

$$3\vartheta^2+a^2X=-a^2\,(b^2-ac)\,I+12\,(b^2-ac)^3,$$

or

$$3\vartheta^2=-a^2X-a^2\,(b^2-ac)\,I+12\,(b^2-ac)^3,$$

and the identity will be true, if

$$(3X^2+9a^2J^2)\left\{-X-(b^2-ac)\,I+12\,\frac{(b^2-ac)^3}{a^2}\right\}$$

$$=-4X^3+36\,\frac{(b^2-ac)^3}{a^2}\,X^2-4\,(b^2-ac)^3\,(I^3-27J^2).$$

This gives

$$(3X^2+9a^2J^2)\,\{-X-(b^2-ac)\,I\}=-4X^3-4\,(b^2-ac)^3\,I^3,$$

or, what is the same thing,

$$(3X^2+9a^2J^2)\,\{\;\;X+(b^2-ac)\,I\}=4\,\{\;X^3+\;(b^2-ac)^3\,I^3\},$$

or, dividing by $X+(b^2-ac)\,I$,

$$3X^2+9a^2J^2=4\,\{X^2-X\,(b^2-ac)\,I+(b^2-ac)^2\,I^2\},$$

and reducing

$$X^2-4X\,(b^2-ac)\,I-9a^2J^2+4\,(b^2-ac)\,I^2=0,$$

or finally

$$\{X-3aJ-2\,(b^2-ac)\,I\}\,\{X+3aJ-2\,(b^2-ac)\,I\}=0,$$

which is true in virtue of

$$X=3aJ+2\,(b^2-ac)\,I,$$

and the identity is thus proved.

The general conclusion is,

if $\quad I^3 - 27J^2$ is positive, the four roots are all real or all imaginary,

viz., all real if $b^2 - ac$ and $3aJ + 2(b^2 - ac)\,I$ are both positive, imaginary if otherwise. But if $I^3 - 27J^2$ is negative, the roots are two of them real, and the other two imaginary.

The following special cases may be noticed,

1°. $b^2 - ac = 0$,

here

$$9(3a^2J^2 + X^2)\,\Theta^2 = -4a^2X^3, \text{ or } X = 3aJ + 2(b^2 - ac)\,I = 3aJ, \text{ is negative,}$$

so that,

if $I^3 - 27J^2$ is $+$, the roots are all imaginary;

if $I^3 - 27J^2$ is $-$, the roots are two real and two imaginary.

2°. $X = 3aJ + 2(b^2 - ac)\;I = 0$,

here

$$27a^3J^2\Theta^2 = -4a^3(b^2 - ac)^3(I^3 - 27J^2),$$

or $b^2 - ac$, $I^3 - 27J^2$ are of opposite signs, and if

$b^2 - ac = -$, $I^3 - 27J^2 = +$, the roots are all imaginary,

$b^2 - ac = +$, $I^3 - 27J^2 = -$, the roots are two real and two imaginary.

3°. $b^2 - ac = 0$, $X = 3aJ + 2(b^2 - ac)\,I = 0$,

here $J = 0$, that is,

$$2bcd - ad^2 - c^3 = 0, \text{ or } (ad - bc)^2 + c^2(ac - b^2) = 0, \text{ or } ad - bc = 0, \text{ and } I^3 - 27J^2 = I^3,$$

$$I = ae - 4bd + 3c^2 = ae - 4\frac{b^4}{a^2} + 3\frac{b^4}{a^2} = ae - \frac{b^4}{a^2} = \frac{1}{a^2}(a^3e - b^4),$$

whence $\quad I = +$, the roots are all imaginary.

$I = -$, the roots are two real and two imaginary.

This is easily verified, in fact $ac - b^2 = 0$, $ad - bc = 0$, give $c = \frac{b^2}{a}$, $d = \frac{bc}{a} = \frac{b^3}{a^2}$, and the equation becomes

$$ax^4 + 4bx^3 + 6\frac{b^2}{a}x^2 + 4\frac{b^3}{a^2}x + e = 0,$$

or, which is the same thing,

$$(ax + b)^4 + (a^3e - b^4) = 0,$$

so that the roots are all imaginary, or two real and two imaginary, according to the sign of $a^3e - b^4$ as above.

It may be noticed that for a quintic equation

$$(a, b, c, d, e, f)(x, 1)^5,$$

if the Sturmian Constants are

$$a, a, C, D, E, F,$$

where as before a is positive, then the roots are real or imaginary as follows: viz.,

C,	D,	E,	F	
+	+	+	+	+, 5 real roots.
−	+	+		+, 1 real root, 4 imaginary roots.
+	−	+		
−	−	+		
+	+	−		
+	−	−		
−	−	−		
+	+	+		−, 3 real roots, 2 imaginary roots.
+	+	−		
+	−	−		
−	−	−		
−	+	−	+	+, case which does not occur.
−	+	+		−, cases which do not occur.
+	−	+		
−	−	+		
−	+	−		

The values of C, D, E, and F are given in my "Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees," *Phil. Trans.*, t. 147 (1857), pp. 733—736, [151], but I have not further examined this case.

2, *Stone Buildings, W.C., September* 29, 1859.

291.

ON THE DEMONSTRATION OF A THEOREM RELATING TO THE MOMENTS OF INERTIA OF A SOLID BODY.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IV. (1861), pp. 25—27.]

CONSIDERING in the first instance the analogous question *in plano*, let $a=\int x^2dm$, $b=\int y^2dm$, $h=\int xydm$, where the integration extends over any closed figure whatever, then it is to be shown that the equations

$$(a,\ h,\ b)(p,\ q)^2=1, \text{ and } (a+b)(p^2+q^2)-(a,\ h,\ b)(p,\ q)^2=1,$$

represent respectively ellipses.

If in the first case, by a transformation of coordinates,

$$(a,\ h,\ b)(p,\ q)^2=a_1p_1^2+b_1q_1^2,$$

then a_1, b_1 are the roots of the quadratic equations,

$$\begin{vmatrix} a-\rho, & h \\ h\ , & b-\rho \end{vmatrix}=0,$$

and if in the second case,

$$(a+b)(p^2+q^2)-(a,\ h,\ b)(p,\ q)^2=a_1p_1^2+b_1q_1^2,$$

then a_1, b_1 are the roots of

$$\begin{vmatrix} b-\rho, & -h \\ -h, & a-\rho \end{vmatrix}=0,$$

the two equations being in fact the same equation,

$$\rho^2-(a+b)\rho+ab-h^2=0,$$

and the conditions that the curve may be an ellipse, are

$$a+b=+,$$
$$ab-h^2=+,$$

the former of which requires no demonstration; to prove the latter, changing merely the variables under the integral sign, I write

$$a'=\int x'^2dm',\quad b'=\int y'^2dm',\quad h'=\int x'y'dm',$$

these quantities being of course respectively equal to a, b, h, we have then

$$ab'+a'b-2hh'=\iint(xy'-x'y)^2\,dmdm'=2\,(ab-h^2),$$

or since the quantity under the integral sign is a square, $ab-h^2$ is positive.

For the analogous problem *in solido*, we have

$$a=\int x^2dm,\ b=\int y^2dm,\ c=\int z^2dm,\ f=\int yzdm,\ g=\int zxdm,\ h=\int xydm,$$

and it is to be shown that the equations

$$(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=1,$$
$$(a+b+c)(p^2+q^2+r^2)-(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2=1,$$

represent respectively ellipsoids.

The conditions in the first problem are

$$\begin{aligned} a+b\ \ +c\qquad\qquad\qquad &=+,\\ bc+ca+ab-f^2-g^2-h^2&=+,\\ abc-af^2-bg^2-ch^2+2fgh&=+,\end{aligned}$$

the first of which is obviously true: as regards the second, the theorem *in plano* shows that each of the quantities $bc-f^2$, $ca-g^2$, $ab-h^2$ is positive, or merely reproducing the investigation, we find

$$2\,(bc+ca+ab-f^2-g^2-h^2)=\iint[(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2]\,dmdm',$$

which proves the theorem, and where it is to be observed that the integral may also be written

$$\iint[(x^2+y^2+z^2)(x'^2+y'^2+z'^2)-(xx'+yy'+zz')^2]\,dmdm';$$

and for the third, we find in a precisely similar manner,

$$6\,(abc-af^2-bg^2-ch^2+2fgh)=\iint\begin{vmatrix} x\,, & y\,, & z\\ x', & y', & z'\\ x'', & y'', & z''\end{vmatrix}^2dmdm',$$

which proves the theorem. The integral may also be written

$$\iint \begin{vmatrix} x^2 + y^2 + z^2, & xx' + yy' + zz', & xx'' + yy'' + zz'' \\ x'x + y'y + z'z, & x'^2 + y'^2 + z'^2, & x'x'' + y'y'' + z'z'' \\ x''x + y''y + z''z, & x''x' + y''y' + z''z', & x''^2 + y''^2 + z''^2 \end{vmatrix} dmdm'.$$

The conditions in the second problem are

$$(b+c)+(c+a)+(a+b)=+,$$
$$(c+a)(a+b)+(a+b)(b+c)+(b+c)(c+a)-f^2-g^2-h^2=+,$$
$$(a+b)(b+c)(c+a)-(b+c)f^2-(c+a)g^2-(a+b)h^2-2fgh=+,$$

the first and second of which are respectively equivalent to

$$a+b+c=+,$$
$$(a+b+c)^2+bc+ca+ab-f^2-g^2-h^2=+,$$

which are already proved. The last may be written

$$(a+b+c)(bc+ca+ab-f^2-g^2-h^2)-(abc-af^2-bg^2-ch^2+2fgh)=+,$$

which, putting for shortness,

$$A=x^2+y^2+z^2,\ B=x'^2+y'^2+z'^2,\ C=x''^2+y''^2+z''^2,$$
$$F=x'x''+y'y''+z'z'',\ G=x''x+y''y+z''z,\ H=xx'+yy'+zz',$$

is by what precedes expressible in the form

$$\tfrac{1}{6}\iint \{A(BC-F^2)+B(CA-G^2)+C(AB-H^2)-(ABC-AF^2-BG^2-CH^2+2FGH)\}\,dmdm'$$
$$=\tfrac{1}{3}\iint (ABC-FGH)\,dmdm',$$

or, since $\sqrt{BC}>F$, $\sqrt{CA}>G$, $\sqrt{AB}>H$, we have $ABC>FGH$, or $ABC-FGH=+$, and therefore the value of the integral is also positive.

2, *Stone Buildings, W.C., 6th March,* 1860.

292.

A THEOREM IN CONICS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IV. (1861), pp. 131—133.]

THE following theorem is given in Todhunter's *Conic Sections*, [Ed. 7, p. 304], "If ellipses be inscribed in a triangle, each with one focus in a fixed straight line, the locus of the other focus is a conic section through the angular points of the triangle." A focus is the intersection of tangents to the conic from the circular points at infinity; and instead of the circular points at infinity we may substitute any two points whatever. This being so, let the equations of the sides of the triangle be $x=0$, $y=0$, $z=0$, and let a pair of tangents to the curve from the points (α, β, γ), $(\alpha', \beta', \gamma')$ meet in the point (ξ, η, ζ), and the other pair of tangents from the same two points meet in the point (X, Y, Z). I find that we have the very simple relation

$$X\xi : Y\eta : Z\zeta = \alpha\alpha' : \beta\beta' : \gamma\gamma',$$

and consequently, when the locus of the point (ξ, η, ζ) is given, that of the point (X, Y, Z) is at once determined by substituting in the equation of the first-mentioned locus, in the place of ξ, η, ζ, the values $\frac{\alpha\alpha'}{\xi}$, $\frac{\beta\beta'}{\eta}$, $\frac{\gamma\gamma'}{\zeta}$, or as we may express it, the second locus is derived from the first by the method of reciprocal trilinear substitutions. And, in particular, when the first locus is a line, the second locus is a conic through the angular points of the triangle, which is Mr Todhunter's theorem. I have considered some of the properties of this substitution in a Memoir "Sur quelques transmutations des Courbes," *Liouville*, t. XIV. (1849), pp. 40—46 and t. XV. (1850), pp. 351—356, [80 and 81].

To demonstrate the theorem, I take for the equation of the conic

$$\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0,$$

and I write for shortness

$$\beta\zeta-\gamma\eta,\ \gamma\xi-\alpha\zeta,\ \alpha\eta-\beta\xi=A\,,\ B\,,\ C,$$
$$\beta'\zeta-\gamma'\eta,\ \gamma'\xi-\alpha'\zeta,\ \alpha'\eta-\beta'\xi=A',\ B',\ C',$$
$$\xi(\beta\gamma'-\beta'\gamma)+\quad\eta(\gamma\alpha'-\gamma'\alpha)+\quad\zeta(\alpha\beta'-\alpha'\beta)=\Delta\,,$$
$$\eta\zeta\alpha\alpha'(\beta\gamma'-\beta'\gamma)+\zeta\xi\beta\beta'(\gamma\alpha'-\gamma'\alpha)+\xi\eta\gamma\gamma'(\alpha\beta'-\alpha'\beta)=\square,$$

so that in fact

$$\Delta=\begin{vmatrix}\xi, & \eta\,, & \zeta\\ \alpha\,, & \beta\,, & \gamma\\ \alpha', & \beta', & \gamma'\end{vmatrix},\quad \square=\alpha\beta\gamma\alpha'\beta'\gamma'\xi\eta\zeta\begin{vmatrix}\frac{1}{\xi}, & \frac{1}{\eta}, & \frac{1}{\zeta}\\ \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma}\\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'}\end{vmatrix},$$

and we have

$$BC'-B'C=\xi\Delta,\quad CA'-C'A=\eta\Delta,\quad AB'-A'B=\zeta\Delta,$$
$$\beta\gamma'BC'-\beta'\gamma B'C=\gamma\alpha'CA'-\gamma'\alpha C'A=\alpha\beta'AB'-\alpha'\beta A'B=\square.$$

The conditions in order that the conic

$$\sqrt{(lx)}+\sqrt{(my)}+\sqrt{(nz)}=0$$

may touch the line through $(\alpha,\ \beta,\ \gamma)$ and $(\xi,\ \eta,\ \zeta)$ is

$$\frac{l}{A}+\frac{m}{B}+\frac{n}{C}=0,$$

and the condition in order that it may touch the line through $(\alpha',\ \beta',\ \gamma')$ and $(\xi,\ \eta,\ \zeta)$ is

$$\frac{l}{A'}+\frac{m}{B'}+\frac{n}{C'}=0,$$

and we thus have

$$l\ :\ m\ :\ n=\frac{1}{BC'}-\frac{1}{B'C}:\frac{1}{CA'}-\frac{1}{C'A}:\frac{1}{AB'}-\frac{1}{A'B},$$

or, what is the same thing,

$$l\ :\ m\ :\ n=\quad AA'\xi\quad:\quad BB'\eta\quad:\quad CC'\zeta,$$

which determine the constants in the equation of the conic.

Consider now the tangents to the conic from the point $(\alpha,\ \beta,\ \gamma)$; if the equation of the tangent is assumed to be

$$px+qy+rz=0,$$

then we have

$$p\alpha+q\beta+r\gamma=0,$$
$$\frac{l}{p}+\frac{m}{q}+\frac{n}{r}=0,$$

and these equations are of course satisfied by $p : q : r = A : B : C$, since the line through (ξ, η, ζ) is a tangent. They are also satisfied by

$$p : q : r = \frac{l}{A\alpha} : \frac{m}{B\beta} : \frac{n}{C\gamma},$$

as is obvious by substitution, we have therefore

$$\frac{l}{A\alpha}x + \frac{m}{B\beta}y + \frac{n}{C\gamma}z = 0,$$

or more simply

$$\frac{A'\xi}{\alpha}x + \frac{B'\eta}{\beta}y + \frac{C'\zeta}{\gamma}z = 0,$$

for the equation of the other tangent through (α, β, γ), and we have in like manner

$$\frac{A\xi}{\alpha'}x + \frac{B\eta}{\beta'}y + \frac{C\zeta}{\gamma'}z = 0,$$

for the equation of the other tangent through $(\alpha', \beta', \gamma')$; the last-mentioned two lines intersect in the point X, Y, Z, that is we have

$$X\xi : Y\eta : Z\zeta = \frac{B'C}{\beta\gamma'} - \frac{BC'}{\beta'\gamma} : \frac{C'A}{\gamma\alpha'} - \frac{CA'}{\gamma'\alpha} : \frac{A'B}{\alpha\beta'} - \frac{AB'}{\alpha'\beta},$$

or attending to an above-mentioned equation, we have

$$X\xi : Y\eta : Z\zeta = \quad \alpha\alpha' \quad : \quad \beta\beta' \quad : \quad \gamma\gamma',$$

which is the property in question. In the particular case, where the points (α, β, γ), $(\alpha', \beta', \gamma')$ are the foci, the theorem is an immediate consequence of the well-known proposition that the product of the perpendiculars let fall from the two foci upon any tangent of the conic is a constant.

2, *Stone Buildings, W.C.,* 17*th March,* 1860.

293.

ON A CERTAIN SYSTEM OF FUNCTIONAL SYMBOLS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IV. (1861), pp. 225—230.]

M. HERMITE, in the Memoir "Sur la Résolution de l'équation du cinquième degré," *Comptes Rendus*, t. XLVI., (1858), p. 508, has given for equations of the fifth degree a solution by means of elliptic functions, analogous to the well-known trigonometrical solution of (the irreducible case of) the cubic equation. He has for this purpose to consider the effect of transforming certain functions $\phi\omega$, $\psi\omega$, connected with the elliptic functions, by replacing therein the argument ω by a linear fraction of the form $\dfrac{c+d\omega}{a+b\omega}$, where a, b, c, d are integer numbers satisfying the condition

$$ad-bc=1,$$

and he finds for $\phi\left(\dfrac{c+d\omega}{a+b\omega}\right)$ (and the like would apply to $\psi\left(\dfrac{c+d\omega}{a+b\omega}\right)$), six different expressions according to the evenness, or values in relation to the modulus 2, of the coefficients a, b, c, d, viz., the congruence

$$ad-bc\equiv 1 \pmod{2}$$

admits of the six solutions shown by the table

	a	b	c	d
I.	1	0	0	1
II.	0	1	1	0
III.	1	1	0	1
IV.	1	1	1	0
V.	1	0	1	1
VI.	0	1	1	1

(where 0 or 1 denotes that the coefficient to which it refers is even or odd), and to each solution there corresponds a distinct value of $\phi\left(\frac{c+d\omega}{a+b\omega}\right)$ or $\psi\left(\frac{c+d\omega}{a+b\omega}\right)$.

It appeared to me that the formulæ in question were not only of the highest importance in the theory of elliptic functions, but that they were very remarkable in and for themselves, and that the question might be looked at in the following manner.

Suppose that A, B are functional symbols, or, so to term them, substitutions, the laws of operation being

$$A(X, Y) = \left(\theta\frac{X}{Y}, \frac{1}{Y}\right),$$

$$B(X, Y) = (Y, X),$$

where θ is a constant. The equations mean that the effect of A is to change X into $\theta\frac{X}{Y}$, and Y into $\frac{1}{Y}$, and the effect of B to change X into Y, and Y into X. We have

$$A^2X = A\,.\,\theta\frac{X}{Y} = \theta\frac{AX}{AY} = \theta\,.\,\theta\frac{X}{Y} \div \frac{1}{Y} = \theta^2X,$$

$$A^2Y = A\,\frac{1}{Y} = \frac{1}{AY} = 1 \div \frac{1}{Y} = Y, \qquad \text{or } A^2(X, Y) = (\theta^2X, Y),$$

and so

$$B^2X = X, \quad B^2Y = Y, \qquad \text{or } B^2(X, Y) = (X, Y);$$

thus B is periodic of the second order; and in the particular case $\theta = 1$, A is also periodic of the second order.

This being so, let K denote any one of the compound symbols A^p, BA^p, A^pB, A^qBA^p, &c.; I say that the *species* of K is determined by connecting with each of the symbols in question, a certain continued fraction as follows:

with

$A,\ p+\omega,$	
$BA^p,\ p-\frac{1}{\omega},$	$A^pB,\ -\frac{1}{p+\omega},$
$A^qBA^p,\ p-\frac{1}{q+\omega},$	$BA^pB,\ -\frac{1}{p-\frac{1}{\omega}},$
$BA^qBA^p,\ p-\frac{1}{q-\frac{1}{\omega}},$	$A^qBA^pB,\ -\frac{1}{p-\frac{1}{q+\omega}},$
$A^rBA^qBA^p,\ p-\frac{1}{q-\frac{1}{r+\omega}},$	&c.,
&c.,	

and when the continued fraction is reduced to the form $\dfrac{c+d\omega}{a+b\omega}$, where, of course, $ad-bc=1$, then, that the corresponding symbol K is of the species I. if $a, b, c, d \equiv 1, 0, 0, 1$ (mod. 2), and in like manner, that the species is II., III., IV., V., or VI., according as a, b, c, d are in regard to the modulus 2 of the form specified in the second, third, fourth, fifth, or sixth line of the foregoing table.

We have then a system of equations which I write in the form

$$K(X, Y) = \left\{\begin{array}{ll} \text{for } K \text{ of the species} & \\ \theta^x \quad \theta^y & \\ \hline X, \ Y & \text{I.} \\ Y, \ X & \text{II.} \\ \dfrac{1}{X}, \ \dfrac{Y}{X} & \text{III.} \\ \dfrac{1}{Y}, \ \dfrac{X}{Y} & \text{IV.} \\ \dfrac{X}{Y}, \ \dfrac{1}{Y} & \text{V.} \\ \dfrac{Y}{X}, \ \dfrac{1}{X} & \text{VI.} \end{array}\right.$$

viz., the first line denotes that $K(X, Y) = (\theta^x X, \theta^y Y)$, and so for the other lines, the values of the indices x, y, being different in the several cases, but I have written the system in the above form, to put in evidence more distinctly the theorem as applied to the most simple case, where $\theta = 1$.

It will be sufficient to indicate how the theorem is proved in this particular case: I will suppose that for

$$K = BA^qBA^p\ldots,$$

we have

$$K(X, Y) = \left\{\begin{array}{ll} \text{for } K \text{ of the species} & \\ X, \ Y & \text{I.} \\ Y, \ X & \text{II.} \\ \dfrac{1}{X}, \ \dfrac{Y}{X} & \text{III.} \\ \dfrac{1}{Y}, \ \dfrac{X}{Y} & \text{IV.} \\ \dfrac{X}{Y}, \ \dfrac{1}{X} & \text{V.} \\ \dfrac{Y}{X}, \ \dfrac{1}{X} & \text{VI.} \end{array}\right.$$

and let it be inquired what is the effect of the symbol $K' = A^rBA^qBA^p\ldots$, which, since $A^2(X, Y) = (X, Y)$, is equivalent to AK or K, according as r is odd or even.

Let $\Theta = \dfrac{c + d\omega}{a + b\omega}$ be the continued fraction which corresponds to K, and $\Theta' = \dfrac{c' + d'\omega}{a' + b'\omega}$ the continued fraction which corresponds to K', then Θ' is deduced from Θ by putting therein $r + \omega$ for ω, or we have

$$\frac{c + d(\omega + r)}{a + b(\omega + r)} = \frac{c' + d'\omega}{a' + b'\omega},$$

or

$$c' = c + dr,\ d' = d,$$
$$a' = a + br,\ b' = b,$$

and therefore when r is even, or $\equiv 0$ (mod. 2), then

$$c' \equiv c,\ d' \equiv d, \quad \text{(mod. 2)},$$
$$a' \equiv a,\ b' \equiv b,$$

or K' is of the *same* species as K. But in this case since $A^2(X, Y) = (X, Y)$ and r is even, we ought to have K' equivalent to K; and hence r being even, the theorem, assumed to be true for K, is also true for K'.

But if r be odd, or $\equiv 1$ (mod. 2), then

$$c' \equiv c + 1,\ d' \equiv d, \quad \text{(mod. 2)},$$
$$a' \equiv a + 1,\ b' \equiv b,$$

whence, according as

K is of species I., II., III., IV., V., VI., so

K' . . . . . V., IV., VI., II., I., III.

But if from $K(X, Y)$ we derive $AK(X, Y)$, we find that of whatever species K may be, the species of AK is identical with the corresponding species of K', and in the case in question, where r is odd, it is clear that we ought to have K' equivalent to AK. Hence in this case also, and therefore generally, if the theorem be true for K it will be also true for K'.

In a very similar manner it may be shown that if the theorem is true for $K' = A^r BA^q BA^p \ldots$, it will be true for $BA^r BA^q BA^p \ldots$, and the foregoing cases include the case where the symbol is of the form $BA^r BA^q B$ or $A^r BA^q B$ (terminating in a B), for it is only necessary to assume that the last index of A is zero; hence the theorem, if true in any case, as it obviously is for $A^0(X, Y) = (X, Y)$, is true universally.

When θ is not equal 1, the only difference is that the terms of $K(X, Y)$ are respectively multiplied by certain powers θ^x, θ^y of θ, and it is not difficult to see that for $K = A^r BA^q BA^p \ldots$ the indices x, y are of the forms

$$x = cr + \mathrm{b}q + \mathrm{a}p + \ldots,$$
$$y = \mathrm{c}'r + \mathrm{b}'q + \mathrm{a}'p + \ldots,$$

where c, b, a, ..., c′, b′, a′, ... have only the values 0, +1, −1, the values of

c, c′ depending on the species of BA^qBA^p...,
b, b′ . . . . . BA^p...,
a, a′ . . . .,
&c.,

in a manner which can easily be determined. In the particular case, where $\theta^{16}=1$ (as it is in M. Hermite's problem), it is only necessary to know the values of x, y in regard to the modulus 16, and these values can be expressed in terms of the numbers (a, b, c, d) which correspond to the symbol K (this is in itself a remarkable theorem), and I find that (to modulus 16)

$$x \equiv \begin{cases} d(c+d)-1 \\ c(c-d)-1 \\ d(d-c)-1 \\ c(d-c)+1 \\ cd \\ -cd \end{cases}, \qquad y = \begin{cases} a(a-b)-1 & \text{for } K \text{ of the species I.} \\ b(a+b)-1 & \text{II.} \\ ab & \text{III.} \\ -ab & \text{IV.} \\ a(b-a)+1 & \text{V.} \\ b(b-a)-1 & \text{VI.} \end{cases}$$

This may be shown by induction in like manner as the theorem in regard to the parts independent of θ. Thus K, K' as before, let K be of the species I., and assume that $K(X, Y)=(\theta^x X, \theta^y Y)$ where

$$x \equiv d(c+d)-1,\ y \equiv a(a-b)-1 \text{ (mod. 16)},$$

then if r be even, since $A^2(X, Y)=(\theta^2 X, Y)$, we ought to have $K'(X, Y)=(\theta^{x+2r} X, \theta^y Y)$. But in this case K' is of the same species with K, viz., of the species I., and by the theorem, $K'(X, Y)=(\theta^{x'} X, \theta^{y'} Y)$ where

$$x' \equiv d'(c'+d')-1,\ y' \equiv a'(a'-b')-1 \text{ (mod. 16)};$$

and we ought to have

$$x' \equiv x+r,\ y' \equiv y \text{ (mod. 16)},$$

that is

$$d'(c'+d')-1 \equiv d(c+d)-1+r, \text{ (mod. 16)},$$
$$a'(a'-b')-1 \equiv a(a-b)-1$$

or substituting for a', b', c', d' their values $a+br$, b, $c+dr$, d, these become

$$d(c+d+dr) \equiv d(c+d)+r, \text{ (mod. 16)},$$
$$(a+br)(a-b+br) \equiv a(a-b)$$

or

$$(d^2-1)r \equiv 0, \qquad \text{(mod. 16)}.$$
$$(2ab-b^2+b^2r)r \equiv 0$$

But r being even, these become

$$d^2-1\equiv 0,\ 2ab-b^2+b^2r\equiv 0 \text{ (mod. 8)};$$

or, since in the case under consideration, $a, b, c, d\equiv 1, 0, 0, 1$ (mod. 2), and therefore $b^2r\equiv 0$ (mod. 8), the two congruences become

$$d^2-1\equiv 0,\ a^2-(a-b)^2\equiv 0 \text{ (mod. 8)},$$

which are satisfied since d, a, and $a-b$ are all odd. And in a similar way the theorem is proved for all the other cases.

In M. Hermite's problem, X Y are replaced by $\phi\omega$, $\psi\omega$, AX, AY are $\phi(\omega+1)$, $\psi(\omega+1)$, and BX, BY are $\phi\left(-\frac{1}{\omega}\right)$, $\psi\left(-\frac{1}{\omega}\right)$, the properties of the functions ϕ, ψ being such that, as in the preceding investigation,

$$A(X,\ Y)=\left(\theta\frac{X}{Y},\ \frac{1}{Y}\right),\quad B(X,\ Y)=(Y,\ X),$$

where $\theta\,(=e^{\frac{1}{8}i\pi})$ is a sixteenth root of unity, and KX, KY are respectively equal to $\phi\left(\frac{c+d\omega}{a+b\omega}\right)$, $\psi\left(\frac{c+d\omega}{a+b\omega}\right)$, and the formulæ are precisely similar to those of the present paper.

2, *Stone Buildings, W.C.*, 5 *May*, 1860.

294.

ON A NEW ANALYTICAL REPRESENTATION OF CURVES IN SPACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. v. (1862), pp. 81—86.]

THE employment of a new kind of coordinates for the analytical representation of curves in space is suggested in my former paper under the same title *Journal*, t. III., pp. 225—236 (1859). The idea was as follows: viz. if (x, y, z, w) are current coordinates of a point in space (ordinary point coordinates), and $(\alpha, \beta, \gamma, \delta)$ the coordinates of a particular point, then taking (p, q, r, s, t, u) to represent the minor determinants formed out of the matrix

$$\left(\begin{array}{cccc} x, & y, & r, & w \\ \alpha, & \beta, & \gamma, & \delta \end{array}\right),$$

viz.

$$\begin{aligned} p &= \gamma y - \beta z, & s &= \delta x - \alpha w, \\ q &= \alpha z - \gamma x, & t &= \delta y - \beta w, \\ r &= \beta x - \alpha y, & u &= \delta z - \gamma w, \end{aligned}$$

values which satisfy identically

$$ps + qt + ru = 0,$$

then the equation of a cone passing through a given curve and having for its vertex the arbitrary point $(\alpha, \beta, \gamma, \delta)$, is of the form

$$V = 0,$$

V being a homogeneous function of the six new coordinates (p, q, r, s, t, u). And it was proposed to consider $V = 0$ as the equation of the curve.

But as remarked in the paper, it is not every function V of the coordinates (p, q, r, s, t, u) which equated to zero, does in fact represent a curve. In order that

the equation $V=0$ may represent a curve, it is necessary, that when any infinitesimal variations whatever are given to the constants $(\alpha, \beta, \gamma, \delta)$, thus converting the equation into $V+\delta V=0$, the two equations $V=0$, $\delta V=0$ (considered as equations in ordinary point coordinates) shall represent one and the same curve, whatever the system of infinitesimal variations attributed to $\alpha, \beta, \gamma, \delta$ may be. Let P, Q, R, S, T, U denote the differential coefficients of V in regard to p, q, r, s, t, u respectively, then the equation $\delta V=0$, breaks up into the equations

$$\begin{aligned} &. - Ry + Qz - Sw = 0, \\ Rx \quad &. - Pz - Tw = 0, \\ - Qx + Py \quad &. - Uw = 0, \\ Sx + Ty + Uz \quad &. = 0, \end{aligned}$$

and the system composed of these four equations and the equation $V=0$ (considered as equations in ordinary point coordinates) must belong to one and the same curve.

The four equations gave

$$PS + QT + RU = 0,$$

a relation between the differential coefficients of V which must be satisfied either identically or in virtue of the equation $V=0$. And this relation existing, any two of the four equations lead to the other two. Attending exclusively to the coordinates (p, q, r, s, t, u) and considering (x, y, z, w) as mere arbitrary multipliers, the above equation

$$PS + QT + RU = 0$$

is the only relation between the differential coefficients of V which is deducible from the four equations.

But it was noticed that the equation $V=0$, even when V is a function such that we have (identically or in virtue of the equation $V=0$) the equation $PS+QT+RU=0$, does not of necessity represent a curve. Some further relation or relations between the differential coefficients of V must therefore exist, either identically or in virtue of the equation $V=0$; and such relations can be found by resorting to the second differential $\delta^2 V$ of the function V. In fact not only the equation $\delta V=0$ but the entire series of relations $\delta^2 V=0$, $\delta^3 V=0, \ldots$ should be satisfied by the coordinates of any point of the curve. I find by means of the equation $\delta^2 V=0$ a plexus of equations, which are consequently necessary, and I am inclined to believe sufficient, in order that the equation $V=0$ may in fact represent a curve; the equations of the plexus are, it will be seen, very numerous, and certainly only a small number of them are independent, but this is a question which I have not as yet investigated.

Attending to the expressions for p, q, r, s, t, u, we have

$$\begin{aligned} d_\alpha &= \quad . - yd_r + zd_q - wd_s = (1), \\ d_\beta &= xd_r \quad . - zd_p - wd_t = (2), \\ d_\gamma &= - xd_q + yd_p \quad . - wd_u = (3), \\ d_\delta &= xd_s + yd_t + zd_u \quad . = (4), \end{aligned}$$

and writing for convenience a, b, c, d instead of d_α, d_β, d_γ, d_δ, we have

$$d = (1)\,\mathrm{a} + (2)\,\mathrm{b} + (3)\,\mathrm{c} + (4)\,\mathrm{d}.$$

It was in effect by operating on V with this symbol and equating to zero the coefficients of a, b, c, d, that the before-mentioned equations

$$\begin{aligned} &. - Ry + Qz - Sw = 0,\\ &Rx \quad . - Pz - Tw = 0,\\ &-Qx + Py \quad . - Uw = 0,\\ &Sx + Ty + Uz \quad . = 0, \end{aligned}$$

were found.

If to these equations we join the equation

$$Ax + By + Cz + Dw = 0,$$

where A, B, C, D are arbitrary multipliers, we can express x, y, z, w in terms of A, B, C, D in such manner as to satisfy the four equations, viz. we have

$$\begin{aligned} x &= \quad . \quad BU - CT + DP,\\ y &= -AU \quad . + CS + DQ,\\ z &= AT - BS \quad . + DR,\\ w &= -AP - BQ - CR \quad . , \end{aligned}$$

and if in the expressions for (1), (2), (3), (4) we substitute for x, y, z, w these values, and form therewith the value of d, which value I will for distinction call $\mathfrak{D}$, we have

$$\mathfrak{D} = \left(\begin{array}{cccc} Ud_r + Td_q + Pd_s, & Qd_s - Sd_q\ , & Rd_s - Sd_r\ , & Rd_q - Qd_r \\ Pd_t - Td_p\ , & Ud_r + Qd_t + Sd_p, & Rd_t - Td_r\ , & Pd_r - Rd_p \\ Pd_u - Ud_p\ , & Qd_u - Ud_q\ , & Rd_u + Sd_p + Td_q, & Qd_p - Pd_q \\ Td_u - Ud_t\ , & Ud_s - Sd_u\ , & Sd_t - Td_s\ , & Pd_s + Qd_t + Rd_u \end{array} \right)$$

$$(A,\ B,\ C,\ D)\ (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d}),$$

viz. $\mathfrak{D}$ is a lineo-linear function of the two sets of indeterminate quantities (A, B, C, D), (a, b, c, d), the coefficients thereof being the operators

$$Ud_r + Td_q + Pd_s,\ Qd_s - Sd_q,\ \&c.$$

It may be remarked that we have identically

$$\mathfrak{D}V = (PS + QT + RU)(A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + D\mathrm{d}),$$

since obviously each term such as $(Qd_s - Sd_q)\,V$, which is equal to $QS - SQ$, vanishes identically. The equation $\mathfrak{D}V = 0$ gives therefore only the before-mentioned equation $PS + QT + RU = 0$, which is as it should be.

The equation $\mathfrak{D}^2 V = 0$, is then to be satisfied independently of the values of (A, B, C, D) and (a, b, c, d), and as $\mathfrak{D}$ contains 16 distinct terms, $\mathfrak{D}^2$ will contain in all $\frac{1}{2}16 . 17$ or 136 distinct terms. The equation $\mathfrak{D}^2 V = 0$ gives therefore a plexus of 136 equations, and the equations in each succeeding plexus, involved in $\mathfrak{D}^3 V = 0$, $\mathfrak{D}^4 V = 0$, &c. will, of course, be still be more numerous.

If $V = 0$ be the plane conic which is the intersection of the surfaces

$$x^2 + y^2 + z^2 + w^2 = 0,$$
$$ax + by + cz + dw = 0,$$

then we have

$$V = \left(\begin{array}{cccccc} b^2+c^2, & -ab, & -ac, & ., & cd, & -bd \\ -ba, & c^2+a^2, & -bc, & -cd, & ., & ad \\ -ca, & -cb, & a^2+b^2, & bd, & -ad, & . \\ ., & -cd, & bd, & a^2+d^2, & ab, & ac \\ cd, & ., & -ad, & ba, & b^2+d^2, & bc \\ -bd, & ad, & ., & ca, & cb, & c^2+d^2 \end{array} \right) (p, q, r, s, t, u)^2.$$

The values of P, Q, R, S, T, U (omitting a common factor 2) are

$$P = (b^2+c^2, -ab, -ac, ., +cd, -bd)(p, q, r, s, t, u),$$
&c.,

and if we proceed to form a term in $\mathfrak{D}^2 V$, say the coefficient of A^2a^2, this is $(Ud_r + Td_q + Pd_s)^2\ V$, or

$$(a^2+b^2, c^2+a^2, a^2+d^2, -cd, +bd, -cb)(U, T, P)^2.$$

The coefficient therein of p^2 is

$$(a^2+b^2, c^2+a^2, a^2+d^2, -cd, +bd, -cb)(-bd, cd, b^2+c^2)^2,$$

that is, it is

$$\begin{aligned} &(a^2+b^2)\, b^2d^2 && -2cd\,.\ cd\,(b^2+c^2) \\ &+(c^2+a^2)\, c^2d^2 && +2bd\,.\,-bd\,(b^2+c^2) \\ &+(a^2+d^2)\,(b^2+c^2)^2 && -2cb\,.\,-bd\,.\,cd, \end{aligned}$$

where the terms in which (b^2+c^2) does not appear as a factor are together equal to

$$a^2d^2\,(b^2+c^2) + d^2\,(b^2+c^2)^2,$$

the entire expression thus divides by b^2+c^2, the quotient being

$$(a^2+d^2)\,(b^2+c^2) - 2c^2d^2 - 2b^2d^2 + a^2d^2 + d^2\,(b^2+c^2),$$

which is equal to $a^2\,(b^2+c^2+d^2)$, or restoring the factor b^2+c^2, we see that in $\mathfrak{D}^2 V$ the coefficient of A^2a^2 is

$$a^2\,(b^2+c^2+d^2)\,(b^2+c^2)\,p^2 + \&c.$$

The complete value must, it is clear, be of the form

$$a^2(b^2+c^2+d^2)\,V + k\,(ps+qt+ru),$$

vanishing in virtue of the equations $V=0$, $ps+qt+ru=0$, and this being so, observing that V contains no term in ps, we have $k=$ coefficient ps in

$$(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(U,\ T,\ P)^2,$$

that is

$$k = 2\,(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(-bd,\ cd,\ b^2+c^2)\,(ca,\ ba,\ 0),$$

or

$$\begin{aligned}\tfrac{1}{2}k = \quad & (a^2+b^2)\,.\,-bd\,.\,ca \ - \ cd\,\{cd\,.\,0+(b^2+c^2)\,ba\}\\ & +(c^2+a^2)\,.\quad cd\,.\,ba \ + \ bd\,\{(b^2+c^2)\,ca-bd\,.\,0\}\\ & +(a^2+d^2)\,.\qquad\quad 0 \ - \ cb\,\{-bd\,.\,ba+cd\,.\,ca\},\end{aligned}$$

which is

$$= abcd\left\{\begin{array}{l} -(a^2+b^2)\ -(b^2+c^2)\\ +(c^2+a^2)\ +(b^2+c^2)\\ \qquad\qquad\ +(b^2-c^2)\end{array}\right\},\ =0.$$

The coefficient k consequently vanishes, and therefore in $\mathfrak{D}^2V$ the coefficient of A^2a^2 is $a^2(b^2+c^2+d^2)\,V$, but I have not worked out the coefficients of the other terms.

2, *Stone Buildings, W.C.*, 30*th October*, 1860.

295.

ON THE CONSTRUCTION OF THE NINTH POINT OF INTERSECTION OF THE CUBICS WHICH PASS THROUGH EIGHT GIVEN POINTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*. vol. v. (1862), pp. 222—233.]

I REPRODUCE with additional developments the solution which has been given of this interesting problem.

The equation of a given cubic may be written

$$qU - pV = 0,$$

where $U=0$, $V=0$ are any two conics meeting the cubic in the same four points; $p=0$ is the line joining the remaining two points of intersection of the cubic with the conic $U=0$; and $q=0$ is the line joining the remaining two points of intersection of the cubic with the conic $V=0$; the relation between the arbitrary constant factors implicitly contained in the functions qU and pV is assumed to be properly determined.

The form is employed by Plücker, *Theorie der algebraischen Curven*, p. 56 (1839), and in connexion therewith he gives some geometrical considerations which, he remarks, contain implicitly the solution of the above-mentioned problem.

The form is also the analytical basis of the investigations of M. Chasles, "Construction de la courbe du troisième ordre par neuf points," *Comptes Rendus*, t. XXXVI. (1853), pp. 942—952. In fact calling to mind the theorem that in the pencil of conics $U - \alpha V = 0$ (where α is an arbitrary multiplier) the anharmonic ratio of the multipliers α is equal to the anharmonic ratio of the tangents at any one of the points of intersection, or (what is the same thing) to the anharmonic ratio of

the polars of any point whatever in regard to the conics, and recollecting that the anharmonic ratio in question is said to be the anharmonic ratio of the conics themselves; then since the equation $qU-pV=0$ is satisfied by the system

$$U-\alpha V=0,\ p-\alpha q=0,$$

(where α is arbitrary) we have at once the theorem, p. 949, viz. that if there be a pencil of lines through a *point*, and corresponding anharmonically thereto, a pencil of conics through the same *four points;* the locus of the intersections of a line by the corresponding conic is a cubic through the five points: and, conversely, that a given cubic may be so generated, the point of the pencil of lines, and the four points of the pencil of conics being any five points whatever of the cubic.

This gives at once the construction (M. Chasles' *first* construction) for the cubic through nine given points. In fact if the points are called 1, 2, 3, 4, 5, 6, 7, 8, 9; then grouping the points in any manner, it is only necessary to find a point x such that

$$x\,(1,\ 2,\ 3,\ 4,\ 5)=6789\ (1,\ 2,\ 3,\ 4,\ 5),$$

that is, such that the pencil of lines $x1$, $x2$, $x3$, $x4$, $x5$ shall correspond anharmonically to the pencil of conics 67891, 67892, 67893, 67894, 67895. The foregoing notation is that employed in M. de Jonquières' "Essai sur la génération des Courbes géométriques, &c.," *Mém. Sav. Etrang.*, t. XVI. (1858), which I take the opportunity of referring to. In fact, if x satisfies the foregoing condition, then taking through the point x any other line, and corresponding anharmonically thereto a conic through the points 6, 7, 8, 9, the locus of the intersections of the line and conic will be a cubic through the nine points. But the condition in question gives

$$x\,(1,\ 2,\ 3,\ 4)=6789\,(1,\ 2,\ 3,\ 4),$$
$$x\,(1,\ 2,\ 3,\ 5)=6789\,(1,\ 2,\ 3,\ 5),$$

which (by the anharmonic property of the points of a conic) show, the first that x is in a certain conic passing through 1, 2, 3, 4, and the second that x is in a certain conic passing through 1, 2, 3, 5; the two conics intersect in the points 1, 2, 3, and in a fourth point which is the required point x. Or we may say that x is given by the condition that the pencils $x\,(1,\ 2,\ 3,\ 4)$ and $x\,(1,\ 2,\ 3,\ 5)$ shall have given anharmonic ratios. It will presently be seen how x can be determined by the ruler alone.

Suppose now that the points 1, 2, 3, 4, 5, 6, 7, 8, 9 are the points of intersection of two cubics; the construction should become indeterminate; this is only the case when the two conics which by their intersection should determine x become one and the same conic. This implies that the conic $x\,(1,\ 2,\ 3,\ 4)$ passes through 5, or that x, 1, 2, 3, 4, 5 are points of the same conic. And then since by the anharmonic property of the points of a conic $x\,(1,\ 2,\ 3,\ 4)=5\,(1,\ 2,\ 3,\ 4)$, we have

$$5\,(1,\ 2,\ 3,\ 4)=6789\,(1,\ 2,\ 3,\ 4).$$

The grouping of the nine points is altogether arbitrary, hence there are in all $(9 \times 70 =)$ 630 such equations, which are really equivalent to only two equations, and which when eight of the points are given, determine the ninth point. Supposing that the given points are 1, 2, 3, 4, 5, 6, 7, 8, the equations for the determination of the remaining point 9 may be taken to be

$$9\ (5,\ 6,\ 7,\ 8) = 1234\ (5,\ 6,\ 7,\ 8),$$
$$9\ (4,\ 6,\ 7,\ 8) = 1235\ (4,\ 6,\ 7,\ 8),$$

which (it is to be remarked) determine 9 in a similar way to that in which x is given in the construction of the cubic through nine points; viz. 9 is the fourth intersection of two conics which pass through the points 5, 6, 7, 8 and the points 4, 6, 7, 8 respectively. Or we may say that 9 is given by the conditions that the pencils 9 (5, 6, 7, 8) and 9 (4, 6, 7, 8) shall have given anharmonic ratios.

The foregoing equations

$$9\ (5,\ 6,\ 7,\ 8) = 1234\ (5,\ 6,\ 7,\ 8),$$
$$9\ (4,\ 6,\ 7,\ 8) = 1235\ (4,\ 6,\ 7,\ 8),$$

are equivalent to and constitute the geometrical interpretation of the equations obtained (previous to M. Chasles' Memoir) by Weddle in the paper "On the construction of the ninth point of intersection of two curves of the third degree when the other eight points are given," *Cambridge and Dublin Mathematical Journal*, t. VI., pp. 83—86 (1851). In fact, reproducing his analysis with only a slight change of notation, let 012 denote the determinant

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

so that $012 = 0$ is the equation of the line through the points 1 and 2; and in like manner let 012345 denote the determinant

$$\begin{vmatrix} x^2, y^2, z^2, yz, zx, xy \\ x_1^2, \text{\&c.} \\ \vdots \end{vmatrix},$$

so that $012345 = 0$ is the equation of the conic through the points 1, 2, 3, 4, 5. Of course 123, 123456, &c. will denote given functions of the coordinates of the points 1, 2, 3, the points 1, 2, 3, 4, 5, 6, &c. This being so

$$012345\,.\,078 = \lambda\,.\,012347\,.\,058$$

is the equation of a particular cubic passing through the points 1, 2, 3, 4, 5, 7, 8, and which if we properly determine λ, viz. if we write

$$\lambda = \frac{612345\,.\,678}{612347\,.\,658}$$

will also pass through the point 6.

And similarly

$$012345 \,.\, 076 = \mu \,.\, 012347 \,.\, 056$$

is the equation of a particular cubic curve passing through the points 1, 2, 3, 4, 5, 6, 7 and which if we properly determine μ, viz. if we write

$$\mu = \frac{812345 \,.\, 876}{812347 \,.\, 856}$$

will also pass through the point 8. Hence the two curves, each of them passing through the points 1, 2, 3, 4, 5, 6, 7, 8 will intersect in the remaining point 9; and writing 9 for 0, and combining the two equations, we have

$$\frac{978 \,.\, 956}{958 \,.\, 976} = \frac{\lambda}{\mu} = \frac{612345}{612347}\,\frac{812347}{812345},$$

or, what is the same thing,

$$\frac{956 \,.\, 978}{958 \,.\, 967} = \frac{123456 \,.\, 123478}{123458 \,.\, 123467},$$

which is Weddle's equation, and is equivalent to the above-mentioned equation

$$9\ (5,\ 6,\ 7,\ 8) = 1234\ (5,\ 6,\ 7,\ 8).$$

To prove this I remark that we have identically

$$012 \,.\, 034 \,.\, 514 \,.\, 523 - 014 \,.\, 023 \,.\, 512 \,.\, 534 = 012345.$$

In fact the left-hand side equated to zero is the equation of the conic through 1, 2, 3, 4, 5, and such left-hand side must therefore, save to a mere numerical factor, be equal to 012345. And to determine this factor it is to be observed that 012345 contains the term

$$+ x_0^2 \,.\, y_1^2 \,.\, z_2^2 \,.\, y_3 z_3 \,.\, z_4 x_4 \,.\, x_5 y_5,$$

but that there is no such term in 012.034.514.523, and that there is in $-$ 014.023.512.534 the equivalent term

$$- x_0 y_1 z_4 \,.\, - x_0 y_3 z_2 \,.\, x_3 y_1 z_2 \,.\, y_3 z_3 x_4,$$

so that the numerical factor is rightly determined.

The foregoing identity written under the form

$$\frac{012 \,.\, 034}{014 \,.\, 023} - \frac{512 \,.\, 534}{514 \,.\, 523} = \frac{012345}{014 \,.\, 023 \,.\, 514 \,.\, 523}$$

shows that, when $012345 = 0$, i.e. if 0 be a point of the conic through 1, 2, 3, 4, 5, then we have

$$0\ (1,\ 2,\ 3,\ 4) = 5\ (1,\ 2,\ 3,\ 4),$$

which is in fact the anharmonic property of the points of a conic. And observing that $012345 = 051234$, and substituting 5, 6, for 0, 5 respectively, the identity becomes

$$\frac{512 \,.\, 534}{514 \,.\, 523} - \frac{612 \,.\, 634}{614 \,.\, 623} = \frac{561234}{514 \,.\, 523 \,.\, 614 \,.\, 623}.$$

The equation $012345 = 0$ may be written

$$012 \,.\, 034 - \frac{512 \,.\, 534}{514 \,.\, 523} 014 \,.\, 023 = 0,$$

and hence the anharmonic ratio of the conics

$$012345 = 0,\ 012346 = 0,\ 012347 = 0,\ 012348 = 0$$

is equal to that of the quantities

$$\frac{512 \,.\, 534}{514 \,.\, 523},\ \frac{612 \,.\, 634}{614 \,.\, 623},\ \frac{712 \,.\, 734}{714 \,.\, 723},\ \frac{812 \,.\, 834}{814 \,.\, 823},$$

or calling these quantities for a moment α, β, γ, δ, it is

$$= \frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \delta)(\beta - \gamma)},$$

where

$$\alpha - \beta = \frac{512 \,.\, 534}{514 \,.\, 523} - \frac{612 \,.\, 634}{614 \,.\, 623} = \frac{561234}{514 \,.\, 523 \,.\, 614 \,.\, 623};$$

and forming in this manner the expressions of each of the four factors $\alpha - \beta$, $\gamma - \delta$, $\alpha - \delta$, $\beta - \gamma$, we have

$$\frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \delta)(\beta - \gamma)} = \frac{561234 \,.\, 781234}{581234 \,.\, 671234},$$

so that in the equation

$$\frac{956 \,.\, 978}{958 \,.\, 967} = \frac{561234 \,.\, 781234}{581234 \,.\, 671234},$$

the right-hand side is

$$= 1234\,(5,\ 6,\ 7,\ 8),$$

and since by what precedes the left-hand side is $= 9\,(5,\ 6,\ 7,\ 8)$, the equation is

$$9\,(5,\ 6,\ 7,\ 8) = 1234\,(5,\ 6,\ 7,\ 8),$$

which is the transformation in question.

Now resuming the two equations

$$9\,(5,\ 6,\ 7,\ 8) = 1234\,(5,\ 6,\ 7,\ 8),$$
$$9\,(4,\ 6,\ 7,\ 8) = 1235\,(4,\ 6,\ 7,\ 8),$$

the right-hand sides are given anharmonic ratios, and as we have seen the question is to find 9 so that the anharmonic ratios $9\,(5,\ 6,\ 7,\ 8)$, $9\,(4,\ 6,\ 7,\ 8)$ shall have given values. But for the geometrical solution by the ruler alone, we have the preliminary question, from the given eight points, without the assistance of the before-mentioned conics, to construct the given anharmonic ratios $1234\,(5,\ 6,\ 7,\ 8)$ and $1235\,(4,\ 6,\ 7,\ 8)$. The solution of both questions is given in Dr Hart's paper, "Construction by the ruler alone to determine the ninth point of intersection of two curves of the third degree," *Cambridge and Dublin Mathematical Journal*, t. VI., pp. 181, 182 (1851).

The Preliminary Question. The anharmonic ratio 1234 (5, 6, 7, 8) is equal to that of the polars of an arbitrary point X in regard to the conics 12345, 12346, 12347, 12348 respectively (these polars all pass through one and the same point). Now to construct the polars of X in regard to these conics, and first in regard to the conic 12345: the fourth harmonics of X in regard to the lines 12, 34, in regard to the lines 13, 42, and in regard to the lines 14, 23, meet in a point; and considering the several combinations 1234, 1235, 1245, 1345, 2345 we have thus five points; these lie on a line which is the required polar of X in regard to the conic 12345. The polars in regard to the other conics are obtained in the same manner; and it is clear that the first above-mentioned point (viz. that deduced from the points 1, 2, 3, 4) is in fact the point of intersection of the four polars, or point of the pencil formed by the polars.

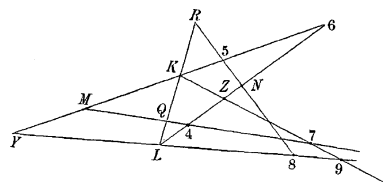
The Principal Question then is, given the points 4, 5, 6, 7, 8 to find the point 9, such that

$$9\,(5,\ 6,\ 7,\ 8) = 1234\,(5,\ 6,\ 7,\ 8),$$

$$9\,(4,\ 6,\ 7,\ 8) = 1235\,(4,\ 6,\ 7,\ 8),$$

where the right-hand sides represent given anharmonic ratios.

For this, let 65, 74 (see the figure) meet in M, and on 74 find a point Q such



that the anharmonic ratio of the points (4, M, 7, Q) may be equal to the given ratio 1235 (4, 6, 7, 8), say

$$(4,\ M,\ 7,\ Q) = 1235\,(4,\ 6,\ 7,\ 8),$$

and let 64, 85 meet in N and on 85 find a point R, such that the anharmonic ratio of the points 5, N, R, 8 is equal to the given anharmonic ratio 1234 (5, 6, 7, 8), say

$$(5,\ N,\ R,\ 8) = 1234\,(5,\ 6,\ 7,\ 8).$$

Join QR meeting 65 in K and 64 in L; then $7K$ and $8L$ will meet in the required point 9.

In fact taking Y as the intersection of the lines $65KM$ and $8L$, and Z as the intersection of the lines $64NL$ and $7K$; then, as is clear from the figure, first, the

anharmonic ratio $9(4, 6, 7, 8)$ is equal to that of the points $(4, 6, Z, L)$ on the line $46ZL$, that is

$$9(4, 6, 7, 8) = (4, 6, Z, L),$$

which is

$$= (4, M, 7, Q),$$

since the lines $6M$, $Z7$, LQ meet in the point K; but by the construction $(4, M, 7, Q) = 1235\,(4, 6, 7, 8)$, that is, we have

$$9(4, 6, 7, 8) = 1235\,(4, 6, 7, 8);$$

and, secondly, the anharmonic ratio $9(5, 6, 7, 8)$ is equal to that of the points $(5, 6, K, Y)$ on the line $56KY$, that is

$$9(5, 6, 7, 8) = (5, 6, K, Y),$$

which is

$$= (5, N, R, 8),$$

since the lines $6N$, KR, $Y8$ meet in the same point L; but by the construction $(5, N, R, 8) = 1234\,(5, 6, 7, 8)$, that is, we have

$$9(5, 6, 7, 8) = 1234\,(5, 6, 7, 8),$$

so that the point 9 satisfies the required conditions.

It has been already remarked that the point x in M. Chasles' theorem for the construction of the cubic through nine points is determined by precisely similar conditions to those which determine the ninth intersection of the two cubics; that is, the foregoing construction by the ruler alone is applicable to the determination of the point x; and when this is once obtained, the remainder of the construction, giving the points of the cubic through the nine given points, can obviously be performed by the ruler alone. The construction for the cubic through nine points gives implicitly the relation between ten points of the cubic and such relation is accordingly expressed by the equation

$$x(1, 2, 3, 4, 5, 10) = 6789\,(1, 2, 3, 4, 5, 10),$$

which is one out of 210 similar forms. But it is possible that some more convenient form of the relation between the ten points may yet be found.

I proceed to further develope the analytical theory. Writing for convenience ω in the place of 10, we have

$$x(1, 2, 3, 4, 5, 6) = 789\omega\,(1, 2, 3, 4, 5, 6),$$

or, what is the same thing,

$$x(1, 2, 3, 4) = 789\omega\,(1, 2, 3, 4),$$
$$x(1, 2, 3, 5) = 789\omega\,(1, 2, 3, 5),$$
$$x(1, 2, 3, 6) = 789\omega\,(1, 2, 3, 6),$$

which belong to three conics each of them passing through 1, 2, 3, and which must have a remaining fourth point of intersection.

The equation of the first conic is

$$012\,.\,034 = \frac{\lambda}{\mu}014\,.\,023,$$

where

$$\lambda = 789\omega 12\,.\,789\omega 34,$$

$$\mu = 789\omega 14\,.\,789\omega 23,$$

and thence, in virtue of an identical equation already referred to,

$$-\lambda - \mu = 789\omega 13\,.\,789\omega 24.$$

But we have identically

$$123\,.\,0pq = 1pq\,.\,023 + 2pq\,.\,031 + 3pq\,.\,012,$$

and thence in particular

$$123\,.\,034 = 134\,.\,023 + 234\,.\,031,$$

$$123\,.\,014 = 214\,.\,031 + 314\,.\,012,$$

and the equation of the conic may therefore be written

$$012\,(134\,.\,023 + 234\,.\,031)\,\mu - 023\,(214\,.\,031 + 314\,.\,012)\,\lambda = 0,$$

that is

$$031\,.\,012\,.\,234\,.\,\mu + 012\,.\,023\,.\,314\,(-\lambda - \mu) + 023\,.\,031\,.\,124\,.\,\lambda = 0,$$

or, substituting for μ, $-\lambda - \mu$, λ, their values, this is

$$\begin{aligned} &031\,.\,012\,.\,234\,.\,789\omega 14\,.\,789\omega 23 \\ &+ 012\,.\,023\,.\,314\,.\,789\omega 13\,.\,789\omega 42 \\ &+ 023\,.\,031\,.\,124\,.\,789\omega 12\,.\,789\omega 34 = 0, \end{aligned}$$

or, what is the same thing,

$$\frac{234\,.\,789\omega 14\,.\,789\omega 23}{023} + \frac{314\,.\,789\omega 13\,.\,789\omega 24}{031} + \frac{124\,.\,789\omega 12\,.\,789\omega 34}{012} = 0,$$

or, making a slight change of form, the equation of the conic is

$$\frac{423\,.\,789\omega 41\,.\,789\omega 23}{023} + \frac{431\,.\,789\omega 42\,.\,789\omega 31}{031} + \frac{412\,.\,789\omega 43\,.\,789\omega 12}{012} = 0.$$

The equations of the other two conics are deduced by writing successively 5 and 6 in the place of 4; and the condition in order that the conics may have a remaining fourth point of intersection is

$$\begin{vmatrix} 423\,.\,789\omega 41, & 431\,.\,789\omega 42, & 412\,.\,789\omega 43 \\ 523\,.\,789\omega 51, & 531\,.\,789\omega 52, & 512\,.\,789\omega 53 \\ 623\,.\,789\omega 61, & 631\,.\,789\omega 62, & 612\,.\,789\omega 63 \end{vmatrix} = 0.$$

This equation, say $\square = 0$, expresses the relation between the coordinates of the ten points 1, 2, 3, 4, 5, 6, 7, 8, 9, ω, of the cubic. Hence if 123456789ω denote the determinant

$$\left|\begin{array}{l} x_1^3,\ y_1^3,\ z_1^3,\ y_1^2z_1,\ z_1^2x_1,\ x_1^2y_1,\ y_1z_1^2,\ z_1x_1^2,\ x_1y_1^2,\ x_1y_1z_1 \\ x_2^3,\ \&c. \\ \vdots \end{array}\right|,$$

123456789ω must be a factor of $\square$, and a little consideration shows that the other factor which is of the order one as regards the coordinates of each of the points 1, 2, 3, and of the order three as regards the coordinates of each of the points 7, 8, 9, ω, must be of the form $123 . 789 . 78\omega . 79\omega . 89\omega$. We must therefore have

$$\square = \epsilon . 123 . 789 . 78\omega . 79\omega . 89\omega . 123456789\omega,$$

where the merely numerical factor ϵ is, I believe, equal to $+1$ or else to -1.

In order to verify the factor $123 . 789 . 78\omega . 79\omega . 89\omega$, observing that the points 7, 8, 9, ω enter symmetrically, it will be sufficient to show that 123, 789 are each of them factors of $\square$, or, what is the same thing, that if $123 = 0$, or if $789 = 0$, then in either case $\square = 0$.

First, if $123 = 0$, we may write

$$x_3 = \lambda x_1 + \mu x_2,$$
$$y_3 = \lambda y_1 + \mu y_2,$$
$$z_3 = \lambda z_1 + \mu z_2$$

equations which give

$$423 = \lambda . 421,\ 431 = \lambda . 421,\ \&c.,$$

and the equation $\square = 0$, thus becomes

$$\left|\begin{array}{lll} 789\omega 41, & 789\omega 42, & 789\omega 43 \\ 789\omega 51, & 789\omega 52, & 789\omega 53 \\ 789\omega 61, & 789\omega 62, & 789\omega 63 \end{array}\right| = 0,$$

or, what is the same thing,

$$\left|\begin{array}{lll} 789\omega 14, & 789\omega 24, & 789\omega 34 \\ 789\omega 15, & 789\omega 25, & 789\omega 35 \\ 789\omega 16, & 789\omega 27, & 789\omega 36 \end{array}\right| = 0.$$

Now if the terms in the same column are multiplied by $789\omega 56$, $789\omega 64$, $789\omega 45$ respectively and added, then for the first column the sum is

$$789\omega 14 . 789\omega 56 + 789\omega 15 . 789\omega 64 + 789\omega 16 . 789\omega 45,$$

which is $= 0$, and the sums for the second and third columns are each $= 0$ in the same manner: wherefore the determinant vanishes as it should do.

Next, if $789=0$, we have identically

$$789\omega 41 = 789\,.\,741\,.\,\omega 81\,.\,\omega 94 + 781\,.\,794\,.\,\omega 89\,.\,\omega 41,$$

which when $789=0$ gives

$$789\omega 41 = 781\,.\,794\,.\,\omega 89\,.\,\omega 41,$$

and in like manner,

$$789\omega 42 = 782\,.\,794\,.\,\omega 89\,.\,\omega 42,$$

$$789\omega 43 = 783\,.\,794\,.\,\omega 89\,.\,\omega 43,$$

in which three equations 4 may be changed into 5 and 6 successively. The equation $\square=0$ thus becomes

$$\begin{vmatrix} 423\,.\,\omega 41, & 431\,.\,\omega 42, & 412\,.\,\omega 43 \\ 523\,.\,\omega 51, & 531\,.\,\omega 52, & 512\,.\,\omega 53 \\ 623\,.\,\omega 61, & 631\,.\,\omega 62, & 612\,.\,\omega 63 \end{vmatrix} = 0,$$

or, what is the same thing,

$$\begin{vmatrix} 423\,.\,41\omega, & 421\,.\,4\omega 3, & 42\omega\,.\,431 \\ 523\,.\,51\omega, & 521\,.\,5\omega 3, & 52\omega\,.\,531 \\ 623\,.\,61\omega, & 621\,.\,6\omega 3, & 69\omega\,.\,631 \end{vmatrix} = 0,$$

and since the sum of the terms in each line of the determinant is $=0$, the determinant is as it should be $=0$.

The foregoing equation

$$\begin{vmatrix} 412\,.\,789\omega 43, & 423\,.\,789\omega 41, & 431\,.\,789\omega 42 \\ 512\,.\,789\omega 53, & 523\,.\,789\omega 51, & 531\,.\,789\omega 52 \\ 612\,.\,789\omega 63, & 623\,.\,789\omega 61, & 631\,.\,789\omega 62 \end{vmatrix}$$

$$= \epsilon\,.\,123\,.\,789\,.\,78\omega\,.\,79\omega\,.\,89\omega\,.\,123456789\omega,$$

(since 412, $789\omega 43$, &c. are interpretable functions of the coordinates) affords a geometrical interpretation of the equation

$$123456789\omega = 0$$

between the coordinates of the ten points of the cubic; but it would be more satisfactory if a similar identical equation could be found, having on the right-hand side the function 123456789ω without the irrelevant factor

$$123\,.\,789\,.\,78\omega\,.\,79\omega\,.\,89\omega.$$

2, *Stone Buildings, W.C., 6th March*, 1862.

296.

ON THE CONICS WHICH PASS THROUGH THE FOUR FOCI OF A GIVEN CONIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. v. (1862), pp. 275—280.]

THE foci of a conic are the points of intersection of the tangents through the circular points at infinity; the pair of tangents through each of the circular points at infinity is a conic through the four foci; and we have thus two conics $P=0$, $Q=0$ passing through the four foci; the equation of any other conic through the four foci is of course $P+\lambda Q=0$; and in particular if λ be suitably determined this equation gives the axes of the conic.

I was led to develope the solution, in seeking to obtain the elegant formulæ given in Mr P. J. Hensley's paper "Determination of the foci of the conic section expressed by trilinear coordinates," *Journal*, t. v., pp. 177—183, (March, 1862).

I take the coordinates to be proportionate to the perpendicular distances of the point from the sides of the fundamental triangle, each distance divided by the perpendicular distance of the side from the opposite angle. This being so, the equation of the line infinity is

$$x+y+z=0,$$

and, α, β, γ denoting the sides of the fundamental triangle, the equation of the circle circumscribed about the triangle is

$$\frac{\alpha^2}{x}+\frac{\beta^2}{y}+\frac{\gamma^2}{z}=0.$$

The foregoing two equations determine the circular points at infinity; and if (x_1, y_1, z_1) are the coordinates, there is no difficulty in obtaining the system of values

$$\begin{aligned} x_1 : y_1 : z_1 &= -\alpha &&: \beta(\cos C+i\sin C) &&: \gamma(\cos B+i\sin B) \\ &= \alpha(\cos C-i\sin C) &&: -\beta &&: \gamma(\cos A+i\sin A) \\ &= \alpha(\cos B+i\sin B) &&: \beta(\cos A-i\sin A) &&: -\gamma \quad , \end{aligned}$$

where as usual $i=\sqrt{(-1)}$, and where A, B, C denote the angles of the triangle, so that the cosines and sines of these angles denote given functions of α, β, γ. The coordinates (x_2, y_2, z_2) of the other circular point at infinity are of course obtained by merely writing $-i$ for i. We find also

$$x_1x_2 : y_1y_2 : z_1z_2 : y_1z_2+y_2z_1 : z_1x_2+z_2x_1 : x_1y_2+x_2y_1$$
$$=-\alpha^2 : -\beta^2 : -\gamma^2 : \beta^2+\gamma^2-\alpha^2 : \gamma^2+\alpha^2-\beta^2 : \alpha^2+\beta^2-\gamma^2,$$

which are the formula chiefly made use of in the sequel.

Suppose now that the equation of the conic is

$$U=(a, b, c, f, g, h)(x, y, z)^2=0;$$

then putting for a moment

$$U_1=(a, b, c, f, g, h)(x_1, y_1, z_1)^2,$$
$$W_1=(a, b, c, f, g, h)(x, y, z)(x_1, y_1, z_1),$$

and the like as regards U_2 and W_2; the tangents from the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively are

$$UU_1-W_1^2=0,$$
$$UU_2-W_2^2=0,$$

which are respectively pairs of lines intersecting in the four foci. And it is moreover clear that the equation of the axes is

$$U_2W_1^2-U_1W_2^2=0.$$

The foregoing equations may be written

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_1-y_1z, zx_1-z_1x, xy_1-x_1y)^2=0,$$
$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_2-y_2z, zx_2-z_2x, xy_2-x_2y)^2=0,$$

where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the inverse system of coefficients.

These may be written

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_1, y_1, z_1)^2=0,$$
$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_2, y_2, z_2)^2=0,$$

where a, b, c, f, g, h are quadric functions of x, y, z, viz.

$$\begin{aligned}
\mathrm{a} &= \mathfrak{B}z^2+\mathfrak{C}y^2-2\mathfrak{F}yz,\\
\mathrm{b} &= \mathfrak{C}x^2+\mathfrak{A}z^2-2\mathfrak{G}zx,\\
\mathrm{c} &= \mathfrak{A}y^2+\mathfrak{B}x^2-2\mathfrak{H}xy,\\
\mathrm{f} &= -\mathfrak{A}yz-\mathfrak{F}x^2+\mathfrak{G}xy+\mathfrak{H}xz,\\
\mathrm{g} &= -\mathfrak{B}zx-\mathfrak{G}y^2+\mathfrak{H}yz+\mathfrak{F}yx,\\
\mathrm{h} &= -\mathfrak{C}xy-\mathfrak{H}z^2+\mathfrak{F}zx+\mathfrak{G}xy,
\end{aligned}$$

and this being so, I combine the two equations as follows:

$$\begin{array}{llll}
x_2^2 & (\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 + x_1^2 & (\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 & = 0, \\
y_2^2 & \quad\quad " \quad\quad + y_1^2 & \quad\quad " & = 0, \\
z_2^2 & \quad\quad " \quad\quad + z_1^2 & \quad\quad " & = 0, \\
y_2 z_2 & (\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 + y_1 z_1 & (\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 & = 0, \\
z_2 x_2 & \quad\quad " \quad\quad + z_1 x_1 & \quad\quad " & = 0, \\
x_2 y_2 & \quad\quad " \quad\quad + x_1 y_1 & \quad\quad " & = 0,
\end{array}$$

any one of which is the equation of a conic passing through the four foci; the current coordinates being always (x, y, z).

The first of these equations is

$$\mathrm{a}\,(-2x_1^2x_2^2) + \mathrm{b}\,(x_2^2y_1^2 + x_1^2y_2^2) + \mathrm{c}\,(x_2^2z_1^2 + x_1^2z_2^2)$$
$$+ 2\mathrm{f}\,(x_2^2y_1z_1 + x_1^2y_2z_2) + 2\mathrm{g}\,(x_2^2z_1x_1 + x_1^2z_2x_2) + 2\mathrm{h}\,(x_2^2x_1y_1 + x_1^2x_2y_2) = 0,$$

where the quantities multiplied by a, b, &c. are all of them easily expressible in terms of x_1x_2, y_1y_2, z_1z_2, $y_1z_2 + y_2z_1$, $z_1x_2 + z_2x_1$, $x_1y_2 + x_2y_1$, which are respectively proportional to given functions of (α, β, γ); and replacing for a, b, &c. their values, the equation is

$$\begin{array}{ll}
\quad (\mathfrak{B}z^2 + \mathfrak{C}x^2 - 2\mathfrak{F}yz) & . \; 2\alpha^4 \\
+ \; (\mathfrak{C}x^2 + \mathfrak{A}y^2 - 2\mathfrak{G}zx) & . \; (\alpha^2 + \beta^2 - \gamma^2)^2 - 2\alpha^2\beta^2 \\
+ \; (\mathfrak{A}y^2 + \mathfrak{B}x^2 - 2\mathfrak{H}xy) & . \; (\gamma^2 + \alpha^2 - \beta^2)^2 - 2\gamma^2\alpha^2 \\
+ 2\,(-\mathfrak{A}yz - \mathfrak{F}x^2 + \mathfrak{G}xy + \mathfrak{H}xz) & . \; \alpha^2(\beta^2 + \gamma^2) - (\beta^2 - \gamma^2)^2 \\
+ 2\,(-\mathfrak{B}zx - \mathfrak{G}y^2 + \mathfrak{H}yz + \mathfrak{F}yz) & . - \alpha^2(\gamma^2 + \alpha^2 - \beta^2) \\
+ 2\,(-\mathfrak{C}xy - \mathfrak{H}z^2 + \mathfrak{F}zx + \mathfrak{G}zy) & . - \alpha^2(\alpha^2 + \beta^2 - \gamma^2) = 0.
\end{array}$$

Now putting for shortness

$$\square = \alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2,$$

so that $-\square$ is equal to sixteen times the square of the area of the fundamental triangle, the coefficient of x^2 is

$$= \mathfrak{B}\,(\square + 2\alpha^2\gamma^2) + \mathfrak{C}\,(\square + 2\alpha^2\beta^2) - 2\mathfrak{F}\,\{-\square - \alpha^2(\beta^2 + \gamma^2 - \alpha^2)\},$$

which is

$$= (\mathfrak{B} + \mathfrak{C} + 2\mathfrak{F})\,\square + 2\alpha^2\,\{\mathfrak{B}\gamma^2 + \mathfrak{C}\beta^2 + \mathfrak{F}\,(\beta^2 + \gamma^2 - \alpha^2)\}.$$

And reducing in a similar manner the other coefficients, the equation is

$$\square\,\{(\mathfrak{B} + \mathfrak{C} + 2\mathfrak{F})\,x^2 + \mathfrak{A}\,(y + z)^2 - 2\,(\mathfrak{H} + \mathfrak{G})\,x\,(y + z)\} + 2\alpha^2\Theta = 0,$$

where for shortness

$$\begin{array}{ll}
\Theta = \; x^2 \, . & \mathfrak{B}\gamma^2 + \mathfrak{C}\,\beta^2 + \mathfrak{F}\,(\beta^2 + \gamma^2 - \alpha^2) \\
+ y^2 \, . & \mathfrak{C}\alpha^2 + \mathfrak{A}\,\gamma^2 + \mathfrak{G}\,(\gamma^2 + \alpha^2 - \beta^2) \\
+ z^2 \, . & \mathfrak{A}\beta^2 + \mathfrak{B}\,\alpha^2 + \mathfrak{H}\,(\alpha^2 + \beta^2 - \gamma^2) \\
+ yz \, . & - 2\mathfrak{F}\alpha^2 + \mathfrak{A}\,(\beta^2 + \gamma^2 - \alpha^2) - \mathfrak{H}\,(\gamma^2 + \alpha^2 - \beta^2) - \mathfrak{G}\,(\alpha^2 + \beta^2 - \gamma^2) \\
+ zx \, . & - 2\mathfrak{G}\beta^2 - \mathfrak{H}\,(\beta^2 + \gamma^2 - \alpha^2) + \mathfrak{B}\,(\gamma^2 + \alpha^2 - \beta^2) - \mathfrak{F}\,(\alpha^2 + \beta^2 - \gamma^2) \\
+ xy \, . & - 2\mathfrak{H}\gamma^2 - \mathfrak{G}\,(\beta^2 + \gamma^2 - \alpha^2) - \mathfrak{F}\,(\gamma^2 + \alpha^2 - \beta^2) + \mathfrak{C}\,(\alpha^2 + \beta^2 - \gamma^2),
\end{array}$$

or, what is the same thing, the equation is

$$\square\ \{(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H})\,x^2-2\,(\mathfrak{A}+\mathfrak{H}+\mathfrak{G})\,x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0,$$

or putting for shortness

$$\begin{aligned}\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H}&=\mathfrak{P}\,,\\ \mathfrak{A}+\mathfrak{H}+\mathfrak{G}&=\mathfrak{L}\,,\\ \mathfrak{H}+\mathfrak{B}+\mathfrak{F}&=\mathfrak{M},\\ \mathfrak{G}+\mathfrak{F}+\mathfrak{C}&=\mathfrak{N}\,,\end{aligned}$$

so that in fact

$$\mathfrak{P}=\mathfrak{L}+\mathfrak{M}+\mathfrak{N},$$

the equation is

$$\square\ \{\mathfrak{P}x^2-2\mathfrak{L}x\,(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0.$$

The equation with y_2z_2, y_1z_1 is in a similar manner found to be

$$\square\ \{\mathfrak{F}x^2-(\mathfrak{C}+\mathfrak{G})\,y^2-(\mathfrak{B}+\mathfrak{H})\,z^2+(\mathfrak{A}+2\mathfrak{F}+\mathfrak{G}+\mathfrak{H})\,yz+(-\mathfrak{B}-\mathfrak{H}+\mathfrak{F})\,zx+(-\mathfrak{C}-\mathfrak{G}+\mathfrak{F})\,xy\}$$
$$-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0,$$

or, what is the same thing,

$$\square\ [(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H})\,yz-\{(\mathfrak{G}+\mathfrak{F}+\mathfrak{C})\,y+(\mathfrak{H}+\mathfrak{B}+\mathfrak{F})\,z\}(x+y+z)+\mathfrak{F}\,(x+y+z)^2]$$
$$-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0,$$

or, finally,

$$\square\ \{\mathfrak{P}yz-(\mathfrak{N}y+\mathfrak{M}z)\,(x+y+z)+\mathfrak{F}\,(x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0.$$

Hence the entire system of equations is

$$\begin{aligned}&\square\ \{\mathfrak{P}x^2 && -2\,\mathfrak{L}\,x\ (x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0,\\ &\square\ \{\mathfrak{P}y^2 && -2\mathfrak{M}y\ (x+y+z)+\mathfrak{B}\,(x+y+z)^2\}+2\beta^2\Theta=0,\\ &\square\ \{\mathfrak{P}z^2 && -2\mathfrak{N}\,z\ (x+y+z)+\mathfrak{C}\ (x+y+z)^2\}+2\gamma^2\Theta=0,\\ &\square\ \{\mathfrak{P}yz && -(\mathfrak{N}y+\mathfrak{M}z)\,(x+y+z)+\mathfrak{F}\ (x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0,\\ &\square\ \{\mathfrak{P}zx && -(\mathfrak{L}z+\mathfrak{N}x)\,(x+y+z)+\mathfrak{G}\ (x+y+z)^2\}-(\gamma^2+\alpha^2-\beta^2)\,\Theta=0,\\ &\square\ \{\mathfrak{P}xy && -(\mathfrak{M}x+\mathfrak{L}\,y)\,(x+y+z)+\mathfrak{H}\ (x+y+z)^2\}-(\alpha^2+\beta^2-\gamma^2)\,\Theta=0,\end{aligned}$$

which are the equations of six conics, each of them passing through the four foci.

From the first three of these, we have

$$\frac{1}{\alpha^2}\{\mathfrak{P}x^2-2\mathfrak{L}x\ \ (x+y+z)+\mathfrak{A}\,(x+y+z)^2\}$$

$$=\frac{1}{\beta^2}\{\mathfrak{P}y^2-2\mathfrak{M}y\,(x+y+z)+\mathfrak{B}\,(x+y+z)^2\}$$

$$=\frac{1}{\gamma^2}\{\mathfrak{P}z^2-2\mathfrak{N}z\ \ (x+y+z)+\mathfrak{C}\ (x+y+z)^2\},$$

which, allowing for the difference of notation, are Mr Hensley's equations: it appears by his investigation that their geometrical signification is as follows; viz. if for shortness we denote the equations by

$$A = B = C,$$

then if we consider the tangents parallel to the x-side of the fundamental triangle, and the tangents parallel to the y-side of the fundamental triangle, the equation $A = B$ is the locus of a point such that the feet of the perpendiculars let fall from it on the four tangents lie in a circle. And similarly for the equations $A = C$, $B = C$.

If we multiply the six equations by 1, 1, 1, 2, 2, 2 and add, we obtain the identical equation $0 = 0$; if we multiply them by a, b, c, $2f$, $2g$, $2h$ and add, then after some easy reductions, we obtain for the equation of a new conic passing through the four foci

$$\square \{\mathfrak{P} U + K(x+y+z)^2\} + 2S\Theta = 0,$$

where

$$U = (a, b, c, f, g, h)(x, y] z)^2,$$

K is the discriminant $abc - af^2 - bg^2 - ch^2 + 2fgh$, and

$$S = a\alpha^2 + b\beta^2 + c\gamma^2 - f(\beta^2 + \gamma^2 - \alpha^2) - g(\gamma^2 + \alpha^2 - \beta^2) - h(\alpha^2 + \beta^2 - \gamma^2),$$

or, what is the same thing,

$$S = (f + a - h - g)\alpha^2 + (g - h + b - f)\beta^2 + (h - g - f + c)\gamma^2.$$

It would be interesting to ascertain the geometrical signification of the six conics and of the last-mentioned new conic.

2, *Stone Buildings, W.C., March* 13*th*, 1862.

297.

ON SOME FORMULÆ RELATING TO THE DISTANCES OF A POINT FROM THE VERTICES OF A TRIANGLE, AND TO THE PROBLEM OF TACTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. v. (1862), pp. 381—384.]

THE relation between the distances of four points 1, 2, 3, 4 in a plane is

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{42}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

where, see my paper "Note on the value of certain Determinants the terms of which are the squared distances of Points in a plane or in space," *Quarterly Journal of Mathematics*, t. III., p. 275 (1859), [286], the determinant is

$$= \Sigma\overline{12}^2 . \overline{23}^2 . \overline{34}^2 - \Sigma\overline{12}^2 . \overline{34}^2 . \overline{43}^2 - \Sigma\overline{12}^2 . \overline{23}^2 . \overline{31}^2,$$

an identity which subsists without the aid of the relations $12 = 21$, &c., and in which the Σ, Σ, Σ contain 24, 12, and 8 terms respectively.

Writing $23 = f$, $31 = g$, $12 = h$, $14 = a$, $24 = b$, $34 = c$, the determinant is

$$\begin{aligned} &= 2\{ \; g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ &\quad + b^2c^2(g^2+h^2) + c^2a^2\;(a^2+f^2) + a^2b^2(f^2+g^2) \\ &\quad - a^2f^2(a^2+f^2) - b^2g^2\;(b^2+g^2) - c^2h^2(c^2+h^2) \\ &\quad - b^2c^2f^2 - c^2a^2g^2 - a^2b^2h^2 - f^2g^2h^2 \qquad \} \\ &= \; -2\square, \end{aligned}$$

if $\square$ denote the function in { } with the signs reversed. The function $\square$ may be expressed in the form

$$\begin{aligned}\square = \quad & a^4f^2 + b^4g^2 + c^4h^2 + f^2h^2g^2 \\ & + (a^2f^2 + b^2c^2)(f^2 - g^2 - h^2) \\ & + (b^2g^2 + c^2a^2)(g^2 - h^2 - f^2) \\ & + (c^2h^2 + a^2b^2)(h^2 - f^2 - g^2),\end{aligned}$$

and also in the form

$$\square = \quad U^2 + (f+g+h)V,$$

if for shortness

$$\begin{aligned}U = \quad & a^2f + b^2g + c^2h + fgh, \\ V = \quad & (a^2f^2 + b^2c^2)(f - g - h) \\ & + (b^2g^2 + c^2a^2)(g - h - f) \\ & + (c^2h^2 + a^2b^2)(h - f - g);\end{aligned}$$

and it may be remarked that since $\square$ is an even function of f, g, h, we may in this last formula change at pleasure the signs of these quantities; we thus obtain in all four similar forms of the function $\square$.

It is clear that considering a triangle, and any point in the plane of the triangle, f, g, h may be taken to denote the sides of the triangle, and a, b, c the distances of the point from the vertices: and the equation $\square = 0$ is the relation connecting the sides and distances.

The equation $f+g+h=0$ denotes that the vertices are *in lined*, and when this equation is satisfied we have

$$U = a^2f + b^2g + c^2h + fgh = 0,$$

which is in fact, as it is easy to see, the relation connecting the distances of a point from any three points *in lined*.

For a, b, c write $a+x$, $b+x$, $c+x$; x will be the radius of a circle touching the circles, radii a, b, c, described about the vertices as centres. The equation $\square = 0$ becomes after all reductions

$$\begin{aligned}& U^2 - (f+g+h)V \\ + x\,[\;& 4U(af+bg+ch) \\ & - 2(f+g+h)\{(af^2 + bc(b+c))(f-g-h) \\ & \qquad + (bg^2 + ca(c+a))(g-h-f) \\ & \qquad + (ch^2 + ab(a+b))(h-f-g)\}] \\ + x^2\,[\;& f^2\{-4a^2 + 6a(b+c) - 6bc\} \\ & + g^2\{-4b^2 + 6b(c+a) - 6ca\} \\ & + h^2\{-4c^2 + 6c(a+b) - 6ab\}] = 0,\end{aligned}$$

which is a quadratic equation only: the two circles thus obtained are those which touch the given circles all three externally or all three internally. But by changing in every possible manner the signs of a, b, c we obtain in all four equations giving the eight tangent circles. It may be noticed that if as before $f+g+h=0$, $U=0$,

then not only the constant term vanishes, but the coefficient of x also vanishes or the equation becomes simply $x^2=0$.

In particular, suppose $f=b+c$, $g=c+a$, $h=a+b$; developing this *de novo*, and putting for shortness

$$\begin{aligned} a+b+c &= p,\\ bc+ca+ab &= q,\\ abc &= r, \end{aligned}$$

we find

$$\begin{aligned} U &= 2\{px^2+2qx+pq-2r\},\\ V &= 2\{px^4+4qx^3+(2pq+12r)x^2+4q^2x+pq^2-4qr\}, \end{aligned}$$

and then the equation $\square = U^2-2pV=0$ gives

$$\begin{aligned} \tfrac{1}{4}\square &= (px^2+2qx+pq-2r)^2-p\{px^4+4qx^3+(2pq+12r)x^2+4q^2x+pq^2-4qr\}\\ &= 4\{(q^2-4pr)x^2-2qrx+r^2\}, \end{aligned}$$

so that we have

$$\tfrac{1}{16}\square = (q^2-4pr)x^2-2qrx+r^2 = (qx-r)^2-4prx = 0,$$

and thence

$$qx-r=\pm x\sqrt{pr}, \text{ or } x=\frac{r}{q\pm},$$

which gives the radii of the circles inscribed in and circumscribed about the three circles radii a, b, c, whereof each touches the two others: a formula given by Descartes, *Epistolæ* (Ed. 2, Franc. 1792), Pars III., p. 261, in a letter to the Princess Elizabeth, viz. Descartes has

$$(d^2e^2+d^2f^2+e^2f^2-2def^2-2d^2ef-2de^2f)x^2-2(de^2f^2+d^2ef^2+d^2e^2f)x+d^2e^2f^2=0,$$

which putting a, b, c for his d, e, f, becomes *ut suprà*

$$x^2(q^2-4pr)-2qrx+r^2=0.$$

In conclusion I notice the following formula which is obtained without difficulty, viz. if as before we have a triangle the sides whereof are f, g, h, and if a, b, c are the distances of a point from the vertices (so that as before $\square=0$) then the perpendicular distances of the point from the sides, each perpendicular distance divided by the perpendicular distance of the opposite vertex from the same side, are as follows: viz. the quotient for the side f is

$$=\frac{1}{16\Delta^2}[(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4],$$

where Δ is the area of the triangle. It is clear that we ought to have

$$\Sigma\{(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4\}=16\Delta^2,$$

and this equation in fact reduces itself to

$$2g^2h^2+2h^2f^2+2f^2g^2-f^4-g^4-h^4=16\Delta^2,$$

which is right.

2, *Stone Buildings, W.C.*, 17*th September*, 1862.

298.

REPORT ON THE PROGRESS OF THE SOLUTION OF CERTAIN SPECIAL PROBLEMS OF DYNAMICS.

[From the *Report of the British Association for the Advancement of Science*, 1862, pp. 184—252.]

My "Report on the Recent Progress of Theoretical Dynamics" was published in the Report of the British Association for the year 1857, [195]. The present Report (which is in some measure supplemental thereto) relates to the *Special Problems* of Dynamics: to give a general idea of the contents, I will at once mention the heads under which these problems are considered; viz., relating to the motion of a particle or system of particles, we have

Rectilinear Motion;
Central Forces, and in particular
Elliptic Motion;
The Problem of two Centres;
The Spherical Pendulum;
Motion as affected by the Rotation of the Earth, and Relative Motion in general;
Miscellaneous Problems;
The Problem of three bodies.

And relating to the motion of a solid body, we have

The Transformation of Coordinates;
Principal Axes, and Moments of Inertia;
Rotation of a Solid Body;
Kinematics of a Solid Body;
Miscellaneous Problems.

As regards the first division of the subject, I remark that the lunar and planetary theories, as usually treated, do not (properly speaking) relate to the problem of three bodies, but to that of disturbed elliptic motion—a problem which is not considered in the present Report. The problem of the spherical pendulum is that of a particle moving on a spherical surface; but, with this exception, I do not much consider the motion of a particle on a given curve or surface, nor the motion in a resisting medium; what is said on these subjects is included under the head Miscellaneous Problems. The first six heads relate exclusively, and the head Miscellaneous Problems relates principally, to the motion of a single particle. As regards the second division of the subject, I will only remark that, from its intimate connexion with the theory of the motion of a solid body, I have been induced to make a separate head of the geometrical subject, "Transformation of Coordinates," and to treat of it in considerable detail.

I have inserted at the end of the present Report a list of the memoirs and works referred to, arranged (not, as in the former Report, in chronological order, but) alphabetically according to the authors' names: those referred to in the former Report formed for the purpose thereof a single series, which is not here the case. The dates specified are for the most part those on the title-page of the volume, being intended to show approximately the date of the researches to which they refer, but in some instances a more particular specification is made.

I take the opportunity of noticing a serious omission in my former Report, viz., I have not made mention of the elaborate memoir, Ostrogradsky "Mémoire sur les équations différentielles relatives au problème des Isopérimètres," *Mém. de St Pét.*, t. IV. (6 sér.) pp. 385—517, 1850, which among other researches contains, and that *in the most general form*, the transformation of the equations of motion from the Lagrangian to the Hamiltonian form, and indeed the transformation of the general isoperimetric system (that is, the system arising from any problem in the calculus of variations) to the Hamiltonian form. I remark also, as regards the memoir of Cauchy referred to in the note p. 12 as an *unpublished* memoir of 1831, there is an "Extrait du Mémoire présenté à l'Académie de Turin le 11 Oct. 1831," published in lithograph under the date Turin, 1832, with an addition dated 6 Mar. 1833. The Extract begins thus:—"§ I. Variation des Constantes Arbitraires. Soient données entre la variable $t, \ldots$ n fonctions de t désignées par $x, y, z \ldots$ et n autres fonctions de t désignées par $u, v, w, \ldots$ $2n$ équations différentielles du premier ordre et de la forme

$$\frac{dx}{dt} = \frac{dQ}{du}, \quad \frac{dy}{dt} = \frac{dQ}{dv}, \quad \frac{dz}{dt} = \frac{dQ}{dw},$$

$$\frac{du}{dt} = -\frac{dQ}{dx}, \quad \frac{dv}{dt} = -\frac{dQ}{dy}, \quad \frac{dw}{dt} = -\frac{dQ}{dz}, \text{ \&c.}"$$

without any explanation as to the origin of these equations; and the formulæ are then given for the variations of the constants in the integrals of the foregoing system; this seems sufficient to establish that Cauchy in the year 1831 was familiar with the Hamiltonian form of the equations of motion.

Bour's "Mémoire sur l'intégration des équations différentielles de la Mécanique," as published, *Mém. prés. à l'Inst.*, t. XIV. pp. 792—821, is substantially the same as the extract thereof in *Liouville's Journal*, referred to in my former Report; but since the date of that Report there have been published in the *Comptes Rendus*, 1861 and 1862, several short papers by the same author; also Jacobi's great memoir, see list, Jacobi, Nova Methodus &c., 1862, edited after his decease by Clebsch; some valuable memoirs by Natani and Clebsch (*Crelle*, 1861 and 1862) relating to the Pfaffian system of equations (which includes those of Dynamics), and Boole "On Simultaneous Differential Equations of the First Order, in which the number of the Variables exceeds by more than one the number of the Equations," *Phil. Trans.*, t. CLII. (1862), pp. 437—454.

Rectilinear Motion. Article Nos. 1 to 5.

1. The determination of the motion of a falling body, which is the case of a constant force, is due to Galileo.

2. A variable force, assumed to be a force depending only on the position of the particle, may be considered as a function of the distance from any point in the line, selected at pleasure as a centre of force; but if, as usual, the force is given as a function of the distance from a certain point, it is natural to take that point for the centre of force. The problem thus becomes a particular case of that of central forces; and it is so treated in the *Principia*, Book I. § 7; the method has the advantage of explaining the paradoxical result which presents itself in the case Force $\propto$(Dist.)$^{-2}$, and in some other cases where the force becomes infinite. According to theory, the velocity becomes infinite at the centre, but the direction of the motion is there abruptly reversed; so that the body in its motion does not pass through the centre, but on arriving there, forthwith returns towards its original position; of course such a motion cannot occur in nature, where neither a force nor a velocity ever is actually infinite.

3. Analytically the problem may be treated separately by means of the equation $\frac{d^2x}{dt^2} = X$, which is at once integrable in the form $\left(\frac{dx}{dt}\right)^2 = C + 2\int X dx$.

4. The following cases may be mentioned:

Force $\propto$ Dist. The law of motion is well known, being in fact the same as for the cycloidal pendulum.

Force $\propto$(Dist.)$^{-2}$, $= \frac{\mu}{x^2}$, which is the case above alluded to.

Assuming that the body falls from rest at a distance a, we have

$$x = a(1 - \cos\phi),$$

where, if $n = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}$, ϕ is given in terms of the time by means of the equation

$$nt = \phi - \sin\phi.$$

If the body had initially a small transverse velocity, the motion would be in a very eccentric ellipse, and the formulæ are in fact the limiting form of those for elliptic motion.

5. There are various laws of force for which the motion may be determined. In particular it can be determined by means of Elliptic Integrals, in the case of a body attracted to two centres, force $\propto$(dist.)$^{-2}$: see Legendre, *Exercices de Cal. Intég.*, t. II. pp. 502—512, and *Théorie des Fonct. Ellip.*, t. I. pp. 531—538.

Central Forces, Article Nos. 6 to 26.

6. The theory of the motion of a body under the action of a given central force was first established in the *Principia*, Book I. §§ 2 and 3: viz. Prop. I. the areas are proportional to the times, that is (using the ordinary analytical notation), $r^2 d\theta = h dt$, and Prop. VI. Cor. 3, $P \propto \frac{1}{SY^2 . PV}$, $= h^2 u^2 \left(\frac{d^2u}{d\theta^2} + u\right)$, so that

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0.$$

7. It is to be noticed that, given the orbit, the law of force is at once determined; and § 2 contains several instances of such determination; thus,

Prop. VII. If a body revolve in a circle, the law of force to a point S is force $\propto \frac{1}{SP^2 . PV^3}$ (P the body, PV the chord through S).

Prop. IX. If a body move in a logarithmic spiral, force $\propto$(dist.)$^{-3}$.

Prop. X. If a body move in an ellipse, force to centre $\propto$dist., and as a particular case, if the body move in a parabola under the action of a force parallel to the axis, the force is constant. The particular case of motion in a parabola had been obtained by Galileo.

And § 3, Props. XI. XII. XIII. If a body move in an ellipse, hyperbola, or parabola under the action of a force tending to the focus, force $\propto$(dist.)$^{-2}$.

8. But Newton had no direct method of solving the inverse problem (which depends on the solution of the differential equation), "Given the force to find the orbit." Thus force $\propto$(dist.)$^{-2}$, after it has been shown that an ellipse, a hyperbola, and a parabola may each of them be described under the action of such a force, the remainder of the solution consists in showing that, given the initial circumstances of the motion, a conic section (ellipse, parabola, or hyperbola, as the case may be) can be constructed, passing through the point of projection, having its tangent in the direction of the initial motion, and such that the velocity of the body describing the conic section under the action of the given central force is equal to the velocity of projection; which being so, the orbit will be the conic section so constructed. This is what is done, Prop. XVII.; it may be observed that the latus rectum is constructed not very elegantly by means of the latus rectum of an auxiliary orbit.

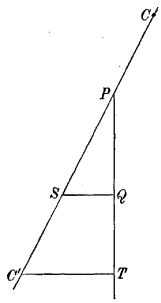
9. A more elegant construction was obtained by Cotes (see the *Harmonia Mensurarum*, pp. 103—105, and demonstration from the author's papers in the Notes by R. Smith, pp. 124, 125), depending on the position of a point C, such that the velocity acquired in falling under the action of the central force from C *directly or through infinity* [1] to P the point of projection, is equal to the given velocity of projection.

10. But Newton's original construction is now usually replaced by a construction which employs the space due to the velocity of projection considered as produced by a constant force equal to the central force at the point of projection.

11. § 9 of Book I. relates to revolving orbits, viz., it is shown that a body may be made to move in an orbit revolving round the centre of force, by adding to the central force required to make the body move in the same orbit at rest, a force $\propto$ (dist.)$^{-3}$. This appears very readily by means of the differential equation (*antè*, No. 6), viz. writing therein $P + cu^3$ for P, and then θ', h' in the place of $\theta\sqrt{1-\frac{c}{h^2}}$, $h\sqrt{1-\frac{c}{h^2}}$ respectively, the equation retains its original form, with θ', h' in the place of θ, h respectively.

12. It may be remarked that when the original central force vanishes, the fixed orbit is a right line (not passing through the centre of force). It thus appears by § 9 that the curve $u = A\cos(n\theta + B)$ may be described under the action of a force $\propto$ (dist.)$^{-3}$. A proposition in § 2, already referred to, shows that a logarithmic spiral may be described under the action of such a force.

13. But the case of a force $\propto$ (dist.)$^{-2}$ was first completely discussed by Cotes in



the *Harmonia Mensurarum*, pp. 31—35, 98—104, and Notes, pp. 117—173. There are in all five cases, according as the velocity of projection is

1. Less than that acquired in falling from infinity, or say equal to that acquired in falling from a point C to P, the point of projection.

[1] In the second case C lies on the radius vector produced beyond the centre, and the body is supposed to fall from rest at C (under the action of the central force considered as repulsive) to infinity, and then from the opposite infinity (with an initial velocity equal to the velocity so acquired) to P.

2. Equal to that acquired in falling from infinity.

3, 4, 5. Greater than that acquired in falling from infinity, or say equal to that acquired in falling from a point C', *through infinity*, to P; viz. PQ being the direction of projection, and SQ, $C'T$ perpendiculars thereon from S and C' respectively,

$$3.\quad SQ < TQ;$$
$$4.\quad SQ = TQ;$$
$$5.\quad SQ > TQ;$$

the equations of the orbits being

1. $u = Ae^{m\theta} + Be^{-m\theta}$, A and B same sign, so that rad. vector is never infinite.

2. $u = Ae^{m\theta}$ or $Be^{-m\theta}$, logarithmic spiral.

3. $u = Ae^{m\theta} + Be^{-m\theta}$, A and B opposite signs, so that rad. vector becomes infinite.

4. $u = A\theta + B$, $m = 0$, reciprocal spiral.

5. $u = A\cos(n\theta + B)$, $m = n\sqrt{-1}$.

14. The before-mentioned equation,

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0,$$

is in effect given (but the equation is encumbered with a tangential force) in Clairaut's *Théorie de la Lune*, 1765. It is given in its actual form, and extensively used (in particular for obtaining the above-mentioned equations for Cotes' spirals) in Whewell's *Dynamics*, 1823. The equation appears to be but little known to continental writers, and (under the form $u'' + u - \alpha^2 y^2 R = 0$) it is given *as new* by Schellbach as late as 1853. The formulæ used in place of it are those which give t and θ each of them in terms of r; viz.

$$dt = \frac{rdr}{\{-h^2 + r^2(C - 2\int Pdr)\}^{\frac{1}{2}}},$$

$$d\theta = \frac{hdr}{r\{-h^2 + r^2(C - 2\int Pdr)\}^{\frac{1}{2}}},$$

which, however, assume that P is a function of r only.

15. Force $\propto$ (dist.)$^{-2}$. The law of motion in the conic sections is implicitly given by Newton's theorem for the equable description of the areas. For the parabola, if α denote the pericentric distance, and f the angle from pericentre or true anomaly, we have

$$t = \frac{\alpha^{\frac{3}{2}}\sqrt{2}}{\sqrt{\mu}}\left(\tan\tfrac{1}{2}f + \tfrac{1}{3}\tan^3\tfrac{1}{2}f\right).$$

For the ellipse we have an angle g, the mean anomaly varying directly as the time $\left(g = nt \text{ if } n = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}\right)$; an auxiliary angle u, the eccentric anomaly, connected with g by the equation

$$g = u - e \sin u;$$

and then the radius vector r and the true anomaly f are given in terms of u by the equations $r = a(1 - e \cos u)$, and

$$\cos f = \frac{\cos u - e}{1 - e \cos u}, \quad \sin f = \frac{\sqrt{1 - e^2} \sin u}{1 - e \cos u}, \text{ and therefore } \tan \tfrac{1}{2} f = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} u.$$

16. It is very convenient to have a notation for $\frac{r}{a}$ and f considered as functions of e, g, and I have elsewhere proposed to write

$$r = a \operatorname{elqr}(e, g), \quad f = \operatorname{elta}(e, g),$$

read elqr elliptic quotient radius, and elta elliptic true anomaly.

17. The formulæ for the hyperbola correspond to those for the ellipse, but they contain exponential in the place of circular functions (see *post*, Elliptic Motion).

18. Euler, in the memoir "Determinatio Orbitæ Cometæ Anni 1742," (1743), p. 16 *et seq.*, obtained an expression for the time of describing a parabolic arc in terms of the radius vectors and the chord; viz. these being f, g, and k, the expression is

$$\text{Time} = \frac{1}{6\sqrt{\mu}} \left\{ \left(f + g + k\right)^{\frac{3}{2}} - \left(f + g - k\right)^{\frac{3}{2}} \right\},$$

which, however, as remarked by Lagrange, *Méc. Anal.*, t. II. (3rd edit. p. 28), is deducible from Lemma X. of the third book of the *Principia*. But the theorem in its actual form is due to Euler.

19. Lambert, in the *Proprietates Insigniores &c.* (1761), Theorem VII. Cor. 2, obtained the same theorem, and in section 4 he obtained the corresponding theorem for elliptic motion; viz. the expression for the time is

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left\{ \phi - \phi' - (\sin \phi - \sin \phi') \right\},$$

if

$$\sin \tfrac{1}{2}\phi = \tfrac{1}{2} \sqrt{\frac{f + g + k}{a}}, \quad \sin \tfrac{1}{2}\phi' = \tfrac{1}{2} \sqrt{\frac{f + g - k}{a}}.$$

The form of the formula is, it will be observed, similar to that for motion in a straight line (*antè*, No. 4), and in fact the motion in the ellipse is, by an ingenious geometrical transformation, made to depend upon that in the straight line. The geometrical theorems upon which the transformation depends are stated, Cayley "On Lambert's Theorem &c." (1861).

20. The theorem was also obtained by Lagrange in the memoir "Recherches &c." (1767) as a corollary to his solution of the problem of two centres; viz. upon making the attractive force of one of the centres equal to zero, and assuming that such centre is situate on the curve, the expression for the time presents itself in the form given by Lambert's theorem.

21. Two other demonstrations of the theorem are given by Lagrange in the memoir "Sur une manière particulière d'exprimer le temps &c." (1778), reproduced in Note V. of the second volume of the last edition (Bertrand's) of the *Mécanique Analytique*. As M. Bertrand remarks, these demonstrations are very complete, very elegant, and very natural, assuming that the theorem is known beforehand.

Demonstrations were also given by Gauss, "Theoria Motus" (1809), p. 119 *et seq.*; Pagani, "Démonstration d'un théorème &c." (1834); and (in connexion with Hamilton's Principal Function) by Sir W. R. Hamilton, "On a General Method &c." (1834), p. 282; Jacobi, "Zur Theorie &c." (1837), p. 122; Cayley, "Note on the Theory of Elliptic Motion" (1856).

22. Connected with the problem of central forces, we have Sir W. R. Hamilton's "Hodograph," which in the paper (*Proc. R. Irish Acad.* 1847) is defined, and the fundamental properties thereof are stated; viz. if in an orbit round a centre of force there be taken on the perpendicular from the centre on the tangent at each point, a length equal to the velocity at that point of the orbit, the extremities of these lengths will trace out a curve which is the hodograph. As the product of the velocity into the perpendicular on the tangent is equal to twice the area swept out in a unit of time ($vp = h$), the hodograph is the reciprocal polar of the orbit with respect to a circle described about the centre of force, radius $= \sqrt{h}$. Whence also the tangent at any point of the hodograph is perpendicular to the radius vector through the corresponding point of the orbit, and the product of the perpendicular on the tangent into the corresponding radius vector is $= h$.

If force $\propto$ (dist.)$^{-3}$, the hodograph, *quà* reciprocal polar of a conic section with respect to a circle described about the focus, is a circle.

23. The following theorem is also given without demonstration; viz. if two circular hodographs, which have a common chord passing or tending through a common centre of force, be both cut at right angles by a third circle, the times of hodographically describing the intercepted arcs (that is, the times of describing the corresponding elliptic arcs) will be equal.

24. Droop, "On the Isochronism &c." (1856), shows geometrically that the last-mentioned property is equivalent to Lambert's theorem; and an analytical demonstration is also given, Cayley, "A demonstration of Sir W. R. Hamilton's Theorem &c." (1857). See also Sir W. R. Hamilton's *Lectures on Quaternions* (1853), p. 614.

25. The laws of central force which have been thus far referred to are force $\propto r$, $\propto \frac{1}{r^2}$, $\propto \frac{1}{r^3}$; and it has been seen that the case of a force $P + \frac{C}{r^3}$ depends upon that

of a force P, so that the motions for the forces $Ar+\frac{C}{r^3}$ and $\frac{B}{r^2}+\frac{C}{r^3}$ are deducible from those for the forces Ar and $\frac{B}{r^2}$ respectively. Some other laws of force, e.g. $\frac{A}{r^2}\pm Br$, $\frac{A}{r^2}+\frac{B}{r^3}+\frac{C}{r^4}+\frac{D}{r^5}$, are considered by Legendre, "Théorie des Fonctions Elliptiques" (1825), being such as lead to results expressible by elliptic integrals, and also the law $\frac{M}{r}$, for which the result involves a peculiar logarithmic integral. But the most elaborate examination of the different cases in which the solution can be worked out by elliptic integrals or otherwise is given in Stader's memoir "De Orbitis &c." (1852), where the investigation is conducted by means of the formulæ which give t and θ in terms of r (*antè*, No. 14).

26. In speaking of a central force, it is for the most part implied that the force is a function of the distance: for some problems in which this is not the case, see *post*, Miscellaneous Problems, Nos. 86 and 87.

It is to be noticed that, although the problem of central forces may be (as it has so far been) considered as a problem *in plano* (viz. the plane of the motion has been made the plane of reference), yet that it is also interesting to consider it as a problem in space; in fact, in this case the integrals, though of course involved in those which belong to the plane problem, present themselves under very distinct forms, and afford interesting applications of the theory of canonical integrals, of the derivation of the successive integrals by Poisson's method, and of other general dynamical theories. Moreover, in the lunar and planetary theories, the problem must of necessity be so treated. Without going into any details on this point, I will refer to Bertrand's memoir, "Sur les Équations différentielles de la Mécanique" (1852), Donkin's memoir "On a Class of Differential Equations &c." (1855), and Jacobi's posthumous memoir, "Nova Methodus &c." (1862).

Elliptic Motion, Article Nos. 27—40.

27. The question of the development of the true anomaly in terms of the mean anomaly (Kepler's problem), and of the other developments which present themselves in the theory of elliptic motion, is one that has very much occupied the attention of geometers. The formulæ on which it depends are mentioned *antè*, No. 15; they involve as an auxiliary quantity the eccentric anomaly u.

28. Consider first the equation

$$g = u - e\sin u,$$

which connects the mean anomaly g with the eccentric anomaly u.

Any function of u, and in particular u itself, and the functions $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, nu$ may be expanded in terms of g by means of Lagrange's theorem (Lagrange, *Mém. de Berlin*, 1768—1769, "Théorie des Fonctions," chap. 16, and "Traité de la Résolution des Équations Numériques," Note 11).

29. Considering next the equation

$$\tan \tfrac{1}{2} f = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} u,$$

which gives the true anomaly in terms of the eccentric anomaly, then, by replacing the circular functions by their exponential values (a process employed by Lagrange, *Mém. de Berlin*, 1776), f can be expressed in terms of u; viz. the result is

$$f = u + 2\lambda \sin u + 2\lambda^2 . \tfrac{1}{2} \sin 2u + 2\lambda^3 . \tfrac{1}{3} \sin 3u + \&c.,$$

where $\lambda = \dfrac{1-\sqrt{1-e^2}}{e} \left(= \dfrac{e}{1+\sqrt{1-e^2}}\right)$. Hence if u, $\sin u$, $\sin 2u$, &c. are expressed in terms of the mean anomaly, f will be obtained in the form $f = g +$ a series of multiple sines of g, the coefficients of the different terms being given in the first instance as functions of e and λ; and to complete the development λ and its powers have to be developed in powers of e. The solution is carried thus far in the *Mécanique Analytique* (1788), and in the *Mécanique Celeste* (1799).

30. We have next Bessel's investigations in the Berlin Memoirs for 1816, 1818, and 1824, and which are carried on mainly by means of the integral

$$2\pi I_k^h = \int_0^{2\pi} \cos (hu - k \sin u)\, du,$$

and various properties are there obtained and applications made of this important transcendant.

31. Relating to this integral we have Jacobi's memoir, "Formulæ transformationis &c." (1836), Liouville, "Sur l'intégrale $\int_0^\pi \cos i (u - x \sin u)\, du$" (1841), and Hansen's "Ermittelung der absoluten Störungen" (1843); the researches of Poisson in the *Connaissance des Temps* for 1825 and 1836 are closely connected with those of Bessel.

32. A very elegant formula, giving the actual expression of the coefficients considered as functions of e and λ, is given by Greatheed in the paper "Investigation of the General Term &c." (1838); viz. this is

$$f = g + 2\Sigma\lambda^r \left\{ \epsilon^{\frac{1}{2}re(\lambda+\lambda^{-1})} + \lambda^{-r} \epsilon^{-\frac{1}{2}(\lambda+\lambda^{-1})} \right\} \frac{\sin rg}{r},$$

where, after developing in powers of λ, the negative powers of λ must be rejected, and the term independent of λ divided by 2. This result is extended to other functions of f, Cayley "On certain Expansions &c." (1842).

33. An expression for the coefficient of the general term as a function of e only is obtained, Lefort, "Expression Numérique &c." (1846). The expression, which, from the nature of the case, is a very complicated one, is obtained by means of Bessel's integral. This is an indirect process which really comes to the combination of the developments of f in terms of u, and u in terms of g; and an equivalent result is obtained directly in this manner, Creedy, "General and Practical Solution &c." (1855).

34. We have also on the subject of these developments the very valuable and interesting researches of Hansen, contained in his *Fundamenta Nova &c.* (1838), in the memoir "Ermittelung der absoluten Störungen &c." (1845), and in particular in the memoir "Entwickelung des Products &c." (1853).

35. But the expression for the coefficient of the general term $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} rg$ in any of these expansions is so complicated that it was desirable to have for the coefficients corresponding to the values $r = 0, 1, 2, 3, \ldots$ the finally reduced expressions in which the coefficient of each power of e is given as a numerical fraction. Such formulæ for the development of $\left(\frac{r}{a} - 1\right)^m \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf$, where j is a general symbol, the expansion being carried as far as e^7, were given, Leverrier, *Annales de l'Observatoire de Paris*, t. I. (1855).

36. And starting from these I deduced the results given in my "Tables of the Developments &c." (1861); viz. these tables give $\left(x = \frac{r}{a} - 1\right)$,

$$(x^1, \ldots x^7),$$

$$(x^0, x^1, \ldots x^7) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf,\ j = 1 \text{ to } j = 7,$$

$$\left(\left(\frac{r}{a}\right)^4, \left(\frac{r}{a}\right)^3 \ldots \log\frac{r}{a}, \left(\frac{r}{a}\right)^{-1}, \ldots \left(\frac{r}{a}\right)^{-5}\right),$$

$$\left(\left(\frac{r}{a}\right)^4, \ldots \left(\frac{r}{a}\right)^1, \left(\frac{r}{a}\right)^{-1}, \ldots \left(\frac{r}{a}\right)^{-5}\right) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jf,\ j = 1 \text{ to } j = 7,$$

all carried to e^7.

37. The true anomaly f has been repeatedly calculated to a much greater extent, in particular by Schubert (*Ast. Théorique*, St Pét. 1822), as far as e^{20}. The expression for $\frac{r}{a}$ as far as e^{13} is given in the same work, and that for $\log\frac{r}{a}$ as far as e^9 was calculated by Oriani, see Introd. to Delambre's *Tables du Soleil*, Paris (1806).

38. It may be remarked that when the motion of a body is referred to a plane which is not the plane of the elliptic orbit, then we have questions of development similar in some measure to those which regard the motion in the orbit; if, for instance, z be the distance from node, ϕ the inclination, and x the reduced distance from node, then $\cos z = \cos\phi \cos x$, from which we may derive $z = x +$ series of multiple sines of x. And there are, moreover, the questions connected with the development of the reciprocal distance of two particles—say $(a^2 + a'^2 - 2aa' \cos\theta)^{-\frac{1}{2}}$—which present themselves in the planetary theory; but this last is a wide subject, which I do not here enter upon. I will, however, just refer to Hansen's memoir, "Ueber die Entwickelung der negativen und ungeraden Potenzen &c." (1854).

39. The question of the convergence of the series is treated in Laplace's memoir of 1823, where he shows that in the series which express r and f in multiple cosines or sines of g, the coefficient of a term $\begin{matrix}\cos\\ \sin\end{matrix} ig$, where i is very great, is at most equal in absolute value to a quantity of the form $\frac{A}{i\sqrt{i}}\left(\frac{e}{\lambda}\right)^i$, A and λ being finite quantities independent of i, whence he concludes that, in order to the convergency of the series, the limiting value of the eccentricity is $e=\lambda$, the numerical value being $e=0{\cdot}66195$.

40. The following important theorem was established by Cauchy, as part of a theory of the convergence of series in general; viz. so long as e is less than $0{\cdot}6627432$, which is the least modulus of e for which the equations

$$\frac{\pi}{2}=u-e\sin u,\quad 1-e\cos u=0,$$

can be satisfied, the development of the true anomaly and other developments in the theory of elliptic motion will be convergent. This was first given in the "Mémoire sur la Mécanique Céleste," read at Turin in 1831, but it is reproduced in the memoir "Considérations nouvelles sur les suites &c.," *Mém. d'Anal. et de Phys. Math.* t. I. (1840); and see also the memoirs in *Liouville's Journal* by Puiseux, and his Note i. to vol. II. of the 3rd ed. of the *Mécanique Analytique* (1855). There are on this subject, and on subjects connected with it, several papers by Cauchy in the *Comptes Rendus*, 1840 *et seq.*, which need not be particularly referred to.

The Problem of two Centres, Article Nos. 41 to 64.

41. The original problem is that of the motion of a body acted upon by forces tending to two centres, and varying inversely as the squares of the distances; but, as will be noticed, the solutions apply with but little variation to more general laws of force.

42. It may be convenient to notice that the coordinates made use of (in the several solutions) for determining the position of the body, are either the sum and difference of the two radius vectors, or else quantities which are respectively functions of the sum and the difference of these radius vectors [1]. If the plane of the motion is not given, then there is a third coordinate, which is the inclination of the plane through the body and the two centres to a fixed plane through the two centres, or say the azimuth of the axial plane, or simply the azimuth.

[1] If v, u are the distances of the body P from the centres A and B, a the distance AB, ζ, η the angles at A and B respectively, and $p=\tan\frac{1}{2}\zeta\tan\frac{1}{2}\eta$, $q=\tan\frac{1}{2}\zeta+\tan\frac{1}{2}\eta$, then, as may be shown without difficulty, $v+u=a\frac{1+p}{1-p}$, $v-u=a\frac{1-q}{1+q}$, so that p and q are functions of $v+u$ and $v-u$ respectively; these quantities p and q are Euler's original coordinates.

43. Calling the first-mentioned two coordinates r and s, and the azimuth ψ, the solution of the problem leads ultimately to equations of the form

$$\frac{dr}{\sqrt{R}} = \frac{ds}{\sqrt{S}}, \quad dt = \frac{\lambda dr}{\sqrt{R}} + \frac{\mu ds}{\sqrt{S}}, \quad \delta\psi = \frac{\rho dr}{\sqrt{R}} + \frac{\sigma ds}{\sqrt{S}},$$

where R and S are rational and integral functions (of the third or fourth degree, in the case of forces varying as $(\text{dist.})^{-2}$) of r, s respectively (but they are not in general the same functions of r, s respectively); λ and ρ are simple rational functions of r, and μ and σ simple rational functions of s; so that the equations give by quadratures, the first of them the curve described in the axial plane, the second the position of the body in this curve at a given time, and the third of them the position of the axial plane. In the ordinary case, where R and S are each of them of the third or the fourth order, the quadratures depend on elliptic integrals [1]; but on account of the presence in the formulæ of the two distinct radicals $\sqrt{R}$, $\sqrt{S}$, it would appear that the solution is not susceptible of an ulterior development by means of elliptic and Jacobian functions [1] similar to those obtained in the problems of Rotation and the Spherical Pendulum.

44. It has just been noticed that when R, S are each of them of the fourth order, the quadratures depend on elliptic integrals; in the particular cases in which the relation between r, s is of the form $\frac{mdr}{\sqrt{R}} = \frac{nds}{\sqrt{S}}$, R and S being the same functions of r, s respectively, and m and n being integers (or more generally for other relations between the forms of R, S given by the theory of elliptic integrals), the equation admits of algebraical integration; but as the relations in question do not in general hold good, the theory of the algebraical integration of the equations plays only a secondary part in the solution of the problem. It is, however, proper to remark that Euler, when he wrote his first two memoirs "On the Problem of the two Centres" (*post*, Nos. 45 and 46), had already discovered and was acquainted with the theory of the algebraic integration of the equation $\frac{mdr}{\sqrt{R}} = \frac{nds}{\sqrt{S}}$ (R, S, m, n, *ut suprà*), although his memoir, "Integratio æquationis

$$\frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}} = \frac{dy}{\sqrt{A + By + Cy^2 + Dy^3 + Ey^4}},"$$

N. Comm. Petrop. t. XII. 1766—1767 ?, bears in fact a somewhat later date.

45. Having made these preliminary remarks, I come to the history of the problem.

It is I think clear that Euler's *earliest* memoir is the one "De Motu Corporis &c." in the Petersburg Memoirs for 1764 (printed 1766). In this memoir the forces vary

[1] The elliptic integrals are Legendre's functions F, E, Π; the elliptic and Jacobian functions are sinam, cosam, Δam, and the higher transcendants, Θ, H.

as (dist.)$^{-2}$, and the body moves in a given plane. The equations of motion are taken to be

$$\frac{d^2x}{dt^2} = 2g\left(-\frac{Ax}{v^3} + \frac{B(a-x)}{u^3}\right),$$

$$\frac{d^2y}{dt^2} = 2g\left(-\frac{Ay}{v^3} - \frac{By}{u^3}\right),$$

which, if ζ, η are the inclinations of the distances v, u to the axis respectively (see foot-note to No. 42), lead to

$$dv^2 + v^2dt^2 = 4gdt^2\left(\frac{A}{v} + \frac{B}{u} + \frac{D+E}{a}\right),$$

$$v^2u^2\,d\zeta\,d\eta = 2gadt^2\,(A\cos\zeta + B\cos\eta + D),$$

where D, E are constants of integration. Substituting for v, u their values in terms of η, ζ and eliminating dt, Euler obtains

$$\frac{d\zeta\sin\eta}{d\eta\sin\zeta} = \frac{P + \sqrt{P^2 - Q^2}}{Q},$$

where

$$A\cos\eta + B\cos\zeta + D\cos\zeta\cos\eta + E\sin\zeta\sin\eta = P,$$

$$A\cos\zeta + B\cos\eta + D = Q.$$

And he then enters into a very interesting discussion of the particular case $A=0$ or $B=0$ (viz. the case where one of the attracting masses vanishes, which was of course known to be integrable); and after arriving at some paradoxical conclusions which he does not completely explain, although he remarks that the explanation depends on the circumstance that the integral found is a *singular solution* of a derivative equation, and as such does not satisfy the original equations of motion,—he proceeds to notice that an inquiry into the cause of the difficulty led him to a substitution by which the variables were separated.

46. But in the memoir "Problème, un Corps &c." in the Berlin Memoirs for 1760 (printed 1767), after obtaining the last-mentioned formulæ, he gives at once, without explaining how he was led to it, the analytical investigation of the substitution in question, viz. in *each* of the two memoirs he in fact writes

$$\frac{d\zeta\sin\eta + d\eta\sin\zeta}{d\zeta\sin\eta - d\eta\sin\zeta} = \sqrt{\frac{P+Q}{P-Q}},$$

$$\tan\tfrac{1}{2}\zeta = f,\quad \tan\tfrac{1}{2}\eta = g,\quad fg = p,\quad \frac{f}{g} = q,$$

that is

$$p = \tan\tfrac{1}{2}\zeta\tan\tfrac{1}{2}\eta;\quad q = \tan\tfrac{1}{2}\zeta \div \tan\tfrac{1}{2}\eta;$$

and in terms of these quantities p, q, the equation becomes

$$\frac{dp}{\sqrt{P}} = \frac{dq}{\sqrt{Q}},$$

where

$$P = (\ \ A + B + D)\,p + 2Ep^2 + (-A - B + D)\,p^3,$$
$$Q = (-A + B - D)\,q + 2Eq^2 + (\ \ A - B - D)\,q^3,$$

so that P and Q are cubic functions (not the same functions) of p and q respectively and the equation for the time is found to be

$$\frac{dt\sqrt{2g}}{a\sqrt{a}} = \frac{pdp}{(1-p)^2\sqrt{P}} + \frac{qdq}{(1+q)^2\sqrt{Q}},$$

which are the formulæ for the solution of the problem, as obtained in Euler's first and second memoirs.

47. In his third memoir, viz. that "De Motu Corporis &c." in the Petersburg Memoirs for 1765 (printed 1767), Euler considers the body as moving in space, the forces being as before as (dist.)$^{-2}$. Assuming that the coordinates y, z are in the plane perpendicular to the axis, there is in this case the equation of areas $y\dfrac{dz}{dt} - z\dfrac{dy}{dt} = \text{const.}$; and writing $y = y' \sin\psi$, $z = y' \cos\psi$, that is, $y' = \sqrt{y^2 + z^2}$, and ψ the azimuth, the integral equations for the motion in the variable plane (coordinates x, y') are not materially different in form from those which belong to the motion in a fixed plane, coordinates x, y (see *post*, No. 56, Jacobi); and the last-mentioned equation, which reduces itself to the form $y'^2\dfrac{d\psi}{dt} = \text{const.}$, gives at once $d\psi$ in a form such as that above alluded to (*antè*, No. 43), and therefore ψ by quadratures. The variables employed by Euler in the memoir in question are

$$v + u,\quad v - u,\ (\text{say } r,\ s),\ \text{and } \psi,$$

v, u being, as above, the distances from the two centres, and ψ the azimuth of the axial plane. The functions of r, s under the radical signs are of the fourth order; this is so, with these variables, even if the motion is in a fixed plane; but this is no disadvantage, since, as is well known, the case of a quartic radical is not really more complicated than that of a cubic radical, the two forms being immediately convertible the one into the other.

48. Lagrange's first memoir (Turin Memoirs, 1766—1769) refers to Euler's three memoirs, but the author mentions that it was composed in 1767 without the knowledge of Euler's third memoir. The coordinates ultimately made use of are $v + u$, $v - u$, (say r, s), and ψ, the same as in Euler's third memoir, and the results consequently present themselves in the like form.

49. If the attractive force of one of the centres is taken equal to zero, then the position of such centre is arbitrary, and it may be assumed that the centre lies on the curve, which is in this case an ellipse (conic section); the expression of the time presents itself as a function of the focal radius vectors and the chord of the arc described; which, as remarked, *antè*, No. 20, leads to Lambert's theorem for elliptic motion.

50. The case presents itself of an ellipse or hyperbola described under the action of the two forces, viz. the equation $\frac{dr}{\sqrt{R}} = \frac{ds}{\sqrt{S}}$ will be satisfied by $r-\alpha=0$, if $r-\alpha$ be a double factor of R, or by $s-\beta=0$, if $s-\beta$ be a double factor of S, a case which is also considered in the *Mécanique Analytique*; and see in regard to the analytical theory, t. II. 3rd ed. Note III. by M. Serret, and "Thèse," *Liouv.* 1848. It is remarked by M. Bonnet, Note IV. and *Liouv.* t. IX. p. 113, (1844), that the result is a mere corollary of a general theorem, which is in effect as follows, viz. if a particle under the separate actions of the forces $F, F', \ldots$ starting in each case from the same point in the same direction but with the initial velocities v, v', &c. respectively, describe the same curve, then such curve will also be described under the conjoint action of all the forces, provided the body start from the same point in the same direction, with the initial velocity $V=\sqrt{v^2+v'^2+\ldots}$.

51. Lagrange's second memoir (same volume of the Turin Memoirs) contains an exceedingly interesting discussion as to the laws of force for which the problem can be solved. Writing U, V, u, v in the place of Lagrange's P, Q, p, q, the equations of motion are

$$\frac{d^2x}{dt^2}+\frac{(x-a)\,U}{u}+\frac{(x-\alpha)\,V}{v}=0,$$

$$\frac{d^2y}{dt^2}+\frac{(y-b)\,U}{u}+\frac{(y-\beta)\,V}{v}=0,$$

$$\frac{d^2z}{dt^2}+\frac{(z-c)\,U}{u}+\frac{(z-\gamma)\,V}{v}=0,$$

where

$$u=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2},$$

$$v=\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2},$$

and putting also $f\left(=\sqrt{(a-\alpha)^2+(b-\beta)^2+(z-\gamma)^2}\right)$ the distance of the centres, and then $u^2=f^2x$, $v^2=f^2y$, $\frac{U}{u}=X$, $\frac{V}{v}=Y$ (x, y are of course not to be confounded with the coordinates originally so represented), Lagrange obtains the equations

$$\tfrac{1}{2}\frac{d^2x}{dt^2}+Xx+\frac{(x+y-1)\,Y}{2}+\int(Xdx+Ydy)=0,$$

$$\tfrac{1}{2}\frac{d^2y}{dt^2}+Yy+\frac{(x+y-1)\,X}{2}+\int(Xdx+Ydy)=0,$$

which he represents by

$$\tfrac{1}{2}\frac{d^2x}{dt^2}+M=0,$$

$$\tfrac{1}{2}\frac{d^2y}{dt^2}+N=0\,;$$

and he then inquires as to the conditions of integrability of these equations, for which purpose he assumes that the equations multiplied by $mdx+ndy$ and $\mu dx+\nu dy$ respectively and added, give an integrable equation.

52. A case satisfying the required conditions is found to be

$$X = 2\alpha + \frac{\beta}{x\sqrt{x}}, \quad Y = 2\alpha + \frac{\gamma}{y\sqrt{y}},$$

or, what is the same thing,

$$U = 2\alpha u + \frac{\beta f^3}{u^2}, \quad V = 2\alpha v + \frac{\gamma f^3}{v^2};$$

that is, besides the forces $\frac{\beta f^3}{u^2}$, $\frac{\gamma f^3}{v^2}$, which vary as (dist.)$^{-2}$, there are the forces $2\alpha u$, $2\alpha v$, varying directly as the distance, and of the same amount at equal distances; or, what is the same thing, there is, besides the forces varying as (dist.)$^{-2}$, a force varying directly as the distance, tending to a third centre midway between the other two, a case which is specially considered in the memoir; it is found that the functions in r, s under the radicals (instead of rising only to the order 4) rise in this case to the order 6.

53. Among other cases are found the following, viz.:

1°. $$U = \alpha u + \frac{7\lambda}{f^2} u^3 + \frac{5\lambda}{f^4} u^5,$$

$$V = \alpha v + \frac{7\lambda}{f^2} v^3 + \frac{5\lambda}{f^4} v^5;$$

2°. $$U = \alpha u + \frac{\beta}{f^2} u^3,$$

$$V = \delta v + \frac{\epsilon}{f^2} v^3,$$

where $\beta = \epsilon$, or else $\alpha\epsilon = \beta\delta = 2\beta\epsilon$.

In regard to the subject of this second memoir of Lagrange, see *post*, Miscellaneous Problems, Liouville's Memoirs, Nos. 100 to 105.

54. In the *Mécanique Analytique* (1st ed. 1788, and 2nd ed. t. II. 1813), Lagrange in effect reproduces his solution for the above-mentioned law of force (say $U = \frac{\alpha}{u^2} + 2\gamma u$, $V = \frac{\beta}{v^2} + 2\gamma v$).[1] There are even in the third edition a few trifling errors of work to be corrected. The remarks above referred to, as made by Lagrange in his first memoir, are also reproduced (see *antè*, Nos. 49 and 50).

55. Legendre, *Exercices de Calcul Intégral*, t. II. (1817), and *Théorie des Fonctions Elliptiques*, t. I. (1825), uses p^2 and q^2 in the place of Euler's p, q; the forces are assumed to vary as (dist.)$^{-2}$, and in consequence of the change Euler's cubic radicals are replaced by quartic radicals involving only even powers of p and q

[1] In the *Mécanique Analytique*, Lagrange's letters are r, q for the distances $r+q=s$, $r-q=u$: the change in the present Report was occasioned by the retention of Euler's variables p, q.

respectively; that is, the radicals are in a form adapted for the transformation to elliptic integrals; in certain cases, however, it becomes necessary to attribute to Legendre's variables p and q imaginary values. The various cases of the motion are elaborately discussed by means of the elliptic integrals; in particular Legendre notices certain cases in which the motion is oscillatory, and which, as he remarks, seem to furnish the first instance of the description by a free particle of only a finite portion of the curve which is analytically the orbit of the particle; there is, however, nothing surprising in this kind of motion, although its existence might easily not have been anticipated.

56. § 26 of Jacobi's memoir "Theoria Novi Multiplicatoris &c." (1845) is entitled "Motus puncti versus duo centra secundum legem Neutonianum attracti." The equations for the motion in space are by a general theorem given in the memoir "De Motu puncti singularis" (1842), reduced to the case of motion in a plane: viz. if x, y are the coordinates, the centre point of the axis being the origin, and y being at right angles to the axis, and if the distance of the centres is $2a$; then the only difference is that to the expression for $\frac{d^2y}{dt^2}$ there is added a term $\frac{a^2}{y^3}$, which arises from the rotation about the axis. Two integrals are obtained, one the integral of *Vis Viva*, and the other of them an integral similar to one of those of Euler's or Lagrange's. And then x', y' being the differential coefficients of x, y with regard to the time, the remaining equation may be taken to be $y'dx - x'dy = 0$, where x', y' are to be expressed as functions of x, y by means of the two given integrals. This being so, the principle of the Ultimate Multiplier[1] furnishes a multiplier of this differential equation, and the integral is found to be

$$\int \frac{y'dx - x'dy}{xy\,(x'^2 - y'^2) + (a^2 - x^2 + y^2)\,x'y'} = \epsilon,$$

the quantity under the integral sign being a complete differential. To verify *à posteriori* that this is so, Jacobi introduces the auxiliary quantities λ', λ'' defined as the roots of the equation $\lambda^2 + \lambda(x^2 + y^2 - a^2) - a^2y^2 = 0$, which in fact, if as before u, v are the distances from the centres, leads to

$$u + v = 2\sqrt{a^2 - \lambda'}, \quad u - v = 2\sqrt{a^2 - \lambda''},$$

so that λ', λ'' are functions of $u+v$, $u-v$ respectively; and the formulæ, as ultimately expressed in terms of λ', λ'', are substantially of the same form with those of Euler and Lagrange.

57. The investigations contained in Liouville's three memoirs "Sur quelques cas particuliers &c." (1846), find their chief application in the problem of two centres, and by leading in the most direct and natural manner to the general law of force for which the integration is possible, they not only give some important extension of the problem, but they in fact exhibit the problem itself and the preceding solutions of it in their true light. But as they do not relate to this problem exclusively, it will be convenient to consider them separately under the head Miscellaneous Problems.

[1] Explained in Jacobi's memoir "Theoria Novi Multiplicatoris &c.," *Crelle*, tt. XXVII., XXVIII., XXIX. 1844–45.

58. In Serret's "Thèse sur le Mouvement &c." (1848), the problem is very elegantly worked out according to the principles of Liouville's memoirs as follows: viz. assuming that the expression of the distance between two consecutive positions of the body is

$$ds^2 = \lambda(m d\mu^2 + n d\nu^2) + \lambda'' d\gamma^2,$$

where m, n are functions of μ, ν respectively, and if the forces can be represented by means of a force-function U, then the motion can be determined, provided only λ, λU, $\frac{\lambda''}{\lambda}$ are of the forms

$$\lambda = \phi\mu - \Phi\nu,$$

$$\lambda U = \psi\mu - \Psi\nu,$$

$$\frac{\lambda}{\lambda''} = \varpi\mu - \Pi\nu,$$

where the functional symbols ϕ, Φ, &c. denote any arbitrary functions whatever.

59. It is then assumed that μ, ν are the parameters of the confocal ellipses and hyperbolas situate in the moveable plane through the axis, viz. that we have

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} = 1,$$

$$\frac{x^2}{\nu^2} - \frac{y^2}{b^2 - \nu^2} = 1,$$

(the origin is midway between the two centres, $2b$ being their distance; $\frac{1}{2}\mu$, $\frac{1}{2}\nu$ are in fact equal to the sum and difference $u+v$, $u-v$ of the two centres respectively); and that the position of the moveable plane is determined by means of γ, the inclination to a fixed plane through the axis, or say, as before, its azimuth. In fact, with these values of the coordinates, the expression of ds^2 is

$$ds^2 = (\mu^2 - \nu^2)\left(\frac{d\mu^2}{\mu^2 - b^2} + \frac{d\nu^2}{b^2 - \nu^2}\right) + \frac{(\mu^2 - b^2)(b^2 - \nu^2)}{b^2} d\gamma^2,$$

which is of the required form. And moreover if the forces to the two centres vary as (dist.)$^{-2}$, and there is besides a force to the middle point varying as the distance, then

$$U = \frac{g}{\mu + \nu} + \frac{g'}{\mu - \nu} + K(\mu^2 + \nu^2 - b^2),$$

whence (observing that $\lambda = \mu^2 - \nu^2$) λU is of the required form. The equations obtained by substituting for U the above value give the ordinary solution of the problem.

60. Liouville's note to the last-mentioned memoir (1848) contains the demonstration of a theorem obtained by a different process in his *second* memoir, but which is in the present note, starting from Serret's formulæ, demonstrated by the more simple method of the *first* memoir, viz., it is shown that the motion can be obtained if the two centres, instead of being fixed, revolve about the point midway between them in

a circle in such manner that the diameter through the two centres always passes through the projection of the body on the plane of the circle. It will be observed that the circular motion of the two centres is neither a uniform nor a given motion, but that they are, as it were, carried along with the moving body.

61. In Desboves's memoir "Sur le Mouvement d'un point matériel &c." (1848), the author developes the solution of the foregoing problem of moving centres, chiefly by the aid of the method employed in Liouville's *second* memoir. And he shows also that the methods of Euler and Lagrange for the case of two fixed centres apply with modification to the more complicated problem of the moving centres.

62. The problem of two centres is considered in Bertrand's "Mémoire sur les équations différentielles &c." (1852), by means of Jacobi's form of the equations of motion, viz., the problem is reduced to a plane one by means of the addition of a force $\propto \frac{1}{y^3}$ (*antè*, No. 56).

63. Cayley's "Note on Lagrange's Solution &c." (1857) is merely a reproduction of the investigation in the *Mécanique Analytique;* the object was partly to correct some slight errors of work, and partly to show what were the combinations of the differential equations, which give at once the integrals of the problem.

64. In § II. of Bertrand's "Mémoire sur quelques unes des formes &c." (1857), the following question is considered, viz., assuming that the dynamical equations

$$\frac{d^2x}{dt^2} = \frac{dU}{dx}, \quad \frac{d^2y}{dt^2} = \frac{dU}{dy},$$

have an integral of the form

$$\alpha = Px'^2 + Qx'y' + Ry'^2 + Sy' + Tx' + K$$

(where α is the arbitrary constant, and $P, Q, \ldots K$ are functions of x and y), it is required to find the form of the force-function U. It is found that U must satisfy a certain partial differential equation of the second order, the general solution of which is not known; but taking U to be a function of the distance from any fixed point (or rather the sum of any number of such functions), it is shown that the *only* case in which the differential equations for the motion of a point attracted to a fixed centre of forces have an integral of the form in question is the above-mentioned one of two centres, each attracting according to the inverse square of the distance, and a third centre midway between them, attracting as the distance.

The Spherical Pendulum, Article Nos. 65 to 73.

65. The problem is obviously the same as that of a heavy particle on the surface of a sphere.

I have not ascertained whether the problem was considered by Euler. Lagrange refers to a solution by Clairaut, *Mém. de l'Acad.*, 1735.

The question was considered by Lagrange, *Méc. Anal.*, 1st edit. p. 283. The angles which determine the position are ψ the inclination of the string to the horizon, ϕ the inclination of the vertical plane through the string to a fixed vertical plane, or say the azimuth. And then forming the equations of motion, two integrals are at once obtained; these are the integrals of *Vis Viva*, and an integral of areas. And these give equations of the form $dt = \text{funct.}\ (\psi)\, d\psi$, $d\phi = \text{funct.}\ (\psi)\, d\psi$; so that t, ϕ are each of them given by a quadrature in terms of ψ, which is the point to which the solution is carried. It is noticed that ψ may have a constant value, which is the case of the conical pendulum.

66. In the second edition, t. XI. p. 197 (1815), the solution is reproduced; only, what is obviously more convenient, the angles are taken to be

ψ, the inclination to the vertical,

ϕ, the azimuth.

It is remarked that ψ will always lie between a greatest value α and a least value β, and the integrals are transformed by introducing therein instead of ψ the angle σ, which is such that

$$\cos\psi = \cos\alpha \sin^2\sigma + \cos\beta \cos^2\sigma,$$

by which substitution they assume a more elegant form, involving only the radical

$$\sqrt{1 + k^2(\cos\beta - \cos\alpha)\cos 2\sigma},$$

where k is a constant depending on $\cos\alpha$, $\cos\beta$; and the integration is effected approximately in the case where $\cos\beta - \cos\alpha$ is small.

M. Bravais has noticed, however, that by reason of some errors in the working out, Lagrange has arrived at an incorrect value for the angle Φ, which is the apsidal angle, or difference of the azimuths for the inclinations α and β: see the 3rd edition (1855), Note VII., where M. Bravais resumes the calculation, and he arrives at the value $\Phi = \frac{1}{2}\pi(1 + \frac{3}{8}\alpha\beta)$, α and β being small.

Lagrange considers also the case where the motion takes place in a resisting medium, the resistance varying as velocity squared.

67. A similar solution to Lagrange's, not carried quite so far, is given in Poisson's *Mécanique*, t. I. pp. 385 *et seq.* (2nd ed. 1833).

A short paper by Puiseux, "Note sur le Mouvement d'un point matériel sur une sphère" (1842), shows merely that the angle Φ is $> \frac{1}{2}\pi$.

68. The ulterior development of the solution consists in the effectuation of the integrations by the elliptic and Jacobian functions. It is proper to remark that the dynamical problem the solution whereof by such functions was first fairly worked out,

is the more difficult one of the rotation of a solid body, as solved by Jacobi (1839), in completion of Rueb's solution (1834), *post*, Nos. 186 and 197.

69. In relation to the present problem we have Gudermann's memoir "De pendulis sphæricis &c." (1849), who, however, does not arrive at the actual expressions of the coordinates in terms of the time; and the perusal of the memoir is rendered difficult by the author's peculiar notations for the elliptic functions [1].

70. It would appear that a solution involving the Jacobian functions was obtained by Durège, in a memoir completed in 1849, but which is still unpublished; see § XX. of his *Theorie der elliptischen Functionen* (1861), where the memoir is in part reproduced. It is referred to by Richelot in the Note presently mentioned.

71. We have next Tissot's *Thèse de Mécanique*, 1852, where the expressions for the variables in terms of the time are completely obtained by means of the Jacobian functions H, Θ, and which appears to be the earliest published one containing a complete solution and discussion of the problem.

72. Richelot, in the Note "Bemerkungen zur Theorie des Raumpendels" (1853), gives also, but without demonstration, the final expressions for the coordinates in terms of the time.

Donkin's memoir "On a Class of Differential Equations &c." (1855) contains (No. 59) an application to the case of the spherical pendulum.

73. The first part of the memoir by Dumas, "Ueber die Bewegung des Raumpendels, &c." (1855), comprises a very elegant solution of the problem of the spherical pendulum based upon Jacobi's theorem of the Principal Function (1837), and which is completely developed by the elliptic and Jacobian functions. The latter part of the memoir relates to the effect of the rotation of the Earth; and we thus arrive at the next division of the general subject.

Motion as affected by the Rotation of the Earth, and Relative Motion in general.
Article Nos. 74 to 85.

74. Laplace (*Méc. Céleste*, Book X. c. 5) investigates the equations for the motion of a terrestrial body, taking account of the rotation of the Earth (and also of the resistance of the air), and he applies them to the determination of the deviations of falling bodies, &c. He does not, however, apply them to the case of the pendulum.

75. We have also the memoir of Gauss, "Fundamental-gleichungen &c." (1804): the equations ultimately obtained are similar to those of Poisson. I have not had the opportunity of consulting this memoir.

[1] The mere use of sn, cn, dn as an abbreviation of the somewhat cumbrous sinam, cosam, Δam of the *Fundamenta Nova* is decidedly convenient.

76. Poisson, in the "Mémoire sur le mouvement des Projectiles &c." (1838), also obtains the general equations of motion, viz. (omitting terms involving n^2), these may be taken to be

$$\frac{d^2x}{dt^2} = X + 2n\left(\frac{dy}{dt}\sin\beta + \frac{dz}{dt}\cos\beta\right),$$

$$\frac{d^2y}{dt^2} = Y + 2n\frac{dx}{dt}\sin\beta,$$

$$\frac{d^2z}{dt^2} = g + Z + 2n\frac{dx}{dt}\cos\beta$$

(see p. 20), where the axes of x, y, z are fixed on the Earth and moveable with it: viz., z is in the direction of gravity; x, y in the directions perpendicular to gravity, viz., y in the plane of the meridian northwards, x westwards; g is the actual force of gravity as affected by the resolved part of the centrifugal force; β is the latitude. There are some niceties of definition which are carefully given by Poisson, but which need not be noticed here.

77. Poisson applies his formulæ incidentally to the motion of a pendulum, which he considers as vibrating in a plane; and after showing that the time of oscillation is not sensibly affected, he remarks that upon calculating the force perpendicular to the plane of oscillation, arising from the rotation of the Earth, it is found to be too small sensibly to displace the plane of oscillation or to have any appreciable influence on the motion—a conclusion which, as is well known, is erroneous. He considers also the motion of falling bodies, but the memoir relates principally to the theory of projectiles.

78. That the motion of the spherical pendulum is sensibly affected by the rotation of the Earth is the well-known discovery of Foucault; it appears by his paper, "Démonstration Physique &c.," *Comptes Rendus*, t. XXXII. 1851, that he was led to it by considering the case of a pendulum oscillating at the pole; the plane of oscillation, if actually fixed in space, will by the rotation of the Earth appear to rotate with the same velocity in the contrary direction; and he remarks that although the case of a different latitude is more complicated, yet the result of an apparent rotation of the plane of oscillation, diminishing to zero at the equator, may be obtained either from analytical or from mechanical and geometrical considerations. Some other Notes by Foucault on the subject are given, *Comptes Rendus*, t. XXXV. (1853).

79. An analytical demonstration of the theorem was given by Binet, *Comptes Rendus*, t. XXXII. (1851), and by Baehr (1853). Various short papers on the subject will be found in the *Philosophical Magazine*, and elsewhere.

80. In regard to the above-mentioned problem of falling bodies, we have a Note by W. S., *Camb. and Dubl. Math. Journ.* t. III. (1848), containing some errors which are rectified in a subsequent paper, "Remarks on the Deviation of Falling bodies, &c." t. IV. (1849), by Dr Hart and Professor W. Thomson.

81. The theory of relative motion is considered in a very general manner in M. Quet's memoir, "Des Mouvements relatifs en général &c." (1853). Suppose that x, y, z are the coordinates of a particle in relation to a set of moveable axes; let ξ', η', ζ' be the coordinates of the moveable origin in reference to a fixed set of axes, and treating the accelerations $\frac{d^2\xi'}{dt^2}$, $\frac{d^2\eta'}{dt^2}$, $\frac{d^2\zeta'}{dt^2}$ as if they were coordinates, let these, when resolved along the moveable axes, give u', v', w': suppose, moreover, that p, q, r denote the angular velocities of the system of the moveable axes (or axes of x, y, z) round the axes of x, y, and z respectively; u', v', w', p, q, r are considered as given functions of the time, and then, if

$$u=\frac{d^2x}{dt^2}+2\left(q\frac{dz}{dt}-r\frac{dy}{dt}\right)+z\frac{dq}{dt}-y\frac{dr}{dt}+q(py-qx)-r(rx-pr)+u',$$

$$v=\frac{d^2y}{dt^2}+2\left(r\frac{dx}{dt}-p\frac{dz}{dt}\right)+x\frac{dr}{dt}-z\frac{dp}{dt}+r(qz-ry)-p(py-qx)+v',$$

$$w=\frac{d^2z}{dt^2}+2\left(p\frac{dy}{dt}-q\frac{dx}{dt}\right)+y\frac{dp}{dt}-x\frac{dq}{dt}+p(rx-pz)-q(qz-ry)+w',$$

it is shown that the equations of motion are to be obtained from the equation

$$\Sigma m\left[(u-X)\,\delta x+(v-Y)\,\delta y+(w-Z)\,\delta z\right]=0,$$

where δx, δy, δz are the virtual velocities of the particle m in the directions of the moveable axes. This equation is in fact obtained as a transformation of the equation

$$\Sigma m\left[\left(\frac{d^2\xi}{dt^2}-X\right)\delta\xi+\left(\frac{d^2\eta}{dt^2}-Y\right)\delta\eta+\left(\frac{d^2\zeta}{dt^2}-Z\right)\delta\zeta\right]=0,$$

which belongs to a set of fixed axes of ξ, η, ζ.

82. The equations for the motion of a free particle are of course $u=X$, $v=Y$, $w=Z$. In the case where the moveable axes are fixed on the Earth, and moveable with it (the diurnal motion being alone attended to), these lead to equations for the motion of a particle in reference to the Earth, similar to those obtained by Gauss and Poisson. The formulæ are applied to the case of the spherical pendulum, which is developed with some care; and Foucault's theorem of the rotation of the plane of oscillation very readily presents itself. The general formulæ are applied to the relative motion of a solid body, and in particular to the question of the gyroscope; the memoir contains other interesting results.

83. The principal memoirs on the motion of the spherical pendulum, as affected by the rotation of the Earth, are those of Hansen, "Theorie der Pendelbewegung &c." (1853), which contains an elaborate investigation of all the physical circumstances (resistance of the air, torsion of the string, &c.) which can affect the actual motion, and the before-mentioned memoir by Dumas, "Ueber der Bewegung des Raumpendels &c." (1855). The investigation is conducted by means of the variation of the constants;

the integrals for the undisturbed problem were, as already noticed, obtained by means of Jacobi's Principal Function, that is, in a form which leads at once to the expressions for the variation of the constants; and the investigation appears to be carried out in a most elaborate and complete manner.

84. In concluding this part of the subject I refer to Mr Worm's work, *The Rotation of the Earth* (1862), where the last-mentioned questions (falling bodies, the pendulum, and the gyroscope) are, in reference to the proofs they afford of the rotation of the Earth, considered as well in an experimental as in a mathematical point of view. The second part of the volume contains the theory (after Laplace and Gauss) of falling bodies, that of the pendulum (after Hansen), and that of the gyroscope (after Yvon Villarceau); and the whole appears to be a complete and satisfactory *résumé* of the experimental and mathematical theories to which it relates.

85. We have also Cohen "On the Differential Coefficients and Determinants of Lines &c." (1862), where the equations for relative motion are obtained in a very elegant manner. The fundamental notion of the memoir may be considered to be the dealing *directly* with lines, velocities, &c., which are variable in direction as well as in magnitude, instead of referring them, as in the ordinary analytical method, to axes fixed in space. The memoir is a highly interesting and valuable one, and the results are brought out with great facility; but I cannot but think that the great care required to apply the method correctly is an objection to it, if used otherwise than by way of interpretation of previously obtained results, and that the ordinary method is preferable.

I may remark that the theory of relative motion connects itself with the lunar and planetary theories as regards the reference of the plane of the orbit to the variable ecliptic, and as regards the variations of the position of the orbit; but this is a subject which I have abstained from entering upon.

Miscellaneous Problems. Articles Nos. 86 to 111 (several subheadings).

Motion of a single particle.

86. Jacobi, in the memoir "De Motu puncti singularis" (1842), notices (§ 5) the case of a body acted on by a central force which is any homogeneous function of the degree -2 of the coordinates; or representing these by $r\cos\phi$, $r\sin\phi$, then the force is $=\frac{\Phi}{r^2}$, where Φ is any function of the angle ϕ. In fact, after integrating by a process different from the ordinary one the case of a central force $\propto\frac{1}{r^2}$, he remarks that the method in fact applies to the more general law of force just mentioned.

87. Jacobi, in the memoir "Theoria Novi Multiplicatoris &c." (1845), considers (§ 25) the case of a body acted on by a central force P a function of the distance, and

besides by forces X and Y, which are homogeneous functions of the degree -3 of the coordinates (x, y); viz. the equations of motion are in this case

$$\frac{d^2x}{dt^2} = -\frac{Px}{r} + X,$$

$$\frac{d^2y}{dt^2} = -\frac{Py}{r} + Y,$$

and there is an integral

$$\tfrac{1}{2}(xy' - x'y)^2 - \int x^2 (xY - yX)\, d\frac{y}{x} = \text{const.}$$

(the function under the integral sign is obviously a function of the degree 0 in (x, y), that is, it is a function of $\frac{y}{x}$). If X, Y are the derived functions of a force-function U of the degree -2 in (x, y), then there is, besides, the integral of *Vis Viva*, and thence a third integral is obtained by means of the theorem of the Ultimate Multiplier. It may be noticed that in the last-mentioned case the force-function is of the form $\frac{\Phi}{r^2}$, so that if we represent also the central force by means of a force-function R (= function of r), then the entire force-function is $R + \frac{\Phi}{r^2}$. The case is a very interesting one; it includes that considered § iv. of Bertrand's "Mémoire sur les équations différentielles de la Mécanique" (1852), where the force-function is of the form $= \frac{\Phi}{r^2}$.

Motion of three mutually attracting bodies in a right line.

88. The problem is considered by Euler in the memoir "De Motu rectilineo &c." (1765), the forces being as the inverse square of the distance; and a solution is obtained for an interesting particular case. Let A, B, C be the masses, and suppose that at the commencement of the motion the distances CB, BA are in the ratio $\alpha : 1$, and that the velocities (assumed to be in the same sense) are proportional to the distances from a fixed point. Then, if α be the real root (there is only one) of the equation of the fifth order

$$C(1 + 3\alpha + 3\alpha^2) = A\alpha^3(\alpha^2 + 3\alpha + 3) + B(\alpha + 1^2)(\alpha^3 - 1),$$

the distances CB, BA will always continue in the ratio $\alpha : 1$. It may be added that the distances CB, BA each of them vary as $r^2 - a^2$, where a is a constant, and r is, according to the initial circumstances, a function of t defined by one or the other of the two equations

$$t = n^3 r\sqrt{r^2 - a^2} - n^3 a^2 \log \frac{r + \sqrt{r^2 - a^2}}{a},$$

$$t = n^3 r\sqrt{a^2 - r^2} + n^3 a^2 \sin^{-1}\frac{r}{a}.$$

89. The bodies are considered as restricted to move in a given line; but it is clear that if the bodies, considered as free points in space, are initially in a line, and the initial velocities are also in this line, then the bodies will always continue in this line, which will be a fixed line in space. But if the distances and velocities are as above, except only that the velocities, instead of being along the line, are parallel to each other in any direction whatever, then the bodies will always continue in a line, which is in this case a moveable line in space (see *post*, No. 93).

90. Euler resumes the problem in the memoir of 1776 in the *Nova Acta Petrop.* The distances AB, BC being p and q, then

$$\frac{d^2p}{dt^2} = -\frac{A+B}{p^2} + \frac{C}{q^2} - \frac{C}{(p+q)^2},$$

$$\frac{d^2q}{dt^2} = \frac{A}{p^2} - \frac{A}{(p+q)^2} - \frac{B+C}{q^2};$$

and in particular he considers the before-mentioned case of a solution of the form $p = nq$; and also the particular problem where one of the masses vanishes, $C = 0$; in this case, introducing (instead of p, q), the new variables u, s, where $q = up$, $dq = sdp$ (a transformation suggested by the homogeneity of the equations), and making, moreover, the particular supposition that the integral of the first equation is $\left(\frac{dp}{dt}\right)^2 = \frac{2(A+B)}{p}$ (viz. making the constant of integration to vanish), he obtains between s and u the equation of the first order

$$2(A+B)\frac{ds}{du}(s-u) = (A+B)s + A - \frac{A}{(1+u)^2} - \frac{B}{u^2},$$

which, however, he is not able to integrate.

91. Jacobi has given in the memoir "Theoria Novi Multiplicatoris &c." (1845) (§ 28, entitled "De Problemate trium corporum in eâdem rectâ motorum. Substitutio Euleriana. Theoremata de viribus homogeneis") a very symmetrical and elegant investigation of the same problem. The centre of gravity being assumed to be at rest, the coordinates x, x_1, x_2 of the three bodies are in the first instance expressed as linear functions of the two variables u, v (being, as Jacobi remarks, the transformation employed in his memoir "Sur l'élimination des Nœuds" (1843), *post*, No. 114), $\frac{d^2u}{dt^2}$ and $\frac{d^2v}{dt^2}$ come out respectively equal to homogeneous functions of the degree -2 of these variables u and v, and the integral of *Vis Viva* exists. The subsequent transformation consists in the introduction of the variables r, ϕ, s, η, where $u = r\cos\phi$, $v = r\sin\phi$, $s = \sqrt{r}\frac{dr}{dt}$, $\eta = \sqrt{r^3}\frac{d\phi}{dt}$; this gives a system of equations independent of r; viz.,

$$d\phi : ds : d\eta = \eta : \tfrac{1}{2}s^2 + \eta^2 - \Phi : -\tfrac{1}{2}s\eta + \Phi',$$

where Φ is a given function of ϕ, and Φ' is the derived function. If these equations were integrated, the equation of *Vis Viva* gives at once $r = \frac{1}{h}\left(\Phi - \tfrac{1}{2}(s^2+\eta^2)\right)$; and

finally the time t would be given by a quadrature. The system of three equations has the multiplier $M = \frac{1}{\sqrt{\phi - \frac{1}{2}(s^2+\eta^2)}}$, hence if one integral were known the other would be at once furnished by the general theory. There is a simplification in the form of the solution if h (the constant of *Vis Viva*) $= 0$. It is remarked that the method is equally applicable when the force varies as *any* power of the distance; and moreover that when the force varies as (dist.)$^{-2}$, then the solution depends upon one quadrature only.

92. The concluding part of the section relates to the very general problem of a system of n particles acted on by any forces homogeneous functions of the coordinates (this includes the case of n particles mutually attracting each other according to a power of the distance), and this more general investigation illustrates the method employed in regard to the three bodies in a line. It may be remarked that in the general theorem for the n particles "sint vires &c.," the constant of *Vis Viva* is supposed to vanish.

Particular cases of the motion of three bodies.

93. In the case of three bodies attracting each other according to the inverse square of the distance, the bodies may move in such manner as to be constantly in a line (a moveable line in space); this appears by the memoir, Euler, "Considérations générales &c." (1764), in which memoir, however (which it will be observed precedes the memoir "De Motu rectilineo &c." (1765), referred to No. 88), Euler assumes that the mass of one of the bodies is so small as not to affect the relative motion of the other two. Calling the bodies the Sun, Earth, and Moon, and taking the masses to be 1, m, 0, then a result obtained is, that in order that the Moon may be perpetually in conjunction, its distance must be to that of the Sun as $\alpha : 1$, where $m(1-\alpha)^2 = 3\alpha^3 - 3\alpha^4 + \alpha^5$, or $\alpha = \sqrt[3]{\frac{1}{3}m}$ nearly. It appears, however (*antè*, No. 88), that the foregoing restriction as to the masses is unnecessary, and, as will be mentioned, the problem has since been treated without such restriction. Euler investigates the motion in the case where the initial circumstances are nearly but not exactly as originally supposed; this assumes, however, that the motion is stable—i.e. that the bodies will continue to move nearly, but not exactly as originally supposed, which is at variance with the conclusions of Liouville's memoir, *post*, No. 95. I have not examined the cause of this discrepancy.

94. In the *Mécanique Céleste* (1799), Book X. c. 6, Laplace considers two cases where the motion can be exactly determined.

1°. Force varies as any function of the distance. It is shown that the motion may be such that the bodies form always an equilateral triangle of variable magnitude—the motion of each body about the centre of gravity being the same as if that point were a centre of force attracting the body according to a similar law.

2°. Force $\propto$(dist.)^n. The motion may be such that the three bodies are always in a right line (moveable in space), the relative distances being in fixed ratios to each

other. In particular, if force $\propto$ (dist.)$^{-2}$, then m, m', m'' being the masses, the quantity z which determines the ratio of the distances $m''m'$, $m'm$ is given by

$$0 = mz^2\left[(1+z)^2 - 1\right] - m'(1+z)^2(1-z^2) - m''\left[(1+z)^3 - z^3\right] = 0,$$

which is, in fact, the formula in Euler's memoir "De Motu rectilineo &c."

95. Liouville's memoir "Sur un cas particulier &c." (1842) has for its object to show that if the initial circumstances are not precisely as supposed in the second of the two cases considered by Laplace, or, what is the same thing, in Euler's memoir "Considérations générales &c.," then the motion is unstable; the instability manifests itself in the usual manner, viz. the expressions for the deviations from the normal positions are found to contain real exponentials which increase indefinitely with the time.

96. It may be proper to refer here to Jacobi's theorem, *Comptes Rendus*, t. III. p. 61 (1836), quoted in the foot-note to No. 31 of my Report of 1857, [195], which relates to the motion of a point *without mass* revolving round the Sun, and disturbed by a planet moving in a circular orbit, and properly belongs (as I have there remarked) to the problem of two centres, one of them moveable and the other revolving round it in a circle with uniform velocity. The theorem (given without demonstration by Jacobi) is proved by Liouville in his last-mentioned memoir, and he remarks that the theorem follows very simply as a corollary of the theorem by Coriolis, "Mémoire sur le principe des forces vives dans les mouvemens relatifs des Machines," *Journ. de l'Ecole Polyt.* t. XIII. pp. 268—302 (1832). There is, however, no difficulty in proving the theorem; another proof is given, Cayley, "Note on a Theorem of Jacobi's &c." (1862).

Motion in a resisting medium.

97. I do not consider the various integrable cases of the motion of a particle in a resisting medium, the resistance varying with the velocity according to some assumed law, the particle being either not acted on by any force, or acted upon by gravity only. Some interesting cases are considered in Jacobi's memoir "De Motu puncti singularis" (1842), §§ 6 and 7 (see *post*, No. 108).

98. In the case of a central force varying as (dist.)$^{-2}$, the effect of a resisting medium ($R \propto v^2$) is considered in reference to the lunar theory, in the *Mécanique Céleste*, Book VII. c. 6. Formulæ for the variations of the elliptic elements are given in the *Mécanique Analytique*, t. II. (2nd edition). But the variations of the elliptic elements are fully worked out by means of elliptic and Jacobian functions in Sohncke's valuable memoir "Motus Corporum &c." (1833).

99. The effect of the resistance of the air on a pendulum has been elaborately considered by Poisson, Bessel, Stokes, and others; as the dimensions of the ball are attended to, the problem is in fact a hydrodynamical one.

The effect on the spherical pendulum is considered in Hansen's memoir "Theorie der Pendelbewegung &c. (1853).

The effect on the motion of a projectile is considered in Poisson's memoirs "Sur le Mouvement des Projectiles &c." (1838).

Liouville's *memoirs "Sur quelques Cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer"* (1846—49).

100. In the *first* memoir (§ 1) the author considers a point moving in a plane or on a given surface, where the principle of *Vis Viva* holds good (or say where there is a force-function U). The coordinates of the point, and the function U, may be expressed in terms of two variables α, β, and it is assumed that these are such that

$$ds^2 = \lambda\,(d\alpha^2 + d\beta^2),$$

where λ is a function of α and β. That is, we have $T = \frac{1}{2}\lambda\,(\alpha'^2 + \beta'^2)$; and the equations of motion are

$$\frac{d\,.\,\lambda\alpha'}{dt} = \tfrac{1}{2}\frac{d\lambda}{d\alpha}(\alpha'^2 + \beta'^2) + \frac{dU}{d\alpha},$$

$$\frac{d\,.\,\lambda\beta'}{dt} = \tfrac{1}{2}\frac{d\lambda}{d\beta}(\alpha'^2 + \beta'^2) + \frac{dU}{d\beta}.$$

One integral of these is

$$\lambda\,(\alpha'^2 + \beta'^2) = 2U + C;$$

and by means of it the equations take the form

$$\frac{d\,.\,\lambda\alpha'}{dt} = \frac{1}{2\lambda}\frac{d\lambda}{d\alpha}(2U + C) + \frac{dU}{d\alpha},$$

$$\frac{d\,.\,\lambda\beta'}{dt} = \frac{1}{2\lambda}\frac{d\lambda}{d\beta}(2U + C) + \frac{dU}{d\beta}.$$

These equations, it is easy to show, may be integrated if

$$(2U + C)\,\lambda = f\alpha - F\beta,$$

and they then in fact give

$$\lambda^2\alpha'^2 = f\alpha - A\ ,$$
$$\lambda^2\beta'^2 = A - F\beta,$$

where A is an arbitrary constant. And we then have

$$\frac{d\alpha}{\sqrt{f\alpha - A}} = \frac{d\beta}{\sqrt{A - F\beta}},$$

which gives the path, and the expression for the time is easily obtained by means of a quadrature.

It is not more general, but it is frequently convenient to employ instead of α, β, two variables μ and ν, such that

$$ds^2 = \lambda\,(m d\mu^2 + n d\nu^2),$$

where m is a function of μ only and n of ν only, while λ contains μ and ν. The geometrical signification of the equation $ds^2 = \lambda(d\alpha^2 + d\beta^2)$, or of the last-mentioned equivalent form, is that the curves

$$\alpha \text{ or } \lambda = \text{const.}, \qquad \beta \text{ or } \mu = \text{const.}$$

intersect at right angles.

The foregoing differential equation of the path, writing $f\mu$, $F\nu$ in the place of $f\alpha$, $F\beta$ respectively, may be expressed in the form

$$f\mu \cos^2 i + F\nu \sin^2 i = A,$$

where i, $90^\circ - i$ are the inclinations of the path at the point (λ, μ) to the two orthotomic curves through this point.

101. The before-mentioned equation

$$(2U + C)\lambda = f\alpha - F\beta$$

may be satisfied independently of C, or else only for a particular value of C. In the former case the law of force is much more restricted, but on the other hand there is no restriction as regards the initial circumstances of the motion; it is the more important one, and is alone attended to in the sequel of the memoir. In the case in question (changing the functional symbols) we must have

$$\lambda = \phi\alpha - \varpi\beta, \quad \lambda U = f\alpha - F\beta;$$

so that the functions denoted above by $f\alpha$, $F\beta$ now are $2f\alpha + C\phi\alpha$, $2F\beta + C\varpi\beta$; the equation of the trajectory is

$$\frac{d\alpha}{\sqrt{2f\alpha + C\phi\alpha - A}} = \frac{d\beta}{\sqrt{A - 2F\beta + C\varpi\beta}},$$

and for the *time* the formula is

$$dt = \frac{\phi\alpha\, d\alpha}{\sqrt{2f\alpha + C\phi\alpha - A}} - \frac{\varpi\beta\, d\beta}{\sqrt{A - 2F\beta + C\varpi\beta}}.$$

It is noticed also that taking B, ϵ to denote two new arbitrary constants, and writing

$$\Theta \int d\alpha \sqrt{2f\alpha + C\phi\alpha - A} + \int d\beta \sqrt{A - 2F\beta + C\varpi\beta},$$

the equation of the trajectory and the expression for the time assume the forms

$$\frac{d\Theta}{dA} = B, \qquad t = 2\frac{d\Theta}{dC} + \epsilon,$$

as is known *à priori* by a theorem of Jacobi's.

If the forces vanish, the path is a geodesic line; and denoting by a the ratio of the constants A, C, we have

$$\frac{d\alpha}{\sqrt{\phi\alpha - a}} = \frac{d\beta}{\sqrt{a - \varpi\beta}};$$

and moreover

$$ds = d\alpha\sqrt{\phi\alpha - a} + d\beta\sqrt{a - \phi\beta},$$

which are geometrical properties relating to the geodesic line.

102. Passing to the applications: in the first place, if α, β are rectangular coordinates of a point *in plano*, then writing instead of them x, y, we have $ds^2 = dx^2 + dy^2$, which is of the required form; but the result obtained is the self-evident one, that the equations may be integrated by quadratures when U is of the form funct. x − funct. y.

But taking instead the elliptic coordinates μ, ν, of a point *in plano*—viz., as employed by the author, these are the semiaxes of the confocal ellipse and hyperbola represented by the equations

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} = 1, \quad \frac{x^2}{\nu^2} - \frac{y^2}{b^2 - \nu^2} = 1,$$

—very interesting results are obtained. The equations give

$$b^2x^2 = \mu^2\nu^2, \quad b^2y^2 = (\mu^2 - b^2)(b^2 - \nu^2),$$

and thence

$$ds^2 = (\mu^2 - \nu^2)\left(\frac{d\mu^2}{\mu^2 - b^2} + \frac{d\nu^2}{\nu^2 - b^2}\right),$$

which is of the proper form, and the corresponding expression of U is

$$U = \frac{f\mu - F\nu}{\mu^2 - \nu^2};$$

so that the force-function having this value ($f\mu$, $F\nu$ being arbitrary functions of μ and ν respectively), the equations of motion may be integrated by quadratures.

103. In particular, if

$$f\mu = g\mu + g'\mu + k(\mu^4 - b^2\mu^2),$$
$$F\nu = g\nu - g'\nu + k(\nu^4 - b^2\nu^2),$$

then

$$U = \frac{g}{\mu + \nu} + \frac{g'}{\mu - \nu} + k(\mu^2 + \nu^2 - b^2).$$

But $\mu+\nu$, $\mu-\nu$ are the distances of the point from the two foci, and $\mu^2+\nu^2-b^2 (= x^2+y^2)$ is the square of the distance from the centre, so that the expression for U is

$$U = \frac{g}{r} + \frac{g'}{r'} + kR^2;$$

and the case is that of forces to the foci varying inversely as the squares of the distances, and a force to the centre varying directly as the distance—the case considered by Lagrange in the problem of two centres. But this is merely one particular case of those given by the general formula.

The cases $g = 0$, $g' = 0$, $k = 0$ (no forces), and $g = 0$, $g' = 0$ (a force to the centre) lead to some interesting results; it is noticed also that the expression for the force-function may be written $U = \dfrac{\text{funct.}\,(r + r') - \text{funct.}\,(r - r')}{rr'}$, and that it may be thereby ascertained (without transforming to elliptic coordinates) whether a given value of the force-function is of the form considered in the theory.

In § 3 the author considers the expression $dx^2+dy^2=\lambda(d\alpha^2+d\beta^2)$, λ being in the first instance any function whatever of α and β; and he shows that the expressions of x, y are given by the equation

$$x+y\sqrt{-1}=\psi(\alpha+\beta\sqrt{-1}),$$

ψ being any real function. If, however, it is besides assumed that λ is of the required form $=f\alpha-F\beta$, then he shows that the system of elliptic coordinates is the only one for which the conditions are satisfied. §§ 4, 5, 6 and 7 relate to the motion of a point on a sphere, an ellipsoid, a surface of revolution, and the skew helicoid respectively; and the concluding § 8 contains only a brief reference to the author's second memoir.

104. Liouville's second and third memoirs may be more briefly noticed. In the *second* memoir the author starts from Jacobi's theorem of the V function, viz., assuming that there is a force-function U independent of the time, then in order to integrate the equations of motion $\left(\frac{d^2x}{dt^2}=\frac{dU}{dx},\ \frac{d^2y}{dt^2}=\frac{dU}{dy},\ \frac{d^2z}{dt^2}=\frac{dU}{dz}\right)$, all that is required is to find a function Θ of x, y, z containing three arbitrary constants A, B, C (distinct from the constant attached to Θ by mere addition) satisfying the differential equation

$$\left(\frac{d\Theta}{dx}\right)^2+\left(\frac{d\Theta}{dy}\right)^2+\left(\frac{d\Theta}{dz}\right)^2=2(U+C);$$

for then the required integrals of the equations of motion are

$$\frac{d\Theta}{dA}=A',\quad \frac{d\Theta}{dB}=B',\quad \frac{d\Theta}{dC}=t+C',$$

A', B', C' being new arbitrary constants. Liouville introduces in place of x, y, z, the elliptic coordinates ρ, μ, ν, which are such that

$$\frac{x^2}{\rho^2}+\frac{y^2}{\rho^2-b^2}+\frac{z^2}{\rho^2-c^2}=1,$$

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}-\frac{z^2}{c^2-\mu^2}=1,$$

$$\frac{x^2}{\nu^2}-\frac{y^2}{b^2-\nu^2}-\frac{z^2}{c^2-\nu^2}=1,$$

or, what is the same thing,

$$x=\frac{\rho\mu\nu}{bc},$$

$$y=\frac{\sqrt{\rho^2-b^2}\sqrt{\mu^2-b^2}\sqrt{b^2-\nu^2}}{b\sqrt{c^2-b^2}},$$

$$z=\frac{\sqrt{\rho^2-c^2}\sqrt{c^2-\mu^2}\sqrt{c^2-\nu^2}}{c\sqrt{c^2-b^2}};$$

and he then finds that the resulting partial differential equation in ρ, μ, ν may be integrated provided that U is of the form

$$U = \frac{(\mu^2 - \nu^2) f\rho + (\rho^2 - \nu^2) F\mu + (\rho^2 - \nu^2) \varpi\nu}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)(\mu^2 - \nu^2)},$$

f, F, ϖ being any functional symbols whatever: viz., the expression for Θ is

$$\Theta = \int d\rho \sqrt{\frac{2f\rho + A + B\rho^2 + 2C\rho^4}{(\rho^2 - b^2)(\rho^2 - c^2)}},$$

$$+ \int d\mu \sqrt{\frac{2F\mu + A + B\mu^2 + 2C\mu^4}{(\mu^2 - b^2)(c^2 - \mu)^2}},$$

$$+ \int d\nu \sqrt{\frac{2\varpi\nu + A + B\nu^2 + 2C\nu^4}{(b^2 - \nu^2)(c^2 - \nu^2)}}.$$

In the case where $U = 0$ we have a particle not acted on by any forces, and the path is of course a straight line. The peculiar form in which these equations are obtained leads to very interesting results in regard to the theory of Abelian integrals, and to that of the geodesic lines of an ellipsoid.

The formulæ require to be modified in certain cases, such as $c = b$ or $b = 0$. The case $b = 0$ leads to the theory developed in the first memoir in relation to the problem of two centres. The case is indicated where $b = 0$, $c = 0$, the ratio $b : c$ remaining finite.

The case is briefly considered of a particle moving on a given surface.

105. The *third* memoir purports to relate to a system of particles, but the formulæ are exhibited under a purely analytical point of view; so much so, that the coordinates of the points (3 for each point) are considered as forming a single system of variables $x_1, x_2, \ldots x_i$. The partial differential equation is

$$\left(\frac{d\Theta}{dx_1}\right)^2 + \left(\frac{d\Theta}{dx_2}\right)^2 \ldots + \left(\frac{d\Theta}{dx_i}\right)^2 = 2(U + h),$$

which is transformed by introducing therein the new variables $\rho_1, \rho_2, \ldots \rho_i$ analogous to the elliptic coordinates of the second memoir. The memoir really belongs rather to the theory of the Abelian integrals (in regard to which it appears to be a very valuable one) than to dynamics.

Memoirs by Jacobi, Bertrand, *and* Donkin, *relating to various Special Problems.*

106. I have inserted this heading for the sake of showing at a single view what are the special problems incidentally considered in the undermentioned memoirs which are referred to in several places in the present Report.

107. Jacobi, "De Motu puncti singularis" (1842).—I call to mind that the memoir chiefly depends on the theorem of the Ultimate Multiplier (the theory in its generality being developed in the later memoir "Theoria Novi Multiplicatoris &c.," 1844—45). § 4 is entitled "The motion of a point on the surface of revolution," which, the principle of the conservation of areas holding good, is reduced to the problem of the motion on the meridian curve, and thus depends upon quadratures only. § 5 is entitled "On the motion of a point about a fixed centre attracted according to a certain law more general than the Newtonian one" (*antè*, No. 85). § 6, "On the motion of a point on a given curve and in a resisting medium" (resistance $=a+be^{cv^2}$, or $=a+bv^2$); and § 7, "On the Ballistic Curve," viz., the forces are gravity and a resistance $=a+bv^n$.

108. In Jacobi's memoir "Theoria Novi Multiplicatoris &c." (1845), § 25 is entitled "On the motion of a point attracted towards a fixed centre" (see *antè*, No 87); § 26, "On the motion of a point attracted towards two fixed centres according to the Newtonian law" (*antè*, No. 56); § 27, "On the rotation of a solid body about a fixed point" (*post*, No. 193); § 28, "On the problem of three bodies moving in a right line; the Eulerian substitution; theorems on homogeneous forces" (*antè*, No. 91); and § 29, "The principle of the ultimate multiplier applied to a free system of material points moving in a resisting medium; on the motion of a comet in a resisting medium about the sun."

109. And in Jacobi's memoir "Nova Methodus &c." (1862), besides § 64 and § 65, which are applications of the method to general dynamical theorems, we have § 66, containing a simultaneous solution of the problem of the motion of a point attracted to a fixed centre and of that of the rotation of a solid body (*post*, No. 206) and § 67, relating to the motion of a point attracted to a fixed centre according to the Newtonian law.

110. Bertrand's "Mémoire sur les intégrales différentielles de la Mécanique" (1852).—§ III. relates to the motion of a point attracted to a fixed centre by a force varying as a function of the distance; § IV. to the case where the forces arise from a force-function $U=\frac{1}{x^2+y^2}\phi\left(\frac{x}{y}\right)$ $\left(\text{or, what is the same thing, } =\frac{\Phi}{r^2}\right)$ (*antè*, No. 87); § V. to the problem of two centres (*antè*, No. 62), and § VI. to the problem of three bodies (*post*, No. 117).

111. Donkin's memoir "On a Class of Differential Equations &c." (1855). Part I. Nos. 27 to 30 relate to the problem of central forces (in space), No. 31 to the rotation of a solid body, and § III. to the same subject, viz. Nos. 40 and 41 to the general case, Nos. 42 to 44 to the particular case $A=B$; and Nos. 45 to 48 to the reduction thereto of the general case by treating the forces which arise from the inequality of A and B as disturbing forces. Part II. Nos. 59 and 60 relate to the spherical pendulum; Nos. 72 and 73 to "Transformation from fixed to moving axes of coordinates," say to Relative Motion; and Nos. 84 to 96 to the problem of three bodies (*post*, No. 120).

The Problem of Three Bodies, Article Nos. 112 to 123.

112. A system of differential equations, such as

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}\dots=\frac{dx_{n+1}}{X_{n+1}}$$

(n equations between $n+1$ variables), may be termed a system of the nth order, or more simply a system of n equations. Let $(u_1, u_2 \dots . u_{n+1})$ be any functions of the original variables $(x_1, x_2 \dots . x_{n+1})$, the system may be transformed into the similar system

$$\frac{du_1}{U_1}=\frac{du_2}{U_2}\dots=\frac{du_{n+1}}{U_{n+1}},$$

and if it happens that we have e.g. U_1 identically equal to zero, then the system becomes

$$du_1=0, \quad \left(\frac{du_2}{U_2}=\frac{du_3}{U_3}\dots=\frac{du_{n+1}}{U_{n+1}}\right),$$

so that we have an integral $u_1=c$, and then in the remaining equations substituting this value, or treating u_1 as constant, the system is reduced to one of $m-1$ equations. Or again, if it happen that we have in the transformed system m equations $(m<n)$, say

$$\frac{du_1}{U_1}=\frac{du_2}{U_2}\dots=\frac{du_{m+1}}{U_{m+1}},$$

which are such that $U_1, U_2 \dots U_{m+1}$ are functions of only the $m+1$ variables $u_1, u_2 \dots u_{m+1}$, then the integration of the proposed system of n equations depends on the integration in the first instance of a system of m equations. It is to be observed that if the system of m equations can be integrated, then the completion of the integration of the original system depends on the integration of a system of $n-m$ equations, and in this sense the original system of n equations may be said to be broken up into two systems of m equations and $n-m$ equations respectively: but *non constat* that the system of m equations admits of integration; and it is therefore more correct to say that, from the original system of the n equations, there has been *separated off* a system of m equations.

113. The bearing of the foregoing remarks on the problem of three bodies will presently appear. It will be seen that whereas the problem as it stood before Jacobi depends on a system of seven equations, it has been shown by him that there may be separated off from this a system of *six* equations.

114. Jacobi's memoir "Sur l'élimination des Nœuds &c." (1843).—The problem of the motion of three mutually attracting bodies is in the first instance reduced to that of the motion of two fictitious bodies (which may be considered as mutually attracting bodies, attracted by a fixed centre of force) [1]. In fact, in the original problem the

[1] This is the effect of Jacobi's reduction; but the explicit statement of the theorem, and actual replacement of the problem of the three bodies by that of the two bodies attracted to a fixed centre, is due to Bertrand (*post*, No. 117).

centre of gravity of the three bodies moves uniformly in a right line, and it may without any real loss of generality be taken to be at rest; that is, if the x-coordinates of the three bodies are ξ_1, ξ_2, ξ_3, then $m_1\xi_1 + m_2\xi_2 + m_3\xi_3 = 0$, or ξ_1, ξ_2, ξ_3 may be taken to be linear functions of two quantities x_1 and x_2. And similarly for the y-coordinates and the z-coordinates respectively. And (x_1, y_1, z_1), (x_2, y_2, z_2) may be regarded as the coordinates of two bodies revolving about a fixed centre of force. Hence representing the differential coefficients in regard to the time by x_1', &c., and treating these as new variables, the equations of motion will assume the form

$$\frac{dx_1}{x_1'} = \frac{dy_1}{y_1'} = \frac{dz_1}{z_1'} = \frac{dx_2}{x_2'} = \frac{dy_2}{y_2'} = \frac{dz_2}{z_2'} = \frac{dx_1'}{X_1} = \frac{dy_1'}{Y_1} = \frac{dz_1'}{Z_1} = \frac{dx_2'}{X_2} = \frac{dy_2'}{Y_2} = \frac{dz_2'}{Z_2} (= dt),$$

where X_1, Y_1, Z_1, X_2, Y_2, Z_2 are forces capable of representation by means of a force-function U. This is a system of twelve equations; but since X_1, Y_1, Z_1, X_2, Y_2, Z_2 are independent of the time, we may omit the equation $(= dt)$, and treat the system as one of eleven equations between the variables x_1, y_1, z_1, x_2, y_2, z_2, x_1', y_1', z_1', x_2', y_2', z_2': if this system were integrated, the determination of the time would then depend on a quadrature only. But for the system of eleven equations we have four integrals, viz., the integral of *Vis Viva* and the three integrals of areas, and the system is thus reducible to one of $(11 - 4 =)$ seven equations. This preliminary transformation in Jacobi's memoir explains the remark that the problem, as it stood before him, depended on a system of seven equations.

115. Jacobi remarks that in the transformed problem the three integrals of areas show (1) that the intersection of the planes of the orbits of the two bodies lie in a fixed plane, the invariable plane of the system; (2) that the inclinations of the planes of the two orbits to this fixed plane, and the parameters of the two orbits considered as variable ellipses, are four elements any two of which rigorously determine the two others.

And then choosing for variables the inclinations of the two orbits to the invariable plane, the two radius vectors, the angles which they form with the intersection of the planes of the two orbits, and lastly the angle between this intersection (being as already mentioned a line in the invariable plane) with a fixed line in the invariable plane, he finds *that the last-mentioned angle entirely disappears from the system of differential equations, and is determined after their integration by a quadrature.* In this new form of the differential equations there is no trace of the nodes. The differential equations which determine the relative motion of the three bodies are reduced to five equations of the first order and one of the second order. The equations in question are the equations I. to VI. given at the end of the memoir. It is to be remarked that the differential dt is not eliminated from these equations; the last of them is $\frac{d^2}{dt^2}(\mu r^2 + \mu_1 r_1^2) = 2U - 2h$; and if to reduce them to a system of equations of the first order we write $\frac{d}{dt}(\mu r^2 + \mu_1 r_1^2) = \theta$, and therefore $\frac{d\theta}{dt} = 2U - 2h$, the system may be presented in the form

$$\frac{du}{U} = \frac{du_1}{U_1} = \frac{di}{I} = \frac{di_1}{I_1} = \frac{dr}{R} = \frac{dr_1}{R_1} = \frac{d\theta}{\Theta} (= dt),$$

which if we do, and then omit the equation $(=dt)$, we have a system of six equations between the seven quantities $u, u_1, i, i_1, r, r_1, \theta$; when this is integrated, the equation $(=dt)$ gives the time by a quadrature; and finally, Jacobi's equation VII. $\left(d\Omega = \tan u \dfrac{di}{\sin i}\right)$ gives by a quadrature the angle before referred to as disappearing from the system of equations I. to VI.

116. But when Jacobi says, "Par suite on a fait cinq intégrations. Les intégrales connues n'étant qu'au nombre de quatre, on pourra donc dire que l'on a fait une intégration de plus dans le système du monde. Je dis dans le système du monde puisque la même méthode s'applique à un nombre quelconque de corps," the language used is not, I think, quite accurate. It in fact appears from the memoir that it is only on the assumption of the integration of the system of six equations that, besides the integral of *Vis Viva* and the integrals of areas, the remaining two integrals are known; in fact, after, but not before, the system of the order six has been integrated, the time t and the angle Ω are each of them given by a quadrature.

117. Bertrand's "Mémoire sur l'intégration des équations différentielles de la Mécanique" (1852).—I have spoken of this memoir in No. 56 of my former Report. The course of investigation is the inquiry as to the integrals, which, combined according to Poisson's theorem with the integral of *Vis Viva* or any other given integral, give rise to an illusory result. But as regards the application made to the problem of three bodies, it will be more convenient to state from a different point of view the conclusions arrived at: and I may mention that when the author says "Je parviens.. à réduire la question à l'intégration de six équations toutes du premier ordre, c'est-à-dire que j'effectue une intégration de plus que ne l'avait fait Jacobi," he seems to have overlooked that Jacobi's system of five equations of the first order and one of the second order really is, as above noticed, a system of the six equations with another equation which then gives the time by a quadrature, and that, at least as appears to me, he has not advanced the solution beyond the point to which it had been carried by Jacobi [1].

118. Presenting Bertrand's results in the slightly different notation in which they are reproduced in Bour's memoir (*post.* No. 122), then if (x, y, z), (x_1, y_1, z_1) are the coordinates of the two bodies (the problem actually considered being, as with Jacobi, that of the motion of two bodies about a fixed centre of force), and representing the functions $x^2+y^2+z^2$, $x_1^2+y_1^2+z_1^2$, $m^2(x'^2+y'^2+z'^2)$, $m_1^2(x_1'^2+y_1'^2+z_1'^2)$, $m(xx'+yy'+zz')$, $m_1(x_1x_1'+y_1y_1'+z_1z_1')$, $m(x_1x'+y_1y'+z_1z')$, $m_1(xx_1'+yy_1'+zz_1')$, $(xx_1+yy_1+zz_1)$, $mm_1(x'x_1'+y'y_1'+z'z_1')$ by $u, u_1, v, v_1, w, w_1, r, r_1, q, s$ respectively, then the last-mentioned quantities are connected by a single geometrical relation, so that any one of them, say s, may be considered as a given function of the remaining nine. And the author *in effect* shows that the equations of motion give a system

$$\frac{du}{U} = \frac{du_1}{U_1} = \frac{dv}{V} = \frac{dv_1}{V_1} = \frac{dw}{W} = \frac{dw_1}{W_1} = \frac{dr}{R} = \frac{dr_1}{R_1} = \frac{dq}{Q} = (dt),$$

[1] These remarks were communicated by me to M. Bertrand—see my letter "Sur l'intégration des équations différentielles de la Mécanique," *Comptes Rendus* (1863)—and, in the answer he kindly sent me, he agrees that they are correct.

where U, U_1, &c. are functions of the quantities u, u_1, v, &c. Omitting from the system the equation $(=dt)$, there are eight equations between nine quantities; but there are two known integrals, viz., the integral of *Vis Viva* and the integral of principal moment (or sum of the squares of the integrals of areas); that is to say, the system is really a system of *six* equations.

119. Painvin, "Recherche du dernier Multiplicateur &c." (1854).—The author investigates the ultimate multiplier for two systems of differential equations:

1°. The system of the equations I. to VI. in Jacobi's memoir "Sur l'élimination des Nœuds &c." (*antè*, No. 114). Writing in the equations $\frac{dr}{dt}=r'$, $\frac{dr_1}{dt}=r_1'$, and treating r', r_1' as new variables, the system may be written in the form

$$\frac{du}{U}=\frac{du_1}{U_1}=\frac{di}{I}=\frac{di_1}{I_1}=\frac{dr}{R}=\frac{dr_1}{R_1}=\frac{dr'}{R'}=\frac{dr_1'}{R_1'}(=dt),$$

which, omitting the equation $(=dt)$, is a system of seven equations between eight variables; and it is for this form of the system that the value of M is determined, the result obtained being the simple and elegant one, $M=\frac{\sin i \sin i_1}{\sin^2 I}$. The system of seven equations has an integral which is in fact the equation V. of the system in Jacobi's form, so that it is really a system of *six* equations (*antè*, No. 115).

2°. The system secondly discussed is Bertrand's system of nine equations (*antè*, No. 118). The multiplier M is obtained under four different forms, $M=\frac{1}{\sqrt{B^2-AC}}=\frac{1}{\sqrt{\alpha\alpha_1}}=\frac{1}{AZ+B}=\frac{1}{mn}$ (I do not stop to explain the notation), the last of them being referred to as a result announced by M. Bertrand in his course. But it is shown by M. Bour in the memoir next referred to (*post*, No. 122), that the multiplier for the system in question can be obtained in a very much more simple manner, almost without calculation.

120. In connexion with Jacobi's theory of the elimination of the Nodes, I may refer to the investigations "Application to the Problem of three Bodies," Nos. 84 to 96 of Donkin's memoir "On a Class of Differential Equations &c." Part II. The author remarks that his differential equations No. 93 afford an example of the so-called elimination of the Nodes, quite different however (in that they are *merely* transformations of the original differential equations of the problem without any integrations) from that effected by Jacobi.

121. It may be right to refer again in this place to the concluding part of § 28 of Jacobi's memoir "Nova Theoria Multiplicatoris &c." (*antè*, No. 92), as bearing on the problem of three bodies.

122. Bour's "Mémoire sur le Problème des Trois Corps" (1856).—The author remarks that Bertrand's system of equations have lost the remarkable form and the properties which characterize the ordinary equations for the solution of a dynamical

problem. But by selecting eight new variables, functions of Bertrand's variables, the system may be brought back to the standard Hamiltonian form

$$\frac{dq_i}{dt} = \frac{dH}{dp_i}, \quad \frac{dp_i}{dt} = -\frac{dH}{dq_i},$$

or to the form adopted by M. Bour, of a partial differential equation

$$\Sigma\left(\frac{dH}{dq_i}\frac{d\zeta}{dp_i} - \frac{dH}{dp_i}\frac{d\zeta}{dq_i}\right) = 0;$$

and guiding himself by a theorem in relation to canonical integrals obtained in his memoir of 1855 (see No. 66 of my former Report), he finds by a somewhat intricate analysis the expressions of the eight new variables p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4. The results ultimately obtained are of a very remarkable and interesting form, viz. $H =$ funct. $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4)$ is equal to the value it would have for motion in a plane, *plus* a term admitting of a simple geometrical interpretation; and he thus arrives at the following theorem as a *résumé* of the whole memoir, viz.,

"In order to integrate in the general case the problem of three bodies, it is sufficient to solve the case of motion in a plane, and then to take account of a disturbing function equal to the product of a constant depending on the areas by the sum of the moments of inertia of the bodies round a certain axis, divided by the square of the triangle formed by the three bodies."

123. It may be remarked that the only given integral of the system of eight equations is the integral of *Vis Viva*, $H =$ const., and that using this equation to eliminate one of the variables, and omitting the equation $(= dt)$, we have, as in the solutions of Jacobi and Bertrand, a system of six equations between seven variables. As the equations are in the standard dynamical form, no investigation is needed of the multiplier M, which is given by Jacobi's general theory, and consequently when any five integrals of the six equations are given, the remaining integral can be obtained by a quadrature.

In the case of three bodies moving in a plane, the solution takes a very simple form, which is given in the concluding paragraph of the memoir.

Transformation of Coordinates, Article Nos. 124 to 141.

124. It may be convenient to remark at once that two sets of rectangular coordinates may be related to each other properly or improperly, viz., the axes to which they belong (considered as drawn from the origin in the positive directions) may be either capable, or else incapable, of being brought into coincidence. The latter relation, although of equal generality with the former one, may for the most part be disregarded; for by merely reversing the directions of the one set of axes, the improper is converted into the proper relation.

125. In the memoir "Problema Algebraicum &c." (1770) Euler proposes to himself the question "Invenire novem numeros ita in quadratum disponendos

$$\begin{matrix} A, & B, & C \\ D, & E, & F \\ G, & H, & I \end{matrix}$$

ut satisfiat duodecem sequentibus conditionibus &c.", viz., substituting for A, B, C, &c. the ordinary letters

$$\begin{matrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \\ \alpha'', & \beta'', & \gamma'', \end{matrix}$$

the twelve conditions are

$$\begin{aligned} &\alpha^2 + \alpha'^2 + \alpha''^2 = 1, && \alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0, \\ &\beta^2 + \beta'^2 + \beta''^2 = 1, && \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0, \\ &\gamma^2 + \gamma'^2 + \gamma''^2 = 1, && \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0, \\ &\alpha^2 + \beta^2 + \gamma^2 = 1, && \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0, \\ &\alpha'^2 + \beta'^2 + \gamma'^2 = 1, && \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0, \\ &\alpha''^2 + \beta''^2 + \gamma''^2 = 1, && \alpha\alpha'' + \beta''\beta + \gamma''\gamma = 0. \end{aligned}$$

And he remarks that this is in fact the problem of the transformation of coordinates, viz., if we have

$$\begin{aligned} X &= \alpha x + \beta y + \gamma z, \\ Y &= \alpha' x + \beta' y + \gamma' z, \\ Z &= \alpha'' x + \beta'' y + \gamma'' z, \end{aligned}$$

then the first equations are such as to give identically

$$X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2.$$

126. Assuming the first six equations, he shows by a direct analytical process that $\alpha^2 = (\beta'\gamma'' - \beta''\gamma')^2$, or $\alpha = \pm(\beta'\gamma'' - \beta''\gamma')$; or taking the positive sign (for, as the numbers may be taken as well positively as negatively, there is nothing lost by doing so) $\alpha = \beta'\gamma'' - \beta''\gamma'$, which gives the system

$$\begin{aligned} \alpha &= \beta'\gamma'' - \beta''\gamma', & \beta &= \gamma'\alpha'' - \gamma''\alpha', & \gamma &= \alpha'\beta'' - \alpha''\beta', \\ \alpha' &= \beta''\gamma - \beta\gamma'', & \beta' &= \gamma''\alpha - \gamma\alpha'', & \gamma' &= \alpha''\beta - \alpha\beta'', \\ \alpha'' &= \beta\gamma' - \beta'\gamma, & \beta'' &= \gamma\alpha' - \gamma'\alpha, & \gamma'' &= \alpha\beta' - \alpha'\beta, \end{aligned}$$

and from these he deduces the second system of six equations. The inverse system of equations

$$\begin{aligned} X &= \alpha x + \alpha' y + \alpha'' z, \\ Y &= \beta x + \beta' y + \beta'' z, \\ Z &= \gamma x + \gamma' y + \gamma'' z \end{aligned}$$

is not explicitly referred to.

127. He then satisfies the equations by means of trigonometrical substitutions, viz., assuming $\alpha = \cos\zeta$, then $\alpha'^2 + \alpha''^2 = \sin^2\zeta$, which is satisfied by $\alpha' = \sin\zeta\cos\eta$, $\alpha'' = \sin\zeta\sin\eta$, &c., and he thus obtains for the coefficients a set of values involving the angles ζ, η, θ, which are the same as those mentioned *post*, No. 130. And he shows how these formulæ may be obtained geometrically by three successive transformations of two coordinates only. The remainder of the memoir relates to the analogous problem of the transformation of four or more coordinates.

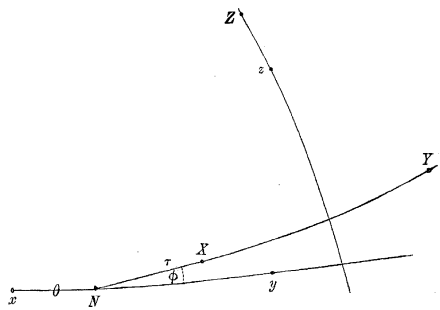
128. I have analysed so much of Euler's memoir in order to show that it contains nearly the whole of the ordinary theory of the transformation of coordinates; the only addition required is the equation

$$\begin{vmatrix} \alpha\,, & \beta\,, & \gamma \\ \alpha'\,, & \beta'\,, & \gamma' \\ \alpha''\,, & \beta''\,, & \gamma'' \end{vmatrix} = \pm 1,$$

where the sign + gives $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c. (*ut supra*), but the sign − would give $\alpha = -(\beta'\gamma'' - \beta''\gamma')$, &c.

129. The distinction of the ambiguous sign is in fact the above-mentioned one of the proper and improper transformations; viz., for the sign + the two sets of axes can, for the sign − they cannot, be brought into coincidence: this very important remark was, I believe, first made by Jacobi in one of his early memoirs in *Crelle's Journal*, but I have lost the reference. As already mentioned, it is allowable to attend only to the proper transformation, and to consider the value of the determinant as being $= +1$; and this is in fact almost always done.

130. Euler's formulæ involving the three angles are those which are ordinarily made use of in the problem of rotation and the problems of physical astronomy generally.



It is convenient to take them as in the figure, viz., θ, the longitude of node,

ϕ, the inclination, τ, the angular distance of X from node, and the formulæ of transformation then are

	X	Y	Z
x	$\cos\tau\cos\theta-\sin\tau\sin\theta\cos\phi$	$-\sin\tau\cos\theta-\cos\tau\sin\theta\cos\phi$	$\sin\theta\sin\phi$
y	$\cos\tau\sin\theta+\sin\tau\cos\theta\cos\phi$	$-\sin\tau\sin\theta+\cos\tau\cos\theta\cos\phi$	$-\cos\theta\sin\phi$
z	$\sin\tau\sin\phi$	$\cos\tau\sin\phi$	$\cos\phi$

The foregoing very convenient algorithm, viz., the employment of

	X	Y	Z
x	α	β	γ
y	α'	β'	γ'
z	α''	β''	γ''

to denote the system of equations

$$x=\alpha X+\beta Y+\gamma Z,$$
$$y=\alpha' X+\beta' Y+\gamma' Z,$$
$$z=\alpha'' X+\beta'' Y+\gamma'' Z,$$

is due to M. Lamé.

131. But previously to the foregoing investigations, viz., in the memoir "Du Mouvement de Rotation &c.," *Mém. de Berlin for* 1758 (pr. 1765), Euler had obtained incidentally a very elegant solution of the problem of the transformation of coordinates; this is in fact identical with the next mentioned one, the letters l, m, n; λ, μ, ν being used in the place of ζ, ζ', ζ''; η, η', η''.

132. In the memoir "Formulæ generales pro translatione &c." (1775), Euler gives the following formulæ for the transformation of coordinates, viz., if the position of the set of axes XYZ in reference to the set xyz is determined by

$$xX,\ yX,\ zX=90^\circ-\zeta,\ 90^\circ-\zeta',\ 90^\circ-\zeta'',\ \angle^{s} YXx,\ YXy,\ YXz=\eta,\ \eta',\ \eta'',$$

then the formulæ of transformation are

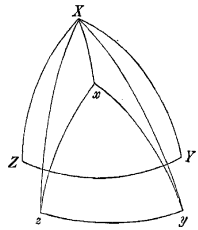
	X	Y	Z
x	$\sin\zeta$	$\cos\zeta\ \sin\eta$	$\cos\zeta\ \cos\eta$
y	$\sin\zeta''$	$\cos\zeta'\ \sin\eta'$	$\cos\zeta'\ \cos\eta'$
z	$\sin\zeta''$	$\cos\zeta''\ \sin\eta''$	$\cos\zeta''\ \cos\eta''$

with the following equations connecting the six angles, viz., if

$$-\Delta^2 = \cos(\eta' - \eta'')\cos(\eta'' - \eta)\cos(\eta - \eta'),$$

then

$$\tan\zeta = \frac{-\Delta}{\cos(\eta' - \eta'')}, \quad \tan\zeta' = \frac{-\Delta}{\cos(\eta'' - \eta)}, \quad \tan\zeta'' = \frac{-\Delta}{\cos(\eta - \eta')}.$$



133. It is right to notice that these values of ζ, ζ', ζ'' give the twelve equations $\alpha^2 + \beta^2 + \gamma^2 = 1$, &c., but they do not give definitely $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c., but only $\alpha = \pm(\beta'\gamma'' - \beta''\gamma')$; that is, in the formulæ in question the two sets of axes are not of necessity displacements the one of the other. In the same memoir Euler considers two sets of rectangular axes, and assuming that they are displacements the one of the other (this assumption is not made as explicitly as it should have been), he remarks that the one set may be made to coincide with the other set by means of a finite rotation about a certain axis (which may conveniently be termed the Resultant Axis). This consideration leads him to an equation which ought to be satisfied by the coefficients of transformation, but which he is not able to verify by means of the foregoing expressions in terms of ζ, ζ', ζ'', η, η', η''.

134. I remark that Euler's equation in fact is

$$\begin{vmatrix} \alpha - 1, & \beta, & \gamma \\ \alpha', & \beta' - 1, & \gamma' \\ \alpha'', & \beta'', & \gamma'' - 1 \end{vmatrix} = 0,$$

or, as it may be written,

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} - (\beta'\gamma'' - \beta''\gamma') - (\gamma''\alpha - \gamma\alpha'') - (\alpha\beta' - \alpha'\beta) + \alpha + \beta' + \gamma'' - 1 = 0,$$

in which form it is an immediate consequence of the equations

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = 1, \ \alpha = \beta'\gamma'' - \beta''\gamma', \ \&c.,$$

which are true for the proper, but not for the improper transformation.

135. In the undated addition to the memoir, Euler states the theorem of the resultant axis as follows:—"Theorema. Quomodocunque sphæra circa centrum suum convertatur, semper assignari potest diameter cujus directio in situ translato conveniat cum situ originali;" and he again endeavours to obtain a verification of the foregoing analytical theorem.

136. The theory of the Resultant Axis was further developed by Euler in the memoir "Nova Methodus Motum &c." (1775), and by Lexell in the memoir "Nonnulla theoremata generalia &c." (1775): the geometrical investigations are given more completely and in greater detail in Lexell's memoir. The result is contained in the following system of formulæ for the transformation of coordinates, viz., if α, β, γ are the inclinations of the resultant axis to the original set, and if ϕ is the rotation about the resultant axis, or say the resultant rotation, then we have

	X	Y	Z
x	$\cos^2\alpha + \sin^2\alpha\cos\phi$	$\cos\alpha\cos\beta(1-\cos\phi)+\cos\gamma\sin\phi$	$\cos\alpha\cos\gamma(1-\cos\phi)-\cos\beta\sin\phi$
y	$\cos\beta\cos\alpha(1-\cos\phi)-\cos\gamma\sin\phi$	$\cos^2\beta+\sin^2\beta\cos\phi$	$\cos\beta\cos\gamma(1-\cos\phi)+\cos\alpha\sin\phi$
z	$\cos\gamma\cos\alpha(1-\cos\phi)+\cos\beta\sin\phi$	$\cos\gamma\cos\beta(1-\cos\phi)-\cos\alpha\sin\phi$	$\cos^2\gamma+\sin^2\gamma\cos\phi$

Euler attempts, but not very successfully, to apply the formulæ to the dynamical problem of the rotation of a solid body: he does not introduce them into the differential equations, but only into the integral ones, and his results are complicated and inelegant. The further simplification effected by Rodrigues was in fact required.

137. Jacobi's paper, "Euleri formulæ &c." (1827), merely cites the last-mentioned result.

138. I find it stated in Lacroix's *Differential Calculus*, t. I. p. 533, that the following system for the transformation of coordinates was obtained by Monge (no reference is given in Lacroix), viz., the system being as above,

$$\begin{vmatrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \\ \alpha'', & \beta'', & \gamma'', \end{vmatrix}$$

and the quantities α, β', γ'' being arbitrary, then putting

$$1+\alpha+\beta'+\gamma''=M,$$
$$1+\alpha-\beta'-\gamma''=N,$$
$$1-\alpha+\beta'-\gamma''=P,$$
$$1-\alpha-\beta'+\gamma''=Q,$$

so that

$$M+N+P+Q=4,$$

we have

$$2\beta = \sqrt{NP} + \sqrt{MQ}, \quad 2\gamma' = \sqrt{PQ} + \sqrt{MN}, \quad 2\alpha'' = \sqrt{QN} + \sqrt{MP},$$
$$2\alpha' = \sqrt{NP} - \sqrt{MQ}, \quad 2\beta'' = \sqrt{PQ} - \sqrt{MN}, \quad 2\gamma = \sqrt{QN} - \sqrt{MP}.$$

These are formulæ very closely connected with those of Rodrigues.

139. The theory was perfected by Rodrigues in the valuable memoir "Des lois géométriques &c." (1840). Using for greater convenience λ, μ, ν in the place of his $\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}p$, he in effect writes

$$\tan \tfrac{1}{2}\phi \cos \alpha = \lambda,$$
$$\tan \tfrac{1}{2}\phi \cos \beta = \mu,$$
$$\tan \tfrac{1}{2}\phi \cos \gamma = \nu,$$

and this being so, the coefficients of transformation are

$$\begin{array}{llll} 1+\lambda^2-\mu^2-\nu^2, & 2(\lambda\mu+\nu) & , & 2(\lambda\nu-\mu) \quad , \\ 2(\mu\lambda-\nu) \quad , & 1-\lambda^2+\mu^2-\nu^2, & & 2(\mu\nu+\lambda) \quad , \\ 2(\nu\lambda+\mu) \quad , & 2\nu\mu-\lambda & , & 1-\lambda^2-\mu^2+\nu^2, \end{array}$$

all divided by the common denominator $1+\lambda^2+\mu^2+\nu^2$. Conversely, if the coefficients of transformation are as usual represented by

$$\begin{array}{lll} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \\ \alpha'', & \beta'', & \gamma'', \end{array}$$

then λ^2, μ^2, ν^2, λ, μ, ν are respectively equal to

$$\begin{array}{lll} 1+\alpha-\beta'-\gamma'', & 1-\alpha+\beta'-\gamma'', & 1-\alpha-\beta'+\gamma'', \\ \gamma'-\beta'' \quad , & \alpha''-\beta \quad , & \beta-\alpha' \quad , \end{array}$$

each of them divided by $1+\alpha+\beta'+\gamma''$.

The memoir contains very elegant formulæ for the composition of finite rotations, and it will be again referred to in speaking of the kinematics of a solid body.

140. Sir W. R. Hamilton's first papers on the theory of quaternions were published in the years 1843 and 1844: the fundamental idea consists in the employment of the imaginaries i, j, k, which are such that

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k,$$

whence also

$$\begin{aligned} &(w+ix+jy+kz)(w'+ix'+jy'+kz') \\ &= \quad ww'-xx'-yy'-zz' \\ &\quad +i(wx'+w'x+yz'-y'z) \\ &\quad +j(wy'+w'y+zx'-z'x) \\ &\quad +k(wz'+w'z+xy'-x'y); \end{aligned}$$

so that representing the right-hand side by

$$W + iX + jY + kZ,$$

we have identically

$$W^2 + X^2 + Y^2 + Z^2 = (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2).$$

It is hardly necessary to remark that Sir W. R. Hamilton in his various publications on the subject, and in the *Lectures on Quaternions*, Dublin, 1853, has developed the theory in detail, and has made the most interesting applications of it to geometrical and dynamical questions; and although the first explicit application of it to the present question may have been made in my own paper next referred to, it seems clear that the whole theory was in its original conception intimately connected with the notion of rotation.

141. Cayley, "On certain Results relating to Quaternions" (1845).—It is shown that Rodrigues' transformation formula may be expressed in a very simple manner by means of quaternions; viz., we have

$$ix + jy + kz = (1 + i\lambda + j\mu + k\nu)^{-1}(iX + jY + kZ)(1 + i\lambda + j\mu + k\nu),$$

where developing the function on the right-hand side, and equating the coefficients of i, j, k, we obtain the formulæ in question. A subsequent paper, Cayley, "On the application of Quaternions to the Theory of Rotation" (1848), relates to the composition of rotations.

Principal Axes, and Moments of Inertia. Article Nos. 142—163.

142. The theorem of principal axes consists herein, that at any point of a solid body there exists a system of axes Ox, Oy, Oz, such that

$$\int yzdm = 0, \quad \int zxdm = 0, \quad \int xydm = 0.$$

But this, the original form of the theorem, is a mere deduction from a general theory of the representation of the integrals

$$\int x^2dm, \quad \int y^2dm, \quad \int z^2dm, \quad \int yzdm, \quad \int zxdm, \quad \int xydm$$

for any axes through the given origin by means of an ellipsoid depending on the values of these integrals corresponding to a given set of rectangular axes through the same origin.

143. If, for convenience, we write as follows, $M = \int dm$ the mass of the body, and

$$A' = \int x^2dm, \quad B' = \int y^2dm, \quad C' = \int z^2dm, \quad F' = \int yzdm, \quad G' = \int zxdm, \quad H' = \int xydm,$$

and moreover

$$A=\int(y^2+z^2)\,dm,\quad B=\int(z^2+x^2)\,dm,\quad C=\int(x^2+y^2)\,dm,$$

$$F=-\int yzdm,\quad G=-\int zxdm,\quad H=-\int xydm,$$ [1]

so that

$$A=B'+C',\quad B=C'+A',\quad C=A'+B',\quad F=-F',\quad G=-G',\quad H=-H',$$

then the ellipsoid which in the first instance presents itself for this purpose, and which Prof. Price has termed the Ellipsoid of Principal Axes, but which I would rather term the "Comomental Ellipsoid," is the ellipsoid

$$(A',\ B',\ C',\ F',\ G',\ H'\between x,\ y,\ z)^2=Mk^4,$$

where k is arbitrary, so that the absolute magnitude is not determined. But it is more usual, and in some respects better to consider in place thereof the "Momental Ellipsoid" (Cauchy, "Sur les Moments d'Inertie," *Exercices de Mathématique*, t. II. pp. 93—103, 1827),

$$(A\ ,\ B\ ,\ C\ ,\ F\ ,\ G\ ,\ H\between x,\ y,\ z)^2=Mk^4,$$

or as it may also be written,

$$(A'+B'+C')(x^2+y^2+z^2)-(A',\ B',\ C',\ F',\ G',\ H'\between x,\ y,\ z)^2=Mk^4,$$

which shows that the two ellipsoids have their axes, and also their circular sections, coincident in direction.

144. And there is besides this a third ellipsoid, the "Ellipsoid of Gyration," which is the reciprocal of the momental ellipsoid in regard to the concentric sphere, radius k. The last-mentioned ellipsoid is given in magnitude, viz., if the body is referred to its principal axes, then putting $A=Ma^2$, $B=Mb^2$, $C=Mc^2$, the equation of the ellipsoid of gyration is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The axes of any one of the foregoing ellipsoids coincide in direction with the principal axes of the body, and the magnitudes of the axes lead very simply to the values of the principal moments A, B, C.

145. The origin has so far been left arbitrary: in the dynamical applications, this origin is in the case of a solid body rotating about a fixed point, the fixed point; and in the case of a free body, the centre of gravity. But the values of the coefficients $(A,\ B,\ C,\ F,\ G,\ H)$, or $(A',\ B',\ C',\ F',\ G',\ H')$, corresponding to any given origin whatever, are very easily expressed in terms of the coordinates of this origin, and the values of the corresponding coefficients for the centre of gravity as origin; or, what is the same thing, any one of the ellipsoids for the given origin may be geometrically con-

[1] I have ventured to make this change instead of writing as usual $F=\int yzdm$, &c.; as in most cases $F=G=H=0$, the formulæ affected by the alteration are not numerous.

structed by means of the ellipsoid for the centre of gravity. The geometrical theory, as regards the magnitudes of the axes, does not appear to have been anywhere explicitly enunciated; as regards their direction, it is comprised in the theorem that the directions at any point are the three rectangular directions at that point in regard to the ellipsoid of gyration for the centre of gravity[1], *post*, No. 159. The notion of the ellipsoids, and of the relation between the ellipsoids at a given point and those at the centre of gravity, once established, the theory of principal axes and moments of inertia becomes a purely geometrical one.

146. The existence of principal axes was first established by Segner in the work *Specimen Theoriæ Turbinum*, Halle (1755), where, however, it is remarked that Euler had said something on the subject in the {Berlin} *Memoirs* for 1749 and 1750 (*post*, No. 167), and had constructed a new mechanical principle, but without pursuing the question. Segner's course of investigation is in principle the same as that now made use of, viz. a principal axis is defined to be an axis such that when a body revolves round it the forces arising from the rotation have no tendency to alter the position of the axes. It is first shown that there are systems of axes x, y, z such that $\int xzdm = 0$, and then, in reference to such a set of axes, the position of a principal axis, say the axis of X, is determined by the conditions $\int XYdm = 0$, $\int XZdm = 0$, viz. the unknown quantities being taken to be $t = \frac{\cos\alpha}{\cos\gamma}$, $\tau = \frac{\cos\beta}{\cos\gamma}$ (α, β, γ being the inclinations of the principal axis to those of x, y, z), and then putting $A = \int x^2dm$, &c. ($F = 0$ by hypothesis), Segner's equations for the determination of t, τ, are

$$G't^2 + (C' - A')\,t - G' - H'\tau = 0,$$
$$(C' - B')\,\tau - G't\tau + H't = 0,$$

the second of which gives

$$\tau = \frac{H't}{B' - C' + G't},$$

and by means of it the first gives

$$G'^2t^3 - G'(A' - B')\,t^2 + \{(B' - C')(C' - A') - G'^2 - H'^2\}\,t + G'(B' - C') = 0,$$

which being a cubic equation shows that there are three principal axes; and it is afterwards proved that these are at right angles to each other.

147. To show the equivalence of Segner's solution to the modern one, I remark that if $u = \int X^2dm$, we have

$$\begin{aligned} (A' - u)\,t + H'\quad\tau + G'\quad &= 0,\\ B'\quad t + (B' - u)\,\tau + F'\quad &= 0,\\ G'\quad t + F'\quad\tau + C' - u &= 0, \end{aligned}$$

[1] The rectangular directions at a point in regard to an ellipsoid are the directions of the axes of the circumscribed cone, or, what is the same thing, they are the directions of the normals to the three quadric surfaces, confocal with the given ellipsoid, which pass through the given point. The theory of confocal surfaces appears to have been first given by Chasles, Note XXXI. of the *Aperçu Historique* (1837).

whence

$$\begin{aligned} t^2 : \tau^2 : 1 : \tau : t : t\tau = \; & B'C' - F'^2 - (B' + C')\,u + u^2 \\ : \; & C'A' - G'^2 - (C' + A')\,u + u^2 \\ : \; & A'B' - H'^2 - (A' + B')\,u + u^2 \\ : \; & G'H' - A'F' \qquad + F'\;u \\ : \; & H'F' - B'G' \qquad + G'\;u \\ : \; & F'G' - C'H' \qquad + H'\;u, \end{aligned}$$

or putting therein $F' = 0$,

$$\begin{aligned} t^2 : \tau^2 : 1 : \tau : t : t\tau = \; & B'C' \qquad - (B' + C')\,u + u^2 \\ : \; & C'A' - G'^2 - (C' + A')\,u + u^2 \\ : \; & A'B' - H'^2 - (A' + B')\,u + u^2 \\ : \; & G'H' \\ : \; & -B'G' \qquad + G'\,u \\ : \; & -C'H' \qquad + H'\,u, \end{aligned}$$

by means of which Segner's equations may be verified. I have given this analysis, as the first solution of such a problem is a matter of interest.

148. There is little if anything added to Segner's results by the memoir, Euler, "Recherches sur la Connaissance Mécanique des Corps" (1758), which is introductory to the immediately following one on Rotation.

149. Relating to the theory of principal axes we have Binet's "Mémoire sur les axes conjugués &c.," (1813). The author proposes to make known the new systems of axes which he calls *conjugate axes*, which, when they are at right angles to each other, coincide with the principal axes; viz. considering the sum of the molecules each into its distance from a plane, such distance being measured in the direction of a line, then (the direction of the line being given) of all the planes which pass through a given point, there is one for which the sum in question is a minimum, and this plane is said to be *conjugate* to the given line, and from the notion of a line and conjugate plane he passes to that of a system of *conjugate* axes. The investigation (which is throughout an elegant one) is conducted analytically; the coordinates made use of are oblique ones, and the formulæ are thus rendered more complicated than they would have been; in referring to them it will be convenient to make the axes rectangular.

150. One of the results is the well-known equation

$$(A' - \Theta)(B' - \Theta)(C' - \Theta) - F'^2(A' - \Theta) - G'^2(B' - \Theta) - H'^2(C' - \Theta) + 2F'G'H' = 0;$$

which, if x_1, y_1, z_1 are the principal axes, has for its roots $\int x_1^2 dm$, $\int y_1^2 dm$, $\int z_1^2 dm$.

And the equations (1), p. 49, taking therein the original axes as rectangular, are

$$\left(\mathfrak{A}' - \frac{K'}{\Theta'}\right)\cos\alpha + \mathfrak{H}' \cos\beta + \mathfrak{G}' \cos\gamma = 0,$$

$$+\mathfrak{H}' \cos\alpha + \left(\mathfrak{B}' - \frac{K'}{\Theta'}\right)\cos\beta + \mathfrak{F}' \cos\gamma = 0,$$

$$+\mathfrak{G}' \cos\alpha + \mathfrak{F}' \cos\beta + \left(\mathfrak{C}' - \frac{K'}{\Theta'}\right)\cos\gamma = 0,$$

where $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{F}'$, $\mathfrak{G}'$, $\mathfrak{H}'$ denote the reciprocal coefficients, $\mathfrak{A}' = B'C' - F'^2$ &c., and K' is the discriminant $= A'B'C' - A'F'^2 - B'G'^2 - C'H'^2 + 2F'G'H'$: this is a symmetrical system of equations for finding $\cos\alpha : \cos\beta : \cos\gamma$, less simple however than the modern form (*post*, No. 154), the identity of which with Binet's may be shown without difficulty.

151. Another result (p. 57) is that if the original axes are principal axes, and if Ox, Oy, Oz are the principal axes through a point the coordinates whereof are f, g, h, and if $\Theta_1' = (\text{say}) \int x_1^2 dm$, then we have

$$\frac{f^2}{\Theta_1' - A'} + \frac{g^2}{\Theta_1' - B'} + \frac{h^2}{\Theta_1' - C'} = \frac{1}{M},$$

(in which I have restored the mass M, which is put equal to unity), so that if Θ_1' have a given constant value, the locus of the point is a quadric surface, the nature whereof will depend on the value of Θ_1. The surfaces in question are confocal with each other {and with the imaginary surface $\frac{x^2}{-A'} + \frac{y^2}{-B'} + \frac{z^2}{-C'} = \frac{1}{M}$, which is similar to the ellipsoid $\frac{x^2}{A'} + \frac{y^2}{B'} + \frac{z^2}{C'} = \frac{1}{M}$, which is the reciprocal of the comomental ellipsoid $A'x^2 + B'y^2 + C'z^2 = Mk^2$ in regard to a concentric sphere, radius k}. The author mentions the ellipsoid $\frac{x^2}{A'} + \frac{y^2}{B'} + \frac{z^2}{C'} = \frac{1}{M}$ (see p. 64), and he remarks that his conjugate axes are in fact conjugate axes in respect to this ellipsoid, and consequently that the principal axes are in direction the principal axes of this ellipsoid: it is noticeable that the ellipsoid thus incidentally considered is not the comomental ellipsoid itself, but, as just remarked, its reciprocal in regard to a concentric sphere.

152. Poisson, *Mécanique* (1st ed. 1811, and indeed 2nd ed. 1833), gives the theory of principal axes in a less complete form than in Binet's memoir; for the directions of the principal axes are obtained in anything but an elegant form.

153. Ampère's Memoir (1823).—The expression *permanent axis* is used in the place of principal axis, which is employed to designate a principal axis through the centre of gravity. The memoir contains a variety of very interesting geometrical theorems, which however, as no ellipsoid is made use of, can hardly be considered as exhibited in their proper connexion. The author arrives incidentally at certain conics, which are in fact the focal conics of the ellipsoid of gyration $\left(\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M}\right)$ for the centre of gravity.

154. Cauchy, in the memoir "Sur les Momens d'Inertie" (1827), considers the momental ellipsoid $(A, B, C, F, G, H \between x, y, z)^2 = 1$, and employs it as well to prove the existence of the principal axes as to determine their direction, and also the magnitudes of the principal moments; the results are obtained in the simplest and best forms; viz. the direction cosines are given by

$$\begin{array}{llll} (A-\Theta)\cos\alpha + H & \cos\beta + G & \cos\gamma = 0, \\ H & \cos\alpha + (B-\Theta)\cos\beta + F & \cos\gamma = 0, \\ G & \cos\alpha + F & \cos\beta + (C-\Theta)\cos\gamma = 0, \end{array}$$

where

$$(A-\Theta)(B-\Theta)(C-\Theta) - (A-\Theta)F^2 - (B-\Theta)G^2 - (C-\Theta)H^2 + 2FGH = 0,$$

Θ being one of the principal moments.

155. Poinsot, "Mémoire sur la Rotation" (1834), defines the "Central Ellipsoid" as an ellipsoid having for its axes the principal axes through the centre of gravity, the squares of the lengths being reciprocally proportional to the principal moments; and he remarks in passing that *the moment about any diameter of the ellipsoid is inversely proportional to the square of this diameter.* It is to be noticed that Poinsot speaks only of the ellipsoid having its centre at the centre of gravity, but that such ellipsoid may be constructed about any point whatever as centre, and that so generalized it is in fact the momental ellipsoid $Ax^2 + By^2 + Cz^2 = Mk^4$; and moreover that Poinsot defines his ellipsoid by reference to the principal axes.

156. Pirie, "On the Principal Axes &c." (1837), obtained analytically in a very elegant manner equations for determining the positions of the principal axes; viz. these are

$$\begin{array}{llll} (A'-\Theta')\cos\alpha + H' & \cos\beta + G' & \cos\gamma = 0, \\ H' & \cos\alpha + (B'-\Theta')\cos\beta + F' & \cos\gamma = 0, \\ G' & \cos\alpha + F' & \cos\beta + (C'-\Theta')\cos\gamma = 0, \end{array}$$

where

$$(A'-\Theta')(B'-\Theta')(C'-\Theta') - (A'-\Theta')F'^2 - (B'-\Theta')G'^2 - (C'-\Theta')F'^2 + 2F'G'H' = 0;$$

viz. these are similar to those of Cauchy, only they belong to the comomental instead of the momental ellipsoid.

157. Maccullagh, in his Lectures of 1844 (see Haughton), considers the momental ellipsoid

$$(A, B, C, F, G, H \between x, y, z)^2 = Mk^4$$

(A, B, C, F, G, H *ut suprà*), which is such that the moment of inertia of the body with respect to any axis passing through the origin is proportional to the square of the radius vector of the ellipsoid; and from the geometrical theorem of the ellipsoid having principal axes he obtained the mechanical theorem of the existence of principal axes of the body; at least I infer that he did so, although the conclusion is not

explicitly stated in Haughton's account; but in all this he had been anticipated by Cauchy. And afterwards, referring the ellipsoid to its principal axes, so that the equation is $Ax^2+By^2+Cz^2=Mk^4$, he writes $A=Ma^2$, $B=Mb^2$, $C=Mc^2$, which reduces the equation to $a^2x^2+b^2y^2+c^2z^2=k^4$, and he considers the reciprocal ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, or, what is the same thing, $\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=\frac{1}{M}$, which is the ellipsoid of gyration.

158. Thomson, "On the Principal Axes of a Solid Body" (1846), shows analytically that the principal axes coincide in direction with the axes of the momental ellipsoid

$$(A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2=Mk^4;$$

but the geometrical theorem might have been assumed: the investigation is really an investigation of the axes of this ellipsoid. And he remarks that the ellipsoid $(A',\ B',\ C',\ F',\ G',\ H' \between x,\ y,\ z)^2=Mk^4$ (the comomental ellipsoid) might equally well have been used for the purpose.

159. He obtains the before-mentioned theorem that the directions of the principal axes at any point are the rectangular directions in regard to the ellipsoid of gyration $\left(\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=\frac{1}{M}\right)$ for the centre of gravity. And for determining the moments of inertia at the given point (say its coordinates are ξ, η, ζ) he obtains the equation

$$\frac{\xi^2}{\xi^2+\eta^2+\zeta^2+\frac{A-P}{M}}+\frac{\eta^2}{\xi^2+\eta^2+\zeta^2+\frac{B-P}{M}}+\frac{\zeta^2}{\xi^2+\eta^2+\zeta^2+\frac{C-P}{M}}=1,$$

where the three roots of the cubic in P are the required moments. Analytically nothing can be more elegant, but, as already remarked, a geometrical construction for the magnitudes of these moments appears to be required.

160. Some very interesting geometrical results are obtained by considering the "equimomental surface", the locus of the points for which one of the moments of inertia is equal to a given quantity Π; the equation is of course

$$\frac{x^2}{x^2+y^2+z^2+\frac{A-\Pi}{M}}+\frac{y^2}{x^2+y^2+z^2+\frac{B-\Pi}{M}}+\frac{z^2}{x^2+y^2+z^2+\frac{C-\Pi}{M}}=1,$$

which includes Fresnel's wave-surface. In particular it is shown that the equimomental surface cuts any surface

$$\frac{x^2}{A+\Theta}+\frac{y^2}{B+\Theta}+\frac{z^2}{C+\Theta}=\frac{1}{M}$$

confocal with the ellipsoid of gyration in a spherical conic and a curve of curvature; a theorem which is also demonstrated, Cayley, "Note on a Geometrical Theorem &c." (1846).

161. Townsend, "On principal Axes &c." (1846).—This elaborate paper is contemporaneous, or nearly so, with Thomson's, and several of the conclusions are common to the two. From the character of the paper, I find it difficult to give an account of it; and I remark that, the theory of principal axes once brought into connexion with that of confocal surfaces, all ulterior developments belong more properly to the latter theory.

162. Haton de la Goupillière's two memoirs, "Sur la Théorie Nouvelle de la Géométrie des Masses" (1858), relate in a great measure to the theory of the integral $\int xydm$, and its variations according to the different positions of the two planes $x=0$ and $y=0$; the geometrical interpretations of the several results appear to be given with much care and completeness, but I have not examined them in detail. The author refers to the researches of Thomson and Townsend.

163. I had intended to show (but the paper has not been completed for publication) how the momental ellipsoid for any point of the body may be obtained from that for the centre of gravity by a construction depending on the "square potency" of a point in regard to the last-mentioned ellipsoid.

The Rotation of a solid body. Article Nos. 164—207.

164. It will be recollected that the problem is the same for a body rotating about a fixed point, and for the rotation of a free body about the centre of gravity; the case considered is that of a body not acted on by any forces. According to the ordinary analytical mode of treatment, the problem depends upon Euler's equations

$$Adp+(C-B)qrdt=0,$$
$$Bdq+(A-C)rpdt=0,$$
$$Cdr+(B-A)pqdt=0,$$

for the determination of p, q, r, the angular velocities about the principal axes; considering p, q, r as known, we obtain by merely geometrical considerations a system of three differential equations of the first order for the determination of the position in space of the principal axes.

165. The solution of these, which constitutes the chief difficulty of the problem, is usually effected by referring the body to a set of axes fixed in space, the position whereof is not arbitrary, but depends on the initial circumstances of the motion; viz. the axis of z is taken to be perpendicular to the so-called *invariable plane.* The solution is obtained *without* this assumption both by Euler and Lagrange, although, as remarked by them, the formulæ are greatly simplified by making it; it is, on the other hand, made in the solution (which may be considered as the received one) by Poisson; and the results depending on it are the starting-point of the ulterior analytical developments of Rueb and Jacobi; the method of Poinsot is also based upon the consideration of the invariable plane.

166. D'Alembert's Principle, which affords a direct and general method for obtaining the equations of motion in any dynamical problem whatever, was given in his "Traité de Dynamique" (1743); and in his memoir of 1749 he applied it to the physical problem of the Precession of the Equinoxes, which is a special case of the problem of Rotation, the motion of rotation about the centre of gravity being in fact similar to that about a fixed point. But, as might be expected in the first attempt at the analytical treatment of so difficult a problem, the equations of motion are obtained in a cumbrous and unmanageable form.

167. They are obtained by Euler in the memoir "Découverte d'un Nouveau Principe de Mécanique," *Berlin Memoirs* for 1750 (1752) (written before the establishment of the theory of principal axes), in a perfectly elegant form, including the ordinary one already mentioned, and, in fact, reducible to it by merely putting the quantities F, G, H (which denote the integrals $\int yzdm$, &c.) equal to zero. But Euler does not in this memoir enter into the question of the integration of the equations.

168. The notion of principal axes having been suggested by Euler, and their existence demonstrated by Segner, we come to Euler's investigations contained in the memoirs "Du Mouvement de Rotation &c.," *Berlin Memoirs* for 1758 (printed 1765) and for 1760 (printed 1767), and the "Theoria Motus Corporum Solidorum &c." (1765). In the memoir of 1760, and in the "Theoria Motus," Euler employs 8, the angular velocity round the instantaneous axis, but not the resolved velocities $8\cos\alpha$, $8\cos\beta$, $8\cos\gamma\,(=p,\ q,\ r)$: these quantities (there called x, y, z) are however employed in the memoir, *Berlin Memoirs* (1758), which must, I apprehend, have been written after the other, and in which at any rate the solution is developed with much greater completeness. It is in fact carried further than the ordinary solutions, and after the angular velocities p, q, r have been found, the remaining investigation for the position in space of the principal axes, conducted, (as above remarked), without the aid of the invariable plane, is one of great elegance.

169. In the last-mentioned memoir Euler starts from the equations given by d'Alembert's principle; viz. the impressed forces being put equal to zero, these are

$$\Sigma dm\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=0,\ \text{\&c.},$$

or, what is the same thing, using u, v, w to denote the velocities of an element in the directions of the axes fixed in space, these are

$$\Sigma dm\left(y\frac{dw}{dt}-z\frac{dv}{dt}\right)=0,$$

$$\Sigma dm\left(z\frac{du}{dt}-x\frac{dw}{dt}\right)=0,$$

$$\Sigma dm\left(x\frac{dv}{dt}-y\frac{dy}{dt}\right)=0.$$

It is assumed that at any moment the body revolves round an instantaneous axis, inclinations α, β, γ, with an angular velocity 8; this gives

$$u = 8(z\cos\beta - y\cos\gamma) = qz - ry,$$
$$v = 8(x\cos\gamma - z\cos\alpha) = rx - pz,$$
$$w = 8(y\cos\alpha - x\cos\beta) = px - qy,$$

if $8\cos\alpha$, $8\cos\beta$, $8\cos\gamma$ are denoted by p, q, r. The values of du, dv, dw are obtained by differentiating these formulæ, treating p, q, r, x, y, z as variable, and replacing dx, dy, dz by udt, vdt, wdt respectively; in the resulting formulæ for $ydw - zdv$, &c., x, y, z are considered as denoting the coordinates of the element in regard to axes fixed in the body and moveable with it, but which at the moment under consideration coincide in position with the axes fixed in space. The expressions for $\Sigma(ydw - zdv)\,dm$ involve the integrals $A = \int(y^2 + z^2)\,dm$, &c., where the coordinates refer to axes fixed in the body; and if these are taken to be principal axes, the expression for $\Sigma(ydw - zdv)\,dm$ is $= Adp + (C - B)\,qrdt$, which gives the three equations

$$Adp + (C - B)\,qrdt = 0,$$
$$Bdq + (A - C)\,rpdt = 0,$$
$$Cdr + (B - A)\,pqdt = 0,$$

already referred to as Euler's equations.

170. Next, as regards the determination of the position in space of the principal axes: if about the fixed point we describe a sphere meeting the principal axes in x_1, y_1, z_1, and if P be an arbitrary point on the sphere and PQ an arbitrary direction through P, the quantities used to determine the positions of x_1, y_1, z_1 are the distances x_1P, y_1P, z_1P $(= l, m, n)$ and the inclinations x_1PQ, y_1PQ, z_1PQ $(= \lambda, \mu, \nu)$ of these arcs to the fixed direction PQ (it is to be observed that the sines and cosines of the differences of λ, μ, ν are given functions of the sines and cosines of l, m, n, and, moreover, that $\cos^2 l + \cos^2 m + \cos^2 n = 1$, so that the number of independent parameters is three). The above is Euler's definition; but if we consider a set of axes fixed in space meeting the sphere in the points X, Y, Z, then if the point X be taken for P, and the arc XY for PQ, it is at once seen that the angles used for determining the relative positions of the two sets of axes are the same as in Euler's memoir "Formulæ Generales &c.," 1775 (*ante*, No. 132), where the formulæ for this transformation of coordinates are considered apart from the dynamical theory.

Euler expresses the quantities p, q, r in terms of an auxiliary variable u, which is such that $du = pqrdt$; p, q, r are at once obtained in terms of u, and then t is given in terms of u by a quadrature. Euler employs also an auxiliary angle U, given in terms of u by a quadrature. And he obtains finite algebraical expressions in u, $\cos U$, $\sin U$ for the cosines or sines of l, m, n; s (the angular distance IP, if I denote the point in which the instantaneous axis meets the sphere), ϕ (the angle IPQ) and $\lambda - \phi$, $\mu - \phi$, $\nu - \phi$. The formulæ, although complicated, are extremely elegant, and they appear to have been altogether overlooked by subsequent writers.

171. Euler remarks, however, that the complexity of his solution is owing to the circumstance that the fixed point P is left arbitrary, and that they may be simplified by taking this point so that a certain relation $G-\mathfrak{D}^2=0$ may be satisfied between the constants of the solution; and he gives the far more simple formulæ corresponding to this assumption. This amounts to taking the point P *in the normal of the invariable plane*, and the resulting formulæ are in fact identical with the ordinary formulæ for the solution of the problem. The expression *invariable plane* is not used by Euler, and seems to have been first employed in Lagrange's memoir "Essai sur le Problème de Trois Corps," *Prix de l'Acad. de Berlin*, t. IX. (1772): the theory in reference to the solar system has been studied by Laplace, Poinsot, and others.

172. Lagrange's solution in the memoir of 1773 is substantially the same with that in the *Mécanique Analytique*. Only he starts from the integral equations of areas and of *Vis Viva*, but in the last-mentioned work from the equations of motion as expressed in the Lagrangian form by means of the *Vis Viva* function $T\left(=\frac{1}{2}\Sigma(x'^2+y'^2+z'^2)\,dm\right)$. The distinctive feature is that he does not refer the body to the principal axes but to any rectangular axes whatever fixed in the body: the expression for T consequently is $T=\frac{1}{2}(A, B, C, F, G, H \between p, q, r)^2$, instead of the more simple form

$$T=\tfrac{1}{2}(Ap^2+Bq^2+Cr^2),$$

which it assumes when the body is referred to its principal axes. And Lagrange effects the integration as well with this more general form of T, as without the simplification afforded by the invariable plane; the employment, however, of the more general form of T seems an unnecessary complication of the problem, and the formulæ are not worked out nearly so completely as in Euler's memoir. It may be observed that p, q, r are expressed as functions of the instantaneous velocity $\omega\,(=\sqrt{p^2+q^2+r^2})$, and thence t obtained by a quadrature as a function of ω.

173. Poisson's Memoir of 1809.—The problem is only treated incidentally for the sake of obtaining the expressions for the variations of the arbitrary constants; the results (depending, as already remarked, on the consideration of the invariable plane) are obtained and exhibited in a very compact form, and they have served as a basis for further developments; it will be proper to refer to them somewhat particularly. The Eulerian equations give, in the first place, the integrals

$$Ap^2+Bq^2+Cr^2=h,$$
$$A^2p^2+B^2q^2+C^2r^2=k^2;$$

and then by means of these, p, q being expressed in terms of r, we have t in terms of r by a quadrature.

174. The position in space of the principal axes is determined by referring them, by means of the angles θ, ϕ, τ, to axes Ox, Oy, Oz fixed in space; if, to fix the ideas, we call the plane of xy the ecliptic (Ox being the origin of longitudes), and the plane of the two principal axes x_1y_1 the equator, then we have

θ, the longitude of node,
ϕ, the inclination,
τ, the hour-angle, or angular distance of Ox_1 from the node,

and α, β, γ the cosine inclinations of Ox_1, α', β', γ' those of Oy_1, and α'', β'', γ'' those of Oz_1 to Ox, Oy, Oz respectively are given functions of θ, ϕ, τ (the values of α'', β'', γ'' depending upon θ, ϕ only), we have

$$pdt = \sin\tau \sin\phi\, d\theta + \cos\tau\, d\phi,$$
$$qdt = \cos\tau \sin\phi\, d\theta - \sin\tau\, d\phi,$$
$$rdt = d\tau + \cos\phi\, d\theta.$$

175. A set of integrals is

$$Ap\alpha + Bq\beta + Cr\gamma = k\cos\lambda,$$
$$Ap\alpha' + Bq\beta' + Cr\gamma' = k\cos\mu,$$
$$Ap\alpha'' + Bq\beta'' + Cr\gamma'' = k\cos\nu,$$

equivalent to two independent equations, the values of λ, μ, ν being such that $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$; but the position of the axis of z may be chosen so that the values on the right-hand sides become 0, 0, k; the axis of z is then perpendicular to the *invariable plane*, the condition in question serving as a definition. And the three equations then give

$$Ap = k\alpha'', \quad Bq = k\beta'', \quad Cr = k\gamma'',$$

where the values of α'', β'', γ'' in fact are

$$\alpha'' = \sin\tau \sin\phi, \quad \beta'' = \cos\tau \sin\phi, \quad \gamma'' = \cos\phi;$$

we have thus τ, ϕ in terms of r. And the equation $rdt = d\tau + \cos\phi d\theta$ then leads to the value of $d\theta$, or θ is determined as a function of r by a quadrature.

176. The constants of integration are h, k, l (the constant attached to t), g (the constant attached to θ); and two constants, say α the longitude of the node, and γ the inclination of the invariable plane in reference to an arbitrary plane of xy and origin x of longitudes therein. I remark in passing that Poisson obtains an elegant set of formulæ for the variations of the constants h, k, g, l, α, γ, not actually in the canonical form, but which may by a slight change be reduced to it.

177. Legendre considers the problem of Rotation in the *Exercices de Calcul Intégral*, t. II. (1817), and the *Théorie des Fonctions Elliptiques*, t. I. pp. 366—410 (1826). He does not employ the quantities p, q, r, but obtains *de novo* a set of differential equations of the second order involving the angles which determine the position of the principal axes with regard to the axes fixed in space: these angles are in fact (calling the plane of the fixed axes x, y the ecliptic) the longitude and latitude of *one* of the principal axes, and the azimuth from the meridian through such principal axis of an arbitrary axis fixed in the body and moveable with it. The solution is developed by means of the elliptic *integrals*. From the peculiar choice of variables there would, it would seem, be considerable labour in comparing the results with those of other writers, and there would be but little use in doing so.

178. Poinsot's "Théorie Nouvelle de la Rotation des Corps."—The 'Extrait' of the memoir was published in 1834, but the memoir itself was not published *in extenso* until the year 1851. The 'Extrait' contains, however, not only the fundamental theorem of the representation of the motion of a body about a fixed point by means of the momental ellipsoid rolling on a fixed tangent plane, but also the geometrical and mechanical reasonings by which this theorem is demonstrated; it establishes also the notions of the Poloid and Serpoloid curves; and it contains incidentally, and without any developments, a very important remark as to the representation of the motion by means of the rolling and sliding motion of an elliptic cone. The whole theory (including that of the last-mentioned representation of the motion) is in the memoir itself also analytically developed, but without the introduction of the elliptic and Jacobian functions: to form a complete theory, it would be necessary to incorporate the memoir with that of Jacobi.

179. The following is an outline of the 'Extrait':

The instantaneous motion of a body about a fixed point is a motion of rotation about an axis (the instantaneous axis); and hence the finite motion is as if there were a cone fixed in the body which rolls (without sliding) upon another cone fixed in space.

The instantaneous motion of a body in space is a motion of rotation about an axis combined with a translation in the direction of this axis: this remark is hardly required for Poinsot's purpose, and he does not further develope the theory of the motion of a body in space. The effect of a couple in a plane perpendicular to a principal axis is to turn the body about this axis with an angular velocity proportional to the moment of the couple divided by the moment of inertia about the axis.

And hence by resolving any couple into couples perpendicular to the principal axes, the effect of such couple may be calculated; but more simply by means of the central ellipsoid (momental ellipsoid $a^2x^2+b^2y^2+c^2z^2=k^4$, if $A, B, C = Ma^2, Mb^2, Mc^2$), viz., if the body is acted on by a couple in a tangent plane of the ellipsoid, the instantaneous axis passes through the point of contact; and reciprocally, given the instantaneous axis, the couple must act in the tangent plane.

180. Considering now a body rotating about a fixed point, and taking as the plane of the couple of impulsion a tangent plane of the ellipsoid, the instantaneous axis is initially the diameter through the point of contact; the centrifugal forces arising from the rotation produce however an accelerating couple, the effect whereof is continually to impress on the body a rotation which is compounded with that about the instantaneous axis, and thus to cause a variation in the position of this axis and in the angular velocity round it. The axis of the accelerating couple is always situate in the plane of the couple of impulsion.

181. Hence also

1°. Throughout the motion the angular velocity is proportional to the length of the instantaneous axis considered as a radius vector of the ellipsoid.

2°. The distance of the tangent plane from the centre is constant; that is, the tangent plane to the ellipsoid at the extremity of the instantaneous axis is a fixed plane in space.

Or, what is the same thing, the motion is such that the ellipsoid remains always in contact with a fixed plane, viz., the body revolves round the radius vector through the point of contact, the angular velocity being always proportional to the length of this radius vector.

It is right to remark that in Poinsot's theory the distance of this plane from the centre depends on the arbitrarily assumed magnitude of the central ellipsoid; the parallel plane through the centre is the invariable plane of the motion.

182. The motion is best understood by the consideration that it is implied in the theorem that the pole of the instantaneous axis describes on the ellipsoid a certain curve, "the Poloid," which is the locus of all the points for which the perpendicular on the tangent plane has a given constant value (the curve in question is easily seen to be the intersection of the ellipsoid by a concentric cone of the second order); and that the instantaneous axis describes on the fixed tangent plane a curve called "the Serpoloid," which is the locus of the points with which the several points of the poloid come successively in contact with the tangent plane, and is a species of undulating curve, viz., the radius vector as it moves through the angles θ to $\theta_1 + 2\Pi$, $\theta_1 + 2\Pi$ to $\theta_1 + 4\Pi$, &c. assumes continually the same series of values. This is in fact evident from the mode of generation; and it is moreover clear that the serpoloid is an algebraical or else a transcendental curve according as Π is or is not commensurable with π.

{Treating the poloid and serpoloid as cones instead of curves, the motion of the body is the rolling motion of the former upon the latter cone, which agrees with a previous remark.}

There is a very interesting special case where the perpendicular distance from the tangent plane is equal to the mean axis of the ellipse.

183. Poinsot remarks that the motion is such that {considering the plane of the couple of impulsion as drawn through the centre of the ellipsoid} the section of the ellipsoid is an ellipse variable in form but of constant magnitude, and that this leads to a new representation of the motion, viz., that it may be regarded as the *motion of an elliptic cone which rolls on the plane of the couple* {the invariable plane} *with a variable velocity, and which slides on this plane with a uniform velocity.*

184. The theory of the last-mentioned cone, say the "rolling and sliding cone," is developed in the memoir, *Liouville*, t. XVI. p. 303, in the chapter entitled "Nouvelle Image de la Rotation des Corps." If a, b, c signify as before (viz., A, B, $C = Ma^2$, Mb^2, Mc^2), and if h be the distance of the centre from Poinsot's fixed tangent plane ($h < a > c$), then the invariable axis describes in the body a cone the equation whereof is

$$(a^2 - h^2)\, x^2 + (b^2 - h^2)\, y^2 + (c^2 - h^2)\, z^2 = 0\,;$$

the cone reciprocal to this, viz. the cone the equation whereof is

$$\frac{x^2}{a^2 - h^2} + \frac{y^2}{b^2 - h^2} + \frac{z^2}{c^2 - h^2} = 0,$$

is the "rolling and sliding cone." The generating line OT of this cone is perpendicular to the plane of the instantaneous axis OI, and of the invariable axis OG; and the analytical expressions for the rolling and sliding velocities follow from the geometrical consideration that the motion at any instant is a rotation round the instantaneous axis OI: that for the sliding velocity is the instantaneous angular velocity into the cosine of the angle IOG, which is in fact constant; that for the rolling velocity is given, but a further explanation of the geometrical signification is perhaps desirable.

185. I may in this place again refer to Cohen's memoir "On the Differential Coefficients and Determinants of Lines &c." (1862), the latter part of which contains an application of the method to finding Euler's equations for the motion of a rotating body.

186. Rueb in his memoir (1834) first applied the elliptic and Jacobian functions to the present problem. Starting from the equations

$$A\,p^2 + B\,q^2 + C\,r^2 = h,$$
$$A^2p^2 + B^2q^2 + C^2r^2 = l^2, \text{ (1)}$$

and

$$dt = \frac{-B\,dq}{(A-C)\,rp},$$

it is easy to perceive that by assuming $q =$ a proper multiple of $\sin\xi$, the expression for dt takes the form $n\,dt = \dfrac{d\xi}{\sqrt{1-k^2\sin^2\xi}}$, so that writing $\xi = \operatorname{am} u$, the integral equation is $nt - \epsilon = u$, or u is an angle varying directly as the time (and corresponding to the mean longitude, or, if we please, to the mean anomaly in the problem of elliptic motion). And then p, q, r are expressed as elliptic functions of u. The value of the modulus k, and that of n ($nt - \epsilon = u$ *ut suprà*) are

$$n = \sqrt{\frac{(B-C)(-l^2+Ah)}{ABC}},$$

$$k = \sqrt{\frac{(A-B)(l^2-Ch)}{ABC}},$$

and then

$$\left.\begin{aligned} p &= \pm\sqrt{\frac{l^2-Ch}{A\,.\,A-C}}\cos\operatorname{am} u, \\ q &= -\sqrt{\frac{l^2-Ch}{B\,.\,B-C}}\sin\operatorname{am} u, \\ r &= \sqrt{\frac{-l^2+Ah}{C\,.\,A-C}}\,\Delta\operatorname{am} u. \end{aligned}\right\}$$

[1] l is Poisson's k, the constant of the principal area; it is the letter afterwards used by Jacobi; Rueb's letter is g. In quoting (*infrà*) the expressions for p, q, r, I have given them with Rueb's signs, but it would be too long to explain how the signs of the radicals are determined.

187. Substituting for p, q, r their values in terms of u, we have $d\theta$, and thence θ (the longitude of the node of the equator on the invariable plane) in the form

$$\theta = -\frac{l}{An}u + i\Pi(u, ia) \quad (i = \sqrt{-1}),$$

which by Jacobi's formulæ for the transformation of the elliptic integral of the third class becomes

$$\theta = \left(-\frac{l}{An} + iZ(ai)\right)u + \tfrac{1}{2}i\log\frac{\Theta(u-ai)}{\Theta(u+ai)},$$

which Rueb reduces to the real form

$$\theta = -n'u + \tan^{-1} W,$$

W being given in the form of a fraction, the numerator and denominator whereof are series in multiple sines and multiple cosines respectively of $\frac{\pi u}{2K}$.

188. Rueb investigates also the values in terms of u of the cosine inclinations of the instantaneous axis to the axes fixed in space; and he obtains a very elegant expression for the angle ζ, which is the angular distance from x of the projection on the plane of xy (the invariable plane) of the instantaneous axis; viz., this is

$$\zeta = \tan^{-1}\left(-\frac{ABn}{(A-B)\,l}\,\frac{\Delta\,\mathrm{am}\,u}{\sin\mathrm{am}\,u\cos\mathrm{am}\,u}\right) - \theta,$$

and there is throughout a careful discussion of the geometrical signification of the results.

189. The advance made was enormous; the result is that we have in terms of the time $\sin\tau\sin\phi$, $\cos\tau\sin\phi$, $\cos\phi$ (the cosine inclinations of the invariable axis to the principal axes), and also θ, the longitude of the node. The cosine inclinations of the axes of x and y to the principal axes could of course be obtained from these, but they would be of a very complicated and unmanageable form; *the reason of this is the presence in the expression for θ of the non-periodic term $-n'u$.* It will presently be seen how this difficulty was got over by Jacobi.

190. Briot's paper of 1842 contains an analytical demonstration of some of the theorems given in the 'Extrait' of Poinsot's memoir of 1834.

191. In Maccullagh's Lectures of 1844 (see Haughton, 1849; Maccullagh, 1847) the problem of the rotation of a solid body is treated in a mode somewhat similar to that of Poinsot; only the ellipsoid of gyration $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ if } A, B, C = Ma^2, Mb^2, Mc^2\right)$ is used instead of *the momental ellipsoid.* Thus, reciprocal to the poloid curve on the momental ellipsoid we have on the ellipsoid of gyration a curve all the points whereof are equidistant from the centre; such curve is of course the intersection of the ellipsoid by a concentric sphere, that is, it is a spherical conic; and the points of this spherical conic come successively to coincide with a fixed point on the invariable axis. This is a theorem stated in Art. VI. of Haughton's memoir: it may be added

that the several tangent planes of the ellipsoid of gyration at the points of the spherical conic as they come to coincide with the fixed point, form a cone reciprocal to Poinsot's serpoloid cone. It is clear that every theorem in the one theory has its reciprocal in the other theory; I have not particularly examined as to how far the reciprocal theorems have been stated in the two theories.

192. Cayley, "On the Motion of Rotation of a Solid Body" (1843).—The object was to apply to the solution of the problem Rodrigues' formulæ for the resultant rotation; viz., if the principal axes, considered as originally coinciding with the axes of x, y, z, can be brought into their actual position at the end of the time t by a rotation θ round an axis inclined at angles f, g, h to the axes of x, y, z, and if $\lambda = \tan\frac{1}{2}\theta\cos f$, $\mu = \tan\frac{1}{2}\theta\cos g$, $\nu = \tan\frac{1}{2}\theta\cos h$, then the principal axes are referred to the axes fixed in space by means of the quantities λ, μ, ν. And these are to be obtained from the equations

$$\begin{aligned}\kappa\, p dt &= 2(\quad d\lambda + \nu d\mu - \mu d\nu),\\ \kappa\, q dt &= 2(-\nu d\lambda + \quad d\mu + \lambda d\nu),\\ \kappa\, r dt &= 2(\quad \mu d\lambda - \lambda d\mu + \quad d\nu),\end{aligned}$$

where $\kappa = 1 + \lambda^2 + \mu^2 + \nu^2$, and p, q, r are to be considered as given functions of t, or of other the variable selected as the independent one. But for effecting the integration it was found necessary to take the axis of z as the invariable axis.

193. The solution by Jacobi, § 27 of the memoir "Theoria Novi Multiplicatoris" (1845), is given as an application of the general theory, the author remarking that, as well in this question as in the problem of the two fixed centres, he purposely employed a somewhat inartificial analysis, in order to show that the principle (of the Ultimate Multiplier) would lead to the result without any special artifices. The principal axes are referred to the axes fixed in space by the ordinary three angles (here called q_1, q_2, q_3), and the solution is carried so far as to give the integral equations, without any reduction of the integrals contained in them to elliptic integrals. The solution is, however, in so far remarkable that the integrations are effected without the aid of the invariable plane.

194. Cayley, "On the Rotation of a Solid Body &c." (1846).—It appeared desirable to obtain the solution by means of the quantities λ, μ, ν, *without the assistance of the invariable plane*, and Jacobi's discovery of the theorem of the Ultimate Multiplier induced me to resume the problem, and at least attempt to bring it so far as to obtain a differential equation of the first order between two variables only, the multiplier of which could be obtained theoretically by Jacobi's discovery. The choice of two new variables to which the equations of the problem led me, enabled me to effect this in a simple manner; and the differential equation which I finally obtained turned out to be integrable *per se*, so that the laborious process of finding the multiplier became unnecessary.

195. The new variables Ω, v have the following geometrical significations: $\Omega = l\tan\frac{1}{2}\theta\cos I$, where l is the principal moment $(A^2p^2 + B^2q^2 + C^2r^2 = l^2)$, θ (as before) the angle of resultant

rotation, and I is the inclination of the resultant axis to the invariable axis; and $v = l^2 \cos^2 \frac{1}{2} J$, where if we imagine a line AQ having the same position relatively to the axes fixed in space that the invariable axis has to the principal axes of the body, then J is the inclination of this line to the invariable axis. It is found that p, q, r are functions of v only, whereas λ, μ, ν contain besides the variable Ω. In obtaining these relations, there occurs a singular relation $\Omega^2 = \kappa v - l^2$, which may also be written $1 + \tan^2 \frac{1}{2}\theta \cos^2 I = \sec^2 \frac{1}{2}\theta \cos^2 \frac{1}{2} J$, where the geometrical significations of the quantities I, J have just been explained. The final results are that the time t, and the arc $\tan^{-1} \frac{\Omega}{l}$ are each of them expressible as the integrals of certain algebraical functions of v. There might be some interest in comparing the results with those of Euler's memoir of 1758, where the principal axes are also referred to an arbitrary system of axes fixed in space; but I was not then acquainted with Euler's memoir.

The concluding part of the paper relates to the determination of the variations of the constants in the disturbed problem.

196. Cayley, "Note on the Rotation of a Solid of Revolution" (1849), shows the simplification produced in the formulæ of the last-mentioned memoir in the case where two of the moments of inertia are equal, say $A = B$.

197. Jacobi's final solution of the problem of Rotation was given without demonstration in the letter to the Academy of Sciences at Paris; the demonstration is added in the memoir, *Crelle*, t. XXXIX. (1849). The fundamental idea consists in taking in the invariable plane, instead of the fixed axes xy, a set of axes xy revolving with uniform velocity, such that the angular distance of the axis of x from its initial position is precisely $= -n'u$; and consequently if θ' be the longitude of the node of the equator on the invariable plane, measured from the moveable axis of x as the origin of longitude, we have

$$\theta' = \theta + n'u = \frac{1}{2i} \log \frac{\Theta(u + ia)}{\Theta(u - ia)}, \quad (i = \sqrt{-1});$$

and in consequence of this form of the expression for θ' $\left(= \frac{1}{2i} \text{ into a logarithmic function}\right)$ in passing to the trigonometrical functions $\sin\theta'$, $\cos\theta'$ the logarithm disappears altogether; and we have in a simple form the expressions for the actual functions $\sin\theta'$, $\cos\theta'$, through which θ' enters into the formulæ, and thus, Jacobi remarks, the barrier is cleared which stands in the way when the expression of an angle is reduced to an elliptic integral of the third class.

198. For the better expression of the results, Jacobi joins to the functions H, Θ, considered in the "Fundamenta Nova," the functions $\Theta_1 u = \Theta(K - u)$, $H_1(u) = H(K - u)$; so that

$$\sqrt{k} \sin \operatorname{am} u = \frac{Hu}{\Theta u}, \quad \sqrt{\frac{k}{k'}} \cos \operatorname{am} u = \frac{H_1 u}{\Theta u}, \quad \frac{1}{\sqrt{k'}} \Delta \operatorname{am} u = \frac{\Theta_1 u}{\Theta u},$$

and then considering the cosine inclinations of the principal axes to the invariable axis and the revolving axes in the invariable plane, these are all fractions which,

neglecting constant factors, have the common denominator Θu; α'', β'', γ'' (as shown by Rueb's formulæ) have the numerators $H_1 u$, Hu, and $\Theta_1 u$ respectively; and α, α' have the numerators $H(u+ia) \pm H(u-ia)$, β, β' the numerators $H_1(u-ia) \pm H_1(u+ia)$, γ, γ' the numerators $\Theta(u+ia) \pm \Theta(u-ia)$ respectively: there are also expressions of a similar form for the angular velocities about the axis of x and y; that about the axis of z (the invariable axis) having, as was known, the constant value $\frac{h}{l}$. The memoir is also very valuable analytically, as completing the systems of formulæ given in the *Fundamenta Nova* in reference to elliptic integrals of the third class.

199. It is worth noticing how the results connect themselves with Poinsot's theorem of the rolling and sliding cone: the velocity of the rolling motion depends only upon the position, on the cone, of the line of contact, so that the same series of velocities recur after any number of complete revolutions of the cone; that is, the total angle described by the line of contact in consequence of the rolling motion, consists of a part varying directly with the time (or say varying as u) and a periodic part; the former part combines with the similar term arising from the sliding motion, and the two together give Jacobi's term $n'u$.

200. Somoff's memoir (1851), written after Jacobi's Note in the *Comptes Rendus*, but before the appearance of the memoir in *Crelle*, gives the demonstration of the greater part of Jacobi's results.

201. Booth's *Theory of Elliptic Integrals &c.* (1851) (contemporaneous with the publication of Poinsot's memoir of 1834) contains various interesting analytical developments, and, as an interpretation of them, the author obtains (p. 93) the theorem of the rolling and sliding cone. The investigations involve the elliptic integrals, but not the elliptic or Jacobian functions.

202. Richelot's two Notes (*Crelle*, tt. XLII. and XLIV.) relate to the solution of the problem of rotation given in his memoir "Eine neue Lösung &c." (1851). This is an application of Jacobi's theorem for the integration of a system of dynamical equations by means of the principal function S (see my "Report" of 1857, art. 34). Retaining Richelot's letters ϕ, ψ, θ, which signify

ψ, the longitude of the node,

θ, the inclination,

ϕ, the hour-angle,

the question is to find a complete solution of the partial differential equation

$$\begin{aligned} 0 = & \frac{1}{2A}\left\{\left(\frac{dV}{d\phi}\cos\theta + \frac{dV}{d\psi}\right)^2 \frac{\sin\phi}{\sin\theta} - \frac{dV}{d\theta}\cos\phi\right\}^2 \\ & + \frac{1}{2B}\left\{\left(\frac{dV}{d\phi}\cos\theta + \frac{dV}{d\psi}\right)\frac{\cos\phi}{\sin\theta} + \frac{dV}{d\theta}\sin\phi\right\}^2 \\ & + \frac{1}{2C}\left(\frac{dV}{d\phi}\right)^2 + \frac{dV}{dt}; \end{aligned}$$

that is, a solution involving (besides the constant attached to V by a mere addition) three arbitrary constants; these are t_1, ψ_1, ρ. Writing in the first place $V = W + tt_1 + \psi\psi_1$, the resulting equation for W may be satisfied by taking W, a function of ϕ and θ, without ψ or t; and it is sufficient to have a solution involving only a single arbitrary constant. This leads to a solution which is as follows:

$$V = tt_1 + \psi\psi_1 - \psi_1 \tan^{-1}\frac{\theta_1}{\sqrt{\rho^2 - \psi_1^2 - \theta_1^2}} + \rho\left\{\tan^{-1}\frac{\psi_1\theta_1}{\rho\sqrt{\rho^2 - \psi_1^2 - \theta_1^2}} + \tan^{-1}\frac{\phi_1\theta_1}{\rho\sqrt{\rho^2 - \phi_1^2 - \theta_1^2}}\right\}$$

$$-\left(\frac{\rho^2}{C} + 2t_1\right)\int\frac{\phi_1^2 d\phi_1}{(\rho^2 - \phi_1^2)\sqrt{\left(\frac{1}{B} - \frac{1}{C}\right)\left(\phi_1^2 - \frac{\rho^2}{B} + 2t_1\right)\left(\left(\frac{1}{C} - \frac{1}{A}\right)\phi_1^2 + \frac{\rho^2}{A} + 2t_1\right)}},$$

where ϕ_1 and θ_1 are certain given functions of t_1, ψ_1, ρ, and of θ and ϕ. The solution of the dynamical problem is then obtained by putting the differential coefficients $\frac{dV}{dt_1}$, $\frac{dV}{d\psi_1}$, $\frac{dV}{d\rho}$ equal to arbitrary constants L, α, G respectively; the results are somewhat more simple than might be expected from the very complicated form of the function V. The foregoing results (although not by themselves very intelligible) will give an idea of the form in which the analytical solution in the first instance presents itself.

203. Richelot proceeds to remark that the solution in question, and the resulting integral equations of the problem, may be simplified in a peculiar manner by the method which he calls "the integration by the spherical triangle." For this purpose he introduces a spherical triangle, the sides and angles whereof are

$$\nu,\ \lambda,\ \mu;\ \mathrm{N},\ \Lambda,\ \mathrm{M},$$

and then assuming

$$\mathrm{N}\ \text{constant},\ \mathrm{M} = \pi - \theta$$

$$\left(\frac{1}{C} - \frac{1}{A}\right)\sin^2\left(\phi + \nu\right)\sin^2\Lambda + \left(\frac{1}{C} - \frac{1}{B}\right)\cos^2(\phi + \nu)\sin^2\Lambda = \frac{1}{C} + \frac{2t_1}{\rho^2},$$

where ρ and t_1 are constant, the solution is

$$V = t_1 t - \rho\,(\psi - \lambda)\cos\mathrm{N} - \rho\mathrm{M} + \rho\int\cos\Lambda\, d\,(\phi + \nu);$$

and that this expression leads to all the results almost without calculation.

204. I have quoted the foregoing results from the Note (*Crelle*, t. XLII.), having seen, but without having studied, the Memoir itself: the results appear very interesting and valuable ones; but they seem to require a more complete geometrical development than they have received in the Memoir; and I am not able to bring them into connexion with the other researches on the subject.

205. The solution, § 3 of Donkin's memoir "On a Class of Differential Equations &c." (part I. 1854), is given as an illustration of the general theory to which the memoir relates; it contains, however, some interesting geometrical developments in regard to the case $(A = B)$ of two equal moments of inertia. I have not compared the results with those in my Note of 1849.

206. The solution of the rotation problem, § 66 of Jacobi's memoir "Nova Methodus &c." (1862), has for its object to show the complete analogy which exists between this problem and the problem of a body attracted to a fixed centre. The section is in fact headed "Solutio simultanea problematis de motu puncti versus centrum attracti atque problematis de rotatione &c."; and Jacobi, after noticing that Poisson, in his memoir of 1816 (*Mém. de l'Inst.* t. I.), had shown that the expressions for the variations of the elements in the two problems could be investigated by a common analysis, remarks, "Sed ipsa problemata duo imperturbata hic primum, quantum credo, amplexus sum." The solution is in fact as follows:—Suppose that in the one problem the position of the point in space, and in the other problem the position of the body in regard to the fixed axes, is determined in any manner by the quantities q_1, q_2, q_3. Let

$$\frac{dq_1}{dt} = q_1', \quad \frac{dq_2}{dt} = q_2', \quad \frac{dq_3}{dt} = q_3',$$

and expressing the *Vis Viva* function T in terms of $q_1, q_2, q_3, q_1', q_2', q_3'$, let

$$\frac{dT}{dq_1'} = p_1, \quad \frac{dT}{dq_2'} = p_2, \quad \frac{dT}{dq_3'} = p_3,$$

and let H be the value of T expressed in terms of $q_1, q_2, q_3, p_1, p_2, p_3$, so that $H = a$ is the integral of *Vis Viva* (this is merely the transformation to the Hamiltonian form). And let $H_1 = a_1$, $\phi = a_1'$, $\psi = a_1''$ be the three integrals of areas (H, H_1, ϕ, ψ are functions of the variables only, not containing the arbitrary constants a, a_1, a_1', a_1''). Then, expressing

$$H,\ H_1,\ H_2 (= \sqrt{H_1^2 + \phi^2 + \psi^2})$$

in terms of $p_1, p_2, p_3, q_1, q_2, q_3$, and by means of the equations

$$H = a, \quad H_1 = a_1, \quad H_2 = a_2$$

(where $a_2 = \sqrt{a_1^2 + a_1'^2 + a_1''^2}$) expressing p_1, p_2, p_3 in terms of q_1, q_2, q_3, we have $p_1 dq_1 + p_2 dq_2 + p_3 dq_3$ a complete differential; and putting

$$\int \left(p_1 dq_1 + p_2 dq_2 + p_3 dq_3\right) = V,$$

then (a, a_1, a_2, b, b_1, b_2 being arbitrary constants) we have

$$H = a, \quad H_1 = a_1, \quad H_2 = a_2,$$

$$\frac{dV}{da} = \int \left(\frac{dp_1}{da} dq_1 + \frac{dp_2}{da} dq_2 + \frac{dp_3}{da} dq_3\right) = t + b,$$

$$\frac{dV}{da_1} = \int \left(\frac{dp_1}{da_1} dq_1 + \frac{dp_2}{da_1} dq_2 + \frac{dp_3}{da_1} dq_3\right) = b_1,$$

$$\frac{dV}{da_2} = \int \left(\frac{dp_1}{da_2} dq_1 + \frac{dp_2}{da_2} dq_2 + \frac{dp_3}{da_2} dq_3\right) = b_2,$$

as the complete integrals of either problem, the last three of them being the final integrals.

And it is added that if in either problem we have $H+\Omega$ instead of H, the expressions for the variations of the elements assume the canonical forms $\frac{da}{dt}=-\frac{d\Omega}{db}$, $\frac{db}{dt}=\frac{d\Omega}{da}$, &c.

The solution is not further developed as regards the rotation problem, but it is so (§ 67) as regards the other problem.

207. It must, I think, be considered that a comprehensive memoir on the Problem of Rotation, embracing and incorporating all that has been done on the subject, is greatly needed.

Kinematics of a solid body. Article Nos. 208 to 215.

208. The general theorem in regard to the infinitesimal motions (rotations and translations) of a solid body is that these are compounded and resolved in the same way as if they were single forces and couples respectively. Thus any infinitesimal rotations and translations are resolvable into a rotation and a translation; the rotation is given as to its magnitude and as to the direction of its axis, but not as to the position of the axis (which may be any line in the given direction): the magnitude and direction of the translation depend on the assumed position of the axis of rotation; in particular this may be taken so that the translation shall be in the direction of the axis of rotation; and the magnitude of the rotation is then a minimum. I remark that the theorem as above stated presupposes the establishment of the theory of couples (of forces) which was first accomplished by Poinsot in his '*Élémens de Statique,*' 1st edit. 1804; it must have been, the whole or nearly the whole of it, familiar to Chasles at the date of his paper of 1830 next referred to (see also Note XXXIV. of the *Aperçu Historique,* 1837); and it is nearly the whole of it stated in the 'Extrait' of Poinsot's memoir on Rotation, 1834.

209. Chasles' paper in the *Bulletin Univ. des Sciences* for 1830.—The corresponding theorem is here given for the finite motions (rotations and translations) of a solid body as follows: viz. if any finite displacement be given to a free solid body in space, there exists always in the body a certain indefinite line which after the displacement remains in its original situation. The theorem is deduced from a more general one relating to two similar bodies. It may be otherwise stated thus: viz., any motions may be represented by a translation and a rotation (the order of the two being indifferent); the rotation is given as regards its magnitude and the direction of its axis, but not as to the position of the axis (which may be any line in the given direction); the magnitude and direction of the translation depend on the assumed position of the axis of rotation; in particular this may be taken so that the translation shall be in the direction of the axis of rotation; the magnitude of the translation is then a minimum.

It may be noticed that a translation may be represented as a couple of rotations; that is, two equal and opposite rotations about parallel axes.

210. It is part of the general theorem that any number of rotations about axes passing through one and the same point may be compounded into a single rotation about an axis through that point; this is, in fact, the theory of the "Resultant Axis" developed in Euler's and Lexell's memoirs of 1775.

211. The following properties are also given, viz., considering two similar solid bodies (or in particular any two positions of a solid body) and joining the corresponding points, the lines which pass through one and the same point form a cone of the second order; and the points of either body form on this cone a curve of the third order (skew cubic). And, reciprocally, the lines, intersections of corresponding planes, which lie in one and the same plane envelope a conic, and such planes of either body envelope a developable surface, which is such that any one of these planes meets it in a conic {or, what is the same thing, the planes envelope a developable surface of the fourth order}.

And also, given in space two equal bodies situate in any manner in respect to each other, then joining the points of the first body to the homologous points of the second body, the middle points of these lines form a body capable of an infinitesimal motion, each point of it along the line on which the same is situate.

212. The entire theory, as well of the infinitesimal as of the finite motions of a solid body, is carefully and successfully treated in Rodrigues' memoir "Des lois géométriques &c." (1840). It may be remarked that for the purpose of compounding together any rotations and translations, each rotation may be resolved into a rotation about a parallel axis and a couple of rotations, that is, a translation; the rotations are thus converted into rotations about axes through one and the same point, and these give rise to a single resultant rotation given as to its magnitude and the direction of the axis, but not as to the position of the axis (which is an arbitrary line in the given direction); the translations are then compounded together into a single translation, and finally the position of the axis of rotation is so determined that the translation shall be in the direction of this axis; the entire system is thus compounded (in accordance with Chasles' theorem) into a rotation and a translation in the direction of the axis of the rotation. The problem of the composition depends therefore on the composition of rotations about axes through one and the same point; that is, upon Euler's and Lexell's theory of the resultant axis. But, as already noticed, the analytical theory of the resultant axis was perfected by Rodrigues in the present memoir (see *ante*, 'Transformation of Coordinates,' Nos. 139—141, as to this memoir and the quaternion representation of the formulæ contained in it.

213. It was remarked in Poinsot's memoir of 1834 that every continuous motion of a solid body about a fixed point is the motion of a cone fixed in the body rolling upon another cone fixed in space. The corresponding theorem for the motion of a solid body in space is given

Cayley, "On the Geometrical Representation &c." (1846): viz. premising that a skew surface is said to be "deformed" if, considering the elements between consecutive generating lines as rigid, these elements be made in any manner to turn round and

slide along the successive generating lines:—and that two skew surfaces can be made to roll and slide one upon the other, only if the one is a deformation of the other—and that then the rolling and sliding motions are perfectly determined—and that such a motion may be said to be a "gliding" motion: the theorem is that any motion whatever of a solid body in space may be represented as the gliding motion of one skew surface upon another skew surface of which it is the deformation.

214. The same paper contains the enunciation and analytical proof of the following theorem supplementary to that of Poinsot just referred to, viz. that when the motion of a solid body round a fixed point is represented as the rolling motion of one cone on another, then "the angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of the curvatures of the two cones at any point in this line is to the reciprocal of the distance of the point from the vertex."

215. There are a great number of theorems relating to the composition of forces and force-couples, which consequently relate also to infinitesimal rotations and translations. See, for instance, Chasles, "Théorèmes généraux &c." (1847), Möbius, "Lehrbuch der Statik" (1837), Steichen's Memoirs of 1853 and 1854, &c. Arising out of some theorems of Möbius in the "Statik," we have Sylvester's theory of the involution of six lines: viz. six lines (given in position) may be such that properly selected forces along them (or if we please, properly selected infinitesimal rotations round them) will counterbalance each other; or, what is the same thing, the six lines may be such that a system of forces, although satisfying for each of the six lines the condition moment = 0, will not of necessity be in equilibrium; such six lines are said to be in involution, and the geometrical theory is a very extensive and interesting one.

Miscellaneous Problems. Article Nos. 216 to 223.

216. As under the foregoing head, "Rotation round a fixed point," I have considered only the case of a body not acted upon by any forces, the case where the body is acted upon by any forces comes under the present head. The forces, whatever they are, may be considered as disturbing forces, and the problem be treated by the method of the variation of the elements; this is at any rate a *separate* part of the theory of rotation round a fixed point, and I have found it convenient to include it under the present head; the only case in which the forces have been treated as principal ones, seems to be that of a heavy body (a solid of revolution) rotating about a point not its centre of gravity. The case of a body suspended by a thread or resting on a plane comes under the present head, as also would (in some at least of the questions connected with it) the gyroscope. But none of these questions are here considered in any detail.

Rotation round a fixed point—Variation of the elements.

217. The forces acting on the body are treated as disturbing forces. Formulæ for the variations of the elements were first obtained by Poisson in the memoir "Sur la Variation des Constantes Arbitraires &c." (1809). The variations are expressed in terms

of the differential coefficients of the disturbing function in regard to the *elements*, and, as the author remarks, they are very similar in their form to, and can be rendered identical with, those for the variations of the elements in the theory of elliptic motion.

218. Cayley, "On the Rotation &c." (1846).—The latter part of the paper relates to the variations of the elements therein made use of, which are different from the ordinary ones.

219. Richelot, "Eine neue Lösung &c." (1851).—The form in which the integrals are obtained by means of a function V, satisfying a partial differential equation, leads at once to a canonical system for the variations of the elements; these formulæ are referred to in the introduction to the memoir, but they are not afterwards considered.

220. Cayley, "On the Rotation of a Solid Body" (1860).—The elements are those ordinarily made use of, with only a slight variation occasioned by the employment of the "departure" of the node. The course of the investigation consists in obtaining the variations in terms of the differential coefficients of the disturbing function in regard to *the coordinates* (formulæ which were thought interesting for their own sake), and in deducing therefrom those in terms of the differential coefficients in terms of the *elements*.

Other cases of the motion of a solid body.

221. In regard to a heavy solid of revolution rotating about a fixed point not its centre of gravity, we have

Poisson, "Mémoire sur un cas particulier &c." (1831), and the elaborate memoir

Lottner, "Reduction der Bewegung &c." (1855), where the solution is worked out by means of the Elliptic and Jacobian functions.

222. As regards a heavy solid suspended by a string,

Pagani, "Mémoire sur l'équilibre &c." (1839).

223. As regards the motion of a body resting on a fixed plane,

Cournot, "Mémoire sur le Mouvement &c." (1830 and 1832).

Puiseux, "Du Mouvement &c." (1848).

To these several others might doubtless be added; but the problems are so difficult, that the solutions cannot, it is probable, be obtained in any very complete form.

In conclusion, I can only regret that, notwithstanding the time which has elapsed since the present Report was undertaken, it is still—both as regards the omission of memoirs and works which should have been noticed, and the merely cursory examination of some of those which are mentioned—far from being as complete as I should have wished to make it. To have reproduced, to any much greater extent than has been

done, the various mathematical investigations, would not have been proper, nor indeed practicable; at the same time, more especially as regards the subjects treated of in the second part of this Report, or say the kinematics and dynamics of a solid body, such a reproduction, incorporating and to some extent harmonizing the original researches, but without ignoring the points of view and methods of investigation of the several authors, would be a work which would well repay the labour of its accomplishment.

List of Memoirs and Works.

Ampère. Mémoire sur quelques propriétés nouvelles des axes permanens de rotation des corps, et des plans directeurs de ces axes. 4to. Paris, 1823.

———. Mémoire sur la Rotation. Mém. de l'Institut, t. v. 1834.

———. Mémoire sur les équations générales du mouvement. Liouv. t. I. pp. 211—228 (1836). (Written 1826.)

Anon. Note on the problem of falling bodies as affected by the earth's rotation. Camb. and Dubl. Math. Journ. t. III. pp. 206—208 (1848).

———. Remarks on the deviation of falling bodies to the east and south of the perpendicular, and corrections of a previously published paper on the same subject. Camb. and Dubl. Math. Journ. t. IV. pp. 96—105 (1849).

Baehr. Notice sur le mouvement du pendule ayant égard à la rotation de la terre. 4to. Middelbourg, 1853.

Bertrand. Mémoire sur l'intégration des équations différentielles de la Mécanique. Liouv. t. XVII. pp. 393—436 (1852).

———. Note sur le Gyroscope de M. Foucault. Liouv. t. I. 2 sér. (1856), pp. 379—382.

———. Mémoire sur quelques-unes des formes les plus simples que puissent présenter les équations différentielles du mouvement d'un point matériel. Liouv. t. II. 2 sér. (1857), pp. 113—140.

Bessel. Analytische Auflösung der Keplerschen Aufgabe. Berl. Abh. 1816—17, pp. 49—55. (Read July 1818.)

———. Ueber die Entwickelung der Functionen zweier Winkeln u und u' in Reihen welche nach der Cosinussen und Sinussen der Vielfachen von u und u' fortgehen. Berl. Abh. 1820—21, pp. 56—60. (Read June 1821.)

———. Untersuchung des Theils der planetarischen Störungen welcher aus der Bewegung der Sonne entsteht. Berl. Abh. 1824, pp. 1—52.

Binet. Mémoire sur la théorie des axes conjugués et des momens d'inertie des corps. Journ. Polyt. t. IX. (cah. 16), pp. 41—67 (1813). (Read May 1811.)

———. Note sur le mouvement du pendule simple en ayant égard à l'influence de la rotation diurne de la terre. Comptes Rendus, t. XXXII. (1851), pp. 157—160 and 197—205.

Bonnet. Note sur un théorème de Mécanique. Liouv. t. IX. p. 113 (1844), and Note IV. of t. II. of the last edition of the Méc. Anal., pp. 329—331 (1855).

Booth. Theory of Elliptic Integrals. 8vo. Lond. 1851.

Bour. Mémoire sur le problème des trois corps. Journ. École Polyt. t. XXI. (cah. 36), pp. 35—58 (1856).

Bravais. Mémoire sur l'influence qu'exerce la rotation de la terre sur le mouvement d'un pendule à oscillations coniques. Liouv. t. XIX. pp. 1—50 (1854).

———. Note sur une formule de Lagrange relative au mouvement pendulaire. Note VII. of t. II. of the last edition of the Méc. Anal., pp. 352—355 (1855).

Briot. Thèse sur le mouvement d'un corps solide autour d'un point fixe. Liouv. t. VII. pp. 70—84 (1842).

Cauchy. Sur les momens d'inertie. Exer. de Math. t. I. pp. 93—103 (1827).

———. Résumé d'un mémoire sur la Mécanique Céleste et sur un nouveau calcul appelé des limites. (Read at Turin Oct. 1831.) Exer. d'Anal. &c. t. II. pp. 41—109 (1841).

Cayley. On certain expansions in multiple sines and cosines. Camb. Math. Journ. t. III. pp. 162—167 (1842), [4].

———. On the motion of rotation of a solid body. Camb. Math. Journ. t. III. pp. 224—232 (1842), [5].

———. On certain results relating to quaternions. Phil. Mag. t. XXVI. (1845), p. 141, [20].

———. On the geometrical representation of the motion of a solid body. Camb. and Dubl. Math. Journ. t. I. pp. 164—167 (1846), [36].

———. On the rotation of a solid body round a fixed point. Camb. and Dubl. Math. Journ. t. I. pp. 167—173 and 264—274 (1846), [37].

———. Note on a geometrical theorem in Prof. Thomson's memoir on the principal axes of a solid body. Camb. and Dubl. Math. Journ. t. I. pp. 207, 208 (1846), [38].

———. On the application of quaternions to the theory of Rotation. Phil. Mag. t. XXXIII. (1848), p. 196, [68].

———. Note on the motion of rotation of a solid of revolution. Camb. and Dubl. Math. Journ. t. IV. pp. 268—271 (1849), [78].

———. Sur les déterminants gauches. Crelle, t. XXXVIII. (1849), pp. 93—96, [69].

———. Note on the theory of Elliptic Motion. Phil. Mag. t. XI. (1856), pp. 425—428, [199].

———. A demonstration of Sir W. R. Hamilton's theorem of the Isochronism of the Circular Hodograph. Phil. Mag. t. XIII. (1857), p. 427, [209].

———. Report on the recent progress of Theoretical Dynamics. Rep. Brit. Assoc. for 1857, pp. 1—42, [195].

Cayley. On Lagrange's solution of the problem of two fixed Centres. Quart. Math. Journ. t. II. pp. 76—82 (1858), [182].

———. Note on the expansion of the true anomaly. Quart. Math. Journ. t. II. pp. 229—232 (1858), [191].

———. Tables in the theory of Elliptic Motion. Mem. R. Astr. Soc. t. XXIX. (1860), pp. 191—306, [216].

———. A Memoir on the problem of the rotation of a solid body. Mem. R. Astr. Soc. t. XXIX. (1860), pp. 307—342, [217].

———. On Lambert's theorem for Elliptic Motion. Monthly Not. R. Astr. Soc. t. XXII. pp. 238—242, (1861), [222].

———. Note on a theorem of Jacobi's in relation to the problem of three bodies. Monthly Not. R. Astr. Soc. t. XI. pp. 76—79 (1861), [220].

Chasles. Note sur les propriétés générales du système de deux corps semblables entr'eux et placés d'une manière quelconque dans l'espace, et sur le déplacement fini ou infiniment petit d'un corps solide libre. (Read Feb. 1831.) Bull. Univ. des Sciences (Férussac), t. XIV. pp. 321—326.

———. Théorèmes généraux sur les systèmes de forces et leurs moments. Liouv. t. XII. pp. 213—224 (1847).

Clairaut. Théorie de la Lune déduite du seul principe de l'attraction réciproquement proportionnelle aux carrés des distances. 4to. St Pét. 1752, and Paris, 1765.

Cohen. On the Differential Coefficients and Determinants of Lines, and their Application to Theoretical Mechanics. Phil. Trans. t. 152 (1862), pp. 469—510.

Cotes. Harmonia mensurarum sive analysis et synthesis per rationum et angulorum mensuras promotæ; accedunt alia opuscula mathematica. 4to. Camb. 1722.

Cournot. Mémoire sur le mouvement d'un corps rigide soutenu par un plan fixe. Crelle, t. V. pp. 133—162 and 223—249 (1830); Suite, t. VIII. pp. 1—12 (1832).

Creedy. General and practical solution of Kepler's Problem. Quart. Math. Journ. t. I. pp. 259—271 (1855).

D'Alembert. Traité de Dynamique. Paris, 1743.

———. Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre. Mém. de Berl. (1749).

Desboves. Thèse sur le mouvement d'un point matériel attiré en raison inverse du carré des distances vers deux centres fixes. Liouv. t. XIII. pp. 369—396 (1848).

Donkin. On an application of the calculus of operations to the transformation of trigonometric series. Quart. Math. Journ. t. II. pp. 1—15 (1858).

———. On a class of Differential Equations, including those which occur in Dynamical Problems. Part I. Phil. Trans. t. CXLIV. (1854), pp. 71—113; Part II. t. CXLV. (1855), pp. 299—358.

Droop. On the Isochronism of the Circular Hodograph. Quart. Math. Journ. t. I. (1856), pp. 374—378.

Dumas. Ueber die Bewegung des Raumpendels mit Rucksicht auf die Rotation der Erde. Crelle, t. L. pp. 52—78 and 126—185 (1855).

Durège. Theorie der elliptischen Functionen. 8vo. Leipzig, 1861. (§ XX. reproduces some results on the spherical pendulum obtained in an unpublished memoir of 1849.)

Euler. Determinatio Orbitæ Cometæ anni 1742. Misc. Berl. t. VII. (1743), p. 1.

———. Theoria motuum planetarum et cometarum. 4to. Berl. 1744.

———. De motu corporis ad duo virium centra attracti. Nov. Comm. Petrop. t. X. for 1764, pub. 1766, pp. 207—242.

———. Problème : un corps étant attiré en raison réciproque carrée des distances vers deux points fixes donnés, trouver les cas où la courbe décrite par ce corps sera algébrique. Mém. de Berl. for 1760, pub. 1767, pp. 228—249.

———. De motu corporis ad duo centra virium fixa attracti. Nov. Comm. Petrop. t. XI. for 1765, pub. 1767, pp. 152—184.

———. Considerationes de motu corporum cœlestium. Nov. Comm. Petrop. t. X. for 1764, pub. 1766, pp. 544—558.

———. De motu rectilineo trium corporum se mutuo attrahentium. Nov. Comm. Petrop. t. XI. for 1765, pub. 1767, pp. 144—151.

———. De motu trium corporum se mutuo attrahentium super eâdem lineâ rectâ. Nov. Acta Petrop. t. III. (1776), pp. 126—141.

———. Problema algebraicum ob affectiones prorsus singulares memorabile. Nov. Comm. Petrop. t. XV. (1770), p. 75; Comm. Arith. Coll. t. I. pp. 427—443.

———. Formulæ generales pro translatione quâcunque corporum rigidorum. Nov. Comm. Petrop. t. XX. 1775, pp. 189—207.

———. Nova methodus motum corporum rigidorum determinandi. Nov. Comm. Petrop. t. XX. (1775), p. 208.

———. Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre. Mém. de Berl. t. V. for 1749, pub. 1751, pp. 326—338. (Euler mentions, t. VI., that this was written after he had seen D'Alembert's memoir.)

———. Découverte d'un nouveau principe de Mécanique. Mém. de Berl. t. VI. for 1750, pub. 1752, pp. 185—217.

———. Recherches sur la connaissance mécanique des corps. Mém. de Berl. for 1758, pub. 1767, pp. 132—153.

———. Du mouvement de rotation des corps solides autour d'une axe variable. Mém. de Berl. for 1758, pub. 1765, pp. 154—193.

Euler. Du mouvement d'un corps solide lorsqu'il tourne autour d'une axe mobile. Mém. de Berl. for 1760, pub. 1767, pp. 176—227.

———. Theoria motus corporum solidorum. 4to. Rostock, 1765.

Foucault. Démonstration physique du mouvement de rotation de la terre au moyen du pendule. Comptes Rendus, t. XXXII. (1851), pp. 135—138.

Gauss. Fundamental-Gleichungen für die Bewegung schwerer Körper auf der rotirenden Erde, 1804.

———. Theoria motus corporum cœlestium. 4to. Hamb. 1809. [*Werke*, t. VII. Gotha, 1871.]

Greatheed. Investigation of the general term of the expansion of the true anomaly in terms of the mean. Camb. Math. Journ. t. I. pp. 228—232 (1838).

Gudermann. De pendulis sphæricis et de curvis quæ ab ipsis describuntur sphæricis. Crelle, t. XXXVIII. pp. 185—215 (1849).

Hamilton, Sir W. R. A theorem of anthodographic isochronism. Proc. R. Irish Acad. 1847, t. III. pp. 465—466.

———. Lectures on Quaternions. 8vo. Dublin, &c. (1853).

Hansen. Fundamenta Nova investigationis orbitæ veræ quam Luna perlustrat. 4to. Gothæ, 1838.

———. Ermittelung der absoluten Störungen in Ellipsen von beliebigen Excentricität und Neigung. Gotha, 1843, pp. 1—167.

———. Entwickelung des Products einer Potenz des Radius-Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie in Reihen die nach den Sinussen oder Cosinussen der Vielfachen der wahren excentrischen oder mittleren Anomalie fortschreiten. Abh. d. K. Sächs. Ges. zu Leipzig, t. II. pp. 183—281 (1853).

———. Entwickelung der negativen und ungeraden Potenzen der Quadratwurzel der Function $r^2 + r'^2 - 2rr'(\cos U \cos U' + \sin U \sin U' \cos J)$. Abh. d. K. Sächs. Ges. zu Leipzig, pp. 286—376 (1854).

———. Theorie der Pendelbewegung. 4to. Dantzig, 1856.

Haton de la Goupillière. Mémoire sur une théorie nouvelle de la géométrie des masses. Journ. École Polyt. t. XXI. (cah. 37), 1858, 1r Mémoire, pp. 35—72; 2d Mémoire, pp. 73—96.

Haughton. On the rotation of a solid body round a fixed point, being an account of the late Professor **Maccullagh's** lectures on that subject, Hilary Term, 1844, in Trinity College, Dublin; compiled by the Rev. S. Haughton. Trans. R. Irish Acad. t. XXII. (1849), pp. 1—18.

Jacobi. Euleri formulæ de transformatione coordinatarum. Crelle, t. II. pp. 188, 189 (1827).

Jacobi. Zur Theorie der Variations-Rechnung und der Differential-Gleichungen. Crelle, t. XVII. (1837), pp. 68—82.

———. Formulæ transformationis integralium definitorum. Crelle, t. XV. pp. 1—26 (1836).

———. De motu puncti singularis. Crelle, t. XXIV. pp. 5—27 (1842).

———. Elimination des nœuds dans le problème des trois corps. Crelle, t. XXVI. (1843), pp. 115—131.

———. Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi (§ 26, two centres). Crelle, t. XXIX. pp. 333—337 (1845).

———. Sur la rotation d'un corps. Extrait d'une lettre adressée à l'Académie des Sciences. Comptes Rendus, t. XXIX. p. 97; and Liouv. t. XIV. pp. 337—344 (1849).

———. ———. (With addition containing the demonstration of the formulæ.) Crelle, t. XXXIX. pp. 293—350 (1850).

———. Nova methodus æquationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi (posthumous, edited by **A. Clebsch**). Crelle, t. LX. pp. 1—181 (1862).

Lagrange. Mécanique Analytique. 1st ed. 1788; 2nd ed. t. I. 1811; t. II. 1815; 3rd ed. 1855.

———. Sur une manière particulière d'exprimer le temps dans les sections coniques décrites par des forces tendantes au foyer et réciproquement proportionnelles aux carrés des distances. Mém. de Berlin for 1778; and Note V. of t. II. of the 3rd edition of the Méc. Anal. pp. 332—349.

———. Recherchès sur le mouvement d'un corps qui est attiré vers deux centres fixes. Premier Mémoire, où l'on suppose que l'attraction est en raison inverse des carrés des distances. Anc. Mém. de Turin, t. IV. (1766—1769), pp. 118—215.

———. ———. Second Mémoire, où l'on applique la méthode précédente à différentes hypothèses d'attraction. Anc. Mém. de Turin, t. IV. (1766—1769), pp. 215—243.

———. Nouvelle solution du problème du mouvement de rotation d'un corps. Mém. de Berl. for 1773.

Lambert. Insigniores Orbitæ Cometarum Proprietates. 8vo. Aug. 1765.

Laplace. Mécanique Céleste, t. I. 1799; t. II. 1799; t. III. 1802; t. IV. 1805; t. V. 1823.

———. Mémoire sur le développement de l'anomalie vraie et du rayon vecteur elliptique en séries ordonnées suivant les puissances de l'excentricité. Mém. de l'Inst. t. VI. (1823), pp. 63—80.

Lefort. Expressions numériques des intégrales définies qui se présentent quand on cherche les termes généraux des développements des coordonnés d'un planète dans son mouvement elliptique. Liouv. t. XI. pp. 142—152 (1846).

Legendre. Exercices de Calcul Intégrale, t. II. (containing the dynamical applications) 1817.

———. Traité des Fonctions Elliptiques, t. I. (1825) (but the dynamical applications are for the most part reproduced from the *Exercices*).

Leverrier. Annales de l'Observatoire de Paris, t. I. (1855).

Lexell. Theoremata nonnulla generalia de translatione corporum rigidorum. Nov. Comm. Petrop. t. XX. (1775), pp. 239—270.

Liouville. Sur l'intégrale $\int_0^\pi \cos i(u - x \sin u)\, du$. Liouv. t. VI. pp. 36, 37 (1841).

———. Extrait d'un mémoire sur un cas particulier du problème de trois corps. Liouv. t. VII. pp. 110—113 (1842).

———. Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer. 3 Memoirs. Liouv. t. XI. pp. 345—379 (1846), t. XII. pp. 410—44 (1847), and t. XIV. pp. 257—300 (1849).

———. Mémoire sur un cas particulier du problème de trois corps. Conn. des Temps for 1845, and Liouv. t. I. (2 sér.), pp. 248—256 (1842).

———. Note ajoutée au mémoire de M. Serret (two revolving centres). Liouv. t. XIII. pp. 34—37 (1848).

Lottner. Reduction der Bewegung eines schweren um einen festen Punct rotirenden Revolutions-Körpers auf die elliptischen Transcendenten. Crelle, t. L. pp. 111—125 (1855).

———. Zur Theorie des Foucaultschen Pendelversuchs. Crelle, t. LII. pp. 52—58 (1856).

Maccullagh, *see* **Haughton.**

Möbius. Ueber die Zusammensetzung unendlich kleiner Drehungen. Crelle, t. XVIII. pp. 189—212 (1838).

———. Der barycentrische Calcul. 8vo. Leipzig, 1827.

———. Lehrbuch der Statik. 8vo. Leipzig, 1837.

———. Die Elemente der Mechanik des Himmels. 8vo. Leipzig, 1843.

Newton (Sir Isaac). Philosophiæ Naturalis Principia Mathematica. 4to. London, 1687.

Pagani. Démonstration d'un théorème de Lambert. Crelle, t. XV. pp. 350—352 (1834).

———. Mémoire sur l'équilibre d'un corps solide suspendu à un cordon flexible. Crelle, t. XXIX. pp. 185—204 (1839).

Painvin. Recherche du dernier multiplicateur pour deux formes spéciales et remarquables des équations différentielles du problème de trois corps. Liouv. t. XIX. pp. 88—111 (1854).

Poinsot. Mémoire sur la rotation. Extrait, 8vo. Paris, 1834.

Poinsot. Théorie nouvelle de la rotation des corps. Liouv. t. XVI. pp. 9—130 and 289—336 (1851).

———. Théorie des cônes circulaires roulants. Liouv. t. XVIII. pp. 41—70 (1853).

———. Questions dynamiques sur la percussion des corps. Liouv. t. II. (2 sér.), pp. 281—329 (1859).

Poisson. Traité de Mécanique. 1st ed. Paris, 1811; 2nd ed. Paris, 1833.

———. Mémoire sur un cas particulier du mouvement de rotation des corps graves. Journ. École Polyt. t. IX. (cah. 16), pp. 247—262 (1813).

———. Sur une nouvelle manière d'exprimer les coordonnés des planètes dans le mouvement elliptique. Conn. des Temps for 1825, pp. 379—386.

———. Sur la développement des coordonnés d'une planète dans son mouvement elliptique et de la fonction perturbative de ce mouvement. Conn. des Temps for 1836, pp. 3—31.

———. Mémoire sur le mouvement d'un corps solide (read 1834). Mém. de l'Inst. t. XIV. pp. 275—432 (1838).

———. Sur le mouvement des projectiles en ayant égard à la rotation de la terre. Journ. École Polyt. t. XVI. pp. 1—68 (1838).

———. Mémoire sur le mouvement des projectiles dans l'air, en ayant égard à leur rotation. Journ. École Polyt. pp. 69—176 (1838).

Prony. Analyse détaillée des différentes questions qui se rapportent au mouvement d'un corps sollicité par des puissances quelconques. Journ. École Polyt. t. IV. (cah. 11), pp. 87—143 (1802).

Puiseux. Note sur le mouvement d'un point matériel pesant sur une sphère. Liouv. t. VII. pp. 517—520 (1842).

———. Du mouvement d'un solide de révolution posé sur un plan horizontal. Liouv. t. XIII. pp. 249—256 (1848).

———. Sur la convergence des séries qui se présentent dans la théorie du mouvement elliptique des planètes. Liouv. t. XIV. pp. 33—39 (1849).

———. Seconde Note sur la convergence des séries du mouvement elliptique. Liouv. t. XIV. pp. 247, 248 (1849).

———. Solution de quelques questions relatives au mouvement d'un corps solide pesant posé sur un plan horizontal. Liouv. t. XVII. pp. 1—30 (1852).

———. Sur la convergence des séries ordonnées suivant les puissances de l'excentricité qui se présentent dans la théorie du mouvement elliptique. Méc. Anal. 3rd ed. t. II. (Note I.), pp. 313—317 (1855).

Quet. Des mouvements relatifs en général, et spécialement des mouvements relatifs sur la terre. Liouv. t. XVIII. pp. 213—298 (1853).

Richelot. Bemerkung über einen Fall der Bewegung eines systems von materiellen Puncten. Crelle, t. XL. pp. 178—182 (1850).

———. Auszug eines Schreibens am Herrn Prof. Jacobi. Crelle, t. XLII. pp. 32—34 (1851).

———. Bemerkungen zur Theorie des Raumpendels. Crelle, t. XLII. pp. 233—238 (1853).

———. Eine neue Lösung des Problems der Rotation eines festen Körpers um einen Punct. (Auszug einer zu Berlin, 1851, erschienenen Schrift.) Crelle, t. XLIV. pp. 60—65.

Rodrigues. Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnés provenants de ces déplacements considérés indépendamment des causes qui peuvent les produire. Liouv. t. III. pp. 380—440 (1840).

Rueb. Specimen inaugurale de motu gyratorio corporis rigidi nullâ vi acceleratrice sollicitati. Utrecht, 1834.

Schellbach. Mathematische Miszellen. No. I.—IV. Ueber die Bewegung eines Puncts der von einem festen Puncte angezogen wird. Crelle, t. XLV. pp. 255—266 (1853).

Schubert. Astronomie Théorique. 4to. St Pét. (1822).

Segner. Specimen theoriæ turbinum. 4to. Halæ, 1755.

Serret. Thèse sur le mouvement d'un point attiré par deux centres fixes en raison inverse du carré des distances. Liouv. t. XIII. pp. 17—33 (1848).

———. Sur la solution particulière que peut admettre le problème du mouvement d'un corps attiré vers deux centres fixes par des forces réciproquement proportionnelles aux carrés des distances. Note III. to t. II. of the 3rd ed. of the Méc. Anal. pp. 327—331 (1855).

Sohncke. Motus corporum cœlestium in medio resistente. Crelle, t. X. pp. 23—40 (1833).

Somoff. Démonstration des formules de M. Jacobi relatives à la théorie de la rotation d'un corps solide. Crelle, t. XLII. pp. 95—116 (1851).

Stader. De orbitis et motibus puncti cujusdam corporei circa centrum attractivum aliis quam Neutonianâ attractionis legibus sollicitati. Crelle, t. XLVI. pp. 262—327 (1852).

Steichen. De la propriété fondamentale du mouvement cycloidal, et de sa liaison avec le principe de la composition des mouvements de rotation autour des axes parallèles et des axes qui se coupent. Crelle, t. XLVI. pp. 24—42 (1853).

———. Mémoire sur la question réciproque du centre de percussion. Crelle, t. XLVIII. pp. 1—68 (1854).

Steichen. Mémoire de Mécanique relatif au mouvement de rotation et au mouvement naissant des corps solides. Crelle, t. XLIII. pp. 161—244 (1852), and Addition t. XLIV. pp. 43—46 (1853).

Sylvester. Sur l'involution des lignes droites dans l'espace considérées comme des axes de rotation. Comptes Rendus, t. LII. (1861), p. 741.

———. Note sur l'involution de six lignes dans l'espace. Comptes Rendus, t. LII. (1861), p. 815.

Thomson. On the principal axes of a solid body. Camb. and Dubl. Math. Journ. t. I. pp. 127—133 and 195—206 (1846).

Tissot. Thèse de Mécanique. Liouv. t. XVII. pp. 88—116 (1852).

Townsend. On the principal axes of a solid body, their moments of inertia and distribution in space. Camb. and Dubl. Math. Journ. t. I. pp. 209—227, and t. II. pp. 19—42, 140—171 and 241—251 (1846—47).

Walton. Instantaneous lines and planes in a body revolving round a fixed point. Camb. and Dubl. Math. Journ. t. VII. pp. 111—113 (1852).

Whewell. A Treatise on Dynamics. 1st ed. Cambridge, 1823.

Worms. The Earth and its Mechanism—being an account of the various proofs of the rotation of the Earth, with a description of the instruments used in the experimental demonstrations: to which is added the theory of Foucault's Pendulum and Gyroscope. 8vo. London, 1862.

299.

MATHEMATICS, RECENT TERMINOLOGY IN.

[From the *English Cyclopædia*, vol. v. (1860), pp. 534—542.]

THE terms intended to be explained in the present article relate to subjects distinct indeed, but intimately connected together, as well logically as historically. *Determinants* were devised as a means to the solution of a system of simple equations, but the principle of their construction is contained in the rule of signs which belongs to the theory of *arrangements* (or permutations): this theory has been studied, as well for its own sake, as in reference to the theory of equations, and in it originated the notion of a *group*, the most outlying term of those which are here explained. Moreover, in a system of simple equations, if the coefficients arranged in the natural square order are considered apart by themselves, this leads to the theory of *matrices*, a theory which indeed might have preceded that of determinants; the matrix, is, so to speak, the matter of a determinant; the rule of signs giving the form. But when the rule of signs is applied to other matter, this leads to the function called *permutants;* these include *commutants* and *intermutants*, and also *Pfaffians*, which however were not originally so arrived at. The theory of elimination (according to one of the two ways in which it may be treated) is essentially dependent upon systems of linear equations, and is thus also connected with determinants. And all, or nearly all, the before-mentioned theories have an application to the theory of rational and integral homogeneous functions, or, as they have been termed, *forms* or *quantics;* they are thus connected with the "Calculus of Forms," and with "Quantics"; the last-mentioned expression (used as a singular), has been defined to denote the entire subject of rational and integral functions, and of the equations and loci to which those give rise. The theory of rational and integral functions was first studied in a general manner in the question of *linear transformations*, and it was this question which led to the discovery of the functions, called originally *hyperdeterminants*, but afterwards *invariants*, and of the more general functions called *covariants:* the theory of covariants is indeed the part which has been chiefly attended to of the Calculus of forms, or of Quantics.

The following list of terms may be convenient:

Rule of signs.
Group.
Determinant.
  Minor,
  Symmetric, skew, skew symmetric.
Commutant.
Pfaffian.
Permutant.
Intermutant.
Cumulant.
Matrix.
Resultant.
Discriminant.
Plexus.
Rational and integral functions (notation and nomenclature of).
  Quantic, quadric, cubic, &c., binary, ternary, &c., facient, tantipartite, lineo-linear.
Emanant.
Linear transformations.
  Modulus of transformation, unimodular.
Hyperdeterminant.
Invariant.
Covariant.
  Contravariant, peninvariant, seminvariant, quadrinvariant, quadricovariant, &c., catalecticant, canonisant.
Canonical form.
Bezoutic matrix, &c.
Tactinvariant, reciprocant.
Functional determinant, Jacobian, Hessian.
Concomitant, cogredient, contragredient.
Combinant.

RULE OF SIGNS.—Any arrangement of a series of terms may be derived (and that in a variety of ways) from any other arrangement by successive interchanges of two terms; but in whatever way this is done, the number of interchanges will be constantly even or constantly odd; and the two arrangements are said to be of the same sign or of contrary signs accordingly. In particular, if any arrangement is selected as the primitive arrangement, and taken to be positive, then any other arrangement will be positive or negative according as it is derivable from the primitive arrangement by an even or an odd number of interchanges. The definition leads to the following theorem: any arrangement is positive or negative, according as the total number of times in which the several elements respectively precede (mediately or immediately) elements posterior to them in the primitive arrangement, is even or odd: it may be added, that the positive and negative arrangements are equal in number. Thus in the case of three terms, the primitive arrangement being 123; the positive arrangements are 123, 231, 312, the

negative arrangements, 132, 213, 321: in the case of four terms, the primitive arrangement being 1234, the arrangements 1234, 2341, 3412, 4123 are respectively positive, negative, positive, negative; there are in all twelve positive and twelve negative arrangements.

GROUP.—The term was originally used as applied to substitutions only, but the more general use of the term is as follows: let θ be a symbol operating on any number of terms $x, y, z, ..$ and producing as the result of the operation the same number of new terms $X, Y, Z, ..$ (where $X, Y, Z ...$ may be each of them functions of all or any of the set, $x, y, z, ..$; if $X, Y, Z, ..$ are merely the terms, $x, y, x, ..$ in a different order, then θ is a substitution, which explains in what sense that term has just been used). Imagine a set of operative symbols $1, \theta, \phi, \chi ..$ (1, as an operative symbol denotes, of course, a symbol which leaves the operand unaltered) such that the result of the operation of *any* two symbols θ, ϕ (the same or different, and if different, then in either order) is identical with that of the operation of *some* symbol χ of the set; as thus, $\theta\phi(x, y, z ..) = \theta(X, Y, Z, ..) = (X', Y', Z', ..) = \chi(x, y, z ..)$, say, $\theta\phi = \chi$; then the symbols $1, \theta, \phi, \chi, ...$ form a *group*. It is to be remarked that 1 belongs to every group, and moreover, that if θ be any symbol of the group, then $\theta^2, \theta^3, \theta^4, ..$ belong to the group: the most simple form of a group (and when the number of terms is prime, the only form) is $1, \theta, \theta^2, ... \theta^{n-1}$ $\{\theta^n = 1\}$. More generally, if there are n terms in the group, then every symbol θ of the group is an operation periodic of the order n (if not of an order a submultiple of n) and thus satisfies the symbolic equation $\theta^n = 1$. The symbols of the group are, so to speak, the symbolic n-th roots of unity, and as in the above-mentioned example, they may, whether n is prime or composite, form a system precisely analogous to that of the ordinary n-th roots of unity; but when n is composite, then upon two grounds this is not of necessity the case. 1°. The symbols of a group need not be convertible (thus $n = 6$, there is a group, $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ $[\alpha^2 = 1, \beta^3 = 1, \beta\alpha = \alpha\beta^2$ and $\therefore \beta^2\alpha = \alpha\beta$, this is in fact, the group of the substitutions of three things). 2°. There may be distinct n-th roots, thus $n = 4$, there is a group, $1, \alpha, \beta, \alpha\beta$ $[\alpha^2 = 1, \beta^2 = 1, \alpha\beta = \beta\alpha]$, in which α, β are distinct square roots of (the symbolical) unity, and which is thus wholly different from the group, $1, \alpha, \alpha^2, \alpha^3$ $[\alpha^4 = 1]$.

The combination of a series of terms in the way of addition or subtraction, according to the rule of signs, gives rise to the class of functions called permutants, which include as a particular but the earliest discovered and most important case, the determinant:

DETERMINANT.—Imagine a square arrangement of terms, for example

$$\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

and taking this as the primitive arrangement, permute in every possible way entire columns (or, what would give the same results, entire lines) and for each such arrange-

ment form the product of the terms in the dexter diagonal (N.W. to S.E.) of the square, giving to such product the sign which belongs to the arrangement of the columns (or lines). The algebraical sum of these products is a determinant, and such determinant is or may be represented as above, by enclosing the terms within two vertical lines. Thus the developed value of the determinant in question is

$$ab'c'' - ab''c' + a'b''c + a''bc' - a''b'c - a'b''c.$$

The rule may be otherwise stated as follows: a determinant is the sum of a series of products each with its proper sign, such that in each product the factors are taken out of each line and out of each column, and if the factors are arranged according to the primitive arrangement of the columns in which they occur, then the sign is that corresponding to the resulting arrangement of the lines (or *vice versâ*): thus in the product $-ab''c'$, the factors a, b'', c' occur in the columns 1, 2, 3 (they are therefore arranged according to the primitive arrangement of the columns) and in the lines 1, 3, 2; such arrangement of the lines considered as derived from the primitive arrangement 1 2 3 is negative, and the product has therefore the sign $-$. A generalisation of this construction will be mentioned under the term commutant.

The word *resultant* was formerly used as synonymous with determinant, but it is now employed and is here explained in a more extended signification. The new synonym *eliminant* seems unnecessary.

A few of the numerous properties of determinants may be stated.

A determinant is a linear function (without constant term) of the terms in each of its columns, and also of the terms in each of its lines, or, more briefly expressed, it is a linear function of each column, and also of each line. Moreover, without altering the value of the determinant, the lines may be made columns, and the columns lines, and all the properties of the function exist equally with respect to the lines and to the columns. The absolute value of the determinant is not altered, but the sign is reversed, by an interchange of two columns, hence also if two columns become identical, the determinant vanishes. Moreover when the columns are permuted in any manner whatever, the absolute value is not altered, but the sign will be that corresponding to the arrangement of the columns. A determinant may be developed as a linear function of the terms in any line, thus

$$\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = a \mid b' \mid - a' \mid b \mid,$$

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + b \begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix},$$

the signs being alternately positive and negative or else all positive, according as the number of columns is even or odd.

The square arrangement of terms out of which a determinant is formed, and generally any square or rectangular arrangement of terms, is called a matrix. Consider a determinant

$$\begin{vmatrix} a\,, & b\,, & c\,, & d \\ a'\,, & b'\,, & c'\,, & d' \\ a''\,, & b''\,, & c''\,, & d'' \\ a'''\,, & b'''\,, & c'''\,, & d''' \end{vmatrix}$$

and partitioning the lines in any manner, form with them the matrices

$$\begin{vmatrix} a\,, & b\,, & c\,, & d \\ a'\,, & b'\,, & c'\,, & d' \end{vmatrix}, \quad \begin{vmatrix} a''\,, & b''\,, & c''\,, & d'' \\ a'''\,, & b'''\,, & c'''\,, & d''' \end{vmatrix},$$

and out of these matrices, with complementary columns thereof, a sum of products

$$\Sigma \pm \begin{vmatrix} a\,, & b \\ a'\,, & b' \end{vmatrix} \begin{vmatrix} c''\,, & d'' \\ c'''\,, & d''' \end{vmatrix},$$

(the sign $\pm$ being that corresponding to the product $ab'c''d'''$ of the terms in the dexter diagonals of the factor determinants, considered as a term of the original determinant), the sum of all the products so obtained in the original determinant.

It has been mentioned that the determinant is a linear function of each column; hence if the terms of any column are $\rho\alpha$, $\rho'\alpha$,.. the determinant is equal to ρ times a determinant in which the corresponding column is α, α',... and similarly if the column is $a+b$, $a'+b'$,.. then the determinant is the sum of two other determinants in which the corresponding columns are a, a',... and b, b',... respectively. This property, in combination with some of those already mentioned, leads very simply to the rule for the multiplication of determinants; for example we have

$$\begin{vmatrix} \rho\,, & \sigma \\ \rho'\,, & \sigma' \end{vmatrix} \begin{vmatrix} \alpha\,, & \beta \\ \alpha'\,, & \beta' \end{vmatrix} = \begin{vmatrix} \rho\alpha+\sigma\beta\,, & \rho'\alpha+\sigma'\beta \\ \rho\alpha'+\sigma\beta'\,, & \rho'\sigma'+\sigma'\beta' \end{vmatrix},$$

from which the law is obvious. The product might also be expressed, and although it appears less simple, there is an advantage in expressing it, in the form

$$\begin{vmatrix} \rho\alpha+\sigma\alpha'\,, & \rho\beta+\sigma\beta' \\ \rho'\alpha+\sigma'\alpha'\,, & \rho'\beta+\sigma'\beta' \end{vmatrix}.$$

If we omit simultaneously any line and any column of a determinant, and with the terms which are left form a determinant, the determinants so obtained are the *first minors* of the given determinant. A similar process, but omitting pairs, triads, &c. of lines and columns, gives the *second minors*, *third minors*, &c. of the given determinant. But the first minors are the most important, and are sometimes spoken of simply as the minors.

A determinant

$$\begin{vmatrix} a, & h, & g, .. \\ h, & b, & f \\ g, & f, & c \\ \vdots & & \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 11, & 12, .. \\ 21, & 22 \\ \vdots & \end{vmatrix},$$

where the corresponding terms on opposite sides of the dexter diagonal are equal to each other (say $rs = sr$) is said to be *symmetrical.*

But if the terms are equal in magnitude only, but have opposite signs (say $rs = -sr$, this relation not extending to the terms *in* the diagonal, for which $s = r$) the determinant is said to be *skew;* and if the relation extends to the case $s = r$, or what is the same thing, if the terms *in* the diagonal vanish, the determinant is said to be *skew symmetrical.* Skew determinants have an intimate connection with the functions called Pfaffians.

COMMUTANT.—The second rule for the construction of a determinant might have been thus stated, viz. for the determinant

$$\begin{vmatrix} 11, & 12, .. 1n \\ 21, & 22 \\ \vdots & \\ n1 & \end{vmatrix}$$

write down the expression

$$\begin{matrix} 1\,1 \\ 2\,2 \\ \vdots \\ n\,n \end{matrix}$$

and permute in every possible way the numbers in the first column, prefixing in each case the sign of the arrangement. Then reading off

$$\begin{matrix} \pm r\,1 \\ s\,2 \\ \vdots \\ z\,n. \end{matrix}$$

as meaning

$$\pm y1 \,.\, s2 \ldots zn$$

the sum of all the terms so obtained is in fact the determinant in question. The same result would be obtained by permuting the numbers in the second column instead of those in the first column. And moreover, if the numbers in both columns are permuted, the sign being the sign $\pm\,\pm$ compounded of the signs corresponding to the separate arrangements, the only difference is, that the determinant will be multiplied by the numerical factor $1 \,.\, 2 \,.\, 3 \ldots n$.

If instead of two we have three or more columns, the resulting function is a *commutant*. But a distinction is to be made according as the number of columns is even or odd. In the former case we may permute all but one of the columns, and it is indifferent which column is left unpermuted; and if all the columns are permuted, the effect is merely to introduce the numerical factor $1.2.3....n$. In the latter case, if all the columns are permuted, the result is zero, and it is therefore essential that one column should remain unpermuted; moreover, different results are obtained according to the column which is left unpermuted, and such column must therefore be distinguished; this is done by placing above it the mark †.

PFAFFIAN.—Suppose that the terms 12, 13, 21, &c., are such that $21=-12$, and generally that $sr=-rs$, then the Pfaffians 1234, 123456, &c., are defined by means of the equations

$$1234=12.34+13.42+14.23,$$

$$123456=12.3456+13.4562+14.5623+15.6234+16.2345,$$

(where of course $3456=34.56+35.64+36.45$, and so for 4562, &c.)

and so on. The functions in question occur in the solution of an important problem (including that of partial differential equations of the first order and of any degree) known as Pfaff's problem, and were named accordingly.

It may be noticed that a skew symmetrical determinant of any odd order is equal to zero; but that a skew symmetrical determinant of any even order is the square of a Pfaffian, e.g. if $12=-21$, &c., as above, then

$$\begin{vmatrix} 0, & 12, & 13, & 14 \\ 21, & 0, & 23, & 24 \\ 31, & 32, & 0, & 34 \\ 41, & 42, & 43, & 0 \end{vmatrix} = (12.34+13.42+14.23)^2.$$

PERMUTANT.—A very simple instance of a permutant is as follows, viz.: V_{123}, V_{213}, &c. being any quantities whatever, then the permutant $((V_{123}))$ denotes the sum

$$V_{123}+V_{231}+V_{312}-V_{132}-V_{213}-V_{321}$$

and in like manner for any number of permutable suffixes, or if instead of a single set of permutable suffixes we have two or more sets of such suffixes. It will be at once obvious how a permutant includes a determinant, commutant, or Pfaffian; thus, if V_{123} denotes $\alpha_1\beta_2\gamma_3$ and therefore $V_{213}=\alpha_2\beta_1\gamma_3$, &c., then we have a determinant, so if V_{1234} denotes $\alpha_{12}.\alpha_{34}$ where $\alpha_{21}=-\alpha_{12}$, we have a Pfaffian.

INTERMUTANT is a special form of permutant which need not be here further explained.

CUMULANT.—The name has been given to the function which is the numerator or denominator of a continued fraction. Such function may be exhibited (and indeed

naturally presents itself) in the form of a determinant, thus the cumulant $(abcd)$ or numerator of the fractions $a+\frac{1}{b}\frac{1}{+c}\frac{1}{+d}$ is

$$\begin{vmatrix} a, & 1, & ., & . \\ -1, & b, & 1, & . \\ ., & -1, & c, & 1 \\ ., & ., & -1, & d \end{vmatrix},$$

and so for a greater number of terms. The developed expression is $abcd+ab+bc+bc+1$ which is formed from the product $abcd$ by successively omitting each product (cd, bc, ab), or set of products (cd, ab) of two consecutive letters; in like manner the cumulant $(abcde)$ is $abcde+abc+abe+a+c+e$.

MATRIX.—The term might be used to denote any arrangement of terms, but in a restricted sense it denotes a square or rectangular arrangement of terms, and it is thus employed in the theory of determinants.

To show further how the notion of a matrix is made use of, it may be remarked that a system of linear equations

$$\begin{aligned} \xi &= a\,x+b\,y+c\,z, \\ \eta &= a'x+b'y+c'z, \\ \zeta &= a''x+b''y+c''z \end{aligned}$$

is in the notation of matrices represented by

$$(\xi,\ \eta,\ \zeta)=\begin{pmatrix} a\ , & b\ , & c\ \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix}(x,\ y,\ z).$$

The corresponding set of equations which give (x, y, z) in terms of (ξ, η, ζ) is represented by

$$(x,\ y,\ z)=\left(\begin{pmatrix} a\ , & b\ , & c\ \\ a', & b', & c' \\ a'', & c'', & c'' \end{pmatrix}\right)^{-1}(\xi,\ \eta,\ \zeta),$$

and we have thus the definition of the *inverse* or *reciprocal* matrix: it follows from the theory of determinants that the terms of the reciprocal matrix are the first minor determinants formed out of the original matrix, each of them divided by the determinant formed out of the original matrix; but in writing down the expression some attention is required with respect to the arrangement and signs of the terms.

Similar considerations lead to the notion of *multiplying* or *compounding* together two or more matrices. As an instance of such composition, take

$$\begin{pmatrix} \rho, & \sigma \\ \rho' & \sigma' \end{pmatrix}\begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix}=\begin{pmatrix} \rho\alpha+\sigma\alpha', & \rho\beta+\sigma'\beta' \\ \rho'\alpha+\sigma'\alpha', & \rho'\beta+\sigma'\beta \end{pmatrix}$$

where it is to be observed that the lines of the first or *further* component matrix are compounded with the columns of the second or *nearer* component matrix to form the lines of the compound matrix. The words further, nearer, are used in reference to a set (x, y) which is, or may be considered to be, understood at the right of each side of the equation. A matrix may be compounded with itself once or oftener, giving rise to a positive power of such matrix; the notion of the negative powers is deducible from that of the inverse or reciprocal matrix, and the same process of generalisation as is employed for powers of a single quantity leads to the notion of the fractional powers of a matrix. As a definition of *addition*, matrices are added together by the addition of their corresponding terms, and as a particular case of the multiplication or composition of matrices we have the multiplication of a matrix by a single quantity, effected by multiplying by such quantity each term of the matrix; all these notions together lead to the notion of *functions* of a matrix.

As an instance of the employment of the notation of matrices for another purpose, take

$$\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)(x, y, z)(\xi, \eta, \zeta)$$

used to denote the lineo-linear function

$$\begin{aligned} &(a\,x + b\,y + c\,z)\,\xi \\ &(a'\,x + b'\,y + c'\,z)\,\eta \\ &+ (a''x + b''y + c''z)\,\zeta \end{aligned}$$

which includes

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & o \end{array}\right)(x, y, z)^2,$$

used to denote the quadric function

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

The last preceding notation is an instance of a *symmetrical* matrix: the terms *skew, skew symmetrical,* already explained with respect to determinants, apply also to matrices.

RESULTANT.—If there be a system of equations between the same number of unknown quantities (it is assumed that the several equations are of the form $U = 0$, where U is a rational and integral homogeneous function), then the function of the coefficients which equated to zero expresses the result of the elimination of the unknown quantities from the several equations, or (what is the same thing) gives the condition for the existence of a set of values satisfying the equations simultaneously—is the *Resultant* of the equations, or of the functions which are thereby put equal to zero. In the case of two (non-homogeneous) equations involving a single unknown quantity,

we may say more briefly that the resultant is the function which equated to zero gives the condition for the existence of a common root. In the particular case of a system of linear equations between as many unknown quantities, the resultant is the determinant formed with the coefficients of the equations.

DISCRIMINANT.—If in a system of equations the functions equated to zero are the derived functions of a single rational and integral homogeneous function with respect to each of the variables thereof, the resultant of the system is said to be the discriminant of the single function. The definition is easily made applicable to the case of a non-homogeneous function, the functions equated to zero are here the function itself and its derived functions with respect to each of the several variables. For a single function, it may be said that the discriminant is the function which equated to zero gives the condition for a pair of equal roots of the equation obtained by putting the function equal to zero.

To fix the precise value of the discriminant of a given function, it is assumed that the coefficient of some one selected term is $+1$. Thus, the discriminant of $ax^2+2\,bxy+cy^2$ is $ac-b^2$: that of

$$ax^3+3\,bx^2y+3\,cxy^2+dy^3 \text{ is } a^2d^2-6\,abcd+4\,ac^3+4\,b^3d-3\,b^2c^2.$$

In quadratic forms (in the theory of numbers) the expression b^2-ac, which is the determinant $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}$ with the sign reversed, is called the determinant of the form $ax^2+2\,bxy+cy^2$. And in like manner for ternary quadratic forms, there is the same reversal of sign. It may be said as a convenient definition, that the determinant is the discriminant taken negatively.

PLEXUS.—It frequently happens, in problems of elimination and in other problems, that a given number of relations existing between a system of quantities can only be completely expressed by means of a greater number of equations. Thus, to take a very simple instance, if the unknown quantities x, y, are to be eliminated between the three equations $ax+by=0$, $a'x+b'y=0$, $a''x+b''y=0$: this implies two relations between the coefficients a, b, a', b', a'', b''; but these relations cannot be completely expressed otherwise than by means of the three equations $ab'-a'b=0$, $a'b''-a''b'=0$, $a''b-ab''=0$; for taking any two of these equations, e.g. the first and second, these would be satisfied by $a'=0$, $b'=0$, which however do not satisfy the third equation and are not a solution. Such a system of equations, or generally the system of equations required for the complete expression of the relations existing between a set of quantities (and which are *in general* more numerous than the relations themselves) is said to be a *Plexus*.

RATIONAL AND INTEGRAL FUNCTIONS (*Notation and Nomenclature of*).—A rational and integral homogeneous function, such as the function $ax^2+2bxy+cy^2$ is denoted by

$$(*)\,(x,\ y)^2$$

where the coefficients are only indicated by the asterisk, but are not expressed. A non-homogeneous rational and integral function is considered as derived from a homo-

geneous function by putting one of the variables thereof equal to unity, and is represented accordingly: thus $ax^2+2bx+c$ is denoted by

$$(*)(x,\ 1)^2.$$

But it is often proper to express the coefficients, and in regard to this the following distinction is made, namely

$$(a,\ b,\ c)\ (x,\ y)^2$$

denotes $ax^2+2bxy+cy^2$; and in like manner $(a, b, c, d)(x, y)^3$ denotes $ax^3+3bx^2y+3cxy^2+dy^3$, &c., the numerical coefficients being those of the successive powers of a binomial. But when such numerical coefficients are not to be inserted, this is denoted by an arrowhead, or other distinctive mark; thus $(a, b, c)\dagger(x, y)^2$ denotes $ax^2+bxy+cy^2$. A rational and integral function of any order is termed a *quantic*, and a function of the orders two, three, four, five, &c., is termed a *quadric, cubic, quartic, quintic*, &c. respectively. The number of variables (the function being homogeneous) is denoted by the words *binary, ternary*, &c. As a correlative term to coefficients, the variables have been termed *facients*. A function which is linear in respect to several distinct sets of variables separately is said to be *tantipartite:* or, when there are two sets only, *lineo-linear*. Thus a determinant is a tantipartite function of the lines or of the columns; the function $axx'+bxy'+cx'y+dyy'$ is a lineo-linear function of (x, y) and (x', y'); a notation for it is

$$\begin{pmatrix} a, & b \\ a', & b' \end{pmatrix}(x,\ y)(x',\ y')$$

such as has been spoken of in regard to matrices.

EMANANT.—The development of an expression such as

$$(*)(\lambda x+\mu x',\ \lambda y+\mu y')^m$$

is naturally written under the form

$$\begin{array}{lll} & (*)(x,\ y)^m & \lambda^m \\ + & \frac{m}{1}(*)(x,\ y)^{m-2}(x',\ y') & \lambda^{m-1}\mu \\ & \vdots & \\ + & (*)(x'\ y')^m & \mu^m, \end{array}$$

and the coefficients of the successive terms λ^m, $\lambda^{m-1}\mu$, &c. are said to be the emanants of the quantic $(*)(x, y)^m$. The coefficients of λ^m, or 0-th emanant, is the quantic itself, and the coefficient of μ^m, or ultimate emanant, is the quantic with (x', y') in the place of (x, y); but the intermediate emanants are functions of (x, y) and (x', y'), homogeneous in respect to the two sets separately. The coefficients may of course be expressed thus, the emanant (first emanant) of $(a, b, c)(x\ y)^2$ is $(a, b, c)(x, y)(x', y')$ which stands for $axx'+b(xy'+x'y)+cyy'$.

LINEAR TRANSFORMATIONS.—In this theory the variables of a function are supposed to be respectively linear functions of a new set of variables, so that the function is transformed into a similar function of these new variables, with of course altered values of the coefficients, and the question was to find the relations which existed between the original and new coefficients and the coefficients of the linear equations. The determinant composed of the coefficients of the linear equations is said to be the *modulus of transformation*, and when this determinant is unity the transformation is said to be *unimodular*. It was observed that a certain function of the coefficients, namely, the discriminant, possessed a remarkable property, found afterwards to belong to it as one of a class of functions called originally *hyperdeterminants*, but now *invariants*, and it was in this manner that the problem of linear transformation led to the general theory of covariants.

INVARIANTS.—An invariant is a function of the coefficients of a rational and integral homogeneous function or quantic, the characteristic property whereof is as follows: namely, if a linear transformation is effected on the quantic, then the new value of the invariant is to a factor *près* equal to the original value; the factor in question (or quotient of the two values) being a power of the modulus of transformation, and the two values being thus equal when the transformation is unimodular. The easiest example is afforded by the quadric function $(a, b, c)(x, y)^2$; effecting upon it a linear transformation, suppose that we have identically

$$(a, b, c)(\alpha x' + \beta y', \gamma x' + \delta y')^2 = (a', b', c')(x', y')^2$$

then it may be easily verified that $a'c' - b'^2 = (\alpha\delta - \beta\gamma)^2(ac - b^2)$. The invariant $ac - b^2$ is however in this case nothing else than the discriminant; as another example take the quartic $(a, b, c, d, e)(x, y)^4$, for which $ae - 4bd + 3c^2$, $ace - ad^2 - b^2d + 2bcd - c^3$ are functions possessed of the like property of remaining to a factor *près* unaltered by the transformation, and are consequently invariants; it may be added that calling them I, J, respectively, the discriminant is here $= I^3 - 27J^2$, a rational and integral function of invariants of a lower degree.

COVARIANT.—Instead of a function of the coefficients only, we may have a function of the coefficients and variables, possessed of the like property of remaining unaltered to a factor *près* by the linear transformation: such function is termed a covariant. Thus, a covariant (the Hessian) of the quartic $(a, b, c, d, e)(x, y)^4$ is

$$(ac - b^2,\ 2ad - 2bc,\ ae + 2bd - 3c^2,\ 2be - 2cd,\ ce - d^2)(x, y)^4.$$

The quantic itself is one of its own covariants. The term covariant may be used in contradistinction to, or as including, invariant. The terms invariant, covariant, have been explained in reference to the simple case of a single quantic containing but one set of variables, but they apply equally to the case of a system of quantics, and to quantics which are homogeneous functions of two or more distinct sets of variables. There is one case which it is proper to mention; if in conjunction with a quantic $(*)(x, y, z, ..)^m$ we consider a linear function $\xi x + \eta y\ \zeta z + ..$, the invariants of the system are functions of the coefficients of the quantic, and of the coefficients $\xi, \eta, \zeta, ...$ of the linear function; and treating these as facients, the invariant is said to be a *contravariant* of the given quantic.

The foregoing definition gives the characteristic property of a covariant, but it does not directly show how the covariants of a given quantic are to be investigated. This is supplied as follows:—For any quantic with arbitrary coefficients, for example $(a, b, c, d)(x, y)^4$, there exist operators involving differentiations in respect to the coefficients, tantamount to the operators xd_y and yd_x in respect to the variables; thus the operator $ad_b + 2bd_c + 3cd_d$ is tantamount to yd_x, and $3bd_a + 2bd_c + cd_d$ is tantamount to xd_y. Or what is the same thing, denoting for shortness these operators by $\{yd_x\}$, $\{xd_y\}$ respectively, the quantic is reduced to zero by each of the operators $\{yd_x\} - yd_x$, $\{xd_y\} - xd_y$. Any function of the coefficients and the variables which, in like manner with the quantic itself, is reduced to zero by these two operators respectively, is said to be a *covariant;* or, if it contains the coefficients only, an *invariant* of the quantic. That the two definitions lead to the same result is of course a theorem to be proved.

The leading coefficient of a covariant, say the coefficient of x^m in any covariant of a binary quantic $(*)(x, y)^m$, possesses the property of being reduced to zero by the operator $\{yd_x\}$, and has been termed a *peninvariant* but a more appropriate term is *seminvariant.* An invariant is a function of a given degree in the coefficients, and a covariant is a function of a given degree in the coefficients and order in the variables, and they may be and are designated accordingly; thus, the above-mentioned invariants I, J of a binary quartic are called respectively the *quadrinvariant* and the *cubinvariant,* and the covariant of the same quartic is termed the *quadricovariant,* or if the distinction were required it would be termed the *quadricovariant quartic.* In these cases the designations are sufficient, but it is to be noticed that in general there is more than one invariant or covariant of the same degree or of the same degree and order, and that any such designation is only a generic, not a specific, name. An invariant or covariant may also be designated by a name referring to the mode of generation—for example, the discriminant. The name *catalecticant* denotes a certain invariant of a binary quantic of an even order: namely, it is a determinant, which, for the above-mentioned function, is

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$$

(being in this particular case the cubinvariant), and the name *canonisant* denotes a certain covariant of a binary quantic of an odd order, namely, it is a determinant the terms whereof are linear functions of the coefficients, and which for the cubic $(a, b, c, d)(x, y)^3$ is

$$\begin{vmatrix} ax + by, & bx + cy \\ bx + cy, & cx + dy \end{vmatrix}$$

(being for the particular case the Hessian or quadricovariant).

Canonical Forms.—A binary quantic of an odd order $2m + 1$ admits of being expressed as a sum of $(m + 1)$ powers of linear functions, for example, the cubic $(a, b, c, d)(x, y)^3$ can be expressed in the form $(lx + my)^3 + (l'x + m'y)^3$—this is the

canonical form of a binary function of an odd order. And there is in like manner a form (not admitting, however, of a simple definition) which is taken as the *canonical form* of a binary quantic of an even order. The catalecticant and the canonisant present themselves in the problem of the reduction of a binary quantic to the canonical form.

BEZOUTIC MATRIX.—If V, W are any two binary quantics of the same order m, and V', W' are what V, W become when the variables (x, y) of the two quantics are changed into (x', y'); then $(VW' - VW') \div (xy' - x'y)$ is a rational and integral homogeneous function of the degree $m-1$ in each of the two sets $(x, y)(x', y')$, and the coefficients taken in their natural square arrangement constitute the Bezoutic matrix. The determinant formed out of this matrix is in fact the resultant of the two functions, or equated to zero it is the equation obtained by the elimination of the variables from the two equations $V=0$, $W=0$. If V, W are the derived functions of one and the same binary quantic of the order m, then the corresponding matrix, being of course of the order $m-2$, is the Bezoutoidal matrix, and the determinant is then the discriminant of the single quantic.

It would be too long to explain the allied terms Bezoutiant, Cobezoutiants, Bezoutoid, Cobezoutoids.

TACINVARIANT, RECIPROCANT.—A definition in the language of analytical geometry will be the most easily intelligible, and it can readily be converted into an analytical form and made applicable to any number of variables. The function of the coefficients which equated to zero expresses that the two curves $U=0$, $V=0$, touch each other, is an invariant, namely, it is the *tacinvariant* of the two functions U, V. And in particular, if, instead of the curve $V=0$, we have the line $\xi x + \eta y + \lambda z = 0$, then the function which equated to zero expresses that this line touches the curve $U=0$, is a contravariant, namely, it is the *reciprocant* of the function U.

FUNCTIONAL DETERMINANT, JACOBIAN, HESSIAN.—If V, W be quantics, then the determinant—

$$\begin{vmatrix} d_x V, & d_y V, .. \\ d_x W, & d_y W, \\ \vdots & \end{vmatrix}$$

is the functional determinant, or Jacobian, of the quantics V, W... And if V, W,.. are the derived functions of $d_x U$, $d_y U$,.. of one and the same quantic U, then the determinant in question is the *Hessian* of the single quantic: the Hessian is in fact to the Jacobian what the discriminant is to the resultant.

CONCOMITANT, COGREDIENT, CONTRAGREDIENT.—The theory of linear transformations has been considered from a different point of view; instead of the variables of a function being *put equal* to the linear functions of a new set of variables, they are considered as being *replaced* by a new set of variables, linear functions of the original variables. Two sets of variables may be so related that when the first set is thus replaced by a set of linear functions of themselves, the second set is also replaced by

a set of linear functions of themselves, the coefficients of the two sets of linear functions being related together in a definite manner; this is *concomitance*, or rather it is (what is alone here spoken of) *simple concomitance*. The two most important kinds of concomitance are, 1st. *Congrediency*, that is, when the substitution on the second set of variables is identical with that upon the first set; 2nd. *Contragrediency*, that is, when the substitution on the second set of variables is the inverse or reciprocal one to that on the first set; it will make the notion of contragrediency clearer to remark that if the variables $x, y, ..$ and $\xi, \eta, ...$ are contragredients, then $x', y'...$ (which are linear functions of $\xi, \eta ..$) are so related that $\xi'x' + \eta'y' + ..$ is $= \xi x + \eta y + ...$ It was from the consideration of contragredient variables that the notion of a contravariant was first derived, but as above remarked, the notion is really included in that of a covariant.

COMBINANT.—A combinant is a covariant (or invariant) of a set of quantics of the same order, which, besides being a covariant in the ordinary sense of the word, is, so to speak, a covariant *quoad* the system, that is, it remains to a factor *près* unaltered, when the quantics of the system are replaced by linear functions of themselves; the factor in question being a power of the determinant formed with the coefficients of the linear functions. For instance, if $U = (a, b, c)(x, y)^2$ and $U' = (a', b', c')(x, y)^2$, then $ac' - 2bb' + ca'$ is a function which, when for the original coefficients are substituted those of $\lambda U + \mu U', \nu U + \rho U'$, is merely changed into $(\lambda\rho - \mu\nu)^2 (ac' - 2bb' + ca')$, and it is therefore a combinant. It would appear that the notion of a combinant might be extended to the case of a system of quantics not of the same order, and that the resultant of the system of quantics could be brought under the extended definition of a combinant, but this is a point which has not been considered.

The principal text-books on the foregoing subjects, are—on determinants:—Spottiswoode's *Elementary Theorems relating to Determinants*, 4to. London, 1851; Brioschi, *Teorica dei Determinanti*, 4to. Pavia, 1854, translated into French by Combescure and into German by Schellbach; Baltzer, *Ueber die Determinanten*, 8vo. Leipzig, 1857 (especially valuable for its references to the original sources). On elimination: Faà de Bruno, *Théorie générale de l'élimination*, 8vo. Paris, 1859. And extending to nearly all the subjects: Salmon, *Lessons introductory to the modern higher Algebra*, 8vo. Dublin, 1858. The memoirs on the different subjects are very numerous, and it was not thought expedient to give a list of them.

NOTES AND REFERENCES.

235. CONTAINS the demonstration and I think the first publication (1858) of Hermite's formula alluded to 135, for the reduction of an elliptic differential to the form

$$\frac{dr}{\sqrt{-J+zI-4z^3}},$$

which is in fact the Weierstrassian form, the theory of which has been of late so extensively developed.

241, 242. Figures of Poinsot's stellated Polyhedra are given, Fouché et Comberousse, *Traité de Geométrie Elementaire*, 8vo. Paris, 1866, and Dostor, "Propriétés générales des polyèdres reguliers etoilés," *Liouv.* t. v. (1879), pp. 209—226.

246. In connexion with this paper, On Contour and Slope Lines (1859), I refer to the earlier paper, Reech, "Démonstration d'une proprieté générale des surfaces fermées," *Jour. École Polyt.* Cah. 37 (1858), pp. 169—178: the contour lines are here considered with reference to a closed surface; the special object is the demonstration of the formula $I+S=M+2$, where I is the number of summits, S the number of imits (the letters I, S being thus interchanged) and M the number of cols. I refer also to the paper, Maxwell, "On Hills and Dales," *Phil. Mag.* t. XL. (1870), pp. 421—427, and *Works* (4to. Cambridge, 1890), t. II. No. XLIII.

259. It would have been proper to distinguish between *ab* and *ba*, and thus for instance to have presented the face-symbols of the polyhedron considered in the form *abcd, aefb, bfge, dcgh, adhe, ehgf* (read *ab, bc, cd, da*, &c.) so as to obtain therein each duad *once* in each of its two forms *ab* and *ba*, &c.: and the like as regards the vertex-symbols. And so as to the edge-symbols, instead of $ab=KL$, it would have been better to write $aKbL$, to be read, in like manner right-handedly, as a face- or vertex-symbol.

264. See the paper, Jenkins, "On Professor Cayley's Extension of Arbogast's Method of Derivations," *Amer. Math. Jour.* t. X. (1888), pp. 29—41.

268. In connexion herewith I refer to the memoirs Mc Clintock, "On the resolution of equations of the fifth degree," *Amer. Math. Jour.* t. VI. (1884), pp. 301—314, and "Analysis of Quintic Equations," *Amer. Math. Jour.* t. VIII. (1886), pp. 45—84: the author considers the *dexter resolvent* equation, which as he remarks is my equation

in ϕ, *ante*, p. 321; the *sinister resolvent* equation deduced from it by reversing the order of the coefficients (a, b, c, d, e, f), this is in fact the equation in χ obtained from my covariant equation, p. 323, for $\Phi = \frac{a}{U}(\phi x - \chi y)$ by writing therein $x = 0$ (viz. ψ, Mc Clintock $= \chi$, Cayley), but he afterwards modifies the form of this sinister resolvent; and a *central resolvent* equation for τ $\left(= \frac{\psi}{\phi}, \text{ Mc Clintock}\right) = \frac{\chi}{\phi}$, Cayley. We obtain this equation by writing in the equation for Φ, $x = \chi$, $y = \phi$, whence $\Phi = 0$, that is the equation becomes simply $AJ - 25D^2 = 0$, where in the covariants A, D, J the original variables x, y are to be replaced by χ, ϕ respectively: viz. the equation thus is

$$(a, b, c, d, e, f \between \chi, \phi)^5 . (J_0, J_1 \between \chi, \phi) - 25\,\{(D_0, D_1, D_2, D_3 \between \chi, \phi)^3\}^2 = 0,$$

or what is the same thing

$$(a, b, c, d, e, f \between \tau, 1)^5 . (J_0, J_1 \between \tau, 1) - 25\,\{(D_0, D_1, D_6, D_3 \between \tau, 1)^3\}^2 = 0,$$

which is the central resolvent equation for τ.

It is proper to remark that the foregoing expression $AJ - 25D^2$ for the last coefficient of the equation in Φ, which as appears p. 324 was not given in my original memoir, was in fact suggested to me by Mc Clintock's formula.

The equation in ϕ is

$$\phi^6 - 100\,B\phi^4 + 2000\,(3B^2 - 4H)\,\phi^2 - 800\,A\sqrt{5Q'}\,\phi + AJ - 25D^2 = 0,$$

viz. getting rid of the radical, we have

$$\{\phi^6 - 100\,B\phi^4 + 2000\,(3B^2 - 4H)\,\phi^2 + AJ - 25D^2\}^2 - 320000\,AQ'\phi^2 = 0,$$

a rational sextic equation in ϕ^2; and we infer that ϕ^2 is expressible as a rational function of τ. But the actual expression is obtained by Mc Clintock, and constitutes a very important and remarkable theorem; viz. we have

$$\frac{\phi^2}{500} = -\frac{(D_0, D_1, D_2, D_3 \between \tau, 1)^3}{(a, b, c, d, e, f \between \tau, 1)^5}, = -\frac{(J_0, J_1 \between \tau, 1)}{25\,(D_0, D_1, D_2, D_3 \between \tau, 1)^3},$$

the two expressions in τ being equal to each other in virtue of the foregoing equation in τ.

I verify this result in the case of the special quintic $(a, 0, 0, 0, e, f \between x, y)^5$. Writing with Mc Clintock $\phi^2 = w$, the equation in ϕ^2, or w, here is

$$(w^3 - 100\,aew^2 + 6000\,a^2e^2w + 40000\,a^3e^3)^2 - 320000\,(a^6f^4 + 256\,a^5e^5)\,w = 0;$$

and it is to be shown that, assuming as a definition of w the foregoing expressions in terms of τ, viz. for the form in question the expressions

$$\frac{w}{500} = \frac{ae^2\tau}{a\tau^5 + 5e\tau + f}, \quad = \frac{ae\tau - af}{25\tau},$$

(implying of course $(a\tau^5 + 5e\tau + f)(e\tau - f) - 25\,e^2\tau^2 = 0$ for the sextic equation in τ) the elimination of τ from these equations leads to the just mentioned sextic equation in w.

The two equations are

$$aw\tau^5 + 5e(w - 100\,ae)\,\tau + wf = 0, \quad \tau(w - 20\,ae) + 20\,af = 0,$$

or, if for greater convenience we write $20\,ae = \theta$, then

$$aw\tau^5 + 5e(w - \theta)\,\tau \qquad + wf = 0, \quad \tau(w - \theta) \qquad + 20\,af = 0,$$

we have

$$\tau = -\frac{20\,af}{w - \theta},$$

and thence

$$aw\,.\,\frac{-3200000\,a^5f^5}{(w-\theta)^5} - \frac{100\,aef(w - 5\theta)}{w - \theta} + wf = 0,$$

that is

$$w(w-\theta)^5 - 5\theta\,(w - 5\theta)(w - \theta)^4 - 3200000\,a^6f^4w = 0,$$

which should be identical with the before mentioned equation in w, that is with

$$(w^3 - 5\theta w^2 + 15\theta^2 w + 5\theta^3)^2 \qquad - (3200000 + 256\,\theta^6)\,w = 0,$$

and it is in fact at once seen that each of these equations is

$$\begin{aligned} 0 = \;& w^6 \\ & + w^5\,.\,-\;10\,\theta \\ & + w^4\,.\quad 55\,\theta^2 \\ & + w^3\,.\,-140\,\theta^3 \\ & + w^2\,.\quad 175\,\theta^4 \\ & + w\;.\,-106\,\theta^5 - 3200000\,a^6f^4 \\ & + w^0\,.\quad 25\,\theta^6\,; \end{aligned}$$

which completes the proof. The proof for the general form $(a, b, c, d, e, f \between x, y)^2$ is similar in principle, viz. treating for the moment ϕ^2 or w as a constant, we have in τ a quintic equation and a cubic equation, in each of which the coefficients contain w linearly; and the elimination of τ leads to the required sextic equation in w, but there would probably be considerable difficulty in effecting the calculations.

It thus appears that assuming the solution of the central resolvent equation for τ, $= \frac{\chi}{\phi}$, we also know ϕ: I recall that for the quintic equation whose roots are x_1, x_2, x_3, x_4, x_5, the significations of these quantities are

$$\tau = \frac{\chi}{\phi} = \frac{(12345) - (24135)}{12345 - 24135}, \quad \phi = 12345 - 24135,$$

where

12345 = 12 + 23 + 34 + 45 + 51, meaning thereby $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$,

(12345) = 123 + 234 + 345 + 451 + 512 „ „ $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$,

viz. from the values just written down, we obtain by permutation of the roots the six values of each of the functions τ and ϕ.

By means of these data, or say of $t, = \tau + a^{-1}b$, and $v, = \dfrac{\phi}{10\sqrt{5}}$ (I write v instead of his $\sqrt{v}$) Mc Clintock completes in a very elegant manner the determination of the roots x_1, x_2, x_3, x_4, x_5 of the quintic equation: the solution contains also the coefficients $\gamma, \delta, \epsilon, \zeta$ which belong to the equation deprived of its second term, viz. for the definition of these, we have

$$(a, b, c, d, e, f \between x, 1)^5 = a\,(1, 0, \gamma, \delta, \epsilon, \zeta \between x + a^{-1}b, 1)^5 = 0.$$

I reproduce this solution: writing as usual

$$\begin{aligned}
5u_1 &= x_1 + \omega\, x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5,\\
5u_2 &= x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega\, x_4 + \omega^3 x_5,\\
5u_3 &= x_1 + \omega^3 x_2 + \omega\, x_3 + \omega^4 x_4 + \omega^2 x_5,\\
5u_4 &= x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega\, x_5,
\end{aligned}$$

(ω an imaginary fifth root of unity), we find the four "Eulerian" equations

$$\begin{aligned}
-2\gamma &= u_1u_4 + u_2u_3\\
-2\delta &= u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4\\
-\epsilon + 4\gamma^2 - 3u_1u_2u_3u_4 &= u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1\\
-\zeta - 20\gamma\delta &= u_1^5 + u_2^5 + u_3^5 + u_4^5 - 10\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_2^3u_1u_3 + u_3^3u_4u_2),
\end{aligned}$$

from which u_1, u_2, u_3, u_4 are to be obtained.

It is found that the first and second equations may be replaced by the two pairs of equations

$$u_1u_4 = -\gamma - v, \qquad u_2u_3 = -\gamma + v$$

$$u_1^2u_3 + u_4^2u_2 = -\delta - tv, \quad u_2^2u_1 + u_3^2u_4 = -\delta + tv.$$

As to the first of these pairs, we find

$$\begin{aligned}
25u_1u_4 = \quad & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\\
&+ (\omega + \omega^4)(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)\\
&+ (\omega^2 + \omega^3)(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2),
\end{aligned}$$

say this is

$$25u_1u_4 = \Sigma x^2 + (\omega + \omega^4)\,12345 + (\omega^2 + \omega^3)\,13524;$$

and then similarly

$$25u_2u_3 = \Sigma x^2 + (\omega^2 + \omega^3)\,12345 + (\omega + \omega^4)\,13524;$$

whence

$$25\,(u_1u_4 - u_2u_3) = (\omega + \omega^4 - \omega^2 - \omega^3)\,(12345 - 13524), = \sqrt{5}\,\phi, = 50\,v,$$

if as above $v = \frac{\phi}{10\sqrt{5}}$; that is

$$u_1u_4 - u_2u_3 = \quad 2v,$$

and combining herewith the equation

$$u_1u_4 + u_2u_3 = -2\gamma,$$

we have the first pair of equations. The second pair of equations is obtained by a similar process, but the work is longer. We have

$$125\, u_1^2u_3 = F + A\omega + B\omega^2 + C\omega^3 + D\omega^4$$
$$125\, u_2^2u_1 = F + A\omega^2 + B\omega^4 + C\omega + D\omega^3$$
$$125\, u_3^2u_4 = F + A\omega^3 + B\omega + C\omega^4 + D\omega^2$$
$$125\, u_4^2u_2 = F + A\omega^4 + B\omega^3 + C\omega^2 + D\omega$$

where

$$F = \Sigma x_1^3 + 2\Sigma x_1x_2x_3$$
$$A = \{24135\} + 2\,\{12345\} + 2\,(24135)$$
$$B = \{54321\} + 2\,\{24135\} + 2\,(54321)$$
$$C = \{12345\} + 2\,\{53142\} + 2\,(12345)$$
$$D = \{53142\} + 2\,\{54321\} + 2\,(53142)$$

where

$$\{24135\} = x_2^2x_4 + x_4^2x_1 + x_1^2x_3 + x_3^2x_5 + x_5^2x_2, \text{ \&c.}:$$

and as before

$$(24135) = x_2x_4x_1 + x_4x_1x_3 + x_1x_3x_5 + x_3x_5x_2 + x_5x_2x_4, \text{ \&c.}$$

Hence

$$125\,(u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1) = (\omega + \omega^4 - \omega^2 - \omega^3)(A + D - B - C) = \sqrt{5}\,(A + D - B - C).$$

Here

$$A + D - B - C = \quad \{12345\} - \{24135\} - \{53142\} + \{54321\}$$
$$- 2\,[(12345) - (24135) - (53142) + (54123)],$$

where, substituting the values, the first line is

$x_1^2\,(x_2 - x_3 - x_4 + x_5) + x_2^2\,(x_3 - x_4 - x_5 + x_1) + x_3^2\,(x_4 - x_5 - x_1 + x_2) + x_4^2\,(x_5 - x_1 - x_2 + x_3) + x_5^2\,(x_1 - x_2 - x_3 + x_4)$, $= \Sigma'$ suppose: and the second line observing that (54123) and (53142) are equal to (12345) and (24135) respectively, is $= -4\,[(12345) - (24135)]$, $= -4\chi$: hence

$$A + D - B - C = \Sigma' - 4\chi.$$

But from the equations

$$a^{-1}\phi = 12345 - 24135, \; = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 - x_2x_4 - x_4x_1 - x_1x_3 - x_3x_5 - x_5x_2,$$

and

$$-5b \qquad = x_1 + x_2 + x_3 + x_4 + x_5,$$

we easily obtain

$$-5a^{-1}b\phi = \Sigma' + \chi,$$

and the last result thus is

$$A + D - B - C = -5a^{-1}b\phi - 5\chi, = -5a^{-1}b\phi - 5\phi\tau, = -5\phi(a^{-1}b + \tau),$$

or substituting for ϕ and $a^{-1}b + \tau$ their values $10\sqrt{5}\,v$, and t, this is $= -50\sqrt{5}\,tv$. Hence the value of $125(u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1)$ is $= -250\,tv$, that is we have

$$u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1 = -2tv,$$

and combining herewith

$$u_1^2u_3 + u_4^2u_2 + u_3^2u_4 + u_2^2u_1 = -2\delta,$$

we have the second pair of equations.

We next write

$$\begin{aligned} 2u_1^2u_3 &= -\delta - tv + 2n_1, \\ 2u_4^2u_2 &= -\delta - tv - 2n_1, \\ 2u_2^2u_1 &= -\delta + tv + 2n_2, \\ 2u_3^2u_4 &= -\delta + tv - 2n_2, \end{aligned}$$

whence

$$\begin{aligned} 4n_1^2 &= (\delta + tv)^2 + 4(\gamma^2 - v^2)(\gamma + v), \\ 4n_2^2 &= (\delta - tv)^2 + 4(\gamma^2 - v^2)(\gamma - v), \end{aligned}$$

which equations determine n_1, n_2, so that $u_1^2u_3$, $u_4^2u_2$, $u_2^2u_1$ and $u_3^2u_4$ are now known; and we then have

$$\begin{aligned} u_1^5 &= (u_1^2u_3)^2(u_3^2u_1) \div (u_3u_3)^2, \\ u_2^5 &= (u_2^2u_1)^2(u_4^2u_2) \div (u_1u_4)^2, \\ u_4^5 &= (u_4^2u_2)^2(u_3^2u_4) \div (u_2u_3)^2, \\ u_3^2 &= (u_3^2u_4)^2(u_1^2u_3)^2 \div (u_1u_4)^4, \end{aligned}$$

which determine u_1, u_2, u_3, u_4. (Compare herewith the formulæ, $A = \dfrac{A'A''}{\beta\gamma}$, &c. p. 54 of my paper "On a solvible quintic Equation," *Amer. Math. Jour.* t. XIII. (1890), pp. 53—58.)

But, as Mc Clintock remarks, it is possible to obtain better formulæ: viz. these are

$$\begin{aligned} u_1^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + \sqrt{s_1 + s_2}, \\ u_2^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + \sqrt{s_1 - s_2}, \\ u_3^5 &= \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - \sqrt{s_1 - s_2}, \\ u_4^5 &= \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - \sqrt{s_1 + s_2}, \end{aligned}$$

where

$$r_1 = -\zeta + 20\,tv^2,$$

$$r_2 = (\gamma^2 - v^2)^{-1}\, v^{-1} \left\{ \begin{array}{l} (-\delta + 2\gamma t - t^2)\, v^4 \\ + \{\delta\epsilon + 2\gamma^2\delta + (-2\gamma\epsilon + \delta^2 + 4\gamma^3)\, t + \gamma\delta t^2\}\, v^2 \\ + \{\gamma^2\delta\epsilon - \gamma\delta^3\} \end{array} \right\},$$

$$s_1 = \tfrac{1}{16} r_1^2 + \tfrac{1}{16} r_2^2 + \gamma^5 + 10\gamma^3 v^2 + 5\gamma v^4,$$

$$s_2 = \tfrac{1}{8} r_1 r_2 \quad - 5\gamma^4 v - 10\gamma^2 v^3 - v^5.$$

To prove these results write for shortness

$2m_1 = -\delta - tv$, $2m_2 = -\delta + tv$, and (as above) $2n_1 = u_1^2 u_3 - u_4^2 u_2$, $2n_2 = u_2^2 u_1 - u_3^2 u_4$, then

$$u_1^2 u_3 = m_2 + n_1, \quad u_4^2 u_2 = m_1 - n_1, \quad u_2^2 u_1 = m_2 + n_2, \quad u_3^2 u_4 = m_2 - n_2;$$

we have $u_1 u_2 u_3 u_4 = u_1 u_4 \,.\, u_2 u_3 = (-\gamma + v)(-\gamma - v)$, $= \gamma^2 - v^2$, and the third Eulerian equation thus becomes

$$\epsilon = \gamma^2 + 3v^2 + (v^2 - \gamma^2)^{-1} \{u_1^4 u_2^2 u_3 u_4 + u_4^4 u_3^2 u_1 u_2 + u_2^4 u_3^2 u_1 u_3 + u_3^4 u_1^2 u_2 u_4\}.$$

The terms within the { } are equal to

$$(m_1 + n_1)(m_2 + n_2)(-\gamma + v), \quad (m_1 - n_1)(m_2 - n_2)(-\gamma + v), \quad (m_1 - n_1)(m_2 + n_2)(-\gamma - v),$$
$$(m_1 + n_1)(m_2 - n_2)(-\gamma - v),$$

respectively, and their sum is $= -4\gamma m_1 m_2 + 4v n_1 n_2$.

Hence putting for shortness $p = 4v n_1 n_2$, $= 4v(u_1^2 u_3 - u_4^2 u_2)(u_2^2 u_1 - u_3^2 u_4)$, the third equation becomes

$$\epsilon = \gamma^2 + 3v + (v^2 - \gamma^2)^{-1} (\gamma t^2 v^2 - \gamma\delta^2 + p),$$

or, what is the same thing,

$$p = \gamma\delta^2 - \gamma t^2 v^2 + (v^2 - \gamma^2)(\epsilon - \gamma^2 - 3v).$$

Again, writing the fourth Eulerian equation in the form

$$\Sigma u^5 = -\zeta - 20\gamma\delta + 10\{(m_1 + n_1) u_1 u_4 + (m_1 - n_1) u_1 u_4 + (m_2 + n_2) u_2 u_3 + (m_2 - n_2) u_2 u_3\},$$

the term in { } is $2m_1 u_1 u_4 + 2m_2 u_2 u_3$, $= (-\delta - tv)(-\gamma + v) + (-\delta + tv)(-\gamma - v)$, $= 2\gamma\delta - 2tv^2$, so that writing $\Sigma u^5 = r_1$, the equation becomes

$$r_1 = -\zeta - 20tv^2,$$

viz. this is the above-mentioned value of r_1.

We then have $r_2 = u_1^5 + u_4^5 - u_2^5 - u_3^5$, and for calculating this, we have

$$(\gamma^2 - v^2)^2\, u_1^5 = (m_1 + n_1)^2 (m_2 + n_2)(\gamma - v)^2,$$
$$(\gamma^2 - v^2)^2\, u_4^5 = (m_1 - n_1)^2 (m_2 - n_2)(\gamma - v)^2,$$
$$(\gamma^2 - v^2)^2\, u_2^5 = (m_2 + n_2)^2 (m_1 - n_1)(\gamma + v)^2,$$
$$(\gamma^2 - v^2)\, u_3^5 = (m_2 - n_2)^2 (m_1 + n_1)(\gamma + v)^2,$$

and thence, instead of $4vn_1n_2$ which occurs on the right-hand side writing its value $=p$, we find

$$\begin{aligned}(\gamma^2-v^2)^2 r_2 = \quad & (m_1+m_2)\,v^{-1}\{p\,(\gamma^2+v^2)-4m_1m_2\gamma\} \\ & +2\,(m_1-m_2) \quad \{m_1m_2\,(\gamma^2+v^2)- \quad p\gamma\} \\ & +2\,(n_1^2m_2-n_2^2m_1) \qquad (\gamma^2+v^2) \\ & -4\,(n_1^2m_1+n_2^2m_1) \qquad \gamma v.\end{aligned}$$

We have

$$4n_1^2=(\delta+tv)^2+4\,(\gamma^2-v^2)\,(\gamma-v), \quad 4n_2^2=(\delta-tv)^2+4\,(\gamma^2-v^2)\,(\gamma+v),$$

p is given as above, and moreover $m_1+m_2=-\delta$, $m_1-m_2=-tv$, $m_1m_2=\frac{1}{4}\,(\delta^2-t^2v^2)$. Substituting these values and dividing out by $(\gamma^2-v^2)^2$, we find after some reductions the value given above for r_2.

Finally we have

$$4\,(\gamma-v)^5=-4\,(u_1u_4)^5=(u_1^5+u_4^5)^2-4\,(u_1u_4)^5-\tfrac{1}{2}\,(r_1+r_2)\,(u_1^5+u_4^5)=(u_1^5-u_4^5)^2-\tfrac{1}{2}\,(r_1+r_2)\,(u_1^5+u_4^5),$$

that is

$$4\,(\gamma-v)^6=4\,(s_1+s_2)-\tfrac{1}{4}\,(r_1+r_2)^2,$$

or say

$$s_1+s_2=(\tfrac{1}{4}r_1+\tfrac{1}{4}r_2)^2+(\gamma-v)^5,$$

and similarly

$$s_1-s_2=(\tfrac{1}{4}r_1-\tfrac{1}{4}r_2)^2+(\gamma+v)^5$$

which equations give the above-mentioned values of s_1 and s_2.

As to Jacobi's Memoir spoken of in the Addition, I refer to the paper Kronecker, "Ueber eine stelle in Jacobi's Aufsatz, Observatiunculæ ad theoriam æquationum pertinentes," *Crelle*, t. CVII. (1891), pp. 349—352, which incorporates some remarks of mine in regard thereto.

284, 294. The fundamental idea of these two papers is *not* that of "the six coordinates of a line," but (as indeed appears from the title) a somewhat different one, viz. I say that a curve in space may be represented by a homogeneous equation $V=0$ between six coordinates (p, q, r, s, t, u) such that $ps+qt+ru=0$; this equation being the equation of a cone of arbitrary vertex passing through the curve in question: taking x, y, z, w to be current point-coordinates and $\alpha, \beta, \gamma, \delta$ to be the point-coordinates of the arbitrary vertex, then p, q, r, s, t, u are the six determinants of the matrix

$$\left(\begin{matrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{matrix}\right),$$

or, what is the same thing, we have

$$\begin{aligned} p&=\gamma y-\beta z, & s&=\delta x-\alpha w, \\ q&=\alpha z-\gamma x, & t&=\delta y-\beta w, \\ r&=\beta x-\alpha y, & u&=\delta z-\gamma w, \end{aligned}$$

values which satisfy $ps+qt+ru=0$. And I accordingly say that the equation of a line in space is

$$Ap+Bq+Cr+Fs+Gt+Hu=0,$$

viz. this is the equation of the cone of arbitrary vertex $(\alpha, \beta, \gamma, \delta)$ (that is, of the plane through the point $(\alpha, \beta, \gamma, \delta)$) which passes through the line in question. But I go on to say that if $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$ are the coordinates of any two points on the given line, or if the line be given as the intersection of the two planes $ax+by+cz+dw=0$, $a'x+b'y+c'z+d'w=0$, then in the first case

$$\begin{aligned} A&=\alpha'\delta''-\alpha''\delta', & F&=\beta'\gamma''-\beta''\gamma', \\ B&=\beta'\delta''-\beta''\delta', & G&=\gamma'\alpha''-\gamma''\alpha', \\ C&=\gamma'\delta''-\gamma''\delta', & H&=\alpha'\beta''-\alpha''\beta', \end{aligned}$$

and in the second case

$$\begin{aligned} A&=bc'-b'c, & F&=ad'-a'd, \\ B&=ca'-c'a, & G&=bd'-b'd, \\ C&=ab'-a'b, & H&=cd'-c'd, \end{aligned}$$

so that in each case $AF+BG+CH=0$. I thus in effect, although not quite explicitly, define (A, B, C, F, G, H) as the "six coordinates of a line"; and after giving in terms of these quantities for any two lines the condition of the intersection of the two lines I say that any other problems in relation to the line, for instance...&c., may also be solved "by means of the new coordinates."

Plücker's Memoir "On a New Geometry of Space" is published *Phil. Trans.* t. CLV. (for 1865), pp. 725—791, the paper being received Dec. 22, 1864, and the Additional Note appended thereto, Dec. 11, 1865. My two papers are referred to in the foot-note p. 784, belonging to this additional note as follows: "In two remarkable papers 'On a New Analytical Representation of Curves in Space' published in the third and fifth volumes of the *Quarterly Journal of Mathematics*, Professor Cayley employed before me in order to represent cones the six coordinates of a right line depending upon any two of its points. Having lately only seen the papers I hasten to mention it now. But besides the coincidence referred to the leading views of Professor Cayley's paper and mine have nothing in common. On this occasion I may state that the principles upon which my paper is based were advanced by me nearly twenty years ago (Geometry of Space, No. 258), but this had entirely escaped my memory when I recurred to Geometry some time since."

In the work referred to, "System der Geometrie des Raumes u. s. w." (4to. Düsseldorf, 1846), No. 258, Plücker remarks that a straight line depends upon four constants, viz. its equations in point-coordinates being $x=\kappa z+\lambda$, $y=\mu z+\nu$, or in line-coordinates being $t=\kappa v+\lambda w$, $u=\mu v+\nu w$, then in either case the constants are $\kappa, \lambda, \mu, \nu$; and he defines these four quantities as the "four coordinates of a line."

The leading idea of Plücker's memoir appears in the first words thereof, "I. On linear Complexes of Right lines." He works at first with the *four* coordinates of a line; as long as these are arbitrary the line is any line whatever; but considering them as connected by a single equation, then there is he says "a Complex," considering them as connected by two equations "a Congruency", and considering them as connected by three equations "a Configuration" or ruled surface. The establishment of these notions of a Complex and a Congruency, and the general idea of regarding the line as an element in the Geometry of Space are absolutely Plücker's, there is no anticipation of them in my two papers. Later on five coordinates r, s, σ, ρ, $s\rho - r\sigma$ are introduced, but the *six* coordinates are first used in the Additional Note, viz. here instead of a non-homogeneous equation or equations between four coordinates we have a homogeneous equation or equations between six quantities connected by a homogeneous quadric equation. It is hardly necessary to refer to the posthumous work *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Leipzig, 1868 and 1869, or to later developments of the theory.

END OF VOL. IV.

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

CAMBRIDGE UNIVERSITY PRESS.

A Treatise on Natural Philosophy. By Sir W. THOMSON, LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Professors Sir W. THOMSON and P. G. TAIT. Demy 8vo. 9*s.*

An Elementary Treatise on Quaternions. By P.G.TAIT,M.A. *3rd Edition.* Demy 8vo. 18*s.*

A Treatise on the Theory of Determinants and their applications in Analysis and Geometry, by R. F. SCOTT, M.A., Fellow of St John's College. Demy 8vo. 12*s.*

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A. Demy 8vo. 12*s.*

Elementary Thermodynamics. By J. PARKER, M.A., Fellow of St John's College, Cambridge. Crown 8vo. 9*s.*

The Analytical Theory of Heat, by JOSEPH FOURIER. Translated, with Notes, by A. FREEMAN, M.A., formerly Fellow of St John's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on the General Principles of Chemistry. By M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A. and D. J. CARNEGIE, M.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

Lectures on the Physiology of Plants. By S. H. VINES, Sc.D., Professor of Botany in the University of Oxford. Demy 8vo. With Illustrations. 21*s.*

A Short History of Greek Mathematics. By J. GOW, Litt.D., Fellow of Trinity College. Demy 8vo. 10*s.* 6*d.*

Diophantos of Alexandria; a Study in the History of Greek Algebra. By T. L. HEATH, M.A., Fellow of Trinity College, Cambridge. Demy 8vo. 7*s.* 6*d.*

PITT PRESS MATHEMATICAL SERIES.

Arithmetic for Schools. By C. SMITH, M.A., Master of Sidney Sussex College, Cambridge. Extra Fcap. 8vo. 3*s.* 6*d.*

Elementary Algebra (with Answers to the Examples). By W. W. ROUSE BALL, M.A., Fellow and Mathematical Lecturer of Trinity College, Cambridge. Extra Fcap. 8vo. 4*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Books I. & II. Extra Fcap. 8vo. 1*s.* 6*d.* Books III. and IV. 1*s.* 6*d.* Books I.—IV. in one Volume. 3*s.*

The Elements of Statics and Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. Extra Fcap. 8vo. 7*s.* 6*d.*

Part I. ELEMENTS OF STATICS. Extra Fcap. 8vo. 4*s.* 6*d.*

Part II. ELEMENTS OF DYNAMICS. Extra Fcap. 8vo. 3*s.* 6*d.*

London: C. J. CLAY AND SONS,

CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,

AVE MARIA LANE.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 4to. 10 vols. Vols. I. II. III. and IV. 25*s.* each. [Vol. V. *In the Press.*

The Electrical Researches of the Hon. H. Cavendish, F.R.S. Written between 1771 and 1781. Edited from the original MSS. in the possession of the Duke of Devonshire, K.G., by the late J. CLERK MAXWELL, F.R.S. Demy 8vo. 18*s.*

The Scientific Papers of the late Prof. J. Clerk Maxwell. Edited by W. D. NIVEN, M.A., formerly Fellow of Trinity College. In 2 vols. Royal 4to. £3. 3*s.* (net).

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.*
Vol. II. Saint-Venant to Sir WILLIAM THOMSON. By the same Editor. [*Nearly ready.*

A History of the Study of Mathematics at Cambridge. By W. W. ROUSE BALL, M.A., Fellow and Lecturer on Mathematics of Trinity College, Cambridge. Crown 8vo. 6*s.*

Mathematical and Physical Papers, by Sir W. THOMSON, LL.D., D.C.L., F.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Demy 8vo. Vol. I. 18*s.* Vol. II. 15*s.* Vol. III. 18*s.*

Mathematical and Physical Papers, by Sir G. G. STOKES, Sc.D., LL.D., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge. Reprinted from the Original Journals and Transactions, with Additional Notes by the Author. Vol. I. Demy 8vo. 15*s.* Vol. II. 15*s.* [Vol. III. *In the Press.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Royal 4to. cloth (Vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half-morocco 28*s.* (net). Vol. IX. New Series for the years 1874—1883. [*Nearly ready.*

Theory of Differential Equations. Part I. Exact Equations and Pfaff's Problem. By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, Cambridge. Demy 8vo. 12*s.*

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. New and Enlarged Edition. Crown 8vo. 7*s.* 6*d.*

Solutions to the Examples in a Treatise on Elementary Dynamics. By the same Author. [*In the Press.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of Peterhouse, Cambridge. Vol. I. Demy 8vo. 14*s.*

A Treatise on Plane Trigonometry. By E. W. HOBSON, M.A., Fellow and Lecturer of Christ's College, Cambridge. Demy 8vo. 12*s.*

A Catalogue of the Portsmouth Collection of Books and Papers written by or belonging to Sir ISAAC NEWTON. Demy 8vo. 5*s.*

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. V.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1892

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 84 papers numbered 300 to 383 published for the most part in the years 1861 to 1866: No. 378, Report on Catalogue of Philosophical Memoirs, was however published in the *British Association Report for* 1856, and No. 379, Notices of Communications to the British Association, were published in the *British Association Reports*, 1854 to 1864: the concluding Paper 383, Problems and Solutions, contains problems for the most part geometrical ones proposed or solved by me in the *Educational Times* in the years 1863 to 1865.

The Table for the five volumes is

Vol.	I.	Numbers	1	to	100.
,,	II.	,,	101	,,	158.
,,	III.	,,	159	,,	222.
,,	IV.	,,	223	,,	299.
,,	V.	,,	300	,,	383.

CONTENTS.

[An Asterisk denotes that the paper is not printed in full.]

PAGE

300. *Note relative aux droites en involution de M. Sylvester* . . 1
Comptes Rendus, Paris, t. LII. (1861), pp. 1039—1042

301. *Sur les cônes du second ordre qui passent par six points donnés* 4
Comptes Rendus, Paris, t. LII. (1861), pp. 1216—1218

302. *Considérations générales sur les courbes en espace* . . . 7
Comptes Rendus, Paris, t. LIV. (1862), pp. 55—60, 396—400, 672—678

303. *Sur le problème du polygone inscrit et circonscrit. Lettre à M. Poncelet* 21
Comptes Rendus, Paris, t. LV. (1862), pp. 700, 701

304. *Sur un mémoire de Jacobi. Extrait d'une lettre à M. J. Bertrand* 23
Comptes Rendus, Paris, t. LVI. (1863), p. 43

305. *Considérations générales sur les courbes en espace. Courbes du cinquième ordre* 24
Comptes Rendus, Paris, t. LVIII. (1864), pp. 994—1000

306. *Sur les coniques qui touchent des courbes d'ordre quelconque. Extrait d'une lettre à M. Chasles* 31
Comptes Rendus, Paris, t. LIX. (1864), pp. 224, 225

307. *Note sur les fonctions* al (x), *&c., de M. Weierstrass* . . . 33
Liouville, t. VII. (1862), pp. 137—142

308. *On the* $\triangle$ *faced Polyacrons, in reference to the problem of the enumeration of Polyhedra* 38
Manchester Memoirs, t. I. (1862), pp. 248—256

PAGE

309. *Note on the Theory of Determinants* 45
Phil. Mag. t. XXI. (1861), pp. 180—185

310. *Note on Mr Jerrard's researches on the Equation of the Fifth Order* 50
Phil. Mag. t. XXI. (1861), pp. 210—214

311. *On a Theorem of Abel's relating to Equations of the Fifth Order* 55
Phil. Mag. t. XXI. (1861), pp. 257—263

312. *On the Partitions of a Close* 62
Phil. Mag. t. XXI. (1861), pp. 424—428

313. *On a Surface of the Fourth Order* 66
Phil. Mag. t. XXI. (1861), pp. 491—495

314. *On the Curves situate on a Surface of the Second Order* . . 70
Phil. Mag. t. XXII. (1861), pp. 35—38

315. *On the Cubic Centres of a Line with respect to three lines and a line* 73
Phil. Mag. t. XXII. (1861), pp. 433—436

*316. *Note on the solution of an Equation of the Fifth Order* . . 77
Phil. Mag. t. XXIII. (1862), pp. 195, 196

317. *Note on the transformation of a certain Differential Equation* 78
Phil. Mag. t. XXIII. (1862), pp. 266, 267

*318. *On a question in the Theory of Probabilities* 80
Phil. Mag. t. XXIII. (1862), pp. 361—365

*319. *Postscript to the paper, On a question in the Theory of Probabilities* 85
Phil. Mag. t. XXIII. (1862), pp. 470, 471

320. *On the Transcendent* $\operatorname{gd} u = \frac{1}{i} \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$ 86
Phil. Mag. t. XXIV. (1862), pp. 19—21

*321. *Final Remarks on Mr Jerrard's theory of Equations of the Fifth Order* 89
Phil. Mag. t. XXIV. (1862), p. 290

PAGE

322. *On a Skew Surface of the Third Order* 90
Phil. Mag. t. XXIV. (1862), pp. 514—519

323. *On a tactical Theorem relating to the Triads of Fifteen Things* 95
Phil. Mag. t. XXV. (1863), pp. 59—61

324. *Note on a Theorem relating to Surfaces* 98
Phil. Mag. t. XXV. (1863), pp. 61, 62

325. *Note on a Theorem relating to a Triangle, Line, and Conic* . 100
Phil. Mag. t. XXV. (1863), pp. 181—183

326. *Theorems relating to the Canonic Roots of a Binary Quantic of an odd order* 103
Phil. Mag. t. XXV. (1863), pp. 206—208

327. *On the Stereographic Projection of the Spherical Conic* . . 106
Phil. Mag. t. XXV. (1863), pp. 350—353

328. *On the delineation of a Cubic Scroll* 110
Phil. Mag. t. XXV. (1863), pp. 528—530

329. *Note on the Problem of Pedal Curves* 113
Phil. Mag. t. XXVI. (1863), pp. 20, 21

330. *On Differential Equations and Umbilici* 115
Phil. Mag. t. XXVI. (1863), pp. 373—379 and 441—452

331. *Analytical Theorem relating to the four Conics inscribed in the same Conic and passing through the same three Points* 131
Phil. Mag. t. XXVII. (1864), pp. 42, 43

332. *Analytical Theorem relating to the sections of a Quadric Surface* 133
Phil. Mag. t. XXVII. (1864), pp. 43, 44

333. *Note on the Nodal Curve of the Developable derived from the Quartic Equation* $(a, b, c, d, e \between t, 1)^4 = 0$ . . . 135
Phil. Mag. t. XXVII. (1864), pp. 437—440

334. *Note on the Theory of Cubic Surfaces* 138
Phil. Mag. t. XXVII. (1864), pp. 493—496

PAGE

335. *Tables des formes quadratiques binaires pour les déterminants négatifs depuis $D=-1$ jusqu'à $D=-100$, pour les déterminants positifs non carrés depuis $D=2$ jusqu'à $D=99$, et pour les treize déterminants négatifs irréguliers qui se trouvent dans le premier millier* 141
Crelle, t. LX. (1862), pp. 357—372

336. *Note sur l'élimination* 157
Crelle, t. LX. (1862), pp. 373, 374

337. *Note sur la réalité des racines d'une équation quadratique* . 160
Crelle, t. LXI. (1863), pp. 367, 368

338. *Nouvelles recherches sur l'élimination et la théorie des courbes* 162
Crelle, t. LXIII. (1864), pp. 34—39

339. *On Skew Surfaces, otherwise Scrolls*. 168
Phil. Trans. t. CLIII. (for 1863), pp. 453—483

340. *A Second Memoir on Skew Surfaces, otherwise Scrolls* . . 201
Phil. Trans. t. CLIV. (for 1864), pp. 559—576

341. *On the Sextactic Points of a Plane Curve* 221
Phil. Trans. t. CLV. (for 1865), pp. 545—578

342. *On the Conics which pass through three given points and touch a given line* 258
Quart. Math. Jour. t. VI. (1864), pp. 24—30

343. *On the Cusp of the second kind or Nodecusp* 265
Quart. Math. Jour. t. VI. (1864), pp. 74, 75

344. *On Certain Developable Surfaces* 267
Quart. Math. Jour. t. VI. (1864), pp. 108—126

345. *On the Inflexions of the Cubical Divergent Parabolas* . . 284
Quart. Math. Jour. t. VI. (1864), pp. 199—203

346. *Note on an expression for the Resultant of two Binary Cubics* 289
Quart. Math. Jour. t. VI. (1864), pp. 380—382

347. *On the Notion and Boundaries of Algebra* 292
Quart. Math. Jour. t. VI. (1864), pp. 382—384

PAGE

348. *On the Theory of Involution* 295
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 21—38

349. *On a case of the Involution of Cubic Curves* 313
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 39—80

350. *On the Classification of Cubic Curves* 354
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 81—128

351. *On Cubic Cones and Curves* 401
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 129—144

352. *Suite des recherches sur l'élimination et la théorie des courbes* 416
Crelle, t. LXIV. (1865), pp. 167—171

353. *Note sur la surface du quatrième ordre de Steiner* . . 421
Crelle, t. LXIV. (1865), pp. 172—174

354. *Note sur les singularités supérieures des courbes planes* . . 424
Crelle, t. LXIV. (1865), pp. 369—371

355. *Sur un théorème relatif à huit points situés sur une conique* . 427
Crelle, t. LXV. (1866), pp. 180—184

356. *Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers* 431
Crelle, t. LXV. (1866), pp. 284—291

357. *A Supplementary Memoir on the Theory of Matrices* . . 438
Phil. Trans. t. CLVI. (for 1866), pp. 25—35

358. *Addition to the Memoir on Tschirnhausen's Transformation* . 449
Phil. Trans. t. CLVI. (for 1866), pp. 97—100

359. *A Supplementary Memoir on Caustics* 454
Phil. Trans. t. CLVII. (for 1867), pp. 7—16

360. *Note on a Quartic Surface* 465
Phil. Mag. t. XXIX. (1865), pp. 19—22

361. *On Quartic Curves* 468
Phil. Mag. t. XXIX. (1865), pp. 105—108

PAGE

362. *Note on Lobatschewsky's Imaginary Geometry* 471
Phil. Mag. t. XXIX. (1865), pp. 231—233

363. *On the theory of the Evolute* 473
Phil. Mag. t. XXIX. (1865), pp. 344—350

364. *On a Theorem relating to Five Points in a plane* . . . 480
Phil. Mag. t. XXIX. (1865), pp. 460—464

365. *On the Intersections of a Pencil of four lines by a Pencil of two lines* 484
Phil. Mag. t. XXIX. (1865), pp. 501—503

366. *Note on the Projection of the Ellipsoid*. 487
Phil. Mag. t. XXX. (1865), pp. 50—52

367. *On a Triangle in-and-circumscribed to a Quartic Curve* . 489
Phil. Mag. t. XXX. (1865), pp. 340—342

368. *On a problem of Geometrical Permutation* 493
Phil. Mag. t. XXX. (1865), pp. 370—372

369. *On a property of Commutants* 495
Phil. Mag. t. XXX. (1865), pp. 411—413

370. *On the signification of an elementary formula of Solid Geometry* 498
Phil. Mag. t. XXX. (1865), pp. 413, 414

371. *On a Formula for the intersections of a Line and Conic, and on an Integral Formula connected therewith* . . 500
Quart. Math. Jour. t. VII. (1866), pp. 1—6

372. *On the Reciprocation of a Quartic Developable* . . . 505
Quart. Math. Jour. t. VII. (1866), pp. 87—92

373. *On a Special Sextic Developable.* 511
Quart. Math. Jour. t. VII. (1866), pp. 105—113

374. *On the Higher Singularities of a Plane Curve* 520
Quart. Math. Jour. t. VII. (1866), pp. 212—223

PAGE

375. *Notes on Polyhedra* 529

Quart. Math. Jour. t. VII. (1866), pp. 304—316

376. *Théorème relatif à l'équilibre de quatre forces* . . . 540

Comptes Rendus, t. LXI. (1865), pp. 829, 830

377. *Note sur la correspondance de deux points sur une courbe* . 542

Comptes Rendus, t. LXII. (1866), pp. 586—590

378. *Report of a Committee appointed by the British Association to consider the formation of a Catalogue of Philosophical Memoirs (A. Cayley, R. Grant, G. G. Stokes)* . . . 546

Report of the British Association (1856), pp. 463, 464

379. *Notices of Communications to the British Association for the Advancement of Science* 549

Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1854 to 1864)

380. *Note on the Rectangular Hyperbola* 554

Oxford, Camb. and Dubl. Messenger of Mathematics, t. I. (1862), p. 77

381. *Note on Bezout's Method of Elimination* 555

Oxford, Camb. and Dubl. Messenger of Mathematics, t. II. (1864), pp. 88, 89

382. *Note on the Tetrahedron* 557

Oxford, Camb. and Dubl. Messenger of Mathematics, t. III. (1866), pp. 8—10

383. *Problems and Solutions* 560

Mathematical Questions with their Solutions from the Educational Times, vols. I. to IV. (1863 to 1865); for contents, see p. 612

Notes and References 613

Plates . . . . . *to face pp.* 44, 400

CLASSIFICATION.

GEOMETRY

The Node-cusp, 343; Higher Singularities of Plane Curves, 354, 374
Curves in space (as defined by Cone and Monoid Surface), 302, 305
Correspondence of Points on Plane Curve, 377
Sextactic Points of Plane Curve, 341
Lines in Involution, 300; Involution, 348, 349
Elimination and Theory of Curves, 338, 352
Conics touching given Curves, 306
Conics, 325, 331, 342, 355, 371, 380; Spherical, 327
Cubic Curves and Cones, 345, 350, 351
Cubic Surfaces 322, 328, 333
Quadric Cones and Surfaces, 301, 314, 332, 366
Quartic Curves, 361, 367
Quartic Surfaces, 313, 360; Steiner's, 353; 16-nodal, 356
Caustics, 359
Equilibrium of Four Forces, 376
Evolutes, 363
In-and-circumscribed Polygon, 303
Surfaces, 324; Developable, 344, 372, 373; Skew, 339, 340
Miscellaneous, 315, 325, 364, 365, 368, 370, 382; and 383, (see contents, p. 612)
Polyhedra, Notes on, 375: the Δ-faced polyacra, 308
Lobatschewsky's Imaginary Geometry, 362
Partitions of Close, 312
Pedal Curves, 329
Umbilici, 330

ANALYSIS

Algebra, the Notion and Boundaries of, 347
Differential Equations, 317; theorem of Jacobi's, 304; for Umbilici, 330
Determinants, 309; Commutants, 369
Binary Quadratic Forms (numerical), tables of, 335
Binary Forms, 326, 346
Roots of quadratic equation, 337
Elimination, 336, 338, 352, 381

Functions al (x) of Weierstrass, 307
Gudermannian, 320
Integral Formula connected with Line and Conic, 371
Matrices, 357
Probabilities, 318*, 319*
Quintic Equations, 310, 311, 316*, 321*
Tschirnhausen's Transformation, 358
Triads of 15 things, 323

Report On Catalogue of Scientific Memoirs, 378
Notices of Communications to the British Association, 378
Problems and Solutions, 383, see contents, p. 612

300.

NOTE RELATIVE AUX DROITES EN INVOLUTION DE M. SYLVESTER.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LII. (*Janvier—Juin*, 1861), pp. 1039—1042.]

LA courbe cubique dans l'espace, représentée par les équations

$$y\,u - z^2 = 0, \quad z\,y - x\,u = 0, \quad x\,z - y^2 = 0,$$

passe par le point $A\,(x = y = z = 0)$ et le point $B\,(y = z = u = 0)$; le plan $x = 0$ est le plan osculant en A, le plan $y = 0$ le plan par la tangente en A et la droite AB; le plan $z = 0$ celui par la droite AB et la tangente en B; et enfin le plan $u = 0$ est le plan osculant en B. Réciproquement, pour une courbe cubique quelconque, en prenant les points A, B, sur la courbe à volonté, et en fixant comme ci-dessus les significations des coordonnées x, y, z, u, les facteurs constants que contiennent implicitement ces valeurs étant convenablement déterminés, les équations de la courbe cubique seront

$$y\,u - z^2 = 0, \quad z\,y - x\,u = 0, \quad x\,z - y^2 = 0.$$

Par un point quelconque de l'espace il passe une droite qui coupe deux fois la courbe cubique; et en prenant (x_1, y_1, z_1, u_1) pour les coordonnées du point dont il s'agit, et en écrivant

$$p_1 = y_1 u_1 - z_1^2, \quad q_1 = z_1 y_1 - x_1 u_1, \quad r_1 = x_1 z_1 - y_1^2,$$

les équations de la droite seront

$$p_1 x + q_1 y + r_1 z = 0, \quad p_1 y + q_1 z + r_1 u = 0.$$

Or, en considérant en général une droite représentée par les équations

$$\alpha x + \beta y + \gamma z + \delta u = 0, \quad \alpha' x + \beta' y + \gamma' z + \delta' u = 0,$$

les six quantités

$$\beta\gamma'-\beta'\gamma,\quad \gamma\alpha'-\gamma'\alpha,\quad \alpha\beta'-\alpha'\beta,\quad \alpha\delta'-\alpha'\delta,\quad \beta\delta'-\beta'\delta,\quad \gamma\delta'-\gamma'\delta,$$

sont ce que je nomme les coordonnées de la droite (en représentant par a, b, c, f, g, h ces coordonnées, on a l'équation identique $af+bg+ch=0$, et les coordonnées d'une droite peuvent être des quantités quelconques qui satisfont à cette équation). La condition pour l'involution de six droites est celle-ci, savoir: le déterminant formé avec les coordonnées des six droites est égal à zéro.

Je reviens à la droite qui coupe deux fois la courbe cubique. En écrivant les équations sous la forme

$$p_1x+q_1y+r_1z+0u=0,\quad 0x+p_1y+q_1z+r_1u=0,$$

les coordonnées de cette droite seront

$$p_1^2,\quad q_1^2-p_1r_1,\quad -p_1q_1,\quad p_1r_1,\quad q_1r_1,\quad r_1^2,$$

savoir, ces coordonnées seront des fonctions linéaires de $(p_1^2,\ q_1^2,\ r_1^2,\ q_1r_1,\ r_1p_1,\ p_1q_1)$. Donc, en considérant six droites dont chacune coupe deux fois la courbe cubique, et en attribuant des significations analogues à $(p_2,\ q_2,\ r_2)$, etc., la condition pour l'involution des six droites se trouve en égalant à zéro le déterminant dont les lignes sont $(p_1^2,\ q_1^2,\ r_1^2,\ q_1r_1,\ r_1p_1,\ p_1q_1)$, $(p_2^2,\ q_2^2,$ etc.), etc.; condition qui exprime que les six droites

$$p_1x+q_1y+r_1z=0,$$

dans le plan $u=0$ (ou si l'on veut les six droites $p_1y+q_1z+r_1u=0$ dans le plan $x=0$) touchent une même conique. Or la droite

$$p_1x+q_1y+r_1z=0$$

est la projection de l'une des six droites sur le plan osculant $u=0$, avec le point $x=y=z=0$ de la courbe cubique comme centre de projection; et si, en prenant un plan osculant quelconque et un point quelconque de la courbe cubique pour plan et centre de projection, nous appelons tout simplement *projection* une telle projection d'une droite quelconque (le plan osculant et le point de la cubique étant toujours les mêmes), on est conduit au théorème que voici, savoir:

Six droites dont chacune coupe deux fois la même courbe cubique seront en involution, si les projections de ces droites touchent une même conique.

Et de même, pour un nombre quelconque de droites, si les projections touchent une même conique, ces droites seront en involution, c'est-à-dire six quelconques des droites seront des droites en involution.

Il convient de remarquer qu'en considérant six droites quelconques, on peut en général trouver une courbe cubique coupée deux fois par chacune des droites: la condition du théorème est donc, comme cela doit être, une seule relation entre les six droites. Je remarque aussi que cette relation ne dépend nullement du plan osculant ni du point de la courbe cubique choisis pour plan et centre de projection. Réci-

proquement, en prenant dans un plan osculant quelconque de la courbe cubique un nombre quelconque (six ou plus) de tangentes d'une même conique, et en reprojetant ces tangentes sur la courbe cubique au moyen d'un point quelconque de la courbe comme centre de projection (de manière à obtenir pour reprojection de chaque tangente une droite qui coupe deux fois la courbe cubique), on obtient un système de droites en involution. Le lieu des droites dont chacune coupe deux fois la courbe cubique, et qui sont en involution, est une surface réglée du quatrième ordre qui a la courbe cubique pour courbe double. En effet, si l'équation en coordonnées tangentielles de la conique enveloppée par les droites $p_1x+q_1y+r_1z=0$, etc. (ou, si l'on veut, par les droites $p_1y+q_1z+r_1u=0$, etc.), est

$$(a,\ b,\ c,\ f,\ g,\ h)\,(p,\ q,\ r)^2=0,$$

cette même équation, en y considérant p, q, r comme dénotant $yu-z^2$, $zy-xu$, $xz-y^2$, autrement dit, l'équation,

$$(a,\ b,\ c,\ f,\ g,\ h)\,(yu-z^2,\ zy-xu,\ xz-y^2)^2=0,$$

sera celle d'une surface du quatrième ordre ayant la courbe cubique pour courbe double. Et cette surface sera une surface réglée; car en menant par un point quelconque de la surface une droite qui coupe deux fois la courbe cubique, chaque point d'intersection avec la courbe cubique doit compter pour deux points d'intersection avec la surface, et la droite coupe la surface en cinq points, c'est-à-dire que cette droite est située entièrement dans la surface.

J'ai remarqué ailleurs (*Camb. and Dubl. Math. Journ.*, t. VII. (1852), p. 172, [107]) qu'il y a sur une surface réglée de l'ordre n une courbe double rencontrée par chaque génératrice en $(n-2)$ points. Cette courbe double sera de l'ordre $(n-2)$ au moins, et de l'ordre $\frac{1}{2}(n-1)(n-2)$ au plus; donc, pour $n=4$, la courbe double sera de l'ordre 2 ou 3, et comme évidemment cette courbe n'est pas une courbe plane, elle sera: ou 1° deux droites qui ne se rencontrent pas; ou 2° une courbe cubique en espace. Cette seconde espèce des surfaces réglées du quatrième ordre est celle qui se présente dans la théorie des droites en involution.

301.

SUR LES CÔNES DU SECOND ORDRE QUI PASSENT PAR SIX POINTS DONNÉS.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LII. (*Janvier—Juin*, 1861), pp. 1216—1218.]

DANS un Mémoire par feu M. Weddle "On the theorems in space analogous to those of Pascal and Brianchon in a plane" (*Camb. and Dubl. Math. Journ.*, t. v. 1850, voir la Note p. 69), on trouve à propos d'un théorème de M. Chasles la remarque que le lieu du sommet d'un cône du second ordre qui passe par six points donnés est une surface du quatrième ordre qui contient la courbe cubique en espace par les six points. Voici comment je démontre ce théorème:

En prenant (X, Y, Z, U) pour les coordonnées courantes, $(\alpha_1, \beta_1, \gamma_1, \delta_1)\ldots(\alpha_6, \beta_6, \gamma_6, \delta_6)$ pour les coordonnées des six points donnés, et (x, y, z, u) pour ceux du sommet, je pose l'équation

$$\begin{vmatrix} \cdot, & X^2, & Y^2, & Z^2, & U^2, & YZ, & ZX, & XY, & XU, & YU, & ZU \\ \lambda & 2x & \cdot & \cdot & \cdot & \cdot & z & y & u & \cdot & \cdot \\ \mu & \cdot & 2y & \cdot & \cdot & z & \cdot & x & \cdot & u & \cdot \\ \nu & \cdot & \cdot & 2z & \cdot & y & x & \cdot & \cdot & \cdot & u \\ \rho & \cdot & \cdot & \cdot & 2u & \cdot & \cdot & \cdot & x & y & z \\ \cdot & \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \beta\gamma & \gamma\alpha & \alpha\beta & \alpha\delta & \beta\delta & \gamma\delta \end{vmatrix} = 0,$$

où la dernière ligne dénote les six lignes qu'on obtient en écrivant successivement $(\alpha_1, \beta_1, \gamma_1, \delta_1)\ldots(\alpha_6, \beta_6, \gamma_6, \delta_6)$ au lieu de $(\alpha, \beta, \gamma, \delta)$, de manière que la fonction au côté gauche est un déterminant de l'ordre onze: les coefficients λ, μ, ν, ρ sont des quantités arbitraires et les points $(\cdot)$ dénotent des zéros.

Cette équation est évidemment celle d'une surface du second ordre qui passe par les six points, et il ne faut qu'une seule condition pour que cette surface soit un cône: la condition sera

$$\begin{vmatrix} 2x, & ., & ., & ., & ., & z, & y, & u, & ., & . \\ . & 2y & . & . & z & . & x & . & u & . \\ . & . & 2z & . & y & x & . & . & . & u \\ . & . & . & 2u & . & . & . & x & y & z \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \beta\gamma & \gamma\alpha & \alpha\beta & \alpha\delta & \beta\delta & \gamma\delta \end{vmatrix} = 0,$$

où la fonction à côté gauche est de même un déterminant de l'ordre dix; cette équation, laquelle est de l'ordre quatre par rapport à (x, y, z, u), sera celle du lieu du sommet.

En effet, pour que la surface du second ordre soit un cône ayant pour sommet le point (x, y, z, u), il faut et il suffit que les équations dérivées par rapport à chacune des coordonnées (X, Y, Z, U), soient satisfaites en y écrivant (x, y, z, u) au lieu de (X, Y, Z, U). Je forme l'équation dérivée par rapport à X, et j'y écris (x, y, z, u) au lieu de (X, Y, Z, U); l'équation est

$$\begin{vmatrix} ., & 2x, & ., & ., & ., & ., & z, & y, & u, & ., & . \\ \lambda & 2x & . & . & . & . & z & y & u & . & . \\ \vdots & & & & & & & & & & \end{vmatrix} = 0.$$

Or on ne change pas la valeur du déterminant en substituant pour la première ligne cette même ligne moins la seconde ligne; l'équation devient ainsi:

$$\begin{vmatrix} -\lambda, & ., & ., & ., & ., & ., & ., & ., & ., & ., & . \\ \lambda & 2x & . & . & . & . & z & y & u & . & . \\ \vdots & & & & & & & & & & \end{vmatrix} = 0;$$

et le déterminant se réduit à $-\lambda$ multiplié par le déterminant de l'ordre dix; donc, en supposant que ce dernier déterminant se réduise à zéro, l'équation dérivée par rapport à X sera satisfaite; et de même, les équations dérivées par rapport à Y, Z, U, en substituant toujours (x, y, z, u) au lieu de (X, Y, Z, U), seront toutes satisfaites si le déterminant de l'ordre dix se réduit à zéro. C. Q. F. D.

Il convient de remarquer que l'on peut sans perte de généralité réduire à zéro trois quelconques des quantités λ, μ, ν, ρ; de là on obtient l'équation du cône en substituant, au lieu de l'une quelconque des premières quatre lignes du déterminant de l'ordre dix, la ligne

$$| X^2, Y^2, Z^2, U^2, YZ, ZX, XY, XU, YU, ZU |.$$

En considérant la courbe cubique par les six points, on peut supposer que les équations de cette courbe soient

$$yu - z^2 = 0, \quad zy - xu = 0, \quad xz - y^2 = 0;$$

cela étant, on aura

$$\beta\delta - \gamma^2 = 0, \quad \beta\gamma - \alpha\delta = 0, \quad \alpha\gamma - \beta^2 = 0$$

pour l'un quelconque des points $(\alpha_1, \beta_1, \gamma_1, \delta_1), \ldots (\alpha_6, \beta_6, \gamma_6, \delta_6)$; et de là, au moyen des propriétés des déterminants, et en écrivant

$$\square = 4(yu - z^2)(xz - y^2) - (zy - xu)^2,$$

on exprime l'équation de la surface comme fonction linéaire par rapport à x, y, z, u et par rapport à $d_x\square$, $d_y\square$, $d_z\square$, $d_u\square$; ces dernières fonctions se réduisent à zéro en vertu des équations

$$yu - z^2 = 0, \quad zy - xu = 0, \quad xz - y^2 = 0,$$

et ainsi, comme cela doit être, la surface passe par la courbe cubique.

Je prends l'occasion de remarquer que le théorème que j'ai donné par rapport aux six droites en involution de M. Sylvester [300], peut s'exprimer dans une forme encore plus simple comme suit:

Soit donnée une courbe cubique en espace, et prenons un point quelconque de la courbe pour sommet d'un cône du second ordre, d'ailleurs arbitraire; un plan tangent du cône rencontre la courbe en deux points, et par ces deux points on peut mener une droite: les droites qui correspondent de cette manière à six plans tangents quelconques du cône sont des droites en involution. Je dois remarquer que l'idée de rattacher ces droites à une surface du quatrième ordre est due à M. Sylvester.

A propos de ce sujet, j'ai considéré le problème de trouver le lieu du sommet d'un cône du second ordre qui touche à six droites données: ce lieu est une surface du huitième ordre; et en représentant les coordonnées de l'une quelconque des droites par (a, b, c, f, g, h), savoir les coordonnées de la première droite, etc., sont

$$(a_1, b_1, c_1, f_1, g_1, h_1), \ldots (a_6, b_6, c_6, f_6, g_6, h_6),$$

les coefficients de l'équation seront des fonctions linéaires des déterminants du sixième ordre formés au moyen de la matrice $(a, b, c, f, g, h)^2$, à six lignes et vingt et une colonnes.

302.

CONSIDÉRATIONS GÉNÉRALES SUR LES COURBES EN ESPACE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LIV. (*Janvier—Juin*, 1862), pp. 55—60, 396—400, 672—678.]

SOIT une courbe donnée du $m^{\text{ième}}$ ordre; je suppose toujours que cette courbe soit une courbe propre, savoir qu'elle n'est pas composée de courbes d'ordres inférieurs. Si nous prenons pour sommet d'un cône qui passe par la courbe un point A *quelconque* qui n'est pas sur la courbe, ce cône sera de l'ordre m; cela est vrai *en général* quelle que soit la courbe; seulement si m est un nombre composé, alors pour de certaines courbes il peut y avoir des positions de A pour lesquelles le cône sera d'un ordre sous-multiple de m; mais en faisant abstraction de ces positions particulières, le cône sera de l'ordre m. Et, cela étant, une droite du cône ne contiendra en général qu'un seul point de la courbe. En employant quatre coordonnées (x, y, z, w) et en supposant qu'au point A on ait

$$x=0, \quad y=0, \quad z=0,$$

l'équation du cône sera $U=0$, où U est une fonction homogène de (x, y, z) de l'ordre m. On peut faire passer par la courbe une surface ayant pour équation

$$Qw-P=0$$

$\left(\text{ou } w=\dfrac{P}{Q}\right)$, où P, Q sont des fonctions homogènes de (x, y, z) des ordres p, $p-1$ respectivement. Et on peut supposer que p soit égal tout au plus à $m-1$: en effet, en prenant $p=m-1$, l'équation contiendrait

$$\tfrac{1}{2}(m-1)m+\tfrac{1}{2}m(m+1)-1,$$

c'est-à-dire m^2-1 constantes arbitraires; et en déterminant convenablement m^2-m+1 de ces quantités, la surface de l'ordre $m-1$ passera par m^2-m+1 points de la courbe de l'ordre m, c'est-à-dire cette surface contiendra la courbe entière. De cette manière, on obtiendrait toujours une surface de l'ordre $m-1$; mais si les fonctions P, Q

ainsi trouvées avaient un facteur commun, ce facteur devrait être écarté; il convient donc de supposer que les degrés de P, Q soient p, $p-1$ respectivement, p étant tout au plus égal à $m-1$. La surface $Qw-P=0$ a au point A un point conique du $(p-1)^{ième}$ ordre; en effet dans le voisinage de ce point l'équation se réduit à $Q=0$, laquelle appartient à un cône du $(p-1)^{ième}$ ordre. J'ajoute que la surface contient les $p(p-1)$ droites $P=0$, $Q=0$ qui passent chacune par le point A; toute autre droite par ce point rencontre la surface dans ce point (lequel compte pour $p-1$ points d'intersection) et encore dans un seul point donné par l'équation

$$w=\frac{P}{Q}.$$

On peut appeler *monoïde* une telle surface; le point A sera le sommet; le cône $P=0$ le cône supérieur; le cône $Q=0$, le cône inférieur; les droites d'intersection de ces deux cônes, les droites de la monoïde.

Or le cône circonscrit $U=0$ et la monoïde $Qw-P=0$ se coupent selon une courbe de l'ordre mp: si $p=1$, cette intersection des deux surfaces sera la courbe du $m^{ième}$ ordre, laquelle sera une courbe plane; mais, dans tout autre cas, la courbe d'intersection sera composée de la courbe du $m^{ième}$ ordre, et d'un autre système de l'ordre $m(p-1)$; ce système ne peut être autre chose que les droites d'intersection du cône circonscrit $U=0$, et du cône inférieur $Q=0$ de la monoïde; c'est-à-dire les équations

$$U=0, \quad Q=0$$

doivent donner $P=0$; car, cela étant, les droites $U=0$, $Q=0$ seront situées sur la monoïde; et ces droites, lesquelles forment un système de l'ordre $m(p-1)$, seront partie de l'intersection de la monoïde et du cône circonscrit $U=0$. Et il est nécessaire que cela soit ainsi, car autrement chaque droite du cône $U=0$ ne contiendrait sur la monoïde que le point A, et le point déterminé par l'équation $w=\frac{P}{Q}$, lequel est un point sur la courbe du $m^{ième}$ ordre; donc cette autre partie de l'intersection de la monoïde et du cône $U=0$ serait, non pas une courbe quelconque, mais le seul point A; ce qui est absurde.

Le cône circonscrit $U=0$ ne peut pas être un cône quelconque à moins que $p=1$; en effet si $p>1$, il est nécessaire que le cône ait au moins $(p-1)m$ droites doubles (en comprenant dans cette locution le cas où le cône a des singularités qui équivalent à $(p-1)m$ droites doubles), car en supposant pour un moment que le cône $U=0$ n'ait pas de singularités, le cône $P=0$ de l'ordre p devrait passer par les $(p-1)m$ droites d'intersection du cône $Q=0$ de l'ordre $(p-1)$ et du cône $U=0$ de l'ordre m; or m est au moins égal à $p+1$, de manière que le cône $P=0$ doit passer au moins par (p^2-1) droites du cône $Q=0$; mais p^2-1 est $>p^2-p$, à moins que $p=1$; donc ce cône $P=0$ serait composé du cône $Q=0$ et d'un plan $P'=0$ par le point A; c'est-à-dire $P=QP'$, et l'équation de la monoïde se réduirait à $w=P'$, ou l'on aurait $p=1$, ce qui est contraire à l'hypothèse. On obtiendra le même résultat à

moins de supposer que le cône $Q=0$ passe par un certain nombre x de droites doubles du cône $U=0$; mais en faisant cette supposition, chacune de ces droites compte pour deux intersections des cônes $Q=0$, $U=0$; il y a encore $(p-1)m-2x$ droites d'intersection; et les $x+\{(p-1)m-2x\}$, c'est-à-dire $(p-1)m-x$ droites peuvent être comprises parmi les $p(p-1)$ droites de la monoïde si x est égal au moins à $(p-1)(m-p)$; c'est-à-dire le cône $U=0$ doit avoir au moins ce nombre de droites doubles. Je remarque que pour m impair, et $p=\frac{1}{2}(m+1)$, le nombre sera $\frac{1}{4}(m^2-2m+1)$, et pour m pair, et $p=\frac{1}{2}m$ ou $\frac{1}{2}m+1$, le nombre sera $\frac{1}{4}(m^2-2m)$; mais pour toute autre valeur de p, le nombre sera moins élevé.

Je résume comme suit:

Toute courbe du $m^{ième}$ ordre est l'intersection d'un cône circonscrit $U=0$, du $m^{ième}$ ordre, et d'une surface monoïde $Qw-P$, de l'ordre $p=m-1$ au plus. L'intersection complète de deux surfaces est composée de la courbe du $m^{ième}$ ordre et des $m(p-1)$ droites d'intersection du cône circonscrit $U=0$, et du cône inférieur $Q=0$ de la monoïde. Ces droites seront $(p-1)(m-p)+\alpha$ droites, chacune répétée deux fois, et $(p-1)(2p-m)-2\alpha$ droites, où α peut être égal à zéro; chacune des $(p-1)(m-p)+\alpha$ droites sera une droite double du cône $U=0$; et les $(p-1)(m-p)+\alpha$ droites et $(p-1)(2p-m)-2\alpha$ droites, ensemble $p(p-1)-\alpha$ droites, seront situées sur le cône supérieur $P=0$ de la monoïde.

Il y a deux circonstances qui empêchent que cette théorie ne conduise tout de suite à une classification des courbes en espace. D'abord, une droite double du cône $U=0$ peut correspondre ou à un point double réel, ou à un point double apparent de la courbe; et de même en supposant que la droite double devienne une droite de rebroussement, cette droite peut ou correspondre à un point de rebroussement (point stationnaire) de la courbe, ou la droite peut être une tangente ordinaire de la courbe, sans qu'il ait sur la courbe aucune singularité qui corresponde à cette droite de rebroussement (voir le Mémoire de M. Salmon: "On the classification of curves of double curvature," *Camb. et Dubl. Math. Journ.*, t. v. pp. 23—46, 1850).

Puis, étant donnée l'équation $U=0$ du cône circonscrit, la monoïde n'est pas une surface déterminée, et il n'est guère facile de voir quel doit être l'ordre de cette surface. En effet, cette équation étant $w=\dfrac{P}{Q}$, il peut y avoir des fonctions P', Q' telles que $PQ'-P'Q=MU$, et, cela étant, puisqu'il ne s'agit que de l'intersection avec le cône $U=0$, on pourrait remplacer l'équation $w=\dfrac{P}{Q}$ par celle-ci, $w=\dfrac{P'}{Q'}$, laquelle peut être d'un ordre inférieur.

Ces difficultés se présentent dès le commencement. En effet soit $m=3$. On a $p=1$ ou $p=2$, mais $p=1$ ne donne que la cubique plane; je suppose donc $p=2$. Le cône $U=0$ du troisième ordre aura une droite double, laquelle peut être une droite de rebroussement. L'équation de la monoïde sera $w=\dfrac{P}{Q}$, où $Q=0$ est l'équation d'un

plan qui passe par le point double ou de rebroussement, et qui coupe ainsi le cône $U=0$ selon une autre droite; et $Qw-P=0$ est l'équation d'un cône du second ordre qui passe par ces deux droites. Mais soit que le cône $U=0$ ait une droite double, soit que cette droite soit de rebroussement, on n'obtient qu'une seule espèce de courbe cubique; au premier cas le sommet n'est pas situé, au deuxième cas ce sommet est situé, sur une tangente de la courbe cubique; voilà toute la différence.

Soit encore $m=4$; on peut avoir $p=1$, 2 ou 3; mais $p=1$ ne donne que les courbes planes du quatrième ordre, je suppose donc $p=2$ ou $p=3$; dans l'un ou l'autre cas, le cône $U=0$ du quatrième ordre doit avoir au moins deux droites doubles. Il peut donc y avoir seulement deux droites doubles; l'une de ces droites peut être une droite de rebroussement, ou toutes les deux peuvent être de telles droites. Ou encore, il peut y avoir trois droites doubles; l'une de ces droites peut être une droite de rebroussement, ou deux droites ou toutes les trois peuvent être de telles droites. Il y a donc un assez grand nombre de cas à considérer; mais on sait qu'il n'y a que quatre espèces en tout, savoir: 1° la courbe d'intersection de deux surfaces du second ordre qui ne se touchent pas, courbe que je nomme *quadriquadrique générale;* 2° les deux surfaces du second ordre peuvent se toucher; la courbe d'intersection sera une *quartique nodale;* 3° les deux surfaces peuvent avoir un contact singulier, la courbe d'intersection sera une *quartique cuspidale;* 4° il y a enfin la courbe du quatrième ordre qui n'est située que sur une seule surface du second ordre, et que l'on n'obtient qu'au moyen d'une surface du troisième ordre: ce sera la courbe *excubo-quartique.* Je remarque en passant que les quartiques nodale et cuspidale sont des sous-espèces tant de l'excubo-quartique que de la quadriquadrique. En supposant que le cône $U=0$ n'ait que deux droites doubles ou de rebroussement, et soit que $p=2$ ou $p=3$, on obtiendra par la théorie actuelle la quadriquadrique générale (cela est évident par les formules du Mémoire cité de M. Salmon). Si le cône $U=0$ a trois droites doubles ou de rebroussement, alors soit que $p=2$ ou $p=3$, on obtiendra, selon les circonstances, ou l'excubo-quartique, ou la quartique nodale, ou la quartique cuspidale (mais non pas cette dernière, à moins qu'il n'y ait au moins une droite de rebroussement). Mais il faudrait pour tout cela une discussion plus approfondie.

Je remarque qu'en prenant le point A sur la courbe du $m^{\text{ième}}$ ordre, l'on aurait eu, au lieu du cône $U=0$ du $m^{\text{ième}}$ ordre, un cône du $(m-1)^{\text{ième}}$ ordre, et l'ordre du cône se réduirait encore si le point A était un point multiple de la courbe. Peut-être il conviendrait de considérer de tels cônes au lieu du cône du $m^{\text{ième}}$ ordre.

En conclusion, je fais les réflexions que voici, savoir: Si $S=0$, $T=0$ sont des surfaces quelconques qui passent par la courbe du $m^{\text{ième}}$ ordre, alors en éliminant entre ces équations la coordonnée w, on obtient une équation

$$\Pi = UV = 0,$$

qui contient comme facteur l'équation $U=0$ du cône du $m^{\text{ième}}$ ordre. Mais il y a plus: la théorie de l'élimination entre deux équations algébriques fait voir que les équations

$S=0$, $T=0$ donnent lieu à un assez grand nombre d'équations de la forme $w=\frac{P}{Q}$ (en représentant deux quelconques de ces équations par

$$w=\frac{P}{Q}, \quad w=\frac{P'}{Q'},$$

on aura toujours $PQ'-P'Q=M\Pi$), c'est-à-dire on obtient par une telle élimination plusieurs surfaces monoïdes dont chacune coupe le cône $\Pi=UV=0$, selon la courbe d'intersection complète de deux surfaces $S=0$, $T=0$. Mais il ne s'ensuit pas (même en admettant que l'on ait de cette manière *toutes* les surfaces monoïdes qui passent par l'intersection complète), que l'on ait *toutes* les surfaces monoïdes qui passent par la courbe du $m^{ième}$ ordre; en effet il peut y avoir des fonctions P', Q' lesquelles, sans donner $PQ'-P'Q=MUV$, donnent cependant $PQ'-P'Q=MU$, et, cela étant, $w=\frac{P'}{Q'}$ serait une surface monoïde qui passerait par la courbe du $m^{ième}$ ordre.

P.S. On déduit sans peine la théorie des courbes situées sur une surface du second ordre (voir ma Note "On the curves situate on a surface of the second order," *Phil. Mag.*, July 1861, [314], et les savantes recherches que M. Chasles vient de publier dans les *Comptes Rendus*). En effet, en supposant que la monoïde soit une surface du second ordre (hyperboloïde) et que son équation soit $w=\frac{xy}{z}$, alors, puisque le cône $U=0$, du $m^{ième}$ ordre, doit rencontrer le plan $z=0$ selon les seules droites $x=0$, $y=0$, il faut que ces droites soient des droites multiples du cône $U=0$, et en prenant p, q des nombres tels que $p+q=m$, on peut supposer que les deux droites soient des droites multiples des ordres p et q respectivement; et cela arrivera si U (fonction homogène du $m^{ième}$ ordre en x, y, z) contient x^p pour la plus haute puissance de x, et y^q pour la plus haute puissance de y. Car en arrangeant selon les puissances descendantes de y, on aura

$$U=y^q(x, z)^p+y^{q-1}z(x, z)^p+\ldots,$$

ce qui fait voir que $x=0$, $z=0$ sera une droite multiple du $p^{ième}$ ordre, et de même $y=0$, $z=0$ sera une droite multiple du $q^{ième}$ ordre. On a donc selon la notation de M. Chasles

$$U=M(x^p, y^q),$$

en se souvenant qu'ici U contient aussi la coordonnée z.

Suite.—Courbes du quatrième ordre.

Toute surface du second ordre est une surface monoïde, et on peut prendre pour sommet un point quelconque de la surface. En effet, en considérant un point quelconque de la surface du second ordre, soient

$$x=0, \quad y=0, \quad z=0,$$

les équations de trois plans quelconques qui passent par ce point; l'équation de la surface sera satisfaite en y écrivant

$$x=0,\quad y=0,\quad z=0;$$

donc cette équation ne contiendra pas de terme en w^2, et elle sera ainsi de la forme

$$wQ-P=0 \quad \text{ou} \quad w=\frac{P}{Q},$$

P et Q étant des fonctions homogènes en x, y, z, du second ordre et du premier ordre respectivement; c'est-à-dire, la surface sera monoïde, ou, si l'on veut, monoïde quadrique.

Or, par une courbe du quatrième ordre (ou courbe quartique) quelconque en espace, on peut faire passer une surface du second ordre, ou monoïde quadrique. Selon la théorie générale, la surface monoïde est tout au plus du troisième ordre, ou monoïde cubique; j'avais tort de supposer que pour la courbe excubo-quartique la surface monoïde fût nécessairement une monoïde cubique. Il arrive comme suit, savoir: pour la courbe quadriquadrique, en prenant pour sommet un point quelconque de l'espace (on suppose toujours que le sommet de la monoïde n'est pas situé sur la courbe), on aura une monoïde quadrique; mais pour la courbe excubo-quartique, pour que la monoïde soit quadrique, il faut que le sommet soit situé sur la surface du second ordre (il n'y a qu'une seule surface) qui passe par la courbe; cela étant, la monoïde quadrique sera cette surface même du second ordre. Mais en prenant pour sommet un point quelconque qui n'est point situé sur la surface du second ordre, la monoïde sera nécessairement une surface cubique.

Ainsi, pour les courbes quartiques, il suffit de considérer ces courbes comme situées sur une monoïde quadrique; il est cependant assez intéressant de les considérer comme situées sur une monoïde cubique. Je suppose donc $U=0$, $w=\frac{P}{Q}$, où $U=0$ est un cône quartique et $w=\frac{P}{Q}$ une monoïde cubique avec le même point $x=0$, $y=0$, $z=0$ pour sommet.

Selon la théorie générale, les huit droites $Q=0$, $U=0$ doivent être comprises parmi les six droites $Q=0$, $P=0$. Or, pour cela, il faut que le cône $U=0$ ait des droites multiples; il y a trois cas à considérer: 1° Le cône passe par les six droites, et une de ces droites est une droite triple du cône; il y aura, comme cela doit être,

$$3+1+1+1+1+1=8$$

droites d'intersection de $Q=0$, $U=0$. 2° Le cône passe par les six droites; deux de ces droites étant des droites doubles, il y aura

$$2+2+1+1+1+1=8$$

droites d'intersection. 3° Le cône passe par cinq des six droites; trois de ces cinq droites étant des droites doubles, il y aura

$$2+2+2+1+1=8$$

droites d'intersection. Or, dans le premier et le second cas, le cône $U=0$ passe par les six droites d'intersection des cônes $P=0$, $Q=0$; il faut donc que l'on ait identiquement

$$U=PQ'-P'Q,$$

P', Q' étant des fonctions homogènes en x, y, z du second ordre et du premier ordre respectivement. Mais en vertu de l'équation

$$U=PQ'-P'Q=0,$$

on a $\frac{P}{Q}=\frac{P'}{Q'}$, c'est-à-dire la courbe est située sur la monoïde quadrique $w=\frac{P'}{Q'}$. La courbe sera quadriquadrique ou excubo-quartique, selon les circonstances.

Reste à considérer le troisième cas. La monoïde cubique est une surface cubique ayant le sommet pour point conique; la théorie des droites sur une telle surface a été examinée par M. Salmon dans son Mémoire: "On the triple tangent planes of a surface of the third order," *Camb. and Dubl. Math. Journ.*, pp. 252—260 (1849). Il y a, en effet, les six droites par le point conique, savoir: les droites $P=0$, $Q=0$, qui comptent pour douze droites, et de plus quinze droites; $6\times 2+15=27$. Chacune des quinze droites est donnée comme troisième intersection de la surface avec un plan qui passe par deux des six droites. Donc, en nommant 1, 2, 3, 4, 5, 6 les six droites, on peut nommer 12 la droite dans le plan mené par les droites 1, 2; et de même pour les droites 13, 23, etc. La droite 1 est rencontrée par les droites 2, 3, 4, 5, 6, 12, 13, 14, 15, 16; la droite 12 par les droites 1, 2; 34, 56; 35, 64; 36, 45; et ainsi pour les autres droites.

Cela étant, je suppose que le cône $U=0$ passe par les droites 2, 3, 4, 5, 6, et que les droites 4, 5, 6 soient droites doubles du cône. Je dis que la courbe sera située sur une surface du second ordre qui passe par les droites 12, 13 (droites qui ne se coupent pas), savoir, ces deux droites et la courbe seront l'intersection complète de la monoïde cubique et de la surface du second ordre; cela fait voir que la courbe est une courbe excubo-cubique. Et, comme il est auparavant dit, en prenant pour sommet un point quelconque de la surface du second ordre, la courbe sera située sur une monoïde quadrique $w=\frac{P'}{Q'}$.

Donc, en partant de la monoïde cubique, on trouve toujours que la courbe du quatrième ordre est située sur une monoïde quadrique.

J'établis comme suit l'existence de la surface du second ordre qui passe par les droites 12, 13. Je remarque en général que l'équation $w=\frac{P}{Q}$ peut s'écrire sous la forme $w+L=\frac{P+LQ}{Q}$, où L est une fonction homogène linéaire quelconque de x, y, z; ou en changeant w, cette équation sera

$$w=\frac{P+LQ}{Q},$$

c'est-à-dire on peut remplacer le cône $P=0$ par un cône quelconque qui passe par les droites d'intersection des cônes $P=0$, $Q=0$. Donc, pour la monoïde cubique, on peut prendre pour $P+LQ=0$ un système de trois plans, et en prenant pour équations de ces plans $x=0$, $y=0$, $z=0$, on peut prendre pour équations de la monoïde cubique

$$w=\frac{xyz}{Q}.$$

Comme les coordonnées x, y, z renferment chacune un multiplicateur indéterminé, on peut écrire

$$Q=x^2+y^2+z^2+\quad 2lyz+\quad 2mzx+\quad 2nxy,$$

ou, en posant $\alpha'=\frac{1}{\alpha}$, $\beta'=\frac{1}{\beta}$, $\gamma'=\frac{1}{\gamma}$, α, β, γ étant des quantités quelconques, on peut écrire

$$Q=x^2+y^2+z^2+(\alpha+\alpha')yz+(\beta+\beta')zx+(\gamma+\gamma')xy,$$

ce qui est la forme la plus commode pour mettre en évidence les droites d'intersection $xyz=0$, $Q=0$. On peut supposer que les équations de ces droites soient

$$\begin{array}{llll}
(1) & x=0, \quad y+\alpha z=0, & (2) & x=0, \quad \alpha y+z=0,\\
(3) & y=0, \quad z+\beta x=0, & (4) & y=0, \quad \beta z+x=0,\\
(5) & z=0, \quad x+\gamma y=0, & (6) & z=0, \quad \gamma x+y=0.
\end{array}$$

Donc, pour les plans 56, 34, 24, on aura les équations

$$(56)\quad z=0,\qquad (34)\quad y=0,\qquad (24)\quad x+\alpha\beta y+\beta z=0;$$

et de là l'équation

$$AQ^2+Qz[By+C(x+\alpha\beta y+\beta z)]+Dz^2y(x+\alpha\beta y+\beta z)=0$$

sera celle d'un cône du quatrième ordre qui passe par les droites 2, 3, 4, 5, 6 et a les droites 4, 5, 6 pour droites doubles; et comme cette équation contient les trois quantités arbitraires $A:B:C:D$, ce sera l'équation la plus générale qui satisfait aux conditions dont il s'agit: c'est-à-dire cette équation sera celle du cône $U=0$.

Les équations de la droite 12 sont $x=0$, $w=0$; pour obtenir celle de la droite 13, j'observe que l'équation du point 13 est

$$\alpha\beta x+y+\alpha z=0,$$

et je forme l'équation identique

$$Q=(\alpha\beta x+y+\alpha z)[(\gamma+\gamma'-\alpha\beta)x+y+\alpha' z]+\beta'(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x(z+\beta x),$$

laquelle se vérifie sans peine. Donc, en écrivant

$$\alpha\beta x+y+\alpha z=0, \text{ ou } y=-\alpha(z+\beta x),$$

l'équation $w=\frac{xyz}{Q}$ devient

$$w=\frac{-\alpha x(z+\beta x)z}{\beta'(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x(z+\beta x)}, =\frac{-\alpha\beta z}{(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')},$$

ou, ce qui est la même chose,

$$w(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')+\alpha\beta z=0,$$

laquelle et l'équation

$$\alpha\beta x+y+\alpha z=0$$

sont les deux équations de la droite 13.

Cela étant,

$$(Ax+Bw)(\alpha\beta x+y+\alpha z)+(Cx+Dw)[\alpha\beta z+(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')w]=0$$

sera l'équation d'une surface du second ordre qui passe par les deux droites 12, 13; et, en éliminant w au moyen de l'équation

$$w=\frac{xyz}{Q},$$

on obtient l'équation du cône du quatrième ordre. En effet, en substituant cette valeur de w, on obtient une équation du sixième ordre laquelle, divisée par $(\alpha\beta x+y+\alpha z)$, devient

$$AQ^2+ByzQ+(CQ+Dyz)z\frac{\alpha\beta Q+(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')xy}{\alpha\beta x+y+\alpha z}=0;$$

or

$$\frac{Q}{\alpha\beta x+y+\alpha z}=(\gamma+\gamma'-\alpha\beta)x+y+\alpha' z+\frac{\beta'(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x(z+\beta x)}{\alpha\beta x+y+\alpha z},$$

donc la partie fractionnelle est

$$\frac{\alpha(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x(z+\beta x)+(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')xy}{\alpha\beta x+y+\alpha z},$$

c'est-à-dire

$$(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x\frac{\alpha(z+\beta x)+y}{\alpha\beta x+y+\alpha z},=(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x,$$

et l'équation devient

$$AQ^2+ByzQ+(CQ+Dyz)z\begin{bmatrix}\alpha\beta(\gamma+\gamma'-\alpha\beta)x+\alpha\beta y+\beta z\\+(1-\alpha\beta\gamma)(1-\alpha\beta\gamma')x\end{bmatrix},$$

ou enfin

$$AQ^2+ByzQ+(CQ+Dyz)z(x+\alpha\beta y+\beta z)=0,$$

ce qui est en effet l'équation ci-dessus trouvée pour le cône $U=0$.

Suite.—Courbes du cinquième ordre.

On pourrait assez bien dénoter les courbes des ordres un, deux, trois, comme suit, savoir :

Courbe du premier ordre, par	1
Courbe du second ordre, par	2
Courbes du troisième ordre, par . . .	3 et $4-1$,

c'est-à-dire que la courbe plane serait 3 et la courbe dans l'espace $4-1$. Mais pour le quatrième ordre, cette notation serait déjà en défaut, et l'on aurait besoin d'une notation telle que celle-ci:

Courbe plane	4.1
Courbe quadriquadrique .	2.2
Courbe excubo-quartique .	$2.3-1-1$.

Cela devient cependant trop complexe, et comme je ne cherche nullement une notation parfaite, il suffit pour le moment de dénoter la courbe plane (dont je n'ai guère à m'occuper) par 4*, la quadriquadrique par 4, et l'excubo-quartique par $6-2$. De même pour le troisième ordre, on peut dénoter la courbe plane par 3* et la courbe dans l'espace par 3.

Cela étant, pour les courbes du cinquième ordre, ou courbes quintiques, il y a cinq espèces, savoir:

			P. D. A.
Courbe plane	ou espèce	5	0
Courbe quadricubique	„	$6-1$	4
Courbe quadriquartique	„	$8-3$	6
Courbe cubicubique (deux espèces)	„	$9-3-1$	6
		$9-6+2$	5

où la colonne P. D. A. fait voir pour chaque espèce le nombre des points doubles apparents (voir le Mémoire de M. Salmon: "On the classification of Curves of double Curvature," *Camb. et Dubl. Math. Journ.*, t. v. 1850). Cette classification est au fond celle du Mémoire cité; seulement M. Salmon a énuméré trois sous-espèces qui n'existent pas, à savoir les sous-espèces quadriquadriques analogues à $V.7$, $V.8$, $V.9$ (p. 42, où M. Salmon parle des courbes algébriques correspondantes à $V.7$, $V.8$, $V.9$, $V.10$, sans attacher des numéros à ces quatre sous-espèces). Je vais à présent expliquer la théorie des cinq espèces.

Courbe plane, ou espèce 5.—Il va sans dire que cette courbe est l'intersection d'une surface quintique par un plan quelconque.

Courbe quadricubique, ou espèce $6-1$.—Cette courbe est l'intersection partielle d'une surface quadrique et d'une surface cubique qui ont en commun une seule droite. En supposant que les équations de la droite soient $x=0$, $y=0$, on peut prendre pour équation de la surface quadrique $xw-yz=0$, et pour celle de la surface cubique $xV-yU=0$, où $U=0$, $V=0$, sont des surfaces quadriques quelconques. Au lieu des deux équations

$$xw-yz=0,$$
$$xV-yU=0,$$

il est permis d'écrire

$$\left\| \begin{matrix} U, & x, & z \\ V, & y, & w \end{matrix} \right\| = 0,$$

ce qui fait voir qu'il passe par la courbe cette nouvelle surface cubique,

$$zV - wU = 0,$$

laquelle a en commun avec la première surface cubique la courbe quadriquadrique $U = 0$, $V = 0$.

La courbe a 4 points doubles apparents; elle peut donc avoir 0, 1 ou 2 points doubles ou de rebroussement; cela donne les sous-espèces

$$V.1,\ V.2,\ V.3,\ V.4,\ V.5,\ V.6,$$

de M. Salmon.

Je remarque en passant qu'en supposant que la surface cubique $xV - yU = 0$ a en commun avec la surface quadrique $xw - yz = 0$, non-seulement la droite $x = 0$, $y = 0$, mais aussi une autre génératrice du même mode de génération, on aura, au lieu de la courbe quintique $6 - 1$, cette nouvelle droite, et une courbe excubo-quartique. C'est là le théorème qui donne une des constructions que M. Chasles a trouvées pour la courbe excubo-quartique.

J'ajoute que la courbe considérée comme courbe située sur une surface quadrique sera de l'espèce (3, 2), ou, selon la notation de M. Chasles, $M(x^3y^2)$. On connaît ainsi un grand nombre des propriétés de cette courbe, et aussi de la courbe d'espèce $8 - 3$ dont nous allons parler, laquelle, considérée comme courbe située sur une surface quadrique, est de l'espèce (4, 1) ou $M(x^4y)$.

Courbe quadriquartique, ou espèce $8 - 3$.—Une telle courbe est l'intersection partielle d'une surface quadrique et d'une surface quartique qui ont en commun trois droites qui ne se rencontrent pas: autrement dit, ces droites seront des génératrices du même mode de génération de la surface quadrique [1].

Soit $xw - yz = 0$ l'équation de la surface quadrique; on peut prendre pour les trois génératrices

$$(x - \lambda y = 0,\quad \lambda w - z = 0),$$
$$(x - \mu y = 0,\quad \mu w - z = 0),$$
$$(x - \nu y = 0,\quad \nu w - z = 0);$$

et cela étant, l'équation de la surface quartique sera

$$(a, \ldots)(x - \lambda y,\ \lambda w - z)(x - \mu y,\ \mu w - z)(x - \nu y,\ \nu w - z) = 0,$$

en représentant de cette manière une fonction linéaire par rapport à $x - \lambda y$ et $\lambda w - z$, par rapport à $x - \mu y$ et $\mu w - z$, et par rapport à $x - \nu y$ et $\nu w - z$, les coefficients $a, \ldots$ étant des fonctions linéaires quelconques de x, y, z, w.

La courbe à 6 points doubles apparents; il n'y a donc pas d'autre singularité: c'est l'espèce analogue à

$$V.10$$

de M. Salmon.

[1] Dans le symbole $8 - 3$ on remarquera que 3 dénote non pas la cubique gauche, mais les trois droites; $8 - 1 - 1 - 1$ serait trop long, et je me suis servi exprès de la notation moins complète; et ainsi il est nécessaire en pareil cas d'expliquer la notation.

Courbe cubicubique, espèce $9-3-1$.—La courbe est l'intersection partielle de deux surfaces cubiques qui ont en commun une courbe cubique gauche et une droite qui ne rencontre pas la courbe cubique.

Soient p, q, r, s, t, u, P, Q des fonctions linéaires quelconques des coordonnées; $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ des fonctions linéaires quelconques de P, Q (autrement dit, $\alpha=0, \beta=0$, etc., seront les équations de six plans quelconques qui passent par la droite $P=0, Q=0$). Cela étant, les surfaces cubiques

$$\begin{vmatrix} p, & s, & \alpha \\ q, & t, & \beta \\ r, & u, & \gamma \end{vmatrix} = 0, \qquad \begin{vmatrix} p, & s, & \alpha' \\ q, & t, & \beta' \\ r, & u, & \gamma' \end{vmatrix} = 0,$$

auront en commun la courbe cubique

$$\left\| \begin{matrix} p, & q, & r \\ s, & t, & u \end{matrix} \right\| = 0$$

(ainsi les surfaces quadriques $pt-sq=0$, $pu-sr=0$ se rencontrent selon la droite $p=0$, $s=0$ et selon la courbe cubique dont il s'agit) et la droite $P=0$, $Q=0$. Il y aura donc encore une intersection qui sera la courbe quintique $9-3-1$.

La courbe a 6 points doubles apparents: il n'y a donc pas d'autre singularité: c'est l'espèce

$$V.10$$

de M. Salmon.

Je remarque en passant que cette courbe quintique $9-3-1$ a avec une certaine courbe sextique une relation semblable à celle qui existe entre la courbe excubo-quartique et la courbe quintique $6-1$. En effet, $p, q, r, s, t, u, \alpha, \beta, \gamma, \alpha', \beta', \gamma'$ étant à présent des fonctions linéaires quelconques des coordonnées, la courbe sextique sera donnée par les équations

$$\left\| \begin{matrix} p, & s, & \alpha, & \alpha' \\ q, & t, & \beta, & \beta' \\ r, & u, & \gamma, & \gamma' \end{matrix} \right\| = 0,$$

ou, ce qui revient à la même chose, elle sera l'intersection partielle des deux surfaces cubiques

$$\begin{vmatrix} p, & s, & \alpha \\ q, & t, & \beta \\ r, & u, & \gamma \end{vmatrix} = 0, \qquad \begin{vmatrix} p, & s, & \alpha' \\ q, & t, & \beta' \\ r, & u, & \gamma' \end{vmatrix} = 0,$$

lesquelles ont en commun la courbe cubique

$$\left\| \begin{matrix} p, & q, & r \\ s, & t, & u \end{matrix} \right\| = 0.$$

Or, en prenant α, β, γ, α', β', γ' des fonctions lineaires de P et Q, nous avons, en effet, réduit la courbe sextique à la droite $P=0$, $Q=0$ et à la courbe quintique $6-3-1$.

Courbe cubicubique, espèce $9-6+2$.—Cette courbe est l'intersection partielle de deux surfaces cubiques qui ont en commun une courbe excubo-quartique. En supposant que cette courbe excubo-quartique soit l'intersection partielle d'une surface quadrique et d'une surface cubique qui ont en commun les deux droites $(x=0,\ y=0)$ et $(z=0,\ w=0)$, on peut prendre pour équation de ces deux surfaces

$$U = xw - yz = 0,$$

$$V = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} (x,\ y)(z,\ w) = 0,$$

en représentant de cette manière la fonction $axz + byz + cxw + dyw$, linéaire par rapport à x, y et par rapport à z, w, avec des coefficients a, b, c, d, lesquels sont des fonctions linéaires quelconques de x, y, z, w.

En écrivant d'abord

$$V = (ax + by)z + (cx + dy)w,$$

$$U = \quad - \ y\ z + \qquad x\ w,$$

on obtient

$$xV - (cx + dy)\ U = z\,[ax^2 + (b + c)\,xy + dy^2].$$

Et de même en écrivant

$$V = (az + cw)x + (bz + dw)y,$$

$$U = \qquad w\ x - \qquad z\ y,$$

on obtient

$$zV + (bz + dw)\ U = x\,[az^2 + (b + c)\,zw + dw^2].$$

Or le premier de ces résultats fait voir qu'en supposant $U=0$, $V=0$, on obtient $ax^2 + (b+c)\,xy + dy^2 = 0$, et le second, qu'en supposant $U=0$, $V=0$, on obtient de même $az^2 + (b+c)\,zw + dw^2 = 0$. Les surfaces $U=0$, $V=0$ se coupent selon la courbe excubo-quartique et les droites $(x=0,\ y=0)$ et $(z=0,\ w=0)$; mais la surface $ax^2 + (b+c)\,xy + dy^2 = 0$ ne passe que par la première, et la surface $az^2 + (b+c)\,zw + dw^2 = 0$ ne passe que par la seconde de ces deux droites; donc les deux surfaces se coupent selon la courbe excubo-quartique, mais non pas selon l'une ou l'autre des deux droites, c'est-à-dire que les deux surfaces cubiques

$$ax^2 + (b+c)\,xy + dy^2 = 0,$$

$$az^2 + (b+c)\,zw + dw^2 = 0,$$

se coupent selon la courbe excubo-quartique, et encore selon une courbe quintique $9-6+2$.

Les deux surfaces cubiques ont chacune une droite double, elles sont done des surfaces réglées. La courbe est donc comprise parmi les courbes décrites sur une surface cubique réglée, pour lesquelles M. Chasles a trouvé dernièrement une construction géométrique très-élégante.

La courbe a cinq points doubles apparents; elle peut donc ne pas avoir d'autre singularité, ou avoir un point double ou de rebroussement: cela donne les trois sous-espèces

$$V.7, \quad V.8, \quad V.9$$

de M. Salmon.

On démontre sans peine que toute courbe quintique est plane, quadricubique, quadriquartique ou cubicubique; mais, pour faire voir qu'il n'existe que les cinq espèces ci-dessus mentionnées, il y a encore plusieurs cas à considérer. Par exemple, pour les courbes cubicubiques, on pourrait supposer que les deux surfaces cubiques avaient en commun une courbe quadriquadrique: si cela était, les équations des deux surfaces seraient de la forme $Vx - Uy = 0$, $Vz - Uw = 0$ (surfaces qui ont en commun la courbe quadriquadrique $U = 0$, $V = 0$), mais dans ce cas la courbe quintique serait située sur la surface quadrique $xw - yz = 0$, et l'on ne fait que retrouver l'espèce quadricubique $6 - 1$. J'ai fait, après M. Salmon, cette revue des différents cas, et je me suis assuré qu'il n'y a que les cinq espèces. Il convient peut-être de remarquer que l'énumération des sous-espèces comprises dans celles-ci n'est pas tout à fait complète, parce que, en certains cas, la courbe peut avoir un point triple, ou autre singularité plus élevée que les points doubles ou de rebroussement. Cela ne présente pas de difficulté, et en effet je n'ai parlé des sous-espèces que pour rapprocher mes résultats de ceux de M. Salmon.

La longueur de cette communication m'empêche de faire voir à présent comment les cinq espèces peuvent se déduire de la théorie générale des courbes dans l'espace considérées comme situées sur une surface monoïde.

303.

SUR LE PROBLÈME DU POLYGONE INSCRIT ET CIRCONSCRIT. LETTRE À M. PONCELET.

[From the *Comptes Rendus de l'Académie des Sciences à Paris*, tom. LV. (*Juillet—Décembre*, 1862), pp. 700, 701.]

J'OSE vous écrire par rapport aux remarques que vous faites p. 483 de l'ouvrage (*Applications d'Analyse etc.* [Paris, t. I. (1862)]) au sujet de mes recherches sur le problème du polygone inscrit et circonscrit [267 and the papers 113, 115, 116 and 128 therein referred to].

Je n'ai nullement voulu attribuer à Fuss le théorème pour les deux cercles. J'ai seulement dit, tout à fait en passant: *The case...of the two circles* (*the original case of the Porism as considered by Fuss*) et en effet Fuss a fait des recherches sur ce cas d'un polygone inscrit et circonscrit à deux cercles. Mais je n'ai jamais imaginé qu'il y eût un géomètre (algébriste ou non) qui ne connût pas tant votre ouvrage classique de 1822, que le mémoire de 1828 de Jacobi, où l'on voit précisément ce que Fuss a fait sur ce problème. Par rapport à mon dernier Mémoire (*Phil. Trans.* 1861) [267], que vous citez et qui résume quelques Notes que j'ai publiées en 1853, permettez-moi de vous mentionner la forme de ma solution: on a une fonction $a+b\xi+c\xi^2+d\xi^3$, où ξ est une quantité indéterminée, et a, b, c, d sont des fonctions données très simples des paramètres qui determinent les deux cercles (ou coniques). On développe la racine carrée de cette fonction dans la forme $A+B\xi+C\xi^2+D\xi^3+E\xi^4+\ldots$, et, cela fait, on a tout de suite l'équation entre les paramètres pour un polygone d'ordre quelconque; savoir pour le triangle, le pentagone, l'heptagone, etc., ces conditions sont

$$C=0,\quad \begin{vmatrix} C, & D \\ D, & E \end{vmatrix}=0,\quad \begin{vmatrix} C, & D, & E \\ D, & E, & F \\ E, & F, & G \end{vmatrix}=0, \text{ etc.,}$$

tandis que pour le quadrangle, l'hexagone, l'octogone, etc., ces conditions sont

$$D=0, \quad \begin{vmatrix} D, & E \\ E, & F \end{vmatrix} = 0, \quad \begin{vmatrix} D, & E, & F \\ E, & F, & G \\ F, & G, & H \end{vmatrix} = 0, \text{ etc.,}$$

de manière que la condition est trouvée explicitement pour un polygone d'ordre quelconque *sans passer par celles qui appartiennent aux polygones d'ordre inférieur.*

Comme j'attache, je l'avoue, un peu d'importance à cette solution (laquelle selon l'explication que je viens de donner ne paraît pas mériter la critique que vous en faites) je serais bien aise si vous voulez bien communiquer cette lettre à l'Académie.

304.

SUR UN MÉMOIRE DE JACOBI. EXTRAIT D'UNE LETTRE À M. J. BERTRAND.

[From the *Comptes Rendus de l'Académie des Sciences à Paris*, tom. LVI. (*Janvier—Juin*, 1863), p. 43. See **298**, foot-note to No. 117.]

PERMETTEZ-MOI de vous soumettre une remarque que je viens de faire par rapport au Mémoire de Jacobi "Sur l'élimination des nœuds dans le problème des trois corps" (*Compte Rendu* 8 août, 1842 [and Crelle, t. XXVI. (1843), pp. 115—131]). Il me semble que quoique Jacobi dise qu'il a fait dépendre le problème d'un système de cinq équations du premier ordre et une seule du second ordre, il a réellement fait plus que cela, savoir qu'il l'a fait dépendre d'un système de six équations du premier ordre et qu'ainsi il est allé aussi loin que vous dans le "Mémoire sur l'intégration de quelques équations différentielles de la Mécanique," *Journal de M. Liouville*, t. XVII. (1852). En effet si dans les équations I,...VI. de Jacobi, pour les réduire à un système d'équations du premier ordre, on écrit

$$\frac{d}{dt}(\mu r^2 + \mu_1 r_1^2) = \theta,$$

le système peut evidemment se présenter sous la forme

$$\frac{di}{I} = \frac{di_1}{I_1} = \frac{du}{U} = \frac{du_1}{U_1} = \frac{dr}{R} = \frac{dr_1}{R_1} = \frac{d\theta}{\Theta} (= dt),$$

et, cela étant, en remarquant que les fonctions I, I_1, U...., ne contiennent pas t, et en omettant l'équation $(=dt)$, on a un système de six équations entre les quantités i, i_1, u, u_1, r, r_1, θ: en supposant que l'intégration soit effectuée, on obtient alors t au moyen d'une quadrature.

Je remarque en passant qu'il ne me paraît pas que Jacobi ait dû dire: "Par suite l'on a fait cinq intégrations;" les seules intégrations qu'il a faites sont: l'intégrale des forces vives, et les trois intégrales des aires: cela étant on obtient au lieu de 12 équations entre 13 variables, 8 équations entre 9 variables, et dans la solution de Jacobi il arrive que ce système de 8 équations contient, comme partie de lui-même, un système de 5 équations entre 7 variables; mais à moins d'intégrer les 6 équations on n'obtient pas d'intégrale nouvelle.

305.

CONSIDÉRATIONS GÉNÉRALES SUR LES COURBES EN ESPACE. COURBES DU CINQUIÈME ORDRE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LVIII. (*Janvier—Juin*, 1864), pp. 994—1000. Continuation of **302**.]

EN considérant une courbe du $m^{\text{ième}}$ ordre représentée au moyen des équations

$$U=0, \quad w=\frac{P}{Q},$$

qui dénotent respectivement un cône du $m^{\text{ième}}$ ordre et une surface monoïde du $p^{\text{ième}}$ ordre, le cône doit passer $m(p-1)$ fois par les $p(p-1)$ droites $(P=0,\ Q=0)$ de la monoïde. J'indique la manière de ce passage au moyen d'un symbole que je nomme la *signature* du système; ce symbole, composé ordinairement des numéros 2, 1, 0, ensemble $p(p-1)$ numéros, fait voir combien des $p(p-1)$ droites de la monoïde sont, par rapport au cône, des droites doubles, des droites simples, ou des droites qui ne sont pas situées sur le cône: par exemple, $m=5$, $p=3$, la signature 222211 fait voir qu'il y a quatre droites doubles, deux droites simples; la signature 222220, qu'il y a cinq droites doubles, une droite qui n'est pas située sur le cône.

Je reviens aux courbes du cinquième ordre; j'ai établi (t. LIV. p. 672) qu'il y a cinq espèces de ces courbes, à savoir:

			P. D. A.
Courbe plane	ou espèce	5	0
Courbe quadricubique	„	$6-1$	4
Courbe quadriquartique	„	$8-3$	6
Courbe cubicubique (deux espèces)	„	$9-3-1$	6
		$9-6+2$	5

Je fais abstraction de la courbe plane, et je cherche à rattacher les quatre autres espèces à la théorie de la surface monoïde. Pour cela je remarque qu'en prenant pour sommet du cône et de la surface monoïde un point *quelconque*, la surface monoïde (ne pouvant pas être de l'ordre 2) sera de l'ordre 3 ou 4.

Je considère d'abord le cas d'une monoïde cubique: la signature sera 222211 ou 222220.

Monoïde cubique, signature 222211.—Ici le cône $U=0$ passe par les six droites de la monoïde; donc U est fonction syzygétique de P et Q: autrement dit, on peut trouver P' et Q' fonctions homogènes de (x, y, z) de manière à avoir identiquement $U=PQ'-P'Q$: P et Q sont des ordres 3 et 2 respectivement, donc P' et Q' seront aussi des ordres 3 et 2 respectivement. En combinant les équations

$$U=PQ'-P'Q=0, \quad w=\frac{P}{Q},$$

on obtient

$$w=\frac{P'}{Q'},$$

ou plus généralement

$$\frac{P+\alpha P'}{Q+\alpha Q'}$$

(où α est un paramètre arbitraire); mais en écrivant cette équation sous la forme $(Qw+P)+\alpha(Q'w-P')=0$, on voit que les monoïdes cubiques que représente cette équation sont toutes en involution avec les deux monoïdes cubiques $Qw-P=0$, $Q'w-P'=0$; on peut donc dire qu'il y a dans le cas dont il s'agit *deux* monoïdes cubiques.

Monoïde cubique, signature 222220.—Ici le cône ne passe pas par les six droites de la monoïde, donc il n'existe pas d'équation identique telle que $U=PQ'-P'Q$, et la monoïde $w=\dfrac{P}{Q}$ est la seule monoïde cubique.

Je passe au cas d'une monoïde quartique; la signature sera

222111111111, ou 222211111110, ou 222221111100, ou 222222111000.

Monoïde quartique, signature 222111111111.—Le cône $U=0$ passe ici par toutes les douze droites de la monoïde, c'est-à-dire on aurait identiquement $U=PQ'-P'Q$, où P, Q seraient des fonctions homogènes de (x, y, z) des ordres 2 et 1 respectivement, et il y aurait une monoïde *quadrique* $w=\dfrac{P'}{Q'}$. Ce cas n'existe donc pas.

Monoïde quartique, signature 222211111110.—Le cône $U=0$ passe par toutes les douze droites de la monoïde, hormis une seule droite; donc en écrivant $M=0$ pour l'équation d'un plan quelconque par cette droite exceptée, le cône $MU=0$ passe par les douze droites de la monoïde: on a donc identiquement $MU=PQ'-P'Q$, où P', Q'

sont des ordres 3 et 2 respectivement, et il passe par la courbe la monoïde cubique $w=\frac{P'}{Q'}$; de plus M contient une constante arbitraire ($M=K+\alpha L$, en prenant $K=0$, $L=0$ pour les équations de deux plans qui passent chacun par la droite mentionnée); donc P', Q' contiennent aussi cette constante arbitraire, autrement dit il y a *deux* monoïdes cubiques. Cela rentre donc dans le cas, monoïde cubique, signature 222211.

Monoïde quartique, signature 222221111100.—Le cône $U=0$ passe par toutes les droites de la monoïde, hormis deux droites; donc en écrivant $M=0$ pour l'équation du plan passant par ces deux droites, le cône $MU=0$ contient toutes les droites; on a donc identiquement $MU=PQ'-P'Q$, où P', Q' sont des ordres 3 et 2 respectivement, et il passe par la courbe la monoïde cubique $w=\frac{P'}{Q'}$. Mais ici $M=0$ est un plan déterminé; donc P' et Q' sont aussi des fonctions déterminées, et il n'y a qu'une seule monoïde cubique. Cela rentre dans le cas, monoïde cubique, signature 222220.

Monoïde quartique, signature 222222111000.—Le cône $U=0$ passe par toutes les droites de la monoïde, hormis trois droites; donc en prenant $M=0$ pour l'équation d'un cône quadrique quelconque qui passe par les droites exceptées, le cône $MU=0$ passe par toutes les droites. On a donc identiquement $MU=PQ'-P'Q$, où P', Q' sont des ordres 4 et 3 respectivement. Cela donne la monoïde quartique $w=\frac{P'}{Q'}$. Mais M contient trois constantes arbitraires: il y a done trois nouvelles monoïdes quartiques $w=\frac{P'}{Q'}$, $w=\frac{P''}{Q''}$, $w=\frac{P'''}{Q'''}$, ou en tout quatre monoïdes quartiques.

On démontre sans peine que pour l'espèce $6-1$, il y a deux monoïdes cubiques, pour l'espèce $9-6+2$ une seule monoïde cubique, et que pour les espèces $8-3$ et $9-3-1$ il n'y a pas de monoïde cubique; on a donc l'identification que voici:

Espèce $6-1$, monoïde cubique, signature	222211,
Espèce $9-6+2$, monoïde cubique, signature	222220,
Espèce $8-3$ / Espèce $9-3-1$, monoïde quartique, signature	222222111000,

et il ne reste qu'à distinguer les deux espèces $8-3$ et $9-3-1$, considérées comme représentées au moyen de cône et monoïde.

Je remarque que le système de cône et monoïde à signature 222222111000 contient 20 constantes. En effet, en prenant $Q=0$ un cône cubique quelconque (9 constantes), on peut prendre à volonté sur ce cône huit droites (8 constantes), et par six de ces droites comme droites doubles et deux de ces droites comme droites simples (20 conditions) faire passer le cône quintique determiné $U=0$; ce cône et le cône cubique $Q=0$ se coupent selon les huit droites (qui comptent pour quatorze droites) et selon une neuvième droite; et par les neuf droites on peut faire passer le cône quartique $P=0$ (5 constantes). Cela donne la monoïde quartique $w=\frac{P}{Q}$, où w contient implicitement

comme facteur une constante ; il y a donc en tout $9+8+5+1=23$ constantes. Mais en combinant l'équation de la monoïde avec l'équation $U=0$ du cône quintique, on obtient la monoïde quartique

$$\omega = \frac{P+\alpha P'+\beta P''+\gamma P'''}{Q+\alpha Q'+\beta Q''+\gamma Q'''},$$

et sans perte de généralité on peut disposer des constantes α, β, γ, de manière à satisfaire à trois conditions quelconques ; on doit donc diminuer de 3 le nombre 23, ce qui donne enfin 20 constantes.

La courbe $8-3$ contient 18 constantes, il faut donc chercher quelle est la particularité qui doit avoir lieu pour que le cas, monoïde quartique à signature 222222111000, donne une courbe $8-3$.

J'ai nommé *droites de la monoïde* les droites $P=0$, $Q=0$ qui passent par le sommet ; en supposant qu'il y ait sur la monoïde des droites qui ne passent pas par le sommet, on peut appeler *transversale* une telle droite. Or, pour l'espèce $8-3$, il doit exister sur la monoïde quartique trois transversales qui ne se rencontrent pas ; car alors, en faisant passer par ces transversales un hyperboloïde, cet hyperboloïde et la monoïde se coupent selon les trois transversales et selon la courbe $8-3$ dont il s'agit. Or, en supposant qu'il existe une transversale, le plan passant par le sommet et cette transversale contient trois des droites $P=0$, $Q=0$. En effet, un plan quelconque par le sommet coupe la monoïde selon une courbe quartique avec un point triple au sommet ; pour le plan mené par une transversale, cette courbe quartique devient la transversale et une courbe cubique avec un point triple au sommet ; cette courbe cubique sera évidemment un système de trois droites, à savoir trois des droites $P=0$, $Q=0$. Et réciproquement, si trois quelconques des droites de la monoïde sont situées dans un plan, ce plan coupe la monoïde selon les trois droites et selon une transversale. S'il y a sur la monoïde une seconde transversale, il y aura de même un second système de trois droites dans un plan ; on démontre que si le premier système est composé de trois droites, et le second système de trois *autres* droites, les deux transversales se coupent ; donc, si les deux transversales ne se coupent pas, les deux systèmes auront une droite commune. S'il y a sur la monoïde une troisième transversale, il y a de même un troisième système de trois droites dans un plan ; et si les trois transversales ne se rencontrent pas, il est de plus nécessaire que deux quelconques des trois plans aient en commun une droite de la monoïde ; cela revient à dire qu'il doit y avoir parmi les douze droites $P=0$, $Q=0$ de la monoïde six droites 7, 8, 9, 7′, 8′, 9′ telles que les droites 7, 8′, 9′, les droites 7′, 8, 9′, et les droites 7′, 8′, 9 soient situées chaque système dans un même plan : cela étant, la monoïde aura trois transversales qui ne se rencontrent pas.

Je prends à volonté par un point quelconque de l'espace un tel système de six droites 7, 8, 9, 7′, 8′, 9′ (9 constantes) ; je fais passer par les six droites un cône cubique quelconque $Q=0$ (3 constantes) et aussi un cône quartique quelconque $P=0$ (8 constantes) ; au moyen des deux cônes je forme l'équation $w=\frac{P}{Q}$ de la surface monoïde ; il y a une constante arbitraire contenue implicitement en w : cela donne en

tout $9+3+8+1=21$ constantes. Les deux cônes $P=0$, $Q=0$ se coupent selon les six droites 7, 8, 9, 7′, 8′, 9′, et selon six autres droites 1, 2, 3, 4, 5, 6 : il suit de la théorie précédente (mais on peut aussi démontrer analytiquement) qu'il existe un cône quintique $U=0$ qui satisfait aux conditions de passer deux fois par chacune des droites 1, 2, 3, 4, 5, 6 (avoir chacune de ces droites pour une droite double, 18 conditions) et une fois par chacune des droites 7, 8, 9 (3 conditions, en tout $18+3=21$ conditions). Et cela étant, on aura la courbe $8-3$ déterminée au moyen du cône $U=0$ et la surface monoïde $w=\frac{P}{Q}$, à signature 222222111000 (à savoir les droites 1, 2, 3, 4, 5, 6 qui sont par rapport au cône des droites doubles, les droites 7, 8, 9 des droites simples, et les droites 7′, 8′, 9′ des droites qui ne sont pas situées sur le cône). Le nombre des constantes est 21, mais au moyen de la transformation

$$\omega=\frac{P+\alpha P'+\beta P''+\gamma P'''}{Q+\alpha Q'+\beta Q''+\gamma Q'''},$$

on réduit comme auparavant ce nombre à $21-3=18$, ce qui est juste.

J'ajoute les considérations que voici : le cône $U=0$ passe deux fois par chacune des droites 1, 2, 3, 4, 5, 6, une fois par chacune des droites 7, 8, 9. Soit $M=0$ l'équation du système des trois plans qui contiennent les droites 7, 8′, 9′, les droites 7′, 8, 9′, et les droites 7′, 8′, 9 respectivement ; le cône $M=0$ contient chacune des droites 7, 8, 9 une fois, et chacune des droites 7′, 8′, 9′ deux fois. Donc le cône $MU=0$ contient chacune des droites 1, 2, 3, 4, 5, 6, 7, 8, 9, 7′, 8′, 9′ deux fois ; ces douze droites sont les droites d'intersection des cônes $P=0$, $Q=0$, et ainsi nous avons identiquement $MU=AP^2+BPQ+CQ^2$, A, B, C étant des fonctions homogènes de (x, y, z) des ordres 0, 1, 2 respectivement. Cela étant, les équations

$$w=\frac{P}{Q},\quad MU=AP^2+BPQ+CQ^2=0$$

donnent

$$Aw^2+Bw+C=0,$$

équation de la surface quadrique sur laquelle est située la courbe $8-3$.

Je passe à la théorie analytique. Soit, pour abréger,

$$\begin{aligned}
\xi &= \quad b\,y+c\,z, & X &= \quad \beta\,y+\gamma\,z, & \Theta &= \lambda x+\mu y+\nu z,\\
\eta &= a'x \quad +c'z, & Y &= \alpha'x \quad +\gamma'z, & &\\
\zeta &= a''x+b''y \quad , & Z &= \alpha''x+\beta''y. & &
\end{aligned}$$

Je prends pour équations des droites 7, 8, 9, 7′, 8′, 9′ :

$$(y=0,\quad z=0),\quad (z=0,\quad x=0),\quad (x=0,\quad y=0),$$
$$(x=0,\quad \xi=0),\quad (y=0,\quad \eta=0),\quad (z=0,\quad \zeta=0),$$

et je forme les équations les plus générales pour le cône cubique et le cône quartique qui passent par ces droites ; ces équations seront

$$Q = yz\xi\delta + zx\eta\delta' + xy\zeta\delta'' + xyz\Theta = 0,$$
$$-P = yz\xi X + zx\eta Y + xy\zeta Z + xyz\Theta = 0.$$

On a de là la surface monoïde $w = \frac{P}{Q}$. En écrivant dans cette équation $x = 0$, on obtient $w = -\frac{X}{\delta}$; et de même, pour $y = 0$, on obtient $w = -\frac{X}{\delta'}$, et pour $z = 0$ on obtient $w = -\frac{Z}{\delta''}$, c'est-à-dire qu'il y a sur la monoïde les trois transversales

$$(x = 0,\ X + \delta W = 0),\quad (y = 0,\ Y + \delta' W = 0),\quad (z = 0,\ Z + \delta'' W = 0),$$

ou, comme on peut écrire ces équations,

$$(x = 0,\quad \beta y + \gamma z + \delta w = 0),$$
$$(y = 0,\quad \alpha' x + \gamma' z + \delta' w = 0),$$
$$(z = 0,\quad \alpha'' x + \beta'' y + \delta'' w = 0).$$

On trouve sans peine l'équation de la surface quadrique qui passe par les transversales ; en écrivant, pour abréger,

$$A = \delta\delta' o\ ,$$
$$B = (\delta'\alpha'' + \delta''\alpha')\,\delta x + (\delta''\alpha + \delta\alpha'')\,\delta' y + \delta''(\delta\alpha' + \delta'\alpha)\,\delta'' z,$$
$$C = \alpha'\alpha''\delta x^2 + \beta''\beta\delta' y^2 + \gamma\gamma'\delta'' z^2 + (\gamma\beta''\delta' + \gamma'\beta\delta'')\,yz + (\alpha'\gamma\delta'' + \alpha''\gamma'\delta)\,zx + (\beta''\alpha'\delta + \beta\alpha''\delta')\,xy,$$

cette équation est

$$Aw^2 + Bw + C = 0,$$

et en éliminant w entre cette équation et l'équation $w = \frac{P}{Q}$, on obtient l'équation

$$AP^2 + BPQ + CQ^2 = 0,$$

laquelle, en vertu de l'identité

$$AP^2 + BPQ + CQ^2 = xyzU,$$

se réduit à $U = 0$, équation d'un cône du cinquième ordre, ce qui donne le système $U = 0$, $w = \frac{P}{Q}$ de cône et monoïde à signature 222222111000. Pour démontrer l'identité dont il s'agit, il convient de remarquer qu'en substituant dans l'expression $AP^2 + BPQ + CQ^2$ les valeurs de P et Q, tous les termes contiennent explicitement le facteur xyz hormis les termes que voici :

$$A\,(y^2z^2\xi^2X^2 + z^2x^2\eta^2Y^2 + x^2y^2\zeta^2Z^2\ ),$$
$$-\,B\,(y^2z^2\xi^2X\delta + z^2x^2\eta^2Y\delta' + x^2y^2\zeta^2Z\delta''),$$
$$+\,C\,(y^2z^2\xi^2\,\delta^2 + z^2x^2\eta^2\,\delta'^2 + x^2y^2\zeta^2\,\delta''^2),$$

et pour démontrer que ces termes exceptés contiennent aussi le facteur xyz, il suffit de faire voir que la fonction $AX^2 - BX\delta + C\delta^2$ contient le facteur x, car alors, par la symétrie, les fonctions $AY^2 - BY\delta' + C\delta'^2$ et $AZ^2 - BZ\delta'' + C\delta''^2$ contiendront respectivement les facteurs y et z, et l'expression entière sera divisible par xyz. Mais en écrivant $x = 0$, on trouve

$$\left.\begin{array}{ll} AX^2 = & \delta\delta'\delta''(\beta y + \gamma z)^2 \\ -BX\delta & -[\delta'\delta''(\beta y + \gamma z) + \delta(\beta''\delta' y + \gamma'\delta'' z)]\,\delta(\beta y + \gamma z) \\ +C\delta^2 & +(\beta y + \gamma z)(\beta''\delta' y + \gamma'\delta'' z)\,\delta^2 \end{array}\right\} = 0,$$

c'est-à-dire $AX^2 - BX\delta + C\delta^2$ contient le facteur x. Donc enfin

$$AP^2 + BPQ + CQ^2$$

contient le facteur xyz, ce qui était le théorème à démontrer.

306.

SUR LES CONIQUES QUI TOUCHENT DES COURBES D'ORDRE QUELCONQUE. EXTRAIT D'UNE LETTRE À M. CHASLES.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LIX. (*Juillet—Décembre*, 1864), pp. 224—225.]

En considérant l'expression

$$S_5\,(S_5 + S_4 + S_3 - 3S_2 + 3S_1)$$

que vous avez donnée (*Comptes Rendus*, t. LVIII. p. 223) pour le nombre des coniques qui touchent cinq courbes d'ordre quelconque, j'ai trouvé qu'elle peut s'écrire sous la forme que voici, savoir: en dénotant les ordres par (m, n, p, q, r), et en mettant $M = m^2 - m, \ldots$, de manière que (M, N, P, Q, R) seront les classes des cinq courbes, l'expression transformée est

$$(M, m)\,(N, n)\,(P, p)\,(Q, q)\,(R, r)\,\{1, 2, 4, 4, 2, 1\};$$

en représentant par cette notation abrégée la fonction

$$\begin{aligned} &1.\ MNPQR \\ &+ 2\Sigma\,(mNPQR) \\ &+ 4\Sigma\,(mn\,PQR) \\ &+ 4\Sigma\,(mn\,p\,QR) \\ &+ 2\Sigma\,(mn\,p\,qR) \\ &+ 1.\ mn\,p\,q\,r. \end{aligned}$$

En écartant les relations $M = m^2 - m, \ldots$, et en supposant seulement que (m, n, p, q, r) soient les ordres, et (M, N, P, Q, R) les classes des cinq courbes, la nouvelle formule s'applique aux courbes avec des points doubles ou de rebroussement; on peut même

supposer que la courbe de la classe M et de l'ordre m se réduise à un système de M points et de m droites, et que les autres courbes se réduisent aussi à des systèmes de points et droites; et cela étant, on obtient une vérification immédiate de la formule. Car en choisissant dans le système qui remplace chaque courbe un élément (point ou droite) à volonté, on obtient

$$\begin{aligned} MNPQR &\text{ systèmes } 5p, \\ \Sigma\, m\, NPQR &\text{ systèmes } 4p,\ 1d, \\ \Sigma\, m\, n\, PQR &\text{ systèmes } 3p,\ 2d, \\ \Sigma\, m\, n\, p\, QR &\text{ systèmes } 2p,\ 3d, \\ \Sigma\, m\, n\, p\, q\, R &\text{ systèmes } 1p,\ 4d, \\ m\, n\, p\, q\, r &\text{ systèmes } \quad 5d. \end{aligned}$$

Or la condition par rapport à la première courbe se réduit à celle de passer par l'un quelconque des M points, ou de toucher l'une quelconque des m droites; et de même pour les autres courbes. Donc (en entendant par le mot *toucher* appliqué à un système de points et de droites, passer par les points et toucher les droites du système) la conique doit toucher l'un quelconque des systèmes $(5p)$, ou $(4p,\ 1d)$, ou $(3p,\ 2d)$, ..., ou $(5d)$; et pour un système de la forme

$$(5p),\quad (4p,\ 1d),\quad (3p,\ 2d),\quad (2p,\ 3d),\quad (1p,\ 4d),\quad \text{ou}\quad (5d),$$

le nombre des coniques est

$$1,\qquad 2,\qquad 4,\qquad 4,\qquad 2,\qquad \text{ou}\quad 1,$$

ce qui donne pour le nombre total des coniques l'expression ci-dessus écrite.

On peut supposer que la conique, au lieu de toucher les deux courbes m, n, ait avec la seule courbe m (1°) un contact du deuxième ordre; (2°) un contact double. Le nombre des coniques qui satisfont à l'une ou l'autre de ces conditions, et qui passent aussi par trois points donnés, a été trouvé par Steiner (Aufgabe und Lehrsätze, *Crelle*, t. XLIX. p. 273), savoir:

$$(1^\circ)\ \text{le nombre} = 3m(m-1);\quad (2^\circ)\ \text{le nombre} = \tfrac{1}{2}(m^2-m)(m^2+3m-6);$$

j'ai vérifié d'une manière particulière ces deux résultats.

En supposant que la conique (au lieu de passer par les trois points donnés) touche les courbes p, q, r, je trouve pour les deux cas respectivement ces résultats,

1° le nombre $= 3(m^2-m) \times (P,\ p)(Q,\ q)(R,\ r)\{1,\ 2,\ 2,\ 1\}$,

2° le nombre $= \tfrac{1}{2}(m^2-m) \times$

$$(P,\ p)(Q,\ q)(R,\ r)\{m^2+3m-6,\ 2m^2+6m-16,\ 4m^2+4m-22,\ 4m^2-15\};$$

formules dans lesquelles la courbe m doit être une courbe sans singularités.

Grasmere, Westmoreland, 37 *Juillet*, 1864.

307.

NOTE SUR LES FONCTIONS al (x), &c., DE M. WEIERSTRASS.

[From the *Journal de Mathématiques* (Liouville), tom. VII. (1862), pp. 137—142.]

LES fonctions al (x), al $(x)_1$, al $(x)_2$, al $(x)_3$ de M. Weierstrass satisfont respectivement aux équations

$$\frac{d^2 \operatorname{al}(x)}{dx^2} + 2k^2x\frac{d \operatorname{al}(x)}{dx} + 2kk'^2\frac{d \operatorname{al}(x)}{dk} + \qquad k^2x^2 \operatorname{al}(x) = 0,$$

$$\frac{d^2 \operatorname{al}(x)_1}{dx^2} + 2k^2x\frac{d \operatorname{al}(x)_1}{dx} + 2kk'^2\frac{d \operatorname{al}(x)_1}{dk} + (k'^2 + k^2x^2) \operatorname{al}(x)_1 = 0,$$

$$\frac{d^2 \operatorname{al}(x)_2}{dx^2} + 2k^2x\frac{d \operatorname{al}(x)_2}{dx} + 2kk'^2\frac{d \operatorname{al}(x)_2}{dk} + (1 \quad + k^2x^2) \operatorname{al}(x)_2 = 0,$$

$$\frac{d^2 \operatorname{al}(x)_3}{dx^2} + 2k^2x\frac{d \operatorname{al}(x)_3}{dx} + 2kk'^2\frac{d \operatorname{al}(x)_3}{dk} + (k^2 + k^2x^2) \operatorname{al}(x)_3 = 0,$$

ou, ce qui est la même chose, les fonctions

$$\operatorname{al}(x), \quad \sqrt{k} \operatorname{al}(x)_1, \quad \frac{\sqrt{k}}{\sqrt{k'}} \operatorname{al}(x)_2, \quad \frac{1}{\sqrt{k'}} \operatorname{al}(x)_3,$$

satisfont chacune à l'équation

$$\frac{d^2z}{dx^2} + 2k^2x\frac{dz}{dx} + 2kk'^2\frac{dz}{dk} + k^2x^2z = 0.$$

Écrivons pour un moment ξ, κ au lieu de x, k; les fonctions al (ξ), etc., satisfont à l'équation

$$\frac{d^2z}{d\xi^2} + 2\kappa^2\xi\frac{dz}{d\xi} + 2\kappa\kappa'^2\frac{dz}{d\kappa} + \kappa^2\xi^2z = 0.$$

Cela étant, en posant

$$\xi = \frac{x}{\sqrt{k}}, \quad \kappa = k,$$

on obtient

$$\frac{dz}{dx} = \frac{1}{\sqrt{k}}\frac{dz}{d\xi}, \quad \frac{d^2z}{dx^2} = \frac{1}{k}\frac{d^2z}{d\xi^2}, \quad \frac{dz}{dk} = -\frac{x}{2k\sqrt{k}}\frac{dz}{d\xi} + \frac{dz}{d\kappa},$$

et de là

$$\frac{dz}{d\xi} = \sqrt{k}\frac{dz}{dx}, \quad \frac{d^2z}{d\xi^2} = k\frac{d^2z}{dx^2}, \quad \frac{dz}{d\kappa} = \frac{dz}{dk} + \frac{x}{2k}\frac{dz}{dx}.$$

L'équation différentielle devient ainsi

$$k\frac{d^2z}{dx^2} + 2k^3x\frac{dz}{dx} + 2kk'^2\left(\frac{dz}{dk} + \frac{x}{2k}\frac{dz}{dx}\right) + kx^2z = 0,$$

c'est-à-dire

$$\frac{d^2z}{dx^2} + \frac{1+k^2}{k}x\frac{dz}{dx} + 2k'^2\frac{dz}{dk} + x^2z = 0,$$

équation qui sera ainsi satisfaite par

$$\mathrm{al}\left(\frac{x}{\sqrt{k}}\right), \quad \sqrt{k}\,\mathrm{al}\left(\frac{x}{\sqrt{k}}\right)_1, \quad \frac{\sqrt{k}}{\sqrt{k'}}\,\mathrm{al}\left(\frac{x}{\sqrt{k}}\right)_2, \quad \frac{1}{\sqrt{k'}}\,\mathrm{al}\left(\frac{x}{\sqrt{k}}\right)_3.$$

Or en écrivant

$$k + \frac{1}{k} = \alpha,$$

l'équation en z devient

$$(1) \qquad \frac{d^2z}{dx^2} + 2\alpha x\frac{dz}{dx} - 2(\alpha^2 - 4)\frac{dz}{d\alpha} + x^2z = 0,$$

laquelle est ce que devient celle-ci

$$(2) \quad (1 - \alpha x^2 + x^4)\frac{d^2z}{dx^2} + (n-1)(\alpha x - 2x^3)\frac{dz}{dx} - 2n(\alpha^2 - 4)\frac{dz}{d\alpha} + n(n-1)x^2z = 0,$$

en y écrivant $\frac{x}{\sqrt{n}}$ au lieu de x, et puis $n = \infty$. L'équation (2), trouvée par Jacobi (*Journal de Crelle*, t. IV. p. 185, 1829), a la propriété que voici, savoir en posant

$$\alpha = k + \frac{1}{k}, \quad x = \sqrt{k}\sin\mathrm{am}\,u,$$

alors l'équation est satisfaite en prenant pour z soit le numérateur, soit le dénominateur, de la fonction rationnelle de x qui donne la valeur de la fonction $\sqrt{\lambda}\sin\mathrm{am}\left(\frac{\mu}{M}, \lambda\right)$, où λ, M sont le module et le multiplicateur qui correspondent à la transformation de l'ordre n (n étant un nombre impair quelconque).

L'équation fut donnée par Jacobi sans démonstration. Je l'ai démontrée (*Camb. and Dubl. Math. Journal*, t. II. p. 256, 1847, [45]) de la manière que voici, savoir en écrivant

$$z = \left(\frac{2Kk'}{\pi}\right)^{\frac{1}{2}(n-1)} \Theta^{-n}(u) \,.\, \Sigma,$$

on obtient pour Σ l'équation

$$\frac{d^2\Sigma}{du^2} - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2nkk'^2\frac{d\Sigma}{dk} = 0.$$

Cette équation, en prenant

$$\omega = \frac{n\pi K'}{K}, \quad v = \frac{n\pi u}{2K},$$

devient

$$\frac{d^2\Sigma}{dv^2} - 4\frac{d\Sigma}{d\omega} = 0,$$

équation mentionnée par Jacobi, laquelle est satisfaite par

$$\Sigma = \Theta\left(nu, \frac{nK'}{K}\right), \text{ ou } \Sigma = \mathrm{H}\left(nu, \frac{nK'}{K}\right);$$

cela donne pour z deux valeurs qui sont le dénominateur et le numérateur de la fraction dont il s'agit. J'ose croire que ce doit être à peu près de cette manière que l'équation fut trouvée par Jacobi.

Or les solutions en question de l'équation (1) peuvent être trouvées au moyen de l'équation de Jacobi; pour cela, au lieu des valeurs ci-dessus données de ω, v, j'écris

$$\omega = \frac{\pi K'}{K}, \quad v = \frac{\sqrt{n}\pi u}{2K},$$

ce qui conduit à la même équation

$$\frac{d^2\Sigma}{dv^2} - 4\frac{d\Sigma}{d\omega} = 0,$$

laquelle sera ainsi satisfaite par

$$\Sigma = \Theta\left(\sqrt{n}u, \frac{K'}{K}\right), \quad \Sigma = \mathrm{H}\left(\sqrt{n}u, \frac{K'}{K}\right),$$

ou, ce qui est la même chose, par

$$\Sigma = \Theta(\sqrt{n}u), \qquad \Sigma = \mathrm{H}(\sqrt{n}u).$$

L'équation (2) sera donc satisfaite par

$$z = \left(\frac{2Kk'}{\pi}\right)^{\frac{1}{2}(n-1)} \Theta^{-n}(u)\,\Theta(\sqrt{n}u),$$

ou, en se souvenant que $\Theta(0)=\sqrt{\frac{2Kk'}{\pi}}$, par

$$z=\frac{\Theta(\sqrt{n}u)}{\Theta(0)}\cdot\frac{\Theta^n(0)}{\Theta^n(u)}.$$

Or en écrivant $\frac{x}{\sqrt{n}}$ au lieu de x, pour faire ensuite $n=\infty$, nous avons

$$\frac{x}{\sqrt{n}}=\sqrt{k}\sin\operatorname{am}u,$$

équation qui se réduit à

$$u=\frac{x}{\sqrt{nk}};$$

cela donne

$$\Theta(\sqrt{n}u)=\Theta\left(\frac{x}{\sqrt{k}}\right),\quad \Theta^n u=\Theta^n(0)\,e^{\frac{1}{2}nu^2\left(1-\frac{E}{K}\right)}=\Theta^n(0)\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)},$$

puisque

$$\Theta(u)=\Theta(0)\,e^{\frac{1}{2}u^2\left(1-\frac{E}{K}\right)-k^2\int_0 du\int_0 du\sin^2\operatorname{am}u};$$

et $n\int_0 du\int_0 du\sin^2\operatorname{am}u$, en y substituant $u=\frac{x}{\sqrt{nk}}$, contient le facteur $\frac{1}{n}$ et se réduit ainsi à zéro. Donc on obtient

$$z=\frac{\Theta\left(\frac{x}{\sqrt{k}}\right)}{\Theta(0)}\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)}$$

comme solution de l'équation (1), qui se déduit de l'équation (2) en y écrivant $\frac{x}{\sqrt{n}}$ au lieu de x et puis $n=\infty$. Et cette valeur de z est précisément la fonction $\operatorname{al}\left(\frac{x}{\sqrt{k}}\right)$ de M. Weierstrass. On obtient de même la solution

$$z=\frac{\mathrm{H}\left(\frac{x}{\sqrt{k}}\right)}{\Theta(0)}\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)},$$

ou, ce qui est la même chose,

$$z=\frac{\mathrm{H}\left(\frac{x}{\sqrt{k}}\right)}{\frac{1}{\sqrt{k}}\mathrm{H}'(0)}\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)},$$

qui est la fonction $\sqrt{k}\operatorname{al}\left(\frac{x}{\sqrt{k}}\right)_1$. Et d'une manière semblable les solutions

$$z=\frac{\mathrm{H}\left(\frac{x}{\sqrt{k}}+K\right)}{\Theta(0)}\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)},\quad z=\frac{\Theta\left(\frac{x}{\sqrt{k}}+K\right)}{\Theta(0)}\,e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)}$$

qui sont les fonctions $\sqrt{\frac{k}{k'}}\operatorname{al}\left(\frac{x}{\sqrt{k}}\right)_2$, $\frac{1}{\sqrt{k'}}\operatorname{al}\left(\frac{x}{\sqrt{k}}\right)_3$.

J'ajoute que dans le Mémoire cité (1847) j'ai donné la suite

$$z = C_0 + C_1 \frac{x^2}{1.2} + C_2 \frac{x^4}{1.2.3.4} + \dots,$$

où

$$\begin{aligned}
C_0 &= 1, \\
C_1 &= 0, \\
C_2 &= -\ 2, \\
C_3 &= +\ 8\alpha, \\
C_4 &= -\ 32\alpha^2 - 4, \\
C_5 &= +\ 128\alpha^3 + 96\alpha, \\
C_6 &= -\ 512\alpha^4 - 960\alpha^2 - 408, \\
C_7 &= +\ 2048\alpha^5 + 7168\alpha^3 + 7584\alpha, \\
C_8 &= -\ 8192\alpha^6 - 46080\alpha^4 - 88320\alpha^2 - 15384, \\
&\vdots
\end{aligned}$$

de façon qu'en général

$$C_{r+2} = -(2r+1)(2r+2)\,C_r - (2r+2)\,\alpha C_{r+1} + 2(\alpha^2 - 4)\frac{dC_{r+1}}{d\alpha},$$

c'est le développement de M. Weierstrass pour la fonction al $\left(\frac{x}{\sqrt{k}}\right)$: seulement je ne connaissais pas alors l'expression finie

$$\text{al}\left(\frac{x}{\sqrt{k}}\right), \quad \text{ou} \quad \frac{1}{\Theta(0)}\,\Theta\left(\frac{x}{\sqrt{k}}\right) e^{\frac{x^2}{2k}\left(1-\frac{E}{K}\right)}$$

de cette suite. Je remarque en passant que pour $\alpha = 2$, la suite se réduit à

$$e^{\frac{1}{2}x^2} \cdot \tfrac{1}{2}\left(e^{x} + e^{-x}\right).$$

308.

ON THE △ FACED POLYACRONS, IN REFERENCE TO THE PROBLEM OF THE ENUMERATION OF POLYHEDRA.

[From the *Memoirs of the Literary and Philosophical Society of Manchester*, vol. I. (1862), pp. 248—256.]

THE problem of the enumeration of polyhedra [1] is one of extreme difficulty, and I am not aware that it has been discussed elsewhere than in Mr Kirkman's valuable series of papers on this subject in the *Memoirs* of the Society and in the *Philosophical Transactions*. A case of the general problem is that of the enumeration of the polyhedra with trihedral summits; and Mr Kirkman in the earliest of his papers, viz. that "On the representation and enumeration of polyhedra" (*Memoirs*, vol. XII. pp. 47—70, 1854), has in fact, by an examination of the particular case, accomplished the enumeration of the octahedra with trihedral summits. A subsequent paper "On the enumeration of x-edra having trihedral summits and an $(x-1)$gonal base," *Phil. Trans.* vol. XLVI. pp. 399—411, 1856), relates, as the title shows, only to a special case of the problem of the polyhedra with trihedral summits, and in this particular case the number of polyhedra is more completely determined; but the later memoirs relate to the problem in all its generality, and the above-mentioned particular problem of the enumeration of the polyhedra with trihedral summits is not, I think, any where resumed. Instead of the polyhedra with trihedral summits, it is really the same thing, but it is rather more convenient to consider the polyacrons with triangular faces, or as these may for shortness be called, the △ faced polyacrons; and it is intended in the present paper to give a method for the derivation of the △ faced polyacrons of a given number of summits from those of the next inferior number of summits, and to exemplify it by finding, in an orderly manner, the △ faced polyacrons

[1] I use with Mr Kirkman the expression "enumeration of polyhedra" to designate the general problem, but I consider that the problem is to find the different polyhedra rather than to count them, and I consequently take the word *enumeration* in the popular rather than the mathematical sense.

up to the octacrons: thus, as regards the examples, stopping at the same point as Mr Kirkman, for although perfectly practicable it would be very tedious to carry them further, and there would be no commensurate advantage in doing so. The epithet △ faced will be omitted in the sequel, but it is to be understood throughout that I am speaking of such polyacrons only; and I shall for convenience use the epithets tripleural, tetrapleural, &c. to denote summits with three, four, &c. edges through them. The number of edges at a summit is of course equal to the number of faces, but it is the edges rather than the faces which have to be considered.

An n-acron has

$$n \text{ summits}, \quad 3n-6 \text{ edges}, \quad 2n-4 \text{ faces},$$

and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.

2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.

3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

$$n = c + d + e + f + g + h + \&c.,$$

$$6n - 12 = 3c + 4d + 5e + 6f + 7g + 8h + \&c.,$$

and therefore

$$12 = 3c + 2d + e + 0f - g - 2h - \&c.,$$

or

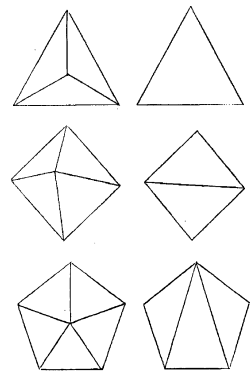
$$3c + 2d + e = 12 + g + 2h + \&c.;$$

whence if $c=0$ and $d=0$, then $e=12$ at least. It appears, moreover (since n cannot be less than e), that any polyacron with less than 12 summits cannot belong to the third class, and must therefore belong to the first or the second class.

An $(n+1)$-acron, by a process which I call the subtraction of a summit, may be reduced to an n-acron; viz., the faces about any summit of the $(n+1)$-acron stand upon a polygon (not in general a plane figure) which may be called the basic polygon, and when the summit with the faces and edges belonging to it is removed, the basic polygon, if a triangle, will be a face of the n-acron; if not a triangle, it can be partitioned into triangles which will be faces of the n-acron. The annexed figures exhibit the process for the cases of a tripleural, tetrapleural and pentipleural summit respectively, which are the only cases which need be considered; these may be called the first, second and third process respectively. It is proper to remark that for the same removed summit the first process can be performed in one way only, the second process in two ways, the third in five ways; these being in fact the numbers of ways of partitioning the basic polygon.

We may in like manner, by the converse process of the addition of a summit, convert an n-acron into an $(n+1)$-acron; viz., it is only necessary to take on the

n-acron a polygon of any number of sides, and make this the basic polygon of the new summit of the $(n+1)$-acron, and for this purpose to remove the faces within the polygon and substitute for them a set of triangular faces standing on the sides of the polygon and meeting in the new summit: the same figures exhibit the process for the cases of a tripleural, tetrapleural and pentipleural summit respectively, which

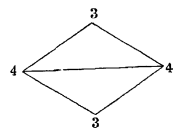


(as for the subtractions) are the only cases which need be considered. It may be noticed that for the same basic polygon the process is in each case a unique one; the process is said to be the first, second, or third process, according as the new summit is tripleural, tetrapleural, or pentipleural.

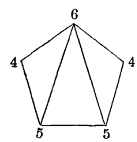
Now, reverting to the before-mentioned division of the polyacrons into three classes, an $(n+1)$-acron of the first class may by the first process of subtraction be reduced to an n-acron, and conversely it can be by the first process of addition derived from an n-acron. An $(n+1)$-acron of the second class, as having a tetrapleural summit, may by the second process of subtraction be reduced to an n-acron, and conversely it can be by the second process of addition derived from an n-acron. And in like manner, an $(n+1)$-acron of the third class, as having a pentipleural summit, may be by the third process of subtraction reduced to an n-acron, and conversely it may be by the third process of addition derived from an n-acron.

Hence all the $(n+1)$-acrons can be by the first, second and third processes of addition respectively derived from the n-acrons. It is to be observed that all the $(n+1)$-acrons of the first class are obtained by the first process; the second process is only required for finding the $(n+1)$-acrons of the second class; and these being all obtained by means of it, the third process is only required for finding the $(n+1)$-acrons of the third class. Hence the second process need only be made use of when the n-acron has no tripleural summit, or when it has only one tripleural summit, or when, having two tripleural summits, they are the opposite summits of two

adjacent faces. In the last-mentioned two cases respectively it is only necessary to consider the basic quadrangles which pass through the single tripleural summit and the basic quadrangle which passes through the two tripleural summits; for with any other basic quadrangle the derived $(n+1)$-acron would retain a tripleural summit, and would consequently be of the first class. The condition is more simply expressed as follows, viz.: The second process need only be employed when there is on the n-acron a basic quadrangle the summits of which are at least of the number of edges shown in the



annexed figure, and all the other summits are at least 4-pleural. Again, by the third process (as already mentioned) we seek only to obtain the $(n+1)$-acrons of the third class; the process need only be applied to the n-acrons for which there exists a basic pentagon the summits of which are at least of the number of edges shown in the



annexed figure, all the other summits being at least 5-pleural; for it is only in this case that the derived $(n+1)$-acron will be of the third class. The condition just referred to obviously implies that the n-acron is of the second or third class. It is to be noticed that in applying the foregoing principles to the formation of the polyacrons as far as the 11-acrons we are only concerned with the first and second processes.

Consider the entire series of n-acrons, say A, B, C, &c., and suppose that the n-acron A gives rise to a certain number, say P, Q, R, S of $(n+1)$-acrons, the $(n+1)$-acron P is of course derivable from the n-acron A, but it may be derivable from other n-acrons, suppose from the n-acrons B and C. Then in considering the $(n+1)$-acrons derived from B, one of these will of course be found to be the $(n+1)$-acron P, and it is only the remaining $(n+1)$-acrons derived from B which are or may be $(n+1)$-acrons not already previously obtained as $(n+1)$-acrons derived from A. And if in this manner, as soon as each $(n+1)$-acron is obtained, we apply to it the process of subtraction so as to ascertain the entire series of n-acrons from which it is derivable, and, in forming the $(n+1)$-acrons derived from these, take account of the $(n+1)$-acrons already previously obtained and found to be derivable from these, we should obtain without any repetitions the entire series of the $(n+1)$-acrons.

For merely finding the number of the $(n+1)$-acrons, a more simple process might be adopted: say that an n-acron is p-wise generating when it gives rise to a number p of $(n+1)$-acrons, and that it is q-wise generable when it can be derived from a number q of $(n+1)$-acrons; and assume that a given n-acron is $(y_1+y_2+y_3+\&c.)$-wise generating, viz. that it gives rise to a number y_1 of $(n+1)$-acrons which are 1-wise generable, a number y_2 of $(n+1)$-acrons which are 2-wise generable, and so on; these forming the sum

$$\Sigma\,(y_1+\tfrac{1}{2}y_2+\tfrac{1}{3}y_3+\ldots)$$

where Σ refers to the entire series of the n-acrons, it is clear that every m-wise generable $(n+1)$-acron will in respect of each of the n-acrons from which it is derivable be reckoned as $\frac{1}{m}$, that is, it will be in the entire sum reckoned as 1, and the sum in question will consequently be the number of the $(n+1)$-acrons.

The figures of the polyacrons comprised in the annexed Tables show the application of the method to the genesis of the polyacrons as far as the octacrons, in which the numbers indicate the nature of the different summits, according to the number of edges through each summit, viz., 3 a tripleural summit, 4 a tetrapleural summit, and so on. It will be noticed that there is only a single case in which this notation is insufficient to distinguish the polyacron, viz., among the octacrons there are two forms each of them with the same symbol 33445566; the inspection of the figures shows at once that these are wholly distinct forms, for in the first of them, viz. that derived from 3344555, each of the tripleural summits stands upon a basic triangle 456, while in the other of them, that from 3444555, each of the tripleural summits stands upon a basic triangle 566. But the symbol is merely generic, and of course in the polyacrons of a greater number of summits it may very well happen that a considerable number of polyacrons are comprised in the same genus.

The following remarks on the derivation of the octacrons from the heptacrons will further illustrate the method:

1. The heptacron 3335556 has three kinds of faces, viz. 355([1]), 356, 555, the first process consequently gives rise to 3 octacrons. As the heptacron has more than two tripleural summits the second process is not applicable.

2. The heptacron 3344466 has three kinds of faces, viz.: 366, 346 and 446, and the first process gives therefore 3 octacrons. The heptacron has only two tripleural summits, and they are disposed in the proper manner; the second process gives therefore 1 octacron.

3. The heptacron 3344556 has five kinds of faces, viz. 345, 346, 356, 456 and 455, and the first process consequently gives 5 octacrons. The heptacron has two tripleural summits, but they are not disposed in such manner as to render the second process applicable.

[1] It is hardly necessary to remark that it must not be imagined that in general all the faces denoted by a symbol such as 355 (which determines only the nature of the summits on the face) are faces of the same kind, but this is so in the cases referred to in the text.

4. The heptacron 3444555 has four kinds of faces, viz. 355, 455, 445 and 444, and the first process gives therefore 4 octacrons. The heptacron has one tripleural summit, and the basic quadrangles 3545 which belong to it are of the same kind; the second process gives therefore 1 octacron.

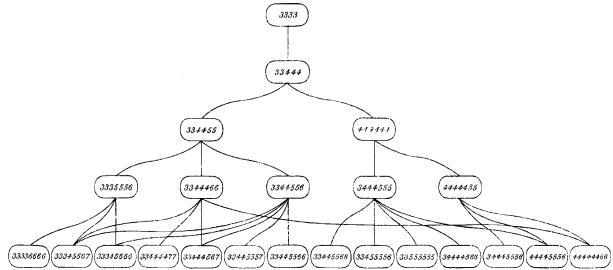
5. The heptacron 4444455 has only one kind of face, viz. 445, and the first process gives therefore 1 octacron. There are two kinds of basic quadrangles, viz. 4545 and 4445, and the second process gives therefore 2 octacrons.

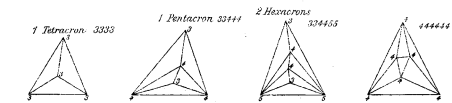
The number of octacrons would thus be 20, but by passing back from the octacrons to the heptacrons, it is found that there are in fact only 14 octacrons. Thus the octacron 33336666 has only one kind of tripleural summit 666 (the summit is here indicated by the symbol of the basic polygon) and the octacron is thus seen to be derivable from a single heptacron only, viz. the heptacron 3335556 from which it was in fact derived. But the octacron 33345567 has three kinds of tripleural summits, viz. 567, 557 and 467, and it is consequently derivable from three heptacrons, viz. the heptacrons 3335556, 3344466 and 3344555, and so on. The passage to the heptacrons from an octacron with one or more tripleural summits is of course always by the first process, but for the last two octacrons, which have no tripleural summits, the passage back to the heptacrons is by the second process: thus for the octacron 44445555 we have but one kind of tetrapleural summit 4555; but as opposite pairs of summits of the basic quadrangle are of different kinds, viz. 45 and 55, we obtain two heptacrons, viz. 3444555 and 4444455. The octacron 44444466 has but one kind of tetrapleural summit, viz. 4646, and the pairs of opposite summits of the basic quadrangle being of the same kind 46, we obtain from it only the heptacron 4444455.

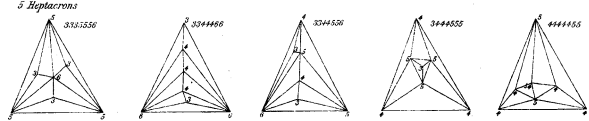
It may be remarked that for the five heptacrons respectively the values of the sum $y_1 + \frac{1}{2}y_2 + \frac{1}{3}y_3 + \ldots$ are

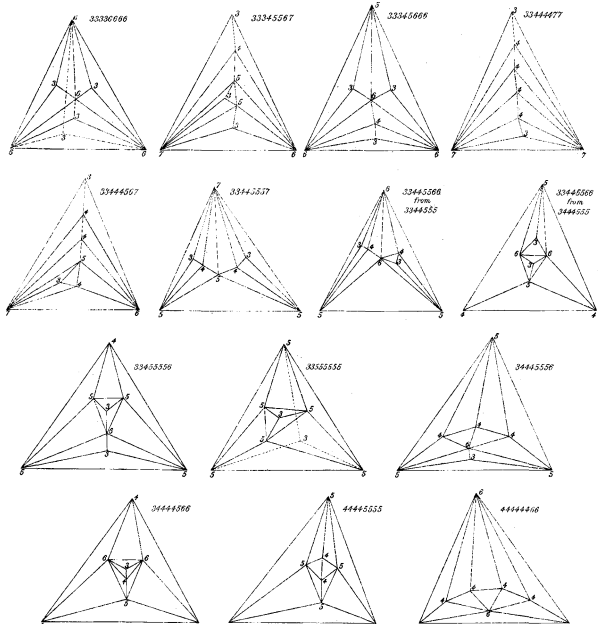
$$1+\tfrac{1}{3}+\tfrac{1}{2},\quad \tfrac{1}{3}+1+\tfrac{1}{2}+\tfrac{1}{2},\quad \tfrac{1}{3}+\tfrac{1}{2}+\tfrac{1}{2}+1+1,\quad 1+1+1+1+\tfrac{1}{2},\quad 1+\tfrac{1}{2}+\tfrac{1}{2},$$

giving for $\Sigma\,(y_1 + \frac{1}{2}y_2 + \frac{1}{3}y_3 + \ldots)$ the value 14, as it should do.


3333
33444
334455
444441
3335556
3344466
3344556
3444555
4444456
33336666
33345507
33345566
33444477
33444667
33445557
33445566
33445556
33453556
33535555
33444569
33445556
44445556
44444468


1 Tetracron. 3333
1 Pentacron 33444
2 Hexacrons
334455
444444


5 Heptacrons
3335556
3344466
3344556
3344555
4444455


33336666
33345567
33345666
33444477
33444567
33445577
33445566
from
3344555
33445566
from
3344555
33455556
33555555
34445556
34444566
44445555
44444466

309.

NOTE ON THE THEORY OF DETERMINANTS.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 180—185.]

THE following mode of arrangement of the developed expression of a determinant had presented itself to me as a convenient one for the calculation of a rather complicated determinant of the fifth order; but I have since found that it is in effect given, although in a much less compendious form, in a paper by J. N. Stockwell, "On the Resolution of a System of Symmetrical Equations with Indeterminate Coefficients," Gould's *Ast. Journal*, No. 139 (Cambridge, U. S., Sept. 10, 1860).

Suppose that the determinant

$$\begin{vmatrix} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{vmatrix}$$

is represented by $\{123\}$, and so for a determinant of any order $\{123 \ldots n\}$.

Let $|1|$, $|2|$, $|12|$, $|123|$, &c., denote as follows: viz.

$$|1| = 11, \quad |2| = 22, \text{ \&c.}$$
$$|12| = 12 . 21,$$
$$|123| = 12 . 23 . 31,$$
$$\text{\&c.,}$$

where it is to be noticed that, with the same two symbols, e.g. 1 and 2, there is but one distinct expression $|12|$ (in fact $|21| = 21 . 12 = |12|$); with the same three symbols, 1, 2, 3, there are two distinct expressions, $|123|$ $(= 12 . 23 . 31)$ and $|132|$ $(= 13 . 32 . 21)$; and generally with the same m symbols $1, 2, 3 \ldots m$, there are $1 . 2 . 3 \ldots (m-1)$ distinct

expressions $|\,123\ldots m\,|$, which are obtained by permuting in every possible manner all but one of the m symbols.

This being so, and writing for greater simplicity $|\,1\,|\,2\,|$ to denote the product $|\,1\,| \times |\,2\,|$, and so in general, the values of the determinants $\{12\}$, $\{123\}$, $\{1234\}$, $\{12345\}$, &c. are as follows: viz.

		No. of terms. +	No. of terms. −
$\{12\} = +$	$\lvert 1 \rvert 2 \rvert$	1	
$-$	$\lvert 1\ 2 \rvert$		1
		$1 +$	$1 = 2$
$\{123\} = +$	$\lvert 1 \rvert 2 \rvert 3 \rvert$ . . .	1	
$-$	$\lvert 1\ 2 \rvert 3 \rvert$ . . .		3
$+$	$\lvert 1\ 2\ 3 \rvert$ . . .	2	
		$3 +$	$3 = 6$
$\{1234\} = +$	$\lvert 1 \rvert 2 \rvert 3 \rvert 4 \rvert$ . .	1	
$-$	$\lvert 1\ 2 \rvert 3 \rvert 4 \rvert$ . .		6
$+$	$\lvert 1\ 2\ 3 \rvert 4 \rvert$ . .	8	
$+$	$\lvert 1\ 2 \rvert 3\ 4 \rvert$ . .	3	
$-$	$\lvert 1\ 2\ 3\ 4 \rvert$ . .		6
		$12 +$	$12 = 24$
$\{12345\} = +$	$\lvert 1 \rvert 2 \rvert 3 \rvert 4 \rvert 5 \rvert$.	1	
$-$	$\lvert 1\ 2 \rvert 3 \rvert 4 \rvert 5 \rvert$.		10
$+$	$\lvert 1\ 2\ 3 \rvert 4 \rvert 5 \rvert$.	20	
$+$	$\lvert 1\ 2 \rvert 3\ 4 \rvert 5 \rvert$.	15	
$-$	$\lvert 1\ 2\ 3\ 4 \rvert 5 \rvert$.		30
$-$	$\lvert 1\ 2\ 3 \rvert 4\ 5 \rvert$.		20
$+$	$\lvert 1\ 2\ 3\ 4\ 5 \rvert$.	24	
		$60 +$	$60 = 120$

where, as regards the signs, it is to be observed that there is a sign $-$ for each compartment $|\ \ |$ containing an even number of symbols; thus in the expression for $\{1234\}$, the terms $|\,1\,2\,|\,3\,4\,|$ have the sign $- - = +$, and the terms $|\,1\,2\,3\,4\,|$ the sign $-$. Or, what comes to the same thing; when n is even, the sign is $+$ or $-$ according as the number of compartments is even or odd; and contrariwise when n is odd. As regards the remaining part of the expression, this merely exhibits the partitions

of a set of n things; and the formulæ for the several determinants up to the determinant of a given order are all of them obtained by means of the form

$$\left|\begin{array}{c|c|c|c|c|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

which is carried up to the order 7, but which can be further extended without any difficulty whatever.

It is perhaps hardly necessary, but I give at full length the expressions of the determinant of the third order: this is

$$\{123\} = \left.\begin{array}{r}
|\,1\,|\,2\,|\,3\,| \\
-\,|\,1\;\;2\,|\,3\,| \\
-\,|\,2\;\;3\,|\,1\,| \\
-\,|\,3\;\;1\,|\,2\,|
\end{array}\right\} \\
\left.\begin{array}{r}
+\,|\,1\;\;2\;\;3\,| \\
+\,|\,1\;\;3\;\;2\,|
\end{array}\right\};$$

and by writing down in like manner the expression for the twenty-four terms of the determinant of the fourth order, the notation will become perfectly clear.

The formula hardly requires a demonstration. The terms of a determinant $\{123\ldots n\}$, for example the determinant $\{1234\}$, are obtained by permuting in every possible manner the symbols in either column, say the second column, of the arrangement

$$\begin{array}{cc}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4
\end{array}$$

and prefixing the sign (+ or −) of the arrangement; and the resulting arrangements, for instance

$$\begin{array}{ccc} +1\ 1, & -1\ 2, & -1\ 2, \\ 2\ 2 & 2\ 1 & 2\ 3 \\ 3\ 3 & 3\ 3 & 3\ 4 \\ 4\ 4 & 4\ 4 & 4\ 1 \end{array}$$

are interpreted either into $+11.22.33.44$, $-12.21.33.44$, $-12.23.34.41$, or in the notation of the formula, into

$$+|1|2|3|4|,\quad -|12|3|4|,\quad -|1\,2\,3\,4|;$$

and so in general.

Suppose that any partition of n contains α compartments each of a symbols, β compartments each of b symbols ... (a, b, ... being all of them different and greater than unity), and ρ compartments each of a single symbol, we have

$$n = \alpha a + \beta b + \ldots + \rho;$$

and writing, as usual, $\Pi a = 1.2.3 \ldots a$, &c., the number of ways in which the symbols 1, 2, . 3, ... n, can be so arranged in compartments is

$$\frac{\Pi n}{(\Pi a)^\alpha (\Pi b)^\beta \ldots \Pi\alpha\, \Pi\beta \ldots \Pi\rho};$$

but each such arrangement gives $\big(\Pi(a-1)\big)^\alpha . \big(\Pi(b-1)\big)^\beta$ terms of the determinant, and the corresponding number of terms therefore is

$$\frac{\Pi n}{a^\alpha b^\beta \ldots \Pi\alpha\, \Pi\beta \ldots \Pi\rho}.$$

The whole number of terms of the determinant is Πn, and we have thus the theorem

$$1 = \Sigma \frac{1}{a^\alpha b^\beta \ldots \Pi\alpha\, \Pi\beta \ldots \Pi\rho},$$

in which the summation corresponds to all the different partitions $n = \alpha a + \beta b, \ldots + \rho$, where a, b, ... are all of them different and greater than unity; a theorem given in Cauchy's *Mémoire sur les Arrangements* &c., 1844. But it is to be noticed also that, the number of the positive and negative terms being equal, we have besides

$$0 = \Sigma \frac{(-)^{\alpha(a-1)+\beta(b-1)+\cdots}}{a^\alpha b^\beta \ldots \Pi\alpha\, \Pi\beta \ldots \Pi\rho};$$

or, what is the same thing,

$$0 = \Sigma \frac{(-)^{n-\alpha-\beta\cdots-\rho}}{a^\alpha b^\beta \ldots \Pi\alpha\, \Pi\beta \,.\,.\, \Pi\rho};$$

and thence also

$$\tfrac{1}{2} = \Sigma \frac{1}{a^\alpha b^\beta \ldots \Pi\alpha\, \Pi\beta \ldots \Pi\rho};$$

where, as before, $n = \alpha a + \beta b \ldots + \rho$ ($a, b, \ldots$ being all different and greater than unity); but the summation is restricted either to the partitions for which $n - \alpha - \beta \ldots - \rho$ is even, or else to those for which $n - \alpha - \beta \ldots - \rho$ is odd.

The formula affords a proof of the fundamental property of skew symmetrical determinants. In such a determinant we have not only $12 = -21$, &c., but also $11 = 0$, &c. Suppose that n, the order of the determinant, is odd; then in each line of the expression

$$\{123 \ldots n\} = |\,1\,|\,2\,| \ldots |\,n\,|$$
$$\pm \text{ \&c.}$$

of the determinant, there is at least one compartment $|\,1\,|$ or $|\,123\,|$ &c. containing an odd number of symbols: let $|\,123\,|$ be such a compartment, then the determinant contains the terms $|\,123\,|\,P$ and $|\,132\,|\,P$ (where P represents the remaining compartments), that is, $12.23.31.P$ and $13.32.21.P$. But in virtue of the relations $12 = -21$, &c., we have

$$12.23.31 = -13.32.21;$$

and so in all similar cases; that is, the terms destroy each other, or the skew symmetrical determinant of an odd order is equal to zero.

The like considerations show that a skew symmetrical determinant of an even order is a perfect square. In fact, considering for greater simplicity the case $n = 4$, any line in the foregoing expression of $\{1234\}$ for which a compartment contains an odd number of symbols, gives rise to terms which destroy each other, and may be omitted. The expression thus reduces itself to

$$\{1234\} = +\,|\,12\,|\,34\,| \quad 3 \text{ terms}$$
$$-\,|\,12 \quad 34\,| \quad 6 \text{ terms},$$

which is in fact the square of

$$12.34 + 13.42 + 14.23;$$

for the square of a term, say 12.34, is $\overline{12}^2.\overline{34}^2$ or $12.21.34.43$, that is, $|\,12\,|\,34\,|$, and the double of the product of two terms, say 12.34 and 13.42, is $2.12.34.13.42$, or $-12.24.43.31 - 13.34.42.21$, that is $-\,|\,1243\,| - |\,1342\,|$, and so for the other similar terms, and we have

$$\{1234\} = (12.34 + 13.42 + 14.23)^2:$$

and so in general, n being any even number, the skew symmetrical determinant $\{123 \ldots n\}$ is equal to the square of the Pfaffian $12 \ldots n$, where the law of these Pfaffian functions is

$$1234 \quad = 12.34 \quad + 13.42 \quad + 14.23$$
$$123456 = 12.3456 + 13.4562 + 14.5623 + 15.6234 + 16.2345,$$

where, in the second equation, 3456, &c. are Pfaffians, viz. $3456 = 34.56 + 35.64 + 36.45$; and so on.

2, *Stone Buildings, W.C., December* 28, 1860.

310.

NOTE ON MR JERRARD'S RESEARCHES ON THE EQUATION OF THE FIFTH ORDER.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 210—214.]

FUNCTIONS of the same set of quantities which are, by any substitution whatever, simultaneously altered or simultaneously unaltered, may be called *homotypical*. Thus all symmetric functions of the same set of quantities are homotypical: $(x+y-z-w)^2$ and $xy+zw$ are homotypical, &c.

It is one of the most beautiful of Lagrange's discoveries in the theory of equations, that, given the value of any function of the roots, the value of any homotypical function may be rationally determined[1]; in other words, that any homotypical function whatever is a rational function of the coefficients of the equation and of the given function of the roots.

The researches of Mr Jerrard are contained in his work, *An Essay on the Resolution of Equations*, London, Taylor and Francis, 1859. The solution of an equation of the fifth order is made to depend on an equation of the sixth order in W; and he conceives that he has shown that one of the roots of this equation is a rational function of another root: "The equation for W will therefore belong to a class of equations of the sixth degree, the resolution of which can, as Abel has shown, be effected by means of equations of the second and third degrees; whence I infer the possibility of solving any proposed equation of the fifth degree by a finite combination of radicals and rational functions."

[1] The *à priori* demonstration shows the cases of failure. Suppose that the roots of a biquadratic equation are 1, 3, 5, 9; then, given $a+b=8$, we know that either $a=3$, $b=5$, or else $a=5$, $b=3$, and in either case $ab=15$; hence in the present case (which represents the general case), $a+b$ being known, the homotypical function ab is rationally determined. But if the roots are 1, 3, 5, 7 (where $1+7=3+5$), then, given $a+b=8$, this is satisfied by $(a, b=3, 5)$ or by $(a, b=1, 7)$, and the conclusion is $ab=15$ or 7; so that here ab is determined, not as before, rationally, but by a quadratic equation.

The above property of rational expressibility, if true for W, will be true for any function homotypical with W; and conversely. I proceed to inquire into the form of the function W.

The function W is derived from the function P, which denotes any one of the quantities p_1, p_2, p_3. And if x_1, x_2, x_3, x_4, x_5 are the roots of the given equation of the fifth order, and if α, β, γ, δ, ϵ represent in an undetermined or arbitrary order of succession the five indices 1, 2, 3, 4, 5, and if ι denote an imaginary fifth root of unity (I conform myself to Mr Jerrard's notation), then p_1, p_2, p_3, and the other auxiliary quantities t, u, are obtained from the system of equations:

$$\begin{aligned}
x_\alpha^3 + p_1 x_\alpha^2 + p_2 x_\alpha + p_3 &= t + u,\\
x_\beta^3 + p_1 x_\beta^2 + p_2 x_\beta + p_3 &= \iota t + \iota^4 u,\\
x_\gamma^3 + p_1 x_\gamma^2 + p_2 x_\gamma + p_3 &= \iota^2 t + \iota^3 u,\\
x_\delta^3 + p_1 x_\delta^2 + p_2 x_\delta + p_3 &= \iota^3 t + \iota^2 u,\\
x_\epsilon^3 + p_1 x_\epsilon^2 + p_2 x_\epsilon + p_3 &= \iota^4 t + \iota u.
\end{aligned}$$

If from these equations we seek for the values of p_1, p_2, p_3, t, u, we have

$$1 : p_1 : p_2 : p_3 : -t : -u = \Pi_1 : \Pi_2 : \Pi_3 : \Pi_4 : \Pi_5 : \Pi_6,$$

where Π_1, Π_2, .. denote the determinants formed out of the matrix

$$\begin{vmatrix}
x_\alpha^3, & x_\alpha^2, & x_\alpha, & 1, & 1, & 1\\
x_\beta^3, & x_\beta^2, & x_\beta, & 1, & \iota, & \iota^4\\
x_\gamma^3, & x_\gamma^2, & x_\gamma, & 1, & \iota^2, & \iota^3\\
x_\delta^3, & x_\delta^2, & x_\delta, & 1, & \iota^3, & \iota^2\\
x_\epsilon^3, & x_\epsilon^2, & x_\epsilon, & 1, & \iota^4, & \iota
\end{vmatrix},$$

i.e., denoting the columns of this matrix by 1, 2, 3, 4, 5, 6, we have $\Pi_1 = 23456$, $\Pi_2 = -34561$, $\Pi_3 = 45612$, &c. In particular, the value of Π_1 is

$$= \begin{vmatrix}
x_\alpha^2, & x_\alpha, & 1, & 1, & 1\\
x_\beta^2, & x_\beta, & 1, & \iota, & \iota^4\\
x_\gamma^2, & x_\gamma, & 1, & \iota^2, & \iota^3\\
x_\delta^2, & x_\delta, & 1, & \iota^3, & \iota^2\\
x_\epsilon^2, & x_\epsilon, & 1, & \iota^4, & \iota
\end{vmatrix},$$

and developing, and putting for shortness $\{\alpha\beta\} = x_\alpha x_\beta (x_\alpha - x_\beta)$, &c., we have

$$\begin{aligned}
\Pi_1 = &(\{\alpha\beta\} + \{\beta\gamma\} + \{\gamma\delta\} + \{\delta\epsilon\} + \{\epsilon\alpha\})(-2\iota + \iota^2 - \iota^3 + 2\iota^4)\\
&+ (\{\alpha\gamma\} + \{\gamma\epsilon\} + \{\epsilon\beta\} + \{\beta\delta\} + \{\delta\alpha\})(+\iota + 2\iota^2 - 2\iota^3 - 2\iota^4);
\end{aligned}$$

and this is also the form of the other determinants, the only difference being as to the meaning of the symbol $\{\alpha\beta\}$, which, however, in each case denotes a function such that $\{\alpha\beta\} = -\{\beta\alpha\}$. Writing for greater shortness,

$$\{\alpha\beta\gamma\delta\epsilon\} = \{\alpha\beta\} + \{\beta\gamma\} + \{\gamma\delta\} + \{\delta\epsilon\} + \{\epsilon\alpha\},$$

and in like manner

$$\{\alpha\gamma\epsilon\beta\delta\} = \{\alpha\gamma\} + \{\gamma\epsilon\} + \{\epsilon\beta\} + \{\beta\delta\} + \{\delta\alpha\},$$

Π_1 is an unsymmetric linear function (without constant term) of $\{\alpha\beta\gamma\delta\epsilon\}$, $\{\alpha\gamma\epsilon\beta\delta\}$; or, what is all that is material, it is an unsymmetric function, containing only odd powers, of $\{\alpha\beta\gamma\delta\epsilon\}$, $\{\alpha\gamma\epsilon\beta\delta\}$.

If for

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

we substitute any one of the five arrangements

$$\begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \epsilon, \\ \beta & \gamma & \delta & \epsilon & \alpha, \\ \gamma & \delta & \epsilon & \alpha & \beta, \\ \delta & \epsilon & \alpha & \beta & \gamma, \\ \epsilon & \alpha & \beta & \gamma & \delta, \end{array}$$

then $\{\alpha\beta\gamma\delta\epsilon\}$ and $\{\alpha\gamma\epsilon\beta\delta\}$ will in each case remain unaltered.

But if we substitute any one of the five arrangements

$$\begin{array}{ccccc} \alpha & \epsilon & \delta & \gamma & \beta, \\ \epsilon & \delta & \gamma & \beta & \alpha, \\ \delta & \gamma & \beta & \alpha & \epsilon, \\ \gamma & \beta & \alpha & \epsilon & \delta, \\ \beta & \alpha & \epsilon & \delta & \gamma, \end{array}$$

then in each case $\{\alpha\beta\gamma\delta\epsilon\}$ and $\{\alpha\gamma\epsilon\beta\delta\}$ will be changed into $-\{\alpha\beta\gamma\delta\epsilon\}$ and $-\{\alpha\gamma\epsilon\beta\delta\}$ respectively. Hence Π_1 remains unaltered by any one of the first five substitutions; and it is changed into $-\Pi_1$ by any one of the second five substitutions. And the like being the case as regards Π_2, &c., it follows that the quotient $\Pi_1 \div \Pi_2$, or say P, remains unaltered by any one of the ten substitutions. Now the 120 permutations of α, β, γ, δ, ϵ can be obtained as follows, viz. by forming the 12 different pentagons which can be formed with α, β, γ, δ, ϵ (treated as five points), and reading each of them off in either direction from any angle. To each of the 12 pentagons there corresponds a distinct value of P, but such value is not altered by the different modes of reading off the pentagon; P is consequently a 12-valued function.

But there is a more simple form of the analytical expression of such a 12-valued function; in fact, if $[\alpha\beta\gamma\delta\epsilon]$ be any function which is not altered by any one of the above ten substitutions—if, for instance, $[\alpha\beta]$ is a symmetrical function of x_α, x_β, and

$$[\alpha\beta\gamma\delta\epsilon] = [\alpha\beta] + [\beta\gamma] + [\gamma\delta] + [\delta\epsilon] + [\epsilon\alpha],$$

and therefore

$$[\alpha\gamma\epsilon\beta\delta] = [\alpha\gamma] + [\gamma\epsilon] + [\epsilon\beta] + [\beta\delta] + [\delta\alpha],$$

then any unsymmetrical function of $[\alpha\beta\gamma\delta\epsilon]$ and $[\alpha\gamma\epsilon\beta\delta]$ will be a 12-valued function homotypical with P.

Mr Jerrard's function W is the sum of two values of his function P; the substitution by which the second is derived from the first can only be that which interchanges the two functions $[\alpha\beta\gamma\delta\epsilon]$ and $[\alpha\gamma\epsilon\beta\delta]$; and hence any symmetrical function of $[\alpha\beta\gamma\delta\epsilon]$ and $[\alpha\gamma\epsilon\beta\delta]$ is a function homotypical with Mr Jerrard's W; such symmetric function is in fact a 6-valued function only. Indeed it is easy to see that the twelve pentagons correspond together in pairs, either pentagon of a pair being derived from the other one by *stellation*, and the six values of the function in question corresponding to the six pairs of pentagons respectively.

Writing with Mr Cockle and Mr Harley,

$$\tau = x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\delta + x_\delta x_\epsilon + x_\epsilon x_\alpha,$$
$$\tau' = x_\alpha x_\gamma + x_\gamma x_\epsilon + x_\epsilon x_\beta + x_\beta x_\delta + x_\delta x_\alpha,$$

then ($\tau + \tau'$ is a symmetrical function of all the roots, and it must be excluded; but) $(\tau - \tau')^2$ or $\tau\tau'$ are each of them 6-valued functions of the form in question, and either of these functions is linearly connected with the Resolvent Product. In Lagrange's general theory of the solution of equations, if

$$fi = x_1 + \iota x_2 + \iota^2 x_3 + \iota^3 x_4 + \iota^4 x_5,$$

then the coefficients of the equation the roots whereof are $(fi)^5$, $(fi^2)^5$, $(fi^3)^5$, $(fi^4)^5$, and in particular the last coefficient $(fi\,fi^2\,fi^3\,fi^4)^5$, are determined by an equation of the sixth degree; and this last coefficient is a perfect fifth power, and its fifth root, or $fi\,fi^2\,fi^3\,fi^4$, is the function just referred to as the Resolvent Product.

The conclusion from the foregoing remarks is that *if the equation for W has the above property of the rational expressibility of its roots*, the equation of the sixth order resulting from Lagrange's general theory has the same property.

I take the opportunity of adding a simple remark on cubic equations. The principle which furnishes what in a foregoing foot-note is called the *à priori* demonstration of Lagrange's theorem is that an equation need never contain extraneous roots; a quantity which has only one value will, if the investigation is properly conducted, be determined in the first instance by a linear equation; one which has two values by a quadratic equation, and so on; there is always enough, and not more than enough, to determine what is required.

Take Cardan's solution of the cubic equation $x^3+qx-r=0$, we have $x=a+b$, and thence $3ab=-q$, $a^3+b^3=r$; and to obtain the solution we write

$$a^3b^3=-\frac{q}{27}, \quad a^3+b^3=r.$$

But *these* two equations are not enough to precisely determine x, they lead to the 9-valued function

$$\sqrt[3]{\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}:$$

in order to precisely determine x, it is (as everybody knows) necessary to use the original equation $ab=-\frac{q}{3}$. But seek for the solution as follows; viz. write $x=ab(a+b)$, which gives

$$3a^3b^3=-q, \quad a^3b^3(a^3+b^3)=r,$$

or what is the same thing,

$$a^3b^3=-\frac{q}{3}, \quad a^3+b^3=-\frac{3r}{q};$$

these equations give $x=ab(a+b)$, where

$$a=\sqrt[3]{-\frac{3r}{2q}+\sqrt{\frac{9r^2}{4q^2}+\frac{q}{3}}}, \quad b=\sqrt[3]{-\frac{3r}{2q}-\sqrt{\frac{9r^2}{4q^2}+\frac{q}{3}}},$$

which is a 3-valued function only, ab in this case being not given.

2, *Stone Buildings, W.C., January* 28, 1861.

311.

ON A THEOREM OF ABEL'S RELATING TO EQUATIONS OF THE FIFTH ORDER.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 257—263.]

THE following is given (Abel, *Œuvres*, vol. II. p. 253 [Ed. 2, vol. II. p. 266]) as an extract of a letter to M. Crelle:

"Si une équation du cinquième degré, dont les coefficients sont des *nombres rationnels*, est résoluble algébriquement, on peut donner aux racines la forme suivante,

$$x = c + Aa^{\frac{1}{5}}a_1^{\frac{2}{5}}a_2^{\frac{4}{5}}a_3^{\frac{3}{5}} + A_1a_1^{\frac{1}{5}}a_2^{\frac{3}{5}}a_3^{\frac{4}{5}}a^{\frac{2}{5}} + A_2a_2^{\frac{1}{5}}a_3^{\frac{2}{5}}a^{\frac{4}{5}}a_1^{\frac{3}{5}} + A_3a_3^{\frac{1}{5}}a^{\frac{2}{5}}a_1^{\frac{4}{5}}a_2^{\frac{3}{5}},$$

où

$$a = m + n\sqrt{1+e^2} + \sqrt{h\,(1+e^2+\sqrt{1+e^2})},$$

$$a_1 = m - n\sqrt{1+e^2} + \sqrt{h\,(1+e^2-\sqrt{1+e^2})},$$

$$a_2 = m + n\sqrt{1+e^2} - \sqrt{h\,(1+e^2+\sqrt{1+e^2})},$$

$$a_3 = m - n\sqrt{1+e^2} - \sqrt{h\,(1+e^2-\sqrt{1+e^2})},$$

$$A = K + K'a + K''a_2 + K''aa_2, \quad A_1 = K + K'a_1 + K''a_3 + K'''a_1a_3,$$

$$A_2 = K + K'a_2 + K''a + K''aa_2, \quad A_3 = K + K'a_3 + K''a_1 + K'''a_1a_3.$$

Les quantités c, h, e, m, n, K, K', K'', K''' sont des *nombres rationnels*.

"Mais de cette manière l'équation $x^5 + ax + b = 0$ n'est pas résoluble tant que a et b sont des quantités quelconques. J'ai trouvé de pareils théorèmes pour les équations du 7ème, 11ème, 13ème, &c. degré. Fribourg, le 14 Mars, 1826."

The theorem is referred to by M. Kronecker (*Berl. Monatsb.* June 20, 1853), but nowhere else that I am aware of.

It is to be noticed that in the expressions for a, a_1, a_2, a_3, the radicals are such that

$$\sqrt{1+e^2}\sqrt{h(1+e^2+\sqrt{1+e^2})}\sqrt{h(1+e^2-\sqrt{1+e^2})}=he(1+e^2),$$

a rational number.

The theorem is given as belonging to numerical equations; but considering it as belonging to literal equations, it will be convenient to change the notation; and in this point of view, and to avoid suffixes and accents, I write

$$x=\theta+A\alpha^{\frac{1}{5}}\beta^{\frac{2}{5}}\gamma^{\frac{4}{5}}\delta^{\frac{3}{5}}+B\beta^{\frac{1}{5}}\gamma^{\frac{2}{5}}\delta^{\frac{4}{5}}\alpha^{\frac{3}{5}}+C\gamma^{\frac{1}{5}}\delta^{\frac{2}{5}}\alpha^{\frac{4}{5}}\beta^{\frac{3}{5}}+D\delta^{\frac{1}{5}}\alpha^{\frac{2}{5}}\beta^{\frac{4}{5}}\gamma^{\frac{3}{5}},$$

where

$$\alpha=m+n\sqrt{\Theta}+\sqrt{p+q\sqrt{\Theta}},$$
$$\beta=m-n\sqrt{\Theta}+\sqrt{p-q\sqrt{\Theta}},$$
$$\gamma=m+n\sqrt{\Theta}-\sqrt{p+q\sqrt{\Theta}},$$
$$\delta=m-n\sqrt{\Theta}-\sqrt{p-q\sqrt{\Theta}};$$

the radicals being connected by

$$\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}}=s,$$

and where

$$A=K+L\alpha+M\gamma+N\alpha\gamma,\quad B=K+L\beta+M\delta+N\beta\delta,$$
$$C=K+L\gamma+M\alpha+N\alpha\gamma,\quad D=K+L\delta+M\beta+N\beta\delta,$$

in which equations θ, m, n, p, q, Θ, s, K, L, M, N are rational functions of the elements of the given quintic equation.

The basis of the theorem is, that the expression for x has only the five values which it acquires by giving to the quintic radicals contained in it their five several values, and does not acquire any new value by substituting for the quadratic radicals their several values. For, this being so, x will be the root of a rational quintic; and conversely.

Now attending to the equation

$$\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}}=s,$$

the different admissible values of the radicals are

$$\begin{array}{rrr}
\sqrt{\Theta}, & \sqrt{p+q\sqrt{\Theta}}, & \sqrt{p-q\sqrt{\Theta}}, \\
-\sqrt{\Theta}, & \sqrt{p-q\sqrt{\Theta}}, & -\sqrt{p+q\sqrt{\Theta}}, \\
\sqrt{\Theta}, & -\sqrt{p+q\sqrt{\Theta}}, & -\sqrt{p-q\sqrt{\Theta}}, \\
-\sqrt{\Theta}, & -\sqrt{p-q\sqrt{\Theta}}, & \sqrt{p+q\sqrt{\Theta}},
\end{array}$$

corresponding to the systems

$$\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \beta, & \gamma, & \delta, & \alpha, \\ \gamma, & \delta, & \alpha, & \beta, \\ \alpha, & \beta, & \gamma, & \delta, \end{array}$$

of the roots $\alpha, \beta, \gamma, \delta$; i.e. the effect of the alteration of the values of the quadratic radicals is merely to cyclically permute the roots $\alpha, \beta, \gamma, \delta$; and observing that any such cyclical permutation gives rise to a like cyclical permutation of A, B, C, D, the alteration of the quadratic radicals produces no alteration in the expression for x.

The quantities $\alpha, \beta, \gamma, \delta$ are the roots of a rational quartic. If, solving the quartic by Euler's method, we write

$$\begin{aligned} \alpha &= m + \sqrt{F} + \sqrt{G} + \sqrt{H}, \qquad \sqrt{FGH} = \nu, \text{ a rational function,} \\ \beta &= m - \sqrt{F} + \sqrt{G} - \sqrt{H}, \\ \gamma &= m + \sqrt{F} - \sqrt{G} - \sqrt{H}, \\ \delta &= m - \sqrt{F} - \sqrt{G} + \sqrt{H}, \end{aligned}$$

then the expressions for F, G, H in terms of the roots are

$$(\alpha + \gamma - \beta - \delta)^2, \quad (\alpha + \beta - \gamma - \delta)^2, \quad (\alpha + \delta - \beta - \gamma)^2,$$

which are the roots of a cubic equation

$$u^3 - \lambda u^2 + \mu u - \nu^2 = 0,$$

where λ, μ, ν are given rational functions of the coefficients of the quartic. We have

$$\sqrt{G} + \sqrt{H} = \sqrt{(\sqrt{G} + \sqrt{H})^2} = \sqrt{G + H + 2\sqrt{GH}} = \sqrt{\lambda - F + \frac{2\nu}{F}\sqrt{F}};$$

so that, taking $\Theta = F$, the last-mentioned expressions for $\alpha, \beta, \gamma, \delta$ will be of the assumed form

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \quad \&c.$$

The equation

$$\sqrt{\Theta}\sqrt{p + q\sqrt{\Theta}}\sqrt{p - q\sqrt{\Theta}} = s$$

thus becomes

$$\sqrt{F}\sqrt{(G - H)^2} = s, \text{ or } F(G - H)^2 = s^2;$$

that is,

$$-F^2 + F(F^2 + G^2 + H^2) - 2FGH = s^2;$$

or, what is the same thing, and putting Θ for F,

$$-\lambda\Theta^2 + (\lambda^2 - \mu)\Theta - 3\nu = s^2.$$

Hence in order that the roots of the quartic may be of the assumed form,

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \text{ \&c.},$$

where m, p, q, Θ are rational, and where also

$$\sqrt{\Theta}\sqrt{p + q\sqrt{\Theta}}\sqrt{p - q\sqrt{\Theta}} = s, \text{ a rational function,}$$

the necessary and sufficient conditions are that the quartic should be such that the reducing cubic

$$u^3 - \lambda u^2 + \mu u - \nu^2 = 0$$

(whose roots are $(\alpha+\beta-\gamma-\delta)^2$, $(\alpha+\gamma-\beta-\delta)^2$, $(\alpha+\delta-\beta-\gamma)^2$) may have *one rational root* Θ, and moreover that the function

$$-\lambda\Theta^2 + (\lambda^2 - \mu)\,\Theta - 3\nu$$

shall be the *square of a rational function* s. This being so, the roots of the quartic will be of the assumed form

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \text{ \&c.};$$

and from what precedes, it is clear that any function of the roots of the quartic which remains unaltered by the cyclical substitution $\alpha\beta\gamma\delta$, or what is the same thing, any function of the form

$$\phi(\alpha, \beta, \gamma, \delta) + \phi(\beta, \gamma, \delta, \alpha) + \phi(\gamma, \delta, \alpha, \beta) + \phi(\delta, \alpha, \beta, \gamma)$$

will be a rational function of m, Θ, p, q, s, and consequently of the coefficients of the quartic. The above are the conditions in order that a quartic equation may be of the Abelian form.

It may be as well to remark that, assuming only the system of equations

$$\begin{aligned}
\alpha &= m + \sqrt{\Theta} + \sqrt{\Upsilon},\\
\beta &= m - \sqrt{\Theta} + \sqrt{\Upsilon'},\\
\gamma &= m + \sqrt{\Theta} - \sqrt{\Upsilon},\\
\delta &= m - \sqrt{\Theta} - \sqrt{\Upsilon'},
\end{aligned}$$

then any rational function of α, β, γ, δ which remains unaltered by the cyclical substitution $\alpha\beta\gamma\delta$ will be a rational function of Θ, $\Upsilon + \Upsilon'$, $\Upsilon\Upsilon'$, $\sqrt{\Upsilon\Upsilon'}(\Upsilon - \Upsilon')$, $\sqrt{\Theta}(\Upsilon - \Upsilon')$, $\sqrt{\Theta}\sqrt{\Upsilon\Upsilon'}$. In fact, suppose such a function contains the term

$$(\sqrt{\Theta})^\alpha (\sqrt{\Upsilon})^\beta (\sqrt{\Upsilon'})^\gamma;$$

then it will contain the four terms

$$\begin{aligned}
&(\sqrt{\Theta})^\alpha (\sqrt{\Upsilon})^\beta (\sqrt{\Upsilon'})^\gamma,\\
&(-\sqrt{\Theta})^\alpha (\sqrt{\Upsilon'})^\beta (-\sqrt{\Upsilon})^\gamma,\\
&(\sqrt{\Theta})^\alpha (-\sqrt{\Upsilon})^\beta (-\sqrt{\Upsilon'})^\gamma,\\
&(-\sqrt{\Theta})^\alpha (-\sqrt{\Upsilon'})^\beta (\sqrt{\Upsilon})^\gamma,
\end{aligned}$$

which together are

$$(\sqrt{\Theta})^{\alpha}\{(1+(-)^{\beta+\gamma}1)(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}+(-)^{\alpha}[(-)^{\beta}1+(-)^{\gamma}1](\sqrt{\Upsilon})^{\gamma}(\sqrt{\Upsilon'})^{\beta}\},$$

an expression which vanishes unless $(-)^{\beta}$, $(-)^{\gamma}$ are both positive or both negative. The forms to be considered are therefore

$(-)^{\alpha}$,	$(-)^{\beta}$,	$(-)^{\gamma}$
+	+	+
−	+	+
+	−	−
−	−	−

The first form is

$$(\sqrt{\Theta})^{\alpha}\{(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}+(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}\},$$

which, α, β, γ being each of them even, is a rational function of Θ, $\Upsilon+\Upsilon'$, $\Upsilon\Upsilon'$.

The second form is

$$(\sqrt{\Theta})^{\alpha}\{(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}-(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}\},$$

which, α being odd and β and γ each of them even, is the product of such a function into $\sqrt{\Theta}(\Upsilon-\Upsilon')$.

The third form is

$$(\sqrt{\Theta})^{\alpha}\{(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}-(\sqrt{\Upsilon})^{\gamma}(\sqrt{\Upsilon'})^{\beta}\},$$

which, α being even and β and γ each of them odd, is the product of such a function into $\sqrt{\Upsilon\Upsilon'}(\Upsilon-\Upsilon')$.

And the fourth form is

$$(\sqrt{\Theta})^{\alpha}\{(\sqrt{\Upsilon})^{\beta}(\sqrt{\Upsilon'})^{\gamma}+(\sqrt{\Upsilon})^{\gamma}(\sqrt{\Upsilon'})^{\beta}\},$$

which, α, β, γ being each of them odd, is the product of such a function into $\sqrt{\Theta}(\Upsilon-\Upsilon')$.

Hence if $\Upsilon=p+q\sqrt{\Theta}$, $\Upsilon'=p-q\sqrt{\Theta}$, and $\sqrt{\Theta}\sqrt{p+q}\sqrt{\Theta}\sqrt{p-q}\sqrt{\Theta}=s$, then

$$\Theta,\quad \Upsilon+\Upsilon'(=2p),\quad \Upsilon\Upsilon'(=p^2-q^2\Theta),\quad \sqrt{\Upsilon\Upsilon'}(\Upsilon-\Upsilon')\left(=\frac{2q}{s}\right),$$
$$\sqrt{\Theta}(\Upsilon-\Upsilon')(=2q\Theta),\text{ and }\sqrt{\Theta}\sqrt{\Upsilon\Upsilon'}(=s)$$

are respectively rational functions. This is the *à posteriori* verification, that with the system of equations

$$\alpha=m+\sqrt{\Theta}+\sqrt{p+q\sqrt{\Theta}},\ \&c.,\quad \sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}}=s,$$

any function

$$\phi(\alpha,\beta,\gamma,\delta)+\phi(\beta,\gamma,\delta,\alpha)+\phi(\gamma,\delta,\alpha,\beta)+\phi(\delta,\alpha,\beta,\gamma)$$

is a rational function.

The coefficients of the quintic equation for x must of course be of the form just mentioned; that is, they must be functions of α, β, γ, δ, which remain unaltered by the cyclic substitution $\alpha\beta\gamma\delta$. To form the quintic equation, I write

$$\theta - x = a,$$

$$A\alpha^{\frac{1}{5}}\beta^{\frac{2}{5}}\gamma^{\frac{4}{5}}\delta^{\frac{3}{5}} = b, \quad D\delta^{\frac{1}{5}}\alpha^{\frac{2}{5}}\beta^{\frac{4}{5}}\gamma^{\frac{3}{5}} = c, \quad B\beta^{\frac{1}{5}}\gamma^{\frac{2}{5}}\delta^{\frac{4}{5}}\alpha^{\frac{3}{5}} = d, \quad C\gamma^{\frac{1}{5}}\delta^{\frac{2}{5}}\alpha^{\frac{4}{5}}\beta^{\frac{3}{5}} = e;$$

then we have

$$0 = a + b + c + d + e,$$

and the quintic equation is

$$f1\, f\omega\, f\omega^2\, f\omega^3\, f\omega^4 = 0,$$

where ω is an imaginary fifth root of unity, and

$$f\omega = a + b\omega + c\omega^2 + d\omega^3 + e\omega^4.$$

We have

$$f\omega\, f\omega^4 = \Sigma a^2 + (\omega + \omega^4)\,\Sigma' ab + (\omega^2 + \omega^3)\,\Sigma' ac,$$

$$f\omega^2 f\omega^3 = \Sigma a^2 + (\omega^2 + \omega^3)\,\Sigma' ab + (\omega^2 + \omega^3)\,\Sigma' ac,$$

where Σ' is Mr Harley's cyclical symbol, viz.

$$\Sigma' ab = ab + bc + cd + de + ea;$$

and so in other cases, the order of the cycle being always $abcde$. This gives

$$f\omega\, f\omega^2 f\omega^3 f\omega^4 = \Sigma a^4 + \Sigma a^2 b^2 - \Sigma a^3 b + 2\Sigma a^2 bc - \Sigma abcd - 5\Sigma' a^3 (be + cd);$$

and multiplying by $f1, = \Sigma a$, and equating to zero, the result is found to be

$$\Sigma a^5 - 5abcde - 5\Sigma' a^3 (be + cd) + 5\Sigma' a\,(b^2e^2 + c^2d^2) = 0;$$

or arranging in powers of a, this is

$$\left.\begin{array}{ll}
 & a^5 \\
+ & a^3 . \quad -5\,(be \quad + cd) \\
+ & a^2 . \quad\quad 5\,(b\,c^2 \quad + c\,e^2 \quad + e\,d^2 + d\,b^2) \\
+ & a\,. \left\{\begin{array}{l} 5\,(b^3c \quad + c^3e \quad + e^3d + d^3\,b) \\ + 5\,(b^2e^2 \quad + c^2d^2 \quad - becd) \end{array}\right. \\
+ & \left\{\begin{array}{l} b^5 + c^5 + e^5 + d^5 \\ - 5\,(b^3de + c^3bd + e^3cb + d^3ec) \\ + 5\,(bd^2e^2 + cb^2d^2 + ec^2b^2 + de^2c^2) \end{array}\right.
\end{array}\right\} = 0,$$

the several coefficients being, it will be observed, cyclical functions to the cycle b, c, e, d.

Putting for a its value $-(x-\theta)$, and for b, c, d, e their values, the quintic equation in x is

$$\left.\begin{array}{ll}
 & (x-\theta)^5 \\
+ & (x-\theta)^3 . \quad -5\,(AC \qquad + BD)\,\alpha\beta\gamma\delta \\
+ & (x-\theta)^2 . \quad -5\,(A^2B\gamma\delta \quad + B^2C\delta\alpha \quad + C^2D\alpha\beta \quad + D^2A\beta\gamma\ )\,\alpha\beta\gamma\delta \\
+ & (x-\theta) \quad \begin{cases} -5\,(A^3D\beta\gamma^2\delta + B^3A\gamma\delta^2\alpha + C^3B\delta\alpha^2\beta + D^3C\alpha\beta^2\gamma)\,\alpha\beta\gamma\delta \\ +5\,(A^2C^2 \qquad + B^2D^2 \qquad - ABCD)\,\alpha^2\beta^2\gamma^2\delta^2 \end{cases} \\
+ & \left\{\begin{array}{l} \quad (A^5\beta\gamma^3\delta^2 \ + B^5\gamma\delta^3\alpha^2 \ + C^5\delta\alpha^3\beta^2 \ + D^5\alpha\beta^3\gamma^2\ )\,\alpha\beta\gamma\delta \\ -5\,(A^3BC\gamma\delta \ + B^3CD\delta\alpha \ + C^3DA\alpha\beta \ + D^3AB\beta\gamma)\,\alpha^2\beta^2\gamma^2\delta^2 \\ +5\,(AB^2C^2\alpha\delta + BC^2D^2\beta\alpha + CD^2A^2\gamma\beta + DE^2A^2\delta\alpha)\,\alpha^2\beta^2\gamma^2\delta^2 \end{array}\right.
\end{array}\right\} = 0,$$

where as before

$$\begin{aligned}
A &= K + L\alpha + M\gamma + N\alpha\gamma, \\
B &= K + L\beta + M\delta + N\beta\delta, \\
C &= K + L\gamma + M\alpha + N\gamma\alpha, \\
D &= K + L\delta + M\beta + N\delta\beta;
\end{aligned}$$

and the coefficients of the quintic equation are, as they should be, cyclical functions with the cycle $\alpha\beta\gamma\delta$.

2, *Stone Buildings, W.C., February* 10, 1861.

312.

ON THE PARTITIONS OF A CLOSE.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 424—428.]

If F, S, E denote the number of faces, summits, and edges of a polyhedron, then, by Euler's well-known theorem,

$$F+S=E+2;$$

and if we imagine the polyhedron projected on the plane of any one face in such manner that the projections of all the summits not belonging to the face fall *within* the face, then we have a partitioned polygon, in which (if P denote the number of component polygons, or say the number of parts) $F=P+1$, or we have

$$P+S=E+1,$$

where S is the number of summits and E the number of edges of the plane figure. I retain for convenience the word *edge*, as having a different initial letter from *summit*.

The formula, however, excludes cases such as that of a polygon divided into two parts by means of an interior polygon wholly detached from it; and in order to extend it to such cases, the formula must be written under the form

$$P+S=E+1+B,$$

where B is the number of breaks of contour, as will be presently explained.

The edges of a polygon are right lines: it might at first sight appear that the theory would not be materially altered by removing this restriction, and allowing the edges to be curved lines; but the fact is that we thus introduce closed figures bounded by two edges, or even by a single edge, or by what I term a mere contour; and we have a new theory, which I call that of the Partitions of a Close.

Several definitions and explanations are required. The words line and curve are used indifferently to denote any path which can be described *currente calamo* without lifting the pen from the paper. A closed curve, not cutting or meeting itself [1], is called a *contour*. An enclosed space, such that no part of it is shut out from any other part of it, or, what is the same thing, such that any part can be joined with any other part by a line not cutting the boundary, is termed a *close*. The boundary of a close may be considered as the limit of a single contour, or of two or more contours lying wholly within the close. The reason for speaking of a limit will appear by an example. Consider a circle, and within it, but wholly detached from it, a figure of eight; the space interior to the circle but exterior to the figure of eight is a close: its boundary may be considered as the limit of two contours,—the first of them interior to the close, and indefinitely near the circle (in this case we might say the circle itself); the second of them an hour-glass-shaped curve, interior to the close (that is, exterior to the figure of eight) and indefinitely near to the figure of eight. The figure of eight, as being a curve which cuts itself, is not a contour; and in the case in question we could not have said that the boundary of the close consisted of two contours. A similar instance is afforded by a circle having within it two circles exterior to each other, but connected by a line not cutting or meeting itself; or even two points, or, as they may be called, summits, connected by a line not cutting or meeting itself; or, again, a single summit: in each of these cases the boundary of the close may be considered as the limit of two contours. But this explanation once given, we may for shortness speak of the close as bounded by a single contour, or by two or more contours; and I shall throughout do so, instead of using the more precise expression of the boundary being the limit of a contour, or of two or more contours. The excess above unity of the number of the contours which form the boundary of a close is the *break of contour* for such close; in the case of a close bounded by a single contour, the break of contour is zero.

Any point whatever on a curve may be considered as the point of meeting of two curves, or, in the case of a closed curve, as the point where the curve meets itself, but it is not of necessity so considered. A point where a curve cuts or meets itself or any other curve, is a *summit;* each point of termination of an unclosed curve is also a *summit;* any isolated point may be taken to be a *summit*. It follows that, in the case of a closed curve not cutting or meeting itself (that is, a contour), any point or points on the curve may be taken to be summits; but the contour need not have upon it any summit: it is in this case termed a *mere contour*. The curve which is the path from a summit to itself, or to any other summit, is an *edge:* the former case is that of a contour having upon it a single summit, the latter that of an edge having, that is terminated by, two summits, and no more. It is hardly necessary to remark that a contour having upon it two or more summits consists of the same number of edges, and, by what precedes, a contour having upon it a single summit is an edge; but it is to be noted that a contour without any summit upon

[1] It is hardly necessary to add, except in so far as any point whatever of the curve may be considered as a point where the curve meets itself.

it, or mere contour, is *not* an edge. It may be added that an edge does not cut or meet itself or any other edge except at the summit or summits of the edge itself.

Consider now a close bounded by $\beta+1$ mere contours: if for any partitioned close we have P the number of parts, S the number of summits, E the number of edges, B the number of breaks of contour; then, for the unpartitioned close, we have $P=1$, $S=0$, $E=0$, $B=\beta$, and therefore

$$P+S+\beta=E+1+B;$$

and it is to be shown that this equation holds good in whatever manner the close is partitioned. The partitionment is effected by the addition, in any manner, of summits and mere contours, and by drawing edges, any edge from a summit to itself or to another summit. The effect of adding a summit is first to increase S by unity: if the summit added be on a contour, E will be thereby increased by unity; for if the contour is a mere contour, it is not an edge, but becomes so by the addition of the summit; if it is not a mere contour, but has upon it a summit or summits, the addition of the summit will increase by unity the number of edges of the contour. If, on the other hand, the summit added be an isolated one, then the addition of such summit causes a break of contour, or B is increased by unity. Hence the addition of a summit increases by unity S; and it also increases by unity E or else B, that is, it leaves the equation undisturbed. The effect of the addition of a mere contour is to increase P by unity, and also to increase B by unity: it is easy to see that this is the case, whether the new mere contour does or does not contain within it any contour or contours. Hence the addition of a mere contour leaves the equation undisturbed. The effect of drawing an edge is first to increase E by unity; if the edge is drawn from a summit to itself, or from a summit on a contour to another summit on the same contour, then the effect is also to increase P by unity; if, however, the edge is drawn from a summit on a contour to a summit on a different contour, then P remains unaltered, but B is diminished by unity. There are a few special cases, which, although apparently different, are really included in the two preceding ones: thus, if the edge be drawn to connect two isolated summits, these are in fact to be considered as summits belonging to two distinct contours, and the like when a summit on a contour is joined to an isolated summit. And so if there be two or more summits connected together in order, and a new edge is drawn connecting the first and last of them, this is the same as when the edge is drawn through two summits of the same contour. The effect of drawing a new edge is thus to increase E by unity, and also to increase P by unity, or else to diminish B by unity; that is, it leaves the equation undisturbed. Hence the equation $P+S+\beta=E+1+B$, which subsists for the unpartitioned close, continues to subsist in whatever manner the close is partitioned, or it is always true.

In particular, if $\beta=0$, that is, if the original close be bounded by a mere contour, $P+S=E+1+B$; and if, besides, $B=0$, then $P+S=E+1$, which is the ordinary equation in the theory of the partitions of a polygon.

If we consider the surface of a plane as bounded by a mere contour at infinity, then for the infinite plane, $\beta = 0$, or we have $P + S = E + 1 + B$: in the case where the infinite plane is partitioned by a mere contour, $P = 2$, $S = 0$, $E = 0$, $B = 1$ (for the exterior part is bounded by the contour at infinity, and the partitioning contour, that is, for it, $B = 1$), and the equation is thus satisfied. And so for a contour having upon it n summits, $P = 2$, $S = n$, $E = n$, $B = 1$, and the equation is still satisfied: this is the case of the plane partitioned into two parts by means of a single polygon.

The case of a spherical surface is very interesting: the entire surface of the sphere must be considered as a close bounded by 0 contour, or we have $\beta = -1$, and the equation thus becomes $P + S = E + 2 + B$. Thus, if the sphere be divided into two parts by a mere contour, $P = 2$, $S = 0$, $E = 0$, $B = 0$, and the equation is satisfied. And in general, when $B = 0$, then $P + S = E + 2$; or writing F for P, then $F + S = E + 2$, which is Euler's equation for a polyhedron.

2, *Stone Buildings, W.C., March* 8, 1861.

313.

ON A SURFACE OF THE FOURTH ORDER.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 491—495.]

LET A, B, C be fixed points; it is required to investigate the nature of the surface, the locus of a point P such that

$$\lambda AP + \mu BP + \nu CP = 0,$$

where λ, μ, ν are given coefficients; the equation depends, it is clear, on the ratios only of these quantities.

The surface is easily seen to be of the fourth order; it is obviously symmetrical in regard to the plane ABC; and the section by this plane, or say the principal section, is a curve of the fourth order, the locus of a point M such that

$$\lambda AM + \mu BM + \nu CM = 0.$$

The curve is considered incidentally by Mr Salmon, p. 125 of his *Higher Plane Curves* [Ed. 3, p. 126 and see also p. 240 *et seq.*]; and he has remarked that the two circular points at infinity are double points on the curve, which is therefore of the eighth class. Moreover, that there are two double foci, since at each of these circular points there are two tangents, each tangent of the one pair intersecting a tangent of the other pair in a double focus; hence, further, that there are four other foci, the points A, B, C, and a fourth point D lying in a circle with A, B, C, and which are such that, selecting any three at pleasure of the points A, B, C, D, the equation of the curve is in respect to such three points of the same form as it is in regard to the points A, B, C.

Consider a given point M, on the principal section, then the equations

$$\frac{BP}{BM} = \frac{CP}{CM}, \quad \frac{CP}{CM} = \frac{AP}{AM}, \quad \frac{AP}{AM} = \frac{BP}{BM}$$

belong respectively to three spheres: each of the spheres passes through the point M. The first of the spheres is such that, with respect to it, B and C are the images

each of the other; that is, the centre of the sphere lies on the line BC, and the product of its distances from B and C is equal to the square of the radius; in like manner the second sphere is such that, with regard to it, C and A are the images each of the other; and the third sphere is such that, with regard to it, A and B are the images each of the other. The three spheres intersect in a circle through M at right angles to the principal plane (that is, the three spheres have a common circular section), and the equations of this circle may be taken to be

$$\frac{AP}{AM} = \frac{BP}{BM} = \frac{CP}{CM}.$$

It is clear that the circle of intersection lies wholly on the surface.

The spheres meet the principal plane in three circles, which are the diametral circles of the spheres; these circles are related to each other and to the points A, B, C, in like manner as the spheres are to each other and to the same points. The circles have thus a common chord; that is, they meet in the point M and in another point M': and MM' is the diameter of the circle, the intersection of the three spheres.

It may be shown that M, M' are the images each of the other in respect to the circle through A, B, C. In fact, consider in the first place the two points A, B, and a circle such that, with respect to it, A, B are the images each of the other; take M a point on this circle, and let O be any point on the line at right angles to AB through its middle point, and join OM cutting the circle in M'; then it is easy to see that M, M' are the images each of the other, in regard to the circle, centre O and radius $OA\,(=OB)$. Hence starting with the points A, B, C and the point M, let O be the centre of the circle through A, B, C, and take M' the image of M in respect to this circle; then considering the circle which passes through M, and in respect to which B, C are images each of the other, this circle passes through M'; and so the circle through M, in respect to which C, A are images each of the other, and the circle through M, in respect to which A, B are images each of the other, pass each of them through M'; that is, the three circles intersect in M'.

It is to be noticed that M', being on the surface, must be on the principal section; that is, the principal section is such that, taking upon it any point M, and taking M' the image of M in regard to the circle through A, B, C, then M' is also on the principal section. It is very easily shown that the curve of the fourth order possesses this property; for M, M' being images each of the other in respect to the circle through A, B, C, then A, B, C are points of this circle, or we have

$$\frac{MA}{M'A} = \frac{MB}{M'B} = \frac{MC}{M'C};$$

that is, the equation

$$\lambda AM + \mu BM + \nu CM = 0$$

being satisfied, the equation

$$\lambda AM' + \mu BM' + \nu CM' = 0$$

is also satisfied.

The points M, M' of the curve, which are images each of the other in respect to the circle through A, B, C, may be called conjugate points of the curve. The above-mentioned circle, the intersection of the three spheres, is the circle having MM' for its diameter; hence the required surface is the locus of a circle at right angles to the principal plane, and having for its diameter MM', where M and M' are conjugate points of the curve.

In the particular case where the equation of the surface is

$$BC.AP + CA.BP + AB.CP = 0,$$

the principal section is the circle through A, B, C, twice repeated. Any point on the circle is its own conjugate, and the radius of the generating circle of the surface is zero; that is, the surface is the annulus, the envelope of a sphere radius 0, having its centre on the circle through A, B, C. Or attending to real points only, the surface reduces itself to the circle through A, B, C. But this last statement of the solution is an incomplete one. The equation of an annulus, the envelope of a sphere radius c, having its centre on a circle radius unity, is

$$\sqrt{x^2+y^2} = 1 \pm \sqrt{c^2-z^2};$$

and hence putting $c=0$, the equation of the surface is,

$$\sqrt{x^2+y^2} = 1 \pm zi$$

(if, as usual, $i=\sqrt{-1}$), or, what is the same thing, it is

$$x^2+y^2+(z \pm i)^2 = 0;$$

that is, the surface is made up of the two spheres, passing through the points A, B, C, and having each of them the radius zero; or say the two *cone-spheres* through the points A, B, C. In other words, the equation

$$BC.AP + CA.BP + AB.CP = 0$$

is the condition in order that the four points A, B, C, P may lie on a sphere radius zero, or cone-sphere. Using 1, 2, 3, 4 in the place of A, B, C, P to denote the four points, the last-mentioned equation becomes

$$12.34 + 13.42 + 14.23 = 0;$$

and considering $\overline{12}$, &c. as quadratic radicals, the rational form of this equation is

$$\square = \begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0.$$

In my paper "On a Theorem in the Geometry of Position," *Camb. Math. Journ.* vol. II. pp. 267—271 (1841), [1], I obtained this equation, the four points being there considered as lying in a plane, as the relation between the distances of four points in a circle, in addition to the relation

$$\begin{vmatrix} & 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ 1, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ 1, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ 1, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0,$$

which exists between the distances of any four points in a plane. The present investigation shows the signification of the equation $\square = 0$ between the distances of four points in space; viz. it expresses that the four points lie in a sphere radius zero, or cone-sphere. But the formula in question is in reality included in that given in the paper for the distances of five points in space. For calling the points 0, 1, 2, 3, 4, the relation between the distances of these five points is

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{01}^2, & \overline{02}^2, & \overline{03}^2, & \overline{04}^2 \\ 1, & \overline{10}^2, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ 1, & \overline{20}^2, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ 1, & \overline{30}^2, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ 1, & \overline{40}^2, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0.$$

Hence if 1, 2, 3, 4 are the centres of spheres radii α, β, γ, δ, and if 0 is the centre of a tangent sphere radius r, we have

$$\overline{01} = r \pm \alpha, \quad \overline{02} = r \pm \beta, \quad \overline{03} = r \pm \gamma, \quad \overline{04} = r \pm \delta;$$

so that, for any given combination of signs, it would at first sight appear that r is determined by a quartic equation; but by means of a simple transformation (indicated to me by Prof. Sylvester) it may be shown that the equation for r is really a quadratic one; moreover, the equation remains unaltered if the signs of α, β, γ, δ and of r, are all reversed; and r^2 has thus in the whole sixteen values. In particular, if α, β, γ, δ are each equal 0, then r^2 is determined by a simple equation (r the radius of the sphere through the four points); and if, moreover, $r = 0$, then we have for the relation between the distances of the four points, the foregoing equation $\square = 0$.

2, *Stone Buildings, W.C., March* 25, 1861.

314.

ON THE CURVES SITUATE ON A SURFACE OF THE SECOND ORDER.

[From the *Philosophical Magazine*, vol. XXII. (1861), pp. 35—38.]

A SURFACE of the second order has on it a double system of generating lines, real or imaginary; and any two generating lines of the first kind form with any two generating lines of the second kind a skew quadrangle. If the equations of the planes containing respectively the first and second, second and third, third and fourth, fourth and first sides of the quadrangle are $x=0$, $y=0$, $z=0$, $w=0$, and if the constant multipliers which are implicitly contained in x, y, z, w respectively are suitably determined, then the equation of the surface of the second order (or say for shortness the quadric surface) is $xw-yz=0$.

Assume $\frac{y}{x}=\frac{\mu}{\lambda}$, $\frac{z}{x}=\frac{\nu}{\rho}$, then $\frac{\mu}{\lambda}$, $\frac{\nu}{\rho}$, or say $(\lambda, \mu, \nu, \rho)$, may be regarded as the coordinates of a point on the quadric surface; we in fact have $x:y:z:w=1:\frac{\mu}{\lambda}:\frac{\nu}{\rho}:\frac{\mu\nu}{\lambda\rho}$, or what is the same thing, $=\lambda\rho:\mu\rho:\nu\lambda:\mu\nu$. The four quantities $(\lambda, \mu, \nu, \rho)$ are for symmetry of notation used as coordinates; but it is to be throughout borne in mind that the absolute magnitudes of λ and μ, and of ν and ρ are essentially indeterminate; it is only the ratios $\lambda:\mu$ and $\nu:\rho$ that we are concerned with.

An equation of the form

$$(*\between\lambda, \mu)^p(\nu, \rho)^q=0,$$

that is, an equation homogeneous of the degree p as regards (λ, μ), and homogeneous of the degree q as regards (ν, ρ), represents a curve on the quadric surface; and this curve is of the order $p+q$. In fact, combining with the equation of the curve the equation of an arbitrary plane

$$Ax+By+Cz+Dw=0,$$

this equation, expressed in terms of the coordinates $(\lambda, \mu, \nu, \rho)$, is

$$A\lambda\rho + B\mu\rho + C\nu\lambda + D\mu\nu = 0;$$

or, as it is more conveniently written,

$$\left(\begin{matrix} C, & D \\ A, & B \end{matrix} \between \lambda, \mu \right)(\nu, \rho) = 0;$$

and if from this and the equation of the curve we eliminate $\lambda : \mu$ or $\nu : \rho$, say the second of these quantities, we obtain

$$(* \between \lambda, \mu)^p (-A\lambda - B\mu, C\lambda + D\mu)^q = 0,$$

which is of the order $p+q$ in (λ, μ); and $\lambda : \mu$ being known, $\nu : \rho$ is linearly determined. There are thus $p+q$ systems of values of the coordinates, or the plane meets the curve in $p+q$ points; that is, the curve is of the order $p+q$.

A linear equation $A\lambda + B\mu = 0$ gives a generating line, say of the first kind, of the quadric surface, and a linear equation $C\nu + D\rho = 0$ gives a generating line of the second kind: and by combining the one or the other of these equations with the equation of the curve, it is at once seen that the curve meets each generating line of the first kind in q points, and each generating line of the second kind in p points.

Consider the curves of the order n: the different solutions of the equation $p+q=n$ give different species of curves. But the solution $(n, 0)$ gives only a system of n generating lines of the first kind, and the solution $(0, n)$ gives only a system of generating lines of the second kind. And in general the solutions (p, q) and (q, p) give species of curves which are related, the one of them to the generating lines of the first and second kinds, in the same way as the other of them to the generating lines of the second and first kinds; and they may be considered as correlative members of the same species. The number of distinct species is thus $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$, according as n is odd or even; for $n=3$ we have the single species (2, 1) or (1, 2); for $n=4$, the two species (1, 3) or (3, 1), and (2, 2); for $n=5$, the two species (4, 1) or (1, 4), and (3, 2) or (2, 3); and so on. Thus for $n=3$, the species (2, 1) is represented by an equation of the form

$$(a, b, c \between \lambda, \mu)^2 \nu + (a', b', c' \between \lambda, \mu)^2 \rho = 0,$$

which belongs to a cubic curve in space. To show *à posteriori* that this is so, I observe that the equation expressed in terms of the original coordinates (x, y, z, w) is

$$x(a, b, c \between x, y)^2 + z(a', b', c' \between x, y)^2 = 0,$$

which by means of the equation $xw - yz = 0$ of the quadric surface is reduced to

$$(a, b, c \between x, y)^2 + a'xz + 2b'yz + c'yw = 0;$$

and this is the equation of a quadric surface intersecting the quadric surface $xw - yz = 0$ in the line $x=0$, $y=0$; and therefore also intersecting it in a cubic curve.

For $n=4$, I take first the species (2, 2) [the quadriquadric curve] which is represented by an equation of the form

$$(a,\ b,\ c\between\lambda,\ \mu)^2\nu^2+2(a',\ b',\ c'\between\lambda,\ \mu)^2\nu\rho+(a'',\ b'',\ c''\between\lambda,\ \mu)^2\rho^2=0,$$

which in fact belongs to a quartic curve, the intersection of two quadric surfaces. For, reverting to the original coordinates, the equation becomes

$$(a,\ b,\ c\between x,\ y)^2x^2+2(a',\ b',\ c'\between x,\ y)^2xz+(a'',\ b'',\ c''\between x,\ y)^2z^2=0,$$

which by means of the equation $xw-yz=0$ of the quadric surface is at once reduced to

$$(a,\ b,\ c\between x,\ y)^2+2a'xz+4b'yz+2c'yw+a''z^2+2b''zw+c''w^2=0,$$

which is the equation of a quadric surface intersecting the given quadric surface $xw-yz=0$ in the curve in question.

Consider next the species (3, 1) [the excubo-quartic curve] represented by an equation of the form

$$(a,\ b,\ c,\ d\between\lambda,\ \mu)^3\nu+(a',\ b',\ c',\ d'\between\lambda,\ \mu)^3\rho=0,$$

which is the other species of quartic curve situate on only a single quadric surface. Reverting to the original coordinates, the equation becomes

$$(a,\ b,\ c,\ d\between x,\ y)^3x+(a',\ b',\ c',\ d'\between x,\ y)^3z=0;$$

and by means of the equation $xw-yz=0$ of the quadric surface this is reduced to

$$(a,\ b,\ c,\ d\between x,\ y)^3+a'x^2z+3b'xyz+3c'y^2z+d'y^3w=0,$$

which is the equation of a cubic surface containing the line $(x=0,\ y=0)$ twice, and therefore along this line touching the quadric surface $xw-yz=0$; and consequently intersecting it besides in a quartic curve. And in like manner for the curves of the fifth and higher orders which lie upon a quadric surface.

The combination of the equations

$$(*\between\lambda,\ \mu)^p(\nu,\ \rho)^q=0,$$
$$(*'\between\lambda,\ \mu)^{p'}(\nu,\ \rho)^{q'}=0,$$

shows at once that two curves on the same quadric surface of the species $(p,\ q)$ and $(p',\ q')$ respectively intersect in a number $(pq'+p'q)$ of points. Thus if the curves are (1, 0) and (1, 0), or (0, 1) and (0, 1), i.e. generating lines of the same kind, the number of intersections is $1.0+0.1=0$; but if the curves are (1, 0) and (0, 1), i.e. generating lines of different kinds, the number of intersections is $1.1+0.0=1$.

The notion of the employment of hyperboloidal coordinates presented itself several years ago to Prof. Plücker (see his paper "Die analytische Geometrie der Curven auf den Flächen zweiter Ordnung und Classe," *Crelle*, vol. XXXIV. pp. 341—359, 1847); but the systems made use of, e.g. $\xi=-\dfrac{d}{\mu}\dfrac{z}{y}$, $\eta=-\dfrac{d}{\mu}\dfrac{z}{x}$, with $z(z+d)+\mu xy=0$ for the equation of the surface of the second order, is less simple; and the question of the classification of the curves on the surface is not entered on.

2, *Stone Buildings, W.C., May* 24, 1861.

315.

ON THE CUBIC CENTRES OF A LINE WITH RESPECT TO THREE LINES AND A LINE.

[From the *Philosophical Magazine*, vol. XXII. (1861), pp. 433—436.]

ON referring to my Note on this subject (*Phil. Mag.* vol. XX. pp. 418—423, 1860 [257]), it will be seen that the cubic centres of the line

$$\lambda x + \mu y + \nu z = 0$$

in relation to the lines $x=0$, $y=0$, $z=0$, and the line $x+y+z=0$, are determined by the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

where θ is a root of the cubic equation

$$\frac{1}{\theta+\lambda} + \frac{1}{\theta+\mu} + \frac{1}{\theta+\nu} - \frac{2}{\theta} = 0;$$

or as it may also be written,

$$\theta^3 - \theta(\mu\nu + \nu\lambda + \lambda\mu) - 2\lambda\mu\nu = 0.$$

Two of the centres will coincide if the equation for θ has equal roots; and this will be the case if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

or, what is the same thing, if λ, μ, $\nu = a^{-3}$, b^{-3}, c^{-3}, where $a+b+c=0$. In fact, if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, and the equation in θ becomes

$$\theta^3 - \frac{3\theta}{a^2b^2c^2} - \frac{2}{a^3b^3c^3} = 0;$$

that is

$$(abc\theta)^2 - 3(abc\theta) - 2 = 0,$$

which is

$$(abc\theta + 1)^2(abc\theta - 2) = 0;$$

so that the values of θ are $\frac{-1}{abc}, \frac{2}{abc}$.

First, if $\theta = -\frac{1}{abc}$, then x, y, z will be the coordinates of the double centre. And we have

$$\theta + \lambda = \frac{1}{a^3} - \frac{1}{abc} = \frac{1}{2a^3bc}(2bc - 2a^2) = \frac{1}{2a^3bc}(-a^2 - b^2 - c^2);$$

or putting for shortness $\square = a^2 + b^2 + c^2$,

$$\theta + \lambda = -\frac{1}{2a^3bc}\square, \quad = -\frac{3}{abc}\cdot\frac{\square}{6a^2},$$

with similar values for $\theta + \mu$, $\theta + \nu$. But $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are proportional to $\theta + \lambda$, $\theta + \mu$, $\theta + \nu$; and we may therefore write

$$\frac{P}{x} = \frac{\square}{6a^2}, \quad \frac{P}{y} = \frac{\square}{6b^2}, \quad \frac{P}{z} = \frac{\square}{6c^2};$$

whence, in virtue of the equation $a + b + c = 0$, we have for the locus of the double centre,

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 0;$$

or this locus is a conic touching the lines $x = 0$, $y = 0$, $z = 0$ harmonically in respect to the line $x + y + z = 0$, a result which was obtained somewhat differently in the paper above referred to.

Next, if $\theta = \frac{2}{abc}$, x, y, z will be the coordinates of the single centre. And we now have

$$\theta + \lambda = \frac{1}{a^3} + \frac{2}{abc} = \frac{1}{2a^3bc}(2bc - 2a^2 + 6a^2) = \frac{1}{2a^3bc}(-\square + 6a^2) = -\frac{3}{abc}\,\frac{\square - 6a^2}{6a^2},$$

with similar values for $\theta + \mu$, $\theta + \nu$. But $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are proportional to $\theta + \lambda$, $\theta + \mu$, $\theta + \nu$, and we may therefore write

$$\frac{P}{x} = \frac{\square - 6a^2}{6a^2}, \quad \frac{P}{y} = \frac{\square - 6b^2}{6b^2}, \quad \frac{P}{z} = \frac{\square - 6c^2}{6c^2},$$

from which equations, and the equation $a + b + c = 0$, the quantities P, a, b, c have to be eliminated. I at first effected the elimination as follows: viz., writing the equations under the form

$$\frac{x}{x+P} = \frac{6a^2}{\square}, \quad \frac{y}{y+P} = \frac{6b^2}{\square}, \quad \frac{z}{z+P} = \frac{6c^2}{\square},$$

we obtain

$$\frac{x}{x+P}+\frac{y}{y+P}+\frac{z}{z+P}=6,$$

$$\sqrt{\frac{x}{x+P}}+\sqrt{\frac{y}{y+P}}+\sqrt{\frac{z}{z+P}}=0,$$

which are easily transformed into

$$\frac{x}{x+P}+\frac{y}{y+P}+\frac{z}{z+P}=6,$$

$$\frac{yz}{(y+P)(z+P)}+\frac{zx}{(z+P)(x+P)}+\frac{xy}{(x+P)(y+P)}=9;$$

or, what is the same thing,

$$6(P+x)(P+y)(P+z)-x(P+y)(P+z)-y(P+z)(P+x)-z(P+x)(P+y)=0,$$

$$9(P+x)(P+y)(P+z)-yz(P+x)\qquad-zx(P+y)\qquad-xy(P+z)=0,$$

which give

$$6P^3+5P^2(x+y+z)+4P(yz+zx+xy)+3xyz=0,$$

$$9P^3+9P^2(x+y+z)+8P(yz+zx+xy)+6xyz=0;$$

or, multiplying the first equation by 2, and subtracting the second,

$$3P+\quad(x+y+z)=0;$$

and we thus obtain for the locus of the single centre the equation

$$\frac{x}{-2x+y+z}+\frac{y}{-2y+z+x}+\frac{z}{-2z+x+y}=2,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+zx^2+xy^2+y^2z+z^2x+x^2y)+3xyz=0,$$

which may also be written,

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0.$$

The same result may also be obtained as follows: viz., observing that

$$\square-6a^2=b^2+c^2-5a^2=-4a^2-2bc,$$

we have

$$\frac{x}{P}=\frac{-3a^2}{2a^2+bc},\quad \frac{y}{P}=\frac{-3b^2}{2b^2+ca},\quad \frac{z}{P}=\frac{-3c^2}{2c^2+ab},$$

and then by means of the equation

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}-1=0,$$

which is identically true in virtue of $a+b+c=0$ (in fact, multiplying out, this gives

$$12a^2b^2c^2+4(b^3c^3+c^3a^3+a^3b^3)+abc(a^3+b^3+c^3)$$
$$-8a^2b^2c^2-4(b^3c^3+c^3a^3+a^3b^3)-2abc(a^3+b^3+c^3)-a^2b^2c^2=0;$$

that is

$$3a^2b^2c^2-abc(a^3+b^3+c^3)=0, \text{ or } abc(a^3+b^3+c^3-3abc)=0,$$

where the second factor divides by $a+b+c$), we find the above-mentioned equation,

$$x+y+z+3P=0.$$

We then have

$$\frac{-x+y+z}{P}=\frac{x+y+z}{P}-\frac{2x}{P}=-3+\frac{6a^2}{2a^2+bc}=-\frac{3bc}{2a^2+bc};$$

that is

$$\frac{-x+y+z}{P}=\frac{-3bc}{2a^2+bc},\quad \frac{x-y+z}{P}=\frac{-3ca}{2b^2+c},\quad \frac{x+y-z}{P}=\frac{-3ab}{2c^2+ab};$$

and forming the product of these functions, and that of the foregoing values of $\frac{x}{P}$, $\frac{y}{P}$, $\frac{z}{P}$, we find as before,

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0$$

for the equation of the locus of the single centre. The equation shows that the locus is a cubic curve which touches the lines $x=0$, $y=0$, $z=0$ at the points where these lines are intersected by the lines $y-z=0$, $z-x=0$, $x-y=0$ (that is, it touches the lines $x=0$, $y=0$, $z=0$ harmonically in respect to the line $x+y+z=0$), and besides meets the same lines $x=0$, $y=0$, $z=0$ at the points in which they are respectively met by the line $x+y+z=0$.

2, *Stone Buildings, W.C., September* 25, 1861.

316.

NOTE ON THE SOLUTION OF AN EQUATION OF THE FIFTH ORDER.

[From the *Philosophical Magazine*, vol. XXIII. (1862), pp. 195, 196.]

This Note was in answer to Mr Jerrard's paper "Remarks on Mr Cayley's Note," *Phil. Mag.* vol. XXI. pp. 348—350, referring to the foregoing paper 310.

317.

NOTE ON THE TRANSFORMATION OF A CERTAIN DIFFERENTIAL EQUATION.

[From the *Philosophical Magazine*, vol. XXIII. (1862), pp. 266, 267.]

THE differential equation

$$(1+\theta^2)\frac{d^2y}{d\theta^2}+\theta\frac{dy}{d\theta}-m^2y=0,$$

if we put therein $i\theta=2x^2+1$ ($i=\sqrt{-1}$ as usual), becomes

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4m^2y=0.$$

In fact an integral of the second equation is $(\sqrt{1+x^2}+x)^{2m}$; this is

$$=\left(\sqrt{(2x^2+1)^2-1}+2x^2+1\right)^m;$$

or putting $i\theta=2x^2+1$, it is

$$=(\sqrt{-\theta^2-1}+i\theta)^m,$$

which is

$$=\{i(\sqrt{\theta^2+1}+\theta)\}^m;$$

so that an integral of the transformed equation in θ is

$$=\quad(\sqrt{\theta^2+1}+\theta)^m;$$

and writing in the second equation θ for x, and $\frac{1}{2}m$ for m, we see that the last-mentioned function, viz. $(\sqrt{\theta^2+1}+\theta)^m$, is an integral of

$$(1+\theta^2)\frac{d^2y}{d\theta^2}+\theta\frac{dy}{d\theta}-m^2y=0;$$

whence the transformed equation in θ must be this very equation, that is, it must be the first equation. I have for shortness used the particular integral $(\sqrt{1+x^2}+x)^{2m}$; but the reasoning should have been applied, and it is in fact applicable, without alteration, to the general integral

$$C(\sqrt{1+x^2}+x)^m + C'(\sqrt{1+x^2}-x)^m.$$

There is of course no difficulty in a direct verification. Thus, starting from the first equation, or equation in θ, the relation $i\theta = 2x^2+1$ gives

$$\frac{dy}{d\theta} = \frac{i}{4x}\frac{dy}{dx}, \quad \frac{d^2y}{d\theta^2} = \frac{i}{4x}\frac{d}{dx}\left(\frac{i}{4x}\frac{dy}{dx}\right) = -\frac{1}{16x^2}\left(\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx}\right),$$

$$1+\theta^2 = -4x^2(1+x^2);$$

so that the equation becomes

$$\tfrac{1}{4}(1+x^2)\left(\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx}\right) + \frac{1+2x^2}{4x}\frac{dy}{dx} - m^2y = 0,$$

or multiplying by 4,

$$(1+x^2)\frac{d^2y}{dx^2} + \left(-\frac{1+x^2}{x} + \frac{1+2x^2}{x}\right)\frac{dy}{dx} - 4m^2y = 0;$$

that is

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4m^2y = 0,$$

the second equation. But the first method shows the reason why the two forms are thus connected together.

2, *Stone Buildings, W.C., February* 19, 1862.

318.

ON A QUESTION IN THE THEORY OF PROBABILITIES.

[From the *Philosophical Magazine*, vol. XXIII. (1862), pp. 361—365.]

The question referred to is that discussed in the paper 121; the remarks on that paper in the Notes and References to volume II. are in a great measure to the same effect as the present and next papers, 318 and 319, the existence of which I had entirely overlooked. In the first part (dated 2, Stone Buildings, W.C., March 1862) of the present paper 318, after referring to the two modes of statement which may be called the *Causation* statement and the *Concomitance* statement, I reproduce nearly as in the Notes and References first my own solution as completed by Dedekind, next Boole's solution of the problem, involving his logical probabilities; and the paper is then continued as follows.

The foregoing paper was submitted to Prof. Boole, who, in a letter dated March 26, 1862, writes:

"The observations which have occurred to me after studying your paper are the following.

1st. I think that your solution is correct under conditions partly expressed and partly implied. The one to which you direct attention is the assumed independence of the causes denoted by A and B. Now I am not sure that I can state precisely what the others are; but one at least appears to me to be the assumed independence of the events of which the probabilities according to your hypothesis are $\alpha\lambda$, $\beta\mu$. Assuming the independence of the causes as to *happening*, I do not think that you are entitled *on that ground* to assume their independence as to *acting;* because, to confine our observations to common experience, we often notice that states of things apparently independent as to their occurrence, may, when concurring, aid or hinder each other in such a manner that the one may be more or less likely to act 'efficiently' in the presence of the other than in its absence. I use the language of your own hypothesis of *efficient* action.

2ndly. When I say that I think your solution correct under certain conditions, I ought to add that it appears to me that such conditions ought to be stated as

part of the original data, and that they ought to be of such a kind that they can be established by experience in the same way as the other data are. For instance, if experience, as embodied in a sufficiently long series of statistical records, establish that

$$\text{Prob. } A = \alpha, \quad \text{Prob. } B = \beta,$$

the very same experience may, by establishing also that

$$\text{Prob. } AB = \alpha\beta,$$

whence in conjunction with the former it follows that

$$\text{Prob. } AB' = \alpha\beta', \quad \text{Prob. } A'B = \alpha'\beta, \quad \text{Prob. } A'B' = \alpha'\beta',$$

enable us to pronounce that A and B are in the long run, as to happening or not happening, in the position of mutually independent events.

3rdly. I think it may be shown to demonstration, from the nature of the result, that the solution you have obtained does not apply simply and generally to the problem under the single modification of the assumption that A and B are independent. The completed data under this assumption are

$$\text{Prob. } A = \alpha, \quad \text{Prob. } B = \beta, \quad \text{Prob. } AB = \alpha\beta,$$

$$\text{Prob. } AE = \alpha p, \quad \text{Prob. } BE = \beta q.$$

You may deduce all these from your Table of Probabilities of 'compound events' given in your paper. Now you may easily satisfy yourself that the sole necessary and sufficient conditions for the *consistency* of these data are the following:

$$\left.\begin{array}{ll} (1) & \alpha p' + \beta q \gtreqless \alpha\beta, \\ (2) & \alpha p + \beta q' \gtreqless \alpha\beta, \\ (3) & \left\{\begin{array}{l} \left.\begin{array}{l}\alpha\\ \beta\end{array}\right\} \lesseqgtr 1, \\ \left.\begin{array}{l}p\\ q\end{array}\right\} \gtreqless 0. \end{array}\right. \end{array}\right\} \qquad \text{(M).}$$

But your solution requires the following conditions to be satisfied, viz.,

$$q - \alpha p \gtreqless 0, \quad p - \beta q \gtreqless 0,$$

together with the system (3). Now (1) and (2) are expressible in the form

$$\beta(q - \alpha p) + \alpha\beta' p' \gtreqless 0,$$

$$\alpha(p - \beta q) + \beta\alpha' q' \gtreqless 0;$$

from which you will see that your conditions are *narrower* than those which the data are really subject to. If your conditions are satisfied, the data will be consistent; but the converse of this proposition does not hold.

4thly. You remark that my solution of the problem, in which the independence of A and B is not assumed, but in which the probabilities are otherwise the same as in yours, is only applicable when

$$\alpha' + \alpha p \gtreqless \beta q, \quad \beta' + \beta q \gtreqless \alpha p;$$

but you do not appear to have noticed that these are actually the conditions of *consistency in the data.* Unless these are satisfied, the data cannot possibly be furnished by experience.

5thly. You remark that I have solved the problem under what you call the '*concomitance*' statement, and not the '*causation*' statement. I think that every problem stated in the 'causation' form admits, if capable of scientific treatment, of reduction to the 'concomitance' form. I admit it would have been better, in stating my problem, not to have employed the word 'cause' at all. But the introduction of the hypothesis of the independence of A and B does not affect the *nature* of the problem.

6thly. The x, s, &c., about the interpretation of which you inquire, are the probabilities of ideal events in an ideal problem connected by a formal relation with the real one. I should fully concede that the auxiliary probabilities which are employed in my method always refer to an ideal problem; but it is one, the form of which, as given by the calculus of logic, is not arbitrary. Nor does its connexion with the real problem appear to me arbitrary. It involves an extension, but as it seems to me, a perfectly scientific extension, of the principles of the ordinary theory of probabilities. On this subject, however, I have but little to add to what I have said, *Transactions of the Royal Society of Edinburgh*, vol. XXI. part 4, "On the Application of the Theory of Probabilities &c."

7thly. The problem, as stated by me, and then modified by the simple introduction of the hypothesis of the independence of A and B, must admit of solution by my method; and that solution ought to impose no restriction beyond the conditions of possible experience noted in (M).

I should be extremely glad if mathematicians would examine the analytical questions connected with the application of my method. There can, I think, after the partial proofs which I have given, exist no doubt that the conditions of applicability of the solutions are always identical with the conditions of consistency in the data, i.e. with what I have called, in the paper above referred to, the conditions of possible experience. The proof of the general proposition would involve the showing that a certain functional determinant consists solely of positive terms, with some connected theorems which appear to me to be of considerable analytical interest.

8thly. I certainly think your paper deserving of publication. If you think proper to add any or the whole of my remarks, you can do so, with of course any comments of your own."

I remark upon Prof. Boole's observations:

1st. I do assume that the causes A and B are absolutely independent of, and uninfluenced by each other; viz. not only the probability of A acting, but also the

probability of its acting efficiently, are each of them the same whether B does not act, or acts inefficiently, or acts efficiently; and the like for B.

2ndly. I do assume that the same experience which establishes

$$\text{Prob. } A = \alpha, \quad \text{Prob. } B = \beta,$$

would in the long run establish

$$\text{Prob. } AB = \alpha\beta;$$

if it does not, *cadit quæstio*, the causes are not independent.

3rdly. I assume not only

$$\text{Prob. } A = \alpha, \quad \text{Prob. } B = \beta, \quad \text{Prob. } AB = \alpha\beta,$$

but also as 1st above stated; and I consider that, inasmuch as the result of the investigation is to show that the conditions $q - \alpha p \nless 0$, $p - \beta q \nless 0$ are necessary and sufficient conditions, it is also a result of the investigation that these are the conditions of *consistency among the data*, viz. the conditions in order that the data may be consistent with the above assumptions as to the independence of the causes. It is clear that since, as just stated, I do assume something beyond the last-mentioned three equations, the conditions of consistency *ought* to be *narrower* than those in Prof. Boole's 3rdly.

4thly. I had not overlooked, but I ought to have stated, that Prof. Boole's conditions were actually the conditions of consistency in the data.

5thly. I contend that the conception of A and B as *causes* does alter the *nature* of the problem. For when A and B are conceived of as causes, there is a definite notion of the efficient or inefficient action of A or B; and in particular when they both act, one of them, say A, may act inefficiently. But according to the concomitance statement, then either there is no such notion as that of the efficient or non-efficient happening of A or B (I believe this to be so), *or else* the only notion of efficient or inefficient happening is happening in concomitance or in non-concomitance with E; but in this view, if A, B, E all happen, then A and B each of them happens efficiently. The argument is to me conclusive as to the diversity of the two problems.

6thly. I do not in anywise assert, or even suppose, that the ideal problem is arbitrary, or that its connexion with the real problem is arbitrary. I simply do not know what the ideal problem is; I do not know the point of view, or the assumed mental state of knowledge or ignorance according to which x, y, s, t are the probabilities of A, B, AE, BE. It is to be borne in mind that x, y, s, t are, in Prof. Boole's solution, determined as numerical quantities included between the limits 0 and 1, i.e. as quantities which are or may be actual probabilities. What I desiderate is, that Prof. Boole should give for his auxiliary quantities x, y, s, t such an explanation of the meaning as I have given for my auxiliary quantities λ, μ. I do not find any such explanation in the memoir referred to.

7thly and 8thly. No remark is necessary.

March 29, 1862.

Prof. Boole, in his reply, dated April 2, writes, "No such explanation as you desiderate of the interpretation of the auxiliary quantities in my method of solution is possible; because they are not of the nature of additional data, and their introduction does not limit the problem as any hypotheses which are of that nature do. I do not see any difficulty whatever in the conception of the ideal problem."

We thus join issue as follows: Prof. Boole says that there is no difficulty in understanding, I say that I do not understand, the *rationale* of his solution.

It may be remarked that the question may be, not to find any actual probability whatever, but only to find a Boolian probability or probabilities. Thus the equations (L), p. 356, omitting the last member, which alone involves u, determine in terms of the data α, β, αp, βq the Boolian probabilities x, y, s, t of the events A, B, AE, BE.

In a subsequent hastily-written letter, Prof. Boole gives an explanation of the equations (L), which appears to me little more than a translation of these equations into ordinary language.

April 16, 1862.

319.

POSTSCRIPT TO THE PAPER "ON A QUESTION IN THE THEORY OF PROBABILITIES."

[From the *Philosophical Magazine*, vol. XXIII. (1862), pp. 470—471.]

See 318. The present paper reproduces Wilbraham's discussion of Boole's Solution; and concludes with the remark "Professor Boole wishes me to mention that he has succeeded in obtaining a demonstration of the analytical theorem arising from his theory referred to in his Reply in my paper." See *ante* p. 82, 7thly.

320.

ON THE TRANSCENDENT $\operatorname{gd} u = \frac{1}{i} \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$.

[From the *Philosophical Magazine*, vol. XXIV. (1862), pp. 19—21.]

THE elliptic functions which correspond to the modulus $k=1$ reduce themselves, as is well known, to circular functions. The case in question plays implicitly an important part in the general theory, and it has been particularly studied by Gudermann, and by Dr Booth in connexion with his theory of parabolic logarithms. But in the absence of a notation corresponding to that used for elliptic functions in general, the theory has not, it appears to me, been exhibited in its proper form. The defect is very easily supplied: using for $\operatorname{am} u$, to the modulus 1, the notation $\operatorname{gd} u$ (Gudermannian of u), then if

$$u = \int_0 \frac{d\phi}{\cos\phi} = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right),$$

we have

$$\phi = \operatorname{gd} u;$$

and hence, observing that the equation between u and ϕ is

$$u = \log\frac{1 + \frac{1}{2}\tan\phi}{1 - \frac{1}{2}\tan\phi},$$

or, what is the same thing,

$$\tan\tfrac{1}{2}\phi = \frac{e^u - 1}{e^u + 1},$$

and that we have

$$\log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}ui\right) = \log\frac{1 + \tan\frac{1}{2}ui}{1 - \tan\frac{1}{2}ui}$$

$$= \log\frac{\cos\frac{1}{2}ui + \sin\frac{1}{2}ui}{\cos\frac{1}{2}ui - \sin\frac{1}{2}ui}$$

$$= \log \frac{e^{\frac{1}{2}u} + e^{-\frac{1}{2}u} + i\,(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})}{e^{\frac{1}{2}u} + e^{-\frac{1}{2}u} - i\,(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})}$$

$$= \log \frac{e^{u} + 1 + i\,(e^{u} - 1)}{e^{u} + 1 - i\,(e^{u} - 1)}$$

$$= \log \frac{1 + i \tan \frac{1}{2}\phi}{1 - i \tan \frac{1}{2}\phi} = \log e^{i\phi} = i\phi,$$

or if

$$u = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

then

$$\phi = \frac{1}{i} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}ui);$$

and substituting for ϕ its value, we obtain

$$\operatorname{gd} u = \frac{1}{i} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}ui),$$

which is the definition of the transcendent $\operatorname{gd} u$. It is to be noticed that $\operatorname{gd} u$, although exhibited in an imaginary form, is a real function of u; and, moreover, that it is an odd function, viz. we have

$$\operatorname{gd}(-u) = -\operatorname{gd}(u),$$

and therefore also

$$\operatorname{gd}(0) = 0.$$

The original equation,

$$u = \quad \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

written under the form

$$u = i\frac{1}{i} \log \tan \left(\tfrac{1}{4}\pi + \tfrac{1}{2}i\frac{\phi}{i}\right),$$

shows that we have

$$u = i \operatorname{gd}\left(\frac{\phi}{i}\right) = i \operatorname{gd}(-i\phi);$$

or substituting for ϕ its value $\operatorname{gd} u$, we have

$$u = i \operatorname{gd}(-i \operatorname{gd} u),$$

which may also be written

$$iu = \operatorname{gd}(i \operatorname{gd} u);$$

so that $\operatorname{gd} u$ is a quasi-periodic function of the second order—a property which has not, at least obviously, any analogue in the general theory. We have

$$\cos \operatorname{gd} u = \tfrac{1}{2}\,(e^{i \operatorname{gd} u} + e^{-i \operatorname{gd} u})$$

$$= \tfrac{1}{2}\left(\frac{1 + \tan \frac{1}{2}ui}{1 - \tan \frac{1}{2}ui} + \frac{1 - \tan \frac{1}{2}ui}{1 + \tan \frac{1}{2}ui}\right)$$

$$= \quad \frac{1 + \tan^2 \frac{1}{2}ui}{1 - \tan^2 \frac{1}{2}ui} = \frac{1}{\cos ui};$$

and in like manner

$$\sin \operatorname{gd} u = \frac{1}{2i}\left(e^{i \operatorname{gd} u} - e^{-i \operatorname{gd} u}\right)$$

$$= \frac{1}{2i}\left(\frac{1+\tan\frac{1}{2}ui}{1-\tan\frac{1}{2}ui} - \frac{1-\tan\frac{1}{2}ui}{1+\tan\frac{1}{2}ui}\right)$$

$$= \frac{2\tan\frac{1}{2}ui}{i\left(1-\tan^2\frac{1}{2}ui\right)} = \frac{\sin ui}{i\cos ui},$$

or, as these equations may also be written,

$$\sec \operatorname{gd} u = \cos ui = \tfrac{1}{2}\left(e^{u} + e^{-u}\right),$$

$$\tan \operatorname{gd} u = \frac{1}{i}\sin ui = \tfrac{1}{2}\left(e^{u} - e^{-u}\right);$$

and from these equations we have

$$\sec \operatorname{gd}(u+v) = \sec \operatorname{gd} u \,.\, \sec \operatorname{gd} v + \tan \operatorname{gd} u \,.\, \tan \operatorname{gd} v,$$
$$\tan \operatorname{gd}(u+v) = \tan \operatorname{gd} u \,.\, \sec \operatorname{gd} v + \tan \operatorname{gd} v \,.\, \sec \operatorname{gd} v;$$

or, what is the same thing,

$$\sin \operatorname{gd}(u+v) = \frac{\sin \operatorname{gd} u + \sin \operatorname{gd} v}{1 + \sin \operatorname{gd} u \,.\, \sin \operatorname{gd} v},$$

$$\cos \operatorname{gd}(u+v) = \frac{\cos \operatorname{gd} u \,.\, \cos \operatorname{gd} v}{1 + \sin \operatorname{gd} u \,.\, \sin \operatorname{gd} v};$$

which forms are at once obtainable from the formulæ

$$\sin \operatorname{am}(u+v) = \frac{\sin \operatorname{am} u \cos \operatorname{am} v\, \Delta \operatorname{am} v + \sin \operatorname{am} v \cos \operatorname{am} u\, \Delta \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

$$\cos \operatorname{am}(u+v) = \frac{\cos \operatorname{am} u \cos \operatorname{am} v - \sin \operatorname{am} u \sin \operatorname{am} v\, \Delta \operatorname{am} u\, \Delta \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

$$\Delta \operatorname{am}(u+v) = \frac{\Delta \operatorname{am} u\, \Delta \operatorname{am} v - k^2 \sin \operatorname{am} u \sin \operatorname{am} v \cos \operatorname{am} u \cos \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

observing that for $k=1$ we have $\Delta \operatorname{am} = \cos \operatorname{am}$, and that the numerators and denominator each of them divide by

$$1 - \sin \operatorname{am} u \sin \operatorname{am} v.$$

2, *Stone Buildings, W.C., May* 7, 1862.

321.

FINAL REMARKS ON MR JERRARD'S THEORY OF EQUATIONS OF THE FIFTH ORDER.

[From the *Philosophical Magazine*, vol. XXIV. (1862), p. 290.]

322.

ON THE SKEW SURFACE OF THE THIRD ORDER.

[From the *Philosophical Magazine*, vol. XXIV. (1862), pp. 514—519.]

THE skew surface of the third order, or "cubic scroll" (disregarding a certain special form), may be considered as generated by a line which always passes through three directrices; viz., a plane cubic having a node, and two lines, one of them meeting the cubic in the node, the other of them meeting the cubic in an ordinary point. The analytical investigation possesses some interest as an illustration of the analytical theory of skew surfaces in general.

Take for the equation of the cubic

$$(\alpha^2+\beta^2)\,xy-(x^2+y^2)\,\alpha\beta=0,$$

which belongs to a cubic having a node at the origin, and passing through the point (α, β); and for the equations of the two lines

$$(x-mz=0,\quad y-nz=0),$$
$$(x-\alpha=0,\quad y-\beta=0).$$

Then, (X, Y, Z) being current coordinates, the equations of the generating line will be

$$X=x+AZ,$$
$$Y=y+BZ;$$

when this meets the line $(X-mZ=0,\ Y-nZ=0)$, we have

$$mZ=x+AZ,$$
$$nZ=y+BZ,$$

and thence

$$x(n-B)-y(m-A)=0;$$

or, what is the same thing,

$$nx - my - Bx + Ay = 0:$$

and when it meets the line $(X - \alpha = 0,\ Y - \beta = 0)$, we have

$$\alpha = x + AZ,$$
$$\beta = y + BZ;$$

and thence

$$B(x - \alpha) - A(y - \beta) = 0.$$

We have thus the system of equations

$$(\alpha^2 + \beta^2)\, xy - (x^2 + y^2)\, \alpha\beta = 0,$$
$$X = x + AZ,$$
$$Y = y + BZ,$$
$$nx - my - Bx + Ay = 0,$$
$$B(x - \alpha) - A(y - \beta) = 0;$$

from which, eliminating $(A,\ B,\ x,\ y)$, we obtain the equation of the surface.

Writing in the last equation

$$B = s(x - \alpha), \quad A = s(y - \beta)$$

(values which give $Bx - Ay = -s(\beta x - \alpha y)$), we find

$$X + \alpha sZ = (1 + sZ)\, x,$$
$$Y + \beta sZ = (1 + sZ)\, y,$$
$$(n + \beta s)\, x - (m + \alpha s)\, y = 0;$$

whence also

$$(n + \beta s)(X + \alpha sZ) - (m + \alpha s)(Y + \beta sZ) = 0,$$

that is

$$nX - mY + (n\alpha - m\beta)\, sZ + s(\beta X - \alpha Y) = 0;$$

or eliminating s from this equation and the two equations

$$x - X + Z(x - \alpha)\, s = 0,$$
$$y - Y + Z(y - \beta)\, s = 0,$$

we have

$$\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\}(x - X) - Z(x - \alpha)(nX - mY) = 0,$$
$$\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\}(y - Y) - Z(y - \beta)(nX - mY) = 0;$$

these give

$$\begin{aligned}\Omega x &= X\{(n\alpha - m\beta)\, Z + \beta X - \alpha Y\} - \alpha Z(nX - mY)\\ &= -mZ\beta X + X(\beta X - \alpha Y) + mZ\alpha Y\\ &= (X - mZ)(\beta X - \alpha Y),\end{aligned}$$

and

$$\begin{aligned}\Omega y &= Y\{(n\alpha - m\beta)Z + \beta X - \alpha Y\} - \beta Z(nX - mY)\\ &= nZ\alpha Y + Y(\beta X - \alpha Y) - nZ\beta X\\ &= (Y - nZ)(\beta X - \alpha Y),\end{aligned}$$

where

$$\begin{aligned}\Omega &= (n\alpha - m\beta)Z + (\beta X - \alpha Y) - Z(nX - mY)\\ &= \beta(X - mZ) - \alpha(Y - nZ) - Z\{n(X - mZ) - m(Y - nZ)\}\\ &= (\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ);\end{aligned}$$

so that

$$x = \frac{(X - mZ)(\beta X - \alpha Y)}{(\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ)},$$

$$y = \frac{(Y - nZ)(\beta X - \alpha Y)}{(\beta - nZ)(X - mZ) - (\alpha - mZ)(Y - nZ)},$$

which equations give the coordinates (x, y) of the point in which the generating line through the point (X, Y, Z) of the surface meets the cubic

$$(\alpha^3 + \beta^3)xy - (x^3 + y^3)\alpha\beta = 0.$$

Substituting these values of (x, y) in the equation of the cubic, we obtain the equation

$$\begin{aligned}(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)\{(\beta - nZ)(x - mZ) - (\alpha - mZ)(Y - nZ)\}\\ - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\} = 0;\end{aligned}$$

or, as it may be written,

$$\begin{aligned}&(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)\{\beta(X - mZ) - \alpha(Y - nZ)\}\\ &\quad + (\alpha^3 + \beta^3)(X - mZ)(Y - nZ)Z(mY - nZ)\\ &\quad - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\} = 0.\end{aligned}$$

This equation contains, however, the extraneous factor

$$\beta(X - mZ) - \alpha(Y - nZ),$$

which, equated to zero, gives the equation of the plane through the node and the line $(x - mz = 0,\ y - nz = 0)$. In fact, assuming

$$\begin{aligned}(\alpha^3 + \beta^3)(X - mZ)(Y - nZ)Z(mY - nZ) - \alpha\beta(\beta X - \alpha Y)\{(X - mZ)^3 + (Y - nZ)^3\}\\ = \{\beta(X - mZ) - \alpha(Y - nZ)\}\Phi(X, Y, Z),\end{aligned}$$

it will presently be shown that Φ is an integral function. Hence, omitting the factor in question, we have

$$(\alpha^3 + \beta^3)(X - mZ)(Y - nZ) + \Phi(X, Y, Z) = 0,$$

which is the equation of the surface. It only remains to find Φ: writing for this purpose $X+mZ$, $Y+nZ$ in the place of X, Y, respectively, and putting for a moment

$$\Phi(X+mZ,\ Y+nZ,\ Z)=\Phi',$$

we have

$$(\alpha^3+\beta^3)XYZ(mY-nZ)-\alpha\beta\{\beta(X+mZ)-\alpha(Y+nZ)\}(X^3+Y^3)=(\beta X-\alpha Y)\Phi';$$

that is

$$(\beta X-\alpha Y)\Phi'=Z\{(\alpha^3+\beta^3)XY(mY-nZ)-(X^3+Y^3)\alpha\beta(m\beta-n\alpha)\}-(\beta X-\alpha Y)\alpha\beta(X^3+Y^3);$$

or, effecting the division,

$$\Phi'=Z\{(X^2\alpha-Y^2\beta)(\alpha n-\beta m)-XY(\alpha^2m+\beta^2n)\}-\alpha\beta(X^3+Y^3),$$

and then writing $X-mZ$, $Y-nZ$ in the place of X, Y respectively, we have

$$\Phi(X,\ Y,\ Z)=Z\{((X-mZ)^2\alpha-(Y-nZ)^2\beta)(\alpha n-\beta m)$$
$$-(X-mZ)(Y-nZ)(\alpha^2m+\beta^2n)\}-\alpha\beta\{(X-mZ)^3+(Y-nZ)^3\}.$$

Hence, finally, the equation of the surface is

$$(\alpha^3+\beta^3)(X-mZ)(Y-nZ)-\alpha\beta\{(X-mZ)^3+(Y-nZ)^3\}$$
$$+Z\{((X-mZ)^2\alpha-(Y-nZ)^2\beta)(\alpha n-\beta m)-(X-mZ)(Y-nZ)(\alpha^2m+\beta^2n)\}=0,$$

which is, as it should be, of the third order.

Arranging in powers of Z and reducing, the equation is found to be

$$(\alpha^3+\beta^3)XY-\alpha\beta(X^3+Y^3)$$
$$+Z\{-(\alpha^3+\beta^3)(mY+nX)+(X^2\alpha+Y^2\beta)(m\beta+n\alpha)+\alpha\beta(mX^2+nY^2)-(\alpha^2m+\beta^2n)XY\}$$
$$+Z^2\{mn(\alpha^3+\beta^3-\alpha^2X-\beta^2Y)(\beta n^2-\alpha m^2)(\beta X-\alpha Y)\}=0.$$

The first form puts in evidence the nodal line

$$(X-mZ=0,\quad Y-nZ=0),$$

and the second form puts in evidence the simple line

$$(X-\alpha=0,\quad Y-\beta=0).$$

But to obtain a more convenient form, write for a moment $X-mZ=P$, $Y-nZ=Q$; the equation is

$$(\alpha^3+\beta^3)PQ-\alpha\beta(P^3+Q^3)+Z\{(P^2\alpha-Q^2\beta)(n\alpha-m\beta)-PQ(m\alpha^2+n\beta^2)\}=0,$$

or, as this may be written,

$$(\alpha^3+\beta^3)PQ+(\alpha^2P-\beta^2Q)Z(Pn-Qm)+\alpha\beta\{-P^3-Q^3-Z(mP^2+nQ^2)\}=0;$$

or, observing that $X = P + mZ$, $Y = Q + nZ$, and thence

$$PY - QX = Z(Pn - Qm), \quad XP^2 + QY^2 = P^3 + Q^3 + Z(mP^2 + nQ^2),$$

the equation becomes

$$(\alpha^3 + \beta^3)PQ + (\alpha^2 P - \beta^2 Q)(PY - QX) - \alpha\beta(P^2 X + Q^2 Y) = 0,$$

or, what is the same thing,

$$(\alpha P^2 - \beta Q^2)(\alpha Y - \beta X) + PQ(\alpha^2 + \beta^2 - \alpha^2 X - \beta^2 Y) = 0\,;$$

whence, making a slight change in the form, and restoring for P, Q their values, the equation is

$$\{\alpha(X - mZ)^2 - \beta(Y - nZ)^2\}\{\alpha(Y - \beta) - \beta(X - \alpha)\}$$
$$- (X - mZ)(Y - nZ)\{\alpha^2(X - \alpha) + \beta^2(Y - \beta)\} = 0,$$

a form which puts in evidence as well the simple line $(X - \alpha = 0,\ Y - \beta = 0)$ as the nodal line $(X - mZ = 0,\ Y - nZ = 0)$.

If $Z = 0$, we have

$$(\alpha X^2 - \beta Y^2)(\alpha Y - \beta X) - XY\{\alpha^2(X - \alpha) + \beta^2(Y - \beta)\} = 0,$$

which is in fact the cubic curve $(\alpha^3 + \beta^3)XY - \alpha\beta(X^3 + Y^3) = 0$.

Reverting to a former system of equations

$$nx - my - Bx + Ay = 0,$$
$$B(x - \alpha) - A(y - \beta) = 0,$$

or, as these may be written,

$$Bx - Ay = nx - my,$$
$$B\alpha - A\beta = nx - my,$$

we find

$$B(\beta x - \alpha y) = (\beta - y)(nx - my),$$
$$A(\beta x - \alpha y) = (\alpha - x)(nx - my)\,;$$

so that we have

$$X = x + \frac{(\alpha - x)(nx - my)}{\beta x - \alpha y} Z,$$

$$Y = y + \frac{(\beta - y)(nx - my)}{\beta x - \alpha y} Z,$$

as the equations of the generating line which passes through the point (x, y) of the cubic curve.

2, *Stone Buildings, W.C., October* 28, 1862.

323.

ON A TACTICAL THEOREM RELATING TO THE TRIADS OF FIFTEEN THINGS.

[From the *Philosophical Magazine*, vol. XXV. (1863), pp. 59—61.]

THE school-girl problem may be stated as follows:—"With 15 things to form 35 triads, involving all the 105 duads, and such that they can be divided into 7 systems, each of 5 triads containing all the 15 things." A more simple problem is, "With 15 things, to form 35 triads involving all the 105 duads."

In the solution which I formerly gave of the school-girl problem (*Phil. Mag.* vol. XXXVII. 1850, [82]), and which may be presented in the form

	a	*b*	*c*	*d*	*e*	*f*	*g*
abc				35	17	82	64
ade		62	84			15	37
afg		13	57	86	42		
bdf	47		16		38		25
bge	58		23	14		67	
cdg	12	78			56	34	
cef	36	45		27			18

(viz. the things being *a*, *b*, *c*, *d*, *e*, *f*, *g*, 1, 2, 3, 4, 5, 6, 7, 8, the first pentad of triads is *abc*, *d* 35, *e* 17, *f* 82, *g* 64, and so for all the seven pentads of triads), there is obviously a division of the 15 things into (7 + 8) things, viz. the 35 triads are composed 7 of

them each of 3 out of the 7 things, and the remaining 28 each of 1 out of the 7 things, and 2 out of the 8 things: or attending only to the 8 things, there are 28 triads each of them containing a duad of the 8 things, but there is no triad consisting of 3 of the 8 things. More briefly, we may say that in the system there is an 8 without 3, that is, there are 8 things such that no triad of them occurs in the system.

I believe, but am not sure, that in all the solutions which have been given of the school-girl problem there is an 8 without 3.

Now, considering the more simple problem, there are of course solutions which have an 8 without 3 (since every solution of the school-girl problem is a solution of the more simple problem): but it is very easy to show that there is no solution which has a 9 without 3. I wish to show that there is in every solution at least a 6 without 3. This being so, there will be (if they all exist) 3 classes of solutions, viz. those which have at most (1) a 6 without 3, (2) a 7 without 3, (3) an 8 without 3. I believe that the first and second classes exist, as well as the third, which is known to do so.

The proposition to be proved is, that given any system of 35 triads involving all the duads of 15 things; there are always 6 things which are a 6 without 3, that is, they are such that no triad of the 6 things is a triad of the system. This will be the case if it is shown that the number of *distinct* hexads which can be formed each of them containing a triad of the system is less than $\left(\frac{15.14.13.12.11.10}{1.2.3.4.5.6} = 5.7.11.13 =\right) 5005$, the entire number of the hexads of 15 things. Now joining to any triad of the system a triad formed out of the remaining 12 things (there are $\frac{12.11.10}{1.2.3} = 4.5.11 = 220$ such triads), we obtain in all $(220 \times 35 =) 7700$ hexads, each of them containing a triad of the system. But these 7700 hexads are not all of them distinct. For, first, considering any triad of the system, there are in the system 16 other triads, each of them having no thing in common with the first-mentioned triad. (In fact if e.g. 123 is a triad of the system, then the system, since it contains all the duads, must have besides 6 triads containing 1, 6 triads containing 2, and 6 triads containing 3, and therefore $35 - 1 - 6 - 6 - 6 = 16$ triads not containing 1, 2, or 3.) Hence we have $\left(\frac{35.16}{2} =\right) 280$ hexads, each of them composed of two triads of the system; and since each of these hexads can be derived from either of its two component triads, these 280 hexads present themselves twice over among the 7700 hexads.

Secondly, there are in the system seven triads containing each of them the same one thing, e.g.

$$123,\quad 145,\quad 167,\quad 189,\quad 1.10.11,\quad 1.12.13,\quad 1.14.15,$$

containing each of them the thing 1. That is, we have $\left(\frac{7.6}{2} =\right) 21$ pairs such as 123, 145 containing the thing 1, and therefore $(15 \times 21 =) 315$ pairs such as $\alpha\beta\gamma$, $\alpha\delta\epsilon$.

And for any such pair, combining with $\alpha\beta\gamma\delta\epsilon$ any one of the remaining 10 things, we have 10 hexads $\alpha\beta\gamma\delta\epsilon\zeta$, each of them derivable from either of the triads $\alpha\beta\gamma$, $\alpha\delta\epsilon$; that is, we have $(315 \times 10 =) 3150$ hexads which present themselves twice over among the 7700 hexads. The hexads not belonging to one or other of the foregoing classes are derived each of them from a single triad only of the system, and they present themselves once among the 7700 hexads. This number is consequently made up as follows, viz.

280	hexads each	twice	=	560
3150	,,	,,	=	6300
840	,,	once	=	840
4270				7700

or there are in all 4270 distinct hexads; and since this is less than 5005, it follows that there are hexads not containing any triad of the system: there must in fact be $(5005 - 4270 =)$ 735 such hexads. The theorem in question is thus shown to be true.

2, *Stone Buildings, W.C., November* 24, 1862.

324.

NOTE ON A THEOREM RELATING TO SURFACES.

[From the *Philosophical Magazine*, vol. xxv. (1863), pp. 61, 62.]

THE following apparently self-evident geometrical theorem requires, I think, a proof; viz. the theorem is—"If every plane section of a surface of the order $m+n$ break up into two curves of the orders m and n respectively, then the surface breaks up into two surfaces of the orders m, n respectively."

To fix the ideas, suppose $n=2$. Imagine any line meeting the surface in $m+2$ points, the section includes a conic which meets the line in two of the $m+2$ points, say the points A, A'[1]. Suppose that the plane revolves round the line AA', the section will always include a conic which passes through *these same two points* A, A'; and it is to be shown that the sheet, the locus of this conic, is a surface of the second order. In fact the conic in question, say APA', by its intersection with an arbitrary plane traces out a branch of the intersection of the given surface with the arbitrary plane. And if $ABA'B'$ be the conic in any particular plane through A, A', and if the arbitrary plane meet this conic in the points B, B', then the branch passes through these points B, B'. Imagine the plane $ABA'B'$ revolving round BB' until it coincides with the arbitrary plane; the section includes a conic passing through the points B, B', and the before-mentioned branch is this conic; that is, the conic APA' by its intersection with an arbitrary plane traces out a conic; or, what is the same thing, the sheet, the locus of the conic APA', is met by an arbitrary plane in a conic, that is, the sheet is a surface of the second order; and the given surface thus includes a surface of the second order, and is therefore made up of two surfaces of the orders m and 2 respectively. The demonstration seems to me to add at least

[1] The figure referred to will be at once understood by considering A, A' as the poles of an ellipsoid, or say of a sphere, $ABA'B'$ the meridian of projection, APA' any other meridian, BPB' the equator or any other great circle.

something to the evidence of the theorem asserted, but I should be glad if a more simple one could be found. Analytically, the theorem is—"If

$$(x, y, z, \alpha x + \beta y + \gamma z)^{m+n},$$

where (α, β, γ) are arbitrary, break up into factors $(x, y, z)^m$, $(x, y, z)^n$, rational in regard to (x, y, z), then $(x, y, z, w)^{m+n}$ breaks up into factors $(x, y, z, w)^m$, $(x, y, z, w)^n$, rational in regard to (x, y, z, w)." It would at first sight appear that (α, β, γ) being arbitrary, these quantities can only enter into the factors of $(x, y, z, \alpha x + \beta y + \gamma z)^{m+n}$ through the quantity $\alpha x + \beta y + \gamma z$; that is, that the factors in question, considered as functions of $(x, y, z, \alpha, \beta, \gamma)$, are of the form

$$(x, y, z, \alpha x + \beta y + \gamma z)^m, \quad (x, y, z, \alpha x + \beta y + \gamma z)^n;$$

and then replacing the arbitrary quantity $\alpha x + \beta y + \gamma z$ by w, the factors of $(x, y, z, w)^{m+n}$ will be $(x, y, z, w)^m$, $(x, y, z, w)^n$. But the objection proves too much; for in a similar way it would follow that if $(x, y, \alpha x + \beta y)^{m+n}$, where α, β are arbitrary, breaks up into the factors $(x, y)^m$, $(x, y)^n$, rational in regard to (x, y) (and quà homogeneous function of two variables *it always does so break up*), then $(x, y, z)^{m+n}$ would in like manner break up into the factors $(x, y, z)^m$, $(x, y, z)^n$, rational in regard to (x, y, z): and a simple example will show that it is not true that the factors of $(x, y, \alpha x + \beta y)^{m+n}$ only contain (α, β) through $\alpha x + \beta y$; in fact, if the function be $= x^2 + y^2 + (\alpha x + \beta y)^2$, then the factor is

$$\frac{1}{\sqrt{\alpha^2 + 1}} \{(\alpha^2 + 1)\, x + (\alpha\beta + i\sqrt{\alpha^2 + \beta^2 + 1})\, y\},$$

which cannot be exhibited as a function of α, β, $\alpha x + \beta y$.

I am not acquainted with any analytical demonstration; the geometrical one cannot easily be exhibited in an analytical form.

2, *Stone Buildings, W.C., November* 26, 1862.

325.

NOTE ON A THEOREM RELATING TO A TRIANGLE, LINE, CONIC.

[From the *Philosophical Magazine*, vol. XXV. (1863), pp. 181—183.]

I FIND, among my papers headed "Generalization of a Theorem of Steiner's," an investigation leading to the following theorem, viz.:

Consider a triangle, a line, and a conic; with each vertex of the triangle join the point of intersection of the line with the polar of the same vertex in regard to the conic; in order that the three joining lines may meet in a point, the line must be a tangent to a curve of the third class; if, however, the conic break up into a pair of lines, or in a certain other case, the curve of the third class will break up into a point, and a conic inscribed in the triangle.

Let the equations of the sides of the triangle be

$$x=0,\quad y=0,\quad z=0,$$

the equation of the conic

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$

and that of the line

$$\lambda x+\mu y+\nu z=0\,;$$

then the polar of the vertex $(y=0,\ z=0)$ has for its equation

$$ax+hy+gz=0\,;$$

it therefore meets the line $\lambda x+\mu y+\nu z=0$ in the point

$$x:y:z=h\nu-g\mu:g\lambda-a\nu:a\mu-h\lambda,$$

and the equation of the line joining this point with the vertex $(y=0,\ z=0)$ is $(a\mu - h\lambda)\,y = (g\lambda - a\nu)\,z$. The equations of the three joining lines therefore are

$$\begin{aligned}(a\mu - h\lambda)\,y &= (g\lambda - a\nu)\,z,\\ (b\nu - f\mu)\,z &= (h\mu - b\lambda)\,x,\\ (c\lambda - g\nu)\,x &= (f\nu - c\mu)\,y,\end{aligned}$$

lines which will meet in a point if

$$(a\mu - h\lambda)\,(b\nu - f\mu)\,(c\lambda - g\nu) - (g\lambda - a\nu)\,(h\mu - b\lambda)\,(f\nu - c\mu) = 0,$$

or, multiplying out and putting as usual

$$\begin{aligned}K &= abc - af^2 - bg^2 - ch^2 + 2fgh,\\ \mathfrak{A} &= bc - f^2,\ \&c.,\end{aligned}$$

if

$$\left.\begin{aligned}&2\,(abc - fgh)\,\lambda\mu\nu\\ &+ a\mathfrak{G}\mu\nu^2 + a\mathfrak{H}\mu^2\nu\\ &+ b\mathfrak{H}\nu\lambda^2 + b\mathfrak{F}\nu^2\lambda\\ &+ c\mathfrak{F}\lambda\mu^2 + c\mathfrak{G}\lambda^2\mu\end{aligned}\right\} = 0,$$

that is, the line must touch a curve of the third class.

If this equation break up into factors, the form must be

$$(\alpha\lambda + \beta\mu + \gamma\nu)\,(A\mu\nu + B\nu\lambda + C\lambda\mu) = 0\,;$$

that is, we must have

$$\begin{aligned}&A\alpha + B\beta + C\gamma = 2\,(abc - fgh),\\ &B\alpha = b\mathfrak{H}\,,\quad C\alpha = c\mathfrak{G}\,,\\ &C\beta = c\mathfrak{F}\,,\quad A\beta = a\mathfrak{H},\\ &A\gamma = a\mathfrak{G},\quad B\gamma = b\mathfrak{F}\,;\end{aligned}$$

and the last six equations give without difficulty

$$\begin{aligned}A &= \frac{ka}{\mathfrak{F}}\,, & \alpha &= \frac{1}{k}\,\mathfrak{G}\mathfrak{H},\\ B &= \frac{kb}{\mathfrak{G}}\,, & \beta &= \frac{1}{k}\,\mathfrak{H}\mathfrak{F},\\ C &= \frac{kc}{\mathfrak{H}}\,, & \gamma &= \frac{1}{k}\,\mathfrak{F}\mathfrak{G},\end{aligned}$$

where k is arbitrary; the first equation then gives

$$\frac{a\mathfrak{G}\mathfrak{H}}{\mathfrak{F}} + \frac{b\mathfrak{H}\mathfrak{F}}{\mathfrak{G}} + \frac{c\mathfrak{F}\mathfrak{G}}{\mathfrak{H}} = 2\,(abc - fgh)\,;$$

or, reducing by the equations $\mathfrak{G}\mathfrak{H} = \mathfrak{A}\mathfrak{F} + aK$, &c., this is

$$\mathfrak{A}a = \mathfrak{B}b + \mathfrak{C}c - 2abc + 2fgh + \left(\frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}}\right)K = 0;$$

which, substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ their values, becomes

$$K\left(1 + \frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}}\right) = 0.$$

Hence if $K = 0$, that is, if the conic break up into a pair of lines, or if

$$1 + \frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}} = 0,$$

in either case the equation of the curve of the third class becomes

$$\left(\frac{\lambda}{\mathfrak{F}} + \frac{\mu}{\mathfrak{G}} + \frac{\nu}{\mathfrak{H}}\right)\left(\frac{a}{\mathfrak{F}}\mu\nu + \frac{b}{\mathfrak{G}}\nu\lambda + \frac{c}{\mathfrak{H}}\lambda\mu\right) = 0;$$

that is, the curve breaks up into a point, and a conic inscribed in the triangle.

In the case where the conic breaks up into a pair of lines, then we have

$$(a, b, c, f, g, h \between x, y, z)^2 = 2(px + qy + rz)(p'x + q'y + r'z),$$

and thence

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between x, y, z)^2 = -\{(qr' - q'r)x + (rp' - r'p)y + (pq' - p'q)z\}^2;$$

so that the equation in (λ, μ, ν) is

$$\{(qr' - q'r)\lambda + (rp' - r'p)\mu + (pq' - p'q)\nu\}$$
$$\{pp'(qr' - q'r)\mu\nu + qq'(rp' - r'p)\nu\lambda + rr'(pq' - p'q)\lambda\mu\} = 0;$$

where the point represented by the equation

$$(qr' - q'r)\lambda + (rp' - r'p)\mu + (pq' - p'q)\nu = 0$$

is, of course, the intersection of the two lines.

326.

THEOREMS RELATING TO THE CANONIC ROOTS OF A BINARY QUANTIC OF AN ODD ORDER.

[From the *Philosophical Magazine*, vol. XXV. (1863), pp. 206—208.]

I CALL to mind Professor Sylvester's theory of the canonical form of a binary quantic of an odd order; viz., the quantic of the order $2n+1$ may be expressed as a sum of a number $n+1$ of $(2n+1)$th powers, the roots of which, or say the *canonic roots* of the quantic, are to constant multipliers *près* the factors of a certain covariant derivative of the order $(n+1)$, called the *Canonizant*. If, to fix the ideas, we take a quintic function, then we may write

$$(a, b, c, d, e, f \between x, y)^5 = A(lx+my)^5 + A'(l'x+m'y)^5 + A''(l''x+m''y)^5$$

(it would be allowable to put the coefficients A each equal to unity; but there is a convenience in retaining them, and in considering that a canonic root $lx+my$ is only given as regards the ratio $l : m$, the coefficients l, m remaining indeterminate); and then the canonic roots $(lx+my)$, &c. are the factors of the Canonizant

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}.$$

It is to be observed that this reduction of the quantic to its canonical form, i.e. to a sum of a number $n+1$ of $(2n+1)$th powers, is a *unique* one, and that the quantic cannot be in any other manner a sum of a number $n+1$ of $(2n+1)$th powers.

Professor Sylvester communicated to me, under a slightly less general form, and has permitted me to publish the following theorems:

1. If the second emanant $(X\partial_x + Y\partial_y)^2 U$ has in common with the quantic U a single canonic root, then all the canonic roots of the emanant are canonic roots of the quantic; and, moreover, if the remaining canonic root of the quantic be $rx+sy$, then (X, Y), the facients of emanation, are $=(s, -r)$, or, what is the same thing, they are given by the equation

$$\text{canont. } U \ (X, \ Y \text{ in place of } x, \ y) = 0.$$

In fact, considering, as before, the quintic $U=(a, b, c, d, e, f \between x, y)^5$, we have

$$U = A\,(lx+my)^5 + A'\,(l'x+m'y)^5 + A''\,(l''x+m''y)^5,$$

and thence

$$(X\partial_x + Y\partial_y)^2 U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3 + B''\,(l''x+m''y)^3,$$

if for shortness

$$B = 6\,.\,5\,(lX+mY)^2 A, \ \&c.$$

Suppose $(X\partial_x + Y\partial_y)^2 U$ has in common with U the canonic root $lx+my$, then

$$(X\partial_x + Y\partial_y)^2 U = C\,(lx+my)^3 + C'\,(px+qy)^3,$$

and thence

$$B'\,(l'x+m'y)^3 + B''\,(l''x+m''y)^3 = (C-B)\,(lx+my)^3 + C'\,(px+qy)^3,$$

which must be an identity; for otherwise we should have the same cubic function expressed in two different canonical forms. And we may write

$$B' = C', \quad l'x+m'y = px+qy, \quad B''=0, \quad C=B,$$

and then we have

$$(X\partial_x + Y\partial_y)^2 U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3;$$

so that all the canonic roots of the emanant are canonic roots of the quantic. Moreover, the condition $B''=0$ gives $l''X+m''Y=0$, that is, $X : Y = m'' : -l''$, or writing $rx+sy$ instead of $l''x+m''y$, $X : Y = s : -r$; and the system is

$$U = A\,(lx+my)^5 + A'\,(l'x+m'y)^5 + A\,(rx+sy)^5,$$

$$(s\partial_x - r\partial_y)^2 U = B\,(lx+my)^3 + B'\,(l'x+m'y)^3,$$

which proves the theorem.

2. The two functions, canont. U, canont. $(X\partial_x + Y\partial_y)^2 U$, have for their resultant $\{\text{canont. } U \ (X, \ Y \text{ in place of } x, \ y)\}^{2n}$, if $2n+1$ be the order of U.

In fact, in order that the equations

$$\text{canont. } U = 0, \quad \text{canont. } (X\partial_x + Y\partial_y)^2 U = 0,$$

may coexist, their resultant must vanish; and conversely, when the resultant vanishes, the equations will have a common root. Now if the equation canont. $(X\partial_x + Y\partial_y)^2 U = 0$ has a common root with the equation canont. $U = 0$, all its roots are roots of canont. $U = 0$; and, moreover, if $rx + sy = 0$ be the remaining root of canont. $U = 0$, then $X : Y = s : -r$, that is, we have

$$\text{canont. } U \ (X,\ Y \text{ in place of } x,\ y) = 0;$$

or the resultant in question can only vanish if the last-mentioned equation is satisfied. It follows that the resultant must be a power of the *nilfactum* of the equation; and observing that canont. U is of the form $(a, \ldots)^{n+1}(x,\ y)^{n+1}$, i.e. that it is of the degree $n+1$ as well in regard to the coefficients as in regard to the variables $(x,\ y)$, it is easy to see that the resultant is of the degree $2n(n+1)$ as well in regard to the coefficients as in regard to $(X,\ Y)$; that is, we have $2n$ as the index of the power in question.

3. In particular, if $Y = 0$, the theorem is that the resultant of the functions canont. U, canont. $\partial_x^2 U$ is equal to the $2n$th power of the first coefficient of canont. U.

Thus for $n = 1$, that is, for the cubic function $(a,\ b,\ c,\ d \between x,\ y)^3$, we have

$$\text{canont. } U = \begin{vmatrix} y^2, & -xy, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix} = (ac - b^2,\quad ad - bc,\quad bd - c^2 \between x,\ y)^2,$$

$$\text{canont. } \partial_x^2 U = \begin{vmatrix} y, & -x \\ a, & b \end{vmatrix} \qquad = ax + by;$$

and the resultant of the two functions is

$$\begin{aligned} &= (ac - b^2,\quad ad - bc,\quad bd - c^2 \between b,\ -a)^2 \\ &= -(ac - b^2)^2, \end{aligned}$$

which verifies the theorem.

The theorems were, in fact, given to me in relation to the quantic U and the second differential coefficient $\partial_x^2 U$; but the introduction instead thereof of the second emanant $(X\partial_x + Y\partial_y)^2 U$ presented no difficulty.

2, *Stone Buildings, W.C., February* 16, 1863.

327.

ON THE STEREOGRAPHIC PROJECTION OF THE SPHERICAL CONIC.

[From the *Philosophical Magazine*, vol. XXV. (1863), pp. 350—353.]

IN order to the tolerable delineation of some figures relating to spherical geometry, I had occasion to consider the stereographic projection of the spherical conic. To fix the ideas, imagine a sphere having its centre in the plane of the paper, and through the centre three rectangular axes, that of x horizontal and that of y vertical, in the plane of the paper, and the axis of z perpendicular to and in front of the plane of the paper. The radius of the sphere is taken equal to unity (so that its intersection by the plane of the paper is the circle radius unity), and the points X, Y, and Z are taken to denote the points where the axes, drawn in the positive direction, meet the surface of the sphere; and the opposite points are called X', Y', and Z'. The eye is supposed to be at Z, and the projection to be made on the plane of the paper. This being so, and supposing that the axes of coordinates are the principal axes of the spherical conic, *the axis of x being the interior axis*, and taking ξ, η, ζ as the coordinates of a point on the spherical conic, its equations are

$$\xi^2+\eta^2 \qquad +\zeta^2=1,$$
$$-\xi^2+\eta^2\cot^2\beta+\frac{\zeta^2}{c^2}=0\,;$$

where it may be remarked that $\tan\beta$, c are the semiaxes of the plane conic which is the gnomonic projection (i.e. the projection by lines through the centre of the sphere) of the spherical conic on the tangent plane at X or X'.

Taking, for a moment, x, y, z as the coordinates of a point on the projecting line (that is, the line through the eye to a point (ξ, η, ζ) on the spherical conic), the equation of this line is

$$\frac{x}{\xi}=\frac{y}{\eta}=\frac{z-1}{\zeta-1};$$

and thence putting $z=0$, x, y will be the coordinates of a point of the projection, and we have

$$\frac{x}{\xi}=\frac{y}{\eta}=\frac{1}{1-\zeta};$$

or, what is the same thing,

$$\xi=x(1-\zeta),\quad \eta=y(1-\zeta);$$

the equations of the spherical conic may be written

$$1-\zeta^2=\quad \xi^2+\eta^2,$$
$$\zeta^2=c^2(\xi^2-\eta^2\cot^2\beta);$$

and by eliminating ξ, η, ζ from the four equations, we obtain the equation of the conic.

Substituting for ξ and η their values, we find

$$1+\zeta=(x^2+y^2)(1-\zeta),$$
$$\zeta^2=c^2(x^2-y^2\cot^2\beta)(1-\zeta)^2;$$

or, observing that the first equation gives

$$\zeta=\frac{x^2+y^2-1}{x^2+y^2+1},$$

and that thence

$$1-\zeta=\frac{2}{x^2+y^2+1},\quad \frac{\zeta}{1-\zeta}=\tfrac{1}{2}(x^2+y^2-1),$$

the equation is

$$(x^2+y^2-1)^2=4c^2(x^2-y^2\cot^2\beta).$$

It is now very easy to trace the curve. We see first that the curve is symmetrical with respect to the axes, and that it meets the axis of y in four imaginary points, but the axis of x in four real points, the coordinates whereof are

$$x=\pm(\sqrt{1+c^2}\pm c),$$

so that the two points on the same side of the centre are the images one of the other in regard to the circle radius unity. Moreover the curve touches the lines

$$y=\pm x\tan\beta$$

at their intersections with the circle. By developing in regard to y, the equation becomes

$$y^4+2(x^2-1+2c^2\cot^2\beta)y^2+(x^2-1)^2-4c^2x^2=0;$$

and putting

$$x=\pm(\sqrt{1+c^2}\pm c),$$

the last term vanishes, and the equation gives $y^2 = 0$, or

$$\begin{aligned} y^2 &= 2(1 - x^2 - 2c^2 \cot^2 \beta) \\ &= 4(-c^2 \mp c\sqrt{1+c^2} - c^2 \cot^2 \beta) \\ &= 4c(-c \operatorname{cosec}^2 \beta \mp \sqrt{1+c^2}), \end{aligned}$$

the upper sign corresponding to the exterior values

$$\pm x = \sqrt{1+c^2} + c,$$

and the lower sign to the interior values

$$\pm x = \sqrt{1+c^2} - c.$$

In the former case the values of y are imaginary; in the latter case they are real if

$$\sqrt{1+c^2} > c \operatorname{cosec}^2 \beta,$$

or, what is the same thing, if

$$\sin^2 \beta > \frac{c}{\sqrt{1+c^2}};$$

that is, if (for a given value of c) β is sufficiently great, but otherwise they are imaginary.

If, as in the annexed figures, $c = \frac{5}{12}$ (and therefore $\sqrt{1+c^2} = \frac{13}{12}$, $\sqrt{1+c^2} + c = \frac{3}{2}$,

Fig. 1.

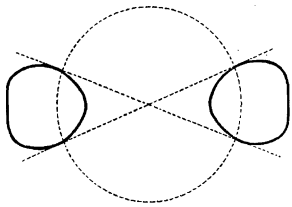
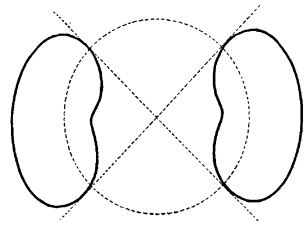


Fig. 2.



$\sqrt{1+c^2} - c = \frac{2}{3}$), then for the limiting value of β we have

$$\sin^2 \beta = \tfrac{5}{13} = \cdot 3846, \ \sin \beta = \cdot 62, \text{ or } \beta = 38^\circ \text{ nearly.}$$

In the first figure β is less, in the second figure greater than this value: the form for the limiting value is obvious from a comparison of the two figures.

I take the opportunity to mention the following theorem, which is perhaps known, but I have not met with it anywhere; viz. any three circles, each two of which meet,

may be considered as the stereographic projections of three great circles of the sphere. In fact suppose, as above, that the projection is made on the plane of a great circle, and calling this the principal circle, the projection of any other great circle meets the principal circle at the extremities of a diameter of the principal circle. It follows that the theorem will be true, if, given any three circles each two of which meet, a circle can be drawn meeting the given circles, each of them at the extremities of a diameter of the circle so to be drawn. It is easy to see that the required circle has for its centre the radical centre (point of intersection of the radical axes) of the given circles, and that the radius is the 'Inner Potency' of the point in question in regard to each of the three given circles. In particular the three circles having for centres the vertices of an equilateral triangle, and the side for radius, may be considered as the stereographic projections of three great circles of a sphere. This is a very ready mode of delineation of a spherical figure depending on three great circles of the sphere.

2, *Stone Buildings, W.C., March* 21, 1863.

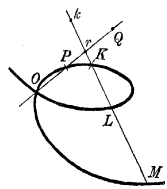
328.

ON THE DELINEATION OF A CUBIC SCROLL.

[From the *Philosophical Magazine*, vol. XXV. (1863), pp. 528—530.]

IMAGINE a cubic scroll (skew surface of the third order) generated by lines each of which meets two given directrix lines. One of these is a nodal (double) line on the surface, and I call it the nodal directrix; the other is a single line on the surface, and I call it the single directrix. The section by any plane is a cubic passing through the points in which the plane meets the directrix lines; i.e. the point on the nodal directrix is a node (double point) of the curve, the point on the single directrix a single point on the curve; the two directrix lines, and the cubic curve, the section by any plane, determine the scroll. Consider the sections by a series of parallel planes. Let one of these planes be called the basic plane, and the section by this plane the basic section or basic cubic; and imagine any other section projected on the basic plane by lines parallel to the nodal directrix: such section may be spoken of simply as 'the section,' and its projection as 'the cubic.' The cubic has a node at the node of the basic cubic; that is, the two curves have at this point *four* points in common. The two curves have, moreover, in common the *three* points at infinity (or, in other words, their asymptotes are parallel); in fact the points at infinity of either curve are the points in which the line at infinity, the intersection of the basic plane and the plane of the section, meets the scroll; and these points are therefore the same for each of the two curves. The remaining *two* points of intersection of the cubic with the basic cubic are also fixed points on the basic cubic, i.e. they are the points of intersection of the basic plane by the two generating lines parallel to the nodal directrix. Hence the cubic meets the basic cubic in *nine* fixed points, viz. the node counting as four points, the three points at infinity, and the two points the feet of the generators parallel to the nodal directrix. It follows that if $U=0$ is the equation of the basic cubic, $V=0$ the equation of some other cubic meeting the basic cubic in the nine points in question, then the equation of 'the cubic' is $U+\lambda V=0$, λ being a parameter the value of which varies according to the position

(in the series of parallel planes) of the plane of the section. Suppose that the basic cubic $U=0$ is given, and suppose for a moment that the cubic $V=0$ is also given, these two cubics having the above-mentioned relations, viz. they have a common node and parallel asymptotes: the cubic $U+\lambda V=0$ might be constructed by drawing through the node (say O) a radius vector meeting the cubics in P, P' respectively, and taking on this radius vector a point Q such that $PQ=\frac{\lambda}{1+\lambda}PP'$, or, what is the same thing, $OQ=\frac{OP+\lambda OP'}{1+\lambda}$; the locus of the point Q will then be the cubic $U+\lambda V=0$. And we may even suppose the cubic $V=0$ to break up into a line and a conic (hyperbola), and then (disregarding the line) use the hyperbola in the construction. In fact, if the hyperbola is determined by the following five conditions, viz. to pass through the node and through the feet of the two generators parallel to the nodal directrix, and to have its asymptotes parallel to two of the asymptotes of the basic cubic, and if the line be taken to be a line through the node parallel to the third asymptote of the basic cubic; then the hyperbola and line form together a cubic curve meeting the basic cubic in the nine points, and therefore satisfying the conditions assumed in regard to the cubic $V=0$. And it is to be noticed that as in general the cubic $V=0$ is the projection of some section of the scroll, so the hyperbola and line are the projection of a section of the scroll, viz. the section through one of the generating lines (there are three such lines) parallel to the basic plane. But it is better to construct 'the cubic' by a different method (using only the basic cubic $U=0$) which results more immediately from the geometrical theory. Taking the basic plane as the plane of the figure, let O be the node, or foot of the nodal directrix, K the foot of the single directrix, Kk the projection of the single directrix,



k being the projection of the point in which the single directrix meets the plane of the section. Drawing through O any radius vector meeting the basic cubic in P, and the line Kk in r, and producing it to a properly determined point Q, then $OPrQ$ will be the projection of the generating line which meets the nodal directrix, the basic cubic, the single directrix, and the section in the points the projections whereof are O, P, r, Q respectively: and the consideration of the solid figure shows easily that the condition for the determination of the point Q is

$$PQ=Kk\,.\,\frac{Pr}{rK}.$$

Hence, starting from the basic cubic and the line Kk, we have a construction for the point Q the locus whereof is the cubic, the projection of a section of the scroll; for the projections of the parallel sections, we have only to vary the length Kk. By what precedes, the construction gives for the locus of Q a cubic having a node at O, and having its asymptotes parallel to those of the basic cubic. As P moves up to K, the distances Pr, rK become indefinitely small; but their ratio is finite, hence the cubic, the locus of Q, does *not* pass through the point K. The construction shows, however, that it does pass through the points L, M, which are the other two intersections of Kk with the basic cubic; these points L, M are in fact the feet of the generators parallel to the nodal directrix.

The general conclusion is, that a series of cubics having each of them at one and the same given point a node—having their asymptotes parallel—and besides passing through the same two given points—may be considered as the projections of a series of parallel sections of a cubic scroll; and such a series of cubics will thus afford a delineation of the scroll.

2, *Stone Buildings, W.C., April* 15, 1863.

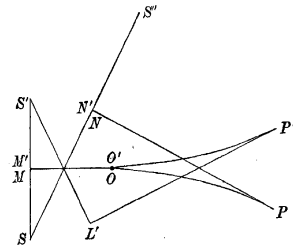
329.

NOTE ON THE PROBLEM OF PEDAL CURVES.

[From the *Philosophical Magazine*, vol. XXVI. (1863), pp. 20, 21.]

It is not, so far as I am aware, generally known that the problem of pedal curves (Steiner's *Fusspuncten-Curve*) was considered by Maclaurin in the *Geometria Organica*, 1720. He appears to have been led to it through an idea such as Sir W. R. Hamilton's *Hodograph*, or at any rate with a view to a dynamical application, for he remarks, p. 95, "Cum vero geometria quæ curvas ad datum centrum relatas contemplatur in philosophia naturali ad motus corporum et vires evolvendas facilius applicari possit,...hac sectione considerabimus curvas tanquam ad punctum quodvis datum relatas ex quo ad omnia circumferentiæ puncta radii undique educuntur, et simul perpendicula in illorum punctorum tangentes demittuntur, et rationem radii ad perpendiculum tanquam curvæ characterem usurpabimus." And accordingly, Props. IX. to XII., he considers the problem: Given a point S in the plane of a given curve, to find the locus of the intersection of a tangent of the curve by the perpendicular let fall upon it from the point S; with some special cases, and deductions from it. In particular if the given curve be a circle, the locus in question (or pedal curve) is a curve of the fourth order having a double point S; viz. if S be inside the circle, this is a conjugate or isolated point; but if outside, a double point with two real branches: if S be on the circle, then instead of the double point we have a cusp: it is shown that in each case the pedal curve is in fact an epicycloid. If the given curve be a parabola, then the locus or pedal curve is a curve of the third order, viz. a defective hyperbola having a double point at S, and with its single asymptote perpendicular to the axis of the parabola: some particular cases are specially noticed. If the curve be an ellipse or hyperbola, then, as in the case of the circle, the locus or pedal curve is a curve of the fourth order having a double point at S. And it is moreover shown, Prop. XII., that for any given curve whatever the locus or pedal curve is, in a generalized sense of the term, an epicycloid. This is in fact seen very easily by a mere inspection of the figure. Imagine the curve $O'P'$, rigidly connected with and

carrying along with it the point S', to roll on the similar and equal fixed curve OP symmetrically situate on the other side of the common tangent OM or OM'; then when P' coincides with P, the point S' is brought to S'', where $SNN'S''$ is the perpendicular from S on the tangent PN or PN', and $SN = N'S''$, that is, $SS'' = 2SN$;



and the curve generated by S'' (that is S'), or say the epicycloid the locus of S', is a curve similar to and similarly situate with the pedal curve the locus of N, but of twice the linear magnitude of the pedal curve. Or, what is the same thing, if instead of the given curve we consider a similar and similarly situated curve of twice the linear magnitude (the point S being the centre of similitude), then the epicycloid the locus of S' is the pedal curve of the substituted curve in relation to the point S. It may be added that, in accordance with a theorem of Dandelin's, if rays proceeding from the point S are reflected at the given curve, then the epicycloid (or pedal) in question is the *secondary caustic*, or an orthogonal trajectory of the reflected rays.

2, *Stone Buildings, W.C., June* 3, 1863.

330.

ON DIFFERENTIAL EQUATIONS AND UMBILICI.

[From the *Philosophical Magazine*, vol. XXVI. (1863), pp. 373—379 and 441—452.]

I.

CONSIDER the integral equation

$$Az^2 + 2Bz + C = 0,$$

where z is the constant of integration: the derived equation is

$$\begin{aligned}\Omega &= (AC' + A'C - 2BB')^2 - 4(AC - B^2)(A'C' - B'^2),\\ &= (CA' - C'A)^2 - 4(AB' - A'B)(BC' - B'C) \quad , = 0;\end{aligned}$$

and if for greater simplicity we write $A = 1$, then the derived equation is

$$\Omega = C'^2 - 4BC'B' + 4CB'^2 = 0,$$

corresponding to the integral equation

$$z^2 + 2Bz + C = 0.$$

Writing the integral equation under the form

$$(z + X)(z + Y) = 0,$$

we have

$$2B = X + Y, \quad C = XY,$$

whence also

$$2B' = X' + Y', \quad C' = XY' + X'Y,$$

and the derived equation becomes

$$\Omega = -(X - Y)^2 X'Y'.$$

Hence if we represent the roots X, Y in the form $P \pm Q\sqrt{\square}$, so that $P = -B$, $Q\sqrt{\square} = \sqrt{B^2 - AC}$, Q^2 being the greatest square factor of $B^2 - AC$, then

$$(X - Y)^2 = 4Q^2\square, \quad X', Y' = P' \pm \left(Q'\sqrt{\square} + \frac{Q\square'}{2\sqrt{\square}}\right),$$

$$X'Y' = P'^2 - \frac{1}{4\square}(2Q'\square + Q\square')^2;$$

and the derived equation is

$$\Omega = -Q^2\{4\square P'^2 - (2Q'\square + Q\square')^2\} = 0.$$

If B, C, &c. are functions of the coordinates (x, y), the equation $z^2 + 2Bz + C = 0$ (z an arbitrary constant) represents a series of curves in the plane of xy; but if we consider z as a coordinate, then the equation represents a surface, and the curves in question are the orthogonal projections on the plane of xy of the sections of the surface by the planes parallel to the plane of xy. To fix the ideas, the plane of xy may be taken to be horizontal, and the ordinates z vertical.

Writing the equation in the form

$$(z + B)^2 - (B^2 - C) = 0,$$

we see that the surface contains upon it the curve $z + B = 0$, $B^2 - C = 0$, which is the line of contact with the circumscribed vertical cylinder: such curve may be termed the envelope, or, when this is necessary, the complete envelope. The equation of the surface has however been taken to be $(z - P)^2 - Q^2\square = 0$ (viz. it has been assumed that $B = -P$, $B^2 - C = Q^2\square$); the envelope thus breaks up into the curve, $z - P = 0$, $Q = 0$, taken twice, and the curve $z - P = 0$, $\square = 0$; the former of these is in general a nodal curve on the surface, and it may be spoken of as the nodal curve; the latter of them is the reduced or proper envelope, or simply the envelope. And the terms nodal curve and envelope may also be applied to the curves $Q = 0$ and $\square = 0$, which are the projections on the plane of xy of the first-mentioned two curves respectively. There is however a case of higher singularity which it is proper to consider: suppose that Q and $\square$ have a common factor K, say $Q = KR$, $\square = K\nabla$; the complete envelope $Q^2\square = R^2K^3\nabla = 0$ here breaks up into the nodal curve $R = 0$ twice, the cuspidal curve $K = 0$ three times, and the reduced or proper envelope $\nabla = 0$ once.

Reverting for a moment to the form $(z + X)(z + Y) = 0$, the derived equation $\Omega = -(X - Y)^2 X'Y' = 0$ is satisfied by $(X - Y)^2 = 0$; this equation, or say the equation of the envelope, being in fact the singular solution of the differential equation. This assumes however that the differential equation is given in the form in which it is immediately obtained by derivation from the integral equation, without the rejection of factors which are functions of the coordinates (x, y) only; it is proper to consider the reduced equation obtained by rejecting such factors. Thus if X and Y are rational functions, the reduced form is $X'Y' = 0$, which is no longer satisfied by the equation

$(X-Y)^2=0$. In the before-mentioned case where the roots are $P \pm Q\sqrt{\square}$ (or $(X-Y)^2=Q^2\square$), P, Q, and $\square$ being rational functions of (x, y), the derived equation

$$\Omega = -Q^2\{4\square P'^2 - (2Q'\square + Q\square')^2\} = 0$$

divides out by the factor Q^2, but it does not divide out by $\square$; the reduced form is therefore

$$4\square P'^2 - (2Q'\square + Q\square')^2 = 0,$$

which is not satisfied by $Q=0$, while it is still satisfied by $\square=0$ (since this gives also $\square'=0$); that is, the nodal curve $Q=0$ is not a solution of the differential equation, but we still have the singular solution $\square=0$, which corresponds to the reduced or proper envelope. In the case $Q=KR$, $\square=K\nabla$ of a cuspidal curve, the above form of the differential equation becomes

$$4K\nabla P'^2 - \{3KK'R\nabla + K^2(2\nabla R' + \nabla' R)\}^2 = 0,$$

which divides out by K; and, when reduced by the rejection of this factor, it is no longer satisfied by the equation $K=0$, which belongs to the cuspidal curve; that is, neither the nodal curve $R=0$ nor the cuspidal curve $K=0$ is a solution of the differential equation, but we still have the singular solution $\nabla=0$, which corresponds to the reduced or proper envelope. It would appear that the conclusion may be extended to singularities of a higher nature, viz. the factor corresponding to any singular curve which presents itself as part of the complete envelope divides out from the derived equation; and such singular curve does not constitute a solution of the reduced equation, but we have a singular solution corresponding to the reduced or proper envelope.

II.

Consider the differential equation

$$y(p^2-1) + 2mxp = 0,$$

where, to fix the ideas, $m >$ or $= 1$; the integral equation may be taken to be

$$z = (mx + \sqrt{m^2x^2+y^2})(mx^2 + y^2 + x\sqrt{m^2x^2+y^2})^{m-1};$$

or rather, writing for shortness $\square = m^2x^2+y^2$, and putting

$$z = (mx + \sqrt{\square})(mx^2 + y^2 + x\sqrt{\square})^{m-1} = P + Q\sqrt{\square},$$

the integral equation is

$$(z-P)^2 - Q^2\square = 0, \text{ or } z^2 - 2Pz + P^2 - Q^2\square = 0,$$

where

$$P^2 - Q^2\square = (m^2x^2 - \square)\{(mx^2+y^2)^2 - x^2\square\}^{m-1} = -y^{2m}\{y^2 + (2m-1)x^2\}^{m-1}.$$

In the particular case $m=1$ the equation is

$$z = x + \sqrt{x^2+y^2}, \text{ or } z^2 - 2zx - y^2 = 0.$$

Before going further, I remark that, m being a positive integer greater than unity, we have

$$z = P + Q\sqrt{\square} = mx(mx^2+y^2)^{m-1} + \{mx^2+y^2+(m-1)mx^2\}(mx^2+y^2)^{m-2}\sqrt{\square} + \&c.,$$

the subsequent terms being divisible, the rational ones by $\square$, and the irrational ones by $\square\sqrt{\square}$. Hence, observing that

$$mx^2+y^2+(m-1)mx^2 = m^2x^2+y^2 = \square,$$

we see that Q contains the factor $\square$, and the equation $\square = 0$ belongs to a cuspidal curve on the surface. If however $m=1$, then the equation is $z = x + \sqrt{\square}$, so that $Q=1$ does not contain the factor $\square$; and $\square = x^2+y^2 = 0$ is not a singular curve on the surface, but is in fact the reduced or proper envelope.

The curve represented by the integral equation will pass through the origin ($x=0$, $y=0$) for the value $z=0$ of the constant of integration. In fact, for this value, the integral equation becomes

$$-y^{2m}\{y^2+(2m-1)x^2\}^{m-1} = 0,$$

which belongs to a set of $2m+(m-1)+(m-1)$ lines coinciding with the lines $y=0$, $y = ix\sqrt{2m-1}$, and $y = -ix\sqrt{2m-1}$ respectively. The directions at the origin are therefore $p=0$, $p = \pm i\sqrt{2m-1}$, which are the same values of p as are obtained from the differential equation; viz. since this is satisfied identically at the point in question, proceeding to the derived equation, we have

$$p(p^2-1)+2mp = 0,$$

that is

$$p(p^2+2m-1) = 0;$$

but it is to be observed that these values of p are different from the values given by the equation $\square = m^2x^2+y^2 = 0$, which are $p = \pm im$. The reason is that the curve $\square = 0$ being, as was shown, a cuspidal curve on the surface, the equation $\square = 0$ is not a solution of the differential equation.

If however $m=1$, then the integral equation gives at the origin no longer three values of p, but only the value $p=0$. The differential equation however gives, as in the general case, three values; viz. we have $p(p^2+1)=0$; and the values $p = \pm i$ obtained from the factor $p^2+1=0$ are precisely the values of p obtained from the equation $\square = x^2+y^2 = 0$, which in the case now under consideration belongs to the reduced or proper envelope of the surface, and *is* therefore the singular solution of the differential equation.

III.

The two curves of curvature which pass through any given point of a surface are distinct curves, not branches of one indecomposable curve. In fact if P, Q are the two curves of curvature for a point A, then for a point A' on P the two curves of

curvature will be P, Q'; and if P, Q were branches of an indecomposable curve, then P, Q' would also be branches of an indecomposable curve, and we should have P a branch of two different indecomposable curves, which is of course impossible. In the case of an umbilicus, the two curves P and Q coincide together; or, as we may express it, the curves of curvature through an umbilicus are the duplication of a single, in general indecomposable, curve; and in general this curve has at the umbilicus a *trifid* node. I use this expression to denote a point at which there are three distinct tangents, or, more accurately, three distinct directions of the curve: an ordinary triple point is of necessity a trifid node, but not conversely. The umbilicus of an ellipsoid or other quadric surface is a peculiar exceptional case.

In support of the foregoing conclusions, consider a surface having an umbilicus at the origin, and take $z=0$ as the equation of the tangent plane at that point; the equation of the surface in the neighbourhood of the umbilicus will be

$$z=\tfrac{1}{2}k(x^2+y^2)+\tfrac{1}{6}(ax^3+3bx^2y+3cxy^2+dy^3);$$

so that, writing as usual p and q for the first, and r, s, t for the second, differential coefficients of z, we have

$$\begin{aligned} p &= kx+\tfrac{1}{2}(ax^2+2bxy+cy^2),\\ q &= ky+\tfrac{1}{2}(bx^2+2cxy+dy^2),\\ r &= k+\quad ax+\quad by,\\ s &= \qquad\quad bx+\quad cy,\\ t &= k+\quad cx+\quad dy. \end{aligned}$$

The differential equation of the curves of curvature projected on the plane of xy is

$$\left(\frac{dy}{dx}\right)^2[(1+q^2)s-pqt]+\frac{dy}{dx}[(1+q^2)r-(1+p^2)t]-[(1+p^2)s-pqr]=0;$$

and substituting therein the foregoing values of p, q, r, s, t, but attending only to the terms of the lowest order in (x, y), and using moreover in the sequel p in the place of $\frac{dy}{dx}$, the equation becomes

$$(bx+cy)(p^2-1)+[(a-c)x+(b-d)y]p=0;$$

which may be taken as the differential equation of the curves of curvature at and in the neighbourhood of the umbilicus. The equation is satisfied identically by the values $x=0$, $y=0$, which correspond to the umbilicus; and to find p, we have to differentiate the equation, and then substitute these values of x and y; we thus obtain

$$(b+cp)(p^2-1)+[(a-c)+(b-d)p]p=0,$$

or, what is the same thing,

$$p(a+2bp+cp^2)-(b+2cp+dp^2)=0,$$

a cubic equation for the determination of p.

I remark that we may without loss of generality write $d=0$: but to simplify the investigation, I suppose in the first instance that we have also $b=0$; this comes to assuming that one of the three planes $ax^3+3bx^2y+3cxy^2+dy^3=0$ bisects the angle formed by the other two planes. The differential equation consequently is

$$cy(p^2-1)+(a-c)xp=0;$$

or, putting for shortness

$$\frac{a-c}{c}=-2m,$$

it is

$$y(p^2-1)+\quad 2mxp=0,$$

which is the differential equation previously considered. Hence, writing now h in the place of z, the equation of the curve of curvature in the neighbourhood of the umbilicus is

$$h=(mx+\sqrt{\square})(mx^2+y^2+\sqrt{\square})^{m-1},\ =P+Q\sqrt{\square},$$

where $\square=m^2x^2+y^2$; or, what is the same thing, the equation is

$$h^2-2Ph+P^2-Q^2\square=0;$$

and the equation (in the neighbourhood of the umbilicus) of the curve through the umbilicus is

$$P^2-Q^2\square=-y^{2m}\{y^2+(2m-1)x^2\}^{m-1}=0;$$

so that the umbilicus is a trifid node. In the case however of an ellipsoid or other quadric surface, we have $m=1$, so that the equation of the curve of curvature in the neighbourhood of the umbilicus is

$$h=x+\sqrt{x^2+y^2},$$

or, what is the same thing,

$$h^2-2hx-y^2=0:$$

and for the curve through the umbilicus, in the neighbourhood of the umbilicus, the equation is $y^2=0$, so that there is only a single direction of the curve of curvature. The differential equation gives, however, at the umbilicus $p(p^2+1)=0$; the value $p=0$ is that which corresponds to the curve of curvature; the other two values $p=\pm i$ correspond to the curve (pair of lines) $x^2+y^2=0$, which is the envelope of the curves of curvature, or, more accurately, the envelope of the projections of the curves of curvature on the tangent plane at the umbilicus.

Blackheath, October 17, 1863.

IV.

The differential equation for the curves of curvature in the neighbourhood of an umbilicus was obtained in a form such as

$$(bx+cy)(p^2-1)+2(fx+gy)p=0;$$

and it was only because this equation did not appear to be readily integrable, that I considered, instead of it, the particular form

$$y(p^2-1)+2mxp=0.$$

But the general equation can be integrated; and the result presents itself in a simple form. For, returning to the differential equation

$$(bx+cy)(p^2-1)+2(fx+gy)p=0,$$

and assuming

$$\frac{bx+cy}{fx+gy}=\frac{-2v}{v^2-1},$$

or

$$(bx+cy)(v^2-1)+2(fx+gy)v=0,$$

we have

$$\frac{p^2-1}{v^2-1}=\frac{p}{v}, \text{ or } (p-v)(vp+1)=0,$$

and we may write

$$p-v=0.$$

Assuming also

$$y=ux, \text{ or } u=\frac{y}{x},$$

the relation between u and v is

$$\frac{b+cu}{f+gu}=\frac{-2v}{v^2-1};$$

or, as this may be written,

$$v^2-1+2\frac{f+gu}{b+cu}v=0,$$

giving

$$v=\frac{-(f+gu)-\sqrt{(b+cu)^2+(f+gu)^2}}{b+cu},$$

where for convenience the radical has been taken with a negative sign. We have moreover

$$u=-\frac{b(v^2-1)+2fv}{c(v^2-1)+2gv}.$$

The equation $p-v=0$, substituting for y its value ux, then becomes

$$x\frac{du}{dx}+u-v=0;$$

or, as this may be written,

$$\frac{dx}{x}+\frac{du}{u-v}=0;$$

or, what is the same thing,

$$\frac{dx}{x}+\frac{dv-du}{v-u}-\frac{dv}{v-u}=0.$$

But

$$v-u=v+\frac{b(v^2-1)+2fv}{c(v^2-1)+2gv}=\frac{V}{c(v^2-1)+2gv},$$

where

$$\begin{aligned}V&=v[c(v^2-1)+2gv]+b^2(v-1)+2fv\\&=(b+cv)(v^2-1)+2(f+gv)v,\end{aligned}$$

and the differential equation takes thus the form

$$\frac{dx}{x}+\frac{dv-du}{v-u}-\frac{[c(v^2-1)+2gv]\,dv}{V}=0;$$

and hence, writing

$$V=(b+cv)(v^2-1)+2(f+gv)v=c(v-\alpha)(v-\beta)(v-\gamma),$$

and

$$\frac{c(v^2-1)+2gv}{V}=\frac{c(v^2-1)+2gv}{c(v-\alpha)(v-\beta)(v-\gamma)}=\frac{A}{v-\alpha}+\frac{B}{v-\beta}+\frac{C}{v-\gamma},$$

so that

$$A=\frac{c(\alpha^2-1)+2g\alpha}{c(\alpha^2-1)+2g\alpha+2\{f+(b+g)\alpha+c\alpha^2\}},$$

with the like values for B and C—values which are such that $A+B+C=1$,—the integral equation is

$$\text{const.}=x(v-u)(v-\alpha)^{-A}(v-\beta)^{-B}(v-\gamma)^{-C},$$

or, substituting for $v-u$ its value, $=\dfrac{c(v-\alpha)(v-\beta)(v-\gamma)}{c(v^2-1)+2gv}$,

$$\text{const.}=x\{c(v^2-1)+2gv\}^{-1}(v-\alpha)^{1-A}(v-\beta)^{1-B}(v-\gamma)^{1-C}.$$

But

$$v=\frac{-(f+gu)-\sqrt{U}}{b+cu};$$

if for shortness $U=(b+cu)^2+(f+gu)^2$, and thence

$$v^2=\frac{2(f+gu)^2+(b+cu)^2+2(f+gu)\sqrt{U}}{(b+cu)^2},$$

and

$$c(v^2-1)+2gv=\frac{2(cf-bg)(f+gu+\sqrt{U})}{(b+cu)^2},$$

$$v-\alpha=\frac{-(f+gu)-\sqrt{U}-\alpha(b+cu)}{b+cu},\ \&c.$$

Substituting these values, and observing that the exponent of $b+cu$ is

$$(-2+1-A+1-B+1-C, =1-A-B-C)=0,$$

the integral equation is

$$\text{const.}=x(f+gu+\sqrt{U})^{-1}\times$$

$$(f+gu+\alpha(b+cu)+\sqrt{U})^{1-A}(f+gu+\beta(b+cu)+\sqrt{U})^{1-B}(f+gu+\gamma(b+cu)+\sqrt{U})^{1-C};$$

or, observing that the exponent 1 of x is

$$=-1+(1-A)+(1-B)+(1-C),$$

and putting for shortness $\square=(fx+gy)^2+(bx+cy)^2$, the integral equation finally is

$$\text{const.}=(fx+gy+\sqrt{\square})^{-1}\times$$

$$(fx+gy+\alpha(bx+cy)+\sqrt{\square})^{1-A}(fx+gy+\beta(bx+cy)+\sqrt{\square})^{1-B}(fx+gy+\gamma(bx+cy)+\sqrt{\square})^{1-C},$$

where the quantities α, β, γ, A, B, C are given by

$$(b+cv)(v^2-1)+2(f+gv)=c(v-\alpha)(v-\beta)(v-\gamma),$$

$$\frac{c(v^2-1)+2gv}{c(v-\alpha)(v-\beta)(v-\gamma)}=\frac{A}{v-\alpha}+\frac{B}{v-\beta}+\frac{C}{v-\gamma}.$$

Consider the curve

$$0=(fx+gy+\alpha(bx+cy)+\sqrt{\square})^{1-A}(fx+gy+\beta(bx+cy)+\sqrt{\square})^{1-B}(fx+gy+\gamma(bx+cy)+\sqrt{\square})^{1-C},$$

which corresponds to the value $=0$ of the constant. If, for instance,

$$fx+gy+\alpha(bx+cy)+\sqrt{\square}=0,$$

this equation gives

$$(bx+cy)\{(bx+cy)(\alpha^2-1)+2(fx+gy)\alpha\}=0;$$

or say

$$(bx+cy)(\alpha^2-1)+2(fx+gy)\alpha=0.$$

But we have

$$(b+c\alpha)(\alpha^2-1)+2(f+g\alpha)\alpha=0,$$

and the equation therefore is

$$(bx+cy)(f+g\alpha)-(fx+gy)(b+c\alpha)=0;$$

that is

$$(cf-bg)(y-\alpha x)=0;$$

or simply $y-\alpha x=0$; that is, the directions of the curve at the origin, or point $x=0$, $y=0$, are given by the equations $y-\alpha x=0$, $y-\beta x=0$, $y-\gamma x=0$. This is right, since from the differential equation we obtain at the origin

$$(b+cp)(p^2-1)+2(f+gp)p, =c(p-\alpha)(p-\beta)(p-\gamma), =0.$$

V.

The particular case of the equation

$$y(p^2-1)+2mxp=0$$

is obtained from the general equation by writing therein $b=0$, $c=1$, $g=0$, $f=m$; we have therefore

$$v(v^2+2m-1)=(v-\alpha)(v-\beta)(v-\gamma),$$

or say

$$\alpha=0,\quad \beta=i\sqrt{2m-1},\quad \gamma=-i\sqrt{2m-1};$$

and thence

$$\frac{v^2-1}{v(v^2+2m-1)}=-\frac{1}{2m-1}\frac{1}{v}+\frac{2m}{2m-1}\frac{v}{v^2+2m-1}=\frac{A}{v}+\frac{B}{v+i\sqrt{2m-1}}+\frac{C}{v-i\sqrt{2m-1}},$$

giving

$$A=-\frac{1}{2m-1},\quad B=C=\frac{m}{2m-1}.$$

The integral equation thus is

$$\text{const.}=(mx-\sqrt{\square})^{-1}(mx+\sqrt{\square})^{\frac{2m}{2m-1}}\{(mx+i\sqrt{2m-1}\,y+\sqrt{\square})(mx-i\sqrt{2m-1}\,y+\sqrt{\square})\}^{\frac{m-1}{2m-1}}$$

where $\square=m^2x^2+y^2$; or, observing that

$$\begin{aligned}&(mx+i\sqrt{2m-1}\,y+\sqrt{\square})(mx-i\sqrt{2m-1}\,y+\sqrt{\square})\\&=(mx+\sqrt{\square})^2+y^2\\&=2m(mx^2+y^2+x\sqrt{\square}),\end{aligned}$$

the integral equation is

$$\text{const.}=(mx+\sqrt{\square})^{\frac{1}{2m-1}}(mx^2+y^2+x\sqrt{\square})^{\frac{m-1}{2m-1}},$$

or, what is the same thing,

$$\text{const.}=(mx+\sqrt{\square})(mx^2+y^2+x\sqrt{\square})^{m-1},$$

the result given in the former part of the present paper.

VI.

I annex the following *à posteriori* verification of the solution

$$\text{const.}=(mx+\sqrt{\square})(mx^2+y^2+x\sqrt{\square})^{m-1}$$

of the particular equation

$$y(p^2-1)+2mxp=0.$$

Putting for shortness

$$A = mx + \sqrt{\square},$$

$$B = mx^2 + y^2 + x\sqrt{\square},$$

where it will be remembered that

$$\square = m^2x^2 + y^2,$$

then we have

$$2mB = A^2 + (2m-1)\,y^2.$$

The integral equation may be written

$$h = P + Q\sqrt{\square} = U = AB^{m-1};$$

and we have

$$\frac{U'}{U} = \frac{A'}{A} + (m-1)\frac{B'}{B} = \frac{\Theta}{\sqrt{\square}},$$

if

$$\Theta = \sqrt{\square}\left\{\frac{A'}{A} + (m-1)\frac{B'}{B}\right\}.$$

But we have

$$A'\sqrt{\square} = m\sqrt{\square} + m^2x + yp = mA + yp,$$

$$\begin{aligned} B'\sqrt{\square} &= (2mx + 2yp)\sqrt{\square} + \square + x(m^2x + yp) \\ &= 2m^2x^2 + y^2 + xyp + (2mx + 2yp)\sqrt{\square} \\ &= A^2 + pyx + 2\sqrt{\square}, \end{aligned}$$

and

$$\frac{1}{B} = \frac{2m}{A^2 + (2m-1)\,y^2},$$

and the value of Θ thus is

$$\begin{aligned} \Theta &= \frac{mA + yp}{A} + (2m^2 - 2m)\frac{A^2 + py(x + 2\sqrt{\square})}{A^2 + (2m-1)\,y^2} \\ &= \frac{1}{A\,[A^2 + (2m-1)\,y^2]}\{(mA + yp)\,[A^2 + (2m-1)\,y^2] + (2m^2 - 2m)\,[A^3 + Apy\,(x + 2\sqrt{\square})]\}, \end{aligned}$$

where the expression in { } is

$$\begin{aligned} &= (2m^2 - m)\,A\,(A^2 + y^2) \\ &\quad + yp\,\{A^2 + (2m-1)\,y^2 + (2m^2 - 2m)\,A\,(x + 2\sqrt{\square})\}. \end{aligned}$$

Here the coefficient of yp is $= (2m^2 - m)(A^2 + y^2)$; in fact we have identically

$$A^2 + y^2 - 2A\sqrt{\square} = 0,$$

and thence

$$(2m^2 - 3m + 1)(A^2 + y^2) - 2(2m-1)(m-1)\,A\sqrt{\square} = 0,$$

that is

$$(2m^2 - m - 1)\,A^2 + (2m^2 - 3m + 1)\,y^2 - (2m-2)\,A\,\{A + (2m-1)\sqrt{\square}\} = 0;$$

or

$$(2m^2-m-1)A^2+(2m^2-3m+1)y^2-(2m^2-2m)A(x+2\sqrt{\square})=0,$$

and therefore

$$A^2+(2m-1)y^2+(2m^2-2m)A(x+2\sqrt{\square})=(2m^2-m)(A^2+y^2).$$

Hence the term in { } is

$$=(2m^2-m)(A^2+y^2)(A+yp);$$

or, what is the same thing, it is $=(4m^2-2m)A\sqrt{\square}(A+yp)$. Hence, restoring for $A^2+(2m-1)y^2$ its value $2mB$, we find

$$\Theta=\frac{(2m-1)\sqrt{\square}}{B}(A+yp),$$

or

$$\frac{U'}{U}=\frac{2m-1}{B}(A+yp).$$

But writing U_1, U_2 to denote the values corresponding to $+\sqrt{\square}$, $-\sqrt{\square}$ respectively, we have

$$U'_1=\frac{(2m-1)U_1}{B_1}(mx+yp+\sqrt{\square}),$$

$$U'_2=\frac{(2m-1)U_2}{B_2}(mx+yp-\sqrt{\square}),$$

and thence

$$U'_1U'_2=\frac{(2m-1)^2U_1U_2}{B_1B_2}\{(mx+yp)^2-\square\}$$

$$=\frac{(2m-1)^2U_1U_2}{B_1B_2}y\{y(p^2-1)+2mxp\}.$$

But we have

$$U'=P'+Q'\sqrt{\square}+\frac{Q\square'}{2\sqrt{\square}}=\frac{1}{2\sqrt{\square}}(2Q'\square+Q\square'+2P'\sqrt{\square}),$$

and thence

$$U'_1U'_2=-\frac{1}{4\square}\{(2Q'\square+Q\square')^2-4P'^2\square\},$$

and moreover

$$U_1U_2=P^2-Q^2\square=A_1A_2(B_1B_2)^{m-1},$$

where

$$AA_2=m^2x^2-\square=-y^2,$$

$$B_1B_2=(mx^2+y^2)^2-\square=y^2\{y^2+(2m-1)x^2\};$$

and we thence find

$$-\frac{1}{4\square}\{(2Q'\square+Q\square')^2-4P'^2\square\}$$

$$=(2m-1)^2A_1A_2(B_1B_2)^{m-2}y^2\{y(p^2-1)+2mxp\}$$

$$=-(2m-1)^2y^{2m-1}[y^2+(2m-1)x^2]^{m-2}\{y(p^2-1)+2mxp\}.$$

Hence, the derived equation being

$$Q^2\{(2Q'\square + Q\square')^2 - 4P'^2\square\} = 0,$$

the last preceding equation becomes

$$Q^2\square y^{2m-1}\{y^2 + (2m-1)x^2\}^{m-2}\{y(p^2-1) + 2mxp\} = 0.$$

Here, besides the factor Q^2 corresponding to the nodal curve, and the factor $\square$ corresponding to the cuspidal curve, we have the factors y^{2m-1} and $\{y^2 + (2m-1)x^2\}^{m-2}$; and, rejecting all these, the differential equation in its reduced form is

$$y(p^2-1) + 2mxp = 0;$$

and the required verification is effected. The occurrence of

$$Q^2\square y^{2m-1}\{y^2 + (2m-1)x^2\}^{m-2}$$

as a factor in the complete derived equation would give rise to some further investigations, but I will not now enter on them.

I remark however that if $m = 1$, viz. if the integral equation be const. $= x + \sqrt{x^2+y^2}$, or say $z = x + \sqrt{x^2+y^2}$, or, what is the same thing,

$$z^2 - 2zx - y^2 = 0,$$

then observing that $y^2 + (2m-1)x^2$ is here $= x^2 + y^2$ which is $= \square$, so that

$$\square\{y^2 + (2m-1)x^2\}^{m-2} = \square . \square^{-1} = 1,$$

the differential equation in its complete form is

$$y(p^2y + 2px - y) = 0;$$

so that we have here the factor y which divides out. The last-mentioned result is most readily obtained directly from the equation

$$\Omega = Q^2(2Q'\square + Q\square')^2 - 4P'^2\square = 0,$$

which is the derived equation corresponding to the integral equation $z = P + Q\sqrt{\square}$. We in fact have $P = x$, $Q = 1$, $\square = x^2 + y^2$, and the derived equation thus is

$$(x + yp)^2 - (x^2 + y^2) = 0,$$

that is, $y(p^2y + 2px - y) = 0$.

I mention also, in connexion with the foregoing investigation, the integral equation

$$z = x + \sqrt{2x^2 - y^2}, \text{ or } z^2 - 2zx - x^2 + y^2 = 0,$$

for which the derived equation in its complete form is

$$(2x - yp)^2 - (2x^2 - y^2) = 0,$$

or, what is the same thing, $y^2p^2 - 4xyp + 2x^2 + y^2 = 0$, and for which therefore there is no factor to divide out.

VII.

The conics confocal with a given conic form a system similar in its properties to that of the curves of curvature of a quadric surface; and the theory of the last-mentioned system may be studied by means of the system of confocal conics. Consider then the equation

$$\frac{x^2}{a^2+z}+\frac{y^2}{b^2+z}=1,$$

which, if z be an arbitrary parameter, belongs to the conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Treating z as a coordinate, the equation represents a surface of the third order, which is such that its section by any plane parallel to the plane of xy is a conic; and the confocal conics are the projections on the plane of xy, by lines parallel to the axis of z, of the sections of the surface.

The sections by the planes of zx, zy are the parabolas $x^2=z+a^2$ and $y^2=z+b^2$ respectively. When $z>-b^2$, the ordinates in each parabola are real, and these ordinates give the semiaxes of the elliptic section. When $z>-a^2<-b^2$, then only the parabola section in the plane of zx has a real ordinate, and the sections are hyperbolic; and when $z<-a^2$, the section is altogether imaginary. The section in the planes $z=-b^2$ is the pair of coincident lines $y^2=0$, $z=-b^2$, and the section in the plane $z=-a^2$ is the pair of coincident lines $z=-a^2$, $x^2=0$; or, in other words, the plane $z+b^2=0$ touches the surface along the line $y=0$, and the plane $z+a^2=0$ touches the surface along the line $x=0$: this at once appears from the integral form

$$(z+a^2)(z+b^2)-x^2(z+b^2)-y^2(z+a^2)=0.$$

The points $(z=-b^2,\ y=0,\ x=\pm\sqrt{a^2-b^2})$ and $(z=-a^2,\ x=0,\ y=\pm\sqrt{b^2-a^2})$ are conical points; the last two are however imaginary points on the surface. To find the nature of the surface about one of the first-mentioned two points, say the point $(z=-b^2,\ y=0,\ x=\sqrt{a^2-b^2})$, taking this point for the origin and writing therefore $\sqrt{a^2-b^2}+x$, y and $-b^2+z$ in the place of x, y, z respectively, the equation becomes

$$(a^2-b^2+z)z-\left((a^2-b^2)+2x\sqrt{a^2-b^2}+x^2\right)z-(a^2-b^2+z)y^2=0,$$

that is

$$z^2-2zx\sqrt{a^2-b^2}-(a^2-b^2)y^2-z(x^2+y^2)=0;$$

so that there is a tangent cone the equation whereof is

$$z^2-2zx\sqrt{a^2-b^2}-(a^2-b^2)y^2=0,$$

or, as it may be written,

$$(z-x\sqrt{a^2-b^2})^2-(a^2-b^2)(x^2+y^2)=0.$$

The equation is that of a cone of the second order, meeting the plane of zx in the lines $z=0$, $z=2x\sqrt{a^2-b^2}$ (and therefore such that its sections parallel to the plane of xy are parabolas), and meeting the plane of yz in the lines $z=\pm y\sqrt{a^2-b^2}$ (the origin being at the vertex of the cone or conical point of the surface).

Returning to the original origin, and to the equation of the surface written in the form

$$z^2+z(a^2+b^2-x^2-y^2)+a^2b^2-b^2x^2-a^2y^2=0,$$

calling this for a moment $z^2+2Bz+C=0$, the differential equation is $C'^2-4BB'C'+4CB'^2=0$; or, substituting, this is

$$(b^2x+a^2yp)^2-(a^2+b^2-x^2-y^2)(x+yp)(b^2x+a^2yp)+(a^2b^2-b^2x^2-a^2y^2)(x+yp)^2=0;$$

or, reducing, this is

$$(a^2-b^2)xy\{xy(p^2-1)-(a^2-b^2-x^2+y^2)p\}=0,$$

or say

$$xy\{xy(p^2-1)-(a^2-b^2-x^2+y^2)p\}=0,$$

where the factor xy arises from the level lines $(z+b^2=0,\ y=0)$ and $(z+a^2=0,\ x=0)$. Throwing out this factor, the equation becomes

$$xy(p^2-1)-(a^2-b^2-x^2+y^2)\ p=0,$$

which is satisfied identically by $z+b^2=0$, $y=0$, $x^2=a^2-b^2$. The first derived equation is

$$(xp+y)(p^2-1)+2(x-yp)p=0,$$

which for the values in question gives

$$p(p^2+1)=0,$$

where the factor $p=0$ corresponds to the section $y=0$ by the plane $z+b^2=0$: and taking the conical point for origin, and observing that the polar of the line $x=0$, $y=0$ in regard to the tangent cone is $z-x\sqrt{a^2-b^2}=0$, then writing the equation of the tangent cone in the form

$$(z-x\sqrt{a^2-b^2})^2-(a^2-b^2)(x^2+y^2)=0,$$

the two tangent planes through $(x=0,\ y=0)$ are given by the equation $x^2+y^2=0$; and for these planes we have $p^2+1=0$. The factor $p^2+1=0$ determines therefore the directions of the envelope at the conical point.

VIII.

In verification of the equation

$$z=\tfrac{1}{2}k(x^2+y^2)+\tfrac{1}{6}ax(x^2+y^2)$$

for a quadric surface in the neighbourhood of the umbilicus, I remark that, starting from the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

of an ellipsoid, and taking α, 0, γ as the coordinates of the umbilicus, and θ as the inclination to the axis of x of the tangent to the principal section through the umbilicus, then transforming to the umbilicus as origin and the new axes through that point, viz. the axes of x, z being the tangent and normal in the plane of ac, and the axis of y being at right angles to this (or in the direction of b), the equation becomes

$$\frac{(\alpha + x\cos\theta - z\sin\theta)^2}{a^2} + \frac{y^2}{b^2} + \frac{(\gamma - x\sin\theta - z\cos\theta)^2}{c^2} = 1,$$

or, expanding,

$$\left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1\right) + 2x\left(\frac{\alpha\cos\theta}{a^2} - \frac{\gamma\sin\theta}{b^2}\right) - 2z\left(\frac{\alpha\sin\theta}{a^2} + \frac{\gamma\cos\theta}{c^2}\right)$$

$$+ x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{c^2}\right) + \frac{y^2}{b^2} + z^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2}\right) - 2zx\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{c^2}\right) = 0.$$

But we have

$$\alpha = a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad \gamma = \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}},$$

$$\tan\theta = \frac{c}{a}\frac{\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}},$$

and thence

$$\sin\theta = \frac{c\sqrt{a^2 - b^2}}{b\sqrt{a^2 - c^2}} = \frac{c}{ba}\alpha, \quad \cos\theta = \frac{a\sqrt{b^2 - c^2}}{b\sqrt{a^2 - c^2}} = \frac{a}{bc}\gamma;$$

and substituting these values, the equation becomes

$$-2z\frac{b}{ca} + \frac{x^2}{b^2} + \frac{y^2}{b^2} + z^2\frac{(a^2 + c^2)b^2 - a^2c^2}{a^2b^2c^2} + 2\frac{\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}}{b^3ca}zx = 0,$$

or, what is the same thing,

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2) + \frac{1}{b^3}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}\,zx + \frac{a^2b^2 + c^2b^2 - a^2b^2}{2b^3ac}z^2;$$

whence approximately

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2),$$

and thence to the third order in x, y,

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2) + \frac{ca}{2b^3}\sqrt{a^2 - b^2}\sqrt{a^2 - c^2}\,x(x^2 + y^2),$$

which is of the form in question.

5, *Downing Terrace, Cambridge, November* 2, 1863.

331.

ANALYTICAL THEOREM RELATING TO THE FOUR CONICS INSCRIBED IN THE SAME CONIC AND PASSING THROUGH THE SAME THREE POINTS.

[From the *Philosophical Magazine*, vol. XXVII. (1864), pp. 42, 43.]

IMAGINE the four conics determined, and, selecting at pleasure any three of them, let their chords of contact with the given conic be taken for the axes of coordinates, or lines $x=0$, $y=0$, $z=0$; then, taking for the equation of the given conic

$$U=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$

the equations of the selected three conics must be of the form $U+lx^2=0$, $U+my^2=0$, $U+nz^2=0$, where l, m, n are to be determined in such manner that these conics may have three common points; the resulting values of l, m, n, and of the coordinates of the three common points, that is, the three given points, will of course be functions of the coefficients $(a,\ b,\ c,\ f,\ g,\ h)$; and the equation of the fourth conic will be of the form $U+(ix+jy+kz)^2=0$.

There is no difficulty in carrying out the investigation: it is found that the coordinates of the given points must be taken to be

$$(-f,\ g,\ h);\quad (f,\ -g,\ h);\quad (f,\ g,\ -h)$$

respectively, and that, writing as usual

$$K=abc-af^2-bg^2-ch^2+2fgh,$$

the equations of the four conics are

$$U+(K-abc)\frac{x^2}{f^2}=0,$$

$$U+(K-abc)\frac{y^2}{g^2}=0,$$

$$U+(K-abc)\frac{z^2}{h^2}=0,$$

$$U+(K-abc)\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)^2=0.$$

It is in fact easy to verify directly that each of these conics passes through the three given points; but the equations may also be exhibited in the form proper for putting this in evidence. Putting for shortness

$$X=\frac{y}{g}+\frac{z}{h},\quad Y=\frac{z}{h}+\frac{x}{f},\quad Z=\frac{x}{f}+\frac{y}{g},$$

the equations of the sides of the triangle formed by the given points are $X=0$, $Y=0$, $Z=0$, and the foregoing equations of the four conics may be expressed in the form

$$\begin{aligned}
(-bg^2-ch^2+2fgh)\,YZ+\quad & bg^2\,.\,ZX+\quad & ch^2\,.\,XY&=0,\\
af^2\,.\,YZ+(-ch^2-af^2+2fgh)\,ZX+\quad & & ch^2\,.\,XY&=0,\\
af^2\,.\,YZ+\quad & bg^2\,.\,ZX+(-af^2-bg^2+2fgh)\,XY & &=0,\\
(-bg^2-ch^2+2fgh)\,YZ+(-ch^2-af^2+2fgh)\,ZX+(-af^2-bg^2+2fgh)\,XY & & &=0,
\end{aligned}$$

which is the required form.

Cambridge, November 28, 1863.

332.

ANALYTICAL THEOREM RELATING TO THE SECTIONS OF A QUADRIC SURFACE.

[From the *Philosophical Magazine*, vol. XXVII. (1864), pp. 43, 44.]

THE four sections $x=0$, $y=0$, $z=0$, $w=0$ of the quadric surface

$$ax^2+by^2+6xy\sqrt{ab}-cz^2-dw^2=0$$

are each of them touched by each of the four sections

$$x\sqrt{2a}+y\sqrt{2b}\pm z\sqrt{c}\pm w\sqrt{d}=0;$$

where it is to be noticed that the radicals $\sqrt{2a}$, $\sqrt{2b}$ are such that their product is $=2\sqrt{ab}$ if $\sqrt{ab}$ be the radical contained in the equation of the surface. There is of course no loss of generality in attributing a definite sign to the radical $\sqrt{2a}$; but upon this being done, the sign of the radical $\sqrt{2b}$ is determined, whereas the signs of $\sqrt{c}$ and $\sqrt{d}$ are severally arbitrary. We may if we please write the equation of any one of the last-mentioned sections in the form

$$x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0,$$

it being understood that the radicals $\sqrt{2a}$, $\sqrt{2b}$ have each a determinate sign, but that the signs of $\sqrt{c}$ and $\sqrt{d}$ are each of them arbitrary.

To prove the theorem in question, it is enough to show (1) that the sections $x=0$, $x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0$; (2) that the sections $z=0$, $x\sqrt{2a}+y\sqrt{2b}+w\sqrt{d}=0$, touch each other.

1. The sections $x=0$, $x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0$ of the quadric surface $ax^2+by^2+6xy\sqrt{ab}-cz^2-dw^2=0$ will touch each other if, combining together the equations

$$x=0,\quad y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0,\quad by^2-cz^2-dw^2=0,$$

these give a twofold value (pair of equal values) for the ratios $y:z:w$. We in fact have

$$\begin{aligned} by^2-cz^2-dw^2 &= \quad by^2-cz^2-(y\sqrt{2b}+z\sqrt{c})^2,\\ &= -by^2-2cz^2-2yz\sqrt{2bc},\\ &= -(y\sqrt{b}+z\sqrt{2c})^2; \end{aligned}$$

and the right-hand side being a perfect square, the condition of contact is satisfied.

2. In like manner we have the system

$$z=0,\quad x\sqrt{2a}+y\sqrt{2b}+w\sqrt{d}=0,\quad ax^2+by^2+6xy\sqrt{ab}-dw^2=0,$$

which gives

$$\begin{aligned} &ax^2+by^2+6xy\sqrt{ab}-dw^2\\ &\quad= \quad ax^2+by^2+6xy\sqrt{ab}-(x\sqrt{2a}+y\sqrt{2b})^2,\\ &\quad= -ax^2-by^2+2xy\sqrt{ab},\\ &\quad= -(x\sqrt{a}-y\sqrt{b})^2; \end{aligned}$$

and here also, the right-hand side being a perfect square, the condition of contact is satisfied.

Cambridge, November 28, 1863.

333.

NOTE ON THE NODAL CURVE OF THE DEVELOPABLE DERIVED FROM THE QUARTIC EQUATION $(a, b, c, d, e \between t, 1)^4 = 0$.

[From the *Philosophical Magazine*, vol. XXVII. (1864), pp. 437—440.]

CONSIDERING the coefficients (a, b, c, d, e) as linear functions of the coordinates x, y, z, w, then the equation

$$\text{Disct. } (a, b, c, d, e \between t, 1)^4 = 0,$$

or, as it may be written,

$$(ae - 4bd + 3c^2)^3 - 27\,(ace + 2bcd - ad^2 - b^2e - e^3)^2 = 0$$

represents, as is known, a developable surface or "torse," having for its edge of regression (or cuspidal curve) the sextic curve the equations whereof are

$$\begin{aligned} ae \;\; - 4bd + 3c^2 &= 0, \\ ace + 2bcd - ad^2 - b^2e - c^3 &= 0\,; \end{aligned}$$

and for its nodal curve, a curve the equations whereof (equivalent to two independent relations between the coordinates) are

$$\frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e}\,;$$

or, as these may also be written,

$$\begin{aligned} a^2d - 3abc + 2b^3 &= 0, \\ a^2e + 2abd - 9ac^2 + 6b^2c &= 0, \\ abe - 3acd + 2b^2d &= 0, \\ ad^2 - b^2e &= 0, \\ ade - 3bce + 2bd^2 &= 0, \\ ae^2 + 2bde - 9c^2e + 6cd^2 &= 0, \\ be^2 - 3cde + 2d^3 &= 0\,; \end{aligned}$$

which curve is in fact an excubo-quartic,—viz. a quartic curve the partial intersection of a quadric surface and a cubic surface, having in common two non-intersecting right lines. To show that this is so, I remark that the coefficients a, b, c, d, e, quà linear functions of the four coordinates, satisfy a linear equation which may be taken to be

$$a+b+c+d+e=0\,;$$

this being so, the first form shows that the curve in question lies on the quadric surface

$$ac-b^2+\tfrac{1}{2}(ad-bc)+\tfrac{1}{6}(ae+2bd-3c^2)+\tfrac{1}{2}(be-cd)+ce-d^2=0,$$

or, as this equation may also be written,

$$c\,(a-\tfrac{1}{2}b-\tfrac{1}{2}c-\tfrac{1}{2}d+e)-b^2+\tfrac{1}{2}ad+\tfrac{1}{6}(ae+2bd)+\tfrac{1}{2}be-d^2=0.$$

Substituting for c its value, this equation is

$$-(a+e+b+d)\,(\tfrac{3}{2}a+\tfrac{3}{2}e)-b^2+\tfrac{1}{2}ad+\tfrac{1}{6}(ae+2bd)+\tfrac{1}{2}be-d^2=0,$$

or, what is the same thing,

$$9\,(a+e+b+d)\,(a+e)+6\,(b^2+d^2)-3\,(ad+be)-(ae+2bd)=0.$$

Hence, finally, the equation of the quadric surface is

$$9a^2+17ae+9e^2+6b^2-2bd+6d^2+9ab+9de+6ad+6be=0\,;$$

and the curve lies also on the cubic surface

$$ad^2-b^2e=0.$$

It only remains to show that these surfaces have in common two right lines, and to find the equations of these lines.

The cubic surface is a skew surface or "scroll" such that the equations of any generating line are $d-\theta b=0$, $e-\theta^2 a=0$, where θ is an arbitrary parameter. But considering the two lines

$$(d-\theta_1 b=0,\quad e-\theta_1^2 a=0),\quad (d-\theta_2 b=0,\quad e-\theta_2^2 a=0),$$

the general equation of the quadric surface through these two lines may be written

$$\begin{aligned}
&A\ .\ (d-\theta_1 b)(d-\theta_2 b)\\
+&B\ .\ (e-\theta_1^2 a)(e-\theta_2^2 a)\\
+&C\ .\ (d-\theta_1 b)(e-\theta_2^2 a)+(d-\theta_2 b)(e-\theta_1^2 a)\\
+&\frac{D}{\theta_1-\theta_2}\{(d-\theta_1 b)(e-\theta_2^2 a)-(d-\theta_2 b)(e-\theta_1^2 a)\}=0
\end{aligned}$$

or, expanding and reducing,

$$\begin{aligned}
&A\{\ d^2 - (\theta_1 + \theta_2)\, bd + \theta_1\theta_2\, b^2\} \\
&+ B\{\ e^2 - (\theta_1^2 + \theta_2^2)\, ea + \theta_1^2\theta_2^2 a^2\} \\
&+ C\{2de - (\theta_1^2 + \theta_2^2)\, ad - (\theta_1 + \theta_2)\, be + \theta_1\theta_2(\theta_1 + \theta_2)\, ab\} \\
&+ D\{\qquad (\theta_1 + \theta_2)\, ad - \qquad be - \qquad \theta_1\theta_2\, ab\} = 0,
\end{aligned}$$

which, if θ_1, θ_2 are the roots of the equation $\theta^2 - \frac{1}{3}\theta + 1 = 0$, and therefore $\theta_1 + \theta_2 = \frac{1}{3}$, $\theta_1\theta_2 = 1$, and $\theta_1^2 + \theta_2^2 = -\frac{17}{9}$, is

$$\begin{aligned}
&A(\ d^2 - \tfrac{1}{3}\, db + b^2) \\
&+ B(\ e^2 + \tfrac{17}{9}\, ae + a^2) \\
&+ C(2de + \tfrac{17}{9}\, ad - \tfrac{1}{3}\, be + \tfrac{1}{3}\, ab) \\
&+ D(\qquad \tfrac{1}{3}\, ad - \ be - \ ab) = 0.
\end{aligned}$$

Putting $A = 6$, $B = 9$, $C = \frac{9}{2}$, $D = -\frac{15}{2}$, this is

$$\begin{aligned}
&9\ (\ a^2 + \tfrac{17}{9}\, ae + e^2) \\
&+ 6\ (\ b^2 - \tfrac{1}{3}\, bd + d^2) \\
&+ \tfrac{9}{2}\,(\tfrac{1}{2} ab + 2\ de + \tfrac{17}{9}\, ad - \tfrac{1}{3}\, be) \\
&+ \tfrac{15}{2}\,(\ ab \qquad - \tfrac{1}{3}\, ad + \ be) = 0,
\end{aligned}$$

which is the before-mentioned quadric surface; hence the quadric surface and the cubic surface intersect in the two lines

$$(d - \theta_1 b = 0,\quad e - \theta_1^2 a = 0),\quad (d - \theta_2 b = 0,\quad e - \theta_2^2 a = 0)$$

(where θ_1, θ_2 are the roots of the quadric equation $\theta^2 - \frac{1}{3}\theta_1 + 1 = 0$); and they consequently intersect also in an excubo-quartic curve, which is the theorem required to be proved.

Blackheath, March 26, 1864.

334.

NOTE ON THE THEORY OF CUBIC SURFACES.

[From the *Philosophical Magazine*, vol. XXVII. (1864), pp. 493—496.]

THE equation

$$AX^3 + BY^3 + 6CRST = 0,$$

where $X + Y + R + S + T = 0$, represents a cubic surface of a special form, viz. each of the planes $R = 0$, $S = 0$, $T = 0$ is a triple tangent plane meeting the surface in three lines *which pass through a point* (1); and, moreover, the three planes $AX^3 + BY^3 = 0$ are triple tangent planes intersecting in a line. It is worth noticing that the equation of the surface may also be written

$$ax^3 + by^3 + c(u^3 + v^3 + w^3) = 0,$$

where $x + y + u + v + w = 0$. In fact, the coordinates satisfying the foregoing linear equations respectively, we have to show that the equation

$$AX^3 + BY^3 + 6CRST = ax^3 + by^3 + c(u^3 + v^3 + w^3)$$

may be identically satisfied. We have

$$\begin{aligned} & ax^3 + by^3 + c\ (u^3 + v^3 + w^3) \\ = & ax^3 + by^3 + c\,[(u + v + w)^3 - 3\ (v + w)(w + u)(u + v)] \\ = & ax^3 + by^3 - c\ (x + y)^3 \qquad - 3c\,(v + w)(w + u)(u + v), \end{aligned}$$

[1] The tangent plane of a surface intersects the surface in a curve having at the point of contact a double point, and in like manner a triple tangent plane intersects the surface in a curve with three double points, viz. each point of contact is a double point; there is not in general any triple tangent plane such that the three points of contact come together, or (what is the same thing) there is not in general any tangent plane intersecting the surface in a curve having at the point of contact a triple point. A surface may, however, have the kind of singularity just referred to, viz. a tangent plane intersecting the surface in a curve having at the point of contact a triple point; such tangent plane may be termed a 'tritom' tangent plane, and its point of contact a 'tritom' point: for a cubic surface the intersection by a tritom tangent plane is of course a system of three lines meeting in the tritom point. The tritom singularity is sibi-reciprocal; it is, I think, a singularity which should be considered in the theory of reciprocal surfaces.

which is to be

$$= AX^3 + BY^3 + 6CRST;$$

and we may find X, Y, R, S, T, linear functions of x, y, u, v, w, so as to satisfy these equations, and so that in virtue of

$$x + y + u + v + w = 0,$$

we shall have also $X + Y + R + S + T = 0$. For, assuming

$$\begin{aligned} AX^3 + BY^3 &= ax^3 + by^3 - c(x+y)^3, \\ X + Y &= x + y, \\ R &= \tfrac{1}{2}(v + w), \quad C = -4c, \\ S &= \tfrac{1}{2}(w + u), \\ T &= \tfrac{1}{2}(u + v), \end{aligned}$$

we have identically

$$\begin{aligned} AX^3 + BY^3 + 6CRST &= ax^3 + by^3 - c(x+y)^3 - 3c(v+w)(w+u)(u+v), \\ X + Y + R + S + T &= x + y + u + v + w; \end{aligned}$$

and thus it only remains to show that we can find X, Y linear functions of x, y, such that

$$\begin{aligned} AX^3 + BY^3 &= ax^3 + by^3 - c(x+y)^3, \\ X + Y &= x + y. \end{aligned}$$

This is always possible; in fact if

$$U = ax^3 + by^3 - c(x+y)^3,$$

then taking Φ for the cubicovariant, and $\square$ for the discriminant of U, we have $\frac{1}{2}(\Phi + \sqrt{\square}\,U)$, $\frac{1}{2}(\Phi - \sqrt{\square}\,U)$ each a perfect cube, say

$$\begin{aligned} \tfrac{1}{2}(\Phi + \sqrt{\square}\,U) &= (\lambda x + \mu y)^3, \\ \tfrac{1}{2}(\Phi - \sqrt{\square}\,U) &= (\nu x + \rho y)^3, \end{aligned}$$

and we then have

$$U = \frac{1}{\sqrt{\square}}\{(\lambda x + \mu y)^3 - (\nu x + \rho y)^3\} = AX^3 + BY^3,$$

which is satisfied by

$$\begin{aligned} X &= l(\lambda x + \mu y), \\ Y &= m(\nu x + \rho y), \end{aligned}$$

if

$$Al^3 = \frac{1}{\sqrt{\square}}, \quad Bm^3 = -\frac{1}{\sqrt{\square}}.$$

The equation $X+Y=x+y$ then gives

$$l\lambda + m\nu = 1,$$
$$l\mu + m\rho = 1,$$

which give the values of l and m, and thence the values of A and B; and collecting all the equations, we have

$$X = \frac{\rho-\nu}{\lambda\rho-\mu\nu}(\lambda x+\mu y), \qquad A = \frac{1}{\sqrt{\square}}\left(\frac{\lambda\rho-\mu\nu}{\rho-\nu}\right)^3,$$

$$Y = -\frac{\mu-\lambda}{\lambda\rho-\mu\nu}(\nu x+\rho y), \qquad B = \frac{1}{\sqrt{\square}}\left(\frac{\lambda\rho-\mu\nu}{\rho-\nu}\right)^3,$$

$$R = \tfrac{1}{2}(v+w), \qquad C = -4c,$$
$$S = \tfrac{1}{2}(w+u),$$
$$T = \tfrac{1}{2}(u+v),$$

where

$$\lambda x+\mu y = \{\tfrac{1}{2}(\Phi+\sqrt{\square}\,U)\}^{\frac{1}{3}},$$
$$\mu x+\rho y = \{\tfrac{1}{2}(\Phi-\sqrt{\square}\,U)\}^{\frac{1}{3}}$$

(Φ, $\square$ being respectively the cubicovariant and the discriminant of $U=ax^3+by^3-c(x+y)^3$), for the formulæ of the transformation

$$AX^3+BY^3+6CRST = ax^3+by^3+c(u^3+v^3+w^3).$$
$$X+Y+R+S+T = x+y+u+v+w.$$

The equation $ax^3+by^3+c(u^3+v^3+w^3)=0$, where

$$x+y+u+v+w=0,$$

presents over the other form the advantage that it is included as a particular case under the equation $ax^3+by^3+cu^3+dv^3+ew^3=0$ (where $x+y+u+v+w=0$) employed by Dr Salmon as the canonical form of equation for the general cubic surface.

5, *Downing Terrace, Cambridge, April* 29, 1864.

335.

TABLES DES FORMES QUADRATIQUES BINAIRES POUR LES DÉTERMINANTS NÉGATIFS DEPUIS $D=-1$ JUSQU'À $D=-100$, POUR LES DÉTERMINANTS POSITIFS NON CARRÉS DEPUIS $D=2$ JUSQU'À $D=99$ ET POUR LES TREIZE DÉTERMINANTS NÉGATIFS IRRÉGULIERS QUI SE TROUVENT DANS LE PREMIER MILLIER.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LX. (1862), pp. 357—372.]

LES tables suivantes sont arrangées de la manière préscrite dans les "*Disquisitiones arithmeticae.*" Dans le mémoire de *Lejeune Dirichlet* "Recherches sur diverses applications de l'analyse à la théorie des nombres," tom. XIX (1839), p. 338 de ce Journal on trouve un tableau dans lequel les règles qui servent à former les caractères des genres sont résumées. Soit $D=PS^2$ ou $2PS^2$, S^2 désignant le plus grand carré que D contient, et P un nombre impair; soient de plus $p, p', p''\ldots$ les facteurs premiers inégaux de P et $r, r', r''\ldots$ les nombres premiers impairs qui divisent S sans diviser P; écrivons enfin pour abréger $\delta=(-)^{\frac{m-1}{2}}$, $\epsilon=(-)^{\frac{m^2-1}{8}}$. Cela posé on trouve à l'endroit cité le tableau suivant:

Premier cas, $D=PS^2$, $P\equiv 1 \pmod 4$

$S\equiv 1 \pmod 2$	$\frac{m}{p}, \frac{m}{p'}, \ldots$	$\frac{m}{r}, \frac{m}{r'}, \ldots$
$S\equiv 2 \pmod 4$	$\frac{m}{p}, \frac{m}{p'}, \ldots$	$\delta, \frac{m}{r}, \frac{m}{r'}, \ldots$
$S\equiv 0 \pmod 4$	$\frac{m}{p}, \frac{m}{p'}, \ldots$	$\delta, \epsilon, \frac{m}{r}, \frac{m}{r'}, \ldots.$

Deuxième cas, $D = PS^2$, $P \equiv 3 \pmod{4}$

$$\begin{array}{l|ll|ll}
S \equiv 1 \pmod{2} & \delta, & \frac{m}{p}, \frac{m}{p'}, \ldots & \frac{m}{r}, \frac{m}{r'}, \ldots \\
S \equiv 2 \pmod{4} & \delta, & \frac{m}{p}, \frac{m}{p'}, \ldots & \frac{m}{r}, \frac{m}{r'}, \ldots \\
S \equiv 0 \pmod{4} & \delta, & \frac{m}{p}, \frac{m}{p'}, \ldots & \epsilon, \frac{m}{r}, \frac{m}{r'}, \ldots.
\end{array}$$

Troisième cas, $D = 2PS^2$, $P \equiv 1 \pmod{4}$

$$\begin{array}{l|ll|ll}
S \equiv 1 \pmod{2} & \epsilon, & \frac{m}{p}, \frac{m}{p'}, \ldots & \frac{m}{r}, \frac{m}{r'}, \ldots \\
S \equiv 0 \pmod{2} & \epsilon, & \frac{m}{p}, \frac{m}{p'}, \ldots & \delta, \frac{m}{r}, \frac{m}{r'}, \ldots.
\end{array}$$

Quatrième cas, $D = 2PS^2$, $P \equiv 3 \pmod{4}$

$$\begin{array}{l|lll|ll}
S \equiv 1 \pmod{2} & \delta\epsilon, & \frac{m}{p}, \frac{m}{p'}, \ldots & \frac{m}{r}, \frac{m}{r'}, \ldots \\
S \equiv 0 \pmod{2} & \delta, \ \epsilon, & \frac{m}{p}, \frac{m}{p'}, \ldots & \frac{m}{r}, \frac{m}{r'}, \ldots.
\end{array}$$

Dans ce tableau la notation $\frac{m}{p}$ dans laquelle j'omets les parenthèses usitées, signifie le caractère d'un nombre quelconque m par rapport au nombre premier impair p, c.-à-d. que m est résidu ou non résidu de p selon que $\frac{m}{p} = +1$ ou $= -1$, de même δ est le caractère de m par rapport au nombre 4, savoir $m \equiv 1$ ou 3 (mod. 4) selon que $\delta = +1$ ou $= -1$, enfin ϵ, $\delta\epsilon$ sont les caractères de m par rapport au nombre 8, savoir $m \equiv 1$ ou 7 (mod. 8) pour $\epsilon = +1$, $\equiv 3$ ou 5 (mod. 8) pour $\epsilon = -1$; $m \equiv 1$ ou 3 (mod. 8) pour $\delta\epsilon = +1$, $\equiv 5$ ou 7 (mod. 8) pour $\delta\epsilon = -1$. Si pour un déterminant donné on veut former au moyen de ce tableau les caractères des genres, on prend la ligne horizontale qui convient à ce déterminant; à tous les caractères $\frac{m}{p}, \frac{m}{p'}, \ldots \frac{m}{r}, \frac{m}{r'}, \ldots$ δ, ϵ, $\delta\epsilon$ qui se trouvent dans la ligne horizontale, on attribue les signes + ou − à volonté, avec cette restriction cependant que le signe composé des signes qui se trouvent dans la première partie de la ligne dont il s'agit soit positif. Si par exemple le déterminant donné est $D = -35$, on a $D = -35 = PS^2$, $P = -35 \equiv 1 \pmod{4}$, $S = 1 \equiv 1 \pmod{2}$, les nombres p, p', ... sont 5, 7, et les signes que l'on doit considérer sont $\frac{m}{5}$, $\frac{m}{7}$. De là on obtient les caractères

$$\begin{array}{cc}
\frac{m}{5}, & \frac{m}{7} \\
+ & + \\
- & -
\end{array}$$

il y a donc deux genres de l'ordre proprement primitif. Dans le cas dont il s'agit (et en général pour $D \equiv 1 \pmod{4}$) il y a un ordre improprement primitif avec des genres qui ont les caractères identiques à ceux des genres de l'ordre proprement primitif, les caractères se rapportant dans ce cas à la moitié d'un nombre quelconque représenté par la forme.

Pour faciliter l'impression des tables j'ai introduit deux nouvelles lettres α et β dont voici la définition. Pour tous les déterminants auxquels se rapportent mes tables, c.-à-d. pour les déterminants négatifs quelconques et positifs non-carrés depuis -100 jusqu'à $+99$ ainsi que pour les déterminants négatifs irréguliers du premier millier, le nombre des facteurs premiers désignés ci-dessus par les lettres $p, p', p'' \ldots, r, r', r'' \ldots$ n'excède pas deux. Soit donc q le plus petit de ces facteurs premiers et q' le plus grand lorsqu'il y en a deux, je désigne par α le caractère $\frac{m}{q}$ et par β le caractère $\frac{m}{q'}$.

Dans la colonne relative à la composition et portant l'inscription Cp, je représente comme à l'ordinaire par l'unité la forme principale, par la lettre c une forme qui produit par la duplication la forme principale, par les lettres $d, e, \ldots$ des formes qui la produisent par la triplication, la quadruplication, etc., de manière que l'on ait $c^2 = 1$, $d^3 = 1$, $e^4 = 1$, $f^5 = 1$, $g^6 = 1$, $h^7 = 1$, $i^8 = 1$, $j^9 = 1$, etc. Les notations d, d_1 par exemple signifient deux formes différentes dont chacune produit par la triplication la forme principale. Je représente de plus par σ la forme principale de l'ordre improprement primitif et par $\sigma c, \sigma d, \ldots$ des formes qui produisent σ par la duplication, la triplication, etc. Dans l'énumération des classes, j'ai toujours écrit en premier lieu l'ordre proprement primitif, en le faisant suivre après un trait de séparation par l'ordre improprement primitif lorsqu'il existe. Dans chacun des deux ordres les divers genres se trouvent séparés les uns des autres par des traits subordonnés.

Pour les déterminants positifs les périodes sont données par une abréviation facile à comprendre. Chaque forme de la période ayant son dernier coefficient égal au premier de la suivante, cette valeur identique n'a été imprimée qu'une fois; de plus les coefficients extérieurs a, c des formes (a, b, c) ont été distingués des coefficients b en imprimant ces derniers en caractères plus petits. Ainsi pour le déterminant 7 la période de la classe principale $(1, 0, -7)$ est donnée par les nombres

$$1,\ {}_2,\ -3,\ {}_1,\ 2,\ {}_1,\ -3,\ {}_2,\ 1$$

qui représentent la série des formes

$$(1, 2, -3), (-3, 1, 2), (2, 1, -3), (-3, 2, 1).$$

Londres, 6 *Novembre*, 1860.

Table I des formes quadratiques binaires ayant pour déterminants les nombres négatifs depuis $D=-1$ jusqu'à $D=-100$.

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-1	1, 0, 1			+			1
-2	1, 0, 2					+	1
-3	1, 0, 3	+					1
	2, 1, 2	+					σ
-4	1, 0, 4			+			1
-5	1, 0, 5	+		+			1
	2, 1, 3	−		−			c
-6	1, 0, 6	+			+		1
	2, 0, 3	−			−		c
-7	1, 0, 7	+					1
	2, 1, 4	+					σ
-8	1, 0, 8			+	+		1
	3, 1, 3			−	−		c
-9	1, 0, 9	+		+			1
	2, 1, 5	−		+			c
-10	1, 0, 10	+				+	1
	2, 0, 5	−				−	c
-11	1, 0, 11	+					1
	3, 1, 4	+					d
	3, −1, 4	+					d^2
	2, 1, 6	+					σ
-12	1, 0, 12	+		+			1
	3, 0, 4	+		−			c
-13	1, 0, 13	+		+			1
	2, 1, 7	−		−			c
-14	1, 0, 14	+			+		1
	2, 0, 7	+			+		e^2
	3, 1, 5	−			−		e
	3, −1, 5	−			−		e^3
-15	1, 0, 15	+	+				1
	3, 0, 5	−	−				c
	2, 1, 8	+	+				σ
	4, 1, 4	−	−				σc
-16	1, 0, 16			+	+		1
	4, 2, 5			+	−		c
-17	1, 0, 17	+		+			1
	2, 1, 9	+		+			e^2
	3, 1, 6	−		−			e
	3, −1, 6	−		−			e^3
-18	1, 0, 18	+				+	1
	2, 0, 9	−				+	c
-19	1, 0, 19	+					1
	4, 1, 5	+					d
	4, −1, 5	+					d^2
	2, 1, 10	+					σ
-20	1, 0, 20	+		+			1
	2, 0, 5	+		+			e^2
	3, 1, 7	−		−			e
	3, −1, 7	−		−			e^3
-21	1, 0, 21	+	+	+			1
	3, 0, 7	+	−	−			c
	2, 1, 11	−	+	−			c_1
	5, 2, 5	−	−	+			cc_1
-22	1, 0, 22	+			+		1
	2, 0, 11	−			−		c
-23	1, 0, 23	+					1
	3, 1, 8	+					d
	3, −1, 8	+					d^2
	2, 1, 12	+					σ
	4, −1, 6	+					σd
	4, 1, 6	+					σd^2
-24	1, 0, 24	+		+	+		1
	3, 0, 8	−		−	−		c
	5, 1, 5	−		+	−		c_1
	4, 2, 7	+		−	+		cc_1
-25	1, 0, 25	+		+			1
	2, 1, 13	−		+			c
-26	1, 0, 26	+				+	1
	3, −1, 9	+				+	g^2
	3, 1, 9	+				+	g^4
	5, 2, 6	−				−	g
	2, 0, 13	−				−	g^3
	5, −2, 6	−				−	g^5
-27	1, 0, 27	+					1
	4, 1, 7	+					d
	4, −1, 7	+					d^2
	2, 1, 14	+					σ
-28	1, 0, 28	+		+			1
	4, 0, 7	+		−			c
-29	1, 0, 29	+		+			1
	5, 1, 6	+		+			g^2
	5, −1, 6	+		+			g^4
	3, 1, 10	−		−			g
	2, 1, 15	−		−			g^3
	3, −1, 10	−		−			g^5
-30	1, 0, 30	+	+		+		1
	2, 0, 15	−	−		+		c
	3, 0, 10	+	−		−		c_1
	5, 0, 6	−	+		−		cc_1
-31	1, 0, 31	+					1
	5, 2, 7	+					d
	5, −2, 7	+					d^2
	2, 1, 16	+					σ
	4, −1, 8	+					σd
	4, 1, 8	+					σd^2
-32	1, 0, 32			+	+		1
	4, 2, 9			+	+		e^2
	3, 1, 11			−	−		e
	3, −1, 11			−	−		e^3
-33	1, 0, 33	+	+	+			1
	2, 1, 17	−	−	+			c
	3, 0, 11	−	+	−			c_1
	6, 3, 7	+	−	−			cc_1
-34	1, 0, 34	+				+	1
	2, 0, 17	+				+	e^2
	5, 1, 7	−				−	e
	5, −1, 7	−				−	e^3
-35	1, 0, 35	+	+				1
	4, 1, 9	+	+				g^2
	4, −1, 9	+	+				g^4
	3, −1, 12	−	−				g
	5, 0, 7	−	−				g^3
	3, 1, 12	−	−				g^5
	2, 1, 18	+	+				σ
	6, 1, 6	−	−				σg
-36	1, 0, 36	+		+			1
	4, 0, 9	+		+			e^2
	5, 2, 8	−		+			e
	5, −2, 8	−		+			e^3
-37	1, 0, 37	+		+			1
	2, 1, 19	−		−			c
-38	1, 0, 38	+			+		1
	6, 2, 7	+			+		g^2
	6, −2, 7	+			+		g^4
	3, 1, 13	−					g
	2, 0, 19	−			−		g^3
	3, −1, 13	−			−		g^5
-39	1, 0, 39	+	+				1
	3, 0, 13	+	+				e^2
	5, 1, 8	−	−				e
	5, −1, 8	−	−				e^3
	2, 1, 20	+	+				σ
	6, 3, 8	+	+				σe^2
	4, −1, 10	−	−				σe
	4, 1, 10	−	−				σe^3

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-40	1, 0, 40	+				+	1
	4, 2, 11	+				+	c
	5, 0, 8	−				−	c_1
	7, 3, 7	−				−	cc_1
-41	1, 0, 41	+		+			1
	5, 2, 9	+		+			i^2
	2, 1, 21	+		+			i^4
	5, −2, 9	+		+			i^6
	3, 1, 14	−		−			i
	6, −1, 7	−		−			i^3
	6, 1, 7	−		−			i^5
	3, −1, 14	−		−			i^7
-42	1, 0, 42	+	+			+	1
	2, 0, 21	−	+			−	c
	3, 0, 14	−	−			+	c_1
	6, 0, 7	+	−			−	cc_1
-43	1, 0, 43	+					1
	4, 1, 11	+					d
	4, −1, 11	+					d^2
	2, 1, 22	+					σ
-44	1, 0, 44	+		+			1
	5, −1, 9	+		+			g^2
	5, 1, 9	+		+			g^4
	3, 1, 15	+		−			g
	4, 0, 11	+		−			g^3
	3, −1, 15	+		−			g^5
-45	1, 0, 45	+	+	+			1
	5, 0, 9	−	+	+			c
	2, 1, 23	−	−	−			c_1
	7, 2, 7	+	−	−			cc_1
-46	1, 0, 46	+			+		1
	2, 0, 23	+			+		e^2
	5, 2, 10	−			−		e
	5, −2, 10	−			−		e^3
-47	1, 0, 47	+					1
	3, 1, 16	+					f
	7, 3, 8	+					f^2
	7, −3, 8	+					f^3
	3, −1, 16	+					f^4
	2, 1, 24	+					σ
	6, 1, 8	+					σf
	4, 1, 12	+					σf^2
	4, −1, 12	+					σf^3
	6, −1, 8	+					σf^4
-48	1, 0, 48	+		+	+		1
	3, 0, 16	+		−	−		c
	7, 1, 7	+		−	+		c_1
	4, 2, 13	+		+	−		cc_1
-49	1, 0, 49	+		+			1
	2, 1, 25	+		+			e^2
	5, 1, 10	−		−			e
	5, −1, 10	−		−			e^3
-50	1, 0, 50	+				+	1
	6, 2, 9	+				+	g^2
	6, −2, 9	+				+	g^4
	3, 1, 17	−				+	g
	2, 0, 25	−				+	g^3
	3, −1, 17	−				+	g^5
-51	1, 0, 51	+	+				1
	4, 1, 13	+	+				g^2
	4, −1, 13	+	+				g^4
	2, 1, 26	+	+				σ
	6, 3, 10	−	−				σg
-52	1, 0, 52	+		+			1
	4, 0, 13	+		+			e^2
	7, 2, 8	−		−			e
	7, −2, 8	−		−			e^3
-53	1, 0, 53	+		+			1
	6, −1, 9	+		+			g^2
	6, 1, 9	+		+			g^4
	3, 1, 18	−		−			g
	2, 1, 27	−		−			g^3
	3, −1, 18	−		−			g^5
-54	1, 0, 54	+			+		1
	7, 3, 9	+			+		g^2
	7, −3, 9	+			+		g^4
	5, 1, 11	−			−		g
	2, 0, 27	−			−		g^3
	5, −1, 11	−			−		g^5
-55	1, 0, 55	+	+				1
	5, 0, 11	+	+				e^2
	7, 1, 8	−	−				e
	7, −1, 8	−	−				e^3
	2, 1, 28	+	+				σ
	8, 3, 8	+	+				σe^2
	4, 1, 14	−	−				σe
	4, −1, 14	−	−				σe^3
-56	1, 0, 56	+		+	+		1
	8, 4, 9	+		+	+		e^2
	5, 2, 12	−		+	−		e
	5, −2, 12	−		+	−		e^3
	4, 2, 15	+		−	+		c
	7, 0, 8	+		−	+		ce^2
	3, 1, 19	−		−	−		ce
	2, −1, 19	−		−	−		ce^3
-57	1, 0, 57	+	+	+			1
	3, 0, 19	+	−	−			c
	2, 1, 29	−	−	+			c_1
	6, 3, 11	−	+	−			cc_1
-58	1, 0, 58	+				+	1
	2, 0, 29	−				−	c
-59	1, 0, 59	+					1
	3, 1, 20	+					j
	7, 2, 9	+					j^2
	4, 1, 15	+					j^3
	5, −1, 12	+					j^4
	5, 1, 12	+					j^5
	4, −1, 15	+					j^6
	7, −2, 9	+					j^7
	3, −1, 20	+					j^8
	2, 1, 30	+					σ
	6, 1, 10	+					σj
	6, −1, 10	+					σj^2
-60	1, 0, 60	+	+	+			1
	3, 0, 20	−	−	−			c
	4, 0, 15	+	+	−			c_1
	5, 0, 12	−	−	+			cc_1
-61	1, 0, 61	+		+			1
	5, −2, 13	+		+			g^2
	5, 2, 13	+		+			g^4
	7, 3, 10	−		−			g
	2, 1, 31	−		−			g^3
	7, −3, 10	−		−			g^5
-62	1, 0, 62	+		+			1
	7, 1, 9	+		+			g^2
	7, −1, 9	+		+			g^4
	6, 2, 11	−		−			g
	2, 0, 31	−		−			g^3
	6, −2, 11	−		−			g^5
-63	1, 0, 63	+	+				1
	7, 0, 9	+	+				e^2
	8, 3, 9	−	+				e
	8, −3, 9	−	+				e^3
	2, 1, 32	+	+				σ
	8, 1, 8	+	+				σe^2
	4, −1, 16	−	+				σe
	4, 1, 16	−	+				σe^3
-64	1, 0, 64			+	+		1
	5, 1, 13			+	+		e^2
	4, 2, 17			+	−		e
	5, −1, 13			+	−		e^3
-65	1, 0, 65	+	+	+			1
	9, 4, 9	+	+	+			e^2
	3, 1, 22	−	+	−			e
	3, −1, 22	−	+	−			e^3
	5, 0, 13	−	−	+			c
	2, 1, 33	−	−	+			ce^2
	6, −1, 11	+	−	−			ce
	6, 1, 11	+	−	−			ce^3

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-66	1, 0, 66	+	+			+	1
	3, 0, 22	+	+			+	e^2
	7, 2, 10	+	−			−	e
	7, −2, 10	+	−			−	e^3
	2, 0, 33	−	−			+	c
	6, 0, 11	−	−			+	ce^2
	5, −2, 14	−	+			−	ce
	5, 2, 14	−	+			−	ce^3
-67	1, 0, 67	+					1
	4, 1, 17	+					d
	4, −1, 17	+					d^2
	2, 1, 34	+					σ
-68	1, 0, 68	+		+			1
	8, 2, 9	+		+			i^2
	4, 0, 17	+		+			i^4
	8, −2, 9	+		+			i^6
	3, 1, 23	+		−			i
	7, −3, 11	+		−			i^3
	7, 3, 11	+		−			i^5
	3, −1, 23	+		−			i^7
-69	1, 0, 69	+	+	+			1
	6, 3, 13	+	+	+			e^2
	5, 1, 14	−	−	+			e
	5, −1, 14	−	−	+			e^3
	2, 1, 35	−	+	−			c
	3, 0, 22	−	+	−			ce^2
	7, −1, 10	+	−	−			ce
	7, 1, 10	+	−	−			ce^3
-70	1, 0, 70	+	+		+		1
	2, 0, 35	−	+		−		c
	5, 0, 14	+	−		−		c_1
	7, 0, 10	−	−		+		cc_1
-71	1, 0, 71	+					1
	3, 1, 24	+					h
	8, −1, 9	+					h^2
	5, −2, 15	+					h^3
	5, 2, 15	+					h^4
	8, 1, 9	+					h^5
	3, −1, 24	+					h^6
	2, 1, 36	+					σ
	6, 1, 12	+					σh
	4, −1, 18	+					σh^2
	8, −3, 10	+					σh^3
	8, 3, 10	+					σh^4
	4, 1, 18	+					σh^5
	6, −1, 12	+					σh^6
-72	1, 0, 72	+		+	+		1
	4, 2, 19	+		−	−		c
	8, 0, 9	−		+	+		c_1
	8, 4, 11	−		−	−		cc_1
-73	1, 0, 73	+		+			1
	2, 1, 37	+		+			e^2
	7, 2, 11	−		−			e
	7, −2, 11	−		−			e^3

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-74	1, 0, 74	+				+	1
	3, −1, 25	+				+	k^2
	9, −4, 10	+				+	k^4
	9, 4, 10	+				+	k^6
	3, 1, 25	+				+	k^8
	5, 1, 15	−				−	k
	6, −2, 13	−				−	k^3
	2, 0, 37	−				−	k^5
	6, 2, 13	−				−	k^7
	5, −1, 15	−				−	k^9
-75	1, 0, 75	+	+				1
	4, −1, 19	+	+				g^2
	4, 1, 19	+	+				g^4
	7, 3, 12	+	−				g
	3, 0, 25	+	−				g^3
	7, −3, 12	+	−				g^5
	2, 1, 38	+	+				σ
	6, 3, 14	+	−				σg
-76	1, 0, 76	+		+			1
	5, −2, 16	+		+			g^2
	5, 2, 16	+		+			g^4
	7, 1, 11	+		−			g
	4, 0, 19	+		−			g^3
	7, −1, 11	+		−			g^5
-77	1, 0, 77	+	+	+			1
	9, 2, 9	+	+	+			e^2
	6, −1, 13	−	−	+			e
	6, −1, 13	−	−	+			e^3
	2, 1, 39	+	−	−			c
	7, 0, 11	+	−	−			ce^2
	3, 1, 26	−	+	−			ce
	3, −1, 26	−	+	−			ce^3
-78	1, 0, 78	+	+		+		1
	2, 0, 39	−	−		+		c
	3, 0, 26	−	+		−		c_1
	6, 0, 13	+	−		−		cc_1
-79	1, 0, 79	+					1
	5, 1, 16	+					f
	8, −3, 11	+					f^2
	8, 3, 11	+					f^3
	5, −1, 16	+					f^4
	2, 1, 40	+					σ
	8, −1, 10	+					σf
	4, 1, 20	+					σf^2
	4, −1, 20	+					σf^3
	8, 1, 10	+					σf^4
-80	1, 0, 80	+		+	+		1
	9, 1, 9	+		+	+		e^2
	3, 1, 27	−		−	−		e
	3, −1, 27	−		−	−		e^3
	5, 0, 16	+		+	−		c
	4, 2, 21	+		+	−		ce^2
	7, −2, 12	−		−	+		ce
	7, 2, 12	−		−	+		ce^3

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-81	1, 0, 81	+		+			1
	9, −3, 10	+		+			g^2
	9, 3, 10	+		+			g^4
	5, 2, 17	−		+			g
	2, 1, 41	−		+			g^3
	5, −2, 17	−		+			g^5
-82	1, 0, 82	+				+	1
	2, 0, 41	+				+	e^2
	7, 3, 13	−				−	e
	7, −3, 13	−				−	e^3
-83	1, 0, 83	+					1
	3, 1, 28	+					j
	9, 4, 11	+					j^2
	4, 1, 21	+					j^3
	7, −1, 12	+					j^4
	7, 1, 12	+					j^5
	4, −1, 21	+					j^6
	9, −4, 11	+					j^7
	3, −1, 28	+					j^8
	2, 1, 42	+					σ
	6, 1, 14	+					σj
	6, −1, 14	+					σj^2
-84	1, 0, 84	+	+	+			1
	4, 0, 21	+	+	+			e^2
	5, 1, 17	−	−	+			e
	5, −1, 17	−	−	+			e^3
	3, 0, 28	+	−	−			c
	7, 0, 12	+	−	−			ce^2
	8, 2, 11	−	+	−			ce
	8, −2, 11	−	+	−			ce^3
-85	1, 0, 85	+	+	+			1
	5, 0, 17	−	−	+			c
	2, 1, 43	−	+	−			c_1
	10, 5, 11	+	−	−			cc_1
-86	1, 0, 86	+			+		1
	6, 2, 15	+			+		k^2
	9, 2, 10	+			+		k^4
	9, −2, 10	+			+		k^6
	6, −2, 15	+			+		k^8
	5, 2, 18	−			−		k
	3, 1, 29	−			−		k^3
	2, 0, 43	−			−		k^5
	3, −1, 29	−			−		k^7
	5, −2, 18	−			−		k^9
-87	1, 0, 87	+	+				1
	7, 2, 13	+	+				g^2
	7, −2, 13	+	+				g^4
	8, 1, 11	−	−				g
	3, 0, 29	−	−				g^3
	8, −1, 11	−	−				g^5
	2, 1, 44	+	+				σ
	8, −3, 12	+	+				σg^2
	8, 3, 12	+	+				σg^4
	4, 1, 22	−	−				σg
	6, 3, 16	−	−				σg^3
	4, −1, 22	−	−				σg^5

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-88	1, 0, 88	+			+		1
	4, 2, 23	+			+		c
	8, 0, 11	−			−		c_1
	8, 4, 13	−			−		cc_1
-89	1, 0, 89	+		+			1
	9, 1, 10	+		+			m^2
	5, 1, 18	+		+			m^4
	2, 1, 45	+		+			m^6
	5, −1, 18	+		+			m^8
	9, −1, 10	+		+			m^{10}
	3, 1, 30	−		−			m
	7, −3, 14	−		−			m^3
	6, −1, 15	−		−			m^5
	6, 1, 15	−		−			m^7
	7, 3, 14	−		−			m^9
	3, −1, 30	−		−			m^{11}
-90	1, 0, 90	+	+			+	1
	9, 0, 10	+	+			+	e^2
	7, 1, 13	+	−			−	e
	7, −1, 13	+	−			−	e^3
	2, 0, 45	−	−			−	c
	5, 0, 18	−	−			−	ce^2
	9, −3, 11	−	+			+	ce
	9, 3, 11	−	+			+	ce^3
-91	1, 0, 91	+	+				1
	4, 1, 23	+	+				g^2
	4, −1, 23	+	+				g^4
	5, −2, 19	−	−				g
	7, 0, 13	−	−				g^3
	5, 2, 19	−	−				g^5
	2, 1, 46	+	+				σ
	10, 3, 10	−	−				σg

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-92	1, 0, 92	+		+			1
	9, 4, 12	+		+			g^2
	9, −4, 12	+		+			g^4
	3, 1, 31	+		−			g
	4, 0, 23	+		−			g^3
	3, −1, 31	+		−			g^5
-93	1, 0, 93	+	+	+			1
	3, 0, 31	+	−	−			c
	2, 1, 47	−	+	−			c_1
	6, 3, 17	−	−	+			cc_1
-94	1, 0, 94	+			+		1
	7, 2, 14	+			+		i^2
	2, 0, 47	+			+		i^4
	7, −2, 14	+			+		i^6
	5, 1, 19	−			−		i
	10, 4, 11	−			−		i^3
	10, −4, 11	−			−		i^5
	5, −1, 19	−			−		i^7
-95	1, 0, 95	+	+				1
	9, −2, 11	+	+				i^2
	5, 0, 19	+	+				i^4
	9, 2, 11	+	+				i^6
	3, 1, 32	−	−				i
	8, 3, 13	−	−				i^3
	8, −3, 13	−	−				i^5
	3, −1, 32	−	−				i^7
	2, 1, 48	+	+				σ
	8, 1, 12	+	+				σi^2
	10, 5, 12	+	+				σi^4
	8, −1, 12	+	+				σi^6
	6, 1, 16	−	−				σi
	4, −1, 24	−	−				σi^3
	4, 1, 24	−	−				σi^5
	6, −1, 16	−	−				σi^7

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
-96	1, 0, 96	+		+	+		1
	4, 2, 25	+		+	+		e^2
	5, 2, 20	−		+	−		e
	5, −2, 20	−		+	−		e^3
	3, 0, 32	−		−	−		c
	11, 5, 11	−		−	−		ce^2
	7, 3, 15	+		−	+		ce
	7, −3, 15	+		−	+		ce^3
-97	1, 0, 97	+		+			1
	2, 1, 49	+		+			e^2
	7, 1, 14	−		−			e
	7, −1, 14	−		−			e^3
-98	1, 0, 98	+				+	1
	9, 1, 11	+				+	i^2
	2, 0, 49	+				+	i^4
	9, −1, 11	+				+	i^6
	3, 1, 33	−				+	i
	6, 2, 17	−				+	i^3
	6, −2, 17	−				+	i^5
	3, −1, 33	−				+	i^7
-99	1, 0, 99	+	+				1
	4, −1, 25	+	+				g^2
	4, 1, 25	+	+				g^4
	5, 1, 20	−	+				g
	9, 0, 11	−	+				g^3
	5, −1, 20	−	+				g^5
	2, 1, 50	+	+				σ
	10, 1, 10	−	+				σg
-100	1, 0, 100	+		+			1
	4, 0, 25	+		+			e^2
	8, 2, 13	−		+			e
	8, −2, 13	−		+			e^3

Table II des formes quadratiques binaires ayant pour déterminants les nombres positifs non-carrés depuis $D = 2$ jusqu'à $D = 99$.

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
2	1, 0, − 2				+		1	1, 1, − 1, 1, 1
3	1, 0, − 3	+		+			1	1, 1, − 2, 1, 1
	− 1, 0, 3	−		−			c	− 1, 1, 2, 1, − 1
5	1, 0, − 5	+					1	1, 2, − 1, 2, 1
	2, 1, − 2	+					σ	2, 1, − 2, 1, 2
6	1, 0, − 6	+				+	1	1, 2, − 2, 2, 1
	− 1, 0, 6	−				−	c	− 1, 2, 2, 2, − 1
7	1, 0, − 7	+		+			1	1, 2, − 3, 1, 2, 1, − 3, 2, 1
	− 1, 0, 7	−		−			c	− 1, 2, 3, 1, − 2, 1, 3, 2, − 1
8	1, 0, − 8			+	+		1	1, 2, − 4, 2, 1
	− 1, 0, 8			−	+		c	− 1, 2, 4, 2, − 1
10	1, 0, − 10	+			+		1	1, 3, − 1, 3, 1
	2, 0, − 5	−			−		c	2, 2, − 3, 1, 3, 2, − 2, 2, 3, 1, − 3, 2, 2
11	1, 0, − 11	+		+			1	1, 3, − 2, 3, 1
	− 1, 0, 11	−		−			c	− 1, 3, 2, 3, − 1
12	1, 0, − 12	+		+			1	1, 3, − 3, 3, 1
	− 1, 0, 12	−		−			c	− 1, 3, 3, 3, − 1
13	1, 0, − 13	+					1	1, 3, − 4, 1, 3, 2, − 3, 1, 4, 3, − 1, 3, 4, 1, − 3, 2, 3, 1, − 4, 3, 1
	2, 1, − 6	+					σ	2, 3, − 2, 3, 2
14	1, 0, − 14	+				+	1	1, 3, − 5, 2, 2, 2, − 5, 3, 1
	− 1, 0, 14	−				−	c	− 1, 3, 5, 2, − 2, 2, 5, 3, − 1
15	1, 0, − 15	+	+	+			1	1, 3, − 6, 3, 1
	− 1, 0, 15	−	+	−			c	− 1, 3, 6, 3, − 1
	2, 1, − 7	−	−	+			c_1	2, 3, − 3, 3, 2
	− 2, 1, 7	+	−	−			cc_1	− 2, 3, 3, 3, − 2
17	1, 0, − 17	+					1	1, 4, − 1, 4, 1
	2, 1, − 8	+					σ	2, 3, − 4, 1, 4, 3, − 2, 3, 4, 1, − 4, 3, 2
18	1, 0, − 18	+			+		1	1, 4, − 2, 4, 1
	− 1, 0, 18	−			+		c	− 1, 4, 2, 4, − 1
19	1, 0, − 19	+		+			1	1, 4, − 3, 2, 5, 3, − 2, 3, 5, 2, − 3, 4, 1
	− 1, 0, 19	−		−			c	− 1, 4, 3, 2, − 5, 3, 2, 3, − 5, 2, 3, 4, − 1
20	1, 0, − 20	+		+			1	1, 4, − 4, 4, 1
	− 1, 0, 20	+		−			c	− 1, 4, 4, 4, − 1
21	1, 0, − 21	+	+				1	1, 4, − 5, 1, 4, 3, − 3, 3, 4, 1, − 5, 4, 1
	− 1, 0, 21	−	−				c	− 1, 4, 5, 1, − 4, 3, 3, 3, − 4, 1, 5, 4, − 1
	2, 1, − 10	+	+				σ	2, 3, − 6, 3, 2
	− 2, 1, 10	−	−				σc	− 2, 3, 6, 3, − 2
22	1, 0, − 22	+				+	1	1, 4, − 6, 2, 3, 4, − 2, 4, 3, 2, − 6, 4, 1
	− 1, 0, 22	−				−	c	− 1, 4, 6, 2, − 3, 4, 2, 4, − 3, 2, 6, 4, − 1
23	1, 0, − 23	+		+			1	1, 4, − 7, 3, 2, 3, − 7, 4, 1
	− 1, 0, 23	−		−			c	− 1, 4, 7, 3, − 2, 3, 7, 4, − 1
24	1, 0, − 24	+		+	+		1	1, 4, − 8, 4, 1
	− 1, 0, 24	−		−	+		c	− 1, 4, 8, 4, − 1
	3, 0, − 8	+		−	−		c_1	3, 3, − 5, 2, 4, 2, − 5, 3, 3
	− 3, 0, 8	−		+	−		cc_1	− 3, 3, 5, 2, − 4, 2, 5, 3, − 3

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
26	1, 0, −26	+			+		1	1, 5, −1, 5, 1
	2, 0, −13	−			−		c	2, 4, −5, 1, 5, 4, −2, 4, 5, 1, −5, 4, 2
27	1, 0, −27	+		+			1	1, 5, −2, 5, 1
	−1, 0, 27	−		−			c	−1, 5, 2, 5, −1
28	1, 0, −28	+		+			1	1, 5, −3, 4, 4, 4, −3, 5, 1
	−1, 0, 28	−		−			c	−1, 5, 3, 4, −4, 4, 3, 5, −1
29	1, 0, −29	+					1	1, 5, −4, 3, 5, 2, −5, 3, 4, 5, −1, 5, 4, 3, −5, 2, 5, 3, −4, 5, 1
	2, 1, −14	+					σ	−1, 5, 4, 3, −5, 2, 5, 3, −4, 5, 1, 5, −4, 3, 5, 2, −5, 3, 4, 5, −1
30	1, 0, −30	+	+			+	1	1, 5, −5, 5, 1
	−1, 0, 30	−	+			−	c	−1, 5, 5, 5, −1
	2, 0, −15	−	−			+	c_1	2, 4, −7, 3, 3, 3, −7, 4, 2
	−2, 0, 15	+	−			−	cc_1	−2, 4, 7, 3, −3, 3, 7, 4, −2
31	1, 0, −31	+		+			1	1, 5, −6, 1, 5, 4, −3, 5, 2, 5, −3, 4, 5, 1, −6, 5, 1
	−1, 0, 31	−		−			c	−1, 5, 6, 1, −5, 4, 3, 5, −2, 5, 3, 4, −5, 1, 6, 5, −1
32	1, 0, −32			+	+		1	1, 5, −7, 2, 4, 2, −7, 5, 1
	−1, 0, 32			−	+		c	−1, 5, 7, 2, −4, 2, 7, 5, −1
33	1, 0, −33	+	+				1	1, 5, −8, 3, 3, 3, −8, 5, 1
	−1, 0, 33	−	−				c	−1, 5, 8, 3, −3, 3, 8, 5, −1
	2, 1, −16	+	+				σ	2, 5, −4, 3, 6, 3, −4, 5, 2
	−2, 1, 16	−	−				σc	−2, 5, 4, 3, −6, 3, 4, 5, −2
34	1, 0, −34	+			+		1	1, 5, −9, 4, 2, 4, −9, 5, 1
	−1, 0, 34	+			+		c^2	−1, 5, 9, 4, −2, 4, 9, 5, −1
	3, −1, −11	−			−		e	3, 5, −3, 4, 6, 2, −5, 3, 5, 2, −6, 4, 3
	−3, −1, 11	−			−		e^2	−3, 5, 3, 4, −6, 2, 5, 3, −5, 2, 6, 4, −3
35	1, 0, −35	+	+	+			1	1, 5, −10, 5, 1
	−1, 0, 35	+	−	−			c	−1, 5, 10, 5, −1
	2, 1, −17	−	+	−			c_1	2, 5, −5, 5, 2
	−2, 1, 17	−	−	+			cc_1	−2, 5, 5, 5, −2
37	1, 0, −37	+					1	1, 6, −1, 6, 1
	3, 1, −12	+					d	3, 4, −7, 3, 4, 5, −3, 4, 7, 3, −4, 5, 3
	3, −1, −12	+					d^2	−3, 4, 7, 3, −4, 5, 3, 4, −7, 3, 4, 5, −3
	2, 1, −18	+					σ	2, 5, −6, 1, 6, 5, −2, 5, 6, 1, −6, 5, 2
38	1, 0, −38	+				+	1	1, 6, −2, 6, 1
	−1, 0, 38	−				−	c	−1, 6, 2, 6, −1
39	1, 0, −39	+	+	+			1	1, 6, −3, 6, 1
	−1, 0, 39	−	+	−			c	−1, 6, 3, 6, −1
	2, 1, −19	−	−	+			c_1	2, 5, −7, 2, 5, 3, −6, 3, 5, 2, −7, 5, 2
	−2, 1, 19	+	−	−			cc_1	−2, 5, 7, 2, −5, 3, 6, 3, −5, 2, 7, 5, −2
40	1, 0, −40	+		+	+		1	1, 6, −4, 6, 1
	−1, 0, 40	+		−	+		c	−1, 6, 4, 6, −1
	3, 1, −13	−		−	−		c_1	3, 4, −8, 4, 3, 5, −5, 5, 3
	−3, 1, 13	−		+	−		cc_1	−3, 4, 8, 4, −3, 5, 5, 5, −3
41	1, 0, −41	+					1	1, 6, −5, 4, 5, 6, −1, 6, 5, 4, −5, 6, 1
	2, 1, −20	+					σ	−1, 6, 5, 4, −5, 6, 1, 6, −5, 4, 5, 6, −1
42	1, 0, −42	+	+		+		1	1 6, −6, 6, 1
	−1, 0, 42	−	−		+		c	−1, 6, 6, 6, −1
	2, 0, −21	−	+		−		c_1	2, 6, −3, 6, 2
	−2, 0, 21	+	−		−		cc_1	−2, 6, 3, 6, −2
43	1, 0, −43	+		+			1	1, 6, −7, 1, 6, 5, −3, 4, 9, 5, −2, 5, 9, 4, −3, 5, 6, 1, −7, 6, 1
	−1, 0, 43	−		−			c	−1, 6, 7, 1, −6, 5, 3, 4, −9, 5, 2, 5, −9, 4, 3, 5, −6, 1, 7, 6, −1

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
44	1, 0, −44	+		+			1	1, 6, −8, 2, 5, 3, −7, 4, 4, 4, −7, 3, 5, 2, −8, 6, 1
	−1, 0, 44	−		−			c	−1, 6, 8, 2, −5, 3, 7, 4, −4, 4, 7, 3, −5, 2, 8, 6, −1
45	1, 0, −45	+	+				1	1, 6, −9, 3, 4, 5, −5, 5, 4, 3, −9, 6, 1
	−1, 0, 45	−	+				c	−1, 6, 9, 3, −4, 5, 5, 5, −4, 3, 9, 6, −1
	2, 1, −22	+	+				σ	2, 5, −10, 5, 2
	−2, 1, 22	−	+				σc	−2, 5, 10, 5, −2
46	1, 0, −46	+				+	1	1, 6, −10, 4, 3, 5, −7, 2, 6, 4, −5, 6, 2, 6, −5, 4, 6, 2, −7, 5, 3, 4, −10, 6, 1
	−1, 0, 46	−				−	c	−1, 6, 10, 4, −3, 5, 7, 2, −6, 4, 5, 6, −2, 6, 5, 4, −6, 2, 7, 5, −3, 4, 10, 6, −1
47	1, 0, −47	+		+			1	1, 6, −11, 5, 2, 5, −11, 6, 1
	−1, 0, 47	−		−			c	−1, 6, 11, 5, −2, 5, 11, 6, −1
48	1, 0, −48	+		+	+		1	1, 6, −12, 6, 1
	−1, 0, 48	−		−	+		c	−1, 6, 12, 6, −1
	3, 0, −16	−		−	−		c_1	3, 6, −4, 6, 3
	−3, 0, 16	+		+	−		cc_1	−3, 6, 4, 6, −3
50	1, 0, −50	+				+	1	1, 7, −1, 7, 1
	2, 0, −25	−				+	c	2, 6, −7, 1, 7, 6, −2, 6, 7, 1, −7, 6, 2
51	1, 0, −51	+	+	+			1	1, 7, −2, 7, 1
	−1, 0, 51	−	+	−			c	−1, 7, 2, 7, −1
	3, 0, −17	+	−	−			c_1	3, 6, −5, 4, 7, 3, −6, 3, 7, 4, −5, 6, 3
	−3, 0, 17	−	−	+			cc_1	−3, 6, 5, 4, −7, 3, 6, 3, −7, 4, 5, 6, −3
52	1, 0, −52	+		+			1	1, 7, −3, 5, 9, 4, −4, 4, 9, 5, −3, 7, 1
	−1, 0, 52	+		−			c	−1, 7, 3, 5, −9, 4, 4, 4, −9, 5, 3, 7, −1
53	1, 0, −53	+					1	1, 7, −4, 5, 7, 2, −7, 5, 4, 7, −1, 7, 4, 5, −7, 2, 7, 5, −4, 7, 1
	2, 1, −26	+					σ	2, 7, −2, 7, 2
54	1, 0, −54	+				+	1	1, 7, −5, 3, 9, 6, −2, 6, 9, 3, −5, 7, 1
	−1, 0, 54	−				−	c	−1, 7, 5, 3, −9, 6, 2, 6, −9, 3, 5, 7, −1
55	1, 0, −55	+	+	+			1	1, 7, −6, 5, 5, 5, −6, 7, 1
	−1, 0, 55	+	−	−			c	−1, 7, 6, 5, −5, 5, 6, 7, −1
	2, 1, −27	−	−	+			c_1	2, 7, −3, 5, 10, 5, −3, 7, 2
	−2, 1, 27	−	+	−			cc_1	−2, 7, 3, 5, −10, 5, 3, 7, −2
56	1, 0, −56	+		+	+		1	1, 7, −7, 7, 1
	−1, 0, 56	−		−	+		c	−1, 7, 7, 7, −1
	4, 2, −13	+		−	−		c_1	4, 6, −5, 4, 8, 4, −5, 6, 4
	−4, 2, 13	−		+	−		cc_1	−4, 6, 5, 4, −8, 4, 5, 6, −4
57	1, 0, −57	+	+				1	1, 7, −8, 1, 7, 6, −3, 6, 7, 1, −8, 7, 1
	−1, 0, 57	−	−				c	−1, 7, 8, 1, −7, 6, 3, 6, −7, 1, 8, 7, −1
	2, 1, −28	+	+				σ	2, 7, −4, 5, 8, 3, −6, 3, 8, 5, −4, 7, 2
	−2, 1, 28	−	−				σc	−2, 7, 4, 5, −8, 3, 6, 3, −8, 5, 4, 7, −2
58	1, 0, −58	+			+		1	1, 7, −9, 2, 6, 4, −7, 3, 7, 4, −6, 2, 9, 7, −1, 7, 9, 2, −6, 4, 7, 3, −7, 4, 6, 2, −9, 7, 1
	2, 0, −29	−			−		c	2, 6, −11, 5, 3, 7, −3, 5, 11, 6, −2, 6, 11, 5, −3, 7, 3, 5, −11, 6, 2
59	1, 0, −59	+		+			1	1, 7, −10, 3, 5, 7, −2, 7, 5, 3, −10, 7, 1
	−1, 0, 59	−		−			c	−1, 7, 10, 3, −5, 7, 2, 7, −5, 3, 10, 7, −1
60	1, 0, −60	+	+	+			1	1, 7, −11, 4, 4, 4, −11, 7, 1
	−1, 0, 60	−	+	−			c	−1, 7, 11, 4, −4, 4, 11, 7, −1
	3, 0, −20	+	−	−			c_1	3, 6, −8, 2, 7, 5, −5, 5, 7, 2, −8, 6, 3
	−3, 0, 20	−	−	+			cc_1	−3, 6, 8, 2, −7, 5, 5, 5, −7, 2, 8, 6, −3
61	1, 0, −61	+					1	1, 7, −12, 5, 3, 7, −4, 5, 9, 4, −5, 6, 5, 4, −9, 5, 4, 7, −3, 5, 12, 7, −1, 7, 12, 5, −3, 7, 4, 5, −9, 4, 5, 6, −5, 4, 9, 5, −4, 7, 3, 5, −12, 7, 1
	2, 1, −30	+					σ	2, 7, −6, 5, 6, 7, −2, 7, 6, 5, −6, 7, 2

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
62	1, 0, −62	+				+	1	1, 7, −13, 6, 2, 6, −13, 7, 1
	−1, 0, 62	−				−	c	−1, 7, 13, 6, −2, 6, 13, 7, −1
63	1, 0, −63	+	+	+			1	1, 7, −14, 7, 1
	−1, 0, 63	−	−	−			c	−1, 7, 14, 7, −1
	2, 1, −31	−	+	+			c_1	2, 7, −7, 7, 2
	−2, 1, 31	+	−	−			cc_1	−2, 7, 7, 7, −2
65	1, 0, −65	+	+				1	1, 8, −1, 8, 8, 8, −1, 8, 1
	5, 0, −13	−	−				c	5, 5, −8, 3, 7, 4, −7, 3, 8, 5, −5, 5, 8, 3, −7, 4, 7, 3, −8, 5, 5
	2, 1, −32	+	+				σ	2, 7, −8, 1, 8, 7, −2, 7, 8, 1, −8, 7, 2
	10, 5, −4	−	−				σc	10, 5, −4, 7, 4, 5, −10, 5, 4, 7, −4, 5, 10
66	1, 0, −66	+	+		+		1	1, 8, −2, 8, 1
	−1, 0, 66	−	−		+		c	−1, 8, 2, 8, −1
	3, 0, −22	−	+		−		c_1	3, 6, −10, 4, 5, 6, −6, 6, 5, 4, −10, 6, 3
	−3, 0, 22	+	−		−		cc_1	−3, 6, 10, 4, −5, 6, 6, 6, −5, 4, 10, 6, −3
67	1, 0, −67	+		+			1	1, 8, −3, 7, 6, 5, −7, 2, 9, 7, −2, 7, 9
	−1, 0, 67	−		−			c	−1, 8, 3, 7, −6, 5, 7, 2, −9, 7, 2, 7, −9
68	1, 0, −68	+		+			1	1, 8, −4, 8, 1
	−1, 0, 68	+		−			c	−1, 8, 4, 8, −1
69	1, 0, −69	+	+				1	1, 8, −5, 7, 4, 5, −11, 6, 3, 6, −11, 5, 4, 7, −5, 8, 1
	−1, 0, 69	−	−				c	−1, 8, 5, 7, −4, 5, 11, 6, −3, 6, 11, 5, −4, 7, 5, 8, −1
	2, 1, −34	+	+				σ	2, 7, −10, 3, 6, 3, −10, 7, 2
	−2, 1, 34	−	−				σc	−2, 7, 10, 3, −6, 3, 10, 7, −2
70	1, 0, −70	+	+			+	1	1, 8, −6, 4, 9, 5, −5, 5, 9, 4, −6, 8, 1
	−1, 0, 70	+	−			−	c	−1, 8, 6, 4, −9, 5, 5, 5, −9, 4, 6, 8, −1
	2, 0, −35	−	+			−	c_1	2, 8, −3, 7, 7, 7, −3, 8, 2
	−2, 0, 35	−	−			+	cc_1	−2, 8, 3, 7, −7, 7, 3, 8, −2
71	1, 0, −71	+		+			1	1, 8, −7, 6, 5, 4, −11, 7, 2, 7, −11, 4, 5, 6, −7, 8, 1
	−1, 0, 71	−		−			c	−1, 8, 7, 6, −5, 4, 11, 7, −2, 7, 11, 4, −5, 6, 7, 8, −1
72	1, 0, −72	+		+	+		1	1, 8, −8, 8, 1
	−1, 0, 72	−		−	+		c	−1, 8, 8, 8, −1
	4, 2, −17	+		−	+		c_1	4, 6, −9, 3, 7, 4, −8, 4, 7, 3, −9, 6, 4
	−4, 2, 17	−		+	+		cc_1	−4, 6, 9, 3, −7, 4, 8, 4, −7, 3, 9, 6, −4
73	1, 0, −73	+					1	1, 8, −9, 1, 8, 7, −3, 8, 3, 7, −8, 1, 9, 8, −1, 8, 9, 1, −8, 7, 3, 8, −3, 7, 8, 1, −9, 8, 1
	2, 1, −36	+					σ	2, 7, −12, 5, 4, 7, −6, 5, 8, 3, −8, 5, 6, 7, −4, 5, 12, 7, −2, 7, 12, 5, −4, 7, 6, 5, −8, 3, 8, 5, −6, 7, 4, 5, −12, 7, 2
74	1, 0, −74	+			+		1	1, 8, −10, 2, 7, 5, −7, 2, 10, 8, −1, 8, 10, 2, −7, 5, 7, 2, −10, 8, 1
	2, 0, −37	−			−		c	2, 8, −5, 7, 5, 8, −2, 8, 5, 7, −5, 8, 2
75	1, 0, −75	+	+	+			1	1, 8, −11, 3, 6, 3, −11, 8, 1
	−1, 0, 75	−	+	−			c	−1, 8, 11, 3, −6, 3, 11, 8, −1
	2, 1, −37	−	−	−			c_1	2, 7, −13, 6, 3, 6, −13, 7, 2
	−2, 1, 37	+	−	+			cc_1	−2, 7, 13, 6, −3, 6, 13, 7, −2
76	1, 0, −76	+		+			1	1, 8, −12, 4, 5, 6, −8, 2, 9, 7, −3, 8, 4, 8, −3, 7, 9, 2, −8, 6, 5, 4, −12, 8, 1
	−1, 0, 76	−		−			c	−1, 8, 12, 4, −5, 6, 8, 2, −9, 7, 3, 8, −4, 8, 3, 7, −9, 2, 8, 6, −5, 4, 12, 8, −1
77	1, 0, −77	+	+				1	1, 8, −13, 5, 4, 7, −7, 7, 4, 5, −13, 8, 1
	−1, 0, 77	−	−				c	−1, 8, 13, 5, −4, 7, 7, 7, −4, 5, 13, 8, −1
	2, 1, −38	+	+				σ	2, 7, −14, 7, 2
	−2, 1, 38	−	−				σc	−2, 7, 14, 7, −2

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
78	1, 0, −78	+	+			+	1	1, 8, −14, 6, 3, 6, −14, 8, 1
	−1, 0, 78	−	+			−	c	−1, 8, 14, 6, −3, 6, 14, 8, −1
	2, 0, −39	−	−			+	c_1	2, 8, −7, 6, 6, 6, −7, 8, 2
	−2, 0, 39	+	−			−	cc_1	−2, 8, 7, 6, −6, 6, 7, 8, −2
79	1, 0, −79	+		+			1	1, 8, −15, 7, 2, 7, −15, 8, 1
	−3, −1, 26	+		+			e^2	−3, 8, 5, 7, −6, 5, 9, 4, −7, 3, 10, 7, −3
	−3, 1, 26	+		+			e^4	−3, 7, 10, 3, −7, 4, 9, 5, −6, 7, 5, 8, −3
	3, 1, −26	−		−			e	3, 7, −10, 3, 7, 4, −9, 5, 6, 7, −5, 8, 3
	−1, 0, 79	−		−			e^3	−1, 8, 15, 7, −2, 7, 15, 8, −1
	3, −1, −26	−		−			e^5	3, 8, −5, 7, 6, 5, −9, 4, 7, 3, −10, 7, 3
80	1, 0, −80	+		+	+		1	1, 8, −16, 8, 1
	−1, 0, 80	+		−	+		c	−1, 8, 16, 8, −1
	4, 2, −19	+		+	−		c_1	4, 6, −11, 5, 5, 5, −11, 6, 4
	−4, 2, 19	+		−	−		cc_1	−4, 6, 11, 5, −5, 5, 11, 6, −4
82	1, 0, −82	+			+		1	1, 9, −1, 9, 1
	2, 0, −41	+			+		e^2	2, 8, −9, 1, 9, 8, −2, 8, 9, 1, −9, 8, 2
	3, −1, −27	−			−		e	3, 8, −6, 4, 11, 7, −3, 8, 6, 4, −11, 7, 3
	−3, −1, 27	−			−		e^3	3, 7, −11, 4, 6, 8, −3, 7, 11, 4, −6, 8, 3
83	1, 0, −83	+		+			1	1, 9, −2, 9, 1
	−1, 0, 83	−		−			c	−1, 9, 2, 9, −1
84	1, 0, −84	+	+	+			1	1, 9, −3, 9, 1
	−1, 0, 84	−	−	−			c	−1, 9, 3, 9, −1
	4, 0, −21	+	+	−			c_1	4, 8, −5, 7, 7, 7, −5, 8, 4
	−4, 0, 21	−	−	+			cc_1	−4, 8, 5, 7, −7, 7, 5, 8, −4
85	1, 0, −85	+	+				1	1, 9, −4, 7, 9, 2, −9, 7, 4, 9, −1, 9, 4, 7, −9, 2, 9, 7, −4, 9, 1
	5, 0, −17	−	−				c	5, 5, −12, 7, 3, 8, −7, 6, 7, 8, −3, 7, 12, 5, −5, 5, 12, 7, −3, 8, 7, 6, −7, 8, 3, 7, −12, 5, 5
	2, 1, −42	+	+				σ	2, 9, −2, 9, 2
	10, 5, −6	−	−				σc	10, 5, −6, 7, 6, 5, −10, 5, 6, 7, −6, 5, 10
86	1, 0, −86	+				+	1	1, 9, −5, 6, 10, 4, −7, 3, 11, 8, −2, 8, 11, 3, −7, 4, 10, 6, −5, 9, 1
	−1, 0, 86	−				−	c	−1, 9, 5, 6, −10, 4, 7, 3, −11, 8, 2, 8, −11, 3, 7, 4, −10, 6, 5, 9, −1
87	1, 0, −87	+	+	+			1	1, 9, −6, 9, 1
	−1, 0, 87	−	+	−			c	−1, 9, 6, 9, −1
	2, 1, −43	−	−	+			c_1	2, 9, −3, 9, 2
	−2, 1, 43	+	−	−			cc_1	−2, 9, 3, 9, −2
88	1, 0, −88	+		+	+		1	1, 9, −7, 5, 9, 4, −8, 4, 9, 5, −7, 9, 1
	−1, 0, 88	−		−	+		c	−1, 9, 7, 5, −9, 4, 8, 4, −9, 5, 7, 9, −1
	4, 2, −21	+		−	−		c_1	4, 6, −13, 7, 3, 8, −8, 8, 3, 7, −13, 6, 4
	−4, 2, 21	−		+	−		cc_1	−4, 6, 13, 7, −3, 8, 8, 8, −3, 7, 13, 6, −4
89	1, 0, −89	+					1	1, 9, −8, 7, 5, 8, −5, 7, 8, 9, −1, 9, 8, 7, −5, 8, 5, 7, −8, 9, 1
	2, 1, −44	+					σ	2, 9, −4, 7, 10, 3, −8, 5, 8, 3, −10, 7, 4, 9, −2, 9, 4, 7, −10, 3, 8, 5, −8, 3, 10, 7, −4, 9, 2
90	1, 0, −90	+	+		+		1	1, 9, −9, 9, 1
	−1, 0, 90	−	+		+		c	−1, 9, 9, 9, −1
	2, 0, −45	−	−		−		c_1	2, 8, −13, 5, 5, 5, −13, 8, 2
	−2, 0, 45	+	−		−		cc_1	−2, 8, 13, 5, −5, 5, 13, 8, −2
91	1, 0, −91	+	+	+			1	1, 9, −10, 1, 9, 8, −3, 7, 14, 7, −3, 8, 9, 1, −10, 9, 1
	−1, 0, 91	−	+	−			c	−1, 9, 10, 1, −9, 8, 3, 7, −14, 7, 3, 8, −9, 1, 10, 9, −1
	2, 1, −45	+	−	−			c_1	2, 9, −5, 6, 11, 5, −6, 7, 7, 7, −6, 5, 11, 6, −5, 9, 2
	−2, 1, 45	−	−	+			cc_1	−2, 9, 5, 6, −11, 5, 6, 7, −7, 7, 6, 5, −11, 6, 5, 9, −2

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp	Périodes
92	1, 0, −92	+		+			1	1, 9, −11, 2, 8, 6, −7, 8, 4, 8, −7, 6, 8, 2, −11, 9, 1
	−1, 0, 92	−		−			c	−1, 9, 11, 2, −8, 6, 7, 8, −4, 8, 7, 6, −8, 2, 11, 9, −1
93	1, 0, −93	+	+				1	1, 9, −12, 3, 7, 4, −11, 7, 4, 9, −3, 9, 4, 7, −11, 4, 7, 3, −12, 9, 1
	−1, 0, 93	−	−				c	−1, 9, 12, 3, −7, 4, 11, 7, −4, 9, 3, 9, −4, 7, 11, 4, −7, 3, 12, 9, −1
	2, 1, −46	+	+				σ	2, 9, −6, 9, 2
	−2, 1, 46	−	−				σc	−2, 9, 6, 9, −2
94	1, 0, −94	+				+	1	1, 9, −13, 4, 6, 8, −5, 7, 9, 2, −10, 8, 3, 7, −15, 8, 2, 8, −15, 7, 3, 8, −10, 2, 9, 7, −5, 8, 6, 4, −13, 9, 1
	−1, 0, 94	−				−	c	−1, 9, 13, 4, −6, 8, 5, 7, −9, 2, 10, 8, −3, 7, 15, 8, −2, 8, 15, 7, −3, 8, 10, 2, −9, 7, 5, 8, −6, 4, 13, 9, −1
95	1, 0, −95	+	+	+			1	1, 9, −14, 5, 5, 5, −14, 9, 1
	−1, 0, 95	+	−	−			c	−1, 9, 14, 5, −5, 5, 14, 9, −1
	2, 1, −47	−	−	+			c_1	2, 9, −7, 5, 10, 5, −7, 9, 2
	−2, 1, 47	−	+	−			cc_1	−2, 9, 7, 5, −10, 5, 7, 9, −2
96	1, 0, −96	+		+	+		1	1, 9, −15, 6, 4, 6, −15, 9, 1
	−1, 0, 96	−		−	+		c	−1, 9, 15, 6, −4, 6, 15, 9, −1
	3, 0, −32	+		−	−		c_1	3, 9, −5, 6, 12, 6, −5, 9, 3
	−3, 0, 32	−		+	−		cc_1	−3, 9, 5, 6, −12, 6, 5, 9, −3
97	1, 0, −97	+					1	1, 9, −16, 7, 3, 8, −11, 3, 8, 5, −9, 4, 9, 5, −8, 3, 11, 8, −3, 7, 16, 9, −1, 9, 16, 7, −3, 8, 11, 3, −8, 5, 9, 4, −9, 5, 8, 3, −11, 8, 3, 7, −16, 9, 1
	2, 1, −48	+					σ	2, 9, −8, 7, 6, 5, −12, 7, 4, 9, −4, 7, 12, 5, −6, 7, 8, 9, −2, 9, 8, 7, −6, 5, 12, 7, −4, 9, 4, 7, −12, 5, 6, 7, −8, 9, 2
98	1, 0, −98	+				+	1	1, 9, −17, 8, 2, 8, −17, 9, 1
	−1, 0, 98	−				−	c	−1, 9, 17, 8, −2, 8, 17, 9, −1
99	1, 0, −99	+	+	+			1	1, 9, −18, 9, 1
	−2, 1, 49	+	+	+			e^2	−2, 9, 9, 9, −2
	5, 2, −19	−	+	+			e	5, 7, −10, 3, 9, 6, −7, 8, 5
	5, −2, −19	−	+	+			e^3	5, 8, −7, 6, 9, 3, −10, 7, 5
	−1, 0, 99	−	−	−			c	−1, 9, 18, 9, −1
	2, 1, −49	−	−	−			ce^2	2, 9, −9, 9, 2
	−5, 2, 19	+	−	−			ce	−5, 7, 10, 3, −9, 6, 7, 8, −5
	−5, −2, 19	+	−	−			ce^3	−5, 8, 7, 6, −9, 3, 10, 7, −5

Table III des formes quadratiques binaires pour les treize déterminants négatifs irréguliers du premier millier.

D	Classes	α	β	δ	ϵ	Cp
-576	1, 0, 576	+		+	+	1
=	9, 0, 64	+		+	+	e^2
$-1(24)^2$	4, 2, 145	+		+	+	e_1^2
	25, 7, 25	+		+	+	$e^2 e_1^2$
	9, 3, 65	−		+	+	e
	9, −3, 65	−		+	+	e^3
	17, 6, 36	−		+	+	ee_1^2
	17, −6, 36	−		+	+	$e^3 e_1^2$
	13, 3, 45	+		+	−	e_1
	16, 4, 37	+		+	−	$e^2 e_1$
	13, −3, 45	+		+	−	e_1^3
	16, −4, 37	+		+	−	$e^3 e_1^3$
	5, 2, 116	−		+	−	ee_1
	20, −2, 29	−		+	−	$e^3 e_1$
	20, 2, 29	−		+	−	ee_1^3
	5, −2, 116	−		+	−	$e^3 e_1^3$
-580	1, 0, 580	+	+	+		1
=	4, 0, 145	+	+	+		e^2
$-145(2)^2$	5, 0, 116	+	+	+		e_1^2
	20, 0, 29	+	+	+		$e^2 e_1^2$
	8, 2, 73	−	−	+		e
	8, −2, 73	−	−	+		e^3
	17, 7, 37	−	−	+		ee_1^2
	17, −7, 37	−	−	+		$e^3 e_1^2$
	19, 3, 31	+	−	−		e_1
	11, 5, 55	+	−	−		$e^2 e_1$
	19, −3, 31	+	−	−		e_1^3
	11, −5, 55	+	−	−		$e^2 e_1^3$
	23, 8, 28	−	+	−		ee_1
	7, −1, 83	−	+	−		$e^3 e_1$
	7, 1, 83	−	+	−		ee_1^3
	23, −8, 28	−	+	−		$e^3 e_1^3$
-820	1, 0, 820	+	+	+		1
=	5, 0, 164	+	+	+		e^2
$-205(2)^2$	20, 0, 41	+	+	+		e_1^2
	4, 0, 205	+	+	+		$e^2 e_1^2$
	13, 5, 65	−	−	+		e
	13, −5, 65	−	−	+		e^3
	17, −8, 52	−	−	+		ee_1^2
	17, 8, 52	−	−	+		$e^3 e_1^2$
	11, 4, 76	+	−	−		e_1
	19, 4, 44	+	−	−		$e^2 e_1$
	11, −4, 76	+	−	−		e_1^3
	19, −4, 44	+	−	−		$e^2 e_1^3$
	23, −10, 40	−	+	−		ee_1
	8, 2, 103	−	+	−		$e^2 e_1$
	8, −2, 103	−	+	−		ee_1^3
	23, 10, 40	−	+	−		$e^3 e_1^3$

D	Classes	α	β	δ	ϵ	Cp
-884	1, 0, 884	+	+	+		1
=	13, 0, 68	+	+	+		i^4
$-221(2)^2$	4, 0, 221	+	+	+		e^2
	17, 0, 52	+	+	+		$e^2 i^4$
	25, −4, 36	+	+	+		i^2
	25, 4, 36	+	+	+		i^6
	9, 4, 100	+	+	+		$e^2 i^2$
	9, −4, 100	+	+	+		$e^2 i^6$
	5, 1, 177	−	−	+		i
	24, −2, 37	−	−	+		i^5
	20, −4, 45	−	−	+		$e^2 i$
	24, 10, 41	−	−	+		$e^2 i^5$
	24, 2, 37	−	−	+		i^3
	5, −1, 177	−	−	+		i^7
	24, −10, 41	−	−	+		$e^2 i^3$
	20, 4, 45	−	−	+		$e^2 i^7$
	8, 2, 111	−	+	−		e
	15, 4, 60	−	+	−		ei^4
	8, −2, 111	−	+	−		e^3
	15, −4, 60	−	+	−		$e^3 i^4$
	19, −3, 47	−	+	−		ei^2
	15, 1, 59	−	+	−		ei^6
	15, −1, 59	−	+	−		$e^3 i^2$
	19, 3, 47	−	+	−		$e^3 i^6$
	27, −13, 39	+	−	−		ei
	3, 1, 295	+	−	−		ei^5
	23, −6, 40	+	−	−		$e^3 i$
	12, 4, 75	+	−	−		$e^3 i^5$
	12, −4, 75	+	−	−		ei^3
	23, 6, 40	+	−	−		ei^7
	3, −1, 295	+	−	−		$e^3 i^3$
	27, 13, 39	+	−	−		$e^3 i^7$
-900	1, 0, 900	+	+	+		1
=	9, 0, 100	+	+	+		e^2
$-1(30)^2$	4, 0, 225	+	+	+		e_1^2
	25, 0, 36	+	+	+		$e^2 e_1^2$
	9, 3, 101	−	+	+		e
	9, −3, 101	−	+	+		e^3
	29, −12, 36	−	+	+		ee_1^2
	29, 12, 36	−	+	+		$e^3 e_1^2$
	17, 1, 53	−	−	+		e_1
	8, −2, 113	−	−	+		$e^2 e_1$
	17, −1, 53	−	−	+		e_1^3
	8, 2, 113	−	−	+		$e^2 e_1^3$
	13, −6, 72	+	−	+		ee_1
	25, −5, 37	+	−	+		$e^3 e_1$
	25, 5, 37	+	−	+		ee_1^3
	13, 6, 72	+	−	+		$e^3 e_1^3$

D	Classes	α	β	δ	ϵ	Cp
-243	1, 0, 243	+				1
=	7, 3, 36	+				d
$-3\,(9)^2$	7, -3, 36	+				d^2
	4, 1, 161	+				d_1
	13, -2, 19	+				dd_1
	9, 3, 28	+				$d^2 d_1$
	4, -1, 161	+				d_1^2
	9, -3, 28	+				dd_1^2
	13, 2, 19	+				$d^2 d_1^2$
	2, 1, 122	+				σ
	14, 3, 18	+				σd
	14, -3, 18	+				σd^2
-307	1, 0, 307	+				1
=	7, 1, 44	+				d
$-307\,(1)$	7, -1, 44	+				d^2
	4, 1, 77	+				d_1
	11, -1, 28	+				dd_1
	17, 4, 19	+				$d^2 d_1$
	4, -1, 77	+				d_1^2
	17, -4, 19	+				dd_1^2
	11, 1, 28	+				$d^2 d_1^2$
	2, 1, 154	+				σ
	14, 1, 22	+				σd
	14, -1, 22	+				σd^2
-339	1, 0, 339	+	+			1
=	7, 2, 49	+	+			d
$-339\,(1)^2$	7, -2, 49	+	+			d^2
	4, 1, 85	+	+			d_1
	15, 6, 25	+	+			dd_1
	13, -5, 28	+	+			$d^2 d_1$
	4, -1, 85	+	+			d_1^2
	13, 5, 28	+	+			dd_1^2
	15, -6, 25	+	+			$d^2 d_1^2$
	3, 0, 113	−	−			c
	20, -9, 21	−	−			cd
	20, 9, 21	−	−			cd^2
	12, -3, 29	−	−			cd_1
	5, 1, 68	−	−			cdd_1
	17, 1, 20	−	−			$cd^2 d_1$
	12, 3, 29	−	−			cd_1^2
	17, -1, 20	−	−			cdd_1^2
	5, -1, 68	−	−			$cd^2 d_1^2$
	2, 1, 170	+	+			σ
	14, -5, 20	+	+			σd
	14, 5, 26	+	+			σd^2
	6, 3, 58	−	−			σc
	10, 1, 34	−	−			σcd
	10, -1, 34	−	−			σcd^2

D	Classes	α	β	δ	ϵ	Cp
-459	1, 0, 459	+	+			1
=	9, 3, 52	+	+			d
$-51\,(3)^2$	9, -3, 52	+	+			d^2
	4, 1, 115	+	+			d_1
	19, -4, 25	+	+			dd_1
	13, 3, 36	+	+			$d^2 d_1$
	4, -1, 115	+	+			d_1^2
	13, -3, 36	+	+			dd_1^2
	19, 4, 25	+	+			$d^2 d_1^2$
	17, 0, 27	−	−			c
	20, -9, 27	−	−			cd
	20, 9, 27	−	−			cd^2
	11, 4, 44	−	−			cd_1
	5, 1, 92	−	−			cdd_1
	20, -1, 23	−	−			$cd^2 d_1$
	11, -5, 44	−	−			cd_1^2
	20, 1, 23	−	−			cdd_1^2
	5, -1, 92	−	−			$cd^2 d_1^2$
	2, 1, 230	+	+			σ
	18, 3, 26	+	+			σd
	18, -3, 26	+	+			σd^2
	22, 5, 22	−	−			σc
	10, 1, 46	−	−			σcd
	10, -1, 46	−	−			σcd^2
-675	1, 0, 675	+	+			1
=	9, 3, 76	+	+			d
$-3\,(15)^2$	9, -3, 76	+	+			d^2
	4, 1, 169	+	+			d_1
	25, -10, 31	+	+			dd_1
	19, 3, 36	+	+			$d^2 d_1$
	4, -1, 169	+	+			d_1^2
	19, -3, 36	+	+			dd_1^2
	25, 10, 31	+	+			$d^2 d_1^2$
	25, 0, 27	+	−			c
	27, -9, 28	+	−			cd
	27, 9, 28	+	−			cd^2
	13, 1, 52	+	−			cd_1
	25, 5, 28	+	−			cdd_1
	7, -2, 97	+	−			$cd^2 d_1$
	13, -1, 52	+	−			cd_1^2
	7, 2, 97	+	−			cdd_1^2
	25, -5, 28	+	−			$cd^2 d_1^2$
	2, 1, 338	+	+			σ
	18, 3, 38	+	+			σd
	18, -3, 38	+	+			σd^2
	26, 1, 26	+	−			σc
	14, -5, 50	+	−			σcd
	14, 5, 50	+	−			σcd^2

D	Classes	α	β	δ	ϵ	Cp
-755	1, 0, 755	+	+			1
=	19, −9, 44	+	+			d
$-755\,(1)^2$	19, 9, 44	+	+			d^2
	4, 1, 189	+	+			d_1
	21, −8, 39	+	+			dd_1
	11, 2, 69	+	+			d^2d_1
	4, −1, 189	+	+			d_1^2
	11, −2, 69	+	+			dd_1^2
	21, 8, 39	+	+			$d^2\,d_1^2$
	5, 0, 151	+	+			e^2
	9, −1, 84	+	+			$e^2\,d$
	9, 1, 84	+	+			$e^2\,d^2$
	20, 5, 39	+	+			$e^2\,d_1$
	29, 12, 31	+	+			$e^2\,dd_1$
	21, −1, 36	+	+			$e^2\,d^2\,d_1$
	20, −5, 39	+	+			$e^2\,d_1^2$
	12, 1, 36	+	+			$e^2\,dd_1^2$
	29, −12, 31	+	+			$e^2\,d^2\,d_1^2$
	27, 1, 28	−	−			e
	15, −5, 52	−	−			ed
	3, −1, 252	−	−			ed^2
	7, −1, 108	−	−			ed_1
	23, −2, 33	−	−			edd_1
	12, 5, 65	−	−			$ed^2\,d_1$
	28, 13, 33	−	−			ed_1^2
	13, 5, 60	−	−			edd_1^2
	12, −1, 63	−	−			$ed^2\,d_1^2$
	27, −1, 28	−	−			e^3
	3, 1, 252	−	−			$e^3\,d$
	15, 5, 52	−	−			$e^3\,d^2$
	28, −13, 33	−	−			$e^3\,d_1$
	12, 1, 63	−	−			$e^3\,dd_1$
	13, −5, 60	−	−			$e^3\,d^2\,d_1$
	7, 1, 108	−	−			$e^3\,d_1^2$
	12, −5, 65	−	−			$e^3\,dd_1^2$
	23, 2, 33	−	−			$e^3\,d^2\,d_1^2$
	2, 1, 378	+	+			σ
	22, 9, 38	+	+			σd
	22, −9, 38	+	+			σd^2
	10, 5, 78	+	+			σe^2
	18, −1, 42	+	+			$\sigma e^2\,d$
	18, 1, 42	+	+			$\sigma e^2\,d^2$
	14, −1, 54	−	−			σe
	26, 5, 30	−	−			σed
	6, −1, 126	−	−			σed^2
	14, 1, 54	−	−			σe^3
	6, 1, 126	−	−			$\sigma e^3\,d$
	26, −5, 30	−	−			$\sigma e^3\,d^2$

D	Classes	α	β	δ	ϵ	Cp
-891	1, 0, 891	+	+			1
=	9, 3, 100	+	+			d
$-11\,(9)^2$	9, −3, 100	+	+			d^2
	4, 1, 223	+	+			d_1
	31, 15, 36	+	+			dd_1
	25, 3, 36	+	+			$d^2\,d_1$
	4, −1, 223	+	+			d_1^2
	25, −3, 36	+	+			dd_1^2
	31, −15, 36	+	+			$d^2\,d_1^2$
	11, 0, 81	+	−			c
	20, −7, 47	+	−			cd
	20, 7, 47	+	−			cd^2
	23, 11, 44	+	−			cd_1
	20, 3, 45	+	−			cdd_1
	5, 2, 179	+	−			$cd^2\,d_1$
	23, −11, 44	+	−			cd_1^2
	5, −2, 179	+	−			cdd_1^2
	20, −3, 45	+	−			$cd^2\,d_1^2$
	2, 1, 446	+	+			σ
	18, 3, 50	+	+			σd
	18, −3, 50	+	+			σd^2
	22, 11, 46	+	+			σc
	10, 3, 90	+	+			σcd
	10, −3, 90	+	+			σcd^2
-974	1, 0, 974	+			+	1
=	18, −4, 55	+			+	d
$-2.487\,(1)^2$	18, 4, 55	+			+	d^2
	31, 7, 33	+			+	d_1
	30, −14, 39	+			+	dd_1
	15, −4, 66	+			+	$d^2\,d_1$
	31, −7, 33	+			+	d_1^2
	15, 4, 66	+			+	dd_1^2
	30, 14, 39	+			+	$d^2\,d_1^2$
	2, 0, 487	+			+	e
	9, −4, 110	+			+	$e^2\,d$
	9, 4, 110	+			+	$e^2\,d^2$
	25, −1, 39	+			+	$e^2\,d_1$
	15, 1, 65	+			+	$e^2\,dd_1$
	30, −4, 33	+			+	$e^2\,d^2\,d_1$
	25, 1, 39	+			+	$e^2\,d_1^2$
	30, 4, 33	+			+	$e^2\,dd_1^2$
	15, −1, 65	+			+	$e^2\,d^2\,d_1^2$
	27, 5, 37	−			−	e
	3, 1, 325	−			−	ed
	6, 2, 163	−			−	ed^2
	5, 1, 195	−			−	ed_1
	11, 4, 90	−			−	edd_1
	13, 1, 75	−			−	$ed^2\,d_1$
	10, 4, 99	−			−	ed_1^2
	26, 12, 43	−			−	edd_1^2
	22, 4, 45	−			−	$ed^2\,d_1^2$
	27, −5, 37	−			−	e^3
	6, −2, 163	−			−	$e^3\,d$
	3, −1, 325	−			−	$e^3\,d^2$
	10, −4, 99	−			−	$e^3\,d_1$
	22, −4, 45	−			−	$e^3\,dd_1$
	26, −12, 43	−			−	$e^3\,d^2\,d_1$
	5, −1, 195	−			−	$e^3\,d_1^2$
	13, −1, 75	−			−	$e^3\,dd_1^2$
	11, −4, 90	−			−	$e^3\,d^2\,d_1^2$

336.

NOTE SUR L'ÉLIMINATION.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LX. (1861), pp. 373—374.]

SOIENT $U=(a, \ldots \between x, y)^m$, $V=(b, \ldots \between x, y)^n$ des fonctions homogènes quelconques des degrés m et n respectivement. Dénotons par $(x, y)^\phi$ la suite entière ou seulement une partie de la suite de termes $x^\phi, x^{\phi-1}y, \ldots y^\phi$, et en prenant $\theta \overset{=}{>} m \overset{=}{>} n$, formons le déterminant

$$\{(x, y)^{\theta-m} U, (x, y)^{\theta-n} V\}.$$

Cette notation signifie qu'en supposant les suites $(x, y)^{\theta-m} U$, $(x, y)^{\theta-n} V$ composées respectivement de p et de q termes et qu'en posant $p+q=s$ on forme le déterminant

$$\begin{vmatrix} (x_1, y_1)^{\theta-m} U_1, & (x_1, y_1)^{\theta-n} V_1 \\ \vdots & \vdots \\ (x_s, y_s)^{\theta-m} U_s, & (x_s, y_s)^{\theta-n} V_s \end{vmatrix}$$

dans lequel les différentes lignes (chacune composée de s termes) sont ce que deviennent $(x, y)^{\theta-m} U$, $(x, y)^{\theta-n} V$, lorsqu'on y substitue (x_1, y_1), $(x_2, y_2), \ldots (x_s, y_s)$ successivement au lieu de (x, y).

Le déterminant que je viens définir est divisible par le déterminant

$$\{(x, y)^{\theta-1}\},$$

notation qui est équivalente à:

$$\begin{vmatrix} (x_1, y_1)^{\theta-1} \\ \vdots \\ (x_s, y_s)^{\theta-1} \end{vmatrix}$$

et dans laquelle $(x, y)^{s-1}$ dénote la suite entière des termes $x^{s-1}, x^{s-2}y, \ldots y^{s-1}$. Nous obtenons ainsi une équation

$$\frac{\{(x, y)^{\theta-m} U, (x, y)^{\theta-n} V\}}{\{(x, y)^{s-1}\}} = (a, \ldots)^p (b, \ldots)^q (x_1, y_1)^{\theta-s+1} \ldots (x_s, y_s)^{\theta-s+1},$$

c.-à-d. que le quotient est du degré p par rapport aux coefficients $(a, \ldots)$, du degré q par rapport aux coefficients $(b, \ldots)$, et du degré $\theta - s + 1$ par rapport à chaque système de variables $(x_1, y_1), \ldots (x_s, y_s)$. Or en supposant $U_1 = 0$, $V_1 = 0$, on obtient

$$0 = (a, \ldots)^p (b, \ldots)^q (x_1, y_1)^{\theta-s+1} \ldots (x_s, y_s)^{\theta-s+1},$$

équation qui subsiste quelles que soient les valeurs des variables $(x_2, y_2), \ldots (x_s, y_s)$; cela donne une suite de $(\theta - s + 2)^{s-1}$ équations chacune de la forme

$$0 = (a, \ldots)^p (b, \ldots)^q (x_1, y_1)^{\theta-s+1}.$$

En considérant un système quelconque de $\theta - s + 2$ de ces équations, pour en éliminer tous les termes de $(x_1, y_1)^{\theta-s+1}$, on obtiendra ou l'équation identique $0 = 0$, ou une équation de la forme

$$F = (a, \ldots)^{p(\theta-s+2)} (b, \ldots)^{q(\theta-s+2)} = 0$$

où F sera un déterminant de l'ordre $\theta - s + 2$, chaque terme étant de la forme $(a, \ldots)^p (b, \ldots)^q$.

Cela posé il est évident que F contiendra comme facteur la fonction

$$\square = (a, \ldots)^n (b, \ldots)^m$$

qui est le résultant des deux équations $U = 0$, $V = 0$. En particulier, on aura les deux cas que voici:

1°. Soit $\theta = m + n - 1$, et supposons que $(x, y)^{n-1} U$, $(x, y)^{m-1} V$, dénotent les suites entières

$$x^{n-1} U, x^{n-2} y U, \ldots y^{n-1} U; \; x^{m-1} V, x^{m-2} y V, \ldots y^{m-1} V,$$

nous aurons $p = n$, $q = m$, $s = m + n$, $\theta - s + 2 = 1$, et de là

$$F = (a, \ldots)^n (b, \ldots)^m,$$

donc $F = \square$. On voit sans peine que l'on obtient de cette manière le résultant $\square$, sous la forme d'un déterminant de l'ordre $m + n$, le même que l'on obtient en éliminant les termes de $(x, y)^{m+n-1}$ entre les équations $(x, y)^{n-1} U = 0$, $(x, y)^{m-1} V = 0$.

2°. En supposant $m \overset{=}{>} n$, on peut prendre $\theta = m$, ce qui donne pour $(x, y)^{\theta-m} U$ le seul terme U. Réduisons aussi $(x, y)^{\theta-n} V$ au seul terme $x^\alpha y^{m-n-\alpha} V$ (α désignant un

nombre entier arbitraire entre 0 et $m-n$), c.-à-d. au terme $x^{m-n}V$ ou $y^{m-n}V$ dans le cas des deux valeurs extrèmes de α. On a ainsi $p=1$, $q=1$, $s=2$, et delà

$$F=(a,\ldots)^m(b,\ldots)^n.$$

Done $F=(a,\ldots)^{m-n}\square$, c.-à-d. que l'on obtient le résultant $\square$ affecté d'un facteur $(a,\ldots)^{m-n}$ qui ne contient que les coefficients de U, et qui est de l'ordre $m-n$ par rapport à ces coefficients. L'expression de ce facteur peut être trouvée assez facilement. Dans le cas du terme $x^{m-n}V$, c.-à-d. pour $\alpha=m-n$, le facteur sera k^{m-n} (k désignant le dernier coefficient de la suite $(a,\ldots)$), et dans le cas du terme $y^{m-n}V$, c.-à-d. pour $\alpha=0$, le facteur sera a^{m-n}. Mais en supposant $m=n$ on a tout simplement $F=\square$, c.-à-d. que l'on obtient le résultant sans facteur étranger. C'est sous cette dernière forme que j'ai présenté la méthode abregée de *Bezout* dans le tome LIII. p. 366 (1857) de ce Journal, [230].

Londres, 17*ième* *Décembre*, 1861.

337.

NOTE SUR LA RÉALITÉ DES RACINES D'UNE ÉQUATION QUADRATIQUE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXI. (1863), pp. 367—368.]

À PROPOS du mémoire que vient de publier M. Hesse (voir ce *Journal* t. LX., p. 305) je remarque que si l'une ou l'autre des deux formes

$$(a, b, c, f, g, h \between)^2, \quad (a', b', c', f', g', h' \between)^2$$

est une forme *définie* (forme qui conserve toujours le même signe pour des valeurs réelles quelconques des variables), l'équation suivante en λ:

$$\begin{vmatrix} a-\lambda a', & h-\lambda h', & g-\lambda g', & x \\ h-\lambda h', & b-\lambda b', & f-\lambda f', & y \\ g-\lambda g', & f-\lambda f', & c-\lambda c', & z \\ x, & y, & z & \end{vmatrix} = 0$$

aura ses deux racines réelles. En écrivant

$$\begin{aligned} &A = bc - f'^2, \quad A' = b'c' - f'^2, \quad A_1 = bc' + b'c - 2ff', \\ &B = ca - g^2, \ldots \\ &\vdots \end{aligned}$$

de manière que $(A, B, C, F, G, H \between)^2$ dénote la forme adjointe (ou réciproque) de $(a, b, c, f, g, h \between)^2$, cette équation prend la forme

$$(A, \ldots \between x, y, z)^2 - \lambda (A_1, \ldots \between x, y, z)^2 + \lambda^2 (A', \ldots \between x, y, z)^2 = 0,$$

et les racines étant réelles, on doit avoir

$$\square = -4 (A, \ldots \between x, y, z)^2 (A', \ldots \between x, y, z)^2 + [(A_1, \ldots \between x, y, z)^2]^2 = +.$$

Or pour démontrer directement cette proposition, il n'est pas ce me semble possible d'exprimer $\square$ comme une somme de carrés; on a besoin de considérer une forme plus génerale, savoir une somme de carrés multipliés chacun par un coefficient litéral positif. Par exemple, en ne faisant attention qu'au coefficient de x^4, on doit avoir

$$\square_0 = -4(bc - f^2)(b'c' - f'^2) + (bc' + b'c - 2ff')^2 = +.$$

Pour en faire la démonstration, on peut exprimer $\square_0$ sous la forme

$$\square_0 = (bc' - b'c)^2 + 4(bf' - b'f)(cf' - c'f),$$

ce qui donne

$$bc\square_0 = (bc - f^2)(bc' - b'c)^2 + [b(cf' - c'f) + c(bf' - b'f)]^2.$$

En effet, en y substituant la seconde expression de $\square_0$, on a l'identité

$$4bc(bf' - b'f)(cf' - c'f) = -f^2(bc' - b'c)^2 + [b(cf' - c'f) + c(bf' - b'f)]^2$$

et l'expression pour $bc\square_0$ est ainsi démontrée. Mais en supposant que $(a, b, c, f, g, h)(\,,,)^2$ soit une forme définie, on a $bc - f^2 = +$, donc aussi $bc = +$, et $bc\square_0 = (bc - f^2)X^2 + Y^2 = +$, donc enfin $\square_0 = +$. Il serait assez intéressant de trouver une démonstration pareille pour l'expression générale de $\square$.

Londres, 23*ième* *Octobre* 1862.

338.

NOUVELLES RECHERCHES SUR L'ÉLIMINATION ET LA THÉORIE DES COURBES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXIII. (1864), pp. 34—39.]

DANS le problème de l'élimination, on cherche la relation qui doit exister entre les coefficients d'une fonction ou système de fonctions pour que quelque circonstance particulière (ou singularité) puisse avoir lieu; par exemple, pour que deux équations puissent avoir une racine commune, ou (comme application géometrique) pour qu'une courbe puisse avoir un point double. En prenant les coefficients comme donnés, tant la relation cherchée que la singularité qu'elle implique n'ont qu'une existence *hypothétique*. Mais on peut transformer la question en supposant que les coefficients d'une ou de plusieurs des fonctions soient de la forme $a = \lambda a' + \mu a''$, $b = \lambda b' + \mu b'', \ldots$ où $a', b', \ldots$, $a'', b'', \ldots$ sont des coefficients donnés, mais λ, μ des quantités arbitraires. On peut alors disposer en sorte que la singularité dont il s'agit existe actuellement, en déterminant, au moyen de la relation donnée par l'élimination, la valeur du rapport $\lambda : \mu$. Ces substitutions $a = \lambda a' + \mu a''$, $b = \lambda b' + \mu b'', \ldots$ changent la fonction U à laquelle se rapportent les coefficients $a, b, \ldots$ en $U = \lambda U' + \mu U''$, où U', U'' sont des fonctions semblables à U, mais avec les coefficients $a', b', \ldots$ ou $a'', b'', \ldots$ au lieu de $a, b, \ldots$: en se servant d'une expression usitée, on peut dire que la fonction U est en involution avec U', U''; et de même en géométrie que la courbe $U = 0$ est en involution avec les courbes $U' = 0$, $U'' = 0$; au reste, pour les courbes, cela veut dire que les trois courbes se coupent dans les mêmes points.

On conçoit comment cette manière d'envisager le problème peut conduire à une interprétation géométrique de résultats qui n'avaient auparavant qu'une signification analytique. Considérons par exemple la proposition suivante, "le discriminant d'une fonction quadratique à trois variables est du degré 3 par rapport aux coefficients," ou ce qui est la même chose, "la fonction qui égalée à zéro exprime que la conique

$U=0$ ait un point double (se réduise à une paire de droites) est du degré 3 par rapport aux coefficients," c'est là une proposition purement analytique, mais si comme ci-dessus on met $\lambda a'+\mu a''$, $\lambda b'+\mu b''$,... au lieu de $a, b,\ldots$ on a le théorème géométrique que voici: "Dans le système de coniques $\lambda U'+\mu U''=0$ en involution avec les coniques données $U'=0$, $U''=0$, il y a 3 coniques à point double (c'est-à-dire, trois paires de droites)." En considérant le cas plus général d'une fonction à trois variables et d'ordre quelconque, la question analytique "quel est le degré du discriminant de la fonction U" peut être remplacée par la question géométrique "dans le système des courbes $\lambda U'+\mu U''=0$ en involution avec les deux courbes données $U'=0$, $U''=0$, quel est le nombre des courbes à point double" ou, ce qui est la même chose, "quel est le nombre des points dont chacun est le point double d'une courbe du système." En considérant la question sous cette dernière forme, non seulement on retrouve la valeur connue $3(n-1)^2$ du degré du discriminant de la fonction $U=(A,\ldots \between x, y, z)^n$, mais on trouve aussi le théorème plus général:

La fonction $U=(A,\ldots \between x, y, z)^n$ étant telle que la courbe $U=0$ ait un nombre α de points doubles et un nombre β de points de rebroussement, son discriminant spécial est du degré $3(n-1)^2-7\alpha-11\beta$.

Sous la désignation de "discriminant special" j'entends la fonction laquelle égalée à zéro donne la condition pour que la courbe $U=0$ ait un point double de plus. Il convient de remarquer par rapport à cette expression que le discriminant de la fonction générale du $n^{\text{ième}}$ ordre, en y substituant, au lieu des valeurs générales, les coefficients de la fonction U dont il s'agit, ne donne nullement le discriminant special de U mais se réduit identiquement à zéro; ce discriminant special est donc tout autre chose que le discriminant de la fonction générale. En parlant tout simplement de *l'ordre* du discriminant special, j'ai voulu désigner l'ordre auquel cette expression s'élève par rapport à des coefficients absolument arbitraires ou éléments $a, b,\ldots$ lesquels sont censés entrer linéairement dans la fonction U. Il est donc nécessaire de démontrer d'abord la proposition auxiliaire que l'équation d'une courbe qui a déjà un nombre donné de points doubles et de rebroussement peut s'exprimer sous la forme signalée, c'est-à-dire linéairement par rapport à des coefficients absolument arbitraires ou éléments $a, b,\ldots$, proposition qui peut être démontrée sans difficulté.

Considérons en effet l'équation générale $U=(A,\ldots \between x, y, z)^n=0$ où les coefficients $A,\ldots$ sont tous arbitraires; dans le cas d'un point double supposons que les coordonnées de ce point, dans le cas d'un point de rebroussement supposons que les coordonnées de ce point et la direction de la tangente soient données: cela établit pour chaque point double trois conditions, et pour chaque point de rebroussement quatre conditions, qui contiennent d'une manière quelconque les paramètres appartenants au point double ou de rebroussement, mais qui sont linéaires par rapport aux coefficients $A,\ldots$: ces coefficients peuvent donc s'exprimer linéairement au moyen d'un nombre convenable de coefficients absolument arbitraires ou éléments $a, b,\ldots$; et c'est de ces éléments $a, b,\ldots$ qu'il s'agit et nullement des paramètres mentionnés ci-dessus qui entrent dans les expressions par lesquelles $A,\ldots$ sont donnés en termes de $a,\ldots$.

Cette proposition auxiliaire peut encore se démontrer de la manière que voici. Concevons que $P=0$ représente une courbe particulière quelconque du même ordre que $U=0$ et telle que pour chaque point double de la courbe $U=0$ elle ait un point double au même point, et que pour chaque point de rebroussement de la courbe $U=0$, elle ait un point de rebroussement au même point et avec la même tangente. Soient de même $Q=0$, $R=0, \ldots$ des équations de courbes qui satisfont aux mêmes conditions. Cela posé, on peut évidemment écrire $U=aP+bQ+cR+\ldots$, c'est-à-dire que l'équation contiendra linéairement les coefficients absolument arbitraires ou éléments $a, b, \ldots$.

Je reviens au théorème dont je suis parti; soit d'abord $U=(a, \ldots \between x, y, z)^n=0$ une courbe sans points doubles ou de rebroussement, de sorte qu'il s'agisse du discriminant ordinaire. En écrivant pour plus de simplicité V, W au lieu de U', U'', on a à considérer la courbe $\lambda V+\mu W=0$ en involution avec les deux courbes $V=0$, $W=0$. Le degré du discriminant de U est égal au nombre des points dont chacun est le point double d'une courbe particulière du système $\lambda V+\mu W=0$. Or pour trouver ces points on n'a qu'à former les équations

$$\lambda\partial_x V+\mu\partial_x W=0,$$
$$\lambda\partial_y V+\mu\partial_y W=0,$$
$$\lambda\partial_z V+\mu\partial_z W=0,$$

qui expriment que la courbe $\lambda V+\mu W=0$ a un point double, et d'éliminer entre ces équations les indéterminées λ, μ. Cela donne le système

$$\left\| \begin{matrix} \partial_x V, & \partial_y V, & \partial_z V \\ \partial_x W, & \partial_y W, & \partial_z W \end{matrix} \right\| = 0$$

qui comprend les deux équations

$$(1) \quad \begin{vmatrix} \partial_x V, & \partial_z V \\ \partial_x W, & \partial_z W \end{vmatrix} = 0, \qquad (2) \quad \begin{vmatrix} \partial_y V, & \partial_z V \\ \partial_y W, & \partial_z W \end{vmatrix} = 0$$

auxquelles on satisfait par $\partial_z V=0$, $\partial_z W=0$, et une troisième équation à laquelle on ne satisfait pas par ce dernier système. Or les courbes (1) et (2) se coupent en $4(n-1)^2$ points, mais parmi ceux-ci on a les $(n-1)^2$ points d'intersection des courbes $\partial_z V=0$, $\partial_z W=0$, et en écartant ces points on obtient $4(n-1)^2-(n-1)^2=3(n-1)^2$ pour le nombre des points, ou ce qui est la même chose pour le degré du discriminant de U.

Je suppose à présent que la courbe $U=0$ ait un point double; les courbes $V=0$, $W=0$ ont chacune un point double à ce même point, et en prenant ce point pour origine des coordonnées x, y les deux courbes seront

$$V=z^{n-2}(a, b, c \between x, y)^2+\text{etc.}=0,$$
$$W=z^{n-2}(a', b', c' \between x, y)^2+\text{etc.}=0,$$

en dénotant par les etc. les termes des ordres plus élevés par rapport à x, y, ou moins élevés par rapport à z.

Cela donne pour la courbe (1)

$$\begin{vmatrix} \partial_x V, & \partial_z V \\ \partial_x W, & \partial_z W \end{vmatrix} = 0 = z^{2n-3}\{(ax+by).(a',\ b',\ c'\between x,\ y)^2 - (a'x+b'y).(a,\ b,\ c\between x,\ y)^2\} + \text{etc.}$$
$$= z^{2n-5}\, y\,\{(ax+by)(b'x+c'y) - (a'x+b'y)(bx+cy)\} + \text{etc.};$$

la courbe (1) a donc à l'origine un point triple, les tangentes étant données par les équations

$$y = 0, \quad (ax+by)(b'x+c'y) - (a'x+b'y)(bx+cy) = 0,$$

et de même la courbe (2) a à l'origine un point triple, les tangentes étant données par les équations

$$x = 0, \quad (ax+by)(b'x+c'y) - (a'x+b'y)(bx+cy) = 0;$$

il y a donc au point triple deux branches de la courbe (1) dont chacune touche une de deux branches de la courbe (2); ce qui donne à l'origine $4+4+3=11$ points d'intersection. De plus il est évident que les deux courbes $\partial_z V = 0$, $\partial_z W = 0$ ont chacune un point double à l'origine, c'est-à-dire elles s'y coupent en $2+2=4$ points.

Par conséquent les courbes (1) et (2) se coupent en $4(n-1)^2$ points, savoir

11 points à l'origine, $4(n-1)^2 - 11$ points autrepart,

les courbes $\partial_z V = 0$, $\partial_z W = 0$ se coupent en $(n-1)^2$ points, savoir

4 points à l'origine, $(n-1)^2 - 4$ points autrepart,

et le système des $3(n-1)^2$ points contient

7 points à l'origine, $3(n-1)^2 - 7$ points autrepart.

En écartant les points à l'origine on a donc $3(n-1)^2 - 7$ points; pour une courbe à point double le degré du discriminant spécial est donc $= 3(n-1)^2 - 7$. Si la courbe $U=0$ a un point de rebroussement, les courbes $V=0$, $W=0$ auront au même point un point de rebroussement avec la même tangente, et en prenant ce point pour origine des coordonnées et la droite $x=0$ pour l'équation de la tangente, les deux courbes seront

$$V = z^{n-2}.\,a\,x^2 + z^{n-3}.(\alpha,\ \beta,\ \gamma,\ \delta\between x,\ y)^3 + \text{etc.} = 0,$$
$$W = z^{n-2}.\,a'x^2 + z^{n-3}.(\alpha',\ \beta',\ \gamma',\ \delta'\between x,\ y)^3 + \text{etc.} = 0.$$

Cela donne pour la courbe (1)

$$0 = \begin{vmatrix} \partial_x V, & \partial_z V \\ \partial_x W, & \partial_z W \end{vmatrix}$$
$$= \{z^{n-2}.\,2ax + z^{n-3}.\,3(\alpha,\ \beta,\ \gamma\between x,\ y)^2 + \text{etc.}\}$$
$$\times\{(n-2)\,z^{n-3}\,a'x^2 + (n-3)\,z^{n-4}(\alpha',\ \beta',\ \gamma',\ \delta'\between x,\ y)^3 + \text{etc.}\}$$
$$-\{z^{n-2}.\,2a'x + z^{n-3}.\,3(\alpha',\ \beta',\ \gamma'\between x,\ y)^2 + \text{etc.}\}$$
$$\times\{(n-2)\,z^{n-3}\,ax^2 + (n-3)\,z^{n-4}(\alpha,\ \beta,\ \gamma,\ \delta\between x,\ y)^3 + \text{etc.}\}$$

$$= z^{2n-6}\{2(n-3)[ax(\alpha', \beta', \gamma', \delta'\between x, y)^3 - a'x(\alpha, \beta, \gamma, \delta\between x, y)^3]$$
$$- 3(n-2)[ax^2(\alpha', \beta', \gamma'\between x, y)^2 - a'x^2(\alpha, \beta, \gamma\between x, y)^2]\} + \text{etc.}$$
$$= z^{2n-6}\{-n(a\alpha' - a'\alpha)x^4 + \text{etc. } x^3y\ldots\} + \text{etc.}$$
$$= z^{2n-6}.x(-n(a\alpha' - a'\alpha), \ldots\between x, y)^3 + \text{etc.};$$

la courbe a donc à l'origine un point quadruple et la droite $x=0$ y est tangente de l'une de ses branches. On a de même pour la courbe (2)

$$0 = \begin{vmatrix} \partial_y V, & \partial_z V \\ \partial_y W, & \partial_z W \end{vmatrix} = \{z^{n-3}.3(\beta, \gamma, \delta\between x, y)^2 + \text{etc.}\} \times \{(n-2)z^{n-3}a'x^2 + \text{etc.}\}$$
$$- \{z^{n-3}.3(\beta', \gamma', \delta'\between x, y)^2 + \text{etc.}\} \times \{(n-2)z^{n-3}ax^2 + \text{etc.}\}$$
$$= z^{2n-6}x^2\{a'(\beta, \gamma, \delta\between x, y)^2 - a(\beta', \gamma', \delta'\between x, y)^2\} + \text{etc.}$$
$$= z^{2n-6}x^2(\alpha'\beta - \alpha\beta', \ldots\between x, y)^2;$$

cette courbe a donc à l'origine un point quadruple et la droite $x=0$ y est tangente commune de deux de ses branches. Cela donne à l'origine $5+4+4+4, =17$ points d'intersection des courbes (1) et (2). D'autre part on a

$$\partial_z V = (n-2)z^{n-3}.ax^2 + (n-3)z^{n-4}(\alpha, \beta, \gamma, \delta\between x, y)^3 + \text{etc.} = 0,$$
$$\partial_z W = (n-2)z^{n-3}.a'x^2 + (n-3)z^{n-4}(\alpha', \beta', \gamma', \delta'\between x, y)^3 + \text{etc.} = 0,$$

et en combinant ces deux équations

$$z^{n-3}.ax^2 + \text{etc.} = 0,$$
$$z^{n-4}(a\alpha' - a'\alpha, \ldots\between x, y)^3 + \text{etc.} = 0.$$

De ces deux équations la première appartient à une courbe qui a un point de rebroussement à l'origine des coordonnées et la seconde à une courbe qui y a un point triple. Pour les deux courbes $\partial_z V = 0$, $\partial_z W = 0$ cela donne $3+3=6$ points d'intersection à l'origine.

Les courbes (1) et (2) se coupent donc en $4(n-1)^2$ points, savoir

17 points à l'origine, $4(n-1)^2 - 17$ points autrepart,

les courbes $\partial_z V = 0$, $\partial_z W = 0$ se coupent en $(n-1)^2$ points, savoir

6 points à l'origine, $(n-1)^2 - 6$ points autrepart

et le système des $3(n-1)^2$ points contient

11 points à l'origine, $3(n-1)^2 - 11$ points autrepart.

En écartant les points à l'origine, on obtient $3(n-1)^2 - 11$ points; pour une courbe avec un point de rebroussement, le degré du discriminant spécial est donc $=3(n-1)^2 - 11$.

Comme résultat final de cette recherche j'obtiens que la réduction du degré est de 7 unités pour un point double et de 11 unités pour un point de rebroussement; et

de là, que pour un nombre α de points doubles et β de points de rebroussement, la réduction est de $7\alpha + 11\beta$ unités; le degré du discriminant spécial sera donc dans ce cas $3(n-1)^2 - 7\alpha - 11\beta$, ce qu'il s'agissait de démontrer.

Dans tout ce qui précède, j'ai supposé que le système soit tel que l'élimination conduise à une seule relation entre les coefficients; si au contraire l'élimination conduit à deux relations, il faut écrire au lieu de $a, b, \ldots$ les valeurs $\lambda a' + \mu a'' + \nu a'''$, $\lambda b' + \mu b'' + \nu b''', \ldots$: et de même pour un plus grand nombre de relations. En supposant par exemple que la courbe $U = 0$ doit avoir un point de rebroussement, ce qui implique deux relations entre les coefficients, la question à resoudre serait celle-ci, "quel est le nombre des points qui sont chacun un point de rebroussement d'une courbe particulière du système $\lambda V + \mu W + \nu X = 0$"; je réserve à une autre occasion la considération de ce problème.

Londres, 22ième *Mai* 1863.

339.

ON SKEW SURFACES, OTHERWISE SCROLLS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLIII. (for the year 1863) pp. 453—483. Received February 3,—Read March 5, 1863.]

IT may be convenient to mention at the outset that, in the paper "On the Theory of Skew Surfaces"(1), I pointed out that upon any skew surface of the order n there is a singular (or nodal) curve meeting each generating line in $(n-2)$ points, and that the class of the circumscribed cone (or, what is the same thing, the class of the surface) is equal to the order n of the surface. In the paper "On a Class of Ruled Surfaces"(2), Dr Salmon considered the surface generated by a line which meets three curves of the orders m, n, p respectively: such surface is there shown to be of the order $=2mnp$; and it is noticed that there are upon it a certain number of double right lines (nodal generators); to determine the number of these, it was necessary to consider the skew surface generated by a line meeting a given right line and a given curve of the order m twice; and the order of such surface is found to be $=\frac{1}{2}m(m-1)+h$, where h is the number of apparent double points of the curve. The theory is somewhat further developed in Dr Salmon's memoir "On the Degree of a Surface reciprocal to a given one"(3), where certain minor limits are given for the orders of the nodal curves on the skew surface generated by a line meeting a given right line and two curves of the orders m and n respectively, and on that generated by a line meeting a given right line and a curve of the order m twice. And in the same memoir the author considers the skew surface generated by a line the equations whereof are $(a, \ldots \between t, 1)^m=0$, $(a', \ldots \between t, 1)^n=0$, where $a, \ldots a', \ldots$ are any linear functions of the coordinates, and t is an arbitrary parameter. And the same theories are reproduced in the "Treatise on the Analytic Geometry of Three Dimensions"(4).

1 *Cambridge and Dublin Math. Journ.* vol. VII. pp. 171—173 (1852), [108].
2 *Ibid.* vol. VIII. pp. 45, 46 (1853).
3 *Trans. Royal Irish Acad.* vol. XXIII. pp. 461—488 (read 1855).
4 Dublin, 1862. [Ed. 4, 1882.]

I will also, though it is less closely connected with the subject of the present memoir, refer to a paper by M. Chasles, "Description des courbes à double courbure de tous les ordres sur les surfaces réglées du troisième et du quatrième ordre"[1].

The present memoir (in the composition of which I have been assisted by a correspondence with Dr Salmon) contains a further development of the theory of the skew surfaces generated by a line which meets a given curve or curves: viz. I consider, 1st, the surface generated by a line which meets each of three given curves of the orders m, n, p respectively; 2nd, the surface generated by a line which meets a given curve of the order m twice, and a given curve of the order n once; 3rd, the surface which meets a given curve of the order m three times; or, as it is very convenient to express it, I consider the skew surfaces, or say the "Scrolls," $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$. The chief results are embodied in the Table given after this introduction, at the commencement of the memoir. It is to be noticed that I attend throughout to the general theory, not considering otherwise than incidentally the effect of any singularity in the system of the given curves, or in the given curves separately: the memoir contains however some remarks as to what are the singularities material to a complete theory; and, in particular as regards the surface $S(m^3)$, I am thus led to mention an entirely new kind of singularity of a curve in space—viz. such a curve has in general a determinate number of "lines through four points" (lines which meet the curve in four points); it may happen that, of the lines through three points which can be drawn through any point whatever of the curve, a certain number will unite together and form a line through four (or more) points, the number of the lines through four points (or through a greater number of points) so becoming infinite.

Notation and Table of Results, Articles 1 to 10.

1. In the present memoir a letter such as m denotes the order of a curve in space. It is for the most part assumed that the curve has no actual double points or stationary points, and the corresponding letter M denotes the class of the curve taken negatively and divided by 2; that is, if h be the number of apparent double points, then $M = -\frac{1}{2}[m]^2 + h$: here and elsewhere $[m]^2$, &c. denote factorials, viz. $[m]^2 = m(m-1)$, $[m]^3 = m(m-1)(m-2)$, &c. It is to be noticed that for the system of two curves m, m', if h, h' represent the number of apparent double points of the two curves respectively, then for the system the number of apparent double points is $= mm' + h + h'$, and the corresponding value of M is therefore $-\frac{1}{2}[m+m']^2 + mm' + h + h'$, which is $= -\frac{1}{2}[m]^2 + h - \frac{1}{2}[m']^2 + h'$, which is $= M + M'$.

2. The use of the combinations (m, n, p, q), (m^2, n, p), &c. hardly requires explanation; it may however be noticed that $G(m, n, p, q)$ denoting the lines which meet the curves m, n, p, q (that is, curves of these orders) each of them once, $G(m^2, n, p)$ will denote the lines which meet the curve m twice and the curves n and p each of them once; and so in all similar cases.

3. The letters G, S, ND, NG, NR, NT (read Generators, Scroll, Nodal Director, Nodal Generator, Nodal Residue, and Nodal Total) are in the nature of functional

[1] *Comptes Rendus*, t. LIII. (1861, 2e Sem.), pp. 884—889.

symbols, used (according to the context) to denote geometrical forms, or else the orders of these forms. Thus $G(m, n, p, q)$ denotes either the lines meeting the curves m, n, p, q each of them once, or else it denotes the order of such system of lines, that is, the number of lines. And so $S(m, n, p)$ denotes the Skew Surface or Scroll generated by a line which meets the curves m, n, p each once, or else it denotes the order of such surface.

4. $G(m, n, p, q)$: the signification is explained above.

5. $S(m, n, p)$: the signification has just been explained; but as the surfaces $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$ are in fact the subject of the present memoir, I give the explanation in full for each of them, viz. $S(m, n, p)$ is the surface generated by a line which meets the curves m, n, p each once; $S(m^2, n)$ is the surface generated by a line which meets the curve m twice and the curve n once; $S(m^3)$ the surface generated by the line which meets the curve m thrice. As already mentioned, these surfaces and their orders are represented by the same symbols respectively.

6. $ND(m, n, p)$. The directrix curves m, n, p of the scroll $S(m, n, p)$ are nodal (multiple) curves on the surface, viz. m is an np-tuple curve, and so for n and p. Reckoning each curve according to its multiplicity, viz. the curve m being reckoned $\frac{1}{2}[np]^2$ times, or as of the order $m \cdot \frac{1}{2}[np]^2$, and so for the curves n and p, the aggregate, or sum of the orders, gives the Nodal Director $ND(m, n, p)$.

7. $NG(m, n, p)$. The scroll $S(m, n, p)$ has the nodal generating lines $G(m^2, n, p)$, $G(m, n^2, p)$, $G(m, n, p^2)$. Each of these is a mere double line, to be reckoned once only, and we have thus the Nodal Generator

$$NG(m, n, p) = G(m^2, n, p) + G(m, n^2, p) + G(m, n, p^2).$$

But to take another example, the scroll $S(m^2, n)$ has the nodal generating lines $G(m^2, n)$, each of which is a triple line to be reckoned $\frac{1}{2}[3]^2$, that is, three times, and also the nodal generating lines $G(m^2, n^2)$, each of them a mere double line to be reckoned once only; whence here $NG(m^2, n) = 3G(m^3, n) + G(m^2, n^2)$. And so for the scroll $S(m^3)$, this has the nodal generating lines $G(m^4)$, each of them a quadruple line to be reckoned $\frac{1}{2}[4]^2$, that is, six times; or we have $NG(m^3) = 6G(m^4)$.

8. $NR(m, n, p)$. The scroll $S(m, n, p)$ has besides the directrix curves m, n, p or Nodal Director, and the nodal generating lines or Nodal Generator, a remaining nodal curve or Nodal Residue, the locus of the intersections of two non-coincident generating lines meeting in a point not situate on any one of the directrix curves. This Nodal Residue, as well for the scroll $S(m, n, p)$ as for the scrolls $S(m^2, n)$ and $S(m^3)$ respectively, is a mere double curve to be reckoned once only; and such curve or its order is denoted by NR, viz. for the scroll $S(m, n, p)$, the Nodal Residue is $NR(m, n, p)$.

9. $NT(m, n, p)$. The Nodal Director, Nodal Generator, and Nodal Residue of the scroll $S(m, n, p)$ form together the Nodal Total $NT(m, n, p)$, that is, we have

$$NT(m, n, p) = ND(m, n, p) + NG(m, n, p) + NR(m, n, p);$$

and similarly for the scrolls $S(m^2, n)$ and $S(m^3)$.

10. I remark that the formulæ are best exhibited in an order different from that in which they are in the sequel obtained, viz. I collect them in the following

Table.

$$
\begin{aligned}
G(m, n, p, q) &= 2mnpq,\\
G(m^2, n, p) &= np([m]^2 + M),\\
G(m^2, n^2) &= \tfrac{1}{2}[m]^2[n]^2 + M \,.\, \tfrac{1}{2}[n]^2 + N \,.\, \tfrac{1}{2}[m]^2 + MN,\\
G(m^2, n) &= n\left(\tfrac{1}{3}[m]^3 + M(m-2)\right),\\
G(m^4) &= \tfrac{1}{12}[m]^4 + m + M\left(\tfrac{1}{2}[m]^2 - 2m + \tfrac{11}{2}\right) + M^2 \,.\, \tfrac{1}{2},\\
S(m, n, p) &= 2mnp,\\
ND(m, n, p) &= \tfrac{1}{2}mnp(mn + mp + np - 3),\\
NG(m, n, p) &= mnp(m + n + p - 3) + Mnp + Nmp + Pmn,\\
NR(m, n, p) &= \tfrac{1}{2}mnp\left(4mnp - (mn + mp + np) - 2(m + n + p) + 5\right),\\
(^1)\, NT(m, n, p) &= \tfrac{1}{2}S^2 - S + Mnp + Nmp + Pmn,\\
&= 2mnp(mnp - 1) + Mnp + Nmp + Pmn;
\end{aligned}
$$

included in which we have

$$
\begin{aligned}
S(1, 1, m) &= 2m,\\
ND(1, 1, m) &= [m]^2,\\
NG(1, 1, m) &= [m]^2 + M,\\
NR(1, 1, m) &= 0,\\
NT(1, 1, m) &= \tfrac{1}{2}S^2 - S + M,\\
&= 2[m]^2 + M,
\end{aligned}
$$

and

$$
\begin{aligned}
S(1, m, n) &= 2mn,\\
ND(1, m, n) &= \tfrac{1}{2}mn(mn + m + n - 3),\\
NG(1, m, n) &= mn(m + n - 2) + Mn + Nm,\\
NR(1, m, n) &= \tfrac{3}{2}[m]^2[n]^2,\\
NT(1, m, n) &= \tfrac{1}{2}S^2 - S + Mn + Nm,\\
&= 2[mn]^2 + Mn + Nm.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
S(m^2, n) &= n\,(\,[m]^2 + M),\\
ND(m^2, n) &= n\left(\tfrac{1}{8}[m]^4 + [m]^3 + M\left(\tfrac{1}{2}[m]^2 - \tfrac{1}{2}\right) + M^2 \,.\, \tfrac{1}{2}\right) + [n]^2\left(\tfrac{1}{2}[m]^3 + \tfrac{1}{2}[m]^2\right),\\
NG(m^2, n) &= n\left(\,[m]^3 + M \,.\, 3(m-2)\right) + [n]^2\left(\tfrac{1}{2}[m]^2 + \tfrac{1}{2}M\right) + N\left(\tfrac{1}{2}[m]^3 + M\right),
\end{aligned}
$$

[1] In the first of the two expressions for $NT(m, n, p)$, S stands for $S(m, n, p)$; and so in the first of the two expressions for $NT(m^2, n)$, &c., S stands for $S(m^2, n)$, &c.

$$\begin{aligned}
NR(m^2, n) &= n\left(\tfrac{3}{8}[m]^4 + M\left(\tfrac{1}{2}[m]^2 - 2m + 3\right)\right) \\
&\qquad + [n]^2\left(\tfrac{1}{2}[m]^4 + \tfrac{3}{2}[m]^3 + [m]^2 + M\left([m]^2 - \tfrac{1}{2}\right) + M^2 . \tfrac{1}{2}\right), \\
NT(m^2, n) &= \tfrac{1}{2}S^2 - S + nM\left(m - \tfrac{5}{2}\right) + N\left(\tfrac{1}{2}[m]^2 + M\right), \\
&= n\left(\tfrac{1}{2}[m]^4 + 2[m]^3 + M\left([m]^2 + m - \tfrac{7}{2}\right) + M^2 . \tfrac{1}{2}\right) \\
&\qquad + [n]^2\left(\tfrac{1}{2}[m]^4 + 2[m]^3 + [m]^2 + M . [m]^2 + M^2 . \tfrac{1}{2}\right) + N\left(\tfrac{1}{2}[m]^2 + M\right);
\end{aligned}$$

included in which we have

$$\begin{aligned}
S(1, m^2) &= [m]^2 + M, \\
ND(1, m^2) &= \tfrac{1}{8}[m]^4 + [m]^3 + M\left(\tfrac{1}{2}[m]^2 - \tfrac{1}{2}\right) + M^2 . \tfrac{1}{2}, \\
NG(1, m^2) &= [m]^3 + M . 3(m - 2), \\
NR(1, m^2) &= \tfrac{3}{8}[m]^4 + M\left(\tfrac{1}{2}[m]^2 - 2m + 3\right), \\
NT(1, m^2) &= \tfrac{1}{2}S^2 - S + M\left(m - \tfrac{5}{2}\right), \\
&= \tfrac{1}{2}[m]^4 + 2[m]^3 + M\left([m]^2 + m - \tfrac{7}{2}\right) + M^2 . \tfrac{1}{2};
\end{aligned}$$

and finally

$$\begin{aligned}
S(m^3) &= \tfrac{1}{3}[m]^3 + (m - 2)M, \\
ND(m^3) &= \tfrac{1}{8}[m]^5 + \tfrac{1}{2}[m]^4 + \tfrac{1}{2}[m]^3 + M\left(\tfrac{1}{2}[m]^3 + \tfrac{1}{2}[m]\right) + M^2 . \tfrac{1}{2}m, \\
NG(m^3) &= \tfrac{1}{2}[m]^4 + 6m + M\left(3[m]^2 - 12m + 33\right) + M^2 . 3, \\
NR(m^3) &= \tfrac{1}{18}[m]^6 + \tfrac{3}{8}[m]^5 - \tfrac{1}{2}[m]^3 + 3m \\
&\qquad + M\left(\tfrac{1}{3}[m]^4 - \tfrac{1}{6}[m]^3 - \tfrac{5}{2}[m]^2 + 8m - 20\right) + M^2\left(\tfrac{1}{2}[m]^2 - 2m\right), \\
NT(m^3) &= \tfrac{1}{2}S^2 - S + 3m + M\left(\tfrac{1}{2}[m]^2 - \tfrac{5}{2}m + 11\right) + M^2, \\
&= \tfrac{1}{18}[m]^6 + \tfrac{1}{2}[m]^5 + [m]^4 + 3m \\
&\qquad + M\left(\tfrac{1}{6}[m]^4 + \tfrac{1}{3}[m]^3 + \tfrac{1}{2}[m^2] - \tfrac{7}{2}m + 13\right) + M^2\left(\tfrac{1}{2}[m]^2 - \tfrac{3}{2}m + 3\right).
\end{aligned}$$

The formulæ are investigated in the following order, ND, G, NG, S, NR, and NT.

The ND formulæ, Articles 11 to 13.

11. $ND(m, n, p)$.—Taking any point on the curve m, this is the vertex of two cones passing through the curves n, p respectively; the cones are of the orders n, p respectively, and they intersect therefore in np lines, which are the generating lines through the point on the curve m; hence this curve is an np-tuple line on the scroll $S(m, n, p)$, and we have thus the term $m . \tfrac{1}{2}[np]^2$ of ND. Hence

$$\begin{aligned}
ND(m, n, p) &= m . \tfrac{1}{2}[np]^2 + n . \tfrac{1}{2}[mp]^2 + p . \tfrac{1}{2}[mn]^2, \\
&= \tfrac{1}{2}mnp\,(mn + mp + np - 3).
\end{aligned}$$

12. $ND(m^2, n)$.—Taking first a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m, and of a cone of the order n through the curve n; the two cones intersect in $(m-1)n$ lines, which are the generating lines through the point on the curve m; that is, the curve m is a $(m-1)n$-tuple line on the scroll $S(m^2, n)$; and we have thus the term $m \cdot \frac{1}{2}[(m-1)n]^2$ of ND. Taking next a point on the curve n, this is the vertex of a cone of the order m through the curve m; such cone has $(h=)\frac{1}{2}[m]^2+M$ double lines, which are the generating lines through the point on the curve n; hence this curve is a $(\frac{1}{2}[m]^2+M)$tuple line on the surface, and we have thus the term $n \cdot \frac{1}{2}\left[\frac{1}{2}[m]^2+M\right]^2$ in ND. And therefore

$$\begin{aligned} ND(m^2, n) &= m \cdot \tfrac{1}{2}[(m-1)n]^2 + n \cdot \tfrac{1}{2}\left[\tfrac{1}{2}[m]^2+M\right]^2, \\ &= \quad n\left(\tfrac{1}{8}[m]^4+[m]^3+M\left(\tfrac{1}{2}[m]^2-\tfrac{1}{2}\right)+M^2 \cdot \tfrac{1}{2}\right)+[n]^2\left(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]^2\right). \end{aligned}$$

13. $ND(m^3)$.—Taking a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m; such cone has $(h-m+2=)\frac{1}{2}[m]^2-m+2+M$ double lines, or the curve m is a $(\frac{1}{2}[m]^2-m+2+M)$tuple line on the scroll $S(m)^3$). Hence we have

$$\begin{aligned} ND(m)^3 &= m \cdot \tfrac{1}{2}\left[\tfrac{1}{2}[m]^2-m+2+M\right]^2, \\ &= \tfrac{1}{8}[m]^5+\tfrac{1}{2}[m]^4+\tfrac{1}{2}[m]^3+M\left(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]\right)+M^2 \cdot \tfrac{1}{2}m. \end{aligned}$$

Preparatory remarks in regard to the G formulæ, the hypertriadic singularities of a curve in space, Articles 14 to 22.

14. It is to be remarked that the generating line of any one of the scrolls $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$ satisfies three conditions; and that it cannot in anywise happen that one of these conditions is implied in the other two. Thus, for instance, as regards the scroll $S(m, n, p)$, if the curves m, n are given, and we take the entire series of lines meeting each of these curves, these lines form a double series of lines, all of them passing of course through the curves m, n, but not all of them passing through any other curve whatever; that is, there is no curve p such that every line passing through the curves m and n passes also through the curve p. And the like as regards the scrolls $S(m^2, n)$ and $S(m^3)$.

15. But (in contrast to this) if the three conditions are satisfied, it may very well happen that a fourth condition is satisfied *ipso facto*. To see how this is, imagine a curve q on the scroll $S(m, n, p)$, or, to meet an objection which might be raised, say a curve q the complete intersection of the scroll $S(m, n, p)$ by a plane or any other surface. Every line whatever which meets the curves m, n, p is a generating line of the scroll $S(m, n, p)$, and as such will meet the curve q; that is, in the case in question, $G(m, n, p, q)$, the lines which meet the curves m, n, p, q, are the entire series of generating lines of the scroll $S(m, n, p)$, and they are thus infinite in number; so that in such case the question does not arise of finding the number of

the lines $G(m, n, p, q)$. The like remarks apply to the lines $G(m^2, n, p)$, $G(m^3, n^2)$, $G(m^3, n)$, and $G(m^4)$; but I will develope them somewhat more particularly as regards the lines $G(m^4)$.

16. Given a curve m, then (as in fact mentioned in the investigation for $ND(m^3)$) through *any point whatever* of the curve there can be drawn

$$(h-m+2=)\,[m]^2+m-2+M$$

lines meeting the curve in two other points, or say $[m]^2+m-2+M$ lines through three points. But in general no one of these lines meets the curve in a fourth point; that is, we cannot through every point of the curve m draw a line through four points; there are, however, on the curve m a *certain* number $(=4G(m^4))$ of points through which can be drawn a line through four points, or line $G(m^4)$.

17. But the curve m may be such that through every point of the curve there passes a line through four points. In fact, assume any skew surface or scroll whatever, and upon this surface a curve meeting each generating line in four points (e.g. the intersection of the scroll by a quartic surface). Taking the curve in question for the curve m, then it is clear that through every point of this curve there passes a line (the generating line of the assumed scroll) which is a line through four points, or line $G(m^4)$.

18. It is to be noticed, moreover, that if we take on the curve m any point whatever, then of the $[m]^2+m-2+M$ lines through three points which can be drawn through this point, three will unite together in the generating line of the assumed scroll (for if 0 be the point on the curve m, and 1, 2, 3 the other points in which the generating line of the assumed scroll meets the curve m, then such generating line unites the three lines 012, 013, 023, each of them a line through three points); and there will be besides $\frac{1}{2}[m]^2+m-5+M$ mere lines through three points. The line through four points generates the assumed scroll taken $(\frac{1}{2}[3]^2=)$ 3 times, or considered as three coincident scrolls; the remaining lines generate a scroll $S'(m^3)$, which is such that the curve m is on this scroll a $(\frac{1}{2}[m]^2+m-5+M)$tuple line; the assumed scroll three times and the scroll $S'(m^3)$ make up the entire scroll $S(m^3)$ derived from the curve m, or say $S(m^3)=3$ (assumed scroll) $+S'(m^3)$.

19. The case just considered is that of a curve m such that through every point of it there passes a line through four points counting as $(\frac{1}{2}[3]^2=)$ 3 lines through three points, and that all the other lines through three points are mere lines through three points. But it is clear that we may in like manner have a line through p points counting as $\frac{1}{2}[p-1]^2$ lines through three points; and more generally if $p, q, \ldots$ are numbers all different and not <3, and if

$$\tfrac{1}{2}[m]^2-m+2+M=\alpha\,.\,\tfrac{1}{2}[p-1]^2+\beta\,.\,\tfrac{1}{2}[q-1]^2+\ldots,$$

then we may have a curve m such that through every point of it there pass α lines each through p points and counting as $\frac{1}{2}[p-1]^2$ lines, β lines each through q points and counting as $\frac{1}{2}[q-1]^2$ lines, &c....: the case $p=3$ gives of course α lines each

through three points and counting as a single line. It is to be added that, in the case just referred to, the α lines will generate a scroll $S'(m^3)$ taken $\frac{1}{6}[p]^3$ times, the β lines will generate a scroll $S''(m^3)$ taken $\frac{1}{6}[q]^3$ times, &c., which scrolls together make up the scroll $S(m^3)$, or say

$$S(m^3) = \tfrac{1}{6}[p]^3 . S'(m^3) + \tfrac{1}{6}[q]^3 . S''(m^3) + \&c.;$$

it may however happen that, e.g. of the α lines, any set or sets or even each line will generate a distinct scroll or scrolls—that is, that the scroll $S'(m^3)$ will itself break up into scrolls of inferior orders.

20. A good illustration is afforded by taking for the curve m a curve on the hyperboloid or quadric scroll[1]; such curves divide themselves into species; viz. we have say the (p, q) curve on the hyperboloid, a curve of the order $p+q$ meeting each generating line of the one kind in p points, and each generating line of the other kind in q points; here

$$m = p+q, \ (h = \tfrac{1}{2}[p]^2 + \tfrac{1}{2}[q]^2, \text{ and } \therefore) \ M = -pq.$$

Assuming for the moment that p, q are each of them not less than 3, it is clear that the lines through three points which can be drawn through any point of the curve are the generating line of the one kind counting as $\frac{1}{2}[p-1]^2$ lines through three points, and the generating line of the other kind counting as $\frac{1}{2}[q-1]^2$ lines through three points, so that

$$\tfrac{1}{2}[m]^2 + m - 2 + M = \tfrac{1}{2}[p-1]^2 + \tfrac{1}{2}[q-1]^2.$$

The complete scroll $S(m^3)$ is made up of the hyperboloid considered as generated by the generating lines of the one kind taken $\frac{1}{6}[p]^3$ times, and the hyperboloid considered as generated by the generating lines of the other kind taken $\frac{1}{6}[q]^3$ times (so that there is in this case the speciality that the surfaces $S'(m^3)$, $S''(m^3)$ are in fact the same surface). And hence we have

$$S(m^3) = \left(2\left(\tfrac{1}{6}[p]^3 + \tfrac{1}{6}[q]^3\right) =\right) \tfrac{1}{3}[p]^3 + \tfrac{1}{3}[q]^3.$$

21. I notice also the case of a system of m lines. Taking here a point on one of the lines, the $(h - m + 2 =)\,[m]^2 - m + 2$ lines through three points which can be drawn through this point are the $\frac{1}{2}[m-1]^2$ lines which can be drawn meeting a pair of the other $(m-1)$ lines, and besides this the line itself counting as one line through three points $(\frac{1}{2}[m-1]^2 + 1 = \frac{1}{2}[m]^2 - m + 2)$; the line itself, thus counting as a single line through three points, is not to be reckoned as a line through four or more points drawn through the point in question, that is, the system is not to be regarded as a curve through every point of which there passes a line through four points: each of the lines is nevertheless to be counted as a single line through four points, and (since there are besides two lines which may be drawn meeting each four of the m lines) the total number of lines through four points is $= \frac{1}{12}[m]^4 + m$.

22. In the following investigations for $G(m, n, p, q)$, &c., the foregoing special cases are excluded from consideration; it may however be right to notice how it is

[1] It is hardly necessary to remark that (*reality* being disregarded) any quadric surface whatever is a hyperboloid or quadric scroll.

that the formulæ obtained are inapplicable to these special cases; for instance, as will immediately be seen, the number of the lines $G(m, n, p, q)$ is obtained as the number of intersections of the surface $S(m, n, p)$ by the curve q, $= 2mnp \times q = 2mnpq$; but if the curve q lie on the surface $S(m, n, p)$, then $G(m, n, p, q)$ is no longer $= 2mnpq$.

The G formulæ, Articles 23 to 34.

23. $G(m, n, p, q)$.—Considering the scroll $S(m, n, p)$ generated by a line which meets each of the curves m, n, p, this meets the curve q in $q\,S(m, n, p)$ points through each of which there passes a line $G(m, n, p, q)$; that is, we have

$$G(m, n, p, q) = q\,S(m, n, p).$$

But from this equation we have

$$S(m, n, p) = G(1, m, n, p) = p\,S(1, m, n);$$

thence also

$$S(1, m, n) = G(1, 1, m, n) = n\,S(1, 1, n),$$

and

$$S(1, 1, m) = G(1, 1, 1, m) = m\,S(1, 1, 1);\quad S(1, 1, 1) = G(1, 1, 1, 1) = 2,$$

since 2 is the number of lines which can be drawn meeting each of four given right lines. Hence ultimately

$$G(m, n, p, q) = mnpq\,G(1, 1, 1, 1) = 2mnpq.$$

24. $G(m^2, n, p)$.—In a precisely similar manner we find

$$G(m^2, n, p) = np\,G(1, 1, m^2) = np\,S(1, m^2),$$

and it is the same question to find $G(1, 1, m^2)$ and to find $S(1, m^2)$. I investigate $G(1, 1, m^2)$ by considering the particular case where the curve m is a plane curve having n double points. The plane of the curve meets the two lines 1, 1 in two points, and the line through these two points meets each of the lines 1, 1, and meets the curve in m points; combining the last-mentioned m points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2$ coincident lines, each of them meeting the lines 1, 1, and also meeting the curve m twice. But we may also through any double point of the curve draw a line meeting each of the lines 1, 1; such line, inasmuch as it passes through a double point, meets the curve twice; and we have h such lines. This gives for the case in question $G(1, 1, m^2) = h + \frac{1}{2}[m]^2$; or, introducing in the place of h the quantity $M(= h - \frac{1}{2}[m]^2)$, so that $h = \frac{1}{2}[m]^2 + M$, we have

$$G(1, 1, m^2) = [m]^2 + M;$$

and, to the double points of the plane curve, there correspond in the general case the apparent double points of the curve m. Admitting the correctness of the result just obtained, we then have

$$G(m^2, n, p) = np([m]^2 + M).$$

25. $G(m^2, n^2)$.—I investigate the value by a process similar to that employed for $G(1, 1, m^3)$. Suppose that the curves m and n are plane curves having respectively h and k double points; then the line of intersection of the two planes meets the curve m in m points, and the curve n in n points; or, combining in every manner the m points two and two together, and the n points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2 . \frac{1}{2}[n]^2$ coincident lines, each meeting the curve m twice and the curve n twice. There are besides the hk lines joining each double point of the curve m with each double point of the curve n. This gives in all $\frac{1}{4}[m]^2[n]^2 + hk$ lines; or, writing $h = \frac{1}{2}[m]^2 + M$, $k = \frac{1}{2}[n]^2 + N$, the number is

$$= \tfrac{1}{2}[m]^2[n]^2 + M . \tfrac{1}{2}[n]^2 + N . \tfrac{1}{2}[m]^2 + MN;$$

which is the value of $G(m^2, n^2)$ given by the investigation.

26. $G(m^3, n)$.—We have

$$G(m^3, n) = n\, G(1, m^3) = n\, S(m^3),$$

and it is in fact the same question to find $G(1, m^3)$ and to find $S(m^3)$. I assume for the present that the value [of $S(m^3)$, see *post* Art. 38] is $= \frac{1}{3}[m]^3 + M(m-2)$; and we then have

$$G(m^3, n) = n\left(\tfrac{1}{3}[m]^3 + M(m-2)\right).$$

27. Before going further, I observe that there are certain functional conditions which must be satisfied by the G formulæ. Thus if the curve m be replaced by the system of the two curves m, m', instead of M we have $M + M'$. Let $G(m)$ denote any one of the functions $G(m, n, p, q)$, $G(m, n^2, p)$, $G(m, n^3)$, we must have

$$G(m + m') = G(m) + G(m').$$

Similarly, if $G(m^2)$ denote either of the functions $G(m^2, n, p)$, $G(m^2, n^2)$, we must have

$$G(m + m')^2 = G(m^2) + G(m, m') + G(m'^2);$$

and so if $G(m^3)$ stand for $G(m^2, n)$, then

$$G(m + m')^3 = G(m^3) + G(m^2, m') + G(m, m'^2) + G(m'^3);$$

and finally

$$G(m + m')^4 = G(m^4) + G(m^3, m') + G(m^2, m'^2) + G(m, m'^3) + G(m'^4).$$

28. The first three equations may be at once verified by means of the above given values of the G functions. But conversely, at least on the assumption that $G(m)$, $G(m^2)$, &c., in so far as they respectively depend on the curve m, are functions of m and M only, we may, by the solution of the functional equations, obtain the values of the G functions. It is to be observed that the first equation is of the form

$$\phi(m + m') = \phi(m) + \phi(m'),$$

the general solution whereof is

$$\phi m = \alpha m + \beta M;$$

the second equation, supposing that $G(m, m')$ is known—the third equation, supposing that $G(m^2, m')$ and $G(m, m'^2)$ are known—and the fourth equation, supposing that $G(m^2, m')$, $G(m^2, m'^2)$, $G(m, m'^3)$ are known, are respectively of the form

$$\phi(m+m') = \phi m + \phi m' + \text{funct. } (m, m');$$

and hence if a particular solution be given, the general solution is

$$\phi(m) = \text{Particular Solution} + \alpha m + \beta M.$$

The values of the constants must in each case be determined by special considerations.

29. The value of $G(m, n, p, q)$ was obtained strictly; that of $G(m^2, n, p)$ was reduced to depend on $G(1, 1, m^2)$, and that of $G(m^3, n)$ on $G(1, m^3)$. I apply therefore the functional equations to the confirmation of the values of $G(1, 1, m^2)$, $G(m^2, n^2)$, and $G(1, m^3)$, and to the determination of the value of $G(m^4)$.

30. First, if $G(m^2)$ denote $G(1, 1, m^2)$, then $G(m, m')$ denotes $G(1, 1, m, m')$, which is $= 2mm'$; hence

$$G(m+m')^2 - G(m^2) - G(m'^2) = 2mm',$$

which is satisfied by $G(m^2) = [m]^2$. This gives

$$G(1, 1, m^2) = [m]^2 + \alpha m + \beta M;$$

but if the curve m be a system of m lines ($m = m$, $M = 0$), then $G(1, 1, m^2) = [m]^2$; and again, if the curve m be a conic ($m = 2$, $M = -1$), then $G(1, 1, m^2) = 1$. This gives $\alpha = 0$, $\beta = 1$, and therefore

$$G(1, 1, m^2) = [m]^2 + M.$$

31. Next, if $G(m^2)$ denote $G(m^2, n^2)$, then $G(m, m')$ denotes $G(m, m', n^2)$, which is $= mm'([n]^2 + N)$. The functional equation is

$$G(m+m')^2 - G(m^2) - G(m'^2) = mm'([n]^2 + N),$$

which is satisfied by $G(m^2) = \frac{1}{2}[m]^2([n]^2 + N)$. Hence we have

$$G(m^2, n^2) = \tfrac{1}{2}[m]^2([n]^2 + N) + \alpha m + \beta M,$$

where α, β are functions of n, N; and observing that $G(m^2, n^2)$ must be symmetrical in regard to the curves m and n, it is easy to see that we may write

$$G(m^2, n^2) = \tfrac{1}{2}[m]^2[n]^2 + M \,.\, \tfrac{1}{2}[n]^2 + N \,.\, \tfrac{1}{2}[m]^2 + \alpha mn + \beta(mN + nM) + \gamma MN,$$

where α, β, γ are absolute constants. To determine them, if the curve m be a pair of lines ($m = 2$, $M = 0$), then

$$G(m^2, n^2) = G(1, 1, n^2) = [n]^2 + N;$$

and if each of the curves m, n be a conic ($m = 2$, $M = -1$, $n = 2$, $N = -1$), then

$$G(m^2, n^2) = 1.$$

These cases give $\alpha=\beta=0$, $\gamma=1$, and therefore

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2[n]^2+M.\tfrac{1}{2}[n]^2+N.\tfrac{1}{2}[m]^2+MN.$$

32. Again, $G(m^3)$ standing for $G(1, m^3)$, then $G(m^3, m')$ and $G(m, m'^2)$ will stand for $G(1, m^2, m')$ and $G(1, m, m'^2)$, the values whereof are $m'([m]^2+M)$ and $m([m']^2+M')$ respectively. We have thus

$$G(m+m')^3-G(m^3)-G(m'^3)=m'([m]^2+M)+m([m']^2+M'),$$

a solution of which is $G(m^3)=\frac{1}{3}[m]^3+mM$. Hence we have

$$G(1, m^3)=\tfrac{1}{3}[m]^3+mM+\alpha m+\beta M.$$

Suppose first that the curve m is a system of lines ($m=m$, $M=0$), then $G(1, m^3)=\frac{1}{3}[m]^3$; and next that the curve m is a cubic in space or skew cubic ($m=3$, $M=-2$), then $G(1, m^3)=0$, since a line can meet the curve in two points only. We thus find $\alpha=0$, $\beta=-2$, and thence

$$G(1, m^3)=\tfrac{1}{3}[m]^3+M(m-2).$$

33. Hence, substituting for $G(m^2, m')$, $G(m^2, m'^2)$, $G(m, m'^3)$ their values $m'\left(\frac{1}{3}[m]^3+M(m-2)\right)$, $\frac{1}{2}[m]^2[m']^2+M.\frac{1}{2}[m']^2+M'.\frac{1}{2}[m]^2+MM'$, and $m\left(\frac{1}{3}[m']^3+M'(m'-2)\right)$ respectively, we find

$$\begin{aligned} G(m+m')^4-G(m^4)-G(m'^4)= \quad & m'\left(\tfrac{1}{3}[m]^3+M(m-2)\right) \\ & +\tfrac{1}{2}[m]^2[m']^2+M.\tfrac{1}{2}[m']^2+M'.\tfrac{1}{2}[m]^2+MM' \\ & + \quad m\left(\tfrac{1}{3}[m']^3+M'(m'-2)\right), \end{aligned}$$

and thence, obtaining first a particular solution, the general solution is

$$G(m^4)=\tfrac{1}{12}[m]^4+M(\tfrac{1}{2}[m]^2-2m)+M^2.\tfrac{1}{2}+\alpha m+\beta M.$$

34. To determine the constants, suppose first that the curve m is a system of lines ($m=m$, $M=0$), we must have $G(m^4)=\frac{1}{12}[m]^4+m$, and thence $\alpha=0$. Next, if the curve m be a conic ($m=2$, $M=-1$), we must have $G(m^4)=0$; and this gives $\beta=\frac{11}{2}$, and consequently

$$G(m^4)=\tfrac{1}{12}[m]^4+m+M(\tfrac{1}{2}[m]^2-2m+\tfrac{11}{2})+M^2.\tfrac{1}{2}.$$

The NG formulæ, Article 35.

35. The NG formulæ are now at once obtained, viz. we have

$$\begin{aligned} NG(m, n, p) &= G(m^2, n, p)+G(m, n^2, p)+G(m, n, p^2), \\ NG(m^2, n) &= 3G(m^2, n) \quad +G(m^2, n^2), \\ NG(m^3) &= 6G(m^3), \end{aligned}$$

which give the values in the Table.

The S formulæ, particular cases, Articles 36 to 40.

36. The S formulæ have in fact been obtained in the investigation of the G formulæ: we have

$$S(m, n, p) = 2mnp,$$
$$S(m^2, n) = n([m]^2 + M),$$
$$S(m^3) = \tfrac{1}{3}[m]^3 + M(m-2).$$

37. In confirmation of the formula $S(1, m^2) = [m]^2 + M$, it is to be remarked that if we take through the line 1 an arbitrary plane, this meets the curve m in m points, and joining these two and two together we have $\frac{1}{2}[m]^2$ lines, each of them meeting the curve m twice and also meeting the line 1; that is, the lines in question are generating lines of the scroll $S(1, m^2)$. The line 1 is, as already mentioned, an $(h=)(\frac{1}{2}[m]^2 + M)$tuple line on the scroll; the section by the arbitrary plane is therefore the line 1 taken $(\frac{1}{2}[m]^2 + M)$ times, together with the before-mentioned $\frac{1}{2}[m]^2$ lines; that is, the order of the surface is $[m]^2 + M$, as it should be. This is in fact the mode in which the order of the scroll $S(1, m^2)$ was originally obtained by Dr Salmon.

38. As regards the formula $S(m^3) = \frac{1}{3}[m]^3 + M(m-2)$, suppose that the curve m is a (p, q) curve on the hyperboloid, we have as before $m = p + q$, $M = -pq$, and the formula becomes

$$S(m^3) = \tfrac{1}{3}[p+q]^3 - pq(p+q-2),$$

which is

$$= \tfrac{1}{3}[p]^3 + \tfrac{1}{3}[q]^3,$$

viz. as already remarked, the surface is in this case the hyperboloid taken $\frac{1}{6}[p]^3 + \frac{1}{6}[q]^3$ times.

39. It is to be noticed also that if the curve m be a system of lines $(m = m, M = 0)$, then the formula gives

$$S(m^3) = \tfrac{1}{3}[m]^3,$$

which is right, since in this case the scroll is made up of the $\frac{1}{6}[m]^3$ hyperboloids generated each of them by a line which meets three out of the m lines.

In the case of a curve m, which is such that the coordinates of any point of the curve are proportional to rational and integral functions of the order m of an arbitrary parameter θ, or say the case of a *unicursal* curve of the order m, we have

$$(h = \tfrac{1}{2}[m-1]^2 \text{ and } \therefore) M = -(m-1),$$

and the formula gives

$$S(m^3) = \tfrac{1}{3}[m-1]^3,$$

for a direct investigation of which see *post*, Annex No. 1.

40. In the case of a curve m, which is the complete intersection of two surfaces of the orders p and q respectively, or say a complete $(p \times q)$ intersection, we have

$$m = pq, \ \left(h = \tfrac{1}{2} pq\,(p-1)\,(q-1) \text{ and } \therefore\right) M = -\tfrac{1}{2} pq\,(p+q-2);$$

and we find

$$\begin{aligned} S\,(m^3) &= \tfrac{1}{6} pq\,(pq-2)\,(2pq-3p-3q+4), \\ &= \tfrac{1}{6} \beta\,(\beta-2)\,(2\beta-3\alpha+4) \end{aligned}$$

if $\alpha = pq$, $\beta = p+q$. The mode of obtaining this result by a direct investigation was pointed out to me by Dr Salmon; see *post*, Annex No. 2.

Particular cases of the formula for $G\,(m^4)$, Articles 41 & 42.

41. In the case of a (p, q) curve on the hyperboloid, putting as before $m = p+q$, $M = -pq$, we find

$$G\,(m^4) = \tfrac{1}{12}\,[p+q]^4 + p + q - pq\left(\tfrac{1}{2}\,[p+q]^2 - 2\,(p+q) + \tfrac{11}{2}\right) + \tfrac{1}{2} p^2 q^2,$$

which is

$$= \tfrac{1}{12}\left([p]^4 + [q]^4\right) - 2q\,[p-1]^3 - 2p\,[q-1]^3,$$

vanishing if p, q are neither of them greater than 3: this is as it should be, since there is then no line which meets the curve four times. The curves for which the condition is satisfied are (1, 1) the conic, (1, 2) the cubic, (2, 2) the quadriquadric, (1, 3) the excubo-quartic, (2, 3) the excubo-quintic (viz. the quintic curve, which is the partial intersection of a quadric surface and a cubic surface having a line in common), and (3, 3) the quadri-cubic, or complete intersection of a quadric surface and a cubic surface. If either p or q exceeds 3, we have the case of a curve through every point whereof there can be drawn a line or lines through four or more points, and the formula is inapplicable.

42. In the case of a complete $(p \times q)$ intersection, we have as before $m = pq$, $M = -\tfrac{1}{2} pq\,(p+q-2)$, and the formula for $G\,(m^4)$ becomes

$$G\,(m^4) = \tfrac{1}{24}\,\beta \left\{ \begin{array}{l} -66\alpha + 144 \\ +\beta\,(3\alpha^2 + 18\alpha - 26) \\ +\beta^2 . -6\alpha \\ +\beta^3 . 2, \end{array} \right\}$$

a formula the direct verification whereof is due to Dr Salmon; see *post*, Annex No. 3.

The formulæ for $NR\,(1, m, n)$ *and* $NR\,(1, m^2)$, Articles 43 to 46.

43. $NR\,(1, m, n)$.—Through the line 1 take any plane meeting the curve m in m points and the curve n in n points; then if m_1, m_2 be any two of the m points, and n_1, n_2 any two of the n points, the lines $m_1 n_1$ and $m_2 n_2$ are generating lines of the

scroll $S(1, m, n)$, and these lines intersect in a point which belongs to the Nodal Residue NR; and in like manner the lines $m_1 n_2$ and $m_2 n_1$ are generating lines of the scroll, and they intersect on a point of NR; we have thus

$$(2 \cdot \tfrac{1}{2}[m]^2 \cdot \tfrac{1}{2}[n]^2 =) \tfrac{1}{2}[m]^2 [n]^2$$

points on NR, that is, the arbitrary plane through the line 1 cuts NR in $\frac{1}{2}[m]^2[n]^2$ points. But the plane also cuts NR in certain points lying on the line 1, and if the number of these be (a), then

$$NR(1, m, n) = \tfrac{1}{2}[m]^2[n]^2 + \mathrm{a}.$$

44. The points (a) are included among the cuspidal points on the line 1. Taking for a moment $x=0$, $y=0$ for the equations of the line 1 (which, as we have seen, is a mn-tuple line on the scroll), the equation of the scroll is of the form $(A, \ldots \between x, y)^{mn} = 0$, where $A, \ldots$ are functions of the coordinates of the degree mn. The entire number of cuspidal points on the line 1 is thus $= 2[mn]^2$; but these include different kinds of cuspidal points, viz. we have

$$2[mn]^2 = 2\mathrm{a} + 2\alpha + 2\alpha' + R,$$

if (a) be the number of points in which the line 1 meets NR,

" α " " " " $S(m^2, n)$,

" α' " " " " $S(m, n^2)$,

" R " " " " Torse (m, n),

where by Torse (m, n) I denote the developable surface or "Torse" generated by a line which meets each of the curves m and n. The order of the Torse in question is

$$R = \big(n([m]^2 - 2h) + m([n]^2 - 2k) =\big) - 2(nM + mN),$$

see *post*, Annex No. 4. And then observing that we have

$$\alpha = S(m^2, n) = n([m]^2 + M),$$
$$\alpha' = S(m, n^2) = m([n]^2 + N),$$

these values give

$$2\alpha + 2\alpha' + R = 2n[m]^2 + 2m[n]^2,$$

and we have

$$\begin{aligned} \mathrm{a} &= \tfrac{1}{2}(2[mn]^2 - 2\alpha - 2\alpha' - R), \\ &= [mn]^2 - n[m]^2 - m[n]^2, \\ &= [m]^2[n]^2, \end{aligned}$$

and thence

$$NR(1, m, n) = \tfrac{3}{2}[m]^2[n]^2.$$

45. $NR(1, m^2)$.—Through the line 1 take any arbitrary plane meeting the curve m in m points; if m_1, m_2, m_3, m_4 be any four of these, then the lines $m_1 m_2$ and $m_3 m_4$ are generating lines of the scroll $S(1, m^2)$, and their intersection is a point of the

nodal residue NR; but in like manner the lines $m_1 m_3$ and $m_2 m_4$ are generating lines of the scroll, and their intersection is a point of NR; and so the lines $m_1 m_4$ and $m_2 m_3$ are generating lines of the scroll, and their intersection is a point of NR. We have thus $(3 \times \frac{1}{24}[m]^4 =) \frac{1}{8}[m]^4$ points of NR on the arbitrary plane through the line 1. But there are besides the points of NR which lie on the line 1; and if the number of these be (a), then

$$NR(1,\ m^2) = \tfrac{1}{8}[m]^4 + \mathrm{a}.$$

46. The points (a) are included among the cuspidal points of the scroll lying on the line 1. Supposing for a moment that $x=0$, $y=0$ are the equations of the line 1, then this line being a $(\frac{1}{2}[m]^2 + M)$tuple line on the scroll, the equation of the scroll is of the form $(A, \ldots \between x, y)^{\frac{1}{2}[m]^2+M} = 0$, where $A, \ldots$ are functions of the coordinates of the degree $\frac{1}{2}[m]^2$: the number of cuspidal points on the line 1 is thus

$$\left(2 \cdot \tfrac{1}{2}[m]^2 (\tfrac{1}{2}[m]^2 - 1 + M) =\right) [m]^2 (\tfrac{1}{2}[m]^2 - 1 + M).$$

But these include cuspidal points of several kinds, viz. we have

$$[m]^2 (\tfrac{1}{2}[m]^2 - 1 + M) = 2\mathrm{a} + 3\beta + R',$$

if (a) be the number of points in which the line 1 meets NR,

" β " " " " $S(m^2)$,

" R' " " " " Torse (m^2),

where Torse (m^2) denotes the developable surface or Torse generated by a line which meets the curve m twice. The order of the Torse in question is

$$R' = -2(m-3)M$$

(see *post*, Annex No. 5); and then since $\beta = S(m^2) = \frac{1}{3}[m]^3 + M(m-2)$, we find

$$\begin{aligned} 2\mathrm{a} &= [m]^2 (\tfrac{1}{2}[m]^2 - 1 + M) - 3\left(\tfrac{1}{3}[m]^3 + M(m-2)\right) + 2M(m-3), \\ &= \tfrac{1}{2}[m]^4 + [m]^3 + M([m]^2 - m), \end{aligned}$$

and thence

$$NR(1,\ m^2) = \tfrac{3}{8}[m]^4 + \tfrac{1}{2}[m]^3 + M(\tfrac{1}{2}[m]^2 - \tfrac{1}{2}m).$$

But I have not succeeded in finding by a like direct investigation the values of

$$NR(m,\ n,\ p),\ NR(m^2,\ n),\ NR(m^2).$$

Formulæ for $NT(1,\ m,\ n)$, $NT(1,\ m^2)$, Articles 47 and 48.

47. We have

$$\begin{aligned} NT(1,\ m,\ n) = \ & NG(1,\ m,\ n) = && mn(m+n-2) + mNnM \\ & + ND(1,\ m,\ n) && + \tfrac{1}{2}mn(mn+m+n-3) \\ & + NR(1,\ m,\ n) && + \tfrac{3}{2}[m]^2[n]^2, \end{aligned}$$

which is

$$= 2\,[mn]^2 + mN + nM,$$
$$= \tfrac{1}{2}\,S^2 - S + mN + nM,$$

where $S = S(1,\ m,\ n) = 2mn$.

48. And moreover

$$\begin{aligned} NT(1,\ m^2) = \quad & ND(1,\ m^2) = \quad \tfrac{1}{8}\,[m]^4 + [m]^3 + M\,(\tfrac{1}{2}\,[m]^2 - \tfrac{1}{2}) + M^2 \,.\, \tfrac{1}{2} \\ & + NG(1,\ m^2) \qquad\qquad + [m]^3 + M\,(3m - 6) \\ & + NR(1,\ m^2) \quad + \tfrac{3}{8}\,[m]^4 \qquad + M\,(\tfrac{1}{2}\,[m]^2 - 2m + 3), \end{aligned}$$

which is

$$\begin{aligned} &= \tfrac{1}{2}\,[m]^4 + 2\,[m]^3 + M\,([m]^2 + m - \tfrac{7}{2}) + M^2 \,.\, \tfrac{1}{2}, \\ &= \tfrac{1}{2}\,S^2 - S + M\,(m - \tfrac{5}{2}), \end{aligned}$$

if $S = S(1,\ m^2) = [m]^2 + M$.

The NT and NR formulæ, Articles 49 to 58.

49. I proceed to find $NT(m,\ n,\ p)$, &c. by a functional investigation, such as was employed for finding $G(1,\ 1,\ m^2)$, &c. Writing $S(m)$ to denote either of the scrolls $S(m,\ n,\ p)$, $S(m,\ n^2)$, and supposing that in place of the curve m we have the aggregate of the two curves $m,\ m'$; then the scroll $S(m+m')$ breaks up into the scrolls $Sm,\ Sm'$, and the intersection of these is part of the nodal total $NT(m+m')$; that is, we have

$$NT(m+m') = NT(m) + NT(m') + S(m)\,.\,S(m');$$

and in like manner, if $S(m^2)$ stands for $S(m^2,\ n)$, then

$$NT(m+m')^2 = NT(m^2) + NT(m,\ m') + NT(m'^2) + C_2\big(S(m^2),\ S(m,\ m'),\ S(m'^2)\big),$$

where C_2 denotes the sum of the combinations two and two together; and so also

$$\begin{aligned} NT(m+m')^3 = NT(m^3) + NT(m^2,\ m') + NT(m,\ m'^2) + NT(m'^3) \\ + C_2\big(S(m^2),\ S(m^2,\ m'),\ S(m,\ m'^2),\ S(m'^3)\big). \end{aligned}$$

50. Instead of assuming

$$NT = \tfrac{1}{2}\,S^2 + \phi,$$

it is the same thing, and it is rather more convenient, to assume

$$NT = \tfrac{1}{2}\,S^2 - S + \phi,$$

viz. $NT(m) = \tfrac{1}{2}\big(S(m)\big)^2 - S(m) + \phi(m)$, &c. Then observing that

$$S(m+m') = S(m) + S(m'),\ \&c.,$$

the foregoing equations for NT give

$$\phi(m+m') = \phi(m) + \phi(m'),$$
$$\phi(m+m')^2 = \phi(m^2) + \phi(m, m') + \phi(m'^2),$$
$$\phi(m+m')^3 = \phi(m^3) + \phi(m^2, m') + \phi(m, m'^2) + \phi(m'^3);$$

and if in the second equation $\phi(m, m')$ and in the third equation $\phi(m^2, m')$ and $\phi(m, m'^2)$ are regarded as known, these are all of them of the form

$$f(m+m') - f(m) - f(m') = \text{Funct. } (m, m');$$

so that, a particular solution being obtained, the general solution is $f(m) =$ Particular Solution $+ \alpha m + \beta M$, at least on the assumption that $f(m)$, in so far as it depends on the curve m, is a function of m and M only.

51. First, if $\phi(m)$ stands for $\phi(m, n, p)$, we obtain $\phi(m, n, p) = \alpha m + \beta M$, or observing that $\phi(m, n, p)$ must be symmetrical in regard to the curves m, n, and p, we may write

$$\phi(m, n, p) = \alpha mnp + \beta(Mnp + Nmp + Pmn) + \gamma(mNP + nMP + pMN) + \delta MNP,$$

and then

$$NT(m, n, p) = \tfrac{1}{2}S^2 - S + \phi(m, n, p),$$
$$= 2mnp(mnp - 1) + \phi(m, n, p).$$

But for $p = 1$ this should reduce itself to the known value of $NT(1, m, n)$; this gives $\alpha = 0$, $\beta = 1$, $\gamma = 0$; we in fact have, as will be shown, *post*, Art. 55, $\delta = 0$; and hence

$$NT(m, n, p) = \tfrac{1}{2}S^2 - S + (Mnp + Nmp + Pmn),$$
$$= 2[mnp]^2 + (Mnp + Nmp + Pmn).$$

52. Next, if $\phi(m^2)$ stand for $\phi(m^2, n)$, then $\phi(m, m')$ stands for $\phi(m, m', n)$, which is $= Nmm' + n(mM' + m'M)$, and the equation is

$$\phi(m+m')^2 - \phi(m^2) - \phi(m'^2) = Nmm' + n(mM' + m'M).$$

A particular solution is $\phi(m^2) = \frac{1}{2}[m]^2 N + nmN$, and we have therefore

$$\phi(m^2, n) = \tfrac{1}{2}[m]^2 N + nmM + \alpha m + \beta M;$$

or observing that $\phi(m^2, n)$ considered as a function of n, satisfies the equation

$$\phi(n+n') = \phi(n) + \phi(n'),$$

and is therefore a linear function of n and N, we may write

$$\phi(m^2, n) = \tfrac{1}{2}[m]^2 N + nmM + \alpha nm + \beta nM + \gamma mN + \delta MN;$$

we then have

$$NT(m^2, n) = \tfrac{1}{2}S^2 - S + \phi(m^2, n),$$

where

$$S = S(m^2, n) = n([m] + M).$$

And then putting $n = 1$, and comparing with the known value of $NT(1, m^2)$, we find $\alpha = 0$, $\beta = -\frac{5}{2}$. It will be shown, *post*, Art. 55, that $\gamma = 0$, $\delta = 0$; and we have therefore

$$\phi(m^2, n) = nM(m - \tfrac{5}{2}) + N(\tfrac{1}{2}[m]^2 + M),$$

and thence

$$\begin{aligned} NT(m^2, n) &= \tfrac{1}{2}S^2 - S + \phi(m^2, n), \\ &= n\left(\tfrac{1}{2}[m]^4 + 2[m]^3 + M([m]^2 + m - \tfrac{7}{2}) + M^2 . \tfrac{1}{2}\right) \\ &\quad + [n]^2\left(\tfrac{1}{2}[m]^4 + 2[m]^3 + [m]^2 + M[m]^3 + M^2 . \tfrac{1}{2}\right) \\ &\quad + N(\tfrac{1}{2}[m]^2 + M). \end{aligned}$$

53. Next for $\phi(m^3)$, substituting for $\phi(m^2, m')$ and $\phi(m, m'^2)$ their values, we have

$$\begin{aligned} \phi(m + m')^3 - \phi(m^3) - \phi(m'^3) &= m'M(m - \tfrac{5}{2}) + M'(\tfrac{1}{2}[m]^2 + M) \\ &\quad + mM'(m' - \tfrac{5}{2}) + M(\tfrac{1}{2}[m']^2 + M'), \end{aligned}$$

which is satisfied by

$$\phi(m^3) = M(\tfrac{1}{2}[m]^2 - \tfrac{5}{2}m) + M^2,$$

and the general value then is

$$\phi(m^3) = M(\tfrac{1}{2}[m]^2 - \tfrac{5}{2}m) + M^2 + \alpha m + \beta M,$$

and we have

$$NT(m^3) \quad = \tfrac{1}{2}S^2 - S + \phi(m^3),$$

where

$$S = S(m^3) = \tfrac{1}{3}[m]^3 + M(m - 2).$$

54. Taking for the curve m the (p, q) curve on the hyperboloid $(m = p + q$, $M = -pq)$, $S(m^3)$ becomes the hyperboloid taken k times, if $k = \frac{1}{6}[p]^3 + \frac{1}{6}[q]^3$; that is, $S(m^3) = 2k$, and $NT(m^3) = 4 . \frac{1}{2}[k]^2 + \phi(m^3)$; $\phi(m^3)$ must vanish if p and q are each not greater than 3, this implies $\alpha = 3$, $\beta = 11$, for with these values the formula gives

$$\phi(m^3) = -\tfrac{1}{2}(q[p-1]^3 + p[q-1]^3).$$

55. I assume the correctness of the value

$$\phi(m^3) = 3m + M(\tfrac{1}{2}[m]^2 - \tfrac{5}{2}m + 11) + M^2$$

so obtained, as being in fact verified by means of the six several curves (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3); and I remark that if the foregoing value of $\phi(m, n, p)$ had been increased by $6\alpha MNP$, then it would have been necessary to increase the value of $\phi(m^2, n)$ by $3\alpha M^2N$, and that of $\phi(m^3)$ by αM^3; and moreover that if the foregoing value of $\phi(m^2, n)$ had been increased by $\gamma mN + \delta MN$, then it would have been

necessary to increase the value of $\phi(m^3)$ by $\gamma mM + \delta M^2$; this is easily seen by writing down the values

$$\begin{aligned}
\phi(m^3) &= \gamma mM + \delta M^2 + \alpha M^3 \;, \\
\phi(m^2, m') &= \gamma mM' + \delta MM' + 3\alpha M^2 M', \\
\phi(m, m'^2) &= \gamma m'M + \delta MM' + 3\alpha MM'^2, \\
\phi(m'^3) &= \gamma m'M' + \delta M'^2 + \alpha M'^3 \;,
\end{aligned}$$

the sum of which is

$$= \gamma(m+m')(M+M') + \delta(M+M')^2 + \alpha(M+M')^3,$$

the corresponding term of $\phi(m^3)$; hence the value of $\phi(m^3)$ being correct without the foregoing addition, we must have $\gamma = 0$, $\delta = 0$, $\alpha = 0$; which confirms the foregoing values of $\phi(m, n, p)$, $\phi(m^2, n)$.

56. The equation

$$NT(m^3) = \tfrac{1}{2}S^2 - S + \phi(m^3),$$

gives

$$\begin{aligned}
NT(m^3) &= \tfrac{1}{2}S^2 - S + 3m + M(\tfrac{1}{2}[m]^2 - \tfrac{5}{2}m + 11) + M^2, \\
&= \tfrac{1}{18}[m]^6 + \tfrac{1}{2}[m]^5 + [m]^4 + 3m \\
&\quad + M(\tfrac{1}{3}[m]^4 + \tfrac{1}{3}[m]^3 + \tfrac{1}{2}[m]^2 - \tfrac{7}{2}m + 13) \\
&\quad + M^2(\tfrac{1}{2}[m]^2 - \tfrac{3}{2}m + 3).
\end{aligned}$$

57. We have

$$\begin{aligned}
NR(m^2, n) &= NT(m^2, n) - ND(m^2, n) - NG(m^2, n), \\
&= n\left(\tfrac{3}{8}[m]^4 + M(\tfrac{1}{2}[m]^2 - 2m + 3)\right) \\
&\quad + [n]^2\left(\tfrac{1}{2}[m]^4 + \tfrac{3}{2}[m]^3 + [m]^2 + M([m]^2 - \tfrac{1}{2}) + M^2 \cdot \tfrac{1}{2}\right).
\end{aligned}$$

58. And moreover

$$\begin{aligned}
NR(m^3) &= NT(m^3) - ND(m^3) - NG(m^3), \\
&= \tfrac{1}{18}[m]^6 + \tfrac{3}{8}[m]^6 - \tfrac{1}{2}[m]^3 + 3m \\
&\quad + M(\tfrac{1}{3}[m]^4 - \tfrac{1}{6}[m]^3 - \tfrac{5}{2}[m]^2 + 8m - 20) + M^2(\tfrac{1}{2}[m]^2 - 2m):
\end{aligned}$$

and the investigation of the series of results given in the Table is thus concluded.

Intersections of a generating line with the Nodal Total, Articles 59 to 63.

59. We may for the scrolls $S(1, m, n)$ and $S(1, m^2)$ verify the theorem that each generating line meets the Nodal Total in a number of points $= S - 2$.

In fact for the scroll $S(1, m, n)$, the directrix curves are respectively multiple curves of the orders mn, n, m, and a generating line meets each of these in a single

point, counting for the three curves respectively as $mn-1$, $n-1$, and $m-1$ points respectively. Moreover the construction (*ante*, Art. 43) for the Nodal Residue $NR(1, m, n)$ shows that a generating line meets this curve in $(m-1)(n-1)$ points; and since the curve is merely a double curve, these count each as a single point; and the generating line does not meet the Nodal Generator $NG(1, m, n)$. The number of intersections therefore is

$$mn-1+(m-1)+(n-1)+(m-1)(n-1),$$

which is

$$=2mn-2, \quad =S-2.$$

60. Similarly for the scroll $S(1, m^2)$; the directrix curves are multiple curves, viz. the line 1 is a $(\frac{1}{2}[m]^2+M)$tuple curve, and the curve m a $(m-1)$tuple curve; the generating line meets the former in a single point, counting as $\frac{1}{2}[m]^2+M-1$ points, and the latter in two points, each counting as $(m-2)$ points. The construction (*ante*, Art. 45) for the Nodal Residue $NR(1, m^2)$ shows that the generating line meets this curve in $\frac{1}{2}[m-2]^2$ points; and since the curve is merely a double curve, these count each as a single point. Finally, the generating line does not meet the Nodal Generator $NG(1, m^2)$. The number of intersections thus is

$$\tfrac{1}{2}[m]^2-1+M+2(m-2)+\tfrac{1}{2}[m-2]^2,$$

which is

$$=[m]^2-2+M, \quad =S-2.$$

In the remaining cases we may use the theorem to find the number of points in which the generating line meets the Nodal Residue. Using Π as the symbol for the points in question ($\Pi(m, n, p)$ for the scroll $S(m, n, p)$, &c.), we find

61. For the scroll $S(m, n, p)$,

$$(mn-1)+(np-1)+(mp-1)+\Pi(m, n, p)=S-2=2mnp-2,$$

which gives

$$\Pi(m, n, p)=2mnp-mn-mp-np+1.$$

This includes the before-mentioned case

$$\Pi(1, m, n)=(m-1)(n-1),$$

and the more particular one

$$\Pi(1, 1, m)=0.$$

62. For the scroll $S(m^2, n)$,

$$\tfrac{1}{2}[m]^2-1+M+2\big((m-1)n-1\big)+\Pi(m^2, n)=S-2=n([m]^2+M)-2,$$

which gives

$$\Pi(m^2, n) \quad =n([m]^2-2m+2+M)-\tfrac{1}{2}[m]^2+1-M.$$

This includes the before-mentioned particular case

$$\Pi(1, m^2) \quad =\tfrac{1}{2}[m-2]^2.$$

63. And lastly for the scroll $S(m^3)$,

$$3\left(\tfrac{1}{2}[m]^2 - m + 1 + M\right) + \Pi(m^3) = S - 2 = \tfrac{1}{3}[m]^3 + (m-2)M - 2,$$

which gives

$$\Pi(m^3) = \tfrac{1}{3}[m]^3 - \tfrac{3}{2}[m]^2 + 3m - 5 + M(m-5).$$

The foregoing expressions for Π might with propriety have been inserted in the Table.

Annex No. 1.—*Investigation of the formula for $S(m^3)$ in the case of the unicursal curve* (referred to, Art. 39).

Consider the unicursal m-thic curve the equations whereof are $x : y : z : w = A : B : C : D$, where A, B, C, D are rational and integral functions of a parameter θ; and let it be required to find the equation of a plane meeting the curve in such manner that three of the points of intersection are *in lineâ*. Taking for the equation of the plane

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

we find between $(\xi, \eta, \zeta, \omega)$ an equation of a certain degree in $(\xi, \eta, \zeta, \omega)$, which is the equation in plane-coordinates of the scroll $S(m^3)$, the degree of the equation is therefore equal to the class of the scroll; but as the class of a scroll is equal to its order, the degree of the equation is equal to the order of the scroll, or say $= S(m^3)$.

Proceeding with the investigation, if θ be determined by the equation

$$\xi A + \eta B + \zeta C + \omega D = 0,$$

then the roots $\theta_1, \theta_2, \ldots \theta_m$ of this equation belong to the points of intersection of the plane and curve; and the corresponding coordinates of these points are (A_1, B_1, C_1, D_1), &c.

Suppose that the points 1, 2, 3 are *in lineâ*, and let λ, μ, ν, ρ be the coordinates of an arbitrary point, then the four points are *in plano*, that is, we have

$$\begin{vmatrix} \lambda, & \mu, & \nu, & \rho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix} = 0;$$

and if we form the equation

$$\Pi \begin{vmatrix} \lambda, & \mu, & \nu, & \rho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix} = 0,$$

where Π denotes the product of the terms belonging to all the triads of the m roots, the result will be symmetrical in regard to all the roots; and replacing the symmetrical functions of the roots by their values in terms of the coefficients, we have the required relation between $(\xi, \eta, \zeta, \omega)$.

Π contains $\frac{1}{6}[m]^3$ terms, whereof $\frac{1}{2}[m-1]^2$ contain the m-thic functions (A_1, B_1, C_1, D_1) of the root θ_1; that is, the form of Π is

$$(\lambda, \mu, \nu, \rho)^{\frac{1}{6}[m]^3}(\theta_1, 1)^{\frac{1}{2}[m]^2}(\theta_2, 1)^{\frac{1}{2}[m]^2}\ldots;$$

or, when the symmetrical functions are expressed in terms of the coefficients, the form is

$$(\lambda, \mu, \nu, \rho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^2}.$$

Now the above-mentioned determinant is divisible by $(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$, or Π is divisible by $\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$; and since this product contains $(3\times\frac{1}{6}[m]^3=)\frac{1}{2}[m]^3$ linear factors, and the product $\zeta(\theta_1, \theta_2, \ldots \theta_m)$ of the squared differences of the roots contains $(2\times\frac{1}{2}[m]^2=)[m]^2$ linear factors, so that we have

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)=\{\zeta(\theta_1, \theta_2, \ldots \theta_m)\}^{\frac{1}{2}(m-2)},$$

where

$$\zeta(\theta_1, \theta_2, \ldots \theta_m)=\text{Disct.}=(\xi, \eta, \zeta, \omega)^{2(m-2)},$$

and consequently

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)=(\xi, \eta, \zeta, \omega)^{[m-1]^2},$$

so that, omitting this factor, the remaining factor of Π is of the form

$$(\lambda, \mu, \nu, \rho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^2-[m-1]^2};$$

but the determinant vanishes if

$$\lambda, \mu, \nu, \rho=(A_1, B_1, C_1, D_1),\quad (A_2, B_2, C_2, D_2),\quad (A_3, B_3, C_3, D_3),$$

or say if

$$(\lambda, \mu, \nu, \rho)=(A, B, C, D),\quad \theta=\theta_1, \theta_2, \text{ or } \theta_3;$$

it follows that the product Π contains the factor

$$(\lambda\xi+\mu\eta+\nu\zeta+\rho\omega)^{\frac{1}{6}[m]^3};$$

or omitting this factor, and observing that

$$\tfrac{1}{2}[m]^3-[m-1]^2-\tfrac{1}{6}[m]^3=\tfrac{1}{3}[m]^3-[m-1]^2=\tfrac{1}{3}[m-1]^3,$$

the remaining factor is of the form

$$(\xi, \eta, \zeta, \omega)^{\frac{1}{3}[m-1]^3};$$

or we have finally

$$S(m^3)=\tfrac{1}{3}[m-1]^3,$$

which is the required expression.

I give the following investigation of the expression $\frac{1}{2}[m-1]^2$ for the number of apparent double points. Imagine through the point $(x=0,\ y=0,\ z=0)$ a line cutting the curve in the two points corresponding to the values θ_1, θ_2 of the parameter. We have

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2},$$

which equations determine θ_1 and θ_2.

Writing the equations under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2}=0,\quad \frac{A_1C_2-A_2C_1}{\theta_1-\theta_2}=0,$$

and treating θ_1 and θ_2 as coordinates, each of these equations belongs to a curve of the order $2(m-1)$, having a $(m-1)$thic point at infinity on each of the axes. The number of intersections thus is

$$=4(m-1)^2-(m-1)^2-(m-1)^2,\ =2(m-1)^2.$$

But among these are included points not belonging to the original system, viz. the points for which $(A_1=0,\ A_2=0)$ other than those for which $\theta_1=\theta_2$; the points so included are in number $=m^2-m$; and omitting them, the number is

$$\big(2(m-1)^2-m(m-1)\big)=[m-1]^2,$$

which is the number of points θ_1 lying *in lined* with the origin and another point θ_2; the number of apparent double points is the half of this, or $h=\frac{1}{2}[m-1]^2$. And thence

$$M=\big(-\tfrac{1}{2}[m]^2+h=\big)-(m-1).$$

I investigate also the number of lines through two points which meet two arbitrary lines; this is in fact $=S(1,\ m^2)$, which for the curve in question is

$$=\big(\tfrac{1}{2}[m]^2-(m-1)=\big)(m-1)^2.$$

Let the equations of the two lines be $(x=0,\ y=0)$ and $(z=0,\ w=0)$; then the conditions to be satisfied are

$$\frac{A_1}{A_2}=\frac{B_1}{B_2},\quad \frac{C_1}{C_2}=\frac{D_1}{D_2};$$

or writing these under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2}=0,\quad \frac{C_1D_2-C_2D_1}{\theta_1-\theta_2}=0,$$

and treating θ_1, θ_2 as coordinates, the number of intersections of these two curves is $=2(m-1)^2$, the same as for the two curves last above considered. And the number of the lines in question is one half of this, or $=(m-1)^2$.

Lemma employed in the following Annexes 2 and 3. *Formulæ for the order and weight of certain systems of equations.*

Let $\alpha_{\alpha'}$ denote a function of the degree α in the *order* variables $(x, y, ..)$, and of the degree α' in the *weight* variables $(x', y', ..)$, and so in other cases; and consider first the equation

$$\begin{vmatrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, \ldots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, \\ \cdot & \\ \cdot & \\ \cdot & \end{vmatrix} = 0,$$

where the matrix is a square; then

$$\text{Order} = \Sigma\alpha + \Sigma A,$$
$$\text{Weight} = \Sigma\alpha' + \Sigma A'.$$

Consider next the system

$$\left\| \begin{matrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & (\alpha+B)_{\alpha'+B'}, \ldots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & (\beta+B)_{\beta'+B'}, \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{matrix} \right\| = 0,$$

where the matrix is a square $+1$, that is, the number of columns exceeds by 1 the number of lines; then

$$\text{Order} = \Sigma AB - \Sigma\alpha\beta + \Sigma\alpha(\Sigma A + \Sigma\alpha),$$
$$\text{Weight} = (\Sigma A + \Sigma\alpha)(\Sigma A' + \Sigma\alpha') - \Sigma AA' + \Sigma\alpha\alpha'.$$

And again, the system

$$\left\| \begin{matrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & (\alpha+B)_{\alpha'+B'}, & (\alpha+C)_{\alpha'+C'}, \ldots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & (\beta+B)_{\beta'+B'}, & (\beta+C)_{\beta'+C'}, \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{matrix} \right\| = 0,$$

where the matrix is a square $+2$, that is, the number of columns exceeds by 2 the number of lines; then

$$\text{Order} = \Sigma ABC + \Sigma\alpha\beta\gamma + \Sigma\alpha(\Sigma AB - \Sigma\alpha\beta) + \left((\Sigma\alpha)^2 - \Sigma\alpha\beta\right)(\Sigma A + \Sigma\alpha),$$
$$\text{Weight} = \{\Sigma AB - \Sigma\alpha\beta + \Sigma\alpha(\Sigma A + \Sigma\alpha)\}(\Sigma A' + \Sigma\alpha') - (\Sigma A + \Sigma\alpha)(\Sigma AA' - \Sigma\alpha\alpha') + \Sigma A^2A' + \Sigma\alpha^2\alpha'.$$

The last formula, for the weight of the square $+2$ system, was communicated to me by Dr Salmon, the others are all in effect given in the Appendix, "On the Order of Systems of Equations," to his Treatise on the Analytic Geometry of Three Dimensions; and in the investigation in the following Annexes 2 and 3, the route which I have followed was completely traced out for me by him, so that I have only supplied the details of the work.

Annex No. 2.—*Investigation of the formula for $S(m^3)$, when the curve m is the pq complete intersection, viz. when it is the intersection of two surfaces of the orders p and q respectively* (referred to, Art. 40).

Let $U=0$, $V=0$ be the equations of the two surfaces of the orders p and q respectively. Take (x, y, z, w) the coordinates of a point on the curve, so that for these coordinates we have $U=0$, $V=0$; and in the equations of the two curves respectively, write for the coordinates $x+\rho x'$, $y+\rho y'$, $z+\rho z'$, $w+\rho w'$; then putting for shortness

$$\Delta = x'\partial_x + y'\partial_y + z'\partial_z + w'\partial_w,$$

the resulting equations may be represented by

$$(\Delta U, \Delta^2 U, \ldots \Delta^p U \between 1, \rho)^{p-1} = 0,$$

$$(\Delta V, \Delta^2 V, \ldots \Delta^q V \between 1, \rho)^{q-1} = 0,$$

where it is to be noticed that besides the expressed literal coefficients there are numerical coefficients (not as the notation usually denotes, the binomial coefficients, but) $=\frac{1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}$, &c.

Supposing that (x', y', z', w') are the current coordinates of a point on the line drawn through the point (x, y, z, w) to meet the curve in two other points, the equations in ρ must have two common roots, and this gives a system equivalent to two equations, or say a plexus of two equations. If from the plexus and the two equations $U=0$, $V=0$ we eliminate (x, y, z, w), we obtain an equation $S'=0$ in (x', y', z', w'), which is in fact the equation of the scroll $S(m^3)$, taken (as is easily seen to be the case) thrice; that is, $S(m^3)=\frac{1}{3}$ Degree of S'. But observing that the coordinates (x', y', z', w') enter into the plexus only and not into the functions U, V, and treating (x', y', z', w') as *weight* variables, Degree of $S'=$ Weight of System $(U=0, V=0,$ Plexus$)=$ Deg. $U\times$ Deg. $V\times$ Weight of Plexus, $=pq\times$ Weight of Plexus; or, writing $pq=\beta$,

$$S(m^3)=\tfrac{1}{3}\beta\times \text{Weight of Plexus.}$$

The plexus in question is the square $+1$ system,

$$\left\| \begin{matrix} \Delta U, & \Delta^2 U, & .. \\ & \Delta U, & .. \\ . & & \\ . & & \\ \Delta V, & \Delta^2 V, & .. \\ & \Delta V, & .. \\ . & & \\ . & & \end{matrix} \right\| = 0,$$

$p+q-3$ columns, $(q-2)+(p-2)=(p+q-4)$ lines; or representing the terms according to their order and weight, that is, degree in (x, y, z, w) and (x', y', z', w') respectively (the order and weight of the evanescent terms being fixed so as that they may form a regular series with the other terms), the system is

$$\left\| \begin{matrix} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{p+q-3 \text{ columns.}} \\ (q-2) \text{ lines.} \left\{ \begin{matrix} (p-1)_1, & (p-2)_2, \ldots \\ p_0 \quad , & (p-1)_1, \\ . & \\ . & \end{matrix} \right. \\ (p-2) \text{ lines.} \left\{ \begin{matrix} (q-1)_1, & (q-2)_2, \\ q_0 \quad , & (q-1)_1, \\ . & \\ . & \end{matrix} \right. \end{matrix} \right\| = 0,$$

so that

$$\begin{array}{lllll} \alpha\ ,\ \beta, .. = & p-1, & p, \ldots\ p+q-4, & q-1,\ q, \ldots & p+q-4, \\ \alpha'\ ,\ \beta', .. = & 1, & 0, \ldots\ -q+4, & 1,\ 0, \ldots & -p+4, \\ A\ ,\ B, .. = & -1, -2, & & & \ldots -(p+q-4), \\ A',\ B', .. = & 1, & 2, & & p+q-4, \end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{lll} \alpha,\ \beta, .. = & p-2+\theta, & q-2+\phi \\ \alpha',\ \beta', .. = & 2-\theta, & 2-\phi \end{array}\right\} \theta = 1 \text{ to } q-2, \text{ and } \phi = 1 \text{ to } p-2.$$

We then find

$$\begin{aligned} \Sigma\alpha &= (q-2)(p-2)+\tfrac{1}{2}(q-2)(q-1)+(p-2)(q-2)+\tfrac{1}{2}(p-2)(p-1), \\ \Sigma\alpha' &= 2(q-2)-\tfrac{1}{2}(q-2)(q-1)+2(p-2)-\tfrac{1}{2}(p-2)(p-1), \\ \Sigma A &= -\Sigma A' = -\tfrac{1}{2}(p+q-4)(p+q-3), \\ \Sigma\alpha\alpha' &= 2(p-2)(q-2)-(p-4).\tfrac{1}{2}(q-2)(q-1)-\tfrac{1}{6}(q-2)(q-1)(2q-3) \\ &\quad +2(q-2)(p-2)-(q-4).\tfrac{1}{2}(p-2)(p-1)-\tfrac{1}{6}(p-2)(p-1)(2p-3), \\ \Sigma AA' &= -\tfrac{1}{6}(p+q-4)(p+q-3)(2p+2q-7), \end{aligned}$$

which putting therein $p+q=\alpha$, $pq=\beta$, give

$$\begin{aligned}
\Sigma\alpha &= \beta+\tfrac{1}{2}\alpha^2-\tfrac{11}{2}\alpha+10,\\
\Sigma\alpha' &= \beta-\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-10,\\
\Sigma A = -\Sigma A' &= -\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-6,\\
\Sigma\alpha\alpha' &= \tfrac{1}{2}\alpha\beta-\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{103}{6}\alpha+26,\\
\Sigma AA' &= -\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{73}{6}\alpha+14,
\end{aligned}$$

and thence

$$\begin{aligned}
\Sigma A+\Sigma\alpha &= \beta-2\alpha+4,\\
\Sigma A'+\Sigma\alpha' &= \beta-4,\\
\Sigma AA'-\Sigma\alpha\alpha' &= -\tfrac{1}{2}\alpha\beta+5\alpha-12,
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{Weight} &= (\beta-2\alpha+4)(\beta-4)+\tfrac{1}{2}\alpha\beta-5\alpha+12\\
&= \beta^2-\tfrac{3}{2}\alpha\beta+6\alpha-8\\
&= \tfrac{1}{2}(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

and consequently

$$\begin{aligned}
S(m^3) &= \tfrac{1}{3}\beta\times\text{weight}\\
&= \tfrac{1}{6}\beta(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

which is right.

Annex No. 3.—*Investigation of $G(m^4)$ in the case where the curve m is a pq complete intersection* (referred to, Art. 42).

Suppose, as before, that $U=0$, $V=0$ are the equations of the two surfaces of the orders p and q respectively; taking also (x, y, z, w) as the coordinates of a point on the curve, and substituting in the equations $x+\rho x'$, $y+\rho y'$, $z+\rho z'$, $w+\rho w'$ in place of the coordinates, then if $\Delta=x'\partial_x+y'\partial_y+z'\partial_z+w\partial_w$, we have as before

$$\begin{aligned}
(\Delta U,\ \Delta^2 U,\ldots\Delta^p U \between 1,\ \rho)^{p-1} &= 0,\\
(\Delta V,\ \Delta^2 V,\ldots\Delta^q V \between 1,\ \rho)^{q-1} &= 0,
\end{aligned}$$

where the numerical coefficients $\frac{1}{1}$, $\frac{1}{1.2}$, $\frac{1}{1.2.3}$, &c. are to be understood as before.

Suppose now that (x, y, z, w) are the coordinates of a point on the curve, through which point there passes a line through three other points, or line $G(m^4)$; and that (x', y', z', w') are the current coordinates of a point on such line; the two equations in ρ must have three equal roots; or we must have a system equivalent to three equations, or say a plexus of three equations. The coordinates (x', y', z', w'), although four in number, are in fact eliminable from this plexus; or what is the same thing, combining with the plexus the equation

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0$$

of an arbitrary plane, and then eliminating (x', y', z', w'), the result is of the form

$$(\alpha x + \beta y + \gamma z + \delta w)^\theta \,\square = 0,$$

where $\square$ is a function of (x, y, z, w) only; and considering (x, y, z, w) as weight variables, $\theta =$ Order of Plexus. But degree in (x, y, z, w) of $(\alpha x + \beta y + \gamma z + \delta w)^\theta \,\square$ is $=$ Weight of Plexus, and therefore Degree of $\square$ is $=$ Weight of Plexus $- \theta$, $=$ (Weight $-$ Order) of Plexus.

The equations $U = 0$, $V = 0$, $\square = 0$ then give the coordinates (x, y, z, w) of the points through which may be drawn a line $G(m^4)$; viz. they give (as it is easy to see) these points four times over. And we therefore have

$$\begin{aligned} G(m^4) &= \tfrac{1}{4} \text{ Order of } (U = 0,\ V = 0,\ \square = 0) \\ &= \tfrac{1}{4} \text{ Deg. } U. \text{ Deg. } V. \text{ Deg. } \square \\ &= \tfrac{1}{4}\, \beta \times (\text{Weight} - \text{Order}) \text{ of Plexus.} \end{aligned}$$

The Plexus is here the square $+2$ system

$$\left\| \begin{matrix} \Delta U, & \Delta^2 U, \ldots \\ . & \Delta\, U, \\ . & \\ . & \\ \Delta V, & \Delta^2 V, \\ . & \Delta\, V, \\ . & \\ . & \end{matrix} \right\| = 0,$$

$(p + q - 4$ columns, $(q - 3) + (p - 3) = p + q - 6$ lines$)$. Or representing the terms by their order and weight (the weight variables being in the present case (x, y, z, w), and the order variables (x', y', z', w')), and attributing as before an order and weight to the evanescent terms, the system is

$$\left\| \begin{array}{c} \\ \\ q-3 \text{ lines.} \\ \\ \\ \\ p-3 \text{ lines.} \\ \\ \\ \end{array} \begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{p+q-3 \text{ columns.}} \\ \left\{ \begin{array}{ll} 1_{p-1}, & 2_{p-2}, \ldots \\ 0_p\ , & 1_{p-1}, \\ . & \\ . & \end{array} \right. \\ \left\{ \begin{array}{ll} 1_{q-1}, & 2_{q-2}, \\ 0_q\ , & 1_{q-1}, \\ . & \\ . & \end{array} \right. \end{array} \right\| = 0,$$

so that we have

$$\begin{array}{lll}
\alpha\,,\ \beta,\,..= & 1,\ 0,\ -1,\,..\ -(q-5), & 1,\ 0,\ \ -1,\,..-(p-5),\\
\alpha'\,,\ \beta',\,..=p-1,\ p,\ p+1,\,..p+\ q-5, & q-1,\ q,\ \ q\ +1,\,..\,q+p-5, & \\
A',\ B,\,..= & 1,\ \ 2,\,\ldots & p+q\ \ -5,\\
A',\ B',\,..= & -1,\ -2,\,\ldots & -(p+q\ \ -5),
\end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{l}\alpha,\ \beta,\,..= \quad 2-\theta,\ 2-\phi\\ \alpha',\ \beta',\,..=p-2+\theta,\ p-2+\phi\end{array}\right\}\ \theta=1 \text{ to } q-3,\ \phi=1 \text{ to } p-3.$$

We then find

$$\begin{aligned}
\Sigma\alpha &= 2(q-3)-\tfrac{1}{2}(q-3)(q-2)+2(p-3)-\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha' &= (p-2)(q-3)+\tfrac{1}{2}(q-3)(q-2)+(q-2)(p-3)+\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha^2 &= 4(q-3)-4.\tfrac{1}{2}(q-3)(q-2)+\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&\quad +4(p-3)-4.\tfrac{1}{2}(p-3)(p-2)+\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^3 &= 8(q-3)-12.\tfrac{1}{2}(q-3)(q-2)+6.\tfrac{1}{6}(q-3)(q-2)(2q-5)-\tfrac{1}{4}(q-3)^2(q-2)^2\\
&\quad +8(p-3)-12.\tfrac{1}{2}(p-3)(p-2)+6.\tfrac{1}{6}(p-3)(p-2)(2p-5)-\tfrac{1}{4}(p-3)^2(p-2)^2,\\
\Sigma\alpha\alpha' &= 2(p-2)(q-3)-(p-4).\tfrac{1}{2}(q-3)(q-2)-\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&\quad +2(q-2)(p-3)-(q-4).\tfrac{1}{2}(p-3)(p-2)-\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^2\alpha' &= 4(p-2)(q-3)-4(p-3).\tfrac{1}{2}(q-3)(q-2)+(p-6).\tfrac{1}{6}(q-3)(q-2)(2q-5)+\tfrac{1}{4}(q-3)^2(q-2)^2,\\
&\quad +4(q-2)(p-3)-4(q-3).\tfrac{1}{2}(p-3)(p-2)+(q-6).\tfrac{1}{6}(p-3)(p-2)(2p-5)+\tfrac{1}{4}(p-3)^2(p-2)^2,\\
\Sigma A &= \tfrac{1}{2}(p+q-5)(p+q-4),\\
\Sigma A^2 &= -\Sigma AA' = \tfrac{1}{6}(p+q-5)(p+q-4)(2p+2q-9),\\
\Sigma A^3 &= -\Sigma A^2A' = \tfrac{1}{4}(p+q-5)^2(p+q-4)^2,
\end{aligned}$$

which, putting therein $p+q=\alpha$, $pq=\beta$, and from the reduced expressions obtaining the values of $\Sigma\alpha\beta$, &c., give

$$\begin{aligned}
\Sigma\alpha &= \beta-\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha-18,\\
\Sigma\alpha^2 &= \beta\,(-\alpha+9)+\tfrac{1}{3}\alpha^3-\tfrac{9}{2}\alpha^2+\tfrac{121}{6}\alpha-58,\\
\Sigma\alpha^3 &= \beta^2(-\tfrac{1}{2})+\beta(\alpha^2-\tfrac{27}{2}\alpha+\tfrac{121}{2})-\tfrac{1}{4}\alpha^4+\tfrac{9}{2}\alpha^3-\tfrac{121}{4}\alpha^2+90\alpha-198,\\
\Sigma\alpha\beta &= \beta^3(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha^2+5\alpha-\tfrac{45}{2})+\tfrac{1}{8}\alpha^4-\tfrac{29}{12}\alpha^3-\tfrac{171}{8}\alpha^2-\tfrac{1092}{12}\alpha+191,\\
\Sigma\alpha\beta\gamma &= \beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\quad -\tfrac{1}{48}\alpha^6+\tfrac{31}{48}\alpha^5-\tfrac{445}{48}\alpha^4+\tfrac{3617}{48}\alpha^3-\tfrac{8969}{24}\alpha^2+1071\alpha-1560,
\end{aligned}$$

$$\Sigma\alpha' \quad = \beta + \tfrac{1}{2}\alpha^2 - \tfrac{15}{2}\alpha + 18,$$

$$\Sigma\alpha\alpha' \quad = \beta\,(-\tfrac{1}{2}\alpha) - \tfrac{1}{3}\alpha^3 + \tfrac{9}{2}\alpha^2 - \tfrac{175}{6}\alpha + 58,$$

$$\Sigma\alpha^2\alpha' \quad = \beta^2(-\tfrac{1}{6}) + \beta\,(-\tfrac{2}{3}\alpha^2 + 9\alpha - \tfrac{121}{6}) + \tfrac{1}{4}\alpha^4 - \tfrac{9}{2}\alpha^3 + \tfrac{121}{4}\alpha^2 - 119\alpha + 198,$$

$$\Sigma A \quad = \tfrac{1}{2}\alpha^3 - \tfrac{9}{2}\alpha + 10,$$

$$\Sigma A^2 \quad = -\Sigma AA' = \tfrac{1}{3}\alpha^3 - \tfrac{9}{2}\alpha^2 + \tfrac{121}{6}\alpha - 30,$$

$$\Sigma A^3 \quad = -\Sigma A^2A' = \tfrac{1}{4}\alpha^4 - \tfrac{3}{2}\alpha^3 + \tfrac{121}{4}\alpha^2 - 90\alpha + 100,$$

$$\Sigma AB \quad = \tfrac{1}{8}\alpha^4 - \tfrac{29}{12}\alpha^3 + \tfrac{139}{8}\alpha^2 - \tfrac{661}{12}\alpha + 65,$$

$$\Sigma ABC = \tfrac{1}{48}\alpha^6 - \tfrac{31}{48}\alpha^5 + \tfrac{397}{48}\alpha^4 - \tfrac{2689}{48}\alpha^3 + \tfrac{5081}{24}\alpha^2 - \tfrac{1270}{3}\alpha + \tfrac{1050}{3},$$

we then find

$$\Sigma A + \Sigma\alpha = \beta - 8,$$

$$\Sigma A' + \Sigma\alpha' = \beta - 3\alpha + 8,$$

$$\Sigma AB - \Sigma\alpha\beta = \beta^2(-\tfrac{1}{2}) + \beta\,(\tfrac{1}{2}\alpha^2 - 5\alpha + \tfrac{45}{2}) - 4\alpha^2 + 36\alpha - 126,$$

$$\Sigma AA' - \Sigma\alpha\alpha' = \beta\,(-\tfrac{1}{2}\alpha) + 9\alpha - 28,$$

$$\Sigma ABC + \Sigma\alpha\beta\gamma = \beta^2(\tfrac{1}{6}) + \beta^2(-\tfrac{1}{4}\alpha^2 + \tfrac{11}{4}\alpha - \tfrac{41}{3}) + \beta\,(\tfrac{1}{8}\alpha^4 - \tfrac{8}{3}\alpha^3 + \tfrac{629}{24}\alpha^2 - \tfrac{749}{6}\alpha + \tfrac{1753}{6})$$
$$- \alpha^4 + \tfrac{58}{3}\alpha^3 - 162\alpha^2 + \tfrac{1943}{3}\alpha - 1210,$$

$$\Sigma A^2A' + \Sigma\alpha^2\alpha' = \beta^2(-\tfrac{1}{6}) + \beta\,(-\tfrac{2}{3}\alpha^2 + 9\alpha - \tfrac{121}{6}) - 29\alpha + 98;$$

and then also

$$\Sigma\alpha(\Sigma A + \Sigma\alpha) = \beta^2 + \beta(-\tfrac{1}{2}\alpha^2 + \tfrac{9}{2}\alpha - 26) + 4\alpha^2 - 36\alpha + 144,$$

$$(\Sigma AB - \Sigma\alpha\beta) + \Sigma\alpha(\Sigma A + \Sigma\alpha) = \beta^3(\tfrac{1}{2}) + \beta(-\tfrac{1}{2}\alpha - \tfrac{7}{2}) + 18,$$

$$\{(\Sigma AB - \Sigma\alpha\beta) + \Sigma\alpha(\Sigma A + \Sigma\alpha)\}(\Sigma A' + \Sigma\alpha') =$$
$$\beta^3(\tfrac{1}{2}) + \beta^2(-2\alpha + \tfrac{1}{2}) + \beta(\tfrac{3}{2}\alpha^2 + \tfrac{13}{2}\alpha - 10) - 54\alpha + 144,$$

$$-(\Sigma A + \Sigma\alpha)(\Sigma AA' - \Sigma\alpha\alpha') = \beta^2(\tfrac{1}{2}\alpha) + \beta(-13\alpha + 28) + 72\alpha - 224,$$

and

$$\Sigma A^2A' + \Sigma\alpha^2\alpha' = (\textit{ut suprà})\ \beta^2(-\tfrac{1}{6}) + \beta(-\tfrac{2}{3}\alpha^2 + 9\alpha - \tfrac{121}{6}) - 29\alpha + 98;$$

whence, adding the last three expressions, we find

$$\text{Weight} = \beta^3(\tfrac{1}{2}) + \beta^2(-\tfrac{3}{2}\alpha + \tfrac{1}{3}) + \beta\,(\tfrac{5}{6}\alpha^2 + \tfrac{5}{2}\alpha - \tfrac{13}{6}) - 11\alpha + 18;$$

and for the order we have

$$(\Sigma\alpha)^2 - \Sigma\alpha\beta = \beta^2(\tfrac{1}{2}) + \beta\,(-\tfrac{1}{2}\alpha^2 + 4\alpha - \tfrac{27}{2}) + \tfrac{1}{8}\alpha^4 - \tfrac{25}{12}\alpha^3 + \tfrac{135}{8}\alpha^2 - \tfrac{851}{12}\alpha + 133;$$

and then

$$\Sigma ABC + \Sigma\alpha\beta\gamma = (ut\ suprà)$$

$$\beta^3(\tfrac{1}{6}) + \beta^2(-\tfrac{1}{4}\alpha^2 + \tfrac{11}{4}\alpha - \tfrac{41}{3}) + \beta(\tfrac{1}{8}\alpha^4 - \tfrac{8}{3}\alpha^3 + \tfrac{629}{24}\alpha^2 - \tfrac{749}{6}\alpha + \tfrac{1753}{6})$$
$$- \alpha^4 + \tfrac{58}{3}\alpha^3 - 162\alpha^2 + \tfrac{1242}{3}\alpha - 1210,$$

$$(\Sigma AB - \Sigma\alpha\beta)\,\Sigma\alpha =$$
$$\beta^3(-\tfrac{1}{2}) + \beta^2(\tfrac{3}{4}\alpha^2 - \tfrac{29}{4}\alpha + \tfrac{63}{2}) + \beta(-\tfrac{1}{4}\alpha^4 + \tfrac{19}{4}\alpha^3 - \tfrac{187}{4}\alpha^2 + \tfrac{209}{4}\alpha - 531)$$
$$+ 2\alpha^4 - 36\alpha^3 + 297\alpha^2 - 1215\alpha + 2268,$$

$$\big((\Sigma\alpha)^2 - \Sigma\alpha\beta\big)(\Sigma A + \Sigma\alpha) =$$
$$\beta^3(\tfrac{1}{2}) + \beta^2(-\tfrac{1}{2}\alpha^2 + 4\alpha - \tfrac{35}{2}) + \beta(\tfrac{1}{8}\alpha^4 - \tfrac{25}{12}\alpha^3 + \tfrac{167}{8}\alpha^2 - \tfrac{1235}{12}\alpha + 241)$$
$$- \alpha^4 + \tfrac{50}{3}\alpha^3 - 135\alpha^2 + \tfrac{1702}{3}\alpha - 1064;$$

whence, adding these three expressions,

$$\text{Order} = \beta^3(\tfrac{1}{6}) + \beta^2(-\tfrac{1}{2}\alpha + \tfrac{1}{3}) + \beta(\tfrac{1}{3}\alpha^2 - \tfrac{1}{2}\alpha + \tfrac{13}{6}) - 6;$$

and by means of the foregoing expression for the weight, we then have

$$\text{Weight} - \text{Order} = \beta^3(\tfrac{1}{3}) + \beta^2(-\ \alpha\ ) + \beta(\tfrac{1}{2}\alpha^2 + 3\alpha - \tfrac{13}{3}) - 11\alpha + 24;$$

and therefore

$$G(m^4) = \tfrac{1}{4}\beta \times (\text{Weight} - \text{Order}),$$
$$= \tfrac{1}{24}\beta\{2\beta^3 + \beta^2(-6\alpha) + \beta(3\alpha^2 + 18\alpha - 26) - 66\alpha + 144\},$$

which is right.

Annex No. 4.—*Order of* Torse (m, n) (referred to, Art. 44).

We have to find the order of the developable or Torse generated by a line meeting two curves of the orders m, n respectively; viz. representing by μ, ν the classes of the two curves respectively, it is to be shown that the expression for the Order is

$$\text{Torse}\ (m, n) = m\nu + n\mu.$$

I remark, in the first place, that, given two surfaces of the orders p and q respectively, the curve of intersection is of the order pq and class $pq(p+q-2)$, or as this may be written, class $= qp(p-1) + pq(q-1)$. Reciprocally for two surfaces of the classes p and q respectively, the Torse enveloped by their common tangent planes is of the class pq and order $qp(p-1) + pq(q-1)$. Now, in the same way that a surface of the order p may degenerate into a Torse of the order p, so a surface of the class p may degenerate into a curve of the class p; and the class of a curve being p, then (disregarding singularities) its order is $= p(p-1)$. Hence replacing p and $p(p-1)$ by μ and m respectively, and in like manner q and $q(q-1)$ by ν and n respectively, we have $m\nu + n\mu$ as the order of the Torse generated by the tangent planes of the curves of the orders m and n respectively; where by tangent plane of a curve is to

be understood a plane passing through a tangent line of the curve. The intersection of two consecutive tangent planes is a line meeting the two curves, which line is the generating line of the Torse, and such Torse is therefore the Torse (m, n) in question.

The foregoing investigation is not very satisfactory, but I confirm it by considering the case of two plane curves, orders m and n, and classes μ and ν, respectively. The tangents of the two curves can, it is clear, only meet on the line of intersection of the planes of the curves; and the construction of the Torse is in fact as follows: from any point of the line of intersection draw a tangent to m and a tangent to n, then the line joining the points of contact of these tangents is a generating line of the Torse. The order of the Torse is equal to the number of generating lines which meet an arbitrary line; and taking for the arbitrary line the line of intersection of the two planes, it is easy to see that the only generating lines which meet the line of intersection are those for which one of the points of contact lies on the line of intersection; that is, they are the generating lines derived from the points in which the line of intersection meets one or other of the two curves; they are therefore in fact the tangents drawn to the curve n from the points in which the line of intersection meets the curve m, together with the tangents drawn to the curve m from the points in which the line of intersection meets the curve n. Now the line meets the curve n in n points, and from each of these there are μ tangents to the curve m; and it meets the curve m in m points, and from each of these there are ν tangents to the curve n; hence the entire number of the tangents in question is $=n\mu+m\nu$, which confirms the theorem.

Annex No. 5.—*Order of* Torse (m^2) (referred to, Art. 46).

We have here to find the order of the developable or Torse generated by a line meeting a curve of the order m twice, viz., the class of the curve being μ, it is to be shown that we have

$$\text{Torse } (m^2) = (m-3)\,\mu.$$

I deduce the expression from the formula given p. 424 of Dr Salmon's 'Geometry of Three Dimensions;' viz. putting in his formula $\beta=0$, and μ for his r, we have

$$\text{Order} = m(\mu-4) - \tfrac{1}{2}\alpha = m\mu - (4m + \tfrac{1}{2}\alpha),$$

where (see p. 234 *et seq.*)

$$\mu = m(m-1) - 2h,$$
$$\tfrac{1}{2}\alpha = (n-m) = 3m(m-2) - 6h - m,$$

and thence

$$3\mu - \tfrac{1}{2}\alpha = 4m, \text{ or } 4m + \tfrac{1}{2}\alpha = 3\mu,$$

so that we have

$$\text{Order} = (m-3)\,\mu.$$

A more complete discussion of the Torses (m, n) and (m^2) is obviously desirable; but as they are only incidentally connected with the subject of the present memoir, I have contented myself with obtaining the required results in the way which most readily presented itself.

340.

A SECOND MEMOIR ON SKEW SURFACES, OTHERWISE SCROLLS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLIV. (for the year 1864), pp. 559—576. Received April 29,—Read May 26, 1864.]

THE principal object of the present memoir is to establish the different kinds of skew surfaces of the fourth order, or Quartic Scrolls; but, as preliminary thereto, there are some general researches connected with those in my former memoir "On Skew Surfaces, otherwise Scrolls"(1), and I also reproduce the theory (which may be considered as a known one) of cubic scrolls; there are also some concluding remarks which relate to the general theory. As regards quartic scrolls, I remark that M. Chasles, in a footnote to his paper, "Description des courbes de tous les ordres situées sur les surfaces réglées du troisième et du quatrième ordres"(2), states, "les surfaces réglées du quatrième ordre.... admettent *quatorze* espèces." This does not agree with my results, since I find only eight species of quartic scrolls; the developable surface or "torse" is perhaps included as a "surface réglée;" but as there is only one species of quartic torse, the deficiency is not to be thus accounted for. My enumeration appears to me complete, but it is possible that there are subforms which M. Chasles has reckoned as distinct species.

On the Degeneracy of a Scroll, Article Nos. 1 to 5.

1. A scroll considered as arising from any geometrical construction, for instance one of the scrolls $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$ considered in my former memoir, or say in general the scroll S, may break up into two or more inferior scrolls S', S'',..; but as long as S', S'',.. are proper scrolls (not torses, and *à fortiori* not cones or planes), no one of these can be considered, apart from the others, as the result of the geometrical

[1] *Philosophical Transactions*, vol. CLIII. (1863), pp. 453—483, [339].
[2] *Comptes Rendus*, t. LIII. (1861), see p. 888.

construction, and we can only say that the scroll S given by the construction is the aggregate of the scrolls S', S'',..; and the like when we have the scrolls S', S'',..., each repeated any number of times, or say when $S = S'^{\alpha} S''^{\beta} \ldots$ Suppose however that the scrolls S', S'',.. are any one or more of them a torse or torses—or, to make at once the most general supposition, say that we have $S = \Sigma S'$, where Σ is a torse, or aggregate of torses ($\Sigma = \Sigma'^{\alpha} \Sigma''^{\beta} \ldots$), and S' is a proper scroll or aggregate of proper scrolls; then, although it is not obligatory to do so, we may without impropriety throw aside the torse-factor Σ, and consider the original scroll S as degenerating into the scroll S', and as suffering a reduction in order accordingly.

2. As an illustration, consider the scroll $S(m, n, p)$ generated by a line which meets three directrix curves of the orders m, n, p respectively; and assume that the curves m, n, p are each of them situate on the same scroll Σ, the curve m meeting each generating line of Σ in α points, the curve n each generating line in β points, and the curve p each generating line in γ points. Each generating line of Σ is $\alpha\beta\gamma$ times a generating line of S, and we have $S = \Sigma^{\alpha\beta\gamma} S'$, where S' may be a proper scroll; it is however to be noticed that if the curves m, n, p any two of them intersect, S' will itself break up and contain cone-factors, as will presently appear. And if Σ, instead of being a proper scroll, be a torse, then we may consider S as degenerating into S', the reduction in order being of course $= \alpha\beta\gamma \times$ order of Σ.

3. But this is not the only way in which the scroll $S(m, n, p)$ may degenerate; for suppose that two of the directrix curves, say n and p, intersect, then the lines from the point of intersection to the curve m form a cone of the order m which will present itself as a factor of S; and generally if the curves n and p intersect in α points, the curves p and m in β points, and the curves m and n in γ points, then we have α cones each of the order m, β cones each of the order n, and γ cones each of the order p, or say $S = CS'$, where C is the aggregate of the cone-factors; and the scroll S degenerates into S', the reduction in order being $= \alpha m + \beta n + \gamma p$. It is hardly necessary to remark that if a point of intersection of two of the curves is a multiple point on either or each of the curves, it is, in reckoning the number of intersections of the two curves, to be taken account of according to its multiplicity in the ordinary manner.

4. There is yet another case to be considered: suppose that the curves n and p lie on a cone, and that the curve m passes through the vertex of this cone; this cone, repeated a certain number of times, is part of the locus, or we have $S = C^{\theta} S'$, so that the scroll S degenerates into S', the reduction in order being $= \theta \times$ order of cone. If, to fix the ideas, the curves n and p are respectively the complete intersections of the cone by two surfaces of the orders g, h respectively (this implies $n = gk$, $p = hk$, if k be the order of the cone), which surfaces do not pass through the vertex of the cone, and if, moreover, the vertex of the cone be an a-tuple point on the curve m, then $\theta = agh$, and the reduction in order is $= aghk$.

5. The foregoing causes of reduction, or some of them, may exist simultaneously; it would require a further examination to see whether the aggregate reduction is in

all cases the sum of the separate reductions. But the aggregate reduction once ascertained, then writing $S(m, n, p)$ for the order of the reduced scroll, we shall have

$$S(m, n, p) = 2mnp - \text{Reduction}.$$

In particular, in the case above referred to, where the curves n and p, p and m, m and n meet in α, β, γ points respectively, but there is no other cause of reduction,

$$S(m, n, p) = 2mnp - \alpha m - \beta n - \gamma p,$$

which is a formula which will be made use of.

The foregoing investigations apply, *mutatis mutandis*, to the scrolls $S(m^2, n)$, $S(m^3)$; but I do not at present enter into the development of them in regard to these scrolls.

Scrolls with two directrix lines, Article Nos. 6 to 11.

6. Consider now a scroll having two directrix lines: it may be assumed that these do not intersect; for if they did, then any generating line, *quà* line meeting the two directrix lines, would either lie in the plane of the two lines, or else would pass through their point of intersection; that is, the scroll would break up into the plane of the two lines, considered as the locus of the tangents of a plane curve, and into a cone having for its vertex the point of intersection of the two lines. Each generating line meets any plane section of the scroll in the point where such generating line meets the plane of the section; the plane section constitutes a third directrix; or the scrolls in question are all included in the form $S(1, 1, m)$, where m is a plane curve. The order of the scroll $S(1, 1, m)$ is in general $=2m$; but if the one line meets the curve α times, that is, in an α-tuple point of the curve, and the other line meets the curve β times, that is, in a β-tuple point of the curve, then by the general formula (*ante*, No. 5) the order of the scroll is $=2m-\alpha-\beta$; and in particular if $\alpha+\beta=m$, then the order is $=m$.

7. We may *without loss of generality* attend only to the last-mentioned case. To show how this is, suppose for a moment that the two lines do not either of them meet the curve; the scroll is then of the order $2m$. Call the point in which each line meets the plane of the curve the foot of this line, then the line joining the two feet meets the curve in m points; and it is in respect of each of these points a generating line of the scroll; that is, it is an m-tuple generating line: the section of the scroll by the plane of the curve m is in fact this line counting m times, and the curve m; $m+m=2m$, the order of the scroll. And in like manner the section by any plane through the m-tuple line is this line counting m times, and a curve of the order m not meeting either of the directrix lines. But the section by any other plane is a curve of the order $2m$ meeting each of the directrix lines in a point which is an m-tuple point of the section (each directrix line is in fact an m-tuple line of the scroll); and by considering, in place of the particular section m, this general section, we have the scroll of the order $2m$ in the form $S(1, 1, 2m)$, where the two directrix lines each meet the section m times; so that the order is $4m-m-m=2m$.

8. And so in general, m being a plane curve, when the scroll $S(1, 1, m)$ is of an order superior to m, say $=m+k$, this only means that the section chosen for the directrix curve m is not the complete section by the plane of such curve, but that the line joining the feet of the two directrix lines is a k-tuple generating line of the scroll, and that the complete section is made up of this line counting k times and of the curve m. So that taking, not the section through the multiple generating line, but the general section, for the plane directrix curve, the only case to be considered is that in which the section is a proper curve of an order equal to that of the scroll; or, what is the same thing, we have only to consider the scrolls $S(1, 1, m)$ for which the order is depressed from $2m$ to m in consequence of the directrix lines meeting the plane section α times and β times, that is, in an α-tuple point and a β-tuple point respectively, where $\alpha+\beta=m$.

9. It is clear that in the case in question the directrix lines are an α-tuple line and a β-tuple line respectively. The generation is as follows: Scroll $S(1, 1, m)$ of the order m; the curve m being a plane curve of the order m having an α-tuple point and a β-tuple point, where $\alpha+\beta=m$: the directrix lines, say 1 and 1′, pass through these points respectively, and they do not intersect each other. The generating lines pass through the directrix lines 1 and 1′ and the curve m, and we have thence the scroll $S(1, 1, m)$. Taking at pleasure any point on the curve m, we can through this point draw a *single* line meeting each of the directrix lines 1, 1′; that is, the curve m is a simple curve on the scroll. Taking at pleasure a point on the directrix line 1, and making this the vertex of a cone standing on the curve m, this cone has an α-tuple line (the line 1) and a β-tuple line (the line joining the vertex with the foot of the line 1′); the line 1′ meets this cone in the foot of the line 1′, counting β times, and besides in $m-\beta, =\alpha$ points; the lines joining the vertex with the last-mentioned points respectively (or, what is the same thing, the lines, other than the β-tuple line, in which the plane through the vertex and the line 1′ meets the cone) are the α generating lines through the assumed point on the line 1; and the line 1 is thus an α-tuple line of the scroll. And in like manner, through an assumed point of the directrix line 1′, we construct β generating lines of the scroll; and the line 1′ is a β-tuple line of the scroll.

10. The scroll $S(1, 1, m)$ now in question has not in general any multiple generating line; in fact a multiple generating line would imply a corresponding multiple point on the section m; and this section, assumed to be a curve having an α-tuple point and a β-tuple point, has not in general any other multiple point. But it *may* have other multiple points; and if there is, for example, a γ-tuple point, then the line from this point which meets the two directrix lines counts γ times, or it is a γ-tuple generating line; and so for all the multiple points of m other than the α-tuple point and the β-tuple point which correspond to the directrix lines respectively. It is to be noticed that the multiplicity γ of any such multiple generating line is at most equal to the smallest of the two numbers α and β; for suppose $\gamma>\alpha$, then, since $\alpha+\beta=m$, we should have $\gamma+\beta>m$, and the line joining the γ-tuple point and the β-tuple point would meet the curve m in $\gamma+\beta$ points, which is absurd. In the

case of several multiple lines, there are other conditions of inequality preventing self-contradictory results [1].

11. The general section is a curve of the order m, having an α-tuple point and a β-tuple point corresponding to the directrix lines respectively, and a γ-tuple point, &c. ... corresponding to the other multiple points (if any). A section through the directrix line 1 is in general made up of this line, counting α times, and of β generating lines passing through one and the same point of the directrix line 1′; if the section pass also through a γ-tuple generating line, then, of the β generating lines in question, γ (which, as has been seen, is $\not> \beta$) unite together in the γ-tuple generating line; and so for the sections through the directrix line 1′. The general section through a γ-tuple generating line is this line counting γ times, and a curve of the order $m-\gamma$, which has an $(\alpha-\gamma)$ tuple point at its intersection with the directrix line 1, and a $(\beta-\gamma)$ tuple point at its intersection with the directrix line 1′; it has a δ-tuple point, &c... at its intersections with the other multiple generating lines, if any.

Scrolls with a twofold directrix line, Article Nos. 12 to 16.

12. But there is a case included indeed as a limiting one in the foregoing general case, but which must be specially considered; viz. the two directrix lines 1 and 1′ may coincide, giving rise to a twofold directrix line. To show how this is, I return for the moment to the case of the scroll $S(1, 1, m)$ with two distinct directrix lines 1 and 1′, and, to fix the ideas, I suppose that the directrix lines do not either of them meet the curve m, so that the order of the scroll is $=2m$. Through the line 1 imagine the series of planes $A, B, C, \ldots$ meeting the line 1′ in the points $a', b', c' \ldots$; the generating lines through the point a' are the lines in the plane A to the points in which this plane meets the curve m; the generating lines through the point b' are the lines in the plane B to the points where this plane meets the curve m; and so for the generating lines through the points $c', d' \ldots$; and it is clear that the points $a', b', c', \ldots$ correspond homographically with the planes $A, B, C, \ldots$ This gives immediately the construction for the case where the two directrix lines come to coincide. In fact, on the twofold directrix line $1=1'$ take the series of points $a, b, c \ldots$, and through the same line, corresponding homographically to these points, the series of planes $A, B, C, \ldots$; the generating lines through the point a are the lines through this point, in the plane A, to the points in which this plane meets the curve m; and so for the entire series of points $b, c, \ldots$ of the line $1=1'$; the resulting scroll, which I will designate as the scroll $S(\overline{1, 1}, m)$, remains of the order $=2m$. If there is given a point of the curve m, then the plane through this point and the directrix line is the plane A; and the point a is then also given by the homographic correspondence of the series of planes and points, and the generating line through the given point on the curve m is the line joining this point with the point a.

[1] Suppose, for example (see next paragraph of the text), that there were a γ-tuple generating line and a δ-tuple generating line lying *in plano* with the line 1; these lines counting as $(\gamma+\delta)$ lines, must be included among the β generating lines through the plane in question; this implies that $\gamma+\delta \not> \beta$, a conclusion which must be obtainable from consideration of the curve m irrespectively of the scroll.

13. We may say that, in regard to any point a of the line 1, the corresponding plane A is the plane of approach of the coincident line 1′; and that in regard to the same point a and to any plane through it, the trace on that plane of the plane of approach is the line of approach of 1′; that is, we may consider that the coincident directrix line 1′ meets the plane through a in a consecutive point on the line of approach. In particular if the point a be the foot of the directrix line 1 (that is, the point where this line meets the plane of the curve m), and the plane through a be the plane of the curve m, then the intersection of the last-mentioned plane by the plane A which corresponds to the point a is the line of approach, and the foot of the coincident directrix line 1′ is the consecutive point to a along the line of approach. The expression "the line of approach," used absolutely, has always the signification just explained, viz. it is the intersection of the plane of the curve m by the plane corresponding to the foot of the directrix line.

14. Suppose now that the line 1 meets the curve m, or, more generally, meets it α times, that is, in an α-tuple point; it might at first sight appear that the coincident line 1′ should also be considered as meeting the curve α times, and that the resulting scroll should be of the order $2m-\alpha-\alpha=2m-2\alpha$. But this is not the case; so long as the direction of the line of approach is arbitrary, the line 1′ must be considered as a line indefinitely near to the line 1, but nevertheless as a line not meeting the curve at all; and the order of the scroll is thus $=2m-\alpha$. If, however, the line of approach is the tangent to a branch through the α-tuple point—that is, if the plane corresponding to the α-tuple point meet the plane of the curve in such tangent, then the coincident line 1′ is to be considered as meeting the curve m in a consecutive point on such branch, and the order of the scroll is $=2m-\alpha-1$. And so if at the multiple point there are β branches having a common tangent, then the coincident line 1′ is to be considered as meeting the curve m in a consecutive point along each of such branches, or say in a consecutive β-tuple point along the branch, and the order of the scroll sinks to $2m-\alpha-\beta$. The point spoken of as the α-tuple point is, it should be observed, more than an α-tuple point with a β-fold tangent; it is really a point of union of an α-tuple point and a β-tuple point, or say a united $\alpha(+\beta)$ tuple point, equivalent to

$$\tfrac{1}{2}\alpha(\alpha-1)+\tfrac{1}{2}\beta(\beta-1)$$

double points or nodes; and the case is precisely analogous to that of the scroll $S(1, 1, m)$, where the two directrix lines pass through an α-tuple point and a β-tuple point of the curve m respectively. It may be added that if at the multiple point in question, besides the β branches having a common tangent, there are γ branches having a common tangent, then the point is, so to speak, a united $\alpha(+\beta, +\gamma)$ tuple point equivalent to $\frac{1}{2}\alpha(\alpha-1)+\frac{1}{2}\beta(\beta-1)+\frac{1}{2}\gamma(\gamma-1)$ double points or nodes; but the order of the scroll is still $=2m-\alpha-\beta$.

15. In the same way as the scrolls $S(1, 1, m)$ are all included in the case where the order of the scroll, instead of being $=2m$, is $=m$, so that the scrolls $S(\overline{1, 1}, m)$ are all included in the case where the order of the scroll, instead of being $=2m$, is $=m$.

That is, we may suppose that the curve m has a united $\alpha(+\beta)$ tuple point $(\alpha+\beta=m)$, and may take the directrix line to pass through this point, and the line of approach to be the common tangent of the β branches; and this being so, the order of the scroll will be $2m-\alpha-\beta, =m$. It may be added that if the curve m has, besides the $\alpha(+\beta)$ tuple point, a γ-tuple point, then the scroll will have a γ-tuple generating line, and so for the other multiple points of the curve m.

16. We may, in the same way as for the scroll $S(1, 1, m)$, consider the different sections of the scroll $S(\overline{1, 1}, m)$ of the order m. The general section is a curve of the order m, having an $\alpha(+\beta)$ tuple point at the intersection with the directrix line, and a γ-tuple point, &c. corresponding to the multiple generating lines, if any. A section through the directrix line is in general made up of this line counting α times, and of β generating lines through the point which corresponds to the plane of the section; if the section pass also through a γ-tuple generating line $(\gamma \not> \beta$, in the same way as for the scroll $S(1, 1, m)$), then, of the β generating lines, γ unite together in the γ-tuple generating line. The general section through a γ-tuple generating line breaks up into this line counting γ times, and a curve of the order $m-\gamma$, which has on the directrix line an $\overline{\alpha-\gamma}(+\overline{\beta-\gamma})$ tuple point and a δ-tuple point, &c. at its intersections with the other multiple generating lines, if any.

Equation of the Scroll $S(1, 1, m)$ of the order m, Article Nos. 17 and 18.

17. Taking for the equations of the directrix lines $(x=0, y=0)$ and $(z=0, w=0)$, and supposing that these are respectively an α-tuple line and a β-tuple line on the scroll $\alpha+\beta=m$, it is obvious that the equation of the scroll is

$$(*\between x, y)^\alpha (z, w)^\beta = 0.$$

In fact starting with this equation, if we consider the section by a plane through the line $(x=0, y=0)$, say the plane $y=\lambda x$, then the equation gives

$$x^\alpha (*\between 1, \lambda)^\alpha (z, w)^\beta = 0;$$

that is, the section is made up of the line $(x=0, y=0)$ reckoned α times, and of β other lines in the plane $y=\lambda x$; and the like for the section by any plane through the line $(z=0, w=0)$, say the plane $z=\nu w$. Hence the assumed equation represents a scroll of the order m, having the two lines for an α-tuple line and a β-tuple line respectively, and conversely such scroll has an equation of the assumed form.

Case of a γ-tuple generating line.

18. The multiple generating line meets each of the lines $(x=0, y=0)$ and $(z=0, w=0)$; and we may take for the equations of the multiple generating line $x+y=0$, $z+w=0$. This being so, the foregoing equation of the scroll may be expressed in the form

$$(\dagger\between x, y)^\alpha (z, z+w)^\beta = 0,$$

or say

$$(U, V, W, ..)(z, z+w)^\beta = 0,$$

where $U, V, W, \ldots$ are functions of the form $(*\between x, y)^{\alpha}$. Hence ($\gamma \not> \alpha$ or β), if the functions $U, V, W, \ldots$ contain respectively the factors $(x+y)^{\gamma}, (x+y)^{\gamma-1}, (x+y)^{\gamma-2}, \ldots$, the equation will be of the form

$$(*\between x+y,\ z+w)^{\gamma}=0$$

(the coefficients being functions of x, y, z and $z+w$, or, what is the same thing, x, y, z, w, of the order $\alpha+\beta-\gamma$), and the scroll will therefore have the line $x+y=0$, $z+w=0$ as a γ-tuple generating line.

Equation of the Scroll $S(\overline{1,\ 1},\ m)$ *of the order* m, Article Nos. 19 to 24.

19. We may take $x=0$, $y=0$ for the equations of the twofold directrix line, $z=0$ for the equation of the plane of the curve m (an arbitrary plane section of the scroll). Then ($\alpha+\beta=m$), if the curve m have at the point ($x=0$, $y=0$), or foot of the directrix line, an $\alpha(+\beta)$tuple point, and if moreover we have $y=0$ for the equation of the common tangent of the β branches (viz. if the plane $y=0$, instead of being an arbitrary plane through the directrix line, be the plane through this line and the common tangent of the β branches), the equation of the curve m will be of the form

$$\Sigma\,(yw)^{\beta'}\,(*\between x,\ y)^{\alpha+\beta-2\beta'}=0,$$

where the summation extends to all integer values of β' from 0 to β, both inclusive.

20. Taking $y=\lambda x$ for the equation of any plane through the directrix line, then the corresponding point on the directrix line will be the intersection of this line ($x=0$, $y=0$) by the plane $z=\theta w$, where $\theta=\dfrac{a\lambda+b}{c\lambda+d}$; the foot of the directrix line is given by the value $\theta=0$, or $\lambda=-\dfrac{b}{a}$, and the equation of the line of approach is therefore $y=-\dfrac{b}{a}x$; this should coincide with the line $y=0$, which is the common tangent of the β branches; that is, we must have $b=0$; I retain, however, for the moment the general value of b.

21. The equations of a generating line will be

$$y=\lambda x,\quad z=\theta w-px;$$

and then taking $X, Y, (Z=0)$ and W for the coordinates of the point of intersection with the curve m, we have

$$Y=\lambda X,\quad 0=\theta W-pX,$$
$$\Sigma\,(YW)^{\beta'}\,(*\between X,\ Y)^{\alpha+\beta-2\beta}=0,$$

and thence

$$\Sigma\left(\frac{\lambda p}{\theta}\right)^{\beta'}(*\between 1,\ \lambda)^{\alpha+\beta-2\beta'}=0,$$

or, what is the same thing,

$$\Sigma\theta^{-\beta'}\,(\lambda p)^{\beta'}\,(*\between 1,\ \lambda)^{\alpha+\beta-2\beta'}=0;$$

which equation, substituting therein for θ its value in terms of λ, gives the parameter p which enters into the equations of the generating line; or, what is the same thing, the equation of the scroll is obtained by eliminating λ, θ, p from the equation just mentioned and the equations

$$y = \lambda x, \quad z = \theta w - px, \quad \theta = \frac{a\lambda + b}{c\lambda + d}.$$

22. These last three equations give

$$\lambda = \frac{y}{x}, \quad \theta = \frac{ay + bx}{cy + dx}, \quad p = \frac{\theta w - z}{x} = \frac{(ay + bx)\, w - (cy + dx)\, z}{x};$$

and substituting these values, we find for the equation of the scroll

$$\Sigma\, (ay + bx)^{\beta - \beta'} y^{\beta'}\, [(ay + bx)\, w - (cy + dx)\, z]^{\beta'}\, (*\between x,\ y)^{\alpha + \beta - 2\beta'} = 0,$$

which is of the order $\alpha + 2\beta, = 2m - \alpha$, so that the $\alpha(+\beta)$ tuple point, in the case actually under consideration, produces only a reduction $= \alpha$. If however the line of approach coincides with the tangent of the β branches, then $b = 0$; the factor y^β divides out, and the equation is

$$\Sigma\, (ayw - cyz - dxz)^{\beta'}\, (*\between x,\ y)^{\alpha + \beta - 2\beta'} = 0,$$

which is of the order $\alpha + \beta, = m$, so that here the reduction caused by the $\alpha(+\beta)$ tuple point is $= \alpha + \beta$. We may without loss of generality substitute ax for $cy + dx$, and then, putting also $a = 1$, we find that when the equation of the curve m is as before

$$\Sigma\, (yw)^{\beta'}\, (*\between x,\ y)^{\alpha + \beta - 2\beta'} = 0,$$

but the plane through the directrix line $(x = 0,\ y = 0)$, and the point on this line, are respectively given by the equations $x = \lambda y$, $z = \lambda w$, the equation of the scroll is

$$\Sigma\, (yw - xz)^{\beta'}\, (*\between x,\ y)^{\alpha + \beta - 2\beta'} = 0.$$

23. The result may be verified by considering the section by any plane $y = \lambda x$ through the directrix line. Substituting for y this value, we find

$$x^\alpha \Sigma x^{\beta - \beta'}\, (\lambda w - z)^{\beta'}\, (*\between 1,\ \lambda)^{\alpha + \beta - \beta'} = 0,$$

which is of the form

$$x^\alpha\, (+\between x,\ \lambda w - z)^\beta = 0;$$

so that the section is made up of the directrix line $(x = 0,\ y = 0)$ reckoned α times and of β lines in the plane $y - \lambda x = 0$, the intersections of the plane $y - \lambda x = 0$ by planes such as $z = \lambda w - px$.

Case of a γ-tuple generating line.

24. The equation of the scroll may be written

$$(U,\ V,\ W, \ldots \between 1,\ yw - xz)^\beta = 0,$$

where $U, V, W, \ldots$ are functions of x, y of the forms

$$(*\between x, y)^m, \quad (*\between x, y)^{m-2}, \quad (*\between x, y)^{m-4}, \ldots;$$

assuming that these contain respectively the factors

$$(y-\kappa x)^\gamma, \quad (y-\kappa x)^{\gamma-1}, \quad (y-\kappa x)^{\gamma-2} \ldots,$$

where $\gamma \not> \frac{1}{2}m$, then the equation takes the form

$$(U', V', W' \ldots \between y-\kappa x, w(y-\kappa x)+x(\kappa w-z))^\gamma = 0,$$

where the coefficients $U', V', W', \ldots$ are functions of x, y, z, w of the orders $m-\gamma$, $m-\gamma-1$, $m-\gamma-2, \ldots$; or, what is the same thing, the equation is

$$(U'', V'', W'', \ldots \between y-\kappa x, \kappa w-z)^\gamma = 0,$$

where $U'', V'', W'', \ldots$ are functions of x, y, z, w of the order $m-\gamma$. The scroll has thus the γ-tuple generating line

$$y-\kappa x = 0, \quad \kappa w - z = 0.$$

Cubic Scrolls, Article Nos. 25 to 35.

25. In the case of a cubic scroll there is necessarily a nodal[1] line; in fact for the m-thic scroll there is a nodal curve which is of the order $m-2$ at least, and of the order $\frac{1}{2}(m-1)(m-2)$ at most, and which for $m=3$ is therefore a right line. And moreover we see at once that every cubic surface having a nodal line is a scroll; in fact any plane whatever through the nodal line meets the surface in this line counting as 2 lines, and in a curve of the order 1, that is, a line; there are consequently on the surface an infinity of lines, or the surface is a scroll. We have therefore to examine the cubic surfaces which have a nodal line.

26. Let the equations of the nodal line be $x=0$, $y=0$; then the equation of the surface is

$$Uz + Vw + Q = 0,$$

where U, V, Q are functions of (x, y) of the orders 2, 2, 3 respectively. Suppose first that U, V have no common factor, then we may write

$$Q = (\alpha x + \beta y)\, U + (\gamma x + \delta y)\, V;$$

and substituting this value, and changing the values of z and w, the equation of the surface is of the form

$$Uz + Vw = 0,$$

or, what is the same thing,

$$(*\between x, y)^2 (z, w) = 0;$$

[1] The nodal line of a cubic scroll is of course a double line, and in regard to these scrolls the epithets 'nodal' and 'double' may be used indifferently.

so that, besides the nodal directrix line $(x=0,\ y=0)$, the scroll has the simple directrix line $(z=0,\ w=0)$: it is clear that the section by any plane whatever is a cubic curve having a node at the foot of the nodal directrix line $(x=0,\ y=0)$, and passing through the foot of the simple directrix line $(z=0,\ w=0)$; that is, it is a cubic scroll of the kind $S(1,\ 1,\ 3)$; and since for $m=3$ the only partition $m=\alpha+\beta$ is $m=2+1$, there is only one kind of cubic scroll $S(1,\ 1,\ 3)$, and we may say *simpliciter* that the scroll in question is the cubic scroll $S(1,\ 1,\ 3)$.

27. If however the functions $U,\ V$ have a common factor, say $(\lambda x+\mu y)$, then $zU+wV$ will contain this same factor, and the remaining factor will be of the form

$$z(\alpha x+\beta y)+w(\gamma x+\delta y),\quad =y(\beta z+\delta w)+x(\alpha z+\gamma w),$$

or, changing the values of z and w, the remaining factor will be of the form $yw-xz$, and the equation of the scroll thus is

$$(\lambda x+\mu y)(yw-xz)+(*\between x,\ y)^3=0,$$

where it is clear that the section by any plane whatever is a cubic curve having a node at the foot of the directrix line $x=0,\ y=0$. The scroll is thus a cubic scroll of the form $S(\overline{1,\ 1},\ 3)$, viz. it is the scroll of the kind where the section is a cubic curve with a $2(+1)$ tuple point (ordinary double point, or node), the line of approach being one of the two tangents at the node; and since for $m=3$ the only partition $m=\alpha+\beta$ is $m=2+1$, there is only one kind of cubic scroll $S(\overline{1,\ 1},\ 3)$, and we may say *simpliciter* that the scroll in question is the cubic scroll $S(\overline{1,\ 1},\ 3)$. The conclusion therefore is that for cubic scrolls we have only the two kinds, $S(1,\ 1,\ 3)$ and $S(\overline{1,\ 1},\ 3)$. The foregoing equations of these scrolls admit however of simplification; and I will further consider the two kinds respectively.

The Cubic Scroll $S(1,\ 1,\ 3)$.

28. Starting from the equation

$$(*\between x,\ y)^2(z,\ w)=0,$$

or, writing it at full length,

$$z(a,\ b,\ c\between x,\ y)^2+w(a',\ b',\ c'\between x,\ y)^2=0,$$

we may find $\theta_1,\ \theta_2$ so that

$$(a,\ b,\ c\between x,\ y)^2+\theta_1(a',\ b',\ c'\between x,\ y)^2=(p_1x+q_1y)^2,$$
$$(a,\ b,\ c\between x,\ y)^2+\theta_2(a',\ b',\ c'\between x,\ y)^2=(p_2x+q_2y)^2,$$

θ_1 and θ_2 being unequal, since by hypothesis $(a,\ b,\ c\between x,\ y)^2$ and $(a',\ b',\ c'\between x,\ y)^2$ have no common factor. This gives

$$(a,\ b,\ c\between x,\ y)^2=\alpha(p_1x+q_1y)^2+\beta(p_2x+q_2y)^2,$$
$$(a',\ b',\ c'\between x,\ y)^2=\gamma(p_1x+q_1y)^2+\delta(p_2x+q_2y)^2;$$

or the equation becomes

$$(\alpha z+\gamma w)(p_1x+q_1y)^2+(\beta z+\delta w)(p_2x+q_2y)^2=0;$$

or changing the values of (x, y) and of (z, w), the equation is

$$x^2z + y^2w = 0,$$

which may be considered as the canonical form of the equation. It may be noticed that the Hessian of the form is x^2y^2.

29. We may of course establish the theory of the surface from the equation $x^2z + y^2w = 0$; the equation is satisfied by $x = \lambda y$, $w = -\lambda^2 z$, which are the equations of a line meeting the line $(x=0,\ y=0)$ (1) and the line $(z=0,\ w=0)$ (1′). The generating line meets also any plane section of the surface; in fact, if the equation of the plane of the section be $\alpha x + \beta y + \gamma z + \delta w = 0$, then we have at once

$$x : y : z : w = \delta\lambda^3 - \gamma\lambda : \delta\lambda^2 - \gamma : \alpha\lambda + \beta : -\alpha\lambda^3 - \beta\lambda^2$$

for the coordinates of the point of intersection.

30. The form of the equation shows that there are on the line 1 two points, viz. the points $(x=0,\ y=0,\ z=0)$ and $(x=0,\ y=0,\ w=0)$, through each of which there passes a pair of coincident generating lines: calling these A and B, then, if the coincident lines through A meet the line 1′ in C, and the coincident lines through B meet the line 1′ in D, it is easy to see that $x=0$, $y=0$, $z=0$, and $w=0$ will denote the equations of the planes BAC, BAD, BCD, and ACD respectively.

31. We obtain also the following construction: take a cubic curve having a node, and from any point K on the curve draw to the curve the tangents Kp, Kq; through the points of contact draw at pleasure the lines pAC and qBD; through the node draw a line meeting these two lines in the points A, B respectively, this will be the line 1; and through the point K a line meeting the same two lines in the points C and D respectively, this will be the line 1′; and, the equations $x=0$, $y=0$, $z=0$, $w=0$ denoting as above, the equation of the surface will be $x^2z + y^2w = 0$.

The points A and B are cuspidal points on the nodal line; any section of the scroll by a plane through one of these points is a cubic curve having at the point in question a cusp.

32. It is to be noticed however that the cuspidal points are not of necessity real; if for x, y we write $x + \iota y$, $x - \iota y$, and in like manner $z + \iota w$, $z - \iota w$ for z, w, then the equation takes the form

$$(x^2 - y^2)\,z - 2xyw = 0,$$

which is a cubic scroll $S(1, 1, 3)$ with the cuspidal points imaginary.

In the last-mentioned case the nodal line is throughout its whole length crunodal; in the case first considered, where the equation is $x^2z + y^2w = 0$, the nodal line is for that part of its length for which z, w have opposite signs, crunodal; and for the remainder of its length, or where z, w have the same sign, acnodal. There are two different forms, according as the line is for the portion intermediate between the cuspidal points crunodal and for the extramediate portions acnodal, or as it is for the intermediate portion acnodal and for the extramediate portions crunodal.

Cubic Scroll $S(\overline{1,\ 1},\ 3)$.

33. Starting from the equation

$$(\lambda x+\mu y)(yw-xz)+(*\between x,\ y)^3=0,$$

then putting $w-\mu z$ for w and λz for z, this may be written

$$(\lambda x+\mu y)\{yw-z(\lambda x+\mu y)\}+(*\between\lambda x+\mu y,\ y)^3=0,$$

or, what is the same thing,

$$x(yw-xz)+(*\between x,\ y)^3=0\,;$$

and then, if $(*\between x,\ y)^3=(\alpha,\ \beta,\ \gamma,\ \delta\between x,\ y)^3$, this may be written

$$x\{y(w+\beta x+\gamma y)-x(z-\alpha x)\}+\delta y^3=0\,;$$

or changing the values of w and z, we have

$$x(yw-xz)+y^3=0$$

for the equation of the scroll $S(\overline{1,\ 1},\ 3)$[1].

34. The Hessian of the form is x^4, and it thus appears that the plane $x=0$ is a determinate plane through the double line. But $y=0$ is not a determinate plane; in fact, if for y we write $y+\lambda x$, the equation is

$$-x^2z+xw(y+\lambda x)+(y+\lambda x)^3=0,$$

that is

$$-x^2(z-\lambda w-3\lambda^2y-\lambda^3x)+xy(w+3\lambda x)+y^3=0,$$

which, changing z and w, is still of the form $x(yw-xz)+y^3=0$.

The planes $z=0$, $w=0$ will alter with the plane $y=0$, but they are not determined even when the plane $y=0$ is determined; in fact we may, without altering the equation, change w, z into $w+\theta y$, $z+\theta x$ respectively.

35. In the equation $x(yw-xz)+y^3=0$, writing $y=\lambda x$, we find for the equations of a generating line, $y=\lambda x$, $z=\lambda w+\lambda^3x$. Considering the section by the plane $\alpha x+\beta y+\gamma z+\delta w=0$, we have

$$x:y:z:w=-\gamma\lambda-\delta:-\gamma\lambda^2-\delta\lambda:-\delta\lambda^3+\beta\lambda^2+\alpha\lambda:\gamma\lambda^3+\beta\lambda+\alpha$$

for the coordinates of the point where the generating line meets the section.

The generating line meets the nodal line at the intersection of the nodal line by the plane $z=\lambda w$; that is, the points $z=\lambda w$ on the nodal line correspond to the planes $y=\lambda x$ through the nodal line. In particular the point $w=0$ on the nodal line corresponds to the plane $x=0$ through the nodal line: the point $\gamma z+\delta w=0$ on the nodal line (that is, the point where this line is met by the plane $\alpha x+\beta y+\gamma z+\delta w=0$) corresponds to the plane $\gamma x+\delta y=0$ through the nodal line; the intersections of the plane $\alpha x+\beta y+\gamma z+\delta w=0$ by this plane $\gamma x+\delta y=0$, and by the plane $x=0$, are the tangents of the section at the node.

[1] It is somewhat more convenient to change the sign of z, and take $x(yw+xz)+y^3=0$ as the canonical form.

Quartic Scrolls, Article Nos. 36 to 50.

36. We may consider, first, the quartic scrolls $S(1, 1, 4)$. The section is a quartic curve having an α-tuple point and a β-tuple point, where $\alpha+\beta=4$; that is, we have $\alpha=2$, $\beta=2$, a quartic with two nodes (double points), or else $\alpha=3$, $\beta=1$, a quartic with a triple point. But the case $\alpha=2$, $\beta=2$ gives rise to two species: viz., in general the quartic has only the two double points, and we have then a scroll with two nodal (2-tuple) directrix lines, and without any nodal generator; the section *may* however have a third double point, and the scroll has then a nodal (double) generator. For the case $\alpha=3$, $\beta=1$, the section admits of no further singularity, and we have a quartic scroll with a triple directrix line and a single directrix line.

37. Next for the quartic scroll $S(\overline{1, 1}, 4)$. The section is here a quartic curve with an $\alpha(+\beta)$ tuple point, where $\alpha+\beta=4$; that is, $\alpha=2$, $\beta=2$, or else $\alpha=3$, $\beta=1$. In the former case the section has a $2(+2)$ tuple point, that is, a double point where the two branches have a common tangent—otherwise, two coincident double points: say the curve has a *tacnode*; the line of approach is the tangent at the tacnode. We have here a scroll with a twofold double line; there are however two cases: viz., in general the section has, besides the tacnode, no other double point; that is, the scroll has no nodal generator: the section *may* however have a third double point, and the scroll has then a nodal (double) generator. In the case $\alpha=3$, $\beta=1$ the section has a triple point, and the line of approach is the tangent at one of the branches at the triple point; the scroll has a twofold, say a $3(+1)$ tuple directrix line: as the section admits of no further singularity, this is the only case. The foregoing enumeration gives three species of quartic scrolls $S(1, 1, 4)$, and three species of quartic scrolls $S(\overline{1, 1}, 4)$, together six species, viz. these are as follows:

Quartic Scroll, First Species, $S(1_2, 1_2, 4)$, with two double directrix lines, and without a nodal generator.

38. Taking $(x=0, y=0)$ and $(z=0, w=0)$ for the equations of the two directrix lines respectively, the equation of the scroll is

$$(*\between x, y)^2(z, w)^2=0.$$

Quartic Scroll, Second Species, $S'(1_2, 1_2, 4)$, with two double directrix lines and with a double generator.

39. This is in fact a specialized form of the first species, the difference being that there is a nodal (double) generator. Supposing as before that the equations of the directrix lines are $(x=0, y=0)$ and $(z=0, w=0)$ respectively; let the equations of the nodal generator be $(x+y=0, z+w=0)$; then, observing that for the first species the equation may be written $(*\between x, y)^2(z, z+w)^2=0$, it is clear that if the terms in z^2 and $z(z+w)$ are divisible by $(x+y)^2$ and $(x+y)$ respectively, the surface will have as a new

double line the line $(x+y=0,\ z+w=0)$, which will be a double generator; and we thus arrive at the equation of the second species of quartic scrolls, viz. this is

$$\big((x+y)^2,\ (x+y)(x,\ y),\ (x,\ y)^2 \between z,\ z+w\big)^2=0.$$

Quartic Scroll, Third Species, $S(1_3,\ 1,\ 4)$, with a triple directrix line and a single directrix line.

40. Taking $(x=0,\ y=0)$ for the equations of the triple directrix line, and $(z=0,\ w=0)$ for the equations of the single directrix line, the equation is

$$(* \between x,\ y)^3\,(z,\ w)=0.$$

Quartic Scroll, Fourth Species, $S(\overline{1_2,\ 1_2},\ 4)$, with a twofold $\big(2\,(+\,2)$ tuple$\big)$ *directrix line, and without a nodal generator.*

41. Taking $(x=0,\ y=0)$ for the equations of the directrix line, $z=0$ for that of a plane section of the scroll, $y=0$ for the equation of a plane through the tangent at the tacnode of the section, and supposing (see *ante*, No. 22) that the plane through the directrix line and the corresponding point on this line are respectively given by the equations $x=\lambda y$ and $z=\lambda w$, the equation of the scroll is

$$(yw-xz)^2+(yw-xz)\,(x,\ y)^2+(x,\ y)^4=0.$$

Quartic Scroll, Fifth Species, $S'(\overline{1_2,\ 1_2},\ 4)$, with a twofold $\big(2\,(+\,2)$ tuple$\big)$ *generating line, and with a double generator.*

42. Let the equations of the double generator be $x+y=0,\ z+w=0$; then the line in question must be a double line on the surface represented by the last-mentioned equation, and this will be the case if only the second and third terms contain the factors $(x+y)$ and $(x+y)^2$ respectively. The equation for the fifth species consequently is

$$(yw-xz)^2+2\,(yw-xz)\,(x+y)\,(x,\ y)+(x+y)^2\,(x,\ y)^2=0.$$

Quartic Scroll, Sixth Species, $S(\overline{1_3,\ 1},\ 4)$, with a twofold $\big(3\,(+\,1)$ tuple$\big)$ *generating line.*

43. Taking $(x=0,\ y=0)$ for the equations of the directrix line, $z=0$ for the equation of a plane section, and assuming that the plane $y=0$ passes through the tangent which is the line of approach, and that the plane through the directrix line and the corresponding point on this line are respectively given by the equations $x=\lambda y$ and $z=\lambda w$, the equation of the scroll is

$$(yw-xz)\,(x,\ y)^2+(x,\ y)^4=0.$$

I refrain on the present occasion from a more particular discussion of the foregoing six species of quartic scrolls. I establish two other species, as follows:

Quartic Scroll, Seventh Species, $S(1, 2, 2)$, with nodal directrix line, and nodal directrix conic which meet, and with a simple directrix conic which meets the nodal conic in two points.

44. We see, *à priori*, that the scroll generated as above will be of the order 4, that is, a quartic scroll. In fact using the formula (*ante*, No. 5),

$$\text{Order} = 2mnp - \alpha m - \beta n - \gamma p,$$

we have here

$$\begin{aligned} &\text{Nodal conic,} \quad m = 2, \quad \alpha = 0, \\ &\text{Simple conic,} \quad n = 2, \quad \beta = 1, \\ &\text{Line} \qquad\quad , \quad p = 1, \quad \gamma = 2, \end{aligned}$$

and hence

$$\text{Order} = 8 - 2 - 2, = 4.$$

45. Take $(x=0, y=0)$ for the equations of the directrix line, $z=0$ for the equation of the plane of the simple conic, $w=0$ for that of the plane of the nodal conic; since the conics intersect in two points, they lie on a quadric surface, say the surface $U=0$; the equations of the simple conic thus are $z=0$, $U=0$; those of the nodal conic are $w=0$, $U=0$. The directrix line $x=0$, $y=0$ meets the nodal conic; that is, U must vanish identically for $x=0$, $y=0$, $w=0$; and this will be the case if only the term in z^2 is wanting; that is, we must have

$$U = (a, b, 0, d, f, g, h, l, m, n \between x, y, z, w)^2.$$

But we may in the first instance omit the condition in question, and write

$$U = (a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2;$$

this would lead to a sextic instead of a quartic scroll.

46. The equations of a generating line (since it meets the directrix line $x=0$, $y=0$) may be taken to be

$$x = \alpha y, \quad z = \beta\left(y - \frac{w}{\theta}\right);$$

the condition in order to the intersection of the generating line with the nodal conic is at once found to be

$$a\alpha^2 + 2h\alpha + b + 2\beta(f + g\alpha) + c\beta^2 = 0,$$

and that for its intersection with the simple conic

$$a\alpha^2 + 2h\alpha + b + 2\theta(m + l\alpha) + d\theta^2 = 0;$$

and writing the equations of the generating line in the form

$$\alpha = \frac{y}{x}, \quad \theta = \frac{\beta w}{\beta y - z},$$

the elimination of α, β, θ from these four equations gives the required equation of the scroll. Writing for a moment

$$\begin{aligned} \Theta &= a\alpha^2 + 2h\alpha + \beta, \\ F &= g\alpha + f \quad , \\ M &= l\alpha + m \quad , \end{aligned}$$

we find

$$\begin{aligned} c\beta^2 \ + 2F\beta \ + \Theta &= 0, \\ (\Theta y^2 + 2Myw + dw^2)\,\beta^2 - 2\,(\Theta yz + Mwz)\,\beta + \Theta z^2 &= 0\,; \end{aligned}$$

or, introducing at this place the condition $c=0$, the first equation gives β linearly, and we thence obtain

$$\Theta\,(\Theta y^2 + 2Myw + dw^2) + 4F\,(\Theta yz + Mwz) + 4F^2z^2 = 0,$$

or, what is the same thing,

$$(\Theta y + 2Fz)^2 + 2Mw\,(\Theta y + 2Fz) + \Theta dw^2 = 0\,;$$

whence, observing that we have

$$\Theta = \frac{ax^2 + 2hxy + by^2}{y^2}, \quad F = \frac{gx + fy}{y}, \quad M = \frac{lx + my}{y},$$

the equation of the scroll is

$$\begin{aligned} &(ax^2 + 2hxy + by^2 + 2gzx + 2fyz)^2 \\ &+ 2\,(ax^2 + 2hxy + by^2 + 2gzx + 2fyz)\,(lx + my)\,w \\ &+ \ (ax^2 + 2hxy + by^2)\,dw^2 = 0. \end{aligned}$$

We see from the equation that the surface contains the line $(x=0,\ y=0)$ as a double line, the conic

$$w = 0, \quad ax^2 + 2hxy + by^2 + 2gzx + 2fyz = 0$$

as a double curve, also the conic

$$z = 0, \quad ax^2 + 2hxy + by^2 + 2lxw + 2myw + dw^2 = 0$$

as a simple curve on the surface, the complete intersection by the plane $z=0$ being in fact the last-mentioned conic, and the pair of lines

$$z = 0, \quad ax^2 + 2hxy + by^2 = 0.$$

Quartic Scroll, Eighth Species, $S(1,\ 3^2)$, with a directrix line, and a directrix skew cubic met twice by each generating line.

47. We see, *à priori*, that the scroll is of the order 4, that is, a quartic scroll; in fact for the quartic scroll $S(1,\ m^2)$ the order is $=[m]^2 + M$ (first memoir, p. 457 [*ante* p. 172]), and we have here $m=3$, $M = h - \frac{1}{2}[m]^2 = 1 - 3 = -2$; that is, order $= 6 - 2$, $= 4$.

48. The equations of the cubic curve may be taken to be

$$\left\| \begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix} \right\| = 0,$$

or, what is the same thing,

$$xz - y^2 = 0, \quad xw - yz = 0, \quad yw - z^2 = 0;$$

those of the directrix line may be represented by

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$
$$\alpha' x + \beta' y + \gamma' z + \delta' w = 0;$$

or, what is the same thing, if

$$\beta\gamma' - \beta'\gamma = a, \quad \alpha\delta' - \alpha'\delta = f,$$
$$\gamma\alpha' - \gamma'\alpha = b, \quad \beta\delta' - \beta'\delta = g,$$
$$\alpha\beta' - \alpha'\beta = c, \quad \gamma\delta' - \gamma'\delta = h,$$

(and therefore identically $af + bg + ch = 0$), the line is defined by means of its "six coordinates" (a, b, c, f, g, h).

49. The equations of the cubic curve are satisfied by writing therein

$$x : y : z : w = 1 : t : t^2 : t^3,$$

and therefore the coordinates of any two points on the curve may be represented by $(1, \theta, \theta^2, \theta^3)$ and $(1, \phi, \phi^2, \phi^3)$; hence, if x, y, z, w are the coordinates of a point in the line joining the last mentioned two points, we have

$$x : y : z : w = l + m : l\theta + m\phi : l\theta^2 + m\phi^2 : l\theta^3 + m\phi^3,$$

which equations, treating therein l, m as indeterminate parameters, give the equations of the line in question. And putting moreover

$$p = yw - z^2, \quad q = yz - xw, \quad r = xz - y^2,$$

we have identically

$$p : q : r = \theta\phi : -(\theta + \phi) : 1.$$

50. In order that the line in question may meet the directrix line, we must have

$$l(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3) + m(\alpha + \beta\phi + \gamma\phi^2 + \delta\phi^3) = 0,$$
$$l(\alpha' + \beta'\theta + \gamma'\theta^2 + \delta'\theta^3) + m(\alpha' + \beta'\phi + \gamma'\phi^2 + \delta'\phi^3) = 0;$$

that is, eliminating l and m, we must have

$$\left| \begin{matrix} \alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3, & \alpha + \beta\phi + \gamma\phi^2 + \delta\phi^3 \\ \alpha' + \beta'\theta + \gamma'\theta^2 + \delta'\theta^3, & \alpha' + \beta'\phi + \gamma'\phi^2 + \delta'\phi^3 \end{matrix} \right| = 0,$$

or, developing,

$$(\alpha\beta' - \alpha'\beta)(\phi - \theta) + (\alpha\gamma' - \alpha'\gamma)(\phi^2 - \theta^2) + (\alpha\delta' - \alpha'\delta)(\phi^3 - \theta^3)$$
$$+ (\beta\gamma' - \beta'\gamma)(\theta\phi^2 - \theta^2\phi) + (\beta\delta' - \beta'\delta)(\theta\phi^3 - \theta^3\phi) + (\gamma\delta' - \gamma'\delta)(\theta^2\phi^3 - \theta^3\phi^2) = 0;$$

the several terms in (θ, ϕ), each divided by $\phi - \theta$, give respectively

$$1,\ \phi + \theta,\ (\phi + \theta)^2 - \phi\theta,\ \theta\phi,\ \theta\phi(\phi + \theta),\ \theta^2\phi^2,$$

which are equal to

$$(r^2,\ -qr,\ \ q^2 - pr\ \ ,\ pr,\ \ -pq\ \ ,\ p^2);$$

hence replacing also $\alpha\beta' - \alpha'\beta$, &c. by their values c, &c., we find

$$(c, -b, f, a, g, h)(r^2, -qr, q^2 - pr, pr, -pq, p^2) = 0,$$

or, what is the same thing,

$$(h, f, c, b, a - f, -g \between p, q, r)^2 = 0,$$

where the coefficients (a, b, c, f, g, h) satisfy the relation $af + bg + ch = 0$; p, q, r stand respectively for $yw - z^2$, $yz - xw$, $xz - y^2$.

Writing for greater convenience

$$(h, f, c, b, a - f, -g) = (\mathrm{a, b, c, 2f, 2g, 2h}),$$

or, what is the same thing,

$$(a, b, c, f, g, h) \qquad = (\mathrm{b + 2g,\ 2f,\ c,\ b,\ -2h,\ a}),$$

then we have

$$af + bg + ch = \mathrm{ac + b^2 + 2bg - 4fh} = 0;$$

and hence finally we have for the equation of the scroll $S(1, 3^2)$,

$$(\mathrm{a, b, c, f, g, h} \between yw - z^2,\ \ yz - xw,\ \ xz - y^2)^2 = 0,$$

where the coefficients satisfy the relation

$$\mathrm{ac + b^2 + 2bg - 4fh} = 0.$$

The equations of the directrix cubic are of course

$$yw - z^2 = 0,\quad yz - xw = 0,\quad xz - y^2 = 0;$$

and the directrix line is given by its six coordinates,

$$(\mathrm{b + 2g,\ 2f,\ c,\ b,\ -2h,\ a}).$$

On the general Theory of Scrolls, Article Nos. 51 to 53.

51. I annex in conclusion the following considerations on the general theory of scrolls. Consider a scroll of the nth order; the intersection by an arbitrary plane, say the plane $w = 0$, is a curve of the nth order $(* \between x, y, x)^n = 0$; any point $(x, y, z, 0)$

where (x, y, z) satisfy the foregoing equation, is the foot of a generating line; and we may imagine this generating line determined by means of the coordinates (X, Y, Z, W), given functions of (x, y, z) of a point on the line. This being so, the "six coordinates," say (p, q, r, s, t, u), of the line are

$$\left\| \begin{matrix} X, & Y, & Z, & W \\ x, & y, & z, & 0 \end{matrix} \right\|$$

viz.

$$\begin{aligned} p &= Yz - Zy, & s &= -Wx, \\ q &= Zx - Xz, & t &= -Wy, \\ r &= Xy - Yx, & u &= -Wz; \end{aligned}$$

or, writing for greater convenience $-v$ in the place of W, the six coordinates of the line are p, q, r, vx, vy, vz, where p, q, r are functions of (x, y, z), connected by the relation $px + qy + vz = 0$; and v is also a function of (x, y, z).

52. Consider the intersection of the surface by an arbitrary line, the six coordinates whereof are (A, B, C, F, G, H); then for the generating lines which meet this line we have

$$v(Ax + By + Cz) + Fp + Gq + Hr = 0,$$

and this equation, together with the equation $(*\between x, y, z)^n = 0$, determines (x, y, z), the coordinates of the foot of a generating line which meets the arbitrary line (A, B, C, F, G, H). Since the order of the scroll is $= n$, the number of such generating lines should be $= n$, that is, there should be n relevant intersections of the two curves,

$$\begin{aligned} &v(Ax + By + Cz) + Fp + Gq + Hr = 0, \\ &(*\between x, y, z)^n = 0; \end{aligned}$$

but if (p, q, r, vx, vy, vz) are each of the order k, the number of actual intersections is $= kn$, which is too many by $(k-1)n$.

53. Suppose that the curves

$$p = 0, \quad q = 0, \quad r = 0, \quad vx = 0, \quad vy = 0, \quad vz = 0,$$

or say the curves

$$p = 0, \quad q = 0, \quad r = 0, \quad v = 0$$

have in common θ intersections, and let these be points of the multiplicities $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_\theta$ on the curve $(*\between x, y, z)^n = 0$ (viz. according as the curve does not pass through any one of the intersections in question, or passes once, twice, &c. through such intersection, we have for that intersection $\alpha_1 = 0, 1, 2$, &c., as the case may be, and so for the other intersections); then the kn points of intersection include the $\alpha_1 + \alpha_2 \ldots + \alpha_\theta$, or say the $\Sigma\alpha$ intersections; but these, being independent of the line (A, B, C, F, G, H) under consideration, are irrelevant points, and the number of relevant points of intersection is $kn - \Sigma\alpha$; that is, if we have $\Sigma\alpha = (k-1)n$, then the scroll in question, viz. the scroll generated by a line which meets the plane $w = 0$ in the curve $(*\between x, y, z)^n = 0$, and which has for its six coordinates (p, q, r, vx, vy, vz), will be a scroll of the nth order.

341.

ON THE SEXTACTIC POINTS OF A PLANE CURVE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLV. (for the year 1865), pp. 545—578. Received November 5,—Read December 22, 1864.]

IT is, in my memoir "On the Conic of Five-pointic Contact at any point of a Plane Curve," *Phil. Trans.* vol. CXLIX. (1859), pp. 371—400, [261], remarked that as in a plane curve there are certain singular points, viz. the points of inflexion, where three consecutive points lie in a line, so there are singular points where six consecutive points of the curve lie in a conic; and such a singular point is there termed a "sextactic point." The memoir in question (here cited as "former memoir") contains the theory of the sextactic points of a cubic curve; but it is only recently that I have succeeded in establishing the theory for a curve of the order m. The result arrived at is that the number of sextactic points is $=m(12m-27)$, the points in question being the intersections of the curve m with a curve of the order $12m-27$, the equation of which is

$$\begin{aligned}&(12m^2-54m+57)\,H\ \text{Jac.}\ (U,\ H,\ \Omega_{\overline{H}})\\ &+(m-2)(12m-27)\,H\ \text{Jac.}\ (U,\ H,\ \Omega_{\overline{V}})\\ &+40\,(m-2)^2\qquad\qquad\ \text{Jac.}\ (U,\ H,\ \Psi\ )=0,\end{aligned}$$

where $U=0$ is the equation of the given curve of the order m, H is the Hessian or determinant formed with the second differential coefficients $(a,\ b,\ c,\ f,\ g,\ h)$ of U, and, $(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})$ being the inverse coefficients $(\mathfrak{A}=bc-f^2$, &c.), then

$$\Omega=(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_x,\ \partial_y,\ \partial_z)^2\,H,$$

$$\Psi=(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\partial_x H,\ \partial_y H,\ \partial_z H)^2;$$

and Jac. denotes the Jacobian or functional determinant, viz.

$$\text{Jac. } (U, H, \Psi) = \begin{vmatrix} \partial_x U, & \partial_y U, & \partial_z U \\ \partial_x H, & \partial_y H, & \partial_z H \\ \partial_x \Psi, & \partial_y \Psi, & \partial_z \Psi \end{vmatrix},$$

and Jac. (U, H, Ω) would of course denote the like derivative of (U, H, Ω); the subscripts $(_{\bar{H}}, {}_{\bar{U}})$ of Ω denote restrictions in regard to the differentiation of this function, viz. treating Ω as a function of U and H,

$$\Omega = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between a', b', c', f', 2f', 2g', 2h')$$

if (a', b', c', f', g', h') are the second differential coefficients of H, then we have

$$\begin{aligned} \partial_x \Omega = \quad & (\partial_x \mathfrak{A}, \ldots \between a', \ldots) \qquad (= \partial_x \Omega_{\bar{H}}) \\ + & (\mathfrak{A}, \ldots \between \partial_x a', \ldots) \qquad (= \partial_x \Omega_{\bar{U}}); \end{aligned}$$

viz. in $\partial_x \Omega_{\bar{H}}$ we consider as exempt from differentiation (a', b', c', f', g', h') which depend upon H, and in $\partial_x \Omega_{\bar{U}}$ we consider as exempt from differentiation $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ which depend upon U. We have similarly

$$\partial_y \Omega = \partial_y \Omega_{\bar{H}} + \partial_y \Omega_{\bar{U}}, \text{ and } \partial_z \Omega = \partial_z \Omega_{\bar{H}} + \partial_z \Omega_{\bar{U}};$$

and in like manner

$$\text{Jac. } (U, H, \Omega) = \text{Jac. } (U, H, \Omega_{\bar{H}}) + \text{Jac. } (U, H, \Omega_{\bar{U}}),$$

which explains the signification of the notations Jac. $(U, H, \Omega_{\bar{H}})$, Jac. $(U, H, \Omega_{\bar{U}})$.

The condition for a sextactic point is in the first instance obtained in a form involving the arbitrary coefficients (λ, μ, ν); viz. we have an equation of the order 5 in (λ, μ, ν) and of the order $12m - 22$ in the coordinates (x, y, z). But writing $\vartheta = \lambda x + \mu y + \nu z$, by successive transformations we throw out the factors $\vartheta^2, \vartheta, \vartheta, \vartheta$, thus arriving at a result independent of (λ, μ, ν); viz. this is the before-mentioned equation of the order $12m - 27$. The difficulty of the investigation consists in obtaining the transformations by means of which the equation in its original form is thus divested of these irrelevant factors.

Articles Nos. 1 to 6.—*Investigation of the Condition for a Sextactic Point.*

1. Following the course of investigation in my former memoir, I take (X, Y, Z) as current coordinates, and I write

$$\Upsilon = (* \between X, Y, Z)^m = 0$$

for the equation of the given curve; (x, y, z) are the coordinates of a particular point on the given curve, viz. the sextactic point; and $U, = (* \between x, y, z)^m$, is what Υ becomes when (x, y, z) are written in place of (X, Y, Z): we have thus $U = 0$ as a condition satisfied by the coordinates of the point in question.

2. Writing for shortness

$$DU = (X\partial_x + Y\partial_y + Z\partial_z)\, U,$$

$$D^2U = (X\partial_x + Y\partial_y + Z\partial_z)^2 U,$$

and taking $\Pi = aX + bY + cZ = 0$ for the equation of an arbitrary line, the equation

$$D^2U - \Pi DU = 0$$

is that of a conic having an ordinary (two-pointic) contact with the curve at the point (x, y, z); and the coefficients of Π are in the former memoir determined so that the contact may be a five-pointic one; the value obtained for Π is

$$\Pi = \tfrac{2}{3}\frac{1}{H} DH + \Lambda DU,$$

where

$$\Lambda = \frac{1}{9H^3}(-3\Omega H + 4\Psi).$$

3. This result was obtained by considering the coordinates of a point of the curve as functions of a single arbitrary parameter, and taking

$$x + dx + \tfrac{1}{2}d^2x + \tfrac{1}{6}d^3x + \tfrac{1}{24}d^4x,\ \ y + \&c.,\ \ z + \&c.$$

for the coordinates of a point consecutive to (x, y, z); for the present purpose we must go a step further, and write for the coordinates

$$x + dx + \tfrac{1}{2}\, d^2x + \tfrac{1}{6}\, d^3x + \tfrac{1}{24}\, d^4x + \tfrac{1}{120}\, d^5x,$$

$$y + dy + \tfrac{1}{2}\, d^2y + \tfrac{1}{6}\, d^3y + \tfrac{1}{24}\, d^4y + \tfrac{1}{120}\, d^5y,$$

$$z + dz + \tfrac{1}{2}\, d^2z + \tfrac{1}{6}\, d^3z + \tfrac{1}{24}\, d^4z + \tfrac{1}{120}\, d^5z.$$

4. Hence if

$$\partial_1 = dx\,\partial_x + dy\,\partial_y + dz\,\partial_z,\ \ \partial_2 = d^2x\,\partial_x + d^2y\,\partial_y + d^2z\,\partial_z,\ \ \&c.,$$

we have, in addition to the equations

$$U = 0,$$

$$\partial_1 U = 0,$$

$$(\partial_1^2 + 2\partial_2)\, U = 0,$$

$$(\partial_1^3 + 3\partial_1\partial_2 + \partial_3)\, U = 0,$$

$$(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2 + \partial_4)\, U = 0,$$

of my former memoir, the new equation

$$(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2 + 5\partial_1\partial_4 + 10\partial_2\partial_3 + \partial_5)\, U = 0,$$

and in addition to the equations, $(P = ax + by + cz)$,

$$
\begin{aligned}
&- \quad (m-2)\,\partial_1^2 U + P\,.\,\tfrac{1}{2}\partial_1^2 U = 0,\\
&- \tfrac{1}{3}\,[(m-1)\,\partial_1^3 + 3\,(m-2)\,\partial_1\partial_2]\,U + P\,.\,\tfrac{1}{6}\,(\partial_1^3 + 3\partial_1\partial_2)\,U + \partial_1 P\,.\,\tfrac{1}{2}\partial_1^2 U = 0,\\
&- \tfrac{1}{12}\,[(m-1)\,(\partial_1^4 + 6\partial_1^2\partial_2) + (m-2)\,(4\partial_1\partial_3 + 3\partial_2^2)]\,U\\
&\qquad + P\,.\,\tfrac{1}{24}\,(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\,U + \partial_1 P\,.\,\tfrac{1}{6}\,(\partial_1^3 + 3\partial_1\partial_2)\,U + \tfrac{1}{2}\partial_2 P\,.\,\tfrac{1}{2}\partial_1^2 U = 0,
\end{aligned}
$$

giving in the first instance

$$
\begin{aligned}
P &= 2\,(m-2),\\
\partial_1 P &= \tfrac{2}{3}\,\frac{\partial_1^3 U}{\partial_1^2 U},\\
\partial_2 P &= \tfrac{1}{2}\,\frac{(\partial_1^4 + 6\partial_1^2\partial_2)\,U}{\partial_1^2 U} - \tfrac{4}{9}\,\frac{\partial_1^3 U}{\partial_1^2 U}\,\frac{(\partial_1^3 + 3\partial_1\partial_2)\,U}{\partial_1^2 U},
\end{aligned}
$$

and leading ultimately to the before-mentioned value of Π, we have the new equation

$$
\begin{aligned}
&- \tfrac{1}{60}\,[(m-1)\,(\partial_1^5 + 10\partial_1^3\partial^2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2) + (m-2)\,(5\partial_1\partial_4 + 10\partial_2\partial_3)]\,U\\
&+ \quad P\,.\,\tfrac{1}{120}\,(\partial_1^6 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2 \qquad + 5\partial_1\partial_4 + 10\partial_2\partial_3)\,U\\
&+ \quad \partial_1 P\,.\,\tfrac{1}{24}\,(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\,U\\
&+ \tfrac{1}{2}\partial_2 P\,.\,\tfrac{1}{6}\,(\partial_1^3 + 3\partial_1\partial_2)\,U\\
&+ \tfrac{1}{6}\partial_3 P\,.\,\tfrac{1}{2}\,\partial_1^2 U = 0.
\end{aligned}
$$

5. This may be written in the form

$$
\begin{aligned}
&- \quad 2\,[(m-1)\,(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2) + (m-2)\,(5\partial_1\partial_4 + 10\partial_2\partial_3)]\,U\\
&+ \quad P\,(\qquad \partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2 \qquad + 5\partial_1\partial_4 + 10\partial_2\partial_3)\,U\\
&+ \quad 5\partial_1 P\,(\qquad \partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\,U\\
&+ 10\partial_2 P\,(\qquad \partial_1^3 + 3\partial_1\partial_2)\,U\\
&+ 10\partial_3 P\,(\qquad \partial_1^2 U) = 0\,;
\end{aligned}
$$

or putting for P its value, $= 2\,(m-2)$, the equation becomes

$$
\begin{aligned}
&- \quad 2\,(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\,U\\
&+ \quad 5\partial_1 P\,(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2\ )\,U\\
&+ 10\partial_2 P\,(\partial_1^3 + 3\partial_1\partial_2)\,U\\
&+ 10\partial_3 P\,.\,\partial_2^2 U = 0\,;
\end{aligned}
$$

or as this may also be written,

$$
\begin{aligned}
&2\,(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\,U\\
&\qquad + 5\partial_1 P\,.\,\partial_4 U + 10\partial_2 P\,.\,\partial_3 U + 10\partial_3 P\,.\,\partial_2 U = 0.
\end{aligned}
$$

6. But the equation

$$\Pi = \tfrac{2}{3}\frac{1}{H}DH + \Lambda DU,$$

which is an identity in regard to (X, Y, Z), gives

$$\partial_1 P = \tfrac{2}{3}\frac{1}{H}\partial_1 H,$$

$$\partial_2 P = \tfrac{2}{3}\frac{1}{H}\partial_2 H + \Lambda\partial_2 U,$$

$$\partial_3 P = \tfrac{2}{3}\frac{1}{H}\partial_3 H + \Lambda\partial_3 U;$$

and substituting these values, the foregoing equation becomes

$$2(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\,U$$
$$+ (5\partial_4 U\partial_1 H + 10\partial_3 U\partial_2 H + 10\partial_2 U\partial_3 H)\,\tfrac{2}{3}\frac{1}{H} + \Lambda\,.\,20\partial_2 U\partial_3 U = 0;$$

or putting for Λ its value, $= \dfrac{1}{9H^3}(-3\Omega H + 4\Psi)$, and multiplying by $\frac{2}{3}H^2$ this is

$$9H^2(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\,U$$
$$+ 15H\,(\partial_4 U\partial_1 H + 2\partial_3 U\partial_2 H + 2\partial_2 U\partial_3 H)$$
$$+ \frac{1}{H}(-3\Omega H + 4\Psi)\,.\,10\partial_2 U\partial_3 U = 0,$$

which is, in its original or unreduced form, the condition for a sextactic point.

Article Nos. 7 and 8.—*Notations and Remarks.*

7. Writing, as in my former memoir, A, B, C for the first differential coefficients of U, we have $B\nu - C\mu$, $C\lambda - A\nu$, $A\mu - B\lambda$ for the values of dx, dy, dz, and instead of the symbol **D** used in my former memoir, I use indifferently the original symbol ∂_1, or write instead thereof ∂, to denote the resulting value

$$\partial_1 (= \partial) = (B\nu - C\mu)\,\partial_x + (C\lambda - A\nu)\,\partial_y + (A\mu - B\lambda)\,\partial_z,$$

and I remark here that for any function whatever Ω, we have

$$\partial\Omega = \begin{vmatrix} A, & B, & C \\ \lambda, & \mu, & \nu \\ \partial_x\Omega, & \partial_y\Omega, & \partial_z\Omega \end{vmatrix} = \text{Jac.}\,(U, \mathfrak{H}, \Omega),$$

where $\mathfrak{H} = \lambda x + \mu y + \nu z$. I write, as in the former memoir,

$$\Phi = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\between\lambda, \mu, \nu)^3;$$

and also

$$\nabla = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\mathfrak{K}\lambda, \mu, \nu\mathfrak{K}\partial_x, \partial_y, \partial_z),$$

which new symbol ∇ serves to express the functions Π, $\square$, occurring in the former memoir; viz. we have $\Pi = 2\nabla\Phi$, $\square = 2\nabla H$, so that the symbols Π, $\square$ are not any longer required.

8. I remark that the symbols ∂, ∇ are each of them a linear function of $(\partial_x, \partial_y, \partial_z)$, with coefficients which are functions of the variables (x, y, z), and this being so, that for any function Π whatever, we have

$$\partial(\nabla\Pi) = (\partial.\nabla)\Pi + \partial\nabla\Pi,$$

viz. in $\partial(\nabla\Pi)$ we operate with ∇ on Π, thereby obtaining $\nabla\Pi$, and then with ∂ on $\nabla\Pi$; in $(\partial.\nabla)\Pi$ we operate with ∂ upon ∇ in so far as ∇ is a function of (x, y, z), thus obtaining a new operating symbol $\partial.\nabla$, a linear function of $(\partial_x, \partial_y, \partial_z)$, and then operate with $\partial.\nabla$ upon Π; and lastly, in $\partial\nabla\Pi$, we simply multiply together ∂ and ∇, thus obtaining a new operating symbol $\partial\nabla$ of the form $(\partial_x, \partial_y, \partial_z)^2$, and then operate therewith on Π; it is clear that, as regards the last-mentioned mode of combination, the symbols ∂ and ∇ are convertible, or $\partial\nabla = \nabla\partial$, that is, $\partial\nabla\Pi = \nabla\partial\Pi$.

It is to be observed throughout the memoir that the point (.) is used (as above in $\partial.\nabla$) when an operation is performed upon a symbol of operation as operand; the mere apposition of two or more symbols of operation (as above in $\partial\nabla$) denotes that the symbols of operation are simply multiplied together; and when $\partial\nabla$ is followed by a letter Π denoting not a symbol of operation, but a mere function of the coordinates, that is in an expression such as $\partial\nabla\Pi$, the resulting operation $\partial\nabla$ is performed upon Π as operand; if instead of the single letter Π we have a compound symbol such as HU or $H\nabla\vartheta$, so that the expression is ∂HU, $\partial H\nabla\vartheta$, $\partial\nabla HU$ or $\partial\nabla H\nabla\vartheta$, then it is to be understood that it is merely the immediately following function H which is operated upon by ∂ or $\partial\nabla$; in the few instances where any ambiguity might arise a special explanation is given.

Article Nos. 9 to 11.—*First transformation.*

9. We have, assuming always $U = 0$, the following formulæ (*see post*, Article Nos. 31 to 33):

$$\begin{aligned}
&(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\, U \\
&\quad = \frac{\vartheta^2}{(m-1)^4}\{(27m^2 - 96m + 81)\, H\partial\Phi + (17m^2 - 56m + 51)\,\Phi\partial H\} \\
&\qquad + \frac{\vartheta^3}{(m-1)^4}\{(-14m - 22)(\partial.\nabla)\, H \quad - (10m - 18)\,\partial\nabla H\} \\
&\qquad + \frac{\vartheta^4}{(m-1)^4}\{\partial\Omega\},
\end{aligned}$$

$$\partial_4 U \partial_1 H + 2\partial_3 U \partial_2 H + 2\partial_2 U \partial_3 H$$

$$= \frac{\vartheta^2}{(m-1)^4}\{(-6m^2+18m-12)\,H^2\partial\Phi + (-17m^2+60m-55)\,H\Phi\partial\Phi\}$$

$$+ \frac{\vartheta^3}{(m-1)^4}\{(2m-2)\,H\,(\partial.\nabla)\,H \qquad + (8m-16)\,\partial H \nabla H\}$$

$$+ \frac{\vartheta^4}{(m-1)^4}\{-\Omega\partial H\},$$

$$\partial_2 U \partial_3 U = \frac{\vartheta^4}{(m-1)^4} H\partial H.$$

10. And by means of these the condition becomes

$$0 = \frac{\vartheta^2 H^2}{(m-1)^4}\{(153m^2-594m+549)\,H\partial\Phi + (-102m^2+396m+366)\,\Phi\partial H\}$$

$$+ \frac{\vartheta^3 H}{(m-1)^4}\{(-96m+168)\,H\,(\partial.\nabla)\,H + (-90m+162)\,H\partial\nabla H + (120m-240)\,\partial H\nabla H\}$$

$$+ \frac{\vartheta^4}{(m-1)^4}\{9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H\},$$

being, as already remarked, of the degree 5 in the arbitrary coefficients (λ, μ, ν), and of the order $12m-22$ in the coordinates (x, y, z).

11. But throwing out the factor ϑ^2, and observing that in the first line the quadric functions of m are each a numerical multiple of $51m^2-198m+183$, the condition becomes

$$0 = \qquad (51m^2-198m+183)\,H^2\,(3H\partial\Phi - 2\Phi\partial H)$$

$$+ \vartheta\,\{(-96m+168)\,H^2\,(\partial.\nabla)\,H + (-90m+162)\,H^2\partial\nabla H + (120m-240)\,\partial H\nabla H\}$$

$$+ \vartheta^2\,\{9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H\}.$$

Article Nos. 12 and 13.—*Second transformation.*

12. We effect this by means of the formula

$$(m-2)\,(3H\partial\Phi - 2\Phi\partial H) = -\vartheta \text{ Jac. } (U, \Phi, H), \qquad \text{(J) (1)}$$

for substituting this value of $(3H\partial\Phi - 2\Phi\partial H)$ the equation becomes divisible by ϑ; and dividing out accordingly, the condition becomes

$$-\frac{51m^2-198m+183}{m-2}\,H^2 \text{ Jac. } (U, \Phi, H)$$

$$+ (-96m+168)\,H^2\,(\partial.\nabla)\,H + (-90m+162)\,H^2\partial\nabla H + (120m-240)\,H\partial H\nabla H$$

$$+ \vartheta\,(9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H) = 0.$$

(J) here and elsewhere refers to the Jacobian Formula, *see post*, Article Nos. 34 and 35.

13. We have (*see post,* Article Nos. 36 to 40)

$$\text{Jac. } (U, \Phi, H) = -(\partial.\nabla) H;$$

and introducing also $\partial . \nabla H$ in place of $\partial \nabla H$ by means of the formula

$$\partial \nabla H = \partial (\nabla H) - (\partial.\nabla) H,$$

the condition becomes

$$\begin{aligned} &\left\{\frac{51m^2 - 198m + 183}{m-2} - (6m-6)\right\} H^2 (\partial.\nabla) H \\ &+ (-90m + 162) H^2 \partial (\nabla H) \qquad + 120 (m-2) H \partial H \nabla H \\ &+ \vartheta (9H^2 \partial \Omega - 45 H \Omega \partial H + 40 \Psi \partial H) = 0, \end{aligned}$$

or, as this may be written,

$$\begin{aligned} &(45m^2 - 180m + 171) H^2 (\partial.\nabla) H \\ &+ (-90m + 162)(m-2) H^2 \partial (\nabla H) + 120 (m-2)^2 H \partial H \nabla H \\ &+ (m-2) \vartheta (9H^2 \partial \Omega - 45 H \Omega \partial H + 40 \Psi \partial H) = 0. \end{aligned}$$

Article Nos. 14 to 17.—*Third transformation.*

14. We have the following formulæ,

$$\vartheta \text{ Jac. } (U, \nabla H, H) - (5m-11) \partial H \nabla H + (3m-6) H \partial (\nabla H) = 0, \qquad \text{(J)}$$

$$\vartheta \text{ Jac. } (U, \nabla, H) H - (2m-4) \partial H \nabla H + (3m-6) H (\partial.\nabla) H = 0, \qquad \text{(J)}$$

in the latter of which, treating ∇ as a function of the coordinates, we first form the symbol Jac. (U, ∇, H), and then operating therewith on H, we have Jac. $(U, \nabla, H) H$; these give

$$H \partial (\nabla H) = \frac{5m-11}{3(m-2)} \partial H \nabla H - \frac{\vartheta}{3(m-2)} \text{Jac. } (U, \nabla H, H),$$

$$H (\partial.\nabla) H = \qquad \tfrac{2}{3} \partial H \nabla H - \frac{\vartheta}{3(m-2)} \text{Jac. } (U, \nabla, H) H;$$

and substituting these values, the resulting coefficient of $H \partial H \nabla H$ is

$$\begin{aligned} &(45m^2 - 180m + 171) \tfrac{2}{3} \\ &+ (-90m + 162) \frac{5m-11}{3} \\ &+ 120 (m-2)^2, \end{aligned}$$

which is $= 0$.

15. Hence the condition will contain the factor ϑ, and throwing out this, and also the constant factor $\frac{1}{m-2}$, it becomes

$$\begin{aligned}&(-15m^2+60m-57)\,H\ \text{Jac.}\ (U,\ \nabla\ ,\ H)\,H\\ &+(30m-54)(m-2)\ \ H\ \text{Jac.}\ (U,\ \nabla H,\ H)\\ &+(m-2)^2(9H^2\partial\Omega-45H\Omega\partial H+40\Psi\partial H)=0.\end{aligned}$$

16. We have

$$\partial_x(\nabla H)=(\partial_x.\nabla)\,H+\partial_x\nabla H,$$

viz. in $(\partial_x.\nabla)\,H$, treating ∇ as a function of $(x,\ y,\ z)$ we operate upon it with ∂_x to obtain the new symbol $\partial_x.\nabla$, and with this we operate on H; in $\partial_x\nabla$ we simply multiply together the symbols ∂_x and ∇, giving a new symbol of the form $(d_x^2,\ \partial_x\partial_y,\ \partial_x\partial_z)$ which then operates on H. We have the like values of $\partial_y(\nabla H)$ and $\partial_z(\nabla H)$; and thence also

$$\text{Jac.}\ (U,\ \nabla H,\ H)=\text{Jac.}\ (U,\ \nabla,\ H)\,H+\text{Jac.}\ (U,\ \overline{\nabla}H,\ H),$$

viz. in the determinant Jac. $(U,\ \nabla,\ H)$ the second line corresponding to ∇ is $\partial_x.\nabla$, $\partial_y.\nabla$, $\partial_z.\nabla$ (∇ being the operand); and the Jacobian thus obtained is a symbol which operates on H giving Jac. $(U,\ \nabla,\ H)\,H$; and in the determinant Jac. $(U,\ \overline{\nabla}H,\ H)$ the second line is $\partial_x\nabla H$, $\partial_y\nabla H$, $\partial_z\nabla H$ (∇ being simply multiplied by ∂_x, ∂_y, ∂_z respectively).

17. Substituting, the condition becomes

$$\begin{aligned}&(-15m^2+60m-57)\,H\ \text{Jac.}\ (U,\ \nabla,\ H)\,H\\ &+(30m-54)(m-2)\,\{H\ \text{Jac.}\ (U,\ \nabla,\ H)\,H+\text{Jac.}\ (U,\ \overline{\nabla}H,\ H)\}\\ &+(m-2)^2\qquad\qquad\{9H^2\partial\Omega-54H\Omega\partial H+40\Psi\partial H\}=0,\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}&(15m^2-54m+51)\,H\ \text{Jac.}\ (U,\ \nabla\ ,\ H)\,H\\ &+(30m-54)(m-2)\,H\ \text{Jac.}\ (U,\ \overline{\nabla}H,\ H)\\ &+(m-2)^2\{9H^2\partial\Omega-45H\Omega\partial H+40\Psi\partial H\}=0.\end{aligned}$$

Article Nos. 18 to 27.—*Fourth transformation, and final form of the condition for a Sextactic Point.*

18. I write

$$\begin{aligned}(5m-12)\,\Omega\partial H-(3m-6)\,H\partial\Omega&=\vartheta\ \text{Jac.}\ (U,\ \Omega,\ H)\qquad\qquad\text{(J)}\\ \Omega\partial H+\qquad\qquad H\partial\Omega&=\qquad\qquad\partial(\Omega H),\end{aligned}$$

and, introducing for convenience the new symbol W,

$$-5\Omega\partial H+\qquad\qquad H\partial\Omega=\ W,$$

so that

$$\begin{vmatrix} 5m-12, & -(3m-6), & \vartheta\,\text{Jac.}\,(U,\ \Omega,\ H) \\ 1\ , & 1\ , & \partial.\Omega H \\ -5\ , & 1\ , & W \end{vmatrix} = 0,$$

or, what is the same thing,

$$(8m-18)\,W + 6\,\partial\,\text{Jac.}\,(U,\ \Omega,\ H) + (10m-18)\,\partial\,(\Omega H) = 0,$$

we have

$$W = H\partial\Omega - 5\Omega\partial H = \frac{-3}{4m-9}\,\vartheta\,\text{Jac.}\,(U,\ \Omega,\ H) - \frac{5m-9}{4m-9}\,\partial\,(\Omega H).$$

19. We have also

$$(8m-18)\,\Psi\partial H - (3m-6)\,H\partial\Psi - \vartheta\,\text{Jac.}\,(U,\ \Psi,\ H) = 0, \qquad \text{(J)}$$

that is

$$\Psi\partial H = \frac{\frac{1}{2}}{4m-9}\,\vartheta\,\text{Jac.}\,(U,\ \Psi,\ H) + \frac{\frac{3}{2}(m-2)}{4m-9}\,H\partial\Psi,$$

and thence

$$\begin{aligned} 9HW + 40\Psi\partial H &= 9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H, \\ &\doteq -\frac{9\,(5m-9)}{4m-9}\,H\partial\,(\Omega H) + \frac{60\,(m-2)}{4m-9}\,H\partial\Psi \\ &\quad + \frac{\vartheta}{4m-9}\{-27H\,\text{Jac.}\,(U,\ \Omega,\ H) + 40\,\text{Jac.}\,(U,\ \Psi,\ H)\}. \end{aligned}$$

20. The condition thus becomes

$$\begin{aligned} &(15m^2-54m+51)\,(4m-9)\,H\,\text{Jac.}\,(U,\ \nabla\ ,\ H)\,H \\ &+ 6\,(5m-9)\,(m-2)\ \ (4m-9)\,H\,\text{Jac.}\,(U,\ \overline{\nabla}H,\ H) \\ &+ 3\,(m-2)\,\{-3\,(5m-9)\,(m-2)\,H\partial\,(\Omega H) + 20\,(m-2)^2\,H\partial\Psi\} \\ &+\ \ (m-2)^2\vartheta\,\{-27H\,\text{Jac.}\,(U,\ \Omega,\ H) + 40\,\text{Jac.}\,(U,\ \Psi,\ H)\} = 0, \end{aligned}$$

which for shortness I represent by

$$3H\Pi + (m-2)^2\,\vartheta\,\{-27H\,\text{Jac.}\,(U,\ \Omega,\ H) + 40\,\text{Jac.}\,(U,\ \Psi,\ H)\} = 0,$$

so that we have

$$\begin{aligned} \Pi = &\ \ (5m^2-18m+17)\,(4m-9)\,\text{Jac.}\,(U,\ \nabla\ ,\ H)\,H \\ &+ 2\,(5m-9)\,(m-2)\,(4m-9)\,\text{Jac.}\,(U,\ \overline{\nabla}H,\ H) \\ &+\ \ (m-2)\,\{-3\,(5m-9)\,(m-2)\,\partial\,(\Omega H) + 20\,(m-2)^2\partial\Psi\}. \end{aligned}$$

21. Write

$$\Psi_1 = (\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}'\between A,\ B,\ C)^2,$$

where $(A,\ B,\ C)$ are as before the differential coefficients of U, and $(a',\ b',\ c',\ f',\ g',\ h')$ being the second differential coefficients of H, $(\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}')$ are the inverse coefficients, viz., $\mathfrak{A}' = b'c' - f'^2$, &c. We have

$$-(m-1)^2\partial\Psi_1 = (3m-6)\,(3m-7)\,\partial\,(\Omega H) - (3m-7)^2\partial\Psi \ \ (\textit{see post}, \text{ Nos. 41 to 46}),$$

that is

$$(3m-6)\,\partial(\Omega H) = (3m-7)\,\partial\Psi - \frac{(m-1)^2}{3m-7}\,\partial\Psi_1,$$

and thence

$$\begin{aligned}\Pi = \quad & (5m^2-18m+17)(4m-9)\,\text{Jac.}\,(U,\ \nabla\ ,\ H)\,H \\ & + 2(5m-9)\ (m-2)(4m-9)\,\text{Jac.}\,(U,\ \overline{\nabla}H,\ H) \\ & + \ (m-2)\left\{(5m^2-18m+17)\,\partial\Psi + \frac{(m-1)^2(5m-9)}{3m-7}\,\partial\Psi_1\right\} = 0.\end{aligned}$$

22. Now

$$\Psi = (\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between A',\ B',\ C')^2,\quad \Psi_1 = (\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}'\between A,\ B,\ C)^2,$$

and writing for shortness

$$\begin{aligned}E\Psi &= (\partial\mathfrak{A},\ ..\between A',\ B',\ C')^2, & F\Psi &= (\mathfrak{A},\ ..\between A',\ B',\ C'\between\partial\mathfrak{A}',\ \partial\mathfrak{B}',\ \partial\mathfrak{C}'),\\ E\Psi_1 &= (\partial\mathfrak{A}',\ ..\between A,\ B,\ C)^2, & F\Psi_1 &= (\mathfrak{A}'\ ..\between A,\ B,\ C\between\partial\mathfrak{A},\ \partial\mathfrak{B},\ \partial\mathfrak{C}),\end{aligned}$$

(we might, in a notation above explained, write $E\Psi = \partial\Psi_{\overline{H}}$, $F\Psi = \frac{1}{2}\partial\Psi_{\overline{U}}$, and in like manner $E\Psi_1 = \partial\Psi_{1\overline{U}}$, $F\psi_1 = \frac{1}{2}\partial\Psi_{1\overline{H}}$), then we have

$$\partial\Psi = E\Psi + 2F\Psi,\quad \partial\Psi_1 = E\Psi_1 + 2F\Psi_1.$$

We have moreover

$$\left.\begin{aligned}\text{Jac.}\,(U,\ \overline{\nabla}H,\ H) &= -\frac{m-1}{3m-7}\,E\Psi_1,\\ \text{Jac.}\,(U,\ \nabla\ ,\ H)\,H &= \qquad -E\Psi\,,\end{aligned}\right\}\ \begin{aligned}&\textit{post}, \text{ Nos. 47 to 50.}\\ &\textit{post}, \text{ Nos. 51 to 53.}\end{aligned}$$

23. The just-mentioned formulæ give

$$\begin{aligned}\Pi = - \quad & (5m^2-18m+17)(4m-9)\,E\Psi \\ & - 2(5m-9)(m-2)(4m-9)\,\frac{m-1}{3m-7}\,F\Psi_1 \\ & + \ (m-2)(5m^2-18m+17)(E\Psi\ + 2F\Psi\ ) \\ & + \ \frac{(5m-9)(m-1)^2(m-2)}{3m-7}\,(E\Psi_1 + 2F\psi_1),\end{aligned}$$

that is

$$\begin{aligned}\Pi = - \quad & (3m-7)(5m^2-18m+17) && E\Psi \\ & + 2(\ m-2)(5m^2-18m+17) && F\Psi \\ & + \ \frac{(5m-9)(m-1)^2(m-2)}{3m-7} && E\Psi_1 \\ & - \frac{2(m-1)(m-2)(3m-8)(5m-9)}{3m-7}\,F\Psi_1,\end{aligned}$$

or, as this may also be written,

$$\begin{aligned}(3m-7)\Pi = &-(5m^2-18m+17)\{-2(m-1)(m-2)F\Psi_1 + (3m-7)^2 E\Psi\} \\ &-(5m-9)(m-2)\{(m-1)(3m-8)F\Psi_1 + (3m-7)(3m-8)F\Psi - (m-1)^2 E\Psi_1\} \\ &+(25m^2-103m+106)(m-2)\{-(m-1)F\Psi_1 + (3m-7)F\Psi\}.\end{aligned}$$

24. But recollecting that

$$\begin{aligned}\Omega &= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\between\partial_x, \partial_y, \partial_z)^2 H \\ &= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\between a', b', c', 2f', 2g', 2h'),\end{aligned}$$

and putting

$$\begin{aligned}E\Omega &= (\partial\mathfrak{A}, \ldots\between a', \ldots) \qquad (=\partial\Omega_{\overline{H}}), \\ F\Omega &= (\mathfrak{A}, \ldots\between\partial a', \ldots) \qquad (=\partial\Omega_{\overline{v}}),\end{aligned}$$

we have, *post*, Nos. 41 to 46,

$$\begin{aligned}-2(m-1)(m-2)F\Psi_1 + (3m-7)^2 E\Psi_1 &= (3m-6)(3m-7)HE\Omega, \\ (m-1)(3m-8)F\Psi_1 + (3m-7)(3m-8)F\Psi - (m-1)^2 E\Psi_2 &= (3m-6)(3m-7)HF\Omega, \\ -(m-1)F\Psi_1 + (3m-7)F\Psi &= (3m-7)\Omega\partial H,\end{aligned}$$

and the foregoing equation becomes

$$\begin{aligned}(3m-7)\Pi = &-(5m^2-18m+17)(3m-6)(3m-7)HE\Omega \\ &-(5m-9)(m-2)(3m-6)(3m-7)HF\Omega \\ &+(m-2)(25m^2-103m-106)(3m-7)\Omega\partial H.\end{aligned}$$

25. But we have

$$\mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{H}}) - (3m-6)HE\Omega + (2m-4)\Omega\partial H = 0, \qquad \text{(J)}$$

$$\mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{v}}) - (3m-6)HF\Omega + (3m-6)\Omega\partial H = 0, \qquad \text{(J)}$$

that is

$$\begin{aligned}3(m-2)HE\Omega &= 2(m-2)\Omega\partial H + \mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{H}}), \\ 3(m-2)HF\Omega &= (3m-8)\Omega\partial H + \mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{v}}),\end{aligned}$$

and we thus obtain

$$\begin{aligned}\Pi = &-(5m^2-18m+17)\{2(m-2)\Omega\partial H + \mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{H}})\} \\ &-(5m-9)(m-2)\{(3m-8)\Omega\partial H + \mathfrak{I}\,\text{Jac.}\,(U, H, \Omega_{\overline{v}})\} \\ &+(25m^2-103m+106)(m-2)\Omega\partial H,\end{aligned}$$

where the coefficient of $(m-2)\Omega\partial H$ is

$$\begin{aligned}&-(10m^2-36m+34) \\ &-(5m-9)(3m-8) \\ &+(25m^2-103m+106),\end{aligned}$$

which is $=0$. Hence

$$\Pi = -(5m^2-18m+17)\,\vartheta\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}})$$
$$-(5m-9)(m-2)\ \vartheta\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{v}}).$$

26. Substituting this in the equation

$$3H\Pi + (m-2)^2\{-27H\,\text{Jac.}\,(U,\ \Omega,\ H) + 40\,\text{Jac.}\,(U,\ \Psi,\ H)\} = 0,$$

the result contains the factor ϑ, and, throwing this out, the condition is

$$3H\{-(5m^2-18m+17)\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}}) - (5m-9)(m-2)\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{v}})\}$$
$$+(m-2)^2\{27H\,\text{Jac.}\,(U,\ H,\ \Omega) - 40\,\text{Jac.}\,(U,\ H,\ \Psi)\} = 0,$$

or, as this may also be written,

$$\begin{aligned} &-(15m^2-54m+51)\,H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}}) - 3(5m-9)(m-2)\,H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{v}}) \\ &+27(m-2)^2 \qquad \{H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}}) + \qquad H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{v}})\} \\ &-40(m-2)^2 \qquad \text{Jac.}\,(U,\ H,\ \Psi\ ) = 0. \end{aligned}$$

27. Hence the condition finally is

$$(12m^2-54m+57)\,H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}}) + (m-2)(12m-27)\,H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{v}})$$
$$-40(m-2)^2\,\text{Jac.}\,(U,\ H,\ \Psi) = 0,$$

or, as this may also be written,

$$-3(m-1)\,H\,\text{Jac.}\,(U,\ H,\ \Omega_{\bar{H}}) + (m-2)(12m-27)\,H\,\text{Jac.}\,(U,\ H,\ \Omega)$$
$$-40(m-2)^2\,\text{Jac.}\,(U,\ H,\ \Psi) = 0,$$

viz. the sextactic points are the intersections of the curve m with the curve represented by this equation; and observing that U, H, $H\Omega$ and Ψ are of the orders m, $3m-6$, $8m-18$ respectively, the order of the curve is as above mentioned $=12m-27$.

Article Nos. 28 to 30.—*Application to a Cubic.*

28. I have in my former memoir, No. 30, shown that for a cubic curve

$$\Omega(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\mathfrak{K}\partial_x,\ \partial_y,\ \partial_z)^2\,H = -2S\,.\,U = 0;$$

this implies Jac. $(U,\ H,\ \Omega) = 0$, and hence if one of the two Jacobians, Jac. $(U,\ H,\ \Omega_{\bar{v}})$, Jac. $(U,\ H,\ \Omega_{\bar{H}})$ vanish, the other will also vanish. Now, using the canonical form

$$U = x^3 + y^3 + z^3 + 6lxyz,$$

we have

$$\begin{aligned} \Omega &= (\mathfrak{A},\ ..\mathfrak{K}a',\ \ldots) \\ &= (yz - l^2x^2\ ,\ \ zx - l^2y^2\ ,\ \ xy - l^2z^2\ ,\ \ l^2yz - lx^2\ ,\ \ l^2zx - ly^2\ ,\ \ l^2xy - lz^2) \\ &\quad (\ \ -3l^2x,\ \ \ -3l^2y,\ \ \ -3l^2z,\ \ (1+2l^3)\,x,\ \ (1+2l^3)\,y,\ \ (1+2l^3)\,z\ ), \end{aligned}$$

the development of which in fact gives the last-mentioned result. But applying this formula to the calculation of Jac. $(U, H, \Omega_{\bar{U}})$, then disregarding numerical factors, we have

$$\begin{aligned}\partial_x\Omega_{\bar{U}} &= (yz - l^2x^2, \ldots l^2yz - lx^2, \ldots, \mathfrak{X} - 3l^2,\ 0,\ 0,\ (1+2l^2),\ 0,\ 0)\\ &= -3l^2 \qquad (yz - l^2x^2)\\ &\quad + (1+2l^3)(l^2yz - lx^2)\\ &= (-l+l^4)(x^2+2lyz),\ = S\partial_x U;\end{aligned}$$

and in like manner $\partial_y\Omega_{\bar{U}} = S\partial_y U$, $\partial_z\Omega_{\bar{U}} = S\partial_z U$, and therefore

$$\text{Jac. } (U, H, \Omega_{\bar{U}}) = S \text{ Jac. } (U, H, U) = 0,$$

whence also

$$\text{Jac. } (U, H, \Omega_{\bar{H}}) = 0;$$

and the condition for a sextactic point assumes the more simple form,

$$\text{Jac. } (U, H, \Psi) = 0.$$

29. Now (former memoir, No. 32) we have

$$\begin{aligned}\Psi = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\,\mathfrak{J}\,\partial_x H, \partial_y H, \partial_z H)^2\\ = \quad (1+8l^3)^2 \qquad (y^3z^3 + z^3x^3 + x^3y^3)\\ + (-9l^6) \qquad (x^3+y^3+z^3)^2\\ + (-2l - 5l^4 - 20l^7)\ (x^3+y^3+z^3)\,xyz\\ + (-15l^2 - 78l^5 + 12l^8)\,x^2y^2z^2,\end{aligned}$$

or observing that $x^3+y^3+z^3$ and xyz, and therefore the last three lines of the expression of Ψ are functions of $U(=x^3+y^3+z^3+6lxyz)$ and $H\left(=-l^2(x^3+y^3+z^3)+(1+2l^3)\,xyz\right)$, and consequently give rise to the term $=0$ in Jac. (U, H, Ψ), we may write

$$\Psi = (1+8l^3)^2(y^3z^3 + z^3x^3 + x^3y^3).$$

30. We have then, disregarding a constant factor,

$$\text{Jac. } (U, H, \Psi) = \text{Jac. } (x^3+y^3+z^3,\ xyz,\ y^3z^3+z^3x^3+x^3y^3),$$

$$= \begin{vmatrix} x^2 & , & y^2 & , & z^2 \\ yz & , & zx & , & xy \\ x^2(y^3+z^3), & & y^2(z^3+x^3), & & z^2(x^3+y^3) \end{vmatrix},$$

$$\begin{aligned}&= x^3(y^6 - z^6) + y^3(z^6 - x^6) + z^3(x^6 - y^6),\\ &= (y^3 - z^3)(z^3 - x^3)(x^3 - y^3),\end{aligned}$$

so that the sextactic points are the intersections of the curve

$$U = x^3 + y^3 + z^3 + 6lxyz = 0,$$

with the curve

$$(y^3 - z^3)(z^3 - x^3)(x^3 - y^3) = 0.$$

Article Nos. 31 to 33.—*Proof of identities for the first transformation.*

31. Calculation of $(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\, U$.

Writing ∂ in place of $\mathbf{D}$, we have (former memoir, No. 20)

$$(\partial_1^4 + 6\partial_1^2\partial_2)\, U = \frac{\vartheta^2}{(m-1)^2}\left(-2\partial_2 H - \partial^2 H + \frac{3m-6}{m-1} H\Phi - \frac{2\vartheta}{m-1}\nabla H\right).$$

But

$$\left.\begin{aligned} -2\partial_2 H &= \frac{6m-12}{m-1} H\Phi - \frac{2\vartheta}{m-1}\nabla H, \\ -\partial^2 H &= \frac{(3m-6)(3m-7)}{(m-1)^2} H\Phi - \frac{6m-14}{(m-1)^2}\vartheta\nabla H + \frac{\vartheta^2}{(m-1)}\Omega, \end{aligned}\right\} \text{ former memoir, Nos. 21 and 22;}$$

and thence

$$\begin{aligned} (\partial_1^4 + 6\partial_1^2\partial_2)\, U = {} & \frac{\vartheta^2}{(m-1)^4}(18m^2 - 66m + 60)\, H\Phi \\ & + \frac{\vartheta^3}{(m-1)^4}(-10m + 18)\nabla H \\ & + \frac{\vartheta^4}{(m-1)^4}(\Omega); \end{aligned}$$

whence operating on each side with ∂_1, $=\partial$, we have

$$\begin{aligned} (\partial_1^5 + 10\partial_1^3\partial_2 + 6\partial_1^2\partial_3 + 12\partial_1\partial_2^2)\, U = {} & \frac{\vartheta^2}{(m-1)^4}(18m^2 - 66m + 60)\,(H\partial\Phi + \Phi\partial H) \\ & + \frac{\vartheta^3}{(m-1)^4}(-10m + 18)\{(\partial\,.\,\nabla)\, H + \partial\nabla H\} \\ & + \frac{\vartheta^4}{(m-1)^4}\partial\Omega. \end{aligned}$$

We have besides (see Appendix, Nos. 69 to 74),

$$\begin{aligned} \partial_1^2\partial_3 U = {} & \frac{\vartheta^2}{(m-1)^3}\{(3m-6)\, H\partial\Phi + (-m+3)\,\Phi\partial H\} \\ & + \frac{\vartheta^3}{(m-1)^3}\{-(\partial\,.\,\nabla)\, H\}, \end{aligned}$$

$$\partial_1\partial_2^2 U = \frac{\vartheta^2}{(m-1)^2}(-H\partial\Phi + \Phi\partial H);$$

and thence

$$(4\partial_1^2\partial_3 + 3\partial_1\partial_2^2)\,U = \frac{\vartheta^2}{(m-1)^3}\{(90m-21)\,H\partial\Phi + (-m+9)\,\Phi\partial H\}$$

$$+\frac{\vartheta^3}{(m-1)^3}\{-4\,(\partial.\nabla)\,H\};$$

and adding this to the foregoing expression for $(\partial_1^5 + 10\partial_1^3\partial_2 + 6\partial_1^2\partial_3 + 12\partial_1\partial_2^2)\,U$, we have

$$(\partial_1^5 + 10\partial_1^3\partial_2 + 10\partial_1^2\partial_3 + 15\partial_1\partial_2^2)\,U =$$

$$\frac{\vartheta^2}{(m-1)^4}\{(27m^2-96m+81)H\partial\Phi + (17m^2-56m+51)\Phi\partial H\}$$

$$+\frac{\vartheta^3}{(m-1)^4}\{(-14m+22)(\partial\,.\,\nabla)\,H + (-10m+18)\,\partial\nabla\,.\,H\}$$

$$+\frac{\vartheta^4}{(m-1)^4}\,\partial\Omega.$$

32. Calculation of

$$\partial_4 U\partial_1 H + 2\partial_3 U\partial_2 H + 2\partial_2 U\partial_3 H.$$

We have

$$\partial_4 U = \frac{\vartheta^2}{(m-1)^2}\,\tfrac{2}{3}\partial_2 H + \partial^2 H - \frac{1}{m-1}\,H\Phi - \frac{\tfrac{2}{3}\vartheta}{m-1}\,\nabla H, \qquad \partial_1 H = \partial H,$$

$$\partial_3 U = \frac{\vartheta^2}{(m-1)^2}\,\partial H, \qquad \partial_2 H = \partial_2 H,$$

$$\partial_2 U = \frac{\vartheta^2}{(m-1)^2}\,H, \qquad \partial_3 H = \frac{1}{m-1}(-3m+6)\,\partial\Phi - \Phi\partial H + \frac{\vartheta}{m-1}\,(\partial.\nabla)\,H,$$

for which values see Appendix, No. 58. And hence the expression sought for is

$$= \frac{\vartheta^2}{(m-1)^3}\Big\{\big((m-1)\,(\tfrac{2}{3}\partial_2 H + \partial^2 H) - H\Phi - \tfrac{2}{3}\vartheta\nabla H\big)\,\partial H$$

$$+ 2\,(m-1)\,\partial H\,\partial_2 H$$

$$+ 2H\,\big((-3m+6)\,H\partial\Phi - \Phi\partial H + \vartheta\,(\partial.\nabla)\,H\big)\Big\},$$

which is

$$= \frac{\vartheta^2}{(m-1)^3}\{\ \tfrac{8}{3}\,(m-1)\,\partial H\,\partial_2 H$$

$$+\ (m-1)\,\partial H\,\partial^2 H$$

$$+\ (-6m+12)\,H^2\partial\Phi - 3H\Phi\partial H\}$$

$$+\frac{\vartheta^3}{(m-1)^3}\{2H\,(\partial.\nabla)\,H - \tfrac{2}{3}\partial H\,\nabla H\}.$$

But we have, former memoir, Nos. 21 and 25,

$$\partial_2 H = -\frac{(3m-6)}{m-1} H\Phi - \frac{\vartheta}{m-1} \nabla H,$$

$$\partial^2 H = -\frac{(3m-6)(3m-7)}{(m-1)^2} H\Phi + \frac{6m-14}{(m-1)^2} \vartheta \nabla H - \frac{\vartheta^2}{(m-1)^2} \Omega,$$

so that the foregoing expression becomes

$$= \frac{\vartheta^2}{(m-1)^3} \{ -(8m-16) H\Phi\partial H + \tfrac{8}{3}\vartheta\partial H \nabla H$$

$$- \frac{(3m-6)(3m-7)}{m-1} H\Phi\partial H + \frac{6m-14}{m-1} \vartheta\partial H \nabla H - \frac{\vartheta^2}{m-1} \Omega\partial H$$

$$- \qquad 3H\Phi\partial H - (6m-12) H^2\partial\Phi \}$$

$$+ \frac{\vartheta^3}{(m-1)^3} \{ 2H(\partial \,.\, \nabla) H - \tfrac{2}{3}\partial H \nabla H \};$$

or finally

$$\partial_4 U \partial_1 H + 2\partial_3 U \partial_2 H + 2\partial_2 U \partial_3 H =$$

$$\frac{\vartheta^3}{(m-1)^4} \{ (-6m^2+18m-12) H^2\partial\Phi + (-17m^2+60m-55) H\Phi\partial H \}$$

$$+ \frac{\vartheta^3}{(m-1)^4} \{ (2m-2) H(\partial \,.\, \nabla) H + (8m-16)\partial H \nabla H \}$$

$$+ \frac{\vartheta^4}{(m-1)^4} \{ -\Omega\partial H \}.$$

33. Calculation of $\partial_2 U \partial_3 U$.

This is

$$= \frac{\vartheta^4}{(m-1)^4} H\partial H.$$

Article Nos. 34 and 35.—*The Jacobian Formula.*

34. In general, if P, Q, R, S be functions of the degrees p, q, r, s respectively, we have identically

$$\begin{vmatrix} pP, & qQ, & rR, & sS \\ \partial_x P, & \partial_x Q, & \partial_x R, & \partial_x S \\ \partial_y P, & \partial_y Q, & \partial_y R, & \partial_y S \\ \partial_z P, & \partial_z Q, & \partial_z R, & \partial_z S \end{vmatrix} = 0,$$

or, what is the same thing,

$$pP \text{ Jac. } (Q, R, S) - qQ \text{ Jac. } (R, S, P) + rR \text{ Jac. } (S, P, Q) - sS \text{ Jac. } (P, Q, R) = 0.$$

Hence in particular if $P = U$, and assuming $U = 0$, we have

$$-qQ \text{ Jac. } (R,\ S,\ U) + rR \text{ Jac. } (S,\ U,\ Q) - sS \text{ Jac. } (U,\ Q,\ R) = 0.$$

If moreover $Q = \mathfrak{H}$, and therefore $q = 1$, we have

$$-\mathfrak{H} \text{ Jac. } (R,\ S,\ U) + rR \text{ Jac. } (S,\ U,\ \mathfrak{H}) - sS \text{ Jac. } (U,\ \mathfrak{H},\ R) = 0;$$

or, as this may also be written,

$$-\mathfrak{H} \text{ Jac. } (U,\ R,\ S) + rR \text{ Jac. } (U,\ \mathfrak{H},\ S) - sS \text{ Jac. } (U,\ \mathfrak{H},\ R) = 0;$$

that is

$$-\mathfrak{H} \text{ Jac. } (U,\ R,\ S) + rR\partial S - sS\partial R = 0.$$

35. Particular cases are

$$\begin{array}{llll}
(2m-4) & \Phi\partial H - (3m-6)\, H\partial\Phi & = \mathfrak{H} \text{ Jac. } (U,\ \Phi\ ,\ H), & \textit{ante}, \text{ No. } 12, \\
(5m-11) & \nabla H\partial H - (3m-6)\, H\partial(\nabla H) & = \mathfrak{H} \text{ Jac. } (U,\ \nabla H,\ H), & \quad ,, \quad 14, \\
(2m-4) & \nabla : \partial H - (3m-6)\, H\partial\,.\,\nabla & = \mathfrak{H} \text{ Jac. } (U,\ \nabla\ ,\ H), & \quad ,, \quad ,, \\
(5m-12) & \Omega\partial H - (3m-6)\, H\partial\Omega & = \mathfrak{H} \text{ Jac. } (U,\ \Omega\ ,\ H), & \quad ,, \quad 18, \\
(8m-18) & \Psi\partial H - (3m-6)\, H\partial\Psi & = \mathfrak{H} \text{ Jac. } (U,\ \Psi\ ,\ H), & \quad ,, \quad 19, \\
(2m-4) & \Omega\partial H - (3m-6)\, HE\Omega & = \mathfrak{H} \text{ Jac. } (U,\ \Omega_{\bar{H}}\ ,\ H), & \quad ,, \quad 25, \\
(3m-8) & \Omega\partial H - (3m-6)\, HF\Omega & = \mathfrak{H} \text{ Jac. } (U,\ \Omega_{\bar{U}}\ ,\ H), & \quad ,, \quad ,,
\end{array}$$

where it is to be observed that in the third of these formulæ I have, in accordance with the notation before employed, written $\partial\,.\,\nabla$ to denote the result of the operation ∂ performed on ∇ as operand. I have also written $\nabla : \partial H$ to show that the operation ∇ is not to be performed on the following ∂H as an operand, but that it remains as an unperformed operation. As regards the last two equations, it is to be remarked that the demonstration in the last preceding number depends merely on the homogeneity of the functions, and the orders of these functions: in the former of the two formulæ, the differentiation of Ω is performed upon Ω in regard to the coordinates $(x,\ y,\ z)$ in so far only as they enter through U, and Ω is therefore to be regarded as a function of the order $2m-4$; in the latter of the two formulæ the differentiation is to be performed in regard to the coordinates in so far only as they enter through H, and Ω is therefore to be regarded as a function of the order $3m-8$. The two formulæ might also be written

$$(2m-4)\,\Omega\partial H - (3m-6)\, H\partial\Omega_{\bar{H}} = \mathfrak{H} \text{ Jac. } (U,\ \Omega_{\bar{H}},\ H),$$

$$(3m-8)\,\Omega\partial H - (3m-6)\, H\partial\Omega_{\bar{U}} = \mathfrak{H} \text{ Jac. } (U,\ \Omega_{\bar{U}},\ H);$$

and it may be noticed that, adding these together, we obtain the foregoing formula,

$$(5m-12)\,\Omega\partial H - (3m-6)\, H\partial\Omega = \mathfrak{H} \text{ Jac. } (U,\ \Omega,\ H).$$

Article Nos. 36 to 40.—*Proof of equation* $(\partial . \nabla) H = \text{Jac.}\ (U,\ H,\ \Phi)$, *used in the second transformation.*

36. We have

$$\nabla = (\mathfrak{A}, \ldots \mathfrak{X}\lambda,\ \mu,\ \nu\mathfrak{X}\partial_x,\ \partial_y,\ \partial_z)$$
$$= (\mathfrak{A}\partial_x + \mathfrak{H}\partial_y + \mathfrak{G}\partial_z,\ \mathfrak{H}\partial_x + \mathfrak{B}\partial_y + \mathfrak{F}\partial_z,\ \mathfrak{G}\partial_x + \mathfrak{F}\partial_y + \mathfrak{C}\partial_z\mathfrak{X}\lambda,\ \mu,\ \nu).$$

Also

$$\partial = (B\nu - C\mu)\,\partial_x + (C\lambda - A\nu)\,\partial_y + (A\mu - B\lambda)\,\partial_z$$
$$= \lambda P + \mu Q + \nu R,$$

if for a moment $P,\ Q,\ R = C\partial_y - B\partial_z,\ A\partial_z - C\partial_x,\ B\partial_x - A\partial_y$.

Hence

$$\partial . \nabla = (P\lambda + Q\mu + R\nu) . (\mathfrak{A}\partial_x + \mathfrak{H}\partial_y + \mathfrak{G}\partial_z,\ \mathfrak{H}\partial_x + \mathfrak{B}\partial_y + \mathfrak{F}\partial_z,\ \mathfrak{G}\partial_x + \mathfrak{F}\partial_y + \mathfrak{C}\partial_z\mathfrak{X}\lambda,\ \mu,\ \nu),$$

viz. coefficient of λ^2

$$= P\mathfrak{A}\partial_x + P\mathfrak{H}\partial_y + P\mathfrak{G}\partial_z,$$

and so for the other terms; whence also in $(\partial . \nabla) H$ the coefficients of λ^2, &c. are

$$(P\mathfrak{A}\partial_x + P\mathfrak{H}\partial_y + P\mathfrak{G}\partial_z)\,H,\ \&c.$$

37. Again, in Jac. $(U,\ H,\ \Phi)$, where $\Phi = (\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\mathfrak{X}\lambda,\ \mu,\ \nu)^2$, the coefficients of λ^2, &c. are Jac. $(U,\ H,\ \mathfrak{A})$, &c.; and hence the assumed equation

$$(\partial . \nabla)\,H = \text{Jac.}\ (U,\ H,\ \Phi),$$

in regard to the term in λ^2, is

$$(P\mathfrak{A}\partial_x + P\mathfrak{H}\partial_y + P\mathfrak{G}\partial_z)\,H = \text{Jac.}\ (U,\ H,\ \mathfrak{A}),$$

and we have

$$\text{Jac.}\ (U,\ H,\ \mathfrak{A}) = \begin{vmatrix} A\ , & B\ , & C \\ \partial_x H, & \partial_y H, & \partial_z H \\ \partial_x\ , & \partial_y\ , & \partial_z \end{vmatrix} \mathfrak{A}$$
$$= [\partial_x H\,(C\partial_y - B\partial_z) + \partial_y H\,(A\partial_z - C\partial_x) + \partial_z H\,(B\partial_x - A\partial_y)]\,\mathfrak{A},$$
$$= (\partial_x H . P + \partial_y H . Q + \partial_z H . R)\,\mathfrak{A};$$

so that the equation is

$$P\mathfrak{A}\partial_x H + P\mathfrak{H}\partial_y H + P\mathfrak{G}\partial_z H,$$
$$= P\mathfrak{A}\partial_x H + Q\mathfrak{A}\partial_y H + R\mathfrak{A}\partial_z H,$$

or, as this may be written,

$$[(B\partial_z - C\partial_y)\,\mathfrak{H} - (C\partial_x - A\partial_z)\,\mathfrak{A}]\,\partial_y H$$
$$+ [(B\partial_z - C\partial_y)\,\mathfrak{G} - (A\partial_y - B\partial_x)\,\mathfrak{A}]\,\partial_z H = 0.$$

38. The coefficient of $\partial_y H$ is

$$= \quad A\partial_z\mathfrak{A} + B\partial_z\mathfrak{H} - C(\partial_x\mathfrak{A} + \partial_y\mathfrak{H}),$$

which, in virtue of the identity, *post*, No. 40, $\partial_x\mathfrak{A} + \partial_y\mathfrak{H} + \partial_z\mathfrak{G} = 0$,

is

$$= \quad A\partial_z\mathfrak{A} + \mathfrak{B}\partial_z\mathfrak{H} + C\partial_z\mathfrak{G};$$

and in like manner the coefficient of $\partial_z H$ is

$$= -(A\partial_y\mathfrak{A} + B\partial_y\mathfrak{H} + C\partial_y\mathfrak{G}),$$

so that the equation is

$$(A\partial_z\mathfrak{A} + B\partial_z\mathfrak{H} + C\partial_z\mathfrak{G})\,\partial_y H - (A\partial_y\mathfrak{A} + B\partial_y\mathfrak{H} + C\partial_z\mathfrak{G})\,\partial_z H = 0.$$

39. But we have

$$\begin{aligned} \mathfrak{A}a \quad &+ \mathfrak{H}h \quad + \mathfrak{G}g \quad = H, \\ \mathfrak{A}h \quad &+ \mathfrak{H}b \quad + \mathfrak{G}f \quad = 0, \\ \mathfrak{A}g \quad &+ \mathfrak{H}f \quad + \mathfrak{G}c \quad = 0, \end{aligned}$$

or multiplying by x, y, z and adding,

$$(m-1)(\mathfrak{A}A \quad + \mathfrak{H}B \quad + \mathfrak{G}C) \quad = xH;$$

whence also

$$(m-1)(\mathfrak{A}h \quad + \mathfrak{H}b \quad + \mathfrak{G}c + A\partial_y\mathfrak{A} + B\partial_y\mathfrak{H} + C\partial_y\mathfrak{G}) = x\partial_y H,$$

that is

$$(m-1)(A\partial_y\mathfrak{A} + B\partial_y\mathfrak{H} + C\partial_y\mathfrak{G}) = x\partial_y H;$$

and in like manner

$$(m-1)(A\partial_z\mathfrak{A} + B\partial_z\mathfrak{H} + C\partial_z\mathfrak{G}) = x\partial_z H,$$

whence the equation in question. The terms in λ^2 are thus shown to be equal, and it might in a similar manner be shown that the terms in $\mu\nu$ are equal; the other terms will then be equal, and we have therefore

$$(\partial\,.\,\nabla)H = \text{Jac.}\,(U,\ H,\ \Phi).$$

40. The identity

$$\partial_x\mathfrak{A} \quad + \partial_y\mathfrak{H} \quad + \partial_z\mathfrak{G} \quad = 0$$

assumed in the course of the foregoing proof is easily proved. We have in fact

$$\partial_x\mathfrak{A} + \partial_y\mathfrak{H} + \partial_z\mathfrak{G} = \partial_x(bc - f^2) + \partial_y(fg - ch) + \partial_z(fh - bg)$$
$$= b(\partial_x c - \partial_z g) + c(\partial_x b - \partial_y h) + f(-2\partial_x f + \partial_y g + \partial_z h) + g(\partial_y f - \partial_z b) + h(-\partial_y c + \partial_z f),$$

where the coefficients of b, c, f, g, h separately vanish: we have of course the system

$$\begin{aligned} \partial_x\mathfrak{A} \quad &+ \partial_y\mathfrak{H} \quad + \partial_z\mathfrak{G} \quad = 0, \\ \partial_x\mathfrak{H} \quad &+ \partial_y\mathfrak{B} \quad + \partial_z\mathfrak{F} \quad = 0, \\ \partial_x\mathfrak{G} \quad &+ \partial_y\mathfrak{F} \quad + \partial_z\mathfrak{C} \quad = 0. \end{aligned}$$

Article Nos. 41 to 46.—*Proof of identities for the fourth transformation.*

41. Consider the coefficients (a, b, c, f, g, h) and the inverse set $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$, and the coefficients (a', b', c', f', g', h'), and the inverse set $(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}, \mathfrak{H}')$; then we have identically

$$\begin{aligned}&(a, \ldots \between x, y, z)^2 (\mathfrak{A}', \ldots \between a, \ldots) - (\mathfrak{A}', \ldots \between ax + hy + gz, \ldots)^2\\ &= (a', \ldots \between x, y, z)^2 (\mathfrak{A}, \ldots \between a', \ldots) - (\mathfrak{A}, \ldots \between a'x + h'y + g'z, \ldots)^2,\end{aligned}$$

where $(\mathfrak{A}', \ldots \between a, \ldots)$ and $(\mathfrak{A}, \ldots \between a', \ldots)$ stand for

$$(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}' \between a, b, c, 2f, 2g, 2h)$$

and

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between a', b', c', 2f', 2g', 2h')$$

respectively.

42. Taking (a, b, c, f, g, h), the second differential coefficients of a function U of the order m, and in like manner (a', b', c', f', g', h'), the second differential coefficients of a function U' of the order m', we have

$$\begin{aligned}&m(m-1)U.(\mathfrak{A}', \ldots \between \partial_x, \partial_y, \partial_z)^2 U' - (m-1)^2 (\mathfrak{A}', \ldots \between \partial_x U, \partial_y U, \partial_z U)^2\\ &= m'(m'-1)U'.(\mathfrak{A}, \ldots \between \partial_x, \partial_y, \partial_z)^2 U - (m'-1)^2 (\mathfrak{A}, \ldots \between \partial_x U', \partial_y U', \partial_z U')^2;\end{aligned}$$

and in particular if U' be the Hessian of U, then $m' = 3m - 6$.

43. Hence writing

$$\begin{aligned}\Omega &= (\mathfrak{A}, \ldots \between \partial_x, \partial_y, \partial_z)^2 H, \quad \Psi = (\mathfrak{A}, \ldots \between \partial_x H, \partial_y H, \partial_z H)^2,\\ \Omega_1 &= (\mathfrak{A}', \ldots \between \partial_x, \partial_y, \partial_z)^2 U, \quad \Psi_1 = (\mathfrak{A}', \ldots \between \partial_x U, \partial_y U, \partial_z U)^2,\end{aligned}$$

we have

$$m(m-1)U\Omega_1 - (m-1)^2 \Psi_1 = (3m-6)(3m-7)H\Omega - (3m-7)^2 \Psi;$$

or if $U = 0$, then

$$-(m-1)^2 \Psi_1 = (3m-6)(3m-7)H\Omega - (3m-7)^2 \Psi;$$

whence also

$$-(m-1)^2 \partial\Psi_1 = (3m-6)(3m-7)(H\partial\Omega + \Omega\partial H) - (3m-7)^2 \partial\Psi,$$

which is the formula, *ante* No. 21.

44. Recurring to the original formula, since this is an actual identity, we may operate on it with the differential symbol ∂ on the three assumptions:

1. (a, b, c, f, g, h), $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are alone variable.
2. (a', b', c', f', g', h'), $(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}')$ are alone variable.
3. (x, y, z) are alone variable.

We thus obtain

$$\begin{aligned}
&(\partial a, ..\between x, y, z)^2 (\mathfrak{A}', ..\between a, ..) && = (a', ..\between x, y, z)^2 (\partial\mathfrak{A}, ..\between a', ..) \\
&+ (a, ..\between x, y, z)^2 (\mathfrak{A}', ..\between \partial a, ..) && \quad - (\partial\mathfrak{A}, ..\between a'x + h'y + g'z, ..)^2, \\
&- 2(\mathfrak{A}', ..\between ax + hy + gz, ..\between x\partial a + y\partial b + z\partial c, ..) && \\
&(a, ..\between x, y, z)^2 (\partial\mathfrak{A}', ..\between a, ..) && = (\partial a', ..\between x, y, z)^2 (\mathfrak{A}, ..\between a', ..) \\
&- (\partial\mathfrak{A}', ..\between ax + hy + gz, ..)^2 && \quad + (a', ..\between x, y, z)^2 (\mathfrak{A}, ..\between \partial a' ..) \\
& && \quad - 2(\mathfrak{A}, ..\between a'x + h'y + g'z, ..\between x\partial a' + y\partial h' + z\partial g', ..), \\
&2(a, ..\between x, y, z\between \partial x, \partial y, \partial z)(\mathfrak{A}', ..\between a, ..) && = 2(a', ..\between x, y, z\between \partial x, \partial y, \partial z)(\mathfrak{A}, ..\between a', ..) \\
&- 2(\mathfrak{A}', .\between ax + hy + gz, .\between a\partial x + h\partial y + g\partial z, .) && \quad - 2(\mathfrak{A}, .\between a'x + h'y + g'z, ..\between a'\partial x + h'\partial y + g'\partial z, ..).
\end{aligned}$$

45. If in these equations respectively we suppose as before that (a, b, c, f, g, h) are the second differential coefficients of a function U of the order m, and (a', b', c', f', g', h') the second differential coefficients of a function U' of the order m'; and that (A, B, C), (A', B', C') are the first differential coefficients of these functions respectively, then after some easy reductions we have

$$\begin{aligned}
&(m-1)(m-2)\partial U(\mathfrak{A}', ..\between a, ..) && = m'(m'-1)\, U'(\partial\mathfrak{A}, ..\between a', ..) \\
&+ m(m-1)\, U(\mathfrak{A}', ..\between \partial a, ..) && \quad - (m'-1)^2 (\partial\mathfrak{A}, ..\between A', B', C')^2, \\
&-2(m-1)(m-2)(\mathfrak{A}', ..\between A, B, C\between \partial A, \partial B, \partial C) && \\
&m(m-1)\, U(\partial A', ..\between a', ..) && = (m'-1)(m'-2)\partial U'(\mathfrak{A}, ..\between a', ..) \\
&- (m-1)^2 (\partial A', ..\between A, B, C)^2 && \quad + m'(m'-1)\, U'(\mathfrak{A}, ..\between \partial a', ..) \\
& && \quad - 2(m'-1)(m'-2)(\mathfrak{A}, ..\between A', B', C'\between \partial A', \partial B', \partial C') \\
&2(m-1)\partial U(\mathfrak{A}', ..\between a, ..) && = 2(m'-1)\partial U'(\mathfrak{A}, ..\between a', ..) \\
&- 2(m-1)(\mathfrak{A}', ..\between A, B, C\between \partial A, \partial B, \partial C) && \quad - 2(m'-1)(\mathfrak{A}, ..\between A', B', C'\between \partial A', \partial B', \partial C'),
\end{aligned}$$

equations which may be verified by remarking that their sum is

$$\begin{aligned}
&m(m-1)\{\partial U(\mathfrak{A}', ..\between a, ..) + U[(\mathfrak{A}', ..\between \partial a, ..) + (\partial\mathfrak{A}', ..\between a, ..)]\} \\
&- (m-1)^2 \{\partial\mathfrak{A}', ..\between A, B, C)^2 + (\mathfrak{A}', ..\between A, B, C\between \partial A, \partial B, \partial C)\} = m'(m'-1) \text{ \&c.},
\end{aligned}$$

viz., this is the derivative with ∂ of the equation

$$m(m-1)\, U(\mathfrak{A}', ..\between a, ..) - (m-1)^2 (A', ..\between A, B, C)^2 = m'(m'-1) \text{ \&c.}$$

46. Taking now $U' = H$, and therefore $m' = 3m - 6$; putting also $U = 0$, $\partial U = 0$, and writing as before

$$\begin{aligned}
E\Psi &= (\partial\mathfrak{A}, ..\between A', B', C')^2, \\
F\Psi &= (\mathfrak{A}, ..\between A', B', C'\between \partial A', \partial B', \partial C'), \\
E\Psi_1 &= (\partial\mathfrak{A}', ..\between A, B, C)^2, \\
F\Psi_1 &= (\mathfrak{A}', ..\between A, B, C\between \partial A, \partial B, \partial C), \\
E\Omega &= (\partial\mathfrak{A}, ..\between a', ..), \\
F\Omega &= (\mathfrak{A}, ..\between \partial a', ..),
\end{aligned}$$

then the three equations are

$$\begin{aligned}
-2(m-1)(m-2)F\Psi_1 &= (3m-6)(3m-7)HE\Omega-(3m-7)^2E\Psi,\\
-(m-1)^2E\Psi &= (3m-7)(3m-8)\Omega\partial H\\
&\quad+(3m-6)(3m-7)HF\Omega-2(3m-7)(3m-8)F\Psi,\\
-2(m-1)F\Psi_1 &= 2(3m-7)\Omega\partial H-2(3m-7)F\Psi,
\end{aligned}$$

whence, adding, we have

$$\begin{aligned}
-(m-1)^2(E\Psi_1+2F\Psi_1) &= -(3m-7)^2(E\Psi+2F\Psi)\\
&\quad+(3m-6)(3m-7)\{\Omega\partial H+H(E\Omega+F\Omega)\},
\end{aligned}$$

(that is

$$-(m-1)^2\partial\Psi_1 = -(3m-7)^2\partial\Psi+(3m-6)(3m-7)\partial\,.\,\Omega H,$$

which is right).

And by linearly combining the three equations, we deduce

$$\begin{aligned}
(3m-6)(3m-7)HE\Omega &= -2(m-1)(m-2)F\Psi_1 &&+(3m-7)^2E\Psi,\\
(3m-7)\Omega\partial H &= -(m-1)F\Psi_1+(3m-7)F\Psi,\\
(3m-6)(3m-7)HF\Omega &= (m-1)(3m-8)F\Psi_1+(3m-7)(3m-8)F\Psi-(m-1)^2E\Psi_1,
\end{aligned}$$

which are the formulæ, *ante*, No. 24.

Article Nos. 47 to 50.—*Proof of an identity used in the fourth transformation*, viz.,

$$\text{Jac. }(U,\ \overline{\nabla}H,\ H) = -\frac{m-1}{3m-7}F\Psi_1,$$

or say

$$\text{Jac. }(U,\ H,\ \overline{\nabla}H) = \frac{m-1}{3m-7}(\mathfrak{A}',\,..\between A,\ B,\ C\between\partial A,\ \partial B,\ \partial C).$$

47. We have

$$\begin{aligned}
\nabla &= (\mathfrak{A},\,..\between\lambda,\ \mu,\ \nu\between\partial_x,\ \partial_y,\ \partial_z)\\
&= \big((\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\between\lambda,\ \mu,\ \nu),\ (\mathfrak{H},\ \mathfrak{B},\ \mathfrak{F}\between\lambda,\ \mu,\ \nu),\ (\mathfrak{G},\ \mathfrak{F},\ \mathfrak{C}\between\lambda,\ \mu,\ \nu)\between\partial_x,\ \partial_y,\ \partial_z\big);
\end{aligned}$$

or, attending to the effect of the bar as denoting the exemption of the $(\mathfrak{A},\,..)$ from differentiation,

$$\begin{aligned}
\text{Jac. }(U,\ H,\ \overline{\nabla}H) &= (\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\between\lambda,\ \mu,\ \nu)\text{ Jac. }(U,\ H,\ \partial_xH)\\
&\quad+(\mathfrak{H},\ \mathfrak{B},\ \mathfrak{F}\between\lambda,\ \mu,\ \nu)\text{ Jac. }(U,\ H,\ \partial_yH)\\
&\quad+(\mathfrak{G},\ \mathfrak{F},\ \mathfrak{C}\between\lambda,\ \mu,\ \nu)\text{ Jac. }(U,\ H,\ \partial_zH).
\end{aligned}$$

48. Now

$$\text{Jac.}\,(U,\ H,\ \partial_x H) = \frac{1}{3m-6}\,\text{Jac.}\,(U,\ x\partial_x H + y\partial_y H + z\partial_z H,\ \partial_x H),$$

and the last-mentioned Jacobian is

$$\begin{aligned} &= \partial_x H\ \text{Jac.}\,(U,\ x,\ \partial_x H) + \partial_y H\ \text{Jac.}\,(U,\ y,\ \partial_x H) + \partial_z H\ \text{Jac.}\,(U,\ z,\ \partial_x H) \\ &\quad + y\ \text{Jac.}\,(U,\ \partial_y H,\ \partial_x H) + z\ \text{Jac.}\,(U,\ \partial_z H,\ \partial_x H), \end{aligned}$$

where the second line is

$$= -y\,\text{Jac.}\,(U,\ \partial_x H,\ \partial_y H) + z\ \text{Jac.}\,(U,\ \partial_z H,\ \partial_x H),$$

or writing $(A',\ B',\ C')$ for the first differential coefficients and $(a',\ b',\ c',\ f',\ g',\ h')$ for the second differential coefficients of H, this is

$$\begin{aligned} &= -y \begin{vmatrix} A, & B, & C \\ a', & h', & g' \\ h', & b', & f' \end{vmatrix} + z \begin{vmatrix} A, & B, & C \\ g', & f', & c' \\ a', & h', & g' \end{vmatrix} \\ &= -y\,(\mathfrak{G}',\ \mathfrak{F}',\ \mathfrak{C}'\between A,\ B,\ C) + z\,(\mathfrak{H}',\ \mathfrak{B}',\ \mathfrak{F}'\between A,\ B,\ C). \end{aligned}$$

The first line is

$$\begin{aligned} &= \begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ a', & h', & g' \end{vmatrix} \\ &= A\,(B'g' - C'h') + B\,(C'a' - A'g') + C\,(A'h' - B'a'), \end{aligned}$$

or reducing by the formulæ,

$$(3m-7)\,(A',\ B',\ C') = (a'x + h'y + g'z,\ h'x + b'y + f'z,\ g'x + f'y + c'z),$$

this is

$$\begin{aligned} &= \frac{1}{3m-7}\,\{A\,(-\mathfrak{G}'y + \mathfrak{H}'z) + B\,(-\mathfrak{F}'y + \mathfrak{B}'z) + C\,(-\mathfrak{C}'y + \mathfrak{F}'z)\} \\ &= \frac{1}{3m-7}\,\{-y\,(\mathfrak{G}',\ \mathfrak{F}',\ \mathfrak{C}'\between A,\ B,\ C) + z\,(\mathfrak{H}',\ \mathfrak{B}',\ \mathfrak{F}'\between A,\ B,\ C)\}. \end{aligned}$$

Hence we have

$$\begin{aligned} \text{Jac.}\,(U,\ H,\ \partial_x H) &= \frac{1}{3m-6}\left(1 + \frac{1}{3m-7}\right)\{-y\,(\mathfrak{G}',\ \mathfrak{F}',\ \mathfrak{C}'\between A,\ B,\ C) + z\,(\mathfrak{H}',\ \mathfrak{B}',\ \mathfrak{F}'\between A,\ B,\ C)\} \\ &= \frac{1}{3m-7}\qquad\qquad \{-y\,(\mathfrak{G}',\ \mathfrak{F}',\ \mathfrak{C}'\between A,\ B,\ C) + z\,(\mathfrak{H}',\ \mathfrak{B}',\ \mathfrak{F}'\between A,\ B,\ C)\}; \end{aligned}$$

and in like manner

$$\text{Jac.}\,(U,\ H,\ \partial_y H) = \frac{1}{3m-7}\qquad \{-z\,(\mathfrak{A}',\ \mathfrak{H}',\ \mathfrak{G}'\between A,\ B,\ C) + x\,(\mathfrak{G}',\ \mathfrak{F}',\ \mathfrak{C}'\between A,\ B,\ C)\},$$

$$\text{Jac.}\,(U,\ H,\ \partial_z H) = \frac{1}{3m-7}\qquad \{-x\,(\mathfrak{H}',\ \mathfrak{B}',\ \mathfrak{F}'\between A,\ B,\ C) + y\,(\mathfrak{A}',\ \mathfrak{H}',\ \mathfrak{G}'\between A,\ B,\ C)\}.$$

49. We thence have

$$\text{Jac.}\,(U, H, \overline{\nabla} H) = \frac{1}{3m-7}\begin{vmatrix} (\mathfrak{A}, \mathfrak{H}, \mathfrak{G}\between\lambda, \mu, \nu), & (\mathfrak{H}, \mathfrak{B}, \mathfrak{F}\between\lambda, \mu, \nu), & (\mathfrak{G}, \mathfrak{F}, \mathfrak{C}\between\lambda, \mu, \nu) \\ (\mathfrak{A}', \mathfrak{H}', \mathfrak{G}'\between A, B, C), & (\mathfrak{H}', \mathfrak{B}', \mathfrak{F}'\between A, B, C), & (\mathfrak{G}', \mathfrak{F}', \mathfrak{C}'\between A, B, C) \\ x\quad, & y\quad, & z, \end{vmatrix},$$

or multiplying the two sides by

$$H, = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

the right-hand side is

$$= \frac{1}{3m-7}\begin{vmatrix} H\lambda\quad, & H\mu\quad, & H\nu \\ X\quad, & Y\quad, & Z \\ (m-1)\,A, & (m-1)\,B, & (m-1)\,C, \end{vmatrix}$$

which is

$$= H\,\frac{m-1}{3m-7}\begin{vmatrix} \lambda\,, & \mu\,, & \nu \\ X, & Y, & Z \\ A, & B, & C, \end{vmatrix},$$

if for a moment

$$X = (\mathfrak{A}', \ldots\between A,\ B,\ C\between a,\ h,\ g),$$
$$Y = (\mathfrak{A}', \ldots\between A,\ B,\ C\between h,\ b,\ f),$$
$$Z = (\mathfrak{A}', \ldots\between A,\ B,\ C\between g,\ f,\ c).$$

50. Hence observing that these equations may be written

$$X = (\mathfrak{A}', \ldots\between A,\ B,\ C\between \partial_x A,\ \partial_x B,\ \partial_x C),$$
$$Y = (\mathfrak{A}', \ldots\between A,\ B,\ C\between \partial_y A,\ \partial_y B,\ \partial_y C),$$
$$Z = (\mathfrak{A}', \ldots\between A,\ B,\ C\between \partial_z A,\ \partial_z B,\ \partial_z C),$$

and that we have

$$\partial = \begin{vmatrix} \lambda\,, & \mu\,, & \nu \\ \partial_x, & \partial_y, & \partial_z \\ A, & B, & C, \end{vmatrix}$$

we obtain for H Jac. $(U, H, \overline{\nabla}, H)$ the value

$$= H\,\frac{m-1}{3m-7}\,(\mathfrak{A}', \ldots\between A,\ B,\ C\between \partial A,\ \partial B,\ \partial C),$$

or throwing out the factor H, we have the required result.

Article Nos. 51 to 53.—*Proof of identity used in the fourth transformation*, viz. Jac. $(U, \nabla, H) H = -E\Psi$, or say Jac. $(U, H, \nabla) H = (\partial\mathfrak{A}, \ldots \between A', B', C')^2$.

51. We have

$$\nabla \quad = \big((\mathfrak{A}, \mathfrak{H}, \mathfrak{G}, \between \lambda, \mu, \nu), (\mathfrak{H}, \mathfrak{B}, \mathfrak{F} \between \lambda, \mu, \nu), (\mathfrak{G}, \mathfrak{F}, \mathfrak{C} \between \lambda, \mu, \nu) \between \partial_x, \partial_y, \partial_z\big),$$

and thence

$$\partial_x . \nabla \quad = \big((\partial_x\mathfrak{A}, \partial_x\mathfrak{H}, \partial_x\mathfrak{G} \between \lambda, \mu, \nu), (\partial_x\mathfrak{H}, \partial_x\mathfrak{B}, \partial_x\mathfrak{F} \between \lambda, \mu, \nu), (\partial_x\mathfrak{G}, \partial_x\mathfrak{F}, \partial_x\mathfrak{C} \between \lambda, \mu, \nu) \between \partial_x, \partial_y, \partial_z\big),$$

and

$$(\partial_x . \nabla) H = \big((\partial_x\mathfrak{A}, \partial_x\mathfrak{H}, \partial_x\mathfrak{G} \between \lambda, \mu, \nu), (\partial_x\mathfrak{H}, \partial_x\mathfrak{B}, \partial_x\mathfrak{F} \between \lambda, \mu, \nu), (\partial_x\mathfrak{G}, \partial_x\mathfrak{F}, \partial_x\mathfrak{C} \between \lambda, \mu, \nu) \between A', B', C'\big),$$

with the like values for $(\partial_y . \nabla) H$ and $(\partial_z . \nabla) H$. And then

$$\text{Jac.}\,(U, H, \nabla) H = \begin{vmatrix} A \;, & B \;, & C \\ A' \;, & B' \;, & C' \\ (\partial_x . \nabla) H, & (\partial_y . \nabla) H, & (\partial_z . \nabla) H, \end{vmatrix}$$

in which the coefficient of A'^2 is

$$= (C\partial_y - B\partial_z)(\mathfrak{A}, \mathfrak{H}, \mathfrak{G} \between \lambda, \mu, \nu);$$

or putting for shortness

$$(C\partial_y - B\partial_z, A\partial_z - C\partial_x, B\partial_x - A\partial_y) = (P, Q, R);$$

the coefficient is

$$(P\mathfrak{A}, P\mathfrak{H}, P\mathfrak{G} \between \lambda, \mu, \nu).$$

52. We have

$$\partial = (P\lambda + Q\mu + R\nu),$$

and thence

$$\text{coefficient } A'^2 - \partial\mathfrak{A} = (P\mathfrak{A}, P\mathfrak{H}, P\mathfrak{G} \between \lambda, \mu, \nu) - (P\mathfrak{A}, Q\mathfrak{A}, R\mathfrak{A} \between \lambda, \mu, \nu)$$

which is

$$\begin{aligned} = \quad & \mu\,\{(C\partial_y - B\partial_z)\,\mathfrak{H} - (\mathfrak{A}\partial_z - \mathfrak{C}\partial_x)\,\mathfrak{A}\} \\ & + \nu\,\{(C\partial_y - B\partial_z)\,\mathfrak{G} - (B\partial_x - \mathfrak{A}\partial_y)\,\mathfrak{A}\}, \end{aligned}$$

where coefficient of μ is

$$\begin{aligned} &= -\;A\partial_z\mathfrak{A} - B\partial_z\mathfrak{H} + C\,(\partial_x\mathfrak{A} + \partial_y\mathfrak{H}) \\ &= -\,(A\partial_z\mathfrak{A} + B\partial_z\mathfrak{H} + C\partial_z\mathfrak{G}) = -\frac{1}{m-1}\,x\partial_z H, \end{aligned}$$

and coefficient of ν is

$$= +\,(A\partial_y\mathfrak{A} + B\partial_y\mathfrak{H} + C\partial_y\mathfrak{G}) = \frac{1}{m-1}\,x\partial_y H,$$

so that

$$\text{coefficient } A'^2 - \partial\mathfrak{A} = -\frac{1}{m-1}\,x\,(\mu\partial_z H - \nu\partial_y H).$$

53. By forming in a similar manner the coefficients of the other terms, it appears that

$$\text{Jac.}\,(U,\ H,\ \nabla)\,H-(\partial\mathfrak{A},\ \ldots\between A',\ B',\ C')^2$$

$$=-\frac{1}{m-1}\,(A'x+B'y+C'z)\begin{vmatrix} A', & B', & C' \\ \lambda, & \mu, & \nu \\ \partial_x H, & \partial_y H, & \partial_z H \end{vmatrix},$$

or since the determinant is

$$\begin{vmatrix} A', & B', & C' \\ \lambda, & \mu, & \nu \\ A', & B', & C' \end{vmatrix}, = 0,$$

we have the required equation,

$$\text{Jac.}\,(U,\ H,\ \nabla)\,H=(\partial\mathfrak{A},\ \ldots\between A',\ B',\ C')^2.$$

This completes the series of formulæ used in the transformations of the condition for the sextactic point.

Appendix, Nos. 54 to 74.

For the sake of exhibiting in their proper connexion some of the formulæ employed in the foregoing first transformation of the condition for a sextactic point, I have investigated them in the present Appendix, which however is numbered continuously with the memoir.

54. The investigations of my former memoir and the present memoir have reference to the operations

$$\begin{aligned} \partial_1 &= dx\,\partial_x + dy\,\partial_y + dz\,\partial_z, \\ \partial_2 &= d^2x\partial_x + d^2y\partial_y + d^2z\partial_z, \\ \partial_3 &= d^3x\partial_x + d^3y\partial_y + d^3z\partial_z, \\ &\&c., \end{aligned}$$

where if $(A,\ B,\ C)$ are the first differential coefficients of a function $U=(*\between x,\ y,\ z)^m$, and $\lambda,\ \mu,\ \nu$ are arbitrary constants, then we have

$$dx=B\nu-C\mu,\quad dy=C\lambda-A\nu,\quad dz=A\mu-B\lambda;$$

so that putting

$$\begin{aligned} \partial &= (B\nu-C\mu)\,\partial_x+(C\lambda-A\nu)\,\partial_y+(A\mu-B\lambda)\,\partial_z \\ &= \begin{vmatrix} A, & B, & C \\ \lambda, & \mu, & \nu \\ \partial_x, & \partial_y, & \partial_z, \end{vmatrix} \end{aligned}$$

we have $\partial_1=\partial$. The foregoing expressions of $(dx,\ dy,\ dz)$ determine of course the values of $(d^2x,\ d^2y,\ d^2z)$, $(d^3x,\ d^3y,\ d^3z)$, &c., and it is throughout assumed that these values are substituted in the symbols ∂_2, ∂_3, &c., so that ∂_1, $=\partial$, and ∂_2, ∂_3, &c. denote each of them an operator such as $X\partial_x+Y\partial_y+Z\partial_z$, where $(X,\ Y,\ Z)$ are functions of the coordinates; such operator, in so far as it is a function of the coordinates, may therefore be made an operand, and be operated upon by itself or any other like operator.

55. Taking (a, b, c, f, g, h) for the second differential coefficients of U, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ for the inverse coefficients, and H for the Hessian, I write also

$$\begin{aligned}
\Phi &= (\mathfrak{A}, \ldots \between \lambda, \mu, \nu)^2, \\
\nabla &= (\mathfrak{A}, \ldots \between \lambda, \mu, \nu \between \partial_x, \partial_y, \partial_z), \\
\square &= (\mathfrak{A}, \ldots \between \partial_x, \partial_y, \partial_z)^2, \\
\vartheta &= \lambda x + \mu y + \nu z, \\
\Omega &= (\mathfrak{A}, \ldots \between \partial_x, \partial_y, \partial_z)^2 H, = \square H, \\
\Psi &= (\mathfrak{A}, \ldots \between \partial_x H, \partial_y H, \partial_z H)^2, \\
\Gamma &= (a, \ldots \between \mu\partial_z - \nu\partial_y, \nu\partial_x - \lambda\partial_z, \lambda\partial_y - \mu\partial_x)^2,
\end{aligned}$$

and I notice that we have

$$\Gamma U = 2\Phi, \quad \nabla U = \frac{\vartheta}{m-1} H, \quad \square U = 3H,$$

$$\nabla\vartheta = \Phi, \quad \nabla^2 U = H\Phi \quad , \quad \nabla . \partial = 0 \ ,$$

the last of which is proved, *post* No. 65; the others are found without any difficulty.

56. I form the Table

$$\begin{aligned}
&\partial_1 U = 0, \\
&\partial_1^2 U = \frac{mU}{m-1}\Phi && + \frac{\vartheta^2}{(m-1)^2}(-H), \\
&\partial_2 U = \frac{mU}{m-1}(-\Phi) && + \frac{\vartheta^2}{(m-1)^2}(H), \\
&\partial_1^3 U = \frac{mU}{m-1}\partial\Phi && + \frac{\vartheta^2}{(m-1)^2}(-\partial H), \\
&\partial_1\partial_2 U = 0, \\
&\partial_3 U = \frac{mU}{m-1}(-\partial\Phi) && + \frac{\vartheta^2}{(m-1)^2}(\partial H), \\
&\partial_1^4 U = \frac{mU}{m-1}\left(\partial^2\Phi - \frac{2m\vartheta}{m-1}\nabla\Phi\right) && + \frac{\vartheta^2}{(m-1)^2}\left(-\partial^2 H - \frac{3m-6}{m-1}H\Phi + \frac{2\vartheta}{m-1}\nabla H\right), \\
&\partial_1^2\partial_2 U = \frac{mU}{m-1}\left(\tfrac{1}{3}\partial_2\Phi + \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla\Phi\right) && + \frac{\vartheta^2}{(m-1)^2}\left(-\tfrac{1}{3}\partial_2 H + \frac{m-2}{m-1}H\Phi - \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla H\right), \\
&\partial_1\partial_3 U = \frac{mU}{m-1}\left(-\tfrac{1}{3}\partial_2\Phi - \Phi_2 - \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla\Phi\right) && + \frac{\vartheta^2}{(m-1)^2}\left(\tfrac{1}{3}\partial_2 H + \frac{1}{m-1}H\Phi + \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla H\right), \\
&\partial_2^2 U = \frac{mU}{m-1}(\Phi^2) && + \frac{\vartheta^2}{(m-1)^2}(-H\Phi), \\
&\partial_4 U = \frac{mU}{m-1}\left(-\tfrac{2}{3}\partial_2\Phi - \partial^2\Phi + \Phi^2 + \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla\Phi\right) + \frac{\vartheta^2}{(m-1)^2}\left(\tfrac{2}{3}\partial_2 H + \partial^2 H - \frac{1}{m-1}H\Phi - \tfrac{2}{3}\frac{\vartheta}{m-1}\nabla H\right) \\
&\partial_2 H = -\frac{3m-6}{m-1}H\Phi + \frac{\vartheta}{m-1}\nabla H,
\end{aligned}$$

and assuming $U=0$,

$$\partial_1^2 H = \partial^2 H = -\frac{(3m-6)(3m-7)}{(m-1)^2} H\Phi + \frac{6m-14}{(m-1)^2}\mathfrak{H}\nabla H - \frac{\mathfrak{H}^2}{(m-1)^2}\Omega,$$

$$(\partial_1 H)^2 = (\partial H)^2 = -\frac{(3m-6)^2}{(m-1)^2} H^2\Phi + \frac{6m-12}{(m-1)^2}\mathfrak{H}H\nabla H - \frac{\mathfrak{H}^2}{(m-1)^2}\Psi,$$

which are for the most part given in my former memoir; the expressions for $\partial_2 U$, $\partial_3 U$, which are not explicitly given, follow at once from the equations

$$(\partial_1^2 + \partial_2)\, U = 0, \quad (\partial_1^3 + 2\partial_1\partial_2 + \partial_3)\, U = 0;$$

those for $\partial_1\partial_3 U$, $\partial_2^2 U$, and $\partial_4 U$ are new, but when the expressions for $\partial_1\partial_3 U$ and $\partial_2^2 U$ are known, that for $\partial_4 U$ is at once found from the equation

$$(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2 + \partial_4)\, U = 0.$$

57. Before going further, I remark that we have identically

$$\begin{aligned}(a, \ldots \between x, y, z)^2 (a, \ldots \between \mu\gamma - \nu\beta, \nu\alpha - \lambda\gamma, \lambda\beta - \mu\alpha)^2 \\ - \begin{vmatrix} ax+hy+gz, & hx+by+fz, & gx+fy+cz \\ \lambda\ , & \mu\ , & \nu \\ \alpha\ , & \beta\ , & \gamma \end{vmatrix}^2 \\ = (\mathfrak{A}, \ldots \between \lambda p - \alpha\mathfrak{H}, \mu p - \beta\mathfrak{H}, \nu p - \gamma\mathfrak{H})^2,\end{aligned}$$

(if for shortness $p = \alpha x + \beta y + \gamma z$, $\mathfrak{H} = \lambda x + \mu y + \nu z$)

$$\begin{aligned} = \quad & p^2\ (\mathfrak{A}, \ldots \between \lambda, \mu, \nu)^2 \\ & - 2p\mathfrak{H}\,(\mathfrak{A}, \ldots \between \lambda, \mu, \nu \between \alpha, \beta, \gamma) \\ & + \mathfrak{H}^2 (\mathfrak{A}, \ldots \between \alpha, \beta, \gamma)^2.\end{aligned}$$

58. If in this equation we take (a, b, c, f, g, h) to be the second differential coefficients of U, and write also $(\alpha, \beta, \gamma) = (\partial_x, \partial_y, \partial_z)$, the equation becomes

$$\begin{aligned} m(m-1)\, U\Gamma - (m-1)^2\, \partial^2 = \quad & \Phi\,(x\partial_x + y\partial_y + z\partial_z)^2 \\ & - 2\mathfrak{H}\,(x\partial_x + y\partial_y + z\partial_z)\,\nabla \\ & + \mathfrak{H}^2\Box,\end{aligned}$$

which is a general equation for the transformation of $\partial^2\,(=\partial_1^2)$.

59. If with the two sides of this equation we operate on U, we obtain

$$\begin{aligned} m(m-1)\, U\Gamma U - (m-1)^2\, \partial^2 U = \quad & m(m-1)\,\Phi U \\ & - 2(m-1)\,\mathfrak{H}\nabla U \\ & + \qquad \mathfrak{H}^2\Box U;\end{aligned}$$

and substituting the values

$$\Gamma U = 2\Phi, \quad \nabla U = \frac{\vartheta}{m-1} H, \quad \square U = 3H,$$

we find the before-mentioned expression of $\partial_1^2 U$.

60. Operating with the two sides of the same equation on a function H of the order m', we find

$$\begin{aligned} m(m-1)\, U\Gamma H - (m-1)^2 \partial^2 H = \quad & m'(m'-1)\,\Phi H \\ & -2\ (m'-1)\,\vartheta\nabla H \\ & + \qquad\qquad \vartheta^2 \square H; \end{aligned}$$

and in particular if H is the Hessian, then writing $m' = 3m-6$, and putting $U=0$, we find the before-mentioned expression for $\partial^2 H$.

61. But we may also from the general identical equation deduce the expression for $(\partial H)^2$. In fact taking H a function of the degree m' and writing

$$(\alpha,\ \beta,\ \gamma) = (\partial_x H,\ \partial_y H,\ \partial_z H),$$

we have

$$\begin{aligned} m(m-1)\, U(a, \ldots \mathfrak{X}\mu\partial_z H - \nu\partial_y H,\ \nu\partial_x H - \lambda\partial_z H,\ \lambda\partial_y H - \mu\partial_x H)^2 - (m-1)^2 (\partial H)^2 \\ = m'^2 \Phi H^2 - 2m'\vartheta H \nabla H + \vartheta^2 (\mathfrak{A}, \ldots \mathfrak{X}\partial_x H,\ \partial_y H,\ \partial_z H)^2; \end{aligned}$$

and if H be the Hessian, then writing $m' = 3m-6$ and putting also $U=0$, we find the before-mentioned expression for $(\partial H)^2$.

62. Proof of equation

$$\partial_2 = -\frac{1}{m-1}(x\partial_x + y\partial_y + z\partial_z) + \frac{\vartheta}{m-1}\nabla.$$

We have

$$\partial_2 = \partial . \partial = \{(B\nu - C\mu)\,\partial_x + (C\lambda - A\nu)\,\partial_y + (A\mu - B\lambda)\,\partial_z\}.$$

$$\big(\lambda(C\partial_y - B\partial_z) + \mu(A\partial_z - C\partial_x) + \nu(B\partial_x - A\partial_y)\big),$$

which is

$$= \lambda(C'\partial_y - B'\partial_z) + \mu(A'\partial_z - C'\partial_x) + \nu(B'\partial_x - A'\partial_y),$$

where

$$\begin{aligned} A' = \partial A &= a(B\nu - C\mu) + h(C\lambda - A\nu) + g(A\mu - B\lambda) \\ &= \lambda(hC - gB) + \mu(gA - aC) + \nu(aB - hA), \end{aligned}$$

with the like values for B' and C'. Substituting the values

$$(m-1)(A,\ B,\ C) = (ax + hy + gz,\ hx + by + fz,\ gx + fy + cz),$$

we have

$$(m-1)\,A' = \lambda(\mathfrak{G}y - \mathfrak{H}z) + \mu(\mathfrak{F}y - \mathfrak{B}z) + \nu(\mathfrak{C}y - \mathfrak{F}z);$$

and similarly

$$\begin{aligned} (m-1)\,B' &= \lambda(\mathfrak{A}z - \mathfrak{G}x) + \mu(\mathfrak{H}z - \mathfrak{F}x) + \nu(\mathfrak{G}z - \mathfrak{C}x), \\ (m-1)\,C' &= \lambda(\mathfrak{H}x - \mathfrak{A}y) + \mu(\mathfrak{B}x - \mathfrak{H}y) + \nu(\mathfrak{F}x - \mathfrak{G}y), \end{aligned}$$

and then

$$\begin{aligned}(m-1)(C'\partial_y - B'\partial_z) = \quad & \lambda\,[(\mathfrak{H}x - \mathfrak{A}y)\partial_y - (\mathfrak{A}z - \mathfrak{G}x)\partial_z] \\ & + \mu\,[(\mathfrak{B}x - \mathfrak{H}y)\partial_y - (\mathfrak{H}z - \mathfrak{F}x)\partial_z] \\ & + \nu\,[(\mathfrak{F}x - \mathfrak{G}y)\partial_y - (\mathfrak{G}z - \mathfrak{C}x)\partial_z] \\ = \quad & \lambda\,[x(\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\,\mathfrak{X}\partial_x,\ \partial_y,\ \partial_z) - \mathfrak{A}\,(x\partial_x + y\partial_y + z\partial_z)] \\ & + \mu\,[x(\mathfrak{H},\ \mathfrak{B},\ \mathfrak{F}\,\mathfrak{X}\partial_x,\ \partial_y,\ \partial_z) - \mathfrak{H}\,(x\partial_x + y\partial_y + z\partial_z)] \\ & + \nu\,[x(\mathfrak{G},\ \mathfrak{F},\ \mathfrak{C}\,\mathfrak{X}\partial_x,\ \partial_y,\ \partial_z) - \mathfrak{G}\,(x\partial_x + y\partial_y + z\partial_z)] \\ = \quad & x(\mathfrak{A},\ \ldots\mathfrak{X}\lambda,\ \mu,\ \nu\,\mathfrak{X}\partial_x,\ \partial_y,\ \partial_z) - (\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\,\mathfrak{X}\lambda,\ \mu,\ \nu)(x\partial_x + y\partial_y + z\partial_z);\end{aligned}$$

that is

$$(m-1)(C'\partial_y - B'\partial_z) = x\nabla - (\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\,\mathfrak{X}\lambda,\ \mu,\ \nu)(x\partial_x + y\partial_y + z\partial_z),$$

and so

$$(m-1)(A'\partial_z - C'\partial_x) = y\nabla - (\mathfrak{H},\ \mathfrak{B},\ \mathfrak{F}\,\mathfrak{X}\lambda,\ \mu,\ \nu)(x\partial_x + y\partial_y + z\partial_z),$$

$$(m-1)(B'\partial_x - A'\partial_y) = z\nabla - (\mathfrak{G},\ \mathfrak{F},\ \mathfrak{C}\,\mathfrak{X}\lambda,\ \mu,\ \nu)(x\partial_x + y\partial_y + z\partial_z);$$

whence

$$\begin{aligned}(m-1)\,\partial_3 &= (\lambda x + \mu y + \nu z)\,\nabla - (\mathfrak{A},\ \ldots\mathfrak{X}\lambda,\ \mu,\ \nu)^2\ (x\partial_x + y\partial_y + z\partial_z), \\ &= \mathfrak{I}\nabla - \Phi\,(x\partial_x + y\partial_y + z\partial_z);\end{aligned}$$

or finally

$$\partial_3 = -\frac{1}{m-1}\,\Phi\,(x\partial_x + y\partial_y + z\partial_z) + \frac{\mathfrak{I}}{m-1}\,\nabla.$$

63. This leads to the expression for $\partial_3^2 U$; we have

$$\begin{aligned}\partial_3^2 = \quad & \frac{1}{(m-1)^2}\,\Phi^2(x\partial_x + y\partial_y + z\partial_z)^2 \\ & - \frac{2\mathfrak{I}}{(m-1)^2}\,\Phi\nabla\,(x\partial_x + y\partial_y + z\partial_z) \\ & + \frac{\mathfrak{I}^2}{(m-1)^2}\,\nabla^2;\end{aligned}$$

and operating herewith on U, we find

$$\begin{aligned}\partial_3^2 U = \quad & \frac{m(m-1)}{(m-1)^2}\,\Phi^2 U \\ & - \frac{2(m-1)\mathfrak{I}}{(m-1)^2}\,\Phi\nabla U \\ & + \frac{\mathfrak{I}^2}{(m-1)^2}\,\nabla^2 U;\end{aligned}$$

or since

$$\nabla U = \frac{\vartheta}{m-1} H, \quad \nabla^2 U = H\Phi,$$

this is

$$\partial_2^2 U = \frac{mU}{(m-1)^2}\Phi^2 + \frac{\vartheta^2}{(m-1)^2} H\Phi.$$

64. We have $\partial_1\partial_2 U = 0$, and thence

$$(\partial_1^2\partial_2 + \partial_1\partial_3 + \partial_2^2)\, U = 0,$$

that is

$$\partial_1\partial_3 U = -\partial_1^2\partial_2 U - \partial_2^2 U;$$

or substituting the values of $\partial_1^2\partial_2 U$ and $\partial_2^2 U$, we find the value of $\partial_1\partial_3 U$ as given in the Table. And then from the equation

$$(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2 + \partial_4)\, U = 0,$$

or

$$\partial_4 U = -(\partial_1^4 + 6\partial_1^2\partial_2 + 4\partial_1\partial_3 + 3\partial_2^2)\, U,$$

we find the value of $\partial_4 U$, and the proof of the expressions in the Table is thus completed.

65. Proof of equation $\nabla . \partial = 0$.

We have

$$\begin{aligned}\nabla . \partial &= \nabla . \big((B\nu - C\mu)\,\partial_x + \quad (C\lambda - A\nu)\,\partial_y + \quad (A\mu - B\lambda)\,\partial_z\big)\\ &= \nabla . \big(A\,(\mu\partial_z - \nu\partial_y) + \quad B\,(\nu\partial_x - \lambda\partial_z) + \quad C\,(\lambda\partial_y - \mu\partial_x)\big)\\ &= \quad \nabla\, A\,(\mu\partial_z - \nu\partial_y) + \nabla B\,(\nu\partial_x - \lambda\partial_z) + \nabla C\,(\lambda\delta_y - \mu\partial_x);\end{aligned}$$

and then

$$\nabla A = (\mathfrak{A}, \ldots \mathfrak{X}\lambda,\ \mu,\ \nu\mathfrak{X} a,\ h,\ g) = H\lambda,$$

$$\nabla B = (\mathfrak{A}, \ldots \mathfrak{X}\lambda,\ \mu,\ \nu\mathfrak{X} h,\ b,\ f) = H\mu,$$

$$\nabla C = (\mathfrak{A}, \ldots \mathfrak{X}\lambda,\ \mu,\ \nu\mathfrak{X} g,\ f,\ c) = H\nu;$$

or substituting these values, we have the equation in question.

66. Proof of the expression for ∂_3.

We have

$$\partial_2 = -\frac{1}{m-1}\Phi\,(x\partial_x + y\partial_y + z\partial_z) + \frac{\vartheta}{m-1}\nabla;$$

and thence operating on the two sides respectively with ∂_1, $= \partial$, we have

$$\partial_3 = -\frac{1}{m-1}\{\partial\Phi\,(x\partial_x + y\partial_y + z\partial_z) + \Phi\partial\,.\,(x\partial_x + y\partial_y + z\partial_z)\}$$

$$+\frac{1}{m-1}\{\partial\vartheta\nabla + \vartheta\partial\,.\,\nabla\};$$

or since

$$\partial\,.\,(x\partial_x + y\partial_y + z\partial_z) = \partial,\ \partial\mathfrak{H} = 0,$$

this is

$$\partial_3 = -\frac{1}{m-1}\,\partial\Phi\,(x\partial_x + y\partial_y + z\partial_z) - \frac{1}{m-1}\,\Phi\partial + \frac{\mathfrak{H}}{m-1}\,\partial\,.\,\nabla.$$

67. Proof of expression for $\partial_3 H$.

Operating with ∂_3 upon H, we have at once

$$\partial_3 H = -\frac{3m-6}{m-1}\,H\partial\Phi - \frac{1}{m-1}\,\Phi\partial H + \frac{\mathfrak{H}}{m-1}\,(\partial\,.\,\nabla)\,H.$$

The remainder of the present Appendix is preliminary, or relating to the investigation of the expressions for $\partial_1\partial_2{}^2 U$ and $\partial_1{}^2\partial_3 U$, used *ante*, No. 31.

68. Proof of equation $\nabla^2\partial U = \Phi\partial H - H\partial\Phi$.

We have identically

$$(\mathfrak{A},\ \ldots\between\lambda,\ \mu,\ \nu)^2(\mathfrak{A},\ \ldots\between\partial_x,\ \partial_y,\ \partial_z)^2 - [(\mathfrak{A},\ \ldots\between\lambda,\ \mu,\ \nu\between\partial_x,\ \partial_y,\ \partial_z)]^2$$
$$= (abc - \&c.)\,(a,\ \ldots\between\nu\partial_y - \mu\partial_z,\ \lambda\partial_z - \nu\partial_x,\ \mu\partial_x - \lambda\partial_y)^2;$$

that is

$$\Phi\square - \nabla^2 = H\Gamma;$$

and then multiplying by ∂, and with the result operating on U, we find

$$\Phi\square\partial U - \nabla^2\partial U = H\Gamma\partial U.$$

Now

$$\square U = (\mathfrak{A},\ \ldots\between\partial_x,\ \partial_y,\ \partial_z)^2\ U$$
$$= (\mathfrak{A},\ \ldots\between a,\ b,\ c,\ 2f,\ 2g,\ 2h);$$

and thence

$$\square\partial U = (\mathfrak{A},\ \ldots\between\partial a,\ \partial b,\ \partial c,\ 2\partial f,\ 2\partial g,\ 2\partial h);$$

and observing that

$$H = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

and thence that

$$\partial H = \begin{vmatrix} \partial a, & \partial h, & \partial g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} + \begin{vmatrix} a, & h, & g \\ \partial h, & \partial b, & \partial f \\ g, & f, & c \end{vmatrix} + \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ \partial g, & \partial f, & \partial c \end{vmatrix},$$
$$= (\mathfrak{A},\ \mathfrak{H},\ \mathfrak{G}\between\partial a,\ \partial h,\ \partial g) + (\mathfrak{H},\ \mathfrak{B},\ \mathfrak{F}\between\partial h,\ \partial b,\ \partial f) + (\mathfrak{G},\ \mathfrak{F},\ \mathfrak{C}\between\partial g,\ \partial f,\ \partial c),$$
$$= (\mathfrak{A},\ \ldots\between\partial a,\ \partial b,\ \partial c,\ 2\partial f,\ 2\partial g,\ 2\partial h,$$

we see that

$$\square\partial U = \partial H.$$

Moreover

$$\begin{aligned}\Gamma U = &\ (a, \ldots \between \nu\partial_y - \mu\partial_z, \ldots)^2\, U,\\ = &\ a\,(\ b\nu^2 + c\mu^2 - 2f\mu\nu\,)\\ &+ b\,(\ c\lambda^2 + a\nu^2 - 2g\nu\lambda\,)\\ &+ c\,(\ a\mu^2 + b\lambda^2 - 2h\lambda\mu)\\ &+ 2f\,(-f\lambda^2 + g\lambda\mu + h\lambda\nu - a\mu\nu)\\ &+ 2g\,(\ f\lambda\mu - g\mu^2 + h\mu\nu - b\nu\lambda\,)\\ &+ 2h\,(\ f\nu\lambda + g\nu\mu - h\nu^2 - c\lambda\mu);\end{aligned}$$

and thence

$$\begin{aligned}\Gamma\partial U = &\ (a, \ldots \between \nu\partial_y - \mu\partial_z, \ldots)^2\, \partial U,\\ = &\ a\,(\nu^2\partial b + \mu^2\partial c - 2\mu\nu\partial f)\\ &+ \&c.\\ = &\ \lambda^2\,(b\partial c + c\partial b - 2f\partial f)\\ &+ \&c.\\ = &\ (\partial\mathfrak{A}, \partial\mathfrak{B}, \partial\mathfrak{C}, \partial\mathfrak{F}, \partial\mathfrak{G}, \partial\mathfrak{H} \between \lambda, \mu, \nu)^2,\end{aligned}$$

that is

$$\Gamma\partial U = \partial\Phi.$$

Hence the equation

$$\Phi\square\partial U - \nabla^2\partial U = H\Gamma\partial U$$

becomes

$$\Phi\partial H - \nabla^2\partial U = H\partial\Phi,$$

that is

$$\nabla^2\partial U = \Phi\partial H - H\partial\Phi.$$

69. Proof of equation $\partial_1\partial_2^2 U = \dfrac{\vartheta^2}{(m-1)^2}(\Phi\partial H - H\partial\Phi)$.

We have

$$\begin{aligned}\partial_2^2 = &\ \frac{1}{(m-1)^2}\,\Phi^2\,(x\partial_x + y\partial_y + z\partial_z)^2\\ &- \frac{2\vartheta}{(m-1)^2}\,\Phi\ (x\partial_x + y\partial_y + z\partial_z)\,\nabla\\ &+ \frac{\vartheta^2}{(m-1)^2}\,\nabla^2;\end{aligned}$$

and thence multiplying by ∂_1, $=\partial$, and with the result operating upon U, we find

$$\partial_1\partial_2^2 U = \frac{(m-1)(m-2)}{(m-1)^2}\,\Phi^2\partial U - \frac{2(m-2)}{(m-1)^2}\,\vartheta\Phi\partial\nabla U + \frac{\vartheta^2}{(m-1)^2}\,\partial\nabla^2 U.$$

But $\partial U = 0$, and thence also $\nabla(\partial U) = 0$, that is $(\nabla.\partial)\,U + \nabla\partial U = 0$; moreover $\nabla.\partial = 0$, and therefore $(\nabla.\partial)\,U = 0$, whence also $\nabla\partial U = 0$. Therefore

$$\partial_1\partial_2^2 U = \frac{\vartheta^2}{(m-1)^2}\,\partial\nabla^2 U;$$

or substituting for $\partial\nabla^2 U$ its value $= \Phi\partial H - H\partial\Phi$, we have the required expression for $\partial_1\partial_2^2 U$.

70. Proof of equation

$$\partial_1^2\partial_3 U = \frac{\vartheta^2}{(m-1)^2}\left((3m-6)H\partial\Phi + (-m+3)\Phi\partial H\right) + \frac{\vartheta^3}{(m-1)^3}\{-(\partial\,.\,\nabla)H\}.$$

We have

$$\partial_3 = -\frac{1}{m-1}\partial\Phi\,(x\partial_x + y\partial_y + z\partial_z) - \frac{1}{m-1}\Phi\partial + \frac{\vartheta}{m-1}\partial\,.\,\nabla,$$

and thence multiplying by $\partial_1^2 = \partial^2$, and operating on U,

$$\partial_1^2\partial_3 U = -\frac{m-2}{m-1}\partial\Phi\partial^2 U - \frac{1}{m-1}\Phi\partial^3 U + \frac{\vartheta}{m-1}(\partial\,.\,\nabla)\partial^2 U.$$

To reduce $(\partial\,.\,\nabla)\partial^2 U$, we have

$$\begin{aligned}\partial(\nabla\partial^2 U) &= \nabla\partial^3 U + (\partial\,.\,\nabla\partial^2)\,U,\\ &= \nabla\partial^3 U + [(\partial\,.\,\nabla)\partial^2 + \nabla\,(\partial\,.\,\partial^2)]\,U,\\ &= \nabla\partial^3 U + (\partial\,.\,\nabla)\partial^2 U + 2\nabla\partial\partial_2 U,\end{aligned}$$

and since

$$\partial_2 = -\frac{1}{m-1}\Phi\,(x\partial_x + y\partial_y + z\partial_z) + \frac{\vartheta}{m-1}\nabla\,;$$

multiplying by $\nabla\partial$, and with the result operating on U, we obtain

$$\nabla\partial\partial_2 U = -\frac{m-2}{m-1}\Phi\nabla\partial U + \frac{\vartheta}{m-1}\nabla^2\partial U;$$

or since $\nabla\partial U = 0$, this is

$$\nabla\partial\partial_2 U = \frac{\vartheta}{m-1}\nabla^2\partial U.$$

Hence

$$\partial(\nabla\partial^2 U) = \nabla\partial^3 U + (\partial\,.\,\nabla)\partial^2 U + \frac{2\vartheta}{m-1}\nabla^2\partial U,$$

that is

$$(\partial\,.\,\nabla)\partial^2 U = \partial(\nabla\partial^2 U) - \nabla\partial^3 U - \frac{2\vartheta}{m-1}\nabla^2\partial U.$$

Substituting this value of $(\partial\,.\,\nabla)\partial^2 U$, we find

$$\begin{aligned}\partial_1^2\partial_3 U = &-\frac{m-2}{m-1}\partial\Phi\partial^2 U - \frac{1}{m-1}\Phi\partial^3 U\\ &+\frac{\vartheta}{m-1}\left(\partial(\nabla\partial^2 U) - \nabla\partial^3 U\right)\\ &+\frac{\vartheta^2}{(m-1)^2}(-2\nabla^2\partial U),\end{aligned}$$

the three lines whereof are to be separately further reduced.

71. For the first line we have

$$\partial^2 U = -\frac{\vartheta^2}{(m-1)^2} H, \quad \partial^3 U = -\frac{\vartheta^2}{(m-1)^2} \partial H,$$

and hence

$$\text{first line of } \partial_1^2 \partial_3 U = \frac{\vartheta^2}{(m-1)^2} \left((m-2) H \partial\Phi + \Phi \partial H\right).$$

72. For the second line, we have

$$\nabla (\partial^2 U) = \nabla \partial^2 U + 2 (\nabla . \partial) \partial U$$

$$= \nabla \partial^2 U, \text{ since } \nabla . \partial = 0, \text{ and therefore } (\nabla . \partial) \partial U = 0;$$

that is

$$\nabla \partial^2 U = \nabla (\partial^2 U) = \nabla \left(\frac{mU}{m-1} \Phi - \frac{\vartheta^2}{(m-1)^2} H\right),$$

$$= \frac{m}{m-1} (U \nabla \Phi + \Phi \nabla U) - \frac{1}{(m-1)^2} (\vartheta^2 \nabla H + 2\vartheta H \nabla \vartheta);$$

or writing

$$U = 0, \quad \nabla U = \frac{\vartheta}{m-1} H, \quad \nabla \partial = \Phi,$$

this is

$$\nabla \partial^2 U = \frac{(m-2)\vartheta}{(m-1)^2} H \Phi - \frac{\vartheta^2}{(m-1)^2} \nabla H,$$

whence also

$$\partial (\nabla \partial^2 U) = \frac{(m-2)\vartheta}{(m-1)^2} (H \partial\Phi + \Phi \partial H) - \frac{\vartheta^2}{(m-1)^2} \partial (\nabla H).$$

Similarly

$$\nabla \partial^3 U = \nabla (\partial^3 U),$$

$$= \nabla \left(\frac{mU}{m-1} \partial\Phi - \frac{\vartheta^2}{(m-1)^2} \partial H\right),$$

$$= \frac{m}{m-1} \left(\nabla U \partial\Phi + U \nabla (\partial\Phi)\right) - \frac{1}{(m-1)^2} \left(\vartheta^2 \nabla (\partial H) + 2\vartheta \nabla \vartheta \partial H\right);$$

or putting

$$U = 0, \quad \nabla U = \frac{\vartheta}{m-1} H, \quad \nabla \vartheta = \Phi,$$

and observing also that $\nabla (\partial H), = \nabla \partial H + (\nabla . \partial) H$ is equal to $\nabla \partial H$, that is to $\partial \nabla H$, we obtain

$$\nabla \partial^3 U = \frac{\vartheta}{(m-1)^2} (mH \partial\Phi - 2\Phi \partial H) - \frac{\vartheta^2}{(m-1)^2} \partial \nabla H;$$

and then from the above value of $\partial (\nabla \partial^2 U)$, we find

$$\partial (\nabla \partial^2 U) - \nabla \partial^3 U = \frac{\vartheta}{(m-1)^2} (-2H \partial\Phi + m\Phi \partial H) + \frac{\vartheta^2}{(m-1)^2} \left(-\partial (\nabla H) + \partial \nabla H\right);$$

or observing that the term multiplied by $\frac{\vartheta^2}{(m-1)^2}$ is $=-(\partial\,.\,\nabla)\,H$, we find

$$\text{second line of } \partial_1^2\partial^3 U = \frac{\vartheta^2}{(m-1)^3}(-2H\partial\Phi + m\Phi\partial H) + \frac{\vartheta^2}{(m-1)^3}\big(-\partial\,.\,\nabla)\,H\big).$$

73. For the third line, substituting for $\nabla^2\partial U$ its value $=\Phi\partial H - H\partial\Phi$, we have

$$\text{third line of } \partial_1^2\partial_3 U = -\frac{2\vartheta^2}{(m-1)^2}(\Phi\partial H - H\partial\Phi).$$

74. Hence, uniting the three lines, we have

$$\begin{aligned}\partial_1^2\partial_3 U = {} & \frac{\vartheta^2}{(m-1)^3}\big((m-2)\,H\partial\Phi + \Phi\partial H\big)\\ & + \frac{\vartheta^2}{(m-1)^3}\big(-2\,H\partial\Phi + m\Phi\partial H\big) + \frac{\vartheta^2}{(m-1)^3}\big(-(\partial\,.\,\nabla)\,H\big)\\ & + \frac{\vartheta^2}{(m-1)^3}\big((2m-2)\,H\partial\Phi + (-2m+2)\,\Phi\partial H\big),\end{aligned}$$

and, reducing, we have the above-mentioned value of $\partial_1^2\partial_3 U$.

342.

ON THE CONICS WHICH PASS THROUGH THREE GIVEN POINTS AND TOUCH A GIVEN LINE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 24—30.]

CONSIDER the system of conics which pass through three given points and touch a given line; if among these we select the conics which touch an assumed line, it is easy to show analytically that there are four such conics, all real or else all imaginary; viz. the three points form a triangle, and if the two lines cut the three conics produced or cut the same two sides and the third side produced, then the conics are all real; but in every other case they are all imaginary. The latter part of the theorem may also be seen geometrically; in fact, if a triangle is inscribed in a conic, say first in an ellipse, or in a parabola, or in one branch of a hyperbola, then all the tangents of the conic (and therefore any two tangents whatever) cut the three sides produced, but if the triangle is inscribed in the two branches of a hyperbola (that is, two vertices on one branch and the remaining vertex on the other branch), then all the tangents of the conic (and therefore any two tangents whatever) cut the same two sides and the third side produced: and thus the only real conics are those which cut the three sides produced, or else the same two sides and the third side produced. The analytical proof referred to is as follows: taking $(x=0,\ y=0,\ z=0)$ for the equations of the sides of the triangle, the equation of a conic through the three points is

$$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0,$$

or, what is the same thing,

$$2fyz+2gzx+2hxy=0,$$

that is

$$(0,\ 0,\ 0,\ f,\ g,\ h \between x,\ y,\ z)^2=0.$$

The inverse coefficients are

$$(-f^2,\ -g^2,\ -h^2,\quad gh,\quad hf,\quad fg),$$

and hence the condition in order that the conic may touch the line $\alpha x+\beta y+\gamma z=0$ is

$$(f^2,\quad g^2,\quad h^2,\ -gh,\ -hf,\ -fg \between \alpha,\ \beta,\ \gamma)^2=0,$$

or, what is the same thing,

$$\sqrt{(\alpha f)}+\sqrt{(\beta g)}+\sqrt{(\gamma h)}=0.$$

Similarly the condition in order that the conic may touch the line $lx+my+nz=0$ is $\sqrt{(lf)}+\sqrt{(mg)}+\sqrt{(nh)}=0$. Hence if the conic touch the two lines, we have

$$\sqrt{(f)} : \sqrt{(g)} : \sqrt{(h)}=\sqrt{(\beta n)}-\sqrt{(\gamma m)} \quad : \sqrt{(\gamma l)}-\sqrt{(\alpha n)} \quad : \sqrt{(\alpha m)}-\sqrt{(\beta l)},$$

or, what is the same thing,

$$f\ :\quad g\ :\quad h\ =\beta n+\gamma m-2\sqrt{(\beta\gamma mn)}\ :\ \gamma l+\alpha n-2\sqrt{(\gamma\alpha nl)}\ :\ \alpha m+\beta l-2\sqrt{(\alpha\beta lm)},$$

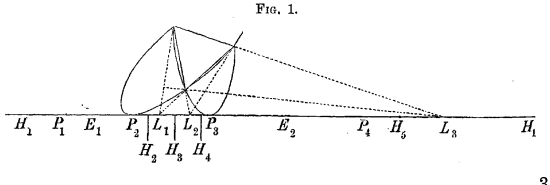
which, since the radicals must be so taken that the product may be $=\alpha\beta\gamma lmn$, gives in all four conics: and these will be all real if the signs of $(l,\ m,\ n)$ are the same with, or opposite to those of $(\alpha,\ \beta,\ \gamma)$ respectively; which proves the theorem.

In particular since infinity is a line meeting the three sides produced; if the given line meet the three sides produced, the system will contain four real parabolas; but, if the given line meets two sides and a side produced, there is not any real parabola. In the latter case, as is obvious geometrically, the conics of the system are all hyperbolas.

Any side of the triangle, and the line joining the opposite vertex with the point of intersection of the side and given line, form a pair of lines passing through the three points and meeting on the given line; such pair of lines is a conic of the system; and we have thus three pairs of lines, each pair a conic of the system.

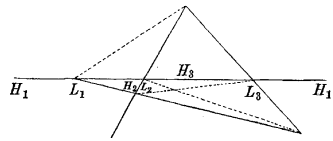
We may by what precedes form some idea of the nature of the system of conics which pass through the three given points and touch the given line. In fact writing at the point of contact the letters H, P, E, L according as the conic is a hyperbola, parabola, ellipse, or pair of lines, then if the given line cut the three sides produced, we have as in fig. 1.

FIG. 1.



Whereas, if the given line cuts two sides and a side produced, we have more simply as in fig. 2.

FIG. 2.



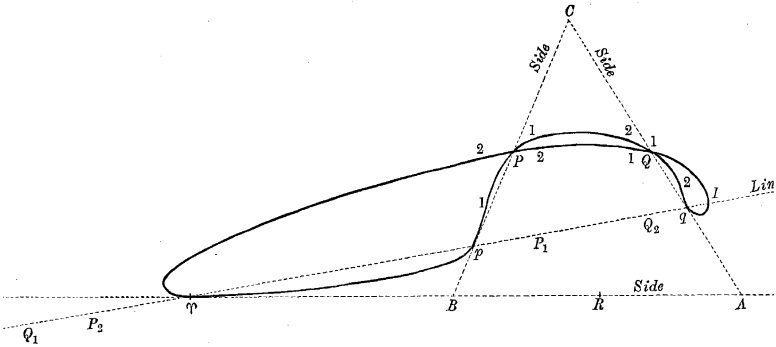
But to gain a more precise knowledge, it is proper to consider the curve which is the locus of the centres of the conics of the system.

Such locus which, as will presently be seen, is a curve of the fourth order, must, it is clear, pass through the points of intersection (L_1, L_2, L_3 in figs. 1 and 2 respectively and p, q, r in fig. 3 presently referred to) of the sides with the given line; and it is not difficult to show geometrically that it *touches*, at these points, the sides of the triangle. It may be shown also that the curve has three nodes (double points), viz. the middle point of each side of the triangle is a node of the curve. In fact if upon any side as base we apply an equal and opposite triangle so as to form with the given triangle a parallelogram, then any conic through the four vertices of the parallelogram will have for its centre the central point of the parallelogram; that is, the middle point of the side in question. But we may through the four vertices describe two conics, each of them touching the given line; that is the middle point of the side is the centre of two different conics of the system, and it is therefore a node upon the curve of centres. And moreover the node will be a crunode or an acnode (i.e. a double point with two real branches, or else a conjugate or isolated point) according as the conics are real or imaginary: and it is easy to see that if the given line does not cut the parallelogram, or if it cuts two opposite sides, the conics will be both real; but if it cuts two adjacent sides the conics will be both imaginary; that is, in the former case we have a crunode, and in the latter an acnode. Through each node may be drawn two tangents to the curve; and it is a known property of curves of the fourth order that the six points of contact lie on a conic; one of the tangents through the node is however the side whereon the node lies, and the points of contact of the three sides lie on a line, viz. the given line: hence the last mentioned conic is composed of the given line, and another line; that is, the three points of contact of the other tangents through the three nodes lie on this other line.

It is proper to add that the points at infinity of the curve of centres are the centres of the four parabolas; that is, there will be four infinite branches, if the parabolas are real, viz. if the given line cuts the three sides produced; but no infinite branch if the parabolas are imaginary, viz. if the given line cut two sides and a side produced.

The triangle and the three triangles applied to the three sides form together a triangle similar to the original triangle but of double the linear magnitude, and the form of the curve of centres depends as has been shown on the position of the given line in regard to the triangle and the double triangle. The cases to be considered are tolerably numerous, but it is easy from the foregoing considerations, to see in any particular case what is the form of the curve of centres; for facility of delineation I select a form without infinite branches, see fig. 3, in which the given line cuts the two sides CA, CB, and the third side AB produced; it is moreover to be observed

FIG. 3.



that as the figure is drawn the given line cuts the two sides CA, CB below their middle points Q and P respectively. By what precedes it appears that the middle points Q, P of these two sides CA, CB are each of them crunodes, but that the middle point R of the remaining side AB is an acnode. And this being so the general form of the curve is at once perceived to be that shown by fig. 3.

It is very interesting to trace the corresponding positions of the point of contact on the given line, and of the centre on the curves of centres. When the point of contact is at ∞, the centre is at I, as the point of contact moves from ∞ to q, the centre moves from I to q, and at q the two coincide; as the point of contact moves from q to a point Q_2, the centre moves from q to Q (along the branch $Q2$); as the point of contact moves from Q_2 to a point P_1, the centre moves from Q to P (along the branch $Q2\,1P$); as the point of contact moves from P_1 to p, the centre moves from P to p (along the branch $P1$) and at p the point of contact and the centre again coincide; as the point of contact moves from p to r, the centre moves from p to r and at r they again coincide; as the point of contact moves from r to a point P_2 the centre moves from r to P (along the branch $2P$); as the point of contact moves from P_2 to a point Q_1, the centre moves from P to Q (along the branch $P2\,1Q$) and finally as the point of contact moves from Q_1 to ∞, the centre moves from Q (along the branch Q_1) to I, thus completing the circuit.

The equation of the curve of centres was given in the late Mr Hearn's "Researches on Curves of the Second Order, &c. London, 1846," viz. if $x=0$, $y=0$, $z=0$ be the equations of the sides of the triangle formed by the given points; $x+y+z=0$ the equation of the line infinity, and $\alpha x+\beta y+\gamma z=0$ the equation of the given line, then the equation of the curve of centres is

$$\sqrt{\{\alpha x(-x+y+z)\}}+\sqrt{\{\beta y(x-y+z)\}}+\sqrt{\{\gamma z(x+y-z)\}}=0,$$

or more generally if $x+y+z=0$ be the equation of an assumed line, then this equation is that of the locus of the pole of the assumed line in regard to the conics passing through the given points and touching the given line, see my paper "Note on a Family of Curves of the Fourth Order," *Cambridge and Dublin Mathematical Journal*, t. v. (1850), pp. 148—152, [85], where I have noticed the above mentioned property, that the conic through the points of contact of the tangents through the nodes breaks up into a pair of lines. It is I think worth while to show how the equation is obtained. The equation of a conic through the given points and touching the given line is

$$(0,\ 0,\ 0,\ f,\ g,\ h\between x,\ y,\ z)^2=0$$

with the condition $\sqrt{(\alpha f)}+\sqrt{(\beta g)}+\sqrt{(\gamma h)}=0$, and this being so, the coordinates of the pole in relation thereto, of the assumed line $x+y+z=0$, are

$$\begin{aligned} x : y : z = \ & (-f+g+h)f \\ : \ & (\ \ f-g+h)g \\ : \ & (\ \ f+g-h)h. \end{aligned}$$

We have thence

$$\begin{aligned} -x+y+z \text{ proportional to } & -(-f+g+h)f \\ & +(\ \ f-g+h)g \\ & +(\ \ f+g-h)h, \end{aligned}$$

that is, to $f^2-(g-h)^2$, which is $=(f-g+h)(f+g-h)$,

and combining with this the equation

$$\alpha x \text{ proportional to } (-f+g+h)f\alpha,$$

we obtain

$$\alpha x(-x+y+z) \text{ proportional to } \alpha f,$$

that is

$$\alpha x(-x+y+z) : \beta y(x-y+z) : \gamma z(x+y-z)=\alpha f : \beta g : \gamma h,$$

so that from the equation $\sqrt{(\alpha f)}+\sqrt{(\beta g)}+\sqrt{(\gamma h)}=0$, we have at once the foregoing equation

$$\sqrt{\{\alpha x(-x+y+z)\}}+\sqrt{\{\beta y(x-y+z)\}}+\sqrt{\{\gamma z(x+y-z)\}}=0.$$

The rationalised form is

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1\between \alpha x(-x+y+z),\ \ \beta y(x-y+z),\ \ \gamma z(x+y-z))^2=0,$$

which shows what has been all along assumed, that the curve is of the fourth order. This equation may be transformed into

$$\begin{aligned}
& x^2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2-2\beta\gamma yz+2\gamma\alpha zx+2\alpha\beta xy) \\
+\ & y^2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2+2\beta\gamma yz-2\gamma\alpha zx+2\alpha\beta xy) \\
+\ & z^2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2+2\beta\gamma yz+2\gamma\alpha zx-2\alpha\beta xy) \\
-\ & 2yz(\alpha x+\beta y+\gamma z)(-\alpha x+\beta y+\gamma z) \\
-\ & 2zx(\alpha x+\beta y+\gamma z)(\ \ \alpha x-\beta y+\gamma z) \\
-\ & 2xy(\alpha x+\beta y+\gamma z)(\ \ \alpha x+\beta y-\gamma z)=0:
\end{aligned}$$

if with this equation we combine the equation $\alpha x+\beta y+\gamma z=0$, we find at the points of intersection with the given line

$$x^2 . 4\beta\gamma yz+y^2 . 4\gamma\alpha zx+z^2 . 4\alpha\beta xy=0,$$

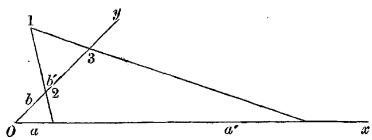
that is

$$xyz(\beta\gamma x+\gamma\alpha y+\alpha\beta z)=0,$$

so that the points in question are the intersections of the given line $\alpha x+\beta y+\gamma z=0$, with the lines $x=0$, $y=0$, $z=0$, $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$. The point $\alpha x+\beta y+\gamma z=0$, $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$ corresponds to the conic which touches the given line at its intersection with the assumed line $x+y+z=0$, the pole in relation to this conic is obviously a point on the given line. The point in question, if $x+y+z=0$ denote the line infinity, is the point I of fig. 3.

It may be proper to mention a far less symmetrical form of the equation of the conic, but which has the advantage of putting in evidence the point of contact; viz. the equation is expressed in terms of the parameter α denoting the distance of the point of contact from a given point in the base line, and which is therefore very convenient for tracing the changes of form of the conic. Assuming as before that the base line cuts the sides produced, then (see fig. 4) if of the three points 1 denote

FIG. 4.



that which is furthest from, and 2 that which is nearest to the base line, and if the base line be taken as the axis of x, and 23 as the axis of y; the equation of the base line is $y=0$, and the equations of the sides 23, 31, 12 are $x=0$, $\frac{x}{a'}+\frac{y}{b'}=1$, $\frac{x}{a}+\frac{y}{b}=1$, where a, b, a', b', $a'-a$, $b'-b$, $a'b-ab'$ are all positive, so that, by choosing

the axes as above, we avoid the consideration of the several cases corresponding to different signs of these quantities. And this being so, if $x=\alpha$ is the coordinate of the point of contact, the equation of the conic is

$$(A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ 1)^2 = 0,$$

where

$$\begin{aligned}
A &= 2bb'\ \ (a'-a),\\
B &= 2\alpha^2\ \ (a'-a),\\
C &= 2\alpha^2 bb'(a'-a),\\
F &= -\ \alpha^2\ \ (a'-a)(b'+b),\\
G &= -2\alpha bb'\ (a'-a),\\
H &= \{(\alpha^2+aa')(b'-b)-2\alpha(ab'-a'b)\},
\end{aligned}$$

and these give

$$\begin{aligned}
AB-H^2 &= -[\{(\alpha-a)\sqrt{(b')}+(\alpha-a')\sqrt{(b)}\}^2-(a'-a)(a'b-ab')]\times\\
&\qquad [\{(\alpha-a)\sqrt{(b')}-(\alpha-a')\sqrt{(b)}\}^2-(a'-a)(a'b-ab')],\\
BC-F^2 &= -\alpha^4(a'-a)^2(b'-b)^2,\\
CA-G^2 &= 0,\\
GH-AF &= -2bb'(a'-a)(b'-b)\alpha(\alpha-a)(\alpha-a'),\\
HF-BG &= -(a'-a)(b'-b)\alpha^2\{(b'+b)(\alpha^2+aa')-2\alpha(ab'+a'b)\},\\
FG-CH &= -2bb'(a'-a)(b'-b)\alpha^2(\alpha-a)(\alpha-a').
\end{aligned}$$

The condition that the conic may be a parabola is $AB-H^2=0$, which gives, as it should do, four real values of α.

2, *Stone Buildings, W.C.*

343.

ON THE CUSP OF THE SECOND KIND OR NODECUSP.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 74, 75.]

THE so-called cusp of the second kind or ramphoid cusp, is not an ordinary singularity of plane curves, but it is a singularity of a higher order. It is however particularly considered in Plücker's *Theorie der Analytischen Curven*, 1839; and it is there, not only in the analytical discussion of the singularities of plane curves, but in the author's theory of the generation of a curve, considered as described and enveloped by a point moving along a line which at the same time rotates round the point; when the motion along the line vanishes, we have a cusp; when the motion round the point vanishes, we have an inflexion; when the two motions vanish together, we have a cusp of the second kind, which thus presents itself as a singularity uniting the characters of a cusp or stationary point, and an inflexion or stationary tangent: (I remark in passing that in this explanation it is not clear what is the independent variable wherewith the motions are compared). But there is another point of view from which the singularity in question may be considered, viz., it may be regarded as a singularity arising from the union and amalgamation of a cusp, and a double point or node; in fact, in the figure, which represents a curve having a cusp

and also a node, we have only to imagine the node approaching nearer and nearer to and ultimately coinciding with the cusp, and it will be at once seen that the point will become a cusp of the second kind; or as it might properly, with reference to

this generation of it, be termed, a "nodecusp." It is to be noticed that in the point-theory of curves, there is between the cusp and the nodecusp the intermediate singularity of the tacnode, which arises from the union and amalgamation of two nodes, and possesses the character of a cusp.

I return to the nodecusp; taking the point in question as the origin, and the tangent for the axis of x, the equation will be a specialised form of the equation

$$\tfrac{1}{2}y^2 + \tfrac{1}{6}(a,\ b,\ c,\ d \between x,\ y)^3 + \tfrac{1}{24}(a',\ b',\ c',\ d',\ e' \between x,\ y)^4 + \&c. = 0,$$

which belongs to the case of a cusp, viz. (see Plücker, p. 165) the conditions satisfied by the special form are $a = 0$, $a' = 3b^2$, or the equation is

$$\tfrac{1}{2}(y + \tfrac{1}{2}bx^2)^2 + \tfrac{1}{6}y^2(3cx + dy) + \tfrac{1}{24}y(4b'x^3 + 6c'x^2y + 4d'xy^2 + e'y^3) + \&c. = 0,$$

which is most easily verified, by observing that (this being so) the expansion of y in terms of x will be of the form

$$y = -\tfrac{1}{2}bx^2 + Ax^{\frac{5}{2}} + \&c.$$

It is now to be shown how the foregoing conditions $a = 0$, $a' = 3b^2$, are obtained by assuming that the curve has, besides the cusp, a node which ultimately coincides with the cusp. Let $(\alpha,\ \beta)$ be the coordinates of the node; we must have

$$\begin{aligned}
\tfrac{1}{2}\beta^2 + \tfrac{1}{6}(a,\ b,\ c,\ d \between \alpha,\ \beta)^3 + \tfrac{1}{24}(a',\ b',\ c',\ d',\ e' \between \alpha,\ \beta)^4 + \&c. &= 0,\\
\tfrac{1}{2}(a,\ b,\ c \between \alpha,\ \beta)^2 \quad + \tfrac{1}{6}(a',\ b',\ c',\ d' \between \alpha,\ \beta)^3 \quad + \&c. &= 0,\\
\beta + \tfrac{1}{2}(b,\ c,\ d \between \alpha,\ \beta)^2 \quad + \tfrac{1}{6}(b',\ c',\ d',\ e' \between \alpha,\ \beta)^3 \quad + \&c. &= 0.
\end{aligned}$$

Assume $\beta = m\alpha^2$, and then let α vanish; the equations become in the first instance

$$\begin{aligned}
\tfrac{1}{6}a\alpha^3 + (\tfrac{1}{2}m^2 + \tfrac{1}{2}bm + \tfrac{1}{24}a')\,\alpha^4 + \&c. &= 0,\\
\tfrac{1}{2}a\alpha^2 + \&c. \qquad\qquad &= 0,\\
(m + \tfrac{1}{2}b)\,\alpha^2 + \&c. \qquad\qquad &= 0,
\end{aligned}$$

the second and third equation give $a = 0$, $m + \tfrac{1}{2}b = 0$, and the first equation then gives

$$\tfrac{1}{2}m^2 + \tfrac{1}{2}bm + \tfrac{1}{24}a' = 0;$$

or substituting for m its value $= -\tfrac{1}{2}b$, this is $a' = 3b^2$, or the required conditions are $a = 0$, $a' = 3b^2$, *ut suprà*. The single condition $a = 0$ corresponds to the case of the tacnode.

2, *Stone Buildings, W.C., September* 17, 1862.

344.

ON CERTAIN DEVELOPABLE SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 108—126.(1)]

IF $U=0$ be the equation of a developable surface, or say a developable, then the Hessian HU vanishes, not identically, but only by virtue of the equation $U=0$ of the surface; that is, HU contains U as a factor, or we may write $HU=U.PU$; the function PU, which for the developable replaces as it were the Hessian HU, is termed the *Prohessian;* and (since if r be the order of U the order of HU is $4r-8$) we have $3r-8$ for the order of the Prohessian. If $r=4$, the order of the Prohessian is also 4, and in fact, as is known, the Prohessian is in this case $=U$. The Prohessian is considered, but not in much detail, in Dr Salmon's *Geometry of Three Dimensions*, (1862), pp. 338 and 426 [Ed. 4 (1882), p. 408]: the theorem given in the latter place is almost all that is known on the subject. I call to mind that the tangent plane along a generating line of the developable meets the developable in this line taken 2 times, and in a curve of the order $r-2$; the line touches the curve at the point of contact, or say the ineunt, on the edge of regression, and besides meets it in $r-4$ points. The ineunt taken 3 times, and the $r-4$ points form a linear system of the order $r-1$, and the Hessian of this system (considered as a curve of one dimension, or binary quantic) is a linear system of $2r-6$ points; viz. it is composed of the ineunt taken 4 times, and of $2r-10$ other points. This being so, the theorem is that the generating line meets the Prohessian in the ineunt taken 6 times, in the $r-4$ points, and in the $2r-10$ points $(6+r-4+2r-10=3r-8)$; it is assumed that $r=5$ at least.

The developables which first present themselves are those which are the envelopes of a plane

$$(a, b, \ldots \between t, 1)^n=0,$$

[1] Presented to the Royal Society and read 27 Nov., 1862, but withdrawn by permission of the Council.

where t is an arbitrary parameter, and the coefficients $(a, b, \ldots)$ are linear functions of the coordinates; the equation of the developable is

$$\text{Disct.}\ (a, b, \ldots \between t, 1)^n = 0,$$

the discriminant being taken in regard to the parameter t. Such developable is in general of the order $2n-2$, but if the second coefficient b is $=0$, or, more generally, if it is a mere numerical multiple of a, then a will divide out from the equation, and we have a developable of the order $2n-3$: the like property of course exists in regard to the last but one, and the last, of the coefficients of the function. We thus obtain developables of the orders 4, 5, and 6, sufficiently simple to allow of the actual calculation of their Prohessians, and the chief object of the present Memoir is to exhibit these Prohessians; but the Memoir contains some other researches in relation to the developables in question.

Quartic Developable, Nos. 1 *to* 6.

1. I consider first the developable of the fourth order

$$U = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2,$$

derived from the cubic function $(a, b, c, d \between t, 1)^3$, and which is in fact the general quartic developable.

2. Taking (a, b, c, d) as coordinates and omitting common numerical factors, the first derived functions are

$$\begin{aligned} &ad^2 - 3bcd + 2c^3\ , \\ &-3acd + 6b^2d - 3bc^2, \\ &-3abd + 6ac^2 - 3b^2c, \\ &a^2d - 3abc + 2b^3\ , \end{aligned}$$

(quantities which, if $(X, Y, Z, W \between t, 1)^3$ denote the cubicovariant of $(a, b, c, d \between t, 1)^3$, are equal to $(-W, 3Z, -3Y, X)$ respectively). And the second derived functions are

$$\begin{array}{llll} d^2\ , & -3cd\ , & -3bd + 6c^2, & 2ad - 3bc, \\ -3cd\ , & 12bd - 3c^2, & -3ad - 6bc, & -3ac + 6b^2, \\ -3bd + 6c^2, & -3ad - 6bc, & 12ac - 3b^2, & -3ab\ , \\ 2ad - 3bc, & -3ac + 6b^2, & -3ab\ , & a^2\ . \end{array}$$

3. Representing these by

$$\begin{array}{llll} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & N, \\ L, & M, & N, & P, \end{array}$$

and expressing the determinant in the partially developed form

$$
\begin{aligned}
= \; & (AM - LH)(FN - CM) \\
& + (AN - LG)(FM - BN) \\
& + (AP - L^2)(BC - F^2) \\
& + (HN - GM)^2 \\
& + (HP - LM)(FG - CH) \\
& + (GP - LN)(HF - BG),
\end{aligned}
$$

and proceeding to the calculation, we find

$AM - LH$	$FN - CM$	$AN - LG$	$FM - BN$	$AP - L^2$	$BC - F^2$
$= 3\times$	$= 9\times$	$= 3\times$	$= 9\times$	$= 3\times$	$= 9\times$
$acd^2 + 1$	$a^2bd + 1$	$abd^2 + 1$	$a^2cd + 1$	$a^2d^2 - 1$	$a^2d^2 - 1$
$b^2d^2 + 2$	$a^2c^2 + 4$	$ac^2d - 4$	$ab^3d + 2$	$abcd + 4$	$abcd + 12$
$bc^2d - 3$	$ab^2c - 7$	$b^2cd - 3$	$abc^2 + 1$	$b^2c^2 - 3$	$ac^3 - 4$
	$b^4 + 2$	$bc^3 + 6$	$b^3c - 4$		$b^3d - 4$
					$b^2c^2 - 3$

$HN - GM$	$HP - LM$	$FG - CH$	$GP - LN$	$HF - BG$
$= 18\times$	$= 3\times$	$= 9\times$	$= 3\times$	$= 9\times$
$ac^3 + 1$	$a^2cd + 1$	$abd^2 + 1$	$a^2bd + 1$	$acd^2 + 1$
$b^3d + 1$	$ab^3d - 4$	$ac^2d + 2$	$a^2c^2 + 2$	$b^2d^2 + 4$
$b^3c^2 - 2$	$abc^3 - 3$	$b^3cd + 1$	$ab^3c - 3$	$bc^3d - 7$
	$b^3c + 6$	$bc^3 - 4$		$c^4 + 2$

4. Hence, forming the six parts and collecting, we find

$27\times$

a^4d^4	$+1$			$+1$		
a^3bcd^3	-12	$+1$	$+1$	-16	$+1$	$+1$
$a^3c^3d^2$	$+8$	$+4$	-4	$+4$	$+2$	$+2$
$a^2b^3d^3$	$+8$	$+2$	$+2$	$+4$	-4	$+4$
$a^2b^2c^2d^2$	$+30$	-2	-10	$+54$	-10	-2
a^2bc^4d	-48	-12	$+2$	-16	-10	-12
a^3c^6	$+16$			$+12$		$+4$
ab^4cd^2	-48	-12	-10	-16	$+2$	-12
ab^3c^3d	$+68$	$+21$	$+25$	-48	$+24$ $+25$	$+21$
ab^2c^5	-24		$+6$	$+12$	-48 $+12$	-6
b^6d^2	$+16$	$+4$		$+12$		
b^5c^3d	-24	-6	$+12$	$+12$	-48 $+6$	
b^4c^4	$+9$		-24	$+9$	$+48$ -24	

where the first column is the Hessian. This is in fact $= U^2$, and hence the Prohessian is

$$PU = U = \begin{vmatrix} a^2d^2 & +1 \\ abcd & -6 \\ ad^3 & +4 \\ b^3c & +4 \\ b^2c^2 & -3 \end{vmatrix}$$

5. To complete the theory it is proper to calculate the inverse coefficients

$$\begin{matrix} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L}, \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M}, \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N}, \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P}. \end{matrix}$$

We have for example

$$\mathfrak{P} = ABC - AF^2 - BG^2 - CH^2 + 2FGH,$$

which is found to be

$$9 \times$$

$$= \begin{vmatrix} a^2d^4 & -1 \\ abcd^3 & +6 \\ ac^3d^2 & -4 \\ b^3d^3 & -16 \\ b^2c^2d^2 & +39 \\ bc^4d & -36 \\ c^6 & +12 \end{vmatrix}$$

which, omitting the factor 9, is

$$= 3(ad^2 - 3bcd + 2c^3)^2 - 4d^2(a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2),$$

that is $= 3W^2 - 4d^2U$; and calculating in like manner the other coefficients, the system is found to be

$$\begin{matrix} 3X^2 - 4a^2U, & 3XY - 4abU, & 3XZ + (2ac - 6b^2)U, & 3XW + (5ad - 9bc)U, \\ 3YX - 4abU, & 3Y^2 - 4acU, & 3YZ - (1ad + 3bc)U, & 3YW + (2bd - 6c^2)U, \\ 3ZX + (2ac - 6b^2)U, & 3ZY - (1ad + 3bc)U, & 3Z^2 - 4bdU, & 3ZW - 4cdU \\ 3WX + (5ad - 9bc)U, & 3WY + (2bd - 6c^2)U, & 3WZ - 4cdU, & 3W^2 - 4d^2U \end{matrix}$$

6. Let $(\lambda, \mu, \nu, \rho)$ be any arbitrary multipliers, and write

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = \begin{pmatrix} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P} \end{pmatrix} (\lambda, \mu, \nu, \rho),$$

then if $\theta = \mathfrak{Z}(\lambda X + \mu Y + \nu Z + \rho W)$, we have

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = (\theta X + \alpha U, \ \theta Y + \beta U, \ \theta Z + \gamma U, \ \theta W + \delta U).$$

The function $(X, Y, Z, W \between t, 1)^3$ is the cubicovariant of $(a, b, c, d \between t, 1)^3$ and if for a moment these functions are represented by v, u respectively, and if we also write $U = a^3 d^3 - \&c. = \tilde{U}(a, b, c, d)$, then

$$(a + \theta X, \ b + \theta Y, \ c + \theta Z, \ d + \theta W \between t, 1)^3 = u + \theta v,$$

and thence

$$\begin{aligned} \tilde{U}(a + \theta X, \ b + \theta Y, \ c + \theta Z, \ d + \theta W) &= \text{Disct.}\,(u + \theta v), \\ &= (1 - \theta^2 U)^2 \, U, \end{aligned}$$

by a formula given in my "Fifth Memoir on Quantics," *Phil. Trans.*, t. CXLVIII. (1858), see p. 442 [156]; the function on the left-hand side thus contains U as a factor, and it at once follows that the function

$$\tilde{U}(a + \mathfrak{X}, \ b + \mathfrak{Y}, \ c + \mathfrak{Z}, \ d + \mathfrak{W});$$

viz., the function obtained from U by writing therein $(a + \mathfrak{X}, \ b + \mathfrak{Y}, \ c + \mathfrak{Z}, \ d + \mathfrak{W})$ in the place of (a, b, c, d) respectively, contains U as a factor, and therefore vanishes if $U = 0$; that is $a + \mathfrak{X}$, $b + \mathfrak{Y}$, $c + \mathfrak{Z}$, $d + \mathfrak{W}$, are the coordinates of a point on the surface $U = 0$; they are in fact the coordinates of a point on the generating line through (a, b, c, d); this is a theorem which applies to any developable whatever, as appears by the following considerations.

Remarks on the General Theory of Developables, Nos. 7 to 9.

7. In general for any surface whatever, taking a point on the surface, the successive polars of this point (the last of them being the tangent plane) all touch at this point; and not only so, but the tangents to the two branches of the curve in which the surface itself (or any of its polars down to the quadric polar) is intersected by the ultimate polar or tangent plane, are respectively coincident. Suppose that for any point on the surface, the quadric polar becomes a cone: the vertex of this cone is *not* the point itself; hence the tangent plane at the point touches the cone along a generating line; that is the tangents to the curve of intersection with the surface, or with any of its polars, coincide with the generating line of the cone—and the curve of intersection of the tangent plane with the surface, or any of its polars, at the point of contact (instead of, as in general, a node) has a cusp. In particular the curve of intersection with the surface has at the point of contact a cusp. The condition that the quadric polar may be a cone is $HU = 0$, and when this differential equation is satisfied in virtue of the equation $U = 0$ (that is, when we have identically $HU = U.PU$), the surface is a developable. Now all that is proved in the first instance by the equation $HU = 0$ is that *every point* of the surface has the above mentioned property; viz., that the tangent plane at the point cuts the surface in a curve having a cusp at the point in question.

8. What really happens in the case of a developable is more than this; viz. the curve of intersection is made up of the generating line taken twice, and of a curve of an order less by 2 than the order of the surface. Let (x, y, z, w) be the coordinates of the point on the developable, $U=0$ the equation of the developable, $(A, B, C, P, F, G, H, L, M, N)$ the second derived functions of U, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N})$ the inverse system, K the determinant formed with the second derived functions, so that we have $K=HU=0$. The coordinates of the vertex of the cone are given by the equations

$$\begin{aligned} \alpha : \beta : \gamma : \delta &= \mathfrak{A} : \mathfrak{H} : \mathfrak{G} : \mathfrak{L}, \\ &= \mathfrak{H} : \mathfrak{B} : \mathfrak{F} : \mathfrak{M}, \\ &= \mathfrak{G} : \mathfrak{F} : \mathfrak{C} : \mathfrak{N}, \\ &= \mathfrak{L} : \mathfrak{M} : \mathfrak{N} : \mathfrak{P}; \end{aligned}$$

these several sets of ratios being equivalent to each other in virtue of the equation $K=0$. Hence $(\lambda, \mu, \nu, \rho)$ being arbitrary multipliers, if we write

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = \left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P} \end{array} \right) (\lambda, \mu, \nu, \rho);$$

the coordinates of the vertex of the cone will be as $\mathfrak{X} : \mathfrak{Y} : \mathfrak{Z} : \mathfrak{W}$, and hence observing that the absolute magnitudes of these quantities are arbitrary, $x+\mathfrak{X} : y+\mathfrak{Y} : z+\mathfrak{Z} : w+\mathfrak{W}$ will represent the coordinates of any point on the line joining the point (x, y, z, w) with the vertex of the cone, that is, the generating line through the point (x, y, z, w); which is the theorem in question, the coordinates being in the present investigation denoted by (x, y, z, w) instead of the (a, b, c, d) of the example.

9. Reverting to the developable $U=a^2d^2-\&c.=0$, the results previously obtained show that the coordinates of the vertex of the cone which is the quadric polar of the point (a, b, c, d) are as $X : Y : Z : W$ (these quantities denoting as above the coefficients of the cubicovariant), and thence also that the coordinates of any point on the generating line will be as $a+\theta X : b+\theta Y : c+\theta Z : d+\theta W$, where θ is arbitrary.

Special Quintic Developable, Nos. 10 *to* 25.

10. We have, secondly, the developable of the fifth order

$$U = a^2e^2 + 6a^2c^2e - 24ab^2ce + 9ac^4 + 16b^4e - 8b^2c^2 = 0,$$

derived from the quartic function $(a, 2b, 3c, 0, -27e\between t, 1)^4$, or, what is the same thing, $at^4 + 8bt^3 + 18ct^2 - 27e = 0$, where it will be observed that, as well in the quartic function as in the equation of the developable, the sum of the numerical coefficients is = zero; it was on this account that the foregoing form of the quartic function was selected in preference to the form $(a, b, c, 0, e\between t, 1)^4$. The last mentioned form has for its discriminant

$$(ae + 3c^2)^3 - 27\,(ace - b^2e - c^3)^2, \quad = e\,(a^3e^2 - 18a^2c^2e + 54ab^2ce + 81ac^4 - 27b^4e - 54b^2c^3),$$

and writing therein in place of (a, b, c, e), the values $(a, 2b, 3c, -27e)$, the second factor divided by 729 gives the foregoing expression for U, belonging to the form

$$(a, 2b, 3c, 0, -27e \between t, 1)^4.$$

11. Taking (a, b, c, e) as coordinates, and omitting common numerical factors, the first derived functions of U are

$$\begin{aligned} &3a^2e^2 + 12ac^2e - 24b^2ce + 9c^4 \,, \\ &-48abce + 64b^3e - 16bc^3 \,, \\ &12a^2ce - 24ab^2e + 36ac^3 - 24b^2c^2, \\ &2a^3e + 6a^2c^2 - 24ab^2c + 16b^4 \,, \end{aligned}$$

and the second derived functions are

$$\begin{array}{llll} 3(ac^2+2c^2e) \,, & -24bce \,, & 6(2ace-2b^2e+3c^3), & 3(a^2e+2ac^2-4b^2c), \\ -24bce \,, & 8(-3ace+12b^2e-c^3), & 24(-abe-bc^2) \,, & 8(-3abc+4b^3) \,, \\ 6(2ace-2b^2e+3c^3), & 24(-abe-bc^2) \,, & 6(a^2e+9ac^2-4b^2c), & 6(a^2c-2ab^3) \,, \\ 3(a^2e+2ac^2-4b^2c), & 8(-3abc+4b^3) \,, & 6(a^2c-2ab^2) \,, & a^2 \,. \end{array}$$

12. Representing these by

$$\begin{array}{llll} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & N, \\ L, & M, & N, & P, \end{array}$$

and expressing the determinant in the partially developed form

$$\begin{aligned} &P(ABC - AF^2 - BG^2 - CH^2 + 2FGH) \\ &- L^2(BC - F^2) - M^2(CA - G^2) - N^2(AB - H^2) \\ &- 2MN(GH - AF) - 2NL(HF - BG) - 2LM(FG - CH), \end{aligned}$$

then, proceeding to the calculation, we have

$BC-F^2$ $=48\times$	$CA-G^2$ $=18\times$	$AB-H^2$ $=24\times$	$GH-AF$ $=72\times$	$HF-BG$ $=48\times$	$FG-CH$ $=144\times$	$ABC-$&c. $=288\times$
a^2ce^2 -3	a^3e^2 $+1$	a^2ce^3 -3	a^2be^3 $+1$	a^2ce^2 $+6$	a^2bce^2 -1	a^4ce^4 -3
a^2c^3e -28	$a^2c^2e^2$ $+3$	ab^2e^3 $+12$	abc^2e^2 -1	ab^2ce^2 -18	ab^3c^2 $+2$	$a^3c^3e^2$ -10
ab^2c^2e $+96$	ab^2ce^2 $+12$	ac^3e^2 -7	b^3ce^2 $+4$	ac^3e $+11$	abc^3e $+4$	$a^2b^2c^2e^3$ $+24$
ac^3 -9	ac^4e -6	c^5e -2	bc^4e -4	b^4e^2 $+24$	b^3c^2e -2	$a^3c^5e^2$ $+15$
b^4ce -48	b^4e^2 -8			b^2c^3e -26	bc^5 -3	ab^3ce^3 $+72$
b^2c^4 -8	b^2c^5e $+16$			c^6 $+3$		$ab^2c^4e^2$ -168
	c^6 -18					ac^7e $+60$
						b^6e^3 -96
						$b^4c^3e^2$ -200
						b^3c^5e -112
						c^9 $+18$

$L^2 = 9\times$	$M^2 = 64\times$	$N^2 = 96\times$	$2MN = 96\times$	$2NL = 36\times$	$2LM = 48\times$
$a^4e^2 + 1$	$a^2b^2c^2 + 9$	$a^4c^2 + 1$	$a^3bc^2 - 3$	$a^4ce + 1$	$a^3bce - 3$
$a^3c^2e + 4$	$ab^4c - 24$	$a^2b^2c - 4$	$a^2b^3e + 10$	$a^3b^2e - 2$	$a^2b^2e + 4$
$a^2b^2ce - 8$	$b^6 + 16$	$a^2b^4 + 4$	$ab^5 - 8$	$a^3c^3 + 2$	$a^2bc^3 - 6$
$a^2c^4 + 4$				$a^2b^2c^2 - 8$	$ab^2c^2 + 20$
$ab^2c^3 - 16$				$ab^4c + 8$	$b^3c - 16$
$b^4c^2 + 16$					

13. It is now easy to form the seven component terms of the determinant, and thence the determinant itself; each of the component terms divides by 576, and omitting this factor, the sum of the seven terms divides by 2; the result is

a^7ce^4	$+ 3$	$- 3$	$+ 9$					
$a^6c^3e^2$	$+ 28$	$- 10$	$+ 120$		$+ 18$		$- 72$	
$a^5b^2c^2e^3$	$- 96$	$+ 24$	$- 360$	$- 72$	$- 144$	$+ 144$	$+ 360$	$- 144$
$a^5c^5e^2$	$+ 90$	$+ 15$	$+ 399$		$+ 42$		$- 276$	
$a^4b^4ce^3$	$+ 24$	$+ 72$	$+ 144$	$+ 192$	$+ 360$	$- 480$	$- 720$	$+ 480$
$a^4b^2c^4e^2$	$- 384$	$- 168$	$- 1944$	$- 216$	$- 168$	$- 144$	$+ 1584$	$+ 288$
a^4c^7e	$+ 108$	$+ 60$	$+ 444$		$+ 12$		$- 300$	
$a^3b^6e^3$	$+ 32$	$- 96$		$- 128$	$- 288$	$+ 384$	$+ 576$	$- 384$
$a^3b^4c^3e^2$	$+ 568$	$+ 200$	$+ 3024$	$- 288$	$- 168$	$+ 1056$	$+ 3504$	$+ 480$
$a^3b^2c^6e$	$- 224$	$- 112$	$- 2616$	$+ 432$	$- 48$	$- 576$	$+ 1752$	$+ 720$
a^3c^9	$+ 27$	$+ 18$	$+ 108$				$- 72$	
$a^2b^6c^2e^2$	$+ 384$		$- 1152$	$+ 2496$		$- 2304$	$+ 4032$	$- 2304$
$a^2b^4c^5e$	$- 696$		$+ 6336$	$- 2304$	$+ 48$	$+ 1920$	$- 3552$	$- 3840$
$a^2b^2c^3$	$+ 192$		$- 336$	$+ 1296$			$+ 288$	$- 864$
ab^8ce^2	$- 1152$			$- 3072$		$+ 1536$	$- 2304$	$+ 1536$
ab^6c^4e	$+ 1440$		$- 6912$	$+ 3840$		$- 1536$	$+ 2496$	$+ 4992$
ab^4c^7	$- 408$		$+ 48$	$- 3456$			$- 288$	$+ 2880$
$b^{10}e^2$	$+ 512$			$+ 1024$				
b^8c^3e	$- 640$		$+ 2304$	$- 2048$				$- 1536$
b^6c^6	$+ 192$		$+ 384$	$+ 2304$				$- 2304$

where the first column is the Hessian.

14. This divides as it should do by

$$U = \boxed{\begin{array}{l} a^3e^2 + 1 \\ a^2c^3e + 6 \\ ab^2ce - 24 \\ ac^4 + 9 \\ b^4e + 16 \\ b^2c^3 - 8 \end{array}}$$

and the quotient, which is the Prohessian, is found to be

$$PU = \boxed{\begin{array}{l} a^4ce^2 + 3 \\ a^3c^3e + 10 \\ a^3b^2c^2e - 24 \\ a^2c^5 + 3 \\ ab^4ce - 24 \\ ab^2c^4 + 24 \\ b^6e + 32 \\ b^4c^3 - 24 \end{array}}$$

15. But before discussing the Prohessian, I will further consider the developable itself. Regarding it as derived from the equation $(a, 2b, 3c, 0, -27e \between t, 1)^4 = 0$, we have

$$eU = (27)^3 \{(ae - c^2)^3 + (-3ace + 4b^3e - c^3)^2\} = 0,$$

and observing that

$$-3ace + 4b^3e - c^3 = c(ae - c^2) - 4e(ac - b^2),$$

it appears that the equations of the cuspidal curve or edge of regression of the developable are $ae - c^2 = 0$, $ac - b^2 = 0$ (so that the cuspidal curve is a curve of the fourth order, the intersection of two quadric surfaces, or say a quadri-quadric curve). This is perhaps better seen by writing the equation of the developable in the form

$$U = a(ae - c^2)^2 - 8c(ae - c^2)(ac - b^2) + 16e(ac - b^2)^2 = 0,$$

or what is the same thing

$$U = (a, c, e \between ae - c^2, 4ac - 4b^2)^2 = 0,$$

where the discriminant of the quadric function is $= ae - c^2$, which vanishes for the curve $(ae - c^2 = 0, ac - b^2 = 0)$.

16. Another form of the equation is

$$U = a(ae + 3c^2)^2 - 8b^2(3ace - 2b^3e + c^3) = 0,$$

which shows that the conic $ae + 3c^2 = 0$, $b = 0$, is a nodal curve on the developable.

And, again, another form is

$$U = c^3(9ac - 8b^2) + e(a^2e + 6a^2c^2 - 24ab^2c + 16b^4) = 0,$$

which shows that the conic $9ac - 8b^2 = 0$, $e = 0$, is a simple line on the developable.

17. In my paper "On the Developable Surfaces which arise from Two Surfaces of the Second Order," *Camb. and Dub. Math. Jour.*, t. v. (1850), pp. 46—57, [84], I considered first the developable having for its edge of regression the intersection of two quadric surfaces; in the general case the developable is of the order 8; but if the two surfaces have an ordinary contact it is of the order 6; and if they have a singular contact (as there explained) it is of the order 5. And in the last mentioned case, if the equations of the two quadric surfaces are taken to be $x^2 - 2wz = 0$, $y^2 - 2zx = 0$, then the equation of the developable was found to be

$$4z^3w^2 + 12z^2x^2w + 9zx^4 - 24zxy^2w - 4x^3y^2 + 8y^4w = 0,$$

which putting therein $z = a$, $y = 2b$, $x = 2c$, $w = 2e$, becomes

$$a^3e^3 + 6a^2c^2e - 24ab^2ce + 9ac^4 + 16b^4e - 8b^2c^3 = 0,$$

which is the before mentioned developable $U = 0$; the two equations $x^2 - 2wz = 0$, $y^2 - 2zx = 0$ become by the same substitution $ae - c^2 = 0$, $ac - b^2 = 0$. We have in fact already seen that the developable $U = 0$ has this curve for its edge of regression.

18. But in the paper just referred to, it is also shown that considering the developable which is the envelope of the common tangent planes of two quadric surfaces; in the general case the developable is of the order 8, but if the two surfaces have an ordinary contact it is of the order 6, and if they have a singular contact it is of the order 5.

In the last mentioned case the surfaces may without loss of generality be reduced to conics, and their equations may be taken to be $(y^2-2zx=0,\ w=0)$ and $(x^2-2zw=0,\ y=0)$[1], and this being so the equation of the developable is

$$32z^3w^2-32z^2x^2w+72zxy^2w+8zx^4-27y^4w-4x^3y^2=0.$$

This is really a developable of the same kind with the first mentioned developable of the order 5; for writing $x=12c,\ y=8b,\ z=3a,\ w=-8e$, the equation becomes

$$a^3c^2+6a^2c^2e-24ab^2ce+9ac^4+16b^4e-8b^2c^3=0,$$

which is the before mentioned developable $U=0$. The equations of the two conics become $(9ac-8b^2=0,\ e=0)$ and $(ae+3c^2=0,\ b=0)$, and the developable is thus the envelope of the common tangent planes of these two conics. It has been seen that the first conic is a simple line, but the second conic a nodal line, on the developable.

19. Recapitulating, the developable of the fifth order $U=0$, which is the envelope of the plane $(a,\ 2b,\ 3c,\ 0,\ -27e)(t,\ 1)^4=0$ is the locus of the tangents of the quadriquadric curve $(ae-c^2=0,\ ac-b^2=0)$, and it is also the envelope of the common tangent planes of the conics $(9ac-8b^2=0,\ e=0)$ and $(ae+3c^2=0,\ b=0)$.

20. Returning now to the Prohessian, its equation may be written in the form

$$PU=(3a^2c,\ -3b^2c,\ ace+2b^3e \between ae-c^2,\ 4ac-4b^2)^2=0,$$

and the discriminant of the quadric function is

$$3a^2ce(ac+2b^2)-9b^4c^2,$$

which is

$$=3c\,\{(a^2e+3b^3c)(ac-b^2)+3ab^2(ae-c^2)\},$$

and recollecting that the equations of the cuspidal curve or edge of regression of the developable are $ae-c^2=0,\ ac-b^2=0$, it thus appears that the curve in question is also a cuspidal curve on the Prohessian.

21. Consider for a moment the surface

$$(3a^2c,\ -3b^2c\theta,\ ace+2\theta b^2e \between ae-c^2,\ 4ac-4b^2)^2=0,$$

[1] These are in fact the conics made use of, p. 57 in the paper above referred to [and p. 495 in the reprint], but the equations are by mistake given as $(x^2-2yz=0,\ w=0)$, $(y^2-2zw=0,\ x=0)$, that is, x and y are interchanged.

where θ is an arbitrary parameter; this is a surface having for a nodal curve the cuspidal curve of the developable; but if the discriminant of the quadric function vanishes, that is if

$$3a^2ce\,(ac+2b^2\theta)-9b^4c^2\theta^2=0,$$

for $ae-c^2=0$, $ac-b^2=0$, then the curve in question will be a cuspidal curve on the surface. But the last mentioned equation is

$$3c\,[(a^2e+3b^2c\theta^2)\,(ac-b^2)+ab^2\,\{(1+2\theta)\,ae-3\theta^2c^2\}]=0,$$

which for $ae-c^2=0$, $ac-b^2=0$, becomes $1+2\theta-3\theta^2=0$, that is $\theta=1$, which gives the Prohessian, or $\theta=-\frac{1}{3}$.

22. For the latter value the surface is

$$(9a^2c,\ 3b^2c,\ 3ace-2b^2e \between ae-c^2,\ 4ac-4b^2)^2=0,$$

or, expanding, the surface $(\theta=-\frac{1}{3})$ is

$$\left|\begin{array}{ll} a^4ce^2 & +\ 9 \\ a^3c^2e & +\ 30 \\ a^2b^2a^2e & -\ 104 \\ a^2c^5 & +\ 9 \\ ab^4ce & +\ 88 \\ ab^2c^4 & -\ 24 \\ b^6e & -\ 36 \\ b^4c^3 & -\ 24 \end{array}\right| =0,$$

the before mentioned discriminant being

$$=c\,\{(3a^2e-b^2c)\,(ac-b^2)+ab^2\,(ae-c^2)\}\,;$$

but I have not further examined the geometrical signification of this surface, or inquired into its relation to the Prohessian.

23. The equation of the Prohessian may be written

$$PU=(ac-b^2)\,\{16e\,(ac-b^2)^2+3a\,(ae-c^2)^2\}+b^2U=0,$$

or what is the same thing

$$PU=(ac-b^2)\,\{a\,(ae+3c^2)\,(3ae+c^2)-32ab^2ce+16b^4e\}+b^2U=0,$$

the latter of which shows that the conic $(ae+3c^2=0,\ b=0)$, which is the nodal line of the developable, is a simple line on the Prohessian.

24. Consider the curve of intersection of the developable and the Prohessian; this is of the order 5×7, $=35$. We have $ac-b^2=0$, $U=0$, or else

$$16e\,(ac-b^2)^2+3a\,(ae-c^2)^2=0,\ \ U=0.$$

Consider for a moment the second system, this is

$$3a\,(ae-c^2)^2+\qquad\qquad 16e\,(ac-b^2)^2=0,$$
$$3a\,(ae-c^2)^2-24c\,(ae-c^2)\,(ac-b^2)+48e\,(ac-b^2)^2=0,$$

which give

$$(ac-b^2)\{3c(ae-c^2)-4e(ac-b^2)\}=0,\ U=0,$$

and these are equivalent to

$$(ac-b^2=0,\ U=0) \text{ and } \{3c(ae-c^2)-4e(ac-b^2)=0,\ U=0\},$$

so that the entire intersection is made up of $(ac-b^2=0,\ U=0)$ twice, and of $\{4e(ac-b^2)-3c(ae-c^2)=0,\ U=0\}$ once.

25. The first part is at once seen to give

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	4	times, order	16
the line	$(a=0,\ b=0)$	4	„ „	4
				20

The second part gives

$$(ae-c^2)\{4c(ac-b^2)+\ a(ae-c^2)\}=0,$$
$$\{4e(ac-b^2)-3c(ae-c^2)\}=0,$$

this consists of 1° the part $ae-c^2=0,\ e(ac-b^2)=0$, viz.

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	once,	order	4
the line	$(c=0,\ e=0)$	twice,	„	2
				6

and 2° the part

$$4c(ac-b^2)+\ a(ae-c^2)=0,$$
$$4e(ac-b^2)-3c(ae-c^2)=0,$$

which contains

the cuspidal curve $(ac-b^2=0,\ ae-c^2=0)$ once, order 4,

and by writing the two equations in the form

$$c(ae+3c^2)-4b^2e=0,$$
$$a(ae+3c^2)-4b^2c=0,$$

it is clear that it contains also

the nodal curve	$(ae+3c^2,\ b=0)$	twice,	order	4
and the line	$(c=0,\ e=0)$	once,	„	1
				9

whence the complete intersection of the developable and the Prohessian is made up as follows, viz.

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	6	times, order	24
the nodal curve	$(ae+3c^2=0,\ b=0)$	2	times, „	4
the line	$(a=0,\ b=0)$	4	times, „	4
the line	$(a=0,\ e=0)$	3	times, „	3
				35

Special Sextic Developable, Nos. 26 *to* 35.

26. Lastly, we have the developable

$$U = a^3e^3 - 12a^3bde^2 - 27a^2d^4 - 6ab^2d^2e - 27b^4e^2 - 64b^3d^3 = 0,$$

derived from the quartic function $(a, b, 0, d, e \between t, 1)^4$; the discriminant is in fact $(ae - 4bd)^3 - 27(-ad^2 - b^2e)^2$, which is equal to the foregoing value of U.

27. Taking a, b, d, e as coordinates, then omitting common numerical factors, the first derived functions are

$$\begin{aligned} a^2e^3 &- 8abde^2 - 18ad^4 - 2b^2d^2e, \\ -4a^2de^2 &- 4abd^3e - 36b^3e^2 - 64b^2d^3, \\ -4a^2be^2 &- 36a^2d^3 - 4ab^2de - 64b^3d^2, \\ a^3e^2 &- 8a^2bde - 2ab^2d^2 - 18b^4e, \end{aligned}$$

and the second derived functions (changing, for greater convenience, the signs) are

$$\begin{aligned} &2(-ae^3 + 4bde^2 + 9d^4), \quad 4(2ad^2 + bd^3e), \quad 4(2abe^2 + 18ad^3 + b^2de), \quad -3a^2e^2 + 16abde + 2b^2d^2, \\ &4(2ade^2 + bd^2e), \quad 4(ad^2e + 27b^2e^2 + 32bd^3), \quad 4(a^2e^2 + 2abde + 48b^2d^2), \quad 4(2a^2de + abd^2 + 18b^3e), \\ &4(2abe^2 + 18ad^3 + b^2de), \quad 4(a^2e^2 + 2abde + 48b^2d^2), \quad 4(27a^2d^2 + ab^2e + 32b^3d), \quad 4(2a^2be + ab^2d), \\ &-3a^2e^2 + 16abde + 2b^2d^2, \quad 4(2a^2de + abd^2 + 18b^3e), \quad 4(2a^2be + ab^2d), \quad 2(-a^3e + 4a^2bd + 9b^4). \end{aligned}$$

28. Representing these by

$$\begin{matrix} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & M, \\ L, & M, & N, & P, \end{matrix}$$

and employing for the determinant the same partially developed form as in the first example, then proceeding to the calculation, we find

$AM - LH$ $= 4\times$	$FN - CM$ $= 16\times$	$AN - LG$ $= 4\times$	$FM - BN$ $= 16\times$	$AP - L^2$ $= 1\times$	$BC - F^2$ $= 16\times$	$HN - GM$ $= 288\times$
a^3de^4 + 2	a^4be^3 + 2	a^2be^4 + 2	a^4de^3 + 2	a^4e^4 − 5	a^4e^4 − 1	a^3d^4e − 2
$a^3bd^2e^3$ − 15	a^4d^3e − 54	$a^2d^3e^2$ + 54	$a^3bd^2e^2$ + 3	a^3bde^3 + 64	a^3bde^3 − 4	a^2bd^5 − 1
a^2d^5e + 36	$a^3b^2de^2$ + 3	$a^2b^2de^3$ − 15	$a^2b^2e^3$ − 36	a^3d^4e − 36	a^2d^4e + 27	ab^4e^3 − 2
ab^3e^4 − 36	a^3bd^4 − 27	a^2bd^4e − 252	$a^2b^2d^3e$ + 33	$a^2b^2d^2e^2$ − 180	$a^2b^2d^2e^2$ + 630	ab^2d^3e − 18
$b^2d^3e^2$ − 12	$a^2b^3d^2e$ − 453	$ab^3d^3e^2$ − 12	ab^4de^2 + 9	a^2bd^5 + 144	a^2bd^5 + 864	b^5de^2 − 1
abd^6 + 18	ab^5e^2 − 18	ab^2d^5 − 18	ab^3d^4 + 16	ab^4e^3 − 36	ab^4e^3 + 27	
b^4de^3 + 144	ab^4d^2 + 16	b^4d^3e − 2	b^5d^3e + 864	ab^3d^2e − 64	ab^3d^3e − 128	
b^3d^4e + 322	b^6de − 576			b^5de^2 + 144	b^5de^2 + 864	
				b^4d^4 + 320	b^4d^4 − 1280	
+ 459	− 1107	− 243	+ 891	+ 351	− 999	− 24

$HP-LM$ $=4\times$	$FG-CH$ $=16\times$	$GP-LN$ $=4\times$	$HF-BG$ $=16\times$
a^4de^3 + 2	a^3be^4 + 2	a^4be^3 + 2	a^3de^4 + 2
$a^3bd^2e^2$ − 15	$a^3d^3e^2$ − 36	a^4d^3e − 36	$a^2bd^3e^3$ + 3
$a^2b^3e^3$ + 54	$a^2b^2de^3$ + 3	$a^2b^3de^2$ − 15	a^2d^5e − 18
$a^2b^3d^3e$ − 12	a^2bd^4e + 9	a^2bd^4 + 144	ab^3e^4 − 54
ab^4de^2 − 252	$ab^3d^2e^2$ + 33	$a^2b^3d^2e$ − 12	$ab^3d^3e^2$ − 453
ab^3d^4 − 2	ab^3d^3 + 864	ab^5e^2 + 36	abd^6 − 576
b^3d^2e − 18	b^4d^3e + 16	ab^4d^3 + 322	b^4de^3 − 27
		b^6de + 18	b^3d^4e + 16
− 243	+ 891	+ 459	− 1107

where, by way of verification, I have annexed to each column the sum of the coefficients.

29. Forming from these the seven component terms of the determinant, and collecting, the result divides by 5, and we have

a^9e^8 + 1			+ 5			
a^7bde^7 + 4	+ 16	+ 16	− 44		+ 16	+ 16
$a^7d^3e^5$ − 135	− 432	+ 432	− 99		− 288	− 288
$a^6b^2d^2e^6$ − 722	− 96	− 96	− 3226		− 96	− 96
$a^5bd^3e^4$ + 432	+ 3312	− 1368	− 2592		+ 2232	+ 576
$a^5d^5e^3$ + 2916	− 7776		− 972	+ 20736		+ 2592
$a^5b^4e^7$ − 135	− 288	− 288	− 99		+ 432	− 432
$a^5b^3d^2e^2$ + 6784	+ 3876	− 7788	+ 41744		− 7788	+ 3876
$a^5b^2d^6e^3$ + 21492	+ 4788	+ 3960	+ 27180		+ 8100	+ 63432
a^5bd^9e + 11664	− 7776		− 27216	+ 20736		+ 72576
$a^4b^5de^5$ + 432	+ 576	+ 2232	− 2592		− 1368	+ 3312
$a^4b^4d^4e^4$ + 352	+ 2524	+ 36220	− 117200	+ 41472	+ 36220	+ 2524
$a^4b^3d^7e^2$ − 26352	− 133272	− 30024	− 61920	+ 373248	− 51984	− 227808
$a^4b^2d^{10}$ − 40824	− 1944		+ 124416	+ 5184		− 331776
$a^3b^6d^2e^5$ + 21492	+ 63432	+ 8100	+ 27180		+ 3960	+ 4788
$a^3b^4d^5e^3$ + 50944	+ 6504	+ 177576	− 154912	+ 41472	+ 177576	+ 6504
$a^3b^4d^8e$ + 11232	− 65088	− 18504	− 19008	+ 186624	− 41544	+ 13680
$a^2b^8e^6$ + 2916	+ 2592		− 972	+ 20736		− 7776
$a^2b^7d^3e^4$ − 26352	− 227808	− 51984	− 61920	+ 373248	− 30024	− 133272
$a^2b^6d^6e^2$ − 140624	− 668472	− 872592	+ 689024	+ 1689984	− 872592	− 668472
$a^2b^3d^9$ − 131328	+ 1152	− 1152	+ 92160		− 6912	− 741888
ab^9de^5 + 11664	+ 72576		− 27216	+ 20736		− 7776
$ab^8d^3e^3$ + 11232	+ 13680	− 41544	− 19008	+ 186624	− 18504	− 65088
ab^7d^7e − 25088	− 20864	− 62336	+ 40960		− 62336	− 20864
$b^{10}d^2e^4$ − 40824	− 331776		+ 124416	+ 5184		− 1944
$b^9d^5e^2$ − 131328	− 741888	− 6912	+ 92160		− 1152	+ 1152
b^8d^8 − 81920			− 409600			
+ 492075	− 2032452	− 866052	+ 350649	+ 2985984	− 866052	− 2032452

where the first column is the Hessian.

30. This divides, as it should do, by

$$U = \boxed{\begin{array}{lr} a^3e^3 & + \quad 1 \\ a^3bde^2 & - \quad 12 \\ a^3d^4 & - \quad 27 \\ ab^2d^3e & - \quad 6 \\ b^4e^2 & - \quad 27 \\ b^3d^3 & - \quad 64 \end{array}}$$

$$-135$$

and the other factor, which is the Prohessian, is

$$PU = \boxed{\begin{array}{lr} a^5e^5 & + \quad 1 \\ a^4bde^4 & + \quad 16 \\ a^4d^4e^2 & - \quad 108 \\ a^3b^2d^2e^3 & - \quad 524 \\ a^3bd^5e & - \quad 432 \\ a^2b^4e^4 & - \quad 108 \\ a^2b^2d^3e^2 & + \quad 656 \\ a^3b^2d^6 & + \quad 1512 \\ ab^5de^3 & - \quad 432 \\ ab^4d^4e & + \quad 272 \\ b^6d^2e^2 & + \quad 1512 \\ b^5d^5 & + \quad 1280 \end{array}}$$

$$+ 3645.$$

31. To simplify this, I first collect the six terms

$$\begin{aligned} &- 108\, a^2e^2 \ (a^2d^4 + b^4e^2) \\ &- 432\, abde\, (a^2d^4 + b^4e^2) \\ &+ 1512\, b^2d^2 \ (a^2d^4 + b^4e^2), \end{aligned}$$

and then putting $a^2d^4 + b^4e^2 = (ad^2 + b^2e)^2 - 2ab^2d^2e$, we have the terms

$$\begin{aligned} &+ 216\, a^3b^2d^2e^3 \\ &+ 864\, a^2b^3d^3e^2 \\ &- 3024\, ab^4d^4e\,, \end{aligned}$$

which combined with the remaining terms give

$$\begin{aligned} &+ \quad 1\, a^5e^5 \\ &+ \quad 16\, a^4bde^4 \\ &- \quad 308\, a^3b^2d^2e^3 \\ &+ 1520\, a^2b^3d^2e^2 \\ &- 2752\, ab^4d^4e \\ &+ 1280\, b^5d^5 \quad ; \end{aligned}$$

which is found to be divisible by $(ae-4bd)^3$: and we thus obtain for PU the form

$$PU=-108\,(a^2e^2+4\,abde-14b^2d^2)\,(ad^2+b^2e)^2+(a^2e^2+28\,abde-20\,b^2d^2)\,(ae-4\,bd)^3,$$

which puts in evidence that the cuspidal line $(ae-4bd=0,\ ad^2+b^2e=0)$ of the developable is also a cuspidal line of the Prohessian.

32. Writing the equations of the developable and the Prohessian under the forms

$$A^3-27\,B^2=0,$$
$$LA^3-108\,MB^2=0,$$

and substituting in the second equation $A^3=27\,B^2$, it becomes $B^2(L-4M)=0$, that is the intersection is made up of $A^3=0$, $B^2=0$, which is the cuspidal curve taken six times (order 36), and of the curve $A^3-27\,B^2=0$, $L-4M=0$ (order 24). But substituting for L, M their values, the equation $L-4M=0$ becomes

$$a^2e^2-4abde-12b^2d^2=0,$$

that is

$$(ae+2bd)\,(ae-6bd)=0,$$

so that the last mentioned curve is composed of the intersections of the developable by the two quadric surfaces

$$ae+2bd=0,\quad ae-6bd=0.$$

33. Now combining with the equation of the developable the equation $ae+2bd=0$, and observing that in consequence of the last mentioned equation we have

$$(ae-4bd)^3=(-6bd)^3=-\,216\,b^3d^3=+\,108\,ab^2d^2e,$$

the equation of the developable gives $(ad^2-b^2e)^2=0$, or we have (taken twice) the curve $ae+2bd=0$, $ad^2-b^2e=0$, which is a curve of the sixth order made up of the lines $(a=0,\ b=0)$, $(d=0,\ e=0)$, and of a quartic curve (an excubo-quartic[1]) the nodal line on the developable. If in like manner with the equation of the developable we combine the equation $ae-6bd=0$, then from this equation we have

$$(ae-4bd)^3=(2bd)^3=8b^3d^3=\tfrac{4}{3}ab^2d^2e,$$

and the equation of the developable then gives

$$(ad^2+b^2e^2)-\tfrac{4}{81}\,ab^2d^2e=0\,;$$

that is,

$$ad^2+\theta b^2e=0,\quad ad^2+\frac{1}{\theta}\,b^2e=0,\quad \text{if}\ \ \theta+\frac{1}{\theta}=2-\tfrac{4}{81}=\tfrac{158}{81}.$$

[1] A quartic curve which is the complete intersection of two quadric surfaces is termed a quadriquadric; a quartic curve of the kind which is not such complete intersection but can only be represented by means of a cubic surface is termed an excubo-quartic.

The curve $ae-6bd=0$, $ad^2+\theta b^2e=0$ is made up of the lines $(a=0,\ b=0)$, $(d=0,\ e=0)$, and of an excubo-quartic, and the curve $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2e=0$ is made up of the same two lines and of an excubo-quartic.

34. Hence we see that the intersection of the developable and the Prohessian which is of the order $(6+10=)\,60$ is made up as follows, viz.,

cuspidal curve $ae-4bd=0$, $ad^2+b^2e=0$,	taken	6	times,	$6\times 6=36$
line $(a=0,\ b=0)$	„	4	„	$1\times 4=\ 4$
line $(d=0,\ e=0)$	„	4	„	$1\times 4=\ 4$
nodal curve (excubo-quartic) $ae+2bd=0$, $ad^2-b^2e=0$	„	2	„	$4\times 2=\ 8$
excubo-quartic $ae-6bd=0$, $ad^2+\theta b^2e=0$	„	1	„	$4\times 1=\ 4$
excubo-quartic $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2e=0$	„	1	„	$4\times 1=\ 4$
				60

35. It is to be added that a generating line of the developable meets the Prohessian in the ineunt on the cuspidal edge taken 6 times, in a point of the nodal line taken 2 times, viz. the $r-4$ points (r being here $=6$) of the general theorem, in a point of the excubo-quartic $ae-6bd=0$, $ad^2+\theta b^2e=0$, and in a point of the excubo-quartic $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2=0$, (these being the $2r-10$ points of the general theorem); we have thus $(6+2+2=)\,10$ points of intersection of the generating line with the Prohessian.

345.

ON THE INFLEXIONS OF THE CUBICAL DIVERGENT PARABOLAS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 199—203.]

THE five divergent parabolas, species 67 to 71, of Newton's *Enumeratio Linearum tertii Ordinis* (1704), are included under the general equation $y^2 = ax^3 + 3bx^2 + 3cx + d$; there are two general forms, or forms without singularities, viz. the *parabola cum ovali*, sp. 67, and the *parabola pura*, sp. 71; two forms having a double point, viz. the *nodata*, sp. 68, and the *punctata*, sp. 69, according as the double point is one with real branches, or is a conjugate or isolated point; and finally the *cuspidata* or semicubical parabola, sp. 70, which has a cusp. In the nomenclature of my short note "On Curves of the Third Order," *British Assoc. Report for the Year* 1861, *Notices &c.* p. 2, the five parabolas are the *complex*, the *simplex*, the *crunodal*, the *acnodal*, and the *cuspidal*; the distinction there made of the simplex kind of cones of the third order into three subspecies, applies to the simplex parabola, and *for this particular case* was, as I have since ascertained, noticed in Murdoch's *Newtoni Genesis Curvarum per Umbras*, 8vo. Lond. 1746, pp. 1—126. It may be remarked that in this very interesting and valuable work the number of species is given as 78, viz. the author includes the four species added by Stirling, and the other two usually considered to have been added by Cramer (one of them the author himself attributes to Cramer), and that the demonstration of Newton's theorem is effected in the most complete way by showing in what manner the five cones are each of them to be cut so as to obtain the 78 species of cubic curves.

The analytical investigation of the points of inflexion of the above-mentioned divergent parabolas, that is, the curves defined by the equation

$$y^3 = ax^3 + 3bx^2 + 3cx + d,$$

is not without interest. The result which should be obtained is, by the general theory of cubic curves, known to be as follows: there is always an inflexion at infinity, in the point where the line infinity is met by the line $x=0$ (or, what is the same thing, by any ordinate of the curve); but disregarding altogether this inflexion at infinity, then in the general case where the curve is without singularity, the remaining eight inflexions (two of them real, six imaginary) lie in pairs on four ordinates of the curve: if however the curve has an acnode, the six imaginary inflexions coincide with the acnode, viz. the three ordinates corresponding to these pass through the acnode, but there are still two real inflexions; if the curve has a crunode, four of the imaginary inflexions and the two real inflexions coincide with the crunode, viz. the three ordinates corresponding to these pass through the crunode, and there is not any real inflexion, although there are still two imaginary inflexions; finally, if the curve has a cusp, then the eight inflexions coincide with the cusp, viz. the four ordinates corresponding to these pass through the cusp, and there is no inflexion real or imaginary.

Proceeding now to the analytical investigation, if in order to form the Hessian we introduce the new coordinate $z=1$, the equation of the curve becomes

$$U=-y^2z+ax^3+3bxz^2+3cxz^2+dz^3,$$

and thence forming the second differential coefficients, and ultimately replacing z by its value, $=1$, we have

$$HU=\begin{vmatrix} -6(ax+b), & 0\ , & -6(bx+c) \\ 0\ , & 2\ , & 2y \\ -6(bx+c), & 2y, & -6(cx+d) \end{vmatrix}=0;$$

whence, developing and dividing by 24, we find

$$3\{(ax+b)(cx+d)-(bx+c)^2\}+(ax+b)y^2=0,$$

or, what is the same thing,

$$3\{(ac+b^2)x^2+(ad-bc)x+(bd-c^2)\}+(ax+b)y^2=0,$$

as the equation of the Hessian curve, meeting the given cubic curve

$$ax^3+3bx^2+3cx+d-y^2=0,$$

in its points of inflexion. Multiplying the last-mentioned equation by b, and adding it to the equation of the Hessian, we obtain

$$abx^3+3acx^2+3adx+4bd-3c^2+axy^2=0,$$

or, what is the same thing,

$$xy^2=-bx^3-3cx^2-3dx+\frac{3c^2-4bd}{a},$$

as the equation of a curve meeting the given curve

$$y^2 = ax^3 + 3bx^2 + 3cx + d,$$

in its points of inflexion; and if for greater simplicity we assume $a = 1$, $d = 0$ (the latter equation means obviously that the origin is taken at one of the three intersections of the curve with the axis of x, say the real one, if the intersections are one real, two imaginary), then the equation of the curve is

$$y^2 = x(x^2 + 3bx + 3c),$$

and the inflexions are given as the intersections of the curve with the curve

$$xy^2 = -bx^3 - 3cx^2 + 3c^2.$$

There is, it is clear, an inflexion at the point at infinity on the line $x = 0$; and eliminating y^2 we find

$$x^4 + 3bx^3 + 3cx^2 = -bx^3 - 3cx^2 + 3c^2,$$

or, what is the same thing,

$$x^4 + 4bx^3 + 6cx^2 - 3c^2 = 0,$$

a quartic equation giving the four ordinates through the remaining eight inflexions.

If the curve has a cuspidal point, then the origin will be at the cusp, and we have $b = 0$, $c = 0$, and the quartic equation becomes $x^4 = 0$; that is, the four ordinates pass through the cusp.

If the curve have a node, then taking the origin at the node we have $c = 0$; the equation of the curve is

$$y^2 = x^2(x + 3b),$$

and the curve has a crunode or an acnode according as b is positive or negative; the quartic equation becomes

$$x^3(x + 4b) = 0,$$

and the factor $x^3 = 0$ gives three ordinates through the node; the remaining factor $x + 4b = 0$ gives the ordinate through the two inflexions; and substituting this value of x in the equation of the cubic, we find

$$y^2 = -16b^3,$$

and the resulting values of y (consequently also the inflexions) are imaginary if b be positive, that is, for the crunodal form; but real if b be negative, that is, for the acnodal form. It is to be observed that the indefinite ordinate $x + 4b = 0$ or $x = -4b$ is real in each of the two cases: in the crunodal case, the ordinate lies outside the curve, that is beyond the loop; in the acnodal case inside the curve, that is on the opposite side to the acnode in regard to the vertex; and using $3b$ to denote the distance (taken positively) of the vertex from the node, (that is, in the crunodal case changing the sign of b), the distance (taken positively) of the ordinate from the vertex is $= 4b - 3b$, $= b$, $= \frac{1}{3}.3b$, that is, it is one-third of the distance of the vertex from the node.

Consider next the case of a curve without singularities; and first the complex case, the condition for which is that the equation $x^2+3bx+3c=0$ may have its roots real, or $c<\frac{3}{4}b^2$. The values of x which give $y=0$ are

$$x=0,\quad x=-\tfrac{3}{2}b\pm\sqrt{3\left(\tfrac{3}{4}b^2-c\right)};$$

and we may without loss of generality assume that b and c are each of them positive; the value $x=0$ will then belong to the vertex of the parabolic portion, and the two *negative* values $x=-\frac{3}{2}b\pm\sqrt{3\left(\frac{3}{4}b^2-c\right)}$ will belong to the vertices of the oval. The limiting values $c=0$ and $c=\frac{3}{4}b^2$ give the acnodal and the crunodal curves respectively, which have been already considered.

In the case in question ($b=+$, $c=+$, $c<\frac{3}{4}b^2$), the equation $x^4+4bx^3+6cx-3c^2=0$ has only two real roots, one of them positive and the other negative; and the positive root substituted in the equation $y^2=x(x^2+3bx+3c)$ gives $y^2=+$, and we have thus the two real inflexions: in order to verify that the negative root gives imaginary inflexions, it must be shown that this negative root does not lie between the two values $x=-\frac{3}{2}b\pm\sqrt{3\left(\frac{3}{4}b^2-c\right)}$, or, what is the same thing, that these values substituted for x in the function

$$x^4+4bx^3+3cx^2-3c^2,$$

give results of the same sign.

To verify this write $c=\frac{3}{4}b^2(1-\theta^2)$ (where $\theta<1$) and $x=\frac{3}{2}b(\xi-1)$; then for the limiting values of x, we have $\frac{3}{2}b(\xi-1)=-\frac{3}{2}b\mp\frac{3}{2}b\theta$, or $\xi=\pm\theta$; and moreover

$$x^4+4bx^3+6cx^2-3c^2=\tfrac{27}{16}b^4\{3(\xi-1)^4+8(\xi-1)^3+6(\xi-1)^2(1-\theta^2)-(1-\theta^2)^2\},$$

where the term in brackets is

$$=3\xi^4-4\xi-6\theta^2\xi^2+12\theta^2\xi-4\theta^2-\theta^4,$$

and writing $\xi=\pm\theta$, this becomes

$$-4\theta^4-4\theta^3\pm8\theta^3,\quad=-4\theta^2(\theta\mp1)^2,$$

so that the two values are each negative, and the theorem is thus proved. It may be added that the curve

$$y=x^4+4bx^3+6cx^2-3c^2$$

cuts the axis in two real points, one of them situate between the oval and the parabolic portion of the cubic parabola, the other within the parabolic portion.

Lastly, for the simplex case, the condition for which is $c>\frac{3}{4}b^2$; the equation $0=x^4+4bx^3+6cx^2-3c^2$ has, as before, two real roots, one positive and the other negative; and since the negative root substituted for x in the equation $y^2=x^3+3bx^2+3cx$ gives a negative value of y^2, it is only the positive root which gives an ordinate through two real inflexions. The curve $y=x^4+4bx^3+6cx^2-3c^2$ meets the axis in two real points, one of them without, the other within the cubic parabola.

I remark that the equation $x^2+2bx+c=0$ gives the level points (i.e. the points where the tangent is parallel to the axis) of the cubic parabola. In the complex case, where $c<\frac{3}{4}b^2$, then *a fortiori* $c<b^2$ or the values of x are both real, one of these values gives y^2 positive, and we have thus the maximum ordinate of the oval; the other value of x gives y^2 negative. In the simplex case, where $c>\frac{3}{4}b^2$, we may have 1°. $c<b^2$, and the two values of x give each of them y^2 positive; the least value of x corresponds to a maximum ordinate, the greatest to a minimum ordinate of the cubic parabola, and between these we have the ordinate through the two real inflexions, the tangents at the inflexions meeting on the axis within the parabola. 2°. We may have $c=b^2$, the two values of x here coincide, giving the ordinate through the two real inflexions, the tangents at the inflexions being horizontal. And 3°. We may have $c>b^2$, the two values of x are then imaginary and we have no real level point. These are, in fact, Murdoch's three forms, which he distinguishes as the *ampullata*, *media*, and *campaniformis*.

2, *Stone Buildings, W.C., June* 2, 1863.

346.

NOTE ON AN EXPRESSION FOR THE RESULTANT OF TWO BINARY CUBICS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 380—382.]

MR WARREN, in his paper "Illustrations of the Theory of Critical Functions," *Quarterly Mathematical Journal*, t. VI. pp. 231—237, (1864), has given for the Resultant of two binary cubic functions, an expression which is in effect as follows; viz. considering the cubic

$$(a,\ b,\ c,\ d \between x,\ y)^3,$$

its Hessian

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c} \between x,\ y)^2,\ = (ac - b^2,\ ad - bc,\ bd - c^2 \between x,\ y)^2,$$

and the cubicovariant

$$(A,\ B,\ C,\ D \between x,\ y)^3,\ = \left\{\begin{array}{r} a^2d - 3abc + 2b^3\ , \\ 3abd - 6ac^2 + 3b^2c, \\ -3acd + 6b^2d - 3bc^2, \\ -\ ad^2 + 3bcd - 2c^3\ , \end{array}\right\} (x,\ y)^3;$$

and in like manner the cubic

$$(a',\ b',\ c',\ d' \between x,\ y)^3,$$

its Hessian

$$(\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}' \between x,\ y)^2,$$

and the cubicovariant

$$(A',\ B',\ C',\ D' \between x,\ y)^3;$$

and writing

$$\begin{aligned} \mathfrak{A} &= ad' - 3bc' - 3b'c - a'd\ , \\ \mathfrak{B} &= \mathrm{ac}' + \mathrm{a'c} - 2\mathrm{bb}'\ , \\ \mathfrak{C} &= AD' - 3BC' + 3B'C - A'D, \end{aligned}$$

then the Resultant is

$$= -2\,\mathfrak{A}^3 + 27\,\mathfrak{A}\mathfrak{B} + 27\,\mathfrak{C},$$

that is, the Resultant is

$$\begin{aligned}
= -\ & 2\,(ad' - a'd - 3bc' + 3b'c)^2 \\
& + 27\,(ad' - a'd - 3bc' + 3b'c) \times \\
& \qquad \{(ac - b^2)(b'c' - d'^2) - \tfrac{1}{2}(ad - bc)(a'd' - b'c') + (bd - c^2)(a'c' - b'^2)\} \\
& + 27\,\{\ \ (a^2d - 3abc + 2b^3\ \ )(-a'd'^2 + 3b'c'd' - 2c'^3\ ) \\
& \qquad - 3\,(abd - 2ac^2 + b^2c\ \ )(-a'c'd' + 2b'^2d' - b'c'^2) \\
& \qquad + 3\,(-acd + 2b^2d - bc^2)(\ \ a'b'd' - 2a'c'^2 + b'^2c\ ) \\
& \qquad - \ (-ad^2 + 3bcd - 2c^3)(\ \ a'^2d' - 3a'b'c' + 2b'^3)\}.
\end{aligned}$$

In particular assume

$$(a',\ b',\ c',\ d' \between x,\ y)^3 = x^3 + y^3,$$

so that

$$(\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}' \between x,\ y)^2 = xy,$$

$$(A',\ B',\ C',\ D' \between x,\ y)^3 = x^3 - y^3,$$

and thus

$$\begin{aligned}
a' &= d' \quad = 1, \quad b' = c' = 0, \\
\mathrm{a}' &= \mathrm{c}' \quad = 0, \quad \mathrm{b}' = \tfrac{1}{2}, \\
A' &= -D' = 1, \quad B' = C' = 0.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A} &= a - d, \\
\mathfrak{B} &= -\mathrm{b} = bc - ad, \\
\mathfrak{C} &= A + D = a^2d - ad^2 - 3abc + 3bcd + 2b^3 - 2c^3,
\end{aligned}$$

or, putting for shortness,

$$a - d = \theta, \text{ and therefore } a = d + \theta,$$

we have

$$\begin{aligned}
\mathfrak{A} &= \theta, \\
\mathfrak{B} &= bc - d\theta - d^2, \\
\mathfrak{C} &= 2\,(b^3 - c^3) - 3bc\theta + d^2\theta + d\theta^2,
\end{aligned}$$

and Resultant is

$$\begin{aligned}
& - \ 2\theta^3 \\
& + 27\theta\,(bc - d^2 - d\theta) \\
& + 27\,\{2\,(b^3 - c^3) - 3bc\theta + d^2\theta + d\theta^2\},
\end{aligned}$$

which is

$$= -\ 2\theta^3 + 54b^3 - 54c^3 - 54bc\theta,$$

or rejecting the factor -2, it is

$$= \quad \theta^3 - 27b^3 + 27c^3 + 27cb\theta.$$

But the two equations are

$$(a,\ b,\ c,\ d \between x,\ y)^3 = 0,$$
$$x^3 + y^3 \qquad\qquad = 0,$$

the last of which gives $y = -x$, $y = -\omega x$, $y = -\omega^2 x$, if ω be an imaginary cube root of unity, and hence the Resultant is

$$= (a - 3b + 3c - d)(a - 3b\omega + 3c\omega^2 - d)(a - 3b\omega^2 + 3c\omega - d),$$

which is

$$= (\theta - 3b + 3c)(\theta - 3b\omega + 3c\omega^2)(\theta - 3b\omega^2 + 3c\omega),$$

or finally is

$$= \theta^3 - 27b^3 + 27c^3 + 27bc\theta,$$

and the formula is thus verified.

If the two cubics are taken to be

$$(a,\ b,\ c,\ d \between x,\ y)^3 = 0,$$
$$(b,\ c,\ d,\ e \between x,\ y)^3 = 0,$$

then the formula gives for the Discriminant of the quartic function $(a,\ b,\ c,\ d,\ e \between x,\ y)^4$ a new expression, which however does not appear to be an elegant one.

347.

ON THE NOTION AND BOUNDARIES OF ALGEBRA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 382—384.]

I DO not admit the assertion, that the idea of number is derived from that of time, it appears to me that it is derived from that of succession in time or space indifferently. But I would rather say that the idea of cardinal number is derived and abstracted from that of ordinal number, viz. (distinguishing the expressions 'set' and 'series,' the latter being used to designate a set of things considered as arranged in a definite order), if we have a series of things a, b, c, d, &c., or say a series of words *first, second, third, fourth*, &c.; then any set of things X, N, Y, P, Q, &c., taking them up one after the other, no matter in what order, and coordinating them with the terms of the series a, b, c, d, &c. or with the words, first, second, third, fourth, &c.—the last of them will be coordinated with a definite term of the series a, b, c, d, &c., or with a definite term of the series first, second, third, fourth, &c.; that is, the set, *whatever* be the assumed order of the terms, or (what is the same thing) without assuming any order therein, will have a certain property; viz. in the set X, N, Y, P, Q, where the last term is coordinated with e or with the word *fifth*, the property is that the set consists of *five* things: and so in general the set consists of a certain (cardinal) number of things, such cardinal number being the number corresponding to the rank in the series a, b, c, d, &c., of the term wherewith is coordinated the last term of the set, or corresponding with the like ordinal number in the series first, second, third, fourth, &c.

The foregoing remarks are made to some extent incidentally, but they have a bearing on the distinction in kind which exists e.g. between the proposition $1+1+1+1=4$, and the proposition which for ordinary purposes would be expressed as $1+1+$ &c.

(n terms) $= n$, but which is better expressed in the form $1_1 + 1_2 + \ldots + 1_n = n$, where the subscript numbers merely distinguish between the different unities which are added altogether.

I use the term Algebra in a wide sense as including, or indeed I might say identical with, Finite Analysis, and excluding Infinite Analysis; but in speaking of it as identical with Finite Analysis I include in that term part of what might be considered Infinite Analysis; viz. many of the theorems relating to infinite series or other successions of operations, e.g.

$$(1-x)(1+x+x^2+\ldots \textit{ad inf.}) = 1,$$

really belong to Finite Analysis; for what is asserted is that the coefficient of the term of indefinite rank, say x^n, is a *finite* series equal in value to zero (this coefficient in fact is $1-1$ which is $=0$). On the other hand the theorem

$$1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c. = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\ldots,$$

the truth whereof depends on the equations

$$\frac{\pi^2}{2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \ldots \textit{ad inf.}, \&c., \&c.,$$

which are not arithmetically verifiable, belongs strictly to Infinite Analysis.

Algebra is an Art and a Science; *quà* Art, it defines and prescribes operations which are either tactical or else logistical; viz. a tactical operation is one relating to the arrangement in any manner of a set of things; a logistical operation (I prefer to use the new expression instead of arithmetical) is the actual performance, so as to obtain for the result a number, of any arithmetical operations (of course, given operations) finite in number, since these alone can be *actually* performed, upon given numbers. And *quà* Science Algebra affirms *à priori*, or predicts, the result of any such tactical or logistical (or tactical and logistical) operations. An equation such as $1+4+10=15$ is not an algebraical theorem; it is merely the assertion that the sum of the numbers 1, 4, 10 is that number, viz. 15, which is the sum of the numbers in question. And, similarly, the equation $1+1+1=3$ is not an algebraical theorem. But on the other hand, the equation $1+1+1+\ldots$ (n terms) $= n$, is an algebraical theorem; in the equivalent form $1_1+1_2+\ldots+1_n=n$, (where $1_k=1$) it is not different in kind from the equation $1+2+3\ldots+n=\frac{1}{2}n(n+1)$, or say $1_1+1_2+\ldots+1_n=\frac{1}{2}n(n+1)$, (where $1_k=k$) which is certainly an algebraical theorem. And this leads to the remark, that every algebraical theorem rests ultimately on a tactical foundation. In fact, whether we prove the last-mentioned theorem in the easiest way by writing

$$\begin{aligned} 1 + \ \ 2 \ \ + \ \ 3 \ \ \ldots + n &= S, \\ n + (n-1) + (n-2) \ldots + 1 &= S, \end{aligned}$$

and therefore $2S = (n+1) + (n+1) + \ldots (n \text{ terms}) = n(n+1)$ or $S = \frac{1}{2}n(n+1)$; or by induction by showing that the theorem, if true for n, is true for $(n+1)$, (this depends on the equation $\frac{1}{2}n(n+1) + (n+1) = (n+1)(\frac{1}{2}n+1) = \frac{1}{2}(n+2)$; the proof is equally a tactical one; it is always tactic which determines what logistical operations are to be performed.

Although it may not be possible absolutely to separate the tactical and logistical operations; for in (at all events) a series of logistical operations, there is always something that is tactical, and in many tactical operations (e.g. in the Partition of Numbers) there is something which is logistical, yet the two great divisions of Algebra are Tactic and Logistic. Or if, as might be done, we separate Tactic off altogether from Algebra, making it a distinct branch of Mathematical Science, then (assuming in Algebra a knowledge of all the Tactic which is required) Algebra will be nothing else than Logistic.

348.

ON THE THEORY OF INVOLUTION.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part I. (1866), pp. 21—38. Read February 22, 1864.]

THREE or more quantics which satisfy identically a linear equation such as

$$\lambda U + \lambda' U' + \lambda'' U'' + \ldots = 0,$$

where λ, λ', λ'', ... are constants, are said to be in Involution. In particular any quantic $U + kV$, where k is a constant, is in involution with the quantics U, V; and the entire system of such quantics, k having any value whatever, is a system in involution with the quantics U, V. And in like manner the equation $U + kV = 0$, or the locus or system of loci thereby represented is said to be in involution, or to form a system in involution with the equations or loci $U = 0$, $V = 0$. If U, V are binary quantics then the equation $U + kV = 0$ may be considered as representing a range of points in involution with the ranges $U = 0$, $V = 0$. And similarly, if U, V are ternary quantics, then the equation $U + kV = 0$ may be considered as representing a curve in involution with the curves $U = 0$, $V = 0$.

In the case of a range $U + kV = 0$, the constant k may be determined so that the range shall have a twofold[1] point. The condition for this may be written

$$\text{Disc}^{\text{t}}.\ (U + kV) = 0,$$

[1] The series of epithets *onefold*, *twofold*, &c., seems preferable to the series *single*, *double*, &c., as avoiding ambiguities which would sometimes be occasioned by the use of these last. The double point of a curve I call a *node*, viz. a *crunode* when it is a double point with two real branches, and an *acnode* when it is a conjugate or isolated point. The subject-matter and context will in general show whether the term node is to be considered as including or as not including a cusp.

it being understood that the discriminant of the function $U+kV$ is taken in regard to the coordinates. And this being so, we may write $\text{Disc}^t.\,\text{Disc}^t.\,(U+kV)$ to denote the discriminant in regard to k of the function $\text{Disc}^t.\,(U+kV)$. The quantity in question (viz. $\text{Disc}^t.\,\text{Disc}^t.\,(U+kV)$, or say for shortness $\square$) is a function of the coefficients of U, V, homogeneous as regards each set of coefficients separately, and it breaks up into factors in the form

$$\square = RQ^3P^2,$$

where $R=0$ is the condition in order that the ranges $U=0$, $V=0$ may have a point in common; or what is the same thing, in order that there shall be a range $U+kV=0$ having a twofold point at a common point of the ranges $U=0$, $V=0$, (R is in fact the resultant of the quantics U, V): $Q=0$ is the condition in order that there may be a range $U+kV=0$ having a threefold point: and $P=0$ is the condition in order that there may be a range $U+kV=0$ having a pair of twofold points.

And similarly, when $U=0$, $V=0$ are curves, then we have the like equation

$$\square = RQ^3P^2,$$

where $R=0$ is the condition in order that the curves $U=0$, $V=0$ may have a point of twofold intersection, that is, that the two curves may touch each other, (R is the Tactinvariant of the quantics U, V); or what is the same thing, it is the condition in order that there may be a curve $U+kV=0$ having a node at a point of twofold intersection of the curves $U=0$, $V=0$; moreover $Q=0$ is the condition in order that there may be a curve $U+kV=0$ having a cusp: and $P=0$ is the condition in order that there may be a curve $U+kV=0$ having a pair of nodes.

The establishment and illustration of the foregoing theorems form the chief object of the present memoir.

Article Nos. 1 to 16, *relating to two Binary Quantics.*

1. Let $U=(a, \ldots \between x, y)^n$, $V=(a', \ldots \between x, y)^n$, be two binary quantics of the same order n; and write $W=U+kV=(a+ka', \ldots \between x, y)^n$, so that $W=U+kV=0$ is the equation of a range in involution with the ranges $U=0$, $V=0$. But for greater distinctness it is in general convenient to retain $U+kV$ instead of replacing it by the single letter W.

2. In order that the range $U+kV=0$ may have a twofold point we must have simultaneously

$$\delta_x(U+kV)=0,$$
$$\delta_y(U+kV)=0,$$

and eliminating (x, y) from these equations we find

$$\text{Disc}^t.\,(U+kV)=0,$$

which is an equation of the order $2(n-1)$ as regards k, and of the same order as regards the coefficients of U and V conjointly. And to each of the $2(n-1)$ values of k there corresponds a point (x, y) satisfying the required conditions; that is, a point which is a twofold point of the range $U+kV=0$. The points in question may be termed the 'critic centres' of the involution.

3. The elimination of k from the before-mentioned two equations gives

$$\begin{vmatrix} \delta_x U, & \delta_y U \\ \delta_x V, & \delta_y V \end{vmatrix} = 0,$$

where the determinant, which for shortness I call J, is the Jacobian of the two functions U, V. The equation $J=0$ gives a range of $2(n-1)$ points which are in fact the critic centres; and for each of these points we have

$$\delta_x U : \delta_x V = \delta_y U : \delta_y V = -k : 1,$$

which gives the value of k corresponding to the point in question.

4. The condition in order that the equation in k may have a twofold root is

$$\text{Disc}^{t}.\ \text{Disc}^{t}.\ (U+kV)=0,$$

or say

$$\square = 0,$$

where $\text{Disc}^{t}.\ \text{Disc}^{t}.\ (U+kV), = \square$, is a function of the degree $2(2n-2)(2n-3)$ in regard to the coefficients of U, V conjointly; but it is separately homogeneous, and therefore of the degree $(2n-2)(2n-3)$ in regard to each of the two sets of coefficients.

5. To each point of the range $J=0$, there corresponds a value of k; hence if the range J have a twofold point, then the equation in k will have a twofold root. Now *first* if the ranges $U=0$, $V=0$ have a common point, then this is a twofold point of the range $V=0$. But *secondly*, without a common point in the ranges $U=0$, $V=0$, the range $J=0$ may have a twofold point; and in this case also we have a twofold root of the equation in k. And *thirdly*, without a twofold point in the range $V=0$, there may be in this range two onefold points giving each of them the same value of k, and so giving a twofold root of the equation in k. And the three suppositions correspond respectively to the cases of there being a range $U+kV=0$ having a twofold point at a common point of the ranges $U=0$, $V=0$, having a threefold point, and having a pair of twofold points.

6. *First*, if the ranges $U=0$, $V=0$ have a common point we may write

$$U=(x-\alpha y)\,U',\quad V=(x-\alpha y)\,V',$$

and these give

$$U+kV=(x-\alpha y)\,(U'+kV').$$

Now in general

$$\text{Disc}^t.\ PQ = \text{Disc}^t.\ P\,.\,\text{Disc}^t.\ Q\,.\,[\text{Result.}\ (P,\ Q)]^2,$$

and hence in the present case

$$\text{Disc}^t.\ (U + kV) = \text{Disc}^t.\ (U' + kV')\,.\,(U_0' + kV_0')^2,$$

where $U_0' + kV_0'$ is what $U' + kV'$ becomes on writing therein $x = \alpha y$ and neglecting the factor y^{n-1} which then presents itself. We see therefore that in this case the equation k has a twofold root $k = -U_0' \div V_0'$; a result which might also have been obtained from the consideration of the Jacobian.

7. The condition in order that the ranges $U = 0$, $V = 0$ may have a common point is

$$\text{Result.}\ (U,\ V) = 0,$$

say $P = 0$. P is of the degree n in regard to the coefficients of U, V respectively.

8. *Secondly*, suppose that the functions U, V are such that there exists a range $U + kV = 0$ having a threefold point. If k_1 be the proper value of k, then we have $U + k_1 V = (x - \alpha y)^3\,\Theta$, and therefore $U = -k_1 V + (x - \alpha y)^3\,\Theta$. Hence forming the Jacobian of U, V, the equation for the determination of the critic centres will be

$$\begin{vmatrix} \delta_x V,\ \delta_x\,.\,(x - \alpha y)^3\,\Theta \\ \delta_y V,\ \delta_y\,.\,(x - \alpha y)^3\,\Theta \end{vmatrix} = 0,$$

which is of the form

$$(x - \alpha y)^2\,\Omega = 0;$$

or we have $(x - \alpha y)^2 = 0$, a twofold critic centre. The corresponding value of k given by the equation $-k : 1 = \delta_x U : \delta_x V$ is $k = k_1$, and we have thus $k = k_1$ as a twofold root of the equation in k.

9. But if the range $W = U + kV = 0$ has a threefold point, or what is the same thing, if the equation $W = 0$ has a threefold root; then we must have between the coefficients of W a plexus of equations equivalent to two relations. Such plexus is known to be of the order $3(n-2)$. This comes to saying that if the coefficients of W are assumed to be of the form $a + ka' + la'', \ldots$ and if between the several equations of the plexus we eliminate k, we obtain for l an equation $Q = 0$ of the degree $3(n-2)$. The equation in question would be of the form Funct. $(a + la'',\ a',\ldots) = 0$. Hence Q is of the degree $3(n-2)$ in the coefficients $(a, \ldots)$ of U. And in a similar manner Q is of the degree $3(n-2)$ in the coefficients $(a', \ldots)$ of V. And omitting altogether the terms in l, or taking the coefficients of W to be $a + ka', \ldots$ if from the equations of the plexus we eliminate k, we find an equation $Q = 0$, where Q is a function of the degree $3(n-2)$ as regards the coefficients of U, and of the same degree as regards the coefficients of V. We have thus found the form of the condition $Q = 0$ which expresses that there may be a range $U + kV = 0$ having a threefold point.

10. It may be proper to remark conversely that given the equation $Q=0$, if in this equation we write $a=(a+ka')-ka', \ldots$ so that Q becomes a function of $a+ka', \ldots a', \ldots k$, then the equation $Q=0$ will be satisfied irrespectively of the values of $a', \ldots k$ by a plexus of equations involving only the coefficients $a+ka', \ldots$ and which is in fact the very plexus (equivalent therefore to two relations) which gives the conditions in order that the equation $W=0$ may have a threefold root.

11. *Thirdly*, suppose that the functions U, V are such that there exists a range $U+kV=0$ having a pair of twofold points. If k_1 be the proper value of k, then we have $U+k_1V=(x-\alpha y)^2(x-\beta y)^2\Theta$, and therefore $U=-k_1V+(x-\alpha y)^2(x-\beta y)^2\Theta$. Hence forming the Jacobian of U, V, we have for the determination of the critic centres the equation

$$\begin{vmatrix} \delta_x V, & \delta_x . (x-\alpha y)^2(x-\beta y)^2\Theta \\ \delta_y V, & \delta_y . (x-\alpha y)^2(x-\beta y)^2\Theta \end{vmatrix} = 0,$$

which is of the form

$$(x-\alpha y)(x-\beta y)\Omega = 0;$$

or, we have $x-\alpha y=0$, or $x-\beta y=0$, a pair of critic centres; and for each of these the corresponding value of k_1 given by the equation $-k : 1 = \delta_x U : \delta_x V$ is $k=k_1$, so that $k=k_1$ is a twofold root of the equation in k.

12. By the like considerations as for the threefold root (observing that if the equation $W=0$ has a pair of twofold roots we must have between the coefficients of W a plexus equivalent to two relations, and of the order $2(n-2)(n-3)$), we see that the condition for the existence of a range $U+kV=0$ having a pair of twofold points is of the form $P=0$, where P is a function of the degree $2(n-2)(n-3)$ as regards the coefficients of U, and of the same degree as regards the coefficients of V; and conversely that, given the equation $P=0$, we may find the plexus.

13. The equation $\square=0$ will be satisfied if $R=0$, or if $Q=0$, or if $P=0$; *and in no other cases.* To prove this, suppose that $x-\alpha y=0$ is the critic centre corresponding to a twofold root k_1 of the equation in k. We have $U=-k_1V+(x-\alpha y)^2\Theta$, and thence the equation for the critic centres is

$$\begin{vmatrix} \delta_x V, & \delta_x (x-\alpha y)^2\Theta \\ \delta_y V, & \delta_y (x-\alpha y)^2\Theta \end{vmatrix} = 0,$$

which is an equation of the form $(x-\alpha y)\Omega=0$; and where, corresponding to the root $x-\alpha y=0$, the equation $-k : 1 = \delta_x U : \delta_x V$ gives $k=k_1$. Since k_1 is a twofold root, there must be another critic centre also giving the value k_1 of k. This new critic centre may be either $x-\alpha y=0$ (the same as the first mentioned critic centre) or it may be a distinct critic centre $x-\beta y=0$. In the former case

$$\begin{vmatrix} \delta_x V, & \delta_x (x-\alpha y)^2\Theta \\ \delta_y V, & \delta_y (x-\alpha y)^2\Theta \end{vmatrix}$$

must contain, instead of the factor $(x-\alpha y)$, the factor $(x-\alpha y)^2$. In order that this may be so, we must have

$$\begin{vmatrix} \delta_x V, & \Theta \\ \delta_y V, & -\alpha\Theta \end{vmatrix},$$

that is, $(\alpha\delta_x V+\delta_y V)\Theta$ divisible by $(x-\alpha y)$, that is, either $\alpha\delta_x V+\delta_y V$, or else Θ, divisible by $x-\alpha y$; or, what is the same thing, either V, or else Θ, divisible by $x-\alpha y$. But if V be divisible by $x-\alpha y$, then U, V have the common factor $x-\alpha y$, and we have the case *first* above considered. And again if Θ contain the factor $x-\alpha y$, then we have

$$U=-k_1 V+(x-\alpha y)^2\Theta',$$

and we have the case *secondly* above considered. Finally if the new critic centre be the distinct centre $x-\beta y=0$, then for $x-\beta y=0$ the equation

$$-k : 1=\delta_x U : \delta_x V=\delta_y U : \delta_y V$$

should give $k=k_1$; but this will only happen if $\delta_x.(x-\alpha y)^2\Theta$, $\delta_y.(x-\alpha y)^2\Theta$ vanish for $x-\beta y=0$, that is, if Θ contains the factor $(x-\beta y)^2$; and when this is so,

$$U=-k_1 V+(x-\alpha y)^2(x-\beta y)^2\Theta',$$

or we have the case *thirdly* above considered.

14. Hence the equation $\square=0$ being satisfied if $R=0$, or else if $Q=0$, or else if $P=0$, and in no other cases, the function $\square$ must be made up of the factors R, Q, P, each taken the proper number of times, and knowing the degrees of the several functions, it follows that we must have

$$\square=RQ^3P^2,$$

in fact, considering the coefficients of either U or V, the comparison of the degrees gives

$$2(n-1)(2n-3)=n+9(n-2)+4(n-2)(n-3),$$

where the function on the right-hand side is

$$\begin{aligned} &= n \\ &\quad + 9n-18 \\ &\quad + 4n^2-20n+24 \\ &= 4n^2-10n+6, \end{aligned}$$

which is the value of the function on the left-hand side.

15. In the very particular case $n=2$, Q and P are each of them of the degree $=0$; and we have simply $\square=R$, that is, the resultant of the two quadric functions

$$U=(a,\ b,\ c\between x,\ y)^2,\ V=(a',\ b',\ c'\between x,\ y)^2$$

is

$$\begin{aligned} &=\text{Disc}^{\text{t}}.\ (U+kV) \\ &=\text{Disc}^{\text{t}}.\ (ac-b^2,\ ac'+a'c-2bb',\ a'c'-b'^2\between 1,\ k)^2, \end{aligned}$$

which is Prof. Boole's ancient theorem referred to in my Fifth Memoir on Quantics[1], but which is now first exhibited in connexion with the general theory to which it belongs.

16. It may be noticed that the condition for a twofold critic centre, or (what is the same thing) a twofold factor of the Jacobian {which condition is of the degree $2(2n-3)$ in regard to the coefficients of U or V} is $RQ=0$; and that we in fact have

$$2(2n-3)=n+3(n-2).$$

This remark is due to Dr Salmon.

Article, Nos. 17 to 42, *relating to two Ternary Quantics.*

17. Suppose now that $U=(a,..\between x, y, z)^n$, $V=(a',..\between x, y, z)^n$ are two ternary quantics of the same order n, and write $W=U+kV=(a+ka',..\between x, y, z)^n$, so that

$$W=U+kV=0$$

is the equation of a curve in involution with the curves $U=0$, $V=0$. But for greater distinctness it is in general proper to retain $U+kV$ in place of W.

18. In order that the curve $U+kV=0$ may have a node, we must have simultaneously

$$\delta_x(U+kV)=0,$$
$$\delta_y(U+kV)=0,$$
$$\delta_z(U+kV)=0,$$

and eliminating (x, y, z) from these equations we have

$$\text{Disc}^{t}.\ (U+kV)=0,$$

which is an equation of the degree $3(n-1)^2$ as regards k, and of the same order as regards the coefficients of U, V conjointly.

19. To each of the $3(n-1)^2$ values of k there corresponds a point satisfying the conditions in question, and which is therefore a node of the corresponding nodal curve

$$U+kV=0;$$

the points in question are the critic centres of the involution.

20. The critic centres may be differently obtained as follows; viz. if from the three equations we eliminate k, we find

$$\left\| \begin{matrix} \delta_x U, & \delta_y U, & \delta_z U \\ \delta_x V, & \delta_y V, & \delta_z V \end{matrix} \right\| = 0,$$

[1] *Phil. Trans.* vol. CXLVIII. (1858), pp. 415—427, [156].

a plexus of three curves, each of them of the order $2(n-1)$; any two of the three curves intersect in $4(n-1)^2$ points; but $(n-1)^2$ of these do not lie on the third curve; the remaining $3(n-1)^2$ of them lie on all three of the curves, and they are the critic centres of the involution.

21. More generally the critic centres lie on any curve whatever of the form

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \delta_x U, & \delta_y U, & \delta_z U \\ \delta_x V, & \delta_y V, & \delta_z V \end{vmatrix} = 0,$$

and any such curve, viz. any curve of the order $2(n-1)$ passing through the $3(n-1)^2$ critic centres, may be termed a diacritic curve.

22. For any one of the critic centres we have

$$\delta_x U : \delta_y U : \delta_z U = \delta_x V : \delta_y V : \delta_z V = k : -1,$$

which gives the value of k corresponding to the point in question.

23. The condition in order that the equation in k may have a twofold root is

$$\text{Disc}^t.\ \text{Disc}^t.\ (U+kV) = 0,$$

or say

$$\square = 0,$$

where $\text{Disc}^t.\ \text{Disc}^t.\ (U+kV), = \square$, is a function of the degree $2 . 3(n-1)^2\{3(n-1)^2-1\}$ in regard to the coefficients of U, V conjointly; but it is separately homogeneous, and therefore of the degree $3(n-1)^2\{3(n-1)^2-1\}$ in regard to each set of coefficients.

24. To each of the critic centres there corresponds a value of k. Hence if two of the critic centres coincide, or say if there is a twofold critic centre, the equation in k will have a twofold root. Now *first* if the curves $U=0$, $V=0$ touch each other (have a point of contact or twofold intersection) then the diacritic curves will all touch (have a point of twofold intersection) at the point in question, which is therefore a twofold critic centre. It may be remarked in passing that the diacritic curves do not at the twofold critic centre touch the curves $U=0$, $V=0$. But *secondly* the diacritic curves may touch at a point which is not a point of contact of the curves $U=0$, $V=0$. Such a point is a twofold critic centre. In each of these two cases the equation in k has a twofold root. Moreover, in the first case the curve $U+kV=0$ corresponding to the twofold root has a node at the point of contact of the two curves $U=0$, $V=0$; in the second case the curve $U+kV=0$ corresponding to the twofold root has the twofold centre (not a mere node but) a cusp. And *thirdly*, without any twofold critic centre, two distinct critic centres may give by the equations

$$\delta_x U : \delta_y U : \delta_z U = \delta_x V : \delta_y V : \delta_z V = k : -1$$

the same value of k, and then the curve $U+kV=0$ corresponding to such value of k is a curve having a node at each of the critic centres in question, that is, it has two nodes.

25. *First*, if the curves $U=0$, $V=0$ touch each other, then, (x, y, z) being the coordinates of the point of contact, we have $U=0$, $V=0$,

$$\delta_x(U+k_1V)=0,$$
$$\delta_y(U+k_1V)=0,$$
$$\delta_z(U+k_1V)=0,$$

where k_1 denotes the value given by the equations

$$\delta_x U : \delta_y U : \delta_z U = \delta_x V : \delta_y V : \delta_z V = k_1 : -1$$

belonging to the point of contact. It at once follows that every diacritic curve passes through the point in question. But it is somewhat more difficult to show that the diacritic curves touch at this point.

26. I represent for shortness the first and second differential coefficients of U by (L, M, N), (a, b, c, f, g, h), and similarly those of V by (L', M', N'), (a', b', c', f', g', h'), these values all belonging to the point of contact: we have therefore

$$L+k_1L'=0,\ M+k_1M'=0,\ N+k_1N'=0.$$

The equation of the diacritic curve is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ L', & M', & N' \end{vmatrix}=0;$$

to find the tangent we must operate on the left-hand side with $X\delta_x+Y\delta_y+Z\delta_z$, where X, Y, Z are current coordinates. Calling the foregoing symbol D, this gives

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ DL', & DM', & DN' \end{vmatrix}+\begin{vmatrix} \alpha, & \beta, & \gamma \\ DL, & DM, & DN \\ L', & M', & N' \end{vmatrix}=0;$$

or, what is the same thing,

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ DL', & DM', & DN' \end{vmatrix}-\begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ DL, & DM, & DN \end{vmatrix}=0;$$

or, substituting in the first determinant for L, M, N their values $-kL'$, $-kM'$, $-kN'$, and transferring the factor k_1 from the second to the third line, we obtain

$$-\begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ k_1DL', & k_1DM', & k_1DN' \end{vmatrix}-\begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ DL, & DM, & DN \end{vmatrix}=0,$$

which may also be written

$$\begin{vmatrix} \alpha & , & \beta & , & \gamma \\ L' & , & M' & , & N' \\ DL + k_1 DL' & , & DM + k_1 DM' & , & DN + k_1 DN \end{vmatrix} = 0$$

or, more symmetrically, θ being any quantity whatever, it is

$$\begin{vmatrix} \alpha & , & \beta & , & \gamma \\ L + \theta L' & , & M + \theta M' & , & N + \theta N' \\ DL + k_1 DL' & , & DM + k_1 DM' & , & DN + k_1 DN' \end{vmatrix} = 0;$$

or, substituting for D its value, the equation of the tangent is

$$\begin{vmatrix} \alpha & , & \beta & , & \gamma \\ L + \theta L' & , & M + \theta M' & , & N + \theta N' \\ (a + k\alpha', \between X, Y, Z), & & (h + kh', \between X, Y, Z), & & (g + kg', \between X, Y, Z) \end{vmatrix} = 0.$$

27. Now if the diacritics touch, this equation should be independent of α, β, γ. Putting for shortness $a + k_1 a' = a$, &c., and also taking as we may do $\theta = 0$, the parts multiplied by α, β, γ respectively are

$$M(gX + fY + cZ) - N(hX + bY + fZ),$$
$$N(aX + hY + gZ) - L(gX + fY + cZ),$$
$$L(hX + bY + fZ) - M(aX + hY + gZ),$$

and we ought therefore to have

$$\begin{aligned} & Mg - Nh : Mf - Nb : Mc - Nf \\ = & Na - Lg : Nh - Lf : Ng - Lc \\ = & Lh - Ma : Lb - Mh : Lf - Mg, \end{aligned}$$

equations which are in fact satisfied; for take any one of them, for example, the equation

$$\frac{Mg - Nh}{Na - Lg} = \frac{Mf - Nh}{Nh - Lf},$$

this is

$$MN(gh - af') - N^2(h^2 - ab) - LM(fg - fg) + LN(fh - bg) = 0,$$

or, omitting the term in LM, and throwing out the factor N,

$$L(hf - bg) + M(gh - af) + N(ab - h^2) = 0.$$

But the equations $L + kL' = 0$, &c., give

$$ax + hy + gz = 0,$$
$$hx + by + fz = 0,$$
$$gx + fy + cz = 0,$$

that is

$$\begin{aligned} x : y : z &= bc - f^2 : fg - ch : hf - bg \\ &= fg - ch : ca - g^2 : gh - af \\ &= hf - bg : gh - af : ah - h^2, \end{aligned}$$

so that the above written equation is $Lx + My + Nz = 0$, which is true in virtue of the equation $U = 0$; and similarly for all the other equations which were to be verified.

28. It is to be noticed that the determination of the tangent of the diacritics depends only on the second differential coefficients (a, b, c, f, g, h), (a', b', c', f', g', h'), of U, V. The tangent in question will be the same if instead of the curves, $U = 0$, $V = 0$ we have the conics $(a, b, c, f, g, h \between x, y, z)^2 = 0$, $(a', b', c', f', g', h' \between x, y, z)^2 = 0$: these conics pass through the point of contact of the two curves, and their tangents are coincident with those of the two curves $U = 0$, $V = 0$ respectively; that is, the conics touch at the point in question. They consequently intersect in two more points; the chord of intersection or line joining the last-mentioned two points, meets the common tangent in a point; the polars of this point in regard to the two conics respectively, pass through the point of contact, and moreover they are one and the same line; this line is the required tangent of the diacritics. The proof will be given, *post*, No. 41.

29. Let $R = 0$ be the condition in order that the two curves $U = 0$, $V = 0$ may touch each other, or say let R be the Tactinvariant of U, V. When the curves U, V are of the degrees m, n respectively, then R is of the degrees $n(2m + n - 3)$, $m(m + 2n - 3)$ in regard to the coefficients of U, V respectively. Hence in the present case where U, V are each of the degree n, R is of the degree $3n(n - 1)$ in regard to each set of coefficients.

30. *Secondly*, if the functions U, V are such that there exists a curve $U + kV = 0$ (say the curve $U + k_1 V = 0$) which has a cusp, then it is to be shown that $k = k_1$ is a twofold root of the equation in k; and to do this it has to be shown that the cusp is a twofold critic centre; or that the diacritic curves touch at the cusp: it may be added that the cuspidal tangent is the common tangent of the diacritic curves. Now the cuspidal curve being $U + k_1 V = 0$, then at the cusp the first derived functions $L + k_1 L'$, $M + k_1 M'$, $N + k_1 N'$ vanish identically; and moreover the second derived functions $a + ka', \ldots$ are such that (X, Y, Z) being any magnitudes whatever, $(a + k_1 a', \ldots \between X, Y, Z)^2$ is a perfect square, $= (\lambda X + \mu Y + \nu Z)^2$ suppose. Now X, Y, Z being current coordinates, and D denoting the operation $D = X\delta_x + Y\delta_y + Z\delta_z$, the equation of the tangent to the diacritic curve (by an investigation similar to that for this same tangent in the case first above considered) is found to be

$$\begin{vmatrix} \alpha & , & \beta & , & \gamma \\ L + \theta L' & , & M + \theta M' & , & N + \theta N' \\ (a + k_1 a', \ldots \between X, Y, Z), & & (h + k_1 h', \ldots \between X, Y, Z), & & (g + k_1 g', \ldots \between X, Y, Z) \end{vmatrix} = 0,$$

and we have

$$(a+k_1a',..\between X,\ Y,\ Z)=\tfrac{1}{2}\delta_x(a+k_1a',..\between X,\ Y,\ Z)^2=\tfrac{1}{2}\delta_x(\lambda X+\mu Y+\nu Z)^2=\lambda(\lambda X+\mu Y+\nu Z):$$

and similarly the values of $(h+k_1h',..\between X,\ Y,\ Z)$ and $(g+k_1g',..\between X,\ Y,\ Z)$ are

$$=\mu(\lambda X+\mu Y+\nu Z) \text{ and } =\nu(\lambda X+\mu Y+\nu Z) \text{ respectively.}$$

Hence the equation of the tangent to the diacritic curve is

$$\lambda X+\mu Y+\nu Z=0,$$

that is, the tangent being independent of the values of $(\alpha,\ \beta,\ \gamma)$ is the same for all the diacritic curves, and is the tangent at the cusp of the cuspidal curve $U+k_1V=0$.

31. The conditions in order that the curve $W=U+kV=0$ may have a cusp are given by a plexus equivalent to three relations between the coefficients $a+ka',..$ of W, and using for a moment β to denote the degree of the plexus or system, then eliminating k between the equations of the plexus we find between the coefficients $a,\ldots$ of U and the coefficients $a',\ldots$ of V an equation $Q=0$ of the degree β in regard to the two sets of coefficients respectively. Conversely, given the equation $Q=0$, we may find the plexus between the coefficients $a+ka',\ldots$ of W. The value of β, as will be shewn *post*, Annex, is

$$=12(n-1)(n-2).$$

32. *Thirdly*, when the functions U, V are such that there exists a curve $U+kV=0$ (suppose the curve $U+k_1V=0$) which has a pair of nodes; each of these nodes is a critic centre, and (by means of the equation $-k:1=\delta_xU:\delta_xV$) gives the value k_1 of k, that is, k_1 is a twofold root of the equation in k.

33. The conditions in order that the curve $W=U+kV=0$ may have a pair of nodes are given by a plexus of the degree α; then the coefficients being $a+ka',\ldots$ if we eliminate k between the equations of the plexus, we find between the coefficients $a,\ldots$ of U and $a',\ldots$ of V an equation $P=0$ of the degree α in the two sets of coefficients respectively. And conversely, given the equation $P=0$, we may find the plexus between the coefficients $a+ka',\ldots$ of W. I have not succeeded in finding directly the value of α, but only derive it from the equation $\square=RQ^3P^2$, which if α had been found independently, would have been verified by means of such value of α; the value is $\alpha=\frac{1}{2}.3(n-1)(n-2)(3n^2-3n-11)$.

34. The equation $\square=0$ is satisfied if $R=0$, or if $Q=0$, or if $P=0$, and it may be seen that it is not satisfied in any other case. Hence $\square$ is made up of the factors P, Q, R, and I assume that its form is the same as in the case of a binary quantic, that is, that we have

$$\square=RQ^3P^2.$$

35. Comparing the degrees of the two sides we have

$$\begin{aligned} 3(n-1)^2(3n^2-6n+2) = \quad & 3n(n-1) \\ & +36(n-1)(n-2) \\ & + \ 3(n-1)(n-2)(3n^2-3n-11); \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} (n-1)(3n^2-6n+2) = \quad & n \\ & +12(n-2) \\ & + \quad (n-2)(3n^2-3n-11), \end{aligned}$$

which is true, but, as just remarked, this equation itself was used to find the value

$$\alpha = \tfrac{1}{2} \,.\, 3(n-1)(n-2)(3n^2-3n-11).$$

36. Recapitulating, the equation in k will have a twofold root

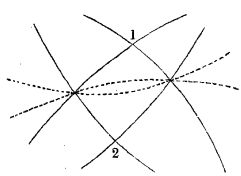
1°, if $R=0$, that is, if the curves $U=0$, $V=0$ touch each other, and in this case there is a twofold critic centre at the point of contact:

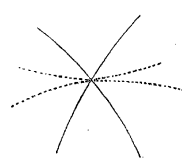
2°, if $Q=0$, that is, if there be a curve $U+kV=0$ having a cusp, and in this case the cusp is a twofold critic centre:

3°, if $P=0$, that is, if there is a curve $U+kV=0$ having a pair of nodes.

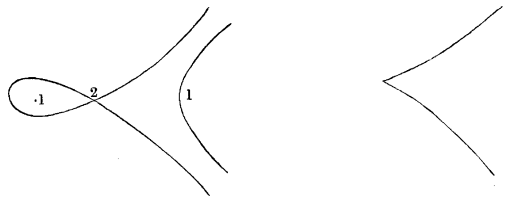
37. The three cases may be geometrically illustrated by supposing that the curves $U=0$, $V=0$ are in the first instance nearly, but not exactly, in the several relations in question.

First, if the curves $U=0$, $V=0$ are about to touch each other, that is, if there are two points of intersection about to coincide with each other. There are here two critic centres in the neighbourhood of the two points of intersection, and which, when the two points of intersection become a point of contact, coincide each with the point of contact.

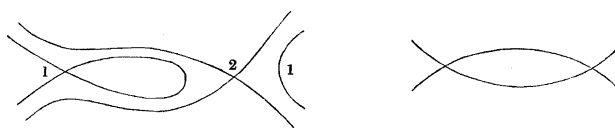




Secondly, when the curves are such that there are two critic centres which become ultimately a twofold centre.



And, *thirdly*, when the curves are such that there are two critic centres which remaining distinct from each other belong ultimately to the same critic curve.



38. The curves 1 and 2 in the left-hand figures respectively represent the nodal curves corresponding to slightly different values of k, which in the right-hand figures respectively give the curve corresponding to a twofold value of k. In the first pair of figures the curves $U=0$ and $V=0$, about to touch in the left-hand figure, touching in the right-hand figure, are shown by dotted lines. It will be observed that in the second case in the left-hand figure the two nodes which give rise to a cusp are the one of them an acnode and the other a crunode; this is in fact the only mode of drawing the figure so that a cusp shall present itself. The transition of form is one of ordinary occurrence in cubic curves and in curves of a higher order; thus if $y^2=(x-a)(x-b)(x-c)$, where $a<b<c$, then if $a=b$, we have an acnodal curve, if $b=c$ a crunodal curve, and if $a=b=c$ a cuspidal curve.

39. In the case of two conics, $n=2$. We have here simply $\square=R$, where $R=0$ is the condition in order that the two conics may touch each other. The nodal curves are of course the three pairs of lines passing through the points of intersection of the two conics, and the nodes of these curves, or critic centres, are the centres of the quadrangle formed by the four points in question; or, what is the same thing, they are the system of conjugate points common to the two conics, viz. the points which are such that the polar of one of them taken with respect to either of the conics is the line joining the other two of them. The diacritics are any conics passing through the three points.

40. If the two conics are

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$
$$(a', b', c', f', g', h' \between x, y, z)^2 = 0,$$

and if the determinant formed with $a + ka'$, &c., is denoted by

$$(K, \Theta, \Theta', K' \between 1, k)^3,$$

so that

$$\begin{aligned} K &= abc \quad - af^2 - bg^2 - ch^2 + 2fgh, \\ \Theta &= a'(bc - f^2) + \&c., \\ \Theta' &= a(b'c' - f'^2) + \&c., \\ K &= a'b'c' - a'f'^2 - b'g'^2 - c'h'^2 + 2f'g'h', \end{aligned}$$

then the before-mentioned equation $\square = R = 0$ which gives the condition that the conics may touch is

$$\text{Disc}^t.\ (K, \Theta, \Theta', K' \between 1, k)^2 = 0,$$

where the left-hand side is of the order 6 in the coefficients of the two conics respectively: this is a known formula.

41. If the equation in k have a twofold root the two conics will touch: two of the critic centres will then coincide at the point of contact, or this point is a twofold critic centre: the remaining or onefold critic centre is the intersection of the common tangent and of the line joining the two points of intersection of the conics. In virtue of the general property, the first-mentioned two centres must be considered as lying on the line which is the polar of the onefold critic centre in regard to either of the conics. The diacritics pass through the critic centres, that is, they pass through the onefold centre, and touch the polar in question at the point of contact of the two conics, or twofold critic centre; this is in fact the property mentioned *ante*, No. 28.

42. The equation

$$\text{Disc}^t.\ \text{Disc}^t.\ (U + kV) = 0,$$

as applied to a conic and a circle leads at once to the equation of the curve parallel to a given conic; such parallel curve is in fact the envelope of the circles of a given constant radius which touch the given conic. This method is in effect due to Dr Salmon, who applied the corresponding theorem *in solido* to the determination of the surface parallel to an ellipsoid.

Annex, referred to No. 31. *Investigation of the order of the plexus or system for the existence of a Cusp.*

Considering for a moment the curve

$$U = (* \between x, y, z)^n = 0, \text{ let } (L, M, N), \ (a, b, c, f, g, h)$$

be the first and second differential coefficients of U; (A, B, C, F, G, H) the inverse system, viz. $A = bc - f^2$, &c. At a cusp we have

$$L = 0,\ M = 0,\ N = 0,\ A = 0,\ B = 0,\ C = 0,\ F = 0,\ G = 0,\ H = 0,$$

a system of equations which is contained in the system, $L=0$, $M=0$, $N=0$, $A=0$. But this system contains besides the cusp system the irrelevant system $L=0$, $M=0$, $N=0$, $x^2=0$. In fact the equations $L=0$, $M=0$, $N=0$ give

$$ax+hy+gz=0,$$
$$hx+by+fz=0,$$
$$gx+fy+cz=0,$$

and thence

$$x^2 : y^2 : z^2 : yz : zx : xy = A : B : C : F : G : H.$$

Hence the equation $A=0$, if $x^2=0$, implies the entire system

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0.$$

But if $L=0$, $M=0$, $N=0$, $x^2=0$, then these equations give $A=0$ (and also $H=0$, $G=0$), but they do not give the remaining equations $B=0$, $C=0$, $F=0$. Or the same thing may be shown in a less symmetrical form, but more clearly thus; we have identically

$$-cx(ax+hy+gz)+(hx-fz)(hx+by+fz)+bz(gx+fy+cz)-(bc-f^2)z^2+(ab-h^2)x^2=0,$$

whence the equations $L=0$, $M=0$, $N=0$, $A=0$ give $Cx^2=0$, that is, $C=0$, or $x^2=0$. But the equations $L=0$, $M=0$, $N=0$, $A=0$, $C=0$ give (as it is easy to show) the entire system $A=0$, $B=0$, $C=0$, $F=0$, $G=0$, $H=0$. That is, the system

$$L=0,\ M=0,\ N=0,\ A=0$$

is made up of the cusp system, and of the system $(L=0,\ M=0,\ N=0,\ A=0,\ x^2=0)$; or since $A=0$ is a consequence of the other equations, the second system is

$$(L=0,\ M=0,\ N=0,\ x^2=0).$$

Consider now the curve $\lambda U+\mu U'+\nu U''=0$, which will have a cusp if the ratios $\lambda:\mu:\nu$ are properly determined. And to each set of values of $\lambda:\mu:\nu$ there corresponds a set of values (x, y, z), the coordinates of a cusp of the curve; so that the number of such sets, that is, the number of points each whereof is the cusp of a corresponding curve $\lambda U+\mu V+\nu W=0$ is precisely equal to the number of sets of values of $\lambda:\mu:\nu$: or it is equal to the order of the system of conditions for the existence of a cusp.

Denoting as before the first and second differential coefficients of U by

$$L,\ M,\ N,\ a,\ b,\ c,\ f,\ g,\ h,$$

and those of U', U'' in a corresponding manner, and taking for the cusp the system before represented by $L=0$, $M=0$, $N=0$, $A=0$, we have

$$\lambda L+\mu L'+\nu L''=0,$$
$$\lambda M+\mu M'+\nu M''=0,$$
$$\lambda N+\mu N'+\nu N''=0,$$
$$(\lambda b+\mu b'+\nu b'')(\lambda c+\mu c'+\nu c'')-(\lambda f+\mu f'+\nu f'')^2=0,$$

which last equation, to denote that it is of the second order in regard to the differential coefficients a, b, &c., a', &c. I represent by

$$((a, b, \ldots)^2 \between \lambda, \mu, \nu)^2 = 0.$$

But this system of four equations contains not only the cusp system, but the system made of the three linear equations and the equation $x^2 = 0$. Eliminating λ, μ, ν, the last-mentioned system is

$$\begin{vmatrix} L, & L', & L'' \\ M, & M', & M'' \\ N, & N', & N'' \end{vmatrix} = 0,\ x^2 = 0,$$

where the first equation is that of a curve of the order $3(n-1)$. And the two equations give together $6(n-1)$ points, viz. the points of intersection of the curve by the line $x = 0$, each reckoned as a twofold point.

Returning to the first-mentioned system, this may be replaced by

$$\begin{vmatrix} L, & L', & L'' \\ M & M' & M'' \\ N & N' & N'' \end{vmatrix} = 0,\ ((a, b, \ldots)^2 \between L'M'' - L''M',\ L''M - LM'',\ LM' - L'M)^2 = 0,$$

which are curves of the orders $3(n-1)$ and $6n-8$ respectively. But each of these curves passes through the $3(n-1)^2$ points given by the equations

$$\left\| \begin{matrix} L, & L', & L'' \\ M, & M', & M'' \end{matrix} \right\| = 0,$$

and these points are moreover nodes on the curve of the order $6n-8$; hence the points in question reckon as $6(n-1)^2$ intersections of the two curves. The number of the remaining intersections is

$$3(n-1)(6n-8) - 6(n-1)^2 = 6(n-1)\big(3n-4-(n-1)\big) = 6(n-1)(2n-3),$$

but among these are included the $6(n-1)$ intersections of the curve of the order $3(n-1)$ by the twofold line $x^2 = 0$; or, subtracting these, the number of the remaining points is

$$6(n-1)(2n-3-1) = 12(n-1)(n-2);$$

which number is consequently the order of the cusp system.

It may be remarked that considering the entire series of equations at first denoted by $(L=0,\ M=0,\ N=0)$, $(A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0)$, the elimination of λ, μ, ν from the three linear equations gives as before

$$\begin{vmatrix} L, & L', & L'' \\ M, & M' & M'' \\ N & N' & N'' \end{vmatrix} = 0,$$

which is a curve of the order $3(n-1)$: and the eliminating of λ^2, μ^2, ν^2, $\mu\nu$, $\nu\lambda$, $\lambda\mu$ from the six quadric equations gives

$$\left|\begin{array}{l} bc-f^2,\ b'c'-f'^2,\ b''c''-f''^2,\ b'c''+b''c'-2f'f'',\ b''c+bc''-2ff'',\ bc'+b'c-2ff' \\ ca-g^2,\ \&c., \\ \vdots \end{array}\right| = 0,$$

which is a curve of the order $12(n-2)$; the two curves would intersect in $36(n-1)(n-2)$ points, but as this is precisely three times the number $12(n-1)(n-2)$, I infer that these are in fact the $12(n-1)(n-2)$ points three times repeated, that is, that each of these is a point of threefold intersection of the two curves.

Cambridge, 7th November, 1863.

349.

ON A CASE OF THE INVOLUTION OF CUBIC CURVES.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part I, (1866), pp. 39—80.—Read 22 February, 1864.]

THE present memoir relates to the involution

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0,$$

viz. treating x, y, z as coordinates, and k as a variable parameter, this equation represents the series of cubic curves passing through the intersections of the two cubics

$$xyz = 0, \quad (x+y+z)^2(\lambda x + \mu y + \nu z) = 0;$$

or, what is the same thing, the line $x+y+z=0$ meets any cubic of the series in three points the tangents at which are $x=0$, $y=0$, $z=0$, and these tangents again meet the cubic in three points lying on the line $\lambda x + \mu y + \nu z = 0$; so that in the language which I have used elsewhere, the lines $x+y+z=0$, $\lambda x + \mu y + \nu z = 0$ are in regard to the cubic a primary and a satellite line respectively. The investigation (which is a development of two short papers already published in the *Philosophical Magazine*)(1) was undertaken in order to applying it to the explanation and discussion of Plücker's Classification of Curves of the Third Order; but such application will properly be made in a separate memoir, *On the Classification of Cubic Curves*, and it has also appeared to me convenient to give therein the discussion of the geometrical forms of certain loci which present themselves in the present memoir.

I remark that the involution intended to be here considered is a case of the more general one $U+kV=0$, where $U=0$, $V=0$ are any two cubic curves whatever. It appears from my memoir *On the Theory of Involution*, [348], that the equation, Disc^t. $(U+kV)=0$, which determines the critic values of k is in the general case of the order 12; the

1 On the Cubic Centres of a Line with respect to Three Lines and a Line.—*Phil. Mag.* t. xx. pp. 418—423 (1860), [257]. *Ditto*, Second Paper, t. XXII. pp. 433—436 (1861), [315].

special case is however in the present memoir treated irrespectively of the general one, and the equation for the critic values of k is found to be of the order 3; this of course means that the equation of the order 12 breaks up into two equations of the orders 9 and 3 respectively, but I have not attempted to show how the decomposition and reduction arise. Moreover, in the general case the equation, Disc^t. Disc^t. $(U+kV)=0$, which is the condition for the existence of a twofold critic value, presents itself in the form $RQ^3P^2=0$, where $R=0$ is the condition that the two cubics $(U=0,\ V=0)$ shall touch each other; $Q=0$ the condition that there shall be in the involution $U+kV=0$ a curve having (not a mere node but) a cusp; and $P=0$ the condition that there shall be a curve having two nodes, or (what is the same thing) breaking up into a line and conic. But in the special case, which, as already noticed, is here considered irrespectively of the general one, the equation Disc^t. Disc^t. $(U+kV)=0$, for the existence of a twofold critic value presents itself in the reduced form $Q=0$, giving the condition, that corresponding to the twofold critic value there shall be a curve having (not a mere node but) a cusp.

Article Nos. 1 to 18, *Explanations, Definitions, and Results.*

1. I consider the involution

$$xyz+k(x+y+z)^2(\lambda x+\mu y+\nu z)=0,$$

where $x=0$, $y=0$, $z=0$, $x+y+z=0$ may be considered as representing any four lines no three of which meet in a point, and $\lambda x+\mu y+\nu z=0$, as representing any fifth line whatever: k is a variable parameter. The lines $x+y+z=0$, $\lambda x+\mu y+\nu z=0$, are a primary line and a satellite line of any cubic of the series, viz. the tangents $x=0$, $y=0$, $z=0$, at the points of intersection with the primary line $x+y+z=0$, meet the cubic in three points lying on the satellite line $\lambda x+\mu y+\nu z=0$.

2. A critic value of k is a value for which the corresponding curve

$$xyz+k(x+y+z)^2(\lambda x+\mu y+\nu z)=0$$

has a node; and such node, or say rather the site of such node, is a critic centre.

3. The critic values of k are in effect determined by a cubic equation, and the coordinates of the critic centre are then given rationally in terms of k; there are consequently three critic values of k; and the same number of critic centres, and of nodal curves: it is however found to be convenient to express as well the critic value of k, as the coordinates of the critic centre, rationally in terms of an auxiliary parameter θ which is given by a cubic equation.

4. The cubic equation in k (or what is the same thing, that in θ) may have a twofold root (pair of equal roots); or, say rather, it may have a twofold root and a one-with-twofold root: corresponding to the twofold value of k we have a twofold critic

centre, which is not a mere node but a cusp on the cubic, or instead of a merely nodal cubic we have a cuspidal cubic; and corresponding to the one-with-twofold value of k we have a one-with-twofold critic centre, being of course a mere node on the nodal cubic.

5. In the case in question of a twofold and one-with-twofold value of k, the line $\lambda x + \mu y + \nu z = 0$, or say the satellite line, envelopes a curve which might be termed the twofold and one-with-twofold envelope, but which is spoken of simply as the envelope.

The locus of the twofold centre is a curve which is called the twofold centre locus.

The locus of the one-with-twofold centre is a curve which is called the one-with-twofold centre locus.

These definitions premised, the following results may be stated;

6. The equation in θ may be represented in the three equivalent forms

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=0,$$

$$\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-2\lambda\mu\nu=0.$$

7. The critic value of k and the coordinates of the critic centre are then given by the equations

$$k=\frac{-\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

$$x:y:z:x+y+z:\lambda x+\mu y+\nu z=\frac{1}{\theta+\lambda}:\frac{1}{\theta+\mu}:\frac{1}{\theta+\nu}:\frac{2}{\theta}:1.$$

8. The condition for a twofold and one-with-twofold value of k is

$$\lambda^{-\frac{1}{3}}+\mu^{-\frac{1}{3}}+\nu^{-\frac{1}{3}}=0,$$

or, what is the same thing,

$$(\mu\nu+\nu\lambda+\lambda\mu)^3-27\lambda^2\mu^2\nu^2=0,$$

which equations may either of them be considered as the line-equation of the envelope. The equation in the coordinates (x, y, z), or point-equation of the envelope is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

or, in its rationalised form,

$$x^4+y^4+z^4-4(yz^3+y^3z+zx^3+z^3x+xy^3+x^3y)$$
$$+6(y^2z^2+z^2x^2+x^2y^2)-124(x^2yz+xy^2z+xyz^2)=0.$$

9. The equation of the twofold centre locus is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or, in its rationalised form,

$$x^2+y^2+z^2-2yz-2zx-2xy=0;$$

the curve is therefore a conic, and it may be spoken of as the twofold centre conic.

10. The equation of the one-with-twofold centre locus is

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

the curve is therefore a cubic, and it may be spoken of as the one-with-twofold centre cubic.

11. The before-mentioned equation $\lambda^{-\frac{1}{3}}+\mu^{-\frac{1}{3}}+\nu^{-\frac{1}{3}}=0$ is satisfied by

$$\lambda : \mu : \nu = \alpha^{-3} : \beta^{-3} : \gamma^{-3},$$

where $\alpha+\beta+\gamma=0$, and it is very convenient to introduce these quantities α, β, γ into the formulæ.

12. The equation of the satellite line giving a twofold and one-with-twofold centre is

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}=0;$$

the coordinates of the point of contact with the envelope are $x : y : z = \alpha^4 : \beta^4 : \gamma^4$.

The equation in θ gives $\theta_1=\theta_2=-\dfrac{1}{\alpha\beta\gamma}$ for the values corresponding to the twofold centre; and $\theta_3=\dfrac{2}{\alpha\beta\gamma}$ for the value corresponding to the one-with-twofold centre.

The coordinates of the twofold centre, or cusp, are $x : y : z = \alpha^2 : \beta^2 : \gamma^2$.

The coordinates of the one-with-twofold centre, or node, are

$$x : y : z = \alpha^2(\beta-\gamma) : \beta^2(\gamma-\alpha) : \gamma^2(\alpha-\beta).$$

The equation of the tangent at the cusp is

$$(\beta-\gamma)\frac{x}{\alpha}+(\gamma-\alpha)\frac{y}{\beta}+(\alpha-\beta)\frac{z}{\gamma}=0.$$

The equation of the line joining the cusp and the node, which line is also one of the tangents at the node is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0.$$

The equation of the other tangent at the node is

$$(2\beta\gamma+\alpha^2)\frac{x}{\alpha}+(2\gamma\alpha+\beta^2)\frac{y}{\beta}+(2\alpha\beta+\gamma^2)\frac{z}{\gamma}=0.$$

13. Considering the critic centre corresponding to a root θ of the cubic equation, the equation of the line joining the other two critic centres is

$$\frac{\lambda x}{\theta+\lambda}+\frac{\mu y}{\theta+\mu}+\frac{\nu z}{\theta+\nu}=0,$$

which is the polar of the critic centre in regard to the twofold centre conic. The critic centres are consequently conjugate poles in regard to the twofold centre conic.

14. The equation of the tangents at the critic centre considered as a node of the corresponding cubic curve is

$$\left(\theta+4\lambda,\ \theta+4\mu,\ \theta+4\nu,\ -\theta-\frac{2\mu\nu}{\theta},\ -\theta-\frac{2\nu\lambda}{\theta},\ -\theta-\frac{2\lambda\mu}{\theta}\right)(x,\ y,\ z)^2=0.$$

15. The last-mentioned formulæ lead to some which involve the three critic centres viz. if $X=0$, $Y=0$, $Z=0$ are the equations of the sides of the triangle formed by the critic centres, then the equations of the tangents at the three critic centres respectively are of the form

$$\begin{aligned} &\ .\quad BY^2+CZ^2=0,\\ &AX^2\quad .\quad +CZ^2=0,\\ &AX^2+BY^2\quad .\quad =0,\end{aligned}$$

so that the tangents in question are in fact the tangents from the three nodes respectively to the conic

$$AX^2+BY^2+CZ^2=0:$$

the three nodes or critic centres being thus conjugate poles in regard to the conic, this is called "the three centre conic."

16. The equation of a nodal cubic is also expressible in a simple form in terms of the new coordinates X, Y, Z. In the formulæ for these transformations, and indeed throughout the memoir, the three roots of the equation in θ are represented by θ_1, θ_2, θ_3, and I write also

$$\begin{aligned} &l_1=\theta_2-\theta_3,\quad l_2=\theta_3-\theta_1,\quad l_3=\theta_1-\theta_2,\\ &\Theta_1=(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu),\\ &\Theta_2=(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu),\\ &\Theta_3=(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu).\end{aligned}$$

17. If $\lambda a+\mu b+\nu c=0$, that is, if $(a,\ b,\ c)$ are the coordinates of a point on the line $\lambda x+\mu y+\nu z=0$, then the critic centres lie on the cubic

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-\frac{2(a+b+c)}{x+y+z}=0,$$

or, what is the same thing, this curve is the locus of the critic centres corresponding to the several lines $\lambda x+\mu y+\nu z=0$ through the point $(a,\ b,\ c)$.

18. In particular, taking in succession for the point (a, b, c) the point of intersection of the line $\lambda x + \mu y + \nu z = 0$, with the lines $x = 0$, $y = 0$, $z = 0$, $x + y + z = 0$, the critic centres lie on the conics

$$\frac{\nu}{y} - \frac{\mu}{z} - \frac{2(\nu - \mu)}{x + y + z} = 0,$$

$$\frac{\lambda}{z} - \frac{\nu}{x} - \frac{2(\lambda - \nu)}{x + y + z} = 0,$$

$$\frac{\mu}{x} - \frac{\lambda}{y} - \frac{2(\mu - \lambda)}{x + y + z} = 0,$$

$$\frac{\mu - \nu}{x} + \frac{\nu - \lambda}{y} + \frac{\lambda - \mu}{z} = 0,$$

which are useful for the construction of the critic centres for a given line $\lambda x + \mu y + \nu z = 0$. The last of the four conics passes through the point (1, 1, 1) which is the harmonic of the line $x + y + z = 0$ in regard to the triangle formed by the lines $x = 0$, $y = 0$, $z = 0$; and I call it the harmonic conic.

Article Nos. 19 to 21. *General Formulæ for the Critic Centres.*

19. I consider the involution

$$xyz + k(x + y + z)^2(\lambda x + \mu y + \nu z) = 0.$$

Writing the equation in the form

$$k(\lambda x + \mu y + \nu z) = \frac{-xyz}{(x + y + z)^2},$$

and differentiating with regard to x, y, z respectively, we obtain

$$-k\lambda(x + y + z)^3 = yz(-x + y + z),$$
$$-k\mu(x + y + z)^3 = zx(\ \ x - y + z),$$
$$-k\nu(x + y + z)^3 = xy(\ \ x + y - z),$$

which determine the coordinate ratios $x : y : z$ of the node or critic centre; and the corresponding value of k.

20. Writing the equations under the form

$$\frac{-k(x + y + z)^3}{xyz} = \frac{-x + y + z}{\lambda x} = \frac{x - y + z}{\mu y} = \frac{x + y - z}{\nu z} = \frac{2}{\theta},$$

where θ is an auxiliary parameter to be determined, we find

$$x\left(-1 - \frac{2\lambda}{\theta}\right) + y + z = 0,$$

that is

$$x+y+z=2x\left(1+\frac{\lambda}{\theta}\right),$$

and consequently

$$x+y+z=2x\left(1+\frac{\lambda}{\theta}\right)=2y\left(1+\frac{\mu}{\theta}\right)=2z\left(1+\frac{\nu}{\theta}\right),$$

or, what is the same thing,

$$x+y+z : x : y : z=\frac{2}{\theta} : \frac{1}{\lambda+\theta} : \frac{1}{\mu+\theta} : \frac{1}{\nu+\theta},$$

and we have thence

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

an equation which may also be written in the form

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=0,$$

or in the form

$$\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-2\lambda\mu\nu=0;$$

and we then have

$$k=-\frac{2}{\theta}\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\div\left(\frac{2}{\theta}\right)^3,$$

$$=-\frac{\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}.$$

21. We see that θ is determined by a cubic equation, and that the ratios $x : y : z$ and the parameter k are rational functions of θ. There are thus three nodes or critic centres, and the like number of nodal curves and of critic values of k.

The form secondly obtained for the equation in θ shows that we may write

$$x : y : z : x+y+z : \lambda x+\mu y+\nu z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} : 1.$$

Article Nos. 22 to 32, *relating to a Twofold and a One-with-Twofold Centre.*

22. If k has a twofold and a one-with-twofold value, then θ will have also a twofold and a one-with-twofold value; and conversely. The equation in θ will have a twofold and a one-with-twofold root if

$$(\mu\nu+\nu\lambda+\lambda\mu)^3-27\lambda^2\mu^2\nu^2=0;$$

or, what is the same thing, if

$$\mu\nu+\nu\lambda+\lambda\mu-3(\lambda\mu\nu)^{\frac{2}{3}}=0,$$

or if

$$(\mu\nu)^{\frac{1}{3}}+(\nu\lambda)^{\frac{1}{3}}+(\lambda\mu)^{\frac{1}{3}}=0,$$

or finally if

$$\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0,$$

so that the condition is satisfied if $\lambda = \alpha^{-3}$, $\mu = \beta^{-3}$, $\nu = \gamma^{-3}$ where $\alpha + \beta + \gamma = 0$. In fact with these values the equation in θ becomes

$$(\alpha\beta\gamma\theta)^2 - 3\alpha\beta\gamma\theta - 2 = 0,$$

that is

$$(\alpha\beta\gamma\theta + 1)^2(\alpha\beta\gamma\theta - 2) = 0,$$

so that the twofold value is $\theta_1 = \theta_2 = -\dfrac{1}{\alpha\beta\gamma}$; and the one-with-twofold value is $\theta_3 = \dfrac{2}{\alpha\beta\gamma}$.

23. It is throughout assumed that the quantities α, β, γ satisfy the condition $\alpha + \beta + \gamma = 0$. The result just obtained shows that the line

$$\frac{x}{\alpha^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3} = 0,$$

is a twofold and one-with-twofold satellite line. From this equation, considering α, β, γ as variable parameters satisfying the condition $\alpha + \beta + \gamma = 0$, we find at once the equation of the curve enveloped by the line in question, which curve is called simply the envelope—viz. the coordinates of the point of contact are found to be $x : y : z = \alpha^4 : \beta^4 : \gamma^4$, and thence the equation of the envelope is

$$\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z} = 0,$$

or rationalising, it is

$$\begin{aligned} x^4 + y^4 + z^4 - 4(yz^3 + y^3z + zx^3 + z^3x + xy^3 + x^3y) \\ + 6(y^2z^2 + z^2x^2 + x^2y^2) - 124(x^2yz + y^2zx + z^2xy) = 0. \end{aligned}$$

The before-mentioned equation $\lambda^{-\frac{1}{3}} + \mu^{-\frac{1}{3}} + \nu^{-\frac{1}{3}} = 0$, or

$$(\mu\nu + \nu\lambda + \lambda\mu)^3 - 27\lambda^3\mu^3\nu^3 = 0,$$

may be considered as the tangential equation; the envelope is thus of the order 4, and the class 6.

24. It is easy to show that the curve has three nodes the coordinates whereof are $(-4, 1, 1)$, $(1, -4, 1)$, $(1, 1, -4)$; and this being known, the equation may be transformed so as to put the nodes in evidence. I effect the transformation synthetically as follows, viz. writing $x + y + z = \sigma$, $yz + zx + xy = q$, $xyz = r$, the equation of the curve is

$$\begin{aligned} &(\sigma^4 - 2q\sigma^2 + 2q^2 + 4r\sigma) \\ -4(&\quad q\sigma^2 - 2q^2 - r\sigma) \\ +6(&\quad q^2 - 2r\sigma) \\ -124(&\quad r\sigma) = 0, \end{aligned}$$

viz. it is

$$\sigma^4 - 6q\sigma^2 + 16q^2 - 128r\sigma = 0,$$

which is

$$(7\sigma^2 + 4q)^2 - 16\sigma(3\sigma^3 + 4q\sigma + 8r) = 0,$$

or, putting for a moment $l = -\frac{6}{5}$, and therefore $5(l-2) = -16$, it is

$$(7\sigma^2 + 4q)^2 + (l-2)\,5\sigma(3\sigma^3 + 4q\sigma + 8r) = 0.$$

Now writing

$$\begin{aligned} x' = \sigma + 2x &= 3x + y + z, \\ y' = \sigma + 2y &= x + 3y + z, \\ z' = \sigma + 2z &= x + y + 3z, \end{aligned}$$

we find

$$\begin{aligned} y'z' + z'x' + x'y' &= 7\sigma^2 + 4q, \\ x' + y' + z' &= 5\sigma, \\ x'y'z' &= 3\sigma^3 + 4q\sigma + 8r, \end{aligned}$$

so that the equation is

$$(y'z' + z'x' + x'y')^2 + (l-2)\,x'y'z'(x' + y' + z') = 0,$$

that is

$$y'^2z'^2 + z'^2x'^2 + x'^2y'^2 + lx'y'z'(x' + y' + z') = 0;$$

or, putting for l its value, the equation is

$$5(y'^2z'^2 + z'^2x'^2 + x'^2y'^2) - 6x'y'z'(x' + y' + z') = 0;$$

or, as this may also be written,

$$(5,\ 5,\ 5,\ -3,\ -3,\ -3)\left(\frac{1}{x'},\ \frac{1}{y'},\ \frac{1}{z'}\right)^2 = 0;$$

a form which shows that the curve has three nodes at the angles of the triangle

$$x' = 0, \quad y' = 0, \quad z' = 0.$$

25. It is easy to see that the curve is touched by the lines $x = 0$, $y = 0$, $z = 0$ at their intersections with the lines $y - z = 0$, $z - x = 0$, $x - y = 0$ respectively, or (what is the same thing) in the points (0, 1, 1), (1, 0, 1), (1, 1, 0) respectively. It may be added that the line $y - z = 0$ meets the curve in the node $(-4, 1, 1)$, being of course a point of twofold intersection, in the point (0, 1, 1) on the line $x = 0$, and besides in the point (16, 1, 1): and the like for the lines $z - x = 0$ and $x - y = 0$.

26. It may be noticed that although any line passing through one of the nodes is in a sense a tangent to the envelope, yet that it is not a proper tangent and does not give rise to a twofold centre. It is in fact shown (*post*, Nos. 73 and 74) that the critic centres for a line $\lambda x + \mu y + \nu z = 0$ passing through the point $(-4, 1, 1)$ are three points lying, one of them on the line $y + z = 0$, and the other two on the conic $x(x + y + z) - 4yz = 0$.

27. Assume that θ has its twofold value $=-\dfrac{1}{\alpha\beta\gamma}$, the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

substituting also therein for λ, μ, ν the values α^{-3}, β^{-3}, γ^{-3}, give for the coordinates of the twofold centre

$$x : y : z = \frac{\alpha^2\beta\gamma}{\beta\gamma-\alpha^2} : \frac{\beta^2\gamma\alpha}{\gamma\alpha-\beta^2} : \frac{\alpha\beta\gamma^2}{\alpha\beta-\gamma^2},$$

but in virtue of the relation $\alpha+\beta+\gamma=0$ we have

$$\beta\gamma-\alpha^2=\beta\gamma+\gamma\alpha+\alpha\beta=\gamma\alpha-\beta^2=\alpha\beta-\gamma^2;$$

or the values are $x : y : z = \alpha^2 : \beta^2 : \gamma^2$. Hence also we have as the equation of the locus of the twofold centre,

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or, what is the same thing,

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

which is a conic touching the lines $x=0$, $y=0$, $z=0$ at their intersections with the lines $y-z=0$, $z-x=0$, $x-y=0$ respectively, or, what is the same thing, in the points (0, 1, 1), (1, 0, 1), (1, 1, 0) respectively.

28. Similarly, if θ has its one-with-twofold value $=\dfrac{2}{\alpha\beta\gamma}$, the equations

$$x : y : z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu},$$

substituting also therein for λ, μ, ν the values α^{-3}, β^{-3}, γ^{-3}, give for the coordinates of the one-with-twofold centre

$$x : y : z = \frac{\alpha^3\beta\gamma}{2\alpha^2+\beta\gamma} : \frac{\beta^3\gamma\alpha}{2\beta^2+\gamma\alpha} : \frac{\gamma^3\alpha\beta}{2\gamma^2+\alpha\beta};$$

but in virtue of $\alpha+\beta+\gamma=0$ we have

$$2\alpha^2+\beta\gamma=\alpha^2-\alpha(\beta+\gamma)+\beta\gamma=(\alpha-\beta)(\alpha-\gamma)=-(\gamma-\alpha)(\alpha-\beta),$$

and similarly

$$2\beta^2+\gamma\alpha=-(\alpha-\beta)(\beta-\gamma), \qquad 2\gamma^2+\alpha\beta=-(\beta-\gamma)(\gamma-\alpha);$$

and thence these values are

$$x : y : z = \alpha^2(\beta-\gamma) : \beta^2(\gamma-\alpha) : \gamma^2(\alpha-\beta),$$

for the coordinates of the one-with-twofold centre.

We thence deduce

$$y+z=\beta^2\gamma-\beta^2\alpha+\gamma^2\alpha-\gamma^2\beta=(\beta\gamma-\alpha\beta-\alpha\gamma)(\beta-\gamma)=(\beta\gamma+\alpha^2)(\beta-\gamma),$$

and consequently

$$-x+y+z=\beta\gamma(\beta-\gamma),$$

that is

$$\begin{aligned} x \quad &: \quad y \quad : \quad z \quad : -x+y+z : x-y+z : x+y-z \\ &= \alpha^2(\beta-\gamma) : \beta^2(\gamma-\alpha) : \gamma^2(\alpha-\beta) : \beta\gamma(\beta-\gamma) : \gamma\alpha(\gamma-\alpha) : \alpha\beta(\alpha-\beta), \end{aligned}$$

and these give

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

as the equation of the locus of the one-with-twofold centre, which locus is thus a cubic curve.

29. The equation

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

of the one-with-twofold centre locus may be transformed as follows, viz. writing for a moment $x+y+z=-w$, we have

$$\begin{aligned} &(9x+4w)(9y+4w)(9z+4w)-w^3 \\ &=729\,xyz+324\,w(yz+zx+xy)-144\,w^3+64w^3-w^3, \\ &=81\ \{9xyz+4w(yz+zx+xy)-w^3\}, \\ &=81\ \{9xyz-12xyz-4(yz^2+\&c.)+(x^3+y^3+z^3)+(3yz^2+\&c.)+6xyz\}, \\ &=81\ \{x^3+y^3+z^3-(yz^2+\&c.)+3xyz\}, \end{aligned}$$

so that the equation may be written

$$(9x+4w)(9y+4w)(9z+4w)-w^3=0,$$

or, what is the same thing,

$$(5x-4y-4z)(-4x+5y-4z)(-4x-4y+5z)+(x+y+z)^3=0,$$

which shows that the intersections of the line $x+y+z=0$, with the sides $x=0$, $y=0$, $z=0$ of the triangle are inflexions on the curve; and that the tangents at these points are respectively

$$5x-4y-4z=0,\quad -4x+5y-4z=0,\quad -4x-4y+5z=0.$$

30. The curve passes through the point (1, 1, 1), which is the harmonic of $x+y+z=0$ in regard to the triangle; and this point is moreover a node on the curve; in fact if the equation be represented by $W=0$, then we have

$$d_xW=3x^2-2x(y+z)-y^2+3yz-z^2,$$

$=0$ for the point in question; and similarly $d_yW=0$, and $d_zW=0$.

31. The equation for the twofold centre conic may also be obtained as follows: viz. the equations

$$\frac{-x+y+z}{\lambda x}=\frac{x-y+z}{\mu y}=\frac{x+y-z}{\nu z}=\frac{2}{\theta},$$

give

$$\frac{(-x+y+z)(x-y+z)(x+y-z)}{xyz}=\frac{8}{\theta^3}\lambda\mu\nu=\frac{8}{\theta^2(\alpha\beta\gamma)^3},$$

which substituting for θ the twofold value $=-\dfrac{1}{\alpha\beta\gamma}$,

gives

$$(-x+y+z)(x-y+z)(x+y-z)+8xyx=0,$$

an equation which may be written

$$(x+y+z)^2(x^2+y^2+z^2-2yz-2zx-2xy)=0,$$

which is the former result affected by the extraneous factor $x+y+z$.

If, instead, we substitute for θ the onefold value $=\dfrac{2}{\alpha\beta\gamma}$, we find

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0,$$

which is the one-with-twofold centre cubic.

32. Recollecting that

$$k=\frac{-\frac{1}{4}\theta^2}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

we deduce for the twofold value of k

$$k_1=k_2=\frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta\gamma-\alpha^2)(\gamma\alpha-\beta^2)(\alpha\beta-\gamma^2)},$$

$$=\frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta\gamma+\gamma\alpha+\alpha\beta)^3},$$

$$=\frac{2\alpha^3\beta^3\gamma^3}{(\alpha^2+\beta^2+\gamma^2)^3},$$

and for the one-with-twofold value,

$$k_3=\frac{-\frac{1}{4}\alpha^3\beta^3\gamma^3}{(2\alpha^2+\beta\gamma)(2\beta^2+\gamma\alpha)(2\gamma^2+\alpha\beta)}$$

$$=\frac{\frac{1}{4}\alpha^3\beta^3\gamma^3}{(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2}.$$

Article Nos. 33 to 38, *relating to the Tangents at a Node or Critic Centre.*

33. I proceed to investigate the equation of the tangents at the node of the curve

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0;$$

it will be recollected that if x, y, z are the coordinates of the node, then we have

$$x : y : z : x+y+z : \lambda x + \mu y + \nu z = \frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} : 1.$$

Representing for a moment the equation of the curve by $U=0$, then the second derived functions of U are

$$\begin{aligned} &k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\lambda, \\ &k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\mu, \\ &k\,.\,2(\lambda x + \mu y + \nu z) + 4k(x+y+z)\nu, \end{aligned}$$

$$\begin{aligned} &x + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\mu+\nu), \\ &y + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\nu+\lambda), \\ &z + k\,.\,2(\lambda x + \mu y + \nu z) + 2k(x+y+z)(\lambda+\mu), \end{aligned}$$

or calling these (a, b, c, f, g, h) respectively, and substituting the values $x = \frac{1}{\theta+1}$, &c., we find

$$a = 2k + \frac{8k\lambda}{\theta}, \quad = \frac{2k}{\theta}(\theta + 4\lambda),$$

with the like values for b, c; and

$$f = \frac{1}{\theta+\lambda} + 2k + \frac{4k}{\theta}(\mu+\nu) = \frac{2k}{\theta}\left(\frac{\theta}{\theta+\lambda}\,\frac{1}{2k} + \theta + 2\mu + 2\nu\right),$$

where the term in ( ) is

$$\begin{aligned} &= \frac{\theta}{\theta+\lambda}\cdot\frac{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}{-\frac{1}{2}\theta^2} + \theta + 2\mu + 2\nu, \\ &= \frac{-2(\theta+\mu)(\theta+\nu)}{\theta} + \theta + 2\mu + 2\nu, \\ &= -2\theta - 2\mu - 2\nu - \frac{2\mu\nu}{\theta} + \theta + 2\mu + 2\nu, \\ &= -\theta - \frac{2\mu\nu}{\theta}, \end{aligned}$$

that is

$$f = \frac{2k}{\theta}\left(-\theta - \frac{2\mu\nu}{\theta}\right),$$

with the like values for g and h; or omitting the common factor $\frac{2k}{\theta}$, we have

$$(a,\ b,\ c,\ f,\ g,\ h) = \left(\theta + 4\lambda,\ \theta + 4\mu,\ \theta + 4\nu,\ -\theta - \frac{2\mu\nu}{\theta},\ -\theta - \frac{2\nu\lambda}{\theta},\ -\theta - \frac{2\lambda\mu}{\theta}\right),$$

and thence, taking now x, y, z as current coordinates, the equation of the tangents at the node is

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0.$$

34. Substituting for λ, μ, ν the values α^{-3}, β^{-3}, γ^{-3}, and for θ the twofold value $-\frac{1}{\alpha\beta\gamma}$, the equation of the tangents at the twofold centre becomes

$$\left(\beta^2\gamma^2(4\beta\gamma - \alpha^2), \ldots,\ \alpha^2\beta\gamma(\beta\gamma + 2\alpha^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

which is at once reduced to

$$\left(\beta^2\gamma^2(\beta - \gamma)^2, \ldots,\ \alpha^2\beta\gamma(\gamma - \alpha)(\alpha - \beta), \ldots\right)(x,\ y,\ z)^2 = 0,$$

or, what is the same thing,

$$\{\beta\gamma(\beta - \gamma)x + \gamma\alpha(\gamma - \alpha)y + \alpha\beta(\alpha - \beta)z\}^2 = 0,$$

which shows that the twofold centre is a cusp, and that the tangent is

$$\beta\gamma(\beta - \gamma)x + \gamma\alpha(\gamma - \alpha)y + \alpha\beta(\alpha - \beta)z,$$

or, what is the same thing,

$$(\beta - \gamma)\frac{x}{\alpha} + (\gamma - \alpha)\frac{y}{\beta} + (\alpha - \beta)\frac{z}{\gamma} = 0.$$

35. Writing in like manner λ, μ, $\nu = \alpha^{-3}$, β^{-3}, γ^{-3}, and θ for the one-with-twofold value $= \frac{2}{\alpha\beta\gamma}$, we find for the equation of the tangents at the one-with-twofold centre

$$\left(2\beta^2\gamma^2(2\beta\gamma + \alpha^2), \ldots,\ -\alpha^2\beta\gamma(2\beta\gamma + \alpha^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

which may be reduced to

$$\left(2\beta^2\gamma^2(2\beta\gamma + \alpha^2), \ldots,\ \alpha^2\beta\gamma(2\gamma\alpha + \beta^2 + 2\alpha\beta + \gamma^2), \ldots\right)(x,\ y,\ z)^2 = 0,$$

or, what is the same thing,

$$(\beta\gamma x + \gamma\alpha y + \alpha\beta z)\{(2\beta\gamma + \alpha^2)\beta\gamma x + (2\gamma\alpha + \beta^2)\gamma\alpha y + (2\alpha\beta + \gamma^2)\alpha\beta z\} = 0.$$

Hence at the one-with-twofold centre the equation of one of the tangents is

$$(2\beta\gamma + \alpha^2)\beta\gamma x + (2\gamma\alpha + \beta^2)\gamma\alpha y + (2\alpha\beta + \gamma^2)\alpha\beta z = 0,$$

or, as this may otherwise be written,

$$(2\beta\gamma + \alpha^2)\frac{x}{\alpha} + (2\gamma\alpha + \beta^2)\frac{y}{\beta} + (2\alpha\beta + \gamma^2)\frac{z}{\gamma} = 0.$$

36. The equation of the other tangent is

$$\beta\gamma x + \gamma\alpha y + \alpha\beta z = 0,$$

or, what is the same thing,

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0.$$

This is in fact equivalent to

$$\begin{vmatrix} x, & y, & z \\ \alpha^2, & \beta^2, & \gamma^2 \\ \alpha^2(\beta-\gamma), & \beta^2(\gamma-\alpha), & \gamma^2(\alpha-\beta) \end{vmatrix} = 0;$$

for, developing the determinant, we find

$$x\,.\,\beta^2\gamma^2(2\alpha-\beta-\gamma) + y\,.\,\gamma^2\alpha^2(2\beta-\gamma-\alpha) + z\,.\,\alpha^2\beta^2(2\gamma-\alpha-\beta) = 0,$$

or, what is the same thing,

$$\alpha^2\beta^2\gamma^2\left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}\right) = 0;$$

hence the line

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0,$$

which is one of the tangents at the one-with-twofold centre, is also the line joining this point with the twofold centre.

37. The equation of the tangents at a critic centre or node may be obtained in a different form, involving, instead of the parameter θ, the coordinates (x, y, z) of the node. We have

$$(\theta+\lambda)x = \frac{2}{\theta}(\ \ x+y+z),$$

or, what is the same thing,

$$\lambda x = \frac{2}{\theta}(-x+y+z),$$

and similarly

$$\mu y = \frac{2}{\theta}(\ \ x-y+z),$$

$$\nu z = \frac{2}{\theta}(\ \ x+y-z),$$

thence also

$$(\theta+4\lambda)x = \theta(x-2x+2y+2z) = \theta(-x+2y+2z),$$

and

$$\left(\theta + \frac{2\mu\nu}{\theta}\right)yz = \theta\{yz + \tfrac{1}{2}(x-y+z)(x+y-z)\},$$

$$= \tfrac{1}{2}\theta\{2yz + x^2 - (y-z)^2\},$$

$$= \tfrac{1}{2}\theta(x^2 - y^2 - z^2 + 4yz)$$

from which we obtain

$$\begin{aligned} a : b : c : f : g : h \\ = \; & 2yz(-x+2y+2z) \\ : \; & 2zx(2x-y+z) \\ : \; & 2xy(2x+2y-z) \\ : \; & -x(x^2-y^2-z^2+4yz) \\ : \; & -y(-x^2+y^2-z^2+4zx) \\ : \; & -z(-x^2-y^2+z^2+4xy), \end{aligned}$$

which are the required new forms.

38. We have

$$\begin{aligned} bc-f^2 &= 4x^2yz(2x-y+z)(2x+2y-z)-x^2(x^2-y^2-z^2+4yz)^2 \\ &= (x+y+z)^2(x^2+y^2+z^2-2yz-2zx-2xy), \end{aligned}$$

which is $=0$, if $x+y+z=0$, or if $x^2-2x(y+z)+(y-z)^2=0$. In the former case, viz. if $x+y+z=0$, we find $a=b=c=f=g=h=-6xyz$, and therefore

$$(a, b, c, f, g, h \between x', y', z')^2 = -6xyz(x'+y'+z')^2,$$

but this corresponds merely to the value $k=\infty$, for which the cubic is

$$(x+y+z)^2(\lambda x+\mu y+\nu z)=0,$$

which is not a proper cuspidal curve. In the latter case, or where

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

or, what is the same thing, $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, we have a proper cuspidal curve.

Article Nos. 39 to 43, *relating to the Triangle of the Critic Centres.*

39. The equation

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

is satisfied by substituting therein

$$x : y : z = \frac{1}{\theta_2+\lambda} : \frac{1}{\theta_2+\mu} : \frac{1}{\theta_2+\nu}, \text{ or } x : y : z = \frac{1}{\theta_3+\lambda} : \frac{1}{\theta_3+\mu} : \frac{1}{\theta_3+\nu};$$

in fact, for the first set of values the equation becomes

$$\frac{\lambda}{(\theta_1+\lambda)(\theta_2+\lambda)}+\frac{\mu}{(\theta_1+\mu)(\theta_2+\mu)}+\frac{\nu}{(\theta_1+\nu)(\theta_2+\nu)}=0,$$

or as this may be written

$$\left(\frac{\lambda}{\theta_1+\lambda}+\frac{\mu}{\theta_1+\mu}+\frac{\nu}{\theta_1+\nu}\right)-\left(\frac{\lambda}{\theta_2+\lambda}+\frac{\mu}{\theta_2+\mu}+\frac{\nu}{\theta_2+\nu}\right)=0,$$

that is, $1-1=0$, and similarly for the second set of values. Hence the equation in question is that of the line joining the critic centres corresponding to the roots θ_2 and θ_3. Hence

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

$$\frac{\lambda x}{\theta_2+\lambda}+\frac{\mu y}{\theta_2+\mu}+\frac{\nu z}{\theta_2+\nu}=0,$$

$$\frac{\lambda x}{\theta_3+\lambda}+\frac{\mu y}{\theta_3+\mu}+\frac{\nu z}{\theta_3+\nu}=0,$$

are the equations of the sides of the triangle formed by the three critic centres.

40. It is to be remarked that the line

$$\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}=0,$$

is the polar of the critic centre $\left(\frac{1}{\theta_1+\lambda}, \frac{1}{\theta_1+\mu}, \frac{1}{\theta_1+\nu}\right)$ in regard to the twofold centre conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0:$$

in fact, forming the equation of the polar in question, this is,

$$\left(-\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}\right)x+\left(\frac{1}{\theta_1+\lambda}-\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}\right)y+\left(\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}-\frac{1}{\theta_1+\nu}\right)z=0;$$

but from the equation in θ,

$$-\frac{1}{\theta_1+\lambda}+\frac{1}{\theta_1+\mu}+\frac{1}{\theta_1+\nu}=\frac{2}{\theta_1}-\frac{2}{\theta_1+\lambda}=\frac{2\lambda}{\theta_1+\lambda},$$

and the like for the coefficients of y and z; this proves the theorem, and it thus appears that the critic centres are conjugate poles in regard to the twofold centre conic.

Article Nos. 41 to 50. *Transformation of the Equation of the Nodal Tangents; the Three-Centre Conic.*

41. Writing as above,

$$l_1=\theta_2-\theta_3,\quad l_2=\theta_3-\theta_1,\quad l_3=\theta_1-\theta_2,$$

$$\Theta_1=(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu),$$

$$\Theta_2=(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu),$$

$$\Theta_3=(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu).$$

I put for greater convenience

$$X = -\frac{\Theta_1}{l_2 l_3}\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right),$$

$$Y = -\frac{\Theta_2}{l_3 l_1}\left(\frac{\lambda x}{\theta_2+\lambda}+\frac{\mu y}{\theta_2+\mu}+\frac{\nu z}{\theta_2+\nu}\right),$$

$$Z = -\frac{\Theta_3}{l_1 l_2}\left(\frac{\lambda x}{\theta_3+\lambda}+\frac{\mu y}{\theta_3+\mu}+\frac{\nu z}{\theta_3+\nu}\right),$$

so that $X=0$, $Y=0$, $Z=0$ are the equations of the sides of the triangle formed by the critic centres.

42. Then X, Y, Z may if we please be considered as new coordinates replacing the original coordinates x, y, z; the relation between the two sets being given by the equations last written down; the values of x, y, z in terms of X, Y, Z are given by the converse system

$$x = \frac{1}{\theta_1+\lambda}X + \frac{1}{\theta_2+\lambda}Y + \frac{1}{\theta_3+\lambda}Z,$$

$$y = \frac{1}{\theta_1+\mu}X + \frac{1}{\theta_2+\mu}Y + \frac{1}{\theta_3+\mu}Z,$$

$$z = \frac{1}{\theta_1+\nu}X + \frac{1}{\theta_2+\nu}Y + \frac{1}{\theta_3+\nu}Z.$$

43. To show the identity of the two systems, I start from the last-mentioned one; this gives

$$\begin{vmatrix} x, & \frac{1}{\theta_2+\lambda}, & \frac{1}{\theta_3+\lambda} \\ y, & \frac{1}{\theta_2+\mu}, & \frac{1}{\theta_3+\mu} \\ z, & \frac{1}{\theta_2+\nu}, & \frac{1}{\theta_3+\nu} \end{vmatrix} = X \begin{vmatrix} \frac{1}{\theta_1+\lambda}, & \frac{1}{\theta_2+\lambda}, & \frac{1}{\theta_3+\lambda} \\ \frac{1}{\theta_1+\mu}, & \frac{1}{\theta_2+\mu}, & \frac{1}{\theta_3+\mu} \\ \frac{1}{\theta_1+\nu}, & \frac{1}{\theta_2+\nu}, & \frac{1}{\theta_3+\nu} \end{vmatrix},$$

where the coefficient of X is

$$= \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_3)}{(\theta_1+\lambda)(\theta_1+\mu)(\theta_1+\nu)(\theta_2+\lambda)(\theta_2+\mu)(\theta_2+\nu)(\theta_3+\lambda)(\theta_3+\mu)(\theta_3+\nu)},$$

or, what is the same thing,

$$= \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)\, l_1 l_2 l_3}{\Theta_1\Theta_2\Theta_3}.$$

The first side is a linear function of x, y, z which vanishes for

$$x : y : z = \frac{1}{\theta_2+\lambda} : \frac{1}{\theta_2+\mu} : \frac{1}{\theta_2+\nu},$$

and for

$$x : y : z = \frac{1}{\theta_3+\lambda} : \frac{1}{\theta_3+\mu} : \frac{1}{\theta_3+\nu};$$

and hence it is of the form

$$K\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right),$$

and by comparing the coefficients of x we have

$$K\frac{\lambda}{\theta_1+\lambda}=\frac{1}{(\theta_2+\mu)(\theta_3+\nu)}-\frac{1}{(\theta_2+\nu)(\theta_3+\mu)}=\frac{(\theta_2-\theta_3)(\mu-\nu)}{(\theta_2+\mu)(\theta_2+\nu)(\theta_3+\mu)(\theta_3+\nu)},$$

$$=\frac{l_1(\mu-\nu)(\theta_2+\lambda)(\theta_3+\lambda)}{\Theta_2\Theta_3},$$

that is

$$K=\frac{l_1(\mu-\nu)(\theta_1+\lambda)(\theta_2+\lambda)(\theta_3+\lambda)}{\lambda\Theta_2\Theta_3},$$

and it is easy to see that

$$(\theta_1+\lambda)(\theta_2+\lambda)(\theta_3+\lambda)=-\lambda(\nu-\lambda)(\lambda-\mu),$$

so that

$$K=\frac{-l_1(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\Theta_2\Theta_3},$$

and the equation becomes

$$X=\frac{-\Theta_1}{l_2l_3}\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right),$$

which is right; and similarly for the values of Y and Z.

44. The equation of the tangents at the node corresponding to the root θ_1 is

$$\left(\theta_1+4\lambda,\ \theta_1+4\mu,\ \theta_1+4\nu,\ -\theta_1-\frac{2\mu\nu}{\theta_1},\ -\theta_1-\frac{2\nu\lambda}{\theta_1},\ -\theta_1-\frac{2\lambda\mu}{\theta_1}\right)(x,\ y,\ z)^2=0;$$

and substituting for x, y, z their values in terms of X, Y, Z, it appears in the first place that the coefficients of X^2, XY, XZ, YZ, all of them vanish.

45. In fact

$$\text{coeff. } X^2=\left(\theta_1+4\lambda,\ldots,\ -\theta_1-\frac{2\mu\nu}{\theta_1},\ldots\right)\left(\frac{1}{\theta_1+\lambda},\ \frac{1}{\theta_1+\mu},\ \frac{1}{\theta_1+\nu}\right),$$

$$=\Sigma\frac{\theta_1+4\lambda}{(\theta_1+\lambda)^2}-2\Sigma\frac{\theta_1+\dfrac{2\mu\nu}{\theta_1}}{(\theta_1+\mu)(\theta_1+\nu)}.$$

First term is

$$=\Sigma\frac{1}{\theta_1+\lambda}+3\Sigma\frac{\lambda}{(\theta_1+\lambda)^2},$$

$$=\frac{2}{\theta_1}+\frac{3}{\Theta_1}\{3\theta_1^2-(\mu\nu+\nu\lambda+\lambda\mu)\},$$

where the value in question for $\Sigma \frac{\lambda}{(\theta_1+\lambda)^2}$ is most readily found from the identical equation

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}-1=-\frac{\theta^3-\theta(\mu\nu+\nu\lambda+\lambda\mu)-\lambda\mu\nu}{\Theta}$$

by differentiating and then writing $\theta=\theta_1$.

Second term is

$$=-\frac{2}{\Theta_1}\Sigma(\theta_1+\lambda)\left(\theta_1+\frac{2\mu\nu}{\theta_1}\right),$$

$$=-\frac{2}{\Theta_1}\Sigma\left(\theta_1^2+\lambda\theta_1+2\mu\nu+\frac{2\lambda\mu\nu}{\theta_1}\right),$$

$$=-\frac{2}{\Theta_1}\left\{3\theta_1^2+(\lambda+\mu+\nu)\theta_1+2(\mu\nu+\nu\lambda+\lambda\mu)+\frac{6\lambda\mu\nu}{\theta_1}\right\}.$$

Whole is $=\frac{1}{\Theta_1}$ multiplied into

$$\frac{2}{\theta_1}\Theta_1+3\{3\theta_1^2-(\mu\nu+\nu\lambda+\lambda\mu)\}$$
$$-2\left\{3\theta_1^2(\lambda+\mu+\nu)\theta_1+2(\mu\nu+\nu\lambda+\lambda\mu)+\frac{6\lambda\mu\nu}{\theta_1}\right\},$$

$$=\frac{5}{\theta_1}\{\theta_1^3-(\mu\nu+\nu\lambda+\lambda\mu)\theta_1-2\lambda\mu\nu\},$$

which is $=0$.

46. We have next

$$\text{coeff. } XY=\left(\theta_1+4\lambda,\ldots,-\theta_1-\frac{2\mu\nu}{\theta_1},\ldots\right)\left(\frac{1}{\theta_1+\lambda},\frac{1}{\theta_1+\mu},\frac{1}{\theta_1+\nu}\right)\left(\frac{1}{\theta_2+\lambda},\frac{1}{\theta_2+\mu},\frac{1}{\theta_2+\nu}\right)$$
$$=\Sigma\frac{\theta_1+4\lambda}{(\theta_1+\lambda)(\theta_2+\lambda)}-\Sigma\left(\theta_1+\frac{2\mu\nu}{\theta_1}\right)\left\{\frac{1}{(\theta_1+\mu)(\theta_2+\nu)}+\frac{1}{(\theta_1+\nu)(\theta_2+\lambda)}\right\}.$$

First term is

$$=\Sigma\frac{1}{\theta_2+\lambda}+3\Sigma\frac{\lambda}{(\theta_1+\lambda)(\theta_2+\lambda)},$$

$$=\Sigma\frac{1}{\theta_2+\lambda}+\frac{3}{\theta_1-\theta_2}\Sigma\left(\frac{\lambda}{\theta_1+\lambda}-\frac{\lambda}{\theta_2+\lambda}\right),$$

$$=\frac{2}{\theta_2},$$

$\left(\text{since } \Sigma\frac{\lambda}{\theta_1+\lambda}=1=\Sigma\frac{\lambda}{\theta_2+\lambda}\right)$.

Second term, writing it out in full and collecting the terms which contain $\frac{1}{\theta_1+\lambda}$, is

$$=-\Sigma\frac{\theta_1}{\theta_1+\lambda}\left(\frac{1}{\theta_2+\mu}+\frac{1}{\theta_2+\nu}\right)-2\Sigma\frac{\lambda}{\theta_1(\theta_1+\lambda)}\left(\frac{\mu}{\theta_2+\mu}+\frac{\nu}{\theta_2+\nu}\right),$$

whereof the first part is

$$=-\theta_1\Sigma\frac{1}{\theta_1+\lambda}\Sigma\frac{1}{\theta_2+\lambda}+\theta_1\Sigma\frac{1}{\theta_1+\lambda}\frac{1}{\theta_2+\lambda},$$

$$=-\theta_1\Sigma\frac{1}{\theta_1+\lambda}\Sigma\frac{1}{\theta_2+\lambda}+\frac{\theta_1}{\theta_1-\theta_2}\Sigma\left(\frac{1}{\theta_2+\lambda}-\frac{1}{\theta_1+\lambda}\right),$$

$$=-\theta_1\cdot\frac{2}{\theta_1}\cdot\frac{2}{\theta_2}+\frac{\theta_1}{\theta_1-\theta_2}\left(\frac{2}{\theta_2}-\frac{2}{\theta_1}\right),\ =-\frac{4}{\theta_2}+\frac{2}{\theta_2},$$

$$=-\frac{2}{\theta_2}:$$

and the second part is

$$=-\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_1+\lambda}\Sigma\frac{\lambda}{\theta_2+\lambda}-\Sigma\frac{\lambda^2}{(\theta_1+\lambda)(\theta_2+\lambda)}\right\},$$

$$=-\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_1+\lambda}\Sigma\frac{\lambda}{\theta_2+\lambda}-\frac{1}{\theta_1-\theta_2}\Sigma\left(\frac{\lambda^2}{\theta_2+\lambda}-\frac{\lambda^2}{\theta_1+\lambda}\right)\right\};$$

and observing that

$$\Sigma\frac{\lambda^2}{\theta_1+\lambda}=\Sigma\frac{\{(\theta_1+\lambda)-\theta_1\}^2}{\theta_1+\lambda}=\Sigma(\theta_1+\lambda)-2\theta_1\Sigma_1+\theta_1^2\Sigma\frac{1}{\theta_1+\lambda},$$

$$=3\theta_1+(\lambda+\mu+\nu)-6\theta_1+\theta_1^2\cdot\frac{2}{\theta_1},$$

$$=-\theta_1+\lambda+\mu+\nu,$$

with the like value for $\Sigma\frac{\lambda^2}{\theta_2+\lambda}$, the second part is

$$=-\frac{2}{\theta_1}\left\{1-\frac{1}{\theta_1-\theta_2}(-\theta_2+\theta_1)\right\},\ =-\frac{2}{\theta_1}(1-1),\ =0.$$

47. Hence the whole second term is $=-\frac{2}{\theta_2}$, and combining the two terms we have

$$\text{coeff. } XY=\frac{2}{\theta_2}-\frac{2}{\theta_2}=0.$$

In the same manner precisely it appears that

$$\text{coeff. } XZ= \qquad 0.$$

48. Next,

$$\text{coeff. } YZ = \left\{\theta_1 + 4\lambda, \ldots, -\left(\theta_1 + \frac{2\mu\nu}{\theta_1}\right), \ldots\right\}\left(\frac{1}{\theta_2+\lambda}, \frac{1}{\theta_2+\mu}, \frac{1}{\theta_2+\nu}\right)\left(\frac{1}{\theta_3+\lambda}, \frac{1}{\theta_3+\mu}, \frac{1}{\theta_3+\nu}\right).$$

First term is

$$= \frac{\theta_1}{\theta_2-\theta_3}\Sigma\left(\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2+\lambda}\right)$$

$$+ \frac{4}{\theta_2-\theta_3}\Sigma\left(\frac{\lambda}{\theta_3+\lambda} - \frac{\lambda}{\theta_2+\lambda}\right),$$

which is

$$= \frac{\theta_1}{\theta_2-\theta_3}\left(\frac{2}{\theta_3} - \frac{2}{\theta_2}\right) + \frac{4}{\theta_2-\theta_3}(1-1), = \frac{2\theta_1}{\theta_2\theta_3}.$$

Second term, writing it out at full length and rearranging the parts, is easily seen to be

$$= -\theta_1\left\{\Sigma\frac{1}{\theta_2+\lambda}\Sigma\frac{1}{\theta_3+\lambda} - \Sigma\frac{1}{(\theta_2+\lambda)(\theta_3+\lambda)}\right\}$$

$$-\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_2+\lambda}\Sigma\frac{\lambda}{\theta_3+\lambda} - \Sigma\frac{\lambda^2}{(\theta_2+\lambda)(\theta_3+\lambda)}\right\},$$

where the first part is

$$= -\theta_1\left\{\Sigma\frac{1}{\theta_2+\lambda}\Sigma\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2-\theta_3}\Sigma\left(\frac{1}{\theta_3+\lambda} - \frac{1}{\theta_2+\lambda}\right)\right\},$$

$$= -\theta_1\left\{\frac{2}{\theta_2}\cdot\frac{2}{\theta_3} - \frac{1}{\theta_2-\theta_3}\left(\frac{2}{\theta_3} - \frac{2}{\theta_2}\right)\right\}, = -\theta_1\left(\frac{4}{\theta_2\theta_3} - \frac{2}{\theta_2\theta_3}\right),$$

$$= -\frac{2\theta_1}{\theta_2\theta_3},$$

and the second part is

$$= -\frac{2}{\theta_1}\left\{\Sigma\frac{\lambda}{\theta_2+\lambda}\Sigma\frac{\lambda}{\theta_3+\lambda} - \frac{1}{\theta_2-\theta_3}\Sigma\left(\frac{\lambda^2}{\theta_3+\lambda} - \frac{\lambda^2}{\theta_2+\lambda}\right)\right\},$$

$$= -\frac{2}{\theta_1}\left\{1 - \frac{1}{\theta_2-\theta_3}(\theta_2-\theta_3)\right\}, = 0,$$

so that the whole second term is

$$= -\frac{2\theta_1}{\theta_2\theta_3},$$

whence combining the two terms we have

$$\text{coeff. } YZ = \frac{2\theta_1}{\theta_2\theta_3} - \frac{2\theta_1}{\theta_2\theta_3} = 0.$$

49. We have now to find the coefficients of Y^2 and Z^2.

$$\text{coeff. } Y^2 = \left(\theta_1 + 4\lambda, \ldots, -\theta_1 - \frac{2\mu\nu}{\theta_1}, \ldots\right)\left(\frac{1}{\theta_2+\lambda}, \frac{1}{\theta_2+\mu}, \frac{1}{\theta_2+\nu}\right)^2,$$

$$= \Sigma \frac{\theta_1 + 4\lambda}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_1 + \dfrac{2\mu\nu}{\theta_1}}{(\theta_2+\mu)(\theta_2+\nu)},$$

$$= \Sigma \frac{\theta_1 - \theta_2 + \theta_2 + 4\lambda}{(\theta_2+\lambda)^3} - 2\Sigma \frac{\theta_1 - \theta_2 + \dfrac{2\mu\nu}{\theta_1} - \dfrac{2\mu\nu}{\theta_2} + \theta_2 + \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)},$$

and observing that the terms

$$\Sigma \frac{\theta_2 + 4\lambda}{(\theta_2+\lambda)^3} - 2\Sigma \frac{\theta_2 + \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)}$$

only differ from those of coeff. X^2 by having θ_2 in the place of θ_1 and are therefore $=0$, we have

$$\text{coeff. } Y^2 = \Sigma \frac{\theta_1 - \theta_2}{(\theta_2+\lambda)^2} - 2\Sigma \frac{\theta_1 - \theta_2 + \dfrac{2\mu\nu}{\theta_1} - \dfrac{2\mu\nu}{\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)},$$

$$= (\theta_1 - \theta_2)\left\{\Sigma \frac{1}{(\theta_2+\lambda)^2} - 2\Sigma \frac{1 - \dfrac{2\mu\nu}{\theta_1\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)}\right\}.$$

Here

$$\Sigma \frac{1}{(\theta_2+\lambda)^2} = \Sigma\left\{\frac{-1}{(\theta_2+\lambda)(\theta_2+\mu)} - \frac{1}{(\theta_2+\lambda)(\theta_2+\nu)} + \frac{2}{\theta_2(\theta_2+\lambda)}\right\}.$$

$$= \frac{1}{\Theta_2}\Sigma\left\{-(\theta_2+\nu) - (\theta_2+\mu) + \frac{2}{\theta_2}(\theta_2+\mu)(\theta_2+\nu)\right\},$$

$$= \frac{1}{\Theta_2}\Sigma\left(\mu + \nu + \frac{2\mu\nu}{\theta_2}\right),$$

$$= \frac{1}{\Theta_2}\left\{2(\lambda+\mu+\nu) + \frac{2(\mu\nu+\nu\lambda+\lambda\mu)}{\theta_2}\right\},$$

and

$$-2\Sigma \frac{1 - \dfrac{2\mu\nu}{\theta_1\theta_2}}{(\theta_2+\mu)(\theta_2+\nu)} = -\frac{2}{\Theta_2}\Sigma\left(1 - \frac{2\mu\nu}{\theta_1\theta_2}\right)(\theta_2+\lambda),$$

$$= -\frac{2}{\Theta_2}\Sigma\left(\theta_2 + \lambda - \frac{2\mu\nu}{\theta_1} - \frac{2\lambda\mu\nu}{\theta_1\theta_2}\right),$$

$$= -\frac{2}{\Theta_2}\left\{3\theta_2 + \lambda + \mu + \nu - \frac{2(\mu\nu+\nu\lambda+\lambda\mu)}{\theta_1} - \frac{6\lambda\mu\nu}{\theta_1\theta_2}\right\},$$

and hence, substituting for $\theta_1 - \theta_2$ its value $= l_3$,

$$\text{coeff. } Y^2 = \frac{2l_3}{\Theta_2}\left\{-3\theta_2 + \frac{\mu\nu + \nu\lambda + \lambda\mu}{\theta_2} + \frac{2(\mu\nu + \nu\lambda + \lambda\mu)}{\theta_1} + \frac{6\lambda\mu\nu}{\theta_1\theta_2}\right\},$$

$$= \frac{2l_3}{\Theta_2\theta_1\theta_2}\{-3\theta_1\theta_2^2 + (\mu\nu + \nu\lambda + \lambda\mu)(\theta_1 + 2\theta_2) + 6\lambda\mu\nu\};$$

but we have $\theta_1 + \theta_2 + \theta_3 = 0$, whence $\theta_1 + 2\theta_2 = \theta_2 - \theta_3 = l_1$; $\mu\nu + \nu\lambda + \lambda\mu = -(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)$, $2\lambda\mu\nu = \theta_1\theta_2\theta_3$, and therefore $-3\theta_1\theta_2^2 + 6\lambda\mu\nu = -3\theta_1\theta_2^2 + 3\theta_1\theta_2\theta_3 = -3\theta_1\theta_2(\theta_2 - \theta_3) = -3l_1\theta_1\theta_2$; and hence

$$\text{coeff. } Y^2 = \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}\{-3\theta_1\theta_2 - \theta_1\theta_2 - \theta_3(\theta_1 + \theta_2)\},$$

$$= \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}\{-4\theta_1\theta_2 + (\theta_1 + \theta_2)^2\},$$

$$= \frac{2l_1l_3}{\Theta_2\theta_1\theta_2}(\theta_1 - \theta_2)^2;$$

that is

$$\text{coeff. } Y^2 = \frac{2l_1l_3^3}{\Theta_2\theta_1\theta_2},$$

and, by merely interchanging θ_2 and θ_3,

$$\text{coeff. } Z^2 = \frac{2l_1l_2^3}{\Theta_3\theta_1\theta_3}.$$

50. Hence the equation of the tangents is

$$\frac{2l_1l_3^3}{\Theta_2\theta_1\theta_2}Y^2 + \frac{2l_1l_2^3}{\Theta_3\theta_1\theta_3}Z^2 = 0,$$

or, what is the same thing,

$$\frac{1}{l_2^3\theta_2\Theta_2}Y^2 + \frac{1}{l_3^3\theta_3\Theta_3}Z^2 = 0,$$

or putting

$$A = \frac{1}{l_1^3\theta_1\Theta_1},\quad B = \frac{1}{l_2^3\theta_2\Theta_2},\quad C = \frac{1}{l_3^3\theta_3\Theta_3},$$

the equation of the tangents at the node corresponding to θ_1 is $BY^2 + CZ^2 = 0$. And hence the equations of the tangents at the three nodes respectively are

$$BY^2 + CZ^2 = 0,$$
$$AX^2 \quad . \quad + CZ^2 = 0,$$
$$AX^2 + BY^2 \quad . \quad = 0;$$

that is, the nodes or critic centres are conjugate poles in regard to a conic

$$AX^2 + BY^2 + CZ^2 = 0,$$

which is the three-centre conic; and the tangents at each node are the tangents from such node to the conic in question.

Article Nos. 51 and 52. *Special Case of the Three-Centre Conic.*

51. Write for a moment

$$X' = \frac{\lambda x}{\theta_1 + \lambda} + \frac{\mu y}{\theta_1 + \mu} + \frac{\nu z}{\theta_1 + \nu},$$

$$Y' = \frac{\lambda x}{\theta_2 + \lambda} + \frac{\mu y}{\theta_2 + \mu} + \frac{\nu z}{\theta_2 + \nu},$$

$$Z' = \frac{\lambda x}{\theta_3 + \lambda} + \frac{\mu y}{\theta_3 + \mu} + \frac{\nu z}{\theta_3 + \nu},$$

so that

$$X = -\frac{\Theta_1}{l_2 l_3} X',\ Y = -\frac{\Theta_2}{l_3 l_1} Y',\ Z = -\frac{\Theta_3}{l_2 l_1} Z',$$

the equation of the three-centre conic expressed in terms of (X', Y', Z') is

$$\frac{\Theta_1}{l_1 \theta_1} X'^2 + \frac{\Theta_2}{l_2 \theta_2} Y'^2 + \frac{\Theta_3}{l_3 \theta_3} Z'^2 = 0,$$

say

$$A' X'^2 + B' Y'^2 + C' Z'^2 = 0.$$

When $\theta_1 = \theta_2$, we have $C' = \infty$, $A' = -B'$, $X' = Y'$; by writing the equation in the form

$$(A' + B') X'^2 + B' (Y'^2 - X'^2) + C' Z'^2 = 0,$$

and observing that in the limit $Y'^2 - X'^2 = 2X'(Y' - X')$, we see that the equation will thus assume the form

$$(\theta_1 - \theta_2) X' S + \frac{1}{\theta_1 - \theta_2} \cdot \frac{\Theta_3}{\theta_3} Z'^2 = 0,$$

where

$$S, = \frac{A' + B'}{\theta_1 - \theta_2} X' + 2B' \frac{Y' - X'}{\theta_1 - \theta_2},$$

is a finite function; $X' = 0$ is the line joining the twofold centre and the one-with-twofold centre, $S = 0$ is the other tangent at the one-with-twofold centre, $Z' = 0$ the tangent at the twofold centre or cusp; the form $X'S + \infty\, Z'^2 = 0$ shows that the three-centre conic reduces itself to a pair of points, viz. the twofold centre or cusp, and the point where the tangent at the cusp is met by the other tangent (that is the tangent not passing through the cusp) at the one-with-twofold centre.

52. To verify the value of S I proceed as follows:

$$\frac{A' + B'}{\theta_1 - \theta_2} = \frac{1}{\theta_1 - \theta_2} \left\{ \frac{\Theta_1}{\theta_1 (\theta_2 - \theta_3)} + \frac{\Theta_2}{\theta_2 (\theta_3 - \theta_1)} \right\},$$

$$= \frac{1}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)} \cdot \frac{1}{\theta_1 - \theta_2} \left\{ \theta_3 \left(\frac{\Theta_1}{\theta_1} - \frac{\Theta}{\theta_2} \right) - (\Theta_1 - \Theta_2) \right\},$$

$$= \frac{1}{9\theta_1^2 (\theta_1 - \theta_2)} \left\{ 2\theta_1 \left(\frac{\Theta_1}{\theta_1} - \frac{\Theta_3}{\theta_2} \right) + (\Theta_1 - \Theta_2) \right\}:$$

$$\frac{\Theta_1}{\theta_1}-\frac{\Theta_2}{\theta_2}=\theta_1^2-\theta_2^2+(\lambda+\mu+\nu)(\theta_1-\theta_2)+\lambda\mu\nu\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right),$$

$$=(\theta_1-\theta_2)\left(\theta_1+\theta_2+\lambda+\mu+\nu-\frac{\lambda\mu\nu}{\theta_1\theta_2}\right),$$

$$=(\theta_1-\theta_2)(\lambda+\mu+\nu+3\theta_1);$$

$$\Theta_1-\Theta_2=\theta_1^3-\theta_2^3+(\lambda+\mu+\nu)(\theta_1^2-\theta_2^2)+(\mu\nu+\nu\lambda+\lambda\mu)(\theta_1-\theta_2),$$

$$=(\theta_1-\theta_2)\{\theta_1^2+\theta_1\theta_2+\theta_2^2+(\lambda+\mu+\nu)(\theta_1+\theta_2)+\mu\nu+\nu\lambda+\lambda\mu\},$$

$$=(\theta_1-\theta_2).\{6\theta_1^2+2(\lambda+\mu+\nu)\theta_1\}:$$

and thence

$$\frac{A'+B'}{\theta_1-\theta_2}=\frac{4}{9\theta_1}(3\theta_1+\lambda+\mu+\nu).$$

Moreover

$$Y'-X'=(\theta_1-\theta_2)\left\{\frac{\lambda x}{(\theta_1+\lambda)(\theta_2+\lambda)}+\frac{\mu y}{(\theta_1+\mu)(\theta_2+\mu)}+\frac{\nu z}{(\theta_1+\nu)(\theta_2+\nu)}\right\},$$

$$=(\theta_1-\theta_2)\left\{\frac{\lambda x}{(\theta_1+\lambda)^2}+\frac{\mu y}{(\theta_1+\mu)^2}+\frac{\nu z}{(\theta_1+\nu)^2}\right\},$$

and hence

$$S=\frac{4}{9\theta_1}(3\theta_1+\lambda+\mu+\nu)\left(\frac{\lambda x}{\theta_1+\lambda}+\frac{\mu y}{\theta_1+\mu}+\frac{\nu z}{\theta_1+\nu}\right)$$

$$-\frac{2\Theta_1}{3\theta_1^2}\left\{\frac{\lambda x}{(\theta_1+\lambda)^2}+\frac{\mu y}{(\theta_1+\mu)^2}+\frac{\nu z}{(\theta_1+\nu)^2}\right\},$$

in which we have only now to substitute $(\lambda, \mu, \nu)=(\alpha^{-3}, \beta^{-3}, \gamma^{-3})$ and $\theta_1=\frac{-1}{\alpha\beta\gamma}$. We have

$$\theta_1+\lambda=\frac{M}{\alpha^2},\quad \theta_1+\mu=\frac{M}{\beta^2},\quad \theta_1+\nu=\frac{M}{\gamma^2},$$

where $M=\frac{1}{\alpha\beta\gamma}(\beta\gamma-\alpha^2)=\frac{1}{\alpha\beta\gamma}(\beta\gamma+\gamma\alpha+\alpha\beta)$, and then observing that

$$3\theta_1+\lambda+\mu+\nu=M\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=M\frac{(\beta\gamma+\gamma\alpha+\alpha\beta)^2}{\alpha^2\beta^2\gamma^2}=M^3,$$

the equation $S=0$ becomes

$$2(\beta\gamma+\gamma\alpha+\alpha\beta)\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}\right)+3(\alpha x+\beta y+\gamma z)=0,$$

or, what is the same thing,

$$(2\beta\gamma+\alpha^2)\frac{x}{\alpha}+(2\gamma\alpha+\beta^2)\frac{y}{\beta}+(2\alpha\beta+\gamma^2)\frac{z}{\gamma}=0,$$

which agrees with a former result.

Article Nos. 53 to 55. *Transformation of the Equation of the Cubic.*

53. Let it be required to express the cubic

$$xyz + k(x+y+z)^2(\lambda x + \mu y + \nu z) = 0$$

in terms of the coordinates X, Y, Z. We have

$$\begin{aligned} x + \ \ y + \ \ z &= 2\left(\frac{1}{\theta_1}X + \frac{1}{\theta_2}Y + \frac{1}{\theta_3}Z\right), \\ \lambda x + \mu y + \nu z &= \quad X + \quad Y + \quad Z, \end{aligned}$$

and the equation therefore is

$$\Pi\left(\frac{X}{\theta_1+\lambda} + \frac{Y}{\theta_2+\mu} + \frac{Z}{\theta_2+\nu}\right) + 4k\left(\frac{X}{\theta_1} + \frac{Y}{\theta_2} + \frac{Z}{\theta_3}\right)^2(X+Y+Z) = 0,$$

where Π denotes the product of the three factors obtained by writing λ, μ, ν successively in the place of λ.

For one of the nodal cubics we have

$$k = k_1 = \frac{-\frac{1}{2}\theta_1^2}{\Theta_1},$$

and the equation multiplied by Θ_1 is

$$\Pi\left\{X + \frac{Y(\theta_1+\lambda)}{\theta_2+\lambda} + \frac{Z(\theta_1+\lambda)}{\theta_3+\lambda}\right\} - \left(X + Y\frac{\theta_1}{\theta_2} + Z\frac{\theta_1}{\theta_3}\right)^2(X+Y+Z) = 0,$$

which it is clear *à priori* must be of the form

$$KX\left(\frac{Y^3}{l_2^3\theta_2\Theta_2} + \frac{Z^3}{l_3^3\theta_3\Theta_3}\right) + \Theta_1\Pi\left(\frac{Y}{\theta_2+\lambda} + \frac{Z}{\theta_3+\lambda}\right) - \theta_1^2\left(\frac{Y}{\theta_2} + \frac{Z}{\theta_3}\right)^2(Y+Z) = 0,$$

and there is in fact no difficulty in verifying that the coefficients of X^3, X^2Y, X^2Z, XYZ all of them vanish. To find K, comparing the coefficients of XY^2 we have

$$K \cdot \frac{1}{l_2^3\theta_2\Theta_2} = \Sigma\frac{(\theta_1+\lambda)(\theta_1+\mu)}{(\theta_2+\lambda)(\theta_2+\mu)} - \frac{\theta_1^2}{\theta_2^2} - 2\frac{\theta_1}{\theta_2},$$

that is

$$\begin{aligned} K \cdot \frac{1}{l_2^3\theta_2} &= \Sigma(\theta_1+\lambda)(\theta_1+\mu)(\theta_2+\nu) - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2, \\ &= \Sigma(\theta_1+\lambda)(\theta_1+\mu)(\theta_2-\theta_1+\theta_1+\nu) - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2, \\ &= \left\{(\theta_2-\theta_1)\frac{2}{\theta_1} + 3\right\}\Theta_1 - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_2, \\ &= \frac{1}{\theta_1}(\theta_1+2\theta_2)\Theta_1 - \frac{\theta_1}{\theta_2^2}(\theta_1+2\theta_2)\Theta_3, \\ &= \frac{l_1}{\theta_1}\Theta_1 - \frac{l_1\theta_1}{\theta_2^2}\Theta_2, \\ &= \frac{l_1}{\theta_1\theta_2^2}(\theta_2^2\Theta_1 - \theta_1^2\Theta_2): \end{aligned}$$

and

$$\begin{aligned}\theta_2^2\Theta_1-\theta_1^2\Theta_2 &= \theta_2^2\theta_1^3-\theta_1^2\theta_2^3+(\theta_2^2\theta_1-\theta_1^2\theta_2)(\mu\nu+\nu\lambda+\lambda\mu)+(\theta_2^2-\theta_1^2)\lambda\mu\nu,\\ &= l_2\{\theta_1^2\theta_2^2-\theta_1\theta_2(\mu\nu+\nu\lambda+\lambda\mu)-(\theta_1+\theta_2)\lambda\mu\nu\},\\ &= l_3\{\theta_1^2\theta_2^2+\theta_1\theta_2(\theta_1\theta_2+\theta_3\overline{\theta_1+\theta_2})-\tfrac{1}{2}(\theta_1+\theta_2)\theta_1\theta_2\theta_3\},\\ &= l_3\theta_1\theta_2\{2\theta_1\theta_2+\tfrac{1}{2}\theta_3(\theta_1+\theta_2)\},\\ &= \tfrac{1}{2}l_3\theta_1\theta_2\{4\theta_1\theta_2-(\theta_1+\theta_2)^2\},\\ &= -\tfrac{1}{2}\theta_1\theta_2 l_3^3;\end{aligned}$$

so that we have

$$K\cdot\frac{1}{l_2^3\theta_2}=\frac{l_1}{\theta_1\theta_2^2}\cdot-\tfrac{1}{2}\theta_1\theta_2 l_3^3=-\tfrac{1}{2}\frac{l_1 l_3^2}{\theta_2},$$

that is

$$K=-\tfrac{1}{2}l_1 l_2^3 l_3^3,$$

and the equation of the nodal cubic is

$$-\tfrac{1}{2}l_1 l_2^3 l_3^3 X\left(\frac{Y^2}{l_2^3\theta_2\Theta_3}+\frac{Z^2}{l_3^3\theta_3\Theta_3)}\right)+\Theta_1\Pi\left(\frac{Y}{\theta_2+\lambda}+\frac{Z}{\theta_3+\lambda}\right)-\theta_1^2\left(\frac{Y}{\theta_2}+\frac{Z}{\theta_3}\right)^2(Y+Z)=0.$$

54. To complete the reduction we have

$$\text{coeff. } Y^3=\frac{\Theta_1}{\Theta_2}-\frac{\theta_1^2}{\theta_2^2}=\frac{1}{\theta_2^2\Theta_2}(\theta_2^2\Theta_1-\theta_1^2\Theta_2);$$

$$\begin{aligned}\text{coeff. } Y^2Z &= \Theta_1\Sigma\frac{1}{(\theta_2+\lambda)(\theta_2+\mu)(\theta_3+\nu)}-\frac{\theta_1^2}{\theta_2^2}-\frac{2\theta_1^2}{\theta_2\theta_3},\\ &= \frac{\Theta_1}{\Theta_2}\Sigma\frac{\theta_2+\lambda}{\theta_3+\lambda}-\frac{\theta_1^2}{\theta_2^2\theta_3}(\theta_3+2\theta_2),\\ &= \frac{\Theta_1}{\Theta_2}\frac{\theta_3+2\theta_2}{\theta_2}-\frac{\theta_1^2}{\theta_2^2\theta_3}(\theta_3+2\theta_2),\\ &= -\frac{l_3}{\theta_3}\frac{\Theta_1}{\Theta_2}+\frac{l_3\theta_1^2}{\theta_2^2\theta_3},\\ &= -\frac{l_3}{\theta_2^2\theta_3\Theta_2}(\theta_2^2\Theta_1-\theta_1^2\Theta_2),\end{aligned}$$

so that substituting for $\theta_2^2\Theta_1-\theta_1^2\Theta_2$ its value $=-\frac{1}{2}\theta_1\theta_2 l_3^3$, the terms in Y^2 and Y^2Z are

$$=-\tfrac{1}{2}\theta_1 l_3^3\frac{1}{\theta_2\Theta_2}\left(Y^3-\frac{l_3}{\theta_2}Y^2Z\right),$$

and in like manner the terms in YZ^2 and Z^3 are

$$=+\tfrac{1}{2}\theta_1 l_2^3\frac{1}{\theta_3\Theta_3}\left(\frac{l_2}{\theta_2}YZ^2+Z^3\right),$$

so that the terms in $(Y, Z)^3$ are

$$= -\tfrac{1}{2}\,\theta_1 l_2^3 l_3^3 \left[\frac{1}{l_2^3\theta_2\Theta_3}\left(Y^3 - \frac{l_3}{\theta_3}\,Y^2 Z\right) - \frac{1}{l_3^3\theta_2\Theta_3}\left(\frac{l_2}{\theta_2}\,YZ^2 + Z^3\right)\right],$$

and the equation, omitting the factor $-\tfrac{1}{2}\,l_1^3 l_2^3$, is

$$l_1 X\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) + \theta_1\left[\frac{1}{l_2^3\theta_2\Theta_2}\left(Y^3 - \frac{l_3}{\theta_3}\,Y^2 Z\right) - \frac{1}{l_3^3\theta_3\Theta_3}\left(\frac{l_2}{\theta_2}\,YZ^2 + Z^3\right)\right] = 0.$$

55. But the term in [ ] is

$$(Y-Z)\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) + \frac{1}{l_2^3\theta_2\Theta_2}\left(1 - \frac{l_3}{\theta_3}\right)Y^2 Z + \frac{1}{l_3^3\theta_3\Theta_3}\left(-1 - \frac{l_2}{\theta_2}\right)YZ^2,$$

which is

$$= (Y-Z)\left(\frac{Y^2}{l_2^3\theta_2\Theta_3} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) - \frac{2\theta_1}{\theta_2\theta_3}\,YZ\left(\frac{1}{l_2^3\Theta_2}\,Y - \frac{1}{l_3^3\Theta_3}\,Z\right),$$

and the equation of the nodal cubic is finally

$$\{l_1 X + \theta_1 (Y-Z)\}\left(\frac{Y^2}{l_2^3\theta_2\Theta_2} + \frac{Z^2}{l_3^3\theta_3\Theta_3}\right) - \frac{2\theta_1^2}{\theta_2\theta_3}\,YZ\left(\frac{Y}{l_2^3\Theta_2} - \frac{Z}{l_3^3\Theta_3}\right) = 0.$$

The lines $Y=0$, $Z=0$, $\dfrac{Y}{l_2^3\Theta_2} - \dfrac{Z}{l_3^3\Theta_3} = 0$ each pass through the node and meet the cubic in a third point; the three points of intersection lie in the line $l_1 X + \theta_1 (Y-Z) = 0$.

Article Nos. 56 to 66. *The Cubic Locus, Harmoconics and Harmonic Conic.*

56. Suppose that the line $\lambda x + \mu y + \nu z = 0$ passes through a given point (a, b, c), then we have

$$\lambda a + \mu b + \nu c = 0;$$

and observing that $\theta+\lambda$, $\theta+\mu$, $\theta+\nu$, θ are proportional to

$$\frac{1}{x},\ \frac{1}{y},\ \frac{1}{z},\ \frac{2}{x+y+z} \text{ respectively,}$$

we find

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} - \frac{2(a+b+c)}{x+y+z} = 0$$

the equation of a cubic curve, the locus of the critic centres corresponding to the several lines $\lambda x + \mu y + \nu z = 0$ which pass through the point (a, b, c). The cubic curve passes, it is clear, through the six points which are the angles of the quadrilateral

$$x=0,\ y=0,\ z=0,\ x+y+z=0.$$

57. If we take for (a, b, c) the coordinates of the point of intersection of the line $\lambda x+\mu y+\nu z=0$ with any one of the lines $x=0$, $y=0$, $z=0$, $x+y+z=0$, then in each case the cubic breaks up into the same line and a conic, viz. we have the conics

$$\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0,$$

$$\frac{\lambda}{z}-\frac{\nu}{x}-\frac{2(\lambda-\nu)}{x+y+z}=0,$$

$$\frac{\mu}{x}-\frac{\lambda}{y}-\frac{2(\mu-\lambda)}{x+y+z}=0,$$

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0;$$

and it is to be noticed that in each case the critic centres all of them lie on the conic. In fact, since the point $(0, \nu, -\mu)$ is an arbitrary point on the line $x=0$, a line $\lambda x+\mu y+\nu z=0$ passing through the point in question is an absolutely arbitrary line, and the corresponding critic centres therefore do not lie on the line $x=0$; that is, they lie on the conic

$$\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0;$$

and it may also be remarked that the elimination of λ, θ, from the system

$$\theta+\lambda : \theta+\mu : \theta+\nu : \theta=\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{2}{x+y+z},$$

or, what is the same thing, the elimination of θ from the system

$$\theta+\mu : \theta+\nu : \theta= \quad \frac{1}{y} : \frac{1}{z} : \frac{2}{x+y+z},$$

gives the last-mentioned equation, unencumbered by the factor $x=0$.

We have thus four conics, each of them passing through the three critic centres which correspond to the line $\lambda x+\mu y+\nu z=0$; as to the signification of the first three of these conics, I remark as follows.

58. The 'harmoconic' of a point A as to the line T in respect of the conic Θ, may be defined as follows; viz. considering the pencil of lines through A, the locus of the fourth harmonic of the point in which a line of the pencil meets T, in regard to the two points in which the same line meets the conic Θ, is a conic which is the harmoconic in question. (In particular, if the line T pass through the point A the harmoconic breaks up into the line T and into the polar of A.) The conic Θ may of course be a pair of lines.

Consider any three lines x, y, z, a line S, and the line T; then the harmoconics being all as to the same line T, we have the theorem

Harmoconic of intersection of x, S in regard to pair of lines y, z,
Ditto " " of y, S " " z, x,
Ditto " " of z, S " " x, y,

all pass through the same three points.

And taking $x=0$, $y=0$, $z=0$ for the equations of the lines x, y, z; $\lambda x+\mu y+\nu z=0$ for the equation of the line S; and $x+y+z=0$ for the equation of the line T, the harmoconics just spoken of are the above-mentioned three conics respectively.

59. In fact, considering the harmoconic of intersection of x, S in regard to the pair y, z; and taking x', y', z' as the coordinates of a point P of the harmoconic, then the equation of the line AP is

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ 0, & \nu, & -\mu \end{vmatrix} = 0,$$

that is

$$x+(\mu y'+\nu z')-x'(\mu y+\nu z)=0,$$

and at the point of intersection with the line T or $x+y+z=0$, we have

$$(y+z)(\mu y'+\nu z')+x'(\mu y+\nu z)=0,$$

or, what is the same thing,

$$y(\mu x'+\mu y'+\nu z')+z(\nu x'+\mu y'+\nu z')=0,$$

which is the line through the last-mentioned point and the point $(y=0,\ z=0)$.

The line from the point A to the point $(y=0,\ z=0)$ is

$$yz'-zy'=0.$$

60. By the definition of the harmoconic, the last-mentioned two lines are harmonics in regard to the lines $y=0$, $z=0$; that is, we have for the equation of the harmoconic in question

$$-y'(\mu x'+\mu y'+\nu z')+z'(\nu x'+\mu y'+\nu z')=0;$$

this equation may also be written

$$(\nu z'-\mu y')(x'+y'+z')-2(\nu-\mu)y'z'=0,$$

or, what is the same thing,

$$\frac{\nu}{y'}-\frac{\mu}{z'}-\frac{2(\nu-\mu)}{x'+y'+z'}=0,$$

whence writing x, y, z in place of x', y', z', we see that this harmoconic is in fact the first of the above-mentioned three conics.

61. The fourth conic through the critic centres is the conic

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0,$$

which it will be observed passes through the vertices of the triangle $x=0$, $y=0$, $z=0$, and also through the point (1, 1, 1) which is the harmonic of the line $x+y+z=0$ in regard to the triangle: I call it the 'harmonic conic.' Representing the equation by

$$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0,$$

or, what is the same thing,

$$2fyz+2gzx+2hxy=0,$$

we have $f=\mu-\nu$, $g=\nu-\lambda$, $h=\lambda-\mu$, and therefore $f+g+h=0$.

62. It is easy to show that the coordinates of the pole of the line $x+y+z=0$ in regard to the harmonic conic are $x:y:z=f^2:g^2:h^2$; these values satisfy the condition $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, that is, the pole in question lies on the twofold centre conic.

63. The equation of the tangents to the harmonic conic at its intersection with the line $x+y+z=0$ (which tangents meet of course in the last-mentioned pole, that is in a point of the twofold centre conic) is found to be

$$2fgh\,(x+y+z)^2+\square\,(2fyz+2gzx+2hxy)=0\,;$$

if for shortness

$$\square=\quad f^2+g^2+h^2-2gh-2hf-2fg,$$

or what is the same thing

$$\square=-4\,(gh+hf+fg),\ =2\,(f^2+g^2+h^2).$$

64. We have identically

$$\begin{aligned}&-6fgh\,(x^2+y^2+z^2-2yz-2zx-2xy)\\&\qquad=2fgh\,(x+y+z)^2+\square\,(2fyz+2gzx+2hxy)-8\,(fx+gy+hz)\,(ghx+hfy+fgz),\end{aligned}$$

so that the tangents in question meet the twofold centre conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

at its intersections with the lines $fx+gy+hz=0$, and $ghx+hfy+fgz=0$: the latter of these is in fact the tangent of the conic at the point (f^2, g^2, h^2) of intersection of the two tangents. Hence the two tangents meet at the point (f^2, g^2, h^2) of the twofold centre conic and they besides meet the conic at its points of intersection with the line $fx+gy+hz=0$.

65. The line $\lambda x+\mu y+\nu z=0$ may be expressed in the form,

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}+h(x+y+z)=0,$$

(where, *ut suprà*, $\alpha+\beta+\gamma=0$). The corresponding values of f, g, h are

$$f:g:h=\alpha^3(\beta^3-\gamma^3):\beta^3(\gamma^3-\alpha^3):\gamma^3(\alpha^3-\beta^3),$$

or, what is the same thing,

$$f:g:h=\alpha^2(\beta-\gamma):\beta^2(\gamma-\alpha):\gamma^2(\alpha-\beta),$$

or, again,

$$f:g:h=\alpha^2(\beta^2-\gamma^2):\beta^2(\gamma^2-\alpha^2):\gamma^2(\alpha^2-\beta^2).$$

The equation $fx+gy+hz=0$ may be written

$$\begin{vmatrix} x, & y, & z \\ 1, & 1, & 1 \\ \frac{1}{\alpha^2}, & \frac{1}{\beta^2}, & \frac{1}{\gamma^2} \end{vmatrix}=0,$$

that is, the line in question is the line joining the harmonic point (1, 1, 1) with the point

$$\left(\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}\right),$$

the inverse of the point $(\alpha^2, \beta^2, \gamma^2)$, which is (*ante*, No. 27) the point of contact of the line

$$\frac{x}{\alpha^3}+\frac{y}{\beta^3}+\frac{z}{\gamma^3}=0$$

with the envelope.

66. The harmonic conic passes through the vertices of the triangle $x=0, y=0, z=0$, through the harmonic point (1, 1, 1), and through the critic centres. Hence if one of the critic centres be given, the harmonic conic passes through five given points and is thus completely determined. But a critic centre being given, the line joining the other two critic centres is the polar of the given centre in regard to the twofold centre conic (*ante*, No. 40), and it is thus completely determined; and the other two critic centres are of course the intersections of this line with the harmonic conic.

Article Nos. 67 to 87. *Miscellaneous Investigations.*

67. I demonstrate by means of the last-mentioned formulæ a theorem already in effect demonstrated by the investigation which led to the three centre conic, viz. that the tangents at a node or critic centre, and the lines drawn to the other two critic centres, form a harmonic pencil.

In fact the tangents at the node or critic centre are given by the equation

$$\left(\theta+4\lambda, \ldots, -\theta-\frac{2\mu\nu}{\theta}, \ldots\right)(x, y, z)^2=0,$$

the other two critic centres are given as the intersection of the line

$$\frac{\lambda x}{\theta+\lambda}+\frac{\mu y}{\theta+\mu}+\frac{\nu z}{\theta+\nu}=0,$$

with the conic

$$\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0,$$

the theorem will be true if the pair of tangents and the last-mentioned conic are cut harmonically by the last-mentioned line. Now in general the condition in order that the line $\xi x+\eta y+\zeta z=0$, may cut harmonically the conics $(a, b, c, f, g, h \between x, y, z)^2$ and $(a', b', c', f', g', h')(x, y, z)^2=0$ is

$$(bc'+b'c-2ff', \ldots, gh'+g'h-af'-a'f, \ldots \between \xi, \eta, \zeta)^2=0,$$

and if $a'=b'=c'=0$, then the condition is

$$(-2ff', \ldots gh'+g'h-af', \ldots \between \xi, \eta, \zeta)^2=0.$$

68. In the present case the equations of the two conics may be written

$$\left(\theta+4\lambda, \ldots -\theta-\frac{2\mu\nu}{\theta}, \ldots\right)(x, y, z)^2=0,$$

$$(0, \ldots, \mu-\nu, \ldots \between x, y, z)^2=0,$$

and we have

$$-2ff'=-2(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right),$$

$$gh'+g'h-af'=-\left(\theta+\frac{2\nu\lambda}{\theta}\right)(\lambda-\mu)-\left(\theta+\frac{2\lambda\mu}{\theta}\right)(\nu-\lambda)-(\mu-\nu)(\theta+4\lambda),$$

$$=-\theta(\lambda-\mu+\mu-\nu+\nu-\lambda)$$

$$+\frac{2}{\theta}(-\nu\lambda^2+\nu\lambda\mu-\lambda\mu\nu-\lambda^2\mu)+4\lambda(\mu-\nu),$$

$$=(\mu-\nu)\left(\frac{2\lambda^2}{\theta}-4\lambda\right),$$

and the condition is

$$\left\{(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right), \ldots (\mu-\nu)\left(\frac{2\lambda^2}{\theta}-4\lambda\right), \ldots\right\}\left(\frac{\lambda}{\theta+\lambda}, \frac{\mu}{\theta+\mu}, \frac{\nu}{\theta+\nu}\right)^2=0.$$

69. Writing this in the form

$$\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left(\frac{\lambda}{\theta+\lambda}\right)^2+\Sigma\, 2(\mu-\nu)\left(\frac{\lambda^2}{\theta}-2\lambda\right)\frac{\mu\nu}{(\theta+\mu)(\theta+\nu)}=0,$$

then observing that

$$\frac{\lambda}{\theta+\lambda}+\frac{\mu}{\theta+\mu}+\frac{\nu}{\theta+\nu}=1,$$

the first part is

$$=\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left(\frac{-\lambda\mu}{(\theta+\lambda)(\theta+\mu)}-\frac{\lambda\nu}{(\theta+\lambda)(\theta+\nu)}+\frac{\lambda}{\theta+\lambda}\right),$$

which is

$$=\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)\left\{-\lambda\mu(\theta+\nu)-\lambda\nu(\theta+\mu)+\lambda(\theta+\mu)(\theta+\nu)\right\},$$

$$=\frac{1}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}\Sigma(\mu+\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right)(\lambda\theta^2-\lambda\mu\nu),$$

and observing that the sum is

$$=\Sigma\lambda(\mu-\nu)(\theta^3+2\mu\nu\theta)-\lambda\mu\nu\,\Sigma(\mu-\nu)\left(\theta+\frac{2\mu\nu}{\theta}\right),$$

$$=-\frac{2\lambda\mu\nu}{\theta}\Sigma\mu\nu(\mu-\nu)=\frac{\theta}{2\lambda\mu\nu}(\mu-\nu)(\nu-\lambda)(\lambda-\mu),$$

the first part is

$$=\frac{2\lambda\mu\nu}{\theta}\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)}.$$

The second part is

$$=\frac{2\lambda\mu\nu}{\theta(\theta+\lambda)(\lambda+\mu)(\theta+\nu)}\Sigma(\mu-\nu)(\lambda-2\theta)(\lambda+\theta),$$

in which the sum is

$$=\Sigma(\mu-\nu)(\lambda^2-\lambda\theta-2\theta^2)=\Sigma\lambda^2(\mu-\nu)=-(\mu-\nu)(\nu-\lambda)(\lambda-\mu),$$

so that the second part is

$$=-\frac{2\lambda\mu\nu}{\theta}\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(\theta+\lambda)(\theta+\mu)(\theta+\nu)},$$

and the sum of the two parts is $=0$, which proves the theorem.

70. Let x_1, y_1, z_1 be the coordinates of a critic centre, then the equation of the polar in regard to the twofold centre conic is

$$(-x_1+y_1+z_1)x+(x_1-y_1+z_1)y+(x_1+y_1-z_1)z=0,$$

and the equation of the conic through the five points is

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

and these equations together determine the remaining two critic centres.

71. I remark in passing that the equation of the one-with-twofold centre locus may also be obtained by means of the equations

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

$$(-x_1+y_1+z_1)x+(x_1-y_1+z_1)y+(x_1+y_1-z_1)z=0,$$

which determine the remaining two critic centres corresponding to a given critic centre (x_1, y_1, z_1); in fact, in order that the centre (x_1, y_1, z_1) may be accompanied by a twofold centre the line must touch the conic; and the analytical condition, substituting therein (x, y, z) in the place of (x_1, y_1, z_1), is found to be

$$xyz\left\{x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz\right\}=0,$$

the three lines $xyz=0$ are not properly part of the locus, but their appearance may be accounted for without difficulty.

72. Assume that the line $\lambda x+\mu y+\nu z=0$ passes successively through the points

$$(x=0,\ y-z=0)\quad(y=0,\ z-x=0),\quad(z=0,\ x-y=0),$$

or, what is the same thing, the points (0, 1, 1), (1, 0, 1), (1, 1, 0): then (*ante*, No. 56) the critic centres are in all these cases respectively on the conics.

$$\frac{1}{y}+\frac{1}{z}-\frac{4}{x+y+z}=0,$$

$$\frac{1}{z}+\frac{1}{x}-\frac{4}{x+y+z}=0,$$

$$\frac{1}{x}+\frac{1}{y}-\frac{4}{x+y+z}=0;$$

or, as these may be written,

$$(y-z)^2+x(y+z)=0,$$

$$(z-x)^2+y(z+x)=0,$$

$$(x-y)^2+z(x+y)=0,$$

the first of which is a conic touching the lines $x=0$, $y+z=0$ at the points of intersection with the line $y-z=0$; and similarly for the other two conics.

73. Suppose that the line $\lambda x+\mu y+\nu z=0$ passes through the point $(4, -1, -1)$, or let $4\lambda-\mu-\nu=0$; we have $(\alpha, \beta, \gamma)=(4, -1, -1)$; and the critic centres lie on the curve

$$\frac{4}{x}-\frac{1}{y}-\frac{1}{z}-\frac{4}{x+y+z}=0,$$

that is

$$\frac{4(y+z)}{x(x+y+z)}-\frac{y+z}{yz}=0,$$

or, as this may be written,

$$(y+z)\left\{x(x+y+x)-4yz\right\}=0,$$

so that the cubic locus breaks up into the line $y+z=0$ and into the conic

$$x(x+y+z)-4yz=0.$$

74. I say that the critic centres lie, one of them on the line, and the other two on the conic.

In fact, putting $\lambda=\frac{1}{4}(\mu+\nu)$ the equation in θ is

$$\theta^3-\theta\left(\mu\nu+\tfrac{1}{4}(\mu+\nu)^2\right)-\tfrac{1}{2}\mu\nu(\mu+\nu)=0,$$

that is

$$\left\{\theta+\tfrac{1}{2}(\mu+\nu)\right\}\left\{\theta^2-\tfrac{1}{2}(\mu+\nu)\theta-\mu\nu\right\}=0,$$

and we have

$$x:y:z=\frac{1}{\theta+\frac{1}{4}(\mu+\nu)}:\frac{1}{\theta+\mu}:\frac{1}{\theta+\nu}.$$

75. Hence if $\theta+\frac{1}{2}(\mu+\nu)=0$, we obtain

$$x:y:z=\frac{1}{-\frac{1}{2}(\mu+\nu)}:\frac{1}{\frac{1}{2}(\mu-\nu)}:\frac{-1}{\frac{1}{2}(\mu-\nu)}$$

$$=\frac{\mu-\nu}{\mu+\nu}:-1:1;$$

whence also

$$\begin{aligned}(\mu+\nu)x+(\mu-\nu)y&=0,\\(\mu+\nu)x-(\mu-\nu)z&=0,\\y+z\qquad\qquad&=0,\end{aligned}$$

so that the corresponding critic centre lies on the line $y+z=0$; the last-mentioned equations, restoring the value 4λ in place of $\mu+\nu$, may also be written

$$\begin{aligned}4\lambda x+(\mu-\nu)y&=0,\\4\lambda x-(\mu-\nu)z&=0,\\y+z\qquad\qquad&=0.\end{aligned}$$

76. If on the other hand

$$\theta^2-\tfrac{1}{2}(\mu+\nu)\theta-\mu\nu=0,$$

or, as this equation may be written,

$$5\theta^2-(\theta+2\mu)(\theta+2\nu)=0,$$

then observing that in general, in virtue of the equation

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\nu}-\frac{2}{\theta}=0,$$

we have

$$y : z : x+y : x+z = \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{2}{\theta} - \frac{1}{\theta+\nu} : \frac{2}{\theta} - \frac{1}{\theta+\mu},$$

$$= \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu} : \frac{\theta+2\nu}{\theta(\theta+\nu)} : \frac{\theta+2\mu}{\theta(\theta+\mu)},$$

and consequently

$$y : x+z = \theta : \theta+2\mu; \qquad z : x+y = \theta : \theta+2\nu,$$

the foregoing equation

$$5\theta^2 - (\theta+2\mu)(\theta+2\nu) = 0$$

gives

$$5yz - (x+z)(x+y) = 0,$$

that is

$$x(x+y+z) - 4yz = 0;$$

or the critic centres corresponding to the two values of θ lie on the conic. The line joining them is the polar of the point $\left(\frac{\mu-\nu}{\mu+\nu}, -1, 1\right)$ in regard to the twofold centre conic; the equation therefore is

$$(\mu-\nu)x - (3\mu+\nu)y + (\mu+3\nu)z = 0.$$

77. Starting with a critic centre on the line $y+z=0$, the other two critic centres lie on the conic $x(x+y+z)-4yz=0$, and they are the intersections of the conic by the polar of the first centre in regard to the twofold centre conic.

78. Starting with a critic centre on the conic $x(x+y+z)-4yz=0$, the other two critic centres lie one on the conic, and the other on the line $y+z=0$; viz. the polar of the first centre in regard to the twofold centre conic meets the line in one point, and the conic in two points; of these one is the harmonic of the point on the line in regard to the twofold centre conic; this point on the conic, and the point on the line, are the other two centres.

79. The point $(4, -1, -1)$ is of course one of a system of three points; viz. these are $(4, -1, -1)$ $(-1, 4, -1)$, $(-1, -1, 4)$; and the corresponding loci of the critic centres are

$$(y+z)\left\{x(x+y+z) - 4xy\right\} = 0,$$

$$(z+x)\left\{y(x+y+z) - 4zx\right\} = 0,$$

$$(x+y)\left\{z(x+y+z) - 4xy\right\} = 0,$$

the three points in question are (*ante*, No. 24) shown to be nodes of the twofold centre envelope.

80. The line $3x+y+z=0$ is the line through the points $(-1, 4, -1)$, $(-1, -1, 4)$, and as such the corresponding critic centres lie

one on the line $z+x=0$, two on the conic $x(x+y+z)-4yz=0$,

one on the line $x+y=0$, two on the conic $y(x+y+z)-4zx=0$.

The two lines meet in the point $(1, -1, -1)$.

The two conics meet in the points $(1, 0, 0)$, $(2, 3, 3)$; and touch at the point $(0, 1, -1)$, the common tangent being $5x+y+z=0$: this appears by writing the equations of the two conics in the forms

$$(y-z)(5x+y+z)+(y+z)(-3x+y+z)=0,$$
$$-(y-z)(5x+y+z)+(y+z)(-3x+y+z)=0,$$

for we have then the four points of intersection put in evidence; viz. these are

$$\begin{array}{llll} y-z=0, & y+z=0, & \text{that is} & (1, 0, 0), \\ y-z=0, & -3x+y+z=0, & \text{,,} & (2, 3, 3), \\ 5x+y+z=0, & y+z=0, & \text{,,} & (0, 1, -1), \\ 5x+y+z=0, & -3x+y+z=0, & \text{,,} & (0, 1, -1). \end{array}$$

The point of intersection $(1, 0, 0)$, which is an angle of the triangle, is not a critic centre; the three critic centres are the other point of intersection $(2, 3, 3)$; the point of contact $(0, 1, -1)$; and the point of intersection $(1, -1, -1)$ of the two lines.

81. To obtain in a different manner the last-mentioned result it may be remarked that for the line $3x+y+z=0$, for which $(\lambda, \mu, \nu)=(3, 1, 1)$, the equation in θ is

$$\theta^3-7\theta-6=(\theta+1)(\theta+2)(\theta-3)=0,$$

so that the values of $\theta+\lambda$, $\theta+\mu$, $\theta+\nu$ are

$$\begin{array}{lrrr} \text{for } \theta=-1, & 2, & 0, & 0, \\ \text{,, } \theta=-2, & 1, & -1, & -1, \\ \text{,, } \theta=\ \ 3, & 6, & 4, & 4, \end{array}$$

and the corresponding values of $x : y : z$ are

$$\begin{array}{lrrrll} =\frac{1}{2} : & \infty : & \infty, & \text{that is,} & (0, & 1, -1), \\ =1 : & -1 : & -1, & \text{,,} & (1, & -1, -1), \\ =\frac{1}{6} : & \frac{1}{4} : & \frac{1}{4}, & \text{,,} & (2, & 3, 3), \end{array}$$

which points are therefore the critic centres for the line $3x+y+z=0$.

The last-mentioned line, it is clear, is one of the system of three lines

$$3x+y+z=0, \quad x+3y+z=0, \quad x+y+3z=0.$$

82. If $\lambda=0$, that is if the line $\lambda x+\mu y+\nu z=0$ pass through an angle $y=0$, $z=0$ of the triangle; then reverting to the original equations

$$\frac{-x+y+z}{\lambda x}=\frac{x-y+z}{\mu y}=\frac{x+y-z}{\nu z},$$

these give $(y=0,\ z=0)$ or else $(-x+y+z=0,\ \nu z^2-\mu y^2=0)$, that is, one of the three critic centres is the angle $(y=0,\ z=0)$ of the triangle; and the other two are the intersections of the line $-x+y+z=0$ with the pair of lines $\nu z^2-\mu y^2=0$.

It should be remarked that, given the critic centre $y=y_1=0,\ z=z_1=0$, the remaining two centres cannot be determined as the intersection of the polar $-x+y+z=0$ with the conic

$$\frac{x_1(y_1-z_1)}{x}+\frac{y_1(z_1-x_1)}{y}+\frac{z_1(x_1-y_1)}{z}=0,$$

inasmuch as the equation of this conic becomes the identity $0=0$.

83. The critic centres for the case in question, $\lambda=0$, may also be determined by means of the equation of the cubic through the three centres; in fact, since $\lambda=0$, the equation $\lambda\alpha+\mu\beta+\nu\gamma=0$ becomes $\mu\beta+\nu\gamma=0$, that is $\beta:\gamma=\nu:-\mu$; and the equation of the cubic therefore is

$$\alpha\left(\frac{1}{x}-\frac{2}{x+y+z}\right)+\frac{\beta}{\nu}\left(\frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}\right)=0,$$

and since the ratio $\alpha:\beta$ is arbitrary we have the two equations

$$\frac{1}{x}-\frac{2}{x+y+z}=0,\quad \frac{\nu}{y}-\frac{\mu}{z}-\frac{2(\nu-\mu)}{x+y+z}=0,$$

which resolve themselves into the above-mentioned two equations, $-x+y+z=0$, $\nu z^2-\mu y^2=0$.

84. Consider a critic centre the coordinates of which are $(0,\ y_1,\ z_1)$, that is, which is an arbitrary point on the side $x=0$ of the triangle: it is to be remarked that there is not any position of the line $\lambda x+\mu y+\nu z=0$, which properly gives rise to such a critic centre.

For writing $x_1=0$ the equations

$$\frac{-x_1+y_1+z_1}{\lambda x_1}=\frac{x_1-y_1+z_1}{\mu y_1}=\frac{x_1+y_1-z_1}{\nu z_1},$$

give $\mu=0,\ \nu=0$, that is, the line $\lambda x+\mu y+\nu z=0$ is found to be $x=0$; but in this case the cubic is $x\left(yz+k(x+y+z)^2\right)=0$, which irrespectively of the value of k has nodes at the points $x=0,\ yz+k(y+z)^2=0$, and which only for the value $k=0$ acquires a third node at the point $y=0,\ z=0$: the case is a singular and exceptional one.

85. If notwithstanding we assume a critic centre at the point $(0,\ y_1,\ z_1)$, then the other two critic centres are by the general theorem given as the intersection of the line

$$(y_1+z_1)x-(y_1-z_1)(y-z)=0$$

with the conic (pair of lines) $x(y-z)=0$, that is, we have a twofold centre $x=0,\ y-z=0$, or what is the same thing a twofold centre $(0,\ 1,\ 1)$.

86. If a critic centre lie on the line $y-z=0$, then of the other two critic centres, one lies on this same line and the other is the point $x=0$, $x+y+z=0$, or say the point $(0, 1, -1)$. And in this case the line $\lambda x+\mu y+\nu z=0$ passes through the last-mentioned point; that is, we have $\mu=\nu$. Conversely, starting from the equation $\mu=\nu$, so that the line $\lambda x+\mu y+\nu z=0$ is $\lambda x+\mu(y+z)=0$, a line through the intersection of the lines $x=0$, $x+y+z=0$, the equation in θ is

$$(\theta+\mu)(\theta^2-\mu\theta-2\lambda\mu)=0,$$

where the factor $\theta+\mu=0$ corresponds to the critic centre $x=0$, $x+y+z=0$, or $(0, 1, -1)$, (it will presently be shown that this is so), and the quadric equation $\theta^2-\mu\theta-2\lambda\mu=0$ corresponds to two critic centres on the line $y-z=0$. We have

$$x : y : z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\mu},$$

and thence $y-z=0$; and $\theta(x-y)=-\lambda x+\mu y$, which substituted in the equation $\theta^2-\mu\theta-2\lambda\mu=0$ gives

$$(\lambda x-\mu y)\{(\lambda+\mu)x-2\mu y\}-2\lambda\mu(x-y)^2=0,$$

and the two critic centres are given as the intersections of this conic by the line $y-z=0$.

87. Consider for a moment the case $\nu=\mu+\epsilon$, where ϵ is ultimately $=0$, the equation in θ is

$$\frac{1}{\theta+\lambda}+\frac{1}{\theta+\mu}+\frac{1}{\theta+\mu+\epsilon}-\frac{2}{\theta}=0;$$

then if a root is $\theta=-\mu+A\epsilon$, we have

$$\frac{1}{A\epsilon+\lambda-\mu}+\frac{1}{A\epsilon}+\frac{1}{(A+1)\epsilon}-\frac{2}{A\epsilon-\mu}=0,$$

so that, ϵ being indefinitely small, we have

$$\frac{1}{A}+\frac{1}{A+1}=0, \text{ that is, } 2A+1=0 \text{ or } A=-\tfrac{1}{2},$$

and then

$$\theta=-\mu-\tfrac{1}{2}\epsilon,\quad \theta+\lambda=\lambda-\mu-\tfrac{1}{2}\epsilon;\quad \theta+\mu=-\tfrac{1}{2}\epsilon;\quad \theta+\nu=+\tfrac{1}{2}\epsilon,$$

which gives

$$x : y : z=\frac{1}{\theta+\lambda} : \frac{1}{\theta+\mu} : \frac{1}{\theta+\nu}=\frac{1}{\lambda-\mu-\frac{1}{2}\epsilon} : -\frac{2}{\epsilon} : +\frac{2}{\epsilon},$$

or, ϵ being indefinitely small, $x : y : z=0 : 1 : -1$, so that the factor $\theta+\mu=0$ corresponds, as mentioned above, to the critic centre $(0, 1, -1)$.

350.

ON THE CLASSIFICATION OF CUBIC CURVES.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part I. (1866), pp. 81—128. Read April 18, 1864.]

THE notion of a curve of a given order may be considered as arising from Descartes' invention of his method of coordinates; and one of the earliest applications of the method was made by Sir Isaac Newton in the *Enumeratio linearum tertii Ordinis* (1706), a work worthy of its author, and which opened a new field of geometrical science. The classification is according to the nature of the infinite branches; there are fourteen genera containing together seventy-two species, but four species were added by Stirling in his *Lineæ tertii Ordinis Newtonianæ; sive Illustratio &c.* (1717), and two more by Murdoch or Cramer[1], making in all seventy-eight species. A new classification was made by Plücker in his *System der Analytischen Geometrie*, 1835; this is likewise according to the nature of the infinite branches, but after his six head divisions, and some subordinate divisions thereof, Plücker establishes the divisions called Groups, which have nothing analogous to them in the Newtonian theory; there are sixty-one groups, and the total number of species is 219.

The present Memoir contains an exposition of the foregoing classifications, and of the principles on which they are founded, in so far as relates to the superior divisions of the two classifications: and in particular I develope more completely than was done by Plücker the theory of the division into groups. I do not however consider otherwise than very slightly the ultimate division into species.

The above-mentioned work of Newton contains, under the heading "Genesis Curvarum per Umbras," the remarkable theorem that the curves of the third order may all of them be considered as the shadows of the five Divergent Parabolas; I reserve for a separate Memoir the whole series of considerations to which this theorem gives rise.

[1] The two additional species are, I believe, first mentioned in Murdoch's *Genesis Curvarum per Umbras* (1746), but one of them is there ascribed to Cramer.

I commence by establishing the theory of the classification of cubic curves according to the nature of their infinite branches, in what appears to me the scientifically correct manner as follows:

The Seven Head Divisions, Article Nos. 1 to 4.

1. A line in general, and therefore the line Infinity, meets a cubic curve in three points, and these may be

Three onefold points,

A twofold point and a onefold (or, as it may also be termed, a one-with-twofold) point,

A threefold point.

2. But in the second case the line Infinity

may be a proper tangent to the curve,

may pass through a node,

may pass through a cusp;

and in the third case the line Infinity

may touch the curve at an inflexion,

may at a node touch one of the two branches,

may touch the curve at a cusp.

3. The first case, the three divisions of the second case, and the three divisions of the third case, give in all seven divisions, which, as will appear in the sequel, fall in with Newton's classification, and can be named in his language, viz.

Three onefold points,	The Hyperbolas.
A onefold and a twofold point;	
Infinity a proper tangent,	The Parabolic Hyperbolas.
Do. through a node,	The Central Hyperbolisms.
Do. through a cusp,	The Parabolic Hyperbolisms.
A threefold point;	
Infinity a tangent at an inflexion,	The Divergent Parabolas.
Do. Do. at a node, to one branch,	The Trident Curve.
Do. Do. at a cusp,	The Cubical Parabola.

4. As regards the signification of these terms, it may be remarked that the Hyperbolas have hyperbolic branches, the Parabolic Hyperbolas, hyperbolic and parabolic branches; where by a hyperbolic branch is meant one having an asymptote, and by

a parabolic branch one not having an asymptote. The hyperbolism of any curve is the curve derived from it by altering the ordinate in the ratio of the abscissa to any given line

$$\left(y' = \frac{m}{x}y, \text{ or say } y' = \frac{y}{x}\right);$$

the expression Central Hyperbolism is used to include Newton's hyperbolisms of the hyperbola and ellipse; and the expression Parabolic Hyperbolism to denote his hyperbolism of the parabola. The Divergent Parabolas are curves the branches of which ultimately diverge from each other as in the semicubical parabola $y^2 = x^3$, which is in fact one of these curves. The names Trident Curve and Cubical Parabola are not generic but specific; it so happens that the genera to which they respectively belong contain each only a single species. The names for the several kinds of curves are not scientifically-devised ones, but it is convenient to have them such as they are.

The foregoing seven divisions, uniting in one the Central Hyperbolisms and the Parabolic Hyperbolisms, are the six head divisions of Plücker.

Asymptotes, &c. Equations for the Seven Head Divisions. Article Nos. 5 to 22.

5. For a Hyperbola there is at each of the points at infinity a tangent, which is an asymptote; and the hyperbola has thus three asymptotes.

6. For a Parabolic Hyperbola there is at the onefold point at infinity a tangent, which is an asymptote. There may be described a conic having with the curve at the twofold point at infinity a five-pointic intersection[1]. Such conic, as having the line infinity for a tangent, is a parabola, and it may be termed the asymptotic parabola: the Parabolic Hyperbola has thus an asymptote and an asymptotic parabola.

7. For a Central Hyperbolism there is at the onefold point at infinity a tangent which is an asymptote, and which for distinction may be called the onefold asymptote; and at the node or twofold point at infinity there is a pair of tangents which are the parallel asymptotes.

8. For a Parabolic Hyperbolism there is at the onefold point at infinity a tangent which is an asymptote, and which may be called the onefold asymptote; and at the cusp or twofold point at infinity a twofold tangent which is an asymptote, and which may be called the twofold asymptote.

9. For a Divergent Parabola there is not any asymptote or asymptotic conic; but we may consider an asymptotic cubic, viz. this will be a semicubical parabola ($y^2 = x^3$), which is in fact one of the divergent parabolas, the cuspidal divergent parabola, and which may be in general so determined as to have at the inflexion or threefold point

[1] I have elsewhere spoken of the conic of five-pointic contact: the expression five-pointic *intersection* is more accurate.

at infinity a seven-pointic intersection. For the asymptotic cubic, in order that it may have a cusp, must satisfy two conditions; it may therefore be made to satisfy seven more conditions, or to have a seven-pointic intersection; and then the original curve having an inflexion or threefold point at infinity, the asymptotic cubic will *ipso facto* have the same point as an inflexion, or threefold point at infinity, and be thus a cuspidal divergent parabola.

10. For the Trident Curve, we have at the node or threefold point at infinity, viz. to the branch which is not touched by the line infinity, a tangent which is an asymptote: this cuts at the node the other branch of the curve, and it is therefore an asymptote of three-pointic intersection. We may describe a conic having at the node a five-pointic intersection with the other branch of the curve; such conic as touching the line infinity is a parabola, and it may be called the asymptotic parabola; since the parabola cuts at the node the first-mentioned branch of the curve, viz. the branch not touched by the line infinity, the parabola is in fact a parabola of six-pointic intersection. The Trident Curve has thus an asymptote and an asymptotic parabola of six-pointic intersection.

11. For the Cubical Parabola there is not any asymptote or asymptotic conic: the curve *qua* curve having a cusp (viz. the cusp or threefold point at infinity) has a single inflexion; and the line joining the cusp with the inflexion, regarded as a threefold line, has with the curve a six-pointic intersection at infinity, and may be considered as an asymptotic cubic.

12. We have in every case a cubic curve $V=0$ having with the original curve an intersection at infinity which is at least six-pointic, and which I call the asymptotic aggregate: viz. the asymptotic aggregate is

For the Hyperbolas; the three asymptotes, intersection six-pointic.

For the Parabolic Hyperbolas; the asymptote and the asymptotic parabola, intersection seven-pointic.

For the Central Hyperbolisms; the onefold asymptote and the parallel asymptotes, intersection eight-pointic.

For the Parabolic Hyperbolisms; the onefold asymptote and the twofold asymptote regarded as a twofold line; intersection eight-pointic.

For the Divergent Parabolas; the asymptotic semicubical parabola, intersection seven-pointic.

For the Trident Curve; the asymptote and the asymptotic parabola, intersection nine-pointic.

For the Cubical Parabola; the line joining the cusp at infinity with the inflexion, regarded as a threefold line, intersection six-pointic.

13. I have said that the intersection at infinity is at least six-pointic; but more than this, the intersection at any onefold point at infinity is at least two-pointic; at

a twofold point at infinity it is at least four-pointic; and at a threefold point at infinity it is at least six-pointic.

It follows that the intersections at infinity of the cubic and the asymptotic aggregate include the six intersections of the cubic by the line infinity considered as a twofold line; and hence the remaining three intersections of the cubic and the asymptotic aggregate must lie in a line $s=0$ (Plücker's line S), which I call the satellite line. And writing $z=0$ for the equation of the line infinity, the equation of the cubic is of the form $U=V+\mu z^2 s=0$. It is to be observed moreover, that when, as for the Hyperbolas and the Cubical Parabola, the intersection at infinity is six-pointic, the line $s=0$ is an arbitrary line; when as for the Parabolic Hyperbolas, and the Divergent Parabolas, the intersection is seven-pointic, the line $s=0$ meets the cubic in a given point at infinity, viz. the twofold or the threefold point at infinity; and when as for the Central Hyperbolisms and the Parabolic Hyperbolisms the intersection is eight-pointic, the line $s=0$ has with the cubic a given twofold intersection at infinity; this however merely implies that the line $s=0$ passes through the node or cusp at infinity, and so imposes only one condition on the line $s=0$. Finally, when as in the Trident Curve the intersection is nine-pointic, the line $s=0$ has with the curve a given threefold intersection at infinity; that is, it coincides with the line infinity, $z=0$.

14. The preceding considerations in regard to the asymptotic aggregate $V=0$, lead very directly to the best analytical form of the function V, and therefore to that of the equation $U=V+\mu z^2 s=0$, of the cubic.

15. For the Hyperbolas; the equations of the asymptotes being $p=0$, $q=0$, $r=0$, then we have $V=pqr=0$ for the asymptotic aggregate; the satellite line is arbitrary, and hence

Equation of the Hyperbolas is

$$pqr+\mu z^2 s=0.$$

16. For the Parabolic Hyperbolas. Imagine parallel to the asymptote a line $p=0$ touching the asymptotic parabola; and let the line joining the point of contact with the twofold point at infinity have for its equation $q=0$; the equation of the asymptote is

$$p+\kappa z=0,$$

that of the asymptotic parabola is $q^2+\lambda pz=0$, and hence the equation of the asymptotic aggregate is $(p+\kappa z)(q^2+\lambda pz)=0$; the satellite line passes through the twofold point at infinity, or its equation is $q+\sigma z=0$; hence

Equation of the Parabolic Hyperbolas is

$$(p+\kappa z)(q^2+\lambda pz)+\mu z^2(q+\sigma z)=0.$$

17. For the Central Hyperbolisms; the equation of the onefold asymptote is taken to be $p=0$, and that of the parallel asymptotes to be $q^2+\kappa z^2=0$; hence the equation

of the asymptotic aggregate is $p(q^2+\kappa z^2)=0$, the satellite line passes through the twofold point at infinity, its equation is $q+\sigma z=0$; hence

Equation of the Central Hyperbolisms is

$$p(q^2+\kappa z^2)+\mu z^2(q+\sigma z)=0.$$

18. For the Parabolic Hyperbolisms: the only difference is that instead of the parallel asymptotes $q^2+\kappa z^2=0$ we have the twofold asymptote $q^2=0$; hence

Equation of the Parabolic Hyperbolisms is

$$pq^2+\mu z^2(q+\sigma z)=0.$$

19. For the Divergent Parabolas: the asymptotic aggregate is a semicubical parabola; let $q=0$ be the equation of the cuspidal tangent, $p=0$ the equation of the line joining the cusp with the inflexion at infinity, then the equation is $p^3+\lambda q^2 z=0$. The satellite line passes through the threefold point at infinity, its equation is $p+\sigma z=0$, hence

Equation of the Divergent Parabolas is

$$p^3+\lambda q^2 z+\mu z^2(p+\sigma z)=0.$$

20. For the Trident Curve: let $p=0$ be the equation of the asymptote, $q=0$ that of the tangent to the asymptotic parabola at the point not at infinity where it is met by the asymptote, then the equation of the parabola is $p^2+\lambda qz=0$, and that of the asymptotic aggregate is $p(p^2+\lambda qz)=0$; the satellite line is the line infinity, $z=0$; hence

Equation of the Trident Curve is

$$p(p^2+\lambda qz)+\mu z^3=0.$$

21. For the Cubical Parabola: let $p=0$ be the equation of the line joining the inflexion with the cusp at infinity, then the asymptotic aggregate is this line taken as a threefold line, or the equation is $p^3=0$; the satellite line is arbitrary; hence

Equation of the Cubical Parabola is

$$p^3+\mu z^2 s=0.$$

22. It is convenient to notice here that for the Hyperbolas the line $s=0$ is determined as follows, viz. the line infinity meets the curve in three points, and the tangents at these points (the asymptotes) again meet the curve in three points lying in a line which is the line in question; in other words, the line $s=0$ is (in the sense in which I have elsewhere used the term) the satellite line of infinity. For the other kinds of cubic curves, the line $s=0$ is *not*, in the sense just referred to, the satellite line of infinity: but in the present Memoir I shall in every case call the line, $s=0$, the satellite line.

The Thirteen Divisions. Article Nos. 23 to 33.

23. The characters of the foregoing seven divisions are irrespective of *reality;* and before going further it may be remarked, that as to the Hyperbolas and the Parabolic Hyperbolas a subdivision also irrespective of reality may be made as follows.

24. For a Hyperbola, the three asymptotes may not meet in a point, or they may meet in a point. For shortness I say that in the former case we have a Hyperbola Δ, in the latter case a Hyperbola $\odot$. I consider more particularly (*post,* No. 41) the special case of a Hyperbola $\odot$.

25. For a Parabolic Hyperbola, the asymptote may meet the asymptotic parabola in two onefold points; or in a twofold point.

26. I come now to the divisions which depend on *reality:* it is assumed that the curve is real.

27. For the Hyperbola the three points at infinity may be all real or else one real, two imaginary. In the former case, the asymptotes are all real, and we have the redundant hyperbola; in the latter case the real point at infinity gives rise to a real asymptote, the imaginary points to imaginary asymptotes: we have in this case the defective hyperbola. It is to be noticed that the imaginary asymptotes meet in a real point, called the asymptote-point; and that such point, if we regard it as an indefinitely small ellipse given as to the position and ratio of its axes, determines the imaginary asymptotes. Combining the division with the Δ, $\odot$, we have four subdivisions of the Hyperbola.

28. For a Hyperbola Δ redundant the three asymptotes form a triangle, and for a Hyperbola $\odot$ redundant they meet in a point. For a Hyperbola Δ defective, the asymptote-point does not lie on the real asymptote; for a Hyperbola $\odot$ defective it does lie on the real asymptote.

29. For a Parabolic Hyperbola: the onefold point and the twofold point at infinity are of necessity real, as are also the asymptote and the asymptotic parabola. If the asymptote meets the asymptotic parabola in two onefold points, these may be both real or both imaginary: if it meets it in a twofold point, this is real. We have thus three subdivisions of the Parabolic Hyperbola. For the Central Hyperbolism, the onefold point, and the node or twofold point at infinity, are both real; the asymptote is also real. But the node may be a crunode or an acnode; that is, the tangents at the node, or parallel asymptotes, may be both real, or both imaginary: we have thus two subdivisions, viz. the Hyperbolism of the hyperbola, and the Hyperbolism of the ellipse.

30. For the Parabolic Hyperbolism, the onefold point and the cusp or twofold point at infinity, and also the onefold asymptote and the twofold asymptote are all real.

31. For the Divergent Parabola, the inflexion or threefold point at infinity is real.

32. For the Trident Curve the node or threefold point at infinity is real, and inasmuch as one of the tangents is the line infinity, the node is a crunode, and the other tangent, or asymptote of the curve, is also real.

33. For the Cubical Parabola, the cusp or threefold point at infinity and the tangent at this point are each real.

Reckoning the hyperbolas as 4, the parabolic hyperbolas as 3, the central hyperbolisms as 2, and the parabolic hyperbolisms, the divergent parabolas, the trident curve, and the cubical parabola, each as 1, we have in all 13 divisions.

The Notion of a Group. Article No. 34.

34. I remark that the characters as well of the 7 divisions as of the 13 divisions have exclusive reference to the form of the asymptotic aggregate $V=0$; we have an ulterior division depending on the relation of the satellite line to the asymptotic aggregate, and which I regard as the proper origin of Plücker's Groups: viz. for a given form of the asymptotic aggregate $V=0$, and corresponding to each characteristically distinct position in relation thereto of the satellite line $s=0$, we have a Group. The determination of the characteristically distinct positions of the satellite line cannot be completely effected *à priori*; for instance, in the case of the Hyperbolas Δ redundant, the distinctions which immediately present themselves are that the satellite line cuts the three sides produced, or two sides and the third side produced, of the triangle formed by the asymptotes, or passes through an angle of the triangle, &c.; but these are not *all* the distinctions which have to be made; to determine them, taking the satellite line as given, we discuss the series of curves represented by the equation $V+\mu z^2 s=0$; for instance (and it is on this that the discussion chiefly turns), we see that the parameter μ may be so determined that the curve shall have a node, but the reality or non-reality of the roots of the equation in μ, and therefore the existence of a real nodal curve or curves will depend on the position of the satellite line $s=0$; and it is thus only by the discussion of the group that we arrive at an enumeration of the different groups.

Osculating Asymptotes and other Specialities. Article Nos. 35 to 41.

35. But Plücker nevertheless, prior to the establishment of his groups, introduces certain intermediate divisions as to osculating asymptotes, &c., which have really reference to the position of the satellite line; an osculating asymptote gives rise to a 'diameter,' and the diameter is a distinctive character in the Newtonian genera; to explain how all this is, I proceed as follows.

36. The parallel asymptotes of a Central Hyperbolism, the twofold asymptote of a Parabolic Hyperbolism and the asymptote of the Trident Curve are singular asymptotes, that is, each of them touches the curve at a node or a cusp, and is thus an asymptote of three-pointic intersection. Excluding these, and using the term asymptote to denote

a non-singular asymptote, an asymptote is in general an ordinary tangent or asymptote of two-pointic intersection; if, however, the point of contact is an inflexion, then the asymptote is an asymptote of three-pointic intersection, or osculating asymptote. In particular for the Hyperbolas, the asymptotes may be all ordinary, or they may be two ordinary and one osculating, or all three osculating; but they cannot be only two of them osculating; for the line through two inflexions meets a cubic curve in a third point which is also an inflexion; that is, if two asymptotes are osculating, the third is also an osculating asymptote. The foregoing remarks apply as well to the defective as the redundant Hyperbolas; it is to be noticed, however, as regards the defective Hyperbolas that the osculating asymptote, when there is only one, is necessarily the real asymptote, and consequently that the cases are—asymptotes ordinary; the real asymptote alone osculating; three osculating asymptotes. For the Parabolic Hyperbolas the asymptote, and for the Central Hyperbolisms and the Parabolic Hyperbolisms the onefold asymptote, may be ordinary or osculating.

37. The distinction of ordinary and osculating asymptotes has reference to the position of the satellite line; viz. for the Hyperbolas, when there is a single osculating asymptote, the satellite line passes through the point at infinity of the osculating asymptote, or what is the same thing, the satellite line is parallel to the osculating asymptote: and when there are three osculating asymptotes, the satellite line coincides with the line infinity. And, conversely, when the satellite line is parallel to an asymptote such asymptote is an osculating one, and when the satellite line is at infinity the three asymptotes are osculating. For the Parabolic Hyperbolas the asymptote, and for the Hyperbolisms the onefold asymptote, is an osculating asymptote when the satellite line is at infinity; and conversely.

38. There is in regard to the Divergent Parabolas a distinction which may be mentioned here; viz. the satellite line may disappear altogether ($\mu = 0$), and the curve thus coincide with the asymptotic semicubical parabola. Or, what is the general case, the satellite line may be distinct from the line infinity,—and it may cut in two real points, touch, or cut in two imaginary points the asymptotic semicubical parabola: or the satellite line may coincide with the line infinity, the asymptotic semicubical parabola being in this case of nine-pointic intersection.

39. The term "diameter" is used by Newton in the *Enumeratio* in two different senses; viz. for any given direction of the ordinates there exists a right line or "diameter," such that measuring the ordinates from this line the sum $y + y' + y''$ of the three ordinates is $= 0$. Such diameter is in fact the second or line polar in regard to the cubic of an arbitrary point on the line infinity. But the term diameter is afterwards and will be here used to denote a diameter *absolutè dictum*, viz. for a direction of the ordinates parallel to a non-singular asymptote there may exist a right line or "diameter" such that the ordinates measured from this point are equal and opposite to each other, or what is the same thing, such that the sum $y + y'$ of the two ordinates is $= 0$; this implies that the asymptote is an osculating asymptote. In fact, the first or conic polar of any inflexion of the cubic breaks up into a pair of lines, one of which is the tangent at the inflexion, the other of them, the 'polar' of

the inflexion, a line which cuts harmonically the chords through the inflexion, and which when the inflexion is at infinity becomes a diameter. The remarks previously made as to osculating asymptotes apply therefore to diameters, viz. the Hyperbolas may have no diameter, a single diameter, or three diameters, &c.

40. Newton speaks also of the "centre" of a cubic curve; viz. there may be a point on the curve such that for any line through this point the two radius vectors are equal and opposite to each other, or that the sum $r+r'$ of the two radius vectors is $=0$. The centre is in fact a point of inflexion which has for its polar the line infinity. The curves which may have a centre are the Hyperbolas (redundant or defective), the Central Hyperbolisms (of the hyperbola or ellipse) and the Cubical Parabola. For the hyperbolas, the three asymptotes and the satellite line must meet in a point of the curve, which point is then the centre; for the central hyperbolisms the onefold asymptote and the satellite line must meet in a point of the curve, which point is then a centre; and for the cubical parabola no condition is required, but the inflexion is a centre. I remark here, in passing, that the notion of a centre as just explained has no place in Plücker's Classification, and that the two Newtonian species 58 and 59 (hyperbolisms of the hyperbola) and the two Newtonian species 61 and 62 (hyperbolisms of the ellipse) which differ, the two of a pair from each other, according as there is no centre or a single centre, form each pair a single species with Plücker; viz. they are 198 and 201 respectively.

41. It has been already remarked that the three asymptotes of a Hyperbola may meet in a point. As to this it is to be noticed that from any point we may draw six tangents to a cubic, the points of contact lie on a conic, the conic polar of the point: if, however the point lie on the Hessian of the cubic, then the conic breaks up into a pair of lines, each of which is a tangent to the Pippian; the two lines meet in a point of the Hessian, which point forms with the first mentioned point a pair of conjugate poles of the cubic[1].

Conversely, any tangent of the Pippian meets the cubic in three points, the tangents at which meet in a point of the Hessian; and from this point we may draw to the cubic three other tangents the points of contact of which lie on a line which is also a tangent of the Pippian, and the two tangents of the Pippian meet in a point of the Hessian; the two points of the Hessian being conjugate poles of the cubic. In particular, if the line infinity is a tangent of the Pippian, then the three asymptotes meet in a point of the Hessian, and the three tangents from this point to the cubic touch the cubic in three points lying on a line which is a tangent of the Pippian, and which meets the line infinity in a point forming with the first mentioned point a pair of conjugate poles of the cubic.

I proceed now to explain the classification of Newton so far as relates to the division into genera, and the classification of Plücker so far as relates to the divisions immediately superior to the groups.

[1] See as to this theory my *Memoir on Curves of the Third Order*. *Phil. Trans.* p. 147 (1856), [145].

Newton's Classification. Article Nos. 42 to 46.

42. Newton establishes in the first instance the following four cases; viz. the equation of a cubic curve is one of the forms

$$\begin{array}{lll} \text{I.} & xy^2 + ey = & ax^3 + bx^2 + cx + d, \\ \text{II.} & xy & = ax^3 + bx^2 + cx + d, \\ \text{III.} & y^2 & = ax^3 + bx^2 + cx + d, \\ \text{IV.} & y & = ax^3 + bx^2 + cx + d. \end{array}$$

It is not, I think, necessary to reproduce here the very interesting reasoning by means of which this most important step in the classification was effected.

43. Starting from the four cases, Newton obtains his 14 genera, viz. Case I gives 11 genera, and Cases II, III, IV give each a single genus. But these genera group themselves as follows, viz. 1, 2, 3, 4, 5, 6 are Hyperbolas; 7 and 8, Parabolic Hyperbolas; 9 and 10, Central Hyperbolisms; 11, Parabolic Hyperbolisms; 12, the Trident Curve; 13, the Divergent Parabolas; and 14, the Cubical Parabola. And the equations are as follows:

$$\begin{array}{lll} \text{the Hyperbolas} & xy^2 + ey = & ax^3 + bx^2 + cx + d, \\ \text{the Parabolic Hyperbolas} & xy^2 + ey = & bx^2 + cx + d, \\ \text{the Central Hyperbolisms} & xy^2 + ey = & cx + d, \\ \text{the Parabolic Hyperbolisms} & xy^2 + ey = & d, \\ \text{the Divergent Parabolas} & y^2 & = ax^3 + bx^2 + cx + d, \\ \text{the Trident Curve} & xy & = ax^3 + bx^2 + cx + d, \\ \text{the Cubical Parabola} & y & = ax^3 + bx^2 + cx + d, \end{array}$$

where it is to be understood that the highest expressed power on the right-hand side of each equation does not vanish.

44. In these equations the axes $x=0$, $y=0$ are not for the most part lines precisely determined in relation to the curve, but it is easy to see as well analytically as geometrically how by a proper transformation of the equations they may be brought into forms such as those previously obtained, in which the several lines $p=0$, &c., stand in a determinate relation to the curve. Thus, taking the equation of the Cubical Parabola, this may be written $y=a\left(x+\frac{b}{3a}\right)^3+c'x+d'$; or, what is the same thing, $y'=ax'^3$. Or geometrically, we see that $x=0$ is a line completely determined as to its direction, it is in fact a line through the cusp at infinity; but that $y=0$ is an arbitrary line in regard to the curve; taking for $y=0$ the tangent at the inflexion, and for $x=0$ the line from the inflexion to the cusp at infinity, then the curve must pass through the point $(x=0, y=0)$, and $y=0$ must give a threefold value of x; the equation thus is $y=ax^3$. And so in other cases.

45. In the division into genera, Newton distinguishes the Hyperbolas into the redundant and defective, and the redundant hyperbolas into those for which the asymptotes form a triangle, and those for which the asymptotes meet in a point. The redundant hyperbolas with asymptotes forming a triangle are distinguished according as they have no diameter, a single diameter, or three diameters. The like distinction might have been, but is not, made as to the redundant hyperbolas with asymptotes meeting in a point; these are in fact included in a single genus; but the distinction presents itself in the species of that genus. As to the defective hyperbolas, Newton attends only to the real asymptote; and the only distinction is according as they have no diameter or a real diameter. The Parabolic Hyperbolas are in like manner divided according as they have no diameter or a diameter. The Central Hyperbolisms, according as the parallel asymptotes are real or imaginary, are the hyperbolisms of the hyperbola or of the ellipse. The hyperbolisms of the hyperbola form a single genus. Each of the Hyperbolisms might have been distinguished according as there is no diameter or a single diameter; this distinction appears in the species. The Trident Curve, the Divergent Parabolas, and the Cubical Parabola, form each a single genus.

46. We have thus the following Table of the Newtonian genera: I show in it the species in each genus, retaining Newton's numbers, and distinguishing by the numbers 10′, 13′, 22′, 22″ the four species added by Stirling, and by 56′ and 56″ the two species added by Murdoch or Cramer: I show also the division of genus 4, according to the number of diameters; and I also show the five species of curves having a centre.

Table of the Newtonian Genera.

1. Redundant Hyperbolas with asymptotes forming a triangle, and without a diameter.

Sp. 1, 2, 3, 4, 5, 6, 7, 8, 9.

2. Redundant Hyperbolas with asymptotes forming a triangle, and with a single diameter.

Sp. 10, 10′, 11, 12, 13, 13′, 14, 15, 16, 17, 18, 19, 20, 21.

3. Redundant Hyperbolas with asymptotes forming a triangle, and with three diameters.

Sp. 22, 22′, 22″, 23.

4. Redundant Hyperbola with asymptotes meeting in a point.

Without a diameter, Sp. 24, 25, 26, 27. With one diameter, Sp. 28, 29, 30, 31. With three diameters, Sp. 32. Sp. 27 has a centre.

5. Defective Hyperbolas without a diameter.

Sp. 33, 34, 35, 36, 37, 38. Sp. 38 has a centre.

6. Defective Hyperbolas with a diameter.

Sp. 39, 40, 41, 42, 43, 44, 45.

7. Parabolic Hyperbolas without a diameter.

Sp. 46, 47, 48, 49, 50, 51, 52.

8. Parabolic Hyperbolas with a diameter.

Sp. 53, 54, 55, 56, 56′, 56″.

9. Hyperbolisms of the hyperbola.

Without a diameter, Sp. 57, 58, 59. With a diameter, Sp. 60. Sp. 59 has a centre.

10. Hyperbolisms of the ellipse.

Without a diameter, Sp. 61, 62. With a diameter, Sp. 63. Sp. 62 has a centre.

11. Hyperbolisms of the parabola.

Without a diameter, Sp. 64. With a diameter, Sp. 65.

12. Trident Curve, Sp. 66.

13. Divergent Parabolas. Sp. 67, 68, 69, 70, 71.

14. Cubical Parabola. Sp. 72.

Plücker's Classification. Article Nos. 47 to 49.

47. Plücker in the first instance obtains by analytical considerations his six head divisions, corresponding to the seven divisions in the present Memoir, the Central Hyperbolisms and the Parabolic Hyperbolisms forming with him a single division. The equations are obtained in the form already mentioned, the only difference being that he writes $z=1$; his six head divisions with their equations thus are

Hyperbolas	$pqr+\mu s=0,$
Parabolic Hyperbolas	$(p+\kappa)(q^2+\lambda p)+\mu(q+\sigma)=0,$
Hyperbolisms	$p(q^2+\kappa)\ +\mu(q+\sigma)=0,$
Divergent Parabolas	$p^3+\lambda q^2+\mu(p+\sigma)=0,$
Trident Curve	$p(p^2+\lambda q)\ +\mu\ =0,$
Cubical Parabola	$p^3\qquad\quad +\mu s=0.$

48. He then divides the Hyperbolas into the redundant and defective. The redundant hyperbolas are then divided as they have no osculating asymptote, one osculating asymptote, or three osculating asymptotes; and each of these according as the asymptotes form a triangle or meet in a point. As regards the defective hyperbolas he attends to the imaginary asymptotes, represented by means of their real point of intersection, the "asymptote-point," and the division is thus similar to that of the redundant hyperbolas, viz. the defective hyperbolas are distinguished according as they have no osculating asymptote, a real osculating asymptote, or three osculating asymp-

totes; and each of these according as the asymptotes form a triangle or meet in a point; that is, according as the asymptote-point does not, or does, lie on the real asymptote.

The Parabolic Hyperbolas are distinguished according as the asymptote is ordinary and the asymptotic parabola one of five-pointic intersection; or, as the asymptote is osculating and the parabola one of six-pointic intersection; and each of these according as the asymptote cuts, touches, or does not cut, the parabola. The Hyperbolisms are distinguished into those of the hyperbola, ellipse, and parabola, and each of these according as the asymptote is ordinary or osculating. The Divergent Parabolas are distinguished in the manner already mentioned; viz. according as the curve is the semicubical parabola; or, as there is a satellite line not at infinity, and an asymptotic semicubical parabola of seven-pointic intersection, and which is cut, touched, or not cut, by the satellite line; or, as the satellite line is at infinity, and the semicubical parabola is of nine-pointic intersection. The Trident Curve and the Cubical Parabola are not divided.

49. I annex the following Table showing the Groups included in each division: for shortness I use the before-mentioned symbols Δ, $\odot$ to denote that the asymptotes form a triangle or meet in a point respectively.

Table of the Plückerian Divisions.

Hyperbolas:

Redundant;

No osculating asymptote,

Δ I, II, III, IV, V, VI
$\odot$ VII, VIII

One osculating asymptote,

Δ IX, X, XI, XII, XIII, XIV
$\odot$ XV

Three osculating asymptotes,

Δ XVI
$\odot$ XVII

Defective;

No osculating asymptote,

Δ XVIII, XIX, XX, XXI, XXII, XXIII
$\odot$ XXIV, XXV, XXVI, XXVII

Real osculating asymptote,

Δ XXVIII, XXIX, XXX, XXXI, XXXII, XXXIII
$\odot$ XXXIV

Three osculating asymptotes,

Δ	XXXV
⊙	XXXVI

Parabolic Hyperbolas:

Ordinary asymptote and five-pointic parabola.

asymptote cuts parabola	XXXVII, XXXVIII, XXXIX, XL, XLI
does not cut „	XLII
touches „	XLIII, XLIV

Osculating asymptote and six-pointic parabola.

asymptote cuts parabola	XLV
does not cut „	XLVI
touches „	XLVII

Hyperbolisms:

Of hyperbola;

asymptote ordinary	XLVIII, XLIX
„ osculating	L

Of ellipse;

asymptote ordinary	LI
„ osculating	LII

Of parabola;

asymptote ordinary	LIII
„ osculating	LIV

Divergent Parabolas:

Semicubical Parabola	LV
Asymptotic ditto, seven-pointic.	
satellite line cuts semic. parab.	LVI
does not cut „	LVII
touches (i.e. passes through the vertex of)	LVIII
Asymptotic ditto, nine-pointic	LIX
Trident Curve	LX
Cubical Parabola	LXI

As to the Theory of Groups. Article No. 50.

50. It has been already remarked that the division into groups depends on the different positions of the satellite line in regard to the asymptotic aggregate, and that the investigation relates in a great measure to the discussion of the values of the

parameter μ in the equation $V+\mu z^2 s=0$, which are such as to give rise to a nodal curve. It is to be noticed that except for the Hyperbolas and for the Cubical Parabola, the satellite $s=0$ is either a line passing through a given point at infinity (determinate, that is, as regards its direction), or in the case of the Trident Curve it is the given line infinity; there is at most only a *single* series of positions to be considered, and the theory is a short and easy one; and for the Cubical Parabola, although the satellite line $s=0$ is here an arbitrary line, yet on account of the cusp at infinity, there is not any critic value of μ, or in fact any distinction of cases. The only other case where the satellite line $s=0$ is an arbitrary line, admitting therefore of a double series of positions, is that of the Hyperbolas; and the division into groups constitutes an extensive and interesting theory, which is insufficiently discussed by Plücker; and it was with a view to the development of this theory that my Memoir, *On a Case of the Involution of two Cubic Curves*, (*ante*, pp. 39 to 81), referred to in the sequel as *Memoir on Involution*, [349], was written. I remark that of the three curves there established as material to the theory, and which are further spoken of in the sequel of the present memoir, viz. the envelope, the twofold centre locus, and the one-with-twofold centre locus, Plücker considers only the twofold centre locus. I proceed to apply the results of that Memoir to the present theory.

As to the Groups of the Hyperbolas Δ. Article Nos. 51 to 53.

51. The assumed form of equation was $pqr+\mu z^2 s=0$, but using now

$$x=0,\ y=0,\ z=0$$

(instead of $p=0$, $q=0$, $r=0$) for the equations of the asymptotes, we may imagine the implicit constants so determined that the line infinity (before represented by $z=0$) shall have for its equation $x+y+z=0$; writing moreover $\lambda x+\mu y+\nu z=0$ for the satellite line $s=0$, and k in the place of μ, the equation becomes

$$xyz+k(x+y+z)^2(\lambda x+\mu y+\nu z)=0,$$

which is the form considered in the Memoir just referred to.

52. It is there shown that for an arbitrary position of the satellite line, the parameter k, or, what is the same thing, the auxiliary parameter θ, may be determined by a cubic equation in such manner that the curve shall have a node; the node, or rather the site of the node is termed a critic centre; and there are consequently three critic centres (all real or else one real, two imaginary). If however the satellite line touches a certain curve called the envelope, then two of the critic centres unite together, forming a twofold centre which is (not a mere node but) a cusp on the corresponding cubic curve; the other critic centre is termed a one-with-twofold centre;

and the loci of the twofold centre and of the one-with-twofold centre respectively are determinate curves, the former a conic, the latter a cubic. The critic centres corresponding to the entire series of satellite lines which pass through a certain fixed point lie on a cubic, which when the fixed point lies on the line $x+y+z=0$, here the line infinity, degenerates into a conic called the "Harmonic Conic," and when one of the critic centres is known the other two are determined as the intersections of the harmonic conic by the polar of the given critic centre in regard to the twofold centre conic.

53. For establishing the theory of the groups of the hyperbolas Δ, it is necessary to consider the geometrical forms of the several curves which have been just referred to; viz. this is to be done, on the assumption always that the line $x+y+z=0$ is the line infinity, for the Redundant Hyperbolas taking the lines $x=0$, $y=0$, $z=0$, to be real lines and for the Defective Hyperbolas, taking them to be one real and the other two imaginary. The formulæ of the memoir above referred to, are in their actual form adapted to the former case, but they can of course be transformed so as to adapt them to the latter case. I proceed to examine the two cases separately.

The hyperbolas Δ *Redundant* (*See fig.* 1). Article Nos. 54 to 71.

54. Every thing is symmetrical with respect to the three asymptotes, and to fix the ideas and without any real loss of generality we may consider the asymptotes as forming an equilateral triangle. Taking the perpendicular distance of a vertex from the opposite side as unity, the absolute magnitudes of the coordinates may be fixed by assuming $x+y+z=1$; x, y, z will then denote the perpendicular distances of a point from the three sides respectively. If the coordinates of a point are proportional to α, β, γ, then the absolute magnitudes are of course

$$\frac{\alpha}{\alpha+\beta+\gamma},\ \frac{\beta}{\alpha+\beta+\gamma},\ \frac{\gamma}{\alpha+\beta+\gamma};$$

the point may be spoken of indifferently as the point (α, β, γ) or the point

$$\left(\frac{\alpha}{\alpha+\beta+\gamma},\ \frac{\beta}{\alpha+\beta+\gamma},\ \frac{\gamma}{\alpha+\beta+\gamma}\right),$$

and it is sometimes convenient to use both of the two notations; thus the Harmonic point (that is, the point the harmonic of infinity in regard to the triangle) is the point $(1, 1, 1)$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The lines $y-z=0$, $z-x=0$, $x-y=0$, which are the lines joining the harmonic point with the three vertices respectively, are in the case of the equilateral triangle the perpendiculars from the vertices on the three sides respectively,

and they may be spoken of simply as the perpendiculars, but it is to be borne in mind that the former is the proper construction of these lines.

55. The equation of the envelope is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

or, what is the same thing,

$$x^4+y^4+z^4-4(yz^3+y^3z+zx^3+z^3x+xy^3+x^3y)$$
$$+6(y^2z^2+z^2x^2+x^2y^2)-124(x^2yz+xy^2z+xyz^2)=0.$$

The curve consists as shown in the figure of a trigonoid branch inscribed in the triangle and of three acnodes outside the triangle.

56. The side $x=0$ touches the curve in the point $(0, 1, 1)$ or $(0, \frac{1}{2}, \frac{1}{2})$, which is its intersection with the perpendicular $y-z=0$; the side $x=0$ has with the curve at the point in question a four-pointic intersection. The last-mentioned line $y-z=0$ meets the curve in the point $(-4, 1, 1)$ or $(2, -\frac{1}{2}, -\frac{1}{2})$, which is one of the acnodes, and therefore a point of twofold intersection; then again in the point $(16, 1, 1)$ or $(\frac{8}{9}, \frac{1}{18}, \frac{1}{18})$ which may be considered as a vertex of the trigonoid branch, and finally in the before-mentioned point $(0, 1, 1)$ or $(0, \frac{1}{2}, \frac{1}{2})$, which is the point of contact with the side $x=0$.

57. The equation of the twofold centre locus is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or in a rational form

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

which is in the case of an equilateral triangle, a *circle* inscribed in the triangle and touching the sides at their midpoints respectively. The circle is shown in the figure.

58. The equation of the one-with-twofold centre locus is

$$-(-x+y+z)(x-y+z)(x+y-z)+xyz=0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0.$$

It is a cubic having the harmonic point $(1, 1, 1)$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, for an acnode, touching the sides of the triangle externally at their midpoints respectively, and having the three asymptotes

$$5x-4y-4z=0,\ -4x+5y-4z=0,\ 4x-4y+5z=0,$$

or, what is the same thing, $x=\frac{4}{9}$, $y=\frac{4}{9}$, $z=\frac{4}{9}$; the form of the curve is shown in the figure.

59. The equation of the harmonic conic corresponding to the satellite line $\lambda x + \mu y + \nu z = 0$, is

$$\frac{\mu - \nu}{x} + \frac{\nu - \lambda}{y} + \frac{\lambda - \mu}{z} = 0,$$

or say

$$2fyz + 2gzx + 2hxy = 0,$$

where $f + g + h = 0$.

It is a hyperbola passing through the angles of the triangle, and through the harmonic point (1, 1, 1) or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Observing that the points in question are the intersections of the two rectangular hyperbolas (pairs of lines) $x(y-z) = 0$, $y(z-x) = 0$, it follows that the harmonic conic is a *rectangular* hyperbola.

60. The coordinates of the centre are f^2, g^2, h^2 and the centre is consequently a point on the circle which is the twofold centre locus.

The asymptotes of course meet in the centre, and they again meet the circle in two points which are the intersections of the circle with the line $fx + gy + hz = 0$.

61. The Harmonic Conic is the same for the satellite lines which have a given direction, and we may to determine it take a satellite line which touches the envelope. If the constants α, β, γ satisfy the condition $\alpha + \beta + \gamma = 0$; then the equation of a satellite line tangent to the envelope is $\frac{x}{\alpha^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3} = 0$: the coordinates of the point of contact with the envelope are as $\alpha^4 : \beta^4 : \gamma^4$; the coordinates of the twofold centre as $\alpha^2 : \beta^2 : \gamma^2$; the coordinates of the one with twofold centre as

$$\alpha^2(\beta - \gamma) : \beta^2(\gamma - \alpha) : \gamma^2(\alpha - \beta).$$

The values of f, g, h are as $\alpha^3(\beta^3 - \gamma^3) : \beta^3(\gamma^3 - \alpha^3) : \gamma^3(\alpha^3 - \beta^3)$, or what is the same thing, as $\alpha^3(\beta - \gamma) : \beta^3(\gamma - \alpha) : \gamma^3(\alpha - \beta)$, or as $\alpha^2(\beta^2 - \gamma^2) : \beta^2(\gamma^2 - \alpha^2) : \gamma^2(\alpha^2 - \beta^2)$: the last-mentioned values show that the line $fx + gy + hz = 0$ passes through the harmonic point (1, 1, 1) or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and also through the point $\left(\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}\right)$, the inverse of the point $(\alpha^2, \beta^2, \gamma^2)$ which is the twofold centre.

62. On account of the symmetry of the figure in regard to the three asymptotes, it is sufficient to construct the harmonic conic for a direction of the satellite line inclined to the base at an angle not $> 30°$, and this is what is accordingly done: it may however be remarked that for the limiting inclination $= 0°$, that is, when the satellite line is parallel to the base, the harmonic conic becomes a pair of right lines, the base and perpendicular; but that for the other limiting inclination $= 30°$, that is, when the satellite line is perpendicular to one of the legs of the triangle, the harmonic conic is still a proper hyperbola, and is situate symmetrically in regard to the leg in question; the two limiting cases will be readily understood by means of the general case shown in the figure.

63. Imagine now the satellite line moving parallel to itself through the series of positions $ABCMDEA'$; to simplify the figure these are not delineated in their proper positions (but they are merely indicated according to their order of succession), and it is to be understood that they have the following positions, viz.

A, at infinity[1],

B, through the vertex B,

C, touching the envelope,

M, through the vertex M_3,

D, through the vertex D_1,

E, through node X of the envelope,

A', at infinity[1];

then the corresponding positions of the critic centres are

On one branch of the hyperbola,	On the other branch,
A_3, at infinity,	A_1, at infinity: A_2, the harmonic point,
B_3,	B_1, B_2,
C_3, a one-with-twofold centre,	C_{12}, a twofold centre,
M_3,	M_1, M_2 are imaginary,
C_3', a one-with-twofold centre,	C_{12}', a twofold centre,
D_3,	D_1, D_2,
E_3,	E_1, E_2,
A_3', at infinity.	A_1', at infinity; A_2, the harmonic point.

64. For the further explanation of the figure it is to be observed that B_2, B_3 lie on the line joining the midpoints of two sides; and in like manner D_2, D_3 on the line joining the midpoints of two sides; (the imaginary points M_1, M_2 are in like manner on the line joining the midpoints of two sides): these relations depend on the theorem, No. 81, of the Memoir on Involution, viz. that for the satellite lines which pass through a vertex (1, 0, 0) of the triangle, one of the critic centres is the vertex (1, 0, 0), and the other two critic centres are points on the line $-x+y+z=0$, or, what is the same thing, $x=\frac{1}{2}$.

65. Again, the point E_3 is on the line $(x=1)$ through the vertex D, parallel to the base, and the points E_1, E_2 are on the hyperbola (indicated by a dotted line in the figure) $(y+\frac{1}{4})(z+\frac{1}{4})=\frac{5}{16}$; this depends on the theorem Nos. 73 and 74 of the

[1] Strictly speaking a line at infinity is the line infinity, and as such has no definite direction; but we may of course consider a line which moves parallel to itself in opposite senses as having for its limit the line infinity.

Memoir on Involution, viz. the critic centres corresponding to the satellite lines through the point

$$(-4,\ 1,\ 1) \text{ or } (2,\ -\tfrac{1}{2},\ -\tfrac{1}{2})$$

lie one of them on the line $y+z=0$, and the other two on the conic $x(x+y+z)-4yz=0$; reducing by the condition $x+y+z=1$, these equations become respectively $x=1$, and

$$(y+\tfrac{1}{4})(z+\tfrac{1}{4})-\tfrac{5}{16}=0.$$

66. The foregoing positions of the satellite line, and the critic centres, as exhibited in the figure, were selected partly for facility of delineation; I wished however to examine the effect of the passage of the satellite line through a node of the envelope; and it appears that such passage does not give rise to any marked peculiarity in regard to the critic centres. The selected positions are sufficient to indicate the circumstances of the critic centres as the satellite line passes from the position A at infinity continuously to the position A' at infinity; in particular we see that as the line passes from A to C, or from C' to A', there are three real centres; but that as the line passes from C to C' there is only one real centre.

67. The case of the satellite line parallel to the asymptote $x=0$, is included (as already mentioned) as a limiting case in the foregoing one; the harmonic conic is here the pair of lines $x(y-z)=0$; and we have two critic centres on the line $y-z=0$, (the perpendicular), and the third (not properly a critic centre) at infinity on the asymptote $x=0$; in fact, starting with a critic centre on the line $y-z=0$, the polar of the centre in regard to the twofold centre conic or circle is a line parallel to the asymptote $x=0$, and which therefore meets the harmonic conic $x(y-z)=0$ in a second centre on the line $y-z=0$, and in the point at infinity on the line $x=0$. But the analytical theory of the case is peculiar and may be specially considered.

68. Writing $\mu=\nu$, the equation of the satellite line is $\lambda x+\mu(y+z)=0$, or putting $x+y+z=0$ this is $x=\dfrac{\mu}{\mu-\lambda}$. The equation in θ (see Memoir on Involution, No. 20) becomes

$$(\theta+\mu)(\theta^2-\theta\mu-2\lambda\mu)=0;$$

or, disregarding the factor $\theta+\mu=0$, which corresponds to the centre at infinity, the equation is

$$\theta^2-\theta\mu-2\lambda\mu=0,$$

which is a quadratic equation, giving therefore two values of θ, and the corresponding critic centres lie on the perpendicular $y-z=0$, the x coordinate being given by the equation

$$x=\frac{1}{\theta+\lambda}\div\left(\frac{1}{\theta+\lambda}+\frac{2}{\theta+\mu}\right)=\frac{1}{\theta+\lambda}\div\frac{2}{\theta}=\tfrac{1}{2}\,\theta\div(\theta+\lambda).$$

We have therefore conversely

$$\theta=\frac{\lambda x}{\frac{1}{2}-x},$$

and thence

$$\lambda x^2 - \mu x\left(\tfrac{1}{2} - x\right) - 2\mu\left(\tfrac{1}{2} - x\right)^2 = 0,$$

or, what is the same thing,

$$(\lambda - \mu)\, x^2 + \tfrac{3}{2}\mu x - \tfrac{1}{2}\mu = 0,$$

so that putting for shortness $\dfrac{\mu}{\mu - \lambda} = \varpi$, ($\varpi$ denotes the distance of the satellite line from the asymptote $x = 0$) then the equation which determines the distance of the critic centres from the asymptote is

$$x^2 - \tfrac{3}{2}\varpi x + \tfrac{1}{2}\varpi = 0,$$

or we have

$$x = \tfrac{1}{4}\left(3\varpi \pm \sqrt{9\varpi^2 - 8\varpi}\right).$$

The condition for a twofold centre is ($\varpi = 0$, which may be disregarded, or else) $\varpi = \frac{8}{9}$, or, what is the same thing, $8\lambda + \mu = 0$.

69. If x_1, x_2 are the coordinates of the two centres, we have

$$2x_1^2 = \varpi\,(3x_1 - 1),$$
$$2x_2^2 = \varpi\,(3x_2 - 1),$$

and thence

$$\frac{x_1^2}{x_2} = \frac{3x_1 - 1}{3x_2 - 1},$$

or, reducing,

$$x_1 + x_2 - 3x_1x_2 = 0,$$

a relation connecting the two values x_1, x_2; this equation however only expresses the known relation that the two centres are harmonics of each other in regard to the twofold centre conic or circle.

70. The foregoing examination of the form of the envelope shows very readily what are the positions of the satellite line which give rise to Plücker's groups for the Hyperbolas Δ Redundant.

We have in fact first,

Hyperbolas Δ Redundant, no osculating asymptote.

The satellite line is not parallel to a side of the triangle; and the different positions give the following six of Plücker's groups, viz.

I.	Satellite line	cuts three sides produced.
II.	"	passes through a vertex and cuts opposite side produced.
III.	"	passes through a vertex and cuts opposite side.
IV.	"	cuts two sides and a side produced, but does not cut the envelope.
V.	"	touches the envelope.
VI.	"	cuts the envelope.

Next,

Hyperbolas Δ Redundant, one osculating asymptote.

The satellite line is parallel to the osculating asymptote, say to the base of the triangle; and the different positions give the following six of Plücker's groups, viz.

IX. Satellite line above the vertex.

X. „ through the vertex.

XI. „ below the vertex, but not cutting envelope.

XII. „ touches envelope.

XIII. „ cuts envelope.

XIV. „ lies below the base.

And finally,

Hyperbolas Δ Redundant, three osculating asymptotes.

The position of the satellite line is here completely determined, giving one of Plücker's groups, viz.

XVI. Satellite line at infinity.

71. It may be remarked that in this enumeration no account is taken of the nodes of the envelope: the enumeration was in fact made by Plücker by considerations relating to the critic centres, but without arriving at or making use of the envelope at all: if account were taken of the nodes of the envelope several of the foregoing groups would have to be subdivided according to the different positions of the satellite line in regard to these nodes: but the effect produced by the passage of the satellite line through a node of the envelope is so slight, that I am inclined to think that the enumeration may be properly effected in the foregoing manner, without any account being taken of these nodes.

The Hyperbolas Δ *Defective* (*See fig.* 2). Article Nos. 72 to 101.

72. If in the formulæ for the Hyperbolas Δ Redundant we write

$$\begin{aligned} &\tfrac{1}{2}(x+yi) \text{ for } x,\\ &\tfrac{1}{2}(x-yi) \quad ,, \quad y,\\ &\lambda-\mu i \quad ,, \quad \lambda,\\ &\lambda+\mu i \quad ,, \quad \mu, \end{aligned}$$

then the equation of the satellite line is

$$\tfrac{1}{2}(\lambda-\mu i)(x+yi)+\tfrac{1}{2}(\lambda+\mu i)(x-yi)+\nu z=0,$$

which is $\lambda x+\mu y+\nu z=0$ as before; the equation of the line infinity is $x+z=0$, and the equation of the curve is

$$\tfrac{1}{4}(x^2+y^2)z+k(x+z)^2(\lambda x+\mu y+\nu z)=0.$$

We may fix the absolute magnitudes of the coordinates by writing $x+z=1$; and the equation then becomes

$$(x^2+y^2)(1-x)+4k\{(\lambda-\nu)x+\mu y+\nu\}=0.$$

The origin is at the asymptote-point, or intersection of the imaginary asymptotes; the equation of the real asymptote is $x=1$; that of the imaginary asymptotes is $x^2+y^2=0$.

If x and y are ordinary rectangular coordinates then the pair of lines represented by this equation will be an indefinitely small circle, and conversely, if the Asymptote-Point be an indefinitely small circle, then x and y will be rectangular coordinates; and we may without loss of generality assume that this is so.

73. The equation of the envelope is

$$\sqrt[4]{\tfrac{1}{2}(x+yi)}+\sqrt[4]{\tfrac{1}{2}(x-yi)}+\sqrt[4]{z}=0;$$

this gives successively

$$\sqrt{\tfrac{1}{2}(x+yi)}+\sqrt{\tfrac{1}{2}(x-yi)}-\sqrt{z}=-\sqrt{2}\sqrt[4]{x^2+y^2},$$

$$x+z-\sqrt{x^2+y^2}=2\sqrt{z}\{\sqrt{\tfrac{1}{2}(x+iy)}+\sqrt{\tfrac{1}{2}(x-iy)}\},$$

$$(x+z)^2+x^2+y^2-2(x+z)\sqrt{x^2+y^2}=4z(x+\sqrt{x^2+y^2}),$$

$$(x-z)^2+x^2+y^2\ =2(x+3z)\sqrt{x^2+y^2},$$

$$\{(x-z)^2+x^2+y^2\}^2=4(x+3z)^2(x^2+y^2);$$

that is

$$(x-z)^4+2(x^2+y^2)\{(x-z)^2-2(x+3z)^2\}+(x^2+y^2)^2=0.$$

Putting $z=1-x$, and therefore $x-z=2x-1$, and $x+3z=3-2x$, this becomes

$$(2x-1)^4+2(x^2+y^2)\{(2x-1)^2-2(2x-3)^2\}+(x^2+y^2)^2=0,$$

that is

$$(x^2+y^2)^2-2(x^2+y^2)(4x^2-20x+17)+(2x-1)^4=0,$$

which may also be written

$$y^4-2y^2(3x^2-20x+17)+9x^4+8x^2-10x^2-8x+1=0,$$

or, what is the same thing,

$$y^4-2y^2(x-1)(3x-17)+(x-1)(9x-1)(x+1)^2=0,$$

for the equation of the envelope. The solution of the equation in y gives

$$y^2=(x-1)(3x-17)\pm(2x-3)\sqrt{-32(x-1)}.$$

74. The original curve $\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0$ had the three nodes

$$(2,\ -\tfrac{1}{2},\ -\tfrac{1}{2}),\quad(-\tfrac{1}{2},\ 2,\ -\tfrac{1}{2}),\quad(-\tfrac{1}{2},\ -\tfrac{1}{2},\ 2);$$

and thence writing

$$\tfrac{1}{2}(x+yi),\ \tfrac{1}{2}(x-yi),\ z, \text{ for } x,\ y,\ z,$$

we find the nodes

$$(x=\tfrac{3}{2},\ y=\tfrac{5}{2}i),\quad (x=\tfrac{3}{2},\ y=-\tfrac{5}{2}i),\quad (x=-1,\ y=0);$$

the first and second of these are acnodes, or the curve has a pair of imaginary acnodes; the third is a crunode; and to find the directions at this point, if in the equation of the curve we write $x-1$ for x, the equation becomes

$$y^4-2y^2(x-2)(3x-20)+(x-2)(9x-10)x^2=0;$$

the lowest terms therefore are $20(-4y^2+x^2)=0$; and we have $y=\pm\frac{1}{2}(x+1)$ for the equation of the tangents at the crunode.

$y=0$ gives $x=1$, $x=\frac{1}{9}$ and (as a twofold value) $x=-1$, which belongs to the crunode.

$x=1$ gives $y^4=0$, or the line $x=1$ is a tangent of four-pointic intersection.

$x=0$ gives $y^4-34y^2+1=0$, that is $y^2=17\pm12\sqrt{2}$, or $y=\pm(3\pm2\sqrt{2})$.

75. The curve has a pair of asymptotic parabolas, and taking for the equation of one of them

$$(y-x\sqrt{3}+\beta)^2=-\tfrac{32}{3}(x+\alpha),$$

this gives

$$y=x\sqrt{3}-\beta+\sqrt{-\tfrac{32}{3}(x+\alpha)},$$

and thence

$$\begin{aligned}y^2=3x^2-2\beta x\sqrt{3}+\beta^2\\ -\tfrac{32}{3}(x+\alpha)\\ +2\left(x-\frac{\beta}{\sqrt{3}}\right)\sqrt{-32(x+\alpha)},\end{aligned}$$

which agrees with the value of y^2 in the envelope as to the terms x^2, $x^{\frac{3}{2}}$, and by properly determining α, β, it may be made to agree as to the terms x and $x^{\frac{1}{2}}$.

We have in the parabola

$$\begin{aligned}y^2=3x^2-2\beta x\sqrt{3}+\beta^2\\ -\tfrac{32}{3}x-\tfrac{32}{3}\alpha\\ +\sqrt{-32}\left\{2x\sqrt{x}+\left(\alpha-\frac{2\beta}{\sqrt{3}}\right)\sqrt{x}\right\},\end{aligned}$$

and in the curve

$$\begin{aligned}y^2=3x^2-20x+17\\ +\sqrt{-32}\,\{2x\sqrt{x}-4\sqrt{x}\}.\end{aligned}$$

We have therefore

$$-2\beta\sqrt{3}-\tfrac{32}{3}=-20,\ \alpha-\frac{2\beta}{\sqrt{3}}=-4,$$

giving

$$\beta=\frac{14}{3\sqrt{3}}=\frac{14\sqrt{3}}{9},\ \alpha=-\tfrac{8}{9},$$

so that the equation of the parabola is

$$\{y-(x-\tfrac{14}{9})\sqrt{3}\}^2=-\tfrac{32}{3}(x-\tfrac{8}{9});$$

that of the other parabola is of course obtained by merely changing the sign of $\sqrt{3}$.

76. From the foregoing results we may trace the curve, but this may be done somewhat more easily by means of polar coordinates, viz. writing $x=r\cos\theta$, $y=r\sin\theta$, and therefore $z=1-r\cos\theta$, we have

$$\sqrt[4]{\tfrac{1}{2}r}\,.\,2\cos\tfrac{1}{4}\theta+\sqrt[4]{1-r\cos\theta}=0,$$

that is

$$r(\cos\theta+8\cos^2\tfrac{1}{4}\theta)=1;$$

or since

$$\cos\theta=1-8\sin^2\tfrac{1}{4}\theta\cos^2\tfrac{1}{4}\theta,$$
$$=1-8\cos^2\tfrac{1}{4}\theta+8\cos^4\tfrac{1}{4}\theta,$$

we have

$$\cos\theta+8\cos^4\tfrac{1}{4}\theta=(1-4\cos^2\tfrac{1}{4}\theta)^2$$
$$=(-1-2\cos\tfrac{1}{2}\theta)^2,$$

and the equation is

$$r=\frac{1}{(1+2\cos\frac{1}{2}\theta)^2};$$

$\theta=0°$ gives $r=\frac{1}{9}$, $\theta=180°$ gives $r=1$, $\theta=240°$ gives $r=\infty$, $\theta=360°$ gives $r=1$,

values which agree with the results obtained by rectangular coordinates.

77. The form is shown in the figure; we see that the curve consists of a lower branch without any singularity; and of an upper branch which cuts itself in the crunode.

78. The equation of the twofold centre locus (making in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, the foregoing transformation) is

$$\sqrt{\tfrac{1}{2}(x+yi)}+\sqrt{\tfrac{1}{2}(x-yi)}+\sqrt{1-x}=0,$$

that is

$$x+\sqrt{x^2+y^2}=1-x,$$

or

$$\sqrt{x^2+y^2}=1-2x,$$

which is

$$3x^2 - 4x - y^2 = -1,$$

or

$$3\left(x - \tfrac{2}{3}\right)^2 - y^2 = \tfrac{1}{3},$$

or finally

$$9\left(x - \tfrac{2}{3}\right)^2 - 3y^2 = 1,$$

which is a hyperbola having its centre at the harmonic point $x = \frac{2}{3}$, $y = 0$; having $x = \frac{1}{3}$, $x = 1$ for the extremities of the transverse axis, and such that the asymptotes are inclined to the axis of x at an angle $= 60°$; this curve is also shown in the figure.

79. Similarly making the transformation in the equation of the one-with-twofold centre locus written under the form $-(-x+y+z)(x-y+z)(x+y-z)+xyz=0$, this becomes

$$-(-yi+z)(yi+z)(x-z)+\tfrac{1}{4}(x+yi)(x-yi)z=0,$$

that is

$$-4(y^2+z^2)(x-z)+(x^2+y^2)z=0,$$

which is

$$-4z^2(x-z)+x^2z+y^2(5z-4x)=0,$$

or, what is the same thing,

$$z(x-2z)^2+y^2(5z-4x)=0,$$

or, putting for z its value $= 1 - x$, this is

$$(1-x)(3x-2)^2+y^2(5-9x)=0,$$

that is

$$y^2 = -\frac{(x-1)(3x-2)^2}{9x-5},$$

which is the equation of the one-with-twofold centre locus.

80. The curve is symmetrical in regard to the axis of x. And moreover

$x < \frac{5}{9}$, y is impossible,

$x = \frac{5}{9}$, $y^2 = \infty$, or the line $x = \frac{5}{9}$ is an asymptote,

$x = \frac{2}{3}$, $y^2 = 0$, which is a crunode,

$x = 1$, $y^2 = 0$,

and

$x > 1$, y is impossible.

The equation of the tangents at the crunode are $y^2 = 3\left(x - \frac{2}{3}\right)$, or the tangents are inclined to the axis of x at angles $= 60°$. The curve consists, as shown in the figure, of a single branch cutting itself in the crunode, and tending on each side towards the asymptote.

81. The equation of the harmonic conic, making the foregoing transformation in the equation $\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0$, becomes

$$\frac{\lambda-\nu+\mu i}{x+yi}-\frac{\lambda-\nu-\mu i}{x-yi}-\frac{\mu i}{z}=0,$$

that is

$$-2(\lambda-\nu)yi+2\mu i\,x-\mu i\frac{x^2+y^2}{z}=0,$$

which is

$$\mu(x^2+y^2)-2z\{\mu x-(\lambda-\nu)y\}=0,$$

or, what is the same thing,

$$\mu(x^2+y^2-2zx)+2(\lambda-\nu)yz=0,$$

which, putting for z its value $=1-x$, is

$$\mu(3x^2+y^2-2x)+2(\lambda-\nu)y(1-x)=0,$$

or developing,

$$3\mu x^2-2(\lambda-\nu)xy+\mu y^2-2\mu x+2(\lambda-\nu)y=0,$$

either of which is the equation of the harmonic conic corresponding to the satellite line $(\lambda-\nu)x+\mu y+\nu=0$, or, since the direction is alone material, to the satellite line $(\lambda-\nu)x+\mu y=0$. The second form shows that the conic is

an ellipse for $3\mu^2>(\lambda-\nu)^2$,
a parabola „ $3\mu^2=(\lambda-\nu)^2$,
a hyperbola „ $3\mu^2<(\lambda-\nu)^2$.

82. The first form shows that the conic passes through the four points which are the intersection of the ellipse $3x^2+y^2-2x=0$, (or, as the equation may also be written, $9(x-\frac{1}{3})^2+3y^2=1$), with the pair of lines $(x-1)y=0$: this is right, for the points in question are the three points $(x=0,\ y=0)$, $(x=1,\ y=i)$, $(x=1,\ y=-i)$, which are the vertices of the triangle formed by the asymptotes $(x^2+y^2)(x-1)=0$; and the point $x=\frac{2}{3}$, $y=0$, which is the harmonic point.

83. Putting for shortness $\lambda-\nu=\kappa$, so that the equation of the satellite line is $\kappa x+\mu y+\nu=0$, and that of the corresponding harmonic conic is

$$\mu(3x^2+y^2-2x)+2\kappa y(1-x)=0,$$

the coordinates of the centre are found from the formulæ

$$\tfrac{1}{2}(x+yi):\tfrac{1}{2}(x-yi):z=(\kappa+\mu i)^2:(\kappa-\mu i)^2:-4\mu^2,$$

whence, since $x+z=1$, we have for the coordinates of the centre

$$x=\frac{\mu^2-\kappa^2}{3\mu^2-\kappa^2},\quad y=\frac{-2\mu\kappa}{3\mu^2-\kappa^2},$$

and it is easy to verify that these belong to a point on the twofold centre conic

$$3x^2-4x-y^2+1=0.$$

84. The asymptotes of the harmonic conic meet at the centre; and they again cut the twofold centre conic in two points, the intersections of the last-mentioned conic with the line

$$(\kappa+\mu i)\tfrac{1}{2}(x+yi)+(-\kappa+\mu i)\tfrac{1}{2}(x-yi)-2\mu iz=0$$

that is

$$\mu x+\kappa y-2\mu z=0,$$

or, what is the same thing,

$$3\mu x+\kappa y-2\mu=0.$$

85. I remark that the equation of the asymptotes of the harmonic conic is

$$(3\mu^2-\kappa^2)[\mu(3x^2+y^2-2x)+2\kappa y(1-x)]+\mu(\mu^2+\kappa^2)=0,$$

and that the theorem for the construction of the asymptotes depends on the identical equation

$$\begin{aligned}(3\mu^2-\kappa^2)[\mu(3x^2+y^2-2x)+2\kappa y(1-x)]+\mu(\mu^2+\kappa^2)+3\mu(\mu^2+\kappa^2)(3x^2-y^2-4x+1)\\ =-2[-(3\mu^2-\kappa^2)x+2\mu\kappa y+\mu^2+\kappa^2](3\mu x+\kappa y-2\mu),\end{aligned}$$

which is easily verified; and where

$$-(3\mu^2-\kappa^2)x+2\mu\kappa y+\mu^2+\kappa^2=0$$

is the equation of the tangent of the twofold centre conic $3x^2-y^2-4x+1=0$ at the centre of the harmonic conic.

86. On account of the symmetry in regard to the axis of x, it will be sufficient to attend to the series of curves corresponding to a direction $y=-\frac{\kappa}{\mu}x$ of the satellite line, for which the ratio $-\frac{\kappa}{\mu}$ has a given sign; and the inclination of the satellite line to the asymptote will pass from 90° to 0° according as the value of the ratio $-\frac{\kappa}{\mu}$ passes from 0 to ∞.

87. First, if $-\frac{\kappa}{\mu}$ is $=0$, that is, if the satellite line be perpendicular to the asymptote, then the harmonic conic is the ellipse

$$3x^2+y^2-2x=0,$$

or, as it may also be written,

$$9(x-\tfrac{1}{3})^2+3y^2=1.$$

As $-\frac{\kappa}{\mu}$ increases from 0 to $\sqrt{3}$, that is, as the inclination of the satellite line diminishes from 90° to 30°, the harmonic conic becomes an ellipse of greater and greater excentricity, and ultimately a parabola.

88. I notice the particular case $-\frac{\kappa}{\mu}=\frac{1}{2}$, which corresponds to the direction parallel to one of the nodal tangents of the envelope: the harmonic conic is in this case the ellipse

$$3x^2+y^2-2x-y+xy=0,$$

which, it will be observed, passes through the point ($x=0$, $y=1$) which is one of the points of intersection of the twofold centre locus or hyperbola $3x^2-4x-y^2=-1$ by the line $x=0$.

89. For the value $-\frac{\kappa}{\mu}=\sqrt{3}$ corresponding to a direction inclined at an angle $=30°$ to the asymptote, the harmonic conic becomes the parabola $(x\sqrt{3}-y)^2-2(x-y\sqrt{3})=0$: this equation may also be written $(x-\frac{1}{2})^2+y^2=\frac{1}{4}(x+y\sqrt{3}-1)^2$, a form which puts in evidence the focus and directrix of the parabola.

90. For a value $-\frac{\kappa}{\mu}>\sqrt{3}$, that is, when the satellite line is inclined to the asymptote at an angle $<30°$, the harmonic conic is a hyperbola, and ultimately when $-\frac{\kappa}{\mu}=\infty$, or the satellite line is parallel to the asymptote, the harmonic conic becomes the pair of lines $y(1-x)=0$.

91. I have in the figure shown the following forms of the harmonic conic; viz.

hyperbola, corresponding to inclination $<30°$ of satellite line to asymptote.

parabola, to inclination $=30°$.

$$\left.\begin{array}{l}\text{ellipse}\\ \text{ellipse}\\ \text{ellipse}\end{array}\right\}\text{to inclination}=\left\{\begin{array}{l}<\\ \\ >\end{array}\right.\text{inclination } (=\tan^{-1}2) \text{ of a nodal tangent of the envelope,}$$

and for these forms respectively the successive positions of the satellite line are indicated as follows:

92. For the inclination $<30°$, the positions are $ACMC'DEA'$, viz.

A, at infinity,

C, touching lower branch of envelope,

M, between C and C',

C', touching upper branch of envelope,

D, passing through asymptote point D_1,

E, passing through crunode of envelope,

A', at infinity.

The corresponding positions of the critic centres on the hyperbola are

On one branch of hyperbola.	On the other branch.
A_1, A_3, each at infinity,	A_2, the harmonic point,
B_1, B_3,	B_2,
C_{13}, a twofold centre,	C_2, a one-with-twofold centre,
	M_2, a real centre,
	C'_3, a one-with-twofold centre and C'_{12}, a twofold centre,
	D_3 and D_1 (Asymptote-Point), D_2,
	E_3 and E_1, E_2,
	A'_3, at infinity, and A'_1, at infinity, A_2, harmonic point.

93. For inclination $=30°$ the positions are $(AC)MC'DE(A'C)$, viz.

AC, at infinity, touches envelope at infinity,
M, between (AC) and C',
C', touching upper branch of envelope,
D, passing through asymptote point D_1,
E, passing through crunode of envelope,
$A'C$, at infinity touches envelope at infinity.

The corresponding positions of the critic centres on the parabola are

A_2, (harmonic point) a one-with-twofold centre; $A_{13}=A'_{13}$ at infinity, a twofold centre,
M_2, real centre, the other two centres being imaginary,
C'_{12}, a twofold centre, C'_3, a one-with-twofold centre,
D_1, (asymptote point), D_3; D_2,
E_1, E_3; E_2.
A_{13}, $=A'_{13}$, at infinity, a one-with-twofold centre; A_2 (harmonic point) a twofold centre.

94. For inclination $=\tan^{-1}2$, the positions are $AMC'D(EC'')NA'$, viz.

A, at infinity,
M, between A and C',
C', touching upper branch of envelope,
D, through asymptote point D_1,
EC''_1, touching upper branch of envelope at crunode N between EC'' and A',
A', at infinity.

And the corresponding positions of the critic centres on the ellipse are

A_2, (harmonic point), the other two centres imaginary,

M_2, real centre, the other two centres imaginary,

C'_{12}, a twofold centre, C'_3 a one-with-twofold centre,

D_1, (asymptote point), $D_2 : D_3$,

$C''E_{13}$, twofold centre, $C''E_2$ one-with-twofold centre,

N_2, real centre, the other two centres imaginary,

A_2, harmonic point; the other two centres imaginary.

95. And for the inclinations $<$ and $> \tan^{-1}2$, the only difference is that the positions are $AC'DEC''A'$, viz.

A, at infinity,

M, between A and C'',

C', touching upper branch of envelope,

D, through asymptote point D_1,

E, through crunode,

C'', touching upper branch of envelope,

N, between C'' and A',

A', at infinity,

and that instead of the points $C''E_{13}$ and $C''E_2$ we have separately the points C''_{13}, C''_2 and E_1, E_3, E_2 as shown in the figure.

96. For the better understanding of the figure it is to be observed that the points D_2 and D_3 lie on the line $x=\frac{1}{2}$: this depends on the theorem No. 81 of the Memoir on Involution, viz. of the critic centres which belong to a satellite line through the vertex (0, 0, 1), one is the vertex itself, the other two lie on the line $x+y-z=0$; or making the transformation $\frac{1}{2}(x+iy)$, $\frac{1}{2}(x-iy)$, z for x, y, z and writing $x+z=1$, of the critic centres which pass through the vertex (0, 0, 1) (the asymptote point), one is this point itself, the other two lie on the line $x-z=0$, that is $x=\frac{1}{2}$.

97. Again it is to be observed that the centre E_3 lies on the line $x=0$, and the centres E_1, E_2 on the circle (indicated by a dotted line in the figure) $(x+\frac{1}{2})^2+y^2=\frac{5}{4}$: this depends on the theorem Nos. 73 and 74 of the Memoir on Involution, viz. of the critic centres for satellite lines through the node (acnode) (1, 1, -4), one lies on the line $x+y=0$, and the other two lie on the conic $z(x+y+z)-4xy=0$; making the substitution

$$\tfrac{1}{2}(x+yi),\quad \tfrac{1}{2}(x-yi),\quad z \quad \text{for } x,\ y,\ z,$$

and writing also $x+z=1$, we find that, for the satellite lines which pass through the crunode (1, 0, 2), the critic centres lie one on the line $x=0$, the other two on the conic

$$z(x+z)-x^2-y^2=0,$$

that is, on the circle $x^2+y^2+x-1=0$, or $(x+\frac{1}{2})^2+y^2=\frac{5}{4}$.

98. The circle in question cuts the twofold centre conic $3x^2-4x-y^2=-1$ at its intersections with the line $x=0$, viz. in the points $x=0$, $y=\pm 1$; and it moreover touches the one-with-twofold centre locus $y^2=\dfrac{-(x-1)(3x-2)}{9x-5}$ at a point where this same circle meets the ellipse $3x^2+xy+y^2-2x-y=0$, which is the harmonic conic corresponding to the inclination $\tan^{-1}2$. In fact, writing down the three equations,

$$x^2+y^2+x-1=0,$$
$$3x^2+xy+y^2-2x-y=0,$$
$$y^2=-\frac{(x-1)(3x-2)^2}{9x-5},$$

the first and third equations give

$$-\frac{(x-1)(3x-2)^2}{9x-5}=1-x-x^2,$$

that is

$$-(x-1)(3x-2)^2+(9x-5)(x^2+x-1)=0,$$

or, reducing,

$$(5x-3)^2=0,$$

that is $x=\frac{3}{5}$, and then from the first or third equation $y^2=\frac{1}{25}$, or $y=\pm\frac{1}{5}$; hence the circle *touches* the one-with-twofold centre locus at the points

$$x=\tfrac{3}{5},\quad y=-\tfrac{1}{5};\quad x=\tfrac{3}{5},\quad y=+\tfrac{1}{5};$$

and by means of the second equation we see that the first of these points, viz. the point $x=\frac{3}{5}$, $y=-\frac{1}{5}$, is a point of the ellipse or harmonic conic $3x^2+xy+y^2-2x-y=0$.

99. I consider the analytical theory of the case where the satellite line is parallel to the asymptote; this is in fact similar to the theory *ante* Nos. 67—69; writing

$$\tfrac{1}{2}(x+yi),\quad \tfrac{1}{2}(x-yi),\ z,\ \lambda-\mu i,\ \lambda+\mu i,\ \nu$$

in the place of x, y, z, λ, μ, ν, and putting afterwards $\mu=0$, that is, in the transformed equation $\lambda x+\mu y+\nu z=0$ writing $\mu=0$, we find for the satellite line $\lambda x+\nu(1-x)=0$; the equation in θ (the factor $\theta+\lambda=0$ being disregarded) is

$$\theta^2-\lambda\theta-2\lambda\nu=0,$$

and the corresponding critic centres lie on the line $y=0$, at the distances given by the equation

$$x:1-x=\frac{2}{\theta+\lambda}:\frac{1}{\theta+\nu},\quad \text{whence } x=\frac{\theta}{\theta+\lambda},\quad z=1-x=\frac{\lambda}{\theta+\lambda};$$

we have then

$$\theta = \frac{\lambda(1-z)}{z},$$

and the values of z are given by the equation

$$\lambda(1-z)^2 - \lambda z(1-z) - 2\nu z^2 = 0,$$

that is

$$2(\lambda-\nu)z^2 - 3\lambda z + \lambda = 0,$$

which, putting $\frac{\nu}{\nu-\lambda} = 1-\varpi$, or $\varpi = \frac{\lambda}{\lambda-\nu}$ (ϖ is the distance of the satellite line from the asymptote $z=0$), becomes

$$2z^2 - 3\varpi z + \varpi = 0,$$

or we have

$$z = \tfrac{1}{4}\{3\varpi \pm \sqrt{\varpi(9\varpi-8)}\}.$$

The condition for a twofold centre is ($\varpi = 0$ which may be disregarded, or else)

$$9\varpi - 8 = 0;$$

or, what is the same thing, $\lambda + 8\nu = 0$.

100. If z_1, z_2 are the coordinates of the two critic centres, then we have

$$2z_1^2 = \varpi(3z_1 - 1),$$

$$2z_2^2 = \varpi(3z_2 - 1),$$

and thence

$$\frac{z^2}{z_2^2} = \frac{3z_1 - 1}{3z_2 - 1};$$

or, reducing,

$$3z_1z_2 - z_1 - z_2 = 0,$$

or in terms of the x-coordinates

$$3x_1x_2 - 2(x_1 + x_2) + 1 = 0,$$

which equation however merely expresses that the two centres are harmonics to each other in regard to the twofold centre conic $3x^2 - 4x - y^2 + 1 = 0$. It is right to remark that the formulæ, although referring to a different system of coordinates, are absolutely identical with those given Nos. 67—69, writing therein ν for μ, and z for x.

101. An inspection of the form of the envelope shows what are the positions of the satellite line which gives rise to Plücker's Groups for the Hyperbolas Δ Defective. We have in fact,

Hyperbolas Δ Defective, asymptote not osculating.

The satellite line is not parallel to the asymptote, and the different positions give the following six of Plücker's Groups, viz.

XVIII. Satellite line cuts upper, cuts lower, branch of envelope.

XX. „ „ „ „ and passes through asymptote point.

XIX. Satellite line touches upper, cuts lower, branch.

XXI. „ does not cut upper, cuts lower, branch.

XXIII. „ „ „ , touches lower branch.

XXII. „ „ „ , does not cut lower branch.

Hyperbolas Δ Defective, asymptote osculating.

The satellite line is parallel to the asymptote, and we have the six groups,

XXVIII. Satellite line cuts upper branch, cuts lower branch of envelope, viz. it lies above the asymptote point.

XXIX. Do, Do, but it passes through the asymptote point.

XXX. Do, Do, but it lies below the asymptote point.

XXXI. Satellite line touches upper branch, cuts lower branch.

XXXII. „ does not cut upper branch, cuts lower branch.

XXXIII. „ „ „ , does not cut lower branch, viz. it lies below the asymptote.

And finally,

Hyperbolas Δ Defective, three osculating asymptotes.

Satellite line at infinity, giving the single group

XXXV.

But the division gives rise to a remark such as is made *ante* No. 71.

As to the Groups of the Hyperbolas ⊙. Article Nos. 102 to 104.

102. Taking $z=0$ as the equation of the line infinity, and $x=0$, $y=0$ as the equation of any two lines through the point of intersection of the asymptotes, or 'asymptote point,' then the equation of the cubic may be taken to be

$$\tfrac{1}{3}(a,\ b,\ c,\ d \between x,\ y)^3 + kz^2(\lambda x + \mu y + \nu z) = 0.$$

103. To determine the critic centres we have

$$(a,\ b,\ c \between x,\ y)^2 + kz^2\lambda = 0,$$
$$(b,\ c,\ d \between x,\ y)^2 + kz^2\mu = 0$$
$$2z(\lambda x + \mu y + \nu z) + z^2\nu = 0,$$

and thence

$$\mu(a,\ b,\ c \between x,\ y)^2 - \lambda(b,\ c,\ d \between x,\ y)^2 = 0,$$

or as it may also be written

$$(\mu a - \lambda b,\ \mu b - \lambda c,\ \mu c - \lambda d \between x,\ y)^2 = 0,$$

and also

$$2(\lambda x + \mu y) - 3\nu z = 0,$$

which two equations determine the critic centres for a given position of the satellite line; the first of them gives a pair of lines through the asymptote point; the latter is a line parallel to the satellite line: there are thus two critic centres.

104. The condition for a twofold centre is

$$(ac - b^2,\ bc - ad,\ bd - c^2 \between \lambda,\ \mu)^2 = 0,$$

so that there are a pair of twofold centres which will be real if

$$(bc - ad)^2 - 4(ac - b^2)(bd - c^2) = +,$$

imaginary if

$$(bc - ad)^2 - 4(ac - b^2)(bd - c^2) = -,$$

that is, the twofold centres will be real or imaginary, according as the equation

$$(a,\ b,\ c,\ d \between x,\ y)^3 = 0$$

has its roots one real and two imaginary, or all three real; viz. the twofold centres are real for the Hyperbolas ⊙ Defective; imaginary for the Hyperbolas ⊙ Redundant. And we see also that for the Hyperbolas ⊙ Redundant the critic centres are always real; but that for the Hyperbolas ⊙ Defective, they may be both real, or both imaginary, or may coincide together, giving a twofold centre. But the two cases are best studied by assuming different special forms for the equation.

The Hyperbolas ⊙ Redundant. Article Nos. 105 to 107.

105. The equation may be taken to be

$$xy(x - y) + kz^2(\lambda x + \mu y + \nu z) = 0,$$

or writing $z = 1$, then the equation is

$$xy(x - y) + k(\lambda x + \mu y + \nu z) = 0.$$

We may, to fix the ideas, consider the case where the three asymptotes are parallel to the sides of an equilateral triangle; x, y, and $x - y$ will then denote the perpendicular distances of the point from the three asymptotes respectively.

106. The critic centres are given by the equations

$$2(\lambda x + \mu y) - 3\nu = 0,$$
$$\lambda x^2 - 2(\lambda + \mu) xy + \mu y^2 = 0;$$

or, what is the same thing, they are the intersections of the line

$$2(\lambda x + \mu y) - 3\nu = 0$$

by the two real lines

$$\lambda x = [(\lambda + \mu) \pm \sqrt{\lambda^2 + \lambda\mu + \mu^2}]\, y.$$

107. The groups are

Hyperbolas ⊙ Redundant. No osculating asymptote.

The satellite line not parallel to any asymptote, that is $\lambda = 0$, $\mu = 0$, $\lambda + \mu = 0$. We have the two groups

VII. Satellite line does not pass through asymptote point (ν not $= 0$).

VIII. Satellite line passes through asymptote point ($\nu = 0$).

Hyperbolas ⊙ Redundant. One osculating asymptote. Satellite line is parallel to an asymptote, suppose to the asymptote $x = 0$; that is, $\mu = 0$, or the satellite line is $\lambda x + \nu = 0$. We have only the group

XV.

Hyperbolas ⊙ Redundant. Three osculating asymptotes. Satellite line lies at infinity, that is, $\lambda = 0$, $\mu = 0$. We have only the group

XVII.

The Hyperbolas ⊙ Defective. Article Nos. 108 to 110.

108. The equation may be taken to be

$$\tfrac{1}{3} x (x^2 + y^2) + kz^2 (\lambda x + \mu y + \nu z) = 0,$$

or writing $z = 1$, then it is

$$\tfrac{1}{3} x (x^2 + y^2) + k (\lambda x + \mu y + \nu) = 0,$$

and if to fix the ideas we take the case where the two imaginary asymptotes are the asymptotes of a circle, then x, y will be ordinary rectangular coordinates.

109. The critic centres are given by the equations

$$2(\lambda x + \mu y) - 3\nu = 0,$$
$$3\mu x^2 - 2\lambda xy + \mu y^2 = 0,$$

that is they are the intersections of the line $2(\lambda x + \mu y) - 3\nu = 0$ (which is a line parallel to the satellite line, on the other side of the asymptote point and at a distance from it $= \frac{3}{2}$ distance of satellite line) by the pair of lines

$$3\mu x = (\lambda \pm \sqrt{\lambda^2 - 3\mu^2})\, y.$$

Hence the critic centres are real if $\lambda^2 > 3\mu^2$, that is, if the satellite line is inclined to the asymptote at an angle $> 60°$; imaginary if $\lambda^2 < 3\mu^2$, that is, if the satellite line is inclined to the asymptote at an angle $< 60°$: and there is a twofold centre if $\lambda^2 = 3\mu^2$, that is, if the inclination is $= 60°$. This assumes, however, that ν is not $= 0$, that is that the satellite line does not pass through the asymptote point; when it does the distinction of the cases disappears. Hence the groups are

110. Hyperbolas ⊙ Defective. No osculating asymptote. The Satellite line is not parallel to the asymptote, and the groups are,

Satellite line not passing through asymptote point.

XXIV. Satellite line inclined to asymptote at angle $> 60°$.

XXV. " " " " $= 60°$.

XXVI. " " " " $< 60°$.

Satellite line passes through asymptote point, the single group

XXVII.

Hyperbolas ⊙ Defective. Real osculating asymptote. The satellite line is parallel to the asymptote, and we have the single group

XXXIV.

Hyperbolas ⊙ Defective. Three osculating asymptotes. Satellite line is at infinity and we have the single group

XXXVI.

The foregoing theory of the hyperbolas Δ and ⊙ completes the enumeration of the groups I. to XXXVI.

As to the groups of the parabolic hyperbolas. Article Nos. 111 to 115.

111. I consider the equation in the form

$$\tfrac{1}{2}x(by^2 + cz^2 + 2gzx) + kz^2(\mu y + \nu z) = 0,$$

viz. the cubic $x(by^2 + cz^2 + 2gzx) = 0$ is made up of a conic $by^2 + cz^2 + 2gzx = 0$, and a line $x = 0$; the other cubic $z^2(\mu y + \nu z) = 0$ is made up of a tangent of the conic, regarded as a twofold line, $z^2 = 0$, and of a line $\mu y + \nu z = 0$ through the point of contact of such tangent.

112. To determine the critic centres we have

$$\begin{aligned} x \,.\, gz + \tfrac{1}{2}(by^2 + cz^2 + 2gzx) &= 0, \\ x \,.\, by + kz \,.\, \mu z \qquad\qquad &= 0, \\ x(cz + gz) + kz(2\mu y + 3\nu z) &= 0\,; \end{aligned}$$

eliminating k from the second and third equations

$$\mu z(cz+gx)-by(2\mu y+3\nu z)=0,$$

that is

$$-2b\mu y^2+c\mu z^2+g\mu zx-3b\nu yz=0;$$

or reducing by means of the first equation written under the form

$$by^2+cz^2+4gzx=0,$$

we find

$$3c\mu z^2+9g\mu zx-3b\nu yz=0,$$

that is $z=0$, which may be rejected, or else

$$c\mu z+3g\mu x-b\nu y=0,$$

or, as it may be written,

$$\mu(3gx+cz)-\nu by=0.$$

Hence the entire series of critic centres lie on the conic

$$by^2+cz^2+4gzx=0,$$

and corresponding to the satellite line $\mu y+\nu z=0$, we have the two critic centres which are the intersections of the conic by the line

$$\mu(3gx+cz)-\nu by=0,$$

the lines pass through the fixed point $3gx+cz=0$, $y=0$, and form a pencil homographic with the satellite lines $\mu y+\nu z=0$.

113. We have a twofold centre if the line touches the conic, the condition for this is

$$(bc,\ -4g^2,\ 0,\ 0,\ -2bg,\ 0\between 3g\mu,\ -b\nu,\ c\mu)^2=0,$$

that is $3bg^2(3c\mu^2+4b\nu^2)=0$, or simply,

$$3c\mu^2+4b\nu^2=0,$$

and from this and the equation $\mu y+\nu z=0$, eliminating μ and ν we find

$$4by^2+3cz^2=0,$$

for the equation of the satellite lines which respectively give rise to a twofold centre; the lines in question are real or imaginary according as the lines $by^2+cz^2=0$ are real or imaginary, that is, according as the line $x=0$ cuts the conic $by^2+cz^2+2gzx=0$ in two real, or in two imaginary, points.

114. Writing now $b=1$, $c=-mn$, $2g=n$ and $x+mz$ in the place of z, the equation is

$$(x+mz)(y^2+nzx)+2kz^2(\mu y+\nu z)=0,$$

and we may consider $z=0$ as the equation of the line infinity: writing in the formulæ $z=1$, the critic centres are given as the intersection of the parabola

$$y^2+2nx+mn=0,$$

with the line

$$\nu y=\tfrac{1}{2}\mu n(3x+m);$$

and the condition for a two-fold centre is

$$4\nu^2-3mn\mu^2=0;$$

the equation of the satellite lines corresponding respectively to a two-fold centre is

$$4y^2-3mn=0;$$

the lines are real or imaginary according as mn is positive or negative, or (observing that the equations $x+m=0$, $y^2+nx=0$ give $y^2-mn=0$), according as the line $x+m=0$ cuts or does not cut the parabola $y^2+nx=0$. Suppose for a moment that the line does cut the parabola and that y_1 is the corresponding value of y, then $y_1^2=mn$; and the equation $4y^2-3mn=0$ of the satellite lines which correspond respectively to the case of a two-fold centre is $y^2=\frac{3}{4}y_1^2$. We have thus $y=\pm y_1$ and $y=\pm\sqrt{\frac{3}{2}}\,y_1$ as special positions of the satellite line $\mu y+\nu=0$. In the case where the line $x+m=0$ touches the parabola $y^2+nx=0$, the value of y_1 is $=0$, and we have only the special position $y=0$; finally, when the line does not cut the parabola there is no special position.

115. Plücker's groups are consequently as follows:

Parabolic Hyperbolas; ordinary asymptote and five-pointic asymptotic parabola, that is the line $\mu y+\nu=0$ is not at infinity.

Asymptote cuts parabola, $mn=+$.

XXXVII. Satellite line lies outside the lines $y=\pm y_1$ which belong to the points of intersection.

XXXVIII. Satellite line passes through a point of intersection, that is, coincides with one of the lines $y=\pm y_1$.

XXXIX. Satellite line lies between the lines $y=\pm y_1$ and $y=\pm\sqrt{\frac{3}{2}}\,y$.

XL. Satellite line coincides with one of the lines $y=\pm\sqrt{\frac{3}{2}}\,y_1$, which give respectively a two-fold centre.

XLI. Satellite line lies between the lines $y=\pm\sqrt{\frac{3}{2}}\,y_1$.

Asymptote touches parabola, viz. $m=0$.

XLIII. Satellite line does not pass through the point of contact.

XLIV. Satellite line passes through point of contact or its equation is $y=0$.

Asymptote does not cut parabola, viz. $mn=-$. This gives the single group

XLII.

Parabolic hyperbolas. Osculating asymptote and six-pointic asymptotic parabola. The satellite line is here at infinity, and there is no new distinction of groups. The groups therefore are

Asymptote cuts parabola,

XLV.

Asymptote does not cut parabola,

XLVI.

Asymptote touches parabola,

XLVII.

As to the Groups of the Central and Parabolic Hyperbolisms. Article No. 116.

116. For the Hyperbolisms, Central and Parabolic, since these have a node or a cusp at infinity, they cannot acquire a new node, and the theory of critic centres does not arise. There is, however, as regards the Hyperbolisms of the Hyperbola a distinction in the position of the satellite line, viz. this may lie outside, or between, the parallel asymptotes. The groups are

Hyperbolisms of the Hyperbola. Ordinary asymptote. The satellite line is not at infinity, and it may lie in either of the positions just mentioned. We have therefore

XLVIII. Satellite line lies between the parallel asymptotes.

XLIX. " " outside " " .

Osculating asymptote; the satellite line is at infinity. We have

L.

Hyperbolisms of the Ellipse. Ordinary asymptote. The satellite line is not at infinity, and we have

LI.

Osculating asymptote. Satellite line is at infinity,

LII.

Hyperbolisms of the Parabola. Ordinary asymptote. Satellite line is not at infinity we have

LIII.

Osculating asymptote: satellite line is at infinity: we have

LIV.

As to the Groups of the Divergent Parabolas. Article Nos. 117 and 118.

117. Taking the equation under the form

$$ax^3 + by^2z + kz^2(\lambda x + \nu z) = 0,$$

we find for a critic centre

$$\begin{aligned} 3ax^2 + kz \,.\, \lambda z \quad &= 0, \\ 2byz \quad &= 0, \\ by^2 + kz(2\lambda x + 3\nu z) &= 0; \end{aligned}$$

hence there is a single critic centre $y = 0$, $2\lambda x + 3\nu z = 0$, the critic value of k is $k = \dfrac{-27a\nu^2}{4\lambda^3}$, and with this value of k the equation in fact is

$$4\lambda^3(ax^3 + by^2z) - 27a\nu^2(\lambda x + \nu z)z^2 = 0,$$

that is

$$a(4\lambda^3x^3 - 27\lambda^2\nu x^2z - 27\nu^2z^3) + 4\lambda^3by^2z = 0,$$

or, as it may be written,

$$a(2\lambda x + 3\nu z)^2(\lambda x - 3\nu z) + 4\lambda^3by^2z = 0,$$

which puts in evidence the critic centre or node of the curve. But, as there is here only a single critic centre, there is of course no further theory of the two-fold centre, &c.

118. The groups are as given *à priori* by the relation of the satellite line $\lambda x + \nu z = 0$, to the semicubical parabola $ax^3 + by^2z = 0$, viz. writing $z = 1$ and changing the constants,

Divergent Parabolas, the semicubical parabola $y^2 = x^3$, which is

LV.

Divergent Parabolas, Asymptotic Semicubical Parabola of seven-pointic contact, viz. equation of the asymptotic parabola being $y^2 - x^3 = 0$, and writing for convenience $k(\lambda x + \nu) = -3ax + 2b = 0$ for the equation of the satellite line, the equation of the curve is $y^2 = x^3 - 3ax + 2b$. And the groups are

LVI. Satellite line cuts asymptotic parabola.

LVII. „ does not cut „ „ .

LVIII. „ passes through vertex of parabola.

And further

Divergent Parabolas. Asymptotic Semicubical Parabola of nine-pointic intersection. The satellite line is at infinity, and the equation is $y^2 = x^3 + 2b$. This is group

LIX.

As to the Trident Curve and the Cubical Parabola. Article Nos. 119 and 120.

119. For the Trident Curve, equation is $x(x^2 + \lambda y) + \mu = 0$, or the satellite line is at infinity, and there is no distinction of groups; we have only group

LX.

120. For the Cubical Parabola this is $x^3 + \mu y = 0$, there is no distinction of groups, and the curve is group

LXI.

As to the Division into Species: Comparison of Newton and Plücker.

Article Nos. 121 and 122.

121. The division into species is obtained without difficulty when the groups are once established; in fact it only remains to trace for each given form of V and s the series of curves $V+\mu s=0$, as μ passes from ∞ to $-\infty$ through the value 0 and the critic values which correspond to nodal curves: I have nothing to add to what has been done by Plücker, and it is unnecessary to reproduce the investigation. It may be remarked that the mere inspection of Plücker's figures is sufficient to show which of his species correspond to the same Newtonian species; the species which do so belong to the same Newtonian species in some instances closely resemble each other in form, although in others the difference of form is apparent enough: but the Plückerian species which correspond to the same Newtonian species belong for the most part to different groups and are thus distinguished from each other by the characters which distinguish the groups to which they respectively belong: thus for instance Newton's Species 1 (a hyperbola Δ Redundant) is characterised as consisting of three hyperbolic branches, one inscribed, one circumscribed, and one ambigene, with an oval. Such a curve may exist with three different positions of the satellite line in regard to the asymptotes, viz. the satellite line may cut the three sides produced, or it may pass through a vertex, cutting the opposite side produced, or it may cut two sides and the third side produced, not cutting the envelope—which are the characters of Plücker's groups I, II, IV, respectively, and there belongs to the Newtonian species 1, a species out of each of these groups, viz. they are I. 1; II. 9, and IV. 18.

122. The correspondence of the Plückerian Species with those of Newton is shown in the following Table.

Newton's Genus 1, contains 9 Species, viz.

1 2 3 4 5 6 7 8 9

corresponding to Plücker's Species

I.	1			2	3	8	7	4	5, 6
II.	9			10	11			12	13, 14
III.									15
IV.	18	17		19	16, 20			21	22, 23
V.			25		24, 26			27	28, 29
VI.					30			31	32, 33

Part of Newton's Genus 4, contains 3 Species, viz.

24 25 26

corresponding to Plücker's Species

VII.	34	35	36
VIII.		37	

Newton's Genus 2, contains 14 Species, viz.

corresponding to Plücker's Species

	10	10′	11	12	13	13′	14	15	16	17	18	19	20	21
IX.	38				39		40			43	42		41	
X.	44				45		46						47	
XI.	50		49		51		48, 52						53	
XII.				55			54, 56						57	
XIII.							58						59	
XIV.		60				61		62	65			63		64

Further part of Newton's Genus 4, contains 4 Species, viz.

corresponding to Plücker's Species

	28	29	30	31
XV.	69	66	67	68

Newton's Genus 3, contains 4 Species, viz.

corresponding to Plücker's Species

	22	22′	22″	23
XVI.	72	70	71	73

Residue of Newton's Genus 4, contains 1 Species, viz.

corresponding to Plücker's Species

	32
XVII.	74

Newton's Genus 5, contains 6 Species, viz.

corresponding to Plücker's Species

	33	34	35	36	37	38
XVIII.	77, 82	78, 81		76	75, 79, 80	
XIX.	88	87	84		83, 85, 86	
XX.	89	90			91	
XXI.	92	93			94, 95	
XXII.	96, 100	97, 101		99	98, 102, 103	
XXIII.	104	105	107		106, 108, 109	
XXIV.	112	113		111	110, 114, 115	
XXV.			117		116, 118, 119	
XXVI.					120	
XXVII.						121

Newton's Genus 6, contains 7 species, viz.

39 40 41 42 43 44 45

corresponding to Plücker's Species

XXVIII.	122	125	123			126	124, 127
XXIX.	128	131	129				130
XXX.	134	137	135		133		132, 136
XXXI.		141		139			138, 140
XXXII.		143					142
XXXIII.	144	148	145				146, 149
XXXIV.		152					150, 153
XXXV.	154		155				156, 157
XXXVI.							158

Newton's Genus 7, contains 7 Species, viz.

46 47 48 49 50 51 52

corresponding to Plücker's Species

XXXVII.	159			160	161, 162	163	164
XXXVIII.	165			166	167, 168		
XXXIX.	171	170		172	169, 173, 174		
XL.			176		175, 177, 178		
XLI.					179		
XLII.					182	181	180
XLIII.					185, 186	184	183
XLIV.					187		

Newton's Genus 8, contains 6 Species, viz.

53 54 55 56 56′ 56″

corresponding to Plücker's Species

XLV.	190			191	188	189
XLVI.	194	193	192	195		
XLVII.	196			197		

Newton's Genus 9, contains 4 Species, viz.

57 $\underbrace{58 \quad 59}$ 60

corresponding to Plücker's Species

XLVIII.		198	
XLIX.	199		
L.			200

Newton's Genus 10, contains 3 Species, viz.

61 62 63

corresponding to Plücker's Species

LI.	201	
LII.		202

Newton's Genus 11, contains 2 Species, viz.

64 65

corresponding to Plücker's Species

LIII.	203	
LIV.		204

Newton's Genus 12, contains 1 Species, viz.

66

corresponding to Plücker's Species

LX.	218

Newton's Genus 13, contains 5 Species, viz.

67 68 69 70 71

corresponding to Plücker's Species

LV.				205	
LVI.	208	207			206, 209
LVII.	212		211		210, 213
LVIII.	214				215
LIX.					216, 217

Newton's Genus 14, contains 1 Species, viz.

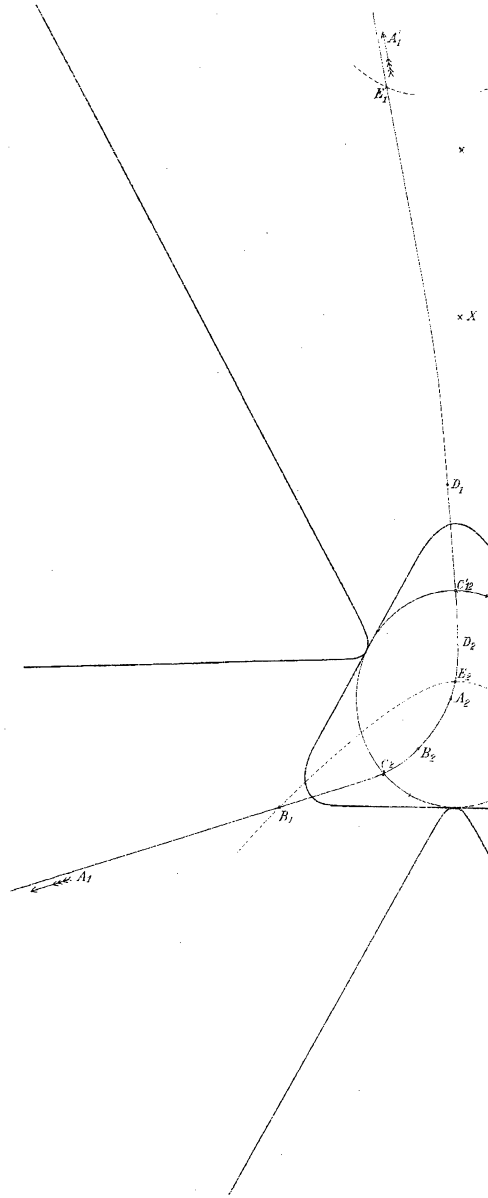
72

corresponding to Plücker's Species

LXI.	219

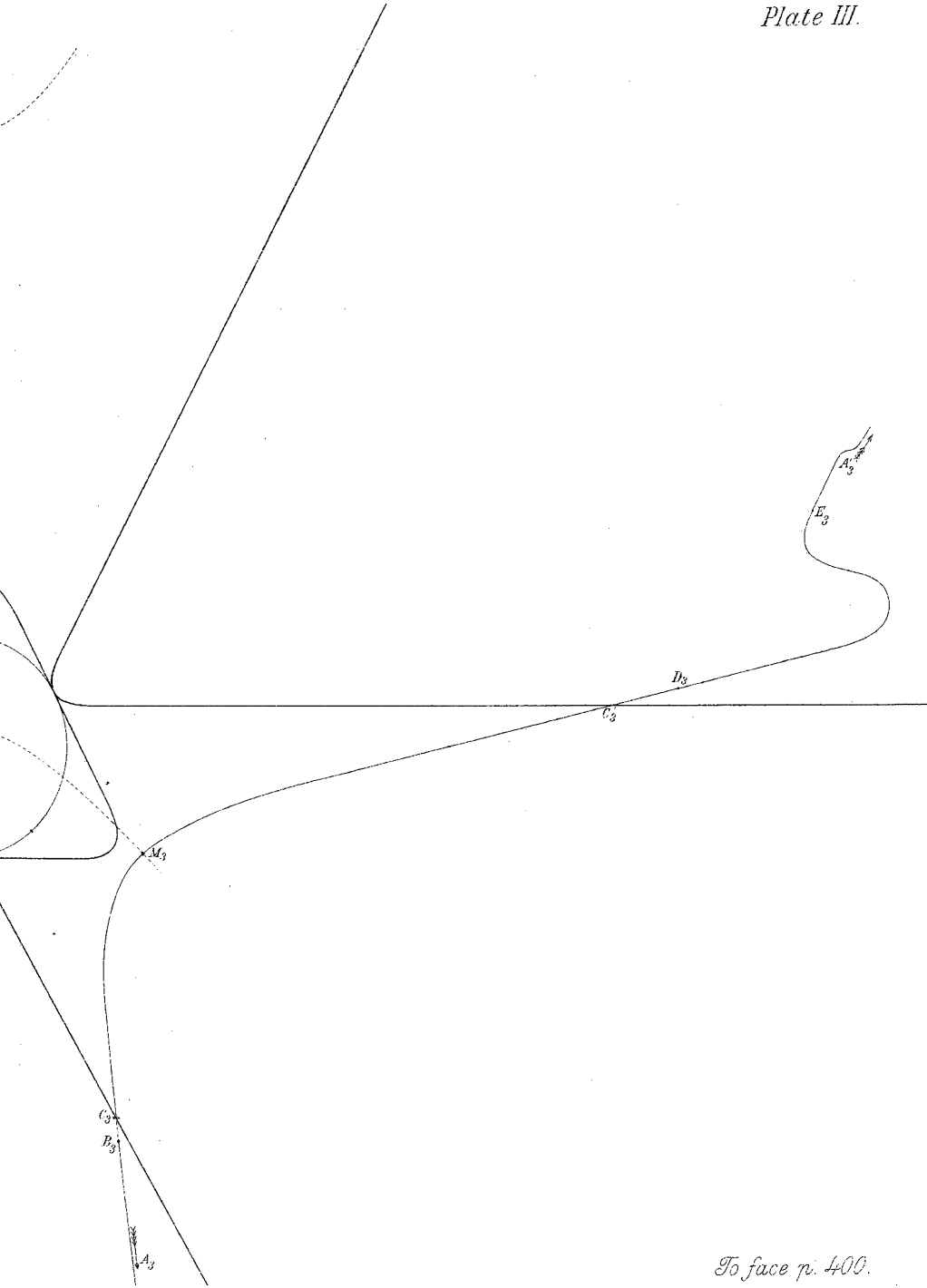
It is to be noticed that (as appears by the Table) Plücker enumerates 13 species of the Divergent Parabola, viz. corresponding to the *Parabola Pura* of Newton we have five species, and to the *Parabola cum Ovali* three species; but to each of the other three Newtonian species (*Nodata, Punctata, Cuspidata*) only a single species. The difference in nowise affects Newton's before-mentioned theorem, that every cubic curve is the shadow of a Divergent Parabola; but (the characters of Plücker's species being unaffected by projection) the number of resulting kinds of cubic curves (or cones) will be five or thirteen according as the one or the other classification is adopted; but this is a subject which I do not enter upon in the present Memoir.

Cambridge, February 8, 1864.

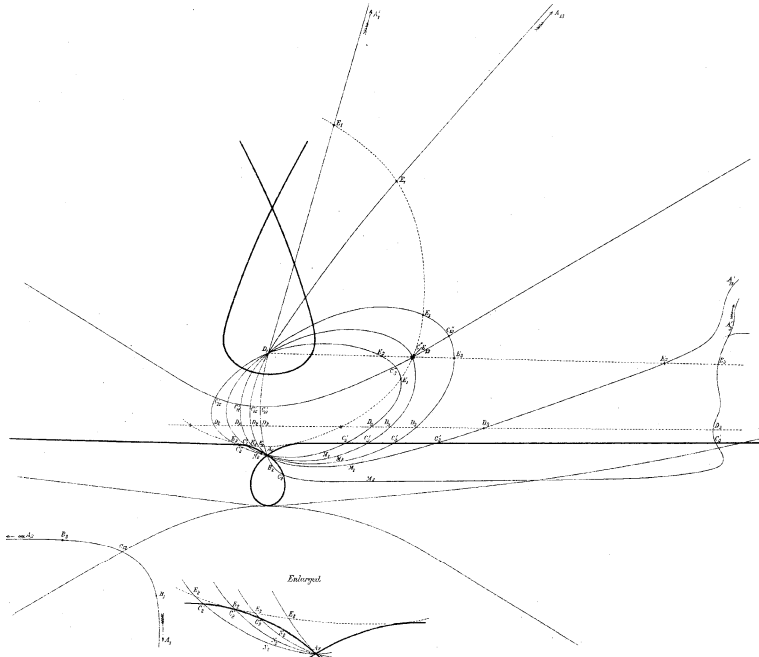


Cayley's Papers V.

Plate III.



To face p. 400.



To face p. 400

351.

ON CUBIC CONES AND CURVES.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part I. (1866), pp. 129—144. Read April 18, 1864.]

THERE is contained in Sir Isaac's Newton's *Enumeratio Linearum tertii Ordinis* (1706), under the heading *Genesis Curvarum per Umbras*, the remarkable theorem that, in the same way as the several curves of the second order may be considered as the shadows of a circle, that is, the sections of a cone having a circular base, so the several curves of the third order, or cubic curves, may be considered as the shadows of the five Divergent Parabolas. It was remarked by Chasles, Note XX. to the *Aperçu Historique* (1837), that they may also be considered as the shadows of the five curves having a centre (the Newtonian Species 27, 38, 59, 62, 72), and that the theorem may be stated as follows, viz. (in the same way that all the curves of the second order are the sections of a single kind of cone of the second order, so) all the curves of the third order may be considered as the sections of five kinds of cones of the third order—and that cutting these in one way we have the five Divergent Parabolas, cutting them in another way the five curves with a centre. The nature of these five kinds of cones, or, what is the same thing, that of the spherical curves in which they are intersected by a concentric sphere, was first pointed out by Möbius in his most interesting Memoir, "Grundformen der Linien dritter Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, 1853. I reproduce in the present memoir the characterisation of these five kinds of cones—which I call the *simplex*, the *complex*, the *acnodal*, the *crunodal*, and the *cuspidal*—and I further develope the geometrical and analytical theory; in particular I arrive at a division of the simplex cones into three subkinds, the simplex *trilateral*, *neutral*, and *quadrilateral*. I have throughout spoken of cones rather than of plane curves, using however, as far as may be, language which is also applicable to a plane curve, thus, instead of lines of inflexion, tangent planes, of a cone, I say inflexions, tangents, &c. But the theory of the cone is of course that of the projective properties

of the curves which are the sections of such cone; it appears to me that the true classification of curves is to divide them according to the cones which give rise to them; and I consider the present memoir as affording in part the materials for such a classification of cubic curves, viz. it seems to me that after, in the first instance, dividing them into the simplex, the complex, the acnodal, the crunodal, and the cuspidal kinds, the simplex kind should be further divided in the above-mentioned manner; and that we should establish, lastly, the divisions which relate to the particular mode in which the cone is to be cut, in order to obtain such and such a curve: in effect, that the principle of classification, according to the nature of the infinite branches adopted by Newton in the work above referred to, and by Plücker in his *System der Analytischen Geometrie* (Berlin, 1835), and to which has reference my Memoir *On the Classification of Cubic Curves*, [350], should be not the primary, but a secondary principle of classification. I remark that as regards the division into five kinds, Murdoch, in his highly interesting work, *Newtoni Genesis Curvarum per Umbras*, [Lond. 1746], has not only distinguished the Newtonian species which arise from each of the Divergent Parabolas, or, what is the same thing, from each of the five kinds of cones (it will presently appear how the mere inspection of Newton's figures is sufficient to enable this), but that he has also shown how the cone must in each case be cut in order to obtain the particular cubic curve. Murdoch also distinguishes the three forms ampullate, campaniform and intermediate, of the simplex Divergent Parabola, which correspond to the simplex quadrilateral, trilateral, and neutral.

I remark also that Plücker in his work above referred to, *Dritter Abschnitt*, 98, has considered the equation of a cubic curve in the form $pqr+\mu s^3=0$, which is in fact equivalent to the form $(X+Y+Z)^3+6kXYZ=0$ used in the sequel, but without arriving at the results obtained in the present Memoir.

The five kinds of Cubic Cones. Nos. 1 to 7.

1. A cone of any order may comprise two distinct forms of sheet, viz. (1) a twin-pair sheet, or sheet which meets a concentric sphere in a pair of closed curves such that each point of the one curve is opposite to a point of the other curve (a cone of the second order affords an example of such a sheet); the twin-pair sheet may be considered as consisting of two sheets, each of which may be called a twin sheet: and (2) a single sheet, viz. a sheet which meets a concentric sphere in a closed curve such that each point of the curve is opposite to another point of the curve: the plane affords an example of such a curve. We have five kinds of cubic cones, viz. the simplex, the complex, the acnodal, the crunodal, and the cuspidal. The cone may consist of a single sheet; it is then of the *simplex* kind. Or it may consist of a single sheet and a twin-pair sheet, it is then of the *complex* kind: these are the non-singular kinds. The remaining kinds are singular ones, which are most easily explained by considering them as limiting forms of the complex kind; the twin-pair sheet may come to unite itself with the single sheet giving rise to a crunodal line, or say a crunode; the cone is then of the *crunodal* kind. Or the twin-pair sheet may dwindle into a

mere line which is an acnodal line, or say an acnode; the cone is then of the *acnodal* kind. Or the two things may happen together, viz. the twin-pair sheet at the instant that it unites itself with the single sheet may dwindle into a mere line, which is then a cuspidal line, or say a cusp; and the cone is then of the *cuspidal* kind. I remark, as regards the crunodal kind, that the cone may be considered as consisting of two portions, one of them corresponding to the single sheet of a complex cone, and which I call the quasi-single sheet; the other of them corresponding to the twin-pair sheet, and which I call the loop-sheet.

2. It is to be added that a cubic cone has in general 9 lines of inflexion, or say inflexions, but of these 6 are always imaginary; the remaining 3, which are real, belong to the single sheet. The plane through any two inflexions meets the cone in a line which is also an inflexion. In particular the three real inflexions lie in a plane.

3. When the cone is acnodal the six imaginary inflexions unite at the acnode; and the single sheet has still 3 real inflexions lying in a plane. But if the cone is crunodal, then 4 imaginary inflexions and 2 real inflexions unite in the crunode; and the cone has 1 real inflexion; there are besides 2 imaginary inflexions, the 3 inflexions lie in a real plane. Finally, if the cone is cuspidal, then 2 of the real inflexions, and the 6 imaginary inflexions unite together in the cusp; the cone has besides 1 real inflexion, but there are not any imaginary inflexions.

4. Suppose that the cone is of one of the non-singular kinds; that is, let it be simplex or complex. From any line of the cone we may draw four tangent planes to the cone—the anharmonic ratio of the four planes is the same whatever may be the line on the cone. As regards reality, the following distinction exists, viz. for the complex kind of cone, the planes are all real or all imaginary; for the simplex kind they are two real and two imaginary. First, as regards the complex kind, if the line be taken on the twin-pair sheet, the four tangent planes are all imaginary; but if it be taken on the single sheet, then there are two real tangent planes to the single sheet and two real tangent planes to the twin-pair sheet, together four real tangent planes. Secondly, as regards the simplex kind, there is only the single sheet, and the line being taken on it, there are two real tangent planes and no more.

5. As regards the singular kinds, assuming always that the line on the cone does not coincide with the node or the cusp (for when it does there are no tangent planes), it may be noticed that for the crunodal kind there are two tangent planes which are real or imaginary according as the line lies on the part corresponding to the single sheet or on the part corresponding to the twin-pair sheet. For the acnodal kind there are two tangent planes which are always real; and for the cuspidal kind there is a single tangent plane which is always real.

6. The foregoing properties of cubic cones apply to the curves which are the sections of these cones; thus a cubic curve is of the simplex, the complex, the crunodal, the acnodal, or the cuspidal kind. As regards the last-mentioned three kinds, or singular kinds, it is of course to be borne in mind that the crunode, acnode, or cusp, may be at infinity; and consequently that all the curves in Newton's genus 9 (the hyperbolisms of the hyperbola) and the curve in his genus 12 (the trident curve) belong to the crunodal kind; the curves in genus 10 (the hyperbolisms of the ellipse) to the

acnodal kind; and those of genus 11 (the hyperbolisms of the parabola) and the curve in genus 14 (the cubical parabola) to the cuspidal kind.

7. In the other genera such of the species as are without a node or a cusp, belong to the simplex or the complex kind: and the mere inspection of the figure (Newton's or Plücker's) is sufficient to show to which of the two kinds the curve belongs; in fact, when from any point of the curve there are four real tangents, or there is else no real tangent, the curve is of the complex kind, but if there are two and only two real tangents the curve is of the simplex kind. And in the former case we see whether a branch arises from the single sheet or the twin-pair sheet of the cone, viz. if from a point on the branch there can be drawn four real tangents to the curve, the branch arises from the single sheet, but if no real tangent can be drawn, the branch arises from the twin-pair sheet. And in the crunodal kind we see which part of the curve arises from the quasi-single sheet, and which part from the loop sheet.

Ulterior Theory leading to the Subkinds of the Simplex Cones. Nos. 8 to 35.

(*Several Subheadings.*)

8. But the division of cubic cones may be carried further: we may in fact subdivide the simplex kind. To show how this is, I consider a cone complex or simplex, but I attend for the moment only to the single sheet. The cone has on the single sheet three (real) inflexions lying in a plane. I call this the equator, and I call the tangent planes at the inflexions simply the tangents; the three tangents do not in general meet in a line, and they divide space into eight regions; of these there are two not divided by the equator, and which remain trilateral; the other six regions are divided by the equator each of them into a trilateral and a quadrilateral region, this gives six trilateral regions and six quadrilateral regions; there are thus in all $2+6=8$ trilateral regions (I distinguish them as the 2 and the 6 such regions) and 6 quadrilateral regions.

9. It is easy to see that for a complex cone the single sheet lies wholly in the 6 trilateral regions, and the twin-pair sheet wholly in the 2 trilateral regions. Imagine the twin-pair sheet to dwindle into a line and then disappear, that is, let the cone pass from the complex, through the acnodal, into the simplex kind; the single sheet of the simplex cone will lie wholly in the 6 trilateral regions; this is one form of the simplex cone; I call it the *simplex trilateral.* But there is a different form, viz. the cone may lie wholly in the 6 quadrilateral regions; this is the *simplex quadrilateral.* And there is an intermediate form, viz. the three tangent planes at the inflexions may meet in a line, the 2 trilateral regions then disappear, and there are only 12 regions, all of them trilateral, which may be considered as forming two systems, each of 6 regions, viz. each system consists of three non-contiguous regions on one side of the equator, and (alternating therewith) three non-contiguous regions on the other side of the equator: the cone lies wholly in the one 6 regions or in the other 6 regions and I say that the cone is *simplex neutral.*

10. A non-singular cubic cone (simplex or complex) may be represented by an equation of the canonical form

$$x^3 + y^3 + z^3 + 6lxyz = 0,$$

where the coordinates x, y, z and the parameter l are all real; the invariants of the form are $S = -l + l^4$, $T = 1 - 20l^3 - 8l^6$. It is to be noticed that the form in question cannot represent a singular cone; we find as the condition that it may do so

$$R = 64S^3 - T^2 = -(1 + 8l^3)^3 = 0,$$

but when this condition is satisfied, the cone breaks up into a system of three planes; thus for the real root $l = -\frac{1}{2}$, we have

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

where ω is an imaginary cube root of unity; and by merely writing ωx, $\omega^2 x$ successively in place of x, we see that the like decomposition occurs from the imaginary roots

$$l = -\tfrac{1}{2}\omega, \quad l = -\tfrac{1}{2}\omega^2.$$

11. The equation $x^3 + y^3 + z^3 + 6lxyz = 0$ is in general transformable into the form

$$(X + Y + Z)^3 + 6kXYZ = 0,$$

where X, Y, Z are linear functions of the original coordinates, such that $X = 0$, $Y = 0$, $Z = 0$ are the equations of the tangent planes at three inflexions in the plane $X + Y + Z = 0$; if however the three tangent planes meet in a line, then X, Y, Z will satisfy identically a certain linear equation, and it is clear *à priori* that the transformation must fail. The condition for the three tangent planes meeting in a line is $S = -l + l^4 = 0$, that is, we have

$$l = 0,\ 1,\ \omega, \text{ or } \omega^2;$$

and attending only to the real roots $l = 0$, $l = 1$, it will be presently seen that for $l = 0$ the tangent planes at the three real inflexions do not, for $l = 1$, they do meet in a line. Hence the simplex neutral cone corresponds to the value $l = 1$, that is, the equation is

$$x^3 + y^3 + z^3 + 6xyz = 0,$$

and this equation is not transformable into the form $(X + Y + Z)^3 + 6kXYZ = 0$, which is that employed in the sequel for the general discussion of the simplex and complex cones. The theory on which the foregoing conclusion depends is as follows.

On the condition $S = 0$. Nos. 12 to 17.

12. A cubic has in general nine inflexions, which lie by threes on twelve planes, viz. denoting the inflexions by 1, 2, 3, 4, 5, 6, 7, 8, 9, the planes may be taken to be

123,	456,	789,
147,	258,	369,
159,	267,	348,
168,	249,	357,

that is, we have four systems, each of three planes passing through the nine inflexions.

The tangent planes, or, say the tangents at the inflexions *in plano*, for instance, at the inflexions 1, 2, 3, form a trilateral, and we have thus corresponding to each of the three planes a trilateral formed by the tangents at the inflexions on such plane; and there are of course four systems, each of three trilaterals formed by the tangents at the nine inflexions.

13. I say that if $S=0$, then in one of the four systems the trilaterals become each of them a line, that is, the tangents at the nine inflexions meet by threes in three lines.

14. This may be shown by means of the before-mentioned canonical form

$$x^3+y^3+z^3+6lxyz=0$$

of the equation of a cubic cone, for then the notation of the inflexions being in accordance with the foregoing scheme, the coordinates may be taken to be

$$\begin{array}{llllllllll}
(1) & x=0, & y+z=0, & (4) & x=0, & y+\omega z=0, & (7) & x=0, & y+\omega^2 z=0, \\
(2) & y=0, & z+x=0, & (5) & y=0, & z+\omega x=0, & (8) & y=0, & z+\omega^2 x=0, \\
(3) & z=0, & x+y=0, & (6) & z=0, & x+\omega y=1, & (9) & z=0, & x+\omega^2 y=0,
\end{array}$$

where ω denotes an imaginary cubic root of unity, and the equations of the tangents are

$$\begin{array}{llllllll}
(1) & -2lx+y+z=0, & (4) & -2lx+\omega y+\omega^2 z=0, & (7) & -2lx+\omega^2 y+\omega z=0, \\
(2) & x-2ly+z=0, & (5) & \omega^2 x-2ly+\omega z=0, & (8) & \omega x-2ly+\omega^2 z=0, \\
(3) & x+y-2lz=0, & (6) & \omega x+\omega^2 y-2lz=0, & (9) & \omega^2 x+\omega y-2lz=0.
\end{array}$$

15. The value of S is $=-l+l^4$, and for each of the values 1, 0, ω, ω^2, of l, which give $S=0$, we have a system of the nine tangents meeting by threes in three lines, viz. the systems are

$$\begin{array}{llllll}
\text{for} & l=1\,, & 123, & 456, & 789, \\
\text{,,} & l=0\,, & 147, & 258, & 369, \\
\text{,,} & l=\omega\,, & 159, & 267, & 348, \\
\text{,,} & l=\omega^2, & 168, & 249, & 357.
\end{array}$$

16. It is proper to notice that starting with the systems in question, or what is the same thing, with a single set of each system, say the sets 123, 147, 159, 168, we obtain as the condition to be satisfied by l, the equation

$$l\,(4l^3-3l-1)\,(4l^3-3l\omega+1)\,(4l^3-3l\omega^2+1)=0,$$

or, as it may otherwise be written,

$$l\,(2l-1)^2\,(l-1)\,(2l\omega-1)^2\,(l\omega-1)\,(2l\omega^2-1)^2\,(l\omega^2-1)=0, \text{ that is, } (-l+l^4)\,(1+8l^3)^2=0\,;$$

it is clear that a factor has dropped out, and that the true form of the condition is

$$(-l+l^4)\,(1+8l^3)^3=0\,;$$

that is, $S(64S^3 - T^2) = 0$; where the equation $64S^3 - T^2 = 0$ would denote the existence of a nodal line, and consequent coincidence therewith of 6 of the 9 inflexions; the equation $S = 0$ being left as the proper condition for the intersection by threes of the tangents at the inflexions in three lines.

17. The investigation shows that the four systems correspond respectively to the four values of l which give $S = 0$; and that (reality being disregarded) there is no distinction between the four systems, or the corresponding values of l; if however we assume that x, y, z, l are all of them real, then the cone has only the three real inflexions 1, 2, 3, lying in the real plane 123; and there is an essential distinction between the real roots $l = 1$, $l = 0$ of the equation $S = -l + l^4 = 0$, viz. for $l = 1$, the tangents at the three real inflexions meet in a line; whereas for $l = 0$ there is not any relation between the tangents at the real inflexions, and there is consequently no visible peculiarity in the form of the cone.

18. I return to the analytical theory of the general case, as depending on the representation of the equation of the cone in the form

$$(X + Y + Z)^3 + 6kXYZ = 0,$$

where the coordinates are real, viz. $X = 0$, $Y = 0$, $Z = 0$ represent the tangent planes at the three (real) inflexions, or, as they have before been called, the tangents; and $X + Y + Z = 0$ represents the plane through the three inflexions, or, as it has before been called, the equator. And we may assume the signs to be such that in one of the 2 trilateral regions the coordinates X, Y, Z shall be each of them positive: this being so the 14 regions will correspond to the following combinations of signs

X	Y	Z	$X+Y+Z$	
+	+	+	+	the 2 trilateral regions,
−	−	−	−	
+	+	−	−	the 6 trilateral regions,
+	−	+	−	
+	−	−	+	
−	+	+	−	
−	+	−	+	
−	−	+	+	
−	+	+	+	the 6 quadrilateral regions.
+	−	+	+	
+	+	−	+	
+	−	−	−	
−	+	−	−	
−	−	+	−	

where it may be noted that the equator $X+Y+Z=0$ does not cut the two trilateral regions $(+++\,+)$ and $(----)$; and further that the line $X=Y=Z$ which is the harmonic of the equator $X+Y+Z=0$ in regard to the system of the three tangents $XYZ=0$, lies wholly in the two trilateral regions $(+++\,+)$ and $(----)$.

19. The equation in question,

$$(X+Y+Z)^3+6kXYZ=0,$$

shows that, as above stated, the cone lies wholly in the 8 trilateral regions, or in the 6 quadrilateral regions, viz. if k be negative, it lies wholly in the 8 trilateral regions, and if k be positive, it lies wholly in the 6 quadrilateral regions. Let k be negative, then the positive quantity $-\dfrac{1}{6k}$, which is

$$=\frac{XYZ}{(X+Y+Z)^3},$$

if we attend only to the values of X, Y, Z which have the same sign (that is, to points in one of the two trilateral regions), has a maximum value $=\frac{1}{27}$ corresponding to $X=Y=Z$. And if $-\dfrac{1}{6k}$ exceeds this value, that is, if $-k<\frac{9}{2}$, or, what is the same thing, if k lie between the values 0 and $-\frac{9}{2}$, then the equation $-\dfrac{1}{6k}=\dfrac{XYZ}{(X+Y+Z)^3}$ cannot be satisfied in the assumed manner, that is, by values of X, Y, Z having the same sign; and thus no portion of the cone lies in the two trilateral regions: in the contrary case, that is, if k lie between the values $-\infty$, $-\frac{9}{2}$, the equation can be so satisfied, and a portion of the cone lies in the two trilateral regions.

Hence k being negative, we have as follows:

k between $-\infty$ and $-\frac{9}{2}$, the cone is complex,

$k=-\frac{9}{2}$, the cone is acnodal,

k between $-\frac{9}{2}$ and 0, the cone is simplex trilateral;

and k being positive, or say

k between 0, ∞, the cone is simplex quadrilateral.

20. It is to be remarked that for $k=0$, the cone as represented by the foregoing equation degenerates into the threefold plane $(X+Y+Z)^3=0$. The value $k=0$ corresponds however to the value $l=1$ of the parameter l in the equation $x^3+y^3+z^3+6lxyz=0$, that is, it corresponds to the *simplex neutral* cone, represented by the equation

$$x^3+y^3+z^3+6xyz=0,$$

which, as already remarked, is not transformable into the form $(X+Y+Z)^3+6kXYZ=0$: this leads to the consideration of the transformation in question.

On the relation of the two forms $x^3+y^3+z^3+6lxyz=0$, *and* $(X+Y+Z)^3+6kXYZ=0$.

Nos. 21 to 24.

21. Starting with the form $x^3+y^3+z^3+6lxyz=0$, and writing

$$\begin{aligned} X &= -2lx + y + z, \\ Y &= x - 2ly + z, \\ Z &= x + y - 2lz, \end{aligned}$$

then we have

$$\begin{aligned} XYZ = &-2l(x^3+y^3+z^3) \\ &+(1-2l+4l^2)(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ &+2(1-3l-4l^2)xyz, \\ X+Y+Z = &\ 2(1-l)(x+y+z), \end{aligned}$$

and thence

$$\begin{aligned} (X+Y+Z)^3 = &\ 8(1-l)^3(x^3+y^3+z^3) \\ &+24(1-l)^3(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ &+48(1-l)^3xyz, \end{aligned}$$

and we thus obtain

$$\begin{aligned} (1-2l+4l^2)(X+Y+Z)^3+24(l-1)^3XYZ \\ = 8(2l+1)^2(l-1)^3(x^3+y^3+z^3+6lxyz); \end{aligned}$$

or, what is the same thing,

$$(X+Y+Z)^3+6kXYZ = \frac{8(2l+1)^2(l-1)^3}{1-2l+4l^2}(x^3+y^3+z^3+6lxyz),$$

if

$$k = \frac{4(l-1)^3}{1-2l+4l^2}.$$

22. For the form

$$(X+Y+Z)^3+6kXYZ=0,$$

we find

$$\begin{aligned} S = &\ (1+k)^4, & T = &-8(1+k)^6 \\ &-6(1+k)^2 & &+72(1+k)^4 \\ &+8(1+k) & &-128(1+k)^3 \\ &-3 & &+72(1+k)^2 \\ & & &-8 \\ = &\ k^3(4+k) & = &-8k^4(6+6k+k^2), \end{aligned}$$

and thence

$$\begin{aligned} R = 64S^3 - T^2 = &\quad 64k^3\{k(k+4)^3 - (k^2+6k+6)^2\}, \\ = &\quad 64k^3(-8k-36), \\ = &\quad -256k^3(2k+9). \end{aligned}$$

It may be right to remark that from the value $k = \dfrac{-4(1-l)^3}{1-2l+4l^2}$ we deduce

$$k+4 = \frac{4l(1+l+l^2)}{1-2l+4l^2},$$

$$k^3+6k+6 = \frac{-2(1-20l^3-8l^6)}{(1-2l+4l^2)^3},$$

and that thence

$$S = \alpha^4 S', \quad T = \alpha^6 T', \quad \frac{T^2}{S^3} = \frac{T'^2}{S'^3},$$

if

$$S' = -l + l^4,$$

$$T' = 1 - 20l^3 - 8l^6,$$

$$\alpha = \frac{4(1-l)^2}{1-2l+4l^2} = \frac{k}{l-1}.$$

23. The equation

$$k = \frac{4(l-1)^3}{4l^2-2l+1},$$

or as it may also be written

$$k = \frac{16(l-1)^3}{16(l-\frac{1}{4})^2+3},$$

gives without difficulty

$$k+\tfrac{9}{4} = \frac{16(l-\frac{1}{4})^3+27(l-\frac{1}{4})}{16(l-\frac{1}{4})^2+3},$$

and

$$k+\tfrac{9}{2} = \frac{16(l+\frac{1}{2})^3}{16(l-\frac{1}{4})^2+3}.$$

24. Hence treating l, k as coordinates, we see that the locus is a cubic curve, viz. a hyperbolism of the ellipse, having a centre (Newton's species 62), the coordinates of the centre being $l=\frac{1}{4}$, $k=-\frac{9}{4}$, and the equation of the asymptote being $k+\frac{9}{4}=l-\frac{1}{4}$, (that is the asymptote passes through the centre and is inclined at an angle $=45°$ to the axis of l). The centre is of course an inflexion, the equation of the tangent at this point is $k+\frac{9}{4}=9(l-\frac{1}{4})$, and for the other two inflexions we have $l=1$, $k=0$, and $l=-\frac{1}{2}$, $k=-\frac{9}{2}$, the tangents at the two inflexions respectively being $k=0$ and $k=-\frac{9}{2}$, that is the tangents at the inflexions are parallel to the axis of l. The curve consists of a single branch lying below the asymptote for large negative values of l, k, crossing the asymptote at the centre and lying above it for large positive values of k, l. For each value of l there is consequently a single value of k and reciprocally; and l, k pass together from $-\infty$ to $+\infty$. There are certain critical values of k and l, the meaning of which will appear from the following article.

On the Anharmonic Property of a Cubic Cone. Nos. 25 to 29.

25. The property in question is the one already referred to, viz. the four tangent planes, or say the four tangents, to the cone from any line of the cone form a system the anharmonic ratios of which are constant. Taking the equations of the tangents to be

$$p-aq=0,\ p-bq=0,\ p-cq=0,\ p-dq=0,$$

and writing for shortness $m=64-\frac{T^2}{S^3}$, then the functions

$$(a-b)(c-d),\ (a-c)(d-b),\ (a-d)(b-c),$$

or say α, β, γ, on which the anharmonic ratios depend, are the roots of the equation $t^3-12t+2\sqrt{m}=0$. The anharmonic ratios are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$, $\frac{\alpha}{\gamma}$, $\frac{\gamma}{\alpha}$, $\frac{\beta}{\gamma}$, $\frac{\gamma}{\beta}$; hence forming the equation $\left(\vartheta-\frac{\alpha}{\beta}\right)\left(\vartheta-\frac{\beta}{\alpha}\right)=0$, and reducing by the conditions,

$$\begin{aligned}\alpha+\beta+\gamma &= 0,\\ \beta\gamma+\gamma\alpha+\alpha\beta &= -12,\\ \alpha\beta\gamma &= -2\sqrt{m},\end{aligned}$$

this is found to be $(\vartheta^2+\vartheta+1)+\frac{6\gamma}{\sqrt{m}}\vartheta=0$, or we have $\gamma=-\frac{\sqrt{m}}{6}\,\frac{\vartheta^2+\vartheta+1}{\vartheta}$, and substituting this value in the equation $\gamma^3-12\gamma+2\sqrt{m}=0$, we find

$$(\vartheta^2+\vartheta+1)^3-(\vartheta^2+\vartheta+1)\frac{432\,\vartheta^2}{m}-\frac{432\,\vartheta^3}{m}=0,$$

which is

$$m(\vartheta^2+\vartheta+1)^3-432\,\vartheta^2(\vartheta+1)^2=0,$$

or, what is the same thing,

$$\frac{(\vartheta^2+\vartheta+1)^3}{\vartheta^2(\vartheta+1)^2}=\frac{432}{m}=\frac{432}{64-\frac{T^2}{S^3}},$$

that is

$$\frac{(\vartheta^2+\vartheta+1)^3}{\vartheta^2(\vartheta+1)^2}=\frac{27}{4\left(1-\frac{T^2}{64S^3}\right)};$$

and as a verification it may be remarked that, θ being a root, the six roots are

$$\theta,\ \frac{1}{\theta},\ -(1+\theta),\ \frac{-1}{1+\theta},\ -\frac{\theta}{1+\theta},\ -\frac{1+\theta}{\theta}:$$

of course the roots are all real or else all imaginary.

26. If $T=0$, the equation becomes

$$(\vartheta^2+\vartheta+1)^3-\tfrac{27}{4}\vartheta^2(\vartheta+1)^2=0;$$

or reducing, this is

$$\{(\vartheta-1)(\vartheta+\tfrac{1}{2})(\vartheta+2)\}^2=0,$$

that is the six roots are $1, -\frac{1}{2}, -2$, each twice: and the four tangents form therefore a harmonic pencil, which is the geometrical interpretation of the condition $T=0$.

27. The function $\dfrac{(\vartheta^2+\vartheta+1)^3}{\vartheta^2(\vartheta+1)^2}$ is constantly positive and it has three equal minima values corresponding to the last-mentioned values $1, -\frac{1}{2}, -2$ of ϑ, viz. this minimum value is $=\frac{27}{4}$. Hence we see that the equation in ϑ will have its six roots all real if $1-\dfrac{T^2}{64S^3}$ is positive and less than unity, that is, if S and $64S^3-T^2$ are each of them positive: but when these conditions are not satisfied the six roots are imaginary: the limiting case $1-\dfrac{T^2}{64S^3}=1$ or $T=0$ gives, as already mentioned, the three roots $1, -\frac{1}{2}, -2$, each twice.

28. The quantities a, b, c, d which determine the four tangents may be all real, or two real and two imaginary, or all four imaginary; but the imaginary values appear as usual as a conjugate pair or conjugate pairs; and this being so, it is easy to see that in general if ϑ be real the quantities a, b, c, d are all real or else all imaginary; but if ϑ is imaginary then a, b, c, d are two of them real, two imaginary: in fact if a, b are real and c and d are conjugate imaginaries $\gamma\pm\delta i$, then we have for one of the six values of ϑ,

$$\vartheta=\frac{(a-b)\,.\,2\delta i}{-(a-\gamma-\delta i)(b-\gamma+\delta i)},$$

which is in general imaginary.

29. But, as might have been foreseen, the limiting values $\vartheta=1, -\frac{1}{2}, -2$, are an exception, viz. for these values a, b, c, d *may be* two of them real the other two imaginary: in fact the last-mentioned value of ϑ is real and $=\dfrac{(a-b)\,.\,2\delta i}{(a-b)\,.\,\delta i}, =2$, if $(a-\gamma)(b-\gamma)+\delta^2=0$, that is $ab+\gamma^2+\delta^2=\gamma(a+b)$, or, as the condition may also be written,

$$2ab+2(\gamma+\delta i)(\gamma-\delta i)=(\gamma+\delta i+\gamma-\delta i)(a+b),$$

that is $2(ab+cd)=(a+b)(c+d)$, or if a, b, c, d form a harmonic system.

The two forms $x^3+y^3+z^3+6lxyz=0$, $(X+Y+Z)^3+6kXYZ=0$; *Enumeration of the Cones comprised therein.* Nos. 30 and 31.

30. I form the following Table:

l	k	S	T	$64S^3-T^2$	$1-\frac{T^2}{64S^3}$,	
$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	0	complex,
$-\frac{1}{2}(1+\sqrt{3})$	$-3-\sqrt{3}$	$\frac{3}{4}(3+2\sqrt{3})$	0	$81(45+26\sqrt{3})$	1	
$-$	$-$	$+$	$+$	$+$	$+$	
$-\frac{1}{2}$	$-\frac{9}{2}$	$\frac{9}{16}$	$\frac{27}{8}$	0	0	acnodal,
$-$	$-$	$+$	$+$	$-$	$-$	simplex trilateral,
0	-4	0	1	-1	$\pm\infty$	
$\frac{1}{4}$	$-\frac{9}{4}$	$-\frac{63}{256}$	$\frac{351}{512}$	$-\frac{729}{512}$	$\frac{512}{343}$	
$\frac{1}{2}(\sqrt{3}-1)$	$-3+\sqrt{3}$	$-\frac{3}{4}(-3+2\sqrt{3})$	0	$-81(-45+26\sqrt{3})$	1	
$+$	$-$	$-$	$-$	$-$	$+$	
1	0	0	-27	-729	∞	simplex neutral,
$+$	$+$	$+$	$-$	$-$	$-$	simplex quadrilateral.
∞	∞	∞	$-\infty$	∞	0	

And I further describe as follows the nature of the cones which correspond to the several values of k and l.

31. l between $-\infty$ and $-\frac{1}{2}$, or k between $-\infty$ and $-\frac{9}{2}$.

The cone is complex. In the series, viz. corresponding to $l=-\frac{1}{2}(1+\sqrt{3})$ or $k=-3-\sqrt{3}$, there is a special form which may be called the complex harmonic, viz. the four tangents from any line of the cone form a harmonic system: but observe, *quà* complex cone, the tangents are all real or all imaginary. $l=-\frac{1}{2}$ (form fails), $k=-\frac{9}{2}$, the cone is acnodal. l between $-\frac{1}{2}$ and 1, or k between $-\frac{9}{2}$ and 0; the cone is simplex trilateral. In the series, viz. corresponding to $l=0$ or $k=-4$, there is a special form which might be called the quasi-neutral, the speciality having however reference to the imaginary inflexions, viz. corresponding to each real inflexion we have two imaginary inflexions such that the three tangents meet in a line.

There is also corresponding to $l=\frac{1}{4}$, or $k=-\frac{9}{4}$, a form which seems to be a special one, though I have not ascertained wherein that speciality consists.

And there is corresponding to $l = \frac{1}{2}(\sqrt{3}-1)$ or $k = -3+\sqrt{3}$, a special form which might be called the simplex harmonic, viz. the tangents from any line of the cone form a harmonic system. It is to be observed that, *quâ* simplex cone, the four tangents are two of them real, two imaginary.

$l = 1$; $k = 0$ (form fails). The cone is simplex neutral.

l between 1 and ∞, or k between 0 and ∞; the cone is simplex quadrilateral.

32. It will be observed that the crunodal and cuspidal kinds of cones do not present themselves in the foregoing investigations; the reason is that the crunodal kind admits of no representation in the form $x^3+y^3+z^3+6lxyz=0$, and (inasmuch as there is only one real inflexion) it admits of no real representation in the other form $(X+Y+Z)^3+6kXYZ=0$; the cuspidal kind admits of no representation in either of the two forms.

I conclude with a discussion not absolutely required for the purpose of the memoir, but which is of interest in regard to the form $(X+Y+Z)^3+6kXYZ=0$.

Reduction of the Hessian to the form $(X'+Y'+Z')^3+6k'X'Y'Z'=0$.

33. The cubic cone

$$x^3+y^3+z^3 + \qquad 6lxyz=0,$$

has for its Hessian

$$-l^2(x^3+y^3+z^3)+(1+2l^3)xyz=0,$$

or say

$$x^3+y^3+z^3 + \qquad 6l'xyz=0,$$

if

$$l' = -\frac{1+2l^3}{6l^2}.$$

Hence writing

$$\begin{aligned} X' &= -2l'x + \quad y + \quad z, \\ Y' &= \quad x - 2l'y + \quad z, \\ Z' &= \quad x + \quad y - 2l'z, \end{aligned}$$

we have

$$(X'+Y'+Z')^3+6k'X'Y'Z' = \frac{8(2l'+1)^2(l'-1)^2}{1-2l'+4l'^2}(x^3+y^3+z^3+6l'xyz).$$

Hence the equation of the Hessian is

$$(X'+Y'+Z')^3+6k'X'Y'Z'=0,$$

where the value of k' is

$$k' = \frac{4(l'-1)^3}{1-2l'+4l'^2}.$$

34. But we have

$$l' - 1 = -\frac{1+2l^3}{6l^2} - 1 = -\frac{1}{6l^2}(1+6l^2+2l^3),$$

$$4(1-2l'+4l'^2) = (4l'-1)^2 + 3 = \frac{1}{9l^4}\left\{(2+3l^3+4l^3)^2 + 27l^4\right\} = \frac{4}{9l^4}(1+l+l^2)^2(1-2l+4l^2),$$

and thence

$$k' = -\tfrac{1}{6}\,\frac{(1+6l^2+2l^3)^3}{l^2(1+l+l^2)^2(1-2l+4l^2)}.$$

But the equation

$$k = \frac{4(l-1)^2}{1-2l+4l^2},$$

gives

$$k+4 = \frac{4l(1+l+l^2)}{1-2l+4l^2}, \quad k+6 = \frac{2(1+6l^2+2l^3)}{1-2l+4l^2},$$

and we thence have

$$k' = -\tfrac{1}{3}\,\frac{(k+6)^3}{(k+4)^2},$$

which determines k' in terms of k.

35. It may be observed that the value $k=-6$ corresponds to $l'=1$, that is, the Hessian is here $x^3+y^3+z^3+6xyz=0$, a simplex neutral form *not* transformable into $(X'+Y'+Z')^3+6k'X'Y'Z'=0$; the corresponding value of l is of course given by the equation $1+6l^2+2l^3=0$; the only speciality of the cone $x^3+y^3+z^3+6lxyz=0$, or $(X+Y+Z)^3-36XYZ=0$, consequently is that the Hessian is a simplex neutral cone.

The value $k=-4$ corresponds to $l=0$, $l'=\infty$, $k'=\infty$; hence $X':Y':Z'=x:y:z$ and the transformation of the Hessian $x^3+y^3+z^3+6l'xyz=0$ into the new form $(X'+Y'+Z')^3+6kX'Y'Z'=0$ degenerates into the mere identity $xyz=xyz$.

Cambridge, 19th Feb. 1865.

352.

SUITE DES RECHERCHES SUR L'ÉLIMINATION ET LA THÉORIE DES COURBES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXIV. (1865), pp. 167—171.]

DANS le mémoire "Recherches sur l'élimination et la théorie des courbes," t. XXXIV. pp. 30—45 de ce Journal (1847), [53], j'ai donné pour une courbe $U=0$ du n-ième ordre sans points doubles ou de rebroussement, les expressions pour les degrés, tant par rapport aux coefficients que par rapport aux variables, des fonctions qui entrent dans l'équation $FFU = KU(PU)^2(QU)^3 U$ qui sert à expliquer comment la réciproque de la réciproque de la courbe $U=0$ se réduit à la courbe originale $U=0$. En partant des principes établis dans le mémoire, "Nouvelles Recherches sur l'élimination et la théorie des courbes," t. LXIII., pp. 34—39 de ce Journal (1864), [338], je suis parvenu à résoudre à peu près cette question pour le cas d'une courbe $U=0$ du n-ième ordre avec α points doubles et β points de rebroussement; mon investigation a cependant par rapport à quelques points besoin de confirmation.

Je commence par rappeler que l'équation d'une courbe avec des points doubles et de rebroussement peut être présentée sous la forme

$$U = aP + bQ + cR + \ldots = 0,$$

où $a, b, c, \ldots$ sont des quantités absolument arbitraires, $P=0$, $Q=0$, $R=0, \ldots$ sont des courbes du n-ième ordre (je suppose toujours que $U=0$ est une courbe du n-ième ordre avec α points doubles et β points de rebroussement) qui ont chacune pour chaque point double de la courbe $U=0$ un point double au même point, et pour chaque point de rebroussement de la courbe $U=0$ un point de rebroussement au même point et avec la même tangente. En parlant des coefficients de U, je désignerai toujours les quantités $(a, b, c, \ldots)$ sans faire attention aux constantes contenues dans les fonctions $(P, Q, R, \ldots)$. La fonction U (voir les Nouvelles Recherches etc.) a un discriminant special KU du degré $3(n-1)^2 - 7\alpha - 11\beta$: il y a en outre une certaine fonction AU des coefficients,

laquelle dépend des points de rebroussement, qui semble jouer un rôle analogue en quelque sorte à celui du discriminant. Car soient pour un moment (x, y, z) les coordonnées d'un des points de rebroussement de la courbe $U=0$; écrivons $D=xd_x+yd_y+zd_z$, et dans la fonction U substituons (x, y, z) au lieu de (x, y, z); l'équation $D^2U=0$ donne le carré de la tangente au point de rebroussement: or $D^2U=aD^2P+bD^2Q+cD^2R+\ldots$, et puisque les courbes $P=0$, $Q=0$, $R=0, \ldots$ ont chacune la même tangente au point de rebroussement, les fonctions D^2P, D^2Q, $D^2R, \ldots$ seront des fonctions de la forme $\lambda\Phi^2$, $\mu\Phi^2$, $\nu\Phi^2, \ldots$ où $\Phi=0$ est l'équation de la tangente, et λ, μ, $\nu\ldots$ sont des quantités constantes qui ne dépendent que des constantes que contiennent les fonctions P, Q, $R, \ldots$. Nous aurons donc $D^2U=(a\lambda+b\mu+c\nu+\ldots)\Phi^2$; et je remarque que l'équation $a\lambda+b\mu+c\nu+\ldots=0$ serait la condition pour qu'il y eût au lieu du point de rebroussement un point triple. On obtient donc l'équation du système des carrés des tangentes aux points de rebroussement sous la forme

$$(a\lambda_1+b\mu_1+c\nu_1\ldots)(a\lambda_2+b\mu_2+c\nu_2\ldots)\ldots(a\lambda_\beta+b\mu_\beta+c\nu_\beta+\ldots)\,\Phi_1^2\Phi_2^2\ldots\Phi_\beta^2=0:$$

le facteur constant $(a\lambda_1+b\mu_1+c\nu_1\ldots)\ldots(a\lambda_\beta+b\mu_\beta+c\nu_\beta\ldots)$, du degré β par rapport aux coefficients, est précisément la dérivée que je nomme AU (de manière que $AU=0$ est la condition pour l'existence d'un point triple): l'autre facteur $\Phi_1^2\Phi_2^2\ldots\Phi_\beta^2$ est du degré 0 par rapport aux coefficients.

Cela étant, je pose d'abord, pour la vérifier plus tard, la table suivante:

	Degrés par rapport aux variables	Degrés par rapport aux coefficients
équation de la courbe, $U=0$	2	1
condition pour un nouveau point double, $KU=0$	0	$3(n-1)^2-7\alpha-11\beta$
condition pour un point triple, $AU=0$	0	β
équation de la courbe réciproque, $FU=0$	$n(n-1)-2\alpha-3\beta$	$2(n-1)$
équation de la courbe des inflexions, $HU=0$	$3(n-2)$	3
équation des tangentes aux points d'inflexion, $QU=0$	$3n(n-2)-6\alpha-8\beta$	$3n(n-2)-3\alpha-4\beta$
équation de la courbe des contacts des tangentes doubles $\Pi U=0$	$(n-2)(n^2-9)$	$(n+4)(n-3)$
équation des tangentes doubles, $PU=0$	$\frac{1}{2}n(n-2)(n^2-9)-(n^2-n-6)(2\alpha+3\beta)$ $+2\alpha(\alpha-1)+6\alpha\beta+\frac{9}{2}\beta(\beta-1)$	$2n(n-2)(n-3)$ $-(2n-6)(2\alpha+3\beta)-\beta$
équation de la courbe réciproque de la réciproque de la courbe, $FFU=0$	$(n^2-n-2\alpha-3\beta)(n^2-n-1-2\alpha-3\beta)$	$2(n^2-n-1-2\alpha-3\beta)\,2(n-1)$.

Et puis on a l'équation

$$FFU = AU\,.\,KU\,.\,(PU)^2.\,(QU)^3.\,U.$$

La comparaison des degrés par rapport aux variables donne

$$\begin{aligned}(n^2-n-2\alpha-3\beta)(n^2-n-1-2\alpha-3\beta) = \\ n(n-2)(n^2-9)-(n^2-n-6)(4\alpha+6\beta)+4\alpha(\alpha-1)+12\alpha\beta+9\beta(\beta-1) \\ +9n(n-2) \qquad -18\alpha-24\beta \\ +\ n\end{aligned}$$

ce qui est exacte. La comparaison des degrés par rapport aux coefficients donne

$$\begin{aligned}4(n-1)(n^2-n-1-2\alpha-3\beta) = \qquad \beta \\ +3(n-1)^2-7\alpha-11\beta \\ +4n(n-2)(n-3)-(4n-12)(2\alpha+3\beta)-2\beta \\ +9n(n-2)-9\alpha-12\beta \\ +1\end{aligned}$$

ce qui de même est exacte.

Les expressions pour les degrés de KU et AU sont déjà démontrées; pour les autres expressions, en considérant d'abord la courbe générale $W=0$ du n-ième ordre, laquelle, en établissant entre les coefficients les relations convenables, se réduit à la courbe $U=0$ avec α points doubles et β points de rebroussement, on sait par la théorie de M. Plücker quels sont les facteurs desquels seront affectés FW, QW, PW, et qu'il faut écarter pour réduire ces fonctions à FU, QU, PU respectivement.

Pour FW ce facteur est A^2B^3, où $A=0$ est l'équation tangentielle des points doubles, et $B=0$, l'équation tangentielle des points de rebroussement: la réduction du degré par rapport aux variables est donc de $2\alpha+3\beta$ unités. En prenant (x, y, z) pour les coordonnées d'un point double quelconque on a $A=\Pi(\xi\mathrm{x}+\eta\mathrm{y}+\zeta\mathrm{z})$, et de même en prenant (x, y, z) pour les coordonnées d'un point de rebroussement quelconque on a $B=\Pi(\xi\mathrm{x}+\eta\mathrm{y}+\zeta\mathrm{z})$; A et B ne contiennent donc pas les coefficients a, b, c,... de U, et une réduction de degré par rapport aux coefficients n'a pas lieu.

Pour QW le facteur est M^3N^4, où $M=0$ est l'équation des tangentes aux points doubles et $N=0$ l'équation des carrés des tangentes aux points de rebroussement: la réduction de degré par rapport aux variables est donc $6\alpha+8\beta$ unités. Soient (x, y, z) les coordonnées d'un point double, $D=xd_x+yd_y+zd_z$; en substituant comme auparavant (x, y, z) au lieu de (x, y, z) dans la fonction U, l'équation des deux tangentes au point double est $D^2U=0$, où D^2U est du degré 1 par rapport aux coefficients: en formant l'équation analogue pour chaque point double on a $M=\Pi(D^2U)=0$, et M sera du degré α par rapport aux coefficients. En prenant (x, y, z) pour les coordonnées d'un point de rebroussement, on a de même $N=\Pi(D^2U)=0$ pour l'équation des carrés des tangentes aux points de rebroussement; N est donc du degré β par rapport aux coefficients. Nous avons vu que l'équation $N=0$ se réduit à la forme $N=AU.\Phi_1^2\Phi_2^2\ldots\Phi_\beta^2$,

j'admets cependant qu'il faut retenir ce facteur constant AU, et considérer ainsi N comme étant effectivement du degré β. Le facteur M^3N^4 est donc du degré $3\alpha+4\beta$, et la réduction de degré par rapport aux coefficients qui a lieu pour QW est donc de $3\alpha+4\beta$ unités.

Pour PW le facteur est R^2S^3T, ou $R=0$ est l'équation du système des tangentes menées à la courbe par les points doubles, $S=0$ l'équation du système des tangentes menées à la courbe par les points de rebroussement, $T=0$ l'équation des droites qui contiennent deux points doubles (chacune de ces droites étant comptée 4 fois) ou qui contiennent un point double et un point de rebroussement (chacune de ces droites étant comptée 6 fois), ou enfin qui contiennent deux points de rebroussement (chacune de ces droites étant comptée 9 fois). Par rapport aux variables le degré de R est égal à $\alpha\{n^2-n-6-2(\alpha-1)-3\beta\}$, celui de S à $\beta\{n^2-n-6-2\alpha-3(\beta-1)\}$: le degré de R^2S^3 est donc égal à $(n^2-n-6)(2\alpha+3\beta)-4\alpha(\alpha-1)-6\alpha\beta-9\beta(\beta-1)$. Le degré de T est égal à $4.\frac{1}{2}\alpha(\alpha-1)+6\alpha\beta+9.\frac{1}{2}\beta(\beta-1)$, le degré de R^2S^3T s'élève donc à $(n^2-n-6)(2\alpha+3\beta)-2\alpha(\alpha-1)-3\alpha\beta-\frac{9}{2}\beta(\beta-1)$, nombre qui exprime la réduction de degré par rapport aux variables qui a lieu pour PW. Par rapport aux coefficients le degré de R est égal à $(2n-6)\alpha$, celui de S à $(2n-6)\beta$, celui de T à zéro: le degré de R^2S^3T s'élève donc à $(2n-6)(2\alpha+3\beta)$. On aurait par conséquent pour PW par rapport aux coefficients une réduction de degré égale à $(2n-6)(2\alpha+3\beta)$ unités; mais d'après un exemple très-particulier (il est vrai) j'admets que PW contiendra encore le facteur constant AU, ce qui donnerait pour le nombre dont il s'agit la valeur $(2n-6)(2\alpha+3\beta)+\beta$.

J'ai dit que par rapport aux coefficients le degré de R est égal à $(2n-6)\alpha$ et celui de S à $(2n-6)\beta$: pour prouver l'exactitude de ces nombres il faut se rappeler que l'équation $\Theta=0$ des tangentes menées par un point quelconque est du degré (n^2-n) par rapport aux variables et du degré $2(n-1)$ par rapport aux coefficients. En prenant pour le point dont il s'agit un point double ou de rebroussement et supposant que dans la courbe il n'y a que ce seul point double ou de rebroussement, le degré par rapport aux variables est (n^2-n-6) et celui par rapport aux coefficients est $2n-6$. Mais dans le cas général Θ contiendra comme facteur G^2H^3, en dénotant par $G=0$ l'équation des droites menées par le point dont il s'agit à tous les points doubles, et par $H=0$ l'équation des droites menées par ce point à tous les points de rebroussement. De cette manière on obtient un abaissement de $2(\alpha-1)+3\beta$, ou de $2\alpha+3(\beta-1)$ unités pour le degré par rapport aux variables, mais le degré par rapport aux coefficients est toujours $(2n-6)$. Donc en considérant les systèmes des points doubles et des points de rebroussement, pour R la réduction est égal à $(2n-6)\alpha$ et pour S à $(2n-6)\beta$ unités.

Les difficultés de cette investigation sont dues aux points de rebroussement: en admettant en FFU l'existence d'un facteur $(AU)^m$, il n'est pas clair que l'on doit avoir $m=1$; et la démonstration pour les valeurs des termes en β, des expressions $3\alpha+4\beta$ et $(2n-6)(2\alpha+3\beta)-\beta$ est imparfaite. Écrivons

$$FFU=(AU)^m.KU.(PU)^2(QU)^3.U$$

et supposons que le nombre qui exprime la réduction de degré par rapport aux coefficients soit donné par la valeur $3\alpha + k\beta$ pour QU et par la valeur $(2n-6)(2\alpha+3\beta)+l\beta$ pour PU. La comparaison des degrés par rapport aux coefficients donne

$$\begin{aligned} 4(n-1)(n^2-n-2\alpha-3\beta) = \quad & m\beta \\ & +3\ (n-1)^2-7\alpha-11\beta \\ & +4n(n-2)(n-3)-(4n-12)(2\alpha+3\beta)-2l\beta \\ & +9n(n-2)-9\alpha-3k\beta \\ & +1, \end{aligned}$$

ce qui établit la relation $m-2l=3k-13$, à laquelle on satisfait en prenant $m=1$, $l=1$, $k=4$ Mais je serais bien aise de prouver ces valeurs par une démonstration plus concluante.

Cambridge, 26 *Mai*, 1864.

353.

NOTE SUR LA SURFACE DU QUATRIÈME ORDRE DE STEINER.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXIV. (1865), pp. 172—174.]

En considérant les deux coniques définies par les équations

$$U = (a, b, c, f, g, h \between x, y, z)^2 = 0$$
$$U' = (a', b', c', f', g', h' \between x', y', z')^2 = 0,$$

on en déduit les trois équations dérivées

$$F = (bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch \between \xi, \eta, \zeta)^2 = 0,$$
$$G = (bc' + b'c - 2ff', \ldots\ldots\ldots\ldots \between \xi, \eta, \zeta)^2 = 0,$$
$$F' = (b'c' - f'^2, \ldots\ldots\ldots\ldots \between \xi, \eta, \zeta)^2 = 0,$$

et l'on sait que $F = 0$ est l'équation tangentielle de la conique $U = 0$ (autrement dit, l'équation qui exprime que cette conique est touchée par la droite $\xi x + \eta y + \zeta z = 0$), que de même $F' = 0$ est l'équation tangentielle de la conique $U' = 0$, et enfin que $G = 0$ est l'équation tangentielle de la conique enveloppée par une droite $\xi x + \eta y + \zeta z = 0$ qui coupe harmoniquement les deux coniques $U = 0$, $U' = 0$.

Or, en considérant les deux surfaces quadriques

$$U = (a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0,$$
$$U' = (a', b', c', d', f', g', h', l', m', n' \between x, y, z, w)^2 = 0,$$

on forme d'une manière analogue les quatre équations dérivées

$$F = (bcd + \text{etc.}, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0,$$
$$G = (b'cd + \text{etc.}, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0,$$
$$G' = (b'c'd + \text{etc.}, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0,$$
$$F' = (b'c'd' + \text{etc.}, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0.$$

$F=0$ est l'équation tangentielle de la surface $U=0$ (et de même $F'=0$ est l'équation tangentielle de la surface $U'=0$). Les deux équations $G=0$, $G'=0$, qui ont des coefficients formés d'après une loi facile à saisir, se changent l'une dans l'autre lorsqu'on échange entre elles les deux surfaces quadriques $U=0$, $U'=0$. L'équation $G'=0$ (celle des deux dont il s'agira dans la suite) est l'équation tangentielle de la surface quadrique enveloppée par un plan $\xi x+\eta y+\zeta z+\omega w=0$ qui coupe les surfaces $U=0$, $U'=0$ selon des coniques $S=0$, $S'=0$ telles qu'il y ait sur la conique $S=0$ une infinité de systèmes de trois points conjugués par rapport à la conique $S'=0$.

En supposant à présent que l'équation $U'=0$ est celle d'un cône, on peut dire que $G'=0$ est l'équation tangentielle de la surface quadrique enveloppée par un plan qui coupe la surface $U=0$ selon une conique $S=0$ telle que par cette conique et par le sommet du cône $U'=0$ on puisse faire passer une infinité de systèmes de trois droites conjuguées par rapport au cône $U'=0$. On peut présenter le théorème sous une autre forme; en faisant passer par le sommet du cône $U'=0$ trois droites conjuguées par rapport à ce cône, et en choisissant à volonté l'un des deux points de rencontre de chacune des droites avec la surface $U=0$, on obtient trois points qui déterminent un plan; en considérant tous les systèmes des trois droites conjuguées, on a pour chaque système un plan, et l'enveloppe de ces plans n'est autre chose que la surface quadrique $G'=0$.

Je suppose que le sommet du cône $U'=0$ soit situé sur la surface $U=0$, et je dis que la surface $G'=0$ se réduira à un système de deux points, à savoir le sommet du cône $U'=0$ et un autre point. Pour démontrer cela, on peut prendre pour coordonnées du sommet $x=0$, $y=0$, $z=0$; les deux équations seront alors

$$U=(a,\ b,\ c,\ 0,\ f,\ g,\ h,\ l,\ m,\ n\between x,\ y,\ z,\ w)^2=0,$$
$$U'=(a',\ b',\ c',\ 0,\ f',\ g',\ h',\ 0,\ 0,\ 0\between x,\ y,\ z,\ w)^2=0,$$

(ou, ce qui est la même chose, $U'=(a',\ b',\ c',\ f',\ g',\ h'\between x,\ y,\ z)^2=0$ et) en substituant ces valeurs on voit sans peine que les coefficients de x^2, y^2, z^2, xz, zx, xy dans la fonction G' se réduiront à zéro, et que l'équation $G'=0$ aura la forme

$$G'=(0,\ 0,\ 0,\ D,\ 0,\ 0,\ 0,\ L,\ M,\ N\between \xi,\ \eta,\ \zeta,\ \omega)^2=0,$$

c'est-à-dire nous aurons

$$G'=\omega\,(D\omega+2L\xi+2M\eta+2N\zeta)=0,$$

équation qui représente en effet le point $\omega=0$ (ou ce qui est la même chose le point $x=0$, $y=0$, $z=0$) et un autre point $D\omega+2L\xi+2M\eta+2N\zeta=0$, ou ce qui est la même chose le point $x:y:z:w=2L:2M:2N:D$.

Dans le cas actuel chacune des trois droites rencontre la surface $U=0$ dans le sommet et de plus dans un seul point, et en prenant ce dernier point pour point de rencontre de la droite avec la surface $U=0$ le plan mené par les trois points ne passe pas par le sommet; ce plan passe donc par le point $x:y:z:w=2L:2M:2N:D$, et on a ainsi

THÉORÈME I. En faisant passer par un point donné de la surface quadrique $U=0$ trois droites conjuguées par rapport au cône $U'=0$ (qui a ce même point pour sommet) le plan mené par les trois points de rencontre des droites avec la surface $U=0$ passe toujours (quel que soit le système des trois droites conjuguées) par un point fixe.

J'ajoute que, lorsque les équations $U=0$, $U'=0$ ont la forme spéciale qui leur a été donnée en dernier lieu, les coordonnées du point seront $x:y:z:w=2L:2M:2N:D$, et il convient de remarquer que ces valeurs L, M, N, D sont des fonctions quadriques par rapport aux coefficients $(a', \ldots)$ du cône $U'=0$.

Au lieu d'un cône donné $U'=0$, considérons le système entier des cônes $\lambda P+\mu Q+\nu R=0$, où $P=0$, $Q=0$, $R=0$ sont des cônes donnés ayant leur sommet commun dans le point $(x=0,\ y=0,\ z=0)$ de la surface et λ, μ, ν des coefficients arbitraires, système qui est celui des cônes en involution avec les cônes donnés $P=0$, $Q=0$, $R=0$. A chaque système des coefficients λ, μ, ν correspond un point fixe, et en conservant pour ses coordonnées la notation antérieure $x:y:z:w=2L:2M:2N:D$, les quantités L, M, N, D sont des fonctions quadriques des quantités arbitraires λ, μ, ν. Le lieu du point dont il s'agit sera évidemment une surface, et on démontre sans peine que cette surface est du quatrième ordre. Car, pour trouver en combien de points la surface est rencontrée par une droite quelconque, il faut combiner avec les équations $x:y:z:w=2L:2M:2N:D$ les équations de la droite dont il s'agit, c'est-à-dire deux équations linéaires en x, y, z, w; cela donne deux équations linéaires en L, M, N, D, ou quadriques en (λ, μ, ν). On a ainsi quatre systèmes de valeurs de (λ, μ, ν); et à chaque système correspond un seul point (x, y, z, w), il y a par conséquent quatre points d'intersection, et la surface est du quatrième ordre. Nous avons donc

THÉORÈME II. En considérant au lieu du cône $U'=0$ le système entier des cônes $\lambda P+\mu Q+\nu R=0$ en involution avec les cônes $P=0$, $Q=0$, $R=0$ qui ont leur sommet commun dans un point de la surface $U=0$, le lieu du point fixe du théorème I. est une surface du quatrième ordre.

Cette surface du quatrième ordre est la surface de Steiner, considérée dernièrement par MM. Kummer, Weierstrass, Schröter, et Cremona.

Cambridge, 2 *Novembre*, 1864.

354.

NOTE SUR LES SINGULARITÉS SUPÉRIEURES DES COURBES PLANES.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXIV. (1865), pp. 369—371.]

DANS un mémoire "On the higher singularities of plane Curves" destiné pour le *Quarterly Mathematical Journal* j'ai cherché à établir qu'une singularité quelconque équivaut à un certain nombre δ' de points doubles, κ' de points de rebroussement, τ' de tangentes doubles, et ι' d'inflexions; et pour déterminer ces nombres, j'ai donné dans le cas d'une singularité simple, où la courbe n'a qu'une seule branche, des formules que je vais reproduire ici. Si la branche est par rapport à ses points de l'indice α, ayant avec elle-même le nombre $\frac{1}{2}M$ de points communs, et par rapport à ses tangentes de l'indice β, ayant avec elle-même le nombre $\frac{1}{2}N$ de tangentes communes, on trouve

$$\begin{aligned} \delta' &= \tfrac{1}{2}[M - 3(\alpha - 1)], \\ \kappa' &= \qquad\quad \alpha - 1 \ , \\ \tau' &= \tfrac{1}{2}[N - 3(\beta - 1)], \\ \iota' &= \qquad\quad \beta - 1 \ . \end{aligned}$$

Pour expliquer ces formules, je remarque que la singularité dont il s'agit est telle que, prenant pour origine le point sur la courbe, on obtient pour l'ordonnée y *une seule suite* de la forme

$$y = Ax^p + Bx^q + \dots ,$$

où la suite est arrangée suivant les puissances ascendantes de x et les coefficients $A, B, \dots$ ont chacun une valeur unique. Si l'axe des y ne touche pas la courbe, aucun des exposants $p, q, \dots$ ne sera inférieur à l'unité, et si de plus l'axe des x touche la

courbe, ce que l'on peut toujours effectuer par un choix convenable de la direction des axes, les exposants $p, q, \ldots$ seront tous supérieurs à l'unité. Cela posé, et les exposants fractionnaires étant exprimés chacun dans ses moindres termes, si α est le plus petit nombre entier divisible par tous les dénominateurs des fractions (de manière que y soit fonction entière de $x^{\frac{1}{\alpha}}$), je dis que la branche est de l'indice α par rapport à ses points. On a donc pour y précisément le nombre α de valeurs, qui s'obtiennent en attribuant à $x^{\frac{1}{\alpha}}$ ses valeurs diverses. A chacune de ses valeurs correspond une "branche partielle" de la courbe, de manière que la branche à l'indice α est composée de α branches partielles; pour $\alpha = 1$ la branche partielle n'est autre chose que la branche même. En considérant deux branches partielles, et en désignant par p le plus petit exposant de x qui se trouve dans la suite par laquelle est exprimée la différence $y_1 - y_2$ des ordonnées des deux branches partielles (ce nombre p pouvant être entier ou fractionnaire), je pose comme définition que les deux branches partielles ont un nombre p de points communs, ou d'intersection. En combinant deux à deux les α branches partielles qui composent la branche de l'indice α, et en formant la somme Σp des nombres p qui correspondent à chaque paire de branches partielles, on obtient le nombre $\frac{1}{2}M$ des points communs de la branche avec elle-même. En se servant des coordonnées tangentielles, on a par rapport aux tangentes de la branche une théorie tout à fait semblable; cette remarque suffit pour expliquer les notions d'une branche de l'indice β par rapport à ses tangentes, et du nombre $\frac{1}{2}N$ des tangentes communes de la branche avec elle-même.

Comme exemple je prends la singularité donnée par l'équation

$$y = x^{\frac{4}{3}} + x^{\frac{5}{2}} + \ldots;$$

dans ce cas les exposants n'ont que les dénominateurs 2 et 3, la branche est de l'indice 6 par rapport à ses points, elle est composée de six branches partielles représentées par les équations

$$\begin{aligned} y_1 &= \, x^{\frac{4}{3}} + x^{\frac{5}{2}} \ldots, & y_4 &= \, x^{\frac{4}{3}} - x^{\frac{5}{2}} \ldots, \\ y_2 &= \omega\, x^{\frac{4}{3}} + x^{\frac{5}{2}} \ldots, & y_5 &= \omega\, x^{\frac{4}{3}} - x^{\frac{5}{2}} \ldots, \\ y_3 &= \omega^2 x^{\frac{4}{3}} + x^{\frac{5}{2}} \ldots, & y_6 &= \omega^2 x^{\frac{4}{3}} - x^{\frac{5}{2}} \ldots, \end{aligned}$$

où ω est une racine cubique imaginaire de l'unité. La branche partielle y_1 coupe les autres branches partielles dans un nombre $\frac{4}{3}, \frac{4}{3}, \frac{5}{2}, \frac{4}{3}, \frac{4}{3}$ de points, ce qui donne pour la branche partielle y_1 le nombre $\frac{16}{3} + \frac{5}{2}, = \frac{47}{6}$ de points; on a ce même nombre $\frac{47}{6}$ pour les autres branches partielles y_2, y_3, y_4, y_5, y_6 respectivement, et de là on trouve, pour le *double* du nombre des intersections de la branche avec elle-même, la valeur $M = 47$, donc $\delta' = \frac{1}{2}(47 - 15) = 16$, $\kappa' = 5$.

En coordonnées tangentielles, la branche $y = x^{\frac{4}{3}} + x^{\frac{5}{2}} + \ldots$ s'exprime par l'équation

$$Z = X^4 + \ldots + X^{\frac{15}{2}} \ldots.$$

Plus généralement, on a pour une branche $y = Ax^p + Bx^q + \ldots$ l'équation en coordonnées tangentielles $Z = A'X^{\frac{q}{p-1}} + B'X^{\frac{q}{p-1}} + \ldots$, la forme générale des exposants étant $\frac{\lambda p + \mu q + \ldots}{p-1}$, où λ, $\mu, \ldots$ sont des entiers positifs, résultat que je ne m'arrête pas à démontrer. Dans le cas particulier qui nous occupe, la branche est donc de l'indice 2 par rapport à ses tangentes. On trouve de suite $N = 15$ et de là $\tau' = \frac{1}{2}(15-3) = 6$, $\iota' = 1$; donc la singularité dont il s'agit équivaut à un nombre 16 de points doubles, 5 de points de rebroussement, 6 de tangentes doubles, et 1 inflexion.

On a un exemple plus simple dans le point de rebroussement de seconde espèce; l'équation est ici $y = x^2 + x^{\frac{5}{2}} \ldots$ et en coordonnées tangentielles on obtient l'équation $Z = X^2 + X^{\frac{5}{2}} \ldots$ de la même forme. De là on trouve $\delta' = 1$, $\kappa' = 1$, $\tau' = 1$, $\iota' = 1$, de manière que cette singularité équivaut à 1 point double, 1 point de rebroussement, 1 tangente double et 1 inflexion. M. Plücker dans son grand ouvrage a trouvé *à posteriori* que cette singularité se compose de $2\frac{1}{2}$ points doubles et de $2\frac{1}{2}$ tangentes doubles, ce qui donne en effet les mêmes réductions pour la classe et les mêmes nombres pour les inflexions et les tangentes doubles, que donnent mes valeurs $\delta' = 1$, $\kappa' = 1$, $\tau' = 1$, $\iota' = 1$; mais il y a à remarquer qu'en considérant par exemple une courbe du quatrième ordre avec un point double et un point de rebroussement de seconde espèce (courbe qui existe), on aurait $\delta + \kappa = 3\frac{1}{2}$, nombre plus grand que le maximum du nombre des points doubles et de rebroussement que peut avoir une courbe du quatrième ordre.

Je n'ai parlé que des singularités simples, où il y a une seule branche de la courbe, mais on étend sans peine la théorie précédente aux singularités composées, où il y a plusieurs branches de la courbe. Cette extension exige la distinction de trois cas différents. Il peut y avoir sur la courbe un point avec une seule tangente, mais avec plusieurs branches qui se touchent,—ou un point avec plusieurs tangentes dont chacune touche une ou plusieurs branches,—ou enfin une tangente avec plusieurs points de contact, dans lesquels la tangente touche une seule ou plusieurs branches de la courbe.

Cambridge, 1 *Juin*, 1865.

355.

SUR UN THÉORÈME RELATIF À HUIT POINTS SITUÉS SUR UNE CONIQUE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXV. (1866), pp. 180—184.]

On sait que le théorème de Pascal peut être déduit du théorème suivant: toute courbe cubique qui passe par 8 des 9 points d'intersection de deux courbes cubiques passe par tous les 9 points.

De même cet autre théorème—toute courbe quartique qui passe par 13 des 16 points d'intersection de deux courbes quartiques passe par tous les 16 points—conduit à un théorème relatif à 8 points situés sur une conique.

En effet si par 8 points donnés et situés sur une conique donnée on fait passer deux systèmes de 4 droites (ces deux systèmes doivent être sans droite commune) les deux systèmes sont des courbes quartiques qui se rencontrent dans les 8 points donnés et de plus dans 8 nouveaux points; donc toute courbe quartique qui passe par 13 des $8+8$ points passe par tous les $8+8$ points. Or la conique donnée passe par les 8 points donnés, et par 5 des 8 nouveaux points on peut faire passer une autre conique: les deux coniques forment ensemble une courbe quartique qui passe par $8+5$ des $8+8$ points, et qui passera ainsi par les $8+8$ points; c'est-à-dire la nouvelle conique passe par les 8 nouveaux points, ou autrement dit, les 8 nouveaux points sont situés sur une conique—c'est là le théorème relatif à 8 points situés sur une conique.

On déduit de là les théorèmes 3, 4, 5 de Steiner (Lehrsätze und Aufgaben, ce journal t. XXX, [1846], pp. 274 et 275). En effet considérons sur une conique donnée n points donnés, et les n tangentes dans ces mêmes points. En combinant deux à

deux les n points on obtient $\frac{1}{2}n(n-1)$ droites G: ces droites se coupent deux à deux dans les n points donnés, qui comptent pour $\frac{1}{2}n(n-1)(n-2)$ intersections, et de plus dans $\frac{1}{8}n(n-1)(n-2)(n-3)$ points r. Chacune des n tangentes rencontre les $\{\frac{1}{2}n(n-1)-(n-1)\}$ droites G qui ne passent pas par le point de contact de cette tangente, dans $\frac{1}{2}(n-1)(n-2)$ points s, ce qui donne en tout $\frac{1}{2}n(n-1)(n-2)$ points s. Enfin les n tangentes se rencontrent deux à deux dans $\frac{1}{2}n(n-1)$ points t.

On a ainsi

$$\begin{aligned} &\tfrac{1}{8}n(n-1)(n-2)(n-3) \text{ points } r,\\ &\tfrac{1}{2}n(n-1)(n-2) \qquad\quad \text{,,} \quad s,\\ &\tfrac{1}{2}n(n-1) \qquad\qquad\quad \text{,,} \quad t,\\ \text{ensemble } &\tfrac{1}{8}n(n-1)(n^2-n+2) \text{ points};\end{aligned}$$

or parmi ces points il y a selon les trois théorèmes de Steiner un grand nombre de systèmes de 8 points sur une conique.

Prenons d'abord sur la conique donnée 4 points quelconques a, b, c, d des n points, et considérons aussi les points consécutifs a', b', c', d'. La figure des 4 points a, b, c, d et des 4 tangentes dans ces mêmes points équivaut à celle des 8 points $a, a', b, b', c, c', d, d'$. Partant de l'arrangement $abcd$ (lisez-le cycliquement et il correspondra à l'un des 3 quadrilatères que l'on peut former avec les 4 points) on forme avec les 8 points les deux systèmes que voici de 4 droites chacun:

système aa', bb', cc', dd', c'est-à-dire les tangentes aux 4 points a, b, c, d;

système $a'b, b'c, c'd, d'a$, c'est-à-dire ab, bc, cd, da;

et ces deux systèmes se rencontrent dans les 8 points $a, a', b, b', c, c', d, d'$ (ou, ce qui est la même chose, dans les points a, b, c, d, chacun compté 2 fois) et dans 8 nouveaux points compris entre les points r, s, t; ces 8 points sont donc situés sur une conique. Comme il y a 3 arrangements $abcd, acdb, adbc$ des 4 points, on obtient de cette manière 3 systèmes de 8 points sur une conique.

Prenons sur la conique 5 points quelconques a, b, c, d, e des n points, et considérons aussi 3 points consécutifs a', b', c'. Partant de l'arrangement $abcde$ (qui correspond à l'un des 12 pentagones que l'on peut former avec les 5 points) on forme avec les points $a, a', b, b', c, c', d, e$ les deux systèmes de 4 droites chacun:

système aa', bb', cc', de, c'est-à-dire les tangentes en a, b, c et la droite de;

système $a'b, b'c, c'd, ea$, c'est-à-dire ab, bc, cd, ea;

et on obtient de là (parmi les points r, s, t) un système de 8 points sur une conique. A cause des 12 arrangements des 5 points, il y a 12 systèmes. Mais au lieu des points consécutifs (a', b', c') on aurait pu prendre toute autre combinaison (a', b', d') etc.; le nombre des combinaisons étant 10, il y a donc $12 \times 10 = 120$ systèmes de 8 points sur une conique.

Prenons de même 6 points quelconques a, b, c, d, e, f des n points. En considérant les points consécutifs a', b' et en partant de l'arrangement $abcdef$, on forme avec les 8 points a, a', b, b', c, d, e, f les deux systèmes de 4 droites:

système aa', bb', cd, ef, c'est-à-dire les tangentes en a, b et les droites cd, ef;

système $a'b, b'c, de, fa$, c'est-à-dire ab, bc, de, fa;

ce qui donne parmi les points r, s, t un système de 8 points sur une conique. Il y a 60 arrangements des points a, b, c, d, e, f et 15 combinaisons (a', b') etc. des points consécutifs; on a donc $60 \times 15 = 900$ systèmes de 8 points sur une conique.

Prenons encore 7 points quelconques a, b, c, d, e, f, g des n points. En considérant le point consécutif a', et en partant de l'arrangement $abcdefg$, on forme avec les points a, a', b, c, d, e, f, g les deux systèmes de 4 droites:

système aa', bc, de, fg, c'est-à-dire la tangente en a, et les droites bc, de, fg;

système $a'b, cd, ef, ga$, c'est-à-dire ab, cd, ef, ga;

et on obtient ainsi parmi les points r, s, t un système de 8 points sur une conique. Il y a 360 arrangements $abcdefg$, etc. et 7 différents points consécutifs a', etc.: cela donne $360 \times 7 = 2520$ systèmes de 8 points sur une conique.

Prenons enfin 8 points quelconques a, b, c, d, e, f, g, h des n points:

partant de l'arrangement $abcdefgh$, on forme avec les 8 points les deux systèmes de 4 droites chacun (ab, cd, ef, gh) et (bc, de, fg, ha), ce qui conduit à un système de 8 points sur une conique. Mais on a 2520 arrangements $abcdefgh$, etc.—il y a ainsi 2520 systèmes de 8 points sur une conique.

On voit que les systèmes de 8 points sur une conique se dérivent de 4, 5, 6, 7 ou 8 des n points sur la conique donnée. En supposant $n = 4$ on n'a que les systèmes qui se dérivent des 4 points; si $n = 5$, on a les systèmes qui se dérivent de 4 points choisis d'une manière quelconque entre les 5 points—et les systèmes qui se dérivent des 5 points: et ainsi de suite; pour $n = 8$ on a les systèmes qui se dérivent de 4, 5, 6 ou 7 points choisis d'une manière quelconque entre les 8 points, et les systèmes qui se dérivent des 8 points. On peut former la table suivante pour montrer dans les différents cas le nombre des systèmes de 8 points sur une conique:

					Nombre des systèmes de 8 points sur une conique				
	$r=$	$s=$	$t=$	$r+s+t=$	4 points 3	5 points 120	6 points 900	7 points 2520	8 points 2520
$n=4$, sys. 3	3	12	6	21	$\times 1 = 3$				
$n=5$, sys. 4	15	30	10	55	$\times 5 = 15$	$\times 1 = 120$			
$n=6$, sys. 5	45	60	15	110	$\times 15 = 45$	$\times 6 = 720$	$\times 1 = 900$		
$n=7$	105	105	21	231	$\times 35 = 105$	$\times 21 = 2520$	$\times 7 = 6300$	$\times 1 = 2520$	
$n=8$	210	168	28	406	$\times 70 = 210$	$\times 56 = 6720$	$\times 28 = 25200$	$\times 8 = 20160$	$\times 1 = 2520$

Le cas $n = 4$ est le théorème 3 de Steiner, il y a 3 systèmes de 8 points sur une conique; le cas $n = 5$ est le théorème 4, il y a $15 + 120$ systèmes; le cas $n = 6$ est le

théorème 5, il y a $45+720+900$ systèmes. Pour $n=7$ il y a $105+2520+6300+2520$ systèmes et pour $n=8$, $210+6720+25200+20160+2520$ systèmes.

Le cas $n=5$ est surtout intéressant: en effet comme une conique est déterminée par 5 points, on a ici 5 points quelconques (a, b, c, d, e), et les cinq tangentes (les droites A, B, C, D, E de Steiner) sont des droites déterminées par les cinq points et que l'on peut construire (avec la règle seulement). C'est là en effet la forme sous laquelle le théorème est présenté par Steiner; il ne parle nullement de la conique qui passe par les 5 points—et il donne pour les 5 droites une construction; à savoir, les 15 points r sont situés deux à deux sur 15 droites L qui ne dépendent chacune que de 4 points, et sur 60 droites H qui dépendent chacune des 5 points; les 60 droites H combinées deux à deux d'une manière convenable se rencontrent dans 30 points s (c'est la définition de ces points) et puis (théorème) on a 5 droites A, B, C, D, E qui contiennent chacune 6 points s et qui passent par les points a, b, c, d, e respectivement—et (théorème) les 30 points s sont aussi situés sur les 10 droites G, 3 points sur chaque droite. Je remarque qu'en prenant sur la conique qui passe par a, b, c, d, e, un point quelconque g, il y aurait 24 hexagones inscrits ayant ag pour côté—et de là 24 droites Pascaliennes—et par le point d'intersection de ag avec l'une quelconque des 6 droites be, etc. on a 4 de ces droites Pascaliennes. Cela posé, en prenant pour g le point consécutif a', les 24 hexagones se confondent deux à deux—on a donc 12 hexagones inscrits et autant de droites Pascaliennes—ces droites sont les 12 droites H lesquelles se rencontrent deux à deux dans les 6 points s situés sur la droite aa', ou A. Steiner dit que les 120 coniques dépendent des 5 points, mais que les 15 coniques *dépendent chacune de 4 points seulement;* en donnant (comme il l'a fait) le théorème comme un théorème par rapport à cinq points quelconques, cela n'est pas exact—en effet les coniques dont il s'agit dépendent chacune de 4 des cinq points, et des 4 droites correspondantes, tangentes dans ces mêmes points à la conique qui passe par les cinq points—ces coniques dépendent ainsi des cinq points.

Je remarque en passant que partant des cinq points donnés a, b, c, d, e, il y a sur chacune des droites A, B, C, D, E un point remarquable, dont Steiner ne parle pas, mais qui aurait pu servir à une construction de cette droite—par exemple il y a sur la droite A le point α qui est l'intersection commune des polaires de a par rapport à toutes les coniques qui passent par les points b, c, d, e—en particulier ce point α est l'intersection commune des polaires (harmonicales) de a par rapport aux trois paires de droites (bc, de), (bd, ec), (be, cd) respectivement.

Cambridge, 16 *Fév.* 1865.

356.

SUR UN CAS PARTICULIER DE LA SURFACE DU QUATRIÈME ORDRE AVEC SEIZE POINTS SINGULIERS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXV. (1866), pp. 284—291.]

DANS la note "sur la surface des ondes" (*Liouville* t. XI., 1846), [47], j'ai étudié sous le nom de *tétraédroïde* la surface du quatrième ordre douée de seize points singuliers, et qu'une transformation homographique fait naître de la surface des ondes. Mon point de départ a été la propriété fondamentale suivante.

"Le tétraédroïde est une surface du quatrième ordre, qui est coupée par les plans d'un certain tétraèdre suivant des paires de coniques par rapport auxquelles les trois sommets du tétraèdre dans ce plan sont des points conjugués. De plus: les seize points d'intersection des quatre paires de coniques sont des points singuliers de la surface, c'est-à-dire des points où, au lieu d'un plan tangent, il y a un cône tangent du second ordre."

Dans la même note j'ai reconnu l'existence de seize plans singuliers qui touchent chacun la surface suivant une conique. Il est intéressant d'examiner de quelle manière mes formules se rattachent à celles de M. Kummer dans ses belles recherches (Monatsbericht der Berliner Akademie für 1864, pp. 246—260 et 495—499) relatives à la surface du quatrième ordre douée de seize points singuliers.

Partant des formules de M. Kummer il convient, pour plus de symmétrie, de changer les signes de a, f; puis en remarquant que dans l'équation (3) p. 250 on doit avoir (voir p. 496) $+\frac{3}{2}cf$ au lieu de $-\frac{3}{2}cf$, l'équation de la surface sera

$$\begin{aligned}
&a^2q^2r^2 + b^2r^2p^2 + c^2p^2q^2 + d^2p^2s^2 + e^2q^2s^2 + f^2r^2s^2\\
&+ 2bcp^2qr + 2cepq^2s - 2bfpr^2s - 2efqrs^2\\
&+ 2capq^2r + 2afqr^2s - 2cdqp^2s - 2fdrps^2\\
&+ 2abpqr^2 + 2bdrp^2s - 2aerq^2s - 2depqs^2 - 4gpqrs = 0.
\end{aligned}$$

Pour donner les équations des seize plans singuliers de cette surface je pose d'abord pour abréger

$$ad = \alpha, \quad be = \beta, \quad cf = \gamma,$$

et je détermine k au moyen de l'équation cubique

$$\gamma k^3 + (-g - \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{3}{2}\gamma)\, k^2 + (-g - \tfrac{3}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)\, k - \alpha = 0\,;$$

puis j'introduis les quantités

$$\begin{aligned}
p' &= \quad . \quad + cq + \frac{b}{k+1} r \quad + \frac{d}{k} s,\\
q' &= \quad - cp \quad . \quad + \frac{a}{k} r - \frac{e}{k+1} s,\\
r' &= - \frac{b}{k+1} p \quad - \frac{a}{k} q \quad . \quad + fs,\\
s' &= \quad - \frac{d}{k} p + \frac{e}{k+1} q \quad - fr \quad . \quad ,
\end{aligned}$$

enfin je dénote par p_1, q_1, r_1, s_1; p_2, q_2, r_2, s_2; p_3, q_3, r_3, s_3 ce que deviennent les quantités p', q', r', s' en y substituant successivement pour k les trois racines k_1, k_2, k_3 de l'équation en k. Cela posé, les seize plans singuliers sont donnés par les équations

$$\begin{aligned}
p &= 0, \quad q = 0, \quad r = 0, \quad s = 0,\\
p_1 &= 0, \quad q_1 = 0, \quad r_1 = 0, \quad s_1 = 0,\\
p_2 &= 0, \quad q_2 = 0, \quad r_2 = 0, \quad s_2 = 0,\\
p_3 &= 0, \quad q_3 = 0, \quad r_3 = 0, \quad s_3 = 0.
\end{aligned}$$

En prenant une ligne quelconque (p_1, q_1, r_1, s_1) et une colonne quelconque (r, r_1, r_2, r_3), puis en omettant le terme commun r_1, on a une des seize combinaisons $(p_1, q_1, s_1, r, r_2, r_3)$ de six plans qui se rencontrent dans un des seize points singuliers.

Supposons que les plans p, s_1, r_2, q_3 se rencontrent dans le même point. Pour que cette circonstance ait lieu il faut que la condition $\dfrac{k_2(k_3+1)}{k_3(k_1+1)} = 1$ ou, ce qui est la même chose, $k_3(k_1 - k_2) - (k_2 - k_3) = 0$ soit remplie; mais si cette condition est remplie, non seulement les plans (p, s_1, r_2, q_3) se rencontront dans le même point, mais aussi les plans (q, p_1, s_2, r_3), les plans (r, q_1, p_2, s_3) et les plans (s, r_1, q_2, p_3) se rencontront dans le même point. L'équation $k_3(k_1 - k_2) - (k_2 - k_3) = 0$ appartient évidemment à un système de six équations, et l'une quelconque de ces équations donnerait un résultat semblable; chacune de ces équations conduit, comme on va voir, à une certaine relation entre les quantités g, α, β, γ (ou g, a, b, c, d, e, f), relation en vertu de laquelle la surface générale du quatrième ordre douée de seize points singuliers se réduit au *tétraédroïde*. Pour former la relation dont il s'agit, il faut égaler à zéro le produit des six fonctions analogues à $k_2(k_1 - k_2) - (k_2 - k_3)$. Je forme d'abord le produit des trois fonctions $k_3(k_1 - k_2) - (k_2 - k_3)$, $k_1(k_2 - k_3) - (k_3 - k_1)$, $k_2(k_3 - k_1) - (k_1 - k_2)$, et en représentant

pour un moment l'équation en k par $\mathfrak{a}k^3+\mathfrak{b}k^2+\mathfrak{c}k+\mathfrak{d}=0$, on trouve que le produit des trois fonctions est égal à

$$P+Q\sqrt{\Delta}=(\mathfrak{b}+\mathfrak{c})(\mathfrak{b}\mathfrak{c}+9\,\mathfrak{a}\mathfrak{d})-6\,(\mathfrak{a}\mathfrak{c}^2+\mathfrak{b}^2\mathfrak{d})$$
$$+(\mathfrak{b}+\mathfrak{c}-2\,\mathfrak{a}-2\,\mathfrak{d})\sqrt{\mathfrak{b}^2\mathfrak{c}^2-4\,\mathfrak{b}^3\mathfrak{d}-4\,\mathfrak{a}\mathfrak{c}^3+18\,\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}-27\,\mathfrak{a}^2\mathfrak{d}^2},$$

et en substituant pour $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ leurs valeurs

$$\mathfrak{a}=\gamma,\quad \mathfrak{b}=-g-\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{3}{2}\gamma,\quad \mathfrak{c}=-g-\tfrac{3}{2}\alpha-\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma,\quad \mathfrak{d}=-\alpha,$$

on trouve, toute réduction faite,

$$\begin{aligned}
P&=-2g^3+\tfrac{1}{2}g\,(\Sigma\alpha^2-10\Sigma\alpha\beta)+2\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta),\\
Q&=-2g,\\
\Delta&=\quad g^4-\tfrac{1}{2}g^3(\Sigma\alpha^2-10\Sigma\alpha\beta)-4g\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\\
&\qquad+\tfrac{1}{16}(\Sigma\alpha^4+12\Sigma\alpha^3\beta-26\Sigma\alpha^2\beta^2+244\Sigma\alpha^2\beta\gamma).
\end{aligned}$$

Cela posé, l'équation cherchée est $P^2-Q^2\Delta=0$, c'est-à-dire

$$\begin{aligned}
0=\tfrac{1}{2}(P^2-Q^2\Delta)=g^3\,.\,4\,(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)&\\
+g^2\,.\,4\,(-\Sigma\alpha^3\beta+4\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma)&\\
+g\;.\;(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\Sigma\alpha^2-10\Sigma\alpha\beta)&\\
+\quad 2\,(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2&;
\end{aligned}$$

cette équation, dans laquelle $\alpha=ad$, $\beta=be$, $\gamma=cf$, constitue la condition sous laquelle la surface de M. Kummer se réduit à un tétraédroïde.

Je passe à présent à mes formules de 1846. En écrivant pour plus de commodité f^2, g^2, h^2, l^2, m^2, n^2 au lieu de f, g, h, l, m, n, mon équation du tétraédroïde est

$$\begin{vmatrix}
. & x^2, & y^2, & z^2, & w^2\\
x^2, & . & h^2, & g^2, & l^2\\
y^2, & h^2, & . & f^2, & m^2\\
z^2, & g^2, & f^2, & . & n^2\\
w^2, & l^2, & m^2, & n^2 & .
\end{vmatrix}=0,$$

ou, ce qui est la même chose,

$$(A,\ B,\ C,\ D,\ F,\ G,\ H,\ L,\ M,\ N\between x^2,\ y^2,\ z^2,\ w^2)^2=0,$$

c'est-à-dire

$$Ax^4+By^4+Cz^4+Dw^4+2Fy^2z^2+2Gz^2x^2+2Hx^2y^2+2Lx^2w^2+2My^2w^2+2Nz^2w^2=0,$$

où les coefficients ont les valeurs

$$\begin{aligned}
&A = 2m^2n^2f^2, \quad B = 2n^2l^2g^2, \quad C = 2l^2m^2h^2, \quad D = 2f^2g^2h^2,\\
&F = l^2\,(\;l^2f^2 - m^2g^2 - n^2h^2), \quad L = f^2\,(\;l^2f^2 - m^2g^2 - n^2h^2),\\
&G = m^2\,(-l^2f^2 + m^2g^2 - n^2h^2), \quad M = g^2\,(-l^2f^2 + m^2g^2 - n^2h^2),\\
&H = n^2\,(-l^2f^2 - m^2g^2 + n^2h^2), \quad N = h^2\,(-l^2f^2 - m^2g^2 + n^2h^2).
\end{aligned}$$

Les coordonnées des seize points singuliers sont

$$(0,\ \pm h,\ \pm g,\ \pm l),\quad (\pm h,\ 0,\ \pm f,\ \pm m),\quad (\pm g,\ \pm f,\ 0,\ \pm n),\quad (\pm l,\ \pm m,\ \pm n,\ 0),$$

et les équations des seize plans singuliers sont

$$\begin{aligned}
&\cdot \quad \pm ny \pm mz \pm fw = 0,\\
&\pm nx \quad \cdot \quad \pm lz \pm gw = 0,\\
&\pm mx \pm ly \quad \cdot \quad \pm hw = 0,\\
&\pm fx \pm gy \pm hz \quad \cdot \quad = 0,
\end{aligned}$$

où l'on donne des valeurs quelconques aux signes $\pm$. Pour comparer ces plans aux plans de M. Kummer j'écris le tableau

$$\begin{array}{c|c|c|c|c}
p\,,q\,,r\,,s & \cdot \quad +ny-mz+fw & -nx \quad \cdot \quad +lz+gw & mx-ly \quad \cdot \quad +hw & -fx-gy-\ lz \quad \cdot \\
p_1,q_1,r_1,s_1 & nx \quad \cdot \quad +\ lz+gw & mx+ly \quad \cdot \quad +hw & -fx-gy+hz \quad \cdot & \cdot \quad ny-mz-fw \\
p_2,q_2,r_2,s_2 & -mx-\ ly \quad \cdot \quad +hw & -fx+gy-\ hz \quad \cdot & \cdot \quad ny+mz+fw & -nx \quad \cdot \quad +\ lz-gw \\
p_3,q_3,r_3,s_3 & fx-\ gy-hz \quad \cdot & \cdot \quad -ny-mz+fw & -nx \quad \cdot \quad -\ lz+gw & mx-ly \quad \cdot \quad -hw
\end{array}$$

et j'obtiens les valeurs suivantes:

$$\begin{aligned}
p &= \quad \cdot \quad ny - mz + fw,\\
q &= -\ nx \quad \cdot \quad +\ lz + gw,\\
r &= \quad mx - ly \quad \cdot \quad +hw,\\
s &= -\ fx - gy - hz \quad \cdot \quad \cdot
\end{aligned}$$

En résolvant ces équations par rapport à x, y, z, w et en posant pour abréger $\theta = lf + mg + nh$, on trouve

$$\begin{aligned}
\theta x &= \quad \cdot \quad -hq + gr - ls,\\
\theta y &= \quad hp \quad \cdot \quad -fr - ms,\\
\theta z &= -gp + fq \quad \cdot \quad -ns,\\
\theta w &= \quad lp + mq + nr \quad \cdot \quad ,
\end{aligned}$$

valeurs qu'il s'agit de substituer dans l'équation

$$U = (A,\ B,\ C,\ D,\ F,\ G,\ H,\ L,\ M,\ N \between x^2,\ y^2,\ z^2,\ w^2)^2 = 0$$

de la surface dont il est question.

De l'expression de U en p, q, r, s je ne considère d'abord que le terme multiplié par p^2q^2. Désignons par $\mathfrak{C}$ le coefficient de p^2q^2 dans $\theta^4 U$, nous aurons

$$\left.\begin{array}{lll}
\mathfrak{C} = & 6f^2g^2 & \times 2l^2m^2h^2 \\
& +6l^2m^2 & \times 2f^2g^2h^2 \\
& +\ h^2f^2 & \times 2l^2\ (\ l^2f^2-m^2g^2-n^2h^2) \\
& +\ g^2h^2 & \times 2m^2\,(-l^2f^2+m^2g^2-n^2h^2) \\
& +\ h^4 & \times 2n^2\ (-l^2f^2-m^2g^2+n^2h^2) \\
& +\ h^2l^2 & \times 2f^2\ (\ l^2f^2-m^2g^2-n^2h^2) \\
& +\ h^2m^2 & \times 2g^2\ (-l^2f^2+m^2g^2-n^2h^2) \\
& +(l^2f^2+m^2g^2-4lmfg) & \times 2h^2\ (-l^2f^2-m^2g^2+n^2h^2)
\end{array}\right\} = 2h^2 \left\{\begin{array}{l}
6i^2j^2 \\
+6i^2j^2 \\
+\ i^2\ (\ i^2-j^2-k^2) \\
+\ j^2\ (-i^2+j^2-k^2) \\
+\ k^2(-i^2-j^2+k^2) \\
+\ i^2\ (\ i^2-j^2-k^2) \\
+\ j^2\ (-i^2+j^2-k^2) \\
+(i^2+j^2-4ij)\,(-i^2-j^2+k^2)
\end{array}\right\},$$

les lettres i, j, k étant introduites pour désigner les produits

$$lf = i, \quad mg = j, \quad nh = k.$$

Après toutes les réductions on obtient

$$\mathfrak{C} = 2h^2\left((i+j)^2-k^2\right)^2, \ = 2h^2(i+j+k)^2(-i-j+k)^2, \ = 2h^2\theta^2(-i-j+k)^2,$$

pour le coefficient de p^2q^2 dans $\theta^4 U$, ou, ce qui est la même chose,

$$h^2(-i-j+k)^2$$

pour le coefficient de p^2q^2 dans $\frac{1}{2}\theta^2 U$.

En calculant de même les autres coefficients de $\frac{1}{2}\theta^2 U$ et en écrivant pour abréger

$$\begin{aligned}
i-j-k &= \ \ lf-mg-nh = a, \\
-i+j-k &= -lf+mg-nh = b, \\
-i-j+k &= -lf-mg+nh = c,
\end{aligned}$$

l'équation transformée sera

$$\begin{aligned}
& f^2a^2q^2r^2 + g^2b^2r^2p^2 + h^2c^2p^2q^2 + l^2a^2p^2s^2 + m^2b^2q^2s^2 + n^2c^2r^2s^2 \\
& + 2ghbcp^2qr + 2hmbcpq^2s - 2gnbcpr^2s - 2mnbcqrs^2 \\
& + 2hfcapq^2r + 2fncaqr^2s - 2hlcaqp^2s - 2nlcarps^2 \\
& + 2fgabpqr^2 + 2glabrp^2s - 2fmabrq^2s - 2lmabpqs^2 \\
& \qquad - (b-c)(c-a)(a-b)\,pqrs = 0.
\end{aligned}$$

En posant

$$\begin{aligned}
& a' = fa, \quad d' = la, \quad -4g' = (b-c)(c-a)(a-b), \\
& b' = gb, \quad e' = mb, \\
& c' = hc, \quad f' = nc,
\end{aligned}$$

les quantités a', b', c', d', e', f', g' sont liées par une relation. Pour en prouver l'existence on n'a qu'à faire

$$a'd' = \alpha', \quad b'e' = \beta', \quad c'f' = \gamma'$$

et à se servir des expressions de a, b, c en f, g, h, l, m, n, alors on obtient

$$\begin{aligned} -2\alpha' &= a^2(b+c), \\ -2\beta' &= b^2(c+a), \\ -2\gamma' &= c^2(a+b), \\ -4g' &= (b-c)(c-a)(a-b), \end{aligned}$$

équations qui impliquent une relation entre α', β', γ', g'; mais en supposant que a', b', c', d', e', f', g' soient des quantités qui satisfont à cette relation, il existe toujours des valeurs correspondantes de f, g, h, l, m, n, c'est-à-dire que l'équation du tétraédroïde est identique avec celle de M. Kummer toutes les fois que les coefficients a, b, c, d, e, f, g de cette dernière sont liés par une certaine relation. Ecrivons comme auparavant $ad = \alpha$, $be = \beta$, $cf = \gamma$, cette relation se trouve en éliminant a, b, c entre les équations

$$\begin{aligned} -2\alpha &= a^2(b+c), \\ -2\beta &= b^2(c+a), \\ -2\gamma &= c^2(a+b), \\ -4g &= (b-c)(c-a)(a-b), \end{aligned}$$

et il ne s'agit que de prouver l'identité de cette relation avec celle que nous avons trouvée ci-dessus par d'autres considérations.

J'introduis les nouvelles notations

$$\begin{aligned} \alpha + \beta + \gamma &= -\tfrac{1}{2}P, & a + b + c &= \mathfrak{p}, \\ \beta\gamma + \gamma\alpha + \alpha\beta &= -\tfrac{1}{4}Q, & bc + ca + ab &= \mathfrak{q}, \\ \alpha\beta\gamma &= -\tfrac{1}{8}R, & abc &= \mathfrak{r}, \end{aligned}$$

je forme l'expression

$$2(\beta - \gamma) = -(bc + ca + ab)(b - c), \; = -\mathfrak{q}(b - c),$$

et les deux expressions analogues pour $2(\gamma - \alpha)$, $2(\alpha - \beta)$; j'en déduis le résultat

$$8(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta) = -\mathfrak{q}^3(b - c)(c - a)(a - b);$$

enfin je note les équations

$$\begin{aligned} (b-c)^2(c-a)^2(a-b)^2 &= -4\mathfrak{q}^3 + \mathfrak{p}^2\mathfrak{q}^2 + 18\mathfrak{p}\mathfrak{q}\mathfrak{r} - 27\mathfrak{r}^2 - 4\mathfrak{p}^3\mathfrak{r}, \\ P &= \mathfrak{p}\mathfrak{q} - 3\mathfrak{r}, \\ Q &= \mathfrak{q}^3 - 2\mathfrak{p}\mathfrak{q}\mathfrak{r} + 3\mathfrak{r}^2, \\ R &= \mathfrak{p}\mathfrak{q}\mathfrak{r}^2 - \mathfrak{r}^3, \end{aligned}$$

qui donnent la transformation de leurs premiers membres en fonction de $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$; cela posé, et à l'aide de ces valeurs, on forme les égalités

$$\begin{aligned}
512(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)g^3 &= (-\ 4\mathfrak{q}^3+\mathfrak{p}^3\mathfrak{q}^2+18\mathfrak{p}\mathfrak{q}\mathfrak{r}-27\mathfrak{r}^2-4\mathfrak{p}^3\mathfrak{r})\mathfrak{q}^3(b-c)^2(c-a)^2(a-b)^2,\\
256(-\Sigma\alpha^3\beta+4\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma)g^2 &= (\ \ 6\mathfrak{q}^3-\mathfrak{p}^3\mathfrak{q}^2-18\mathfrak{p}\mathfrak{q}\mathfrak{r}+27\mathfrak{r}^2+2\mathfrak{p}^3\mathfrak{r})\mathfrak{q}^3(b-c)^2(c-a)^2(a-b)^2,\\
128(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\Sigma\alpha^2-10\Sigma\alpha\beta)g &= (-12\mathfrak{q}^3+\mathfrak{p}^3\mathfrak{q}^2+18\mathfrak{p}\mathfrak{q}\mathfrak{r}-27\mathfrak{r}^2\qquad)\mathfrak{q}^3(b-c)^2(c-a)^2(a-b)^2,\\
64(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 &= (\qquad\mathfrak{q}^3\qquad\qquad\qquad\qquad\qquad\qquad)\mathfrak{q}^3(b-c)^2(c-a)^2(a-b)^2,
\end{aligned}$$

lesquelles, multipliées par 1, 2, 1, 4, ajoutées ensemble et divisées par 128, conduisent à l'équation finale

$$\begin{aligned}
&\ \ g^3.4(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\\
&+g^2.4(-\Sigma\alpha^3\beta+4\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma)\\
&+g\ .\ \ (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\Sigma\alpha^2-10\Sigma\alpha\beta)\\
&+\qquad 2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=0,
\end{aligned}$$

identique avec celle que l'on a trouvée ci-dessus, ce qui achève la démonstration que l'on avait en vue.

Cambridge, 18 *Mai*, 1865.

357.

A SUPPLEMENTARY MEMOIR ON THE THEORY OF MATRICES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVI. (for the year 1866), pp. 25—35. Received October 24,—Read December 7, 1865.]

M. HERMITE, in a paper "Sur la théorie de la transformation des fonctions Abéliennes," *Comptes Rendus*, t. XL. (1855), pp. 249, &c., establishes incidentally the properties of the matrix for the automorphic linear transformation of the bipartite quadric function $xw' + yz' - zy' - wx'$, or transformation of this function into one of the like form, $XW' + YZ' - ZY' - WX'$. These properties are (as will be shown) deducible from a general formula in my "Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function," *Phil. Trans.* vol. CXLVIII. (1858), pp. 39—46, [153]; but the particular case in question is an extremely interesting one, the theory whereof is worthy of an independent investigation. For convenience the number of variables is taken to be *four*; but it will be at once seen that as well the demonstrations as the results are in fact applicable to any *even* number whatever of variables.

Article Nos. 1 and 2. *Notation and Remarks.*

1. I use throughout the notation and formulæ contained in my "Memoir on the Theory of Matrices," *Phil. Trans.* vol. CXLVIII. (1858), pp. 17—37, [152], and in the above-mentioned memoir on the Automorphic Transformation. With respect to the composition of matrices, the rule of composition is as follows, viz., any *line* of the compound matrix is obtained by combining the corresponding *line* of the first or further component matrix with the several *columns* of the second or nearer component matrix; it is very convenient to indicate this by the algorithm,

$$\begin{array}{ccc|c} & & & (\alpha, \alpha', \alpha''), (\beta, \beta', \beta''), (\gamma', \gamma', \gamma'') \\ \hline \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'' & b'', & c'' \end{array}\right) \between \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right) & = & \begin{array}{c} (a, b, c) \\ (a', b', c') \\ (a'', b'', c'') \end{array} & \begin{array}{ccc} \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \end{array} \end{array}$$

which exhibits very clearly the terms which are to be combined together; thus in the upper left-hand corner we have $(a, b, c \between \alpha, \alpha', \alpha'')$, and so for the other places in the compound matrix.

2. It is not in the Memoir on Matrices explicitly remarked, but it is easy to see that sums of matrices, all the matrices being of the same order, may be multiplied together by the ordinary rule; thus

$$(A+B)(C+D) = AC + AD + BC + BD:$$

this remark will be useful in the sequel.

Article Nos. 3 to 13. *First Investigation.*

3. We have to consider the formulæ for the automorphic linear transformation of the function $xw' + yz' - zy' - wx'$, that is, of the function

$$\left(\begin{array}{cccc} 0, & 0, & 0, & -1 \\ 0, & 0, & -1, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \end{array}\right) \between x, y, z, w \between x', y', z', w')$$

$$= (\Omega \between x, y, z, w \between x', y', z', w'),$$

viz., if the variables are transformed by the formulæ

$$(x, y, z, w) = (\Pi \between X, Y, Z, W),$$
$$(x', y', z', w') = (\Pi \between X', Y', Z', W'),$$

then the matrix (Π) is such that we have identically

$$(\Omega \between x, y, z, w \between x', y', z', w') = (\Omega \between X, Y, Z, W \between X', Y', Z', W');$$

the expression for (Π) is given in my memoir [153] above referred to; viz. observing that the matrix (Ω) is skew symmetrical, then (No. 13) we have

$$\Pi = \Omega^{-1}(\Omega - \Upsilon)(\Omega + \Upsilon)^{-1}\Omega,$$

where Υ is an arbitrary symmetrical matrix.

4. I propose to compare with the matrix Π the inverse matrix Π^{-1}. Recollecting that in the theory of matrices $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$, we have

$$\Pi^{-1} = \Omega^{-1}(\Omega + \Upsilon)(\Omega - \Upsilon)^{-1}\Omega;$$

and it is to be shown that Π and Π^{-1} are composed of terms which (except as to their signs) are the same in each, so that either of these matrices is derivable from the other by a peculiar form of transposition. It is to be borne in mind throughout that Υ is symmetrical, Ω skew symmetrical.

5. I write for greater convenience

$$-\Pi = \Omega^{-1}(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1}\Omega,$$
$$-\Pi^{-1} = \Omega^{-1}(\Upsilon+\Omega)(\Upsilon-\Omega)^{-1}\Omega,$$

and I compare in the first instance the matrices $(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1}$ and $(\Upsilon+\Omega)(\Upsilon-\Omega)^{-1}$.

6. Any matrix whatever, and therefore the matrix $(\Upsilon+\Omega)^{-1}$, may be exhibited as the sum of a symmetrical matrix and a skew symmetrical matrix; that is, we may write

$$(\Upsilon+\Omega)^{-1} = \Upsilon' + \Omega',$$

where Υ' is symmetrical, Ω' is skew symmetrical. We have then

$$(\Upsilon+\Omega)(\Upsilon+\Omega)^{-1} = (\Upsilon+\Omega)(\Upsilon'+\Omega'), = 1,$$

where, here and in what follows, 1 denotes the matrix unity. Moreover

$$\Upsilon-\Omega = \text{tr.}\ (\Upsilon+\Omega),$$

and thence

$$(\Upsilon-\Omega)^{-1} = \left(\text{tr.}\,(\Upsilon+\Omega)\right)^{-1} = \text{tr.}\ (\Upsilon+\Omega)^{-1} = \text{tr.}\ (\Upsilon'+\Omega') = \Upsilon'-\Omega';$$

that is

$$(\Upsilon-\Omega)^{-1} = \Upsilon'-\Omega';$$

and thence also

$$(\Upsilon-\Omega)(\Upsilon-\Omega)^{-1} = (\Upsilon-\Omega)(\Upsilon'-\Omega') = 1.$$

We have therefore

$$(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1} = (\Upsilon+\Omega-2\Omega)(\Upsilon'+\Omega') = 1-2\Omega(\Upsilon'+\Omega'),$$
$$(\Upsilon+\Omega)(\Upsilon-\Omega)^{-1} = (\Upsilon-\Omega+2\Omega)(\Upsilon'-\Omega') = 1+2\Omega(\Upsilon'-\Omega').$$

7. Suppose for a moment that

$$\Upsilon'+\Omega' = \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix}$$

and therefore

$$\Upsilon'-\Omega' = \begin{pmatrix} a, & e, & i, & m \\ b, & f, & j, & n \\ c, & g, & k, & o \\ d, & h, & l, & p \end{pmatrix}.$$

8. We have

$$-\Omega(\Upsilon'+\Omega') = \left(\begin{array}{cccc} . & . & . & 1 \\ . & . & 1 & . \\ . & -1 & . & . \\ -1 & . & . & . \end{array}\right)\left(\begin{array}{cccc} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{array}\right)$$

$$= \begin{array}{c} \left(\begin{array}{cccc} . & . & . & 1 \end{array}\right) \\ \left(\begin{array}{cccc} . & . & 1 & . \end{array}\right) \\ \left(\begin{array}{cccc} . & -1 & . & . \end{array}\right) \\ \left(\begin{array}{cccc} -1 & . & . & . \end{array}\right) \end{array} \left|\begin{array}{cccc} (a, e, i, m), & (b, f, j, n), & (c, g, k, o), & (d, h, l, p) \\ \text{"} & \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} & \text{"} \end{array}\right.$$

$$= \left(\begin{array}{cccc} m, & n, & o, & p \\ i, & j, & k, & l \\ -e, & -f, & -g, & -h \\ -a, & -b, & -c, & -d \end{array}\right).$$

9. And similarly,

$$\Omega(\Upsilon'-\Omega') = \left(\begin{array}{cccc} . & . & . & -1 \\ . & . & -1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{array}\right)\left(\begin{array}{cccc} a, & e, & i, & m \\ b, & f, & j, & n \\ c, & g, & k, & o \\ d, & h, & l, & p \end{array}\right)$$

$$= \begin{array}{c} \left(\begin{array}{cccc} . & . & . & -1 \end{array}\right) \\ \left(\begin{array}{cccc} . & . & -1 & . \end{array}\right) \\ \left(\begin{array}{cccc} . & 1 & . & . \end{array}\right) \\ \left(\begin{array}{cccc} 1 & . & . & . \end{array}\right) \end{array} \left|\begin{array}{cccc} (a, b, c, d), & (e, f, g, h), & (i, j, k, l), & (m, n, o, p) \\ \text{"} & \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} & \text{"} \end{array}\right.$$

$$= \left(\begin{array}{cccc} -d, & -h, & -l, & -p \\ -c, & -g, & -k, & -o \\ b, & f, & j, & n \\ a, & e, & i, & m \end{array}\right).$$

10. Hence also

$$(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1} = \left(\begin{array}{cccc} 1+2m, & 2n, & 2o, & 2p \\ 2i, & 1+2j, & 2k, & 2l \\ -2e, & -2f, & 1-2g, & -2h \\ -2a, & -2b, & -2c, & 1-2d \end{array}\right),$$

and

$$(\Upsilon+\Omega)(\Upsilon-\Omega)^{-1}=\begin{pmatrix} 1-2d, & -2h, & -2l, & -2p \\ -2c, & 1-2g, & -2k, & -2o \\ 2b, & 2f, & 1+2j, & 2n \\ 2a, & 2e, & 2i, & 1+2m \end{pmatrix},$$

so that these matrices are composed of terms which, except as to the signs, are the same in each.

11. Now in general if

$$\Theta=\begin{pmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ \alpha'', & \beta'', & \gamma'', & \delta'' \\ \alpha''', & \beta''', & \gamma''', & \delta''' \end{pmatrix},$$

then it is easy to see that

$$\Omega^{-1}\Theta\Omega=\begin{pmatrix} \delta''', & \gamma''', & -\beta''', & -\alpha''' \\ \delta'', & \gamma'', & -\beta'', & -\alpha'' \\ -\delta', & -\gamma', & \beta', & \alpha' \\ -\delta, & -\gamma, & \beta, & \alpha \end{pmatrix},$$

and hence, from the foregoing values of $(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1}$ and $(\Upsilon+\Omega)(\Omega-\Upsilon)^{-1}$, we find

$$\Pi=-\Omega^{-1}(\Upsilon-\Omega)(\Upsilon+\Omega)^{-1}\Omega=\begin{pmatrix} -1+2d, & 2c, & -2b, & -2a \\ 2h, & -1+2g, & -2f, & -2e \\ 2l, & 2k, & -1-2j, & -2i \\ 2p, & 2o, & -2n, & -1-2m \end{pmatrix},$$

and

$$\Pi^{-1}=-\Omega^{-1}(\Upsilon+\Omega)(\Upsilon-\Omega)^{-1}\Omega=\begin{pmatrix} -1-2m, & -2i, & 2e, & 2a \\ -2n, & -1-2j, & 2f, & 2b \\ -2o, & -2k, & -1+2g, & 2c \\ -2p, & -2l, & +2h, & -1+2d \end{pmatrix};$$

this shows that the matrix Π for the automorphic transformation of the function $xw'+yz'-zy'-wx'$ is such that writing

$$\Pi=\begin{pmatrix} A, & B, & C, & D \\ E, & F, & G, & H \\ I, & J, & K, & L \\ M, & N, & O, & P \end{pmatrix} \text{ we have } \Pi^{-1}=\begin{pmatrix} P, & L, & -H, & -D \\ O, & K, & -G, & -C \\ -N, & -J, & F, & B \\ -M, & -I, & E, & A \end{pmatrix},$$

which is the theorem in question.

12. I remark in reference to the foregoing proof that writing

$$\Upsilon = \begin{pmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{pmatrix}$$

then the actual value of

$$(\Upsilon + \Omega)^{-1}, = \begin{pmatrix} a & , & h & , & g & , & l-1 \\ h & , & b & , & f-1, & & m \\ g & , & f+1, & & c & , & n \\ l+1, & & m & , & n & , & d \end{pmatrix}^{-1}$$

is

$$= \frac{1}{\Delta} \begin{pmatrix} A+d & , & H+n+\nu, & G-m-\mu, & L-l+\rho \\ H+n-\nu, & & B+c & , & F-f+\lambda, & M-g+\sigma \\ G-m+\mu, & & F-f-\lambda, & & C+b & , & N+h+\tau \\ L-l-\rho & , & M-g-\sigma, & & N+h-\tau, & & D+a \end{pmatrix}$$

where

$$\begin{pmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{pmatrix}$$

is the matrix formed with the first minors of

$$\begin{pmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{pmatrix};$$

moreover

$$\begin{aligned} \lambda &= ad - l^2 + nh - mg + 1, & \rho &= bc - f^2 + nh - mg + 1, \\ \mu &= bn - mf + dh - ml \quad , & \sigma &= fg - ch + gl - na \quad , \\ \nu &= dg - nl + nf - cm \quad , & \tau &= hf - bg + ma - lh \quad , \end{aligned}$$

and Δ is the determinant

$$\begin{pmatrix} a & , & h & , & g & , & l+1 \\ h & , & b & , & f+1, & & m \\ g & , & f-1, & & c & , & n \\ l-1, & & m & , & n & , & d \end{pmatrix}$$

viz., this is

$$= \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} + ad - l^2 + bc - f^2 + 2(nh - mg) + 1.$$

13. The expression for $(\Upsilon - \Omega)^{-1}$ is obtained from that of $(\Upsilon + \Omega)^{-1}$ by merely transposing the terms of the matrix, or, what is the same thing, by changing the signs of λ, μ, ν, ρ, σ, τ. And it would be easy by means of these developed values to verify the foregoing comparison of $(\Upsilon - \Omega)(\Upsilon + \Omega)^{-1}$ and $(\Upsilon + \Omega)(\Upsilon - \Omega)^{-1}$.

Article Nos. 14 to 22. *Second Investigation.*

14. I consider from a different point of view the theory of a matrix

$$\Pi = \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix} \text{ such that } \Pi^{-1} = \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix},$$

or, as we may call it, a Hermitian matrix.

15. *Lemma.* The determinant

$$\nabla = \begin{vmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{vmatrix}$$

may be expressed, and that in two different ways, as a Pfaffian.

16. In fact multiplying the determinant into itself thus,

$$\nabla^2 = \begin{vmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{vmatrix} \text{ tr. } \begin{vmatrix} d, & c, & -b, & -a \\ h, & g, & -f, & -e \\ l, & k, & -j, & -i \\ p, & o, & -n, & -m \end{vmatrix},$$

we find

$$\nabla^2 = \begin{array}{r|cccc} & (d, c, -b, -a), & (h, g, -f, -e), & (l, k, -j, -i), & (p, o, -n, -m) \\ \hline (a, b, c, d) & \text{"} & \text{"} & \text{"} & \text{"} \\ (e, f, g, h) & \text{"} & \text{"} & \text{"} & \text{"} \\ (i, j, k, l) & \text{"} & \text{"} & \text{"} & \text{"} \\ (m, n, o, p) & \text{"} & \text{"} & \text{"} & \text{"} \end{array} = \begin{vmatrix} s_{11}, & s_{12}, & s_{13}, & s_{14} \\ s_{21}, & s_{22}, & s_{23}, & s_{24} \\ s_{31}, & s_{32}, & s_{33}, & s_{34} \\ s_{41}, & s_{42}, & s_{43}, & s_{44} \end{vmatrix},$$

viz. we have $s_{11}=(a, b, c, d \between d, c, -b, -a)$, $s_{12}=(a, b, c, d \between h, g, -f, -e)$, &c.: we see at once that $s_{11}=0$, $s_{12}+s_{21}=0$, &c., viz. the determinant in s is a skew determinant, that is, the square of a Pfaffian. We have therefore

$$\nabla^2=(s_{12}\,s_{34}+s_{13}\,s_{42}+s_{14}\,s_{23})^2,$$

or extracting the square root of each side, and determining the sign by a comparison of any single term, we have

$$\nabla \;=s_{12}\,s_{34}+s_{13}\,s_{42}+s_{14}\,s_{23},$$

which is one of the required forms of ∇.

17. And in the same manner

$$\nabla^2=\text{tr.}\begin{vmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{vmatrix}\cdot\begin{vmatrix} m, & n, & o, & p \\ i, & j, & k, & l \\ -e, & -f, & -g, & -h \\ -a, & -b, & -c, & -d \end{vmatrix},$$

which is equal to the determinant

$$\begin{array}{c|c|cccc}
 & & (m, i, -e, -a), & (n, j, -f, -b), & (o, k, -g, -c), & (p, l, -h, -d) \\ \hline
\begin{vmatrix} t_{11}, & t_{12}, & t_{13}, & t_{14} \end{vmatrix} & =(a, e, i, m) & \text{"} & \text{"} & \text{"} & \text{"} \\
\begin{vmatrix} t_{21}, & t_{22}, & t_{23}, & t_{24} \end{vmatrix} & (b, f, j, n) & \text{"} & \text{"} & \text{"} & \text{"} \\
\begin{vmatrix} t_{31}, & t_{32}, & t_{33}, & t_{34} \end{vmatrix} & (c, g, k, o) & \text{"} & \text{"} & \text{"} & \text{"} \\
\begin{vmatrix} t_{41}, & t_{42}, & t_{43}, & t_{44} \end{vmatrix} & (d, h, l, p) & \text{"} & \text{"} & \text{"} & \text{"}
\end{array}$$

viz. $t_{11}=(a, e, i, m \between m, i, -e, -a)$, &c.; this is likewise a skew determinant, and we have

$$\nabla^2=(t_{12}\,t_{34}+t_{13}\,t_{42}+t_{14}\,t_{23})^2,$$

or extracting the square root of each side, and determining the sign by the comparison of any single term, we have

$$\nabla \;=t_{12}\,t_{34}+t_{13}\,t_{42}+t_{14}\,t_{23},$$

which is the other of the required forms of ∇.

18. Consider now the matrix

$$\begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix}$$

which is such that

$$\begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix}^{-1} = \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix};$$

this gives

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} = \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix} \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix}$$

$$= \begin{array}{c|cccc} & (p, o, -n, -m), & (l, k, -j, -i), & (-h, -g, f, e), & (-d, -c, b, a) \\ \hline (a, b, c, d) & \text{"} & \text{"} & \text{"} & \text{"} \\ (e, f, g, h) & \text{"} & \text{"} & \text{"} & \text{"} \\ (i, j, k, l) & \text{"} & \text{"} & \text{"} & \text{"} \\ (m, n, o, p) & \text{"} & \text{"} & \text{"} & \text{"} \end{array}$$

which is in fact

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} = \begin{pmatrix} s_{14}, & s_{13}, & -s_{12}, & -s_{11} \\ s_{24}, & s_{23}, & -s_{22}, & -s_{21} \\ s_{34}, & s_{33}, & -s_{32}, & -s_{31} \\ s_{44}, & s_{43}, & -s_{42}, & -s_{41} \end{pmatrix},$$

and the two matrices will be equal, term by term, if only

$$1 = s_{14} = s_{23},\ 0 = s_{13} = s_{12} = s_{24} = s_{34},$$

that is, if six conditions are satisfied.

19. But we have also (a matrix and its reciprocal being convertible)

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} = \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix} \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix}$$

$$= \begin{array}{c|cccc} & (a, e, i, m), & (b, f, j, n), & (c, g, k, o), & (d, h, l, p) \\ \hline (p, l, -h, -d) & \text{"} & \text{"} & \text{"} & \text{"} \\ (o, k, -g, -c) & \text{"} & \text{"} & \text{"} & \text{"} \\ (-n, -j, f, b) & \text{"} & \text{"} & \text{"} & \text{"} \\ (-m, -i, e, a) & \text{"} & \text{"} & \text{"} & \text{"} \end{array}$$

which is in fact

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 1, & 1 \end{pmatrix} = \begin{pmatrix} t_{14}, & t_{24}, & t_{34}, & t_{44} \\ t_{13}, & t_{23}, & t_{33}, & t_{43} \\ -t_{12}, & -t_{22}, & -t_{32}, & -t_{42} \\ -t_{11}, & -t_{21}, & -t_{31}, & -t_{41} \end{pmatrix},$$

and we obtain for the equality of the two matrices the six conditions

$$1 = t_{14} = t_{23}, \quad 0 = t_{13} = t_{12} = t_{24} = t_{34},$$

equivalent to the former set of six conditions.

20. We obtain from either set of conditions, for the determinant the value

$$\nabla = \begin{vmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{vmatrix} = \kappa^2.$$

21. Write

$$(x, y, z, w) = \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix} \between (X, Y, Z, W); \quad (x', y', z', w') = \begin{pmatrix} a, & b, & c, & d \\ e, & f, & g, & h \\ i, & j, & k, & l \\ m, & n, & o, & p \end{pmatrix} \between (X', Y', Z', W'),$$

then substituting for (x, y, z, w) (x', y', z', w') their values, we find

$$xw' + yz' - zy' - wx' = -\begin{pmatrix} t_{11}, & t_{12}, & t_{13}, & t_{14} \\ t_{21}, & t_{22}, & t_{23}, & t_{24} \\ t_{31}, & t_{32}, & t_{33}, & t_{34} \\ t_{41}, & t_{42}, & t_{43}, & t_{44} \end{pmatrix} \between (X, Y, Z, W) \between (X', Y', Z', W'),$$

$$= \begin{pmatrix} . & . & . & -1 \\ . & . & -1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix} \between (X, Y, Z, W) \between (X', Y', Z', W'),$$

$$= XW' + YZ' - ZY' - WX';$$

and similarly writing

$$(X, Y, Z, W) = \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix} \between (x, y, z, w); \quad (X', Y', Z', W') = \begin{pmatrix} p, & l, & -h, & -d \\ o, & k, & -g, & -c \\ -n, & -j, & f, & b \\ -m, & -i, & e, & a \end{pmatrix} \between (x', y', z', w'),$$

we obtain with the s coefficients the equivalent result,

$$XW' + YZ' - ZY' - WX' = xw' + yz' - zy' - wx'.$$

We thus see conversely that the Hermitian matrix is in fact the matrix for the automorphic transformation of the function $xw' + yz' - zy' - wx'$.

22. Considering any two or more matrices for the automorphic transformation of such a function, the matrix compounded of these is a matrix for the automorphic transformation of the function—or, theorem, the matrix compounded of two or more Hermitian matrices is itself Hermitian.

Article No. 23. *Theorem on a Form of Matrices.*

23. I take the opportunity of mentioning a theorem relating to the matrices which present themselves in the arithmetical theory of the composition of quadratic forms. Writing

$$(X) = \begin{pmatrix} . & , & \alpha & , & a & , & b+\beta \\ -\alpha & , & . & , & b-\beta, & & c \\ -a & , & -(b-\beta), & & . & , & \gamma \\ -(b+\beta), & & -c & , & -\gamma, & & . \end{pmatrix} \text{ and } \therefore (X)^{-1} = \frac{1}{D-\Delta}\begin{pmatrix} . & , & \gamma & , & -c & , & b-\beta \\ -\gamma & , & . & , & b+\beta, & & -a \\ c & , & -(b+\beta), & & . & , & \alpha \\ -(b-\beta), & & a & , & -\alpha, & & . \end{pmatrix}$$

where $D = ac - b^2$, $\Delta = \alpha\gamma - \beta^2$; and similarly,

$$X' = \begin{pmatrix} . & , & \alpha' & , & a' & , & b'+\beta' \\ -\alpha' & , & . & , & b'-\beta', & & c' \\ -a' & , & -(b'-\beta'), & & . & , & \gamma' \\ -(b'+\beta'), & & -c' & , & -\gamma', & & . \end{pmatrix} \text{ and } \therefore (X')^{-1} = \frac{1}{D'-\Delta'}\begin{pmatrix} . & , & \gamma' & , & -c' & , & b'-\beta' \\ -\gamma' & , & . & , & b'+\beta', & & -a' \\ c' & , & -(b'+\beta'), & & . & , & \alpha' \\ -(b'-\beta'), & & a' & , & -\alpha, & & . \end{pmatrix}$$

where $D' = a'c' - b'^2$, $\Delta' = \alpha'\gamma' - \beta'^2$; then

$$(X \between X') + (D-\Delta)(D'-\Delta')(X'^{-1} \between X^{-1}),$$

or, what is the same thing,

$$(X \between X') + (D-\Delta)(D'-\Delta')\big((X \between X')\big)^{-1}$$

is to a factor *près* equal to the matrix unity; viz. writing

$$\Lambda = a\alpha + 2b\beta + c\gamma + a'\alpha' + 2b'\beta' + c'\gamma',$$

the foregoing expression is

$$= \Lambda \begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}.$$

The theorem is verified without difficulty by merely forming the expressions of the compound matrices $(X \between X')$ and $(X'^{-1} \between X^{-1})$.

358.

ADDITION TO THE MEMOIR ON TSCHIRNHAUSEN'S TRANSFORMATION.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVI. (for the year 1866), pp. 97—100. Received October 24,—Read December 7, 1866.]

IN the memoir "On Tschirnhausen's Transformation," *Philosophical Transactions*, vol. CLII. (1862), pp. 561—568, [275], I considered the case of a quartic equation: viz. it was shown that the equation

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0$$

is, by the substitution

$$y = (ax+b)\,B + (ax^2+4bx+3c)\,C + (ax^3+4bx^2+6cx+3d)\,D,$$

transformed into

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4 = 0$$

where $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ have certain given values. It was further remarked that $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ were expressible in terms of U', H', Φ', invariants of the two forms $(a, b, c, d, e \between X, Y)^4$, $(B, C, D \between Y, -X)^2$, of I, J, the invariants of the first, and of Θ', $= BD - C^2$, the invariant of the second of these two forms, viz. that we have

$$\mathfrak{C} = 6H' - 2I\Theta',$$

$$\mathfrak{D} = 4\Phi',$$

$$\mathfrak{E} = IU'^3 - 3H'^2 + I^2\Theta'^2 + 12J'\Theta'U' + 2I'\Theta'H';$$

and by means of these I obtained an expression for the quadrinvariant of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4;$$

viz. this was found to be

$$= IU'^2 + \tfrac{4}{3}I^2\Theta'^2 + 12J\Theta'U'.$$

But I did not obtain an expression for the cubinvariant of the same function: such expression, it was remarked, would contain the square of the invariant Φ'; it was probable that there existed an identical equation,

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2,$$

which would serve to express Φ'^2 in terms of the other invariants; but, assuming that such an equation existed, the form of the factor M remained to be ascertained; and until this was done, the expression for the cubinvariant could not be obtained in its most simple form. I have recently verified the existence of the identical equation just referred to, and have obtained the expression for the factor M; and with the assistance of this identical equation I have obtained the expression for the cubinvariant of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\,\mathfrak{Y}y,\ 1)^4.$$

The expression for the quadrinvariant was, as already mentioned, given in the former memoir: I find that the two invariants are in fact the invariants of a certain linear function of U, H; viz. the linear function is $= U'U + \frac{2}{3}\Theta'H$; so that, denoting by I^*, J^*, the quadrinvariant and the cubinvariant respectively of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\,\mathfrak{Y}y,\ 1)^4,$$

we have

$$I^* = \tilde{I}\,(U'U + 4\Theta'H),$$
$$J^* = \tilde{J}\,(U'U + 4\Theta'H),$$

where $\tilde{I}$, $\tilde{J}$ signify the functional operations of forming the two invariants respectively. The function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\,\mathfrak{Y}y,\ 1)^4$, obtained by the application of Tschirnhausen's transformation to the equation

$$(a,\ b,\ c,\ d,\ e\,\mathfrak{Y}x,\ 1)^4 = 0,$$

has thus the *same invariants* with the function

$$U'U + 4\Theta'H = U'\,(a, b, c, d, e\,\mathfrak{Y}x,\ 1)^4 + 4\Theta'\,(ac - b^2,\ ad - bc,\ ae + 2bd - 3c^2,\ be - cd,\ ce - d^2\,\mathfrak{Y}x,\ 1)^4,$$

and it is consequently a linear transformation of the last-mentioned function; so that the application of Tschirnhausen's transformation to the equation $U = 0$ gives an equation linearly transformable into, and thus virtually equivalent to, the equation

$$U'U + 4\Theta'H = 0,$$

which is an equation involving the single parameter $\frac{4\Theta'}{U'}$: this appears to me a result of considerable interest. It is to be remarked that Tschirnhausen's transformation, wherein y is put equal to a rational and integral function of the order $n-1$ (if n be the order of the equation in x), is not really less general than the transformation wherein y is put equal to any rational function $\frac{V}{W}$ whatever of x; such rational function may, in fact, by means of the given equation in x, be reduced to a rational and integral function of the order $n-1$; hence in the present case, taking V, W to

be respectively of the order $n-1$, $=3$, it follows that the equation in y obtained by the elimination of x from the equations

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0,$$

$$y = \frac{(\alpha,\ \beta,\ \gamma,\ \delta \between x,\ 1)^3}{(\alpha',\ \beta',\ \gamma',\ \delta' \between x,\ 1)^3},$$

is a mere linear transformation of the equation $AU + BH = 0$, where A, B are functions (not as yet calculated) of $(a,\ b,\ c,\ d,\ e,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \alpha',\ \beta',\ \gamma',\ \delta')$.

Article Nos. 1, 2, 3. *Investigation of the identical equation*

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2.$$

1. It is only necessary to show that we have such an equation, M being an invariant, in the particular case $a = e = 1$, $b = d = 0$, $c = \theta$, that is for the quartic function $(1,\ 0,\ \theta,\ 0,\ 1 \between x,\ 1)^4$; for, this being so, the equation will be true in general. Writing the equation in the form

$$-M\Theta' = U'^2(JU' - IH') + 4H'^3 + \Phi'^2,$$

and observing that we have

$$\begin{aligned}
U' &= (B^2 + D^2) + 2\theta BD + 4\theta C^2,\\
H' &= \theta(B^2 + D^2) + (1 + \theta^2)BD - 4\theta^2C^2,\\
\Theta' &= BD - C^2,\\
\Phi' &= (1 - 9\theta^2)C(B^2 - D^2),\\
I &= 1 + 3\theta^2,\\
J &= \theta - \theta^3,
\end{aligned}$$

and thence

$$JU' - IH' = -4\theta^3(b^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4)BD + (8\theta^2 + 8\theta^4)C^2,$$

the equation becomes

$$\begin{aligned}
-(BD - C^2)M = {} & \\
& [-4\theta^3(B^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4)BD + (8\theta^2 + 8\theta^4)C^2] \times \{B^2 + D^2 + 2\theta BD + 4\theta C^2\}^2\\
& + 4\{\theta(B^2 + D^2) + (1 + \theta^2)BD - 4\theta^2C^2\}^2\\
& + \ (1 - 9\theta^2)^2C^2\{(B^2 + D^2)^2 - 4B^2D^2\}.
\end{aligned}$$

2. It is found by developing that the right-hand side is in fact divisible by $BD - C^2$, and that the quotient is

$$\begin{aligned} = \quad & (-1 + 10\theta^2 - 9\theta^4)(B^2 + D^2)^2 \\ & + (8\theta + 16\theta^3 - 24\theta^5)(B^2 + D^2)\,BD \\ & + (4 + 8\theta^2 + 4\theta^4 - 16\theta^6)\,B^2D^2 \\ & + (-64\theta^3 - 192\theta^5)(B^2 + D^2)\,C^2 \\ & + (16\theta^2 - 416\theta^4 - 112\theta^6)\,BDC^2 \\ & + (-128\theta^4 + 128\theta^6)\,C^4. \end{aligned}$$

3. This is found to be

$$\begin{aligned} = & -I^2U'^2 + 12JU'H' + 4IH'^2 \\ & - 8IJU'\Theta' \\ & - 16J^2\Theta'^2, \end{aligned}$$

which is consequently the value of $-M$. We have therefore

$$\begin{aligned} -\Phi'^2 = \quad & JU'^3 - IU'^2H' + 4H'^3 \\ + \quad & (I^2U'^2 - 12JU'H' - 4IH'^2)\,\Theta' \\ + \quad & 8IJU'\Theta'^2 \\ + \quad & 16J^2\Theta'^3, \end{aligned}$$

which is the required identical equation.

Article No. 4. *Calculation of the Cubinvariant.*

4. We have

$$\begin{aligned} J^* = & \tfrac{1}{6}\,\mathfrak{C}\,.\,\mathfrak{E} - (\tfrac{1}{6}\,\mathfrak{E})^3 - (\tfrac{1}{4}\mathfrak{D})^2 \\ = \quad & (H - \tfrac{1}{3}I\Theta')\,\{IU'^2 - 3H'^2 + (12JU' + 2IH')\,\Theta' + I^2\Theta'^2\} \\ & - (H - \tfrac{1}{3}I\Theta')^3 \\ & - \Phi'^2, \end{aligned}$$

whence, substituting for $-\Phi'^2$ its value and reducing, we find

$$J^* = JU'^3 + \Theta'\,.\,\tfrac{2}{3}\,I^2U'^2 + \Theta'^2\,(4IJU') + \Theta'^3\,(16J^2 - \tfrac{8}{27}\,I^3).$$

Article No. 5. *Final expressions of the two Invariants.*

The value of I^* has been already mentioned to be $I^* = IU'^2 + \Theta' 12\,JU' + \Theta'^2\,.\,\tfrac{4}{3}I^2$, and it hence appears that the values of the two invariants may be written

$$\begin{aligned} I^* &= (I,\ 18J,\ 3I^2\between U',\ \tfrac{2}{3}\Theta')^2, \\ J^* &= (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2\between U',\ \tfrac{2}{3}\Theta')^3. \end{aligned}$$

But we have (see Table No. 72 in my "Seventh Memoir on Quantics," *Philosophical Transactions*, vol. CLI. (1861), pp. 277—292, [269])

$$\tilde{I}(\alpha U + 6\beta H) = (I,\ 18J,\ 3I\between\alpha,\ \beta)^2,$$
$$\tilde{J}(\alpha U + 6\beta H) = (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2\between\alpha,\ \beta)^3;$$

so that, writing $\alpha = U'$, $\beta = \frac{2}{3}\Theta'$, we have

$$I^* = \tilde{I}(U'U + 4\Theta'H),$$
$$J^* = \tilde{J}(U'U + 4\Theta'H);$$

or the function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4$ obtained from Tschirnhausen's transformation of the equation $U = 0$ has the same invariants with the function $U'U + 4\Theta'H$; or, what is the same thing, the equation $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4 = 0$ is a mere linear transformation of the equation $U'U + 4\Theta H = 0$; which is the above-mentioned theorem.

359.

A SUPPLEMENTARY MEMOIR ON CAUSTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVII. (for the year 1867), pp. 7—16. Received November 15,—Read November 22, 1866.]

IT is near the conclusion of my "Memoir on Caustics," *Philosophical Transactions*, vol. CXLVII. (1857), pp. 273—312, [145], remarked that for the case of parallel rays refracted at a circle, the ordinary construction for the secondary caustic cannot be made use of (the entire curve would in fact pass off to an infinite distance), and that the simplest course is to measure off the distance GQ from a line through the centre of the refracting circle perpendicular to the direction of the incident rays. The particular secondary caustic, or orthogonal trajectory of the refracted rays, obtained on the above supposition was shown to be a curve of the order 8; and it was further shown (by consideration of the case wherein the distance GQ is measured off from an arbitrary line perpendicular to the incident rays) that the general secondary caustic or orthogonal trajectory of the refracted rays was a curve of the same order 8. The last-mentioned curve in the case of reflexion, or for $\mu=-1$, degenerates into a curve of the order 6; and I propose in the present supplementary memoir to discuss this sextic curve, viz. the sextic curve which is the general secondary caustic or orthogonal trajectory of parallel rays reflected at a circle.

1. For parallel rays refracted at a circle, taking the equation of the circle to be $x^2+y^2=1$, and the incident rays to be parallel to the axis of x, then if $x=m$ be an arbitrary line perpendicular to the direction of the incident rays, the secondary caustic is the envelope of the circle

$$\mu^2\{(x-\alpha)^2+(y-\beta)^2\}-(x-m)^2=0,$$

where (α, β) are the coordinates of a variable point on the refracting circle, and as such satisfy the equation $\alpha^2+\beta^2=1$. Or, what is the same thing, writing $\alpha=\cos\theta$, $\beta=\sin\theta$, the secondary caustic is the envelope of the circle

$$\mu^2\{(x-\cos\theta)^2+(y-\sin\theta)^2\}-(x-m)^2=0,$$

where θ is a variable parameter.

2. The equation may be written

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

where

$$\begin{aligned} A &= 1, \\ B &= 0, \\ C &= 4\mu^2 x - 4m, \\ D &= 4\mu^2 y, \\ E &= -2\mu^2(x^2+y^2) - 2\mu^2 + 1 + 2m^2, \end{aligned}$$

and which in the case of reflexion, or for $\mu = -1$, become

$$\begin{aligned} A &= 1, \\ B &= 0, \\ C &= 4x - 4m, \\ D &= 4y, \\ E &= -2(x^2+y^2) - 1 + 2m^2, \end{aligned}$$

viz. the equation of the variable circle is in this case

$$\cos 2\theta + 4(x-m)\cos\theta + 4y\sin\theta + 2m^2 - 1 - 2(x^2+y^2) = 0.$$

3. Now in general for the equation

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

where the coefficients are any functions whatever of the coordinates (x, y), the equation of the envelope is $S^3 - T^2 = 0$, where

$$\begin{aligned} S &= 12(A^2+B^2) - 3(C^2+D^2) + 4E^2, \\ -T &= 27A(C^2-D^2) + 54BCD - \big(72(A^2+B^2) + 9(C^2+D^2)\big)E + 8E^3. \end{aligned}$$

4. Hence, substituting for A, B, C, D, E the above reflexion values, we find

$$\begin{aligned} S &= \quad 12 - 48\big((x-m)^2+y^2\big) + 4(2m^2-1-2x^2-2y^2)^2, \\ -T &= \quad 432\big((x-m)^2-y^2\big) \\ &\quad - 72\big(12 + 144\big((x-m)^2+y^2\big)\big)(2m^2-1-2x^2-2y^2) \\ &\quad + \quad 8(2m^2-1-2x^2-2y^2)^3. \end{aligned}$$

Writing in these equations

$$\begin{aligned} (x-m)^2 + y^2 &= x^2 + y^2 - 2mx + m^2, \\ (x-m)^2 - y^2 &= 2x^2 - 2mx + m^2 - (x^2+y^2), \end{aligned}$$

then after some simple reductions, we find

$$S = 16\,\{(x^2+y^2-m^2-1)^2+6m\,(x-m)\},$$
$$T = 32\,\{2\,(x^2+y^2-m^2-1)^3+18m\,(x-m)\,(x^2+y^2-m^2-1)-27\,(x-m)^2\},$$

and thence

$$S^3-T^2 = 1024\,(x-m)^2\,U,$$

where

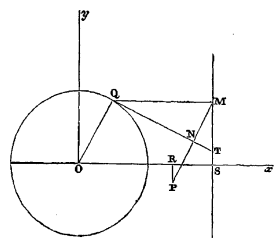
$$\begin{aligned} U = \quad & 4\,(x^2+y^2-m^2-1)^3 \\ & + 4m^2\,(x^2+y^2-m^2-1)^2 \\ & + 36m\,(x^2+y^2-m^2-1)\,(x-m) \\ & - 27\,(x-m)^2 \\ & + 32m^3\,(x-m), \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} U = \quad & 4\,(x^2+y^2)^3 \\ & - (8m^2+12)\,(x^2+y^2)^2 \\ & + (36mx+4m^4-20m^2+12)\,(x^2+y^2) \\ & - 27x^2+(-4m^2+18)\,mx+m^2-4\,; \end{aligned}$$

so that the equation of the secondary caustic is $U=0$, or the secondary caustic is, as stated above, a sextic curve.

5. It is easy to see that the foregoing envelope may be geometrically constructed as follows: viz. if from the point Q (coordinates $\cos\theta$, $\sin\theta$) on the reflecting circle we draw QM perpendicular to the line $x-m=0$, and then from the point M draw



MN perpendicular to QT, the tangent at T, and produce MN to a point P such that $PN=NM$, then P is a point of the envelope; and we thence obtain for the coordinates (x, y) of a point P of the envelope the values

$$x = \quad m-2\,(m-\cos\theta)\cos^2\theta,$$
$$y = \sin\theta-2\,(m-\cos\theta)\cos\theta\sin\theta,$$

or, what is the same thing,

$$x = 2\cos^2\theta - m(2\cos^2\theta - 1),$$
$$y = \sin\theta(2\cos^2\theta + 1) - 2m\sin\theta\cos\theta,$$

or, as these equations may also be written,

$$x = \tfrac{3}{2}\cos\theta - m\cos 2\theta + \tfrac{1}{2}\cos 3\theta,$$
$$y = \tfrac{3}{2}\sin\theta - m\sin 2\theta + \tfrac{1}{2}\sin 3\theta.$$

6. This result may be verified by showing that these values satisfy the equation

$$\cos 2\theta + 4(x - m)\cos\theta + 4y\sin\theta + 2m^2 - 1 - 2(x^2 + y^2) = 0,$$

and also the derived equation

$$\sin 2\theta + 2(x - m)\sin\theta - 2y\cos\theta = 0.$$

We in fact have

$$x\sin\theta - y\cos\theta = \quad m\sin\theta - \tfrac{1}{2}\sin 2\theta,$$
$$x\cos\theta + y\sin\theta = \tfrac{3}{2} - m\cos\theta + \tfrac{1}{2}\cos 2\theta,$$

and thence

$$(x - m)\sin\theta - y\cos\theta = \qquad -\tfrac{1}{2}\sin 2\theta,$$

which is one of the equations to be verified; and also

$$(x - m)\cos\theta + y\sin\theta = \tfrac{3}{2} - 2m\cos\theta + \tfrac{1}{2}\cos 2\theta.$$

We have moreover

$$x^2 + y^2 = \qquad \tfrac{5}{2} + m^2 - 4m\cos\theta + \tfrac{3}{2}\cos 2\theta;$$

and, by means of these last equations, the other equation

$$\cos 2\theta + 4(x - m)\cos\theta + 4y\sin\theta + 2m^2 - 1 - 2(x^2 + y^2) = 0,$$

is also verified.

7. The foregoing values of (x, y) give

$$dx = (-\tfrac{3}{2}\sin\theta + 2m\sin 2\theta - \tfrac{3}{2}\sin 3\theta)\,d\theta, \; = -\sin 2\theta(3\cos\theta - 2m)\,d\theta,$$
$$dy = (\;\tfrac{3}{2}\cos\theta - 2m\cos 2\theta + \tfrac{3}{2}\cos 3\theta)\,d\theta, \; = \;\cos 2\theta(3\cos\theta - 2m)\,d\theta,$$

or, what is the same thing, $dx : dy = -\sin 2\theta : \cos 2\theta$.

Hence taking for a moment (X, Y) as the current coordinates of a point in the tangent of the envelope, the equation of the tangent of the envelope is

$$Xdy \quad - Ydx = xdy - ydx,$$

or, substituting for x, y, dx, dy their values, this equation takes the very simple form

$$X\cos 2\theta - Y\sin 2\theta - 2\cos\theta + m = 0,$$

or writing (x, y) in place of (X, Y), that is taking now (x, y) as the current coordinates of a point in the tangent, the equation of the tangent is

$$x \cos 2\theta - y \sin 2\theta - 2 \cos \theta + m = 0;$$

whence observing that this equation may be expressed as a rational equation of the fourth order in terms of the parameter $\tan \frac{1}{2}\theta$ (or $\cos \theta + \sqrt{-1} \sin \theta$), it appears that the class of the secondary caustic is $= 4$.

8. The secondary caustic may be considered as the envelope of the tangent, and the equation be obtained in this manner. Comparing with the general equation

$$A \cos 2\theta + D \sin 2\theta + C \cos \theta + D \sin \theta + E = 0,$$

we have

$$\begin{aligned} A &= x, \\ B &= -y, \\ C &= -2, \\ D &= 0, \\ E &= m, \end{aligned}$$

and thence

$$\begin{aligned} S &= 4\{3(x^2+y^2)+m^2-3\}, \\ T &= 4\{18m(x^2+y^2)-27x-2m^3+9m\}, \end{aligned}$$

giving

$$S^3 - T^2 = 16V,$$

if for a moment

$$V = 4\{3(x^2+y^2)+m^2-3\}^3 - \{18m(x^2+y^2)-27x-2m^3+9m\}^2.$$

The equation of the curve is thus obtained in the form $V=0$; this should of course be equivalent to the before-mentioned equation $U=0$; and by developing V, and comparing with the second of the two expressions of U, it appears that we in fact have $V = 27U$.

9. Taking as parameter $\tan \frac{1}{2}\theta$, or if we please $\cos \theta + \sqrt{-1} \sin \theta$, the foregoing values of (x, y) in terms of θ give $(x, y, 1)$ proportional to rational and integral functions of the degree 6 in the parameter; so that not only the curve is a sextic curve, but it is a unicursal sextic, or curve of the order 6 with the maximum number, $=10$, of nodes and cusps; that is, if δ be the number of nodes and κ the number of cusps, we have $\delta + \kappa = 10$. Moreover, introducing the same parameter into the equation of the tangent, this equation is seen to be of the degree 4 in the parameter; that is, the class of the curve is $=4$: this implies $2\delta + 3\kappa = 26$, and we have therefore $\delta = 4$, $\kappa = 6$. To verify these numbers, it is to be remarked that it appears by the equation of the curve that there is at each of the circular points at infinity a triple point in the nature of the point $x=0$, $y=0$ on the curve $y^3 = x^4$;

such a point is in fact equivalent to a node and two cusps, and we have thus the two circular points at infinity counting together as 2 nodes and 4 cusps; there should therefore besides be 2 nodes and 2 cusps, and I proceed to establish the existence of these by means of the expressions for (x, y) in terms of θ.

10. To find the cusps, we have

$$\frac{dx}{d\theta} = -\sin 2\theta\,(3\cos\theta - 2m) = 0,$$

$$\frac{dy}{d\theta} = \cos 2\theta\,(3\cos\theta - 2m) = 0,$$

which are each of them satisfied if only $3\cos\theta - 2m = 0$, or $\cos\theta = \frac{2}{3}m$; the corresponding values of (x, y) are found to be

$$x = m - \tfrac{8}{27}m^3, \quad y = \pm\,(1 - \tfrac{4}{9}m^2)^{\frac{3}{2}},$$

and we have thus two cusps situate symmetrically in regard to the axis of x; the cusps are real if $m < \frac{3}{2}$, imaginary if $m > \frac{3}{2}$; for $m = \frac{3}{2}$, the two cusps unite together at the point $x = \frac{1}{2}$ on the axis of x, giving rise to a higher singularity, which will be further examined, *post*, No. 12.

11. The curve is symmetrical in regard to the axis of x, and hence any intersection with the axis of x, not being a point where the curve cuts the axis at right angles, will be a node. Hence, in order to find the nodes, writing $y = 0$, this is

$$\sin\theta\,(1 - 2m\cos\theta + 2\cos^2\theta) = 0,$$

giving $\sin\theta = 0$, that is,

$$\theta = 0, \quad x = \quad 2 - m;$$

or

$$\theta = \pi, \quad x = -2 - m;$$

but these are each of them ordinary points on the axis of x; or else giving

$$1 - 2m\cos\theta + 2\cos^2\theta = 0,$$

that is

$$\cos\theta = \tfrac{1}{2}(m \pm \sqrt{m^2 - 2}).$$

The corresponding values of x are

$$x = \cos\theta\,(2\cos^2\theta - 2m\cos\theta) + m, \; = m - \cos\theta, \; = \tfrac{1}{2}(m \mp \sqrt{m^2 - 2});$$

each of the points in question, viz. the points

$$x = \tfrac{1}{2}(m \mp \sqrt{m^2 - 2}), \quad y = 0,$$

is a node on the axis of x.

12. It is to be observed that for $m<\sqrt{2}$ the nodes are both imaginary; for $m=\sqrt{2}$ they coincide together at the point $x=\frac{1}{\sqrt{2}}$; for $m>\sqrt{2}$ they are both real: it is to be further noticed that

$$\text{node, } x=\tfrac{1}{2}(m+\sqrt{m^2-2}), \text{ corresponds to } \cos\theta=\tfrac{1}{2}(m-\sqrt{m^2-2}),$$

where (m being $>\sqrt{2}$) the point ($\cos\theta$, $\sin\theta$) is a real point on the circle $x^2+y^2=1$; in fact for $m<\frac{3}{2}$ (that is, $m=\sqrt{2}$ to $m=\frac{3}{2}$) we have $\frac{1}{2}(m-\sqrt{m^2-2})<\frac{1}{2}m$, that is, $\cos\theta<\frac{3}{4}$; but m = or $>\frac{3}{2}$, then $\cos\theta=\frac{1}{2}(m-\sqrt{m^2-2})=\dfrac{1}{m+\sqrt{m^2-2}}$ is = or $<\frac{1}{2}$, and

$$\text{node, } x=\tfrac{1}{2}(m-\sqrt{m^2-2}), \text{ corresponds to } \cos\theta=\tfrac{1}{2}(m+\sqrt{m^2-2}),$$

where (m being $>\sqrt{2}$) the point ($\cos\theta$, $\sin\theta$) is a real point on the circle $x^2+y^2=1$ so long as m is not $>\frac{3}{2}$, that is, from $m=\sqrt{2}$ to $m=\frac{3}{2}$; but if $m>\frac{3}{2}$, then the point in question is an imaginary point on the circle—whence also the node $x=\frac{1}{2}(m-\sqrt{m^2-2})$ is an acnode or isolated point.

In the case $m=\frac{3}{2}$ we have

$$\text{node, } x=1, \text{ corresponding to } \cos\theta=\tfrac{1}{2} \text{ or } \theta=60^\circ,$$

$$\text{"} \quad x=\tfrac{1}{2}, \quad \text{"} \quad \cos\theta=1 \text{ or } \theta=0^\circ,$$

the last-mentioned point $x=\frac{1}{2}$ being in fact the point of union of two cusps in the case $m=\frac{3}{2}$ now in question. Hence in this case we have at ($x=\frac{1}{2}$, $y=0$) a triple point equivalent to two cusps and a node; visibly, there is only a single branch cutting the axis of x at right angles.

In the case $m=\sqrt{2}$, the nodes coincide as above mentioned at the point $x=\dfrac{1}{\sqrt{2}}$ on the axis; for this value of m the coordinates of the cusps are

$$x=\tfrac{11}{27}\sqrt{2}\,(=\tfrac{22}{27}\div\sqrt{2}, \text{ which is } <1\div\sqrt{2});\ y=\pm\tfrac{1}{27}.$$

13. Starting from the equation $1024(x-m)^2U=S^3-T^2=0$, it is clear that the cusps are included among the intersections of the curves $S=0$, $T=0$; these two curves intersect in 24 points which lie $9+9$ at the circular points at infinity, $2+2$ at the points $x=m$, $y^2-1=0$, and $1+1$ are the cusps, or points $x=m-\frac{8}{27}m^2$, $y^2=(1-\frac{4}{9}m^2)^2$. To verify this, writing for a moment

$$S'=\ (x^2+y^2-m^2-1)^2+\ 6m(x-m),$$

$$T'=2(x^2+y^2-m^2-1)^3+18m(x-m)(x^2+y^2-m^2-1)-27(x-m)^2,$$

then we have

$$T'-2(x^2+y^2-m^2-1)S'=6m(x-m)(x^2+y^2-m^2-1)-27(x-m)^2,$$

$$=3(x-m)\{2m(x^2+y^2-m^2-1)-9(x-m)\};$$

so that the equations $S=0$, $T=0$, or, what is the same thing, $S'=0$, $T'=0$ give

$$(x-m)\{2m(x^2+y^2-m^2-1)-9(x-m)\}=0,$$

that is, $x-m=0$, or else $x^2+y^2-m^2-1=\frac{9}{2m}(x-m)$. And combining herewith the equation $S'=(x^2+y^2-m^2-1)^2+6m(x-m)=0$, we have $x-m=0$, $(y^2-1)^2=0$, or else

$$(x^2+y^2-m^2-1)^2=\frac{81}{4m^2}(x-m)^2=6m(x-m),$$

and therefore

$$(x-m)\frac{3}{4m^2}\{27(x-m)-8m^3\}=0,$$

the second factor of which gives $x=m-\frac{8}{27}m^3$, and thence $x^2+y^2-m^2-1=-\frac{4}{3}m^2$, that is, $x^2+y^2=1-\frac{1}{3}m^2$, and therefore $y^2=(1-\frac{1}{3}m^2)-(m-\frac{8}{27}m^3)^2$, $=(1-\frac{4}{9}m^2)^3$, that is, we have

$$x=m-\tfrac{8}{27}m^3,\quad y^2=(1-\tfrac{4}{9}m^2)^3,$$

which, as appears above, gives the two cusps.

14. Similarly, in the equation $16V=S^3-T^2=0$, the intersections of the curves $S=0$, $T=0$ must include the cusps; the curves in question are the two circles

$$3(x^2+y^2)+m^2-3=0,$$
$$18m(x^2+y^2)-27x-2m^3+9m=0,$$

meeting in the circular points at infinity, and in the two cusps. It is to be added that the tangent at the cusps coincides with the tangent of the last-mentioned circle,

$$18m(x^2+y^2)-27x-2m^3+9m=0,$$

or, as this may also be written,

$$\left(x-\frac{3}{4m}\right)^2+y^2=\left(\frac{4m^2-9}{12m}\right)^2.$$

15. The axis of x meets the secondary caustic in the two nodes counting as 4 intersections, and besides in 2 points, viz. the points $x=2-m$, $x=-2-m$; these correspond to the values $\theta=0$ and $\theta=\pi$ respectively. But to verify them by means of the equation

$$16V=S^3-T^2=0$$

of the curve, it may be remarked that for $y=0$ we have

$$S=4(3x^2+m^2-3),\quad T=4(18mx^2-27x-2m^3+9m);$$

and writing herein $x=\pm 2-m$, we find

$$S=4(2m\mp 3)^2,\qquad T=8(2m\mp 3)^3,$$

values which satisfy the equation $S^3-T^2=0$.

16. In the equation $U=0$ of the curve, writing $x-m=0$, the equation becomes

$$4(y^2-1)^2+4m^2(y^2-1)^2=0,$$

that is

$$4(y^2-1)^2(y^2-1+m^2)\ =0,$$

and the line $(x-m)=0$ is thus a double tangent to the curve, touching it at the points $x=m$, $y=\pm 1$, and besides meeting it at the points $x=m$, $y=\pm\sqrt{1-m^2}$, that is, at the intersections of the line $x-m=0$, with the circle $x^2+y^2=1$.

17. The maximum or minimum values of y correspond to the values $\theta=\frac{1}{4}\pi$, $\theta=\frac{3}{4}\pi$, $\theta=\frac{5}{4}\pi$, $\theta=\frac{7}{4}\pi$ of θ; and we have for

$$\theta=\tfrac{1}{4}\pi,\quad x=\ \tfrac{1}{2}\sqrt{2},\quad y=\ \sqrt{2}-m,$$

$$\theta=\tfrac{3}{4}\pi,\quad x=-\tfrac{1}{2}\sqrt{2},\quad y=\ \sqrt{2}+m,$$

$$\theta=\tfrac{5}{4}\pi,\quad x=-\tfrac{1}{2}\sqrt{2},\quad y=-\sqrt{2}-m,$$

$$\theta=\tfrac{7}{4}\pi,\quad x=\ \tfrac{1}{2}\sqrt{2},\quad y=-\sqrt{2}+m.$$

18. It is now easy to trace the secondary caustic; we may without loss of generality assume that m is positive, and the values to be considered are

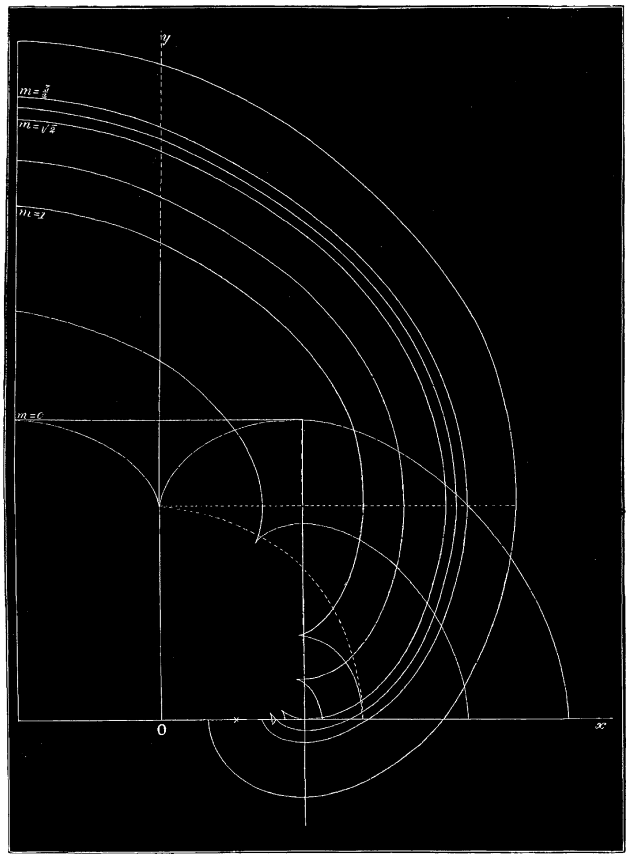
$$m=0,\quad m=1,\quad m=\sqrt{2},\quad m=\tfrac{3}{2},$$

with the intermediate values $m>0<1$, &c. ... and $m>\frac{3}{2}$. I have for convenience delineated in the figure only a portion of each curve, viz. the figure is terminated at the negative value $x=-\frac{1}{2}\sqrt{2}$, which corresponds to the maximum value $y=\sqrt{2}+m$; as x increases negatively, the value of the ordinate y diminishes continuously from this maximum value, becoming $=0$ for the value $x=-2-m$, and the curve at this point cutting the axis of x at right angles; this is a sufficient explanation of the form of the curves beyond the limits of the figure. Moreover the curve is symmetrical in regard to the axis of x, and I have within the limits of the figure delineated only one of the two halves of the curve.

19. For $m>\frac{3}{2}$ the cusps are both imaginary, the nodes both real, but one of them is an isolated point or acnode (shown in the figure by a small cross). The curve has an interior loop, as shown in the figure, and there is also the acnode lying within the loop.

For $m=\frac{3}{2}$, there is still an interior loop, but the acnode has united itself to the loop, the point of union, although presenting no visible singularity, being really a triple point equivalent to a node and two cusps. And in all the cases which follow there are two real cusps.

For $m = < \frac{3}{2} > \sqrt{2}$, the loop has altered its form in such wise as to exhibit the node and two cusps, the curve has therefore two real nodes.



For $m = \sqrt{2}$, the two nodes unite together into a tacnode, so that the loop is on the point of disappearing; and for $m < \sqrt{2} > 1$ the nodes are imaginary, and there is thus no longer any loop.

In all the above forms the double tangent $x=m$ touches the curve at the points $y=\pm 1$, but the other two intersections of the double tangent with the curve are imaginary.

For $m=1$, the double tangent has the two coincident real intersections $y=0$, or it is in fact a triple tangent.

For $m<1>0$, the double tangent has with the curve two real intersections, viz. they are the points where the double tangent meets the circle $x^2+y^2=1$.

And finally, for $m=0$, the points in question unite themselves with the points of contact, the double tangent $x=0$ being in this case the common tangent at the two cusps $x=0$, $y=\pm 1$.

Added May 13, 1867.

20. As remarked in the original memoir, p. 312, the secondary caustic, in the last-mentioned case $m=0$, is a curve similar to and double the magnitude of the caustic itself (viz. the caustic for parallel rays reflected at a circle), the position of the two curves differing by a right angle.

The secondary caustics corresponding to the different values of m form, it is clear, a system of parallel curves; and, by the remark just referred to, it appears that this system is similar to the system of curves parallel to the caustic for parallel rays reflected at a circle.

360.

NOTE ON A QUARTIC SURFACE.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 19—22.]

It would, I think, be worth while to study in detail the quartic surface which is the envelope of a sphere having its centre on a given conic, and passing through a given point. The equations of the conic being $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the coordinates of a point on the conic may be taken to be $x=a\cos\theta$, $y=b\sin\theta$, $z=0$, whence, if (α, β, γ) be the coordinates of the given point, the equation of the sphere is

$$(x-a\cos\theta)^2+(y-b\sin\theta)^2+z^2=(\alpha-a\cos\theta)^2+(\beta-b\sin\theta)^2+\gamma^2,$$

or, what is the same thing,

$$x^2+y^2+z^2-\alpha^2-\beta^2-\gamma^2-2(x-\alpha)a\cos\theta-2(y-\beta)b\sin\theta=0;$$

and hence the equation of the surface is at once seen to be

$$(x^2+y^2+z^2-\alpha^2-\beta^2-\gamma^2)^2=4a^2(x-\alpha)^2+4b^2(y-\beta)^2.$$

If $a=b$ (that is, if the conic be a circle), then we may without loss of generality write $\beta=0$, and the equation then is

$$(x^2+y^2+z^2-\alpha^2-\gamma^2)^2=4a^2\{(x-\alpha)^2+y^2\}.$$

This may be written

$$(x^2+y^2+z^2-\alpha^2-\gamma^2-2a^2)^2=-8a^2\alpha\left(x-\frac{a^2+2\alpha^2+\gamma^2-z^2}{2\alpha}\right),$$

which, considering z as a constant, is of the form

$$(x^2+y^2-\alpha)^2=16A(x-m);$$

that is, the section of the surface by a plane parallel to the plane of the conic is a Cartesian.

If a and b are unequal, but if we still have $\beta=0$, the equation of the surface is

$$(x^2+y^2+z^2-\alpha^2-\gamma^2)^2=4a^2(x-\alpha)^2+4b^2y^2.$$

There are here two planes parallel to the plane of the conic, each of them meeting the surface in a pair of circles. In fact, writing $x^2+y^2=\rho$, and therefore also $y^2=\rho-x^2$, putting moreover $z^2-\alpha^2-\gamma^2=k$, we have

$$(\rho+k)^2=4a^2x^2-8a^2\alpha x+4a^2\alpha^2+4b^2(\rho-x^2);$$

that is,

$$\rho^2+4(b^2-a^2)x^2+k^2-4a^2\alpha^2+8a^2\alpha x+(2k-4b^2)\rho=0,$$

or, as this may also be written,

$$(1,\ 4(b^2-a^2),\quad k^2-4a^2\alpha^2,\quad 4a^2\alpha,\quad k-2b^2,\quad 0\between\rho,\ x,\ 1)^2=0,$$

which is of the form

$$(a,\quad b\ ,\quad c\ ,\quad f\ ,\quad g\ ,\quad 0\between\rho,\ x,\ 1)^2=0;$$

and the left-hand side will break up into factors, each of the form $\rho+Ax+B$ (so that, equating either factor to zero, we have $\rho+Ax+B=0$, that is, $x^2+y^2+Ax+B=0$, the equation of a circle), if only

$$abc-af^2-bg^2=0.$$

Writing this under the form $b(ac-g^2)-af^2=0$, and substituting for a, b, c, f, g their values, we have

$$b=4(b^2-a^2),\quad ac-g^2=k^2-4a^2\alpha^2-(k-2b^2)^2,\ =4(b^2k-b^4-a^2\alpha^2),\quad af^2=16a^4\alpha^2,$$

and therefore the condition is

$$(b^2-a^2)(b^2k-b^4-a^2\alpha^2)-a^4\alpha^2=0;$$

that is,

$$b^2\{(b^2-a^2)(k-b^2)-a^2\alpha^2\}=0.$$

If $b^2=0$, the surface is a pair of spheres; rejecting this factor, we have $(b^2-a^2)(k^2-b^2)-a^2\alpha^2=0$; or putting for k its value, the condition becomes

$$(b^2-a^2)(z^2-\alpha^2-\gamma^2-b^2)-a^2\alpha^2=0;$$

that is, for each of the values of z given by this equation, the section by a plane parallel to the plane of the conic will be a pair of circles.

The planes in question will coincide with the plane of the conic, if only

$$(b^2-a^2)(\alpha^2+\gamma^2+b^2)+a^2\alpha^2=0,$$

or, what is the same thing,

$$b^2\alpha^2-(a^2-b^2)\gamma^2=b^2(a^2-b^2);$$

that is, if the point $(\alpha,\ 0,\ \gamma)$ be situated on the hyperbola $y=0$, $\dfrac{x^2}{a^2-b^2}-\dfrac{z^2}{b^2}=1$. The hyperbola in question and the ellipse $z=0$, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, are, it is clear, conics in planes

at right angles to each other, having the transverse axes coincident in direction, and being such that each curve passes through the foci of the other curve; or, what is the same thing, they are a pair of focal conics of a system of confocal ellipsoids.

The surface in the case in question, viz. when the parameters a, b, α, β are connected by the equation

$$\frac{\alpha^2}{a^2-b^2}-\frac{\gamma^2}{b^2}=1,$$

is in fact the "Cyclide" of Dupin. It is to be noticed that we have here

$$(\alpha-a\cos\theta)^2+b^2\sin^2\theta+\gamma^2=\alpha^2+\gamma^2+b^2-2a\alpha\cos\theta+(a^2-b^2)\cos^2\theta;$$

which, observing that $\alpha^2+\gamma^2+b^2$ is $=\dfrac{a^2\alpha^2}{a^2-b^2}$, gives

$$(\alpha-a\cos\theta)^2+b^2\sin^2\theta+\gamma^2=\left(\sqrt{a^2-b^2}\cos\theta-\frac{a\alpha}{\sqrt{a^2-b^2}}\right)^2;$$

so that the radius of the variable sphere is

$$=\sqrt{a^2-b^2}\cos\theta-\frac{a\alpha}{\sqrt{a^2-b^2}}.$$

If the variable sphere, instead of passing through the point $(\alpha, 0, \gamma)$ on the hyperbola, be drawn so as to touch a sphere of radius l, having its centre at the point in question, then the radius of the variable sphere would be

$$=\sqrt{a^2-b^2}\cos\theta-\frac{a\alpha}{\sqrt{a^2-b^2}}-l,$$

which is in fact

$$=\sqrt{a^2-b^2}\cos\theta-\frac{a\alpha'}{\sqrt{a^2-b^2}},$$

if only $\alpha'=\alpha+\dfrac{l\sqrt{a^2-b^2}}{a}$; hence if γ' be the corresponding value of γ, the variable sphere passes through the point $(\alpha', 0, \gamma')$ on the hyperbola, and the envelope is still a cyclide. The cyclide as derived from the foregoing investigation is thus the envelope of a sphere having its centre on the ellipse, and touching a fixed sphere having its centre on the hyperbola. It also appears that there are, having their centres on the hyperbola, an infinite series of spheres each touched by the spheres which have their centre on the ellipse; if, instead of one of these spheres we take any four of them, this will imply that the centre of the variable sphere is on the ellipse, and it is thus seen that the cyclide as obtained above is identical with the cyclide according to the original definition, viz. as the envelope of a sphere touching four given spheres.

Cambridge, December 5, 1864.

361.

ON QUARTIC CURVES.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 105—108.]

THE expression 'an oval' is used, in regard to the plane, to denote a closed curve without nodes or cusps; and, in regard to the sphere, it is assumed moreover that the oval is a curve which is not its own opposite, and does not meet the opposite curve[1]—that is, that the oval is one of a pair of non-intersecting twin ovals. I say that every spherical curve of the fourth order (or spherical quartic) without nodes or cusps may be considered as composed of an oval or ovals lying wholly in one hemisphere (that is, not cutting or touching the bounding circle of the hemisphere), and of the opposite oval or ovals lying wholly in the opposite hemisphere; or, disregarding the opposite curves, that it consists of an oval or ovals lying wholly in one hemisphere. And this being so, the quartic cone having its vertex at the centre of the sphere is met by a plane parallel to that of the bounding circle in a plane quartic curve consisting of an oval or ovals; and thence every plane quartic is either a finite curve consisting of an oval or ovals, or else the projection of such a curve.

Considering first the case of the plane, a line in general meets the oval in an *even* number of points (the number may of course be $=0$); hence as the point of contact of a tangent reckons for two points, the tangent at any point of the oval again intersects the oval in an even number of points (this number may of course be $=0$). The number of points of intersection by the tangent (the point of contact being always excluded) is either evenly even, and the point is then situate on a *convex* portion of the oval; or it is oddly even, and the point is then situate on a *concave* portion of the oval. Now imagine that the oval is (or is part of) a quartic curve; the number of points

[1] The notions of opposite curves, &c. are fully developed in the excellent Memoir of Möbius, "Ueber die Grundformen der Linien der dritter Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, vol. I. (1852), to which I have elsewhere frequently referred.

of intersection by the tangent is $=0$ or else $=2$; and there is at least one portion of the oval for which the number of intersections is $=0$; for otherwise the oval would be *concave* at every point, which is impossible. Hence there is a tangent which does not meet the oval (except at the point of contact), and we may in the immediate neighbourhood of the tangent draw a line which does not meet the oval at all.

Precisely the same considerations apply to the case of an oval which is part of a spherical quartic, the tangent being of course a great circle; and the conclusion arrived at is that there exists a great circle which does not meet the oval at all; that is, the oval lies wholly in one hemisphere.

I remark that the demonstration would, as it ought to do, fail, if we attempted to apply it to an oval portion of a spherical sextic; the tangent circle meets the oval in a number of points which is $=0$, 2, or 4; and the number cannot be for every tangent circle whatever $=2$; but there is nothing to prevent it from being for every tangent circle whatever $=2$ or 4. Hence we cannot, for every spherical sextic, obtain a tangent circle not meeting the oval except at the point of contact; and consequently we do not obtain in the immediate neighbourhood of the tangent a circle which does not meet the oval at all. And in fact such circle does not in every case exist; that is, *the oval portion of a spherical sextic does not in every case lie in a hemisphere.*

It has been shown that the oval portion of a spherical quartic lies in a hemisphere; but we have to consider the case where the quartic consists of two or more ovals. To fix the ideas, let A, A' be a pair of opposite ovals, and B, B' another pair of opposite ovals, components of the same spherical quartic. If there exists a tangent circle of A which does not meet B, then there exists in the immediate neighbourhood of the tangent circle a circle which does not meet either A or B; and we may assume that A and B lie on the same side of this circle; for if B were on the side opposite to A, then B' would be on the same side with A; and we have only, instead of B, to consider the opposite oval B'. Hence we may consider that the ovals A and B lie on the same side of the circle; that is, we have a spherical quartic consisting of or comprising the ovals A and B in the same hemisphere: the two ovals are, it is clear, external each to the other.

But every tangent of A may meet B in two points; consider the whole spherical figure, and suppose that the tangent (or say, the tangent circle) of A, A' meets the ovals B, B' in the points K, L and the opposite points K', L': then considering the tangent circle as moving round A, A' until it returns to its original position, the points K, L, K', L' are always four distinct points; and K and some one (say L) of the two points L, L' will describe the same oval, say the oval B; while the opposite points K', L' will describe the opposite oval B'. We have here the oval A included in the oval B (and of course the opposite oval A' included in the opposite oval B'). But the oval B, *quà* portion of a spherical quartic, lies wholly in one hemisphere; hence the two ovals A, B lie wholly in one hemisphere. It is easy to see that there is not in this case any other portion of the spherical quartic, but that the two ovals A, B are the entire curve.

Reverting to the case where we have in one hemisphere the two ovals A, B external to each other, the spherical quartic may comprise as part of itself another oval C. The ovals A and B, *quà* ovals external to each other, have a common tangent circle (a double tangent of the spherical quartic) which cannot meet the oval C (for if it did we should have six points of intersection); hence in the immediate neighbourhood thereof we have a circle not meeting any one of the ovals A, B, C. We may consider A, B, C as lying on the same side of this circle; for if B were on the opposite side to A, then B' would be on the same side; and so if C be on the opposite side, then C' will be on the same side; that is, we have the three ovals A, B, C external to each other, and in the same hemisphere.

There may be a fourth oval, D, and it would be shown in a similar manner that we have then the four ovals A, B, C, D external to each other and in the same hemisphere. But there cannot be a fifth oval, E; the proof is precisely the same as for the theorem *in plano*; viz. taking within each of the five ovals a point, and through these points drawing a conic, the conic would meet each oval in two points, and therefore the plane quartic in ten points, which is impossible.

Passing from the sphere to the plane, the foregoing investigation shows that every plane quartic without nodes or cusps is either a finite curve, or else the projection of a finite curve, of one of the following forms:

1. a single oval.

2. two ovals external to each other.

3. two ovals, one inside the other.

4. three ovals external to each other.

5, 6. four ovals external to each other.

The last case has been called (5, 6) for the sake of the following subdivision, viz.:

5. the four ovals are so situate as to be intersected, each in two points, by the same ellipse.

6. they are so situate as not to be intersected by any one ellipse whatever—the distinction being similar to that which exists between four points, which may be either such as to have passing through them as well ellipses as hyperbolas, or else to have passing through them hyperbolas only.

I remark that the limitation of the theorem to the case of a quartic curve without nodes or cusps is necessary, at any rate as regards the nodes. We may in fact find a quartic curve having a single node which is met by every line in at least two real points, and which is therefore not the projection of any finite curve; for if we imagine two hyperbolas so situate that each branch of the one cuts each branch of the other, then it may be seen that there exists a quartic curve approaching everywhere very nearly to the system of two hyperbolas, but having, instead of the four nodes of the system, only a single node, which is such that every line meets it in at least two points.

Cambridge, December 15, 1864.

362.

NOTE ON LOBATSCHEWSKY'S IMAGINARY GEOMETRY.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 231—233.]

WRITING down the equations

$$\frac{1}{\cos a'} = \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

where A, B, C are real positive angles each $< \frac{1}{2}\pi$: *first*, if $A+B+C > \pi$, then a, b, c are real positive angles each less than $\frac{1}{2}\pi$ (this is in fact the case of a real acute-angled spherical triangle), but a', b', c' are pure imaginaries of the form $p'i$, $q'i$, $r'i$ (where p', q', r' are real positive quantities; and *secondly*, if $A+B+C < \pi$, then a, b, c are pure imaginaries of the form pi, qi, ri (where p, q, r are real positive quantities), but a', b', c' are real positive angles each less than $\frac{1}{2}\pi$. Hence assuming $A+B+C < \pi$ and writing ai, bi, ci in place of a, b, c, the system is

$$\frac{1}{\cos a'} = \cos ai = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \cos bi = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \cos ci = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

which equations (if only we write therein $\frac{1}{2}\pi - a'$, $\frac{1}{2}\pi - b'$, $\frac{1}{2}\pi - c'$ in place of a', b', c' respectively) are in fact the equations given under a less symmetrical form in the curious paper "Géométrie Imaginaire" by N. Lobatschewsky, Rector of the University of Kasan, *Crelle*, vol. XVII. (1837), pp. 295—320. The view taken of them by the author is hard to be understood. He mentions that in a paper published five years previously in a scientific journal at Kasan, after developing a new theory of parallels, he had endeavoured to prove that it is only experience which obliges us to assume that in a rectilinear triangle the sum of the angles is equal to two right angles, and that a geometry may exist, if not in nature at least in analysis, on the hypothesis that the sum of the angles is less than two right angles; and he accordingly attempts to establish such a geometry, viz. a, b, c being the sides of a rectilinear triangle, wherein the sum of the angles $A + B + C$ is $< \pi$, and the angles a', b', c' being calculated from the sides by the formulæ

$$\cos a' = \frac{1}{\cos ai}, \quad \cos b' = \frac{1}{\cos bi}, \quad \cos c' = \frac{1}{\cos ci}.$$

(I have, as mentioned above, replaced Lobatschewsky's a', b', c' by their complements): the relation between the angles A, B, C and the subsidiary quantities a', b', c' which replace the sides, is given by the formulæ

$$\frac{1}{\cos a'} = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

I do not understand this; but it would be very interesting to find a *real* geometrical interpretation of the last-mentioned system of equations, which (if only A, B, C are positive real quantities such that $A + B + C < \pi$; for the condition, A, B, C each $< \frac{1}{2}\pi$, may be omitted) contains only the *real* quantities A, B, C, a', b', c'; and is a system correlative to the equations of ordinary Spherical Trigonometry.

It is hardly necessary to remark that the equation

$$\frac{1}{\cos a'} = \cos ai$$

is Jacobi's imaginary transformation in the Theory of Elliptic Functions. See, as to this, my paper "On the Transcendent $\operatorname{gd}.u = \frac{1}{i}\log\tan(\frac{1}{4}\pi + \frac{1}{2}ui)$," *Phil. Mag.* vol. XXIV. (1862), pp. 19—22, [320].

Cambridge, January 21, 1865.

363.

ON THE THEORY OF THE EVOLUTE.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 344—350.]

ACCORDING to the generalized notion of geometrical magnitude, two lines are said to be at right angles to each other when they are harmonics in regard to a certain conic called the Absolute; this being so, the normal at any point of a curve is the line at right angles to the tangent, and the Evolute is the envelope of the normals.

Let the equation of the absolute be

$$\Theta = (a, b, c, f, g, h \between x, y, z)^2 = 0,$$

and suppose, as usual, that the inverse coefficients are (A, B, C, F, G, H). Consider a given curve $U = (* \between x, y, z)^m = 0$, and suppose, for shortness, that the first differential coefficients of U are denoted by L, M, N. Then we have to find the equation of the normal at the point (x, y, z) of the curve $U = 0$.

The condition that any two lines are harmonics in regard to the absolute, is equivalent to this, viz. each line passes through the pole of the other line in regard to the absolute. Hence the normal at the point (x, y, z) is the line joining this point with the pole of the tangent. Now, taking (X, Y, Z) as current coordinates, the equation of the tangent is

$$LX + MY + NZ = 0,$$

the coordinates of the pole of the tangent are therefore

$$(A, H, G \between L, M, N) : (H, B, F \between L, M, N) : (G, F, C \between L, M, N),$$

and the equation of the normal is

$$\begin{vmatrix} X & , & Y & , & Z \\ x & , & y & , & z \\ (A, H, G \between L, M, N), & & (H, B, F \between L, M, N), & & (G, F, C \between L, M, N) \end{vmatrix} = 0.$$

The formula in this form will be convenient in the sequel; but there is no real loss of generality in taking the equation of the absolute to be $x^2+y^2+z^2=0$; the values of (A, B, C, F, G, H) are then $(1, 1, 1, 0, 0, 0)$, and the formula becomes

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ L, & M, & N \end{vmatrix} = 0;$$

where it will be remembered that (L, M, N) denote the derived functions $(\partial_x U, \partial_y U, \partial_z U)$.

The evolute is therefore the envelope of the line represented by the foregoing equation, say the equation $\Omega=0$, considering therein (x, y, z) as variable parameters connected by the equation $U=0$.

As an example, let it be required to find the evolute of a conic; since the axes are arbitrary, we may without loss of generality assume that the equation of the conic is $xz-y^2=0$. The values of (L, M, N) here are $(z, -2y, x)$. Moreover the equation is satisfied by writing therein $x : y : z = 1 : \theta : \theta^2$; the values of (L, M, N) then become $(\theta^2, -2\theta, 1)$ and the equation is

$$\begin{vmatrix} 1 & , & \theta & , & \theta^2 \\ X & , & Y & , & Z \\ (A, H, G \between \theta, -1)^2, & & (H, B, F \between \theta, -1)^2, & & (G, F, C \between \theta, -1)^2 \end{vmatrix} = 0;$$

or, developing, this is

$$\begin{aligned} & X \begin{pmatrix} G\theta^3 - 2F\theta^2 + C\theta \\ -H\theta^4 + 2B\theta^3 - F\theta^2 \end{pmatrix} \\ & + Y \begin{pmatrix} A\theta^4 - 2H\theta^3 + G\theta^2 \\ - G\theta^2 + 2F\theta - C \end{pmatrix} \\ & + Z \begin{pmatrix} H\theta^2 - 2B\theta + F \\ -A\theta^3 + 2H\theta^2 - G\theta \end{pmatrix} = 0, \end{aligned}$$

which I leave in this form in order to show the origin of the different terms, and in particular in order to exhibit the destruction of the term θ^2 in the coefficient of Y. But the equation is, it will be observed, a quartic equation in θ, with coefficients which are linear functions of the current coordinates (X, Y, Z).

The equation shows at once that the evolute is of the class 4; in fact treating the coordinates (X, Y, Z) as given quantities, we have for the determination of θ an equation of the order 4, that is, the number of normals through a given point (X, Y, Z), or, what is the same thing, the class of the evolute, is $=4$.

The equation of the evolute is obtained by equating to zero the discriminant of the foregoing quartic function of θ; the order of the evolute is thus $=6$. There are no inflexions, and the diminution of the order from $4.3, =12$, to 6 is caused by three double tangents.

I consider the particular case where the conic touches the absolute. There is no loss of generality in assuming that the contact takes place at the point $(y=0,\ z=0)$, the common tangent being therefore $z=0$; the conditions for this are $a=0,\ h=0$, and we have thence $C=0,\ F=0$. Substituting these values, the equation contains the factor θ; and, throwing this out, it is

$$\begin{array}{l} X\left(-H\theta^3+(B+2G)\,\theta^2 \qquad\qquad\qquad\qquad\right) \\ Y\left(\ A\theta^3-\qquad 2H\,\theta^2 \qquad\qquad\qquad\right) \\ +Z\left(\qquad\qquad -\ A\,\theta^2+3H\theta-(B+2G)\right)=0, \end{array}$$

or, what is the same thing,

$$\begin{array}{l} \theta^3\left(\qquad -H\ X+\ AY \qquad\qquad\right) \\ +\theta^2\left((B+2G)\,X-2HY-\qquad AZ\right) \\ +\theta\ \left(\qquad\qquad\qquad\qquad 3HZ\right) \\ +\ \ \left(\qquad\qquad\qquad -(B+2G)\,Z\right)=0, \end{array}$$

where it will be observed that the constant term and the coefficient of θ have the same variable factor Z, where $Z=0$ is the equation of the common tangent of the conic and the absolute. The evolute is in this case of the class 3. It at once appears that the line $Z=0$ is a stationary tangent of the evolute, the point of contact (or inflexion on the evolute) being given by the equations $Z=0,\ (B+2G)\,X-2HY=0$. The equation of the evolute is found by equating to zero the discriminant of the cubic function; the equation so obtained has the factor Z, and throwing this out the order is $=3$. The evolute is thus a curve of the class 3 and order 3, the reduction in the order from $3\,.\,2,\ =6$, to 3 being caused by the existence of an inflexion. Comparing with the former case, we see that the effect of the contact of the conic with the absolute is to give rise to an inflexion of the evolute, and to cause a reduction $=1$ in the class, and a reduction $=3$ in the order.

I return now to the general case of a curve

$$U=(*\between x,\ y,\ z)^m=0.$$

Using, for greater simplicity, the equation $x^2+y^2+z^2=0$ for the absolute, the equation of the normal is

$$\Omega=\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ \partial_x U, & \partial_y U, & \partial_z U \end{vmatrix}=0;$$

we may at once find the class of the evolute; in fact, treating $(X,\ Y,\ Z)$ as the coordinates of a given point, the two equations $U=0,\ \Omega=0$ determine the values $(x,\ y,\ z)$ of the coordinates of a point such that the normal thereof passes through

the point (X, Y, Z); the number of such points is the number of normals which can be drawn through a given point (X, Y, Z), viz. it is equal to the class of the evolute. The points in question are given as the intersections of the two curves $U=0$, $\Omega=0$, which are respectively curves of the order m, hence the number of intersections is $=m^2$. It is to be observed, however, that if the curve $U=0$ has nodes or cusps, then the curve $\Omega=0$ passes through each node of the curve $U=0$, and through each cusp, the two curves having at the cusp a common tangent; that is, each node reckons for two intersections, and each cusp for three intersections. Hence, if the curve $U=0$ has δ nodes and κ cusps, the number of the remaining points of intersection is $=m^2-2\delta-3\kappa$. The class of the evolute is thus $=m^2-2\delta-3\kappa$. The number of inflexions is in general $=0$. If, however, the given curve touches the absolute, then it has been seen in a particular case that the effect is to diminish the class by 1, and to give rise to an inflexion, the stationary tangent being in fact the common tangent of the curve and the absolute: I assume that this is the case generally. Suppose that there are θ contacts, then there will be a diminution $=\theta$ in the class, or this will be $=m^2-2\delta-3\kappa-\theta$; and there will be θ inflexions; there may however be special circumstances giving rise to fresh inflexions, and I will therefore assume that the number of inflexions is $=\iota'$.

Suppose in general that for any curve we have

m,	the order,		
n,	"	class,	
δ,	"	number of nodes,	
κ,	"	"	cusps,
τ,	"	"	double tangents,
ι,	"	"	inflexions.

Then Plücker's equations give

$$\iota-\kappa=3(n-m), \quad \tau-\delta=\tfrac{1}{2}(n-m)(n+m-9);$$

and we thence have

$$\iota-\kappa+\tau-\delta=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(m-1)(m-2),$$

or, what is the same thing,

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa=\tfrac{1}{2}(n-1)(n-2)-\tau-\iota.$$

Now M. Clebsch in his recent paper "Ueber die Singularitäten algebraischer Curven," *Crelle*, vol. LXIV. (1864), pp. 98—100, has remarked (as a consequence of the investigations of Riemann in the Integral Calculus) that whenever from a given curve another curve is derived in such manner that to each point (or tangent) of the given curve there corresponds a *single* tangent (or point) of the derived curve, then the expression

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa, \quad =\tfrac{1}{2}(n-1)(n-2)-\tau-\iota,$$

has the same value in the two curves respectively, or that, writing m', n', δ', κ', τ', ι' for the corresponding quantities in the second curve, then we have

$$\begin{aligned} \tfrac{1}{2}(m-1)(m-2)-\delta-\kappa &= \tfrac{1}{2}(n-1)(n-2)-\tau-\iota, \\ = \tfrac{1}{2}(m'-1)(m'-2)-\delta'-\kappa', &= \tfrac{1}{2}(n'-1)(n'-2)-\tau'-\iota'; \end{aligned}$$

and consequently that, knowing any *two* of the quantities m', n', δ', κ', τ', ι', the remainder of them can be determined by means of this relation and of Plücker's equations. The theorem is applicable to the evolute according to the foregoing generalized definition[1]; and starting from the values

$$\begin{aligned} n' &= m^2-2\delta-3\kappa-\theta, \\ \iota' &= \iota', \end{aligned}$$

we find in the first instance

$$\tau' = \tfrac{1}{2}(n'-1)(n'-2)-\tfrac{1}{2}(m-1)(m-2)+\delta+\kappa-\iota';$$

and substituting in the equation

$$m' = n'(n'-1)-2\tau'-3\iota',$$

we find

$$m' = 2(n'-1)+(m-1)(m-2)-2\delta-2\kappa-\iota';$$

and the equation $\iota'-\kappa' = 3(n'-m')$ gives also

$$\kappa' = -3(n'-m')+\iota',$$

whence, attending to the value of n', we find the following system of equations for the singularities of the evolute, viz.

$$\begin{aligned} n' &= m^2 \qquad\qquad -2\delta-3\kappa-\theta, \\ m' &= 3m(m-1)-6\delta-8\kappa-2\theta-\iota', \\ \iota' &= \iota', \\ \kappa' &= 3m(2m-3)-12\delta-15\kappa-3\theta-2\iota', \end{aligned}$$

and the values of τ' and δ' may then also be found from the equations

$$\begin{aligned} m' &= n'(n'-1)-2\tau'-3\iota', \\ n' &= m'(m'-1)-2\delta'-3\kappa'. \end{aligned}$$

I have given the system in the foregoing form, as better exhibiting the effect of the inflexions; but as each of the θ contacts with the absolute gives an inflexion, we

[1] M. Clebsch in fact applies it to the evolute in the ordinary sense of the term, but by inadvertently assuming $\iota'=\kappa$ instead of $\iota'=0$ he is led to some incorrect results.

may write $\iota' = \theta + \iota''$, where, in the absence of special circumstances giving rise to any more inflexions, $\iota'' = 0$. The system thus becomes

$$\begin{aligned} n' &= m^2 - 2\delta - 3\kappa - \theta, \\ m' &= 3m(m-1) - 6\delta - 8\kappa - 3\theta - \iota'', \\ \iota' &= \theta + \iota'', \\ \kappa' &= 3m(2m-3) - 12\delta - 15\kappa - 5\theta - 2\iota'', \end{aligned}$$

so that each contact with the absolute diminishes the class by 1, the order by 3, and the number of cusps by 5.

I remark that when the absolute becomes a pair of points, a contact of the given curve m means one of two things: either the curve touches the line through the two points, or else it passes through one of the two points: the effect of a contact of either kind is as above stated. Suppose that the two points are the circular points at infinity, and let $m = 2$, the evolute in question is then the evolute of a conic, in the ordinary sense of the word evolute. We have, in general, class $= 4$, order $= 6$; but if the conic touches the line infinity (that is, in the case of the parabola), the reductions are 1 and 3, and we have class $= 3$, order $= 3$, which is right. If the conic passes through one of the circular points of infinity, then in like manner the reductions are 1 and 3; and therefore if the conic passes through each of the circular points at infinity (that is, in the case of a circle), the reductions are 2 and 6, and we have class $= 2$, order $= 0$, which is also right; for the evolute is in this case the centre, regarded as a pair of coincident points. That this is so, or that the class is to be taken to be (not $= 1$ but) $= 2$, appears by the consideration that the number of normals to the circle from a given point is in fact $= 2$, the two normals being, however, coincident in position.

To complete the theory in the general case where the absolute is a proper conic, I remark that, besides the inflexions which arise from contacts of the given curve with the absolute, there will be an inflexion, first, for each stationary tangent of the given curve which is also a tangent of the absolute; secondly, for each cusp of the given curve situate on the absolute. Hence, if the number of such stationary tangents be $= \lambda$, and the number of such cusps be $= \mu$, we may write $\iota'' = \lambda + \mu$, and therefore also $\iota' = \theta + \lambda + \mu$.

I remark also that we have

$$\begin{aligned} -2\delta - 3\kappa &= -m(m-1) + n, \\ -6\delta - 8\kappa &= -3m(m-2) + \iota, \end{aligned}$$

and therefore also

$$-12\delta - 15\kappa = -6m^2 + 15m - 3n + 3\iota.$$

The general formulæ thus become

$$\begin{aligned} n' &= m + n - \theta, \\ m' &= 3m + \iota - 2\theta - \iota', \\ \iota' &= \iota', \\ \kappa' &= 6m - 3n + 3\iota - 3\theta - 2\iota'. \end{aligned}$$

If instead of the given curve we consider its reciprocal in regard to the absolute, then

$$m,\ n,\ \delta,\ \kappa,\ \tau,\ \iota;\quad \theta,\ \lambda,\ \mu;\quad \iota',\ =\theta+\lambda+\mu$$

are changed into

$$n,\ m,\ \tau,\ \iota,\ \delta,\ \kappa;\quad \theta,\ \mu,\ \lambda;\quad \iota',\ =\theta+\mu+\lambda$$

respectively.

Hence for the evolute of the reciprocal curve we have

$$\begin{aligned} n' &= n+m \qquad -\theta \quad , \\ m' &= 3n \qquad +\kappa-2\theta-\iota', \\ \iota' &= \qquad\qquad\qquad\qquad \iota', \\ \kappa' &= 6n-3m+3\kappa-3\theta-\iota', \end{aligned}$$

which, attending to the relation $\iota-\kappa=3(n-m)$, are in fact the same as the former values; that is, the evolute of the given curve, and the evolute of the reciprocal curve are curves of the same class and order, and which have the same singularities.

Cambridge, February 22, 1865.

364.

ON A THEOREM RELATING TO FIVE POINTS IN A PLANE.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 460—464.]

Two triangles, ABC, $A'B'C'$ which are such that the lines AA', BB', CC' meet in a point, are said to be in perspective; and a triangle $A'B'C'$, the angles A', B', C' of which lie in the sides BC, CA, AB respectively, is said to be inscribed in the triangle ABC; hence, if A', B', C' are the intersections of the sides by the lines AO, BO, CO respectively (where O is any point whatever), the triangle $A'B'C'$ is said to be perspectively inscribed in the triangle ABC, viz. it is so inscribed by means of the point O.

We have the following theorem, relating to any triangle ABC, and two points O, O'. If in the triangle ABC, by means of the point O, we inscribe a triangle $A'B'C'$, and in the triangle $A'B'C'$, by means of the point O', we inscribe a triangle $\alpha\beta\gamma$, then the triangles ABC, $\alpha\beta\gamma$ are in perspective, viz. the lines $A\alpha$, $B\beta$, $C\gamma$ will meet in a point.

This is very easily proved analytically; in fact, taking $x=0$, $y=0$, $z=0$ for the equations of the lines $B'C'$, $C'A'$, $A'B'$ respectively, and (X, Y, Z) for the coordinates of the point O, then the coordinates of (A, B, C) are found to be $(-X, Y, Z)$, $(X, -Y, Z)$, $(X, Y, -Z)$ respectively. Moreover, if (X', Y', Z') are the coordinates of the point O', then the coordinates of (α, β, γ) are found to be

$$(0, Y', Z'),\ (X', 0, Y),\ (X', Y', 0)$$

respectively. Hence the equations of the lines $A\alpha$, $B\beta$, $C\gamma$ are respectively

$$\begin{vmatrix} x, & y, & z \\ -X, & Y, & Z \\ 0, & Y', & Z' \end{vmatrix} = 0,\quad \begin{vmatrix} x, & y, & z \\ X, & -Y, & Z \\ X', & 0, & Z' \end{vmatrix} = 0,\quad \begin{vmatrix} x, & y, & z \\ X, & Y, & -Z \\ X', & Y', & 0 \end{vmatrix} = 0;$$

that is

$$\begin{aligned}
x(\ YZ' - Y'Z) + y(\qquad\quad Z'X) + z(-XY' \qquad\quad) &= 0,\\
x(-YZ' \qquad\quad) + y(\ ZX' - Z'X) + z(\qquad\quad X'Y) &= 0,\\
x(\qquad + Y'Z) + y(-ZX' \qquad\quad) + z(\ XY' - X'Y) &= 0,
\end{aligned}$$

which are obviously the equations of three lines which meet in a point.

But the theorem may be exhibited as a theorem relating to a quadrangle 1234 and a point O'; for writing 1, 2, 3, 4 in place of A, B, C, O, the triangle $A'B'C'$ is in fact the triangle formed by the three centres 41.23, 42.31, 43.12 of the quadrangle 1234, hence the triangle in question must be similarly related to each of the four triangles 423, 431, 412, 123; or, forming the diagram

	P	Q	R	S
41.23	4	3	2	1
42.31	3	4	1	2
43.12	2	1	4	3,

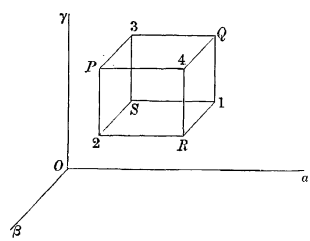
we have the following form of the theorem: viz. the lines

$\alpha 4$, $\beta 3$, $\gamma 2$ meet in a point P,
$\alpha 3$, $\beta 4$, $\gamma 1$ " " Q,
$\alpha 2$, $\beta 1$, $\gamma 4$ " " R,
$\alpha 1$, $\beta 2$, $\gamma 3$ " " S,

or, what is the same thing, we have with the points 1, 2, 3, 4 and the point O' constructed the four points P, Q, R, S such that

$1S$, $2R$, $3Q$, $4P$ meet in a point α,
$2S$, $1R$, $4Q$, 3β " " β,
$3S$, $4R$, $1Q$, $2P$ " " γ.

The eight points 1, 2, 3, 4, P, Q, R, S form a figure such as the perspective representation of a parallelopiped, or, if we please, a cube; and not only so, but the



plane figure is really a certain perspective representation of the cube; this identification depends on the following two theorems:

1. Considering the four summits 1, 2, 3, 4, which are such that no two of them belong to the same edge, then, if through any point O we draw

the line OA' meeting the lines 41, 23,
" OB' " " 42, 31,
" OC' " " 43, 12,

and the lines $O\alpha$, $O\beta$, $O\gamma$ parallel to the three edges of the cube respectively, the three planes $(OA', O\alpha)$, $(OB', O\beta)$, $(OC', O\gamma)$ will meet in a line.

2. For a properly selected position of the point O,

the lines OB', OC', $O\alpha$ will lie in a plane,
" OC', OA', $O\beta$ " " ,
" OA', OB', $O\gamma$ " " .

In fact for such a position of O, projecting the whole figure on any plane whatever, the lines $O1$, $O2$, $O3$, $O4$, OP, OQ, OR, OS, $O\alpha$, $O\beta$, $O\gamma$, OA', OB', OC' meet the plane of projection in the points 1, 2, 3, 4, P, Q, R, S, α, β, γ, A', B', C' related to each other as in the last-mentioned form of the plane theorem. To prove the two solid theorems, take O for the origin, $O\alpha$, $O\beta$, $O\gamma$ for the axes, (α, β, γ) for the coordinates of the summit S, and 1 for the edge of the cube,

the coordinates of 1 are $\alpha+1$, β, γ,
" 2 " α, $\beta+1$, γ,
" 3 " α, β, $\gamma+1$,
" 4 " $\alpha+1$, $\beta+1$, $\gamma+1$.

The equations of the line OA', or say of the line $O(41, 23)$, are those of the planes $O41$, $O23$, viz. these are

$$\begin{vmatrix} x, & y, & z \\ \alpha+1, & \beta, & \gamma \\ \alpha+1, & \beta+1, & \gamma+1 \end{vmatrix} = 0, \quad \begin{vmatrix} x, & y, & z \\ \alpha, & \beta+1, & \gamma \\ \alpha, & \beta, & \gamma+1 \end{vmatrix} = 0;$$

that is

$$x(\beta-\gamma) \quad -(\alpha+1)(y-z) = 0,$$
$$x(\beta+\gamma+1)-\alpha \quad (y+z) = 0.$$

Writing for shortness

$$M = \alpha+\beta+\gamma+1,$$

these equations give

$$x : y : z = \frac{2\alpha(\alpha+1)}{(M+2\gamma\alpha)(M+2\alpha\beta)} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta};$$

or, completing the system,

for line OA' we have

$$x : y : z = \frac{2\alpha(\alpha+1)}{(M+2\gamma\alpha)(M+2\alpha\beta)} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta};$$

for line OB' we have

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{2\beta(\beta+1)}{(M+2\alpha\beta)(M+2\beta\gamma)} : \frac{1}{M+2\alpha\beta};$$

for line OC' we have

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{1}{M+2\gamma\alpha} : \frac{2\gamma(\gamma+1)}{(M+2\beta\gamma)(M+2\gamma\alpha)}.$$

The equations of the lines $O\alpha$, $O\beta$, $O\gamma$ are of course $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$ respectively; and we therefore see at once that the planes $(OA',\ O\alpha)$, $(OB',\ O\beta)$, $(OC',\ O\gamma)$ meet in a line, viz. in the line which has for its equations

$$x : y : z = \frac{1}{M+2\beta\gamma} : \frac{1}{M+2\gamma\alpha} : \frac{1}{M+2\alpha\beta}.$$

The lines OB', OC', $O\alpha$ will lie in a plane, if only

$$1 = \frac{4\beta\gamma(\beta+1)(\gamma+1)}{(M+2\beta\gamma)^2};$$

that is

$$(M+2\beta\gamma)^2 = 4\beta\gamma(\beta+1)(\gamma+1),$$

or, as this may be written,

$$M^2 + 4\beta\gamma(\alpha+\beta+\gamma+1+\beta\gamma) = 4\beta\gamma(\beta\gamma+\beta+\gamma+1);$$

that is

$$M^2 + 4\alpha\beta\gamma = 0,$$

or, what is the same thing,

$$(\alpha+\beta+\gamma+1)^2 + 4\alpha\beta\gamma = 0;$$

and from the symmetry of this equation we see that, when it is satisfied,

the lines OB', OC', $O\alpha$ will lie in a plane,
" OC', OA', $O\beta$ " " ,
" OA', OB', $O\gamma$ " " ;

viz. this will be the case when the point O is situate in the cubic surface represented by the last-mentioned equation; this completes the demonstration of the solid theorems.

It is clear that considering five points 1, 2, 3, 4, 5 in a plane, then, since any one of these may be taken for the point O' of the foregoing theorem, the theorem exhibited in the first instance as a theorem relating to a triangle and two points, and afterwards as a theorem relating to a quadrangle and a point, is really a theorem relating to five points in a plane. There are, of course, five different systems of points (P, Q, R, S), corresponding to the different combinations of four out of the five points.

Cambridge, March 6, 1865.

365.

ON THE INTERSECTIONS OF A PENCIL OF FOUR LINES BY A PENCIL OF TWO LINES.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 501—503.]

PLÜCKER has considered ("Analytisch-geometrische Aphorismen," *Crelle*, vol. XI. (1834) pp. 26—32) the theory of the eight points which are the intersections of a pencil of four lines by any two lines, or say the intersections of a pencil of *four* lines by a pencil of *two* lines: viz., the eight points may be connected two together by twelve new lines; the twelve lines meet two together in forty-two new points; and of these, six lie on a line through the centre of the two-line pencil, twelve lie four together on three lines through the centre of the four-line pencil, and twenty-four lie two together on twelve lines, also through the centre of the four-line pencil.

The first and third of these theorems, viz. (1) that the six points lie on a line through the centre of the two-line pencil, and (3) that the twenty-four points lie two together on twelve lines through the centre of the four-line pencil, belong to the more simple theory of the intersections of a pencil of *three* lines by a pencil of *two* lines; the second theorem, viz. (2) the twelve points lie four together on three lines through the centre of the four-line pencil, is the only one which properly belongs to the theory of the intersections of a pencil of *four* lines by a pencil of *two* lines. The theorem in question (proved analytically by Plücker) may be proved geometrically by means of two fundamental theorems of the geometry of position: these are the theorem of two triangles in perspective, and Pascal's theorem for a line-pair. I proceed to show how this is.

Consider a pencil of two lines meeting a pencil of four lines in the eight points (a, b, c, d), (a', b', c', d'); so that the two lines are $abcd$, $a'b'c'd'$, meeting suppose in

Q; and the four lines are aa', bb', cc', dd', meeting suppose in P; then the twelve points are

$$\begin{array}{llll} a'd\,.\,c'b, & ad'\,.\,cb', & a'c\,.\,d'b, & ac'\,.\,db' \text{ lying in a line through } P, \\ a'b\,.\,d'c, & ab'\,.\,dc', & a'd\,.\,b'c, & ad'\,.\,bc' \qquad \text{,,} \qquad \text{,,} \qquad , \\ a'c\,.\,b'd, & ac'\,.\,bd', & a'b\,.\,c'd, & ab'\,.\,cd' \qquad \text{,,} \qquad \text{,,} \qquad ; \end{array}$$

where the combinations are most easily formed as follows; viz., for the first four points starting from the arrangement $\begin{matrix} a & c \\ d & b \end{matrix}$ (or any other arrangement having the diagonals $ab\,.\,cd$), and thence writing down the four expressions

$$\begin{array}{llll} a'c', & ac, & a'c, & ac' \\ db, & d'b', & d'b, & db', \end{array}$$

we read off from these the symbols of the four points; and the like for the other two sets of four points.

Now, considering the points (a, b, c) and (a', b', c'), the points $ab'\,.\,a'b$, $ac'\,.\,a'c$, $bc'\,.\,b'c$ lie in a line through Q; and similarly the points $ab'\,.\,a'b$, $ad'\,.\,a'd$, $bd'\,.\,b'd$ lie in a line through Q; which lines, inasmuch as they each contain the points Q and $ab'\,.\,a'b$, must be one and the same line; considering the combinations (b, c, d), (b', c', d'), the line in question also passes through $cd'\,.\,c'd$; that is, the six points $ab'\,.\,a'b$, $ac'\,.\,a'c$, $ad'\,.\,a'd$, $bc'\,.\,b'c$, $bd'\,.\,b'd$, $cd'\,.\,c'd$ lie in a line through Q, which is in fact the before-mentioned first theorem. Hence the points $ab'\,.\,a'b$ and $cd'\,.\,c'd$ lie in a line through Q; or, calling these points M and N respectively, the triangles Maa', Mbb', Ncc', Ndd' are in perspective. Hence, considering the two triangles Maa', Ndd' (or, if we please, the complementary set Mbb', Ncc'), the corresponding sides are

$$\begin{array}{lll} Ma,\ Nd & \text{meeting in} & ab'\,.\,dc', \\ Ma',\ Nd' & \text{,,} & a'b\,.\,d'c, \\ aa',\ dd' & \text{,,} & P\ ; \end{array}$$

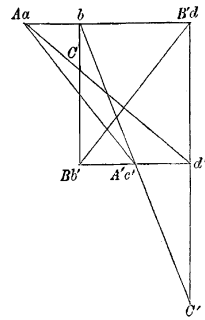
that is, the points $ab'\,.\,dc'$, $a'b\,.\,d'c$ lie in a line through P.

Similarly $ad'\,.\,a'd$ and $bc'\,.\,b'c$ lie in a line through Q; or, calling these points H, I respectively, the triangles Haa', Hdd', Ibb', Icc' are in perspective; and considering the combination Hdd', Ibb' (or, if we please, the complementary set Haa', Icc'), the corresponding sides are

$$\begin{array}{lll} Ha,\ Ib & \text{meeting in} & ad'\,.\,bc', \\ Ha',\ Ib' & \text{,,} & a'd\,.\,cb', \\ aa',\ bb' & \text{,,} & P\ ; \end{array}$$

that is, the points $a'd\,.\,c'b$, $ad'\,.\,cb'$ lie in a line through P.

It remains to be shown that the two lines through P, viz. the line containing $ab'.dc'$ and $a'b.d'c$, and the line containing $ad'.bc'$ and $a'd.cb'$, are one and the same line. This will be the case if, for instance, $ab'.dc'$ and $ad'.bc'$ also lie in a line through P.



We have the points (a, b, d) in a line, and the points (b', c', d') in a line; the points a, d, b', c' are also called A, B', B, A' respectively; ad', bb' meet in C, and bc', dd' meet in C'; hence, considering the hexagon $ad'db'bc'$, the lines

$$\begin{array}{lll} ad', b'b \text{ meet in} & C & , \\ d'd, bc' \quad ,, & C' & , \\ db', ca' \quad ,, & AA'.BB' & ; \end{array}$$

and hence these three points lie in a line; or, what is the same thing, the lines AA', BB', and CC' meet in a point; that is, the triangles ABC, $A'B'C'$ are in perspective: the corresponding sides are

$$\begin{array}{llll} AB, A'B', \text{ that is,} & ab', c'd, \text{ meeting in} & ab'.c'd, \\ BC, B'C' \quad ,, & b'b, d'd, \quad ,, & P \quad , \\ CA, C'A' \quad ,, & ad', bc', \quad ,, & ad'.bc'; \end{array}$$

and these three points lie in a line; that is, the points $ab'.dc'$ and $ad'.bc'$ lie in a line through P. Hence the line through $ab'.dc'$ and $a'b.d'c$ and the line through $ad'.bc'$ and $a'd.cb'$ are one and the same line; that is,

the points $ab'.dc'$, $a'b.d'c$, $ad'.bc'$, $a'd.b'c$ lie in a line through P.

This proves the existence of one of the lines through P; and that of the other two lines follows from the symmetry of the figure; it thus appears that the twelve points lie four together on three lines through P.

Cambridge, April 11, 1865.

366.

NOTE ON THE PROJECTION OF THE ELLIPSOID.

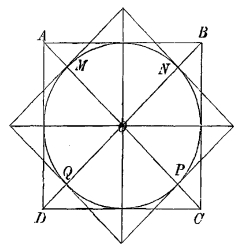
[From the *Philosophical Magazine*, vol. XXX. (1865), pp. 50—52.]

CONSIDER an ellipsoid, situate any way whatever in regard to the eye and the plane of the picture; the apparent contour of the ellipsoid is an ellipse, the intersection of the plane of the picture by the tangent cone having the eye for vertex; this cone touches the ellipsoid along a plane curve (the intersection of the ellipsoid by the polar plane of the eye), which may be called the contour section; and the apparent contour is thus the projection of the contour section. Consider any other plane section; the projection thereof has double contact (real or imaginary) with the projection of the contour section: the common tangents are the intersections with the plane of the picture of the tangent planes of the tangent cone which pass through the pole of the section; or, what is the same thing, they are the tangents to the projection of the contour section, or to the projection of the section, from the point which is the projection of the pole of the section. The projection of the pole lies in the line which is the projection of the diameter conjugate to the plane of the section; and in particular, if the section is central, that is, if the plane thereof passes through the centre of the ellipsoid, then the pole is the point at infinity on the conjugate diameter; whence also if the eye be at an infinite distance, so that the projection is a projection by parallel rays, then the projection of the pole is the point at infinity on the projection of the conjugate diameter; and therefore the common tangents of the projections of the section and the contour section are in this case parallel to the projection of the diameter conjugate to the plane of the section.

Suppose that the plane of the picture is parallel to a principal plane of the ellipsoid, and that the projection is by parallel rays; then if OA, OB, OC are the projections of the semiaxes (OA, OC will be at right angles to each other if the plane parallel to the plane of the picture is that of xz), the projections of the principal sections are the ellipses having for conjugate semidiameters OB, OC; OC, OA; OA, OB

respectively. Hence to the ellipse OB, OC drawing the two tangents which are parallel to OA, to the ellipse OC, OA the two tangents which are parallel to OB, and to the ellipse OA, OB the two tangents which are parallel to OC, we have on each of these ellipses the two points which are the points of contact therewith of the ellipse which is the projection of the contour section, or apparent contour of the ellipsoid; that is, we know six points, and at each of these points the tangent, of the last-mentioned ellipse; and the ellipse in question, or apparent contour of the ellipsoid, can thus be traced by hand accurately enough for ordinary purposes.

In connexion with what precedes, I may notice a convenient construction for the projection of a circle. Suppose that we have given the projection of the circumscribed square $ABCD$; then if we know the projection of one of the points M, N, P, Q, say of the point M, the projections of all the points and lines of the figure can be obtained



graphically by the ruler only with the utmost facility; that is, in the ellipse which is the projection of the circle we have *eight* points, and the tangent at each of them, and the ellipse may then be drawn by hand. And to find the projection of the point M, it is only necessary to remark that in the figure the anharmonic ratio $\dfrac{AM \,.\, OC}{AC \,.\, MO}$ of the points A, M, O, C is $= \frac{1}{2}(\sqrt{2}-1)$; hence the corresponding anharmonic ratio of the projections of the four points is also $= \frac{1}{2}(\sqrt{2}-1)$; and the projections of A, B, C, D, and consequently those of A, C, O, being known, the projection of M is thus also known.

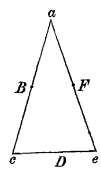
Cambridge, June 15, 1865.

367.

ON A TRIANGLE IN-AND-CIRCUMSCRIBED TO A QUARTIC CURVE.

[From the *Philosophical Magazine*, vol. XXX. (1865), pp. 340—342.]

THE quartic curve $(x^2-a^2)^2+(y^2-b^2)^2=c^4$ presents a simple example of a triangle in-and-circumscribed to a single curve, viz. such that each angle of the triangle is situate on, and each side touches, the curve. Assuming that the triangle is symmetrically situate in regard to the axis of y, viz. if it be the isosceles triangle ace, the sides whereof touch the curve in the points B, D, F respectively, then we must have a single relation between the constants a, b, c of the curve; or if (as may



be done without loss of generality) we write $a=1$, then there must be a single relation between b and c. The relation in question is most conveniently expressed by putting b and c equal to certain functions of a parameter ϕ, which is in fact $=\tan^2\theta$, if θ be the angle at the base of the triangle; the equation of the curve is thus obtained in the form

$$(x^2-1)^2+\left(y^2-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)^2=1+\frac{(\phi^2-1)^4}{16\phi^2(\phi^2+1)^2};$$

and the coordinates of a, c, e, B, D, F are as follows:

Coordinates of	a are	0,	$\sqrt{\tfrac{1}{2}\phi}$,
"	c, e, "	$\pm\sqrt{2}$,	$-\sqrt{\tfrac{1}{2}\phi}$,
"	D, "	0,	$-\sqrt{\tfrac{1}{2}\phi}$,
"	B, F "	$\pm\dfrac{\phi^2-1}{\sqrt{2}(\phi^2+1)}$,	$\dfrac{2\sqrt{\phi}}{\sqrt{2}(\phi^2+1)}$.

It is easy to verify that the points a, c, e, D are points of the curve, and it is obvious that the tangent at D is the horizontal line ce. It only remains to be shown that B and F are points of the curve, and that the tangents at these points are the lines ac and ea respectively. It is sufficient to consider one of the two points, say the point F; and taking its coordinates to be

$$\xi=\frac{\phi^2-1}{\sqrt{2}(\phi^2+1)},\quad \eta=\frac{2\sqrt{\phi}}{\sqrt{2}(\phi^2+1)},$$

we have to show that (ξ, η) is a point of the curve, and that the equation of the tangent at this point is $X\sqrt{\phi}+Y=\sqrt{\tfrac{1}{2}\phi}$, where (X, Y) are current coordinates.

First, to show that (ξ, η) is a point of the curve, the equation to be verified may be written

$$(\xi^2-1)^2+\left(\eta^2-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)^2=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2(\phi^2+1)^2},$$

and we have

$$\xi^2-1=-\frac{\phi^4+6\phi^2+1}{2(\phi^2+1)^2},\qquad \eta^2-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}=-\frac{(\phi^2-1)(\phi^4+6\phi^2+1)}{4\phi(\phi^2+1)^2},$$

so that the equation becomes

$$\frac{(\phi^4+6\phi^2+1)^2}{4(\phi^2+1)^4}+\frac{(\phi^2-1)^2(\phi^4+6\phi^2+1)^2}{16\phi^2(\phi^2+1)^4}=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2(\phi^2+1)^2};$$

that is

$$4\phi^2+(\phi^2-1)^2=(\phi^2+1)^2,$$

which is right.

Next, the equation of the tangent at the point (ξ, η) is

$$(\xi^2-a^2)(\xi X-a^2)+(\eta^2-b^2)(\eta Y-b^2)-c^4=0;$$

that is

$$(\xi^2-1)(\xi X-1)+\left(\eta^2-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)\left(\eta Y-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)=\frac{(\phi^4+6\phi^2+1)^2}{16\phi^2(\phi^2+1)^2};$$

or, substituting for ξ, η, ξ^2-1, and $\eta^2-\dfrac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}$, their values, and throwing out a factor $\dfrac{\phi^4+6\phi^2+1}{16\phi^2(\phi^2+1)^2}$, the equation becomes

$$-8\phi^2\left(X\frac{\phi^2-1}{\sqrt{2}(\phi^2+1)}-1\right)-4\phi(\phi^2-1)\left(Y\frac{2\sqrt{\phi}}{\sqrt{2}(\phi^2+1)}-\frac{\phi^4+4\phi^2-1}{4\phi(\phi^2+1)}\right)=\phi^4+6\phi^2+1,$$

or, what is the same thing,

$$-8\phi^2\{X(\phi^2-1)-\sqrt{2}(\phi^2+1)\}-(\phi^2-1)\{Y.8\phi\sqrt{\phi}-\sqrt{2}(\phi^4+4\phi^2-1)\}$$
$$=\sqrt{2}(\phi^2+1)(\phi^4+6\phi^2+1);$$

that is

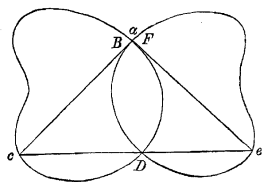
$$\begin{aligned}&(\phi^2-1)(-8\phi^2X-8\phi\sqrt{\phi}\,Y)\\ &=\sqrt{2}(\phi^2+1)(\phi^4+6\phi^2+1)-\sqrt{2}(\phi^2+1).8\phi^2-\sqrt{2}(\phi^2-1)(\phi^4+4\phi^2-1),\\ &=\sqrt{2}(\phi^2+1)(\phi^2-1)^2-\sqrt{2}(\phi^2-1)(\phi^4+4\phi^2-1),\\ &=\sqrt{2}(\phi^2-1)\{(\phi^4-1)-(\phi^4+4\phi^2-1)\},\\ &=-4\sqrt{2}(\phi^2-1)\phi^2,\end{aligned}$$

whence, finally,

$$X\sqrt{\phi}+Y=\sqrt{\tfrac{1}{2}}\,\phi,$$

which is the required equation.

It may be remarked that for $\phi=1$, the equation of the curve is $(x^2-1)^2+(y^2-\frac{1}{2})^2=1$, which is the binodal form $a^2>b^2$, $c^4=a^4$. We have in this case $\xi=0$, $\eta=\sqrt{\frac{1}{2}}$, and the curve and triangle are as shown in the figure, viz. the base ce of the triangle, instead



of being a proper tangent, is a line through the node D. For any other value of ϕ, the curve consists of an exterior oval (pinched in at the sides and the top and bottom) and of an interior oval; the angles a, c, e lie in the exterior oval, the sides ac, ea touch the interior oval, and the base ce touches the exterior oval.

If, to fix the ideas, we assume $\phi > 1$, then we have always $c^4 > a^4 < a^4 + b^4$: for $\phi = 1$ we have, as appears above, $b^2 = \frac{1}{2}$, which is $< a^2$; but for a certain value of ϕ between 3 and 4, b^2 becomes $= a^2$, and for any greater value of ϕ we have $b^2 > a^2$. The condition for the equality of a^2 and b^2 is

$$\phi^4 + 4\phi^2 - 1 = 4\phi(\phi^2 + 1), \text{ or } \phi^4 - 4\phi^3 + 4\phi^2 - 4\phi - 1 = 0;$$

this equation may be written $2\phi(\phi - 2)(\phi^2 + 1) = (\phi^2 - 1)^2$, and we thence obtain

$$\frac{(\phi^2 - 1)^4}{16\phi^2(\phi^2 + 1)^2} = \tfrac{1}{4}(\phi - 2)^2;$$

or the equation of the curve is $(x^2 - 1)^2 + (y^2 - 1)^2 = 1 + \frac{1}{4}(\phi - 2)^2$, where ϕ is determined by the equation just referred to. The curve is in this case symmetrical in regard to the two axes; and there are in fact four triangles, each in-and-circumscribed to the curve.

Cambridge, June 16, 1865.

368.

ON A PROBLEM OF GEOMETRICAL PERMUTATION.

[From the *Philosophical Magazine*, vol. xxx. (1865), pp. 370—372.]

It is required to find in how many modes the nine points of inflexion of a cubic curve can be denoted by the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, in such wise that the twelve lines, each containing three points of inflexion, shall be in every case denoted by the same triads of figures, say by the triads

123,	147,	159,	168,
456,	258,	267,	249,
789,	369,	348,	357.

We may imagine the inflexions so denoted in one particular way, which may be called the primitive denotation; then in any other mode of denotation, a figure, for example 1, is either affixed to the inflexion to which it originally belonged, and it is then said to be *in loco*, or it is affixed to some other point of inflexion. This being so, the total number of modes is $=432$; viz. this number is made up as follows:

9 figures *in loco*	1
3 „ „	60
1 figure „	243
0 „ „	128
	432

There is of course only one mode wherein the nine figures remain *in loco*. It may be seen without much difficulty that there is not any mode in which 8, 7, 6, 5, or 4 figures remain *in loco*. There is no mode in which only 2 figures remain *in loco;*

for any two inflexions are in a line with a third inflexion; and if the figures which belong to the first two inflexions are *in loco*, then the figure belonging to the third inflexion will be *in loco;* that is, there will be 3 figures *in loco*. The only remaining modes are therefore those which have 3 figures, 1 figure, or 0 figure *in loco*.

First, if three figures are *in loco*, these, as just seen, will be the figures which belong to three inflexions in a line. Suppose the figures are 1, 2, 3; then the inflexion originally denoted, say by the figure 4, may be denoted by any one of the remaining figures 5, 6, 7, 8, 9; but when the figure is once fixed upon, then the remaining inflexions can be denoted only in one manner. Hence when the figures 1, 2, 3 remain *in loco* there are 5 modes; and consequently the number of modes wherein 3 figures remain *in loco* is $5 \times 12, = 60$.

Next, if only a single figure, suppose 1, remains *in loco*, the triads which belong to the figure 1 are 123, 147, 159, 168; and there is 1 mode in which we simultaneously interchange all the pairs (2, 3), (4, 7), (5, 9), (6, 8). (Observe that the triads 123, 147, 159, 168 here denote the same lines respectively as in the primitive denotation, the figure 1 remains *in loco*, but the figures belonging to the other two inflexions on each of the four lines are interchanged.) There are, besides this, 2 modes in which the figures (2, 3), but not any other two figures, are interchanged; similarly 2 modes in which the figures (4, 7), 2 modes in which the figures (5, 9), 2 modes in which the figures (6, 8), but in each case no other two figures, are interchanged; this gives in all $1+2+2+2+2, = 9$ modes. There are besides, the figure 1 still remaining *in loco*, 18 modes where there are no two figures (2, 3), (4, 7), (5, 9), or (6, 8) which are interchanged: viz. the figure 2 may be made to denote any one of the inflexions originally denoted by 4, 5, 6, 7, 8, or 9. Suppose the inflexion originally denoted by 4; 3 will then denote the inflexion originally denoted by 7: it will be found that of three of the remaining six inflexions, any one may be denoted by the figure 4, and that the scheme of denotation can then in each case be completed in one way only. This gives $6 \times 3, = 18$, as above, for the number of the modes in question; and we have then $9 + 18, = 27$, for the number of the modes in which the figure 1 remains *in loco;* and $9 \times 27, = 243$, for the number of modes in which some one figure remains *in loco*.

Finally, if no figure remains *in loco*, the figure 1 will then denote some one of the inflexions originally denoted by 2, 3, 4, 5, 6, 7, 8, 9. Suppose it to denote that originally denoted by 2; 2 cannot then denote the inflexion originally denoted by 1, for if it did, 3 would remain *in loco:* 2 must therefore denote the inflexion originally denoted by 3, or else some one of the inflexions originally denoted by 4, 5, 6, 7, 8, 9. It appears, on examination, that in the first case there are 4 ways of completing the scheme, and in each of the latter cases 2 ways; there are therefore in all $1 \times 4 + 6 \times 2, = 16$ ways; that is, 16 modes in which (no figure remaining *in loco*) the figure 1 is used to denote the inflexion originally denoted by 2; and therefore $8 \times 16, = 128$ modes, for which no figure remains *in loco*. This completes the investigation of the numbers 1, 60, 243, and 128, which together make up the total number 432 of the modes of denotation of the nine inflexions.

369.

ON A PROPERTY OF COMMUTANTS.

[From the *Philosophical Magazine*, vol. XXX. (1865), pp. 411—413.]

I CALL to mind the definition of a commutant, viz. if in the symbol

$$\left[\begin{array}{ccc} \dagger & & \\ 1 & 1 & 1 \; (\theta) \\ 2 & 2 & 2 \\ \vdots & \vdots & \vdots \\ p & p & p \end{array}\right]$$

we permute independently in every possible manner the numbers $1, 2, \ldots p$ of each of the θ columns except the column marked ($\dagger$), giving to each permutation its proper sign, $+$ or $-$, according as the number of inversions is even or odd, thus

$$\pm_s \pm_t \ldots A_{\substack{1\, s_1\, t_1 \ldots (\theta) \\ 2\, s_2\, t_2 \\ \vdots\; \vdots\; \vdots \\ p\, s_p\, t_p}}$$

which is to be read as meaning

$$\pm_s \pm_t \ldots A_{1\, s_1\, t_1 \ldots} \quad A_{2\, s_2\, t_2 \ldots} \quad \ldots A_{p\, s_p\, t_p \ldots},$$

the sum of all the $(1.2.3\ldots p)^{\theta-1}$ terms so obtained is the commutant denoted by the above-mentioned symbol. In the particular case $\theta = 2$, the commutant is of course a determinant: in this case, and generally if θ be even, it is immaterial which of the columns is left unpermuted, so that the ($\dagger$) instead of being placed over any column may be placed on the left hand of the A; but when θ is odd, the function has different values according as one or another column is left unpermuted, and the position of the ($\dagger$) is therefore material. It may be added that if *all* the columns are permuted, then, if θ be even, the sum is $1.2\ldots p$ into the commutant obtained by leaving any one column unpermuted; but if θ is odd, then the sum is $= 0$.

The property in question is a generalization of a property of determinants, viz. we have

$$\begin{vmatrix} 2\lambda\lambda' & , & \lambda\mu'+\lambda'\mu, & \lambda\nu'+\lambda'\nu, \ldots \\ \lambda\mu'+\lambda'\mu, & & 2\mu\mu' & , & \mu\nu'+\mu'\nu, \ldots \\ \lambda\nu'+\lambda'\nu, & & \mu\nu'+\mu'\nu, & & 2\nu\nu' & , \\ \vdots & & & & & \end{vmatrix} = 0$$

whenever the order of the determinant is greater than 2.

To enunciate the corresponding property of commutants, let

$$\begin{Bmatrix} \lambda_{11}, & \lambda_{12} \ldots \\ \lambda_{21}, & \lambda_{22} \\ \cdot & \end{Bmatrix}$$

or, in a notation analogous to that of a commutant,

$$\begin{bmatrix} {}^{+}\lambda^{+} \\ 1\ 1 \\ 2\ 2 \\ \vdots\ \vdots \\ p\ p \end{bmatrix}$$

denote a function formed precisely in the manner of a determinant (or commutant of two columns), except that the several terms (instead of being taken with a sign + or − as above) are taken with the sign +: thus

$$\begin{Bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{Bmatrix} \text{ or } \begin{bmatrix} {}^{+}\lambda^{+} \\ 1\ 1 \\ 2\ 2 \end{bmatrix}$$

each denote

$$\lambda_{11}\lambda_{22}+\lambda_{12}\lambda_{21}.$$

This being so, the theorem is that the commutant

$$\begin{bmatrix} A \\ 1\ 1\ 1 \ldots (\theta) \\ 2\ 2\ 2 \\ \vdots\ \vdots\ \vdots \\ p\ p\ p \end{bmatrix}$$

where

$$A_{r\,s\,t\,\ldots\,(\theta)} = \begin{Bmatrix} \lambda_{1r}, & \lambda_{1s} \ldots (\theta) \\ \lambda_{2r}, & \lambda_{2s} \quad . \\ \cdot & \\ \cdot & \\ \lambda_{\theta r}, & \lambda_{\theta s} \quad . \end{Bmatrix} = \begin{bmatrix} {}^{+}\lambda^{+} \\ r\ 1 \\ s\ 2 \\ t\ 3 \\ \vdots\ \vdots \\ .\ p \end{bmatrix}$$

whenever $p > \theta$, is $= 0$.

To prove this, consider the general term of the commutant, viz. this is

$$\pm_s \pm_t \ldots A_{1s't'\ldots} A_{2s''t''\ldots} \ldots A_{ps^pt^p\ldots}$$

the general term of $A_{rst..}$ is $\lambda_{ar}\lambda_{bs}\lambda_{ct}..$, where $a, b, c, ..$ represent some permutation of the numbers $1, 2, 3..\theta$. Substituting the like values for each of the factors $A_{1s't'..}$, $A_{2s''t''..}$, &c., the general term of the commutant is

$$= \pm_s \pm_t \ldots \lambda_{a'1}\lambda_{b's'}\lambda_{c't'}..\lambda_{a''2}\lambda_{b''s''}\lambda_{c''t''}..\lambda_{a^p p}\lambda_{b^p s^p}\lambda_{c^p t^p}\ldots,$$

Taking the sum of this term with respect to the quantities $s', s'', ..s^p$, which denote any possible permutation of the numbers $1, 2 \ldots p$; again, with respect to the quantities $t', t'', ..t^p$, which denote any possible permutation of the numbers $1, 2, \ldots p$; and the like for each of the $(\theta - 1)$ series of quantities, the sum in question is

$$\lambda_{a'1}\lambda_{a''2}\ldots\lambda_{a^p p}\Sigma \pm_s \lambda_{b's'}\lambda_{b''s''}..\lambda_{b^p s^p}\Sigma \pm_t \lambda_{c't'}\lambda_{c''t''}\ldots\lambda_{c^p t^p}..,$$

which is

$$= \lambda_{a'1}\lambda_{a''2}\ldots\lambda_{a^p p}\begin{bmatrix} \lambda^\dagger \\ b'\,1 \\ b''\,2 \\ \vdots\;\vdots \\ b^p p \end{bmatrix}\begin{bmatrix} \lambda^\dagger \\ c'\,1 \\ c''\,2 \\ \vdots\;\vdots \\ c^p p \end{bmatrix}\ldots;$$

but p being greater than θ, since the numbers $b', b'', \ldots b^p$ are all of them taken out of the series $1, 2 \ldots \theta$, some of these numbers must necessarily be equal to each other, and we have therefore

$$\begin{bmatrix} \lambda^\dagger \\ b'\,1 \\ b''\,2 \\ \vdots\;\vdots \\ b^p p \end{bmatrix} = 0;$$

whence finally the commutant is $= 0$.

In the case where $p = \theta = 2$, we have for a determinant of the order 2 the theorem

$$\begin{vmatrix} 2\lambda\lambda', & \lambda\mu' + \lambda'\mu \\ \lambda\mu' + \lambda'\mu, & 2\mu\mu' \end{vmatrix} = -\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}^2;$$

and it is probable that there exists a corresponding theorem for the commutant

$$\begin{bmatrix} A^\dagger \\ 1\,1\,1..(p) \\ 2\,2\,2 \\ \vdots\;\vdots\;\vdots \\ p\;p\;p \end{bmatrix},$$

where

$$A_{r\,s\,t\,..\,(p)} = \left\{\begin{matrix} \lambda_{1r}, & \lambda_{1s} \ldots (p) \\ \lambda_{2r}, & \lambda_{2s} \quad . \\ \vdots & \\ \lambda_{pr}, & \lambda_{ps} \quad . \end{matrix}\right\} = \begin{bmatrix} {}^\dagger\lambda^+ \\ r\,1 \\ s\,2 \\ t\,3 \\ \vdots\;\vdots \\ .\;p \end{bmatrix},$$

but I have not ascertained what this theorem is.

Cambridge, October 26, 1865.

370.

ON THE SIGNIFICATION OF AN ELEMENTARY FORMULA OF SOLID GEOMETRY.

[From the *Philosophical Magazine*, vol. XXX. (1865), pp. 413, 414.]

THE expression for the perpendicular distance of a point (x, y, z) from a line through the origin inclined at the angles (α, β, γ) to the three axes respectively, is

$$\begin{aligned} p^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 \\ = \quad (y\cos\gamma - z\cos\beta)^2 \\ + (z\cos\alpha - x\cos\gamma)^2 \\ + (x\cos\beta - y\cos\alpha)^2; \end{aligned}$$

and the remark in reference to it is that, if at the given point P we draw, perpendicular to the plane through P and the given line, a distance PK equal to the distance of P from the given line, then the expressions

$$y\cos\gamma - z\cos\beta, \quad z\cos\alpha - x\cos\gamma, \quad x\cos\beta - y\cos\alpha,$$

which enter into the preceding formula, denote respectively the coordinates of the point K referred to P as origin.

If the given line instead of passing through the origin pass through the point x_0, y_0, z_0, then the corresponding expressions are of course

$$(y - y_0)\cos\gamma - (z - z_0)\cos\beta, \quad (z - z_0)\cos\alpha - (x - x_0)\cos\gamma, \quad (x - x_0)\cos\beta - (y - y_0)\cos\gamma,$$

and if we denote the "six coordinates" of the given line, viz.

$$\cos\alpha, \quad \cos\beta, \quad \cos\gamma, \quad y_0\cos\gamma - z_0\cos\beta, \quad z_0\cos\alpha - x_0\cos\gamma, \quad x_0\cos\beta - y_0\cos\gamma,$$

by

$$a\ , \quad b\ , \quad c\ , \quad f\ , \quad g\ , \quad h$$

respectively (so that $af+bg+ch=0$), then the three expressions become

$$cy-bz-f\,, \qquad az-cx-g, \qquad bx-ay-h$$

respectively.

It is moreover clear that if the point P be moved to P' by an infinitesimal rotation ω about the given line, then P' lies on the line PK at a distance $PP', =\omega PK$, from the point P, and the displacements of P in the directions of the axes are consequently equal to

$$\omega\,(cy-bz-f), \quad \omega\,(az-cx-g), \quad \omega\,(bx-ay-h)$$

respectively, which is a fundamental formula in the theory of the infinitesimal rotations of a solid body.

Cambridge, October 26, 1865.

371.

ON A FORMULA FOR THE INTERSECTIONS OF A LINE AND CONIC, AND ON AN INTEGRAL FORMULA CONNECTED THEREWITH.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VII. (1866), pp. 1—6.]

In a letter to me, dated 15 May, 1862, Mr Spottiswoode has extracted from an unpublished Memoir, and he has kindly permitted me to communicate, the following formula for the points of intersection of a line and conic; viz. if the equations of the line and conic are

$$\xi x + \eta y + \zeta z = 0,$$

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0\,;$$

and if

$$\theta^2 = \begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi, & a, & h, & g \\ \eta, & h, & b, & f \\ \zeta, & g, & f, & c \end{vmatrix},$$

or, what is the same thing, if

$$\theta^2 = -(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2,$$

where $A = bc - f^2$, &c. as usual; then the coordinates $(x,\ y,\ z)$ of a point of intersection of the line and conic are found from the linear equations

$$(g\eta - h\zeta - \theta)\,x + (f\eta - b\zeta\quad)\,y + (c\eta - f\zeta\quad)\,z = 0,$$

$$(a\zeta - g\xi\quad)\,x + (h\zeta - f\xi - \theta)\,y + (g\zeta - c\xi\quad)\,z = 0,$$

$$(h\xi - a\eta\quad)\,x + (b\xi - h\eta\quad)\,y + (f\xi - g\eta - \theta)\,z = 0,$$

equivalent of course to two equations, and giving by the elimination of (x, y, z), the equation

$$\theta\,[-(A, \ldots \between \xi, \eta, \zeta)^2 - \theta^2] = 0,$$

that is, giving for θ the foregoing value. And the linear equations then give

$$\begin{aligned}
x \quad & : y \quad & : z, \\
= \xi \frac{d\theta}{d\xi} + g\eta - h\zeta + \theta \; & : \; \xi \frac{d\theta}{d\eta} + a\zeta - g\xi \quad & : \; \xi \frac{d\theta}{d\zeta} + h\xi - a\eta, \\
= \eta \frac{d\theta}{d\xi} + f\zeta - b\xi \quad & : \; \eta \frac{d\theta}{d\eta} + h\xi - f\eta + \theta \; & : \; \eta \frac{d\theta}{d\zeta} + b\eta - h\zeta, \\
= \zeta \frac{d\theta}{d\xi} + c\xi - f\eta \quad & : \; \zeta \frac{d\theta}{d\eta} + g\eta - c\zeta \quad & : \; \zeta \frac{d\theta}{d\zeta} + f\zeta - g\xi + \theta,
\end{aligned}$$

where obviously

$$-\theta \frac{d\theta}{d\xi} = A\xi + H\eta + G\zeta, \quad -\theta \frac{d\theta}{d\eta} = H\xi + B\eta + F\zeta, \quad -\theta \frac{d\theta}{d\zeta} = G\xi + F\eta + C\zeta.$$

By changing the sign of θ, we have of course the coordinates of the other point of intersection. The formulæ which, singularly enough, have since been given incidentally by M. Aronhold [1], may be easily obtained as follows.

Writing for shortness

$$\begin{aligned}
P &= ax + hy + gz, \\
Q &= hx + by + fz, \\
R &= gx + fy + cz,
\end{aligned}$$

then the equation of the conic gives

$$Px + Qy + Rz = 0,$$

and combining with this the equation

$$\xi x + \eta y + \zeta z = 0,$$

we have

$$x : y : z = Q\zeta - R\eta : R\xi - P\zeta : P\eta - Q\xi,$$

or what is the same thing, taking an indeterminate multiplier θ,

$$\begin{aligned}
-\theta x + R\eta - Q\zeta &= 0, \\
-\theta y + P\zeta - R\xi &= 0, \\
-\theta z + Q\xi - P\eta &= 0,
\end{aligned}$$

[1] In his interesting Memoir "Ueber eine neue algebraische Behandlungsweise der Integrale irrationaler Differentiale von der Form $\Pi(x, y)\,dx$, in welcher $\Pi(x, y)$ eine beliebige rationale Function ist, und zwischen x und y eine allgemeine Gleichung zweiter Ordnung besteht." *Crelle*, t. LXI. (1862).

which are, in fact, Mr Spottiswoode's linear equations, and which lead, as before, to the value

$$\theta^2 = -(A, \ldots \between \xi, \eta, \zeta)^2.$$

But this value of θ is obtained in a different manner by expressing (x, y, z) as linear functions of P, Q, R; viz. putting as usual $K = abc - af^2 - bg^2 - ch^2 + 2fgh$, the linear equations thus become

$$AP + HQ + GR + \frac{K}{\theta}(\zeta Q - \eta R) = 0,$$

$$HP + BQ + FR + \frac{K}{\theta}(\xi R - \zeta P) = 0,$$

$$GP + FQ + CR + \frac{K}{\theta}(\eta P - \xi Q) = 0,$$

or eliminating (P, Q, R), we have

$$\begin{vmatrix} A, & H + \frac{K\zeta}{\theta}, & G - \frac{K\eta}{\theta} \\ H - \frac{K\zeta}{\theta}, & B, & F + \frac{K\xi}{\theta} \\ G + \frac{K\eta}{\theta}, & F - \frac{K\xi}{\theta}, & C \end{vmatrix} = 0,$$

that is

$$ABC - A\left(F^2 - \frac{K^2\xi^2}{\theta^2}\right) - B\left(G^2 - \frac{K^2\eta^2}{\theta^2}\right) - C\left(H^2 - \frac{K^2\zeta^2}{\theta^2}\right)$$
$$+ \left(F + \frac{K\xi}{\theta}\right)\left(G + \frac{K\eta}{\theta}\right)\left(H + \frac{K\zeta}{\theta}\right)$$
$$+ \left(F - \frac{K\xi}{\theta}\right)\left(G - \frac{K\eta}{\theta}\right)\left(H - \frac{K\zeta}{\theta}\right) = 0;$$

or, reducing,

$$ABC - AF^2 - BG^2 - CH^2 + 2FGH + \frac{K^2}{\theta^2}(A, \ldots \between \xi, \eta, \zeta)^2 = 0,$$

that is

$$\theta^2 + (A, \ldots \between \xi, \eta, \zeta)^2 = 0,$$

as before.

I reproduce, as follows, a fundamental formula of Aronhold's Memoir. Consider the function

$$\varpi = \frac{1}{\sqrt{\{-(A, \ldots \between u, v, w)^2\}}} \log \frac{(a, \ldots \between x_1, y_1, z_1 \between x, y, z)}{ux + vy + wz},$$

where x_1, y_1, z_1 (corresponding to x, y, z in the former part of this paper) are determined by the conditions

$$(a, \ldots \between x_1, y_1, z_1)^2 = 0,$$
$$ux_1 + vy_1 + wz_1 = 0,$$

so that putting

$$\theta = \sqrt{\{-(A, \ldots \between u, v, w)^2\}},$$

and

$$P_1 = ax_1 + hy_1 + gz_1,$$
$$Q_1 = hx_1 + by_1 + fz_1,$$
$$R_1 = gx_1 + fy_1 + cz_1,$$

we now have

$$-\theta x_1 + R_1 v - Q_1 w = 0,$$
$$-\theta y_1 + P_1 w - R_1 u = 0,$$
$$-\theta z_1 + Q_1 u - P_1 v = 0,$$

and the value of ϖ is

$$\varpi = \frac{1}{\theta} \log \frac{P_1 x + Q_1 y + R_1 z}{ux + vy + wz}.$$

Treating (x, y, z) as independent variables and differentiating, we have

$$d\varpi = \frac{1}{\theta}\left\{\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 x + Q_1 y + R_1 z} - \frac{udx + vdy + wdz}{ux + vy + wz}\right\}$$

$$= \frac{1}{\theta} \frac{(xdy - ydx)(Q_1 u - P_1 v) + (ydz - zdy)(R_1 v - Q_1 w) + (zdx - xdz)(P_1 w - R_1 u)}{(P_1 x + Q_1 y + R_1 z)(ux + vy + wz)}$$

$$= \frac{x_1(ydz - zdy) + y_1(zdx - xdz) + z_1(xdy - ydx)}{(P_1 x + Q_1 y + R_1 z)(ux + vy + wz)},$$

or, what is the same thing, if

$$P = ax + hy + gz,$$
$$Q = hx + by + fz,$$
$$R = gx + fy + cz,$$

so that

$$P_1 x + Q_1 y + R_1 z = (a, \ldots \between x, y, z \between x_1, y_1, z_1) = Px_1 + Qy_1 + Rz_1,$$

then we have

$$d\varpi = \frac{(ydz - zdy)x_1 + (zdx - xdz)y_1 + (xdy - ydx)z_1}{(ux + vy + wz)(Px_1 + Qy_1 + Rz_1)}.$$

Suppose now that (x, y, z) are connected by the equation

$$(a, \ldots \between x, y, z)^2 = 0,$$

we have

$$P\,x + Q\,y + R\,z = 0,$$
$$Pdx + Qdy + Rdz = 0,$$

and thence

$$ydz - zdy = \Theta P,$$
$$zdx - xdz = \Theta Q,$$
$$xdy - ydx = \Theta R,$$

and consequently

$$d\varpi = \frac{\Theta}{ux + vy + wz} = \frac{ydz - zdy}{(ux + vy + wz)\,P} = \frac{zdx - xdz}{(ux + vy + wz)\,Q} = \frac{xdy - ydx}{(ux + vy + wz)\,R};$$

or selecting the value

$$d\varpi = \frac{zdx - xdz}{(ux + vy + wz)\,Q} = \frac{zdx - xdz}{(ux + vy + wz)(hx + by + fz)},$$

and writing

$$\frac{x}{z} = X, \quad \frac{y}{z} = Y,$$

we have

$$d\varpi = \frac{z^2 dX}{(ux + vy + wz)(hx + by + fz)}$$
$$= \frac{dX}{(uX + vY + w)(hX + bY + f)},$$

where X and Y are connected by the equation

$$(a, \ldots \between X,\ Y,\ 1)^2 = 0,$$

that is, Y is a given quadric radical function of X. Hence integrating and restoring for ϖ its original value, but writing therein $\frac{x}{z} = X$ and $\frac{y}{z} = Y$, we have

$$\int \frac{dX}{(uX + vY + w)(hX + bY + f)} = \frac{1}{\sqrt{\{-(A, \ldots \between u,\ v,\ w)^2\}}} \log \frac{(a, \ldots \between x_1,\ y_1,\ z_1 \between X,\ Y,\ 1)}{uX + vY + w},$$

where, as just mentioned, Y is a given quadric radical function of X determined by the equation

$$(a,\ b,\ c,\ f,\ g,\ h \between X,\ Y,\ 1)^2 = 0,$$

and the constants x_1, y_1, z_1 are such that

$$(a, \ldots \between x_1,\ y_1,\ z_1)^2 = 0,$$
$$ux_1 + vy_1 + wz_1 = 0,$$

the ratios of these quantities being therefore determinate; there would, it is clear, be no loss of generality in assuming $z_1 = 1$. This is Aronhold's Theorem I.

2, *Stone Buildings, W.C., October* 23, 1862.

372.

ON THE RECIPROCATION OF A QUARTIC DEVELOPABLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VII. (1866), pp. 87—92.]

IT is interesting to consider in a particular case the system of equations which shows *à posteriori* that the reciprocal of a torse (developable surface) is a curve (curve in space); and the reciprocal system which shows that the reciprocal of a curve is a torse.

Using (a, b, c, d) and (x, y, z, w) for the reciprocal coordinates, it will be convenient to collect the different equations as follows:

$$a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2 = 0, \tag{1}$$

$$\left.\begin{aligned} ad^2 - 3\,bcd + 2\,c^3 \;\; + \lambda x &= 0,\\ -3\,acd + 6\,b^2d \;\; - 3\,bc^2 + \lambda y &= 0,\\ -3\,abd + 6\,ac^2 \;\; - 3\,b^2c - \lambda z &= 0,\\ a^2d \;\; - 3\,abc + 2\,b^3 \;\; + \lambda w &= 0, \end{aligned}\right\} \tag{2}$$

$$ax + by + cz + dw = 0, \tag{3}$$

$$3xz - y^2 = 0, \quad yz - 9xw = 0, \quad 3yw - z^2 = 0, \tag{4}$$

$$\left.\begin{aligned} 3\,pz - 9\,qw \qquad\qquad + a &= 0,\\ -2\,py + \;\; qz + 3\,rw + b &= 0,\\ 3\,px + \;\; qy - 2\,rz + c &= 0,\\ -9\,qx + 3\,ry + d &= 0, \end{aligned}\right\} \tag{5}$$

This being so, the equation (1) belongs to a quartic torse, the reciprocal whereof is the skew cubic determined by the equations (4): and we have to show *à posteriori* that this is so.

First if the torse is given, then the reciprocal figure is the envelope of the plane $ax+by+cz+dw=0$, (3), where (x, y, z, w) are the coordinates and (a, b, c, d) are regarded as variable parameters connected by the equation (1); we thence obtain the equations (2), where λ is an arbitrary multiplier; and from the equations (1), (2), and (3), we have to eliminate a, b, c, d, λ. The equation (3) is at once seen to be included in the equations (1) and (2); and the elimination would give only a single equation between the (x, y, z, w)—since however the equation (1) is that of a torse, we know that the elimination must give two equations, or more accurately a two-fold relation (represented, as in the present case it happens, by three equations) between the coordinates (x, y, z, w).

Putting for shortness

$$\square = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2:$$

and substituting for $\lambda x, \lambda y, \lambda z, \lambda w$, their values from the equations (1), we have identically

$$\begin{aligned}
\lambda^2(\ xz - \tfrac{1}{3}y^2\ ) &= -(bd-c^2)\,\square,\\
\lambda^2(\tfrac{1}{3}yz - \ xw) &= -(ad-bc)\,\square,\\
\lambda^2(\ yw - \tfrac{1}{3}z^2\ ) &= -(ac-b^2)\,\square;
\end{aligned}$$

and hence (since $\square = 0$) we have between (x, y, z, w) the equations (4), showing that the reciprocal figure is a skew cubic.

Secondly, let the skew cubic be given; then the reciprocal figure is the envelope of the plane $ax+by+cz+dw=0$, (3), where now (a, b, c, d) are the coordinates and (x, y, z, w) are regarded as variable parameters connected by the equations (4): we thence obtain the equations (5) containing the arbitrary multipliers p, q, r: and from the equations (3), (4), (5) we have to eliminate x, y, z, w, p, q, r. The equation (3) is at once seen to be included in the equations (4) and (5): since however the equations (4) are those of a curve, we know that the elimination must give a single equation between (a, b, c, d).

The equations (4) are satisfied if we write therein

$$x : y : z : w = \tfrac{1}{3} : \theta : \theta^2 : \tfrac{1}{3}\theta^3,$$

and substituting these values of x, y, z, w in the equations (5), these equations give (a, b, c, d) in terms of θ, p, q, r, and we thence find identically

$$\begin{aligned}
ac-b^2 &= -\ \theta^2(p-2q\theta+r\theta^2)^2,\\
bc-ad &= -2\theta\ (p-2q\theta+r\theta\ )^2,\\
bd-c^2 &= -\quad (p-2q\theta+r\theta^2)^2,
\end{aligned}$$

that is

$$ac-b^2 : ad-bc : bd-c^2 = \theta^2 : -2\theta : 1,$$

or we have

$$(ad-bc)^2-4(ac-b^2)(bd-c^2)=0,$$

which is in fact the equation

$$\square = a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2=0, \tag{1}$$

and thus the reciprocal is in fact the quartic torse, given by the equation (1).

The equations (3), (4), (5) lead to

$$x\delta a+y\delta b+z\delta c+w\delta d=0;$$

but (a, b, c, d) are in consequence of these equations connected by a single equation, viz. the equation (1); the equation just obtained is thus the only relation between the differentials $(\delta a, \delta b, \delta c, \delta d)$: it is clear from the equations (2) that this is in fact the equation

$$\delta\square=\delta_a\square\delta a+\delta_b\square\delta b+\delta_c\square\delta c+\delta_d\square\delta d=0.$$

The equations (1), (2), (3) lead in like manner to

$$a\delta x+b\delta y+c\delta z+d\delta w=0,$$

which in virtue of the equations (5) is equivalent to

$$p\delta(3xz-y^2)+q\delta(yz-9xw)+r\delta(3yw-z^2)=0,$$

viz. it is a consequence of

$$\delta(3xz-y^2)=0,\quad \delta(yz-9xw)=0,\quad \delta(3yw-z^2)=0.$$

But inasmuch as (x, y, z, w) are connected by a two-fold relation, the equations (1), (2), (3) must lead to one other linear relation between the differentials $(\delta x, \delta y, \delta z, \delta w)$: and I proceed to show that this is so.

Differentiating we have

$$\begin{aligned}
d^2\,\delta a &- 3cd\,\delta b+(-3bd+6c^2)\,\delta c+(2ad-3bc)\,\delta d+\lambda dx+x\delta\lambda=0,\\
-3cd\,\delta a &+(12bd-3c^2)\,\delta b+(-3ad-6bc)\,\delta c+(-3ac-6b^2)\,\delta d+\lambda dy+yd\lambda=0,\\
(-3bd+6c^2)\,\delta a &+(-3ad-6bc)\,\delta b+(12ac-3b^2)\,\delta c- 3ab\,\delta d+\lambda dz+zd\lambda=0,\\
(2ad-3bc)\,\delta a &+(-3ac+6b^2)\,\delta b- 3ab\,\delta c+ a^2\,\delta d+\lambda dw+wd\lambda=0.
\end{aligned}$$

Consider the matrix

$$\begin{vmatrix}
d^2, & -3cd, & -3bd+6c^2, & 2ad-3bc\\
-3cd, & 12bd-3c^2, & -3ad-6bc, & -3ac+6b^2\\
-3bd+6c^2, & -3ad-6bc, & 12ac-3b^2, & -3ab\\
2ad-3bc, & -3ac+6b^2, & -3ab, & a^2
\end{vmatrix}$$

and suppose for a moment that the determinant formed therewith is $=K$; suppose also that the reciprocal matrix is

$$\left|\begin{matrix} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{matrix}\right|.$$

Then we have

$$K\delta a + d\lambda\,(\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + \mathfrak{L}w) + \lambda\,(\mathfrak{A}dx + \mathfrak{H}dy + \mathfrak{G}dz + \mathfrak{L}dw) = 0,$$

with the similar equations involving b, c, d and $(\mathfrak{H}, \mathfrak{B}, \mathfrak{F}, \mathfrak{M})$, $(\mathfrak{G}, \mathfrak{F}, \mathfrak{C}, \mathfrak{N})$, $(\mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{D})$ respectively.

But, substituting for (x, y, z, w) their values from the equations (2), it is easy to see that we have

$$\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + \mathfrak{L}w = -\frac{1}{3\lambda}Ka,$$

the last-mentioned equation thus is

$$K\left(\delta a - \tfrac{1}{3}\frac{a}{\lambda}\delta\lambda\right) + \lambda\,(\mathfrak{A}\delta x + \mathfrak{H}\delta y + \mathfrak{G}\delta z + \mathfrak{L}\delta w) = 0.$$

But K is $=27\,\square$ (see my paper "On Certain Developable Surfaces," *Quarterly Mathematical Journal*, t. VI. 1864, pp. 108—126, [344]), which is $=0$, and we thus have

$$\mathfrak{A}\delta x + \mathfrak{H}\delta y + \mathfrak{G}\delta z + \mathfrak{L}\delta w = 0,$$

and similarly

$$\mathfrak{H}\delta x + \mathfrak{B}\delta y + \mathfrak{F}\delta z + \mathfrak{M}\delta w = 0,$$
$$\mathfrak{G}\delta x + \mathfrak{F}\delta y + \mathfrak{C}\delta z + \mathfrak{N}\delta w = 0,$$
$$\mathfrak{L}\delta x + \mathfrak{M}\delta y + \mathfrak{N}\delta z + \mathfrak{D}\delta w = 0.$$

But observing that in virtue of the equation $K=0$, we have

$$\mathfrak{A} : \mathfrak{H} : \mathfrak{G} : \mathfrak{L} = \mathfrak{H} : \mathfrak{B} : \mathfrak{F} : \mathfrak{M} = \mathfrak{G} : \mathfrak{F} : \mathfrak{C} : \mathfrak{N} = \mathfrak{L} : \mathfrak{M} : \mathfrak{N} : \mathfrak{D},$$

these are, as they should be, one and the same equation.

The values of the coefficients $\mathfrak{A}$, $\mathfrak{B}$, &c. are given, p. 112 of the paper just referred to, viz. writing $\square$ in place of U, we have $\mathfrak{A} = 3X^2 - 4a^2\square$, &c. where

$$X = \quad a^2d - 3abc + 2b^3,$$
$$Y = \quad abd - 2ac^2 + b^2c,$$
$$Z = -acd + 2b^2d - bc^2,$$
$$W = -ad^2 + 3bcd - 2c^2,$$

and writing $\square = 0$, the values each divided by 3, are simply

$$\begin{array}{llll} X^2\ , & XY, & XZ, & XW, \\ YX\ , & Y^2\ , & YZ, & YW, \\ ZX\ , & ZY, & Z^2\ , & ZW\ , \\ WX, & WY, & WZ, & W^2\ , \end{array}$$

so that each of the four equations in fact, becomes

$$X\delta x + Y\delta y + Z\delta z + W\delta w = 0,$$

or multiplying by 3, and attending to the equations (2), this is

$$-3w\delta x + z\delta y - y\delta z + 3x\delta w = 0.$$

This should be a consequence of the equations

$$\delta(3xz - y^2) = 0, \quad \delta(yz - 9xw) = 0, \quad \delta(3yw - z^2) = 0,$$

that is, we should be able from the first three to deduce the fourth equation in the system

$$\begin{aligned} 3z\ \delta x - 2y\,\delta y + 3x\,\delta z \qquad\qquad &= 0, \\ -9w\,\delta x + \ z\,\delta y + \ y\,\delta z - 9x\,\delta w &= 0, \\ 3w\,\delta y - 2z\,\delta z + 3y\,\delta w &= 0, \\ -3w\,\delta x + \ z\,\delta y - \ y\,\delta z + 3x\,\delta w &= 0, \end{aligned}$$

or we ought to have

$$\begin{vmatrix} 3z, & -2y, & 3x, & \\ -9w, & z, & y, & -9x \\ & 3w, & -2z, & 3y \\ -3w, & z, & -\ y, & 3x \end{vmatrix} = 0\,;$$

but expanding, this is

$$-6\,(81x^2w^2 - 54xyzw + 12xz^3 + 12y^3w - 3y^2z^2) = 0,$$

or

$$(yz - 9xw)^2 - 4\,(xz - y^2)\,(yw - z^2) = 0,$$

which is true in virtue of the relations (4). Or what is the same thing, we may show without difficulty that the equation

$$-3w\delta x + z\delta y - y\delta z + 3x\delta w = 0,$$

is satisfied by writing therein $x : y : z : w = \frac{1}{3} : \theta : \theta^2 : \frac{1}{3}\theta^3$.

I remark that in general, if $\square = \phi(a, b, c, d) = 0$ is the equation of a torse, then for finding the reciprocal curve, we have

$$\begin{aligned} \square = 0, \quad & ax + by + cz + dw = 0, \\ & \delta_a \square + \lambda x = 0, \\ & \delta_b \square + \lambda y = 0, \\ & \delta_c \square + \lambda z = 0, \\ & \delta_d \square + \lambda w = 0, \end{aligned}$$

and that from these equations we deduce not only

$$a\delta x + b\delta y + c\delta z + d\delta w = 0,$$

but also the equation

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}\between \alpha, \beta, \gamma, \delta\between \delta x, \delta y, \delta z, \delta w) = 0,$$

where $\mathfrak{A}, \ldots$ are the inverse system derived from the second differential coefficients of $\square$: $(\alpha, \beta, \gamma, \delta)$ are arbitrary coefficients, introduced only for symmetry, and there is no real loss of generality in reducing all but one of them to zero, and so reducing the equation for example to the form

$$\mathfrak{A}\delta x + \mathfrak{H}\delta y + \mathfrak{G}\delta z + \mathfrak{L}\delta w = 0.$$

The existence of the two linear equations between $(\delta x, \delta y, \delta z, \delta w)$ *proves* that (x, y, z, w) are connected by a two-fold relation, that is, that the reciprocal of the given torse is a curve.

Cambridge, January 26, 1865.

373.

ON A SPECIAL SEXTIC DEVELOPABLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VII. (1866), pp. 105—113.]

THE present paper contains some investigations in relation to the special sextic developable or torse

$$(ae-4bd)^3-27(-ad^3-b^3e)^2=0,$$

considered Nos. 26 to 35 of my paper "On Certain Developable Surfaces," *Quarterly Mathematical Journal*, t. VI. (1864), pp. 108—126, [344].

The cuspidal curve is

$$ae-4bd=0,\quad ad^2+b^3e=0,$$

and the nodal curve is

$$ae+2bd=0,\quad ad^2-b^3e=0,$$

viz. to put this in evidence, the equation is to be written in the form

$$(ae+2bd)^2(ae-16bd)-27(ad^2-b^3e)^2=0.$$

The coordinates of a point on the cuspidal curve may be taken to be

$$a=2,\quad b=-t,\quad d=+t^3,\quad e=-2t^4,$$

and then if A, B, D, E are current coordinates, and α, β, δ, ϵ arbitrary coefficients, the equation of a plane through the tangent line is

$$\begin{vmatrix} A, & B, & D, & E \\ 2, & -t, & +t^3, & -2t^4 \\ . & -1, & +3t^2, & -8t^3 \\ \alpha, & \beta, & \delta, & \epsilon \end{vmatrix}=0,$$

which is

$$\left.\begin{array}{l}(A\beta - B\alpha)(-\ t^6)\\ +(A\delta - D\alpha)(-3t^4)\\ +(A\epsilon - E\alpha)(-\ t^3)\\ +(B\delta - D\beta)(-8t^3)\\ +(B\epsilon - E\beta)(-3t^2)\\ +(D\epsilon - E\delta)(-1\ )\end{array}\right\} = 0,$$

or, what is the same thing,

$$\left.\begin{array}{l}\alpha(\ \ Bt^6 + 3Dt^4 + Et^3)\\ +\beta(-\ At^6 + 8Dt^3 + 3Et^2)\\ +\delta(-3At^4 - 8Bt^3 + E\ )\\ +\epsilon(-\ At^3 - 3Bt^2 - D\ )\end{array}\right\} = 0,$$

and by equating to zero the coefficients α, β, δ, ϵ, we have four equations which are easily seen to reduce themselves to two equations only, and which are in fact the equations of the tangent line, the equations of this line may therefore be taken to be

$$\left.\begin{array}{l}At^3 + 3Bt^2 + D = 0\\ 3At^4 + 8Bt^3 - E = 0\end{array}\right\}.$$

The coordinates of a point in the nodal curve may be taken to be

$$a = \sqrt{2},\quad b = \tau,\quad d = -\tau^3,\quad e = \sqrt{2}\,\tau^4,$$

and substituting these values in the place of A, B, D, E in the equations of the tangent line, we have

$$\sqrt{2}\,t^3 + 3t^2\tau - \tau^3 \quad = 0,$$

$$3\sqrt{2}\,t^4 + 8t^3\tau - \sqrt{2}\,\tau^4 = 0,$$

or, what is the same thing,

$$\tau^3 - 3t^2\tau \quad + \sqrt{2}\,t^3 = 0, \text{ i.e. } \{\tau + \sqrt{2}\,t \quad\}\{\tau^2 - \sqrt{2}\,t\tau - t^2\} = 0,$$

$$\tau^4 - 4\sqrt{2}\,t^3\tau - 3t^3 \quad = 0, \qquad \{\tau^2 + \sqrt{2}\,t\tau - t^2\}\{\tau^2 - \sqrt{2}\,t\tau - t^2\} = 0,$$

so that the equations are satisfied by the values of τ given by the equation

$$\tau^2 - \sqrt{2}\,t\tau - t^2 = 0,$$

that is, by the values

$$\tau = \frac{1 \pm \sqrt{3}}{\sqrt{2}}\,t,$$

which belong to the points where the tangent line meets the nodal curve. Call these values τ_1 and τ_2; then considering a, b as current coordinates, the values of $a : b$

belonging to the point where the tangent line meets the cuspidal curve considered as *three* coincident points, and to the points where it meets the nodal curve, are given by the equation

$$(2b+at)^2\{b\sqrt{2}-a\tau_1\}(b\sqrt{2}-a\tau_2\}=0,$$

that is

$$(2b+at)^3(2b^2-2abt-a^2t^2)=0,$$

or say

$$(at+2b)^3(a^2t^2+2abt-2b^2)=0.$$

I proceed to find the intersections of the tangent with the Prohessian: for this purpose putting for a moment in the last-mentioned equation x for at and y for b, this is

$$(x+2y)^3(x^2+2xy-2y^2)=0,$$

or, if in the place of $(x+y)$ we write x, this is

$$(x+y)^3(x^2-3y^2)=0,$$

and the Hessian of this is easily found to be

$$(x+y)^4(3x^2+8xy+4y^2);$$

whence, replacing x by $(x+y)$, the Hessian of

$$(x+2y)^3(\ x^2+\ \ 2xy-\ \ 2y^2),$$

is

$$(x+2y)^4(3x^2+14xy+18y^2).$$

We have thus

$$3x^2+14xy+18y^2=0\,;$$

that is

$$3x+\{7\pm\sqrt{-5}\}\,y=0,$$

or

$$3at+\{7\pm\sqrt{-5}\}\,b=0\,;$$

and therefore

$$\frac{b}{a}=\frac{-3}{7\pm\sqrt{-5}}\,t=\frac{-3\{7\mp\sqrt{-5}\}}{54}\,t=-\frac{7\mp\sqrt{-5}}{18}\,t\,;$$

or putting

$$n_1=-\frac{7+\sqrt{-5}}{18},$$

$$n_2=-\frac{7-\sqrt{-5}}{18};$$

so that $n_1+n_2=-\frac{7}{9}$, $n_1n_2=\frac{1}{6}$, and n_1, n_2 are the roots of the equation $18n^2+14n+3=0$, then we have $\frac{b}{a}=n_1t$ or n_2t, say $\frac{b}{a}=n_1t$, or assuming $a=1$, then $b=n_1t$.

But the equations of the tangent line being *ut suprâ*

$$at^3 + 3bt^2 + d = 0,$$
$$3at^4 + 8bt^3 - e = 0,$$

we have thus

$$\begin{array}{l|l} a = 1, & a = 1, \\ b = n_1 t, & b = n_2 t, \\ d = (-1 - 3n_1)\, t^3, & d = (-1 - 3n_2)\, t^3, \\ e = (\ \ 3 + 8n_1)\, t^4, & e = (\ \ 3 + 8n_2)\, t^4, \end{array}$$

as the coordinates of the required points, viz. the tangent line meets the Prohessian, in the point on the cuspidal edge considered as 6 points, in two points on the nodal curve and in the last-mentioned 2 points; $6 + 2 + 2 = 10$ the order of the Prohessian.

The foregoing equations give

$$ae - 6bd = 18n_1^2 + 14n_1 + 3 = 0,$$

$$\frac{ad^2}{b^2e} = \frac{(1 + 3n_1)^2}{n_1^2(8n_1 + 3)} = \tfrac{1}{81}(144\, n_1 - 23),$$

$$(\text{in virtue of } 18n_1^2 + 14n_1 + 3 = 0),$$

so that the two points in question are the intersections of the tangent line with the surface $ae - 6bd = 0$.

If we consider the intersection of this surface with the torse

$$(ae - 4bd)^3 - 27\,(-ad^2 - b^2e)^3 = 0,$$

the equation $ae - 6bd = 0$, gives

$$(ae - 4bd)^3 = (2bd)^3 = 8b^3d^3bd = \tfrac{3}{4}\, aeb^3d^2,$$

and thence

$$4ab^3d^3e - 81\,(ad^2 + b^2e)^2 = 0;$$

that is

$$81a^2d^4 - 158ab^3d^2e + 81b^4e^2 = 0,$$

an equation which should agree with

$$\frac{ad^2}{b^2e} = \tfrac{1}{81}(144\, n_1 - 23).$$

In fact writing

$$x = \tfrac{1}{81}(144\, n_1 - 23),$$

the equation $18n_1^2 + 14n_1 + 3 = 0$ is $(18n_1 + 7)^2 + 5 = 0$; that is, $(144\, n_1 + 56)^2 + 320 = 0$, but $144\, n_1 + 56 = 81x + 79$, or the equation becomes $(81x + 79)^2 + 320 = 0$, that is

$$\overline{81}^2x^2 + 81\,.\,158\, x + \overline{81}^2 = 0, \text{ or } 81x^2 + 158\, x + 81 = 0,$$

which is right.

Consider in like manner the intersection of the torse with the surface $ae-\lambda bd=0$, where λ is a given constant coefficient; we have

$$(ae-4bd)^3=(\lambda-4)^3\,b^3d^3=\frac{(\lambda-4)^3}{\lambda}\,aeb^3d^3,$$

and therefore

$$(\lambda-4)^3\,aeb^3d^3-27\lambda\,(ad^2+b^2e)^3=0,$$

that is

$$27\lambda a^3d^4+[54\lambda-(\lambda-4)^3]\,ab^3d^3e+27\lambda b^3e^4=0,$$

which gives

$$ad^2-\theta_1b^2e=0,\ \text{or}\ ad^2-\theta_2b^2e=0,$$

if θ_1, θ_2 are the roots of

$$27\lambda\theta^2+[54\lambda-(\lambda-4)^3]\,\theta+27\lambda=0.$$

The surfaces $ae-\lambda bd=0$, $ad^2-\theta_1b^2e=0$ have in common the two lines $(a=0,\ b=0)$ and $(d=0,\ e=0)$, and they intersect besides in a quartic curve. And so for the surfaces $ae-\lambda bd=0$, $ad^2-\theta_2b^2e=0$. That is, the surface $ae-\lambda bd=0$ intersects the torse $(ae-4bd)^3-27\,(-ad^2-b^2e)^3=0$, in the line $a=0$, $b=0$ twice, in the line $d=0$, $e=0$ twice, and in two quartic (excubo-quartic) curves. The two quartic curves become identical, if

$$(54\lambda)^2=\{54\lambda-(\lambda-4)^3\}^2,$$

that is

$$\pm\,54\lambda=\ 54\lambda-(\lambda-4)^3,$$

and therefore, if either

$$(\lambda-4)^3\qquad\qquad=0,$$

which gives the cuspidal curve; or else if

$$(\lambda-4)^3-108\,\lambda=0,$$

that is

$$\lambda^3-12\lambda^2-60\lambda-64=(\lambda+2)^2(\lambda-16)=0.$$

$(\lambda+2)^2=0$ or $\lambda=-2$ gives the nodal curve: $\lambda-16=0$ gives $ae-16bd=0$, a surface which intersects the developable in the line $a=0$, $b=0$ twice, in the line $d=0$, $e=0$ twice, and in the two coincident quartic (excubo-quartic) curves given by the equations $ae-16bd=0$, $ad^2-b^2e=0$. As a verification, I remark, that the surface $ad^2-b^2e=0$ combined with the developable gives

$$(ae-4bd)^3-27\,(ad^2+b^2e)^3=(ae-4bd)^3-108\,ab^3d^3e=0,$$

that is $(ae+2bd)^2(ae-16bd)=0$, or it meets the developable in its curve of intersection with $ae+2bd=0$ twice, and in its curve of intersection with $ae-16bd=0$; that is, in the line $a=0$, $b=0$ three times, in the line $d=0$, $e=0$ three times, in the nodal quartic $ae+2bd=0$, $ad^2-b^2e=0$ twice, and in the quartic $ae-16bd=0$, $ad^2-b^2e=0$ once; $3+3+8+4=18$, the order of the complete intersection.

Greenwich, January 4, 1864.

In my theory of the singularities of curves and torses, *Liouville*, t. x. (1845) pp. 245—250, [30], translated under the title "On Curves of Double Curvature and Developable Surfaces," *Cambridge and Dublin Mathematical Journal*, t. v. (1850), pp. 18—22, [83], I omitted to take account of a noteworthy singularity, viz. this is, the stationary tangent line; or when the system has three consecutive points in a line, or, what is the same thing, three consecutive planes through a line. I reproduce the theory with this addition as follows. We have

m,	the order of the system,	= order of the curve,	
r,	„ rank of the system,	= class of curve,	= order of torse,
n,	„ class of the system,	= class of torse.	
α,	„ number of stationary planes,		
β	„ „	stationary points,	
ϑ	„ „	stationary lines,	
g	„ „	lines in two planes,	
h	„ „	lines through two points,	
x	„ „	points in two lines,	
y	„ „	planes through two lines.	

This being so, the section of the torse by an arbitrary plane is a plane curve for which

r	is the order,		
n	„	class,	
x	„	number of nodes,	
$m+\vartheta$	„	„	cusps,
g	„	„	double tangents,
α	„	„	inflexions;

and we have thence Plücker's six equations, which may be considered as included in the three equations

$$n = r(r-1) - 2x - 3(m+\vartheta),$$
$$\alpha = 3r(r-2) - 6x - 8(m+\vartheta),$$
$$r = n(n-1) - 2g - 3\alpha.$$

Similarly considering the cone standing on the curve and having an arbitrary point for vertex, then for this cone

m	is the order,		
r	„	class,	
h	„	number of nodal lines,	
β	„	„	cuspidal lines,
y	„	„	double tangent planes,
$n+\vartheta$	„	„	inflexions;

and we have Plücker's six equations, which may be considered as included in the three equations

$$m = r(r-1) - 2y - 3(n+\vartheta),$$
$$\beta = 3r(r-2) - 6y - 8(n+\vartheta),$$
$$r = m(m-1) - 2h - 3\beta.$$

These two systems constitute together a system of six equations between the ten quantities $m, r, n, \alpha, \beta, \vartheta, g, h, x, y$. Considering m, r, x, ϑ as arbitrary, the six equations determine the remaining quantities $n, \alpha, \beta, h, x, y$.

The curve

$$ae - 4bd + 3c^2 = 0, \quad ace + 2bcd - ad^2 - b^2e - c^3 = 0,$$

is a sextic curve, the edge of regression of the sextic torse

$$(ae - 4bd + 3c^2)^3 - 27(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0,$$

and we have in this case, as is well known,

$$m, r, n, \alpha, \beta, \vartheta, g, h, x, y$$
$$= 6, 6, 4, 0, 4, 0, 3, 6, 4, 6.$$

But putting as above $c = 0$, then instead of the sextic curve we have the excubo-quartic curve $ae - 4bd = 0$, $ad^2 + b^2e = 0$, which is a curve having two stationary tangents, viz. these are the lines $(a = 0, b = 0)$ and $(d = 0, e = 0)$, which are in fact given along with the curve, by the foregoing equations $ae - 4bd = 0$, $ad^2 + b^2e = 0$. We have in this case $\vartheta = 2$, and the system is thus found to be

$$m, r, n, \alpha, \beta, \vartheta, g, h, x, y$$
$$= 4, 6, 4, 0, 0, 2, 3, 3, 4, 4,$$

it was in fact the consideration of this case which led me to take account of the new singularity of the stationary tangent lines.

I take the opportunity of referring to a most valuable and interesting paper by Schwarz, "De superficiebus in planum explicabilibus primorum septem ordinum," *Crelle*, t. LXIV. (1864), pp. 1—16. The author, after referring to my paper "On the developable derived from an equation of the fifth order," *Cambridge and Dublin Mathematical Journal*, t. V. (1850), pp. 152—159, [86], enters into the enquiry there suggested as to the means of ascertaining the degree of the 'planarity' of a developable surface. He starts from certain theorems derived from Riemann's theory of transcendental functions, viz.: If an algebraical (plane) curve of the order r has $\frac{1}{2}(r-1)(r-1) - \rho$ double points (nodes or cusps), then the coordinates of a point of the curve may be expressed rationally

If $\rho = 0$, that is, if the curve has the maximum number of double points, by a single parameter.

If $\rho = 1$, by a single parameter, and the square root of a cubic or quartic function of this parameter.

If $\rho = 2$, by a single parameter, and the square root of a quintic or sextic function of this parameter.

If $\rho > 2$, by a parameter ξ, and an algebraical function thereof η; where ξ, η are connected by an equation of the order $\frac{1}{2}(\rho+3)$ or $\frac{1}{2}(\rho+2)$ according as ρ is odd or even.

These principles establish a division of plane curves into algebraical classes; all plane curves (other than the generating lines) situate on a ruled surface, belong to the same algebraical class, and the surface itself belongs to the same class. Hence, if on a ruled surface there is either a right line which is not a generating line (this cannot be the case for developables) or a conic, or a cubic having a double point, or any other plane curve having the maximum number of double points, the surface belongs to the class for which $\rho = 0$; and in the case of a developable surface the equation of the tangent plane may be rationally expressed by means of a single parameter; that is, the degree of the planarity is $= 1$, or the surface is *planar*. This leads to the conclusion, that the developable surfaces or torses of the orders 4, 5, 6 and 7 are all of them planar.

The author points out that the 'special quintic developable' of my paper first above referred, (viz. that obtained by writing $b = 0$ in the equation of the sextic developable) is in fact the *general* developable of the fifth order, or quintic torse.

The foregoing theorem, that for a curve which has the maximum number of double points, the coordinates may be expressed rationally by a single parameter, admits of a very simple algebraical proof, as is shown in the paper by Clebsch "Ueber Curven deren coordinaten rationale Functionen eines Parameters sind," *Crelle*, t. LXIV. (1864), pp. 43—65. In another paper by the same author, "Ueber die Singularitäten algebraischer Curven," pp. 98—100, it is remarked that if in any plane curve we have m the order, n the class, δ the number of nodes, κ of cusps, τ of double tangents, ι of inflexions, then as a deduction from Riemann's principles, but also at once obtainable from Plücker's equations, we have

$$\tfrac{1}{2}(m-1)(m-2) - \delta - \kappa = \tfrac{1}{2}(n-1)(n-2) - \tau - \iota;$$

and moreover if from a given curve we derive in any manner another curve, such that to each tangent (or point) of the first curve there corresponds a *single* point (or tangent) of the second curve, then in the second curve the expression

$$\tfrac{1}{2}(m'-1)(m'-2) - \delta' - \kappa', = \tfrac{1}{2}(n'-1)(n'-2) - \tau' - \iota',$$

has the same value as in the first curve.

The like property exists for curves in space—viz. taking account as above of the new singularity of the stationary lines, then we have

$$\begin{aligned}
&\tfrac{1}{2}(m-1)(m-2)-h-\beta,\\
=&\tfrac{1}{2}(r-1)(r-2)-y-n-\vartheta,\\
=&\tfrac{1}{2}(r-1)(r-2)-x-m-\vartheta,\\
=&\tfrac{1}{2}(n-1)(n-2)-g-\alpha,
\end{aligned}$$

which equations are in fact at once deducible from the above-mentioned system of six equations between the quantities m, r, n, α, β, ϑ, g, h, x, y, and may if we please be taken for equations of the system.

If from a given curve and torse we derive a second curve and torse, in such manner that to each point (or plane) of the first figure there corresponds a *single* plane (or point) of the second figure—then the corresponding expressions $\frac{1}{2}(m'-1)(m'-2)-h'-\beta'$, &c., have the same value for the second as for the first figure.

Cambridge, April 11, 1865.

374.

ON THE HIGHER SINGULARITIES OF A PLANE CURVE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VII. (1866), pp. 212—223.]

THE theory of the singularities of a plane curve was first established by Plücker in his great work the *Theorie der Algebraischen Curven*, (1839), where he establishes, in regard to the ordinary singularities, a system of six equations; viz. if we have

m, the order of the curve,
n, „ class,
δ, „ number of double points,
κ, „ „ cusps,
τ, „ „ double tangents
κ, „ „ inflexions,

then Plücker's six equations are

$$\begin{aligned}
n &= m(m-1)-2\delta-3\kappa,\\
\iota &= 3m(m-2)-6\delta-8\kappa,\\
\tau &= \tfrac{1}{2}m(m-2)(m^2-9)-(m^2-m-6)(2\delta+3\kappa)+2\delta(\delta-1)+6\delta\kappa+\tfrac{9}{2}\kappa(\kappa-1),\\
m &= n(n-1)-2\tau-3\iota,\\
\kappa &= 3n(n-2)-6\tau-\delta\iota,\\
\delta &= \tfrac{1}{2}n(n-2)(n^2-9)-(n^2-n-6)(2\tau+3\iota)+2\tau(\tau-1)+6\tau\iota+\tfrac{9}{2}\iota(\iota-1),
\end{aligned}$$

equivalent to three equations; thus m and (within proper limits) δ and κ may be considered arbitrary, and the first three equations then give n, ι, τ; and in like manner n and (within proper limits) τ and ι may be considered as arbitrary, and the equations then give m, κ, δ.

I have used the ordinary expressions double points, cusps, double tangents, inflexions; but (using as I have elsewhere done *ineunt* as the correlative of tangent) it would be more precise and symmetrical to say, double ineunts, stationary ineunts, double tangents, and stationary tangents. The double ineunt is called also a node, viz. it is a crunode, or an acnode, according as the tangents are real or imaginary; and the stationary ineunt, or cusp, considered as (what in the theory of point-coordinates it in fact is) a particular case of the double ineunt, is a spinode; to render this notation symmetrical, we require certain new terms, say link, as the correlative to node, and flex as the correlative to cusp; then the double tangent is a link, viz. it is a colink, or an allink according as the ineunts upon it (points of contact) are real or imaginary; and the stationary tangent (inflexion) or flex, considered as (what in the theory of line-coordinates it in fact is) a particular case of the double tangent, is a relink. The ordinary singularities of a plane curve would thus be the node, the cusp, the link, and the flex; but I shall retain the above-mentioned more usual expressions.

Deducible from the six equations, we have

$$n-m=\tfrac{1}{3}(\iota-\kappa),$$

$$(n-m)(n+m-9)=2(\tau-\delta),$$

which are noticed by Plücker; and also the equation

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa=\tfrac{1}{2}(n-1)(n-2)-\tau-\iota,$$

recently noticed by M. Clebsch, in connection with Riemann's investigations on the Abelian Integrals; a curve of the order m may have $\frac{1}{2}(m-1)(m-2)$ double points, reckoning the cusp as a double point, and so a curve of the class n may have $\frac{1}{2}(n-1)(n-2)$ double tangents, reckoning the inflexion as a double tangent; the two sides of this equation exhibit therefore, the right-hand side the deficiency of the actual from the possible number of double tangents, and the left-hand side the deficiency of the actual from the possible number of double points; and these two numbers are equal. We have a division into families based on the value of the expressions in question, or say on that of $\frac{1}{2}(m-1)(m-2)-\delta-\kappa$; when this is $=0$, that is, when the curve has its maximum number of double points (reckoning the cusp as a double point), the coordinates x, y are expressible rationally in terms of a parameter θ; when the number is $=1$, they can be expressed rationally in terms of θ and of the square root of a cubic or a quartic function of θ, &c. &c. It thus appears that as well the number $\delta+\kappa$, as the combinations $2\delta+3\kappa$ and $6\delta+8\kappa$ which enter into Plücker's equations, plays an important part in the theory of the curve; the bearing of this remark will be seen in the sequel.

Plücker considers also some of the higher singularities; it will be convenient to mention two of his results.

No. 76, p. 216. If two branches of a curve touch each other, or more generally have a g-pointic intersection, the point in question is equivalent to g double points,

and the tangent at this point to g double tangents; hence, if there is no other point singularity, the equations give

$$\begin{aligned} n &= m(m-1)-2g, \\ \iota &= 3m(m-2)-6g, \\ \delta+g &= \tfrac{1}{2}m(m-2)(m^2-9)-(m^2-m-6)\,2g+2g(g-1), \end{aligned}$$

the last of which may also be written

$$\delta=\tfrac{1}{2}m(m-2)(m^2-9)-(m^2-m-g-4\tfrac{1}{2})\,2g.$$

And Nos. 77—82, pp. 217—222. For a cusp of the second kind, we have

$$\begin{aligned} n &= m(m-1)-5, \\ \iota &= 3m(m-2)-15, \\ \delta &= \tfrac{1}{2}m(m-2)(m^2-9)-(m^2-m-7)\,5; \end{aligned}$$

these equations Plücker establishes by an independent algebraical investigation, and having done so, he remarks that they are deducible from the foregoing ones by writing therein $g=2\frac{1}{2}$; that is, that the cusp of the second kind may be considered as equivalent to $2\frac{1}{2}$ double points, and the tangent at the cusp to $2\frac{1}{2}$ double tangents. And he thence passes to the cusp of a higher cusp equivalent to $h+\frac{1}{2}$ double points and $h+\frac{1}{2}$ double tangents. The results in this general case (although not, as in the original case, $g=2\frac{1}{2}$, established independently) is perfectly correct; but the theory is open to a grave objection.

I remark, that assuming a certain singularity to be equivalent to the numbers δ' of double points, κ' of cusps, τ' of double tangents, and ι' of inflexions, we have in the first instance to determine δ', κ', τ' and ι' in such manner as to give in the class n, and in the numbers ι of inflexions and τ of double tangents, the reductions actually given by the singularity in question. Thus in the case of the cusp of the second kind, we ought to have

$$\begin{aligned} &2\delta'+3\kappa'=5, \\ &6\delta'+8\kappa'+\iota'=15, \\ &(m^2-m-6)(2\delta'+3\kappa')-2\delta'(\delta'-1)-6\delta'\kappa'-\tfrac{9}{2}\kappa'(\kappa'-1)+\tau'=(m^2-m-7)\,5, \end{aligned}$$

or, what is the same thing,

$$2\delta'(\delta'-1)+6\delta'\kappa'+\tfrac{9}{2}\kappa'(\kappa'-1)-\tau'=5;$$

and so in general there are, for the determination of the four quantities δ', κ', τ', ι', three equations. In the particular case these are satisfied by the values $\delta'=2\frac{1}{2}$, $\kappa'=0$, $\tau'=2\frac{1}{2}$, $\iota'=0$, which are Plücker's values; they are also satisfied by the values $\delta'=1$, $\kappa'=1$, $\tau'=1$, $\iota'=1$, which have the advantage of being integer instead of fractional.

But there is really a further condition to be satisfied, viz. the number $\delta'+\kappa'$ must have a certain definite value dependent on the nature of the singularity; for

the case in hand, this is $\delta' + \kappa' = 2$ (I obtain this by the consideration of a quartic curve, having a cusp of the second kind, and also a double point; $\delta + \kappa$ has here its maximum value $= 3$; and as the double point gives $\delta = 1$, the cusp of the second kind gives $\delta' + \kappa' = 2$); and joining to the former conditions this new condition, we have definitely $\delta' = 1$, $\kappa' = 1$, $\tau' = 1$, $\iota' = 1$. I have elsewhere noticed, [343], that the cusp of the second kind was equivalent to a double point and cusp, and accordingly proposed to call it the node-cusp; but I had not then remarked that it was also necessary to treat the tangent as equivalent to a double tangent and a stationary tangent (or inflexion).

It appears from the foregoing considerations, that any singularity whatever is to be regarded as equivalent, and that in a perfectly definite manner, to a certain number δ' of double points, κ' of cusps, τ' of double tangents, and ι' of inflexions; we have only to ascertain how for any given singularity the values of these numbers are to be ascertained; and when this is done, Plücker's equations will be applicable to any singularities whatever of a plane curve.

At any point of a plane curve there is either one branch, or any number of branches, touching or not touching each other: taking the given point as origin, then for each branch the equation of the curve gives for the ordinate y an expression of the form

$$y = Ax^p + Bx^q + \dots,$$

where the series is arranged in ascending powers of x, and where the coefficients $A, B, \dots$ have definite unique values; and, conversely, that which is given by such expression of y is a branch of the curve. It is assumed that the axis of y, or line $x = 0$, is not a tangent to the curve; this implies that the exponents $p, q, \dots$ are none of them inferior to 1, or, what is the same thing, that the lowest exponent p is $= 1$ at least: it is for the most part convenient to take the axis of x, or line $y = 0$, a tangent to the branch; the lowest exponent p is then > 1.

The exponents may be all integer, and the branch is then said to be *linear;* or else the exponents or some of them may be fractional, and the branch is then *superlinear;* viz. in the latter case, assuming that the fractional exponents are all of them in their least terms, and that α is the least common multiple of the denominators (so that the expression for y is a rational function of $x^{\frac{1}{\alpha}}$), then the branch is quadric, cubic, &c. according as we have $\alpha = 2$, $\alpha = 3$, &c. It is clear that the expression for y has precisely α values, viz. the values obtained by attributing to the radical $x^{\frac{1}{\alpha}}$ each of its α values. Corresponding to each of these α values, we have what I term a partial branch of the curve, so that the quadric branch is made up of two partial branches, the cubic branch of three partial branches and so on; for a linear branch or when $\alpha = 1$, a partial branch is nothing else than the branch itself; and the expression a partial branch will accordingly include the case of a linear branch.

Suppose that at any point of the curve we have two partial branches, belonging or not belonging to the same branch; let these be referred to the same axes, the

axis of y not being a tangent to either branch, so that the exponents are none of them <1. If in the series for $y_1 - y_2$, (the difference of the two ordinates) the least exponent is $=P$, then (whether P is integer or fractional) the two partial branches are said to have, at the given point, P common points, or, more briefly, to intersect in P points. We may from this definition calculate the number of intersections of two branches with each other, or of a branch with itself; for instance, suppose that at any point of the curve we have ($\alpha = 6$) the sextic branch

$$y = x^{\frac{4}{3}} + x^{\frac{8}{3}} + \dots,$$

we have the six partial branches

$$\begin{aligned} y_1 &= \, x^{\frac{4}{3}} + x^{\frac{8}{3}} + \dots, & y_4 &= \, x^{\frac{4}{3}} - x^{\frac{8}{3}} + \dots, \\ y_2 &= \omega\, x^{\frac{4}{3}} + x^{\frac{8}{3}} + \dots, & y_5 &= \omega\, x^{\frac{4}{3}} - x^{\frac{8}{3}} + \dots, \\ y_3 &= \omega^2 x^{\frac{4}{3}} + x^{\frac{8}{3}} + \dots, & y_6 &= \omega^2 x^{\frac{4}{3}} - x^{\frac{8}{3}} + \dots; \end{aligned}$$

hence calculating (what is most convenient) *twice* the number of intersections of the branch with itself, the partial branch y_1 intersects the other partial branches in $\frac{4}{3}, \frac{4}{3}, \frac{5}{2}, \frac{4}{3}, \frac{4}{3}$ points respectively, giving the sum $\frac{16}{3} + \frac{5}{2} = \frac{47}{6}$; each other partial branch intersects the remaining five branches in the same number of points; and therefore twice the number of intersections is $=47$.

For the singularity $y = x^{\frac{4}{3}} + x^{\frac{8}{3}} + \dots$ in question, I say that if this be equivalent as above to δ' double points, κ' cusps, τ' double tangents and ι' inflexions, then that the number 47 just obtained is the value of $2\delta' + 3\kappa'$, and moreover, that the value of κ' is $\kappa' = \alpha - 1 = 5$; that is, we have $2\delta' + 3\kappa' = 47$ and $\kappa' = 5$; or, what is the same thing, $\delta' = 16$; $\kappa' = 5$. For the determination of the numbers τ', ι', it is to be observed that the foregoing theory of branches is a theory of the points of a branch, by means of point-coordinates: there is a precisely similar theory of the tangents of a branch by means of line-coordinates, and we may inquire as to the number of the common tangents of two partial branches; and thence as to the number of common tangents of two branches, or of a branch with itself—it will appear that the line-equation of the branch is $Z = X^4 \dots + X^{\frac{14}{3}} \dots$, so that the branch (which is as to its points sextic, $\alpha = 6$) is as to its tangents quadric, $\beta = 2$, the two partial branches have with each other the number $= \frac{15}{2}$ of common tangents, or twice this number is $=15$; that is, we have $2\tau' + 3\iota' = 15$, and moreover $\iota' = \beta - 1 = 1$, that is $\tau' = 6$, $\iota' = 1$; or finally for the singularity in question, the numbers δ', κ', τ', ι' are $=16, 5, 6, 1$ respectively.

And so generally in the case of a branch which is as to its points α-ic, having with itself a number $=\frac{1}{2}M$ of common points; and as to its tangents β-ic, having with itself a number $=\frac{1}{2}N$ of common tangents, we have $2\delta' + 3\kappa' = M$, $\kappa' = \alpha - 1$, $2\tau' + 3\iota' = N$, $\iota' = \beta - 1$, or, what is the same thing, the values of δ', κ', τ', ι' are

$$\begin{aligned} \delta' &= \tfrac{1}{2}\left[M - 3(\alpha - 1)\right], \\ \kappa' &= \alpha - 1 \quad , \\ \tau' &= \tfrac{1}{2}\left[N - 3(\beta - 1)\right], \\ \iota' &= \beta - 1 \quad . \end{aligned}$$

I say that a singularity is simple when we have one branch, compound when we have more than one branch; the case above considered is that of a simple singularity, viz. we have on the curve one point, one tangent, one branch.

We may have a compound singularity where the branches all touch, that is we may have one point, one tangent, several branches. It may be seen that if $\frac{1}{2}M$ denote the number of common points of all the branches (that is of each branch with itself, and of every two branches with each other), and in like manner if $\frac{1}{2}N$ denote the number of common tangents of all the branches (that is of each branch with itself, and of every two branches with each other), then the formulæ are

$$\begin{aligned}
\delta' &= \tfrac{1}{2}[M - 3\Sigma(\alpha - 1)],\\
\kappa' &= \Sigma(\alpha - 1),\\
\tau' &= \tfrac{1}{2}[N - 3\Sigma(\beta - 1)],\\
\iota' &= \Sigma(\beta - 1),
\end{aligned}$$

where the signs Σ refer to the different branches.

Again, we may have a compound singularity, one point, several tangents with to each of them a branch or branches; here if M denote the number of the common points of all the branches, and N the number of the common tangents of all the branches belonging to any one tangent, then the formulæ are

$$\begin{aligned}
\delta' &= \tfrac{1}{2}\ [M - 3\Sigma'\Sigma(\alpha - 1)],\\
\kappa' &= \Sigma'\Sigma(\alpha - 1),\\
\tau' &= \tfrac{1}{2}\Sigma'[N - 3\Sigma\ (\beta - 1)],\\
\iota' &= \Sigma'\Sigma(\beta - 1),
\end{aligned}$$

where the signs Σ refer to all the branches belonging to the same tangent, and the signs Σ' to the different tangents. It is to be remarked, that the point on the curve is equivalent to the δ' double points and κ' cusps; each tangent is equivalent to $\frac{1}{2}[N - 3\Sigma(\beta - 1)]$ double tangents, and $\Sigma(\beta - 1)$ inflexions, the numbers N, β referring of course to the tangent in question.

Lastly, we may have a compound singularity, one tangent, several points (of contact), with to each of them a branch or branches; here if N denote the number of the common tangents of all the branches, M the number of the common points of all the branches belonging to any one point of contact, the formulæ are

$$\begin{aligned}
\delta' &= \tfrac{1}{2}\Sigma'[(M - 3\Sigma\ (\alpha - 1)],\\
\kappa' &= \Sigma'\Sigma(\alpha - 1),\\
\tau' &= \tfrac{1}{2}\ [(N - 3\Sigma\Sigma'(\beta - 1)],\\
\iota' &= \Sigma'\Sigma(\beta - 1),
\end{aligned}$$

where the signs Σ refer to all the branches belonging to the same point of contact, and the signs Σ' to the different points of contact; it is to be remarked that the

tangent of the curve is equivalent to the τ' double tangents and ι' inflexions; each point of contact is equivalent to $\frac{1}{2}[M-3\Sigma(\alpha-1)]$ double points and $\Sigma(\alpha-1)$ cusps, the numbers M, α referring of course to the point of contact in question.

There is no difficulty in passing to the case of the compound singularity when the formulæ for the simple singularity, one point, one tangent, one branch, are once obtained, and I now go back to the consideration of this case.

The class of a curve is equal to the number of tangents which can be drawn through an arbitrary point: the points of contact of these tangents are given as the intersections of the curve with a certain curve, the polar of the arbitrary point in regard to the curve; this polar passes through each double point and cusp, the double point counting as two points of intersection, and the cusp as three points of intersection (this is in fact the theory by which is found the reduction $=2\delta+3\kappa$ in the class of the curve). Hence, if the curve has a singularity $(\delta', \kappa', \tau', \iota')$, which to fix the ideas may be assumed to be a simple singularity, 'one point, one tangent, one branch'; then the polar passes through the singular point, the number of intersections being $2\delta'+3\kappa'$, or if the actual number of intersections be M, then we have $M=2\delta'+3\kappa'$. It is to be shown that the number M is equal to twice the number of common points which the curve has with itself at the singular point, so that the last-mentioned number is $=\frac{1}{2}M$. Suppose in the first instance that there is only a single branch, and let the branch be given by the equation

$$P=y \quad +Ax^p \quad +Bx^q+\ldots=0,$$

or introducing for homogeneity the third coordinate z, let this equation be

$$P=yz^{-1}+Ax^pz^{-p}+Bx^qz^{-q}\ldots=0,$$

and let $P_1=0$, $P_2=0, \ldots P_\alpha=0$, be the corresponding equations for the component partial branches; it is allowable to write $P_1P_2\ldots P_\alpha=0$ for the equation of the curve[1]. Hence if (α, β, γ) be the coordinates of the arbitrary point, or putting in the first instance $\gamma=1$, if $(\alpha, \beta, 1)$ be the coordinates, then writing $\Delta=\alpha\delta_x+\beta\delta_y+\delta_z$, the equation of the polar is $\Delta P_1P_2\ldots P_\alpha=0$, or, what is the same thing,

$$P_2P_3\ldots P_\alpha\Delta P_1+P_1P_3\ldots P_\alpha\Delta P_2+\&c.=0,$$

and we have

$$\Delta P=\alpha(pAx^{p-1}z^{-p}+qBx^{q-1}z^{-q}\ldots)+\beta z^{-1}-(pAx^pz^{-p-1}+qBx^qz^{-q-1}\ldots),$$

or putting $z=1$, this is

$$\Delta P=\alpha(pAx^{p-1} \quad +qBx^{q-1} \quad \ldots)+\beta \quad -(pAx^p \quad +qBx^q \quad \ldots),$$

and we have thence the values of ΔP_1, $\Delta P_2\ldots\Delta P_\alpha$; the thing to be observed is, that the equation $\Delta P=0$ is *not* satisfied (and therefore also each of the equations $\Delta P_1=0, \ldots \Delta P_\alpha=0$ is not satisfied) by the coordinates $x=0$, $y=0$ of the singular point. We have now with the equation $\Delta P_1P_2\ldots P_\alpha=0$ of the polar to combine the

[1] Of course this is not the equation in its rational and integral form, and on this account the reasoning of the text is not free from difficulty; the same remark applies to a subsequent equation.

equation $P_1P_2 \dots P_\alpha = 0$: the last-mentioned equation breaks up into the equations $P_1 = 0$, $P_2 = 0, \dots P_\alpha = 0$; and selecting for example the equation $P_1 = 0$, this gives the system $P_1 = 0$, $P_2P_3 \dots P_\alpha \Delta P_1 = 0$, or since we require only the intersections at the singular point, and $\Delta P_1 = 0$ does not pass through this point, this may be replaced by $P_1 = 0$, $P_2P_3 \dots P_\alpha = 0$. The complete system is thus $(P_1 = 0,\ P_2P_3 \dots P_\alpha = 0)$, $(P_2 = 0,\ P_1P_3 \dots P_\alpha = 0), \dots (P_\alpha = 0,\ P_1P_2 \dots P_{\alpha-1} = 0)$; or, what is the same thing, we have each pair $(P_r = 0,\ P_s = 0)$ taken twice. To eliminate y from these equations, we have merely to write $P_r - P_s = 0$, or, what is the same thing, we have $\zeta(P_1,\ P_2 \dots P_\alpha) = 0$, ζ denoting the product of the squares of the differences of the functions $(P_1,\ P_2 \dots P_\alpha)$. Suppose that any two partial branches $P_r = 0$, $P_s = 0$ intersect (according to the above-mentioned definition) in p points; then $P_r - P_s$ contains the factor x^p, and hence the product $\zeta(P_1,\ P_2 \dots P_\alpha)$ contains as a factor x to the power $2\Sigma p$, that is, the equation in x has $2\Sigma p$ roots each $= 0$. Whence if $\Sigma p = \frac{1}{2}M$, then the equation in x has M roots each $= 0$, or the curve and polar have at the singular point M intersections, that is $M = 2\delta' + 3\kappa'$.

I have no complete proof to offer of the remaining equation $\kappa' = \alpha - 1$, it was obtained from the consideration of a particular case as follows. Consider the linear branch $y = Ax^p + \dots$, where the exponents are all positive integers, and taking the axis of x to be the tangent, the least exponent p is greater than unity; if $p = 2$ there is at the origin no inflexion, if $p = 3$ there is a single inflexion, and generally the number of inflexions is $= p - 2$. Now it will presently appear that in line-coordinates the equation of the branch is $Z = A'X^{\frac{p}{p-1}}$, or replacing Z, X by the original point-coordinates y, x the branch $y = A'x^{\frac{p}{p-1}} + \dots$ has at the origin $p - 2$ cusps; but in the branch in question we have $\alpha = p - 1$, and the number of cusps is thus $= \alpha - 1$; this result is confirmed by other particular instances, and I assume in general that we have $\kappa' = \alpha - 1$; whence in the case of a simple singularity, or where there is only one branch we have $M = 2\delta' + 3\kappa'$, $\kappa' = \alpha - 1$, or, what is the same thing, $\delta' = \frac{1}{2}[M - 3(\alpha - 1)]$, $\kappa' = \alpha - 1$. The reasoning is easily adapted to the case of a compound singularity.

I consider the branch

$$y \quad + Ax^p \quad + Bx^q \quad + \dots = 0,$$

(where it is assumed that the axis of x is a tangent to the branch, and therefore that the lowest exponent p is greater than unity), introducing the coordinate z for homogeneity, this becomes

$$yz^{-1} + Ax^p z^{-p} + Bx^q z^{-q} + \dots = 0,$$

and I proceed to find the corresponding equation in line coordinates, taking these to be X, Y, Z, we have

$$\begin{aligned}
\lambda X &= pAx^{p-1}z^{-p} + \quad Bqx^{q-1}z^{-q} + \dots, \\
\lambda Y &= \qquad\quad z^{-1}, \\
\lambda Z &= - \qquad yz^{-2} - pAx^p z^{-p-1} - qBx^q z^{-q-1} + \dots,
\end{aligned}$$

or writing $z=1$, $Y=1$, we find $\lambda=1$, and therefore

$$\begin{aligned} X &= \quad pAx^{p-1} + \quad qBx^{q-1} + \dots, \\ Z &= -y - pAx^p \; - \quad qBx^q \; + \dots; \end{aligned}$$

here substituting for $-y$ its value $= Ax^p + Bx^q + \dots$, we have

$$\begin{aligned} X &= \quad pAx^{p-1} + \quad qBx^{q-1} + \dots, \\ Z &= (1-p)\,Ax^p \; + (1-q)\,Bx^q \; + \dots. \end{aligned}$$

Hence writing $pAx^{p-1} = \theta$, the equations are

$$X = \theta - B''\theta^{\frac{q-1}{p-1}} - \dots,$$

$$Z = \; - A'\theta^{\frac{p}{p-1}} - B'\theta^{\frac{q}{p-1}} - \dots,$$

so that eliminating θ, we have

$$Z = \quad A'X^{\frac{p}{p-1}} + B'X^{\frac{q}{p-1}} + \dots,$$

and it is easy to see by Lagrange's theorem, that the general form of the exponents in the series on the right-hand side is $\dfrac{p + f(q-p) + g(r-p) + \dots}{p-1}$, where $f, g, \dots$ are positive integers, zero included. The equation in line-coordinates being known, the subsequent investigation is precisely the same as that for the point-coordinates, and hence in the case of one branch, if this be in regard to its tangents β-ic, and have $\frac{1}{2}N$ common tangents with itself, then $2\tau' + 3\iota' = N$, $\iota' = \beta - 1$, or, what is the same thing, $\tau' = \frac{1}{2}[N - 3(\beta - 1)]$, $\iota' = \beta - 1$. The investigation in the case of a simple singularity of the values of δ', κ', τ', ι' is thus completed.

375.

NOTES ON POLYHEDRA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VII. (1866), pp. 304—316.]

Axial Properties. Article 1 to 18.

1. A POLYHEDRON may have a q-axis, viz. a line about which if it is made to rotate through an angle $=\frac{2\pi}{q}$ (but not through any sub-multiple of this angle), it will occupy the same portion of space. It is then clear that when the rotation is repeated any number of times the body will still occupy the same portion of space; or if Θ denote the rotation through the angle $\frac{2\pi}{q}$, then we have the rotations $1, \Theta, \Theta^2, \ldots \Theta^{q-1}$, and finally $\Theta^q = 1$, that is, when the rotation is q-times repeated, the body will resume its original position. Similarly for any number of axes ($\Theta^q = 1$, $\Theta'^{q'} = 1, \ldots$, where the indices $q, q', \ldots$ may be the same or different) we have the rotations $1, \Theta, \Theta^2, \ldots \Theta^{q-1}$, $\Theta', \Theta'^2, \ldots \Theta'^{q'-1}, \ldots$; and if $\Theta, \Theta', \ldots$ be the entire system of the axes of the body, these rotations will form a *group*. The rotations in question are in fact the entire series of those which leave unaltered the portion of space occupied by the body, and since any two rotations combine together into a single rotation, any two of the rotations in question must combine together into some one of these rotations, that is, the rotations in question form a group. Some analytical consequences of this theorem will be obtained in the sequel.

2. The number of axes may be denoted by $\Sigma 1$ and the number of rotations by $1 + \Sigma(q-1)$; we may say that $\Sigma 1$ is the number, and $1 + \Sigma(q-1)$ the efficiency or *weight*, of the axes.

3. For any one of the regular polyhedra, E being the number of edges, then the number of axes or $\Sigma 1$ is $=E+1$, and their weight or $1+\Sigma(q-1)$ is $=2E$. In fact, if as usual S denote the number of summits, F the number of faces, and if there be m edges to a face, and n edges to a summit, then $S+F=E+2$, $mF=nS=2E$. Now in all the polyhedra except the tetrahedron, we have a number $\frac{1}{2}F$ of m-axes passing through the centres of opposite faces (amphihedral axes as Mr Kirkman has termed them) and a number $\frac{1}{2}S$ of n-axes passing through opposite summits (amphigonal axes); and we have besides a number $\frac{1}{2}E$ of 2-axes passing through the mid-points of opposite edges (amphigrammic axes): the entire number of axes is thus $\frac{1}{2}(S+F+E)$, which is $=E+1$: and the weight is $1+\frac{1}{2}F(m-1)+\frac{1}{2}S(n-1)+\frac{1}{2}E$, which is $=1+\frac{1}{2}mF+\frac{1}{2}nS-\frac{1}{2}(F+S-E)$, $=1+E+E-1$, $=2E$. In the case of the tetrahedron $S=F=4$, $m=n=3$, and the only difference is that instead of the $\frac{1}{2}F$ amphihedral m-axes and the $\frac{1}{2}S$ amphigonal n-axes, we have a number $(F=S=)\frac{1}{2}(F+S)$ of $(n=)m$-gonal axes each through a summit and the centre of an opposite face (gonohedral axes).

4. The theorem that the weight $1+\Sigma(q-1)=2E$, or say $1+\Sigma(q-1)=mF$, may be extended so as to apply to any polyhedron whatever. In fact considering any face A of the polyhedron, let F be the number of faces homologous to (and inclusive of) A; and, taking a any edge of the face A, let m be the number of edges of A homologous to (and inclusive of) a: then we have $1+\Sigma(q-1)=mF$. This is almost a truism when the signification of the term "homologous" is explained. Imagine the polyhedron placed on a plane, say the table, and draw on the table a polygon equal to the polygonal face A, and in this polygon select some one edge corresponding to the edge a. The polyhedron may be placed on the table with the face A coinciding with the polygon, or say the face A may be superimposed on the polygon, and that in m different ways, viz. any one of the edges homologous to a may be made to coincide with the assumed edge: and in like manner there are F different faces (viz. the faces homologous to A) which may be superimposed on the polygon, each of them in m different ways; that is there are in all mF different positions of the polyhedron for each of which it occupies the same portion of space. And we have thus the required theorem $1+\Sigma(q-1)=mF$.

5. As an example, take the regular pyramid on a square base; there is here a single axis, viz. a 4-axis, and we have $1+\Sigma(q-1)=1+3=4$. If for the face A we take the square base, then there is no other face homologous thereto and therefore $F=1$; but the four sides are homologous to each other or $m=4$, and we have $mF=4$. Similarly taking for A one of the triangular faces, since these are homologous to each other, then $F=4$; and if we take for the side a the base of the triangle, then there is no other side homologous to this, or $m=1$; and therefore $mF=4$. It might at first sight appear that the two equal sides of the triangle were homologous to each other, and therefore that taking for the edge a one of these sides we should have $m=2$; but in fact although the two sides in question are homologously related to the pyramid, yet according to the definition they are not homologous sides of the triangular face, and we still have $m=1$, and therefore $mF=4$.

6. Of course in the case of a regular polyhedron the faces are all homologous, and the edges of a face are all homologous, that is F will denote the entire number of faces, and m the number of edges to a face: so that for this case the theorem gives $1+\Sigma(q-1)=2E$ as above.

7. Returning to the regular polyhedra the axial systems are

Tetrahedron	$4L^3$, $3L^2$.
Cube and Octahedron	$3L^4$, $4L^3$, $6L^2$.
Dodecahedron and Icosahedron	$6L^5$, $10L^3$, $15L^2$,

where L^3 denotes a 3-axis, &c.; this is in accordance with the notation of M. Bravais in the memoir subsequently referred to.

8. The regular polyhedra may be exhibited in connexion with each other as follows: Imagine the polyhedron projected on a concentric sphere by lines through the centre; so that the summits become points on the sphere, the edges arcs of great circles, and the faces spherical polygons. Starting from the dodecahedron, the centres of the pentagonal faces are the summits of the icosahedron, and conversely for the icosahedron the centres of the triangular faces are the summits of the dodecahedron: moreover each edge of the dodecahedron cuts at right angles an edge of the icosahedron and the two edges have the same mid-point. Again if in any face of the dodecahedron we draw one of the five diagonals (arcs through two non-adjacent summits) there is in the face a single edge not met by this diagonal; and in the other face through this edge a single diagonal not met by the edge; joining the extremities of the two diagonals we have a spherical square, the face of the cube; it is to be observed that the summits of the cube are eight out of the twenty summits of the dodecahedron, and that the centres of the faces of the cube are the mid-points of six out of the thirty edges of the dodecahedron or the icosahedron. The cube given by the foregoing construction is of course one out of five different cubes. The centres of the faces of the cube are the summits of the octahedron; and conversely the centres of the faces of the octahedron are the summits of the cube; moreover each edge of the cube cuts at right angles an edge of the octahedron; and the two edges have the same mid-point. Finally, taking four non-adjacent summits of the cube (which can be done in two different ways), these are the summits of the tetrahedron, and the mid-points of the edges of the tetrahedron are the summits of the octahedron.

9. Considering the polyhedra in the foregoing mutual connexion, all the axes of the tetrahedron are axes of the cube and octahedron, viz. the 2-axes of the tetrahedron are the 4-axes of the cube and octahedron; and the 3-axes of the tetrahedron are the 3-axes of the cube and octahedron; moreover the 3-axes of the cube and octahedron are included among the 3-axes of the dodecahedron and icosahedron and the 4-axes of the cube and octahedron are included among the 2-axes of the dodecahedron and icosahedron; but the 2-axes of the cube and octahedron are not included among the axes of the dodecahedron and icosahedron. The 4-axes of the cube and

octahedron form thus a system of rectangular axes common to all the polyhedra, and representing these axes (or say the summits of the corresponding rectangular spherical triangle) by X, Y, Z, we have a convenient system of coordinate axes to which to refer all the other axes of the polyhedron, viz. if P be the extremity (chosen at pleasure) of the axis in question, then the position of the axis may be determined by its distance PZ and azimuth XPZ (measured in the direction from X to Y), or by its distances PX, PY, PZ, or say X, Y, Z from the three rectangular axes (we have, it is clear, $\cos X = \sin$ dist. $\cos$ azim., $\cos Y = \sin$ dist. $\sin$ azim., $\cos Z = \cos$ dist.). The rotation angle of a q-axis is $=\frac{2\pi}{q}$ (i.e. this is the angle through which if the body be turned about the axis, it still occupies the same portion of space) and the half-rotation angle is therefore $=\frac{\pi}{q}$. Moreover if i, j, k are Sir W. R. Hamilton's quaternion symbols, then the "rotation symbol" of the axis is

$$\cos\frac{\pi}{q}+\sin\frac{\pi}{q}(i\cos X+j\cos Y+k\cos Z),$$

the application of which will be presently explained.

10. The angular coordinates of the different axes may be found by spherical trigonometry without much difficulty; and we are then able to form the following axial tables of the several polyhedra: the extremity of each axis is chosen in such manner that the distance PZ is not $>90°$.

Axial System of the Tetrahedron.

Distances angle	cos	sin	Azimuths angle	cos	sin	cos X	cos Y	cos Z	Rot. Symbols
4 3-axes, $\frac{1}{2}$ Rot. angle $=60°$, $\cos=\frac{1}{2}$, $\sin=\frac{1}{2}\sqrt{3}$.									
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{2}(1+i+j+k)$
,,	,,	,,	135°	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$\frac{1}{2}(1-i+j+k)$
,,	,,	,,	225°	− ,,	− ,,	− ,,	− ,,	+ ,,	$\frac{1}{2}(1-i-j+k)$
,,	,,	,,	315°	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}(1+i-j+k)$
3 2-axes, $\frac{1}{2}$ Rot. angle $=90°$, $\cos=0$, $\sin=1$.									
0°	1	0	*	*	*	0	0	1	k
90°	0	1	0°	1	0	1	0	0	i
,,	,,	,,	90°	0	1	0	1	0	j

Axial System of the Cube and Octahedron.

Distances angle	cos	sin	Azimuths angle	cos	sin	cos X	cos Y	cos Z	Rot. Symbols
4 3-axes, $\frac{1}{2}$ Rot. angle $= 60°$, $\cos = \frac{1}{2}$, $\sin = \frac{1}{2}\sqrt{3}$.									
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{2}(1+i+j+k)$
,,	,,	,,	135°	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$\frac{1}{2}(1-i+j+k)$
,,	,,	,,	225°	− ,,	− ,,	− ,,	− ,,	+ ,,	$\frac{1}{2}(1-i-j+k)$
,,	,,	,,	315°	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}(1+i-j+k)$
3 4-axes, $\frac{1}{2}$ Rot. angle $= 45°$, $\cos = \frac{1}{\sqrt{2}}$, $\sin = \frac{1}{\sqrt{2}}$.									
0°	1	0	*	*	*	0	0	1	$\frac{1}{\sqrt{2}}(1+k)$
90°	0	1	0°	1	0	1	0	0	$\frac{1}{\sqrt{2}}(1+i)$
,,	,,	,,	90°	0	1	0	1	0	$\frac{1}{\sqrt{2}}(1+j)$
6 2-axes, $\frac{1}{2}$ Rot. angle $= 90°$, $\cos = 0$, $\sin = 1$.									
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0°	1	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(i+k)$
,,	,,	,,	90°	0	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(j+k)$
,,	,,	,,	180°	− 1	0	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(-i+k)$
,,	,,	,,	270°	0	−1	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(-j+k)$
90°	0	1	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}(i+j)$
,,	,,	,,	135°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}(-i+j)$

Axial System of the Dodecahedron and Icosahedron.

angle	Distances cos	Distances sin	angle	Azimuths cos	Azimuths sin	cos X	cos Y	cos Z	Rot. Symbols
6 5-axes, $\frac{1}{5}$ Rot. angle $=36°$, $\cos=\frac{\sqrt{5}+1}{4}$, $\sin=\frac{\sqrt{(10-2\sqrt{5})}}{4}$.									
31°44′	$\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	0°	1	0	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	0	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$\frac{\sqrt{5}+1}{4}+\frac{\sqrt{5}-1}{4}i+\frac{1}{2}k$
,,	,,	,,	180°	-1	0	$-$,,	,,	$+$,,	$\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}i+\frac{1}{2}k$
58°17′	$\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	90°	0	1	0	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$\frac{\sqrt{5}+1}{4}+\frac{1}{2}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	270°	0	-1	,,	$-$,,	$+$,,	$\frac{\sqrt{5}+1}{4}-\frac{1}{2}j+\frac{\sqrt{5}-1}{4}k$
90°	0	1	31°43′	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	0	$\frac{\sqrt{5}+1}{4}+\frac{1}{2}i+\frac{\sqrt{5}-1}{4}j$
,,	,,	,,	148°17′	$-$,,	$+$,,	$-$,,	$+$,,	,,	$\frac{\sqrt{5}+1}{4}-\frac{1}{2}i+\frac{\sqrt{5}-1}{4}j$
10 3-axes, $\frac{1}{3}$ Rot. angle $=60°$, $\cos=\frac{1}{2}$, $\sin=\frac{1}{2}\sqrt{3}$.									
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{3}(1+i+j+k)$
,,	,,	,,	135°	$-$,,	$+$,,	$-$,,	$+$,,	$+$,,	$\frac{1}{3}(1-i+j+k)$
,,	,,	,,	225°	$-$,,	$-$,,	$-$,,	$-$,,	$+$,,	$\frac{1}{3}(1-i-j+k)$
,,	,,	,,	315°	$+$,,	$-$,,	$+$,,	$-$,,	$+$,,	$\frac{1}{3}(1+i-j+k)$
20°55′	$\frac{\sqrt{5}+1}{2\sqrt{3}}$	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	90°	0	1	0	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$\frac{1}{2}+\frac{\sqrt{5}-1}{4}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	270°	0	-1	0	$-$,,	$+$,,	$\frac{1}{2}-\frac{\sqrt{5}-1}{4}j+\frac{\sqrt{5}+1}{4}k$
69°5′	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{\sqrt{5}+1}{2\sqrt{3}}$	0°	1	0	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	0	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{1}{2}+\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	180°	-1	0	$-$,,	0	$+$,,	$\frac{1}{2}-\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}k$
90°	0	1	69°5′	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	0	$\frac{1}{2}+\frac{\sqrt{5}-1}{4}i+\frac{\sqrt{5}+1}{4}j$
,,	,,	,,	110°55′	$-$,,	$+$,,	$-$,,	$+$,,	0	$\frac{1}{2}-\frac{\sqrt{5}-1}{4}i+\frac{\sqrt{5}+1}{4}j$

Axial System of the Dodecahedron and Icosahedron (concluded).

Distances angle	Distances cos	Distances sin	Azimuths angle	Azimuths cos	Azimuths sin	cos X	cos Y	cos Z	Rot. Symbols
15 2-axes, $\frac{1}{2}$ Rot. angle = 90°, cos = 0, sin = 1.									
0°	1	0	*	*	*	0	0	1	k
90°	0	1	0°	1	0	1	0	0	i
,,	,,	,,	90°	0	1	0	1	0	j
36°	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{(10-2\sqrt{5})}}{4}$	62°40′	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{\sqrt{5}-1}{4}$	$+\frac{1}{2}$	$+\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{5}-1}{4}i+\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	297°20′	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{\sqrt{5}-1}{4}i-\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	242°40′	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{\sqrt{5}-1}{4}i-\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	117°20′	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{\sqrt{5}-1}{4}i+\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	20°55′	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{4}$	$+\frac{\sqrt{5}-1}{4}$	$+\frac{1}{2}$	$\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	339°5′	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{\sqrt{5}+1}{4}i-\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	200°55′	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{\sqrt{5}+1}{4}i-\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	159°5′	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
72°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{(10+2\sqrt{5})}}{4}$	62°40′	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{1}{2}$	$+\frac{\sqrt{5}+1}{4}$	$+\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}i+\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	297°20′	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}i-\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	242°40′	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{1}{2}i-\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	117°20′	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{1}{2}i+\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$

11. Before proceeding further I remark that exclusively of the foregoing axial systems of the regular polyhedra the only cases are as follows:

A. A polyhedron may have a single q-axis, say Λ^q: taking this as the axis of Z the table is

X	Y	Z	Rot. Symbol
90°	90°	0°	$\cos\frac{\pi}{q} + \sin\frac{\pi}{q} . k$

B. It may have a single q-axis, and (symmetrically arranged in a plane at right angles thereto) q 2-axes, say Λ^q, qL^2. Taking the q-axis as the axis of Z and some one of the 2-axes as the axis of X, the table is

X	Y	Z	Rot. Symbols
One q-axis, $\frac{1}{2}$ Rot. angle $= \frac{\pi}{q}$.			
90°	90°	0°	$\cos\frac{\pi}{q} + \sin\frac{\pi}{q} . k$
q 2-axes, $\frac{1}{2}$ Rot. angle $= 90°$.			
0°	90°	90°	i
$\frac{\pi}{q}$	$90° - \frac{\pi}{q}$	90°	$i\cos\frac{\pi}{q} + j\sin\frac{\pi}{q}$
⋮	⋮	⋮	⋮
$(q-1)\frac{\pi}{q}$	$90° - \frac{(q-1)\pi}{q}$	90°	$i\cos\frac{(q-1)\pi}{q} + j\sin\frac{(q-1)\pi}{q}$

and in particular if $q = 2$, the axes are $3L^2$ and the table is

X	Y	Z	Rot. Symbols
3 2-axes, $\frac{1}{2}$ Rot. angle $= 90°$			
90°	90°	0°	k
0°	90°	90°	i
90°	0°	90°	j

This in fact appears, Bravais, "Mémoire sur les polyèdres de forme symétrique," *Liouville*, t. XIV., pp. 141—180 (1843), observing that for the present purpose there is no distinction between his three cases

$$\Lambda^{2q+1},\ (2q+1)\,L^2;\quad \Lambda^{2q},\ qL^2,\ qL'^2;\quad \Lambda^{2q},\ 2qL^2.$$

12. The meaning of the rotation symbol is as follows: viz. if in general we have a rotation θ about an axis inclined at the angles X, Y, Z to any three rectangular axes, and if Π be the rotation symbol,

$$\Pi = \cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\,(i\cos X + j\cos Y + k\cos Z),$$

then if x, y, z are the original coordinates of any point of the body, and x', y', z' the coordinates of the same point after the rotation; the values of x', y', z' are given in terms of x, y, z by the formula

$$ix' + jy' + kz' = \Pi\,(ix + jy + kz)\,\Pi^{-1}.$$

This is in fact the form under which, in the paper "On certain results relating to Quaternions," *Phil. Mag.*, vol. XXVI. (1845), p. 141, [20], I exhibited the rotation formulæ of Euler and Rodrigues. See also my paper "On the application of Quaternions to the Theory of Rotation," *Phil. Mag.*, vol. XXXIII. (1848), p. 196, [68].

13. We have, it is clear,

$$\Pi^s = \cos s\theta + \sin s\theta\,(i\cos X + j\cos Y + k\cos Z)$$

which shows that Π^s is the symbol for the rotation Π repeated s times: (more generally performing on the body, first the rotation Π and then the rotation Φ about any axis, the same or different, the symbol of the resultant rotation is $=\Phi\Pi$). If Π be the symbol for a rotation through the angle $\dfrac{2\pi}{q}$, then the rotation which corresponds to the symbol Π^q is a rotation through 360°, that is the body returns to its original position; it might at first sight appear that we ought to have $\Pi^q = 1$, and that the symbols $1, \Pi, \Pi^2, \ldots \Pi^{q-1}$ would form a *group;* this however is not so, for we have not $\Pi^q = 1$, but $\Pi^q = -1$; in fact, it is to be observed that to pass from $ix + jy + kz$ to $ix' + jy' + kz'$, we have to multiply by $\Pi(\ )\Pi^{-1}$, so that the symbol of the rotation is indifferently $\pm\Pi$, and that the rotation symbol -1 is thus equivalent to the rotation symbol $+1$. But as regards the formation of the group, the only difference is that it is not $1, \Pi, \Pi^2, \ldots \Pi^{q-1}$ which form a group of q symbols, but ± 1, $\pm\Pi$, $\pm\Pi^2, \ldots \pm\Pi^{q-1}$ which form a group of $2q$ symbols. And so in the axial system of any polyhedron, if Π be the rotation symbol of any q-axis, then taking for each axis of the polyhedron the set of symbols $\pm\Pi$, $\pm\Pi^2, \ldots \pm\Pi^{q-1}$, and besides the two symbols ± 1, the whole series of symbols form together a group.

14. Thus in the before-mentioned case $B\,(q=2)$ we have the eight symbols

$$\pm 1,\ \pm i,\ \pm j,\ \pm k$$

forming (as they obviously do) a group. In the general case B, putting for shortness

$$\Theta = \cos\frac{\pi}{q} + \sin\frac{\pi}{q}.\,k \text{ and } \Phi_s = i\cos\frac{s\pi}{q} + j\sin\frac{s\pi}{q},$$

the group consists of the $4q$ symbols

$$\pm 1,\ \pm \Theta, \ldots \pm \Theta^{q-1};\quad \pm \Phi_1,\ \pm \Phi_2, \ldots \pm \Phi_{q-1},$$

(to verify that this is so, it is only necessary to form the equations

$$\Theta^r\Theta^s = \Theta^{r+s},\quad \Phi_s^2 = -1,\quad \Theta^r\Phi_s = \Phi_{s+r},\quad \Phi^s\Theta^r = \Phi_{s-r},\quad \Phi_r\Phi_s = -\Theta^{r-s},$$

which are at once seen to be true).

15. The $\pm$ general case A gives merely the group of the $2q$ symbols

$$\pm 1,\ \pm \Theta, \ldots \pm \Theta^{q-1},$$

which has been already mentioned.

16. The tetrahedron gives the group of 24 symbols,

$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$	16 cube roots of ± 1
$\pm i,\ \pm j,\ \pm k$	6 square „ „ „
± 1	2 terms
	24

(the signs $\pm$ being all independent).

17. The cube and octahedron give the group of 48 symbols

$\frac{1}{\sqrt{2}}(\pm 1 \pm i),\ \frac{1}{\sqrt{2}}(\pm 1 \pm j),\ \frac{1}{\sqrt{2}}(\pm 1 \pm k)$	12 fourth roots of ± 1
$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$	16 cube „ „ „
$\pm i,\ \pm j,\ \pm k,\ \frac{1}{\sqrt{2}}(\pm j \pm k),\ \frac{1}{\sqrt{2}}(\pm k \pm i),\ \frac{1}{\sqrt{2}}(\pm i \pm j)$	18 square „ „ „
± 1	2 terms
	48

(the signs $\pm$ being all independent).

18. The dodecahedron and icosahedron give the group of 120 symbols

$$\left.\begin{array}{l} \pm \dfrac{\sqrt{5} \pm 1}{4} \pm \frac{1}{2} j \pm \dfrac{\sqrt{5} \mp 1}{4} k, \\[2ex] \pm \dfrac{\sqrt{5} \pm 1}{4} \pm \frac{1}{2} k \pm \dfrac{\sqrt{5} \mp 1}{4} i, \\[2ex] \pm \dfrac{\sqrt{5} \pm 1}{4} \pm \frac{1}{2} i \pm \dfrac{\sqrt{5} \mp 1}{4} j, \end{array}\right\} \text{48 fifth roots of } \pm 1$$

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

$$\left.\begin{array}{l} \pm\tfrac{1}{2}\ \pm\dfrac{\sqrt{5}-1}{4}j \pm \dfrac{\sqrt{5}+1}{4}k, \\ \pm\tfrac{1}{2}\ \pm\dfrac{\sqrt{5}-1}{4}k \pm \dfrac{\sqrt{5}+1}{4}i, \\ \pm\tfrac{1}{2}\ \pm\dfrac{\sqrt{5}-1}{4}i \pm \dfrac{\sqrt{5}+1}{4}j, \end{array}\right\} \text{40 cube roots of } \pm 1$$

$$\pm i,\ \pm j,\ \pm k,$$

$$\left.\begin{array}{l} \pm\tfrac{1}{2}i \pm \dfrac{\sqrt{5}+1}{4}j \pm \dfrac{\sqrt{5}-1}{4}k, \\ \pm\tfrac{1}{2}j \pm \dfrac{\sqrt{5}+1}{4}k \pm \dfrac{\sqrt{5}-1}{4}i, \\ \pm\tfrac{1}{2}k \pm \dfrac{\sqrt{5}+1}{4}i \pm \dfrac{\sqrt{5}-1}{4}j, \end{array}\right\} \text{30 square } \quad ,, \quad ,,$$

$$\pm 1 \qquad\qquad \text{2 terms}$$

$$\overline{120}$$

(The signs $\pm$ are all independent, except that in each of the three expressions in the top line the signs in $\sqrt{5} \pm 1$, $\sqrt{5} \mp 1$, are opposite to each other, so that each of the three expressions has 16 values.)

It is to be remarked that in the groups of 24 and 48, the group is not altered by any permutation whatever of the symbols i, j, k; whereas the group of 120 is not altered by the cyclical permutation of these symbols, but it is altered by the interchange of any two of them; the geometrical reason of this difference may be perceived without difficulty.

P.S. I found accidentally, *Gergonne*, t. xv., p. 40, (1824—25), the following problem: "De combien de manières m couleurs différentes les unes des autres peuvent-elles être appliquées sur les faces d'un polyèdre régulier; m représentant tour à tour les nombres 4, 6, 8, 12, 20?"

Instead of the m of the problem, writing as before F for the number of faces, and writing also E for the number of edges; then if different positions of the same polyhedron were reckoned as different polyhedra, the number of ways would of course be $\Pi(F)$ $(=1.2.3\ldots F)$; and since by what precedes the same polyhedron can be placed in $2E$ positions, the required number of ways is $\dfrac{1}{2E}\Pi(F)$.

Thus for the tetrahedron, if the colours are black, white, red, green, we may place it with the black face on the table and the white face in front; the only variation in the disposition of the colours, is according as the right hand and the left hand faces are coloured red and green or else green and red respectively; and the number of ways therefore is $=2$, which agrees with the formula.

2, *Stone Buildings, W.C.*, 30 *January*, 1863.

376.

THÉORÈME RELATIF À L'ÉQUILIBRE DE QUATRE FORCES.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXI. (*Juillet—Décembre* 1865), pp. 829—830.]

On sait que si quatre forces qui agissent sur un corps solide se tiennent en équilibre, alors (théorème de M. Möbius) les droites suivant lesquelles ces forces agissent sont quatre génératrices d'un même hyperboloïde: et de plus en représentant chaque force par une longueur proportionnelle sur la direction de cette force, alors (théorème de M. Chasles) le tetraèdre formé par deux quelconques des forces est égal au tetraèdre formé par les deux autres forces.

En cherchant les valeurs des quatre forces lesquelles en agissant selon quatre génératrices données d'un même hyperboloïde se tiennent en équilibre, j'ai réussi à trouver pour ces valeurs une expression assez remarquable qui comprend comme corollaire le théorème de M. Chasles.

Je nomme *moment de deux droites* la distance perpendiculaire de ces droites multipliée par le sinus de leur inclinaison mutuelle. Cela étant, en considérant quatre droites 1, 2, 3, 4 génératrices d'un même hyperboloïde, je dénote par ces mêmes symboles 1, 2, 3, 4 les forces qui agissent selon ces quatre droites respectivement, et par 12 le moment des droites 1 et 2, et de même pour les autres combinaisons de deux droites.

Or je dis que les forces 1, 2, 3, 4 qui se tiennent en équilibre ont les valeurs proportionnelles que voici, à savoir en prenant les radicaux avec des signes convenables:

$$1 = \sqrt{23 \,.\, 34 \,.\, 42},$$

$$2 = \sqrt{34 \,.\, 41 \,.\, 13},$$

$$3 = \sqrt{41 \,.\, 12 \,.\, 24},$$

$$4 = \sqrt{12 \,.\, 23 \,.\, 31}.$$

On déduit de là, en écrivant pour abréger

$$\nabla = \sqrt{23 . 31 . 12 . 14 . 24 . 34},$$

les équations

$$\nabla \sqrt{23 . 14} = 2 . 3 . 23 = 1 . 4 . 14,$$

$$\nabla \sqrt{31 . 24} = 3 . 1 . 31 = 2 . 4 . 24,$$

$$\nabla \sqrt{12 . 34} = 1 . 2 . 12 = 3 . 4 \ 34,$$

où par exemple l'équation $1 . 2 . 12 = 3 . 4 . 34$ exprime que le produit des forces 1 et 2 par le moment 12 des droites suivant lesquelles ces forces agissent est égal au produit des forces 3 et 4 par le moment 34 des droites selon lesquelles ces forces agissent.

J'ajoute que l'on a, en prenant les radicaux avec les signes convenables,

$$\sqrt{23 . 14} + \sqrt{31 . 24} + \sqrt{12 . 34} = 0,$$

équation qui subsiste non seulement pour quatre génératrices quelconques d'un même hyperboloïde, mais pour quatre droites liées par une relation géométrique plus générale, à savoir, pour quatre droites telles que les deux droites qui rencontrent ces quatre droites se réduisent à une seule droite: ou (ce qui est la même chose) telles que chacune des quatre droites touche l'hyperboloïde qui passe par les trois autres droites.

377.

NOTE SUR LA CORRESPONDANCE DE DEUX POINTS SUR UNE COURBE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXII. (*Janvier—Juin*, 1866), pp. 586—590.]

DANS la théorie à laquelle se rapporte cette Note, un point de rebroussement, s'il était nécessaire d'en parler, serait censé un cas particulier du point double; mais, pour simplifier, je ne ferai attention qu'aux courbes sans point de rebroussement.

Une courbe de l'ordre m peut avoir au plus $\frac{1}{2}(m-1)(m-2)$ points doubles; la différence entre ce nombre et le nombre actuel δ des points doubles d'une courbe donnée, savoir le nombre

$$D = \tfrac{1}{2}(m-1)(m-2) - \delta,$$

que je nomme le *défaut* (en anglais, *deficiency*), joue, comme on sait, un rôle important dans la théorie de la courbe. En particulier, pour une courbe de l'ordre m avec le défaut $D=0$, ou, comme je dis, pour une courbe *unicursale* de l'ordre m, les coordonnées (x, y, z) d'un point quelconque de la courbe (je me sers toujours des coordonnées homogènes) sont proportionnelles à des fonctions rationnelles et entières du degré m d'un paramètre variable θ.

Cela étant, le théorème de M. Chasles: "Lorsque sur une droite deux séries de points P, P' se correspondent de manière qu'à un point donné P correspondent α points P', et qu'à un point donné P' correspondent α' points P, alors le nombre des points P qui coïncident avec les points correspondants P' est $\alpha+\alpha'$;" ce théorème, dis-je, s'étend sans changement à des points correspondants situés sur une courbe *unicursale* quelconque; et l'on peut énoncer le théorème comme il suit:

Lorsque, sur une courbe unicursale, il y a deux séries de points qui ont une correspondance (α, α'), le nombre des points unis est $\alpha+\alpha'$.

Cela donne lieu au théorème: "Lorsque, sur une courbe, avec le défaut D, il y a deux séries de points qui ont une correspondance (α, α'), le nombre des points unis est $\alpha + \alpha' + 2kD$," où $2k$ est un facteur qu'il s'agit de déterminer. Cela peut se faire, sinon toujours, au moins dans la plupart des cas, au moyen du théorème que voici, tiré d'une induction qui me paraît suffisante:

En considérant sur la courbe $U=0$ un point donné P', et puis les intersections de la courbe $U=0$ par une courbe $\Theta=0$ dont l'équation contient d'une manière quelconque les coordonnées (x', y', z') du point donné P'; s'il y a k intersections qui coïncident avec le point P', et que les autres intersections forment un système de points P qui correspondent au point donné P', et si cette correspondance est une correspondance (α, α'), alors le nombre des points unis est $\alpha + \alpha' + 2kD$.

Je donne quatre exemples de ce théorème:

1°. *Recherche de la classe.*—Si les points correspondants P, P' sont situés en ligne droite avec un point donné O, alors les points unis sont les points de contact des tangentes menées par le point O; donc le nombre des points unis est égal à la classe de la courbe. La courbe $\Theta=0$ est ici la droite OP', il y a donc une seule intersection P'; donc $k=1$, et nous avons entre les points P, P' une correspondance $(m-1, m-1)$. Donc nous avons pour la classe M de la courbe l'expression

$$M = 2(m-1) + 2D,$$

où, en substituant pour D la valeur

$$D = \tfrac{1}{2}(m-1)(m-2) - \delta,$$

nous trouvons

$$M = m^2 - m - 2\delta,$$

comme cela doit être.

2°. *Recherche du nombre des inflexions.*—Si les points P sont les points de rencontre avec la courbe de la tangente au point P', alors les points unis seront les points d'inflexion. La courbe $\Theta=0$ est ici la tangente au point P'; il y a ainsi deux intersections au point P; donc $k=2$; de plus, à chaque point P' correspondent $(m-2)$ points P, et à chaque point P correspondent $M-2$ points P'. On a donc pour le nombre des inflexions

$$i = (m + M - 4) + 4D,$$

ou, en substituant pour M, D, leurs valeurs,

$$i = 3m(m-2) - 6\delta,$$

ce qui est juste.

Avant d'aller plus loin, il convient de généraliser le théorème, en remarquant que les intersections des courbes $U=0$, $\Theta=0$ peuvent former plusieurs systèmes simples ou multiples de points: les intersections peuvent être le point P' (k fois), un système de points P (p fois), un système de points Q (q fois), etc. Cela étant, s'il y a entre

les points P' et P une correspondance (α, α'), et si le nombre des points unis de ce système est a; s'il y a entre les points P' et Q une correspondance (β, β'), et si le nombre des points unis de ce système est b, et ainsi de suite; alors le théorème prend la forme

$$p\mathrm{a} + q\mathrm{b} + \ldots = p(\alpha + \alpha') + q(\beta + \beta') + \ldots + 2kD;$$

c'est la forme applicable à l'exemple qui suit.

3°. *Recherche du nombre des tangentes doubles.*—Prenons pour la courbe $\Theta = 0$ le système des $(M-2)$ tangentes menées à la courbe par le point donné P'; on a ici les points P qui sont les points de contact de ces tangentes, et les points Q qui sont les autres intersections de la courbe par ces tangentes; les intersections sont le point P' $(M-2)$ fois (donc $k = M-2$), le système des points P (2 fois) et le système des points Q (1 fois). Le système P, P' est précisément celui qui donne les points d'inflexion. On a done

$$\alpha = \alpha' = m - 1;$$

a est égal au nombre de points d'inflexion (mais, pour plus de commodité, je retiens le symbole a); $p = 2$. Le système P, Q est un système qui a pour points unis les points de contact des tangentes doubles, le nombre b des points unis sera donc 2τ, en dénotant par τ le nombre des tangentes doubles. On a pour la correspondance (β, β') entre les points P' et Q

$$\beta = \beta' = (m-3)(M-2);$$

enfin

$$q = 1.$$

Le théorème donne ainsi

$$2\mathrm{a} + \mathrm{b} = 2(m + M - 4) + 2(m-3)(M-2) + 2(M-2)D;$$

mais nous avons ci-dessus trouvé

$$\mathrm{a} = (m + M - 4) + 4D;$$

donc enfin

$$\mathrm{b} = 2\tau = 2(m-3)(M-2) + 2(M-6)D,$$

où, en substituant pour M et D leurs valeurs, on retrouve la formule ordinaire

$$2\tau = m(m-2)(m^2-9) - (m^2 - m - 6)4x + 4x(x-1).$$

Parmi les intersections des courbes $U = 0$, $\Theta = 0$, il peut y avoir un système simple ou multiple de points fixes, c'est-à-dire indépendants de la position du point P'; disons un système de λ points A (l fois). Il y aura dans ce cas, entre les points P', A, une correspondance $(0, \lambda)$, et les points unis du système sont les points A mêmes; le nombre des points unis est donc λ; les deux côtés de l'équation contiendront les termes égaux $l\lambda$ et $l(0+\lambda)$ respectivement, qui se détruisent, ce qui fait voir qu'il est permis de négliger les points fixes A, et ne faire attention qu'aux points d'intersection variables.

Il est assez remarquable que le théorème général peut s'écrire sous cette forme plus simple

$$p\mathrm{a} + q\mathrm{b} + \ldots = p(\alpha + \alpha') + q(\beta + \beta') + \ldots,$$

en comprenant parmi les systèmes formés par les intersections des courbes $U = 0$, $\Theta = 0$, le système du point P' (k fois), et en posant pour ce système

$$\mathrm{a} = 0, \quad \alpha = \alpha' = D;$$

le système du point P (k fois) donne ainsi un terme $= 0$ au côté gauche, un terme $= 2kD$ au côté droit de l'équation.

Comme dernier exemple appartenant à la formule simple

$$\mathrm{a} = \alpha + \alpha' + 2kD,$$

je prends:

4°. *Recherche du nombre des points sextactiques*, c'est-à-dire des points qui sont tels, que par chacun passe une conique qui a dans ce point un contact du cinquième ordre avec la courbe.—Il faut prendre pour les points P les intersections avec la courbe de la conique qui a au point P' un contact du quatrième ordre; les points unis seront ceux dont il s'agit. La courbe $\Theta = 0$ est la conique qui a au point P' un contact du quatrième ordre. On a ainsi, parmi les intersections, le point P' 5 fois; donc $k = 5$. A chaque point P' correspondent $2m - 5$ points P; à chaque point P, $(10m^2 - 20m - 5 - 20\delta)$ points P' (j'emprunte le terme -20δ d'une formule que vient de donner M. Zeuthen); donc la formule donne pour le nombre des points unis

$$10m^2 - 18m - 10 - 20\delta + 10D,$$

c'est-à-dire

$$15m^2 - 33m - 30\delta.$$

Mais cette expression comprend le nombre $3m(m-2) - 6\delta$ des inflexions; en effet, pour un point d'inflexion, la conique avec contact du quatrième ordre se réduit à la tangente prise deux fois, ce qui est une conique avec contact du cinquième ordre. Donc enfin le nombre des points sextactiques sera

$$m(12m - 27) - 24\delta,$$

ou, pour une courbe sans points doubles,

$$m(12m - 27):$$

ce qui s'accorde avec la valeur que j'ai trouvée par d'autres moyens, [341].

378.

REPORT OF A COMMITTEE APPOINTED BY THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, TO CONSIDER THE FORMATION OF A CATALOGUE OF PHILOSOPHICAL MEMOIRS.

[From the *Report of the British Association for the Advancement of Science*, (1856), pp. 463—464.]

THE Committee were appointed—on the occasion of a communication from Professor Henry of Washington, containing a proposal for the publication of Philosophical Memoirs scattered throughout the Transactions of Societies in Europe and America, with the offer of cooperation on the part of the Smithsonian Institute, to the extent of preparing and publishing, in accordance with the general plan which might be adopted by the British Association, a Catalogue of all the American Memoirs on Physical Science—to consider the best system of arrangement, and to report thereon to the Council.

The Committee are desirous of expressing their sense of the great importance and increasing need of such a Catalogue.

They understand the proposal of the Smithsonian Institute to be, that a separate Catalogue should be prepared and published for America.

In the opinion of the Committee,

The Catalogue should embrace the Mathematical and Physical Sciences, but should exclude Natural History and Physiology, Geology, Mineralogy, and Chemistry, which would properly form the subject-matter of a distinct Catalogue or Catalogues. The difficulty of drawing the line would perhaps be greatest with regard to Chemistry and Geology; but the Committee would admit into the Catalogue memoirs not purely Chemical or Geological, but having a direct bearing upon the subjects of the Catalogue.

The Catalogue should not be restricted to memoirs in Transactions of Societies, but should comprise also memoirs in the Proceedings of Societies, in Mathematical and Scientific Journals, in Ephemerides and volumes of Observations, and in other collections not coming under any of the preceding heads. The Catalogue would not comprise separate works.

The Catalogue should begin from the year 1800.

There should be a Catalogue according to the names of authors, and also a Catalogue according to subjects; the title of the memoir, date, and other particulars to be in each case given in full, so as to avoid the necessity of a reference from the one Catalogue to the other.

The Catalogue should, in referring to a memoir, give the number as well of the last as of the first page, so as to show the length of the memoir.

The Catalogue should give in every case the date of a memoir (the year only), namely, in the case of memoirs published in the Transactions of a Society, the date of reading, and in other cases the date on the title-page of the volume. Such date should be inserted as a distinct fact, even in the case of a volume of transactions referred to by its date.

The Catalogue should contain a list of volumes indexed, showing the complete title; in the case of transactions, the year to which the volume belongs, and the year of publication; and in other cases, the year of publication, and the abbreviated reference to the work.

The references to works should be given in a form sufficiently full to be easily intelligible without turning to the explanation of such reference.

The author's name and the date should be printed in a distinctive type, so as to be conspicuous at first sight; and generally the typographical execution should be such as to facilitate as much as possible the use of the Catalogue.

As to the Catalogue according to the authors' names, the memoirs of the same author should be arranged according to their dates.

As to the Catalogue according to subjects, the question of the arrangement is one of very great difficulty. It appears to the Committee that the scheme of arrangement cannot be fixed upon according to any *à priori* classification of subjects, but must be determined after some progress has been made in the preliminary work of collecting the titles of the memoirs to be catalogued. The value of this part of the Catalogue will materially depend upon the selection of a proper principle of arrangement, and the care and accuracy with which such principle is carried out. The arrangement of the memoirs in the ultimate subdivisions should be according to their dates.

The most convenient method of making the Catalogue would appear to be, that each volume to be indexed should be gone through separately, and a list formed of

all the memoirs which come within the plan of the proposed Catalogue. Such list should be in triplicate, one copy for reference, a second copy to be cut up and arranged for the Catalogue according to authors' names, and another copy to be cut up and arranged for the Catalogue according to subjects.

The Committee have endeavoured to form an estimate of the space which the Catalogue would occupy. The number of papers in a volume of transactions is in general small, but there are works, such as the *Comptes Rendus*, the *Astronomische Nachrichten*, the *Philosophical Magazine*, &c., containing a very great number of papers, the titles of which would consequently occupy a considerable space in the Catalogue. Upon the whole, the Committee consider that, excluding America, they may estimate the number of papers to be entered at 125,000; or since each paper would be entered twice, the number of entries would be 250,000. The number of entries that could conveniently be brought into a page 4to. (double columns) would be about 30, so that, according to the above estimate, the Catalogue would occupy ten quarto volumes of rather more than 800 pages each.

It appears to the Committee that there should be paid Editors, who should be familiar with the several great branches respectively of the Sciences to which the Catalogue relates; but that the general scheme of arrangement and details of the Catalogue should be agreed upon between all the Editors, and that they should be jointly responsible for the execution. It would of course be necessary that the Editors should have the assistance of an adequate staff of clerks.

The principal scientific transactions and works would be accessible in England at the Library of the British Museum, and the libraries of the Royal Society and other Philosophical Societies. It would be the duty of the Editors to ascertain all the different works which ought to be catalogued, and to procure information as to the contents of such of them as may not happen to be accessible.

The Catalogue according to authors' names would be the most readily executed, and this Catalogue, if it should be found convenient, might be first published. The time of bringing out the two Catalogues would of course depend upon the sufficiency of the assistance at the command of the Editors; but if the Catalogue be undertaken, it is desirable that the arrangement should be such, that the complete work might be brought out within a period not exceeding three years.

A. CAYLEY.
R. GRANT.
G. G. STOKES.

13*th June*, 1856.

379.

NOTICES OF COMMUNICATIONS TO THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

[From the *Reports of the British Association for the Advancement of Science*, 1854 to 1860, *Notices and Abstracts of miscellaneous Communications to the Sections.*]

1. *On the Solution of Cubic and Biquadratic Equations.* Report, 1854, p. 1.

---

2. *On the Porism of the In-and-circumscribed Triangle.* Report, 1855, p. 1.

THE porism of the in-and-circumscribed triangle in its most general form relates to a triangle the angles of which lie in fixed curves, and the sides of which touch fixed curves, but at present I consider only the case in which the angles lie in one and the same fixed curve which for greater simplicity I consider to be a conic. We have therefore a triangle ABC the angles of which lie in a fixed conic $\mathfrak{S}$ and the sides of which touch the fixed curves $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. And if we consider the conic $\mathfrak{S}$ and the curves $\mathfrak{A}$, $\mathfrak{B}$ as given, the curve $\mathfrak{C}$ will be the envelope of the side AB of the triangle. Suppose that the curves $\mathfrak{A}$, $\mathfrak{B}$ are of the classes m, n respectively; there is no difficulty in showing that the curve $\mathfrak{C}$ is of the class $2mn$. But the curve $\mathfrak{C}$ has in general double tangents forming two distinct groups, the first group arising from the quadrilaterals inscribed in the conic $\mathfrak{S}$ and such that two opposite sides touch the curve $\mathfrak{A}$, and the other two opposite sides the curve $\mathfrak{B}$; the second group arising from quadrilaterals such that two adjacent sides touch the curve $\mathfrak{A}$ and the other two adjacent sides touch the curve $\mathfrak{B}$. The number of double tangents of the first group is $= mn(mn-1)$, and the number of double tangents of the second group is $= mn(mn-m-n+1)$; the number of double tangents of the two groups is therefore $= mn(2mn-m-n)$. The curve $\mathfrak{C}$ has not in general any inflexions, hence, being of the class $2mn$ with $mn(2mn-m-n)$ double tangents, it will be of the order $2mn(m+n-1)$.

When the curves $\mathfrak{A}$ and $\mathfrak{B}$ are conics, the curve $\mathfrak{C}$ is therefore of the class 8, with 16 double tangents but no inflexions, consequently of the order 24. But there are two remarkable cases in which the order is further diminished.

First when each of the conics $\mathfrak{A}$, $\mathfrak{B}$ has double contact with the conic $\mathfrak{S}$. The four points of contact give rise to 8 new double tangents or there are in all 24 double tangents, the curve $\mathfrak{C}$ is therefore of the degree 8: and being of the class 8 with 24 double tangents, it must of necessity break up into 4 curves each of the class 2, i.e. into 4 conics. Each of these has double contact with the conic $\mathfrak{S}$, or attending to only one of the four conics we have the well-known theorem which I call the porism (homographic) of the in-and-circumscribed triangle, viz. "there are an infinity of triangles inscribed each in a conic, and such that the sides touch conics having each of them double contact with the circumscribed conic."

Secondly, the conics $\mathfrak{A}$ and $\mathfrak{B}$ may intersect the conic $\mathfrak{S}$ in the same four points. Here every tangent of the curve $\mathfrak{C}$ is in fact a double tangent belonging to the first-mentioned group, the curve $\mathfrak{C}$ in fact consists of two coincident curves: each of them is therefore of the class 4. But this curve of the class 4 has itself four double tangents arising from the common points of intersection of the conics $\mathfrak{A}$, $\mathfrak{B}$ with the conic $\mathfrak{S}$; it must therefore break up into two curves each of the class 2, i.e. into two conics: each of these intersects the conic $\mathfrak{S}$ in the same four points in which it is intersected by the conics $\mathfrak{A}$, $\mathfrak{B}$. Attending only to one of the two conics we have the other well-known theorem which I call the porism (allographic) of the in-and-circumscribed triangle, viz. "there exist an infinity of triangles inscribed in a conic, and such that the sides touch conics, each of them meeting the circumscribed conic in the same four points."

---

3. *On the Notion of Distance in Analytical Geometry.* Report, 1858, p. 3.

THE author remarks that the principles of Modern Geometry show that any metrical property whatever is really based upon a purely descriptive property, and that these principles contain in fact a theory of distance—but that such theory has not been disengaged from its applications and stated in a distinct and explicit form. The paper contains an account of the theory in question, viz. it is shown that in any system of geometry of two dimensions the notion of distance can be arrived at from descriptive principles by means of a conic called the Absolute, and which in ordinary geometry degenerates into a pair of points.

---

4. *On Curves of the Fourth Order having Three Double Points.* Report, 1860, p. 4.

THE paper is a short notice only of researches which the Author is engaged in with reference to curves of the fourth order having three double points. A curve of the kind in question is derived from a conic by the well-known transformation of substituting for the original trilinear coordinates their reciprocals: and the species of the curve of the fourth order depends on the position of the conic with respect to the fundamental triangle.

---

5. *On Curves of the Third Order.* Report, 1861, p. 2.

A CURVE of the third order or cubic curve is a section of a cubic cone and such cone is intersected by a concentric sphere in a spherical cubic. It is an obvious consequence of a theorem of Sir Isaac Newton's that there are five principal kinds of cubic cones, or what is the same thing five principal kinds of spherical cubics—but the nature of these five kinds of spherical cubics was first distinctly explained by Möbius. They may be designated the *simplex*, the *complex*, the *crunodal*, the *acnodal* and the *cuspidal:* where crunode, acnode, denote respectively the two species of double points (nodes), viz. the double point with two real branches, and the conjugate or isolated point. The foregoing results are known: the special object of the paper is to establish a subdivision of the simplex kind of spherical cubics. The simplex kind is a continuous reentering curve cutting a great circle, to fix the ideas say the equator, in three pairs of opposite points, which are the three real inflexions of the curve. The three great circles which are the tangents at the inflexions and the equator divide the entire surface of the sphere into fourteen regions whereof eight are trilateral and the remaining six are quadrilateral. The curve may be entirely in six out of the eight trilateral regions, and it is in this case said to be *simplex trilateral;* or it may lie entirely in the six quadrilateral regions, and it is in this case said to be *simplex quadrilateral;* and there is an intermediate form, the *simplex neutral;* viz. in this case the three great circles tangents at the inflexions meet in a pair of opposite points and there are in all only twelve regions all of them trilateral; the curve lies entirely in six of these regions.

---

6. *On a Certain Curve of the Fourth Order.* Report, 1862, p. 3.

THE curve in question is the locus of the centres of the conics which pass through three given points and touch a given line; if the equations of the sides of the triangle formed by the three points are $x=0$, $y=0$, $z=0$, these coordinates being such that $x+y+z=0$ is the equation of the line infinity, and if $\alpha x+\beta y+\gamma z=0$ be the equation of the given line, then (as is known) the equation of the curve is

$$\sqrt{\alpha x(y+z-x)}+\sqrt{\beta y(z+x-y)}+\sqrt{\gamma z(x+y-z)}=0.$$

The special object of the communication was to exhibit the form of the curve in the case where the line cuts the triangle, and to point out the correspondence of the positions of the centre upon the curve, and the point of contact on the given line.

---

7. *On the Representation of a Curve in Space by means of a Cone and Monoid Surface.* Report, 1862, p. 3.

THE author gave a short account of his researches recently published in the *Comptes Rendus.* The difficulty as to the representation of a curve in space is, that such a curve is not in general the complete intersection of two surfaces; any two surfaces passing through the curve intersect not only in the curve itself, but in a certain companion curve, which cannot be got rid of; this companion curve is in the proposed mode of representation reduced to the simplest form, viz. that of a system of lines passing through one and the same point. The two surfaces employed for the representation of a curve of the nth order are, a *cone* of the nth order having for its vertex an arbitrary point (say the point $x=0$, $y=0$, $z=0$), and a *monoid* surface with the same vertex, viz. a surface the equation whereof is of the form $Qw-P=0$, P and Q being homogeneous functions of (x, y, z) of the degrees p and $p-1$ respectively (where p is at most $=n-1$). The monoid surface contains upon it $p(p-1)$ lines given by the equations $(P=0, Q=0)$; and, the cone passing through $n(p-1)$ of these lines (if, as above supposed, $p \not> n-1$, this implies that some of these lines are multiple lines of the cone), the monoid surface will besides intersect the cone in a curve of the nth order.

---

8. *On a Formula of M. Chasles relating to the Contact of Conics.* Report, 1864, p. 1.

THE author gave an account of the recent investigations of M. Chasles in relation to the theory of conics, viz., M. Chasles has found that the properties of a system of conics, containing one arbitrary parameter, depend upon two quantities called by him the *characteristics* of the system; these are, μ, the number of conics of the system which pass through a given point, and, ν, the number of conics of the system which touch a given line; or, say, μ is the *parametric order*, ν the *parametric class*, of the system. And he exhibited a transformation obtained by him of a formula of M. Chasles for the number of conics which touch five given curves, viz., if (M, m) (N, n) (P, p) (Q, q) (R, r) be the orders and classes of the five given curves respectively, then the number of curves is

$$=(1, 2, 4, 4, 2, 1)(M, m)(N, n)(P, p)(Q, q)(R, r),$$

where the notation stands for $1 . MNPQR + 2\Sigma mNPQR + 4\Sigma mnPQR + \&c.$ The transformed formula in question was communicated by the author to M. Chasles, and had appeared in the *Comptes Rendus;* but it is, in fact, included in a very beautiful and general theorem given in the same Number by M. Chasles himself.

---

9. *On the Problem of the In-and-circumscribed Triangle.* Report, 1864, p. 1.

THE general problem of the in-and-circumscribed triangle may be thus stated, viz., to find a triangle the angles whereof severally lie in, and the sides severally touch, a given curve or curves; and we may, in the first instance, inquire as to the number of such triangles. The first and easiest case is when the curves are all distinct; here, if the angles lie in curves of the *orders* m, n, p, respectively, and the sides touch curves of the *classes* Q, R, S, respectively, then the number of triangles is $=2mnpQRS$. The number may be obtained for some other cases; but the author has not yet considered the final and most difficult case, viz. that in which the angles severally lie in, and the sides severally touch, one and the same given curve.

The foregoing notices relate to verbal communications upon questions with which I was at the time occupied and which are for the most part more fully discussed in papers printed elsewhere. I remark upon them as follows:

1. I have no remembrance as to this; I think no paper printed or written.

2. See 175.

3. See 158.

4. No paper printed. The intention was to consider the different forms of trinodal quartic curves, in particular those with real nodes, as obtained from the inversion of a conic according to the different relations of the conic to the fundamental triangle. Thus according as the conic cuts in two real points, touches, or cuts in two imaginary points, a side of the triangle, the tangents at the corresponding node are real, coincident, or imaginary; viz. the node is a crunode, cusp, or acnode. And in the case of real intersections there is a further distinction according as the intersections lie each or either of them on the side itself, or on the side produced in one or other of the two directions. By considering the different relations of the conic to the fundamental triangle we thus obtain the different forms of the trinodal quartic.

5. See 351.

6. I think no paper printed or written.

7. See 302 and 305.

8. See 306.

9. The question is considered in a memoir On the Problem of the In-and-circumscribed Triangle, *Phil. Trans.* t. CLXI. (for 1871), pp. 369—412.

380.

NOTE ON THE RECTANGULAR HYPERBOLA.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. I. (1862), p. 77.]

EVERY conic which passes through the points of intersection of two rectangular hyperbolas is a rectangular hyperbola. In fact if a conic be referred to rectangular axes, the condition that it may be a rectangular hyperbola is *Coeff. of* $x^2 = -$ *Coeff. of* y^2. Hence if U, V be any two quadratic functions of x, y, and if λ be a constant, the condition in question being satisfied for each of the functions U, V, is satisfied for the function $U+\lambda V$: and the equation of any conic through the points of intersection of the conics $U=0$, $V=0$ is $U+\lambda V=0$: which proves the theorem in question.

In particular if from two of the angles of a triangle perpendiculars are let fall on the opposite sides, and if the point of intersection of the perpendiculars and the third angle be joined: then since the first side and the perpendicular upon it are a rectangular hyperbola, and the second side and the perpendicular upon it are a rectangular hyperbola; the third side and the joining line must be a rectangular hyperbola: that is, these two lines must be at right angles to each other. We have thus the well-known theorem that the perpendiculars let fall from the angles of a triangle on the opposite sides meet in a point.

The theorem as to the hyperbolas is a particular case of the theorem that three conics which pass through the same four points are met by any line whatever in six points forming a system in involution. In fact a rectangular hyperbola is a conic meeting the line at infinity in two points harmonically related to the circular points at infinity: hence two of the conics being rectangular hyperbolas, the foci of the involution are the circular points at infinity: hence these points and the points in which the line at infinity meets the third conic are harmonically related to each other; that is, the third conic is a rectangular hyperbola.

381.

NOTE ON BEZOUT'S METHOD OF ELIMINATION.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. II. (1864), pp. 88, 89.]

LET U, U' be any two rational and integral functions of x of the same order; to fix the ideas let them be the cubic functions

$$U = ax^3 + bx^2 + cx + d,$$
$$U' = a'x^3 + b'x^2 + c'x + d'.$$

Write

$$A = \begin{vmatrix} U, & U' \\ a, & a' \end{vmatrix}, \quad P = \begin{vmatrix} U, & U' \\ \alpha, & \alpha' \end{vmatrix},$$

$$B = \begin{vmatrix} U, & U' \\ b, & b' \end{vmatrix}, \quad Q = \begin{vmatrix} U & , U' \\ ax + b, & a'x + b' \end{vmatrix},$$

$$C = \begin{vmatrix} U, & U' \\ c, & c' \end{vmatrix}, \quad R = \begin{vmatrix} U & , U' \\ ax^2 + bx + c, & a'x^2 + b'x + c' \end{vmatrix},$$

$$D = \begin{vmatrix} U, & U' \\ d, & d' \end{vmatrix}, \quad S = \begin{vmatrix} U & , U' \\ ax^3 + bx^2 + cx + d, & a'x^3 + b'x^2 + c'x + d' \end{vmatrix}, = \begin{vmatrix} U, & U' \\ U, & U' \end{vmatrix}, = 0,$$

then we have

$$P = A,$$
$$Q = Ax + B,$$
$$R = Ax^2 + Bx + C,$$
$$S = Ax^3 + Bx^2 + Cx + D, = 0,$$

and thence

$$\begin{aligned} A &= P, \\ B &= Q - Px, \\ C &= R - Qx, \\ D &= S - Rx, = -Rx. \end{aligned}$$

Let α be an arbitrary quantity and write

$$\square z = \begin{vmatrix} U & , U' \\ a\alpha^3 + b\alpha^2 + c\alpha + d', & a'\alpha^3 + b'\alpha^2 + c'\alpha + d' \end{vmatrix};$$

we have it is clear

$$\begin{aligned} \square &= A\alpha^3 + B\alpha^2 + C\alpha + D, \\ &= \alpha^3 P + \alpha^2 (Q - Px) + \alpha (R - Qx), = Rx, \\ &= (\alpha^3 - \alpha^2 x) P + (\alpha^2 - \alpha x) Q + (\alpha - x) R, \end{aligned}$$

and thence

$$\frac{\square}{\alpha - x} = \alpha^2 P + \alpha Q + R.$$

The equations $P = 0$, $Q = 0$, $R = 0$ are respectively quadratic equations in x, the equations which are used in Bezout's method of elimination; and representing them by

$$\begin{aligned} P &= Lx^2 + Mx + N, = 0, \\ Q &= L'x^2 + M'x + N', = 0, \\ R &= L''x^2 + M''x + N'', = 0, \end{aligned}$$

we have

$$\begin{vmatrix} L, & M, & N \\ L', & M', & N' \\ L'', & M'', & N'' \end{vmatrix} = 0$$

as the equation resulting from the elimination of x from the equations $U = 0$, $U' = 0$. The foregoing investigation shows that the functions P, Q, R are obtained as the coefficients of α^2, α, 1 in the development of

$$\frac{1}{\alpha - x} \begin{vmatrix} U & , U' \\ a\alpha^3 + b\alpha^2 + c\alpha + d, & a'\alpha^3 + b'\alpha^2 + c'\alpha + d' \end{vmatrix};$$

or more generally, taking U, U' to be any two functions of the order n, that the n functions P, Q, R, &c. each of the order $n-1$ are obtained as the coefficients of α^{n-1}, α^{n-2}, ... α, 1 in the development of

$$\frac{1}{\alpha - x} \begin{vmatrix} U, & U' \\ U_\alpha, & U_\alpha' \end{vmatrix},$$

where U_α, U_α' are what U, U' become when x is replaced therein by α: and we have thus a simple *à posteriori* verification of the form in which, several years ago, I presented Bezout's Method of Elimination.

2, *Stone Buildings, W.C., March* 5, 1863.

382.

NOTE ON THE TETRAHEDRON.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, t. III. (1866), pp. 8—10.]

THE following simple properties of a tetrahedron seem worth noticing.

In the tetrahedron $ABCD$ if $AC = BD$ and $AD = BC$, then the line joining the middle points of $AB + CD$, or say the points $\frac{1}{2}AB$ and $\frac{1}{2}CD$, cuts at right angles these lines AB and CD.

If $AB = CD$, then the line joining the points $\frac{1}{2}AC$, $\frac{1}{2}BD$, and the line joining the points $\frac{1}{2}AD$, $\frac{1}{2}BC$ (lines which in any tetrahedron meet each other), cut each other at right angles.

In fact if A, B, C, D have for their coordinates $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$: then the coordinates of the point $\frac{1}{2}AB$ are $\frac{1}{2}(\alpha_1 + \alpha_2)$, $\frac{1}{2}(\beta_1 + \beta_2)$, and so for the points $\frac{1}{2}CD$, &c.: the equations of the line through the points $\frac{1}{2}AB$, $\frac{1}{2}CD$ therefore are

$$\frac{x - \frac{1}{2}(\alpha_1 + \alpha_2)}{\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4} = \frac{y - \frac{1}{2}(\beta_1 + \beta_2)}{\beta_1 + \beta_2 - \beta_3 - \beta_4} = \frac{z - \frac{1}{2}(\gamma_1 + \gamma_2)}{\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4},$$

and I observe in passing that this line passes through the point whose coordinates are

$$\tfrac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),\ \tfrac{1}{4}(\beta_1 + \beta_2 + \beta_3 + \beta_4),\ \tfrac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4);$$

the other two similar lines pass through the same point, and the above-mentioned property of the general tetrahedron is thus proved.

The condition that the foregoing line may cut at right angles the line AB, the equations whereof are

$$\frac{x - \alpha_1}{\alpha_1 - \alpha_2} = \frac{y - \beta_1}{\beta_1 - \beta_2} = \frac{z - \gamma_1}{\gamma_1 - \gamma_2},$$

is at once seen to be

$$\Sigma\,(\alpha_1-\alpha_2)(\alpha_1+\alpha_2-\alpha_3-\alpha_4)=0,$$

where Σ denotes the sum of the corresponding terms in α, β, γ. And so the condition that the same line may cut at right angles the line CD is

$$\Sigma\,(\alpha_3-\alpha_4)(\alpha_1+\alpha_2-\alpha_3-\alpha_4)=0.$$

But the conditions $AC=BD$, $AD=BC$ give respectively

$$\Sigma\,\{(\alpha_1-\alpha_3)^2-(\alpha_2-\alpha_4)^2\}=0,\ \Sigma\,\{(\alpha_1-\alpha_4)^2-(\alpha_2-\alpha_3)^2\}=0,$$

or writing these in the form

$$\Sigma\,(\alpha_1-\alpha_2-\alpha_3+\alpha_4)(\alpha_1+\alpha_2-\alpha_3-\alpha_4)=0,$$

$$\Sigma\,(\alpha_1-\alpha_3+\alpha_2-\alpha_4)(\alpha_1+\alpha_2-\alpha_3-\alpha_4)=0,$$

we obtain, by successively adding and subtracting, the two required equations.

The equations of the line through $\frac{1}{2}AC$, $\frac{1}{2}BD$ are

$$\frac{x-\frac{1}{2}(\alpha_1+\alpha_3)}{\alpha_1+\alpha_3-\alpha_2-\alpha_4}=\frac{y-\frac{1}{2}(\beta_1+\beta_3)}{\beta_1+\beta_3-\beta_2-\beta_4}=\frac{z-\frac{1}{2}(\gamma_1+\gamma_3)}{\gamma_1+\gamma_3-\gamma_2-\gamma_4},$$

and those of the line through $\frac{1}{2}AD$, $\frac{1}{2}BC$ are

$$\frac{x-\frac{1}{2}(\alpha_1+\alpha_4)}{\alpha_2+\alpha_3-\alpha_1-\alpha_4}=\frac{y-\frac{1}{2}(\beta_1+\beta_4)}{\beta_2+\beta_3-\beta_1-\beta_4}=\frac{z-\frac{1}{2}(\gamma_1+\gamma_4)}{\gamma_2+\gamma_3-\gamma_1-\gamma_4};$$

and the condition that these may cut at right angles is

$$\Sigma\,(\alpha_1+\alpha_3-\alpha_2-\alpha_4)(\alpha_1+\alpha_4-\alpha_2-\alpha_3)=0,$$

that is

$$\Sigma\,\{(\alpha_1-\alpha_2)^2-(\alpha_3-\alpha_4)^2\}=0,$$

which is in fact the condition $AB=CD$.

Combining the two theorems we see that if in a tetrahedron the pairs of opposite sides are respectively equal, then the line joining the centres of opposite sides cuts these sides at right angles, and moreover the three joining lines cut each other at right angles.

A tetrahedron of the form in question may be constructed as follows: viz. taking a parallelogram $ABCD$, whereof the diagonals AC, BD are unequal, then bending the parallelogram about its shorter diagonal AC in such manner that in the solid figure BD becomes equal to AC, we have a tetrahedron the opposite sides whereof are respectively equal.

Or it may be constructed even more simply as follows: viz. if $AB'CD'$ and $A'BC'D$ be parallel faces of any rectangular parallelopiped (the angles A and A', B and B', C and C', D and D' being respectively opposite to each other), then $ABCD$ or $A'B'C'D'$ is a tetrahedron of the form in question. The consideration of the rectangular parallelopiped puts in evidence the foregoing geometrical property.

In such a tetrahedron the line joining the centres of a pair of opposite sides is in the language of Bravais, see his "Mémoire sur les polyèdres de forme symétrique," *Liouville*, t. XIV. (1849), pp. 141—180, a binary axis of symmetry: viz. the figure is not altered by turning it round such axis through an angle $=\frac{1}{2}360°$. There are thus three such axes at right angles to each other, but the figure has not any centre of symmetry, nor (assuming that it is not further particularised) any plane of symmetry: each of the three axes is a principal axis, and the figure belongs to the sixth of Bravais' twenty-three classes of polyhedra, see the table p. 179. It was in fact by seeking to construct a figure of this class that I was led to the investigation.

383.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. I. to IV., 1863 to 1865.]

[Vol. I. (June 1863 to June 1864), pp. 18, 19.]

1373. (By T. T. WILKINSON, F.R.A.S.)—Given a circle (C) and any point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that, if LEM be drawn cutting the circle in L, M, we may always have $AE^2 = LE \,.\, EM \pm BA \,.\, AD$.

Solution by PROFESSOR CAYLEY.

Consider a circle centre O and radius OA, and in relation thereto a point M either outside or inside the circle, and suppose that

$(OA)^2 - (OM)^2$, or the "squared inner potency" of M is denoted by $\square i \,.\, M$,

and

$(OM)^2 - (OA)^2$, or the "squared outer potency" of M is denoted by $\square o \,.\, M$,

so that, for an outside point, $\square o \,.\, M$, $= - \square i \,.\, M$, is the square of the tangential distance of M from the circle; and, for an inside point, $\square i \,.\, M$, $= - \square o \,.\, M$, is the square of the shortest semi-chord through M.

Suppose now that M is a given point; the proposed question is in effect to find the locus of a point P such that $\pm \square o \,.\, P \pm \square o \,.\, M = (MP)^2$; but we have thus in reality four different questions according as the signs are assumed to be $++$, $+-$, $-+$, or $--$; the case $++$, or when $\square o \,.\, P + \square o \,.\, M = (MP)^2$, is perhaps the most interesting.

Taking the radius as unity, (α, β) as the coordinates of M, and (x, y) as the coordinates of P, we have here

$$(x^2+y^2-1)+(\alpha^2+\beta^2-1)=(x-\alpha)^2+(y-\beta)^2, \text{ or } \alpha x+\beta y-1=0\,;$$

that is, the locus of P is a right line, the polar of M in regard to the circle.

It may be remarked, that, when M is an inside point, then throughout the locus P is an outside point; and, replacing the negative quantity $\square o\,.\,M$ by its value, $=-\square\, i\,.\,M$, we have $\square o\,.\,P-\square i\,.\,M=(MP)^2$. If, however, M is an outside point, then in part of the locus P is an outside point, and we have $\square o\,.\,P+\square o\,.\,M=(MP)^2$, while in the remainder of the locus P is an inside point, and, replacing the negative quantity $\square o\,.\,P$ by its value, $=-\square i\,.\,P$, we have $-\square i\,.\,P+\square o\,.\,M=(MP)^2$. For the case $+-$, the locus of P is a right line, but for each of the other two cases $-+$ and $--$ the locus is a circle; the discussion of the several cases presents no particular difficulty.

[Vol. I. pp. 43—45.]

1387. (By W. K. CLIFFORD.)—1. Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be opposite intersections of the tangents, prove that OA and OB are equally inclined to the tangent at O to the ellipse.

2. If a straight line A join the poles of B with respect to two conics, prove that the lines joining AB to a pair of opposite intersections of common tangents, form, with A, B, an harmonic pencil.

3. If a point A be the intersection of the polars of B with respect to two conics, and AB be cut by a pair of common chords in C, D, prove that $ACBD$ is an harmonic range.

---

2. *Solution by* PROFESSOR CAYLEY.

This elegant theorem is included as a particular case in the known theorem, "Given three conics inscribed in the same quadrilateral, the tangents from any point to these conics form a pencil in involution."

Mr Clifford's theorem is in fact as follows: viz., Four common tangents are drawn to a circle and an ellipse which passes through the centre O of the circle; if A, B be opposite intersections of the tangents, then OA, OB are equally inclined to the tangent at O to the ellipse.

This comes to saying that the tangent at O to the ellipse, say OT, is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ drawn from O to the circular points at infinity; and if we replace the circle by an arbitrary conic S, and the line at infinity by an arbitrary line IJ, the theorem will be as follows:

Consider a conic S; a line meeting this conic in the points I, J; and the point O, the intersection of the tangents at I, J, or (what is the same thing) the pole of the line IJ in regard to the conic. If through the point O there be drawn any other conic Θ, and if A, B be opposite intersections of the common tangents of the conics S, Θ; then the tangent OT at the point O to the conic Θ is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ; or, as we may also express it, the lines OT, OT, the lines OA, OB, and the lines OI, OJ form a pencil in involution.

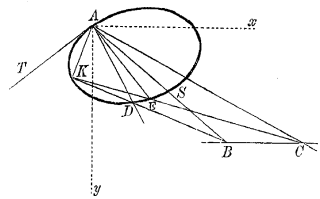
Now, considering the two points or point-pair (A, B) as a conic inscribed in the quadrilateral formed by the common tangents of the conics S and Θ, the conics S and Θ and the point-pair (A, B) are a system of three conics inscribed in the same quadrilateral; and hence, by the general theorem above referred to, if O' be any point whatever, the tangents from O to the conic S, the tangents from O' to the conic Θ, and the tangents from O' to the point-pair (that is, the two lines $O'A$, $O'B$) form a pencil in involution. But, if O' coincide with O, then the tangents to the conic S are the lines OI, OJ; and the tangents to the conic Θ are the coincident lines OT, OT; and we have thence the theorem in question; viz., that the lines OT, OT, the lines OI, OJ, and the lines OA, OB form a pencil in involution.

[Vol. I. pp. 77—79.]

1409. (By W. K. CLIFFORD.)—For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point K on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary.

Solution by PROFESSOR CAYLEY.

I find that this very elegant theorem depends on the lemma to be presently stated, and that it is intimately connected with Newton's theorem for the organic description of a conic, or, what is the same thing, with the theorem of the anharmonic relation of the points of a conic.



LEMMA. If AT be the tangent, and AS any other line through a point A of a conic, and if two lines equally inclined to AT and AS respectively meet the conic

in the points K and D (viz., if $\angle TAK = SAD$, the two angles being measured in opposite directions from AT, AS respectively); then the line KD meets AS in a fixed point B, that is, a point the position of which is independent of the magnitude of the equal angles.

To prove this, take A for the origin, and the bisectors of the angle TAS for the axes of x and y: then the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2fy + 2gx = 0;$$

the equation of the tangent at the origin, that is, the line AT, is $gx + fy = 0$; and hence the equation of the line AS is $gx - fy = 0$. Taking $y = \alpha x$ for the equation of the line AK, we have, for the coordinates x_1, y_1 of the point K where this meets the conic,

$$(a + 2h\alpha + b\alpha^2)\, x_1 + 2\,(f\alpha + g) = 0, \quad y_1 = \alpha x_1;$$

and then the equation of the line AD will be $y = -\alpha x$, and we shall have, for the coordinates x_2, y_2 of the point D where this meets the conic,

$$(a - 2h\alpha + b\alpha^2)\, x_2 + 2\,(f\alpha + g) = 0, \quad y_2 = -\alpha x_2.$$

The equation of the line KD is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & \alpha x_1, & 1 \\ x_2, & -\alpha x_2, & 1 \end{vmatrix} = 0,$$

that is

$$\alpha x\,(x_1 + x_2) + y\,(x_2 - x_1) - 2\alpha x_1 x_2 = 0;$$

and for the coordinates of the point B where this meets the line AS, the equation whereof is $gx - fy = 0$, we have

$$x\,\{f\alpha\,(x_1 + x_2) + g\,(x_2 - x_1)\} - 2f\alpha x_1 x_2 = 0,$$

or, as this may be written,

$$x\left\{\frac{f\alpha + g}{x_1} - \frac{-f\alpha + g}{x_2}\right\} - 2f\alpha = 0.$$

But we have

$$\frac{f\alpha + g}{x_1} = -\tfrac{1}{2}\,(a + 2h\alpha + b\alpha^2), \quad \frac{-f\alpha + g}{x_2} = -\tfrac{1}{2}\,(a - 2h\alpha + b\alpha^2);$$

and hence the equation is

$$x\,(-2h\alpha) - 2f\alpha = 0,$$

giving $x = -\dfrac{f}{h}$, and thence $y = -\dfrac{g}{h}$, for the coordinates of the point B; and, these being independent of α, the lemma is seen to be true.

Consider now the points A, K as fixed points on the conic; then, revolving about A the constant angle DAB, and about K the constant (zero) angle DKB, the locus of B is (by the theorem of the anharmonic relation of the points of a conic) given in the first instance as a conic through the points A, K; but, observing that a position of the angle DAB is TAK, and that the corresponding position of DKB is AKA, the line AK is part of the locus; and the locus is made up of this line and a line BC. And, conversely, given the fixed points A, K, and the line BC, the original conic is, by Newton's theorem, described by means of the constant angles DAB, DKB revolving about these points in such a manner that the arms AB, KB generate by their intersections the line BC. This being so, the other two arms AD, KD generate by their intersections the conic.

And then, considering the two positions DAB, EAC of the angle DAB (so that D, B are in a line with K, and E, C are also in a line with K), we have $\angle DAB = \angle EAC$, that is, $\angle DAE = \angle BAC$, which is Mr Clifford's theorem.

It has been seen that, A being given, the same line BC is obtained whatever be the position of the point K; and, taking AK for the normal at A, it at once appears geometrically that (as remarked by Mr Clifford) the line BC is the polar of the point Θ of intersection of all the chords which subtend a right angle at A.

{Professor Cayley's lemma may be otherwise proved, as follows:

The trilinear equation of the conic, referred to two tangents (α at A, β at S) and their chord of contact (γ or AS), is $U = \lambda\alpha\beta - \gamma^2 = 0$; and the equation of two straight lines (AK, AD) equally inclined to α, γ is

$$(\alpha - \mu\gamma)(\mu\alpha - \gamma) = 0, \quad \text{or} \quad V = \alpha^2 + \gamma^2 - (\mu + \mu^{-1})\alpha\gamma = 0;$$

also $U + V = 0$ denotes a conic passing through the intersections of U and V; but $U + V$ is resolvable into $\alpha = 0$, or the tangent AT, and

$$\alpha + \lambda\beta - (\mu + \mu^{-1})\gamma = 0,$$

which is, therefore, the equation of the chord KD; whence we see that KD meets AS (or γ) in a point B (given by $\gamma = 0$, $\alpha + \lambda\beta = 0$) whose position is independent of μ, that is, of the equal angles SAD, TAK.}

[Vol. I. pp. 125—127.]

1478. (By J. McDowell, M.A.)—(α) Two sides of a given triangle always pass through two fixed points; prove that the third side always touches a fixed circle.

(β) Two sides of a given triangle touch two fixed circles; prove that the third side also touches a fixed circle.

(γ) Two sides of a given polygon touch fixed circles; prove that all the remaining sides also touch fixed circles.

---

3. *Solution by* PROFESSOR CAYLEY.

Since the theorem (γ) follows at once from (β), and (α) is included in (β), it is only necessary to prove (β). Consider three given circles, and let it be proposed to construct a triangle the sides whereof touch the given circles, and which is similar to a given triangle; the direction of one side may be assumed at pleasure, and then the triangle is determined. Impose now on the triangle the condition that the area is equal to a given quantity; we obtain for the given area an expression involving the angle θ which fixes the direction of one of the sides, and we have thus an equation for the determination of the angle θ. But, for a properly determined relation between the data of the problem, the expression for the area becomes independent of the angle θ, that is, every triangle, the sides whereof touch the three circles, and which is similar to a given triangle, is of the same area, or say, the area of every such triangle is equal to a given quantity Δ; and, this being so, it is clear that, if we construct a triangle similar to a given triangle and of the given area Δ (that is, a triangle equal to a given triangle), in such manner that two of the sides touch two of the given circles, then the envelope of the remaining side will be the remaining given circle; which is in fact the theorem (β).

It only remains therefore to show that the foregoing porismatic case of the problem exists.

For the first circle, let the coordinates of the centre be a, b, and the radius be c; and suppose in like manner that we have a', b', and c' for the second circle, and a'', b'', and c'' for the third circle. Let λ, λ', λ'' be the inclinations to the axis of x of the perpendiculars on the sides which touch these circles respectively; then the equations of the three sides respectively are

$$(x-a)\cos\lambda+(y-b)\sin\lambda-c=0,\ (x-a')\cos\lambda'+(y-b')\sin\lambda'-c'=0,$$
$$(x-a'')\cos\lambda''+(y-b'')\sin\lambda''-c''=0.$$

If the triangle be similar to a given triangle, then the differences of the angles λ, λ', λ'' will be given angles, or, what is the same thing, we may write

$$\lambda=\theta+\xi,\quad \lambda'=\theta'+\xi,\quad \lambda''=\theta''+\xi,$$

where θ, θ', θ'' are given angles, and ξ is a variable angle. Let Δ be the area of the triangle, then (disregarding a merely numerical factor) we have

$$\begin{aligned}\sqrt{\Delta}=&\sin(\lambda'-\lambda'')(a\cos\lambda+b\sin\lambda+c)\\&+\sin(\lambda''-\lambda)(a'\cos\lambda'+b'\sin\lambda'+c')+\sin(\lambda-\lambda')(a''\cos\lambda''+b''\sin\lambda''+c'');\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}\sqrt{\Delta}=&\ \sin(\theta'-\theta'')\{a\cos(\theta+\xi)+b\sin(\theta+\xi)+c\}\\&+\sin(\theta''-\theta)\{a'\cos(\theta'+\xi)+b'\sin(\theta'+\xi)+c'\}\\&+\sin(\theta-\theta')\{a''\cos(\theta''+\xi)+b''\sin(\theta''+\xi)+c''\}.\end{aligned}$$

It is now clear that the right-hand side will be independent of ξ, if only

$$\sin(\theta'-\theta'')(a\cos\theta+b\sin\theta)+\sin(\theta''-\theta)(a'\cos\theta'+b'\sin\theta')$$
$$+\sin(\theta-\theta')(a''\cos\theta''+b''\sin\theta'')=0,$$

$$\sin(\theta'-\theta'')(-a\sin\theta+b\cos\theta)+\sin(\theta''-\theta)(-a'\sin\theta'+b'\cos\theta')$$
$$+\sin(\theta-\theta')(-a''\sin\theta''+b''\cos\theta'')=0;$$

equations which show that, given the form of the triangle and the centres of two of the circles, the centre of the third circle (in the porismatic case) is a determinate unique point: and the theorem is thus proved.

[Vol. I. pp. 137—141.]

1273. (By the EDITOR [W. J. MILLER, B.A.].)—In a given triangle let three triangles be inscribed, by joining the points of contact of the inscribed circle, the points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the corresponding sides of these three triangles pass through the same point; also the triangle formed by the three points of intersection will be a circumscribed co-polar to the original triangle, and the pole will be on the straight line in which the sides of the given triangle meet the bisectors of its exterior angles.

---

1. *Solution by* PROFESSOR CAYLEY.

The theorem is, in fact, included in the following more general

THEOREM. Let the points O, O', O'', ... lie on a conic circumscribed about a triangle ABC; then first the polars of the points O, O', O'', ... in regard to the triangle (see Note at the end of the Solution) pass through a fixed point Ω. And secondly, if by means of the point O, joining it with the vertices A, B, C, and taking the intersections of these lines with the sides BC, CA, AB, respectively, we form a triangle inscribed in the triangle ABC; and the like for the points O', O'', ...; the corresponding sides of the inscribed triangles meet in three points forming a triangle circumscribed about the original triangle ABC, and such that the lines joining the corresponding vertices of the last-mentioned two triangles meet in the point Ω.

But, in order to see that the proposed theorem 1273 is in fact included under the foregoing more general one, it is necessary to state the following

SUBSIDIARY THEOREM. Consider a conic inscribed in the triangle ABC, and passing through the points I, J.

Take O the pole of the line IJ in regard to the conic; O' the point of intersection of the lines joining the vertices of the triangle with the points of contact on the opposite sides respectively; O'' the point of intersection of the lines Al, Bm, Cn, where l is a point on BC such that the lines lA, lBC, lI, lJ form a harmonic pencil, and the like for the points m and n respectively.

Then the points O, O', O'' lie on a conic circumscribed about the triangle ABC.

In fact, if in the subsidiary theorem the inscribed conic be a circle, and the points I, J be the circular points at infinity, the point O will be the centre of the circle, that is, the point of intersection of the interior bisectors of the angles; O' will be the point of intersection of the lines to the points of contact of the inscribed circle; and O'' the point of intersection of the perpendiculars on the sides of the triangle; and, these three points being on a conic circumscribed about the triangle, the general theorem will apply to the three points in question.

I first prove the subsidiary theorem. Taking $x=0$, $y=0$, $z=0$ for the equations of the sides of the triangle and (α, β, γ), $(\alpha', \beta', \gamma')$ for the coordinates of the points I, J respectively; the equation of the inscribed conic is

$$\begin{vmatrix} \sqrt{x}, & \sqrt{y}, & \sqrt{z} \\ \sqrt{\alpha}, & \sqrt{\beta}, & \sqrt{\gamma} \\ \sqrt{\alpha'}, & \sqrt{\beta'}, & \sqrt{\gamma'} \end{vmatrix} = 0,$$

or say

$$a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0,$$

where

$$a=\sqrt{\beta\gamma'}-\sqrt{\beta'\gamma}=p-p_1, \quad b=\sqrt{\gamma\alpha'}-\sqrt{\gamma'\alpha}=q-q_1, \quad c=\sqrt{\alpha\beta'}-\sqrt{\alpha'\beta}=r-r_1, \text{ suppose.}$$

The coordinates of the point of intersection of the lines from the vertices to the points of contact on the opposite sides are

$$x : y : z = \frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2},$$

that is,

$$= \frac{1}{(p-p_1)^2} : \frac{1}{(q-q_1)^2} : \frac{1}{(r-r_1)^2}.$$

The equation of the line IJ is

$$(\beta\gamma'-\beta'\gamma)\,x+(\gamma\alpha'-\gamma'\alpha)\,y+(\alpha\beta'-\alpha'\beta)\,z=0;$$

or, what is the same thing,

$$(p^2-p_1^2)\,x+\ (q^2-q_1^2)\,y+\ (r^2-r_1^2)\,z=0.$$

Representing this for a moment by $\lambda x+\mu y+\nu z=0$, the coordinates of the pole of this line, in regard to the inscribed conic $a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0$, are as

$$c^2\mu+b^2\nu : a^2\nu+c^2\lambda : b^2\lambda+a^2\mu.$$

Now

$$\begin{aligned} c^2\mu+b^2\nu &= (r-r_1)^2(q^2-q_1^2)+(q-q_1)^2(r^2-r_1^2), \\ &= (r-r_1)(q-q_1)[(r-r_1)(q+q_1)+(q-q_1)(r+r_1)], \\ &= 2(r-r_1)(q-q_1)(qr-q_1r_1), \end{aligned}$$

but, observing that $pqr = p_1q_1r_1$, we have

$$qr - q_1r_1 = \left(\frac{p_1}{p} - 1\right) q_1r_1 = -\frac{(p-p_1)\,q_1r_1}{p} = -\frac{(p-p_1)\,p_1q_1r_1}{pp_1};$$

hence

$$c^2\mu + b^2\nu = -\frac{2\,(p-p_1)\,(q-q_1)\,(r-r_1)\,p_1q_1r_1}{pp_1};$$

and we have the like values for $a^2\nu + c^2\lambda$ and $b^2\lambda + a^2\mu$ respectively; hence, omitting the symmetrical factor, we have, for the coordinates of the point in question,

$$x : y : z = \frac{1}{pp_1} : \frac{1}{qq_1} : \frac{1}{rr_1}.$$

Taking the equation of the line Al to be $Qy + Rz = 0$, those of the lines lI, lJ will be

$$x = \lambda\,(Qy + Rz),\ x = \lambda'\,(Qy + Rz),$$

where

$$\lambda = \frac{\alpha}{Q\beta + R\gamma},\quad \lambda' = \frac{\alpha'}{Q\beta' + R\gamma'};$$

and the harmonic condition gives $\lambda + \lambda' = 0$, that is,

$$Q\,(\alpha\beta' + \alpha'\beta) + R\,(\alpha\gamma' + \alpha'\gamma) = 0;$$

the equation of the line Al is thus found to be

$$(\gamma\alpha' + \gamma'\alpha)\,y = (\alpha\beta + \alpha'\beta)\,z;$$

and, since we have the like forms for the equations of the lines Bm and Cn, we have for the coordinates of the point of intersection of these three lines

$$x : y : z = \frac{1}{\beta\gamma' + \beta'\gamma} : \frac{1}{\gamma\alpha' + \gamma'\alpha} : \frac{1}{\alpha\beta' + \alpha'\beta},$$

that is

$$= \frac{1}{p^2 + p_1^2} : \frac{1}{q^2 + q_1^2} : \frac{1}{r^2 + r_1^2}.$$

The equation of a conic circumscribed about the triangle ABC is

$$\frac{\lambda}{x} + \frac{\mu}{y} + \frac{\nu}{z} = 0,$$

where λ, μ, ν are arbitrary coefficients; and the condition for the three points being in the conic is thus found to be

$$\begin{vmatrix} (p-p_1)^2, & (q-q_1)^2, & (r-r_1)^2 \\ pp_1\ , & qq_1\ , & rr_1 \\ p^2+p_1^2, & q^2+q_1^2, & r^2+r_1^2 \end{vmatrix} = 0,$$

but, in virtue of the relations

$$(p-p_1)^2 = -2pp_1 + (p^2+p_1^2), \text{ \&c.},$$

this equation is identically true, and the subsidiary theorem is thus proved.

Passing now to the general theorem, I prove the first part of it as follows:

The equation of a conic circumscribed about the triangle $x=0$, $y=0$, $z=0$ is

$$\frac{A}{x}+\frac{B}{y}+\frac{C}{z}=0;$$

hence, if (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ are the coordinates of any three points on the conic, we have

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0, \quad \frac{A}{\alpha'}+\frac{B}{\beta'}+\frac{C}{\gamma'}=0, \quad \frac{A}{\alpha''}+\frac{B}{\beta''}+\frac{C}{\gamma''}=0,$$

and thence

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix} = 0,$$

which is the condition for the intersection in a point of the three lines

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0,$$

$$\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}=0,$$

$$\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}=0;$$

and the theorem in question is thus proved. I remark, in passing, that the theorem might also be stated as follows:—The locus of a point O, such that its polar in regard to the triangle ABC passes through a fixed point Ω, is a conic circumscribed about the triangle.

To prove the second part of the theorem, take for the coordinates of the points O, O', O'' respectively (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; then

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix} = 0,$$

and if X, Y, Z are the coordinates of the point Ω, then we have

$$\frac{X}{\alpha}+\frac{Y}{\beta}+\frac{Z}{\gamma}=0,$$

$$\frac{X}{\alpha'}+\frac{Y}{\beta'}+\frac{Z}{\gamma'}=0,$$

$$\frac{X}{\alpha''}+\frac{Y}{\beta''}+\frac{Z}{\gamma''}=0.$$

The equations of the sides of the inscribed triangle obtained by means of the point O are

$$-\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0, \qquad \frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}=0, \qquad \frac{x}{\alpha}+\frac{y}{\beta}-\frac{z}{\gamma}=0,$$

and the like for the triangles obtained by means of the points O' and O'' respectively. Hence, for a set of corresponding sides of the three triangles, we have, e.g.,

$$-\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0, \quad -\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}=0, \quad -\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}=0,$$

and it is clear that these equations are simultaneously satisfied by the values

$$x : y : z = -X : Y : Z,$$

and we have the like expressions for the other sets of corresponding sides; that is, we have for the coordinates of the vertices of the resulting triangle

$$(-X : Y : Z), \quad (X : -Y : Z), \quad (X : Y : -Z);$$

and hence also the equations of the sides of the triangle in question are

$$\frac{y}{Y}+\frac{z}{Z}=0, \quad \frac{z}{Z}+\frac{x}{X}=0, \quad \frac{x}{X}+\frac{y}{Y}=0,$$

that is, it is a triangle circumscribed about the triangle ABC. The equations of the lines joining the corresponding vertices of the two triangles are

$$\frac{y}{Y}=\frac{z}{Z}, \quad \frac{z}{Z}=\frac{x}{X}, \quad \frac{x}{X}=\frac{y}{Y},$$

and these lines meet in the point $(X : Y : Z)$, which is the point Ω, the intersection of the polars of O, O', O''; the demonstration of the theorem is thus completed.

[The expression Polar of a point in regard to a triangle denotes a line constructed as follows:—viz., O being the point and ABC the triangle, then, taking on BC a point a, the harmonic in regard to the points B and C of the intersection of BC by AO; and in like manner on CA and AB the points b and c respectively, the three points a, b, c lie on a line which is the polar of the point O. If the equations of the sides are $x=0$, $y=0$, $z=0$, and the coordinates of the point are (α, β, γ), then

the equation of the polar is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$; the equation may also be written $(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)^2\,xyz=0$, and it thus appears that the line just defined as the polar is in fact the second or line polar of the point in regard to the three lines BC, CA, AB considered as forming a cubic curve.]

[Vol. II. July to December 1864, pp. 6—9.]

1505. (Proposed by Professor CAYLEY.)—If P, Q, 1, 2, 3, 4 be points on a conic, then the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ lie on a conic passing through the points P and Q.

Solution by the PROPOSER.

This is an immediate consequence of the theorem of the anharmonic property of the points of a conic. For if $(P1, P2, P3, P4)$ denote the anharmonic ratio of the lines $P1$, $P2$, $P3$, $P4$, and so in other cases; then

$$(P2,\ P1,\ P4,\ P3)=(P1,\ P2,\ P3,\ P4)=(Q1,\ Q2,\ Q3,\ Q4);$$

that is

$$(P2,\ P1,\ P4,\ P3)=(Q1,\ Q2,\ Q3,\ Q4),$$

which proves the theorem.

In particular, if P, Q are the circular points at infinity, then the conic is a circle. Moreover the points $P1$, $Q2$; $P2$, $Q1$ are the antifocal points of 1, 2; viz., calling these $1'$, $2'$, then 12 and $1'2'$ are lines at right angles to each other, having a common centre O, but such that $1'2'=i.12$, ($i=\sqrt{-1}$, as usual); or, what is the same thing, $O1=O2=i.O1'=i.O2'$. And the theorem is as follows: viz., if 1, 2, 3, 4 are points on a circle, and

$1'$, $2'$ are the antifocal points of 1, 2,
$3'$, $4'$ " " " 3, 4,

then $1'$, $2'$, $3'$, $4'$ are points on a circle.

As an *à posteriori* proof, take the centre of the given circle as origin, so that (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4) being the coordinates of 1, 2, 3, 4, and the radius being taken as unity, we have

$$\alpha_1^2+\beta_1^2=\alpha_2^2+\beta_2^2=\alpha_3^2+\beta_3^2=\alpha_4^2+\beta_4^2=1.$$

Suppose for a moment that x, y are the coordinates of the antifocal points of 1, 2; we have

$$x-\alpha_1\pm i\,(y-\beta_1)=0,\quad x-\alpha_2\mp i\,(y-\beta_2)=0,$$

that is

$$x+iy=\alpha_1+i\beta_1,\quad x-iy=\alpha_2-i\beta_2,$$

for the coordinates of the one point; and similarly

$$x - iy = \alpha_1 - i\beta_1, \quad x + iy = \alpha_2 + i\beta_2,$$

for the coordinates of the other point.

Hence, taking the new coordinates

$$X = x + iy, \quad Y = x - iy,$$

and similarly $A_1 = \alpha_1 + i\beta_1$, $B_1 = \alpha_1 - i\beta_1$, &c.; the coordinates of the antifocal points 1′, 2′ are (A_1, B_2) and (A_2, B_1) respectively; but we have $A_1B_1 = \alpha_1^2 + \beta_1^2 = 1$, $A_2B_2 = \alpha_2^2 + \beta_2^2 = 1$; so that $B_1 = \frac{1}{A_1}$, $B_2 = \frac{1}{A_2}$; and the coordinates are $\left(A_1, \frac{1}{A_2}\right)$, $\left(A_2, \frac{1}{A_1}\right)$ respectively. Similarly the coordinates of the antifocal points (3′, 4′) are $\left(A_3, \frac{1}{A_4}\right)$, $\left(A_4, \frac{1}{A_3}\right)$ respectively.

Take as the equation of the circle through the two pairs of antifocal points

$$x^2 + y^2 + 2\lambda x + 2\mu y + \nu = 0,$$

or, what is the same thing,

$$XY + \lambda(X + Y) - i\mu(X - Y) + \nu = 0,$$

that is

$$XY + LY + MX + N = 0,$$

if

$$L = \lambda + i\mu, \; M = \lambda - i\mu, \; N = \nu.$$

We ought then to have

$$\frac{A_1}{A_2} + L\frac{1}{A_2} + MA_1 + N = 0,$$

$$\frac{A_2}{A_1} + L\frac{1}{A_1} + MA_2 + N = 0,$$

$$\frac{A_3}{A_4} + L\frac{1}{A_4} + MA_3 + N = 0,$$

$$\frac{A_4}{A_3} + L\frac{1}{A_3} + MA_4 + N = 0;$$

and these will exist simultaneously, if

$$\begin{vmatrix} \frac{A_1}{A_2}, & \frac{1}{A_2}, & A_1, & 1 \\ \frac{A_2}{A_1}, & \frac{1}{A_1}, & A_2, & 1 \\ \frac{A_3}{A_4}, & \frac{1}{A_4}, & A_3, & 1 \\ \frac{A_4}{A_3}, & \frac{1}{A_3}, & A_4, & 1 \end{vmatrix} = 0,$$

an identical equation which is easily verified. It, in fact, gives

$$\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)-\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)+\left(\frac{A_1}{A_2}-\frac{A_2}{A_1}\right)\left(1-1\right)+$$

$$\left(1-1\right)\left(\frac{A_3}{A_4}-\frac{A_4}{A_3}\right)-\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)+\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)=0,$$

which is obviously true. The equation may also be written

$$\begin{vmatrix} 1, & A_1, & A_2, & A_1A_2 \\ 1, & A_2, & A_1, & A_1A_2 \\ 1, & A_3, & A_4, & A_3A_4 \\ 1, & A_4, & A_3, & A_3A_4 \end{vmatrix}=0,$$

and in this form it expresses the known theorem of the equality of the anharmonic ratios of (A_1, A_2, A_3, A_4) and (A_2, A_1, A_4, A_3).

But, in order to actually find the circle, we may write

$$\begin{aligned} XY+LY+MX \quad &+N \quad =0, \\ A_1 \quad +L \quad &+MA_1A_2+NA_2=0, \\ A_2 \quad +L \quad &+MA_1A_2+NA_1=0, \\ A_3 \quad +L \quad &+MA_3A_4+NA_4=0, \end{aligned}$$

and eliminating L, M, N, the equation of the circle is

$$\begin{vmatrix} XY, & Y, & X \quad , & 1 \\ A_1 \ , & 1 \ , & A_1A_2, & A_2 \\ A_2 \ , & 1 \ , & A_1A_2, & A_1 \\ A_3 \ , & 1 \ , & A_3A_4, & A_4 \end{vmatrix}=0,$$

or, reducing, this is

$$\begin{aligned} (A_2-A_1)\,[XY(A_3A_4-A_1A_2)+Y\{A_1A_2(A_3+A_4)-A_3A_4(A_1+A_2)\} \\ +X(A_1+A_2-A_3-A_4)+(A_3A_4-A_1A_2)]=0, \end{aligned}$$

or say

$$\begin{aligned} XY(A_1A_2-A_3A_4)+Y\{A_3A_4(A_1+A_2)-A_1A_2(A_3+A_4)\} \\ +X\{A_3+A_4-A_1-A_2\}+(A_1A_2-A_3A_4)=0\,; \end{aligned}$$

that is

$$\begin{vmatrix} XY+1 \ , & X \quad , & Y \\ A_1 \ +A_2, & A_1A_2, & 1 \\ A_3 \ +A_4, & A_3A_4, & 1 \end{vmatrix}=0,$$

which is the required equation; or, transforming to the original axes, we have $x+iy=X$, $x-iy=Y$, &c., and therefore $XY=x^2+y^2$; and the equation becomes

$$\begin{vmatrix} x^2+y^2+1 & , \; x+iy & , \; x-iy \\ \alpha_1+\alpha_2+i(\beta_1+\beta_2), & (\alpha_1+i\beta_1)(\alpha_2+i\beta_2), & 1 \\ \alpha_3+\alpha_4+i(\beta_3+\beta_4), & (\alpha_3+i\beta_3)(\alpha_4+i\beta_4), & 1 \end{vmatrix} = 0,$$

which is the equation of the circle through the two pairs of antifocal points.

{NOTE. The *second* form of the equation of the circle may be otherwise deduced from the *first*, without expanding the determinants, by the following method:

$$\begin{vmatrix} XY, & Y, & X, & 1 \\ A_1, & 1, & A_1A_2, & A_2 \\ A_2, & 1, & A_1A_2, & A_1 \\ A_3, & 1, & A_3A_4, & A_4 \end{vmatrix} = \begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ A_1+A_2, & 1, & A_1A_2, & A_1 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} =$$

$$\begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ 0, & 0, & 0, & A_1-A_2 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} = (A_1-A_2) \begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix};$$

therefore

$$\begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix} = 0.$$

ED. [W. J. M.]}

[Vol. II. pp. 22—24.]

1513. (Proposed by the Rev. J. BLISSARD, B.A.)—Prove the following formulæ:

$$(1) \qquad \frac{(x-1)(x-2)\,..\,(x-n)}{x(x+1)\,..\,(x+n-1)} =$$

$$1+(-)^n\left\{n\cdot\frac{1}{x}-\frac{n(n^2-1^2)}{1^2}\cdot\frac{1}{x+1}+\frac{n(n^2-1^2)(n^2-2^2)}{1^2.\,2^2}\cdot\frac{1}{x+2}-\&c.\right\}$$

(2) The above formula expressed as

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\Gamma(x+n)}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1\,.\,2\,.\,3}\cdot\frac{1}{x(x+1)(x+2)}+\&c.$$

and show that this equation is subject to the sole restriction that when n is not integral x must not be negative.

Solution by PROFESSOR CAYLEY; *and* X. U. J.

Let n be a positive integer, and suppose that $[x]^n$ denotes as usual the factorial $x(x-1)\ldots.(x-n+1)$; then we have

$$[x+k]^n=(1+\Delta)^k[x]^n=\left(1+k\Delta+\frac{k(k-1)}{1\,.\,2}\Delta^2+\&c.\right)[x]^n$$

$$=[x]^n+\frac{kn}{1}[x]^{n-1}+\frac{k(k-1)n(n-1)}{1\,.\,2}[x]^{n-2}+\&c.;$$

or putting $k=-n$ we have

$$[x-n]^n=[x]^n-\frac{n^2}{1}[x]^{n-1}+\frac{n^2(n^2-1^2)}{1\,.\,2}[x]^{n-2}-\&c.$$

Writing herein $(x+n-1)$ for x, and dividing by $[x+n-1]^n$, we have

$$\frac{[x-1]^n}{[x+n-1]^n}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\&c.;$$

or, what is the same thing,

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\Gamma(x+n)}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\&c.,$$

which is the formula (2). The foregoing demonstration applies to the case of n a positive integer; but as the two sides are respectively unaltered when n is changed into $-n$, it is clear that the formula holds good also for n a negative integer. The right hand side is the hypergeometric series $F(n, -n, x, 1)$ and the formula therefore is

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\Gamma(x+n)}=F(n, -n, x, 1),$$

a particular case of the known formula

$$\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}=F(\alpha, \beta, \gamma, 1),$$

which when α or β is a positive integer is a mere identity, true therefore for all values of γ; but if neither α nor β is a positive integer, then the right hand side is an infinite series which is only convergent for $\gamma>\alpha+\beta$. In the particular case we

have $\alpha = n$, $\beta = -n$, $\gamma = x$; hence if n be a positive or negative integer, the formula is an identity, but if n be fractional, the condition of convergency is $x > 0$, that is, x must be positive.

To prove the formula (1) it is only necessary to remark, that (n being a positive integer) the quantity $\frac{[x-1]^n}{[x+n-1]^n}$ is a rational fraction, the numerator and denominator whereof are of the same degree n, and which becomes $=1$ for $x=\infty$. Hence, decomposing it into simple fractions, we may write

$$\frac{[x-1]^n}{[x+n-1]^n} = 1 + S_r \cdot \frac{A_r}{x+r},$$

where the summation extends from $r=0$ to $r=n-1$ both inclusive. And we have

$$A_r = \left\{\frac{(x+r)[x-1]^n}{[x+n-1]^n}\right\}_{x=-r},$$

or, observing that $[x+n-1]^n = [x+n-1]^{n-r-1}(x+r)[x+r-1]^r$, we have

$$A_r = \left\{\frac{[x-1]^n}{[x+n-1]^{n-r-1}[x+r-1]^r}\right\}_{x=-r} = \frac{[-r-1]^n}{[n-r-1]^{n-r-1}[-1]^r}$$

$$= \frac{(-)^n[n+r]^n}{[n-r-1]^{n-r-1}(-)^r[r]^r} = (-)^{n+r}\frac{[n+r]^{n+r}}{[n-r-1]^{n-r-1}[r]^r[r]^r} = (-)^{n+r}\frac{[n+r]^{2r+1}}{[r]^r \cdot [r]^r}.$$

Hence the formula is

$$\frac{[x-1]^n}{[x+n-1]^n} = 1 + (-)^n \cdot S_r (-)^r \cdot \frac{[n+r]^{2r+1}}{[r]^r[r]^r}\,\frac{1}{x+r},$$

or, as this may also be written,

$$\frac{(x-1)(x-2)\,..\,(x-n)}{x(x+1)\,..\,(x+n-1)} = 1 + (-)^n\left\{n \cdot \frac{1}{x} - \frac{n(n^2-1^2)}{1^2}\cdot\frac{1}{x+1} + \frac{n(n^2-1^2)(n^2-2^2)}{1^2 \cdot 2^2}\cdot\frac{1}{x+2} - \&c.\right\}$$

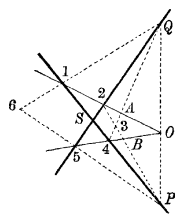
which is the formula in question.

[Vol. II. pp. 51, 52.]

1512. (Proposed by Professor CAYLEY.)—It is possible to construct a hexagon 123456, inscribed in a conic, and such that the diagonals 14, 25, 36 pass respectively through the Pascalian points (intersections of opposite sides) 23, 56; 34, 61; 45, 12. Given the points 1, 2; 4, 5; to construct the hexagon.

Solution by the PROPOSER.

Let 12, 45, meet in O, and through O draw at pleasure a line meeting 14 in P, and 25 in Q; let $P2$, $Q4$ meet in 3, and $P5$, $Q1$ in 6; then the line 36 will pass through O, and this being so, the hexagon 123456 satisfies the required conditions.



We have to show that 36 passes through O. Let $Q4$ meet $O12$ in A, and $P2$ meet $O45$ in B; then the points 6, 3, O, are the intersections of corresponding sides of the triangles $A1Q$, $B5P$; and in order that these points may lie in a line, the lines joining the corresponding vertices must meet in a point, that is, we have to show that the lines 15, AB, PQ meet in a point. The property is in fact as follows; viz., given the points 2, 4; and also the points Q, O, P lying in a line; then constructing the points 1, 5, A, B, which are the respective intersections of $P4$, $O2$; $Q2$, $O4$; $Q4$, $O2$; $P2$, $O4$; the lines 15, AB, PQ will meet in a point. Take $x=0$, $y=0$, $z=0$ for the respective equations of $P2$, $Q4$, PQ; then O is an arbitrary point in the line PQ, say that for the point O we have $z=0$, $ax+by=0$; also $O2$, $O4$ are arbitrary lines through O: say that their equations are $ax+by+\lambda z=0$; $ax+by+\mu z=0$; then we have for the points A and B, respectively, $ax+by+\mu z=0$, $y=0$; $ax+by+\mu z=0$, $x=0$; hence the equation of AB is $\mu ax+\lambda by+\lambda\mu z=0$. The equation of $P4$ is $ax+\mu z=0$, and that of $Q2$ is $by+\lambda z=0$; the point 1 is therefore given by $ax+\mu z=0$, $ax+by+\lambda z=0$; and 5 by $by+\lambda z=0$, $ax+by+\mu z=0$; hence the equation of 15 is $\mu ax+\lambda by+(\mu^2-\mu\lambda+\lambda^2)z=0$; and the equation of PQ being $z=0$, it is clear that the three lines AB, 15, PQ intersect in the point given by the equations $\mu ax+\lambda by=0$, $z=0$.

OBS. 1. By inspection of the figure we see that $3PQ$ is a triangle whereof the sides $3P$, $3Q$, PQ pass respectively through the fixed points 2, 4, O; while the vertices P and Q lie in the fixed lines 14, 25; the locus of the vertex 3 is consequently a conic; and the like as regards the triangle $6PQ$.

OBS. 2. The regular hexagon projects into a hexagon inscribed in a conic and circumscribed about another conic having double contact therewith; in the hexagon so obtained (as appears at once by the consideration of the regular hexagon) the

above-mentioned property holds; but the in-and-circumscribed hexagon has the additional property that the three diagonals meet in a point, and it is therefore a less general figure than the hexagon of the foregoing theorem. It would, I think, be worth while to study further the hexagon of the theorem.

{NOTE. In the solution of Question 1548 it is shown that if two pairs of opposite sides of *any hexagon* intersect each on a diagonal produced, so likewise will the third pair.

A slight variation of Professor Cayley's proof may be obtained by finding the equations of $P5$, $Q1$, and thence of 36, which are respectively

$$ax-(\lambda-\mu)z=0,\ bx+(\lambda-\mu)z=0,\ ax+by=0,$$

showing that 36 passes through O. ED. [W. J. M].}.

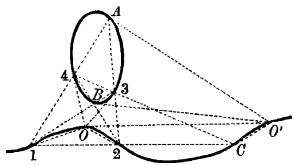
[Vol. II. pp. 70—72.]

1562. (Proposed by F. D. THOMSON, M.A.)—Find the locus of the points of contact of tangents drawn from a given point to a conic circumscribing a given quadrangle. The quadrangle being supposed convex, trace the changes of form of the locus for different positions of the given point.

Solution by PROFESSOR CAYLEY; *and the* PROPOSER.

Let O be the given point; 1, 2, 3, 4 the vertices of the given quadrangle; A, B, C the centres of the quadrangle, viz., A the intersection of the lines 14, 23; B of 24, 31; C of 34, 12. The polars of O in regard to the several circumscribed conics intersect in a point O'. This being so, the locus is a cubic passing through the nine points 1, 2, 3, 4, A, B, C, O, O', and which is moreover such that the tangents at the four points 1, 2, 3, 4 meet the cubic in the point O, and the tangents at the four points A, B, C, O meet the cubic in the point O'. It is to be remarked that the nine points are so related to each other that a cubic through any eight of these points passes through the remaining ninth point; say a cubic through 1, 2, 3, 4, A, B, C, O passes through O'; the nine points consequently do not determine the cubic; but the cubic will be determined, e.g., by the conditions that it passes through 1, 2, 3, 4, A, B, C, O, and has $O1$ for the tangent at 1. The series of cubics corresponding to different positions of the point O is identical with the series of cubics passing through the seven points 1, 2, 3, 4, A, B, C

Conversely any given cubic curve may be taken to be a cubic of the series; and the points 1, 2, 3, 4 will then be determined as follows, viz., 1, 2, 3, 4 are the points of contact of the tangents to the cubic from an arbitrary point O on the cubic; and



then taking as before A, B, C for the intersections of 14, 23, of 24, 31 and of 34, 12, respectively, the points A, B, C will lie on the cubic, and the tangents at A, B, C, O will meet the cubic in a point O'. I call to mind that a cubic curve without singularities is either *complex* or simplex; in the simplex kind there can be drawn from any point of the curve two, and only two, real tangents to the curve; in the complex kind, there can be drawn four real tangents or else no real tangent, viz. from any point on a certain branch of the curve there can be drawn four real tangents, from a point on the remaining portion of the curve no real tangent. Hence, in the foregoing construction, in order that the points 1, 2, 3, 4 may be real, the given cubic must be of the complex kind, and the point O must be taken on the branch which has through each of its points four real tangents.

The foregoing results may be established *geometrically* or *analytically;* but for brevity I merely indicate the analytical demonstration. Suppose first, that the points 1, 2, 3, 4 are given as the intersections of the conics $U=0$, $V=0$; let α, β, γ be the coordinates of the point O, and write $D=\alpha\delta_x+\beta\delta_y+\gamma\delta_z$, so that $DU=0$ and $DV=0$ are the equations of the polars of O in regard to the conics $U=0$, $V=0$ respectively. The equation of any conic through the four points is $U+kV=0$; and the equation of the polar of O in regard thereto is $DU+kDV=0$; eliminating k from these equations, we have $UDV-VDU=0$, which is the equation of the given locus. We see at once that it is a cubic curve passing through the points ($U=0$, $V=0$), that is, the points 1, 2, 3, 4; and through the point $DU=0$, $DV=0$, that is, the point O'; it also follows without difficulty that the curve passes through the point O. But for the remaining results it is better to particularize the conics $U=0$, $V=0$. Let the equations of 12, 23, 34, 41 be $x=0$, $y=0$, $z=0$, $w=0$ respectively, (where $x+y+z+w=0$); and in the same system, let α, β, γ, δ be the coordinates of O ($\alpha+\beta+\gamma+\delta=0$), then $xz=0$, $yw=0$ are each of them a conic (pair of lines) passing through the four points; and we may therefore write $U=yw$, $V=xz$; the equation $UDV-VDU=0$ thus becomes $yw(\alpha z+\gamma x)-xz(\beta w+\delta y)=0$, or, as this equation may also be written,

$$\frac{\alpha}{x}-\frac{\beta}{y}+\frac{\gamma}{z}-\frac{\delta}{w}=0,$$

which is the equation of the cubic curve; and from this form the several above-mentioned results may be obtained without difficulty.

To give an idea of the form of the curve corresponding to a given convex quadrangle 1234, and given position of the point O, I suppose that O is situate *within* the quadrangle, for instance in the triangle $B12$. The mere inspection of the figure, and consideration of the conditions which are to be satisfied by the cubic curve, is enough to show that this is of the form described by Newton as *anguinea cum ovali*, viz., the oval passes through the points 3, 4, A, B, and the serpentine branch through the points 1, 2, C, O, O'. But the complete discussion of the different cases would be somewhat laborious.

{A *geometrical* investigation of the locus is given on p. 124 of Cremona's *Teoria Geometrica delle Curve Piane.* ED. [W. J. M.].}

[Vol. II. pp. 89, 90.]

1533. (Proposed by Professor CAYLEY.)—If on the sides of a triangle there are taken three points, one on each side; and if through the three points and the three vertices of the triangle there are drawn a cubic curve and a quartic curve, intersecting in six other points; then there exists a quintic curve passing through each of the three points, and having each of the six points for a double point.

Solution by the PROPOSER.

Let $P=0$ be the equation of the quartic curve, $Q=0$ the equation of the cubic curve, $M=0$ the equation of the three sides of the triangle; then if we can find A, B, C functions of the orders 0, 1, 2 respectively, and U a function of the fifth order, such that we have identically $MU = AP^2 + BPQ + CQ^2$; we have $MU=0$, a curve of the eighth order, having a double point at each of the points $(P=0,\ Q=0)$, which points are the three vertices of the triangle, the three points, and the six points; but the curve $MU=0$ is made up of the curve $M=0$ (the three sides of the triangle, being a cubic curve having each of the vertices for a double point, and passing through each of the three points) and of a certain quintic curve $U=0$; hence the quintic curve must pass through each of the three points, and have a double point at each of the six points; or there exists a quintic curve satisfying the conditions of the theorem.

I take $x=0$, $y=0$, $z=0$ for the equations of the three sides of the triangle, and then (the constants being all of them arbitrary) writing for shortness

$$\begin{aligned}
\xi &= \ .\quad by + cz, & X &= \ .\quad \beta y + \gamma z, & \Theta &= \lambda x + \mu y + \nu z,\\
\eta &= a'x \quad . + c'z, & Y &= \alpha' x \quad . + \gamma' z, & &\\
\zeta &= a''x + b''y \quad . , & Z &= \alpha'' x + \beta'' y \quad . , & &
\end{aligned}$$

I assume that the three points are given by the equations $(x=0,\ \xi=0)$, $(y=0,\ \eta=0)$, $(z=0,\ \zeta=0)$, respectively. This being so, we may write

$$Q = yz\xi\delta + zx\eta\delta' + xy\zeta\delta'' + xyz\epsilon = 0, \qquad -P = yz\xi X + zx\eta Y + xy\zeta Z + xyz\Theta = 0,$$

for the equations of the cubic curve and the quartic curve respectively. We have of course $M = xyz = 0$ for the equation of the three sides of the triangle, and the identity to be satisfied is $xyzU = AP^2 + BPQ + CQ^2$.

I was led to the values of A, B, C by considerations founded on the theory of curves in space. We have

$$A = \delta\delta'\delta'', \qquad B = (\delta'\alpha'' + \delta''\alpha)\,\delta x + (\delta''\beta + \delta\beta'')\,\delta' y + (\delta\gamma' + \delta'\gamma)\,\delta'' z,$$

$$C = \alpha'\alpha''\delta x^2 + \beta''\beta\delta' y^2 + \gamma\gamma'\delta'' z^2 + (\gamma\beta''\delta' + \gamma'\beta\delta'')\,yz + (\alpha'\gamma\delta'' + \alpha''\gamma'\delta)\,zx + (\beta''\alpha'\delta + \beta\alpha''\delta')\,xy;$$

and with these values it is easy to show that the function $AP^2 + BPQ + CQ^2$ contains the factor xyz; for substituting the values of P, Q, all the terms of $AP^2 + BPQ + CQ^2$ contain explicitly the factor xyz, except the terms

$$A\,(y^2z^2\xi^2X^2 + z^2x^2\eta^2Y^2 + x^2y^2\zeta^2Z^2) - B\,(y^2z^2\xi^2X\delta + z^2x^2\eta^2Y\delta' + x^2y^2\zeta^2Z\delta')$$
$$+ C\,(y^2z^2\xi^2\delta^2 + z^2x^2\eta^2\delta'^2 + x^2y^2\zeta^2\delta''^2);$$

and these terms will contain the factor xyz, if only the expressions $AX^2 - BX\delta + C\delta^2$, $AY^2 - BY\delta' + C\delta'^2$, $AZ^2 - BZ\delta'' + C\delta''^2$ contain respectively the factors x, y, z. But $AX^2 - BX\delta + C\delta^2$ will contain the factor x, if only the expression vanishes for $x=0$; and for $x=0$ we have

$$AX^2 - BX\delta + C\delta^2 = 0 =$$
$$\delta\delta'\delta''\,(\beta y + \gamma z)^2 - [\delta'\delta''\,(\beta y + \gamma z) + \delta\,(\beta''\delta' y + \gamma'\delta'' z)]\,\delta\,(\beta y + \gamma z) + (\beta y + \gamma z)\,(\beta''\delta' y + \gamma'\delta'' z)\,\delta^2;$$

that is, $AX^2 - BX\delta + C\delta^2$ contains the factor x; and by symmetry the other two expressions contain the factors y and z respectively. The excepted terms contain therefore the factor xyz; and there exists therefore a quintic function $U = (AP^2 + BPQ + CQ^2) \div xyz$; which proves the theorem.

The values of A, B, C were obtained by considering the surface $w = \dfrac{P}{Q}$, which, as is at once seen, contains upon itself the three lines

$$\left(x=0,\ w=-\frac{X}{\delta}\right),\ \left(y=0,\ w=-\frac{Y}{\delta'}\right),\ \left(z=0,\ w=-\frac{Z}{\delta''}\right)$$

or as these equations may be written

$$\begin{aligned}
&(x=0, \quad . \quad \beta y + \gamma z + \delta w = 0),\\
&(y=0, \quad \alpha' x \quad . + \gamma' z + \delta' w = 0),\\
&(z=0, \quad \alpha'' x + \beta'' y \quad . + \delta'' w = 0);
\end{aligned}$$

and then seeking for the equation of the hyperboloid which passes through the three lines, this is found to be $Av^2+Bw+C=0$, where A, B C have the before-mentioned values.

If in the foregoing theorem the cubic is considered as a given cubic curve, and the three points as three arbitrary points on the cubic, the question then arises to find the triangle; or we have the problem proposed as Question 1607.

[Vol. II. p. 91.]

1542. (Proposed by Professor CAYLEY.)—If a given line meet two given conics in the points (A, B) and (A', B') respectively; and if (A'', B'') be the sibi-conjugate points (or foci) of the pairs (A, A') and (B, B'), or of the pairs (A, B') and (A', B), then (A'', B'') lie on a conic passing through the four points of intersection of the two given conics.

[Vol. II. pp. 97—100.]

1606. (Proposed by the EDITOR, [W. J. M.]).—Solve the following problems:

(α) Through three given points to draw a conic whose foci shall lie in two given lines.

(β) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

(γ) Through two given points to draw a circle such that its chords of intersection with a given circle shall pass through a given point.

Solution by PROFESSOR CAYLEY.

(α) Through three given points to draw a conic whose foci shall lie in two given lines.

The focus of a conic is a point such that the lines joining it with the two circular points at infinity (say the points I, J) are tangents to the conic. Hence the question is, in a given line to find a point A, and in another given line to find a point B, such that there exists a conic touching the four lines AI, AJ, BI, BJ (where I, J are any given points) and besides passing through three given points. More generally, instead of the lines from A, B to the given points I, J, we may consider the tangents from A, B, to a given conic Θ; the question then is, in a given line to find a point A, and in another given line to find a point B, such that

there exists a conic touching the tangents from A, B to a given conic Θ, and besides passing through three given points. It is rather more convenient to consider the reciprocal question—though it is to be borne in mind that for any two reciprocal questions a solution of the one question by means of coordinates (x, y, z) regarded as point-coordinates is in fact a solution of the other question by means of the same coordinates (x, y, z) regarded as line-coordinates. The reciprocal question is: through a given point to draw a line A, and through another given point to draw a line B, such that there exists a conic passing through the intersections of these lines with a given conic Θ, and besides touching three given lines. The given points may be taken to be $(x=0,\ z=0)$, $(y=0,\ z=0)$; this determines the line $z=0$, but not the lines $x=0$, $y=0$, so that the point $(x=0,\ y=0)$ may without loss of generality be supposed to lie on the conic Θ; the equation of this conic will therefore be

$$(a,\ b,\ 0,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0.$$

I take $\alpha z + \gamma x = 0$ for the equation of the line A, $\mu y + \nu z = 0$ for the equation of the line B (so that the quantities to be determined are the ratios $\alpha : \gamma$ and $\mu : \nu$); this being so, the required conic passes through the intersections of these lines with the conic Θ; its equation will therefore be

$$(a,\ b,\ 0,\ f,\ g,\ h \between x,\ y,\ z)^2 + 2(\alpha x + \gamma z)(\mu y + \nu z) = 0\,;$$

or what is the same thing

$$(a,\ b,\ 2\nu\gamma,\ f+\mu\gamma,\ g+\nu\alpha,\ h+\mu\alpha \between x,\ y,\ z)^2 = 0\,;$$

where α, γ, μ, ν have to be determined in such manner that this conic may touch three given lines. It is to be observed that α, γ, μ, ν, enter into the equation through the combinations $\alpha\mu$, $\alpha : \gamma$, and $\mu : \nu$, so that there are really only three disposable quantities.

The condition in order that the conic may touch a line $\xi x + \eta y + \zeta z = 0$ is

$$\left\{\begin{array}{l} 2b\nu\gamma - (f+\mu\gamma)^2,\ 2a\nu\gamma - (g+\nu\alpha)^2,\ ab - (h+\mu\alpha)^2, \\ (g+\nu\alpha)(h+\mu\alpha) - a(f+\mu\gamma), \\ (h-\mu\alpha)(f+\mu\gamma) - b(g+\nu\alpha), \\ (f+\mu\gamma)(g+\nu\alpha) - 2\nu\gamma(h+\mu\alpha) \end{array}\right\} \between \xi,\ \eta,\ \zeta)^2 = 0,$$

that is, putting for shortness $C = ab - h^2$, $F = gh - af$, $G = hf - bg$, and reversing the sign of the whole expression,

$$\begin{array}{l} \{\ f^2\xi^2 + \quad g^2\eta^2 - C\zeta^2 - \quad 2F\eta\zeta - \quad 2G\zeta\xi - \quad 2fg\xi\eta\} \\ + 2\mu\{\ f\gamma\xi^2 \quad + h\alpha\zeta^2 + (a\gamma - g\alpha)\eta\zeta - (h\gamma + f\alpha)\zeta\xi - \quad g\gamma\xi\eta\} \\ + 2\nu\{-b\gamma\xi^2 - (a\gamma - g\alpha)\eta^2 \quad - \quad h\alpha\eta\zeta + \quad b\alpha\zeta\xi + (h\gamma - \alpha f)\xi\eta\} \\ + \ \mu^2\{(\gamma\xi - \alpha\zeta)^2\} + 2\mu\nu\{\alpha\eta(\gamma\xi - \alpha\zeta)\} + \nu^2\{\alpha^2\eta^2\} \qquad = 0\,; \end{array}$$

or what is the same thing

$$\{\nu\alpha\eta + \mu(\gamma\xi - \alpha\zeta)\}^2 + 2\nu(p\alpha + q\gamma) + 2\mu(r\alpha + s\gamma) + t = 0;$$

where p, q, r, s, t are given functions of (ξ, η, ζ).

I write for greater convenience

$$\nu = \frac{1}{X}, \quad \mu = \frac{1}{Y}, \quad \alpha = W, \ \gamma = Z,$$

(so that the quantities to be determined will be the ratios $X : Y : Z : W$); the foregoing equation then becomes

$$\left\{\eta\frac{W}{X} + \frac{1}{Y}(\xi Z - \zeta W)\right\}^2 + \frac{2}{X}(pW + qZ) + \frac{2}{Y}(rW + sZ) + t = 0,$$

or what is the same thing

$$\{\eta YW + X(\xi Z - \zeta W)\}^2 + 2XY^2(pW + qZ) + 2X^2Y(rW + sZ) + tX^2Y^2 = 0.$$

Hence, considering in place of the line $\xi x + \eta y + \zeta z = 0$, the three given lines $\xi_1 x + \eta_1 y + \zeta_1 z = 0$, $\xi_2 x + \eta_2 y + \zeta_2 z = 0$, $\xi_3 x + \eta_3 y + \zeta_3 z = 0$ successively, we have the three equations

$$\begin{aligned}
&\{\eta_1 YW + X(\xi_1 Z - \zeta_1 W)\}^2 + 2XY^2(p_1 W + q_1 Z) + 2X^2Y(r_1 W + s_1 Z) + t_1 X^2Y^2 = 0,\\
&\{\eta_2 YW + \&c. \qquad\qquad \}^2 + \&c. \qquad\qquad\qquad\qquad\qquad\qquad = 0,\\
&\{\eta_3 YW + \&c. \qquad\qquad \}^2 + \&c. \qquad\qquad\qquad\qquad\qquad\qquad = 0;
\end{aligned}$$

which, treating X, Y, Z, W as the coordinates of a point in space, are each of them the equation of a quartic surface having the line $(X = 0, Y = 0)$ for a cuspidal line. The required values of X, Y, Z, W are the coordinates of a point of intersection of the three surfaces, and these being found the equation of the conic satisfying the conditions of the question is

$$(a, b, 0, f, g, h \between x, y, z)^2 + 2(Wx + Zz)\left(\frac{y}{Y} + \frac{z}{Z}\right) = 0.$$

As to the intersection of surfaces having a common line, see Salmon's *Solid Geometry*, p. 257; but the case of a cuspidal line not having been hitherto discussed, I am not able to say now how many points of intersection there are of the three surfaces, nor consequently what is the number of the solutions of the question in hand. It of course appears that 64 is a superior limit.

(β) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

Let the four points be given as the intersections of the conics $U = 0$, $V = 0$, and let $W = 0$ be the equation of the given conic, (α, β, γ) the coordinates of the given point.

The equation of the required conic may be taken to be $\Theta = \lambda U + \mu V = 0$, and this being so, the equation of *any* conic passing through the points of intersection of the conic $\Theta = 0$ and the given conic $W = 0$, will be $\lambda U + \mu V + \nu W = 0$; and if ν be properly determined, viz. by the equation

$$\text{Disct. } (\lambda U + \mu V + \nu W) = 0,$$

which it will be observed is a cubic equation in (λ, μ, ν), then $\lambda U + \mu V + \nu W = 0$ will be the equation of a pair of the chords of intersection of the conics $\Theta = 0$, $W = 0$. The chord which passes through the given point (α, β, γ) may be taken to be one of the pair of chords; the pair of chords, regarded as a conic, then passes through the given point (α, β, γ); or if U_0, V_0, W_0 are what U, V, W become on substituting therein the values (α, β, γ) for the coordinates, we have

$$\lambda U_0 + \mu V_0 + \nu W_0 = 0,$$

which is a linear equation in (λ, μ, ν); and combining it with the before-mentioned cubic equation,

$$\text{Disct. } (\lambda U + \mu V + \nu W) = 0,$$

the two equations give the ratios $(\lambda : \mu : \nu)$, and the equation of the required conic is $\lambda U + \mu V = 0$. There are three systems of ratios $\lambda : \mu : \nu$, and consequently three conics satisfying the conditions of the Question.

Suppose that the conics $U = 0$, $V = 0$, $W = 0$, have a common chord, then the conics $\Theta = \lambda U + \mu V = 0$, $W = 0$, have this common chord, say the fixed chord; and they have besides another chord of intersection, say the proper chord, which is the line joining the two points of intersection not on the fixed chord. It follows that, in the equation $\lambda U + \mu V + \nu W = 0$, ν may be so determined that this equation shall represent the fixed and proper chords; the required value of ν is one of those given by the before-mentioned cubic equation, which will then have a single rational factor of the form $a\lambda + b\mu + c\nu$, and the value of ν is that obtained by means of this factor, that is, by the equation $a\lambda + b\mu + c\nu = 0$.

{The value in question may, however, be found independently of the cubic equation; thus, if the three conics have the common chord $P = 0$, then their equations may be taken to be $U = 0$, $U + PQ = 0$, $U + PR = 0$; we have then $\Theta = \lambda U + \mu(U + PQ)$, and the value of ν is at once seen to be $\nu = -(\lambda + \mu)$, for then

$$\lambda U + \mu V + \nu W = \lambda U + \mu(U + PQ) - (\lambda + \mu)(U + PR) = 0,$$

that is, $P[\mu Q - (\lambda + \mu) R] = 0$, which is the conic made up of the fixed chord $P = 0$ and the proper chord $\mu Q - (\lambda + \mu) R = 0$.}

But returning to the equations $U = 0$, $V = 0$, $W = 0$, the value of ν is given by the equation $a\lambda + b\mu + c\nu = 0$, obtained by equating to zero the rational factor of the cubic equation. Suppose now that the *proper chord* passes through the given point (α, β, γ), then, as before, the equation $\lambda U + \mu V + \nu W = 0$ is satisfied by these values

of the coordinates, or we have $\lambda U_0 + \mu V_0 + \nu W_0 = 0$; which, with the before-mentioned linear equation $a\lambda + b\mu + c\nu = 0$, determines the ratios $\lambda : \mu : \nu$; and the required conic is $\lambda U + \mu V = 0$; there is, then, in the present case only one conic satisfying the conditions of the Question.

(γ) Through two given points to draw a circle such that its chord of intersection with a given circle shall pass through a given point.

The foregoing discussion of the case of three conics having a common chord is of course directly applicable to the present question, the common chord being the line infinity; it is therefore sufficient to give the final analytical result; viz., if the given points are $y=0$, $x=\pm 1$, and if the given circle is $x^2+y^2+c+2fy+2gx=0$, and the point through which passes the chord is $x=\alpha$, $y=\beta$, then the equation of the required circle is

$$x^2+y^2-1+\frac{1}{\beta}(2g\alpha+2f\beta+1+c)y=0.$$

The equation of the chord of intersection is, in fact,

$$1+c-\frac{1}{\beta}(2g\alpha+2f\beta+1+c)y+2gx+2fy=0,$$

which is satisfied, as it should be, by $x=\alpha$, $y=\beta$.

But the geometrical solution is even more simple. Let A, B, be the given points, C the point through which passes the chord of intersection; then, joining C, A, and taking on this line a point H such that $CA \,.\, CH$ is equal to the square on the tangential distance of C from the given circle, it is at once seen that *any* circle through A, H is such that its chord of intersection with the given circle passes through C; hence the required circle is that drawn through the three points A, H, B.

[Vol. III. January to July, 1865, p. 29.]

1607. (Proposed by Professor CAYLEY.)—In a given cubic curve to inscribe a triangle such that the three sides shall pass respectively through three given points on the curve.

[Vol. III. pp. 60—63.]

1647. (Proposed by Professor CAYLEY.)—Find the locus of the foci of an ellipse of given major axis, passing through three given points.

[In connexion with the problem the Proposer remarks as follows:

Let A, B, C be the given points; take P an arbitrary point (not in general in the plane of the three given points), then we may find a point Q (not in general

in the plane of the three given points) such that $QA + AP = QB + BP = QC + CP =$ given major axis. And this being so, if the locus of P be a given surface, then we shall have a certain surface, the locus of Q; and so if the locus of P be a given curve in space, then we shall have a given curve in space, the locus of Q. In particular, if the locus of P be the plane of the three given points, then the locus of Q will be a certain surface, cutting the plane in a curve which is the locus in the foregoing problem; and when Q is situate on this curve, then also P will be situate on the same curve. Or if the locus of P be the curve in question, then the locus of Q will be the same curve. Say, in general, that the loci of P and Q are reciprocal loci, then the curve in the problem *is its own reciprocal.* And we may propose the following question:

Find the curve or surface, the locus of P, which is its own reciprocal.

We have also analogous to the original problem the following question in Solid Geometry:

Given the four points A, B, C, D in space, to find the locus of the points P, Q such that

$$PA + AQ = PB + BQ = PC + CQ = PD + DQ = \text{ a given line.}\}$$

Solution by the PROPOSER.

In general if a, b, c be the sides of a triangle, and f, g, h the distances of any point from the angles of the triangle (or, what is the same thing, if (a, b, c, f, g, h) be the distances of any four points in a plane from each other), then we have a certain relation

$$\phi(a, b, c, f, g, h) = 0.$$

Hence if r, s, t be the distances of the one focus from the angles of the triangle, and the major axis is $= 2\lambda$, then the distances for the other focus are $2\lambda - r$, $2\lambda - s$, $2\lambda - t$; and considering the three angles and the other focus as a system of four points, we have

$$\phi(a, b, c, 2\lambda - r, 2\lambda - s, 2\lambda - t) = 0,$$

which is a relation between the distances r, s, t of the first focus from the angles of the triangle, and which, treating these distances as coordinates (of course in a generalised sense of the term "Coordinate"), may be regarded as the equation of the required locus. It is to be observed, that we have identically

$$\phi(a, b, c, r, s, t) = 0,$$

and the equation may be expressed in the simplified form

$$\phi(a, b, c, 2\lambda - r, 2\lambda - s, 2\lambda - t) - \phi(a, b, c, r, s, t) = 0.$$

To develope the solution, I notice that the expression for the equation $\phi(a, b, c, f, g, h)=0$ is

$$\begin{aligned}
&b^2c^2(g^2+h^2)+c^2a^2(h^2+f^2)+a^2b^2(f^2+g^2)\\
&+g^2h^2(b^2+c^2)+h^2f^2(c^2+a^2)+f^2g^2(a^2+b^2)\\
&-a^2f^2(a^2+f^2)-b^2g^2(b^2+g^2)-c^2h^2(c^2+h^2)\\
&-a^2g^2h^2-b^2h^2f^2-c^2f^2g^2-a^2b^2c^2=0\,;
\end{aligned}$$

see my paper, "Note on the value of certain determinants &c.," *Quart. Math. Journ.* vol. III. (1860), pp. 275—277, [286]. Or, as this may also be written

$$\Sigma\{(b^2+c^2-a^2)(g^2h^2+a^2f^2)-a^2f^4\}-a^2b^2c^2=0,$$

where Σ refers to the simultaneous cyclical permutation of (a, b, c) and of (f, g, h). Hence we have only in this equation to write $2\lambda-r$, $2\lambda-s$, $2\lambda-t$ in place of (f, g, h), and to omit the terms independent of λ, being in fact those which are equal to $\phi(a, b, c, r, s, t)$. Observing that we have

$$\begin{aligned}
g^2h^2+a^2f^2&=\{4\lambda^2-2\lambda(s+t)+st\}^2+a^2(2\lambda-r)^2\\
&=16\lambda^4-16\lambda^3(s+t)+4\lambda^2(s^2+t^2+4st+a^2)-4\lambda[st(s+t)+a^2r]+s^2t^2+a^2r^2\,;\\
f^4&=(2\lambda-r)^4=16\lambda^4-32\lambda^3r+24\lambda^2r^2-8\lambda r^3+r^4,
\end{aligned}$$

the equation becomes

$$\begin{aligned}
&16\lambda^4\{\Sigma(b^2+c^2-a^2)-\Sigma a^2\}\\
-\;&16\lambda^3\{\Sigma(b^2+c^2-a^2)(s+t)-2\Sigma a^2r\}\\
+\;&4\lambda^2\{\Sigma(b^2+c^2-a^2)(s^2+t^2+4st+a^2)-6\Sigma a^2r^2\}\\
-\;&4\lambda\{\Sigma(b^2+c^2-a^2)[st(s+t)+a^2r]-2\Sigma a^2r^3\}=0,
\end{aligned}$$

where the Σ's refer to the simultaneous cyclical permutation of the (a, b, c) and the (r, s, t). The coefficients of λ^4 and λ^3 are, it is easy to see, each $=0$; and in the coefficient of λ^2 the terms $\Sigma(b^2+c^2-a^2)(s^2+t^2)-6\Sigma a^2r^2$ are $=-4\Sigma a^2r^2$; hence dividing the whole equation by 4λ, we find

$$\lambda\{\Sigma(b^2+c^2-a^2)(4st+a^2)-4\Sigma a^2r^2\}-\{\Sigma(b^2+c^2-a^2)[st(s+t)+a^2r]-2\Sigma a^2r^3\}=0,$$

which is the required relation between (r, s, t).

It may be noticed that, expressing the distances r, s, t in terms of Cartesian or trilinear coordinates (x, y) or (x, y, z), then r^2, s^2, t^2 are rational and integral functions of the coordinates, and the form of the equation therefore is

$$A_2+B_2r+C_2s+D_2t+E_0st+F_0tr+G_0rs=0,$$

where the subscript numbers denote the degrees in regard to the coordinates. Multiplying this equation successively by 1, r, s, t, st, tr, rs, rst, we have eight equations linear in the last-mentioned eight quantities, the coefficients being of known degrees respectively;

and eliminating the eight quantities, we have the rationalised equation expressed in the form, determinant (of order 8) = 0; viz. this is

$$\begin{vmatrix} A_2, & B_2, & C_2, & D_2, & E_0, & F_0, & G_0, & 0 \\ B_2r^2, & A_2, & G_0r^2, & F_0r^2, & 0, & D_2, & C_2, & E_0 \\ C_2s^2, & G_0s^2, & A_2, & E_0s^2, & D_2, & 0, & B_2, & F_0 \\ D_2t^2, & F_0t^2, & E_0t^2, & A_2, & C_2, & B_2, & 0, & G_0 \\ E_0s^2t^2, & 0, & D_2t^2, & C_2s^2, & A_2, & G_0s^2, & F_0t^2, & B_2 \\ F_0t^2r^2, & D_2t^2, & 0, & B_2r^2, & G_0r^2, & A_2, & E_0t^2, & C_2 \\ G_0r^2s^2, & C_2s^2, & B_2r^2, & 0, & F_0r^2, & E_0s^2, & A_2, & D_2 \\ 0, & E_0s^2t^2, & F_0t^2r^2, & G_0r^2s^2, & B_2r^2, & C_2s^2, & D_2t^2, & A_2 \end{vmatrix} = 0.$$

This seems to be of the degree 18 in the coordinates, but it is probable that the real degree is lower.

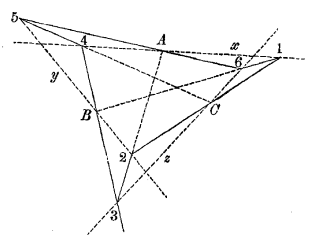
[Vol. III. pp. 63—65.]

1652. (Proposed by W. K. CLIFFORD.)—Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R; RB meets Aa in P; PC meets Bb in Q; QA meets Cc in r; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Show that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane.

Solution by PROFESSOR CAYLEY.

1. The theorem may be thus stated: Given three lines x, y, z, and in these lines respectively the points A, B, C; then there exist an infinity of hexagons, such that



the pairs of opposite angles lie in the lines x, y, z, respectively, and that the pairs of opposite sides pass through the points A, B, C, respectively.

2. The demonstration is as follows: We have to show that, starting from an arbitrary point 1 in the line x, and constructing in the prescribed manner (as shown successively in the figure) the points 2, 3, 4, 5, 6, the last side 61 of the hexagon 123456 will pass through B. By the construction, we have A, 2, 3 in a line, and likewise C, 4, 5; hence, by Pascal's theorem, applied to the six points in a pair of lines, the points of intersection of the lines (25, 34), ($3C$, $A5$), ($A4$, $C2$), that is, the points B, 6, 1, lie in a line; which is the required theorem.

3. More generally suppose that the points A, B, C are not on the lines x, y, z, respectively. I remark that it is not in general possible to describe a hexagon such that the opposite angles lie in the lines x, y, z, respectively, and the opposite sides pass through the points A, B, C, respectively; but if there exists one hexagon (viz., a proper hexagon, not a triangle twice repeated), then there exists an infinity of such hexagons.

4. In fact, if it be required to find a polygon, the angles whereof lie in given lines respectively, and the sides whereof pass through given points respectively; the problem is either indeterminate or admits of only *two* solutions. If therefore in any particular case there are three or more solutions, the problem is indeterminate, and has an infinity of solutions. Now, in the above-mentioned case of the three lines and the three points, there exist *two* triangles, the angles whereof lie in the given lines, and the sides pass through the given points; and each triangle, taking the angles twice over in the same order 123123, is a hexagon satisfying the conditions of the problem; hence, if we have besides a proper hexagon satisfying the conditions of the problem, there are really *three* solutions, and the problem is therefore indeterminate.

5. Suppose that the three lines x, y, z, and also two of the three points, say the points A and B, are given; we may construct geometrically a locus, such that, taking for C any point of this locus, the problem shall be indeterminate: in fact, starting with the point 4, and constructing successively the points 3, 2; taking an arbitrary direction for the line 21, and constructing successively the points 1, 6, 5; then the intersection of the lines 21 and 54 is a position of the point C: and by taking any number of directions for the line 21, we obtain for each of them a different position of the point C; and so construct the locus.

6. The locus in question is, as will be shown, a line; and if the point A is on the line x, and the point B on the line y, then the locus of C will be the line z; that is, C being any point of the line z, the problem is indeterminate; which is Mr Clifford's theorem.

7. To prove this, consider the lines x, y, z, and also the points A, B, C, as given; the point 1 is an arbitrary point on the line x, linearly determined by means of a parameter u; and for every position of the point 1 we have a corresponding position of the point 4; let u' be the corresponding parameter for the point 4; the series of points 1 is homographic with the series of points 4; that is, the parameters u, u' are connected by an equation of the form $auu' + bu + cu + d = 0$, (where of course a, b, c, d are functions of the parameters which determine the given lines x, y, z and

points A, B, C). But if the problem be indeterminate, then starting from the point 1 and constructing the point 4, and again starting from the point 4 and making the very same construction, we arrive at the original point 1, that is, u must be the same function of u' that u' is of u; and this will be the case if $b=c$; hence $b=c$ is the condition in order that the problem may be indeterminate.

8. To effect the calculation, take $x=0$, $y=0$, $z=0$ for the equations of the lines x, y, z respectively; and let (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be the coordinates of the points A, B, C respectively. Let 1 and 4 be given as the intersections of the line $x=0$ with the lines $y-uz=0$, $y-u'z=0$, respectively; and assume that for the point 2 we have $y=0$, $z-vx=0$, and for the point 3, $z=0$, $x-wy=0$. Then 1, C, 2 are in a line; as are also 2, A, 3; 3, B, 4; hence we obtain

$$v=\frac{\gamma''u-\beta''}{\alpha''u}, \quad w=\frac{\alpha v-\gamma}{\beta v}, \quad u'=\frac{\beta' w-\alpha}{\gamma' w};$$

therefore, eliminating v and w, we have

$$(\alpha\gamma''-\alpha''\gamma)\gamma' uu'-\alpha\beta''\gamma' u'-(\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma)u-\beta''(\alpha'\beta-\alpha\beta')=0.$$

The required condition, therefore, is

$$\alpha\beta''\gamma'=\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma, \quad \text{or} \quad \alpha\beta'\gamma''-\alpha\beta''\gamma'-\alpha'\beta\gamma''-\alpha''\beta'\gamma=0;$$

which is linear in regard to each of the three sets (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, separately; that is, two of the points A, B, C being given, the locus of the remaining point is a line. In particular, if $\alpha=0$, $\beta'=0$; then the equation becomes $\alpha'\beta\gamma''=0$, and assuming that neither $\alpha'=0$, or $\beta=0$, then the equation becomes $\gamma''=0$, that is, A, B being arbitrary points on the lines $x=0$, $y=0$ respectively, the locus of C is the line $z=0$.

9. Mr Clifford's theorem is clearly its own reciprocal. I do not know the *precise* analogues of his special form of the theorem; but the analogue of the more general theorem stated in (6) is as follows: viz., we may have in the plane n lines $x, y, z, \ldots$ and n points $A, B, C, \ldots$, such that there exist an infinity of $2n$-gons whereof the pairs of opposite angles lie in the given lines respectively; and the pairs of opposite sides pass through the given points respectively; and if the n lines and $n-1$ of the n points be assumed at pleasure, then the locus of the remaining point is a line. It is moreover clear by the principle of reciprocity, that if the n points and $n-1$ of the n lines be assumed at pleasure, then the envelope of the remaining line is a point.

There exists also an analogue in space; viz. we may have n lines $x, y, z, \ldots$ and n lines $A, B, C, \ldots$ such that there exist an infinity of (skew) $2n$-gons whereof the pairs of opposite angles lie in the given lines $x, y, z, \ldots$ respectively; and the pairs of opposite sides meet in the given lines $A, B, C, \ldots$ respectively. It may be added, that if all but one of the $2n$ lines $x, y, z, \ldots A, B, C \ldots$ are given, then the 'six coordinates' of the remaining line satisfy a certain linear equation, but I do not stop to explain the geometrical interpretation of this theorem.

10. Referring to the foregoing figure, if instead of the point 1 we take on the line x, a point $1'$, and construct therewith the hexagon $1'2'3'4'5'6'$; then if α, α' be the (foci or) sibi-conjugate points of the range 1, 4, $1'$, $4'$ on the line x; β, β' the sibi-conjugate points of the range 2, 5, $2'$, $5'$ on the line y; and γ, γ' the sibi-conjugate points of the range 3, 6, $3'$, $6'$ on the line z; the points in question form two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, such that for each triangle the angles lie in the given lines and the sides pass through the given points. This is an elegant geometrical construction for the problem of the in-and-circumscribed triangle, in the particular case where the given points A, B, C lie in the given lines x, y, z, respectively.

11. The points 1, 2, 3, 4, 5, 6, A, B, C constitute a system of 9 points which lie in 9 lines of 3 each. The points α, β, γ, α', β', γ', A, B, C constitute a radically distinct system of 9 points lying in 9 lines of 3 each; viz., in the former system there are 3 sets of 3 lines which contain all the 9 points; in the latter system there is only the set of lines $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$ which contains all the nine points. The last-mentioned system may be constructed as follows: The points β, β' and γ, γ' are arbitrary: A is the intersection of the lines $\beta\gamma$ and $\beta'\gamma'$; and then joining A with the point of intersection of the lines $\beta\gamma'$ and $\beta'\gamma$ we have α an arbitrary point on the joining line; the lines $\alpha\gamma$ and $\beta\beta'$ meet in the point B, the lines $\alpha\beta$ and $\gamma\gamma'$ in the point C; the lines $C\beta'$ and $B\gamma'$ will then meet in a point α' on the line $A\alpha$; and we have thus the figure of the nine points α, β, γ, α', β', γ', A, B, C.

[Vol. III. pp. 78, 79.]

1667. (Proposed by Professor SYLVESTER.)

Show that the discriminant of the form

$$ax^5 + b\lambda x^4 y + c\lambda^2 x^3 y^2 + c\mu^2 x^2 y^3 + b\mu x y^4 + a y^5$$

will be a rational integral function of the quantities a, b, c, $\lambda\mu$, $\lambda^5 + \mu^5$, and of the second degree only in respect to the last of them.

Solution by PROFESSOR CAYLEY.

In general

$$\text{Disct.}\,(a, b, c, d, e, f \between \lambda x + \mu y, \lambda' x + \mu' y)^5 = (\lambda\mu' - \lambda'\mu)^{20}\ \text{Disct.}\,(a, b, c, d, e, f \between x, y)^5.$$

Hence first, if $(\lambda, \mu, \lambda', \mu') = (0, 1, 1, 0)$, then

$$\text{Disct.}\,(a, b, c, d, e, f \between y, x)^5 = \text{Disct.}\,(a, b, c, d, e, f \between x, y)^5;$$

and secondly, if ω be an imaginary fifth root of unity and $(\lambda, \mu, \lambda', \mu') = (\omega, 0, 0, 1)$, then

$$\text{Disct.}\,(a, b, c, d, e, f \between \omega x, y)^5 = \text{Disct.}\,(a, b, c, d, e, f \between x, y)^5.$$

These two results may also be written,

$$\text{Disct.}\,(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5 = \text{Disct.}\,(f,\ e,\ d,\ c,\ b,\ a\between x,\ y)^5,$$

$$\text{Disct.}\,(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5 = \text{Disct.}\,(a,\ b\omega^4,\ c\omega^3,\ d\omega^2,\ e\omega,\ f\between x,\ y)^5;$$

that is, the discriminant of $(a, b, c, d, e, f\between x, y)^5$ is not altered by taking the coefficients in a reverse order, or by multiplying the several coefficients by the powers $\omega^5, \omega^4, \omega^3, \omega^2, \omega$, of an imaginary fifth root of unity. Applying these theorems to the form $(a, b\lambda, c\lambda^2, c\mu^2, b\mu, a\between x, y)^5$, the discriminant is not altered by changing the coefficients into $(a, b\mu, c\mu^2, c\lambda^2, b\lambda, a)$; that is, by the interchange of λ and μ; nor by changing the coefficients into

$$(a,\ b\omega^4\lambda,\ c\omega^3\lambda^2,\ c\omega^2\mu^2,\ b\omega\mu,\ a), \quad \text{or} \quad \{a,\ b\,(\lambda\omega^4),\ c\,(\lambda\omega^4)^2,\ c\,(\mu\omega)^2,\ b\,(\mu\omega),\ a\};$$

that is, the discriminant is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. The discriminant is therefore a rational and integral function, symmetrical in regard to λ, μ, and which is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. In virtue of the second property the discriminant is a rational integral function of $(\lambda\mu,\ \lambda^5,\ \mu^5)$, and then in virtue of the first property it is a rational integral function of $(\lambda\mu,\ \lambda^5\mu^5,\ \lambda^5+\mu^5)$, that is, of $\lambda\mu,\ \lambda^5+\mu^5$. For the general form $(a, b, c, d, e, f\between x, y)^5$, if a term of the discriminant be $a^\alpha b^\beta c^\gamma d^\delta e^\epsilon f^\phi$, then we have $\alpha+\beta+\gamma+\delta+\epsilon+\phi=8$, $5\alpha+4\beta+3\gamma+2\delta+\epsilon=20$; hence attending only to the indices α, β, γ we have $5\alpha+4\beta+3\gamma>20$, and therefore *à fortiori* $3\beta+3\gamma>20$, so that $\beta+\gamma$ is $=6$ at most. It follows that for the form $(a,\ b\lambda,\ c\lambda^2,\ c\mu^2,\ b\mu,\ a\between x,\ y)^5$, the sum of the indices of $b\lambda$, $c\lambda^2$ is $=6$ at most, and therefore, even if the index of $c\lambda^2$ is $=6$, the index of λ will be only $=12$, that is, the discriminant contains no power of λ higher than λ^{12}: hence considered as a function of $\lambda\mu$, $\lambda^5+\mu^5$, the highest power of $\lambda^5+\mu^5$ is $(\lambda^5+\mu^5)^2$; which completes the theorem.

[Vol. III. p. 90.]

1687. (Proposed by Professor CAYLEY.)—To describe a spherical triangle such that the angles thereof and of the polar triangle lie on a spherical conic.

On the sphere, the locus of a point such that the perpendiculars from it upon the sides of a given spherical triangle have their feet on a line (great circle), is in general a spherical cubic; if however the triangle be such as is mentioned in the above Problem, then the locus breaks up into a line (great circle) and into the conic through the angles of the given and polar triangles.

[Vol. III. pp. 92—96.]

1690. (Proposed by W. A. WHITWORTH, M.A.)—If ABC be the triangle formed by the three diagonals aa', bb', cc' of a complete quadrilateral $aa'bb'cc'$, then a conic can be found having double contact in the chord aa' with the critical conic of the quadrilateral $bb'cc'$, double contact in the chord bb' with the critical conic of the quadrilateral $cc'aa'$, and double contact in the chord cc' with the critical conic of the quadrilateral $aa'bb'$.

The same conic will also intersect in the chord $a'b'c'$, the three conics which pass through the intersection of Aa, Bb, Cc and touch any two sides of the triangle abc at the extremities of the third side.

It will intersect in the chord $a'bc$ the three conics which pass through the intersection of Aa, Bb', Cc' and touch any two sides of the triangle $ab'c'$ at the extremities of the third side.

It will intersect in the chord $ab'c$ the three conics which pass through the intersection of Aa', Bb, Cc' and touch any two sides of the triangle $a'bc'$ at the extremities of the third side.

It will intersect in the chord abc' the three conics which pass through the intersection of Aa', Bb', Cc and touch any two sides of the triangle $a'b'c$ at the extremities of the third side.

DEF. *The critical conic of any quadrilateral is a circumscribed conic such that the tangent at any angular point forms a harmonic pencil with the sides and diagonal meeting at that point.*

It is obvious that if the quadrilateral be projected into a square, the critical conic will become the circumscribed circle.

---

3. *Solution by* PROFESSOR CAYLEY.

1. The equations of the sides of the quadrilateral may be taken to be respectively $x=0$, $y=0$, $z=0$, $w=0$, where the implicit constants are so determined that we have *identically*

$$x+y+z+w=0;$$

this being so, the equations of the three diagonals are respectively

$$\begin{array}{llll} x+y=0, & \text{or} \quad z+w=0, & \text{or} \quad x+y-z-w=0 & \text{(three equivalent forms)} \\ x+z=0, & \text{or} \quad y+w=0, & \text{or} \quad x-y+z-w=0 & (\quad \text{,,} \quad \text{,,} \quad \text{,,} \quad) \\ x+w=0, & \text{or} \quad y+z=0, & \text{or} \quad x-y-z+w=0 & (\quad \text{,,} \quad \text{,,} \quad \text{,,} \quad) \end{array}$$

and the equations of the critical conics are respectively

$$xy+zw=0, \quad xz+yw=0, \quad xw+yz=0.$$

Hence we see that the equation of the required conic is

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0.$$

In fact this equation may be written

$$\Delta = (x+y-z-w)^2-4(xy+zw)=0,$$
$$\Delta = (x-y+z-w)^2-4(xz+yw)=0,$$
$$\Delta = (x-y-z+w)^2-4(xw+yz)=0,$$

equations which put in evidence the double contact with the three critical conics respectively. We have also, identically,

$$\Delta = (x+y+z+w)(x+y-3z-w) - 2w(x+y-z-w) + 4(z^2-xy),$$

and the equation $\Delta = 0$ may therefore be written

$$\Delta = -2w(x+y-z-w) + 4(z^2-xy) = 0,$$

a form which shows that the conic $z^2-xy=0$ meets the line $w=0$ in the same two points in which it is met by the conic $\Delta=0$. And it hence appears by symmetry that the conics

$\Delta=0,\ x^2-yz=0,\ y^2-zx=0,\ z^2-xy=0$ meet the line $w=0$ in the same two points,
$\Delta=0,\ w^2-yz=0,\ y^2-zw=0,\ z^2-wy=0$ meet the line $x=0$ in the same two points,
$\Delta=0,\ w^2-xz=0,\ x^2-zw=0,\ z^2-wx=0$ meet the line $y=0$ in the same two points,
$\Delta=0,\ w^2-xy=0,\ x^2-yw=0,\ y^2-wx=0$ meet the line $z=0$ in the same two points,

which are the relations constituting the latter part of the proposed theorem.

2. The analogous theorems in space may be briefly referred to. Taking $x=0$, $y=0$, $z=0$, $w=0$ as the equations of the faces of a tetrahedron $ABCD$, then the implicit constants may be so determined that the coordinates of a given arbitrary point O shall be (1, 1, 1, 1). We may by lines drawn from the vertices of the tetrahedron project the point O on the faces, so as to obtain a point in each of the four faces; and then in each face, by lines drawn from the vertices of the face, project the point in that face upon the edges of the face; the two points thus obtained on each edge of the tetrahedron are (it is easy to see) one and the same point; that is, we have on each edge of the tetrahedron a point; and there exists a quadric surface

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0$$

touching the edges of the tetrahedron in these six points respectively.

The surface in question has plane contact with

the hyperboloid $xy+zw=0$ along the intersection with $x+y-z-w=0$,
" " $xz+yw=0$ " " " $x-y+z-w=0$,
" " $xw+yz=0$ " " " $x-y-z+w=0$,

and moreover the surfaces

$\Delta=0,\ x^2-yz=0,\ y^2-zx=0,\ z^2-xy=0$ meet the line $w=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ w^2-yz=0,\ y^2-zw=0,\ z^2-wy=0$ meet the line $x=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ w^2-xz=0,\ x^2-zw=0,\ z^2-wx=0$ meet the line $y=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ w^2-xy=0,\ x^2-yw=0,\ y^2-wx=0$ meet the line $z=0,\ x+y+z+w=0$ in the same two points.

With respect to the construction of the four planes,

$$x+y-z-w=0,\quad x-y+z-w=0,\quad x-y-z+w=0,\quad x+y+z+w=0,$$

it is to be observed that if through any edge of the tetrahedron, for instance the edge $x=0$, $y=0$, we draw the plane $x-y=0$ through the point O, then the harmonic of this in regard to the planes $x=0$, $y=0$ is the plane $x+y=0$; we have thus six planes, one through each edge of the tetrahedron, viz., these are $y+z=0$, $z+x=0$, $x+y=0$, $x+w=0$, $y+w=0$, $z+w=0$; the six planes being the faces of a hexahedron, which is such that the vertices of the tetrahedron are four of the eight vertices of the hexahedron. The pairs of opposite faces of the hexahedron meet in three lines lying in the plane $x+y+z+w=0$, and consequently forming a triangle such that through each side of the triangle there pass two opposite faces of the hexahedron; the planes $x+y-z-w=0$, $x-y+z-w=0$, $x-y-z+w=0$ are the harmonics of the plane $x+y+z+w=0$ in respect of the pairs of opposite faces of the hexahedron; viz., the plane $x+y-z-w=0$ is the harmonic of the plane $x+y+z+w=0$ in respect to the planes $x+y=0$, $z+w=0$; and the like for the other two planes $x-y+z-w=0$ and $x-y-z+w=0$ respectively.

[Vol. IV. July to December, 1865, pp. 17, 18.]

1710. (Proposed by Professor CAYLEY.)—Trace the curve $y^4-2y^2zx-z^4=0$, where the coordinates are such that $x+y+z=0$ is the line *infinity*.

Solution by the PROPOSER.

We have $x=\dfrac{y^4-z^4}{2y^2z}$; or writing $y=\theta z$, then $x=\dfrac{\theta^4-1}{2\theta^2}z$, that is

$$x:y:z=\theta^4-1:2\theta^3:2\theta^2.$$

Hence, we see that y, z are indefinitely small in comparison of x,

$$\text{if } \theta=\infty, \text{ and then } x:y:z=\theta^4:2\theta^3:2\theta^2, \text{ that is } y^2=2zx;$$

$$\text{or, if } \theta=0, \text{ and then } x:y:z=-1:2\theta^3:2\theta^2, \text{ that is } z^3=-2y^2x;$$

so that in the neighbourhood of the point $(y=0,\ z=0)$ there are two branches coinciding with the parabola $y^2=2zx$ and with the semicubical parabola $z^3=-2y^2xz$, respectively.

To find the points at infinity we have $x+y+z=0$, that is $\theta^4+2\theta^3+2\theta^2-1=(\theta+1)(\theta^3+\theta^2+\theta-1)=0$; and observing that the equation $\theta^3+\theta^2+\theta-1=0$ has one real root, say $\theta=k$, if k be the real root of the equation $k^3+k^2+k-1=0$ ($k=\cdot505$ nearly),—there are two real points at infinity, viz., corresponding to $\theta=-1$, we have the point $(0,\ -1,\ 1)$, and corresponding to $\theta=k$ the point $(-1-k,\ k,\ 1)$.

The equation of the tangent at a point $(\alpha,\ \beta,\ \gamma)$ is

$$x(-\beta^2\gamma)+y(2\beta^3-2\alpha\beta\gamma)+z(-\alpha\beta^2-2\gamma^3)=0,$$

and hence writing $(\alpha, \beta, \gamma) = (0, -1, 1)$ we have the asymptote $x + 2y + 2z = 0$: to find where this meets the curve, we have $\theta^4 + 4\theta^3 + 4\theta^2 - 1 = 0$, that is $(\theta + 1)^2(\theta^2 + 2\theta - 1) = 0$, or at the points of intersection $\theta^2 + 2\theta - 1 = 0$, that is $\theta = -1 \pm \sqrt{2}$, or there are two real points of intersection.

Again writing $(\alpha, \beta, \gamma) = (-1-k, k, 1)$ we find the asymptote $k^2x - 2y + (k+1)z = 0$: to find where this meets the curve, we have $k^2(\theta^4 - 1) - 4k\theta^3 + (2k+2)\theta^2 = 0$, that is $k^2\theta^4 - 4k\theta^3 + (2k+2)\theta^2 - k^2 = (\theta - k)^2\{k^2\theta^2 - 2(k^2 + k + 1)\theta - 1\} = 0$; or for the points of intersection $k^2\theta^2 - 2(k^2 + k + 1)\theta - 1 = 0$, an equation in θ with two real roots, hence the points of intersection are real.

It is now easy to lay down the curve; viz., if, to fix the ideas, the fundamental triangle is taken to be equilateral, and the coordinates x, y, z are considered to be positive for points *within* the triangle, then the curve is as shown in the annexed figure 1.

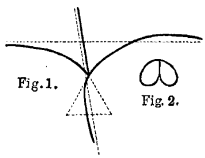


Fig. 1. Fig. 2.

It may be remarked that the curve is met by every real line in two real points at least, and consequently that it is not the projection of any finite curve whatever. By a modification of the constants of the equation, we might obtain curves which are finite, such as the curve in figure 2; or curves with two or four infinite branches, which are the projections of such a finite curve.

[Vol. IV. pp. 32—37.]

1744. (Proposed by W. S. BURNSIDE, B.A.)—It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3 + y_1^3 + z_1^3}{x_1 y_1 z_1} = \frac{x^3 + y^3 + z^3}{xyz}.$$

Solution by PROFESSOR CAYLEY.

The Solution is in fact given in my "Memoir on Curves of the Third Order," *Philosophical Transactions*, vol. CXLVII. (1857), pp. 415—446, [146].

Write $\dfrac{x^3 + y^3 + z^3}{xyz} = -6l$; then, taking (X, Y, Z) as current coordinates, (x, y, z) are, it is clear, the coordinates of a point on the cubic curve $X^3 + Y^3 + Z^3 + 6lXYZ = 0$;

and if (x_1, y_1, z_1) are the coordinates of any other point on the same cubic curve, then we shall have

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1}=-6l=\frac{x^3+y^3+z^3}{xyz},$$

so that (x_1, y_1, z_1) will satisfy the condition in question. Hence, if from a given point (x, y, z) on the cubic curve we obtain by any geometrical construction another point on the curve, the coordinates of this new point will be functions (and, if the construction is such as to lead to a single point only, they will be *rational* functions) of (x, y, z), satisfying the condition in question.

For instance, if the point (x, y, z) be joined with any point (α, β, γ) on the curve, the joining line will again meet the curve in a single point, which may be taken to be the point (x_1, y_1, z_1). This assumes that we know on the cubic curve a point (α, β, γ); but such a point at once presents itself, viz., we may write $(\alpha, \beta, \gamma)=(1, -1, 0)$; which gives only the self-evident solution $(x_1, y_1, z_1)=(y, x, z)$. The point $(1, -1, 0)$ is clearly one of the nine points of inflexion of the cubic curve, and by using these in any manner whatever, viz., joining the point (x, y, z) with any point of inflexion, and then the new point with any other point of inflexion, and so on indefinitely, we obtain in connexion with the given point (x, y, z) seventeen other points on the curve, in all a system of eighteen points: these are

$$\begin{array}{lll|lll}
(x, y, z), & (x, \omega y, \omega^2 z), & (x, \omega^2 y, \omega z) & (x, z, y), & (x, \omega z, \omega^2 y), & (x, \omega^2 z, \omega y) \\
(y, z, x), & (\omega y, \omega^2 z, x), & (\omega^2 y, \omega z, x) & (z, y, x), & (\omega z, \omega^2 y, x), & (\omega^2 z, \omega y, x) \\
(z, x, y), & (\omega^2 z, x, \omega y), & (\omega z, x, \omega^2 y) & (y, x, z), & (\omega^2 y, x, \omega z), & (\omega y, x, \omega^2 z)
\end{array}$$

possessing remarkable geometrical properties; and of course each of the seventeen new points furnishes a (self-evident) solution of the given identity.

But we may take $(\alpha, \beta, \gamma)=(x, y, z)$; the point (x_1, y_1, z_1) is here the point of intersection of the cubic by the tangent at the point (x, y, z); or say it is the "tangential" of the point (x, y, z). The values thus obtained for (x_1, y_1, z_1) are

$$(x_1, y_1, z_1)=\{x(y^3-z^3),\quad y(z^3-x^3),\quad z(x^3-y^3)\},$$

which (excluding the above-mentioned self-evident solutions) is in fact the most simple solution of the proposed identity. In order to verify that the last-mentioned values of (x_1, y_1, z_1) are in fact the coordinates of the tangential of (x, y, z), I observe that this will be the case if only we have

$$(x^2+2lyz)x_1+(y^2+2lzx)y_1+(z^2+2lxy)z_1=0,\quad x_1^3+y_1^3+z_1^3+6lx_1y_1z_1=0,$$

the first of which is obviously satisfied by the values in question; and for the verification of the second equation,

$$\begin{aligned}
x_1^3+y_1^3+z_1^3 &= x^3(y^3-z^3)^3+y^3(z^3-x^3)^3+z^3(x^3-y^3)^3,\\
&= -x^9(y^3-z^3)-y^9(z^3-x^3)-z^9(x^3-y^3),\\
&= (x^3+y^3+z^3)(y^3-z^3)(z^3-x^3)(x^3-y^3),\\
x_1y_1z_1 &= xyz(y^3-z^3)(z^3-x^3)(x^3-y^3),
\end{aligned}$$

therefore

$$x_1^3+y_1^3+z_1^3+6lx_1y_1z_1=(x^3+y^3+z^3+6lxyz)(y^3-z^3)(z^3-x^3)(x^3-y^3), \quad =0$$

if $x^3+y^3+z^3+6lxyz=0$; the same equations verify at once the identity

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1}=\frac{x^3+y^3+z^3}{xyz}.$$

Another solution is as follows: viz., if we take the third intersection with the cubic of the line joining the points (y, x, z) and $\{x(y^3-z^3), y(z^3-x^3), z(x^3-y^3)\}$, the coordinates of the line in question are

$$\begin{aligned} x_1 : y_1 : z_1 = \; & x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3 \\ : \; & x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3 \\ : \; & xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3). \end{aligned}$$

According to a very beautiful theorem of Professor Sylvester's in relation to the theory of cubic curves, the coordinates of a point which depends linearly on a given point of the curve are necessarily rational and integral functions *of a square degree* of the coordinates (x, y, z) of the given point; and moreover that (considering as *one* solution those which can be derived from each other by a mere permutation of the coordinates, or change of x into ωx, &c.), there is only one solution of a given square degree m^2; the solutions of the degrees 4 and 9 are given above. The tangential of the tangential, or second tangential of the point (x, y, z), gives the solution of the degree 16; joining this second tangential with the original point (x, y, z), we have the solution of the degree 25; and the same solution is also given as the sixth point of intersection with the cubic, of the conic of 5-pointic intersection at the point (x, y, z). See my memoir "On the conic of 5-pointic contact at any point of a plane curve," *Phil. Trans.* vol. CXLIX. (1859), pp. 371—400, [261].

Addition to the foregoing Solution. On a system of Eighteen Points on a Cubic Curve.

Considering the cubic curve $x^3+y^3+z^3+6lxyz=0$, we have the nine points of inflexion, which I represent as follows:

$$\begin{aligned} a &= (0, 1, -1), & d &= (-1, 0, 1), & g &= (1, -1, 0), \\ b &= (0, 1, -\omega), & e &= (-\omega, 0, 1), & h &= (1, -\omega, 0), \\ c &= (0, 1, -\omega^2), & f &= (-\omega^2, 0, 1), & i &= (1, -\omega^2, 0), \end{aligned}$$

viz., ω being an imaginary cube root of unity, the coordinates of a are $(0, 1, -1)$, those of b, $(0, 1, -\omega)$, &c.

The points of inflexion lie (as is known) by threes on twelve lines; viz., the lines are

$$\begin{array}{llll} abc, & afh, & bfg, & cfi, \\ adg, & bdi, & cdh, & def, \\ aei, & beh, & ceg, & ghi. \end{array}$$

Consider now a point on the curve, the coordinates whereof are (x, y, z), where of course $x^3+y^3+z^3+6kxyz=0$; this is one of a system of eighteen points on the curve, which may be represented as follows:

$$\begin{array}{lll} A=(x,\ y,\ z), & D=(\ x,\ \omega y,\ \omega^2 z), & G=(\ x,\ \omega^2 y,\ \omega z), \\ B=(y,\ z,\ x), & E=(\omega y,\ \omega^2 z,\ x), & H=(\omega^2 y,\ \omega z,\ x), \\ C=(z,\ x,\ y), & F=(\omega^2 z,\ x,\ \omega y), & I=(\omega z,\ x,\ \omega^2 y), \\ \\ J=(x,\ z,\ y), & M=(\ x,\ \omega z,\ \omega^2 y), & P=(\ x,\ \omega^2 z,\ \omega y), \\ K=(z,\ y,\ x), & N=(\omega z,\ \omega^2 y,\ x), & Q=(\omega^2 z,\ \omega y,\ x), \\ L=(y,\ x,\ z), & O=(\omega^2 y,\ x,\ \omega z), & R=(\omega y,\ x,\ \omega^2 z), \end{array}$$

viz., the coordinates of A are (x, y, z); those of B are (y, z, x), &c.

The tangent at A meets the curve in a point, "the tangential of A," the coordinates whereof are $x(y^3-z^3)$, $y(z^3-y^3)$, $z(x^3-y^3)$; which point may be called A'. And we have thus the eighteen tangentials

$$A',\ B',\ C',\ D',\ E',\ F',\ G',\ H',\ I',\ J',\ K',\ L',\ M',\ N',\ O',\ P',\ Q',\ R'.$$

The eighteen points A, B, &c., have the following property; viz., the line joining any two of them meets the cubic in a third point, which is either one of the nine points of inflexion, or one of the eighteen tangentials; there are through each point of inflexion 9 such lines, and through each tangential 4 such lines; $(9\times 9)+(18\times 4)=153=\frac{1}{2}(18.17)$, the number of pairs of points AB, AC, &c. The lines through the inflexions are the 81 lines obtained by joining any one of the points $(A, B, C, D, E, F, G, H, I)$ with any one of the points $(J, K, L, M, N, O, P, Q, R)$, as shown in the following Table:

	A	B	C	D	E	F	G	H	I
J	a	d	g	c	f	i	b	e	h
K	d	g	a	f	i	c	e	h	b
L	g	a	d	i	c	f	h	b	e
M	c	f	i	b	e	h	a	d	g
N	f	i	c	e	h	b	d	g	a
O	i	c	f	h	b	e	g	a	d
P	b	e	h	a	d	g	c	f	i
Q	e	h	b	d	g	a	f	i	c
R	h	b	e	g	a	d	i	c	f

viz., the line AJ passes through a, the line AK through d, &c.; the proof that AJ passes through a depends on the identical equation

$$\begin{vmatrix} x, & y, & z \\ x, & z, & y \\ 0, & 1, & -1 \end{vmatrix} = 0;$$

and the like for the other lines AK, AL, &c.

The lines through the tangentials are the 36 lines obtained by joining any two of the points $(A, B, C, D, E, F, G, H, I)$ and the 36 lines obtained by joining any two of the points $(J, K, L, M, N, O, P, Q, R)$; and these 72 lines pass through the tangentials, as shown by the table

ABC,	BDI,	CEG,	JKL,	KMR,	LNP,
ADG,	BEH,	CFI,	JMP,	KNQ,	LOR,
AEI,	BFG,	DEF,	JNR,	JOP,	MNO,
AFH,	CDH,	GHI,	JOQ,	LMQ,	PQR,

viz., in the triad ABC, BC passes through A', CA through B', AB through C'; and the like for the other triads. The proof that BC passes through A depends on the identical equation

$$\begin{vmatrix} y, & z, & x \\ z, & x, & y \\ x(x^3-z^3), & y(z^3-x^3), & z(x^3-y^3) \end{vmatrix} = 0;$$

and the like for the other combinations of points.

If we attend only to the points A, B, C and their tangentials A', B', C'; then we have on the cubic three points A, B, C, such that the line joining any two of them passes through the tangential of the third point. And the figure may be constructed by means of the three real points of inflexion a, d, g, as follows, viz., joining these with any point J on the cubic, the lines so obtained respectively meet the cubic in three new points which may be taken for the points A, B, C. Or if one of these points, say A, be given, then joining it with one of the three real inflexions, this line again meets the cubic in the point J, and from it by means of the other two real inflexions we obtain the remaining points B and C; it is clear that, A being given, the construction gives three points, say J, K, L, each of them leading to the same two points B and C.

We may consider the question from a different point of view. Let A, B, C be given points, and let there be given also three lines passing through these three points respectively; through the given points, touching at these points the given lines respectively, describe a cubic; and let the given lines again meet the cubic in the points A', B', C' respectively. The equation of the cubic contains three arbitrary

parameters; but when two of these are properly determined, the points A, B, C and their tangentials A', B', C' will be related as in the theorem; viz., the line through any two of the points will pass through the tangential of the third point. The analytical investigation is as follows:

Let the equations of the three tangents be $x=0$, $y=0$, $z=0$, and suppose that, for the points A, B, C respectively, we have

$$(x=0,\ y=\lambda z),\quad (y=0,\ z=\mu x),\quad (z=0,\ x=\nu y),$$

then the equation of a cubic touching the three lines at the three points respectively will be

$$(y-\lambda z)^2(\nu^2By+Cz)+(z-\mu x)^2(\lambda^2Cz+Ax)+(x-\nu y)^2(\mu^2Ax+By)$$
$$-\mu^2Ax^3-\nu^2By^3-\lambda^2Cz^3+Kxyz=0,$$

where A, B, C, K are arbitrary coefficients; but if $A:B:C=\lambda:\mu:\nu$, then the equation is

$$(y-\lambda z)^2\nu(\mu\nu y+z)+(z-\mu x)^2\lambda(\nu\lambda z+x)+(x-\nu y)^2\mu(\lambda\mu x+y)$$
$$-\lambda\mu^2x^3-\mu\nu^2y^3-\nu\lambda^2z^3+Kxyz=0,$$

where

A, A'	are the intersections of	$x=0$,	by	$y-\lambda z=0$,	$\mu\nu y+z=0$	respectively,
B, B'	„ „	$y=0$,	„	$z-\mu x=0$,	$\nu\lambda z+x=0$	„ ,
C, C'	„ „	$z=0$,	„	$x-\nu y=0$,	$\lambda\mu x+y=0$	„ ;

the equations of BC, CA, AB thus are

$$-\mu x+\mu\nu y+z=0,\quad x-\nu y+\nu\lambda z=0,\quad \lambda\mu x+y-\lambda z=0,$$

which pass through A', B', and C' respectively.

If we consider along with the points A, B, C the points J, K, L, and their respective tangentials, then we have inscribed in the cubic a hexagon $ALBJCK$ which has the following properties, viz., the pairs of opposite sides and the three diagonals pass through the three real inflexions *in lineâ*, viz.,

AL,	JC,	BK,	through	g
LB,	CK,	JA,	„	a
BJ,	KA,	CL,	„	d.

This shows that the six points A, B, C, J, K, L are the intersections of the cubic by a conic; and moreover, considering the triangles ABC, JKL formed by the alternate vertices, then in each triangle the sides pass through the tangentials of the opposite vertices respectively.

In what precedes we have in effect found the coordinates (z, x, y) of the third point of intersection with the cubic, of the line joining the points (y, z, x) and

$\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\}$. The coordinates of the same point may be otherwise found by a direct investigation, as follows: Write

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3); \quad x_1 : y_1 : z_1 = y : z : x.$$

If in the equation of the curve we substitute for x, y, z, the values ux_1+vx_2, uy_1+vy_2, uz_1+vz_2, we find

$$u\{x_1^2x_2+y_1^2y_2+z_1^2z_2+2l(x_2y_1z_1+y_2z_1x_1+z_2x_1y_1)\}$$
$$+v\{x_1x_2^2+y_1y_2^2+z_1z_2^2+2l(x_1y_2z_2+y_1z_2x_2+z_1x_2y_2)\}=0,$$

say $uP+vQ=0$; we may therefore write $u=Q$, $v=-P$, and the coordinates of the third point are Qx_1-Px_2, Qy_1-Py_2, Qz_1-Pz_2. Now

$$\begin{aligned}
Qx_1-Px_2 &= y_1y_2(x_1y_2-x_2y_1)+z_1z_2(x_1z_2-x_2z_1)+2l(x_1^2y_2z_2-x_2^2y_1z_1)\\
&= \quad yz(z^3-x^3)\{y^3(z^3-x^3)-zx(y^3-z^3)\}\\
&\quad +zx(x^3-y^3)\{yz(x^3-y^3)-x^2(y^3-z^3)\}\\
&\quad +2l\{y^2.yz.(z^3-x^3)(x^3-y^3)-x^2(y^3-z^3)^2zx\}.\\
&= (x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3)z\\
&\quad +xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3)z\\
&\quad -2l(x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3)z;
\end{aligned}$$

so that we have $Qx_1-Px_2=\Pi z$; and in like manner $Qy_1-Py_2=\Pi x$, $Qz_1-Pz_2=\Pi y$; and therefore $Qx_1-Px_2 : Qy_1-Qy_2 : Qz_1-Pz_2=z : x : y$, which proves the theorem.

I consider in like manner the following question; viz., if (y, x, z) be joined with the tangential of (x, y, z); to find the third point of intersection. We have here

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3); \quad x_1 : y_1 : z_1 = y : x : z;$$

and P, Q as before; and the coordinates of the third point are

$$Qx_1-Px_2 : Qy_1-Py_2 : Qz_1-Pz_2;$$

also

$$\begin{aligned}
Qx_1-Px_2 = \ & xy(z^3-x^3)\{y^2(z^3-x^3)-x^2(y^3-z^3)\}\\
&+z^2(x^3-y^3)\{yz(x^3-y^3)-zx(y^3-z^3)\}\\
&+2l\{y^3z(z^3-x^3)(x^3-y^3)-x^3(y^3-z^3)^2zx\},\\
= \ & \quad x\{y^3(z^3-x^3)^2-z^3(x^3-y^3)(y^3-z^3)\}\\
&+\ y\{z^3(x^3-y^3)^2-x^3(y^3-z^3)(z^3-x^3)\}\\
&+2lz\{y^3(z^3-x^3)(x^3-y^3)-x^3(y^3-z^3)^2\},
\end{aligned}$$

that is $\quad Qx_1-Px_2=(x+y-2lz)(x^3z^6+y^3x^6+z^3y^6-3x^3y^3z^3)$;

similarly $\quad Qy_1-Py_2=(x+y-2lz)(y^3z^3+y^6z^3+z^3x^6-3x^3y^3z^3)$;

also $\quad Qz_1-Pz_2=(x+y-2lz)(x^6+y^6+z^6-y^3z^3-x^3y^3)xyz$;

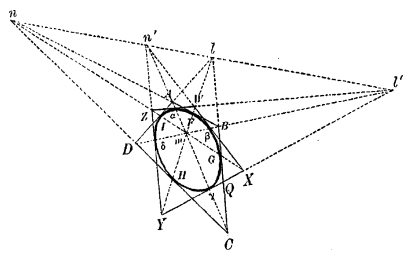
and we hence have the values

$$X : Y : Z = x^6y^3 + y^6z^3 + z^6x^3 - 3x^3y^3z^3 : x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3$$
$$: xyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)$$

for the coordinates of the point in question.

[Vol. IV. pp. 38, 39.]

1751. (Proposed by Professor CAYLEY.)—Let $ABCD$ be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\frac{BG}{BC} \cdot \frac{CQ}{GQ} = \frac{1}{\sqrt{2}}$; and complete the construction as shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure.



{Professor Cayley remarks that if $ABCD$ is the perspective representation of a square, then the ellipse is the perspective representation of the inscribed circle; the theorem gives eight points and the tangent at each of them; and the ellipse may therefore be drawn by hand with an accuracy quite sufficient for practical purposes. The demonstration is immediate, by treating the figure as a perspective representation: the gist of the theorem is the very convenient construction in perspective which it furnishes.}

[Vol. IV. pp. 65—67.]

1775. (Proposed by W. K. CLIFFORD.)—If a straight line meet the faces of the tetrahedron $ABCD$ in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

Solution by PROFESSOR CAYLEY.

Let the given line be taken for the axis of z; the axes of x, y being any rectangular axes in the plane perpendicular thereto; the equations of the given line are therefore $(x=0,\ y=0)$. Take $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ for the coordinates of the points A, B, C, D respectively; and $(0, 0, c_1)$, $(0, 0, c_2)$, $(0, 0, c_3)$, $(0, 0, c_4)$ for the coordinates of the points a, b, c, d respectively. Then, to determine c_1, the equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

and cutting this by the line $x=0$, $y=0$, we have

$$\begin{vmatrix} 0, & 0, & c_1, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

with similar equations for c_2, c_3, c_4 respectively. The four equations may be united into the single equation

$$\begin{vmatrix} c_1p_1, & 1, & \alpha_1, & \beta_1 \\ c_2p_2, & 1, & \alpha_2, & \beta_2 \\ c_3p_3, & 1, & \alpha_3, & \beta_3 \\ c_4p_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} p_1, & \alpha_1, & \beta_1, & \gamma_1 \\ p_2, & \alpha_2, & \beta_2, & \gamma_2 \\ p_3, & \alpha_3, & \beta_3, & \gamma_3 \\ p_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

where p_1, p_2, p_3, p_4 are arbitrary multipliers. Hence, writing successively $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ and $(p_1, p_2, p_3, p_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, we have *first*

$$\begin{vmatrix} c_1, & 1, & \alpha_1, & \beta_1 \\ c_2, & 1, & \alpha_2, & \beta_2 \\ c_3, & 1, & \alpha_3, & \beta_3 \\ c_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} 1, & \alpha_1, & \beta_1, & \gamma_1 \\ 1, & \alpha_2, & \beta_2, & \gamma_2 \\ 1, & \alpha_3, & \beta_3, & \gamma_3 \\ 1, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4 \end{vmatrix} = 0;$$

and *secondly*,

$$\begin{vmatrix} c_1\gamma_1, & 1, & \alpha_1, & \beta_1 \\ c_2\gamma_2, & 1, & \alpha_2, & \beta_2 \\ c_3\gamma_3, & 1, & \alpha_3, & \beta_3 \\ c_4\gamma_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} \gamma_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \gamma_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \gamma_3, & \alpha_3, & \beta_3, & \gamma_3 \\ \gamma_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4\gamma_4 \end{vmatrix} = 0;$$

and these two results may be united into the single formula

$$\left\| \begin{matrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4, & c_4\gamma_4 \end{matrix} \right\| = 0.$$

Now the equation of a sphere having for the extremities of a diameter the points (α, β, γ) and (a, b, c) is

$$[x-\tfrac{1}{2}(a+\alpha)]^2+[y-\tfrac{1}{2}(b+\beta)]^2+[z-\tfrac{1}{2}(c+\gamma)]^2=\tfrac{1}{4}[(a-\alpha)^2+(b-\beta)^2+(c-\gamma)^2],$$

or

$$(x-a)(x-\alpha)+(y-b)(y-\beta)+(z-c)(z-\gamma)=0,$$

or

$$x^2+y^2+z^2-(a+\alpha)x-(b+\beta)y-(c+\gamma)z+a\alpha+b\beta+c\gamma=0;$$

therefore, when the two points are (α, β, γ) and $(0, 0, c)$, the equation is

$$x^2+y^2+z^2-\alpha x-\beta y-(c+\gamma)z+c\gamma=0.$$

Hence, putting for shortness $P=-\alpha x-\beta y-(c+\gamma)z+c\gamma$, viz., $P_1=-\alpha_1 x-\beta_1 y-(c_1+\gamma_1)z+c_1\gamma_1$, &c., the equations of the four spheres are

$$x^2+y^2+z^2+P_1=0,\quad x^2+y^2+z^2+P_2=0,\quad x^2+y^2+z^2+P_3=0,\quad x^2+y^2+z^2+P_4=0,$$

and the four spheres will have a common radical axis, if for proper values of the multipliers μ, ν, ρ we have

$$\mu(P_1-P_2)+\nu(P_1-P_3)+\rho(P_1-P_4)=0,$$

or what is the same thing, if for proper values of λ, μ, ν, ρ we have

$$\lambda P_1+\mu P_2+\nu P_3+\rho P_4=0,\quad \lambda+\mu+\nu+\rho=0;$$

that is, if

$$\begin{aligned}
&\lambda + \mu + \nu + \rho &&= 0,\\
&\lambda\alpha_1 + \mu\alpha_2 + \nu\alpha_3 + \rho\alpha_4 &&= 0,\\
&\lambda\beta_1 + \mu\beta_2 + \nu\beta_3 + \rho\beta_4 &&= 0,\\
&\lambda(c_1+\gamma_1) + \mu(c_2+\gamma_2) + \nu(c_3+\gamma_3) + \rho(c_4+\gamma_4) &&= 0,\\
&\lambda c_1\gamma_1 + \mu c_2\gamma_2 + \nu c_3\gamma_3 + \rho c_4\gamma_4 &&= 0;
\end{aligned}$$

and eliminating from these equations (λ, μ, ν, ρ), we find the above-mentioned relation between α_1, β_1, γ_1, c_1, &c.; which proves the theorem.

[Vol. IV. pp. 70, 71.]

1771. (Proposed by Professor CAYLEY.)—Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature.

---

2. *Solution by the* PROPOSER.

Let $x^2+y^2-1=0$ be the equation of the given circle, $x=m$ that of the given line. Taking on the circle an arbitrary point $(\cos\theta, \sin\theta)$, we may find a parabola having the given line for its directrix, and touching the circle at the last-mentioned point; viz., the equation of the parabola is found to be

$$y^2 - 2y\sin\theta(1+2\cos^2\theta - 2m\cos\theta) - 4x\cos^2\theta(\cos\theta - m) + 1 + 3\cos^2\theta - 4m\cos\theta = 0.$$

{There is no difficulty in *verifying* that this parabola has for its directrix the line $x-m=0$, that the equation is satisfied by the values $x=\cos\theta$, $y=\sin\theta$, and that the derived equation is satisfied by the values $x=\cos\theta$, $y=\sin\theta$, $\dfrac{dy}{dx}=-\cot\theta$.}

Representing for a moment the left-hand side of the equation by U, we have identically

$$\begin{aligned}
U - \cos^2\theta(x^2+y^2-1)\\
&= y^2\sin^2\theta - x^2\cos^2\theta - 2y\sin\theta(1+2\cos^2\theta - 2m\cos\theta) - 4x\cos^2\theta(\cos\theta - m)\\
&\qquad + 1 + 4\cos^2\theta - 4m\cos\theta,\\
&= (y\sin\theta + x\cos\theta - 1)(y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta).
\end{aligned}$$

Hence to find the intersections of the parabola with the circle, we have first

$$x^2+y^2-1=0, \quad y\sin\theta + x\cos\theta - 1 = 0,$$

giving the point $(\cos\theta, \sin\theta)$ *twice*, since $y\sin\theta + x\cos\theta - 1 = 0$ is the equation of the tangent to the circle at the point in question; and secondly

$$x^2+y^2-1=0, \quad y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta = 0,$$

giving the remaining two points of intersection. If the circle be a circle of curvature, one of these must coincide with the point $(\cos\theta, \sin\theta)$, that is the equation $y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta = 0$, must be satisfied by the values $x = \cos\theta$, $y = \sin\theta$; this will be the case if $-6\cos^2\theta + 4m\cos\theta = 0$, that is $\cos\theta = 0$, giving for the parabola $y^2 \pm 2y + 1 = 0$, which is not a proper Solution, or else $\cos\theta = \frac{2}{3}$, giving $\sin\theta = \pm(1 - \frac{4}{9}m^2)^{\frac{1}{2}}$, so that there are *two* parabolas satisfying the conditions of the problem; if to fix the ideas we take the upper sign, the equation of the corresponding parabola is

$$y^2 - 2(1 - \tfrac{4}{9}m^2)^{\frac{3}{2}}y + \tfrac{16}{27}m^2x + 1 - \tfrac{4}{3}m^2 = 0;$$

and it may be added that the coordinates of the focus are

$$x = m - \tfrac{8}{27}m^3, \quad y = (1 - \tfrac{4}{9}m^2)^{\frac{3}{2}}.$$

The equation of the other parabola and the coordinates of the focus are of course found by merely changing the sign of the radical. The parabolas are real if $m < \frac{3}{2}$; if $m = \frac{3}{2}$ we have a single parabola, the point of contact being in this case the vertex of the parabola; and if $m > \frac{3}{2}$ the parabolas are imaginary.

{Professor Cayley states that he was led to the foregoing problem by the consideration of the curve (proposed for investigation in Quest. 1812) which is the envelope of a variable circle having its centre in the given circle and touching the given line. The required curve (which is of the sixth order) has two cusps which, it is easy to see geometrically, are the foci of the parabolas in the problem. Taking $(\cos\theta, \sin\theta)$ for the coordinates of the centre of the variable circle, we shall have

$$x = \tfrac{3}{2}\cos\theta - m\cos 2\theta + \tfrac{1}{2}\cos 3\theta, \quad y = \tfrac{3}{2}\sin\theta - m\sin 2\theta + \tfrac{1}{2}\sin 3\theta,$$

for the coordinates of a point on the envelope.}

[Vol. IV. pp. 81—83.]

1790. (Proposed by Professor SYLVESTER.)—(1) If a set of six points be respectively represented by the six permutations of $a : b : c$, show that they lie in a conic, and write down its equation.

(2) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic.

Solution by PROFESSOR CAYLEY.

1. That the six points,

$$1 = (a, b, c), \quad 2 = (b, c, a), \quad 3 = (c, a, b),$$
$$4 = (a, c, b), \quad 5 = (b, a, c), \quad 6 = (c, b, a),$$

are situate on a conic, appears at once by writing down its equation: viz.,

$$(bc+ca+ab)(x^2+y^2+z^2)-(a^2+b^2+c^2)(yz+zx+xy)=0,$$

which is satisfied by the coordinates of each of the six points.

2. It is interesting to remark that the six points on the conic form, not a general inscribed hexagon, but a hexagon such as is mentioned in Prob. 1512 (vol. II. p. 51), viz., one in which the three diagonals pass respectively through the Pascalian points (intersections of opposite sides): in fact, in the hexagon 143526, forming the equations of the sides and diagonals, these are

$$\begin{array}{llll}
14. & (b+c)x-ay-az=0, & 25. & (c+a)x-by-bz=0, \\
15. & -cx-cy+(a+b)z=0, & 26. & -ax-ay+(b+c)z=0, \\
16. & -bx+(c+a)y-bz=0, & 24. & -cx+(a+b)y-cz=0,
\end{array}$$

$$\begin{array}{ll}
36. & (a+b)x-cy-cz=0, \\
34. & -bx-by+(c+a)z=0, \\
35. & -ax+(b+c)y-az=0;
\end{array}$$

so that the lines 14, 25, 36 meet in the point $x=0,\ y+z=0$,
" 16, 24, 35 " $y=0,\ z+x=0$,
" 15, 26, 34 " $z=0,\ x+y=0$.

3. It is further to be remarked that the six points lie on the cubic curve

$$\frac{x^3+y^3+z^3}{a^3+b^3+c^3}-\frac{xyz}{abc}=0,$$

and are consequently the six points of intersection of this cubic by the above mentioned conic.

4. The points $(x=0,\ y+z=0)$, $(y=0,\ z+x=0)$, $(z=0,\ x+y=0)$ are the three real inflexions of the cubic; hence, attending only to the cubic, and starting from the arbitrary point $(a,\ b,\ c)$ on this curve, it appears by what precedes, that we may, by means of the three real inflexions of the cubic, construct the system of six points, (the construction is, in fact, identical with that given in my Solution of Problem 1744, vol. IV. pp. 32—37, [*ante* p. 597] the six points being six out of the therein mentioned eighteen points); and it further appears, that these six points lie on a conic.

5. As regards the second part of the proposed Problem, consider the cubic curve $x^3+y^3+z^3+6lxyz=0$; the three real lines containing the nine points of inflexion are the lines $x=0,\ y=0,\ z=0$; and the points $A,\ B,\ C$ are therefore $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$ respectively. From each of these points we may draw to the curve six tangents, and we have thus on the curve eighteen points, which are a particular case of the system in the Solution of Prob. 1744. Or if from each of the points we draw two properly selected tangents, (when the cubic has an oval these

may be the two tangents to the oval,) then we obtain a system of six points, (part of the system of eighteen points); viz., the coordinates of the six points are of the form (a, b, c), (b, c, a), (c, a, b), (a, c, b), (b, a, c), (c, b, a) and therefore the six points are in a conic.

6. To verify this, if we take $y=\theta x$ for the equation of a tangent from the point $(x=0,\ y=0)$, the equation $(1+\theta^3)x^3+6l\theta x^2z+z^3=0$ must have a pair of equal roots, giving for θ the equation $(1+\theta^3)^2+32l^3\theta^3=0$; and we then find $z=-\dfrac{1+\theta^3}{4l\theta}x$, that is, θ being determined by the foregoing equation, the coordinates of the point of contact are $x:y:z=1:\theta:-\dfrac{1+\theta^3}{4l\theta}$. The roots of the equation in θ are of the form $\theta_1,\ \theta_2,\ \theta_3,\ \dfrac{1}{\theta_1},\ \dfrac{1}{\theta_2},\ \dfrac{1}{\theta_3}$; and assuming that the curve has an oval, there are two real roots $\theta_1,\ \dfrac{1}{\theta_1}$. Hence, writing $x:y:z=1:\theta_1:-\dfrac{1+\theta_1^3}{4l\theta_1}=a:b:c$, the substitution $\dfrac{1}{\theta_1}$ for θ, gives $x:y:z=b:a:c$, that is, the coordinates of the points of contact of the tangents to the oval, from the point $(x=0,\ y=0)$ are (a, b, c) and (b, a, c) respectively; and writing successively (y, z, x) and (z, x, y) in place of (x, y, z), the coordinates for the tangents from $(y=0,\ z=0)$ are (b, c, a), (c, b, a); and those for the tangents from $(z=0,\ x=0)$ are (c, a, b) and (a, c, b); so that the coordinates of the six points of contact are a system of the form in question.

[Vol. IV. p. 107.]

1812. (Proposed by Professor CAYLEY.)—Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. {See Quest. 1771.}

[Vol. IV. pp. 108, 109.]

1816. (Proposed by R. BALL, M.A.)—Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\{(ac-b^2)x^4+2(ad-bc)x^3$$
$$+(ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\}^2=0,$$

in terms of the roots α, β, γ, δ of $x^4+4bx^3+6cx^2+4dx+e=0$.

Solution by PROFESSOR CAYLEY.

Writing

$$U=(a,\ b,\ c,\ d,\ e\between x,\ y)^4=a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y),$$
$$H=(ac-b^2,\ \tfrac{1}{2}(ad-bc),\ \tfrac{1}{6}(ae+2bd-3c^2),\ \tfrac{1}{2}(be-cd),\ ce-d^2\between x,\ y)^4,$$
$$I=ae-4bd+3c^2;$$

then considering $x : y$ as the unknown quantity, it is required to find the roots of the equation $IU^2 - 3H^2 = 0$ in terms of the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $U = 0$; or, what is the same thing, it is required to find the linear factors of the function $IU^2 - 3H^2$. The function in question is the product of four quadratic factors, rational functions of $(\alpha, \beta, \gamma, \delta)$; and these being known, the four pairs of linear factors can be determined each of them by the solution of a quadratic equation. In fact, writing

$$\begin{aligned}
\Theta_\alpha &= a\{(\beta-\alpha)(x-\gamma y)(x-\delta y)+(\gamma-\alpha)(x-\delta y)(x-\beta y)+(\delta-\alpha)(x-\beta y)(x-\gamma y)\},\\
\Theta_\beta &= a\{(\gamma-\beta)(x-\delta y)(x-\alpha y)+(\delta-\beta)(x-\alpha y)(x-\gamma y)+(\alpha-\beta)(x-\gamma y)(x-\delta y)\},\\
\Theta_\gamma &= a\{(\delta-\gamma)(x-\alpha y)(x-\beta y)+(\alpha-\gamma)(x-\beta y)(x-\delta y)+(\beta-\gamma)(x-\delta y)(x-\alpha y)\},\\
\Theta_\delta &= a\{(\alpha-\delta)(x-\beta y)(x-\gamma y)+(\beta-\delta)(x-\gamma y)(x-\alpha y)+(\gamma-\delta)(x-\alpha y)(x-\beta y)\},
\end{aligned}$$

we have identically $256(IU^2 - 3H^2) = \Theta_\alpha\Theta_\gamma\Theta_\beta\Theta_\delta$; so that the quadratic factors of $IU^2 - 3H^2$ are Θ_α, Θ_β, Θ_γ, Θ_δ. To show that this is so, it is to be remarked that the product $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is a symmetrical function of the roots $\alpha, \beta, \gamma, \delta$, and consequently a rational and integral function of the coefficients (a, b, c, d, e) of U; moreover Θ_α, Θ_β, Θ_γ, Θ_δ being each of them a covariant (an irrational one) of U, the product in question must be a covariant. But a covariant is completely determined when the leading coefficient is given; hence it will be sufficient to show that the leading coefficients, or coefficients of x^8, in the functions $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ and $256(IU^2 - 3H^2)$ are equal to each other. Writing for a moment $\Sigma\alpha = p$, $\Sigma\alpha\beta = q$, $\Sigma\alpha\beta\gamma = r$, $\alpha\beta\gamma\delta = s$, the coefficient of x^8 in $a^{-1}\Theta_\alpha$ is $\beta+\gamma+\delta-3\alpha$, which $= p - 4\alpha$; we have thence the product $(p-4\alpha)(p-4\beta)(p-4\gamma)(p-4\delta)$, which is $= p^4 - 4p^3 . p + 16p^2 . q - 64p . r + 256s$, $= 256s - 64pr + 16p^2q - 3p^4$.

Hence, restoring the omitted factor a^4, and observing that we have p, q, r, s equal to $-4b$, $6c$, $-4d$, e, each divided by a, the coefficient of x^8 in $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is

$$256\,(a^3e - 4a^2bd + 6ab^2c - 3b^4), \text{ or } 256\,\{(ae - 4bd + 3c^2)\,a^2 - 3\,(ac - b^2)^2\},$$

and is consequently equal to the coefficient of x^8 in $256(IU^2 - 3H^2)$; which proves the theorem.

It may be remarked that the leading coefficient of $IU^2 - 3H^2$ is $= a^{-1}(a, b, c, d, e \between b, -a)^4$; and that for a quantic $U = (a, b, \ldots)(x, y)^n$ of the order n we have a corresponding covariant of the order $n(n-2)$, the leading coefficient of which is $= a^{-1}(a, b, \ldots \between b, -a)^n$. For $n = 2$, this is the invariant (discriminant) $ac - b^2$; for $n = 3$ it is the cubicovariant $(a^2d - 3abc + 2b^3, \ldots \between x, y)^3$; for $n = 4$ it is, as we have seen, the covariant $IU^2 - 3H^2$. For $n = 5$, the leading coefficient $a^4f - 5a^3be + 10a^2b^2d - 10ab^3c + 4b^5$ is $= a^2(a^2f - 5abe + 2acd + 8b^3d - 6bc^2) - 2(ac - b^2)(a^2d - 3abc + 2b^3)$, which shows that the covariant in question (of the order 15) is $= U^2(\text{No. } 17) - 2(\text{No. } 15)(\text{No. } 18)$, where the Nos. refer to the Tables of my Second Memoir on Quantics, *Phil. Trans.*, vol. CXLVI. (1856), pp. 101—126, [141; in the notation there explained, the expression for the covariant is $A^2E - 2CF$].

{The roots of $\Theta_\alpha = 0$ are readily found to be

$$\frac{\alpha(\beta+\gamma+\delta) - (\gamma\delta+\delta\beta+\beta\gamma) \pm \{\frac{1}{2}[(\alpha-\beta)^2(\gamma-\delta)^2+(\alpha-\gamma)^2(\delta-\beta)^2+(\alpha-\delta)^2(\beta-\gamma)^2]\}^{\frac{1}{2}}}{3\alpha-(\beta+\gamma+\delta)}$$

these then, with three similar pairs, express the eight roots as required.}

In the following Contents, the Problems are referred to, each by its No. and the Proposer's name, and the subject is briefly indicated: an Asterisk shows that no solution was given.

			PAGE
1373	Wilkinson	Circle and Points	560
1387	Clifford	Circle and Ellipse	561
1409	Clifford	Conic	562
1478	McDowell	Triangle and Circle	564
1273	Miller	Triangles	566
1505	Cayley	Four Points on a Conic	571
1513	Blissard	Factorials	574
1512	Cayley	Conic and Hexagon	576
1562	Thomson	Tangents of Conic	578
1533	Cayley	Quintic Curve in connexion with Cubic and Quartic	580
*1542	Cayley	Intersection of Conics	582
1606	Miller	Conics	ib.
*1607	Cayley	Cubics	586
1647	Cayley	Foci of Ellipse	ib.
1652	Clifford	Triangle and Polygons	589
1667	Sylvester	Discriminant of Quintic	592
*1687	Cayley	Spherical Triangle	593
1690	Whitworth	Triangle and Conics	ib.
1710	Cayley	Quartic Curve	596
1744	Burnside	Cubic Identity	597
1751	Cayley	Quadrilateral and Ellipse	604
1771	Cayley	Line, Circle, and Parabola	607
1790	Sylvester	Conic and Cubic	608
1812	Cayley	Envelope of Circles	610
1816	Ball	Roots of Quartic	ib.

NOTES AND REFERENCES.

302, 305. The theory of curves in space was proposed as the subject of the Prize-question of the Steiner Foundation by the Academy of Sciences of Berlin in the year 1881 "irgend eine auf die Theorie der höheren algebraischen Raumcurven sich beziehende Frage von wesentlicher Bedeutung vollstandig zu erledigen," and the prize was divided between the two memoirs

Halphen, "Mémoire sur la classification des courbes gauches algébriques." *Jour. École Polyt.* Cah. LII. (1882), pp. 1—200, and

Nöther, "Zur Grundlegung der Theorie der algebraischen Raumcurven." *Abh. der Akad. zu Berlin vom Jahre* 1882, pp. 1 to 120; both treating of the classification of curves in space.

We have also the valuable memoir

Valentiner, "Zur Theorie der Raumcurven," *Acta Mathematica*, t. II., 1883, pp. 136—230, which relates less directly to the question of classification.

The three authors all refer to these papers in the *Comptes Rendus*, and make considerable use of my conception of the monoid surface. It would be out of place to attempt to give any account here of these memoirs: I only refer to such remarks or theorems contained in them as stand in immediate connection with the remarks which follow.

The question of classification is much simplified by excluding from consideration the curves with singular points (that is actual double points and stationary points), and this is in fact done both by Halphen and Nöther and in the present Note. The curves considered are thus curves with only apparent double points (adps.) viz. for a curve of the order d (I use Halphen's letters) with h apparent double points, taking an arbitrary point as vertex, the cone through the curve is a cone of the order d, with h nodal lines, each of these meeting the curve in two (real or imaginary) non-coincident points. Such a curve is the partial intersection of the cone in question say $U, =(x, y, z)^d, =0$ with a monoid surface $w, =\dfrac{(x, y, z)^{k+1}}{(x, y, z)^k}, =\dfrac{Q}{P}$, where the inferior cone $P, =(x, y, z)^k, =0$, of the monoid surface, and the superior cone $Q, =(x, y, z)^{k+1}, =0$, of the monoid surface each of them pass through all the h

nodal lines of the cone, and besides through θ lines of the cone: the complete intersection of the cone and monoid surface is thus made up of the curve once, the h nodal lines each twice, and the θ lines each once; and if as before the order of the curve is $=d$, then we thus have $(k+1)\,d=d+2h+\theta$, viz. we must have $kd=2h+\theta$, as the condition to be satisfied in order that the curve of the order d may be the partial intersection of the cone and monoid surface.

In my papers in the *Comptes Rendus* I endeavoured to find, and Halphen and Nöther both endeavour to find, the surfaces of lowest order which have the curve of order d for their complete or partial intersection. This (although, as will presently appear, the theory may be considered in a more complete form) is an important and interesting question; but upon further reflection it appears to me that it is a question beside that which first presents itself and ought to be in the first instance considered, viz. this is the question of the classification of curves in space according to the foregoing representation of any such curve as the partial intersection of a cone and monoid surface. Supposing it effected, and a kind of curve completely defined according to this mode of representation, then there arises the further question to which I have referred (Salmon's *Solid Geometry*, Ed. 3, (1874), p. 285, and Ed. 4, (1882), p. 281), viz. we may have passing through any given curve a *complete* system of surfaces, that is a system $U=0$, $V=0$, $W=0, \ldots$ where these functions are not connected by any such equation as $U=NV+PW+\ldots$, and where every other surface which passes through the curve is expressible in the form $MU+NV+PW+\ldots=0$. It is not easy to prove (but as to this see Hilbert "Zur Theorie der algebraischen Gebilde," *Göttingen Nachrichten*, 1888, p. 454), but it may be safely assumed that for a curve of any given order whatever, the number of equations in such a complete system is finite, and we have thus the representation of a curve in space by means of a complete system of surfaces passing through it. Obviously the curve is here the partial (or if the system consists of only two surfaces then the complete) intersection say of the two surfaces $U=0$, $V=0$ of lowest order passing through it, which is the question above referred to.

Reverting to the representation by the cone and monoid surface, Halphen gives the capital theorem, that if we have any particular inferior cone $P=0$ passing through the curve, then we may without loss of generality take the equation of the monoid surface to be $w=\frac{Q}{P}$: viz. if instead hereof the equation of the monoid surface is taken to be $w=\frac{Q'}{P'}$, then this equation in virtue of the equation $U=0$ of the cone is always reducible to the first mentioned form $w=\frac{Q}{P}$; that is in virtue of the equation $U=0$, we have $w=\frac{Q'}{P'}=\frac{Q}{P}$, or what is the same thing, $\frac{Q'}{P'}=\frac{Q}{P}$ in virtue of $U=0$, that is $Q'P-QP'=MU$, where M is a rational and integral function $(x, y, z)^\lambda$ of the degree λ, $=k+n+1-d$, if k be the degree of P' and n that of P.

It thus appears that if n be the order of the cone of lowest order which passes through the h nodal lines of the cone $U=0$, then we have always functions Q, P

of the orders $n+1$, n respectively, such that the equation of the monoid surface is $w=\frac{Q}{P}$. Or what is the same thing, we have always a monoid surface of the order $n+1$: we thus arrive at the notion of Halphen's characteristic n.

Instead of the foregoing equation $kd=2h+\theta$, we thus have $nd=2h+\theta$, and for given values of d, h there is thus a minimum value of n (viz. nd must be at least $=2h$); there is also a maximum value of n, viz. this is the least value for which $\frac{1}{2}n(n+3)$ is $=$ or $<h$, for with such a value of n there is always through the h nodal lines a cone of the order n.

For a given value of d, we have $h=$ at most $\frac{1}{2}(d-1)(d-2)$, and Halphen shows that h must be at least $=[\frac{1}{4}(d-1)^2]$, if we denote in this manner the integral part of the expression within the brackets. And then, h having any value between these limits, for any given values of d, h we have by what precedes a certain number of values of n.

We thus have *primâ facie* curves in space of the several forms (d, h, n): but it may very well be, and in fact Halphen finds that when d is $=9$ or upwards, then for certain values of h, n as above, there is not any curve (d, h, n): thus $d=9$, $h=17$, the values of n are $n=4$ or 5, but there is not any curve $d=9$, $h=17$, for either of these values of n; or say the curves (9, 17, 4) and (9, 17, 5) are non-existent.

And I notice further that in certain cases for which Halphen finds a curve (d, h, n) such curve does not exist except for special configurations of the h nodal lines not determined by the mere definition of n as the order of the cone of lowest order which passes through the h nodal lines: for instance $d=9$, $h=16$, $n=4$ for which Halphen gives a curve, I find that it is not enough that the 16 nodal lines are situate on a quartic cone, but that they must be the 16 lines of intersection of two quartic cones.

I remark moreover that Halphen does not carry out the foregoing principle of classification according to the values of (d, h, n): thus $d=9$, $h=22$, the values of n are 6 and 5; viz. the 22 nodal lines are in general on a sextic cone but they may be on a quintic cone; the curves (9, 22, 6) and (9, 22, 5) exist each of them, but he gives only the former of the two forms. The form (9, 22, 6) has a capacity 36 (depends upon 36 constants) but (9, 22, 5) a capacity 35 only, and I assume that Halphen considered it as a particular case of (9, 22, 6), (there is it seems to me a want of precision in his definition of a family)—but I consider that this is an abandonment of the principle—the two curves differ *ipso facto* in that in the first form the 22 nodal lines are not, in the second form they are, on a quintic curve. In Nöther's theory the characteristic n does not present itself.

Resuming the general theory, and considering d, h, n as given, we start from the cone $U=0$ of the order d, with h nodal lines lying in a cone of the order n: we take $P=0$ a cone of the order n passing through the h nodal lines, and besides meeting the cone $U=0$ in θ lines; $nd=2h+\theta$, (where θ may be $=0$). And we then have $Q=0$ a cone of the order $n+1$ passing through the h lines and the θ lines;

and this being so we have $w=\frac{Q}{P}$ for the equation of the monoid surface, and consequently $U=0$ and $w=\frac{Q}{P}$ for the equations of the curve, viz. the cone $U=0$ and the monoid surface of the order $n+1$ meet in the h lines each twice, in the θ lines, and in the curve of the order d; $(n+1)d=2h+\theta+d$. Observe here that the cone $Q=0$ as a cone of the order $n+1$ subjected only to the conditions of passing through the h lines and the θ lines has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$; this number should be $=3$ at least, for if it were $=2$, we should have $Q=(x+\beta y+\gamma z)P$ (since $P=0$ is a cone of the next inferior order through the same $h+\theta$ lines), and thus the curve would be a plane curve. Observe further that the cone $U=0$, quà cone of the order d with h nodal lines has in general a capacity $=\frac{1}{2}d(d+3)-h$; the cone $P=0$, by what precedes may be regarded as determinate, and the cone $Q=0$ as just appearing has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$; there is a term $+1$ for the implicit constant factor in the function Q, and we thus find for the capacity of the curve the expression $\frac{1}{2}d(d+3)-h+1+\frac{1}{2}(n+1)(n+4)-h-\theta$, viz. this is $=\frac{1}{2}d(d+3)+\frac{1}{2}(n^2+5n)+3-nd$, $=\frac{1}{2}(d-n)^2+\frac{1}{2}(3d+5n)+3$, which putting for a moment $d-n=\alpha$ is $=\frac{1}{2}\alpha^2+\frac{1}{2}(8d-5\alpha)+3$, $=4d+\frac{1}{2}(\alpha-2)(\alpha-3)$; hence restoring for α its value, we find capacity of curve $=4d+\frac{1}{2}(d-2-n)(d-3-n)$: in particular if $n=d-2$ or $d-3$, the capacity is $=4d$.

We are thus able in the case where $\frac{1}{2}(n+1)(n+4)-h-\theta=3$ or more, say $\frac{1}{2}n(n+5)$ $=$ or $>$ $h+\theta+1$, actually to construct the equation of a curve (d, h, n), having in the case where $n=d-2$ or $d-3$ a capacity $=4d$: the conditions in question for any given value of d, are satisfied by the considerable number of curves which form Halphen's "premier groupe."

For instance $d=9$, then the complete table of the values of h, n, θ is

d	h	n	θ	Cap.
9	16	4	4	38
		5	13	0
	17	4	2	0
		5	11	0
	18	4	0	36
		5	9	36
	19	5	7	36
	20	5	5	36
	21	5	3	36
		6	12	0
	22	5	1	35
		6	10	36†
	23	6	8	36†
	24	6	6	36†
	25	6	4	36†
	26	6	2	36†
	27	6	0	36†
	28	7	7	36†

and the conditions are satisfied for those values of (d, h, n) against which I have set the capacity 36†. I do not explain the remaining figures of the column of capacities, but remark only that 0 means that the curve is non-existent, and that 35 refers to the curve (9, 22, 5) which is alluded to above as not specified by Halphen.

It is important to remark that if the above-mentioned condition $\frac{1}{2}n(n+5) =$ or $> h+\theta+1$, or restoring it to the original form $\frac{1}{2}(n+1)(n+4)-h-\theta=3$ at least, is *not* satisfied, then it by no means follows, and it is not in general the case, that the curve is non-existent: I have said only that the cone $Q=0$ has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$, but the configuration of the $h+\theta$ lines may be such as not to impose on the cone $Q=0$ which passes through them so many as $h+\theta$ conditions, and the capacity of the cone may thus be greater than $\frac{1}{2}(n+1)(n+4)-h-\theta$, and may thus be $=3$ at least; moreover supposing that in such a case the curve exists, the capacity of the cone $U=0$ instead of being $=\frac{1}{2}d(d+3)-h$, may very well have, and presumably has, a greater value, and the reasoning by which the capacity of the curve was found to be $=4d+\frac{1}{2}(d-2-n)(d-3-n)$ ceases to be applicable. The theory, as depending upon special configurations of the h lines and the θ lines, is a complicated and difficult one, and I do not attempt to enter upon it.

In conclusion I wish to refer to an important theorem given by Valentiner and also by Halphen and Nöther. Considering in connexion with the curve of the order d, a surface of the order m, then since the capacity hereof (or number of constants contained in its equation) is $=\frac{1}{6}(m+1)(m+2)(m+3)-1$ or $\frac{1}{6}m(m^2+6m+11)$, it is obvious that if this be greater than md, the surface can be made to pass through more than md points of the curve, and thus that the curve will lie upon a surface of the order m. But the condition which has really to be satisfied in order that the curve may lie upon a surface of the order m is a less stringent one: if p be the deficiency of the curve, $=\frac{1}{2}(d-1)(d-2)-h$, if as before the curve is without actual singularities, and h be the number of its apparent double points, then the condition is $\frac{1}{6}m(m^2+6m+11)$ greater than $md-p$, viz. the surface of the order m being made to pass through $md+1-p$ points assumed at pleasure on the curve will *ipso facto* pass through p determinate points of the curve, that is in all through $md+1$ points of the curve, or it will contain the curve. The theorem is true subject only to the limitation $m=$ or $>d-2$. The most simple form of statement is perhaps that given by Valentiner, p. 194 (changing only his letters), viz. if m be $=$ or $>d-2$, the intersections of a surface of the order m with a curve of the order d with h apparent double points are determined by means of

$$dm-\tfrac{1}{2}(d-1)(d-2)+h\,(=dm-p)$$

of these intersections.

312. The generalisation which is here given of Euler's theorem $S+F=E+2$, is a first step towards the theory developed in Listing's Memoir "Census räumlicher Complexe oder Verallgemeinerung des Euler'schen Satzes von den Polyedern." Göttingen Abh. t. x. (1862).

320. The transcendent $i\,\mathrm{gd}(-iu)$, with a pure imaginary argument is the function $\log\tan(\frac{1}{4}\pi+\frac{1}{2}u)$ (hyperbolic logarithm) tabulated by Legendre, *Exer. de Calcul Intégral*,

t. II. (1816), Table IV. and *Traité des Fonctions Elliptiques*, t. II. (1826), Table IV. at intervals of 30′ from 0° to 90°, to twelve decimals and fifth differences. But the march of the function is somewhat disguised by the argument being taken in degrees and minutes and the function in abstract number. I have in the paper "On the orthomorphosis of the circle into the parabola," *Quart. Math. Jour.* vol. XX. (1885), pp. 213—220, see p. 220, given the table (at intervals of 1° to seven decimals) exhibiting the argument and the function each of them in degrees and minutes and also in abstract number.

335. Besides the 13 numbers mentioned by Gauss it appears by the paper, Perott, "Sur la formation des déterminants irreguliers," *Crelle*, t. XCV. (1883), pp. 232—236, that in the first thousand the determinants -468 and -931 are irregular.

341. Consider the equation of a curve as given in the form $y-f(x)=0$; then in the notation of Reciprocants $(t=y', a=\frac{1}{2}y'', b=\frac{1}{6}y''', c=\frac{1}{24}y'''', d=\frac{1}{120}y'''''$, where the accents denote differentiation in regard to x) the equation of the conic of five-pointic contact at the point (x, y) of the curve is

$$\begin{aligned} & a^4(X-x)^2 \\ & +a^2b(X-x)\{Y-y-t(X-x)\} \\ & +(ac-b^2)\quad\{Y-y-t(X-x)\}^2 \\ & -a^3\qquad\qquad\{Y-y-t(X-x)\}=0, \end{aligned}$$

which I verify as follows: writing $X=x+\theta$, we have

$$Y=y+t\theta+a\theta^2+b\theta^3+c\theta^4+d\theta^5,$$

and thence

$$Y-y-t(X-x)=\qquad a\theta^2+b\theta^3+c\theta^4+d\theta^5.$$

Substituting these values and developing as far as θ^5 we find

$$\begin{aligned} & a^4\theta^2 \\ & +a^2b\theta(a\theta^2+b\theta^3+c\theta^4) \\ & +(ac-b^2)(a^2\theta^4+2ab\theta^5) \\ & -a^3(a\theta^2+b\theta^3+c\theta^4+d\theta^5)=0, \end{aligned}$$

viz. this is

$$0\theta^2+0\theta^3+0\theta^4-a(a^2d-3abc+2b^3)\theta^5=0.$$

The equation is thus satisfied as far as θ^4, showing that the conic is a conic of 5-pointic contact; and it will be satisfied as far as θ^5 if only $a(a^2d-3abc+2b^3)=0$. The value $a=0$ belongs to an inflexion, and reduces the equation of the conic to $\{Y-y-t(X-x)\}^2=0$, viz. this is the stationary tangent taken twice, which is in an improper sense a conic of six-pointic contact: the other factor determines a sextactic point, viz. we have $a^2d-3abc+2b^3=0$ as the condition of a sextactic point.

We might from this form, which belongs to the curve as given by the equation $y-f(x)=0$, pass to the form belonging to the curve as given by the equation

$U, =(x, y, z)^m, =0$, and thus obtain the form given in the memoir, and the process would I can well imagine be a more simple one, but it would certainly be very complicated: as an illustration take the simple case of an inflexion: the condition for this, for the equation $y-f(x)=0$ of the curve is $a=0$, that is $\frac{d^2y}{dx^2}=0$. Passing first to the form $U, =(x, y, 1)^m=0$, we have

$$\frac{dU}{dx}+\frac{dU}{dy}\frac{dy}{dx}=0,$$

and thence

$$\frac{d^2U}{dx^2}+2\frac{d^2U}{dxdy}\frac{dy}{dx}+\frac{d^2U}{dy^2}\left(\frac{dy}{dx}\right)^2+\frac{dU}{dy}\frac{d^2y}{dx^2}=0,$$

viz. substituting for $\frac{dy}{dx}$ its value from the first equation the condition $\frac{d^2y}{dx^2}=0$, becomes

$$\left(\frac{dU}{dy}\right)^2\frac{d^2U}{dx^2}-2\frac{dU}{dy}\frac{dU}{dx}\frac{d^2U}{dxdy}+\left(\frac{dU}{dx}\right)^2\frac{d^2U}{dy^2}=0,$$

and we can then make the further transformation to the form $U=(x, y, z)^m, =0$, and so obtain but not *very* easily the result $H(U)=0$: but in the transformations for the sextactic point, besides the differential coefficient a of the second order we have the coefficients b, c, d of the orders 3, 4 and 5 respectively; and the complication is thus very much greater.

343, 354, 374. The principal paper is 374; 354 is a mere résumé of this; and 343 relates to the higher singularity which first presented itself, and which is there shown to arise from the coalescence of a node and a cusp, but in 374 (where it is considered more fully) it is shown to be equivalent to a node, a cusp, a double tangent and an inflexion.

On the general subject, and founded on 374, we have

Smith, H. J. S., "On the Higher Singularities of Plane Curves," *Proc. Lond. Math. Soc.* vol. VI. (1875), pp. 153—182. The author refers to the two following enquiries:

(1) It is important to prove that the indices of singularity as defined by Professor Cayley satisfy the equations of Plücker; and that the "genus" or "deficiency" of the plane curve is correctly given by these indices.

(2) It is also of interest to examine whether any given singularity can be actually formed by the coalescence of the ordinary singularities to which it is regarded as equivalent: in other words whether a singularity of which the indices are δ, τ, κ, i and which is therefore regarded as equivalent to δ double points, τ double tangents, κ cusps and i inflexions possesses a penultimate form in which all these singularities exist distinct from one another but infinitely close together.

The paper relates chiefly to the first of these enquiries, the second being reserved for a further communication which was never made.

See also Halphen's "Étude sur lès points singuliers des courbes algébriques planes," published as an Appendix, pp. 537—648, to the translation of Salmon's *Higher*

Plane Curves, "Traité de Géométrie Analytique," par G. Salmon traduit par O. Chemin, 8vo. Paris, 1884, and the list of Memoirs given, p. 538.

347. I attach some importance to this short paper as giving my own general views of the subject to which it relates, and in particular as to the line of separation between finite and transcendental analysis.

378. I have printed this Report as it was in some measure in connexion therewith that the Royal Society of London undertook the very important work, their Catalogue of Scientific Papers. I do not remember by whom the Report was drafted but some of the recommendations contained in it are due to me. The Catalogue is on a more extensive plan than that recommended in the Report, inasmuch as it is not limited to Physics and Mathematics but extends to all branches of Natural Knowledge—but it is interesting to compare the extent of it with the estimate in the Report—vols. I. to VI. (1800 to 1863) contain together 5743 pages: vols. VII. and VIII. (1864 to 1873) contain together 2357 pages—the number of entries on a page is about $=30$; and we thus have, 1800 to 1863, about 173,000 entries, and 1864 to 1873, about 71,000 entries.

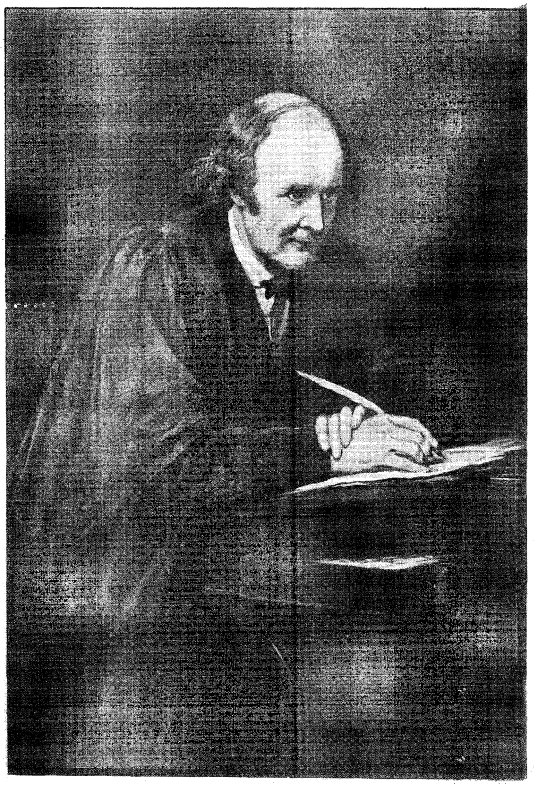
END OF VOL. V.

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.



THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. VI.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1893

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 33 papers numbered 384 to 416 published for the most part in the years 1865 to 1872; the last paper 416, of the year 1882, is inserted in the present volume on account of its immediate connexion with the papers 411 and 415 on Reciprocal Surfaces.

The Table for the six volumes is

Vol.	I.	Numbers	1	to	100.
„	II.	„	101	„	158.
„	III.	„	159	„	222.
„	IV.	„	223	„	299.
„	V.	„	300	„	383.
„	VI.	„	384	„	416.

CONTENTS.

PAGE

384. *On the Transformation of Plane Curves* 1
Proc. London Math. Society, t. I. (1865—66), No. III. pp. 1—11

385. *On the Correspondence of Two Points on a Curve* . . . 9
Proc. London Math. Society, t. I. (1865—66), No. VII. pp. 1—7

386. *On the Logarithms of Imaginary Quantities* 14
Proc. London Math. Society, t. II. (1866—69), pp. 50—54

387. *Notices of Communications to the London Mathematical Society* . 19
Proc. London Math. Society, t. II. (1866—69), pp. 6—7, 25—26, 29, 61—63, 103—104, 123—125

388. *Note on the Composition of Infinitesimal Rotations* . . . 24
Quart. Math. Jour. t. VIII. (1867), pp. 7—10

389. *On a Locus derived from Two Conics* 27
Quart. Math. Jour. t. VIII. (1867), pp. 77—84

390. *Theorem relating to the four Conics which touch the same two lines and pass through the same four points* . . 35
Quart. Math. Jour. t. VIII. (1867), pp. 162—167

391. *Solution of a Problem of Elimination* 40
Quart. Math. Jour. t. VIII. (1867), pp. 183—185

392. *On the Conics which pass through two given Points and touch two given Lines* 43
Quart. Math. Jour. t. VIII. (1867), pp. 211—219

PAGE

393. *On the Conics which touch three given Lines and pass through a given Point* 51
Quart. Math. Jour. t. VIII. (1867), pp. 220—222

394. *On a Locus in relation to the Triangle* 53
Quart. Math. Jour. t. VIII. (1867), pp. 264—277

395. *Investigations in connexion with Casey's Equation* . . . 65
Quart. Math. Jour. t. VIII. (1867), pp. 334—341

396. *On a certain Envelope depending on a Triangle inscribed in a Circle* 72
Quart. Math. Jour. t. IX. (1868), pp. 31—41 and 175—176

397. *Specimen Table* $M \equiv a^a b^\beta$ *(Mod. N) for any prime or composite Modulus* 83
Quart. Math. Jour. t. IX. (1868), pp. 95—96 and plate

398. *On a certain Sextic Developable and Sextic Surface connected therewith* 87
Quart. Math. Jour. t. IX. (1868), pp. 129—142 and 373—376

399. *On the Cubical Divergent Parabolas* 101
Quart. Math. Jour. t. IX. (1868), pp. 185—189

400. *On the Cubic Curves inscribed in a given Pencil of Six Lines* 105
Quart. Math. Jour. t. IX. (1868), pp. 210—221

401. *A Notation of the Points and Lines in Pascal's Theorem* . 116
Quart. Math. Jour. t. IX. (1868), pp. 268—274

402. *On a Singularity of Surfaces* 123
Quart. Math. Jour. t. IX. (1868), pp. 332—338

403. *On Pascal's Theorem* 129
Quart. Math. Jour. t. IX. (1868), pp. 348—353

404. *Reproduction of Euler's Memoir of* 1758 *on the Rotation of a Solid Body* 135
Quart. Math. Jour. t. IX. (1868), pp. 361—373

405. *An Eighth Memoir on Quantics* 147
Phil. Trans. t. CLVII. (for 1867), pp. 513—554

PAGE

406. *On the Curves which satisfy given Conditions* 191
Phil. Trans. t. CLVIII. (for 1868), pp. 75—143

407. *Second Memoir on the Curves which satisfy given Conditions; the Principle of Correspondence* 263
Phil. Trans. t. CLVIII. (for 1868), pp. 145—172

408. *Addition to Memoir on the Resultant of a System of two Equations* 292
Phil. Trans. t. CLVIII. (for 1868), pp. 173—180

409. *On the Conditions for the existence of three equal Roots or of two pairs of equal Roots of a Binary Quartic or Quintic* 300
Phil. Trans. t. CLVIII. (for 1868), pp. 577—588

410. *A Third Memoir on Skew Surfaces, otherwise Scrolls* . . 312
Phil. Trans. t. CLIX. (for 1869), pp. 111—126

411. *A Memoir on the Theory of Reciprocal Surfaces* 329
Phil. Trans. t. CLIX. (for 1869), pp. 201—229

412. *A Memoir on Cubic Surfaces* 359
Phil. Trans. t. CLIX. (for 1869), pp. 231—326

413. *A Memoir on Abstract Geometry* 456
Phil. Trans. t. CLX. (for 1870), pp. 51—63

414. *On Polyzomal Curves, otherwise the Curves* $\sqrt{U}+\sqrt{V}+\&c.=0$. 470
Trans. R. Soc. Edinburgh, t. XXV. (for 1868), pp. 1—110

415. *Corrections and Additions to the Memoir on the Theory of Reciprocal Surfaces* 577
Phil. Trans. t. CLXII. (for 1872), pp. 83—87

416. *On the Theory of Reciprocal Surfaces* 582
Addition to Salmon's Analytic Geometry of Three Dimensions, 4th ed. (1882), pp. 592—604

Notes and References 593
Plates . . . . . *to face pp.* 52, 122, 190
Portrait . . . . . *to face Title.*

CLASSIFICATION.

GEOMETRY:

Abstract Geometry, 413
Notices of Communications to London Mathematical Society, 387
Transformation of Plane Curves, 384; Correspondence, 385
Curves satisfying given Conditions, 406, 407
Polyzomal Curves, 414
Reciprocal Surfaces, 411, 415, 416
Cubic Surfaces, 412
Skew Surfaces, 410; Developable, 398
Singularity of Surfaces, 402
Locus and Envelope in regard to a Triangle, 394, 396
Casey's Equation, 395
Conics, 389, 390, 392, 393; Pascal's Theorem, 401, 403
Cubical divergent Parabolas, 399
Cubic Curves in pencil of six lines, 400

ANALYSIS:

Logarithms of Imaginary Quantities, 386
Theory of Numbers, Specimen Table $M \equiv a^{\alpha} b^{\beta}$ (Mod. N), 397
Symmetric Functions, Elimination, &c., 391, 408, 409
Quantics, eighth Memoir, 405

Infinitesimal Rotations, 388
Euler's Memoir of 1758 on Rotation of a Solid Body, 404

384.

ON THE TRANSFORMATION OF PLANE CURVES.

[From the *Proceedings of the London Mathematical Society*, vol. I. (1865—1866), No. III. pp. 1—11. Read Oct. 16, 1865.]

1. THE expression a "double point," or, as I shall for shortness call it, a "dp," is to be throughout understood to include a cusp: thus, if a curve has δ nodes (or double points in the restricted sense of the expression) and κ cusps, it is here regarded as having $\delta+\kappa$ dps.

2. It was remarked by Cramer, in his "Théorie des Lignes Courbes" (1750), that a curve of the order n has at most $\frac{1}{2}(n-1)(n-2)$, $=\frac{1}{2}(n^2-3n)+1$, dps.

3. For several years past it has further been known that a curve such that the coordinates $(x:y:z)$ of any point thereof are as rational and integral functions of the order n of a variable parameter θ, is a curve of the order n having this maximum number $\frac{1}{2}(n-1)(n-2)$ of dps.

4. The converse theorem is also true, viz.: in a curve of the order n, with $\frac{1}{2}(n-1)(n-2)$ dps, the coordinates $(x:y:z)$ of any point are as rational and integral functions of the order n of a variable parameter θ—or, somewhat less precisely, the coordinates are expressible rationally in terms of a parameter θ.

5. The foregoing theorem, as a particular case of Riemann's general theorem, to be presently referred to, dates from the year 1857; but it was first explicitly stated only last year (1864) by Clebsch, in the Paper, "Ueber diejenigen ebenen Curven deren Coordinaten rationale Functionen eines Parameters sind," *Crelle*, t. LXIV. (1864), pp. 43—63.

6. The demonstration is, in fact, very simple; it depends merely on the remark that we may, through the $\frac{1}{2}(n-1)(n-2)$ dps, and through $2n-3$ other points on the given curve of the order n, together $\frac{1}{2}(n^2+n)-2$, $=\frac{1}{2}(n-1)(n+2)-1$, points, draw

a series of curves of the order $n-1$, given by an equation $U+\theta V=0$, containing an arbitrary parameter θ; any such curve intersects the given curve in the dps, each counting as two points, in the $2n-3$ points, and in *one* other point; hence, as there is only one variable point of intersection, the coordinates of this point, viz., the coordinates of an arbitrary point on the given curve, are expressible rationally in terms of the parameter θ. The demonstration may also be effected in a similar manner by means of curves of the order $n-2$.

7. Before going further, it will be convenient to introduce the term "Deficiency," viz., a curve of the order n with $\frac{1}{2}(n-1)(n-2)-D$ dps, is said to have a deficiency $=D$: the foregoing theorem is that for curves with a deficiency $=0$, the coordinates are expressible rationally in terms of a parameter θ. Since in such a curve the different points succeed each other in a certain definite order, viz., in the order obtained by giving to the parameter its different real values from $-\infty$ to ∞, the curve may be termed a *unicursal* curve.

8. Riemann's general theorem, as applied to plane curves, is stated, but not in its complete form, by Schwarz, in the Paper, "De superficiebus in planum explicabilibus primorum septem ordinum," *Crelle*, t. LXIV. (1864), pp. 1—17: to complete the enunciation it is necessary to refer to page 137 of Riemann's own Paper, "Theorie der Abelschen Functionen," *Crelle*, t. LIV. (1857), pp. 115—155, viz., the enunciation will be:

9. For a curve of any order with a given deficiency D, the coordinates may be expressed as follows:

$D=0$, rationally in terms of a parameter θ, or what comes to the same thing, rationally in terms of the parameters (ξ, η), connected by an equation of the form $(1, \xi)(1, \eta)=0$.

$D>0$, rationally in terms of the parameters (ξ, η) connected by an equation of a certain form, viz.:

$D=1$, the equation is $(1, \xi)^2(1, \eta)^2=0$, or (what comes to the same thing) η is the square root of a quartic function of ξ.

$D=2$, the equation is $(1, \xi)^3(1, \eta)^2=0$, or (what comes to the same thing) η is the square root of a sextic function of ξ.

$D>2$, viz.:

D odd, $=2\mu-3$, the equation is $(1, \xi)^\mu(1, \eta)^\mu=0$, and is besides such, that treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-2)^2$ dps.

D even, $=2\mu-2$, the equation is $(1, \xi)^\mu(1, \eta)^\mu=0$, and is besides such, that treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-1)(\mu-3)$ dps.

10. To see more clearly the meaning of this, write $\frac{\xi}{\eta}, \frac{\eta}{\zeta}$, in place of (ξ, η), so that the coordinates $(x : y : z)$ are expressible rationally and homogeneously in terms

of (ξ, η, ζ), connected by an equation of the form $(\zeta, \xi)^\mu (\zeta, \eta)^\mu = 0$. Such an equation, treating therein (ξ, η, ζ) as coordinates, belongs to a curve of the order 2μ, with a μ-tuple point at $(\xi = 0, \zeta = 0)$, a μ-tuple point at $(\eta = 0, \zeta = 0)$, and which has besides $(\mu - 2)^2$ or $(\mu - 1)(\mu - 3)$ dps, according as $D = 2\mu - 3$, or $2\mu - 2$. The coordinates $(x : y : z)$ of a point of the given curve are expressible rationally in terms of the coordinates $(\xi : \eta : \zeta)$ of a point on the new curve; and we may say that the original curve is by means of the equations which give $(x : y : z)$ in terms of $(\xi : \eta : \zeta)$ *transformed* into the new curve.

11. A curve of the order 2μ may have $\frac{1}{2}(2\mu - 1)(2\mu - 2)$, $= 2\mu^2 - 3\mu + 1$ dps; hence in the new curve, observing that the μ-tuple points each count for $\frac{1}{2}(\mu^2 - \mu)$ dps, we have

In the case $D = 2\mu - 3$,

$$\begin{aligned} \text{Deficiency} = {} & 2\mu^2 - 3\mu + 1 \\ & -\mu^2 + \mu \\ & -\mu^2 + 4\mu - 4 \\ \hline = {} & 2\mu - 3, = D \end{aligned}$$

In the case $D = 2\mu - 2$,

$$\begin{aligned} \text{Deficiency} = {} & 2\mu^2 - 3\mu + 1 \\ & -\mu^2 + \mu \\ & -\mu^2 + 4\mu - 3 \\ \hline = {} & 2\mu - 2, = D \end{aligned}$$

Moreover for $D = 0$, the transformed curve is a conic, with 0 dps, and therefore with deficiency $= 0$; in the case $D = 1$, it is a quartic with 2 dps, and therefore deficiency $= 2$; in the case $D = 2$ it is a quintic with a triple point $= 3$, and a double point $= 1$, together 4 dps, and therefore deficiency $= 2$. Hence in every case the new curve has the same deficiency as the original curve.

12. The theorem thus is that the given curve of the order n, with deficiency D, may be rationally transformed into a curve of an order depending only on the deficiency, and having the same deficiency with the given curve, viz.: $D = 0$, the new curve is of the order $2 (= D + 2)$; $D = 1$, it is of the order $4 (= D + 3)$; $D = 2$, it is of the order $5 (= D + 3)$; and $D > 2$, it is for D odd, of the order $D + 3$; and for D even, of the order $D + 2$. It will presently appear that these are not the lowest values which it is possible to give to the order of the new curve. Riemann's object was, not that the order of the transformed curve might be as low as possible, but that the equation in (ξ, η) might be in each of these parameters separately of the lowest possible order; and this he effected by giving to the transformed curve the two μ-tuple points.

13. It is to be noticed that the theorem that for any rational transformation of one curve into another the two curves have the same deficiency is in effect given (as a consequence of Riemann's theory) by Clebsch in the Paper, "Ueber die Singularitäten algebraischer Curven," *Crelle*, t. LXIV., pp. 98—100. I have, by the assistance of a formula communicated to me by Dr Salmon, obtained a direct analytical demonstration of this theorem.

14. I remark that (x, y, z) being connected by an equation, if $(x : y : z)$ are given rationally in terms of $(\xi : \eta : \zeta)$, then it follows that $(\xi : \eta : \zeta)$ are also

expressible rationally in terms of $(x : y : z)$: it is convenient to consider the transformation from this point of view, and I now proceed to the independent development of the theory, as follows:

15. We have a given curve $U=(x, y, z)^n=0$, with deficiency D, which is by the transformation $\xi : \eta : \zeta = P : Q : R$ (where P, Q, R are given functions $(x, y, z)^k$ each of the same order k) transformed into the curve $\Upsilon=(\xi, \eta, \zeta)^\nu=0$. The transformed curve has, as we know, the same deficiency D as the original curve.

16. To find the order of the transformed curve, we must find the number of its intersections with an arbitrary line $a\xi+b\eta+c\zeta=0$. Writing in this equation $\xi : \eta : \zeta = P : Q : R$, we obtain the equation $aP+bQ+cR=0$, and combining therewith the equation $U=0$, the two equations, being of the orders k and n respectively, give kn systems of values of $(x : y : z)$, and to each of these, in virtue of the equations $\xi : \eta : \zeta = P : Q : R$, there corresponds a single set of values of $(\xi : \eta : \zeta)$, and therefore a single point of intersection; hence the number of intersections, that is, the order of the transformed curve, is $=kn$.

17. If, however, the curves $P=0$, $Q=0$, $R=0$, meet in an ordinary point of the curve $U=0$, then it is easy to see there is a reduction $=1$ in the foregoing value; and so if they meet in a dp of the curve $U=0$, then there is a reduction $=2$. More generally if the curves $P=0$, $Q=0$, $R=0$ each pass through the same α dps and β ordinary points of the curve $U=0$, then there is a reduction $=2\alpha+\beta$. In fact the curve $aP+bQ+cR=0$, meets the curve $U=0$, in kn points; but among these are included the α dps, each counting as 2 intersections, and the β points; the number of the remaining intersections is $=kn-2\alpha-\beta$, and the order of the transformed curve is equal to this number.

I assume that we have $k<n$:

18. A curve of the order k may be made to pass through $\frac{1}{2}k(k+3)$ points; it is moreover known that if any three curves, $P=0$, $Q=0$, $R=0$, of the order k each pass through the same $\frac{1}{2}k(k+3)-1$ points, then the three curves have all their intersections common, the equations being, in fact, connected by an identical relation of the form $\alpha P+\beta Q+\gamma R=0$. To make the order of the transformed curve as low as possible, we must make the curves $P=0$, $Q=0$, $R=0$, meet on the curve $U=0$ in as many points as possible, and it appears from the remark just made, that the greatest possible number is $=\frac{1}{2}k(k+3)-2$; in particular, for $k=n-1$, $n-2$, $n-3$, the number of points on the curve $U=0$ will be at most equal to $\frac{1}{2}(n^2+n)-3$, $\frac{1}{2}(n^2-n)-3$, $\frac{1}{2}(n^2-3n)-2$, respectively.

19. Hence, considering the curve $U=0$ with deficiency D, or with $\frac{1}{2}(n^2-3n)-D+1$ dps, *first* if $k=n-1$, we may assume that the transforming curves $P=0$, $Q=0$, $R=0$ of the order $n-1$, each pass

$$\begin{array}{lll} \text{through the} & \frac{1}{2}(n^2-3n)-D+1 & \text{dps,} \\ \text{and through} & 2n+D-4 & \text{other points,} \\ \hline \text{together} & \frac{1}{2}(n^2+n) \qquad -3 & \text{points of the curve } U=0. \end{array}$$

This being so, each of the three curves will meet the curve $U=0$

in the dps, counting as	$n^2-3n-2D+2$	points,
in the	$2n+D-4$	points,
and in	$D+2$	other points,
together	n^2-n	points;

whence the order of the transformed curve is $=D+2$.

20. In precisely the same manner, *secondly*, if $k=n-2$, then we may assume that the transforming curves $P=0$, $Q=0$, $R=0$, of the order $n-2$, each pass

through the	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$n+D-4$	other points,
together	$\frac{1}{2}(n^2-n)\quad-3$	points of the curve $U=0$;

and this being so, each of the three curves will meet the curve $U=0$

in the dps counting as	$n^2-3n-2D+2$	points,
in the	$n+D-4$	points,
and in	$D+2$	other points,
together	n^2-2n	points;

whence the order of the transformed curve is also in this case $=D+2$.

21. I was under the impression that the order of the transformed curve could not be reduced below $D+2$, but it was remarked to me by Dr Clebsch, that in the case $D>2$, the order might be reduced to $D+1$. In fact, considering, *thirdly*, the case $k=n-3$, we see that the transforming curves $P=0$, $Q=0$, $R=0$ of the order $n-3$ may be made to pass

through the	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$D-3$	other points,
together	$\frac{1}{2}(n^2-3n)\quad-2$	points of the curve $U=0$;

and this being so, each of the three curves meets the curve $U=0$,

in the dps counting as	$(n^2-3n)-2D+2$	points,
in the	$D-3$	points,
and in	$D+1$	other points,
together	n^2-3n	points;

whence the order of the transformed curve is in this case $=D+1$.

22. The general theorem thus is that a curve of the order n with deficiency D, can be, by a transformation of the order $n-1$ or $n-2$, transformed into a curve of

the order $D+2$; and if $D>2$, then the given curve can be by a transformation of the order $n-3$ transformed to a curve of the order $D+1$: the transformed curve having in each case the same deficiency D as the original curve.

23. In particular, if $D=1$, a curve of the order n with deficiency 1, or with $\frac{1}{2}(n^2-3n)$ dps, can be transformed into a cubic curve with the same deficiency, that is with 0 dps; or the given curve can be transformed into a cubic. This case is discussed by Clebsch in the Memoir "Ueber diejenigen Curven deren Coordinaten elliptische Functionen eines Parameters sind," *Crelle*, t. LXIV., pp. 210—271. And he has there given in relation to it a theorem which I establish as follows:

24. Using the transformation of the order $n-1$, if besides the $2n+D-4\,(=2n-3)$ points on the given curve $U=0$, we consider another point O on the curve, then we may, through the $\frac{1}{2}(n^2-3n)$ dps, the $2n-3$ points and the point O, draw a series of curves of the order $n-1$, viz., if P_0, Q_0, R_0, are what the functions P, Q, R, become on substituting therein for (x, y, z), the coordinates (x_0, y_0, z_0) of the given point O, then the equation of any such curve will be $aP+bQ+cR=0$, with the relation $aP_0+bQ_0+cR_0$ between the parameters a, b, c; or (what is the same thing) eliminating c, the equation will be $a(PR_0-P_0R)+b(QR_0-Q_0R)=0$, which contains the single arbitrary parameter $a:b$. In the cubic which is the transformation of the given curve we have a point O' corresponding to O and if (ξ_0, η_0, ζ_0) be the coordinates of this point, then corresponding to the series of curves of the order $n-1$, we have a series of lines through the point O' of the cubic, viz., the lines $a\xi+b\eta+c\zeta=0$ with the relation $a\xi_0+b\eta_0+c\zeta_0=0$ between the parameters; or, what is the same thing, we have the series of lines $a(\xi\zeta_0-\zeta\xi_0)+b(\eta\zeta_0-\zeta\eta_0)=0$, containing the same single parameter $a:b$. By determining this parameter, the curves of the order $n-1$, will be the curves of this order through the dps, the $2n-3$ points, and the point O, which touch the given curve $U=0$; and the lines will be the tangents to the cubic from the point O'; as the number of tangents to a cubic from a point on the cubic is $=4$, it is clear that the values of the parameter $a:b$ will be determined by a certain quartic equation; and there will of course be 4 tangent curves of the order $n-1$ corresponding to the 4 tangents to the cubic. Now by Dr Salmon's anharmonic property of the tangents of a cubic, if on the cubic we vary the position of the point O', the absolute invariant $I^3\div J^2$ of the quartic in $(a:b)$ remains unaltered; that is the absolute invariant $I^3\div J^2$ of the quartic which determines the 4 tangent curves of the order $n-1$ is independent of the position of the point O on the given curve $U=0$, and since the tangent curves in question have the same relation to each of the $2n-3$ points and to the point O, it follows that the invariant is also independent of the position of each of the $2n-3$ points; that is, we have the following theorem, viz.:

25. Considering a curve of the order n with deficiency $=1$; we may, through the $\frac{1}{2}(n^2-3n)$ dps, and through any $2n-2$ points on the curve, draw so as to touch the curve, four curves of the order $n-1$; viz., these are given by an equation $aP'+bQ'=0$, where the ratio $a:b$ is determined by a certain quartic equation $(*\between a, b)^4=0$; then *theorem*, the absolute invariant $I^3\div J^2$ of the quartic function, is independent of the

positions of the $2n-2$ points on the curve $U=0$, and it is consequently a function of only the coefficients of the curve $U=0$, being, as is obvious, an absolute invariant of the curve $U=0$.

26. And, moreover, if the curve $U=0$ is by a transformation of the order $n-1$, by means of $2n-3$ points on the curve as above, transformed into a cubic, then the absolute invariant $I^3 \div J^2$ of the quartic equation which determines the tangents to the cubic from any point O' on the cubic (or, what is the same thing, the absolute invariant $S^3 \div T^2$ of the cubic, taken with a proper numerical multiplier) is independent of the positions of the $2n-3$ points on the curve $U=0$, being in fact equal to the above-mentioned absolute invariant of the curve $U=0$. The like results apply to the transformation of the order $n-2$.

27. Suppose now that we have $D>2$, and consider a curve of the order n with the deficiency D, that is with $\frac{1}{2}(n^2-3n)-D+1$ dps, transformed by a transformation of the order $n-3$ into a curve of the order $D+1$ with deficiency D; then, assuming the truth of the subsidiary theorem to be presently mentioned, it may be shown by very similar reasoning to that above employed, that the absolute invariants of the transformed curve of the order $D+1$ (the number of which is $=4D-6$), will be independent of the positions of the $D-3$ points used in the transformation, and will be equal to absolute invariants[1] of the given curve $U=0$.

28. The subsidiary theorem is as follows: consider a curve of the order $D+1$, with deficiency D, that is, with $\frac{1}{2}D(D-1)-D=\frac{1}{2}(D^2-3D)$ dps; the number of tangents to the curve from any point O' on the curve is $=(D+1)D-(D^2-3D)-2$, $=4D-2$, (this assumes however, that the dps are proper dps, not *cusps*,) the pencil of tangents has $4D-5$ absolute invariants, and of these all but one, that is, $4D-6$, absolute invariants of the pencil are independent of the position of the point O' on the curve, and are respectively equal to absolute invariants of the curve.

29. To establish it, I observe that a curve of the order $D+1$ with deficiency D, or with $\frac{1}{2}(D^2-3D)$ dps, contains $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)$, $=4D+2$ arbitrary constants, and it may therefore be made to satisfy $4D+2$ conditions. Now imagine a given pencil of $4D-2$ lines, and let a curve of the form in question be determined so as to pass through the centre of the pencil, and touch each of the $4D-2$ lines; the curve thus satisfies $4D-1$ conditions, and its equation will contain $4D+2-(4D-1)$, $=3$ arbitrary constants. But if we have any particular curve satisfying the $4D-1$ conditions, then by transforming the whole figure homologously, taking the centre of the pencil as pole and any arbitrary line as axis of homology, so as to leave the pencil of lines unaltered (analytically if at the centre of the pencil $x=0$, $y=0$, then by writing $\alpha x+\beta y+yz$ in place of z) the transformed curve still satisfies the $4D-1$ conditions, and we have by the homologous transformation introduced into its equation 3 arbitrary constants, that is, we have obtained the most general curve which satisfies the conditions in question. The absolute invariants of the general curve are independent of the

[1] It is right to notice that the absolute invariants spoken of here, and in what follows, are not in general rational ones.

3 arbitrary constants introduced by the homologous transformation; and they are consequently functions of only the coefficients of the given pencil of $4D-2$ lines; this being so, it is obvious that they will be respectively equal to absolute invariants of the pencil of $4D-2$ lines. The number of the absolute invariants of the general curve of the order $D+1$ is $=\frac{1}{2}(D+1)(D+4)+1-9$, but there is a reduction $=1$, for each of the dps, hence in the present case the number is $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)-8$, $=4D-6$; and there are thus $4D-6$ absolute invariants of the curve, each of them equal to an absolute invariant of the pencil; that is, of the $4D-5$ absolute invariants of the pencil, there are $4D-6$, each of them equal to an absolute invariant of the curve, and consequently independent of the position of the point O' on the curve; which is the theorem which was to be proved. I believe the reasoning is quite correct, but there are some points in it which require further examination, it is therefore given subject to any correction which may hereafter appear to be necessary.

30. The general subject may be illustrated by considerations belonging to solid geometry. If we imagine the original curve and the transformed curve as situate in different planes, then joining each point of the original curve with the corresponding point on the transformed curve, we have a series of lines forming a scroll (skew surface): if the two curves are of the orders n, n' respectively, then the complete section by the plane of the original curve is made up of this curve of the order n, and of n' generating lines; and similarly the complete section by the plane of the transformed curve is made up of this curve of the order n', and of n generating lines. Conversely, given a scroll of the order $n+n'$, any two sections of this scroll, being in general curves of the same order $n+n'$, are rational transformations the one of the other; but for the *general* scroll of the order $n+n'$, it is not possible to find sections breaking up as above.

385.

ON THE CORRESPONDENCE OF TWO POINTS ON A CURVE.

[From the *Proceedings of the London Mathematical Society*, vol. I. (1865—1866), No. VII. pp. 1—7. Read April 16, 1866.]

1. In a unicursal curve the coordinates (x, y, z) of any point of the curve are proportional to rational and integral functions of a variable parameter θ. Hence, if two points of the curve correspond in suchwise that to a given position of the first point there correspond α' positions of the second point, and to a given position of the second point α positions of the first point, the number of points which correspond each to itself is $=\alpha+\alpha'$. For let the two points be determined by their parameters θ, θ' respectively—then to a given value of θ there correspond α' values of θ', and to a given value of θ' there correspond α values of θ; hence the relation between (θ, θ') is of the form $(\theta, 1)^{\alpha}(\theta', 1)^{\alpha'}=0$; and writing therein $\theta'=\theta$, then for the points which correspond each to itself, we have an equation $(\theta, 1)^{\alpha+\alpha'}=0$ of the order $\alpha+\alpha'$; that is, the number of these points is $=\alpha+\alpha'$.

2. Hence for a unicursal curve we have a theorem similar to that of M. Chasles' for a line, viz., the theorem may be thus stated:

If two points of a unicursal curve have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'$.

3. But a unicursal curve is nothing else than a curve with a deficiency $D=0$, and we thence infer

Theorem. If two points of a curve with deficiency D have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'+2kD$; in which theorem $2k$ is a coefficient to be determined.

4. Suppose that the corresponding points are P, P' and imagine that when P is given, the corresponding points P' are the intersections of the given curve by a

curve Θ (the equation of the curve Θ will of course contain the coordinates of P as parameters, for otherwise the position of P' would not depend upon that of P). I find that if the curve Θ has with the given curve k intersections at the point P, then in the system of (P, P'), the number of united points is

$$\mathrm{a} = \alpha + \alpha' + 2kD,$$

whence in particular, if the curve Θ does not pass through the point P, then the number of united points is $= \alpha + \alpha'$, as in a unicursal curve.

4*. The foregoing theorem is easily proved in the particular case where the k intersections at the point P take place in consequence of the curve θ having a k-tuple point at P. Taking $U = (x, y, z)^m = 0$ as the equation of the given curve (which for greater simplicity is assumed to be a curve without singularities), then if we suppose (x, y, z) to be the coordinates of the point P, and (x', y', z') to be the coordinates of the point P', write $U = (x, y, z)^m$, $U' = (x', y', z')^m$, U' being what U becomes on writing therein (x', y', z') in place of (x, y, z); and

$$\Theta = (x, y, z)^a (x', y', z')^{a'} (yz' - y'z, zx' - z'x, xy' - x'y)^k,$$

viz., Θ is a function of the order k in $yz' - yz'$, $zx' - z'x$, $x'y - x'y$, the coefficients of the several powers and products of these quantities being functions of the order a in (x, y, z) and of the order a' in (x', y', z'), which functions are such that they do not all of them vanish, identically, or in virtue of the equation $U = 0$, on writing therein $(x', y', z') = (x, y, z)$. Taking for a moment (x, y, z) as current coordinates, suppose that the equation of the given curve is $U = 0$; then if (x, y, z) are the coordinates of the point P, we have $U = 0$, and similarly if (x', y', z') are the coordinates of the point P' we have $U' = 0$. The equation $\Theta = 0$, considering therein (x, y, z) as the coordinates of the given point P (and so as parameters satisfying the equation $U = 0$) and (x', y', z') as current coordinates, will be a curve having a k-tuple point at P, we have thus the case above supposed; and P being given, the corresponding points P' are given as the intersections of the curves $U' = 0$, $\Theta = 0$, which are respectively of the orders m and $a' + k$; the total number of intersections is thus $= m(a' + k)$, but inasmuch as the curve $\Theta = 0$ has a k-tuple point at P, k of these intersections coincide with the point P, and the number of the remaining intersections, that is the number of positions of the point P', is $= ma' + (m-1)k$. Similarly when P' is given, the number of positions of the point P is $= ma + (m-1)k$: and we have therefore

$$\alpha + \alpha' = m(a + a') + 2(m-1)k.$$

To find the united points, it is to be observed, that upon writing $(x', y', z') = (x, y, z)$, the function Θ becomes identically $= 0$; but if we suppose, in the first instance, that P', P, are consecutive points on the curve $U = 0$, then we have

$$yz' - y'z : zx' - z'x : xy' - x'y = \delta_x U;\ \delta_y U;\ \delta_z U;$$

and the equation $\Theta = 0$ assumes the form

$$\Theta = (x, y, z)^a (x, y, z)^{a'} (\delta_x U, \delta_y U, \delta_z U)^k = 0,$$

which, (x, y, z) being current coordinates, is the equation of a curve of the order $\alpha+\alpha'+(m-1)k$; the united points are the intersection of this curve with the given curve $U=0$, and the number of the united points is thus

$$\mathrm{a}=m(\alpha+\alpha')+m(m-1)k.$$

Hence attending to the above-mentioned value of $\alpha+\alpha'$, we have

$$\mathrm{a}=\alpha+\alpha'+(m-1)(m-2)k.$$

But in the case of a curve $U=0$, without singularities, we have $2D=(m-1)(m-2)$, and we have thus the required formula

$$\mathrm{a}=\alpha+\alpha'+2kD.$$

The investigation in the case where the k intersections at P arise wholly or in part from a contact of the curve Θ, or any branch or branches thereof, with the given curve U, is more difficult, and I abstain from entering upon it.

I apply the theorem to some examples:

5. Investigation of the class of a curve of the order m with δ dps. Take as corresponding points on the curve two points, such that the line joining them passes through a fixed point O: the united points will be the points of contact of the tangents through O; that is, the number of the united points is equal to the class of the curve. The curve Θ is here the line OP, which has with the given curve a single intersection at P; that is, we have $k=1$. The points P' corresponding to a given position of P are the remaining $m-1$ intersections of OP with the curve; that is, we have $\alpha'=m-1$; and in like manner $\alpha=m-1$. Hence the class is $=2(m-1)+2D$, viz., this is $=(2m-2)+(m^2-3m+2-2\delta)$, which is $=m^2-m-2\delta$, as it should be.

6. It is in the foregoing example assumed that the dps are none of them cusps; if the curve has $\delta+\kappa$ dps, κ of which are cusps (or what is the same thing, δ nodes and κ cusps); then the number of united points is equal $2(m-1)+2D, =m^2-m-2\delta-2\kappa$; but in this case each of the cusps is reckoned as a united point, and we have, therefore, class $+\kappa=m^2-m-2\delta-2\kappa$, that is, class $=m^2-m-2\delta-3\kappa$. This will serve as an instance of the special considerations which are required in the case of a curve with cusps, but in what follows, I shall assume that the dps are none of them cusps and thus attend to the case of a curve of the order m, with δ dps, and therefore of the class $n=m^2-m-2\delta$, and of the deficiency $D=\frac{1}{2}(m-1)(m-2)-\delta, =\frac{1}{2}(n-2m+2)$.

7. Investigation of the number of inflexions. Taking the point P' to be a tangential of P (that is, an intersection of the curve by the tangent at P), the united points are the inflexions, and the number of the united points is equal to the number of inflexions. The curve Θ is here the tangent at P, having with the given curve two intersections at P; that is $k=2$. P' is any one of the $m-2$ tangentials of P, hence $\alpha'=m-2$; and P is the point of contact of any one of the $n-2$

tangents from P' to the curve, that is, $\alpha = n-2$. Hence the number of inflexions is $=(m-2)+(n-2)+4D$, $=m+n-4+2(n-2m+2)$, $=3(n-m)$, which is right.

8. For the purpose of the next example it is necessary to present the fundamental equation $\mathrm{a}=\alpha+\alpha'+2kD$ under a more general form. The curve Θ may intersect the given curve in a system of points P' each p times, a system of points Q' each q times, &c., in such manner that the points (P, P'), the points (P, Q'), &c., are pairs of points corresponding to each other according to distinct laws; and we shall then have the numbers $(\mathrm{a}, \alpha, \alpha')$, $(\mathrm{b}, \beta, \beta')$, &c., belonging to these pairs respectively; viz. (P, P') are points having an (α, α') correspondence, and the number of united points is $=\mathrm{a}$; similarly (P, Q') are points having a (β, β') correspondence, and the number of united points is $=\mathrm{b}$; and so on. The theorem then is

$$p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\ldots=2kD.$$

9. Investigation of the number of double tangents:—Take P', an intersection with the curve of a tangent drawn from P to the curve (or what is the same thing, P, P' cotangentials of any point of the curve); the united points are here the points of contact of the several double tangents of the curve; or if τ be the number of double tangents, then the number of united points is $=2\tau$. The curve Θ is the system of the $n-2$ tangents from P to the curve; each tangent has with the curve 1 intersection at P, that is, $k=n-2$; each tangent, besides, meets the curve in the point of contact Q' twice, and in $(m-3)$ points P'. Hence, if $(\mathrm{a}, \alpha, \alpha')$ refer to the points (P, Q'), and $(2\tau, \beta, \beta')$ to the points (P, P'), we have

$$2(\mathrm{a}-\alpha-\alpha')+2\tau-\beta-\beta'=2(n-2)D.$$

Moreover, from the last example the value of $\mathrm{a}-\alpha-\alpha'$ is $=4D$, and the formula thus becomes

$$2\tau-\beta-\beta'=2(n-6)D;$$

but from above it appears that we have $\beta=\beta'=(n-2)(m-3)$, whence

$$\begin{aligned}2\tau &= 2(n-2)(m-3)+2(n-6)D,\\ &= 2(n-2)(m-3)+(n-6)(n-2m+2),\\ &= n^2-10n+8m,\end{aligned}$$

which is right; in fact, observing that ι (the number of inflexions) is $=3n-3m$, the formula is equivalent to $2\tau+3\iota=n^2-n-m$, that is, $m=n^2-n-2\tau-3\iota$.

In the foregoing examples the curve Θ is a line or system of lines; but I give an example in which Θ is a system of conics, and in which, as will appear, we have to consider the two characteristics (μ, ν) of the system.

10. Investigation of the number of conics which can be drawn, satisfying any four conditions, and touching the given curve; or say of the number of the conics $(4Z)$ (1). Take P', an intersection of the given curve by a conic $(4Z)$ passing through the point P, then the number of the united points is equal to that of the conics $(4Z)$ (1). The curve Θ is here the system of the conics $(4Z)$ which pass through P;

hence, if (μ, ν) be the characteristics of the system of conics $(4Z)$, the number of the conics through P is $=\mu$; each of these has with the given curve 1 intersection at P, and consequently $k=\mu$. Moreover, each of the conics besides meets the curve in $(2m-1)$ points, and consequently $\alpha=\alpha'=\mu(2m-1)$. Hence the formula gives the number of united points

$$=2\mu(2m-1)+\mu(n-2m+2),$$
$$=\mu(n+2m);$$

or, as this may be written,

$$=\mu n+\nu m+m(2\mu-\nu).$$

But the system of conics $(4Z)$ contains $(2\mu-\nu)$ point-pairs (*coniques infiniment aplaties*), each of which, regarded as a pair of coincident lines, meets the given curve in m pairs of coincident points; that is, the point-pair is to be considered as a conic *touching* the given curve in m points; and there is on this account a reduction $=m(2\mu-\nu)$ in the number of the united points; whence, finally, the number of the conics $(4Z)$ (1) is $=\mu n+\nu m$. It is hardly necessary to remark that it is assumed that the conditions $(4Z)$ are conditions having no special relation to the given curve.

11. As a final example, suppose that the point P on a given curve of the order m, and the point Q on a given curve of the order m', have an (α, α') correspondence, and let it be required to find the class of the curve enveloped by the line PQ. Take an arbitrary point O, join OQ, and let this meet the curve m in P', then (P, P') are points on the curve m, having a $(m'\alpha, m\alpha')$ correspondence; in fact, to a given position of P there correspond α' positions of Q, and to each of these m positions of P', that is, to each position of P there correspond $m\alpha'$ positions of P'; and similarly to each position of P' there correspond $m'\alpha$ positions of P. The curve Θ is the system of the lines drawn from each of the α' positions of Q to the point O, hence the curve Θ does not pass through P, and we have $k=0$. Hence the number of the united points (P, P'), that is, the number of the lines PQ which pass through the point O, is $=m\alpha'+m'\alpha$, or this is the class of the curve enveloped by PQ.

12. It may be remarked, that if the two curves are curves in space (plane or of double curvature), then the like reasoning shows that the number of the lines PQ which meet a given line O is $=m\alpha'+m'\alpha$, that is, the order of the scroll generated by the line PQ is $=m\alpha'+m'\alpha$.

386.

ON THE LOGARITHMS OF IMAGINARY QUANTITIES.

[From the *Proceedings of the London Mathematical Society*, vol. II. (1866—1869), pp. 50—54. Read Dec. 12, 1867.]

THE theory of the logarithms of imaginary quantities admits of a remarkably simple representation.

Let P denote at pleasure the imaginary quantity $x+iy$, or else the point the coordinates of which are (x, y); viz., P regarded as a quantity will denote $x+iy$; but we may also speak of the point P.

Writing thus

$$P = x + iy,$$

and similarly

$$P' = x' + iy',$$

we have of course

$$\frac{P}{P'} = \frac{x+iy}{x'+iy'},$$

an imaginary quantity $X+iY$; and the point $\frac{P}{P'}$ will be the point the coordinates of which are (X, Y).

We have

$$P = re^{i\theta},$$

viz., r is $=\sqrt{x^2+y^2}$, the radical being positive, and θ is an arc such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}},$$

and moreover θ may be taken to be an arc between the limits $-\pi$, $+\pi$. The arc so defined may be denoted by $\tan^{-1}\frac{y}{x}$, so that we have $\theta = \tan^{-1}\frac{y}{x}$.

It is to be observed that θ has always a determinate unique value, except in the single case $y=0$, x negative, where we have indeterminately $\theta=\pm\pi$.

It is further to be remarked that, taking A for the origin of coordinates, we have $\theta=$ angle xAP, considered as positive or as negative according as P lies above or below the axis of x.

Starting from the equation

$$P=re^{i\theta},$$

we have similarly

$$P'=r'e^{i\theta'},$$

and

$$\frac{P}{P'}=\frac{r}{r'}e^{i\phi},$$

where ϕ is derived from $\frac{P}{P'}$ in the same way as θ from P, or θ' from P'.

Consequently

$$e^{i(\theta-\theta'-\phi)}=1,$$

and therefore $\theta-\theta'-\phi$ a multiple of 2π, say

$$\theta-\theta'-\phi=2m\pi,$$

and in this equation the value of m is determined by the limiting conditions above imposed on the values of θ, θ', ϕ. To see how this is, suppose in the first instance that the finite line or chord $P'P$, considered as drawn from P' to P, cuts the negative part of the axis of x upwards; P is then above, P' below, the axis of x; that is, θ, $-\theta'$ are each positive; and drawing the figure, it at once appears that the sum $\theta+(-\theta')$, that is $\theta-\theta'$, is a positive quantity greater than π. And in this case the angle ϕ will be equal to $2\pi-(\theta-\theta')$ taken negatively, that is, $\phi=-\{2\pi-(\theta-\theta')\}$, or $\theta-\theta'-\phi=2\pi$. But, in like manner, if $P'P$ cut the negative part of the axis of x downwards, P will be below, P' above, the axis of x; $-\theta$ and θ' are here each positive, and the figure shows that the sum $-\theta+\theta'$ is greater than π; and in this case the angle ϕ is $=2\pi-(-\theta+\theta')$; that is, we have $\theta-\theta'-\phi=-2\pi$. In every other case, (that is, if the chord $P'P$ either does not meet the axis of x, or if it meets the positive part of the axis of x,) $\theta-\theta'$ and ϕ are each in absolute magnitude less than π, and we have $\theta-\theta'-\phi=0$. So that we see that, according as the chord $P'P$, considered as drawn from P' to P, meets the negative part of the axis of x upwards or downwards, or as it does not meet the negative part of the axis of x, the value of $\theta-\theta'-\phi$ is $=2\pi$, $=-2\pi$, or $=0$.

Taking now $\log r$ to represent the real logarithm of the positive real quantity r, we may, as a definition of the logarithm of the imaginary quantity $P\,(=x+iy)$, write

$$\log P=\log r+i\theta.$$

The value so defined is of course one out of the infinite number of values of the logarithm, and it may for distinction be termed the "selected" value. In all that follows, the symbol "log" is to be understood to denote the selected value. We have

$$\log P=\log r+i\theta,$$

and similarly

$$\log P' = \log r' + i\theta',$$

and

$$\log \frac{P}{P'} = \log \frac{r}{r'} + i\phi.$$

Hence

$$\log P - \log P' = \log \frac{P}{P'} + i(\theta - \theta' - \phi),$$

so that, by what precedes, $\log P - \log P'$, if the chord $P'P$, considered as drawn from P' to P, cuts the negative part of the axis of x upwards, is $= \log \frac{P}{P'} + 2i\pi$; if the chord cuts the negative part of the axis of x downwards, it is $= \log \frac{P}{P'} - 2i\pi$, and in every other case it is $= \log \frac{P}{P'}$.

It is to be remarked that $\log P$, as above defined, is a continuous function of $P(=x+iy)$, with the single exception that, if the point P move from above to below or from below to above the negative part of the axis of x, the imaginary part of the logarithm changes from $+i\pi$ to $-i\pi$, or from $-i\pi$ to $+i\pi$, in the two cases respectively. And we are thus led to another mode of looking at the question.

Consider the integral

$$\int_{P'}^{P} \frac{dz}{z}.$$

The value of the integral may depend on the series of values assumed by the variable z as it passes from the limit $z = P'$ to the limit $z = P$, or say it may depend on the path of the variable z; in order to give the notation a precise signification, we must therefore fix the path of the variable z; and I do this by taking the path to be the right line $P'P$. Write now $z = P'.u$, we have $\frac{dz}{z} = \frac{du}{u}$; $z = P'$ gives $u = 1$; $z = P$ gives $u = \frac{P}{P'}$; and it is easy to see that, the path of z being along the right line P' to P, that of u is along the right line 1 to $\frac{P}{P'}$ $\left(\text{that is, from the point the coordinates whereof are } x = 1,\ y = 0, \text{ to the point } \frac{P}{P'}\right)$.

We have thus

$$\int_{P'}^{P} \frac{dz}{z} = \int_{1}^{P \div P'} \frac{du}{u},$$

the path in each case being a right line as above. The indefinite integral $\int \frac{du}{u} = \log u$; and as u passes from 1 to $\frac{P}{P'}$, there is no discontinuity in the value of $\log u$; the

value of the right-hand side is thus $=\log\frac{P}{P'}$. As regards the left-hand side, the indefinite integral is in like manner $=\log z$; but here, if the chord $P'P$ cuts the negative part of the axis of x, there is a discontinuity in the value of $\log z$, viz., if the chord $P'P$, considered as drawn from P' to P, cuts the negative part of the axis of x upwards, there is an abrupt change in the value of $\log z$ from $-i\pi$ to $+i\pi$; and similarly, if the chord cut the negative part of the axis of x downwards, there is an abrupt change from $+i\pi$ to $-i\pi$; in the former case, by taking the definite integral to be $\log P-\log P'$, we take its value too large by $2i\pi$, in the latter case we take it too small by $2i\pi$; that is, the true value of the definite integral is in the former case $=\log P-\log P'-2i\pi$, in the latter case it is $=\log P-\log P'+2i\pi$. But if the chord PP' does not cut the negative part of the axis of x, then there is not any discontinuity, and the true value of the definite integral is $=\log P-\log P'$. We have thus in the three cases respectively

$$\begin{aligned}\log P-\log P' &= \log\frac{P}{P'}+2i\pi,\\ &= \log\frac{P}{P'}-2i\pi,\\ &= \log\frac{P}{P'},\end{aligned}$$

which agrees with the previous results.

It may be remarked, that it is merely in consequence of the particular definition adopted that there is in the value of $\log P$ a discontinuity at the passage over the negative part of the axis of x; with a different definition of the logarithm, there would be a discontinuity at the passage over some other line from the origin; but a discontinuity somewhere there must be. For if, as above, the chord $P'P$ meet the negative part of the axis of x, then forming a closed quadrilateral by joining by right lines the points 1 to P, P to P', P' to $\frac{P}{P'}$, and $\frac{P}{P'}$ to 1; the only side meeting the negative part of the axis of x is the side $P'P$; the integral $\int\frac{dz}{z}$, taken through the closed circuit in question, or say the integral

$$\left(\int_P^1+\int_{P'}^P+\int_{P\div P'}^{P'}+\int_1^{P\div P'}\right)\frac{dz}{z}$$

has, by what precedes, a value in consequence of the discontinuity in passing from P' to P; viz., this is $=-2i\pi$ or $=2i\pi$, according as the chord $P'P$, considered as drawn from P' to P, cuts the negative part of the axis of x upwards or downwards; but this value $-2i\pi$ or $+2i\pi$ must be altogether independent of the definition of the logarithm; whereas if, by any alteration in the definition, the discontinuity could be avoided, the value of the integral, instead of being as above, would be $=0$. The foregoing value $-2i\pi$ or $+2i\pi$ is in fact that of the integral taken along (in the one

or the other direction) any closed curve surrounding the point $z=0$ for which the function $\frac{1}{z}$ under the integral sign becomes infinite: but in obtaining the value as above, no use is made of the principles relating to the integration of functions which thus become infinite.

The equation

$$\log P = \log r + i\theta$$

gives

$$P^m = e^{m \log P} = r^m e^{im\theta},$$

or say

$$(x+iy)^m = r^m e^{im\theta},$$

where, m being any real quantity whatever, r^m denotes the positive real value of r^m. We have thus a definition of the value of $(x+iy)^m$, and the value so defined may be called the selected value. And similarly, for an imaginary exponent $m=p+qi$, we have

$$\begin{aligned}(x+iy)^{p+qi} &= e^{(p+qi)(\log r+i\theta)} \\ &= e^{p\log r - q\theta + i(p\theta + q\log r)} \\ &= r^p e^{-q\theta} \cdot e^{i(p\theta+q\log r)},\end{aligned}$$

which is the selected value of $(x+iy)^{p+qi}$.

It may be remarked, in illustration of the advantage (or rather the necessity) of having a selected value, that in an integral $\int Zdz$, taken between given limits along a given path, it is necessary that we know, for the real or imaginary value of z corresponding to each point of the path, the value of the function Z; and consequently, if Z is a function involving $\log z$ or z^m, the indeterminateness which presents itself in these symbols (considered as belonging to a single value of z) is, so to speak, indefinitely multiplied, and $\int Zdz$ is really an unmeaning combination of symbols, unless by selecting, as above or otherwise, a unique value of $\log z$ or z^m, we render the function to be integrated a determinate function of the variable.

387.

NOTICES OF COMMUNICATIONS TO THE LONDON MATHEMATICAL SOCIETY.

[From the *Proceedings of the London Mathematical Society*, vol. II. (1866—1869), pp. 6—7, 25—26, 29, 61—63, 103—104, 123—125.]

December 13, 1866. pp. 6—7.

PROF. CAYLEY exhibited and explained some geometrical drawings. Thinking that the information might be convenient for persons wishing to make similar drawings, he noticed that the paper used was a tinted drawing paper, made in continuous lengths up to 24 yards, and of the breadth of about 56 inches[1]; the half-breadth being therefore sufficient for ordinary figures, and the paper being of a good quality and taking colour very readily. Among the drawings was one of the conics through four points forming a convex quadrangle. The plane is here divided into regions by the lines joining each of the six pairs of points, and by the two parabolas through the four points; and the regions being distinguished by different colours, the general form of the conics of the system is very clearly seen. (Prof. Cayley remarked that it would be interesting to make the figures of other systems of conics satisfying four conditions; and in particular for the remaining elementary systems of conics, where the conics pass through a number 3, 2, 1 or 0 of points and touch a number 1, 2, 3 or 4 of lines: the construction of some of these figures is, however, practically a great deal more difficult.) Other figures related to Cartesians and Bicircular Quartics. One of these was a figure of a system of triconfocal Cartesians; and derived from this by inversion in regard to a circle, there was a figure of a system of quadriconfocal bicircular quartics: in the assumed position of the inverting circle, each quartic consists (like the Cartesian which gives rise to it) of an exterior and an interior continuous curve, and the general aspect of the figure is that of a distortion of the original figure of the Cartesians. Another figure was that of the bicircular quartic, for which the

[1] Sold at Messrs Lechertier-Barbe's, Regent Street, at 6*d*. per yard, or 9*s*. the piece.

algebraical sum of the distances of a point thereof from three given foci is $= 0$ (this was selected for facility of construction, by the intersections of circles and confocal conics). The quartic consists of two equal and symmetrically situated pear-shaped curves, exterior to each other, and including the one of them two of the three given foci, the other of them the third given focus, and a fourth focus lying in a circle with the given foci: by inversion in regard to a circle having its centre at a focus the two pear-shaped curves became respectively the exterior and the interior ovals of a Cartesian. There was also a figure of the two circular cubics, having for foci four given points on a circle; and a figure (coloured in regions) in preparation for the construction of the analogous sextic curve derived from four given points not in a circle.

March 28, 1867. pp. 25—26.

Professor Cayley mentioned a theorem included in Prof. Sylvester's theory of derivation of the points of a cubic curve. Writing down the series of numbers 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, &c., viz., all the numbers not divisible by 3, then (repetitions of the same number being permissible) taking any two numbers of the series, we have in the series a third number, which is the sum or else the difference of the two numbers (for example, 2, 2 give their sum 4, but 2, 7 give their difference 5), and we have thus a series of triads, in each of which one number is the sum of the other two. The theorem is, that it is possible on a cubic curve to construct a series of points, such that denoting them by the above numbers respectively, then for any triad of numbers as aforesaid the points denoted by the three numbers respectively lie *in lineâ*. And the theorem gives its own construction: in fact the series of triads is 112, 224, 145, 257, 178, 248, &c. Take 1, an arbitrary point on the cubic, then (by the theorem) the triad 112 shows that 2 is the tangential of 1; 224 shows that 4 is the tangential of 2; 145 that 5 is the third point of 1 and 4; 257 that 7 is the third point of 2 and 5. So far we have no theorem; we have merely, starting from the point 1, constructed by an arbitrary process the points 2, 4, 5, and 7. But going a step further; 178 and 448 show, the first of them, that 8 is the third point of 1 and 7, the second of them, that 8 is the tangential of 4. We have here the theorem that the third point of 1 and 7 is also the tangential of 4. Similarly, 10, 11, 13 are each of them (like 8) determined by two constructions; 14, 16, 17, 19, each of them by three constructions, and so on; the number of constructions increasing by unity for each group of four numbers. And the theorem is, that these constructions, 2, 3, or more, as the case may be, give always one and the same point. Prof. Cayley mentioned that on a large figure of a cubic curve he had, in accordance with the theorem, constructed the series of points 1, 2, 4, 5, 7, 8, 10, 11, 13, 14.

April 15, 1867. p. 29.

Prof. Cayley communicated a theorem relating to the locus of the ninth of the points of intersection of two cubics, seven of these points being fixed, while the eighth moves on a straight line.

March 26, 1868. pp. 61—63.

Prof. Cayley made some remarks on a mode of generation of a sibi-reciprocal surface, that is, a surface the reciprocal of which is of the same order and has the same singularities as the original surface.

If a surface be considered as the envelope of a *plane* varying according to given conditions, this is a mode of generation which is essentially not sibi-reciprocal; the reciprocal surface is given as the locus of a *point* varying according to the reciprocal conditions. But if a surface be considered as the envelope of a *quadric surface* varying according to given conditions, then the reciprocal surface is given as the envelope of a *quadric surface* varying according to the reciprocal conditions; and if the conditions be sibi-reciprocal, it follows that the surface is a sibi-reciprocal surface. For instance, considering the surface which is the envelope of a quadric surface touching each of 8 given lines; the reciprocal surface is here the envelope of a quadric surface touching each of 8 given lines; that is, the surface is sibi-reciprocal. So again, when a quadric surface is subjected to the condition that 4 given points shall be in regard thereto a conjugate system, this is equivalent to the condition that 4 given planes shall be in regard thereto a conjugate system—or the condition is sibi-reciprocal; analytically the quadric surface $ax^2+by^2+cz^2+dw^2=0$ is a quadric surface subjected to a sibi-reciprocal system of six conditions. Impose on the quadric surface two more sibi-reciprocal conditions,—for instance, that it shall pass through a given point and touch a given plane,—the envelope of the quadric will be a sibi-reciprocal surface. It was noticed that in this case the envelope was a surface of the order (=class) 12, and having (besides other singularities) the singularities of a conical point with a tangent cone of the class 3, and of a curve of plane contact of the order 3. In the foregoing instances the number of conditions imposed upon the quadric surface is 8; but it may be 7, or even a smaller number. An instance was given of the case of 7 conditions, viz.,—the quadric surface is taken to be $ax^2+by^2+cz^2+dw^2=0$ (6 conditions) with a relation of the form

$$Abc+Bca+Cab+Fad+Gbd+Hcd=0$$

between the coefficients (1 condition); this last condition is at once seen to be sibi-reciprocal; and the envelope is consequently a sibi-reciprocal surface—viz., it is a surface of the order (=class) 4, with 16 conical points and 16 conics of plane contact. It is the surface called by Prof. Cayley the "tetrahedroid," (see his paper "Sur la surface des ondes," *Liouv.* tom. XI. (1846), pp. 291—296 [47]), being in fact a homographic transformation of Fresnel's Wave Surface.

{Prof. Cayley adds an observation which has since occurred to him. If the quadric surface $ax^2+by^2+cz^2+dw^2=0$, be subjected to touch a given line, this imposes on the coefficients a, b, c, d, a relation of the above form, viz., the relation is

$$A^2bc+B^2ca+C^2ab+F^2ad+G^2bd+H^2cd=0;$$

where A, B, C, F, G, H are the "six coordinates" of the given line, and satisfy therefore the relation $AF+BG+CH=0$. It is easy to see that there are 8 lines for which the squared coordinates have the same values A^2, B^2, C^2, F^2, G^2, H^2; these 8 lines are symmetrically situate in regard to the tetrahedron of coordinates, and

moreover they lie in a hyperboloid. The quadric surface, instead of being defined as above, may, it is clear, be defined by the equivalent conditions of touching each of the 8 given lines: that is, we have the envelope of a quadric surface touching each of 8 given lines; these lines not being arbitrary lines, but being a system of a very special form. By what precedes, the envelope is a quartic surface. It appears, however, that in virtue of the relation $AF+BG+CH=0$, this is no longer a proper quartic surface, but that it resolves itself into the above-mentioned hyperboloid taken twice. That is, restoring the original A, B, &c., in place of A^2, B^2, &c., the envelope of the quadric $ax^2+by^2+cz^2+dw^2=0$, where a, b, c, d vary, subject to the condition $Abc+Bca+Cab+Fad+Gbd+Hcd=0$, (which is in general a tetrahedroid), is when A, B, C, F, G, H are the squared coordinates of a line (or, what is the same thing, when $\sqrt{AF}+\sqrt{BG}+\sqrt{CH}=0$) a hyperboloid taken twice, viz., this is the hyperboloid passing through the given line and through the symmetrically situate seven other lines.}

November 12, 1868. pp. 103, 104.

Professor Cayley gave an account to the Meeting of a Memoir by Herr Listing, "Census räumlicher Complexe oder Verallgemeinerung des Euler'schen Satzes von den Polyedern," published in the *Göttingen Transactions* for 1862. The fundamental theorem is a relation $a-(b-\kappa)+(c-\kappa'+\pi)-(d-\kappa''+\pi'-\omega)=0$ existing in any figure whatever between a the number of points, b the number of lines, c the number of areas, d the number of spaces, and certain supplementary quantities κ, κ', κ'', π, π', ω. In an extensive class of figures these last are each $=0$, and the relation is $a-b+c-d=0$; thus, in a closed box, $a=8$, $b=12$, $c=6$, $d=2$ (viz., there is the finite space inside, and the infinite space outside, the box): if the box be opened, $a=10$, $b=15$, $c=6$, $d=1$; if the lid be taken away, $a=8$, $b=12$, $c=5$, $d=1$; in each case, $a-b+c-d=0$. If the bottom be also taken away, $a=8$, $b=12$, $c=4$, $d=1$; but here one of the supplementary quantities comes in, $\kappa''=1$, and the theorem is $a-b+c-(d-\kappa'')=0$. The chief difficulty and interest of the Memoir consist in the determination of the supplementary quantities κ, κ', κ'', π, π', ω.

December 10, 1868. pp. 123—125. Appended to Paper by Mr T. Cotterill "On a Correspondence of Points etc."

Observations by Professor Cayley and Mr W. K. Clifford on the connexion of the transformation with Cremona's general theory, and the analytical formulæ.

According to Cremona's general theory,—taking (x_1, y_1, z_1) and (x_2, y_2, z_2) as current coordinates in the two planes respectively,—if we take in the first plane, any three points 1, 2, 3, and any other three points 4′, 5′, 6′, then if $X_1=0$, $Y_1=0$, $Z_1=0$ are quartic curves, each having the double points 1, 2, 3, and the simple points 4′, 5′, 6′, we have a transformation $x_2 : y_2 : z_2 = X_1 : Y_1 : Z_1$ leading to a converse system

$$x_1 : y_1 : z_1 = X_2 : Y_2 : Z_2$$

of the like form; viz., there will be in the second plane three points 4, 5, 6, and three other points 1′, 2′, 3′, such that $X_2=0$, $Y_2=0$, $Z_2=0$, are quartics having the double points 4, 5, 6, and the simple points 1′, 2′, 3′.

Analytically, Cremona's transformation is obtained by assuming the reciprocals of x_2, y_2, z_2 to be proportional to linear functions of the reciprocals of x_1, y_1, z_1—(of course, this being so, the reciprocals of x_1, y_1, z_1 will be proportional to linear functions of the reciprocals of x_2, y_2, z_2). Solving this under the theory as above explained, write

$$\left.\begin{matrix} \dfrac{1}{x_2} \\ : \dfrac{1}{y_2} \\ : \dfrac{1}{z_2} \end{matrix}\right\} = \left\{\begin{matrix} \dfrac{a}{x_1}+\dfrac{b}{y_1}+\dfrac{c}{z_1} \\ : \dfrac{d}{x_1}+\dfrac{e}{y_1}+\dfrac{f}{z_1} \\ : \dfrac{g}{x_1}+\dfrac{h}{y_1}+\dfrac{i}{z_1} \end{matrix}\right\} = \left\{\begin{matrix} P_1 \\ : Q_1 \\ : R_1 \end{matrix}\right.$$

if

$$P_1 = ay_1z_1 + bz_1x_1 + cx_1y_1,$$
$$Q_1 = dy_1z_1 + ez_1x_1 + fx_1y_1,$$
$$R_1 = gy_1z_1 + hz_1x_1 + ix_1y_1.$$

Hence

$$x_2 : y_2 : z_2 = Q_1R_1 : R_1P_1 : P_1Q_1.$$

$Q_1R_1 = 0$, &c., are quartics, or generally $\alpha Q_1R_1 + \beta R_1P_1 + \gamma P_1Q_1 = 0$ is a quartic, having three double points ($y_1 = 0$, $z_1 = 0$), ($z_1 = 0$, $x_1 = 0$), ($x_1 = 0$, $y_1 = 0$), and having besides the three points which are the remaining points of intersection of the conics ($Q_1 = 0$, $R_1 = 0$), ($R_1 = 0$, $P_1 = 0$), ($P_1 = 0$, $Q_1 = 0$) respectively; viz., these last are the points

$$\frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1} = ei - hf : fg - id : dh - ge, \text{ \&c. \&c.}$$

The double and simple points are fixed points (that is, independent of α, β, γ), and the formulæ come under Cremona's theory. It is, however, necessary to show that if the points 4′, 5′, 6′ are in a line, the points 1′, 2′, 3′ are also in a line. This may be done as follows:

Let there be three planes A, B, C, and let the points of the first two correspond by ordinary triangular inversion in respect of the triangle α_1 on the plane A, and β_1 on the plane B. Let also the planes B, C correspond by ordinary triangular inversion in respect of the triangle β_2 on the plane B, and γ_2 on the plane C. Then the correspondence between A and C is the one considered, the points 1 2 3 forming the triangle α_1 and the points 4 5 6 forming the triangle γ_2. The points 4′5′6′ and 1′2′3′ in the planes A, C respectively correspond to the triangles β_1, β_2; and the conditions that 4′, 5′, 6′ shall be in a line and that 1′, 2′, 3′ shall be in a line, are the same condition, namely, that the triangles β_1, β_2 shall be inscribed in the same conic. Analogous properties must apparently belong to Cremona's other transformations, and the investigation of them will form an interesting part of the theory.

It is important, also, to notice the relation of the transformation to Hesse's "Uebertragungsprincip," *Crelle*, tom. LXVI. p. 15, which establishes a correspondence between the points of a plane and the point-pairs on a line. If $Ax^2 + 2Bxy + Cy^2 = 0$ is the equation of a point-pair, the coordinates in the plane are taken by Hesse *directly*, but in the present Paper *inversely* proportional to A, B, C.

388.

NOTE ON THE COMPOSITION OF INFINITESIMAL ROTATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 7—10.]

THE following is a solution of a question proposed by me in the last Smith's Prize Examination:

"Show that infinitesimal rotations impressed upon a solid body may be compounded together according to the rules for the composition of forces."

DEFINITION. The "six coordinates" of a line passing through the point (x_0, y_0, z_0), and inclined at angles (α, β, γ), to the axes, are

$$\begin{aligned} a &= \cos\alpha, & f &= y_0 \cos\gamma - z_0 \cos\beta, \\ b &= \cos\beta, & g &= z_0 \cos\alpha - x_0 \cos\gamma, \\ c &= \cos\gamma, & h &= x_0 \cos\beta - y_0 \cos\alpha. \end{aligned}$$

I use, throughout, the term rotation to denote an infinitesimal rotation; this being so,

LEMMA 1. A rotation ω round the line (a, b, c, f, g, h), produces in the point (x, y, z), rigidly connected with the line, the displacements

$$\begin{aligned} \delta x &= \omega(\quad . \quad cy - bz + f), \\ \delta y &= \omega(-cx \quad . \quad + az + g), \\ \delta z &= \omega(\quad bx - ay \quad . \quad + h). \end{aligned}$$

LEMMA 2. Considering in a solid body the point (x, y, z), situate in the line (a, b, c, f, g, h), then for any infinitesimal motion of the solid body, the displacement of the point in the direction of the line is

$$= al + bm + cn + fp + gq + hr,$$

where l, m, n, p, q, r are constants depending on the infinitesimal motion of the solid body.

Hence, *first*, for a system of rotations

$$\begin{array}{lll} \omega_1 & \text{about the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ \omega_2 & \quad\text{"} \quad\quad\text{"} & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&\text{c.} & & \end{array}$$

the displacements of the point (x, y, z), are

$$\begin{aligned} \delta x &= \quad . \quad y\Sigma c\omega - z\Sigma b\omega + \Sigma f\omega, \\ \delta y &= -x\Sigma c\omega \quad . \quad + z\Sigma a\omega + \Sigma g\omega, \\ \delta z &= \quad x\Sigma b\omega + y\Sigma a\omega \quad . \quad + \Sigma h\omega; \end{aligned}$$

and when the rotations are in equilibrium, the displacements $(\delta x, \delta y, \delta z)$ of any point (x, y, z) whatever must each of them vanish; that is, we must have

$$\Sigma\omega a = 0, \quad \Sigma\omega b = 0, \quad \Sigma\omega c = 0, \quad \Sigma\omega f = 0, \quad \Sigma\omega g = 0, \quad \Sigma\omega h = 0,$$

which are therefore the conditions for the equilibrium of the rotations ω_1, ω_2, &c.

Secondly, for a system of forces

$$\begin{array}{lll} P_1 & \text{along the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ P_2 & \quad\text{"} \quad\quad\text{"} & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&\text{c.} & & \end{array}$$

the condition of equilibrium as given by the principle of virtual velocities is

$$\Sigma P\,(al + bm + cn + fp + gq + hr) = 0;$$

or, what is the same thing, we must have

$$\Sigma Pa = 0, \quad \Sigma Pb = 0, \quad \Sigma Pc = 0, \quad \Sigma Pf = 0, \quad \Sigma Pg = 0, \quad \Sigma Ph = 0,$$

which are therefore the conditions for the equilibrium of the forces P_1, P_2, &c.

Comparing the two results we see that the conditions for the equilibrium of the rotations ω_1, ω_2, &c. are the same as those for the equilibrium of the forces P_1, P_2, &c.; and since, for rotations and forces respectively, we pass at once from the theory of equilibrium to that of composition; the rules of composition are the same in each case.

Demonstration of Lemma 1.

Assuming for a moment that the axis of rotation passes through the origin, then for the point P, coordinates (x, y, z), the square of the perpendicular distance from the axis is

$$\begin{aligned} = \;& (\quad . \quad - y\cos\gamma + z\cos\beta)^2 \\ &+ (\quad x\cos\gamma \quad . \quad - z\cos\alpha)^2 \\ &+ (-x\cos\beta + y\cos\alpha \quad . \quad)^2, \end{aligned}$$

and the expressions which enter into this formula denote as follows; viz. if through the point P, at right angles to the plane through P and the axis of rotation, we draw a line PQ, = perpendicular distance of P from the axis of rotation, then the coordinates of Q referred to P as origin are

$$\begin{array}{ccc} . & -y\cos\gamma + z\cos\beta, \\ x\cos\gamma & . & -z\cos\alpha, \\ -x\cos\beta + y\cos\alpha & . & , \end{array}$$

respectively. Hence the foregoing quantities each multiplied by ω are the displacements of the point P in the directions of the axes, produced by the rotation ω. Suppose that the axis of rotation (instead of passing through the origin) passes through the point (x_0, y_0, z_0); the only difference is that we must in the formulæ write $(x-x_0, y-y_0, z-z_0)$ in place of (x, y, z): and attending to the significations of the six coordinates (a, b, c, f, g, h) it thus appears that the displacements produced by the rotation are equal to ω into the expressions

$$\begin{array}{ccc} . & -cy + bz + f, \\ cx & . & -az + g, \\ -bx + ay & . & +h, \end{array}$$

respectively.

Demonstration of Lemma 2.

For any infinitesimal motion whatever of a solid body, the displacements of the point (x, y, z) in the directions of the axes are

$$\begin{aligned} \delta x &= l \quad . \quad -ry + qz, \\ \delta y &= m + rx \quad . \quad -pz, \\ \delta z &= n - qx + py \quad . \quad , \end{aligned}$$

and hence the displacement in the direction of the line (α, β, γ), is

$$\delta x\cos\alpha + \delta y\cos\beta + \delta z\cos\gamma,$$

which, attending to the signification of the six coordinates (a, b, c, f, g, h), is

$$= al + bm + cn + fp + gq + hr,$$

which is the required expression.

It is proper to remark that the last-mentioned expressions of $(\delta x, \delta y, \delta z)$ are in fact the displacements produced by a translation and a rotation. If we *assume* that every infinitesimal motion of a solid body can be resolved into a translation and a rotation, then, since a translation can be produced by two rotations, every infinitesimal motion of a solid body can be resolved into rotations alone, and the foregoing expressions for the displacements produced by a rotation, combining any number of them and writing $(\Sigma\omega a, \Sigma\omega b, \Sigma\omega c, \Sigma\omega f, \Sigma\omega g, \Sigma\omega h) = (-p, -q, -r, l, m, n)$ respectively, lead to the expressions for the displacements $\delta x, \delta y, \delta z$ produced by the infinitesimal motion of the solid body.

389.

ON A LOCUS DERIVED FROM TWO CONICS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 77—84.]

REQUIRED the locus of a point which is such that the pencil formed by the tangents through it to two given conics has a given anharmonic ratio.

Suppose, for a moment, that the equation of the tangents to the first conic is $(x-ay)(x-by)=0$, and that of the tangents to the second conic is $(x-cy)(x-dy)=0$, and write

$$A=(a-b)(c-d),$$
$$B=(a-c)(d-b),$$
$$C=(a-d)(b-c),$$

so that

$$A+B+C=0,$$

write also

$$k_1=\frac{B}{A},\quad k_2=\frac{C}{A},$$

then the anharmonic ratio of the pencil will have a given value k if

$$(k-k_1)(k-k_2)=0;$$

that is, if

$$k^2+k+\frac{BC}{A^2}=0,$$

or, what is the same thing, if

$$A^2(2k+1)^2+4BC-A^2=0;$$

that is, if

$$A^2(2k+1)^2-(B-C)^2=0,$$

where

$$A^2 = (a-b)^2(c-d)^2,$$
$$B - C = (a+b)(c+d) - 2(ab+cd),$$

are each of them symmetrical in regard to a, b, and in regard to c, d, respectively.

Let the equations of the two conics be

$$U = (a, b, c, f, g, h \between x, y, z)^2 = 0,$$
$$U' = (a', b', c', f', g', h' \between x, y, z)^2 = 0,$$

and let (α, β, γ) be the coordinates of the variable point. Putting as usual

$$(A, B, C, F, G, H) = (bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch),$$
$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

the equation of the tangents to the first conic is

$$(A, B, C, F, G, H \between X, Y, Z)^2 = 0,$$

where

$$X = \gamma y - \beta z, \quad Y = \alpha z - \gamma x, \quad Z = \beta x - \alpha y,$$

and therefore

$$\alpha X + \beta Y + \gamma Z = 0.$$

Hence substituting for Z the value $-\frac{1}{\gamma}(\alpha X + \beta Y)$, we find, for the equation of the tangents, an equation of the form $\mathrm{a}X^2 + 2\mathrm{h}XY + \mathrm{b}Y^2 = 0$, which has, in effect, been taken to be $(X - aY)(X - bY) = 0$; that is, we have

$$1 : a+b : ab = \mathrm{a} : -2\mathrm{h} : \mathrm{b};$$

and, in like manner, if the accented letters refer to the second conic

$$1 : c+d : cd = \mathrm{a}' : -2\mathrm{h}' : \mathrm{b}'.$$

Substituting for a, h, b their values, and for a′, h′, b′ the corresponding values, we find

$$\begin{aligned} 1 : a+b : ab \\ &= A\gamma^2 - 2G\gamma\alpha + C\alpha^2 \\ &: -2(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta) \\ &: B\gamma^2 - 2F\beta\gamma + C\beta^2. \end{aligned}$$

$$\begin{aligned} 1 : c+d : cd \\ &= A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2 \\ &: -2(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta) \\ &: B'\gamma^2 - 2F'\beta\gamma + C'\beta^2. \end{aligned}$$

We then have

$$\begin{aligned} (a-b)^2 &= (a+b)^2 - 4ab, \\ &= 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)^2 \\ &\quad - 4(A\gamma^2 - 2G\gamma\alpha + C\alpha^2)(B\gamma^2 - 2F\beta\gamma + C\beta^2), \\ &= -4\gamma^2(BC - F^2, \ldots \between \alpha, \beta, \gamma)^2, \\ &= -4\gamma^2 K(a, \ldots \between \alpha, \beta, \gamma)^2, \end{aligned}$$

and similarly

$$(c-d)^2 = -4\gamma^2 K'(a', \ldots \between \alpha, \beta, \gamma)^2.$$

We have, moreover,

$$\begin{aligned}
&(a+b)(c+d) - 2(ab+cd) \\
&\quad = \quad 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta) \\
&\qquad - 2(B\gamma^2 - 2F\beta\gamma + C\beta^2 \qquad)(A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2) \\
&\qquad - 2(B'\gamma^2 - 2F'\beta\gamma + C'\beta^2 \qquad)(A\gamma^2 - 2G\gamma\alpha + C\alpha^2), \\
&\quad = -2\gamma^2(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2,
\end{aligned}$$

and substituting the foregoing values, we find

$$4(2k+1)^2 KK'(a, \ldots \between \alpha, \beta, \gamma)^2 (a', \ldots \between \alpha, \beta, \gamma)^2 - \{(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2\}^2 = 0,$$

or putting for shortness

$$\Theta = (BC' + B'C - 2FF', \ldots GH' + G'H - AF' - A'F, \ldots \between \alpha, \beta, \gamma)^2,$$

the equation of the locus is

$$4(2k+1)^2 KK' . UU' - \Theta^2 = 0,$$

where (α, β, γ) are current coordinates. The locus is thus a quartic curve having quadruple contact with each of the conics $U=0$, $U'=0$; viz. it touches them at their points of intersection with the conic $\Theta = 0$, *which is the locus of the point such that the four tangents form a harmonic pencil.*

The equation may be written somewhat more elegantly under the form

$$4(2k+1)^2 . KU . K'U' - \Theta^2 = 0;$$

viz. in this equation we have

$$\begin{aligned}
KU &= (BC - F^2, \ldots \qquad \between \alpha, \beta, \gamma)^2, \\
K'U' &= (B'C' - F'^2, \ldots \qquad \between \alpha, \beta, \gamma)^2, \\
\Theta &= (BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2.
\end{aligned}$$

In the last form the equation is expressed in terms of the coefficients $(A, \ldots)$, $(A', \ldots)$ of the *line* equations of the conics, viz. these may be taken to be

$$(A, \ldots \between \xi, \eta, \zeta)^2 = 0, \quad (A', \ldots \between \xi, \eta, \zeta)^2 = 0.$$

In particular, if each of the conics break up into a pair of points, viz. (l, m, n) and (p, q, r) for the first conic, (l', m', n') and (p', q', r') for the second conic, then the line equations are

$$2(l\xi + m\eta + n\zeta)(p\xi + q\eta + r\zeta) = 0,$$

$$2(l'\xi + m'\eta + n'\zeta)(p'\xi + q'\eta + r'\zeta) = 0,$$

so that

$$A = 2lp, \;..\; F = mr + nq, \;..$$
$$A' = 2l'p', \;..\; F' = m'r' + n'q', \;..$$

$$(BC - F^2, \ldots) = -(mr - nq\,,\; np - lr\,,\; lq - mp)^2,$$
$$(B'C' - F'^2, \ldots) = -(m'r' - n'q',\; n'p' - l'r',\; l'q' - m'p')^2,$$
$$BC' + B'C - 2FF' = 2\,\{(mn' - m'n)(qr' - q'r) - (mr' - nq')(m'r - n'q),\; \ldots\},$$

and substituting these values the equation is

$$(2k+1)^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix}^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}^2 - \left\{ \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ p, & q, & r \\ p', & q', & r' \end{vmatrix} - \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix} \right\}^2 = 0,$$

which, if A, B, C denote

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ p', & q', & r' \\ p, & q, & r \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix}$$

respectively, $(A + B + C = 0)$ is, in fact, the equation

$$(2k+1)^2 A^2 - (B - C)^2 = 0,$$

or, what is the same thing,

$$\left(k - \frac{B}{A}\right)\left(k - \frac{C}{A}\right) = 0,$$

that is

$$k = \frac{B}{A} \text{ or } k = \frac{C}{A},$$

either of which expresses the anharmonic property of the points of a conic in the form given by the theorem *ad quatuor lineas.*

Reverting to the case of two conics, then if these be referred to a set of conjugate axes, the equations will be

$$a\,x^2 + b\,y^2 + c\,z^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 = 0,$$

we have $K = abc$, $K' = a'b'c'$,

$$\Theta = (bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2,$$

and the equation of the quartic curve is

$$4\,(2k+1)^2\,abca'b'c'\,(ax^2 + by^2 + cz^2)\,(a'x^2 + b'y^2 + c'z^2)$$
$$- \{(bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2\}^2 = 0.$$

I suppose in particular that the two conics are

$$x^2 + my^2 - 1 = 0,$$
$$mx^2 + \quad y^2 - 1 = 0,$$

the equation of the quartic is

$$4(2k+1)^2 m^2 (x^2 + my^2 - 1)(mx^2 + y^2 - 1) - \{(m^2+m)(x^2+y^2) - m^2 - 1\}^2 = 0;$$

or putting $\lambda = \dfrac{(m+1)^2}{4(2k+1)^2}$, this is

$$\lambda\left(x^2 + y^2 - \frac{m^2+1}{m^2+m}\right)^2 - (x^2 + my^2 - 1)(mx^2 + y^2 - 1) = 0.$$

To fix the ideas, suppose that m is positive and >1, so that each of the conics is an ellipse, the major semi-axis being $=1$, and the minor semi-axis being $=\dfrac{1}{\sqrt{(m)}}$. For any real value of k the coefficient λ is positive, and it may accordingly be assumed that λ is positive.

We have $\dfrac{m^2+1}{m(m+1)} > \dfrac{1}{m} < 1$, or the radius of the circle is intermediate between the semi-axes of the ellipses, hence the points of contact on each ellipse are real points.

Writing for shortness

$$\alpha = \frac{m^2+1}{m^2+m},$$

the equation is

$$(x^2 + my^2 - 1)(mx^2 + y^2 - 1) - \lambda(x^2 + y^2 - \alpha)^2 = 0.$$

For the points on the axis of x, we have

$$(x^2 - 1)(mx^2 - 1) - \lambda(x^2 - \alpha)^2 = 0,$$

that is

$$(m - \lambda)x^4 + \{-(1+m) + 2\lambda\alpha\}x^2 + (1 - \lambda\alpha^2) = 0,$$

and thence

$$(m-\lambda)x^2 = \tfrac{1}{2}(1+m) - \lambda\alpha \pm \tfrac{1}{2}\sqrt{\{(m-1)^2 + 4\lambda(1-\alpha)(1-m\alpha)\}},$$

or, substituting for α its value, this is

$$(m-\lambda)x^2 = \tfrac{1}{2}(m+1) - \frac{\lambda\left(m + \dfrac{1}{m}\right)}{m+1} \pm \frac{\tfrac{1}{2}(m-1)}{m+1}\sqrt{\{(m+1)^2 - 4\lambda\}}.$$

Remarking that the values $\dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2}$, m, $\tfrac{1}{4}(m+1)^2$ are in the order of increasing magnitude,

and considering successive values of λ; first the value $\lambda = \frac{1}{\alpha^2}$, $= \frac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, we have

$$(m-\lambda)\,x^2 = \tfrac{1}{2}(m+1) - \frac{m+1}{m+\frac{1}{m}} \pm \frac{\frac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)}$$

$$= \frac{(m+1)\frac{1}{2}\left(m+\frac{1}{m}-2\right) \pm \frac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)};$$

or observing that

$$(m+1)\left(m+\frac{1}{m}-2\right) = (m+1)\frac{1}{m}(m-1)^2 = \frac{1}{m}(m-1)(m^2-1) = (m-1)\left(m-\frac{1}{m}\right),$$

this is

$$(m-\lambda)\,x^2 = 0, \text{ or } \frac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}},$$

or, what is the same thing,

$$\frac{(m-1)(m^3+2m^2-1)}{m\left(m+\frac{1}{m}\right)^2}x^2 = 0, \text{ or } \frac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}}, \quad x^2 = 0, \text{ or } \frac{\left(m^2-\frac{1}{m^2}\right)m}{m^3+2m^2-1}.$$

The next critical value is $\lambda = m$. The curve here is

$$(x^2+my^2-1)(mx^2+y^2-1) - m(x^2+y^2-\alpha)^2 = 0,$$

that is

$$m(x^4+y^4) + (1+m^2)x^2y^2 - (m+1)(x^2+y^2) + 1$$
$$- m(x^4+y^4) - \quad 2m \quad x^2y^2 + \quad 2m\alpha \quad (x^2+y^2) - m\alpha^2 = 0,$$

that is

$$(m-1)^2x^2y^2 + (2m\alpha - m - 1)(x^2+y^2) + 1 - m\alpha^2 = 0,$$

or, substituting for α its value,

$$2m\alpha - m - 1 = \frac{2m^2+2}{m+1} - (m+1) = \frac{(m-1)^2}{m+1},$$

$$1 - m\alpha^2 = 1 - \frac{(m^2+1)^2}{m(m+1)^2} = -\frac{(m-1)^2(m^2+m+1)}{m(m+1)^2};$$

the equation is

$$x^2y^2 + \frac{1}{m+1}(x^2+y^2) - \frac{m^2+m+1}{m(m+1)^2} = 0,$$

or, as this may also be written,

$$\left(x^2+\frac{1}{m+1}\right)\left(y^2+\frac{1}{m+1}\right) - \frac{1}{m} = 0,$$

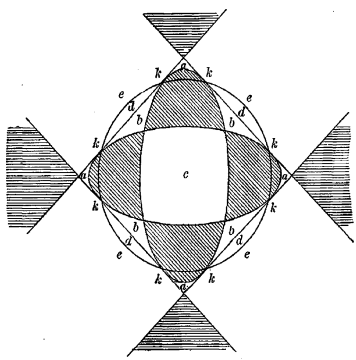
which has a pair of imaginary asymptotes parallel to the axis of x, and a like pair parallel to the axis of y, or what is the same thing, the curve has two isolated points at infinity, one on each axis.

The next critical value is $\lambda = \frac{1}{4}(m+1)^2$; the curve here reduces itself to the four lines

$$\left\{(x+y)^2 - \frac{m+1}{m}\right\}\left\{(x-y)^2 - \frac{m+1}{m}\right\} = 0;$$

and it is to be observed that when λ exceeds this value, or say $\lambda > \frac{1}{4}(m+1)^2$, the curve has no real point on either axis; but when $\lambda = \infty$, the curve reduces itself to $(x^2+y^2-\alpha)^2 = 0$, i.e. to the circle $x^2+y^2-\alpha = 0$ twice repeated, having in this special case real points on the two axes.

It is now easy to trace the curve for the different values of λ. The curve lies in every case within the unshaded regions of the figure (except in the limiting cases after-mentioned); and it also touches the two ellipses and the four lines at the eight points k, at which points it also cuts the circle; but it does not cut or touch the



four lines, the two ellipses, or the circle, except at the points k. Considering λ as varying by successive steps from 0 to ∞;

$\lambda = 0$, the curve is the two ellipses.

$\lambda < \dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2}$, the curve consists of two ovals, an exterior sinuous oval lying in the four regions a and the four regions b; and an interior oval lying in the region c.

$\lambda = \dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2}$, there is still a sinuous oval as above, but the interior oval has dwindled to a conjugate point at the centre.

$\lambda > \dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2} < m$; $\lambda = m$; $\lambda > m < \dfrac{(m+1)^2}{4}$; there is no interior oval, but only a sinuous oval as above; which, as λ increases, approaches continually nearer to the four sides of the square. For the critical value $\lambda = m$, there is no change in the general form, but the curve has for this value of λ, two conjugate points, one on each axis at infinity.

$\lambda = \frac{1}{4}(m+1)^2$, the curve becomes the four lines.

$\lambda > \frac{1}{4}(m+1)^2$, the curve lies wholly in the four regions a and the four regions e, consisting thereof of four detached sinuous ovals. As λ deviates less from the value $\frac{1}{4}(m+1)^2$, each oval approaches more nearly to the infinite trilateral formed by the side and infinite line-portions which bound the regions d, e to which the oval belongs. And as λ departs from the limit $\frac{1}{4}(m+1)^2$, and approaches to ∞, each sinuous oval approaches more nearly to the circular arc which separates the two regions d, e, which contains the sinuous oval.

Finally, $\lambda = 0$, the curve is the circle twice repeated.

390.

THEOREM RELATING TO THE FOUR CONICS WHICH TOUCH THE SAME TWO LINES AND PASS THROUGH THE SAME FOUR POINTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 162—167.]

THE sides of the triangle formed by the given points meet one of the given lines in three points, say P, Q, R; and on this same line we have four points of contact, say A_1, A_2, A_3, A_4; any two pairs, say A_1, A_2; A_3, A_4, form with a properly selected pair, say Q, R, out of the above-mentioned three points, an involution; and we have thus the three involutions

$$(A_1, A_2;\ A_3, A_4;\ Q, R),$$
$$(A_1, A_3;\ A_4, A_2;\ R, P),$$
$$(A_1, A_4;\ A_2, A_3;\ P, Q).$$

To prove this, let $x=0$, $y=0$ be the equations of the given lines, and take for the equations of the sides of the triangle formed by the given points

$$b\,x+a\,y-a\,b=0,$$
$$b'x+a'y-a'b'=0,$$
$$b''x+a''y-a''b''=0:$$

the equation of any one of the four conics may be written

$$\frac{Lab}{bx+ay-ab}+\frac{L'a'b'}{b'x+a'y-a'b'}+\frac{L''a''b''}{b''x+a''y-a''b''}=0,$$

and if this touches the axis of x, say at the point $x=\alpha$, then we must have

$$\frac{La}{x-a}+\frac{L'a'}{x-a'}+\frac{L''a''}{x-a''}=\frac{-K(x-\alpha)^2}{(x-a)(x-a')(x-a'')};$$

or, assuming as we may do, $K=-(a'-a'')(a''-a)(a-a')$, this gives

$$L\,a=(a-\alpha)^2(a'-a''),$$
$$L'a'=(a'-\alpha)^2(a''-a),$$
$$L''a''=(a''-\alpha)^2(a-a').$$

But in the same manner, if the conic touch the axis of y, say at the point $y=\beta$, we have

$$L\,b=(b-\beta)^2(b'-b''),$$
$$L'b'=(b'-\beta)^2(b''-b),$$
$$L''b''=(b''-\beta)^2(b-b');$$

and thence

$$b(a-\alpha)^2(a'-a'') : b'(a'-\alpha)^2(a''-a) : b''(a''-\alpha)^2(a-a')$$
$$=a(b-\beta)^2(b'-b'') : a'(b'-\beta)^2(b''-b) : a''(b''-\beta)^2(b-b').$$

Putting

$$P=a\,b\,(a'-a'')(b'-b''),$$
$$P'=a'b'\,(a''-a)(b''-b),$$
$$P''=a''b''(a-a')(b-b'),$$

we have

$$(a-\alpha)^2\frac{P}{a^2} : (a'-\alpha)^2\frac{P'}{a'^2} : (a''-\alpha)^2\frac{P''}{a''^2}=(b-\beta)^2(b'-b'')^2 : (b'-\beta)^2(b''-b)^2 : (b''-\beta)^2(b-b')^2;$$

and thence

$$(a-\alpha)\frac{\sqrt{(P)}}{a} : (a'-\alpha)\frac{\sqrt{(P')}}{a'} : (a''-\alpha)\frac{\sqrt{(P'')}}{a''}$$
$$=(b-\beta)(b'-b'') : (b'-\beta)(b''-b) : (b''-\beta)(b-b'),$$

which gives

$$(a-\alpha)\frac{\sqrt{(P)}}{a}+(a'-\alpha)\frac{\sqrt{(P')}}{a'}+(a''-\alpha)\frac{\sqrt{(P'')}}{a''}=0,$$

and we have in like manner

$$(b-\beta)\frac{\sqrt{(P)}}{b}+(b'-\beta)\frac{\sqrt{(P')}}{b'}+(b''-\beta)\frac{\sqrt{(P'')}}{b''}=0,$$

but the first of these equations is alone required for the present purpose. Putting for shortness

$$P=a^2X,\qquad P'=a'^2X',\qquad P''=a''^2X'',$$

the equation is

$$(a-\alpha)\sqrt{(X)}+(a'-\alpha)\sqrt{(X')}+(a''-\alpha)\sqrt{(X'')},$$

and by attributing the signs + and − to the radicals, we have, corresponding to the four conics, the equations

$$\begin{aligned}
(a-\alpha_1)\sqrt{(X)}+(a'-\alpha_1)\sqrt{(X')}+(a''-\alpha_1)\sqrt{(X'')}&=0,\\
-(a-\alpha_2)\sqrt{(X)}+(a'-\alpha_2)\sqrt{(X')}+(a''-\alpha_2)\sqrt{(X'')}&=0,\\
(a-\alpha_3)\sqrt{(X)}-(a'-\alpha_3)\sqrt{(X')}+(a''-\alpha_3)\sqrt{(X'')}&=0,\\
(a-\alpha_4)\sqrt{(X)}+(a'-\alpha_4)\sqrt{(X')}-(a''-\alpha_4)\sqrt{(X'')}&=0,
\end{aligned}$$

where α_1, α_2, α_3, α_4 are the values of α for the four conics respectively.

Eliminating a'' we obtain the system of three equations

$$\begin{aligned}
(2a-\alpha_1-\alpha_2)\sqrt{(X)}+\quad(\alpha_2-\alpha_1)\sqrt{(X')}\quad+(\alpha_2-\alpha_1)\sqrt{(X'')}&=0,\\
(\alpha_3-\alpha_1)\sqrt{(X)}+\quad(2a'-\alpha_1-\alpha_3)\sqrt{(X')}\quad+(\alpha_3-\alpha_1)\sqrt{(X'')}&=0,\\
(\alpha_1+\alpha_2-\alpha_3-\alpha_4)\sqrt{(X)}+(\alpha_1+\alpha_3-\alpha_2-\alpha_4)\sqrt{(X')}+(\alpha_1+\alpha_4-\alpha_2-\alpha_3)\sqrt{(X'')}&=0,
\end{aligned}$$

and then eliminating the radicals we have

$$\begin{vmatrix}
2a-\alpha_1-\alpha_2, & \alpha_2-\alpha_1, & \alpha_2-\alpha_1\\
\alpha_3-\alpha_1, & 2a'-\alpha_1-\alpha_3, & \alpha_3-\alpha_1\\
\alpha_1+\alpha_2-\alpha_3-\alpha_4, & \alpha_1+\alpha_3-\alpha_2-\alpha_4, & \alpha_1+\alpha_4-\alpha_2-\alpha_3
\end{vmatrix}=0,$$

which is in fact

$$-4\begin{vmatrix}
1, & a+a', & aa'\\
1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4\\
1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3
\end{vmatrix}=0,$$

as may be verified by actual expansion; the transformation of the determinant is a peculiar one.

The foregoing result was originally obtained as follows, viz. writing for a moment

$$\begin{aligned}
a\sqrt{(X)}+a'\sqrt{(X')}+a''\sqrt{(X'')}&=\Theta,\\
\sqrt{(X)}+\quad\sqrt{(X')}+\quad\sqrt{(X'')}&=\Phi,
\end{aligned}$$

the four equations are

$$\begin{aligned}
\Theta-\alpha_1\Phi&=0,\\
\Theta-\alpha_2\Phi&=2(a-\alpha_2)\sqrt{(X)},\\
\Theta-\alpha_3\Phi&=2(a'-\alpha_3)\sqrt{(X')},\\
\Theta-\alpha_4\Phi&=2(a''-\alpha_4)\sqrt{(X'')};
\end{aligned}$$

these give

$$\begin{aligned}
(\alpha_1-\alpha_2)\Phi&=2(a-\alpha_2)\sqrt{(X)},\\
(\alpha_1-\alpha_3)\Phi&=2(a'-\alpha_3)\sqrt{(X')},\\
(\alpha_1-\alpha_4)\Phi&=2(a''-\alpha_4)\sqrt{(X'')}.
\end{aligned}$$

From the last equation we have

$$(\alpha_1-\alpha_4)\Phi = 2\{\Theta - a\sqrt{(X)} - a'\sqrt{(X')}\} - 2\alpha_4\{\Phi - \sqrt{(X)} - \sqrt{(X')}\}$$
$$= 2(\alpha_1-\alpha_4)\Phi - 2(a-\alpha_4)\sqrt{(X)} - 2(a'-\alpha_4)\sqrt{(X')};$$

that is

$$(\alpha_1-\alpha_4)\Phi - 2(a-\alpha_4)\sqrt{(X)} - 2(a'-\alpha_4)\sqrt{(X')} = 0;$$

or substituting for $\sqrt{(X)}$, $\sqrt{(X')}$ their values in terms of Φ, we find

$$\alpha_1-\alpha_4 - \frac{(a-\alpha_4)(\alpha_1-\alpha_2)}{a-\alpha_2} - \frac{(a'-\alpha_4)(\alpha_1-\alpha_3)}{a'-\alpha_3} = 0,$$

which may be written

$$\alpha_1-\alpha_4-(\alpha_1-\alpha_2)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)-(\alpha_1-\alpha_3)\left(1+\frac{\alpha_3-\alpha_4}{a'-\alpha_3}\right)=0,$$

that is

$$\alpha_2+\alpha_3-\alpha_1-\alpha_4+\frac{(\alpha_2-\alpha_1)(\alpha_2-\alpha_4)}{a-\alpha_2}+\frac{(\alpha_3-\alpha_1)(\alpha_3-\alpha_4)}{a'-\alpha_3}=0;$$

or again

$$(\alpha_2-\alpha_1)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)+(\alpha_3-\alpha_4)\left(1+\frac{\alpha_3-\alpha_1}{a'-\alpha_3}\right)=0,$$

that is

$$(\alpha_2-\alpha_1)\frac{a-\alpha_4}{a-\alpha_2}+(\alpha_3-\alpha_4)\frac{a'-\alpha_1}{a'-\alpha_3}=0;$$

or finally

$$(\alpha_2-\alpha_1)(a-\alpha_3)(a'-\alpha_3)+(\alpha_3-\alpha_3)(a-\alpha_2)(a'-\alpha_1)=0,$$

which is a known form of the relation

$$\begin{vmatrix} 1, & a+a', & aa' \\ 1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4 \\ 1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3 \end{vmatrix} = 0,$$

which gives the involution of the quantities a, a'; α_1, α_4; α_2, α.

We have in like manner

$$\begin{vmatrix} 1, & a'+a'', & a'a'' \\ 1, & \alpha_1+\alpha_2, & \alpha_1\alpha_2 \\ 1, & \alpha_3+\alpha_4, & \alpha_3\alpha_4 \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} 1, & a''+a, & a''a \\ 1, & \alpha_1+\alpha_3, & \alpha_1\alpha_3 \\ 1, & \alpha_2+\alpha_4, & \alpha_2\alpha_4 \end{vmatrix} = 0,$$

which give the involutions of the systems a', a''; α_1, α_2; α_3, α_4 and a'', a; α_1, α_3; α_2, α_4 respectively.

It may be remarked that the equation of the conic passing through the three points and touching the axis of x in the point $x=\alpha$ is

$$\frac{(a-\alpha)^2(a'-a'')b}{bx+ay-ab}+\frac{(a'-\alpha)^2(a''-a)b'}{b'x+a'y-a'b'}+\frac{(a''-\alpha)^2(a-a')b''}{b''x+a''y-a''b''}=0,$$

and when this meets the axis of y we have

$$\frac{\frac{b}{a}(a-\alpha)^2(a'-a'')}{y-b}+\frac{\frac{b'}{a'}(a'-\alpha)^2(a''-a)}{y-b'}+\frac{\frac{b''}{a''}(a''-\alpha)^2(a-a')}{y-b''}=0.$$

Hence, if this touches the axis of y in the point $y=\beta$, the left-hand side must be

$$=\frac{\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right](y-\beta)^2}{(y-b)(y-b')(y-b'')},$$

and equating the coefficients of $\frac{1}{y^2}$, we have

$$\frac{b^2}{a}(a-\alpha)^2(a'-a'')+\frac{b'^2}{a'}(a'-\alpha)^2(a''-a)+\frac{b''^2}{a''}(a''-\alpha)^2(a-a')$$

$$=\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right](b+b'+b''-2\beta),$$

or what is the same thing,

$$\frac{b(b'+b'')}{a}(a-\alpha)^2(a'-a'')+\frac{b'(b''+b)}{a'}(a'-\alpha)^2(a''-a)+\frac{b''(b+b')}{a''}(a''-\alpha)^2(a-a')$$

$$=2\beta\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right],$$

which gives β in terms of α, that is β_1, β_2, β_3, β_4 in terms of α_1, α_2, α_3, α_4 respectively.

Cambridge, 30 *November*, 1863.

391.

SOLUTION OF A PROBLEM OF ELIMINATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 183—185.]

IT is required to eliminate x, y from the equations

$$\left\| \begin{matrix} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{matrix} \right\| = 0.$$

This system may be written

$$\begin{aligned} x^4 &= \Sigma\lambda a, \\ x^3y &= \Sigma\lambda b, \\ x^2y^2 &= \Sigma\lambda c, \\ xy^3 &= \Sigma\lambda d, \\ y^4 &= \Sigma\lambda e; \end{aligned}$$

if for shortness

$$\Sigma\lambda a = \lambda a + \lambda' a' + \lambda'' a'', \text{ \&c.}$$

Or putting

$$\frac{x}{y} = -k,$$

we have

$$\begin{aligned} \Sigma\lambda\,(a + kb) &= 0, \\ \Sigma\lambda\,(b + kc) &= 0, \\ \Sigma\lambda\,(c + kd) &= 0, \\ \Sigma\lambda\,(d + ke) &= 0; \end{aligned}$$

or, what is the same thing,

$$\begin{aligned}
&\lambda(a+kb)+\lambda'(a'+kb')+\lambda''(a''+kb'')=0,\\
&\lambda(b+kc)+\lambda'(b'+kc')+\lambda''(b''+kc'')=0,\\
&\lambda(c+kd)+\lambda'(c'+kd')+\lambda''(c''+kd'')=0,\\
&\lambda(d+ke)+\lambda'(d'+ke')+\lambda''(d''+ke'')=0;
\end{aligned}$$

and representing the columns

$$\begin{array}{cccccc}
a, & b, & a', & b', & a'', & b'',\\
b, & c, & b', & c', & b'', & c'',\\
c, & d, & c', & d', & c'', & d'',\\
d, & e, & d', & e', & d'', & e'',
\end{array}$$

by

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,$$

each equation is of the type

$$\lambda(1+k2)+\lambda'(3+k4)+\lambda''(5+k6)=0.$$

Multiplying the several equations by the minors of 135, each with its proper sign, and adding, the terms independent of k disappear, the equation divides by k, and we find

$$\lambda\, 2135+\lambda'\, 4135+\lambda''\, 6135=0;$$

operating in a similar manner with the minors of 246, the terms in k disappear, and we find

$$\lambda\, 1246+\lambda'\, 3246+\lambda''\, 5246=0;$$

again, operating with the minors of $(146+236+245+k246)$, we find

$$\begin{aligned}
&\lambda\,\{1236+1245+k(2146+1246)\}\\
&+\lambda'\,\{3146+3245+k(4236+3246)\}\\
&+\lambda''\,\{5146+5236+k(6245+5246)\}=0,
\end{aligned}$$

where the terms in k disappear, and this is

$$\lambda(1236+1245)+\lambda'(3146+3245)+\lambda''(5146+5236)=0.$$

We have thus three linear equations, which written in a slightly different form are

$$\begin{aligned}
&\lambda\ 1235 && +\lambda'\ 3451 && +\lambda''\ 5613 && =0,\\
&\lambda(1236+1245) && +\lambda'(3452+3461) && +\lambda''(5614+5623) && =0,\\
&\lambda\ 1246 && +\lambda'\ 3462 && +\lambda''\ 5624 && =0,
\end{aligned}$$

and thence eliminating λ, λ', λ'', we have

$$\begin{vmatrix}
1235, & 1236+1245, & 1246\\
3451, & 3452+3461, & 3462\\
5613, & 5614+5623, & 5624
\end{vmatrix}=0,$$

which is the required result. It may be remarked that the second and third column are obtained from the first by operating on it with Δ, $\frac{1}{2}\Delta^2$, if $\Delta = 2\delta_1 + 4\delta_3 + 6\delta_5$. Or say the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2)\begin{vmatrix} 1235 \\ 3451 \\ 5613 \end{vmatrix} = 0.$$

In like manner for the system

$$\left\| \begin{matrix} x^5, & x^4y, & x^3y^2, & x^2y^3, & xy^4, & y^5 \\ a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \\ a''', & b''', & c''', & d''', & e''', & f''' \end{matrix} \right\| = 0,$$

if the columns are

$$\begin{matrix} a\,b, & a'\,b', & a''\,b'', & a'''\,b''', \\ b\,c, & b'\,c', & b''\,c'', & b'''\,c''', \\ c\,d, & c'\,d', & c''\,d'', & c'''\,d''', \\ d\,e, & d'\,e', & d''\,e'', & d'''\,e''', \\ e\,f, & e'\,f', & e''\,f'', & e'''\,f''', \\ = 1,\ 2, & 3,\ 4, & 5,\ 6, & 7,\ 8; \end{matrix}$$

then the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2,\ \tfrac{1}{6}\Delta^3)\begin{vmatrix} 12357 \\ 34571 \\ 56713 \\ 78135 \end{vmatrix} = 0,$$

where

$$\Delta = 2\delta_1 + 4\delta_3 + 6\delta_5 + 8\lambda_7.$$

392.

ON THE CONICS WHICH PASS THROUGH TWO GIVEN POINTS AND TOUCH TWO GIVEN LINES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 211—219.]

LET $x=0$, $y=0$ be the equations of the given lines; $z=0$ the equation of the line joining the given points. We may, to fix the ideas, imagine the implicit constants so determined that $x+y+z=0$ shall be the equation of the line infinity.

Take $x-my=0$, $x-ny=0$ as the equations of the lines which by their intersection with $z=0$ determine the given points. The equation of the conic is

$$\{\sqrt{(m)}+\sqrt{(n)}\}\sqrt{(xy)}=x+y\sqrt{(mn)}+\gamma z,$$

or, what is the same thing,

$$(x-my)(x-ny)+2\{x+y\sqrt{(mn)}\}\gamma z+\gamma^2 z^2=0,$$

so that there are two distinct series of conics according as $\sqrt{(mn)}$ is taken with the positive or the negative sign.

The equation of the chord of contact is

$$x+y\sqrt{(mn)}+\gamma z=0,$$

which meets $z=0$ in the point $\{x+y\sqrt{(mn)}=0,\ z=0\}$ that is in one of the centres of the involution formed by the lines $(x=0,\ y=0)$, $(x-my=0,\ x-ny=0)$. It is to be observed that the conic is only real when mn is positive, that is (the lines and points being each real) the two points must be situate in the same region or in opposite regions of the four regions formed by the two lines: there are however other real cases; e.g. if the lines $x=0$, $y=0$ are real, but the quantities m, n are conjugate imaginaries; included in this we have the circles which touch two real lines.

To fix the ideas I take m and n each positive and $mn > 1$; also I attend first to the series where $\sqrt{(mn)}$ is taken positively. At the points where the conic meets infinity, we have

$$\{\sqrt{(m)} + \sqrt{(n)}\}\sqrt{(xy)} = x + y\sqrt{(mn)} - \gamma(x+y),$$

which gives two coincident points, that is the conic is a parabola, if

$$(1-\gamma)\{\sqrt{(mn)} - \gamma\} = \tfrac{1}{4}\{\sqrt{(m)} + \sqrt{(n)}\}^2,$$

that is

$$\gamma^2 - \gamma\{1 + \sqrt{(mn)}\} = \tfrac{1}{4}\{\sqrt{(m)} - \sqrt{(n)}\}^2,$$

or

$$\gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} \pm \sqrt{\{(1+m)(1+n)\}}],$$

where it is to be noticed that

$$\gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} + \sqrt{\{(1+m)(1+n)\}}]$$

is a positive quantity greater than $\sqrt{(mn)}$, say $\gamma = p$,

$$\gamma = \tfrac{1}{2}[1 + \sqrt{(mn)} - \sqrt{\{(1+m)(1+n)\}}]$$

is a negative quantity, say $\gamma = -q$, q being positive.

The order of the lines is as shown in fig. 1, see plate facing p. 52.

$\gamma = -\infty$ to $\gamma = -q$, curve is ellipse; $\gamma = -q$, parabola P_2,
$\gamma = -q$ to p, curve is hyperbola; $\gamma = p$, parabola P_1,
$\gamma = p$ to $\gamma = \infty$, ellipse.

Resuming the equation

$$(x - my)(x - ny) + 2\{x + y\sqrt{(mn)}\}\gamma z + \gamma^2 z^2 = 0,$$

the coefficients are

$$(a, b, c, f, g, h) = \{1, mn, \gamma^2, \gamma\sqrt{(mn)}, \gamma, -\tfrac{1}{2}(m+n)\},$$

and thence the inverse coefficients are

$$(A, B, C, F, G, H) =$$
$$[0, 0, -\tfrac{1}{4}(m-n)^2, -\tfrac{1}{2}\gamma\{\sqrt{(m)} + \sqrt{(n)}\}^2, -\tfrac{1}{2}\gamma\sqrt{(mn)}\{\sqrt{(m)} + \sqrt{(n)}\}^2, \tfrac{1}{2}\gamma^2\{\sqrt{(m)} + \sqrt{(n)}\}^2],$$
$$K = -\tfrac{1}{4}\gamma^2\{\sqrt{(m)} + \sqrt{(n)}\}^4,$$

or, omitting a factor, the inverse coefficients are

$$(A, B, C, F, G, H) = \left[0, 0, \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2, 1, \sqrt{(mn)}, -\gamma\right].$$

Considering the line

$$\lambda x + \mu y + \nu z = 0,$$

the coordinates of the pole of this line are

$$\begin{aligned} x : y : z &= \quad -\gamma\mu + \sqrt{(mn)}\,\nu \\ &: \quad -\gamma\lambda \quad + \quad \nu \\ &: \sqrt{(mn)}\,\lambda + \mu + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2\nu, \end{aligned}$$

or (what is the same thing) introducing the arbitrary coefficient k, we have

$$kx + \gamma\mu - \nu\sqrt{(mn)} = 0,$$

$$ky + \gamma\lambda - \nu \qquad = 0,$$

$$kz - \lambda\sqrt{(mn)} - \mu - \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2\nu = 0;$$

the first two equations give

$$k : \gamma : -1 = \nu\{\mu - \lambda\sqrt{(mn)}\} : \nu\{y\sqrt{(mn)} - x\} : \lambda x - \mu y,$$

that is

$$k = \frac{-\nu\{\mu - \lambda\sqrt{(mn)}\}}{\lambda x - \mu y}, \quad \gamma = \frac{-\nu\{y\sqrt{(mn)} - x\}}{\lambda x - \mu y},$$

or, substituting this value of γ in the third equation,

$$\frac{\nu\{\mu - \lambda\sqrt{(mn)}\}z}{\lambda x - \mu y} + \{\mu + \lambda\sqrt{(mn)}\} + \frac{\lambda x - \mu y}{x - y\sqrt{(mn)}}\,\frac{\{\sqrt{(m)} - \sqrt{(n)}\}^2}{2} = 0,$$

that is

$$(\lambda x - \mu y)^2 . \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 + \{x - y\sqrt{(mn)}\}(\lambda x - \mu y)\{\mu + \lambda\sqrt{(mn)}\}$$
$$+ z\{x - y\sqrt{(mn)}\}\nu\{\mu - \lambda\sqrt{(mn)}\} = 0,$$

which is the equation of the curve, the locus of the pole of the line $\lambda x + \mu y + \nu z = 0$ in regard to the conic

$$(x - my)(x - ny) + 2\{x + y\sqrt{(mn)}\}\gamma z + \gamma^2 z^2 = 0.$$

In particular, if $\lambda = \mu = \nu = 1$, then for the coordinates of the centre of the conic, we have

$$x : y : z = -\gamma + \sqrt{(mn)} : -\gamma + 1 : \sqrt{(mn)} + 1 + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2;$$

and for the locus of the centre,

$$(x - y)^2 . \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 + (x - y)\{x - y\sqrt{(mn)}\}\{1 + \sqrt{(mn)}\} + z\{x - y\sqrt{(mn)}\}\{1 - \sqrt{(mn)}\} = 0,$$

so that the locus is a conic, and it is obvious that this conic is a hyperbola. Putting for greater simplicity

$$x - y \qquad = X,$$

$$x - y\sqrt{(mn)} = Y,$$

$$z \qquad = Z,$$

the equation of the curve of centres is

$$X^2 . \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 + XY\{1 + \sqrt{(mn)}\} + YZ\{1 - \sqrt{(mn)}\} = 0,$$

or, writing this under the form

$$Y[X\{1 + \sqrt{(mn)}\} + Z\{1 - \sqrt{(mn)}\}] + \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2 X^2 = 0,$$

the equation is

$$YQ + X^2 = 0,$$

where

$$X = x - y,$$

$$Y = x - y\sqrt{(mn)},$$

$$Q = \frac{2}{\{\sqrt{(m)} - \sqrt{(n)}\}^2}\left[\{1 + \sqrt{(mn)}\}(x - y) + \{1 - \sqrt{(mn)}\}z\right]:$$

these values give

$$\begin{aligned} x - y \qquad &= X, \\ x - y\sqrt{(mn)} &= Y, \\ \{1 - \sqrt{(mn)}\}z &= \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q + 2\{1 + \sqrt{(mn)}\}X, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} \{1 - \sqrt{(mn)}\}x &= \quad -\sqrt{(mn)}\,X + Y, \\ \{1 - \sqrt{(mn)}\}y &= \qquad\quad - X + Y, \\ \{1 - \sqrt{(mn)}\}z &= 2\{1 + \sqrt{(mn)}\}X \qquad + \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q, \end{aligned}$$

whence also

$$\{1 - \sqrt{(mn)}\}(x + y + z) = \{1 + \sqrt{(mn)}\}X + 2Y + \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q,$$

or the equation of the line infinity is

$$\{1 + \sqrt{(mn)}\}X + 2Y + \{\sqrt{(m)} - \sqrt{(n)}\}^2 Q = 0,$$

a formula which may be applied to finding the asymptotes and thence the centre of the conic

$$YQ + X^2 = 0.$$

In fact we have identically

$$\{2kx + 2ky - (2k + 1)z\}^2 - (1 + 4k)(2kx - z)^2 = 4k^2(x + y + z)^2 - 4k(1 + 4k)(kx^2 + yz),$$

that is

$$-4k(1 + 4k)(kx^2 + yz) = \{2kx + 2ky - (2k + 1)z\}^2 - (1 + 4k)(2kx - z)^2 - 4k^2(x + y + z)^2,$$

which, if $x + y + z = 0$ is the equation of the line infinity, puts in evidence the asymptotes of the conic $kx^2 + yz = 0$. Hence writing αx, βy, γz in the place of x, y, z respectively, and $\frac{k\alpha^2}{\beta\gamma} = k'$, that is, $k = \frac{\beta\gamma}{\alpha^2}k'$, we have

$$-\frac{4\beta\gamma}{\alpha^2}k'\left(1 + \frac{4\beta\gamma}{\alpha^2}k'\right)\beta\gamma(k'x^2 + yz) = \left\{\frac{2\beta\gamma}{\alpha}k'x + \frac{2\beta^2\gamma}{\alpha^2}k'y - \left(2\frac{\beta\gamma}{\alpha^2}k' + 1\right)\gamma z\right\}^2$$

$$-\left(1 + \frac{4\beta\gamma}{\alpha^2}k'\right)\left(\frac{2\beta\gamma}{\alpha}k'x - \gamma z\right)^2 - 4\frac{\beta^2\gamma^2}{\alpha^4}k'^2(\alpha x + \beta y + \gamma z)^2,$$

that is

$$-4\beta^2\gamma^2k'(\alpha^2+4\beta\gamma k')(k'x^2+yz)=\{2\alpha\beta\gamma k'x+2\beta^2\gamma k'y-(2\beta\gamma k'+\alpha^2)\gamma z\}^2$$
$$-(\alpha^2+4\beta\gamma k')(2\beta\gamma k'x-\alpha\gamma z)^2-4\beta^2\gamma^2k'^2(\alpha x+\beta y+\gamma z)^2,$$

or, what is the same thing,

$$-4\beta^3k'(\alpha^2+4\beta\gamma k')(k'x^2+yz)=\{2\alpha\beta k'x+2\beta^2k'y-(2\beta\gamma k'+\alpha^2)z\}^2$$
$$-(\alpha^2+4\beta\gamma k')(2\beta k'x-\alpha z)^2-4\beta^3k'^2(\alpha x+\beta y+\gamma z)^2,$$

which, when $\alpha x+\beta y+\gamma z=0$ is the equation of the line infinity, puts in evidence the asymptotes of the conic $k'x^2+yz=0$.

Now writing X, Y, Q in the place of x, y, z; $k'=1$, and $\alpha=\{1+\sqrt{(mn)}\}$, $\beta=2$, $\gamma=\{\sqrt{(m)}-\sqrt{(n)}\}^2$, we have

$$\begin{aligned}-16\,[\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,(YQ+X^2)\\ =\ [4\,\{1+\sqrt{(mn)}\}\,X+8Y-(4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2)\,Q]^2\\ -[\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,[4X-\{1+\sqrt{(mn)}\}\,Q]^2\\ -16\,[\{1+\sqrt{(mn)}\}\,X+2Y+\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q]^2,\end{aligned}$$

and the asymptotes are

$$4\,\{1+\sqrt{(mn)}\}\,X+8Y-[4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2]\,Q$$
$$=\pm\sqrt{\{1+\sqrt{(mn)}\}^2+8\,\{\sqrt{(m)}-\sqrt{(n)}\}^2}\,[4X-\{1+\sqrt{(mn)}\}\,Q].$$

At the centre

$$4\,\{1+\sqrt{(mn)}\}\,X+8Y-[4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2+\{1+\sqrt{(mn)}\}^2]\,Q=0,$$
$$4X-\{1+\sqrt{(mn)}\}\,Q=0,$$

but the first equation is

$$\{1+\sqrt{(mn)}\}\,[4X-Q\,\{1+\sqrt{(mn)}\}]+8Y-4\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q=0,$$

so that we have

$$4X=\{1+\sqrt{(mn)}\}\,Q,\quad 2Y=\{\sqrt{(m)}-\sqrt{(n)}\}^2\,Q,$$

the first of these is

$$2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,(x-y)-\{1+\sqrt{(mn)}\}^2\,(x-y)-(1-mn)\,z=0,$$

and the two together give

$$2X\,\{\sqrt{(m)}-\sqrt{(n)}\}^2-\{1+\sqrt{(mn)}\}\,Y=0,$$

so that we have

$$2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2\,(x-y)-\{1+\sqrt{(mn)}\}\,\{x-y\sqrt{(mn)}\}=0,$$
$$[\{1+\sqrt{(mn)}\}^2-2\,\{\sqrt{(m)}-\sqrt{(n)}\}^2]\,(x-y)+(1-mn)\,z=0,$$

to determine the coordinates of the centre.

The equation of the chord of contact is

$$x + y\sqrt{(mn)} + \gamma z = 0,$$

which for $\gamma = 1$ is parallel to $y = 0$ and for $\gamma = \sqrt{(mn)}$ is parallel to $x = 0$. But the coordinates of the centre are

$$x : y : z = -\gamma + \sqrt{(mn)} : -\gamma + 1 : \sqrt{(mn)} + 1 + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2,$$

which for $\gamma = 1$ give

$$y = 0,\ x : z = -1 + \sqrt{(mn)} : \sqrt{(mn)} + 1 + \tfrac{1}{2}\{\sqrt{(m)} - \sqrt{(n)}\}^2, = -2 + 2\sqrt{(mn)} : 2 + m + n,$$

and for $\gamma = \sqrt{(mn)}$ give

$$x = 0,$$

$$y : z = 1 - \sqrt{(mn)} : \sqrt{(mn)} + 1 + \frac{1}{2\sqrt{(mn)}}\{\sqrt{(m)} - \sqrt{(n)}\}^2, = 2 - 2\sqrt{(mn)} : 2\sqrt{(mn)} + \frac{m+n}{\sqrt{(mn)}}.$$

The line drawn from the fixed point on the chord of contact to the centre has for its equation

$$x + y\sqrt{(mn)} + \gamma' z = 0,$$

where, writing for x, y, z the coordinates of the centre, we have

$$-\gamma\{1 + \sqrt{(mn)}\} + 2\sqrt{(mn)} + \gamma'\left[\sqrt{(mn)} + 1 + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2\right] = 0,$$

that is

$$\gamma' = \frac{\gamma\{1 + \sqrt{(mn)}\} - 2\sqrt{(mn)}}{1 + \sqrt{(mn)} + \frac{1}{2\gamma}\{\sqrt{(m)} - \sqrt{(n)}\}^2},$$

or, what is the same thing,

$$\gamma' - \gamma = \frac{-\gamma\{\sqrt{(m)} + \sqrt{(n)}\}^2}{\{\sqrt{(m)} - \sqrt{(n)}\}^2 + 2\gamma\{1 + \sqrt{(mn)}\}},$$

and consequently $\gamma' = \gamma$ only for $\gamma = 0$.

It is now easy to trace the corresponding positions of the chord of contact through the fixed point $\{x + y\sqrt{(mn)} = 0,\ z = 0\}$, and of the centre on the hyperbola which is the curve of centres: see fig. 2 in the plate facing p. 52.

The lines OP_2, OL, $O\Theta$, OP_1, OX, OG, OH are positions of the chord of contact, and the points P_2, L, Θ, P_1, X, G, H, on the hyperbola which is the curve of centres are the corresponding positions of the centre.

Chord of Contact.	*Centre.*
OP_2.	P_2, at infinity on hyperbola.
$OL\,(z=0)$.	L, $(z=0,\ x-y=0)$.
$O\Theta$.	Θ, the line joining this with O being always behind $O\Theta$.
OP_1.	P_1, at infinity on hyperbola.
OX $\{x+y\sqrt{(mn)}=0\}$.	X $(x=0,\ y=0)$.
OG (parallel to $y=0$).	G (on line $y=0$).
OH (parallel to $x=0$) and so back to	H (on line $x=0$) and so on to
OP_2.	P_2.

I have treated separately the case $\sqrt{(mn)}=1$.

Consider the conics which touch the lines $y-x=0$, $y+x=0$ and pass through the points

$$\{x=1,\ y=\sqrt{(1-c^2)}\},\ \{x=1,\ y=-\sqrt{(1-c^2)}\}.$$

The equation is of the form

$$y^2-x^2+k(x-\alpha)^2=0,$$

and to determine k, we have

$$1-c^2-1+k(1-\alpha)^2=0, \text{ and therefore } k=\frac{c^2}{(1-\alpha)^2}.$$

The equation thus becomes

$$(1-\alpha)^2(y^2-x^2)+c^2(x-\alpha)^2=0,$$

that is

$$(1-\alpha)^2y^2+\{c^2-(1-\alpha)^2\}x^2-2c^2\alpha x+c^2\alpha^2=0,$$

or as this may be written

$$(1-\alpha)^2y^2+\{c^2-(1-\alpha)^2\}\left\{x-\frac{c^2\alpha}{c^2-(1-\alpha)^2}\right\}^2-\frac{c^2\alpha^2(1-\alpha)^2}{c^2-(1-\alpha)^2}=0.$$

Hence the nature of the conic depends on the sign of $c^2-(1-\alpha)^2$, viz. if this be positive, or α between the limits $1+c$, $1-c$, the curve is an ellipse,

$$x\text{-coordinate of centre}=\frac{c^2\alpha}{c^2-(1-\alpha)^2},$$

which is positive,

$$x\text{-semi-axis}=\frac{\pm c\alpha(1-\alpha)}{c^2-(1-\alpha)^2},$$

$$y\text{-semi-axis}=\frac{c\alpha}{\sqrt{\{c^2-(1-\alpha)^2\}}}.$$

The coordinate of centre for $\alpha=1+c$ is $=+\infty$ (the curve being in this case a parabola P_1) and for $\alpha=1-c$ it is also $=+\infty$ (the curve being in this case a parabola P_2). The coordinate has a minimum value corresponding to $\alpha=\sqrt{(1-c^2)}$, viz. this is $=\frac{1}{2}\{1+\sqrt{(1-c^2)}\}$.

Hence as (α) passes from $1+c$ to $\sqrt{(1-c^2)}$, the coordinate of the centre passes from ∞ to its minimum value $\frac{1}{2}\{1+\sqrt{(1-c^2)}\}$; in the passage we have $\alpha=1$ giving the coordinate $=1$, the conic being in this case a pair of coincident lines $(x-1)^2=0$. And as (α) passes from the foregoing value $\sqrt{(1-c^2)}$ to $1-c$, the coordinate of the centre passes from the minimum value $\frac{1}{2}\{1+\sqrt{(1-c^2)}\}$ to ∞.

The curve is a hyperbola if α lies without the limits $1+c$, $1-c$,

$$x\text{-coordinate of centre} \quad = \frac{-c^2\alpha}{(1-\alpha)^2-c^2},$$

which has the sign of $-\alpha$,

$$x\text{-semi-axis} \quad = \frac{\pm c\alpha(1-\alpha)}{(1-\alpha)^2-c^2},$$

$$y\text{-semi-axis} \quad = \frac{\pm c\alpha}{\sqrt{\{(1-\alpha)^2-c^2\}}},$$

$$\text{semi-aperture of asymptotes} = \tan^{-1}\sqrt{\left\{1-\frac{c^2}{(1-\alpha)^2}\right\}},$$

which for $\alpha=1\pm c$ is $=0$ (parabola), but increases as $1-\alpha$ increases positively or negatively, becoming $=45°$ for $\alpha=\pm\infty$ (the asymptotes being in this case the pair of lines $y^2-x^2=0$):

$$\alpha=+\infty, \text{ coordinate of centre is } = 0,$$
$$\alpha=1+c, \quad \text{"} \quad \text{"} \quad = -\infty,$$

so that α diminishing from ∞ to $1+c$, the coordinate of the centre moves constantly in the same direction from 0 to $-\infty$,

$$\alpha=1-c, \text{ coordinate of centre is } = -\infty,$$
$$\alpha=0, \quad \text{"} \quad \text{"} \quad = 0,$$

the hyperbola being in this case the pair of lines $y^2=(1-c^2)x^2$.

α negative, the coordinate of centre becomes positive, viz. as α passes from $\alpha=0$ to $\alpha=-\sqrt{(1-c^2)}$, the coordinate of centre passes from 0 to a maximum positive value $\frac{1}{2}\{1-\sqrt{(1-c^2)}\}$, and then as α passes from $-\sqrt{(1-c^2)}$ to $-\infty$, the coordinate of centre diminishes from $\frac{1}{2}\{1-\sqrt{(1-c^2)}\}$ to 0. It is to be remarked that α being negative, the lines $y^2-x^2=0$ are touched by the branch on the negative side of the origin, that is the branch not passing through the two points $x=1$, $y=\pm\sqrt{(1-c^2)}$.

393.

ON THE CONICS WHICH TOUCH THREE GIVEN LINES AND PASS THROUGH A GIVEN POINT.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 220—222.]

CONSIDER the triangles which touch three given lines; the three lines form a triangle, and the lines joining the angles of the triangle with the points of contact of the opposite sides respectively meet in a point S: conversely given the three lines and the point S, then joining this point with the angles of the triangle the joining lines meet the opposite sides respectively in three points which are the points of contact with the three given lines respectively of a conic; such conic is determinate and unique. Suppose now that the conic passes through a given point; the point S is no longer arbitrary, but it must lie on a certain curve; and this curve being known, then taking upon it any point whatever for the point S, and constructing as before the conic which corresponds to such point, the conic in question will pass through the given point, and will thus be a conic touching the three given lines and passing through the given point. And the series of such conics corresponds of course to the series of points on the curve.

I proceed to find the curve which is the locus of the point S.

We may take $x=0$, $y=0$, $z=0$ for the equations of the given lines, and $x:y:z=1:1:1$ for the coordinates of the given point. The equation of a conic touching the three given lines is

$$a\sqrt{(x)}+b\sqrt{(y)}+c\sqrt{(z)}=0,$$

and the coordinates of the corresponding point S are as $\frac{1}{a^2}:\frac{1}{b^2}:\frac{1}{c^2}$, that is, taking (x, y, z) for the coordinates of the point in question, we have

$$a:b:c=\frac{1}{\sqrt{(x)}}:\frac{1}{\sqrt{(y)}}:\frac{1}{\sqrt{(z)}},$$

the condition in order that the conic may pass through the given point is $a+b+c=0$, and we thus find for the curve, which is the locus of the point S, the equation

$$\frac{1}{\sqrt{(x)}}+\frac{1}{\sqrt{(y)}}+\frac{1}{\sqrt{(z)}}=0,$$

or, what is the same thing,

$$\sqrt{(yz)}+\sqrt{(zx)}+\sqrt{(xy)}=0,$$

the rationalised form of which is

$$y^2z^2+z^2x^2+x^2y^2-2xyz\,(x+y+z)=0.$$

This is a quartic curve with three cusps, viz. each angle of the triangle is a cusp; and by considering for example the cusp $(y=0,\ z=0)$ and writing the equation under the form

$$x^2(y-z)^2-2x(yz^2+y^2z)+y^2z^2=0,$$

we see that the tangent at the cusp in question is the line $y-z=0$; that is, the tangents at the three cusps are the lines joining these points respectively with the given point (1, 1, 1). Each cuspidal tangent meets the curve in the cusp counting as three points and in a fourth point of intersection, the coordinates whereof in the case of the tangent $y-z=0$, are at once found to be $x:y:z=1:4:4$, or say this is the point (1, 4, 4); the point on the tangent $z-x=0$ is of course (4, 1, 4), and that on the tangent $x-y=0$ is (4, 4, 1). To find the tangents at these points respectively, I remark that the general equation of the tangent is

$$(X\delta_x+Y\delta_y+Z\delta_z)\left\{\frac{1}{\sqrt{(x)}}+\frac{1}{\sqrt{(y)}}+\frac{1}{\sqrt{(z)}}\right\}=0,$$

that is

$$\frac{X}{x^{\frac{3}{2}}}+\frac{Y}{y^{\frac{3}{2}}}+\frac{Z}{z^{\frac{3}{2}}}=0,$$

or for the point (1, 4, 4) the equation of the tangent is $8X+Y+Z=0$, or say $8x+y+z=0$; that is, the tangent passes through the point $x=0$, $x+y+z=0$, being the point of intersection of the line $x=0$ with the line $x+y+z=0$, which is the harmonic of the given point (1, 1, 1) in regard to the triangle; the tangents at the points (1, 4, 4), (4, 1, 4), (4, 4, 1) respectively pass through the points of intersection of the harmonic line $x+y+z=0$ with the three given lines respectively.

In the case where the given point lies within the triangle, the curve the locus of S lies wholly within the triangle, and is of the form shown in fig. 3 in the plate opposite; it is clear that in this case the conics of the system are all of them ellipses; there are however three limiting forms, viz. the line joining the given point with any angle of the triangle, such line being regarded as a twofold line or pair of coincident lines, is a conic of the system. The discussion of the two cases in which the given point lies outside the triangle, viz. in the infinite space bounded by two sides produced, or in the infinite space bounded by a side and two sides produced, may be effected without much difficulty.

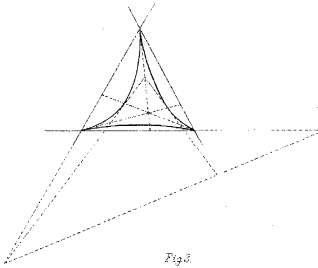


Fig 3.

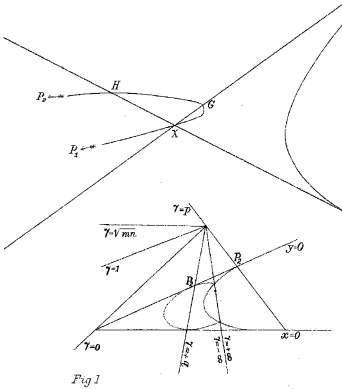


Fig 1

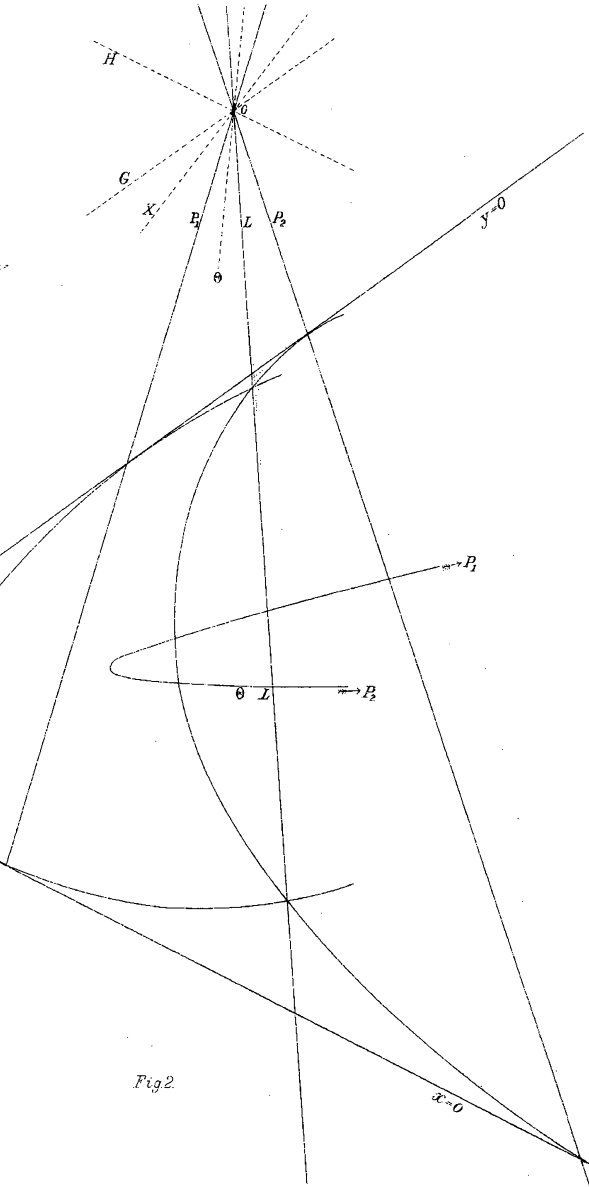


Fig. 2.

To face p. 52.

394.

ON A LOCUS IN RELATION TO THE TRIANGLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 264—277.]

IF from any point of a circle circumscribed about a triangle perpendiculars are let fall upon the sides, the feet of the perpendiculars lie in a line; or, what is the same thing, the locus of a point, such that the perpendiculars let fall therefrom upon the sides of a given triangle have their feet in a line, is the circle circumscribed about the triangle.

In this well known theorem we may of course replace the circular points at infinity by any two points whatever; or the Absolute being a point-pair, and the terms perpendicular and circle being understood accordingly, we have the more general theorem expressed in the same words.

But it is less easy to see what the corresponding theorem is, when instead of being a point-pair, the Absolute is a proper conic; and the discussion of the question affords some interesting results.

Take $(x=0,\ y=0,\ z=0)$ for the equations of the sides of the triangle, and let the equation of the Absolute be

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$

then any two lines which are harmonics in regard to this conic (or, what is the same thing, which are such that the one of them passes through the pole of the other) are said to be perpendicular to each other, and the question is:

Find the locus of a point, such that the perpendiculars let fall therefrom on the sides of the triangle have their feet in a line.

Supposing, as usual, that the inverse coefficients are $(A,\ B,\ C,\ F,\ G,\ H)$, and that K is the discriminant, the coordinates of the poles of the three sides respectively are

(A, H, G), (H, B, F), (G, F, C). Hence considering a point P, the coordinates of which are (x, y, z), and taking (X, Y, Z) for current coordinates, the equation of the perpendicular from P on the side $X=0$ is

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ A, & H, & G \end{vmatrix} = 0,$$

and writing in this equation $X=0$, we find

$$(\quad 0 \quad , Ay - Hx, Az - Gx)$$

for the coordinates of the foot of the perpendicular. For the other perpendiculars respectively, the coordinates are

$$(Bx - Hy, \quad 0 \quad , Bz - Fy),$$

and

$$(Cx - Gz, Cy - Fz, \quad 0 \quad),$$

and hence the condition in order that the three feet may lie in a line is

$$\begin{vmatrix} 0 \quad , & Ay - Hx, & Az - Gx \\ Bx - Hy, & 0 \quad , & Bz - Fy \\ Cx - Gz, & Cy - Fz, & 0 \end{vmatrix} = 0;$$

or, what is the same thing,

$$(Ay - Hx)(Bz - Fy)(Cx - Gz) + (Az - Gx)(Bx - Hy)(Cy - Fz) = 0,$$

that is

$$\begin{aligned} & 2(ABC - FGH)xyz \\ & + A(FH - BG)yz^2 + A(FG - CH)y^2z \\ & + B(FG - CH)zx^2 + B(GH - AF)z^2x \\ & + C(GH - AF)xy^2 + C(HF - BG)x^2y = 0, \end{aligned}$$

which is the equation of the locus of P; the locus is therefore a cubic. Writing for a moment

$$(BC - F^2, CA - G^2, AB - H^2, GH - AF, HF - BG, FG - CH) = (A', B', C', F', G', H'),$$

and K' for the discriminant $ABC - AF^2 -$ &c., the equation is

$$2(ABC - FGH)xyz + Ayz(H'y + G'z) + Bzx(H'x + F'z) + Cxy(G'x + F'y) = 0,$$

or as this may also be written

$$\frac{2}{F'G'H'}(ABC - FGH)xyz + \frac{A}{F'}yz\left(\frac{y}{G'} + \frac{z}{H'}\right) + \frac{B}{G'}zx\left(\frac{x}{F'} + \frac{z}{H'}\right) + \frac{C}{H'}xy\left(\frac{x}{F'} + \frac{y}{G'}\right) = 0,$$

that is

$$\left[\frac{2}{F'G'H'}(ABC-FGH)-\frac{A}{F'^2}-\frac{B}{G'^2}-\frac{C}{H'^2}\right]xyz+\left(\frac{A}{F'}yz+\frac{B}{G'}zx+\frac{C}{H'}xy\right)\left(\frac{x}{F'}+\frac{y}{G'}+\frac{z}{H'}\right)=0,$$

and the cubic will therefore break up into a line and conic if only

$$\frac{2}{F'G'H'}(ABC-FGH)-\frac{A}{F'^2}-\frac{B}{G'^2}-\frac{C}{H'^2}=0,$$

and it is easy to see that conversely this is the necessary and sufficient condition in order that the cubic may so break up.

The condition is

$$\Omega=2F'G'H'(ABC-FGH)-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2=0,$$

we have

$$AA'+BB'+CC'=3ABC-AF'^2-BG'^2-CH'^2,\ =K'+2(ABC-FGH),$$

and thence

$$\Omega=F'G'H'(AA'+BB'+CC'-K')-AG'^2H'^2-BH'^2F'^2-CF'^2G'^2,$$

that is

$$\begin{aligned}\Omega&=-AG'H'(G'H'-A'F')-BH'F'(H'F'-B'G')-CF'G'(F'G'-C'H')-K'F'G'H',\\&=-AG'H'K'F-BH'F'K'G-CF'G'K'H-K'F'G'H',\\&=-K'(AFG'H'+BGH'F'+CHF'G'+F'G'H'),\end{aligned}$$

so that the condition $\Omega=0$ is satisfied if $K'=0$, that is if the equation

$$(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2=0,$$

which is the line-equation of the Absolute breaks up into factors; that is, if the Absolute be a point-pair.

In the case in question we may write

$$(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2=2(\alpha\xi+\beta\eta+\gamma\zeta)(\alpha'\xi+\beta'\eta+\gamma'\zeta),$$

that is

$$(A,\ B,\ C,\ F,\ G,\ H)=(2\alpha\alpha',\ 2\beta\beta',\ 2\gamma\gamma',\ \beta\gamma'+\beta'\gamma,\ \gamma\alpha'+\gamma'\alpha,\ \alpha\beta'+\alpha'\beta),$$

whence also, putting for shortness,

$$(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)=\ (\lambda,\ \mu,\ \nu),$$

we have

$$(A',\ B',\ C',\ F',\ G',\ H')=-(\lambda^2,\ \mu^2,\ \nu^2,\ \mu\nu,\ \nu\lambda,\ \lambda\mu),$$

and also

$$K'=0,\ 2(ABC-FGH)=AA'+BB'+CC',\ =-2(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2).$$

The original cubic equation is

$$(\alpha\alpha'\lambda^2+\beta\beta'\mu^2+\gamma\gamma'\nu^2)\,xyz+\alpha\alpha'\lambda yz\,(\mu y+\nu z)+\beta\beta'\mu zx\,(\lambda x+\nu z)+\gamma\gamma'\nu xy\,(\lambda x+\mu y)=0,$$

and this in fact is

$$(\alpha\alpha'\lambda yz+\beta\beta'\mu zx+\gamma\gamma'\nu xy)\,(\lambda x+\mu y+\nu z)=0.$$

The equation $\lambda x+\mu y+\nu z=0$ is that of the line through the two points which constitute the Absolute; the other factor gives

$$\alpha\alpha'\lambda yz+\beta\beta'\mu zx+\gamma\gamma'\nu xy=0,$$

which is the equation of a conic through the angles of the triangle ($x=0$, $y=0$, $z=0$), and which also passes through the two points of the Absolute; in fact, writing (α, β, γ) for (x, y, z) the equation becomes $\alpha\beta\gamma(\alpha'\lambda+\beta'\mu+\gamma'\nu)=0$, and so also writing $(\alpha', \beta', \gamma')$ for (x, y, z) it becomes $\alpha'\beta'\gamma'(\alpha\lambda+\beta\mu+\gamma\nu)=0$, which relations are identically satisfied by the values of (λ, μ, ν). Hence we see that the Absolute being a point-pair, the locus is the conic passing through the angles of the triangle, and the two points of the Absolute; that is, it is the *circle* passing through the angles of the triangle.

But assuming that K' is not $=0$, or that the Absolute is a proper conic, the equation $\Omega=0$ will be satisfied if

$$AFG'H'+BGH'F'+CHF'G'+F'G'H'=0,$$

we have $F', G', H'=Kf, Kg, Kh$ respectively, or omitting the factor K^2, the equation becomes

$$AFgh+BGhf+CHfg+Kfgh=0,$$

which is

$$f^3g^2h^2-bcg^2h^2-cah^2f^2-abf^2g^2+2abcfgh=0,$$

or, as it may also be written,

$$abcf^2g^2h^2\left(\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}\right)=0.$$

I remark that we have $ABC-FGH=K(abc-fgh)$; substituting also for F', G', H' the values Kf, Kg, Kh, the equation of the cubic curve is

$$2\,(abc-fgh)\,xyz+Ayz\,(hy+gz)+Bzx\,(hx+fy)+Cxy\,(gx+fy)=0,$$

and the transformed form is

$$\left[\frac{2}{fgh}(abc-fgh)-\frac{A}{f^2}-\frac{B}{g^2}-\frac{C}{h^2}\right]xyz+\left(\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy\right)\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)=0;$$

we have

$$\frac{2}{fgh}(abc-fgh)-\frac{A}{f^2}-\frac{B}{g^2}-\frac{C}{h^2}=abc\left(\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}\right),$$

so that the foregoing condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

being satisfied, the cubic breaks up into the line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$, and the conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0.$$

It is to be remarked that in general a triangle and the reciprocal triangle are in perspective; that is, the lines joining corresponding angles meet in a point, and the points of intersections of opposite sides lie in a line; this is the case therefore with the triangle $(x=0,\ y=0,\ z=0)$, and the reciprocal triangle

$$(ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0);$$

and it is easy to see that the line through the points of intersection of corresponding sides is in fact the above mentioned line $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$. It is to be noticed also that the coordinates of the point of intersection of the lines joining the corresponding angles are $(F,\ G,\ H)$. The conic

$$\frac{A}{f}yz+\frac{B}{g}zx+\frac{C}{h}xy=0$$

is of course a conic passing through the angles of the triangle $(x=0,\ y=0,\ z=0)$; it is *not*, what it might have been expected to be, a conic having double contact with the Absolute $(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2$.

I return to the condition

$$\frac{1}{abc}-\frac{1}{af^2}-\frac{1}{bg^2}-\frac{1}{ch^2}+\frac{2}{fgh}=0,$$

this can be shown to be the condition in order that the sides of the triangle $(x=0,\ y=0,\ z=0)$, and the sides of the reciprocal triangle $(ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0)$ touch one and the same conic; in fact, using line coordinates, the coordinates of the first three sides are $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(0,\ 0,\ 1)$ respectively, and those of the second three sides are $(a,\ h,\ g)$, $(h,\ b,\ f)$, $(g,\ f,\ c)$ respectively; the equation of a conic touching the first three lines is

$$\frac{L}{\xi}+\frac{M}{\eta}+\frac{N}{\zeta}=0,$$

and hence making the conic touch the second three sides, we have three linear equations from which eliminating $L,\ M,\ N$, we find

$$\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix}=0,$$

which is the equation in question.

We know that if the sides of two triangles touch one and the same conic, their angles must lie in and on the same conic. The coordinates of the angles are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (A, H, G), (H, B, F), (G, F, C) respectively, and the angles will be situate in a conic if only

$$\begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix} = 0,$$

an equation which must be equivalent to the last preceding one; this is easily verified. In fact, writing for shortness

$$\nabla = \begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & \frac{1}{g} \\ \frac{1}{h}, & \frac{1}{b}, & \frac{1}{f} \\ \frac{1}{g}, & \frac{1}{f}, & \frac{1}{c} \end{vmatrix}, \quad \square = \begin{vmatrix} \frac{1}{A}, & \frac{1}{H}, & \frac{1}{G} \\ \frac{1}{H}, & \frac{1}{B}, & \frac{1}{F} \\ \frac{1}{G}, & \frac{1}{F}, & \frac{1}{C} \end{vmatrix},$$

we have

$$-\square = \frac{1}{ABCF^2}(BC - F^2) + \frac{1}{CFGH^2}(FG - CH) + \frac{1}{BFG^2H}(HF - BG),$$

$$= \frac{K}{ABCF^2G^2H^2}(aG^2H^2 + hABFG + gCAHF),$$

and the second factor is

$$= aGH(AF + Kf) + AFhBG + AFgCH,$$
$$= AF(aGH + hBG + gCH) + KafGH.$$

But

$$\begin{aligned} aGH + hBG + gCH = \quad & G(aH + hB) + gCH = G - gF + gCH, \\ = \quad & G - gF + gCH, \\ = & -g(FG - CH), \\ = & -ghK, \end{aligned}$$

so that the second factor is

$$= K(afGH - ghAF),$$

which is

$$= K(f^2g^2h^2 - bcg^2h^2 - cah^2f^2 - abf^2g^2 + 2abcfgh),$$

$$= Kabcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right),$$

$$= Kabcf^2g^2h^2\nabla,$$

so that we have identically

$$-ABCF^2G^2H^2\,\square = K^2abcf^2g^2h^2\nabla,$$

and the conditions $\nabla = 0$, $\square = 0$ are consequently equivalent.

The condition

$$\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh} = 0,$$

is the condition in order that the function

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)\!\!\!

contains the angles of the reciprocal triangle, and is thus in fact the conic in which are situate the angles of the two triangles. For the coordinates of one of the angles of the reciprocal triangle are (A, H, G); we should therefore have

$$\frac{A}{f}HG + \frac{B}{g}GA + \frac{C}{h}AH = 0,$$

which is

$$\frac{A}{fgh}(GHgh + BGhf + CHfg) = 0,$$

or attending only to the second factor and writing

$$GH = Kf + AF,$$

the condition is

$$Kfgh + AFgh + BGhf + CHfg = 0,$$

or substituting for K, A, B, C, F, G, H their values and reducing, this is

$$-abcf^2g^2h^2\left(\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh}\right) = 0,$$

which is satisfied: hence the three angles of the reciprocal triangle lie on the conic in question.

Partially recapitulating the foregoing results, we see in the case where the Absolute is not a point-pair, that the locus of a point such that the perpendiculars from it on the sides of the triangle have their feet in a line, is in general a *cubic curve* passing through the angles of the triangle: if, however, the condition

$$\frac{1}{abc} - \frac{1}{af^2} - \frac{1}{bg^2} - \frac{1}{ch^2} + \frac{2}{fgh} = 0$$

be satisfied, that is, if the triangle be such that the angles thereof and of the reciprocal triangle lie in a conic (or, what is the same thing, if the sides touch a conic) then the cubic locus breaks up into the line $\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0$, which is the line through the points of intersection of the corresponding sides of the two triangles, and into the conic

$$\frac{A}{f}yz + \frac{B}{g}zx + \frac{C}{h}xy = 0,$$

which is the conic through the angles of the two triangles.

The question arises, given a conic (the Absolute) to construct a triangle such that its angles, and the angles of the reciprocal triangle in regard to the given conic, lie in a conic.

I suppose that two of the angles of the triangle are given, and I enquire into the locus of the remaining angle. To fix the ideas, let A, B, C be the angles of the triangle, A', B', C' those of the reciprocal triangle; and let the angles A and B be given. We have to find the locus of the point C: I observe however, that the lines AA', BB', CC' meet in a point O, and I conduct the investigation in such manner as to obtain simultaneously the loci of the two points C and O. The lines $C'B'$, $C'A'$ are the polars of A, B respectively, let their equations be $x=0$, and $y=0$, and let the equation of the line AB be $z=0$; this being so, the equation of the given conic will be of the form

$$(a,\ b,\ c,\ 0,\ 0,\ h\between x,\ y,\ z)^2=0.$$

I take $(\alpha,\ \beta,\ \gamma)$ for the coordinates of O and $(x,\ y,\ z)$ for those of C; the coordinates of either of these points being of course deducible from those of the other.

Observing that the inverse coefficients are

$$(bc,\ ca,\ ab-h^2,\ 0,\ 0,\ -ch),$$

we find

$$\begin{aligned} &\text{coordinates of } A \text{ are } (\ b,\ -h,\ 0),\\ &\quad\text{,,}\qquad\quad B\ \text{,,}\ (-h,\ \ a,\ 0). \end{aligned}$$

The points A' and B' are then given as the intersections of AO with $C'A'\,(y=0)$ and of BO with $C'B'\,(x=0)$; we find

$$\begin{aligned} &\text{coordinates of } A' \text{ are } (h\alpha+b\beta,\quad 0\quad,\ h\gamma),\\ &\quad\text{,,}\qquad\quad B'\ \text{,,}\ (\ \ 0\ \ ,\ \ a\alpha+h\beta,\ h\gamma). \end{aligned}$$

Moreover,

$$\begin{aligned} &\text{coordinates of } C' \text{ are } (0,\ 0,\ 1),\\ &\quad\text{,,}\qquad\quad C\ \text{,,}\ (x,\ y,\ z). \end{aligned}$$

The six points A, B, C, A', B', C' are to lie in a conic; the equations of the lines $C'A$, $C'B$, AB are $hX+bY=0$, $aX+hY=0$, $Z=0$, and hence the equation of a conic passing through the points C', A, B is

$$\frac{L}{aX+hY}+\frac{M}{hX+bY}+\frac{N}{Z}=0.$$

Hence, making the conic pass through the remaining points A', B', C, we find

$$\frac{L}{a(h\alpha+b\beta)}+\frac{M}{h(h\alpha+b\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{h(a\alpha+h\beta)}+\frac{M}{b(a\alpha+h\beta)}+\frac{N}{h\gamma}=0,$$

$$\frac{L}{ax+hy}+\frac{M}{hx+by}+\frac{N}{z}=0,$$

and eliminating the L, M, N, we find

$$\begin{vmatrix} \frac{1}{a}, & \frac{1}{h}, & h\alpha+b\beta \\ \frac{1}{h}, & \frac{1}{b}, & a\alpha+h\beta \\ \frac{1}{ax+hy}, & \frac{1}{hx+by}, & \frac{hy}{z} \end{vmatrix} = 0,$$

or developing and reducing, this is

$$-\frac{(ab-h^2)}{hab}\frac{y}{z}+\frac{1}{h}\frac{a\alpha+h\beta}{ax+hy}+\frac{1}{h}\frac{h\alpha+b\beta}{hx+by}$$
$$-\frac{1}{a}\frac{a\alpha+h\beta}{hx+by}-\frac{1}{b}\frac{h\alpha+b\beta}{ax+hy}=0.$$

We have still to find the relation between (α, β, γ) and (x, y, z); this is obtained by the consideration that the line $A'B'$, through the two points A', B' the coordinates of which are known in terms of (α, β, γ), is the polar of the point C, the coordinates of which are (x, y, z). The equation of $A'B'$ is thus obtained in the two forms

$$(a\alpha+h\beta)X+(h\alpha+b\beta)Y-\frac{(a\alpha+h\beta)(h\alpha+b\beta)}{hy}Z=0,$$

and

$$(ax+hy)X+(hx+by)Z+\qquad cz\qquad Z=0,$$

and comparing these, we have

$$x:y:z=\alpha:\beta:\frac{-(a\alpha+h\beta)(h\alpha+b\beta)}{chy},$$

or what is the same thing

$$\alpha:\beta:\gamma=x:y:\frac{-(ax+hy)(hx+by)}{chz},$$

(where it is to be observed that the equation $\alpha:\beta=x:y$ is the verification of the theorem that the lines AA', BB', CC' meet in a point O).

We may now from the above found relation eliminate either the (α, β, γ) or the (x, y, z); first eliminating the (α, β, γ), we find

$$-\frac{ab-h^2}{hab}\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\frac{ax+hy}{hx+by}-\frac{1}{b}\frac{hx+by}{ax+hy}=0,$$

where

$$\frac{Y}{Z}=-\frac{(ax+hy)(hx+by)}{chz^2},$$

or, completing the elimination,

$$\frac{ab-h^2}{ch}\frac{(ax+hy)^2(hx+by)^2}{z^2}=(hb,\ -ab,\ ha \between ax+hy,\ hx+by)^2=0,$$

which is a quartic curve having a node at each of the points

$$(z=0,\ ax+hy=0),\ (z=0,\ hx+by=0),\ (ax+hy=0,\ hx+by=0),$$

that is, at each of the points B, A, C'. The right-hand side of the foregoing equation is

$$=-(ab-h^2)(ha,\ ab,\ hb\between x,\ y)^2,\ =-(ab-h^2)\,h\left(ax^2+by^2+\frac{2ab}{h}xy\right),$$

so that the equation may also be written

$$(ax+hy)^2(hx+by)^2+ch^2z^2\left(ax^2+by^2+\frac{2ab}{h}xy\right)=0.$$

Secondly, to eliminate the $(x,\ y,\ z)$, we have

$$-\frac{ab-h^2}{hab}\frac{Y}{Z}+\frac{2}{h}-\frac{1}{a}\frac{a\alpha+h\beta}{h\alpha+b\beta}-\frac{1}{b}\frac{h\alpha+b\beta}{a\alpha+h\beta}=0,$$

where

$$\frac{Y}{Z}=-\frac{chy^2}{(a\alpha+h\beta)(h\alpha+b\beta)},$$

or, completing the elimination,

$$\begin{aligned}(ab-h^2)\,chy^2 &= (hb,\ -ab,\ ha\between a\alpha+h\beta,\ h\alpha+b\beta)^2\\ &=-(ab-h^2)\,h\left(a\alpha^2+b\beta^2+\frac{2ab}{h}\alpha\beta\right),\end{aligned}$$

that is

$$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h}\between\alpha,\ \beta,\ \gamma\right)^2=0.$$

Writing $(x,\ y,\ z)$ in place of $(\alpha,\ \beta,\ \gamma)$, the locus of the point O is the conic

$$\left(a,\ b,\ c,\ 0,\ 0,\ \frac{ab}{h}\between x,\ y,\ z\right)^2=0,$$

which is a conic intersecting the Absolute

$$(a,\ b,\ c,\ 0,\ 0,\ h\between x,\ y,\ z)^2=0,$$

at its intersections with the lines $x=0$, $y=0$, that is the lines $C'B'$ and $C'A'$.

In regard to this new conic, the coordinates of the pole of $C'B'\,(x=0)$ are at once found to be $(-h,\ a,\ 0)$, that is, the pole of $C'B'$ is B; and similarly the coordinates of the pole of $C'A'\,(y=0)$ are $(b,\ -h,\ 0)$, that is, the pole of $C'A'$ is A. We may consequently construct the conic the locus of O, viz. given the Absolute and the points A and B, we have $C'A'$ the polar of B, meeting the Absolute in two points $(a_1,\ a_2)$, and $C'B'$ the polar of A meeting the Absolute in the points $(b_1$ and $b_2)$; the lines $C'A'$ and $C'B'$ meet in C'. This being so, the required conic passes through the points a_1, a_2, b_1, b_2, the tangents at these points being Aa_1, Aa_2, Bb_1, Bb_2 respectively; eight conditions, five of which would be sufficient to determine the conic. It is to be remarked that the lines $C'B'$, $C'A'$ (which in regard to the Absolute are the polars of A, B respectively) are in regard to the required conic the polars of B, A respectively.

The conic the locus of O being known, the point O may be taken at any point of this conic, and then we have A' as the intersection of $C'A'$ with AO, B' as the intersection of $C'B'$ with BO, and finally, C as the pole of the line $A'B'$ in regard

to the Absolute, the point so obtained being a point on the line $C'O$. To each position of O on the conic locus, there corresponds of course a position of C; the locus of C is, as has been shown, a quartic curve having a node at each of the points C', A, B.

The foregoing conclusions apply of course to spherical figures; we see therefore that on the sphere the locus of a point such that the perpendiculars let fall on the sides of a given spherical triangle have their feet in a line (great circle), is a spherical cubic. If, however, the spherical triangle is such that the angles thereof and the poles of the sides (or, what is the same thing, the angles of the polar triangle) lie on a spherical conic; then the cubic locus breaks up into a line (great circle), which is in fact the circle having for its pole the point of intersection of the perpendiculars from the angles of the triangle on the opposite sides respectively, and into the before-mentioned spherical conic. Assuming that the angles A and B are given, the above-mentioned construction, by means of the point O, is applicable to the determination of the locus of the remaining angle C, in order that the spherical triangle ABC may be such that the angles and the poles of the sides lie on the same spherical conic, but this requires some further developments. The lines $C'B'$, $C'A'$ which are the polars of the given angles A, B respectively, are the cyclic arcs of the conic the locus of O, or say for shortness the conic O; and moreover these same lines $C'B'$, $C'A'$ are in regard to the conic O, the polars of the angles B, A respectively. If instead of the conic O we consider the polar conic O', it follows that A, B are the foci, and $C'A'$, $C'B'$ the corresponding directrices of the conic O'. The distance of the directrix $C'A'$ from the centre of the conic, measuring such distance along the transverse axis is clearly $=90^\circ -$ distance of the focus A; it follows that the transverse semi-axis is $=45^\circ$, or what is the same thing, that the transverse axis is $=90^\circ$; that is, the conic O' is a conic described about the foci A, B with a transverse axis (or sum or difference of the focal distances) $=90^\circ$. Considering any tangent whatever of this conic, the pole of the tangent is a position of the point O, which is the point of intersection of the perpendiculars let fall from the angles of the spherical triangle on the opposite sides; hence, to complete the construction, we have only through A and B respectively to draw lines AC, BC perpendicular to the lines BO, CO respectively; the lines in question will meet in a point C, which is such that CO will be perpendicular to AB, and which point C is the required third angle of the spherical triangle ABC. In order to ascertain whether a given spherical triangle ABC has the property in question (viz. whether it is such that the angles thereof and of the polar triangle lie in a spherical conic), we have only to construct as before the conic O' with the foci A, B and transverse axis $=90^\circ$, and then ascertain whether the polar of the point O, the intersections of the perpendiculars from the angles of the triangle on the opposite sides respectively, is a tangent of the conic O'. It is moreover clear, that given a triangle ABC having the property in question, if with the foci A, B and transverse axis $=90^\circ$ we describe a conic, and if in like manner with the foci A, C and the same transverse axis, and with the foci B, C and the same transverse axis, we describe two other conics; then that the three conics will have a common tangent the pole whereof will be the point of intersection of the perpendiculars from the angles of the triangle ABC on the opposite sides respectively.

395.

INVESTIGATIONS IN CONNEXION WITH CASEY'S EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 334—341.]

In a paper read April 9, 1866, and recently published in the *Proceedings of the Royal Irish Academy*, Mr Casey has given in a very elegant form the equation of a pair of circles touching each of three given circles, viz. if $U=0$, $V=0$, $W=0$ be the equations of the three given circles respectively, and if considering the common tangents of $(V=0,\ W=0)$, of $(W=0,\ U=0)$, and of $(U=0,\ V=0)$ respectively, these common tangents being such that the centres of similitude through which they respectively pass lie in a line (viz. the tangents are all three direct, or one is direct and the other two are inverse), then if f, g, h are the lengths of the tangents in question, the equation

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

belongs to a pair of circles, each of them touching the three given circles. (There are, it is clear, four combinations of tangents, and the theorem gives therefore the equations of four pairs of circles, that is of the eight circles which touch the three given circles.)

Generally, if $U=0$, $V=0$, $W=0$ are the equations of any three curves of the same order n, and if f, g, h are arbitrary coefficients, then the equation

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

is that of a curve of the order $2n$, touching each of the curves $U=0$, $V=0$, $W=0$, n^2 times, viz. it touches

$U=0$, at its n^2 intersections with $gV-hW=0$,
$V=0$ " " $hW-fU=0$,
$W=0$ " " $fU-gV=0$.

If however the curves $U=0$, $V=0$, $W=0$ have a common intersection, then the curve in question has a node at this point, and besides touches each of the three curves in n^2-1 points; and similarly, if the curves $U=0$, $V=0$, $W=0$ have k common intersections, then the curve in question has a node at each of these points, and besides touches each of the three curves in n^2-k points.

In particular, if $U=0$, $V=0$, $W=0$ are conics having two common intersections, then the curve is a quartic having a node at each of the common intersections, and besides touching each of the given conics in two points; whence, if the coefficients f, g, h (that is, their ratios) are so determined that the quartic may have two more nodes, then the quartic, having in all four nodes, will break up into a pair of conics, each passing through the common intersections, and the pair touching each of the given conics in two points; that is, the component conics will each of them touch each of the given conics once. Taking the circular points at infinity for the common intersections, the conics will be circles, and we thus see that Casey's theorem is in effect a determination of the coefficients f, g, h, in such wise that the curve

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

(which when $U=0$, $V=0$, $W=0$ are circles, is by what precedes a bicircular quartic) shall have two more nodes, and so break up into a pair of circles.

The question arises, given $U=0$, $V=0$, $W=0$, curves of the same order n, it is required to determine the ratios $f:g:h$ in such wise that the curve

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

may have two nodes; or we may simply inquire as to the number of the sets of values of $(f:g:h)$, which give a binodal curve, $\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0$.

I had heard of Mr Casey's theorem from Dr Salmon, and communicated it together with the foregoing considerations to Prof. Cremona, who, in a letter dated Bologna, March 3, 1866, sent me an elegant solution of the question as to the number of the binodal curves. This solution is in effect as follows:

LEMMA. Given the curves $U=0$, $V=0$, $W=0$ of the same order n; consider the point (f, g, h), and corresponding thereto the curve $fU+gV+hW=0$. As long as the point (f, g, h) is arbitrary, the curve $fU+gV+hW=0$, will not have any node, and in order that this curve may have a node, it is necessary that the point (f, g, h) shall lie on a certain curve Σ; this being so, the node will lie on a curve J, the Jacobian of the curves U, V, W; and the curves J and Σ will correspond to each other, point to point; viz. taking for (f, g, h) any point whatever on the curve Σ, the curve $fU+gV+hW=0$ will be a curve having a node at some one point on the curve J; and conversely, in order that the curve $fU+gV+hW=0$ may be a curve having a node at a given point on the curve J, it is necessary that the point (f, g, h) shall be at some one point of the curve Σ. The curve Σ has however nodes and cusps; each node of Σ corresponds to two points of J, viz. the point (f, g, h) being at a node of Σ, the curve $fU+gV+hW=0$, is a binodal curve having a node at

each of the corresponding points on J; and each cusp of Σ corresponds to two coincident points of J, viz. the point (f, g, h) being at a cusp of Σ, the curve $fU+gV+hW=0$ is a cuspidal curve having a cusp at the corresponding point of J. The number of the binodal curves $fU+gV+hW=0$ is thus equal to the number of the nodes of Σ, and the number of the cuspidal curves $fU+gV+hW=0$ is equal to the number of the cusps of Σ. The curve Σ is easily shown to be a curve of the order $3(n-1)^2$ and class $3n(n-1)$; and quà curve which corresponds point to point with J, it is a curve having the same deficiency as J, that is a deficiency $=\frac{1}{2}(3n-4)(3n-5)$; we have thence the Plückerian numbers of the curve Σ, viz.:

$$\begin{aligned}
&\text{Order is} &&= 3(n-1)^2,\\
&\text{Class} &&= 3n(n-1),\\
&\text{Cusps} &&= 12(n-1)(n-2),\\
&\text{Nodes} &&= \tfrac{3}{2}(n-1)(n-2)(3n^2-3n-11),\\
&\text{Inflexions} &&= 3(n-1)(4n-5),\\
&\text{Double tangents} &&= \tfrac{3}{2}(n-1)(n-2)(3n^2+3n-8).
\end{aligned}$$

Remarks. The consideration of the foregoing curve Σ is, I believe, first due to Prof. Cremona, it is a curve related to the three distinct curves $U=0$, $V=0$, $W=0$, in the same way precisely as Steiner's curve P_0 is related to the three curves $d_xU=0$, $d_yU=0$, $d_zU=0$. (Steiner, "Allgemeine Eigenschaften der algebraischen Curven," *Crelle*, t. XLVII. (1854), pp. 1—6; see also Clebsch, "Ueber einige von Steiner behandelte Curven," *Crelle*, t. LXIV. (1865), pp. 288—293), and the Plückerian numbers of P_0 (writing therein $n+1$ for n) are identical with those of Σ. The foregoing expressions $\frac{3}{2}(n-1)(n-2)(3n^2-3n-11)$ and $12(n-1)(n-2)$ for the numbers of the binodal and cuspidal curves $fU+gV+hW=0$, are given in my memoir "On the Theory of Involution," *Cambridge Philosophical Transactions*, t. XI. (1866), pp. 21—38, see p. 32, [348]; but the employment of the curve Σ very much simplifies the investigation.

Passing now to the proposed question, we have as before the curves $U=0$, $V=0$, $W=0$, of the same order n; and we may consider the point (f, g, h), and corresponding thereto the curve $\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0$, say for shortness the curve Ω, which is a curve of the order $2n$, having n^2 contacts with each of the given curves U, V, W. As long as the point (f, g, h) is arbitrary, the curve Ω has not any node; and in order that this curve may have a node, it is necessary that the point (f, g, h) shall lie on a certain curve Δ; this being so, the node will lie on the foregoing curve J, the Jacobian of the given curves U, V, W; and the curves J and Δ will correspond to each other, point to point, viz. taking for (f, g, h) any point whatever on the curve Δ, the curve Ω will have a node at some one point of J; and conversely, in order that the curve Ω may be a curve having a node at a given point of J, it is necessary that the point (f, g, h) shall be at some one point of the curve Δ. The curve Δ has however nodes and cusps; each node of Δ corresponds to two points of J, viz. for (f, g, h) at a node of Δ, the curve Ω is a binodal curve having a node at each of the corresponding points of J; each cusp of Δ corresponds to two coincident points of J, viz. for (f, g, h) at a cusp of Δ, the curve Ω is a cuspidal curve having a cusp at the corresponding

point of J. The number of the binodal curves Ω is consequently equal to that of the nodes of Δ, and the number of the cuspidal curves Ω is equal to that of the cusps of Δ; we have consequently to find the Plückerian numbers of the curve Δ; and this Prof. Cremona accomplishes by bringing it into connexion with the foregoing curve Σ, and making the determination depend upon that of the number of the conics which satisfy certain conditions of contact in regard to the curve Σ.

Consider, as corresponding to any given point (f, g, h) whatever, the conic $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$ which passes through three fixed points, the angles of the triangle $x=0$, $y=0$, $z=0$. For points (f, g, h) which lie in an arbitrary line $Af+Bg+Ch=0$, the corresponding conics pass through the fourth fixed point $x:y:z=A:B:C$. Assume for the moment that to the points (f, g, h) which lie on the foregoing curve Δ, correspond conics which touch the foregoing curve Σ. Then 1°. to the points of intersection of the curve Δ with an arbitrary line, correspond the conics which pass through four arbitrary points and touch the curve Σ; or the order of the curve Δ is equal to the number of the conics which can be drawn through four arbitrary points to touch the curve Σ; viz. if m be the order, n the class of Σ, the number of these conics is $=2m+n$, or substituting for m, n the values $3(n-1)^2$ and $3n(n-1)$ respectively, the number of these conics, that is the order of Δ, is $=3(n-1)(3n-2)$. 2°. To the nodes of Δ correspond the conics which pass through three arbitrary points and have two contacts with Σ, viz. if m be the order, n the class, and κ the number of cusps of Σ, then the number of these conics is $=\frac{1}{2}(2m+n)^2-2m-5n-\frac{3}{2}\kappa$, or substituting for m, n their values as above, and for κ its value $=12(n-1)(n-2)$, the number of these conics, that is, the number of the nodes of Δ, is found to be

$$=\tfrac{3}{2}(n-1)(27n^3-63n^2+22n+16).$$

3°. To the cusps of Δ correspond the conics which pass through three arbitrary points, and have with Σ a contact of the second order; the number of these (m, n, κ as above) is $=3n+\kappa$, or substituting for n and κ their values as above, the number of these conics, that is the number of the cusps of Δ, is $=3(n-1)(7n-8)$. We have thence all the Plückerian numbers of the curve Δ, viz. these are

Order	$= 3(n-1)(3n-2)$,
Class	$= 6(n-1)^2$,
Nodes	$= \frac{3}{2}(n-1)(27n^3-63n^2+22n+16)$,
Cusps	$= 3(n-1)(7n-8)$,
Double tangents	$= \frac{3}{2}(n-1)(12n^3-36n^2+19n+16)$,
Inflexions	$=12(n-1)(n-2)$,

and as a verification it is to be observed, that the deficiency of the curve Δ is equal to that of the curve J, viz. it has the value $\frac{1}{2}(3n-4)(3n-5)$. The foregoing numbers include the result that the number of the binodal curves

$$\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0,$$

is

$$=\tfrac{3}{2}(n-1)(27n^3-63n^2+22n+16).$$

The proof depended on the assumption, that to the points (f, g, h) which lie on the curve Δ, correspond the conics $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$ which touch the curve Σ; this M. Cremona proves in a very simple manner: the points of J correspond each to each with the points of Σ, or if we please they correspond each to each with the tangents of Σ. To the $6n(n-1)$ intersections of J with any curve Ω (viz. $\sqrt{(fU)}+\sqrt{(gV)}+\sqrt{(hW)}=0$) correspond the $6n(n-1)$ common tangents of Σ and the conic $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$; if Ω has a node, two of the $6n(n-1)$ intersections coincide, and the corresponding two tangents will also coincide, that is Ω having a node (or the point (f, g, h) being on the curve Δ), the conic touches the curve Σ. But it is not uninteresting to give an independent analytical proof. Write for shortness

$$\begin{aligned} dU &= Adx + Bdy + Cdz, \\ dV &= A'dx + B'dy + C'dz, \\ dW &= A''dx + B''dy + C''dz, \end{aligned}$$

and let (x, y, z) be the coordinates of a point on J, (X, Y, Z) those of the corresponding point on Σ, (f, g, h) those of the corresponding point on Δ. Write also for shortness

$$BC'-B'C,\ CA'-C'A,\ AB'-A'B=P : Q : R,$$

then we have

$$\begin{aligned} AX + BY + CZ &= 0, \\ A'X + B'Y + C'Z &= 0, \\ A''X + B''Y + C''Z &= 0, \end{aligned}$$

$$\begin{aligned} A\sqrt{\left(\frac{f}{U}\right)} + B\sqrt{\left(\frac{g}{V}\right)} + C\sqrt{\left(\frac{h}{W}\right)} &= 0, \\ A' \quad \text{,,} \quad + B' \quad \text{,,} \quad + C' \quad \text{,,} \quad &= 0, \\ A'' \quad \text{,,} \quad + B'' \quad \text{,,} \quad + C'' \quad \text{,,} \quad &= 0, \end{aligned}$$

giving $\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix} = 0$, which is in fact the equation of the curve J; and moreover giving $X : Y : Z = P : Q : R$, to determine the point (X, Y, Z) on Σ; and

$$\sqrt{\left(\frac{f}{U}\right)} : \sqrt{\left(\frac{g}{V}\right)} : \sqrt{\left(\frac{h}{W}\right)} = P : Q : R,$$

or, what is the same thing, $f : g : h = P^2U : Q^2V : R^2W$, to determine the point (f, g, h) on Δ. Treating now (f, g, h) as constants, and (X, Y, Z) as current coordinates, the conic $\frac{f}{X}+\frac{g}{Y}+\frac{h}{Z}=0$, will touch the curve Σ at the point (P, Q, R), if only the

equation of the conic is satisfied by these values and by the consecutive values $P+dP$, $Q+dQ$, $R+dR$; or what is the same thing, if we have

$$\frac{f}{P} + \frac{g}{Q} + \frac{h}{R} = 0,$$

$$\frac{fdP}{P^2} + \frac{gdQ}{Q^2} + \frac{hdR}{R^2} = 0,$$

that is

$$\frac{f}{P^2} : \frac{g}{Q^2} : \frac{h}{R^2} = QdR - RdQ : RdP - PdR : PdQ - QdP.$$

If the functions on the right-hand side are as $U : V : W$, then these equations give

$$f : g : h = P^2U : Q^2V : R^2W,$$

that is (f, g, h) will be a point on the curve Δ. It is therefore only necessary to show that in virtue of the equation $J=0$ of the curve J, and of the derived equation $dJ=0$, we have

$$QdR - RdQ : RdP - PdR : PdQ - QdP = U : V : W.$$

Take for instance the equation

$$V(QdR - RdQ) - U(RdP - PdR) = 0,$$

that is

$$dR(UP + VQ + WR) - R(UdP + VdQ + WdR) = 0,$$

and this, and the other two equations will be satisfied if only $UP + VQ + WR = 0$, $UdP + VdQ + WdR = 0$; we have, neglecting a numerical factor,

$$U = Ax + A'y + A''z,$$
$$V = Bx + B'y + B''z,$$
$$W = Cx + C'y + C''z,$$

whence, attending to the values of P, Q, R, we have

$$UP + VQ + WR = zJ = 0;$$

hence also

$$UdP + VdQ + WdR + (PdU + QdV + RdW) = 0,$$

so that

$$UdP + VdQ + WdR = 0,$$

if only

$$PdU + QdV + RdW = 0,$$

and substituting for P, Q, R, dU, dV, dW their values, the left-hand side is $=-Jdz$, which is $=0$; hence the equations in question are proved, and (f, g, h) is a point on the curve Δ.

It is to be noticed, that the two curves Σ, Δ are geometrically connected through the three arbitrary points as follows: viz. taking as axes the sides of the triangle formed by these three points, then starting from any point (f, g, h) of Δ, we take the inverse point $\left(\frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)$, the harmonic line thereof $fx+gy+hz=0$, and finally the inverse conic $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$, which by what precedes touches Σ in the point corresponding to the assumed point (f, g, h) of Δ: and conversely starting with an assumed point on Σ, we take the conic $\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0$ which passes through the angles of the triangle and touches Σ at the assumed point; the inverse line $fx+gy+hz=0$; the harmonic point $\left(\frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)$ of this line; and finally the inverse point (f, g, h), which will be on the curve Δ, the point corresponding to the assumed point on the curve Σ.

396.

ON A CERTAIN ENVELOPE DEPENDING ON A TRIANGLE INSCRIBED IN A CIRCLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 31—41 and 175—176.]

CONSIDERING a triangle and the circumscribed circle, and from any point of the circle drawing perpendiculars to the sides of the triangle; the feet of the three perpendiculars lie on a line; and (regarding the point as a variable point on the circle) the envelope of the line is a curve of the third class, having the line infinity for a double tangent, and being therefore a curve of the fourth order with three cusps, see Steiner's paper "Ueber eine besondere Curve dritter Klasse und vierten Grades," *Crelle*, t. LIII. (1857), pp. 231—237, which contains a series of very beautiful geometrical properties.

Mr Greer, in a paper in the last volume of the *Journal*, has expressed the equation of the line in a very elegant form, viz. if α, β, γ are the perpendicular distances of the point from the sides of the triangle; A, B, C the angles of the triangle; $(\lambda, \mu, \nu) = \left(\frac{\cos A}{\alpha}, \frac{\cos B}{\beta}, \frac{\cos C}{\gamma}\right)$; and (X, Y, Z) certain current coordinates, viz. these are the perpendicular distances from the sides, multiplied by $\sin A \tan A$, $\sin B \tan B$, $\sin C \tan C$ respectively; then the equation of the line is

$$X\lambda(\lambda-\mu)(\lambda-\nu) + Y\mu(\mu-\nu)(\mu-\lambda) + Z\nu(\nu-\lambda)(\nu-\mu) = 0,$$

where the parameters λ, μ, ν are connected by the equation $\lambda \tan A + \mu \tan B + \nu \tan C = 0$, or say by the equation

$$\lambda \mathrm{a} + \mu \mathrm{b} + \nu \mathrm{c} = 0.$$

We have a cubic equation in (λ, μ, ν) with coefficients which are linear functions of (X, Y, Z), and the required equation is that obtained by equating to zero the reciprocant of this cubic function, the facients of the reciprocant being the (a, b, c) of the linear relation; the reciprocant is of the degree 6 in (a, b, c) and of the degree 4 in the coefficients of the cubic function, that is in (X, Y, Z). But I remark that the equation in (λ, μ, ν), regarding these quantities as coordinates, is that of a cubic curve having a node at the point $\lambda=\mu=\nu$, or say the point (1, 1, 1); the corresponding value of $\lambda\mathrm{a}+\mu\mathrm{b}+\nu\mathrm{c}$ is $=\mathrm{a}+\mathrm{b}+\mathrm{c}$, and the reciprocant consequently contains the factor $(\mathrm{a}+\mathrm{b}+\mathrm{c})^2$, or dividing this out, the equation is only of the degree 4 in (a, b, c). The equation of the curve thus is

$$\frac{1}{(\mathrm{a}+\mathrm{b}+\mathrm{c})^2}\,\text{recip.}\,\{X\lambda(\lambda-\mu)(\lambda-\nu)+Y\mu(\mu-\nu)(\mu-\lambda)+Z\nu(\nu-\lambda)(\nu-\mu)\}=0,$$

being of the degree 4 in (a, b, c), and also of the degree 4 in (X, Y, Z), that is, treating (X, Y, Z) as current coordinates, the envelope is as above stated a curve of the fourth order.

A symmetrical method for finding the reciprocant of a cubic function was given by Hesse, see my paper "On Homogeneous Functions of the Third Order with Three Variables," *Camb. and Dubl. Math. Jour.*, vol. I. (1846), pp. 97—104, [35]; the developed expression there given for the reciprocant is however erroneous; the correct value is given in my "Third Memoir on Quantics," *Phil. Trans.*, vol. CXLVI. (1856), see the Table 67, p. 644, [144] and we have only in the table to substitute for (ξ, η, ζ) the quantities (a, b, c), and for $(a, b, c, f, g, h, i, j, k, l)$ the coefficients of the cubic function of (λ, μ, ν), viz. multiplying by 6 in order to avoid fractions, these are

$$\begin{aligned}&(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l)\\ &=(6X,\ 6Y,\ 6Z,\ -2Y,\ -2Z,\ -2X,\ -2Z,\ -2X,\ -2Y,\ X+Y+Z)\end{aligned}$$

respectively. The substitution might be performed as follows, viz. for the coefficient of a^6, we have

$$\left.\begin{array}{lll} b^2c^2 & +1\,.\,1296\ Y^2Z^2 & +1296\\ bcfi & -6\,.\ \ 144\ Y^2Z^2 & -\ \ 864\\ bi^3 & +4\,.\,-48\ YZ^3 & -\ \ 192\\ cf^3 & +4\,.\,-48\ Y^3Z & -\ \ 192\\ f^2i^2 & -3\,.\ \ 16\ Y^2Z^2 & -\ \ 48 \end{array}\right\}=-192YZ(Y-Z)^2,$$

and so for the other coefficients; but I have not gone through the labour of performing the calculation. Omitting the numerical factor -192, the coefficients of a^6, b^6, c^6 are of course

$$YZ(Y-Z)^2,\ ZX(Z-X)^2,\ XY(X-Y)^2;$$

and I find also that the coefficient of b^5c (the factor -192 being omitted) is

$$= ZX(3X^2 + 3Z^2 + 3YZ - 6ZX + 5XY),$$

whence those of the terms c^5a, &c. are also known.

I denote the result as follows:

$$(YZ(Y-Z)^2,\ ZX(Z-X)^2,\ XY(X-Y)^2,\ \ldots \between a,\ b,\ c)^6 = 0;$$

this equation divides as we have seen by $(a+b+c)^2$, and the quotient is

$$(YZ(Y-Z)^2,\ ZX(Z-X)^2,\ XY(X-Y)^2,\ \ldots \between a,\ b,\ c)^4 = 0;$$

and it may be remarked that the coefficient of b^3c in this quartic function of (a, b, c) is

$$= ZX(X^2 + Z^2 + 3YZ - 2ZX + 5XY.$$

The last mentioned equation, if the calculation were completed, would be analytically the best form for the equation of the envelope; but in view of what follows, I will change it by writing ax, by, cz in place of (X, Y, Z); x is therefore $= \frac{1}{\tan A} X$, that is, it is = perpendicular distance $\times \sin A$; or, what is the same thing, the new coordinates (x, y, z) are proportional to the perpendicular distances from the sides, each distance divided by the perpendicular distance of the side from the opposite angle, the equation of the line infinity is thus $x+y+z=0$. I write also $(a, b, c) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, that is, we have $(a, b, c) = (\cot A, \cot B, \cot C)$. The system of equations is therefore

$$\frac{x}{a}\lambda(\lambda-\mu)(\lambda-\nu) + \frac{y}{b}\mu(\mu-\nu)(\mu-\lambda) + \frac{z}{c}\nu(\nu-\lambda)(\nu-\mu) = 0,$$

$$\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c} = 0,$$

giving for the envelope the equation

$$bcyz(cy-bz)^2 + cazx(az-cx)^2 + abxy(bx-ay)^2 + \&c. = 0;$$

and in this function, corresponding to the term

$$b^3cZX(X^2 + Z^2 + 3YZ - 2ZX + 5XY),$$

we have the term

$$azx(bc^2x^2 + a^2bz^2 + 3a^2cyz - 2abczx + 5ac^2xy).$$

It may be noticed that, arranging in powers of (x, y, z), the several portions of each coefficient are distinct literal functions; thus we see that the coefficient of z^2x is $= a^3c + a^2b +$ other combinations of (a, b, c): this is material in order to the comparison of the foregoing equation of the envelope in a different form which will be presently mentioned.

I proceed to find the tangential equation of the envelope. Representing the equation of the line by

$$\xi x + \eta y + \zeta z = 0,$$

we have

$$\xi : \eta : \zeta = \frac{1}{a}\lambda(\lambda-\mu)(\lambda-\nu) : \frac{1}{b}\mu(\mu-\nu)(\mu-\lambda) : \frac{1}{c}\nu(\nu-\lambda)(\nu-\mu),$$

or, what is the same thing,

$$\xi : \eta : \zeta = \frac{1}{a}\,\frac{\lambda}{\mu-\nu} : \frac{1}{b}\,\frac{\mu}{\nu-\lambda} : \frac{1}{c}\,\frac{\nu}{\lambda-\mu},$$

where as before

$$\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c} = 0,$$

and eliminating λ, μ, ν, we find

$$a\xi(\eta-\zeta)^2 + b\eta(\zeta-\xi)^2 + c\zeta(\xi-\eta)^2 = 0.$$

In fact we find at once

$$\begin{aligned} a\xi(\eta-\zeta)^2 : b\eta(\zeta-\xi)^2 : c\zeta(\xi-\eta)^2 = \; & (\mu-\nu)\lambda\left\{\frac{1}{b}\mu(\lambda-\mu) - \frac{1}{c}\nu(\nu-\lambda)\right\} \\ & : (\nu-\lambda)\mu\left\{\frac{1}{c}\nu(\mu-\nu) - \frac{1}{a}\lambda(\lambda-\mu)\right\} \\ & : (\lambda-\mu)\nu\left\{\frac{1}{a}\lambda(\nu-\lambda) - \frac{1}{b}\mu(\mu-\nu)\right\}, \end{aligned}$$

and the sum of the three expressions on the right-hand side is

$$= -(\mu-\nu)(\nu-\lambda)(\lambda-\mu)\left(\frac{\lambda}{a} + \frac{\mu}{b} + \frac{\nu}{c}\right) = 0,$$

which verifies the result just obtained.

The tangential equation of the envelope is thus

$$a\xi(\eta-\zeta)^2 + b\eta(\zeta-\xi)^2 + c\zeta(\xi-\eta)^2 = 0,$$

or the envelope is a curve of the third class having as a double tangent the line $\xi = \eta = \zeta$, that is the line infinity; in fact for these values the equation $\xi x + \eta y + \zeta z = 0$ becomes $x + y + z = 0$, which is the equation of the line infinity. The curve is *therefore* a curve of the fourth order, the equation of which is

$$\frac{1}{(x+y+z)^2}\,\text{recip.}\,\{a\xi(\eta-\zeta)^2 + b\eta(\zeta-\xi)^2 + c\zeta(\xi-\eta)^2\} = 0,$$

where the reciprocant in question may be calculated from the before mentioned table 67, viz. multiplying by 3 in order to avoid fractions, the coefficients of the table are

$$\begin{aligned}&(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\quad l\quad)\\ =&(0,\ 0,\ 0,\ c,\ a,\ b,\ b,\ c,\ a,\ -a-b-c)\end{aligned}$$

respectively, and for the facients $(\xi,\ \eta,\ \zeta)$ of the table we have to write $(x,\ y,\ z)$. The expression of the reciprocant is

$$= b^2c^2x^6 + c^2a^2y^6 + a^2b^2z^6 + \&c.,$$

and dividing by $(x+y+z)^2$ we have the equation of the envelope in the form

$$b^2c^2x^4 + c^2a^2y^4 + a^2b^2z^4 + \&c. = 0,$$

which must of course be identical with the former result

$$bcyz\,(cy-bz)^2 + cazx\,(az-cx)^2 + abxy\,(bx-ay)^2 + \&c. = 0.$$

Instead of discussing the curve of the third class

$$a\xi\,(\eta-\zeta)^2 + b\eta\,(\zeta-\xi)^2 + c\zeta\,(\xi-\eta)^2 = 0,$$

it will be convenient to write $(x,\ y,\ z)$ in place of $(\xi,\ \eta,\ \zeta)$, and discuss the curve of the third order, or cubic curve

$$U = ax\,(y-z)^2 + by\,(z-x)^2 + cz\,(x-y)^2 = 0,$$

which is of course a curve having a node at the point $(x=y=z)$, or say at the point (1, 1, 1), and having therefore three inflexions lying in a line. The equation of the tangents at the node is found to be

$$a\,(y-z)^2 + b\,(z-x)^2 + c\,(x-y)^2 = 0,$$

that is, at the node the second derived functions of U are proportional to

$$(b+c,\ c+a,\ a+b,\ -a,\ -b,\ -c).$$

The equation of the Hessian may be found directly, or by means of the table, No. 61, in my memoir above referred to. It is as follows:

$$\begin{aligned}&(b+c)\,\{a\,(b+c)+2bc\}\,x^3\\ &+(c+a)\,\{b\,(c+a)+2ca\}\,y^3\\ &+(a+b)\,\{c\,(a+b)+2ab\}\,z^3\\ &-(3a^2+2bc+2ca+2ab)\,(cy^2z+byz^2)\\ &-(3b^2+2bc+2ca+2ab)\,(az^2x+czx^2)\\ &-(3c^2+2bc+2ca+2ab)\,(bx^2y+axy^2)\\ &+\{4\,(bc^2+b^2c+ca^2+c^2a+ab^2+a^2b)+6abc\}\,xyz = 0.\end{aligned}$$

I find that in the function $b^2c^2x^6 + \&c.$ the term in z^5x is

$$= a\,(4a^2b + 4a^2c + 4ab^2 + 4abc - 2b^3c),$$

whence in the function $b^3c^3x^4 + \&c.$ the term in z^2x is

$$= z^2xa\,(4a^2b + 4a^2c + 2ab^2 + 4abc - 2b^3c),$$

a portion whereof is $= 4z^2x\,(a^3b + a^3c)$; and we thus obtain the numerical factor $= 4$, and thence the identity

$$b^3c^3x^4 + c^3a^3y^4 + a^3b^3z^4 + \&c. = 4bcyz\,(cy - bz)^2 + \ldots + \&c.$$

which equation I represent by

$$Ax^3 + By^3 + Cz^3 + 3\,(Fy^2z + Gz^2x + Hx^2y + Iyz^2 + Jzx^2 + Kxy^2) + 6\,Lxyz = 0,$$

or

$$(A,\ B,\ C,\ F,\ G,\ H,\ I,\ J,\ K,\ L \between x,\ y,\ z)^3 = 0,$$

viz. writing for shortness

$$M = bc + ca + ab,$$

the values of the coefficients are as follows:

$$\begin{array}{lll} A = 3\,(b+c)\,(bc+M), & F = -c\,(3a^2+M), & I = -b\,(3a^2+M), \\ B = 3\,(c+a)\,(ca+M), & G = -a\,(3b^2+M), & J = -c\,(3b^2+M), \\ C = 3\,(a+b)\,(ab+M), & H = -b\,(3c^2+M), & K = -a\,(3c^2+M), \end{array}$$

$$L = 2\,(a+b+c)\,M - 3abc.$$

I remark that the cubic having a node at the point (1, 1, 1), the Hessian has at this point a node with the same tangents. The second derived functions for the Hessian are therefore at the node proportional to those of the cubic; it is easy to verify that we have in fact

$$\begin{array}{ll} A + H + J = (b+c)\,M, & L + F + I = -aM, \\ K + B + F = (c+a)\,M, & J + L + G = -bM, \\ G + I + C = (a+b)\,M, & H + K + L = -cM, \end{array}$$

these values give also

$$\begin{aligned} A + K + G + 2L + 2J + 2H &= 0, \\ H + B + I + 2F + 2L + 2K &= 0, \\ J + F + C + 2I + 2G + 2L &= 0, \end{aligned}$$

equations which merely express that the first derived functions vanish at the node. If, by these equations we express A, B, C in terms of the other coefficients, and substitute these values in the equation of the Hessian, this may be expressed in the form

$$\begin{aligned} &(y-z)^2\,\{\qquad\qquad L\,x + (2F + I + L)\,y + (F + 2I + L)\,z\} \\ &+ (z-x)^2\,\{(G + 2J + L)\,x + \qquad\qquad L\,y + (2G + J + L)\,z\} \\ &+ (x-y)^2\,\{(2H + K + L)\,x + (H + 2K + L)\,y + \qquad\qquad L\,z\} = 0, \end{aligned}$$

a form which puts in evidence the node (1, 1, 1).

I write

$$X = y - z, \quad Y = z - x, \quad Z = x - y,$$

so that we have identically

$$X + Y + Z = 0,$$

and that the equation of the tangents at the node is

$$aX^2 + bY^2 + cZ^2 = 0.$$

I write also for shortness

$$\begin{aligned} F' &= 2F + I + L, & I' &= F + 2I + L, \\ G' &= 2G + J + L, & J' &= G + 2J + L, \\ H' &= 2H + K + L, & K' &= H + 2K + L, \end{aligned}$$

the equation of the cubic is then

$$U = axX^2 + byY^2 + czZ^2, = 0,$$

and that of the Hessian is

$$\tilde{H}U = X^2(Lx + F'y + I'z) + Y^2(J'x + Ly + G'z) + Z^2(H'x + K'y + Lz).$$

Now observing that we have

$$\begin{aligned} L + 3aM &= L - 3(I + F + L) = -F' - I', \\ L + 3bM &= L - 3(G + J + L) = -G' - J', \\ L + 3cM &= L - 3(H + K + L) = -H' - K', \end{aligned}$$

we find

$$\begin{aligned} \tilde{H}U + 3MU = \quad & X^2(I'Y - F'Z) \\ & + Y^2(J'Z - G'X) \\ & + Z^2(K'X - H'Y), \end{aligned}$$

which shows that the function $\tilde{H}U + 3MU$ is a cubic function of $y - z$, $z - x$, $x - y$, decomposable therefore into three linear factors; and the equation $\tilde{H}U + 3MU = 0$, is consequently that of the three lines drawn from the node to the three inflexions of the cubic (or the Hessian). We know also that the Hessian of the three lines is the pair of tangents at the node [1], viz. that regarding any one of the variables X, Y, Z as a linear function of the third of them (in virtue of the equation $X + Y + Z = 0$), then that the cubic function of X, Y, Z has $aX^2 + bY^2 + cZ^2$ for its Hessian.

[1] Taking as the canonical form of a nodal cubic $U = x^3 + y^3 + 6lxyz = 0$, then we have $\tilde{H}U = x^3 + y^3 - 2lxyz = 0$; $x^3 + y^3 = 0$ is the equation of the lines from the node to the inflexions, and the Hessian of the binary cubic $x^3 + y^3$ is xy, where $xy = 0$ is the equation of the tangents at the node. We obtain as the only linear functions of U, $\tilde{H}U$ which are decomposable, $x^3 + y^3$ and xyz, the equation $xyz = 0$ gives $xy = 0$ which belongs to the tangents at the node or else $z = 0$, which is the equation of the line through the three inflexions: this line is so obtained a little further on in the text.

It is interesting to verify this; I write $Z = -X - Y$, the cubic function then assumes the form

$$(\alpha, \beta, \gamma, \delta \between X, Y)^3,$$

where $(\alpha, \beta, \gamma, \delta)$ have the values presently given.

The Hessian is

$$(2\alpha\gamma - 2\beta^2) X^2 + (\alpha\delta - \beta\gamma) . 2XY + (2\beta\delta - 2\gamma^2) Y^2,$$

or writing $2XY = Z^2 - X^2 - Y^2$, this is

$$= (2\alpha\gamma - 2\beta^2 - \alpha\delta + \beta\gamma) X^2 + (2\beta\delta - 2\gamma^2 - \alpha\delta - \beta\gamma) Y^2 + (\alpha\delta - \beta\gamma) Z^2.$$

We find after some easy reductions,

$$\begin{aligned}
\tfrac{1}{3}\alpha &= K' + F', & &= -3\ (a+c)(ac+M),\\
\beta &= -H' + 2K' + I' + F', & &= 3a(a+c)(b\ \ +c\ ),\\
-\ \gamma &= -K' + 2H' + J' + G', & &= 3b(a+c)(b\ \ +c\ ),\\
-\tfrac{1}{3}\delta &= H' + J', & &= -3\ (b+c)(bc+M),
\end{aligned}$$

and hence

$$\alpha\delta - \beta\gamma = -81(a+c)(b+c)\{(ac+M)(bc+M) - ab(a+c)(b+c)\},$$

where the expression in { } is

$$\begin{aligned}
&= (ab + 2ac + bc)(ab + ac + 2bc) - ab(ab + ac + bc + c^2),\\
&= c\{ab(3a + 3b) + c(b + 2a)(a + 2b) - ab(a + b + c)\},\\
&= 2c(a + b)(bc + ca + ab),
\end{aligned}$$

and therefore

$$\alpha\delta - \beta\gamma = -162(b + c)(c + a)(a + b)Mc;$$

the other coefficients may be similarly calculated, and omitting the merely numerical factor, we have

$$\text{Hessian} = (b + c)(c + a)(a + b)M(aX^2 + bY^2 + cZ^2),$$

which is right.

I write next

$$\begin{aligned}
\tilde{H}U + 3MU - \vartheta U = \ & X^2(-a\vartheta x + I'Y - F'Z)\\
& + Y^2(-b\vartheta y + J'Z - G'X)\\
& + Z^2(-a\vartheta z + K'X - H'Y),
\end{aligned}$$

or writing $x = z - Y$, $y = z + X$, this is

$$\begin{aligned}\tilde{H}U + 3MU - \mathfrak{H}U = & -\mathfrak{H}z(aX^2 + bY^2 + cZ^2) \\ & + X^2\{(I' + a\mathfrak{H})Y - F'Z\} \\ & + Y^2\{J'Z - (G' + b\mathfrak{H})X\} \\ & + Z^2\{K'X - H'Y \qquad\}, \end{aligned}$$

we may determine $\mathfrak{H}$, so that the cubic function of X, Y, Z contains the factor $aX^2 + bY^2 + cZ^2$; writing $Z = -X - Y$, then

	Contains the factor	Quotient is
$X^3\ (\ K' + F' \qquad\qquad)$	$(a+c)\ X^3$	$-3(ac+M)X$
$+X^2Y(-H' + 2K' + F' + I' + a\mathfrak{H})$	$+\ 2c\ XY$	$+3(bc+M)Y.$
$+XY^2(\ K' - 2H' - G' - J' - b\mathfrak{H})$	$+(b+c)\ Y^2$	
$+Y^3\ (-J' - H' \qquad\qquad)$		

We have seen that

$$K' + F' = -3(a+c)(ac+M),$$
$$J' + H' = -3(b+c)(bc+M),$$

whence the quotient is, as above stated,

$$= -3(ac+M)X + 3(bc+M)Y.$$

Comparing the coefficients of X^2Y, we have

$$\begin{aligned} a\mathfrak{H} &= -(-H' + 2K' + F' + I') + 3(a+c)(bc+M) - 6c(ac+M), \\ &= 9a(a+c)(b+c) + 3(a+c)(ab+ac+2bc) - 6c(ab+2ac+bc), \\ &= 12a(bc+ca+ab) = 12aM, \end{aligned}$$

that is $\mathfrak{H} = 12M$; and the same value would have been obtained by comparing the coefficients of XY^2. Hence $\tilde{H}U - \mathfrak{H}MU$ divides by $aX^2 + bY^2 + cZ^2$, the quotient being

$$-12Mz - 3(ac+M)X + 3(bc+M)Y,$$

which is

$$= -12Mz - 3(ac+M)(y-z) + 3(bc+M)(z-x),$$

or, finally it is

$$= -3\{(bc+M)x + (ca+M)y + (ab+M)z\},$$

and we thus have

$$\tilde{H}U - \mathfrak{H}MU = -3(aX^2 + bY^2 + cZ^2) \times \{(bc+M)x + (ca+M)y + (ab+M)z\},$$

so that the three inflexions are the intersections of the cubic curve by the line

$$(bc+M)x + (ca+M)y + (ab+M)z = 0.$$

It may be noticed, that if we write

$$\begin{aligned} x + y + z &= u, \\ bcx + cay + abz &= -Mu, \\ ax + by + cz &= v, \end{aligned}$$

then x, y, z will be as

$$\begin{aligned} &(b-c)\{(2M-bc)u-av\} \\ :\ &(c-a)\{(2M-ca)u-bv\} \\ :\ &(a-b)\{(2M-ab)u-cv\}, \end{aligned}$$

and substituting these values in the equation

$$ax(y-z)^2+by(z-x)^2+cz(x-y)^2=0$$

of the cubic, we have a cubic equation for the ratio $(u:v)$; and thence the values (x, y, z) for the coordinates of the inflexions.

It may be added, that we have

$$\begin{aligned} 12MU=-3(aX^2+bY^2+cZ^2)\{&(bc+M)x+(ca+M)y+(ab+M)z\} \\ &+\{X^2(I'Y-F'Z)+Y^2(J'Z-G'X)+Z^2(K'X-H'Y)=0, \end{aligned}$$

which is the equation of the cubic expressed in the canonical form.

Pp. 175—179. Effecting the process indicated p. 73, but writing for greater convenience (x, y, z) in place of (X, Y, Z), so that the substitution to be made is

$$\begin{aligned} &(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l\ ) \\ =&(6x,\ 6y,\ 6z,\ -2y,\ -2z,\ -2x,\ -2z,\ -2x,\ -2y,\ x+y+z), \end{aligned}$$

respectively (where I have corrected a misprint in the formula as originally given) I find the equation of the envelope to be

$$\begin{aligned} &4yz\,(y-z)^2\,\mathrm{a}^4 \\ +\ &4zx\,(z-x)^2\,\mathrm{b}^4 \\ +\ &4xy\,(x-y)^2\,\mathrm{c}^4 \\ +\ &4zx\,(z^2+x^2+3yz-2zx+5xy)\,\mathrm{b}^3\mathrm{c} \\ +\ &4xy\,(x^2+y^2+3zx-2xy+5yz)\,\mathrm{c}^3\mathrm{a} \\ +\ &4yz\,(y^2+z^2+3xy-2yz+5zx)\,\mathrm{a}^3\mathrm{b} \\ +\ &4xy\,(x^2+y^2+3yz-2xy+5zx)\,\mathrm{bc}^3 \\ +\ &4yz\,(y^2+z^2+3zx-2yz+5xy)\,\mathrm{ca}^3 \\ +\ &4zx\,(z^2+x^2+3xy-2zx+5yz)\,\mathrm{ab}^3 \\ +\ &x\,(x^3-2x^2y-2x^2z+xy^2+38xyz+xz^2+12y^3z+12yz^2)\,\mathrm{b}^3\mathrm{c}^2 \\ +\ &y\,(y^3-2y^2z-2y^2x+yz^2+38xyz+yx^2+12z^2x+12zx^2)\,\mathrm{c}^3\mathrm{a}^2 \\ +\ &z\,(z^3-2z^2x-2z^2y+zx^2+38xyz+zy^2+12x^2y+12xy^2)\,\mathrm{a}^3\mathrm{b}^2 \\ +\ &2yz\,(11x^2+y^2+z^2-2yz+24xy+24zx)\,\mathrm{a}^2\mathrm{bc} \\ +\ &2zx\,(11y^2+z^2+x^2-2zx+24yz+24xy)\,\mathrm{b}^2\mathrm{ca} \\ +\ &2xy\,(11z^2+x^2+y^2-2xy+24zx+24yz)\,\mathrm{c}^2\mathrm{ab}=0. \end{aligned}$$

The function on the left-hand side is the quotient by $-48(\mathrm{a}+\mathrm{b}+\mathrm{c})^2$ of the sextic function, Table 67 of my third Memoir on Quantics, [144]; the foregoing quotient was calculated without using the coefficient of the term in $\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$ $(\xi\eta^2\zeta^2)$ of the table, but by way of verification, I calculated from the table the term in question, and found it to be

$$\begin{aligned} & (x+y+z)^4 \\ - \quad & 2(x+y+z)^2(yz+zx+xy) \\ + \; & 296(x+y+z)xyz \\ - \quad & 8(y^2z^2+z^2x^2+x^2y^2), \end{aligned}$$

and this should consequently be equal to the coefficient of $\mathrm{a}^2\mathrm{b}^2\mathrm{c}^2$ in the product of $(\mathrm{a}+\mathrm{b}+\mathrm{c})^2$ into the foregoing quartic function of (a, b, c) that is, it should be

$$\begin{aligned} = \quad & x\,(x^3-2x^2y-2x^2z+xy^2+38xyz+xz^2+12y^2z+12yz^2) \\ + \quad & y\,(y^3-2y^2z-2y^2x+yz^2+38xyz+yx^2+12z^2x+12zx^2) \\ + \quad & z\,(z^3-2z^2x-2z^2y+zx^2+38xyz+zy^2+12x^2y+12xy^2) \\ + \; & 4yz\,(11x^2+y^2+z^2-2yz+24xy+24zx) \\ + \; & 4zx\,(11y^2+z^2+x^2-2zx+24yz+24xy) \\ + \; & 4xy\,(11z^2+x^2+y^2-2xy+24zx+24yz), \end{aligned}$$

which is accordingly found to be the case.

397.

SPECIMEN TABLE $M \equiv a^\alpha b^\beta$ (MOD. N) FOR ANY PRIME OR COMPOSITE MODULUS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 95—96 and plate.]

If N be a prime number, and a one of its primitive roots, then any number M prime to N, or what is the same thing, any number in the series $1, 2, \ldots N-1$, may be exhibited in the form $M \equiv a^\alpha$ (Mod. N); where α is said to be the index of M in regard to the particular root a. Jacobi's *Canon Arithmeticus* (Berlin, 1839), contains a series of tables, giving the indices of the numbers $1, 2, 3 \ldots N-1$ for every prime number N less than 1000, and giving conversely for each such prime number the numbers M which correspond to the indices $\alpha = 1, 2, \ldots (N-1)$ (*Tabulæ Numerorum ad Indices datos pertinentium et Indicum Numero dato correspondentium*). A similar theory applies, it is well known, to the composite numbers; the only difference is, that in order to exhibit for a given composite number N the different numbers less than N and prime to it, we require not a single root a, but two or more roots $a, b, \ldots$ and that in terms of these we have $M = a^\alpha b^\beta \ldots$ (Mod. N). For each root a there is an index A (or say the Indicator of the root), such that $a^A \equiv 1$ (Mod. N), A being the least index for which this equation is satisfied; and the indices $a, b, \ldots$ extend from 1 to $A, B, \ldots$ respectively; the number of different combinations or the product $AB\ldots$, being precisely equal to $\phi(N)$, the number of integers less than N and prime to it. The least common multiple of $A, B\ldots$, is termed the Maximum Indicator, and representing it by I, then for any number M not prime to N, we have $M^I \equiv 1$ (Mod. N), a theorem made use of by Cauchy for the solution of indeterminate equations of the first order. Thus $N = 20$, the roots may be taken to be 3, 11; the corresponding exponents are 4, 2 (viz. $3^4 \equiv 1$ (Mod. 20) $11^2 \equiv 1$ (Mod. 20)), and the product of these is 8, the number of integers less than 20 and prime to it; the series [*go to p.* 86]

NOS.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
ROOTS		1	2	3	2	5	3	3, 5	2	3	2	5, 7	6	3	2, 11	3, 7	10	5	10	3, 11	2, 13	7	10	5, 7, 13	2	7	2	3, 13	10	7, 11
IND.		1	2	2	4	2	6	2, 2	6	4	10	2, 2	12	6	4, 2	4, 2	16	6	18	4, 2	6, 2	10	22	2, 2, 2	20	12	18	6, 2	28	4, 2
M. I.		1	2	2	4	2	6	2	6	4	10	2	12	6	4	4	16	6	18	4	6	10	22	2	20	12	18	6	28	4
ϕ	0	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8	12	10	22	8	20	12	18	12	28	8
	1	0	0	0	0	0	0	0, 0	0	0	0	0, 0	0	0	0, 0	0, 0	0	0	0	0, 0	0, 0	0	0	0, 0, 0	0	0	0	0, 0	0	0, 0
		2	1		1		2		1		1		5		1, 0		10		17		1, 0		8		1		1		11	
			3	1	3		1	1, 0		1	8		8	1		1, 0	11		5	1, 0		4	20		7	8		1, 0	27	
				4	2		4		2		2		10		2, 0		4		16		2, 0		16		2		2		22	
					5	1	5	0, 1	5		4	1, 0	9	5		1, 1	7	1	2		1, 1	2	15	1, 0, 0		3	5	2, 1	18	
						6	3				9		1				5		4				6		8				10	
							7	1, 1	4	3	7	0, 1	7		1, 1	0, 1	9	2	12	3, 0		1	21	0, 1, 0	5	1	16		20	1, 0
								8	3		3		3		3, 0		14		15		3, 0		2		3		3		5	
									9	2	6		4	2		2, 0	6		10	2, 0		8	18		14	4		2, 0	26	
										10	5		2				1		1		2, 1		1				6		1	
											11	1, 1	11	4	0, 1	3, 0	13	5	6	0, 1	5, 0		3	1, 1, 0	16	5	13	1, 1	23	0, 1
												12	6				15		3				14		9				21	
													13	3	3, 1	3, 1	12	4	13	1, 1	0, 1	3	12	0, 0, 1	19		8	0, 1	2	3, 0
														14	2, 1		3		11				7		6		14		3	
															15	2, 1	2		7			6	13			11		3, 1	17	
																16	8		14		4, 0		10		4		4		16	
																	17	3	8	3, 1	5, 1	7	17	1, 0, 1	13	10	15	4, 1	7	1, 1
																		18	9				4		15				9	
																			19	2, 1	4, 1	9	5	0, 1, 1	8	7	12	5, 0	15	
																				20	3, 1		9				7		12	2, 0
																					21	5	19		12	9			19	
																						22	11		17		14		6	
																							23	1, 1, 1	11	2	11	5, 1	24	3, 1
																								24	10				4	
																									25	6	10	4, 0	8	
																										26	9		13	
																											27	3, 0	25	
																												28	14	
																													29	2, 1
																														30

31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	
17	3, 15	2, 10	3	2, 6	5, 19	5	3	2, 14	3, 11, 21	6	5, 13	28	3, 21	2, 26	5	10	5, 7, 17	3	3	
30	8, 2	10, 2	16	12, 2	6, 2	36	18	12, 2	4, 2, 2	40	6, 2	42	10, 2	12, 2	22	46	4, 2, 2	42	20	
30	8	10	16	12	6	36	18	12	4	40	6	42	10	12	22	46	4	42	20	
30	16	20	16	24	12	36	18	24	16	40	12	42	20	24	22	46	16	42	20	
0	0, 0	0, 0	0	0, 0	0, 0	0	0	0, 0	0, 0, 0	0	0, 0	0	0, 0	0, 0	0	0	0, 0, 0	0	0	1
12		1, 0		1, 0		11		1, 0		26		39		1, 0		30		26		2
13	1, 0		1	11, 0		34	1		1, 0, 0	15		17	1, 0		16	18		1	1	3
24		2, 0		2, 0		22		2, 0		12		36		2, 0		14		10		4
20	7, 1	9, 1	5		1, 0	1	4	9, 0		22	1, 0	5	8, 0		1	17	1, 0, 0	29		5
25				0, 1		9				1		14				2		27		6
4	2, 1	2, 1	11		2, 1	28	6	11, 1	3, 0, 1	39		7	9, 1	1, 1	19	38	0, 1, 0		15	7
6		3, 0		3, 0		33		3, 0		38		33		3, 0		44		36		8
26	2, 0		2	10, 0		32	2		2, 0, 0	30		34	2, 0		10	36		2	2	9
2		0, 1				12		10, 0		8		2				1		13		10
29	7, 0		7	8, 0	5, 1	6	12	7, 0	0, 1, 0	3	5, 1	6		4, 1	9	27	3, 1, 0	40	8	11
7				1, 1		20				27		11				32		11		12
23	1, 1	6, 1	4	3, 1	4, 0	13	17		1, 1, 1	31	0, 1	40	2, 1	11, 1	14	3	3, 0, 1	33	17	13
16		3, 1				3		0, 1		25		4		2, 1		22				14
3	0, 1		6			35	5			37		22	9, 0		17	35		30		15
18		4, 0		4, 0		8		4, 0		24		30		4, 0		28		20		16
1	4, 0	9, 0		5, 1	3, 0	5	16	2, 1		33	5, 0	16	8, 1	9, 0	7	42	0, 0, 1	25	19	17
8				11, 0		7				16		31				20		28		18
22	5, 0	8, 1	14	10, 1	0, 1	25		5, 1	2, 1, 0	9	4, 1	29	1, 1	6, 0	15	29	1, 1, 1	35	14	19
14		1, 1				23		11, 0		34		41				31		39		20
17	3, 1		12			26	7		0, 0, 1	14		24	0, 1		13	10			16	21
11				9, 0		17		8, 0		29		3		5, 1		11		24		22
21	6, 1	5, 1	15	7, 0	1, 1	21	14	10, 1	1, 0, 1	36	1, 1	20	5, 0	11, 0		39	0, 1, 1	38	13	23
19				2, 1		31				13		8				16		37		24
10	6, 0	8, 0	10		2, 0	2	8	6, 0		4	2, 0	10	6, 0		2	34	2, 0, 0	16		25
5		7, 1		4, 1		24				17		37		0, 1		33		17		26
9	3, 0		3	9, 1		30	3		3, 0, 0	5		9	3, 0		4	8		3	3	27
28		4, 1				14		1, 1		11		1		3, 1		6				28
27	5, 1	7, 0	13	6, 0	5, 0	15	11	4, 1	2, 0, 1	7	3, 1	25	4, 1	10, 1	18	43	3, 0, 0	18	6	29
15						10				23		19				19		14		30
31	4, 1	6, 0	9	8, 1	4, 1	27	15	9, 1	0, 1, 1	28	2, 1	32	7, 0	8, 0	6	5	2, 1, 0	7	4	31
	32	5, 0		5, 0		19		5, 0		10		27		5, 0		12		4		32
		33	8	7, 1		4	13		1, 1, 0	18		23			3	45		41	9	33
			34	6, 1		16		3, 1		19		13		10, 0		26		9		34
				35	31	29	10	8, 1		21		12	7, 1		20	9	1, 1, 0			35
					36	18				2		28				4		12		36
						37	9	7, 1	3, 1, 1	32	4, 0	35	4, 0	9, 1	21	24	1, 0, 1	32	7	37
							38	6, 1		35		26		7, 0		13		19		38
								39	2, 1, 1	6		15	3, 1		8	21		34	18	39
									40	20		38				15		23		40
										41	3, 0	18	6, 1	8, 1	12	25	2, 0, 1	15	12	41
											42	21				40				42
												43	5, 1	7, 1	5	37	3, 1, 1	6	5	43
													44	6, 1		41		8		44
														45	11	7		31		45
															46			22		46
																47	2, 1, 1	5	11	47
																	48	21		48
																		49	10	49
																			50	50

[*from p.* 83] of these is in fact 1, 3, 7, 9, 11, 13, 17, 19, each of which is expressible in the required form, viz. $1 \equiv 3^0 . 11^0$, $3 \equiv 3^1 . 11^0$, $7 = 3^3 . 11^0$, &c. (Mod. 20): the maximum indicator is 4; viz. $1^4 \equiv 1$, $3^4 \equiv 1$, $7^4 \equiv 1$, &c. (Mod. 20).

The table pp. 84, 85 gives the Indices for the numbers less than N and prime to it, for all values of N from 1 to 50; the arrangement may be seen at a glance; of the five lines which form a heading, the first contains the numbers N; the second the root or roots belonging to each number N, the third the indicators of these roots, the fourth the maximum indicator, the fifth the number $\phi(N)$. The remaining lines contain the index or indices of each of the ϕN numbers M less than N and prime to it, the number corresponding to such index or indices, being placed outside in the same horizontal line. For example, 30 has the roots 7, 11, indices 4, 2 respectively; the Maximum Indicator is 4, and the number of integers less than 30 and prime to it is 8; taking any such number, say 17, the indices are 1, 1, that is, we have $17 = 7^1 . 11^1$ (Mod. 30).

The foregoing corresponds to the *Tabulæ Indicum Numero dato correspondentium* of Jacobi; on account of multiplicity of roots there does not appear to be any mode of forming a single table corresponding to the *Tabulæ Numerorum ad Indices datos pertinentium;* and there would be no adequate advantage in forming for each number N a separate table in some such form as

$N = 20$.

Roots 3	Roots 11	Nos.
0	0	1
0	1	11
1	0	3
1	1	13
2	0	9
2	1	19
3	0	7
3	1	17

which I have written down in the form of a table of single entry; for although (whenever, as in the present case, the number of roots is only two) it might have been better exhibited as a table of double entry, when the number of roots is three or more it could not of course be exhibited as a table of corresponding multiple entry.

398.

ON A CERTAIN SEXTIC DEVELOPABLE, AND SEXTIC SURFACE CONNECTED THEREWITH.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 129—142 and 373—376.]

I PROPOSE to consider [first] the sextic developable derived from a quartic equation, viz. taking this to be $(a, b, c, d, e \between t, 1)^4 = 0$, where (a, b, c, d, e) are any linear functions of the coordinates (x, y, z, w), the equation of the developable in question is

$$(ae - 4bd + 3c^2)^3 - 27(ace - ad^2 - b^2e + 2bcd - c^3)^2 = 0.$$

I have already, in the paper "On a Special Sextic Developable," *Quarterly Journal of Mathematics*, vol. VII. (1866), pp. 105—113, [373], considered a particular case of this surface, viz. that in which c was $=0$, the geometrical peculiarity of which is that the cuspidal edge is there an excubo-quartic curve (of a special form, having two stationary tangents), whereas in the general case here considered it is a sextic curve. There was analytically the convenience that the linear functions being only the four functions a, b, d, e, these could be themselves taken as coordinates, whereas in the present case we have the five linear functions a, b, c, d, e.

The developable

$$(ae - 4bd + 3c^2)^3 - 27(ace - ad^2 - b^2e + 2bcd - c^3)^2 = 0$$

is a sextic developable having for its cuspidal curve the sextic curve

$$ae - 4bd + 3c^2 = 0,$$

$$ace - ad^2 - b^2e + 2bcd - c^3 = 0,$$

(say $I=0$, $J=0$, as usual), and having besides a nodal curve the equations of which may be written

$$\begin{array}{ccccccccccc} & 6(ac-b^2) & : & 3(ad-bc) & : & ae+2bd-3c^2 & : & 3(be-cd) & : & 6(ce-d^2) & : & 9J \\ = & a & : & b & : & c & : & d & : & e & : & I, \end{array}$$

viz. these equations are really equivalent to two equations, and they represent a curve of the fourth order which is an excubo-quartic. We may in fact find the equations of the nodal curve by assuming $(a, b, c, d, e \between t, 1)^4$ to be a perfect square, say to avoid fractions that it is $=3(\alpha t^2+2\beta t+\gamma)^2$, then we have

$$a : b : c : d : e = 3\alpha^2 : 3\alpha\beta : \alpha\gamma+2\beta^2 : 3\beta\gamma : 3\gamma^2,$$

which equations as involving the two arbitrary parameters $\alpha : \beta : \gamma$, give two equations between (a, b, c, d, e), and we may at once by means of them verify the above-mentioned equations of the nodal curve. It also hereby appears that the nodal curve is as stated an excubo-quartic curve; viz. we have between a, b, c, d, e a single linear relation, that is a quadric relation between α, β, γ, and this equation may be satisfied identically by taking for α, β, γ properly determined quadric functions of a variable parameter θ; whence a, b, c, d, e are proportional to quartic functions of the variable parameter θ, or the curve is an excubo-quartic.

The equations of the nodal curve may be presented under a somewhat different form; viz. the cubi-covariant of $(a, b, c, d, e \between t, 1)^4=0$ being

$$\left\{\begin{array}{l} -\ a^2d + 3abc - 2b^3 \\ -\ a^2e - 2abd + 9ac^2 - 6b^2c \\ -\ 5abe + 15acd - 10b^2d \\ +\ 10ad^2 - 10b^2e \\ +\ 5ade + 10bd^2 - 15bce \\ +\ ae^2 + 2bde - 9c^2e + 6cd^2 \\ +\ be^2 - 3cde + 2d^3 \end{array}\right\} (t, 1)^6 = 0,$$

say this function, multiplied by 6 to avoid fractions, is

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g} \between t, 1)^6,$$

that is

$$\begin{aligned} \mathrm{a} &= 6(-a^2d+3abc-2b^3), \\ \mathrm{b} &= 1(-a^2e-2abd+9ac^2-6b^2c), \\ \mathrm{c} &= 2(-abe+3acd-2b^2d), \\ \mathrm{d} &= 3(+ad^2-b^2e), \\ \mathrm{e} &= 2(+ade+2bd^2-3bce), \\ \mathrm{f} &= 1(+ae^2+2bde-9c^2e+6cd^2), \\ \mathrm{g} &= 6(+be^2-3cde+2d^3), \end{aligned}$$

then the equations of the nodal curve may be written

$$\mathrm{a}=0,\ \mathrm{b}=0,\ \mathrm{c}=0,\ \mathrm{d}=0,\ \mathrm{e}=0,\ \mathrm{f}=0,\ \mathrm{g}=0.$$

It may be mentioned that we have identically

$$\begin{aligned} \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2 &= 0, \\ \mathrm{af} - 3\mathrm{be} + 2\mathrm{cd} &= 0, \\ \mathrm{ag} - 9\mathrm{ce} + 8\mathrm{d}^2 &= 0, \\ \mathrm{bg} - 3\mathrm{cf} + 2\mathrm{de} &= 0, \\ \mathrm{bf} - 4\mathrm{ce} + 3\mathrm{d}^2 &= 0, \end{aligned}$$

and moreover

$$\begin{aligned} &\mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2 \\ = \; & -6(\mathrm{bf} - 4\mathrm{ce} + 3\mathrm{d}^2), = +6(I^3 - 27J^2), \end{aligned}$$

so that the equation of the developable may be written in the form

$$\mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2 = 0,$$

or in the more simple form

$$\mathrm{bf} - 4\mathrm{ce} + 3\mathrm{d}^2 = 0,$$

each of which puts in evidence the nodal curve on the surface.

The nodal and cuspidal curves meet in the points

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e},$$

being, as it is easy to show, a system of four points. The four points in question form a tetrahedron, the equations of the faces of which may be taken to be $x=0$, $y=0$, $z=0$, $w=0$; and the equation of the surface may be expressed in this system of quadriplanar coordinates.

We introduce these coordinates *ab initio*, by taking the quartic function of t to be

$$(a, b, c, d, e \between t, 1)^4 = x(t+\alpha)^4 + y(t+\beta)^4 + z(t+\gamma)^4 + w(t+\delta)^4,$$

that is, by writing

$$\begin{aligned} a &= x + y + z + w, \\ b &= \alpha x + \beta y + \gamma z + \delta w, \\ c &= \alpha^2 x + \beta^2 y + \gamma^2 z + \delta^2 w, \\ d &= \alpha^3 x + \beta^3 y + \gamma^3 z + \delta^3 w, \\ e &= \alpha^4 x + \beta^4 y + \gamma^4 z + \delta^4 w. \end{aligned}$$

Observe that (t_1, t_2, t_3, t_4) being any constant quantities, we thence have

$$\begin{aligned} & e - d\Sigma t_1 + c\Sigma t_1 t_2 - b\Sigma t_1 t_2 t_3 + a t_1 t_2 t_3 t_4 \\ &\quad = x(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4) \\ &\quad + y(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4) \\ &\quad + z(\gamma - t_1)(\gamma - t_2)(\gamma - t_3)(\gamma - t_4) \\ &\quad + w(\delta - t_1)(\delta - t_2)(\delta - t_3)(\delta - t_4), \end{aligned}$$

and thence in particular

$$e - d\Sigma\alpha + c\Sigma\alpha\beta - b\Sigma\alpha\beta\gamma + a\,\alpha\beta\gamma\delta = 0,$$

viz. this is the linear relation which subsists identically between (a, b, c, d, e), the five linear functions of the coordinates (x, y, z, w).

Starting from the above values of (a, b, c, d, e), we find without difficulty

$$I = (\alpha-\beta)^4 xy + (\alpha-\gamma)^4 xz + (\alpha-\delta)^4 xw + (\beta-\gamma)^4 yz + (\beta-\delta)^4 yw + (\gamma-\delta)^4 zw,$$

$$J = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 xyz + (\alpha-\beta)^2(\beta-\delta)^2(\delta-\alpha)^2 xyw$$
$$+ (\alpha-\gamma)^2(\gamma-\delta)^2(\delta-\alpha)^2 xzw + (\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2 yzw,$$

but we thus see the convenience of introducing constant multipliers into the expressions of the four coordinates respectively, viz. writing

$$x = (\beta\gamma\delta)^2 x',$$
$$y = (\gamma\delta\alpha)^2 y',$$
$$z = (\delta\alpha\beta)^2 z',$$
$$w = (\alpha\beta\gamma)^2 w',$$

where for shortness

$$(\beta\gamma\delta) = (\beta-\gamma)(\gamma-\delta)(\delta-\beta), \text{ \&c.},$$

or what is the same thing, taking the quartic to be

$$(a, b, c, d, e \between t, 1)^4 = x'(\beta\gamma\delta)^2(t+\alpha)^4 + y'(\gamma\delta\alpha)^2(t+\beta)^4 + z'(\delta\alpha\beta)^2(t+\gamma)^4 + w'(\alpha\beta\gamma)^2(t+\delta)^4,$$

we find

$$J = (\lambda'\mu'\nu')^2(x'y'z' + y'z'w' + z'x'w' + x'y'w'),$$
$$I = (\lambda'\mu'\nu')\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(z'w' + x'y')\},$$

where for shortness

$$\lambda' = (\alpha-\delta)^2(\beta-\gamma)^2,$$
$$\mu' = (\beta-\delta)^2(\gamma-\alpha)^2,$$
$$\nu' = (\gamma-\delta)^2(\alpha-\beta),$$

or writing

$$\sqrt{(\lambda')} = (\alpha-\delta)(\beta-\gamma),$$
$$\sqrt{(\mu')} = (\beta-\delta)(\gamma-\alpha),$$
$$\sqrt{(\nu')} = (\gamma-\delta)(\alpha-\beta),$$

we have

$$\sqrt{(\lambda')} + \sqrt{(\mu')} + \sqrt{(\nu')} = 0,$$

and the equation of the developable is thus

$$\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(x'w' + x'y')\}^2 - 27\lambda'\mu'\nu'(x'y'z' + y'z'w' + z'x'w' + x'y'w')^2 = 0.$$

Observe that $J = 0$ is a cubic surface passing through each edge of the tetrahedron, and having at each summit a conical point; $I = 0$ is a quadric surface passing through each summit of the tetrahedron, and at each of these points the tangent

plane of the quadric surface touches the tangent cone of the cubic surface: to show this it is only necessary to observe that at the point ($x'=0$, $y'=0$, $z'=0$) the tangent cone is $y'z'+z'x'+x'y'=0$, and the tangent plane is $\lambda'x'+\mu'y'+\nu'z'=0$, and that these touch in virtue of the above-mentioned relation $\sqrt{(\lambda')}+\sqrt{(\mu')}+\sqrt{(\nu')}=0$. It follows that on the curve of intersection, or cuspidal edge of the developable, each of the summits is a cuspidal or stationary point, that is, the cuspidal curve has four stationary points; this agrees with the character of the curve as given, Salmon "On the Classification of Curves of Double Curvature," *Camb. and Dubl. Math. Jour.* vol. v. (1850), p. 39, viz. the character is there given

$$a=6,\ m=6,\ n=4,\ r=6,\ g=3,\ h=6,\ \alpha=0,\ \beta=4,\ x=4,\ y=6,$$

($\beta=4$, that is, there are as stated 4 stationary points).

To find the equations of the nodal curve, instead of transforming the equations as given in terms of (a, b, c, d, e), it is better to deduce these from the equation of the surface; viz. if there is a nodal curve, we must have

$$\begin{aligned} &\delta_{x'}I : \delta_{y'}I : \delta_{z'}I : \delta_{w'}I : 18J \\ =\ &\delta_{x'}J : \delta_{y'}J : \delta_{z'}J : \delta_{w'}J : \quad I. \end{aligned}$$

Writing these under the form $\delta_{x'}I+\theta'\delta_{x'}J=0$, &c., where θ' is regarded as an arbitrary parameter [1], we have

$$\begin{aligned} \lambda'w'+\mu'z'+\nu'y'+\theta'(y'z'+y'w'+z'w')&=0,\\ \lambda'z'+\mu'w'+\nu'x'+\theta'(z'x'+z'w'+x'w')&=0,\\ \lambda'y'+\mu'x'+\nu'w'+\theta'(x'y'+x'w'+y'w')&=0,\\ \lambda'x'+\mu'y'+\nu'z'+\theta'(y'z'+z'x'+x'y')&=0, \end{aligned}$$

which equations (eliminating θ') must be equivalent to two equations only.

I remark that the first three equations may be regarded as a set of linear equations in 1, w', θ', $\theta'w'$; and determining from them the ratios of these quantities, we have, suppose,

$$1 : -w' : \theta' : -\theta'w' = A : B : C : D,$$

where

$$A=\begin{vmatrix} \lambda', & y'z', & y'+z' \\ \mu', & z'x', & z'+x' \\ \nu', & x'y', & x'+y' \end{vmatrix},\qquad B=\begin{vmatrix} y'z', & y'+z', & \mu'z'+\nu'y' \\ z'x', & z'+x', & \nu'x'+\lambda'z' \\ x'y', & x'+y', & \lambda'y'+\mu'x' \end{vmatrix},$$

$$C=\begin{vmatrix} y'+z', & \mu'z'+\nu'y', & \lambda' \\ z'+x', & \nu'x'+\lambda'z', & \mu' \\ x'+y', & \lambda'y'+\mu'x', & \nu' \end{vmatrix},\qquad D=\begin{vmatrix} \mu'z'+\nu'y', & \lambda', & y'z' \\ \nu'x'+\lambda'z', & \mu', & z'x' \\ \lambda'y'+\mu'x', & \nu', & x'y' \end{vmatrix}.$$

[1] The value of θ' is in fact $=-\dfrac{18J}{I}$, that is, instead of the four equations involving an arbitrary parameter θ', we have really four determinate equations.

We have thence $AD-BC=0$ and (substituting in the fourth equation) $A(\lambda'x'+\mu'y'+\nu'z')+C(y'z'+z'x'+x'y')=0$; each of these equations must contain the equation of the cone having ($x'=0$, $y'=0$, $z'=0$) for its vertex, and passing through the nodal curve. The two equations are of the orders 6 and 4 respectively; and as the curve is a quartic curve passing through the vertex in question, the equation of the cone is of the order 3. I have not effected the reduction of the sextic equation, but for the quartic equation, substituting for A, C their values, this is

$$\begin{aligned}&-(\lambda'x'+\mu'y'+\nu'z')[\lambda'x'^2(y'-z')+\mu'y'^2(z'-x')+\nu'z'^2(z'-x')]\\&+(y'z'+z'x'+x'y')[\lambda'^2x'(y'-z')+\mu'^2y'(z'-x')+\nu'z'(x'-y')]\\&+\{\mu'\nu'(y'-z')+\nu'\lambda'(z'-x')+\lambda'\mu'(x'-y')\}(x'+y'+z')=0,\end{aligned}$$

which is easily reduced to

$$\begin{aligned}&\lambda'^2x'(-x'^2+y'z'+z'x'+x'y')(y'-z')\\&+\mu'^2y'(-y'^2+y'z'+z'x'+x'y')(z'-x')\\&+\nu'^2z'(-z'^2+y'z'+z'x'+x'y')(x'-y')\\&+\mu'\nu'[(x'+y'+z')(y'z'+z'x'+x'y')+x'y'z'](y'-z')\\&+\nu'\lambda'[(x'+y'+z')(y'z'+z'x'+x'y')+x'y'z'](z'-x')\\&+\lambda'\mu'[(x'+y'+z')(y'z'+z'x'+x'y')+x'y'z'](x'-y')=0;\end{aligned}$$

and I have found that this is transformable into

$$\begin{aligned}2\{x'\sqrt{(\lambda')}+y'\sqrt{(\mu')}+z'\sqrt{(\nu')}\}\times[&y'z'\sqrt{(\lambda')}(\mu'y'-\nu'z')z'x'\sqrt{(\mu')}(\nu'z'-\lambda'x')x'y'\sqrt{(\nu')}(\lambda'x'-\mu'y')\\&-x'y'z'\{\sqrt{(\mu')}-\sqrt{(\nu')}\}\{\sqrt{(\nu')}-\sqrt{(\lambda')}\}\{\sqrt{(\lambda')}-\sqrt{(\mu')}\}]=0,\end{aligned}$$

viz. the two functions are equivalent in virtue of the relation $\sqrt{(\lambda')}+\sqrt{(\mu')}+\sqrt{(\nu')}=0$, or, what is the same thing, they only differ by a function $(x', y', z')^4$ into the evanescent factor $\lambda'^2+\mu'^2+\nu'^2-2\mu'\nu'-2\nu'\lambda'-2\lambda'\mu'$. The function in { } equated to zero is therefore the equation of the cubic cone.

I do not stop to give the steps of the investigation in the above form, as the investigation may be very much simplified as follows: by linear combinations of the four equations in x', y', z', w', θ', we deduce

$$\begin{aligned}&(\lambda'-\mu'-\nu')(x'+w'-y'-z')+2\theta'(y'z'-x'w')=0,\\&(-\lambda'+\mu'-\nu')(y'+w'-z'-x')+2\theta'(z'x'-y'w')=0,\\&(-\lambda'-\mu'+\nu')(z'+w'-x'-y')+2\theta'(x'y'-z'w')=0,\\&(\lambda'+\mu'+\nu')(x'+y'+z'+w')+2\theta'(y'z'+z'x'+x'y'+x'w'+y'w'+z'w')=0.\end{aligned}$$

Hence writing

$$\begin{aligned}\lambda&=\lambda'-\mu'-\nu', & x&=w'+x'-y'-z',\\ \mu&=-\lambda'+\mu'-\nu', & y&=w'-x'+y'-z',\\ \nu&=-\lambda'-\mu'+\nu', & z&=w'-x'-y'+z',\\ & & w&=w'+x'+y'+z',\end{aligned}$$

we find

$$\begin{aligned}\mu\nu &= \lambda'^2 - \mu'^2 - \nu'^2 + 2\mu'\nu',\\ \nu\lambda &= -\lambda'^2 + \mu'^2 - \nu'^2 + 2\nu'\lambda',\\ \lambda\mu &= -\lambda'^2 - \mu'^2 + \nu' + 2\lambda'\mu',\end{aligned}$$

and thence

$$\mu\nu + \nu\lambda + \lambda\mu = -(\lambda'^2 + \mu'^2 + \nu'^2 - 2\mu'\nu' - 2\nu'\lambda' - 2\lambda'\mu'), = 0,$$

that is

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 0,$$

the relation which connects the new constants λ, μ, ν. Moreover

$$\begin{aligned}yz - xw &= 4(y'z' - x'w'),\\ zx - yw &= 4(z'x' - y'w'),\\ xy - zw &= 4(x'y' - z'w'),\\ 3w^2 - x^2 - y^2 - z^2 &= 8(y'z' + z'x' + x'y' + x'w' + y'w' + z'w'),\end{aligned}$$

and writing for greater convenience $\theta = -\frac{2}{\theta'}$, the equations are transformed into

$$\begin{aligned}\theta\lambda x &= xw - yz,\\ \theta\mu y &= yw - zx,\\ \theta\nu z &= zw - xy,\\ 2\theta(\lambda + \mu + \nu)w &= 3w^2 - x^2 - y^2 - z^2,\end{aligned}$$

where

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 0,$$

viz. these equations, eliminating θ, give the equations of the nodal curve.

From the first three equations eliminating θ, we deduce

$$\begin{aligned}yzw(\mu - \nu) &= x(\mu y^2 - \nu z^2),\\ zxw(\nu - \lambda) &= y(\nu z^2 - \lambda x^2),\\ xyw(\lambda - \mu) &= z(\lambda x^2 - \mu y^2),\end{aligned}$$

or, as these equations may be written,

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu - \nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu - \lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda - \mu},$$

which equations, from the mode in which they are obtained, are it is clear equivalent to two equations only. Using the fourth equation, and eliminating θ by substituting therein for $\theta\lambda$, $\theta\mu$, $\theta\nu$ their values from the first three equations, we find

$$2w^2\left(3w - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}\right) = 3w^2 - x^2 - y^2 - z^2,$$

that is

$$3w^2+x^2+y^2+z^2=2w\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right),$$

or, what is the same thing,

$$xyz\,(3w^2+x^2+y^2+z^2)-2w\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

we have to show that this is in fact included in the former system, for then the four equations with θ eliminated will it is clear give two equations only.

Observe that the former system may be written

$$w=\frac{x\,(\mu y^2-\nu z^2)}{(\mu-\nu)\,yz},$$

$$(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2=0,$$

and that we have thus to show that substituting for w the value $w=\dfrac{x\,(\mu y^2-\nu z^2)}{\mu-\nu}$ in the equation

$$xyz\,(3w^2+x^2+y^2+z^2)-2w\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

the result is

$$(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2=0.$$

The substitution in question gives

$$\frac{3x^2\,(\mu y^2-\nu z^2)^2}{(\mu-\nu)^2\,yz}+yz\,(x^2+y^2+z^2)-\frac{2\,(\mu y^2-\nu z^2)}{(\mu-\nu)\,yz}\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

that is

$$3x^2\,(\mu y^2-\nu z^2)^2+(\mu-\nu)^2\,y^2z^2\,(x^2+y^2+z^2)-2\,(\mu-\nu)\,(\mu y^2-\nu z^2)\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

which is in fact

$$-\mu^2y^2\,(y^2-z^2)\,(z^2-x^2)+2\mu\nu x^2\,(y^2-z^2)^2-\nu^2z^2\,(y^2-z^2)\,(x^2-y^2)=0,$$

that is, throwing out the factor y^2-z^2, it is

$$-\mu^2y^2\,(z^2-x^2)+2\mu\nu x^2\,(y^2-z^2)-\nu^2z^2\,(x^2-y^2)=0.$$

But in virtue of the equation $\dfrac{1}{\lambda}+\dfrac{1}{\mu}+\dfrac{1}{\nu}=0$, we have

$$\frac{\mu\nu}{\lambda}\,\{(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2\}$$

$$=\frac{\mu\nu}{\lambda}\,[\lambda x^2\,(y^2-z^2)+\mu y^2\,(z^2-x^2)+\nu z^2\,(x^2-y^2)],$$

$$=\mu\nu\,x^2\,(y^2-z^2)-(\mu+\nu)\,[\mu y^2\,(z^2-x^2)+\nu z^2\,(x^2-y^2)],$$

$$=-\mu^2y^2\,(z^2-x^2)+2\mu\nu\,x^2\,(y^2-z^2)-\nu^2z^2\,(x^2-y^2),$$

and the required property thus holds good.

We thus see that the equations of the nodal curve are

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu - \nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu - \lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda - \mu},$$

the nodal curve is thus the partial intersection of the two cubic scrolls (skew surfaces)

$$(\mu - \nu)\, wyz = x(\mu y^2 - \nu z^2), \quad (\nu - \lambda)\, wzx = y(\nu z^2 - \lambda x^2),$$

viz. taking A, B, C, D to be the summits of the tetrahedron the faces whereof are $x=0$, $y=0$, $z=0$, $w=0$, the first of these has AD for a nodal directrix, BC for a single directrix, BD, CD for generators; the second has BD for a nodal directrix, AC for a single directrix, AD, CD for generators; the surfaces intersect in the line AD twice, the line BD twice, and the line CD; the order of the residual curve, or nodal curve of the developable, is thus $9-(2+2+1)$, $=4$ as it should be.

I remark that the equation

$$(\mu - \nu)\, y^2z^2 + (\nu - \lambda)\, z^2x^2 + (\lambda - \mu)\, x^2y^2 = 0,$$

is the equation of the cone having its vertex at the point D, $(x=0,\ y=0,\ z=0)$, and passing through the nodal curve; the lines DA, DB, DC are each of them a nodal line of the cone, or "line through two points" of the curve; for an excubo-quartic curve the number of lines through two points which pass through a given point not on the curve is in fact $=3$.

It remains to introduce the coordinates (x, y, z, w) into the equation of the developable. We have

$$\begin{aligned} 4x' &= w + x - y - z, \\ 4y' &= w - x + y - z, \\ 4z' &= w - x - y + z, \\ 4w' &= w + x + y + z, \end{aligned}$$

and thence

$$\begin{aligned} 16y'z' &= (w-x)^2 - (y-z)^2, \\ 16x'w' &= (w+x)^2 - (y+z)^2, \end{aligned}$$

giving

$$8(x'w' + y'z') = w^2 + x^2 - y^2 - z^2$$

and similarly

$$8(y'w' + z'x') = w^2 - x^2 + y^2 - z^2,$$

and

$$8(z'w' + x'y') = w^2 - x^2 - y^2 + z^2.$$

Moreover

$$\begin{aligned} 16(y'z' + z'x' + x'y') = {} & (w-x)^2 - (y-z)^2 \\ & + (w-y)^2 - (z-x)^2 \\ & + (w-z)^2 - (x-y)^2, \\ = {} & 3w^2 - 2w(x+y+z) \\ & - x^2 - y^2 - z^2 + 2yz + 2zx + 2xy. \end{aligned}$$

Consequently

$$64w'(y'z'+z'x'+x'y') = (w+x+y+z)$$
$$\times\{3w^2-2w(x+y+z)-x^2-y^2-z^2+2yz+2zx+2xy\},$$

$$64x'y'z' = (w+x-y-z)(w-x+y-z)(w-x-y-z),$$
$$\begin{aligned} &= w^3 \\ &\quad -w^2(x+y+z) \\ &\quad -w(x^2+y^2+z^2-2yz-2zx-2xy) \\ &\quad +x^3+y^3+z^3-yz^2-y^2z-zx^2-z^2x-xy^2-x^2y+2xyz. \end{aligned}$$

Putting for shortness

$$p = x+y+z, \quad \nabla = x^2+y^2+z^2-2yz-2zx-2xy,$$

the two expressions are

$3w^3$	w^3
$+\ w^2.+p$	$+w^2.-p$
$+\ w.-2p^2-\nabla$	$-w.\nabla$
$-p\nabla$	$+x^3+y^3+z^3-y^2z-yz^2-z^2x-zx^2-x^2y-xy^2+2xyz$

or observing that $-p\nabla$ is

$$= -x^3-y^3-z^3+y^2z+yz^2+z^2x+zx^2+x^2y+xy^2+6xyz,$$

we have

$$64(w'y'z'+w'z'x'+w'x'y'+x'y'z')$$
$$\begin{aligned} &= 4w^3-2w(p^2+\nabla)+8xyz, \\ &= 4w^3 \\ &\quad -4w(x^2+y^2+z^2) \\ &\quad +8xyz, \end{aligned}$$

that is

$$16(w'y'z'+w'z'x'+w'x'y'+x'y'z') = w^3$$
$$-w(x^2+y^2+z^2)$$
$$+2xyz.$$

Moreover

$$8\{\lambda'(x'w'+y'z')+\mu'(y'x'+z'x')+\nu'(z'w'+x'y')\}$$
$$\begin{aligned} &= \lambda'(w^2+x^2-y^2-z^2) \\ &\quad +\mu'(w^2-x^2+y^2-z^2) \\ &\quad +\nu'(w^2-x^2-y^2+z^2), \\ &= -(\lambda+\mu+\nu)w^2+\lambda x^2+\mu y^2+\nu z^2; \end{aligned}$$

and we have

$$\lambda = 2(\beta-\delta)(\gamma-\delta)(\gamma-\alpha)(\alpha-\beta),$$
$$\mu = 2(\gamma-\delta)(\alpha-\delta)(\alpha-\beta)(\beta-\gamma),$$
$$\nu = 2(\alpha-\delta)(\beta-\delta)(\beta-\gamma)(\gamma-\alpha),$$

whence $\lambda\mu\nu = 8\lambda'\mu'\nu'$.

Hence finally λ, μ, ν denoting as just mentioned, and therefore satisfying $\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{\nu}=0$, the equation of the developable is

$$\lambda\mu\nu\{w^3 - w(x^2+y^2+z^2) + 2xyz\}^2 + 108\{(\lambda+\mu+\nu)w^2 - \lambda x^2 - \mu y^2 - \nu z^2\}^2 = 0$$

(say this is $\lambda\mu\nu T^2 + 108S^2 = 0$), and this surface (which has obviously the cuspidal curve $S=0$, $T=0$) has also the nodal curve

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu-\nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu-\lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda-\mu}.$$

I will show *à posteriori* that this is actually a nodal curve on the surface. Introducing an arbitrary parameter θ, the equations of the curve may be written *ut suprà*

$$\theta\lambda x = xw - yz,$$
$$\theta\mu y = yw - zx,$$
$$\theta\nu z = zw - xy,$$
$$2\theta(\lambda+\mu+\nu)w = 3w^2 - x^2 - y^2 - z^2,$$

and we have thence, as before,

$$2w\left(3w - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}\right) = 3w^2 - x^2 - y^2 - z^2.$$

Hence

$$2\theta = \frac{3w^2 - x^2 - y^2 - z^2}{(\lambda+\mu+\nu)w} = \frac{-2w(x^2+y^2+z^2) + 6xyz}{-\lambda x^2 - \mu y^2 - \nu z^2},$$
$$= \frac{3w^2 - w(x^2+y^2+z^2)}{(\lambda+\mu+\nu)w^2},$$
$$= \frac{3\{w^3 - w(x^2+y^2+z^2) + 2xyz\}}{(\lambda+\mu+\nu)w^2 - \lambda x^2 - \mu y^2 - \nu z^2},$$
$$= \frac{3T}{S}.$$

Hence writing

$$\theta(-2\lambda x) - (-2wx + 2yz) = 0,$$
$$\theta(-2\mu y) - (-2wy + 2zx) = 0,$$
$$\theta(-2\nu z) - (-2wz + 2xy) = 0,$$
$$\theta . 2(\lambda+\mu+\nu)w - (3w^2 - x^2 - y^2 - z^2) = 0,$$

substituting for θ its value $=\frac{3T}{2S}$, and attending to the significations of S and T, we have

$$3T\delta_x S - 2S\delta_x T = 0,$$
$$3T\delta_y S - 2S\delta_y T = 0,$$
$$3T\delta_z S - 2S\delta_z T = 0,$$
$$3T\delta_w S - 2S\delta_w T = 0,$$

which are in fact the conditions to be satisfied in order that the point (x, y, z, w) may belong to a nodal curve of the surface $\lambda\mu\nu\, T^2 + 108\, S^3 = 0$.

It is to be noticed that the coordinates of the before mentioned four points of intersection of the cuspidal and the nodal curves (being as already mentioned stationary points on the cuspidal curve) may be written $x, y, z, w = (1, 1, 1, 1), (1, -1, -1, 1), (-1, 1, -1, 1), (-1, -1, 1, 1)$.

We have thus far considered the developable, or torse, the equation of which is

$$\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(z'w' + x'y')\}^3$$
$$- 27\lambda'\mu'\nu'(x'y'z' + x'y'w' + x'z'w' + y'z'w')^2 = 0,$$

where $\sqrt{(\lambda')} + \sqrt{(\mu')} + \sqrt{(\nu')} = 0$; or, what is the same thing, writing a, b, c, in place of $\sqrt{(\lambda')}, \sqrt{(\mu')}, \sqrt{(\nu')}$ respectively, the torse

$$\{a^2(x'w' + y'z') + b^2(y'w' + z'x') + c^2(z'w' + x'y')\}^3 - 27a^2b^2c^2(x'y'z' + x'y'w' + x'z'w' + y'z'w')^2 = 0,$$

where $a + b + c = 0$.

Inverting this by the equations $x', y', z', w' = \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w}$, we obtain a sextic surface

$$\{a^2(xw + yz) + b^2(yw + zx) + c^2(zw + xy)\}^3 - 27a^2b^2c^2\, xyzw\,(x + y + z + w)^2 = 0,$$

where $a + b + c = 0$; which surface I propose [secondly] to consider in the present paper.

The surface has evidently the singular tangent planes $x = 0$, $y = 0$, $z = 0$, $w = 0$, each osculating the surface in a conic, that is, meeting it in the conic taken thrice, viz.,

$x = 0$, in a conic on the quadric cone $a^2yz + b^2yw + c^2zw = 0$,
$y = 0$, " " " $a^2xw + b^2zx + c^2zw = 0$,
$z = 0$, " " " $a^2xw + b^2yw + c^2xy = 0$,
$w = 0$, " " " $a^2yz + b^2zx + c^2xy = 0$;

and it has also a cuspidal conic, the intersection of the plane $x+y+z+w=0$ with the quadric surface

$$a^2(xw+yz)+b^2(yw+zx)+c^2(zw+yz)=0;$$

it may be observed that the four conics of osculation are also sections of this surface.

The surface has also a nodal curve, the equations of which might be obtained by inversion of those of the nodal curve of the sextic torse above referred to; but I prefer to obtain them independently, in a synthetical manner, as follows:

Take α, β, γ arbitrary, and write

$$\begin{aligned}
-A &= (b-c)\alpha+b\beta-c\gamma, & F &= b\gamma-c\beta,\\
-B &= (c-a)\beta+c\gamma-a\alpha, & G &= c\alpha-a\gamma,\\
-C &= (b-c)\gamma+a\alpha-b\beta, & H &= a\beta-b\alpha,\\
M &= (b-c)\alpha+(c-a)\beta+(a-b)\gamma,\\
Q &= a^2(b-c)\alpha+b^2(c-a)\beta+c^2(a-b)\gamma:
\end{aligned}$$

then it is to be shown, that not only the equation of the surface is satisfied, but that also each of the derived equations is satisfied, by the values

$$x:y:z:w=aAGHQ:bBHFQ:cCFGQ:abcFGHM;$$

each of the quantities A, B, C, M, Q is linearly expressible in terms of F, G, H, which are themselves connected by the equation $aF+bG+cH=0$; the foregoing values of x, y, z, w are consequently proportional to quartic functions of a single variable parameter, say $F\div G$; and there is thus an excubo-quartic nodal curve.

To establish the foregoing result, we have

$$\begin{aligned}
aA+bH+cG &= 0,\\
aH+bB+cF &= 0,\\
aG+bF+cC &= 0,\\
aA+bB+cC &= 0,\\
aF+bG+cH &= 0,\\
F+G+H &= -M,\\
bcF+caG+abH &= Q,\\
2bcF &= a^2A-b^2B-c^2C,\\
2caG &= -a^2A+b^2B-c^2C,\\
2abH &= -a^2A-b^2B+c^2C,\\
aAGH+bHBF+cFG &= abcM(\alpha+\beta+\gamma)^2,\\
aGH+bHF+cFG &= -abc(\alpha+\beta+\gamma)^2,\\
a^2AGH+b^2HBF+c^2CFG &= -abc\{Q(\alpha+\beta+\gamma)^2+3FGH\},\\
aBCF+bCAG+cABH &= 2Mabc(\alpha+\beta+\gamma)^3,\\
Q(\alpha+\beta+\gamma)^2+FGH &= ABC,
\end{aligned}$$

which are all of them identical equations; but as to some of them the verification is rather complex.

Hence we have

$$x+y+z=Q(aAGH+bBHF+cCFG)$$
$$=abcMQ(\alpha+\beta+\gamma)^2,$$
$$w=abcMFGH,$$

and thence

$$x+y+z+w=abcM\{Q(\alpha+\beta+\gamma)^2+FGH\}$$
$$=abcMABC.$$

Moreover

$$xyzw=(abc)^2ABC(FGH)^3Q^3M,$$

and

$$27a^2b^2c^2xyzw(x+y+z+w)^2=27(abc)^6(ABCFGHMQ)^3\ (*).$$

Again

$$(a^2x+b^2y+c^2z)w=abcFGHMQ(a^3AGH+b^3BHF+c^3CFG)$$
$$=(abc)^2FGHMQ\{Q(\alpha+\beta+\gamma)^2+3FGH\}$$
$$a^2yz+b^2zx+c^2xy=abcFGHQ^2(aBCF+bCAG+cABH)$$
$$=(abc)^2FGHMQ\,.\,2Q(\alpha+\beta+\gamma)^2,$$

and thence

$$a^2(xw+yz)+b^2(yw+zx)+c^2(xy+zw)$$
$$=3(abc)^2FGHMQ\{Q(\alpha+\beta+\gamma)^2+FGH\}$$
$$=3(abc)^2ABCFGHMQ \qquad (*),$$

and the two equations marked (*) verify the equation of the surface.

To verify the derived equations, write for a moment $P=a^2(yz+xw)+b^2(zx+yw)+c^2(xy+zw)$, so that the equation of the surface is $P^3-27a^2b^2c^2xyzw(x+y+z+w)^2=0$, and the derived equation with respect to x is

$$\frac{3}{P}\frac{dP}{dx}=\frac{1}{x}+\frac{2}{x+y+z+w};$$

or substituting for P and $x+y+z+w$ their values, this is

$$\frac{dP}{dx}=\frac{(abc)^2ABCFGHMQ}{x}+2abcQFGH,$$

and similarly for y, z, and w. In particular, considering the derived equation in respect to w, this is

$$a^2x+b^2y+c^2z=abcABCQ+2abcQFGH$$
$$=abcQ(ABC+2FGH),$$

and we have as before

$$a^2x+b^2y+c^2z=Q(a^3AGH+b^3BHF+c^3CFG)$$
$$=abcQ\{(\alpha+\beta+\gamma)^2+3FGH\}$$
$$=abcQ(ABC+2FGH),$$

which is thus verified; the verification of the derived equations for y, z, w can be effected, but not quite so easily.

The existence of the excubo-quartic nodal curve is thus established.

399.

ON THE CUBICAL DIVERGENT PARABOLAS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 185—189.]

NEWTON reckons five forms, viz. these are the *simplex*, the *complex*, the *crunodal*, the *acnodal*, and the *cuspidal*, but as noticed by Murdoch, the simplex has three different forms, the *ampullate*, the *neutral*, and the *campaniform*. We have thus the 8 forms at once distinguishable by the eye.

Plücker has in all 13 species, the division into species being established or completed geometrically by reference to the asymptotic cuspidal curve (or asymptotic semi-cubical parabola), and analytically as follows, viz. writing the equation in the form

$$y^2 = x^3 - 3cx + 2d,$$

the different species are

simplex,	$y^2 = x^3 - 3cx + 2d$	$c^3 < d^2$,	ampullate,
"	$y^2 = x^3 - 3cx - 2d$		campaniform,
"	$y^2 = x^3 + 2d$,		neutral,
"	$y^2 = x^3 - 2d$,		campaniform,
"	$y^2 = x^3 + 3cx + 2d$,		"
"	$y^2 = x^3 + 3cx$,		"
"	$y^2 = x^3 + 3cx - 2d$,		"
complex,	$y^2 = x^3 - 3cx + 2d$	$c^3 > d^2$,	
"	$y^2 = x^3 - 3cx - 2d$		
"	$y^2 = x^3 - 3cx$,		
acnodal,	$y^2 = x^3 - 3cx - 2c\sqrt{(c)}$,		
crunodal,	$y^2 = x^3 - 3cx + 2c\sqrt{(c)}$,		
cuspidal,	$y^2 = x^3$;		

but of the simplex species, there are five which are to the eye campaniform, and the three complex species have with each other a close resemblance in form.

I remark as regards the simplex forms, that the tangents at the two inflexions meet in a point R on the axis, and that the ampullate, the neutral, and campaniform forms are distinguished from each other according to the position of R, viz. for the ampullate form, R lies within the curve, for the campaniform form R lies without the curve, and for the neutral form, R is at infinity. It is to be observed, as regards the complex forms, that here R always lies without the curve, between the infinite branch and the oval.

The further division of the simplex and complex forms so as to obtain the $7+3$ species of Plücker, may be effected by considering in conjunction with the point R a certain other point I on the axis; it is to be remarked that excluding the inflexion at infinity the cubical divergent parabola has in all eight inflexions, two real and six imaginary, viz. the inflexions lie by pairs on four ordinates, or if x be the abscissa corresponding to an inflexion, x is determined by a quartic equation; this equation has always two real and two imaginary roots, each of the imaginary roots gives a pair of imaginary inflexions; one of the real roots gives a positive value for y^2 and therefore two real inflexions, the tangents at these meet in the above-mentioned point R on the axis; the other real root gives a negative value for y^2 and therefore two imaginary inflexions, but the tangents at these meet in a real point on the axis, and this I call the point I. It is clear that for each of the four pairs of inflexions the tangents at the two inflexions meet at a point on the axis, so that if X be the abscissa of such point, then X is determined by a quartic equation; two of the roots of this equation are imaginary, the other two roots are real, and correspond to the points R and I respectively.

The equation of the curve being as above

$$y^2 = x^3 - 3cx + 2d,$$

then the coordinate x belonging to a pair of inflexions is found by the equation

$$x^4 - 6cx^2 + 8dx - 3c^2 = 0,$$

or what is the same thing,

$$(1,\ 0,\ -c,\ 2d,\ -3c^2 \between x,\ 1)^4 = 0,$$

(the invariant I is $=0$, and hence the discriminant, $=-27J^2$, is negative, or the roots are two real, two imaginary, as already mentioned): the corresponding value of X is easily found to be

$$X = \frac{x^2 + 3cx - 4d}{3(x^2 - c)},$$

and we thence obtain

$$3cX^4 - 4dX^3 - 6c^2X^2 + 12cdX - (c^3 + 4d^2) = 0,$$

or what is the same thing,

$$(3c,\ -d,\ -c^2,\ 3cd,\ -c^3 - 4d^2 \between X,\ 1)^4 = 0,$$

for the equation in X; the quadrinvariant I is $=0$, and hence the discriminant, $=-27J^2$, is negative; that is, the roots are two real, two imaginary, as already mentioned.

Considering the simplex forms, first, if $c=0$, then for the curve

$$y^2=x^3+2d,$$

it appears that R lies at infinity, I within the curve; and for the curve

$$y^2=x^3-2d,$$

that R lies without the curve, I at infinity.

It further appears that when $d=0$, or for the curve,

$$y^2=x^3+3cx,$$

R, I lie equidistant from the vertex, R without, I within the curve.

Hence in the curve

$$y^2=x^3+3cx+2d,$$

since, when $d=0$, the points R, I are equidistant from the vertex, and for $c=0$, the point R is at infinity, it is easy to infer by continuity that the points R, I lie R without, I within the curve, I being nearer to the vertex.

And similarly in the curve

$$y^2=x^3+3cx-2d,$$

that the points R, I lie R without, I within the curve, R being nearer to the vertex.

Again, in the curve

$$y^2=x^3-3cx+2d,$$

since, in the curve $y^2=x^3+3cx+2d$, R is without, I within the curve, and as c becomes $=0$, R passes off to infinity, it appears that c having changed its sign, or for the curve now in question, R having passed through infinity, will be situate within the curve; that is, R, I lie each of them within the curve.

And similarly for the curve

$$y^2=x^3-3cx-2d,$$

it appears that R, I lie each without the curve.

Hence, finally, for the simplex forms, we have the 7 species of Plücker, viz.

$$y^2=x^3-3cx+2d,\ c^3<d^2,$$

simplex ampullate, R, I within the curve;

$$y^2=x^3-3cx-2d,\ c^3<d^2,$$

simplex campaniform, R, I without the curve;

$$y^2=x^3+2d,$$

simplex neutral, I within the curve, R at infinity;

$$y^2 = x^3 - 2d,$$

simplex campaniform quasi-neutral, R without the curve, I at infinity;

$$y^2 = x^3 + 3cx + 2d,$$

simplex campaniform, R without and further from, I within and nearer to the curve;

$$y^2 = x^3 + 3cx,$$

simplex campaniform equidistant, viz. R and I are equidistant from the curve, R without and I within;

$$y^2 = x^3 + 3cx - 2d,$$

simplex campaniform, R without and nearer to, I within and further from the curve.

Passing to the complex forms, suppose for a moment that α is the diameter of the oval and β the distance of the oval from the vertex of the infinite branch; the equation of the curve then is $y^2 = x(x-\alpha)(x-\alpha-\beta)$, or changing the origin so as to make the term in x^2 to vanish, this is

$$y^2 = (x + \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta)(x - \tfrac{1}{3}\alpha + \tfrac{1}{3}\beta)(x - \tfrac{1}{3}\alpha - \tfrac{2}{3}\beta),$$

or, what is the same thing,

$$y^2 = x^3 - \tfrac{2}{3}(\alpha^2 + \alpha\beta + \beta^2)x - \tfrac{1}{27}(\alpha - \beta)(2\alpha + \beta)(\alpha + 2\beta),$$

or comparing this with $y^2 = x^3 - 3cx + 2d$, d is $= +$, 0 or $-$, as $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$, or say as the oval is smaller, mean, or larger; viz. the magnitude of the oval is estimated by the relation which the diameter thereof bears to the distance of the oval from the infinite branch. In the case $d = 0$, or for the curve $y^2 = x^3 - 3cx$ it appears (as for the corresponding simplex form $y^2 = x^3 + 3cx$) that the points R, I are equidistant from the point $x = 0$, which is in the present case the middle vertex, or vertex of the oval which vertex is nearest to the infinite branch. As the oval diminishes, so that the curve becomes ultimately acnodal, I remaining within the oval ultimately coincides with the acnode; and as the oval increases so that the curve becomes ultimately crunodal, R remaining between the oval and the infinite branch, ultimately coincides with the crunode; and it hence easily appears by continuity that for a smaller oval I is nearer to, R further from the middle vertex; while for a larger oval, I is further from, R nearer to the middle vertex. Hence for the complex forms the species are

$$y^2 = x^3 - 3cx + 2d,$$

smaller oval, I nearer to, R further from the middle vertex;

$$y^2 = x^3 - 3cx,$$

mean oval, R and I equidistant from the middle vertex;

$$y^2 = x^3 - 3cx - 2d,$$

larger oval, I further from, R nearer to the middle vertex: and the division into species is thus completed.

Cambridge, June 16, 1865.

400.

ON THE CUBIC CURVES INSCRIBED IN A GIVEN PENCIL OF SIX LINES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 210—221.]

WE have to consider a pencil of six lines, that is, six lines meeting in a point, and a cubic curve touching each of the six lines. As a cubic curve may be made to satisfy nine conditions, the cubic curve will involve three arbitrary parameters; but if we have any particular curve touching the six lines, then transforming the whole figure homologously, the centre of the pencil being the pole and any line whatever the axis of homology, the pencil of lines remains unaltered, and the new curve touches the six lines of the pencil; the transformation introduces three arbitrary constants, and the general solution is thus given as such homologous transformation of a particular solution. To show the same thing analytically, take $(x=0,\ y=0,\ z=0)$ for the axes of coordinates, the lines $x=0$, $y=0$ being any two lines through the centre of the pencil, so that the equation of the pencil is $(*\between x,\ y)^6=0$, then if $\phi(x,\ y,\ z)=0$ is the equation of a cubic curve touching the six lines, the equation of the general curve touching the six lines will be $\phi(x,\ y,\ \alpha x+\beta y+\gamma z)=0$; or what is the same thing, considering the coordinate z as implicitly containing three arbitrary constants, viz. an arbitrary multiplier and the two arbitrary parameters of the line $z=0$, then the equation $\phi(x,\ y,\ z)=0$ may be taken to be that of the cubic touching the six lines.

Now the given binary sextic $(*\between x,\ y)^6$ may be expressed in the form P^2+Q^3, where P is a cubic function, Q a quadric function, of the coordinates $(x,\ y)$; or, what is the same thing, but introducing for homogeneity a constant c, we may write

$$(*\between x,\ y)^6=c\left[(a,\ h,\ k,\ b\between x,\ y)^3\right]^2+4\left[(j,\ l,\ f\between x,\ y)^2\right]^3;$$

in fact, comparing the two sides of this equation, we have each of the seven coefficients of the sextic equal to a function of the seven quantities $a\sqrt{(c)}$, $h\sqrt{(c)}$, $k\sqrt{(c)}$, $b\sqrt{(c)}$, j, l, f; so that conversely, these seven quantities are determinable (not however rationally) in terms of the coefficients of the given sextic. And when the sextic is expressed in the foregoing form, then it will presently be shown that we have

$$(a,\ h,\ k,\ b\between x,\ y)^3 + 3z(j,\ l,\ f\between x,\ y)^2 + cz^3 = 0,$$

or, what is the same thing,

$$(a,\ b,\ c,\ f,\ 0,\ h,\ 0,\ j,\ k\between x,\ y,\ z)^3 = 0,$$

as the equation of a cubic curve touching the six given lines; and by what precedes, it appears that this may be taken to be the equation of the general cubic curve which touches the six given lines. On account of the arbitrary constant c, it is sufficient to replace z by $\alpha x + \beta y + z$, or, what is the same thing, to consider $z = 0$ as the equation of an arbitrary line, but without introducing therein an arbitrary multiplier.

To sustain the foregoing result, consider the cubic

$$(a,\ b,\ c,\ f,\ g,\ h,\ i,\ j,\ k,\ l\between x,\ y,\ z)^3 = 0,$$

then in general if $A = (,,\between x,\ y,\ z)^3$, $B = (,,\between x,\ y,\ z)^2(\alpha,\ \beta,\ \gamma)$, $C = (,,\between x,\ y,\ z)(\alpha,\ \beta,\ \gamma)^2$, $D = (,,\between \alpha,\ \beta,\ \gamma)^3$, the equation of the pencil of tangents drawn from the point $(\alpha,\ \beta,\ \gamma)$ to the curve is

$$A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2 = 0,$$

but writing for shortness

$$(,,\between x,\ y,\ z)^3 = (A',\ B',\ C',\ D'\between 1,\ z)^3,$$

so that

$$\begin{aligned} A' &= (a,\ h,\ k,\ b\between x,\ y)^3, \\ B' &= \quad (j,\ l,\ f\between x,\ y)^2, \\ C' &= \qquad (g,\ i\between x,\ y), \\ D' &= \qquad\quad c \quad , \end{aligned}$$

then for the tangents from the point $(x = 0,\ y = 0)$, writing $(\alpha,\ \beta,\ \gamma) = (0,\ 0,\ 1)$, we have

$$\begin{aligned} A &= (A',\ B',\ C',\ D'\between 1,\ z)^3, \\ B &= \quad (B',\ C',\ D'\between 1,\ z)^2, \\ C &= \qquad (C',\ D'\between 1,\ z), \\ D &= \qquad\quad D' \quad , \end{aligned}$$

and thence the equation of the pencil of tangents is

$$A'^2D'^2 - 6A'B'C'D' + 4A'C'^3 + 4B'^3D' - 3B'^2C'^2 = 0.$$

Hence for the curve

$$(a,\ b,\ c,\ f,\ 0,\ h,\ 0,\ j,\ k,\ l\between x,\ y)^3 = 0,$$

we have $g=0$, $i=0$, and therefore $C'=0$; the equation of the pencil of tangents is $A'^2D'^2+4B'^2D'=0$, or throwing out the constant factor D', and then replacing A', B', D' by their values, the equation of the pencil of tangents is

$$c\,[(a,\ h,\ k,\ b\between x,\ y)^3]^2+4\,[(j,\ l,\ f\between x,\ y)^2]^3=0,$$

which is the before-mentioned result.

The coefficients $a\sqrt{(c)}$, $h\sqrt{(c)}$, $k\sqrt{(c)}$, $b\sqrt{(c)}$, j, l, f, or (as we may call them) the coefficients of the cubic curve, are, it has been seen, functions of the coefficients of the given sextic $(*\between x,\ y)^6$; hence the invariants S and T of the cubic curve are also functions of the coefficients of the sextic, and it is easy to see that they are in fact invariants (not however rational invariants) of the sextic. To verify this, it is only necessary to show that the invariants S and T are functions of the invariants of the functions $\sqrt{(c)}\,.\,(a,\ h,\ k,\ b\between x,\ y)^3$ and $(j,\ l,\ f\between x,\ y)^2$; for if this be so, they will be invariants of the function

$$[c\,(a,\ h,\ k,\ b\between x,\ y)^3]^2+4\,[(j,\ l,\ f\between x,\ y)^2]^3,$$

that is of the sextic. We have in fact the general theorem, that if P, Q, R, ... be any quantics in $(x,\ y,\ \ldots)$, and $\phi(P,\ Q,\ R,\ \ldots)$ a function of these quantics, homogeneous in regard to $(x,\ y,\ \ldots)$, then any function of the coefficients of ϕ, which is an invariant of the quantics P, Q, R, ... is also an invariant of ϕ.

Considering for greater convenience the function

$$(a,\ h,\ k,\ b\between x,\ y)^3$$

in place of $\sqrt{(c)}\,.\,(a,\ h,\ k,\ b\between x,\ y)^3$, the invariants of the two functions $(a,\ h,\ k,\ b\between x,\ y)^3$ and $(j,\ l,\ f\between x,\ y)^2$ are as follows:

$$\begin{aligned}
\square &= a^2b^2-6abhk+4ak^3+4bh^3-3h^2k^2,\\
\nabla &= fj-l^2,\\
\Theta &= j\,(bh-k^2)+l\,(hk-ab)+f\,(ak-h^2),\\
R &= +\ 1\ a^2f^3\\
&\quad +\ 6\ abflj\\
&\quad -\ 6\ ahf^2l\\
&\quad -\ 6\ akf^2j\\
&\quad +12\ akfl^2\\
&\quad +\ 1\ b^2j^3\\
&\quad -\ 6\ bhfj^2\\
&\quad +12\ bhjl^2\\
&\quad -\ 6\ bkj^2l\\
&\quad +\ 9\ h^2f^2j\\
&\quad -18\ hkfjl\\
&\quad +\ 9\ k^2fj^2\\
&\quad -\ 8\ abl^3,
\end{aligned}$$

viz. $\square$, ∇ are the discriminants of the two functions respectively, and Θ, R are simultaneous invariants of the two functions, R being in fact the resultant. The corresponding invariants of the functions $\sqrt{(c)}\,.\,(a, h, k, b \between x, y)^3$, and $(j, l, f \between x, y)^2$ are obviously $c^2\square$, ∇, $c\Theta$ and cR.

The values of S and T are obtained from the Tables 62 and 63 of my "Third Memoir on Quantics," *Phil. Trans.* vol. CXLVI. (1856), pp. 627—647, [144], by merely writing therein $g = i = 0$. It appears that they are in fact functions of $c^2\square$, ∇, $c\Theta$ and cR; viz. we have

$$S = \nabla^2 + c\Theta,$$

$$T = 8\nabla^3 + c(4R + 12\nabla\Theta) + c^2\square.$$

The invariants of the sextic $(* \between x, y)^6$, if for a moment the coefficients of this sextic are taken to be (a, b, c, d, e, f, g), that is, if the sextic be represented by $(a, b, c, d, e, f, g \between x, y)^6$ are the quadrinvariant $(= ag - 6bf + 15ce - 10d^2)$, Table No. 31 and Salmon's A., p. 203(1), the quartinvariant, No. 34, and Salmon's B., p. 203, the sextinvariant No. 35, and Salmon's C., p. 204, and the discriminant, which is a function of the tenth order $= a^5g^5 + \&c.$ recently calculated for the general form, Salmon, pp. 205—207, say these invariants are Q_2, Q_4, Q_6 and Q_{10}. These several invariants are functions of the above-mentioned expressions $c^2\square$, ∇, $c\Theta$ and cR; whence, conversely, these quantities are functions of the four invariants Q_2, Q_4, Q_6, Q_{10}; and the invariants S, T of the cubic curve, being functions of $c^2\square$, ∇, $c\Theta$ and cR, are also, as they should be, functions of the invariants Q_2, Q_4, Q_6 and Q_{10} of the sextic pencil $(* \between x, y)^6$.

To effect the calculation of Q_2, Q_4 and Q_6, I remark that inasmuch as by a linear transformation, the quadric $(j, l, f \between x, y)^2$ may be reduced to the form $2lxy$, and that the invariants of $(a, h, k, b \between x, y)^3$ and $2lxy$ are

$$\begin{aligned} \square &= a^2b^2 - 6abhk + 4ak^3 + 4bh^3 - 3h^2k^2, \\ \nabla &= -l^2, \\ \Theta &= -l(ab - hk), \\ R &= -8l^3ab, \end{aligned}$$

hence, writing $j = 0$, $f = 0$, and writing also $c = 1$, we may consider the sextic

$$[(a, h, k, b \between x, y)^3]^2 + 32l^3x^3y^3,$$

that is

$$(a^2,\ ah,\ \tfrac{1}{3}(2ak + 3h^2),\ \tfrac{1}{10}(ab + 9hk + 16l^3),\ \tfrac{1}{5}(2bh + 3k^2),\ bk,\ b^2 \between x, y)^6,$$

the invariants whereof are found to be functions of the last mentioned values of $\square$, ∇, Θ, R; to pass to the given sextic $(* \between x, y)^6$, put equal to

$$c\,[(a, h, k, b \between x, y)^3]^2 + 4\,[(j, l, f \between x, y)^2]^3,$$

we have only to consider $\square$, ∇, Θ, R as having their before-mentioned general values, and to restore the coefficient c by the principle of homogeneity.

[1] The pages refer to Salmon's Lessons Introductory to the Modern Higher Algebra (Second Edition, 1866). In the Fourth Edition, 1885, the values are given, pp. 260—265.

As regards the discriminant Q_{10}, this as already remarked, has been calculated for the general form, but for the present purpose it is easier, by dealing directly with the form $[(a, h, k, b \between x, y)^3]^2 + 32l^2x^2y^3$, and then interpreting $\square$, ∇, Θ, R and restoring the coefficient c as above, to obtain the discriminant Q_{10} of the function

$$[c\,(a, h, k, b \between x, y)^3]^2 + 4\,[(j, l, f \between x, y)^2]^3$$

in the required form, as a function of $c^2\square$, ∇, $c\Theta$, cR.

I find after some laborious calculations

$$\begin{aligned}
Q_2 = \quad 10 \text{ No. } 31 = \; & c^2 \{ \quad 9\,\square \\
& + c \left\{ \begin{array}{l} \quad 40\,R \\ + \quad 288\,\nabla\Theta \end{array} \right. \\
& + \quad \{ \quad 256\,\nabla^3 \\
Q_4 = \quad 10000 \text{ No. } 34 = \; & c^4 \{ - \quad 99\,\square^2 \\
& + c^3 \left\{ \begin{array}{l} - \quad 400\,R\square \\ + \quad 2304\,\nabla\Theta\square \\ + \quad 8640\,\Theta^3 \end{array} \right. \\
& + c^2 \left\{ \begin{array}{l} \quad 12800\,R\nabla\Theta \\ + \quad 82944\,\nabla^2\Theta^2 \\ + \quad 4608\,\nabla^3\square \end{array} \right. \\
& + c \left\{ \begin{array}{l} \quad 20480\,R\nabla^3 \\ + \quad 147456\,\nabla^4\Theta \end{array} \right. \\
& + \quad 65536\,\nabla^6 \\
Q_6 = 1000000 \text{ No. } 35 = \; & c^6 \{ + \quad 7992\,\square^3 \\
& + c^5 \left\{ \begin{array}{l} + \quad 72000\,R\square^2 \\ + \quad 145152\,\nabla\Theta^2\square \\ - \quad 622080\,\Theta^3\square \end{array} \right. \\
& + c^4 \left\{ \begin{array}{l} \quad 160000\,R^2\square \\ + \quad 691200\,R\nabla\Theta\square \\ + \quad 3456000\,R\Theta^3 \\ + \quad 3815424\,\nabla^2\Theta^2\square \\ + \quad 36080640\,\nabla\Theta^4 \\ + \quad 635904\,\nabla^3\square^2 \end{array} \right. \\
& + c^3 \left\{ \begin{array}{l} + \quad 33177600\,R\nabla^2\Theta^2 \\ + \quad 4669440\,R\nabla^3\square \\ + 217645056\,\nabla^3\Theta^3 \\ + \quad 23003136\,\nabla^4\Theta\square \end{array} \right.
\end{aligned}$$

$$
\begin{aligned}
&+ c^2 \begin{cases} + \quad 8192000\, R^3 \nabla^3 \\ + 110100480\, R \nabla^4 \Theta \\ + 509607936\, \nabla^5 \Theta^2 \\ + \quad 14155776\, \nabla^6 \square \end{cases} \\
&+ c \begin{cases} \quad 62914560\, R \nabla^6 \\ + 452984832\, \nabla^7 \Theta \end{cases} \\
&+ \qquad 134217728\, \nabla^9
\end{aligned}
$$

Q_{10} = multiple of discriminant

$$
\begin{aligned}
= \quad & c^7 \{ - \quad R^3 \square^2 \\
&+ c^6 \begin{cases} - \ 8\, R^4 \square \\ - 24\, R^3 \nabla \Theta \square \\ + 64\, R^3 \Theta^3 \end{cases} \\
&+ c^5 \begin{cases} - 16\, R^5 \\ - 96\, R^4 \nabla \Theta \\ + 48\, R^4 \nabla^2 \Theta^2 \\ - 16\, R^4 \nabla^3 \square \end{cases} \\
&+ c^4 \{ - 64\, R^4 \nabla^3,
\end{aligned}
$$

to which may be joined

$$
\begin{aligned}
Q_2^2 = \quad & c^4 \{ \qquad 81\, \square^2 \\
&+ c^3 \begin{cases} \quad 720\, R \square \\ + \quad 5184\, \nabla \Theta \square \end{cases} \\
&+ c^2 \begin{cases} \quad 1600\, R^2 \\ + \ 23040\, R \nabla \Theta \\ + \ 82944\, \nabla^2 \Theta^2 \\ + \quad 4608\, \nabla^3 \square \end{cases} \\
&+ c \begin{cases} \quad 20480\, R \nabla^3 \\ + 147456\, \nabla^4 \Theta \end{cases} \\
&+ \qquad 65536\, \square^6
\end{aligned}
$$

$$
\begin{aligned}
Q_2^2 - Q_4 = \quad & c^4 \{ \qquad 180\, \square^2 \\
&+ c^3 \begin{cases} \quad 1120\, R \square \\ + \ 2880\, \nabla \Theta \square \\ - \ 8640\, \Theta^3 \end{cases} \\
&+ c^2 \begin{cases} \quad 1600\, R^2 \\ + 10240\, R \nabla \Theta \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{1}{8} (Q_6 - 8\, Q_2^3) = \quad & c^6 \{ \qquad 270\, \square^3 \\
&+ c^5 \begin{cases} - \quad 720\, R \square^2 \\ - \ 51840\, \nabla \Theta \square^2 \\ - \ 77760\, \Theta^3 \square \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&+c^4\begin{cases}
-\quad 23200\,R^2\square\\
-\quad 535680\,R\Theta\nabla\square\\
+\quad 432000\,R\Theta^3\\
-1762560\,\nabla^2\Theta^2\square\\
+4510080\,\nabla\Theta^4\\
+\quad 17280\,\nabla^3\square^2
\end{cases}\\
&+c^3\begin{cases}
-\quad 64000\,R^2\\
-1382400\,R^2\nabla\Theta\\
-5806080\,R\nabla^2\Theta^2\\
+\quad 30720\,R\nabla^3\square\\
+3317760\,\nabla^3\Theta^3\\
-1105920\,\nabla^4\Theta\square
\end{cases}\\
&+c^2\begin{cases}
-\quad 204800\,R^2\nabla^3\\
-3932160\,R\nabla^4\Theta.
\end{cases}
\end{aligned}
$$

The foregoing values of S and T give

$$
\begin{aligned}
T^2-64S^3=\quad & c^4\left\{\quad \square^2\right.\\
&+c^3\begin{cases}
\quad 8\,R\square\\
+24\,\nabla\Theta\square\\
-64\,\Theta^3
\end{cases}\\
&+c^2\begin{cases}
\quad 16\,R^2\\
+96\,R\nabla\Theta\\
-48\,\nabla^2\Theta^4\\
+16\,\nabla^3\square
\end{cases}\\
&+c\qquad 64\,R\nabla^3,
\end{aligned}
$$

so that

$$Q_{10}=-c^2R^3\left(T^2-64S^3\right),$$

$$64S^3-T^2=\frac{Q_{10}}{c^3R^3},$$

$$S=\nabla^2+c\Theta,$$

and therefore

$$64-\frac{T^2}{S^3}=\frac{Q_{10}}{c^3R^3\left(\nabla^2+c\Theta\right)^3}=\frac{Q_{10}}{\left\{cR\left(\nabla^2+c\Theta\right)\right\}^3},$$

formulæ which are interesting in the theory.

We have

$$c\,\Theta = S - \nabla^2,$$
$$c^2\square = T - 12\nabla S + 4\nabla^3 - 4cR,$$

and if by means of these values we eliminate $c\Theta$ and $c^2\square$, we obtain Q_2, Q_4, Q_6 and Q_{10} as functions of S, T, ∇ and cR. Choosing instead of Q_4 and Q_6 the combinations $Q_2^2 - Q_4$ and $Q_6 - 8Q_2^3$, and forming also the expression for the combination $Q_2(Q_2^2 - Q_4)$, we have thus the system of formulæ

$$\begin{aligned}
Q_2 = {} & 9\,T + 180\nabla S + 4\nabla^3 + 4cR,\\
\tfrac{1}{20}(Q_2^2 - Q_4) = {} & +\ 9\,T^2\\
& -\ 432\,S^2\\
& -\ 72\,T\nabla S\\
& -\ 72\,T\nabla^3\\
& -\ 16\,TcR\\
& +\ 864\,\nabla^2 S^2\\
& +\ 144\,\nabla^4 S\\
& +\ 128\,\nabla ScR,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{80}(Q_6 - 8Q_2^3) = {} & \\
& +\ 27\,T^3\\
& -\ 4212\,T^2\nabla S\\
& +\ 6588\,T^2\nabla^3\\
& +\ 252\,T^2cR\\
& -\ 7776\,TS^2\\
& -\ 16848\,T\nabla^2 S^2\\
& +\ 3456\,T\nabla^4 S\\
& -\ 2592\,T\nabla ScR\\
& -\ 1296\,T\nabla^6\\
& -\ 1824\,T\nabla^3 cR\\
& -\ 448\,Tc^2R^2\\
& +\ 544320\,\nabla S^4\\
& -\ 461376\,\nabla^3 S^3\\
& +\ 74304\,cRS^2\\
& +\ 15552\,\nabla^5 S^2\\
& -\ 10368\,\nabla^2 S^2 cR\\
& -\ 10728\,\nabla^7 S\\
& -\ 3264\,\nabla^4 ScR\\
& -\ 1536\,\nabla Sc^2R^2,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{20}\,Q_2(Q_2^2 - Q_4) = {} & \\
& +\ 81\,T^2\\
& +\ 972\,T^2\nabla S\\
& -\ 612\,T^2\nabla^3\\
& -\ 108\,T^2cR\\
& -\ 3888\,TS^2\\
& -\ 5184\,T\nabla^2 S^2\\
& -\ 11952\,T\nabla^4 S\\
& -\ 2016\,T\nabla ScR\\
& -\ 288\,T\nabla^6\\
& -\ 352\,T\nabla^3 cR\\
& -\ 64\,Tc^2R^2\\
& -\ 77760\,\nabla S^4\\
& +\ 153792\,\nabla^3 S^3\\
& -\ 1728\,cRS^2\\
& +\ 29376\,\nabla^5 S^2\\
& +\ 26996\,\nabla^2 S^2 cR\\
& +\ 576\,\nabla^7 S\\
& +\ 1088\,\nabla^4 ScR\\
& +\ 512\,\nabla Sc^2R^2,
\end{aligned}$$

and, as mentioned above,

$$Q_{10} = c^3R^3(-T^2 + 64S^3).$$

The just-mentioned value of Q_{10} should, I think, admit of being established *a priori*, and if this be so, then the substitution of the values of S and T in terms of $c^2\square$, ∇, $c\Theta$, cR, would be the easiest way of arriving at the before-mentioned expression of Q_{10} in terms of these same quantities. The calculation by which this expression was arrived at, is however not without interest, and it will be as well to indicate the mode in which it was effected.

Calculation of Q_{10}.

We have to find the discriminant of

$$c\,[(a,\ h,\ k,\ b \between x,\ y)^3]^2 + 32l^3x^2y^3.$$

Consider for a moment the more general form $P^2 + 4Q^3$, then to find the discriminant, we have to eliminate between the equations

$$P\frac{dP}{dx} + 6Q^2\frac{dQ}{dx} = 0,$$

$$P\frac{dP}{dy} + 6Q^2\frac{dQ}{dy} = 0,$$

these are satisfied by the system $P = 0$, $Q^2 = 0$, and it follows that if R be the resultant of the equations $P = 0$, $Q = 0$, then the discriminant in question contains the factor R^2. For the other factor we may reduce the system to

$$P\frac{dP}{dx} + 6Q^2\frac{dQ}{dx} = 0,$$

$$\frac{dP}{dx}\frac{dQ}{dy} - \frac{dP}{dy}\frac{dQ}{dx} = 0.$$

Now writing $Q = 2lxy$, these equations become

$$P\frac{dP}{dx} + 48l^3x^2y^3 = 0,$$

$$l\left(x\frac{dP}{dx} - y\frac{dP}{dy}\right) = 0,$$

the resultant of which is $= l^3$ into resultant of the system

$$P\frac{dP}{dx} + 48l^3x^2y^3 = 0,$$

$$x\frac{dP}{dx} - y\frac{dP}{dy} = 0,$$

but in virtue of the second equation, we have

$$P = \tfrac{1}{3}\left(x\frac{dP}{dx} + y\frac{dP}{dy}\right) = \tfrac{2}{3}y\frac{dP}{dy},$$

which reduces the first equation to

$$2y\frac{dP}{dx}\frac{dP}{dy} + 144l^3x^2y^3 = 0,$$

or omitting the factor $2y$, to

$$\frac{dP}{dx}\frac{dP}{dy} + 72l^3x^2y^3 = 0.$$

Hence, writing $P = \sqrt{(c)}\,.\,(a,\ h,\ k,\ b \between x,\ y)^3$, and therefore $\dfrac{dP}{dx} = 3\sqrt{(c)}\,.\,(a,\ h,\ k \between x,\ y)^2$, $\dfrac{dP}{dy} = 3\sqrt{(c)}\,.\,(h,\ k,\ b \between x,\ y)^2$; writing also $y = 1$, the two equations become

$$c\,(a,\ h,\ k \between x,\ 1)^2(h,\ k,\ b \between x,\ 1)^2 + 8l^3x^2 = 0,$$

$$(a,\ h,\ k \between x,\ 1)^2 x - (h,\ k,\ b \between x,\ 1)^2 \quad = 0,$$

the second of which is more simply written

$$(a,\ h,\ -k,\ -b \between x,\ 1)^3 = 0.$$

Hence, restoring the factor l^3, and also to avoid fractions introducing the factor $8a^4$, the resultant of the two equations is

$$= 8l^3a^4\Pi\,\{8l^3x^2 + c\,(a,\ h,\ k \between x,\ 1)^2(h,\ k,\ b \between x,\ 1)^2\},$$

where Π denotes the product of the factors corresponding to the three roots x_1, x_2, x_3 of the equation

$$(a,\ h,\ -k,\ -b \between x,\ 1)^3 = 0,$$

or what is the same thing,

$$ax^3 + hx^2 - kx - b = 0,$$

so that the symmetric functions are to be found from

$$\Sigma x_1 = -\frac{h}{a},\ \Sigma x_1x_2 = -\frac{k}{a},\ x_1x_2x_3 = \frac{b}{a}.$$

The required discriminant is the foregoing resultant multiplied by R, or say by c^2R^2, that is the discriminant Q_{10} is

$$= c^2R^2\,.\,8l^3a^4\Pi\,(8l^3x^2 + \Omega),$$

if for shortness we write

$$\Omega = (a,\ h,\ k \between x,\ 1)^2\,.\,(a,\ k,\ b \between x,\ 1)^3,$$

and when the symmetric functions have been expressed in terms of the coefficients, the result is to be expressed as a function of $\square$, ∇, Θ, R by means of the values

$$\begin{aligned} c^2\square &= a^2b^2 + 4ak^3 + 4bh^3 - 6adhk - 3h^3k^2, \\ \nabla &= -l^2, \\ c\Theta &= -l(ab - hk), \\ cR &= -8l^3ab. \end{aligned}$$

Thus, for instance, the first term of the result is

$$= c^2R^2 \,.\, 8l^3a^4 \,.\, 512l^3x_1^2x_2^2x_3^2,$$

which is

$$\begin{aligned} &= c^2R^2 \,.\, 4096l^{12}a^2b^2, \\ &= c^2R^2 \,.\, -64c^2R^2\nabla^3, \\ &= -64c^4R^4\nabla^3, \end{aligned}$$

which is a term in the before-mentioned expression for Q_{10}.

401.

A NOTATION OF THE POINTS AND LINES IN PASCAL'S THEOREM.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 268—274.]

TAKING six points 1, 2, 3, 4, 5, 6 on a conic; let $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O$ denote each a combination of three lines, thus

$$\left|\begin{array}{c|c|c} 12.34.56 = A & 12.35.64 = F & 12.36.45 = K \\ 13.45.62 = B & 13.46.25 = G & 13.42.56 = L \\ 14.56.23 = C & 14.52.36 = H & 14.53.62 = M \\ 15.62.34 = D & 15.63.42 = I & 15.64.23 = N \\ 16.23.45 = E & 16.24.53 = J & 16.25.34 = O \end{array}\right|$$

then any hexagon formed with the six points may be represented by a combination of some two of the letters A, B, &c., viz. the three alternate sides are the lines represented by one letter, and the other three alternate sides the lines represented by the other letter: for example, the hexagon 123456 is AE; and so for the other hexagons. Any duad AE thus representing a hexagon may be termed a hexagonal duad; the number of such duads is sixty. Each Pascalian line may be denoted by the symbol of the hexagon to which it belongs; thus, the line which belongs to the hexagon AE, is the line AE.

I form the following combinations:

$IMO.DHJ$	each involving	all the duads 12, &c.	except those of	123.456,
$DEG.BNO$	"	"	"	124.356,
$ELM.BCJ$	"	"	"	125.346,
$HLN.CGI$	"	"	"	126.345,
$EFI.JKN$	"	"	"	134.256,
$AEH.CKO$	"	"	"	135.246,
$AMN.CDF$	"	"	"	136.245,
$AGJ.ELO$	"	"	"	145.236,
$ABI.DKL$	"	"	"	146.235,
$GKM.BFH$	"	"	"	156.234,

and also the combinations:

AEGMI	involving all the duads 12, 13, &c.,	
ABHJN	"	"
BCFIO	"	"
CDGJK	"	"
DEFHL	"	"
KLMNO	"	"

which I call respectively the ten-partite and six-partite arrangements. It is to be remarked that (considering *IMO*.*DHJ* as standing for the six duads *IM, IO, MO DH, DJ, HJ,* and so for the others) the ten-partite arrangement contains all the sixty hexagonal duads: and in like manner, (considering *AEGMI* as standing for the ten duads *AE, AG, AM, AI, EG, EM, EI, GM, GI, MI,* and so for the others) the six-partite arrangement contains all the sixty hexagonal duads.

The 60 Pascalian lines intersect by 4's in the 45 Pascalian points p, by 3's in 20 points g and in 60 points h, and by 2's in 90 points m, 360 points r, 360 points t, 360 points z, and 9 points w.

The intersections of the Pascalian lines thus are

$45\,p$	counting	as	270
$20\,g$	"	"	60
$60\,h$	"	"	180
$90\,m$	"	"	90
$360\,r$	"	"	360
$360\,t$	"	"	360
$360\,z$	"	"	360
$90\,w$	"	"	90
			$1770 = \frac{1}{2}60\,.\,59,$

and the intersections on each Pascalian line are

$3\,p$	counting	as	9
$1\,g$	"	"	2
$3\,h$	"	"	6
$3\,m$	"	"	3
$12\,r$	"	"	12
$12\,t$	"	"	12
$12\,z$	"	"	12
$3\,w$	"	"	3
			59.

For the ten-partite arrangement, any double triad such as $ABI.DKL$ gives 15 intersections; $10 \times 15 = 150$; and any pair of double triads such as $ABI.DKL$ and $AEH.CKO$ gives 36 intersections; $45 \times 36 = 1620$; and these are

$$10 \times \begin{cases} 6\,g & 60\,g \\ 9\,m & 90\,m \end{cases}$$
$$\text{---}15 \qquad \text{---}\ 150$$

$$45 \times \begin{cases} 6\,p & 270\,p \\ 4\,h & 180\,h \\ 8\,r & 360\,r \\ 8\,t & 360\,t \\ 8\,z & 360\,z \\ 2\,w & 90\,w \end{cases}$$
$$\text{---}36 \qquad \text{---}\ 1620$$
$$1770.$$

For the six-partite arrangement any pentad such as $ABHJN$ gives 45 intersections; $6 \times 45 = 270$; and any two pentads such as $ABHJN$ and $AEGMI$ give 100 intersections; $15 \times 100 = 1500$; and these are

$$6 \times \begin{cases} 30\,h & 180\,h \\ 15\,m & 90\,m \end{cases}$$
$$\text{---}\ 45 \qquad \text{---}\ 270$$

$$15 \times \begin{cases} 4\,g & 60\,g \\ 18\,p & 270\,p \\ 24\,r & 360\,r \\ 24\,t & 360\,t \\ 24\,z & 360\,z \\ 6\,w & 90\,w \end{cases}$$
$$\text{---}\ 100 \qquad \text{---}\ 1500$$
$$1770.$$

I analyse the intersections of a Pascalian line, say AE, by the remaining 59 Pascalian lines as follows:

Observe that AE belongs to the triad AEH, the complementary triad whereof is CKO; it also belongs to the pentad $AEIMG$. We thus obtain, corresponding to AE, the arrangement

$$\begin{array}{cc|ccc} & & H & H & H \\ \hline H & A & B & N & J \\ H & E & F & L & D \\ \hline & & I & M & G \\ & & K & C & O \end{array}$$

viz. *HABNJ*, is the pentad which contains *HA*, the arrangement of the last three letters *B, N, J* thereof being arbitrary; *HEFLD* is the pentad that contains *HE*, but the last three letters are so arranged that the columns *HBF, HNL, HJD* are each of them a triad, *IMG* is then the residue of the pentad *AEIMG*, and *KCO* is the complementary triad to *AEH*, but the arrangement of the letters *IMG*, and of the letters *KCO*, are each of them determinate; viz. these are such that we have *BFICO, NLMKO, JDGCK*, each of them a pentad.

And this being so we derive from the arrangement

$2g$	*AH, EH*;
$3m$	*KC, KO, CO*;
$6h$	*AI, AM, AG*; *EI, EM, EG*;
$12z$	*IB, IF, MN, ML, GJ, GD*; *HB, HF, HN, HL, HJ, HD*;
$9p$	*AB, AN, AJ*; *EF, EL, ED*; *BF, NL, JD*;
$12r$	*CB, CF, CJ, CD*; *OB, OF, ON, OL*; *KN, KL, KJ, KD*;
$12t$	*FL, FD, LD*; *BN, BJ, NJ*; *IC, IO*; *MK, MO*; *GK, GC*;
$3w$	*IM, IG, MG*;
59	

viz. the line *AE* in question meets *AH, EH* each of them in a point g; *KC, KO, CO* each in a point m; and so on. By constructing in the same way an arrangement for each of the lines *AH*, &c., we find the nature of the point of intersection of any two of the lines *AB, AE, AH*, &c.; and we may then present the results in a table (see Plate), which shows at a glance what is the point of intersection (whether a point g, m, h, z, p, r, t, or w) of any two of the Pascalian lines.

I further remark that representing the 45 Pascalian points as follows:

$12.34=a$	$13.24=g$	$14.23=m$	$15.23=s$	$16.23=y$
$12.35=b$	$13.25=h$	$14.25=n$	$15.24=t$	$16.24=z$
$12.36=c$	$13.26=i$	$14.26=o$	$15.26=u$	$16.25=\alpha$
$12.45=d$	$13.45=j$	$14.35=p$	$15.34=v$	$16.34=\beta$
$12.46=e$	$13.46=k$	$14.36=q$	$15.36=w$	$16.35=\gamma$
$12.56=f$	$13.56=l$	$14.56=r$	$15.46=x$	$16.45=\delta$

$23.45=\epsilon$	$25.34=\lambda$	$34.56=\rho$
$23.46=\zeta$	$25.36=\mu$	$35.46=\sigma$
$23.56=\eta$	$25.46=\nu$	$36.45=\tau$
$24.35=\theta$	$26.34=\xi$	
$24.36=\iota$	$26.35=\omega$	
$24.56=\kappa$	$26.45=\pi$	

the sixty hexagons and their Pascalian lines then are

AE	123456	12.45 23.56 34.61	*dηβ*
AH	125634	12.63 25.34 56.41	*cλr*
EH	145236	14.23 45.36 52.61	*mτα*
CK	123654	12.65 23.54 36.41	*fεq*
CO	143256	14.25 43.56 32.61	*nρy*
KO	125436	12.43 25.36 54.61	*aμδ*
AM	126534	12.53 26.34 65.41	*bξr*
AG	125643	12.64 25.43 56.31	*eλl*
AI	124365	12.36 24.65 43.51	*cκv*
EG	132546	13.54 32.46 25.61	*jζa*
DF	126435	12.43 26.35 64.51	*aωx*
FL	124653	12.65 24.53 46.31	*fθk*
DL	134265	13.26 34.65 42.51	*iρt*
BN	132645	13.64 32.45 26.51	*kεu*
BJ	135426	13.42 35.26 54.61	*gωδ*
JN	153246	15.24 53.46 32.61	*tσy*
GK	125463	12.46 25.63 54.31	*eμj*
KM	126354	12.35 26.54 63.41	*bπq*
IO	152436	15.43 52.36 24.61	*vμz*
MO	143526	14.52 43.26 35.61	*nξγ*
EM	145326	14.32 45.26 53.61	*mπγ*
EI	154236	15.23 54.36 42.61	*sτz*
AN	123465	12.46 23.65 34.51	*eηv*
AJ	124356	12.35 24.56 43.61	*bκβ*
AB	126543	12.54 26.43 65.31	*dζl*
DE	154326	15.32 54.26 43.61	*sπβ*
EL	132456	13.45 32.56 24.61	*jηz*
EF	123546	12.54 23.46 35.61	*dζγ*
CD	143265	14.26 43.65 32.51	*oρs*
CF	123564	12.56 23.64 35.41	*fζp*

CG	132564	13.56 32.64 25.41	*lζn*
CI	142365	14.36 42.65 23.51	*qκs*
MN	146235	14.23 46.35 62.51	*mσu*
GJ	135246	13.24 35.46 52.61	*gσα*
BI	136245	13.24 36.45 62.51	*gτu*
DG	134625	13.62 34.25 46.51	*iλx*
LM	135624	13.62 35.24 56.41	*iθr*
FI	124635	12.63 24.35 46.51	*cθx*
BH	136254	13.25 36.54 62.41	*hτo*
FH	125364	12.36 25.64 53.41	*cνp*
FO	125346	12.34 25.46 53.61	*aνγ*
LO	134256	13.25 34.56 42.61	*hρz*
DK	126345	12.34 26.45 63.51	*aπw*
KL	124563	12.56 24.63 45.31	*fιj*
BO	134526	13.52 34.26 45.61	*hξδ*
NO	152346	15.34 52.46 23.61	*vνy*
BC	132654	13.65 32.54 26.41	*lεo*
CJ	142356	14.35 42.56 23.61	*pκy*
JK	124536	12.53 24.36 45.61	*bιδ*
KN	123645	12.64 23.45 36.51	*eεw*
DH	143625	14.62 43.25 36.51	*oλw*
HJ	142536	14.53 42.36 25.61	*pια*
HL	136524	13.52 36.24 65.41	*hιr*
HN	146325	14.32 46.25 63.51	*mνw*
BF	126453	12.46 26.53 64.31	*dωk*
DJ	153426	15.42 53.26 34.61	*tωβ*
LN	132465	13.46 32.65 24.51	*kηt*
GM	135264	13.26 35.64 52.41	*iσn*
IM	142635	14.63 42.35 26.51	*qθu*
GI	136425	13.42 36.25 64.51	*gμx*

Each Pascalian point belongs to four different hexagons; viz. a to the hexagons KD, KO, FD, FO; and so for the other points, thus:

a	$(K, F)(D, O)$	x	$(D, I)(F, G)$
b	$(A, K)(M, J)$	y	$(C, N)(J, O)$
c	$(A, F)(H, I)$	z	$(E, O)(I, L)$
d	$(A, F)(B, E)$	α	$(E, J)(G, H)$
e	$(A, K)(G, N)$	β	$(A, D)(E, J)$
f	$(C, L)(K, F)$	γ	$(E, O)(F, M)$
g	$(B, G)(I, J)$	δ	$(B, K)(J, O)$
h	$(B, L)(H, O)$	ϵ	$(B, K)(C, N)$
i	$(D, M)(G, L)$	ζ	$(C, E)(F, G)$
j	$(E, K)(G, L)$	η	$(A, L)(E, N)$
k	$(B, L)(F, N)$	θ	$(F, M)(I, L)$
l	$(A, C)(B, G)$	ι	$(H, K)(J, L)$
m	$(E, N)(H, M)$	κ	$(A, C)(I, J)$
n	$(C, M)(G, O)$	λ	$(A, D)(G, H)$
o	$(B, D)(C, H)$	μ	$(G, O)(I, K)$
p	$(C, H)(F, J)$	ν	$(F, N)(H, O)$
q	$(C, M)(I, K)$	ξ	$(A, O)(B, M)$
r	$(A, L)(H, M)$	ω	$(B, D)(F, J)$
s	$(C, E)(D, I)$	π	$(D, M)(E, K)$
t	$(J, L)(D, N)$	ρ	$(C, L)(D, O)$
u	$(B, M)(I, N)$	σ	$(G, N)(J, M)$
v	$(A, O)(N, I)$	τ	$(B, E)(H, I)$
w	$(D, N)(H, K)$		

I have constructed on a very large scale a figure of the sixty Pascalian lines, and the forty-five Pascalian points, marking them according to the foregoing notation; but the figure is from its complexity, and the inconvenient way in which the points are either crowded together or fly off to a great distance, almost unintelligible.

	AB	AE	AG	AH	AI	AJ	AM	AN	BC	BF	BH	BI	BJ	BN	BO	CD	CF	CG	CI	CJ	CK	CO	DE	DF	DG	DH	DJ	DK	DL	EF
AB		p	p	h	g	h	p	h	p	p	h	g	h	h	p	r	t	p	z	z	r	t	r	r	r	t	t	m	m	p
AE	p		h	g	h	p	h	p	r	p	z	z	t	t	r	r	r	t	r	r	m	m	p	t	z	z	p	r	t	p
AG	p	h		p	h	g	h	p	p	r	t	z	z	t	r	t	r	p	z	z	t	r	z	r	p	p	z	t	r	t
AH	h	g	p		p	h	p	h	t	z	h	z	w	w	t	r	r	r	r	t	m	m	z	t	p	p	z	r	t	z
AI	g	h	h	p		p	h	p	z	z	z	g	z	z	z	r	l	z	p	p	r	t	t	r	t	r	r	m	m	z
AJ	h	p	g	h	p		p	h	z	t	w	z	h	w	t	t	r	z	p	p	t	r	p	r	z	z	p	t	r	r
AM	p	h	h	p	h	p		g	r	r	t	z	t	z	p	m	m	l	t	r	r	r	t	m	t	r	r	r	r	t
AN	h	p	p	h	p	h	g		l	t	w	z	w	h	z	m	m	r	r	t	r	r	r	m	r	t	t	r	r	r
BC	p	r	p	l	z	z	r	t		h	p	h	g	p	h	p	h	p	h	g	p	h	r	z	t	p	z	l	r	t
BF	p	p	r	z	z	t	r	t	h		g	h	p	p	h	z	h	t	w	z	t	w	t	p	r	z	p	r	l	p
BH	h	z	t	h	z	w	t	w	p	g		p	h	h	p	p	z	r	t	z	r	t	t	z	r	p	z	r	t	z
BI	g	z	z	z	g	z	z	z	h	h	p		p	p	h	t	w	z	h	z	t	w	r	t	r	r	r	m	m	z
BJ	h	t	z	w	z	h	t	w	g	p	h	p		h	p	z	z	z	z	g	z	z	r	p	t	z	p	t	r	r
BN	h	t	t	w	z	w	z	h	p	p	h	p	h		g	r	t	r	t	z	p	z	m	r	m	t	t	r	r	r
BO	p	r	r	t	z	t	p	z	h	h	p	h	p	g		t	w	t	w	z	z	h	m	t	m	r	r	r	r	t
CD	r	r	t	r	r	t	m	m	p	z	p	t	z	r	t		g	h	p	h	h	p	p	g	h	p	h	h	p	z
CF	t	r	r	r	t	r	m	m	h	h	z	w	z	t	w	g		p	h	p	p	h	z	g	z	z	z	z	z	p
CG	p	t	p	r	z	z	t	r	p	t	r	z	z	r	t	h	p		g	h	h	p	z	z	h	t	w	w	t	p
CI	z	t	z	r	p	p	t	r	h	w	t	h	z	t	w	p	h	g		p	p	h	p	z	z	r	t	t	r	z
CJ	z	r	z	t	p	p	r	t	g	z	z	z	g	z	z	h	p	h	p		h	p	t	z	w	z	h	w	t	r
CK	r	m	t	m	r	t	r	r	p	t	r	t	z	p	z	h	p	h	p	h		g	t	z	w	t	w	h	z	r
CO	t	m	r	m	t	r	r	r	h	w	t	w	z	z	h	p	h	p	h	p	g		r	z	t	r	t	z	p	t
DE	r	p	z	z	t	p	t	r	r	t	t	r	r	m	m	p	z	z	p	t	t	r		h	g	h	p	p	h	h
DF	r	t	r	t	r	r	m	m	z	p	z	t	p	r	t	g	g	z	z	z	z	z	h		p	h	p	p	h	h
DG	r	z	p	p	t	z	t	r	t	r	r	r	t	m	m	h	z	h	z	w	w	t	g	p		p	h	h	p	z
DH	t	z	p	p	r	z	r	t	p	z	p	r	z	t	r	p	z	t	r	z	t	r	h	h	p		g	p	h	w
DJ	t	p	z	z	r	p	r	t	z	p	z	r	p	t	r	h	z	w	t	h	w	t	p	h	h	g		h	p	t
DK	m	r	t	r	m	t	r	r	t	r	r	m	t	r	r	h	z	w	t	w	h	z	p	p	h	p	h		g	t
DL	m	t	r	t	m	r	r	r	r	t	t	m	r	r	r	p	z	t	r	t	z	p	h	h	p	h	p	g		w
EF	p	p	t	z	z	r	t	r	t	p	z	z	r	r	t	z	p	p	z	r	r	t	h	h	z	w	t	t	w	
EC	t	h	h	z	w	z	w	t	r	r	r	t	r	m	m	z	p	p	z	t	t	r	g	z	g	z	z	z	z	p
EN	z	g	z	g	z	z	z	z	r	z	p	p	t	t	r	t	t	r	r	r	m	m	h	w	z	h	z	t	w	h
EI	z	h	w	z	h	t	w	t	t	z	p	p	r	r	t	p	z	z	p	r	r	t	p	z	z	t	r	r	t	g
EL	r	p	t	z	t	r	z	p	m	t	t	r	m	r	r	t	t	r	r	m	r	r	h	w	z	w	t	z	h	h
EM	t	h	w	z	w	t	h	z	m	r	r	t	m	r	r	r	r	t	t	m	r	r	p	t	z	t	r	p	z	p
FI	z	z	t	p	p	r	t	r	w	h	z	h	t	t	w	z	h	z	h	t	t	w	z	p	p	t	r	r	t	g
FH	z	z	r	p	p	t	r	t	z	g	g	z	z	z	z	z	p	r	t	p	r	t	w	h	t	h	z	t	w	h
FL	r	t	m	t	r	m	r	r	t	p	z	t	r	p	z	z	p	r	t	r	p	z	w	h	t	w	t	z	h	h
FO	t	r	m	r	t	m	r	r	w	h	z	w	t	z	h	z	h	t	w	t	z	h	t	p	r	t	r	p	z	p
GI	z	w	h	t	h	z	w	t	z	t	r	p	p	r	t	z	z	g	g	z	z	z	z	p	p	r	t	t	r	z
GJ	z	z	g	z	z	g	z	z	z	r	t	p	p	t	r	w	t	h	z	h	w	t	z	t	h	z	h	w	t	r
GK	r	t	p	r	t	z	z	p	t	m	m	r	t	r	r	w	t	h	z	w	h	z	z	h	t	w	h	z	r	
GM	t	w	h	t	w	z	h	z	r	m	m	t	r	r	r	t	r	p	z	t	z	p	z	r	p	r	t	z	p	t
HJ	w	z	z	h	t	h	t	w	z	z	h	t	h	w	t	z	p	t	r	p	t	r	z	z	z	g	g	z	z	t
HL	t	z	r	p	r	t	p	z	r	z	p	r	t	z	p	t	t	m	m	r	r	r	w	t	h	z	z	h	w	
HN	w	z	t	h	t	w	z	h	t	z	h	t	w	h	z	r	r	m	m	t	r	r	t	r	p	z	p	z	t	
IM	z	w	w	t	h	t	h	z	t	t	r	p	r	p	z	r	t	z	p	r	p	z	t	r	t	m	t	r	r	z
IO	z	t	t	r	p	r	z	p	w	w	t	h	t	z	h	t	w	z	h	t	z	h	r	t	r	m	m	r	r	z
JH	t	r	z	t	r	p	p	z	z	r	t	r	p	z	p	w	t	w	t	h	h	z	t	t	w	z	h	h	z	m
JN	w	t	z	w	t	h	z	h	z	t	w	t	h	h	z	t	r	t	r	p	z	p	r	r	t	z	p	z	p	m
KL	m	r	r	r	m	r	t	t	r	r	r	m	r	t	t	z	p	t	r	t	p	z	z	z	z	z	z	g	g	t
KM	r	t	z	r	t	p	p	z	r	m	m	r	r	t	t	t	r	z	p	t	p	z	p	r	z	r	t	p	z	r
KN	t	r	p	t	r	z	z	p	p	r	t	r	z	p	z	t	r	t	r	z	p	z	r	r	t	p	z	p	z	m
KO	r	m	r	m	r	r	t	t	z	t	r	t	p	z	p	z	z	z	z	z	g	g	r	p	t	r	t	p	z	r
LM	r	z	t	p	t	r	p	z	m	r	r	r	m	t	t	r	r	r	r	m	t	t	z	t	p	t	r	z	p	z
LN	t	p	r	z	r	t	z	p	r	p	z	r	t	p	z	r	r	m	m	r	t	t	t	t	r	z	p	z	p	t
LO	r	r	m	r	r	m	t	t	t	z	p	t	r	z	p	p	z	r	t	r	z	p	t	z	r	t	r	z	p	z
MN	z	z	z	z	z	z	g	g	r	r	t	p	t	p	z	m	m	r	r	r	t	t	r	m	r	r	r	t	t	r
MO	p	t	t	r	z	r	p	z	t	t	r	z	r	z	p	r	t	p	z	r	z	p	r	r	r	m	m	t	t	p
NO	z	r	r	t	p	t	z	p	z	z	z	z	z	g	g	r	t	r	t	p	z	p	m	r	m	r	r	t	t	r

EG	EH	EI	EL	EM	FI	FH	FL	FO	GI	GJ	GK	GM	HJ	HL	HN	IM	IO	JK	JN	KL	KM	KN	KO	LM	LN	LO	MN	MO	NO	
t	z	z	r	t	z	z	r	t	z	z	r	t	w	t	w	z	z	t	w	m	r	t	r	r	t	r	z	p	z	AB
h	g	h	p	h	z	z	t	r	w	z	t	w	z	z	z	w	r	t	t	r	t	r	m	z	p	r	z	t	r	AE
h	z	w	t	w	t	r	m	m	h	g	p	h	z	r	t	w	t	z	z	r	z	p	r	t	r	m	z	t	r	AG
z	g	z	z	z	p	p	t	r	t	z	r	t	h	p	h	t	r	t	w	r	r	t	m	p	z	r	z	r	t	AH
w	z	h	t	w	p	p	r	t	h	z	t	w	t	r	t	h	p	r	t	m	t	r	r	t	r	r	z	z	p	AI
z	z	t	r	t	r	t	m	m	z	g	z	z	h	t	w	t	r	p	h	r	p	z	r	r	t	m	z	r	t	AJ
w	z	w	z	h	t	r	r	r	w	z	z	h	t	p	z	h	z	p	z	t	p	z	t	p	z	t	g	p	z	AM
t	z	t	p	z	r	t	r	r	t	z	p	z	w	z	h	z	p	z	h	t	z	p	t	z	p	t	g	z	p	AN
r	r	t	m	m	w	z	t	w	z	z	t	r	z	r	t	t	w	z	z	r	r	p	z	m	r	t	r	t	z	BC
r	z	z	t	r	h	g	p	h	t	r	m	m	z	z	z	t	w	r	t	r	m	r	t	r	p	z	r	t	z	BF
r	p	p	t	r	z	g	z	z	r	t	m	m	h	p	h	r	t	t	w	r	m	t	r	r	z	p	t	r	z	BH
t	p	p	r	t	h	z	t	w	p	p	r	t	t	r	t	p	h	r	t	m	r	r	t	r	r	t	p	z	z	BI
r	t	r	m	m	t	z	r	t	p	p	t	r	h	t	w	r	t	p	h	r	r	z	p	m	t	r	t	r	z	BJ
m	t	r	r	r	t	z	p	z	r	t	r	r	w	z	h	p	z	z	h	t	t	p	z	t	p	z	p	z	g	BN
m	r	t	r	r	w	z	z	h	t	r	r	r	t	p	z	z	h	p	z	t	t	z	p	t	z	p	z	p	g	BO
z	t	p	t	r	z	z	z	z	z	w	w	t	z	t	r	r	t	w	t	z	t	z	r	r	p	m	r	r		CD
p	t	z	t	r	h	p	p	h	z	t	t	r	p	t	r	t	w	t	r	p	r	r	z	r	r	z	m	t	t	CF
p	r	z	r	t	z	r	r	t	g	h	h	p	t	m	m	z	z	w	t	t	z	t	z	r	m	r	r	p	r	CG
z	r	p	r	t	h	t	t	w	g	z	z	z	r	m	m	p	h	t	r	r	p	r	z	r	m	t	r	z	t	CI
t	r	r	m	m	t	p	r	t	z	h	w	t	p	r	t	r	t	h	p	t	t	z	z	m	r	r	r	r	p	CJ
t	m	r	r	r	t	r	p	z	z	w	h	z	t	r	r	p	z	h	z	p	p	p	g	t	t	z	t	z	z	CK
r	m	t	r	r	w	t	z	h	z	t	z	p	r	r	r	z	h	z	p	z	z	z	g	t	t	p	t	p	p	CO
g	h	p	h	p	z	w	w	t	z	z	z	z	w	t	t	r	t	r	z	z	p	r	r	z	t	t	r	r	m	DE
z	w	z	w	t	p	h	h	p	p	t	t	r	z	w	t	r	t	t	r	z	r	r	p	t	t	z	m	r	r	DF
g	z	z	z	z	p	t	t	r	p	h	h	p	z	t	r	t	r	w	t	z	z	t	t	p	r	r	r	r	m	DG
z	h	t	w	t	t	h	w	t	r	z	t	r	g	h	p	m	m	z	z	z	r	p	r	t	z	t	r	m	r	DH
z	z	r	t	r	r	z	t	r	t	h	w	t	g	z	z	t	m	h	p	z	t	z	t	r	p	r	r	m	r	DJ
z	t	r	z	p	r	t	z	p	t	w	h	z	z	z	p	r	r	h	z	g	p	p	p	z	z	z	t	t	t	DK
z	w	t	h	z	t	w	h	z	r	t	z	p	z	h	z	r	r	z	p	g	z	z	z	p	p	p	t	t	t	DL
p	h	g	h	p	g	h	h	p	z	r	r	t	t	w	t	z	z	m	m	t	r	m	r	z	t	z	r	p	r	EF
	p	h	p	h	z	t	t	r	h	p	p	h	p	t	r	w	t	t	r	p	z	r	r	z	r	r	t	z	m	EG
p		p	h	p	z	h	w	t	t	p	r	t	p	h	p	t	r	r	t	t	r	r	m	z	z	t	p	r	r	EH
h	p		p	h	g	z	z	z	h	t	t	w	r	t	r	h	p	m	m	r	t	m	r	z	r	p	t	z	r	EI
p	h	p		g	z	w	h	z	t	r	p	z	t	h	z	z	p	r	r	p	z	t	t	g	p	p	z	z	t	EL
h	p	h	g		z	t	z	p	w	t	z	h	r	z	p	h	z	r	r	z	p	t	t	g	z	z	p	p	t	EM
z	z	g	z	z		p	p	h	p	r	r	t	r	t	r	p	h	m	m	r	r	m	t	p	r	z	r	z	t	FI
t	h	z	w	t	p		h	p	r	r	m	m	p	h	p	r	t	r	t	t	m	r	r	t	z	z	r	r	p	FH
t	w	z	h	z	p	h		g	r	m	r	r	t	h	z	p	z	r	r	p	t	t	z	p	p	g	t	z	z	FL
r	t	z	z	p	h	p	g		t	m	r	r	r	z	p	z	h	r	r	z	t	t	p	z	z	g	t	p	p	FO
h	t	h	t	w	p	r	r	t		p	p	h	r	m	m	h	p	t	r	r	z	r	p	t	m	r	t	z	r	GI
p	p	t	r	t	r	r	m	m	p		h	p	p	r	t	m	r	h	p	t	z	z	t	r	r	m	p	r	r	GJ
p	r	t	p	z	r	m	r	r	p	h		g	t	r	r	z	p	h	z	p	g	p	p	z	t	t	z	z	t	GK
h	t	w	z	h	l	m	r	r	h	p	g		r	r	r	h	z	z	p	z	g	z	z	p	t	t	p	p	t	GM
p	p	r	t	r	r	p	t	r	r	p	t	r		p	h	m	m	p	h	p	r	z	r	r	z	r	t	m	t	HJ
t	h	t	h	z	t	h	h	z	m	r	r	r	p		g	r	r	p	z	p	t	z	t	p	g	p	z	t	z	HL
r	p	r	z	p	r	p	z	p	m	t	r	r	h	g		r	r	z	h	z	t	p	t	z	g	z	p	t	p	HN
w	t	h	z	h	p	r	p	z	h	m	z	h	m	r	r		g	r	r	t	p	t	z	p	t	z	p	g	z	IM
t	r	p	p	z	h	t	z	h	p	r	p	z	m	r	r	g		r	r	t	z	t	p	z	t	p	z	g	p	IO
t	r	m	r	r	m	r	r	r	t	h	h	z	p	p	z	r	r		g	p	p	g	p	l	z	t	z	t	z	JK
r	t	m	r	r	m	t	r	r	r	p	z	p	h	z	h	r	r	g		z	z	g	z	t	p	t	p	t	p	JN
p	t	r	p	z	r	t	p	z	r	t	p	z	p	p	z	t	t	p	z		h	h	h	h	h	h	w	w	w	KL
z	r	t	z	p	r	m	t	t	z	z	g	g	r	t	t	p	z	p	z	h		h	h	h	w	w	h	h	w	KM
r	r	m	t	t	m	r	t	t	r	z	p	z	z	z	p	t	t	g	g	h	h		h	w	h	w	h	w	h	KN
r	m	r	t	t	t	r	z	p	p	t	p	z	r	t	t	z	p	p	z	h	h	h		w	w	h	w	h	h	KO
z	z	z	g	g	t	p	z	t	r	z	p	r	p	z	p	z	t	t	h	h	h	w	w		h	h	h	h	w	LM
r	z	r	p	z	r	z	p	z	m	r	t	t	z	g	g	t	t	z	p	h	w	h	w	h		h	h	w	h	LN
r	t	p	p	z	z	z	g	g	r	m	t	t	r	p	z	z	p	t	t	h	w	w	h	h	h		w	h	h	LO
t	p	t	z	p	r	r	t	t	t	p	z	p	t	z	p	p	z	z	p	w	h	h	w	h	h	w		h	h	MN
t	r	z	z	p	z	r	z	p	z	r	z	p	m	t	t	g	g	t	t	w	h	w	h	h	w	h	h		h	MO
m	r	r	t	t	t	p	z	p	r	r	t	t	t	z	p	z	p	z	p	w	w	h	h	w	h	h	h	h		NO

EG EH EI EL EM FI FH FL FO GI GJ GK GM HJ HL HN IM IO JK JN KL KM KN KO LM LN LO MN MO NO

between pp. 122-123.

402.

ON A SINGULARITY OF SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 332—338.]

A SURFACE having a nodal line has in general on this nodal line points where the two tangent planes coincide, or as I propose to term them "pinch-points." Thus, if the nodal line be the curve of complete intersection of any two surfaces $P=0$, $Q=0$, then the equation of the general surface having this curve for a nodal line is $(a, b, c \between P, Q)^2=0$ (where a, b, c are any functions of the coordinates), and the pinch-points are given as the intersections of the nodal line $P=0$, $Q=0$ with the surface $ac-b^2=0$. Consider the case where the nodal curve is a curve of partial intersection represented by the equations $\left\| \begin{matrix} P, & Q, & R \\ P', & Q', & R' \end{matrix} \right\| = 0$, or say by the equations $p=0$, $q=0$, $r=0$ (viz. p, q, r denote the functions $QR'-Q'R$, $RP'-R'P$, $PQ'-P'Q$ respectively), and consequently we have identically

$$(P, Q, R \between p, q, r)=0,$$
$$(P', Q', R' \between p, q, r)=0,$$

or what is the same thing, (λ, μ) being arbitrary,

$$(\lambda P+\mu P', \lambda Q+\mu Q', \lambda R+\mu R' \between p, q, r)=0.$$

The general surface having the curve in question for its nodal line is represented by the equation

$$(a, b, c, f, g, h \between p, q, r)^2=0,$$

(where (a, b, c, f, g, h) are any functions of the coordinates), and it is easy to see that the condition for a pinch-point is the same as that which (considering p, q, r as coordinates and all the other quantities as constants), expresses that the line

$$(\lambda P+\mu P', \lambda Q+\mu Q', \lambda R+\mu R' \between p, q, r)=0,$$

touches the conic

$$(a,\ b,\ c,\ f,\ g,\ h)(p,\ q,\ r)^2 = 0,$$

viz. A, B, C, F, G, H being the inverse coefficients, $A = bc - f^2$, &c., this condition is

$$(A,\ B,\ C,\ F,\ G,\ H \between \lambda P + \mu P',\ \lambda Q + \mu Q',\ \lambda R + \mu R')^2 = 0,$$

or what is the same thing, the pinch-points are given as the common intersections of the nodal line $p = 0$, $q = 0$, $r = 0$ with each of the three surfaces

$$\begin{aligned} &(A,\ B,\ C,\ F,\ G,\ H)(P,\ Q,\ R)^2 && = 0, \\ &(A,\ B,\ C,\ F,\ G,\ H)(P,\ Q,\ R)\ (P',\ Q',\ R') && = 0, \\ &(A,\ B,\ C,\ F,\ G,\ H) \qquad\qquad (P',\ Q',\ R')^2 && = 0, \end{aligned}$$

these last three equations in fact, adding only a single relation to the relations expressed by the equations

$$p = 0,\ q = 0,\ r = 0.$$

If the functions P, Q, R, P', Q', R' are linear functions of the coordinates, then the curve $(p = 0,\ q = 0,\ r = 0)$ is a cubic curve in space, or skew cubic; and if moreover $(a,\ b,\ c,\ f,\ g,\ h)$ are constants, then the equation

$$(a,\ b,\ c,\ f,\ g,\ h \between p,\ q,\ r)^2 = 0,$$

belongs to a quartic surface having the skew cubic for a nodal line: this surface is (it may be observed) a ruled surface, or scroll. With a view to ulterior investigations, I propose to study the theory of the pinch-points in regard to this particular surface; and to simplify as much as possible, I fix the coordinates as follows:

Considering the skew cubic as given, let any point O on the cubic be taken for the origin; let $x = 0$ be the equation of the osculating plane at O; $y = 0$ that of any other plane through the tangent line at O; $z = 0$, that of any other plane through O, not passing through the tangent line; and $w = 0$ that of a fourth plane; then the equation of the cubic will be

$$\left\| \begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix} \right\| = 0,$$

or what is the same thing, the values of p, q, r are $yw - z^2$, $zy - xw$, and $xz - y^2$ respectively. And conversely, the cubic being thus represented, the point $(x = 0,\ y = 0,\ z = 0)$ may be considered as standing for any point whatever on the skew cubic; the osculating plane at this point being $x = 0$, and the tangent line being $x = 0$, $y = 0$. For the purpose of the present investigations, we may without loss of generality write $w = 1$; and for convenience I shall do this; the values of p, q, r thus become $y - z^2$, $yz - x$, $xz - y^2$, and the equation of the surface is

$$(a,\ b,\ c,\ f,\ g,\ h \between y - z^2,\ yz - x,\ xz - y^2)^2 = 0.$$

At a pinch-point, we have

$$\begin{aligned}
&(A, B, C, F, G, H)(x, y, z)^2 &&= 0,\\
&(A, B, C, F, G, H)(x, y, z)\,(y, z, 1) &&= 0,\\
&(A, B, C, F, G, H) \qquad (y, z, 1)^2 &&= 0,
\end{aligned}$$

and hence the origin will be a pinch-point if $C=0$, that is, if $ab-h^2=0$. This however appears more readily by remarking, that the equation of the pair of tangent planes at the origin is

$$(a, b, c, f, g, h\between y, -x, 0)^2=0,$$

or what is the same thing,

$$(a, h, b\between y, -x)^2=0;$$

the two tangent planes therefore coincide, or there is a pinch-point, if only $ab-h^2=0$.

By what precedes, it appears that if we wish to study the form of the quartic surface, 1°, in the neighbourhood of an arbitrary point on the nodal line; 2°, in the neighbourhood of a pinch-point; it is sufficient in the first case to consider the general surface

$$(a, b, c, f, g, h\between y-z^2, yz-x, xz-y^2)^2=0,$$

in the neighbourhood of the origin; and in the second case, to study the special surface for which $ab-h^2=0$, or writing for convenience $a=1$, and therefore $b=h^2$, the surface

$$(1, h^2, c, f, g, h\between y-z^2, yz-x, xz-y^2)^2=0,$$

in the neighbourhood of the origin.

Consider first the surface

$$(a, b, c, f, g, h)(y-z^2, yz-x, xz-y^2)^2=0.$$

A plane through the origin is either a plane not passing through the tangent line ($x=0$, $y=0$), and the equation $z=0$ will serve to represent any such plane; or if it pass through the tangent line, then it is either a non-special plane, which may be represented by the equation $y=0$; or it is a special plane: viz. either the osculating plane $x=0$ of the nodal line, or else one or the other of the two tangent planes $(a, h, b\between y, -x)^2=0$ of the surface. I consider therefore the sections of the surface by these planes $z=0$, $y=0$, $x=0$, $(a, h, b\between y, -x)^2=0$ respectively.

Section by the non-special plane $z=0$.

The equation is

$$(a, b, c, f, g, h\between y, -x, -y^2)^2=0,$$

which represents a curve having at the origin an ordinary node, the equations of the two tangents being $(a, h, b\between y, -x)^2=0$, viz. these are the intersections of the two tangent planes by the plane $z=0$.

Section by the non-special plane through the tangent line, viz. the plane $y=0$.

The equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between -z^2,\ -x,\ xz)^2=0,$$

or what is the same thing,

$$bx^2-2fx^2z+2hxz^2+cx^2z^2-2gxz^3+az^4=0.$$

Writing as usual $x=Az^\mu+\&c.$ we have

$$\mu=2,\quad bA^2+2hA+a=0,$$

and since $ab-h^2$ is by hypothesis not $=0$, A has two unequal values; we have at the origin two branches $x=A_1z^2+B_1z^3+\&c.$, $x=A_2z^2+B_2z^3+\&c.$, having the common tangent $x=0$ (viz. this is the tangent $x=0$, $y=0$ of the nodal curve), and with a two-pointic intersection of the two branches, that is, the point at the origin is an ordinary tacnode.

Section by the osculating plane $x=0$.

The equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between y-z^2,\ yz,\ -y^2)^2=0.$$

We may write $y=z^2+Az^\mu+\&c.$, we at once find $\mu=3$, and then

$$(a,\ b,\ c,\ f,\ g,\ h \between Az^3+\&c.,\ z^3+\&c.,\ -z^4+\&c.)^2=0,$$

that is

$$(a,\ h,\ b \between A,\ 1)^2=0.$$

A has two unequal values, and the branches through the origin are

$$y=z^2+A_1z^3+B_1z^4+\&c.,\quad y=z^2+A_2z^3+B_2z^4+\&c.\ \ldots,$$

viz. the branches have the common tangent line $y=0$ (the tangent $x=0$, $y=0$ of the nodal curve), but in the present case a three-pointic intersection.

Section by one of the tangent planes $(a,\ h,\ b \between y,\ -x)^2=0$.

Writing $y=-mx$, and therefore $(a,\ h,\ b \between m,\ -1)^2=0$, the equation is

$$(a,\ b,\ c,\ f,\ g,\ h \between -mx-z^2,\ -x-mzx,\ xz-m^2x^2)^2=0,$$

which represents of course the projection of the section on the plane $z=0$, $x=0$, but which (since there is no alteration in the singularities) may be considered as representing the section itself. Developing, the coefficient of x^2 is $am^2+2hm+b$, which is $=0$, and the equation becomes

$$\begin{array}{rrrl}
 & 2m^2(gm+f) & x^3 + & cm^4 \qquad x^4 \\
+ & 2\{hm^2+(b-g)m-f\} & x^2z + & 2m^2(fm-c) \qquad x^3z \\
+ & 2(am+h) & xz^2 + & (b+2g)m^2-2fm+c \quad x^2z^2 \\
 & & + & 2(hm-g) \qquad xz^3 \\
 & & + & a \qquad z^4=0,
\end{array}$$

so that the curve has at the origin a triple point, the tangent to one branch being the line $x=0$ (the tangent $x=0$, $y=0$ of the nodal curve).

Consider next the surface

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y-z^2,\ yz-x,\ xz-y^2)^2=0,$$

being as already remarked, the general surface referred to a pinch-point as origin.

Section by the non-special plane $z=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y,\ -x,\ -y^2)^2=0,$$

where, attending only to the terms of the lowest order, we find $(1,\ h,\ h^2\between y,\ -x)^2=0$, that is $(y-hx)^2=0$, showing that the origin is a cusp.

Section by the non-special plane through the tangent line, viz. the plane $y=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between -z^2,\ -x,\ xz)^2=0,$$

or what is the same thing,

$$h^2x^2+2hxz^2-2fx^2z+cx^2z^2-2gxz^3+z^4=0,$$

that is

$$(hx+z^2)^2-2fx^2z+cx^2z^2-2gxz^3=0,$$

writing $hx=-z^2+Ax^\mu$, we find at once $\mu=\frac{5}{2}$, and then $A^2=\dfrac{2f}{h^2}-\dfrac{2g}{h}$, so that the branches are $hx=-z^2\pm Ax^{\frac{5}{2}}$; whence we have at the origin a cusp of the second order or node cusp.

Section by the osculating plane $x=0$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between y-z^3,\ yz,\ -y^2)^2=0;$$

writing $y=z^2-hz^3+Az_\mu$, we easily find $\mu=\frac{7}{2}$, and then

$$(1,\ h^2,\ c,\ f,\ g,\ h\between -hz^3+Az^{\frac{7}{2}},\ z^3-hz^4,\ -z^4)^2=0,$$

where the terms in z^6, and $z^{6+\frac{1}{2}}$ disappear of themselves, the terms in z^7 give $A^2+2gh=0$, and the branches are

$$y=z^2-hz^3\pm Az^{\frac{7}{2}}\ \&c.,$$

viz. there is a cusp of a superior order.

Section by the tangent plane $y=hx$.

The equation is

$$(1,\ h^2,\ c,\ f,\ g,\ h\between hx-z^2,\ -x+hzx,\ xz-h^2x^2)^2=0,$$

representing the projection on the plane of zx. Developing, the equation is

$$\begin{aligned} 2(f-gh)x^2(h^2x-z) \quad & +x^4ch^4 \\ & +x^3z \quad -2h^2(hf+c) \\ & +x^2z^2 \quad (h^4+2g^2h^2+2fh+c) \\ & +xz^3 \quad -2(h^2+g) \\ & +z^4 \quad 1 \qquad\qquad =0, \end{aligned}$$

and there is at the origin a triple point (= cusp + 2 nodes) arising from the passage of an ordinary branch through a cusp; the tangent at the cusp being it will be noticed the line $x=0$, that is the tangent $x=0$, $y=0$ to the nodal curve at the pinch-point.

The results of the investigation may be presented in a tabular form as follows:

	Nature of Section.	
Plane of Section.	Origin, an ordinary point.	Origin, a Pinch-point.
Non-special.	Node.	Cusp.
Ditto, through tangent line of nodal curve.	Tacnode = 2 nodes.	Node-cusp, = node + cusp.
Osculating plane of nodal curve.	$y = z^3 + \left.\begin{matrix} A_1 \\ A_2 \end{matrix}\right\} z^3 + \&c.$	$y = z^3 + hz^3 \pm Az^{\frac{7}{2}}$ &c.
Either of the two tangent planes.	Triple point, one branch touching the tangent of nodal line.	
The single tangent plane.		Triple point, = cusp + 2 nodes; the cuspidal branch touching the tangent of the nodal line.

I have not considered the special cases where one of the two tangent planes, or (as the case may be) the single tangent plane of the surface coincides with the osculating plane of the nodal curve.

403.

ON PASCAL'S THEOREM.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 348—353.]

I CONSIDER the following question: to find a point such that its polar plane in regard to a given system of three planes is the same as its polar plane in regard to another given system of three planes.

The equations of any six planes whatever may be taken to be $X=0$, $Y=0$, $Z=0$, $U=0$, $V=0$, $W=0$, where

$$X+Y+Z+U+V+W=0,$$
$$aX+bY+cZ+fU+gV+hW=0,$$

and so also any quantities X, Y, Z, U, V, W satisfying these relations may be regarded as the coordinates of a point in space; we pass to the ordinary system of quadriplanar coordinates by merely substituting for V, W their values as linear functions of X, Y, Z, U.

This being so, the equations of the given systems of three planes may be taken to be

$$XYZ=0, \qquad UVW=0,$$

and if we take for the coordinates of the required point (x, y, z, u, v, w), where

$$x+y+z+u+v+w=0,$$
$$ax+by+cz+fu+gv+hw=0,$$

then the equations of the two polar planes are

$$\frac{X}{x}+\frac{Y}{y}+\frac{Z}{z}=0, \qquad \frac{U}{u}+\frac{V}{v}+\frac{W}{w}=0,$$

respectively, and we have to find (x, y, z, u, v, w), such that these two equations may represent the same plane, or that the two equations may in virtue of the linear relations between (X, Y, Z, U, V, W) be the same equation.

The ordinary process by indeterminate multipliers gives

$$\frac{1}{x}+\lambda+\mu a=0,$$

$$\frac{1}{y}+\lambda+\mu b=0,$$

$$\frac{1}{z}+\lambda+\mu c=0,$$

$$\frac{k}{u}+\lambda+\mu f=0,$$

$$\frac{k}{v}+\lambda+\mu g=0,$$

$$\frac{k}{w}+\lambda+\mu h=0,$$

and we have the before-mentioned linear relations between (x, y, z, u, v, w); these last are satisfied by the values

$$(x, y, z, u, v, w)=\left(\frac{1}{a-\theta}, \frac{1}{b-\theta}, \frac{1}{c-\theta}, -\frac{1}{f-\theta}, -\frac{1}{g-\theta}, -\frac{1}{h-\theta}\right),$$

if only

$$\frac{1}{a-\theta}+\frac{1}{b-\theta}+\frac{1}{c-\theta}-\frac{1}{f-\theta}-\frac{1}{g-\theta}-\frac{1}{h-\theta}=0;$$

in fact, θ satisfying this equation, the relation

$$x+y+z+u+v+w=0$$

is obviously satisfied; and observing that we have

$$ax=\frac{a}{a-\theta}=1+\frac{\theta}{a-\theta}, \ldots, fu=\frac{-f}{f-\theta}=-1-\frac{\theta}{f-\theta}, \ldots,$$

we have

$$\begin{aligned} & ax+by+cz+fu+gv+hw \\ &=\Sigma\left(1+\frac{\theta}{a-\theta}\right)-\Sigma\left(1+\frac{\theta}{f-\theta}\right), \\ &=\theta\left(\Sigma\frac{1}{a-\theta}-\Sigma\frac{1}{f-\theta}\right), =0, \end{aligned}$$

so that the relation $ax+by+cz+fu+gv+hw=0$ is also satisfied. Substituting the foregoing values of (x, y, z, u, v, w) the six equations containing k, λ, μ, will be all of them satisfied if only

$$\mu=-1, \lambda=\theta, k=-1.$$

The coordinates of the required point thus are

$$\left(\frac{1}{a-\theta}, \frac{1}{b-\theta}, \frac{1}{c-\theta}, -\frac{1}{f-\theta}, -\frac{1}{g-\theta}, -\frac{1}{h-\theta}\right),$$

where

$$\frac{1}{a-\theta}+\frac{1}{b-\theta}+\frac{1}{c-\theta}-\frac{1}{f-\theta}-\frac{1}{g-\theta}-\frac{1}{h-\theta}=0;$$

and, the equation in θ being of the fourth order, there are thus four points, say the points O_1, O_2, O_3, O_4, which have each of them the property in question.

It will be convenient to designate the planes $X=0$, $Y=0$, $Z=0$, $U=0$, $V=0$, $W=0$ as the planes a, b, c, f, g, h respectively; the line of intersection of the planes $X=0$, $Y=0$ will then be the line ab, and the point of intersection of the planes $X=0$, $Y=0$, $Z=0$ the point abc; and so in other cases.

I say that from any one of the points O it is possible to draw

a line meeting the lines	$af\,.\,bg\,.\,ch$	(1),
" "	$ag\,.\,bh\,.\,cf$	(2),
" "	$ah\,.\,bf\,.\,cg$	(3),
" "	$af\,.\,bh\,.\,cg$	(4),
" "	$ag\,.\,bf\,.\,ch$	(5),
" "	$ah\,.\,bg\,.\,cf$	(6),

and consequently, that the four points O are the four common points of the six hyperboloids passing through these triads of lines respectively.

In fact, considering θ as determined by the foregoing quartic equation, and writing for shortness

$$(a-\theta)X=A, \qquad (f-\theta)U=F,$$
$$(b-\theta)Y=B, \qquad (g-\theta)V=G,$$
$$(c-\theta)Z=C, \qquad (h-\theta)W=H,$$

so that

$$A+B+C+F+G+H=0,$$

the equations $A+F=0$, $B+G=0$, $C+H=0$, are equivalent to two equations only, and it is at once seen, that these are in fact the equations of a line through the point O meeting the three lines af, bg, ch respectively.

The equation $A+F=0$, is in fact satisfied by the values $X : U=\frac{1}{a-\theta} : -\frac{1}{f-\theta}$, and by $X=0$, $U=0$; it is consequently the equation of the plane through O and the line af; similarly, $B+G=0$ is the equation of the plane through O and the line

bg; and $C+H=0$ is the equation of the plane through O and the line ch; and the three equations being equivalent to two equations only, the planes have a common line which is the line in question.

The equations of the six lines thus are:

$$\begin{array}{lll}
(1) & A+F=0, & B+G=0, \quad C+H=0, \\
(2) & A+G=0, & B+H=0, \quad C+F=0, \\
(3) & A+H=0, & B+F=0, \quad C+G=0, \\
(4) & A+F=0, & B+H=0, \quad C+G=0, \\
(5) & A+G=0, & B+F=0, \quad C+H=0, \\
(6) & A+H=0, & B+G=0, \quad C+F=0.
\end{array}$$

It is further to be noticed, that if in any one of these systems, for instance in the system $A+F=0$, $B+G=0$, $C+H=0$, we consider θ as an arbitrary quantity, then the equations are those of any line whatever cutting the lines af, bg, ch; and hence eliminating θ, we have the equation of the hyperboloid through the three lines af, bg, ch; the equations of the six hyperboloids are thus found to be

$$\begin{array}{lll}
(1) & \dfrac{ax+fu}{x+u} = \dfrac{by+gv}{y+v} = \dfrac{cz+hw}{z+w}, \\[2ex]
(2) & \dfrac{ax+gv}{x+v} = \dfrac{by+hw}{y+w} = \dfrac{cz+fu}{z+u}, \\[2ex]
(3) & \dfrac{ax+hw}{x+w} = \dfrac{by+fu}{y+u} = \dfrac{cz+gv}{z+v}, \\[2ex]
(4) & \dfrac{ax+fu}{x+u} = \dfrac{by+hw}{y+w} = \dfrac{cz+gv}{z+v}, \\[2ex]
(5) & \dfrac{ax+gv}{x+v} = \dfrac{by+fu}{y+u} = \dfrac{cz+hw}{z+w}, \\[2ex]
(6) & \dfrac{ax+hw}{x+w} = \dfrac{by+gv}{y+v} = \dfrac{cz+fu}{z+u},
\end{array}$$

respectively; the equations in the same line being of course equivalent to a single equation.

For each one of the six lines we have

$$(A,\ B,\ C)=(-F,\ -G,\ -H)$$

in some order or other, and it is thus seen that the six lines lie on a cone of the second order, the equation whereof is

$$A^2+B^2+C^2-F^2-G^2-H^2=0.$$

Consider now the six planes a, b, c, f, g, h, and taking in the first instance an arbitrary point of projection, and a plane of projection which is also arbitrary—the line of intersection ab of the planes a and b will be projected into a line ab, and the point of intersection of the planes a, b, c into a point abc; and so in other cases. We have thus a plane figure, consisting of the fifteen lines $ab, ac, \ldots gh$, and of the twenty points $abc, abf, \ldots fgh$; and which is such, that on each of the lines there lie four of the points, and through each of the points there pass three of the lines, viz. the points abc, abf, abg, abh lie on the line ab; and the lines bc, ca, ab meet in the point abc, and so in other cases. If now the point of projection instead of being arbitrary, be one of the above-mentioned four points O, then the projections of the lines af, bg, ch meet in a point, and the like for each of the six triads of lines; that is in the plane figure we have six points 1, 2, 3, 4, 5, 6, each of them the intersection of three lines as shown in the diagram,

$$1 = af \,.\, bg \,.\, ch,$$

$$2 = ag \,.\, bh \,.\, cf,$$

$$3 = ah \,.\, bf \,.\, cg,$$

$$4 = af \,.\, bh \,.\, cg,$$

$$5 = ag \,.\, bf \,.\, ch,$$

$$6 = ah \,.\, bg \,.\, cf,$$

and these six points lie in a conic. It is clear that the lines af, ag, ah; bf, bg, bh; cf, cg, ch are the lines 14, 25, 36; 35, 16, 24; 26, 34, 15 respectively.

Conversely, starting from the points 1, 2, 3, 4, 5, 6 on a conic, and denoting the lines 14, 25, 36; 35, 16, 24; 26, 34, 15 (being, it may be noticed, the sides and diagonals of the hexagon 162435) in the manner just referred to, then it is possible to complete the figure of the fifteen lines $ab, ac, \ldots gh$ and of the twenty points $abc, abf, \ldots fgh$, such that each line contains upon it four points, and that through each point there pass three lines, in the manner already mentioned.

Of the fifteen lines, nine, viz. the lines af, ag, ah; bf, bg, bh; cf, cg, ch are, as has been seen, lines through two of the six points 1, 2, 3, 4, 5, 6; the remaining lines are bc, ca, ab; gh, hf, fg. These are Pascalian lines,

bc	of the hexagon	162435,
ca	"	152634,
ab	"	142536,
gh	"	152436,
hf	"	142635,
bg	"	162534,

which appears thus, viz.

$$\begin{aligned} \text{line } bc \text{ contains points } & bcf\ , \quad bcg\ , \quad bch\ , \\ & = bf\,.\,cf,\ \ bg\,.\,cg,\ \ bh\,.\,ch, \\ & = 35\,.\,26,\ \ 16\,.\,34,\ \ 24\,.\,15; \end{aligned}$$

that is, bc is the Pascalian line of the hexagon 162435; and the like for the rest of the six lines.

The twenty points abc, $abf, \ldots fgh$ are as follows, viz. omitting the two points abc, fgh, the remaining eighteen points are the Pascalian points (the intersections of pairs of lines each through two of the points 1, 2, 3, 4, 5, 6) which lie on the Pascalian lines bc, ca, ab, gh, hf, fg respectively; the point abc is the intersection of the Pascalian lines bc, ca, ab, and the point fgh is the intersection of the Pascalian lines gh, hf, fg, the points in question being two of the points P (Steiner's twenty points, each the intersection of three Pascalian lines).

We thus see that we have two triads of hexagons such that the Pascalian lines of each triad meet in a point, and that the two points so obtained, together with the eighteen points on the six Pascalian lines, form a system of twenty points lying four together on fifteen lines, and which points and lines are the projections of the points and lines of intersection of six planes; or, say simply that the figure is the projection of the figure of six planes.

It is to be added, that if the planes are a, b, c, f, g, h, then the point of projection is any one of the four points which have the same polar plane in regard to the system of the planes a, b, c, and in regard to the system of the planes f, g, h. The consideration of the solid figure affords a demonstration of the existence as well of the six Pascalian lines as of the two points each the intersection of three of these lines.

404.

REPRODUCTION OF EULER'S MEMOIR OF 1758 ON THE ROTATION OF A SOLID BODY.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 361—373.]

EULER'S Memoir "Du mouvement de rotation des corps solides autour d'un axe variable," *Mém. de Berlin*, 1758, pp. 154—193 (printed in 1765), seems to have been written subsequently to the memoir with a similar title in the Berlin *Memoirs* for 1760, and to the "Theoria Motus Corporum Solidorum &c.," *Rostock*, 1765, and there are contained in the first-mentioned memoir some very interesting results which appear to have escaped the notice of later writers on the subject; viz. Euler succeeds in integrating the equations of motion *without the assistance furnished by the consideration of the invariable plane.* In reproducing these results I make the following alterations in Euler's notation, viz. instead of x, y, z I write p, q, r; instead of Ma^2, Mb^2, Mc^2 (where M is the mass) I write A, B, C, these quantities denoting the principal moments, and in some equations where the omission or insertion of the factor M is really immaterial I write A, B, C in the place of a^2, b^2, c^2; moreover instead of Euler's A, B, C $\left(\text{which denote respectively } \frac{b^2-c^2}{a^2}, \frac{c^2-a^2}{b^2}, \frac{a^2-b^2}{c^2}\right)$ I write L, M, N; but in other respects Euler's notation is preserved. The equations of motion are

$$Adp + (C-B)\,qrdt = 0,$$
$$Bdq + (A-C)\,rpdt = 0,$$
$$Cdr + (B-A)\,pqdt = 0;$$

so that putting for shortness

$$L = \frac{B-C}{A}, \quad M = \frac{C-A}{B}, \quad N = \frac{A-B}{C},$$

and introducing the auxiliary quantity u such that $du = pqrdt$, we have

$$p^2 = \mathfrak{A} + 2Lu,$$
$$q^2 = \mathfrak{B} + 2Mu,$$
$$r^2 = \mathfrak{C} + 2Nu,$$

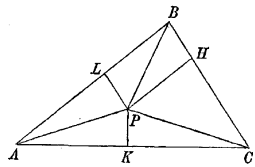
where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are constants of integration, and thence

$$t = \int \frac{du}{\sqrt{\{(\mathfrak{A} + 2Lu)(\mathfrak{B} + 2Mu)\}(\mathfrak{C} + 2Nu)}},$$

where the integral may without loss of generality be taken from $u = 0$; u, and consequently p, q, r, are thus given functions of t; and it is moreover clear that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are the initial values of p^2, q^2, r^2. We have also if ω be the angular velocity round the instantaneous axis

$$\omega^2 = \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + 2(L + M + N)u.$$

Euler then assumes that the position in space of the principal axes is geometrically determined as follows, viz. (treating the axes as points on a sphere) it is assumed that the distances from a fixed point P of the sphere are respectively l, m, n, and that



the inclinations of these distances to a fixed arc PQ are respectively λ, μ, ν. We have then the geometrical relations

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

$$\sin(\mu - \nu) = \frac{\cos l}{\sin m \sin n}, \quad \cos(\mu - \nu) = -\frac{\cos m \cos n}{\sin m \sin n},$$

$$\sin(\nu - \lambda) = \frac{\cos m}{\sin n \sin l}, \quad \cos(\nu - \lambda) = -\frac{\cos n \cos l}{\sin n \sin l},$$

$$\sin(\lambda - \mu) = \frac{\cos n}{\sin l \sin m}, \quad \cos(\lambda - \mu) = \frac{\cos l \cos m}{\sin l \sin m};$$

whence also

$$\sin \mu = \frac{-\cos\lambda \cos n - \sin\lambda \cos l \cos m}{\sin l \sin m},$$

$$\cos \mu = \frac{\sin\lambda \cos n - \cos\lambda \cos l \cos m}{\sin l \sin m},$$

$$\sin \nu = \frac{\cos\lambda \cos m + \sin\lambda \cos l \cos n}{\sin l \sin n},$$

$$\cos \nu = \frac{-\sin\lambda \cos m - \cos\lambda \cos l \cos n}{\sin l \sin n}.$$

The geometrical equations connecting the resolved angular velocities p, q, r with the differentials of l, m, n, λ, μ, ν are

$$\begin{aligned} dl \sin l &= dt(q\cos n - r\cos m), & d\lambda \sin^2 l &= -dt(q\cos m + r\cos n),\\ dm \sin m &= dt(r\cos l - p\cos n), & d\mu \sin^2 m &= -dt(r\cos n + p\cos l),\\ dn \sin n &= dt(p\cos m - q\cos l), & d\nu \sin^2 n &= -dt(p\cos l + q\cos m). \end{aligned}$$

Multiplying the equations of motion respectively by $\cos l$, $\cos m$, $\cos n$, and adding, we obtain an equation which is reducible to the form

$$d(Ap\cos l + Bq\cos m + Cr\cos n) = 0,$$

whence integrating

$$Ap\cos l + Bq\cos m + Cr\cos n = \mathfrak{D},$$

$\mathfrak{D}$ being a constant of integration. One other integral equation is necessary for the determination of the angles l, m, n. The expressions for dl, dm, dn give at once

$$p\,dl\sin l + q\,dm\sin m + r\,dn\sin n = 0.$$

Instead of the arcs l, m, n, Euler introduces a new variable v, such that

$$v = p\cos l + q\cos m + r\cos n;$$

by means of the last preceding equation, we find

$$dv = dp\cos l + dq\cos m + dr\cos n,$$

and then, substituting for dp, dq, dr, their values,

$$dv = \left(\frac{L\cos l}{p} + \frac{M\cos m}{q} + \frac{N\cos n}{r}\right)du,$$

from which the relation between v and u is to be determined. We have

$$\begin{aligned} \cos^2 l + \cos^2 m + \cos^2 n &= 1,\\ Ap\cos l + Bq\cos m + Cr\cos n &= \mathfrak{D},\\ p\cos l + q\cos m + r\cos n &= v, \end{aligned}$$

which give $\cos l$, $\cos m$, $\cos n$ in terms of u, v; the resulting formulæ contain the radical

$$\sqrt{\left\{\begin{aligned}&(L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2) - \mathfrak{D}^2(a^2+y^2+z^2)\\ &\qquad + 2\mathfrak{D}v(Ap^2 + Bq^2 + Cr^2) - v^2(A^2p^2 + B^2q^2 + C^2r^2)\end{aligned}\right\}},$$

which for shortness is represented by $\sqrt{\{(\cdot\cdot)\}}$. We then have

$$\cos l = \frac{\mathfrak{D}p(NCq^2 - MBr^2) + BCpv(Mr^2 - Nq^2) + LAqr\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

$$\cos m = \frac{\mathfrak{D}q(LAr^2 - NCp^2) + CAqv(Np^2 - Lr^2) + MBrp\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

$$\cos n = \frac{\mathfrak{D}r(MBp^2 - LAq^2) + ABrv(Lq^2 - Mp^2) + NCpq\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

and substituting these values in the differential equation

$$\frac{dv}{du}=\frac{L\cos l}{p}+\frac{M\cos m}{q}+\frac{N\cos n}{r},$$

the equation to be integrated becomes

$$\frac{dv}{du}(L^2A^2q^2r^2+M^2B^2r^2p^2+N^2C^2p^2q^2)=LMN\mathfrak{D}(Ap^2+Bq^2+Cr^2)-LMNv(A^2p^2+B^2q^2+C^2r^2)$$
$$+\frac{1}{pqr}(L^2Aq^2r^2+M^2Br^2p^2+N^2Cp^2q^2).$$

Now substituting for p, q, r their values, we have

$$L^2A^2q^2r^2+M^2B^2r^2p^2+N^2C^2p^2q^2=L^2A^2\mathfrak{B}\mathfrak{C}+M^2B^2\mathfrak{C}\mathfrak{A}+N^2C^2\mathfrak{A}\mathfrak{B}-2LMNu(\mathfrak{A}A^2+\mathfrak{B}B^2+\mathfrak{C}C^2),$$

$$L^2Aq^2r^2+M^2Br^2p^2+N^2Cp^2q^2=L^2A\mathfrak{B}\mathfrak{C}+M^2B\mathfrak{C}\mathfrak{A}+N^2C\mathfrak{A}\mathfrak{B}-2LMNu(\mathfrak{A}A+\mathfrak{B}B+\mathfrak{C}C),$$

$$p^2+q^2+r^2=\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2(L+M+N)u,$$

$$Ap^2+Bq^2+Cr^2=\mathfrak{A}A+\mathfrak{B}B+\mathfrak{C}C,$$

$$A^2p^2+B^2q^2+C^2r^2=\mathfrak{A}A^2+\mathfrak{B}B^2+\mathfrak{C}C^2:$$

and writing for shortness

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}=E,$$

$$\mathfrak{A}A+\mathfrak{B}B+\mathfrak{C}C=F,$$

$$\mathfrak{A}A^2+\mathfrak{B}B^2+\mathfrak{C}C^2=G,$$

$$L^2A\,\mathfrak{B}\mathfrak{C}+M^2B\mathfrak{C}\mathfrak{A}+N^2C\mathfrak{A}\mathfrak{B}=H,$$

$$L^2A^2\mathfrak{B}\mathfrak{C}+M^2B^2\mathfrak{C}\mathfrak{A}+N^2C^2\mathfrak{A}\mathfrak{B}=K,$$

where $K=EG-F^2$, substituting these values and observing that

$$L+M+N=-LMN,$$

the radical of the formula becomes

$$\sqrt{\{(\cdot\cdot)\}}=\sqrt{(K-2LMNGu+2\mathfrak{D}^2LMNu-\mathfrak{D}^2E+2\mathfrak{D}Fv-Gv^2)},$$

and the differential equation becomes

$$\frac{dv}{du}(K-2LMNGu)=LMN\mathfrak{D}F-LMNGv+\frac{1}{pqr}(H-2LMNFu)\sqrt{\{(\cdot\cdot)\}},$$

which can be reduced to the form

$$\frac{Kdv-LMNF\mathfrak{D}du-2LMNGudv+LMNGvdu}{\sqrt{\{K-\mathfrak{D}^2E+2LMN(\mathfrak{D}^2-G)u+2\mathfrak{D}Fv-Gv^2\}}}=\frac{Hdu-2LMNFudu}{\sqrt{\{(2Lu+\mathfrak{A})(2Mu+\mathfrak{B})(2Nu+\mathfrak{C})\}}}.$$

Euler remarks that as the right-hand side of the equation contains only the variable u, the solution will be effected if we can find a function of u, a multiplier of the left-hand side; he had elsewhere explained the method of finding such multipliers, and applying it to the equation in hand, the multiplier of the left-hand side, and therefore of the equation itself, is found to be $\frac{1}{K-2LMNGu}$, or what is the same thing $\frac{\sqrt{(G)}}{K-2LMNGu}$.

Multiplying by this quantity, the right-hand side may for shortness be represented by dU, so that

$$dU=\frac{(H-2LMNFu)\sqrt{(G)}\,du}{(K-2LMNGu)\sqrt{\{(2Lu+\mathfrak{A})(2Mu+\mathfrak{B})(2Nu+\mathfrak{C})\}}},$$

and U may be considered as a given function of u, or what is the same thing of t.

As regards the left-hand side, attending to the equation $K=EG-F^2$, the radical multiplied into $\sqrt{(G)}$ may be presented under the form

$$\sqrt{[\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}]};$$

and consequently the left-hand side becomes

$$\frac{(K-2LMNGu)\,Gdv+LMNG\,(Gv-\mathfrak{D}F)\,du}{(K-LMNGu)\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}}},$$

which putting for the moment $K-2LMNGu=p^2$, $Gv-\mathfrak{D}F=q$, $G-\mathfrak{D}^2=f^2$, becomes $\frac{pdq-qdp}{p\sqrt{(f^2p^2-q^2)}}$, the integral of which is $\sin^{-1}\frac{q}{fp}$; hence restoring the values of p, q, f, the integral is

$$\sin^{-1}\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{(K-2LMNGu)}}.$$

Hence considering the constant of integration as included in U, or writing

$$U=\mathfrak{E}+\int\frac{(H-2LMNFu)\sqrt{(G)}\,du}{(K-2LMNGu)\sqrt{\{(2Lu+\mathfrak{A})(2Mu+\mathfrak{B})(2Nu+\mathfrak{C})\}}},$$

we have for the required integral of the differential equation

$$\sin^{-1}\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=U,$$

whence also

$$\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=\sin U,$$

and

$$\frac{\sqrt{[\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}]}}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=\cos U,$$

so that the value of the original radical is

$$\sqrt{\{(\cdot\cdot)\}} = \frac{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}{\sqrt{(G)}}\cos U.$$

Substituting in the expressions for the cosines of the arcs l, m, n, these values of v and the radical; the formulæ after some reductions become

$$\cos l = \frac{\mathfrak{D}Ap}{G} + \frac{BCp\,(M\mathfrak{C}-N\mathfrak{B})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{LAqr\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

$$\cos m = \frac{\mathfrak{D}Bq}{G} + \frac{CAq\,(N\mathfrak{A}-L\mathfrak{C})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{MBrp\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

$$\cos n = \frac{\mathfrak{D}Cr}{G} + \frac{ABr\,(L\mathfrak{B}-M\mathfrak{A})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{NCpq\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

where for shortness p, q, r are retained in place of their values $\sqrt{(2Lu+\mathfrak{A})}$, $\sqrt{(2Mu+\mathfrak{B})}$, $\sqrt{(2Nu+\mathfrak{C})}$.

The values of l, m, n being known, that of λ could be determined by the differential equation

$$d\lambda = -\frac{dt\,(q\cos m + z\cos n)}{\sin^2 l},$$

and then the values of μ, ν would be determined without any further integration; but it is better to consider, in the place of any one of the principal axes in particular, the instantaneous axis, which is a line inclined to these at angles α, β, γ, the cosines of which are $\frac{p}{\omega}$, $\frac{q}{\omega}$, $\frac{r}{\omega}$ (if as before $\omega^2 = p^2+q^2+r^2$). Considering the instantaneous axis as a point of the sphere, let j denote the distance OP from the fixed point P, and ϕ the inclination OPQ of this distance to the fixed arc PQ. We have

$$\cos j = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n,$$

$$\sin j\cos\phi = \cos\alpha\sin l\cos\lambda + \cos\beta\sin m\cos\mu + \cos\gamma\sin n\cos\nu,$$

$$\sin j\sin\phi = \cos\alpha\sin l\sin\lambda + \cos\beta\sin m\sin\mu + \cos\gamma\sin n\sin\nu,$$

$$\cos(\lambda-\phi) = \frac{\cos\alpha - \cos l\cos j}{\sin l\sin j}, \quad \sin(\lambda-\phi) = \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin l\sin j},$$

$$\cos(\mu-\phi) = \frac{\cos\beta - \cos m\cos j}{\sin m\sin j}, \quad \cos(\mu-\phi) = \frac{\cos\alpha\cos n - \cos\gamma\cos l}{\sin m\sin j},$$

$$\cos(\nu-\phi) = \frac{\cos\gamma - \cos n\cos j}{\sin n\sin j}, \quad \cos(\nu-\phi) = \frac{\cos\beta\cos l - \cos\alpha\cos m}{\sin n\sin j},$$

so that λ, μ, ν are determined in terms of j and ϕ. These expressions give

$$d\phi = \frac{1}{\gamma^2 \sin^2 j} \{\cos l\,(qdr - rdq) + \cos m\,(rdp - pdr) + \cos n\,(pdq - qdp)\},$$

which is reducible to Euler's equation

$$d\phi = dt\,\frac{p\,(M\mathfrak{C} - N\mathfrak{B}) \cos l + q\,(N\mathfrak{A} - L\mathfrak{C}) \cos m + r\,(L\mathfrak{B} - M\mathfrak{A}) \cos n}{E - 2LMNu - v^2},$$

and thence, substituting for $\cos l$, $\cos m$, $\cos n$ their values, and observing that

$$Ap^2(M\mathfrak{C} - N\mathfrak{B}) + Bq^2(N\mathfrak{A} - L\mathfrak{C}) + Cr^2(L\mathfrak{B} - M\mathfrak{A}) = -(H - 2LMNFu),$$

$$BCp^2(M\mathfrak{C} - N\mathfrak{B})^2 + CAq^2(N\mathfrak{A} - L\mathfrak{C})^2 + ABr^2(L\mathfrak{B} - M\mathfrak{A})^2 = F(H - 2LMNFu),$$

$$LA\ (M\mathfrak{C} - N\mathfrak{B}) + MB\ (N\mathfrak{A} - L\mathfrak{C}) + NC\ (L\mathfrak{B} - M\mathfrak{A}) = LMNF,$$

the equation becomes

$$d\phi\,(E - 2LMNu - v^2) \div dt = \frac{-\mathfrak{D}\,(H - 2LMFu)}{G} + \frac{F\,(H - 2LMNFu)\sqrt{(G - \mathfrak{D}^2)}}{G\sqrt{(K - 2LMNGu)}} \sin U$$
$$+ \frac{LMNFpqr\sqrt{(G - \mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K - 2LMNGu)}} \cos U,$$

where it is to be remarked that

$$G^2(E - 2LMNu - v^2)$$
$$= (G - \mathfrak{D}^2)F^2 + G\,(K - 2LMNGu) - (G - \mathfrak{D}^2)(K - 2LMNGu)\sin^2 U$$
$$- 2\mathfrak{D}F\sqrt{(G - \mathfrak{D}^2)}\sqrt{\{(K - 2LMNGu)\}}\sin U.$$

Now

$$dU = \frac{dt\,(H - 2LMNFu)\sqrt{(G)}}{K - 2LMNGu},\quad du = pqrdt,$$

the differential $d\phi$ can be expressed as a fraction, the numerator whereof is

$$-\mathfrak{D}dU\,(K - 2LMNGu)\sqrt{(G)} + FdU\sqrt{\{G\,(G - \mathfrak{D}^2)(K - 2LMNGu)\}}\sin U$$
$$+ \frac{LMNFGdu\sqrt{\{G\,(G - \mathfrak{D}^2)\}}}{\sqrt{(K - 2LMNGu)}} \cos U,$$

and the denominator

$$(G - \mathfrak{D}^2)F^2 + G\,(K - 2LMNGu) - 2\mathfrak{D}F\sqrt{(G - \mathfrak{D}^2)(K - 2LMNGu)}\sin U$$
$$-(G - \mathfrak{D}^2)(K - 2LMNGu)\sin^2 U.$$

To simplify, write

$$\sqrt{(K - 2LMNGu)} = s,\quad \sqrt{(G - \mathfrak{D}^2)} = h,$$

the numerator is

$$-\mathfrak{D}s^2\,dU\sqrt{(G)}+Fhs\,dU\sqrt{(G)}\sin U-Fhs\sqrt{(G)}\cos U,$$

and the denominator

$$h^2F^2+Gs^2-2\mathfrak{D}Fhs\sin U-h^2s^2\sin^2 U,$$

which, observing that $h^2=G-\mathfrak{D}^2$, is equal to

$$(Fh-\mathfrak{D}s\sin U)^2+Gs^2\cos^2 U,$$

and we have

$$d\phi=\frac{-\mathfrak{D}s^2dU+Fhs\sin U\,dU-Fhds\cos U}{(Fh-\mathfrak{D}s\sin U)^2+Gs^2\cos^2 U}\sqrt{(G)},$$

the integral of which is

$$\phi+\mathfrak{F}=\tan^{-1}\frac{Fh-\mathfrak{D}s\sin U}{s\cos U\sqrt{(G)}},$$

where $\mathfrak{F}$ is the constant of integration, or substituting for h, s their values, the equation is

$$\tan(\phi+\mathfrak{F})=\frac{F\sqrt{(G-\mathfrak{D}^2)}-\mathfrak{D}\sin U\sqrt{(K-2LMNGu)}}{\cos U\sqrt{\{G(K-2LMNGu)\}}}.$$

It may be added that

$$\omega\cos j=v=\frac{1}{G}\left[\mathfrak{D}F+\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U\right],$$

and therefore

$$\cos j=\frac{\mathfrak{D}F+\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U}{G\sqrt{(E-2LMNu)}}.$$

Euler remarks that the complexity of the solution owing to the circumstance that the fixed point P is left arbitrary; and that the formulæ may be simplified by taking this point so that $G-\mathfrak{D}^2=0$, and he gives the far more simple formulæ corresponding to this assumption; this is in fact taking the point P in the *direction of the normal to the invariable plane*, and the resulting formulæ are identical with the ordinary formulæ for the solution of the problem. The term *invariable plane* is not used by Euler, and seems to have first occurred in Lagrange's "Essai sur le problème de trois corps," *Prix de l'Acad. de Berlin*, t. IX., 1772.

To prove the before-mentioned equation for $d\phi$; starting from the equations

$$\cos j=\cos\alpha\cos l+\cos\beta\cos m+\cos\gamma\cos n=\frac{v}{\omega},$$

$$\sin j\cos\phi=\cos\alpha\sin l\cos\lambda+\cos\beta\sin m\cos\mu+\cos\gamma\sin n\cos\nu$$

$$\sin j\sin\phi=\cos\alpha\sin l\cos\lambda+\cos\beta\sin m\sin\mu+\sin\gamma\sin n\sin\nu,$$

we have

$$\cos j\,dj\cos\phi-\sin j\sin\phi\,d\phi$$

$$=-\sin\alpha\,d\alpha\sin l\cos\lambda-\&c.+\cos\alpha\cos\lambda\cos l\,dl+\&c.-\cos\alpha\sin l\sin\lambda\,d\lambda+\&c.,$$

the second term is

$$\frac{p}{\omega}\cos\lambda\cot l\,(q\cos n - r\cos m)$$

$$+\frac{q}{\omega}\cos\mu\cot m\,(r\cos l - p\cos n)$$

$$+\frac{r}{\omega}\cos\nu\cot n\,(p\cos m - q\cos l),$$

and the third term is

$$+\frac{p}{\omega}\sin\lambda\,\mathrm{cosec}\,l\,(q\cos m + r\cos n)$$

$$+\frac{q}{\omega}\sin\mu\,\mathrm{cosec}\,m\,(r\cos n + p\cos l)$$

$$+\frac{r}{\omega}\sin\nu\,\mathrm{cosec}\,n\,(p\cos l + q\cos m).$$

Hence the second and third terms together are

$$=\frac{pq}{\omega}\left(\cos\lambda\frac{\cos l\cos n}{\sin l} - \cos\mu\frac{\cos m\cos n}{\sin m} + \sin\lambda\frac{\cos m}{\sin l} + \sin\mu\frac{\cos l}{\sin m}\right) + \&c.,$$

$$=\frac{pq}{\omega}\left\{\begin{matrix}-\cos\lambda\sin n\cos(\nu-\lambda) + \sin\lambda\sin n\sin(\nu-\lambda)\\ +\cos\mu\sin n\cos(\mu-\nu) + \sin\mu\sin n\sin(\mu-\nu)\end{matrix}\right\} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\left\{\begin{matrix}-\cos\lambda\cos(\nu-\lambda) + \sin\lambda\sin(\nu-\lambda)\\ +\cos\mu\cos(\mu-\nu) + \sin\mu\sin(\mu-\nu)\end{matrix}\right\} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\left\{\begin{matrix}-\cos\{\lambda+(\nu-\lambda)\}\\ +\cos\{\mu-(\mu-\nu)\}\end{matrix}\right\} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\,(-\cos\nu + \cos\nu) + \&c., = 0;$$

we have therefore

$$\cos j\,dj\cos\phi - \sin j\sin\phi\,d\phi$$

$$= -\sin\alpha\,d\alpha\sin l\cos\lambda - \sin\beta\,d\beta\sin m\cos\mu - \sin\gamma\,d\gamma\sin n\cos\nu,$$

$$= d\frac{p}{\omega}.\sin l\cos\lambda + d\frac{q}{\omega}.\sin m\cos\mu + d\frac{r}{\omega}.\sin n\sin\nu$$

$$= +\frac{1}{\omega}\;(\sin l\cos\lambda\,dp + \sin m\cos\mu\,dq + \sin n\cos\nu\,dr)$$

$$-\frac{d\omega}{\omega^2}(\sin l\cos\lambda\,p + \sin m\cos\mu\,q + \sin n\cos\nu\,r)$$

$$= -\cot j\cos\phi\,d\frac{v}{\omega} - \sin j\sin\phi\,d\phi.$$

Hence therefore

$$\sin j \sin \phi d\phi = -\cot j \cos \phi d \frac{v}{\omega}$$

$$-\frac{1}{\omega}\ (\sin l \cos \lambda\, dp + \sin m \cos \mu\, dq + \sin n \cos \nu\, dr)$$

$$+\frac{d\omega}{\omega^2}(\sin l \cos \lambda\, p + \sin m \cos \mu\, q + \sin n \cos \nu\, r)$$

$$= -\cot j \cos \phi \frac{1}{\omega}(\cos l\, dp + \cos m\, dq + \cos n\, dr)$$

$$+\cot j \cos \phi \frac{d\omega}{\omega^2}(p \cos l + q \cos m + r \cos n)$$

$$-\frac{1}{\omega}\ (\sin l \cos \lambda\, dp + \sin m \cos \mu\, dq + \sin n \cos \nu\, dr)$$

$$+\frac{d\omega}{\omega^2}(\sin l \cos \lambda\,.\, p + \sin m \cos \mu\,.\, q + \sin n \cos \nu\,.\, r)$$

$$= \frac{1}{\omega}\ \{(-\cot j \cos \phi \cos l - \sin l \cos \lambda)\, dp + \&c.\}$$

$$+\frac{d\omega}{\omega^2}\{(\ \ \cot j \cos \phi \cos l + \sin l \cos \lambda)\, p\ \ + \&c.\}.$$

But we have

$$\cos(\lambda - \phi) = \frac{\cos \alpha - \cos l \cos j}{\sin l \sin j},$$

$$= \frac{\cos \alpha}{\sin l \sin j} - \cot l \cot j,$$

$$\sin(\lambda - \phi) = \frac{\cos \gamma \cos m - \cos \beta \cos n}{\sin l \sin j},$$

and thence

$$\cos \lambda = \cos\{(\lambda - \phi) + \phi\} = \frac{\cos \phi\,(\cos \alpha - \cos l \cos j) - \sin \phi\,(\cos \gamma \cos m - \cos \beta \cos n)}{\sin l \sin j},$$

whence also

$$\cot j \cos \phi \cos l + \sin l \cos \lambda$$

$$= \frac{1}{\sin j}\{\cos \phi \cos l + \cos \phi\,(\cos \alpha - \cos l \cos j) - \sin \phi\,(\cos \gamma \cos m - \cos \beta \cos n)\},$$

$$= \frac{1}{\sin j}\{\cos \alpha \cos \phi - \sin \phi\,(\cos \gamma \cos m - \cos \beta \cos n)\},$$

$$= \frac{1}{\omega \sin j}\{p \cos \phi - \sin \phi\,(r \cos m - q \cos n)\}.$$

Hence the expression for $\sin j \sin \phi \, d\phi$ is

$$= -\frac{1}{\omega^2 \sin j} [\{p \cos \phi - \sin \phi (r \cos m - q \cos n)\} \, dp + \ldots]$$

$$+ \frac{d\omega}{\omega^3 \sin j} [\{p \cos \phi - \sin \phi (r \cos m - q \cos n)\} \; p + \ldots]$$

$$= -\frac{1}{\omega^2 \sin j} [\omega \, d\omega \cos \phi - \sin \phi \{(r \cos m - q \cos n) \, dp + \ldots\}]$$

$$+ \frac{d\gamma}{\omega^2 \sin j} \cdot \omega^2 \cos \phi \qquad = \sin j \sin \phi \, d\phi,$$

or finally

$$\sin j \sin \phi \, d\phi = \frac{1}{\omega^2} \frac{\sin \phi}{\sin j} [(r \cos m - q \cos n) \, dp + \&c.],$$

that is

$$d\phi = \frac{1}{\omega^2 \sin^2 j} \left\{ \begin{array}{l} (r \cos m - q \cos n) \, dp \\ + (p \cos n - r \cos l) \, dq \\ + (q \cos l - p \cos m) \, dr \end{array} \right\},$$

which is the required expression for $d\phi$.

Recapitulating, A, B, C, p, q, r denote as usual,

$$L = \frac{B - C}{A}, \; M = \frac{C - A}{B}, \; N = \frac{A - B}{C}, \; du = pqr dt,$$

$$p = \sqrt{(\mathfrak{A} + 2Lu)},$$
$$q = \sqrt{(\mathfrak{B} + 2Mu)},$$
$$r = \sqrt{(\mathfrak{C} + 2Nu)};$$

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = E,$$
$$\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C = F,$$
$$\mathfrak{A}A^2 + \mathfrak{B}B^2 + \mathfrak{C}C^2 = G;$$

$$L^2 A \mathfrak{B} \mathfrak{C} + M^2 B \mathfrak{C} \mathfrak{A} + N^2 C \mathfrak{A} \mathfrak{B} = H,$$
$$L^2 A^2 \mathfrak{B} \mathfrak{C} + M^2 B^2 \mathfrak{C} \mathfrak{A} + N^2 C^2 \mathfrak{A} \mathfrak{B} = K;$$

so that

$$K = EG - F^2,$$

$$U = \mathfrak{C} + \int \frac{(H - 2LMNFu) \, du \sqrt{(G)}}{(K - 2LMNGu) \sqrt{\{(\mathfrak{A} + 2Lu)(\mathfrak{B} + 2Mu)(\mathfrak{C} + 2Nu)\}}},$$

$$\cos l = \frac{\mathfrak{D}Ap}{G} + \frac{BCp(B\mathfrak{C} - C\mathfrak{B})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{LAqr\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{\{G(K-2LMNGu)\}}}\cos U,$$

$$\cos m = \frac{\mathfrak{D}Bq}{G} + \frac{CAq(C\mathfrak{A} - A\mathfrak{C})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{MBrp\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{\{G(K-2LMNGu)\}}}\cos U,$$

$$\cos n = \frac{\mathfrak{D}Cr}{G} + \frac{ABr(A\mathfrak{B} - B\mathfrak{A})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{NCpq\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{\{G(K-2LMNGu)\}}}\cos U,$$

$$\omega^2 = E - 2LMNu,$$

$$\cos j = \frac{\mathfrak{D}F + \sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U}{G\sqrt{(E-2LMNu)}},$$

$$v = p\cos l + q\cos m + r\cos n$$

$$= \frac{1}{G}[\mathfrak{D}F + \sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U],$$

$$\tan(\phi + \mathfrak{F}) = \frac{F\sqrt{(G-\mathfrak{D}^2)} - \mathfrak{D}\sin U\sqrt{(K-2LMNGu)}}{\cos U\sqrt{\{G(K-2LMNGu)\}}}.$$

[The angles which determine the position of the body are thus expressed in terms of u, which is given as a function of t by the foregoing equation $du = pqrdt$, where p, q, r denote given functions of u.]

405.

AN EIGHTH MEMOIR ON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVII. (for the year 1867). Received January 8,—Read January 17, 1867.]

THE present Memoir relates mainly to the binary quintic, continuing the investigations in relation to this form contained in my Second, Third, and Fifth Memoirs on Quantics, [141], [144], [156]; the investigations which it contains in relation to a quantic of any order are given with a view to their application to the quintic. All the invariants of a binary quintic (viz. those of the degrees 4, 8, 12, and 18) are given in the Memoirs above referred to, and also the covariants up to the degree 5; it was interesting to proceed one step further, viz. to the covariants of the degree 6; in fact, while for the degree 5 we obtain 3 covariants and a single syzygy, for the degree 6 we obtain only 2 covariants, but as many as 7 syzygies; one of these is, however, the syzygy of the degree 5 multiplied into the quintic itself, so that, excluding this derived syzygy, there remain $(7-1=)$ 6 syzygies of the degree 6. The determination of the two covariants (Tables 83 and 84 *post*) and of the syzygies of the degree 6, occupies the commencement of the present Memoir. [These covariants 83, 84 are the covariants M and N of the paper 143, "Tables of the covariants M to W of the binary quintic", and they are accordingly not here reproduced.]

The remainder of the Memoir is in a great measure a reproduction (with various additions and developments) of researches contained in Professor Sylvester's Trilogy, and in a recent memoir by M. Hermite[1]. In particular, I establish in a more general form (defining for that purpose the functions which I call "Auxiliars") the theory which is the basis of Professor Sylvester's criteria for the reality of the roots of a quintic equation, or, say, the theory of the determination of the character of an equation of any order. By way of illustration, I first apply this to the quartic equation; and

[1] Sylvester "On the Real and Imaginary Roots of Algebraical Equations; a Trilogy," *Phil. Trans.* vol. CLIV. (1864), pp. 579—666. Hermite, "Sur l'Équation du 5e degré," *Comptes Rendus*, t. LXI. (1866), and in a separate form, Paris, 1866.

I then apply it to the quintic equation, following Professor Sylvester's track, but so as to dispense altogether with his amphigenous surface, and making the investigation to depend solely on the discussion of the bicorn curve, which is a principal section of this surface. I explain the new form which M. Hermite has given to the Tschirnhausen transformation, leading to a transformed equation the coefficients whereof are all invariants; and, in the case of the quintic, I identify with my Tables his cubicovariants $\phi_1(x, y)$ and $\phi_2(x, y)$. And in the two new Tables, 85 and 86, I give the leading coefficients of the other two cubicovariants $\phi_3(x, y)$ and $\phi_4(x, y)$, [these are now also identified with my Tables]. In the transformed equation the second term (or that in z^4) vanishes, and the coefficient $\mathfrak{A}$ of z^3 is obtained as a quadric function of four indeterminates. The discussion of this form led to criteria for the character of a quintic equation, expressed like those of Professor Sylvester in terms of invariants, but of a different and less simple form; two such sets of criteria are obtained, and the identification of these, and of a third set resulting from a separate investigation, with the criteria of Professor Sylvester, is a point made out in the present memoir. The theory is also given of the canonical form which is the mechanism by which M. Hermite's investigations were carried on. The Memoir contains other investigations and formulæ in relation to the binary quintic; and as part of the foregoing theory of the determination of the character of an equation, I was led to consider the question of the imaginary linear transformations which give rise to a real equation: this is discussed in the concluding articles of the memoir, and in an Annex I have given a somewhat singular analytical theorem arising thereout.

The paragraphs and Tables are numbered consecutively with those of my former Memoirs on Quantics. I notice that in the Second Memoir, p. 126, we should have No. $26 = (\text{No. } 19)^2 - 128$ (No. 25), viz. the coefficient of the last term is 128 instead of 1152. [This correction is made in the present reprint, 141, where the equation is given in the form $Q' = G^2 - 128Q$.]

Article Nos. 251 to 254.—*The Binary Quintic, Covariants and Syzygies of the degree* 6.

251. The number of asyzygetic covariants of any degree is obtained as in my Second Memoir on Quantics, *Philosophical Transactions*, vol. CXLVI. (1856), pp. 101—126, [141], viz. by developing the function

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)},$$

as shown p. 114, and then subtracting from each coefficient that which immediately precedes it; or, what is the same thing, by developing the function

$$\frac{1-x}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)},$$

which would lead directly to the second of the two Tables which are there given; the Table is there calculated only up to z^8, but I have since continued it up to z^{18}, so as to show the number of the asyzygetic covariants of every order in the variables up to the degree 18 in the coefficients, being the degree of the skew invariant, the highest of the irreducible invariants of the quintic. The Table is, for greater convenience, arranged in a different form, as follows:

Table No. 81.

Table for the number of the Asyzygetic Covariants of any order, to the degree 18.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
0	1		0		1		0		2		0		3		0		4		1	0
1		0		0		1		1		1		2		3		3		4		1
2			1		0		2		1		4		2		6		4		9	2
3		0		1		1		2		3		4		5		7		8		3
4			0		2		1		4		3		7		6		11		9	4
5		1		1		2		3		5		6		8		10		13		5
6			1		1		4		3		7		7		12		11		18	6
7				1		2		4		5		8		10		13		16		7
8			0		2		2		6		6		11		11		18		18	8
9				1		3		4		7		9		13		16		20		9
10			1		1		4		5		10		10		17		18		26	10
11				1		2		5		7		11		14		19		23		11
12					2		3		7		8		15		16		24		26	12
13				0		2		4		8		11		16		20		27		13
14					1		4		5		11		13		21		23		33	14
15				1		2		5		8		13		17		24		29		15
16					1		2		7		9		16		19		29		32	16
17						2		4		8		12		19		24		32		17
18					0		3		5		11		14		23		27		39	18
19						1		4		7		13		18		26		33		19
20					1		2		6		9		17		21		32		37	20
21						1		3		8		12		19		26		36		21
22							2		4		10		14		24		29		42	22
23						0		3		6		12		18		27		35		23
24							1		5		8		16		21		33		40	24
25						1		2		6		11		19		26		37		25
26							1		3		9		13		23		30		44	26
27								2		5		11		17		27		36		27
28							0		3		6		14		20		32		40	28
29								1		5		9		17		25		37		29
30							1		2		7		12		22		29		44	30
31								1		3		9		15		25		35		31
32									2		5		12		18		31		40	32
33								0		3		7		15		23		35		33
34									1		5		9		19		27		42	34
35								1		2		7		13		23		33		35
36									1		3		10		16		28		38	36
37										2		5		12		20		33		37
38									0		3		7		16		24		39	38
39										1		5		10		20		30		39
40									1		2		7		13		25		35	40
41										1		3		10		17		29		41
42											2		5		13		21		36	42
43										0		3		7		16		26		43
44											1		5		10		21		31	44
45										1		2		7		14		26		45
46											1		3		10		17		31	46
47												2		5		13		22		47
48											0		3		7		17		27	48
49												1		5		10		21		49
50											1		2		7		14		27	50
51												1		3		10		18		51
52												0		2		5		13		52
53													1		3		7		22	53
54												1		2		5		17		54
55													1		3		7		14	55
56													0		2		10		18	56
57														1		3		13		57
58													1		2		7		17	58
59														1		5		10		59
60														1		3		7		60
61															2		10		14	61
62														0		3		5		62
63															1		7		13	63
64														1		2		5		64
65															1		7		10	65
66																3		5		66
67															0		2		10	67
68																1		3		68
69															1		5		7	69
70																1		2		70
71																	3		7	71
72																0		2		72
73																1		3		73
74																	1		5	74
75																	1		2	75
76																	0		3	76
77																		1		77
78																	1		3	78
79																		1		79
80																			2	80
81																		0		81
82																		1		82
83																			1	83
84																			1	84
85																			0	85
86																			1	86
87																				87
88																				88
89																				89
90																				90

[In regard to this table No. 81 it is hardly necessary to notice that for any column with an even heading the numbers of the column correspond to the even outside numbers, while for any column with an odd heading the numbers of the column correspond to the odd outside numbers. The table is in fact a table of the differences of the numbers of the *af*-table, 142; thus in this table writing down cols. 5 and 6 and in each of them forming the differences by subtracting from each number the number immediately below it, we have cols. 5 and 6 of the table No. 81, viz.:

1	1	2	3	5	7	9	11	14	16	18	19	20	col. 5, 13—12 of *af*-table.
1	0	1	1	2	2	2	2	3	2	2	1	1	col. 5 of table No. 81.

1	1	2	3	5	7	10	12	16	19	23	25	29	30	32	32	col. 6, 15 of *af*-table.
1	0	1	1	2	2	3	2	4	3	4	2	4	1	2	0	col. 6 of table No. 81.]

252. The interpretation up to the degree 6 is as follows:

[In the following Table No. 82 as originally printed, the heading of the fourth column was "Constitution. Nos. in ( ) refer to Tables in former Memoirs except (83) and (84) which are given *post*," and the covariants were referred to by their Nos. accordingly.]

Table No. 82.

Degree.	Order.	No.	Constitution. Notation is the alphabetic notation of 143, A the quintic itself, B, C quadricovariants, &c.	N=new covt. S=syzygy.
0	0	1	viz. the absolute constant unity.	
1	5	1	A	N.
2	10	1	A^2	
,,	6	1	C	N.
,,	2	1	B	N.
3	15	1	A^3	
,,	11	1	AC	
,,	9	1	F	N.
,,	7	1	AB	
,,	5	1	E	N.
,,	3	1	D	N.
4	20	1	A^4	
,,	16	1	A^2C	
,,	14	1	AF	
,,	12	2	C^2, A^2B	
,,	10	1	AE	
,,	8	2	AD, BC	
,,	6	1	I	N.
,,	4	2	H, B^2	N.
,,	0	1	G	N.
5	25	1	A^5	
,,	21	1	A^3C	
,,	19	1	A^2F	
,,	17	2	A^3B, AC^2	
,,	15	2	A^2E, CF	
,,	13	2	A^2D, ABC	
,,	11	2	$AI+BF-CE=0$	S.
,,	9	3	AB^2, AH, CD	
,,	7	2	BE, L	N.
,,	5	2	AG, BD	
,,	3	1	K	N.
,,	1	1	J	N.
6	30	1	A^6	
,,	26	1	A^4C	
,,	24	1	A^3F	
,,	22	2	A^3C^2, A^4B	
,,	20	2	ACF, A^3E	
,,	18	3	$E^2+4C^3+A^3D-A^2BC=0$	S.
,,	16	2	$A\{AI+BF-CE\}=0$	S'.
,,	14	4	$-6ACD-1EF-4BC^2+A^2H=0$, A^2B^2	S.
,,	12	3	$AL+3DF-2CI=0$, ABE	S.
,,	10	4	$4B^3C+12ABD-A^2G+E^2=0$, CH	S.
,,	8	2	$AK+2BI-3DE=0$	S.
,,	6	4	$AJ+2BH-B^3-CG-9D^2=0$	S.
,,	4	1	N	N.
,,	2	1	M	N.

253. For the explanation of this I remark that the Table No. 81 shows that we have for the degree 0 and order 0 one covariant; this is the absolute constant unity; for the degree 1 and order 5, 1 covariant, this is the quintic itself, A; for degree 2 and order 10, 1 covariant; this is the square of the quintic, A^2; for same degree and order 6, 1 covariant, which had accordingly to be calculated, viz. this is the covariant C; and similarly whenever the Table No. 81 indicates the existence of a covariant of any degree and order, and there does not exist a product of the covariants previously calculated, having the proper degree and order, then in each such case (shown in the last preceding Table by the letter N) a new covariant had to be calculated. On coming to degree 5, order 11, it appears that the number of asyzygetic invariants is only $=2$, whereas there exist of the right degree and order the 3 combinations AI, BF, CE; there is here a syzygy, or linear relation, between the combinations in question; which syzygy had to be calculated, and was found to be as shown, $AI+BF-CE=0$, a result given in the Second Memoir, p. 126. Any such case is indicated by the letter S. At the place degree 6, order 16, we find a syzygy between the combinations A^2I, A^2BF, ACE; as each term contains the factor A, this is only the last-mentioned syzygy multiplied by A, not a new syzygy, and I have written S' instead of S. The places degree 6, orders 18, 14, 12, 10, 8, 6 indicate each of them a syzygy, which syzygies, as being of the degree 6, were not given in the Second Memoir, and they were first calculated for the present Memoir. It is to be noticed that in some cases the combinations which might have entered into the syzygy do not all of them do so; thus degree 6, order 14, the syzygy is between the four combinations ACD, EF, BC^2, A^2H, and does not contain the remaining combination A^2B. The places degree 6, orders 4, 2, indicate each of them a new covariant, and these, as being of the degree 6, were not given in the Second Memoir, but had to be calculated for the present Memoir.

254. I notice the following results:

$$\begin{aligned}
&\text{Quadrinvt. } 6H &&= 3G^2,\\
&\text{Cubinvt. } 6H &&= -G^3+54GQ,\\
&\text{Disct. } (\alpha B+\beta M) &&= (-G,\ Q,\ -3U\between\alpha,\ \beta)^2,\\
&\text{Jac. } (B,\ H) &&= 6M,\\
&\text{Hess. } 3D &&= N,
\end{aligned}$$

the last two of which indicate the formation of the covariants given in the new Tables $M=$ No. 83 and $N=$ No. 84: viz. if to avoid fractions we take 3 times the covariant D, being a cubic $(a,\ldots)^3(x,\ y)^3$, then the Hessian thereof is a covariant $(a,\ldots)^6(x,\ y)^6$, which is given in Table, M No. 83; and in like manner if we form the Jacobian of the Tables B and H which are respectively of the forms $(a,..)^2(x,\ y)^2$, and $(a,..)^6(x,\ y)^4$, this is a covariant $(a,..)^6(x,\ y)^4$, and dividing it by 6 to obtain the coefficients in their lowest terms, we have the new Table, N No. 84. I have in these, for greater distinctness, written the numerical coefficients *after* instead of *before*, the literal terms to which they belong.

The two new Tables are:

Table No. 83. $M = (*\between x, y)^2$. See 143.

Table No. 84. $N = (*\between x, y)^4$. See 143.

Article No. 255.—*Formulæ for the canonical form* $ax^5 + by^5 + cz^5 = 0$, *where* $x + y + z = 0$.

255. The quintic $(a, b, c, d, e, f \between x, y)^5$ may be expressed in the form

$$ru^5 + sv^5 + tw^5,$$

where u, v, w are linear functions of (x, y) such that $u + v + w = 0$. Or, what is the same thing, the quintic may be represented in the canonical form

$$ax^5 + by^5 + cz^5,$$

where $x + y + z = 0$; this is $= (a - c, -c, -c, -c, -c, b - c \between x, y)^5$, and the different covariants and invariants of the quintic may hence be expressed in terms of these coefficients (a, b, c).

For the invariants we have

$$\begin{aligned} G &= J = b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a + b + c), \\ Q &= K = a^2b^2c^2\,(bc + ca + ab), \\ -U &= L = a^4b^4c^4, \\ W &= I = 4a^5b^5c^5\,(b - c)(c - a)(a - b). \end{aligned}$$

[Observe that throughout the present Memoir, the invariants, instead of being called $G, Q, -U, W$ are called I, J, K, L, viz. the I, J, K, L in all that follows denote the invariants, and not the covariants denoted by these letters in 142, 143. Moreover D is used to denote the invariant Q', which is in fact the discriminant of the quintic.]

Hence, writing for a moment

$$\begin{aligned} a + b + c &= p, \text{ and therefore } J = q^2 - 4pr, \\ bc + ca + ab &= q \qquad\qquad K = r^2q, \\ abc &= r \qquad\qquad L = r^4, \end{aligned}$$

we have

$$(a - b)^2(b - c)^2(c - a)^2 = p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2,$$

and thence

$$I^2 = 16r^{10}(p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2),$$

and

$$\begin{aligned} J(K^2 - JL)^2 + 8K^3L - 72JKL^2 - 432L^3 \\ &= r^{10}\{(q^2 - 4pr)\,16p^2 + 8q^3 - (q^2 - 4pr)\,72q - 432r^2\}, \\ &= 8r^{10}\{(q^2 - 4pr)(2p^2 - 9q) + q^3 - 54r^2\}, \\ &= 16r^{10}\{p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2\}, \end{aligned}$$

that is,

$$I^2 = J(K^2 - JL)^2 + 8K^3L - 72JKL^2 - 432L^3,$$

which is the simplest mode of obtaining the expression for the square of the 18-thic or skew invariant I in terms of the invariants J, K, L of the degrees 4, 8, 12 respectively.

If instead of the invariant K of the degree 8 we consider the invariant D [$= Q'$ as before-mentioned] of the same degree, this is

$$\begin{aligned} Q' = D &= \{b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a+b+c)\}^2 - 128a^2b^2c^2\,(bc+ca+ab), \\ &= q^4 - 8q^2pr - 128qr^2 + 16p^3r^2, \\ D &= \text{Norm}\,\left((bc)^{\frac{1}{2}} + (ca)^{\frac{1}{2}} + (ab)^{\frac{1}{2}}\right); \end{aligned}$$

and we have also the following covariants:

$$\begin{aligned} B &= (-ac,\ ab-ac-bc,\ -bc\between x,\ y)^2, \\ &= bcyz + cazx + abxy. \\ C &= (-ac,\ -3ac,\ -3ac,\ ab-ac-bc,\ -3bc,\ -3bc,\ -bc\between x,\ y)^6 \\ &= bcy^3z^3 + caz^2x^3 + abx^3y^3. \\ D &= (0,\ -abc,\ -abc,\ 0\between x,\ y)^3 = abc\,xyz. \end{aligned}$$

$$\begin{aligned} E = \ & (\ a^2b - \ ac^2 + \ bc^2 - a^2c - 2abc)\,x^5 \\ & + (\ \ - 5ac^2 + 5bc^2 \ \ - 5abc)\,x^4y \\ & + (\ \ - 10ac^2 + 10bc^2 \ \ - 2abc)\,x^3y^2 \\ & + (\ \ - 10ac^2 + 10bc^2 \ \ + 2abc)\,x^2y^3 \\ & + (\ \ - 5ac^2 + 5bc^2 \ \ + 5abc)\,xy^4 \\ & + (-ab^2 - \ ac^2 + \ bc^2 + b^2c + 2abc)\,y^5 \\ = \ & (b-c)\,a^2x^5 + (c-a)\,b^2y^5 + (a-b)\,c^2z^5 \\ & \qquad - abc\,(y-z)\,(z-x)\,(x-y)\,(yz+zx+xy). \end{aligned}$$

Article No. 256.—*Expression of the* 18-*thic Invariant in terms of the roots.*

256. It was remarked by Dr Salmon, that for a quintic $(a, b, c, d, e, f\between x, y)^5$ which is linearly transformable into the form $(a, 0, c, 0, e, 0\between x, y)^5$, the invariant I is $= 0$. Now putting for convenience $y = 1$, and considering for a moment the equation

$$x\,(x-\beta)\,(x-\gamma)\,(x-\delta)\,(x-\epsilon) = 0,$$

then writing herein $\dfrac{x}{mx+n}$ for x, the transformed equation is

$$x\,(x-\beta')\,(x-\gamma')\,(x-\delta')\,(x-\epsilon') = 0,$$

where

$$\beta' = \frac{n\beta}{1-m\beta},\ \gamma' = \frac{n\gamma}{1-m\gamma},\ \&c.;$$

hence m may be so determined that $\beta'+\gamma'$ may be $=0$; viz. this will be the case if $\beta+\gamma=2m\beta\gamma$, or $m=\frac{\beta+\gamma}{2\beta\gamma}$. In order that $\delta'+\epsilon'$ may be $=0$, we must of course have $m=\frac{\delta+\epsilon}{2\delta\epsilon}$, and hence the condition that simultaneously $\beta'+\gamma'=0$ and $\delta'+\epsilon'=0$ is $\frac{\beta+\gamma}{2\beta\gamma}=\frac{\delta+\epsilon}{2\delta\epsilon}$; that is, $(\beta+\gamma)\delta\epsilon-\beta\gamma(\delta+\epsilon)=0$. Or putting $x-\alpha$ for x and $\beta-\alpha$, $\gamma-\alpha$, &c. for β, γ, &c., we have the equation

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon)=0,$$

which is by the transformation $x-\alpha$ into $\frac{x-\alpha}{m(x-\alpha)+n}$ changed into

$$(x-\alpha')(x-\beta')(x-\gamma')(x-\delta')(x-\epsilon')=0$$

(where $\alpha'=\alpha$), and the condition in order that in the new equation it may be possible to have simultaneously $\beta'+\gamma'-2\alpha'=0$, $\delta'+\epsilon'-2\alpha'=0$, is

$$(\beta+\gamma-2\alpha)(\delta-\alpha)(\epsilon-\alpha)-(\delta+\epsilon-2\alpha)(\beta-\alpha)(\gamma-\alpha)=0,$$

or, as this may be written,

$$\begin{vmatrix} 1, & 2\alpha\ , & \alpha^2 \\ 1, & \beta+\gamma, & \beta\gamma \\ 1, & \delta+\epsilon, & \delta\epsilon \end{vmatrix}=0.$$

Hence writing $x+\alpha'$ for x, the last-mentioned equation is the condition in order that the equation

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon)=0$$

may be transformable into

$$x(x-\beta')(x-\gamma')(x-\delta')(x-\epsilon')=0,$$

where $\beta'+\gamma'=0$, $\delta'+\epsilon'=0$, that is, into the form $x(x^2-\beta'^2)(x^2-\delta'^2)=0$. Or replacing y, if we have

$$(a, b, c, d, e, f \between x, y)^5 = a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\epsilon y),$$

then the equation in question is the condition in order that this may be transformable into the form $(a', 0, c', 0, e', 0 \between x, y)^5$; that is, in order that the 18-thic invariant I may vanish. Hence observing that there are 15 determinants of the form in question, and that any root, for instance α, enters as α^2 in 3 of them and in the simple power α in the remaining 12, we see that the product

$$a^{18}\Pi \begin{vmatrix} 1, & 2\alpha\ , & \alpha^2 \\ 1, & \beta+\gamma, & \beta\gamma \\ 1, & \delta+\epsilon, & \delta\epsilon \end{vmatrix}$$

contains each root in the power 18, and is consequently a rational and integral function of the coefficients of the degree 18, viz. save as to a numerical factor it is equal to the invariant I. And considering the equation $(a, \ldots \between x, y)^3 = 0$ as representing a range of points, the signification of the equation $I = 0$ is that, the pairs (β, γ) and (δ, ϵ) being properly selected, the fifth point α is a focus or sibiconjugate point of the involution formed by the pairs (β, γ) and (δ, ϵ).

Article Nos. 257 to 267.—*Theory of the determination of the Character of an Equation; Auxiliars; Facultative and Non-facultative space.*

257. The equation $(a, b, c \ldots \between x, y)^n = 0$ is a *real* equation if the ratios $a : b : c, \ldots$ of the coefficients are all real. In considering a given real equation, there is no loss of generality in considering the coefficients $(a, b, c \ldots)$ as being themselves real, or in taking the coefficient a to be $= 1$; and it is also for the most part convenient to write $y = 1$, and thus to consider the equation under the form $(1, b, c \ldots \between x, 1)^n = 0$. It will therefore (unless the contrary is expressed) be throughout assumed that the coefficients (including the coefficient a when it is not put $= 1$) are all of them real; and, in speaking of any functions of the coefficients, it is assumed that these are rational and integral real functions, and that any values attributed to these functions are also real.

258. The equation $(1, b, c \ldots \between x, 1)^n = 0$, with α real roots and 2β imaginary roots, is said to have the character $\alpha r + 2\beta i$; thus a quintic equation will have the character $5r$, $3r + 2i$, or $r + 4i$, according as its roots are all real, or as it has a single pair, or two pairs, of imaginary roots.

259. Consider any m functions $(A, B, \ldots K)$ of the coefficients, $(m = \text{ or } < n)$. For given values of $(A, B, \ldots K)$, *non constat* that there is any corresponding equation (that is, the corresponding values of the coefficients $(b, c, \ldots)$ may be of necessity imaginary), but attending only to those values of $(A, B, \ldots K)$ which have a corresponding equation or corresponding equations, let it be assumed that the equations which correspond to a given set of values of $(A, B, \ldots K)$ have a determinate character (one and the same for all such equations): this assumption is of course a condition imposed on the form of the functions $(A, B, \ldots K)$; and any functions satisfying the condition are said to be "auxiliars." It may be remarked that the n coefficients $(b, c, \ldots)$ are themselves auxiliars; in fact for given values of the coefficients there is only a single equation, which equation has of course a determinate character. To fix the ideas we may consider the auxiliars $(A, B, \ldots K)$ as the coordinates of a point in m-dimensional space, or say in m-space.

260. Any given point in the m-space is either "facultative," that is, we have corresponding thereto an equation or equations (and if more than one equation then by what precedes these equations have all of them the same character), or else it is "non-facultative," that is, the point has no corresponding equation.

261. The entire system of facultative points forms a region or regions, and the entire system of non-facultative points a region or regions; and the m-space is thus divided into facultative and non-facultative regions. The surface which divides the

facultative and non-facultative regions may be spoken of simply as the bounding surface, whether the same be analytically a single surface, or consist of portions of more than one surface.

262. Consider the discriminant D, and to fix the ideas let the sign be determined in such wise that D is $+$ or $-$ according as the number of imaginary roots is $\equiv 0$ (mod. 4), or is $\equiv 2$ (mod. 4); then expressing the equation $D=0$ in terms of the auxiliars $(A, B, \ldots K)$, we have a surface, say the discriminatrix, dividing the m-space into regions for which D is $+$, and for which D is $-$, or, say, into positive and negative regions.

263. A given facultative or non-facultative region may be wholly positive or wholly negative, or it may be intersected by the discriminatrix and thus divided into positive and negative regions. Hence taking account of the division by the discriminatrix, but attending only to the facultative regions, we have positive facultative regions and negative facultative regions. Now using the simple term region to denote indifferently a positive facultative region or a negative facultative region, it appears from the very notion of a region as above explained that we may pass from any point in a given region to any other point in the same region without traversing either the bounding surface or the discriminatrix; and it follows that the equations which correspond to the several points of the same region have each of them one and the same character; that is, to a given region there correspond equations of a given character.

264. It is proper to remark that there may very well be two or more regions which have corresponding to them equations with the same character; any such regions may be associated together and considered as forming a kingdom; the number of kingdoms is of course equal to the number of characters, viz. it is $=\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ according as n is even or odd; and this being so, the general conclusion from the preceding considerations is that the whole of facultative space will be divided into kingdoms, such that to a given kingdom there correspond equations having a given character; and conversely, that the equations with a given character correspond to a given kingdom. Hence (the characters for the several kingdoms being ascertained) knowing in what kingdom is situate a point $(A, B, \ldots K)$, we know also the character of the corresponding equations.

265. Any conditions which determine in what kingdom is situate the point $(A, B, \ldots K)$ which belongs to a given equation $(1, b, c \ldots \between x, 1)^n = 0$, determine therefore the character of the equation. It is very important to notice that the form of these conditions is to a certain extent indeterminate; for if to a given kingdom we attach any portion or portions of non-facultative space, then any condition or conditions which confine the point $(A, B, \ldots K)$ to the resulting aggregate portion of space, in effect confine it to the kingdom in question; for of the points within the aggregate portion of space it is only those within the kingdom which have corresponding to them an equation, and therefore, if the coefficients $(b, c, \ldots)$ of the given equation are such as to give to the auxiliars $(A, B, \ldots K)$ values which correspond to a point situate within the above-mentioned aggregate portion of space, such point will of necessity be within the kingdom.

266. In the case where the auxiliars are the coefficients $(b, c, \ldots)$, to any given values of the auxiliars there corresponds an equation, that is, all space is facultative space. And the division into regions or kingdoms is effected by means of the discriminatrix, or surface $D=0$, alone. Thus in the case of the quadric equation $(1, x, y \between \theta, 1)^2=0$ the m-space is the plane. We have $D=x^2-y$, and the discriminatrix is thus the parabola $x^2-y=0$. There are two kingdoms, each consisting of a single region, viz. the positive kingdom or region $(x^2-y=+)$ outside the parabola, and the negative kingdom or region $(x^2-y=-)$ inside the parabola, which have the characters $2r$ and $2i$, or correspond to the cases of two real roots and two imaginary roots, respectively. And the like as regards the cubic $(1, x, y, z \between \theta, 1)^3=0$; the m-space is here ordinary space, $D=-4x^3z+3x^2y^2+6xyz-4y^3-z^2$, and the division into kingdoms is effected by means of the surface $D=0$; but as in this case there are only the two characters $3r$ and $r+2i$, there can be only the two kingdoms $D=+$ and $D=-$ having these characters $3r$ and $r+2i$ respectively, and the determination of the character of the cubic equation is thus effected without its being necessary to proceed further, or inquire as to the form or number of the regions determined by the surface $D=0$: I believe that there are only two regions, so that in this case also each kingdom consists of a single region. But proceeding in the same manner, that is, with the coefficients themselves as auxiliars, to the case of a quartic equation, the m-space is here a 4-dimensional space, so that we cannot by an actual geometrical discussion show how the 4-space is by the discriminatrix or hypersurface $D=0$ divided into kingdoms having the characters $4r$, $2r+2i$, $4i$ respectively. The employment therefore of the coefficients themselves as auxiliars, although theoretically applicable to an equation of any order whatever, can in practice be applied only to the cases for which a geometrical illustration is in fact unnecessary.

267. I will consider in a different manner the case of the quartic, chiefly as an instance of the actual employment of a surface in the discussion of the character of an equation; for in the case of a quintic the auxiliars are in the sequel selected in such manner that the surface breaks up into a plane and cylinder, and the discussion is in fact almost independent of the surface, being conducted by means of the curve (Professor Sylvester's Bicorn) which is the intersection of the plane and cylinder.

Article Nos. 268—273.—*Application to the Quartic equation.*

268. Considering then the quartic equation $(a, b, c, d, e \between \theta, 1)^4=0$ (I retain for symmetry the coefficient a, but suppose it to be $=1$, or at all events positive), then if I, J signify as usual, and if for a moment

$$\vartheta = a^2d-3abc+2b^3,$$
$$X = 3aJ+2(b^2-ac)I,$$

we have identically

$$\tfrac{9}{4}(3a^2J^2+X^2)\vartheta^2 = 9(b^2-ac)^3X^2-a^2(b^2-ac)^3(I^3-27J^2)-a^2X^3$$

(see my paper, "A discussion of the Sturmian Constants for Cubic and Quartic Equations," *Quart. Math. Journ.*, vol. IV. (1861) pp. 7—12), [290]. And I write

$$x = b^2 - ac,$$
$$y = 3aJ + 2(b^2 - ac)I,$$
$$z = I^3 - 27J^2 \; (= D).$$

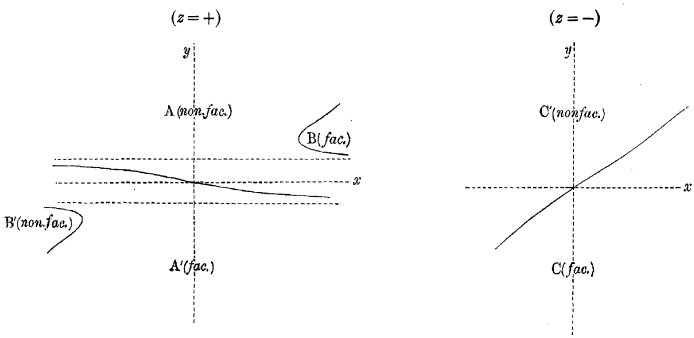
269. I borrow from Sturm's theorem the conclusion (but nothing else than this conclusion) that (x, y, z) possess the fundamental property of auxiliars (that is, that the quartic equations (if any) corresponding to a given system of values of (x, y, z) have one and the same character). The foregoing equation gives $9x^2y^2 - x^3z - y^3 =$ a square function, and therefore positive; that is, the facultative portion of space is that for which $9x^2y^2 - x^3z - y^3$ is $= +$. And the equation

$$x^3(9y^2 - z) - y^3 = 0$$

is that of the bounding surface, dividing the facultative and non-facultative portions of space.

270. To explain the form of the surface we may imagine the plane of xy to be that of the paper, and the positive direction of the axis of z to be in front of the paper. Taking z constant, or considering the sections by planes parallel to that of xy,

$z = 0$, gives $y^2(9x^3 - y) = 0$, viz. the section is the line $y = 0$, or axis of x twice, and the cubical parabola $y = x^3$.



$z = +$, the curve $x^3 = \dfrac{y^2}{9y^2 - z}$ has two asymptotes $y = \pm\frac{1}{3}\sqrt{z}$, parallel to and equidistant from the axis of x, and consists of a branch included between the two parallel asymptotes, and two other portions branches outside the asymptotes, as shown in the figure $(z = +)$.

$z=-$, the curve $x^3=\dfrac{y^3}{9y^3-z}$ has no real asymptote, and consists of a single branch, resembling in its appearance the cubical parabola as shown in the figure $(z=-)$.

Taking x as constant, or considering the sections by planes parallel to that of zy, the equation of the section is $z=9y^3-\dfrac{y^3}{x^3}$, which is a cubical parabola, meeting the plane of xy in a point on the cubical parabola $y=9x^3$, and also in a twofold point on the axis of x, that is, touching the plane of xy at the last-mentioned point.

271. The surface consists of a single sheet extending to infinity, the form of which is most easily understood by considering the sections by a system of spheres having the origin of coordinates for their common centre. These sections have all of them the same general form; and one of them is shown fig. 1 of the Plate at the end of the present Memoir, the projection being orthogonal on the plane of xy or plane of the paper, and the spherical curve being shown, the portion of it above the plane of the paper by a continuous line, that below it by a dotted line (the double point in the figure is thus of course only an apparent one): the same figure shows also the sections by planes parallel to that of xy previously shown in the figures $(z=+)$ and $(z=-)$.

272. Now considering the discriminatrix $D=0$, in this case the plane $z=0$, it appears that the bounding surface and this plane divide space into six regions, viz. above the plane of the paper we have the four regions, A non-facultative, B facultative, A' facultative, B' non-facultative, and below it the two regions, C facultative, C' non-facultative. There are thus in all three facultative regions A', B, C, and since A' and B correspond to $D=+$, these must have the characters $4r$ and $4i$, and it is easy by considering a particular case to show that B has the character $4r$, and A' the character $4i$; C corresponds to $D=-$, and can therefore only have the character $2r+2i$. Hence, for any given equation, (x, y, z) will lie in one of the regions (B, A', C), and if (x, y, z)

is in the region B, the character is $4r$,
" A', " $4i$,
" C, " $2r+2i$.

273. It is right to notice that the determination of the character is really made in what precedes; the determination of the analytical criteria of the different characters is a mere corollary; to obtain these it is only necessary to remark that

$z=+,\ x=+,\ y=+$ includes the whole of facultative region B,

that is, (x, y, z) being each positive, the character is $4r$;

$$\left.\begin{array}{r} z=+,\ x=+,\ y=- \\ x=-,\ y=- \\ x=-,\ y=- \end{array}\right\}\text{ include each a part and together the whole of facultative region } A',$$

that is, z being $+$, but (x, y) not each positive, the character is $4i$;

$$\left.\begin{array}{ll} z=-,\ x=+,\ y=+ \\ \quad\ \text{,,}\quad x=+,\ y=- \\ \quad\ \text{,,}\quad x=-,\ y=- \end{array}\right\}\ \text{include each a part and together the whole of facultative region } C,$$

$$z=-,\ x=-,\ y=+ \quad \text{does not include any facultative space,}$$

that is, z being $-$, the character is $2r+2i$; and the combination of signs $z=-$, $x=-$, $y=+$ is one which does not exist.

The results thus agree with those furnished by Sturm's theorem; and in particular the impossibility of $z=-$, $x=-$, $y=+$ appears from Sturm's theorem, inasmuch as his combination would give a gain instead of a loss of changes of sign.

Article Nos. 274 to 285.—*Determination of the characters of the quintic equation.*

274. Passing now to the case of the quintic, I write

$$\begin{aligned} J &= G, \\ K &= Q, \\ D &= Q', \\ L &= -U, \\ I &= W; \end{aligned}$$

viz. J is the quartinvariant, K and D are octinvariants (D the discriminant), L is 12-thic invariant, and I is the 18-thic or skew invariant. Hence also J, D, $2^{11}L-J^3$ are invariants of the degrees 4, 8, 12 respectively; and forming the combinations

$$x=\frac{2^{11}L-J^3}{J^3},\ y=\frac{D}{J^2},\ z=J,$$

I assume that (x, y, z) are auxiliars, reserving for the concluding articles of the present memoir the considerations which sustain this assumption.

275. The separation into regions is effected as follows:—We have identically (see *ante*, No. 255)

$$16I^2=JK^4+8LK^3-2J^2LK^2-72JL^2K-432L^3+J^3L^2;$$

or putting for K its value $=\frac{1}{128}(J^2-D)$, this is

$$\begin{aligned} 2^{32}I^2 = \quad & J(J^2-D)^4+\&c., \\ = \quad & (J^3-2^{11}L)^2(J^3-3^3\,.\,2^{10}L) \\ & +DJ(-4J^6+61\,.\,2^{10}J^3L+144\,.\,2^{20}L^2) \\ & +D^3J^2(\ \ 6J^3-2^{10}\,.\,29L) \\ & +D^3\ \ (-4J^3-\ \ \ 2^{10}L) \\ & +D^2J; \end{aligned}$$

or writing as above

$$x = \frac{2^{11} L - J^3}{J^3}, \quad y = \frac{D}{J^2},$$

whence also

$$1 + x = \frac{2^{11} L}{J^3},$$

this is

$$\begin{aligned} 2^{22} \frac{I^2}{J^9} = & - x^2 \{ \tfrac{8}{2} (1+x) - 1 \} \\ & + y \{ 36 (1+x)^2 - \tfrac{61}{2} (1+x) - 4 \} \\ & - y^2 \{ \tfrac{22}{2} (1+x) - 6 \} \\ & + y^3 \{ - \tfrac{1}{2} (1+x) - 4 \} \\ & + y^4, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} 2 . 2^{22} \frac{I^2}{J^9} = & - 3x^3 - x^2 \\ & + y (72x^2 + 205x + 125) \\ & + y^2 (-29x - 17) \\ & + y^3 (-x - 9) \\ & + y^4 . 2 \\ = & \quad \phi (x, y) \text{ suppose.} \end{aligned}$$

276. Hence also writing $z = J$, we have

$$z\phi (x, y) = 2 . 2^{22} \frac{I^2}{J^8} = +,$$

or the equation of the bounding surface may be taken to be

$$z\phi (x, y) = 0,$$

that is, the bounding surface is composed of the plane $z = 0$, and the cylinder $\phi (x, y) = 0$. Taking the plane of the paper for the plane $z = 0$, the cylinder meets this plane in a curve $\phi (x, y) = 0$, which is Professor Sylvester's Bicorn: this curve divides the plane into certain regions, and if we attend to the solid figure and instead of the curve consider the cylinder, then to each region of the plane there correspond *in solido* two regions, one in front of, the other behind the plane region, and of these regions *in solido*, one is facultative, the other is non-facultative (viz. for given values of (x, y), whatever be the sign of $\phi (x, y)$, then for a certain sign of z, $z\phi (x, y)$ will be positive or the solid region will be facultative, and for the opposite sign of z, $z\phi (x, y)$ will be negative or the region will be non-facultative). It hence appears that we may attend only to the plane regions, and that (the proper sign being attributed to z, that is to J) each of these may be regarded as facultative. It is to be added that the discriminatrix is in the present case the plane $y = 0$, or, if we attend only to the plane figure, it is the line $y = 0$; so that in the plane figure the separation into regions is effected by means of the Bicorn and the line $y = 0$.

277. Reverting to the equation of the Bicorn, and considering first the form at infinity, the intersections of the curve by the line infinity are given by the equation $y^3(2y-x)=0$, viz. there is a threefold intersection $y^3=0$, and a simple intersection $2y-x=0$; the equation $y^3=0$ indicates that the intersection in question is a point of inflexion, the tangent at the inflexion (or stationary tangent) being of course the line infinity; the visible effect is, however, only that the direction of the branch is ultimately parallel to the axis of x. The equation $2y-x=0$ indicates an asymptote parallel to this line, and the equation of the asymptote is easily found to be $2y-x+5=0$.

278. The discussion of the equation would show that the curve has an ordinary cusp; and a cusp of the second kind, or node-cusp, equivalent to a cusp and node; the curve is therefore a unicursal curve, or the coordinates are expressible rationally in terms of a parameter ϕ; we in fact have

$$x=\frac{-(\phi+2)(\phi^3-\phi^2+2\phi-4)}{\phi^2(\phi+1)},\quad y=\frac{(\phi+2)^2(\phi-3)}{\phi^2(\phi+1)},$$

whence also

$$\frac{dy}{dx}=\tfrac{1}{2}\phi(\phi+2).$$

279. The curve may be traced from these equations (see Plate, fig. 2, where the bicorn is delineated along with a cubic curve afterwards referred to): as ϕ extends from an indefinitely small positive value ϵ through infinity to $-1-\epsilon$, we have the upper branch of the curve, viz.

$\phi=\epsilon$, gives $x=\infty$, $y=-\infty$, point at infinity, the tangent being horizontal,

$\phi=\infty$, gives $x=-1$, $y=+$, the node-cusp, tangent parallel to axis of y,

$\phi=-2$, gives $x=0$, $y=0$, the tangent at this point being the axis of x,

$\phi=-1-\epsilon$, gives $x=\infty$, $y=+$, point at infinity along the asymptote;

and as ϕ extends from $x=-1+\epsilon$ to $x=-\epsilon$, we have the lower branch, viz.

$\phi=-1+\epsilon$, gives $x=-\infty$, $y=-\infty$, point at infinity along the asymptote,

$\phi=-\frac{9}{4}$, $x=-76\frac{23}{27}$, $y=-41\frac{2}{3}$; the cusp, shown in the figure out of its proper position (observe that for $x=-76\frac{23}{27}$, we have for the asymptote $y=-40\frac{25}{27}$, so that the distance below the asymptote is $=\frac{20}{27}$; Professor Sylvester's value $y=-25$ for the ordinate of the cusp is an obvious error of calculation).

$\phi=-\epsilon$, $x=-\infty$, $y=-\infty$, point at infinity, the tangent being horizontal.

The class of the curve is $=4$.

280. The node-cusp counts as a node, a cusp, an inflexion, and a double tangent; the node-cusp absorbs therefore $(6+8+1=)$ 15 inflexions, and the other cusp 8 inflexions; there remains therefore $(24-15-8=)$ 1 inflexion, viz. this is the inflexion at infinity, having the line infinity for tangent; there is not, besides the tangent at the node-cusp, any other double tangent of the curve.

281. The form of the Bicorn, so far as it is material for the discussion, is also shown in the Plate, fig. 3, and it thereby appears that it divides the plane into three regions; viz. these are the regions PQR and S, for each of which $\phi(x, y)$ is $=-$, and the region TU, for which $\phi(x, y)$ is $=+$; that is, for PQR and S we must have $J=-$, and for TU we must have $J=+$. Hence in connexion with the bicorn, considering the line $y=0$, we have the six regions P, Q, R, S, T, U. It has just been seen that for P, Q, R, S we have $J=-$, and for T, U we have $J=+$; and the sign of J being given, the equations $x=\frac{2^{11}L-J^3}{J^3}$, $y=\frac{D}{J^3}$, then fix for the several regions the signs of $2^{11}L-J^3$ and D, as shown in the subjoined Table; by what precedes each of the six regions has a determinate character, which for R, S, and U (since here D is $=-$) is at once seen to be $3r+2i$, and which, as will presently appear, is ascertained to be $5r$ for P and $r+4i$ for Q and T.

282. We have thus the Table

$$\begin{array}{llll} P, & D=+, & J=-, & 2^{11}L-J^3=+ \} 5r, \\ Q, & D=+, & J=-, & 2^{11}L-J^3=- \\ T, & D=+, & J=+, & 2^{11}L-J^3=\pm \} r+4i. \\ R, & D=-, & J=-, & 2^{11}L-J^3=\pm \\ S, & D=-, & J=-, & 2^{11}L-J^3=+ \} 3r+2i; \\ U, & D=-, & J=+, & 2^{11}L-J^3=\pm \end{array}$$

so that we have the kingdom $5r$ consisting of the single region P, the kingdom $r+4i$ consisting of the regions Q and T, and the kingdom $3r+2i$ consisting of the regions R, S, and U.

283. For a given equation if D is $=-$, the character is $3r+2i$; if $D=+$, $J=+$, the character is $r+4i$; if $D=+$, $J=-$, then, according as $2^{11}L-J^3$ is $=+$ or is $=-$, the character is $5r$ or $r+4i$. But in the last case the distinction between the characters $5r$ and $r+4i$ may be presented in a more general form, involving a parameter μ, arbitrary between certain limits. In fact drawing upwards from the origin, as in Plate, fig. 3, the lines $x-2y=0$ and $x+y=0$, and between them any line whatever $x+\mu y=0$, the point (x, y), assumed to lie in the region P or Q, will lie in the one or the other region according as it lies on the one side or the other side of the line in question, viz. in the region P if $x+\mu y$ is $=-$, in the region Q if $x+\mu y$ is $=+$. But we have

$$x+\mu y=\frac{2^{11}L-J^3+\mu JD}{J^3},$$

and J being by supposition negative, the sign of $2^{11}L-J^3+\mu JD$ is opposite to that of $x+\mu y$. The region is thus P or Q according to the sign of $2^{11}L-J^3+\mu JD$; and completing the enunciation, we have, finally, the following criteria for the number of real roots of a given quintic equation, viz.

If $D=-$, the character is $3r+2i$,
If $D=+$, $J=+$, then it is $r+4i$.

But if $D=+$, $J=-$, then μ being any number at pleasure between the limits $+1$ and -2, both inclusive, if

$$2^{11}L-J^3+\mu JD=+, \text{ the character is } 5r,$$
$$2^{11}L-J^3+\mu JD=-, \quad \text{"} \quad \text{"} \quad \text{"} \quad r+4i.$$

284. The characters $5r$ of the region P and $r+4i$ of the regions Q and T may be ascertained by means of the equation $(a, 0, c, 0, e, 0 \between \theta, 1)^5=0$, that is

$$\theta(a\theta^4+10c\theta^2+5e)=0;$$

there is always the real root $\theta=0$, and the equation will thus have the character $5r$ or $r+4i$ according as the reduced equation $a\theta^4+10c\theta^2+5e=0$ has the character $4r$ or $4i$. It is clear that (a, e) must have the same sign, for otherwise θ^2 would have two real values, one positive, the other negative, and the character would be $2r+2i$. And (a, e) having the same sign, then the character will be $4r$, if θ^2 has two real positive values, that is, if $ae-5c^2$ is $=-$, and the sign of c be opposite to that of a and e, or, what is the same thing, if ce be $=-$; but if these two conditions are not satisfied, then the values of θ^2 will be imaginary, or else real and negative, and in either case the character will be $4r$.

285. Now, for the equation in question, putting in the Tables $b=d=f=0$, we find

$$\begin{aligned} D &= 256\, ae^3(ae-5c^2)^2, \\ J &= 16\, ce(ae+3c^2), \\ 2^{11}L-J^3 &= 2^{12}ce^3\left[2(ae-c^2)^4-c^2(ae+3c^2)^3\right] \\ &= 2^{13}ce^3(ae-5c^2)(2a^3e^3+a^2c^2e^2+8ac^4e+5c^6). \end{aligned}$$

We have by supposition $D=+$, that is, $ae=+$; hence J has the same sign as ce; whence if $J=+$, then also $ce=+$, and the character is $4i$; that is the character of the region T is $r+4i$. But if $J=-$, then also $ce=-$. But ae being $=+$, the sign of $2^{11}L-J^3$ is the same as that of $ce(ae-5c^2)$, and therefore the opposite of that of $ae-5c^2$: hence $D=+$, $J=-$, the quartic equation has the character $4r$ or $4i$ according as $2^{11}L-J^3$ is $=+$ or $=-$. Hence the region P has the character $5r$ and the region Q the character $r+4i$; and the demonstration is thus completed.

Article Nos. 286 to 293.—HERMITE'S *new form of* TSCHIRNHAUSEN'S *transformation, and application thereof to the quintic.*

286. M. Hermite demonstrates the general theorem, that if $f(x, y)$ be a given quantic of the n-th order, and $\phi(x, y)$ any covariant thereof of the order $n-2$, then considering the equation $f(x, 1)=0$, and writing

$$z=\frac{\phi(x, 1)}{f_x'(x, 1)}$$

(where $f'_x(x, 1)$ is the derived function of $f(x, 1)$ in regard to x), then eliminating x, we have an equation in z, the coefficients whereof are all of them invariants of $f(x, y)$.

287. In particular for the quintic $f(x, y) = (a, b, c, d, e, f \between x, y)^5$, if

$$\phi_1(x, y), \quad \phi_2(x, y), \quad \phi_3(x, y), \quad \phi_4(x, y)$$

are any four covariant cubics, writing

$$z = \frac{t\,\phi_1(x, 1) + u\,\phi_2(x, 1) + v\,\phi_3(x, 1) + w\,\phi_4(x, 1)}{f'_x(x, 1)}$$

(viz. the numerator is a covariant cubic involving the indeterminate coefficients t, u, v, w) then, in the transformed equation in z, the coefficients are all of them invariants of the given quintic. Conducting the investigation by means of a certain canonical form, which will be referred to in the sequel, he fixes the signification of his four covariant cubics, these being respectively covariant cubics of the degrees 3, 5, 7, and 9, defined as follows; viz. starting with the form

$$-3 \begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a\,, & b\,, & c\,, & d \\ b\,, & c\,, & d\,, & e \\ c\,, & d\,, & e\,, & f \end{vmatrix},$$

$$= -3D, \; = -3(A, B, C, D \between x, y)^3, \text{ or } (-3A, -B, -C, -3D \between x, y)^3, \text{ suppose,}$$

and considering also the quadric covariant

$$(\alpha, \beta, \gamma \between x, y)^2, = B,$$

then $\phi_1, \phi_2, \phi_3, \phi_4$ are derived from the form

$$(A, B, C, D \between \zeta x - \eta(\beta x + 2\gamma y), \; \zeta y + \eta(2\alpha x + \beta y))^3,$$

viz. we have

$$\begin{aligned} \phi_1(x, y) &= -3(A, B, C, D \between x, y)^3, \\ \phi_2(x, y) &= +3(A, B, C, D \between x, y)^2(-\beta x - 2\gamma y, \; 2\alpha x + \beta y), \\ \{\phi_3(x, y)\} &= -3(A, B, C, D \between x, y)\,(-\beta x - 2\gamma y, \; 2\alpha x + \beta y)^2, \\ \{\phi_4(x, y)\} &= +3(A, B, C, D \between \quad -\beta x - 2\gamma y, \; 2\alpha x + \beta y)^3, \end{aligned}$$

where $\{\phi_3(x, y)\}$ and $\{\phi_4(x, y)\}$ are the functions originally called by him $\phi_3(x, y)$ and $\phi_4(x, y)$: those ultimately so called by him are

$$\begin{aligned} (^1)\ \phi_3(x, y) &= 4\{\phi_3(x, y)\} + J\phi_1(x, y), \qquad (J = G), \\ \phi_4(x, y) &= 4\{\phi_4(x, y)\} + 3J\phi_2(x, y) + 96\psi_1(x, y), \end{aligned}$$

[1] M. Hermite, p. 17, has erroneously written $\phi_3(x, y) + 4A\phi_1(x, y)$, instead of $4\phi_3(x, y) + A\phi_1(x, y)$; the latter expression is that which he really makes use of, and the formula in the text is correct.

where $\psi_1(x, y)$ is the cubicovariant $(-27A^2D+9ABC-2B^3, \ldots \between x, y)^3$ of $\phi_1(x, y)$, $=(-3A, -B, -C, -3D \between x, y)^3$, *ut suprà*.

The covariant $\phi_3(x, y)$ has the property that if the given quintic $(a, \ldots \between x, y)^5$ contains a square factor $(lx+my)^2$, then $\phi_3(x, y)$ contains the factor $lx+my$: $\{\phi_3(x, y)\}$ and $\{\phi_4(x, y)\}$ are covariants not possessing the property in question, and they were for this reason replaced by $\phi_3(x, y)$ and $\phi_4(x, y)$ which possess it, viz. $\phi_3(x, y)$ contains the factor $lx+my$, and $\phi_4(x, y)$ contains $(lx+my)^3$, being thus a perfect cube when the given quintic contains a square factor.

288. The covariants $\phi_1(x, y)$ and $\phi_2(x, y)$ are included in my Tables, viz. we have

$$\phi_1(x, y) = -3D \text{ of } 142, 143,$$
$$\phi_2(x, y) = -\ K \quad \text{,,} \quad \text{,,}$$

(observe that in K the first coefficient vanishes if $a=0$, $b=0$, which is the property just referred to of $\phi_2(x, y)$); the other two covariants, as being of the degree 7 and 9, are not included in my Tables, but I have calculated the leading coefficients of these covariants respectively, viz.

Table No. 85 gives leading coefficient (or that of x^3) in $\phi_3(x, y)$, and

Table No. 86 gives leading coefficient (or that of x^3) in $\phi_4(x, y)$ [and by means thereof we have the values of the covariants in question].

The coefficients in question vanish for $a=0$, $b=0$, that is, $\phi_3(x, y)$ and $\phi_4(x, y)$ then each of them contain the factor y; if the remaining coefficients of $\phi_4(x, y)$ were calculated, it should then appear that for $a=0$, $b=0$, those of x^2y, xy^2 would also vanish, and thus that $\phi_4(x, y)$ would be a mere constant multiple of y^3.

Table No. 85 [= leading coefficient of $16BJ-15DG$].

a^3cef^2 + 1	$a^2b^3ef^2$ − 1	ab^3df^2 + 64	b^4gf^2 ...
$a^3d^2f^2$ + 15	a^2bcdf^2 − 94	ab^3e^2f − 54	b^4def − 144
a^3de^2f − 32	a^2bce^2f + 86	$ab^2c^2f^2$ − 48	b^4e^3 + 135
a^3e^4 + 16	a^2bd^2ef + 106	ab^2cdef + 184	b^3c^2ef + 108
	a^2bde^3 − 96	ab^2ce^3 − 135	b^3cd^2f + 288
	$a^2c^2f^2$ + 63	ab^2d^3f − 272	b^3cde^2 − 450
	a^2c^2def − 188	$ab^2d^2e^2$ + 243	b^3d^2e + 80
	$a^2c^2e^3$ + 32	abc^2ef − 66	b^2c^2df − 360
	a^2cd^3f + 60	abc^2d^2f + 212	$b^2c^2e^2$ + 135
	$a^2cd^2e^2$ + 68	abc^2de^2 + 148	$b^2c^2d^2e$ + 360
	a^2d^4e − 36	$abcd^3e$ − 412	b^2cd^4 − 160
		abd^5 + 144	bc^3f + 108
		ac^4df − 36	bc^4de − 180
		ac^4e^2 − 48	bc^3d^3 + 80
		ac^3d^2e + 124	
		ac^2d^4 − 48	
± 32	± 415	± 1119	± 1294

Table No. 86 [= leading coefficient of S'].

a^4cef^3	+ 9	a^3bef^3	− 9	$a^2b^2df^3$	+ 120	ab^4ef^3	− 576	b^5f^3	+ 192
$a^4d^2f^2$	+ 21	a^3bcdf^3	− 162	$a^2b^2e^2f^2$	− 21	ab^4def^2	+ 672	b^5cef^2	− 1440
a^4def^2	− 78	$a^3bce^2f^2$	+ 99	$a^2b^2c^2f^3$	+ 486	ab^4e^2f	− 359	$b^5d^2f^2$	− 192
a^4e^4f	+ 48	$a^3bd^2ef^2$	+ 309	$a^2b^2cdef^2$	− 2160	$ab^3c^2ef^2$	+ 3456	b^5de^2f	− 1080
		a^3bde^3f	+ 12	$a^2b^2ce^2f$	+ 1023	$ab^3cd^2f^2$	− 864	b^5e^4	+ 2025
		a^3be^3	− 240	$a^2b^3d^2f^2$	+ 120	ab^2cde^2f	+ 2094	$b^4c^2df^2$	+ 2592
		$a^3c^3f^2$	− 81	$a^2b^2d^2e^2f$	− 1053	ab^2ce^4	− 3915	$b^4c^2e^2f$	+ 3546
		$a^3c^2def^2$	+ 1026	$a^2b^2de^4$	+ 1314	ab^2d^2ef	+ 528	b^4cd^2ef	+ 5280
		$a^3c^2e^2f$	− 768	$a^2bc^2ef^2$	− 1863	ab^2de^3	− 45	b^4cde^3	− 13500
		$a^3cd^2f^2$	− 738	$a^2bc^2d^2f^2$	+ 2538	$ab^2c^2df^2$	− 2592	b^4d^4f	− 4800
		$a^3cd^2e^2f$	− 564	$a^2bc^2de^2f$	+ 2340	$ab^2c^2e^2f$	− 9747	$b^4d^2e^2$	+ 7800
		a^3cde^4	+ 1056	$a^2bc^2e^4$	+ 672	$ab^2c^2d^2ef$	− 8496	$b^3c^4f^2$	− 648
		a^3d^4ef	+ 756	a^2bcd^2ef	+ 2820	$ab^2c^2de^3$	+ 26610	b^3c^3def	− 14040
		$a^3d^3e^3$	− 696	$a^2bcd^2e^2$	− 7812	ab^2cd^4f	+ 8544	$b^3c^3e^3$	+ 3075
				a^2bd^3f	− 3024	ab^2cd^4e	− 16650	$b^3c^2d^3f$	+ 9120
				$a^2bd^3e^2$	+ 4572	ab^2d^5e	+ 720	$b^3c^2d^2e^2$	+ 16350
				$a^2c^4df^2$	− 324	abc^5f^2	+ 972	b^3cd^4e	− 19200
				$a^2c^4e^2f$	+ 3888	abc^4def	+ 24048	b^3d^6	+ 4800
				$a^2c^3d^2ef$	− 8748	abc^4e^3	− 4464	b^2c^5ef	+ 4860
				$a^2c^3de^3$	− 4800	abc^3d^3f	− 15984	$b^2c^4d^2f$	− 3240
				$a^2c^2d^4f$	+ 4248	$abc^3d^2e^2$	− 30108	$b^2c^4de^2$	− 8100
				$a^2c^2d^3e^2$	+ 14520	abc^2d^4e	+ 35088	$b^2c^3d^5e$	+ 9000
				a^2cd^5e	− 11448	$abcd^6$	− 8640	bc^2d^6	− 2400
				a^2d^7	+ 2592	ac^6ef	− 7776		
						ac^5d^2f	+ 5184		
						ac^5de^3	+ 12960		
						ac^4d^3e	− 14400		
						ac^3d^5	+ 3840		
	± 78		± 3258		± 41253		± 124716		± 68640

[The values thus are $\phi_5(x, y) = 16BJ - 15DG$; $\phi_4(x, y) = S'$.]

289. The equation in z is of the form

$$z^5 + \frac{\mathfrak{A}}{D}z^3 + \frac{\mathfrak{B}}{D}z^2 + \frac{\mathfrak{C}}{D}z + \frac{\mathfrak{D}}{D} = 0,$$

where D is the discriminant of the quintic and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ denote rational and integral functions of the coefficients (a, b, c, d, e, f). And the covariants $\phi_1(x, y)$, $\phi_2(x, y)$, $\phi_3(x, y)$, $\phi_4(x, y)$ having the values given to them above, the actual value of $\mathfrak{A}$ is obtained as a quadric function of the indeterminates (t, u, v, w), viz. this is

$$= [D_1t^2 - 6BDtv - D(D_1 - 10AB)v^2] + D[-Bu^2 + 2D_1uw + 9(BD - 10AD_1)w^2],$$

where $D_1 = 25AB + 16C$, these quantities, and the quantity $N(= D_1^2 - 10ABD_1 + 9B^2D)$ afterwards spoken of, being in the notation of the present Memoir as follows:

$$\begin{aligned}
A &= J && (= G),\\
B &= -K && (= -Q),\\
C &= 9L + JK && (= -9U + GQ),\\
D &= D && (= Q'),\\
D_1 &= 9(16L - JK),\\
N &= 1152(18L^2 - JKL - K^3).
\end{aligned}$$

290. If by establishing two linear relations between the coefficients (t, u, v, w) the equation $\mathfrak{A}=0$ can be satisfied (which in fact can be done by the solution of a quadric equation), then these quantities can be by means of the relations in question expressed as linear functions of any two of them, say of v and w; and then the next coefficient $\mathfrak{B}$ will be a cubic function $(v, w)^3$, and the equation $\mathfrak{B}=0$ will be satisfied by means of a cubic equation $(v, w)^3=0$, that is, the transformed equation in z can be by means of the solution of a quadric and a cubic equation reduced to the trinomial form

$$z^5+\frac{\mathfrak{C}}{D}z+\frac{\mathfrak{D}}{D}=0,$$

and M. Hermite shows that the equation $\mathfrak{A}=0$ can be satisfied as above very simply, and that in two different ways, viz.

291. 1°. $\mathfrak{A}=0$ if

$$D_1t^2-6BDtv-(\quad D_1-10AB)\,v^2=0,$$
$$Bv^2-2D_1uw-(9BD\ -10AD_1)\,w^2=0,$$

that is, N denoting as above, if

$$t=\frac{3BD+\sqrt{ND}}{D_1}v,\qquad u=\frac{D_1+\sqrt{N}}{B}w.$$

292. 2°. Writing the expression for $\mathfrak{A}$ in the form

$$D_1(t^2-Dv^2+2Duw-10ADw^2)+BD(10Av^2-6tv-u^2+9Dw^2),$$

then $\mathfrak{A}=0$, if

$$t^2-Dv^2+2Duw-10ADw^2=0,$$
$$10Av^2-6tv-u^2+\ 9\ \ Dw^2=0.$$

These equations, writing therein

$$t=\frac{1}{\sqrt{2}}\sqrt{D}T,\quad u=U+5AW,\quad v=\frac{1}{\sqrt{2}}V,\quad w=W,$$

become

$$T^2-V^2+4UW=0,$$
$$-5AV^2+3\sqrt{D}TV+U^2+10AUW+(25A^2-9D)\,W^2=0,$$

the first of which is satisfied by the values

$$T=\rho\,W-\frac{1}{\rho}U,\qquad V=\rho\,W+\frac{1}{\rho}U;$$

and then substituting for T and V, the second equation will be also satisfied if only

$$\rho^2=5A+3\sqrt{D}.$$

Article Nos. 293 to 295.—HERMITE'S *application of the foregoing results to the determination of the Character of the quintic equation.*

293. By considerations relating to the form

$$\frac{1}{D}\left\{[D_1t^2-6BDtv-D(D_1-10AB)v^2]+D[-Bu^2+2D_1uw+9BD-10AD_1w^2]\right\},$$

M. Hermite obtains criteria for the character of the quintic equation $f(x, 1)=0$.

294. If $D=-$, the character is $3r+2i$, but if $D=+$, then expressing the foregoing form as a sum of four squares affected with positive or negative coefficients, the character will be $5r$ or $2+4i$, according as the coefficients are all positive, or are two positive and two negative. Whence, if N denote as above, then for

$$D=+,\ N=-,\quad D_1=+,\ B=-,\ \text{character is } 5r,$$

and

$$\left.\begin{array}{l} D=+,\ N=-,\ BD_1=+ \\ \\ D=+,\ N=+ \end{array}\right\}\quad \text{character is } r+4i;$$

and further, the combination $D=+$, $N=-$, $D_1=-$, $B=+$ cannot arise (Hermite's first set of criteria).

295. Again, from the equivalent form

$$\frac{1}{D}\left\{D_1(t^2-Dv^2+2Duw-10Aw^2)+BD(10Av^2-6tv-u^2+9Dw^2)\right\},$$

which, if ω, ω' are the roots of the equation $9\theta^2-10A\theta+D=0$, is

$$=\frac{1}{D}\left\{\frac{D_1\omega-BD}{\omega-\omega'}\left[(t-3\omega'v)^2-\omega'\left(u-\frac{D}{\omega'}w\right)^2\right]+\frac{D_1\omega'-BD}{\omega'-\omega}\left[(t-3\omega v)^2-\omega\left(u-\frac{D}{\omega}w\right)\right]^2\right\};$$

then by similar reasoning it is concluded that

$$D=+,\ 25A^2-9D=+,\ A=-,\ N=-,\ \text{character is } 5r,$$

$$\left.\begin{array}{l} D=+,\ 25A^2-9D=+,\ A=-,\ N=+, \\ D=+,\ 25A^2-9D=+,\ A=+, \\ D=+,\ 25A^2-9D=-, \end{array}\right\}\quad \text{,,}\qquad r+4i.$$

(Hermite's second set of criteria.)

Article Nos. 296 to 303.—*Comparison with the Criteria* No. 283: *the Nodal Cubic.*

296. For the discussion of Hermite's results, it is to be observed that in the notation of the present Memoir we have

$$\begin{aligned} A &= J, \\ B &= -K = -\tfrac{1}{128}(J^2 - D), \\ D &= D, \\ D_1 &= 16L - JK = \tfrac{1}{128}(2^{11}L - J^3 + JD), \\ N &= 18L^2 - JKL - K^3 \\ &= \frac{1}{2^{21}}\{3^3 . 2^{22} L^2 - 14JL(J^2 - D) - (J^2 - D)^3\}, \end{aligned}$$

or, putting as above,

$$x = \frac{2^{11}L - J^3}{J^2},\ y = \frac{D}{J^2},\ \text{and therefore}\ 1 + x = \frac{2^{11}L}{J^3},\ 1 - y = \frac{J^2 - D}{J^2},$$

we have

$$\begin{aligned} A &= J, \\ B &= \tfrac{1}{128}J^2(y-1), \\ D &= J^2 y, \\ D_1 &= \tfrac{1}{128}J^3(x+y), \\ N &= \frac{1}{2^{21}}J^6\{9(1+x)^2 - 8(1+x)(1-y) - (1-y)^3\}, \\ &= \frac{1}{2^{21}}J^6 . \{y^3 - 3y^2 + 8xy + 9x^2 + 11y + 10x\}. \end{aligned}$$

It thus becomes necessary to consider the curve

$$\psi(x, y) = y^3 - 3y^2 + 8xy + 9x^2 + 11y + 10x = 0,$$

the equation whereof may also be written

$$9x + 4y + 5 = (y-1)\sqrt{25 - 9y}.$$

297. This is a cubic curve, viz. it is a divergent parabola having for axis the line $9x + 4y + 5 = 0$, and its ordinates parallel to the axis of x; and having moreover a node at the point $x = -1$, $y = +1$, that is, at the node-cusp of the bicorn; the curve is thus a nodal cubic; we may trace it directly from the equation, but it is to be noticed that *quà* nodal cubic it is a unicursal curve; the coordinates x, y are therefore rationally expressible in terms of a parameter ψ; and it is easy to see that we in fact have

$$\begin{aligned} 81(x+1) &= \psi^2(\psi - 8), \\ 9(y-1) &= -\psi(\psi - 8), \end{aligned}$$

whence also

$$\frac{dy}{dx} = \frac{-18(\psi - 4)}{\psi(3\psi - 16)}.$$

298. We see that

$\psi=\infty$, gives	$x=\infty$,	$y=-\infty$,	point at infinity, the direction of the curve parallel to axis of x.
$\psi=9$, "	$x=0$,	$y=0$,	the origin.
$\psi=8$, "	$x=-1$,	$y=+1$,	the node, tangent parallel to axis of y.
$\psi=\frac{16}{3}$, "	$x=\frac{4325}{2187}$,	$y=\frac{209}{81}$,	tangent parallel to the axis of y.
$\psi=4$, "	$x=-\frac{145}{81}$,	$y=\frac{25}{9}$,	tangent parallel to axis of x.
$\psi=0$, "	$x=-1$,	$y=+1$,	the node.
$\psi=-1$, "	$x=-\frac{10}{9}$,	$y=0$.	
$\psi=-16$, "	$x=-76\frac{23}{27}$,	$y=-41\frac{2}{3}$,	the cusp of the bicorn.
$\psi=-\infty$, "	$x=-\infty$,	$y=-\infty$,	point at infinity, direction of curve parallel to axis of x.

299. The Nodal Cubic is shown along with the Bicorn, Plate, fig. 2; it consists of one continuous line, passing from a point at infinity, through the cusp of the bicorn, on to the node-cusp, then forming a loop so as to return to the node-cusp, again meeting the bicorn at the origin, and finally passing off to a point at infinity, the initial and ultimate directions of the curve being parallel to the axis of x.

300. It may be remarked that, inasmuch as one of the branches of the cubic touches the bicorn at the node-cusp, the node-cusp counts as $(4+2=)$ 6 intersections; the intersections of the cubic with the bicorn are therefore the cusp, the node-cusp, and the origin, counting together as $(2+6+1=)$ 9 intersections, and besides these the point at infinity on the axis of x, counting as 3 intersections. This may be verified by substituting in the equation of the cubic the bicorn ϕ-values of x and y. But to include all the proper factors, we must first write the equation of the cubic in the homogeneous form

$$(9x+8y+5z)^2 z-(y-z)^2(25z-9y)=0,$$

and herein substitute the values

$$x : y : z=-(\phi+2)(\phi^3-\phi^2+2\phi-4) : (\phi+2)^2(\phi-3)\phi : (\phi+1)\phi^2;$$

the result is found to be

$$\phi^3\{(\phi+1)(4\phi^2+6\phi-9)^2-(2\phi+3)^2(4\phi^3+4\phi^2+18\phi+27)\}=0,$$

that is

$$-9\phi^3(\phi+2)(4\phi+3)^2=0;$$

and considering this as an equation of the order 12, the roots are $\phi=0$, 3 times, $\phi=-2$, 1 time; $\phi=-\frac{3}{4}$, 2 times, and $\phi=\infty$, 6 times.

301. The cubic curve divides the plane into 3 regions, which may be called respectively the loop, the antiloop, and the extra cubic; for a point within the loop or antiloop, $\psi(x, y)$ is $=-$, for a point in the extra cubic $\psi(x, y)$ is $=+$. If in conjunction with the cubic we consider the discriminatrix, or line $y=0$, then we have in all six regions, viz. y being $=+$, three which may be called the loop, the triangle,

and the upper region; and y being $=-$, three which may be called the right, left, and under regions respectively; the triangle and the other region form together the antiloop.

302. It is now easy to discuss Hermite's two sets of criteria; the first set becomes

$$\begin{array}{llll} y=+, & y-1=-, & J(x+y) \quad =+, & \psi(x,\ y)=-, \text{ character } 5r, \\ y=+, & & J(y-1)(x+y)=+, & \left.\begin{array}{l}\psi(x,\ y)=-\\ \psi(x,\ y)=+\end{array}\right\} \text{ character } r+4i, \\ y=+, & & & \\ y=+, & y-1=+, & J(x+y) \quad =-, & \psi(x,\ y)=-, \text{ cannot exist.} \end{array}$$

Referring to the Plate, fig. 4, which shows a portion of the cubic and the bicorn, then 1° the conditions $y=+, \psi(x,\ y)=-$ imply that the point $(x,\ y)$ is within the loop or within the triangle of the cubic; the condition $y-1=-$ brings it to be within the triangle, and for any point within the triangle we have $x+y=-$, whence also the condition $J(x+y)=+$ becomes $J=-$; hence the conditions amount to $J=-$, $(x,\ y)$ within the triangle; but by the general theory $(x,\ y)$, being within the triangle, that is, in the region P or T, if $J=-$, will of necessity be within the region P; so that the conditions give $J=-$, $(x,\ y)$ within the region P; the corresponding character being $5r$, which is right.

2°. $y=+$, $\psi(x,\ y)=-$, the point $(x,\ y)$ must be within the loop, or within the triangle; if $(x,\ y)$ is within the loop, then $y-1=+$, $x+y=1$, and the condition $J(y-1)(x+y)=+$ becomes $J=-$, that is, we have $J=-$ and $(x,\ y)$ within the loop, that is, in the region T. And again, if $(x,\ y)$ be within the triangle, then $y-1=-$, $x+y=+$, and the condition $J(y-1)(x+y)=+$ still gives $J=-$; but $J=-$, and $(x,\ y)$ within the triangle, that is, in the region T or P, will of necessity be in the region T; so that in either case we have $J=-$, $(x,\ y)$ in the region T, which agrees with the character $r+4i$.

3°. $y=+$, $\psi(x,\ y)=+$, $(x,\ y)$ is in the upper region, that is, in the region Q or T; if $(x,\ y)$ is in the region Q, then of necessity $J=-$, and if in the region T, then of necessity $J=+$, that is, we have

$$J=-,\ (x,\ y) \text{ in the region } Q, \text{ or}$$
$$J=+,\ (x,\ y) \text{ in the region } P,$$

which agrees with the character $r+4i$.

And it is to be observed that the portions of T under 2° and 3° respectively make up the whole of the region T, and that 3° relates to the whole of the region Q, so that the conditions allow the point $(x,\ y)$ to be anywhere in Q or T, which is right.

4°. $y=+$, $\psi(x,\ y)=-$, $(x,\ y)$ is in the loop or the triangle, and then $y-1=+$ implies that it is in the loop, whence $x+y=+$, and the condition $J(x+y)=-$ becomes $J=-$; we should therefore if the combination existed have $J=-$, $(x,\ y)$ within the loop, that is, in the region T; but this is impossible.

303. Hermite's second set of criteria are

$$\begin{array}{llll} y=+, & \frac{25}{9}-y=+, & J=-, & \psi(x,y)=-, \text{ character } 5r, \\ y=+, & \frac{25}{9}-y=+, & J=-, & \psi(x,y)=+ \\ y=+, & \frac{25}{9}-y=+, & J=+, & \\ y=+, & \frac{25}{9}-y=-, & & \end{array} \Bigg\} \text{ character } r+4i.$$

1°. If $y=+$, $\psi(x, y)=-$, then the point (x, y) must be situate within the loop or within the triangle; and recollecting that at the highest point of the loop we have $y=\frac{25}{9}$, the condition $\frac{25}{9}-y=+$ is satisfied for every such point, and may therefore be omitted. The conditions therefore are $J=-$, (x, y) within the loop, that is, in the region T, or within the triangle, that is, in the region P or the region T; but for any point of T the general theory gives $J=+$, and the conditions are therefore $J=-$, (x, y) within the region P; which agrees with the character $5r$.

2°. $y=+$, $\psi(x, y)=+$, that is, (x, y) is within the upper region, that is, in the region Q or T; and $\frac{25}{9}-y=+$, (x, y) will be within the portions of Q and T which lie beneath the line $y=\frac{25}{9}$; but $J=-$, and therefore (x, y) cannot lie in the region T; hence the conditions amount to $J=-$, (x, y) within that portion which lies beneath the line $y=\frac{25}{9}$ of the region Q.

3°. $y=+$, $\frac{25}{9}-y=+$, (x, y) lies beneath the line $y=\frac{25}{9}$, viz. in one of the regions P, Q or T; but $J=+$, (x, y) cannot lie in the region P or Q; hence the conditions give $J=+$, (x, y) within the portion which lies beneath the line $y=\frac{25}{9}$ of the region T.

4°. $y=+$, $\frac{25}{9}-y=-$, that is, (x, y) lies above the line $y=\frac{25}{9}$, and therefore in one of the regions T or Q; and by the general theory, according as (x, y) lies in T or in Q, we shall have $J=+$ or $J=-$, hence the conditions give

$J=-$, (x, y) within the portion which lies above the line $y=\frac{25}{9}$, of the region Q.

$J=+$, (x, y) within the portion which lies above the line $y=\frac{25}{9}$, of the region T.

2°, 3°, and 4°, each of them agree with the character $r+4i$, and together they imply $J=-$, (x, y) anywhere in the region Q, or else $J=+$, (x, y) anywhere in the region T; which is right.

Article Nos. 304 to 307.—HERMITE'S *third set of Criteria; comparison with* No. 283, *and remarks.*

304. In the concluding portion of his memoir, M. Hermite obtains a third set of criteria for the character of a quintic equation; this is found by means of the equation for the function

$$a^4(\theta_0-\theta_1)(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_4)(\theta_4-\theta_0)$$

of the roots $(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$ of the given quintic equation $(a, b, c, d, e, f \between \theta, 1)^5=0$. The function in question has 12 pairs of equal and opposite values, or it is determined

by an equation of the form $(u^2, 1)^{12}=0$, which equation is decomposable, not rationally but by the adjunction thereto of the square root of the discriminant, into two equations of the form $(u^3, 1)^8=0$; viz. one of these is

$$\begin{aligned}
& u^{12} \\
& + u^{10}(\mathrm{a}+3\sqrt{\Delta}) \\
& + u^8\,[\tfrac{1}{4}(\mathrm{a}-\sqrt{\Delta})^2+\Delta] \\
& - u^6\ \mathrm{d} \\
& + u^4\,[\tfrac{1}{4}(\mathrm{a}+\sqrt{\Delta})^2+\Delta]\,\Delta \\
& + u^2\,(\mathrm{a}-3\sqrt{\Delta})\,\Delta^2 \\
& + \quad \Delta^3=0,
\end{aligned}$$

and the other is of course derived from it by reversing the sign of $\sqrt{\Delta}$. I have in the equation written (a, d) instead of Hermite's writing capitals A, D; the sign $-$ of the term in u^6 instead of $+$, as printed in his memoir, is a correction communicated to me by himself. The signification of the symbols is in the author's notation

$$\begin{aligned}
\mathrm{a} &= 5^4 A, \\
\mathrm{d} &= 4\,.\,5^9(AD-\tfrac{80}{9}D_1), \\
\Delta &= 5^5 D,
\end{aligned}$$

whence, in the notation of the present memoir, the expressions of these symbols are

$$\begin{aligned}
\mathrm{a} &= \ \ 5^4 J, \\
\mathrm{d} &= -\tfrac{1}{2}5^{10}(2^{11}L-J^3-\tfrac{3}{5}JD), \\
\Delta &= \ \ 5^5 D.
\end{aligned}$$

305. From the equation in u, taking therein the radical $\sqrt{\Delta}$ as positive, M. Hermite obtains ($\mathrm{d}<0$ a mistake for $\mathrm{d}>0$) the following as the necessary and sufficient conditions for the reality of all the roots,

$$\Delta=+,\quad \mathrm{a}+3\sqrt{\Delta}=-,\quad \mathrm{d}=+,\quad \text{character } 5r$$

(Hermite's third set of criteria).

306. It is clear that $\mathrm{a}+3\sqrt{\Delta}=-$ is equivalent to ($\mathrm{a}=-$ and $\mathrm{a}^2-9\Delta=+$), and we have $\mathrm{a}^2-9\Delta=5^5(125J^2-9D)$, so that these conditions for the character $5r$ are

$$D=+,\quad J=-,\quad 125J^2-9D=+,\quad 2^{11}L-J^3-\tfrac{3}{5}JD=+.$$

Now, writing as above,

$$x=\frac{2^{11}L-J^3}{J^3},\quad y=\frac{D}{J^2},$$

these are $y=+$, $J=-$, $\frac{125}{9}-y=+$, $x-\frac{3}{5}y=-$; the conditions $y=+$, $J=-$ imply that (x, y) is in the region P or the region Q; and the condition $x-\frac{3}{5}y=-$ (observe the

line $x-\frac{3}{5}y=0$ lies between the lines $x+y=0$, $x-2y=0$, and so does not cut either the region P or the region Q) restricts (x, y) to the region P; and for every point of P y is at most $=1$, and the condition $\frac{125}{9}-y=+$ is of course satisfied. The condition, $125J^2-9D=+$, is thus wholly unnecessary, and omitting it, the conditions are

$$D=+,\quad J=-,\quad 2^{11}L-J^3-\tfrac{3}{5}JD=0,\quad \text{character } 5r,$$

which, $-\frac{3}{5}$ being an admissible value of μ, agrees with the result *ante*, No. 283.

307. It may be remarked in passing that if 12345 is a function of the roots $(x_1, x_2, x_3, x_4, x_5)$ of a quintic equation, which function is such that it remains unaltered by the cyclical permutation 12345 into 23451, and also by the reversal (12345 into 15432) of the order of the roots, so that the function has in fact the 12 values

$$\begin{aligned}
\alpha_1 &= 12345, & \beta_1 &= 24135,\\
\alpha_2 &= 13425, & \beta_2 &= 32145,\\
\alpha_3 &= 14235, & \beta_3 &= 43125,\\
\alpha_4 &= 21435, & \beta_4 &= 13245,\\
\alpha_5 &= 31245, & \beta_5 &= 14325,\\
\alpha_6 &= 41325, & \beta_6 &= 12435,
\end{aligned}$$

then $\phi(\alpha, \beta)$ being any unsymmetrical function of (α, β), the equation having for its roots the six values of $\phi(\alpha, \beta)$ (viz. $\phi(\alpha_1, \beta_1), \phi(\alpha_2, \beta_2) \ldots \phi(\alpha_6, \beta_6)$) can be expressed rationally in terms of the coefficients of the given quintic equation and of the square root of the discriminant of this equation. In fact, v being arbitrary, write

$$L=\Pi_6\{v-\phi(\alpha, \beta)\},\quad M=\Pi_6\{v-\phi(\beta, \alpha)\},$$

then the interchange of any two roots of the quintic produces merely an interchange of the quantities L, M; that is,

$$L+M \text{ and } (L-M)\div\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4, x_5)$$

are each of them unaltered by the interchange of any two roots, and are consequently expressible as rational functions of the coefficients; or observing that $\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4, x_5)$ is a multiple of $\sqrt{D}$, we have L a function of the form $P+Q\sqrt{D}$; the equation $L=0$, the roots whereof are $v=\phi(\alpha_1, \beta_1) \ldots v=\phi(\alpha_6, \beta_6)$, is consequently an equation of the form $P+Q\sqrt{D}=0$, viz. it is a sextic equation $(*\between v, 1)^6=0$, the coefficients of which are functions of the form in question. Hence in particular

$$u^2=12345=(x_1-x_2)^2(x_2-x_3)^2(x_3-x_4)^2(x_4-x_5)^2(x_5-x_1)^2$$

is determined as above by an equation $(*\between u^2, 1)^6=0$. Another instance of such an equation is given by my memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," *Phil. Trans.* vol. CLI. (1861), pp. 263—276, [268].

Article Nos. 308 to 317.—HERMITE'S *Canonical form of the quintic.*

308. It was remarked that M. Hermite's investigations are conducted by means of a canonical form, viz. if $A\,(=J,\ =G)$ be the quartinvariant of the given quintic $(a, b, c, d, e, f \between x, y)^5$, then he in fact finds (X, Y) linear functions of (x, y) such that we have

$$(a, b, c, d, e, f \between x, y)^5 = (\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda' \between X, Y)^5$$

(viz. in the transformed form the two mean coefficients are equal; this is a convenient assumption made in order to render the transformation completely definite, rather than an absolutely necessary one); and where moreover the quadricovariant B of the transformed form is

$$= \sqrt{A}\,XY,$$

or, what is the same thing, the coefficients $(\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda')$ of the transformed form are connected by the relations

$$\left.\begin{aligned} \lambda\mu' - 4\mu\sqrt{k} + 3k &= 0, \\ \lambda'\mu - 4\mu'\sqrt{k} + 3k &= 0, \\ \lambda\lambda' - 3\mu\mu' \quad + 2k &= \sqrt{A}, \end{aligned}\right\}$$

the advantage is a great simplicity in the forms of the several covariants, which simplicity arises in a great measure from the existence of the very simple covariant operator $\dfrac{d}{dX} \cdot \dfrac{d}{dY}$ (viz. operating therewith on any covariant we obtain again a covariant).

309. Reversing the order of the several steps, the theory of M. Hermite's transformation may be established as follows:

Starting from the quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

and considering the quadricovariant thereof

$$(\alpha, \beta, \gamma \between x, y)^2 \qquad B$$

((α, β, γ) are of the degree 2), and also the linear covariant

$$Px + Qy \qquad J$$

((P, Q) are of the degree 5), we have

$$\beta^2 - 4\alpha\gamma = A, \qquad G$$

and moreover

$$(\alpha, \beta, \gamma \between Q, -P)^2 = -C,$$

viz. the expression on the left hand, which is of the degree 12, and which is obviously an invariant, is $= -C$, where C is (*ut suprà*)

$$C = 9L + JK = -9U + GM.$$

The Jacobian of the two forms, viz.

$$\begin{vmatrix} 2\alpha x+\beta y, & \beta x+2\gamma y \\ P \quad , & Q \end{vmatrix},$$

$$= x(2\alpha Q-\beta P)+y(\beta Q-2P\gamma),$$

is a linear covariant of the degree 7, say it is

$$= P'x+Q'y,$$

and it is to be observed that the determinant $PQ'-P'Q$ of the two linear forms is $=-2(\alpha, \beta, \gamma \between Q, -P)^2$, that is, it is $=2C$.

310. Hence writing

$$T=\frac{1}{2\sqrt{C}}(Px+Qy)=\frac{1}{2\sqrt[4]{A}}(\quad X+Y),$$

$$U=\frac{1}{2\sqrt{C}}(P'x+Q'y)=\frac{\sqrt[4]{A}}{2}\quad(-X+Y),$$

whence also

$$X=T\sqrt[4]{A}-\frac{U}{\sqrt[4]{A}},$$

$$Y=T\sqrt[4]{A}+\frac{U}{\sqrt[4]{A}},$$

the determinant of substitution from (X, Y) to (T, U) is $=2$, that from (T, U) to (x, y) is $\frac{1}{4C}2C, =\frac{1}{2}$, and consequently that from (X, Y) to (x, y) is $=1$.

We have

$$AT^2-U^2=\frac{1}{4C}\{(\beta^2-4\alpha\gamma)(Px+Qy)^2-(P'x+Q'y)^2\};$$

or putting for P', Q' their values, this is $=\frac{1}{4C}$ into $4(\alpha, \beta, \gamma \between Q, -P)^2(\alpha x^2+2\beta xy+\gamma y^2)$, that is, we have

$$AT^2-U^2=\alpha x^2+\beta xy+\gamma y^2;$$

and we have also

$$AT^2-U^2=\tfrac{1}{4}\sqrt{A}\,[(X+Y)^2-(X-Y)^2]=\sqrt{A}XY,$$

consequently

$$\alpha x^2+\beta xy+\gamma y^2=AT^2-U^2=\sqrt{A}XY.$$

311. We have

$$x=\frac{1}{\sqrt{C}}(\quad Q'T-QU),$$

$$y=\frac{1}{\sqrt{C}}(-P'T+PU),$$

so that, pausing a moment to consider the transformation from (x, y) to (T, U), we have

$$(a, b, c, d, e, f \between x, y)^5 = \frac{1}{\sqrt{C^5}}(a, b, c, d, e, f \between Q'T - QU, \quad -P'T + PU)^5$$

$$= \frac{1}{\sqrt{C^5}}(\mathrm{a, b, c, d, e, f} \between T, U)^5 \text{ suppose,}$$

where (a, b, c, d, e, f) are *invariants*, of the degrees 36, 34, 32, 30, 28, 26 respectively; it follows that b, d, f each of them contain as a factor the 18-thic invariant I, the remaining factors being of the orders 16, 12, 8 respectively.

312. That (a, b, c, d, e, f) are invariants is almost self-evident; it may however be demonstrated as follows. Writing

$$\{y\partial_x\} = \quad a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + 5e\partial_f, \ = \delta \text{ suppose,}$$
$$\{x\partial_y\} = 5b\partial_a + 4c\partial_b + 3d\partial_c + 2e\partial_d + f\partial_e, \ = \delta_1 \quad ,,$$

then $Px + Qy$, $P'x + Q'y$ being covariants, we have $\delta P = 0$, $\delta Q = P$, $\delta P' = 0$, $\delta Q' = P'$, whence, treating T, U as constants, $\delta(Q'T - QU) = P'T - PU$, $\delta(-P'T + PU) = 0$. Hence

$$\begin{aligned}
&\delta(a, b, c, d, e, f \between Q'T - QU, -P'T + PU)^5 \\
&= \ 5(a, b, c, d, e \between Q'T - QU, -P'T + PU)^4 . (-P'T + PU) \\
&\quad + 5(a, b, c, d, e \between \quad ,, \quad ,, \quad)^4 . (\ P'T - PU) \\
&\quad + 5(b, c, d, e, f \between \quad ,, \quad ,, \quad)^4 \ . \ 0,
\end{aligned}$$

the three lines arising from the operation with δ on the coefficients (a, b, c, d, e, f) and on the facients $Q'T - QU$ and $-P'T + PU$ respectively; the third line vanishes of itself, and the other two destroy each other, that is,

$$\delta\,(a, b, c, d, e, f \between Q'T - QU, -P'T + PU)^5 = 0, \text{ and similarly}$$
$$\delta_1(a, b, c, d, e, f \between Q'T - QU, -P'T + PU)^5 = 0,$$

or the function $(a, b, c, d, e, f \between Q'T - QU, -PT + PU)^5$, treating therein T and U as constants, is an invariant, that is, the coefficients of the several terms thereof are all invariants.

313. The expressions for the coefficients (a, b, c, d, e, f) are in the first instance obtained in the forms

$$\begin{aligned}
\mathrm{a} &= \ 2(L + 5MC + 10C^2), \\
\mathrm{b} &= -2(L' + 3M'C)A, \\
\mathrm{c} &= \ 2(L + MC - 2C^2)A^{-1}, \\
\mathrm{d} &= -2(L' - M'C), \\
\mathrm{e} &= \ 2(L - 3MC + 2C^2)A^{-2}, \\
\mathrm{f} &= -2(L' - 5M'C)A^{-1},
\end{aligned}$$

where, developing M. Hermite's expressions,

$72L =$	$24M =$	$24L' =$	$24M' =$
$A^7B + 1$	$A^4B - 1$	$ABI + 1$	$I + 1$
$A^6C^2 + 1$	$A^3C - 1$	$CI + 5$	
$A^5B^2 + 6$	$A^2B^2 - 3$		
$A^4BC - 24$	$ABC + 12$		
$A^3B^3 + 9$	$C^2 + 24$		
$A^3C^2 - 39$			
$A^2B^2C + 9$			
$ABC^2 + 108$			
$C^3 + 72$			

and substituting these values, we find

36a =	36b =	36c =	36d =	36e =	36f =
$A^7B + 1$	$A^2BI - 3$	$A^6B + 1$	$ABI - 3$	$A^5B + 1$	$BI - 3$
$A^6C^2 + 1$	$ACI - 24$	$A^5C + 1$	$CI - 12$	$A^4C + 1$	
$A^5B^2 + 6$		$A^4B^2 + 6$		$A^3B^2 + 6$	
$A^4BC - 39$		$A^3BC - 27$		$A^2BC - 15$	
$A^3B^3 + 9$		$A^2B^3 + 9$		$AB^3 + 9$	
$A^3C^2 - 54$		$A^2C^2 - 42$		$AC^2 - 30$	
$A^2BC - 36$		$BC^2 + 144$		$B^2C + 36$	
$ABC^2 + 288$					
$C^3 + 1152$					

I have not thought it worth while to make in these formulæ the substitutions $A = J$, $B = -K$, $C = 9L + JK$, which would give the expressions for (a, b, c, d, e, f) in terms of J, K, L.

314. Substituting for (x, y) their values in terms of (X, Y), we have

$$(a, b, c, d, e, f \between x, y)^5$$

$$= (a, b, c, d, e, f \between \frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}} + Q\sqrt[4]{A}\right) X + \frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}} - Q\sqrt[4]{A}\right) Y,$$

$$\frac{1}{2\sqrt{C}}\left(\frac{-P'}{\sqrt[4]{A}} - P\sqrt[4]{A}\right) X + \frac{1}{2\sqrt{C}}\left(-\frac{P'}{\sqrt[4]{A}} + P\sqrt[4]{A}\right) Y$$

$$= (\lambda, \mu, \nu, \nu', \mu', \lambda' \between X, Y)^5 \text{ suppose,}$$

and by what precedes

$$\alpha x^2 + \beta xy + \gamma y^2 = \sqrt{A} XY;$$

this gives

$$\alpha\partial_y^2 - \beta\partial_y\partial_x + \gamma\partial_x^2 = -\sqrt{A}\partial_X\partial_Y,$$

and thence

$$(\alpha\partial_y^2 - \beta\partial_y\partial_x + \gamma\partial_x^2)^2 (a, b, c, d, e, f \between x, y)^5$$

$$= A\partial_X^2\partial_Y^2 (\lambda, \mu, \nu', \mu', \nu, \lambda' \between X, Y)^5$$

$$= 120A (\nu X + \nu' Y);$$

the left-hand side is a linear covariant of the degree 5, it is consequently a mere numerical multiple of $Px+Qy$; and it is easy to verify that it is $=120(Px+Qy)$. (In fact writing $b=d=e=0$, the expression is $(3c^2\partial_y^2 - af^2\partial_y\partial_x)^2(ax^5+10cx^2y^2+fy^5)$, and the only term which contains x is $a^2f^2.\partial_y^2\partial_x^2.10cx^2y^2 = 120a^2cf^2.x$; but for $b=d=e=0$, Table J gives $Px=a^2cf^2x$, and the coefficient 120 is thus verified.) But $Px+Qy$ is $=\frac{\sqrt{C}}{\sqrt[4]{A}}(X+Y)$, and we have thus $A\nu = A\nu' = \frac{\sqrt{C}}{\sqrt[4]{A}}$, whence not only $\nu=\nu'$, $=\sqrt{k}$ suppose, but we have further $k=\frac{C}{\sqrt[4]{A^5}}$, a result given by M. Hermite.

315. Substituting for $\nu=\nu'$ the value $\sqrt{k}$, we have

$$(a, b, c, d, e, f \between x, y)^5$$

$$=(a, b, c, d, e, f \between \frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}}+Q\sqrt[4]{A}\right)X+\frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}}-Q\sqrt[4]{A}\right)Y,$$

$$\frac{1}{2\sqrt{C}}\left(\frac{-P'}{\sqrt[4]{A}}-P\sqrt[4]{A}\right)X+\frac{1}{2\sqrt{C}}\left(-\frac{P'}{\sqrt[4]{A}}+P\sqrt[4]{A}\right)Y)^5,$$

$$=(\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda' \between X, Y)^5,$$

and we have then $\alpha x^2+\beta xy+\gamma y^2=\sqrt{A}XY$, viz. the left-hand side being the quadricovariant of $(a, b, c, d, e, f \between x, y)^5$, the equation shows that the quadricovariant of the form $(\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda' \between X, Y)^5$ is $=\sqrt{A}XY$, and we thus arrive at the starting-point of Hermite's theory.

316. The coefficients $(\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda')$ of Hermite's form are by what precedes *invariants;* they are consequently expressible in terms of the invariants A, B, C (and I). M. Hermite writes

$$\lambda\lambda'=g, \quad \mu\mu'=h,$$

and he finds

$$\sqrt{A}=g-3h+2k, \quad \frac{B}{\sqrt{A^3}}=h-k, \quad \frac{C}{\sqrt{A^5}}=k,$$

or, what is the same thing,

$$g=\frac{A^2+3AB+C}{\sqrt{A^5}}, \quad h=\frac{AB+C}{\sqrt{A^3}}, \quad k=\frac{C}{\sqrt{A^5}},$$

which give g, h, k in terms of A, B, C, and then putting

$$\Delta=(9k^2+16hk-gh)^2-24hk^3, \quad =\frac{I^2}{A^7}$$

(the equation $I^2=A^7\Delta$ is in fact equivalent to the before-mentioned expression of I^2

in terms of the other invariants), the coefficients $(\lambda, \mu, \mu', \lambda')$ are expressed in terms of g, h, k, that is of A, B, C, viz. we have

$$\begin{cases} 72\sqrt{k^5}\lambda = h(g-16k)^2 - 9k(g+16k) + (g-16k)\sqrt{\Delta}, \\ 24\sqrt{k^3}\mu = 9k^2 + 16hk - gh - \sqrt{\Delta}, \\ 24\sqrt{k^3}\mu' = 9k^2 + 16hk - gh + \sqrt{\Delta}, \\ 72\sqrt{k^5}\lambda' = h(g-16k)^2 - 9k(g+16k) - (g-16k)\sqrt{\Delta}; \end{cases}$$

these values of $(\lambda, \mu, \mu', \lambda')$ could of course be at once expressed in terms of (J, K, L), but I have not thought it necessary to make the transformation.

317. It has been already noticed that the linear covariant $(C, = Px + Qy)$, was

$$= \sqrt{A}\,(\sqrt{k}, \sqrt{k}\between X, Y),$$

it is to be added that the septic covariant $(P'x + Q'y)$ is

$$= \sqrt{A^3}\,(\sqrt{k}, -\sqrt{k}\between X, Y),$$

and that the canonical forms of the cubicovariants $\phi_1(x, y)$, &c. are as follows:

$$\begin{aligned} \phi_1(X, Y) &= \sqrt{A}\,(\mu, \quad 3\sqrt{k}, \quad 3\sqrt{k}, \quad \mu'\between X, Y)^3, \\ \phi_2(X, Y) &= \quad A\,(\mu, \quad \sqrt{k}, -\sqrt{k}, -\mu'\between X, Y)^3, \\ \{\phi_3(X, Y)\} &= \sqrt{A^3}\,(\mu, -\sqrt{k}, -\sqrt{k}, \quad \mu'\between X, Y)^3, \\ \{\phi_4(X, Y)\} &= \quad A^3\,(\mu, -3\sqrt{k}, \quad 3\sqrt{k}, -\mu'\between X, Y)^3, \end{aligned}$$

$$\psi_1(X, Y) = \sqrt{A^3}\left\{\begin{matrix} (2\sqrt{k^3} - 3\mu k \quad + \mu'\mu^2), \\ 3(\sqrt{k^3} + \mu\mu'\sqrt{k} \quad - 2\mu k), \\ -3(\sqrt{k^3} + \mu\mu'\sqrt{k} \quad - 2\mu' k), \\ -(2\sqrt{k^3} - 3\mu' k \quad + \mu\mu'^2), \end{matrix}\right\}\between X, Y)^3,$$

$$\phi_3(X, Y) = \sqrt{A^3}\,(5\mu, -\sqrt{k}, \sqrt{k}, 5\mu'\between X, Y)^3,$$

$$\phi_4(X, Y) = \sqrt{A^3}\left\{\begin{matrix} (7\sqrt{A}\mu \quad + 96(2\sqrt{k^3} - 3\mu k \quad + \mu'\mu^2)), \\ -3(3\sqrt{A}\sqrt{k} - 96(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu k)), \\ +3(3\sqrt{A}\sqrt{k} - 96(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu' k)), \\ -(7\sqrt{A}\mu' \quad + 96(2\sqrt{k^3} - 3\mu' k \quad + \mu\mu'^2)) \end{matrix}\right\}\between X, Y)^3$$

or, as the last formula may also be written,

$$\phi_4(X, Y) = \sqrt{A^3}\left\{\begin{matrix} ((7g - 53h + 110k)\mu \quad - 64\lambda\mu'\sqrt{k}), \\ -3((3g + 151h - 90k)\sqrt{k} \quad - 64\lambda'\mu^2 \quad), \\ +3((3g + 151h - 90k)\sqrt{k} \quad - 64\lambda\mu'^2 \quad), \\ -((7g - 53h + 110k)\mu' \quad - 64\lambda'\mu\sqrt{k}) \end{matrix}\right\}\between X, Y)^3.$$

It is in fact by means of these comparatively simple canonical expressions that M. Hermite was enabled to effect the calculation of the coefficient $\mathfrak{A}$.

Article Nos. 318 to 326.—*Theory of the imaginary linear transformations which lead to a real equation.*

318. An equation $(a, b, c, \ldots \between x, y)^n = 0$ is real if the ratios $a : b : c$, &c. of the coefficients are all real. In speaking of a given real equation there is no loss of generality in assuming that the coefficients $(a, b, c, \ldots)$ are all real; but if an equation presents itself in the form $(a, b, c, \ldots \between x, y)^n = 0$ with imaginary coefficients, it is to be borne in mind that the equation may still be real; viz. the coefficients may contain an imaginary common factor in such wise that throwing this out we obtain an equation with real coefficients.

In what follows I use the term *transformation* to signify a linear transformation, and speak of equations connected by a linear transformation as *derivable* from each other. An imaginary transformation will in general convert a real into an imaginary equation; and if the proposition were true universally,—viz. if it were true that the transformed equation was always imaginary—it would follow that a real equation derivable from a given real equation could then be derivable from it only by a real transformation, and that the two equations would have the same character. But any two equations having the same absolute invariants are derivable from each other, the two real equations would therefore be derivable from each other by a real transformation, and would thus have the same character; that is, all the equations (if any) belonging to a given system of values of the absolute invariants would have a determinate character, and the absolute invariants would form a system of auxiliars.

But it is not true that the imaginary transformation leads always to an imaginary equation; to take the simplest case of exception, if the given real equation contains only even powers or only odd powers of x, then the imaginary transformation $x : y$ into $ix : y$ gives a real equation. And we are thus led to inquire in what cases an imaginary transformation gives a real equation.

319. I consider the imaginary transformation $x : y$ into

$$(a+bi)\,x+(c+di)\,y \;:\; (e+fi)\,x+(g+hi)\,y,$$

or, what is the same thing, I write

$$x=(a+bi)\,X+(c+di)\,Y,$$
$$y=(e+fi)\,X+(g+hi)\,Y,$$

and I seek to find P, Q real quantities such that $Px+Qy$ may be transformed into a linear function $RX+SY$, wherein the ratio $R : S$ is real, or, what is the same thing, such that $RX+SY$ may be the product of an imaginary constant into a real linear function of (X, Y). This will be the case if

$$Px+Qy=(1+\theta i)\,\{P\,(aX+cY)+Q\,(eX+gY)\},$$

that is if

$$P(bX+dY)+Q(fX+hY)=\theta\{P(aX+cY)+Q(eX+fY)\},$$

which implies the relations

$$bP+fQ=\theta(aP+eQ),$$
$$dP+hQ=\theta(cP+gQ),$$

or, what is the same thing,

$$(b-a\theta)P+(f-e\theta)Q=0,$$
$$(d-c\theta)P+(h-g\theta)Q=0,$$

and if the resulting value of $P:Q$ be real, the last-mentioned equations give

$$(ag-ce)\theta^2-(ah+bg-cf-de)\theta+bh-df=0,$$

and θ being known, the ratio $P:Q$ is determined rationally in terms of θ.

320. The equation in θ will have its roots real, equal, or imaginary, according as

$$(ah+bg-cf-de)^2-4(ag-ce)(bh-df),$$

that is

$$\begin{aligned}&a^2h^2+b^2g^2+c^2f^2+d^2e^2\\&-2ahbg-2ahcf-2ahde-2bgcf-2bgde-2cfde\\&+4adfgh+4bceh\end{aligned}$$

is $=+$, $=0$, or $=-$; and I say that the transformation is subimaginary, neutral, and superimaginary in these three cases respectively. In the subimaginary case there are two functions $Px+Qy$ which satisfy the prescribed conditions; in the neutral case a single function; in the superimaginary case no such function. But in the last-mentioned case there are two conjugate imaginary functions, $Px+Qy$, which contain as factors thereof respectively two conjugate imaginary functions $UX+VY$.

321. Hence replacing the original x, y, X, Y by real linear functions thereof, the subimaginary transformation is reduced to the transformation $x:y$ into $kX:Y$, where k is imaginary; and the superimaginary transformation is reduced to $x+iy:x-iy$ into $k(X+iY):(X-iY)$, where k is imaginary. As regards the neutral transformation, it appears that this is equivalent to

$$x=(a+bi)X+(c+di)Y,$$
$$y=\qquad\qquad(g+hi)Y,$$

with the condition $0=(ah+bg)^2-4agbh$, $=(ah-bg)^2$, that is, we have $ah-bg=0$, or without any real loss of generality $g=a$, $h=b$, or the transformation is

$$x=(a+bi)X+(c+di)Y,$$
$$y=\qquad\qquad(a+bi)Y,$$

that is, $x:y=X+kY:Y$, where k is imaginary.

322. The original equation after any real transformation thereof, is still an equation of the form

$$(a, \ldots \between x, y)^n = 0;$$

and if we consider first the neutral transformation, the transformed equation is

$$(a, \ldots \between X + kY, Y)^n = 0;$$

this is not a real equation except in the case where k is real.

323. For the superimaginary transformation, starting in like manner from $(a, \ldots \between x, y)^n = 0$, this may be expressed in the form

$$(\alpha + \beta i, \gamma + \delta i, \ldots, \gamma - \delta i, \alpha - \beta i \between x + iy, x - iy)^n = 0,$$

viz. when in a real equation $(x, y)^n = 0$ we make the transformation $x : y$ into $x + iy : x - iy$, the coefficients of the transformed equation will form as above pairs of conjugate imaginaries. Proceeding in the last-mentioned equation to make the transformation $x + iy : x - iy$ into $k(X + iY) : X - iY$, I throw k into the form

$$\cos 2\phi + i \sin 2\phi, \quad = (\cos\phi + i\sin\phi) \div (\cos\phi - i\sin\phi)$$

(of course it is not here assumed that ϕ is real), or represent the transformation as that of $x + iy : x - iy$ into $(\cos\phi + i\sin\phi)(X + iY) : (\cos\phi - i\sin\phi)(X - iY)$; the transformed equation thus is

$$(\alpha + \beta i, \ldots \alpha - \beta i \between (\cos\phi + i\sin\phi)(X + iY), \quad (\cos\phi - i\sin\phi)(X - iY))^n = 0.$$

The left-hand side consists of terms such as $(X^2 + Y^2)^{n-2s}$ into

$$(\gamma + \delta i)(\cos s\phi + i\sin s\phi)(X + iY)^s + (\gamma - \delta i)(\cos s\phi - i\sin s\phi)(X - iY)^s,$$

viz. the expression last written down is

$$\begin{aligned} &= (\gamma\cos s\phi - \delta\sin s\phi)\{(X + iY)^s + (X - iY)^s\} \\ &- (\gamma\sin s\phi + \delta\cos s\phi)\left\{\frac{(X + iY)^s - (X - iY)^s}{s}\right\}, \end{aligned}$$

and observing that the expressions in { } are real, the transformed equation is only real if $(\gamma\cos s\phi - \delta\sin s\phi) \div (\gamma\sin s\phi + \delta\cos s\phi)$ be real, that is, in order that the transformed equation may be real, we must have $\tan s\phi =$ real; and observing that if $\tan s\phi$ be equal to any given real quantity whatever, then the values of $\tan\phi$ are all of them real, and that $\tan\phi$ real gives $\cos\phi$ and $\sin\phi$ each of them real, and therefore also ϕ real, it appears that the transformed equation is only real for the transformation

$$x + iy : x - iy = (\cos\phi + i\sin\phi)(X + iY) : (\cos\phi - i\sin\phi)(X - iY),$$

wherein ϕ is real; and this is nothing else than the *real* transformation $x : y$ into $X\cos\phi - Y\sin\phi : X\sin\phi + Y\cos\phi$. Hence neither in the case of the neutral transformation or in that of the superimaginary transformation can we have an imaginary transformation leading to a real equation.

324. There remains only the subimaginary transformation, viz. into this has been reduced to $x : y$ into $kX : Y$, the transformed equation is

$$(a, \ldots \between kX, Y)^n = 0,$$

and this will be a real equation if some power k^p of k (p not greater than n) be real, and if the equation $(a, \ldots \between x, y)^n = 0$ contain only terms wherein the index of x (or that of y) is a multiple of p. Assuming that it is the index of y which is a multiple, the form of the equation is in fact $x^\alpha (x^p, y^p)^m = 0$, $(n = mp + \alpha)$, and the transformed equation is $X^\alpha (k^p X^p, Y^p)^m = 0$, which is a real equation.

325. It is to be observed that if p be odd, then writing $k^p = K$ (K real) and taking k' the real p-th root of K, then the very same transformed equation would be obtained by the real transformation $x : y$ into $k'X : Y$; so that the equation obtained by the imaginary transformation, being also obtainable by a real transformation, has the same character as the original equation.

326. Similarly if p be even, if K be real and positive, the equation $k^p = K$ has a real root k' which may be substituted for the imaginary k, and the transformed equation will have the same character as the original equation; but if K be negative, say $K = -1$ (as may be assumed without loss of generality), then there is no real transformation equivalent to the imaginary transformation, and the equation given by the imaginary transformation has not of necessity the same character as the original equation; and there are in fact cases in which the character is altered. Thus if $p = 2$, and the original equation be $x(x^2, y^2)^m = 0$, or $(x^2, y^2)^m = 0$, then making the transformation $x : y$ into $iX : Y$, the transformed equation will be $X(X^2, -Y^2)^m = 0$ or $(X^2, -Y^2)^m = 0$, giving imaginary roots $X^2 + aY^2 = 0$ corresponding to real roots $x^2 - ay^2 = 0$.

Article No. 327.—*Application to the auxiliars of a quintic.*

327. Applying what precedes to a quintic equation $(a, \ldots \between x, y)^5 = 0$, this after any real transformation whatever will assume the form $(a', \ldots \between x', y')^5 = 0$; and the only cases in which we can have an imaginary transformation producing a real equation of an altered character is when this equation is $(a', 0, c', 0, e', 0 \between x', y')^5 = 0$ (c' not $= 0$), or when it is $(a', 0, 0, 0, e', 0 \between x', y')^5 = 0$, viz. when it is $x'(a'x'^4 + 10c'x'^2y'^2 + 5e'y'^4) = 0$, or $x'(a'x'^4 + 5e'y'^4) = 0$. In the latter case the transformation x', y' into $X\sqrt[4]{-1} : Y$ gives the real equation $X(a'X^4 - 5e'Y^4) = 0$. I observe however that for the form $(a', 0, 0, 0, e', 0 \between x, y)^4$, and consequently for the form $(a, \ldots \between x, y)^5$ from which it is derived we have $J = 0$; this case is therefore excluded from consideration. The remaining case is $(a', 0, c', 0, e', 0 \between x', y')^5 = 0$, which is by the imaginary transformation $x' : y'$ into $iX : Y$ converted into $(a', 0, -c', 0, e', 0 \between X, Y)^5 = 0$; for the first of the two forms we have $J = 16\, a'c'e'^2$, and for the second of the two forms $J = -16\, a'c'e'^2$, that is, the two values of J have opposite signs. Hence considering an equation $(a, b, c, d, e, f \between x, y)^5 = 0$ for which J is not $= 0$, whenever this is by an imaginary transformation converted into a real equation, the sign of J is reversed; and it follows that, given the values of the absolute invariants and the value of J (or what is sufficient, the sign of J), the

different real equations which correspond to these data must be derivable one from another by real transformations, and must consequently have a determinate character; that is, the Absolute Invariants, and J, constitute a system of auxiliars.

ANNEX.—*Analytical Theorem in relation to a Binary Quantic of any Order.*

The foregoing theory of the superimaginary transformation led me to a somewhat remarkable theorem. Take for example the function

$$(a,\ b,\ c \between x+k,\ 1-kx)^2,$$

or, as this may be written,

$$\begin{array}{c|ccc} & k^2 & k & 1 \\ \hline x^2 & c, & 2b, & a \\ x & 2b, & 2a-2c, & -2b \\ 1 & a, & -2b, & c, \end{array} \quad \text{or} \quad \left(\begin{array}{ccc} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{array} \right) \between (k,\ 1)^2 (x,\ 1)^2,$$

then the determinant

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix}$$

is a product of linear functions of the coefficients $(a,\ b,\ c)$; its value in fact is

$$= -2(a+c)(a+2bi+ci^2)(a-2bi+ci^2),\ = -2(a+c)[(a-c)^2+4b^2].$$

To prove this directly, I write

$$\begin{aligned} a' &= a-2bi+ci^2, \\ b' &= a \qquad\ \ -ci^2, \\ c' &= a+2bi+ci^2, \end{aligned}$$

and we then have

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix} \begin{vmatrix} 1, & 2\ , & 1 \\ i, & 0\ , & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}$$

$$= \begin{array}{c|c} & (1,\ i,\ i^2),\ (2,\ 0,\ -2i^2),\ (1,\ -i,\ i^2) \\ \hline \begin{array}{c} (\ c, \quad 2b, \quad a) \\ (2b, \quad 2a-2c, \quad -2b) \\ (\ a, \quad -2b, \quad c) \end{array} & \end{array},$$

$$= \begin{vmatrix} i^2a', & -2i^2b', & i^2c' \\ 2ia', & 0b', & -2ic' \\ a', & 2b', & c' \end{vmatrix}, \quad = a'b'c' \begin{vmatrix} i^2, & -2i^2, & i^2 \\ 2i, & 0\ , & -2i \\ 1, & 2\ , & 1 \end{vmatrix},$$

whence observing that the determinants

$$\begin{vmatrix} 1, & 2\ , & 1 \\ i\ , & 0\ , & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}, \quad \begin{vmatrix} i^2, & -2i^2, & i^2 \\ 2i, & 0\ , & -2i \\ 1\ , & 2\ , & 1 \end{vmatrix}$$

are as $1 : -2$, we have the required relation,

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix} = -2a'b'c', \ = -2(a+c)\{(a-c)^2+4b^2\}.$$

It is to be remarked that the determinant

$$\begin{vmatrix} 1, & 2\ , & 1 \\ i\ , & 0\ , & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}, \text{ taken as the multiplier of } \begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix}$$

is obtained by writing therein $a=b=c, \ =1$; and multiplying the successive lines thereof by $1, \frac{1}{2}i, i^2$ $(1, \frac{1}{2}, 1$ are the reciprocals of the binomial coefficients $1, 2, 1)$, the proof is the same, and the multiplier is obtained in the like manner for a function of any order; thus for the cubic $(a, b, c, d \between k+x, 1-kx)^3$,

$$\begin{array}{r|cccc} & k^3 & k^2 & k & 1 \\ \hline = x^3 & -d, & 3c, & -3b, & a \\ x^2 & 3c, & -6b+3d, & 3a-6c, & 3b \\ x & -3b, & 3a-6c, & 6b-3d, & 3c \\ 1 & a, & 3b, & 3c, & d \end{array}$$

the multiplier is obtained from the determinant by writing therein $a=b=c=d=1$, and multiplying the successive lines by $1, \frac{1}{3}i, \frac{1}{3}i^2, i^3$, viz. the multiplier is

$$\begin{vmatrix} -1, & 3, & -3, & 1 \\ i, & -i, & -i, & i \\ -i^2, & -i^2, & i^2, & i^2 \\ i^3, & 3i^3, & 3i^3, & i^3 \end{vmatrix}$$

and the value of the determinant is found to be

$$9(a-3bi+3ci^2-di^3)(a-bi-ci^2+di^3)(a+bi-ci^2-di^3)(a+3bi+3ci^2+di^3),$$
$$= 9\left((a-3c)^2+(3b-d)^2\right)\left((a+c)^2+(b+d)^2\right).$$

But the theory may be presented under a better form; take for instance the cubic, viz. writing $\frac{x}{y}$ and $\frac{k}{l}$ for x and k respectively, we have $(a, b, c, d \between ky+lx, ly-kx)^3$

$$
\begin{array}{rc|cccc}
 & & k^3 & k^2l & kl^2 & l^3 \\
\hline
= & x^3 & -d, & 3c, & -3b, & a \\
 & x^2y & 3c, & -6b+3d, & 3a-6c, & 3b \\
 & xy^2 & -3b, & 3a-6c, & 6b-3d, & 3c \\
 & y^3 & a, & 3b, & 3c, & d
\end{array}
$$

a bipartite cubic function $(*\between k, l)^3(x, y)^3$; and the determinant formed out of the matrix is at once seen to be an invariant of this bipartite cubic function.

Assume now that we have identically

$$(a, b, c, d \between x, y)^3 = (a', b', c', d' \between \tfrac{1}{2}(x+iy), \tfrac{1}{2}(x-iy))^3,$$

viz. this equation written under the equivalent form

$$(a', b', c', d' \between X, Y)^3 = (a, b, c, d \between X+Y, \ i(X-Y))^3,$$

determines (a', b', c', d') as linear functions of (a, b, c, d), it in fact gives

$$
\begin{aligned}
a' &= (a, b, c, d \between 1, -i)^3 &&= a - 3bi + 3ci^2 - di^3, \\
b' &= (a, b, c, d \between 1, -i)^2(1, i) &&= a - bi - ci^2 + di^3, \\
c' &= (a, b, c, d \between 1, -i)\,(1, i)^2 &&= a + bi - ci^2 - di^3, \\
d' &= (a, b, c, d \between 1, i)^3 &&= a + 3bi + 3ci^2 + di^3,
\end{aligned}
$$

then observing that $ky+lx \pm i(ly-kx) = (x \pm iy)(\mp ik+l)$, we have

$$(a, b, c, d \between ky+lx, ly-kx)^3 = (a', b', c', d' \between \tfrac{1}{2}(x+iy)(-ik+l), \tfrac{1}{2}(x-iy)(ik+l))^3,$$

and if in the expression on the right-hand side we make the linear transformations

$$
\begin{aligned}
x+iy &= x'\sqrt{2}, & -ik+l &= k'\sqrt{2}, \\
x-iy &= -iy'\sqrt{2}, & ik+l &= -il'\sqrt{2},
\end{aligned}
$$

which are respectively of the determinant $+1$, the transformed function is

$$= (a', b', c', d' \between k'x', -l'y')^3,$$

that is, we have

$$(a, b, c, d \between ky+lx, ly-kx)^3 = (a', b', c', d' \between k'x', -l'y')^3.$$

The last-mentioned function is

$$\begin{array}{c|cccc} & k'^3 & k'^2l' & k'l'^2 & l'^3 \\ \hline x'^3 & a' & . & . & . \\ x'^2y' & . & -3b' & . & . \\ x'y'^2 & . & . & +3c' & . \\ y'^3 & . & . & . & -d' \end{array}$$

and (from the invariantive property of the determinant) the original determinant is equal to the determinant of this new form, viz. we have

$$\begin{vmatrix} -d, & 3c, & -3b, & a \\ 3c, & -6b+3d, & 3a-6c, & 3b \\ -3b, & 3a-6c, & 6b-3d, & 3c \\ a, & 3b, & 3c, & d \end{vmatrix} = 9a'b'c'd',$$

$$= 9\,[(a-3c)^2+(3b-d)^2]\,[(a+c)^2+(b+d)^2],$$

which is the required theorem. And the theorem is thus exhibited in its true connexion, as depending on the transformation

$$(a, \ldots \between x, y)^n = (a', \ldots \between \tfrac{1}{2}(x+iy), \tfrac{1}{2}(x-iy))^n.$$

ADDITION, *7th October*, 1867.

Since the present Memoir was written, there has appeared the valuable paper by MM. Clebsch and Gordan "Sulla rappresentazione tipica delle forme binarie," *Annali di Matematica*, t. I. (1867) pp. 23—79, relating to the binary quintic and sextic. On reducing to the notation of the present memoir the formula 95 for the representation of the quintic in terms of the covariants α, β, which should give for (a, b, c, d, e, f) the values obtained *ante*, No. 312, I find a somewhat different system of values; viz. these are

36a =	36b =	36c =	36d =	36e =	35f =
A^7B + 1	$*A^4I$ − 1	A^6B + 1	$*A^2I$ − 1	A^6B + 1	$*A^2I$ − 1
A^6C + 1	A^2BI − 3	A^5C + 1	ABI − 3	A^4C + 1	ABI − 3
A^5B^2 + 6	$*ACI$ + 24	A^4B^2 + 6	$*CI$ + 12	A^3B^2 + 6	
A^4BC − 39		A^3BC − 27		A^2BC − 15	
A^3B^3 + 9		A^3B^3 + 9		AB^3 + 9	
A^3C^2 − 54		A^2C^2 − 42		AC^2 − 30	
A^2B^2C − 126		$*AB^2C$ − 90		$*B^2C$ − 54	
ABC^2 + 288		BC^2 + 144			
C^3 + 1152					

where I have distinguished with an asterisk the terms which have different coefficients in the two formulæ. I cannot at present explain this discrepancy.

Fig. 1.

Fig. 2.

The lower cusp of the Bicorn is drawn out of its true position which is much further off along the asymptote, the co-ordinates in fact are $x=-76\frac{2}{3}$, $y=-44\frac{2}{3}$ (the co-ordinates of the upper or node-cusp being −1, 1.

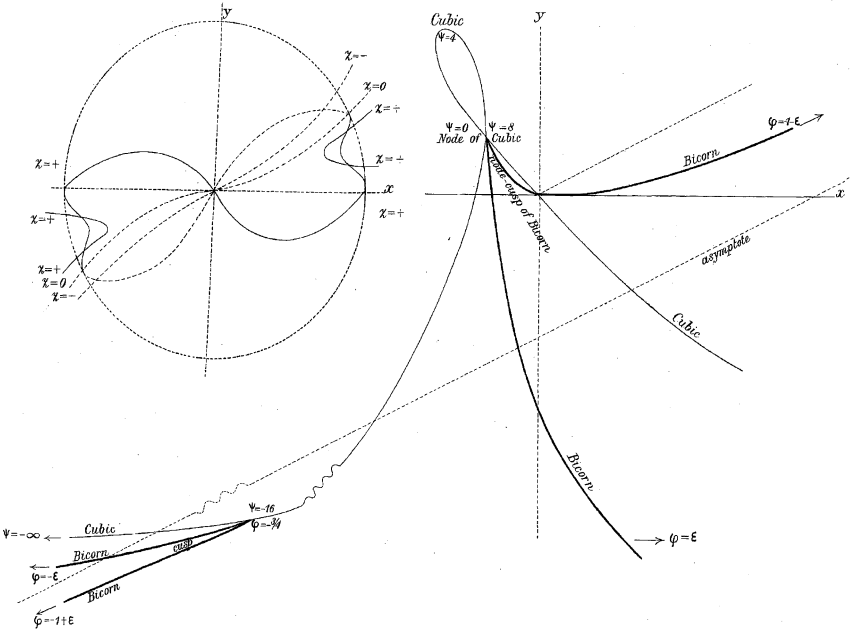


Fig. 4.

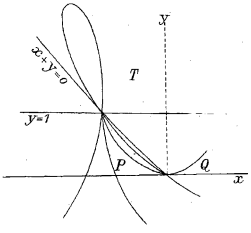
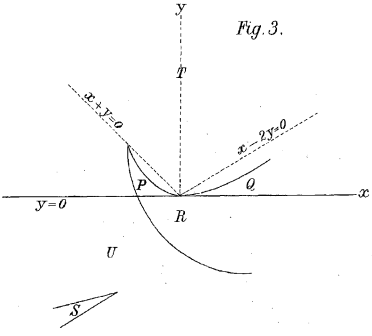


Fig. 3.



406.

ON THE CURVES WHICH SATISFY GIVEN CONDITIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVIII. (for the year 1868), pp. 75—143. Received April 18,—Read May 2, 1867.]

THE present Memoir relates to portions only of the subject of the curves which satisfy given conditions; but any other title would be too narrow: the question chiefly considered is that of finding the number of the curves which satisfy given conditions; the curves are either curves of a determinate order r (and in this case the conditions chiefly considered are conditions of contact with a given curve), or else the curves are conics; and here (although the conditions chiefly considered are conditions of contact with a given curve or curves) it is necessary to consider more than in the former case the theory of conditions of any kind whatever. As regards the theory of conics, the Memoir is based upon the researches of Chasles and Zeuthen, as regards that of the curves of the order r, upon the researches of De Jonquières: the notion of the quasi-geometrical representation of conditions by means of loci in hyper-space is employed by Salmon in his researches relating to the quadric surfaces which satisfy given conditions. The papers containing the researches referred to are included in the subjoined list. I reserve for a separate Second Memoir the application to the present question, of the Principle of Correspondence.

List of Memoirs and Works relating to the Curves which satisfy given conditions, with remarks

De Jonquières: "Théorèmes généraux concernant les courbes géométriques planes d'un ordre quelconque," *Liouv.* t. VI. (1861), pp. 113—134. In this valuable memoir is established the notion of a series of curves of the *index* N; viz. considering the curves of the order n which satisfy $\frac{1}{2}n(n+3)-1$ conditions, then if N denotes how many there are of these curves which pass through a given arbitrary point, the series is said to be of the index N.

In Lemma IV it is stated that all the curves C_n of a series of the index N can

be analytically represented by an equation $F(y, x) = 0$, which is rational and integral of the degree N in regard to a variable parameter λ: this is not the case; see Annex No. 1.

Chasles: Various papers in the *Comptes Rendus*, t. LVIII. *et seq.* 1864—67. The first of them (Feb. 1864), entitled "Détermination du nombre des sections coniques qui doivent toucher cinq courbes données d'ordre quelconque, ou satisfaire à diverses autres conditions," establishes the notion of the two characteristics (μ, ν) of a system of conics which satisfy four conditions; viz. μ is the number of these conics which pass through a given arbitrary point, and ν the number of them which touch a given arbitrary line. The Principle of Correspondence for points on a line is established in the paper of June—July 1864. Many of the leading points of the theory are reproduced in the present Memoir. The series of papers includes one on the conics in space which satisfy seven conditions (Sept. 1865), and another on the surfaces of the second order which satisfy eight conditions (Feb. 1866).

Salmon: "On some Points in the Theory of Elimination," *Quart. Math. Journ.* t. VII. pp. 327—337 (Feb. 1866); "On the Number of Surfaces of the Second Degree which can be described to satisfy nine Conditions," Ibid. t. VIII. pp. 1—7 (June 1866),—which two papers are here referred to on account of the notion which they establish of the quasi-geometrical representation of conditions by means of loci in hyper-space.

Zeuthen: Nyt Bidrag ... Contribution to the Theory of Systems of Conics which satisfy four conditions, 8°. pp. 1—97 (Copenhagen, Cohen, 1865), translated, with an addition, in the *Nouvelles Annales.*

The method employed depends on the determination of the line-pairs and point-pairs, and of the numerical coefficients by which these have to be multiplied, in the several systems of conics which satisfy four conditions of contact with a given curve or curves. It is reproduced in detail, with the enumeration called "Zeuthen's Capitals," in the present Memoir.

Cayley: "Sur les coniques déterminées par cinq conditions d'intersection avec une courbe donnée," *Comptes Rendus*, t. LXIII. pp. 9—12, July 1866. Results reproduced in the present Memoir.

De Jonquières: Two papers, *Comptes Rendus*, t. LXIII. Sept. 1866, reproduced and further developed in the "Mémoire sur les contacts multiples d'ordre quelconque des courbes du degré r qui satisfont à des conditions données de contact avec une courbe fixe du degré m; suivi de quelques réflexions sur la solution d'un grand nombre de questions concernant les propriétés projectives des courbes et des surfaces algébriques," *Crelle*, t. LXVI. (1866), pp. 289—322,—contain a general formula for the number of curves C^r having contacts of given orders $a, b, c, ..$ with a given curve U^m, which formula is referred to and considered in the present Memoir.

De Jonquières: Recherches sur les séries ou systèmes de courbes et de surfaces algébriques d'ordre quelconque; suivies d'une réponse &c. 4°. Paris, Gauthier Villars, 1866 [1].

[1] The foregoing list is not complete, and the remarks are not intended to give even a sketch of the contents of the works comprised therein, but only to show their bearing on the present Memoir.

Article Nos. 1 to 23.—*On the quasi-geometrical representation of Conditions.*

1. A condition imposed upon a subject gives rise to a relation between the parameters of the subject; for instance, the subject may be, as in the present Memoir, a plane curve of a given order, and the parameters be any arbitrary parameters contained in the equation of the curve. The condition may be onefold, twofold, ... or, generally, k-fold, and the corresponding relation is onefold, twofold, ... or k-fold accordingly. Two or more conditions, each of a given manifoldness, may be regarded as forming together a single condition of a higher manifoldness, and the corresponding relations as forming a single relation; and thus, though it is often convenient to consider two or more conditions or relations, this case is in fact included in that of a k-fold condition or relation. In dealing with such a condition or relation it is assumed that the number of parameters is at least $=k$; for otherwise there would not in general be any subject satisfying the condition: when the number of parameters is $=k$, the number of subjects satisfying the condition is in general determinate.

2. A subject which satisfies a given condition may for shortness be termed a solution of the condition; and in like manner any set of values of the parameters satisfying the corresponding relation may be termed a solution of the relation. Thus for a k-fold condition or relation, and the same number k of parameters, the number of solutions is in general determinate.

3. A condition may in some cases be satisfied in more than a single way, and if a certain way be regarded as the ordinary and proper one, then the others are *special* or *improper*: the two epithets may be used conjointly, or either of them separately, almost indifferently. For instance, the condition that a curve shall touch a given curve (have with it a two-pointic intersection) is satisfied if the curve have with the given curve a proper contact; or if it have on the given curve a node or a cusp (or, more specially, if it be or comprise as part of itself two coincident curves); or if it pass through a node or a cusp of the given curve: the first is regarded as the ordinary and proper way of satisfying the condition; the other two as special or improper ways; and the corresponding solutions are ordinary and proper solutions, or special or improper ones accordingly. This will be further explained in speaking of the locus which serves for the representation of a condition.

4. A set of any number, say ω, of parameters may be considered as the coordinates of a point in ω-dimensional space; and if the parameters are connected by a onefold, twofold, ... or k-fold relation, then the point is situate on a onefold, twofold, ... or k-fold locus accordingly; to the relation made up of two or more relations corresponds the locus which is the intersection or common locus of the loci corresponding to the several component relations respectively. A locus is at most ω-fold, viz. it is in this case a point-system. The relation made up of a k-fold relation, an l-fold relation, &c., is in general $(k+l+\&c.)$ fold, and the corresponding locus is $(k+l+\&c.)$ fold accordingly.

5. The order of a point-system is equal to the number of the points thereof, where, of course, coincident points have to be attended to, so that the distinct points of the system may have to be reckoned each its proper number of times. The locus

corresponding to any linear j-fold relation between the coordinates is said to be a j-fold *omal* locus; and if to any given k-fold relation we join an arbitrary $(\omega-k)$ fold linear relation, that is, intersect the k-fold locus by an arbitrary $(\omega-k)$ fold omal locus, so as to obtain a point-system, the order of the k-fold relation or locus is taken to be equal to the number of points of the point-system, that is, to the order of the point-system. And this being so, if a k-fold relation, an l-fold relation, &c. are *completely* independent, that is, if they are not satisfied by values which satisfy a less than $(k+l+\&c.)$ fold relation, or, what is the same thing, if the k-fold locus, the l-fold locus, &c., have no common less than $(k+l+\&c.)$ fold locus, then the relations make up together a $(k+l+\&c.)$ fold relation, and the loci intersect in a $(k+l+\&c.)$ fold locus, the orders whereof are respectively equal to the product of the orders of the given relations or loci. In particular if we have $k+l+\&c.=\omega$, then we have an ω-fold relation, and corresponding thereto a point-system, the orders whereof are respectively equal to the product of the orders of the given relations or loci.

6. A k-fold relation, an l-fold relation, &c., if they were together equivalent to a less than $(k+l+\&c.)$ fold relation, would not be independent; but the relations, assumed to be independent, may yet *contain* a less than $(k+l+\&c.)$ fold relation, that is, they may be satisfied by the values which satisfy a certain less than $(k+l+\&c.)$ fold relation (say the common relation), and exclusively of these, only by the values which satisfy a proper $(k+l+\&c.)$ fold relation, which is, so to speak, a residual equivalent of the given relations. This is more clearly seen in regard to the loci; the k-fold locus, the l-fold locus, &c. may have in common a less than $(k+l+\&c.)$ fold locus, and besides intersect in a residual $(k+l+\&c.)$ fold locus. (It is hardly necessary to remark that such a connexion between the relations is precisely what is excluded by the foregoing definition of *complete* independence.) In particular if $k+l+\&c.=\omega$, the several loci may intersect, say in an $(\omega-j)$ fold locus, and besides in a residual ω-fold locus, or point-system. The order (in any such case) of the residual relation or locus is equal to the product of the orders of the given relations or loci, *less* a reduction depending on the nature of the common relation or locus, the determination of the value of which reduction is often a complex and difficult problem.

7. Imagine a curve of given order, the equation of which contains ω arbitrary parameters: to fix the ideas, it may be assumed that these enter into the equation rationally, so that the values of the parameters being given, the curve is uniquely determined. Suppose, as above, that the parameters are taken to be the coordinates of a point in ω-dimensional space; so long as the curve is not subjected to any condition, the point in question, say the parametric point, is an arbitrary point in the ω-dimensional space; but if the curve be subjected to a onefold, twofold, ... or k-fold condition, then we have a onefold, twofold, ... or k-fold relation between the parameters, and the parametric point is situate on a onefold, twofold, ... or k-fold locus accordingly: to each position of the parametric point on the locus there corresponds a curve satisfying the condition, that is, a solution of the condition. In the case where the condition is ω-fold, the locus is a point-system, and corresponding to each point of the point-system we have a solution of the condition; the number of solutions is equal to the number of points of the point-system.

8. Considering the general case where the condition, and therefore also the locus, is k-fold, it is to be observed that every solution whatever, and therefore each special solution (if any), corresponds to some point on the k-fold locus; we may therefore have on the k-fold locus what may be termed "special loci," viz. a special locus is a locus such that to each point thereof corresponds a special solution. A special locus may of course be a point-system, viz. there are in this case a determinate number of special solutions corresponding to the several points of this point-system. We may consider the other extreme case of a special k-fold locus, viz. the k-fold locus of the parametric point may break up into two distinct loci, the special k-fold locus, and another k-fold locus the several points whereof give the ordinary solutions: we can in this case get rid of the special solutions by attending exclusively to the last-mentioned k-fold locus and regarding it as the proper locus of the parametric point. But if the special locus be a more than k-fold locus, that is, if it be not a part of the k-fold locus itself, but (as supposed in the first instance) a locus on this locus, then the special solutions cannot be thus got rid of: we have the k-fold locus of the parametric point, a locus such that to every point thereof there corresponds a proper solution, save and except that to the points lying on the special locus there correspond special or improper solutions. It is to be noticed that the special locus may be, but that is not in every case, a singular locus on the k-fold locus.

9. Suppose that the conditions to be satisfied by the curve are a k-fold condition, an l-fold condition, &c. of a total manifoldness $=\omega$. If the conditions are *completely* independent (that is, if the corresponding relations, *ante*, No. 5, are completely independent), we have a k-fold locus, an l-fold locus, &c., having no common locus other than the point-system of intersection, and the number of curves which satisfy the given conditions, or (as this has been before expressed) the number of solutions, is equal to the number of points of the point-system, or to the order of the point-system, viz. it is equal to the product of the orders of the loci which correspond to the several conditions respectively; among these we may however have special solutions, corresponding to points situate on the special loci upon any of the given loci; but when this is the case the number of these special solutions can be separately calculated, and the number of proper solutions is equal to the number obtained as above, less the number of the special solutions.

10. If, however, the given conditions are not completely independent (that is, if the corresponding relations are not completely independent), then the k-fold locus, the l-fold locus, &c. intersect in a common $(\omega-j)$ fold locus, and besides in a residual point-system. The several points of the $(\omega-j)$ fold locus give special solutions—in fact the very notion of the conditions being *properly* satisfied by a curve implies that the curve shall satisfy a true $(k+l+\&c.)$ fold, that is, a true ω-fold condition; the proper solutions are therefore comprised among the solutions given by the residual point-system, and the number of them is as before equal to the order of the point-system, or number of the points thereof, *less* the number of points which give special solutions: the order of the point-system is, as has been seen, equal to the product of the orders of the k-fold locus, the l-fold locus, &c., *less* a reduction depending on the nature of

the common $(\omega - j)$ fold locus, and the difficulty is in general in the determination of the value of this reduction.

11. In all that precedes, the number of the parameters has been taken to be ω; but if the parameters are taken to be contained in the equation of the curve homogeneously, then the parameters before made use of are in fact the ratios of these homogeneous parameters; and using the term henceforward as referring to the homogeneous parameters, the numbers of the parameters will be $= \omega + 1$.

12. I assume also that the equation of the curve contains the parameters linearly: this being so, the condition that the curve shall pass through a given arbitrary point implies a linear relation between the parameters; and the condition that the curve shall pass through j given points, a j-fold linear relation between the parameters. It follows that the number of the curves which satisfy a given k-fold condition, and besides pass through $\omega - k$ given points, is equal to the order of the k-fold relation, or of the corresponding k-fold locus; and thus if we define the order of the k-fold condition to be the number of the curves in question, the condition, relation, and locus will be all of the same order, and in all that precedes we may (in place of the order of the relation or of the locus) speak of the order of the condition. Thus, subject to the modifications occasioned by common loci and special solutions as above explained, the order of the $(k + l + \&c.)$ fold condition made up of a k-fold condition, an l-fold condition, &c., is equal to the product of the orders of the component conditions; and in particular if $k + l + \&c. = \omega$, then the order of the ω-fold condition, or number of the solutions thereof, is equal to the product of the orders of the component conditions.

13. The conditions to be satisfied by the curve may be conditions of contact with a given curve or curves. In particular if the curve touch a given curve, the parametric point is then situate on a onefold locus. It is to be noticed in reference hereto that if the given curve have nodes or cusps, then we have special solutions, viz. if the sought for curve passes through a node or a cusp of the given curve; and each such node or cusp gives rise to a special onefold locus, presenting itself in the first instance as a factor of the onefold locus of the parametric point; this is, however, a case where the special locus is of the same manifoldness as the general locus (*ante*, No. 8), and is consequently separable; throwing off therefore all these special loci, we have a onefold locus which no longer comprises the points which correspond to curves passing through a node or a cusp of the given curve; the onefold locus, so divested of the special onefold factors, may be termed the "contact-locus" of the given curve. To each point of the contact-locus there corresponds a curve having with the given curve a two-pointic intersection, viz. this is either a proper contact, or it is a special contact, consisting in that the sought for curve has on the given curve a node or cusp, or (which is a higher speciality) in that the sought for curve is or contains as part of itself two or more coincident curves (*ante*, No. 3). To a point in general on the contact-locus there corresponds a curve having a proper contact with the given curve, save and except that to each point on any one of certain special loci on the contact-locus there corresponds a curve having some kind of special contact as above with the given curve. To fix the ideas, it may be mentioned that for the curves of

the order r which touch a given curve of the order m and class n, the order of the contact-locus is $= n + (2r - 2)\, m$.

14. If, then, the curve touch a given curve, the parametric point is situate on the contact-locus of that curve. If it touch a second given curve, the parametric point is in like manner situate on the contact-locus of the second given curve, that is, it is situate on the twofold locus which is the intersection of the two contact-loci; and the like in the case of any number of contacts each with a distinct given curve. But if the curve, instead of ordinary contacts with distinct given curves, has either a contact of the second, or third, or any higher order, or has two or more ordinary or other contacts with the same given curve, then if the total manifoldness be $= k$, the parametric point is situate on a k-fold locus, which is given as a singular locus of the proper kind on the onefold contact-locus; so that the theory of the contact-locus corresponding to the case of a single contact with a given curve, contains in itself the theory of any system whatever of ordinary or other contacts with the same given curve, viz. the last-mentioned general case depends on the discussion of the singular loci which lie on the contact-locus. And similarly, if the curve has any number of ordinary or other contacts with each of two or more given curves, we have here to consider the intersections of singular loci lying on the contact-loci which correspond to the several given curves respectively, or, what is the same thing, to the singular loci on the intersection of these contact-loci; that is, the theory depends on that of the contact-loci which belong to the given curves respectively.

15. Suppose that the curve which has to satisfy given conditions is a line; the equation is $ax + by + cz = 0$, and the parameters (a, b, c) are to be taken as the coordinates of a point in a plane. Any onefold condition imposed upon the line establishes a onefold relation between the coordinates (a, b, c), and the parametric point is situate on a curve; a second onefold condition imposed on the line establishes a second onefold relation between the coordinates (a, b, c), and the parametric point is thus situate on a second curve; it is therefore determined as a point of intersection of two ascertained curves. In particular if the condition imposed on the line is that it shall touch a given curve, the locus of the parametric point is a curve, the contact-locus; (this is in fact the ordinary theory of geometrical reciprocity, the locus in question being the reciprocal of the given curve;) and the case of the twofold condition of a contact of the second order, or of two contacts, with the given curve, depends on the singular points of the contact-locus, or reciprocal of the given curve; in fact according as the line has a contact of the second order, or has two contacts with the given curve (that is, as it is an inflexion-tangent, or a double tangent of the given curve), the parametric point is a cusp or a node on its locus, the reciprocal curve: this is of course a fundamental notion in the theory of reciprocity, and it is only noticed here in order to show the bearing of the remark (*ante*, No. 14) upon the case now in hand where the curve considered is a line.

16. If the curve which has to satisfy given conditions is a conic

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

we have here six parameters (a, b, c, f, g, h), which are taken as the coordinates of a

point in 5-dimensional space. It may be remarked that in this 5-dimensional space we have the onefold cubic locus $abc - af^2 - bg^2 - ch^2 + 2fgh = 0$, which is such that to any position of the parametric point upon it there corresponds not a proper conic but a line-pair; this may be called the discriminant-locus. We have also the threefold locus the relation of which is expressed by the six equations

$$(bc - f^2 = 0,\quad ca - g^2 = 0,\quad ab - h^2 = 0,\quad gh - af = 0,\quad hf - bg = 0,\quad fg - ch = 0),$$

which is such that to any position of the parametric point thereon, there corresponds not a proper conic but a coincident line-pair. I call this the Bipoint-locus[1], and I notice that its order is $= 4$; in fact to find the order we must with the equations of the Bipoint combine two arbitrary linear relations,

$$(* \between a,\ b,\ c,\ f,\ g,\ h) = 0,$$
$$(*' \between a,\ b,\ c,\ f,\ g,\ h) = 0;$$

the equations of the locus are satisfied by

$$a : b : c : f : g : h = \alpha^2 : \beta^2 : \gamma^2 : \beta\gamma : \gamma\alpha : \alpha\beta$$

(where $\alpha : \beta : \gamma$ are arbitrary); and substituting these values in the linear relations, we have two quadric equations in (α, β, γ), giving four values of the set of ratios $(\alpha : \beta : \gamma)$; that is, the order is $= 4$, or the Bipoint is a threefold quadric locus.

17. The discriminant-locus does not in general present itself except in questions where it is a condition that the conic shall have a node (reduce itself to a line-pair); thus for the conics which have a node and touch a given curve (m, n), or, what is the same thing, for the line-pairs which touch a given curve (m, n), the parametric point is here situate on a twofold locus, the intersection of the discriminant-locus with the contact-locus. It may be noticed that this twofold locus is of the order $3(n + 2m)$, but that it breaks up into a twofold locus of the order $3n$, which gives the proper solutions; viz. the nodal conics which touch the given curve properly, that is, one of the two lines of the conic touches the curve; and into a twice repeated twofold locus of the order $3m$ which gives the special solutions, viz. in these the nodal conic has with the given curve a special contact, consisting in that the node or intersection of the two lines lies on the given curve. By way of illustration see Annex No. 2. But the consideration of the Bipoint-locus is more frequently necessary.

18. Suppose that the conic satisfies the condition of touching a given curve; the parametric point is then situate on a onefold contact-locus $(a, b, c, f, g, h)^q = 0$ (to fix the ideas, if the given curve is of the order m and class n, then the order q of the contact-locus is $= n + 2m$). The contact-locus of any given curve whatever passes through the Bipoint-locus; in fact to each point of the Bipoint-locus there corresponds a coincident line-pair, that is, a conic which (of course in a special sense) touches the given curve whatever it be; and not only so, but inasmuch as we have a special

[1] In framing the epithet Bipoint, the coincident line-pair is regarded as being really a point-pair: see *post*, No. 30.

contact at each of the points of intersection of the given curve with the coincident line-pair regarded as a single line, that is, in the case of a given curve of the m-th order, m special contacts, the Bipoint-locus is a multiple curve on the corresponding contact-locus.

19. If the conic has simply to touch a given curve of the order m_1 and class n_1, then the order of the condition (or number of the conics which satisfy the condition, and besides pass through four given points) is equal to the order of the contact-locus, that is, it is $=n_1+2m_1$. If the conic has also to touch a second given curve of the order m_2 and class n_2, then the order of the twofold condition (or number of the conics which satisfy the twofold condition, and besides pass through three given points) is equal to the order of the intersection or common locus of the two contact-loci; and these being of the orders n_1+2m_1 and n_2+2m_2 respectively, the order of the intersection and therefore that of the twofold condition is $=(n_1+2m_1)(n_2+2m_2)$. But in the next succeeding case it becomes necessary to take account of the singular locus.

20. If the conic has to touch three given curves of the order and class (m_1, n_1), (m_2, n_2), (m_3, n_3) respectively, we have here three contact-loci of the orders n_1+2m_1, n_2+2m_2, n_3+2m_3 respectively; these intersect in a threefold locus, but since each of the contact-loci passes through the threefold Bipoint-locus, this is part of the intersection of the three contact-loci; and not only so, but inasmuch as they pass through the Bipoint-locus m_1, m_2, m_3 times respectively, the Bipoint-locus must be counted $m_1m_2m_3$ times, and its order being $=4$, the intersection of the contact-locus is made up of the Bipoint reckoning as a threefold locus of the order $4m_1m_2m_3$, and of a residual threefold locus of the order

$$\begin{aligned}&(n_1+2m_1)(n_2+2m_2)(n_3+2m_3)-4m_1m_2m_3,\\ &=n_1n_2n_3+2(n_1n_2m_3+\&c.)+4(n_1m_2m_3+\&c.)+4m_1m_2m_3;\end{aligned}$$

and the order of the threefold condition (or number of the conics which touch the three given curves, and besides pass through two given points) is equal to the order of the residual threefold locus, and has therefore the value just mentioned.

21. In going on to the cases of the conics touching four or five given curves, the same principles are applicable; the contact-loci have the Bipoint (a certain number of times repeated) as a common threefold locus, and they besides intersect in a residual fourfold or (as the case is) fivefold locus, and the order of the condition is equal to the order of this residual locus; but the determination of the order of the residual locus presents the difficulties alluded to, *ante*, No. 10. I do not at present further examine these cases, nor the cases of the conics which have with a given curve or curves contacts of the second or any higher order, or more than a single contact with the same given curve.

22. The equation of the conic has been in all that precedes considered as containing the six parameters (a, b, c, f, g, h); but if the question as originally stated relates only to a class of conics the equation whereof contains linearly 2, 3, 4, or 5 parameters, or if, reducing the equation by means of any of the given conditions, it

can be brought to the form in question, then in the latter case we may employ the equation in such reduced form, attending only to the remaining conditions; and in either case we have the equation of a conic containing linearly 2, 3, 4, or 5 parameters, which parameters are taken as the coordinates of a point in 1-, 2-, 3-, or 4-dimensional space, and the discussion relates to loci in such dimensional space. This is in fact what is done in Annex No. 2 above referred to, where the conics considered being the conics which pass through three given points, the equation is taken to be $fyz+gzx+hxy=0$, and we have only the three parameters (f, g, h); and also in Annex No. 3, where the conics pass through two given points, and are represented by an equation containing the four parameters (a, b, c, h): I give this Annex as a somewhat more elaborate example than any which is previously considered, of the application of the foregoing principles, and as an investigation which is interesting for its own sake. See also Annexes 4 and 5, which contain other examples of the theory. The remark as to the number of parameters is of course applicable to the case where the curve which satisfies the given conditions is a curve of any given order r; the number of the parameters is here at most $=\frac{1}{2}(r+1)(r+2)$, and the space therefore at most $\frac{1}{2}r(r+3)$ dimensional; but we may in particular cases have $\omega+1$ parameters, the coordinates of a point in ω-dimensional space, where ω is any number less than $\frac{1}{2}r(r+3)$.

23. I do not at present consider the case of a curve of the order r, or further pursue these investigations; my object has been, not the development of the foregoing quasi-geometrical theory, so as to obtain thereby a series of results, but only to sketch out the general theory, and in particular to establish the notion of the order of condition, and to show that, as a rule (though as a rule subject to very frequent exceptions), the order of a compound condition is equal to the product of the orders of the component conditions. The last-mentioned theorem seems to me the true basis of the results contained in a subsequent part of this paper in connexion with the formulæ of De Jonquières, *post*, No. 74 *et seq.* But I now proceed to a different part of the general subject.

Article Nos. 24 to 72.—*Reproduction and Development of the Researches of* CHASLES *and* ZEUTHEN.

24. The leading points of Chasles's theory are as follows: he considers the conics which satisfy four conditions $(4X)$, and establishes the notion of the *characteristics* (μ, ν) of such a system, viz. $\mu, =(4X\cdot)$, denotes the number of conics in the system which pass through a given (arbitrary) point, and $\nu, =(4X/)$, the number of conics in the system which touch a given (arbitrary) line. We may say that μ is the parametric order, and ν the parametric class of the system.

25. The conics

$$(::), \quad (\cdot\cdot/), \quad (:/\!/), \quad (\cdot/\!/\!/), \quad (/\!/\!/\!/)$$

which pass through four given points, or which pass through three given points and touch a given line, &c., ... or touch four given lines, have respectively the characteristics

$$(1, 2), \quad (2, 4), \quad (4, 4), \quad (4, 2), \quad (2, 1).$$

26. A single condition (X) imposed upon a conic has two representative numbers, or simply representatives, (α, β); viz. if $(4Z)$ be an *arbitrary* system of four conditions, and (μ, ν) the characteristics of $(4Z)$, then the number of the conics which satisfy the five conditions $(X, 4Z)$ is $= \alpha\mu + \beta\nu$.

27. As an instance of the use of the characteristics, if X, X', X'', X''', X'''' be any five independent conditions, and $(\alpha, \beta), \ldots (\alpha'''', \beta'''')$ the representatives of these conditions respectively, then the number of the conics which satisfy the five conditions $(X, X', X'', X''', X'''')$ is

$$=(1, 2, 4, 4, 2, 1 \between \alpha, \beta)(\alpha', \beta')(\alpha'', \beta'')(\alpha''', \beta''')(\alpha'''', \beta'''')$$

viz. this notation stands for $1\alpha\alpha'\alpha''\alpha'''\alpha'''' + 2\Sigma\alpha\alpha'\alpha''\alpha'''\beta'''' \ldots + 1\beta\beta'\beta''\beta'''\beta''''$.

28. In particular if X be the condition that a conic shall touch a given curve of the order m and class n, then the representatives of this condition are (n, m), whence the number of the conics which touch each of five given curves $(m, n), \ldots (m'''', n'''')$ is

$$=(1, 2, 4, 4, 2, 1 \between n, m)(n', m')(n'', m'')(n''', m''')(n'''', m'''').$$

29. A system of conics $(4X)$ having the characteristics (μ, ν), contains

$2\nu - \mu$ line-pairs, that is, conics each of them a pair of lines; and

$2\mu - \nu$ point-pairs, that is, conics each of them a pair of points (*coniques infiniment aplaties*).

30. I stop to further explain these notions of the line-pair and the point-pair; and also the notion of the line-pair-point.

A conic is a curve of the second order and second class; *quà* curve of the second order it may degenerate into a pair of lines, or line-pair (but the class is then $=0$): *quà* curve of the second class it may degenerate into a pair of points, or point-pair (but the order is then $=0$). The two lines of a line-pair may be coincident, and we have then a coincident line-pair; such a line-pair (it must I think be postulated) ordinarily arises, not from a line-pair the two lines of which become coincident, but from a proper conic, flattening by the gradual diminution of its conjugate axis, while its transverse axis remains constant or approaches a limit different from zero; the conic thus tends (not to an indefinitely extended but) to a terminated line[1]; in other words, the tangents of the conic become more and more nearly lines through two fixed points, the terminations of the terminated line; and these terminating points, which continue to exist up to the instant when the conjugate axis takes its limiting value $=0$, are regarded as still existing at this instant, and the coincident line-pair as being in fact the point-pair formed by the two terminating points. Similarly the two points of a point-pair may be coincident, and we have then a coincident point-

[1] A line is regarded as extending from any point A thereof to B, and then in the same direction, from B through infinity to A; it thus consists of two portions separated by these points; and considering either portion as removed, the remaining portion is a terminated line.

pair; such a point-pair (it must in like manner be postulated) ordinarily arises, not from a point-pair the two points of which become coincident, but from a proper conic sharpening itself to coincide with its asymptotes, and so becoming ultimately a pair of lines through the coincident point-pair; and the coincident point-pair is regarded as being in fact the line-pair formed by some two lines through the coincident point-pair.

31. In accordance with the foregoing notions we may with propriety, and it will in the sequel be found convenient to speak of a point-pair as a line terminated by two points on this line, and similarly to speak of a line-pair as a point terminated (that is, the pencil of lines through the point is terminated) by two lines through the point.

32. If in a point-pair thus considered as a line terminated by two points the two points become coincident (the line continuing to exist as a definite line), or, what is the same thing, if in a line-pair thus considered as a point terminated by two lines, the two lines become coincident (the point continuing to exist as a definite point), we have a "line-pair-point;" viz. this is at once a coincident line-pair and a coincident point-pair; it may also be regarded as the limit of a conic the axes of which, and the ratio of the conjugate to the transverse axis, all ultimately vanish: it may be described as a line terminated each way at a point thereof, or as a point terminated each way at a line through it. The notion of a line-pair-point first presents itself in Zeuthen's researches, as will presently appear; but it may be noticed here that line-pair-points, and these the same line-pair-points, may present themselves among the $2\nu-\mu$ line-pairs, and among the $2\mu-\nu$ point-pairs of the system of conics $4X$.

33. Returning to the foregoing theory of characteristics, I remark that the fundamental notion may be taken to be, not the characteristics (μ, ν) of the conics which satisfy four conditions, but in every case the number of the conics which satisfy five conditions. Thus for the conics not subjected to any condition, we may consider the symbols

$$(\therefore),\quad (\because/),\quad (\therefore\cdot//),\quad (\because///),\quad (\cdot////),\quad (/////)$$

denoting the number of the conics which pass through five given points, or which pass through four given points and touch a given line, &c. ..., or which touch five given lines; these numbers are respectively

$$=1,\quad 2,\quad 4,\quad 4,\quad 2,\quad 1.$$

So for the conics which satisfy a given condition X, or two conditions $2X$, ..., or five conditions $5X$, we have respectively the numbers

$$\begin{array}{llllll}
X, & (\because), & (\therefore\cdot/), & (\because//), & (\cdot///), & (////) \\
2X, & (\therefore), & (\because/), & (\cdot//), & (///) & \\
3X, & (\because), & (\cdot/), & (//) & & \\
4X, & (\cdot), & (/) & & & \\
5X, & & & & &
\end{array}$$

where the X, $2X$, &c. belong to the symbols which follow: read $(X::)$, $(X\therefore/)$, &c., or, as we may for shortness represent them,

$$\begin{array}{lllll} \mu''', & \nu''', & \rho''', & \sigma''', & \tau''' \\ \mu'', & \nu'', & \rho'', & \sigma'' & \\ \mu', & \nu', & \rho' & & \\ \mu, & \nu & & & \\ \mu_0 & & & & \end{array}$$

viz. the single condition X has the five characteristics $(\mu''', \ldots \tau''')$, ...; the four conditions $4X$, the characteristics (μ, ν) as in the original theory; and the five conditions $5X$ a single characteristic μ_0.

34. We thus see the origin of the notion of the representatives (α, β) of a single condition X; for considering the arbitrary four conditions $4Z$, the characteristics whereof are (μ, ν), and assuming that the single characteristic, or number of the conics $(X, 4Z)$, is $=\alpha\mu+\beta\nu$, and taking for $(4Z)$ successively the conditions

$$(::), \quad (\therefore/), \quad (:/\!/), \quad (\cdot/\!/\!/), \quad (/\!/\!/\!/),$$

having respectively the characteristics

$$(1, 2), \quad (2, 4), \quad (4, 4), \quad (4, 2), \quad (2, 1),$$

we have

$$\begin{aligned} \mu''' &= 1\alpha + 2\beta, \\ \nu''' &= 2\alpha + 4\beta, \\ \rho''' &= 4\alpha + 4\beta, \\ \sigma''' &= 4\alpha + 2\beta, \\ \tau''' &= 2\alpha + 1\beta, \end{aligned}$$

that is, the characteristics $(\mu''', \nu''', \rho''', \sigma''', \tau''')$ of a single condition X are not independent, but are representable as above by means of two independent quantities (α, β); or, what is the same thing, we have

$$\nu''' = 2\mu''', \quad \sigma''' = 2\tau''', \quad \rho''' = \tfrac{2}{3}(\nu''' + \sigma'''),$$

which being satisfied, the representatives (α, β) are given by

$$\alpha = \tfrac{1}{3}(2\tau''' - \mu'''), \quad \beta = \tfrac{1}{3}(2\mu''' - \tau''').$$

35. I find that a like property exists as to the characteristics $(\mu'', \nu'', \rho'', \sigma'')$ of the two conditions $2X$, viz. these are not independent but are connected by a single linear relation,

$$\mu'' - \tfrac{3}{2}\nu'' + \tfrac{3}{2}\rho'' - \sigma'' = 0.$$

This may be proved in the case where the conditions $2X$ are two separate conditions (X, X'); viz. let the representatives of these be (α, β), (α', β') respectively, then

combining with them the three arbitrary conditions X'', X''', X'''' having respectively the representatives (α'', β''), (α''', β'''), (α'''', β''''), we have the general equation

$$(X, X', X'', X''', X'''') = (1, 2, 4, 4, 2, 1 \between \alpha, \beta)(\alpha', \beta')(\alpha'', \beta'')(\alpha''', \beta''')(\alpha'''', \beta'''');$$

taking herein

$$(X'', X''', X'''') = (\cdot\cdot), (\cdot/), (\cdot//), (///)$$

successively, and observing that the representatives of $(\cdot)$ are $(1, 0)$ and those of $(/)$ are $(0, 1)$, we thus obtain for $(\mu'', \nu'', \rho'', \sigma'')$, characteristics of (X, X'), the values

$$\begin{aligned}
\mu'' &= (1, 2, 4 \between \alpha, \beta)(\alpha', \beta'),\\
\nu'' &= (2, 4, 4 \between \alpha, \beta)(\alpha', \beta'),\\
\rho'' &= (4, 4, 2 \between \alpha, \beta)(\alpha', \beta'),\\
\sigma'' &= (4, 2, 1 \between \alpha, \beta)(\alpha', \beta'),
\end{aligned}$$

(viz. $\mu'' = 1\alpha\alpha' + 2(\alpha\beta' + \alpha'\beta) + 4\beta\beta'$, &c.), and these values give identically

$$2\mu'' - 3\nu'' + 3\rho'' - 2\sigma'' = 0,$$

which is the foregoing equation. And I assume that the theorem extends to the case of two inseparable conditions $2X$, but in this case I do not even know where the proof is to be sought for.

The characteristics (μ', ν', ρ') of the three conditions $3X$ are in general independent.

36. It has been mentioned that if (α, β) are the representatives of the condition X, and (μ, ν) the characteristics of the conditions $4Z$, then

$$(X, 4Z) = \alpha\mu + \beta\nu;$$

this is the most convenient form of the theorem, but as (α, β) are known functions of the characteristics $(\mu''', \nu''', \rho''', \sigma''', \tau''')$ of the condition X, the equation is in effect an expression for $(X, 4Z)$ in terms of the characteristics of X and $4Z$ respectively.

There is, similarly, an expression for $(2X, 3Z)$ in terms of the characteristics $(\mu', \nu', \rho', \sigma')$ of $3Z$ (satisfying the relation $\mu' - \frac{3}{2}\nu' + \frac{3}{2}\rho' - \sigma' = 0$) and the characteristics (μ, ν, ρ) of $2X$, viz. we have

$$\begin{aligned}
(2X, 3Z) = \quad & \mu\,(\qquad\qquad\qquad - \tfrac{1}{4}\rho' + \tfrac{1}{2}\sigma')\\
& + \nu\,(-\tfrac{3}{8}\mu' + \tfrac{5}{16}\nu' + \tfrac{7}{16}\rho' - \tfrac{3}{8}\sigma')\\
& + \rho\,(\ \tfrac{1}{2}\mu' - \tfrac{1}{4}\nu' \qquad\qquad\qquad).
\end{aligned}$$

This may be easily proved in the case where the conditions $2X$ are two separable conditions X, X' having the representatives (α, β), (α', β') respectively, and the conditions $3Z$ three separable conditions Z'', Z''', Z'''' having the representatives (α'', β''), (α''', β'''), (α'''', β'''') respectively; we have, in fact,

$$\begin{aligned}
\mu' &= (1, 2, 4 \between \alpha, \beta)(\alpha', \beta'), & \mu &= (1, 2, 4, 4 \between \alpha'', \beta'')(\alpha''', \beta''')(\alpha'''', \beta''''),\\
\nu' &= (2, 4, 4 \between \ \text{„} \quad \text{„} \ ), & \nu &= (2, 4, 4, 2 \between \ \text{„} \quad \text{„} \quad \text{„} \ ),\\
\rho' &= (4, 4, 2 \between \ \text{„} \quad \text{„} \ ), & \rho &= (4, 4, 2, 1 \between \ \text{„} \quad \text{„} \quad \text{„} \ ),\\
\sigma' &= (4, 2, 1 \between \ \text{„} \quad \text{„} \ ); & &
\end{aligned}$$

and with these values the function

$$(X,\ X',\ Z'',\ Z''',\ Z'''')\ =(1,\ 2,\ 4,\ 4,\ 2,\ 1\between\alpha,\ \beta)(\alpha',\ \beta')(\alpha'',\ \beta'')(\alpha''',\ \beta'''),\ (\alpha'''',\ \beta'''')$$

is found to be expressible as above in terms of $(\mu,\ \nu,\ \rho)$, $(\mu',\ \nu',\ \rho',\ \sigma')$; but I do not know how to conduct the proof for the inseparable conditions $2X$ and $3Z$.

37. It may be remarked by way of verification that writing successively

$$(3Z)=(\therefore),\ (:/),\ (\cdot//),\ (///),$$

that is,

$$(\mu,\ \nu,\ \rho)=(1,\ 2,\ 4),\ (2,\ 4,\ 4),\ (4,\ 4,\ 2),\ (4,\ 4,\ 1),$$

we have in the first case

$$\begin{aligned}(2X\,\therefore)= &\qquad\qquad -\tfrac{1}{4}\rho'+\tfrac{1}{2}\sigma'\\ &-\tfrac{3}{4}\mu'+\tfrac{5}{8}\nu'+\tfrac{5}{8}\rho'-\tfrac{3}{4}\sigma'\\ &+2\mu'-\ \nu'\\ = &\quad \mu'+\tfrac{1}{4}(\mu'-\tfrac{3}{2}\nu'+\tfrac{3}{2}\rho'-\sigma'),\ =\mu',\end{aligned}$$

and similarly in the other three cases,

$$(2X\cdot//)=\nu',\quad (2X\cdot//)=\rho',\quad (2X\,///)=\sigma'.$$

38. Let $(\mu,\ \nu,\ \rho,\ \sigma)$ be the characteristics of $2Z$, $(\mu-\frac{3}{2}\nu+\frac{3}{2}\rho-\sigma=0)$, and $(\mu',\ \nu',\ \rho',\ \sigma')$ the characteristics of $2X$, $(\mu'-\frac{3}{2}\nu'+\frac{3}{2}\rho'-\sigma'=0)$. Then in the formula for $(2X,\ 3Z)$, writing successively for $3X$

$$(2X\cdot),\ \text{characteristics}\ (\mu,\ \nu,\ \rho),$$

and

$$(2X\,/),\qquad ,,\qquad (\nu,\ \rho,\ \sigma),$$

we obtain expressions for the characteristics $(2X,\ 2Z\cdot)$ and $(2X,\ 2Z\,/)$ of $(2X,\ 2Z)$, viz. eliminating from the formulæ, first the $(\sigma,\ \sigma')$ and secondly the $(\mu,\ \mu')$, each of these may be expressed in two different forms as follows:

$$\begin{aligned}&(2X,\ 2Z\cdot)\\ =\ &\tfrac{1}{2}\mu\mu'\\ &+\tfrac{7}{8}\nu\nu'\\ &-\tfrac{3}{4}(\mu\nu'+\mu'\nu)\\ &+\tfrac{1}{2}(\mu\rho'+\mu'\rho)\\ &-\tfrac{1}{4}(\nu\rho'+\nu'\rho)\\ =\ &\tfrac{1}{2}\sigma\sigma'\\ &-\tfrac{3}{8}\rho\rho'\\ &-\tfrac{1}{4}\nu\nu'\\ &-\tfrac{1}{4}(\rho\sigma'+\rho'\sigma)\\ &+\tfrac{1}{2}(\nu\rho'+\nu'\rho),\end{aligned}$$

$$\begin{aligned}&(2X,\ 2Z\,/)\\ =\ &\tfrac{1}{2}\mu\mu'\\ &-\tfrac{3}{8}\nu\nu'\\ &-\tfrac{1}{4}\rho\rho'\\ &-\tfrac{1}{4}(\mu\nu'+\mu'\nu)\\ &+\tfrac{1}{2}(\nu\rho'+\nu'\rho)\\ =\ &\tfrac{1}{2}\sigma\sigma'\\ &+\tfrac{7}{8}\rho\rho'\\ &-\tfrac{3}{4}(\rho\sigma'+\rho'\sigma)\\ &+\tfrac{1}{2}(\nu\sigma'+\nu'\sigma)\\ &-\tfrac{1}{4}(\nu\rho'+\nu'\rho),\end{aligned}$$

the two expressions of the same quantity being of course equivalent in virtue of the relations between (μ, ν, ρ, σ) and $(\mu', \nu', \rho', \sigma')$ respectively.

The characteristics of (X, Z), $(X, 2Z)$, $(X, 3Z)$ are at once deducible from the before-mentioned expression $\alpha\mu + \beta\nu$ of $(X, 4Z)$.

39. Zeuthen's investigations are based upon the before-mentioned theorem, that in a system of conics $(4X)$, characteristics (μ, ν), there are $2\mu - \nu$ point-pairs and $2\nu - \mu$ line-pairs. If in the given system the number of point-pairs is $= \lambda$ and the number of line-pairs is $= \varpi$, then, conversely, the characteristics of the system are

$$\mu = \tfrac{1}{3}(2\lambda + \varpi), \quad \nu = \tfrac{1}{3}(\lambda + 2\varpi).$$

And by means of this formula he investigates the characteristics of the several systems of conics which satisfy four conditions $(4X)$ of contact with a given curve or curves, viz. these are the conics

(1)(1)(1)(1),	(1, 1)(1)(1),	(1, 1)(1, 1),	(1, 1, 1)(1), (1, 1, 1, 1),
(2)(1)(1) ,	(2)(1, 1) ,	(2, 1)(1) ,	(2, 1, 1),
(2)(2) ,	(2, 2) ,		
(3)(1) ,	(3, 1) ,		
(4) ,			

where (1) denotes contact of the first order, (2) of the second order, (3) of the third order, (4) of the fourth order, with a given curve; (1)(1) denotes contacts of the first order with each of two given curves, (1, 1) two such contacts with the same given curve, and so on. A given curve is in every case taken to be of the order m and class n, with δ nodes, κ cusps, τ double tangents, and ι inflexions (m_1, n_1, δ_1, κ_1, τ_1, ι_1; m_2, n_2, &c., as the case may be). The symbols (1), &c. might be referred to the corresponding curves by a suffix; thus $(1)_m$ would denote that the contact is with a given curve of the order m (class n, &c.); but this is in general unnecessary.

40. In a system of conics satisfying four conditions of contact, as above, it is comparatively easy to see what are the point-pairs and line-pairs in these several systems respectively; but in order to find the values of λ and ϖ, each of these point-pairs and line-pairs has to be counted not once, but a proper number of times; and it is in the determination of these multiplicities that the difficulty of the problem consists. I do not enter into this question, but give merely the results.

41. For the statement of these I introduce what I call the notation of Zeuthen's Capitals. We have to consider several classes of point-pairs and the reciprocal classes of line-pairs. A point-pair may be described (*ante*, No. 31) as a terminated line, and a line-pair as a terminated point; and we have first the following point-pairs, viz.:

A, line terminated each way in the intersection of two curves or of a curve with itself (node).

B, tangent to a curve, terminated in a curve, and in the intersection of two curves or of a curve with itself.

C, common tangent of two curves, or double tangent of a curve, terminated each way in a curve.

D, inflexion tangent of a curve terminated each way in a curve:

and the corresponding line-pairs, viz.:

A', point terminated each way in the common tangent of two curves or the double tangent of a curve.

B', point of a curve terminated by the tangent of a curve, and by the common tangent of two curves or double tangent of a curve.

C', intersection of two curves, or of a curve with itself (node), terminated each way by the tangent to a curve.

D', cusp of a curve terminated each way by the tangent to a curve:

all which is further explained by what follows; thus in the case $(1)(1)(1)(1)$, $=(1)_{m_1}(1)_{m_2}(1)_{m_3}(1)_{m_4}$, the value of A is given as $\Sigma m_1 m_2 \,.\, m_3 m_4 \,(= 3 m_1 m_2 m_3 m_4)$. Here A is the number of the point-pairs terminated one way in the intersection of any two m_1, m_2 of the four curves, and the other way in the intersection of the remaining two m_3, m_4 of the four curves. But in the case $(1, 1)(1)(1)$, $=(1, 1)_m(1)_{m_1}(1)_{m_2}$, the value of A is given as $=\delta m_1 m_2 + m m_1 \,.\, m m_2$. Here A denotes the number of the point-pairs, which are either $(\delta m_1 m_2)$ terminated one way at a node of m, and the other way at an intersection of m_1, m_2, or else $(m m_1 \,.\, m m_2)$ terminated one way at an intersection of m, m_1, and the other way at an intersection of m, m_2: and so in other cases.

42. This being so, we have

$(1)(1)(1)(1), = (1)_{m_1}(1)_{m_2}(1)_{m_3}(1)_{m_4}$.

$A = \Sigma m_1 m_2 \;.\, m_3 m_4 \;(= 3\; m_1 m_2 m_3 m_4)$,	1	$A' = \Sigma n_1 n_2 \;.\, n_3 n_4 \;(= 3\; n_1 n_2 n_3 n_4)$,
$B = \Sigma m_1 m_2 \;.\, m_3 \,.\, n_4 (= 3 \Sigma m_1 m_2 m_3 n_4)$,	2	$B' = \Sigma n_1 n_2 \;.\, n_3 \,.\, m_4 (= 3 \Sigma n_1 n_2 n_3 m_4)$,
$C = \Sigma m_1 \,.\, m_2 \,.\, n_3 n_4 \;(= \Sigma m_1 m_2 n_3 n_4)$.	4	$C' = \Sigma n_1 \,.\, n_2 \,.\, m_3 m_4 \;(= \Sigma n_1 n_2 m_3 m_4)$.

$(1, 1)(1)(1), = (1, 1)_m (1)_{m_1} (1)_{m_2}$.

$A = \delta m_1 m_2 + m m_1 \,.\, m m_2$,	1	$A' = \tau n_1 n_2 + n n_1 \,.\, n n_2$,
$B = \delta n_1 m_2 + \delta n_2 m_1$	2	$B' = \tau m_1 n_2 + \tau m_2 n_1$
$+ m m_1 (n-2) m_2 + m m_2 (n-2) m_1$		$+ n n_1 (m-2) n_2 + n n_2 (m-2) n_1$
$+ m m_1 n_2 (m-1) + m m_2 n_1 (m-1)$		$+ n n_1 m_2 (n-1) + n n_2 m_1 (n-1)$
$+ m_1 m_2 n (m-2)$,		$+ n_1 n_2 m (n-2)$,
$C = \tau m_1 m_2$	4	$C' = \delta n_1 n_2$
$+ n m_1 (m-2) m_2 + n n_2 (m-2) m_1$		$+ m m_1 (n-2) n_2 + m m_2 (n-2) n_1$
$+ n_1 n_2 \,.\, \frac{1}{2} m (m-1)$,		$+ m_1 m_2 \,.\, \frac{1}{2} n (n-1)$,
$D = \iota m_1 m_2$.	3	$D' = \kappa n_1 n_2$.

$(1,\ 1)(1,\ 1), = (1,\ 1)_m(1,\ 1)_{m_1}.$

$A = \delta\delta_1 + \frac{1}{2}mm_1(mm_1 - 1),$	1	$A' = \tau\tau_1 + \frac{1}{2}nn_1(nn_1 - 1),$	
$B = \delta n_1(m_1 - 2) + \delta_1 n(m - 2) + mm_1(n - 2)(m_1 - 1) + mm_1(n_1 - 2)(m - 1),$	2	$B' = \tau m_1(n - 2) + \tau_1 m(n - 2) + nn_1(m - 2)(n_1 - 1) + nn_1(m_1 - 2)(n - 1),$	
$C = \tau . \frac{1}{2}m_1(m_1 - 1) + \tau_1 . \frac{1}{2}m(m - 1) + nn_1(m - 2)(m_1 - 2),$	4	$C' = \delta . \frac{1}{2}n_1(n_1 - 2) + \delta_1 . \frac{1}{2}n(n - 1) + mm_1(n - 2)(n_1 - 2),$	
$D = \iota . \frac{1}{2}m_1(m_1 - 1) + \iota_1 . \frac{1}{2}m(m - 1).$	3	$D' = \kappa . \frac{1}{2}n_1(n_1 - 1) + \kappa_1 . \frac{1}{2}n(n - 1).$	

$(1,\ 1,\ 1)(1) = (1,\ 1,\ 1)_m(1)_{m_1}.$

$A = \delta mm_1,$	1	$A' = \tau nn_1,$	
$B = \delta(n - 4)m_1 + \delta n_1(m - 2) + mm_1(n - 2)(m - 3),$	2	$B' = \tau(m - 4)n_1 + \tau m_1(n - 2) + nn_1(m - 2)(n - 3),$	
$C = \tau(m - 4)m_1 + nn_1 . \frac{1}{2}(m - 2)(m - 3),$	4	$C' = \delta(n - 4)n_1 + mm_1 . \frac{1}{2}(n - 2)(n - 3),$	
$D = \iota(m - 3)m_1.$	3	$D' = \kappa(n - 3)n_1.$	

$(1,\ 1,\ 1,\ 1), = (1,\ 1,\ 1,\ 1)_m.$

$A = \frac{1}{2}\delta(\delta - 1),$	1	$A' = \frac{1}{2}\tau(\tau - 1),$	
$B = \delta(n - 4)(m - 4),$	2	$B' = \tau(m - 4)(n - 4),$	
$C = \tau . \frac{1}{2}(m - 4)(m - 5),$	3	$C' = \delta . \frac{1}{2}(n - 4)(n - 5),$	
$D = \iota . \frac{1}{2}(m - 3)(m - 4).$	4	$D' = \kappa . \frac{1}{4}(n - 3)(n - 4).$	

43. Secondly, we have the point-pairs:

E, tangent to curve from intersection of two curves or of a curve with itself (node), and terminated at the point of contact and the last-mentioned point.

F, tangent to a curve at intersection with another curve or with itself, and terminated there and at a curve.

G, common tangent of two curves or double tangent of a curve, terminated at one of the points of contact and at a curve.

D, *ut suprà*.

H, line joining cusp of a curve with intersection of two curves or of a curve with itself, and terminated at these points.

I, line from cusp of a curve touching a curve, and terminated at the cusp and at a curve.

J, Inflexion tangent of a curve, terminated there and at a curve:

and the corresponding line-pairs, viz.

E', point on a curve in common tangent of two curves or double tangent of a curve, and terminated by this tangent and by tangent to a curve.

F', point on a curve in common tangent of this and another curve or in double tangent of this curve, and terminated by this tangent and by tangent to a curve.

D', *ut suprà*.

H', intersection of inflexion tangent of a curve with common tangent of two curves or double tangent of a curve, and terminated by these lines.

I', intersection of inflexion tangent of a curve with a curve, and terminated by this tangent and by tangent of a curve:

and this being so,

$(2)(1)(1), = (2)_m(1)_{m_1}(1)_{m_2}$.

$E = n\,.\,m_1m_2,$	3	$E' = m\,.\,n_1n_2,$
$F = mm_1\,.\,m_2 + mm_2\,.\,m_1,$	3	$F' = nn_1\,.\,m_2 + nn_2\,.\,m_1,$
$G = nm_2\,.\,m_1 \;\; + nm_1\,.\,m_2,$	6	$G' = mm_2\,.\,n_1 + mm_1\,.\,n_2,$
$D = \iota m_1m_2,$	2	$D' = \kappa n_1n_2,$
$H = \kappa m_1m_2,$	1	$H' = \iota n_1n_2,$
$I = \kappa n_1m_2 \;\; + \kappa n_2m_1.$	2	$I' = \iota m_2n_1 \;\; + \iota m_2n_1.$

$(2)(1,\ 1) = (2)_m(1,\ 1)_{m_1}$.

$E = \delta_1 n,$	3	$E' = \tau_1 m,$
$F = m\,.\,m_1(m_1 - 1),$	3	$F' = n\,.\,n_1\ (n_1 - 1),$
$G = nm_1 \quad (m_1 - 2),$	6	$G' = mm_1\ (n_1 - 2),$
$D = \iota\,.\,\frac{1}{2}m_1(m_1 - 1),$	2	$D' = \kappa\,.\,\frac{1}{2}n_1(n_1 - 1),$
$H = \kappa\delta_1,$	1	$H' = \iota\tau_1,$
$I = \kappa n_1 \quad (m_1 - 2).$	2	$I' = \iota m_1 \quad (n_1 - 2).$

$(2,\ 1)(1), = (2,\ 1)_m(1)_{m_1}$.

$E = (n - 2)\,.\,mm_1,$	3	$E' = (m - 2)\,.\,nn_1,$
$F = mm_1(m - 2) + 2\delta m_1,$	3	$F' = nn_1\ (n - 2) \;\; + 2\tau n_1,$
$G = \;\; nn_1(m - 2) + 2\tau m_1,$	6	$G' = mm_1(n - 2) \;\; + 2\delta n_1,$
$D = \iota\,(m - 3)\,m_1,$	2	$D' = \kappa\,(n - 3)\,n_1,$
$H = \kappa mm_1,$	1	$H' = \iota nn_1,$
$I = \kappa\,(n - 3)\,m_1 + \kappa n_1(m - 2),$	2	$I' = \iota\,(m - 3)\,n_1 + \iota m_1(n - 2),$
$J = \iota m_1.$	5	$J' = \kappa n_1.$

$(2, 1, 1), = (2, 1, 1)_m.$

$E = \delta(n-4),$	3	$E' = \tau(m-4),$
$F = 2\delta(m-3),$	3	$F' = 2\tau(n-3),$
$G = 2\tau(m-4),$	6	$G' = 2\delta(n-4),$
$D = \iota.\frac{1}{2}(m-3)(m-4),$	2	$D' = \kappa.\frac{1}{2}(n-3)(n-4),$
$H = \delta\kappa,$	1	$H' = \iota\tau,$
$I = \kappa(n-3)(m-4),$	2	$I' = \iota(m-3)(n-4),$
$J = \iota(m-3).$	5	$J' = \kappa(n-3).$

44. Thirdly, we have the point-pairs:

K, common tangent of two curves or double tangent of a curve, terminated at points of contact.

L, line from cusp of a curve touching a curve, and terminated at cusp and point of contact.

M, line joining cusp of a curve with cusp of a curve, and terminated by the two cusps.

N, inflexion tangent terminated each way at inflexion, viz. this is a *line-pair-point.*

O, cuspidal tangent terminated each way at cusp, viz. this is a *line-pair-point:*

and the corresponding line-pairs:

K', intersection of two curves or of curve with itself (node), and terminated by the two tangents.

L', intersection of inflexion tangent of a curve with a curve, and terminated by the inflexion tangent and the tangent at the intersection.

M', intersection of inflexion tangent of a curve with inflexion tangent of a curve, and terminated by the two inflexion tangents.

$N', = 0$, *line-pair-point* as above.

$O', = N$, *line-pair-point* as above:

which being so, we have

$(2)(2), = (2)_m(2)_{m_1}.$

$K = nn_1,$	9	$K' = mm_1,$
$L = \kappa n_1 + \kappa_1 n,$	3	$L' = \iota m_1 + \iota_1 m,$
$M = \kappa\kappa_1.$	1	$M' = \iota\iota_1.$

$(2, 2), = (2, 2)_m$.

$K = \tau,$	9	$K' = \delta,$
$L = \kappa(n-3),$	3	$L' = \iota(m-3),$
$M = \frac{1}{2}\kappa(\kappa-1),$	1	$M' = \frac{1}{2}\iota(\iota-1),$
$N = \iota,$	2	$N' = \kappa,$
$O = \kappa.$	1	$O' = \iota.$

45. Fourthly, we have the point-pairs:

P, tangent of a curve at its intersection with another curve or itself, terminated each way at the point of contact—*line-pair-point.*

Q, common tangent of two curves or double tangent of a curve, terminated each way at one of the points of contact—*line-pair-point.*

J, *ut suprà.*

R, cuspidal tangent terminated at cusp and at a curve:

and the corresponding line-pairs:

P', $= Q$, *line-pair-point.*

Q', $= P$, *line-pair-point.*

J', *ut suprà.*

R', inflexion of curve terminated by the inflexion tangent and by tangent to a curve:

which being so, we have.

$(3)(1), = (3)_m(1)_{m_1}$.

$P = mm_1,$	2	$P' = nn_1,$
$Q = nn_1,$	2	$Q' = mm_1,$
$J = \iota m_1,$	5	$J' = \kappa n_1,$
$R = \kappa m_1.$	4	$D' = \iota n_1.$

$(3, 1), = (3, 1)_m$.

$P = 2\delta,$	2	$Q' = 2\tau,$
$Q = 2\tau,$	2	$P' = 2\delta,$
$J = \iota(m-3),$	5	$J' = \kappa(n-3),$
$R = \kappa(m-3).$	4	$R' = \iota(n-3).$

46. And lastly, we have the point-pairs N, O (*line-pair-points*) and the line-pairs N', O' (*line-pair-points*), *ut suprà*, and

$(4), = (4)_m$.

$N = \iota,$	4	$N' = \kappa,$
$O = \kappa.$	2	$O' = \iota.$

47. Where in all cases the central column of figures gives the numerical factors which multiply the corresponding capitals, thus we have

for (1)(1)(1)(1)

$$\lambda = 2\nu - \mu = A + 2B + 4C,$$
$$\varpi = 2\mu - \nu = A' + 2B' + 4C';$$

for (1, 1)(1)(1),

$$\lambda = 2\nu - \mu = A + 2B + 4C + 3D,$$
$$\varpi = 2\mu - \nu = A' + 2B' + 4C' + 3D',$$

and so on.

48. The elements $(m, n, \delta, \kappa, \tau, \iota)$ of a curve satisfy Plücker's six equations, and Zeuthen uses these equations, in a somewhat unsystematic way, to simplify the form of his results.

It is convenient in his formulæ to write $3m + \iota, = 3n + \kappa, = \alpha$, and to express everything in terms of (m, n, α), viz. we have for this purpose

$$2\delta = m^2 - m + 8n - 3\alpha,$$
$$2\tau = n^2 - 8m - n - 3\alpha.$$

But I make another alteration in the form of his results; he gives, for instance, the characteristics of (1, 1)(1)(1) as

$$\mu = \mu''' m_1 m_2 + \mu'' (m_1 n_2 + m_2 n_1) + \mu' n_1 n_2,$$
$$\nu = \nu''' m_1 m_2 + \nu'' (m_1 n_2 + m_2 n_1) + \nu' n_1 n_2,$$

where

$$\mu' = 2m(m + n - 3) + \tau, = (1, 1 \therefore),$$
$$\mu'' = \nu' = 2n(m + 2n - 5) + 2\tau, = (1, 1 : /),$$
$$\mu''' = \nu'' = 2n(2m + n - 5) + 2\delta, = (1, 1 \cdot //),$$
$$\nu''' = 2n(m + n - 3) + \delta, = (1, 1 ///),$$

viz. the four components have really the significations (1, 1 ∴) set opposite to them respectively; and accordingly, instead of giving the formulæ for the two characteristics of (1, 1)(1)(1), I give those for the four characteristics (1, 1 ∴), &c. of (1, 1), thus in every case obtaining formulæ which relate to a single curve only. Subject to the last-mentioned variation of form, I give Zeuthen's original expressions in Annex 6; but here in the text I express them as above in terms of (m, n, α), viz.

49. We have the formulæ

(1)

$$(\,::\,) = n + 2m,$$
$$(\,\therefore/\,) = 2n + 4m,$$
$$(\,:/\!/\,) = 4n + 4m,$$
$$(\cdot/\!/\!/\,) = 4n + 2m,$$
$$(\,/\!/\!//\,) = 2n + 2m;$$

(1, 1)

$$(\therefore) = 2m^2 + 2mn + \tfrac{1}{2}n^2 - 2m - \tfrac{1}{2}n - \tfrac{3}{2}\alpha,$$
$$(:/) = 2m^2 + 4mn + n^2 - 2m - n - 3\alpha,$$
$$(\cdot//) = m^2 + 4mn + 2n^2 - m - 2n - 3\alpha,$$
$$(///) = \tfrac{1}{2}m^2 + 2mn + 2n^2 - \tfrac{1}{2}m - 2n - \tfrac{3}{2}\alpha.$$

(1, 1, 1)

$$(:) = \tfrac{2}{3}m^3 + 2m^2n + mn^2 + \tfrac{1}{6}n^3 - 2m^2 - 3mn - \tfrac{1}{2}n^2 - \tfrac{20}{3}m - \tfrac{29}{3}n + \alpha(-3m - \tfrac{3}{2}n + 13),$$
$$(\cdot/) = \tfrac{1}{3}m^3 + 2m^2n + 2mn^2 + \tfrac{1}{3}n^3 - m^2 - 4mn - n^2 - \tfrac{46}{3}m - \tfrac{46}{3}n + \alpha(-3m - 3n + 20),$$
$$(//) = \tfrac{1}{6}m^3 + m^2n + 2mn^2 + \tfrac{2}{3}n^3 - \tfrac{1}{2}m^2 - 3mn - 2n^2 - \tfrac{29}{3}m - \tfrac{20}{3}n + \alpha(-\tfrac{3}{2}m - 3n + 13);$$

(1, 1, 1, 1)

$$\begin{aligned}(\cdot) = {} & \tfrac{1}{12}m^4 + \tfrac{2}{3}m^3n + m^2n^2 + \tfrac{1}{3}mn^2 + \tfrac{1}{24}m^4 \\ & - \tfrac{1}{2}m^3 - 3m^2n - 2mn^2 - \tfrac{1}{4}n^3 - \tfrac{181}{12}m^2 - 21mn - \tfrac{229}{24}n^2 + \tfrac{191}{2}m + \tfrac{403}{4}n \\ & + \alpha(-\tfrac{3}{2}m^2 - 3mn - \tfrac{3}{4}n^2 + \tfrac{43}{2}m + \tfrac{55}{4}n - \tfrac{357}{4}) + \alpha \cdot \tfrac{9}{8},\end{aligned}$$

$$\begin{aligned}(/) = {} & \tfrac{1}{24}m^4 + \tfrac{1}{3}m^3n + m^2n^2 + \tfrac{2}{3}mn^3 + \tfrac{1}{12}n^4 \\ & - \tfrac{1}{4}m^3 - 2m^2n - 3mn^2 - \tfrac{1}{2}n^3 - \tfrac{229}{24}m^2 - 21mn - \tfrac{181}{2}n^2 + \tfrac{403}{4}m + \tfrac{191}{2}n \\ & + \alpha(-\tfrac{3}{4}m^2 - 3mn - \tfrac{3}{2}n^2 + \tfrac{55}{4}m + \tfrac{43}{2}n - \tfrac{357}{4}) + \alpha^2 \cdot \tfrac{9}{8};\end{aligned}$$

(2)

$$(\therefore) = \alpha,$$
$$(:/) = 2\alpha,$$
$$(\cdot//) = 2\alpha,$$
$$(///) = \alpha;$$

(2, 1)

$$(:) = 12m + 12n + (2m + n - 14)\alpha,$$
$$(\cdot/) = 24m + 24n + (2m + 2n - 24)\alpha,$$
$$(//) = 12m + 12n + (m + 2n - 14)\alpha;$$

(2, 1, 1)

$$(\cdot) = 24m^2 + 36mn + 12n^2 - 168m - 168n + \alpha(m^2 + 2mn + \tfrac{1}{2}n^2 - 25m - \tfrac{29}{2}n + 138) - \tfrac{3}{2}\alpha^2,$$
$$(/) = 12m^2 + 36mn + 24n^2 - 168m - 168n + \alpha(\tfrac{1}{2}m^2 + 2mn + n^2 - \tfrac{29}{2}m - 25n + 138) - \tfrac{3}{2}\alpha^2;$$

(2, 2)

$$(\cdot) = 27m + 24n - 20\alpha + \tfrac{1}{2}\alpha^2,$$
$$(/) = 24m + 27n - 20\alpha + \tfrac{1}{2}\alpha^2;$$

(3)

$$(:) = -4m - 3n + 3\alpha,$$
$$(\cdot/) = -8m - 8n + 6\alpha,$$
$$(//) = -3m - 4n + 3\alpha;$$

(3, 1)

$$(\cdot) = -8m^2 - 12mn - 3n^2 + 56m + 53n + \alpha(6m + 3n - 39),$$

$$(/) = -3m^2 - 12mn - 8n^2 + 53m + 56n + \alpha(3m + 6n - 39);$$

(4)

$$(\cdot) = -10m - 8n + 6\alpha,$$

$$(/) = -8m - 10n + 6\alpha.$$

50. By means of the foregoing formulæ I obtain, as will presently be shown, the following formulæ for the number of the conics which satisfy five conditions, viz.:

$$(5) = -15m - 15n + 9\alpha;$$

$$(4, 1) = -8m^2 - 20mn - 8n^2 + 104m + 104n + \alpha(6m + 6n - 66);$$

$$(3, 2) = 120m + 120n + \alpha(-4m - 4n - 78) + 3\alpha^2;$$

$$\begin{aligned}(3, 1, 1) = {} & -\tfrac{3}{2}m^3 - 10m^2n - 10mn^2 - \tfrac{3}{2}n^3 + \tfrac{109}{2}m^2 + 116mn + \tfrac{109}{2}n^2 - 434m - 434n \\ & + \alpha(\tfrac{3}{2}m^2 + 6mn + \tfrac{3}{2}n^2 - \tfrac{69}{2}m - \tfrac{69}{2}n + 291) - \tfrac{9}{2}\alpha^2;\end{aligned}$$

$$\begin{aligned}(2, 2, 1) = {} & 24m^2 + 54mn + 24n^2 - 468m - 468n \\ & + \alpha(-8m - 8n + 327) + \alpha^2(\tfrac{1}{2}m + \tfrac{1}{2}n - 12);\end{aligned}$$

$$\begin{aligned}(2, 1, 1, 1) = {} & 6m^3 + 30m^2n + 30mn^2 + 6n^3 - 174m^2 - 348mn - 174n^2 + 1320m + 1320n \\ & + \alpha(\tfrac{1}{6}m^3 + m^2n + mn^2 + \tfrac{1}{6}n^3 - \tfrac{15}{2}m^2 - 26mn - \tfrac{15}{2}n^2 + \tfrac{358}{3}m + \tfrac{358}{3}n - 960) \\ & + \alpha^2(-\tfrac{3}{2}m - \tfrac{3}{2}n + 28);\end{aligned}$$

$$\begin{aligned}(1, 1, 1, 1, 1)(^1) = {} & \tfrac{1}{120}m^5 + \tfrac{1}{12}m^4n + \tfrac{1}{3}m^3n^2 + \tfrac{1}{3}m^2n^3 + \tfrac{1}{12}mn^4 + \tfrac{1}{120}n^5 \\ & - \tfrac{1}{12}m^4 - \tfrac{5}{6}m^3n - 2m^2n^2 - \tfrac{5}{6}mn^3 - \tfrac{1}{12}n^4 \\ & - \tfrac{113}{12}m^3 - \tfrac{209}{12}m^2n - \tfrac{209}{12}mn^2 - \tfrac{113}{12}n^3 \\ & + \tfrac{1267}{12}m^2 + \tfrac{593}{3}mn + \tfrac{1267}{12}n^2 - \tfrac{3159}{5}m - \tfrac{3159}{5}n \\ & + \alpha(-\tfrac{1}{4}m^3 - \tfrac{3}{2}m^2n - \tfrac{3}{2}mn^2 - \tfrac{1}{4}n^3 + \tfrac{29}{4}m^2 + 23mn + \tfrac{29}{4}n^2 - \tfrac{337}{4}m - \tfrac{337}{4}n + 486) \\ & + \alpha^2(\tfrac{9}{8}m + \tfrac{9}{8}n - 15).\end{aligned}$$

51. I observe that by means of the above-mentioned expressions of $(X, 4Z)$ and $(2X, 3Z)$, the foregoing results, other than those for (5), (4, 1), &c., may be presented in a somewhat different form, viz. we have

$$(4Z)(1) = n(\cdot) + m(/),$$

where $(\cdot)$ denotes $(4Z\,\cdot)$, $(/)$ denotes $(4Z\,/)$, and so in other cases, the understood term being $3Z$ or $2Z$, as the case may be.

[1] In my paper in the *Comptes Rendus*, I gave erroneously the coefficients $-\frac{3259}{5}m - \frac{3259}{5}n \ldots + \alpha(\ldots + \frac{1321}{2})$.

$$
\begin{aligned}
(3Z)(2) &= (\cdot/)\tfrac{1}{2}\alpha;\\
(3Z)(1,\,1) &= (\;:\;)(\tfrac{1}{2}n^2-\tfrac{1}{2}n)\\
&\quad+(\cdot/)(mn-\tfrac{3}{4}\alpha)\\
&\quad+(\;//)(\tfrac{1}{2}m^2-\tfrac{1}{2}m);\\
(2Z)(3) &= (\therefore)(\;\tfrac{3}{2}m+\;n-\tfrac{3}{4}\alpha)\\
&\quad+(:/)(-\tfrac{7}{4}m-\tfrac{3}{2}n+\tfrac{9}{8}\alpha)\\
&\quad+(\cdot//)(-\tfrac{3}{2}m-\tfrac{7}{4}n+\tfrac{9}{8}\alpha)\\
&\quad+(///)(\;\;m+\tfrac{3}{2}n-\tfrac{3}{4}\alpha);\\
(2Z)(2,\,1) &= (\therefore)\{-3m-3n+\alpha(-\tfrac{1}{4}m+\tfrac{1}{4}n+2)\}\\
&\quad+(\;:/)\{\;\tfrac{9}{2}m+\tfrac{9}{2}n+\alpha(\;\tfrac{3}{8}m+\tfrac{1}{8}n-4)\}\\
&\quad+(\cdot//)\{\;\tfrac{9}{2}m+\tfrac{9}{2}n+\alpha(\;\tfrac{1}{8}m+\tfrac{3}{8}n-4)\}\\
&\quad+(\;///)\{-3m-3n+\alpha(\;\tfrac{1}{4}m-\tfrac{1}{4}n+2)\};
\end{aligned}
$$

$$
\begin{aligned}
&(2Z)(1,\,1,\,1)=\\
&(\therefore)\{-\tfrac{1}{24}m^3-\tfrac{1}{4}m^2n+\tfrac{1}{4}mn^2+\tfrac{5}{24}n^3+\tfrac{1}{8}m^2+0mn-\tfrac{5}{8}n^2+\tfrac{11}{12}m+\tfrac{29}{12}n+\alpha(\;\tfrac{3}{8}m-\tfrac{3}{8}n-1)\}\\
&+(\;:/)\{\;\tfrac{1}{16}m^3+\tfrac{3}{8}m^2n+\tfrac{1}{8}mn^2-\tfrac{1}{16}n^3-\tfrac{3}{16}m^2-\tfrac{1}{2}mn+\tfrac{3}{16}n^2-\tfrac{19}{8}m-\tfrac{25}{8}n+\alpha(-\tfrac{9}{16}m-\tfrac{3}{16}n+3)\}\\
&+(\cdot//)\{\;\tfrac{1}{16}m^3+\tfrac{1}{8}m^2n+\tfrac{3}{8}mn^2+\tfrac{1}{16}n^3+\tfrac{3}{16}m^2-\tfrac{1}{2}mn-\tfrac{3}{16}n^2-\tfrac{25}{8}m-\tfrac{19}{8}n+\alpha(-\tfrac{3}{16}m-\tfrac{9}{16}n+3)\}\\
&+(\;///)\{\;\tfrac{5}{24}m^3+\tfrac{1}{4}m^2n-\tfrac{1}{4}mn^2-\tfrac{1}{24}n^3-\tfrac{5}{8}m^2+0mn+\tfrac{1}{8}n^2+\tfrac{29}{12}m+\tfrac{11}{12}n+\alpha(-\tfrac{3}{8}m+\tfrac{3}{8}n-1)\};
\end{aligned}
$$

in all which formulæ it is to be recollected that we have

$$(\therefore)-\tfrac{3}{2}(:/)+\tfrac{3}{2}(\cdot//)-(///)=0,$$

to which may be joined

$$(Z)(4X)=a(4X\cdot)+b(4X/),$$

where a, b are the representatives of the condition (Z), and where $(4X)$ is to be considered as standing successively for (4), (3, 1), (2, 2), (2, 1, 1), and (1, 1, 1, 1), the values of $(4X\cdot)$ and $(4X/)$ being in each case given by the foregoing Table.

52. The formulæ are very convenient for the calculation of the numbers of the conics which satisfy five conditions of contact with two given curves; thus if, for example, $(3Z)$, $=(3)_{m_1}$, denotes the condition of a contact of the third order with a given curve (m_1), then writing for symmetry $(2)_m$ in place of (2), we have

$$
\begin{aligned}
(3)_{m_1}(2)_m &= \tfrac{1}{2}\alpha(3\cdot)_{m_1}\\
&= \alpha(-4m_1-4n_1+3\alpha_1).
\end{aligned}
$$

53. To obtain the foregoing expressions of (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), and (1, 1, 1, 1, 1), I assume that the given curve breaks up into two curves (m, n, α) and (m', n', α'), or, as we may for shortness express it, into two curves m and m'.

We have then

$$(5)_{m+m'} = (5)_m + (5)_{m'},$$

viz. the conics which have contact of the 5th order with the aggregate curve $m+m'$ are made up of the conics which have this contact with the curve m and the conics which have this contact with the curve m'. Writing this under the form

$$(5)_{m+m'} - (5)_m - (5)_{m'} = 0,$$

and observing that $(5)_m$ is a function $\phi(m, n, \alpha)$, and that consequently this is a functional equation $\phi(m+m', n+n', \alpha+\alpha') - \phi(m, n, \alpha) - \phi(m', n', \alpha') = 0$, the solution is

$$\phi(m, n, \alpha) = am + bn + c\alpha,$$

where a, b, c are arbitrary constants; but as the solution should be symmetrical in regard to m, n, we have $a = b$, or the solution is $\phi(m, n, \alpha) = a(m+n) + c\alpha$.

54. Similarly we have

$$(4, 1)_{m+m'} - (4, 1)_m - (4, 1)_{m'} = (4)_m (1)_{m'} + (4)_{m'} (1)_m,$$

viz. the conics which have with the aggregate curve $m+m'$ the contacts (4, 1) are made up of the conics which have the two contacts 4 and 1 with the one curve or with the other curve, or the contact 4 with the one curve and the contact 1 with the other curve. The expression on the right-hand side is a known function of (m, n, α), (m', n', α'); hence the form of the functional equation is

$$\phi(m+m', n+n', \alpha+\alpha') - \phi(m, n, \alpha) - \phi(m', n', \alpha') = F(m, n, \alpha, m', n', \alpha');$$

and any particular solution of this equation being obtained, the general solution is found by adding to it the term $am + bn + c\alpha$. Assuming that the particular solution is symmetrical in regard to (m, n), then the term to be added is as before $= a(m+n) + c\alpha$. And similarly for (3, 2), (3, 1, 1), &c.; that is, in every case we have a solution containing two arbitrary constants a, c, which remain to be determined.

55. Now in every case except $(5)_m$ the number of intersections of the conic with the curve is > 6 (viz. for $(4, 1)_m$ and $(3, 2)_m$ the number is 7, for (3, 1, 1) and (2, 2, 1), it is 8, and for the remaining two cases it is 9 and 10 respectively); hence if the given curve m be a cubic, the number of conics satisfying the prescribed conditions is $= 0$; and since a cubic may be the general cubic or a nodal or a cuspidal cubic, we have the three cases $(m, n, \alpha) = (3, 6, 18)$, $(3, 4, 12)$, and $(3, 3, 10)$. We have thus in each case three conditions for the determination of the constants a, c; so that there is in each case a verification of the resulting formula.

56. In the omitted case $(5)_m$, when the curve m is a cubic, the theory of the conics $(5)_m$ is a known one, viz. the points of contact of these conics, or the "sextactic" points of the cubic, are the points of contact of the tangents from the points of inflexion; the number of the conics $(5)_m$ is thus $= (n-3)\iota$, viz. in the three cases

respectively it is $=27$, 3, and 0. Hence for determining the constants we have the three equations

$$9a+18c=27,$$
$$7a+12c=\ 3,$$
$$6a+10c=\ 0,$$

which are satisfied by $a=-15$, $c=9$, and the resulting formula is

$$(5)=-15m-15n+9\alpha.$$

In the particular case of a curve without nodes or cusps, this is $(5)=12n-15m$, $=m(12m-27)$, which agrees with the result obtained in my memoir "On the Sextactic Points of a Plane Curve," *Phil. Trans.* vol. CLV. (1865), pp. 545—578, [341].

57. The subsidiary results required for the remaining cases (4, 1), &c. are at once obtained from the foregoing formulæ for $(4Z)(1)$, $(3Z)(2)$, &c.; for example, we have

$$\begin{aligned}(4)_m(1)_{m'} &= n'(-10m-\ 8n+6\alpha)\\ &\quad +m'(-\ 8m-10n+6\alpha),\end{aligned}$$

with like expressions for $(3,1)_m(1)_{m'}$, &c.,

$$\begin{aligned}(3)_m(2)_{m'} &= \tfrac{1}{2}\alpha'\ (-8m-8n+6\alpha),\\ (3)_m(1,1)_{m'} &= (\tfrac{1}{2}n'^2-\tfrac{1}{2}n')(-4m-3n+3\alpha)\\ &\quad +(m'n'-\tfrac{3}{4}\alpha')(-8m-8n+6\alpha)\\ &\quad +(\tfrac{1}{2}m'^2-\tfrac{1}{2}m')(-3m-4n+3\alpha);\end{aligned}$$

with like expressions for $(2,1)_m(2)_{m'}$, $(2,1)_m(1,1)_{m'}$, &c. &c.

58. Calculation of (4, 1). We have

$$\begin{aligned}(4,1)_{m+m'}-(4,1)_m-(4,1)_{m'} &= (4)_m(1)_{m'}+(4)_{m'}(1)_m,\\ &= -16mm'-20(mn'+m'n)-16nn'+6(\alpha n'+\alpha' n)+6(\alpha m'+\alpha' m),\end{aligned}$$

the integral of which is

$$(4,1)_m=-8m^2-20mn-8n^2+\alpha(m+n)+\alpha(6m+6n+c).$$

The particular cases $(m, n, \alpha)=(3, 6, 18)$, $(3, 4, 12)$, $(3, 3, 10)$ give respectively

$$0=252+9a+18c,$$
$$0=\ 64+7a+12c,$$
$$0=\ 36+6a+10c,$$

satisfied by $a=104$, $c=-66$.

59. Calculation of (3, 2). We have

$$(3,\ 2)_{m+m'} - (3,\ 2)_m - (3,\ 2)_{m'} = (3)_m\,(2)_{m'} + (3)_{m'}\,(2)_m,$$
$$= -4\,(m\alpha' + m'\alpha) - 4\,(n\alpha' + n'\alpha) + 6\alpha\alpha':$$

the integral is

$$(3,\ 2)_m = a\,(m+n) + \alpha\,(-4m - 4n + c) + 3\alpha^2,$$

and, as before,

$$324 + 9a + 18c = 0,$$
$$96 + 7a + 12c = 0,$$
$$60 + 6a + 10c = 0,$$

satisfied by $a = 120$, $c = -78$.

60. For the calculation of (3, 1, 1) we have similarly

$$(3,\ 1,\ 1)_{m+m'} - (3,\ 1,\ 1)_m - (3,\ 1,\ 1)_{m'} = (3)_m\,(1,\ 1)_{m'} + (3)_{m'}\,(1,\ 1)_m$$
$$+ (3,\ 1)_m\,(1)_{m'} + (3,\ 1)_{m'}\,(1)_m.$$

The function on the right-hand side was of course calculated from the values of $(3)_m\,(1,\ 1)_m$, &c.; but there is no use in this (and the more complicated cases which follow) in actually writing down the values of the function in question; it can in each case be calculated *backwards* from the foregoing expressions of (3, 1, 1), &c., and the values so obtained be verified by actual substitution. But assuming it to be known, the solution of the functional equation gives of course the foregoing expression for (3, 1, 1), except that the terms in $m+n$ and α are therein $a\,(m+n) + c\alpha$; and I shall in this and the subsequent cases give only the three equations which determine the constants. In the present case these are

$$-332 + 9a + 18c = 0,$$
$$-454 + 7a + 12c = 0,$$
$$-306 + 6a + 10c = 0,$$

satisfied by $a = -434$, $c = 291$.

61. The remaining cases are (2, 2, 1), (2, 1, 1, 1) and (1, 1, 1, 1, 1). We have

$$(2,\ 2,\ 1)_{m+m'} - (2,\ 2,\ 1)_{m'} = (2,\ 2)_m\,(1)_{m'} + (2,\ 2)_{m'}\,(1)_m$$
$$+ (2,\ 1)_m\,(2)_{m'} + (2,\ 1)_{m'}\,(2)_m,$$

and

$$-1674 + 9a + 18c = 0,$$
$$-648 + 7a + 12c = 0,$$
$$-462 + 6a + 10c = 0,$$

satisfied by $a = -468$, $c = 327$.

Again,

$$(2,\ 1,\ 1,\ 1)_{m+m'} - (2,\ 1,\ 1,\ 1)_m - (2,\ 1,\ 1,\ 1)_{m'} = (2,\ 1,\ 1)_m\,(1)_{m'} + (2,\ 1,\ 1)_{m'}\,(1)_m$$
$$+ (2,\ 1)_m\,(1,\ 1)_{m'} + (2,\ 1)_{m'}\,(1)_m$$
$$+ (2)_m\,(1,\ 1,\ 1)_{m'} + (2)_{m'}\,(1,\ 1,\ 1)_m,$$

and

$$5400 + 9a + 18c = 0,$$
$$2280 + 7a + 12c = 0,$$
$$1680 + 6a + 10c = 0,$$

satisfied by $a = 1320$, $c = -960$; and finally,

$$(1, 1, 1, 1, 1)_{m+m'} - (1, 1, 1, 1, 1)_m - (1, 1, 1, 1, 1)_{m'} = (1, 1, 1, 1)_m (1)_{m'} + (1, 1, 1, 1)_{m'} (1)_m + (1, 1, 1)_m (1, 1)_{m'} + (1, 1, 1)_{m'} (1, 1)_m,$$

and

$$-30618 + 90a + 180c = 0,$$
$$-14094 + 70a + 120c = 0,$$
$$-10692 + 60a + 120c = 0,$$

satisfied by $10a = 6318$, $10c = 4860$, that is, $a = -\frac{3159}{5}$, $c = 486$.

62. The contacts of a conic with a given curve which have been thus far considered are contacts at unascertained points of the curve; but a conic may have with the given curve *at a given point thereof* a contact of the first order, the condition will be denoted by $(\bar{2})$; or a contact of the second order, the condition will be denoted by $(\bar{3})$, and so on. It is to be observed that the conditions $(\bar{2})$, $(\bar{3})$, &c. are sibireciprocal, the contact at a given point of the curve is the same thing as contact with a given tangent of the curve; but if we write $(\bar{1})$ to denote the condition of passing through a given point of the curve, this is *not* the same thing as the condition of touching a given tangent of the curve; and this last condition, if it were necessary to deal with it, might be denoted by $(\underline{1})$. But I attend only to the condition $(\bar{1})$. The expressions for the number of conics which satisfy such conditions as $(\bar{1})$, $(\bar{2})$, &c. are obtainable in several ways.

63. (1°) When the total number of conditions is 4, the question may be solved by Zeuthen's method, viz. by determining the line-pairs and point-pairs of the system $4Z$, with the proper numerical coefficients, and thence deducing the values of the characteristics $(4Z\cdot)$ and $(4Z/)$. A few cases are in fact thus solved in Zeuthen's work.

64. (2°) By the foregoing functional method. It is to be observed that there is a difference in the form of the functional equation, and that the general solution is always given in the form, Particular Solution + Constant, so that there is only a single constant to be determined by special considerations. To take the simplest example, let it be required to find the number of the conics $(3Z)(\bar{1}, 1)$: writing for shortness in place hereof $(\bar{1}, 1)$, or (in order to mark the curve (m) to which the symbol has reference) $(\bar{1}, 1)_m$, let the curve (m) be the aggregate of the curves (m) and (m'). Regarding the point $\bar{1}$ as a given point on the curve (m), that is, an arbitrary point in regard to the curve (m'), we have thus the equation

$$(\bar{1}, 1)_{m+m'} - (\bar{1}, 1)_m = (\cdot\, 1)_{m'},$$

where the right-hand side is known; and so in general the form of the functional equation is always $\phi(m+m') - \phi(m) =$ *given value*, that is,

$$\phi(m+m', n+n', \alpha+\alpha') - \phi(m, n, \alpha) = \text{given function of } (m, n, \alpha, m', n', \alpha');$$

whence, as stated, the general solution is Particular Solution + Constant. In the case in hand, taking successively $(3Z) = (\therefore)$, $(:/)$, $(\cdot//)$, and $(///)$, we have in the first of these cases

$$(\bar{1},\ 1)_{n+m'} - (\bar{1},\ 1)_m = n' + 2m',$$

whence $(\bar{1},\ 1)_m = n + 2m + \text{const.} = (\bar{1},\ 1)(\therefore)$; and the value of the constant being in any way ascertained to be $= -2$, we have $(\bar{1},\ 1)(\therefore) = n + 2m - 2$; and the like for the other three cases.

65. (3°) The expressions for the number of conics which satisfy such conditions as $(\bar{1})$, $(\bar{2})$, &c. are deducible with more or less facility from the corresponding expressions wherein $(\bar{1})$, $(\bar{2})$, &c. are replaced by $(\cdot)$, $(:)$, &c.; thus from [1]$(::1) = n + 2m$ we deduce

$$(\therefore \bar{1},\ 1) = (::/) - 2(\therefore \bar{2}) = n + 2m - 2,$$

viz. if one of the four arbitrary points of $(::/)$ becomes a point on the curve, then the condition $(::/)$ is satisfied specially by the conic $(\therefore \bar{2})$ which passes through the remaining three points and touches the curve at the point in question; 2 of the conics $(::/)$ coincide with the conic in question. We have thus a reduction $2(\therefore \bar{2})$, $= 2$, and the number of the conics $(\therefore \bar{1},\ 1)$ is $= n + 2m - 2$. Similarly, we have the system

$$\begin{aligned}
(\therefore \bar{1},\ 1\quad\quad\quad) &= n + 2m - 2,\\
(\ :\ \bar{1},\ \bar{1},\ 1\quad) &= n + 2m - 4,\\
(\ \cdot\ \bar{1},\ \bar{1},\ \bar{1},\ 1) &= n + 2m - 6,\\
(\bar{1},\ \bar{1},\ \bar{1},\ \bar{1},\ 1) &= n + 2m - 8.
\end{aligned}$$

Again, two or even three of the given points on the curve may come together without any reduction being thereby caused, that is, we have

$$\begin{aligned}
(:\ \bar{2},\ 1\quad\quad\quad) &\qquad\qquad\qquad = n + 2m - 4,\\
(\cdot\ \bar{2},\ \bar{1},\ 1\quad) &= (\cdot\ \bar{3},\ 1\quad) = n + 2m - 6,\\
(\quad \bar{2},\ \bar{1},\ \bar{1},\ 1) &= (\ \bar{3},\ \bar{1},\ 1) = n + 2m - 8;
\end{aligned}$$

but if the four points on the curve coincide in pairs, or, what is the same thing, if in $(\bar{2},\ \bar{1},\ \bar{1},\ 1)$ the points $\bar{1}$ and $\bar{1}$ come to coincide, then there is a special reduction, and we have

$$(\bar{2},\ \bar{2},\ 1) = n + 2m - 8\,[-(m-2)] = m + n - 6,$$

viz. here $(m-2)$ of the conics come to coincide with the two points considered as a point-pair or infinitely thin conic. If the points $\bar{2}$ and $\bar{2}$ come to coincide, that is, if the four given points on the curve all coincide, there is no further reduction, but we have

$$(\bar{4},\ 1) = m + n - 6.$$

[1] I write indifferently $(1)(::)$, $(1::)$ or $(::1)$; and so in other cases.

66. The expressions involving a single $(\bar{1})$ may in every case be reduced by the foregoing method to depend upon other expressions; thus we have

$$\begin{array}{lll}
(3Z\ )(\bar{1},\ 1) & =(\cdot 1) & -2(\bar{2}) \quad , \\
(2Z\ )(\bar{1},\ 2) & =(\cdot 2) & -3(\bar{3}) \quad , \\
\quad ,, \quad (\bar{1},\ 1,\ 1) & =(\cdot 1,\ 1) & -2(\bar{2},\ 1) \quad , \\
(\ Z\ )(\bar{1},\ 1,\ 2) & =(\cdot 1,\ 2) & -2(\bar{2},\ 2)\ -3(1,\ \bar{3}), \\
\quad ,, \quad (\bar{1},\ 1,\ 1,\ 1) & =(\cdot 1,\ 1,\ 1) & -2(\bar{2},\ 1,\ 1), \\
\quad ,, \quad (\bar{1},\ 3) & =(\cdot 3) & -4(\bar{4}) \quad , \\
(\bar{1},\ 4) & =(\cdot 4) & -5(\bar{5}) \quad , \\
\&c., & &
\end{array}$$

where, comparing for example the equations for $(Z)(\bar{1},\ 1,\ 2)$ and $(2Z)(\bar{1},\ 1,\ 1)$, it will be observed that in the first case the contacts 1, 2 of the symbol $(\bar{1},\ 1,\ 2)$ successively coalesce with the point $\bar{1}$, giving respectively $2(\bar{2},\ 2)$ and $3(1,\ \bar{3})$, the exterior factor being in each case the barred number, whereas the second case, where the contacts 1, 1 of the symbol $(\bar{1},\ 1,\ 1)$ are of the *same* order, we do not consider each of these symbols separately (thus obtaining $2(\bar{2},\ 1)+2(1,\ \bar{2}),\ =4(\bar{2},\ 1)$), but the identical symbol is taken only once, giving $2(\bar{2},\ 1)$. Thus we have also

$$(\bar{1},\ 1,\ 1,\ 1,\ 1)=(\cdot 1,\ 1,\ 1,\ 1)-2(\bar{2},\ 1,\ 1,\ 1).$$

67. The value of a symbol involving $(\bar{2})$, say the symbol $(3Z)(\bar{2})$, is connected with that of $\frac{1}{2}(3Z\cdot/)$; but as an instance of the correction which is sometimes required I notice the equation

$$(\bar{2},\ 1,\ 1,\ 1)=\tfrac{1}{2}(1,\ 1,\ 1\cdot/)-\{\tfrac{1}{2}(m-2)(m-3)+\tfrac{1}{2}(n-2)(n-3)+3(\bar{3},\ 1,\ 1)+2(\bar{4},\ 1)\},$$

which I have verified by other considerations.

68. We obtain the series of results:

$(\bar{1})$

$$\begin{aligned}
(\ ::\ ) &= 1, \\
(\ \therefore/) &= 2, \\
(\ :\,//) &= 4, \\
(\cdot///) &= 4, \\
(////) &= 2;
\end{aligned}$$

$(\bar{1},\ 1)$

$$\begin{aligned}
(\ \therefore\ ) &= n+2m-2, \\
(\ :/\ ) &= 2n+4m-4, \\
(\cdot//\ ) &= 4n+4m-4, \\
(\ ///\ ) &= 4n+2m-2;
\end{aligned}$$

$(\bar{1}, 2)$

$$(:) = \alpha - 3,$$
$$(\cdot /) = 2\alpha - 6,$$
$$(//) = 2\alpha - 3;$$

$(\bar{1}, 1, 1)$

$$(:) = 2m^2 + 2mn + \tfrac{1}{2}n^2 - 6m - \tfrac{5}{2}n + 8 - \tfrac{3}{2}\alpha,$$
$$(\cdot /) = 2m^2 + 4mn + n^2 - 6m - 5n + 12 - 3\alpha,$$
$$(//) = m^2 + 4mn + 2n^2 - 3m - 6n + 8 - 3\alpha;$$

$(\bar{1}, 3)$

$$(\cdot) = -4m - 3n - 4 + 3\alpha,$$
$$(/) = -8m - 8n - 4 + 6\alpha;$$

$(\bar{1}, 1, 2)$

$$(\cdot) = 6m + 9n + 30 + \alpha(2m + n - 16),$$
$$(/) = 21m + 18n + 30 + \alpha(2m + 2n - 26);$$

$(\bar{1}, 1, 1, 1)$

$$(\cdot) = \tfrac{2}{3}m^3 + 2m^2n + mn^2 + \tfrac{1}{6}n^3 - 4m^2 - 7mn - \tfrac{3}{2}n^2 + \tfrac{22}{3}m - \tfrac{2}{3}n - 36 + \alpha(-3m - \tfrac{3}{2}n + 16),$$
$$(/) = \tfrac{1}{3}m^3 + 2m^2n + 2mn^2 + \tfrac{1}{6}n^3 - 2m^2 - 8mn - 3n^2 - \tfrac{19}{3}m - \tfrac{4}{3}n - 36 + \alpha(-3m - 3n + 23);$$

$(\bar{2})$

$$(\therefore) = 1,$$
$$(: /) = 2,$$
$$(\cdot //) = 2,$$
$$(///) = 1;$$

$(\bar{2}, 1)$

$$(:) = 2m + n - 4,$$
$$(\cdot /) = 2m + 2n - 6,$$
$$(//) = m + 2n - 4;$$

$(\bar{2}, 2)$

$$(\cdot) = \alpha - 6,$$
$$(/) = \alpha - 6;$$

$(\bar{2}, 1, 1)$

$$(\cdot) = m^2 + 2mn + \tfrac{1}{2}n^2 - 7m - \tfrac{9}{2}n + 18 - \tfrac{3}{2}\alpha,$$
$$(/) = \tfrac{1}{2}m^2 + 2mn + n^2 - \tfrac{9}{2}m - 7n + 18 - \tfrac{3}{2}\alpha;$$

$(\bar{3})$

$$(:) = 1,$$
$$(\cdot /) = 2,$$
$$(//) = 1;$$

$(\bar{3}, 1)$

$$(\cdot) \quad = n + 2m - 6,$$
$$(/) \quad = 2n + m - 6;$$

$(\bar{4})$

$$(\cdot) \quad = 1,$$
$$(/) \quad = 1;$$

which are the several cases for the conics which satisfy not more than four conditions, and

69. For the conics satisfying 5 conditions, we have

$$(\bar{5}) \quad = 1,$$
$$(\bar{4}, 1) \quad = m + n - 6,$$
$$(\bar{3}, 2) \quad = -9 + \alpha,$$
$$(\bar{3}, 1, 1) \quad = \tfrac{1}{2}m^2 + 2mn + \tfrac{1}{2}n^2 - \tfrac{13}{2}m - \tfrac{13}{2}n + 27 - \tfrac{3}{2}\alpha,$$
$$(\bar{2}, 3) \quad = -4m - 4n - 6 + 3\alpha,$$
$$(\bar{2}, 2, 1) \quad = 6m + 6n + 54 + \alpha(m + n - 15),$$
$$(\bar{2}, 1, 1, 1) \quad = \tfrac{1}{6}m^3 + m^2n + mn^2 + \tfrac{1}{6}n^3 - \tfrac{5}{2}m^2 - 8mn - \tfrac{5}{2}n^2 + \tfrac{37}{3}m + \tfrac{37}{3}n - 75 + \alpha\left(-\tfrac{3}{2}m - \tfrac{3}{2}n + \tfrac{29}{2}\right),$$
$$(\bar{1}, 4) \quad = -10m - 8n - 5 + 6\alpha,$$
$$(\bar{1}, 1, 3) \quad = -8m^2 - 12mn - 3n^2 + 60m + 57n + 36 + \alpha(6m + 3n - 45),$$
$$(\bar{1}, 2, 2) \quad = 27m + 24n + 27 - 23\alpha + \tfrac{1}{2}\alpha^2,$$
$$(\bar{1}, 1, 1, 2) \quad = \tfrac{45}{2}m^2 + 30mn + \tfrac{21}{2}n^2 - \tfrac{321}{2}m - \tfrac{321}{2}n - 189 + \alpha\left(m^2 + 2mn + \tfrac{1}{2}n^2 - 27m - \tfrac{33}{2}n + \tfrac{345}{2}\right) - \tfrac{3}{2}\alpha^2,$$
$$(\bar{1}, 1, 1, 1, 1) = \tfrac{1}{12}m^4 + \tfrac{2}{3}m^3n + m^2n^2 + \tfrac{1}{3}mn^3 + \tfrac{1}{24}n^4 - \tfrac{5}{6}m^3 - 5m^2n - 4mn^2 - \tfrac{7}{2}n^3 - \tfrac{121}{12}m^2 - 5mn - \tfrac{209}{4}n^2 + \tfrac{425}{6}m + \tfrac{913}{4}n + 150 + \alpha\left(-\tfrac{3}{2}m^2 - 3mn - \tfrac{3}{4}n^2 + \tfrac{43}{2}m + \tfrac{67}{4}n - \tfrac{472}{4}\right) + \tfrac{9}{8}\alpha^2.$$

70. The given point on the curve to which the symbols $\bar{1}$, $\bar{2}$, &c. refer may be a singular point, and in particular it is proper to consider the case where the point is a cusp. I use in this case an appropriate notation; a conic which simply passes through a cusp, in fact meets the curve at the cusp in two points; and I denote the condition of passing through the cusp by $\overline{1\kappa 1}$; similarly, a conic which touches the curve at the cusp, in fact there meets it in three points, and I denote the condition by $\overline{2\kappa 1}$; $\overline{1\kappa 1}$, $\overline{2\kappa 1}$ are thus special forms of $\bar{1}$, $\bar{2}$, and the annexed $\bar{1}$ indicates the additional point of intersection arising *ipso facto* from the point $\bar{1}$ or $\bar{2}$ being a cusp. Similarly, we should have the symbols $\overline{3\kappa 1}$, $\overline{4\kappa 1}$, $\overline{5\kappa 1}$; but it is to be observed that at a cusp of the curve there is no *proper* conic having a higher contact than

$\overline{2\kappa 1}$; thus if the symbol contains $\overline{3\kappa 1}$, or *à fortiori*, if it contain $\overline{4\kappa 1}$ or $\overline{5\kappa 1}$, the number of the conics is in every case $=0$; it is thus only the cases $\overline{1\kappa 1}$ and $\overline{2\kappa 1}$ which need to be considered.

71. The several modes of investigation which apply to the case of contact at a given ordinary point of the curve are applicable to the case of contact at a cusp: we may if we please employ the functional method; we have here a functional equation of the foregoing form, $\phi(m+m')-\phi m=$ given value (that is, $\phi(m+m',\ n+n',\ \alpha+\alpha')-\phi(m,\ n,\ \alpha)=$ given function of $(m,\ n,\ \alpha,\ m',\ n',\ \alpha')$), and the general solution is as before $=$ Particular Solution $+$ Constant; so that there is in each case a single arbitrary constant to be determined by special considerations. The determination of the constant is in some instances conveniently effected by means of the case of the cuspidal cubic: see Annexes Nos. 4 and 5.

The formation of the functional equation itself is similar to that in the corresponding case where the given point on the curve is an ordinary point. For example, we have

$$\begin{aligned}(2Z)(1,\ \bar{1},\ 1)_{m+m'}-(1,\ \bar{1},\ 1)_m = \ & (1,\ \bar{1})_m(1)_{m'} \ = \ n'(\bar{1},\ 1\cdot)_m+m'(\bar{1},\ 1/)_m \\ & +(\bar{1})_m(1,\ 1)_{m'} \quad +\tfrac{1}{2}(n'^2-n')(\bar{1}:)_m \\ & \qquad\qquad\qquad\quad + \ (m'n'-\tfrac{3}{4}\alpha')(\bar{1}\cdot/)_m \\ & \qquad\qquad\qquad\quad +\tfrac{1}{2}(m'^2-m')(\bar{1}//)_m,\end{aligned}$$

and we may herein simply change $\bar{1}$ intc $\overline{1\kappa 1}$. Writing successively $2Z=(:)$, $(\cdot/)$ and $(//)$, we find

$$(\overline{1\kappa 1},\ 1,\ 1:)_{m+m'}-(\ :\ )_m=n'(\ n+2m-3)+m'(2n+2m-6)+(\tfrac{1}{2}n'^2-\tfrac{1}{2}n')1+(m'n'-\tfrac{3}{4}\alpha')2+(\tfrac{1}{2}m'^2-\tfrac{1}{2}m')4,$$

$$(\qquad\cdot/)_{m+m'}-(\cdot/\ )_m=n'(2n+4m-6)+m'(4n+4m-6)+(\tfrac{1}{2}n'^2-\tfrac{1}{2}n')2+(m'n'-\tfrac{3}{4}\alpha')4+(\tfrac{1}{2}m'^2-\tfrac{1}{2}m')4,$$

$$(\qquad //)_{m+m'}-(\ //)_m=n'(4n+4m-6)+m'(4n+2m-3)+(\tfrac{1}{2}n'^2-\tfrac{1}{2}n')4+(m'n'-\tfrac{3}{4}\alpha')4+(\tfrac{1}{2}m'^2-\tfrac{1}{2}m')4,$$

which only differ from the corresponding expressions with $\bar{1}$ in that they contain

$$n+2m-3,\ 2n+4m-6,\ 4n+4m-6,\ 4m+2n-3$$

in place of

$$n+2m-2,\ 2n+4m-4,\ 4n+4m-4,\ 4m+2n-2$$

respectively, and they lead to the expressions for $(\overline{1\kappa 1},\ 1,\ 1:)$, &c., the arbitrary constant being in each case properly determined.

72. We have

$(\overline{1\kappa 1})$

$$\begin{aligned}(\ ::\ )&=1,\\ (\ \therefore/\ )&=2,\\ (:/\!/\ )&=4,\\ (\cdot/\!//)&=4,\\ (/\!/\!/\!/)&=2;\end{aligned}$$

$(\overline{1\kappa}1,\ 1)$

$$(\ \therefore\ ) = n + 2m - 3,$$
$$(\ :/\ ) = 2n + 4m - 6,$$
$$(\ \cdot//\ ) = 4n + 4m - 6,$$
$$(\ ///\ ) = 4n + 2m - 3;$$

$(\overline{1\kappa\overline{1}},\ 2)$

$$(\ :\ ) = \alpha - 4,$$
$$(\ \cdot/\ ) = 2\alpha - 8,$$
$$(\ //\ ) = 2\alpha - 4;$$

$(\overline{1\kappa\overline{1}},\ 1,\ 1)$

$$(\ :\ ) = 2m^2 + 2mn + \tfrac{1}{2}n^2 - 8m - \tfrac{7}{2}n + 13 - \tfrac{3}{2}\alpha,$$
$$(\ \cdot/\ ) = 2m^2 + 4mn + n^2 - 8m - 7n + 18 - 3\alpha,$$
$$(\ //\ ) = m^2 + 4mn + 2n^2 - 4m - 8n + 12 - 3\alpha;$$

$(\overline{1\kappa\overline{1}},\ 3)$

$$(\ \cdot\ ) = -4m - 3n - 5 + 3\alpha,$$
$$(\ /\ ) = -8m - 8n - 6 + 6\alpha;$$

$(\overline{1\kappa\overline{1}},\ 1,\ 2)$

$$(\ \cdot\ ) = 4m + 8n + 44 + \alpha(2m + n - 17),$$
$$(\ /\ ) = 20m + 16n + 42 + \alpha(2m + 2n - 27);$$

$(\overline{1\kappa\overline{1}},\ 1,\ 1,\ 1)$

$$(\ \cdot\ ) = \tfrac{2}{3}m^3 + 2m^2n + mn^2 + \tfrac{1}{3}n^3 - 5m^2 - 9mn - 2n^2 + \tfrac{43}{3}m + \tfrac{29}{6}n - 57 + \alpha(-3m - \tfrac{3}{2}n + \tfrac{35}{2}),$$
$$(\ /\ ) = \tfrac{1}{3}m^3 + 2m^2n + 2mn^2 + \tfrac{1}{3}n^3 - \tfrac{5}{2}m^2 - 10mn - 4n^2 - \tfrac{11}{6}m + \tfrac{17}{3}n - 54 + \alpha(-3m - 3n + \tfrac{49}{2});$$

$(\overline{2\kappa\overline{1}})$

$$(\ \therefore\ ) = 1,$$
$$(\ :/\ ) = 2,$$
$$(\ \cdot//\ ) = 2,$$
$$(\ ///\ ) = 1;$$

$(\overline{2\kappa\overline{1}},\ 1)$

$$(\ :\ ) = 2m + n - 5,$$
$$(\ \cdot/\ ) = 2m + 2n - 6,$$
$$(\ //\ ) = m + 2n - 4;$$

$(\overline{2\kappa\overline{1}},\ 2)$

$$(\ \cdot\ ) = \alpha - 7,$$
$$(\ /\ ) = \alpha - 6;$$

$(\overline{2\kappa 1}, 1, 1)$

$$(\ \cdot\ ) \quad = m^2 + 2mn + \tfrac{1}{2}n^2 - 7m - \tfrac{11}{2}n + 21 - \tfrac{3}{2}\alpha,$$

$$(\ /\ ) \quad = \tfrac{1}{2}m^2 + 2mn + n^2 - \tfrac{3}{2}m - 7n + 18 - \tfrac{3}{2}\alpha.$$

73. The remainder of this table, being the part where the symbols $(\cdot)$ and $(/)$ do not occur, I present under a somewhat different form as follows:

$(\overline{5\kappa 1})$	$= 0,$	
$(\overline{4\kappa 1}, 1)$	$= 0,$	
$(\overline{3\kappa 1}, 2)$	$= 0,$	
$(\overline{3\kappa 1}, 1, 1)$	$= 0,$	
$(\bar{2}, 3)$	$-(\overline{2\kappa 1}, 3)$	$= 0,$
$(\bar{2}, 2, 1)$	$-(\overline{2\kappa 1}, 2, 1)$	$= n - 3,$
$(\bar{2}, 1, 1, 1)$	$-(\overline{2\kappa 1}, 1, 1, 1)$	$= \tfrac{1}{2}(n-3)(n-4),$
$(\bar{1}, 4)$	$-(\overline{1\kappa 1}, 4)$	$= 1,$
$(\bar{1}, 1, 3)$	$-(\overline{1\kappa 1}, 1, 3)$	$= (\overline{2\kappa 1}, 3) + (n-3),$
$(\bar{1}, 2, 2)$	$-(\overline{1\kappa 1}, 2, 2)$	$= 3(n-3) + \kappa - 1,$
$(\bar{1}, 1, 1, 2)$	$-(\overline{1\kappa 1}, 1, 1, 2)$	$= (\overline{2\kappa 1}, 1, 2) + \tfrac{1}{2}(n-3)(n-4) + \delta + 2\overline{n-3}\,\overline{m-4},$
$(\bar{1}, 1, 1, 1, 1)$	$-(\overline{1\kappa 1}, 1, 1, 1, 1)$	$= (\overline{2\kappa 1}, 1, 1, 1).$

These results relating to a cusp, are useful for the investigations contained in the Second Memoir.

It will be noticed that the symbols which contain $\overline{2\kappa 1}$ are not, like those which contain $\bar{2}$, symmetrical in regard to (m, n): the interchange of (m, n) would of course imply the change of a cusp into an inflexion, and would therefore give rise to a new symbol such as $\underline{2\iota 1}$; but I have not thought it necessary to consider the formulæ which contain this new symbol.

Investigations in extension of those of DE JONQUIÈRES *in relation to the contacts of a Curve of the order* r *with a given curve.* Article Nos. 74 to 93.

74. De Jonquières has given a formula for the number of curves C^r of the order r which have with a given curve U^m of the mth order t contacts of the orders a, b, c, &c. respectively, which besides pass through p points distributed at pleasure on the curve U^m (this includes the case of contacts of any orders at given points of the curve U^m), and which moreover satisfy any other $\tfrac{1}{2}r(r+3) - (a + b + c + \&c.) - p$

conditions; viz. the number of the curves C^r is $=\mu(a+1)(b+1)(c+1)\ldots$ into

$$\left\{\begin{array}{l} \quad [rm-(a+b+c\,..)-p \quad\quad]^t \\ +[rm-(a+b+c\,..)-p-1]^{t-1}(a\ +b\ +c\ ..)[D]^1 \\ +[rm-(a+b+c\,..)-p-2]^{t-2}(ab+ac+bc\,..)[D]^2 \\ \quad\quad\vdots \\ +[rm-(a+b+c\,..)-p-t]^0\ \ (abc\ldots \quad\quad\quad\quad)[D]^t, \end{array}\right.$$

where the curve U^m is a curve *without cusps*, and having therefore a deficiency $D=\frac{1}{2}(m-1)(m-2)-\delta$; the numbers $a, b, c, ..$ are assumed to be all of them unequal, but if we have α of them each $=a$, β of them each $=b$, &c., then the foregoing expression is to be divided by $[\alpha]^\alpha[\beta]^\beta\ldots$; and μ denotes the number of the curves C^r which satisfy the system of conditions obtained from the given system by replacing the conditions of the t contacts of the orders a, b, c, &c. respectively by the condition of passing through $a+b+c\ldots$ arbitrary points. In order that the formula may give the number of the *proper curves* C^r which satisfy the prescribed conditions, it is sufficient that the $\frac{1}{2}r(r+3)-(a+b+c\,..)-p$ conditions shall include the conditions of passing through at least a certain number T of arbitrary points: this restriction applies to all the formulæ of the present section.

75. I will for convenience consider this formula under a somewhat less general form, viz. I will put $p=0$, and moreover assume that the $\frac{1}{2}r(r+3)-(a+b+c\,..)$ conditions are the conditions of passing through this number of arbitrary points; whence $\mu=1$.

. We have thus a curve C^r having with the given curve U^m t contacts of the orders $a, b, c..$ respectively, and besides passing through $\frac{1}{2}r(r+3)-(a+b+c..)$ arbitrary points; and the number of such curves is by the formula $=(a+1)(b+1)(c+1),\ldots$ into

$$\left\{\begin{array}{l} \quad [rm-(a+b+c\,..) \quad\quad]^t \\ +[rm-(a+b+c\,..)-1]^{t-1}(a\ +b\ +c\ ..)[D]^1 \\ +[rm-(a+b+c\,..)-2]^{t-2}(ab+ac+bc\,..)[D]^2 \\ \quad\quad\vdots \\ +[rm-(a+b+c\,..)-t]^0\ \ (abc\ldots \quad\quad\quad\quad)[D]^t, \end{array}\right.$$

where, as before, in the case of any equalities between the numbers $a, b, c, \ldots$, the expression is to be divided by $[\alpha]^\alpha[\beta]^\beta\ldots$.

76. I have succeeded in extending the formula to the case of a curve with cusps: instead of writing down the general formula, I will take successively the cases of a single contact a, two contacts a, b, three contacts a, b, c, &c.; and then denoting the numbers of the curves C^r by (a), (a, b), (a, b, c), &c. in these cases respectively, I say that we have

$$(a)= \quad (a+1)\left\{\begin{array}{l} rm-a \\ +aD \end{array}\right\} \\ \quad -a\ \ .\ \ .\ \ \kappa;$$

$$
\begin{aligned}
(a,\ b) = \quad & (a+1)(b+1)\left\{\begin{array}{ll} & [rm-a-b]^2 \\ + & [rm-a-b-1]^1\,(a+b)\,[D]^1 \\ + & \qquad\qquad ab \quad [D]^2 \end{array}\right\} \\
& -\left\{\begin{array}{l} a\,(b+1)\left\{\begin{array}{ll} & [rm-a-b-1]^1 \\ + & \qquad\qquad bD \end{array}\right\} \\ + b\,(a+1)\left\{\begin{array}{ll} & [rm-a-b-1]^1 \\ + & \qquad\qquad aD \end{array}\right\} \end{array}\right\}[\kappa]^1 \\
& + ab \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\kappa]^2,
\end{aligned}
$$

$$
\begin{aligned}
(a,\ b,\ c) = (a+1)(b+1)(c+1) & \left\{\begin{array}{ll} & [rm-a-b-c\quad]^3 \\ + & [rm-a-b-c-1]^2\,(a+\ b+\ c)\,[D]^1 \\ + & [rm-a-b-c-2]^1\,(ab+ac+bc)\,[D]^2 \\ + & \qquad\qquad abc \qquad\quad [D]^3 \end{array}\right\} \\
- [\Sigma c\,(a+1)(b+1) & \left\{\begin{array}{ll} & [rm-a-b-c-1]^2 \\ + & [rm-a-b-c-2]^1\,(a+b)\,[D]^1 \\ + & \qquad\qquad ab \quad [D]^2 \end{array}\right\}][\kappa]^1 \\
+ [\Sigma bc\,(a+1) & \left\{\begin{array}{ll} & [rm-a-b-c-2]^1 \\ + & \qquad\qquad aD \end{array}\right\}][\kappa]^2 \\
- abc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & [\kappa]^3.
\end{aligned}
$$

77. The foregoing examples are sufficient to exhibit the law; but as I shall have to consider the cases of four and five contacts, I will also write down the formula for $(a,\ b,\ c,\ d)$, putting therein for shortness

$$
\begin{aligned}
& a+b+c+d=\alpha,\ ab+\ldots+cd=\beta,\ abc\ldots+bcd=\gamma,\ abcd=\delta, \\
& a+b+c=\alpha',\ ab+ac+bc=\beta',\ abc=\gamma',\ a+b=\alpha'',\ ab=\beta'',\ a=\alpha''';
\end{aligned}
$$

and also the formula for $(a,\ b,\ c,\ d,\ e)$, putting therein in like manner

$$
(\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon),\ (\alpha',\ \beta',\ \gamma',\ \delta'),\ (\alpha'',\ \beta'',\ \gamma''),\ (\alpha''',\ \beta'''),\ (\alpha'''')
$$

for the combinations of $(a,\ b,\ c,\ d,\ e)$, $(a,\ b,\ c,\ d)$, $(a,\ b,\ c)$, $(a,\ b)$ and (a) respectively. We have

$$
(a,\ b,\ c,\ d) = \quad (a+1)(b+1)(c+1)(d+1)\left\{\begin{array}{ll} & [rm-\alpha\quad]^4 \\ + & [rm-\alpha-1]^3\,\alpha\,[D]^1 \\ + & [rm-\alpha-2]^3\,\beta\,[D]^2 \\ + & [rm-\alpha-3]^1\,\gamma\,[D]^3 \\ + & \qquad\qquad \delta\,[D]^4 \end{array}\right\}
$$

$$
\begin{aligned}
&[\Sigma d\,(a+1)(b+1)(c+1)\left\{\begin{array}{l}[rm-\alpha-1]^3\\ +[rm-\alpha-2]^2\,\alpha'\,[D]^1\\ +[rm-\alpha-3]^1\,\beta'\,[D]^2\\ +\qquad\qquad\gamma'\,[D]^3\end{array}\right\}][\kappa]^1\\
&+[\Sigma cd\,.\,(a+1)(b+1)\left\{\begin{array}{l}[rm-\alpha-2]^2\\ +[rm-\alpha-3]^1\,\alpha''\,[D]^1\\ +\qquad\qquad\beta''\,[D]^2\end{array}\right\}][\kappa]^2\\
&-[\Sigma bcd\,(a+1)\left\{\begin{array}{l}[rm-\alpha-3]^1\\ +\qquad\qquad\alpha'''\,[D]^1\end{array}\right\}][\kappa]^3\\
&+abcd\quad\ldots\ldots\ldots\ldots\quad[\kappa]^4,
\end{aligned}
$$

$$
\begin{aligned}
(a,\,b,\,c,\,d,\,e)=&(a+1)(b+1)(c+1)(d+1)(e+1)\left\{\begin{array}{l}[rm-\alpha\quad]^5\\ +[rm-\alpha-1]^4\,\alpha\quad[D]^1\\ +[rm-\alpha-2]^3\,\beta\quad[D]^2\\ +[rm-\alpha-3]^2\,\gamma\quad[D]^3\\ +[rm-\alpha-4]^1\,\delta\quad[D]^4\\ +\qquad\qquad\epsilon\quad[D]^5\end{array}\right\}\\
&-[\Sigma e\,(a+1)(b+1)(c+1)(d+1)\left\{\begin{array}{l}[rm-\alpha-1]^4\\ +[rm-\alpha-2]^3\,\alpha'\quad[D]^1\\ +[rm-\alpha-3]^2\,\beta'\quad[D]^2\\ +[rm-\alpha-4]^1\,\gamma'\quad[D]^3\\ +\qquad\qquad\delta'\quad[D]^4\end{array}\right\}][\kappa]^1\\
&+[\Sigma de\,(a+1)(b+1)(c+1)\left\{\begin{array}{l}[rm-\alpha-2]^3\\ +[rm-\alpha-3]^2\,\alpha''\quad[D]^1\\ +[rm-\alpha-4]^1\,\beta''\quad[D]^2\\ +\qquad\qquad\gamma''\quad[D]^3\end{array}\right\}][\kappa]^2\\
&-[\Sigma cde\,(a+1)(b+1)\left\{\begin{array}{l}[rm-\alpha-3]^2\\ +[rm-\alpha-4]^1\,\alpha'''\quad[D]^1\\ +\qquad\qquad\beta'''\quad[D]^2\end{array}\right\}][\kappa]^3\\
&+[\Sigma bcde\,(a+1)\left\{\begin{array}{l}[rm-\alpha-4]^1\\ +\qquad\qquad\alpha''''\quad[D]^1\end{array}\right\}][\kappa]^4\\
&-abcde\quad\ldots\ldots\ldots\ldots\quad[\kappa]^5.
\end{aligned}
$$

78. In all these formulæ there is, as before, a numerical divisor in the case of any equalities among the numbers a, b, c, &c. And D denotes, as before, the deficiency, viz. its value now is $D=\frac{1}{2}(m-1)(m-2)-\delta-\kappa$; or observing that the class n is $=m^2-m-2\delta-3\kappa$, we have $D=\frac{1}{2}n-m+1+\frac{1}{2}\kappa$, or say $D=1-m+\frac{1}{2}n+\frac{1}{2}\kappa$, $=1+\Delta$ if $\Delta=-m+\frac{1}{2}n+\frac{1}{2}\kappa$.

79. It is to be observed with reference to the applicability of these formulæ within certain limits only, that the formulæ are the *only* formulæ which are *generally* true; thus taking the simplest case, that of a single contact a, the only algebraical expression for the number of the curves C^r which have with a given curve U^m a contact of the order a, and besides pass through the requisite number $\frac{1}{2}r(r+3)-a$ of arbitrary points, is that given by the formula, viz.

$$(a)=(a+1)(rm-a+aD)-a\kappa.$$

Considering the curve U^m and the order r of the curve C^r as given, if a has successively the values 1, 2,... up to a limiting value of a, the formula gives the number of the proper curves C^r which have with the given curve U^m a contact of the required order a: beyond this limiting value the formula no longer gives the number of the proper curves C^r which satisfy the required condition, and it thus ceases to be applicable; but there is no algebraic function of a which would give the number of the proper curves C^r as well beyond as up to the foregoing limiting value of a.

80. The formulæ are applicable provided only the conditions include the conditions of passing through a sufficient number of arbitrary points; viz. when the number of arbitrary points is sufficiently great, it is not possible to satisfy the conditions specially by means of improper curves C^r, being or comprising a pair of coincident curves. Thus to take a simple example, suppose it is required to find the number of the conics which touch a given curve t times and besides pass through $5-t$ given points: if the number of the given points be 4 or 3 there is no coincident line-pair through the given points, and therefore no coincident line-pair satisfying the given conditions; if the number of the given points is $=2$, then the line joining these points gives a coincident line-pair having at each of its m intersections with the given curve a special contact therewith, that is, having in $\frac{1}{6}m(m-1)(m-2)$ ways three special contacts with the given curve; if the number of the given points is 1 or 0, then in the first case any line whatever through the given point, and in the second case any line whatever, regarded as a coincident line-pair, has m special contacts with the given curve; and so in general there is a certain value for the number of given points, for which value the conditions of contact may be satisfied by a determinate number of improper curves C^r, and for values inferior to it the conditions may be satisfied by infinite series of improper curves C^r. It is by such considerations as these that De Jonquières has determined the minimum value T of the number of arbitrary points to which the conditions should relate in order that the formulæ may be applicable: I refer for his investigation and results to paragraphs XVII and XVIII of his memoir. I remark that in the case where the number of improper solutions is finite, the formula can be corrected so as to give the number of proper solutions by simply

subtracting the number of the improper solutions: but this is not so when the improper solutions are infinite in number; the mode of obtaining the approximate formula is here to be sought in the considerations contained in the first part of the present Memoir; see in particular *ante*, Nos. 8, 9 and 10.

81. The expressions for (a), (a, b), &c. may be considered as functions of rm, $1+\Delta$, and κ, and they vanish upon writing therein $rm=0$, $\Delta=0$, $\kappa=0$; they are consequently of the form $(rm, \Delta, \kappa)^1+(rm, \Delta, \kappa)^2+$ &c., and I represent by $[a]$, $[a, b]$, &c. the several terms $(rm, \Delta, \kappa)^1$, which are the portions of (a), (a, b), &c. respectively, linear in rm, Δ, and κ. The terms in question are obtained with great facility; thus, to fix the ideas, considering the expressions for (a, b, c, d),

1°. To obtain the term in rm, we may at once write $D=1$, $\kappa=0$, the expression is thus reduced to

$$(a+1)(b+1)(c+1)(d+1)\{[rm-\alpha]^4+[rm-\alpha-1]^3\alpha\},$$

and the factor in { } being $=rm\,[rm-\alpha-1]^3$, the coefficient of rm is

$$(a+1)(b+1)(c+1)(d+1)[-\alpha-1]^3,$$

which is

$$=-(a+1)(b+1)(c+1)(d+1).(\alpha+1)(\alpha+2)(\alpha+3).$$

2°. To obtain the term in Δ, writing $rm=0$, $\kappa=0$, and observing that

$$[D]^1=\Delta+1,\quad [D]^2=(\Delta+1)\Delta,\quad [D]^3=(\Delta+1)\Delta(\Delta-1),\quad [D]^4=(\Delta+1)\Delta(\Delta-1)(\Delta-2),$$

&c. give the terms Δ, Δ, $-\Delta$, $+2\Delta$, -6Δ, &c. respectively, the term in Δ is

$$(a+1)(b+1)(c+1)(d+1)\left\{\begin{array}{ll} & [-\alpha-1]^3\alpha.\quad 1 \\ + & [-\alpha-2]^2\beta.\quad 1 \\ + & [-\alpha-3]^1\gamma.-1 \\ + & \delta.\quad 2 \end{array}\right\}\Delta$$

$$=(a+1)(b+1)(c+1)(d+1)\left\{\begin{array}{lll} - & \alpha(\alpha+1)(\alpha+2)(\alpha+3) & \\ + & \beta & (\alpha+2)(\alpha+3) \\ + & \gamma & (\alpha+3) \\ + & 2\delta & \end{array}\right\}\Delta.$$

3°. For the term in κ, writing $rm=0$, $D=1$, and observing that $[\kappa]^1$, $[\kappa]^2$, $[\kappa]^3$, $[\kappa]^4$ give respectively the terms κ, $-\kappa$, 2κ, -6κ, this is

$$=\left[\begin{array}{lll} -\Sigma d & (a+1)(b+1)(c+1) & \{[-\alpha-1]^3+[-\alpha-2]^2\alpha'\}.\quad 1 \\ +\Sigma cd & (a+1)(b-1) & \{[-\alpha-2]^3+[-\alpha-3]^1\alpha''\}.-1 \\ -\Sigma bcd & (a-1) & \{[-\alpha-3]^1+\qquad\alpha'''\}.\quad 2 \\ +\quad abcd & & .-6 \end{array}\right]\kappa,$$

where the terms in { } are

$$-(\alpha+1-\alpha')(\alpha+2)(\alpha+3),\quad (\alpha+2-\alpha'')(\alpha+3) \text{ and } -(\alpha+3-\alpha'''),$$

that is,

$$-(d+1)(\alpha+2)(\alpha+3),\quad (c+d+2)(\alpha+3) \text{ and } -(b+c+d+3)$$

respectively; whence the whole expression is

$$=\left\{\begin{array}{ll} \Sigma d & (a+1)(b+1)(c+1)(d+1).(\alpha+2)(\alpha+3) \\ -\ \Sigma cd & (c+d+2)(a+1)(b+1). \qquad (\alpha+3) \\ +\ 2\Sigma bcd & (b+c+d+3)(a+1) \\ -\ 6\ abcd & \end{array}\right\}\kappa,$$

the expression multiplying $(\alpha+2)(\alpha+3)$ is

$$(a+1)(b+1)(c+1)(d+1)\Sigma d,\ =(a+1)(b+1)(c+1)(d+1)\,\alpha;$$

and we have moreover

$$(a+1)(b+1)(c+1)(d+1)=(1+\alpha+\beta+\gamma+\delta);$$

the other lines are of course expressible in terms of $(\alpha, \beta, \gamma, \delta)$, but as the law of their formation would then be hidden, I abstain from completing the reduction.

82. The series of formulæ is

$$\begin{aligned}[a] \quad &= (a+1)\,rm \\ &+(a+1)\,a\Delta \\ &- \qquad a\kappa,\end{aligned}$$

$$\begin{aligned}[a,\ b] &= -(a+1)(b+1)(\alpha+1) \qquad ..\,rm \\ &-(a+1)(b+1)\left\{\begin{array}{ll} & \alpha(\alpha+1) \\ - & \qquad\quad \beta \end{array}\right\}\Delta \\ &+ \qquad\qquad\qquad \left\{\begin{array}{ll} & \Sigma b(a+1)(b+1) \\ - & ab \end{array}\right\}\kappa,\end{aligned}$$

where $\alpha=a+b$, $\beta=ab$; and coeff. of κ expressed in terms of α, β is $=\alpha(1+\alpha+\beta)-\beta$.

$$\begin{aligned}[a,\ b,\ c] = &\ (a+1)(b+1)(c+1)(\alpha+1)(\alpha+2) \qquad ..\,rm \\ &+(a+1)(b+1)(c+1)\left\{\begin{array}{ll} & \alpha(\alpha+1)(\alpha+2) \\ - & \qquad\quad \beta(\alpha+2) \\ - & \qquad\qquad\quad \gamma \end{array}\right\}\Delta \\ &+\left\{\begin{array}{ll} -\Sigma & c(a+1)(b+1)(c+1)(\alpha+2) \\ +\Sigma bc & (b+c+2)(a+1) \\ -2abc & \end{array}\right\}\kappa,\end{aligned}$$

where $\alpha=a+b+c$, $\beta=ab+ac+bc$, $\gamma=abc$; and the coefficient of κ expressed in terms of α, β, γ is $=-\alpha^3-\alpha^2\beta-\alpha^2\gamma-3\alpha^2-\alpha\beta-2\alpha+2\beta+\gamma$.

$$
\begin{aligned}
[a,\ b,\ c,\ d] = & -(a+1)(b+1)(c+1)(d+1)(\alpha+1)(\alpha+2)(\alpha+3) \quad \ldots rm \\
& -(a+1)(b+1)(c+1)(d+1)\left\{\begin{array}{lll} & \alpha(\alpha+1)(\alpha+2)(\alpha+3) & \\ -\ \beta & & (\alpha+2)(\alpha+3) \\ -\ \gamma & & (\alpha+3) \\ -2\delta & & \end{array}\right\}\Delta \\
& +\left\{\begin{array}{lll} +\ \Sigma & d(a+1)(b+1)(c+1)(d+1)(\alpha+2)(\alpha+3) & \\ -\ \Sigma & cd(c+d+2) \quad (a+1)(b+1) & (\alpha+3) \\ +2\Sigma & bcd(b+c+d+3)(a+1) & \\ -6 & abcd & \end{array}\right\}\kappa,
\end{aligned}
$$

where $\alpha = a+b+c+d, \ldots \delta = abcd.$

$$
\begin{aligned}
[a,\ b,\ c,\ d,\ e] = & \ (a+1)(b+1)(c+1)(d+1)(e+1)(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4) \quad \ldots rm \\
& +(a+1)(b+1)(c+1)(d+1)(e+1)\left\{\begin{array}{lll} & \alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4) & \\ -\ \beta & & (\alpha+2)(\alpha+3)(\alpha+4) \\ -\ \gamma & & (\alpha+3)(\alpha+4) \\ -2\delta & & (\alpha+4) \\ -6\epsilon & & \end{array}\right\}\Delta \\
& +\left\{\begin{array}{lll} -\ \Sigma e & (a+1)(b+1)(c+1)(d+1)(e+1)(\alpha+2)(\alpha+3)(\alpha+4) & \\ +\ \Sigma de & (d+e+2)(a+1)(b+1)(c+1) & (\alpha+3)(\alpha+4) \\ -2\Sigma cde & (c+d+e+3)(a+1)(b+1) & (\alpha+4) \\ +6\Sigma bcde & (b+c+d+e+4)(a+1) & \\ -24abcde & & \end{array}\right\}\kappa,
\end{aligned}
$$

where $\alpha = a+b+c+d+e,\ \beta = \&c., \ldots \epsilon = abcde.$

83. The complete functions (a), $(a,\ b)$, $(a,\ b,\ c)$, &c. may be expressed by means of the linear terms $[a]$, $[a,\ b]$, $[a,\ b,\ c]$, &c. as follows, viz. we have

$$
\begin{aligned}
(a) \quad & = \quad [a], \\
(a,\ b) \quad & = \quad [a][b] \\
& \quad + \quad [a,\ b], \\
(a,\ b,\ c) \quad & = \quad [a][b][c] \\
& \quad + \quad [a][b,\ c] + [b][a,\ c] + [c][a,\ b] \\
& \quad + \quad [a,\ b,\ c], \\
(a,\ b,\ c,\ d) & = \quad [a][b][c][d] \\
& \quad + \Sigma[a][b][c,\ d] \\
& \quad + \Sigma[a,\ b][c,\ d] \\
& \quad + \Sigma[a][b,\ c,\ d] \\
& \quad + \quad [a,\ b,\ c,\ d],
\end{aligned}
$$

and so on: this is easily verified for (a, b), and without much difficulty for (a, b, c), but in the succeeding cases the actual verification would be very laborious.

84. The theoretical foundation is as follows. Writing for greater distinctness $(a)_m$ in place of (a), we have $(a)_m$ to denote the number of the curves C^r which have with a given curve U^m a contact of the order a, and which besides pass through $\frac{1}{2}r(r+3)-a$ points. Let the curve U^m be the aggregate of two curves of the orders m, m' respectively, or say let the curve U^m be the two curves m, m', then we have

$$(a)_{m+m'} = (a)_m + (a)_{m'},$$

a functional equation, the solution of which is

$$(a)_m = [a]_m,$$

where $[a]_m$ is a linear function of n, m, κ, or, what is the same thing, of m, Δ, κ. I assume for the moment that when the coefficients are determined $[a]_m$ would be found to have the value $=[a]$.

Similarly, if $(a, b)_m$ denote the number of the curves C^r which have with the given curve U^m contacts of the orders a and b respectively, and which besides pass through $\frac{1}{2}r(r+3)-a-b$ points, then if the given curve break up into the curves m, m', then we have

$$(a, b)_{m+m'} - (a, b)_m - (a, b)_{m'} = \{(a)_m (b)_{m'}\} + \{(a)_{m'} (b)_m\},$$

where $\{(a)_m (b)_{m'}\}$ is the number of the curves C^r which have with m a contact of the order a and with m' a contact of the order b, and which pass through the $\frac{1}{2}r(r+3)-a-b$ points; and the like for $\{(a)_{m'} (b)_m\}$. Then, not universally, but for values of a and b which are not too great, the order of the aggregate condition is equal to the product of the orders of the component conditions (*ante*, No. 12), that is, we have

$$\{(a)_m (b)_{m'}\} = (a)_m \cdot (b)_{m'} = [a]_m [b]_{m'},$$
$$\{(a)_{m'} (b)_m\} = (a)_{m'} \cdot (b)_m = [a]_{m'} [b]_m,$$

and thence the functional equation

$$(a, b)_{m+m'} - (a, b)_m - (a, b)_{m'} = [a]_m [b]_{m'} + [a]_{m'} [b]_m.$$

But $[a]_m$, &c. being linear functions of m, Δ, κ, we have

$$[a]_{m+m'} = [a]_m + [a]_{m'}, \quad [b]_{m+m'} = [b]_m + [b]_{m'},$$

and thence a particular solution of the equation is at once seen to be $[a]_m [b]_m$; the general solution is therefore

$$(a, b)_m = [a]_m [b]_m + [a, b]_m,$$

where $[a, b]_m$ is an arbitrary linear function of m, Δ, κ. Hence, assuming for the present that if determined its value would be found to be $=[a, b]$, we have the required formula $(a, b) = [a][b] + [a, b]$.

The investigation of the expression for $(a, b, c)_m$ depends in like manner on the assumption that we have

$$\{(a)_m (b, c)_{m'}\} = (a)_m \,.\, (b, c)_{m'} = [a]_m \{[b]_{m'} [c]_{m'} + [b, c]_{m'}\},$$

and so in the succeeding cases; and we thus, within the limits in which these assumptions are correct, obtain the series of formulæ for (a, b), (a, b, c)....

85. It is to be observed in the investigation of (a, b) that if $a = b$, the two terms $[a]_m [b]_{m'}$ and $[a]_{m'} [b]_m$ become equal, and the equal value must be taken not twice but only once, that is, the functional equation is

$$(a, a)_{m+m'} - (a, a)_m - (a, a)_{m'} = [a]_m [a]_{m'},$$

and the solution, writing $\frac{1}{2}[a, a]_m$ for the arbitrary linear function, is

$$(a, a)_m = \tfrac{1}{2}[a]_m [a]_m + \tfrac{1}{2}[a, a]_m,$$

in which solution it would appear, by the determination of the arbitrary function, that $[a, a]$ has the value obtained from $[a, b]$ by writing therein $b = a$. Writing the equation in the form

$$(a, a) = \tfrac{1}{2}[a][a] + \tfrac{1}{2}[a, a],$$

and comparing with the equation for (a, b), we see that $[a, b]$ is *not* to be considered as acquiring any divisor when b is put $= a$, but that the divisor is introduced as a divisor of the whole right-hand side of the equation in virtue of the remark as to the divisor of the functions (a, b), (a, b, c)... in the case of any equalities between the numbers $(a, b, c \ldots)$. This is generally the case, and the foregoing expressions for $[a, b]$, $[a, b, c]$, &c. are thus to be regarded as true without modification even in the case of any equalities among the numbers a, b, c....

86. To complete according to the foregoing method the determination of the expressions for (a), (a, b),.., we have to determine the linear functions $[a]$, $[a, b]$, &c., which are each of them of the form $fm + g\Delta + h\kappa$, where (f, g, h) are functions of r and of a, b, &c.; and I observe that the determination can be effected if we know the values of (a), (a, b), &c. in the cases of a unicursal curve without cusps and with a single cusp respectively. Thus assume that in these two cases respectively we have

$$(a) = (a+1)(rm - a),$$

$$(a) = (a+1)(rm - a) - a.$$

Writing first $\Delta = -1$, $\kappa = 0$, and secondly $\Delta = -1$, $\kappa = 1$, we have

$$(a+1)(rm - a) \quad = fm - g,$$

$$(a+1)(rm - a) - a = fm - g + h,$$

whence

$$f = (a+1)r, \quad g = (a+1)a, \quad h = -a,$$

giving the foregoing value

$$[a] = (a+1)\,rm + (a+1)\,a\Delta - a\kappa.$$

Similarly, for two contacts assume that we have in the two cases respectively

$$(a,\ b) = (a+1)(b+1)[rm - a - b]^2,$$

$$(a,\ b) = (a+1)(b+1)[rm - a - b]^2 - \{a(b+1) + b(a-1)\}[rm - a - b - 1]^1.$$

Starting here from the formula $[a,\ b] = (a,\ b) - [a][b] = fm + g\Delta + h\kappa$, and writing successively $\Delta = -1$, $\kappa = 0$, and $\Delta = -1$, $\kappa = 1$, we have

$$(a+1)(b+1)[rm - a - b]^2 - \{(a+1)(rm - a)\}\{(b+1)(rm - b)\} = fm - g,$$

$$(a+1)(b+1)[rm - a - b]^2 - \{a(b+1) + b(a+1)\}[rm - a - b - 1]^1$$
$$- \{(a+1)(rm - a) - a\}\{(b+1)(rm - b) - b\} = fm - g + h;$$

the first of which, putting therein $a + b = \alpha$, $ab = \beta$, is at once reduced to

$$(a+1)(b+1)\{rm(-\alpha - 1) + \alpha(\alpha+1) - \beta\} = fm - g,$$

whence $f = -(a+1)(b+1)(\alpha+1)r$, $g = -(a+1)(b+1)(\alpha(\alpha+1) - \beta)$. And taking the difference of the two equations, we have

$$-\{a(b+1) + b(a+1)\}(rm - a - b - 1)$$
$$+ a(b+1)(rm - b) + b(a+1)(rm - a) - ab = h,$$

that is $h = (a+1)(b+1)(a+b) - ab$; whence $[a,\ b]$ has the value above assigned to it.

87. The actual calculation of $[a,\ b,\ c]$ would be laborious, and that of the subsequent terms still more so; but it is clear that the principle applies, and that the foregoing values, assuming them to be correct, would be obtained if only we know, for a unicursal curve without cusps, that

$$(a,\ b,\ c, \ldots) = (a+1)(b+1)(c+1)\ldots[rm - (a+b+c, \ldots)]^t$$

(t the number of contacts a, b, c, ...), and for a unicursal curve with a single cusp, that

$$(a,\ b,\ c, \ldots) = (a+1)(b+1)(c+1)\ldots[rm - (a+b+c\ldots)]^t$$
$$- \Sigma a(b+1)(c+1)\ldots[rm - (a+b+c\ldots) - 1]^{t-1},$$

viz. that the diminution of $(a,\ b,\ c, \ldots)$ occasioned by the single cusp is

$$= [rm - (a+b+c, \ldots) - 1]^{t-1} . \Sigma\,[a(b+1)(c+1)\ldots].$$

88. Consider a unicursal curve U^m, and a curve C^r having therewith t contacts of the orders $a, b, c, \ldots$ respectively. The coordinates (x, y, z) of any point of the unicursal curve are given as functions of the order m of a variable parameter θ; and substituting these values in the equation of the curve C^r, we have an equation of the degree rm in θ, but containing the coefficients of C^r linearly; this equation gives of course the values of θ which correspond to the rm intersections of the two curves. Hence in order that the curve C^r may have the prescribed contacts with U^m, the equation of the degree rm in θ must have t systems of equal roots, viz. a system of a equal roots, another system of b equal roots, &c.: this implies between the coefficients of the equation an $(a+b+c\ldots)$ fold relation, which may be shown to be of the order $(a+1)(b+1)(c+1)\ldots[rm-(a+b+c\ldots)]^t$; and since the coefficients in question are linear in regard to the coefficients in the equation of the curve C^r, the order of the relation between the last-mentioned coefficients has the same value; that is, the number of the curves C^r which have the prescribed contacts with the unicursal curve U^m and besides pass through the requisite number of given points, is $=(a+1)(b+1)(c+1)\ldots[rm-(a+b+c\ldots)]^t$.

89. The reduction in the case of a cusp appears to be caused as follows:—Consider on the curve U^m a points indefinitely near to the cusp, and let the condition of the curve C^r having the contact of the a-th order be replaced by the condition of passing through the a points; that is, consider the curves C^r which have with the curve U^m $(t-1)$ contacts of the orders $b, c, \ldots$ respectively, which pass through the a points on the curve U^m in the neighbourhood of the cusp, and which also pass through the requisite number of arbitrary points. The number of these curves is $=(b+1)(c+1)\ldots[rm-a-(b+c+..)]^{t-1}$ (the term $rm-a$ instead of rm, on account of the given a points on the curve: compare herewith De Jonquières' formula containing $rm-p$). Each of these curves, in that it passes through a points in the neighbourhood of the cusp, will *ipso facto* pass through $a+1$ points (viz. a curve which simply passes through the cusp of a cuspidal curve meets the cuspidal curve there in two points, a curve which touches the cuspidal tangent meets the curve in three points, &c.), and be consequently, in an improper sense, a curve having a contact of the a-th order with the given curve U^m. I assume that it counts as such curve a times, and this being so, we have, on account of the curves in question, a reduction $=a(b+1)(c+1)\ldots[rm-(a+b+c\ldots)]^{t-1}$. We have in like manner for the curves passing through b points in the neighbourhood of the cusp a reduction $=b(a+1)(c+1)\ldots[rm-(a+b+c\ldots)]^{t-1}$, &c., and hence when the given unicursal curve U^m has a cusp, the total reduction on account of the cusp in the number of the curves C^r which have with the given curve the t contacts of the orders $a, b, c, \ldots$ and besides pass through the requisite number of given points, is

$$=\{a(b+1)(c+1)\ldots+b(a+1)(c+1)+\&c.\}[rm-(a+b+c\ldots)]^{t-1},$$

which is the auxiliary theorem in question; some of the steps require however to be more completely made out.

90. I have calculated the following numerical results, wherein, as before, $\Delta=D-1=-m+\frac{1}{2}n+\frac{1}{2}\kappa$. I find also their values in the case where the curve C^r

is a conic (that is $r=2$) first in terms of m, n, κ, and finally in terms of m, n, α ($\alpha = 3n+\kappa$, as above). The results are

					for $r=2$, that is, curve C^r a conic.					
[1]=	$2rm+$	$2\Delta-$	κ	=	$2m+$	n		=	$2m+$	n	
[2]=	$3rm+$	$6\Delta-$	2κ	=		$3n+$	κ	=			α
[3]=	$4rm+$	$12\Delta-$	3κ	= −	$4m+$	$6n+$	3κ	= −	$4m-$	$3n+$	3α
[4]=	$5rm+$	$20\Delta-$	4κ	= −	$10m+$	$10n+$	6κ	= −	$10m-$	$8n+$	6α
[5]=	$6rm+$	$30\Delta-$	5κ	= −	$18m+$	$15n+$	10κ	= −	$18m-$	$15n+$	10α
[1, 1]= −	$12rm-$	$20\Delta+$	7κ	= −	$4m-$	$10n-$	3κ	= −	$4m-$	$n-$	3α
[1, 2]= −	$24rm-$	$60\Delta+$	16κ	=	$12m-$	$30n-$	14κ	=	$12m+$	$12n-$	14α
[1, 3]= −	$40rm-$	$136\Delta+$	29κ	=	$56m-$	$68n-$	39κ	=	$56m+$	$49n-$	39α
[1, 4]= −	$60rm-$	$260\Delta+$	46κ	=	$140m-$	$130n-$	84κ	=	$140m+$	$122n-$	84α
[2, 2]= −	$45rm-$	$144\Delta+$	32κ	=	$54m-$	$72n-$	40κ	=	$54m+$	$48n-$	40α
[2, 3]= −	$72rm-$	$288\Delta+$	54κ	=	$144m-$	$144n-$	90κ	=	$144m+$	$126n-$	90α
[1, 1, 1]=	$160rm+$	$352\Delta-$	98κ	= −	$32m+$	$176n+$	78κ	= −	$32m-$	$58n+$	78α
[1, 1, 2]=	$360rm+$	$1056\Delta-$	240κ	= −	$336m+$	$528n+$	288κ	= −	$336m-$	$336n+$	288α
[1, 1, 3]=	$672rm+$	$2528\Delta-$	478κ	= −	$1184m+$	$1264n+$	786κ	= −	$1184m-$	$1094n+$	786α
[1, 2, 2]=	$756rm+$	$2700\Delta+$	530κ	= −	$1188m+$	$1350n+$	820κ	= −	$1188m-$	$1110n+$	820α
[1, 1, 1, 1]= −	$3360rm-$	$8928\Delta+$	2106κ	=	$2208m-$	$4464n-$	2358κ	=	$2208m+$	$2610n-$	2358α
[1, 1, 1, 2]= −	$8064rm-$	$26784\Delta+$	5376κ	=	$10656m-$	$13392n-$	8016κ	=	$10656m+$	$10656n-$	8016α
[1, 1, 1, 1, 1]=	$96768rm+$	$296448\Delta-$	61464κ	= −	$102912m+$	$148224n+$	86760κ	= −	$10912m-$	$112056n+$	86760α

(It may be noticed as a curious circumstance that in the last column in the expressions of [2], [1, 2], [1, 1, 2] and [1, 1, 1, 2] respectively, the coefficients of m and n are in each case equal.)

91. In the case of the conic, (1), (2), &c. are the expressions denoted in the former part of this Memoir by (1 ::), (2 ∴), &c., the number of points being in each case such as to make in all five conditions; calculating these functions by means of the formulæ $(a)=[a]$, &c., the comparison of the resulting values with the values previously obtained will show *à posteriori* the limits within which the formulæ are applicable; where they cease to be applicable I find the difference, and annex it as a correction to the formula value: I have in some cases given what seems to be the proper theoretical form of this difference. We have

$$(1\ ::) \quad = \quad 2m+\ n;$$

$$(2\ \therefore) \quad = \quad \alpha;$$

$$(3\ :) \quad = -\ 4m-\ 3n+\ 3\alpha;$$

$$(4\ \cdot) \quad = -10m-\ 8n+\ 6\alpha;$$

$$(5) \quad = -18m-15n+10\alpha-[-3m+\alpha] \quad (=-[\iota]);$$

$$
\begin{aligned}
2\,(1,\ 1\ \therefore) \quad &= \quad (2m+n)^2 \\
&\quad - 4m - n - 3\alpha\,; \\
(1,\ 2\ :) \quad &= \quad (2m+n)\,\alpha \\
&\quad + 12m + 12n - 14\alpha\,; \\
(1,\ 3\ \cdot) \quad &= \quad (2m+n)(-4m-3n+3\alpha) \\
&\quad + 56m + 49n - 39\alpha\,; \\
(1,\ 4) \quad &= \quad (2m+n)(-10m-8n+6\alpha) \\
&\quad + 140m + 122n - 84\alpha \\
&\quad - [(m-3)(-12m-6n+6\alpha)] \quad (= -[(m-3)(4\iota+2\kappa)])\,; \\
2\,(2,\ 2\ \cdot) \quad &= \quad \alpha^2 \\
&\quad + 54m + 48n - 40\alpha\,; \\
(2,\ 3) \quad &= \quad \alpha(-4m-3n+3\alpha) \\
&\quad + 144m + 126n - 90\alpha \\
&\quad - [24m + 6n + (n-12)\,\alpha] \quad (= -[6\tau + (n-3)\,\kappa])\,; \\
6\,(1,\ 1,\ 1\ :) \quad &= \quad (2m+n)^3 \\
&\quad + 3\,(2m+n)(-4m-n-3\alpha) \\
&\quad - 32m - 58n + 78\alpha \\
&\quad - [4m(m-1)(m-2)]\,; \\
2\,(2,\ 1,\ 1\ \cdot) \quad &= \quad (2m+n)^2\,\alpha \\
&\quad + 2\,(2m+n)(12m+12n-14\alpha) + \alpha(-4m-n-3\alpha) \\
&\quad - 336m - 336n + 288\alpha \\
&\quad - [2\alpha(m-2)(m-3)] \quad (= -[6n(m-2)(m-3) + 2\kappa(m-2)(m-3)])\,; \\
2\,(3,\ 1,\ 1) \quad &= \quad (2m+n)^3(-4m-3n+3\alpha) \\
&\quad + 2\,(2m+n)(56m+49n-39\alpha) \\
&\quad + (-4m-n-3\alpha)(-4m-3n+3\alpha) \\
&\quad - 1184m - 1094n + 786\alpha \\
&\quad - \begin{bmatrix} -13m^3 - 8m^2n + 4mn^2 + 131m^2 + 92mn - 8n^2 - 316m - 226n \\ + \alpha\,(9m^2 - 87m - 3n + 204) \end{bmatrix}; \\
2\,(2,\ 2,\ 1) \quad &= \quad (2m+n)\,\alpha^2 \qquad\qquad (= -\begin{bmatrix} \delta\,(2n + 10m - 38) \\ + \kappa\,(3m^2 - 19m + 30) \\ + \iota\,(6m^2 - 41m + 69) \\ + \tau\,(8m - 32) \end{bmatrix}) \\
&\quad + 2\alpha\,(12m + 12n - 14\alpha) \\
&\quad + (2m+n)(54m + 48n - 40\alpha) \\
&\quad - 1188m - 1110n - 820\alpha
\end{aligned}
$$

$$-\begin{bmatrix} 60m^2+42mn-252m-174n \\ +\alpha\,(-40m+166) \\ +\alpha^2(m-4) \end{bmatrix} \left(=-\begin{bmatrix} (m-4)(\kappa^2-\kappa) \\ +\ 6(m-4)(n-3)\kappa \\ +18(m-4)\tau \\ +\ (m-3)(4\iota+2\kappa) \end{bmatrix}\right);$$

$$\begin{aligned} 24\ (1,\ 1,\ 1,\ 1\cdot) = \quad & (2m+n)^4 \\ & +6(2m+n)^2(-4m-n-3\alpha) \\ & +3(-4m-n-3\alpha)^2 \\ & +4(2m+n)(-32m-58n+78\alpha) \\ & +2208m+2610n-2358\alpha \\ & -\begin{bmatrix} 14m(m-1)(m-2)(m-3) \\ +16n\ (m+2)(m-2)(m-3) \\ -36\alpha\ (m-2)(m-3) \end{bmatrix} \\ & (=-[(m-2)(m-3)(14m^2+16mn-14m-76n-36\kappa)]); \end{aligned}$$

$$\begin{aligned} 6\ (2,\ 1,\ 1,\ 1) \quad = \quad & (2m+n)^2\alpha \\ & +3(2m+n)^2(12m+12n-14\alpha) \\ & +3(2m+n)\alpha(-4m-n-3\alpha) \\ & +3(2m+n)(-336m-336n+288\alpha) \\ & +\alpha(-32m-58n+78\alpha) \\ & +3(-4m-n-3\alpha)(12m+12n-14\alpha) \\ & +10656m+10656n-8016\alpha \\ & -\begin{bmatrix} 108m^3+108m^2n-1116m^2-1116mn+2736m+2736n \\ +\alpha\,(7m^3+6m^2n-147m^2-30mn+1040m+24n-2256) \\ +\alpha^2(-9m+36), \end{bmatrix}; \end{aligned}$$

where the correction is

$$=-(m-4)\begin{pmatrix} 108m^2+108mn-684m-684n \\ +\alpha\left(7m^2-119m+564+6n(m-1)\right) \\ -9\alpha^2 \end{pmatrix},$$

$$=-(m-4)\begin{pmatrix} 2\delta(21n-36) \\ +2\tau(18m-126) \\ +\ \kappa(7m^2+6mn-65m-13n+165)-9(\kappa^2-\kappa) \\ +\ \iota\ (+16n-96) \end{pmatrix};$$

$$
\begin{aligned}
120\,(1,1,1,1,1) = \quad & (2m+n)^5 \\
& + 10\,(2m+n)^3(-4m-n-3\alpha) \\
& + 10\,(2m+n)^2(-32m-58n+78\alpha) \\
& + 10\,(-4m-n-3\alpha)(-32m-58n+78\alpha) \\
& + \;\; 5\,(2m+n)(2208m+2610n-2358\alpha) \\
& + 15\,(2m+n)(-4m-n-3\alpha)^2 \\
& - 102912m - 112056n + 86760\alpha \\
& - \left(\begin{array}{l}
\quad 31m^5 + \quad 70m^4n + \quad 40m^3n^2 \\
- \quad 310m^4 - \quad 460m^3n - 120m^2n^2 \\
- \quad 235m^3 - \; 1030m^2n - 400m\,n^2 \\
+ 10690m^2 + 16060mn \; + 960 \quad n^2 \\
\quad + \left(\begin{array}{l}
- \quad 210m^3 - 180m^2n \\
+ \quad 2970m^2 + 900mn \\
- 15630m \; - 720n \\
+ 28440
\end{array}\right) \\
+ \alpha^2(135m - 540),
\end{array}\right);
\end{aligned}
$$

where the correction is

$$
= -(m-4)\left(\begin{array}{l}
\quad 31m^4 - 186m^3 - 979m^2 + 6774m \\
+ n\,(70m^3 - 180m^2 - 1750m + 9060) \\
+ 40n^2(m+3)(m-2) \\
+ \alpha\left(\begin{array}{l}
- 210m^2 + 2130m - 7110 \\
- 180n\,(m-1)
\end{array}\right) \\
+ \alpha^2 . 135,
\end{array}\right),
$$

which is

$$
= -(m-4)\left(\begin{array}{l}
31\,(2\delta+3\kappa)^2 + 110\,(2\delta+3\kappa)(2\tau+3\iota) \\
+ \left(\tfrac{70}{3}m^3 + 1142m + 3174 \quad\right)(\;\kappa - \;\iota) \\
+ \left(-14m - 638n - 1524 \quad\right)(2\delta+3\kappa) \\
+ \left(-390m + 110n + 4272\;\right)(2\tau+3\iota) \\
+ \left(-210m^2 - 180mn + 2130m + 990n - 7110\right)\kappa + 135\kappa^2
\end{array}\right)
$$

but I have not sought to further reduce this expression, not knowing the proper form in which to present it.

92. The question which ought now to be considered is to determine the corrections or supplements which should be applied to the foregoing expressions (a), (a, b), &c., or to their equivalents $[a]$, $[a][b]+[a, b]$, &c. in order to obtain formulæ for the cases

beyond the limits within which the present formulæ are applicable; but this I am not in a position to enter upon. If the extended formulæ were obtained, it would of course be an interesting verification or application of them to deduce from them the complete series of expressions $(1::)$, $(2.\cdot.)\ldots(1, 1, 1, 1, 1)$ for the number of the conics which satisfy given conditions of contact with a given curve, and besides pass through the requisite number of given points. It will be recollected that throughout these last investigations, I have put De Jonquières' $p=0$; that is, I have not considered the case of the curves C^r which (among the conditions satisfied by them) have with the curve U^m contacts of given orders at given points of the curve; it is probable that the general formulæ containing the number p admit of extensions and transformations analogous to the formulæ in which p is put $=0$, but this is a question which I have not considered.

93. The set of equations $(a)=[a]$, $(a, b)=[a][b]+[a, b]$, &c., considered irrespectively of the meaning of the symbols contained therein, gives rise to an analytical question which is considered in Annex No. 7.

The question of the conics satisfying given conditions of contact is considered from a different point of view in my Second Memoir above referred to.

Annex No. 1 (referred to in the notice of DE JONQUIÈRES' memoir of 1861).—*On the form of the equation of the curves of a series of given index.*

To obtain the general form of the equation of the curves C^n of a series of the index N, it is to be observed that the equation of any such curve is always included in an equation of the order n in the coordinates, containing linearly and homogeneously certain parameters $a, b, c, \ldots$; this is universally the case, as we may, if we please, take the parameters $(a, b, c, \ldots)$ to be the coefficients of the general equation of the order n; but it is convenient to make use of any linear relations between these coefficients so as to reduce as far as possible the number of the parameters. Assume that the number of the parameters is $=\omega+1$, then in order that the curve should form a series (that is, satisfy $\frac{1}{2}n(n+3)-1$ conditions), we must have a $(\omega-1)$ fold relation between the parameters, or, what is the same thing, taking the parameters to be the coordinates of a point in ω-dimensional space, say the parametric point, the point in question must be situate on a $(\omega-1)$ fold locus. Moreover, the condition that the curve shall pass through a given point establishes between the parameters a linear relation (viz. that expressed by the original equation of the curve regarding the coordinates therein as belonging to the given point, and therefore as constants); that is, when the curve passes through a given point, the corresponding positions of the parametric point are given as the intersections of the $(\omega-1)$ fold locus by an omal onefold locus; the number of the curves is therefore equal to the number of these intersections, that is, to the order of the $(\omega-1)$ fold locus; or the index of the series being assumed to be $=N$, the order of the $(\omega-1)$ fold locus must be also $=N$. That is, the general form of the equation of the curves C^n which form a series of the index N, is that of an equation

of the order n containing linearly and homogeneously the $\overline{\omega+1}$ coordinates of a certain $(\omega-1)$ fold locus of the order N. It is only in a particular case, viz. that in which the $(\omega-1)$ fold locus is unicursal, that the coordinates of a point of this locus can be expressed as rational and integral functions of the order N of a variable parameter θ; and consequently only in this same case that the equation of the curves C^n of the series of the index N can be expressed by an equation $(*\between x, y, z)^n = 0$, or $(*\between x, y, 1)^n = 0$, rational and integral of the degree N in regard to a variable parameter θ.

If in the general case we regard the coordinates of the parametric point as irrational functions of a variable parameter θ, then rationalising in regard to θ, we obtain an equation rational of the order N in θ, but the order in the coordinates instead of being $=n$, is equal to a multiple of n, say qn. Such an equation represents not a single curve but q distinct curves C^n, and it is to be observed that if we determine the parameter by substituting therein for the coordinates their values at a given point, then to each of the N values of the parameter there corresponds a system of q curves, only one of which passes through the given point, the other $\overline{q-1}$ curves are curves not passing through the given point, and having no proper connexion with the curves which satisfy this condition.

Returning to the proper representation of the series by means of an equation containing the coordinates of the parametric point, say an equation $(*\between x, y, 1)^n = 0$, involving the two coordinates (x, y), it is to be noticed that forming the derived equation and eliminating the coordinates of the parametric point, we obtain an equation rational in the coordinates (x, y), and also rational of the degree N in the differential coefficient $\frac{dy}{dx}$; in fact since the number of curves through any given point (x_0, y_0) is $=N$, the differential equation must give this number of directions of passage from the point (x_0, y_0) to a consecutive point, that is, it must give this number of values of $\frac{dy}{dx}$, and must consequently be of the order N in this quantity.

Conversely, if a given differential equation rational in $x, y, \frac{dy}{dx}$, and of the degree N in the last-mentioned quantity $\frac{dy}{dx}$, admit of an algebraical general integral, the curves represented by this integral equation may be taken to be irreducible curves, and this being so they will be curves of a certain order n forming a series of the index N; whence the general integral (assumed to be algebraical) is given by an equation of the above-mentioned form, viz. an equation rational of a certain order n in the coordinates, and containing linearly and homogeneously the $\omega+1$ coordinates of a variable parametric point situate on an $(\omega-1)$ fold locus. The integral equation expressed in the more usual form of an equation rational of the order N in regard to the parameter or constant of integration, will be in regard to the coordinates of an order equal to a multiple of n, say $=qn$, and for any given value of the parameter will represent not a single curve C^n, but a system of q such curves: the first-mentioned form is, it is clear, the one to be preferred.

Annex No. 2 (referred to, No. 17).—*On the line-pairs which pass through three given points and touch a given conic.*

Taking the given points to be the angles of the triangle formed by the lines $(x=0, y=0, z=0)$, we have to find (f, g, h) such that the conic $(0, 0, 0, f, g, h \between x, y, z)^2=0$, or, what is the same thing, $fyz+gzx+hxy=0$, shall reduce itself to a line-pair, and shall touch a given conic $(1, 1, 1, \lambda, \mu, \nu \between x, y, z)^2=0$. The condition for a line-pair is that one of the quantities f, g, h shall vanish, viz. it is $fgh=0$; the condition for the contact of the two conics is found in the usual manner by equating to zero the discriminant of the function $1-(\lambda+\theta f)^2-(\mu+\theta g)^2-(\nu+\theta h)^2+2(\lambda+\theta f)(\mu+\theta g)(\nu+\theta h)=(a, b, c, d \between \theta, 1)^3$ suppose; the values of a, b, c, d being

$$\begin{aligned} a &= 2fgh, \\ b &= -\tfrac{1}{3}(f^2+g^2+h^2-2\lambda gh-2\mu hf-2\nu fg), \\ c &= \tfrac{2}{3}\big((\mu\nu-\lambda)f+(\nu\lambda-\mu)g+(\lambda\mu-\nu)h\big), \\ d &= 1-\lambda^2-\mu^2-\nu^2+2\lambda\mu\nu. \end{aligned}$$

Hence considering (f, g, h) as the coordinates of the parametric point, we have the discriminant-locus $a=0$, and the contact-locus

$$a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd=0,$$

and at the intersection of the two loci, $a=0$, $b^2(4bd-3c^2)=0$, equations breaking up into the system $(a=0, b=0)$ twice, and the system $a=0$, $4bd-3c^2=0$; the former of these is

$$fgh=0, \quad f^2+g^2+h^2-2\lambda gh-2\mu hf-2\nu fg=0,$$

which expresses that the intersection of the two lines of the line-pair intersect on the given conic; in fact the system is satisfied by $f=0$, $g^2+h^2-2\lambda gh=0$, giving a line-pair $x(hy+gz)=0$, the two lines whereof intersect on the conic $(1, 1, 1, \lambda, \mu, \nu \between x, y, z)^2=0$; and similarly, if $g=0$, then $h^2+f^2-2\mu hf=0$, or if $h=0$, then $f^2+g^2-2\lambda fg=0$. As noticed above this system occurs twice.

The second system is

$$fgh=0, \qquad (f^2+g^2+h^2-2\lambda gh-2\mu hf-2\nu fg)(1-\lambda^2-\mu^2-\nu^2+2\lambda\mu\nu) \\ +\big((\mu\nu-\lambda)f+(\nu\lambda-\mu)g+(\mu\lambda-\nu)h\big)^2=0,$$

or, as the second equation may also be written,

$$f^2(1-\mu^2)(1-\nu^2)+g^2(1-\nu^2)(1-\lambda^2)+h^2(1-\lambda^2)(1-\mu^2) \\ +2gh(1-\lambda^2)(\mu\nu-\lambda)+2hf(1-\mu^2)(\nu\lambda-\mu)+2fg(1-\nu^2)(\lambda\mu-\nu)=0,$$

which expresses that a line of the line-pair touches the conic; in fact the system is satisfied by $f=0$, $g^2(1-\nu^2)+h^2(1-\mu^2)+2gh(\mu\nu-\lambda)=0$, viz. we have here the line-pair $x(hy+gz)=0$, in which the line $hy+gz=0$ touches the conic $(1, 1, 1, \lambda, \mu, \nu \between x, y, z)^2=0$ and the like if $g=0$, or if $h=0$. This system it has been seen occurs only once.

Annex No. 3 (referred to, No. 22).—*On the conics which pass through two given points and touch a given conic.*

Consider the conics which pass through two given points and touch a given conic. We may take $Z=0$ as the equation of the line through the two given points, and then taking the pole of this line in regard to the given conic and joining it with the two given points respectively, the equations of the joining lines may be taken to be $X=0$ and $Y=0$ respectively. This being so, we have for the given points $(X=0, Z=0)$ and $(Y=0, Z=0)$ respectively, and for the given conic

$$aX^2+bY^2+2hXY+cZ^2=0\,;$$

and since the required conic is to pass through the two given points its equation will be of the form

$$wX^2+2xYZ+2yZX+2zXY=0,$$

where (x, y, z, w) are variable parameters which must satisfy a single condition in order that the last-mentioned conic may touch the given conic. The condition is at once seen to be that obtained by making the equation

$$\begin{aligned}&(a+\lambda w)\,bc\\ &-(a+\lambda w)\,(h+\lambda z)^2\\ &-b\lambda^2y^2\\ &-c\lambda^2z^2\\ &+2\lambda^2xy\,(h+\lambda z)=0,\end{aligned}$$

considered as a cubic equation in λ, have a pair of equal roots; or if we write

$$\begin{aligned}A&=3c\,(ab-h^2),\\ B&=(ab-h^2)\,w-2chz,\\ C&=-ax^2-by^2-cz^2+2h\,(xy-zw),\\ D&=3z\,(2xy-wz),\end{aligned}$$

then the required condition is

$$A^2D^2+4AC^3+4B^3D-6ABCD-3B^2C^2=0.$$

Hence the conic

$$wX^2+2xYZ+2yZX+2zXY=0$$

satisfies the prescribed conditions, if only the parameters (x, y, z, w) satisfy the last-mentioned equation, that is, if (x, y, z, w) are the coordinates of a point on the sextic surface represented by this equation.

The surface has upon it a cuspidal curve the equations whereof are

$$\left\|\begin{matrix}A, & B, & C\\ B, & C, & D\end{matrix}\right\|=0\,;$$

this may be considered as the intersection of the quadric surface $AC-B^2=0$ and the cubic surface $AD-BC=0$; and the cuspidal curve is consequently a sextic.

The surface has also a nodal curve made up of two conics; to prove this I write for shortness $k=h-\sqrt{ab}$, $k_1=h+\sqrt{ab}$; the values of A, B, C, D then are

$$\begin{aligned} A &= -3ckk_1, \\ B &= -kk_1w-2chz, \\ C &= -ax^2-by^2-cz^2+2h(xy-zw), \\ D &= \quad 3z(2xy-zw); \end{aligned}$$

and it is in the first place to be shown that the surface contains the conic

$$x:y:z:w=\theta\sqrt{b}:\theta\sqrt{a}:1:-k\theta^2+\frac{c}{k},$$

where θ is a variable parameter. Substituting these values, we have

$$\begin{aligned} A &= -3ckk_1, \\ B &= k^2k_1\theta^2-c(3h+\sqrt{ab}), \\ C &= 2kk_1\theta^2-\frac{c}{k}(3h-\sqrt{ab}), \\ D &= 3\left(k_1\theta^2-\frac{c}{k}\right); \end{aligned}$$

and hence

$$\begin{aligned} AD-BC &= -2k\left(kk_1\theta^2+\frac{2c\sqrt{ab}}{k}\right)^2, \\ AC-B^2 &= -k^2\left(kk_1\theta^2+\frac{2c\sqrt{ab}}{k}\right)^2, \\ BD-C^2 &= -\left(kk_1\theta^2+\frac{2c\sqrt{ab}}{k}\right)^2, \end{aligned}$$

values which satisfy identically the equation of the surface written under the form

$$(AD-BC)^2-4(AC-B^2)(BD-C^2)=0.$$

Moreover, proceeding to form the derived equation, and to substitute therein the foregoing values of (x, y, z, w), we have

$$\partial A:\partial B:\partial C:\partial D=0:k^2:2k:3,$$

and then the derived equation is

$$\begin{aligned} &(AD-BC)(\ 3A-2kB-k^2C) \\ &-2(AC-B^2\ )(\ 3B-4kC+k^2D) \\ &-2(BD-C^2\ )(2kA-2k^2B\qquad\ )=0, \end{aligned}$$

that is,

$$\begin{aligned} &-k\,(\,3A-2kB-k^2C) \\ &+k^2(\qquad 3B-4kC+k^2D) \\ &+\ \ (2kA-2k^2B), \end{aligned}$$

or finally

$$-k\,(A-3Bk+3k^2C-k^3D)=0,$$

which is satisfied by the foregoing values of A, B, C, D; hence the conic is a nodal curve on the sextic; and by merely changing the sign of one of the radicals $\sqrt{a}$, $\sqrt{b}$ (and therefore interchanging k, k_1) we obtain another conic which is also a nodal curve on the surface, that is, we have as nodal curves the two conics

$$\begin{cases} x:y:z:w=\theta\sqrt{b}:\ \ \theta\sqrt{a}:1:-k\,\theta^2+\dfrac{c}{k}, \text{ and} \\ x:y:z:w=\theta\sqrt{b}:-\theta\sqrt{a}:1:-k_1\theta^2+\dfrac{c}{k_1}. \end{cases}$$

It is to be remarked that each of the nodal conics meets the cuspidal curve in two points, viz. writing for shortness $\Theta=\dfrac{1}{k}\sqrt{\dfrac{-2c\sqrt{ab}}{k_1}}$, $\Theta_1=\dfrac{1}{k_1}\sqrt{\dfrac{2c\sqrt{ab}}{k}}$, for the intersections of the first conic we have

$$x:y:z:w=\Theta\sqrt{a}:\ \ \Theta\sqrt{b}:1:\frac{c}{k_1} \quad\text{and}\quad =-\Theta\sqrt{a}:-\Theta\sqrt{b}:1:\frac{c}{k_1},$$

and for the intersections with the second conic

$$x:y:z:w=\Theta_1\sqrt{a}:-\Theta_1\sqrt{b}:1:\frac{c}{k} \quad\text{and}\quad =-\Theta_1\sqrt{a}:\ \ \Theta_1\sqrt{b}:1:\frac{c}{k}.$$

The condition of passing through any arbitrary point establishes a linear relation between the parameters (x, y, z, w). Hence, if the conic in addition to the prescribed conditions passes through two other given points, the point (x, y, z, w) is given as the intersection of a line with the sextic surface; the number of intersections is $=6$. If (x, y, z, w) is situate on the cuspidal curve, then the conic instead of simply touching the given conic will have with it a contact of the second order, and if we besides suppose that the conic passes through a given point, then the point (x, y, z, w) is given as the intersection of the cuspidal curve with a plane; the number is $=6$. Similarly, if the conic has two contacts with the given conic, and besides passes through a given point, then the point (x, y, z, w) is given as the intersection of the nodal curve by a plane; the number is $=4$. Finally (observing that in the case in question of the contacts of a conic with a conic we cannot have three simple contacts, or a simple contact and one of the second order), a point of intersection of the nodal and cuspidal curves answers to a contact of the third order; and the number is $=4$. That is, the theory

of the sextic surface leads to the following values (agreeing with those obtained from the formulæ by writing therein $m=n=2$, $\alpha=6$), viz.

$$\begin{aligned}
(1::) &= 6, &&= 2m+n,\\
(1,\ 1\ \therefore) &= 4, &&= 2m^2+2mn+\tfrac{1}{2}n^2-2m-\tfrac{1}{2}n-\tfrac{3}{2}\alpha,\\
(2\ \therefore) &= 6, &&= \alpha,\\
(3:) &= 4, &&= -4m-3n+3\alpha.
\end{aligned}$$

I remark that the section by an arbitrary plane is a sextic curve having 6 cusps and 4 nodes; it is therefore a *unicursal* sextic; this suggests the theorem that the sextic surface is also *unicursal*, viz. that the coordinates are expressible rationally in terms of two parameters; I have found that this is in fact the case. In doing this there is no loss of generality in supposing that $a=b=c=1$; and assuming that this is so, and putting also $-1+h=k$, $1+h=k_1$, and therefore $2h=k+k_1$, we have

$$\begin{aligned}
A &= -3kk_1,\\
B &= -kk_1w-(k+k_1)z,\\
C &= -x^2-y^2-z^2+(k+k_1)(xy-zw),\\
D &= \ \ 3z(2xy-zw).
\end{aligned}$$

The equation of the sextic surface being, as before,

$$A^2D^2+4AC^3+4B^3D-3B^2C^2-6ABCD=0,$$

I say that this equation is satisfied on writing therein

$$\begin{aligned}
x+y &= \sqrt{-\frac{2}{k_1}}\,(1-k_1\alpha)\sin\phi,\\
x-y &= \ \ \sqrt{\frac{2}{k}}\,(1+k\ \alpha)\cos\phi,\\
z &= 1,\\
w &= \left(2\alpha-\frac{1}{k}\right)\cos^2\phi+\left(2\alpha-\frac{1}{k_1}\right)\sin^2\phi,
\end{aligned}$$

where $(\alpha,\ \phi)$ are arbitrary. In fact these values give

$$\begin{aligned}
\tfrac{1}{3}A &= -kk_1 && \cos^2\phi-kk_1 && \sin^2\phi,\\
B &= -k(2\alpha k_1+1)\cos^2\phi-k_1 && (2\alpha k+1)\sin^2\phi,\\
C &= -k(\ \alpha k_1+2)\cos^2\phi-k_1\alpha(\ \alpha k+2)\sin^2\phi,\\
\tfrac{1}{3}D &= -k\alpha^2 && \cos^2\phi-k_1\alpha^2 && \sin^2\phi,
\end{aligned}$$

whence, ω being arbitrary, we have

$$\begin{aligned}
&\tfrac{1}{3}(A,\ B,\ C,\ D\between\omega,\ 1)^3\\
&\qquad = -\left[k\cos^2\phi\,(k_1\omega+1)+k_1\sin^2\phi\,(k\omega+1)\right](\omega+\alpha)^2,
\end{aligned}$$

viz. the equation $(A, B, C, D \between \omega, 1)^3 = 0$, considered as a cubic equation in ω, has the twofold root $\omega = -\alpha$, that is, we have the above relation between (A, B, C, D). Whence also writing $\sin\phi = \frac{2\lambda}{1+\lambda^2}$, $\cos\phi = \frac{1-\lambda^2}{1+\lambda^2}$, the equation of the surface is satisfied by the values

$$\begin{aligned} x+y : x-y : z : w &= \sqrt{-\frac{2}{k_1}}(1-k_1\alpha)\,2\lambda\,(1+\lambda^2) \\ &: \sqrt{\frac{2}{k}}(1+k\alpha) \qquad (1-\lambda^2) \\ &: \qquad\qquad\qquad (1+\lambda^2)^2 \\ &: \left(2\alpha - \frac{1}{k}\right)(1-\lambda^2)^2 + \left(2\alpha - \frac{1}{k_1}\right)4\lambda^2, \end{aligned}$$

or the coordinates are expressed rationally in terms of α, λ.

Annex No. 4 (referred to, Nos. 22 and 71).—*On the Conics which touch a cuspidal cubic.*

In the cuspidal cubic, if $x=0$ be the equation of the tangent at the cusp, $y=0$ that of the line joining the cusp with the inflexion, and $z=0$ that of the tangent at the cusp, then the equation of the curve is $y^2 = x^3 z$; the coordinates of a point on the cubic are given by $x : y : z = 1 : \theta : \theta^3$, where θ is a variable parameter; and we have, at the cusp $\theta = \infty$, at the inflexion $\theta = 0$. In the cubic, $m = n = 3$, $\alpha(=3n+\kappa) = 10$.

Considering now the conic

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

this meets the cubic in the 6 points the parameters of which are determined by the equation

$$(a, b, c, f, g, h \between 1, \theta, \theta^3)^2 = 0,$$

or, what is the same thing,

$$(c, 0, 2f, 2g, b, 2h, a \between \theta, 1)^6 = 0.$$

The discriminant of this sextic function contains the factor c, hence equating the residual factor to zero, we obtain the equation of the contact-locus in the form

$$(c, f, g, b, h, a)^9 = 0.$$

It follows that the number of the conics $(1::)$ is $=9$, which agrees with the general value $(1::) = 2m+n$. If the conic pass through the cusp we have $c=0$, and the equation in θ is reduced to a quartic; it is convenient to alter the letters in such wise that the quartic equation may be obtained in the standard form $(a, b, c, d, e \between \theta, 1)^4 = 0$; viz. this will be the case if the equation of the conic is taken to be

$$(e, 6c, 0, \tfrac{1}{2}a, 2b, 2d \between x, y, z)^2 = 0,$$

and we then obtain the equation of the contact-locus in the form

$$(ae-4bd+3c^2)^3-27\,(ace+2bcd-ad^2-b^2e-c^3)^2=0,$$

which is a onefold locus of the order 6. It follows that we have

$$(\overline{1\kappa 1},\ 1\,\therefore)=6,\ \text{agreeing with}\ (\overline{1\kappa 1},\ 1\,\therefore)=n+2m-3.$$

The condition in order that the conic may touch a given line is given by an equation of the form

$$(*\,\between a^2,\ ab,\ b^2,\ 2ce-3d^2,\ ae-8bd,\ ad-12bc)^1=0,$$

which is a onefold locus of the order 2; it at once follows that we have

$$(\overline{1\kappa 1},\ 1\,:/)=12,\ \text{agreeing with}\ (\overline{1\kappa 1},\ 1\,:/)=2n+4m-6.$$

It is a matter of some difficulty to show that we have

$$(\overline{1\kappa 1},\ 1\,\cdot//)=18,\ \text{agreeing with}\ (\overline{1\kappa 1},\ 1\,\cdot//)=4n+4m-6\,;$$

but I proceed to effect this, first remarking that I do not attempt to prove the remaining case

$$(\overline{1\kappa 1},\ 1\,///)=15,\ \text{agreeing with}\ (\overline{1\kappa 1},\ 1\,///)=4n+2m-3.$$

Investigation of the value $(\overline{1\kappa 1},\ 1\cdot//)=18$:

We have the sextic locus

$$(ae-4bd+3c^2)^3-27\,(ace+2bcd-ad^2-b^2e-c^3)^2=0,$$

and combined therewith two quadric loci,

$$(*\,\between a^2,\ ab,\ b^2,\ 2ce-3d^2,\ ae-8bd,\ ad-12bc)^1=0,$$
$$(*\,'\between a^2,\ ab,\ b^2,\ 2ce-3d^2,\ ae-8bd,\ ad-12bc)^1=0,$$

which intersect in a threefold locus of the order 24; it is to be shown that this contains as part of itself the quadric threefold locus $(a=0,\ b=0,\ 2ce-3d^2=0)$ taken three times, leaving a residual locus of the order $24-6,=18$.

We may imagine the coordinates a, b, c, d, e expressed as linear functions of any four coordinates, and so reduce the problem from a problem in 4-dimensional space to one in ordinary 3-dimensional space. We have thus a sextic surface, and two quadric surfaces; the sextic is a developable surface or torse, having for one of its generating lines the line $a=0,\ b=0$, and for the tangent plane along this line the plane $a=0$; the two quadric surfaces meet in a quartic curve passing through the two points $(a=0,\ b=0,\ 3ce-2d^2=0)$, which are points on the torse; it is to be shown that each of these points counts three times among the intersections of the torse with the quartic curve, the number of the remaining intersections being therefore $24-6,=18$; and in order thereto it is to be shown that each of the points in question $(a=0,\ b=0,\ 3ce-2d^2=0)$

is situate on the nodal line of the torse, and that the quartic curve touches there the sheet which is not touched by the tangent plane $a=0$; for this being so the quartic curve touching one sheet and simply meeting the other sheet meets the torse in three consecutive points, or the two points of intersection count each of them three times.

The torse has the cuspidal line

$$S = ae - 4bd + 3c^2 = 0, \quad T = ace + 2bcd - ad^2 - b^2e - c^3 = 0,$$

and the nodal line

$$\left\| \begin{matrix} 6(ac-b^2), & 3(ad-bc), & ae+2bd-3c^2, & 3(be-cd), & 6(ce-d^2) \\ a \quad , & b \quad , & c \quad , & d \quad , & e \end{matrix} \right\|$$

and the equations of the nodal line are satisfied by the values $(a=0,\ b=0,\ 3ce-2d^2=0)$ of the coordinates of the points in question. To find the tangent planes at these points, starting from the equation $S^3 - 27T^2 = 0$ of the torse, taking (A, B, C, D, E) as current coordinates, and writing

$$\partial = A\partial_a + B\partial_b + C\partial_c + D\partial_d + E\partial_e,$$

then the equation of the tangent plane is in the first instance given in the form $S^2\partial S - 18T\partial T = 0$, which writing therein $(a=0,\ b=0,\ 3ce-2d^2=0)$ assumes, as it should do, the form $0=0$; the left-hand side is in fact found to be $9c^3(3ce-2d^2)A$. Proceeding to the second derived equation, this is $S^2\partial^2 S + 2S(\partial S)^2 - 18T\partial^2 T - 18(\partial T)^2 = 0$, or substituting the values of the several terms, the equation is

$$\begin{aligned} & 9c^4(AE - 4BD + 3C^2) \\ + \ & 3c^2(eA - 4dB + 6cC)^2 \\ + \ & 18c^3\{e(AC-B^2) + 2d(BC-AD) + c(AE+2BD-3C^2)\} \\ - \ & 9\ \{(ce-d^2)A + 2cdB - 3c^2C\}^2 = 0; \end{aligned}$$

the terms in BC, BD, C^2 vanish identically, that in B^2 is $(48-36=)\,12c^2d^2 - 18c^3e$, $= -6c^2(3ce-2d^2)B^2$, which also vanishes; hence there remain only the terms divisible by A, giving first the tangent plane $A=0$, and secondly the other tangent plane,

$$\begin{aligned} & A(-\ 6c^3e^2 + 18cd^2e - 9d^4) \\ + \ & B(-\ 60c^3de + 36cd^3) \\ + \ & C(\ 108c^3e - 54c^2d^2) \\ + \ & D(-\ 36c^3d) \\ + \ & E\,.\,27c^4 \qquad\qquad = 0. \end{aligned}$$

Taking the equations of the quadric surfaces to be

$$(\lambda, \mu, \nu, \rho, \sigma, \tau \between a^2, b^2, ab, 3ce-2d^2, ae-8bd, ad-12bc) = 0,$$
$$(\lambda', \mu', \nu', \rho', \sigma', \tau' \between \quad " \quad " \quad " \quad " \quad) = 0,$$

the equations of the tangent planes are

$$\rho\ (3cE + 3eC - 4dD) + \sigma\ (eA - 8dB) + \tau\ (dA - 12cB) = 0,$$
$$\rho'\ (\qquad \text{,,} \qquad) + \sigma'\ (\quad \text{,,} \quad) + \tau'\ (\quad \text{,,} \quad) = 0,$$

in all which equations we have $3ce - 2d^2 = 0$; and if to satisfy this equation we write $c : d : e = 2 : 3\beta : 3\beta^2$, then the equations of the tangent planes become

$$\beta^3 (A\beta - 8B) + 8\,(3C\beta^2 - 4D\beta + 2E) = 0,$$
$$\rho\ (3CB^2 - 4D\beta + 2E) + (\sigma\beta + \tau)(A\beta - 8B) = 0,$$
$$\rho'\ (\qquad \text{,,} \qquad) + (\sigma'\beta - \tau)\ (\quad \text{,,} \quad) = 0,$$

or the three tangent planes intersect in the line $A\beta - 8B = 0$, $3C\beta^2 - 4D\beta + 2E = 0$, which completes the proof.

Reverting to the sextic locus,

$$(ae + 4bd - 3c^2)^2 - 27\,(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0,$$

considered as a locus in 4-dimensional space depending on the five coordinates (a, b, c, d, e), this has upon it the twofold locus

$$ae - 4bd + 3c^2 = 0, \quad ace + 2bcd - ad^2 - b^2e - c^3 = 0,$$

say the cuspidal locus, of the order 6, and the twofold locus

$$\left\| \begin{matrix} 6\,(ac - b^2), & 3\,(ad - bc), & ae + 2bd - 3c^2, & 3\,(be - cd), & 6\,(ce - d^2) \\ a \quad , & b \quad , & c \quad , & d \quad , & e \end{matrix} \right\| = 0,$$

say the nodal locus, of the order 4: there is also a threefold locus,

$$\left\| \begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \end{matrix} \right\| = 0,$$

say the supercuspidal locus, of the order 4. We thence at once infer

$(\overline{1\kappa 1},\ 2\ \ :) = 6$, agreeing with $(\overline{1\kappa 1},\ 2\ \ :) = \alpha - 4$,

$(\overline{1\kappa 1},\ 1,\ 1\ :) = 4$, ,, ,, $(\overline{1\kappa 1},\ 1,\ 1\ :) = 2m^2 + 2mn + \frac{1}{2}n^2 - 8m - \frac{7}{2}n + 13 - \frac{3}{2}\alpha$,

$(\overline{1\kappa 1},\ 3\ \ :) = 4$, ,, ,, $(\overline{1\kappa 1},\ 3\ \ :) = -4m - 3n - 5 + 3\alpha$;

but I have not investigated the application to the symbols with $\cdot/$ or $//$.

If the conic, instead of simply passing through the cusp, touches the cuspidal tangent, then in the equation $(a, b, 0, f, g, h \between x, y, z)^2 = 0$ of the conic we have $f = 0$, or, what is the same thing, in the equation $(e, 6c, 0, \frac{1}{2}a, 2b, 2d \between x, y, z)^2 = 0$ of the conic we have $a = 0$. The equation in θ is thus reduced to $4b\theta^3 + 6c\theta^2 + 4d\theta + e = 0$. For the independent discussion of this case it is convenient to alter the coefficients so that the equation in θ may be in the standard form $(a, b, c, d \between \theta, 1)^3 = 0$, viz. we

assume the equation of the conic to be $(d, 3b, 0, 0, \frac{1}{2}a, \frac{3}{2}c \between x, y, z)^2 = 0$. The equation of the contact-locus then is

$$a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^3c^2 = 0,$$

viz. this is a developable surface, or torse, of the order 4, and we at once infer

$$(\overline{2\kappa 1}, 1 :) = 4, \text{ agreeing with } (\overline{2\kappa 1}, 1 :) = 2m + n - 5.$$

I will show also that we have

$$(2\kappa\overline{1}, 1 \cdot /) = 6, \text{ agreeing with } (\overline{2\kappa 1}, 1 \cdot /) = 2m + 2n - 6,$$

and

$$(\overline{2\kappa 1}, 1 //) = 5, \quad \text{"} \quad \text{"} \quad (\overline{2\kappa 1}, 1 //) = m + 2n - 4.$$

The condition that the conic may touch an arbitrary line $\alpha x + \beta y + \gamma z = 0$, is in fact

$$(0, -\tfrac{1}{4}a^2, \tfrac{3}{4}(4bd - 3c^2), \tfrac{3}{4}ac, -\tfrac{3}{2}ab, 0 \between \alpha, \beta, \gamma)^2 = 0,$$

which, considering therein (a, b, c, d) as coordinates, is the equation of a quadric surface passing through the conic $a = 0$, $4bd - 3c^2 = 0$; the quartic torse also passes through this conic; hence the quadric surface and the torse intersect in this conic, which is of the order 2, and in a residual curve of the order 6; and the number of the conics $(\overline{2\kappa 1}, 1 \cdot /)$ is equal to the order of this residual curve, that is, it is $= 6$.

If the conic touch a second arbitrary line $\alpha' x + \beta' y + \gamma' z = 0$, then we have in like manner the quadric surface

$$(0, -\tfrac{1}{4}a^2, \tfrac{3}{4}(4bd - 3c^2), \tfrac{3}{4}ac, -\tfrac{3}{2}ab, 0 \between \alpha', \beta', \gamma')^2 = 0;$$

that is, we have the quartic torse and two quadric surfaces, each passing through the conic $a = 0$, $4bd - 3c^2 = 0$, and it is to be shown that the number of intersections not on this conic is $= 5$. The two quadric surfaces intersect in the conic and in a second conic; this second conic meets the torse in 8 points, but 2 of these coincide with the point $a = 0$, $b = 0$, $c = 0$, which is one of the intersections of the two conics (the point $a = 0$, $b = 0$, $c = 0$ is in fact a point on the cuspidal edge of the torse, and, the conic passing through it, reckons for 2 intersections), and 1 of the 8 points coincides with the other of the intersections of the two conics; there remain therefore $8 - 2 - 1, = 5$ intersections, or we have $(2\kappa\overline{1}, 1 //) = 5$.

Annex No. 5 (referred to, Nos. 22 and 71). *On the Conics which have contact of the third order with a given cuspidal cubic, and two contacts (double contact) with a given conic.*

Let the equation of the cuspidal cubic be $x^2z - y^3 = 0$ ($x = 0$ tangent at cusp, $z = 0$ tangent at inflexion, $y = 0$ line joining cusp and inflexion; equation satisfied by

$$x : y : z = 1 : \theta : \theta^2);$$

and let the equation of the given conic be

$$U = (a, b, c, f, g, h \between x, y, z)^2 = 0;$$

then writing

$$\Theta = (a,\ b,\ c,\ f,\ g,\ h \between 1,\ \theta,\ \theta^3)^2$$
$$= c\theta^6 + 2f\theta^4 + 2g\theta^3 + b\theta^2 + 2h\theta + c,$$

the equation of a conic having with the given cubic at a given point $(1,\ \theta,\ \theta^3)$ contact of the second order, and having double contact with the given conic, is

$$\begin{vmatrix} \sqrt{U}, & x, & y, & z \\ \sqrt{\Theta}, & 1, & \theta, & \theta^3 \\ (\sqrt{\Theta})' & . & 1, & 3\theta^2 \\ (\sqrt{\Theta})'' & . & . & 6\theta \end{vmatrix} = 0,$$

viz. in the rational form this is

$$36\theta^2 U - \begin{vmatrix} & x, & y, & z \\ \sqrt{\Theta}, & 1, & \theta, & \theta^3 \\ (\sqrt{\Theta})' & . & 1, & 3\theta^2 \\ (\sqrt{\Theta})'' & . & . & 6\theta \end{vmatrix}^2 = 0,$$

and this will have at the point $(1,\ \theta,\ \theta^3)$ a contact of the third order if θ be determined by

$$\begin{vmatrix} \sqrt{\Theta}, & 1, & \theta, & \theta^3 \\ (\sqrt{\Theta})' & . & 1, & 3\theta^2 \\ (\sqrt{\Theta})'' & . & . & 6\theta \\ (\sqrt{\Theta})''' & . & . & 6 \end{vmatrix} = 0,$$

viz. this is

$$\Theta\,(\sqrt{\Theta})''' - (\sqrt{\Theta})'' = 0\,;$$

or developing and multiplying by $\Theta^{\frac{5}{2}}$, this is

$$\theta\left\{\Theta^2\Theta''' - \tfrac{3}{2}\Theta\Theta'\Theta'' + \tfrac{3}{4}\Theta'^3\right\} - \left(\Theta^2\Theta'' - \tfrac{1}{2}\Theta\Theta'^2\right) = 0,$$

or, what is the same thing,

$$\Theta^2\,(\theta\Theta''' - \Theta'') + \Theta\Theta'\left(-\tfrac{3}{2}\theta\Theta'' + \tfrac{1}{2}\Theta'\right) + \Theta'^2 \,.\, \tfrac{3}{4}\theta\Theta' = 0\,;$$

and substituting for Θ its value, this is

$$(c\theta^6 + 2f\theta^4 + 2g\theta^3 + b\theta^2 + 2h\theta + a)^2\,(45c\theta^4 + 12f\theta^2 - b)$$
$$+ (c\theta^6 + 2f\theta^4 + 2g\theta^3 + b\theta^2 + 2h\theta + a)\ (\ 3c\theta^5 + \ 4f\theta^3 + 3g\theta^2 + b\theta + h)$$
$$(-\ 42c\theta^3 - 32f\theta^2 - 15g\theta^2 - 2b\theta + h)$$
$$+ 3\theta\,(3c\theta^5 + 4f\theta^3 + 3g\theta^2 + b\theta + h)^2 = 0.$$

The coefficients of the powers 16, 15, 14, 13 of θ all vanish, so that this is in fact an equation of the twelfth order $(*\between\theta, 1)^{12}=0$; and putting, as usual,

$$(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch)=(A,\ B,\ C,\ F,\ G,\ H),$$

the equation is found to be

$$\begin{array}{lll}
-\ 4cA\ \theta^{12} & +\ 72hA & +\ 45aB \\
+\ 30cH\ \theta^{11} & +\ 9gB\ \Big\}\ \theta^{7} & -\ 20fC\ \Big\}\ \theta^{4} \\
-\ 36cB & -\ 22bH & +\ 10hG \\
+\ 16fA\ \Big\}\ \theta^{10} & +\ 40aA & +\ 5hF \\
-\ 10cG & -\ 130hH & +\ 20aG\ \Big\}\ \theta^{3} \\
+\ 40gA\ \Big\}\ \theta^{9} & +\ 10gG\ \Big\}\ \theta^{6} & -\ 4bC \\
+\ 20bA & +\ 40fF & -\ 12aF\ \Big\}\ \theta^{2} \\
-\ 60fB & +\ 33hB & -\ 5hC\ \theta \\
-\ 90gH\ \Big\}\ \theta^{8} & +\ 2bG\ \Big\}\ \theta^{5} & -\ aC\ =0, \\
 & -\ 108aH &
\end{array}$$

where the form of the coefficients may be modified by means of the identical equations

$$\begin{aligned}
&(A,\ H,\ G\between a,\ h,\ g)=K,\\
&(H,\ B,\ F\between\ \ ,,\ \ )=0,\\
&(G,\ F,\ C\between\ \ ,,\ \ )=0,\\
&(A,\ H,\ G\between h,\ b,\ f)=0,\\
&(H,\ B,\ F\between\ \ ,,\ \ )=K,\\
&(G,\ F,\ C\between\ \ ,,\ \ )=0,\\
&(A,\ H,\ G\between g,\ f,\ c)=0,\\
&(H,\ B,\ F\between\ \ ,,\ \ )=0,\\
&(G,\ F,\ C\between\ \ ,,\ \ )=K.
\end{aligned}$$

There is consequently a conic answering to each value of θ given by this equation, or we have in all 12 conics.

In the case where the given conic breaks up into a pair of lines, or say,

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=2\,(\lambda x+\mu y+\nu z)\,(\lambda' x+\mu' y+\nu' z),$$

then, writing for shortness

$$\mu\nu'-\mu'\nu,\ \nu\lambda'-\nu'\lambda,\ \lambda\mu'-\lambda'\mu=X,\ Y,\ Z,$$

we have

$$(A,\ B,\ C,\ F,\ G,\ H)=(X^2,\ Y^2,\ Z^2,\ YZ,\ ZX,\ XY).$$

Substituting these values, but retaining (a, b, c, f, g, h) as standing for their values $a = 2\lambda\lambda'$, &c., the equation in θ is found to contain the cubic factor $2X\theta^3 - 3Y\theta^2 + Z$, where it is to be observed that this factor equated to zero determines the values of θ which correspond to the points of contact with the cuspidal cubic of the tangents from the point (X, Y, Z), which is the intersection of the lines $\lambda x + \mu y + \nu z = 0$, and $\lambda' x + \mu' y + \nu' z = 0$; and omitting the cubic factor, the residual equation is found to be

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 2cX & -12cY & -8fX & \begin{array}{l}-20gX\\-12fY\\+\ 4cZ\end{array} & \begin{array}{l}-10bX\\+\ 3gY\end{array} & \begin{array}{l}-40hX\\-\ 8bY\end{array} & \begin{array}{l}-20aX\\+17hY\\+\ 7gZ\end{array} & \begin{array}{l}+15aY\\+\ 4bZ\end{array} & +5hZ & +aZ\\\hline\end{array}\right)(\theta, 1)^9 = 0,$$

where the form of the coefficients may be modified by means of the identical equations

$$aX + hY + gZ = 0,$$
$$hX + bY + fZ = 0,$$
$$gX + fY + cZ = 0.$$

The equation is of the 9th order, and there are consequently 9 conics.

Annex No. 6 (referred to, No. 48).—*Containing, with the variation referred to in the text,* ZEUTHEN'S *forms for the characteristics of the conics which satisfy four conditions.*

(1)

$$(\ ::\ ) = n + 2m,$$
$$(\ \therefore /\ ) = 2n + 4m,$$
$$(\ :/\!/\ ) = 4n + 4m,$$
$$(\ \cdot/\!/\!/\ ) = 4n + 2m,$$
$$(\ /\!/\!/\!/\ ) = 2n + m;$$

(1, 1)

$$(\ \therefore\ ) = 2m(\ m + n - 3) + \tau,$$
$$(\ :/\ ) = 2m(\ m + 2n - 5) + 2\tau,$$
$$(\ \cdot/\!/\ ) = 2n\ (2m + n - 5) + 2\delta,$$
$$(\ /\!/\!/\ ) = 2n\ (\ m + n - 3) + \delta;$$

(1, 1, 1)

$$(\ :\ ) = \tfrac{1}{3}[2m^3 + 6m^2n - n^3 - 30m^2 - 18mn + 13n^2 + 84m - 42n + (6m + 3n - 26)\tau],$$
$$(\ \cdot/\ ) = \tfrac{1}{6}[(m+n)\big(-(m+n)^2 - 7(m+n) + 48\big) + 4mn(3m + 3n - 13) + 2(3m + 3n - 20)(\delta + \tau)],$$
$$(\ /\!/\ ) = \tfrac{1}{3}[-m^3 + 6mn^2 + 2n^3 + 13m^2 - 18mn - 30n^2 - 42m + 84n + (3m + 6n - 26)\delta];$$

(1, 1, 1, 1)

$$\begin{aligned}(\ \cdot\ ) = \tfrac{1}{6}\{\ & 2(m-3)(m-4)(n^2-m-n)+(n-3)(n-4)(m^2-m-n)\\ & +4(m^2-11m+28)\tau \qquad +2(n^2-11n+28)\delta\\ & +\big(4(n-4)(m-4)-1\big)(2\delta+\tau)+2\delta^2+\tau^2\},\end{aligned}$$

$$\begin{aligned}(\ /\ ) = \tfrac{1}{6}\{\ & (m-3)(m-4)(n^2-m-n)+2(n-3)(n-4)(m^2-m-n)\\ & +2(m^2-11m+28)\tau \qquad +4(n^2-11n+28)\delta\\ & +\big(4(n-4)(m-4)-1\big)(\delta+2\tau)+\delta^2+2\tau^2\};\end{aligned}$$

(2)

$$\begin{aligned}(\ \therefore\ ) &= 3m+\iota,\\ (:/) &= 2(3m+\iota),\\ (\cdot//) &= 2(3m+\iota),\\ (///) &= 3m+\iota,\end{aligned}$$

(2, 1)

$$\begin{aligned}(\ :\ ) &= 3(2mn+n^2+4m-10n)+(2m+n-14)\kappa,\\ (\cdot/) &= 2(3m+\iota)(m+n-12)+24(m+n),\\ (//) &= 3(m^2+2mn-10m+4n)+(m+2n-14)\iota;\end{aligned}$$

(2, 1, 1)

$$\begin{aligned}(\ \cdot\ ) = \ & (2m+n-7)\big(6\tau+(n-3)\kappa\big)\\ & +\big((m-n)(m+n-5)+\tau\big)(3m+\iota-36)\\ & +12(m-n)(m+n-3),\end{aligned}$$

$$\begin{aligned}(\ /\ ) = \ & (m+2n-7)\big(6\delta+(m-3)\iota\big)\\ & +\big((n-m)(m+n-5)+\delta\big)(3m+\iota-36)\\ & +12(n-m)(m+n-3);\end{aligned}$$

2, 2)

$$\begin{aligned}(\ \cdot\ ) &= \tfrac{1}{2}(3m+\iota)^2-3(3m+\iota)-9\tau-8\delta,\\ (\ /\ ) &= \tfrac{1}{2}(3m+\iota)^2-3(3m+\iota)-8\tau-9\delta;\end{aligned}$$

(3)

$$\begin{aligned}(\ :\ ) &= 6n-4m+3\kappa = 5m-3n+3\iota,\\ (\cdot/) &= 10n-8m+6\kappa = 10m-8n+6\iota,\\ (//) &= 5n-3m+3\kappa = 6m-4n+3\iota;\end{aligned}$$

(1, 3)

$$\begin{aligned}(\ \cdot\ ) &= 2(-4m^2+3mn+3n^2+28m-32n)+3(2m+n-13)\kappa,\\ (\ /\ ) &= 2(3m^2+3mn-4m^2-32m+28n)+3(m+2n-13)\iota;\end{aligned}$$

(4)

$$\begin{aligned}(\ \cdot\ ) &= 10n-10m+6\kappa = 8m-8n+6\iota,\\ (\ /\ ) &= 8n-8m+6\kappa = 10m-10n+6\iota.\end{aligned}$$

Annex No. 7 (referred to, No. 93).

In connexion with De Jonquières' formula, I have been led to consider the following question.

Given a set of equations:

$$\begin{aligned}
\mathrm{a} &= a \quad (\text{viz. } \mathrm{b} = b,\ \mathrm{c} = c,\ \&\mathrm{c.}),\\
\mathrm{ab} &= ab \quad \left(\text{viz. } \mathrm{ac} = ac,\ \&\mathrm{c.}, \text{ and the like in all the subsequent equations}\right),\\
&\quad +(11)\, a.b \qquad\qquad +(11)\, a.c,\\
\mathrm{abc} &= abc\\
&\quad +(12)(a.bc + b.ac + c.ab)\\
&\quad +(111)\ a.b.c,\\
\mathrm{abcd} &= abcd\\
&\quad +(13)(a.bcd + \&\mathrm{c.})\\
&\quad +(22)(ab.cd + \&\mathrm{c.})\\
&\quad +(112)(a.b.cd + \&\mathrm{c.})\\
&\quad +(1111)\ a.b.c.d,
\end{aligned}$$

and so on indefinitely (where the $(\cdot)$ is used to denote multiplication, and ab, abc, &c., and also ab, abc, &c. are so many separate and distinct symbols not expressible in terms of a, b, c &c., a, b, c &c.), then we have conversely a set of equations

$$\begin{aligned}
a &= \mathrm{a} \quad (\text{viz. } b = \mathrm{b},\ c = \mathrm{c}\ \&\mathrm{c.}),\\
ab &= \mathrm{ab} \quad \left(\text{viz. } ac = \mathrm{ac}\ \&\mathrm{c.}, \text{ and the like in all the subsequent equations}\right),\\
&\quad +[11]\ \mathrm{a.b} \qquad\qquad +[11]\,\mathrm{a.c},\\
abc &= \mathrm{abc}\\
&\quad +[12](\mathrm{a.bc} + \mathrm{b.ac} + \mathrm{c.ab})\\
&\quad +[111]\ \mathrm{a.b.c},\\
abcd &= \mathrm{abcd}\\
&\quad +[13](\mathrm{a.bcd} + \&\mathrm{c.})\\
&\quad +[22](\mathrm{ab.cd} + \&\mathrm{c.})\\
&\quad +[112](\mathrm{a.b.cd} + \&\mathrm{c.})\\
&\quad +[1111]\ \mathrm{a.b.c.d},
\end{aligned}$$

and so on; and it is required to find the relation between the coefficients ( ) and [ ]; we find, for example,

$$
\begin{aligned}
[11] &= - \quad (11),\\
[12] &= - \quad (12),\\
[111] &= \quad 3 \quad (11)(12)\\
&\quad - \quad (111),\\
[13] &= - \quad (13),\\
[22] &= - \quad (22),\\
[112] &= \quad 2 \quad (13)(12)\\
&\quad + \quad (22)(11)\\
&\quad - \quad (112),\\
[1111] &= - 12 \quad (13)(12)(11)\\
&\quad + 4 \quad (13)(111)\\
&\quad - 3 \quad (22)(11)(11)\\
&\quad + 6 \quad (112)(11)\\
&\quad - \quad (1111);
\end{aligned}
$$

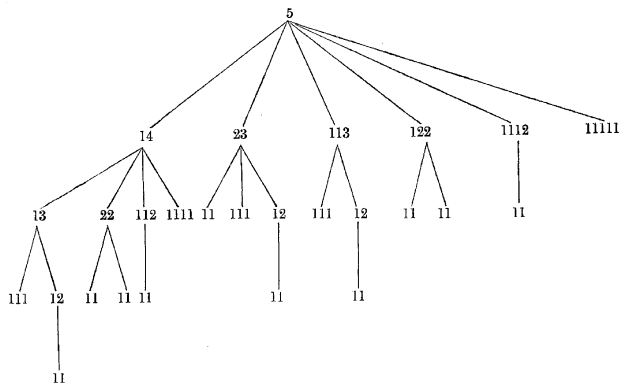
and it is to be noticed that, conversely, the coefficients ( ) are given in terms of the coefficients [ ] by the like equations with the very same numerical coefficients; in fact from the last set of equations, this is at once seen to be the case as far as (112); and for the next term (1111) we have

$$
\begin{aligned}
(1111) = &+ 12\,[13]\,[12]\,[11] && = (12 - 12 - 12 =) - 12 \quad [13]\,[12]\,[11]\\
&- 4\,[13]\,\{3\,[12]\,[11] - [111]\} && \qquad\qquad\qquad\quad + 4 \quad [13]\,[111]\\
&+ 3\,[22]\,[11]\,[11] && + (3 - 6 = \qquad) - 3 \quad [22]\,[11]\,[11]\\
&- 6\,[11] \left\{ \begin{matrix} 2\,[13]\,[12] \\ + [22]\,[11] \\ - [112] \end{matrix} \right\} && \qquad\qquad\qquad\quad + 6 \quad [112]\,[11]\\
&- [1111] && \qquad\qquad\qquad\quad - \quad [1111]
\end{aligned}
$$

having the same coefficients -12, $+4$, -3, $+6$, -1 as in the formula for [1111] in terms of the coefficients ( ); it is easy to infer that the property holds good generally.

To explain the law for the expression of the coefficients of either set in terms of

the other set, I consider, for example, the case where the sum of the numbers in the ( ), or [ ] is = 5; and I form a kind of tree as follows:



the formation of which is obvious; and I derive from it in the manner about to be explained the expressions for the coefficients [14], [23] &c. in terms of the corresponding coefficients in ( ); viz. we have

$$
\begin{aligned}
[14] &= - \quad (14),\\
[23] &= - \quad (23),\\
[113] &= \quad 2\ (14)(13)\\
&\quad + \quad (23)(11)\\
&\quad - \quad (113),\\
[122] &= \quad (14)(22)\\
&\quad + \quad 2\ (23)(12)\\
&\quad - \quad (112),\\
[1112] &= - \quad 6\ (14)(13)(12)\\
&\quad - \quad 3\ (14)(22)(11)\\
&\quad + \quad 3\ (14)(112)\\
&\quad - \quad 6\ (23)(12)(11)\\
&\quad + \quad 3\ (113)(12)\\
&\quad + \quad 1\ (23)(111)\\
&\quad + \quad 3\ (122)(11)\\
&\quad - \quad 1\ (1112),
\end{aligned}
$$

$$
\begin{aligned}
[11111] = &+ 60 \quad (14)(13)(12)(11) \\
&- 20 \quad (14)(13)(111) \\
&+ 15 \quad (14)(22)(11)(11) \\
&- 30 \quad (14)(112)(11) \\
&+ 5 \quad (14)(1111) \\
&+ 30 \quad (23)(12)(11)(11) \\
&- 10 \quad (23)(111)(11) \\
&- 30 \quad (113)(12)(11) \\
&+ 10 \quad (113)(111) \\
&- 15 \quad (122)(11)(11) \\
&+ 10 \quad (1112)(11) \\
&- 1 \quad (11111).
\end{aligned}
$$

To form the symbolic parts, we follow each branch of the tree to each point of its course: thus from the branch 113 we have

(113)	belonging to	[113],
(113)(111)	„	[11111],
(113)(12)	„	[1112],
(113)(12)(11)	„	[11111];

viz. (113) belongs to [113]; (113)(111), read 11 (3 replaced by) 111, belongs to [11111]; (113)(12), read 11 (3 replaced by) 12, belongs to 1112; (113)(12)(11), read 11 (3 replaced by) 1 (2 replaced by) 11, belongs to [11111].

And observe that where (as, for example, with the symbol 122) there are branches derived from two or more figures, we pursue each such branch separately, and also all or any of them simultaneously to every point in the course of such branch or branches; thus for the branch 122 we have

(122)	belonging to	[122],
(122)(11) } (same twice)	„	[1112],
(122)(11) }		
(122)(11)(11)	„	[11111].

Similarly for the branch 23 we have

(23)	belonging to	[23],
(23)(111)	„	[1112],
(23)(12)	„	[122],
(23)(12)(11) (same as *infrà*)	„	[1112],
(23)(11)(111)	„	[11111],
(23)(11)(12) (same as *suprà*)	„	[1112],
(23)(11)(12)(11)	„	[11111].

We thus obtain the symbolic parts of the several expressions for [14], [23].... [11111] respectively: the sign of each term is + or − according as the number of factors in ( ) is even or odd; thus in the expression for [11111], the term (14)(13)(12)(11) has four factors, and is therefore +, the term (113)(12)(11) has three factors, and is therefore −.

The numerical coefficients are obtained as follows. There is a common factor derived from the expression in [ ] on the left-hand side of the equation; viz. for [11111], which contains five equal symbols, this factor is $1.2.3.4.5, =120$; for [1112], which contains three equal symbols, it is $1.2.3, =6$; and so on (for a symbol such as [11222] containing two equal symbols, and three equal symbols, the factor would be $1.2.1.2.3, =12$, and so in other similar cases). In any term on the right-hand side of the equation, we must for a factor such as (11), which contains two equal symbols, multiply by $\frac{1}{2}$; for a factor such as (111), which contains three equal symbols, multiply by $\frac{1}{6}$, and so on. And in the case where a term (as, for example, the term (122)(11) or (23)(12)(11), *vide suprà*) occurs more than once, the term is to be taken account of each time that it occurs; or, what is the same thing, since the coefficient obtained as above is the same for each occurrence, the coefficient obtained as above is to be multiplied by the number of the occurrences of the term. For example, taking in order the several terms of the expression for [1112], the common factor is $=6$, and the several coefficients are

$$6,\quad 6.\tfrac{1}{2},\quad 6.\tfrac{1}{2},\quad 6.\tfrac{1}{2}\times 2,\quad 6.\tfrac{1}{2},\quad 6.\tfrac{1}{6},\quad 6.\tfrac{1}{2}.\tfrac{1}{2}\times 2,\quad 6.\tfrac{1}{6};$$

and similarly in the expression for [11111] the common factor is 120, and the coefficients taken in order are

$$120.\tfrac{1}{2},\quad 120.\tfrac{1}{6},\quad 110.\tfrac{1}{2}.\tfrac{1}{2}.\tfrac{1}{2},\ \&c.,$$

without there being in this case any coefficient with a factor arising from the plural occurrence of the term.

The foregoing result was established by induction, and I have not attempted a general proof.

I observe by way of a convenient numerical verification, that in each equation the sum of the coefficients (taken with their proper signs) is $(-)^{n-1}1.2..(n-1)$; if n be the number of parts in the [ ] ($n=5$ for [11111], $=4$ for [1112] &c.), and moreover, that the sum of these sums each multiplied by the proper polynomial coefficient and the whole increased by unity is $=0$; viz. for

[14], [23], [113], [122], [1112], [11111],

the sums of the coefficients are

$$-1,\quad -1,\quad +2,\quad +2,\quad -6,\qquad +24 \text{ respectively,}$$

and we have

$$1+5(-1)+10(-1)+10(2)+15(2)+10(-6)+1(24),\ =75-75,\ =0.$$

If we have any five distinct things (a, b, c, d, e), then the polynomial coefficients 5, 10, 10, 15, 10, 1 denote respectively the number of ways in which these can be partitioned in the forms 14, 23, 113, 122, 1112, 11111 respectively, and the last-mentioned theorem is thus a theorem in the Partition of Numbers.

407.

SECOND MEMOIR ON THE CURVES WHICH SATISFY GIVEN CONDITIONS; THE PRINCIPLE OF CORRESPONDENCE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVIII. (for the year 1868), pp. 145—172. Received April 18,—Read May 2, 1867.]

In the present Memoir I reproduce with additional developments the theory established in my paper "On the Correspondence of two points on a Curve" (*London Math. Society*, No. VII., April 1866), [385]; and I endeavour to apply it to the determination of the number of the conics which satisfy given conditions; viz. these are conditions of contact with a given curve, or they may include arbitrary conditions Z, $2Z$, &c. If, for a moment, we consider the more general question where the Principle is to be applied to finding the number of the curves C^r of the order r, which satisfy given conditions of contact with a given curve, there are here two kinds of special solutions; viz., we may have proper curves C^r touching (specially) the given curve at a cusp or cusps thereof, and we may have improper curves, that is, curves which break up into two or more curves of inferior orders. In the case where the curves C^r are lines, there is only the first kind of special solution, where the sought for lines touch at a cusp or cusps. But in the case to which the Memoir chiefly relates, where the curves C^r are conics, we have the two kinds of special solutions, viz., proper conics touching at a cusp or cusps, and conics which are line-pairs or point-pairs. In the application of the Principle to determining the number of the conics which satisfy any given conditions, I introduce into the equation a term called the "Supplement" (denoted by the abbreviation "Supp."), to include the special solutions of both kinds. The expression of the Supplement should in every case be furnished by the theory; and this being known, we should then have an equation leading to the number of the conics which properly satisfy the prescribed conditions; but in thus finding the expression of the Supplements, there are difficulties which I am unable to overcome; and I have contented myself with the reverse course, viz., knowing in each case the

number of the proper solutions, I use these results to determine *à posteriori* in each case the expression of the Supplement; the expression so obtained can in some cases be accounted for readily enough, and the knowledge of the whole series of them will be a convenient basis for ulterior investigations.

The Principle of Correspondence for points in a line was established by Chasles in the paper in the *Comptes Rendus*, June—July 1864, referred to in my First Memoir; it is extended to unicursal curves in a paper of the same series, March 1866, "Sur les courbes planes ou à double courbure dont les points peuvent se déterminer individuellement—Application du Principe de Correspondance dans la théorie de ces courbes," but not to the case of a curve of given deficiency D considered in my paper of April 1866 above referred to. The fundamental theorem in regard to unicursal curves, viz. that in a curve of the order m with $\frac{1}{2}(m-1)(m-2)$ double points (nodes or cusps) the coordinates (x, y, z) are proportional to rational and integral functions of a variable parameter θ,—as a case of a much more general theorem of Riemann's—dates from the year 1857, but was first explicitly stated by Clebsch in the paper "Ueber diejenigen ebenen Curven deren Coordinaten rationale Functionen eines Parameters sind," *Crelle*, t. LXIV. (1864), pp. 43—63. See also my paper "On the Transformation of Plane Curves," *London Mathematical Society*, No. III., Oct. 1865, [384].

The paragraphs of the present Memoir are numbered consecutively with those of the First Memoir.

Article Nos. 94 to 104.—*On the Correspondence of two points on a Curve.*

94. In a unicursal curve the coordinates (x, y, z) of any point thereof are proportional to rational and integral functions of a variable parameter θ. Hence if two points of the curve correspond in such wise that to a given position of the first point there correspond α' positions of the second point, and to a given position of the second point α positions of the first point, the number of points which correspond each to itself is $=\alpha+\alpha'$. For let the two points be determined by their parameters θ, θ' respectively, then to a given value of θ there correspond α' values of θ', and to a given value of θ' there correspond α values of θ; hence the relation between (θ, θ') is of the form $(\theta, 1)^{\alpha}(\theta', 1)^{\alpha'}=0$; and writing therein $\theta'=\theta$, then for the points which correspond each to itself, we have an equation $(\theta, 1)^{\alpha+\alpha'}=0$, of the order $\alpha+\alpha'$; that is, the number of these points is $=\alpha+\alpha'$.

Hence for a unicursal curve we have a theorem similar to that of M. Chasles' for a line, viz. the theorem may be thus stated:

If two points of a unicursal curve have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'$. But a unicursal curve is nothing else than a curve with a deficiency $D=0$, and we thence infer:

THEOREM. If two points of a curve with deficiency D have an (α, α') correspondence, the number of united points is $=\alpha+\alpha'+2kD$; in which theorem $2k$ is a coefficient to be determined.

95. Suppose that the corresponding points are P, P' and imagine that when P is given the corresponding points P' are the intersections of the given curve by a curve Θ (the equation of the curve Θ will of course contain the coordinates of P as parameters, for otherwise the position of P' would not depend upon that of P). I find that if the curve Θ has with the given curve k intersections at the point P, then in the system of points (P, P') the number of united points is

$$\mathrm{a} = \alpha + \alpha' + 2kD,$$

whence in particular if the curve Θ does not pass through the point P, then the number of united points is $= \alpha + \alpha'$, as in the case of a unicursal curve. (I have in the paper of April 1866 above referred to, proved this theorem in the particular case where the k intersections at the point P take place in consequence of the curve Θ having a k-tuple point at P, but have not gone into the more difficult investigation for the case where the k intersections arise wholly or in part from a contact of the curve Θ, or any branch or branches thereof, with the given curve at P.)

96. It is to be observed that the general notion of a united point is as follows: taking the point P at random on the given curve, the curve Θ has at this point k intersections with the given curve; the remaining intersections are the corresponding points P'; if for a given position of P one or more of the points P' come to coincide with P, that is, if for the given position of P the curve Θ has at this point more than k intersections with the given curve, then the point in question is a united point.

It might at first sight appear that if for a given position of P a number 2, 3,.. or j of the points P' should come to coincide with P, then that the point in question should reckon, for 2, 3,... or j (as the case may be) united points: but this is not so. This is perhaps most easily seen in the case of a unicursal curve; taking the equation of correspondence to be $(\theta, 1)^\alpha (\theta', 1)^{\alpha'} = 0$, then we have $\alpha + \alpha'$ united points corresponding to the values of θ which satisfy the equation $(\theta, 1)^\alpha (\theta, 1)^{\alpha'} = 0$; if this equation has a j-tuple root $\theta = \lambda$, the point P which answers to this value λ of the parameter is reckoned as j united points. But starting from the equation $(\theta, 1)^\alpha (\theta', 1)^{\alpha'} = 0$, if on writing in this equation $\theta = \lambda$, the resulting equation $(\lambda, 1)^\alpha (\theta', 1)^{\alpha'} = 0$ has a root $\theta' = \lambda$, it follows that the equation $(\theta, 1)^\alpha (\theta, 1)^{\alpha'} = 0$ has a root $\theta = \lambda$, and that the point which belongs to the value $\theta = \lambda$ is a united point; if on writing in the equation $\theta = \lambda$, the resulting equation $(\lambda, 1)^\alpha (\theta', 1)^{\alpha'} = 0$ has a j-tuple root $\theta' = \lambda$, *it does not follow* that the equation $(\theta, 1)^\alpha (\theta, 1)^{\alpha'} = 0$ has a j-tuple root $\theta = \lambda$, nor consequently that the point answering to $\theta = \lambda$ in anywise reckons as j united points.

97. This may be further illustrated by regarding the parameters θ, θ' as the coordinates of a point in a plane; the equation $(\theta, 1)^\alpha (\theta', 1)^{\alpha'} = 0$ is that of a curve of the order $\alpha + \alpha'$, having an α-tuple point at infinity on the axis $\theta = 0$, and an α'-tuple point at infinity on the axis $\theta' = 0$; the united points are given as the intersections of the curve with the line $\theta = \theta'$; a j-fold intersection, whether arising from a multiple point of the curve or from a contact of the line $\theta = \theta'$ with the curve,

gives a point which reckons as j united points. But if $\theta=\lambda$ gives the j-fold root $\theta'=\lambda$, this shows that the line $\theta=\lambda$ has with the curve j intersections at the point $\theta=\theta'=\lambda$; *not* that the line $\theta=\theta'$ has with the curve j intersections at the point in question.

98. Reverting to the notion of a united point as a point P which is such that one or more of the corresponding points P' come to coincide with P; in the case where P is at a node of the given curve, it is necessary to explain that the point P must be considered as belonging to one or the other of the two branches through the node, and that the point P is not to be considered as a united point unless we have *on the same branch of the curve* one or more of the corresponding points P' coming to coincide with the point P. If, to fix the ideas, $k=1$, that is, if the curve Θ simply pass through the point P, then if P be at a node the curve Θ passes through the node and has therefore at this point two intersections with the given curve; but the second intersection belongs to the other branch, and the node is not a united point; in order to make it so, it is necessary that the curve Θ should at the node touch the branch to which the point P is considered to belong. The thing appears very clearly in the case of a unicursal curve; we have here two values $\theta=\lambda$, $\theta=\lambda'$ answering to the node according as it is considered as belonging to one or the other branch of the curve; and in the equation of correspondence $(\theta, 1)^{\alpha}(\theta', 1)^{\alpha'}=0$, writing $\theta=\lambda$, we have an equation $(\lambda, 1)^{\alpha}(\theta', 1)^{\alpha'}=0$ satisfied by $\theta'=\lambda'$ but not by $\theta'=\lambda$, and the equation $(\theta, 1)^{\alpha}(\theta, 1)^{\alpha'}=0$ is thus not satisfied by the value $\theta=\lambda$. The conclusion is that a node *qua* node is not a united point.

99. But it is otherwise as regards a cusp. When the point P is at a cusp, the curve Θ (which has in general with the given curve k intersections at P) has here more than k intersections, and (as in this case there is no distinction of branch) the cusp reckons as a united point. In the case of a unicursal curve, there is at the cusp a single value $\theta=\lambda$ of the parameter, and the equation $(\theta, 1)^{\alpha}(\theta, 1)^{\alpha'}=0$ is satisfied by the value $\theta=\lambda$. But for the very reason that the cusp *qua* cusp reckons as a united point, the cusp is a united point only in an improper or special sense, and it is to be rejected from the number of true united points. We may include the cusps, along with any other special solutions which may present themselves, under a head "Supplement," and instead of writing as above $\mathrm{a}-\alpha-\alpha'=2kD$, write $\mathrm{a}-\alpha-\alpha'+\mathrm{Supp.}=2kD$.

Before going further I apply the theorem to some examples in which the curve Θ is a system of lines.

100. Investigation of the class of a curve of the order m with δ nodes and κ cusps. Take as corresponding points on the given curve two points such that the line joining them passes through a fixed point O; the united points will be the points of contact of the tangents through O; that is, the number of the united points will be equal to the class of the curve. The curve Θ is here the line OP which has with the given curve a single intersection at P; that is, we have $k=1$. The points P' corresponding to a given position of P are the remaining $m-1$ intersections of OP with the curve, that is, we have $\alpha'=m-1$; and in like manner $\alpha=m-1$. Each of the

cusps is (specially) a united point, and counts once, whence the Supplement is $=\kappa$. Hence, writing n for the class, we have $n+2(m-1)+\kappa=2D$, or writing for $2D$ its value $=m^2-3m+2-2\delta-2\kappa$, we have $n=m^2-m-2\delta-3\kappa$, which is right.

101. Investigation of the number of inflexions. Taking the point P' to be a tangential of P (that is, an intersection of the curve by the tangent at P), the united points are the inflexions; and the number of the united points is equal to the number of the inflexions. The curve Θ is the tangent at P having with the given curve two intersections at this point; that is, $k=2$; P' is any one of the $m-2$ tangentials of P, that is, $\alpha'=m-2$; and P is the point of contact of any one of the $n-2$ tangents from P' to the curve, that is, $\alpha=n-2$. Each cusp is (specially) a united point, and counts once, whence the Supplement is $=\kappa$. Hence, writing ι for the number of inflexions, we have

$$\iota-(m-2)-(n-2)+\kappa=4D;$$

or substituting for $2D$ its value expressed in the form $n-2m+2+\kappa$, we have

$$\iota=3n-3m+\kappa,$$

which is right.

102. For the purpose of the next example it is necessary to present the fundamental equation under a more general form. The curve Θ may intersect the given curve in a system of points P', each p times, a system of points Q', each q times, &c. in such manner that the points (P, P'), the points (P, Q'), &c. are pairs of points corresponding to each other according to distinct laws; and we shall then have the numbers $(\mathrm{a}, \alpha, \alpha')$, $(\mathrm{b}, \beta, \beta')$, &c., corresponding to these pairs respectively, viz. (P, P') are points having an (α, α') correspondence, and the number of united points is $=\mathrm{a}$; (P, Q') are points having a (β, β') correspondence, and the number of united points is $=\mathrm{b}$, and so on. The theorem then is

$$p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\&\text{c.}+\text{Supp.}=2kD,$$

being in fact the most general form of the theorem for the correspondence of two points on a curve, and that which will be used in all the investigations which follow.

103. Investigation of the number of double tangents. Take P' an intersection of the curve with a tangent from P to the curve (or, what is the same thing, P, P' cotangentials of any point of the curve): the united points are here the points of contact of the several double tangents of the curve; or if τ be the number of double tangents, then the number of united points is $=2\tau$. The curve Θ is the system of the $n-2$ tangents from P to the curve; each tangent has with the curve a single intersection at P, that is, $k=n-2$; each tangent besides meets the curve in the point of contact Q' twice, and in $(m-3)$ points P'; hence if $(\mathrm{a}, \alpha, \alpha')$ refer to the points (P, Q'), and $(2\tau, \beta, \beta')$ to the points (P, P'), we have

$$2\{\mathrm{a}-\alpha-\alpha'\}+\{2\tau-\beta-\beta'\}+\text{Supp.}=2(n-2)D.$$

From the foregoing example the value of $\mathrm{a}-\alpha-\alpha'$ is $=4D-\kappa$. In the case where

the point P is at a cusp, then the $n-2$ tangents become the $n-3$ tangents from the cusp, and the tangent at the cusp; hence the curve Θ meets the given curve in $2(n-3)+3, =2n-3$ points, that is, $(n-2)+(n-1)$ points; this does not prove (*ante*, No. 96), but the fact is, that the cusp counts in the Supplement $(n-1)$ times, and the expression of the Supplement is $=(n-1)\kappa$. It is clear that we have $\beta=\beta'=(n-2)(m-3)$, so that the equation is

$$8D-2\kappa+2\tau-2(n-2)(m-3)+(n-1)\kappa=(n-2)2D,$$

that is

$$2\tau=2(n-2)(m-3)+(n-6)2D+(-n+3)\kappa;$$

or substituting for $2D$ its value $=n-2m+2+\kappa$ and reducing, this is

$$2\tau=n^2+8m-10n-3\kappa,$$

which is right.

104. As another example, suppose that the point P on a given curve of the order m and the point Q on a given curve of the order m' have an (α, α') correspondence, and let it be required to find the class of the curve enveloped by the line PQ. Take an arbitrary point O, join OQ, and let this meet the curve m in P'; then (P, P') are points on the curve m having a $(m'\alpha, m\alpha')$ correspondence; in fact to a given position of P there correspond α' positions of Q, and to each of these m positions of P'; that is, to each position of P there correspond $m\alpha'$ positions of P'; and similarly to each position of P' there correspond $m'\alpha$ positions of P. The curve Θ is the system of the lines drawn from each of the α' positions of Q to the point O, hence the curve Θ does not pass through P, and we have $k=0$. Therefore the number of the united points (P, P'), that is, the number of the lines PQ which pass through the point O, is $=m\alpha'+m'\alpha$, or this is the class of the curve enveloped by PQ.

It is to be noticed that if the two curves are curves in space (plane, or of double curvature), then the like reasoning shows that the number of the lines PQ which meet a given line O is $=m\alpha'+m'\alpha$, that is, the order of the scroll generated by the line PQ is $=m\alpha'+m'\alpha$.

Article Nos. 105 to 111.—*Application to the Conics which satisfy given conditions, one at least arbitrary.*

105. Passing next to the equations which relate to a conic, we seek for $(4Z)(1)$, the number of the conics which satisfy any four conditions $4Z$ and besides touch a given curve, $(3Z)(2)$ and $(3Z)(1, 1)$, the number of the conics which satisfy three conditions, and besides have with the given curve a contact of the second order, or (as the case may be) two contacts of the first order; and so on with the conditions $2Z$, Z, and then finally (5), $(4, 1)$, ... $(1, 1, 1, 1, 1)$, the numbers of the conics which have with the given curve a contact of the fifth order, or a contact of the fourth and also of the first order ..., or five contacts of the first order.

106. As regards the case $(4Z)(1)$, taking P an arbitrary point of the given curve m, and for the curve Θ the system of the conics $(4Z)(\bar{1})$ which pass through the given point P and besides satisfy the four conditions, then the curve Θ has with the given curve $(4Z)(\bar{1})$ intersections at P, and the points P' are the remaining $(2m-1)(4Z)(\bar{1})$ intersections: in the case of a united point (P, P'), some one of the system of conics becomes a conic $(4Z)(1)$; and the number of the united points is consequently equal to that of the conics $(4Z)(1)$; we have thus the equation

$$\{(4Z)(1) - 2(2m-1)(4Z)(\bar{1})\} + \text{Supp.}\,(4Z)(\bar{1}) = (4Z)(\bar{1}).\,2D.$$

107. It is in the present case easy to find *à priori* the expression for the Supplement. 1°. The system of conics $(4Z)$ contains $2(4Z\cdot) - (4Z/)$ point-pairs[1]; each of these, regarded as a line, meets the given curve in m points, and each of these points is (specially) a united point (P, P'); this gives in the Supplement the term $m\{2(4Z\cdot) - (4Z/)\}$. 2°. The number of the conics $(4Z)$ which can be drawn through a cusp of the given curve is $=(4Z\cdot)$; and the cusp is in respect of each of these conics a united point; we have thus the term $\kappa(4Z\cdot)$, and the Supplement is thus $= m\{2(4Z\cdot) - (4Z/)\} + \kappa(4Z\cdot)$. We have moreover $(4Z)(\bar{1}) = (4Z\cdot)$, $2D = n - 2m + 2 + \kappa$; and substituting these values, we find

$$\begin{aligned}(4Z)(1) = \;& (4m-2)(4Z\cdot) \\ & - m\{2(4Z\cdot) - (4Z/)\} - \kappa(4Z\cdot) \\ & + (n - 2m + 2 + \kappa)(4Z\cdot) \\ = \;& n(4Z\cdot) + m(4Z/),\end{aligned}$$

which is right.

108. It is clear that if, instead of finding as above the expression of the Supplement, the value of $(4Z)(1)$, $= n(4Z\cdot) + m(4Z/)$, had been taken as known, then the equation would have led to

$$\text{Supp.}\,(4Z)(\bar{1}) = m\{2(4Z\cdot) - (4Z/)\} + \kappa(4Z\cdot);$$

and this, as in fact already remarked, is the course of treatment employed in the remaining cases. It is to be observed also that the equation may for shortness be written in the form

$$\begin{aligned}(4Z)\quad & \{(1) - 2(2m-1)(\bar{1})\} \\ & + \text{Supp.}\,(\bar{1}) = (\bar{1})\,2D;\end{aligned}$$

viz. the $(4Z)$ is to be understood as accompanying and forming part of each symbol; and the like in other cases.

109. We have the series of equations

$$\begin{aligned}(4Z)\quad & \{(1) - (\bar{1})(2m-1) - (\bar{1})(2m-1)\} \\ & + \text{Supp.}\,(\bar{1}) \qquad\qquad = (\bar{1})\,2D;\end{aligned}$$

[1] The expression a point-pair is regarded as equivalent to and standing for that of a coincident line-pair: see First Memoir, No. 30.

$$(3Z)\quad \{(2)-(\bar{2})(2m-2)-(\bar{1},\ 1)\}$$
$$+\ \text{Supp. }(\bar{2}) \qquad = 2(\bar{2})\,2D;$$

$$(3Z)\quad 2\{(2)-(\bar{1},\ 1)-(\bar{2})(2m-2)\}$$
$$+\ \{2(1,\ 1)-(\bar{1},\ 1)(2m-3)-(\bar{1},\ 1)(2m-3)\}$$
$$+\ \text{Supp. }(\bar{1},\ 1) \qquad = (\bar{1},\ 1)\,2D;$$

$$(2Z)\quad \{(3)-(\bar{3})(2m-3)-(\bar{1},\ 2)\}$$
$$+\ \text{Supp. }(\bar{3}) \qquad = 3(\bar{3})\,2D;$$

$$(2Z)\quad 2\{(3)-(\bar{2},\ 1)-(\bar{2},\ 1)\}$$
$$+\ \{(2,\ 1)-(\bar{2},\ 1)(2m-4)-(\bar{1},\ 1,\ 1)\,2\}$$
$$+\ \text{Supp. }(\bar{2},\ 1) \qquad = 2(\bar{2},\ 1)\,2D;$$

$$(2Z)\quad 3\{(3)-(\bar{1},\ 2)-(\bar{3})(2m-3)\}$$
$$+\ \{(1,\ 2)-(\bar{1},\ 2)(2m-4)-(\bar{1},\ 2)(2m-4)\}$$
$$+\ \text{Supp. }(\bar{1},\ 2) \qquad = (\bar{1},\ 2)\,2D;$$

$$(2Z)\quad 2\{(2,\ 1)-(\bar{1},\ 1,\ 1)\,2-(\bar{2},\ 1)(2m-4)\}$$
$$+\ \{3(1,\ 1,\ 1)-(\bar{1},\ 1,\ 1)(2m-5)-(\bar{1},\ 1,\ 1)(2m-5)\}$$
$$+\ \text{Supp. }(1,\ 1,\ 1) \qquad = (\bar{1},\ 1,\ 1)\,2D;$$

$$(Z)\quad \{(4)-(\bar{4})(2m-4)-(\bar{1},\ 3)\}$$
$$+\ \text{Supp. }(\bar{4}) \qquad = 4(\bar{4})\,2D;$$

$$(Z)\quad 2\{(4)-(\bar{3},\ 1)-(\bar{2},\ 2)\}$$
$$+\ \{(3,\ 1)-(\bar{3},\ 1)(2m-5)-(\bar{1},\ 1,\ 2)\}$$
$$+\ \text{Supp. }(\bar{3},\ 1) \qquad = 3(\bar{3},\ 1)\,2D;$$

$$(Z)\quad 3\{(4)-(\bar{2},\ 2)-(3,\ \bar{1})\}$$
$$+\ \{2(2,\ 2)-(\bar{2},\ 2)(2m-5)-(\bar{1},\ 1,\ 2)\}$$
$$+\ \text{Supp. }(\bar{2},\ 2) \qquad = 2(\bar{2},\ 2)\,2D;$$

$$(Z)\quad 2\{(3,\ 1)-(\bar{2},\ 1,\ 1)\,2-(\bar{2},\ 1,\ 1)\,2\}$$
$$+\ \{(2,\ 1,\ 1)-(\bar{2},\ 1,\ 1)(2m-6)-(\bar{1},\ 1,\ 1,\ 1)\,3\}$$
$$+\ \text{Supp. }(\bar{2},\ 1,\ 1) \qquad = 2(\bar{2},\ 1,\ 1)\,2D;$$

$$(Z)\quad 4\{(4)-(\bar{1},\ 3)-(\bar{4})(2m-4)\}$$
$$+\ \{(1,\ 3)-(\bar{1},\ 3)(2m-5)-(\bar{1},\ 3)(2m-5)\}$$
$$+\ \text{Supp. }(\bar{1},\ 3) \qquad = (\bar{1},\ 3)\,2D;$$

$$
\begin{aligned}
(Z)\quad & 3\{(3,\ 1)-(\bar{1},\ 1,\ 2)-(\bar{3},\ 1)(2m-5)\} \\
& +2\{2(2,\ 2)-(\bar{1},\ 1,\ 2)-(\bar{2},\ 2)(2m-5)\} \\
& +\ \{2(1,\ 1,\ 2)-(\bar{1},\ 1,\ 2)(2m-6)-(\bar{1},\ 1,\ 2)(2m-6)\} \\
& +\ \text{Supp.}\ (\bar{1},\ 1,\ 2) \qquad\qquad =(\bar{1},\ 1,\ 2)\,2D;
\end{aligned}
$$

$$
\begin{aligned}
(Z)\quad & 2\{(2,\ 1,\ 1)-(\bar{1},\ 1,\ 1,\ 1)\,3-(\bar{2},\ 1,\ 1)(2m-6)\} \\
& +\ \{4(1,1,1,1)-(\bar{1},1,1,1)(2m-7)-(\bar{1},1,1,1)(2m-7)\} \\
& +\ \text{Supp.}\ (\bar{1},\ 1,\ 1,\ 1) \qquad\qquad =(\bar{1},\ 1,\ 1,\ 1)\,2D.
\end{aligned}
$$

110. I content myself with giving the expressions of only the following supplements.

$$
\begin{aligned}
\text{Supp.}\ (4Z)(\bar{1}) &= m\,[2\,(\cdot)-(/)]+\kappa\,(\cdot). \\
\text{Supp.}\ (3Z)(\bar{2}) &= \tfrac{1}{2}n\,[2\,(:)-(\cdot/)]+\tfrac{1}{2}\kappa\,(\cdot/). \\
\text{Supp.}\ (3Z)(\bar{1},\ 1) &= (\quad 2mn-3n^2-\ n+n\alpha)\,(:) \\
&\quad +(2m^2-4mn \qquad -2m+2n+(m-\tfrac{1}{2})\,\alpha)\,(\cdot/) \\
&\quad +(-m^2 \qquad\qquad +\ m \qquad\qquad)\,(//).
\end{aligned}
$$

$$
\begin{aligned}
\text{Supp.}\ (2Z)(\bar{3}) &= -\tfrac{1}{4}m\,[2\,(\cdot\!:\!\cdot)-(:/)] \\
&\quad +\tfrac{1}{4}n\,[2\,(\cdot\!:\!\cdot)-(:/)+2\,(2\,(:/)-(\cdot//))] \\
&\quad +\tfrac{1}{2}\kappa\,(:/). \\
\text{Supp.}\ (\ Z)(\bar{4}) &= a\kappa+b\,(2\kappa+2\iota),
\end{aligned}
$$

where a, b are the representatives of the condition Z.

It may be added that we have in general

$$\text{Supp.}\ (Z)(4X) = a\,\text{Supp.}\ (4X\cdot)+b\,\text{Supp.}\ (4X/),$$

where $(4X)$ stands for any one of the symbols $(\bar{4})$, $(\bar{3},\ 1)\ldots.(\bar{1},\ 1,\ 1,\ 1)$.

111. The expression of Supp. $(4Z)(\bar{1})$ has been explained *suprà*, No. 108. That of Supp. $(3Z)(2)$ may also be explained. 1°. The point-pairs of the system of conics $(3Z)$, regarding each point-pair as a line, are a set of lines enveloping a curve; the class of this curve is equal to the number of the lines which pass through an arbitrary point, that is, as at first sight would appear, to the number of point-pairs in the system $(3Z\cdot)$, or to $2\,(3Z:)-(3Z\cdot/)$: it is, however, necessary to admit that the number of distinct lines, and therefore the class of the curve, is one-half of this, or $=\frac{1}{2}[2\,(3Z:)-(3Z\cdot/)]$; which being so, the number of the point-pairs $(3Z)$ which, regarded as lines, touch the given curve (of the order m and class n) is $=\frac{1}{2}n\,[2\,(3Z:)-(3Z\cdot/)]$. The point of contact of any one of these lines with the given curve is (specially) a united point, and we have thus the term $\frac{1}{2}n\,[2(3Z:)-(3Z\cdot/)]$ of the Supplement. 2°. The number of the conics $(3Z)$ which touch the given curve at a given cusp thereof, or, say, the conics $(3Z)(\overline{2\kappa 1})$, is $=\frac{1}{2}(3Z\cdot/)$, and the cusp is in respect of each of these conics a united point; we have thus the remaining term $\frac{1}{2}\kappa(3Z\cdot/)$ of the Supplement.

Article Nos. 112 to 135.—*Application to the Conics which satisfy five conditions of contact with a given Curve.*

112. We have twelve equations, which I first present in what I call their original forms; viz. these are—

First equation:

$$\begin{aligned} &\{(5)-(\bar{5})(2m-5)-(\bar{1},\ 4)\} \\ +\ &\text{Supp. }(\bar{5}) \qquad\qquad = 5\,(\bar{5})\,2D. \end{aligned}$$

Second equation:

$$\begin{aligned} &2\,\{(5)-(\bar{4},\ 1)-(\bar{2},\ 3)\} \\ +\ &\{(4,\ 1)-(\bar{4},\ 1)(2m-6)-(\bar{1},\ 1,\ 3)\} \\ +\ &\text{Supp. }(\bar{4},\ 1) \qquad\qquad = 4\,(\bar{4},\ 1)\,2D. \end{aligned}$$

Third equation:

$$\begin{aligned} &3\,\{(5)-2\,(\bar{3},\ 2)-2\,(\bar{3},\ 2)\} \\ +\ &\{(3,\ 2)-(\bar{3},\ 2)(2m-6)-2\,(\bar{1},\ 2,\ 2)\} \\ +\ &\text{Supp. }(\bar{3},\ 2) \qquad\qquad = 3\,(\bar{3},\ 2)\,2D. \end{aligned}$$

Fourth equation:

$$\begin{aligned} &2\,\{(4,\ 1)-2\,(\bar{3},\ 1,\ 1)-(\bar{2},\ 2,\ 1)\} \\ +\ &\{(3,\ 1,\ 1)-(\bar{3},\ 1,\ 1)(2m-7)-(\bar{1},\ 1,\ 1,\ 2)\} \\ +\ &\text{Supp. }(\bar{3},\ 1,\ 1) \qquad\qquad = 3\,(\bar{3},\ 1,\ 1)\,2D. \end{aligned}$$

Fifth equation:

$$\begin{aligned} &4\,\{(5)-(\bar{2},\ 3)-(\bar{4},\ 1)\} \\ +\ &\{(3,\ 2)-(\bar{2},\ 3)(2m-6)-(\bar{1},\ 1,\ 3)\} \\ +\ &\text{Supp. }(\bar{2},\ 3) \qquad\qquad = 2\,(\bar{2},\ 3)\,2D. \end{aligned}$$

Sixth equation:

$$\begin{aligned} &3\,\{(4,\ 1)-2\,(\bar{2},\ 2,\ 1)-2,\ (\bar{3},\ 1,\ 1)\} \\ +\ &2\,\{(3,\ 2)-(\bar{2},\ 2,\ 1)-(\bar{2},\ 2,\ 1)\} \\ +\ &\{2\,(2,\ 2,\ 1)-(\bar{2},\ 2,\ 1)(2m-7)-2\,(\bar{1},\ 1,\ 1,\ 2)\} \\ +\ &\text{Supp. }(\bar{2},\ 2,\ 1) \qquad\qquad = 2\,(\bar{2},\ 2,\ 1)\,2D. \end{aligned}$$

Seventh equation:

$$\begin{aligned} &2\,\{(3,\ 1,\ 1)-3\,(\bar{2},\ 1,\ 1,\ 1)-3\,(\bar{2},\ 1,\ 1,\ 1)\} \\ +\ &\{(2,\ 1,\ 1,\ 1)-(\bar{2},\ 1,\ 1,\ 1)(2m-8)-4\,(\bar{1},\ 1,\ 1,\ 1,\ 1)\} \\ +\ &\text{Supp. }(\bar{2},\ 1,\ 1,\ 1) \qquad\qquad = 2\,(\bar{2},\ 1,\ 1,\ 1)\,2D. \end{aligned}$$

Eighth equation:

$$\begin{aligned} & 5\{(5)-(\bar{1},\ 4)-(\bar{5})(2m-5)\} \\ + \quad & \{(4,\ 1)-(\bar{1},\ 4)(2m-5)-(\bar{1},\ 4)(2m-5)\} \\ + \quad & \text{Supp. } (\bar{1},\ 4) \qquad\qquad = (\bar{1},\ 4)\,2D. \end{aligned}$$

Ninth equation:

$$\begin{aligned} & 4\{(4,\ 1)-(\bar{1},\ 1,\ 3)-(\bar{4},\ 1)(2m-6)\} \\ + & 2\{(3,\ 2)-(\bar{1},\ 1,\ 3)-(\bar{2},\ 3)(2m-6)\} \\ + \quad & \{2(3,\ 1,\ 1)-(\bar{1},\ 1,\ 3)(2m-7)-(\bar{1},\ 1,\ 3)(2m-7)\} \\ + \quad & \text{Supp. } (\bar{1},\ 1,\ 3) \qquad\qquad = (\bar{1},\ 1,\ 3)\,2D. \end{aligned}$$

Tenth equation:

$$\begin{aligned} & 3\{(3,\ 2)-2(\bar{1},\ 2,\ 2)-(\bar{3},\ 2)(2m-6)\} \\ + \quad & \{(2,\ 2,\ 1)-(\bar{1},\ 2,\ 2)(2m-7)-(\bar{1},\ 2,\ 2)(2m-7)\} \\ + \quad & \text{Supp. } (\bar{1},\ 2,\ 2) \qquad\qquad = (\bar{1},\ 2,\ 2)\,2D. \end{aligned}$$

Eleventh equation:

$$\begin{aligned} & 3\{(3,\ 1,\ 1)-(\bar{1},\ 1,\ 1,\ 2)-(\bar{3},\ 1,\ 1)(2m-7)\} \\ + & 2\{2(2,\ 2,\ 1)-2(\bar{1},\ 1,\ 1,\ 2)-(\bar{2},\ 2,\ 1)(2m-7)\} \\ + \quad & \{3(2,\ 1,\ 1,\ 1)-(\bar{1},\ 1,\ 1,\ 2)(2m-8)-(\bar{1},\ 1,\ 1,\ 2)(2m-8)\} \\ + \quad & \text{Supp. } (\bar{1},\ 1,\ 1,\ 2) \qquad\qquad = (\bar{1},\ 1,\ 1,\ 2)\,2D. \end{aligned}$$

Twelfth equation:

$$\begin{aligned} & 2\{(2,\ 1,\ 1,\ 1)-4(\bar{1},\ 1,\ 1,\ 1,\ 1)-(\bar{2},\ 1,\ 1,\ 1)(2m-8)\} \\ + \quad & \{5(1,\ 1,\ 1,\ 1,\ 1)-(\bar{1},\ 1,\ 1,\ 1,\ 1)(2m-9)-(\bar{1},\ 1,\ 1,\ 1,\ 1)(2m-9)\} \\ + \quad & \text{Supp. } (\bar{1},\ 1,\ 1,\ 1,\ 1) \qquad\qquad = (\bar{1},\ 1,\ 1,\ 1,\ 1)\,2D. \end{aligned}$$

113. I alter the forms of these equations by substituting for $2D$ its value $= n-2m+2+\kappa$, and by writing for the expressions with $(\bar{1})$ their values,

$$(\bar{1},\ 4) = (\cdot\,4) - 5(\bar{5}),\ \&c.,$$

and except in the terms $\{\text{Supp.}(\bar{5})-\kappa(\bar{5})\}$, &c., by writing for κ its value $-3n+\alpha$. The resulting equations, if the Supplements were known, would serve to determine the values of (5), (4, 1), &c.; but I assume instead that the last-mentioned expressions are known (First Memoir, No. 50), and use the equations to determine the Supplements, or, what comes to the same thing, the values of the terms in { } which contain these Supplements. We have thus the twelve reduced equations, with resulting values of the supplements.

114. First equation:

$$
\begin{array}{l|l}
 & = 0 = \\
(5) & -15m-15n+9\alpha \\
+\{\text{Supp. }(\bar{5})-\kappa(\bar{5})\} & -3m \qquad\quad +\alpha \\
+(\bar{5})(8m+7n-4\alpha) & \quad 8m+7n-4\alpha \\
-(\cdot\,4) & +10m+8n-6\alpha.
\end{array}
$$

(that is, we have

$$\text{Supp. }(\bar{5})-\kappa(\bar{5})=-3m+\alpha,$$

and so in the subsequent cases, the equation gives the value of the term in { } which contains the Supplement).

115. Second equation:

$$
\begin{array}{l|l}
 & = 0 = \\
2(5) & -30m-30n+\alpha(\qquad\qquad 18) \\
+(4,\ 1) & -8m^2-20mn-8n^2+104m+104n+\alpha(\ 6m+6n-66) \\
+\{\text{Supp. }(\bar{4},\ 1)-\kappa(\bar{4},\ 1)\} & -6m^2-3mn\qquad+18m+9n+\alpha(\ 3m\qquad-9) \\
+(\bar{4},\ 1)(6m+5n-3\alpha) & +6m^2+11mn+5n^2-36m-30n+\alpha(-3m-3n+18) \\
-(\cdot\,1,\ 3) & +8m^2+12mn+3n^2-56m-53n+\alpha(-6m-3n+39).
\end{array}
$$

I stop for a moment to notice a very convenient verification of the term in { }; putting therein $\alpha=3n$, the term is

$$-6m^2-3mn+18m+9n+(9mn-27n);$$

and if in this we write $m=n=1$, $m^2=mn=n^2=2$, and when any higher terms enter $m^3=m^2n=mn^2=n^3=4$, $m^4=m^3n=m^2n^2=mn^3=n^4=8$, &c., the value is $-12-6+18+9+18-27$, $=0$, viz. we should always obtain a sum $=0$. The reason is that the term in question should always admit of being expressed in the form $p\delta+q\kappa+r\tau+s\iota$; the reduction to this form might be effected by the substitutions $m=\frac{1}{2}(m+n)+\frac{1}{6}(\kappa-\iota)$, $n=\frac{1}{2}(m+n)-\frac{1}{6}(\kappa-\iota)$, $m^2=2\,.\,\frac{1}{2}(m+n)+2\delta+3\kappa$, $n^2=2\,.\,\frac{1}{2}(m+n)+2\tau+3\iota$, giving a result $=A\,(m+n)+$ terms in $(\delta,\ \kappa,\ \tau,\ \iota)$, where A is a numerical coefficient calculable as above by simply writing $m=n=1$, $m^2=mn=n^2=2$, &c., and which is $=0$ when the term is of the proper form $p\delta+q\kappa+r\tau+s\iota$. The complete reduction to the form in question is material in the sequel, but I advert to the point here only for the sake of the numerical verification.

116. Third equation:

$$
\begin{array}{l|l}
 & = 0 = \\
3(5) & -45m-45n+\alpha(\qquad\qquad +27) \\
+(3,\ 2) & +120m+120n+\alpha(-4m-4n-78)+3\alpha^2 \\
+\{\text{Supp. }(\bar{3},\ 2)-\kappa(\bar{3},\ 2)\} & +15m\qquad\quad+\alpha(\qquad n-7) \\
+(\bar{3},\ 2)(4m+3n-2\alpha) & -36m-27n+\alpha(\ 4m+3n+18)-2\alpha^2 \\
-2(\cdot\,2,\ 2) & -54m-48n+\alpha(\qquad\qquad +40)-\alpha^3.
\end{array}
$$

Verification is $15+3(1\,.\,2-7)=0$.

117. Fourth equation:

$$= 0 =$$

$2\,(4,\ 1)$	$-\ 16m^2 -\ 40mn -\ 16n^2$	(1)
$+\,(3,\ 1,\ 1)$	$-\frac{3}{2}m^3 - 10m^2n - 10mn^2 - \frac{3}{2}n^3 + \frac{109}{2}m^2 + 116mn + \frac{109}{2}n^2$	(2)
$+\,\{\text{Supp. }(\bar{3},\ 1,\ 1) - \kappa\,(\bar{3},\ 1,\ 1)\}$	$-\frac{1}{2}m^3 +\ \frac{1}{2}m^2n +\ 2mn^2 \quad +\ \frac{23}{2}m^2 +\ \frac{11}{2}mn -\ 7n^2$	(3)
$+\,(\bar{3},\ 1,\ 1)\,(4m + 3n - 2\alpha)$	$2m^3 + \frac{19}{2}m^2n +\ 8mn^2 + \frac{5}{2}n^3 -\ 26m^2 -\ \frac{91}{2}mn -\ \frac{39}{2}n^2$	(4)
$-\,(\cdot\,2,\ 1,\ 1)$	$-\ 24m^2 -\ 36mn -\ 12n^2$	(5)

(1) $+\ 208m + 208n + \alpha\,(\qquad\qquad + 12m + 12n - 132)$

(2) $-\ 434m - 434n + \alpha\,(\ \frac{3}{2}m^2 + 6mn + \frac{3}{2}n^2 - \frac{69}{2}m - \frac{69}{2}n + 291) - \frac{9}{2}\alpha^2$

(3) $-\ 50m -\ 23n + \alpha\,(\ \frac{1}{2}m^2 \qquad - \frac{19}{2}m -\ \frac{1}{2}n +\ 33)$

(4) $+\ 108m +\ 81n + \alpha\,(-\ m^2 - 4mn -\ n^2 +\ 7m + \frac{17}{2}n -\ 54) + 3\alpha^2$

(5) $+\ 168m + 168n + \alpha\,(-\ m^2 - 2mn - \frac{1}{2}n^2 + 25m + \frac{29}{2}n - 138) + \frac{3}{2}\alpha^2.$

Verification is $(-\frac{1}{2} + \frac{1}{2} + 2)\,4 + (\frac{23}{2} + \frac{11}{2} - 7)\,2 - 50 - 23 + 3\,(\frac{1}{2}\,.\,4 - (\frac{19}{2} + \frac{1}{2})\,2 + 33) = 0.$

118. Fifth equation:

$$= 0 =$$

$4\,(5)$	$-\ 60m -\ 60n + \alpha\,(\qquad\qquad 36)$	
$+\,(3,\ 2)$	$+\ 120m + 120n + \alpha\,(-\ 4m - 4n - 78) + 3\alpha^2$	
$+\,\{\text{Supp. }(\bar{2},\ 3) - \kappa\,(\bar{2},\ 3)\}$	$n^2 +\ 8m -\ n + \alpha\,(\qquad\qquad - 3)$	
$+\,(\bar{2},\ 3)\,(2m + n - \alpha)$	$-\ 8m^2 - 12mn - 4n^2 -\ 12m -\ 6n + \alpha\,(\ 10m + 7n +\ 6) - 3\alpha^2$	
$-\,(\cdot\,1,\ 3)$	$+\ 8m^2 + 12mn + 3n^2 -\ 56m -\ 53n + \alpha\,(-\ 6m - 3n + 39).$	

Verification is $2 + 8 - 1 + 3\,(-3) = 0.$

119. Sixth equation:

$$= 0 =$$

$3\,(4,\ 1)$	$-\ 24m^2 -\ 60mn - 24n^2 + 312m + 312n$	(1)
$+\ 2\,(3,\ 2)$	$+\ 240m + 240n$	(2)
$+\ 2\,(2,\ 2,\ 1)$	$+\ 48m^2 + 108mn + 48n^2 - 936m - 936n$	(3)
$+\,\{\text{Supp. }(\bar{2},\ 2,\ 1) - \kappa\,(\bar{2},\ 2,\ 1)\}$	$+\ 12m^2 +\ 6mn -\ 6n^2 -\ 60m -\ 6n$	(4)
$+\ \ (\bar{2},\ 2,\ 1)\,(2m + n - \alpha)$	$+\ 12m^2 +\ 18mn +\ 6n^2 + 108m +\ 54n$	(5)
$-\,2\,(\cdot\,2,\ 1,\ 1)$	$-\ 48m^2 -\ 72mn - 24n^2 + 336m + 336n$	(6)

$$
\begin{array}{lll}
(1) & +\alpha(& +18m+18n-198) \\
(2) & +\alpha(& -\ 8m-\ 8n-156)+\alpha^2(\qquad\quad +\ 6) \\
(3) & +\alpha(& -16m-16n+654)+\alpha^2(\ m+n-24) \\
(4) & +\alpha(\qquad mn & -\ 8m-\ 2n+\ 30) \\
(5) & +\alpha(\ 2m^2+3mn+n^2-36m-21n-\ 54)+\alpha^2(-m-n+15) \\
(6) & +\alpha(-2m^2-4mn-n^2+50m+29n-276)+\alpha^2(\qquad\quad +\ 3).
\end{array}
$$

Verification is $(12+6-6)\,2-60-6+3\left(4-(8+2)\,2+30\right)=0.$

120. Seventh equation:

$$=0=$$

$$
\begin{array}{l|ll}
2\,(3,\ 1,\ 1) & & (1) \\
+(2,\ 1,\ 1,\ 1) & & (2) \\
+\{\text{Supp. }(\bar{2},\ 1,\ 1,\ 1)-\kappa\,(\bar{2},\ 1,\ 1,\ 1)\} & \tfrac{1}{2}m^3n+\ m^2n^2 & (3) \\
+(\bar{2},\ 1,\ 1,\ 1)\,(2m+n-\alpha) & \tfrac{1}{3}m^4+\tfrac{13}{6}m^3n+3m^3n^2+\tfrac{4}{3}mn^3+\tfrac{1}{6}n^4 & (4) \\
-(\cdot 1,\ 1,\ 1,\ 1) & -\tfrac{1}{3}m^4-\ \tfrac{8}{3}m^3n-4m^2n^2-\tfrac{4}{3}mn^3-\tfrac{1}{6}n^4 & (5)
\end{array}
$$

$$
\begin{array}{ll}
(1) & -3m^3-20m^2n-20mn^2-3n^3+109m^3+232mn+109n^2-\ 868m-868n \qquad (1) \\
(2) & 6m^3+30m^2n+30mn^2+6n^3-174m^2-348mn-174n^2+1320m \qquad (2) \\
(3) & -\ \tfrac{7}{2}m^3n-\ 5mn^2-\tfrac{3}{2}n^3-\ 20m^2-\ 5mn+\ \tfrac{39}{2}n^2+\ 80m+\ 26n \qquad (3) \\
(4) & -5m^3-\tfrac{37}{2}m^2n-13mn^2-\tfrac{5}{2}n^3+\ \tfrac{74}{3}m^2+\ 37mn+\ \tfrac{37}{2}n^2-\ 150m-\ 75n \qquad (4) \\
(5) & +2m^3+12m^2n+\ 8mn^2+\ n^3+\tfrac{181}{3}m^2+\ 84mn+\tfrac{229}{6}n^2-\ 382m-403n \qquad (5)
\end{array}
$$

$$
\begin{array}{lll}
(1) & +\alpha(& 3m^2+12mn+\ 3n^2-\ 69m-\ 69n+582)+\alpha^2(\qquad\quad -\ 9) \\
(2) & +\alpha(& \tfrac{1}{6}m^3+m^2n+mn^2+\tfrac{1}{6}n^3-\tfrac{15}{2}m^2-26mn-\tfrac{15}{2}n^2+\tfrac{358}{3}m+\tfrac{358}{3}n-960)+\alpha^2(-\tfrac{3}{2}m-\tfrac{3}{2}n+28) \\
(3) & +\alpha(& -\ m^2-\ \tfrac{3}{2}mn+\ \tfrac{1}{2}n^2+\ 19m+\ \tfrac{5}{2}n-\ 54) \\
(4) & +\alpha(-\tfrac{1}{6}m^3-m^2n-mn^2-\tfrac{1}{6}n^3-\tfrac{1}{2}m^2+\ \tfrac{7}{2}mn+\ n^2+\ \tfrac{50}{3}m+\tfrac{50}{3}n+\ 75)+\alpha^2(\ \tfrac{3}{2}m+\tfrac{3}{2}n-\tfrac{29}{2}) \\
(5) & +\alpha(& 6m^2+12mn+\ 3n^2-\ 86m-\ 55n+357)+\alpha^2(\qquad\quad -\ \tfrac{9}{2}).
\end{array}
$$

Verification is

$$(\tfrac{1}{2}+1)\,8+(-\tfrac{7}{2}-5-\tfrac{3}{2})\,4+(-20-5+\tfrac{29}{2})\,2+80+26+3\left((-1-\tfrac{3}{2}+\tfrac{1}{2})\,4+(19+\tfrac{5}{2})\,2-54\right)=0.$$

121. Eighth equation:

$$= 0 =$$

$$\begin{array}{l|l}
5\,(5) & -75m - 75n + \alpha(\quad 45) \\
+\,(4,\,1) & -8m^2 - 20mn - 8n^2 + 104m + 104n + \alpha(6m + 6n - 66) \\
+\left\{\begin{array}{l}\text{Supp. }(\bar{1},\,4) - \kappa(\bar{1},\,4) \\ -(m - \frac{5}{3})\left(2\,(\cdot\,4) - (/\,4)\right)\end{array}\right\} & +\,1m - 4n + \alpha(\quad 1) \\
+\,(-n + \frac{5}{3})(\cdot\,4) & 10mn + 8n^2 - \frac{50}{3}m - \frac{40}{3}n + \alpha(\quad -6n + 10) \\
+\,(-m + \frac{5}{3})(/\,4) & 8m^2 + 10mn - \frac{40}{3}m - \frac{50}{3}n + \alpha(-6m \quad +10) \\
+\,5n\,(\bar{5}) & +5n.
\end{array}$$

Verification is $1 - 4 + 3\,.\,1 = 0$.

122. Ninth equation:

$$= 0 =$$

$$\begin{array}{l|ll}
4\,(4,\,1) & -32m^2 - 80mn - 32n^2 & (1) \\
+\,2\,(3,\,2) & & (2) \\
+\,2\,(3,\,1,\,1) & -3m^3 - 20m^2n - 20mn^2 - 3n^3 + 109m^2 + 232mn + 109n^2 & (3) \\
+\left\{\begin{array}{l}\text{Supp. }(\bar{1},\,1,\,3) - \kappa(\bar{1},\,1,\,3) \\ -(m-2)\left(2\,(\cdot\,1,\,3) - (/\,1,\,3)\right)\end{array}\right\} & -2m^2 + 12mn + 2n^2 & (4) \\
+\,(-n+2)(\cdot\,1,\,3) & +8m^2n + 12mn^2 + 3n^3 - 16m^2 - 80mn - 59n^2 & (5) \\
+\,(-m+2)(/\,1,\,3) & +3m^3 + 12m^2n + 8mn^2 - 59m^2 - 80mn - 16n^2 & (6) \\
+\,n\left(4\,(\bar{4},\,1) + 2\,(\bar{2},\,3)\right) & -4mn - 4n^2 & (7)
\end{array}$$

$$\begin{array}{lll}
(1) & +416m + 416n + \alpha(\quad +24m + 24n - 264) & \\
(2) & +240m + 240n + \alpha(\quad -8m - 8n - 156) + 6\alpha^2 & \\
(3) & -868m - 868n + \alpha(3m^2 + 12mn + 3n^2 - 69m - 69n + 582) - 9\alpha^2 & \\
(4) & -6m + 30n + \alpha(\quad -4m - 10n - 6) + 3\alpha^2 & \\
(5) & +112m + 106n + \alpha(\quad -6mn - 3n^2 + 12m + 45n - 78) & \\
(6) & +106m + 112n + \alpha(-3m^2 - 6mn \quad +45m + 12n - 78) & \\
(7) & -36n + \alpha(\quad +6n \quad). &
\end{array}$$

Verification is $(-2 + 12 + 2)\,2 - 6 + 30 + 3\left((-4 - 10)\,2 - 6\right) + 3\,.\,9\,.\,2 = 0$.

123. Tenth equation:

$$= 0 =$$

$3\,(3,\ 2)$	$+360m+360n$	(1)
$+\,(2,\ 2,\ 1)$	$24m^2+54mn+24n^2-468m-468n$	(2)
$+\left\{\begin{array}{l}\text{Supp. }(\bar{1},\ 2,\ 2)-\kappa\,(\bar{1},\ 2,\ 2)\\ -(m-2)\big(2\,(\cdot\,2,\ 2)-(/\,2,\ 2)\big)\end{array}\right\}$	$+\ 6m+\ 33n$	(3)
$+\,(-n+2)(\cdot\,2,\ 2)$	$-27mn-24n^2+\ 54m+\ 48n$	(4)
$+\,(-m+2)(/\,2,\ 2)$	$-24m^2-27mn\ +\ 48m+\ 54n$	(5)
$+\ 3n\,(\bar{3},\ 2)$	$-\ 27n$	(6)

$$\begin{array}{lll}
(1) & +\alpha\,(-12m-12n-234)+\alpha^2\,(\qquad\qquad+9) \\
(2) & +\alpha\,(-\ 8m-\ 8n+327)+\alpha^2\,(\tfrac{1}{2}m+\tfrac{1}{2}n-12) \\
(3) & +\alpha\,(\qquad\quad-\ 3n-\ 13)+\alpha^2\,(\qquad\qquad+1) \\
(4) & +\alpha\,(\qquad\quad+20n-\ 40)+\alpha^2\,(\quad-\tfrac{1}{2}n+\ 1) \\
(5) & +\alpha\,(\ 20m\qquad\quad-\ 40)+\alpha^2\,(\tfrac{1}{2}m\qquad\quad+\ 1) \\
(6) & +\alpha\,(\qquad\quad+\ 3n\qquad\quad).
\end{array}$$

Verification is $+33+3\,(-3\,.\,2-13)+9\,.\,2=0$.

124. Eleventh equation:

$$= 0 =$$

$3\,(3,\ 1,\ 1)$	$-\ \frac{9}{2}m^3-30m^2n-30mn^2-\ \frac{9}{2}n^3$	(1)
$+\,4\,(2,\ 2,\ 1)$		(2)
$+\,3\,(2,\ 1,\ 1,\ 1)$	$+18m^3+90m^2n+90mn^2+18n^3$	(3)
$+\left\{\begin{array}{l}\text{Supp. }(\bar{1},\ 1,\ 1,\ 2)-\kappa\,(\bar{1},\ 1,\ 1,\ 2)\\ -(m-\frac{7}{3})\big(2\,(\cdot\,1,\ 1,\ 2)-(/\,1,\ 1,\ 2)\big)\end{array}\right\}$	$-\ \frac{3}{2}m^3-\ \frac{3}{2}m^2n-\ 6mn^2-\ 3n^3$	(4)
$+\,(-n+\frac{7}{3})(\cdot\,1,\ 1,\ 2)$	$-24m^2n-36mn^2-12n^3$	(5)
$+\,(-m+\frac{7}{3})(/\,1,\ 1,\ 2)$	$-12m^3-36m^2n-24mn^2$	(6)
$+\,n\,\big(3\,(\bar{3},\ 1,\ 1)+2\,(\bar{2},\ 2,\ 1)\big)$	$+\ \frac{3}{2}m^2n+\ 6mn^2+\ \frac{3}{2}n^3$	(7)

$$
\begin{array}{lllllll}
(1) & +\frac{387}{2}m^2+ & 348mn+ & \frac{327}{2}n^2- & 1302m- & 1302n+\alpha\,(& & (1)\\
(2) & +\ 96m^2+ & 216mn+ & 96n^2- & 1872m- & 1872n+\alpha\,(& & (2)\\
(3) & -\ 522m^2- & 1044mn- & 522n^2+ & 3960m+ & 3960n+\alpha\,(& \frac{1}{2}m^3+3m^2n+3mn^2+\frac{1}{2}n^3 & (3)\\
(4) & +\ \frac{21}{2}m^2- & \frac{33}{2}mn+ & 18n^2- & 2m- & 191n+\alpha\,(& & (4)\\
(5) & +\ 56m^2+ & 252mn+ & 196n^2- & 392m- & 392n+\alpha\,(& -\ m^2n-2mn^2-\frac{1}{2}n^3 & (5)\\
(6) & +\ 196m^2+ & 252mn+ & 56n^2- & 392m- & 392n+\alpha\,(& -\frac{1}{2}m^3-2m^2n-mn^2 & (6)\\
(7) & & -\ \frac{15}{2}mn- & \frac{15}{2}n^2 & & +\ 189n+\alpha\,(& & (7)
\end{array}
$$

$$
\begin{array}{lll}
(1) & \frac{3}{2}m^2+18mn+\frac{3}{2}n^2-\frac{207}{2}m-\frac{207}{2}n+873)+\alpha^2\,(& -\frac{27}{2})\\
(2) & -\ 32m-32n+1308)+\alpha^2\,(& 2m+2n-48)\\
(3) & -\frac{45}{2}m^2-78mn-\frac{45}{2}n^2+358m+358n-2880)+\alpha^2\,(& -\frac{9}{2}m-\frac{9}{2}n+84)\\
(4) & -\ \frac{4}{3}mn-2n^2+\frac{23}{3}m+\frac{253}{6}n+55)+\alpha^2\,(& m+n-\frac{31}{2})\\
(5) & +\ \frac{7}{3}m^2+\frac{89}{3}mn+\frac{47}{3}n^2-\frac{175}{3}m-\frac{1031}{6}n+322)+\alpha^2\,(& \frac{3}{2}m-\frac{7}{2})\\
(6) & +\frac{47}{3}m^2+\frac{89}{3}mn+\frac{7}{3}n^2-\frac{1031}{6}m-\frac{175}{3}n+322)+\alpha^2\,(& +\frac{3}{2}n-\frac{7}{2})\\
(7) & 2mn+2n^2 & \frac{69}{2}n\qquad).
\end{array}
$$

Verification is $4(-\frac{3}{2}-\frac{3}{2}-6-3)+2(\frac{21}{2}-\frac{33}{2}+18)-2-191$

$$+3\left((-\tfrac{4}{3}-2)\,4+(\tfrac{23}{3}+\tfrac{253}{6})\,2+55\right)+9\left((1+1)\,4-\tfrac{31}{2}\,.\,2\right)=0.$$

125. Twelfth equation:

$$
\begin{array}{ll}
 & =0= \\
2\,(\bar{2},\ 1,\ 1,\ 1) & (1)\\
+\,5\,(1,\ 1,\ 1,\ 1,\ 1) & \frac{1}{24}m^5+\frac{5}{12}m^4n+\frac{5}{3}m^3n^2+\frac{5}{3}m^2n^3+\frac{5}{12}mn^4+\frac{1}{24}n^5 \quad (2)\\
+\left\{\begin{array}{l}\text{Supp.}\,(\bar{1},1,1,1,1)-\kappa\,(\bar{1},1,1,1,1)\\ -(m-\frac{8}{3})\big(2\,(\cdot\,1,1,1,1)-(/\,1,1,1,1)\big)\end{array}\right\} & (3)\\
+\ (-n+\frac{8}{3})(\cdot\,1,\ 1,\ 1,\ 1) & -\frac{1}{12}m^4n-\frac{2}{3}m^3n^2-m^2n^3-\frac{1}{3}mn^4-\frac{1}{24}n^5 \quad (4)\\
+\ (-m+\frac{8}{3})(/\,1,\ 1,\ 1,\ 1) & -\frac{1}{24}m^5-\frac{1}{3}m^4n-m^3n^2-\frac{2}{3}m^2n^3-\frac{1}{12}mn^4 \quad (5)\\
+\,2n\,(\bar{2},\ 1,\ 1,\ 1) & (6)
\end{array}
$$

$$
\begin{array}{lrrr

127. Referring to Nos. 41 to 47 of the First Memoir, for convenience I collect the capitals which belong to a single curve, giving the values in terms of m, n, α as follows.

(1, 1, 1, 1)

$$
\begin{array}{lll}
(1) & A = \frac{1}{2}\delta(\delta-1) & = \frac{1}{8}m^4 - \frac{1}{4}m^3 + 2m^2n - \frac{1}{8}m^2 - 2mn + 8n^2 + \frac{1}{4}m - 2n \\
 & & \quad + \alpha(-\frac{2}{3}m^2 + \frac{3}{4}m - 6n + \frac{3}{4}) + \frac{9}{8}\alpha^2; \\
(2) & B = \delta(n-4)(m-4) & = \frac{1}{2}m^3n - 2m^3 - \frac{5}{2}m^2n + 4mn^2 + 10m^2 - 14mn - 16n^2 - 8m + 64n \\
 & & \quad + \alpha(-\frac{3}{2}mn + 6m + 6n - 24); \\
(4) & C = \tau.\frac{1}{2}(m-4)(m-5) & = \frac{1}{4}m^2n^2 + 2m^3 - \frac{1}{4}m^2n - \frac{9}{4}mn^2 - 18m^2 + \frac{9}{4}mn + 5n^2 + 40m - 5n \\
 & & \quad + \alpha(-\frac{3}{4}m^2 + \frac{27}{4}m - 15); \\
(3) & D = \iota.\frac{1}{2}(m-3)(m-4) & = -\frac{3}{2}m^3 + \frac{21}{2}m^2 - 18m \\
 & & \quad + \alpha(\frac{1}{2}m^2 - \frac{7}{2}m + 6).
\end{array}
$$

(2, 1, 1)

$$
\begin{array}{lll}
(3) & E = \delta(n-4) & = \frac{1}{2}m^2n - 2m^2 - \frac{1}{2}mn + 4n^2 + 2m - 16n \\
 & & \quad + \alpha(-\frac{3}{2}n + 6); \\
(3) & F = 2\delta(m-3) & = m^3 - 4m^2 + 8mn + 3m - 24n \\
 & & \quad + \alpha(-3m + 9); \\
(6) & G = 2\tau(m-4) & = mn^2 + 8m^2 - mn - 4n^2 - 32m + 4n \\
 & & \quad + \alpha(-3m + 12); \\
(2) & D\,[= \iota.\frac{1}{2}(m-3)(m-4)\ \textit{supra}]; & \\
(1) & H = \delta\kappa & = -\frac{3}{2}m^2n + \frac{3}{2}mn - 12n^2 \\
 & & \quad + \alpha(\frac{1}{2}m^3 - \frac{1}{2}m + \frac{17}{2}n) - \frac{3}{2}\alpha^2; \\
(2) & I = \kappa(n-3)(m-4) & = -3mn^2 + 9mn + 12n^2 - 36n \\
 & & \quad + \alpha(mn - 3m - 4n + 12); \\
(5) & J = \iota(m-3) & = -3m^2 + 9m + \alpha(m-3).
\end{array}
$$

(2, 2)

$$
\begin{array}{lll}
(9) & K = \tau & = \frac{1}{2}n^2 + 4m - \frac{1}{2}n + \alpha(-\frac{3}{2}); \\
(3) & L = \kappa(n-3) & = -3n^2 + 9n + \alpha(n-3); \\
(1) & M = \frac{1}{2}\kappa(\kappa-1) & = \frac{9}{2}n^2 + \frac{3}{2}n + \alpha(-3n - \frac{1}{2}) + \frac{1}{2}\alpha^2; \\
(2) & N = \iota & = -3m + \alpha(1); \\
(1) & O = \kappa & = -3n + \alpha(1).
\end{array}
$$

(3, 1)

$$\begin{array}{l|l}
(2) & P = 2\delta \qquad\qquad = m^2 \qquad\qquad - m + 8n + \alpha(\quad -3); \\
(2) & Q = 2\tau \qquad\qquad = \qquad\qquad n^2 + 8m - n + \alpha(\quad -3); \\
(5) & J\ [= \iota(m-3)\ \textit{suprà}]; \\
(4) & R = \kappa(m-3) \qquad = -3mn \qquad\qquad + 9n + \alpha(m-3).
\end{array}$$

(4)

$$\begin{array}{l|l}
(4) & N\ [= \iota\ \textit{suprà}]; \\
(2) & O\ [= \kappa\ \textit{suprà}].
\end{array}$$

128. I make the following calculations, serving to express in terms of Zeuthen's Capitals, the terms in { } contained in the twelve equations respectively.

$$\begin{array}{l}
N = -3m + \alpha \\
\hline
-3m + \alpha \text{ (first equation).}
\end{array}$$

$$\begin{array}{rl}
2J = & -6m^2 \qquad + 18m \qquad + \alpha(2m-6) \\
+R = & -3mn \qquad + 9n \qquad + \alpha(m-3) \\
\hline
 & -6m^2 - 3mn + 18m + 9n + \alpha(3m-9) \text{ (second equation).}
\end{array}$$

$$\begin{array}{rl}
6K = & 3n^2 + 24m - 3n + \alpha(\quad -9) \\
+L = & -3n^2 \qquad + 9n + \alpha(n-3) \\
+3N = & \qquad - 9m \qquad + \alpha(\quad 3) \\
+2O = & \qquad\qquad - 6n + \alpha(\quad 2) \\
\hline
 & \qquad 15m \qquad + \alpha(n-7) \text{ (third equation).}
\end{array}$$

$$\begin{array}{rl}
E = & \tfrac{1}{2}m^2n \qquad - 2m^2 - \tfrac{1}{2}mn + 4n^2 + 2m - 16n + \alpha(\qquad\qquad - \tfrac{3}{2}n + 6) \\
+F = & m^3 \qquad\qquad - 4m^2 + 8mn \qquad + 3m - 24n + \alpha(\quad - 3m \qquad + 9) \\
+2G = & \qquad\qquad 2mn^2 + 16m^2 - 2mn - 8n^2 - 64m + 8n + \alpha(\quad - 6m \qquad + 24) \\
+D = & -\tfrac{3}{2}m^3 \qquad + \tfrac{21}{2}m^2 \qquad\qquad - 18m \qquad + \alpha(\tfrac{1}{2}m^2 - \tfrac{7}{2}m \qquad + 6) \\
+3J = & \qquad\qquad - 9m^2 \qquad\qquad + 27m \qquad + \alpha(\qquad 3m \qquad - 9) \\
+J' = & \qquad\qquad\qquad\qquad - 3n^2 + \qquad 9n + \alpha(\qquad\qquad n - 3) \\
\hline
 & -\tfrac{1}{2}m^3 + \tfrac{1}{2}m^2n + 2mn^2 + \tfrac{23}{2}m^2 + \tfrac{11}{2}mn - 7n^2 - 50m - 23n + \alpha(\tfrac{1}{2}m^2 - \tfrac{19}{2}m - \tfrac{1}{2}n + 33)
\end{array}$$

(fourth equation)

$$Q = n^2 + 8m - n - 3\alpha$$

$n^2 + 8m - n - 3\alpha$ (fifth equation).

$$\begin{array}{rl}
3G = & 3mn^2 + 24m^2 - 3mn - 12n^2 - 96m + 12n + \alpha(\quad - 9m \quad + 36) \\
+\ I = & -3mn^2 \quad + 9mn + 12n^2 \quad - 36n + \alpha(mn - 3m - 4n + 12) \\
+ 4J = & \quad - 12m^2 \quad + 36m \quad + \alpha(\quad 4m \quad - 12) \\
+ 2J' = & \quad - 6n^2 \quad + 18n + \alpha(\quad 2n - 6) \\
\hline
 & 12m^2 + 6mn - 6n^2 - 60m - 6n + \alpha(mn - 8m - 2n + 30)
\end{array}$$

(sixth equation).

$$\begin{array}{rlr}
B = & \tfrac{1}{3}m^3n \quad - 2m^3 - \tfrac{5}{2}m^2n + 4mn^2 \quad + 10m^2 - 14mn - 16n^2 - \quad 8m + 64n & (1) \\
+ 4C = & m^2n^2 + 8m^3 - m^2n - 9mn^2 \quad - 72m^2 + 9mn + 20n^2 + 160m - 20n & (2) \\
+ 4D = & \quad - 6m^3 \quad + 42m^2 \quad - 72m & (3) \\
+\ D' = & \quad - \tfrac{3}{2}n^3 \quad + \tfrac{21}{2}n^2 \quad - 18n & (4) \\
\hline
 & \tfrac{1}{2}m^3n + m^2n^2 \quad - \tfrac{7}{2}m^2n - 5mn^2 - \tfrac{3}{2}n^3 - 20m^2 - 5mn + \tfrac{29}{2}n^2 + 80m + 26n & (5)
\end{array}$$

$$\begin{array}{rl}
(1) & + \alpha(\quad - \tfrac{3}{2}mn \quad + 6m + 6n - 24) \\
(2) & + \alpha(-3m^2 \quad + 27m \quad - 60) \\
(3) & + \alpha(\quad 2m^2 \quad - 14m \quad + 24) \\
(4) & + \alpha(\quad \tfrac{1}{2}n^2 \quad - \tfrac{7}{2}n + 6) \\
\hline
(5) & + \alpha(-m^2 - \tfrac{3}{2}mn + \tfrac{1}{2}n^2 + 19m + \tfrac{5}{2}n - 54)
\end{array}$$

(seventh equation).

$$\begin{array}{rl}
-\tfrac{1}{3}N = & m \quad - \tfrac{1}{3}\alpha \\
+\tfrac{4}{3}O = & \quad - 4n + \tfrac{4}{3}\alpha \\
\hline
 & m - 4n + \alpha
\end{array}$$

(eighth equation).

$$\begin{array}{rl}
(\bar{2},\ 3) = & -4m - 4n - 6 + 3\alpha \\
+ 2(\bar{4},\ 1) = & 2m + 2n - 12 \\
\hline
 & -2m - 2n - 18 + 3\alpha
\end{array}$$

(used *infrà*)

$$\begin{array}{rl}
-2P & = -2m^2 \quad + 2m - 16n + \alpha(\quad 6) \\
-\ Q & = \quad - n^2 - 8m + n + \alpha(\quad 3) \\
-2R & = \quad 6mn \quad - 18n + \alpha(-2m \quad + 6) \\
+\ J' & = \quad - 3n^2 \quad + 9n + \alpha(\quad n - 3) \\
\kappa\{(\bar{2},\ 3) + 2(\bar{4},\ 1)\} & = \quad 6mn + 6n^2 \quad + 54n + \alpha(-2m - 11n - 18) + 3\alpha^2 \\
\hline
 & -2m^2 + 12mn + 2n^2 - 6m + 30n + \alpha(-4m - 10n - 6) + 3\alpha^2
\end{array}$$

(ninth equation).

$$
\begin{array}{rcl}
3L & = & -9n^2 + 27n + \alpha(3n - 9) \\
+2M & = & 9n^2 + 3n + \alpha(-6n - 1) + \alpha^2 \\
-2N & = & +6m + \alpha(-2) \\
-O & = & 3n + \alpha(-1) \\
\hline
 & & 6m + 33n + \alpha(-3n - 13) + \alpha^2 \text{ (tenth equation).}
\end{array}
$$

$$
\begin{array}{rcll}
\kappa(\bar{2}, 2, 1) & = & -18mn - 18n^2 - 162n & (1) \\
-\frac{1}{3}D & = & \frac{1}{2}m^3 - \frac{7}{2}m^2 + 6m & (2) \\
-2E & = & -m^2n + 4m^2 + 1mn - 8n^2 - 4m + 32n & (3) \\
-2F & = & -2m^3 + 8m^2 - 16mn - 6m + 48n & (4) \\
-G & = & -mn^2 - 8m^2 + 1mn + 4n^2 + 32m - 4n & (5) \\
+\frac{1}{3}H & = & -\frac{1}{2}m^2n + \frac{1}{2}mn - 4n^2 & (6) \\
+\frac{5}{3}I & = & -5mn^2 + 15mn + 20n^2 - 60n & (7) \\
-\frac{10}{3}J & = & +10m^2 - 30m & (8) \\
+2D' & = & -3n^3 + 21n^2 - 36n & (9) \\
-J' & = & 3n^2 - 9n & (10) \\
\hline
 & & -\frac{3}{2}m^3 - \frac{3}{2}m^2n - 6mn^2 - 3n^3 + \frac{21}{2}m^2 - \frac{33}{2}mn + 18n^2 - 2m - 191n & (11)
\end{array}
$$

$$
\begin{array}{lll}
(1) & & +\alpha(-3mn - 3n^2 + 6m + 51n + 54) + \alpha^2(m + n - 15) \\
(2) & & +\alpha(-\frac{1}{6}m^2 + \frac{7}{6}m - 2) \\
(3) & & +\alpha(3n - 12) \\
(4) & & +\alpha(6m - 18) \\
(5) & & +\alpha(3m - 12) \\
(6) & & +\alpha(\frac{1}{6}m^2 - \frac{1}{6}m + \frac{17}{6}n) + \alpha^2(-\frac{1}{2}) \\
(7) & & +\alpha(\frac{5}{3}mn - 5m - \frac{20}{3}n + 20) \\
(8) & & +\alpha(-\frac{10}{3}n + 10) \\
(9) & & +\alpha(1n^2 - 7n + 12) \\
(10) & & +\alpha(-n + 3) \\
\hline
(11) & & +\alpha(-\frac{4}{3}mn - 2n^2 + \frac{23}{3}m + \frac{253}{6}n + 55) + \alpha^2(m + n - \frac{31}{2})
\end{array}
$$

(eleventh equation).

$$\begin{array}{lll}
\kappa(\bar{2},1,1,1) = & -\tfrac{1}{2}m^3n - 3m^2n^2 - 3mn^3 - \tfrac{1}{2}n^4 & + \tfrac{15}{2}m^2n + 24mn^2 + \tfrac{15}{2}n^3 - 37mn - 37n^2 \quad (1)\\
-\tfrac{4}{3}A = & -\tfrac{1}{6}m^4 & + \tfrac{1}{3}m^3 - \tfrac{3}{3}m^2n + \tfrac{1}{6}m^2 + \tfrac{8}{3}mn - \tfrac{32}{3}n^2 \quad (2)\\
-\tfrac{5}{3}B = & -\tfrac{5}{6}m^3n & + \tfrac{10}{3}m^3 + \tfrac{25}{6}m^2n - \tfrac{20}{3}mn^2 - \tfrac{50}{3}m^2 + \tfrac{70}{3}mn + \tfrac{80}{3}n^2 \quad (3)\\
-\tfrac{4}{3}C = & -\tfrac{1}{3}m^2n^2 & - \tfrac{8}{3}m^3 + \tfrac{1}{3}m^2n + 3mn^2 + 24m^2 - 3mn - \tfrac{20}{3}n^2 \quad (4)\\
-2D = & & 3m^3 - 21m^2 \quad (5)\\
-D' = & & \tfrac{3}{2}n^3 - \tfrac{21}{2}n^2 \quad (6)\\
\hline
 & -\tfrac{1}{6}m^4 - \tfrac{4}{3}m^3n - \tfrac{10}{3}m^2n^2 - 3mn^3 - \tfrac{1}{2}n^4 & + 4m^3 + \tfrac{28}{3}m^2n + \tfrac{61}{3}mn^2 + 9n^3 - \tfrac{27}{2}m^2 - 14mn - \tfrac{229}{6}n^2 \quad (7)
\end{array}$$

$$\begin{array}{lll}
(1) & & + 225n + \alpha(\tfrac{1}{6}m^3 + m^2n + mn^2 + \tfrac{1}{6}n^3 - \tfrac{5}{2}m^2 - \tfrac{7}{2}mn + 2n^2 + \tfrac{37}{3}m - \tfrac{187}{6}n - 75) + \alpha^2(-\tfrac{3}{2}m - \tfrac{3}{2}n + \tfrac{29}{2})\\
(2) & -\tfrac{1}{3}m + \tfrac{8}{3}n + \alpha(& m^2 - m + 8n - 1) + \alpha^2(-\tfrac{3}{2})\\
(3) & +\tfrac{40}{3}m - \tfrac{320}{3}n + \alpha(& +\tfrac{5}{2}mn - 10m - 10n + 40)\\
(4) & -\tfrac{160}{3}m + \tfrac{20}{3}n + \alpha(& m^2 - 9m + 20)\\
(5) & +36m + \alpha(& -m^2 + 7m - 12)\\
(6) & +18n + \alpha(& -\tfrac{1}{2}n^2 + \tfrac{7}{2}n - 6)\\
\hline
(7) & -\tfrac{13}{3}m + \tfrac{437}{3}n + \alpha(\tfrac{1}{6}m^3 + m^2n + mn^2 + \tfrac{1}{6}n^3 - \tfrac{3}{2} & m^2 - mn + \tfrac{3}{2}n^2 - \tfrac{9}{2}m - \tfrac{89}{3}n - 34) + \alpha^2(-\tfrac{3}{2}m - \tfrac{3}{2}n + 13)
\end{array}$$

(twelfth equation).

129. We have consequently, by means of the results just obtained,

$$\begin{aligned}
\text{Supp. } (\bar{5}) &= \kappa(\bar{5}) \\
&\quad + N && \text{(first equation).}\\
\text{Supp. } (\bar{4}, 1) &= \kappa(\bar{4}, 1) \\
&\quad + 2J + R && \text{(second equation).}\\
\text{Supp. } (\bar{3}, 2) &= \kappa(\bar{3}, 2) \\
&\quad + 6K + L + 3N + 2O && \text{(third equation).}\\
\text{Supp. } (\bar{3}, 1, 1) &= \kappa(\bar{3}, 1, 1) \\
&\quad + D + E + F + 2G + 3J + J' && \text{(fourth equation).}\\
\text{Supp. } (\bar{2}, 3) &= \kappa(\bar{2}, 3) \\
&\quad + Q && \text{(fifth equation).}\\
\text{Supp. } (\bar{2}, 2, 1) &= \kappa(\bar{2}, 2, 1) \\
&\quad + 3G + I + 4J + 2J' && \text{(sixth equation).}\\
\text{Supp. } (\bar{2}, 1, 1, 1) &= \kappa(\bar{2}, 1, 1, 1) \\
&\quad + B + 4C + 4D + D' && \text{(seventh equation).}
\end{aligned}$$

$$\text{Supp. } (\bar{1}, 4) = \kappa(\bar{1}, 4) + (m - \tfrac{5}{3})(4N + 2O) - \tfrac{1}{3}N + \tfrac{4}{3}O \qquad \text{(eighth equation).}$$

$$\begin{aligned}\text{Supp. } (\bar{1}, 1, 3) &= \kappa(\bar{1}, 1, 3) + \kappa(\bar{2}, 3) + \kappa \,.\, 2(\bar{4}, 1) \\ &\quad + (m-2)(2P + 2Q + 5J + 4R) \\ &\quad - 2P - Q - 2\mathrm{R} + J' \qquad \text{(ninth equation).}\end{aligned}$$

$$\begin{aligned}\text{Supp. } (\bar{1}, 2, 2) &= \kappa(\bar{1}, 2, 2) \\ &\quad + (m-2)(9K + 3L + M + 2N + O) \\ &\quad + 3L + 2M - 2N - O \qquad \text{(tenth equation).}\end{aligned}$$

$$\begin{aligned}\text{Supp. } (\bar{1}, 1, 1, 2) &= \kappa(\bar{1}, 1, 1, 2) + \kappa(\bar{2}, 2, 1) \\ &\quad + (m - \tfrac{7}{3})(3E + 3F + 6G + 2D + H + 2I + 5J) \\ &\quad - 2E - 2F - G - \tfrac{1}{3}D + \tfrac{1}{3}H + \tfrac{5}{3}I - \tfrac{10}{3}J + 2D' - J' \\ &\qquad \text{(eleventh equation).}\end{aligned}$$

Observe that

$$G - 2E' = 0, \quad G' - 2E = 0, \quad 3G + I + 8J = 3G' + I' + 8J',$$

relations which may be used to modify the form of the last preceding result.

$$\begin{aligned}\text{Supp. } (\bar{1}, 1, 1, 1, 1) &= \kappa(\bar{1}, 1, 1, 1, 1) + \kappa(\bar{2}, 1, 1, 1) \\ &\quad + (m - \tfrac{8}{3})(A + 2B + 4C + 3D) \\ &\quad - \tfrac{4}{3}A - \tfrac{5}{3}B - \tfrac{4}{3}C - D' \qquad \text{(twelfth equation).}\end{aligned}$$

130. We may in these equations introduce on the right-hand sides in place of a symbol such as $\bar{p}$ the symbol $\overline{p\kappa 1}$: for example, in the fifth equation, writing

$$(\bar{2}, 3) = (\overline{2\kappa 1}, 3) + [(\bar{2}, 3) - (\overline{2\kappa 1}, 3)],$$

and therefore also

$$\kappa(\bar{2}, 3) = \kappa(\overline{2\kappa 1}, 3) + \kappa[(\bar{2}, 3) - (\overline{2\kappa 1}, 3)],$$

the second term $\kappa[(\bar{2}, 3) - (\overline{2\kappa 1}, 3)]$ can be expressed in terms of Zeuthen's Capitals. The remark applies to all the twelve equations; only as regards the first four of them, inasmuch as $(\overline{5\kappa 1}) = 0, \ldots (\overline{3\kappa 1}, 1, 1) = 0$, it is the whole original terms $\kappa(\bar{5}) \ldots \kappa(\bar{3}, 1, 1)$ which are thus expressible by means of Zeuthen's Capitals. By the assistance of the formulæ (First Memoir, Nos. 69 and 73) we readily obtain

		Referring to
$\kappa(\bar{5})$	$= \kappa = O$	(first equation).
$\kappa(\bar{4}, 1)$	$= \kappa(m + n - 6)$	
	$= R + J'$	(second equation).

$\kappa(\bar{3},\ 2) \quad = \kappa(-9+\alpha) = \kappa\big(3(n-3)+\kappa-1+1\big)$ Referring to

$= 3L + 2M + O$ (third equation).

$\kappa(\bar{3},\ 1,\ 1) \quad = \kappa(\tfrac{1}{2}m^2 + 2mn + \tfrac{1}{2}n^2 - \tfrac{13}{2}m - \tfrac{13}{2}n + 27 - \tfrac{3}{2}\alpha)$

$= H + 2I + D' + J'$ (fourth equation).

$$
\begin{array}{llllll}
\text{viz. } \kappa^{-1}\ H & = \tfrac{1}{2}m^2 & & -\ \tfrac{1}{2}m & +\ 4n & -\ \tfrac{3}{2}\alpha \\
\kappa^{-1}.\,2I & = & 2mn & -\ 6m & -\ 8n & +\ 24 \\
\kappa^{-1}.\,D' & = & \tfrac{1}{2}n^2 & & -\ \tfrac{7}{2}n & +\ 6 \\
\kappa^{-1}\ J' & = & & & n & -\ 3 \\
\hline
 & \tfrac{1}{2}m^2 + 2mn + \tfrac{1}{2}n^2 & & -\ \tfrac{13}{2}m & -\ \tfrac{13}{2}n & +\ 27 - \tfrac{3}{2}\alpha
\end{array}
$$

$\kappa(\bar{2},\ 3) \quad = \kappa(\overline{2\kappa 1},\ 3)$ (fifth equation).

$\kappa(\bar{2},\ 2,\ 1) \quad = \kappa(\overline{2\kappa 1},\ 2,\ 1) + \kappa(n-3)$

$= \kappa(\overline{2\kappa 1},\ 2,\ 1) + J'$ (sixth equation).

$\kappa(\bar{2},\ 1,\ 1,\ 1) = \kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1) + \kappa.\tfrac{1}{2}(n-3)(n-4)$

$= \kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1) + D'$ (seventh equation).

$\kappa(\bar{1},\ 4) \quad = \kappa(\overline{1\kappa 1},\ 4) + \kappa$

$= \kappa(\overline{1\kappa 1},\ 4) + O$ (eighth equation).

$\kappa(\bar{1},\ 1,\ 3) + \quad \kappa(\bar{2},\ 3) + \kappa\, 2(\bar{4},\ 1)$

$= \kappa(\overline{1\kappa 1},\ 1,\ 3) + \kappa(\overline{2\kappa 1},\ 3) \quad + \kappa(n-3)$

$+ \kappa(\overline{2\kappa 1},\ 3)$

$+ \qquad \kappa(2m + 2n - 6)$

$= \kappa(\overline{1\kappa 1},\ 1,\ 3) + 2\kappa(\overline{2\kappa 1},\ 3) + 2R + 3J'$ (ninth equation).

$\kappa(\bar{1},\ 2,\ 2) \quad = \kappa(\overline{1\kappa 1},\ 2,\ 2) + \kappa\{3(n-3) + \kappa - 1\}$

$= \kappa(\overline{1\kappa 1},\ 2,\ 2) + 3L + 2M$ (tenth equation).

$\kappa(\bar{1},\ 1,\ 1,\ 2) + \kappa(\bar{2},\ 2,\ 1)$

$= \kappa(\overline{1\kappa 1},\ 1,\ 1,\ 2) + \kappa(\overline{2\kappa 1},\ 2,\ 1) + \kappa\{\tfrac{1}{2}(n-3)(n-4) + \delta + 2\,\overline{n-3}\,\overline{m-4}\}$

$+ \kappa(\overline{2\kappa 1},\ 2,\ 1) + \kappa(n-3)$

$= \kappa(\overline{1\kappa 1},\ 1,\ 1,\ 2) + 2\kappa(\overline{2\kappa 1},\ 1,\ 2)$

$+ D' + H + 2I + J'$ (eleventh equation).

$$\kappa(\bar{1},\ 1,\ 1,\ 1,\ 1)+\kappa(\bar{2},\ 1,\ 1,\ 1)$$ Referring to

$$=\kappa(\overline{1\kappa 1},\ 1,\ 1,\ 1,\ 1)+\ \kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1)$$

$$+\ \kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1)+\kappa\,.\,\tfrac{1}{2}(n-3)(n-4)$$

$$=\kappa(\overline{1\kappa 1},\ 1,\ 1,\ 1,\ 1)+2\kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1)+D'$$ (twelfth equation).

131. Hence, substituting in the expressions of the several Supplements, we have

Supp. $(\bar{5})$ $= O$
$+N$ (first equation).

Supp. $(\bar{4},\ 1)$ $= R+J'$
$+2J+R$ (second equation).

Supp. $(\bar{3},\ 2)$ $= 3L+2M+O$
$+6K+L+3N+2O$ (third equation).

Supp. $(\bar{3},\ 1,\ 1)$ $= H+2I+D'+J'$
$+E+F+2G+D+3J+J'$ (fourth equation).

Supp. $(\bar{2},\ 3)$ $= \kappa(\overline{2\kappa 1},\ 3)$
$+Q$ (fifth equation).

Supp. $(\bar{2},\ 2,\ 1)$ $= \kappa(\overline{2\kappa 1},\ 1,\ 1)+J'$
$+3G+I+4J+2J'$ (sixth equation).

Supp. $(\bar{2},\ 1,\ 1,\ 1)$ $= \kappa(\overline{2\kappa 1},\ 1,\ 1,\ 1)+D'$
$+B+4C+4D+D'$ (seventh equation).

Supp. $(\bar{1},\ 4)$ $= \kappa(\overline{1\kappa 1},\ 4)+O$
$+(m-\tfrac{5}{2})(4N+O)$
$-\tfrac{1}{3}N+\tfrac{4}{3}O$ (eighth equation).

Supp. $(\bar{1},\ 1,\ 3)$ $= \kappa(\overline{1\kappa 1},\ 1,\ 3)+2\kappa(\overline{2\kappa 1},\ 3)+2R+3J'$
$+(m-2)(2P+2Q+5J+4R)$
$-2P-\ Q-2R+J'$ (ninth equation).

Supp. $(\bar{1},\ 2,\ 2)$ $= \kappa(\overline{1\kappa 1},\ 2,\ 2)+3L+2M$
$+(m-2)(9K+3L+\ M+2N+O)$
$+3L+2M-2N-O$ (tenth equation).

$$\begin{aligned}\text{Supp. }(\bar{1}, 1, 1, 2) = {} & \kappa(\overline{1\kappa 1}, 1, 1, 2) + 2\kappa(\overline{2\kappa 1}, 1, 2) \\ & + H + 2I + D' + J' \\ & + (m - \tfrac{7}{3})(3E + 3F + 6G + 2D + H + 2I + 5J) \\ & \quad - 2E - 2F - G - \tfrac{1}{3}D + \tfrac{1}{3}H + \tfrac{5}{3}I - \tfrac{10}{3}J + 2D' - J'\end{aligned}$$

(eleventh equation).

$$\begin{aligned}\text{Supp. }(\bar{1}, 1, 1, 1, 1) = {} & \kappa(\overline{1\kappa 1}, 1, 1, 1, 1) + 2\kappa(\overline{2\kappa 1}, 1, 1, 1) + D' \\ & + (m - \tfrac{8}{3})(A + 2B + 4C + 3D) \\ & \quad - \tfrac{4}{3}A - \tfrac{5}{3}B - \tfrac{4}{3}C - 2D - D'.\end{aligned}$$

(twelfth equation).

132. Hence finally, merely collecting the terms, we have the following expressions of the Supplements in the twelve equations respectively.

$$\text{Supp. }(\bar{5}) = N + O \qquad \text{(first equation).}$$

$$\text{Supp. }(\bar{4}, 1) = 2J + 2R + J' \qquad \text{(second equation).}$$

$$\text{Supp. }(\bar{3}, 2) = 6K + 4L + 2M + 3N + 3O \qquad \text{(third equation).}$$

$$\text{Supp. }(\bar{3}, 1, 1) = D + E + F + 2G + H + 2I + 3J + D' + 2J' \qquad \text{(fourth equation).}$$

$$\text{Supp. }(\bar{2}, 3) = \kappa(\overline{2\kappa 1}, 3) + Q \qquad \text{(fifth equation).}$$

$$\text{Supp. }(\bar{2}, 2, 1) = \kappa(\overline{2\kappa 1}, 2, 1) + 3G + I + 4J + 3J' \qquad \text{(sixth equation).}$$

$$\text{Supp. }(\bar{2}, 1, 1, 1) = \kappa(\overline{2\kappa 1}, 1, 1, 1) + B + 4C + 4D + 2D' \qquad \text{(seventh equation).}$$

$$\text{Supp. }(\bar{1}, 4) = \kappa(\overline{1\kappa 1}, 4) + (4m - 7)N + (2m - 1)O \qquad \text{(eighth equation).}$$

$$\begin{aligned}\text{Supp. }(\bar{1}, 1, 3) = {} & \kappa(\overline{1\kappa 1}, 1, 3) + 2\kappa(\overline{2\kappa 1}, 3) \\ & + (2m-6)P + (2m-5)Q + (5m-10)J + (4m-8)R + 4J' \qquad \text{(ninth equation).}\end{aligned}$$

$$\begin{aligned}\text{Supp. }(\bar{1}, 2, 2) = {} & \kappa(\overline{1\kappa 1}, 2, 2) \\ & + (9m-18)K + 3mL + (m+2)M + (2m-6)N + (m-3)O \qquad \text{(tenth equation).}\end{aligned}$$

$$\begin{aligned}\text{Supp. }(\bar{1}, 1, 1, 2) = {} & \kappa(\overline{1\kappa 1}, 1, 1, 2) + 2\kappa(\overline{2\kappa 1}, 1, 2) \\ & + (2m - 5)D + (3m - 9)E + (3m - 9)F + (6m - 15)G \\ & + (m - 1)H + (2m - 1)I + (5m - 15)J + 3D' \qquad \text{(eleventh equation).}\end{aligned}$$

$$\begin{aligned}\text{Supp. }(\bar{1}, 1, 1, 1, 1) = {} & \kappa(\overline{1\kappa 1}, 1, 1, 1, 1) + 2\kappa(\overline{2\kappa 1}, 1, 1, 1) \\ & + (m - 4)A + (2m - 7)B + (4m - 12)C + (3m - 10)D, \qquad \text{(twelfth equation).}\end{aligned}$$

where I recall the remark, *ante*, No. 126, that in each equation the Capitals belong to the system obtained by diminishing the barred number by unity and removing the bar; (4) for the first equation, (3, 1) for the second, and so on.

133. These are, I think, the true theoretical forms of the Supplements, viz. (attending to the signification of the Capitals) the expressions actually exhibit how the Supplement arises, whether from proper conics passing through or touching at a cusp, or from point-pairs (coincident line-pairs) or line-pairs (including of course in these terms line-pair-points). Thus, for instance, Supp. $(\bar{5}) = N + O$. Referring to the explanations, First Memoir, Nos. 41 to 47, $N(=\iota)$ is the number of the line-pair-points described as "inflexion tangent terminated each way at inflexion," and $O\ (=\kappa)$ the number of the line-pair-points described as "cuspidal tangent terminated each way at cusp," or in what is here the appropriate point of view, we have as a coincident line-pair each inflexion tangent and each cuspidal tangent. Reverting to the generation of the first equation, when the point P is a point in general of the given curve, the curve Θ is the conic (5), having with the curve 5 intersections at P, and besides meeting it in the $2m-5$ points P'. When the point P is at an inflexion, the curve Θ becomes the coincident line-pair formed by the tangent taken twice, the number of intersections at P is therefore $=6$, and the inflexion is therefore (specially) a united point. Similarly, when the point P is at a cusp, the curve Θ becomes the coincident line-pair formed by the tangent taken twice, the number of intersections at P is therefore $=6$, and the cusp is thus (specially) a united point: we have thus the total number of special united points $=\kappa+\iota$, agreeing with the foregoing *à posteriori* result, Supp. $(\bar{5}) = N + O$.

134. Or to take another example; for the fifth equation we have

$$\text{Supp. } (\bar{2},\ 3) = \kappa\,(\overline{2\kappa 1},\ 3) + Q;$$

$Q\,(=2\tau)$ is the number of the line-pair-points described as "double tangent terminated each way at point of contact," or, in the point of view appropriate for the present purpose, we have each double tangent as a coincident line-pair in respect to the one of its points of contact, and also as a coincident line-pair in respect to the other of its points of contact. Reverting to the generation of the equation, when the point P is a point in general on the given curve, the curve Θ is the system of conics $(\bar{2},\ 3)$ touching the curve at P, and having besides with it a contact of the third order; since for each conic the number of intersections at P is $=2$, the total number of intersections at P is $=2\,(\bar{2},\ 3)$, and the remaining $(2m-2)\,(\bar{2},\ 3)$ intersections are the points P'. Suppose that the point P is taken at the point of contact of a double tangent; of the $(\bar{2},\ 3)$ conics, 1 (I assume this is so) becomes the coincident line-pair formed by the double tangent taken twice, and gives therefore 4 intersections at P, the remaining $(\bar{2},\ 3)-1$ conics are proper conics, giving therefore $2\,(\bar{2},\ 3)-2$ intersections at P, or the total number of intersections at P is $2\,(\bar{2},\ 3)+2$ intersections; or there is a gain of 2 intersections. As remarked (No. 96), this does not of necessity imply that the point in question is to be considered as being (specially) 2 united points; I do not know how to decide *à priori* whether it is to be regarded as being 2 united points or as 1 united point, but it is in fact to be regarded as being (specially) only 1 united point; and as the points in question are the 2τ points of contact of the double tangents, we have thus the number 2τ of special united points. Again, when the point P is at a cusp, all the $(\bar{2},\ 3)$ conics remain proper conics $((\overline{2\kappa 1},\ 3) = (\bar{2},\ 3)$,

First Memoir, No. 73), but each of these (*quâ* conic touching the cuspidal tangent) has with the given curve at the cusp not 2 but 3 intersections, so that the total number of intersections at P is $3(\overline{2\kappa 1}, 3)$, $=3(\bar{2}, 3)$, and there is a gain of $(\bar{2}, 3) = (\overline{2\kappa 1}, 3)$ intersections. Each cusp counts (specially) as $(\overline{2\kappa 1}, 3)$ united points, and together the cusps count as $\kappa(\overline{2\kappa 1}, 3)$ united points; we have thus the total number $\kappa(\overline{2\kappa 1}, 3) + 2\tau$ of special united points, agreeing with the expression, Supp. $(\bar{2}, 3) = \kappa(\overline{2\kappa 1}, 3) + Q$.

135. As appears from the preceding example, or generally from the remark, *ante*, No. 96, I have not at present any *à priori* method of determining the proper numerical multipliers of the Capitals contained in the expressions of the several Supplements. I will only further remark, that the reason is obvious why (while in the first seven equations the multipliers are mere numbers) in the eighth and following equations the multipliers are linear functions of m; in fact in these last equations the barred symbol is $\bar{1}$, that is, when P is a point in general on the given curve, each of the conics which make up the curve Θ has with the given curve not a contact of any order, but an ordinary intersection at P. Imagine a position of P for which one of these conics becomes a coincident line-pair; this regarded as a single line has with the given curve $(m-\alpha)$ ordinary intersections (α a number, $=4$ at most, depending on the contacts which the line may have with the curve); for each of the $m-\alpha$ points, taken as a position of P, one of the conics which make up the curve Θ becomes the coincident line-pair, and there are in respect of this conic two intersections at P instead of one intersection only. We have thus in respect of the particular coincident line-pair a group of $(m-\alpha)$ special united points, viz. these are the $m-\alpha$ ordinary intersections of the coincident line-pair regarded as a single line with the given curve, and we thus understand in a general way how it is that the order m of the given curve enters into the expressions of the multipliers of the several Capitals in the last five equations. The object of the present Memoir was, however, the *à posteriori* derivation of the expressions (*ante*, No. 132) of the twelve Supplements; and having accomplished this, but being unable to discuss the results with any degree of completeness, I abstain from a further discussion of them.

408.

ADDITION TO MEMOIR ON THE RESULTANT OF A SYSTEM OF TWO EQUATIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVIII. (for the year 1868), pp. 173—180. Received August 6,—Read November 21, 1867.]

THE elimination tables in the Memoir on the Resultant of a System of two Equations, *Phil. Trans.* 1857, pp. 703—715, [148], relate to equations of the form $(a, b \ldots \between x, y)^m = 0$, *without* numerical coefficients; but it is, I think, desirable to give the corresponding tables for equations in the form $(a, b, \ldots \between x, y)^m = 0$ *with* numerical coefficients, which is the standard form in quantics. The transformation can of course be effected without difficulty, and the results are as here given. It is easy to see *à priori* that the sum of the numerical coefficients in each table ought to vanish; these sums do in fact vanish, and we have thus a verification as well of the tables of the present Addition as of the tables of the original memoir, by means whereof the present tables were calculated.

Table (2, 2).
Resultant of
$(a, b, c \between x, y)^2$,
$(p, q, r \between x, y)^2$.

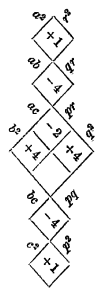


Table (3, 2).
Resultant of
$(a, b, c, d \between x, y)^3$,
$(p, q, r \between x, y)^2$.

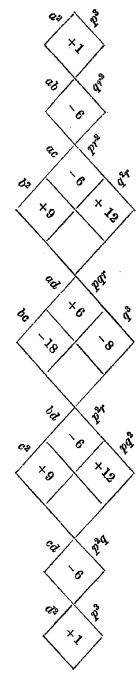


Table (4, 2).
Resultant of
$(a, b, c, d, e \between x, y)^4$,
$(p, q, r \between x, y)^2$.

a^2	r^4							
$+1$								
ab	qr^3							
-8								
ac	pr^3							
-12								
b^2	q^2r^2							
$+16$		-24						
ad	pqr^2							
$+24$								
bc	q^3r							
-48		-32						
ae	p^2r^2							
$+2$								
bd	pqr^2							
-32		-16						
c^2	q^4							
$+36$		$+64$		$+16$				
be	p^3qr							
$+24$								
cd	pq^3							
-48		-32						
ce	p^3r							
-12								
d^2	p^2q^2							
$+16$		$+24$						
de	p^3q							
-8								
e^2	p^4							
$+1$								

Table (3, 3)*.
Resultant of
$(a, b, c, d \between x, y)^3$,
$(p, q, r, s \between x, y)^3$.

a^3	r^3							
$+1$								
a^2b	r^2s							
-9								
a^2c	q^2r							
-18								
ab^2	r^2p							
$+27$		$+27$						
a^2d	p^2qr							
-3								
abc	qr^2s							
$+27$		$+27$						
b^3	r^3							
-27		-81		-27				
abd	pr^2s							
-9								
ac^2	q^2r^2							
-54		-54						
b^2c	qr^3s							
$+81$		$+81$		$+81$				
acd	pqr^2							
$+9$								
b^2d	pr^2s							
$+54$		$+54$						
bc^2	q^2r							
-81		-81		-81				

ad^2	p^2qs							
$+3$								
bcd	pqr^2							
-27		-27						
c^3	q^3r							
$+27$		$+81$		$+27$				
bd^2	p^2r							
$+18$								
c^2d	pq^2r							
-27		-27						
cd^2	p^2qr							
$+9$								
d^3	p^3s							
-1								

* N.B. In the corresponding table of the memoir, there is an error in the signs of the last two terms; they should be

cd^2	p^2q	
$+1$		
d^3	p^3	
-1		

Table (4, 3).
Resultant of
$(a, b, c, d, e \between x, y)^4$,
$(p, q, r, s \between x, y)^3$.

| | | |

Table (4, 4).
Resultant of
$(a, b, c, d, e \between x, y)^4$,
$(p, q, r, s, t \between x, y)^4$.

a^4								p^4
+1								
a^3b								p^3q
−16								
a^3c								p^3r
−72								
a^2c^2								p^2s^2
+96	+96							
a^3d								p^3s
−48								
a^2bc								p^2qr
+288	+288							
ad^2								pt^2
−256	−576	−256						
a^2e								p^2t
−4								
a^2bd								p^2qs
+64	+64							
a^2c^2								p^2r^2
+72	−256	+72						
ab^2c								pq^2r
+72	−1152	−1152	−384					
b^4								q^4
+256	+1536	+1296	+1536	+256				

Table (4, 4) continued:

a^3d_e								p^2r^2
a^2cd							-16	
ab^2d						-480		qr^2
abc^2					$+256$		-480	
b^2c				$+1728$		$+576$		q^2s
			-1536		$+3072$		$+256$	
				-3456		$+3072$		p^2s
					-4096		$+1728$	
						-3456		p^3
							-1536	

a^3cd									p^2r^2
ab^2e								$+72$	
a^2cd							$+192$		q^2r
$abcd$						-288		-288	
b^2d					-288		$+768$		p^2qs
ac^3				$+2304$		$+768$		$+192$	
b^2c			-3072		-4608		-256		qrs
		-2592		0		$+768$		$+2304$	
	$+3456$		$+3456$		-4608		$+4608$		p^2r
				$+4096$		-4608		-2592	
					$+9216$		0		q^3s
						$+3456$		-3072	
							$+4096$		p^2s^2
								$+3456$	

a^2de	pqr^2													
$abce$	prs^2t	+80												
b^3e	q^2s^2t	+192	+192											
abd^2	qr^2t^2	−768	−4608	−1280										
ac^2d	ps^3	−1280	+4608	+576	+576									
b^2cd	qr^3s^2	+576	+1536	+1536	−768	−768								
bc^3	p^2s^4	+4608	+6912	0	+6912	+4608	+4608							
		−8456	−9216	+8192	+8192	0	−9216	−9216	−8456					
			−9216	−9216	−9216	−9216	−4096	−9216						

a^2e^2	p^2q^2													
$abde$	pqs^2t	+6												
ac^2e	pr^2t^2	−128	−128											
b^2ce	q^2r^2t	−144	+2560	−144										
acd^2	ps^2r^2	+384	0	0	+384									
b^2d^2	qs^2t^2	+384	−1536	+5184	−1536	+384								
bc^2d	qr^3s^2	+512	−1536	−6912	−6192	−1536	+512							
c^4	r^4s^4	−2304	−8192	−6912	0	−6912	−8192	−2304						
		+1296	+9216	+9216	+9216	+9216	+9216	+9216	+1296					

Table (4, 4) concluded:

$abce^2$	p^2qr													
	+80	$aade^2$	pqr^2											
		+192	+192	b^2de	q^2r									
			−1920	−4608	−768	bc^2e	pq^2							
				+1536	+4608	−1920	acd^2	pr^2						
					+576	+1536	+6912	bcd^2	q^2s					
						0	−768	+4608	+576	e^3d	qr^2			
							+6912	+4608	+8192	−3456	−9216	q^3		
								0	−9216	−9216	−4096	−9216	−3456	

$acde^2$	p^2qr													
	+72	b^2e^3	pq^2r											
		−288	−192	ade^2	p^2q^2									
			−288	+192	+2304	$bcde$	pqr^2							
				+768	−256	+2304	c^3e	q^3r						
					+768	−2592	−4608	bd^3	pr^3					
						−4608	−3072	0	−2592	e^2d^2	r^4s^2			
							−4608	−4608	+4096	+3456	+3456	+3456		
								+9216	+4096	+3456	+3456	+3456		

cd^3	c^2de	bd^2e	bc^2e	cde^2	p^3q^3	p^2q^4	pq^5	q^6
−1536	+1728	+256	−480	−16				
	−3456	+3072	+576	−480				
		−4096	+3072	+256				
			−3456	+1728				
				−1536				

d^4e	cde^2	be^3	p^4q^2	p^3q^3	p^2q^4	pq^5
−256	+288	−48				
	−576	+288				
		−256				

d^2e^2	ce^3	p^4q^2	p^3q^3	p^2q^4
+96	−72			
	+96			

de^3	p^4q	p^3q^2	pq^4
−16			
	−16		

e^4	p^4	p^3q	q^4
+1			

d^4	cd^2e	c^2de	bde^2	ce^3	p^5q	p^4q^2	p^3q^3	pq^4	q^5
+256	−384	+72	+64	−4					
	+1536	−1152	−256	+64					
		+1296	−1152	+72					
			+1536	−384					
				+256					

409.

ON THE CONDITIONS FOR THE EXISTENCE OF THREE EQUAL ROOTS, OR OF TWO PAIRS OF EQUAL ROOTS, OF A BINARY QUARTIC OR QUINTIC.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLVIII. (for the year 1868), pp. 577—588. Received November 26, 1867,—Read January 9, 1868.]

[IT is remarked, *Proc. R. Soc.* vol. XVII. p. 314, that the above title is a misnomer: I had in fact in regard to the quintic considered *not* the twofold relations belonging to the root-systems 311 and 221 respectively, but the threefold relations belonging to the root-systems 41 and 32 respectively. The proper title would have been "On the conditions for the existence of certain systems of equal roots of a binary quartic or quintic."]

In considering the conditions for the existence of given systems of equalities between the roots of an equation, we obtain some very interesting examples of the composition of relations. A relation is either onefold, expressed by a single equation $U=0$, or it is, say k-fold, expressed by a system of k or more equations. Of course, as regards onefold relations, the theory of the composition is well known: the relation $UV=0$ is a relation compounded of the relations $U=0$, $V=0$; that is, it is a relation satisfied if, and not satisfied unless, one or the other of the two component relations is satisfied. The like notion of composition applies to relations in general; viz., the compound relation is a relation satisfied if, and not satisfied unless, one or the other of the two component relations is satisfied. I purposely refrain at present from any further discussion of the theory of composition. I say that the conditions for the existence of given systems of equalities between the roots of an equation furnish instances of such composition; in fact, if we express that the function $(*\between x, y)^n$, and its first-derived function in regard to x, or, what is the same thing, that the first-derived functions in regard to x, y respectively, have a common quadric factor, we obtain between the coefficients a certain twofold relation, which implies either that the equation $(*\between x, y)^n=0$ has three equal roots, or else that it has two pairs of equal roots; that is, the relation in question is satisfied if, and it is not satisfied

unless, there is satisfied either the relation for the existence of three equal roots, or else the relation for the existence of two pairs of equal roots; or the relation for the existence of the quadric factor is compounded of the last-mentioned two relations. The relation for the quadric factor, for any value whatever of n, is at once seen to be expressible by means of an oblong matrix, giving rise to a series of determinants which are each to be put $=0$; the relation for three equal roots and that for two pairs of equal roots, in the particular cases $n=4$ and $n=5$, are given in my "Memoir on the Conditions for the existence of given Systems of Equalities between the roots of an Equation," *Phil. Trans.* vol. CXLVII. (1857), pp. 727—731, [150]; and I propose in the present Memoir to exhibit, for the cases in question $n=4$ and $n=5$, the connexion between the compound relation for the quadric factor with the component relations for the three equal roots and for the two pairs of equal roots respectively.

Article Nos. 1 to 8, *the Quartic.*

1. For the quartic function

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4,$$

the condition for three equal roots, or, say, for a root system 31, is that the quadrinvariant and the cubinvariant each of them vanish, viz. we must have

$$\begin{aligned} I &= ae - 4bd + 3c^2 = 0, \\ J &= ace - ad^2 - b^2e + 2bcd - c^3 = 0. \end{aligned}$$

2. The condition for two pairs of equal roots, or for a root system 22, is that the cubicovariant vanishes identically, viz. representing this by

$$(A,\ B,\ 5C,\ 10D,\ 5E,\ F,\ G \between x,\ y)^6 = 0,$$

we must have

$$\begin{aligned} A &= a^2d - 3abc + 2b^3 &&= 0, \\ B &= a^2e + 2abd - 9ac^2 + 6b^2c &&= 0, \\ C &= abe - 3acd + 2b^2d &&= 0, \\ D &= -ad^2 + b^2e &&= 0, \\ E &= -ade + 3bce - 2bd^2 &&= 0, \\ F &= -ae^2 - 2bde + 9c^2e - 6cd^2 &&= 0, \\ G &= -be^2 + 3cde - 2d^3 &&= 0. \end{aligned}$$

3. But the condition for the common quadric factor is

$$\left\| \begin{matrix} & a, & 3b, & 3c, & d \\ & b, & 3c, & 3d, & e \\ a, & 3b, & 3c, & d & \\ b, & 3c, & 3d, & e & \end{matrix} \right\| = 0,$$

and the determinants formed out of this matrix must therefore vanish for $(I,\ J)=0$, and also for $(A,\ B,\ C,\ D,\ E,\ F,\ G)=0$, that is, the determinants in question must be syzygetically related to the functions $(I,\ J)$, and also to the functions $(A,\ B,\ C,\ D,\ E,\ F,\ G)$.

4. The values of the determinants are

$1234 = 3\times$	$1235 = 3\times$	$1245 =$	$1345 = 3\times$	$2345 = 3\times$
$a^2ce + 1$	$a^2de - 1$	$a^2e^2 - 1$	$abe^2 - 1$	$ace^2 + 1$
$a^2d^2 - 3$	$abce + 4$	$abde + 2$	$acde + 4$	$ad^2e - 1$
$ab^2e - 1$	$abd^2 + 1$	$ac^2e + 9$	$ad^3 - 3$	$b^2e^2 - 3$
$abcd + 14$	$ac^2d - 3$	$acd^2 - 9$	$b^2de + 1$	$bcde + 14$
$ac^3 - 9$	$b^2e - 3$	$b^2ce - 9$	$bc^2e - 3$	$bd^3 - 8$
$b^3d - 8$	$b^3cd + 2$	$b^3d^2 + 8$	$bcd^2 + 2$	$c^2e - 9$
$b^2c^2 + 6$				$c^2d^2 + 6$

5. The syzygetic relation with (I, J) is given by means of the identical equation

$$\begin{vmatrix} y^4, & -4xy^3, & 6x^2y^2, & -4x^3y, & x^4 \\ & a\ , & 3b\ , & 3c\ , & d \\ & b\ , & 3c\ , & 3d\ , & e \\ a\ , & 3b\ , & 3c\ , & d\ , & \\ b\ , & 3c\ , & 3d\ , & e\ , & \end{vmatrix} = -6I\,.\,\tilde{H}U + 9J\,.\,U,$$

or, as this may be written,

$$(1234,\ 1235,\ 1245,\ 1345,\ 2345 \between x,\ y)^4 = -6I\,.\,\tilde{H}U + 9J\,.\,U,$$

where $\tilde{H}U$ is the Hessian of U,

$$= \left(\begin{array}{c|c|c|c|c} ac+1 & ad+2 & ae+1 & be+2 & ce+1 \\ b^2-1 & bc-2 & bd+2 & cd-2 & d^2-1 \\ & & c^2-3 & & \end{array} \right) \between x,\ y)^4.$$

6. That is, we have

$$\begin{aligned} 1234 &= (\ ac - b^2\ ,\ a \between -6I,\ 9J), \\ 4\,.\,1235 &= (2ad - 2bc\ ,\ 4b \between -6I,\ 9J), \\ 6\,.\,1245 &= (\ ae + 2bd - 3c^2,\ 6c \between -6I,\ 9J), \\ 4\,.\,1345 &= (2be - 2cd\ ,\ 4d \between -6I,\ 9J), \\ 2345 &= (\ ce - d^2\ ,\ e \between -6I,\ 9J). \end{aligned}$$

7. The determinants thus vanish if $(I, J) = 0$, that is, for the root system 31; they will also vanish without this being so, if only

$$\left(\frac{3J}{2I} =\right) \frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e};$$

and we may omit the first member $\left(\frac{3J}{2I} \doteq\right)$, since if the remaining terms are equal

to each other they will also be $=\frac{3J}{2I}$. The equations may then be written

$$\left\| \begin{matrix} ac-b^2, & ad-bc, & ae+2bd-3c^2, & be-cd, & ce-d^2 \\ a\ , & 2b\ , & 6c\ , & 2d\ , & e \end{matrix} \right| = 0,$$

and the ten equations of this system reduce themselves (as it is very easy to show) to the seven equations

$$(A,\ B,\ C,\ D,\ E,\ F,\ G) = 0,$$

which, as above mentioned, are the conditions for the root system 22.

8. It may be added that we have

	A	B	C	D	E	F	G
$\frac{1}{3}.1234 =$		c	$-4b$	$+3a$			
$\frac{1}{3}.1235 =$			c	$-3b$	$+a$		
$0 =$		d	$-3c$		$+a$		
$1245 =$		$-e$	$+4d$	$-3c$			
$0 =$		$-e$		$+6c$		$-a$	
$0 =$			$-d$	$+3c$	$-b$		
$\frac{1}{3}.1345 =$			$-e$	$+3d$	$-c$		
$0 =$			$-e$		$+3c$	$-b$	
$\frac{1}{3}.2345 =$				$-3e$	$+4d$	$-c$	

where it is to be noticed that the four equations having the left-hand side $=0$, give $B : C : D : E : F$ proportional to the determinants of the matrix

$$\left| \begin{matrix} d, & -3c, & .\ , & a & \\ -e, & .\ , & 6c, & .\ , & -a \\ & -d\ , & 3c, & -b & \\ & -e\ , & .\ , & +3c, & -b \end{matrix} \right|;$$

the determinants in question contain each the factor c, and omitting this factor, the system shows that B, C, D, E, F are proportional to their before-mentioned actual values.

Article Nos. 9 to 15, *the Quintic.*

9. For the quintic function

$$(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5,$$

the condition of a root system 41 is that the covariant, $[B=]$ No. 14, shall vanish, viz. we must have

$$\begin{aligned} A &= 2\,(ae-4bd+3c^2) = 0, \\ B &= \quad af-3be+2cd = 0, \\ C &= 2\,(bf-4ce+3d^2) = 0. \end{aligned}$$

10. The condition of a root system 32 is that the following covariant, viz.

$$[3A^2B-25C^2,\ =]\ 3\,(\text{No. } 13)^2\,(\text{No. } 14) - 25\,(\text{No. } 15)^2,$$

shall vanish, where

$[A =]$ No. 13 $= (a,\ b,\ c,\ d,\ e,\ f\ \between x,\ y)^6$, the quintic itself.

$[B =]$ No. 14 $= ($

ae	af	bf
$bd - 4$	$be - 3$	$ce - 4$
$c^2 + 3$	$cd + 2$	$d^2 + 3$

$\between x,\ y)^5$.

$[C =]$ No. 15 $= ($

ac	$ad\ \ 3$	$ae\ \ 3$	af	$bf\ \ 3$	$cf\ \ 3$	df
$b^2 -$	$bc - 3$	$bd + 3$	$be + 7$	$ce + 3$	$de - 3$	$e^2 -$
		$c^2 - 6$	$cd - 8$	$d^2 - 6$		

$\between x,\ y)^4$.

11. The developed expression of the foregoing function is as follows:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{J}$	$\mathfrak{I}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	
$($ $a^3e + 3$	$a^2f + 3$	$a^2bf + 33$	$a^2cf + 10$	$a^2df - 90$	$a^2ef - 114$	$a^2f^2 - 19$	$abf^2 - 114$	$acf^2 - 90$	$adf^2 + 10$	$aef^2 + 33$	$af^2 + 3$	$bf^2 + 3$	$\between x,\ y)^{13}$
$a^2bd - 12$	$a^2be + 21$	$a^2ce - 102$	$a^2de - 390$	$a^2e^2 - 195$	$abdf - 264$	$abef - 608$	$acef - 264$	$adef + 360$	$ae^2f + 155$	$bdf^2 - 102$	$bef^2 + 21$	$cef^2 - 12$	
$a^2c^2 - 16$	$a^2cd - 144$	$a^2d^2 - 216$	$ab^2f + 155$	$abcf + 360$	$abe^2 - 990$	$acdf + 537$	$ad^2f + 468$	$ae^3 + 225$	$bcf^2 - 390$	$be^2f + 135$	$cdf^2 - 144$	$df^2 - 16$	
$ab^2e + 50$	$ab^2d + 30$	$ab^2e + 135$	$abce + 100$	$abde - 1500$	$ac^2f + 468$	$ace^2 - 245$	$ade^2 + 900$	$b^2f^2 - 195$	$bdef + 100$	$c^2f^2 - 216$	$cef + 30$	$def + 50$	
$b^4 - 25$	$abc^2 + 240$	$abcd + 120$	$abd^2 - 600$	$ac^2e + 900$	$acde + 1320$	$ad^2e + 1740$	$b^2ef - 990$	$bcef - 1500$	$be^3 + 125$	$cdef + 120$	$d^2ef + 240$	$e^4 - 25$	
	$b^3c - 150$	$ac^3 + 480$	$ac^2d + 1600$	$acd^3 + 1800$	$ad^2 + 1080$	$b^2cf - 245$	$bcdf + 1320$	$bd^2f + 900$	$c^2ef - 600$	$ce^2 - 150$	$de^2 - 150$		
		$b^2d - 150$	$b^2e + 125$	$b^2f + 225$	$b^2ce - 900$	$b^2de - 2700$	$bce^2 - 2700$	$c^2df + 1800$	$cd^2f + 1600$	$d^3f + 480$			
		$b^2c^2 - 300$	$b^2cd - 1000$	$b^2d^2 - 1500$	$b^2e^2 - 1700$	$bc^2f + 1740$	$bd^2e + 900$	$c^2e^2 - 1500$	$cde^2 - 1000$	$d^2e^2 - 300$			
					$bc^2e + 900$	$bcde - 2000$	$c^3f + 1080$						
					$bcd^2 - 600$	$bd^3 + 600$	$c^2de - 600$						
						$c^3e + 600$							
						$c^2d^2 - 400$							

12. The conditions for the common [cubic] factor are

$$\left\|\begin{matrix} & a, & 4b, & 6c, & 4d, & e \\ a, & 4b, & 6c, & 4d, & e & \\ & b, & 4c, & 6d, & 4e, & f \\ b, & 4c, & 6d, & 4e, & f & \end{matrix}\right\| = 0,$$

the several determinants whereof are given in Table No. 27 of my "Third Memoir on Quantics," *Philosophical Transactions*, vol. CXLVI. (1856), pp. 627—647, [144].

13. These determinants must therefore vanish, for $(A, B, C) = 0$, and also for $(\mathfrak{A}, \mathfrak{B}, \ldots \mathfrak{L}, \mathfrak{M}) = 0$, that is, they must be syzygetically connected with (A, B, C), and also with $(\mathfrak{A}, \mathfrak{B}, \ldots \mathfrak{L}, \mathfrak{M})$. The relation to (A, B, C) is in fact given in the Table appended to Table No. 27, viz. this is

	$C\times$	$+B\times$	$+A\times$
1234 =	$+6a^2$	$-12ab$	$+16ac-10b^2$
1235 =	$+6ab$	$-2ac-10b^2$	$+6ad$
1236 =	$-2ac+8b^2$	$+6ad-18be$	$-2df+8e^2$
1245 =	$+18ac$	$-6ad-30bc$	$+8ae+10bd$
1246 =	$+12bc$	$+4ae-4bd-24c^2$	$+4be+8cd$
1345 =	$+24ad$	$-8ae-40bd$	$+4af+20be$
1256 =	$-1ae+4bd+3c^2$	$+1af+5be-18cd$	$-1bf+4ce+3d^2$
2345 =	$+20ae+40bd-30c^2$	$-80be+20cd$	$+20bf+40ce-30d^2$
1346 =	$+4ae+8bd+6c^2$	$-36cd$	$+4bf+8ce+6d^2$
2346 =	$+4af+20be$	$-8bf-4ce$	$+24cf$
1356 =	$+4be+8cd$	$+4bf-4ce-24d^2$	$+12de$
2356 =	$+8bf+10ce$	$-6cf-30de$	$+18df$
1456 =	$+6ce$	$+6cf-18de$	$-2df+8e^2$
2456 =	$+6cf$	$-2df-10e^2$	$+6ef$
3456 =	$+16df-10e^2$	$-12ef$	$+6f^2$

14. Between the expressions $\mathfrak{A}$, $\mathfrak{B}$, &c., and 1234, 1235, &c., there exist relations the form of which is indicated by the following Table:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{J}$	$\mathfrak{J}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	1234	1235	1236	1245	1246	1345	1256	2345	1346	2346	1356	2356	1456	2456	3456
c	b	a											a														
d	c	b	a										b	a													
e	d	c	b	a									c	b	a	a											
f	e	d	c	b	a								d	c	b	b	a	a									
	f	e	d	c	b	a							e	d	c	c	b	b	a	a	a						
		f	e	d	c	b	a						f	e	d	d	c	c	b	b	b	a	a				
			f	e	d	c	b	a						f	e	e	d	d	c	c	c	b	b	a	a		
				f	e	d	c	b	a						f	f	e	e	d	d	d	c	c	b	b	a	
					f	e	d	c	b	a							f	f	e	e	e	d	d	c	c	b	a
						f	e	d	c	b	a								f	f	f	e	e	d	d	c	b
							f	e	d	c	b	a										f	f	e	e	d	c
								f	e	d	c	b												f	f	e	d
									f	e	d	c														f	e
										f	e	d															f

viz. these relations are of the form

$$(\ )\,c\mathfrak{A} + (\ )\,b\mathfrak{B} + (\ )\,a\mathfrak{C} \qquad + (\ )\,a\,.\,1234 \qquad = 0,$$

$$(\ )\,d\mathfrak{A} + (\ )\,c\mathfrak{B} + (\ )\,b\mathfrak{C} + (\ )\,a\mathfrak{D} \qquad + (\ )\,b\,.\,1234 + (\ )\,a\,.\,1235 = 0,$$

&c.

where the brackets ( ) denote numerical coefficients, determinate as to their ratios.

15. Assuming the existence of these relations, we have for the determination of the numerical coefficients in each relation a set of linear equations, which are shown by the following Tables, viz. referring to the Table headed $c\mathfrak{A}$, $b\mathfrak{B}$, $a\mathfrak{C}$, $a.1234$, [first of the seven tables *infrà*] if the multipliers of the several terms respectively be A, B, C, X, then the Table denotes the system of linear equations

$$\begin{array}{llllll} 0 & A & +3B & +33C & +0X & =0, \\ 3 & A & +0B & -102C & -16X & =0, \\ \&c., & & & & & \end{array}$$

that is, nine equations to be satisfied by the ratios of the coefficients A, B, C, X, and which are in fact satisfied by the values at the foot of the Table, viz.

$$A : B : C : X = +66 : -11 : +1 : +6.$$

There would be in all fourteen Tables, but as those for the second seven would be at once deducible by symmetry from the first seven, I have only written down the seven Tables; the solutions for the first and second Tables were obtained without difficulty, but that for the third Table was so laborious to calculate, and contains such extraordinarily high numbers, that I did not proceed with the calculation, and it is accordingly only the first, second, and third Tables which have at the foot of them respectively the solutions of the linear equations.

16. The results given by these three Tables are, of course,

$$66\,c\mathfrak{A} - 11\,b\mathfrak{B} + 1\,a\mathfrak{C} + 6a.1234 = 0,$$

$$330\,d\mathfrak{A} + 110\,c\mathfrak{B} - 55\,b\mathfrak{C} + 9\,a\mathfrak{D} - 105\,a.1235 = 0,$$

$$\begin{array}{rl} +266478575 & e\mathfrak{A} \\ -617359490 & d\mathfrak{B} \\ +144200810 & c\mathfrak{C} \\ +9656911 & b\mathfrak{D} \\ +9090785 & a\mathfrak{E} \\ -721004050 & c.1234 \\ +90914175 & b.1235 \\ -160758675 & a.1245 \\ +11559295 & a.1236 = 0. \end{array}$$

It is to be noticed that the nine coefficients of this last equation were obtained from, and that they actually satisfy, a system of fourteen linear equations; so that the correctness of the result is hereby verified.

17. The seven Tables are

First Table.

	$c\mathfrak{A}$	$b\mathfrak{B}$	$a\mathfrak{C}$	$a.1234$
a^3bf		+ 3	+ 33	
a^3ce	+ 3		− 102	− 16
a^3d^2			− 216	+ 36
a^2be^2		+ 21	+ 135	+ 16
a^2bcd	− 12	− 144	+ 120	− 152
a^2c^3	− 16		+ 480	+ 96
ab^3d		+ 30	− 150	+ 80
ab^2c^2	+ 50	+ 240	− 300	− 60
b^4c	− 25	− 150		
	+ 66	− 11	+ 1	+ 6

Second Table.

	$d\mathfrak{A}$	$c\mathfrak{B}$	$b\mathfrak{C}$	$a\mathfrak{D}$	$a.1235$	$b.1234$
a^3cf		+ 3		+ 10	− 4	
a^3de	+ 3			− 390	+ 24	
a^2b^2f			+ 33	+ 155	+ 4	
a^2bce		+ 21	− 102	+ 100	− 84	− 16
a^2bd^2	− 12		− 216	− 600	− 24	+ 36
a^2c^2d	− 16	− 144		+ 1600	+ 64	+ 16
ab^3e			+ 135	+ 125	+ 60	
ab^2cd	+ 50	+ 30	+ 120	− 1000	− 40	− 152
abc^3		+ 240	+ 480			+ 96
b^4d	− 25		− 150			+ 80
b^3c^2		− 150	− 300			− 60
	+ 330	+ 110	− 55	+ 9	− 105	0

Third Table.

	$e\mathfrak{A}$	$d\mathfrak{B}$	$c\mathfrak{C}$	$b\mathfrak{D}$	$a\mathfrak{E}$	$c.1234$	$b.1235$	$a.1245$	$a.1236$
a^3df		+ 3			− 90			− 6	+ 6
a^3e^2	+ 3				− 195			+ 16	
a^2bcf			+ 33	+ 10	+ 360		− 4	+ 6	− 22
a^2bde	− 12	+ 21		− 390	− 1500		+ 24	− 26	− 6
a^2c^2e	− 16		− 102		+ 900	− 16		− 96	+ 16
a^2cd^2		− 144	− 216		+ 1800	+ 36		+ 96	
ab^3f				+ 155	+ 225		+ 4		+ 16
ab^2ce	+ 50		+ 135	+ 100		+ 16	− 84	+ 90	− 10
ab^2d^2		+ 30		− 600	− 1500		− 24	− 80	
abc^2d		+ 240	+ 120	+ 1600		− 152	+ 64		
ac^4			+ 480			+ 96			
b^4e	− 25			+ 125			+ 60		
b^3cd		− 150	− 150	− 1000		+ 80	− 40		
b^2c^3			− 300			− 60			
	+ 266478575	− 617359490	+ 144200810	+ 9656911	+ 9090785	− 721004050	+ 90914175	− 160758675	+ 11559295

Fourth Table.

	$f\mathfrak{A}$	$e\mathfrak{B}$	$d\mathfrak{C}$	$c\mathfrak{D}$	$b\mathfrak{E}$	$a\mathfrak{F}$	$d.1234$	$c.1235$	$b.1236$	$b.1245$	$a.1246$	$a.1345$
a^3ef	+ 3	+ 3				− 114					+ 4	
a^2bdf	− 12		+ 33		− 90	− 264			+ 6	− 6	− 4	− 24
a^2be^2		+ 21			− 195	− 990				+ 16	− 4	+ 64
a^2c^2f	− 16			+ 10		+ 468		− 4			− 24	+ 24
a^2cde		− 144	− 102	− 390		+ 1320	− 16	+ 24			+ 24	− 208
a^2d^3			− 216			+ 1080	+ 36					+ 144
ab^2cf	+ 50			+ 155	+ 360	+ 900		+ 4	− 22	+ 6	+ 24	
ab^2de		+ 30	+ 135		− 1500	− 2700	+ 16		− 6	− 26	− 20	− 40
abc^2e		+ 240		+ 100	+ 900	+ 900		− 84	+ 16	− 96		+ 60
$abcd^2$			+ 120	− 600	+ 1800	− 600	− 152	− 24		+ 96		− 40
ac^3d			+ 480	+ 1600			+ 96	+ 64				
b^4f	− 24				+ 225				+ 16			
b^3ce		− 150		+ 125				+ 60	− 10	+ 90		
b^3d^2			− 150		− 1500		+ 80			− 80		
b^2c^2d			− 300	− 1000			− 60	− 40				

Fifth Table.

	$f\mathfrak{B}$	$e\mathfrak{C}$	$d\mathfrak{D}$	$c\mathfrak{E}$	$b\mathfrak{F}$	$a\mathfrak{G}$	$e.1234$	$d.1235$	$c.1236$	$c.1245$	$b.1246$	$b.1345$	$a.1256$	$a.2345$	$a.1346$
a^2f^2	+ 3					− 19							+ 1		
a^2bef	+ 21	+ 33			− 114	− 608					+ 4		− 2		+ 16
a^2cdf	− 144		+ 10	− 90		+ 537		− 4	+ 6	− 6			− 16	+ 20	− 36
a^2ce^2		− 102		− 195		− 245	− 16			+ 16			+ 16	− 80	− 16
a^2d^2e		− 216	− 390			+ 1740	+ 36	+ 24					+ 16	+ 60	+ 36
ab^2df	+ 30		+ 155		− 264	− 245		+ 4			− 4	− 24	− 15	− 80	− 16
ab^2e^2		+ 135			− 990	− 1700	+ 16				− 4	+ 64		+ 240	
abc^2f	+ 240			+ 360	+ 468	+ 1740			− 22	+ 6	− 24	+ 24		+ 60	+ 36
$abcde$		+ 120	+ 100	− 1500	+ 1320	− 2000	− 152	− 84	− 6	− 26	+ 24	− 208		− 860	− 20
abd^3			− 600		+ 1080	+ 600		− 24				+ 144		+ 960	
ac^3e		+ 480		+ 900		+ 600	+ 96		+ 16	− 96				+ 960	
ac^2d^2			+ 1600	+ 1800		− 400		+ 64		+ 96				− 320	
b^3cf	− 150			+ 225	+ 900				+ 16		+ 24				
b^3de		− 150	+ 125		− 2700		+ 80	+ 60			− 20	− 40			
b^2c^2e		− 300			+ 900		− 60		− 10	+ 90		+ 60			
b^2cd^2			− 1000	− 1500	− 600			− 40		− 80		− 40			

Sixth Table.

	$f\mathfrak{A}$	$e\mathfrak{D}$	$d\mathfrak{E}$	$c\mathfrak{F}$	$b\mathfrak{G}$	$a\mathfrak{H}$	$f.1234$	$e.1235$	$d.1236$	$d.1245$	$c.1246$	$c.1345$	$b.1256$	$b.2345$	$b.1346$	$a.2346$	$a.1356$
a^2bf^2	+ 33				− 19	− 114							+ 1				+ 4
a^3cef	− 102	+ 10		− 114		− 264	− 16	− 4			+ 4					− 24	− 4
a^3d^2f	− 216		− 90			+ 468	+ 36		+ 6	− 6						+ 24	− 24
a^2de^2		− 390	− 195			+ 900		+ 24		+ 16							+ 24
ab^2ef	+ 135	+ 155			− 698	− 990	+ 16	+ 4					− 2		+ 16	+ 64	− 4
$abcdf$	+ 120		+ 360	− 264	+ 537	+ 1320	− 152		− 22	+ 6	− 4	− 24	− 16	+ 20	− 36	− 208	+ 24
$abce^2$		+ 100		− 990	− 245	− 2700		− 84			− 4	+ 64	+ 16	− 80	− 16	− 40	− 20
abd^2e		− 600	− 1500		+ 1740	+ 900		− 24	− 6	− 26			+ 16	+ 60	+ 36	+ 60	
ac^3f	+ 480			+ 468		+ 1080	+ 96				− 24	+ 24				+ 144	
ac^2de		+ 1600	+ 900	+ 1320		− 600		+ 64	+ 16	− 96	+ 24	− 208				− 40	
acd^3			+ 1800	+ 1080						+ 96		+ 144					
b^3df	− 150		+ 225		− 245		+ 80		+ 16				− 80	− 80	− 16		
b^3e^2		+ 125			− 1700			+ 60						+ 240			
b^2c^2f	− 300			+ 900	+ 1740		− 60				+ 24			+ 60	+ 36		
b^2cde		− 1000		− 2700	− 2000			− 40	− 10	+ 90	− 20	− 40		− 860	− 20		
b^2d^3			− 1500		+ 600					− 80				+ 960			
bc^3e				+ 900	+ 600							+ 60		+ 960			
bc^2d^2				− 600	− 400							− 40		− 320			

Seventh Table.

	$f\mathfrak{D}$	$e\mathfrak{E}$	$d\mathfrak{F}$	$c\mathfrak{G}$	$b\mathfrak{H}$	$a\mathfrak{J}$	$f.1235$	$e.1236$	$e.1245$	$d.1246$	$d.1345$	$c.1256$	$c.2345$	$c.1346$	$b.2346$	$b.1356$	$a.2356$	$a.1456$
a^2ef^2	+ 10			− 19		− 90	− 4					+ 1					− 6	+ 6
a^2def	− 390	− 90	− 114			+ 360	+ 24	+ 6	− 6	+ 4							+ 6	− 22
a^2e^3		− 195				+ 225			+ 16									+ 16
ab^2f^2	+ 155				− 114	− 195	+ 4									+ 4	+ 16	
$abcef$	+ 100	+ 360		− 608	− 264	− 1500	− 84	− 22	+ 6			− 2		+ 16	− 24	− 4	− 26	− 6
abd^2f	− 600		− 264		+ 468	+ 900	− 24			− 4	− 24				+ 24	− 24	− 96	+ 16
$abde^2$		− 1500	− 990		+ 900			− 6	− 26	− 4	+ 64					+ 24	+ 90	− 10
ac^3df	+ 1600		+ 468	+ 537		+ 1800	+ 64			− 24	+ 24	− 16	+ 20	− 36			+ 96	
ac^2e^2		+ 900		− 245		− 1500		+ 16	− 96			+ 16	− 80	− 16			− 80	
acd^2e		+ 1800	+ 1320	+ 1740					+ 96	+ 24	− 208	+ 16	+ 60	+ 36				
ad^4			+ 1080								+ 144							
b^3ef	+ 125	+ 225			− 990		+ 60	+ 16							+ 64	− 4		
b^2cdf	− 1000		+ 900	− 245	+ 1320		− 40			+ 24		− 15	− 80	− 16	− 208	+ 24		
b^2ce^2				− 1700	− 2700			− 10	+ 90				+ 240		− 40	− 20		
b^2d^2e		− 1500	− 2700		+ 900				− 80	− 20	− 40				+ 60			
bc^3f				+ 1740	+ 1080								+ 60	+ 36	+ 144			
bc^2de			+ 900	− 2000	− 600						+ 60		− 860	− 20	− 40			
bcd^3			− 600	+ 600							− 40		+ 960					
c^4e				+ 600									+ 960					
c^3d^2				− 400									− 320					

And the remaining seven Tables might of course be deduced from these by writing (f, e, d, c, b, a) instead of (a, b, c, d, e, f), and making the corresponding alterations in the top line of each Table.

18. The equations $\mathfrak{A}=0$, $\mathfrak{B}=0, \ldots.$, $\mathfrak{M}=0$ consequently establish between the fifteen functions 1234, 1235, ... 3456 a system of fourteen equations, viz. the first and last three of these are

$$\begin{aligned}
&1234 = 0,\\
&1235 = 0,\\
-160758675 &. 1245\\
+\ 11559295 &. 1236 = 0,\\
&\vdots\\
+\ 11559295 &. 1456\\
-160758675 &. 2356 = 0,\\
&2456 = 0,\\
&3456 = 0.
\end{aligned}$$

To complete the proof that in virtue of the equations $\mathfrak{A}=0$, $\mathfrak{B}=0, \ldots$, $\mathfrak{M}=0$ all the fifteen functions 1234, 1235, ... 3456 vanish, it is necessary to make use of the identical relations subsisting between these quantities 1234, &c.; thus we have

$$a\,.\,1345 + 4b\,.\,1245 + 6c\,.\,1235 + 4d\,.\,1234 = 0,$$

$$b\,.\,1345 + 4c\,.\,1245 + 6d\,.\,1235 + 4e\,.\,1234 = 0,$$

which, in virtue of the above equations $1234=0$ and $1235=0$, become

$$a\,.\,1345 + 4b\,.\,1245 = 0,$$

$$b\,.\,1345 + 4c\,.\,1245 = 0,$$

giving (unless indeed $ac-b^2=0$) $1245=0$, $1345=0$; the equation $1245=0$ then reduces the third of the above equations to $1236=0$, and so on until it is shown that the fifteen quantities all vanish.

410.

A THIRD MEMOIR ON SKEW SURFACES, OTHERWISE SCROLLS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLIX. (for the year 1869), pp. 111—126. Received May 30,—Read June 18, 1868.]

THE present Memoir is supplementary to my "Second Memoir on Skew Surfaces, otherwise Scrolls," *Phil. Trans.* vol. CLIV. (1864), pp. 559—577, [340], and relates also to the theory of skew surfaces of the fourth order, or quartic scrolls. It was pointed out to me by Herr Schwarz(1), in a letter dated Halle, June 1, 1867, that in the enumeration contained in my Second Memoir I have given only a particular case of the quartic scrolls which have a directrix skew cubic; viz. my eighth species, $S(1, 3^2)$, where there is also a directrix line. And this led me to observe that I had in like manner mentioned only a particular case of the quartic scrolls with a triple directrix line; viz. my third species, $S(1_3, 1, 4)$, where there is also a simple directrix line. The omitted species, say, *ninth species*, $S(1_3)$, *with a triple directrix line*, and *tenth species*, $S(3^2)$, *with a directrix skew* cubic, are considered in the present Memoir; and in reference to them I develope a theory of the reciprocal relations of these scrolls, which has some very interesting analytical features.

The paragraphs of the present Memoir are numbered consecutively with those of my Second Memoir above referred to.

Quartic Scroll, Ninth Species, $S(1_3)$, with a triple directrix line.

54. Consider a line the intersection of two planes, and let the equation of the one plane contain in the order 3, that of the second plane contain linearly, a variable parameter θ; the equations of the two planes may be taken to be

$$(p, q, r, s \between \theta, 1)^3 = 0, \quad (u, v \between \theta, 1) = 0,$$

[1] I take the opportunity of referring to his paper on Quintic Scrolls, Schwarz, "Ueber die geradlinigen Flächen fünften Grades," *Crelle*, t. LXVII. (1867), pp. 23—57.

where (p, q, r, s, u, v) are any linear functions whatever of the coordinates (x, y, z, w). Hence eliminating θ we have as the equation of the scroll generated by the line in question

$$(p, q, r, s \between v, -u)^3 = 0,$$

viz. this is a quartic scroll having the line $u=0$, $v=0$ for a triple line; that is, the line in question is a triple directrix line.

55. Taking $x=0$, $y=0$ for the equations of the directrix line, or writing $u=x$, $v=y$, and moreover expressing (p, q, r, s) as linear functions of the coordinates (x, y, z, w), the equation of the scroll takes the form

$$(* \between x, y)^4 + z(* \between x, y)^3 + w(*' \between x, y)^3 = 0;$$

and we may, by changing the values of z and w, make the term in $(x, y)^4$ to be

$$(* \between x, y)^4 + (\alpha x + \beta y)(* \between x, y)^3 + (\gamma x + \delta y)(*' \between x, y)^3,$$

where the arbitrary constants α, β, γ, δ may be so determined as to reduce this to a monomial kx^4, kx^3y, or kx^2y^2.

56. The coefficient k may vanish, and the equation of the scroll then is

$$z(* \between x, y)^3 + w(*' \between x, y)^3 = 0,$$

or, what is the same thing, it is

$$(* \between x, y)^3 (z, w) = 0,$$

viz. the scroll has in this particular case the simple directrix line $z=0$, $w=0$, thus reducing itself to the *third species*, $S(1_3, 1, 4)$, *with a triple directrix line and a single directrix line.* It is proper to exclude this, and consider the ninth species, $S(1_3)$, as having a triple directrix line, but no simple directrix line.

57. The scroll $S(1_3)$ may be considered as a scroll $S(m, n, p)$ generated by a line which meets each of three given directrices; viz. these may be taken to be the directrix line, and any two plane sections of the scroll. The section by any plane is a quartic curve having a triple point at the intersection with the directrix line; moreover the sections by any two planes meet in four points, the intersections of the scroll by the line of intersection of the two planes. Conversely, taking any line and two quartics related as above (that is, each quartic has a triple point at its intersection with the line, and the two quartics meet in four points lying in a line), the lines which meet the three curves generate a quartic scroll $S(1_3)$. This appears from the formula

$$S(m, n, p) = 2mnp - \alpha m - \beta n - \gamma p \text{ (Second Memoir, No. 5)};$$

we have in the present case

$$m=1,\ n=4,\ p=4,\quad \alpha=4,\ \beta=3,\ \gamma=3,$$

and the order of the scroll is $32-4-12-12, =4$, that is, the scroll is a quartic scroll; there is no difficulty in seeing that through each point of the line there pass

three generating lines, but through each point of either of the plane quartics only a single generating line; that is, that the line is a triple directrix line, but each of the plane quartics a simple directrix curve.

58. We may instead of the section by any plane, consider the section by a plane through a generating line, or by a plane through two of the three generating lines which meet at any point of the directrix line; if (to consider only the most simple case) each of the planes be thus a plane through two generating lines, the section by either of these planes is made up of the two generating lines, and of a conic passing through the directrix line; the directrices are thus the line and two conics each of them meeting the line; we have therefore in the foregoing formula

$$m=1,\ n=2,\ p=2,\quad \alpha=0,\ \beta=1,\ \gamma=1,$$

and the order of the scroll is $8-2-2,\ =4$ as before.

Quartic Scroll, Tenth Species, (3^2), *with a directrix skew cubic met twice by each generating line*[1].

59. Consider a line, the intersection of two planes; and let the equation of each plane contain in the order 2 a variable parameter θ; the equations of the two planes may be taken to be

$$(p,\ q,\ r \between \theta,\ 1)^2=0,\quad (p',\ q',\ r' \between \theta,\ 1)^2=0,$$

where $(p,\ q,\ r,\ p',\ q',\ r')$ are linear functions of the coordinates $(x,\ y,\ z,\ w)$; hence eliminating θ, we have as the equation of the scroll generated by the line in question, $\square=0$, where $\square$ is the resultant of the two quadric functions. The equation may be written

$$4(pq'-p'q)(rq'-r'q)-(pr'-p'r)^2=0;$$

and the scroll has thus as a nodal (double) line the skew cubic determined by the equations

$$\left\| \begin{matrix} p, & q, & r \\ p', & q', & r' \end{matrix} \right\| = 0.$$

It is easy to see (and indeed it will be shown presently) that this curve is met twice by each generating line of the scroll, and that the scroll is consequently a quartic scroll as described above.

[1] I have worded this heading in accordance with that of the eighth species, Second Memoir, No. 47, but the two headings might be expressed more completely thus:

Eighth Species, $S(1,\ 3_2^2)$, *with a directrix line and a double directrix skew cubic met twice by each generating line;*

Tenth Species, $S(3_2^2)$, *with a double directrix skew cubic met twice by each generating line;*

viz. the subscript 2 would indicate that the skew cubic is a nodal (double) line on the scroll, the exponent 2 indicating that it is met twice by each generating line.

60. The coordinates (x, y, z, w) may be fixed in such manner that the equations of the skew cubic shall be

$$\left\| \begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix} \right\| = 0,$$

or, what is the same thing,

$$yw - z^2 = 0, \quad zy - xw = 0, \quad xz - y^2 = 0;$$

each of the equations $pq' - p'q = 0$, $rq' - r'q = 0$, $pr' - p'r = 0$ is then the equation of a quadric surface passing through the skew cubic, or, what is the same thing, each of the functions $pq' - p'q$, $rq' - r'q$, $pr' - p'r$ is a linear function of $yw - z^2$, $zy - xw$, $xz - y^2$; and the equation of the scroll is given as a quadric equation in the last-mentioned quantities. It will be convenient to represent the equation in the form

$$(H, F, C, B, A - F, -G \between yw - z^2, zy - xw, xz - y^2)^2 = 0,$$

or, writing for shortness

$$yw - z^2, zy - xw, xz - y^2 = p, q, r,$$

which letters (p, q, r) are used henceforward in this signification only, the equation will be

$$(H, F, C, B, A - F, -G \between p, q, r)^2 = 0,$$

viz. this is a quadric equation in (p, q, r), with arbitrary coefficients.

61. Comparing with the result, Second Memoir, Nos. 47 to 50, we see that in the particular case where the coefficients (A, B, C, F, G, H) satisfy the relation $AF + BG + CH = 0$, we have the *eighth species*, $S(1, 3^2)$, *with a directrix line and a directrix skew cubic met twice by each generating line.* We exclude this particular case, and in the tenth species consider the relation $AF + BG + CH = 0$ as not satisfied, and therefore the scroll as not having a directrix line.

62. I consider how the scroll may be obtained as a scroll $S(m^2, n)$ generated by a line meeting a curve of the order m twice and a curve of the order n once. The first curve will be the skew cubic, that is $m = 3$; the second curve may be any plane section of the scroll; such a section will be a quartic curve having three nodes, one at each intersection of its plane with the skew cubic. Conversely, if we have a skew cubic, and a plane quartic meeting the skew cubic in three points, each of them a node on the quartic, then the scroll generated by the lines which meet the skew cubic twice and the quartic once will be a quartic scroll. In fact (see First Memoir, No. 10, [339], and Second Memoir, No. 5) the order of the scroll is given by the formula $S(m^2, n) = n([m]^2 + M) - \text{reduction}$, $= 16 - \text{reduction}$. And in the present case the reduction arises (Second Memoir, No. 4) from the cones having their vertices at the intersections of the skew cubic and the quartic, and passing through the skew cubic. Each cone is of the order 2, and each intersection *quà* double point on the quartic gives a reduction $2 \times \text{order of cone}$, $= 4$; that is, the reduction arising from the three intersections is $= 12$; or the order of the scroll is $16 - 12$, $= 4$.

63. We may, instead of the section by a plane in general, consider the section by a plane through a generating line; the section is here made up of the generating line and of a plane cubic passing through each of the two points of intersection of the generating line with the skew cubic, and having a node at the remaining intersection of its plane with the skew cubic. Or we may consider the section by a plane through the two generating lines at any point of the skew cubic; the section is here made up of the two generating lines and of a conic passing through the second intersections of the two generating lines with the skew cubic; that is, meeting the skew cubic twice.

64. Conversely, consider a skew cubic, and a conic meeting it twice; the lines which meet the skew cubic twice, and also the conic, generate a quartic scroll; this appears by the before-mentioned formula $S(m^2, n) = n([m]^2 + M) - \text{reduction}$; viz. we have $m = 3$, $n = 2$, and the order is $= 8 - \text{reduction}$; the reduction arises from the cones having their vertices at the intersections of the skew cubic and the conic. Each cone is of the order 2, and (*quà* simple point on the conic) each intersection gives a reduction $=$ order of the cone; that is, the total reduction is $= 4$, and the order of the scroll is $8 - 4$, $= 4$ as above.

65. But a more elegant mode of generation of the scroll may be obtained by means of the skew cubic alone; viz. considering the system of lines which are in involution with five given lines, or say simply the lines which belong to an involution[1], I say that the locus of a line belonging to the involution, and meeting the skew cubic twice is the quartic scroll, *tenth species*, $S(3^2)$. In the particular case where the line (instead of belonging to a proper involution) meets a given line, the locus is a quartic scroll, *eighth species*, $S(1, 3^2)$.

66. The analysis is almost identical with that given (Second Memoir, Nos. 47 to 50) in regard to the scroll $S(1, 3^2)$. Considering a line defined by its "six coordinates" (a, b, c, f, g, h), the condition which expresses that the line shall belong to an involution is

$$(A, B, C, F, G, H \between a, b, c, f, g, h) = 0,$$

where (A, B, C, F, G, H) are arbitrary coefficients; if they are the coordinates of a line, that is, if $AF + BG + CH = 0$, then the condition expresses that the line (a, b, c, f, g, h), instead of belonging to a proper involution, meets the line (F, G, H, A, B, C).

[1] The theory is explained in my memoir "On the Six Coordinates of a Line," *Camb. Phil. Trans.* vol. XI. 1868, [348]. In explanation of the subsequent analytical investigations of the present memoir, it is convenient to remark that if on a given line we have the two points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$, and through the given line two planes $Ax + By + Cz + Dw = 0$ and $A'x + B'y + C'z + D'w = 0$; then we have

$$\begin{aligned} &\beta\gamma' - \beta'\gamma : \gamma\alpha' - \gamma'\alpha : \alpha\beta' - \alpha'\beta : \alpha\delta' - \alpha'\delta : \beta\delta' - \beta'\delta : \gamma\delta' - \gamma'\delta \\ &= AD' - A'D : BD' - B'D : CD' - C'D : BC' - B'C : CA' - C'A : AB' - A'B; \end{aligned}$$

and denoting either of these sets of equal ratios by

$$a : b : c : f : g : h,$$

then (a, b, c, f, g, h) satisfy identically the relation $af + bg + ch = 0$, and are said to be the six coordinates of the line.

We have to determine the locus of the line (a, b, c, f, g, h) the coordinates whereof satisfy the relation

$$(A, B, C, F, G, H \between a, b, c, f, g, h) = 0,$$

and which besides meets the skew cubic $yw - z^2 = 0$, $yz - xw = 0$, $xz - y^2 = 0$.

The equations of the skew cubic are satisfied by writing therein

$$x : y : z : w = 1 : t : t^2 : t^3;$$

and hence taking θ, ϕ for the parameters of the points of intersection of the line (a, b, c, f, g, h) with the skew cubic, we have

$$\begin{array}{llll} 1, & \theta, & \theta^2, & \theta^3; \\ 1, & \phi, & \phi^2, & \phi^3, \end{array}$$

as the coordinates of two points on the line in question; whence forming the expressions of the six coordinates of the line, and omitting the common factor $\phi - \theta$, these are

$$(a, b, c, f, g, h) = \theta\phi, -(\theta + \phi), 1, \theta^2 + \theta\phi + \phi^2, \theta\phi(\theta + \phi), \theta^2\phi^2,$$

and hence the condition of involution gives between the parameters θ, ϕ the equation

$$(A, B, C, F, G, H \between \theta\phi, -\theta - \phi, 1, \theta^2 + \theta\phi + \phi^2, \theta\phi(\theta + \phi), \theta^2\phi^2).$$

Moreover the coordinates of any point on the line in question are given by

$$x : y : z : w = l + m : l\theta + m\phi : l\theta^2 + m\phi^2 : l\theta^3 + m\phi^3;$$

and writing as above $p, q, r = yw - z^2, yz - xw, xz - y^2$, we thence find, omitting the common factor $(\theta - \phi)^2$,

$$p : q : r = \theta\phi : -(\theta + \phi) : 1;$$

and eliminating $\theta\phi$, $\theta + \phi$, we at once obtain

$$(A, B, C, F, G, H \between pr, qr, r^2, q^2 - pr, -pq, p^2) = 0,$$

or, what is the same thing,

$$(H, F, C, B, A - F, -G \between p, q, r)^2 = 0$$

as the equation of the scroll generated by the line in involution which meets the given skew cubic twice.

Reciprocal of the Quartic Scroll $S(3^2)$.

67. I propose to reciprocate in regard to the quadric surface $x^2 + y^2 + z^2 + t^2 = 0$ the foregoing scroll

$$(H, F, C, B, A - F, -G \between p, q, r)^2 = 0.$$

If the coordinates (a, b, c, f, g, h) of a line satisfy the condition of involution

$$(A, B, C, F, G, H \between a, b, c, f, g, h) = 0,$$

then the coordinates (a, b, c, f, g, h) of the reciprocal curve will satisfy the condition of involution

$$(F, G, H, A, B, C \between a, b, c, f, g, h) = 0.$$

The reciprocal of the before-mentioned skew cubic $x : y : z : w = 1 : t : t^2 : t^3$ is the quartic torse having for its edge of regression the skew cubic $3XZ - Y^2 = 0$, $YZ - 9XW = 0$, $3YW - Z^2 = 0$; or, what is the same thing, the skew cubic $X : Y : Z : W = 1 : 3t : 3t^2 : t^3$; see my paper "On the Reciprocation of a Quartic Developable," *Quart. Math. Journ.* vol. VII. (1866), pp. 87—92, [372].

68. Hence the reciprocal of the quartic scroll is the scroll generated by a line (a, b, c, f, g, h) the coordinates of which satisfy the condition of involution

$$(F, G, H, A, B, C \between a, b, c, f, g, h) = 0,$$

and which is moreover the intersection of two osculating planes of the skew cubic $X : Y : Z : W = 1 : 3t : 3t^2 : t^3$. For the point the parameter whereof is t, the equation of the osculating plane is

$$\begin{vmatrix} X, & Y, & Z, & W \\ 1, & 3t, & 3t^2, & t^3 \\ & 1, & 2t, & t^2 \\ & & 1, & t \end{vmatrix} = 0,$$

or, what is the same thing, the equation is

$$(t^3, -t^2, t, -1 \between X, Y, Z, W) = 0.$$

Hence for the line which is the intersection of the two osculating planes

$$(\theta^3, -\theta^2, \theta, -1 \between X, Y, Z, W) = 0,$$

$$(\phi^3, -\phi^2, \phi, -1 \between X, Y, Z, W) = 0,$$

forming the expressions of the six coordinates, but omitting the common factor $\phi - \theta$, these are

$$a, b, c, f, g, h = \theta^2 + \theta\phi + \phi^2, -\theta - \phi, 1, \theta\phi, \theta\phi(\theta + \phi), \theta^2\phi^2;$$

we have thus between the parameters θ, ϕ the relation

$$(F, G, H, A, B, C \between \theta^2 + \theta\phi + \phi^2, -\theta - \phi, 1, \theta\phi, \theta\phi(\theta + \phi), \theta^2\phi^2) = 0;$$

and the equation of the scroll is obtained by eliminating θ, ϕ between this equation and the last-mentioned two equations satisfied by θ, ϕ respectively.

69. We see that θ, ϕ are two of the roots of the equation

$$(X, -Y, Z, -W \between u, 1)^3 = 0;$$

let ρ be the third root, then we have

$$\theta+\phi+\rho \quad =\frac{Y}{X},$$

$$\theta\phi+\rho(\theta+\phi)=\frac{Z}{X},$$

$$\theta\phi.\rho \quad =\frac{W}{X},$$

and thence

$$\theta+\phi=\frac{1}{X}(Y-\rho X),\quad \theta\phi=\frac{1}{X}(Z-\rho Y+\rho^2 X)=\frac{1}{\rho X}W,$$

$$(X,\ -Y,\ Z,\ -W \between \rho,\ 1)^3=0.$$

Substituting for $\theta+\phi$ and $\theta\phi$ their values in terms of ρ, we find

$$F\rho\{Y^2-ZX-\rho XY\}-G\rho X(Y-\rho X)+H\rho X^2$$
$$+W\{(AX+BY+CZ)-\rho(BX+CY)+\rho^2 CX\}=0,$$

or, what is the same thing,

$$\rho^2 X(GX-FY+CW)$$
$$-\rho\{F(Y^2-ZX)-GXY+HX^2-BXW-CYW\}$$
$$+W(AX+BY+CZ)=0;$$

from which and the equation

$$(X,\ -Y,\ Z,\ -W \between \rho,\ 1)^3=0,$$

we have to eliminate ρ.

70. Writing for shortness

$$(\ .,\ \ H,\ -G,\ A \between X,\ Y,\ Z,\ W)=\alpha,$$
$$(-H,\ \ .,\ \ F,\ B \between \quad ,, \quad)=\beta,$$
$$(\ G,\ -F,\ \ .,\ C \between \quad ,, \quad)=\gamma,$$
$$(-A,\ -B,\ -C,\ .\between \quad ,, \quad)=\delta,$$

and therefore $\alpha X+\beta Y+\gamma Z+\delta W=0$: the two equations are

$$\rho^2 X\gamma+\rho(-\gamma Y-\beta X)-\delta W=0,$$
$$\rho^3 X-\rho^2 Y+\rho Z \qquad\qquad -W=0.$$

Writing the first equation in the form

$$\gamma(\rho^2 X-\rho Y+Z)-\beta(\rho X-Y)+\alpha X=0,$$

multiplying by $-\rho$, and reducing by the other equation,

$$\beta(\rho^2 X-\rho Y)-\rho\alpha X-\gamma W=0,$$

or, as this may be written,

$$\beta(\rho^2 X-\rho Y+Z)-\alpha(\rho X-Y)-\alpha Y-\beta Z-\gamma W=0.$$

From this and the preceding equation we deduce the values of $\rho^2 X - \rho Y + Z$ and $\rho X - Y$; viz. writing for shortness

$$\beta\delta - \gamma^2,\ \beta\gamma - \alpha\delta,\ \alpha\gamma - \beta^2 = \mathrm{p},\ \mathrm{q},\ \mathrm{r},$$

we find

$$\rho^2 X - \rho Y + Z : \rho X - Y : 1 = -\mathrm{r}Z + \mathrm{q}W : \mathrm{r}Y - \mathrm{p}W : -\mathrm{r},$$

or, what is the same thing,

$$\begin{aligned} \rho^2 X - \rho Y + Z &= Z - \frac{\mathrm{q}}{\mathrm{r}} W, \\ \rho X - Y &= -Y + \frac{\mathrm{p}}{\mathrm{r}} W; \end{aligned}$$

whence also

$$\begin{aligned} \rho^2 X - \rho^2 Y + \rho Z - W &= 0, \\ \rho^2 X - \rho Y &= -\frac{\mathrm{q}}{\mathrm{r}} W, \\ \rho X &= \frac{\mathrm{p}}{\mathrm{r}} W, \end{aligned}$$

and thence

$$\begin{aligned} \rho\left(Z - \frac{\mathrm{q}}{\mathrm{r}} W\right) &= W, \\ \rho\left(Y - \frac{\mathrm{p}}{\mathrm{r}} W\right) &= \frac{\mathrm{q}}{\mathrm{r}} W, \\ \rho X &= \frac{\mathrm{p}}{\mathrm{r}} W, \end{aligned}$$

and we have therefore

$$\left(\frac{W}{\rho} =\right) \quad Z - \frac{\mathrm{q}}{\mathrm{r}} W = \frac{\mathrm{r}}{\mathrm{q}}\left(Y - \frac{\mathrm{p}}{\mathrm{r}} W\right) = \frac{\mathrm{r}}{\mathrm{p}} X,$$

or omitting the first equation, we have (independent of ρ) a system which it is clear must be equivalent to a single equation.

71. I take any one of these equations, for instance the equation

$$Z - \frac{\mathrm{q}}{\mathrm{r}} W = \frac{\mathrm{r}}{\mathrm{q}}\left(Y - \frac{\mathrm{p}}{\mathrm{r}} W\right),$$

or, what is the same thing,

$$\mathrm{qr}Z - \mathrm{r}^2 Y + (\mathrm{pr} - \mathrm{q}^2)\, W = 0,$$

and I proceed to reduce it so as to obtain the result in a symmetrical form. For this purpose I observe that from the values of α, β, γ, δ, if only $AF + BG + CH$ not $= 0$, we have

$$\begin{aligned} X : Y : Z : W = (\ .,\ & -C,\ & B,\ & -F \between \alpha, \beta, \gamma, \delta) \\ : (\ C,\ & .,\ & -A,\ & -G \between \quad ,, \quad) \\ : (-B,\ & A,\ & .,\ & -H \between \quad ,, \quad) \\ : (\ F,\ & G,\ & H,\ & . \between \quad ,, \quad); \end{aligned}$$

and substituting these values, the equation in question becomes

$$\begin{aligned} &\mathrm{qr}\ (-B\alpha + A\beta - H\delta) \\ &-\mathrm{r}^2\ (\ C\alpha - A\gamma - G\delta) \\ &+(\mathrm{pr}-\mathrm{q}^2)(\ F\alpha + G\beta + H\gamma) = 0. \end{aligned}$$

This becomes

$$\begin{array}{lll} A\mathrm{r}(\mathrm{q}\beta + \mathrm{r}\gamma) & = & A\mathrm{r}(-\mathrm{p}\alpha) \quad = 0 \\ -B\mathrm{qr}\alpha & & -B\mathrm{qr}\alpha \\ -C\mathrm{r}^2\alpha & & -C\mathrm{r}^2\alpha \\ +F(\mathrm{pr}-\mathrm{q}^2)\alpha & & +F(\mathrm{pr}-\mathrm{q}^2)\alpha \\ +G\{\mathrm{r}^2\delta + (\mathrm{pr}-\mathrm{q}^2)\beta\} & & +G\mathrm{pq}\alpha \\ +H\{-\mathrm{qr}\delta + (\mathrm{pr}-\mathrm{q}^2)\gamma\} & & -H\mathrm{p}^2\alpha, \end{array}$$

viz. the whole equation divides by α; and, omitting this factor, the equation is

$$A\mathrm{pr} + B\mathrm{qr} + C\mathrm{r}^2 + F(\mathrm{q}^2 - \mathrm{pr}) - G\mathrm{pq} + H\mathrm{p}^2 = 0,$$

or, what is the same thing, it is

$$(H,\ F,\ C,\ B,\ A-F,\ -G \between \mathrm{p},\ \mathrm{q},\ \mathrm{r})^2 = 0,$$

where I recall that we have

$$\mathrm{p},\ \mathrm{q},\ \mathrm{r} = \beta\delta - \gamma^2,\ \beta\gamma - \alpha\delta,\ \alpha\gamma - \beta^2,$$

α, β, γ, δ being linear functions of the current coordinates $(X,\ Y,\ Z,\ W)$, viz. we have

$$\begin{array}{lllllll} \alpha = (& .\ , & H, & -G, & A \between X, & Y,\ Z,\ W), \\ \beta = (- & H, & .\ , & F, & B \between & ,, &), \\ \gamma = (& G, & -F, & .\ , & C \between & ,, &), \\ \delta = (- & A, & -B, & -C, & . \between & ,, &). \end{array}$$

72. It thus appears that when $AF + BG + CH$ is not $=0$, the reciprocal of the scroll

$$(H,\ F,\ C,\ B,\ A-F,\ -G \between p,\ q,\ r)^2 = 0$$

has an equation of the very same form,

$$(H,\ F,\ C,\ B,\ A-F,\ -G \between \mathrm{p},\ \mathrm{q},\ \mathrm{r})^2 = 0; \qquad \text{(Rec. I.)}$$

so that in fact the scroll, *tenth species*, $S(3^2)$, defined as the scroll generated by a line in involution which passes through two points of a skew cubic, may be reciprocally defined as the scroll generated by a line in involution which lies in two osculating planes of a skew cubic.

73. If for $(\alpha,\ \beta,\ \gamma,\ \delta)$ we substitute their values in terms of $(X,\ Y,\ Z,\ W)$, the foregoing equation of the reciprocal scroll is obtained as an equation of the fourth order in the coordinates $(X,\ Y,\ Z,\ W)$, and (in the first instance) of the fifth degree in the coefficients $(A,\ B,\ C,\ F,\ G,\ H)$. It is a remarkable circumstance that the whole

equation contains the constant factor $AF+BG+CH$, so that throwing this out, the reduced equation will be only of the third degree in the coefficients.

74. The transformation is a very troublesome one, but I will indicate the steps by which I succeeded in accomplishing it. Each of the functions (p, q, r) is a quadric function of (X, Y, Z, W), say,

$$\begin{aligned} \mathrm{p} &= (a, b, c, d, f, g, h, l, m, n \between X, Y, Z, W)^2, \\ \mathrm{q} &= (a', \ldots \between \quad \text{"} \quad)^2, \\ \mathrm{r} &= (a'', \ldots \between \quad \text{"} \quad)^2; \end{aligned}$$

we have to form the value of

$$(H, F, C, B, A-F, -G \between \mathrm{p}, \mathrm{q}, \mathrm{r})^2,$$

viz. representing this for shortmess by

$$(H, F, C, B, A-F, -G) \begin{pmatrix} a, b, c, d, f, g, h, l, m, n \\ a', \ldots \\ a'', \ldots \end{pmatrix}^2,$$

the coefficient of X^4 is

$$(H, F, C, B, A-F, -G \between a^2, a'^2, a''^2, a'a'', a''a, aa''),$$

that of X^3Y is

$$(H, F, C, B, A-F, -G \between 2af, 2a'f', 2a''f'', a'f''+a''f', a''f+af'', af'+a'f),$$

and so on, the successive terms a^2, a'^2, &c., $2af$, $2a'f'$, &c. being derived by an obvious law from the first terms a^2, $2af$, &c.; and these first terms are merely the coefficients of the terms X^4, X^3, Y, &c. in the development of

$$p^2, = \{(a, b, c, d, f, g, h, l, m, n \between X, Y, Z, W)^2\}^2;$$

viz. this is

X^4	X^3Y	X^3Z	X^3W	X^2Y^2	X^2YZ	X^2YW	X^2Z^2,	X^2ZW,	X^2W^2,	XY^3,	XY^2Z,	XY^2W,	XYZ^2
a^2	$2af$	$2ag$	$2al$	$2ab$ $+h^2$	$2af$ $+2gh$	$2am$ $+2hl$	$2ac$ $+g^2$	$2an$ $+2gl$	$2ad$ $+l^2$	$2bh$	$2bg$ $+2fh$	$2bl$ $+2hm$	$2ch$ $+2fg$

$XYZW$,	XYW^2,	XZ^3,	XZ^2W,	XZW^2,	XW^3,	Y^4,	Y^3Z,	Y^3W,	Y^2Z^2,	Y^2ZW,	Y^2W^2,	YZ^3,
$2lf$ $+2mg$ $+2nh$	$2dh$ $+2lm$	$2cg$	$2cl$ $+2gn$	$2dg$ $+2ln$	$2dl$,	b^2,	$2bf$,	$2bm$,	$2bc$, $+f^2$	$2bn$, $+2fm$	$2bd$, $+m^2$	$2cf$

YZ^2W,	YZW^2,	YW^3,	Z^4,	Z^3W,	Z^2W^2,	ZW^3,	W^4
$2cm$ $+2fn$	$+2df$ $+2mn$	$2dm$,	c^2,	$2cn$,	$2cd$, $+n^2$	$2dn$,	d^2

;

and the values of the coefficients $a, b, \ldots$ which enter into the formulæ are given by means of the following values of p, q, r; viz. these are

$$\begin{array}{lllllllllll} & X^2 & Y^2 & Z^2 & W^2 & YZ & ZX & XY & XW & YW & ZW \\ \mathrm{p} = \Big(& AH, & -F^2, & -CF, & -C^2, & -BF, & -AF, & BH, & -AB, & -B^2, & -BC \between X, Y, Z, W)^2, \\ & -G^2 & & & & & +CH & +2FG & -2CG & +2CF & \end{array}$$

$$\begin{array}{lllllllllll} \mathrm{q} = (-GH, & BH, & -CG, & BC, & -BG, & -AG, & AH, & A^2, & AB, & AC & \between \quad ,, \quad)^2, \\ & & & & +CH & +FG & +FH & +BG & -BF & +CF & \\ & & & & -F^2 & & & -CH & & & \end{array}$$

$$\begin{array}{lllllllllll} \mathrm{r} = (-H^2, & -FH, & -F^2, & AC, & FG, & 2FH, & GH, & 2AG, & -AF, & -2BF & \between \quad ,, \quad)^2. \\ & & & -B^2 & & -G^2 & & +BH & +CH & -CG & \end{array}$$

75. As an instance of the calculation of a single term, the coefficient of X^4 is

$$(H, F, C, B, A-F, -G \between AH-G^2, -GH, -H^2)^2;$$

viz. this is

$$\begin{array}{lll} H(AH-G^2)^3 & = & A^3H^3-2AG^2H^2+G^4H^2 \\ +FH^2G^2 & = & FG^2H^2 \\ +CH^4 & = & CH^4 \\ +BGH^3 & = & BGH^2 \\ +(A-F)(-AH^3+G^2H^2) & = & -A^3H^3+AG^2H^2 \\ & & +AFH^3-FG^2H^2 \\ -G(-AGH^2+G^3H) & = & AG^2H^2-G^4H^2; \end{array}$$

the whole term is thus $=(AF+BG+CH)H^3$, viz. there is the factor $AF+BG+CH$ as mentioned above.

76. Throwing out the factor in question, $AF+BG+CH$, the equation of the reciprocal scroll is found to be

$$\begin{array}{lll} 0 = & X^4 . H^3 & \text{(Rec. II.)} \\ & +X^3Y \quad . -2GH^2 & \\ & +X^3Z \quad . \quad AH^2-3FH^2+G^2H & \\ & +X^3W \quad . -3AGH-3BH^2+G^3 & \\ & +X^2Y^2 \quad . \quad 2FH^2+G^2H & \\ & +X^2YZ \quad . -AGH+BH^2+2FGH-G^3 & \\ & +X^2YW \quad . \quad A^2H+3AFH+AG^2+BGH-2CH^2-3FG^2 & \\ & +X^2Z^2 \quad . -2AFH+AG^2+CH^2-FG^2+3F^2H & \\ & +X^2ZW \quad . -2A^2G-2ABH+3AFG+6BFH-BG^2-CGH & \\ & +X^2W^2 \quad . \quad A^3+3ABG-3ACH+3B^2H+3CG^2 & \end{array}$$

$$
\begin{array}{ll}
+XY^3 & -2FGH \\
+XY^2Z & AFH-BGH-3F^2H+2FG^2 \\
+XY^2W & 2ABH-2AFG-BFH+2CGH+3F^2G \\
+XYZ^2 & -AFG-2BFH+BG^2-CGH \\
+XYZW & A^2F-3AF^2-3ABG+ACH-2B^2H+BFG+5CFH-2CG^2 \\
+XYW^2 & 2A^2B-3ABF+2ACG+B^2G+BCH-6CFG \\
+XZ^3 & AF^2-2CFH+CG^2-F^3 \\
+XZ^2W & 2ABF-2ACG-2BCH-3BF^2+CFG \\
+XZW^2 & A^2C+AB^2+3ACF-3B^2F+BCG-2C^2H \\
+XW^3 & 3ABC-B^3+3C^2G \\
+Y^4 & F^2H \\
+Y^3Z & BFH-F^2G \\
+Y^3W & AF^2+B^2H-2CFH-F^3 \\
+Y^2Z^2 & -BFG+CFH+F^3 \\
+Y^2ZW & ABF-B^2G+BCH+2CFG \\
+Y^2W^2 & AB^2-2ACF-B^2F+C^2H+3CF^2 \\
+YZ^3 & BF^2-CFG \\
+YZ^2W & ACF+2B^2F-BCG-3CF^2 \\
+YZW^2 & ABC+B^3-2BCF-C^2G \\
+YW^3 & AC^2+B^2C-3C^2F \\
+Z^4 & CF^2 \\
+Z^3W & 2BCF \\
+Z^2W^2 & B^2C+2C^2F \\
+ZW^3 & 2BC^2 \\
+W^4 & C^3,
\end{array}
$$

where, in regard to the symmetry of this equation, it is to be observed that we may interchange X and W, and Y and Z, leaving A, F unaltered but interchanging B and $-G$, and also C and H; thus the coefficient of X^3Z being $AH^2-3FH^2+G^2H$, that of YW^3 is $AC^2-3FC^2+B^2C$, $=AC^2+B^2C-3C^2F$. Or, again, the coefficient of Y^2Z being $BFH-F^2G$, that of YZ^2 is $-GFC+F^2B$, $=BF^2-CFG$.

77. But the equation may be written in the much more simple form

$$
\begin{array}{ll}
 & X(-\alpha^2\delta+3\alpha\beta\gamma-2\beta^3) \\
+ & Y(-\alpha\beta\delta+2\alpha\gamma^2-\beta^2\gamma) \\
+ & Z(\alpha\gamma\delta-2\beta^2\delta+\beta\gamma^2) \\
+ & W(\alpha\delta^2-3\beta\gamma\delta+2\gamma^3)=0,
\end{array}
\qquad \text{(Rec. III.)}
$$

or, what is the same thing,

$$-\tfrac{1}{6}(3X\partial_\delta - Y\partial_\gamma + Z\partial_\beta - 3W\partial_\alpha)(\alpha^2\delta^2 - 6\alpha\beta\gamma\delta + 4\alpha\gamma^3 + 4\beta^3\delta - 3\beta^2\gamma^2) = 0, \qquad \text{(Rec. III.)}$$

as may be verified by actual substitution of the values of the coordinates.

78. By what precedes, substituting for p, q, r their values in terms of α, β, γ, δ, it appears that we have the remarkable identity

$$(H,\ F,\ C,\ B,\ A-F,\ -G \between \beta\delta-\gamma^2,\ \beta\gamma-\alpha\delta,\ \alpha\gamma-\beta^2)^2$$
$$= (AF+BG+CH)\times \left\{\begin{array}{l} X\,(-\alpha^2\delta + 3\alpha\beta\gamma - 2\beta^3) \\ +\,Y\,(-\alpha\beta\delta + 2\alpha\gamma^2 - \beta^2\gamma) \\ +\,Z\,(\alpha\gamma\delta - 2\beta^2\delta + \beta\gamma^2) \\ +\,W\,(\alpha\delta^2 - 3\beta\gamma\delta + 2\gamma^3) \end{array}\right\}.$$

79. In the case above considered of the tenth species, $S(3^2)$, for which $AF+BG+CH$ not $=0$, the three forms of the reciprocal equation are of course absolutely equivalent to each other. The first form has the advantage of putting in evidence the fact that the reciprocal scroll is also of the tenth species; the other two forms do not, at least obviously, put in evidence any special property of the reciprocal scroll.

Reciprocals of Eighth Species, $S(1,\ 3^2)$, and Ninth Species, $S(1^3)$.

80. If $AF+BG+CH=0$, then the equation

$$(H,\ F,\ C,\ B,\ A-F,\ -G \between p,\ q,\ r)^2 = 0$$

is a scroll of the eighth species, $S(1,\ 3^2)$. The first form of the reciprocal equation becomes identically $0=0$, on account of the evanescent factor $AF+BG+CH$, but the second and third forms continue to subsist, and either of them may be taken as the equation of the reciprocal scroll. Taking the third form, and calling to mind the significations of (α, β, γ, δ), viz.

$$\begin{array}{lllllll} \alpha = (\ .\ , & H, & -G, & A \between X, & Y, & Z, & W), \\ \beta = (-H, & .\ , & F, & B \between & \text{,,} & &), \\ \gamma = (\ G, & -F, & .\ , & C \between & \text{,,} & &), \\ \delta = (-A, & -B, & -C, & .\ \between & \text{,,} & &), \end{array}$$

it is to be observed that $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ are the equations of four planes passing through a common line, viz. the line whose coordinates are (A, B, C, F, G, H), and the equation thus puts in evidence that this line is a triple line on the reciprocal

scroll; that is, the reciprocal scroll is a scroll of the ninth species, $S(1^3)$. Or stating the theorem more completely: For the scroll, *eighth species*, $S(1, 3^2)$,

$$(H, F, C, B, A-F, -G \between \mathrm{p}, \mathrm{q}, \mathrm{r})^2 = 0,$$

generated by a line meeting the line (F, G, H, A, B, C), and the skew cubic $\mathrm{p}=0$ $\mathrm{q}=0$, $\mathrm{r}=0$ twice, the reciprocal scroll is of the *ninth species*, $S(1^3)$,

$$\begin{aligned} & X(-\alpha^2\delta + 3\alpha\beta\gamma - 2\beta^3) \\ + & Y(-\alpha\beta\delta + 2\alpha\gamma^2 - \beta^2\gamma) \\ + & Z(\alpha\gamma\delta - 2\beta^2\delta + \beta\gamma^2) \\ + & W(\alpha\delta^2 - 3\beta\gamma\delta + 2\gamma^3), \end{aligned}$$

having for its triple line the reciprocal line (A, B, C, F, G, H).

81. It should of course be possible, starting from the equation

$$(*' \between X, Y)^4 + Z(* \between X, Y)^3 + W(*' \between X, Y)^3 = 0$$

of a scroll $S(1^3)$, to obtain the equation of the reciprocal scroll $S(1, 3^2)$. But I content myself with a very particular case. I consider the equation

$$Y^2Z^2 - Y^3W - Z^3X = 0,$$

which belongs to a scroll $S(1^3)$ having the line $Y=0$, $Z=0$ for its triple line. To find the equation of the reciprocal scroll, write

$$\begin{aligned} -Z^3 + \lambda x &= 0, \\ 2YZ^2 - 3Y^2W + \lambda y &= 0, \\ 2Y^2Z - 3Z^2X + \lambda z &= 0, \\ -Y^3 + \lambda w &= 0, \end{aligned}$$

we find without difficulty, reducing by means of the equation of the scroll,

$$\begin{aligned} \lambda^2(yw - z^2) &= -3Z^2\{Y^4 + 3XZ(XZ - Y^2)\}, \\ \lambda^2(xw - yz) &= 3Y^2Z^2\{YZ - 3WX\}, \\ \lambda^2(xz - y^2) &= -3Y^2\{Z^4 + 3YW(YW - Z^2)\}. \end{aligned}$$

Hence writing for a moment

$$\Omega = \{Y^4 + 3XZ(XZ - Y^2)\}\{Z^4 + 3YW(YW - Z^2)\} - Y^2Z^2(YZ - 3WX)^2,$$

we have

$$\begin{aligned} \Omega = \; & Y^4Z^4 + 3Y^5W(YW - Z^2) + 3Z^5X(XZ - Y^2) + 9XYZW(Y^2Z^2 - Y^3W - Z^3X + XYZW) \\ & - Y^4Z^4 + 6Y^3Z^3XW - 9Y^2Z^2X^2W^2, \end{aligned}$$

that is

$$\begin{aligned}
\tfrac{1}{3}\Omega = \quad & Y^2W(YW-Z^2)+X^3X(XZ-Y^2)+2Y^3Z^3XW,\\
&= -Y^2Z^2(XZ-Y^2)(YW-Z^2)-Y^3Z^2(YW-Z^2)(XZ-Y^2)+2Y^3Z^3XW,\\
&= -2Y^2Z^2\{(XZ-Y^2)(YW-Z^2)-XYZW\},\\
&= -2Y^2Z^2\{Y^2Z^2-Y^3W-Z^3X\},\\
&= 0, \text{ by the equation of the scroll;}
\end{aligned}$$

and we thus see that the equation of the reciprocal scroll is

$$(yw-z^2)(xz-y^2)-(yz-xw)^2=0,$$

or say $q^2-pr=0$, viz. it is a scroll $S(1, 3^2)$ generated by a line meeting the line $x=0$, $w=0$, and the cubic curve $p=0$, $q=0$, $r=0$ twice. The equation is obviously included in the general equation

$$(H, F, C, B, A-F, -G \between p, q, r)^2=0,$$

where $AF+BG+CH=0$; viz. writing $A=B=C=G=H=0$, this becomes $F(q^2-pr)=0$.

82. Returning to the general case of the scroll, *eighth species*, $S(1, 3^2)$, it is proper to show geometrically how it is that the reciprocal is a scroll, *ninth species*, $S(1^3)$. Consider in the scroll $S(1, 3^2)$ any plane through the directrix line; this contains three generating lines of the scroll, viz. these are the sides of the triangle formed by the three points of intersection of the plane with the skew cubic: hence in the reciprocal figure we have a directrix line such that at each point of it there are three generating lines; that is, we have a scroll $S(1^3)$ with a triple directrix line. Conversely, starting with the scroll $S(1^3)$, each plane through the triple directrix line meets the scroll in this line three times, and in a single generating line; whence there is in the reciprocal scroll a simple directrix line; but in order to show that it is a scroll $S(1, 3^2)$, we have yet to show that there is, as a nodal directrix, a skew cubic met twice by each generating line; this implies that, reciprocally, in the scroll $S(1^3)$ each generating line is the intersection of two osculating planes of a skew cubic (tangent planes of a quartic torse), each such plane containing two generating lines of the scroll—a geometrical property which is far from obvious; and similarly in the scroll, ninth species, $S(3^2)$, where the reciprocal scroll is of the same form, the property that each generating line is a line joining two points of a skew cubic leads to the property that each line is also the intersection of two osculating planes of a skew cubic (or, what is the same thing, two tangent planes of a quartic torse).

Addition, *May* 18, 1869.

Since the foregoing Memoir was written I received from Professor Cremona a letter dated Milan, November 20, 1868, in which (besides the ninth and tenth species considered above) he refers to two other species of quartic scrolls. He remarks that

there is a bitangent torse which should in the classification be considered along with the nodal curve; and he enumerates in all 12 species as follows:

Deficiency.	No. of species.	Nodal curve.	Bitangent torse.	Corresponding to my species.
$p=0$	1	Γ_3	Σ_3	10
	2	H_2+R_1	K_2+R_1	7
	3	R_1^3	K_2+R_1	— (say, 12)
	4	H_2+R_1	R_1^3	— (say, 11)
	5	$R_1+R_1'+S_1$	$R_1+R_1'+S_1$	2
	6	$R_1^2+S_1$	$R_1^2+S_1$	5
	7	Γ_3	R_1^3	8
	8	R_1^3	Σ_3	9
	9	R_1^3	$R_1'^3$	3
	10	R_1^3	R_1^3	6
$p=1$	11	R_1+R_1'	R_1+R_1'	1
	12	R_1^2	R_1^2	4

where Γ_3 denotes a skew cubic, Σ_3 a torse of the 3rd class (or quartic torse), H_2 a conic, K_2 a quadric cone, R_1, R_1', S_1 different right lines, R_1^2, R_1^3 a line counted twice or three times, &c. I have in the last column added the references to my species 9 and 10; Professor Cremona notices (what I knew, but did not recollect) that the species 10 had been considered by M. Chasles, *Comptes Rendus*, June 3, 1861.

I have not yet examined the two new species mentioned in this enumeration; viz. these are (Cremona 3), say *twelfth species*, a scroll having a triple line, but a bitangent torse made up of a quadric cone and a line; and (Cremona 4), say *eleventh species*, a scroll having a nodal conic and line, but for its bitangent torse a triple line: the two species are, it is clear, reciprocal to each other; although properly treated as distinct, species 11 may be considered as a subform of 8, and species 12 as a subform of 9.

411.

A MEMOIR ON THE THEORY OF RECIPROCAL SURFACES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLIX. (for the year 1869), pp. 201—229. Received November 12, 1868,—Read January 14, 1869.]

THE present Memoir contains some extensions of Dr Salmon's theory of Reciprocal Surfaces. I wish to put the formulæ on record, in order to be able to refer to them in a "Memoir on Cubic Surfaces," [412], but without at present attempting to completely develope the theory.

Article Nos. 1 to 5. *Extension of* SALMON'S *Fundamental Equations.*

1. The notation made use of is that of Salmon's *Geometry*, [2nd Ed.] pp. 450—459, [but reproduced in the later editions, see Ed. 4. (1882), pp. 580—592], with the additions presently referred to; the significations of all the symbols are explained by way of recapitulation at the end of the Memoir. I remark that my chief addition to Salmon's theory consists in a modification of his fundamental formulæ (A) and (B); these in their original form are

$$\begin{aligned}
a(n-2) &= \kappa + \rho + 2\sigma,\\
b(n-2) &= \rho + 2\beta + 3\gamma + 3t,\\
c(n-2) &= 2\sigma + 4\beta + \gamma,\\
a(n-2)(n-3) &= 2\delta + 3[ac] + 2[ab],\\
b(n-2)(n-3) &= 4k + [ab] + 3[bc],\\
c(n-2)(n-3) &= 6h + [ac] + 2[bc],
\end{aligned}$$

where

$$\begin{aligned}
[ab] &= ab - 2\rho,\\
[ac] &= ac - 3\sigma,\\
[bc] &= bc - 3\beta - 2\gamma - i.
\end{aligned}$$

2. I take account of conical and biplanar nodes, or, as I call them, cnicnodes, and binodes; of pinch-points[1] on the nodal curve; and of close-points and off-points on the cuspidal curve: viz. I assume that there are

C, cnicnodes,
B, binodes,
j, pinch-points,
χ, close-points,
θ, off-points,

deferring for the present the explanation of these singularities. The same letters, accented, refer to the reciprocal singularities. Or using "trope" as the reciprocal term to node, these will be

C', cnictropes,
B', bitropes,
j', pinch-planes,
χ', close-planes,
θ', off-planes;

but these present themselves, not in the equations above referred to, but in the reciprocal equations.

3. The resulting alterations are that we must in the formulæ write $\kappa - B$, $\delta - C$ in place of κ, δ respectively; and change the formulæ for $c(n-2)$, $[ab]$, $[bc]$, into

$$c(n-2) = 2\sigma + 4\beta + \gamma + \theta,$$
$$[ab] = ab - 2\rho - j,$$
$$[ac] = ac - 3\sigma - \chi,$$

respectively.

4. Making these changes, and substituting for $[ab]$, $[ac]$, $[bc]$ their values, the formulæ become

$$\begin{aligned}
a(n-2) &= \kappa - B + \rho + 2\sigma,\\
b(n-2) &= \rho + 2\beta + 3\gamma + 3t,\\
c(n-2) &= 2\sigma + 4\beta + \gamma + \theta,\\
a(n-2)(n-3) &= 2(\delta - C) + 3(ac - 3\sigma - \chi) + 2(ab - 2\rho - j),\\
b(n-2)(n-3) &= 4k + (ab - 2\rho - j) + 3(bc - 3\beta - 2\gamma - i),\\
c(n-2)(n-3) &= 6h + (ac - 3\sigma - \chi) + 2(bc - 3\beta - 2\gamma - i),
\end{aligned}$$

which replace the original formulæ (A) and (B).

[1] This addition to the theory is in fact indicated in Salmon, see the note, p. 445; the i there employed, which is of course different from the i of his text, is the j of the present Memoir.

5. For convenience I annex the remaining equations; viz. these are

$$a' = n(n-1) - 2b - 3c,$$
$$\kappa' = 3n(n-2) - 6b - 8c,$$
$$\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{8}{2}c(c-1);$$

the equations

$$q = b^2 - b - 2k - 3\gamma - 6t,$$
$$r = c^2 - c - 2h - 3\beta,$$

(q, r in place of Salmon's R, S respectively); the equation

$$a = a';$$

and the corresponding equations, interchanging the accented and unaccented letters, in all 23 equations between the 42 quantities

$$n,\ a,\ \delta,\ \kappa;\quad b,\ k,\ t,\ q,\ \rho,\ j;\quad c,\ h,\ r,\ \sigma,\ \theta,\ \chi;\quad \beta,\ \gamma,\ i;\quad B,\ C,$$
$$n',\ a',\ \delta',\ \kappa';\quad b',\ k',\ t',\ q',\ \rho',\ j';\quad c',\ h',\ r',\ \sigma',\ \theta',\ \chi';\quad \beta',\ \gamma',\ i';\quad B',\ C'.$$

Article Nos. 6 to 12. *Developments.*

6. We have

$$(a - b - c)(n-2) = (\kappa - B - \theta) - 6\beta - 4\gamma - 3t,$$
$$(a - 2b - 3c)(n-2)(n-3) = 2(\delta - C)$$
$$- 8k - 18h - 6(bc - 3\beta - 2\gamma - i);$$

and substituting these values of δ, κ in the formula

$$n' = a(a-1) - 2\delta - 3\kappa,$$

and for a its value, $= n(n-1) - 2b - 3c$, we find

$$n' = n(n-1)^2 - n(7b + 12c) + 4b^2 + 8b + 9c^2 + 15c$$
$$- 8k - 18h + 18\beta + 12\gamma + 12i - 9t$$
$$- 2C - 3B - 3\theta,$$

where the foregoing equations for $a - b - c$ and $a - 2b - 3c$ show clearly the origin of the new terms $-2C - 3B - 3\theta$; these express that there is in the value of n' a reduction $= 2$ for each cnicnode, $= 3$ for each binode, and $= 3$ for each off-point.

7. We have $(n-2)(n-3) = n^2 - n + (-4n+6) = a + 2b + 3c + (-4n+6)$; and making this substitution in the equations which contain $(n-2)(n-3)$, these become

$$a(-4n+6) = 2(\delta - C) - a^2 - 4\rho - 9\sigma - 2j - 3\chi,$$
$$b(-4n+6) = 4k - 2b^2 - 9\beta - 6\gamma - 3i + 2\rho - j,$$
$$c(-4n+6) = 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma - \chi,$$

(Salmon's equations (C)); and adding to each equation 4 times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2-2a &= 2(\delta-C)+4(\kappa-B)-\sigma-2j-3\chi, \\ 2b^2-2b &= 4k-\beta+6\gamma+12t-3i+2\rho-j, \\ 3c^2-2c &= 6h+10\beta+4\theta-2i+5\sigma-\chi. \end{aligned}$$

Writing in the first of these $a^2-2a=a(a-1)-a, =n'+2\delta+3\kappa-a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n'-a &= -2C-4B+\kappa-\sigma-2j-3\chi, \\ 2q+\beta+3i+j &= 2\rho, \\ 3r+c+2i+\chi &= 5\sigma+\beta+4\theta, \end{aligned}$$

(Salmon's equations (D)).

I attend in particular to the first of these, or rather to the reciprocal equation, which will be

$$\sigma' = a-n+\kappa'-2j'-3\chi'-2C'-4B',$$

which, writing therein $a=n(n-1)-2b-3c$, and $\kappa=3n(n-2)-6b-8c$, becomes

$$\sigma' = 4n(n-2)-8b-11c-2j'-3\chi'-2C'-4B'.$$

The singularity σ' is not *explicitly* defined in Salmon; σ' is the reciprocal of σ, and (as such) it denotes the number of common tangent planes of the spinode torse and of the torse generated by the tangent planes along a plane section of the surface; or, what is the same thing, it is the number of the spinode planes which touch the plane section; that is, it is equal to the number of points of intersection of the spinode curve and the plane section; or, finally, σ' is the order of the spinode curve. The spinode curve is in fact for a surface of the order n without singularities the intersection of the surface by the Hessian surface of the order $4(n-2)$, and is thus a curve of the order $4n(n-2)$, which agrees with the formula.

8. But the formula shows that there is in the order a reduction $8b+11c$ arising from the nodal and cuspidal curves of the surface, or, what is the same thing, that the Hessian surface meets the surface in the nodal curve taken 8 times, and in the cuspidal curve taken 11 times—a result which I had arrived at by other means, and also as appears *post*, No. 44. The formula shows further that there is a reduction $2j'+3\chi'+2C'+4B'$, or say there are reductions $=2, 3, 2, 4$, for the reciprocals of a pinch-point, a close-point, a cnicnode, and a binode respectively. Geometrically this must signify that the surface and its Hessian partially intersect in certain curves which are not regarded as belonging to the spinode curve. It will at once suggest itself that for the reciprocal of a cnicnode this curve is a conic, and for the reciprocal of a binode it is a line counting 4 times; while for the reciprocal of a pinch-point it is a line counting 2 times, and for the reciprocal of a close-point, a line counting 3 times.

9. It is clear that ρ' will in like manner denote the order of the node-couple curve.

10. I express in terms of

$$n,\ b,\ c,\ h,\ k,\ \beta,\ \gamma,\ j,\ \theta,\ \chi,\ C,\ B$$

such quantities and combinations of quantities as can be so expressed. We have

$$a = a' = n(n-1) - 2b - 3c,$$

$$\kappa' = 5n(n-2) - 6b - 8c,$$

$$\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{9}{2}c(c-1),$$

$$4i = 12h + c(5n-6) - 6c^2 - 5\gamma + 3\theta - 2\chi,$$

$$24t = (-8n+8)b + (15n-18)c + 8b^2 - 18c^2 - 2(8k-18h) + 20\beta - 15\gamma + 4j + 9\theta + 6\chi,$$

$$q = b^2 - b - 2k - 3\gamma - 6t, \quad (t \text{ suprà}),$$

$$r = c^2 - c - 2h - 3\beta,$$

$$2\sigma = c(n-2) - (4\beta+\gamma) - \theta,$$

$$8\rho = (16n-24)b + (-15n+18)c - 8b^2 + 18c^2 + 2(8k-18h) - 9(4\beta+\gamma) - 4j - 9\theta - 6\chi,$$

$$\begin{aligned} 8\kappa = {} & 8n(n-1)(n-2) + b(-32n+56) + c(-17n+46) + 8b^2 - 18c^2 \\ & - 2(8k-18h) + 17(4\beta+\gamma) + 4j + 17\theta + 6\chi + 8B, \end{aligned}$$

$$\begin{aligned} 2\delta = {} & n(n-1)(n-2)(n-3) + b(-4n^2+20n-24) + c(-6n^2+15n-18) + 12bc + 18c^2 \\ & + (8k-18h) - 9(4\beta+\gamma) - 9\theta + 2C, \end{aligned}$$

$$\begin{aligned} 8n' = {} & 8n(n-1)^2 + (-32n+40)b + (-21n+30)c + 8b^2 - 18c^2 \\ & - 2(8k-18h) + 21(4\beta+\gamma) - 12j + 21\theta - 18\chi - 16C - 24B, \end{aligned}$$

$$\begin{aligned} c' = {} & 4n(n-1)(n-2) + (-16n+28)b + (-10n+26)c + 4b^2 - 9c^2 \\ & - (8k-18h) + 10(4\beta+\gamma) - 4j + 10\theta - 6\chi - 6C - 8B, \end{aligned}$$

$$2b' = -a + n'(n'-1) - 3c', \quad (n',\ c' \text{ suprà}),$$

$$\sigma' + 2j' + 3\chi' + 2C' + 4B' = 4n(n-2) - 8b - 11c,$$

$$\rho' - 4j' - 6\chi' - 4C' - 9B' = -11n(n-2) + a(n'-2) + 22b + 30c, \quad (n',\ a \text{ suprà}),$$

$$2\sigma' + 4\beta' + \gamma' + \theta' = c'(n'-2), \quad (n',\ c' \text{ suprà}),$$

$$4k' - 3(i' + 3\beta' + 2\gamma') - 2\rho' - j' = (-4n'+6)b' + 2b'^2, \quad (n',\ b' \text{ suprà}),$$

$$6h - 2(i' + 3\beta' + 2\gamma') - 3\sigma' - \chi' = (-4n'+6)c' + 3c'^2, \quad (n',\ c' \text{ suprà});$$

{or in place of either of these,

$$8k' - 18h' - 4\rho' + 9\sigma' - 2j' + 3\chi' = (2b'-3c')\{(n'-2)(n'-3) - a\}, \quad (n',\ b',\ c',\ a \text{ suprà})\},$$

$$\rho' + 2\beta' + 3\gamma' + 3t' = b'(n'-2), \quad (n',\ b' \text{ suprà}),$$

$$2q' + \beta' + 3i' + j' - 2\rho' = 0,$$

$$3r' + 2i' + \chi' - 5\sigma' - \beta' - 4\theta' = c', \quad (c' \text{ suprà}),$$

(twenty-three equations, being a transformation of the original system of twenty-three equations).

11. Forming the combinations $4i+6r$, $24t-8q+18r$ (the last of which introduces on the opposite side the term $+48t$), we obtain

$$4i+6r = c(5n-12)-5\gamma-18\beta+3\theta-2\chi,$$
$$-24t-8q+18r = -(8n-16)b+(15n-36)c-34\beta+9\gamma+4j+9\theta+6\chi,$$

equations which are used *post*, No. 53.

12. I remark that if there be on a surface a right line which is such that the tangent plane is different at different points of the line, the line is said to be *scrolar:* the section of the surface by any plane through the line contains the line *once.* But if there is at each point of the line one and the same tangent plane, then the section of the surface by the tangent plane contains the line at least twice; if it contain it twice only, the line is *torsal*; if three times the line is *oscular*; and the tangent plane containing the torsal or oscular line may in like manner be termed a torsal or an oscular tangent plane. These epithets, scrolar, torsal and oscular, will be convenient in the sequel.

Article Nos. 13 to 39. *Explanation of the New Singularities.*

I proceed to the explanation of the new singularities.

13. The cnicnode, or singularity $C=1$, is an ordinary conical point; instead of the tangent plane we have a proper quadricone.

14. The cnictrope, or reciprocal singularity $C'=1$, is also a well known one; it is in fact the conic of plane contact, or say rather the plane of conic contact, viz. the cnictrope is a plane touching a surface, not at a single point, but along a conic.

15. Consider a surface having the cnicnode $C=1$, and the reciprocal surface having the cnictrope $C'=1$. There are on the quadricone of the cnicnode six directions of closest contact(1), and reciprocal thereto we have six tangents of the cnictrope conic, touching it at six points. The plane of the cnictrope meets the surface in the conic twice, and in a residual curve which touches the conic at each of the six points. It would appear that these six contacts are part of the notion of the cnictrope.

16. We may of course have a surface with a conic of plane contact, but such that the residual curve of intersection in the plane of the conic does not touch the conic six times or at all; for instance the general equation of a surface with a conic of plane contact is $PM+V^2N=0$, where $P=0$ is a plane, $V=0$ a quadric surface; and here the conic $P=0$, $V=0$ does not touch the residual curve $P=0$, $N=0$. The reciprocal surface will in this case have a cnicnode, but there is some special circumstance doing away with the six directions of closest contact which in general belong thereto. I do not further pursue this inquiry.

[1] Taking for greater simplicity coordinates x, y, z, 1, then for a surface having a cnicnode at the origin, the equation is $U_2+U_3+\&c.=0$, the suffixes showing the degree in the coordinates; the equation of the quadricone is $U_2=0$, and the six directions are given as the lines of intersection of the two cones $U_2=0$, $U_3=0$.

17. For a surface having the cnictrope $C'=1$, the Hessian surface passes through the conic, which is thus thrown off from the spinode curve; or there is a reduction $=2$ in the order of the curve, which agrees with a foregoing result.

18. The binode, or singularity $B=1$, is a biplanar node, where instead of the proper quadricone we have two planes; these may be called the biplanes, and their line of intersection, the edge of the binode. The biplanes form a plane-pair.

19. The bitrope, or reciprocal singularity $B'=1$, is the plane of point-pair contact; but this needs explanation.

20. Consider a surface having a binode, and the reciprocal surface having a bitrope. We have the bitrope, a plane the reciprocal of the binode; in this plane a line, the reciprocal of the edge; in the line two points, or say a point-pair, the reciprocal of the biplanes: these points may be called the bipoints. There are in each biplane three directions of closest contact; the reciprocals of these are in the bitrope three directions through each of the two points. The section of the reciprocal surface by the bitrope is made up of the line counting three times (or the line is oscular), and of a curve passing in the three directions (having therefore a triple point) through each of the two bipoints. The bitrope contains thus an oscular line; but it is part of the notion that there are on this line two points each a triple point on the residual curve of intersection.

21. We may however have on a surface an oscular line without upon it two or any triple points of the residual curve of intersection. Such a surface is $Mx+Ny^2=0$; the intersections of the line $x=0$, $y=0$ with the curve $x=0$, $N=0$ will be all of them ordinary points. The reciprocal surface will have a binode, but there will be some special circumstance doing away with the existence of the directions of closest contact in the two biplanes respectively. I do not at present pursue the question.

22. For a surface having a bitrope $B'=1$, it appears from what precedes, that the oscular line must count 4 times in the intersection of the surface with the Hessian; for only in this way can the reduction 4 in the order of the spinode curve arise.

23. The pinch-point, or singularity $j=1$, is in fact mentioned in Salmon; it is a point on the nodal curve such that the two tangent planes coincide, or say it is a cuspidal point on the nodal curve. If, to fix the ideas, we take the nodal curve to be a complete intersection $P=0$, $Q=0$, then the equation of the surface is $(A, B, C \between P, Q)^2=0$ (A, B, C functions of the coordinates); we have a surface $AC-B^2=0$, which may be called the critic surface, intersecting the nodal curve in the points $P=0$, $Q=0$, $AC-B^2=0$, which are the pinch-points thereof; or if there be a cuspidal curve, then such of these points as are not situate on the cuspidal curve are the pinch-points: see my paper "On a Singularity of Surfaces," *Quart. Math. Journ.* vol. IX. (1868) pp. 332—338, [402]. The single tangent plane at the pinch-point meets the surface (see p. 338) in a curve having at the pinch-point a triple point, = cusp + 2 nodes, viz. there is a cuspidal branch the tangent to which coincides with that of the nodal curve; and there is a simple branch the tangent to which may be called the cotangent

at the pinch-point. In the particular case where the nodal curve is a right line the section is the line twice (representing the cuspidal branch), and a residual curve of the order $n-2$, the tangent to which is the cotangent.

24. The pinch-plane, or reciprocal singularity $j'=1$, is in fact a torsal plane touching the surface along a line, or meeting it in the line twice and in a residual curve. Let the line and curve meet in a point P; for the reason that the section by the plane is the line twice and the residual curve, the section has at P two coincident nodes; that is, the plane is a node-couple plane with two coincident nodes. The plane meets the consecutive node-couple plane in a line μ passing through P and touching at this point the residual curve. Considering now the reciprocal figure, the reciprocal of the pinch-plane is thus a point of the nodal curve, and is a pinch-point; the tangent plane at the pinch-point is the reciprocal of the point P; the tangent to the nodal curve is the reciprocal of the line μ, that is, of the tangent at P to the residual curve; and the cotangent at the pinch-point is the reciprocal of the torsal line.

25. There is in this theory the difficulty that for a surface of the order n, the torsal plane meets the residual curve of intersection in $(n-2)$ points P, and if each of these be a point on the node-couple curve, then in the reciprocal figure the pinch-point would be a multiple point on the nodal curve. I apprehend that starting with a pinch-point, a simple point on the nodal curve, we have in the reciprocal figure a pinch-plane or torsal plane as above, but with some speciality in virtue of which only one of the $(n-2)$ points of intersection of the torsal line with the residual plane curve is a point of the node-couple curve of the reciprocal surface. In the case of a pinch-plane or torsal plane of a cubic surface, $n-2$ is $=1$, and the question of multiplicity does not arise.

26. For a surface with a pinch-plane or torsal plane as above ($j'=1$), the Hessian surface not only passes through the torsal line, but it touches the surface along this line, causing, as already mentioned, a reduction $=2$ in the order of the spinode curve. That the surfaces *touch* along the line is an important theorem[1], and I annex a proof.

27. Let $x=0$, $y=0$ be the torsal line, $x=0$ being the torsal plane; the equation of the surface therefore is $x\phi+y^2\psi=0$; and if A, B, C, D be the first derived functions of ϕ, $(a, b, c, d, f, g, h, l, m, n)$ the second derived functions, and if (A', B', C', D'), $(a', b', c', d', f', g', h', l', m', n')$ refer in like manner to ψ, then the equation of the Hessian is

$$0=\begin{vmatrix} 2A+xa+y^2a', & B+xh+2yA'+y^2h', & C+xg+y^2g', & D+xl+y^2l' \\ B+xh+2yA'+y^2h', & xb+2\psi+4yB'+y^2b', & xf+2yC'+y^2f', & xm+2yD'+y^2m' \\ C+xg+y^2g', & xf+2yC'+y^2f', & xc+y^2c', & xn+y^2n' \\ D+xl+y^2l', & xm+y^2n'+2yD', & xn+y^2n', & xd+y^2d'; \end{vmatrix}$$

[1] See Salmon, p. 218, where it is only stated that the Hessian passes through the line.

and representing this for a moment by

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

then in the developed equation

$$D(ABC - AF^2 - BG^2 - CH^2 + 2FGH)$$
$$-(BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH \between L,\ M,\ N)^2 = 0,$$

observing that C, F, M, N, D are of the first order in x, y, the only terms of the first order are contained in $B(-DG^2 - CL^2 + 2NGL)$; and since C, D, N are of the first order, we obtain all the terms of the first order by reducing B, G, L to the values 2ψ, C, D; viz. the terms of the first order are

$$2\psi(-C^2dx - D^2cx + 2CDnx), \quad = -2\psi(C^2d + D^2c - 2CDn)x.$$

Hence the complete equation is of the form

$$-2\psi(C^2d + D^2c - 2CDn)x + (x,\ y)^2 = 0,$$

or, what is the same thing, $x\Phi + y^2\Psi = 0$; the Hessian has therefore along the line $x = 0$, $y = 0$ the same tangent plane $x = 0$ as the surface; or it *touches* the surface along this line; that is, the line counts twice in the intersection of the two surfaces.

28. If instead of the right line we have a plane curve, say if the equation be $x\phi + P^2\psi = 0$, then the value of the Hessian is $x\Phi + P\Psi = 0$ (viz. the second term divides by P only, not by P^2), so that, as before mentioned in regard to a conic of contact, the surface and the Hessian merely cut (but do not touch) along the curve $x = 0$, $P = 0$. To show this in the most simple manner take the equation to be $x\phi + \frac{1}{2}P^2 = 0$; let A', B', C', D' be the first derived functions of ϕ, and (A, B, C, D), $(a, b, c, d, f, g, h, l, m, n)$ the first and second derived functions of P; then if in the equation of the Hessian we write for greater simplicity $x = 0$, the equation is

$$\begin{vmatrix} 2A' + Pa + A^2, & B' + Ph + AB, & C' + Pg + AC, & D' + Pl + AD \\ B' + Ph + AB, & Pb + B^2, & Pf + BC, & Pm + BD \\ C' + Pg + AC, & Pf + BC, & Pc + C^2, & Pn + CD \\ D' + Pl + AD, & Pm + BD, & Pn + CD, & Pd + D^2 \end{vmatrix} = 0.$$

The equation contains for example the term

$$-(D' + Pl + AD)^2\{P^2(bc - f^2) + P(bC^2 + cB^2 - 2fBC)\},$$

dividing as it should do by P, but not dividing by P^2; and considering the portion hereof $-D'^2P(bC^2 + cB^2 - 2fBC)$, there are no other terms in D'^2P which can destroy this, and to make the whole equation divide by P^2; which proves the required negative.

29. For the off-point or singularity $\theta = 1$; this is a point on the cuspidal curve at which the second derived functions all of them vanish. In further explanation hereof consider a surface $U = 0$, and the second polar of an arbitrary point $(\alpha, \beta, \gamma, \delta)$; viz. this is $(\alpha\partial_x + \beta\partial_y + \gamma\partial_z + \delta\partial_w)^2 U = 0$, or say for shortness $\Delta^2 U = 0$, where the coefficients of the powers and products of $(\alpha, \beta, \gamma, \delta)$ are of course the second derived functions of U; this equation, when reduced by means of the equations of the cuspidal curve, may acquire a factor Λ, thus assuming the form $\Lambda(\alpha P + \beta Q + \gamma R + \delta S)^2 = 0$, and if so the intersections of the cuspidal curve with the second polar $(= 2\sigma + \theta$, if, as for simplicity is supposed, there is no nodal curve) will be made up of the intersections of the cuspidal curve with the surface $\Lambda = 0$, and of those with the surface $\alpha P + \beta Q + \gamma R + \delta S = 0$ each twice; the latter of these, depending on the coordinates $(\alpha, \beta, \gamma, \delta)$ of the arbitrary points, are the points σ each twice; the former of them, or intersections of the cuspidal curve with the surface $\Lambda = 0$, are the points θ, or off-points of the cuspidal curve. If there is a nodal curve, the only difference is that the off-points are such of the above points as do not lie on the nodal curve.

30. As the most simple instance of the manner in which this singularity may present itself, consider a surface $FP^2 + GQ^3 = 0$, where the degrees of the functions are f, p, g, q, and therefore $n = f + 2p = g + 3q$, if n be the order of the surface. This has a cuspidal curve $P = 0$, $Q = 0$ of the order pq; the equation $\Delta^2(FP^2 + GQ^3) = 0$ of the second polar, when reduced by the equations $P = 0$, $Q = 0$ of the cuspidal curve, becomes simply $F(\Delta P)^2 = 0$; and we have thus the off-points $F = 0$, $P = 0$, $Q = 0$, consequently $\theta = fpq$.

31. But suppose, as before, the case of a surface $(A, B, C \between P, Q)^2 = 0$ having a cuspidal curve $P = 0$, $Q = 0$, and therefore $AC - B^2$ being $= 0$ for $P = 0$, $Q = 0$. The equation of the second polar, writing therein $P = 0$, $Q = 0$, becomes $(A, B, C \between \Delta P, \Delta Q)^2 = 0$, and if for any given surface this assumes the form $\Lambda(M\Delta P + N\Delta Q)^2 = 0$ (observe that M, N may be fractional provided only the $M\Delta P + N\Delta Q$ is integral), then there will be on the cuspidal curve the off-points $\Lambda = 0$, $P = 0$, $Q = 0$.

32. An interesting example is afforded by a surface which presents itself in the Memoir on Cubic Surfaces: the surface

$$\begin{aligned} & 4y^6 \\ & - 4y^3x(x^2 + 3zw) \\ & + \ zw(3x^2 + zw)^2 = 0 \end{aligned}$$

has the cuspidal conic $y = 0$, $3x^2 + zw = 0$, and (as coming under the form $FP^2 + GQ^3 = 0$) has the off-points $zw = 0$, $y = 0$, $3x^2 + zw = 0$; that is, the points $(x = 0, y = 0, z = 0)$, $(x = 0, y = 0, w = 0)$ each twice; $\theta = 4$.

But writing the same equation in the form

$$(4, 6x, 8x^2 + zw \between y^3 - 2x^3, x^2 - zw)^2 = 0,$$

where

$$4 \,.\, (8x^2 + zw) - (6x)^2 = -4(x^2 - zw),$$

it appears that there are also the three cuspidal conics $y^3 - 2x^3 = 0$, $x^2 - zw = 0$. Reducing by means of these two equations, the equation of the second polar is at first obtained in the form

$$(4,\ 6x,\ 8x^2 + zw \between 3y^2\Delta y - 6x^2\Delta x,\ 2x\Delta x - z\Delta\omega - w\Delta z)^2 = 0;$$

but further reducing by the same equations and writing for this purpose $y = \omega x\ (\omega^3 = 2)$, the equation becomes

$$(4,\ 6x,\ 9x^2 \between x^2\,(3\omega^2\Delta y - 6\Delta x),\ 2x\Delta x - z\Delta w - w\Delta z^2)^2 = 0,$$

that is

$$x^2\,[2x\,(3\omega^2\Delta y - 6\Delta x) + 3\,(2x\Delta x - z\Delta w - w\Delta z)]^2 = 0,$$

and we have thus the off-points $x^2 = 0$, $y^3 - 2x^3 = 0$, $x^2 - zw = 0$, in fact the before-mentioned two points each 6 times; and the complete value of θ is $\theta = (4 + 12 =)\,16$; viz. the off-points are the points $(x = 0,\ y = 0,\ z = 0)$, $(x = 0,\ y = 0,\ w = 0)$ each 8 times. On account of this union of points the singularity is really one of a higher order, but equivalent to $\theta = 16$.

I am not at present able to explain the off-plane or reciprocal singularity $\theta' = 1$.

33. As to the close-point or singularity $\chi = 1$. I remark that at an ordinary point of the cuspidal curve the section by the tangent plane *touches*, at the point of contact, the cuspidal curve: the point of contact is on the curve of section a singular point {in the nature of a triple point, viz. taking the point of contact as origin, the form of the branch in the vicinity thereof is $y^3 - x^4 = 0$, where $y = 0$ is the equation of the tangent to the cuspidal curve}, such that the point of contact counts 4 times in the intersection of the cuspidal curve with the curve of section. At a close-point the form of the curve of section is altered; viz. the point of contact is here in the nature of a quadruple point with two distinct branches, one of them a triple branch of the form $y^3 = x^4$, but such that the tangent thereof, $y = 0$, is not the tangent of the cuspidal curve; the other of them a simple branch, the tangent of which is also distinct from the tangent of the cuspidal branch: the point of contact counts $3 + 1$ times, that is 4 times, as before, in the intersection of the cuspidal curve and the curve of section. The tangent to the simple branch may conveniently be termed the cotangent at the close-point; that of the other branch the cotriple tangent.

34. We may look at the question differently thus: to fix the ideas, let the cuspidal curve be a complete intersection $P = 0$, $Q = 0$; the equation of the surface is $(A,\ B,\ C \between P,\ Q)^2 = 0$, where $AC - B^2 = 0$, in virtue of the equations $P = 0$, $Q = 0$ of the cuspidal curve, that is, $AC - B^2$ is $= MP + NQ$ suppose. We have (as in the investigation regarding the pinch-point) a critic surface $AC - B^2 = 0$, this meets the surface in the cuspidal curve and in a residual curve of intersection; the residual curve by its intersection with the cuspidal curve determines the close-points; the tangent at the close-point is I believe the tangent of the residual curve. Analytically the close-points are given by the equations $P = 0$, $Q = 0$, $(A,\ B,\ C \between N,\ -M)^2 = 0$. It is proper to remark that if besides the cuspidal curve there be a nodal curve, only such of the points so determined as do not lie on the nodal curve are the close-points.

35. I take as an example a surface which is substantially the same as one which presents itself in the Memoir on Cubic Surfaces, viz. the surface $(1, w, xy \between w^2 - xy, z)^2 = 0$, having the cuspidal conic $w^2 - xy = 0$, $z = 0$. Since in the present case $AC - B^2 = P$, we have $M = 1$, $N = 0$, and the close-points are given by $P = 0$, $Q = 0$, $C = 0$; that is, they are the points $(z = 0, w = 0, x = 0)$ and $(z = 0, w = 0, y = 0)$.

36. I first however consider an ordinary point on the cuspidal curve, or conic $w^2 - xy = 0$, $z = 0$; the coordinates of any point on the conic are given by $x : y : z : w = 1 : \theta^2 : 0 : \theta$, where θ is an arbitrary parameter; we at once find $\theta^2 x + y - \theta(z + 2w) = 0$ for the equation of the tangent plane of the surface or cuspidal tangent plane at the point $(1, \theta^2, 0, \theta)$. Proceeding to find the intersection of this plane with the surface, the elimination of z gives

$$(\theta^2, \theta w, xy \between w^2 - xy, \theta^2 x + y, -2\theta w)^2 = 0,$$

which is of course the cone, vertex $(x = 0, y = 0, w = 0)$, which passes through the required curve of intersection. In place of the coordinates x, y take the new coordinates $\theta^2 x - y = 2p$, and $\theta^2 x + y - 2\theta w = 2q$; we have

$$\begin{aligned} \theta^2 x &= \theta w + p - q, \\ -y &= -\theta w + p - q, \end{aligned}$$

and thence

$$\begin{aligned} -\theta^2 xy &= p^2 - (q + \theta w)^2 = p^2 - q^2 - 2\theta qw - \theta^2 w^2, \\ \theta^2 (w^2 - xy) & = p^2 - q^2 - 2\theta qw, \end{aligned}$$

and the equation thus is

$$(\theta^4, \theta^3 w, -p^2 + q^2 + 2\theta qw + \theta^2 w^2 \between p^2 - q^2 - 2\theta qw, 2\theta^2 q) = 0,$$

or, what is the same thing,

$$(1, \theta w, -p^2 + q^2 + 2\theta qw + \theta^2 w^2 \between p^2 - q^2 - 2\theta qw, 2q)^2 = 0;$$

viz. this is

$$(p^2 - q^2 - 2\theta qw)^2 + 4\theta qw\,(p^2 - q^2 - 2\theta qw) + 4q^2(-p^2 + q^2 + 2\theta qw + \theta^2 w^2) = 0;$$

or reducing, it is

$$(p^2 - q^2)(p^2 - 5q^2) + 8\theta q^3 w = 0,$$

the equation of the section in terms of the coordinates p, q, w. The equation is satisfied by the values $p = 0$, $q = 0$ which belong to the assumed point $(1, \theta^2, 0, \theta)$ of the conic, and in the vicinity of this point we have $p^4 + 8\theta q^3 w = 0$, which is a triple branch of the form $y^3 = x^4$, the tangent $q = 0$ being, it will be observed, the tangent of the conic. But at the close-points, or when $\theta = 0$ or $\theta = \infty$, the transformation fails; and these points must be considered separately.

37. At the first of these, viz. the point $z = 0$, $w = 0$, $x = 0$, the tangent plane of the surface or cuspidal tangent plane is $x = 0$, and this meets the surface in the curve $x = 0$,

$w^3(w+2z)=0$, that is in the line $x=0$, $w=0$ three times, and in the line $x=0$, $w+2z=0$ (that the section consists of right lines is of course a speciality, and it is clear that considering in a more general surface the section as defined by an equation in (w, z, y), the line $w=0$ represents the tangent to a triple branch $w^3=z^4+$ &c., and the line $w+2z=0$ the tangent to a simple branch); these lines are each of them, it will be observed, distinct from the tangent to the cuspidal conic, which is $x=0$, $z=0$. And similarly the tangent plane at the other of the two points is $y=0$, meeting the surface in the curve $y=0$, $w^3(w+2z)=0$, that is in the line $y=0$, $w=0$ three times, and in the line $y=0$, $w+2z=0$.

38. The close-plane or reciprocal singularity $\chi'=1$ is (like the pinch-plane) a torsal plane, meeting the surface in a line twice and in a residual curve; the distinction is that the line and curve have an intersection P lying on the spinode curve; the close-plane is thus a spinode plane; it meets the consecutive spinode plane in a line μ passing through P, and which is *not* the tangent of the residual curve. In the reciprocal figure, the reciprocal of the close-plane is on the cuspidal curve, and is a close-point; the reciprocal of the point P is the cuspidal tangent plane; that of the line μ the tangent of the cuspidal curve; that of the tangent of the residual curve the cotriple tangent; that of the torsal line the cotangent.

39. The torsal line of a close-plane is not a mere torsal line; in fact by what precedes it appears that the surface and the Hessian intersect in this line, counting not twice but three times, and it is thus that the reduction in the order of the spinode curve caused by the close-plane is $=3$.

Article Nos. 40 and 41. *Application to a Class of Surfaces.*

40. Consider the surface $FP^2+GR^2Q^3=0$, where f, p, g, r, q being the degrees of the several functions, and n the order of the surface, we have of course $n=f+2p=g+2r+3q$.

There is here a nodal curve, the complete intersection of the two surfaces $P=0, R=0$; hence $b=pr$, $k=\frac{1}{2}pr(p-1)(r-1)$, $=\frac{1}{2}b(b-p-r+1)$; $t=0$; whence $(q)=pr(p+r-2)$. There is also a cuspidal curve the complete intersection of the two surfaces $P=0$, $Q=0$; hence $c=pq$, $h=\frac{1}{2}pq(p-1)(q-1)$, $=\frac{1}{2}c(c-p-q+1)$; whence $(r)=pq(p+q-2)$: I have written for distinction (q), (r), to denote the q, r of the fundamental equations. The two curves intersect in the pqr points $P=0$, $Q=0$, $R=0$, which are not stationary points on either curve; that is, $\beta=0$, $\gamma=0$, $i=pqr$.

There are on the nodal curve the $j=(f+g)pr$ pinch-points $F=0$, $P=0$, $R=0$, and $G=0$, $P=0$, $R=0$. There are on the cuspidal curve $\theta=fpq$ off-points $F=0$, $P=0$, $Q=0$; and there the gpq singular points $G=0$, $P=0$, $Q=0$. I find that these last, and also the θ points each three times, must be considered as close-points, that is, that we have $\chi=(g+3f)pq$.

41. We ought then to have

$$\begin{aligned} b(n-2) &= \rho, \\ c(n-2) &= 2\sigma + \theta; \\ 2(q) + 3i + j &= 2\rho, \\ 3(r) + c + 2i + \chi &= 5\sigma + 4\theta; \end{aligned}$$

the first two of which give ρ, σ, and then, substituting their values, the other two equations should become identities. In fact, attending to the values $pr = b$, $pq = c$, the equations become

$$\begin{aligned} 2b(p+r-2) + 3bq + b(f+g) &= 2b(n-2), \\ 3c(p+q-2) + c + 2cr + b(g+3f) &= \tfrac{5}{2}\{c(n-2) - cf\} + 4cf. \end{aligned}$$

The first of these is

$$2n = 2p + 2r + 3q + f + g, \quad = (2p+f) + (2r+3q+g),$$

and the second is

$$\tfrac{5}{2}n = 3p + 3q + 2r + g + \tfrac{3}{2}f, \quad = \tfrac{3}{2}(2p+f) + (2r+3q+g),$$

so that the equations are satisfied.

Article No. 42. *The Flecnodal Curve.*

42. A point on a surface may be flecnodal, viz. the tangent plane may meet the surface in a curve having at the point a flecnode, that is, a node with an inflexion on one of the branches. Salmon has shown that, for a surface of the order n without singularities, the locus of the flecnodal points, or flecnodal curve, is the complete intersection of the surface by a surface of the order $11n - 24$, which may be called the flecnodal surface, the order of the curve being thus $= n(11n - 24)$. I have succeeded in showing, in a somewhat peculiar way by consideration of a surface of revolution, that if the surface of the order n has a nodal curve of the order b, and a cuspidal curve of the order c, then that the order of the flecnodal curve is $= n(11n - 24) - 22b - 27c$; before giving this investigation, I will by the like principles demonstrate the above-mentioned theorem that the order of the spinode curve is $= 4n(n-2) - 8b - 11c$.

Article Nos. 43 to 47. *Surfaces of Revolution, in connexion with the Spinode Curve and the Flecnodal Curve.*

43. Consider a plane curve of the order m with δ nodes and κ cusps, and let this be made to revolve about an axis in its own plane, so as to generate a surface of revolution. The complete meridian section is made up of the given curve and of an equal curve situate symmetrically therewith on the other side of the axis; the

order of the surface is thus $=2m$. The two curves intersect in m points on the axis and in m^2-m points, forming $\frac{1}{2}(m^2-m)$ pairs of points, situate symmetrically on opposite sides of the axes; these last generate $\frac{1}{2}(m^2-m)$ circles, nodal curves on the surface; the nodes generate δ circles, which are nodal curves on the surface, and the cusps generate κ circles, cuspidal curves on the surface. There are $m^2-m-2\delta-3\kappa$ circles of plane contact corresponding in the plane curve to the tangents perpendicular to the axis. Each of the m points on the axis gives in the surface a pair of (imaginary) lines; and we have thus two sets each of m lines, such that along the lines of each set the surface is touched by an (imaginary) meridian plane; viz. these are the circular planes $x+iy=0$, $x-iy=0$ passing through the axis. I assume without stopping to show it that these $2m$ lines are lines not j' but χ', that is, that they each reduce the order of the spinode curve by 3 (1). The inflexions generate $3m^2-6m-6\delta-8\kappa$ circles which constitute the spinode curve on the surface.

44. And we can thus verify that the complete intersection of the surface with the Hessian is made up in accordance with the foregoing theory; viz.

Order of surface $=2m$,		
Order of Hessian $=4(2m-2)$,		
whence order of intersection		$=16m^2-16m$
Nodal curve, $\frac{1}{2}(m^2-m)+\delta$ circles,	8 times	$8m^2-8m+16\delta$
Cuspidal curve, κ circles,	11 times	$+22\kappa$
Circles of contact $m^2-m-2\delta-3\kappa$,		$2m^2-2m-4\delta-6\kappa$
Lines $2m$,	3 times	$+6m$
Spinode curve, $3m^2-6m-6\delta-8\kappa$ circles,		$6m^2-12m-12\delta-16\kappa$
		$16m^2-16m.$

45. We may by a similar reasoning show that the surface and the flecnode surface intersect in the nodal curve taken 22 times, and in the cuspidal curve taken 27 times; and consequently that the order of the residual intersection or flecnodal curve is

$$=n(11n-24)-22b-27c.$$

To effect this, observe that at any point whatever of a quadric surface the tangent plane meets the surface in a pair of lines, that is, in a curve having at the point of contact a node with an inflexion on each branch, or say, a fleflecnode. Imagine in the plane figure a conic having its centre on the axis of rotation and its axis coincident therewith, and the conic having with the curve of the order m a 4-pointic intersection at any point P; the point P generates a circle, such that along this circle the surface is osculated by a quadric surface of revolution in such wise that the meridian sections have a four-pointic contact; the circle in question is thus on the surface a fleflecnode circle; and I assume that it counts twice as a flecnode circle. Hence if the number of the points P be $=\theta$, we have on the surface θ fleflecnode circles, $=2\theta$ flecnode circles, that is, a flecnode curve of the order 4θ. I wish to show that we have $\theta=5m^2-9m-10\delta-12\kappa$.

[1] Observe that the terms in m cannot be got rid of in a different manner, by any alteration of the numbers 8 and 11 to which the present investigation relates.

46. The problem is as follows: given a curve of the order m with δ nodes and κ cusps; it is required to find the number of the conics, centre on a given line, and an axis coincident in direction with this line, which have with the given curve a 4-pointic intersection, or contact of the third order. This may be solved by means of formulæ contained in my "Memoir on the Curves which satisfy given Conditions," *Phil. Trans.* vol. CLVIII. (1868), pp. 75—144; see p. 88; [406].

Taking $x=0$ for the given line, the conic $(a, b, c, f, g, h \between x, y, 1)^2=0$ will have its centre on the given line and an axis coincident therewith, if only $h=0$, $g=0$; and denoting these two conditions by $2X$, it is easy to see that we have

$$(2X\,\therefore)=1,\quad (2X:/)=2,\quad (2X\cdot//)=2,\quad (2X\,///)=1.$$

But in general if the conic satisfy any other three conditions $3Z$, then the number of the conics $(2X, 3Z)$ is

$$\begin{aligned} = \quad & \alpha'\,(\qquad\qquad -\tfrac{1}{4}\gamma+\tfrac{1}{2}\delta) \\ & +\beta'\,(-\tfrac{3}{8}\alpha+\tfrac{5}{16}\beta+\tfrac{5}{16}\gamma-\tfrac{3}{8}\delta) \\ & +\gamma'\,(\ \tfrac{1}{2}\alpha-\tfrac{1}{4}\beta\qquad\qquad), \end{aligned}$$

where α, β, γ, δ denote $(2X\,\therefore)$, $(2X:/)$, $(2X\cdot//)$, $(2X\,///)$, viz. in the present case the values are 1, 2, 2, 1 respectively, and where α', β', γ' denote $(3Z:)$, $(3Z\cdot/)$, $(3Z//)$ respectively.

47. Substituting for α, β, γ, δ their values, the number of the conics in question is $=\frac{1}{2}\beta'$, that is $=\frac{1}{2}(3Z\cdot/)$. Suppose that $3Z$, or say 3, denotes the condition of a contact of the third order with a given curve (m, δ, κ), or say with a given curve (m, n, α) (m the order, n the class $=m^2-m-2\delta-3\kappa$, $\alpha=3n+\kappa$), then we have

$$\begin{aligned} (3:\ \ ) &= -4m-3n+3\alpha, \\ (3\cdot/) &= -8m-8n+6\alpha, \\ (3\,//\,) &= -3m-4n+3\alpha; \end{aligned}$$

and from the second of these the number of the conics in question is $=-4m-4n+3\alpha$, that is, it is $=-4m+5n+3\kappa$, or finally it is $=5m^2-9m-10\delta-12\kappa$.

Hence, assuming that the $2m$ lines each counts 6 times[1],

Order of surface $=2m$	
Order of flecnode surface $=11\,(2m-24)$ or $22m-24$	
Order of intersection	$=44m^2-48m$
Nodal curve, $\frac{1}{2}(m^2-m)+\delta$ circles, 11 times	$22m^2-22m+44\delta$
Cuspidal curve κ circles, 27 times	$+54\kappa$
Circles of contact $m^2-m-2\delta-3\kappa$,	$2m^2-\ 2m-\ 4\delta-\ 6\kappa$
Lines of contact $2m$, 6 times	$+12m$
Flecnodal curve, $5m^2-9m-10\delta-12\kappa$ circles each twice	$20m^2-36m-40\delta-48\kappa$
	$44m^2-48m.$

[1] See foot-note p. 343: the like remark applies to the present terms in m, which cannot be got rid of by an alteration of the numbers 22 and 27 to which the investigation relates.

Article Nos. 48 and 49. *The Flecnodal Torse.*

48. Starting from

$$\begin{aligned} 22b' + 27c' &= 6(6b' + 8c') - 7(2b' + 3c') \\ &= 6(3n'^2 - 6n' - \kappa) - 7(n'^2 - n' - \delta) \\ &= 11n'^2 - 29n' + 7 - 6\kappa, \end{aligned}$$

that is

$$11n'^2 - 24n' - 22b' - 27c' = 5n' - 7\delta + 6\kappa,$$

I find

$$\begin{aligned} & n'(11n' - 24) - 22b' - 27c' \\ &= n(n-1)(11n-24) + b(-59n + 96) + c(-94n + 156) + 26b^2 + 87c^2 \\ &\quad - 52k - 114h + 141\beta + 94\gamma + 77i + 3j + 4\chi - 15\theta - 45t - 10C - 9B. \end{aligned}$$

49. For a surface of the order n without singularities this equation is

$$n'(11n' - 24) - 22b' - 27c' = n(n-1)(11n-24);$$

to explain the meaning of it, I say that the reciprocal of a flecnode is a flecnodal plane, and *vice versâ:* the reciprocal of the flecnodal torse of the surface n (viz. the torse generated by the flecnodal planes of the surface) is thus the flecnodal curve of the reciprocal surface n'; and the class of the torse must therefore be equal to the order of the curve. The flecnodal torse is generated by the tangents of the surface n along the curve of intersection with a surface of the order $11n - 24$; the number of tangent planes which pass through an arbitrary point, or class of the torse, is at once found to be $n(n-1)(11n-24)$; for the reciprocal surface the order of the flecnodal curve is by what precedes $n'(11n' - 24) - 22b' - 27c'$; and the equation thus expresses that the order of the curve is equal to the class of the torse.

Article No. 50. *The general Surface of the Order n without Singularities.*

50. In the general surface of the order n without singularities, we have

$$\begin{aligned} n &= n, \\ a &= n^2 - n, \\ \delta &= \tfrac{1}{2}n(n-1)(n-2)(n-3), \\ \kappa &= n(n-1)(n-2), \\ b &= 0, \\ k &= 0, \\ t &= 0, \\ q &= 0, \end{aligned}$$

$\rho = 0,$

$j = 0,$

$c = 0,$

$h = 0,$

$r = 0,$

$\sigma = 0,$

$\theta = 0,$

$\chi = 0,$

$C = 0,$

$B = 0,$

$\beta = 0,$

$\gamma = 0,$

$i = 0,$

$n' = n(n-1)^2,$

$a' = n(n-1),$

$\delta' = \frac{1}{2}n(n-2)(n^2-9),$

$\kappa' = 3n(n-2),$

$b' = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12),$

$k' = \frac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464),$

$t' = \frac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960),$

$q' = n(n-2)(n-3)(n^2+2n-4),$

$p' = n(n-2)(n^3-n^2+n-12),$

$j' = 0,$

$c' = 4n(n-1)(n-2),$

$h' = \frac{1}{2}n(n-2)(16n^4-64n^3+80n^2-108n+156),$

$r' = 2n(n-2)(3n-4),$

$\sigma' = 4n(n-2),$

$\theta' = 0,$

$\chi' = 0,$

$C' = 0,$

$B' = 0,$

$\beta' = 2n(n-2)(11n-24),$

$\gamma' = 4n(n-2)(n-3)(n^3+3n-16),$

$i' = 0.$

Article Nos. 51 to 64. *Investigation of Formula for* β'.

51. The value $\beta' = 2n(n-2)(11n-24)$ for a surface without singularities was obtained by Salmon by independent geometrical considerations, viz. he obtains

$$2\beta' = 4n(n-2)(11n-24)$$

as the number of intersections of the spinode curve $\big(\text{order} = 4n(n-2)\big)$ by the flecnode surface of the order $11n-24$.

52. The value of β' must be obtainable in the case of a surface with singularities, and I have been led to conclude that we have

$$\begin{aligned}\beta' = \;& 2n(n-2)(11n-24)\\ & -(110n-272)\,b+44q\\ & -(116n-303)\,c+\tfrac{63}{2}r\\ & +\tfrac{691}{2}\beta+248\gamma+198t\\ & +\text{linear function }(i,\,j,\,\theta,\,\chi,\,C,\,B,\,i',\,j',\,\theta',\,\chi',\,C',\,B'),\end{aligned}$$

but I have not yet completely determined the coefficients of the linear function. The reciprocal formula in the case of a surface of the order n without singularities, $i, j, \theta, \chi, C, B, i', j', \theta', \chi', C', B'$ then all vanishing, is the identity

$$\begin{aligned}0 = \;& 2n'(n'-2)(11n'-24)\\ & -(110n'-272)\,b'+44q'\\ & -(116n'-303)\,c'+\tfrac{63}{2}r'\\ & +\tfrac{691}{2}\beta'+248\gamma'+198t'\end{aligned}$$

($n', b', q', c', r', \beta', \gamma', t'$ having the values in the foregoing Table). It was by assuming for β an expression of the above form but with indeterminate coefficients, and then determining these in such wise that the reciprocal equation should be an identity, that the foregoing formula for β' was arrived at.

53. I assume

$$\begin{aligned}\beta' = \;& 2n(n-2)(11n-24)\\ & -b(An-B)+Cq\\ & -c(Dn-E)+Fr\\ & -G\beta-H\gamma-It\\ & +\text{linear function }(i,\,j,\,\theta,\,\chi,\,C,\,B,\,i',\,j',\,\theta',\,\chi',\,C',\,B'),\end{aligned}$$

where it is to be remarked that, in virtue of the equations obtained No. 11, two of the coefficients of this form are really arbitrary: I cannot recall the considerations which led me to write $D=116$, $E=303$.

54. Forming the reciprocal equation

$$\begin{aligned}\beta = \;& 2n'(n'-2)(11n'-24)\\ & -b'(An'-B)+C'q\\ & -c'(Dn'-E)+F'r\\ & -G\beta'-H\gamma'-It'\\ & +\text{linear function } (i', j', \theta', \chi', C', B', i, j, \theta, \chi, C, B),\end{aligned}$$

and substituting herein the values which belong to the surface of the order n without singularities, we should have identically

$$\begin{aligned}0 = \;& 2n(n-1)^2(n-2)(n^2+1)(11n^3-22n^2+11n-24)\\ & -\tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)[An(n-1)^2-B]\\ & +\;n(n-2)(n-3)(n^2+2n-4)C\\ & -4n(n-1)(n-2)[Dn(n-1)^2-E]\\ & +2n(n-2)(3n-4)F\\ & -2n(n-2)(11n-24)G\\ & -4n(n-2)(n-3)(n^3+3n-16)H\\ & -\tfrac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960)I;\end{aligned}$$

or dividing the whole by $n(n-2)$, this is

$$\begin{aligned}0 = \;& 2(n-1)^2(n^2+1)(11n^3-22n^2+11n-24)\\ & -\tfrac{1}{2}(n-1)(n^3-n^2+n-12)[An(n-1)^2-B]\\ & \qquad +(n-3)(n^2+2n-4)C\\ & \qquad\qquad -4(n-1)[Dn(n-1)^2-E]\\ & \qquad\qquad +2(3n-4)F\\ & \qquad\qquad -2(11n-24)G\\ & \qquad -4(n-3)(n^3+3n-16)H\\ & -\tfrac{1}{6}(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960)I.\end{aligned}$$

55. And then, expanding in powers of n and equating to zero the coefficients of the several powers $n^7, \ldots n^0$, we obtain

$$\begin{array}{rrrrrrrr}
22 & -88 & +154 & -224 & +250 & -184 & +118 & -48\\
-\tfrac{1}{2}A & +2A & -\tfrac{7}{2}A & +\tfrac{19}{2}A & -20A & +\tfrac{37}{2}A & -6A & \\
 & & & +\tfrac{1}{2}B & -B & +B & -\tfrac{13}{2}B & +6B\\
 & & & & +C & -C & -10C & +12C\\
 & & & -4D & +12D & -12D & +4D & \\
 & & & & & & +4E & -4E\\
 & & & & & & +6F & -8F\\
 & & & & & & -22G & +48G\\
 & & & -4H & +12H & -12H & +100H & -192H\\
-\tfrac{1}{6}I & +\tfrac{2}{3}I & -\tfrac{7}{6}I & +\tfrac{15}{2}I & -19I & +\tfrac{37}{2}I & -\tfrac{274}{3}I & +160I\\
\| & \| & \| & \| & \| & \| & \| & \|\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

viz. the equations are read vertically downwards. The first, second, and third equations, and the sum of the fourth and fifth, all give the same relation, $132-3A-I=0$; there are consequently, inclusive of this, five independent relations. By combining the equations so as to simplify the numbers, I find these to be

$$\begin{aligned}
&3A+I-132=0,\\
&4A-B-2C-80=0,\\
&7A-B+2E+2F+2G-476=0,\\
&26A-B+8D+8H-1532=0,\\
&6A-E-2F+12G-48H+40I-132=0.
\end{aligned}$$

56. I found, as presently mentioned, $A=110$, $B=272$, $C=44$; values which satisfy (as they should do) the second equation; and then assuming $D=116$ and $E=303$, we have $F=\frac{63}{2}$, $G=-\frac{691}{2}$, $H=-248$, $I=-198$; and the formula is

$$\begin{aligned}
\beta' = \quad & 2n(n-2)(11n-24)\\
&-(110n-272)b+44q\\
&-(116n-303)c+\tfrac{63}{2}r\\
&+\tfrac{691}{2}\beta+248\gamma+198t\\
&+\text{linear function } (i, j, \theta, \chi, C, B, i', j', \theta', \chi', C', B'),
\end{aligned}$$

the process not enabling the determination of the coefficients of the linear function.

57. The values of A, B, C were found from the general theorem that if three surfaces of the orders μ, ν, ρ respectively intersect in a curve of the order m and class r which is α-tuple on μ, β-tuple on ν, and γ-tuple on ρ, then the number of the points of intersection of the three surfaces is

$$=\mu\nu\rho-m(\beta\gamma\mu+\gamma\alpha\nu+\alpha\beta\rho-2\alpha\beta\gamma)+\alpha\beta\gamma r.$$

Apply this to the case of a surface of the order n with a nodal curve of the order b and class q, intersecting the Hessian and flecnodal surfaces, we have

	Order.	Passing through (b, q), times
Surface	n	2
Hessian	$4n-8$	4
Flecnodal	$11n-24$	11

whence number of intersections is

$$\begin{aligned}
=4n(n-2)(11n-24)&-b\{n.4.11+(4n-8)11.2+(11n-24)2.4-2.2.4.11\}\\
&+2.4.11q,
\end{aligned}$$

that is

$$=4n(n-2)(11n-24)-(220n-544)b-88q;$$

and the value of β' is one half of this,

$$= 2n(n-2)(11n-24) - (110n-272)b + 44q.$$

I have not succeeded in applying the like considerations to the cuspidal curve.

58. As regards the general theorem, we know (Salmon, p. 274) that if two surfaces of the orders μ, ν partially intersect in a curve of the order m and class r, and besides in a curve of the order m', then the curves m, m' meet in $m(\mu+\nu-2)-r$ points.

Suppose that the curve m is α-tuple on the surface μ; then to find the number I of the intersections of the curves m and m', we may imagine through m a surface of the order ρ; the surfaces μ, ν intersect in the curve m α times, and in a residual curve of the order $\mu\nu - m\alpha$, this last meets the surface ρ in $\rho(\mu\nu - m\alpha)$ points, and thence the three surfaces meet in $\mu\nu\rho - m\alpha\rho - I$ points. But since m is a simple curve on each of the surfaces ν, ρ, the three surfaces meet in $\mu(\nu\rho - m) - \alpha[m(\nu+\rho-2)-r]$ points, whence equating the two values

$$I = m(\mu + \alpha\nu - 2\alpha) - \alpha r.$$

Next, let the curve m be α-tuple on the surface μ, β-tuple on the surface ν. Considering the new surface ρ through m, then μ, ν intersect in the curve m $\alpha\beta$ times, and in a residual curve of the order $\mu\nu - m\alpha\beta$; this last meets the surface ρ in $\rho(\mu\nu - m\alpha\beta)$ points; whence the three surfaces meet in $\rho(\mu\nu - m\alpha\beta) - I$ points. But the curve m being a β-tuple curve on ν, and a simple curve on ρ, these meet in the curve m β times and in a residual curve of the order $\nu\rho - \beta m$, whence the three surfaces meet in

$$\mu(\nu\rho - \beta m) - \alpha[m(\nu + \beta\rho - 2\beta) - \beta r]$$

points; and equating the two values, we have

$$I = m(\beta\mu + \alpha\nu - 2\alpha\beta) - 2\alpha\beta r.$$

Lastly, if the curve m be γ-tuple on ρ, then the surfaces μ, ρ meet in m $\alpha\gamma$ times and in a residual curve of the order $\mu\rho - m\alpha\gamma$; this last meets ν in

$$\nu(\mu\rho - \alpha\gamma m) - \beta[m(\gamma\mu + \alpha\rho - 2\alpha\gamma) - \alpha\gamma r]$$

points, that is, the number of points of intersection of the three surfaces is

$$= \mu\nu\rho - m(\beta\gamma\mu + \gamma\alpha\nu + \alpha\beta\rho - 2\alpha\beta\gamma) + \alpha\beta\gamma r.$$

59. I represent the complete value of β' by

$$\begin{aligned}\beta' = \;& 2n(11n-24)\\ & -(110n-272)b + 44q\\ & -(116n-303)c + \tfrac{83}{2}r\\ & + \tfrac{691}{2}\beta + 248\gamma + 198t\\ & - hC - gB - xi - \lambda j - \mu\chi - \nu\theta\\ & - h'C' - g'B' - x'i' - \lambda' j' - \mu'\chi' - \nu'\theta',\end{aligned}$$

and (observing that the Table of Singularities in my Memoir on Cubic Surfaces was obtained without the aid of the formula now in question) I endeavour by means of the results therein contained to find the values of the unknown coefficients h, g, x, λ, μ, ν, h', g', x', λ', μ', ν'.

60. For a cubic surface $n=3$, and for a cubic surface without singular lines (in fact for all the cases except the cubic scrolls XXII and XXIII), the formula is

$$\beta' = 54 - hC - gB - \lambda' j' - \mu'\chi' - \nu'\theta' - h'C' - g'B';$$

and applying this to the several cases of cubic surfaces as grouped together in the Table, and referred to by the affixed roman numbers, the resulting equations are

$$\begin{array}{ll}
54 = 54, & \text{(I)} \\
30 = 54 - h, & \text{(II)} \\
18 = 54 - g - 16\nu', & \text{(III)} \\
13 = 54 - 2h - \lambda', & \text{(IV)} \\
6 = 54 - h - g - \mu' - 8\nu', & \text{(VI)} \\
3 = 54 - 3h - 3\lambda', & \text{(VIII)} \\
0 = 54 - 2g - 16\nu' - g', & \text{(IX)} \\
1 = 54 - 2h - g - \lambda' - 2\mu', & \text{(XIII)} \\
0 = 54 - 4h - 6\lambda', & \text{(XVI)} \\
0 = 54 - h - 2g - 2\mu' - g', & \text{(XVII)} \\
0 = 54 - 3g - 3g', & \text{(XXI)}
\end{array}$$

which are all satisfied if only

$$\begin{array}{l}
h = 24, \\
g + 16\nu' = 36, \\
g + 2\mu' = 12, \\
g + g' = 18, \\
\lambda' = -7.
\end{array}$$

61. If we apply to the same surfaces the reciprocal equation for β, or, what is the same thing, apply the original equation to the reciprocal surfaces, as given by interchanging the upper and lower halves of the Table of Singularities, we have another series of equations, viz. this is

$$\begin{array}{llcrl}
0 = 54432 - 54432, & & & & \text{(I)} \\
0 = 27851 - 27846 - h', & h' & = & 5, & \text{(II)} \\
0 = 18180 - 18318 - g' - 16\nu, & g' + 16\nu & = & -138, & \text{(III)} \\
0 = 11765 - 11756 - 2h' - \lambda, & 2h' + \lambda & = & 9, & \text{(IV)} \\
0 = 6917 - 6584 - h' - g' - \mu - 8\nu, & h' + g' + \mu + 8\nu & = & 45, & \text{(VI)} \\
0 = 3534 - 3522 - 3h' - 3\lambda, & 3h' + 3\lambda & = & 12, & \text{(VIII)} \\
0 = 3024 - 3144 - 2g' - 16\nu - g, & 2g' + 16\nu + g & = & -120, & \text{(IX)} \\
0 = 1433 - 1386 - 2h' - g' - \lambda - 2\mu, & 2h' + g' + \lambda + 2\mu & = & 47, & \text{(XIII)} \\
0 = 518 - 504 - 4h' - 6\lambda, & 4h' + 6\lambda & = & 14, & \text{(XVI)} \\
0 = 383 - 322 - h' - 2g' - g - 2\mu, & h' + 2g' + g + 2\mu & = & 61, & \text{(XVII)} \\
0 = 54 - 3g - 3g', & 3g + 3g' & = & 54, & \text{(XXI)}
\end{array}$$

all satisfied if only

$$\begin{aligned} h' &= 5, \\ g'+16\nu &= -138, \\ g'+2\mu &= 38, \\ g+g' &= 18, \\ \lambda &= -1. \end{aligned}$$

62. I remark however that the cubic scroll XXII or XXIII gives

$$0 = 54 - (330 - 272) - 2(\lambda + \lambda'),$$

that is, $\lambda + \lambda' = -2$, instead of $\lambda + \lambda' = -8$. The investigation is in fact really inapplicable to a scroll, for every point of a scroll has the property of a flecnode; whence if $U=0$ be the equation of the scroll, that of the flecnodal surface is $M \,.\, U = 0$, containing U as a factor, and there is not any definite curve of intersection constituting the flecnodal curve; but I am nevertheless surprised at the numerical contradiction.

63. Combining the two sets of results, we find

$$\begin{aligned} h &= 24, \\ g &= g, \\ x &= x, \\ \lambda &= -1, \\ \mu &= 10 + \tfrac{1}{2}g, \\ \nu &= -\tfrac{39}{4} + \tfrac{1}{16}g, \\ h' &= 5, \\ g' &= 18 - g, \\ x' &= x', \\ \lambda' &= -7, \\ \mu' &= 6 - \tfrac{1}{2}g, \\ \nu' &= \tfrac{9}{4} - \tfrac{1}{16}g; \end{aligned}$$

and the formula thus is

$$\begin{aligned} \beta' = \;& 2n(n-2)(11n-24) \\ & -(110n-272)b + 44q \\ & -(116n-303)c + \tfrac{63}{2}r \\ & + \tfrac{691}{2}\beta + 248\gamma + 198t \\ & - 24C + j - 10\chi + \tfrac{39}{4}\theta - 5C' - 18B' - 6\chi' - \tfrac{9}{4}\theta' \\ & - xi - x'i' + \tfrac{1}{16}g(-16B - 8\chi - \theta + 16B' - 8\chi' - \theta'), \end{aligned}$$

where x, x', g are constants which remain to be determined. The cubic surfaces fail to determine them, for the reason that in all of them we have $i=0$, $i'=0$; and $16B+8\chi+\theta=16B'+8\chi'+\theta'$: this last is a very remarkable relation, for the existence of which I do not perceive any *à priori* reason.

Substituting herein for q, r their values from No. 11, this may be written in the form

$$\begin{aligned}\beta' = \;& 2n(n-2)(11n-24)+b(-66n+184)+c(-\tfrac{362}{4}n+240)\\ &+141\beta+\tfrac{352}{4}\gamma+66t\\ &-\quad x'\,i'+7j'-6\chi'-\tfrac{9}{4}\theta'-5C'-18B'\\ &-(x+87)i-21j-\tfrac{41}{2}\chi+\tfrac{51}{2}\theta-24C\\ &+\tfrac{1}{16}g(-16B-8\chi-\theta+16B'+8\chi'+\theta').\end{aligned}$$

64. We have of course by interchanging the unaccented and accented letters, the reciprocal equation giving the value of β.

Article Nos. 65 to 68. *Recapitulation.*

65. In recapitulation, I say that we have between the 42 quantities

$$\begin{aligned}&n,\ a,\ \delta,\ \kappa\ ;\ b,\ k,\ t,\ q,\ \rho,\ j\ ;\ c,\ h,\ r,\ \sigma,\ \theta,\ \chi\ ;\ \beta,\ \gamma,\ i\ ;\ B,\ C,\\ &n',\ a',\ \delta',\ \kappa'\ ;\ b',\ k',\ t',\ q',\ \rho',\ j'\ ;\ c',\ h',\ r',\ \sigma',\ \theta',\ \chi'\ ;\ \beta',\ \gamma',\ i'\ ;\ B',\ C',\end{aligned}$$

in all 25 equations, viz. these are

$$\begin{aligned}a &= a',\\ a' &= n(n-1)-2b-3c,\\ \kappa' &= 3n(n-2)-6b-8c,\\ \delta' &= \tfrac{1}{2}n(n-2)(n^2-9)-(n^2-n-6)(2b+3c)+2b(b-1)+6bc+\tfrac{3}{2}c(c-1),\\ a(n-2) &= \kappa - B + \rho+2\sigma,\\ b(n-2) &= \rho+2\beta+3\gamma+3t,\\ c(n-2) &= 2\sigma+4\beta+\gamma+\theta,\\ a(n-2)(n-3) &= 2(\delta-C)+3(ac-3\sigma-\chi)+2(ab-2\rho-j),\\ b(n-2)(n-3) &= 4k \qquad + (ab-2\rho-j)+3(bc-3\beta-2\gamma-i),\\ c(n-2)(n-3) &= 6h \qquad + (ac-3\sigma-\chi)+2(bc-3\beta-2\gamma-i),\\ q &= b^2-b-2k-3\gamma-6t,\\ r &= c^2-c-2h-3\beta,\\ a &= n'(n'-1)-2b'-3c',\\ \kappa &= 3n'(n'-2)-6b'-8c',\\ \delta &= \tfrac{1}{2}n'(n'-2)(n'^2-9)-(n'^2-n'-6)(2b'+3c')+2b'(b'-1)+6b'c'+\tfrac{3}{2}c'(c'-1),\end{aligned}$$

$$a'(n'-2) = \kappa' - B' + \rho' + 2\sigma',$$
$$b'(n'-2) = \rho' + 2\beta' + 3\gamma' + 3t',$$
$$c'(n'-2) = 2\sigma' + 4\beta' + \gamma' + \theta',$$
$$a'(n'-2)(n'-3) = 2(\delta' - C') + 3(a'c' - 3\sigma' - \chi') + 2(a'b' - 2\rho' - j'),$$
$$b'(n'-2)(n'-3) = 4k' \quad + (a'b' - 2\rho' - j') + 3(b'c' - 3\beta' - 2\gamma' - i'),$$
$$c'(n'-2)(n'-3) = 6h' \quad + (a'c' - 3\sigma' - \chi') + 2(b'c' - 3\beta' - 2\gamma' - i'),$$
$$q' = b'^2 - b' - 2k' - 3\gamma' - 6t',$$
$$r' = c'^2 - c' - 2h' - 3\beta',$$

together with the equations for β and β'.

66. The symbols signify as follows; viz.

n, order of the surface.

a, order of the tangent cone drawn from any point to the surface.

δ, number of nodal edges of the cone.

κ, number of its cuspidal edges.

b, order of nodal curve.

k, number of its apparent double points.

t, number of its triple points.

q, its class.

ρ, number of points where nodal curve is met by curve of contact of tangent cone.

j, number of pinch-points.

c, order of cuspidal curve.

h, number of its apparent double points.

r, its class.

σ, number of points where cuspidal curve is met by curve of contact of tangent cone.

θ, number of off-points.

χ, number of close-points.

β, number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve.

γ, number of intersections, stationary points on nodal curve.

i, number of intersections, not stationary on either curve.

B, number of binodes of surface.

C, number of cnicnodes.

67. And the accented letters have the like significations in regard to the reciprocal surface; or, referring them to the original surface, we have

n', class of the surface.

a', class of curve of intersection by any plane.

δ', number of double tangents of curve of intersection.

κ', number of its inflexions.

b', class of node-couple torse.

k', number of its apparent double planes.

t', number of its triple planes.

q', its order.

ρ', order of node-couple curve.

j', number of pinch-planes.

c', class of spinode torse.

h', number of its apparent double planes.

r', its order.

σ', order of spinode curve.

θ', number of off-planes.

χ', number of close-planes.

β', number of common planes of node-couple and spinode torses, stationary planes of the spinode torse.

γ', number of common planes, stationary planes of node-couple torse.

i', number of common planes, not stationary planes of either torse.

B', number of bitropes of surface.

C', number of its cnictropes.

68. It is hardly necessary to recall that a spinode plane is a tangent plane meeting the surface in a curve having at the point of contact a spinode or cusp; the envelope of the spinode planes is the spinode torse, and the locus of their points of contact the spinode curve. And similarly a node-couple plane is a double tangent plane, or plane meeting the surface in a curve having two nodes; the envelope of the planes is the node-couple torse, and the locus of the points of contact the node-couple curve; the other terms made use of are all explained in the present Memoir.

ADDITION, *August* 3, 1869.

As in the theory of Curves, so in that of Surfaces, there are certain functions of the order, class, &c. and singularities which have the same values in the original and the reciprocal figures respectively; for convenience I represent any such identity by means of the symbol Σ, viz. $\phi(n, a, b, \ldots) = \Sigma$ denotes that the function $\phi(n, a, b, \ldots)$ is equal to the same function $\phi(n', a', b', \ldots)$ of the accented letters. By what precedes we have $a = \Sigma$; and it is moreover clear that any function of the unaccented letters which is $= 0$, or which is equal to a symmetrical function of any of the accented and unaccented letters, or to a function of a, is $= \Sigma$; for instance, from the equations of No. 5 we have $3a' - \kappa' = 3n - c$, and thence $3n - c - \kappa = 3a' - \kappa - \kappa'$, that is, $3n - c - \kappa = \Sigma$;

and from one of the equations of No. 11 we have $n-2C-4B+\kappa-\sigma-2j-3\chi=n+n'-a, =\Sigma$; we have thus the system of eight equations,

$$\begin{aligned}
&a && =\Sigma,\\
&3n-c-\kappa && =\Sigma,\\
&a(n-2)-\kappa+B-\rho-2\sigma && =\Sigma,\\
&b(n-2)-\rho-2\beta-3\gamma-3t && =\Sigma,\\
&c(n-2)-2\sigma-4\beta-\gamma-\theta && =\Sigma,\\
&n+\kappa-\sigma-2C-4B-2j-3\chi && =\Sigma,\\
&2q-2\rho+\beta+3i+j && =\Sigma,\\
&3r+c-5\sigma-\beta-4\theta+2i+\chi && =\Sigma;
\end{aligned}$$

or if from these we eliminate κ, ρ, σ, then the system of five equations,

$$\begin{aligned}
&a && =\Sigma,\\
&n(c-8)-4\beta-\gamma-\theta+4C+8B+6\chi+4j && =\Sigma,\\
&(a-b)(n-2)-11n+3c+4C+9B+2\beta+3\gamma+3t+6\chi+4j && =\Sigma,\\
&3r-20n+6c-\beta+2i+10C+20B+16\chi+10j-4\theta && =\Sigma,\\
&2q-2b(n-2)+5\beta+6\gamma+6t+3i+j && =\Sigma.
\end{aligned}$$

By means of a theorem of Dr Clebsch's I was led to the following expression for the "deficiency" of a surface of the order n having the singularities considered in the foregoing Memoir:

$$\text{Deficiency } =\tfrac{1}{6}(n-1)(n-2)(n-3)-(n-3)(b+c)+\tfrac{1}{2}(q+r)+2t+\tfrac{7}{2}\beta+\tfrac{5}{2}\gamma+i-\tfrac{1}{8}\theta.$$

This should be equal to the deficiency of the reciprocal surface, viz. we must have

$$2(n-1)(n-2)(n-3)-12(n-3)(b+c)+6q+6r+24t+42\beta+30\gamma+12i-\tfrac{3}{8}\theta=\Sigma;$$

but from a combination of the last-mentioned five equations we have

$$\begin{aligned}
-2n^3+6n^2+4n+(12n-36)b+(12n-48)c-6q-6r-24t&\\
-41\beta-30\gamma-13i-7j-8\chi+2\theta-4C-10B&=\Sigma;
\end{aligned}$$

and adding to the last preceding equation we have

$$26n-12c+\beta-i-7j-8\chi+\tfrac{1}{2}\theta-4C-10B=\Sigma.$$

Substituting for Σ its value in terms of the accented letters, we obtain for β' the value

$$\begin{aligned}
\beta'=\beta+26n-12c+i'+7j'+8\chi'-\tfrac{1}{2}\theta'+4C'+10B'&\\
-26n'+12c'-i-7j-8\chi+\tfrac{1}{2}\theta-4C-10B.&
\end{aligned}$$

We have

$$c'=-3a+\kappa+3n',$$

and thence

$$12c'-26n'=-36a+12\kappa+10n';$$

writing herein

$$n' = a + \kappa - \sigma - 2C - 4B - 2j - 3\chi - \sigma,$$

the value is

$$= -26a + 22\kappa - 20C - 40B - 20j - 30\chi - 10\sigma.$$

Substituting for κ its value $= a(n-2) + B - \rho - 2\sigma$, we have

$$12c' - 26n' = a(22n - 70) - 20C - 18B - 20j - 30\chi - 22\rho - 44\sigma;$$

or substituting for a, ρ, σ their values, this is

$$\begin{aligned} &= \{n(n-1) - 2b - 3c\}(22n - 70) - 20C - 18B - 20j - 30\chi \\ &\quad - 22b(n-2) + 44\beta + 66\gamma + 66t \\ &\quad - 27c(n-2) + 108\beta + 27\gamma \qquad + 27\theta, \end{aligned}$$

and adding hereto the remaining terms,

$$\begin{aligned} &\beta + 26n - 12c + i' + 7j' + 8\chi' - \tfrac{1}{2}\theta' + 4C' + 10B' \\ &\qquad - i - 7j - 8\chi + \tfrac{1}{2}\theta - 4C - 10B, \end{aligned}$$

we have

$$\begin{aligned} \beta' = 2n(n-2)(11n - 24) &+ b(-66n + 184) + c(-93n + 252) + 153\beta + 93\gamma + 66t \\ &+ i' + 7j' + 8\chi' - \tfrac{1}{2}\theta' + 4C' + 10B' \\ &- i - 27j - 38\chi + \tfrac{55}{2}\theta - 24C - 28B. \end{aligned}$$

Comparing this with the value of β', No. 63 of the foregoing Memoir, we should have

$$\begin{aligned} 0 = \tfrac{13}{4}cn - 12c - 12\beta - \tfrac{13}{4}\gamma \\ - (x' + 1)i' \qquad - 14\chi' - \tfrac{7}{4}\theta' - 9C' - 28B' \\ - (x + 86)i + 6j + \tfrac{35}{2}\chi - 2\theta \qquad + 28B \\ + \tfrac{1}{16}g(-16B - 8\chi - \theta + 16B' + 8\chi' + \theta'), \end{aligned}$$

or, what is the same thing,

$$0 = 13cn - 48c - 48\beta - 13\gamma + \Phi,$$

if for shortness

$$\begin{aligned} \Phi = -(4x' + 4)i' \qquad - 56\chi' - 7\theta' - 36C' - 112B' \\ - (4x + 344)i + 24j + 70\chi - 8\theta \qquad + 112B \\ + \tfrac{1}{4}g(-16B - 8\chi - \theta + 16B' + 8\chi' + \theta'). \end{aligned}$$

I do not attempt to verify this equation, but I will partially verify a result deducible from it; viz. if Φ' is the like function of the accented letters, then we have

$$\Phi - \Phi' = \Pi - \Pi',$$

where

$$\begin{aligned} \Pi = (4x' - 4x - 340)i + 24j + 126\chi + 224B + 36C - \theta \\ - \tfrac{1}{2}g(16B + 8\chi + \theta); \end{aligned}$$

and Π' is the like function of the accented letters. And this being so, we should have

$$13cn - 48c - 48B - 13\gamma + \Pi = 13c'n' - 48c' - 48\beta' - 13\gamma' + \Pi',$$

or, as this may be written,

$$13cn - 48c - 48\beta - 13\gamma + \Pi = \Sigma.$$

We have

$$26n - 12c + \beta - i - 7j - 8\chi + \tfrac{1}{2}\theta - 4C - 10B = \Sigma;$$

and multiplying by -4 and adding, the equation to be verified is

$$13n(c-8) - 13(4\beta + \gamma) + \Pi + 4i + 28j + 32\chi - 2\theta + 16C + 40B = \Sigma.$$

But we have from the Memoir

$$-13n(c-8) + 13(4\beta + \gamma) \qquad - 52j - 78\chi + 13\theta - 52C - 104B = \Sigma,$$

which reduces the equation to

$$\Pi + 4i - 24j - 46\chi + 11\theta - 36C - 64B = \Sigma;$$

or substituting for Π its value, this is

$$(4x' - 4x - 336)\, i + 80\chi + 10\theta + 160B - 2\lambda(16B + 8\chi + \theta) = \Sigma,$$

that is

$$4(x' - x - 84)\, i - (\tfrac{1}{2}g - 10)(16B + 8\chi + \theta) = \Sigma,$$

an equation which is satisfied if

$$i' = i, \quad x' = x,$$

and

$$g = 20, \text{ or else } 16B + 8\chi + \theta = 16B' + 8\chi' + \theta'.$$

412.

A MEMOIR ON CUBIC SURFACES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLIX. (for the year 1869), pp. 231—326. Received November 12, 1868,—Read January 14, 1869.]

THE present Memoir is based upon, and is in a measure supplementary to that by Professor Schläfli, "On the Distribution of Surfaces of the Third Order into Species, in reference to the presence or absence of Singular Points, and the reality of their Lines," *Phil. Trans.* vol. CLIII. (1863), pp. 193—241. But the object of the Memoir is different. I disregard altogether the ultimate division depending on the reality of the lines, attending only to the division into (twenty-two, or as I prefer to reckon it) twenty-three cases depending on the nature of the singularities. And I attend to the question very much on account of the light to be obtained in reference to the theory of Reciprocal Surfaces. The memoir referred to furnishes in fact a store of materials for this purpose, inasmuch as it gives (partially or completely developed) the equations in plane-coordinates of the several cases of cubic surfaces, or, what is the same thing, the equations in point-coordinates of the several surfaces (orders 12 to 3) reciprocal to these respectively. I found by examination of the several cases, that an extension was required of Dr Salmon's theory of Reciprocal Surfaces in order to make it applicable to the present subject; and the preceding "Memoir on the Theory of Reciprocal Surfaces," [411], was written in connexion with these investigations on Cubic Surfaces. The latter part of the Memoir is divided into sections headed thus:—"Section I = 12, equation $(X, Y, Z, W)^3 = 0$" &c. referring to the several cases of the cubic surface; but the paragraphs are numbered continuously throughout the Memoir.

Article Nos. 1 to 13. *The twenty-three Cases of Cubic Surfaces—Explanations and Table of Singularities.*

1. I designate as follows the twenty-three cases of cubic surfaces, adding to each of them its equation:

$$\begin{array}{ll} \text{I} \quad = 12, & (X, Y, Z, W)^3 = 0, \\ \text{II} \quad = 12 - C_2, & W(a, b, c, f, g, h \between X, Y, Z)^2 + 2kXYZ = 0, \\ \text{III} \quad = 12 - B_3, & 2W(X+Y+Z)(lX+mY+nZ) + 2kXYZ = 0 \end{array}$$

IV	$=12-2C_2,$	$WXZ+Y^2(\gamma Z+\delta W)+(a,\ b,\ c,\ d \between X,\ Y)^3=0,$
V	$=12-B_4,$	$WXZ+(X+Z)(Y^2-aX^2-bZ^2)=0,$
VI	$=12-B_3-C_2,$	$WXZ+Y^2Z+(a,\ b,\ c,\ d \between X,\ Y)^3=0,$
VII	$=12-B_5,$	$WXZ+Y^2Z+YX^2-Z^3=0,$
VIII	$=12-3C_2,$	$Y^3+Y^2(X+Z+W)+4aXZW=0,$
IX	$=12-2B_3,$	$WXZ+(a,\ b,\ c,\ d \between X,\ Y)^3=0,$
X	$=12-B_4-C_2,$	$WXZ+(X+Z)(Y^2-X^2)=0,$
XI	$=12-B_6,$	$WXZ+Y^2Z+X^3-Z^3=0,$
XII	$=12-U_6,$	$W(X+Y+Z)^2+XYZ=0,$
XIII	$=12-B_3-2C_2,$	$WXZ+Y^2(X+Y+Z)=0,$
XIV	$=12-B_5-C_2,$	$WXZ+Y^2Z+YX^2=0,$
XV	$=12-U_7,$	$WX^2+XZ^2+Y^2Z=0,$
XVI	$=12-4C_2,$	$W(XY+XZ+YZ)+XYZ=0,$
XVII	$=12-2B_3-C_2,$	$WXZ+XY^2+Y^3=0,$
XVIII	$=12-B_4-2C_2,$	$WXZ+(X+Z)Y^2=0,$
XIX	$=12-B_6-C_2,$	$WXZ+Y^2Z+X^3=0,$
XX	$=12-U_8,$	$WX^2+XZ^2+Y^3=0,$
XXI	$=12-3B_3,$	$WXZ+Y^3=0,$
XXII	$=3,\ S(1,\ 1),$	$WX^2+ZY^2=0,$
XXIII	$=3,\ S(\overline{1,\ 1}),$	$X(WX+YZ)+Y^3=0;$

2. Where C_2 denotes a conic-node diminishing the class by 2; B_3, B_4, B_5, B_6 a biplanar node diminishing (as the case may be) the class by 3, 4, 5, or 6; and U_6, U_7, U_8 a uniplanar node diminishing (as the case may be) the class by 6, 7, or 8. The affixed explanation, which I shall usually retain in connexion with the Roman number, shows therefore in each case what the class is, and also the singularities which cause the reduction: thus XIII $=12-B_3-2C_2$ indicates that there is a biplanar node, B_3, diminishing the class by 3, and two conic-nodes, C_2, each diminishing the class by 2; and thus that the class is $12-3-2.2,=5$. As regards the cases XXII and XXIII, these are surfaces having a nodal right line, and are consequently scrolls, each of the class 3, viz. XXII is the scroll $S(1,\ 1)$ having a simple directrix right line distinct from the nodal line, and XXIII is the scroll $S(\overline{1,\ 1})$ having a simple directrix right line coincident with the nodal line: see as to this my "Second Memoir on Skew Surfaces, otherwise Scrolls," *Phil. Trans.* vol. CLIV. (1864), pp. 559—577, [340].

3. The nature of the points C_2, B_3, B_4, B_5, B_6, U_6, U_7, U_8 requires to be explained.

$C(=C_2)$ is a conic-node, where, instead of the tangent plane, we have a proper quadric cone.

$B(=B_3,\ B_4,\ B_5$ or $B_6)$ is a biplanar-node, where the quadric cone becomes a plane-pair (two distinct planes): the two planes are called the biplanes, and their line of intersection is the edge:

In B_3, the edge is not a line on the surface—in the other cases it is; this implies that the surface is touched along the edge by a plane, viz. in B_4, B_5 the edge is torsal, in B_6 it is oscular:

In B_4, the tangent plane is distinct from each of the biplanes:

In B_5, the tangent plane coincides with one of the biplanes; we have thus an ordinary biplane, and a torsal biplane:

In B_6, the tangent plane coinciding with one of the biplanes becomes oscular; we have thus an ordinary biplane, and an oscular biplane.

$U\,(=U_6,\ U_7$ or $U_8)$ is a uniplanar-node, where the quadric cone becomes a coincident plane-pair; say, the plane is the uniplane. It is to be observed that there is not in this case any edge. The uniplane meets the cubic surface in three lines, or say "rays," passing through the uniplanar-node, viz.

In U_6, the rays are three distinct lines:

In U_7, two of them coincide:

In U_8, they all three coincide.

4. To connect these singular points with the theory of the preceding Memoir, it is to be observed that they are respectively equivalent to a certain number of the cnicnodes $C\,(=C_2)$ and binodes $B\,(=B_3)$, viz. we have

$$\begin{aligned}
C_2 &= C,\\
B_3 &= \qquad B,\\
B_4 &= 2C,\\
B_5 &= C + B,\\
\begin{cases} B_6 = 3C,\\ U_6 = 3C,\end{cases}\\
U_7 &= 2C + B,\\
U_8 &= C + 2B.
\end{aligned}$$

5. I take the opportunity of remarking that although the expressions cnicnode and binode properly refer to the simple singularities C and B, yet as $C_2 = C$, C_2 is properly spoken of as a cnicnode, and we may (using the term binode as an abbreviation for biplanar-node) speak of any of the singularities B_3, B_4, B_5, B_6 as a binode. Thus the surface $\mathrm{X} = 12 - B_4 - C_2$ has a binode B_4 and a cnicnode C_2; although theoretically the binode B_4 is equivalent to two cnicnodes, and the surface belongs to those with three cnicnodes, or for which $C = 3$. I use also the expression unode for shortness, instead of uniplanar-node, to denote any of the singularities U_6, U_7, U_8.

6. The foregoing equations (substantially the same as Schläfli's) are *Canonical forms;* the reduction of the equation of any case of surface to the above form is not always obvious. It would appear that each equation is from its simplicity in the form best adapted to the separate discussion of the surface to which it belongs; there is the disadvantage that the equations do not always (when from the geometrical connexion of the surfaces they ought to do so) lead the one to the other; for instance, $\mathrm{V} = 12 - B_4$ includes $\mathrm{VII} = 12 - B_5$, but we cannot from the equation $WXZ + (X+Z)(Y^2 - aX^2 - bZ^2) = 0$ of the former pass to the equation $WXZ + Y^2Z + YX^3 - X^3 = 0$ of the latter. This would be a serious imperfection if the object were to form a theory of the quaternary function $(X, Y, Z, W)^3$; but the equations are in the present Memoir used only as means to an end, the establishment of the geometrical theory of the surfaces to which they respectively belong, and the imperfection is not material.

7. I have used the capital letters (X, Y, Z, W) in place of Schläfli's (x, y, z, w), reserving these in place of his (p, q, r, s) for plane-coordinates of the cubic surfaces, or (what is the same thing) point-coordinates of the reciprocal surfaces; but I have in several cases interchanged the coordinates (X, Y, Z, W) so that they do not *in this order* correspond to Schläfli's (x, y, z, w): this has been done so as to obtain a greater uniformity in the representation of the surfaces. To explain this, let A, B, C, D be the vertices of the tetrahedron formed by the coordinate planes $A = YZW$, $B = ZWX$, $C = WXY$, $D = XYZ$; the coordinate planes have been chosen so that determinate vertices of the tetrahedron shall correspond to determinate singularities of the surface.

8. Consider first the surfaces which have no nodes B or U. It is clear that the nodes C_2 might have been taken at any vertices whatever of the tetrahedron; they are taken thus: there is always a node C_2 at D; when there is a second node C_2, this is at C, the third one is at A, and the fourth at B.

9. Consider next the surfaces which have a binode B_3, B_4, B_5, or B_6; this is taken to be at D, and the biplanes to be $X=0$, $Z=0$ [1] (the edge being therefore DB), viz. in B_5 or B_6, where the distinction arises, $X=0$ is the ordinary biplane, $Z=0$ the torsal or (as the case may be) oscular biplane. If there is a second node, this of necessity lies in an ordinary biplane; it may be and is taken to be in the biplane $X=0$, at C. I suppose for a moment that this is a node C_2. It is only when the binode is B_3 or B_4 that there can be a third node, for it is only in these cases that there is a second ordinary biplane $Z=0$; but in these cases respectively the third node, a C_2, may be and is taken to be in the biplane $Z=0$, at A.

10. The only case of two binodes is when each is a B_3. Here the first is as above at D, its biplanes being $X=0$, $Z=0$; and the second is as above in the biplane $X=0$, at C; the biplanes thereof are then $X=0$ (which is thus a biplane common to the two binodes, or say a common biplane), and a remaining biplane which may be and is taken to be $W=0$. If there is a third node, this may be either C_2 or B_3, but it will in either case lie in the biplane $Z=0$ of the first binode, and also in the biplane $W=0$ of the second binode, that is, in the line BA; and it may be and is taken to be at A; if a binode, then its biplanes are of necessity $Z=0$, $W=0$; and the plane $Y=0$ will be the plane through the three binodes D, C, A.

11. If there is a unode, then this may be and is taken to be at D, and its uniplane may be taken to be $X=0$; in the surface $\text{XII}=12-U_6$ the uniplane is, however, taken to be $X+Y+Z=0$. There is never, besides the unode, any other node.

12. The result is that the nodes, in the order of their speciality, are in the equations taken to be at D, C, A, B respectively; and that (except in the case $\text{III}=12-B_3$) the biplanes of the first binode are $X=0$, $Z=0$ (for a binode B_5 or B_6, $X=0$ being the ordinary biplane, $Z=0$ the special biplane), those of the second binode $X=0$, $W=0$, those of the third binode $Z=0$, $W=0$, and that (except in the case $\text{XII}=12-U_6$) the uniplane is $X=0$. For example, in the surface $\text{XVII}=12-2B_3-C_2$, as represented by its equation $WXZ+Y^2Z+X^3=0$, we have a B_3 at D, the biplanes being $X=0$, $Z=0$, a B_3 at C, the biplanes being $X=0$, $W=0$ (therefore $X=0$ the common biplane), and a C_2 at A.

[1] In the case, however, of a single B_3, $\text{III}=12-B_3$, the biplanes are taken to be $X+Y+Z=0$, $lX+mY+nZ=0$.

13. It will be convenient (anticipating the results of the investigations contained in the present Memoir) to give at once the following Table of Singularities; the several symbols have of course the significations explained in the former Memoir.

	I $=12.$	II $=12-C_2.$	III $=12-B_3.$	IV $=12-2C_2.$ V $=12-B_4.$	VI $=12-B_3-C_2.$ VII $=12-B_6.$	VIII $=12-3C_2.$ X $=12-B_4-C_2.$ XI $=12-B_6.$ XII $=12-U_8.$	IX $=12-2B_3.$	XIII $=12-B_3-2C_2.$ XIV $=12-B_5-C_2.$ XV $=12-U_7.$	XVI $=12-4C_2.$ XVIII $=12-B_4-2C_2.$ XIX $=12-B_6-C_2.$	XVII $=12-2B_3-C_2.$ XX $=12-U_8.$	XXI $=12-3B_3.$	XXII $=S(1,1).$ XXIII $=S(\overline{1},\overline{1}).$	
n	3	3	3	3	3	3	3	3	3	3	3	3	n
a	6	6	6	6	6	6	6	6	6	6	6	4	a
δ	0	1	0	2	1	3	0	2	4	1	0	0	δ
κ	6	6	7	6	7	6	8	7	6	8	9	3	κ
b	0	0	0	0	0	0	0	0	0	0	0	1	b
k	0	0	0	0	0	0	0	0	0	0	0	0	k
t	0	0	0	0	0	0	0	0	0	0	0	0	t
q	0	0	0	0	0	0	0	0	0	0	0	0	q
ρ	0	0	0	0	0	0	0	0	0	0	0	1	ρ
j	0	0	0	0	0	0	0	0	0	0	0	2	j
c	0	0	0	0	0	0	0	0	0	0	0	0	c
h	0	0	0	0	0	0	0	0	0	0	0	0	h
r	0	0	0	0	0	0	0	0	0	0	0	0	r
σ	0	0	0	0	0	0	0	0	0	0	0	0	σ
θ	0	0	0	0	0	0	0	0	0	0	0	0	θ
χ	0	0	0	0	0	0	0	0	0	0	0	0	χ
β	0	0	0	0	0	0	0	0	0	0	0	0	β
γ	0	0	0	0	0	0	0	0	0	0	0	0	γ
i	0	0	0	0	0	0	0	0	0	0	0	0	i
C	0	1	0	2	1	3	0	2	4	1	0	0	C
B	0	0	1	0	1	0	2	1	0	2	3	0	B
	I	II	III	IV V	VI VII	VIII X XI XII	IX	XIII XIV XV	XVI XVIII XIX	XVII XX	XXI	XXII XXIII	
n'	12	10	9	8	7	6	6	5	4	4	3	3	n'
a'	6	6	6	6	6	6	6	6	6	6	6	4	a'
δ'	0	0	0	0	0	0	0	0	0	0	0	0	δ'
κ'	9	9	9	9	9	9	9	9	9	9	9	3	κ'
b'	27	15	9	7	3	3	0	1	3	0	0	1	b'
k'	216	60	18	12	3	0	0	0	0	0	0	0	k'
t'	45	15	6	3	0	1	0	0	1	0	0	0	t'
q'	0	0	0	0	0	0	0	0	0	0	0	0	q'
ρ'	27	15	9	7	3	3	0	1	3	0	0	1	ρ'
j'	0	0	0	1	0	3	0	1	6	0	0	2	j'
c'	24	18	16	12	10	6	8	4	0	2	0	0	c'
h'	180	96	72	38	24	6	12	2	0	0	0	0	h'
r'	30	24	42	17	24	9	32	5	0	2	0	0	r'
σ'	12	12	12	10	9	6	8	4	0	2	0	0	σ'
θ'	0	0	16	0	8	0	16	0	0	0	0	0	θ'
χ'	0	0	0	0	1	0	0	2	0	2	0	0	χ'
β'	54	30	18	13	6	3	0	1	0	0	0	0	β'
γ'	0	0	0	0	0	0	0	0	0	0	0	0	γ'
i'	0	0	0	0	0	0	0	0	0	0	0	0	i'
C'	0	0	0	0	0	0	0	0	0	0	0	0	C'
B'	0	0	0	0	0	0	1	0	0	1	3	0	B'

Article Nos. 14 to 19. *Explanation in regard to the Determination of the Number of certain Singularities.*

14. In the several cases I to XXI, we have a cubic surface ($n=3$), with singular points C and B but without singular lines. The section by an arbitrary plane is thus a curve, order $n=3$, that is, a cubic curve, without nodes or cusps, and therefore of the class $a'=6$, having $\delta'=0$ double tangents and $\kappa'=9$ inflexions. The tangent cone with an arbitrary point as vertex is a cone of the order $a=6$, having in the case $\mathrm{I}=12$, $\delta=0$ nodal lines and $\kappa=6$ cuspidal lines, but with (in the several other cases) C nodal lines and B cuspidal lines (or rather singular lines tantamount to C double lines and B cuspidal lines): the class of the cone, or order of the reciprocal surface, is thus $n'=6\,.\,5-2(0+C)-3(6+B)=12-2B-3C$.

15. In the general case $\mathrm{I}=12$, there are on the cubic surface 27 lines, lying by 3's in 45 planes; these 27 lines constitute the node-couple curve of the order $\rho'=27$, and the node-couple torse consists of the pencils of planes through these lines respectively, being thus of the class $\rho'=b'=27$; the 45 planes are triple tangent planes of the node-couple torse, which has thus $t'=45$ triple tangent planes. But in the other cases it is only certain of the 27 lines, say the "facultative lines" (as will be explained), which constitute the node-couple curve of the order ρ': the pencils of planes through these lines constitute the node-couple torse of the class $b'=\rho'$; the t' planes, each containing three facultative lines, are the triple tangent planes of the node-couple torse. Or if (as is somewhat more convenient) we refer the numbers b', t' to the reciprocal surface, then the lines, reciprocals of the facultative lines, constitute the nodal curve of the order b'; and the points t', each containing three of these lines, are the triple points of the nodal curve. Inasmuch as the nodal curve consists of right lines, the number k' of its apparent double points is given by the formula $2k'=b'^2-b'-6t'$; and comparing with the formula $q'=b'^2-b'-2k'-3\gamma'-6t'$, we have $q'+3\gamma'=0$, that is, $q'=0$ (q' the class of the nodal curve), and also $\gamma'=0$.

16. In the general case $\mathrm{I}=12$, the spinode curve is the complete intersection of the cubic surface by the Hessian surface of the order 4, and it is thus of the order $\sigma'=12$; but in the other cases the complete intersection consists of the spinode curve together with certain right lines not belonging to the curve, and the spinode curve is of an order σ' less than 12: this will be further explained, and the reduction accounted for (see *post*, Nos. 24 *et seq.*).

17. Again, in the general case $\mathrm{I}=12$, each of the 27 lines is a double tangent of the spinode curve, and the tangent planes of the surface at the points of contact are common tangent planes of the spinode torse and the node-couple torse, stationary planes of the spinode torse; or we have $\beta'=2\rho'=54$. In the other cases, however, instead of the 27 lines we must take only the facultative lines, each of which is or is not a double or a single tangent of the spinode curve; and the tangent planes of the surface at the points of contact are the common tangent planes as above—that is, the number of contacts gives β' not in general $=2\rho'$.

18. There are not, except as above, any common tangent planes of the two torses, that is, not only $\gamma' = 0$ as already mentioned, but also $i' = 0$. I do not at present account *à priori* for the values $\theta' = 16$, 8, and 16, which present themselves in the Table. The cubic surface cannot have a plane of conic contact, and we have thus in every case $C' = 0$; but the value of B' is not in every case $= 0$.

19. In what precedes we see how a discussion of the equation of the cubic surface should in the several cases respectively lead to the values $b', t', \rho', \sigma', \beta', j', \chi', B'$, and how in the reciprocal surface the nodal curve of the order b' is known by means of the facultative lines of the original cubic surface. The cuspidal curve c' might also be obtained as the reciprocal of the spinode-torse; but this would in general be a laborious process, and it is the less necessary, inasmuch as the equation of the reciprocal surface is in each case obtained in a form putting in evidence the cuspidal curve.

Article Nos. 20 to 23. *The Lines and Planes of a Cubic Surface; Facultative Lines; Explanation of Diagrams.*

20. In the general surface $I = 12$, we have 27 lines and 45 triple-tangent planes, or say simply, planes: through each line pass 5 planes, in each plane lie 3 lines. For the surfaces II to XXI (the present considerations do not of course apply to the Scrolls) several of the lines come to coincide with each other, and several of the planes also come to coincide with each other; but the number of the lines is always reckoned as 27, and that of the planes as 45. If we attend to the distinct lines and the distinct planes, each line has a multiplicity, and the sum of these is $= 27$; and so each plane has a multiplicity, and the sum of these is $= 45$. Again, attending to a particular line in a particular plane, the line has a frequency 1, 2, or 3, that is, it represents 1, 2, or 3 of the 3 lines in the plane (this is in fact the distinction of a scrolar, torsal, or oscular line); and similarly, the plane has a frequency 1, 2, 3, 4, or 5, according to the number which it represents of the 5 planes through the line. It requires only a little consideration to perceive that the multiplicity of the plane into its frequency in regard to the line is equal to the multiplicity of the line into its frequency in regard to the plane. Observe, further, that if M be the multiplicity of the plane, then, considering it in regard to the lines contained therein, we get the products (M, M, M), $(2M, M)$, or $3M$, according as the three lines are or are not distinct, but that the sum of the products is always $= 3M$, and that in regard to all the planes the total sum is $3 \times 45, = 135$. And so if M' be the multiplicity of the line, then, considering it in regard to the planes which pass through it, we get the products (M', M', M', M', M'), $(2M', M', M', M')$, ...$(5M')$, as the case may be, but that the sum of the products is $= 5M'$, and that in regard to all the lines the sum is $5 \times 27, = 135$, as before.

21. The mode of coincidence of the lines and planes, and the several distinct lines and planes which are situate in or pass through the several distinct planes and lines respectively, are shown in the annexed diagrams I to XXI[1]: the multiplicity

[1] See the commencements of the several sections.

of each line appears by the upper marginal line, and that of each plane by the left-hand marginal column (thus in diagram I, $27 \times 1 = 27$ and $45 \times 1 = 45$, 1 is the multiplicity of each line, and it is also the multiplicity of each plane); the frequencies of a line and plane in regard to each other appear by the dots in the square opposite to the line and plane in question, these being read, for the frequency of the line vertically, and for the frequency of the plane horizontally; thus $\begin{smallmatrix} \cdot & \cdot \\ & \cdot \end{smallmatrix}$ indicates that the frequency of the line is $=3$, and the frequency of the plane is $=2$. There should be and are in every line of the diagram 3 dots, and in every column of the diagram 5 dots (a symbol $\begin{smallmatrix} \cdot & \cdot \\ \cdot & \end{smallmatrix}$ being read as just explained, 2 dots in the line, 3 dots in the column).

22. For the surface $\mathrm{I} = 12$, there is of course no distinction between the lines, but these form only a single class, and the like for the planes; but for the other surfaces the lines and planes form separate classes, as shown in the diagrams by the lower marginal explanation of the lines, and the right-hand marginal explanation of the planes. I use here and elsewhere "ray" to denote a line passing through a single node; "axis" to denote a line joining two nodes; "edge" (as above) to denote the edge of a binode; any other line is a "mere line." An axis is always torsal or oscular; when it is torsal, the plane touching along the axis contains a third line which is the "transversal" of such axis; but a transversal may be a mere line, a ray, or an axis; in the case $\mathrm{XVI} = 12 - 4C_2$, each transversal is a transversal in regard to two axes.

23. In the general case $\mathrm{I} = 12$, each of the 27 lines is, as already mentioned, part of the node-couple curve; and the node-couple curve is made up of the 27 lines, and is thus a curve of the order 27. In fact each plane through a line meets the cubic surface in this line, and in a conic; the line and conic meet in two points, and the plane (that is in any plane) through the line is thus a double tangent plane touching the surface at the two points in question; the locus of the points of contact, that is the line itself, is thus part of the node-couple curve. But in the other cases, II to XXI, certain of the lines do not belong to the node-couple curve (this will be examined in detail in the several cases respectively); but I wish to show here how in a general way a line passing through a node, say a nodal ray, is not part of the node-couple curve. To fix the ideas, consider the surface $\mathrm{II} = 12 - C_2$; there are here through C_2 six lines, or say rays: attending to any one of these, a plane through the ray meets the surface in the ray itself and in a conic; the ray and the conic meet as before in two points, one of them being the point C_2: the plane touches the surface at the other point, *but it does not touch the surface at* C_2. (I am not sure, and I leave it an open question, whether we ought to say that at a node C_2 there is *no* tangent plane, or to say that only the tangent planes of the nodal cone are tangent planes of the surface; but, at any rate, an arbitrary plane through C_2 is *not* a tangent plane.) The plane through the ray is only a single tangent plane, not a double tangent plane; and the ray is not part of the node-couple curve. We say that a line of the surface is or is not "facultative" according as it does or does not form part of the node-couple curve.

Article Nos. 24 to 26. *Axis; the different kinds thereof.*

24. A line joining two nodes is an axis; such a line is always a line, and it is a torsal or oscular line, of the surface. But some further distinctions are requisite; using the expressions in their strict sense, cnicnode $=C$, binode $=B$, an axis is a CC-axis joining two cnicnodes, or it is a CB-axis joining a cnicnode and a binode, or it is a BB-axis joining two binodes. A CC-axis is torsal, the transversal being a mere line, not a ray through either of the cnicnodes; a CB-axis is torsal, the transversal being a ray of the binode; a BB-axis is oscular. The distinction is of course carried through as regards the higher biplanar nodes B_4, B_5, B_6, and the uniplanar nodes U_6, U_7, U_8: thus ($B_3=B$) the edge of a binode B_3 is not an axis at all, but ($B_4=2C$) the edge of a binode B_4 is a CC-axis; ($B_5=B+C$) the edge of a binode B_5 is a CB-axis; ($B_6=3C$) the edge of a binode B_6 is a thrice-taken CC-axis; ($U_6=3C$) each of the rays is regarded as a CC-axis; ($U_7=B+2C$) the double ray is regarded as a twice-taken CB-axis, and the single ray as a CC-axis; ($U_8=2B+C$) the ray is regarded as a BB-axis + a twice-taken CB-axis.

25. It has been mentioned that the intersection of the surface with the Hessian consists of the spinode curve, together with certain right lines; these lines are in fact the axes—viz. the examination of the several cases shows that in the complete intersection each CC-axis presents itself twice, each CB-axis 3 times, and each BB-axis 4 times. We thus see that a CC-axis, or rather the torsal plane along such axis, is the pinch-plane or singularity $j'=1$; the CB-axis, or rather the torsal plane along such axis, the close-plane or singularity $\chi'=1$; and the BB-axis, or oscular plane along such axis, the bitrope or singularity $B'=1$; for a cubic surface with singular lines the expression of σ' being in fact $\sigma'=12-2j'-3\chi'-4B'$. There are, however, some cases requiring explanation; thus for the case $\text{VIII}=12-B_5$, where the edge is by what precedes a CB-axis, the complete intersection is made up of the edge 4 times and of an octic curve; the consideration of the reciprocal surface shows, however, that the edge taken once is really part of the spinode curve (viz. that this curve is made up of the edge taken once and of the octic curve, its order being thus $\sigma'=9$); and the interpretation then of course is that the intersection is made up of the edge taken 3 times (as for a CB-axis it should be) and of the spinode curve.

26. I remark in further explanation, that in the several sections, in showing how the complete intersection of the cubic surface with the Hessian is made up, I have not referred to the axes in the above precise significations; thus $\text{XIV}=12-B_5-C_2$, the binode B_5 is $C+B$, and the edge is thus a CB-axis, while the axis B_5C_2 is a CB-axis + a CC-axis ($\chi'=1+1$, $=2$, $j'=1$). The complete intersection should therefore consist of the spinode curve, + edge (as a CB-axis) 3 times + axis (as a CB-axis + a CC-axis) $2+3$, $=5$ times: it is in the section stated (in perfect consistency herewith, but without the full explanation) that the intersection is made up of the axis 5 times, the edge 4 times, and a cubic curve—which cubic curve together with the edge once constitutes the spinode curve; and so in other cases: this explanation will, I think, remove all difficulty.

Article Nos. 27 to 32. *On the Determination of the Reciprocal Equation.*

27. Consider in general the cubic surface $(*\between X, Y, Z, W)^3 = 0$, and in connexion therewith the equation $Xx + Yy + Zz + Ww = 0$, which regarding therein X, Y, Z, W as current coordinates, and x, y, z, w as constants, is the equation of a plane. If from the two equations we eliminate one of the coordinates, for instance W, we obtain

$$(*\between Xw, Yw, Zw, -(Xx + Yy + Zz))^3 = 0,$$

which, (X, Y, Z) being current coordinates, is obviously the equation of the cone, vertex $(X = 0, Y = 0, Z = 0)$, which stands on the section of the cubic surface by the plane. Equating to zero the discriminant of this function in regard to (X, Y, Z), we express that the cone has a nodal line; that is, that the section has a node, or, what is the same thing, that the plane $xX + yY + zZ + wW = 0$ is a tangent plane of the cubic surface; and we thus by the process in fact obtain the equation of the cubic surface in the reciprocal or plane coordinates (x, y, z, w). Consider in the same equation x, y, z, w as current coordinates, (X, Y, Z) as given parameters, the equation represents a system of three planes, viz. these are the planes $xX + yY + zZ + wW' = 0$, where W' has the three values given by the equation $(*\between X, Y, Z, W')^3 = 0$, or, what is the same thing, X, Y, Z, W' are the coordinates of any one of the three points of intersection of the cubic surface by the line $\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z}$; (X, Y, Z, W') belongs to a point on the surface, and

$$xX + yY + zZ + wW' = 0$$

is the polar plane of this point in regard to a quadric surface $X^2 + Y^2 + Z^2 + W^2 = 0$; the equation

$$(*\between Xw, Yw, Zw, -(Xx + Yy + Zz))^3 = 0$$

is thus the equation of a system of 3 planes, the polar planes of three points of the cubic surface (which three points lie on an arbitrary line through the point $x = 0$, $y = 0$, $z = 0$). In equating to zero the discriminant in regard to (X, Y, Z), we find the envelope of the system of three planes, or say of a plane, the polar plane of an arbitrary point on the cubic surface,—or we have the equation of the reciprocal surface, being, as is known, the same thing as the equation of the cubic surface in the reciprocal or plane coordinates (x, y, z, w). In what precedes we have the explanation of an ordinary process of finding the equation of the reciprocal surface, this equation being thereby given by equating to zero the discriminant of a function $(*\between X, Y, Z)^3$, that is, of a ternary cubic function.

28. The process, as last explained, is a special one, viz. the position of a point on the surface is determined by means of certain two parameters, the ratios $X : Y : Z$ which fix the position of the line joining this point with the point $(x = 0, y = 0, z = 0)$. More generally we may consider the position of the point as determined by means of any two parameters; the equation of the polar plane then contains the two parameters, and by taking the envelope in regard to the two parameters considered as variable, we have the equation of the reciprocal surface.

29. But let the parameters, say θ, ϕ, be regarded as varying successively; if ϕ alone vary, we have on the surface a curve Θ, the equation whereof contains the parameter θ, and when θ varies this curve sweeps over the surface. The envelope in regard to ϕ of the polar plane of a point of the surface is a torse, the reciprocal of the curve Θ, and the envelope of the torse is the reciprocal surface. In particular the curve Θ may be the plane section by any plane through a fixed line, say, by the plane $P-\theta Q=0$; the section is a cubic curve, the reciprocal is a sextic cone having its vertex in a fixed line (the reciprocal of the line $P=0$, $Q=0$), and the reciprocal surface is thus obtained as the envelope of this cone; assuming that the equation of the sextic cone has been obtained, this is an equation of a certain order in the parameter θ; or writing $\theta=P:Q$, we obtain the equation of the reciprocal surface by equating to zero the discriminant of a *binary* function of (P, Q).

30. With a variation, this process is a convenient one for obtaining the reciprocal of a cubic surface: we take the fixed line to be one of the lines on the cubic surface; the curve Θ is then a conic, its reciprocal is a quadricone, and the envelope of this quadricone is the required reciprocal surface. This is really what Schläfli does (but the process is not explained) in the several instances in which he obtains the equation of the reciprocal surface by means of a binary function. I remark that it would be very instructive, for each case of surface, to take the variable plane successively through the several kinds of lines on the particular surface; the equation of the reciprocal surface would thus be obtained under different forms, putting in evidence the relation to the reciprocal surface of the fixed line made use of. But this is an investigation which I do not enter upon: I adopt in each case Schläfli's process, without explanation, and merely write down the ternary or (as the case may be) binary function by means of which the equation of the reciprocal surface is obtained.

31. It is to be mentioned that there is a reciprocal process of obtaining the equation of the reciprocal surface; we may imagine, touching the cubic surface along any curve, a series of planes; that is, a torse circumscribed about the surface, and the equation whereof contains a variable parameter θ; the reciprocal figure is a curve, the equations whereof contain the parameter θ; the locus of this curve is the reciprocal surface; that is, the equation of the reciprocal surface is obtained by eliminating θ from the equations of the curve. In particular let the torse be the circumscribed cone having its vertex at any point of a fixed line; the reciprocal figure is then a plane curve, the plane of which passes through the line which is the reciprocal of the fixed line; it is moreover clear that if the position of the vertex on the fixed line be determined by the parameter θ *linearly* (for instance if the vertex be given as the intersection of the fixed line by a plane $P-\theta Q=0$), then the equation of the plane of the curve will be of the form $P'=\theta Q'$, containing the parameter θ linearly; the other equation of the plane curve will contain θ rationally, and the elimination will be at once effected by substituting in this other equation for θ its value, $=P'\div Q'$. And observe moreover that if the fixed line be a line on the cubic surface, then the cone is a quadricone having for its reciprocal a conic; the reciprocal surface is thus given as the locus of a variable conic, the plane of which always passes through a fixed line; there are thus on the reciprocal surface

series of such conics. It would be very instructive and interesting to carry out the investigation in detail.

32. The equation of the reciprocal surface is found by equating to zero the discriminant of a ternary or a binary function[1], viz. this is a ternary cubic, or a binary quartic, cubic, or quadric. The equation as given in the form disct. $=0$, contains a factor which for the adopted forms of equations is always a power or product of powers of w, z, x[2] known *à priori*, and which is thrown out without difficulty, the equation being thereby reduced to the proper order. There is the singular advantage that the process puts in evidence the cuspidal curve of the resulting reciprocal surface, viz. for a ternary cubic, the form obtained is $S^3-T^2=0$, and for a binary quartic it is the equivalent form $I^3-27J^2=0$; but for the factor thrown out as just mentioned, we should have simply $(S=0,\ T=0)$, or, as the case may be, $(I=0,\ J=0)$ for equations of the cuspidal curve; the existence of the factor occasions however a modification, viz. the intersection of the two surfaces is not an indecomposable curve, and the cuspidal curve is in most cases, not the complete intersection, but a partial intersection of the two surfaces. In several cases it thus happens that the cuspidal curve is obtained as a curve $\left\| \begin{matrix} P, & Q, & R \\ P', & Q', & R' \end{matrix} \right\| = 0$, without or with further speciality. Similarly when the equation of the reciprocal surface is obtained by means of a binary cubic; if the coefficients hereof (functions of course of the coordinates x, y, z, w) be A, B, C, D, then the surface is

$$(AD-BC)^2-4(AC-B^2)(BD-C^2)=0,$$

having the cuspidal curve $\left\| \begin{matrix} A, & B, & C \\ B, & C, & D \end{matrix} \right\| = 0$, subject however to modification in the case of a thrown out factor.

Article Nos. 33 and 34. *Explanation as to the Sections of the Memoir.*

33. As regards the following Sections I to XXIII, it is to be observed that for the general surface $\mathrm{I}=12$, I do not attempt to form the equation of the reciprocal surface, and in some of the other cases, $\mathrm{II}=12-C_2$ &c., the equation of the reciprocal surface is either not obtained in a completely developed form, or it is too complicated to allow of its being dealt with, for instance so as to put in evidence the nodal curve of the surface. Portions of the theory given in the latter sections are consequently omitted in the earlier ones, and in particular in the Section I there is given only the diagram of the 27 lines and the 45 planes (with however developments as to notation and otherwise which have no place in the subsequent sections), and with the analytical expressions for the several lines and planes, although from the

[1] In some easy cases, for instance $\mathrm{XVI}=12-4C_2$, the equation of the reciprocal surface is obtained otherwise by a direct elimination.

[2] The factor is in general a power or product of powers of the linear functions which, equated to zero, give the equations of the planes reciprocal to the several nodes of the surface.

want of the equation of the reciprocal surface these analytical expressions have no present application. And so in some of the next following sections, no application is made of the analytical expressions of the lines and planes.

34. I call to mind that if a line be given as the intersection of the two planes

$$AX+BY+CZ+DW=0,\quad A'X+B'Y+C'Z+D'W=0,$$

then the six coordinates of the line are

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h \\ =AD'-A'D, & BD'-B'D, & CD'-C'D, & BC'-B'C, & CA'-C'A, & AB'-A'B, \end{array}$$

and that in terms of its six coordinates the line is given as the common intersection of the four planes

$$\left(\begin{array}{cccc} . & h, & -g, & a \\ -h, & . & f, & b \\ g, & -f, & . & c \\ -a, & -b, & -c, & . \end{array}\right)(X,\ Y,\ Z,\ W)=0.$$

and that (reciprocating as usual in regard to $X^2+Y^2+Z^2+W^2=0$) the coordinates of the reciprocal line are $(f,\ g,\ h,\ a,\ b,\ c)$; that is, this is the common intersection of the four planes

$$\left(\begin{array}{cccc} . & c, & -b, & f \\ -c, & . & a, & g \\ b, & -a, & . & h \\ -f, & -g, & -h, & . \end{array}\right)(x,\ y,\ z,\ w)=0.$$

It is in some cases more convenient to consider a line as determined as the intersection of two planes rather than by means of its six coordinates; thus, for instance, to speak of the line $X=0$, $Y=0$ rather than of the line (0, 0, 0, 1, 0, 0); and in some of the sections I have preferred not to give the expressions of the six coordinates of the several lines.

Article Nos. 35 to 46. § I = 12, Equation $(X,\ Y,\ Z,\ W)^2=0$.

35. There is in the system of the 27 lines and the 45 planes a complicated and many-sided symmetry which precludes the existence of any unique notation: the notation can only be obtained by starting from some arrangement which is not unique, but one of a system of several like arrangements. The notation employed in my original paper "On the Simple Tangent Planes of Surfaces of the Third Order," *Camb. and Dub. Math. Journ.* vol. IV. 1849, pp. 118—132, [76], and which is shown in the right hand and lower margins of the diagram, starts from such an arrangement; but

it is so complicated that it can hardly be considered as at all putting in evidence the relations of the lines and planes; that of Dr Hart (Salmon, "On the Triple Tangent Planes of a Surface of the Third Order," same volume, pp. 252—260), depending on an arrangement of the 27 lines according to a cube of 3 each way, is a singularly elegant one, and will be presently reproduced.

36. But the most convenient one is Schläfli's, starting from a double-sixer; viz. we can (and that in 36 different ways) select out of the 27 lines two systems each of six lines, such that no two lines of the same system intersect, but that each line of the one system intersects all but the corresponding line of the other system; or, say, if the lines are

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & 5, & 6 \\ 1', & 2', & 3', & 4', & 5', & 6', \end{array}$$

then these have the thirty intersections

	1′,	2′,	3′,	4′,	5′,	6′
1		.	.	.	.	.
2	.		.	.	.	.
3	.	.		.	.	.
4	.	.	.		.	.
5	.	.	.	.		.
6	.	.	.	.	.	

Any two lines such as 1, 2′ lie in a plane which may be called 12′; similarly the lines 1′, 2 lie in a plane which may be called 1′2; these two planes meet in a line 12; and any three lines such as 12, 34, 56 meet in pairs, lying in a plane 12.34.56. We have thus the entire system of the 27 lines and 45 planes, as in effect completely explained by what has been stated, but which is exhibited in full in the diagram.

37. The diagram of the lines and planes is

Planes.	Lines. 56	46	45	36	35	34	26	25	24	23	16	15	14	13	12	6′	5′	4′	3′	2′	1′	6	5	4	3	2	1	
I=12	$\overline{27}$														27×1=27													
	$\overline{27}$																											
12′															•					•							•	w
13′														•					•								•	ζ
14′													•					•									•	z
15′												•					•										•	z
16′										•						•											•	$\bar{z}$
21′															•						•					•		ξ
23′										•									•							•		θ
24′									•									•								•		$\bar{f}$
25′								•									•									•		$\bar{p}_1$
26′							•									•										•		p
31′														•							•				•			$\bar{\theta}$
32′										•										•					•			η
34′						•												•							•			g
35′					•												•								•			q
36′				•												•									•			q_1
41′													•								•			•				f
42′									•											•				•				y
43′						•													•					•				$\bar{h}$
45′			•														•							•				m
46′	•															•								•				m_1
51′												•									•		•					p
52′								•												•			•					y
53′					•														•				•					$\bar{r}$
54′			•															•					•					n_1
56′	•															•							•					l
61′											•										•	•						p_1
62′							•													•		•						$\bar{y}$
63′				•															•			•						$\bar{r}$
64′		•																•				•						n
65′	•																•					•						l_1
12.34.56	•					•									•													x
12.35.46		•			•										•													x
12.36.45			•	•											•													x
13.24.56	•								•					•														h
13.25.46		•						•						•														r
13.26.45			•				•							•														$\bar{r}_1$
14.23.56		•								•			•															$\bar{g}$
14.25.36				•				•					•															$\bar{n}$
14.26.35					•		•						•															$\bar{n}_1$
15.23.46		•								•		•																$\bar{q}_1$
15.24.36				•					•			•																$\bar{m}_1$
15.26.34						•	•					•																$\bar{l}_1$
16.23.45			•							•	•																	$\bar{q}$
16.24.35					•				•		•																	$\bar{m}$
16.25.34						•		•			•																	$\bar{l}_1$
45×1=45	$\overline{45}$																											
	$\overline{45}$																											
	c_5	c_7	c_9	c_8	c_6	c_4	b_8	b_6	b_4	b_2	c_8	c_6	c_4	c_2	c_1	c_9	c_7	c_5	c_3	b_1	b_3	b_5	b_7	b_9	c_3	c_1	c_2	

38. It has been mentioned that the number of double-sixers was $=36$, these are as follows:

1, 2, 3, 4, 5, 6	Assumed primitive	1
1′, 2′, 3′, 4′, 5′, 6′		
1, 1′, 23, 24, 25, 26	Like arrangements	15
2, 2′, 13, 14, 15, 16		
1, 2, 3, 56, 46, 45	Like arrangements	20
23, 13, 12, 4, 5, 6		36

where, if we take any column $\frac{1}{2}$ of two lines, we have the complete number 216 of pairs of non-intersecting lines (each line meets 10 lines, there are therefore $27-1-10$, $=16$, which it does not meet, and the number of non-intersecting pairs is thus $\frac{1}{2}.27.16=216$).

39. We can out of the 45 planes select, and that in 120 ways, a trihedral-pair, that is, two triads of planes, such that the planes of the one triad, intersecting those of the other triad, give 9 of the 27 lines. Analytically if $X=0$, $Y=0$, $Z=0$ and $U=0$, $V=0$, $W=0$ are the equations of the six planes, then the equation of the cubic surface is $XYZ+kUVW=0$. See as to this *post*, No. 44.

The trihedral plane pairs are:

12′, 23′, 31′			
1′2, 2′3, 3′1		No. is =	20
12′, 34′, 14.23.56			
2′3, 4′1, 12.34.56		=	90
14.25.36, 35.16.24, 26.34.15			
14.35.26, 25.16.34, 36.24.15		=	10
			120

The construction of the last set is most easily effected by the diagram

$$\begin{array}{ccc} 1\ 2\ 3 & \times & 4\ 5\ 6 \\ 3\ 1\ 2 & & 5\ 6\ 4 \\ 2\ 3\ 1 & & 6\ 4\ 5 \end{array}$$

$$\parallel$$

$$\begin{array}{ccc} 14 & 25 & 36 \\ 35 & 16 & 24 \\ 26 & 34 & 15 \end{array}$$

It is immaterial how the two component triads 123 and 456 are arranged, we obtain always the same trihedral pair.

40. Dr Hart arranges the 27 lines, cubically, thus:

$$\begin{array}{ccc|ccc|ccc} A_1 & B_1 & C_1 & a_1 & b_1 & c_1 & \alpha_1 & \beta_1 & \gamma_1 \\ A_2 & B_2 & C_3 & a_2 & b_2 & c_2 & \alpha_2 & \beta_2 & \gamma_2 \\ A_3 & B_3 & C_3 & a_3 & b_3 & c_3 & \alpha_3 & \beta_3 & \gamma_3 \end{array}$$

where letters of the same alphabet denote lines in the same plane, if only the letters are the same or the suffixes the same; thus A_1, A_2, A_3 lie in a plane $A_1A_2A_3$; A_1, B_1, C_1 lie in a plane $A_1B_1C_1$. Letters of different alphabets denote lines which meet according to the Table

a_1	b_2	c_3	b_1	c_2	a_3	c_1	a_2	b_3
	A_1			B_1			C_1	
α_1	β_2	γ_3	β_1	γ_2	α_3	γ_1	α_2	β_3
c_2	a_3	b_1	a_2	b_3	c_1	b_2	c_3	a_1
	A_2			B_2			C_2	
β_3	γ_1	α_2	γ_3	α_1	β_2	α_3	β_1	γ_2
b_3	c_1	a_2	c_3	a_1	b_2	a_3	b_1	c_2
	A_3			B_3			C_3	
γ_2	α_3	β_1	α_2	β_3	γ_1	β_2	γ_3	α_1

where the letter in the centre of the square denotes a line lying in the same plane with the lines denoted by the letters of each vertical pair in the same square. Thus A_1 lies in the planes $A_1a_1\alpha_1$, $A_1b_2\beta_2$, $A_1c_3\gamma_3$ (and in the before-mentioned two planes $A_1A_2A_3$, $A_1B_1C_1$).

41. I find that one way in which this may be identified with the double-sixer notation is to represent the above arrangement by

$$\begin{array}{ccc|ccc|ccc} 1, & 2', & 12 & 3', & 4, & 34 & 13, & 24, & 56 \\ 14, & 25, & 36 & 2, & 6', & 26 & 1', & 16, & 6 \\ 4', & 5, & 45 & 23, & 46, & 15 & 3, & 35, & 5' \end{array}$$

and then the identification may apparently be effected in $(720 \times 36 =)$ 25920 ways, viz. we may first in any way permute the $\frac{1}{1'}$, $\frac{2}{2'}$, $\frac{3}{3'}$, $\frac{4}{4'}$, $\frac{5}{5'}$, $\frac{6}{6'}$, by this means not altering the double-sixer $\frac{1\,2\,3\,4\,5\,6}{1'\,2'\,3'\,4'\,5'\,6'}$, and then upon the arrangements so obtained make any of the substitutions which permute *inter se* the 36 double-sixers.

42. The equations of the 45 planes are obtained in my paper last referred to, viz. taking the equation of the surface to be

$$W\left(1,\ 1,\ 1,\ 1,\ mn+\frac{1}{mn},\ nl+\frac{1}{nl},\ lm+\frac{1}{lm},\ l+\frac{1}{l},\ m+\frac{1}{m},\ n+\frac{1}{n}\right)(X,\ Y,\ Z,\ W)^2+kXZY=0,$$

where $$k=\frac{p^2-\beta^2}{2(p-\alpha)},\quad \alpha=lmn+\frac{1}{lmn},\quad \beta=lmn-\frac{1}{lmn},$$

then the equations of the planes are:

$$W=0, \qquad [12'=w]$$

$$lX+mY+nZ+W\left[1+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0, \qquad [23'=\theta]$$

$$\frac{X}{l}+\frac{Y}{m}+\frac{Z}{n}+W\left[1-\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0, \qquad [31'=\bar{\theta}]$$

$$X=0, \qquad [12.34.56=x]$$

$$Y=0, \qquad [42' \quad =y]$$

$$Z=0, \qquad [14' \quad =z]$$

$$X+\frac{1}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)W=0, \qquad [21'=\xi]$$

$$Y+\frac{1}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)W=0, \qquad [32'=\eta]$$

$$Z+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)W=0, \qquad [13'=\zeta]$$

$$lX+\frac{Y}{m}+\frac{Z}{n}+W=0, \qquad [41'=\mathrm{f}]$$

$$\frac{X}{l}+mY+\frac{Z}{n}+W=0, \qquad [34'=\mathrm{g}]$$

$$\frac{X}{l}+\frac{Y}{m}+Z+W=0, \qquad [13.24.56=\mathrm{h}]$$

$$\frac{X}{l}+mY+nZ+W=0, \qquad [24'=\bar{\mathrm{f}}]$$

$$lX+\frac{Y}{m}+nZ+W=0, \qquad [14.25.36=\bar{\mathrm{g}}]$$

$$lX+mY+\frac{Z}{n}+W=0, \qquad [43'=\bar{\mathrm{h}}]$$

$$X+\frac{l(p-\alpha)+2mn}{p+\beta}W=0, \qquad [12.35.46=\mathrm{x}]$$

$$Y+\frac{m(p-\alpha)+2nl}{p+\beta}W=0, \qquad [52'=\mathrm{y}]$$

$$Z+\frac{n(p-\alpha)+2lm}{p+\beta}W=0, \qquad [15'=\mathrm{z}]$$

$$X+\frac{\frac{1}{l}(p-\alpha)+\frac{2}{mn}}{p+\beta}W=0, \qquad [12.36.45=\bar{\mathrm{x}}]$$

$$Y+\frac{\frac{1}{m}(p-\alpha)+\frac{2}{nl}}{p+\beta}W=0, \qquad [62'=\bar{\mathrm{y}}]$$

$$Z+\frac{\frac{1}{n}(p-\alpha)+\frac{2}{lm}}{p+\beta}W=0, \qquad [16'=\bar{\mathrm{z}}]$$

$$-\frac{2n}{m(p-\alpha)}X+\frac{1}{m}Y+nZ+W=0, \qquad [56'=\mathrm{l}]$$

$$lX-\frac{2l}{n(p-\alpha)}Y+\frac{1}{n}Z+W=0, \qquad [45'=\mathrm{m}]$$

$$\frac{1}{l}X+mY-\frac{2m}{l(p-\alpha)}Z+W=0, \qquad [64'=\mathrm{n}]$$

$$-\frac{2m}{n(p-\alpha)}X+mY+\frac{1}{n}Z+W=0, \qquad [15.26.34=\bar{\mathrm{l}}]$$

$$\frac{1}{l}X-\frac{2n}{l(p-\alpha)}Y+nZ+W=0, \qquad [16.24.35=\bar{\mathrm{m}}]$$

$$lX+\frac{1}{m}Y-\frac{2l}{m(p-\alpha)}Z+W=0, \qquad [14.25.36=\bar{\mathrm{n}}]$$

$$-\frac{n(p-\alpha)}{2m}X+\frac{Y}{m}+nZ+W=0, \qquad [65'=\mathrm{l}_1]$$

$$lX-\frac{l(p-\alpha)}{2n}Y+\frac{1}{n}Z+W=0, \qquad [46'=\mathrm{m}_1]$$

$$\frac{1}{l}X+mY-\frac{m(p-\alpha)}{2l}Z+W=0, \qquad [54'=\mathrm{n}_1]$$

$$\frac{m(p-\alpha)}{2n}X+mY+\frac{1}{n}Z+W=0, \qquad [16.25.34=\bar{\mathrm{l}}_1]$$

$$\frac{1}{l}X-\frac{n(p-\alpha)}{2l}Y+nZ+W=0, \qquad [15.24.36=\bar{\mathrm{m}}_1]$$

$$lX+\frac{1}{m}Y-\frac{l(p-\alpha)}{2m}Z+W=0, \qquad [14.26.35=\bar{\mathrm{n}}_1]$$

$$-\frac{2X}{p-\alpha}+nY+mZ+\left(mn(p-\alpha)-2l(1-m^2-n^2)\right)\frac{W}{p+\beta}=0, \qquad [51'=\mathrm{p}]$$

$$nX-\frac{2Y}{p-\alpha}+lZ+\left(nl(p-\alpha)-2m(1-n^2-l^2)\right)\frac{W}{p+\beta}=0, \qquad [35'=\mathrm{q}]$$

$$mX+lY-\frac{2Z}{p-\alpha}+\left(lm(p-\alpha)-2n(1-l^2-m^2)\right)\frac{W}{p+\beta}=0, \qquad [13.25.46=\mathrm{r}]$$

$$-\frac{2X}{p-\alpha}+\frac{1}{n}Y+\frac{1}{m}Z+\left(\frac{1}{mn}(p-\alpha)-\frac{2}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [26'=\bar{\mathrm{p}}]$$

$$\frac{1}{n}X+\frac{2Y}{p-\alpha}+\frac{1}{l}Z+\left(\frac{1}{nl}(p-\alpha)-\frac{2}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [16.23.45=\bar{\mathrm{q}}]$$

$$\frac{1}{m}X+\frac{1}{l}Y-\frac{2Z}{p-\alpha}+\left(\frac{1}{lm}(p-\alpha)-\frac{2}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [36'=\bar{\mathrm{r}}]$$

$$-\frac{p-\alpha}{2}X+\frac{Y}{n}+\frac{Z}{m}-lmn\left(\frac{1}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)(p-\alpha)-\frac{2}{mn}\right)\frac{W}{p+\beta}=0, \quad [25'=\mathrm{p}_1]$$

$$\frac{X}{n}-\frac{p-\alpha}{2}Y+\frac{Z}{l}-lmn\left(\frac{1}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)(p-\alpha)-\frac{2}{nl}\right)\frac{W}{p+\beta}=0, \quad [15.23.46=\mathrm{q}_1]$$

$$\frac{X}{m}+\frac{Y}{l}-\frac{p-\alpha}{2}Z-lmn\left(\frac{1}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)(p-\alpha)-\frac{2}{lm}\right)\frac{W}{p+\beta}=0, \quad [53'=\mathrm{r}_1]$$

$$-\frac{p-\alpha}{2}X+nY+mZ-\frac{1}{lmn}\left(l(1-m^2-n^2)(p-\alpha)-2mn\right)\frac{W}{p-\beta}=0, \quad [61'=\bar{\mathrm{p}}_1]$$

$$nX-\frac{p-\alpha}{2}Y+lZ-\frac{1}{lmn}\left(m(1-n^2-l^2)(p-\alpha)-2nl\right)\frac{W}{p-\beta}=0, \quad [36'=\bar{\mathrm{q}}_1]$$

$$mX-lY-\frac{p-\alpha}{2}Z-\frac{1}{lmn}\left(n(1-l^2-m^2)(p-\alpha)-2lm\right)\frac{W}{p-\beta}=0, \quad [13.26.45=\bar{\mathrm{r}}_1]$$

43. The coordinates of the 27 lines are then found to be as follows:

(a)	(b)	(c)
1 0 0	0 1 0	0 0 1
$1-\frac{1}{kl}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $-\frac{l}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $-\frac{l}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{m}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $1-\frac{1}{km}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $-\frac{m}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{n}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $-\frac{n}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $1-\frac{1}{kn}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$
$1-\frac{l}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $-\frac{1}{lk}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $-\frac{1}{lk}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{1}{mk}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $1-\frac{m}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $-\frac{1}{mk}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{1}{nk}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$ $-\frac{1}{nk}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$ $1-\frac{n}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$
1 0 0	0 1 0	0 0 1

(f)	(g)	(h)	
0	0	0	$(12 = a_1)$
0	0	0	$(2' = b_1)$
0	0	0	$(1 = c_1)$
0	$-n$	m	$(2 = a_2)$
n	0	$-l$	$(23 = b_2)$
$-m$	l	0	$(3' = c_2)$
0	$-\frac{1}{n}$	$\frac{1}{m}$	$(1' = a_3)$
$\frac{1}{n}$	0	$-\frac{1}{l}$	$(3 = b_3)$
$-\frac{1}{m}$	$\frac{1}{l}$	0	$(13 = c_3)$
0	$-\frac{1}{n}$	m	$(34 = a_4)$
n	0	$-\frac{1}{l}$	$(24 = b_4)$
$-\frac{1}{m}$	l	0	$(14 = c_4)$

(a)	(b)	(c)
1 0 0	0 1 0	0 0 1
$2m\left(l-\frac{1}{l}\right)$ $-m(p-\alpha)\left(1+\frac{2nl}{m(p-\alpha)}\right)$ $2\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$2\left(1+\frac{2mn}{l(p-\alpha)}\right)$ $2n\left(m-\frac{1}{m}\right)$ $-n(p-\alpha)\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$-l(p-\alpha)\left(1+\frac{2mn}{l(p-\alpha)}\right)$ $2\left(1+\frac{2nl}{m(p-\alpha)}\right)$ $2l\left(n-\frac{1}{n}\right)$
$2n\left(l-\frac{1}{l}\right)$ $2\left(1+\frac{2nl}{m(p-\alpha)}\right)$ $-n(p-\alpha)\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$-l(p-\alpha)\left(1+\frac{2mn}{l(p-\alpha)}\right)$ $2l\left(m-\frac{1}{m}\right)$ $2\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$2\left(1+\frac{2mn}{l(p-\alpha)}\right)$ $-m(p-\alpha)\left(1+\frac{2nl}{m(p-\alpha)}\right)$ $2m\left(n-\frac{1}{n}\right)$
$\frac{2}{n}\left(l-\frac{1}{l}\right)$ $-2\left(1+\frac{2m}{nl(p-\alpha)}\right)$ $\frac{p-\alpha}{n}\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$\frac{p-\alpha}{l}\left(1+\frac{2l}{mn(p-\alpha)}\right)$ $\frac{2}{l}\left(m-\frac{1}{m}\right)$ $-2\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$-2\left(1+\frac{2l}{mn(p-\alpha)}\right)$ $\frac{p-\alpha}{m}\left(1+\frac{2m}{nl(p-\alpha)}\right)$ $\frac{2}{m}\left(n-\frac{1}{n}\right)$
$\frac{2}{m}\left(l-\frac{1}{l}\right)$ $\frac{p-\alpha}{m}\left(1+\frac{2m}{nl(p-\alpha)}\right)$ $-2\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$-2\left(1+\frac{2l}{mn(p-\alpha)}\right)$ $\frac{2}{n}\left(m-\frac{1}{m}\right)$ $\frac{p-\alpha}{n}\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$\frac{p-\alpha}{l}\left(1+\frac{2l}{mn(p-\alpha)}\right)$ $-2\left(1+\frac{2m}{nl(p-\alpha)}\right)$ $\frac{2}{l}\left(n-\frac{1}{n}\right)$

(f)	(g)	(h)	
0	$-n$	$\frac{1}{m}$	$(56 = a_5)$
$\frac{1}{n}$	0	$-l$	$(4 = b_5)$
$-m$	$\frac{1}{l}$	0	$(4' = c_5)$
0	$-(p+\beta)$	$-\frac{2(p+\beta)}{l(p-\alpha)}$	$(35 = a_6)$
$-\frac{2(p+\beta)}{m(p-\alpha)}$	0	$-(p+\beta)$	$(25 = b_6)$
$-(p+\beta)$	$-\frac{2(p+\beta)}{n(p-\alpha)}$	0	$(15 = c_6)$
0	$\frac{2(p+\beta)}{l(p-\alpha)}$	$p+\beta$	$(46 = a_7)$
$p+\beta$	0	$\frac{2(p+\beta)}{m(p-\alpha)}$	$(5 = b_7)$
$\frac{2(p+\beta)}{n(p-\alpha)}$	$p+\beta$	0	$(5' = c_7)$
0	$-\frac{2l(p-\beta)}{p-\alpha}$	$-(p-\beta)$	$(36 = a_8)$
$-(p-\beta)$	0	$-\frac{2m(p-\beta)}{p-\alpha}$	$(26 = b_8)$
$-\frac{2n(p-\beta)}{p-\alpha}$	$-(p-\beta)$	0	$(16 = c_8)$
0	$p-\beta$	$\frac{2l(p-\beta)}{p-\alpha}$	$(45 = a_9)$
$\frac{2m(p-\beta)}{p-\alpha}$	0	$p-\beta$	$(6 = b_9)$
$p-\beta$	$\frac{2n(p-\beta)}{p-\alpha}$	0	$(6' = c_9)$

44. We have $X=0$, $Y=0$, $Z=0$, $W=0$ for the equations of the planes

$$(12.34.56=x),\quad (42'=y),\quad (14'=z),\quad (12'=w);$$

and representing by $\mathrm{f}=lX+\frac{1}{m}Y+\frac{1}{n}Z+W=0$ the equation of any other plane $(41'=\mathrm{f})$ the equation of the cubic surface may be presented in the several forms:

$$\begin{aligned}
0=U &= W\mathrm{f}\bar{\mathrm{f}} &&+\mathrm{k}\xi YZ,\\
&= W\mathrm{g}\bar{\mathrm{g}} &&+\mathrm{k}\eta ZX,\\
&= W\mathrm{h}\bar{\mathrm{h}} &&+\mathrm{k}\zeta XY,\\
&= W\theta\bar{\theta} &&+\mathrm{k}\xi\eta\zeta,\\
&= W\mathrm{l}\bar{\mathrm{l}}_1 &&+\mathrm{k}\mathrm{y}\bar{\mathrm{z}}\mathrm{x},\\
&= W\mathrm{m}\bar{\mathrm{m}}_1 &&+\mathrm{k}\mathrm{z}\bar{\mathrm{x}}\mathrm{y},\\
&= W\mathrm{n}\bar{\mathrm{n}}_1 &&+\mathrm{k}\mathrm{x}\bar{\mathrm{y}}\mathrm{z},\\
&= W\mathrm{l}_1\bar{\mathrm{l}} &&+\mathrm{k}\bar{\mathrm{y}}\mathrm{z}\mathrm{x},\\
&= W\mathrm{m}_1\bar{\mathrm{m}} &&+\mathrm{k}\bar{\mathrm{z}}\mathrm{x}\mathrm{y},\\
&= W\mathrm{n}_1\bar{\mathrm{n}} &&+\mathrm{k}\bar{\mathrm{x}}\mathrm{y}\mathrm{z},\\
&= W\mathrm{p}\mathrm{p}_1 &&+\mathrm{k}\xi\mathrm{y}\mathrm{z},\\
&= W\mathrm{q}\mathrm{q}_1 &&+\mathrm{k}\eta\mathrm{z}\mathrm{x},\\
&= W\mathrm{r}\mathrm{r}_1 &&+\mathrm{k}\zeta\mathrm{x}\mathrm{y},\\
&= W\bar{\mathrm{p}}\bar{\mathrm{p}}_1 &&+\mathrm{k}\bar{\xi}\bar{\mathrm{y}}\bar{\mathrm{z}},\\
&= W\bar{\mathrm{q}}\bar{\mathrm{q}}_1 &&+\mathrm{k}\bar{\eta}\bar{\mathrm{z}}\bar{\mathrm{x}},\\
&= W\bar{\mathrm{r}}\bar{\mathrm{r}}_1 &&+\mathrm{k}\bar{\zeta}\bar{\mathrm{x}}\bar{\mathrm{y}},
\end{aligned}$$

which are the 16 forms containing W, out of the complete system of 120 trihedral-pair forms.

45. The 27 lines are each of them facultative; we have therefore $b'=\rho'=27$; $t'=45$; moreover each of the lines is a double tangent of the spinode curve, and therefore $\beta'\ (=2\rho')=54$.

46. The equation of the reciprocal surface is not here investigated; its form is

$$S^3-T^2=0,$$

where $S=(*\between x, y, z, w)^4$, $T=(*\between x, y, z, w)^6$; wherefore $n'=12$.

The nodal curve is composed of the lines which are the reciprocals of the original 27 lines ($b'=27$, $t'=45$ *ut suprà*). It may be remarked that the reciprocal

of a double-sixer is a double-sixer. Hence the 27 lines of the reciprocal surface may be (and that in 36 different ways) represented by

$$1, 2, 3, 4, 5, 6$$
$$1', 2', 3', 4', 5', 6'$$
$$12, 13, \ldots 56,$$

where 12 is now the line joining the points 12′ and 1′2; and so for the other lines. The lines 12, 34, 56 meet in a point 12.34.56; the 30 points 12′, 1′2 ... 56′, 5′6, and the fifteen points 12.34.56 make up the 45 points t'.

The above equation, $S^3 - T^2 = 0$, shows that the cuspidal curve is a complete intersection 6×4; $c' = 24$.

Section II = 12 − C_2.

Article Nos. 47 to 59. Equation $W(a, b, c, f, g, h)(X, Y, Z)^2 + 2kXYZ = 0$.

47. It may be remarked that the system of lines and planes is at once deduced from that belonging to I = 12, by supposing that in the double-sixer the corresponding lines 1 and 1′, &c. severally coincide; the line 12, instead of being given as the intersection of the planes 12′, 1′2, is given as the third line in the plane 12, which in fact represents the coincident planes 12′ and 1′2.

48. The diagram is

Planes.	$\Pi = 12 - C_2$	Lines. $\overline{56}$	$\overline{46}$	$\overline{45}$	$\overline{36}$	$\overline{35}$	$\overline{34}$	$\overline{26}$	$\overline{25}$	$\overline{24}$	$\overline{23}$	$\overline{16}$	$\overline{15}$	$\overline{14}$	$\overline{13}$	$\overline{12}$	6	5	4	3	2	1	
		$\overline{21}$							$15 \times 1 = 15$								$6 \times 2 = 12$						
12	$15 \times 2 = 30$															•					•	•	Biradial planes, through each pair of rays.
13															•	•				•		•	
14														•	•				•			•	
15													•	•				•				•	
16												•	•				•					•	
23											•	•								•	•		
24										•	•								•		•		
25									•	•								•			•		
26								•	•								•				•		
34							•	•											•	•			
35						•	•											•		•			
36					•	•											•			•			
45				•	•													•	•				
46			•	•													•		•				
56		•	•														•	•					
12.34.56	$15 \times 1 = 15$	•					•									•							Planes each containing three mere lines.
12.35.46			•			•										•							
12.36.45				•	•											•							
13.24.56		•								•					•								
13.25.46			•						•						•								
13.26.45				•				•							•								
14.23.56		•									•			•									
14.25.36					•				•					•									
14.26.35						•		•						•									
15.23.46			•								•		•										
15.24.36					•					•			•										
15.26.34							•	•					•										
16.23.45				•							•	•											
16.24.35						•				•		•											
16.25.34		$\overline{30}$		$\overline{45}$			•		•			•											
									Mere lines, one in each biradial ray.								Rays, through the node situate in a quadric cone.						

49. Putting the equation of the surface in the form

$$W\left(1,\ 1,\ 1,\ l+\frac{1}{l},\ m+\frac{1}{m},\ n+\frac{1}{n}\right)(X,\ Y,\ Z)^2+\frac{\alpha\beta\gamma\delta}{p}XYZ=0,$$

where for shortness

$$\begin{aligned}\alpha&=mn\ -l,\\ \beta&=nl\ \ -m,\\ \gamma&=lm\ \ -n,\\ \delta&=lmn-1,\\ p&=lmn,\end{aligned}$$

then taking $X=0$ as the equation of the plane [12], $Y=0$ as that of the plane [34], $Z=0$ as that of the plane [56], the equations of the 30 distinct planes are found to be

$$\begin{aligned}
X&=0, && [12]\\
Y&=0, && [34]\\
Z&=0, && [56]\\
m\ \ X+l\ \ Y+Z&=0, && [23]\\
m^{-1}X+l\ \ Y+Z&=0, && [24]\\
m\ \ X+l^{-1}Y+Z&=0, && [13]\\
m^{-1}X+l^{-1}Y+Z&=0, && [14]\\
X+n\ \ Y+m\ \ Z&=0, && [45]\\
X+n^{-1}Y+m\ \ Z&=0, && [46]\\
X+n\ \ Y+m^{-1}Z&=0, && [35]\\
X+n^{-1}Y+m^{-1}Z&=0, && [36]\\
n\ \ X+Y+l\ \ Z&=0, && [16]\\
n^{-1}X+Y+l\ \ Z&=0, && [15]\\
n\ \ X+Y+l^{-1}Z&=0, && [26]\\
n^{-1}X+Y+l^{-1}Z&=0, && [25]\\
W&=0, && [12.34.56]\\
X+\beta\gamma\ W&=0, && [12.36.45]\\
X-\alpha\delta\ W&=0, && [12.35.46]\\
Y+\alpha\gamma\ W&=0, && [16.25.34]\\
Y-\beta\delta\ W&=0, && [15.26.34]\\
Z+\alpha\beta\ W&=0, && [14.23.56]\\
Z-\gamma\delta\ W&=0, && [13.24.56]\\
mnX+nl\ \ Y+lm\ \ Z+\alpha\beta\gamma\delta\ W&=0, && [16.23.45]\\
pX+n\ \ Y+m\ \ Z+\beta\gamma\delta\ W&=0, && [13.26.45]\\
nX+p\ \ Y+l\ \ Z+\gamma\alpha\delta\ W&=0, && [16.24.35]\\
mX+l\ \ Y+p\ \ Z+\alpha\beta\delta\ W&=0, && [15.23.46]\\
X+lm\ Y+ln\ Z-\beta\gamma\delta\ W&=0, && [15.24.36]\\
lmX+\ \ Y+mn\ Z-\gamma\alpha\delta\ W&=0, && [13.25.46]\\
nlX+mn\ Y+\ \ Z-\alpha\beta\delta\ W&=0, && [14.26.35]\\
lX+m\ \ Y+n\ \ Z-\alpha\beta\gamma\ W&=0, && [14.25.36]
\end{aligned}$$

50. And the coordinates of the 21 distinct lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be taken to be
l	0	0	0	-1	1	(1) $X=0,\ Y+lZ=0$
0	m	0	1	0	-1	(3) $Y=0,\ Z+mX=0$
0	0	n	-1	1	0	(5) $Z=0,\ X+nY=0$
l^{-1}	0	0	0	-1	1	(2) $X=0,\ Y+l^{-1}Z=0$
0	m^{-1}	0	1	0	-1	(4) $Y=0,\ Z+m^{-1}X=0$
0	0	n^{-1}	-1	1	0	(6) $Z=0,\ X+n^{-1}Y=0$
1	n	m	0	$\frac{m}{\beta\gamma}$	$-\frac{n}{\beta\gamma}$	(45) $X+nY+mZ=0,\ X+\beta\gamma W=0$
n	1	l	$-\frac{l}{\gamma\alpha}$	0	$\frac{n}{\gamma\alpha}$	(16) $Y+lZ+nX=0,\ Y+\gamma\alpha W=0$
m	l	1	$\frac{l}{\alpha\beta}$	$-\frac{m}{\alpha\beta}$	0	(23) $Z+mX+lY=0,\ Z+\alpha\beta W=0$
1	$\frac{1}{n}$	m	0	$-\frac{m}{\alpha\delta}$	$\frac{1}{n\alpha\delta}$	(46) $X+n^{-1}Y+mZ=0,\ X-\alpha\delta W=0$
n	1	$\frac{1}{l}$	$\frac{1}{l\beta\delta}$	0	$-\frac{n}{\beta\delta}$	(26) $Y+l^{-1}Z+nX=0,\ Y-\beta\delta W=0$
$\frac{1}{m}$	l	1	$-\frac{l}{\gamma\delta}$	$\frac{1}{m\gamma\delta}$	0	(24) $Z+m^{-1}X+lY=0,\ Z-\gamma\delta W=0$
1	n	$\frac{1}{m}$	0	$-\frac{1}{m\alpha\delta}$	$\frac{n}{\alpha\delta}$	(35) $X+nY+m^{-1}Z=0,\ X-\alpha\delta W=0$
$\frac{1}{n}$	1	l	$\frac{l}{\beta\delta}$	0	$-\frac{1}{n\beta\delta}$	(15) $Y+lZ+n^{-1}X=0,\ Y-\beta\delta W=0$
m	$\frac{1}{l}$	1	$-\frac{1}{l\gamma\delta}$	$\frac{m}{\gamma\delta}$	0	(13) $Z+mX+l^{-1}Y=0,\ Z-\gamma\delta W=0$
1	$\frac{1}{n}$	$\frac{1}{m}$	0	$\frac{1}{m\beta\gamma}$	$-\frac{1}{n\beta\gamma}$	(36) $X+n^{-1}Y+m^{-1}Z=0,\ X+\beta\gamma W=0$
$\frac{1}{n}$	1	$\frac{1}{l}$	$-\frac{1}{l\gamma\alpha}$	0	$\frac{1}{n\gamma\alpha}$	(25) $Y+l^{-1}Z+n^{-1}X=0,\ Y+\gamma\alpha W=0$
$\frac{1}{m}$	$\frac{1}{l}$	1	$\frac{1}{l\alpha\beta}$	$-\frac{1}{m\gamma\alpha}$	0	(14) $Z+m^{-1}X+l^{-1}Y=0,\ Z+\alpha\beta W=0$
1	0	0	0	0	0	(12) $X=0,\ W=0$
0	1	0	0	0	0	(34) $Y=0,\ W=0$
0	0	1	0	0	0	(56) $Z=0,\ W=0$

51. The six nodal rays are not, the fifteen mere lines are, facultative. Hence

$$b' = \rho' = 15\,;\quad t' = 15.$$

52. Resuming the equation $W(a, b, c, f, g, h \between X, Y, Z)^2 + 2kXYZ = 0$, the equation of the Hessian surface is found to be

$$\begin{aligned}
&KW^2(a, b, c, f, g, h \between X, Y, Z)^2 \\
&+ 2kW\{(a, b, c, f, g, h \between X, Y, Z)^2(FX + GY + HZ) - 3KXYZ\} \\
&- k^2\{a^2X^4 + b^2Y^4 + c^2Z^4 - 2bcY^2Z^2 - 2caZ^2X^2 - 2abX^2Y^2 \\
&\qquad - 4XYZ[(af + gh)X + (bg + hf)Y + (ch + fg)Z]\} = 0,
\end{aligned}$$

where

$$\begin{aligned}
(A, B, C, F, G, H) &= (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch), \\
K &= abc - af^2 - bg^2 - ch^2 + 2fgh.
\end{aligned}$$

The Hessian and the cubic intersect in an indecomposable curve, which is the spinode curve; that is, spinode curve is a complete intersection 3×4; $\sigma' = 12$.

The equations of the spinode curve may be written in the simplified form

$$\begin{aligned}
&W(a, b, c, f, g, h \between X, Y, Z)^2 + 2kXYZ = 0, \\
&- 8KXYZW \\
&+ 8kXYZ(afX + bgY + chZ) \\
&- k^2\{a^2X^4 + b^2Y^4 + c^2Z^4 - 2bcY^2Z^2 - 2caZ^2X^2 - 2abX^2Y^2\} = 0\,;
\end{aligned}$$

and it appears hereby that the node C_2 is a sixfold point on the curve, the tangents of the curve in fact coinciding with the six rays.

Each of the 15 lines touches the spinode curve twice; in fact, for the line 12 we have $X = 0$, $W = 0$; and substituting in the equations of the spinode curve, we have $(bY^2 - cZ^2) = 0$; that is, we have the two points of contact $X = 0$, $W = 0$, $Y\sqrt{b} = \pm Z\sqrt{c}$. Hence $\beta' = 30$.

Reciprocal Surface.

53. The equation is found by equating to zero the discriminant of the ternary cubic function

$$(Xx + Yy + Zz)(a, b, c, f, g, h \between X, Y, Z)^2 - 2kwXYZ,$$

viz. the discriminant contains the factor w^2 which is to be thrown out, thus reducing the order to $n' = 10$.

The ternary cubic, multiplying by 3 to avoid fractions, is

$$X^3,\ Y^3,\ Z^3,\ 3Y^2Z,\ 3Z^2X,\ 3X^2Y,\ 3YZ^2,\ 3ZX^2,\ 3XY^2,\ 6XYZ,$$
$$3ax,\ 3by,\ 3cz,\ bz + 2fy,\ cx + 2gz,\ ay + 2hx,\ cy + 2fz,\ az + 2gx,\ bx + 2hy,\ fx + gy + hz - kw.$$

Write as before (A, B, C, F, G, H) for the inverse coefficients $(A = bc - f^2$, &c.), and $K = abc - af^2 - bg^2 - ch^2 + 2fgh$; and moreover

$$\begin{aligned}
\Phi &= (A, B, C, F, G, H \between x, y, z)^2,\\
P &= Ax + Hy + Gz,\\
Q &= Hx + By + Fz,\\
R &= Gx + Fy + Cz,\\
t &= fx + gy + hz,\\
U &= afyz + bgzx + chxy,\\
V &= 2Kxyz - aPyz - bQzx - cRxy\\
&= -a\,Hy^2z - b\,Fz^2x - c\,Gx^2y\\
&\quad - a\,Gyz^2 - b\,Hzx^2 - c\,Fxy^2\\
&\quad + (-abc - af^2 - bg^2 - ch^2 + 4fgh)\,xyz,\\
W &= (A, B, C, F, G, H \between ayz, bzx, cxy)^3,\\
L &= k^2w^2 - 2ktw - \Phi,\\
M &= kw\,U + V,\\
N &= 2kabc\,xyzw + W:
\end{aligned}$$

54. Then the invariants of the ternary cubic are

$$\begin{aligned}
S &= L^2 - 12kw\,M,\\
T &= L^3 - 18kw\,LM - 54k^2w^2N;
\end{aligned}$$

and the required equation of the reciprocal surface is

$$\frac{1}{108w^2}\{(L^2 - 12kwM)^3 - (L^3 - 18kwLM - 54k^2w^2N)^2\} = 0,$$

viz. this is

$$\begin{aligned}
0 = \quad & L^3N && = (k^2w^2 - 2ktw - \Phi)^3\,(2kabc\,xyzw + W)\\
& + L^2M^2 && + (k^2w^2 - 2ktw - \Phi)^2\,(kwU + V)^2\\
& - 18kwLMN && - 18kw\,(k^2w^2 - 2ktw - \Phi)\,(kwU + V)\,(2kabc\,xyzw + W)\\
& - 16kwM^3 && - 16kw\,(kwU + V)^3\\
& - 27k^2w^2N^2 && - 27k^2w^2\,(2kabc\,xyzw + W)^2,
\end{aligned}$$

which, arranged in powers of kw, is as follows; viz. we have

$$\begin{aligned}
\text{Coeff. } (kw)^7 &= 2abc\,xyz,\\
(kw)^6 &= 2abc\,xyz\,(-6t) + W\\
&\quad + U^2,\\
(kw)^5 &= 2abc\,xyz\,(-3\Phi + 12t^2) + W\,(-6t)\\
&\quad + U^2\,(-4t) + 2UV\\
&\quad - 36abc\,xyz\,U,
\end{aligned}$$

$$
\begin{aligned}
(kw)^4 = \quad & 2abc\,xyz\,(12t\Phi - 8t^3) + W(-3\Phi + 12t^2) \\
& + U^2(-2\Phi + 4t^3) + 2UV(-4t) + V^2 \\
& - 36abc\,xyz\,V - 18UW + 72abc\,xyztU \\
& - 16U^3 \\
& - 108a^2b^2c^2x^2y^2z^2,
\end{aligned}
$$

$$
\begin{aligned}
(kw)^3 = \quad & 2abc\,xyz\,(3\Phi^2 - 12t^2\Phi) + W(12t\Phi - 8t^3) \\
& + U^2 4t\Phi + 2UV(-2\Phi + 4t^2) + V^2(-4t) \\
& - 18VW + 72abc\,xyztV + 36tUW - 36abc\,xyz\Phi U \\
& - 48U^2V \\
& - 108abc\,xyz\,W,
\end{aligned}
$$

$$
\begin{aligned}
\text{,,}\quad (kw)^2 = \quad & 2abc\,xyz\,(-6t\Phi^2) + W(3\Phi^2 - 12t^2\Phi) \\
& + U^2\Phi^2 + 2UV(4t\Phi) + V^2(-2\Phi + 4t^2) \\
& + 36tVW - 36abc\,xyz\Phi V - 18\Phi UW \\
& - 48UV^2 \\
& - 27W^2,
\end{aligned}
$$

$$
\begin{aligned}
\text{,,}\quad (kw)^1 = \quad & 2abc\,xyz\,(-\Phi^3) + W(-6t\Phi)^2 \\
& + 2UV\Phi^2 + V^2(4t\Phi) \\
& - 18\Phi VW \\
& - 16V^3,
\end{aligned}
$$

$$
\begin{aligned}
\text{,,}\quad (kw)^0 = \quad & W(-\Phi^3) \\
& + V^2\Phi^2;
\end{aligned}
$$

but I have not carried the ultimate reduction further than in Schläfli, viz. I give only the terms in $(kw)^7$, $(kw)^6$, $(kw)^5$, and $(kw)^0$.

55. I present the result as follows; the coefficients deducible from those which precede, by mere cyclical permutations of the letters a, b, c and f, g, h, are indicated by (,,).

$0 = (kw)^7 \,.\, 2abc\,xyz$

	y^2z^2	z^2x^2	x^2y^2	x^2yz	xy^2z	xyz^2
$+(kw)^6 \,.$	$a^2bc + 1$	,,	,,	$abcf - 14$ $gebh + 4$	,,	,,

	y^3z^2	y^2z^3	z^3x^2	z^2x^3	x^3y^2	x^2y^3	x^3yz	xy^3z	xyz^3	xy^2z^2	x^2yz^2	x^2y^2z
$+(kw)^5 \,.$	$a^2bcg - 6$ $a^2cfh + 2$	$a^2bch - 6$ $a^2bfg + 2$	,,	,,	,,	,,	$ab^2c^2 - 6$ $abcf^2 + 42$ $b^2cg^2 + 2$ $bc^2h^2 + 2$ $bcfgh - 24$	,,	,,	$a^2bcf - 32$ $abcfgh + 64$ $abfg^2 - 24$ $acfh^2 - 24$ $af^2gh + 8$	,,	,,

$\vdots$

$$+(kw)^0 \,.\, -K[(A, B, C, F, G, H \between x, y, z)^2]^2 (cy^2 - 2fyz + bz^2)(az^2 - 2gzx + bx^2)(bx^2 - 2hxy + ay^2).$$

56. In explanation of the discussion of the reciprocal surface, it is convenient to remark that we have

Node C_2, $X=0$, $Y=0$, $Z=0$.

Tangent cone is

$$(a, b, c, f, g, h \between X, Y, Z)^2 = 0.$$

Nodal rays are sections of cone by planes $X=0$, $Y=0$, $Z=0$ respectively, viz. equations of the rays are

$$\begin{aligned} X=0,\quad & bY^2+2fYZ+cZ^2=0,\\ Y=0,\quad & cZ^2+2gZX+aX^2=0,\\ Z=0,\quad & aX^2+2hXY+bY^2=0. \end{aligned}$$

Reciprocal plane is $w=0$.

Conic of contact is

$$(A, B, C, F, G, H \between x, y, z)^2 = 0,\ w=0.$$

Lines are tangents of this conic from points

$$(y=0,\ z=0),\ (z=0,\ x=0),\ (x=0,\ y=0)$$

respectively, viz. equations are

$$\begin{aligned} w=0,\quad & cy^2-2fyz+bz^2=0,\\ w=0,\quad & az^2-2gzx+cx^2=0,\\ w=0,\quad & bx^2-2hxy+ay^2=0. \end{aligned}$$

57. The equation shows that the section by the plane $w=0$ is made up of the conic $(A, B, C, F, G, H \between x, y, z)^2=0$, twice, and of the six lines, tangents to this conic, viz. the lines

$$\begin{aligned} w=0,\quad & cy^2-2fyz+bz^2=0,\\ w=0,\quad & az^2-2gzx+cx^2=0,\\ w=0,\quad & bx^2-2hxy+ay^2=0, \end{aligned}$$

each once; the lines in question (reciprocals of the nodal rays) are thus mere scrolar lines on the reciprocal surface.

58. I do not attempt to put in evidence the nodal curve of the surface; by what precedes it is made up of 15 lines, intersecting 3 together in 15 points; and if we denote the six tangents of the conic just referred to by

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

then the fifteen lines are respectively lines passing through the intersections of each pair of these tangents; viz. through the intersection of the tangents 1 and 2, we have a line 12; and so in other cases; that is, the 15 lines are 12, 13 .... 56. The lines 12 and 34 meet; and the lines 12, 34, 56 meet in a point; we have thus the 15 points 12.34.56, triple points of the nodal curve.

59. As regards the cuspidal curve, the equation of the surface may be written

$$\begin{aligned} &(L^2-12kwM)(4M^2+3LN)-(LM+9kwN)^2\\ &\quad=3(L^2M^2+L^3N-18kwLMN-16kwM^3-27k^2w^2N^2)=0, \end{aligned}$$

and we thus have

$$\begin{aligned} 4M^2+3LN&=0,\\ LM+9kwN&=0,\\ L^2-12kwM&=0, \end{aligned}$$

or, what is the same thing,

$$\left\| \begin{matrix} L, & 12M, & -9N \\ kw, & L, & M \end{matrix} \right\| = 0$$

(equivalent to two equations) for the equations of the cuspidal curve. Attending to the second and third equations, the cuspidal curve may be considered as the residual intersection of the quartic and quintic surfaces $L^2-12kwM=0$, $LM+9kwN=0$, which partially intersect in the conic $w=0$, $L=0$; or say it is a curve $4\times5-2$; $c'=18$.

Section III $= 12 - B_3$.

Article Nos. 60 to 72. Equation $2W(X+Y+Z)(lX+mY+nZ)+2kXYZ=0$.

60. The system of lines and planes is at once deduced from that belonging to $\mathrm{II} = 12 - C_2$, by supposing the tangent cone to reduce itself to the pair of biplanes; 3 of the planes (a) of $\mathrm{II} = 12 - C_2$ thus coming to coincide with the one biplane, and three of them with the other biplane.

61. The diagram is

III $= 12 - B_3$		Lines. 36	35	34	26	25	24	16	15	14	6	5	4	3	2	1	
		$\overline{15}$ $\overline{27}$				$9\times1=9$							$6\times3=18$				
123	$2\times6=12$													•	•	•	Biplanes.
456											•	•	•	•	•	•	
14	$9\times3=27$									•			•			•	Biradial planes each containing a ray 1, 2, or 3 of the one biplane, and a ray 4, 5, or 6 of the other biplane.
15									•			•				•	
16								•			•					•	
24							•						•		•		
25						•						•			•		
26					•						•				•		
34				•									•	•			
35			•									•		•			
36		•									•			•			
14.25.36	$6\times1=6$	•				•				•							Planes each containing three mere lines.
14.26.35			•		•					•							
15.26.34				•	•				•								
15.24.36		•					•		•								
16.24.35			•				•	•									
16.25.34	$\overline{17}$ $\overline{45}$			•		•		•									
		Merelines in each biradial plane, one.									Rays 1, 2, 3 and 4, 5, 6, in the two biplanes respectively.						

62. Taking $X+Y+Z=0$ for the biplane that contains the rays 1, 2, 3, and $lX+mY+nZ=0$ for that which contains the rays 4, 5, 6, we may take $X=0$, $Y=0$, $Z=0$ for the equations of the planes [14], [25], [36] respectively; and then writing for shortness

$$m-n,\ n-l,\ l-m=\lambda,\ \mu,\ \nu,$$

and assuming, as we may do, $k=\lambda\mu\nu$, so that the equation of the surface is

$$W(X+Y+Z)(lX+mY+nZ)+(m-n)(n-l)(l-m)XYZ=0,$$

the equations of the 17 distinct planes are

$X=0,$	[14]
$Y=0,$	[25]
$Z=0,$	[36]
$X+Y+Z=0,$	[123]
$lX+mY+nZ=0,$	[456]
$lX+nY+nZ=0,$	[15]
$lX+nY+nZ=0,$	[16]
$lX+mY+lZ=0,$	[25]
$nX+mY+nZ=0,$	[26]
$mX+mY+nZ=0,$	[35]
$lX+lY+nZ=0,$	[36]
$W=0,$	[14.25.36]
$W+l\lambda X=0,$	[14.26.35]
$W+m\mu Y=0,$	[16.25.34]
$W+n\nu Z=0,$	[15.24.36]
$lmX+mnY+nlZ+W=0,$	[15.26.34]
$nlX+lmY+mnZ-W=0,$	[16.24.35]

63. And the coordinates of the fifteen distinct lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be written
0	0	0	0	-1	1	(1) $X=0,\ Y+Z=0$
0	0	0	1	0	-1	(2) $Y=0,\ Z+X=0$
0	0	0	-1	1	0	(3) $Z=0,\ X+Y=0$
0	0	0	0	$-n$	m	(4) $X=0,\ mY+nZ=0$
0	0	0	n	0	$-l$	(5) $Y=0,\ nZ+lX=0$
0	0	0	$-m$	l	0	(6) $Z=0,\ lX+mY=0$
1	0	0	0	0	0	(14) $X=0,\ W=0$
0	1	0	0	0	0	(25) $Y=0,\ W=0$
0	0	1	0	0	0	(36) $Z=0,\ W=0$
l	n	n	$n^2\nu$	$-nl\nu$	0	(15) $lX+nY+nZ=0,\ W+n\nu Z=0$
l	m	m	$-m^2\mu$	0	$lm\mu$	(16) $lX+mY+mZ=0,\ W+m\mu Y=0$
l	m	l	0	$l^2\lambda$	$-lm\mu$	(26) $lX+mY+lZ=0,\ W+l\lambda X=0$
n	m	n	$mn\nu$	$-n^2\nu$	0	(24) $nX+mY+nZ=0,\ W+n\nu Z=0$
m	m	n	$-mn\mu$	0	$m^2\mu$	(34) $mX+mY+nZ=0,\ W+m\mu Y=0$
l	l	n	0	$nl\lambda$	$-l^2\lambda$	(35) $lX+lY+nZ=0,\ W+l\lambda X=0$

64. The rays are not, the mere lines are, facultative; hence $b'=\rho'=9:\ t'=6$.

65. The equation of the Hessian surface is

$$\begin{aligned}&-W(X+Y+Z)(lX+mY+nZ)(\mu\nu X+\nu\lambda Y+\lambda\mu Z)\\&-k(l^2X^4+m^2Y^4+n^2Z^4-2mnY^2Z^2-2nlZ^2X^2-2lmX^2Y^2)\\&+kXYZ\{(l^2+3lm+3ln+mn)X+(m^2+3mn+3ml+nl)Y+(n^2+3nl+3nm+lm)Z\}=0.\end{aligned}$$

The Hessian and cubic surfaces intersect in an indecomposable curve, which is the spinode curve; that is, spinode curve is a complete intersection 3×4; $\sigma'=12$.

The equations may be written in the simplified form

$$\begin{aligned}&W(X+Y+Z)(lX+mY+nZ)+kXYZ=0,\\&l^2X^4+m^2Y^4+n^2Z^4-2mnY^2Z^2-2nlZ^2X^2-2lmX^2Y^2\\&\qquad-4XYZ\{l(m+n)X+m(n+l)Y+n(l+m)Z\}=0.\end{aligned}$$

We may also obtain the equation

$$\begin{aligned}&k^2(X+Y+Z)(lX+mY+nZ)\{lX^2+mY^2+nZ^2-(m+n)YZ-(n+l)ZX-(l+m)XY\}\\&\qquad+\lambda^2Y^2Z^2+\mu^2Z^2X^2+\nu^2X^2Y^2-2XYZ(\mu\nu X+\nu\lambda Y+\lambda\mu Z)=0,\end{aligned}$$

which shows that there is at B_3 an eightfold point, the tangents being given by

$$(X+Y+Z)(lX+mY+nZ)=0,$$

$$(\lambda^2,\ \mu^2,\ \nu^2,\ -\mu\nu,\ -\nu\lambda,\ -\lambda\mu \between YZ,\ ZX,\ XY)^2=0.$$

Each of the facultative lines is a double tangent of the spinode curve; whence $\beta'=18$.

Reciprocal Surface.

66. The equation may be deduced from that for $\mathrm{II}=12-C_2$, viz. writing

$$(a,\ b,\ c,\ f,\ g,\ h \between X,\ Y,\ Z)^2=2(X+Y+Z)(lX+mY+nZ),$$

that is

$$(a,\ b,\ c,\ f,\ g,\ h) \quad = \quad (2l,\ 2m,\ 2n,\ m+n,\ n+l,\ l+m),$$

we have

$$(A,\ B,\ C,\ F,\ G,\ H)=-(\lambda^2,\ \ \mu^2,\ \ \nu^2,\ \ \mu\nu\ ,\ \ \nu\lambda\ ,\ \ \lambda\mu\ );\ K=0.$$

Writing also

$$\lambda,\ \mu,\ \nu=m-n,\ n-l,\ l-m \text{ as before,}$$

$$\lambda x+\mu y+\nu z=\sigma,$$

$$lmn\,xyz=\theta,$$

$$l(m+n)\,yz+m(n+l)\,zx+n(l+m)\,xy=v,$$

$$l\lambda\,yz+\quad m\mu\,zx+\quad n\nu\,xy=\psi,$$

$$(m+n)\,x+\quad (n+l)\,y+\quad (l+m)\,z=t,$$

we have

$$U=2v,\quad V=2\sigma\psi,\quad W=-4\psi^2,$$

and then

$$L=k^2w^2-2ktw+\sigma^2,\quad M=2(kwv+\sigma\psi),\quad N=4(4lmnkxyzw-\psi^2);$$

so that the equation is

$$\begin{array}{rcl}
0= \quad L^3N & = & 4(k^2w^2-2kwt+\sigma^2)^3(4kw\theta-\psi^2) \\
+L^2M^2 & + & 4(k^2w^2-2kwt+\sigma^2)^2(kwv+\sigma\psi)^2 \\
-18kwLMN & & -144\ kw(k^2w^2-2kwt+\sigma^2)(kwv+\sigma\psi)(kw\theta-\psi^2) \\
-16kw\quad M^3 & & -128\ kw(kwv+\sigma\psi)^3 \\
-27k^2w^2\ N^2 & & -432\ k^2w^2(kw\theta-\psi^2);
\end{array}$$

or reducing the first two terms so as to throw out from the whole equation the factor kw, the equation is

$$4L^3\{\theta L+(v^2-\psi^2)kw+2\psi(t\psi+v\sigma)\}-18LMN-16M^3-27kwN^2=0$$

or, what is the same thing, it is

$$\begin{aligned}&(k^2w^2-2kwt+\sigma^2)^2\{k^2w^2\theta+kw(-2t\theta+v^2-\psi^2)+\sigma^2\theta+2\sigma v\psi+2t\psi^2\}\\&\quad-36(k^2w^2-2kwt+\sigma^2)(kwv+\sigma\psi)(4kw\theta-\psi^2)\\&\quad-32(kwv+\sigma\psi)^3\\&\quad-108kw(4kw\theta-\psi^2)^2=0.\end{aligned}$$

67. This is

$$\begin{aligned}&(kw)^6.\quad\theta\\&+(kw)^5.-\psi^2-6t\theta+v^2\\&+(kw)^4.\quad\sigma^2.3\theta+\sigma\psi.2v+\psi^2.6t+12t^2\theta-4tv^2-144\theta v\\&+(kw)^3.-2\sigma^2\psi^2+\sigma^2(2v^2-10t\theta)+\sigma\psi(-8tv-144\theta)+\psi^2(-12t^2+36v)\\&\qquad-8t^3\theta+4t^2v^2+288tv\theta-32v^3-1728\theta^2\\&+(kw)^2.\quad\sigma^4.3\theta+\sigma^3\psi.4v+\sigma^2\psi^2.12t+\sigma\psi^3.37\\&\qquad+\sigma^2(12t^2\theta-4tv^2-144\theta v)+\sigma\psi(+8t^2v+288t\theta-96v^2)+\psi^2(8t^3-72tv+864\theta)\\&+(kw).-\sigma^4\psi^2+\sigma^4(-6t\theta+v^2)+\sigma^2\psi(-8tv-144\theta)\\&\qquad+\sigma^2\psi^2(-8t-90v)+\sigma\psi^3.-72+\psi^4.-108\\&+(kw)^0.\quad\sigma^3(\theta,\ 2v,\ 2t,\ 4\between\sigma,\ \psi)^3=0,\end{aligned}$$

which, reducing the last term, is

$$\begin{aligned}&(kw)^6\,lmnxyz\\&\quad\vdots\\&\quad-4\sigma^3\lambda\mu\nu(y-z)(z-x)(x-y)(ny-mz)(lz-nx)(mx-ly)=0.\end{aligned}$$

68. I verify the last term in the particular case $z=0$ as follows: the coefficient of σ^3 is

$$(0,\ 2n(l+m)xy,\ 2(m+n)x+2(n+l)y,\ 4\between\lambda x+\mu y,\ \nu xy)^3,$$

which is

$$\begin{aligned}&=2n^2\nu x^2y^2\{(l+m)(\lambda x+\mu y)^2+[(m+n)x+(n+l)y](\nu\lambda x+\mu\nu y)+2n\nu^2xy\}\\&=2n^2\nu x^2y^2\{[(l+m)\lambda+(m+n)\nu]\lambda x^2\\&\qquad+[2(l+m)\lambda\mu+(m+n)\mu\nu+(n+l)\nu\lambda+2n\nu]xy\\&\qquad+[(l+m)\mu+(n+l)\nu]\mu y^2\},\end{aligned}$$

which, substituting for λ, μ, ν their values $m-n$, $n-l$, $l-m$, is

$$=2n^2\nu x^2y^2.-2\lambda\mu(x-y)(mx-ly);$$

or for $z=0$ the coefficient of σ^3 is

$$=-4\lambda\mu\nu\,n^2x^2y^2(x-y)(mx-ly),$$

agreeing with the general value

$$-4\lambda\mu\nu(y-z)(z-x)(x-y)(ny-mz)(lz-nx)(lx-my).$$

69. In the discussion of the equation it is convenient to write down the relations of the two surfaces, thus:

Cubic surface.	Reciprocal surface.
B_3, $X=0,\ Y=0,\ Z=0$	Plane $w=0$,
Biplanes $X+\ Y+\ Z=0$	Points in $w=0$, viz.
$lX+mY+nZ=0$,	$x=y=z$ and $x:y:z=l:m:n$,
intersecting in edge.	in line $(m-n)x+(n-l)y+(l-m)z=0$,
	that is, $\lambda x+\mu y+\nu z=0$, or $\sigma=0$.
Rays in first biplane,	Lines in plane $w=0$, and through first point, viz.
$X=0,\ Y+Z=0;\ Y=0,\ Z+X=0,$	$y-z=0,\ z-x=0,\ x-y=0;$
$Z=0,\ X+Y=0;$	lines through second biplane, viz.
rays in second biplane,	$ny-mz=0,\ nz-lx=0,\ lx-my=0.$
$X=0,\ mY+nZ=0;\ Y=0,\ nZ+lX=0,$	
$Z=0,\ lX+mY=0.$	

70. The equation puts in evidence the section by the plane $w=0$, viz. this is the line $\sigma=0$ (reciprocal of the edge) three times, and the six lines (reciprocals of the rays) each once. Observe that the edge is *not* a line on the cubic; but its reciprocal is a line, and that an oscular line on the reciprocal surface; the six lines (reciprocals of the rays) are mere scrolar lines on the reciprocal surface; they pass, three of them, through the point $x=y=z$, and the other three through the point $x:y:z=l:m:n$; that is, they are six tangents of the point-pair (reciprocal of the pair of biplanes) formed by these two points.

71. I do not attempt to put in evidence the nodal curve on the surface; by what precedes it consists of 9 lines, reciprocals of the mere lines. If we denote by 1, 2, 3 and 4, 5, 6 the lines which pass through the points $x=0$, $y=0$, $z=0$ and through the point $x:y:z=l:m:n$ respectively, then these intersect in the nine points 14, 15, 16, 24, 25, 26, 34, 35, 36; and through each of these there passes a nodal line which may be represented by the same symbol; that is, we have the nodal lines 14, ... 36. Two lines such as 14, 25 meet; and three lines such as 14, 25, 36 meet in a point; we have thus the six points 14.25.36 &c. triple points on the nodal curve; as before, $b'=9$, $t'=6$.

72. The cuspidal curve is given by the equations

$$\left\| \begin{matrix} k^2w^2-2kwt+\sigma^2, & 24(kwv+\sigma\psi), & -36(4lmnk\,xyzw-\psi^2) \\ kw\quad, & k^2w^2-2kwt+\sigma^2, & 2(kwv+\sigma\psi) \end{matrix} \right\| = 0.$$

Writing down the two equations,

$$(k^2w^2-2kwt+\sigma^2)^2-24kw(kwv+\sigma\psi)=0,$$
$$(k^2w^2-2kwt+\sigma^2)(kwv+\sigma\psi)+18w(lmnk\,xyzw-\psi^2)=0,$$

these are respectively of the orders 4 and 5; but they intersect in the line $w=0$, $\sigma=0$ taken four times, or say, the cuspidal curve is a partial intersection $4.5-4$; $c'=16$.

Section IV = 12 − 2C_2.

Article Nos. 73 to 84. Equation $WXZ + Y^2(\gamma Z + \delta W) + (a, b, c, d \between X, Y)^2 = 0$.

73. The diagram of the lines is

Planes.	IV = 12 − 2C_2.		Lines. 34 . 12′	14 . 23′	23 . 14′	24 . 13′	13 . 24′	12 . 34′	4′	3′	2′	1′	4	3	2	1	5	0	
			$\overline{12}$ 97	6 × 1 = 6					8 × 2 = 16								1 × 1 = 1	1 × 4 = 4	
[0]	1 × 2 = 2																·	· ·	Plane touching along axis.
11′	4 × 4 = 16															·		·	Planes through axis, each containing a ray of the one node and a ray of the other node.
22′															·	·		·	
33′														·	·			·	
44′													·	·				·	
12	12 × 2 = 24							·							·	·			Biradial planes, each containing two rays of the one node or two rays of the other node.
13							·	·						·		·			
14						·		·					·			·			
23					·							·			·				
24				·							·		·		·				
34			·	·						·			·	·					
1′2′			·									·							
1′3′				·							·								
1′4′				·						·									
2′3′					·					·									
2′4′					·				·										
3′4′						·			·										
12 . 34	3 × 1 = 3		·					·									·		Planes each containing three mere lines.
13 . 24				·			·										·		
14 . 23		20 45			·	·											·		
			More lines, each the intersection of a radial plane of this one node and a bi-radial plane of the other node.						Rays 1, 2, 3, 4, and 1′, 2′, 3′, 4′ through the two nodes respectively.								Transversal in the tangent plane along the axis.	Axis through the two nodes.	

74. Writing $X(a, b, c, d \between X, Y)^3 - \gamma\delta Y^4 = -\gamma\delta(f_1X - Y)(f_2X - Y)(f_3X - Y)(f_4X - Y)$, the 20 planes are

$$X = 0, \qquad [\ 0]$$
$$X - f_1Y = 0, \qquad [11']$$
$$X - f_2Y = 0, \qquad [22']$$
$$X - f_3Y = 0, \qquad [33']$$
$$X - f_4Y = 0, \qquad [44']$$

$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_2)Y\} - \mathrm{f}_1\mathrm{f}_2Z = 0, \qquad [12]$$
$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_3)Y\} - \mathrm{f}_1\mathrm{f}_3Z = 0, \qquad [13]$$
$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_4)Y\} - \mathrm{f}_1\mathrm{f}_4Z = 0, \qquad [14]$$
$$\delta\{X - (\mathrm{f}_2 + \mathrm{f}_3)Y\} - \mathrm{f}_2\mathrm{f}_3Z = 0, \qquad [23]$$
$$\delta\{X - (\mathrm{f}_2 + \mathrm{f}_4)Y\} - \mathrm{f}_2\mathrm{f}_4Z = 0, \qquad [24]$$
$$\delta\{X - (\mathrm{f}_3 + \mathrm{f}_4)Y\} - \mathrm{f}_3\mathrm{f}_4Z = 0, \qquad [34]$$

$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_2)Y\} - \mathrm{f}_1\mathrm{f}_2W = 0, \qquad [1'2']$$
$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_3)Y\} - \mathrm{f}_1\mathrm{f}_3W = 0, \qquad [1'3']$$
$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_4)Y\} - \mathrm{f}_1\mathrm{f}_4W = 0, \qquad [1'4']$$
$$\gamma\{X - (\mathrm{f}_2 + \mathrm{f}_3)Y\} - \mathrm{f}_2\mathrm{f}_3W = 0, \qquad [2'3']$$
$$\gamma\{X - (\mathrm{f}_2 + \mathrm{f}_4)Y\} - \mathrm{f}_2\mathrm{f}_4W = 0, \qquad [2'4']$$
$$\gamma\{X - (\mathrm{f}_3 + \mathrm{f}_4)Y\} - \mathrm{f}_3\mathrm{f}_4W = 0, \qquad [3'4']$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_2} + \frac{1}{\mathrm{f}_3\mathrm{f}_4}\right)X + dy + \gamma Z + \delta W = 0, \qquad [12.34]$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_3} + \frac{1}{\mathrm{f}_2\mathrm{f}_4}\right)X + dy + \gamma Z + \delta W = 0, \qquad [13.24]$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_4} + \frac{1}{\mathrm{f}_2\mathrm{f}_3}\right)X + dy + \gamma Z + \delta W = 0, \qquad [14.23]$$

75. And the 16 lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be written	
0	0	0	0	0	1	(0)	$X = 0,\ Y = 0$
δ	0	0	0	$-\gamma$	d	(5)	$X = 0,\ dY + \gamma Z + \delta W = 0$
0	0	0	f_1^2	f_1	$-\delta$	(1)	$X = \mathrm{f}_1Y = 0,\ \delta Y + \mathrm{f}_1Z = 0$
0	0	0	f_2^2	f_2	$-\delta$	(2)	,, ,,
0	0	0	f_3^2	f_3	$-\delta$	(3)	,, ,,
0	0	0	f_4^2	f_4	$-\delta$	(4)	,, ,,

(a)	(b)	(c)	(f)	(g)	(h)		
$\mathfrak{f}_1$	$-\mathfrak{f}_1^2$	0	0	0	γ	(1′)	$X-\mathfrak{f}_1 Y=0,\ \gamma Y+\mathfrak{f}_1 W=0$
$\mathfrak{f}_2$	$-\mathfrak{f}_2^2$	0	0	0	γ	(2′)	,, ,,
$\mathfrak{f}_3$	$-\mathfrak{f}_3^4$	0	0	0	γ	(3′)	,, ,,
$\mathfrak{f}_4$	$-\mathfrak{f}_4^2$	0	0	0	γ	(4′)	,, ,,
$-\dfrac{\delta}{\mathfrak{f}_1\mathfrak{f}_2}$	$\delta\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_2}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_3}+\dfrac{1}{\mathfrak{f}_4}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_3\mathfrak{f}_4}$	$\dfrac{1}{\mathfrak{f}_3\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_2}\right)-\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_2}\left(\dfrac{1}{\mathfrak{f}_3}+\dfrac{1}{\mathfrak{f}_4}\right)$	(12 . 3′4′)*
$-\dfrac{\delta}{\mathfrak{f}_1\mathfrak{f}_3}$	$\delta\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_3}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_4}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_2\mathfrak{f}_4}$	$\dfrac{1}{\mathfrak{f}_2\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_3}\right)-\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_3}\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_4}\right)$	(13 . 2′4′) ,,
$-\dfrac{\delta}{\mathfrak{f}_1\mathfrak{f}_4}$	$\delta\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_4}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_3}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_2\mathfrak{f}_3}$	$\dfrac{1}{\mathfrak{f}_2\mathfrak{f}_3}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_4}\right)-\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_3}\right)$	(14 . 2′3′) ,,
$-\dfrac{\delta}{\mathfrak{f}_2\mathfrak{f}_3}$	$\delta\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_3}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_4}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_1\mathfrak{f}_4}$	$\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_3}\right)-\dfrac{1}{\mathfrak{f}_2\mathfrak{f}_3}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_4}\right)$	(23 . 1′4′) ,,
$-\dfrac{\delta}{\mathfrak{f}_2\mathfrak{f}_4}$	$\delta\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_4}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_3}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_1\mathfrak{f}_3}$	$\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_3}\left(\dfrac{1}{\mathfrak{f}_2}+\dfrac{1}{\mathfrak{f}_4}\right)-\dfrac{1}{\mathfrak{f}_2\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_3}\right)$	(24 . 1′3′) ,,
$-\dfrac{\delta}{\mathfrak{f}_3\mathfrak{f}_4}$	$\delta\left(\dfrac{1}{\mathfrak{f}_3}+\dfrac{1}{\mathfrak{f}_4}\right)$	1	$-\gamma\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_2}\right)$	$-\dfrac{\gamma}{\mathfrak{f}_1\mathfrak{f}_2}$	$\dfrac{1}{\mathfrak{f}_1\mathfrak{f}_2}\left(\dfrac{1}{\mathfrak{f}_3}+\dfrac{1}{\mathfrak{f}_4}\right)-\dfrac{1}{\mathfrak{f}_3\mathfrak{f}_4}\left(\dfrac{1}{\mathfrak{f}_1}+\dfrac{1}{\mathfrak{f}_2}\right)$	(34 . 1′2′) ,,

*equations are

$$\delta\{X-(\mathfrak{f}_1+\mathfrak{f}_2)Y\}-\mathfrak{f}_1\mathfrak{f}_2 Z=0,\quad \gamma\{X-(\mathfrak{f}_3+\mathfrak{f}_4)Y\}-\mathfrak{f}_3\mathfrak{f}_4 W=0,$$

[and similarly for each of the remaining five lines].

76. To verify the equations of the line 12 . 3′4′, observe that the two equations give

$$\gamma Z+\delta W=\gamma\delta\left\{X\left(\frac{1}{\mathfrak{f}_1\mathfrak{f}_2}+\frac{1}{\mathfrak{f}_3\mathfrak{f}_4}\right)-Y\left(\frac{1}{\mathfrak{f}_1}+\frac{1}{\mathfrak{f}_2}+\frac{1}{\mathfrak{f}_3}+\frac{1}{\mathfrak{f}_4}\right)\right\},$$

$$ZW=\frac{\gamma\delta}{\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_3\mathfrak{f}_4}\{X-(\mathfrak{f}_1+\mathfrak{f}_2)YX-(\mathfrak{f}_3+\mathfrak{f}_4)Y\}:$$

the equation of the surface, multiplying by X and observing that $-\gamma\delta=a\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_3\mathfrak{f}_4$, becomes

$$X^2ZW+XY^2(\gamma Z+\delta W)+\gamma\delta Y^4-\frac{\gamma\delta}{\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_3\mathfrak{f}_4}(X-\mathfrak{f}_1Y)(X-\mathfrak{f}_2Y)(X-\mathfrak{f}_3Y)(X-\mathfrak{f}_4Y)=0;$$

and substituting the values just obtained, this is

$$X^2[X-(\mathfrak{f}_1+\mathfrak{f}_2)Y][X-(\mathfrak{f}_3+\mathfrak{f}_4)Y]+XY^2[X(\mathfrak{f}_1\mathfrak{f}_2+\mathfrak{f}_3\mathfrak{f}_4)-Y(\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_3+\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_4+\mathfrak{f}_1\mathfrak{f}_3\mathfrak{f}_4+\mathfrak{f}_2\mathfrak{f}_3\mathfrak{f}_4)]$$
$$+\mathfrak{f}_1\mathfrak{f}_2\mathfrak{f}_3\mathfrak{f}_4Y_4-(X-\mathfrak{f}_1Y)(X-\mathfrak{f}_2Y)(X-\mathfrak{f}_3Y)(X-\mathfrak{f}_4Y)=0,$$

which is in fact an identity.

77. The facultative lines are the transversal and the six mere lines; $b'=\rho'=7$; $t'=3$.

78. The equation of the Hessian surface is found to be

$$\begin{aligned}&(\gamma Z+\delta W)\,XZW+Y^2(\gamma Z-\delta W)^2+3\,(cX+dY)\,XZW+12\gamma\delta XY^2(aX+bY)\\&-(\gamma Z+\delta W)\,(3aX^3+9bX^2Y+6cXY^2)\\&-9X^2\{(ac-b^2)\,X^2+(ad-bc)\,XY+(bd-c^2)\,Y^2\}=0.\end{aligned}$$

79. Combining with the foregoing the equation of the surface

$$XZW+Y^2(\gamma Z+\delta W)+(a,\ b,\ c,\ d\between X,\ Y)^3=0,$$

it appears that these have along the line $X=0$, $Y=0$ the common tangent plane $X=0$, or, what is the same thing, that they meet in the line $X=0$, $Y=0$ (the axis) twice, and in a residual curve of the tenth order, which is the spinode curve; the equations may be presented in the somewhat more simple form

$$\begin{aligned}&XZW+Y^2(\gamma Z+\delta W)+(a,\ b,\ c,\ d\between X,\ Y)^3=0,\\&-4\gamma\delta Y^3ZW-4\,(\gamma Z+\delta W)\,(a,\ b,\ c,\ d\between X,\ Y)^3+12\gamma\delta XY^2(aX+bY)\\&\qquad+X^4(-12ac+9b^2)-3d\,(4aX^3Y+6bX^2Y^2+4cXY^3+dY^4)=0,\end{aligned}$$

which, however, still contain the line $X=0$, $Y=0$ twice. The spinode curve, as just mentioned, is of the tenth order; that is, we have $\sigma'=10$.

Each of the 6 mere lines is a double tangent to the spinode curve, but the transversal is only a single tangent: to show this, observe that the equations of the transversal are $X=0$, $\gamma Z+\delta W+dY=0$; substituting in the equations of the curve the first equation, that of the cubic surface is of course satisfied identically; for the second equation, writing $X=0$, this becomes $Y^2\{-4\gamma\delta ZW-4dY(\gamma Z+\delta W)-3d^2Y^2\}=0$; or writing herein $dY=-(\gamma Z+\delta W)$, it becomes $Y^2(\gamma Z-\delta W)^2=0$. The value $Y^2=0$ gives $X=0$, $Y=0$, $\gamma Z+\delta W=0$, viz. this is a point on the axis $X=0$, $Y=0$ not belonging to the spinode curve; the value $(\gamma Z-\delta W)^2=0$ gives a point of contact $X=0$, $\gamma Z+\delta W+dY=0$, $\gamma Z-\delta W=0$; and the transversal is thus a single tangent. Hence the number of contacts is $2\,.\,6+1,\ =13$; that is, we have $\beta'=13$.

Reciprocal Surface.

80. The equation is found by equating to zero the discriminant of the binary quartic

$$\{xX^2+yXY-(\delta z+\gamma w)\,Y^2\}+4Zw\,\{X\,(a,\ b,\ c,\ d\between X,\ Y)^3-\gamma\delta Y^4\},$$

or say this is $(*\between X,\ Y)^4$, where the coefficients are

$$\begin{aligned}&6x^2 && +24azw,\\&3xy && +18bzw,\\&y^2-2\,(\delta z+\gamma w)\,x && +12czw,\\&-3\,(\delta z+\gamma w)\,y && +\ 6dzw,\\&6\,(\delta z-\gamma w)^2.\end{aligned}$$

81. Forming the invariants, these are

$$\tfrac{1}{8}I = \Lambda^2 + \quad 24Uzw + 144\mu z^2w^2,$$
$$-J = \Lambda^3 + 36\Lambda Uzw + 216Vz^2w^2 + 864\nu z^3w^3,$$

where

$$\begin{aligned}
\Lambda &= y^2 + 4(\delta z + \gamma w)x,\\
U &= 2\gamma\delta x^2 + 2a(\delta z - \gamma w)^2 + 3by(\delta z + \gamma w) + c[y^2 - 2(\delta z + \gamma w)x] - dxy,\\
V &= \quad (-8ac + 9b^2)(\delta z - \gamma w)^2\\
&\quad + (2c^2 - bd)[y^2 - 2(\delta z + \gamma w)x]\\
&\quad + (-4ad + 6bc)y(\delta z + \gamma w)\\
&\quad - 2cd\,xy\\
&\quad + \quad d^2x^2\\
&\quad + 4\gamma\delta(2cx^2 - 3bxy + ay^2),\\
\mu &= c^2 - bd,\\
\nu &= ad^2 - 2bcd + 2c^3,
\end{aligned}$$

and the equation is

$$\frac{1}{432z^3w^2}\{(\Lambda^2 + 24Uzw + 144\mu z^2w^2)^3 - (\Lambda^3 + 36\Lambda Uzw + 216Vz^2w^2 + 864\nu z^3w^3)^2\} = 0;$$

or, expanding, this is

$$\begin{aligned}
&\Lambda^4\mu - \Lambda^3V + \Lambda^2U^2\\
+ \quad & 4zw(\quad -\Lambda^3\nu + 12\Lambda^2U\mu - 9\Lambda UV + 8U^3)\\
+ \quad & 36z^2w^2(\qquad 4\Lambda^2\mu^2 - 4\Lambda U\nu + 16U^2\mu - 3V^2)\\
+ \quad & 864z^3w^3(\qquad 4U\mu^2 - V\nu)\\
+ \quad & 1728z^4w^4(\qquad 4\mu^3 - \nu^2) = 0,
\end{aligned}$$

where observe that the value of

$$4\mu^3 - \nu^2, = 4(bd - c^2)^3 - (ad^2 - 3bcd + 2c^3)^2 \text{ is } = -d^2(a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd).$$

82. It is convenient to modify the form of the equation as follows; write

$$U_1 = U + 8a\gamma\delta zw,\ V_1 = V + (-8ac + 9b^2)\gamma\delta zw,$$

so that

$$\begin{aligned}
\Lambda &= y^2 + 4(\delta z + \gamma w)x,\\
U_1 &= -2\gamma\delta x^2 + 2a(\delta z + \gamma w)^2 + 3by(\delta z + \gamma w) + c[y^2 - 2(\delta z + \gamma w)x] - dxy,\\
V_1 &= \quad (-8ac + 9b^2)(\delta z + \gamma w)^2\\
&\quad + (2c^2 - bd)[y^2 - 2(\delta z + \gamma w)x]\\
&\quad + (-4ad + 6bc)y(\delta z + \gamma w)\\
&\quad - 2cdxy\\
&\quad + d^2x^2\\
&\quad + 4\gamma\delta(2cx^2 - 3bxy + ay^2),\\
\mu &= \quad c^2 - bd,\\
\nu &= \quad ad^2 - 2bcd + 2c^3,
\end{aligned}$$

Λ, U_1, V_1 being, it will be observed, functions of x, y, $\delta z+\gamma w$. The transformed equation is

$$\Lambda^2(\Lambda^2\mu-\Lambda V_1+U_1^2)+\Omega zw=0,$$

where the term Ω may be calculated without difficulty: the first term of this is

$$=\{y^2+4(\delta z+\gamma w)\,x\}^2\,.\,4\gamma^2\delta^2\,[x+\mathrm{f}_1 y-\mathrm{f}_1^2(\delta z+\gamma w)]\,..\,[x+\mathrm{f}_4 y-\mathrm{f}_4^2(\delta z+\gamma w)],$$

the developed expressions of $\frac{1}{4}(\Lambda^2\mu-\Lambda V_1+U_1^2)$ and of $\gamma^2\delta^2$ into the product of the linear factors being in fact each

$$\begin{aligned}
= \quad & x^4\,.\,\gamma^2\delta^2+x^3y\,.\,d\gamma\delta+x^2y^2\,.-3c\gamma\delta+xy^3\,.\,3b\gamma\delta+y^4\,.-a\gamma\delta\\
&+[x^3(-d^2-6c\gamma\delta)+x^2y\,(3cd+9b\gamma\delta)+xy^2\,(-3bd-4a\gamma\delta)+y^3\,.\,ad](\delta z+\gamma w)\\
&+[x^2(9c^2-6bd-2a\gamma)+xy\,(3ad-9bc)+y^2\,.\,3ac](\delta z+\gamma w)^2\\
&+[x(6ac-9b^2)+y\,.\,3ab](\delta z+\gamma w)^3\\
&+a^2\delta^4\,.\,(\delta z+\gamma w)^4.
\end{aligned}$$

The form puts in evidence the section by the plane $w=0$, which is the reciprocal of the node D, viz. this is a conic (the reciprocal of the tangent cone) twice, and four lines, the reciprocals of the nodal rays, each once. And similarly for the section by the plane $z=0$.

83. The nodal curve is made up of the lines which are the reciprocals of the six mere lines and the transversal; viz. we have three pairs of lines and a seventh line, the lines of each pair intersecting at a point of the seventh line, and these three points being the triple points of the nodal curve; $t'=3$ as before.

84. The equations of the cuspidal curve are at once reduced to the form

$$\begin{aligned}
\Lambda^2 \quad &+24Uzw+144\mu z^2w^2=0,\\
\Lambda U &+(18V-12\mu\Lambda)\,zw+72\nu z^2w^2=0,
\end{aligned}$$

which are two quartic surfaces having in common the conics $z=0$, $\Lambda=0$, and $w=0$, $\Lambda=0$; or we may say that the cuspidal curve is a curve $4\,.\,4-2-2$; that is $c'=12$.

Section $V = 12 - B_4$.

Article Nos. 85 to 94. Equation $WXZ + (X+Z)(Y^2 - aX^2 - bZ^2) = 0$.

85. The diagram of the lines and planes is

Planes. $V = 12 - B_4$.		Lines. 22′	21′	12′	11′	2′	1′	2	1	4	3	
		$\overline{10}$ $\overline{27}$	4 × 1 = 4				4 × 4 = 16			1 × 1 = 1	1 × 6 = 6	
12								•	•		•	Biplanes containing rays 1, 2 and 1′, 2′ respectively.
1′2′	2 × 12 = 24					•			•		•	
0	1 × 3 = 3									•	•	Plane touching along edge and containing the transversal.
11′					•		•		•			Biradial planes each containing a ray of the one and a ray of the other biplane.
12′				•		•			•			
21′	4 × 4 = 16		•				•	•				
22′		•				•		•				
11′ . 22′	2 × 1 = 2	•			•					•		Planes each through the transversal.
12′ . 21′	9 45		•	•						•		
			More lines.				Rays 1, 2 and 1′, 2′ in the two biplanes respectively.			Transversal in the plane touching along the edge.	Edge.	

86. The planes are

$$X = 0, \qquad [12]$$
$$Z = 0, \qquad [1'2']$$
$$X + Z = 0, \qquad [0]$$
$$-X\sqrt{a} + Y - Z\sqrt{b} = 0, \qquad [11']$$
$$X\sqrt{a} + Y - Z\sqrt{b} = 0, \qquad [12']$$
$$-X\sqrt{a} + Y + Z\sqrt{b} = 0, \qquad [21']$$
$$X\sqrt{a} + Y + Z\sqrt{b} = 0, \qquad [22']$$
$$\sqrt{ab}(X+Z) + W = 0, \qquad [11'.22']$$
$$-2\sqrt{ab}(X+Z) + W = 0, \qquad [12'.21'].$$

87. And the lines are

a	b	c	f	g	h	equations may be written	
0	0	0	0	1	0	(3)	$X=0,\ Z=0$
1	0	1	0	0	0	(4)	$X+Z=0,\ W=0$
0	0	0	0	$\sqrt{b}$	1	(1)	$X=0,\quad Y-Z\sqrt{b}=0$
0	0	0	0	$-\sqrt{b}$	1	(2)	$X=0,\quad Y+Z\sqrt{b}=0$
0	0	0	1	$\sqrt{a}$	0	(1′)	$Z=0,\ -X\sqrt{a}+Y=0$
0	0	0	1	$-\sqrt{a}$	0	(2′)	$Z=0,\quad X\sqrt{a}+Y=0$
$-\frac{1}{\sqrt{b}}$	$\frac{1}{\sqrt{ab}}$	$-\frac{1}{\sqrt{a}}$	2	$2(\sqrt{a}-\sqrt{b})$	-2	(11′)	but for the other lines the coordinate expressions are the more convenient.
$-\frac{1}{\sqrt{b}}$	$-\frac{1}{\sqrt{ab}}$	$\frac{1}{\sqrt{a}}$	2	$2(-\sqrt{a}-\sqrt{b})$	-2	(12′)	
$\frac{1}{\sqrt{b}}$	$-\frac{1}{\sqrt{ab}}$	$-\frac{1}{\sqrt{a}}$	2	$2(\sqrt{a}+\sqrt{b})$	-2	(21′)	
$\frac{1}{\sqrt{b}}$	$\frac{1}{\sqrt{ab}}$	$\frac{1}{\sqrt{a}}$	2	$2(-\sqrt{a}+\sqrt{b})$	-2	(22′)	

88. The four mere lines and the transversal are each facultative; *the edge is also facultative, counting twice;* $\rho'=b'=7,\ t'=3$.

That the edge is as stated a facultative line counting twice, I discovered, and accept, *à posteriori*, from the circumstance that on the reciprocal surface the reciprocal of the edge is (as will be shown) a tacnodal line, that is, a double line with coincident tangent planes, counting twice as a nodal line. Reverting to the cubic surface, I notice that the section by an arbitrary plane through the edge consists of the edge and of a conic touching the edge at the biplanar point; by what precedes it appears that the arbitrary plane is to be considered, and that twice, as a node-couple plane of the surface: I do not attempt to further explain this.

89. Hessian surface. The equation is

$$(X+Z)XZW+(X-Z)^2Y^2+(X+Z)(3a,\ -a,\ -b,\ 3b\between X,\ Z)^3=0.$$

Combining with the equation

$$XZW+(X+Z)(Y^2-aX^2-bZ^2)=0,$$

and observing that from the two equations we deduce

$$-XZY^2+(X+Z)(aX^3+bZ^3)=0,$$

it appears that the complete intersection of the Hessian and the surface is made up of the line $X=0,\ Z=0$ (the edge) twice (that is, the two surfaces touch along the edge), and of a curve of the tenth order, which is the spinode curve; $c'=10$.

The equations of the spinode curve may be presented in the form

$$\left\| \begin{matrix} XZ, & aX^2+bZ^2-Y^2, & aX^3+bZ^3 \\ X+Z, & W \quad , & Y^2 \end{matrix} \right\| = 0;$$

it is a curve $3.4-2$, the partial intersection of a quartic and a cubic surface which touch along a line.

The binode is on the spinode curve a singular point; through it we have two branches represented in the vicinity thereof by the equations

$$\left(\frac{X}{W} = -\tfrac{1}{2}\left(\frac{Y}{W}\right)^2,\ \frac{Z}{W} = -\left(\frac{1}{2b}\right)^{\frac{1}{2}}\left(\frac{Y}{W}\right)^{\frac{4}{3}}\right) \text{ and } \left(\frac{Z}{W} = -\tfrac{1}{2}\left(\frac{Y}{W}\right)^2,\ \frac{X}{W} = -\left(\frac{1}{2a}\right)^{\frac{1}{3}}\left(\frac{Z}{W}\right)^{\frac{4}{3}}\right)$$

respectively.

90. The edge counted once is regarded as a double tangent of the spinode curve (I do not understand this, there is apparently a higher tangency); each of the four mere lines is a double tangent; the transversal is a single tangent; hence $\beta' = 2.2+2.4+1, = 13$.

Reciprocal Surface.

91. The equation is found by equating to zero the discriminant of the binary quartic

$$y^2X^2Z^2 + 4w(Xx+Zz)XZ(X+Z) + 4w^2(aX^2+bZ^2)(X+Z)^2,$$

viz. multiplying by 6 to avoid fractions, and calling the function $(*\between X, Z)^4$, the coefficients are

$$\begin{aligned} &24aw^2, \\ &6w(x+2aw), \\ &y^2+4(x+z)w+4(a+b)w^2, \\ &6w(z+2bw), \\ &24bw^2; \end{aligned}$$

and then writing

$$\begin{aligned} L &= y^2+4(x+z)w+4(a+b)w^2, \\ M &= \ \ 4\{xz+2(bx+az)w\}, \\ N &= 16aby^2-bx^2-ay^2, \end{aligned}$$

we find

$$\begin{aligned} \tfrac{1}{3}I &= L^2-12w^2M, \\ -J &= L^3-18w^2LM-54w^4N, \end{aligned}$$

and then the equation is

$$\frac{1}{108w^4}\{(L^3-12w^2M)^3-(L^3-18w^2LM-54w^4N)^2\} = 0,$$

viz. it is

$$L^3N+L^2M^2-18w^2LMN-16w^2M^3-27w^4N^2 = 0.$$

92. This, completely developed, is

$$
\begin{aligned}
& 64w^6 . ab(a+b)^2 \{(a+b)y^2-(x-z)^2\} \\
& + 32w^5 . 2ab \left\{ \begin{array}{l} 3(a+b)[(a-2b)x+(-2a+b)z]y^2 \\ +(x-z)^2[(-3a+5b)x+(5a-3b)z] \end{array} \right\} \\
& + 16w^4 \left\{ \begin{array}{l} 3ab(a^2-7ab+b^2)y^4 \\ +[b(9a^2+26ab-b^2)x^2-26ab(a+b)xz+a(-a^2+26ab+9b^2)z^2]y^2 \\ +(x-z)^2[b(-12a+b)x^2+22abxz+a(a-12b)z^2] \end{array} \right\} \\
& + 8w^3 \left\{ \begin{array}{l} 3ab[(2a-b)x+(-a+2b)z]y^4 \\ +[b(-2a+5b)x^3+b(3a-2b)x^2z+a(-2a+3b)xz^2+a(5a-2b)z^3]y^2 \\ +2(x-z)^2[-2bx^3+ba^2z+axz^2-2ay^3] \end{array} \right\} \\
& + 4w^2 \left\{ \begin{array}{l} 3ab(a+b)y^6 \\ +[b(9a-2b)x^2+8abxz+a(-2a+9b)z^2]y^4 \\ +2[-6bx^4+bx^3z-(a+b)x^2z^2+axz^3-6az^4]y^2 \\ +4x^2z^2(x-z)^2 \end{array} \right\} \\
& + 2w \left\{ \begin{array}{l} 2ab(x+z)y^6 \\ -[3bx^3+2bx^2z+2axz+3az^3]y^4 \\ +4x^2z^2(x+z)y^2 \end{array} \right\} \\
& + \qquad y^4(ay^2-x^2)(cy^2-z^2)=0,
\end{aligned}
$$

where we see that the section by the plane $w=0$ (reciprocal of B_4) is made up of the line $w=0$, $y=0$ (reciprocal of the edge) four times, and of the lines $w=0$, $ay^2-x^2=0$; $w=0$, $by^2-z^2=0$ (reciprocals of the rays) each once.

93. The surface contains the line $y=0$, $w=0$ (reciprocal of the edge); and if we attend only to the terms of the lowest order in y, w, viz.

$$x^2z^2\{16(x-z)^2w^2+8(x+z)y^2w+y^4\},$$

which terms equated to zero give

$$w=-\tfrac{1}{4}\frac{1}{(\sqrt{x}\pm\sqrt{z})^2}y^2,$$

we see that the line in question ($y=0$, $w=0$) is a tacnodal line on the surface, the tacnodal plane being $w=0$, *a fixed plane for all points of the line:* it has already been seen that this plane meets the surface in the line taken 4 times; every other plane through the line meets the surface in the line taken twice. We have in what precedes the *à posteriori* proof that in the cubic surface the edge is a facultative line to be counted twice.

94. Cuspidal curve. The equation of the surface may be written

$$(L^2-12w^2M)(4M^2+3LN)-(LM+9w^2N)^2=0,$$

and we thus have

$$4M^3 + 3LN = 0,$$
$$LM + w^2N = 0,$$
$$L^2 - 12w^2M = 0,$$

or, what is the same thing,

$$\left\| \begin{matrix} L, & 12M, & -9N \\ w^3, & L, & M \end{matrix} \right\| = 0$$

for the equation of the cuspidal curve. Attending to the second and third equations, these are quartics having in common $w^2 = 0$, $L = 0$, that is, the line $y = 0$, $w = 0$ four times; or the cuspidal curve is a partial intersection $4 \times 4 - 4 : c' = 12$.

Section VI $= 12 - B_3 - C_2$.

Article Nos. 95 to 102. Equation $WXZ + Y^2Z + (a, b, c, d \between X, Y)^3 = 0$.

95. The diagram of the lines and planes is

Planes.		Lines. 14 . 2'3'	13 . 2'4'	12 . 3'4'	4'	3'	2'	4	3	2	1	0	
VI $= 12 - B_3 - C_2$.		$\overline{11}$ $\overline{27}$	$3 \times 1 = 3$			$3 \times 2 = 6$			$3 \times 3 = 9$		$1 \times 3 = 3$	$1 \times 6 = 6$	
0	$1 \times 6 = 6$										• •	• •	Biplane touching along axis, and containing transversal ray.
00	$1 \times 6 = 6$							• •	• •	• •			Other biplane.
22'	$3 \times 6 = 18$						• • •			• •		•	Planes each through the axis and containing a ray of the binode and a ray of the cnicnode.
33'						• • •			• •			•	
44'					• • •			• •				•	
12	$3 \times 3 = 9$			• • •						•	• •		Biradial planes of the binode, each containing ray of axial biplane and a ray of other biplane.
13			• • •						•				
14		• • •						•					
2'3'	$3 \times 2 = 6$	• •				•	•						Biradial planes of the cnicnode.
2'4'			• •		•		•						
3'4'				• •	•	•							
$\overline{11}$	$\overline{45}$	Mere lines.			Cnicnodal rays.			Biplanar rays of other biplane.			Biplanar ray of axial biplane, being a transversal ray.	Axis, joining the two nodes.	

96. Writing $(a, b, c, d \between X, Y)^3 = -d(\theta_2 X - Y)(\theta_3 X - Y)(\theta_4 X - Y)$, the planes are

$$\begin{aligned}
X &= 0, && [0] \\
Z &= 0, && [00] \\
\theta_2 X - Y &= 0, && [22'] \\
\theta_3 X - Y &= 0, && [33'] \\
\theta_4 X - Y &= 0, && [44'] \\
d(\theta_2 X - Y) - Z &= 0, && [12] \\
d(\theta_3 X - Y) - Z &= 0, && [13] \\
d(\theta_4 X - Y) - Z &= 0, && [14] \\
X\theta_3\theta_4 - Y(\theta_2 + \theta_3) - W &= 0, && [2'3'] \\
X\theta_2\theta_4 - Y(\theta_3 + \theta_4) - W &= 0, && [2'4'] \\
X\theta_3\theta_4 - Y(\theta_3 + \theta_4) - W &= 0, && [3'4']
\end{aligned}$$

97. And the lines are

a	b	c	f	g	h	equations may be written	
0	0	0	0	0	1	(0)	$X = 0,\ Y = 0$
0	0	0	0	-1	d	(1)	$X = 0,\ dY + Z = 0$
0	0	0	1	θ_2	0	(2)	$\theta_2 X - Y = 0,\ Z = 0$
0	0	0	1	θ_3	0	(3)	$\theta_3 X - Y = 0,\ Z = 0$
0	0	0	1	θ_4	0	(4)	$\theta_4 X - Y = 0,\ Z = 0$
θ_2	-1	0	0	0	θ_2^2	(2')	$\theta_2 X - Y = 0,\ \theta_2^2 X + W = 0$
θ_3	-1	0	0	0	θ_3^2	(3')	$\theta_3 X - Y = 0,\ \theta_3^2 X + W = 0$
θ_4	-1	0	0	0	θ_4^2	(4')	$\theta_4 X - Y = 0,\ \theta_4^2 X + W = 0$
$-d\theta_2$	d	1	$-(\theta_3 + \theta_4)$	$-\theta_3\theta_4$	$d(\theta_3\theta_4 - \theta_2\theta_3 - \theta_2\theta_4)$	(12 . 3'4')	but for the remaining lines the coordinate expressions are more convenient.
$-d\theta_3$	d	1	$-(\theta_2 + \theta_4)$	$-\theta_2\theta_4$	$d(\theta_2\theta_4 - \theta_3\theta_2 - \theta_3\theta_4)$	(13 . 2'4')	
$-d\theta_4$	d	1	$-(\theta_2 + \theta_3)$	$-\theta_2\theta_3$	$d(\theta_2\theta_3 - \theta_4\theta_2 - \theta_4\theta_3)$	(14 . 2'3')	

The mere lines are each of them facultative; $b' = \rho' = 3$; $t' = 0$.

98. Hessian surface. The equation is

$$\begin{aligned}
&\{Z + 3(cX + dY)\}\{XZW + Y^2 Z + (a, b, c, d \between X, Y)^3\} \\
&\quad - 4Z(a, b, c, d \between X, Y)^3 \\
&\quad - 3(4ac - 3b^2, ad, bd, cd, d^2 \between X, Y)^4 = 0;
\end{aligned}$$

and it is thence easy to see that the complete intersection is made up of the line $X = 0$, $Y = 0$ (the axis) three times, and of a curve of the ninth order, which is the spinode curve; $\sigma' = 9$.

99. The equations of the spinode curve may be written in the simplified form

$$XZW + Y^2Z + (a,\ b,\ c,\ d\between X,\ Y)^3 = 0,$$

$$4Z(a,\ b,\ c,\ d\between X,\ Y)^3 + 3(4ac - 3b^2,\ ad,\ bd,\ cd,\ d^2\between X,\ Y)^4 = 0,$$

the line $X=0$, $Y=0$ here appearing as a triple line on the second surface; the curve is a partial intersection, $3 \times 4 - 3$.

The node C_2 is a triple point on the curve, the tangents being the nodal rays.

The node B_3 is a quintuple point, one tangent being $X=0$, $3dY+4Z=0$, and the other tangents being given by $Z=0$, $(4ac-3b^2,\ ad,\ bd,\ cd,\ d^2\between X,\ Y)^4=0$.

Each of the facultative lines is a double tangent to the curve, or we have $\beta'=6$.

Reciprocal Surface.

100. Comparing the equation of the cubic surface with that for $\mathrm{IV} = 12 - 2C_2$, it appears that the equation of $\mathrm{VI} = 12 - B_3 - C_2$ is obtained by substituting in that equation the values $\delta=0$, $\gamma=1$. But instead of making this substitution in the final formula, it is convenient to make it in the binary quartic $(*\between X,\ Y)^4$, thus in fact working out the reciprocal surface by means of the function

$$(xX^2 + yXY - wY^2)^2 + 4zwX(a,\ b,\ c,\ d\between X,\ Y)^3,$$

the coefficients whereof (multiplying by 6 to avoid fractions) are

$$\begin{aligned} &6x^2 + 24azw, \\ &3xy + 18bzw, \\ &y^2 - 2xw + 12czw, \\ &-3yw + 6dzw, \\ &6w^2. \end{aligned}$$

We find

$$\begin{aligned} \tfrac{1}{3}I &= L^2 - 12zwM, \\ -J &= L^3 - 18zwLM - 54z^2w^2N, \end{aligned}$$

where

$$\begin{aligned} L &= y^2 + 6(x+3cz)w, \\ M &= 2dxy + 6(2cx - by + 2bdz)w - 4aw^2, \\ N &= -4d^2x^2 - 8d(3bx - 2ay + 2adz)w - 12(3b^2 - 4ac)w^2. \end{aligned}$$

The equation is

$$\frac{1}{108z^2w^2}\{(L^2 - 12zwM)^3 - (L^3 - 18zwLM - 54z^2w^2N)^2\} = 0,$$

viz. it is

$$L^2(LN + M^2) - 18zwLMN - 16zwM^3 - 27z^2w^2N^2 = 0,$$

where however $LN + M^2$ contains the factor w, $= wP$ suppose; the equation thus is

$$L^2P - 18zLMN - 16zM^3 - 27z^2wN^2 = 0.$$

Write

$$\begin{aligned} A &= 4x + 12cz, \\ B &= 6cx - 3by + 6bdz - 2aw, \\ C &= 6bdx - 4ady + 4ad^2z + 3(3b^2 - 4ac)w, \end{aligned}$$

and therefore

$$\begin{aligned} L &= \quad y^2 + \ Aw, \\ M &= \ 2dxy + 2Bw, \\ N &= -4d^2x^2 - 4Cw, \end{aligned}$$

then we have

$$\begin{aligned} P &= \frac{1}{w}\{-(y^2+Aw)(4d^2x^2+4Cw)+(2dxy+2Bw)^2\} \\ &= -4\{Cy^2-2Bdxy+Ad^2x^2+w(AC-B^2)\}, \end{aligned}$$

or the equation is

$$4L^2\{Cy^2-2Bdxy+Ad^2x^2+w(AC-B^2)\}+18zLMN+16zM^3+27z^2wN^2=0.$$

101. Consider the section by the plane $w=0$, we have $L=y^2$, $M=2dxy$, $N=-4d^2x^2$, and the equation becomes $4y^4(Cy^2-2Bdxy+Ad^2x^2)+(128-144=)-16d^3x^3y^3z=0$; which substituting for A, B, C the values

$$\begin{aligned} A &= 4x+12cz, \\ B &= 6cx-3by+6bdz, \\ C &= 6bdx-4ady+4ad^2z, \end{aligned}$$

becomes $16dy^3(y-dz)(dx^3-3cx^2y+3cxy^2-ay^3)=0$; which is in fact the line $w=0$, $y=0$ (reciprocal of the edge) three times, and the lines $w=0$, $(y-dz)(d, -c, b, -a \between x, y)^3=0$ (reciprocals of the biplanar rays) each once. Observe that the edge $(X=0, Z=0)$ is not a line of the cubic surface, but the reciprocal line $y=0$, $w=0$ presents itself as an oscular line of the reciprocal surface.

102. The equations of the cuspidal curve are in the first instance obtained in the form

$$\left\| \begin{matrix} L, & M, & 3N \\ 12zw, & L, & -4M \end{matrix} \right\| = 0.$$

Consider the two equations

$$\begin{aligned} L^2 \ -12zwM &= 0, \\ LM + \ 9zwN &= 0, \end{aligned}$$

each of the fourth order, but which are satisfied by $zw=0$, $L=0$; that is, by $(w=0,\ y^2=0)$, $(z=0,\ y^2+4xw=0)$. The line $(w=0,\ y=0)$ however presents itself in the intersection of the two surfaces, not twice only, but 4 times. To show this, observe that the line in question is a nodal line on the surface $L^2-12zwM=0$; in fact, attending only to the terms of the second order in y, w, we find

$$\{(4x+12cz)^2-144cxz-144bdz^2\}w^2-24dxzyw=0,$$

giving the two sheets

$$\{(4x+12cz)^2-144cxz-144bdz^2\}w-24dxzy \quad =0 \text{ and } w=0;$$

in regard to the last-mentioned sheet the form in the vicinity thereof is given by $w=Ay^3$, viz. we have approximately $L=y^2$, $M=2dxy$, and thence $y^4-12z\,.\,Ay^3\,.\,2dxy=0$, that is, $A=\dfrac{1}{24dxz}$ or $w=\dfrac{1}{24dxz}y^3$; the line is thus a flecnodal line on the surface

$L^2 - 12zwM = 0$. Next as regards the surface $LM + 9zwN = 0$; the line $y = 0$, $w = 0$ is a simple line on the surface, the terms of the lowest order being $9zw(-4d^2x^2) = 0$; that is, we have $w = 0$, and for a next approximation $w = Ay^3$, viz. $L = y^2$, $M = -2dxy$, $N = -4d^2x^2$, and therefore $-2dxy^2 + 9z \,.\, Ay^3(-4d^2x^2) = 0$, that is, $A = -\dfrac{1}{18dxz}$, or $w = -\dfrac{1}{18dxz}y^3$; there is thus a threefold intersection with one sheet and a simple intersection with the other sheet of the surface $L^2 - 12zwM = 0$. The surfaces intersect, as has been mentioned in the conic $z = 0$, $y^2 + 4xw = 0$; or we have the line $y = 0$, $w = 0$ four times, the conic once, and a residual cuspidal curve of the order $4 \,.\, 4 - 4 - 2$, $= 10$; that is, $c' = 10$.

Section VII $= 12 - B_5$.

Article Nos. 103 to 116. Equation $WXZ + Y^2Z + YX^2 - Z^3 = 0$.

103. The diagram of lines and planes[1] is

VII = 12 − B_5.		Lines. 13′ 6 / 27	12′ 2 × 1 = 2	3′	2′ 2 × 5 = 10	1 1 × 5 = 5	0 1 × 10 = 10	
Planes. 01	1 × 15 = 15					⋮	⋮	Torsal biplane.
00	1 × 20 = 20			⋮	⋮		⋮	Ordinary biplane.
12′ / 13′	2 × 5 = 10	⋮	⋮	⋮	⋮	⋮		Planes each containing a mere line.
$\overline{4}$	$\overline{45}$	⋮						
		Mere lines.		Ray of ordinary biplane.		Ray of torsal biplane.	Edge.	

[1] The marginal symbols in the preceding diagrams constitute a real notation of the lines and planes; but here, and still more so in some of the following diagrams, they are mere marks of reference, showing which are the lines and planes to which the several equations respectively belong.

104. The planes are

		The lines are		
$Z=0,$	[10]	$X=0,$	$Z=0,$	(0)
$X=0,$	[00]	$Y=0,$	$Z=0,$	(1)
$Y+Z=0,$	[12′]	$X=0,$	$Y+Z=0,$	(2′)
$Y-Z=0,$	[13′]	$X=0,$	$Y-Z=0,$	(3′)
		$X-W=0,$	$Y+Z=0,$	(12′)
		$X+W=0,$	$Y-Z=0,$	(13′).

105. The two mere lines are facultative, and the edge is also facultative; $\rho'=b'=3$; $t'=0$.

106. Hessian surface. The equation is

$$Z(WXZ+Y^2Z+YX^2-Z^3)-4X^2YZ+X^4+4Z^4=0.$$

The complete intersection with the surface is thus given by the equations

$$WXZ+Y^2Z+YX^2-Z^3=0,\ -4X^2YZ+X^4+4Z^4=0,$$

which is made up of the line $X=0$, $Z=0$ (the edge) four times and a curve of the eighth order. To see this, observe that the last-mentioned surfaces have in common the line $X=0$, $Z=0$, which is on the first surface a torsal line (equation in vicinity being $Z=-\frac{1}{Y}X^2$), and on the second surface a triple line (equations in vicinity being $Z=\frac{1}{Y}X^3$ and $X^2=\frac{1}{Y}Z^3$). But $Z=-\frac{1}{Y}X^2$ *touches* $Z=\frac{1}{Y}X^2$, and the line counts thus $(2+2=)$ 4 times.

107. I say that the complete intersection is the line $(X=0,\ Z=0)$ three times together with a spinode curve made up of this same line once and of the curve of the eighth order; and that thus $\sigma'=9$.

The discussion of the reciprocal surface in fact shows that the reciprocal of the edge is a singular line thereof, counting once as a nodal and twice as a cuspidal line thereof; the cuspidal tangent planes are the reciprocals of the several points of the edge, and the edge is thus part of the spinode curve. The reasoning may appear to show that the edge should be counted twice, but it must be counted once only, making the order $=9$ as mentioned.

108. I find that the octic component of the spinode curve is a unicursal curve, the equations of which may be written

$$X:Y:Z:W=16\theta^2:4\theta+16\theta^5:16\theta^3:-5-8\theta^4-16\theta^8;$$

the values of θ at the binode B_5 are $\theta=0$, $\theta=\infty$, and we thus obtain in the neighbourhood thereof the two branches

$$\frac{Y}{W}=-5\left(\frac{Z}{W}\right)^2,\ \frac{X}{W}=\tfrac{25}{4}\left(\frac{Z}{W}\right)^3 \text{ and } \frac{X}{W}=\left(\frac{Z}{W}\right)^{\frac{5}{3}},\ \frac{Y}{W}=-\left(\frac{Z}{W}\right)^2.$$

109. Each of the lines $(X-W=0,\ Y+Z=0)$ and $(X+W=0,\ Y-Z=0)$ is a double tangent of the spinode octic; in fact for the first of these lines we have

$$16\theta^8+8\theta^4+16\theta^2+5=0,\quad 16\theta^5+16\theta^3+4\theta=0,$$

that is,

$$(2\theta^2+1)^2(4\theta^4-4\theta^2+5)=0,\quad 4\theta\,(2\theta^2+1)^2=0,$$

so that the line touches at the two points given by $2\theta^2+1=0$; and similarly the other line touches at the two points given by $2\theta^2-1=0$.

The edge $X=0$, $Z=0$ has apparently a higher contact with the spinode octic, viz. the equations $X=0$, $Z=0$ are satisfied by $\theta=0$ twice, $\theta=\infty$ five times; but it must be reckoned only as a double tangent. Hence $\beta'=2\,.\,2+2,\ =6$.

Reciprocal Surface.

110. The equation is obtained by equating to zero the discriminant of the binary quartic

$$X^2(yZ-wX)^2+4wZ^2(wZ^2+zZX+xX^2),$$

viz. calling this $(*\between X,\ Z)^4$, the coefficients (multiplying by 6) are

$$(6w^2,\ -3yw,\ y^2+4xw,\ 6zw,\ 24w^2);$$

and then writing

$$\begin{aligned} L &= \quad y^2+4xw,\\ M &= -2yz-4w^2,\\ N &= -4z^2+16xw, \end{aligned}$$

we have

$$\begin{aligned} \tfrac{1}{3}I &= L^2-12w^2M,\\ -J &= L^3-18w^2LM-54w^4N, \end{aligned}$$

and the equation is, as in former cases,

$$L^2(LN+M^2)-18w^2LMN-16w^2M^3-27w^4N^2=0;$$

but $LN+M^2$ and therefore the whole equation divides by w, and we thus obtain

$$16L^2\left(-xz^2+y^2x+w\,(yz+4x^2)+w^3\right)-18wLMN-16wM^3-27w^3N^2=0;$$

or, completely developed, this is

$$\begin{aligned} & w^7\,.\,64\\ &+w^5\,.\,32\ (\ 3yz-4x^2)\\ &+w^4\,.\,16x\,(\ 5y^2+9z^2)\\ &+w^3\,.\quad (\ \ y^4+30y^2z^2+160yzx^2-27z^4+64x^4)\\ &+w^2\,.\ 4x\,(11y^2z+12y^2x^2-9yz^3-4z^2x^2)\\ &+w\ .\ \ y^2(\ \ y^3z+12y^2x^2-yz^3-8z^2x^2)\\ &+\quad y^4x\,(\ \ y^2-z^2)=0. \end{aligned}$$

111. To transform the equation so as to put in evidence the nodal curve, I collect the terms according to their degrees in (y, z) and (x, w); viz. the equation thus becomes

$$\begin{aligned}
& 64x^4w^3 - 128x^3w^5 + 64w^7 \\
+\ & z^2(\ -16x^3w^2 + 144xw^4) \\
+\ & zy\,(\ \ 160x^2w^3 + \ 96w^5) \\
+\ & y^2\,(\ \ \ 48x^3w^2 + \ 80xw^4) \\
+\ & z^4\ \ .-27w^3 \\
+\ & z^3y\ .-36xw^2 \\
+\ & z^2y^2\,.-8x^2w + 30w^3 \\
+\ & zy^3\ .44xw^2 \\
+\ & y^4\ \ .12x^2w + w^3 \\
+\ & z^3y^3\,.-w \\
+\ & z^2y^4\,.-x \\
+\ & zy^5\ \ .w \\
+\ & y^3\ \ .x \qquad\qquad = 0\,;
\end{aligned}$$

and if for a moment we write $z = \alpha + \gamma$, $y = \alpha - \gamma$ and collect, ultimately replacing α, γ by their values $\frac{1}{2}(z+y)$, $\frac{1}{2}(z-y)$, the equation can be expressed in the form

$$\begin{aligned}
& 64w^3(x^2 - w^2)^2 \\
+\ & 8w^2(z+y)^2(x+w)^2(x+3w) \\
+\ & 8w^2(z-y)^2(x-w)^2(x-3w) \\
-\ & 32w^2(z^2-y^2)\ (x^2-w^2)\,x \\
+\ & \tfrac{1}{4}w\ (z+y)^4(x+w)^2 \\
-\ & \ \ w\ (z+y)^3(z-y)(x+w)(3x+7w) \\
+\ & \tfrac{1}{2}w\ (z^2-y^2)^2(11x^2-27w^2) \\
-\ & \ \ w\ (z+y)\ (z-y)^3(x-w)(3x-7w) \\
+\ & \tfrac{1}{4}w\ (z-y)^4(x-w)^2 \\
-\ & \ \ y^3\ (z^2-y^2)\ (zw+xy) = 0,
\end{aligned}$$

and observing that we have

$$\begin{aligned}
zw + xy &= -z(x-w) + x(z+y) \\
&= \ \ z(x+w) - x(z-y),
\end{aligned}$$

we see that every term of the equation is at least of the second order in $z+y$ and $x-w$ conjointly; and also at least of the second order in $z-y$ and $x+w$ conjointly;

that is, the surface has the nodal lines $(z+y=0,\ x-w=0)$ and $(z-y=0,\ x+w=0)$, which are the reciprocals of the lines 12′ and 13′ respectively. The nodal curve is made up of these two lines and of the line $y=0,\ w=0$ (reciprocal of edge), as will presently appear; so that we have $b'=3$.

112. The equations of the cuspidal curve are

$$\begin{aligned} L^2 - 12w^2M &= 0, \\ LM + \ 9w^2N &= 0, \\ 4M^2 + \ 3LN &= 0. \end{aligned}$$

Attending to the two equations

$$\begin{aligned} L^2 - 12w^2M &= y^4 + 8y^2xw + 16x^2w^2 + 24yzw^2 + 48w^4 = 0, \\ LM + 9w^2N &= y^3z + 2y^2w^2 + 4xyzw + (8-72=) - 64xw^3 + 18z^2w^2 = 0, \end{aligned}$$

these surfaces are each of the order 4, and the order of their intersection is $=16$. But the two surfaces contain in common the line $(y=0,\ w=0)$ 7 times; in fact on the first surface this is a cusp-nodal line $4xw+y^2+Ay^{\frac{5}{2}}=0$; and on the second surface it is a nodal line $w(4xy+18zw)=0$; the sheet $w=0$ is more accurately $4xw+y^2+By^3\ldots=0$; whence in the intersection with the first surface the line counts 5 times in respect of the first sheet and 2 times in respect of the second sheet; together $(5+2=)$ 7 times, and the residual curve is of the order $(16-7=)$ 9.

113. I say that the cuspidal curve is made up of this curve of the 9th order, and of the line $y=0,\ w=0$ (reciprocal of the edge) once; so that $c'=10$. In fact, considering the line in question $y=0,\ w=0$ in relation to the surface, the equation of the surface (attending only to the lowest terms in $y,\ w$) may be written

$$-xz^2(y^2+4xw)^2 + w(-y^3z^2) + w^2(-36xyz^3) + \&c. = 0,$$

giving in the vicinity of the line

$$4xw + y^2 = Ay^{\frac{5}{2}},$$

and then

$$-xz^2A^2 + \frac{z^3}{x}\left(\tfrac{1}{4} - \tfrac{36}{16}\right) = 0,$$

that is, $A^2 = -2\dfrac{z}{x^2}$ or $4xw+y^2 = \sqrt{-2}\,.\,\dfrac{z^{\frac{1}{2}}}{x}y^{\frac{5}{2}}$; wherefore the line is a cusp-nodal line, counting once as a nodal and once as a cuspidal line; and so giving the foregoing results $b'=3,\ c'=10$.

114. I revert to the equation which exhibits the nodal lines $(x-w=0,\ y+z=0)$, $(x+w=0,\ y-z=0)$ for the purpose of showing that they have respectively no pinch-points; that is, that in regard to each of them we have $j'=0$. In fact for the first

of these lines, neglecting the terms which contain $x-w$, $y+z$ conjointly in an order above the second, the equation may be written

$$\begin{array}{lll}
 & 64w^3(x+w)^2 & (x-w)^2 \\
+ & 8w^2(x+w)^2(x+3w)(z+y)^2 & \\
+ & 8w^2(z-y)^2(x-3w)(x-w)^2 & \\
- & 32w^2(z-y)(x+w)x & (x-w)(z+y) \\
+ & \frac{1}{2}w(z-y)^2(11x^2-27w^2)(z+y)^2 & \\
- & w(z-y)^3(3x-7w) & (x-w)(z+y) \\
+ & \frac{1}{4}w(z-y)^4 & (x-w)^2 \\
+ & y^3z(z-y) & (x-w)(z+y) \\
- & y^3x(z-y) & (z+y)^2=0,
\end{array}$$

viz. this is

$$(A,\ B,\ C\between x-w,\ z+y)^2=0,$$

where, collecting the terms and reducing the values by means of the equations $x-w=0$, $z+y=0$, or say by writing $x=w$, $-y=z$, we have

$$\begin{array}{lll}
A= & 64w^2(x+w)^2 & =256w^5 \\
 & +8w^2(z-y)^2(x-3w) & -64w^3z^2 \\
 & +\frac{1}{4}w^2(z-y)^4 & +4wz^4 \\
 & & =4w(z^2-8w^2)^2, \\
B= & -32(z-y)(x+w)xw^2 & =-128w^4z \\
 & -w(z-y)^3(3x-7w) & +32w^2z^3 \\
 & +y^3z(z-y) & -2z^5 \\
 & & =-2z(z^2-8w^2)^2, \\
C= & 8w^2(x+w^2)(x+3w) & =128w^5 \\
 & +\frac{1}{2}w(z-y)^3(11x^2-27w^2) & -32w^2z^2 \\
 & -xy^3(z-y) & +2wz^4 \\
 & -xy^3(z-y) & +2wz^4 \\
 & & =2w(z^2-8w^2)^2.
\end{array}$$

Hence the condition $4AC-B^2=0$ of a pinch-point is $(z^2-8w^2)^3=0$, so that the pinch-points (if any) would be at the points $x-w=0$, $y+z=0$, $z^2-8w^2=0$; or say at x, y, z, w $=1, -2\sqrt{2}, 2\sqrt{2}, 1$. But these values give $L, M, N=12, 12, -16$; values which satisfy the equations $L^2-12w^2M=0$, $LM+9w^2N=0$, $4M^2+3LN=0$, and as the points in question are obviously not on the line $y=0$, $w=0$, they lie on the ninthic component of the cuspidal curve, being in fact points β', and not pinch-points.

The line $y=0$, $w=0$ *quà* nodal line would have every point a pinch-point, but being part of the cuspidal curve, no point thereof is regarded as a pinch-point; that is, in regard to this line also we have $j'=0$. And therefore for the entire nodal curve $j'=0$.

115. The cuspidal ninthic curve is a unicursal curve, the equations of which can be very readily obtained by considering it as the reciprocal of the spinode torse; we in fact have

$$x : y : z : w = ZW + 2XY : 2YZ + X^2 : WX + Y^2 - 3Z^2 : ZX,$$

or substituting for X, Y, Z, W their values $(=16\theta^2,\ 4\theta+16\theta^5,\ 16\theta^3,\ -5-8\theta^4-16\theta^8)$ and omitting a common factor $16\theta^2$, we find for the cuspidal curve

$$x : y : z : w = 3\theta + 24\theta^5 - 16\theta^9 : 24\theta^2 + 32\theta^6 : -4 - 48\theta^4 : 16\theta^3$$

(values which verify the equation $Xx + Yy + Zz + Ww = 0$); the spinode curve being thus of the order $=9$ as mentioned.

For $\theta = \infty$ we have the singular point $(y=0,\ z=0,\ w=0)$ (reciprocal of torsal biplane), and in the vicinity thereof $x : y : z : w = 1 : -2\theta^{-3} : 3\theta^{-5} : -\theta^{-6}$, therefore

$$\left(\frac{y}{x}\right)^2 = -4\frac{w}{x},\quad \left(\frac{y}{x}\right)^5 = -\tfrac{32}{27}\left(\frac{z}{x}\right)^3.$$

For $\theta = 0$ we have the singular point $x=0,\ y=0,\ w=0$ (reciprocal of the other biplane), and in the vicinity thereof $x : y : z : w = -\frac{3}{4}\theta : -6\theta^2 : 1 : -4\theta^3$, therefore

$$\frac{y}{z} = -\tfrac{32}{3}\left(\frac{x}{z}\right)^2,\quad \frac{w}{z} = \tfrac{256}{27}\left(\frac{x}{z}\right)^3.$$

116. The section of the surface by the plane $z=0$ is an interesting curve. Writing $z=0$ in the equation of the surface, I find that the resulting equation may be written

$$(64w^3,\ 144xw^2,\ w^3 + 76x^2w + xy^2 \between w^2 + 27x^2,\ y^2 - 32xw)^2 = 0,$$

where observe that

$$64w^3(w^3 + 76x^2w + xy^2) - (72xw^2)^2 = 64w^3[w(w^3 + 27x^2) + x(y^2 - 32xw)];$$

so that the curve has the four cusps $w^2 + 27x^2 = 0$, $y^2 - 32xw = 0$; the plane $z=0$ intersects the cuspidal ninthic curve in the point $(y=0,\ z=0,\ w=0)$ counting 5 times, and in the last-mentioned four points: in fact, writing in the equations of the ninthic curve $z=0$, that is $1 + 12\theta^4 = 0$, we find $x,\ y,\ w = \frac{8}{9}\theta,\ \frac{64}{3}\theta^2,\ 16\theta^3$, and thence $w^2 + 27x^2 = \frac{64}{3}\theta^2(1 + 12\theta^4) = 0$, $y^2 - 32xw = 0$.

The curve has also nodes at the points $(y=0,\ x+w=0;\ y=0,\ x-w=0)$, viz. these are the intersections of the plane $z=0$ with the nodal lines $(y-z=0,\ x+w=0)$ and $(y+z=0,\ x-w=0)$, and it has at the point $(y=0,\ w=0)$ (intersection of its plane with the cusp-nodal line $y=0,\ w=0$, and quintic intersection with the cuspidal ninthic) a singular point $=2$ cusps $+7$ nodes; hence the curve has cusps $=(4+2=)6$; nodes $(2+7=)9$; or 2 nodes $+3$ cusps $=36$; class $=6$, as it should be.

Section VIII $= 12 - 3C_2$.

Article Nos. 117 to 125. Equation $Y^3 + Y(X+Z+W) + 4aXZW = 0$.

117. The diagram of the lines and planes is

VIII $= 12 - 3C_2$.		Lines. 6	5	4	3	2	1	9	8	7	9	8	7	
		$\overline{12}$ $\overline{27}$			$6 \times 2 = 12$				$3 \times 1 = 3$			$3 \times 4 = 12$		
Planes. 7	$3 \times 2 = 6$									•			• •	Planes each touching along an axis and containing the corresponding transversal.
8									•	•		• •		
9								• •			• •			
12	$3 \times 2 = 6$					•	•			•				Biradial planes each containing two rays of the same node.
34				•	•				•	•				
56		•	•					• •						
13	$6 \times 4 = 24$				•		•					•		Planes each containing an axis, and two rays through the terminal nodes respectively.
24				•		•						•		
16		•		•		•	•				•			
25		•	•			•	•				•			
46		•	•	•		•							•	
35		•	•	•	•								•	
789	$1 \times 8 = 8$										•	•	•	Plane through the three axes.
$\overline{789}$	$1 \times 1 = 1$										•	•	•	Plane through the three transversals.
	$\overline{14}$ $\overline{45}$													
		Rays, 1, 2 through node 89. 3, 4 " 97. 5, 6 " " 78.						Transversals, meeting the three axes respectively.			Axes, each joining two nodes.			

118. Take m_1, m_2 as the roots of the equation $(m-1)^2 = 4am$, so that $m_1 + m_2 = 2 + 4a$, $m_1 m_2 = 1$, then the planes are

$$X = 0, \qquad [\ 7]$$
$$Y = 0, \qquad [\ 8]$$
$$Z = 0, \qquad [\ 9]$$
$$Y + Z + X = 0, \qquad [12]$$
$$Y + X + W = 0, \qquad [34]$$
$$Y + Z + W = 0, \qquad [56]$$
$$Y = (m_1 - 1)\,X, \qquad [13]$$
$$Y = (m_2 - 1)\,X, \qquad [24]$$
$$Y = (m_2 - 1)\,Z, \qquad [16]$$
$$Y = (m_1 - 1)\,Z, \qquad [25]$$
$$Y = (m_1 - 1)\,W, \qquad [46]$$
$$Y = (m_2 - 1)\,W, \qquad [35]$$
$$Y = 0, \qquad [789]$$
$$Y + X + Z + W = 0, \qquad [\overline{789}]$$

119. And the lines are

a	b	c	f	g	h	equations may be written
0	0	0	0	0	1	(7) $X = 0,\ Y = 0$
0	0	0	1	0	0	(8) $Z = 0,\ Y = 0$
0	1	0	0	0	0	(9) $W = 0,\ Y = 0$
1	1	1	0	0	0	$(\bar{7})$ $Y + Z + X = 0,\ W = 0$
0	0	1	-1	1	0	$(\bar{8})$ $Y + X + W = 0,\ Z = 0$
1	0	0	0	-1	1	$(\bar{9})$ $Y + Z + W = 0,\ X = 0$
0	0	0	$\frac{1}{m_1 - 1}$	1	$\frac{1}{m_2 - 1}$	(1) $Y = (m_1 - 1)\,X = (m_2 - 1)\,Z$
0	0	0	$\frac{1}{m_2 - 1}$	1	$\frac{1}{m_1 - 1}$	(2) $Y = (m_2 - 1)\,X = (m_1 - 1)\,Z$
-1	$\frac{1}{m_1 - 1}$	0	0	0	$\frac{1}{m_2 - 1}$	(3) $Y = (m_2 - 1)\,W = (m_1 - 1)\,X$
-1	$\frac{1}{m_2 - 1}$	0	0	0	$\frac{1}{m_1 - 1}$	(4) $Y = (m_1 - 1)\,W = (m_2 - 1)\,X$
0	$-\frac{1}{m_1 - 1}$	1	$\frac{1}{m_2 - 1}$	0	0	(5) $Y = (m_1 - 1)\,Z = (m_2 - 1)\,W$
0	$-\frac{1}{m_2 - 1}$	1	$\frac{1}{m_1 - 1}$	0	0	(6) $Y = (m_2 - 1)\,Z = (m_1 - 1)\,W$

120. The three transversals are each facultative; $\rho' = b' = 3$; $t' = 0$.

121. Hessian surface. The equation is

$$4aXZW(3Y + X + Z + W) + Y^2(X^2 + Z^2 + W^2 - 2XZ - 2XW - 2ZW) = 0.$$

The complete intersection with the cubic surface is made up of the lines ($Y=0$, $X=0$), ($Y=0$, $Z=0$), ($Y=0$, $W=0$) (the axes) each twice, and of a sextic curve which is the spinode curve; $\sigma' = 6$.

The spinode curve is a complete intersection 2×3; the equations may in fact be written

$$Y^3 + Y^2(X + Z + W) + 4aXZW = 0,$$
$$3Y^2 + 4Y(X + Z + W) + 4(XZ + XW + ZW) = 0;$$

the nodes D, C, A are nodes (double points) of the curve, the tangents at each node being the nodal rays.

Each of the transversals is a single tangent of the spinode curve; in fact for the transversal $Y + Z + X = 0$, $W = 0$, these equations of course satisfy the equation of the cubic surface; and substituting in the equation of the Hessian, we have $Y^2(X - Z)^2 = 0$. But $Y + Z + X = 0$, $W = 0$, $Y = 0$ is a point on the axis $W = 0$, $Y = 0$, not belonging to the spinode curve; we have only the point of contact $Y + X + Z = 0$, $W = 0$, $X - Z = 0$. Hence $\beta' = 3$.

Reciprocal Surface.

122. The equation is found by means of the binary cubic,

$$aT(T - yU)^2 + (T - xU)(T - zU)(T - wU),$$

viz. writing for shortness

$$\beta = x + z + w,$$
$$\gamma = xz + xw + zw,$$
$$\delta = xzw;$$

this is a binary cubic $(*\between T, U)^3$, the coefficients whereof are

$$3(a+1),\ -2ay - \beta,\ ay^2 + \gamma,\ -3\delta,$$

and the equation is hence found to be

$$\begin{aligned}
&4a^3y^3(y^3 - \beta y^2 + \gamma y - \delta) \\
&+ a^2\{(12\gamma - \beta^2)y^4 - (8\beta\gamma + 36\delta)y^3 + (30\beta\delta + 8\gamma^2)y^2 - 36\gamma\delta y + 27\delta^2\} \\
&+ 2a\{(6\gamma^2 - \beta^2\gamma - 9\beta\delta)y^2 + (12\beta^2\delta - 2\beta\gamma^2 - 18\gamma\delta)y + 2\gamma^3 + 27\delta^2 - 9\beta\gamma\delta\} \\
&- (\beta^2\gamma^2 + 18\beta\gamma\delta - 4\beta^3\delta - 4\gamma^3 - 27\delta^2) = 0;
\end{aligned}$$

or substituting for β, γ, δ in the first and last lines their values

$$(=x+z+w,\ xz+xw+zw,\ xzw),$$

this is

$$\begin{aligned}
&4a^2y^3(y-x)(y-z)(y-w)\\
&+a^2\{(12\gamma-\beta^2)y^4-(8\beta\gamma+36\delta)y^3+(30\beta\delta+8\gamma^2)y^2-36\gamma\delta y+27\delta^2\}\\
&+2a\{(6\gamma^2-\beta^2\gamma-9\beta\delta)y^3+(12\beta^2\delta-2\beta\gamma^2-18\gamma\delta)y+2\gamma^3+27\delta^2-9\beta\gamma\delta\}\\
&-(x-z)^2(x-w)^2(z-w)^2=0.
\end{aligned}$$

123. The nodal curve is made up of the lines $(y=x=z)$, $(y=x=w)$, $(y=z=w)$, reciprocals of the three transversals.

To show this I remark that, writing

$$\begin{aligned}
\beta' &= (x-y)+(z-y)+(w-y),\\
\gamma' &= (x-y)(z-y)+(x-y)(w-y)+(z-y)(w-y),\\
\delta' &= (x-y)(z-y)(w-y),
\end{aligned}$$

the equation of the surface may be written

$$\begin{aligned}
&4a^2y^3(y-x)(y-z)(y-w)\\
&+a^2\{y^2(12\beta'\delta'-\gamma'^2)+x\,.\,18\gamma'\delta'+27\delta'^2\}\\
&+2a\{y(-6\beta'^2\delta'+2\beta'\gamma'^2+9\gamma'\delta')+2\gamma'^3+27\delta'^2-9\beta'\gamma'\delta'\}\\
&-(x-z)^2(x-w)^2(z-w)^2=0,
\end{aligned}$$

whence observing that γ' is of the order 1 and δ' of the order 2 in $(x-y)$, $(z-y)$ conjointly, each term of the equation is at least of the second order in $(x-y)$, $(z-y)$ conjointly; or we have $y=x=z$, a nodal line; and similarly the other two lines are nodal lines.

124. The foregoing transformed equation is most readily obtained by reverting to the cubic in T, U, viz. writing $p=x-y$, $r=z-y$, $s=w-y$, and therefore $x=y+p$, $z=y+r$, $w=y+s$, the cubic function (putting therein $T=V+yU$) becomes

$$a(V+yU)V^2+(V-pU)(V-rU)(V-sU);$$

writing β', γ', $\delta'=p+r+s$, $pr+ps+rs$, prs, the coefficients are $(3(a+1),\ ay-\beta',\ \gamma',\ -3\delta')$, and the equation of the surface is thus obtained in the form

$$\begin{aligned}
&27(a+1)^2\delta'^2\\
&+18(a+1)(ay-\beta')\gamma'\delta'\\
&+\ \ 4(a+1)\gamma'^3\\
&-\ \ 4(ay-\beta')^3\delta'\\
&-\ \ \ (ay-\beta')^2\gamma'^2=0,
\end{aligned}$$

which, arranging in powers of a, and reversing the sign, is the foregoing transformed result.

125. The cuspidal curve is given by the equations

$$\left\| \begin{matrix} 3(a+1), & -2ay-\beta, & ay^2+\gamma \\ -2ay-\beta, & ay^2+\gamma, & -\delta \end{matrix} \right\| = 0,$$

or say by the equations

$$3(a+1)(ay^2+\gamma)-(2ay+\beta)^2=0,$$

that is

$$a(a-3)y^2+4a\beta y-3(a+1)\gamma=0,$$

and

$$-3(a+1)\delta+(2ay+\beta)(ay^2+\gamma)=0,$$

consequently $c'=6$. It is to be added that the cuspidal curve is a complete intersection, 2×3.

Section IX $= 12-2B_3$.

Article Nos. 126 to 136. Equation $WXZ+(a,\ b,\ c,\ d\between X,\ Y)^3=0$.

126. The diagram of the lines and planes is

Planes.		Lines. 6 5 4 3 2 1	0	
IX = 12 − 2B_3.		7 / 27 ; 6×3 = 18	1×9 = 9	
0	1×6 = 6		. . . .	Common biplane, osculer along the axis.
7, 8	2×6 = 12	. . . . .		Other biplanes of the two binodes respectively.
14, 25, 36	3×9 = 27	. . . . .	. . .	Planes each through the axis and containing rays of the two binodes respectively.
6	45	. . . . .		
		Rays, 1, 2, 3 in the non-axial biplane 7 of the one binode, and 4, 5, 6 in the non-axial biplane 8 of the other binode.	Axis joining the two binodes.	

127. Writing $(a, b, c, d \between X, Y)^3 = -d(f_1X - Y)(f_2X - Y)(f_3X - Y)$, the planes are

$$\begin{aligned} X &= 0, && [\ 0] \\ Z &= 0, && [\ 7] \\ W &= 0, && [\ 8] \\ f_1X - Y &= 0, && [14] \\ f_2X - Y &= 0, && [25] \\ f_3X - Y &= 0, && [36]; \end{aligned}$$

and the lines are

$$\begin{aligned} X &= 0, & Y &= 0, & (0) \\ f_1X - Y &= 0, & Z &= 0, & (1) \\ f_2X - Y &= 0, & Z &= 0, & (2) \\ f_3X - Y &= 0, & Z &= 0, & (3) \\ f_1X - Y &= 0, & W &= 0, & (4) \\ f_2X - Y &= 0, & W &= 0, & (5) \\ f_3X - Y &= 0, & W &= 0, & (6). \end{aligned}$$

128. There is no facultative line; $\rho' = b' = 0$; $t' = 0$; and hence also $\beta' = 0$.

129. Hessian surface. The equation is

$$X\{ZW(cX + dY) - 3X(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2\} = 0,$$

so that the Hessian breaks up into the plane $X = 0$ (axial or common biplane) and into a cubic surface.

The complete intersection of the Hessian with the cubic surface is made up of the line $X = 0$, $Y = 0$ (the axis) four times; and of a system of four conics, which is the spinode curve; $c' = 8$.

In fact combining the equations

$$WXZ + (a, b, c, d \between X, Y)^3 = 0$$

and

$$ZW(cX + dY) - 3X(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2 = 0,$$

these intersect in the axis once, and in a curve of the eighth order which breaks up into four conics; for we can from the two equations deduce

$$(a, b, c, d \between X, Y)^3(cX + dY) + 3X^2(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2 = 0,$$

that is

$$(4ac - 3b^2,\ ad,\ bd,\ cd,\ d^2 \between X,\ Y)^4 = 0,$$

a system of four planes each intersecting the cubic $XZW + (a, b, c, d \between X, Y)^3 = 0$ in the axis and a conic; whence, as above, spinode curve is four conics.

It is easy to see that the tangent planes along any conic on the surface pass through a point, and form therefore a quadric cone; hence in particular the spinode torse is made up of the quadric cones which touch the surface along the four conics respectively.

Reciprocal Surface.

130. The equation is obtained by means of the binary cubic

$$X\,(xX+yY)^2+4zw\,(a,\ b,\ c,\ d\between X,\ Y)^3,$$

viz. calling this $(*\between X,\ Y)^3$ the coefficients are

$$(3x^2+12azw,\ 2xy+12bzw,\ y^2+12czw,\ 12dzw).$$

The equation is found to be

$$\begin{aligned}
&432\,(a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2)\,z^2w^2\\
+\ &216\,[(ad^2-3bcd+2c^3)\,x^2+(-2acd+4b^3d-2bc^2)\,xy+(-abd+2ac^2-b^2c)\,y^2]\,z^2w^2\\
+\ &\ \ 9\,[3d^2x^4-12cdx^3y+(10bd+8c^2)\,x^2y^2-(4ad+8bc)\,xy^3+(4ac-b^2)\,y^4]\,zw\\
-\ &\ \ y^3\,(dx^3-3cx^2y+3bxy^2-ay^3)=0.
\end{aligned}$$

The section by the plane $w=0$ (reciprocal of $B_3=D$) is the line $w=0,\ y=0$ (reciprocal of edge) three times, and the lines $w=0,\ dx^3-3cx^2y+3bxy^2-ay^3=0$ (reciprocals of the biplanar rays). And similarly for the section by the plane $z=0$ (reciprocal of $B_3=C$).

The section by the plane $y=0$ is made up of the lines $(y=0,\ z=0)$, $(y=0,\ w=0)$ each once, and of two conics, $y=0$,

$$\begin{aligned}
&16\,(a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2)\,z^2w^2\\
+\ &\ \ 8\,(ad^2-3bcd+2c^2)\,x^2zw\\
+\ &\ \ d^2x^4=0.
\end{aligned}$$

131. There is not any nodal curve; $b'=0$.

132. Cuspidal curve. The equations may be written

$$\left\|\begin{matrix} 3x^2+12azw, & 2xy+12bzw, & y^2+12czw \\ 2xy+12bzw, & y^2+12czw, & 12dzw \end{matrix}\right\|=0.$$

Forming the equations

$$\begin{aligned}
(bd-c^2)\,.\,144z^2w^2+2\,(dxy-cy^2)\,.\,12zw\ -y^4&=0,\\
(ad-bc)\,.\,144z^2w^2+(3dx^2-2cxy-by^2)\,.\,12zw\ -2xy^3&=0,
\end{aligned}$$

these are two quartic surfaces having in common the lines $(y=0,\ w=0)$, $(y=0,\ z=0)$: attending to the line $(y=0,\ z=0)$, this is on the second surface an oscular line, $z=\dfrac{1}{18dxw}y^3$; on the first surface it is a nodal line, the one tangent plane being $6\,(bd-c^2)\,w\,.\,z+dxz\,.\,y=0$, the other tangent plane being $z=0$, but the line being in regard to this sheet an oscular line, $z=\dfrac{1}{24dxw}y^3$. Hence in the intersection of the two surfaces the line counts $(1+3=)$ 4 times; similarly the line $y=0,\ w=0$ counts $(1+3=)$ four times; and there is a residual intersection of the order $(16-4-4=)$ 8, which is the cuspidal curve; $c'=8$.

133. The cuspidal curve is a system of four conics; in fact from the preceding equations written in the forms

$$(bd-c^2,\ 2(dxy-cy^2),\ -y^4 \between 12zw,\ 1)^2=0,$$
$$(ad-bc,\ 3dx^2-2cxy-by^2,\ -2xy^2 \between 12zw,\ 1)^2=0,$$

eliminating zw, we obtain

$$\left\{\begin{array}{l} 3(bd-c^2), \\ 2(-ad^2-3bcd+4c^3), \\ 6(acd+b^2d-2bc^2), \\ 6(bc-ad)\,b, \\ a^2d-b^3, \end{array}\right\} x,\ y)^4=0,$$

which shows that the cuspidal curve lies in four planes, and it hence consists of four conics; these are of course the reciprocals of the quadric cones which touch the cubic surface along the four conics which make up the spinode curve.

134. The equation of the surface, attending only to the terms of the second order in y, z, w, is $27d^2x^4zw=0$; it thus appears that the point $y=0$, $z=0$, $w=0$ (reciprocal of the plane $X=0$) (which is oscular along the axis joining the two binodes, or BB-axis) is a binode on the reciprocal surface, the biplanes being $z=0$, $w=0$, viz. these are the planes reciprocal to the binodes $(X=0,\ Y=0,\ W=0)$ and $(X=0,\ Y=0,\ Z=0)$ of the cubic surface; we have thus $B'=1$.

It is proper to remark that the binode $y=0$, $z=0$, $w=0$ is not on the cuspidal curve, as its being so would probably imply a higher singularity.

135. A simple case, presenting the same singularities as the general one, is when $a=d$, $b=c=0$: to diminish the numerical coefficients assume $a=d=\frac{1}{12}$, the cubic surface is thus $12XZW+X^3+Y^3=0$, and the equation of the reciprocal surface, multiplying it by 4, becomes

$$\begin{aligned} & z^3w^3 \\ & +6x^3z^2w^2 \\ & +(9x^4-12x^3y)\,zw \\ & -4y^3(x^3-y^3)=0, \end{aligned}$$

viz. this is the surface

$$\begin{aligned} & 4y^6 \\ & -4y^3x(x^2+3zw) \\ & +zw(3x^2+zw)^2=0 \end{aligned}$$

considered in the Memoir "On the Theory of Reciprocal Surfaces." The cuspidal curve is, as there shown, composed of the four conics $y=0$, $3x^2+zw=0$ and $y^3-2x^3=0$, $x^2-zw=0$; and it is there shown that the two points $(x=0,\ y=0,\ z=0)$, $(x=0,\ y=0,\ w=0)$, each reckoned eight times, are to be considered as off-points of the reciprocal surface.

136. The like investigation applies to the general surface, and we have thus $\theta' = 16$; the points in question are still the points $(x=0, y=0, z=0)$, $(x=0, y=0, w=0)$; viz. these are the points of intersection of the surface by the line $(x=0, y=0)$, which points are also the common points of intersection of the four conics which compose the cuspidal curve, that is, they are quadruple points on the cuspidal curve; it does not appear that the points are on this account, viz. *quà* quadruple points of the cuspidal curve, off-points of the surface, nor does this even show that the points should be reckoned each eight times. As already remarked, the singularity requires a more complete investigation.

Section $X = 12 - B_4 - C_2$.

Article Nos. 137 to 143. Equation $WXZ + (X+Z)(Y^2 - X^2) = 0$.

137. The diagram of the lines and planes is

Planes.		Lines.					
	$X=12-B_4-C_2$.	12	1′, 2′	1, 2	3	0	
		$1\times1=1$ / 7 / 27	$2\times2=4$	$2\times4=8$	$1\times6=6$	$1\times8=8$	
0	$1\times12=12$				··	·····	Biplane touching along axis, and containing edge.
3	$1\times12=12$			·····	··		Other biplane.
11′	$2\times8=16$		·····	··		·	Planes each through the axis and containing a biplanar ray and a cnicnodal ray.
22′			·····	··		·	
3′	$1\times3=3$	···			··		Plane touching along the edge and containing the mere line.
1′2′	$1\times2=2$ / 6 / 45	··	· ·				Biradial plane through the two cnicnodal rays.
		Mere line, being a transversal.	Cnicnodal rays.	Biplanar rays in the non-axial biplane.	Edge of binode, being a transversal.	Axis, through the two nodes.	

138. The planes are and the lines are

$X=0,$	[0]	$X=0,\ Y=0,$	(0)
$Z=0,$	[3]	$X=0,\ Z=0,$	(3)
$X-Y=0,$	[11′]	$X-Y=0,\ Z=0,$	(1)
$X+Y=0,$	[22′]	$X+Y=0,\ Z=0,$	(2)
$W=0,$	[3′]	$X-Y=0,\ W=0,$	(1′)
$X+Z=0,$	[1′2′]	$X+Y=0,\ W=0,$	(2′)
		$X+Z=0,\ W=0,$	(12).

139. The facultative lines are the edge counting twice, and the mere line;

$$\rho=b'=3;\ t'=1.$$

140. Hessian surface. The equation is

$$X(X+Z)(ZW+3X^2-XZ)+Y^2(X-Z)^2=0.$$

The complete intersection with the surface consists of the line $(X=0,\ Y=0)$, the axis, four times; the line $(X=0,\ Z=0)$, the edge, twice; and a sextic curve, which is the spinode curve; $c'=6$.

Writing the equations of the surface and the Hessian in the form

$$X(ZW+Y^2)-X^3+Z(Y^2-X^2)=0,$$
$$X(X+Z)(ZW+Y^2)+(Z-3X)\{-X^3+Z(Y^2-X^2)\}=0,$$

we see that the equations of the spinode curve may be written

$$ZW+Y^2=0,$$
$$-X^3+Z(Y^2-X^2)=0,$$

viz. the curve is a complete intersection, 2×3.

There is at B_4 a triple point $\frac{Y}{W}=-\left(\frac{Z}{W}\right)^2,\ \frac{X}{W}=-\left(\frac{Z}{W}\right)^{\frac{3}{2}}$; and at C_2 a double point, the tangents coinciding with the nodal rays $W=0,\ Y^2-X^2=0$.

The edge and the mere line are each of them single tangents of the spinode curve. But the edge counting twice in the nodal curve, its contact with the spinode curve will also count twice, that is, we have $\beta'=2.1+1,\ =3$.

Reciprocal Surface.

141. The equation is obtained by means of the binary cubic

$$4w^2X(X+Z)^2+4wZ(X+Z)(xX+zZ)+y^2XZ^2;$$

or calling this $(*\between X,\ Z)^3$, the coefficients are

$$(12w^2,\ 8w^2+4wx,\ 4w^2+4wx+4wz+y^2,\ 12wz),$$

and thence the equation is found to be

$$\begin{aligned}
&16w^4[y^2-(x-z)^2]\\
+\,&16w^3[(2x-5z)y^2-2(x-2z)(x-z)^2]\\
+\,&8w^2[y^4+(x^2-xz+6z^2)y^2-2x^2(x-z)^2]\\
+\,&4w[(2x+3z)y^4-2x^2(x+z)y^2]\\
+\,&y^4(y^2-x^2)=0,
\end{aligned}$$

where the section by the plane $w=0$ (reciprocal of binode) is $y^4(y^2-x^2)=0$, viz. this is the line $w=0$, $y=0$ (reciprocal of the edge) four times, and the lines $w=0$, $y^2-x^2=0$ (reciprocals of the biplanar rays).

The section by the plane $z=0$ is found to be $(y^2-x^2)(y^2+4xw+4w^2)^2=0$, viz. this is the two lines $z=0$, $y^2-x^2=0$ (reciprocals of the nodal rays), and the conic $z=0$, $y^2+4xw+4w^2=0$ (reciprocal of the nodal cone $WX+Y^2-X^2=0$) twice.

142. Nodal curve. The equation shows that the line $y=0$, $x-z=0$ (reciprocal of the line $W=0$, $X+Z=0$) is a nodal line on the surface.

It also shows that the line $y=0$, $w=0$ (reciprocal of the edge) is a tacnodal line ($=2$ nodal lines) on the surface; in fact attending only to the lowest terms in y, w, we have

$$-x^2[16(x-z)^2w^2+8(x+z)wy^2+y^4]=0,$$

that is,

$$4(x-z)w+\frac{\sqrt{x}\pm\sqrt{z}}{\sqrt{x}\mp\sqrt{z}}y^2=0,$$

two values, $w=Ay^2$, $w=By^2$, which indicates a tacnodal line.

The nodal curve is thus made up of the line $y=0$, $x-z=0$ once, and the line $y=0$, $w=0$ twice; $b'=3$.

143. Cuspidal curve. The equations

$$\left\|\begin{matrix} 12w^2, & 8w^2+4wx, & 4w^2+4wx+4wz+y^2 \\ 8w^2+4wx, & 4w^2+4wx+4wz+y^2, & 12wz \end{matrix}\right\|=0$$

give

$$\begin{aligned}
&(4w+2x)^2-3(4w^2+4wx+4wz+y^2)=0,\\
&-36w^2z+(2w+x)(4w^2+4wx+4wz+y^2)=0,
\end{aligned}$$

or, as these are more simply written,

$$\begin{aligned}
&4w^2+4wx-12wz+4x^2-3y^2=0,\\
&8w^3+12w^2x-28w^2z+w(4x^2+4xz+2y^2)+xy^2=0,
\end{aligned}$$

so that the cuspidal curve is a complete intersection 2×3; $c'=6$.

Section $\mathrm{XI} = 12 - B_6$.

Article Nos. 144 to 149. Equation $WXZ + Y^2Z + X^3 - Z^3 = 0$.

144. The diagram of the lines and planes is

					Lines are $X=0,\ Z=0$
			$X=0,\ Y-Z=0$	$X=0,\ Y+Z=0$	
			2	1	0
$\mathrm{XI} = 12 - B_6$. Planes are			$\frac{2}{3} \times 6 = \frac{12}{27}$		$1 \times 15 = 15$
$X=0$	0	$1 \times 15 = 15$			
$X=0$	3	$1 \times 30 = \frac{30}{2}$ $\frac{45}{}$			
			Rays in the ordinary biplane.		Edge.

			Osculer biplane.
			Ordinary biplane.

where the equations of the lines and planes are shown in the margins of the diagram.

145. The edge is a facultative line counting three times; this will appear from the discussion of the reciprocal surface. Therefore $\rho' = b' = 3$; $t' = 1$.

146. Hessian surface. This is

$$Z(WXZ + Y^2Z - 3X^3 - 3Z^3) = 0,$$

breaking up into $Z = 0$, the oscular biplane, and into a cubic surface (itself a surface $\mathrm{XI} = 12 - B_6$). The complete intersection with the cubic surface is made up of the line $X = 0$, $Z = 0$ (the edge) six times, and of a residual sextic ($= 3$ conics), which is the spinode curve; $c' = 6$.

The equations of the sextic are in fact $XZ + Y^2 = 0$, $X^3 + Z^3 = 0$, so that this consists of three conics, each in a plane passing through the edge.

The edge touches each of the three conics at the point $X = 0$, $Z = 0$, $Y = 0$; but it must be reckoned as a single tangent of the spinode curve, and then counting it three times, $\beta' = 3$.

Reciprocal Surface.

147. The equation is obtained by means of the binary cubic

$$(12w^2,\ 4zw,\ y^2+4xw,\ -12w^2 \between Z,\ X)^3,$$

viz. it is

$$\begin{aligned} & 432w^6 \\ + & \ 72w^2z\,(4xw+y^2) \\ - & \ 64w^2z^2 \\ + & \qquad (4xw+y^2)^2 \\ - & \qquad z^2\,(4xw+y^2)^2 = 0, \end{aligned}$$

or, completely developed, it is

$$\begin{aligned} & w^6 \,.\, 432 \\ & + w^4 \,.\, 288xz \\ & + w^3 \,.\, 72y^2z + 64x^3 - 64z^3 \\ & + w^2 \,.\, 48x^2y^2 - 16x^2z^2 \\ & + w \,.\, 12xy^4 - 8xy^2z^2 \\ & + \quad y^4\,(y^2-z^2) = 0\,; \end{aligned}$$

the section by the plane $w=0$ (reciprocal of B_6) is $w=0$, $y=0$ (reciprocal of edge) four times, together with $w=0$, $y^2-z^2=0$, reciprocals of the two rays.

148. The nodal curve is the line $y=0$, $w=0$ (reciprocal of edge counting as three lines); $h'=3$. In fact the form of the surface in the vicinity is given by $w=-\frac{1}{4x}y^2 \pm \frac{1}{4}\sqrt{\frac{z}{x^2}}\,y^3$, viz. there are two sheets osculating along the line in question, that is intersecting in this line taken three times.

149. For the cuspidal curve we have

$$\left\| \begin{matrix} 12w^2, & 4zw, & y^2+4xw \\ 4zw, & y^2+4xw, & -12w^2 \end{matrix} \right\| = 0,$$

giving

$$\begin{aligned} 12xw + 3y^2 \quad - 4z^2 &= 0, \\ 36w^2 + 4wxz + y^2z &= 0\,; \end{aligned}$$

or multiplying the first by $3z$ and subtracting the second, we have $108w^3+4z^3=0$. Hence the equations are

$$\begin{aligned} z^3 + 27w^3 &= 0, \\ 12xw + 3y^2 - 4z^2 &= 0, \end{aligned}$$

viz. the cuspidal curve is made up of three conics lying in planes through the line $z=0$, $w=0$.

The curve may be put in evidence by writing the equation of the surface in the form

$$(3y^2+5z^2+12xw,\ 24z,\ 16 \between 3y^2-4z^2+12xw,\ z^2+27w^2)^2 = 0,$$

where

$$16\,(3y^2+5z^2+12xw) - 144z^2 = 16\,(3y^2-4z^2+12xw).$$

Section XII $= 12 - U_6$.

Article Nos. 150 to 156. Equation $W(X+Y+Z)^2 + XYZ = 0$.

150. The diagram of the lines and planes is

Planes.		Lines. 3′	2′	1′	3	2	1	
XII $= 12 - U_6$.		$\overline{6}$ $\overline{27}$	$3 \times 1 = 3$			$3 \times 8 = 24$		
0	$1 \times 32 = 32$				⋮	⋮	⋮	Uniplane.
1				⋮			⋮	Planes each touching along a ray, and containing a mere line.
2	$3 \times 4 = 12$		⋮		⋮			
3		⋮			⋮			
1′2′3′	$1 \times 1 = 1$							Plane through the three mere lines.
	$\overline{5}$ $\overline{45}$							
			Mere lines.			Rays in the plane.		

151. The planes are

$X + Y + Z = 0$,	[0]
$X = 0$,	[1]
$Y = 0$,	[2]
$Z = 0$,	[3]
$W = 0$,	[1′2′3′]

The lines are

$X = 0,\ Y + Z = 0$,	(1)
$Y = 0,\ Z + X = 0$,	(2)
$Z = 0,\ X + Y = 0$,	(3)
$X = 0,\ W = 0$,	(1′)
$Y = 0,\ W = 0$,	(2′)
$Z = 0,\ W = 0$,	(3′).

152. The three mere lines are each facultative: $\rho' = b' = 3$; $t' = 1$.

153. Hessian surface. The equation is

$$(X+Y+Z)^2(X^2+Y^2+Z^2-2YZ-2ZX-2XY) = 0,$$

viz. the surface consists of the uniplane $X+Y+Z=0$ twice, and of a quadric cone having its vertex at U_6, and touching each of the planes $X=0$, $Y=0$, $Z=0$.

The complete intersection with the cubic surface is made up of the rays each twice and of a residual sextic, which is the spinode curve; $\sigma' = 6$.

The equations of the spinode curve are

$$W(X+Y+Z)^2 + XYZ = 0,$$
$$X^2 + Y^2 + Z^2 - 2YZ - 2ZX - 2XY = 0,$$

viz. the curve is a complete intersection, 2×3.

Each of the mere lines is a single tangent (as at once appears by writing for instance $W=0$, $X=0$, which gives $(Y-Z)^2 = 0$); that is, $\beta' = 3$.

Reciprocal Surface.

154. The equation is found by means of the binary cubic

$$4(T-xU)(T-yU)(T-zU) + wT^2U,$$

viz. writing for shortness

$$\beta = x+y+z,$$
$$\gamma = yz+zx+xy,$$
$$\delta = xyz,$$

then the cubic function is

$$(12,\ w-4\beta,\ 4\gamma,\ -12\delta \between T,\ U)^3,$$

and the equation of the reciprocal surface is found to be

$$\begin{aligned} & 432\,\delta^2 \\ + & \ 64\,\gamma^3 \\ - & \quad (w-4\beta)^3\,\delta \\ + & \ 72\,(w-4\beta)\,\gamma\delta \\ - & \quad (w-4\beta)^2\,\gamma^2 = 0\,; \end{aligned}$$

expanding, this is

$$\begin{aligned} & w^3 \,.\, -\delta \\ + & \quad w^2 \,.\, 12\beta\delta - \gamma^2 \\ + & \ 8w \,.\, -6\beta^3\delta + \beta\gamma^2 + 9\gamma\delta \\ + & \qquad 16\,(4\beta^3\delta - \beta^2\gamma^2 - 18\beta\gamma\delta + 4\gamma^3 + 27\delta^2) = 0\,; \end{aligned}$$

or substituting for β, γ, δ in the first and last lines, this is

$$\begin{aligned} & w^3 \,.\, -xyz \\ + & \quad w^2 \,.\, (12\beta\delta - \gamma^2) \\ + & \ 8w \,.\, -6\beta^2\delta + \beta\gamma^2 + 9\gamma\delta \\ + & \qquad 16\,(y-z)^2(z-x)^2(x-y)^2 = 0 \end{aligned}$$

(where β, γ, $\delta = x+y+z$, $yz+zx+xy$, xyz). The section by the plane $w=0$ (reciprocal of the unode) is made up of the lines $w=0$, $y-z=0$; $w=0$, $z-x=0$; $w=0$, $x-y=0$ (reciprocals of the rays) each twice.

155. The nodal curve is at once seen to consist of the lines $(y=0, z=0)$, $(z=0, x=0)$, $(x=0, y=0)$, reciprocals of the facultative lines; in fact, in regard to (y, z) conjointly γ is of the order 1, and δ is of the order 2; hence every term of the equation is of the order 2 in y, z; and the like as to the other two lines: $b' = 3$ as above.

156. For the cuspidal curve we have

$$\left\| \begin{matrix} 12\ , & w-4\beta, & 4\gamma \\ w-4\beta, & 4\gamma\ , & -12\delta \end{matrix} \right\| = 0,$$

or say

$$48\gamma - (w-4\beta)^2 = 0,$$
$$36\delta + \gamma(w-4\beta) = 0,$$

whence the cuspidal curve is a complete intersection 2×3; $c' = 6$.

Section XIII $= 12 - B_3 - 2C_2$.

Article Nos. 157 to 164. Equation $WXZ + Y^2(Y+X+Z) = 0$.

157. The diagram of the lines and planes is

Planes.		Lines. 012	3 4	1 2	0	5 6	
XIII $= 12 - B_3 - 2C_2$.		$1 \times 1 = 1$ $\overline{8}$ $\overline{27}$	$2 \times 2 = 4$	$2 \times 3 = 6$	$1 \times 4 = 4$	$2 \times 6 = 12$	
1 2	$2 \times 6 = 12$						Biplanes.
056	$1 + 12 = 12$						Plane through the three axes.
5 6	$2 \times 6 = 12$						Planes each through an axis joining the binode with a cnicnode.
34	$1 \times 4 = 4$						Plane through the axis joining the two cnicnodes.
12	$1 \times 3 = 3$						Planes through the biplanar rays.
0	$1 \times 2 = 2$ $\overline{8}$ $\overline{45}$						Plane touching along the axis which joins the two cnicnodes.
		Transversal.	Cnicnodal rays, one through each cnicnode.	Biplanar rays, one in each biplane, and being each a transversal.	Axis joining the two cnicnodes.	Axes, each joining the binode with a cnicnode.	

158. The planes are The lines are

$X=0,$	[1]	$X=0,\ Y=0,$	(5)
$Z=0,$	[2]	$Z=0,\ Y=0,$	(6)
$Y=0,$	[056]	$Y=0,\ W=0,$	(0)
$Y+X=0,$	[5]	$X=0,\ Y+Z=0,$	(1)
$Y+Z=0,$	[6]	$Z=0,\ Y+X=0,$	(2)
$Y-W=0,$	[34]	$W=Y=-Z,$	(3)
$X+Y+Z=0,$	[12]	$W=Y=-X,$	(4)
$W=0,$	[0]	$W=0;\ X+Y+Z=0,$	(012)

159. The transversal is facultative; $\rho'=b'=1,\ t'=0.$

160. The Hessian surface is

$$WXZ(3Y+X+Z)+Y^2(Z-X)^2=0.$$

The complete intersection with the surface is made up of the line $Y=0,\ X=0$ (CB-axis) three times; the line $Y=0,\ Z=0$ (CB-axis) three times; line $Y=0,\ W=0$ (CC-axis) twice, and of a residual quartic, which is the spinode curve; $\sigma'=4.$

161. Representing the two equations by $U=0,\ H=0$, we have

$$(3Y+X+Z)\,U-H=Y^2(3Y^2+4YX+4YZ+4XZ),\ =MY^2 \text{ suppose,}$$

and

$$27\,(X+Z)\,U+9H=9\,WXZ(3Y+4X+4Z)+36\,Y^2(X^2+XZ+Z^2)+27\,Y^3(X+Z);$$

but

$$\big(-9\,(X+Z)\,Y+16XZ\big)\,M=$$
$$64X^2Z^2+28\,YXZ\,(X+Z)-Y^2(36X^2+28XZ+36Z^2)-27\,Y^3(X+Z),$$

whence

$$27\,(X+Z)\,U+9H+(-9XY-9ZY+16XZ)\,M$$
$$=ZX\,\{12Y^2+28\,Y\overline{X+Z}+64XZ+9\,W(3Y+4X+4Z)\};$$

or, as this may also be written,

$$\begin{aligned} &27Y^2(X+Z)\,U && +9Y^2H \\ &+(-9YX-9YZ+16XZ)(3Y+X+Z)\,U && +(9Y\overline{X+Z}-16XZ)\,H, \end{aligned}$$

that is,

$$\{-9Y(X+Z)^2+48YXZ+16XZ(X+Z)\}U+\{9Y^2+9Y\overline{X+Z}-16XZ\}\,H$$
$$=Y^2ZX\,\{12Y^2+28Y(Z+X)+64XZ+9\,W(3Y+4X+4Z)\}=0;$$

and we thus obtain the equation of the residual quartic, or spinode curve, in the form

$$3Y^2+\ 4Y(X+Z)+\ 4XZ=0,$$
$$12Y^2+28\,Y(X+Z)+64XZ+9\,W(3Y+4X+4Z)=0.$$

The spinode curve is thus a complete intersection, 2×2; and since the first surface is a cone having its vertex on the second surface, we see moreover that the spinode curve is a nodal quadriquadric. Instead of the last equation we may write more simply

$$4Y(X+Z)+16XZ+3W(3Y+4X+4Z)=0.$$

The equations of the transversal are $W=0$, $X+Y+Z=0$, and substituting in the equations of the spinode curve we obtain from each equation $(X-Z)^2=0$, that is, the transversal is a single tangent of the spinode curve; $\beta'=1$.

Reciprocal Surface.

162. The equation of the cubic is derived from that belonging to $\mathrm{VI}=12-B_3-C_2$ by writing therein $a=b=0$, $c=\frac{1}{2}$, $d=1$. Making this change in the formulæ for the reciprocal surface of the case just referred to, we have

$$\begin{aligned} L &= y^2+4(x+z)w, \\ M &= 2x(y+2w), \\ N &= -4x^2, \\ P &= 16x^2(y+w-x-z); \end{aligned}$$

and substituting in the equation

$$L^2P+8zM^3-9zLMN-27z^2wN^2=0,$$

the equation divides by x^2; or throwing this out, the equation is

$$\begin{aligned} &(y^2+4xw+4zw)^2(y+w-x-z) \\ &-8xz(y+2w)^2 \\ &+9xz(y^2+4xw+4zw)(y+2w) \\ &-27x^2z^2w=0; \end{aligned}$$

reducing, this is

$$\begin{aligned} &w^3.16(x-z)^2 \\ &+w^2\left\{\begin{array}{l} y^2(x+z) \\ +2y(x^2-4xz+z^2) \\ +(x+z)(2x-z)(-x+2z) \end{array}\right\} \\ &+w\left\{\begin{array}{l} y^4 \\ +8y^3(x+z) \\ -2y^2(4x^2+23xz+4z^2) \\ +36xyz(x+z) \\ -27x^2z^2 \end{array}\right\} \\ &+y^3(y-x)(y-z)=0. \end{aligned}$$

The section by the plane $w=0$ (reciprocal of B_3) is $w=0$, $y=0$ (the edge) three times; and $w=0$, $y-x=0$; $w=0$, $y-z=0$ (reciprocals of the CB-axes).

163. Nodal curve. This is the line $y=x=z$; wherefore $b'=1$. To put the line in evidence, write for a moment $x=y+\alpha$, $z=y+\gamma$, then the equation is readily converted into

$$\begin{aligned} & w^3 . 16(\alpha-\gamma)^2 \\ & + w^2 \left\{\begin{matrix} -y(\alpha^2-4\alpha\gamma+\gamma^2) \\ +(\alpha+\gamma)(2\alpha-\gamma)(-\alpha+2\gamma) \end{matrix}\right\} \\ & + w \left\{\begin{matrix} y^2(\alpha^2-10\alpha\gamma+\gamma^2) \\ -18y\alpha\gamma(\alpha+\gamma) \\ -27\alpha^2\gamma^2 \end{matrix}\right\} \\ & \quad + y^3\alpha\gamma=0, \end{aligned}$$

which, each term being at least of the second order in α, $\gamma\,(=x-y,\ z-y)$, exhibits the nodal line in question.

164. Cuspidal curve. Multiplying by 27, the equation may be written

$$\begin{aligned} (7y-3x-3z-5w,\ -y+6w,\ -w\,\&\,y^2+16yw-12xw-12zw+16w^2, \\ -20y^3+24yx+24yz-27xz-8yw+16w^2)^2=0, \end{aligned}$$

where

$$4w(7y-3x-3z-5w)+(-y+6w)^2=y^2+16yw-12(x+z)w+16w^2;$$

and we have thus in evidence the cuspidal curve,

$$\begin{aligned} y^2+16yw-12(x+z)w+16w^2=0, \\ -20y^2+24y(x+z)-27xz-8yw+16w^2=0, \end{aligned}$$

which is a complete intersection, 2×2, or quadriquadric curve; $c'=4$.

Section XIV $= 12 - B_5 - C_2$.

Article Nos. 165 to 171. Equation $WXZ + Y^2Z + YX^2 = 0$.

165. The diagram of the lines and planes is

XIV $= 12 - B_5 - C_2$.			$W=0,\ Y=0$	$Z=0,\ Y=0$	$X=0,\ Z=0$	$X=0,\ Y=0,$	Lines are
			3	2	1	0	
			$1\times\frac{1}{4}\ 2=\frac{2}{27}$	$1\times 5=5$	$1\times 10=10$	$1\times 10=10$	
Planes are $Z=0$	12	$1\times 15=15$		. . .	. . . . .		Torsal biplane.
$X=0$	01	$1\times 20=20$			. . .	. . . . .	Ordinary biplane.
$Y=0$	023	$1\times 10=10$ $\overline{3}\quad\overline{45}$	. . . . .	. .		.	Plane through axis and the two rays.
			Cuionodal ray.	Biplanar ray in torsal biplane.	Edge.	Axis.	

where the equations of the planes and lines are shown in the margins.

166. The edge is a facultative line, as will appear from the discussion of the reciprocal surface: $\rho' = b' = 1$; $t' = 0$.

167. Hessian surface. The equation is

$$WXZ^2 + Y^2Z^2 - 3X^2YZ + X^4 = 0.$$

The complete intersection with the surface is made up of the line $X=0$, $Y=0$ (the axis) five times, the line $X=0$, $Z=0$ (the edge) four times, and a skew cubic, the equations of which may be written

$$\left\| \begin{matrix} X, & Y, & W \\ 4Z, & X, & -5Y \end{matrix} \right\| = 0.$$

In fact from the equations $U=0$, $H=0$ we deduce $H - ZU = X^2(X^2 + 4YZ) = 0$; and if in $U=0$ we write $X^2 = 4YZ$, it becomes $Z(XW + 5Y^2) = 0$; and then in $5U = 0$, writing $5Y^2 = -XW$, we have

$$5WXZ + Z(-XW) + 5X^2Y = X(5XY + 4ZW) = 0.$$

168. I say that the spinode curve is made up of the edge $X=0$, $Z=0$ once, and of the cubic curve; and therefore $\sigma'=4$.

In fact in the reciprocal surface the cuspidal curve is made up of the skew cubic, and of a line the reciprocal of the axis, being a cusp-nodal line, and so counting once as part of the cuspidal curve: the pencil of planes through the line is thus part of the cuspidal torse; and reverting to the original cubic surface, we have the axis as part of the spinode curve: I assume that it counts *once*.

The edge is a single tangent of the spinode curve; $\beta'=1$.

Reciprocal Surface.

169. The equation is obtained by means of the binary cubic

$$4wZ^2(Xx+Zz)+X(YZ-wX)^2,$$

or, as this may be written,

$$(3w^2,\ -2yw,\ y^2+4xw,\ 12zw \between X,\ Z)^3.$$

The equation is in the first instance obtained in the form

$$\begin{aligned} & 108w^6z^2 \\ - & \ 32w^4y^3z \\ + & \ 36w^4yz\,(y^2+4xw) \\ + & \quad w^2\ \ (y^2+4xw)^3 \\ - & \quad w^2y^2\,(y^2+4xw)^2=0\,; \end{aligned}$$

but the last two terms being together $=4w^3x\,(y^2+4xw)^2$, the whole divides by $4w^3$, and it then becomes

$$\begin{aligned} & 27w^3z^2 \\ - & \ 8wy^3z \\ + & \ 9wyz\,(y^2+4xw) \\ + & \quad x\,(y^2+4xw)^2=0\,; \end{aligned}$$

or, expanding, it is

$$\begin{aligned} & w^3\,.\,27z^2 \\ + & \ w^2\,.\,36xyz+16x^3 \\ + & \ w\ .\quad y^2z+\ 8x^2y^2 \\ + & \quad xy^4=0. \end{aligned}$$

The section by the plane $w=0$ (reciprocal of B_5) is $w=0$, $y=0$ (reciprocal of edge) four times, together with $w=0$, $x=0$ (reciprocal of biplanar ray).

The section by the plane $z=0$ (reciprocal of C_2) is $x\,(y^2+4xw)^2=0$, viz. this is $z=0$, $y^2+4xw=0$ (reciprocal of nodal cone) twice, together with $z=0$, $x=0$ (reciprocal of nodal ray).

170. Nodal curve. This is the line $w=0$, $y=0$, reciprocal of edge. The equation in the vicinity is $y=-\dfrac{1}{4x}w\pm\sqrt{-\dfrac{z}{8x^4}}\,w^{\frac{3}{2}}$, showing that the line is a cusp-nodal line counting once in the nodal and once in the cuspidal curve: wherefore $b'=1$.

171. Cuspidal curve. The equation of the surface may be written

$$(x,\ -y,\ 3w\between 12xw-y^2,\ 9zw+8xy)^2=0,$$

where $4x\,.\,3w-y^2=12xw-y^2$. This exhibits the cuspidal curve $12xw-y^2=0$, $9zw+8xy=0$, breaking up into the line $w=0$, $y=0$ (reciprocal of edge) and a skew cubic; the line is really part of the cuspidal curve, or $c'=4$.

The equations of the cuspidal cubic may be written in the more complete form

$$\left\| \begin{matrix} 12x, & y, & z \\ y, & w, & -8x \end{matrix} \right\| = 0.$$

Section XV $=12-U_7$.

Article Nos. 172 to 176. Equation $WX^2+XZ^2+Y^2Z=0$.

172. The diagram of the lines and planes is

XV=12 − U_7.					
			$Z=0,\ W=0$ 3	$X=0,\ Z=0$ 2	$X=0,\ Y=0$ 1
			$1\times 1=1$ 3 27	$1\times 10=10$	$1\times 16=16$
Planes are $X=0$	21	$1\times 40=40$			Uniplane.
$Z=0$	23	$1\times 5=5$ 2 45			Plane touching along the single ray.
			Mere line.	Single ray.	Torsal ray.

where the equations of the lines and planes are shown in the margins.

173. The mere line is facultative: $\rho'=b'=1$; $t'=0$.

174. The Hessian surface is

$$X^2(XZ-Y^2)=0,$$

viz. this is the uniplane $X=0$ twice, and a quadric cone having its vertex at U_7.

The complete intersection with the surface is made up of $X=0$, $Y=0$ (torsal ray) six times; $X=0$, $Z=0$ (single ray) twice; and of a residual quartic, which is the spinode curve; $\sigma'=4$.

The equations of the spinode curve are $XZ-Y^2=0$, $XW+2Z^2=0$; the first surface is a cone having its vertex on the second surface; and the curve is thus a nodal quadriquadric.

The mere line is a single tangent of the spinode curve; $\beta'=1$.

Reciprocal Surface.

175. The equation is obtained by means of the binary cubic

$$(-3y^2,\ 2yz,\ 4xw,\ 6yw \between X,\ Y)^3,$$

viz. throwing out the factor y, the equation is

$$w^2(-64x^3)+w(-16x^2z^2+72xy^2z+27y^4)+16y^2z^3=0.$$

The section by the plane $w=0$ (reciprocal of U_7) is $w=0$, $z=0$ (reciprocal of torsal ray) three times, and $w=0$, $y=0$ (reciprocal of single ray) twice.

Nodal curve. This is the line $x=0$, $y=0$, reciprocal of the mere line: $b'=1$.

Cuspidal curve. The equation of the surface may be written

$$(64x,\ -16z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0,$$

where

$$4\,.\,64x(-3w)-256z^2=-256\,(z^2+3xw).$$

This exhibits the cuspidal curve $z^2+3xw=0$, $9y^2+4zx=0$, where the surfaces are each of them cones; the vertex of the second cone is on the first cone, and the two cones have at this point a common tangent plane; the curve is thus a cuspidal quadriquadric.

176. {The equation

$$(64x,\ -16z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0$$

resembles that of a quintic torse, viz. the equation of a quintic torse is

$$(\ \ x,\ -4z,\ \ 8w \between z^2-2wx,\ \ y^2-2zx)^2=0,$$

which equation, writing $9y$ for y, $-2x$ for x, and $\frac{3}{4}w$ for w, becomes

$$(-2x,\ -4z,\ \ 6w \between z^2+3xw,\ 9y^2+4zx)^2=0,$$

or, what is the same thing,

$$(\ \ x,\ \ 2z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0;$$

and developing, this is

$$\begin{aligned} &x^3\,.\,w^2\\ &+x^2\,.\,-2z^2w\\ &+x\,.\,-18y^2zw+z^4\\ &\qquad -27y^4w+2y^2z^3=0, \end{aligned}$$

which, however, differs from the equation of the reciprocal surface, not only in the numerical coefficients, but by the presence of a term xz^4.}

Section XVI $= 12 - 4C_2$.

Article No. 177 to 180. Equation $W(XY + XZ + YZ) + XYZ = 0$.

177. The diagram of the lines and planes is

Lines are	12	13	14	23	24	34	12.34	13.24	14.23
	$Z=0,\ W=0$	$Y=0,\ W=0$	$Y=0,\ Z=0$	$X=0,\ W=0$	$X=0,\ Z=0$	$X=0,\ Y=0$	$X+Y=0,\ Z+W=0$	$X+Z=0,\ Y+W=0$	$X+W=0,\ Y+Z=0$
XVI $= 12 - 4C_2$.	$6 \times 4 = 24$						$3 \times 1 = 3$		$\overline{9}$ $\overline{27}$
Planes are									
$Z+W=0$ 12	• •						•		
$Y+W=0$ 13		• •						•	
$Y+Z=0$ 14			• •						•
$X+W=0$ 23				• •					•
$X+Z=0$ 24					• •			•	
$X+Y=0$ 34						• •	•		
$6 \times 2 = 12$									
$X=0$ 1				•	•	•			
$Y=0$ 2		•	•	•		•			
$Z=0$ 3	•		•		•	•			
$W=0$ 4	•	•		•	•	•			
$4 \times 8 = 32$									
$X+Y+Z+W=0$ 1234									
$1 \times 1 = 1$	•	•	•				•	•	•
$\overline{11}$ $\overline{45}$	•	•	•				•	•	•
	Planes each touching along an axis, and containing a transversal.			Axes each through two nodes.			Transversals each a transversal in respect of two opposite axes.		
	Planes each through three axes.								
	Plane through the three transversals.								

where the equations of the lines and planes are shown in the margins.

178. The transversals are each facultative: $\rho' = b' = 3$; $t' = 1$.

179. Hessian surface. The equation is

$$4XYZW - (X+Y+Z+W)(WXY+WXZ+WYZ+XYZ) = 0,$$

or, what is the same thing,

$$\begin{aligned} & X^2(YZ + YW + ZW) \\ + & Y^2(ZW + ZX + WX) \\ + & Z^2(WX + WY + XY) \\ + & W^2(XY + XZ + YZ) = 0. \end{aligned}$$

The complete intersection with the cubic surface is made up of the six axes each twice, and there is no spinode curve; $\sigma' = 0$, whence also $\beta' = 0$.

Reciprocal Surface.

180. The equation is immediately obtained in the irrational form

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 0,$$

or rationalizing, it is

$$(x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw)^2 - 64xyzw = 0;$$

so that this is in fact Steiner's quartic surface.

Nodal curve. This consists of the lines $x-y=0$, $z-w=0$; $x-z=0$, $y-w=0$; $x-w=0$, $y-z=0$; so that $b'=3$.

To put any one of these, for instance the line $x-y=0$, $z-w=0$, in evidence, we may write the equation of the surface in the form

$$[(x-y)^2+(z-w)^2-2(x+y)(z+w)]^2 - 64xyzw = 0,$$

that is

$$\begin{aligned} & \{(x-y)^2+(z-w)^2\}\{(x-y)^2+(z-w)^2-4(x+y)(z+w)\} \\ & \quad + 4[(x+y)^2(z+w)^2 - 16xyzw] = 0, \end{aligned}$$

or finally

$$\begin{aligned} & \{(x-y)^2+(z-w)^2\}\{(x-y)^2+(z-w)^2-4(x+y)(z+w)\} \\ & \quad + 4\{(x-y)^2(z-w)^2 + 4xy(z-w)^2 + 4zw(x-y)^2\} = 0, \end{aligned}$$

where each term is at least of the second order in $x-y$, $z-w$.

There is no cuspidal curve; $c' = 0$.

Section XVII $= 12 - 2B_3 - C_2$.

Article Nos. 181 to 185. Equation $WXZ + XY^2 + Y^3 = 0$.

181. The diagram of the lines and planes is

Planes are		XVII $= 12 - 2B_3 - C_2$.	4 $W=0,\ X+Y=0$	3 $Z=0,\ X+Y=0$	2 $Y=0,\ W=0$	1 $Y=0,\ Z=0$	0 Lines are $X=0,\ Y=0$	
			$\overline{5}$ $\overline{27}$	$2\times 3 = 6$	$2\times 6 = 12$		$1\times 9 = 9$	
$X=0$	0	$1\times 6 = 6$					. . . .	Common biplane, through the axis joining the two binodes.
$Z=0$	13			. . . .	. . . .			Remaining biplanes, one for each binode.
$W=0$	24	$2\times 6 = 12$	. . . .		. . . .			
$Y=0$	012	$1\times 18 = 18$			.	.	. .	Plane through the three axes.
$X+Y=0$	034	$1\times 9 = 9$	. .				.	Plane through the axis joining the two binodes.
		$\overline{5}$ $\overline{45}$	Biplanar rays, one for each binode.		Axes each through the cinemode and a binode.		Axis joining the two binodes.	

where the equations of the lines and planes are shown in the margins.

182. There is no facultative line; $b' = \rho' = 0$, $t' = 0$.

183. The Hessian surface is

$$X(WXZ + 3YZW + XY^2) = 0,$$

viz. this breaks up into $X = 0$ (the common biplane), and into a cubic surface.

The complete intersection with the cubic surface is made up of $X=0$, $Y=0$ (BB-axis) four times, of $Y=0$, $Z=0$ and $Y=0$, $W=0$ (CB-axes) each three times; and of a residual conic, which is the spinode curve; $\sigma'=2$. The equations of the spinode curve are $Y^2-3ZW=0$, $4X+3Y=0$; viz. it lies in a plane passing through the BB-axis; since there is no facultative line, $\beta'=0$.

Reciprocal Surface.

184. The equation is found to be

$$(y^2+4zw)^2-xy^3-36xyzw+27x^2zw=0,$$

or say this is

$$16z^2w^2+(8y^2-36xy+27x^2)\,zw+y^3(y-x)=0.$$

The section by plane $w=0$ (reciprocal of $B_3=D$) is $w=0$, $y^3(y-x)=0$, viz. this is the line $w=0$, $y=0$ (reciprocal of edge) three times, and the line $w=0$, $y-x=0$ (reciprocal of ray) once; and the like as to section by plane $z=0$.

The section by plane $x=0$ (reciprocal of $C_2=A$) is $x=0$, $(y^2+4zw)^2=0$, viz. this is the conic (reciprocal of nodal cone) twice.

There is no nodal curve; $b'=0$.

185. Cuspidal curve. The equation of the surface may be written

$$(1,\ -y,\ 3zw \between y^2-12zw,\ 9x-8y)^2=0,$$

where $4\,.\,1\,.\,3zw-y^2=-(y^2-12zw)$; and there is thus a cuspidal conic $y^2-12zw=0$, $9x-8y=0$: wherefore $c'=2$.

Attending only to the terms of the second order in y, z, w, the equation becomes $x^2zw=0$; that is, the point $y=0$, $z=0$, $w=0$ (reciprocal of the common biplane) is a binode of the surface; or there is the singularity $B'=1$.

Section XVIII $= 12 - B_4 - 2C_2$.

Article Nos. 186 to 189. Equation $WXZ + Y^2(X+Z) = 0$.

186. The diagram of the lines and planes are

XVIII $= 12 - B_4 - 2C_2$.			$W=0,\ X+Z=0$	$X=0,\ Z=0$	$Y=0,\ W=0$	$Y=0,\ Z=0$	$Y=0,\ X=0$	Lines are
			4	3	0	2	1	
			$1\times1=1$	$1\times6=6$	$1\times4=4$	$2\times8=16$		
			5					
			27					
Planes are								
$Y=0$	0	$1\times16=16$						Plane through the three axes.
$X=0$	01	$2\times12=24$						Biplanes.
$Z=0$	02							
$X+Z=0$	34	$1\times3=3$						Plane touching along edge and containing the mere line.
$W=0$	04	$1\times2=2$						Plane touching along axis through the two onicnodes and containing the mere line.
		5 45						
			Mere line, being also a transversal.	Edge of the binode.	Axis through the two onicnodes.	Axes each through the binode and a onicnode.		

where the equations of the lines and planes are shown in the margins.

187. The mere line is facultative; the edge is also facultative counting twice (this will appear from the discussion of the reciprocal surface): $b' = \rho' = 3$, $t' = 1$.

188. The Hessian surface is

$$(X+Z)\,WXZ+(X-Z)^2\,Y^2=0.$$

The complete intersection with the cubic surface is $Y=0$, $Z=0$ and $Y=0$, $X=0$ (the CB-axes) each four times; $Y=0$, $W=0$ (BB-axis) twice; and $X=0$, $Z=0$ (the edge) twice. There is no spinode curve, $\sigma'=0$; wherefore also $\beta'=0$.

Reciprocal Surface.

189. The equation is obtained from the binary quadric $4w(X+Z)(Xx+Zz)+y^2XZ$, or say

$$(8wx,\ 4w(x+z)+y^2,\ 8wz \between X,\ Z)^2.$$

The equation is thus

$$(y^2+4wx+4wz)^2-64w^2xz=0,$$

or in an irrational form

$$iy+2\sqrt{wx}+2\sqrt{wz}=0.$$

The section by the plane $w=0$ (reciprocal of B_4) is $w=0$, $y=0$ (reciprocal of edge) four times.

The section by the plane $z=0$ (reciprocal of $C_2=C$) is $z=0$, $y^2+4wx=0$ (reciprocal of nodal cone) twice; and similarly for the section by $x=0$ (reciprocal of $C_2=A$).

Nodal curve. Writing the equation in the form

$$y^4+8wy^2(z+x)+16w^2(x-z)^2=0,$$

we have a nodal line $y=0$, $x-z=0$, reciprocal of the mere line:

and writing the equation in the form

$$w=\frac{1}{4(\sqrt{x}\pm\sqrt{z})^2}\,y^2,$$

we have $y=0$, $w=0$ (reciprocal of edge), a tacnodal line counting as two lines; $b'=3$:

There is no cuspidal curve; $c'=0$.

Section XIX $= 12 - B_6 - C_2$.

Article Nos. 190 to 193. Equation $WXZ + Y^2Z + X^3 = 0$.

190. The diagram of the lines and planes is

XIX $= 12 - B_6 - C_2$. Planes are		$X=0,\ Z=0$ $\overline{2}$ $1\times 15 = 15$ $\overline{27}$	Lines are $X=0,\ Z=0$ $1\times 12 = 12$	
$Z=0$	$1\times 15 = 15$	. . . . .		Oscular biplane.
$X=0$	$1\times 30 = 30$ $\overline{2}$ $\overline{45}$	. .	. . . . .	Ordinary biplane.
		Edge of the binode.	Axis joining the binode and the cnicnode.	

where the equations of the lines and planes are shown in the margins.

191. The axis is a facultative line counting three times (as will appear from the reciprocal surface); $\rho' = b' = 3$, $t' = 1$.

192. The Hessian surface is

$$Z(WXZ + Y^2Z - 3X^3) = 0,$$

viz. this is the oscular biplane $Z = 0$ and a cubic surface.

The complete intersection with the cubic surface is made up of $X = 0$, $Z = 0$ (the edge) six times, and $X = 0$, $Y = 0$ (the axis) six times. There is no spinode curve, $\sigma' = 0$; whence also $\beta' = 0$.

Reciprocal Surface.

193. The equation is at once found to be

$$64zw^3 + (y^2 + 4xw)^2 = 0.$$

The section by the plane $w=0$ (reciprocal of B_6) is $w=0$, $y=0$ (reciprocal of edge) four times. The section by the plane $z=0$ (reciprocal of C_2) is $z=0$, $y^2+4xw=0$ (reciprocal of nodal cone) twice.

Nodal curve. The equation gives

$$w=-\frac{1}{4x}y^3 \pm \frac{i\sqrt{z}}{x^{\frac{3}{2}}}y^3+\&c.,$$

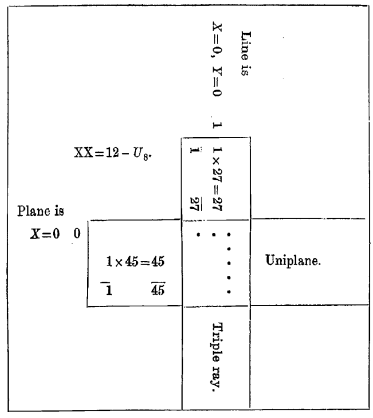
showing that the line $w=0$, $y=0$ (reciprocal of edge) is an oscnodal line counting as three lines; $b'=3$.

There is no cuspidal curve; $c'=0$.

Section $XX=12-U_8$.

Article Nos. 194 to 197. Equation $X^2W+XZ^2+Y^3=0$.

194. The diagram of the lines and planes is



where the equations of the line and plane are shown in the margins.

195. There is no facultative line; $b'=\rho'=0$, $t'=0$.

196. The Hessian surface is $X^3Y=0$, viz. this is the uniplane $X=0$, three times, and the plane $Y=0$ through the ray. The complete intersection with the cubic surface is made up of $X=0$, $Y=0$ (the ray) ten times, and of a residual conic, which is the spinode curve; $\sigma'=2$.

The equations of the spinode conic are $Y=0$, $XW+Z^2=0$, viz. the plane of the conic passes through the ray. Since there is no facultative line, $\beta'=0$.

Reciprocal Surface.

197. The equation is at once found to be

$$27\,(z^2+4xw)^2-64w^3y=0.$$

The section by the plane $w=0$ (reciprocal of the Unode) is $w=0$, $z=0$ (reciprocal of ray) four times.

There is no nodal curve; $b'=0$. But there is a cuspidal conic, $y=0$, $z^2+4xw=0$.

The point $y=0$, $z=0$, $w=0$ (reciprocal of the uniplane $X=0$) is a point which must be considered as uniting the singularities $B'=1$, $\chi'=2$.

I give in an Annex a further investigation in reference to this case of the cubic surface.

Section XXI $=12-3B_3$.

Article Nos. 198 to 201. Equation $WXZ+Y^3=0$.

198. The diagram of the lines and planes is

Planes are		XXI $=12-3B_3$.	Lines are $Y=0$, $X=0$	$Y=0$, $Z=0$	$Y=0$, $W=0$	
				$3\times 9=27$	3 / 27	
$Y=0$	0	$1\times 27=27$	• • •	• • •	• • •	Common biplane containing the three axes.
$X=0$	1		• • •			Remaining biplanes, one for each binode.
$Z=0$	2	$3\times 6=18$		• • •		
$W=0$	3	4 / 45			• • •	
				Axes each joining two binodes.		

where the equations of the lines and planes are shown in the margins.

199. There is no facultative line; $\rho'=b'=0$, $t'=0$.

200. The Hessian surface is $XYZW=0$, the common biplane and the other biplanes each once. The complete intersection with the surface consists of the axes each four times; there is no spinode curve, $\sigma'=0$; whence also $\beta'=0$.

Reciprocal Surface.

201. This is $27xzw - y^3 = 0$, viz. it is a cubic surface of the form $\text{XXI} = 12 - 3B_3$. There is no nodal curve, $b' = 0$, and no cuspidal curve, $c' = 0$. Moreover $B' = 3$.

Article No. 202. *Synopsis for the foregoing sections.*

202. I annex the following synopsis, for the several cases, of the facultative lines (or node-couple curve) and of the spinode curve of the cubic surface; also of the nodal curve and the cuspidal curve of the reciprocal surface. It is to be observed that in designating a curve, for instance, as $18 = 4 \times 5 - 2$, this means that it is a curve of the order 18, the partial intersection of a quartic surface and a quintic surface, but without any explanation of the nature of the common curve 2 which causes the reduction, viz. without explaining whether this is a conic or a pair of lines, and so in other cases; this may be seen by reference to the proper section of the Memoir.

	Facultative lines.	Nodal curve.	Spinode curve.	Cuspidal curve.
$\text{I} = 12$	27	27	$12 = 3 \times 4$	$24 = 6 \times 4$
$\text{II} = 12 - C_2$	15	15	$12 = 3 \times 4$	$18 = 4 \times 5 - 2$
$\text{III} = 12 - B_3$	9	9	$12 = 3 \times 4$	$16 = 4 \times 5 - 4$
$\text{IV} = 12 - 2C_2$	7	7	$10 = 3 \times 4 - 2$	$12 = 4 \times 4 - 2 - 2$
$\text{V} = 12 - B_4$	$7 = 5 +$ edge twice	$7 = 5 +$ rec. of edge twice, rec. of edge tacnodal	$10 = 3 \times 4 - 2$	$12 = 4 \times 4 - 4$
$\text{VI} = 12 - B_3 - C_2$	3	3	$9 = 3 \times 4 - 3$	$10 = 4 \times 4 - 4 - 2$
$\text{VII} = 12 - B_5$	$3 = 2 +$ edge	$3 = 2 +$ rec. of edge, rec. of edge is cuspnodal	$9 =$ edge + unicursal 8-thic	$10 =$ rec. of edge + unicursal 9-thic, rec. of edge is cuspnodal
$\text{VIII} = 12 - 3C_2$	3	3	$6 = 2 \times 3$	$6 = 2 \times 3$
$\text{IX} = 12 - 2B_3$	none	none	$8 = 4$ conics	$8 = 4$ conics
$\text{X} = 12 - B_4 - C_2$	$3 = 1 +$ edge twice	$3 = 1 +$ rec. of edge twice, rec. of edge is tacnodal	$6 = 2 \times 3$	$6 = 2 \times 3$
$\text{XI} = 12 - B_6$	$3 =$ edge 3 times	$3 =$ rec. of edge 3 times, rec. of edge is oscnodal	$6 = 3$ conics	$6 = 3$ conics
$\text{XII} = 12 - U_6$	3	3	$6 = 2 \times 3$	$6 = 2 \times 3$
$\text{XIII} = 12 - B_3 - 2C_2$	1	1	$4 = 2 \times 2$, nodal quadriquadric	$4 = 2 \times 2$ quadriquadric
$\text{XIV} = 12 - B_5 - C_2$	$1 =$ edge	$1 =$ rec. of edge, rec. of edge is cuspnodal	$4 = 3 +$ edge	$4 = 3 +$ rec. of edge, rec. of edge is cuspnodal
$\text{XV} = 12 - U_7$	1	1	$4 = 2 \times 2$, nodal quadriquadric	$4 = 2 \times 2$ cuspidal quadriquadric
$\text{XVI} = 12 - 4C_2$	3	3	none	none
$\text{XVII} = 12 - 2B_3 - C_2$	none	none	$2 =$ conic	$2 =$ conic
$\text{XVIII} = 12 - B_4 - 2C_2$	$3 = 1 +$ edge twice	$1 +$ rec. of edge twice, rec. of edge tacnodal	none	none
$\text{XIX} = 12 - B_6 - C_2$	$3 =$ axis 3 times	$3 =$ rec. of axis 3 times, rec. of axis oscnodal	none	none
$\text{XX} = 12 - U_8$	none	none	$2 =$ conic	$2 =$ conic
$\text{XXI} = 12 - 3B_3$	none	none	none	none

I pass now to the two cases of cubic scrolls.

Article No. 203. Section XXII $= S(1, 1)$. Equation $X^2W + Y^2Z = 0$.

203. As this is a scroll there is here no question of the 27 lines and 45 planes; there is a nodal line $X=0$, $Y=0$, $(b=1)$ and a single directrix line, $Z=0$, $W=0$.

The Hessian surface is $X^2Y^2=0$; the complete intersection with the cubic surface is made up of $X=0$, $Y=0$ (the nodal line) eight times, and of the lines $X=0$, $Z=0$, and $Y=0$, $W=0$, each twice.

The reciprocal surface is $x^2z - y^2w = 0$; viz. this is a like scroll, XXII $= S(1, 1)$; $c'=0$, $b'=1$.

Article No. 204. Section XXIII $= S(\overline{1, 1})$. Equation $X(XW + YZ) + Y^3 = 0$.

204. This is also a scroll; there is a nodal line $X=0$, $Y=0$, and a single directrix line united therewith.

The Hessian surface is $X^4=0$; the complete intersection with the cubic surface is $X=0$, $Y=0$ (the nodal line) twelve times.

The reciprocal surface is $w(xw + yz) - z^3 = 0$; viz. this is a like scroll, XXIII $= S(\overline{1, 1})$; $c'=0$, $b'=1$.

Annex containing Additional Researches in regard to the case XX $= 12 - U_8$; *equation* $WX^2 + XZ^2 + Y^3 = 0$.

Let the surface be touched by the line (a, b, c, f, g, h), that is, the line the equations whereof are

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array}\right)(X, Y, Z, W) = 0.$$

Writing the equation in the form $cW.cX^2 + X(cZ)^2 + c^2Y^3 = 0$, and substituting for cW, cZ their values in terms of X, Y, we have

$$(-gX + fY)cX^2 + X(aX + bY)^2 + c^2Y^3 = 0,$$

that is

$$(a^2 - cg, 2ab + cf, b^2, c^2)(X, Y)^3 = 0,$$

or say

$$(3(a^2 - cg), 2ab + cf, b^2, 3c^2)(X, Y)^3 = 0,$$

viz. the condition of contact is obtained by equating to zero the discriminant of the cubic function. We have thus

$$\begin{aligned} & 27c^4(a^2 - cg)^2 \\ + & 4b^6(a^2 - cg) \\ + & 4c^2(2ab + cf)^3 \\ - & b^4(2ab + cf)^2 \\ - & 18b^2c^2(a^2 - cg)(2ab + cf) = 0, \end{aligned}$$

viz. this is

$$\begin{aligned}
&+ 27\, a^4c^2 \\
&- 4\, a^3b^3 \\
&+ 30\, a^3b^2cf \\
&- 54\, a^2c^3g \\
&+ 36\, ab^3cg \\
&+ 24\, abc^2f^2 \\
&+ 4\, b^5h \\
&- 1\, b^4f^2 \\
&+ 18\, b^2c^2fg \\
&+ 27\, c^4g^2 \\
&+ 4\, c^3f^3 = 0,
\end{aligned}$$

which is the condition in order that the line (a, b, c, f, g, h) may touch the surface $X^2W + XZ^2 + Y^2 = 0$; and if we unite thereto the conditions that the line shall pass through a given point $(\alpha, \beta, \gamma, \delta)$, we have in effect the equation of the circumscribed cone, vertex $(\alpha, \beta, \gamma, \delta)$.

Writing (f, g, h, a, b, c) in place of (a, b, c, f, g, h), we obtain

$$\begin{aligned}
&\ 27\, f^4h^2 \\
&- 4\, f^3g^3 \\
&+ 30\, f^2g^2ha \\
&- 54\, f^3h^3b \\
&+ 36\, fg^3hb \\
&+ 24\, fgh^2a^2 \\
&+ 4\, g^5c \\
&- 1\, g^4a^2 \\
&+ 18\, g^2h^2ab \\
&+ 27\, h^4f^2 \\
&+ 4\, h^3a^3 = 0
\end{aligned}$$

as the condition that the line (a, b, c, f, g, h) shall touch the reciprocal surface

$$27\,(4\,xw + z^2)^2 + 64\, y^3w = 0\,;$$

and if we consider a, b, c, f, g, h as standing for

$$\gamma y - \beta z,\ \alpha z - \gamma x,\ \beta x - \alpha y,\ \delta x - \alpha w,\ \delta y - \beta w,\ \delta z - \gamma w,$$

values which satisfy the relation

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array}\right)(\alpha, \beta, \gamma, \delta) = 0,$$

then the equation in (a, b, c, f, g, h) is that of the circumscribed cone, vertex $(\alpha, \beta, \gamma, \delta)$; the order being (as it should be) $a' = 6$.

The cuspidal conic is $y = 0$, $4xw + z^2 = 0$, and we at once obtain $a^2 - 4cg = 0$ as the condition that the line (a, b, c, f, g, h) shall pass through the cuspidal cone. Hence attributing to (a, b, c, f, g, h) the foregoing values, we have

$$a^2 - 4cg = 0$$

for the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the cuspidal conic; this is of course a quadric cone, $c' = 2$. I proceed to determine the intersections of the two cones.

Representing by $\Theta = 0$ the foregoing equation of the circumscribed cone, and putting for shortness

$$X = 27h^2(f^2 - bh) - 2g^2(2fg + ah),$$

I find that we have identically

$$\Theta - (f^2 - bh)X + (g^4 - 4ah^3 - 8fgh^2)(a^2 - 4cg) - (32fg^3h + 16agh^2)(af + bg + ch) = 0:$$

whence in virtue of the relation $af + bg + ch = 0$, we see that the equations $\Theta = 0$, $a^2 - 4cg = 0$, are equivalent to

$$(f^2 - bh)X = 0, \quad a^2 - 4cg = 0,$$

or the twelve lines of intersection break up into the two systems

$$f^2 - bh = 0, \quad a^2 - 4cg = 0,$$

and

$$(X =) \quad 27h^2(f^2 - bh) - 2g^2(2fg + ah) = 0, \quad a^2 - 4cg = 0.$$

To determine the lines in question, observe that we have

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array}\right)(\alpha, \beta, \gamma, \delta) = 0;$$

and we can by the first three of these express a, b, c linearly in terms of f, g, h; the equations $f^2 - bh = 0$, $a^2 - 4cg = 0$, $27h^2(f^2 - bh) - 2g^2(2fg + ah) = 0$ become thus homogeneous equations in (f, g, h); the equations may in fact be written

$$\delta^2(a^2 - 4cg) = (\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf = 0,$$
$$\delta\ (f^2 - bh) = \delta f^2 - \alpha h^2 + \gamma hf = 0,$$
$$\delta X \qquad = 27h^2(\delta f^2 - \alpha h^2 + \gamma hf) + 2g^2(\beta h^2 - \gamma gh - 2\delta fg) = 0,$$

viz. interpreting (f, g, h) as coordinates *in plano*, the first equation represents a conic, the second a pair of lines, and the third a quartic.

We have identically

$$\{2\beta\delta f - (\gamma^2 + 4\alpha\delta)g + \beta\gamma h\}^2 - 4\beta^2\delta(\delta f^2 - \alpha h^2 + \gamma hf)$$
$$= (\gamma^2 + 4\alpha\delta)\{(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf\};$$

and it thus appears that the two conics touch at the points given by the equations

$$\delta f^2 - \alpha h^2 + \gamma hf = 0,$$
$$2\beta\delta f - (\gamma^2 + 4\alpha\delta)g + \beta\gamma h = 0:$$

we have moreover

$$-(\gamma^2 + 4\alpha\delta)(\beta h^2 - \gamma gh - 2\delta fg) = 4\beta\delta(\delta f^2 - \alpha h^2 + \gamma hf)$$
$$+ (-2\delta f - \gamma h)[2\beta\delta f - (\gamma^2 + 4\alpha\delta)g + \beta\gamma h],$$

hence at the last-mentioned two points $-\beta h^2 + \gamma gh + 2\delta fg$ is $= 0$; and the quartic $X = 0$ thus passes through these two points.

The conic $(a^2 - cg) = 0$ and the quartic $X = 0$ intersect besides (as is evident) in the point $g = 0$, $h = 0$ reckoning as two points, since it is a node of the quartic; and they must consequently intersect in four more points: to obtain these in the most simple manner, write for a moment

$$\Omega = -(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2,$$

then we have identically

$$16\beta^2\delta g^2(\delta f^2 - \alpha h^2 + \gamma hf) - \Omega^2 = -[(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2] + 4\beta^2g^2(\gamma h + 2\delta f)^2,$$
$$= -\{(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 - 2\beta\gamma gh - 2\beta\delta fg\}\{(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 + 2\beta\gamma gh + 4\beta\delta fg\},$$
$$= -\delta(a^2 - 4cg)\{(\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 + 2\beta\gamma gh + 4\beta\delta fg\};$$

and moreover

$$2\beta(\beta h^2 - 2\delta fg - \gamma gh) - \Omega = (\gamma^2 + 4\alpha\delta)g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta fg = \delta(a^2 - 4cg).$$

Hence when $a^2 - 4cg = 0$, we have

$$\delta f^2 - \alpha h^2 + \gamma hf = \frac{\Omega^2}{16\beta^2\delta g^2},\ \beta h^2 - 2\delta fg - \gamma gh = \frac{\Omega}{2\beta};$$

and substituting these values in the equation $X=0$, it becomes

$$27h^2\cdot\frac{\Omega^2}{16\beta^2\delta}+2g^2\cdot\frac{\Omega}{2\beta}=0,$$

viz. multiplying by $16\beta^2\delta$, and omitting the factor Ω, this is

$$27h^2\Omega+16\beta\delta g^4=0,$$

or finally

$$16\beta\delta g^4-27(\gamma^2+4\alpha\delta)g^2h^2+27\beta^2h^4=0,$$

a pencil of four lines, each passing through the point $g=0$, $h=0$, and therefore intersecting the conic

$$(\gamma^2+4\alpha\delta)g^2+\beta^2h^2-2\beta\gamma gh-4\beta\delta hf=0$$

at that point and at one other point; and we have thus four points of intersection, which are the required four points.

Recapitulating, the conic $a^2-4cg=0$ meets the sextic $(f^2-bh)X=0$ in the two points

$$\begin{cases}\delta f^2-ah^2+\gamma hf=0,\\ 2\beta\delta f-(\gamma^2+4\alpha\delta)g+\beta\gamma h=0\end{cases}$$

each three times, in the point $g=0$, $h=0$ twice, and in the four points

$$\begin{cases}16\beta\delta g^4-27(\gamma^2+4\alpha\delta)g^2h^2+27\beta^2h^4=0,\\ (\gamma^2+4\alpha\delta)g^2+\beta^2h^2-2\beta\gamma gh-4\beta\delta hf=0\end{cases}$$

each once. Or reverting to the proper significations of (a, b, c, f, g, h), instead of points we have 2 lines each three times, a line twice, and 4 lines each once; the line $g=0$, $h=0$, that is, $g=0$, $h=0$, $a=0$, being, it will be observed, the line $\frac{y}{\beta}=\frac{z}{\gamma}=\frac{w}{\delta}$ drawn from $(\alpha, \beta, \gamma, \delta)$ to the point $y=0$, $z=0$, $w=0$, which is the reciprocal of the uniplane $X=0$: the twelve lines are the $a'c'$ lines of intersection of the circumscribed cone a' with the cuspidal cone c', viz. $a'c'=[a'c']+3\sigma'+\chi'$; $[a'c']=4$ referring to the last-mentioned four lines; $\sigma'=2$ to the two lines; and $\chi'=2$ to the line $g=0$, $h=0$, $a=0$, which it thus appears must in the present case be reckoned twice.

413.

A MEMOIR ON ABSTRACT GEOMETRY.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLX. (for the year 1870), pp. 51—63. Received October 14,—Read December 16, 1869.]

I SUBMIT to the Society the present exposition of some of the elementary principles of an Abstract m-dimensional Geometry. The science presents itself in two ways,—as a legitimate extension of the ordinary two- and three-dimensional geometries; and as a need in these geometries and in analysis generally. In fact whenever we are concerned with quantities connected together in any manner, and which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as the coordinates of a point in a plane or in space: for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; but this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions: the conditions imply relations between the coefficients in the equation of the curve; and for the better understanding of these relations it was expedient to consider the coefficients as the coordinates of a point in a space of the proper dimensionality.

A fundamental notion in the general theory presents itself, slightly in plane geometry, but already very prominently in solid geometry; viz. we have here the difficulty as to the form of the equations of a curve in space, or (to speak more accurately) as to the expression by means of equations of the twofold relation between the coordinates of a point of such curve. The notion in question is that of a k-fold relation,—as distinguished from any system of equations (or onefold relations) serving for the expression of it, and as giving rise to the problem how to express such relation by means of a system of equations (or onefold relations). Applying to the case of solid geometry

my conclusion in the general theory, it may be mentioned that I regard the twofold relation of a curve in space as being completely and precisely expressed by means of a system of equations ($P=0$, $Q=0$, ... $T=0$), when no one of the functions P, Q, ... T is a linear function, with constant or variable *integral* coefficients, of the others of them, and when *every surface whatever* which passes through the curve has its equation expressible in the form $U=AP+BQ\ldots+KT$, with constant or variable integral coefficients, A, B, ... K. It is hardly necessary to remark that all the functions and coefficients are taken to be rational functions of the coordinates, and that the word integral has reference to the coordinates.

Article Nos. 1 to 36. *General Explanations; Relation, Locus, &c.*

1. Any m quantities may be represented by means of $m+1$ quantities as the ratios of m of these to the remaining $(m+1)$th quantity, and thus in place of the absolute values of the m quantities we may consider the ratios of $m+1$ quantities.

2. It is to be noticed that we are *throughout* concerned with the ratios of the $m+1$ quantities, not with the absolute values; this being understood, any mention of the ratios is in general unnecessary; thus I shall speak of a relation between the $m+1$ quantities, meaning thereby a relation as regards the ratios of the quantities; and so in other cases. It may also be noticed that in many instances a limiting or extreme case is sometimes included, sometimes not included, under a general expression; the general expression is intended to include whatever, having regard to the subject-matter and context, can be included under it.

3. Postulate. We may conceive between the $m+1$ quantities a *relation*[1].

4. A relation is either *regular*, that is, it has a definite manifoldness, or, say, it is a k-fold relation; or else it is *irregular*, that is, composed of relations not all of the same manifoldness. As to the word "composed," see *post*, No. 14.

5. The ratios are determined (not in general uniquely) by means of a m-fold relation; and a relation cannot really be more than m-fold. But the notion of a more than m-fold relation has nevertheless to be considered. A relation may be, either in mere appearance or else according to a provisional conception thereof, more than m-fold, and be really m-fold or less than m-fold. Thus a relation expressed by $m+1$ or more

[1] The whole difficulty of the subject is (so to speak) in the analytical representation of a relation; without solving it, the theories of the text cannot be exhibited analytically with equivalent generality; and I have for this reason presented them in an abstract form without analytical expression or commentary. But it is perfectly easy to obtain analytical illustrations; a onefold relation is expressed by an equation $P=0$; and (although a k-fold relation is not in general expressible by k equations) any k independent equations $P=0$, $Q=0$, &c. constitute a k-fold relation. Thus, No. 4, an instance of an irregular relation is $MP=0$, $MQ=0$, viz. this is satisfied by the satisfaction either of the onefold relation $M=0$, or of the twofold relation $P=0$, $Q=0$. And *post*, Nos. 14 and 21, the relation composed of the two onefold relations $P=0$ and $Q=0$ is the onefold relation $PQ=0$; the relation aggregated of the same two relations is the twofold relation $P=0$, $Q=0$.

equations is in general and *primâ facie* more than m-fold; but if the equations are not independent, and equivalent to m or fewer equations, then the relation will be m-fold or less than m-fold. Or the given relation may depend on parameters, and so long as these are arbitrary be really more than m-fold; but the parameters may have to be, and be accordingly, so determined that the relation shall be m-fold or less than m-fold. A more than m-fold relation is said to be superdeterminate.

6. A system of any number of onefold relations, whether independent or dependent, and if more than m of them, whether compatible or incompatible, is termed a 'Plexus,' viz. if the number of onefold relations be $=\theta$, then the plexus is θ-fold. A θ-fold plexus constitutes a relation which is at most θ-fold, but which may be less than θ-fold.

7. Every relation whatever is expressible, and that *precisely*, by means of a plexus; but for the expression of a k-fold relation we may require a more than k-fold plexus.

8. Postulate. We may conceive a m-dimensional space, the indetermination of the ratios of $m+1$ coordinates, and *locus in quo* of the point, the unique determination of these ratios. More generally we may conceive any number of spaces, each of its own dimensionality, and existing apart by itself.

9. Conversely, any $m+1$ quantities may be taken as the coordinates of a point in a m-dimensional space.

10. The $m+1$ coordinates may have a k-fold relation; it appears (*ante*, No. 5) that the case $k>m$, or where the relation is more than m-fold, is not altogether excluded; but this is not now under consideration. The two limiting cases $k=0$ and $k=m$ will be presently mentioned; the remaining case is $k>0<m$; the system of points the coordinates of which satisfy such a relation constitutes a k-fold or $(m-k)$-dimensional locus. And k is the manifoldness, $m-k$ the dimensionality, of the locus.

11. If $k=m$, that is, if the ratios are determined, we have the point-system, which, if the determination be unique, is a single point. The expression "a locus" may extend to include the point-system, and therefore also the point. If $k=0$, that is, if the coordinates are not connected by any relation, we have the original m-dimensional space.

12. We may say that the m-dimensional space is the *locus in quo* not only of the points in such space, but of the locus determined by any relation whatever between the coordinates; and in like manner that any $(m-k)$dimensional locus in such space is a $(m-k)$dimensional space, a *locus in quo* of the points thereof, and of every locus determined by a relation between the coordinates, implying the k-fold relation which corresponds to the $(m-k)$dimensional locus.

13. There is not any locus corresponding to a relation which is really more than m-fold; hence in speaking of the locus corresponding to a given relation, we either assume that the relation is not more than m-fold, or we mean the locus, *if any*, corresponding to such relation.

14. Any two or more relations may be *composed* together, and they are then factors of a single *composite* relation; viz. the composite relation is a relation satisfied if, and not satisfied unless, some one of the component relations be satisfied.

15. The foregoing notion of composition is, it will be noticed, altogether different from that which would at first suggest itself. The definition is defective as not explaining the composition of a relation any number of times with itself, or elevation thereof to power; which however must be admitted as part of the notion of composition.

16. A k-fold relation which is not satisfied by any other k-fold relation, and which is not a power, is a *prime* relation. A relation which is not prime is composite, viz. it is a relation composed of prime factors each taken once or any other number of times; in particular, it may be the power of a single prime factor. Any prime factor is single or multiple according as it occurs once or a greater number of times.

17. A relation which is either prime, or else composed of prime factors each of the same manifoldness, is a regular relation; a k-fold relation is *ex vi termini* regular. An irregular relation is a composite relation the prime factors whereof are not all of the same manifoldness.

18. A prime k-fold relation cannot be implied in any prime k-fold relation different from itself. But a prime k-fold relation may be implied in a prime more-than-k-fold relation,—or in a composite relation, regular or irregular, each factor whereof is more than k-fold; and so also a composite relation, regular or irregular, each factor whereof is at most k-fold, may be implied in a composite relation, regular or irregular, each factor whereof is more than k-fold. In a somewhat different sense, each factor of a composite relation implies the composite relation.

19. A composite relation is satisfied if any particular one of the component relations is satisfied; but in order to exclude this case we may speak of a composite relation as being satisfied *distributively*; viz. this will be the case if, in order to the satisfaction of the composite relation, it is *necessary* to consider *all* the factors thereof, or, what is the same thing, when the reduced relation obtained by the omission of any one factor whatever is *not always satisfied*. And when the composite relation is satisfied distributively, the several factors thereof are satisfied *alternatively*; viz. there is no one which is throughout unsatisfied.

20. A composite onefold relation is never distributively implied in a prime k-fold relation—that is, a prime k-fold relation implies only a prime onefold relation, or at least only implies a composite onefold relation improperly, in the sense that it implies a certain prime factor of such composite onefold relation. Conversely, every k-fold relation which implies distributively a composite onefold relation is composite.

21. Any two or more relations may be *aggregated* together into, and they are then constituents of, a single *aggregate* relation; viz. the aggregate relation is only satisfied when all the constituent relations are satisfied. The aggregate relation implies each of the constituent relations.

22. There is no meaning in aggregating a relation with itself; such aggregation only occurs accidentally when two relations aggregated together become one and the same relation; and the aggregate of a relation with itself is nothing else than the original relation.

23. A onefold relation is not an aggregate, but is its own sole constituent; a more than onefold relation may always be considered as an aggregate of two or more constituent relations. The constituent relations determine, they in fact constitute, the aggregate relation; but the aggregate relation does not in any wise determine the constituent relations. Any relation implied in a given relation may be considered as a constituent of such given relation.

24. The aggregate of a k-fold and a l-fold relation is in general and at most a $(k+l)$fold relation; when it is a $(k+l)$fold relation, the constituent relations are independent, but otherwise, viz. if the aggregate relation is, or has for factor, a less than $(k+l)$fold equation, the constituent relations are dependent or interconnected.

25. Passing from relations to loci, we may say that the composition of relations corresponds to the *congregation* of loci, and the aggregation of relations to the *intersection* of loci.

26. For, first, the locus (if any) corresponding to a given composite relation is the congregate of the loci corresponding to the several prime factors of the given relation, the locus corresponding to a single factor being taken once, and the locus corresponding to a multiple factor being taken a number of times equal to the multiplicity of the factor.

27. And, secondly, the locus (if any) corresponding to a given aggregate relation is the locus common to and contained in each of the loci corresponding to the several constituent relations respectively; or, what is the same thing, it is the intersection of these several loci.

28. It may be remarked that a k-fold locus and a l-fold locus where $k+l>m$ (or where the aggregate relation is more than m-fold) have not in general any common locus.

29. Any onefold relation implied in a given k-fold relation is said to be in *involution* with the k-fold relation, and so in a system of onefold relations, if any relation be implied in the other relations, or, what is the same thing, in the relation aggregated of the other relations, then the system is said to be in *involution*; a system not in involution is said to be *asyzygetic*.

30. Consider a given k-fold relation, and, in conjunction therewith, a system of any number of onefold relations each implied in the given k-fold relation. We may omit from the system any relation implied in the remaining relations, and so successively until we arrive at an asyzygetic system. Consider now any other onefold relation implied in the given k-fold relation; this is either implied in the system of onefold relations, and it is then to be rejected, or if it is not implied in the system, it is to

be added on to and made part of the system. It may happen that, in the system thus obtained, some one relation of the original system is implied in the remaining relations of the new system; but if this is so the implied relation is to be rejected; the new system will in this case contain only as many relations as the original system, and in any case the new system will be asyzygetic. Treating in the same manner every other onefold relation implied in the given k-fold relation, we ultimately arrive at an asyzygetic system of onefold relations, such that every onefold relation implied in the given k-fold relation is implied in the asyzygetic system. The number of onefold relations will be at least equal to k (for if this were not so we should have the given k-fold relation as an aggregate of less than k onefold relations); but it may be greater than k, and it does not appear that there is any [assignable] superior limit to the number of onefold relations of the asyzygetic system.

31. The system of onefold relations is a precise equivalent of the given k-fold relation. Every set of values of the coordinates which satisfies the given k-fold relation satisfies the system of onefold relations; and reciprocally every set of values which satisfies the system of onefold relations satisfies the given k-fold relation. But if we omit any one or more of the onefold relations, then the reduced system so obtained is not a precise equivalent of the given k-fold relation; viz. there exist sets of values satisfying the reduced system, but not satisfying the given k-fold relation.

32. In fact consider a k-fold relation the aggregate of less than all of the onefold relations of the asyzygetic system, and in connexion therewith an omitted onefold relation; this omitted relation is not implied in the aggregate, and it constitutes with the aggregate not a $(k+1)$fold, but only a k-fold relation. This happens as follows, viz. the omitted relation is a factor of a composite onefold relation distributively implied in the aggregate; hence the aggregate is composite, and it implies distributively a composite onefold relation composed of the omitted relation and of an associated onefold relation; that is, the aggregate will be satisfied by values which satisfy the omitted relation, and also by values which (not satisfying the omitted relation) satisfy the associated relation just referred to.

33. Selecting at pleasure any k of the onefold relations of the asyzygetic system, being such that the aggregate of the k relations is a k-fold relation, we have a composite k-fold relation wherein each of the remaining onefold relations is alternatively implied; viz. each remaining onefold relation is a factor of a composite onefold relation implied distributively in the composite k-fold relation. Hence considering the $k+1$ onefold relations, viz. any $k+1$ relations of the asyzygetic system, each one of these is implied alternatively in the aggregate of the remaining k relations; and we may say that the $k+1$ onefold relations are in *convolution*.

34. More generally any $k+1$ or more, or all the relations of the asyzygetic system are in *convolution*, that is, any relation of the system is alternatively implied in the aggregate of the remaining relations, or indeed in the aggregate of any k relations (not being themselves in convolution) of the remaining relations of the asyzygetic

system. It may be added that, besides the relations of the system, there is not any onefold relation alternatively implied in the asyzygetic system.

35. The foregoing theory has been stated without any limitation as to the value of k, and it has I think a meaning even when k is $>m$; but the ordinary case is $k \not> m$. Considering the theory as applying to this case, I remark that the last proposition, viz. that no reduced system is a precise equivalent of the given k-fold relation, is generally true only on the assumption of the existence or quasi-existence of sets of values satisfying a more than m-fold relation. For let k be $\not> m$, and, on the contrary, assume, as we usually do, that it is not in general possible to satisfy a more than m-fold relation between the coordinates; the number of relations in the system may be $>m+1$; and if this is so, then selecting any $m+1$ relations of the system, it may very well happen that the given k-fold relation is not satisfied by any sets of values other than those which satisfy the $m+1$ relations,—that is, that the $m+1$ relations are a precise equivalent of the given k-fold relation. But even in this case the consideration of the entire system of the onefold relations is not the less advantageous; and I say in general that the given k-fold relation has for its precise and complete equivalent the asyzygetic system of onefold relations.

36. {In illustration of the foregoing Nos. 29 to 35, I remark that, for the functions or equations $P=0$, $Q=0$, $R=0$, &c., if we have identically $AP+BQ+CR+\ldots=0$, where the factors A, B, C, ... are integral functions of the coordinates, and where some one of these factors, say, A, is a constant (or if we please $=1$), then the system of functions or equations is in involution; or, to speak more accurately, the function or equation $P=0$ is in involution with the remaining functions or equations $Q=0$, $R=0$, ... But when the factors A, B, C, ... are no one of them constant, then we have a convolution. If $P=0$ is in involution with the remaining equations $Q=0$, $R=0$, ..., then $P=0$ is implied in these equations, and the relations $(Q=0,\ R=0,\ldots)$ and $(P=0,\ Q=0,\ R=0,\ldots)$ are equivalent to each other. But in the case of a convolution where

$$AP+BQ+CR+\ldots=0,$$

then the relation the equations $Q=0$, $R=0$, ... imply $AP=0$, that is, $A=0$ or else $P=0$; or, what is the same thing, the relation $(Q=0,\ R=0,\ldots)$ is a relation composed of the two relations $(A=0,\ Q=0,\ R=0,\ldots)$ and $(P=0,\ Q=0,\ R=0,\ldots)$. In the k-fold relation expressed by the more than k equations $(P=0,\ Q=0,\ R=0,\ldots)$, selecting any k of these equations which are not in convolution, and uniting thereto any one of the remaining equations, we have a convolution of $k+1$ equations; and when a k-fold relation is precisely expressed by means of a system of k or more equations $(P=0,\ Q=0,\ldots)$, then every equation $\Omega=0$ implied in the given relation, or, what is the same thing, the equation of any onefold locus passing through the locus given by the k-fold relation is in involution with the equations $P=0$, $Q=0$, ..., that is, we have identically $\Omega=AP+BQ+CR+\ldots$, A, B, C, ... being integral functions of the coordinates.}

Article Nos. 37 to 42. *Omal Relation; Order.*

37. A k-fold relation may be linear or omal. If $k=m$, the corresponding locus is a point; if $k<m$ the locus is a k-fold, or $(m-k)$dimensional omaloid; the expression omaloid used absolutely denotes the onefold or $(m-1)$dimensional omaloid; the point may be considered as a m-fold omaloid.

38. A m-fold relation which is not linear or omal is of necessity composite, composed of a certain number M of m-fold linear or omal relations; viz. the m-fold locus corresponding to the m-fold relation is a point-system of M points, each of which may be considered as given by a separate m-fold linear or omal relation; each which relation is a factor of the original m-fold relation. The given m-fold relation, and the point-system corresponding thereto, are respectively said to be of the order M.

39. The order of a point-system of M points is thus $=M$, but it is of course to be borne in mind that the points may be single or multiple points; and that if the system consists of a point taken α times, another point taken β times, &c., then the number of points and therefore the order M of the system is considered to be $=\alpha+\beta+\ldots$.

40. If to a given k-fold relation $(k<m)$ we unite an absolutely arbitrary $(m-k)$fold linear relation, so as to obtain for the aggregate a m-fold relation, then the order M of this m-fold relation (or, what is the same thing, the number M of points in the corresponding point-system) is said to be the order of the given k-fold relation. The notion of order does not apply to a more than m-fold relation.

41. The foregoing definition of order may be more compendiously expressed as follows: viz.

Given between the $m+1$ coordinates a relation which is at most m-fold; then if it is not m-fold, join to it an arbitrary linear relation so as to render it m-fold; we have a m-fold relation giving a point-system; and the order of the given relation is equal to the number of points of the point-system.

42. The relation aggregated of two or more given relations, when the notion of order applies to the aggregate relation, that is, when it is not more than m-fold, is of an order equal to the product of the orders of the constituent relations; or, say, the orders of the given relations being $\mu, \mu', \ldots$, the order of the aggregate relation is $=\mu\mu'\ldots$.

Article Nos. 43 and 44. *Parametric Relations.*

43. We have considered so far relations which involve only the coordinates $(x, y, \ldots)$ [1]; the coefficients are purely numerical, or, if literal, they are absolute constants, which either do or do not satisfy certain conditions; if they do not, the relation assumed in the first instance to be k-fold is really k-fold, or, as we may express it, the relation is

[1] The only exception is *ante*, No. 5, where, in illustration of the notion of a more than m-fold relation, mention is made of "parameters."

really as well as formally k-fold; if they do satisfy certain relations in virtue whereof the formally k-fold relation is really less than k-fold, say, it is $(k-l)$fold, then the relation is in fact to be considered *ab initio* as a $(k-l)$fold relation: there is no question of a relation being in general k-fold and becoming less than k-fold, or suffering any other modification in its form; and the notion of a more than m-fold relation is in the preceding theory meaningless.

44. But a relation between the coordinates $(x, y, \ldots)$ may involve parameters, and so long as these remain arbitrary it may be really as well as formally k-fold; but when the parameters satisfy certain conditions, it may become $(k-l)$fold, or may suffer some other modification in its form. And we have to consider the theory of a relation between the coordinates $(x, y, \ldots)$, involving besides parameters which may satisfy certain conditions, or, say simply, a relation involving variable parameters. If the number of the parameters be m', then these parameters may be regarded as the ratios of m' quantities to a remaining $m'+1$th quantity, and the relation may be considered as involving *homogeneously* the $m'+1$ parameters $(x', y', \ldots)$. And these may, if we please, be regarded as coordinates of a point in their own m'-dimensional space, or we have to consider relations between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ (parameters or) coordinates $(x', y', \ldots)$. It is to be added that a relation may involve distinct sets of parameters, say, we have besides the original set of parameters, a set of $m''+1$ parameters $(x'', y'', \ldots)$ involved homogeneously. But this is a generalization the necessity for which has hardly arisen.

Article Nos. 45 to 55. *Quantics, Notation, &c.*

45. A homogeneous function of the coordinates $(x, y, \ldots)$ is represented by a notation such as

$$(*\between x, y, \ldots)^{(\cdot)}$$

(where $(*)$ indicates the coefficients and $(\cdot)$ the degree), and it is said to be a quantic; and in reference to the quantic the quantities or coordinates $(x, y, \ldots)$ are also termed *facients*. More generally a quantic involving two or more sets of coordinates, or facients, is represented by the similar notation

$$(*\between x, y, \ldots)^{(\cdot)}(x', y', \ldots)^{(:)}\ldots.$$

46. The quantic is unipartite, bipartite, tripartite, &c., according as the number of sets is one, two, three, &c.; and with respect to any set of coordinates, it is binary, ternary, quaternary, ... $(m+1)$ary, according as the number of the coordinates is two, three, four, or $m+1$; and it is linear, quadric, cubic, quartic, ..., according as the degree in regard to the coordinates in question is 1, 2, 3, 4, ...

47. A quantic involving two or more sets of coordinates, and linear in regard to each of them, is said to be tantipartite; or, in particular, when there are only two sets, it is said to be lineo-linear; we may even extend the epithet lineo-linear to the case of any number of sets.

48. Instead of the general notation

$$(*\between x, y, \ldots)^{(\cdot)}(x', y', \ldots)^{(\cdot)}\ldots$$

we may write

$$(a, \ldots\between x, y, \ldots)^{\mu}(x', y', \ldots)^{\mu'}, \ldots,$$

where the coefficients are now indicated by $(a, \ldots)$, and the degrees are $\mu, \mu', \ldots$

49. In the cases where the particular values of the coefficients have to be attended to, we write down the entire series of coefficients, or at least refer thereto by the notation $(a, \ldots)$; and it is to be understood that the coefficients expressed or referred to are each to be multiplied by the appropriate numerical coefficient, viz. for the term $x^{\alpha}y^{\beta}\ldots x'^{\alpha'}y'^{\beta'}\ldots$ this numerical coefficient is

$$=\frac{[\mu]^{\mu}[\mu']^{\mu'}\ldots}{[\alpha]^{\alpha}[\beta]^{\beta}\ldots[\alpha']^{\alpha'}[\beta']^{\beta'}\ldots}.$$

50. It is sometimes convenient not to introduce these numerical multipliers, and we then use the notation

$$(a, \ldots\between x, y, \ldots)^{\mu}(x', y', \ldots)^{\mu'}\ldots,$$

or

$$(a, \ldots\between x, y, \ldots)^{\mu}(x', y', \ldots)^{\mu'}\ldots.$$

In particular $(a, b, c\between x, y)^2$, $(a, b, c, d\between x, y)^3$ &c. denote respectively

$$ax^2+2bxy+cy^2,$$
$$ax^3+3bx^2y+3cxy^2+dy^3,$$
&c.;

but $(a, b, c\between x, y)^2$, $(a, b, c, d\between x, y)^3$, &c. denote

$$ax^2+bxy+cy^2,$$
$$ax^3+bx^2y+cxy^2+dy^3,$$
&c.,

and so $(a, b, c, f, g, h\between x, y, z)^2$ and $(a, b, c, f, g, h\between x, y, z)^2$ denote respectively

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

and

$$ax^2+by^2+cz^2+\ fyz+\ gzx+\ hxy.$$

51. To show which are the coefficients that belong to the several terms respectively, it is obviously proper that the quantic should be once written out at full length; thus, in speaking of a ternary cubic function, we say let $U=(a, \ldots\between x, y, z)^3$

$$\begin{aligned}&=(a, b, c, f, g, h, i, j, k, l\between x, y, z)^3\\&=ax^3+by^3+cz^3\\&\quad+3(fy^2z+gz^2x+hx^2y+lyz^2+jzx^2+kxy^2)\\&\quad+6lxyz,\end{aligned}$$

and the like in other cases.

52. A onefold relation between the coordinates is expressible by means of an equation of the form

$$(*\mathfrak{X}x, y, \ldots)^{(\cdot)} = 0.$$

53. The expression "an equation" used without explanation may be taken to mean an equation of the form in question, viz. the equation obtained by putting a quantic equal to zero; the quantic is said to be the *nilfactum* of the equation. We may consequently say simply that a onefold relation between the coordinates is always expressible by an equation.

54. It is frequently convenient to denote the quantic or nilfactum by a single letter, and to use a locution such as "the equation $U = (*\mathfrak{X}x, y\ldots,)^{(\cdot)} = 0$," which really means that the single letter U stands for the quantic $(*\mathfrak{X}x, y, \ldots)^{(\cdot)}$, so that we are afterwards at liberty to write $U = 0$ as an abbreviated expression for $(*\mathfrak{X}x, y, \ldots)^{\mu} = 0$. We may also speak of the equation or function $U = 0$, meaning thereby the equation $U = 0$, or the function U.

55. A k-fold relation between the coordinates is (as has been shown) equivalent to a system of k or more onefold relations; each of these is expressible by an equation $U = 0$, and the k-fold relation is thus expressible by a system of k or more such equations. Representing by $((U))$ the system of functions which are the nilfacta of these equations respectively, the k-fold relations may be represented thus, $((U)) = 0$; or more completely, the relation being k-fold, and the number of equations being $= s$, by the notation

$$((U)\,s)\,(k\text{-fold}) = 0.$$

We may also speak of the system or relation $((U)) = 0$, meaning thereby the system of functions $((U))$, or the relation $((U)) = 0$.

Article Nos. 56 to 62. *Resultant, Discriminant, &c.*

56. In the case $k > m$, a given k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$ and the parameters $(x', y', \ldots)$ leads to a $(k-m)$fold relation between the parameters. This is termed the *resultant relation* of the given k-fold relation, or when the additional specification is necessary, the *resultant relation* obtained by elimination of the coordinates $(x, y, \ldots)$.

57. Consider a k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x', y', \ldots)$. If $k \not> m$, then, considering the $(x, y, \ldots)$ as coordinates and the $(x', y', \ldots)$ as parameters, we have corresponding to the given relation a k-fold locus in the m-space; and so if $k \not> m'$, then, considering the $(x', y', \ldots)$ as coordinates, but the $(x, y, \ldots)$ as parameters, we have corresponding to the given relation a k-fold locus in the m'-space.

58. If $k > m$, but if the $(k-m)$fold resultant relation is satisfied, then the given k-fold relation becomes a m-fold linear relation between the coordinates $(x, y, \ldots)$, and is consequently satisfied by a single set of values of the coordinates. Hence, considering the given k-fold relation as implying the $(k-m)$fold resultant relation, the k-fold relation will represent a single point in the m-space, say, the *common point*.

59. A m-fold relation, or the locus, or point-system thereby represented, may have a *double* or *nodal* point, viz. two of the points of the point-system may be coincident. More generally a k-fold relation $(k \not> m)$, or the locus thereby represented, may have a *double* or *nodal* point; for let the relation if less than m-fold be made m-fold by adjoining to it a linear $(m-k)$fold relation satisfied by the coordinates of the point in question but otherwise arbitrary, then, if the point in question be a double or nodal point of the m-fold relation, or of the point-system thereby represented, the point is said to be a double or nodal point of the original k-fold relation, or of the locus thereby represented.

60. A given-fold relation $(k \not> m)$ between the $m+1$ coordinates, or the locus thereby represented, has not in general a nodal point. But if the relation involve the $m'+1$ parameters $(x', y', \ldots)$, then, if a certain onefold relation be satisfied between the parameters, there will be a nodal point. The onefold relation between the parameters is the *discriminant relation* of the given k-fold relation.

61. In the case in question, $k \not> m$, the discriminant relation is the resultant relation of a $(m+1)$fold relation which is the aggregate of the given k-fold relation with a certain relation called the *Jacobian relation*, or when the distinction is required, the *Jacobian relation* in regard to the $(x, y, \ldots)$.

62. Consider a k-fold relation $(k \not> m, \not> m')$ between the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x', y', \ldots)$. It has been seen that to a given set of values of the $(x', y', \ldots)$ or, say, to a given point in the m'-space, there corresponds a k-fold locus in the m-space, and that to a given set of values of the $(x, y, \ldots)$, or to a given point in the m-space, there corresponds a k-fold locus in the m'-space. The k-fold locus in the m'-space may have a nodal point; this will be the case if there is satisfied between the $(x, y, \ldots)$ a certain one-fold relation, the discriminant relation of the given k-fold relation in regard to the $(x', y', \ldots)$. This onefold relation represents in the m-space a onefold locus, the *envelope* of the k-fold loci in the m-space corresponding to the several points of the m'-space. The property of the envelope is that to each point thereof there corresponds in the m'-space a k-fold locus having a nodal point.

Article Nos. 63—69. *Consecutive Points; Tangent Omals.*

63. As the notions of proximity and remoteness have been thus far altogether ignored, it seems necessary to make the following

Postulate. We may conceive a point consecutive (or indefinitely near) to a given point.

64. If the coordinates of the given point are $(x, y, \ldots)$, those of the consecutive point may be assumed to be $(x+\delta x, y+\delta y, \ldots)$, where $\delta x, \delta y, \ldots$ are indefinitely small in regard to $(x, y, \ldots)$.

65. It may be remarked that, taking the coordinates to be $(x+X, y+Y, \ldots)$, there is no obligation to have $(X, Y, \ldots)$ indefinitely small; in fact whatever the magnitudes of these quantities are, if only $X : Y : \ldots = x : y : \ldots$, then the point $(x+X, y+Y, \ldots)$ will be the very same with the original point, and it is therefore clear that a consecutive point may be represented in the same manner with magnitudes, however large, of $X, Y, \ldots$ But we *may* assume them indefinitely small, that is, the ratios $x+\delta x : y+\delta y, \ldots$, where $\delta x, \delta y, \ldots$ are indefinitely small in regard to $(x, y, \ldots)$, will represent any set of ratios indefinitely near to the ratios $(x : y, \ldots)$.

The foregoing quantities $(\delta x, \delta y, \ldots)$ are termed the increments.

66. Consider a k-fold relation between the $m+1$ coordinates $(x, y, \ldots)$, $k \not> m$; the increments $(\delta x, \delta y, \ldots)$ are connected by a linear k-fold relation.

The linear k-fold relation is satisfied if we assume the increments proportional to the coordinates—this is, in fact, assuming that the point remains unaltered. We may write $(\delta x, \delta y, \ldots) = (x, y, \ldots)$, since in such an equation only the ratios are attended to. But it may be preferable to write $(\delta x, \delta y, \ldots) = \lambda(x, y, \ldots)$. In particular if $k = m$, then the increments are connected by a linear m-fold relation; that is, the ratio of the increments is uniquely determined; and as the relation is satisfied by taking the increments proportional to the coordinates, it is clear that the values which the linear m-fold relation gives for the increments are in fact proportional to the coordinates: viz. there is not in this case any consecutive point.

67. Considering the k-fold relation as belonging to a k-fold locus in the m-space, so that $(x, y, \ldots)$ are the coordinates of a point on this locus, then if in the linear k-fold relation between the increments these increments are replaced by the coordinates $(\mathrm{x}, \mathrm{y}, \ldots)$ of a point in the m-space, then considering the original coordinates $(x, y, \ldots)$ as parameters, the locus of the point $(\mathrm{x}, \mathrm{y}, \ldots)$ is a k-fold omal locus: it is to be observed that, by what precedes, the linear k-fold relation is satisfied by writing therein the values $\mathrm{x} : \mathrm{y}, \ldots = x : y, \ldots$, that is, the k-fold omal locus passes through the original point $(x, y, \ldots)$; the k-fold omal locus is said to be the *tangent-omal* of the original k-fold locus at the point $(x, y, \ldots)$, which point is said to be the *point of contact*.

68. If in the original k-fold locus we replace $(x, y, \ldots)$ by $(\mathrm{x}, \mathrm{y}, \ldots)$, and combine therewith the k-fold linear relation, we have between the coordinates $(\mathrm{x}, \mathrm{y}, \ldots)$ a $2k$-fold relation (containing as parameters the coordinates $(x, y, \ldots)$); these parameters satisfy the original k-fold relation, and in virtue hereof the $2k$-fold relation (whether $2k$ is or is not greater than m) is satisfied by the values $\mathrm{x}, \mathrm{y}, \ldots = x : y : \ldots$; and not only so, but the point in question is a nodal or double point on the $2k$-fold locus. It also follows that the tangent-omal locus, considering in the k-fold linear relation $(x, y, \ldots)$ as parameters satisfying the original k-fold relation, has for its envelope the k-fold locus.

69. We thus arrive at the notion of the double generation of a k-fold locus, viz. such locus is the locus of the points, or, say, of the *ineunt-points* thereof; and it is also the envelope of the tangent-omals thereof. We have thus a theory of duality; I do not at present attempt to develope the theory, but it is necessary to refer to it, in order to remark that this theory is essential to the systematic development of a m-dimensional geometry; the original classification of loci as onefold, twofold,... $(m-1)$fold is incomplete, and must be supplemented with the loci reciprocally connected with these loci respectively. And moreover the theory of the singularities of a locus can only be systematically established by means of the same theory of duality; the singularities in regard to the ineunt-point must be treated of in connexion with the singularities in regard to the tangent-omal. These theories (that is, the classification of loci, and the establishment and discussion of the singularities of each kind of locus), vast as their extent is, should in the logical order precede that for which other reasons it may be expedient next to consider, the theory of Transformation, as depending on relations involving simultaneously the $m+1$ coordinates $(x, y, \ldots)$ and the $m'+1$ coordinates $(x' y', \ldots)$.

414.

ON POLYZOMAL CURVES, OTHERWISE THE CURVES $\sqrt{U}+\sqrt{V}+\&c.=0$.

[From the *Transactions of the Royal Society of Edinburgh*, vol. XXV. (1868), pp. 1—110. Read 16th December 1867.]

If U, V, &c., are rational and integral functions $(*\between x, y, z)^r$, all of the same degree r, in regard to the coordinates (x, y, z), then $\sqrt{U}+\sqrt{V}+\&c.$ is a polyzome, and the curve $\sqrt{U}+\sqrt{V}+\&c.=0$ a polyzomal curve. Each of the curves $\sqrt{U}=0$, $\sqrt{V}=0$, &c. (or say the curves $U=0$, $V=0$, &c.) is, on account of its relation of circumscription to the curve $\sqrt{U}+\sqrt{V}+\&c.=0$, considered as a girdle thereto (ζώμα), and we have thence the term "zome" and the derived expressions "polyzome," "zomal," &c. If the number of the zomes $\sqrt{U}$, $\sqrt{V}$, &c. be $=\nu$, then we have a ν-zome, and corresponding thereto a ν-zomal curve; the curves $U=0$, $V=0$, &c., are the zomal curves or zomals thereof. The cases $\nu=1$, $\nu=2$, are not, for their own sake, worthy of consideration; it is in general assumed that ν is $=3$ at least. It is sometimes convenient to write the general equation in the form $\sqrt{lU}+\&c.=0$, where l, &c. are constants. The Memoir contains researches in regard to the general ν-zomal curve; the branches thereof, the order of the curve, its singularities, class, &c.; also in regard to the ν-zomal curve $\sqrt{l(\Theta+L\Phi)}+\&c.=0$, where the zomal curves $\Theta+L\Phi=0$, all pass through the points of intersection of the same two curves $\Theta=0$, $\Phi=0$ of the orders r and $r-s$ respectively; included herein we have the theory of the depression of order as arising from the ideal factor or factors of a branch or branches. A general theorem is given of "the decomposition of a tetrazomal curve," viz. if the equation of the curve be $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}+\sqrt{pT}=0$; then if U, V, W, T are in involution, that is, connected by an identical equation $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0$, and if l, m, n, p, satisfy the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, the tetrazomal curve breaks up into

two trizomal curves, each expressible by means of any three of the four functions U, V, W, T; for example, in the form $\sqrt{l'U}+\sqrt{m'V}+\sqrt{p'T}=0$. If, in this theorem, we take $p=0$, then the original curve is the trizomal $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, T is any function $=-\frac{1}{\mathrm{d}}(\mathrm{a}U+\mathrm{b}V+\mathrm{c}W)$, where, considering l, m, n as given, a, b, c are quantities subject only to the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, and we have the theorem of "the variable zomal of a trizomal curve," viz. the equation of the trizomal $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, may be expressed by means of any two of the three functions U, V, W, and of a function T determined as above, for example in the form $\sqrt{l'U}+\sqrt{m'V}+\sqrt{n'T}=0$; whence also it may be expressed in terms of three new functions T, determined as above. This theorem, which occupies a prominent position in the whole theory, was suggested to me by Mr Casey's theorem, presently referred to, for the construction of a bicircular quartic as the envelope of a variable circle.

In the ν-zomal curve $\sqrt{l(\Theta+L\Phi)}+\&c.=0$, if $\Theta=0$ be a conic, $\Phi=0$ a line, the zomals $\Theta+L\Phi=0$, &c. are conics passing through the same two points $\Theta=0$, $\Phi=0$, and there is no real loss of generality in taking these to be the circular points at infinity—that is, in taking the conics to be circles. Doing this, and using a special notation $\mathfrak{A}^\circ=0$ for the equation of a circle having its centre at a given point A, and similarly $\mathfrak{A}=0$ for the equation of an evanescent circle, or say of the point A, we have the ν-zomal curve $\sqrt{l\mathfrak{A}^\circ}+\&c.=0$, and the more special form $\sqrt{l\mathfrak{A}}+\&c.=0$. As regards the last-mentioned curve, $\sqrt{l\mathfrak{A}}+\&c.=0$, the point A to which the equation $\mathfrak{A}=0$ belongs, is a focus of the curve, viz. in the case $\nu=3$, it is an ordinary focus, and in the case $\nu>3$, it is a special kind of focus, which, if the term were required, might be called a foco-focus; the Memoir contains an explanation of the general theory of the foci of plane curves. For $\nu=3$, the equation $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0$ is really equivalent to the apparently more general form $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$. In fact, this last is in general a bicircular quartic, and, in regard to it, the before-mentioned theorem of the variable zomal becomes Mr Casey's theorem, that "the bicircular quartic (and, as a particular case thereof, the circular cubic) is the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle." This theorem is a sufficient basis for the complete theory of the trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$; and it is thereby very easily seen that the curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$ can be represented by an equation $\sqrt{l'\mathfrak{A}'}+\sqrt{m'\mathfrak{B}'}+\sqrt{n'\mathfrak{C}'}=0$. But for $\nu>3$ this is not so, and the curve $\sqrt{l\mathfrak{A}}+\&c.=0$ is only a particular form of the curve $\sqrt{l\mathfrak{A}^\circ}+\&c.=0$; and the discussion of this general form is scarcely more difficult than that of the special form $\sqrt{l\mathfrak{A}}+\&c.=0$, included therein. The investigations in relation to the theory of foci, and in particular to that of the foci of the circular cubic and bicircular quartic, precede in the Memoir the theories of the trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$, and the tetrazomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}+\sqrt{p\mathfrak{D}^\circ}=0$, to which the concluding portions relate. I have accordingly divided the Memoir into four parts, viz. these are—Part I., On Polyzomal Curves in general; Part II., Subsidiary

Investigations; Part III., On the Theory of Foci; and Part IV., On the Trizomal and Tetrazomal Curves where the zomals are circles. There is, however, some necessary intermixture of the theories treated of, and the arrangement will appear more in detail from the headings of the several articles. The paragraphs are numbered continuously through the Memoir. There are four Annexes, relating to questions which it seemed to me more convenient to treat of thus separately.

It is right that I should explain the very great extent to which, in the composition of the present Memoir, I am indebted to Mr Casey's researches. His Paper "On the Equations and Properties (1) of the System of Circles touching three circles in a plane; (2) of the System of Spheres touching four spheres in space; (3) of the System of Circles touching three circles on a sphere; (4) on the System of Conics inscribed in a conic and touching three inscribed conics in a plane," was read to the Royal Irish Academy, April 9, 1866, and is published in their "Proceedings." The fundamental theorem for the equation of the pairs of circles touching three given circles was, previous to the publication of the paper, mentioned to me by Dr Salmon, and I communicated it to Professor Cremona, suggesting to him the problem solved in his letter of March 3, 1866, as mentioned in my paper, "Investigations in connexion with Casey's Equation," *Quarterly Math. Journ.* vol. VIII. 1867, pp. 334—341, [395], and as also appears, Annex No. IV of the present Memoir.

In connexion with this theorem, I communicated to Mr Casey, in March or April 1867, the theorem No. 164 of the present Memoir, that for any three given circles, centres A, B, C, the equation $\overline{BC}\sqrt{\mathfrak{A}^\circ}+\overline{CA}\sqrt{\mathfrak{B}^\circ}+\overline{AB}\sqrt{\mathfrak{C}^\circ}=0$ (where $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, denote the mutual distances of the points A, B, C) belongs to a Cartesian. Mr Casey, in a letter to me dated 30th April, 1867, informed me of his own mode of viewing the question as follows:—"The general equation of the second order $(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2=0$, where α, β, γ are circles, is a bicircular quartic. If we take the equation $(a, b, c, f, g, h \between \lambda, \mu, \nu)^2=0$ in tangential coordinates (that is, when λ, μ, ν are perpendiculars let fall from the centres of α, β, γ on any line), it denotes a conic; denoting this conic by F, and the circle which cuts α, β, γ orthogonally by J, I proved that, if a variable circle moves with its centre on F, and if it cuts J orthogonally, its envelope will be the bicircular quartic whose equation is that written down above;" and among other consequences, he mentions that the foci of F are the double foci of the quartic, and the points in which J cuts F single foci of the quartic, and also the theorem which I had sent him as to the Cartesian, and he refers to his Memoir on Bicircular Quartics as then nearly finished. An Abstract of the Memoir as read before the Royal Irish Academy, 10th February, 1867, and published in their *Proceedings*, pp. 44, 45, contains the theorems mentioned in the letter of 30th April, and some other theorems. It is not necessary that I should particularly explain in what manner the present Memoir has been, in the course of writing it, added to or altered in consequence of the information which I have thus had of Mr Casey's researches; it is enough to say that I have freely availed myself of such information, and that there is no question as to Mr Casey's priority in anything which there may be in common in his memoir on Bicircular Quartics and in the present Memoir.

PART I. (Nos. 1 to 55).—ON POLYZOMAL CURVES IN GENERAL.

Article Nos. 1 to 4. *Definition and Preliminary Remarks.*

1. As already mentioned, U, V, &c. denote rational and integral functions $(*\between x, y, z)^r$, all of the same degree r in the coordinates (x, y, z), and the equation

$$\sqrt{U}+\sqrt{V}+\&c.=0$$

then belongs to a polyzomal curve, viz., if the number of the zomes $\sqrt{U}$, $\sqrt{V}$, &c. is $=\nu$, then we have a ν-zomal curve. The radicals, or any of them, may contain rational factors, or be of the form $P\sqrt{Q}$; but in speaking of the curve as a ν-zomal, it is assumed that any two terms, such as $P\sqrt{Q}+P'\sqrt{Q}$, involving the same radical $\sqrt{Q}$, are united into a single term, so that the number of distinct radicals is always $=\nu$; in particular (r being even), it is assumed that there is only one rational term P. But the ordinary case, and that which is almost exclusively attended to, is that in which the radicals $\sqrt{U}$, $\sqrt{V}$, &c. are distinct irreducible radicals without rational factors.

2. The curves $U=V=0$, &c. are said to be the zomal curves, or simply the zomals of the polyzomal curve $\sqrt{U}+\sqrt{V}+\&c.=0$; more strictly, the term zomal would be applied to the functions U, V, &c. It is to be noticed, that although the form $\sqrt{U}+\sqrt{V}+\&c.=0$ is equally general with the form $\sqrt{lU}+\sqrt{mV}+\&c.=0$ (in fact, in the former case, the functions U, V, &c. are considered as implicitly containing the constant factors l, m, &c., which are expressed in the latter case), yet it is frequently convenient to express these factors, and thus write the equation in the form $\sqrt{lU}+\sqrt{mV}+\&c.$ For instance, in speaking of any given curves $U=0$, $V=0$, &c., we are apt, disregarding the constant factors which they may involve, to consider U, V, &c. as given functions; but in this case the general equation of the polyzomal with the zomals $U=0$, $V=0$, &c., is of course $\sqrt{lU}+\sqrt{mV}+\&c.=0$.

3. Anticipating in regard to the cases $\nu=1$, $\nu=2$, the remark which will be presently made in regard to the ν-zomal, that $\sqrt{U}+\sqrt{V}+\&c.=0$ is the curve represented by the rationalised form of this equation, the monozomal curve $\sqrt{U}=0$ is merely the curve $U=0$, viz., this is any curve whatever $U=0$ of the order r; and similarly, the bizomal curve $\sqrt{U}+\sqrt{V}=0$ is merely the curve $U-V=0$, viz. this is any curve whatever $\Omega=0$, of the order r; the zomal curves $U=0$, $V=0$, taken separately, are not curves standing in any special relation to the curve in question $\Omega=0$, but $U=0$ may be any curve whatever of the order r, and then $V=0$ is a curve of the same order r, in involution with the two curves $\Omega=0$, $U=0$; we may, in fact, write the equation $\Omega=0$ under the bizomal form $\sqrt{U}+\sqrt{\Omega+U}=0$. In the case r even, we may, however, notice the bizomal curve $P+\sqrt{U}=0$ (P a rational function of the degree $\frac{1}{2}r$); the rational equation is here $\Omega=U-P^2=0$, that is $U=\Omega+P^2$, viz., P is any curve whatever of the order $\frac{1}{2}r$, and $U=0$ is a curve of the order r, touching the given curve $\Omega=0$ at each of its $\frac{1}{2}r^2$ intersections with the curve $P=0$. I further

remark that the order of the ν-zomal curve $\sqrt{V}+\&c.=0$ is $=2^{\nu-2}r$; this is right in the case of the bizomal curve $\sqrt{U}+\sqrt{V}=0$, the order being $=r$, but it fails for the monozomal curve $\sqrt{U}=0$, the order being in this case r, instead of $\frac{1}{2}r$, as given by the formula. The two unimportant and somewhat exceptional cases $\nu=1$, $\nu=2$, are thus disposed of, and in all that follows (except in so far as this is in fact applicable to the cases just referred to), ν may be taken to be $=3$ at least.

4. It is to be throughout understood that by the curve $\sqrt{U}+\sqrt{V}+\&c.=0$ is meant the curve represented by the rationalised equation

$$\text{Norm}(\sqrt{U}+\sqrt{V}+\&c.)=0,$$

viz. the Norm is obtained by attributing to all but one of the zomes $\sqrt{U}$, $\sqrt{V}$, &c., each of the two signs $+$, $-$, and multiplying together the several resulting values of the polyzome; in the case of a ν-zomal curve, the number of factors is thus $=2^{\nu-1}r$ (whence, as each factor is of the degree $\frac{1}{2}r$, the order of the curve is $2^{\nu-1}\cdot\frac{1}{2}r$, $=2^{\nu-2}r$, as mentioned above). I expressly mention that, as regards the polyzomal curve, we are not in any wise concerned with the signs of the radicals, which signs are and remain essentially indeterminate; the equation $\sqrt{U}+\sqrt{V}+\&c.=0$, is a mere symbol for the rationalised equation, Norm $(\sqrt{U}+\sqrt{V}+\&c.)=0$.

Article Nos. 5 to 12. *The Branches of a Polyzomal Curve.*

5. But we may in a different point of view attend to the signs of the radicals; if for all values of the coordinates we take the symbol $\sqrt{}$, and consider $\sqrt{U}$, $\sqrt{V}$, &c. as signifying determinately, say the *positive* values of $\sqrt{U}$, $\sqrt{V}$, &c.; then each of the several equations $\pm\sqrt{U}\pm\sqrt{V}+\&c.=0$, or, fixing at pleasure one of the signs, suppose that prefixed to $\sqrt{U}$, then each of the several equations $\sqrt{U}\pm\sqrt{V}\pm\&c.=0$, will belong to a *branch* of the polyzomal curve: a ν-zomal curve has thus $2^{\nu-1}$ branches corresponding to the $2^{\nu-1}$ values respectively of the polyzome. The separation of the branches depends on the precise fixation of the significations of $\sqrt{U}$, $\sqrt{V}$, &c., and in regard hereto some further explanation is necessary.

6. When U is real and positive, $\sqrt{U}$ may be taken to be, in the ordinary sense, the positive value of $\sqrt{U}$, and so when U is real and negative, $\sqrt{U}$ may be taken to be $=i$ into the positive value of $\sqrt{-U}$; and the like as regards $\sqrt{V}$, &c. The functions U, V, &c. are assumed to be real functions of the coordinates; hence, for any real values of the coordinates, U, V, &c. are real positive or negative quantities, and the significations of $\sqrt{U}$, $\sqrt{V}$, &c. are completely determined.

7. But the coordinates may be imaginary. In this case the functions U, V, &c. will for any given values of the coordinates acquire each of them a determinate, in general imaginary, value. If for all real values whatever of α, β, we select once for

all one of the two opposite values of $\sqrt{\alpha+\beta i}$, *calling* it the *positive* value, and representing it by $\sqrt{\overline{\alpha+\beta i}}$, then, for any particular values of the coordinates, U being $=\alpha+\beta i$, the value of $\sqrt{\overline{U}}$ may be taken to be $=\sqrt{\overline{\alpha+\beta i}}$; and the like as regards $\sqrt{\overline{V}}$, &c. $\sqrt{\overline{U}}$, $\sqrt{\overline{V}}$, &c. have thus each of them a determinate signification for any values whatever, real or imaginary, of the coordinates. The coordinates of a given point on the curve $\sqrt{\overline{U}}+\sqrt{\overline{V}}+\&c.=0$, will in general satisfy only one of the equations $\sqrt{\overline{U}}\pm\sqrt{\overline{V}}\pm\&c.=0$; that is, the point will belong to one (but in general only one) of the $2^{\nu-1}$ branches of the curve; the entire series of points the coordinates of which satisfy any one of the $2^{\nu-1}$ equations, will constitute the branch corresponding to that equation.

8. The signification to be attached to the expression $\sqrt{\overline{\alpha+\beta i}}$ should agree with that previously attached to the like symbol in the case of a positive or negative real quantity; and it should, as far as possible, be subject to the condition of continuity, viz., as $\alpha+\beta i$ passes continuously to $\alpha'+\beta' i$, so $\sqrt{\overline{\alpha+\beta i}}$ should pass continuously to $\sqrt{\overline{\alpha'+\beta' i}}$; but (as is known) it is not possible to satisfy universally this condition of continuity; viz., if for facility of explanation we consider (α, β) as the coordinates of a point in a plane, and imagine this point to describe a closed curve surrounding the origin or point (0, 0), then it is not possible so to define $\sqrt{\overline{\alpha+\beta i}}$ that this quantity, varying continuously as the point moves along the curve, shall, when the point has made a complete circuit, resume its original value. The signification to be attached to $\sqrt{\overline{\alpha+\beta i}}$ is thus in some measure arbitrary, and it would appear that the division of the curve into branches is affected by a corresponding arbitrariness, but this arbitrariness relates only to the imaginary branches of the curve: the notion of a real branch is perfectly definite.

9. It would seem that a branch may be impossible for any series whatever of points real or imaginary. Thus, in the bizomal curve $\sqrt{\overline{U}}+\sqrt{\overline{V}}=0$, the branch $\sqrt{\overline{U}}+\sqrt{\overline{V}}=0$ is impossible. In fact, for any point whatever, real or imaginary, of the curve, we have $U=V$, and therefore $\sqrt{\overline{U}}=\sqrt{\overline{V}}$; the point thus belongs to the other branch $\sqrt{\overline{U}}-\sqrt{\overline{V}}=0$, not to the branch $\sqrt{\overline{U}}+\sqrt{\overline{V}}=0$; the only points belonging to the last-mentioned branch are the isolated points for which simultaneously $\sqrt{\overline{U}}=0$, $\sqrt{\overline{V}}=0$; viz., the points of intersection of the two curves $U=0$, $V=0$.

10. It is not clear to me whether the case is the same in regard to the branch $\sqrt{\overline{U}}+\sqrt{\overline{V}}+\sqrt{\overline{W}}=0$ of a trizomal curve. In fact, for each point of the curve $\sqrt{\overline{U}}+\sqrt{\overline{V}}+\sqrt{\overline{W}}=0$ we have $(U-V-W)^2=4VW$, and therefore, $U-V-W=\pm 2\sqrt{\overline{V}}\sqrt{\overline{W}}$; there may very well be points for which the sign is $+$; that is, points for which $U=V+W+2\sqrt{\overline{V}}\sqrt{\overline{W}}$, and for these points we have $\pm\sqrt{\overline{U}}=\sqrt{\overline{V}}+\sqrt{\overline{W}}$; for real values of the coordinates the sign on the left hand must be $+$ (for otherwise the two sides

would have opposite signs), but there is no apparent reason, or at least no obviously apparent reason, why this should be so for imaginary values of the coordinates, and if the sign be in fact $-$, then the point will belong to the branch $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$.

11. But the branch in question is clearly impossible for any series of real points; so that, leaving it an open question whether the epithet "impossible" is to be understood to mean impossible for any series of real points (that is, as a mere synonym of imaginary), or whether it is to mean impossible for any series of points, real or imaginary, whatever, I say that in a ν-zomal curve some of the branches are or may be impossible, and that there is at least one impossible branch; viz., the branch $\sqrt{U}+\sqrt{V}+\&c.=0$.

12. For the purpose of referring to any branch of a polyzomal curve it will be convenient to consider $\sqrt{U}$ as signifying determinately $+\sqrt{U}$, or else $-\sqrt{U}$; and the like as regards $\sqrt{V}$, &c., but without any identity or relation between the signs prefixed to the $\sqrt{U}$, $\sqrt{V}$, &c., respectively; the equation $\sqrt{U}+\sqrt{V}+\&c.=0$, so understood, will denote determinately some one (that is, any one at pleasure) of the equations $\sqrt{U}\pm\sqrt{V}\pm\&c.=0$, and it will thus be the equation of some one (that is, any one at pleasure) of the branches of the polyzomal curve – all risk of ambiguity which might otherwise exist will be removed if we speak either of the *curve* $\sqrt{U}+\sqrt{V}$, &c. $=0$, or else of the *branch* $\sqrt{U}+\sqrt{V}+\&c.=0$. Observe that by the foregoing convention, when only one branch is considered, we avoid the necessity of any employment of the sign $\pm$, or of the sign $-$; but when two or more branches are considered in connection with each other, it is necessary to employ the sign $-$ with one or more of the radicals $\sqrt{U}$, $\sqrt{V}$, &c.; thus in the trizomal curve $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, we may have to consider the branches $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, $\sqrt{U}+\sqrt{V}-\sqrt{W}=0$; viz., either of these equations apart from the other denotes any one branch at pleasure of the curve, but when the branch represented by the one equation is fixed, then the branch represented by the other equation is also fixed.

Article Nos. 13 to 17. *The Points common to Two Branches of a Polyzomal Curve.*

13. I consider the points which are situate simultaneously on two branches of the ν-zomal curve $\sqrt{U}+\sqrt{V}+\&c.=0$. The equations of the two branches may be taken to be

$$\sqrt{U}+\&c.+(\sqrt{W}+\&c.)=0,$$
$$\sqrt{U}+\&c.-(\sqrt{W}+\&c.)=0,$$

viz., fixing the significations of $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, &c. in such wise that in the equation of one branch these shall each of them have the sign $+$, we may take $\sqrt{U}$, &c. to be those radicals which, in the equation of the other branch, have the sign $+$, and

$\sqrt{W}$, &c. to be those radicals which have the sign $-$. The foregoing equations break up into the more simple equations

$$\sqrt{U}+\&c.=0,\quad \sqrt{W}+\&c.=0,$$

which are the equations of certain *branches* of the *curves* $\sqrt{U}+\&c.=0$, and $\sqrt{W}+\&c.=0$, respectively, and conversely each of the intersections of these two curves is a point situate simultaneously on some two branches of the original ν-zomal curve $\sqrt{U}+\sqrt{V}+\&c.=0$. Hence, partitioning in any manner the ν-zome $\sqrt{U}+\sqrt{V}+\&c.$ into an α-zome, $\sqrt{U}=\&c.$ and a β-zome $\sqrt{W}+\&c.$ $(\alpha+\beta=\nu)$, and writing down the equations

$$\sqrt{U}+\&c.=0,\quad \sqrt{W}+\&c.=0$$

of an α-zomal curve and a β-zomal curve respectively, each of the intersections of these two curves is a point situate simultaneously on two branches of the ν-zomal curve; and the points situate simultaneously on two branches of the ν-zomal curve are the points of intersection of the several pairs of an α-zomal curve and a β-zomal curve, which can be formed by any bipartition of the ν-zome.

14. There are two cases to be considered:—First, when the parts are $1, \nu-1$ ($\nu-1$ is >1, except in the case $\nu=2$, which may be excluded from consideration), or say when the ν-zome is partitioned into a *zome* and *antizome*. Secondly, when the parts α, β, are each >1 (this implies $\nu=4$ at least), or say when the ν-zome is partitioned into a pair of *complementary parazomes*.

15. To fix the ideas, take the tetrazomal curve $\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T}=0$, and consider first a point for which $\sqrt{U}=0$, $\sqrt{V}+\sqrt{W}+\sqrt{T}=0$. The Norm is the product of $(2^3=)$ 8 factors; selecting hereout the factors

$$\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T},$$
$$\sqrt{U}-\sqrt{V}-\sqrt{W}-\sqrt{T},$$

let the product of these

$$=U-(\sqrt{V}+\sqrt{W}+\sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation of the curve is therefore $FG=0$. The derived equation is $GdF+FdG=0$; at the point in question $\sqrt{U}=0$, $\sqrt{V}+\sqrt{W}+\sqrt{T}=0$; G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F=0,\ dF=dU-(\sqrt{V}+\sqrt{W}+\sqrt{T})(dV\div\sqrt{V}+dW\div\sqrt{W}+dT\div\sqrt{T}),\ =dU,$$

and the derived equation is thus $GdU=0$, or simply $dU=0$. It thus appears that the point in question is an ordinary point on the tetrazomal curve; and, further, that the tetrazomal curve is at this point touched by the zomal curve $U=0$. And similarly, each of the points of intersection of the two curves $\sqrt{U}=0$, $\sqrt{V}+\sqrt{W}+\sqrt{T}=0$, is an ordinary point on the tetrazomal curve; and the tetrazomal curve is at each of these points touched by the zomal curve $U=0$.

16. Consider, secondly, a point for which $\sqrt{U}+\sqrt{V}=0$, $\sqrt{W}+\sqrt{T}=0$; to form the Norm, taking in this case the two factors

$$\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T},$$
$$\sqrt{U}+\sqrt{V}-\sqrt{W}-\sqrt{T},$$

let their product

$$=(\sqrt{U}+\sqrt{V})^2-(\sqrt{W}+\sqrt{T})^2$$

be called F, and the product of the remaining six factors be called G; the rationalised equation is $FG=0$, and the derived equation is $FdG+GdF=0$. At the point in question G and dG are each of them finite (that is, they neither vanish nor become infinite), but we have

$$F=0,\ dF=(\sqrt{U}+\sqrt{V})(dU\div\sqrt{U}+dV\div\sqrt{V})-(\sqrt{W}+\sqrt{T})(dW\div\sqrt{W}+dT\div\sqrt{T}),\ =0,$$

that is, the derived equation becomes identically $0=0$; the point in question is thus a singular point, and it is easy to see that it is in fact a node, or ordinary double point, on the tetrazomal curve. And similarly, each of the points of intersection of the two curves $\sqrt{U}+\sqrt{V}=0$, $\sqrt{W}+\sqrt{T}=0$ is a node on the tetrazomal curve.

17. The proofs in the foregoing two examples respectively are quite general, and we may, in regard to a ν-zomal curve, enunciate the results as follows, viz., in a ν-zomal curve, the points situate simultaneously on two branches are either the intersections of a zomal curve and its antizomal curve, or else they are the intersections of a pair of complementary parazomal curves. In the former case, the points in question are ordinary points on the ν-zomal, but they are points of contact of the ν-zomal with the zomal; it may be added, that the intersections of the zomal and antizomal, each reckoned twice, are *all* the intersections of the ν-zomal and zomal. In the latter case, the points in question are nodes of the ν-zomal; it may be added, that the ν-zomal has not, *in general*, any nodes other than the points which are thus the intersections of a pair of complementary parazomals, and that it has not *in general* any cusps.

Article Nos. 18 to 21. *Singularities of a ν-zomal Curve.*

18. It has been already shown that the order of the ν-zomal curve is $=2^{\nu-2}r$. Considering the case where ν is $=3$ at least, the curve, as we have just seen, has contacts with each of the zomal curves, and it has also nodes. I proceed to determine the number of these contacts and nodes respectively.

19. Consider first the zomal curve $U=0$, and its antizomal $\sqrt{V}+\sqrt{W}+\&c.=0$, these are curves of the orders r and $2^{\nu-3}r$ respectively, and they intersect therefore in $2^{\nu-3}r^2$ points. Hence the ν-zomal touches the zomal in $2^{\nu-3}r^2$ points, and reckoning each of these twice, the number of intersections is $=2^{\nu-2}r^2$, viz., these are all the intersections of the ν-zomal with the zomal $U=0$. The number of contacts of the ν-zomal with the several zomals $U=0$, $V=0$, &c., is of course $=2^{\nu-2}r^2\nu$.

20. Considering next a pair of complementary parazomal curves, an α-zomal and a β-zomal respectively ($\alpha+\beta=\nu$), these are of the orders $2^{\alpha-2}r$ and $2^{\beta-2}r$ respectively, and they intersect therefore in $2^{\alpha+\beta-4}r^2=2^{\nu-4}r^2$ points, nodes of the ν-zomal. This number is independent of the particular partition (α, β), and the ν-zomal has thus this same number, $2^{\nu-4}r^2$, of nodes in respect of each pair of complementary parazomals; hence the total number of nodes is $=2^{\nu-4}r^2$ into the number of pairs of complementary parazomals. For the partition (α, β) the number of pairs is $=[\nu]^\nu \div [\alpha]^\alpha [\beta]^\beta$, or when $\alpha=\beta$, which of course implies ν even, it is one-half of this; extending the summation from $\alpha=2$ to $\alpha=\nu-2$, each pair is obtained twice, and the number of pairs is thus $=\frac{1}{2}\Sigma\{[\nu]^\nu \div [\alpha]^\alpha [\beta]^\beta\}$; the sum extended from $\alpha=0$ to $\alpha=\nu$ is $(1+1)^\nu$, $=2^\nu$, but we thus include the terms 1, ν, ν, 1, which are together $=2\nu+2$, hence the correct value of the sum is $=2^\nu-2\nu-2$, and the number of pairs is the half of this $=2^{\nu-1}-\nu-1$. Hence the number of nodes of the ν-zomal curve is $=(2^{\nu-1}-\nu-1)\,2^{\nu-4}r^2$.

21. The ν-zomal is thus a curve of the order $2^{\nu-2}r$, with $(2^{\nu-1}-\nu-1)\,2^{\nu-4}r^2$ nodes, but without cusps; the class is therefore

$$=2^{\nu-3}r\,[(\nu+1)\,r-2],$$

and the deficiency is

$$=2^{\nu-4}r\,[(\nu+1)\,r-6]+1.$$

These are the general expressions, but even when the zomal curves $U=0$, $V=0$, &c., are given, then writing the equation of the ν-zomal under the form $\sqrt{lU}+\sqrt{mV}+\&c.=0$, the constants $l:m:$ &c., may be so determined as to give rise to nodes or cusps which do not occur in the general case; the formulæ will also undergo modification in the particular cases next referred to.

Article Nos. 22 to 27. *Special Case where all the Zomals have a Common Point or Points.*

22. Consider the case where the zomals $U=0$, $V=0$ have all of them any number, say k, of common intersections—these may be referred to simply as the common points. Each common point is a $2^{\nu-2}$-tuple point on the ν-zomal curve; it is on each zomal an ordinary point, and on each antizomal a $2^{\nu-3}$-tuple point, and on any α-zomal parazomal a $2^{\alpha-2}$-tuple point. Hence, considering first the intersections of any zomal with its antizomal, the common point reckons as $2^{\nu-3}$ intersections, and the k common points reckon as $2^{\nu-3}k$ intersections; the number of the remaining intersections is therefore $=2^{\nu-3}(r^2-k)$, and the zomal touches the ν-zomal in each of these points. The intersections of the zomal with the ν-zomal are the k-common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore reckoning together as $2^{\nu-2}k$ intersections; and the $2^{\nu-3}(r^2-k)$ points of contact, each reckoning twice, and therefore together as $2^{\nu-2}(r^2-k)$ intersections $\left(2^{\nu-2}k+2^{\nu-2}(r^2-k)=2^{\nu-2}r^2,\ =r\,.\,2^{\nu-2}r\right)$; the total number of contacts with the zomals $U=0$, $V=0$, &c., is thus $=2^{\nu-3}(r^2-k)\,\nu$.

23. Secondly, considering any pair of complementary parazomals, an α-zomal and a β-zomal, each of the common points, being a $2^{\alpha-2}$-tuple point and a $2^{\beta-2}$-tuple point on the two curves respectively, counts as $2^{\alpha+\beta-4}$, $=2^{\nu-4}$ intersections, and the k common points count as $2^{\nu-4}k$ intersections; the number of the remaining intersections is therefore $=2^{\nu-4}(r^2-k)$, each of which is a node on the ν-zomal curve; and we have thus in all $2^{\nu-4}(2^{\nu-1}-\nu-1)(r^2-k)$ nodes.

24. There are, besides, the k common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and therefore each reckoning as $\frac{1}{2}2^{\nu-2}(2^{\nu-2}-1)$, $=2^{2\nu-5}-2^{\nu-3}$ double points, or together as $(2^{2\nu-5}-2^{\nu-3})k$ double points. Reserving the term node for the above-mentioned nodes or proper double points, and considering, therefore, the double points (dps.) as made up of the nodes and of the $2^{\nu-2}$-tuple points, the total number of dps. is thus

$$2^{\nu-4}(2^{\nu-1}-\nu-1)(r^2-k)+(2^{2\nu-5}-2^{\nu-3})k,$$

$$=2^{\nu-4}(2^{\nu-1}-\nu-1)\,r^2+\{(\nu+1)\,2^{\nu-4}-2^{\nu-3}\}\,k;$$

or finally this is

$$=2^{\nu-4}\{(2^{\nu-1}-\nu-1)\,r^2+(\nu-1)\};$$

so that there is a gain $=2^{\nu-4}(\nu-1)k$ in the number of dps. arising from the k common points. There is, of course, in the class a diminution equal to twice this number, or $2^{\nu-3}(\nu-1)k$; and in the deficiency a diminution equal to this number, or $2^{\nu-4}(\nu-1)k$.

25. The zomal curves $U=0$, $V=0$, &c., may all of them pass through the same ν^2 points; we have then $k=r^2$, and the expression for the number of dps. is $=(2^{2\nu-5}-2^{\nu-3})r^2$, viz., this is $=\frac{1}{2}2^{\nu-2}(2^{\nu-2}-1)r^2$. But in this case the dps. are nothing else than the r^2 common points, each of them a $2^{\nu-2}$-tuple point, the ν-zomal curve in fact breaking up into a system of $2^{\nu-2}$ curves of the order r, each passing through the r^2 common points. This is easily verified, for if $\Theta=0$, $\Phi=0$ are some two curves of the order r, then, in the present case, the zomal curves are curves in involution with these curves; that is, they are curves of the form $l\Theta+l'\Phi=0$, $m\Theta+m'\Phi=0$, &c., and the equation of the ν-zomal curve is

$$\sqrt{l\Theta+l'\Phi}+\sqrt{m\Theta+m'\Phi}+\&c.=0.$$

The rationalised equation is obviously an equation of the degree $2^{\nu-2}$ in Θ, Φ, giving therefore a constant value for the ratio $\Theta:\Phi$; calling this q, or writing $\Theta=q\Phi$, we have

$$\sqrt{lq+l'}+\sqrt{mq+m'}+\&c.=0,$$

viz., the rationalised equation is an equation of the degree $2^{\nu-2}$ in q, and gives therefore $2^{\nu-2}$ values of q. And the ν-zomal curve thus breaks up into a system of $2^{\nu-2}$ curves each of the form $\Theta-q\Phi=0$, that is, each of them in involution with the curves $\Theta=0$, $\Phi=0$. The equation in q may have a multiple root or roots, and the system of curves so contain repetitions of the same curve or curves; an instance of this (in relation to the trizomal curve) will present itself in the sequel; but I do not at present stop to consider the question.

26. A more important case is when the zomal curves are each of them in involution with the same two given curves, one of them of the order r, the other of an inferior order. Let $\Theta = 0$ be a curve of the order r, $\Phi = 0$ a curve of an inferior order $r - s$; $L = 0$, $M = 0$, &c., curves of the order s; then the case in question is when the zomal curves are of the form $\Theta + L\Phi = 0$, $\Theta + M\Phi = 0$, &c., the equation of the ν-zomal is

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \&c. = 0,$$

where l, m, &c. are constants. This is the most convenient form for the equation, and by considering the functions L, M, &c. as containing implicitly the factors l^{-1}, m^{-1}, &c. respectively, we may take it to include the form $\sqrt{l\Theta + L\Phi} + \sqrt{m\Theta + M\Phi} + \&c. = 0$, which last has the advantage of being immediately applicable to the case where any one or more of the constants l, m, &c. may be $= 0$.

27. In the case now under consideration we have the $r(r-s)$ points of intersection of the curves $\Theta = 0$, $\Phi = 0$ as common points of all the zomals. Hence, putting in the foregoing formula $k = r(r-s)$, we have a ν-zomal curve of the order $2^{\nu-2}r$, having with each zomal $2^{\nu-2}rs$ contacts, or with all the zomals $2^{\nu-2}rs\nu$ contacts, having a node at each of the $2^{\nu-4}rs$ intersections (not being common points $\Theta = 0$, $\Phi = 0$) of each pair of complementary parazomals; that is, together $2^{\nu-4}(2^{\nu-1} - \nu - 1)rs$ nodes, and having, besides, at each of the $r(r-s)$ common points, a $2^{\nu-2}$-tuple point, counting as $2^{2\nu-3} - 2^{\nu-3}$ dps., together as $(2^{2\nu-5} - 2^{\nu-3})r(r-s)$ dps.; whence, taking account of the nodes, the total number of dps. is $= 2^{\nu-4}r[(2^{\nu-1} - 2)r - (\nu - 1)s]$.

Article Nos. 28 to 37. *Depression of Order of the ν-zomal Curve from the Ideal Factor of a Branch or Branches.*

28. In the case of the $r(r-s)$ common points as thus far considered, the order of the ν-zomal curve has remained throughout $= 2^{\nu-2}r$, but the order admits of depression, viz., the constants l, m, &c., and those of the functions L, M, &c., may be such that the Norm contains the factor Φ^ω; the ν-zomal curve then contains as part of itself ($\Phi^\omega = 0$) the curve $\Phi = 0$ taken ω times, and this being so, if we discard the factor in question, and consider the residual curve as being the ν-zomal, the order of the ν-zomal will be $= 2^{\nu-2}r - \omega(r-s)$.

29. To explain how such a factor Φ^ω presents itself, consider the polyzome $\sqrt{l(\Theta + L\Phi)} + \&c.$, or, what is the same thing, $\sqrt{l}\sqrt{\Theta + L\Phi} + \&c.$, belonging to any particular branch of the curve, we may, it is clear, take $\sqrt{\Theta + L\Phi}$, &c. each in a fixed signification as equivalent to $\sqrt{\overline{\Theta + L\Phi}}$, &c., respectively, and the particular branch will then be determined by means of the significations attached to $\sqrt{l}$, $\sqrt{m}$, &c. Expanding the several radicals, the polyzome is

$$\sqrt{l}\left\{\sqrt{\Theta} + \tfrac{1}{2}L\frac{\Phi}{\sqrt{\Theta}} - \tfrac{1}{8}L^2\frac{\Phi^2}{\Theta\sqrt{\Theta}} + \&c.,\right\} + \&c.;$$

or, what is the same thing, it is

$$\sqrt{\Theta}(\sqrt{l}+\&c.)+\tfrac{1}{2}\frac{\Phi}{\sqrt{\Theta}}(L\sqrt{l}+\&c.)-\tfrac{1}{8}\frac{\Phi^2}{\Theta\sqrt{\Theta}}(L^2\sqrt{l}+\&c.)+\&c.$$

which expansion may contain the factor Φ, or a higher power of Φ. For instance, if we have $\sqrt{l}+\&c.=0$, the expansion will then contain the factor Φ; and if we also have $L\sqrt{l}+\&c.=0$ (observe this implies as many equations as there are asyzygetic terms in the whole series of functions L, M, &c.; thus, if L, M, &c., are each of them of the form $aP+bQ+cR$, with the same values of P, Q, R, but with different values of the coefficients a, b, c, then it implies the three equations $a\sqrt{l}+\&c.=0$, $b\sqrt{l}+\&c.=0$, $c\sqrt{l}+\&c.=0$; and so in other cases), if I say $L\sqrt{l}+\&c.$ be also $=0$, then the expansion will contain the factor Φ^2, and so on; the most general supposition being, that the expansion contains as factor a certain power Φ^α of Φ. Imagine each of the polyzomes expanded in this manner, and let certain of the expansions contain the factors Φ^α, Φ^β, &c., respectively. The produce of the expansions is identically equal to the product of the unexpanded polyzomes—that is, it is equal to the Norm; hence, if $\alpha+\beta+\&c.=\omega$, the Norm will contain the factor Φ^ω.

30. It has been mentioned that the form $\sqrt{l(\Theta+L\Phi)}$ is considered as including the form $\sqrt{l\Theta+L\Phi}$, that is, when $l=0$, the form $\sqrt{L\Phi}$. If in the equation of the ν-zomal curve there is any such term—for instance, if the equation be $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.=0$, the radical $\sqrt{L\Phi}$ contains the factor $\Phi^{\frac{1}{2}}$; but if L contains as factor an odd or an even power of Φ, then $\sqrt{L\Phi}$ will contain the factor Φ^α where α is either an integer, or an integer $+\frac{1}{2}$. Consider the polyzome $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.$, belonging to any particular branch of the curve; the radical $\sqrt{L\Phi}$ contains, as just mentioned, the factor Φ^α, and if the remaining terms $\sqrt{m(\Theta+M\Phi)}+\&c.$, are such that the expansion contains as factor the same or any higher power of Φ, then the expansion of the polyzome $\sqrt{L\Phi}+\sqrt{m(\Theta+M\Phi)}+\&c.$, belonging to the particular branch will contain the factor Φ^α; and similarly we may have branches containing the factors Φ^α, Φ^β, &c., whence, as before, if $\omega=\alpha+\beta+\&c.$, the Norm will contain the factor Φ^ω; the only difference is, that now α, β, &c., instead of being of necessity all integers, are each of them an integer, or an integer $+\frac{1}{2}$; of course, in the latter case the integer may be zero, or the index be $=\frac{1}{2}$. It is clear that ω must be an integer, and it is, in fact, easy to see that the fractional indices occur in pairs; for observe that α being fractional, the expansion of $\sqrt{m(\Theta+M\Phi)}+\&c.$, will contain not Φ^α, but a higher power, $\Phi^{\alpha+q}$, where $\alpha+q$ is an integer; whence *each* of the polyzomes $\sqrt{L\Phi}\pm\left(\sqrt{m(\Theta+M\Phi)}+\&c.\right)$ will contain the factor Φ^α.

31. Observe that in every case the factor Φ^α presents itself as a factor of the expansion of the polyzome corresponding to a particular branch of the curve; the polyzome itself does not contain the factor Φ^α, and we cannot in anywise say that the corresponding branch contains as factor the curve $\Phi^\alpha=0$; but we may, with great propriety of expression, say that *the branch ideally contains the curve* $\Phi^\alpha=0$; and this

being so, the general theorem is, that if we have branches ideally containing the curves $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively, then the ν-zomal curve contains not ideally but actually the factor $\Phi^\omega = 0$ ($\omega = \alpha + \beta +$ &c.), the order of the ν-zomal being thus reduced from $2^{\nu-2}r$ to $2^{\nu-2}r - \omega(r-s)$; and conversely, that any such reduction in the order of the ν-zomal arises from the factors $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c., ideally contained in the several branches of the ν-zomal.

32. It is worth while to explain the notion of an ideal factor somewhat more generally; an irrational function, taking the irrationalities thereof in a determinate manner, may be such that, as well the function itself as all its differential coefficients up to the order $\alpha - 1$, vanish when a certain parameter Φ contained in the function is put $= 0$; this is only saying, in other words, that the function expanded in ascending powers of Φ contains no power lower than Φ^α; and, in this case, we say that the irrational function contains *ideally* the factor Φ^α. The rationalised expression, or Norm, in virtue of the irrational function (taken determinately as above) thus ideally containing Φ^α, will actually contain the factor Φ^α; and if any other values of the irrational function contain respectively Φ^β, &c., then the Norm will contain the factor $\Phi^{\alpha+\beta+\&c.}$.

33. A branch ideally containing $\Phi^\alpha = 0$ may for shortness be called integral or fractional, according as the index α is an integer or a fraction; by what precedes the fractional branches present themselves in pairs. If for a moment we consider integral branches only, then if the ν-zomal contain $\Phi = 0$, this can happen in one way only, there must be some one branch ideally containing $\Phi = 0$; but if the ν-zomal contain $\Phi^2 = 0$, then this may happen in two ways,—either there is a single branch ideally containing $\Phi^2 = 0$, or else there are two branches, each of them ideally containing $\Phi = 0$. And generally, if the ν-zomal contain $\Phi^\omega = 0$, then forming any partition $\omega = \alpha + \beta +$ &c. (the parts being integral), this may arise from there being branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. The like remarks apply to the case where we attend also to fractional branches,—thus, if the ν-zomal contain $\Phi = 0$, this may arise (not only, as above mentioned, from a branch ideally containing $\Phi = 0$, but also) from a pair of branches, each ideally containing $\Phi^{\frac{1}{2}} = 0$. And so in general, if the ν-zomal contain $\Phi^\omega = 0$, the partition $\omega = \alpha + \beta +$ &c. is to be made with the parts integral or fractional ($= \frac{1}{2}$ or integer $+ \frac{1}{2}$ as above), but with the fractional terms in pairs; and then the factor $\Phi^\omega = 0$ may arise from branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively.

34. Any zomal, antizomal, or parazomal of a ν-zomal curve, $\sqrt{l(\Theta + L\Phi)} +$ &c. $= 0$, is a polyzomal curve (including in the term a monozomal curve) of the same form as the ν-zomal; and may in like manner contain $\Phi = 0$, or more generally, $\Phi^\omega = 0$, viz., if $\omega = \alpha + \beta +$ &c. be any partition of ω as above, this will be the case if the zomal, antizomal, or parazomal has branches ideally containing $\Phi^\alpha = 0$, $\Phi^\beta = 0$, &c. respectively. It is to be observed that if a zomal, antizomal, or parazomal contain $\Phi = 0$, or any higher power $\Phi^\omega = 0$, this does not in anywise imply that the zomal contains even $\Phi = 0$. But if (attending only to the most simple case) a zomal and its antizomal, or a pair of complementary parazomals, each contain $\Phi = 0$ inseparably (that

is, through a single branch ideally containing $\Phi=0$), then the ν-zomal will have two branches, each ideally containing $\Phi=0$, and it will thus contain $\Phi^2=0$. In fact, if in the zomal and antizomal, or in the complementary parazomals, the branches which ideally contain $\Phi=0$ are

$$\sqrt{l(\Theta+L\Phi)}+\&c.=0,\quad \sqrt{n(\Theta+N\Phi)}+\&c.=0$$

respectively (for a zomal, the $+\&c.$ should be omitted, and the first equation be written $\sqrt{l(\Theta+L\Phi)}=0$), then in the ν-zomal there will be the two branches

$$\left(\sqrt{l(\Theta+L\Phi)}+\&c.\right)\pm\left(\sqrt{n(\Theta+N\Phi)}+\&c.\right)=0,$$

each ideally containing $\Phi=0$.

Conversely, if a ν-zomal contain $\Phi^2=0$ by reason that it has two branches each ideally containing $\Phi=0$, then either a zomal and its antizomal will each of them, or else a pair of complementary parazomals will each of them, inseparably contain $\Phi=0$.

35. Reverting to the case of the ν-zomal curve

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\&c.=0,$$

which does not contain $\Phi=0$, each of the $r(r-s)$ common points $\Theta=0$, $\Phi=0$ is a $2^{\nu-2}$-tuple point on the ν-zomal; each of these counts therefore for $2^{\nu-2}$ intersections of the ν-zomal with the curve $\Phi=0$, and we have thus the complete number $2^{\nu-2}r(r-s)$ of intersections of the two curves, viz., the curve $\Phi=0$ meets the ν-zomal in the $r(r-s)$ common points, each of them a $2^{\nu-2}$-tuple point on the ν-zomal, and in no other point.

36. But if the ν-zomal contains $\Phi^\omega=0$, then each of the $r(r-s)$ common points is still a $2^{\nu-2}$-tuple point on the aggregate curve; the aggregate curve therefore passes $2^{\nu-2}$ times through each common point; but among these passages are included ω passages of the curve $\Phi=0$ through the common point. The residual curve—say the ν-zomal—passes therefore only $2^{\nu-2}-\omega$ times through the common point; that is, each of the $r(r-s)$ common points is a $(2^{\nu-2}-\omega)$ tuple point on the ν-zomal. The curve $\Phi=0$ meets the ν-zomal in $\{2^{\nu-2}r-\omega(r-s)\}(r-s)$ points, viz., these include the $r(r-s)$ common points, each of them a $(2^{\nu-2}-\omega)$ tuple point on the ν-zomal, and therefore counting together as $(2^{\nu-2}-\omega)r(r-s)$ intersections; there remain consequently $\omega s(r-s)$ other intersections of the curve $\Theta=0$ with the ν-zomal.

37. In the case where the ν-zomal contains the factor $\Phi^\omega=0$, then throughout excluding from consideration the $r(r-s)$ common points $\Theta=0$, $\Phi=0$, the *remaining* intersections of any zomal with its antizomal are points of contact of the zomal with the ν-zomal, and the *remaining* intersections of each pair of complementary parazomals are nodes of the ν-zomal, it being understood that if any zomal, antizomal, or parazomal contain a power of $\Phi=0$, such powers of $\Phi=0$ are to be discarded, and only the residual curves attended to. The number of contacts and of nodes may in any particular case be investigated without difficulty, and some instances will present themselves in the sequel, but on account of the different ways in which the factor

$\Phi^{\omega}=0$ may present itself, ideally in a single branch, or in several branches, and the consequent occurrence in the latter case of powers of $\Phi=0$ in certain of the zomals, antizomals, or parazomals, the cases to be considered would be very numerous, and there is no reason to believe that the results could be presented in any moderately concise form; I therefore abstain from entering on the question.

Article Nos. 38 and 39. *On the Trizomal Curve and the Tetrazomal Curve.*

38. The trizomal curve

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0$$

has for its rationalised form of equation

$$U^2+V^2+W^2-2VW-2WU-2UV=0;$$

or as this may also be written,

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1)(U,\ V,\ W)^2=0;$$

and we may from this rational equation verify the general results applicable to the case in hand, viz., that the trizomal is a curve of the order $2r$, and that

$$\begin{array}{llll} U=0, & \text{at each of its } r^2 \text{ intersections with} & V-W=0, \\ V=0, & \quad\text{"} \qquad\qquad\text{"} & W-U=0, \\ W=0, & \quad\text{"} \qquad\qquad\text{"} & U-V=0, \end{array}$$

respectively touch the trizomal. There are not, in general, any nodes or cusps, and the order being $=2r$, the class is $=2r(2r-1)$.

39. The tetrazomal curve

$$\sqrt{U}+\sqrt{V}+\sqrt{W}+\sqrt{T}=0$$

has for its rationalised form of equation

$$(U^2+V^2+W^2+T^2-2UV-2UW-2UT-2VW-2VT-2WT)^2-64UVWT=0,$$

and we may hereby verify the fundamental properties, viz., that the tetrazomal is a curve of the order $4r$, touched by each of the zomals $U=0$, $V=0$, $W=0$, $T=0$ in $2r^2$ points, viz., by $U=0$ at its intersections with $\sqrt{U}+\sqrt{W}+\sqrt{T}=0$, that is, $V^2+W^2+T^2-2VW-2VT-2WT=0$; (and the like as regards the other zomals), and having $3r^2$ nodes, viz., these are the intersections of $(\sqrt{U}+\sqrt{V}=0,\ \sqrt{W}+\sqrt{T}=0)$, $(\sqrt{U}+\sqrt{W}=0,\ \sqrt{V}+\sqrt{T}=0)$, $(\sqrt{U}+\sqrt{T}=0,\ \sqrt{V}+\sqrt{W}=0)$, or, what is the same thing, the intersections of $(U-V=0,\ W-T=0)$, $(U-W=0,\ V-T=0)$, $(U-T=0,\ V-W=0)$. There are not in general any cusps, and the class is thus $=4r(4r-1)-6r^2=10r^2-4r$.

Article Nos. 40 and 41. *On the Intersection of two ν-Zomals having the same Zomal Curves.*

40. Without going into any detail, I may notice the question of the intersection of two ν-zomals which have the same zomal curves—say the two trizomals $\sqrt{U}+\sqrt{V}+\sqrt{W}=0$, $\sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}=0$, or two similarly related tetrazomals. For the trizomals, writing the equations under the form

$$\sqrt{U}+\sqrt{V}+\sqrt{W}=0,\quad \sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}=0,$$

then, when these equations are considered as existing simultaneously, we may, without loss of generality, attribute to the radicals $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, the same values in the two equations respectively; but doing so, we must in the second equation successively attribute to all but one of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, each of its two opposite values. For the intersections of the two curves we have thus

$$\sqrt{U}:\sqrt{V}:\sqrt{W}=\sqrt{m}-\sqrt{n}:\sqrt{n}-\sqrt{l}:\sqrt{l}-\sqrt{m},$$

viz., this is one of a system of four equations, obtained from it by changes of sign, say in the radicals $\sqrt{m}$ and $\sqrt{n}$. Each of the four equations gives a set of r^2 points; we have thus the complete number, $=4r^2$, of the points of intersection of the two curves.

41. But take, in like manner, two tetrazomal curves; writing their equations in the form

$$\sqrt{U}+\quad\sqrt{V}+\quad\sqrt{W}+\quad\sqrt{T}=0,$$
$$\sqrt{l}\sqrt{U}+\sqrt{m}\sqrt{V}+\sqrt{n}\sqrt{W}+\sqrt{p}\sqrt{T}=0,$$

then $\sqrt{U}$, $\sqrt{V}$, $\sqrt{W}$, $\sqrt{T}$ may be considered as having the same values in the two equations respectively, but we must in the second equation attribute successively, say to $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$, each of their two opposite values. For the intersections of the two curves we have

$$(\sqrt{m}-\sqrt{l})\sqrt{V}+(\sqrt{n}-\sqrt{l})\sqrt{W}+(\sqrt{p}-\sqrt{l})\sqrt{T}=0,$$
$$(\sqrt{l}-\sqrt{m})\sqrt{U}\qquad+(\sqrt{n}-\sqrt{m})\sqrt{W}+(\sqrt{p}-\sqrt{m})\sqrt{T}=0,$$

viz., this is one of a system of eight similar pairs of equations, obtained therefrom by changes of sign of the radicals $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$. The equations represent each of them a trizomal curve, of the order $2r$; the two curves intersect therefore in $4r^2$ points, and if each of these was a point of intersection of the two tetrazomals, we should have in all $8\times 4r^2=32r^2$ intersections. But the tetrazomals are each of them a curve of the order $4r$, and they intersect therefore in only $16r^2$ points. The explanation is, that not all the $4r^2$ points, but only $2r^2$ of them are intersections of the tetrazomals. In fact, to find *all* the intersections of the two trizomals, it is necessary in their two equations to attribute opposite signs to one of the radicals $\sqrt{W}$, $\sqrt{T}$; we obtain $2r^2$ intersections from the equations as they stand, the remaining $2r^2$ intersections from the two equations after we have in the second equation reversed the sign, say of $\sqrt{T}$.

Now, from the two equations as they stand we can pass back to the two tetrazomal equations, and the first-mentioned $2r^2$ points are thus points of intersection of the two tetrazomal curves—from the two equations after such reversal of the sign of $\sqrt{T}$, we cannot pass back to the two tetrazomal equations, and the last-mentioned $2r^2$ points are thus not points of intersection of the two tetrazomal curves. The number of intersections of the two curves is thus $8 \times 2r^2$, $= 16r^2$, as it should be.

Article Nos. 42 to 45. *The Theorem of the Decomposition of a Tetrazomal Curve.*

42. I consider the tetrazomal curve—

$$\sqrt{lU} + \sqrt{mV} + \sqrt{nW} + \sqrt{pT} = 0,$$

where the zomal curves are in involution,—that is, where we have an identical relation,

$$\mathrm{a}U + \mathrm{b}V + \mathrm{c}W + \mathrm{d}T = 0;$$

and I proceed to show that if l, m, n, p satisfy the relation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0,$$

the curve breaks up into two trizomals. In fact, writing the equation under the form

$$(\sqrt{lU} + \sqrt{mV} + \sqrt{nW})^2 - pT = 0,$$

and substituting for T its value, in terms of U, V, W, this is

$$(l\mathrm{d} + p\mathrm{a})\, U + (m\mathrm{d} + p\mathrm{b})\, V + (n\mathrm{d} + p\mathrm{c})\, W$$
$$+ 2\sqrt{mn\mathrm{d}}\sqrt{VW} + 2\sqrt{nl\mathrm{d}}\sqrt{WU} + 2\sqrt{lm\mathrm{d}}\sqrt{UV} = 0;$$

or, considering the left-hand side as a quadric function of $(\sqrt{U}, \sqrt{V}, \sqrt{W})$, the condition for its breaking up into factors is

$$\begin{vmatrix} l\mathrm{d} + p\mathrm{a}, & \mathrm{d}\sqrt{lm}, & \mathrm{d}\sqrt{ln} \\ \mathrm{d}\sqrt{ml}, & m\mathrm{d} + p\mathrm{b}, & \mathrm{d}\sqrt{mn} \\ \mathrm{d}\sqrt{nl}, & \mathrm{d}\sqrt{nm}, & n\mathrm{d} + p\mathrm{c} \end{vmatrix} = 0,$$

that is

$$p^2(l\mathrm{bcd} + m\mathrm{cda} + n\mathrm{dab} + p\mathrm{abc}) = 0,$$

or finally, the condition is

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0.$$

43. Multiplying by $l\mathrm{d} + p\mathrm{a}$, and observing that in virtue of the relation we have

$$(l\mathrm{d} + p\mathrm{a})(m\mathrm{d} + p\mathrm{b}) = lm\mathrm{d}^2 - \frac{\mathrm{abd}}{\mathrm{c}}\, pn,$$

$$(l\mathrm{d} + p\mathrm{a})(n\mathrm{d} + p\mathrm{c}) = ln\mathrm{d}^2 - \frac{\mathrm{acd}}{\mathrm{b}}\, pm,$$

the equation becomes

$$\left((ld+pa)\sqrt{U}+d\sqrt{lm}\sqrt{V}+d\sqrt{ln}\sqrt{W}\right)^2=\frac{ad}{bc}p\left(b\sqrt{n}\sqrt{V}-c\sqrt{m}\sqrt{W}\right)^2,$$

or as this is more conveniently written

$$\left(\left(\sqrt{l}+\frac{ap}{d\sqrt{l}}\right)\sqrt{U}+\sqrt{mV}+\sqrt{nW}\right)^2=\frac{a}{bcd}\frac{p}{l}\left(b\sqrt{nV}-c\sqrt{mV}\right)^2,$$

an equation breaking up into two equations, which may be represented by

$$\sqrt{l_1}\sqrt{U}+\sqrt{m_1}\sqrt{V}+\sqrt{n_1}\sqrt{W}=0,\quad \sqrt{l_2}\sqrt{U}+\sqrt{m_2}\sqrt{V}+\sqrt{n_2}\sqrt{W}=0,$$

where

$$\sqrt{l_1}=\sqrt{l}+\frac{a}{d}\frac{p}{\sqrt{l}}\quad,\quad \sqrt{l_2}=\sqrt{l}+\frac{a}{d}\frac{p}{\sqrt{l}}$$

$$\sqrt{m_1}=\sqrt{m}-\sqrt{\frac{a}{bcd}\frac{p}{l}}\,b\sqrt{n}\quad,\quad \sqrt{m_2}=\sqrt{m}+\sqrt{\frac{a}{bcd}\frac{p}{l}}\,b\sqrt{n}$$

$$\sqrt{n_1}=\sqrt{n}+\sqrt{\frac{a}{bcd}\frac{p}{l}}\,c\sqrt{m}\quad,\quad \sqrt{n_2}=\sqrt{n}-\sqrt{\frac{a}{bcd}\frac{p}{l}}\,c\sqrt{m},$$

where, in the expressions for $\sqrt{l}$, &c., the signs of the radicals

$$\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{\frac{a}{bcd}\frac{p}{l}},$$

may be taken determinately in any way whatever at pleasure; the only effect of an alteration of sign would in some cases be to interchange the values of $(\sqrt{l_1}, \sqrt{m_1}, \sqrt{n_1})$ with those of $(\sqrt{l_2}, \sqrt{m_2}, \sqrt{n_2})$. The tetrazomal curve thus breaks up into two trizomals.

44. It is to be noticed that we have

$$\begin{aligned}\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}= &\ \frac{l}{a}+\frac{ap^2}{d^2l}+2\frac{p}{d}\\ &+\frac{m}{b}+\frac{a}{cd}\frac{np}{l}\\ &+\frac{n}{c}+\frac{a}{bd}\frac{mp}{l}\\ = &\ \left(l+\frac{ap}{dl}\right)\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}\right);\end{aligned}$$

that is

$$\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}=0;$$

and that similarly we have

$$\frac{l_2}{a}+\frac{m_2}{b}+\frac{n_2}{c}=0.$$

The meaning is, that, taking the trizonal curve $\sqrt{l_1 U}+\sqrt{m_1 V}+\sqrt{n_1 W}=0$, this regarded as a tetrazomal curve, $\sqrt{l_1 U}+\sqrt{m_1 V}+\sqrt{n_1 W}+\sqrt{0T}=0$, satisfies the condition $\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}+\frac{0}{d}=0$; and the like as to the trizomal curve $\sqrt{l_2 U}+\sqrt{m_2 V}+\sqrt{n_2 W}=0$.

45. The equation by which the decomposition was effected is, it is clear, one of twelve equivalent equations; four of these are

$$\left(\sqrt{l}+\frac{ap}{d\sqrt{l}},\ \sqrt{m}\ ,\ \sqrt{n}\ ,\ 0\right)\left(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T}\right)^2=\frac{a}{bcd}\frac{p}{l}\left(b\sqrt{nW}-c\sqrt{mW}\right)^2,$$

$$\left(0\ ,\ \sqrt{m}+\frac{bl}{a\sqrt{m}},\ \sqrt{n}\ ,\ \sqrt{p}\right)\left(\quad ,,\quad\right)^2=\frac{b}{cda}\frac{l}{m}\left(c\sqrt{pW}-d\sqrt{nT}\right)^2,$$

$$\left(\sqrt{l}\ ,\ 0\ ,\ \sqrt{n}+\frac{cm}{b\sqrt{n}},\ \sqrt{p}\right)\left(\quad ,,\quad\right)^2=\frac{c}{dab}\frac{m}{n}\left(d\sqrt{lT}-a\sqrt{pU}\right)^2,$$

$$\left(\sqrt{l}\ ,\ \sqrt{m}\ ,\ 0\ ,\ \sqrt{p}+\frac{dn}{c\sqrt{p}}\right)\left(\quad ,,\quad\right)^2=\frac{d}{abc}\frac{n}{p}\left(a\sqrt{mU}-b\sqrt{lV}\right)^2,$$

and the others may be deduced from these by a cyclical permutation of (U, V, W), (a, b, c), (l, m, n), leaving T, d, p unaltered.

Article Nos. 46 to 51. *Application to the Trizomal; the Theorem of the Variable Zomal.*

46. I take the last equation written under the form

$$(a\sqrt{mU}-b\sqrt{lV})^2=\frac{abc}{dn}\left(\sqrt{lpU}+\sqrt{mpV}+\left(p+\frac{dn}{c}\right)\sqrt{T}\right)^2,$$

which, putting therein $p=0$, is

$$(a\sqrt{mU}-b\sqrt{lV})^2=\frac{abd}{c}nT,$$

which is in fact the trizomal curve,

$$a\sqrt{mU}-b\sqrt{lV}+\sqrt{\frac{abd}{c}nT}=0,$$

viz., the trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$,—if a, b, c be any quantities connected by the equation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

(the ratios a, b, c thus involving a single arbitrary parameter); and if we take T a function such that $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0$; that is, $T=0$, any one of the series of curves $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W=0$, in involution with the given curves $U=0$, $V=0$, $W=0$,—has its equation expressible in the form

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT}=0;$$

that is, we have the curve $T=0$ (the equation whereof contains a variable parameter) as a zomal of the given trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$; and we have thus from the theorem of the decomposition of a tetrazomal deduced the theorem of the variable zomal of a trizomal. The analytical investigation is somewhat simplified by assuming $p=0$ *ab initio*, and it may be as well to repeat it in this form.

47. Starting, then, with the trizomal curve

$$\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0,$$

and writing

$$\mathrm{a}U+\ \mathrm{b}V+\ \mathrm{c}W+\mathrm{d}T=0$$

as the definition of T, the coefficients being connected by

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

the equation gives

$$lU+\ mV+2\sqrt{lmUV}-nW=0;$$

or substituting in this equation for W its value in terms of U, V, T, we have

$$(\mathrm{a}n+\mathrm{c}l)\,U+(\mathrm{b}n+\mathrm{c}m)\,V+2\mathrm{c}\sqrt{lmUV}+\mathrm{d}nT=0,$$

which by the given relation between a, b, c, is converted into

$$-\frac{\mathrm{ac}}{\mathrm{b}}mU-\frac{\mathrm{bc}}{\mathrm{a}}lV+2\mathrm{c}\sqrt{lmUV}+\mathrm{d}nT=0;$$

that is

$$\mathrm{a}^2mU+\mathrm{b}^2lV-2\mathrm{ab}\sqrt{lmUV}\ =\frac{\mathrm{abd}}{\mathrm{c}}nT,$$

viz., this is

$$(\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV})^2\qquad\qquad =\frac{\mathrm{abd}}{\mathrm{c}}nT,$$

or finally

$$\mathrm{a}\sqrt{mU}-\mathrm{b}\sqrt{lV}+\sqrt{\frac{\mathrm{abd}}{\mathrm{c}}nT}=0.$$

48. The result just obtained of course implies that when as above

$$\mathrm{a}U+\mathrm{b}V+\mathrm{c}W+\mathrm{d}T=0,\quad \frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

the trizomal curve $\sqrt{l}U+\sqrt{m}V+\sqrt{n}W=0$ can be expressed by means of any three of the four zomals U, V, W, T, and we may at once write down the four forms

$$\begin{pmatrix} \cdot\,, & \sqrt{\frac{n}{\mathrm{c}^2}}\,, & -\sqrt{\frac{m}{\mathrm{b}^2}}\,, & -\sqrt{\frac{l\mathrm{d}}{\mathrm{abc}}} \end{pmatrix}\left(\sqrt{U},\ \sqrt{V},\ \sqrt{W},\ \sqrt{T}\right)=0,$$

$$\begin{vmatrix} -\sqrt{\frac{n}{\mathrm{c}^2}}\,, & \cdot\,, & \sqrt{\frac{l}{\mathrm{a}^2}}\,, & -\sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}} \\ \sqrt{\frac{m}{\mathrm{b}^2}}\,, & -\sqrt{\frac{l}{\mathrm{a}^2}}\,, & \cdot\,, & -\sqrt{\frac{n\mathrm{d}}{\mathrm{abc}}} \\ \sqrt{\frac{l\mathrm{d}}{\mathrm{abc}}}, & \sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}}, & \sqrt{\frac{n\mathrm{d}}{\mathrm{abc}}}, & \cdot \end{vmatrix}$$

the last of which is the original equation $\sqrt{l}U+\sqrt{m}V+\sqrt{n}W=0$. It may be added that if the first equation be represented by $\sqrt{m_1}V+\sqrt{n_1}W+\sqrt{p_1}T=0$,—that is, if we have

$$\sqrt{m_1}=\sqrt{\frac{n}{\mathrm{c}^2}},\quad \sqrt{n_1}=-\sqrt{\frac{m}{\mathrm{b}^2}},\quad \sqrt{p_1}=\sqrt{\frac{l\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}+\frac{p_1}{\mathrm{d}}=\frac{l}{\mathrm{bc}}\left(\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}\right),=0;$$

or if the second equation be represented by $\sqrt{l_2}U+\sqrt{n_2}W+\sqrt{p_2}T=0$,—that is, if we have

$$\sqrt{l_2}=-\sqrt{\frac{n}{\mathrm{c}^2}},\quad \sqrt{n_2}=\sqrt{\frac{l}{\mathrm{a}^2}},\quad \sqrt{p_2}=\sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{l_2}{\mathrm{a}}+\frac{n_2}{\mathrm{c}}+\frac{p_2}{\mathrm{d}}=0;$$

or if the third equation be represented by $\sqrt{l_3}U+\sqrt{m_3}V+\sqrt{p_3}T=0$,—that is, if we have

$$\sqrt{l_3}=\sqrt{\frac{m}{\mathrm{b}^2}},\quad \sqrt{m_3}=-\sqrt{\frac{l}{\mathrm{a}^2}},\quad \sqrt{p_3}=\sqrt{\frac{m\mathrm{d}}{\mathrm{abc}}},$$

and therefore

$$\frac{l_3}{\mathrm{a}}+\frac{m_3}{\mathrm{b}}+\frac{p_3}{\mathrm{d}}=0,$$

then the equation of the trizomal may also be expressed in the forms

$$\left(\begin{array}{cccc} \cdot\,, & \sqrt{m_1}\,, & \sqrt{n_1}\,, & \sqrt{p_1} \\ -\sqrt{m_1}, & \cdot\,, & \sqrt{\frac{p_1\mathrm{bc}}{\mathrm{ad}}}, & -\sqrt{\frac{n_1\mathrm{bd}}{\mathrm{ac}}} \\ -\sqrt{n_1}\,, & -\sqrt{\frac{p_1\mathrm{bc}}{\mathrm{ad}}}, & \cdot\,, & \sqrt{\frac{m_1\mathrm{cd}}{\mathrm{ab}}} \\ -\sqrt{p_1}\,, & \sqrt{\frac{n_1\mathrm{bd}}{\mathrm{ac}}}, & -\sqrt{\frac{m_1\mathrm{cd}}{\mathrm{ab}}}, & \cdot \end{array}\right)(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T}) = 0,$$

$$\left(\begin{array}{cccc} \cdot\,, & -\sqrt{l_2}\,, & -\sqrt{\frac{p_2\mathrm{ac}}{\mathrm{bd}}}, & \sqrt{\frac{n_2\mathrm{ad}}{\mathrm{bc}}} \\ \sqrt{l_2}\,, & \cdot\,, & \sqrt{n_2}\,, & \sqrt{p_2} \\ \sqrt{\frac{p_2\mathrm{ac}}{\mathrm{bd}}}, & -\sqrt{n_2}\,, & \cdot\,, & -\sqrt{\frac{l_2\mathrm{cd}}{\mathrm{ab}}} \\ -\sqrt{\frac{n_2\mathrm{ad}}{\mathrm{bc}}}, & -\sqrt{p_2}\,, & -\sqrt{\frac{l_2\mathrm{cd}}{\mathrm{ab}}}, & \cdot \end{array}\right)\left(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T}\right) = 0,$$

and

$$\left(\begin{array}{cccc} \cdot\,, & \sqrt{\frac{p_3\mathrm{ab}}{\mathrm{cd}}}, & -\sqrt{l_3}\,, & -\sqrt{\frac{m_3\mathrm{ad}}{\mathrm{bc}}} \\ -\sqrt{\frac{p_3\mathrm{ab}}{\mathrm{cd}}}, & \cdot\,, & -\sqrt{m_3}, & -\sqrt{\frac{l_3\mathrm{bd}}{\mathrm{ac}}} \\ \sqrt{l_3}\,, & \sqrt{m_3}\,, & \cdot\,, & \sqrt{p_3} \\ \sqrt{\frac{m_3\mathrm{ad}}{\mathrm{bc}}}, & -\sqrt{\frac{l_3\mathrm{bd}}{\mathrm{ac}}}, & -\sqrt{p_3}\,, & \cdot \end{array}\right)\left(\sqrt{U}, \sqrt{V}, \sqrt{W}, \sqrt{T}\right) = 0.$$

49. These equations may, however, be expressed in a much more elegant form. Write

$$\mathrm{a}' = \frac{a}{(\beta\gamma\delta)}, \quad \mathrm{b}' = -\frac{b}{(\gamma\delta\alpha)}, \quad \mathrm{c}' = \frac{c}{(\delta\alpha\beta)}, \quad \mathrm{d}' = \frac{-d}{(\alpha\beta\gamma)},$$

where, for shortness, $(\beta\gamma\delta) = (\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c.; (α, β, γ) being arbitrary quantities: or, what is the same thing,

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \mathrm{a}'(\beta\gamma\delta) : -\mathrm{b}'(\gamma\delta\alpha) : \mathrm{c}'(\delta\alpha\beta) : -\mathrm{d}'(\alpha\beta\gamma).$$

Assume

$$l : m : n \quad = \rho\mathrm{a}'(\beta-\gamma)^2 : \sigma\mathrm{b}'(\gamma-\alpha)^2 : \tau\mathrm{c}'(\alpha-\beta)^2;$$

then the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$ takes the form

$$\rho(\beta-\gamma)(\alpha-\delta) + \sigma(\gamma-\alpha)(\beta-\delta) + \tau(\alpha-\beta)(\gamma-\delta),$$

and the four forms of the equation are found to be

$$\left(\begin{array}{cccc} \cdot\ , & \sqrt{\tau}(\delta-\gamma), & \sqrt{\sigma}(\beta-\delta), & \sqrt{\rho}(\gamma-\beta) \\ \sqrt{\tau}(\gamma-\delta), & \cdot\ , & \sqrt{\rho}(\delta-\alpha), & \sqrt{\sigma}(\alpha-\gamma) \\ \sqrt{\sigma}(\delta-\beta), & \sqrt{\rho}(\alpha-\delta), & \cdot\ , & \sqrt{\tau}(\beta-\alpha) \\ \sqrt{\rho}(\beta-\gamma), & \sqrt{\sigma}(\gamma-\alpha), & \sqrt{\tau}(\alpha-\beta), & \cdot \end{array}\right)\left(\sqrt{\mathrm{a}'U}, \sqrt{\mathrm{b}'V}, \sqrt{\mathrm{c}'W}, \sqrt{\mathrm{d}'T}\right)=0,$$

viz., these are the equivalent forms of the original equation assumed to be

$$(\beta-\gamma)\sqrt{\rho\mathrm{a}'U}+(\gamma-\alpha)\sqrt{\sigma\mathrm{b}'V}+(\alpha-\beta)\sqrt{\tau\mathrm{c}'W}=0.$$

50. I remark that the theorem of the variable zomal may be obtained as a transformation theorem—viz., comparing the equation $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$ with the equation $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$; this last belongs to a conic touched by the three lines $x=0$, $y=0$, $z=0$; the equation of the same conic must, it is clear, be expressible in a similar form by means of any other three tangents thereof, but the equation of any tangent of the conic is $\mathrm{a}x+\mathrm{b}y+\mathrm{c}z=0$, where a, b, c are any quantities satisfying the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$; whence, writing $\mathrm{a}x+\mathrm{b}y+\mathrm{c}z+\mathrm{d}w=0$, we may introduce $w=0$ along with any two of the original zomals $x=0$, $y=0$, $z=0$, or, instead of them, any three functions of the form w; and then the mere change of x, y, z, w into U, V, W, T gives the theorem. But it is as easy to conduct the analysis with (U, V, W, T) as with (x, y, z, w), and, so conducted, it is really the same analysis as that whereby the theorem is established *ante*, No. 47.

51. It is worth while to exhibit the equation of the curve

$$\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0,$$

in a form containing three new zomals. Observe that the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$ is satisfied by $\mathrm{a}=l\phi\chi$, $\mathrm{b}=m\chi\theta$, $\mathrm{c}=n\theta\phi$, if only $\theta+\phi+\chi=0$; or say, if $\theta=a'-a''$, $\phi=a''-a$, $\chi=a-a''$. The equation

$$\begin{aligned} &\lambda\sqrt{(a-a')(a-a'')lU+(a'-a'')(a'-a)mV+(a''-a)(a''-a')nW} \\ &+\mu\sqrt{(b-b')(b-b'')lU+(b'-b'')(b'-b)mV+(b''-b)(b''-b')nW} \\ &+\nu\sqrt{(c-c')(c-c'')lU+(c'-c'')(c'-c)mV+(c''-c)(c''-c')nW}=0 \end{aligned}$$

is consequently an equation involving three zomals of the proper form; and we can determine λ, μ, ν in suchwise as to identify this with the original equation $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}$, viz., writing successively $U=0$, $V=0$, $W=0$, we find

$$\begin{aligned} (a'-a'')\lambda+(b'-b'')\mu+(c'-c'')\nu&=0, \\ (a''-a\ )\lambda+(b''-b\ )\mu+(c''-c\ )\nu&=0, \\ (a\ -a')\lambda+(b\ -b')\mu+(c\ -c')\nu&=0, \end{aligned}$$

equations which are, as they should be, equivalent to two equations only, and which give

$$\lambda : \mu : \nu = \begin{vmatrix} 1, & 1, & 1 \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix} : \begin{vmatrix} 1, & 1, & 1 \\ c, & c', & c'' \\ a, & a', & a'' \end{vmatrix} : \begin{vmatrix} 1, & 1, & 1 \\ a, & a', & a'' \\ b, & b', & b'' \end{vmatrix},$$

and the equation, with these values of λ, μ, ν substituted therein, is in fact the equation of the trizomal curve $\sqrt{l}U + \sqrt{m}V + \sqrt{n}W = 0$ in terms of three new zomals. It is easy to return to the forms involving one new zomal and any two of the original three zomals.

Article No. 52. *Remark as to the Tetrazomal Curve.*

52. I return for a moment to the case of the tetrazomal curve, in order to show that there is not, in regard to it in general, any theorem such as that of the variable zomal. Considering the form $\sqrt{lx} + \sqrt{my} + \sqrt{nz} + \sqrt{pw} = 0$ (the coordinates x, y, z, w are of course connected by a linear equation, but nothing turns upon this), the curve is here a quartic touched twice by each of the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$ (viz., each of these is a double tangent of the curve), and having besides the three nodes $(x = y,\ z = w)$, $(x = z,\ y = w)$, $(x = w,\ y = z)$. But a quartic curve with three nodes, or trinodal quartic, has only four double tangents—that is, besides the lines $x = 0$, $y = 0$, $z = 0$, $w = 0$, there is no line $\alpha x + \beta y + \gamma z + \delta w = 0$ which is a double tangent of the curve; and writing U, V, W, T in place of x, y, z, w, then if U, V, W, T are connected by a linear equation (and, *à fortiori*, if they are not so connected), there is not any curve $\alpha U + \beta V + \gamma W + \delta T = 0$ which is related to the curve in the same way with the lines $U = 0$, $V = 0$, $W = 0$, $T = 0$; or say there is not (besides the curves $U = 0$, $V = 0$, $W = 0$, $T = 0$), any other zomal $\alpha U + \beta V + \gamma W + \delta T = 0$, of the tetrazomal curve. The proof does not show that for special forms of U, V, W, T there may not be zomals, not of the above form $\alpha U + \beta V + \gamma W + \delta T = 0$, but belonging to a separate system. An instance of this will be mentioned in the sequel.

Article Nos. 53 to 56. *The Theorem of the Variable Zomal of a Trizomal Curve resumed.*

53. I resume the foregoing theorem of the variable zomal of the trizomal curve $\sqrt{l}U + \sqrt{m}V + \sqrt{n}W = 0$. The variable zomal $T = 0$ is the curve $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$, where a, b, c are connected by the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$; that is, it belongs to a single series of curves selected in a certain manner out of the double series $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$ (a double series, as containing the *two* variable parameters a : b : c). These are the whole series of curves in involution with the given curves $U = 0$, $V = 0$, $W = 0$, or being such that the Jacobian of any three of them is identical with the Jacobian of the three given curves; in particular, the Jacobian of any one of the curves $\mathrm{a}U + \mathrm{b}V + \mathrm{c}W = 0$,

and of two of the three given curves, is identical with the Jacobian of the three given curves. I call to mind that, by the Jacobian of the curves $U=0$, $V=0$, $W=0$, is meant the curve

$$J(U,\ V,\ W)=\frac{d(U,\ V,\ W)}{d(x\ ,\ y\ ,\ z\ )}=\begin{vmatrix} d_xU, & d_yU, & d_zU \\ d_xV, & d_yV, & d_zV \\ d_xW, & d_yW, & d_zW \end{vmatrix}=0,$$

viz., the curve obtained by equating to zero the Jacobian or functional determinant of the functions U, V, W. Some properties of the Jacobian, which are material as to what follows, are mentioned in the Annex No. I.

For the complete statement of the theorem of the variable zomal, it would be necessary to interpret geometrically the condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, thereby showing how the single series of the variable zomal is selected out of the double series of the curves $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W=0$ in involution with the given curves. Such a geometrical interpretation of the condition may be sought for as follows, but it is only in a particular case, as afterwards mentioned, that a convenient geometrical interpretation is thereby obtained.

54. Consider the fixed line $\Omega=px+qy+rz=0$, and let it be proposed to find the locus of the $(r-1)^2$ poles of the line $\Omega=0$ in regard to the series of curves $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W=0$, where $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$. Take $(x,\ y,\ z)$ as the coordinates of any one of the poles in question, then in order that $(x,\ y,\ z)$ may belong to one of the $(r-1)^2$ poles of the line $\Omega=px+qy+rz=0$ in regard to the curve $\mathrm{a}U+\mathrm{b}V+\mathrm{c}W=0$, we must have

$$d_x(\mathrm{a}U+\mathrm{b}V+\mathrm{c}W)\ :\ d_y(\mathrm{a}U+\mathrm{b}V+\mathrm{c}W)\ :\ d_z(\mathrm{a}U+\mathrm{b}V+\mathrm{c}W)=p\ :\ q\ :\ r;$$

or, what is the same thing,

$$=d_x\Omega\ :\ d_y\Omega\ :\ d_z\Omega;$$

and these equations give without difficulty

$$\mathrm{a}\ :\ \mathrm{b}\ :\ \mathrm{c}=J(V,\ W,\ \Omega)\ :\ J(W,\ U,\ \Omega)\ :\ J(U,\ V,\ \Omega),$$

whence, substituting in the equation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, we have

$$\frac{l}{J(V,\ W,\ \Omega)}+\frac{m}{J(W,\ U,\ \Omega)}+\frac{n}{J(U,\ V,\ \Omega)}=0$$

as the locus of the $(r-1)^2$ poles in question. Each of the Jacobians is a function of the order $2r-2$, and the order of the locus is thus $=4r-4$. As the given curves $U=0$, $V=0$, $W=0$ belong to the single series of curves, it is clear that the locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the fixed line in regard to the curves $U=0$, $V=0$, $W=0$ respectively.

55. In the case where the given trizomal is

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\sqrt{n(\Theta+N\Phi)}=0,$$

$s=r-1$, that is, where the zomals $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$ are each of them curves of the order r, passing through the r intersections of the line $\Phi=0$ with the curve $\Theta=0$, then, taking this line $\Phi=0$ for the fixed line $\Omega=0$, we have

$$J(V,\ W,\ \Omega)=J(\Theta+M\Phi,\ \Theta+N\Phi,\ \Phi)=\Phi\,\{M,\ N\},$$

if, for shortness, $\{M,\ N\}=J(M-N,\ \Theta,\ \Phi)+\Phi\,J(M,\ N,\ \Phi)$, and the like as to the other two Jacobians, so that, attaching the analogous significations to $\{N,\ L\}$ and $\{L,\ M\}$, the equation of the locus is

$$\frac{l}{\{M,\ N\}}+\frac{m}{\{N,\ L\}}+\frac{n}{\{L,\ M\}}=0,$$

where observe that each of the curves $\{M,\ N\}=0$, $\{N,\ L\}=0$, $\{L,\ M\}=0$ is a curve of the order $2r-3$; the order of the locus is thus $=4r-6$, and (as before) this locus passes through the $3(r-1)^2$ points which are the $(r-1)^2$ poles of the line $\Phi=0$ in regard to the curves $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$ respectively.

56. In the case $r=2$, the trizomal is

$$\sqrt{l(\Theta+L\Phi)}+\sqrt{m(\Theta+M\Phi)}+\sqrt{n(\Theta+N\Phi)}=0,$$

where the zomals are the conics $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$, each passing through the same two points $\Theta=0$, $\Phi=0$; the locus of the pole of the line $\Phi=0$, in regard to the variable zomal, is the conic

$$\frac{l}{\{M,\ N\}}+\frac{m}{\{N,\ L\}}+\frac{n}{\{L,\ M\}}=0,$$

viz., $\{M,\ N\}=0$, $\{N,\ L\}=0$, $\{L,\ M\}=0$, are here the lines passing through the poles of the line $\Phi=0$ in regard to the second and third, the third and first, and the first and second of the given conics respectively: treating l, m, n as arbitrary, the locus is clearly *any* conic through the poles of the line $\Phi=0$ in regard to the three conics respectively. The Jacobian of the three given conics is a conic related in a special manner to the three given conics, and which might be called the Jacobian conic thereof, and it would be easy to give a complete enunciation of the theorem for the case in hand. (See as to this, Annex No. I, above referred to.) But if, in accordance with the plan adopted in the remainder of the memoir, we at once assume that the points $\Theta=0$, $\Phi=0$ are the circular points at infinity, then the theorem can be enunciated under a more simple form—viz., if $\mathfrak{A}^\circ=0$, $\mathfrak{B}^\circ=0$, $\mathfrak{C}^\circ=0$ are the equations of any three circles, then in the trizomal

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0,$$

the variable zomal is any circle whatever of the series of circles cutting at right angles the orthotomic circle of the three given circles, and having its centre on a certain conic which passes through the centres of the given circles. Moreover, if the

coefficients l, m, n are not given in the first instance, but are regarded as arbitrary, then the last-mentioned conic is any conic whatever through the three centres, and there belongs to such conic and the series of zomals derived therefrom as above, a trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0$. This is obviously the theorem, that if a variable circle has its centre on a given conic, and cuts at right angles a given circle, then the envelope of the variable circle is a trizomal curve $\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}$, where $\mathfrak{A}^\circ=0$, $\mathfrak{B}^\circ=0$, $\mathfrak{C}^\circ=0$ are any three circles, positions of the variable circle, and l, m, n are constant quantities depending on the selected three circles.

Part II. (Nos. 57 to 104). Subsidiary Investigations.

Article Nos. 57 and 58. *Preliminary Remarks.*

57. We have just been led to consider the conics which pass through two given points. There is no real loss of generality in taking these to be the circular points at infinity, or say the points I, J—viz., every theorem which in anywise explicitly or implicitly relates to these two points, may, without the necessity of any change in the statement thereof, be understood as a theorem relating instead to any two points P, Q. I call to mind that a circle is a conic passing through the two points I, J, and that lines at right angles to each other are lines harmonically related to the pair of lines from their intersection to the points I, J respectively, so that when (I, J) are replaced by any two given points whatever, the expression a circle must be understood to mean a conic passing through the two given points; and in speaking of lines at right angles to each other, it must be understood that we mean lines harmonically related to the pair of lines from their intersection to the two given points respectively. For instance, the theorem that the Jacobian of any three circles is their orthotomic circle, will mean that the Jacobian of any three conics which each of them passes through the two given points is the orthotomic conic through the same two points, that is, the conic such that at each of its intersections with any one of the three conics, the two tangents are harmonically related to the pair of lines from this intersection to the two given points respectively. Such extended interpretation of any theorem is applicable even to the theorems which involve distances or angles—viz., the terms "distance" and "angle" have a determinate signification when interpreted in reference (not to the circular points at infinity, but instead thereof) to any two given points whatever (see as to this my "Sixth Memoir on Quantics," Nos. 220, *et seq.*). *Phil. Trans.*, vol. CXLIX. (1859), pp. 61—90; see p. 86; [158]. And this being so, the theorem can, without change in the statement thereof, be understood as referring to the two given points.

58. I say then that any theorem (referring explicitly or implicitly) to the circular points at infinity I, J, may be understood as a theorem referring instead to any two given points. We might of course give the theorems in the first instance in terms explicitly referring to the two given points—(viz., instead of a circle, speak of a conic through the two given points, and so in other instances); but, as just explained, this is not really more general, and the theorems would be given in a less concise and

familiar form. It would not, on the face of the investigations, be apparent that in treating of the polyzomal curves

$$\sqrt{l(\Theta + L\Phi)} + \sqrt{m(\Theta + M\Phi)} + \&c. = 0,$$

($\Theta = 0$ a conic, $\Phi = 0$ a line, as above), that we were really treating of the curves the zomals whereof are circles, and therein of the theories of foci and focofoci as about to be explained. And for these reasons I shall consider the two points $\Theta = 0$, $\Phi = 0$, to be the circular points at infinity I, J, and in the investigations, &c., make use of the terms circle, right angles, &c., which, in their ordinary significations, have implicit reference to these two points.

The present Part does not explicitly relate to the theory of polyzomal curves, but contains a series of researches, partly analytical and partly geometrical, which will be made use of in the following Parts III. and IV. of the Memoir.

Article Nos. 59 to 62. *The Circular Points at Infinity; Rectangular and Circular Coordinates.*

59. The coordinates made use of (except in the cases where the general trilinear coordinates (x, y, z), or any other coordinates, are explicitly referred to), will be either the ordinary rectangular coordinates x, y, or else, as we may term them, the circular coordinates ξ, η $(= x + iy,\ x - iy$ respectively, $i = \sqrt{-1}$ as usual), but in either case I shall introduce for homogeneity the coordinate z, it being understood that this coordinate is in fact $= 1$, and that it may be retained or replaced by this its value, in different investigations or stages of the same investigation, as may for the time being be most convenient. In more concise terms, we may say that the coordinates are either the rectangular coordinates x, y, and z $(= 1)$, or else the circular coordinates ξ, η, and z $(= 1)$. The equation of the line infinity is $z = 0$; the points I, J are given by the equations $(x + iy = 0,\ z = 0)$ and $(x - iy = 0,\ z = 0)$, or, what is the same thing, by the equations $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$ respectively; or in the rectangular coordinates the coordinates of these points are $(-i, 1, 0)$ and $(i, 1, 0)$ respectively, and in the circular coordinates they are $(1, 0, 0)$ and $(0, 1, 0)$ respectively. It is, of course, only for points at infinity that the coordinate z is $= 0$ (and observe that for any such point the x and y or ξ and η coordinates may be regarded as finite); for every point whatever not at infinity the coordinate z is, as stated above, $= 1$.

60. Consider a point A, whose coordinates (rectangular) are $(a, a', 1)$ and (circular) $(\alpha, \alpha', 1)$, viz., $\alpha = a + a'i$, $\alpha' = a - a'i$; then the equations of the lines through A to the points I, J, are

$$x - az + i(y - a'z) = 0, \quad x - az - i(y - a'z) = 0$$

respectively, or they are

$$\xi - \alpha z = 0 \qquad , \quad \eta - \alpha' z = 0$$

respectively. These equations, if (a, a') or (α, α') are arbitrary, will, it is clear, be the equations of any two lines through the points I, J, respectively.

61. We have from either of the equations in (x, y, z)

$$(x-az)^2+(y-a'z)^2=0,$$

that is, the distance from each other of any two points $(x, y, 1)$, and $(a, a', 1)$ in a line through I or J is $=0$. And in particular, if $z=0$, then $x^2+y^2=0$; that is, the distance of the point $(a, a', 1)$ from I or J is in each case $=0$.

62. Consider for a moment any three points P, Q, A; the perpendicular distance of P from QA is $=2$ triangle $PQA \div$ distance QA; if Q be any point on the line through A to either of the points I, J, and in particular if Q be either of the points I, J, then the triangle PQA is finite, but the distance QA is $=0$: that is, the perpendicular distance of P from the line through A to either of the points I, J, that is, from any line through either of these points, is $=\infty$. But, as just stated, the triangle PQA is finite, or say the triangles PIA, PJA are each finite; viz., the coordinates (rectangular) of P, A being $(x, y, z=1)$, $(a, a', 1)$ or (circular) $(\xi, \eta, z=1)$, $(\alpha, \alpha', 1)$, the expressions for the doubles of these triangles respectively are

$$\begin{vmatrix} x, & y, & z \\ -i, & 1, & 0 \\ a, & a', & 1 \end{vmatrix} \quad , \quad \begin{vmatrix} x, & y, & z \\ i, & 1, & 0 \\ a, & a', & 1 \end{vmatrix}$$

that is, they are (rectangular coordinates) $x-az+i(y-a'z)$, $x-az-i(y-a'z)$, or (circular coordinates) $\xi-\alpha z$, $\eta-\alpha' z$.

Representing the double areas by PIA, PJA, respectively, and the squared distance of the points A, P, by $\mathfrak{A}$, we have—

$$\begin{aligned}\mathfrak{A} &= (x-az)^2+(y-a'z)^2 \\ &= (\xi-\alpha z)(\eta-\alpha' z), = PIA \,.\, PJA.\end{aligned}$$

Article No. 63. *Antipoints; Definition and Fundamental Properties.*

63. Two pairs of points (A, B) and (A_1, B_1) which are such that the lines AB, A_1B_1 bisect each other at right angles in a point O in such wise that $OA=OB=i\,OA_1=i\,OB_1$, are said to be antipoints, each of the other. In rectangular coordinates, taking the coordinates of $(AB,)$ to be $(a, 0, 1)$ and $(-a, 0, 1)$, those of (A_1, B_1) will be $(0, ai, 1)$ and $(0, -ai, 1)$ respectively, whence joining the points (A, B) with the points (I, J), the points A_1, B_1 are given as the intersections of the lines AI and BJ, and of the lines AJ and BI respectively. Or, what is the same thing, in any quadrilateral wherein I, J are opposite angles, the remaining pairs (A, B) and (A_1, B_1) are antipoints each of the other.

64. In circular coordinates, if the coordinates of A are $(\alpha, \alpha', 1)$, and those of B are $(\beta, \beta', 1)$, then the equations of

$$\begin{aligned} &AI,\ AJ \text{ are } \xi-\alpha z=0, \quad \eta-\alpha' z=0, \\ &BI,\ BJ \quad \text{,,} \quad \xi-\beta z=0, \quad \eta-\beta' z=0, \end{aligned}$$

whence the equations of

$$A_1I,\ A_1J \text{ are } \xi - \alpha z = 0,\quad \eta - \beta' z = 0,$$
$$B_1I,\ B_1J \ \text{ ,, } \ \xi - \beta z = 0,\quad \eta - \alpha' z = 0.$$

65. Considering any point P the coordinates of which are ξ, η, z $(=1)$, let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$, $\mathfrak{B}_1$ be its squared distances from the points A, B, A_1, B_1 respectively; then by what precedes

$$\mathfrak{A} = (\xi - \alpha z)(\eta - \alpha' z),$$
$$\mathfrak{B} = (\xi - \beta z)(\eta - \beta' z),$$
$$\mathfrak{A}_1 = (\xi - \alpha z)(\eta - \beta' z),$$
$$\mathfrak{B}_1 = (\xi - \beta z)(\eta - \alpha' z),$$

and thence

$$\mathfrak{A}.\mathfrak{B} = \mathfrak{A}_1.\mathfrak{B}_1;$$

that is, the product of the squared distances of a point P from any two points A, B, is equal to the product of the squared distances of the same point P from the two antipoints A_1, B_1. This theorem, which was, I believe, first given by me in the *Educational Times* (see reprint, vol. VI. 1866, p. 81), is an important one in the theory of foci. It is to be further noticed that we have

$$\mathfrak{A} + \mathfrak{B} - \mathfrak{A}_1 - \mathfrak{B}_1 = (\alpha - \beta)(\alpha' - \beta')z^2,\ = Kz^2 = K,$$

if $K, = (\alpha - \alpha')(\beta - \beta')$, be the squared distance of the points A, B, $= -$squared distance of points A_1, B_1.

Article No. 66. *Antipoints of a Circle.*

66. A similar notion to that of two pairs of antipoints is as follows, viz., if from the centre of a circle perpendicular to its plane and in opposite senses, we measure off two distances each $= i$ into the radius, the extremities of these distances are antipoints of the circle. It is clear that the antipoints of the circle and the extremities of any diameter thereof are (in the plane of these four points) pairs of antipoints. It is to be added that each antipoint is the centre of a sphere radius zero, or say of a cone sphere, passing through the circle: the circle is thus the intersection of the two cone spheres having their centres at the two antipoints respectively.

Article No. 67. *Antipoints in relation to a Pair of Orthotomic Circles.*

67. It is a well-known property that if any circle pass through the points (A, B), and any other circle through the antipoints (A_1, B_1), then these two circles cut at right angles. Conversely if a circle pass through the points A, B, then all the orthotomic circles which have their centres on the line AB pass through the antipoints A_1, B_1. In particular, if on AB as diameter we describe a circle and on A_1B_1 as diameter a circle, then these two circles—being, it is clear, concentric circles with their radii in the ratio $1 : i$, and as concentric circles touching each other at the points (I, J)—cut each other at right angles; or say they are concentric orthotomic circles.

Article Nos. 68 to 71. *Forms of the Equation of a Circle.*

68. In rectangular coordinates the equation of a circle, coordinates of centre $(a, a', 1)$ and radius $=a''$, is

$$\mathfrak{A}^\circ = (x-az)^2 + (y-a'z)^2 - a''^2z^2 = 0\,;$$

and in circular coordinates, the coordinates of the centre being $(\alpha, \alpha', 1)$, and radius $=a''$ as before, the equation is

$$\mathfrak{A}^\circ = (\xi - \alpha z) \quad (\eta - \alpha' z) - a''^2 z^2 = 0.$$

69. I observe in passing, that the origin being at the centre and the radius being $=1$, then writing also $z=1$, the equation of the circle is $\xi\eta = 1$, that is the circular coordinates of any point of the circle, expressed by means of a variable parameter θ, are $\left(\theta, \frac{1}{\theta}, 1\right)$.

70. Consider a current point P, the coordinates of which (rectangular) are $x, y, z(=1)$, and (circular) are $\xi, \eta, z(=1)$, then the foregoing expression

$$\begin{aligned}\mathfrak{A}^\circ &= (x-az)^2 + (y-a'z)^2 - a''^2z^2 \\ &= (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2\end{aligned}$$

denotes, it is clear, the square of the tangential distance of the point P from the circle $\mathfrak{A}^\circ = 0$.

71. But there is another interpretation of this same function $\mathfrak{A}^\circ$, viz., writing therein $z=1$, and then

$$\mathfrak{A}^\circ = (x-a)^2 + (y-a')^2 + (a''i)^2,$$

we see that $\mathfrak{A}^\circ$ is the squared distance of P from either of the antipoints of the circle (points lying, it will be recollected, out of the plane of the circle), and we have thus the theorem that the square of the tangential distance of any point P from the circle is equal to the square of its distance from either antipoint of the circle.

Article Nos. 72 to 77. *On a System of Sixteen Points.*

72. Take (A, B, C, D) any four concyclic points, and let the antipoints of

$$\begin{array}{llll} (B, C), & (A, D) & \text{be} & (B_1, C_1), \ (A_1, D_1), \\ (C, A), & (B, D) & \text{,,} & (C_2, A_2), \ (B_2, D_2), \\ (A, B), & (C, D) & \text{,,} & (A_3, B_3), \ (C_3, D_3), \end{array}$$

then each of the three new sets (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) will be a set of four concyclic points.

73. Let O be the centre of the circle through (A, B, C, D), say of the circle O, and then, the lines BC, AD meeting in R, the lines CA, BD in S, and the lines AD, CD in T, let each of these points be made the centre of a circle orthotomic to O, viz., let these new circles be called the circles R, S, T respectively.

As regards the circle R, since its centre lies in BC, the circle passes through (B_1, C_1); and since the centre lies in AD, the circle passes through (A_1, D_1), that is, the four points (A_1, B_1, C_1, D_1) lie in the circle R. Similarly (A_2, B_2, C_2, D_2) lie in the circle S, and (A_3, B_3, C_3, D_3) in the circle T.

74. The points R, S, T are conjugate points in relation to the circle O; that is, ST, TR, RS are the polars of R, S, T respectively in regard to this circle; and they are, consequently, at right angles to the lines OR, OS, OT respectively; viz., the four centres O, R, S, T are such that the line joining any two of them cuts at right angles the line joining the other two of them, and we see that the relation between the four sets is in fact a symmetrical one; this is most easily seen by consideration of the circular points at infinity I, J, the four sets of points may be arranged thus:

$$\begin{array}{llll} A\,, & A_3, & A_2, & A_1, \\ B_3, & B\,, & B_1, & B_2, \\ C_2, & C_1\,, & C\,, & C_3\,, \\ D_1, & D_2, & D_3, & D\,, \end{array}$$

in such wise that any four of them in the same vertical line pass through I, and any four in the same horizontal line pass through J; and this being so, starting for instance with (A_3, B_3, C_3, D_3) we have antipoints

$$\begin{array}{llll} \text{of } (B_3\,, C_3\,), & (A_3, D_3) & \text{are} & (B_2, C_2\,), \quad (A_2, D_2), \\ \text{,, } (C_3\,, A_3), & (B_3, D_3) & \text{,,} & (C_1, A_1), \quad (B_1, D_1), \\ \text{,, } (A_3, B_3), & (C_3, D_3) & \text{,,} & (A\,, B\,), \quad (C\,, D\,), \end{array}$$

and similarly if we start from (A_1, B_1, C_1, D_1) or (A_2, B_2, C_2, D_2).

75. I return for a moment to the construction of (A_1, B_1, C_1, D_1); these are points on the circle R, and (B_1, C_1) are the antipoints of (B, C); that is, they are the intersections of the circle R by the line at right angles to BC from its middle point, or, what is the same thing, by the perpendicular on BC from O. Similarly (A_1, D_1) are the antipoints of (A, D); that is, they are the intersections of the circle R by the perpendicular on AD from O. And the like as to (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively.

76. Hence, starting with the points A, B, C, D on the circle O, and constructing as above the circles P, Q, R, and constructing also the perpendiculars from O on the six chords AB, AC, &c.,

$$\begin{array}{llll} \text{the perpendiculars on } BC, AD & \text{meet circle } R & \text{in } (B_1, C_1), & (A_1, D_1), \\ \text{,,} \quad CA, BD & \text{,,} \quad \text{,,} \; S & \text{,,} \; (C_2, A_2), & (B_2, D_2), \\ \text{,,} \quad AB, CD & \text{,,} \quad \text{,,} \; T & \text{,,} \; (A_3, B_3), & (C_3, D_3), \end{array}$$

so that the whole system is given by means of the circles P, Q, R, and the six perpendiculars.

77. If to fix the ideas (A, B, C, D) are real points taken in order on the real circle O, then the points R, S, T are each of them real; but R and T lie outside, S inside the circle O. The circles R and T are consequently real, but the circle S imaginary, viz., its radius is $=i$ into a real quantity; the imaginary points (A_1, B_1, C_1, D_1) are thus given as the intersections of a real circle by a pair of real lines, and the like as to the imaginary points (A_3, B_3, C_3, D_3); but the imaginary points (A_2, B_2, C_2, D_2) are only given as the intersections of an imaginary circle (centre real and radius a pure imaginary) by a pair of real lines. The points (C_2, A_2) *quâ* antipoints of (C, A) are easily constructed as the intersections of a real circle by a real line, and the like as to the points (B_2, D_2) *quâ* antipoints of (B, D), but the construction for the two pairs of points cannot be effected by means of the same real circle.

Article Nos. 78 to 80. *Property in regard to Four Confocal Conics.*

78. All the conics which pass through the four concyclic points A, B, C, D, have their axes in fixed directions; but three such conics are the line-pairs (BC, AD), (CA, BD), and (AB, CD), whence the directions of the axes are those of the bisectors of the angles formed by any one of these pairs of lines; hence, in particular, considering either axis of a conic through the four points, the lines AB and CD are equally inclined on opposite sides to this axis, and this leads to the theorem that the antipoints $(A_3, B_3)(C_3, D_3)$ are in a conic confocal to the given conic through (A, B, C, D); whence, also, considering any given conic whatever through (A, B, C, D), the points (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3) lie severally in three conics, each of them confocal with the given conic.

79. To prove this, consider any two confocal conics, say an ellipse and a hyperbola, and let F be one of their four intersections; join F with the common centre O, and let OT, ON be parallel to the tangent and normal respectively of the ellipse at the point F. OF, OT are in direction conjugate axes of the ellipse, and OF, ON are in direction conjugate axes of the hyperbola; and if they are also the axes in magnitude, that is, if the points T, N are the intersections of OT with the ellipse and of ON with the hyperbola respectively, then it is easy to show that $\overline{OT}^2 + \overline{ON}^2 = 0$. And this being so, imagine on the ellipse any two points A, B such that the chord AB is parallel to OT, that is conjugate to OF; AB is bisected by OF, say in a point K, or we have parallel to OT the semichords or ordinates $KA = KB$; and we may, perpendicularly to this or parallel to ON, draw through K in the hyperbola a chord A_3B_3, which chord will be bisected in K, or we shall have $KA_3 = KB_3$. Hence KA, KA_3 are in the ellipse and the hyperbola respectively ordinates conjugate to the same diameter OF, and the semi-diameters conjugate to OF being OT, ON respectively, we have $KA^2 (= KB^2) : \overline{KA_3}^2 (= KB_3^2) = \overline{OT}^2 : \overline{ON}^2$, this is, $\overline{KA}^2 = \overline{KB}^2 = -\overline{KA_3}^2 = -\overline{KB_3}^2$; or (A_3, B_3) will be the antipoints of (A, B).

80. Conversely, if in the ellipse we have the two points (A, B), then drawing the diameter OF conjugate to AB, and through its extremity F, the confocal hyperbola, then the antipoints (A_3, B_3) will lie on the hyperbola. And similarly, if on the

ellipse we have the two points (C, D), then drawing the diameter OG conjugate to CD, and through its extremity G a confocal hyperbola, the antipoints (C_3, D_3) will lie on the hyperbola. Suppose (A, B, C, D) are concyclic, then, as noticed, AB and CD will be equally inclined on opposite sides to the transverse axis of the ellipse—the conjugate diameters OF, OG will therefore be equally inclined on opposite sides of the transverse axis—and the points F and G will therefore be situate symmetrically on opposite sides of the transverse axis, that is, the points F and G will respectively determine the same confocal hyperbola, and we have thus the required theorem, viz., if (A, B, C, D) are any four concyclic points on an ellipse, or say on a conic, and if (A_3, B_3) are the antipoints of (A, B), and (C_3, D_3) the antipoints of (C, D), then (A_3, B_3, C_3, D_3) will lie on a conic confocal with the given conic.

Article Nos. 81 to 85. *System of the Sixteen Points, the Axial Case.*

81. The theorems hold good when the four points A, B, C, D are in a line; the antipoints (B_1, C_1) of (B, C), &c., are in this case situate symmetrically on opposite sides of the line, so that it is evident at sight that we have (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), each set in a circle; and that the centres R, S, T of these circles lie in the line. The construction for the general case becomes, however, indeterminate, and must therefore be varied. If in the general case we take any circle through (B, C), and any circle through (A, D), then the circle R cuts at right angles these two circles, and has, consequently, its centre R in the radical axis of the two circles; whence, when the four points are in a line, taking any circle through (B, C), or in particular the circle on BC as diameter, and any circle through (A, D), or in particular the circle on AD as diameter,—the radical axis of these two circles intersects the line in the required centre R, and the circle R is the circle with this centre cutting at right angles the two circles respectively; the circles S and T are, of course, obtained by the like construction in regard to the combinations $(C, A; B, D)$ and $(A, B; C, D)$, respectively. It may be added, that we have

$$\left.\begin{matrix} R \\ S \\ T \end{matrix}\right\} \text{ centre and } \left\{\begin{matrix} \text{extremities } R \\ \text{of diameter } S \\ \text{of circles } T \end{matrix}\right\} \text{ sibiconjugate points of involutions } \left\{\begin{matrix} B, C;\ A, D, \\ C, A;\ B, D, \\ A, B;\ C, D, \end{matrix}\right.$$

and that (as in the general case) the circles R, S, T intersect each pair of them at right angles; and they are evidently each intersected at right angles by the line $ABCD$ (or axis of the figure), which replaces the circle O in the general case.

82. If the points A, B, C, D are taken in order on the line, then the points R, S, T are all real, viz., the point R is situate, on one side or the other, outside AD, but the points S and T are each of them situate between B and C; the circles R and T are real, but the circle S has its radius a pure imaginary quantity.

83. If one of the four points, suppose D, is at infinity on the line, then the antipoints of (A, D), of (B, D), and of (C, D) are each of them the two points (I, J).

It would at first sight appear that the only conditions for the circles R, S, T were the conditions of passing through the antipoints of (B, C), of (C, A), and of (A, B) respectively, and that these circles thus became indeterminate; but in fact the definition of the circles is then as follows, viz., R has its centre at A, and passes through the antipoints of (B, C): (whence squared radius $= AB \,.\, AC$). And similarly, S has its centre at B, and passes through antipoints of (C, A), (squared radius $= BA \,.\, BC$); and T has its centre at C, and passes through antipoints of (A, B). (squared radius $= CA \,.\, CB$); these three circles cut each other at right angles. As before, A, B, C being in order on the line, the circles R, T are real, but the circle S has its radius a pure imaginary quantity.

84. That the circles are as just mentioned appears as follows: taking the line as axis of x, and a, b, c, d for the x coordinates of the four points respectively, then the coordinates of A_1, D_1 are

$$\tfrac{1}{2}(a+d),\ \pm\tfrac{1}{2}i(a-d);$$

whence, m being arbitrary, the general equation of a circle through A_1, D_1 is

$$x^2+y^2-2mxz+[m(a+d)-ad]\,z^2=0,$$

writing herein

$$m=a-\frac{k^2}{d}$$

this becomes

$$x^2+y^2-2\left(a-\frac{k^2}{d}\right)xz+\left(a^2-k^2-\frac{ak^2}{d}\right)z^2=0,$$

viz., for $d=\infty$ it is

$$(x-az)^2+y^2-k^2z^2=0,$$

which is a circle having A for its centre, and its radius an arbitrary quantity k. If the circle passes through the antipoints of B, C, the coordinates of these are

$$\tfrac{1}{2}(b+c),\ \pm\tfrac{1}{2}i(b-c),$$

and we find

$$k^2=[\tfrac{1}{2}(b+c)-a]^2-\tfrac{1}{4}(b-c)^2=(a-b)(a-c).$$

85. Reverting to the general case of four points A, B, C, D on a line, the theorem as to the confocal conics holds good under the form that, drawing any conic whatever through (A_1, B_1, C_1, D_1), the points (A_2, B_2, C_2, D_2), and (A_3, B_3, C_3, D_3) lie in confocal conics, these conics have their centre on the line, and axes in the direction of and perpendicular to the line. When D is at infinity, the confocal conics become any three concentric circles through (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively.

Article Nos. 86 to 91. *The Involution of Four Circles.*

86. Consider any four points A, B, C, D, the centres of circles denoted by these same letters, and let $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$ signify as usual, viz., if (in orthogonal coordinates) $(a, a', 1)$ are the coordinates of the centre, and a'' the radius of the circle A, then $\mathfrak{A}^\circ$ stands for $(x-az)^2+(y-a'z)^2-a''^2z^2$, and the like for $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$. Write also

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = BCD : -CDA : DAB : -ABC,$$

where BCD, &c., are the triangles formed by the points (B, C, D), &c.; the analytical expressions are

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \begin{vmatrix} b, & b', & 1 \\ c, & c', & 1 \\ d, & d', & 1 \end{vmatrix} : - \begin{vmatrix} c, & c', & 1 \\ d, & d', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} d, & d', & 1 \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix} : - \begin{vmatrix} a, & a', & 1 \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix}$$

so that

$$\begin{aligned} \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} &= 0, \\ \mathrm{a}a + \mathrm{b}b + \mathrm{c}c + \mathrm{d}d &= 0, \\ \mathrm{a}a' + \mathrm{b}b' + \mathrm{c}c' + \mathrm{d}d' &= 0; \end{aligned}$$

this being so, it is clear that we have

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ =$$

$$z^2 [\mathrm{a}(a^2 + a'^2 - a''^2) + \mathrm{b}(b^2 + b'^2 - b''^2) + \mathrm{c}(c^2 + c'^2 - c''^2) + \mathrm{d}(d^2 + d'^2 - d''^2)] = Kz^2, = K,$$

a constant.

87. I am not aware that in the general case there is any convenient expression for this constant K; it is $=0$ when the four circles have the same orthotomic circle; in fact, taking as origin the centre of the orthotomic circle, and its radius to be $=1$, we have

$$a^2 + a'^2 - a''^2 = 1, \text{ \&c.},$$

whence

$$K = \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d} = 0;$$

that is, if the circles A, B, C, D have the same orthotomic circle, then $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$, a, b, c, d, signifying as above, we have

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

and, in particular, if the circles reduce themselves to the points A, B, C, D respectively, then (writing as usual $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ in place of $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$) if the four points A, B, C, D are on a circle, we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = 0.$$

88. This last theorem may be regarded as a particular case of the theorem

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = Kz^2 = K,$$

viz., the four circles reducing themselves to the points A, B, C, D, we can find for the constant K an expression which will of course vanish when the points are on a circle. For this purpose, let the lines BC, AD meet in R, the lines CA, BD in S, and the lines AB, CD in T; we may, to fix the ideas, consider $ABCD$ as forming a convex quadrilateral, R and T will then be the exterior centres, S the interior centre; a, b, c, d, may be taken *equal* to BCD, $-CDA$, DAB, $-ABC$, where the areas BCD, &c., are each taken positively. The expression $\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D}$ has the same value, whatever is the position of the point P $(x, y, z=1)$; taking this point at R, and writing for a moment

$$RA = \alpha, \quad RB = \beta, \quad RC = \gamma, \quad RD = \delta,$$

then

$$BCD = (RCD - RBD) = \tfrac{1}{2} RD\,(RC - RB) \sin R = (\gamma - \beta)\,\delta \sin R,$$

with similar expressions for the other triangles; and we thus have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = \tfrac{1}{2} z^2 . \sin R \left\{ \begin{matrix} \alpha^3 (\gamma - \beta)\,\delta \\ -\beta^3 (\delta - \alpha)\,\gamma \\ +\gamma^3 (\delta - \alpha)\,\beta \\ -\delta^3 (\gamma - \beta)\,\alpha \end{matrix} \right\} = \tfrac{1}{2} z^2 \sin R\,(\beta\gamma - \alpha\delta)\,(\gamma - \beta)\,(\delta - \alpha),$$

that is, replacing α, β, γ, δ, by their values, and writing also $z = 1$, we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = \tfrac{1}{2} \sin R\,.\,(RB\,.\,RC - RA\,.\,RD)\,BC\,.\,AD,$$

where $\frac{1}{2} \sin R\,.\,BC\,.\,AD$ is in fact the area of the quadrilateral $ABCD$; we have thus

$$\begin{aligned} \mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} &= (RB\,.\,RC - RA\,.\,RD)\,\square \\ &= (SC\,.\,SA - SB\,.\,SD)\,\square \\ &= (TA\,.\,TB - TC\,.\,TD)\,\square \end{aligned}$$

where it is to be observed that SA, SC being measured in opposite directions from S, must be considered, one as positive, the other as negative, and the like as regards SB, SD. This expression for the value of the constant is due to Mr Crofton. In the particular case where A, B, C, D, are on a circle, we have as before

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = 0.$$

89. If the four points A, B, C, D, are on a circle, then, taking as origin the centre of this circle and its radius as unity, the circular coordinates of the four points will be

$$\left(\alpha, \frac{1}{\alpha}, 1\right), \quad \left(\beta, \frac{1}{\beta}, 1\right), \quad \left(\gamma, \frac{1}{\gamma}, 1\right), \quad \left(\delta, \frac{1}{\delta}, 1\right),$$

the corresponding forms of $\mathfrak{A}^\circ$, &c., being

$$\mathfrak{A}^\circ = (\xi - \alpha z)\left(\eta - \frac{1}{\alpha} z\right) - \alpha'^{1/2} z^2, \text{ \&c.}$$

the expressions for a, b, c, d, observing that we have

$$\begin{vmatrix} \beta, & \beta^{-1}, & 1 \\ \gamma, & \gamma^{-1}, & 1 \\ \delta, & \delta^{-1}, & 1 \end{vmatrix} = \frac{1}{\beta\gamma\delta} \begin{vmatrix} 1, & \beta, & \beta^2 \\ 1, & \gamma, & \gamma^2 \\ 1, & \delta, & \delta^2 \end{vmatrix} = \frac{1}{\beta\gamma\delta}\,(\beta\gamma\delta), \text{ \&c.}$$

if $(\beta\gamma\delta)$, &c. denote $(\beta - \gamma)(\gamma - \delta)(\delta - \beta)$, &c., become

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \alpha(\beta\gamma\delta) : -\beta(\gamma\delta\alpha) : \gamma(\delta\alpha\beta) : -\delta(\alpha\beta\gamma),$$

which are convenient formulæ for the case in question.

90. If the points A, B, C, D, are on a line, then taking this line for the axis of x, we may write $\mathfrak{A}^\circ = (x - az)^2 + y^2 - a''^2 z^2$, &c. It is to be remarked here that we can, without any relation whatever between the radii of the circles, satisfy the equation

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0;$$

in fact this will be the case if we have

$$\begin{array}{llll} \mathrm{a} & +\mathrm{b} & +\mathrm{c} & +\mathrm{d} & = 0, \\ \mathrm{a}a & +\mathrm{b}b & +\mathrm{c}c & +\mathrm{d}d & = 0, \\ \mathrm{a}(a^2 - a''^2) & +\mathrm{b}(b^2 - b''^2) & +\mathrm{c}(c^2 - c''^2) & +\mathrm{d}(d^2 - d''^2) & = 0, \end{array}$$

equations which determine the ratios $\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}$. In the case where the circles reduce themselves to the points A, B, C, D, these equations become

$$\begin{array}{llll} \mathrm{a} & +\mathrm{b} & +\mathrm{c} & +\mathrm{d} & = 0, \\ \mathrm{a}a & +\mathrm{b}b & +\mathrm{c}c & +\mathrm{d}d & = 0, \\ \mathrm{a}a^2 & +\mathrm{b}b^2 & +\mathrm{c}c^2 & +\mathrm{d}d^2 & = 0, \end{array}$$

giving

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = (bcd) : -(cda) : (dab) : -(abc);$$

if for shortness (bcd), &c. stand for $(b-c)(c-d)(d-b)$, &c.; and for these values, we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = 0.$$

91. A very noticeable case is when the four circles are such that the foregoing values of (a, b, c, d) also satisfy the equation

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0;$$

the condition for this is obviously

$$\mathrm{a}a''^2 + \mathrm{b}b''^2 + \mathrm{c}c''^2 + \mathrm{d}d''^2 = 0;$$

or, as it may also be written,

$$\frac{a''^2}{(a-b)(a-c)(a-d)} + \frac{b''^2}{(b-c)(b-d)(b-a)} + \frac{c''^2}{(c-d)(c-a)(c-b)} + \frac{d''^2}{(d-a)(d-b)(d-c)} = 0.$$

Article No. 92. *On a Locus connected with the foregoing Properties.*

92. If, as above, A, B, C, D are any four points, and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are the squared distances of a current point P from the four points respectively, then the locus of the foci of the conics which pass through the four points is the tetrazomal curve

$$\mathrm{a}\sqrt{\mathfrak{A}} + \mathrm{b}\sqrt{\mathfrak{B}} + \mathrm{c}\sqrt{\mathfrak{C}} + \mathrm{d}\sqrt{\mathfrak{D}} = 0.$$

In fact the sum $\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D}$ has, it has been seen, a constant value for all positions of the point P; taking P to be the other focus, its squared distances are $(k - \sqrt{A})^2$, &c., whence for the first-mentioned focus we have

$$\mathrm{a}\mathfrak{A} + \mathrm{b}\mathfrak{B} + \mathrm{c}\mathfrak{C} + \mathrm{d}\mathfrak{D} = \mathrm{a}(k - \sqrt{\mathfrak{A}})^2 + \mathrm{b}(k - \sqrt{\mathfrak{B}})^2 + \mathrm{c}(k - \sqrt{\mathfrak{C}})^2 + \mathrm{d}(k - \sqrt{\mathfrak{D}})^2;$$

or recollecting that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, it follows that we have for the locus in question $\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0$; this locus will be discussed in the sequel. I remark here, that in the case where the four points are on a circle, then (as mentioned above), the axes of the several conics are in the same fixed directions; there are thus two sets of foci, those on the axis in one direction, and those on the axis in the other direction; it might therefore be anticipated, and it will appear, that in this case the tetrazomal breaks up into two trizomal curves.

Article Nos. 93 to 98. *Formulæ as to the two Sets* (A, B, C, D), *and* (A_1, B_1, C_1, D_1), *each of four Concyclic Points.*

93. Consider the four points A, B, C, D on a circle, then taking, as before, their circular coordinates to be $(\alpha, \alpha', 1)$, $(\beta, \beta', 1)$, $(\gamma, \gamma', 1)$, $(\delta, \delta', 1)$, the condition that the points may be on a circle is

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix} = 0,$$

viz., this equation may be written

$$\begin{aligned} &(\beta-\gamma)(\alpha-\delta) : (\gamma-\alpha)(\beta-\delta) : (\alpha-\beta)(\gamma-\delta) \\ =&(\beta'-\gamma')(\alpha'-\delta') : (\gamma'-\alpha')(\beta'-\delta') : (\alpha'-\beta')(\gamma'-\delta'); \end{aligned}$$

or if, for shortness, we take

$$\begin{aligned} a&=\beta-\gamma, & f&=\alpha-\delta, & a'&=\beta'-\gamma', & f'&=\alpha'-\delta', \\ b&=\gamma-\alpha, & g&=\beta-\delta, & b'&=\gamma'-\alpha', & g'&=\beta'-\delta', \\ c&=\alpha-\beta, & h&=\gamma-\delta, & c'&=\alpha'-\beta', & h'&=\gamma'-\delta', \end{aligned}$$

and consequently

$$\begin{aligned} &af+bg+ch=0, & &a'f'+b'g'+c'h'=0, \\ &a &&=g-h, & &a' &&=g'-h', \\ &b &&=h-f, & &b' &&=h'-f', \\ &c &&=f-g, & &c' &&=f'-g', \\ &a+b+c &&=0, & &a'+b'+c' &&=0, \end{aligned}$$

then the equation is

$$af : bg : ch = a'f' : b'g' : c'h'.$$

94. Let a, b, c, d, denote as before $(\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = BCD : -CDA : DAB : -ABC)$, then we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = \begin{vmatrix} \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \\ \delta, & \delta', & 1 \end{vmatrix} : -\begin{vmatrix} \gamma, & \gamma', & 1 \\ \delta, & \delta', & 1 \\ \alpha, & \alpha', & 1 \end{vmatrix} : \begin{vmatrix} \delta, & \delta', & 1 \\ \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \end{vmatrix} : -\begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix}$$

and we may write

$$\begin{aligned}
\mathrm{a} &= \quad . \quad , \quad ah' - a'h, \quad ag' - a'g, \quad gh' - g'h, \\
\mathrm{b} &= bh' - b'h, \quad . \quad , \quad bf' - b'f, \quad hf' - h'f, \\
\mathrm{c} &= cg' - c'g, \quad cf' - c'f, \quad . \quad , \quad fg' - f'g, \\
\mathrm{d} &= cb' - c'b, \quad ac' - a'c, \quad ba' - b'a, \quad . \quad ,
\end{aligned}$$

viz., the expressions in the same horizontal line are equal, and a, b, c, d are proportional to the expressions in the four lines respectively.

95. I say that we have

$$\frac{c'f'}{ah}\,\mathrm{a} = \frac{c'g'}{bh}\,\mathrm{b} = \frac{a'f'}{af}\,\mathrm{c} = \frac{f'g'}{ab}\,\mathrm{d},$$

viz., this will be the case if

$$\begin{aligned}
bc'\mathrm{a} &= hg'\,\mathrm{d}, \\
ac'\mathrm{b} &= hf'\mathrm{d}, \\
a'b\mathrm{c} &= fg'\,\mathrm{d},
\end{aligned}$$

and selecting the convenient expressions for a, b, c, d, these equations become

$$\begin{aligned}
bc'\,(gh' - g'h) &= g'h\,(cb' - c'b), \\
ac'\,(hf' - h'f) &= f'h\,(ac' - a'c), \\
a'b\,(fg' - f'g) &= fg'\,(ba' - b'a),
\end{aligned}$$

viz., these equations are respectively $bgc'h' = b'g'ch$, $cha'f' = c'h'af$, $af'b'g' = a'f'bg$, and are consequently satisfied. It thus appears that the equation

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$$

is transformable into

$$\frac{c'f'}{ah}\,l + \frac{c'g'}{bh}\,m + \frac{a'f'}{af}\,n + \frac{f'g'}{ab}\,p = 0,$$

which is of course one of a system of similar forms.

96. Take (A_1, D_1) the antipoints of (A, D); (B_1, C_1) the antipoints of (B, C); or say that the circular coordinates of A_1, B_1, C_1, D_1 are $(\alpha, \delta', 1)$, $(\beta, \gamma', 1)$, $(\gamma, \beta', 1)$, $(\delta, \alpha', 1)$ respectively; the points A_1, B_1, C_1, D_1 are, as above mentioned, on a circle, the condition that this may be so being in fact

$$\begin{vmatrix} 1, & \alpha, & \delta', & \alpha\delta' \\ 1, & \beta, & \gamma', & \beta\gamma' \\ 1, & \gamma, & \beta', & \gamma\beta' \\ 1, & \delta, & \alpha', & \delta\alpha' \end{vmatrix} = 0,$$

equivalent to

$$af : bg : ch = a'f' : b'g' : c'h'.$$

97. Let (a_1, b_1, c_1, d_1) be the corresponding quantities to (a, b, c, d), viz., $a_1 : b_1 : c_1 : d_1 = B_1C_1D_1 : -C_1D_1A_1 : D_1A_1B_1 : -A_1B_1C_1$; we have

$$a_1 : b_1 : c_1 : d_1 = \begin{vmatrix} \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \end{vmatrix} : - \begin{vmatrix} \gamma, & \beta', & 1 \\ \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \end{vmatrix} : \begin{vmatrix} \delta, & \alpha', & 1 \\ \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \end{vmatrix} : - \begin{vmatrix} \alpha, & \delta', & 1 \\ \beta, & \gamma', & 1 \\ \gamma, & \beta', & 1 \end{vmatrix}$$

giving rise to a similar set of forms

$$\begin{aligned} a_1 &= \quad . \quad , -ac' + ha', \quad a'g + b'a, -c'g - b'h, \\ b_1 &= -c'b - g'h, \quad . \quad , -f'b - g'f, -f'h + c'f, \\ c_1 &= \quad b'c + h'g, -f'c + h'f, \quad . \quad , \quad f'g + g'f, \\ d_1 &= \quad g'c + h'b, -h'a + a'c, -a'b - g'a, \quad . \quad , \end{aligned}$$

and leading to

$$\frac{cf}{a'c'} a_1 = -\frac{cg}{c'g'} b_1 = \frac{af}{a'f'} c_1 = -\frac{fg}{a'g'} d_1,$$

so that the equation

$$\frac{l_1}{a_1} + \frac{m_1}{b_1} + \frac{n_1}{c_1} + \frac{p_1}{d_1} = 0,$$

is transformable into

$$\frac{cf}{a'c'} l_1 - \frac{cg}{c'g'} m_1 + \frac{af}{a'f'} n_1 - \frac{fg}{a'g'} p_1 = 0.$$

98. Let A, B, C, D, be, as above, points on a circle; (A_1, D_1) and (B_1, C_1) the antipoints of (A, D), (B, C) respectively. Write

$$\begin{aligned} \mathfrak{A} &= (\xi - \alpha z)(\eta - \alpha' z), & \mathfrak{A}_1 &= (\xi - \alpha z)(\eta - \delta' z), \\ \mathfrak{B} &= (\xi - \beta z)(\eta - \beta' z), & \mathfrak{B}_1 &= (\xi - \beta z)(\eta - \gamma' z), \\ \mathfrak{C} &= (\xi - \gamma z)(\eta - \gamma' z), & \mathfrak{C}_1 &= (\xi - \gamma z)(\eta - \beta' z), \\ \mathfrak{D} &= (\xi - \delta z)(\eta - \delta' z), & \mathfrak{D}_1 &= (\xi - \delta z)(\eta - \alpha' z); \end{aligned}$$

then we have identically

$$\begin{aligned} (\delta - \alpha)(\delta' - \alpha')\mathfrak{B} &= (\beta - \delta)(\beta' - \delta')\mathfrak{A} + (\beta - \alpha)(\beta' - \alpha')\mathfrak{D} - (\beta - \delta)(\beta' - \alpha')\mathfrak{A}_1 - (\beta - \alpha)(\beta' - \delta')\mathfrak{D}_1, \\ (\delta - \alpha)(\delta' - \alpha')\mathfrak{C} &= (\gamma - \delta)(\gamma' - \delta')\mathfrak{A} + (\gamma - \alpha)(\gamma' - \alpha')\mathfrak{D} - (\gamma - \delta)(\gamma' - \alpha')\mathfrak{A}_1 - (\gamma - \alpha)(\gamma' - \delta')\mathfrak{D}_1, \\ (\delta - \alpha)(\delta' - \alpha')\mathfrak{B}_1 &= (\beta - \delta)(\gamma' - \delta')\mathfrak{A} + (\beta - \alpha)(\gamma' - \alpha')\mathfrak{D} - (\beta - \delta)(\gamma' - \alpha')\mathfrak{A}_1 - (\beta - \alpha)(\gamma' - \delta')\mathfrak{D}_1, \\ (\delta - \alpha)(\delta' - \alpha')\mathfrak{C}_1 &= (\gamma - \delta)(\beta' - \delta')\mathfrak{A} + (\gamma - \alpha)(\beta' - \alpha')\mathfrak{D} - (\gamma - \delta)(\beta' - \alpha')\mathfrak{A}_1 - (\gamma - \alpha)(\beta' - \delta')\mathfrak{D}_1, \end{aligned}$$

or, in the foregoing notation,

$$\begin{aligned} ff'\mathfrak{B} &= gg'\mathfrak{A} + cc'\mathfrak{D} + gc'\mathfrak{A}_1 + cg'\mathfrak{D}_1, \\ ff'\mathfrak{C} &= hh'\mathfrak{A} + bb'\mathfrak{D} - hb'\mathfrak{A}_1 - bh'\mathfrak{D}_1, \\ ff'\mathfrak{B}_1 &= gh'\mathfrak{A} - cb'\mathfrak{D} - gb'\mathfrak{A}_1 + ch'\mathfrak{D}_1, \\ ff'\mathfrak{C}_1 &= hg'\mathfrak{A} - bc'\mathfrak{D} + hc'\mathfrak{A}_1 - bg'\mathfrak{D}_1. \end{aligned}$$

Article Nos. 99 to 104. *Further Properties in relation to the same Sets* (A, B, C, D) *and* (A_1, B_1, C_1, D_1).

99. It is to be shown that in virtue of these equations, and if moreover $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0$, then it is possible to find l_1, m_1, n_1, p_1, such that we have identically

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}-p\mathfrak{D}+l_1\mathfrak{A}_1-m_1\mathfrak{B}_1-n_1\mathfrak{C}_1+p_1\mathfrak{D}_1=0.$$

This equation will in fact be identically true if only

$$\begin{array}{llll} -ff'l+gg'm+hh'n & . & -gh'm_1-g'hn_1 & =0, \\ cc'm+bb'n-ff'p & . & +cb'm_1+bc'n_1 & =0, \\ gc'm-hb'n & +ff'l_1 & +gb'm_1-hc'n_1 & =0, \\ cg'm-bh'n & . & +ch'm_1+bg'n_1+ff'p_1 & =0. \end{array}$$

From the first and second equations eliminating m_1 or n_1, the other of these quantities disappears of itself, and we thus obtain two equations which must be equivalent to a single one, viz., we have

$$bc'ff'l+c'g'afm+bh\,a'f'n+g'hff'p=0,$$
$$b'cff'l+cga'f'm+b'h'afn+gh'ff'p=0;$$

which equations may also be written

$$\frac{c'f'}{ah}l+\frac{c'g'}{bh}m+\frac{a'f'}{af}n+\frac{f'g'}{ab}p=0,$$

$$\frac{cf}{a'h'}l+\frac{cg}{b'h'}m+\frac{af}{a'f'}n+\frac{fg}{a'b'}p=0;$$

and it thus appears that the equations are equivalent to each other, and to the assumed relation

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0.$$

100. Similarly, from the third and fourth equations eliminating m or n, the other of these quantities disappears of itself, and we find

$$cg'ff'l_1-cga'f'm_1+afc'g'n_1-c'gff'p_1=0,$$
$$bh'ff'l_1-afb'h'm_1+bha'f'n_1-b'hff'p_1=0,$$

equations which may be written

$$\frac{cf}{a'c'}l-\frac{cg}{c'g'}m+\frac{af}{a'f'}n-\frac{fg}{g'a'}p=0,$$

$$\frac{f'h'}{ah}l-\frac{b'h'}{bg}m+\frac{a'f'}{af}n-\frac{b'g'}{ab}p=0,$$

where we see that the two equations are equivalent to each other and to the equation

$$\frac{l_1}{a_1}+\frac{m_1}{b_1}+\frac{n_1}{c_1}+\frac{p_1}{d_1}=0.$$

It thus appears that the quantities l_1, m_1, n_1, p_1, must satisfy this last equation. It is to be observed that the first and second equations being, as we have seen, equivalent to a single equation, either of the quantities m_1, n_1, may be assumed at pleasure, but the other is then determined; the third and fourth equations then give l_1, p_1; and the quantities l_1, m_1, n_1, p_1, so obtained, satisfy *identically* the equation $\frac{l_1}{a_1}+\frac{m_1}{b_1}+\frac{n_1}{c_1}+\frac{p_1}{d_1}=0$.

101. Now writing

$$ff'l_1=-g(c'm+b'm_1)+h(b'n+c'n_1),$$
$$ff'p_1=-c(g'm-h'm_1)+b(h'n-g'n_1),$$

and

$$ff'p=\quad c(c'm+b'm_1)+b(b'n+c'n_1),$$
$$ff'l=\quad g(g'm-h'm_1)+h(h'n-g'n_1),$$

we find

$$\begin{aligned} f^2f'^2(l_1p_1-lp) &= -(bg+ch)[(c'm+b'm_1)(h'n\ -g'n_1)+(g'm-h'm_1)(b'n+c'n_1)],\\ &= (bg+ch)\ (b'g'+c'h'\ )(m_1n_1-mn),\\ &= aa'ff'(m_1n_1-mn), \end{aligned}$$

that is

$$ff'(l_1p_1-lp)=aa'(m_1n_1-mn)$$

viz., this equation is satisfied identically by the values of l_1, m_1, n_1, p_1 determined as above.

102. Hence if $m_1n_1=mn$, we have also $l_1p_1=lp$, and we can determine m_1, n_1, so that m_1n_1 shall $=mn$, viz., in the first or second of the four equations (these two being equivalent to each other, as already mentioned), writing $m_1=\theta n$, and therefore $n_1=\frac{1}{\theta}m$, we have

$$-ff'l+gg'm+hh'n-gh'n\theta-g'hm\frac{1}{\theta}=0,$$

$$cc'm+bb'n\ -ff'p+cb'n\theta+bc'm\frac{1}{\theta}=0,$$

which are, in fact, the same quadric equation in θ, viz., we have

$$\frac{-ff'l+gg'm+hh'n}{cc'm+bb'n-ff'p}=-\frac{gh'}{cb'}=-\frac{g'h}{bc'}.$$

The final result is that there are two sets of values of l_1, m_1, n_1, p_1, each satisfying the identity

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}-p\mathfrak{D}+l_1\mathfrak{A}_1-m_1\mathfrak{B}_1-n_1\mathfrak{C}_1+p_1\mathfrak{D}_1=0,$$

and for each of which we have

$$\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}+\frac{p_1}{\mathrm{d}_1}=0,\quad l_1p_1=lp,\ m_1n_1=mn.$$

103. Consider, in particular, the case where $p=0$; the relation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0,$$

here becomes

$$l=-\frac{ag'}{bf'}m-\frac{a'h}{c'f}n.$$

The equation in θ is

$$(cc'm+bb'n)\,\theta+cb'n\theta^2+bc'm=0,$$

viz., this is

$$(c\theta+c'm)(b'n\theta+b)=0,$$

giving

$$\theta=-\frac{b}{c}\ ,\quad m_1=-\frac{bn}{c}\ ,\quad n_1=-\frac{cm}{b},$$

or else

$$\theta=-\frac{c'm}{b'n},\quad m_1=-\frac{c'm}{b'},\quad n_1=-\frac{b'n}{c'}.$$

Since in the present case $l_1p_1=0$, we have either $l_1=0$, or else $p_1=0$, and as might be anticipated, the two values of θ correspond to these two cases respectively, viz., proceeding to find the values of l_1, p_1, the completed systems are

$$\theta=-\frac{b}{c}\ ,\quad l_1=\frac{a}{bcf'}\left(cc'm-bb'n\right),\quad m_1=-\frac{bn}{c}\ ,\quad n_1=-\frac{cm}{b}\ ,\quad p_1=0,$$

$$\theta=-\frac{c'm}{b'n},\quad l'_1=0\qquad\qquad ,\quad m'_1=-\frac{c'm}{b'},\quad n'_1=-\frac{b'n}{c'},\quad p'_1=\frac{a'}{b'c'f}\left(cc'm-bb'n\right),$$

so that for the first system we have

$$\frac{l_1}{\mathrm{a}_1}+\frac{m_1}{\mathrm{b}_1}+\frac{n_1}{\mathrm{c}_1}=0,\quad m_1n_1=mn,\quad -l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}_1,$$

and for the second system

$$\frac{m'_1}{\mathrm{b}_1}+\frac{n'_1}{\mathrm{c}_1}+\frac{p'_1}{\mathrm{d}_1}=0,\quad m'_1n'_1=mn,\quad -l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-p'_1\mathfrak{D}_1+m'_1\mathfrak{B}_1+n'_1\mathfrak{C}_1.$$

104. The whole of the foregoing investigation would have assumed a more simple form if the circular coordinates had been taken with reference to the centre of the circle $ABCD$ as origin, and the radius of this circle been put $=1$; we should then have $\alpha'=\frac{1}{\alpha}$, &c., and consequently

$$a'=-\frac{1}{\beta\gamma}a,\quad b'=-\frac{1}{\gamma\alpha}b,\quad c'=-\frac{1}{\alpha\beta}c,\quad f'=-\frac{1}{\alpha\delta}f,\quad g'=-\frac{1}{\beta\delta}g,\quad h'=-\frac{1}{\gamma\delta}h;$$

but the symmetrical relation of the circles $ABCD$ and $A_1B_1C_1D_1$ would not have been so clearly shown.

I will however give the investigation in this simplified form, for the identity $-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}+m_1\mathfrak{B}+n_1\mathfrak{C}$; viz., in this case we have

$$\frac{l}{\alpha}=-\frac{m}{\beta}\frac{(\beta-\gamma)(\beta-\delta)}{(\gamma-\alpha)(\alpha-\delta)}-\frac{n}{\gamma}\frac{(\beta-\gamma)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)},$$

and the identity to be satisfied is

$$\begin{aligned}-l\,(\xi-\alpha z)\left(\eta-\frac{1}{\alpha}z\right) &= -l_1\,(\xi-\alpha z)\left(\eta-\frac{1}{\delta}z\right)\\ +m\,(\xi-\beta z)\left(\eta-\frac{1}{\beta}z\right) &\quad +m_1(\xi-\beta z)\left(\eta-\frac{1}{\gamma}z\right)\\ +n\,(\xi-\gamma z)\left(\eta-\frac{1}{\gamma}z\right) &\quad +n_1\,(\xi-\gamma z)\left(\eta-\frac{1}{\beta}z\right);\end{aligned}$$

writing $\xi=\alpha z$, $\eta=\frac{1}{\beta}z$, we find m_1, and writing $\xi=\alpha z$, $\eta=\frac{1}{\gamma}z$, we find n_1, and it is then easy to obtain the value of l_1, viz., the results are

$$\frac{l_1}{\delta}=\frac{m}{\beta}\frac{(\alpha-\beta)(\beta-\gamma)}{(\gamma-\alpha)(\alpha-\delta)}+\frac{n}{\gamma}\frac{(\beta-\gamma)(\gamma-\alpha)}{(\alpha-\beta)(\alpha-\delta)},\quad m_1=-n\frac{\gamma-\alpha}{\alpha-\beta},\quad n_1=-m\frac{\alpha-\beta}{\gamma-\alpha},$$

and therefore $m_1n_1=mn$; it may be added that we have

$$-\frac{l_1}{\delta}=\frac{\beta-\gamma}{\alpha-\delta}\left(\frac{m_1}{\gamma}+\frac{n_1}{\delta}\right),$$

viz., this is the form assumed by the equation $\frac{l_1}{a_1}+\frac{m_1}{b_1}+\frac{n_1}{c_1}=0$.

Part III. (Nos. 105 to 157). On the Theory of Foci.

Article Nos. 105 to 110. *Explanation of the General Theory.*

105. If from a focus of a conic we draw two tangents to the curve, these pass respectively through the two circular points at infinity, and we have thence the generalised definition of a focus as established by Plücker, viz., in any curve a focus is a point such that the lines joining it with the two circular points at infinity are respectively tangents to the curve; or, what is the same thing, if from each of the circular points at infinity, say from the points I, J, tangents are drawn to the curve, the intersections of each tangent from the one point with each tangent from the other point are the foci of the curve. A curve of the class n has thus in general n^2 foci. It is to be added that, as in the conic the line joining the points of contact of the two tangents from a focus is the directrix corresponding to that focus, so in general the line joining the points of contact of the tangents from the focus through the points I, J respectively is the directrix corresponding to the focus in question.

106. A circular point at infinity I or J, may be an ordinary or a singular point on the curve, and the tangent at this point then counts, or, in the case of a multiple point, the tangents at this point count a certain number of times, say q times, among the tangents which can be drawn to the curve from the point; the number of the remaining tangents is thus $=n-q$. In particular, if the circular point at infinity be an ordinary point, then the tangent counts twice, or we have $q=2$; if it be a node, each of the tangents counts twice, or $q=4$; if it be a cusp, the tangent counts three times, or $q=3$. Similarly, if the other circular point an infinity be an ordinary or a singular point on the curve, the tangent or tangents there count a certain number of times, say q' times, among the tangents to the curve from this point; the number of the remaining tangents is thus $=n-q'$. And if as usual we disregard the tangents at the two points I, J respectively, and attend only to the remaining tangents, the number of the foci is $=(n-q)(n-q')$.

107. Among the tangents from the point I or J there may be a tangent which, either from its being a multiple tangent (that is, a tangent having ordinary contact at two or more distinct points), or from being an osculating tangent at one or more points, counts a certain number of times, say r, among the tangents from the point in question. Similarly, if among the tangents from the other point J or I, there is a tangent which counts r' times, then the foci are made up as follows, viz. we have

Intersections of the two singular tangents counting as .	$r'r$ foci.
Intersections of the first singular tangent with each of the ordinary tangents from the other circular point at infinity, as	$(n-q'-r')r$ „
Do. for second singular tangent, ...	$(n-q-r)r'$ „
Intersections of the ordinary tangents	$(n-q-r)(n-q'-r')$ „
Giving together the	$(n-q)(n-q')$ foci:

and the like observation applies to the more general case where the tangents from each of the points I, J include more than one singular tangent.

108. There is yet another case to be considered; the line infinity may be an ordinary or a singular tangent to the curve: assuming that it counts s times among the tangents from either of the circular points at infinity, the numbers of the remaining tangents are $n-q-s$, $n-q'-s$ from the two points I, J respectively, and the number of foci is $=(n-q-s)(n-q'-s)$.

109. In the case of a real curve the two points I, J are related in the same manner to the curve, and we have therefore $q=q'$; the singular tangents (if any) from the two points respectively being the same as well in character as in number. Writing $n-q-s=n-q'-s$, $=p$, and not for the present attending to the case of singular tangents, I shall assume that the number of tangents to the curve from each of the two points is $=p$; the number of foci is thus $=p^2$; and to each focus there corresponds a directrix, viz., this is the line through the points of contact of the tangents from the focus to the two points I, J respectively.

110. Consider any two foci A, B not *in lineâ* with either of the points I, J, then joining these with the points I, J, and taking A_1, B_1 the intersections of AI, BJ and of AJ, BI (A_1, B_1 being therefore by a foregoing definition the antipoints of (A, B)), then A_1, B_1 are, it is clear, foci of the curve. We may out of the p^2 foci select, and that in $1.2..p$ different ways, a system of p foci such that no two of them lie *in lineâ* with either of the points I, J; and this being so, taking the antipoints of each of the $\frac{1}{2}p(p-1)$ pairs out of the p foci, we have, inclusively of the p foci, in all $p+2.\frac{1}{2}p(p-1)$, that is p^2 foci, the entire system of foci.

Article Nos. 111 to 117. *On the Foci of Conics.*

111. A conic is a curve of the class 2, and the number of foci is thus $=4$. Taking as foci any two points A, B, the remaining two foci will be the antipoints A_1, B_1. In order that a given point A may be a focus, the conic must touch the lines AI, AJ; similarly, in order that a given point B may be a focus, the conic must touch the lines BI, BJ; the equation of a conic having the given points A, B for foci contains therefore a single arbitrary parameter.

112. In the case, however, of the parabola the curve touches the line infinity; there is consequently from each of the points I, J only a single tangent to the curve, and consequently only one focus: the parabola having a given point A for its focus is a conic touching the line infinity and the lines AI, AJ, or say the three sides of the triangle AIJ; its equation contains therefore two arbitrary parameters.

113. Returning to the general conic, there are certain trizomal forms of the focal equation, not of any great interest, but which may be mentioned. Using circular coordinates, and taking $(\alpha, \alpha', 1)$ and $(\beta, \beta', 1)$ for the coordinates of the given foci A, B respectively, the conic touches the lines $\xi-\alpha z=0$, $\eta-\alpha' z=0$, $\xi-\beta z=0$, $\eta-\beta' z=0$; the equation of a conic touching the first three lines is

$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\eta-\alpha' z)}=0,$$

where l, m, n are arbitrary, and it is easy to obtain, in order that the conic may touch the fourth line $\eta-\beta' z=0$, the condition

$$n=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l).$$

114. In fact, n having this value, the equation gives

$$l(\xi-\alpha z)+m(\xi-\beta z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)\big(\eta-\beta' z+(\beta'-\alpha')z\big),$$

and taking over the term

$$\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\beta'-\alpha')z, \quad =(\beta-\alpha)(m-l)z,$$

this gives

$$l(\xi-\beta z)+m(\xi-\alpha z)+2\sqrt{lm(\xi-\alpha z)(\xi-\beta z)}=-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\beta' z),$$

which puts in evidence the tangent $\eta-\beta'z$. It is easy to see that the equation may be written in any one of the four forms

$$\sqrt{l\,(\xi-\alpha z)}+\sqrt{m\,(\xi-\beta z)}+\sqrt{-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\alpha' z)}=0,$$

$$\sqrt{m\,(\xi-\alpha z)}+\sqrt{l\,(\xi-\beta z)}+\sqrt{-\frac{\beta-\alpha}{\beta'-\alpha'}(m-l)(\eta-\beta' z)}=0,$$

$$\sqrt{l\,(\eta-\alpha' z)}+\sqrt{m(\eta-\beta' z)}+\sqrt{-\frac{\beta'-\alpha'}{\beta-\alpha}(m-l)(\xi-\alpha z)}=0,$$

$$\sqrt{m\,(\eta-\alpha' z)}+\sqrt{l\,(\eta-\beta' z)}+\sqrt{-\frac{\beta'-\alpha'}{\beta-\alpha}(m-l)(\xi-\beta z)}=0,$$

viz., in forms containing any three of the four radicals $\sqrt{\xi-\alpha z}$, $\sqrt{\xi-\beta z}$, $\sqrt{\eta-\alpha' z}$, $\sqrt{\eta-\beta' z}$. The conic is thus expressed as a trizomal curve, the zomals being each a line, viz., they are any three out of the four focal tangents; the order of the curve, as deduced from the general expression $2^{r-2}r$, is $=2$; so that there is here no depression of order.

115. But the ordinary form of the focal equation is a more interesting one; viz., $\mathfrak{A}$, $\mathfrak{B}$ being as usual the squared distances of the current point from the two given foci respectively, say

$$\mathfrak{A}=(\xi-\alpha z)(\eta-\alpha' z),$$

$$\mathfrak{B}=(\xi-\beta z)(\eta-\beta' z),$$

then $2a$ being an arbitrary parameter, the equation is

$$2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0,$$

viz., the equation is here that of a trizomal curve, the zomals being curves of the second order, that is, the zomals are $(z^2=0)$ the line infinity twice, and the line-pairs AI, AJ and BI, BJ respectively: the general expression $2^{r-2}r$ gives therefore the order $=4$; but in the present case there are two branches, viz., the branches

$$2az+\sqrt{\mathfrak{A}}-\sqrt{\mathfrak{B}}=0,\quad 2az-\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0,$$

each ideally containing $(z=0)$ the line infinity; the curve contains therefore $(z^2=0)$ the line infinity twice, and omitting this factor the order is $=2$, as it should be.

116. To express the equation by means of the other two foci A_1, B_1, writing the equation under the form

$$\mathfrak{A}+\mathfrak{B}+2\sqrt{\mathfrak{A}\mathfrak{B}}-4a^2z^2=0,$$

and then if $\mathfrak{A}_1$, $\mathfrak{B}_1$ are the squared distances of the current point from A_1, B_1 respectively, we have (*ante*, No. 65),

$$\mathfrak{A}\mathfrak{B}=\mathfrak{A}_1\mathfrak{B}_1,$$

$$\mathfrak{A}+\mathfrak{B}-\mathfrak{A}_1-\mathfrak{B}_1=kz^2,$$

where k is the squared distance of the foci A, B, $=4a^2e^2$ suppose: whence putting $a^2(1-e^2)=b^2$, the equation becomes

$$\mathfrak{A}_1+\mathfrak{B}_1+2\sqrt{\mathfrak{A}_1\mathfrak{B}_1}-4b^2z^2=0,$$

that is

$$\sqrt{\mathfrak{A}_1}+\sqrt{\mathfrak{B}_1}+2bz=0,$$

which is the required new form. It is hardly necessary to remark that the equation $2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, putting therein $z=1$, and expressing $\mathfrak{A}$, $\mathfrak{B}$ in rectangular coordinates measured along the axes, is the ordinary focal equation $2a=\sqrt{(x-ae)^2+y^2}+\sqrt{(x+ae)^2+y^2}$.

117. I remark that the equation $2az+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$ gives rise to $4a^2z^2+\mathfrak{A}-\mathfrak{B}+4az\sqrt{\mathfrak{A}}=0$, but here $\mathfrak{A}-\mathfrak{B}=-4aexz$, so that the equation contains $z=0$, and omitting this it becomes $(az-ex)+\sqrt{\mathfrak{A}}=0$, a bizomal form, being a curve of the order $=2$, as it should be; this is in fact the ordinary equation in regard to a focus and its directrix.

Article Nos. 118 to 123. *Theorem of the Variable Zomal as applied to a Conic.*

118. The equation $2kz+\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$ is in like manner that of a conic; in fact, this would be a curve of the order $=4$, but there are as before the two branches $2kz+\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ}=0$, $2kz-\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$, each ideally containing $(z=0)$ the line infinity, and the order is thus reduced to be $=2$. Each of the circles $\mathfrak{A}^\circ=0$, $\mathfrak{B}^\circ=0$ is a circle having double contact with the conic (this of course implies that the centre of the circle is on an axis of the conic). We may if we please start from the form $2kz+\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, and then by means of the theorem of the variable zomal introduce into the equation one, two, or three such circles.

119. It is in this point of view that I will consider the question, viz., adapting the formula to the case of the ellipse, and starting from the form

$$2az+\sqrt{(x-aez)^2+y^2}+\sqrt{(x+aez)^2+y^2}=0,$$

the equation of the variable zomal or circle of double contact may be taken to be

$$\frac{4a^2z^2}{-2}+\frac{(x-aez)^2+y^2}{1-q}+\frac{(x+aez)^2+y^2}{1+q}=0,$$

where q is an arbitrary parameter; writing for greater simplicity $z=1$, and reducing, the equation is

$$(x-qae)^2+y^2=b^2(1-q^2).$$

120. If $q<1$, then writing $q=\sin\theta$, we obtain the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

as the envelope of the variable circle

$$(x-ae\sin\theta)^2+y^2=b^2\cos^2\theta,$$

viz., of a circle having its centre on the major axis at a distance $=ae\sin\theta$ from the centre, and its radius $=b\cos\theta$. (I notice, in passing, that this gives in practice a very convenient graphical construction of the ellipse.) It may be remarked that for $\theta=\pm\sin^{-1}e$, the circle becomes

$$\left(x\pm\left(a-\frac{b^2}{a}\right)\right)^2+y^2=\frac{b^4}{a^2},$$

viz., this is the circle of curvature at one or other of the extremities of the major axis; as θ passes from 0 to $\pm\sin^{-1}e$ we have a series of real circles, which, by their continued intersection, generate the ellipse; as θ increases from $\theta=\pm\sin^{-1}e$ to $\pm 90°$, the circles continue real, but the consecutive circles no longer intersect in any real point,—and ultimately for $\theta=\pm 90°$, the circles become evanescent at the two foci respectively.

121. In the case $q>1$, we have a real representation of

$$(x-qae)^2+y^2+b^2(q^2-1),$$

as the squared distance of the point (x, y) from a point $(X, 0, Z)$ out of the plane of the figure, viz., putting this $=(x-X)^2+y^2+Z^2$,
we have

$$qae=X,\quad Z^2=b^2(q^2-1),$$

whence

$$Z^2=b^2\left(\frac{X^2}{a^2e^2}-1\right),$$

or, what is the same thing,

$$\frac{X^2}{a^2-b^2}-\frac{Z^2}{b^2}=1;$$

that is, the locus is the focal hyperbola, viz., a hyperbola in the plane of zx, having its vertices at the foci, and its foci at the vertices of the ellipse.

122. If instead of the form first considered, we start from the trizomal form

$$2bz+\sqrt{x^2+(y-aeiz)^2}+\sqrt{x^2+(y+aeiz)^2}=0,$$

then we have the zomal or circle of double contact under the form

$$x^2+(y-qaei)^2=a^2(1-q^2);$$

or putting herein $q=-i\tan\phi$, this is

$$x^2+(y-ae\tan\phi)^2=a^2\sec^2\phi;$$

so that we have the ellipse as the envelope of a variable circle having its centre on the minor axis of the ellipse, distance from the centre $=ae\tan\phi$, and radius $=a\sec\phi$. This is, in fact, Gergonne's theorem, according to which the ellipse is the secondary caustic or orthogonal trajectory of rays issuing from a point and refracted at a right line into a rarer medium. It is to be remarked that for $\tan\phi=\pm\frac{ae}{b}$, the equation of the circle is

$$x^2+\left(y\pm\left(b-\frac{a^2}{b}\right)\right)^2=\frac{a^4}{b^2},$$

viz., this is the circle of curvature at one or other extremity of the minor axis; from $\phi=0$ to $\phi=\pm\tan^{-1}\frac{ae}{b}$, the intersections of the consecutive circles are real, and give the entire real ellipse; from $\phi=\pm\tan^{-1}\frac{ae}{b}$ to $\phi=\pm 90°$, the circles are still real, but the intersections of consecutive circles are imaginary.

123. If in the equation of the generating circle we interchange x, y, a, b, the equation becomes

$$(x-aei\tan\phi)^2+y^2=b^2\sec^2\phi,$$

which is (as it should be) equivalent to the former equation

$$(x-ae\sin\theta)^2+y^2=b^2\cos\theta,$$

the identity being established by means of the equation

$$\cos\theta=\frac{1}{\cos\phi},\text{ and therefore }\sin\theta=i\tan\phi,\ \tan\theta=i\sin\phi,$$

which is Jacobi's imaginary transformation in the theory of Elliptic Functions.

Article Nos. 124 to 126. *Foci of the Circular Cubic and the Bicircular Quartic.*

124. For a cubic curve, the class is in general $=6$, and the number of the foci is $=36$. But a specially interesting case is that of a circular cubic, viz., a cubic passing through each of the circular points at infinity. Here, at each of the circular points at infinity, the tangent at this point reckons twice among the tangents to the curve from the point; the number of the remaining tangents is thus $=4$, and the number of the foci is $=16$. If from any two points whatever on the curve tangents be drawn to the curve, then the two pencils of tangents are, and that in four different ways, homologous to each other, viz., if the tangents of the first pencil are (1, 2, 3, 4), and those of the second pencil, taken in a proper order, are (1′, 2′, 3′, 4′), then we have (1, 2, 3, 4) homologous with each of the arrangements (1′, 2′, 3′, 4′), (2′, 1′, 4′, 3′), (3′, 4′, 1′, 2′), (4′, 3′, 2′, 1′). And in each case the intersections of the four corresponding tangents lie on a conic passing through the two given points on the curve [1].

[1] It may be remarked that if the equation of the first pencil of lines be

$$(x-ay)(x-by)(x-cy)(x-dy)=0,$$

and that of the second pencil

$$(z-aw)(z-bw)(z-cw)(z-dw)=0,$$

then the equations of four conics are

$$xw-yz=0,$$

$$(a+d-b-c)\,xz+(bc-ad)(xw+yz)+\big(ad(b+c)-bc(a+d)\big)\,yw=0,$$

$$(b+d-c-a)\,xz+(ca-bd)(xw+yz)+\big(bd(c+a)-ca(b+d)\big)\,yw=0,$$

$$(c+d-a-b)\,xz+(ab-cd)(xw+yz)+\big(cd(a+b)-ab(c+d)\big)\,yw=0.$$

125. Hence taking the points on the curve to be the circular points at infinity, we have the sixteen foci lying in fours upon four different circles—that is, we have four tetrads of concyclic foci. Let any one of these tetrads be A, B, C, D, then if

$$\begin{array}{llll} \text{Antipoints of } (B, C)(A, D) & \text{are} & (B_1, C_1), & (A_1, D_1), \\ \quad\text{,,} \quad (C, A)(B, D) & \text{,,} & (C_2, A_2), & (B_2, D_2), \\ \quad\text{,,} \quad (A, B)(C, D) & \text{,,} & (A_3, B_3), & (C_3, D_3), \end{array}$$

the four tetrads of concyclic foci are

$$\begin{array}{llll} A, & B, & C, & D; \\ A_1, & B_1, & C_1, & D_1; \\ A_2, & B_2, & C_2, & D_2; \\ A_3, & B_3, & C_3, & D_3. \end{array}$$

It is to be observed that if A, B, C, D are any four points on a circle, then if, as above, we pair these in any manner, and take the antipoints of each pair, the four antipoints lie on a circle, and thus the original system A, B, C, D, of four points on a circle, leads to the remaining three systems of four points on a circle. The theory is in fact that already discussed *ante*, No. 72 *et seq.*

126. The preceding theory applies without alteration to the bicircular quartic, viz., the quartic curve which has a node at each of the circular points at infinity. The class is here $=8$, but among the tangents from a node each of the two tangents at the node is to be reckoned twice, and the number of the remaining tangents is $=4$: the number of foci is $=16$. And, by the general theorem that in a binodal quartic the pencils of tangents from the two nodes respectively are homologous, the sixteen foci are related to each other precisely in the manner of the foci of the circular cubic. The latter is in fact a particular case of the former, viz., the bicircular quartic may break up into the line infinity, and a circular cubic.

Article Nos. 127 to 129. *Centre of the Circular Cubic, and Nodo-Foci, &c. of the Bicircular Quartic.*

127. The tangents at I, J have not been recognised as tangents from I, J, giving by their intersection a focus, but it is necessary in the theory to pay attention to the tangents in question. It is clear that these tangents are in fact asymptotes—viz., in the case of the circular cubic they are the two imaginary asymptotes of the curve, and in the case of a bicircular quartic, the two pairs of imaginary parallel asymptotes; but it is convenient to speak of them as the tangents at I, J.

128. In the case of a circular cubic, the tangents at I and J meet in a point which I call the centre of the curve, viz., this is the intersection of the two imaginary asymptotes.

129. In the case of a bicircular quartic, the two tangents at I and the two tangents at J meet in four points, which (although not recognising them as foci) I call the nodo-foci; these lie in pairs on two lines, diagonals of the quadrilateral formed by the four tangents (the third diagonal is of course the line IJ), which diagonals I call the "nodal axes;" and the point of intersection of the two nodal axes is the "centre" of the curve. The nodo-foci are four points, two of them real, the other two imaginary, viz., they are two pairs of antipoints, the lines through the two pairs respectively being, of course, the nodal axes; these are consequently real lines bisecting each other at right angles in the centre (with the relation $1 : i$ between the distances). The centre may also be defined as the intersection of the harmonic of IJ in regard to the tangents at I, and the harmonic of this same line in regard to the tangents at J. Speaking of the tangents as asymptotes, the nodo-foci are the angles of the rhombus formed by the two pairs of parallel asymptotes; the nodal axes are the diagonals of this rhombus, and the centre is the point of intersection of the two diagonals; as such it is also the intersection of the two lines drawn parallel to and midway between the lines forming each pair of parallel asymptotes.

Article No. 130. *Circular Cubic and Bicircular Quartic; the Axial or Symmetrical Case.*

130. In a circular cubic or bicircular quartic, the pencil of the tangents from I and that of the tangents through J, considered as corresponding to each other in some one of the four arrangements, may be such that the line IJ considered as belonging to the two pencils respectively shall correspond to itself, and when this is so, the four foci, A, B, C, D, which are the intersections of the corresponding tangents in question, will lie in a line (viz., the conic which exists in the general case will break up into a line-pair consisting of the line IJ and another line). The line in question may be called the focal axis; it will presently be shown that in the case of the circular cubic it passes through the centre, and that in the case of the bicircular quartic it not only passes through the centre, but coincides with one or other of the nodal axes, viz., with that passing through the real or the imaginary nodo-foci; that is, the curve may have on the focal axis two real or else two imaginary nodo-foci. The focal axis contains, as has been mentioned, four foci—the remaining twelve foci are situate symmetrically, six on each side of the focal axis, the arrangement of the sixteen foci being as mentioned *ante*, No. 81 *et seq.*; the focal axis is in fact an axis of symmetry of the curve, and if preferred it may be named the axis of symmetry, transverse axis, or simply the axis. And the curve (circular cubic, or bicircular quartic) is in this case a "symmetrical" or "axial" curve.

Article Nos. 131 to 140. *Circular Cubic and Bicircular Quartic: Singular Forms.*

131. The circular cubic may have a node or a cusp. If this were at one of the points I, J the curve would be imaginary, and I do not attend to the case; and for the same reason, for the bicircular quartic I do not attend to the case where *one* of

the points I, J is a cusp. There remain then for the circular cubic and for the bicircular quartic the cases where there is a node or a cusp at a real point of the curve; and for the bicircular quartic the case where each of the points I, J is a cusp—in general the curve has no other node or cusp, but it may besides have a node or cusp at a real point thereof.

132. I consider first the case of the bicircular quartic where each of the points I, J is a cusp. The curve is in this case of necessity symmetrical[1]—it is in fact a Cartesian; viz., the Cartesian may be taken by definition to be a quartic curve having a cusp at each of the circular points at infinity. But in this case, as distinguished from the general case of the bicircular quartic, there is an essential degeneration of all the focal properties, and it is necessary to explain what these become. The centre is evidently the intersection of the cuspidal tangents; the nodofoci (so far as they can be said to exist) coalesce with the centre, and they do not in so coalescing determine any definite directions for the nodal axes; that is, there are no nodal axes, and the only theorem in regard to the focal axis or axis of symmetry is, that it passes through the centre. Of the four tangents through the point I, one has come to coincide with the line IJ; and similarly, of the four tangents through the point J one has come to coincide with the line JI: there remain only three tangents through I and three tangents through J, and these by their intersections determine nine foci—viz., three foci A, B, C on the axis, and besides (B_1, C_1) the antipoints of (B, C): (C_2, A_2) the antipoints of (C, A) and (A_3, B_3) the antipoints of (A, B).

133. The remaining seven foci have disappeared, viz., we may consider that one of them has gone off to infinity on the focal axis, and that three pairs of foci have come to coincide with the points I, J respectively. The circle O (as in the general case of a symmetrical quartic) has become a line, the focal axis; the circles R, S, T (contrary to what might at first sight appear) continue to be determinate circles, viz., these have their centres at A, B, C respectively, and pass through the points (B_1, C_1), (C_2, A_2), and (A_3, B_3) respectively, see *ante*, No. 83. But on each of these circles we have not more than two proper foci, and it is only on the axis as representing the circle O that we have three proper foci, the axial foci A, B, C: in regard hereto it is to be remarked that the equation of the curve can be expressed not only by means of these three foci in the form $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0$; but by means of any two of them in the form $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+K=0$, where K is a constant, or, what is the same thing (z being introduced for homogeneity in the expressions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively), in the form $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+Kz^2=0$.

134. Using for the moment the expression "twisted" as opposed to symmetrical—

[1] It will appear, *post* Nos. 161—164, that if starting with three given points as the foci of a bicircular quartic, we impose the condition that the nodes at I, J shall be each of them a cusp, then either the quartic will be the circle through the three points taken twice, in which case the assumed focal property of the given three points disappears altogether, or else the three points must be *in lineâ*, and thus the curve be symmetrical, that is, a Cartesian.

(viz., the curve is twisted when there is not any axis of symmetry, but the foci lie only on circles)—then the classification is

Circular Cubics, twisted,

" " symmetrical,

Bicircular Quartics, twisted,

" " symmetrical, $\begin{cases}\text{Ordinary,}\\ \text{Bicuspidal} = \text{Cartesian,}\end{cases}$

and each of these kinds may be general, nodal, or cuspidal—viz., for the two last mentioned kinds there may be a node or a cusp at a real point of the curve.

135. In the case of a node, say the point N; first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D we have two, suppose B and C, coinciding with N; and the sixteen foci are as follows, viz.

B, C, A, D are N, N, A, D;

B_1, C_1, A_1, D_1 " N, N, Antipoints of (A, D);

C_2, A_2, B_2, D_2 " Antipoints of (N, A), Antipoints of (N, D);

A_3, B_3, C_3, D_3 " Do. do.

viz., we have the points (A, D) each once, the node N four times, the antipoints of (A, D) once, and the antipoints of (N, A) and of (N, D), each pair twice. But properly there are only four foci, viz., the points A, D and their antipoints. The circle O subsists as in the general case, and so does the circle $R(BC, AD)$, viz., this has for centre the intersection of the line AD by the tangent at N to the circle O, and it passes through the point N, of course cutting the circle O at right angles: the circles S and T each reduce themselves each to the point N considered as an evanescent circle, or what is the same thing to the line-pair NI, NJ.

136. The case is nearly the same if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian: viz., we have on the axis the foci B, C coinciding at N, and the other two foci A, D; the sixteen foci are as above—and the circle R is determined by the proper construction as applied to the case in hand, viz., the centre R is the intersection of the axis by the radical axis of the point N (considered as an evanescent circle) and the circle on AD as diameter; that is $\overline{RN}^2 = RA \,.\, RD$. And the circles S and T reduce themselves each to the point N considered as an evanescent circle.

137. Next if we have a cusp, say the point K: first if the curve (circular cubic or bicircular quartic) be twisted—then of the four foci A, B, C, D, three, suppose A, B, C, coincide with K; and the sixteen foci are as follows, viz.,

B, C, A, D are K, K, K, D,

B_1, C_1, A_1, D_1 " K, K, Antipoints of (K, D),

C_2, A_2, B_2, D_2 " Do. do.

A_3, B_3, C_3, D_3 " Do. do.

viz., we have the point D once, the point K nine times, and the antipoints of K, D three times. But properly the point D is the only focus. The circle O is, it would appear, *any* circle through K, D, but possibly the particular circle which touches the cuspidal tangent may be a better representative of the circle O of the general case—the circles R, S, T reduce themselves each to the point K considered as an evanescent point.

138. The like is the case if the curve be symmetrical, but in the case of the bicircular quartic excluding the Cartesian; the circle O is here the axis, which is in fact the cuspidal tangent.

139. For the Cartesian, if there is a node N; then of the three foci A, B, C, two, suppose B and C, coincide with N; the nine foci are A once, N four times, and the antipoints of N, A twice: but properly the point A is the only focus. And if there be a cusp K; then all the three foci A, B, C coincide with K; and the nine foci are K nine times; but in fact there is no proper focus.

140. A circular cubic cannot have two nodes unless it break up into a line and circle; and similarly a bicircular quartic cannot have two nodes (exclusive of course of the points I, J) unless it break up into two circles; the last-mentioned case will be considered in the sequel in reference to the problem of tactions.

Article No. 141. *As to the Analytical Theory for the Circular Cubic and the Bicircular Quartic respectively.*

141. It may be remarked in regard to the analytical theory about to be given, that although the investigation is very similar for the circular cubic and for the bicircular quartic, yet the former cannot be deduced from the latter case. In fact if for the bicircular quartic, using a form somewhat more general than that which is ultimately adopted, we suppose that for the two nodes respectively $(\xi=0,\ z=0)$ and $(\eta=0,\ z=0)$, then if $l\xi+mz=0$, $l'\xi+m'z=0$, $n\eta+pz=0$, $n'\eta+p'z=0$ are the tangents at the two nodes respectively, the equation will be

$$(l\xi+mz)(l'\xi+m'z)(n\eta+pz)(n'\eta+p'z)+ez^2\xi\eta+z^2(a\xi+b\eta)+cz^4=0,$$

and if (in order to make this equation divisible by z, and the curve so to break up into the line $z=0$ and a cubic) we write $l=0$ or $n=0$, then the curve will indeed break up as required, but we shall have, not the general cubic through the two points $(\xi=0,\ z=0)$, $(\eta=0,\ z=0)$, but in each case a nodal cubic, viz., if $l=0$ there will be a node at the point $(\eta=0,\ z=0)$, and if $n=0$ a node at the point $(\xi=0,\ z=0)$.

Article Nos. 142 to 144. *Analytical Theory for the Circular Cubic.*

142. I consider then the two cases separately; and first the circular cubic. The equation may be taken to be

$$\xi\eta(p\xi+q\eta)+ez\xi\eta+z^2(a\xi+b\eta+cz)=0,$$

or, what is the same thing,

$$\xi\eta\,(p\xi+q\eta+ez)+z^2(a\xi+b\eta+cz)=0,$$

viz. $(\xi,\ \eta,\ z)$ being any coordinates whatever, this is the general equation of a cubic passing through the points $(\xi=0,\ z=0)$, $(\eta=0,\ z=0)$, and at these points touched by the lines $\xi=0$, $\eta=0$ respectively. And if $(\xi,\ \eta,\ z=1)$ be circular coordinates, then we have the general equation of a circular cubic having the lines $\xi=0$, $\eta=0$ for its asymptotes, or say the point $\xi=0$, $\eta=0$ for its centre; the equation of the remaining asymptote is evidently $p\xi+q\eta+ez=0$; to make the curve real we must have $(p,\ q)$ and $(a,\ b)$ conjugate imaginaries, e and c real.

143. Taking in any case the points I, J to be the points $\xi=0$, $z=0$ and $\eta=0$, $z=0$ respectively, for the equation of a tangent from I write $p\xi=\theta z$; then we have

$$\theta\eta\,(\theta z+q\eta+ez)+z\,(a\theta z+bp\eta+cpz)=0,$$

that is

$$z^2(a\theta+cp)+\eta z\,(\theta^2+e\theta+bp)+\eta^2.\,q\theta=0,$$

and the line will be a tangent if only

$$(\theta^2+e\theta+bp)^2-4q\theta\,(a\theta+cp)=0,$$

that is, the four tangents from I are the lines $p\xi=\theta z$, where θ is any root of this equation; similarly the four tangents from J are the lines $q\eta=\phi z$, where ϕ is any root of the equation

$$(\phi^2+e\phi+aq)^2-4p\phi\,(b\phi+cq)=0.$$

Writing the two equations under the forms

$$\left\{\begin{matrix}6,\\ 3e,\\ e^2+2bp-4aq\ ,\\ 3ebp\quad\ -6cpq,\\ 6b^2p^2,\end{matrix}\right\}(\theta,\ 1)^4=0,\qquad \left\{\begin{matrix}6,\\ 3e,\\ e^2+2aq-4bp\ ,\\ 3eaq\quad\ -6cpq,\\ 6a^2q^2,\end{matrix}\right\}(\phi,\ 1)^4=0,$$

the equations have the same invariants; viz., for the first equation the invariants are easily found to be

$$I=\quad 3\,(e^2-4bp-4aq)^2+72\,(ce-2ab)\,pq,$$

$$J=-\quad (e^2-4bp-4aq)^3-36\,(ce-2ab)\,pq\,(e^2-4bp-4aq)-216\,c^2p^2q^2,$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3\div J^2$ has therefore the same value in the two equations; that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the two pencils are homographic.

144. The two equations will be satisfied by $\theta=\phi$, if only $bp=aq$; that is, if $p=\frac{a}{k}$, $q=\frac{b}{k}$; putting for convenience $\frac{e}{k}$ in place of e, the equation of the curve is then

$$\xi\eta\,(a\xi+b\eta+ez)+kz^2\,(a\xi+b\eta+cz)=0.$$

In this case the pencils of tangents are $a\xi = k\theta z$, $b\eta = k\theta z$, where θ is determined by a quartic equation, or taking the corresponding lines (which by their intersections determine the foci A, B, C, D) to be $(a\xi = k\theta_1 z,\ b\eta = k\theta_1 z)$, &c., these four points lie in the line $a\xi - b\eta = 0$, which is a line through the centre of the curve, or point $\xi = 0$, $\eta = 0$: the formulæ just obtained belong therefore to the symmetrical case of the circular cubic. Passing to rectangular coordinates, writing $z = 1$, and taking $y = 0$ for the equation of the axis, it is easy to see that the equation may be written

$$(x^2 + y^2)(x - a) + k(x - b) = 0;$$

or, changing the origin and constants,

$$xy^2 + (x - a)(x - b)(x - c) = 0.$$

Article Nos. 145 to 149. *Analytical Theory for the Bicircular Quartic.*

145. The equation for the bicircular quartic may be taken to be

$$k(\xi^2 - \alpha^2 z^2)(\eta^2 - \beta^2 z^2) + ez^2\xi\eta + z^2(a\xi + b\eta) + cz^4 = 0,$$

viz. (ξ, η, z) being any coordinates whatever, this is the equation of a quartic curve having a node at each of the points $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$: the equations of the two tangents at the one node are $\xi - \alpha z = 0$, $\xi + \alpha z = 0$; and those of the two tangents at the other node are $\eta - \beta z = 0$, $\eta + \beta z = 0$; $\xi = 0$ is thus the harmonic of the line $z = 0$ in regard to the tangents at $(\xi = 0,\ z = 0)$, and $\eta = 0$ is the harmonic of the same line $z = 0$ in regard to the tangents at $(\eta = 0,\ z = 0)$. If $(\xi, \eta, z = 1)$ be circular coordinates, then we have the general equation of the bicircular quartic having the lines $\xi + \alpha z = 0$, $\xi - \alpha z = 0$ for one pair, and the lines $\eta - \beta z = 0$, $\eta + \beta z = 0$ for the other pair of parallel asymptotes; and therefore the point $\xi = 0$, $\eta = 0$ for centre, and the lines $\beta\xi - \alpha\eta = 0$, $\beta\xi + \alpha\eta = 0$ for nodal axes. In order that the curve may be real we must have (α, β), (a, b) conjugate imaginaries, k, e, c real. The points $(\xi = 0,\ z = 0)$ and $(\eta = 0,\ z = 0)$ are as before the points I, J. If $\alpha = 0$, the node at I becomes a cusp, and so if $\beta = 0$, the node at J becomes a cusp; the form thus includes the case of a bicuspidal or Cartesian curve.

146. To find the tangents from I, writing in the equation of the curve $\xi = \theta\alpha z$ we have

$$k\alpha^2(\theta^2 - 1)(\eta^2 - \beta^2 z^2) + e\alpha\theta\eta z + z(a\alpha\theta z + b\eta) + cz^2 = 0;$$

that is

$$\begin{aligned} &\eta^2 \,.\, k\alpha^2(\theta^2 - 1), \\ &+ \eta z \,.\, e\alpha\theta + b, \\ &+ z^2 \,.\, - k\alpha^2\beta^2(\theta^2 - 1) + a\alpha\theta + c = 0, \end{aligned}$$

and the condition of tangency is

$$4k(\theta^2 - 1)\{k\alpha^2\beta^2(\theta^2 - 1) - a\alpha\theta - c\} + \left(e\theta + \frac{b}{\alpha}\right)^2 = 0;$$

viz., the tangents from I are $\xi=\theta\alpha z$, where θ is any root of this equation. Similarly, if we have

$$4k(\phi^2-1)\{k\alpha^2\beta^2(\phi^2-1)-b\beta\phi-c\}+\left(e\phi+\frac{a}{\beta}\right)^2=0,$$

the tangents from J are $\eta=\phi\beta z$, where ϕ is any root of this equation.

147. The two equations may be written

$$\left\{\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6ka\alpha, \\ -\ 8k^2\alpha^2\beta^2-\ 4kc+e^2, \\ 6a\alpha \qquad\qquad +3e\dfrac{b}{\alpha}, \\ 24k^2\alpha^2\beta^2+24kc+6\dfrac{b^2}{\alpha^2} \end{array}\right\}(\theta,\ 1)^4=0, \qquad \left\{\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6kb\beta, \\ -\ 8k^2\alpha^2\beta^2-\ 4kc+e^2, \\ 6kb\beta \qquad\qquad +3e\dfrac{a}{\beta}, \\ 24k^2\alpha^2\beta^2+24kc+6\dfrac{a^2}{\beta^2} \end{array}\right\}(\phi,\ 1)^4=0,$$

which equations have the same invariants; in fact for the first equation the invariants are found to be as follows, viz., if for shortness $C=-8k^2\alpha^2\beta^2-4kc+e^2$, then

$$I=576k^4\alpha^4\beta^4+576k^3c\alpha^2\beta^2+144k^2(a^2\alpha^2+b^2\beta^2)+72kab+3C^2,$$

$$J=C\{576k^4\alpha^4\beta^4+576k^3c\alpha^2\beta^2+144k^2(a^2\alpha^2+b^2\beta^2)+36ke\alpha\beta-C^2\}$$
$$-864k^3eab\alpha^2\beta^2-216k^2e^2(a^2\alpha^2+b^2\beta^2)-216k^2a^2b^2,$$

and then by symmetry the other equation has the same invariants. The absolute invariant $I^3\div J^2$ has thus the same value in the two equations, that is, the equations are linearly transformable the one into the other, which is the before-mentioned theorem that the pencils are homographic.

148. The equations will be satisfied by $\theta=\phi$ if only $a\alpha=b\beta$, that is, if $a,\ b=m\beta,\ m\alpha$; or by $\theta=-\phi$ if only $a\alpha=-b\beta$, that is, if $a,\ b=m\beta,\ -m\alpha$: the equation of the curve is in these two cases respectively

$$k(\xi^2-\alpha^2z^2)(\eta^2-\beta^2z^2)+ez^2\xi\eta+mz^2(\beta\xi+\alpha\eta)+cz^4=0,$$

$$k(\xi^2-\alpha^2z^2)(\eta^2-\beta^2z^2)+ez^2\xi\eta+mz^2(\beta\xi-\alpha\eta)+cz^4=0.$$

If to fix the ideas we attend to the first case, then the equation in θ is

$$\left\{\begin{array}{l} 24k^2\alpha^2\beta^2, \\ -\ 6km\alpha\beta, \\ -\ 8k^2\alpha^2\beta^2-\ 4kc+e^2, \\ 6km\alpha\beta \qquad\quad +3me, \\ 24k^2\alpha^2\beta^2+24kc+6m^2 \end{array}\right\}(\theta,\ 1)^4=0;$$

and we may take as corresponding tangents through the two nodes respectively $\xi=\theta\alpha z$, $\eta=\theta\beta z$; the foci A, B, C, D, which are the intersections of the pairs of lines ($\xi=\theta_1\alpha z$,

$\eta = \theta_1\beta z$), &c., lie, it is clear, in the line $\beta\xi - \alpha\eta = 0$, which is one of the nodal axes of the curve. Similarly, in the second case, if θ be determined by the foregoing equation, we may take as corresponding tangents through the two nodes respectively $\xi = \theta\alpha z$, $\eta = -\theta\beta z$; the foci (A, B, C, D), which are the intersections of the pairs of lines ($\xi = \theta_1\alpha z$, $\eta = -\theta_1\beta z$), &c., lie in the line $\beta\xi + \alpha\eta = 0$, which is the other of the nodal axes of the curve. In either case the foci A, B, C, D lie in a line, that is, we have the curve symmetrical; and, as we have just seen, the focal axis, or axis of symmetry, is one or other of the nodal axes.

149. In the case of the Cartesian, or when $\alpha = 0$, $\beta = 0$, viz., the equation $a\alpha = b\beta$ is satisfied identically, and this seems to show that the Cartesian is symmetrical; it is to be observed, however, that for $\alpha = 0$, $\beta = 0$ the foregoing formulæ fail, and it is proper to repeat the investigation for the special case in question. Writing $\alpha = 0$, $\beta = 0$, the equation of the curve is

$$k\xi^2\eta^2 + ez^2\xi\eta + z^3(a\xi + b\eta) + cz^4 = 0,$$

and then, taking $\xi = \theta bz$ for the equation of the tangent from I, we have

$$\begin{aligned} &\eta^2 \,.\, kb^2\theta^2 \\ &+ \eta z \,.\, b(e\theta + 1) \\ &+ z^2 \,.\, ab\theta + c = 0, \end{aligned}$$

and the condition of tangency is

$$4k\theta^2(ab\theta + c) - (e\theta + 1)^2 = 0;$$

viz., we have here a cubic equation. Similarly, if we have $\eta = \theta az$ for the equation of a tangent from J, then

$$4k\phi^2(ab\phi + c) - (e\phi + 1)^2 = 0.$$

Hence θ being determined by the cubic equation as above, we may take $\phi = \theta$, and consequently the equations of the corresponding tangents will be $\xi = \theta bz$, $\eta = \theta az$, viz., the foci A, B, C will be given as the intersections of the pairs of lines ($\xi = \theta_1 bz$, $\eta = \theta_1 az$), &c. The foci lie therefore in the line $a\xi - b\eta = 0$; or the curve is symmetrical, the focal axis, or axis of symmetry, passing through the centre.

Article Nos. 150 to 158. *On the Property that the Points of Contact of the Tangents from a Pair of Concyclic Foci lie in a Circle.*

150. We have seen that the sixteen foci form four concyclic sets (A, B, C, D), (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), (A_3, B_3, C_3, D_3), that is, A, B, C, D are in a circle. We may, if we please, say that any one focus is concyclic—viz., it lies in a circle with three other foci; but any two foci taken at random are not concyclic; it is only a pair such as (A, B) taken out of a set of four concyclic foci which are concyclic, viz., there exist two other foci lying with them in a circle. The number of such pairs is, it is clear $= 24$. Let A, B be any two concyclic foci, I say that the points of contact of the tangents AI, AJ, BI, BJ, lie in a circle.

151. Consider the case of the bicircular quartic, and take as before $(\xi=0,\ z=0)$, and $(\eta=0,\ z=0)$ for the coordinates of the points I, J respectively. Let the two tangents from the focus A be $\xi-\alpha z=0$, $\eta-\alpha' z=0$, say for shortness $p=0$, $p'=0$, then the equation of the curve is expressible in the form $pp'U=V^2$ [1], where $U=0$, $V=0$ are each of them a circle, viz., U and V are each of them a quadric function containing the terms z^2, $z\eta$, $z\xi$, and $\xi\eta$. Taking an indeterminate coefficient λ, the equation may be written

$$pp'(U+2\lambda V+\lambda^2 pp')=(V+\lambda pp')^2,$$

and then λ may be so determined that $U+2\lambda V+\lambda^2 pp'=0$, shall be a 0-circle, or pair of lines through I and J. It is easy to see that we have thus for λ a cubic equation, that is, there are three values of λ, for each of which the function $U+2\lambda V+\lambda^2 pp'$ assumes the form $(\xi-\beta z)(\eta-\beta' z)$, $=qq'$ suppose: taking any one of these, and changing the value of V so as that we may have V in place of $V+\lambda pp'$, the equation is $pp'qq'+V^2$, where $V=0$ is as before a circle, the equation shows that the points of contact of the tangents $p=0$, $p'=0$, $q=0$, $q'=0$ lie in this circle $V=0$. The circumstance that λ is determined by a cubic equation would suggest that the focus $q=0$, $q'=0$ is one of the three foci B, C, D concyclic with A; but this is the very thing which we wish to prove, and the investigation, though somewhat long, is an interesting one.

152. Starting from the form $pp'qq'=V^2$, then introducing as before an arbitrary coefficient λ, the equation may be written

$$pp'(qq'+2\lambda V+\lambda^2 pp')=(V+\lambda pp')^2,$$

and we may determine λ so that $qq'+2\lambda V+\lambda^2 pp'=0$ shall be a pair of lines. Writing $V=H\xi\eta-L\eta z-L'\xi z+Mz^2$, and substituting for pp' and qq' their values $(\xi-\alpha z)(\eta-\alpha' z)$ and $(\xi-\beta z)(\eta-\beta' z)$, the equation in question is

$$(1+2\lambda H+\lambda^2)\,\xi\eta-(\beta+2\lambda L+\lambda^2\alpha)\,\eta z-(\beta'+2\lambda L'+\lambda^2\alpha')\,\xi z+(\beta\beta'+2\lambda M+\lambda^2\alpha\alpha')\,z^2=0,$$

and the required condition is

$$(1+2\lambda H+\lambda^2)(\beta\beta'+2\lambda M+\lambda^2\alpha\alpha')=(\beta+2\lambda L+\lambda^2\alpha)(\beta'+2\lambda L'+\lambda^2\alpha');$$

or reducing, this is

$$\begin{aligned}&(2M+2H\beta\beta'-2L'\beta-2L\beta)\\&+\lambda\left((\alpha-\beta)(\alpha'-\beta')+4HM-4LL'\right)\\&+\lambda^2(2M+2H\alpha\alpha'-2L'\alpha-2L\alpha')=0,\end{aligned}$$

viz., λ is determined by a quadric equation. Calling its roots λ_1, and λ_2, the foregoing equation, substituting therein successively these values, becomes $(\xi-\gamma z)(\eta-\gamma' z)=0$, and $(\xi-\delta z)(\eta-\delta' z)=0$ respectively, say $rr'=0$ and $ss'=0$.

[1] This investigation is similar to that in Salmon's *Higher Plane Curves*, p. 196, in regard to the double tangents of a quartic curve.

153. We have to show that the four foci $(p=0,\ p'=0)$, $(q=0,\ q'=0)$, $(r=0,\ r'=0)$, $(s=0,\ s'=0)$ are a set of concyclic foci; that is, that the lines $p=0$, $q=0$, $r=0$, $s=0$ correspond homographically to the lines $p'=0$, $q'=0$, $r'=0$, $s'=0$; or, what is the same thing, that we have

$$\begin{vmatrix} 1, & \alpha, & \alpha', & \alpha\alpha' \\ 1, & \beta, & \beta', & \beta\beta' \\ 1, & \gamma, & \gamma', & \gamma\gamma' \\ 1, & \delta, & \delta', & \delta\delta' \end{vmatrix} = 0,$$

or, as it will be convenient to write this equation,

$$\frac{\alpha-\beta}{\alpha'-\beta'}\,\frac{\gamma-\delta}{\gamma'-\delta'} = \frac{\alpha-\delta}{\alpha'-\delta'}\,\frac{\beta-\gamma}{\beta'-\gamma'}.$$

154. We have

$$\gamma = \frac{\beta+2\lambda_1 L+\lambda_1^2\alpha}{1+2H\lambda_1+\lambda_1^2},\quad \gamma' = \frac{\beta'+2\lambda_1 L'+\lambda_1^2\alpha'}{1+2H\lambda_1+\lambda_1^2},$$

$$\delta = \frac{\beta+2\lambda_2 L+\lambda_2^2\alpha}{1+2H\lambda_2+\lambda_2^2},\quad \delta' = \frac{\beta'+2\lambda_2 L'+\lambda_2^2\alpha'}{1+2H\lambda_2+\lambda_2^2}.$$

The expressions of $\alpha-\delta$, &c., are severally fractions, the denominators of which disappear from the equation; the numerators are

$$\begin{aligned}
\text{for } \alpha-\delta, &= \alpha(1+2\lambda_2 H+\lambda_2^2)-(\beta+2\lambda_2 L+\alpha\lambda_2^2),\\
&= \alpha-\beta+2\lambda_2(\alpha H-L);\\
\text{for } \beta-\gamma, &= \beta(1+2\lambda_1 H+\lambda_1^2)-(\beta+2\lambda_1 L+\alpha\lambda_1^2),\\
&= \lambda_1\{2(\beta H-L)(\alpha-\beta)\};\\
\text{for } \gamma-\delta, &= (\beta+2L\lambda_1+\alpha\lambda_1^2)(1+2H\lambda_2+\lambda_2^2)\\
&\quad -(\beta+2L\lambda_2+\alpha\lambda_2^2)(1+2H\lambda_1+\lambda_1^2),\\
&= (\alpha'-\beta')\{2H^2\alpha\beta-2HL(\alpha+\beta)+2L^2+\tfrac{1}{2}(\alpha-\beta)^2\};
\end{aligned}$$

and it hence easily appears that the equation to be verified is

$$\frac{2H^2\alpha\beta-2HL(\alpha+\beta)+2L^2+\frac{1}{2}(\alpha-\beta)^2}{2H^2\alpha'\beta'-2HL'(\alpha'+\beta')+2L'^2+\frac{1}{2}(\alpha'-\beta')^2} = \frac{\alpha-\beta+2(\alpha H-L)\lambda_2}{\alpha'-\beta'+2(\alpha' H-L')\lambda_2}\cdot\frac{2(\beta H-L)-(\alpha-\beta)\lambda_1}{2(\beta' H-L')-(\alpha'-\beta')\lambda_2}.$$

155. This is

$$\frac{B-C}{B'-C'} = \frac{A+B\lambda_1+C\lambda_2+D\lambda_1\lambda_2}{A'+B'\lambda_1+C'\lambda_2+D'\lambda_1\lambda_2},$$

if for shortness

$$\begin{aligned}
A &= 2(\alpha-\beta)(\beta H-L), & A' &= 2(\alpha'-\beta')(\beta' H-L'),\\
B &= -(\alpha-\beta)^2, & B' &= -(\alpha'-\beta')^2,\\
C &= 4(\alpha H-L)(\beta H-L), & C' &= 4(\alpha' H-L')(\beta' H-L'),\\
D &= -2(\alpha-\beta)(\alpha H-L), & D' &= -2(\alpha'-\beta')(\alpha' H-L'),
\end{aligned}$$

and the equation then is

$$AB'-A'B+CA'-C'A-(\lambda_1+\lambda_2)(BC'-B'C)+\lambda_1\lambda_2\big(CD'-C'D-(BD'-B'D)\big).$$

156. Calculating $AB'-A'B$, $CA'-C'A$, $CD'-C'D$, $BD'-B'D$, these are at once seen to divide by $\{(\alpha\beta'-\alpha'\beta)H+L(\alpha'-\beta')-L'(\alpha-\beta)\}$; we have, moreover,

$$\begin{aligned}BC'-B'C &= -4(\alpha-\beta)^2(\alpha'H-L')(\beta'H-L')+4(\alpha'-\beta')^2(\alpha H-L)(\beta H-L),\\ &= -\{(\alpha\alpha'-\beta\beta')H-L(\alpha'-\beta')-L'(\alpha-\beta)\}\{\alpha\beta'-\alpha'\beta)H+L(\alpha'-\beta')-L'(\alpha-\beta)\},\end{aligned}$$

viz., this also contains the same factor; and omitting it, the equation is found to be

$$\begin{aligned}&\{(\alpha-\beta)(\alpha'-\beta')-4(\beta H-L)(\beta'H-L')\}\\ -2&\{(\alpha\alpha'-\beta\beta')H-L(\alpha'-\beta')-L'(\alpha-\beta)\}(\lambda_1+\lambda_2)\\ +&\{-(\alpha-\beta)(\alpha'-\beta')+4(\alpha H-L)(\alpha'H-L')\}\lambda_1\lambda_2 \quad =0;\end{aligned}$$

viz., substituting for $\lambda_1+\lambda_2$ and $\lambda_1\lambda_2$ their values, this is

$$\begin{aligned}&\{(\alpha-\beta)(\alpha'-\beta')-4(\beta H-L)(\beta'H-L')\}(M+H\alpha\alpha'-L\alpha'-L'\alpha)\\ -&\{(\alpha\alpha'-\beta\beta')H-L(\alpha'-\beta')\}\{(\alpha-\beta)(\alpha'-\beta')+4HM-4LL'\}\\ +&\{-(\alpha-\beta)(\alpha'-\beta')+4(\alpha H-L)(\alpha'H-L')\}\{M+H\beta\beta'-L\beta'-L'\beta\}=0,\end{aligned}$$

which should be identically true. Multiplying by H, and writing in the form

$$\begin{aligned}&\{(\alpha-\beta)(\alpha'-\beta')-4(\beta H-L)(\beta'H-L')\}\big(HM-LL'+(\alpha H-L)(\alpha'H-L')\big)\\ -&\{(\alpha H-L)(\alpha'H-L')-(\beta H-L)(\beta'H-L')\}\big((\alpha-\beta)(\alpha'-\beta')+4(HM-LL')\big)\\ +&\{-(\alpha-\beta)(\alpha'-\beta')+4(\alpha H-L)(\alpha'H-L')\}\big(HM-LL'+(\beta H-L)(\beta'H-L')\big)=0,\end{aligned}$$

we at once see that this is so, and the theorem is thus proved, viz., that the equation being $pp'qq'=V^2$, the foci $(p=0,\ p'=0)$ and $(q=0,\ q'=0)$ are concyclic.

157. By what precedes, λ being a root of the foregoing quadric equation, we may write

$$qq'+2\lambda V+\lambda^2pp'=K^2rr',$$

where the focus $r=0$, $r'=0$ is concyclic with the other two foci; but from the equation of the curve $V=\sqrt{pp'qq'}$, that is we have

$$qq'+2\lambda\sqrt{pp'qq'}+\lambda^2pp'=Krr',$$

or, what is the same thing,

$$\lambda\sqrt{pp'}+\sqrt{qq'}+K\sqrt{rr'}=0,$$

viz., this is a form of the equation of the curve; substituting for p, p', q, q', r, r' their values, writing also

$$\begin{aligned}\mathfrak{A}&=(\xi-\alpha z)(\eta-\alpha' z),\\ \mathfrak{B}&=(\xi-\beta z)(\eta-\beta' z),\\ \mathfrak{C}&=(\xi-\gamma z)(\eta-\gamma' z),\end{aligned}$$

and changing the constants λ, K (viz. $\lambda:1:K=\sqrt{l}:\sqrt{m}:\sqrt{n}$) the equation is

$$\sqrt{l}\mathfrak{A}+\sqrt{m}\mathfrak{B}+\sqrt{n}\mathfrak{C}=0,$$

viz., we have the theorem that for a bicircular quartic if $(\xi - \alpha z = 0,\ \eta - \alpha' z = 0)$, $(\xi - \beta z = 0,\ \eta - \beta' z = 0,\ (\xi - \gamma z = 0),\ \eta - \gamma' z = 0)$ be any three concyclic foci, then the equation is as just mentioned; that is, the curve is a trizomal curve, the zomals being the three given foci regarded as 0-circles. The same theorem holds in regard to the circular cubic, and a similar demonstration would apply to this case.

158. It may be noticed that we might, without proving as above that the two foci $(p=0,\ p'=0)$, $(q=0,\ q'=0)$ were concyclic, have passed at once from the form $pp'qq' = V^2$, to the form $\lambda\sqrt{pp'} + \sqrt{qq'} + K\sqrt{rr'} = 0$ (or $\sqrt{l\mathfrak{A}} = \sqrt{m\mathfrak{B}} = \sqrt{n\mathfrak{C}} = 0$), and then by the application of the theorem of the variable zomal (thereby establishing the existence of a fourth focus concyclic with the three) have shown that the original two foci were concyclic. But it seemed the more orderly course to effect the demonstration without the aid furnished by the reduction of the equation to the trizomal form.

PART IV. (Nos. 159 to 206). ON TRIZOMAL AND TETRAZOMAL CURVES WHERE THE ZOMALS ARE CIRCLES.

Article Nos. 159 to 165. *The Trizomal Curve—The Tangents at I, J, &c.*

159. I consider the trizomal

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

where A, B, C being the centres of three given circles, $\mathfrak{A}^\circ$, &c. denote as before, viz., in rectangular and in circular coordinates respectively, we have

$$\begin{aligned}
\mathfrak{A}^\circ &= (x - az)^2 + (y - a'z)^2 - a''^2 z^2, &&= (\xi - \alpha z)(\eta - \alpha' z) - a''^2 z^2,\\
\mathfrak{B}^\circ &= (x - bz)^2 + (y - b'z)^2 - b''^2 z^2, &&= (\xi - \beta z)(\eta - \beta' z) - b''^2 z^2,\\
\mathfrak{C}^\circ &= (x - cz)^2 + (y - c'z)^2 - c''^2 z^2, &&= (\xi - \gamma z)(\eta - \gamma' z) - c''^2 z^2.
\end{aligned}$$

By what precedes, the curve is of the order $=4$, touching each of the given circles twice, and having a double point, or node, at each of the points I, J; that is, it is a bicircular quartic: but if for any determinate values of the radicals $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, we have

$$\sqrt{l} + \sqrt{m} + \sqrt{n} = 0,$$

then there is a branch

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

containing $(z=0)$ the line infinity; and the order is here $=3$: viz., the curve here passes through each of the points I, J and through another point at infinity (that is, there is an asymptote), and is thus a circular cubic.

160. I commence by investigating the equations of the nodal tangents at the points I, J respectively; using for this purpose the circular coordinates $(\xi,\ \eta,\ z=1)$, it is to be observed that, in the rationalised equation, for finding the tangents at $(\xi=0,\ z=0)$ we have only to attend to the terms of the second order in $(\xi,\ z)$, and

similarly for finding the tangents at $(\eta=0,\ z=0)$ we have only to attend to the terms of the second order in $(\eta,\ z)$. But it is easy to see that any term involving a'', b'', or c'' will be of the third order at least in $(\xi,\ z)$, and similarly of the third order at least in $(\eta,\ z)$; hence for finding the tangents we may reject the terms in question, or, what is the same thing, we may write a'', b'', c'' each $=0$, thus reducing the three circles to their respective centres. The equation thus becomes

$$\sqrt{l(\xi-\alpha z)(\eta-\alpha' z)}+\sqrt{m(\xi-\beta z)(\eta-\beta' z)}+\sqrt{n(\xi-\gamma z)(\eta-\gamma' z)}=0.$$

For finding the tangents at $(\xi=0,\ z=0)$ we have in the rationalised equation to attend only to the terms of the second order in $(\xi,\ z)$; and it is easy to see that any term involving α', β', γ' will be of the third order at least in $(\xi,\ z)$, that is, we may reduce α', β', γ' each to zero; the irrational equation then becomes divisible by $\sqrt{\eta}$, and throwing out this factor, it is

$$\sqrt{l(\xi-\alpha z)}+\sqrt{m(\xi-\beta z)}+\sqrt{n(\xi-\gamma z)}=0,$$

viz., this equation which evidently belongs to a pair of lines passing through the point $(\xi=0,\ z=0)$ gives the tangents at the point in question; and similarly the tangents at the point $(\eta=0,\ z=0)$ are given by the equation

$$\sqrt{l(\eta-\alpha' z)}+\sqrt{m(\eta-\beta' z)}+\sqrt{n(\eta-\gamma' z)}=0.$$

161. To complete the solution, attending to the tangents at $(\xi=0,\ z=0)$, and putting for shortness

$$\begin{aligned}\lambda &= l-m-n,\\ \mu &= -l+m-n,\\ \nu &= -l-m+n,\\ \Delta &= l^2+m^2+n^2-2mn-2nl-2lm,\end{aligned}$$

the rationalised equation is easily found to be

$$\begin{aligned}&\xi^2.\Delta\\ &-2\xi z(l\lambda\alpha+m\mu\beta+n\nu\gamma)\\ &+\ z^2(l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta)=0;\end{aligned}$$

and it is to be noticed that in the case of the circular cubic or when $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, then $\Delta=0$, so that the equation contains the factor z, and throwing this out, the equation gives a single line, which is in fact the tangent of the circular cubic.

162. Returning to the bicircular quartic, we may seek for the condition in order that the node may be a cusp: the required condition is obviously

$$\Delta(l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta)-(l\lambda\alpha+m\mu\beta+n\nu\gamma)^2=0,$$

or observing that

$$\Delta-\lambda^2=-4mn,\ \&c.$$

$$\Delta+\mu\nu=-2l\lambda,\ \&c.$$

this is

$$l\alpha^2+m\beta^2+n\gamma^2+\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta=0,$$

or substituting for λ, μ, ν, their values, it is

$$l(\alpha-\beta)(\alpha-\gamma)+m(\beta-\gamma)(\beta-\alpha)+n(\gamma-\alpha)(\gamma-\beta)=0,$$

or, as it is more simply written,

$$\frac{l}{\beta-\gamma}+\frac{m}{\gamma-\alpha}+\frac{n}{\alpha-\beta}=0.$$

163. If the node at $(\eta=0,\ z=0)$ be also a cusp, then we have in like manner

$$\frac{l}{\beta'-\gamma'}+\frac{m}{\gamma'-\alpha'}+\frac{n}{\alpha'-\beta'}=0.$$

Now observing that

$$(\gamma-\alpha)(\alpha'-\beta')-(\gamma'-\alpha')(\alpha-\beta),=\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix}$$

$$=(\alpha-\beta)(\beta'-\gamma')-(\alpha'-\beta')(\beta-\gamma),$$
$$=(\beta-\gamma)(\gamma'-\alpha')-(\beta'-\gamma')(\gamma-\alpha),$$

$=\Omega$ suppose: the two equations give

$$l:m:n=\Omega(\beta-\gamma)(\beta'-\gamma'):\Omega(\gamma-\alpha)(\gamma'-\alpha'):\Omega(\alpha-\beta)(\alpha'-\beta');$$

or if Ω is not $=0$, then

$$l:m:n=\ (\beta-\gamma)(\beta'-\gamma'):\quad(\gamma-\alpha)(\gamma'-\alpha'):\quad(\alpha-\beta)(\alpha'-\beta').$$

164. If

$$\Omega=\begin{vmatrix}\alpha, & \alpha', & 1\\ \beta, & \beta', & 1\\ \gamma, & \gamma', & 1\end{vmatrix},=0,$$

or, what is the same thing, if

$$\begin{vmatrix}a, & a', & 1\\ b, & b', & 1\\ c, & c', & 1\end{vmatrix}=0,$$

the centres A, B, C are in a line; taking it as the axis of x, we have $\alpha=\alpha'=a$, $\beta=\beta'=b$, $\gamma=\gamma'=c$; and the conditions for the cusps at I, J respectively reduce themselves to the single condition

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

so that this condition being satisfied, the curve

$$\sqrt{l\{(x-az)^2+y^2-a''^2z^2\}}+\sqrt{m\{(x-bz)^2+y^2-b''^2z^2\}}+\sqrt{n\{(x-cz)^2+y^2-c''^2z^2\}}=0$$

is a Cartesian; viz., given any three circles with their centres on a line, there are a singly infinite series of Cartesians, each touched by the three circles respectively;

the line of centres is the axis of the curve, but the centres A, B, C are not the foci, except in the case $a''=0$, $b''=0$, $c''=0$, where the circles vanish. The condition for l, m, n is satisfied if $l:m:n=(b-c)^2:(c-a)^2:(a-b)^2$; these values, writing $\sqrt{l}:\sqrt{m}:\sqrt{n}=b-c:c-a:a-b$, give not only $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, but also $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}=0$; these are the conditions for a branch containing $(z^2=0)$ the line infinity twice; the equation

$$(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2}+(c-a)\sqrt{(x-bz)^2+y^2-b''^2z^2}+(a-b)\sqrt{(x-cz)^2+y^2-c''^2z^2}=0,$$

is thus that of a conic, and if $a''=0$, $b''=0$, $c''=0$, then the curve reduces itself to $y^2=0$, the axis twice.

165. If Ω is not $=0$, then we have

$$l:m:n=(\beta-\gamma)(\beta'-\gamma'):(\gamma-\alpha)(\gamma'-\alpha'):(\alpha-\beta)(\alpha'-\beta'),$$

viz., l, m, n are as the squared distances $\overline{BC}^2$, $\overline{CA}^2$, $\overline{AB}^2$, say as $f^2:g^2:h^2$; or when the centres of the given circles A, B, C are not in a line, then f, g, h being the distances BC, CA, AB of these centres from each other, we have, touching each of the given circles twice, the *single* Cartesian

$$f\sqrt{\mathfrak{A}^\circ}+g\sqrt{\mathfrak{B}^\circ}+h\sqrt{\mathfrak{C}^\circ}=0,$$

which, in the particular case where the radii a'', b'', c'' are each $=0$, becomes

$$f\sqrt{\mathfrak{A}}+g\sqrt{\mathfrak{B}}+h\sqrt{\mathfrak{C}}=0,$$

viz., this is the circle through the points A, B, C, say the circle ABC, twice.

Article Nos. 166 to 169. *Investigation of the Foci of a Conic represented by an Equation in Areal Coordinates.*

166. I premise as follows: Let A, B, C be any given points, and in regard to the triangle ABC let the *areal* coordinates of a current point P be u, v, w; that is, writing PBC, &c., for the areas of these triangles, take the coordinates to be

$$u:v:w=PBC:PCA:PAB,$$

or, what is the same thing in the rectangular coordinates $(x, y, z=1)$, if

$$(a, a', 1),\ (b, b', 1),\ (c, c', 1),$$

be the coordinates of A, B, C respectively, take

$$u:v:w=\begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix}:\begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix}:\begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix},$$

or in the circular coordinates $(\xi, \eta, z=1)$, if $(\alpha, \alpha', 1)$, $(\beta, \beta', 1)$, $(\gamma, \gamma', 1)$ be the coordinates of the three points respectively, then

$$u : v : w = \begin{vmatrix} \xi, & \eta, & z \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \gamma, & \gamma', & 1 \\ \alpha, & \alpha', & 1 \end{vmatrix} : \begin{vmatrix} \xi, & \eta, & z \\ \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \end{vmatrix}.$$

167. For the point I we have $(\xi, \eta, z)=(0, 1, 0)$, and hence if its areal coordinates be (u_0, v_0, w_0), we have

$$u_0 : v_0 : w_0 = \beta-\gamma : \gamma-\alpha : \alpha-\beta,$$

and hence also, (u, v, w) referring to the current point P, we find

$$\begin{aligned} v_0w-w_0v = &(\gamma-\alpha)[(\alpha'-\beta')(\xi-\alpha z)-(\alpha-\beta)(\eta-\alpha' z)] \\ &-(\alpha-\beta)[(\gamma'-\alpha')(\xi-\alpha z)-(\gamma-\alpha)(\eta-\alpha' z)], \; = \Omega(\xi-\alpha z), \end{aligned}$$

if

$$\Omega = (\gamma-\alpha)(\alpha'-\beta')-(\alpha-\beta)(\gamma'-\alpha'), \; = \begin{vmatrix} \alpha, & \alpha', & 1 \\ \beta, & \beta', & 1 \\ \gamma, & \gamma', & 1 \end{vmatrix};$$

whence

$$v_0w-w_0v : w_0u-wu_0 : u_0v-uv_0 = \xi-\alpha z : \xi-\beta z : \xi-\gamma z,$$

and in precisely the same manner, if u_0', v_0', w_0' refer to the point J, then

$$u_0' : v_0' : w_0' = \beta'-\gamma' : \gamma'-\alpha' : \alpha'-\beta',$$

and

$$v_0'w-w_0'v : w_0'u-wu_0' : u_0'v-uv_0' = \eta-\alpha' z : \eta-\beta' z : \eta-\gamma' z.$$

168. Consider the conic

$$(a, b, c, f, g, h \between u, v, w)^2 = 0,$$

where u, v, w are any trilinear coordinates whatever; and take the inverse coefficients to be (A, B, C, F, G, H) $(A=bc-f^2$, &c.), then for any given point the coordinates of which are (u_0, v_0, w_0), the equation of the tangents from this point to the conic is, as is well known,

$$(A, B, C, F, G, H \between v_0w-w_0v, w_0u-u_0w, u_0v-v_0u)^2 = 0;$$

consequently for the conic

$$(a, b, c, f, g, h \between u, v, w)^2 = 0,$$

where (u, v, w) are areal coordinates referring, as above, to any three given points A, B, C, the equation of the pair of tangents from the point I to the conic is

$$(A, B, C, F, G, H \between \xi-\alpha z, \xi-\beta z, \xi-\gamma z)^2 = 0,$$

and that of the pair of tangents from J is

$$(A, B, C, F, G, H \between \eta-\alpha' z, \eta-\beta' z, \eta-\gamma' z)^2 = 0,$$

these two line-pairs intersecting, of course, in the foci of the conic.

169. In particular, if the conic is a conic passing through the points A, B, C, then taking its equation to be

$$lvw + mwu + nuv = 0,$$

the inverse coefficients are as $(l^2, m^2, n^2, -2mn, -2nl, -2lm)$, and we have for the equations of the two line-pairs

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0,$$

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0.$$

Article No. 170. *The Theorem of the Variable Zomal.*

170. Consider the four circles

$$\mathfrak{A}^\circ = 0,\ \mathfrak{B}^\circ = 0,\ \mathfrak{C}^\circ = 0,\ \mathfrak{D}^\circ = 0\ \left(\mathfrak{A}^\circ = (x - az)^2 + (y - a'z)^2 - a''^2 z^2, \text{ \&c.}\right),$$

which have a common orthotomic circle; so that as before

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

where

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = BCD : -CDA : DAB : -ABC.$$

I consider the first three circles as given, and the fourth circle as a variable circle cutting at right angles the orthotomic circle of the three given circles; this being so, attending only to the ratios a : b : c, we may write

$$\mathrm{a} : \mathrm{b} : \mathrm{c} = DBC : DCA : DAB,$$

that is, (a, b, c) are proportional to the areal coordinates of the centre of the variable circle in regard to the triangle ABC.

171. Suppose that the centre of the variable circle is situate on a given conic, then expressing the equation of this conic in areal coordinates in regard to the triangle ABC, we have between (a, b, c) the equation obtained by substituting these values for the coordinates in the equation of the conic; that is, the equation of the variable circle is

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ = 0,$$

where (a, b, c) are connected by an equation

$$(a, b, c, f, g, h \between \mathrm{a}, \mathrm{b}, \mathrm{c})^2 = 0.$$

Hence (A, B, C, F, G, H) being the inverse coefficients, the equation of the envelope of the variable circle is

$$(A, B, C, F, G, H \between \mathfrak{A}^\circ, \mathfrak{B}^\circ, \mathfrak{C}^\circ)^2 = 0,$$

and, in particular, if the conic be a conic passing through the points A, B, C, and such that its equation in the areal coordinates (u, v, w) in regard to the triangle ABC is

$$lvw + mwu + nuv = 0,$$

then the equation of the envelope is

$$(l^2,\ m^2,\ n^2,\ -mn,\ -nl,\ -lm \between \mathfrak{A}^\circ,\quad \mathfrak{B}^\circ,\quad \mathfrak{C}^\circ)^2 = 0;$$

that is, it is

$$(1,\ \ 1,\ 1,\ -\ \ 1,\ -\ \ 1,\ -1 \between l\mathfrak{A}^\circ,\ m\mathfrak{B}^\circ,\ n\mathfrak{C}^\circ)^2 = 0,$$

or, what is the same thing, it is

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m}\mathfrak{B}^\circ + \sqrt{n}\mathfrak{C}^\circ = 0.$$

172. It has been seen that the equations of the nodal tangents at the points I, J respectively are respectively

$$\sqrt{l(\xi - \alpha z)} + \sqrt{m(\xi - \beta z)} + \sqrt{n(\xi - \gamma z)} = 0,$$

$$\sqrt{l(\eta - \alpha' z)} + \sqrt{m(\eta - \beta' z)} + \sqrt{n(\eta - \gamma' z)} = 0,$$

and that these are the equations of the tangents to the conic $lvw + mwu + nuv = 0$ from the points I, J respectively. We have thus Casey's theorem for the generation of the bicircular quartic as follows:—The envelope of a variable circle which cuts at right angles the orthotomic circle of three given circles $\mathfrak{A}^\circ = 0$, $\mathfrak{B}^\circ = 0$, $\mathfrak{C}^\circ = 0$, and has its centre on the conic $lvw + mwu + nuv = 0$ which passes through the centres of the three given circles is the bicircular quartic, or trizomal

$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0,$$

which has its nodo-foci coincident with the foci of the conic.

173. To complete the analytical theory, it is proper to express the equation of the orthotomic circle by means of the areal coordinates (u, v, w). Writing for shortness $a^2 + a'^2 - a''^2 = a\backprime$, &c., and therefore

$$\mathfrak{A}^\circ = x^2 + y^2 - 2axz - 2a'yz - a\backprime z^2, \text{ \&c.},$$

then if as before

$$u : v : w = \begin{vmatrix} x, & y, & z \\ b, & b', & 1 \\ c, & c', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ c, & c', & 1 \\ a, & a', & 1 \end{vmatrix} : \begin{vmatrix} x, & y, & z \\ a, & a', & 1 \\ b, & b', & 1 \end{vmatrix},$$

and therefore

$$x : y : z = au + bv + cw : a'u + b'v + c'w : u + v + w,$$

the equation of the orthotomic circle is

$$\begin{vmatrix} x - az, & y - a'z, & ax + a'y - a\backprime z \\ x - bz, & y - b'z, & bx + b'y - b\backprime z \\ x - cz, & y - c'z, & cx + c'y - c\backprime z \end{vmatrix} = 0,$$

viz., throwing out the factor z, this is

$$u(ax + a'y - a\backprime z) + v(bx + b'y - b\backprime z) + w(cx + c'y - c\backprime z) = 0,$$

or, what is the same thing, it is

$$(au+bv+cw)\,x+(a'u+b'v+c'w)\,y-(a`u+b`v+c`w)\,z \qquad =0,$$

viz., it is

$$(au+bv+cw)^2 \;+(a'u+b'v+c'w)^2 \;-(a`u+b`v+c`w)(u+v+w)=0,$$

that is, substituting for $a`$, $b`$, $c`$ their values, it is

$$\begin{aligned} & a''^2u^2 \quad + b''^2v^2 \quad + c''^2w^2 \\ & +\left(b''^2+c''^2-(b-c)^2-(b'-c')^2\right)vw \\ & +\left(c''^2+a''^2-(c-a)^2-(c'-a')^2\right)wu \\ & +\left(a''^2+b''^2-(a-b)^2-(a'-b')^2\right)uv=0, \end{aligned}$$

and it may be observed that using for a moment α, β, γ to denote the angles at which the three circles taken in pairs respectively intersect, then we have $2b''c''\cos\alpha = b''^2+c''^2-(b-c)^2-(b'-c')^2$, &c., and the equation of the orthotomic circle thus is

$$(1,\ 1,\ 1,\ \cos\alpha,\ \cos\beta,\ \cos\gamma\between a''u,\ b''v,\ c''w)^2=0.$$

174. We have in the foregoing enunciation of the theorem made use of the three given circles A, B, C, but it is clear that these are in fact *any* three circles in the series of the variable circle, and that the theorem may be otherwise stated thus:

The envelope of a variable circle which has its centre in a given conic, and cuts at right angles a given circle, is a bicircular quartic, such that its nodo-foci are the foci of the conic.

Article Nos. 175 to 177. *Properties depending on the relation between the Conic and Circle.*

175. I refer to the conic of the theorem simply as the conic, and to the fixed circle simply as the circle, or when any ambiguity might otherwise arise, then as the orthotomic circle. This being so, I consider the effect in regard to the trizomal curve, of the various special relations which may exist between the circle and the conic.

If the conic touch the circle, the curve has a node at the point of contact.

If the conic has with the circle a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle lie on an axis of the conic, then the four intersections lie in pairs symmetrically in regard to this axis, or the curve has this axis as an axis of symmetry.

If the conic has double contact with the circle (this implies that the centre of the circle is situate on an axis of the conic) the curve has a node at each of the points of contact, viz., it breaks up into two circles intersecting in these two points.

The centres of the two circles respectively are the two foci of the conic, which foci lie on the axis in question. Observe that in the general case there are at each of the circular points at infinity two tangents, without any correspondence of the tangents of the one pair singly to those of the other pair, and there are thus four intersections, the four foci of the conic; in the present case, where the curve is a pair of circles, the two tangents to the same circle correspond to each other, and intersect in the two foci on the axis in question. The other two foci, or antipoints of these, are each of them the intersection of a tangent of the one circle by a tangent of the other circle.

If the conic has with the circle a contact of the third order (this implies that the circle is a circle of maximum or minimum curvature, at the extremity of an axis of the conic), then the curve has at this point a tacnode, viz., it breaks up into two circles touching each other and the conic at the point in question, and having their centres at the two foci situate on that axis of the conic respectively.

176. If the conic is a parabola, then the curve is a circular cubic having the four intersections of the parabola and circle for a set of concyclic foci, and having the focus of the parabola for *centre*. The like particular cases arise, viz.,

If the circle touch the parabola, the curve has a node at the point of contact.

If the circle has, with the parabola, a contact of the second order, the curve has a cusp at the point of contact.

If the centre of the circle is situate on the axis of the parabola, then the four intersections are situate in pairs symmetrically in regard to this axis, and the curve has this axis for an axis of symmetry.

If the circle has double contact with the parabola (which, of course, implies that the centre lies on the axis), then the curve has a node at each of the points of contact, viz., the curve breaks up into a line and circle intersecting at the two points of contact, and the circle has its centre at the focus of the parabola.

If the circle has with the parabola a contact of the third order (this implies that the circle is the circle of maximum curvature, touching the parabola at its vertex), then the curve has a tacnode, viz., it breaks up into a line and circle touching each other and the parabola at the vertex, that is, the line is the tangent to the parabola at its vertex, and the circle is the circle having the focus of the parabola for its centre, and passing through the vertex, or what is the same thing, having its radius $=\frac{1}{2}$ of the semi-latus rectum of the parabola.

177. If the conic be a circle, then the curve is a bicircular quartic such that its four nodo-foci coincide together at the centre of the circle; viz., the curve is a Cartesian having the centre of the conic for its cuspo-focus, that is, for the intersection of the cuspidal tangents of the Cartesian. The intersections of the conic with the other circle, or say with the orthotomic circle, are a pair of non-axial foci of the Cartesian; viz., the antipoints of these are two of the axial foci. The third axial focus is the centre of the orthotomic circle.

Article No. 178. *Case of Double Contact, Casey's Equation in the Problem of Tactions.*

178. In the case where the conic has double contact with the orthotomic circle, then (as we have seen) the envelope of the variable circle is a pair of circles, each touching the variable circle; or, if we start with three given circles and a conic through their centres, then the envelope is a pair of circles, each of them touching each of the three given circles; that is, we have a solution of the problem of tactions. Multiplying by 2, the equation found *ante*, No. 173, for the variable circle, and then for the moment representing it by $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between u, v, w)^2 = 0$; then attributing any signs at pleasure to the radicals $\sqrt{\mathrm{a}}$, $\sqrt{\mathrm{b}}$, $\sqrt{\mathrm{c}}$, the equation of a conic through the centres of the given circles, and having double contact with the orthotomic circle, will be

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between u, v, w)^2 - (u\sqrt{\mathrm{a}} + v\sqrt{\mathrm{b}} + w\sqrt{\mathrm{c}})^2 = 0,$$

viz., representing this equation as before by

$$lvw + mwu + nuv = 0,$$

we have

$$l : m : n = \mathrm{f} - \sqrt{\mathrm{bc}} : \mathrm{g} - \sqrt{\mathrm{ca}} : \mathrm{h} - \sqrt{\mathrm{ab}},$$

that is, substituting for a, b, c, f, g, h their values, and taking, for instance, a, b, c $= a''\sqrt{2}$, $b''\sqrt{2}$, $c''\sqrt{2}$, we find

$$\begin{aligned} l : m : n = \; & (b'' - c'')^2 - (b - c)^2 - (b' - c')^2 \\ & : (c'' - a'')^2 - (c - a)^2 - (c' - a')^2 \\ & : (a'' - b'')^2 - (a - b)^2 - (a' - b')^2, \end{aligned}$$

that is, l, m, n are as the squares of the tangential distances (direct) of the three circles taken in pairs, and this being so, the equation of a pair of circles touching each of the three given circles is $\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0$. It is clear that, instead of taking the three direct tangential distances, we may take one direct tangential distance and two inverse tangential distances, viz., the tangential distances corresponding to any three centres of similitude which lie in a line; we have thus in all the equations of four pairs of circles, viz., of the eight circles which touch the three given circles. This is Casey's theorem in the problem of tactions.

Article No. 179. *The Intersections of the Conic and Orthotomic Circle are a set of four Concyclic Foci.*

179. The conic of centres intersects the orthotomic circle in four points, and for each of these the radius of the variable circle is $= 0$, that is, the points in question are a set of four concyclic foci (A, B, C, D) of the curve. Regarding the foci as given, the circle which contains them is of course the orthotomic circle; and there are a singly infinite series of curves, viz., these correspond to the singly infinite series of conics which can be drawn through the given foci. As for a given curve there are

four sets of concyclic foci, there are four different constructions for the curve, viz., the orthotomic circle may be any one of the four circles O, R, S, T, which contain the four sets of concyclic foci respectively; and the conic of centres is a conic through the corresponding set of four concyclic foci. We have thus four conics, but the foci of each of them coincide with the nodo-foci of the curve, that is, the conics are confocal; that such confocal conics exist has been shown, *ante*, Nos. 78 to 80.

Article Nos. 180 and 181. *Remark as to the Construction of the Symmetrical Curve.*

180. It is to be observed that in applying as above the theorem of the variable zomal to the construction of a symmetrical curve, the orthotomic circle made use of was one of the circles R, S, T, not the circle O, which is in this case the axis; in fact, we should then have the conic and the orthotomic circle each of them coinciding with the axis. And the variable circle, *quà* circle having its centre on the axis, cuts the axis at right angles whatever the radius may be; that is, the variable circle is no longer sufficiently determined by the theorem. The curve may nevertheless be constructed as the envelope of a variable circle having its centre on the axis; viz., writing $\mathfrak{A}^\circ = (x - az)^2 + y^2 - a''^2 z^2$, &c., and starting with the form

$$\sqrt{l}\mathfrak{A}^\circ + \sqrt{m}\mathfrak{B}^\circ + \sqrt{n}\mathfrak{C}^\circ = 0,$$

then recurring to the demonstration of the theorem (*ante*, No. 47), the equation of the variable circle is $\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ = 0$, where a, b, c are any quantities satisfying $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, or, what is the same thing, taking q an arbitrary parameter, and writing $\frac{l}{\mathrm{a}} = 1 + q$, $\frac{m}{\mathrm{b}} = 1 - q$, $\frac{n}{\mathrm{c}} = -2$, the equation of the variable circle is

$$\frac{1}{1+q}\, l\mathfrak{A}^\circ + \frac{1}{1-q}\, m\mathfrak{B}^\circ - \tfrac{1}{2} n\mathfrak{C}^\circ = 0.$$

Compare Nos. 118—123 for the like mode of construction of a conic; but it is proper to consider this in a somewhat different form.

181. Assume that the equation of the variable circle is

$$\mathfrak{D}^\circ = (x - dz)^2 + y^2 - d''^2 z^2 = 0\,;$$

we have therefore identically

$$\mathrm{a}\mathfrak{A}^\circ + \mathrm{b}\mathfrak{B}^\circ + \mathrm{c}\mathfrak{C}^\circ + \mathrm{d}\mathfrak{D}^\circ = 0,$$

viz., this gives

$$\begin{aligned} \mathrm{a} \quad + \mathrm{b} \quad + \mathrm{c} &= -\mathrm{d}\,, \\ \mathrm{a}a + \mathrm{b}b + \mathrm{c}c &= -\mathrm{d}d, \\ \mathrm{a}\,(a^2 - a''^2) + \mathrm{b}\,(b^2 - b''^2) + \mathrm{c}\,(c^2 - c''^2) &= -\mathrm{d}\,(d^2 - d''^2), \end{aligned}$$

and from these equations we obtain a, b, c equal respectively to given multiples of d; substituting these values in the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, d divides out, and we have an

equation involving the parameters of the given circles, and also d, d'', the parameters of the variable circle; viz., an equation determining d'', the radius of the variable circle, in terms of d, the coordinate of its centre. I consider in particular the case where the given circles are points; that is, where the given equation is

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0.$$

The equations here are

$$\begin{aligned} \mathrm{a} + \mathrm{b} + \mathrm{c} &= -\mathrm{d}, \\ \mathrm{a}a + \mathrm{b}b + \mathrm{c}c &= -\mathrm{d}d, \\ \mathrm{a}a^2 + \mathrm{b}b^2 + \mathrm{c}c^2 &= -\mathrm{d}\,(d^2 - d''^2), \end{aligned}$$

and from these we obtain

$$\begin{aligned} \mathrm{a}\,(a-b)\,(a-c) &= -\mathrm{d}\,\big((d-b)\,(d-c) - d''^2\big) \\ \mathrm{b}\,(b-c)\,(b-a) &= -\mathrm{d}\,\big((d-c)\,(d-a) - d''^2\big) \\ \mathrm{c}\,(c-a)\,(c-b) &= -\mathrm{d}\,\big((d-a)\,(d-b) - d''^2\big), \end{aligned}$$

so that the equation $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$ becomes

$$\frac{l\,(a-b)\,(a-c)}{(d-b)\,(d-c) - d''^2} + \frac{m\,(b-c)\,(b-a)}{(d-c)\,(d-a) - d''^2} + \frac{n\,(c-a)\,(c-b)}{(d-a)\,(d-b) - d''^2} = 0,$$

or, as this is more conveniently written,

$$\frac{l}{b-c}\,\frac{1}{(d-b)\,(d-c) - d''^2} + \frac{m}{c-a}\,\frac{1}{(d-c)\,(d-a) - d''^2} + \frac{n}{a-b}\,\frac{1}{(d-a)\,(d-b) - d''^2} = 0,$$

viz., considering d, d'' as the abscissa and ordinate of a point on a curve, and representing them by x, y respectively, the equation of this curve is

$$\frac{l}{b-c}\,\frac{1}{(x-b)\,(x-c) - y^2} + \frac{m}{c-a}\,\frac{1}{(x-c)\,(x-a) - y^2} + \frac{n}{c-a}\,\frac{1}{(x-a)\,(x-b) - y^2} = 0,$$

which is a certain quartic curve; and we have the original curve

$$\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} = 0,$$

as the envelope of a variable circle having for its diameter the double ordinate of this quartic curve.

Write for shortness $\frac{l}{b-c}$, $\frac{m}{c-a}$, $\frac{n}{a-b} = L$, M, N respectively, then the equation of the quartic curve may be written

$$\Sigma L\,[(x-a)^2\,(x-b)\,(x-c) - y^2\,(x-a)\,(2x-b-c) + y^4] = 0,$$

viz., this is

$$\begin{aligned} \Sigma L\,[&x\,(x-a)\,(x-b)\,(x-c) \\ &- y^2\,\big(2x^2 - (a+b+c)\,x + (ab+ac+bc)\big) + y^4 \\ &- a\,(x-a)\,(x-b)\,(x-c) + y^2\,(ax+bc)] = 0, \end{aligned}$$

or what is the same thing, the equation is

$$\begin{aligned}(L+M+N)\,[x(x-a)(x-b)(x-c)-y^2\left(2x^2-(a+b+c)\,x+ab+ac+bc\right)+y^4]\\ -(La+Mb+Nc)(x-a)(x-b)(x-c)\\ +y^3\{(La+Mb+Nc)\,x+Lbc+Mca+Nab\}=0.\end{aligned}$$

In the particular case where $L+M+N=0$, that is, where

$$\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0,$$

the quartic curve becomes a cubic, viz., putting for shortness

$$-\delta=\frac{Lbc+Mca+Nab}{La+Mb+Nc},$$

the equation of the cubic is

$$y^2=\frac{(x-a)(x-b)(x-c)}{x-\delta},$$

viz., this is a cubic curve having three real asymptotes, and a diameter at right angles to one of the asymptotes, and at the inclinations $+45°$, $-45°$ to the other two asymptotes respectively—say that it is a "rectangular" cubic. The relation $\frac{l}{b-c}+\frac{m}{c-a}+\frac{n}{a-b}=0$ implies that the curve $\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0$ is a Cartesian, and we have thus the theorem that the envelope of a variable circle having for diameter the double ordinate of a rectangular cubic is a Cartesian.

I remark that using a particular origin, and writing the equation of the rectangular cubic in the form $y^2=x^2-2mx+\alpha+\frac{2A}{x}$, the equation of the variable circle is

$$(x-d)^2+y^2=d^2-2md+\alpha+\frac{2A}{d},$$

that is

$$x^2+y^2-\alpha-2d\,(x-m)-\frac{2A}{d}=0,$$

where d is the variable parameter. Forming the derived equation in regard to d, we have

$$x-m=\frac{A}{d^2},$$

and thence

$$x^2+y^2-\alpha=\frac{4A}{d},$$

$$(x^2+y^2-\alpha)^2=\frac{16A^2}{d^2}=16A\,(x-m),$$

that is, the equation of the envelope is $(x^2+y^2-\alpha)^2=16A\,(x-m)=0$, which is a known form of the equation of a Cartesian.

Article Nos. 182 and 183. *Focal Formulæ for the General Curve.*

182. Considering any three circles centres A, B, C, and taking $\mathfrak{A}^\circ$, &c., to denote as usual, let the equation of the curve be

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}=0\,;$$

then considering a fourth circle, centre D, a position of the variable circle, and having therefore the same orthotomic circle with the given circles, so that as before

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0,$$

the formulæ No. 47 (changing only U, V, W, T into $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$) are at once applicable to express the equation of the curve in terms of any three of the four circles A, B, C, D.

In particular, the circles may reduce themselves to the four points A, B, C, D, a set of concyclic foci, and here, the equation being originally given in the form

$$\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0,$$

the same formulæ are applicable to express the equation in terms of any three of the four foci.

183. It is to be observed that in this case if the positions of the four foci are given by means of the circular coordinates $\left(\alpha,\ \frac{1}{\alpha},\ 1\right)$, &c., which refer to the centre of the circle $ABCD$ as origin, and with the radius of this circle taken as unity, then the values of a, b, c, d (*ante*, No. 90), are given in the form adapted to the formulæ of No. 49, viz., we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d}=\alpha(\beta\gamma\delta) : -\beta(\gamma\delta\alpha) : \gamma(\delta\alpha\beta) : -\delta(\alpha\beta\gamma),$$

where $(\beta\gamma\delta)=(\beta-\gamma)(\gamma-\delta)(\delta-\beta)$, &c. The relation $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, putting therein $l : m : n=\rho\alpha(\beta-\gamma)^2 : \sigma\beta(\gamma-\alpha)^2 : \tau\gamma(\alpha-\beta)^2$, (or, what is the same thing, taking the equation of the curve to be given in the form $(\beta-\gamma)\sqrt{\rho\alpha\mathfrak{A}}+(\gamma-\alpha)\sqrt{\sigma\beta\mathfrak{B}}+(\alpha-\beta)\sqrt{\tau\gamma\mathfrak{C}}=0$), becomes

$$\rho(\beta-\gamma)(\alpha-\delta)+\sigma(\gamma-\alpha)(\beta-\delta)+\tau(\alpha-\beta)(\gamma-\delta)=0,$$

viz., this equation, considering ρ, σ, τ, α, β, γ as given, determines the position of the fourth focus D, or when A, B, C, D are given, it is the relation which must exist between ρ, σ, τ; and the four forms of the equation are

$$\left(\begin{array}{cccc} . & \sqrt{\tau}(\delta-\gamma), & \sqrt{\sigma}(\beta-\delta), & \sqrt{\rho}(\gamma-\beta) \\ \sqrt{\tau}(\gamma-\delta), & . & \sqrt{\rho}(\delta-\alpha), & \sqrt{\sigma}(\alpha-\gamma) \\ \sqrt{\sigma}(\delta-\beta), & \sqrt{\rho}(\alpha-\delta), & . & \sqrt{\tau}(\beta-\alpha) \\ \sqrt{\rho}(\beta-\gamma), & \sqrt{\sigma}(\gamma-\alpha), & \sqrt{\tau}(\alpha-\beta), & . \end{array}\right)(\sqrt{\alpha\mathfrak{A}},\ \sqrt{\beta\mathfrak{B}},\ \sqrt{\gamma\mathfrak{C}},\ \sqrt{\delta\mathfrak{D}})=0,$$

viz., the curve is represented by means of any one of these four equations involving each of them three out of the four given foci A, B, C, D.

Article Nos. 184 and 185. *Case of the Circular Cubic.*

184. In the case of a circular cubic, we must have

$$\rho(\beta-\gamma)(\alpha-\delta)+\sigma(\gamma-\alpha)(\beta-\delta)+\tau(\alpha-\beta)(\gamma-\delta)=0,$$
$$\sqrt{\alpha\rho}\,(\beta-\gamma)\qquad+\sqrt{\beta\sigma}\,(\gamma-\alpha)\qquad+\sqrt{\gamma\tau}\,(\alpha-\beta)\qquad=0,$$

which, when the foci A, B, C, D are given, determine the values of $\rho:\sigma:\tau$ in order that the curve may be a circular cubic. We see at once that there are two sets of values, and consequently two circular cubics having each of them the given points A, B, C, D for a set of concyclic foci. The two systems may be written

$$\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\tau}=\sqrt{\alpha\delta}-\sqrt{\beta\gamma}:\sqrt{\beta\delta}-\sqrt{\gamma\alpha}:\sqrt{\gamma\delta}-\sqrt{\alpha\beta},$$

viz., it being understood that $\sqrt{\alpha\delta}$ means $\sqrt{\alpha}.\sqrt{\delta}$, &c., then, according as $\sqrt{\delta}$ has one or other of its two opposite values, we have one or other of the two systems of values of $\rho:\sigma:\tau$. To verify this, observe that writing the equation under the form

$$\sqrt{\alpha\rho}:\sqrt{\beta\sigma}:\sqrt{\gamma\tau}=\alpha\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\beta\sqrt{\delta}-\sqrt{\alpha\beta\gamma}:\gamma\sqrt{\delta}-\sqrt{\alpha\beta\gamma},$$

the second equation is verified; and that writing them under the form

$$\rho:\sigma:\tau=-(\beta+\gamma)(\alpha+\delta)+M:-(\gamma+\alpha)(\beta+\delta)+M:-(\alpha+\beta)(\gamma+\delta)+M,$$

where

$$M=\beta\gamma+\alpha\delta+\gamma\alpha+\beta\delta+\alpha\beta+\gamma\delta-2\sqrt{\alpha\beta\gamma\delta},$$

the second equation is also verified.

185. If we assume for a moment $\alpha=\cos a+i\sin a=e^{ia}$, &c., viz., if a, b, c, d be the inclinations to any fixed line of the radii through A, B, C, D respectively, then we have

$$\sqrt{\alpha\delta}\pm\sqrt{\beta\gamma}=e^{\frac{1}{2}(a+b+c+d)i}\left\{e^{\frac{1}{2}(a+d-b-c)i}\pm e^{-\frac{1}{2}(a+d-b-c)i}\right\},$$
$$\sqrt{\alpha}\,(\beta-\gamma)\;=e^{\frac{1}{2}(a+b+c)i}\left\{e^{\frac{1}{2}(b-c)i}\qquad-e^{-\frac{1}{2}(b-c)i}\qquad\right\},$$

and thence

$$\begin{aligned}\sqrt{\alpha\rho}\,(\beta-\gamma):\sqrt{\beta\sigma}\,(\gamma-\alpha):\sqrt{\gamma\tau}\,(\alpha-\beta)=&\;\cos\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\cos\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\cos\tfrac{1}{4}(c+d-a-b)\sin\tfrac{1}{2}(a-b);\end{aligned}$$

or else

$$\begin{aligned}=&\;\sin\tfrac{1}{4}(a+d-b-c)\sin\tfrac{1}{2}(b-c)\\ &:\sin\tfrac{1}{4}(b+d-c-a)\sin\tfrac{1}{2}(c-a)\\ &:\sin\tfrac{1}{4}(c+d-a-b)\sin\tfrac{1}{2}(a-b).\end{aligned}$$

Putting in these formulæ,

$$\begin{aligned}&\tfrac{1}{4}(a-b-c)=A, &&\text{then we have}\quad B-C=\tfrac{1}{2}(b-c),\\ &\tfrac{1}{4}(b-c-a)=B, &&\qquad\text{,,}\qquad C-A=\tfrac{1}{2}(c-a),\\ &\tfrac{1}{4}(c-a-b)=C, &&\qquad\text{,,}\qquad A-B=\tfrac{1}{2}(a-b),\end{aligned}$$

and for either set of values the verification of the relation

$$\sqrt{\alpha\rho}\,(\beta-\gamma)+\sqrt{\beta\sigma}\,(\gamma-\alpha)+\sqrt{\gamma\tau}\,(\alpha-\beta)=0,$$

will depend on the two identical equations

$$\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)=0,$$
$$\cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)=0:$$

although the foregoing solution for the case of a circular cubic is the most elegant one, I will presently return to the question and give the solution in a different form.

Article No. 186. *Focal Formulæ for the Symmetrical Curve.*

186. In the symmetrical case, where the foci A, B, C, D are on a line, then if, as usual, a, b, c, d denote the distances from a fixed point, we have the expressions of (a, b, c, d) in a form adapted to the formulæ of No. 49, viz.,

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=(b-c)(c-d)(d-b):-(c-d)(d-a)(a-c):(d-a)(a-b)(b-d):-(a-b)(b-c)(c-a),$$

so that, assuming

$$l:m:n=\rho(b-c)^2:\sigma(c-a)^2:\tau(a-b)^2,$$

the equation

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0,$$

becomes

$$\rho(b-c)(a-d)+\sigma(c-a)(b-d)+\tau(a-b)(c-d)=0,$$

and the equation of the curve may be presented under any one of the four forms

$$\left(\begin{array}{cccc} . & \sqrt{\tau}\,(d-c), & \sqrt{\sigma}\,(b-d), & \sqrt{\rho}\,(c-b) \\ \sqrt{\tau}\,(c-d), & . & \sqrt{\rho}\,(d-a), & \sqrt{\sigma}\,(a-c) \\ \sqrt{\sigma}\,(d-b), & \sqrt{\rho}\,(a-d), & . & \sqrt{\tau}\,(b-a) \\ \sqrt{\rho}\,(b-c), & \sqrt{\sigma}\,(c-a), & \sqrt{\tau}\,(a-b), & . \end{array}\right)(\sqrt{\mathfrak{A}},\sqrt{\mathfrak{B}},\sqrt{\mathfrak{C}},\sqrt{\mathfrak{D}})=0.$$

Article No. 187. *Case of the Symmetrical Circular Cubic.*

187. For a circular cubic we must have

$$\rho(b-c)(a-d)+\sigma(c-a)(b-d)+\tau(a-b)(c-d)=0,$$
$$\sqrt{\rho}\,(b-c)\quad+\sqrt{\sigma}\,(c-a)\quad+\sqrt{\tau}\,(a-b)\quad=0.$$

These equations give $\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\tau}=1:1:1$ (values which obviously satisfy the two equations), or else

$$\sqrt{\rho}:\sqrt{\sigma}:\sqrt{\rho}=a+d-b-c:b+d-c-a:c+d-a-b.$$

In fact, these values obviously satisfy the second equation; and to see that they satisfy the first equation, we have only to write them under the form

$$\rho : \sigma : \tau = M - 4(b+c)(a+d) : M - 4(c+a)(b+d) : M - 4(a+b)(c+d),$$

where $M = (a+b+c+d)^2$. The first set gives for the curve

$$(b-c)\sqrt{\mathfrak{A}} + (c-a)\sqrt{\mathfrak{B}} + (a-b)\sqrt{\mathfrak{C}} = 0,$$

but this contains the line $z=0$ not once only, but twice; it in fact is $(y^2=0)$, the axis taken twice; the only proper cubic with the foci A, B, C, D *in lined* is therefore

$$(b-c)(a+d-b-c)\sqrt{\mathfrak{A}} + (c-a)(b+d-c-a)\sqrt{\mathfrak{B}} + (a-b)(c+d-a-b)\sqrt{\mathfrak{C}} = 0,$$

the equation of which is, of course, expressible in each of the other three forms.

Article Nos. 188 to 192. *Case of the General Circular Cubit.*

188. Returning to the general case of the circular cubic, the lines BC, AD meet in R, and if we denote by a_1, b_1, c_1, d_1, the distances from R of the four points respectively, so that $b_1c_1 = a_1d_1 = \text{rad.}^2 R$, then observing that a, b, c, d are proportional to the triangles BCD, CDA, DAB, ABC, with signs such that $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, we find

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = -d_1(b_1-c_1) : c_1(a_1-d_1) : -b_1(a_1-d_1) : a_1(b_1-c_1);$$

and this being so, the equations $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = 0$, $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$, give two systems of values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$, viz., these are

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = b_1 - c_1 : c_1 - a_1 : \quad a_1 - b_1,$$

and

$$= b_1 - c_1 : c_1 + a_1 : -a_1 - b_1.$$

(To verify this, observe that for the first set we have

$$\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} = \frac{(b_1-c_1)^2}{-d_1(b_1-c_1)} + \frac{(c_1-a_1)^2}{c_1(a_1-d_1)} + \frac{(a_1-b_1)^2}{-b_1(a_1-d_1)},$$

$$= \frac{b_1-c_1}{-d_1} + \frac{1}{a_1-d_1}\left(c_1 + \frac{a_1^2}{c_1} - b_1 - \frac{a_1^2}{b_1}\right),$$

$$= \frac{b_1-c_1}{-d_1} + \frac{b_1-c_1}{a_1-d_1}\left(\frac{a_1^2}{b_1c_1} - 1\right),$$

$$= -\frac{b_1-c_1}{d_1} + \frac{b_1-c_1}{a_1-d_1}\left(\frac{a_1}{d_1} - 1\right), = 0;$$

and the like as regards the second set.)

189. These values of $\sqrt{l} : \sqrt{m} : \sqrt{n}$ give the equations of the two circular cubics with the foci (A, B, C, D), the equation of each of them under a fourfold form, viz., we have

$$\left(\begin{array}{cccc} \cdot\,, & d_1-c_1, & b_1-d_1, & c_1-b_1 \\ c_1-d_1, & \cdot\,, & d_1-a_1, & a_1-c_1 \\ d_1-b_1, & a_1-d_1, & \cdot\,, & b_1-a_1 \\ b_1-c_1, & c_1-a_1, & a_1-b_1, & \cdot \end{array}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \qquad \text{(first curve)},$$

and

$$\left(\begin{array}{cccc} \cdot\,, & -c_1-d_1, & d_1+b_1, & -b_1+c_1 \\ d_1+c_1, & \cdot\,, & a_1-d_1, & c_1-a_1 \\ -b_1-d_1, & d_1-a_1, & \cdot\,, & a_1+b_1 \\ b_1-c_1, & -c_1+a_1, & -a_1-b_1, & \cdot \end{array}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \qquad \text{(second curve)}.$$

190. Similarly CA and BD meet in S, and if we denote by a_2, b_2, c_2, d_2 the distances from S of the four points respectively, so that $c_2a_2 = b_2d_2 = \text{rad.}^2 S$ (observe that if as usual A, B, C, D are taken in order on the circle O, then A, C are on opposite sides of S, and similarly B, D are on opposite sides of S, so that taking a_2, b_2 positive c_2, d_2 will be negative), we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = c_2(b_2-d_2) : d_2(c_2-a_2) : -a_2(b_2-d_2) : -b_2(c_2-a_2),$$

and then the equations $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, are satisfied by the two sets of values

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = \quad b_2-c_2 : c_2-a_2 : a_2-b_2,$$

and

$$= -b_2-c_2 : c_2-a_2 : a_2+b_2,$$

and we have the equations of the same two cubic curves, each equation under a fourfold form, viz., these are

$$\left(\begin{array}{cccc} \cdot\,, & -c_2+d_2, & -d_2+b_2, & -b_2+c_2 \\ c_2-d_2, & \cdot\,, & d_2-a_2, & -c_2+a_2 \\ -b_2+d_2, & a_2-d_2, & \cdot\,, & -a_2+b_2 \\ b_2-c_2, & c_2-a_2, & a_2-b_2, & \cdot \end{array}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \qquad \text{(first curve)},$$

and

$$\left(\begin{array}{cccc} \cdot\,, & c_2+d_2, & -d_2+b_2, & -b_2-c_2 \\ -d_2-c_2, & \cdot\,, & a_2+d_2, & c_2-a_2 \\ -b_2+d_2, & -d_2-a_2, & \cdot\,, & a_2+b_2 \\ b_2+c_2, & -c_2+a_2, & -a_2-b_2, & \cdot \end{array}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0 \qquad \text{(second curve)}.$$

191. And again AB and CD meet in T, and denoting by a_3, b_3, c_3, d_3 the distances from T of the four points respectively, so that $a_3b_3 = c_3d_3 = \text{rad.}^2 T$, we have

$$\mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d} = b_3(c_3-d_3) : -a_3(c_3-d_3) : -d_3(a_3-b_3) : c_3(a_3-b_3);$$

and the equations $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, $\sqrt{l}+\sqrt{m}+\sqrt{n}=0$, then give for $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$ two sets of values, viz., these are

$$\sqrt{l} : \sqrt{m} : \sqrt{n} = b_3 - c_3 : \quad c_3 - a_3 : a_3 - b_3,$$

and

$$= b_3 + c_3 : -c_3 - a_3 : a_3 - b_3;$$

and we again obtain the equations of the two cubics, each equation under a fourfold form, viz., these are

$$\left(\begin{vmatrix} . \;, & -c_3+d_3, & -d_3+b_3, & c_3-b_3 \\ -d_3+c_3, & . \;, & -a_3+d_3, & a_3-c_3 \\ -b_3+d_3, & -d_3+a_3, & . \;, & b_3-a_3 \\ b_3-c_3, & c_3-a_3, & a_3-b_3, & . \end{vmatrix}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0,$$

and

$$\left(\begin{vmatrix} . \;, & c_3-d_3, & d_3+b_3, & -c_3-b_3 \\ d_3-c_3, & . \;, & -a_3-d_3, & a_3+c_3 \\ -b_3-d_3, & d_3+a_3, & . \;, & b_3-a_3 \\ b_3+c_3, & -c_3-a_3, & a_3-b_3, & . \end{vmatrix}\right)(\sqrt{\mathfrak{A}}, \sqrt{\mathfrak{B}}, \sqrt{\mathfrak{C}}, \sqrt{\mathfrak{D}})=0.$$

192. The three systems have been obtained independently, but they may of course be derived each from any other of them: to show how this is, recollecting that we have

$$\begin{aligned} RA,\ RB,\ RC,\ RD &= a_1,\ b_1,\ \ c_1,\ \ d_1, \\ SA,\ SB,\ SC,\ SD &= a_2,\ b_2,\ -c_2,\ -d_2, \\ TA,\ TB,\ TC,\ TD &= a_3,\ b_3,\ \ c_3,\ \ d_3; \end{aligned}$$

then to compare

$$(a_1,\ b_1,\ c_1,\ d_1) \text{ and } (a_2,\ b_2,\ c_2,\ d_2);$$

the similar triangles

$$\begin{aligned} &SBC \quad \text{give} \quad b_1 - c_1 : -c_2 : b_2, \\ &SAD \qquad\quad\ = a_1 - d_1 : -d_2 : a_2, \end{aligned}$$

and the similar triangles

$$\begin{aligned} &RAC \ \text{give} \quad a_2 - c_2 : \quad c_1 : a_1, \\ &RBD \qquad\quad\ = b_2 - d_2 : \quad d_1 : b_1; \end{aligned}$$

using these equations to determine the ratios of a_2, b_2, c_2, d_2 we have

$$\frac{a_2-c_2}{b_2-d_2}=\frac{c_1}{d_1}, \text{ or } d_1a_2 - d_1c_2 - c_1b_2 + c_1d_2 = 0;$$

that is

$$b_2\left\{-c_1 + d_1\frac{a_1-d_1}{b_1-c_1}\right\} + c_2\left\{-d_1 + c_1\frac{a_1-d_1}{b_1-c_1}\right\} = 0;$$

and hence

$$b_2(-b_1c_1+c_1^2+a_1d_1-d_1^2)+c_2(-b_1d_1+c_1d_1+a_1c_1-c_1d_1)=0,$$

that is

$$b_2(c_1^2-d_1^2)+c_2(a_1c_1-b_1d_1)=0,$$

but

$$a_1c_1-b_1d_1=\frac{b_1}{d_1}(c_1^2-d_1^2),$$

or the equation gives $b_2+\frac{b_1}{d_1}c_2=0$, or say $b_2:c_2=b_1:-d_1$, and this with $\frac{b_1-c_1}{a_1-d_1}=\frac{c_2}{d_2}=\frac{b_2}{a_2}$, gives all the ratios, or we have

$$a_2:b_2:c_2:d_2=b_1(a_1-d_1):b_1(b_1-c_1):-d_1(a_1-d_1):-d_1(b_1-c_1).$$

We have then for example

$$b_2-c_2:c_2-a_2:a_2-b_2=b_1-c_1:c_1-a_1:a_1-b_1;\ \&c.,$$

showing the identity of the forms in (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2).

Article No. 193. *Transformation to a New Set of Concyclic Foci.*

193. Consider the equation

$$\sqrt{l\mathfrak{A}}+\sqrt{m\mathfrak{B}}+\sqrt{n\mathfrak{C}}=0,$$

which refers to the foci A, B, C, and taking D the fourth concyclie focus, let (A_1, D_1) be the antipoints of (A, D) and (B_1, C_1) the antipoints of (B, C); so that (A_1, B_1, C_1, D_1) are another set of concyclic foci. We have $\mathfrak{B}_1.\mathfrak{C}_1=\mathfrak{B}.\mathfrak{C}$, and it appears, *ante* No. 104, that we can find l_1, m_1, n_1, such that identically

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}=-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}_1$$

and that $m_1n_1=mn$. The equation of the curve gives

$$-l\mathfrak{A}+m\mathfrak{B}+n\mathfrak{C}+2\sqrt{mn\mathfrak{B}\mathfrak{C}}=0,$$

we have therefore

$$-l_1\mathfrak{A}_1+m_1\mathfrak{B}_1+n_1\mathfrak{C}+2\sqrt{m_1n_1\mathfrak{B}_1\mathfrak{C}_1}=0,$$

that is,

$$\sqrt{l_1\mathfrak{A}_1}+\sqrt{m_1\mathfrak{B}_1}+\sqrt{n_1\mathfrak{C}_1}=0,$$

viz., this is the equation of the curve expressed in terms of the concyclic foci A_1, B_1, C_1.

Article No. 194. *The Tetrazomal Curve, Decomposable or Indecomposable.*

194. I consider the tetrazomal curve

$$\sqrt{l\mathfrak{A}^\circ}+\sqrt{m\mathfrak{B}^\circ}+\sqrt{n\mathfrak{C}^\circ}+\sqrt{p\mathfrak{D}^\circ}=0,$$

where the zomals are circles described about any given points A, B, C, D as centres.

There is not, in general, any identical equation $a\mathfrak{A}^\circ + b\mathfrak{B}^\circ + c\mathfrak{C}^\circ + d\mathfrak{D}^\circ = 0$, but when such relation exists, and when we have also $\frac{l}{a} + \frac{m}{b} + \frac{n}{c} + \frac{p}{d} = 0$, then the curve breaks up into two trizomals. When the conditions in question do not subsist, the curve is indecomposable. But there may exist between l, m, n, p relations in virtue of which a branch or branches ideally contain $(z^2 = 0)$ the line infinity a certain number of times, and which thus cause a depression in the order of the curve. The several cases are as follows:

Article No. 195. *Cases of the Indecomposable Curve.*

195. I. The general case; l, m, n, p not subjected to any condition. The curve is here of the order $=8$; it has a quadruple point at each of the points I, J (and there is consequently no other point at infinity); it is touched four times by each of the circles A, B, C, D; and it has six nodes, viz., these are the intersections of the pairs of circles

$$\sqrt{m\mathfrak{B}^\circ} + \sqrt{n\mathfrak{C}^\circ} = 0, \quad \sqrt{l\mathfrak{A}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0,$$
$$\sqrt{n\mathfrak{C}^\circ} + \sqrt{l\mathfrak{A}^\circ} = 0, \quad \sqrt{m\mathfrak{B}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0,$$
$$\sqrt{l\mathfrak{A}^\circ} + \sqrt{m\mathfrak{B}^\circ} = 0, \quad \sqrt{n\mathfrak{C}^\circ} + \sqrt{p\mathfrak{D}^\circ} = 0;$$

the number of dps. is $6 + 2.6, = 18$, and there are no cusps, hence the class is $= 20$; and the deficiency is $= 3$.

II. We may have

$$\sqrt{l} + \sqrt{m} + \sqrt{n} + \sqrt{p} = 0;$$

there is in this case a single branch ideally containing $(z = 0)$ the line infinity; the order is $=7$. Each of the points I, J is a triple point, there is consequently one other point at infinity; viz., this is a real point, or the curve has a real asymptote. There are 6 nodes as before; dps. are $6 + 2.3, = 12$; class $= 18$, deficiency $= 3$.

III. We may have

$$\sqrt{l} + \sqrt{m} = 0, \quad \sqrt{n} + \sqrt{p} = 0;$$

there are then two branches each ideally containing $(z = 0)$ the line infinity; the order is $=6$. Each of the points I, J is a double point, and there are therefore two more points at infinity. These may be real or imaginary; viz., the curve may have (besides the asymptotes at I, J) two real or imaginary asymptotes. The circles $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} = 0$, $\sqrt{n\mathfrak{C}} + \sqrt{p\mathfrak{D}} = 0$, each contain $(z = 0)$ the line infinity, or they reduce themselves to two lines, so that in place of two nodes we have a single node at the intersection of these lines; number of nodes is $= 5$. Hence dps. are $5 + 2.1, = 7$. Class $= 16$, deficiency $= 3$.

IV. We may have

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = a : b : c : d;$$

there is here a single branch containing ($z^2=0$) the line infinity twice; the order is $=6$. Each of the points I, J is a double point, and there are therefore two more points at infinity, that is (besides the asymptotes at I, J), there are two (real or imaginary) asymptotes. The number of nodes, as in the general case, is $=6$. Hence dps. are $6+2.1, =8$; class is $=14$; deficiency $=2$.

I notice the included particular case where the circles reduce themselves to their centres; viz., we have here the curve

$$\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0,$$

which (see *ante* No. 93) is in fact the curve which is the locus of the foci of the conics which pass through the four points A, B, C, D. It is at present assumed that the four points are not a circle; this case will be considered *post* No. 199. If we have BC, AD meeting in R; CA, BD in S, and AB, CD in T, then these points R, S, T are three of the six nodes. In fact, writing down the equations of the two circles

$$\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}=0,\quad \mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{d}\sqrt{\mathfrak{D}}=0,$$

and observing that when the current point is taken at R, we have $\mathfrak{B}:\mathfrak{C}=\overline{RB}^2:\overline{RC}^2=(BAD)^2:(CAD)^2=\mathrm{c}^2:\mathrm{b}^2$, and similarly $\mathfrak{A}:\mathfrak{D}=\overline{RA}^2:\overline{RD}^2=(ABC)^2:(DBC)^2=\mathrm{d}^2:\mathrm{a}^2$, we see that each of the two circles passes through the point R, or this point is a node. Similarly, the points S and T are each of them a node.

V. If

$$\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p},$$

there are here three branches, each ideally containing ($z=0$) the line infinity; the order is thus $=5$. Each of the points I, J is an ordinary point on the curve; there are besides at infinity three points, all real, or one real and two imaginary; that is (besides the asymptotes at I, J) there are three asymptotes, all real, or one real and two imaginary. Each of the circles $\sqrt{\mathfrak{A}}+\sqrt{\mathfrak{B}}=0$, &c., contains the line infinity, and is thus reduced to a line; the number of nodes is therefore $=3$. Hence also, dps. $=3$; class $=14$; deficiency $=3$.

Article No. 196. *Cases of the Indecomposable Curve, the Centres being in a Line.*

196. There are some peculiarities in the case where the centres A, B, C, D are on a line; taking as usual (a, b, c, d) for the x-coordinates or distances of the four centres from a fixed point on the line, I enumerate the cases as follows:

I. No relation between l, m, n, p; corresponds to I. *supra*.

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; corresponds to II. *supra*.

III. $\sqrt{l}+\sqrt{m}=0$, $\sqrt{n}+\sqrt{p}=0$; corresponds to III. *supra*.

IV. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$, $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$; corresponds to IV. *supra*, viz., there is a branch ideally containing $(z^2=0)$ the line infinity twice. But, observe that whereas in IV. *supra*, in order that this might be so, it was necessary to impose on l, m, n, p three conditions giving the definite systems of values $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$ $=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, in the present case only two conditions are imposed, so that a single arbitrary parameter is left.

V. $\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p}$; corresponds to V. *supra*.

VI. $\sqrt{l}+\sqrt{m}=0$, $\sqrt{n}+\sqrt{p}=0$, $a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$, or what is the same thing, $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=c-d:d-c:b-a:a-b$; the equation is thus $(c-d)(\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ})-(a-b)(\sqrt{\mathfrak{A}^\circ}-\sqrt{\mathfrak{B}^\circ})=0$. There is here one branch ideally containing $(z^2=0)$ the line infinity twice, and another branch ideally containing $(z=0)$ the line infinity once; order is $=5$. Each of the points I, J is an ordinary point on the curve, the remaining points at infinity are a node $(\mathfrak{A}^\circ=\mathfrak{B}^\circ,\ \mathfrak{C}^\circ=\mathfrak{D}^\circ)$, as presently mentioned, counting as three points, viz., one branch has for its tangent the line infinity, and the other branch has for its tangent a line perpendicular to the axis; or what is the same thing, there is a hyperbolic branch having an asymptote perpendicular to the axis, and a parabolic branch ultimately perpendicular to the axis. The number of nodes is $=5$, viz., there is the node $\mathfrak{A}^\circ=\mathfrak{B}^\circ$, $\mathfrak{C}^\circ=\mathfrak{D}^\circ$ just referred to; and the two pairs of nodes $\left((c-d)\sqrt{\mathfrak{A}^\circ}-(a-b)\sqrt{\mathfrak{C}^\circ}=0,\ -(c-d)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{D}^\circ}=0\right)$ and $\left((c-d)\sqrt{\mathfrak{A}^\circ}+(a-b)\sqrt{\mathfrak{D}^\circ}=0,\ (c-d)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}=0\right)$, each pair symmetrically situate in regard to the axis. Hence also dps. $=5$; class $=10$; deficiency $=1$.

And there is apparently a seventh case, which, however, I exclude from the present investigation, viz., this would be if we had

$$\left(\begin{array}{cccc} 1, & 1, & 1, & 1, \\ a, & b, & c, & d, \\ a^2, & b^2, & c^2, & d^2, \\ a''^2, & b''^2, & c''^2, & d''^2, \end{array}\right)(\sqrt{l},\ \sqrt{m},\ \sqrt{n},\ \sqrt{p})=0,$$

that is, a, b, c, d denoting as before, if we had

$$\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d},\quad\text{and}\quad \mathrm{a}a''^2+\mathrm{b}b''^2+\mathrm{c}c''^2+\mathrm{d}d''^2=0.$$

For observe that in this case we have

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0,\quad\text{and}\quad \frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0;$$

that is, the supposition in question belongs to the decomposable case.

Article No. 197. *The Decomposable Curve.*

197. We have next to consider the decomposable case, viz., when we have

$$\mathrm{a}\mathfrak{A}^\circ+\mathrm{b}\mathfrak{B}^\circ+\mathrm{c}\mathfrak{C}^\circ+\mathrm{d}\mathfrak{D}^\circ=0;$$

see *ante*, Nos. 87 *et seq.*—it there appears that (unless the centres A, B, C, D are in a line) the condition signifies that the four circles have a common orthotomic circle; and when we have also

$$\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0.$$

The formulæ for the decomposition are given *ante*, Nos. 42 *et seq.* Writing therein $\mathfrak{A}^\circ$, $\mathfrak{B}^\circ$, $\mathfrak{C}^\circ$, $\mathfrak{D}^\circ$ in place of U, V, W, T respectively, it thereby appears that the tetrazomal curve $\sqrt{l}\mathfrak{A}^\circ+\sqrt{m}\mathfrak{B}^\circ+\sqrt{n}\mathfrak{C}^\circ+\sqrt{p}\mathfrak{D}^\circ=0$, breaks up into the two trizomal curves

$$\sqrt{l_1}\mathfrak{A}^\circ+\sqrt{m_1}\mathfrak{B}^\circ+\sqrt{n_1}\mathfrak{C}^\circ=0,\quad \sqrt{l_2}\mathfrak{A}^\circ+\sqrt{m_2}\mathfrak{B}^\circ+\sqrt{n_2}\mathfrak{C}^\circ=0,$$

where

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},\qquad \sqrt{l_2}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},$$

$$\sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\frac{p}{l}\mathrm{b}\sqrt{n},\quad \sqrt{m_2}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\frac{p}{l}\mathrm{b}\sqrt{n},$$

$$\sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\frac{p}{l}\mathrm{c}\sqrt{m},\quad \sqrt{n_2}=\sqrt{n}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\frac{p}{l}\mathrm{c}\sqrt{m},$$

and where we have

$$\frac{l_1}{\mathrm{a}}+\frac{m_1}{\mathrm{b}}+\frac{n_1}{\mathrm{c}}=0,\quad \frac{l_2}{\mathrm{a}}+\frac{m_2}{\mathrm{b}}+\frac{n_2}{\mathrm{c}}=0.$$

Article Nos. 198 to 203. *Cases of the Decomposable Curve, Centres not in a line.*

198. I assume, in the first instance, that the centres of the circles are not in a line; we have the following cases:

I. No further relation between l, m, n, p; the order of the tetrazomal is $=8$; the order of each of the trizomals is $=4$, that is each of them is a bicircular quartic.

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; the order of the tetrazomal is $=7$, that of one of the trizomals must be $=3$.

To verify this, observe that we have

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=\sqrt{l}+\sqrt{m}+\sqrt{n}+\frac{\mathrm{a}p}{\mathrm{d}\sqrt{l}}+\frac{\sqrt{p}}{\sqrt{l}}\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n}),$$

or substituting for $\sqrt{l}+\sqrt{m}+\sqrt{n}$ the value $-\sqrt{p}$, this is

$$=\frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left\{\mathrm{a}\sqrt{p}-\mathrm{d}\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n})\right\},$$

and similarly for $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}$, the only change being in the sign of the radical $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}$. But from the two conditions satisfied by l, m, n, p it is easy to deduce

$$(\mathrm{a}\sqrt{p}-\mathrm{d}\sqrt{l})^2-\frac{\mathrm{ad}}{\mathrm{bc}}(\mathrm{c}\sqrt{m}-\mathrm{b}\sqrt{n})^2=0,$$

and hence one or other of the two functions

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1},\ \sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2} \text{ is } =0;$$

that is, one of the trizomal curves is a cubic.

III. $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$; order of the tetrazomal is $=6$; and hence order of each of the trizomals is $=3$. To verify this, observe that here

$$l\left(\frac{1}{\mathrm{a}}+\frac{1}{\mathrm{d}}\right)+m\left(\frac{1}{b}+\frac{1}{c}\right)=0,$$

which since $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=0$, gives $\frac{l}{m}=\frac{\mathrm{ad}}{\mathrm{bc}}$; so that, properly fixing the sign of the radical, we may write $\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}=0$. We have then

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l},\quad \sqrt{m_1}+\sqrt{n_1}=\sqrt{\frac{\mathrm{a}}{\mathrm{abc}}}(\mathrm{b}+\mathrm{c})\sqrt{m};$$

which last equation, using $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}$ to denote as above, but properly selecting the signification of $\pm$, may be written

$$\sqrt{m_1}+\sqrt{n_1}=\pm\frac{\mathrm{b}+\mathrm{c}}{\mathrm{d}}\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}.$$

Hence

$$\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})=\frac{1}{\mathrm{a}}\left\{(\mathrm{a}+\mathrm{d})\sqrt{l}+(\mathrm{b}+\mathrm{c})\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\}$$

$$=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\left\{\sqrt{l}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}\right\},=0,$$

viz., $\sqrt{l_1}\mp(\sqrt{m_1}+\sqrt{n_1})$ with a properly selected signification of the sign $\mp$ is $=0$; and similarly $\sqrt{l_2}\mp(\sqrt{m_2}+\sqrt{n_2})$ with a properly selected signification of the sign $\mp$ is $=0$; that is, each of the trizomals is a cubic.

199. IV. $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$ $\left(\text{values which, be it observed, satisfy of themselves the above assumed equation } \frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0\right)$; the order of the tetrazomal is $=6$; and the order of each of the trizomals is here again $=3$. We in fact have $\sqrt{l_1}=\mathrm{a}+\mathrm{d}$, $\sqrt{m_1}+\sqrt{n_1}=\mathrm{b}+\mathrm{c}$, and therefore $\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0$; and similarly $\sqrt{l_2}+\sqrt{m_2}+\sqrt{n_2}=0$; that is, each of the trizomals is a cubic.

I attend, in particular, to the case where the four circles reduce themselves to the points A, B, C, D; these four points are then in a circle; and the curve under consideration is

$$\mathrm{a}\sqrt{\mathfrak{A}}+\mathrm{b}\sqrt{\mathfrak{B}}+\mathrm{c}\sqrt{\mathfrak{C}}+\mathrm{d}\sqrt{\mathfrak{D}}=0;$$

in the general case where the points A, B, C, D are not on a circle, this is, as has been seen, a sextic curve, the locus of the foci of the conics which pass through the four given points; in the case where the points are in a circle then the sextic breaks up into two cubics (viz., observing that the curve under consideration is $\sqrt{l}\mathfrak{A}+\sqrt{m}\mathfrak{B}+\sqrt{n}\mathfrak{C}+\sqrt{p}\mathfrak{D}=0$, where $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, these values do of themselves satisfy the condition of decomposability $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$), that is, the locus of the foci of the conics which pass through four points on a circle is composed of two circular cubics, each of them having the four points for a set of concyclic foci. It is easy to see why the sextic, thus defined as a locus of foci, must break up into two cubics; in fact, as we have seen, the conics which pass through the four concyclic points A, B, C, D have their axes in two fixed directions; there is consequently a locus of the foci situate on the axes which are in one of the fixed directions, and a separate locus of the foci situate on the axes which lie in the other of the fixed directions; viz., each of these loci is a circular cubic.

200. Adopting the notation of No. 188, or writing

$$RA=a_1,\ RB=b_1,\ RC=c_1,\ RD=d_1,$$

(and therefore $b_1c_1=a_1d_1$) we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}=-d_1(b_1-c_1):c_1(a_1-d_1):-b_1(a_1-d_1):a_1(b_1-c_1).$$

Moreover

$$\sqrt{l_1}=\mathrm{a}+\mathrm{d}\quad,\quad \sqrt{l_2}=\mathrm{a}+\mathrm{d},$$

$$\sqrt{m_1}=\mathrm{b}+\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},\quad \sqrt{m_2}=\mathrm{b}-\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

$$\sqrt{n_1}=\mathrm{c}-\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},\quad \sqrt{n_2}=\mathrm{c}+\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

and we have

$$\frac{\mathrm{bcd}}{\mathrm{a}}=(a_1-d_1)^2\frac{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1}{d_1}=a_1^3(a_1-d_1)^2,\quad \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}=-a_1(a_1-d_1)\text{ suppose};$$

and thence

$$\sqrt{l_1}=(a_1-d_1)(b_1-c_1),\quad \sqrt{l_2}=(a_1-d_1)(\ \ b_1-c_1)$$

$$\sqrt{m_1}=(a_1-d_1)(c_1-a_1),\quad \sqrt{m_2}=(a_1-d_1)(\ \ c_1+a_1)$$

$$\sqrt{n_1}=(a_1-d_1)(a_1-b_1),\quad \sqrt{n_2}=(a_1-d_1)(-a_1-b_1),$$

that is

$$\sqrt{l_1}:\sqrt{m_1}:\sqrt{n_1}=b_1-c_1:c_1-a_1:\ \ a_1-b_1,$$

$$\sqrt{l_2}:\sqrt{m_2}:\sqrt{n_2}=b_1-c_1:c_1+a_1:-a_1-b_1,$$

agreeing with the formulæ No. 188.

The tetrazomal curve

$$-d_1(b_1-c_1)\sqrt{\mathfrak{A}}+c_1(a_1-d_1)\sqrt{\mathfrak{B}}-b_1(a_1-d_1)\sqrt{\mathfrak{C}}+a_1(b_1-c_1)\sqrt{\mathfrak{D}}=0$$

is thus decomposed into the two trizomals

$$(b_1 - c_1)\sqrt{\mathfrak{A}} + (c_1 - a_1)\sqrt{\mathfrak{B}} + (a_1 - b_1)\sqrt{\mathfrak{C}} = 0,$$
$$(b_1 - c_1)\sqrt{\mathfrak{A}} + (c_1 + a_1)\sqrt{\mathfrak{B}} - (a_1 + b_1)\sqrt{\mathfrak{C}} = 0.$$

201. Observe that the tetrazomal equation is a consequence of either of the trizomal equations; taking for instance the first trizomal equation, this gives the tetrazomal equation, and consequently any combination of the trizomal equation and the tetrazomal equation is satisfied if only the trizomal equation is satisfied. Multiply the trizomal equation by $-a_1 + d_1$ and add it to the tetrazomal equation; the resulting equation contains the factor a_1, and omitting this, it is

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) + (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0,$$

where observe that $b_1 - c_1$ is the distance BC, and $a_1 - d_1$ the distance AD. But in like manner multiplying the second trizomal equation by $-a_1 + d_1$, and adding it to the original tetrazomal equation, the resulting equation, omitting the factor a_1, is

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) - (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0;$$

viz., it is in fact the same tetrazomal equation as was obtained by means of the first trizomal equation.

The new tetrazomal equation, say

$$(b_1 - c_1)(-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}) + (a_1 - d_1)(\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}) = 0,$$

is thus equivalent to the original tetrazomal equation; observe that it is an equation of the form $\sqrt{l\mathfrak{A}} + \sqrt{m\mathfrak{B}} + \sqrt{n\mathfrak{C}} + \sqrt{p\mathfrak{D}} = 0$, where

$$\sqrt{l} = -(b_1 - c_1), \quad \sqrt{m} = a_1 - d_1, \quad \sqrt{n} = (a_1 - d_1), \quad \sqrt{p} = b_1 - c_1,$$

and where consequently $\sqrt{l} + \sqrt{p} = 0$, $\sqrt{m} + \sqrt{n} = 0$, that is an equation of the form (198) III., decomposable, as it should be, into the equations of two circular cubics. Writing

$$\frac{-\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{D}}}{a_1 - d_1} = \theta, \quad \frac{\sqrt{\mathfrak{B}} - \sqrt{\mathfrak{C}}}{b_1 - c_1} = \theta,$$

where θ is an arbitrary parameter, the curve is obtained as the locus of the intersections of two similar conics having respectively the foci (A, D) and the foci (B, C) (see Salmon, *Higher Plane Curves*, p. 174): whence we have the theorem, that if A, B, C, D are any four points on a circle, the two circular cubics which are the locus of the foci of the conics which pass through the four points A, B, C, D, are also the locus of the intersections of the similar conics, which have for their foci (A, D) and (B, C) respectively; and of the similar conics with the foci (B, D) and (C, A) respectively; and of the similar conics with the foci (C, D) and (A, B) respectively.

202. V. $\sqrt{l} = \sqrt{m} = \sqrt{n} = \sqrt{p}$. The order of the tetrazomal is $=5$, whence those of the trizomals should be $=3$ and $=2$ respectively. To verify this observe that the

equation $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0$ gives $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$, and combining with $a+b+c+d=0$, these are only satisfied by one of the systems $(a+b=0,\ c+d=0)$, $(a+c=0,\ b+d=0)$, $(a+d=0,\ b+c=0)$. Selecting to fix the ideas the first of these, or writing

$$(a,\ b,\ c,\ d)=(a,\ -a,\ c,\ -c),$$

so that we have identically

$$a(A^\circ-B^\circ)+c(C^\circ-D^\circ)=0,$$

an equation which signifies that the radical axis of the circles A, B is also the radical axis of the circles C, D; then, writing as we may do, $\sqrt{\frac{a}{bcd}\frac{p}{l}}\left(=\sqrt{\frac{1}{c^2}}\right)=\frac{1}{c}$, we have

$$\sqrt{l_1}=1-\frac{a}{c},\qquad \sqrt{l_2}=1-\frac{a}{c},$$

$$\sqrt{m_1}=1+\frac{a}{c},\qquad \sqrt{m_2}=1-\frac{a}{c},$$

$$\sqrt{n_1}=1+1,\ =2,\quad \sqrt{n_2}=1-1,\ =0.$$

Here $\sqrt{l_1}+\sqrt{m_1}-\sqrt{n_1}=0$, which gives one of the trizomals a cubic, viz., this is the trizomal

$$\left(1-\frac{a}{c}\right)\sqrt{\mathfrak{A}^\circ}+\left(1+\frac{a}{c}\right)\sqrt{\mathfrak{B}^\circ}+2\sqrt{\mathfrak{C}^\circ}=0.$$

The other trizomal reduces itself to the bizomal $\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ}=0$, which regarded as a trizomal, or written under the form $(\sqrt{\mathfrak{A}^\circ}+\sqrt{\mathfrak{B}^\circ})^2=0$, is the line $\mathfrak{A}^\circ-\mathfrak{B}^\circ=0$ twice, viz., this is the radical axis of the circles A_1, B_1 twice; and the order is thus $=2$. By what precedes, the line in question is in fact the common radical axis of the circles A, B and of the circles C, D.

Article Nos. 203 to 205. *Cases of the Decomposable Curve, the Centres in a Line.*

203. We have yet to consider the decomposable case when the centres A, B, C, D are on a line; the equation $a\mathfrak{A}^\circ+b\mathfrak{B}^\circ+c\mathfrak{C}^\circ+d\mathfrak{D}^\circ=0$ here subsists universally, whatever be the radii a'', b'', c'', d''. We establish as before the relation $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}+\frac{p}{d}=0$. The cases are as follows:

I. No further relation between l, m, n, p; order of tetrazomal $=8$, of trizomals 4 and 4.

II. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$; order of tetrazomal $=7$; of trizomals $=4$ and 3; same as II. *supra*.

III. $\sqrt{l}+\sqrt{p}=0$, $\sqrt{m}+\sqrt{n}=0$; order of tetrazomal $=6$; of trizomals 3 and 3; same as III. *supra*.

204. IV. $\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0$, $\mathrm{a}\sqrt{l}+\mathrm{b}\sqrt{m}+\mathrm{c}\sqrt{n}+\mathrm{d}\sqrt{p}=0$; order of tetrazomal $=6$; this is a remarkable case, the orders of the trizomals are either 3, 3 or else 4, 2.

To explain how this is, it is to be noticed that in the absence of any special relation between the radii, the above conditions combined with $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$ give $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$[1]; when l, m, n, p have these values, the case is the same as IV. *supra*, and the orders of the trizomals are 3, 3. But if the radii of the circles satisfy the condition

$$\begin{vmatrix} 1\,, & 1\,, & 1\,, & 1 \\ a\,, & b\,, & c\,, & d \\ a^2\,, & b^2\,, & c^2\,, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{vmatrix}=0,$$

then the two conditions satisfy of themselves the remaining condition $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}+\frac{p}{\mathrm{d}}=0$, and the ratios $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$ instead of being determinate as above, depend on an arbitrary parameter.

We have

$$\sqrt{l_1}=\sqrt{l}+\frac{\mathrm{a}}{\mathrm{d}}\frac{p}{\sqrt{l}},\quad \sqrt{m_1}=\sqrt{m}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{b}\sqrt{n},\quad \sqrt{n_1}=\sqrt{n}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}\frac{p}{l}}\,\mathrm{c}\sqrt{m},$$

and between l, m, n, p only the relations

$$\sqrt{l}+\sqrt{m}+\sqrt{n}+\sqrt{p}=0,\quad a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0.$$

We find first

$$\begin{aligned}\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}&=\sqrt{l}+\sqrt{m}+\sqrt{n}\\ &\quad+\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{\mathrm{a}}{\mathrm{d}}\sqrt{p}-\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\}\\ &=-\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(\mathrm{d}\sqrt{l}-\mathrm{a}\sqrt{p})-\sqrt{\frac{\mathrm{a}}{\mathrm{bc}}}\,(\mathrm{b}\sqrt{n}-\mathrm{c}\sqrt{m})\right\},\end{aligned}$$

[1] Writing x^2, y^2, z^2, w^2 in place of $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$, we have to find x, y, z, w from the conditions

$$\begin{aligned} x+y+z+w&=0,\\ ax+by+cz+dw&=0,\\ \frac{x^2}{\mathrm{a}}+\frac{y^2}{\mathrm{b}}+\frac{z^2}{\mathrm{c}}+\frac{w^2}{\mathrm{d}}&=0,\end{aligned}$$

where the constants are connected by the relation

$$a\mathrm{a}+b\mathrm{b}+c\mathrm{c}+d\mathrm{d}=0.$$

It readily appears that the line represented by the first two equations *touches* the quadric surface in the point $x:y:z:w=\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d}$, so that these are in general the only values of $\sqrt{l}:\sqrt{m}:\sqrt{n}:\sqrt{p}$. In the case next referred to in the text the line lies in the surface, and the values are not determined.

and then

$$(d-a)\sqrt{l} = (b-d)\sqrt{m} + (c-d)\sqrt{n},$$

$$(d-a)\sqrt{p} = (a-b)\sqrt{m} + (a-c)\sqrt{n},$$

whence

$$d\sqrt{l} - a\sqrt{p} = \frac{b-c}{d-a}(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m}),$$

and we have thus

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} - \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m});$$

and similarly

$$\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} + \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m}):$$

(observe that in the case not under consideration $\mathrm{b}\sqrt{n} - \mathrm{c}\sqrt{m} = 0$, and therefore $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0$, $\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = 0$).

In the present case we have

$$\mathrm{a}:\mathrm{b}:\mathrm{c}:\mathrm{d} = (b-c)(c-d)(d-b) : -(c-d)(d-a)(a-c) : (d-a)(a-b)(b-d) : -(a-b)(b-c)(c-a),$$

and thence

$$\frac{\mathrm{ad}}{\mathrm{bc}} = \frac{(b-c)^2}{(d-a)^2},$$

so that only one of the two sums $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1}$, $\sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2}$ is $= 0$, viz., assuming

$$\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}} = \frac{b-c}{d-a},$$

we have $\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0$.

We have then also

$$\begin{aligned} a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} &= a\sqrt{l} + b\sqrt{m} + c\sqrt{n} \\ &\quad + \frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{a\mathrm{a}\sqrt{p}}{\mathrm{d}} - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m})\right\} \\ &= -\frac{\sqrt{p}}{\sqrt{l}}\left\{\frac{1}{\mathrm{d}}(d\mathrm{d}\sqrt{l} - a\mathrm{a}\sqrt{p}) - \sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m})\right\}; \end{aligned}$$

but we find

$$d\mathrm{d}\sqrt{l} - a\mathrm{a}\sqrt{p} = \frac{b-c}{d-a}(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m}),$$

and thence

$$a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} = \frac{\sqrt{p}}{\mathrm{d}\sqrt{l}}\left(\frac{b-c}{d-a} - \sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\right)(b\mathrm{b}\sqrt{n} - c\mathrm{c}\sqrt{m}), = 0,$$

in virtue of $\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}} = \frac{b-c}{d-a}$. Hence $\sqrt{l_1} : \sqrt{m_1} : \sqrt{n_1}, = b-c : c-a : a-b$, or the corresponding trizomal is a conic, but the other trizomal is a quartic.

205. V. $\sqrt{l}=\sqrt{m}=\sqrt{n}=\sqrt{p}$; order of tetrazomal is $=5$; orders of trizomals $=3,\ 2$; same as V. *supra*.

VI. $\sqrt{l}+\sqrt{p}=0,\ \sqrt{m}+\sqrt{n}=0,\ a\sqrt{l}+b\sqrt{m}+c\sqrt{n}+d\sqrt{p}=0$; order of tetrazomal $=5$; orders of trizomals are 3, 2.

We have here

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l},$$

$$\sqrt{m_1}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{b}\sqrt{m},$$

$$\sqrt{n_1}=\sqrt{m}+\sqrt{\frac{\mathrm{a}}{\mathrm{bcd}}}\,\mathrm{c}\sqrt{m},$$

or writing the values of $\sqrt{m_1}$, $\sqrt{n_1}$ in the form

$$\sqrt{m_1}=\sqrt{m}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{b}}{\mathrm{d}}\sqrt{m},$$

$$\sqrt{n_1}=-\sqrt{m}+\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\,\frac{\mathrm{c}}{\mathrm{d}}\sqrt{m},$$

then observing that as before $l=\frac{\mathrm{ad}}{\mathrm{bc}}m$, if to fix the ideas we assume $\sqrt{l}=\sqrt{\frac{\mathrm{ad}}{\mathrm{bc}}}\sqrt{m}$, the equations are

$$\sqrt{l_1}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l} \text{ and similarly } \sqrt{l_2}=\frac{\mathrm{a}+\mathrm{d}}{\mathrm{d}}\sqrt{l}$$

$$\sqrt{m_1}=\sqrt{m}+\frac{\mathrm{b}}{\mathrm{d}}\sqrt{l}, \qquad \sqrt{m_2}=\sqrt{m}-\frac{\mathrm{b}}{\mathrm{d}}\sqrt{l},$$

$$\sqrt{n_1}=-\sqrt{m}+\frac{\mathrm{c}}{\mathrm{d}}\sqrt{l}, \qquad \sqrt{n_2}=\sqrt{m}-\frac{\mathrm{c}}{\mathrm{d}}\sqrt{l},$$

whence

$$\sqrt{l_1}+\sqrt{m_1}+\sqrt{n_1}=0,\quad \sqrt{l_2}-\sqrt{m_2}-\sqrt{n_2}=0.$$

We have moreover

$$a\sqrt{l_1}=\frac{a\mathrm{a}+d\mathrm{d}}{\mathrm{d}}\sqrt{l},$$

$$b\sqrt{m_1}+c\sqrt{n_1}=(b-c)\sqrt{m}+\frac{b\mathrm{b}+c\mathrm{c}}{\mathrm{d}}\sqrt{l},$$

and thence

$$a\sqrt{l_1}+b\sqrt{m_1}+c\sqrt{n_1}=(a-d)\sqrt{l}+(b-c)\sqrt{m}=0,$$

so that

$$\sqrt{l_1}:\sqrt{m_1}:\sqrt{n_1}=b-c:c-a:a-b;$$

the corresponding trizomal is thus a conic, and it has been seen that the other trizomal is a cubic.

VII. If we have $\begin{vmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a^2, & b^2, & c^2, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{vmatrix} = 0$, and $\begin{pmatrix} 1, & 1, & 1, & 1 \\ a, & b, & c, & d \\ a^2, & b^2, & c^2, & d^2 \\ a''^2, & b''^2, & c''^2, & d''^2 \end{pmatrix} (\sqrt{l}, \sqrt{m}, \sqrt{n}, \sqrt{p}) = 0,$

the tetrazomal has a branch ideally containing $(z^3 = 0)$ the line infinity 3 times; order is $= 5$; orders of the trizomals are 3, 2. We have here

$$\sqrt{l} : \sqrt{m} : \sqrt{n} : \sqrt{p} = \mathrm{a} : \mathrm{b} : \mathrm{c} : \mathrm{d},$$

and thence

$$\sqrt{l_1} = \mathrm{a} + \mathrm{d}, \quad \sqrt{l_2} = \mathrm{a} + \mathrm{d},$$

$$\sqrt{m_1} = \mathrm{b} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{m_2} = \mathrm{b} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

$$\sqrt{n_1} = \mathrm{c} + \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}}, \quad \sqrt{n_2} = \mathrm{c} - \sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}},$$

which give

$$\sqrt{l_1} + \sqrt{m_1} + \sqrt{n_1} = 0, \quad \sqrt{l_2} + \sqrt{m_2} + \sqrt{n_2} = 0.$$

Moreover

$$\begin{aligned} a\sqrt{l_1} + b\sqrt{m_1} + c\sqrt{n_1} &= a(\mathrm{a} + \mathrm{d}) + b\mathrm{b} + c\mathrm{c} \\ &\quad - (b - c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= (a - d)\mathrm{d} - (b - c)\sqrt{\frac{\mathrm{bcd}}{\mathrm{a}}} \\ &= \mathrm{d}\left\{(a - d) - (b - c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\}, \end{aligned}$$

and similarly

$$a\sqrt{l_2} + b\sqrt{m_2} + c\sqrt{n_2} = \mathrm{d}\left\{(a - d) + (b - c)\sqrt{\frac{\mathrm{bc}}{\mathrm{ad}}}\right\};$$

whence in virtue of

$$\frac{\mathrm{ad}}{\mathrm{bc}} = \frac{(b - c)^2}{(d - a)^2},$$

one of the two expressions is $= 0$; and the trizomals are thus a conic and a cubic.

Article No. 206. *The Decomposable Curve; Transformation to a different set of Concyclic Foci.*

206. Consider the decomposable case of

$$\sqrt{l}\mathfrak{A} + \sqrt{m}\mathfrak{B} + \sqrt{n}\mathfrak{C} + \sqrt{p}\mathfrak{D} = 0;$$

viz., the points A, B, C, D lie here in a circle, and we have $\frac{l}{\mathrm{a}} + \frac{m}{\mathrm{b}} + \frac{n}{\mathrm{c}} + \frac{p}{\mathrm{d}} = 0$. Taking (A_1, D_1) the antipoints of (A, D); (B_1, C_1) the antipoints of (B, C); then

$\mathfrak{A}_1\mathfrak{D}_1 = \mathfrak{A}\mathfrak{D}$, $\mathfrak{B}_1\mathfrak{C}_1 = \mathfrak{B}\mathfrak{C}$ (No. 65) and referring to the formulæ, *ante*, Nos. 100 *et seq.*, it appears that we can find l_1, m_1, n_1, p_1 such that identically

$$-l\mathfrak{A} + m\mathfrak{B} + n\mathfrak{C} - p\mathfrak{D} = -l_1\mathfrak{A}_1 + m_1\mathfrak{B}_1 + n_1\mathfrak{C}_1 - p_1\mathfrak{D}_1,$$

and moreover that $lp = l_1p_1$, $mn = m_1n_1$.

The equation of the curve gives

$$-l\mathfrak{A} + m\mathfrak{B} + n\mathfrak{C} - p\mathfrak{D} - 2\sqrt{lp\mathfrak{A}\mathfrak{D}} + 2\sqrt{mn\mathfrak{B}\mathfrak{C}} = 0,$$

which may consequently be written

$$-l_1\mathfrak{A}_1 + m_1\mathfrak{B}_1 + n_1\mathfrak{C}_1 - p_1\mathfrak{D}_1 - 2\sqrt{l_1p_1\mathfrak{A}_1\mathfrak{D}_1} + 2\sqrt{m_1n_1\mathfrak{B}_1\mathfrak{C}_1} = 0;$$

viz., this is

$$\sqrt{l_1\mathfrak{A}_1} + \sqrt{m_1\mathfrak{B}_1} + \sqrt{n_1\mathfrak{C}_1} + \sqrt{p_1\mathfrak{D}_1} = 0;$$

that is, the two trizomals expressed by the original tetrazomal equation involving the set of concyclic foci (A, B, C, D) are thus expressed by a new tetrazomal equation involving the different set of concyclic foci (A_1, B_1, C_1, D_1); and we might of course in like manner express the equation in terms of the other two sets of concyclic foci (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3) respectively. It might have been anticipated that such a transformation existed, for we could as regards each of the component trizomals separately pass from the original set to a different set of concyclic foci, and the two trizomal equations thus obtained would, it might be presumed, be capable of composition into a single tetrazomal equation; but the direct transformation of the tetrazomal equation is not on this account less interesting.

Annex I. *On the Theory of the Jacobian.*

Consider any three curves $U=0$, $V=0$, $W=0$, of the same order r, then writing

$$J(U, V, W) = \frac{d(U, V, W)}{d(x, y, z)} = \begin{vmatrix} d_xU, & d_xV, & d_xW \\ d_yU, & d_yV, & d_yW \\ d_zU, & d_zV, & d_zW \end{vmatrix},$$

we have the Jacobian curve $J(U, V, W) = 0$, of the order $3r-3$.

A fundamental property is that if the curves $U=0$, $V=0$, $W=0$ have any common point, this is a point on the Jacobian, and not only so, but it is a node, or double point, that is, for the point in question we have $J=0$, and we have also $d_xJ=0$, $d_yJ=0$, $d_zJ=0$.

It follows that for the three curves $l\Theta + L\Phi = 0$, $m\Theta + M\Phi = 0$, $n\Theta + N\Phi = 0$ ($\Theta=0$ of the order $r-s'$, $\Phi=0$ of the order $r-s$, $l=0$, $m=0$, $n=0$ each of the order s', $L=0$, $M=0$, $N=0$ each of the order s) which have in common the

$(r-s')(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$, each of these points is a node on the Jacobian, and hence that the Jacobian must be of the form

$$J(l\Theta+L\Phi,\ m\Theta+M\Phi,\ n\Theta+N\Phi)=A\Theta^2+2B\Theta\Phi+C\Phi^2=0,$$

where obviously the degrees of A, B, C must be $r+2s'-3$, $r+s+s'-3$, $r+2s-3$ respectively. In the particular case where $s'=0$, that is where l, m, n are constants, we have $A=0$; the Jacobian curve then contains as a factor $(\Phi=0)$, and throwing this out, the curve is $B\Theta+C\Phi=0$, viz., this is a curve of the order $2r+s-3$ passing through each of $r(r-s)$ points of intersection of the curves $\Theta=0$, $\Phi=0$.

In particular, if $r=2$, $s=1$, that is, if the curves are the conics $\Theta+L\Phi=0$, $\Theta+M\Phi=0$, $\Theta+N\Phi=0$, passing through the two points of intersection of the conic $\Theta=0$ by the line $\Phi=0$, then the Jacobian is a conic passing through these same two points, viz., its equation is of the form $\Theta+\Omega\Phi=0$. This intersects any one of the given conics, say $\Theta+L\Phi=0$ in the points $\Theta=0$, $\Phi=0$, and in two other points $\Theta+\Omega\Phi=0$, $\Omega-L=0$; at *each* of the last-mentioned points, the tangents to the two curves, and the lines drawn to the two points $\Theta=0$, $\Phi=0$, form a harmonic pencil.

Although this is, in fact, the known theorem that the Jacobian of three circles is their orthotomic circle, yet it is, I think, worth while to give a demonstration of the theorem as above stated in reference to the conics through two given points.

Taking $(z=0,\ x=0)$, $(z=0,\ y=0)$ for the two given points $\Theta=0$, $\Phi=0$, the general equation of a conic through the two points is a quadric equation containing terms in z^2, zx, zy, xy; taking any two such conics

$$\begin{aligned} cz^2+2fyz+2gzx+2hxy&=0,\\ Cz^2+2Fyz+2Gzx+2Hxy&=0,\end{aligned}$$

these intersect in the two points $(x=0,\ z=0)$, $(y=0,\ z=0)$ and in two other points; let (x, y, z) be the coordinates of either of the last-mentioned points, and take (X, Y, Z) as current coordinates, the equations of the lines to the fixed points and of the two tangents are

$$\begin{aligned} Xz-Zx&=0, \qquad Yz-Zy=0,\\ (hy+gz)(Xz-Zx)+(hx+fz)(Yz-Zy)&=0,\\ (Hy+Gz)(Xz-Zx)+(Hx+Fz)(Yz-Zy)&=0,\end{aligned}$$

whence the condition for the harmonic relation is

$$(hy+gz)(Hx+Fz)+(hx+fz)(Hy+Gz)=0,$$

that is

$$(fG+gF)z^2+(hF+fH)yz+(gH+hG)zx+2hHxy=0,$$

but from the equations of the two conics multiplying by $\frac{1}{2}H$, $\frac{1}{2}h$ and adding, we have

$$\tfrac{1}{2}(cH+hC)z^2+(hF+fH)yz+(gH+hG)zx+2hHxy=0;$$

viz., the condition is thus reduced to

$$cH+hC-2(fG+gF)=0,$$

so that this condition being satisfied for one of the points in question, it will be satisfied for the other of them. Now for the three conics

$$cz^2 + 2f\ yz + 2g\ zx + 2h\ xy = 0,$$
$$c'z^2 + 2f'\ yz + 2g'\ zx + 2h'xy = 0,$$
$$c''z^2 + 2f''yz + 2g''zx + 2h''xy = 0,$$

forming the Jacobian, and throwing out the factor z, we may write the equation in the form

$$Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

where the values are

$$\begin{aligned} C &= g\,(f'c'' - f''c') + g'\,(f''c - fc'') + g''\,(fc' - f'c), \\ H &= g\,(h'f'' - h''f') + g'\,(h''f - hf'') + g''\,(hf' - h'f), \\ 2F &= h\,(f'c'' - f''c') + h'\,(f''c - fc'') + h''\,(fc' - f'c), \\ 2G &= h\,(c'g'' - c''g') + h'\,(c''g - cg'') + h''\,(cg' - c'g)\,; \end{aligned}$$

and we thence obtain

$$\begin{aligned} cH + hC &= -\,(fg' - f'g)\,(c''h - ch'') + (f''g - fg'')\,(ch' - c'h) \\ &= \ \ 2\,(fG + gF), \end{aligned}$$

viz., the condition is satisfied in regard to the Jacobian and the first of the three conics; and it is therefore also satisfied in regard to the Jacobian and the other two conics respectively.

I do not know any general theorem in regard to the Jacobian which gives the foregoing theorem of the orthotomic circle. It may be remarked that the use in the Memoir of the theorem of the orthotomic circle is not so great as would at first sight appear: it fixes the ideas to speak of the orthotomic circle of three given circles rather than of their Jacobian, but we are concerned with the orthotomic circle less as the circle which cuts at right angles the given circles than as a circle standing in a known relation to the given circles.

ANNEX II. *On* CASEY'S *Theorem for the Circle which touches three given Circles.*

The following two problems are identical:

1. To find a circle touching three given circles.

2. To find a cone-sphere (sphere the radius of which is $=0$) passing through three given points in space.

In fact, in the first problem if we use z to denote a given constant (which may be $=0$), then taking a, a' and $i(z-a'')$ for the coordinates of the centre and for the radius of one of the given circles; and similarly b, b', $i(z-b'')$; c, c', $i(z-c'')$ for the

other two given circles; and S, S', $i(z-S'')$ for the required circle; the equations of the given circles will be

$$(x-a)^2+(y-a')^2+(z-a'')^2=0,$$
$$(x-b)^2+(y-b')^2+(z-b'')^2=0,$$
$$(x-c)^2+(y-c')^2+(z-c'')^2=0,$$

and that of the required circle will be

$$(x-S)^2+(y-S')^2+(z-S'')^2=0.$$

In order that this may touch the given circles, the distances of its centre from the centres of the given circles must be $i(S''-a'')$, $i(S''-b'')$, $i(S''-c'')$ respectively; the conditions of contact then are

$$(S-a)^2+(S'-a')^2+(S''-a'')^2=0,$$
$$(S-b)^2+(S'-b')^2+(S''-b'')^2=0,$$
$$(S-c)^2+(S'-c')^2+(S''-c'')^2=0,$$

or we have from these equations to determine S, S', S''. But taking (a, a', a''), (b, b', b''), (c, c', c'') for the coordinates of three given points in space, and (S, S', S'') for the coordinates of the centre of the cone-sphere through these points, we have the very same equations for the determination of (S, S', S''), and the identity of the two problems thus appears.

I will presently give the direct analytical solution of this system of equations. But to obtain a solution in the form required, I remark that the equation of the cone-sphere in question is nothing else than the relation that exists between the coordinates of any four points on a cone-sphere; to find this, consider any five points in space, 1, 2, 3, 4, 5; and let $\overline{12}$, &c. denote the distances between the points 1 and 2, &c.; then we have between the distances of the five points the relation

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ 1, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ 1, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ 1, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ 1, & \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0;$$

whence taking 5 to be the centre of the cone-sphere through the points 1, 2, 3, 4, we have $\overline{15}=\overline{25}=\overline{35}=\overline{45}=0$; and the equation becomes

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0,$$

which is the relation between the distances of any four points on a cone-sphere; this equation may be written under the irrational form

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0.$$

Taking (a, a', a''), (b, b', b''), (c, c', c''), (x, y, z) for the coordinates of the four points respectively, we have

$$\overline{23}=\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2},\quad \overline{14}=\sqrt{(x-a)^2+(y-a')^2+(z-a'')^2},$$
$$\overline{31}=\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2},\quad \overline{24}=\sqrt{(x-b)^2+(y-b')^2+(z-b'')^2},$$
$$\overline{12}=\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2},\quad \overline{34}=\sqrt{(x-c)^2+(y-c')^2+(z-c'')^2},$$

or the symbols having these significations, we have

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0$$

for the equation of the cone-sphere through the three points; or rather (since the rational equation is of the order 4 in the coordinates (x, y, z)) this is the equation of the pair of cone-spheres through the three given points; and similarly it is in the first problem the equation of a pair of circles each touching the three given circles respectively.

In the first problem the radii of the given circles were $i(z-a'')$, $i(z-b'')$, $i(z-c'')$ respectively; denoting these radii by α, β, γ, or taking the equations of the given circles to be

$$(x-a)^2+(y-a')^2-\alpha^2=0,$$
$$(x-b)^2+(y-b')^2-\beta^2=0,$$
$$(x-c)^2+(y-c')^2-\gamma^2=0,$$

the symbols then are

$$\overline{23}=\sqrt{(b-c)^2+(b'-c')^2-(\beta-\gamma)^2},\quad \overline{14}=\sqrt{(x-a)^2+(y-a')^2-\alpha^2},$$
$$\overline{31}=\sqrt{(c-a)^2+(c'-a')^2-(\gamma-\alpha)^2},\quad \overline{24}=\sqrt{(x-b)^2+(y-b')^2-\beta^2},$$
$$\overline{12}=\sqrt{(a-b)^2+(a'-b')^2-(\alpha-\beta)^2},\quad \overline{34}=\sqrt{(x-c)^2+(y-c')^2-\gamma^2},$$

and the equation of the pair of circles is as before

$$\overline{23}\,.\,\overline{14}+\overline{31}\,.\,\overline{24}+\overline{12}\,.\,\overline{34}=0;$$

where it is to be noticed that $\overline{23}$, $\overline{31}$, $\overline{12}$ are the tangential distances of the circles 2 and 3, 3 and 1, 1 and 2 respectively; viz., if α, β, γ are the radii taken positively, then these are the direct tangential distances. By taking the radii positively or negatively at pleasure, we obtain in all four equations—the tangential distances being all direct as above, or else any one is direct, and the other two are inverse; we have thus the four pairs of tangent circles.

The cone-spheres which pass through a given circle are the two spheres which have their centres in the two antipoints of the given circle; and it is easy to see that the foregoing investigation gives the following (imaginary) construction of the

tangent circles; viz., given any three circles A, B, C in the same plane, to draw the tangent circles. Taking the antipoints of the three circles, then selecting any three antipoints (one for each circle) so as to form a triad, we have in all four complementary pairs of triads. Through a triad, and through the complementary triad draw two circles, these are situate symmetrically on opposite sides of the plane; and combining each antipoint of the first circle with the symmetrically situated antipoint of the second circle, we have two pairs of points, the points of each pair being symmetrically situate in regard to the plane, and having therefore an anticircle in this plane; these two anticircles are a pair of tangent circles; and the four pairs of complementary triads give in this manner the four pairs of tangent circles.

I return to the equations

$$\begin{aligned}
(x-S)^2 &+ (y-S')^2 + (z-S'')^2 = 0,\\
(a-S)^2 &+ (a'-S')^2 + (a''-S'')^2 = 0,\\
(b-S)^2 &+ (b'-S')^2 + (b''-S'')^2 = 0,\\
(c-S)^2 &+ (c'-S')^2 + (c''-S'')^2 = 0;
\end{aligned}$$

by eliminating (S, S', S'') from these equations we shall obtain the equation of the pair of cone-spheres through the points (a, a', a''), (b, b', b''), (c, c', c''). Write $x-S$, $y-S'$, $z-S''=X$, Y, Z, then we have $X^2+Y^2+Z^2=0$, and, putting for shortness

$$\begin{aligned}
\mathfrak{A} &= (a-x)^2 + (a'-y)^2 + (a''-z)^2,\\
\mathfrak{B} &= (b-x)^2 + (b'-y)^2 + (b''-z)^2,\\
\mathfrak{C} &= (c-x)^2 + (c'-y)^2 + (c''-z)^2,
\end{aligned}$$

then, by means of the equation just obtained, the other three equations become

$$\begin{aligned}
\mathfrak{A} + 2\left[(a-x)X + (a'-y)Y + (a''-z)Z\right] &= 0,\\
\mathfrak{B} + 2\left[(b-x)X + (b'-y)Y + (b''-z)Z\right] &= 0,\\
\mathfrak{C} + 2\left[(c-x)X + (c'-y)Y + (c''-z)Z\right] &= 0.
\end{aligned}$$

These last equations give

$$\begin{aligned}
X : Y : Z = \ & \lambda\mathfrak{A} + \mu\mathfrak{B} + \nu\,\mathfrak{C}\\
: & \lambda'\mathfrak{A} + \mu'\mathfrak{B} + \nu'\,\mathfrak{C}\\
: & \lambda''\mathfrak{A} + \mu''\mathfrak{B} + \nu''\mathfrak{C},
\end{aligned}$$

where

$$\begin{aligned}
\lambda &= b'c'' - b''c' + (c' - b')z - (c'' - b'')y,\\
\mu &= c'a'' - c''a' + (a' - c')z - (a'' - c'')y,\\
\nu &= a'b'' - a''b' + (b' - a')z - (b'' - a'')y,\\
\lambda' &= b''c - bc'' + (c'' - b'')x - (c - b)z,\\
\mu' &= c''a - ca'' + (a'' - c'')x - (a - c)z,\\
\nu' &= a''b - ab'' + (b'' - a'')x - (b - a)z,\\
\lambda'' &= bc' - b'c + (c - b)y - (c' - b')x,\\
\mu'' &= ca' - c'a + (a - c)y - (a' - c')x,\\
\nu'' &= ab' - a'b + (b - a)y - (b' - a')x;
\end{aligned}$$

and the result of the elimination then is

$$(\lambda\mathfrak{A}+\mu\mathfrak{B}+\nu\mathfrak{C})^2+(\lambda'\mathfrak{A}+\mu'\mathfrak{B}+\nu'\mathfrak{C})^2+(\lambda''\mathfrak{A}+\mu''\mathfrak{B}+\nu''\mathfrak{C})^2=0.$$

But substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ their values, and writing, for shortness,

$$\begin{aligned}
-i &= b'c''-b''c'+c'a''-c''a'+a'b''-a''b,\\
-j &= b''c-bc''+c''a-ca''+a''b-ab'',\\
-k &= bc'-b'c+ca'-c'a+ab'-a'b,\\
\Delta &= a(b'c''-b''c)+a'(b''c-bc'')+a''(bc'-b'c),\\
-p &= (b'c''-b''c')(a^2+a'^2+a''^2)+(c'a''-c''a')(b^2+b'^2+b''^2)+(a'b''-a''b')(c^2+c'^2+c''^2),\\
-q &= (b''c-bc'')(a^2+a'^2+a''^2)+(c''a-ca'')(b^2+b'^2+b''^2)+(a''b-ab'')(c^2+c'^2+c''^2),\\
-r &= (bc'-b'c)(a^2+a'^2+a''^2)+(ca'-c'a)(b^2+b'^2+b''^2)+(ab'-ab')(c^2+c'^2+c''^2),\\
-l &= (c-b)(a^2+a'^2+a''^2)+(a-c)(b^2+b'^2+b''^2)+(b-a)(c^2+c'^2+c''^2),\\
-m &= (c'-b')(a^2+a'^2+a''^2)+(a'-c')(b^2+b'^2+b''^2)+(b'-a')(c^2+c'^2+c''^2),\\
-n &= (c''-b'')(a^2+a'^2+a''^2)+(a''-c'')(b^2+b'^2+b''^2)+(b''-a'')(c^2+c'^2+c''^2),
\end{aligned}$$

we find

$$\begin{aligned}
\lambda\mathfrak{A}+\mu\mathfrak{B}+\nu\mathfrak{C}\\
&= -\ i(x^2+y^2+z^2)\\
&\quad+2i(x^2+y^2+z^2)-2x(ix+jy+kz)-2\Delta x+ny-mz-p,
\end{aligned}$$

with similar expressions for $\lambda'\mathfrak{A}+\mu'\mathfrak{B}+\nu'\mathfrak{C}$, $\lambda''\mathfrak{A}+\mu''\mathfrak{B}+\nu''\mathfrak{C}$, and the result is

$$\begin{aligned}
&\{i(x^2+y^2+z^2)-2x(ix+jy+kz)-2\Delta x+ny-mz-p\}^2\\
&+\{j(x^2+y^2+z^2)-2y(ix+jy+kz)-nx-2\Delta y+lz-q\}^2\\
&+\{k(x^2+y^2+z^2)-2z(ix+jy+kz)+mx-ly-2\Delta z-r\}^2=0,
\end{aligned}$$

viz., this is

$$\begin{aligned}
&(x^2+y^2+z^2)^2(i^2+j^2+k^2)\\
&+(x^2+y^2+z^2)\{4\Delta(ix+jy+kz)+2\big(i(ny-mz)+j(lz-nx)+k(mx-ly)\big)\\
&\qquad\qquad+4\Delta^2-2(ip+jq+kr)+(l^2+m^2+n^2)\}\\
&-(lx+my+nz)^2+4(ix+jy+kz)(px+qy+rz)\\
&+4\Delta(px+qy+rz)-2\big(p(ny-mz)+q(lz-nx)+r(mx-ly)\big)\\
&+p^2+q^2+r^2=0,
\end{aligned}$$

viz., this is in the rational form the equation of the pair of cone-spheres. The function on the left-hand side must, it is clear, be save to a numerical factor the norm of

$$\begin{aligned}
&\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}\,.\,\sqrt{(x-a)^2+(y-a')^2+(z-a'')^2}\\
&+\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}\,.\,\sqrt{(x-b)^2+(y-b')^2+(z-b'')^2}\\
&+\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\,.\,\sqrt{(x-c)^2+(y-c')^2+(z-c'')^2},
\end{aligned}$$

the numerical factor of the expression in question is in fact $=-4$, that is, the norm is

$$=-4(x^2+y^2+z^2)^2(i^2+j^2+k^2)+\&c.;$$

so that attending only to the highest powers in (x, y, z) we ought to have

$$\text{Norm}\,\{\sqrt{(b-c)^2+(b'-c')^2+(b''-c'')^2}+\sqrt{(c-a)^2+(c'-a')^2+(c''-a'')^2}+\sqrt{(a-b)^2+(a'-b')^2+(a''-b'')^2}\}$$
$$=-4(i^2+j^2+k^2).$$

It is easy to see that the norm is in fact composed of the terms

$$\begin{aligned}&2(b'-c')^2\{\ (b-c)^2-(c-a)^2-(a-b)^2\},\\ &+2(c'-a')^2\{-(b-c)^2+(c-a)^2-(a-b)^2\},\\ &+2(a'-b')^2\{-(b-c)^2-(c-a)^2+(a-b)^2\},\end{aligned}$$

and of the similar terms (a, b, c), (a'', b'', c''), and in (a', b', c'), (a'', b'', c''); the above written terms are $=-4$ into

$$(b'-c')^2(a-b)(a-c)+(c'-a')^2(b-c)(b-a)+(a'-b')^2(c-a)(c-b),$$

which is

$$\begin{aligned}&=a'^2(b-c)^2+b'^2(c-a)^2+c'^2(a-b)^2\\ &\quad+2b'c'(a-b)(c-a)+2c'a'(b-c)(a-b)+2a'b'(c-a)(b-c),\\ &=\{a'(b-c)+b'(c-a)+c'(a-b)\}^2\\ &=k^2;\end{aligned}$$

and the value of the norm is thus $=-4(i^2+j^2+k^2)$, as it should be.

ANNEX III. *On the Norm of* $(b-c)\sqrt{\mathfrak{A}^\circ}+(c-a)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}$, *when the Centres are in a Line.*

The norm of $\sqrt{U}+\sqrt{V}+\sqrt{W}$ is

$$=(1, 1, 1, -1, -1, -1\between U, V, W)^2,$$

whence that of $\sqrt{U+U'}+\sqrt{V+V'}+\sqrt{W+W'}$ is

$$\begin{aligned}&=(1, 1, 1, -1, -1, -1\between U, V, W)^2\\ &+(1, 1, 1, -1, -1, -1\between U', V', W')^2\\ &+2(1, 1, 1, -1, -1, -1\between U, V, W\between U', V', W'),\end{aligned}$$

where the last term is $=2$ into

$$U'(U-V-W)+V'(-U+V-W)+W'(-U-V+W);$$

and the norm of $\sqrt{U+U'+U''}+\sqrt{V+V'+V''}+\sqrt{W+W'+W''}$ is obviously composed in a similar manner.

Now, applying the formula to obtain the norm of

$$(b-c)\sqrt{a^2+\theta+\alpha}+(c-a)\sqrt{b^2+\theta+\beta}+(a-b)\sqrt{c^2+\theta+\gamma},$$

the expression contains six terms, two of which are at once seen to vanish; and writing for shortness $(\,,,)$ in place of $(1, 1, 1, -1, -1, -1)$ the remaining terms are

$$\begin{aligned}
&(\,,,)\left((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\right)^2\\
&+2\ (\,,,)\left((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\right)\left((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\right)\\
&+2\theta\,(\,,,)\left((b-c)^2\alpha,\ (c-a)^2\beta,\ (a-b)^2\gamma\right)\left((b-c)^2\ ,\ (c-a)^2\ ,\ (a-b)^2\ \right)\\
&+2\theta\,(\,,,)\left((b-c)^2a^2,\ (c-a)^2b^2,\ (a-b)^2c^2\right)\left((b-c)^2\ ,\ (c-a)^2\ ,\ (a-b)^2\ \right);
\end{aligned}$$

the first of these terms requires no reduction; the second, omitting the factor 2, is

$$\begin{aligned}
&(b-c)^2\alpha\,[\ \ (b-c)^2a^2-(c-a)^2b^2-(a-b)^2c^2]\\
&+(c-a)^2\beta\,[-(b-c)^2a^2+(c-a)^2b^2-(a-b)^2c^2]\\
&+(a-b)^2\gamma\,[-(b-c)^2a^2-(c-a)^2b^2+(a-b)^2c^2];
\end{aligned}$$

which is

$$=2(a-b)(b-c)(c-a)\,[bc(b-c)\alpha+ca(c-a)\beta+ab(a-b)\gamma].$$

Similarly the third term, omitting the factor 2θ, is

$$\begin{aligned}
&(b-c)^2\alpha\,[\ \ (b-c)^2-(c-a)^2-(a-b)^2]\\
&+(c-a)^2\beta\,[-(b-c)^2+(c-a)^2-(a-b)^2]\\
&+(a-b)^2\gamma\,[-(b-c)^2-(c-a)^2+(a-b)^2],
\end{aligned}$$

which is

$$=2(a-b)(b-c)(c-a)\,[(b-c)\alpha+(c-a)\beta+(a-b)\gamma],$$

and for the last term, omitting the factor 2θ, this may be deduced therefrom by writing (a^2, b^2, c^2) in place of (α, β, γ), viz., it is

$$=-2(a-b)^2(b-c)^2(c-a)^2.$$

Hence, restoring the omitted factors, and collecting, we find

$$\begin{aligned}
&\text{Norm}\ \{(b-c)\sqrt{a^2+\theta+\alpha}+(c-a)\sqrt{b^2+\theta+\beta}+(a-b)\sqrt{c^2+\theta+\gamma}\}\\
&=(b-c)^4\alpha^2+(c-a)^4\beta^2+(a-b)^4\gamma^2-2(c-a)^2(a-b)^2\beta\gamma-2(a-b)^2(b-c)^2\gamma\alpha-2(b-c)^2(c-a)^2\alpha\beta\\
&\quad+4\theta\,(a-b)\ (b-c)\ (c-a)\ [\ \ (b-c)\alpha+\ \ (c-a)\beta+\ \ (a-b)\gamma]\\
&\quad+4\ \ (a-b)\ (b-c)\ (c-a)\ [bc(b-c)\alpha+ca(c-a)\beta+ab(a-b)\gamma]\\
&\quad-4\theta\,(a-b)^2(b-c)^2(c-a)^2.
\end{aligned}$$

Hence, first writing $a-x$, $b-x$, $c-x$ in place of a, b, c; then y^2 for θ, and $(-a''^2, -b''^2, -c''^2)$ for (α, β, γ); and finally introducing z for homogeneity, we find

$$\begin{aligned}
&\text{Norm } \{(b-c)\sqrt{(x-az)^2+y^2-a''^2z^2}+(c-a)\sqrt{}+(a-b)\sqrt{}\}=z^2 \text{ into}\\
&\quad z^2\big((b-c)^4a''^4+(c-a)^4b''^4+(a-b)^4c''^4\\
&\qquad -2(c-a)^2(a-b)^2b''^2c''^2-2(a-b)^2(b-c)^2c''^2a''^2-2(b-c)^2(c-a)^2a''^2b''^2\big)\\
&-4y^2\ (b-c)(c-a)(a-b)[\ (b-c)a''^2+(c-a)b''^2+(a-b)c''^2]\\
&-4\ \ (b-c)(c-a)(a-b)\{\ (b-c)a''^2(z^2bc-zx(b+c)+x^2)\\
&\qquad\qquad +(c-a)b''^2(z^2ca-zx(c+a)+x^2)\\
&\qquad\qquad +(a-b)c''^2(z^2ab-zx(a+b)+x^2)\}\\
&-4y\ \ (b-c)^2(c-a)^2(a-b)^2,
\end{aligned}$$

so that the equation $(b-c)\sqrt{\mathfrak{A}^\circ}+(c-a)\sqrt{\mathfrak{B}^\circ}+(a-b)\sqrt{\mathfrak{C}^\circ}=0$, in its rationalised form, contains $(z^2=0)$ the line infinity twice, and the curve is thus a conic. If $a''^2=b''^2=c''^2=k''^2$, then the expression of the norm is

$$=z^2 \text{ into } -4(a-b)^2(b-c)^2(c-a)^2(y^2-k''^2z^2),$$

viz., when the three circles have each of them the same radius k'', the curve is the pair of parallel lines $y^2-k''^2z^2=0$; and in particular when $k''=0$, or the circles reduce themselves each to a point, then the curve is $y^2=0$, the axis twice.

ANNEX IV. *On the Trizomal Curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, which have a Cusp, or two Nodes.*

The trizomal curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, has not in general any nodes or cusps: in the particular case where the zomal curves are circles, we have however seen how the ratios $l:m:n$ may be determined so that the curve shall acquire a node, two nodes, or a cusp; viz., regarding a, b, c as current areal coordinates, we have here a conic $\frac{l}{\mathrm{a}}+\frac{m}{\mathrm{b}}+\frac{n}{\mathrm{c}}=0$, the locus of the centres of the variable circle, and the solution depends on establishing a relation between this conic and the orthotomic circle or Jacobian of the three given circles. I have in my paper "Investigations in connection with Casey's Equation," *Quart. Math. Jour.* vol. VIII. (1867), pp. 334—342, [395] given, after Professor Cremona, a solution of the general question to find the number of the curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, which have a cusp, or which have two nodes, and I will here reproduce the leading points of the investigation. I remark, that although one of the loci involved in it is the same as that occurring in the case of the three circles (viz., we have in each case the Jacobian of the given curves), the other two loci Σ and Δ, which present themselves, seem to have no relation to the conic of centres which is made use of in the particular case.

We have the curves $U=0$, $V=0$, $W=0$, each of the same order r; and considering a point the coordinates whereof are (l, m, n), we regard as corresponding to this point the curve $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, say for shortness, the curve Ω, being as above a curve of the order $2r$, having r^2 contacts with each of the given curves $U=0$, $V=0$, $W=0$. As long as the point (l, m, n) is arbitrary, the curve Ω has not any node, and in order that this curve may have a node, it is necessary that the point (l, m, n) shall lie on a certain curve Δ; this being so, the node will, it is easy to see, lie on the curve J, the Jacobian of the three given curves; and the curves J and Δ will correspond to each other point to point, viz., taking for (l, m, n) any point whatever on the curve Δ, the curve Ω will have a node at some one point of J; and conversely, in order that the curve Ω may be a curve having a node at a given point of J, the point (l, m, n) must be at some one point of the curve Δ. The curve Δ has, however, nodes and cusps; each node of Δ corresponds to two points of J, viz., for (l, m, n) at a node of Δ, the curve Ω is a binodal curve having a node at each of the corresponding points of J; each cusp of Δ corresponds to two coincident points of J, viz. for (l, m, n) at a cusp of Δ, the curve Ω has a node at the corresponding point of J. The number of the binodal curves Ω is thus equal to the number of the nodes of Δ, and the number of the cuspidal curves Ω is equal to the number of the cusps of Δ; and the question is to find the Plückerian numbers of the curve Δ. This Professor Cremona accomplished in a very ingenious manner, by bringing the curve Δ into connexion with another curve Σ (viz., Σ is the locus of the nodes of those curves $lU+mV+nW=0$ which have a node), and the result arrived at is that for the curve Δ

$$\begin{aligned}
&\text{Order} &&= 3(r-1)(3r-2),\\
&\text{Class} &&= 6(r-1)^2,\\
&\text{Nodes} &&= \tfrac{3}{2}(r-1)(27r^2-63r^2+22r+16),\\
&\text{Cusps} &&= 3(r-1)(7r-8),\\
&\text{Double tangents} &&= \tfrac{3}{2}(r-1)(12r^3-36r^2+19r+16),\\
&\text{Inflexions} &&= 12(r-1)(r-2);
\end{aligned}$$

so that, finally, the number of the cuspidal curves $\sqrt{lU}+\sqrt{mV}+\sqrt{nW}=0$, is found to be $=3(r-1)(7r-8)$, and the number of the binodal curves of the same form is found to be $=\frac{3}{2}(r-1)(27r^3-63r^2+22r+16)$. When the given curves are conics, or for $r=2$, these numbers are $=18$ and 36 respectively; but the formulæ are not applicable to the case where the conics have a point or points of intersection in common; nor, consequently, to the case of the three circles.

415.

CORRECTIONS AND ADDITIONS TO THE MEMOIR ON THE THEORY OF RECIPROCAL SURFACES (*Phil. Trans.* vol. CLIX. 1869, [411]).

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXII. (for the year 1872), pp. 83—87. Received July 22,—Read November 16, 1871.]

1. I AM indebted to Dr Zeuthen for the remark that although the "off-points" and "off-planes," as explained in the memoir, are real singularities, they are not the singularities to which the θ, θ' of the formulæ refer. The most convenient way of correcting this is to retain all the formulæ with θ, θ' as they stand, but to write ω, ω' for the number of "off-points" and "off-planes" respectively; viz. we thus have

ω, off-points,

θ, unexplained singular points,

and

ω', off-planes,

θ', unexplained singular planes,

the formulæ as they stand, taking account of the unexplained singularities θ and θ', but not taking any account at all of the off-points and off-planes ω, ω'. The extended formulæ in which these are taken into account are:

$$\begin{aligned}
a\,(n-2) &= \kappa - B + \rho + 2\sigma + 3\omega,\\
b\,(n-2) &= \qquad\quad \rho + 2\beta + 3\gamma + 3t,\\
c\,(n-2) &= 2\sigma + 4\beta + \gamma + \theta + \omega,\\
a\,(n-2)(n-3) &= 2\,(\delta - C - 3\omega) + 3\,(ac - 3\sigma - \chi - 3\omega) + 2\,(ab - 2\rho - j),\\
b\,(n-2)(n-3) &= \quad 4k \quad + \quad (ab - 2\rho - j) \quad + 3\,(bc - 3\beta - 2\gamma - i),\\
c\,(n-2)(n-3) &= \quad 6h \quad + \quad (ac - 3\sigma - \chi - 3\omega) + 2\,(bc - 3\beta - 2\gamma - i),
\end{aligned}$$

which replace Salmon's original formulæ (A) and (B).

2. In the formulæ

$$q = b^2 - b - 2k - 3\gamma - 6t,$$
$$r = c^2 - c - 2h - 3\beta,$$

it is assumed that the nodal curve has no actual multiple points other than the t triple points, and no stationary points other than the γ points which lie on the cuspidal curve; and similarly that the cuspidal curve has no actual multiple points, and no stationary points other than the β points which lie on the nodal curve; and this being so, q is the class of the nodal curve and r that of the cuspidal curve. But we may take the formulæ as *universally* true; viz. q may be considered as standing for $b^2 - b - 2k - 3\gamma - 6t$, and r as standing for $c^2 - c - 2h - 3\beta$; only then q and r are not in all cases the classes of the two curves respectively.

3. In the formulæ No. 6 *et seq.*, introducing the new singularity ω, we have as follows:

$$(a - b - c)(n - 2) \quad = (\kappa - B - \theta + 2\omega) - 6\beta - 4\gamma - 3t,$$
$$(a - 2b - 3c)(n - 2)(n - 3) = 2(\delta - C - 3\omega) - 8k - 18h - 12(bc - 3\beta - 2\gamma - i);$$

and substituting these in $n' = a(a-1) - 2b - 3c$, and writing for n' its value $= a(a-1) - 2\delta - 3\kappa$, we have, as in the memoir,

$$\begin{aligned} n' = n(n-1)^2 - n(7b + 12c) + 4b^2 + 8b + 9c^2 + 15c \\ - 8k - 8h + 18\beta + 12\gamma + 12i - 9t \\ - 2C - 3B - 3\theta; \end{aligned}$$

viz. there is no term in ω.

Writing $(n-2)(n-3) = a + 2b + 3c + (-4n + 6)$ in the equations which contain $(n-2)(n-3)$, these become

$$\begin{aligned} a(-4n+6) &= 2(\delta - C) - a^2 - 4\rho - 9\sigma - 2j - 3\chi - 15\omega, \\ b(-4n+6) &= 4k - 2b^2 - 9\beta - 6\gamma - 3i - 2\rho - j, \\ c(-4n+6) &= 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma - \chi - 3\omega, \end{aligned}$$

(Salmon's equations (C)); and adding to each equation four times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2 - 2a &= 2(\delta - C) + 4(\kappa - B) - \sigma - 2j - 3\chi - 3\omega, \\ 2b^2 - 2b &= 4k - \beta + 6\gamma + 12t - 3i + 2\rho - j, \\ 3c^2 - 2c &= 6h + 10\beta + 4\theta - 2i + 5\sigma - \chi + \omega. \end{aligned}$$

Writing in the first of these $a^2 - 2a = n' + 2\delta + 3\kappa - a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n' - a &= -2C - 4B + \kappa - \sigma - 2j - 3\chi - 3\omega, \\ 2q + \beta + 3i + j &= 2\rho, \\ 3r + c + 2i + \chi &= 5\sigma + \beta + 4\theta + \omega. \end{aligned}$$

The reciprocal of the first of these is

$$\sigma' = a - n + \kappa' - 2j' - 3\chi' - 2C' - 4B' - 3\omega';$$

viz. writing $a = n(n-1) - 2b - 3c$, and $\kappa = 3n(n-2) - 6b - 8c$, this is

$$\sigma' = 4n(n-2) - 8b - 11c - 2j' - 3\chi' - 2C' - 4B' - 3\omega';$$

and it thus appears that the order σ' of the spinode curve is reduced by 3 for each off-plane ω'.

4. As to the other two equations, writing for ρ, σ their values, these become

$$j + 6t + 3i + 5\beta + 6\gamma = b(2n - 4) - 2q,$$
$$2\chi + 3\omega + 4i + 18\beta + 5\gamma = c(5n - 12) - 6r + 3\theta,$$

equations which admit of a geometrical interpretation. In fact, when there is only a nodal curve, the first equation is

$$j + 6t = b(2n - 4) - 2q,$$

which we may verify when the nodal curve is a complete intersection, $P = 0$, $Q = 0$; for if the equation of the surface is $(A, B, C \between P, Q)^2 = 0$, where the degrees of A, B, C, P, Q are $n - 2f$, $n - f - g$, $n - 2g$, f, g respectively, then the pinch-points are given by the equations $P = 0$, $Q = 0$, $AC - B^2 = 0$, and the number j of pinch-points is thus

$$= fg(2n - 2f - 2g), = (2n - 4)fg - 2fg(f + g - 2);$$

but for the curve $P = 0$, $Q = 0$ we have $t = 0$, and its order and class are $b = fg$, $q = fg(f + g - 2)$, or the formula is thus verified.

Similarly, when there is only a cuspidal curve, the second equation is

$$2\chi + 3\omega = c(5n - 12) - 6r + 3\theta,$$

which may be verified when the cuspidal curve is a complete intersection, $P = 0$, $Q = 0$; the equation of the surface is here $(A, B, C \between P, Q)^2 = 0$, where $AC - B^2 = MP + NQ$, and the points χ, ω are given as the intersections of the curve with the surface $(A, B, C \between N, -M)^2 = 0$.

Now $AC - B^2$ vanishing for $P = 0$, $Q = 0$ we must have $A = \Lambda\alpha^2 + A'$, $B = \Lambda\alpha\beta + B'$, $C = \Lambda\beta^2 + C'$, where A', B', C' vanish for $P = 0$, $Q = 0$; and thence $M = \Lambda M' + M''$, $N = \Lambda N' + N''$, where M'', N'' vanish for $P = 0$, $Q = 0$. The equation

$$(A, B, C \between N, -M)^2 = 0,$$

writing therein $P = 0$, $Q = 0$, thus becomes $\Lambda^3(N'\alpha - M'\beta)^2 = 0$; and its intersections with the curve $P = 0$, $Q = 0$ are the points $P = 0$, $Q = 0$, $\Lambda = 0$ each three times, and the points $P = 0$, $Q = 0$, $N'\alpha - M'\alpha = 0$ each twice; viz. they are the points $2\chi + 3\omega$.

But if the degree of Λ is $= \lambda$, then the degrees of N', M', α^2, $\alpha\beta$, β^2 are $2n - 3f - 2g - \lambda$, $2n - 2f - 3g - \lambda$, $n - 2f - \lambda$, $n - f - g - \lambda$, $n - 2g - \lambda$, whence the degree of $\Lambda^3(N'\alpha - M'\beta)$ is $= 5n - 6f - 6g$, and the number of points is $= fg(5n - 6f - 6g)$, viz. this is

$$= fg(5n - 12) - 6fg(f + g - 2),$$

or it is $= c(5n - 12) - 6r$; so that θ being $= 0$, the equation is verified.

5. It was also pointed out to me by Dr Zeuthen that in the value of $24t$ given in No. 10 the term involving χ should be -6χ instead of $+6\chi$, and that in consequence the coefficients of χ are erroneous in several others of the formulæ. Correcting these, and at the same time introducing the terms in ω, and writing down also the terms in θ as they stand, we have

$$\begin{aligned}
4i &= \ldots - 2\chi + 3\theta - 3\omega,\\
24t &= \ldots - 6\chi + 9\theta - 9\omega,\\
2\sigma &= \ldots \qquad\quad - \theta - \omega,\\
8\rho &= \ldots + 6\chi - 9\theta + 9\omega,\\
8\kappa &= \ldots - 6\chi + 17\theta - 25\omega,\\
2\delta &= \ldots + 6\chi - 9\theta + 15\omega,\\
8n' &= \ldots - 30\chi + 21\theta - 45\omega,\\
c' &= \ldots - 12\chi + 10\theta - 20\omega.
\end{aligned}$$

The equations of No. 11, used afterwards, No. 53, should thus be

$$\begin{aligned}
4i + 6r &= (5n-12)\,c - 18\beta - 5\gamma - 2\chi + 3\theta - 3\omega,\\
-24t - 8q + 18r &= (-8n+16)\,b + (15n-36)\,c - 34\beta + 9\gamma + 4j - 6\chi + 9\theta - 9\omega;
\end{aligned}$$

and from these I deduce

$$44q + \tfrac{63}{2}r = (44n-88)\,b + (\tfrac{105}{4}n - 63)\,c - \tfrac{409}{2}\beta - \tfrac{633}{4}\gamma - 132\,t - 87i - 22j - \tfrac{21}{2}\chi + \tfrac{63}{4}\theta.$$

6. In No. 32 we have (without alteration) $\theta = 16$; but in the application (Nos. 40 and 41) to the surface $FP^2 + GR^2Q^3 = 0$ we have $\theta = 0$, and there are $\omega = fpq$ off-points, $F=0$, $P=0$, $Q=0$, and $\chi = gpq$ close-points, $G=0$, $P=0$, $Q=0$. The new equations involving ω are thus satisfied.

7. I have ascertained that the value of β' obtained, Nos. 51 to 64 of the memoir, is inconsistent with that obtained in the "Addition" by consideration of the deficiency, and that it is in fact incorrect. The reason is that, although, as stated No. 53, the values of two of the coefficients D, E may be assumed at pleasure, they cannot, in conjunction with a given system of values of A, B, C, be thus assumed at pleasure; viz. A, B, C being $=110$, 272, 44 respectively, the values of D, E are really determinate. I have no direct investigation, but by working back from the formula in the Addition I find that we must have $D = \frac{477}{4}$, $E = 315$; the values of the remaining coefficients then are

$$F = \tfrac{63}{2},\ G = -\tfrac{715}{2},\ H = -\tfrac{1005}{4},\ I = -198;$$

or the formula is

$$\begin{aligned}
\beta' = {} & 2n(n-2)(11n-24)\\
& -(110n-272)\,b + 44q\\
& -(\tfrac{477}{4}n - 315)\,c + \tfrac{63}{2}r\\
& +\tfrac{715}{2}\beta + \tfrac{1005}{4}\gamma + 198\,t\\
& -hC - gB - xi - \lambda j - \mu\chi - \nu\theta - f\omega\\
& -h'C' - g'B' - x'i' - \lambda' j' - \mu'\chi' - \nu'\theta' - f'\omega';
\end{aligned}$$

but I have not as yet any means of determining the coefficients f, f' of the terms in ω, ω'.

From the several cases of a cubic surface we obtain as in the memoir; but applying to the same surfaces the reciprocal equation for β, instead of the results of the memoir, we find

$$\begin{aligned} h' &= -\ 4, \\ g' + 16\nu &= -198, \\ g' + 2\mu &= 45, \\ g + g' &= 18, \\ \lambda &= 5 \end{aligned}$$

(so that now $\lambda + \lambda' = -2$, as is also given by the cubic scroll). And combining the two sets of results, we have

$$\begin{aligned} h &= 24, \\ \lambda &= 5, \\ \mu &= \tfrac{27}{2} + \tfrac{1}{29} g, \\ \nu &= -\tfrac{27}{2} + \tfrac{1}{16} g, \\ h' &= -\ 4, \\ g' &= 18 - g, \\ \lambda' &= -\ 7, \\ \mu' &= 6 - \tfrac{1}{29} g, \\ \nu' &= \tfrac{9}{4} - \tfrac{1}{16} g; \end{aligned}$$

but the coefficients g, x, x', f, f' are still undetermined. To make the result agree with that of the Addition, I assume $x = -86$, $x' = -1$, $g = +28$; whence we have

$$\begin{aligned} \beta' = {} & 2n(n-2)(11n-24) \\ & -(110n-272)\,b + 44q \\ & -(\tfrac{477}{4}n - 315)\,c + \tfrac{63}{2} r \\ & + \tfrac{715}{2}\beta + \tfrac{1005}{4}\gamma + 198t \\ & - 24C - 28B + 86i - 5j - \tfrac{55}{2}\chi + \tfrac{47}{4}\theta - f\omega \\ & + 4C' + 10B' + i' + 7j' + 8\chi' - \tfrac{1}{2}\theta' - f'\omega'; \end{aligned}$$

and if we substitute herein the foregoing value of $44q + \frac{63}{2} r$, we obtain

$$\begin{aligned} \beta' = {} & 2n(n-2)(11n-24) \\ & + (-66n + 184)\,b \\ & + (-93n + 252)\,c \\ & + 153\beta + 93\gamma + 66t \\ & - 24C - 28B - i - 27j - 38\chi + \tfrac{55}{2}\theta - f\omega \\ & + 4C' + 10B' + i' + 7j' + 8\chi' - \tfrac{1}{2}\theta' - f'\omega', \end{aligned}$$

which, except as to the terms in ω, ω', the coefficients of which are not determined, agrees with the value given in the Addition.

Dr Zeuthen considers that in general $i' = i$; I presume this is so, but have not verified it.

416.

ON THE THEORY OF RECIPROCAL SURFACES.

[*Published as an* Addition by Prof. CAYLEY *in* Dr SALMON'S *Treatise on the Analytic Geometry of Three Dimensions*, 4th Ed. (8vo. Dublin, 1882), pp. 592—604.]

620. IN further developing the theory of reciprocal surfaces it has been found necessary to take account of other singularities, some of which are as yet only imperfectly understood. It will be convenient to give the following complete list of the quantities which present themselves:

n, order of the surface.

a, order of the tangent cone drawn from any point to the surface.

δ, number of nodal edges of the cone.

κ, number of its cuspidal edges.

ρ, class of nodal torse.

σ, class of cuspidal torse.

b, order of nodal curve.

k, number of its apparent double points.

f, number of its actual double points.

t, number of its triple points.

j, number of its pinch-points.

q, its class.

c, order of cuspidal curve.

h, number of its apparent double points.

θ, number of its points of an unexplained singularity.

χ, number of its close-points.

ω, number of its off-points.

r, its class.

β, number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve.

γ, number of intersections, stationary points on nodal curve.

i, number of intersections, not stationary points on either curve.

C, number of cnicnodes of surface.

B, number of binodes.

And corresponding reciprocally to these:

n', class of surface.

a', class of section by arbitrary plane.

δ', number of double tangents of section.

κ', number of its inflexions.

ρ', order of node-couple curve.

σ', order of spinode curve.

b', class of node-couple torse.

k', number of its apparent double planes.

f', number of its actual double planes.

t', number of its triple planes.

j', number of its pinch-planes.

q', its order.

c', class of spinode torse.

h', number of its apparent double planes.

θ', number of its planes of a certain unexplained singularity.

χ', number of its close-planes.

ω', number of its off-planes.

r', its order.

β', number of common planes of node-couple and spinode torse, stationary planes of spinode torse.

γ', number of common planes, stationary planes of node-couple torse.

i', number of common planes, not stationary planes of either torse.

C', number of cnictropes of surface.

B', number of its bitropes.

In all 46 quantities.

621. In part explanation, observe that the definitions of ρ and σ agree with those given, Art. 609: the nodal torse is the torse enveloped by the tangent planes along the nodal curve; if the nodal curve meets the curve of contact a, then a tangent plane of the nodal torse passes through the arbitrary point, that is, ρ will be the number of these planes which pass through the arbitrary point, viz. the class of the torse. So also the cuspidal torse is the torse enveloped by the tangent planes along the cuspidal curve; and σ will be the number of these tangent planes which pass through the arbitrary point, viz. it will be the class of the torse. Again, as regards ρ' and σ': the node-couple torse is the envelope of the bitangent planes of the surface, and the node-couple curve is the locus of the points of contact of these planes; similarly, the spinode torse is the envelope of the *parabolic* planes of the surface, and the spinode curve is the locus of the points of contact of these planes; viz. it is the curve UH of intersection of the surface and its Hessian; the two curves are the reciprocals of the nodal and cuspidal torses respectively, and the definitions of ρ', σ' correspond to those of ρ and σ.

622. In regard to the nodal curve b, we consider k the number of its apparent double points (excluding actual double points); f the number of its actual double points (each of these is a point of contact of two sheets of the surface, and there is thus at the point a single tangent plane, viz. this is a plane f', and we thus have $f'=f$); t the number of its triple points; and j the number of its pinch-points—these last are not singular points of the nodal curve *per se*, but are singular in regard to the curve as nodal curve of the surface; viz. a pinch-point is a point at which the two tangent planes are coincident. The curve is considered as not having any stationary points other than the points γ, which lie also on the cuspidal curve; and the expression for the class consequently is $q=b^2-b-2k-2f-3\gamma-6t$.

623. In regard to the cuspidal curve c we consider h the number of its apparent double points; and upon the curve, not singular points in regard to the curve *per se*, but only in regard to it as cuspidal curve of the surface, certain points in number θ, χ, ω respectively. The curve is considered as not having any actual double or other multiple points, and as not having any stationary points except the points β, which lie also on the nodal curve; and thus the expression for the class is $r=c^2-c-2h-3\beta$.

624. The points γ are points where the cuspidal curve with the two sheets (or say rather half-sheets) belonging to it are intersected by another sheet of the surface; the curve of intersection with such other sheet belonging to the nodal curve of the surface has evidently a stationary (cuspidal) point at the point of intersection.

As to the points β, to facilitate the conception, imagine the cuspidal curve to be a semi-cubical parabola, and the nodal curve a right line (not in the plane of the curve) passing through the cusp; then intersecting the two curves by a series of parallel planes, any plane which is, say, above the cusp, meets the parabola in two real points and the line in one real point, and the section of the surface is a curve with two real cusps and a real node; as the plane approaches the cusp, these

approach together, and, when the plane passes through the cusp, unite into a singular point in the nature of a triple point (= node + two cusps); and when the plane passes below the cusp, the two cusps of the section become imaginary, and the nodal line changes from crunodal to acnodal.

625. At a point i the nodal curve crosses the cuspidal curve, being on the side away from the two half-sheets of the surface acnodal, and on the side of the two half-sheets crunodal, viz. the two half-sheets intersect each other along this portion of the nodal curve. There is at the point a single tangent plane, which is a plane i'; and we thus have $i = i'$.

626. As already mentioned, a cnicnode C is a point where, instead of a tangent plane, we have a tangent quadricone; and at a binode B the quadricone degenerates into a pair of planes. A cnictrope C' is a plane touching the surface along a conic; in the case of a bitrope B', the conic degenerates into a flat conic or pair of points.

627. In the original formulæ for $a(n-2)$, $b(n-2)$, $c(n-2)$, we have to write $\kappa - B$ instead of κ, and the formulæ are further modified by reason of the singularities θ and ω. So in the original formulæ for $a(n-2)(n-3)$, $b(n-2)(n-3)$, $c(n-2)(n-3)$, we have instead of δ to write $\delta - C - 3\omega$; and to substitute new expressions for $[ab]$, $[ac]$, $[bc]$, viz. these are

$$[ab] = ab - 2\rho - j,$$
$$[ac] = ac - 3\sigma - \chi - \omega,$$
$$[bc] = bc - 3\beta - 2\gamma - i.$$

The whole series of equations thus is

(1) $$a' = a.$$

(2) $$f' = f.$$

(3) $$i' = i.$$

(4) $$a = n\ (n-1) - 2b - 3c.$$

(5) $$\kappa' = 3n(n-2) - 6b - 8c.$$

(6) $$\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{3}{2}c(c-1).$$

(7) $$a(n-2) = \kappa - B + \rho + 2\sigma + 3\omega.$$

(8) $$b(n-2) = \rho + 2\beta + 3\gamma + 3t.$$

(9) $$c(n-2) = 2\sigma + 4\beta + \gamma + \theta + \omega.$$

(10) $$a(n-2)(n-3) = 2(\delta - C - 3\omega) + 3(ac - 3\sigma - \chi - 3\omega) + 2(ab - 2\rho - j\quad).$$

(11) $$b(n-2)(n-3) = 4k + (ab - 2\rho - j\quad) + 3(bc - 3\beta - 2\gamma - i).$$

(12) $$c(n-2)(n-3) = 6h + (ac - 3\sigma - \chi - 3\omega) + 2(bc - 3\beta - 2\gamma - i).$$

(13) $$q = b^2 - b - 2k - 2f - 3\gamma - 6t.$$

(14) $$r = c^2 - c - 2h - 3\beta.$$

Also, reciprocal to these

(15) $a' = n'(n'-1) - 2b' - 3c'$.

(16) $\kappa = 3n'(n'-2) - 6b' - 8c'$.

(17) $\delta = \frac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \frac{9}{2}c'(c'-1)$.

(18) $a'(n'-2) = \kappa' - B' + \rho' + 2\sigma' + 3\omega'$.

(19) $b'(n'-2) = \rho' + 2\beta' + 3\gamma' + 3t'$.

(20) $c'(n'-2) = 2\sigma' + 4\beta' + \gamma' + \theta' + \omega'$.

(21) $a'(n'-2)(n'-3) = 2(\delta' - C' - 3\omega') + 3(a'c' - 3\sigma' - \chi' - 3\omega') + 2(a'b' - 2\rho' - j')$.

(22) $b'(n'-2)(n'-3) = 4k' + (a'b' - 2\rho' - j') + 3(b'c' - 3\beta' - 2\gamma' - i')$.

(23) $c'(n'-2)(n'-3) = 6h' + (a'c' - 3\sigma' - \chi' - 3\omega') + 2(b'c' - 3\beta' - 2\gamma' - i')$.

(24) $q' = b'^2 - b' - 2k' - 2f' - 3\gamma' - 6t'$.

(25) $r' = c'^2 - c' - 2h' - 3\beta'$,

together with one other independent relation, in all 26 relations between the 46 quantities.

628. The new relation may be presented under several different forms, equivalent to each other in virtue of the foregoing 25 relations; these are

(26) $2(n-1)(n-2)(n-3) - 12(n-3)(b+c) + 6q + 6r + 24t + 42\beta + 30\gamma - \frac{3}{2}\theta = \Sigma$;

(27) $26n - 12c - 4C - 10B + \beta - 7j - 8\chi + \frac{1}{3}\theta - 4\omega = \Sigma'$,

in each of which two equations Σ is used to denote the same function of the accented letters that the left-hand side is of the unaccented letters.

(28)
$$\begin{aligned}\beta' + \tfrac{1}{2}\theta' = {} & 2n(n-2)(11n-24)\\ & + (-66n+184)b\\ & + (-93n+252)c\\ & + 22(2\beta+3\gamma+3t)\\ & + 27(4\beta+\gamma+\theta)\\ & + \beta + \tfrac{1}{2}\theta\\ & - 24C - 28B - 27j - 38\chi - 73\omega\\ & + 4C' + 10B' + 7j' + 8\chi' - 4\omega'.\end{aligned}$$

Or, reciprocally,

(29)
$$\begin{aligned}\beta + \tfrac{1}{2}\theta = {} & 2n'(n'-2)(11n'-24)\\ & + (-66n'+184)b'\\ & + (-93n'+252)c'\\ & + 22(2\beta'+3\gamma'+3t')\\ & + 27(4\beta'+\gamma'+\theta')\\ & + \beta' + \tfrac{1}{2}\theta'\\ & - 24C' - 28B' - 27j' - 38\chi' - 73\omega'\\ & + 4C + 10B + 7j + 8\chi - 4\omega.\end{aligned}$$

The foregoing equation (26) in fact expresses that the surface and its reciprocal have the same deficiency; viz. the expression for the deficiency is

$$(30)\quad \text{Deficiency} = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + i - \tfrac{1}{8}\theta,$$
$$= \tfrac{1}{6}(n'-1)(n'-2)(n'-3) - \&c.$$

629. The equation (28) (due to Prof. Cayley) is the correct form of an expression for β', first obtained by him (with some errors in the numerical coefficients) from independent considerations, but which is best obtained by means of the equation (26); and (27) is a relation presenting itself in the investigation. In fact, considering a as standing for its value $n(n-1)-2b-3c$, we have from the first 25 equations

6	a	$=\Sigma$
+ 2	$3n-c-\kappa$	$=\Sigma$
− 2	$a(n-2)-\kappa+B-\rho-2\sigma-3\omega$	$=\Sigma$
− 4	$b(n-2)-\rho-2\beta-3\gamma-3t$	$=\Sigma$
− 6	$c(n-2)-2\sigma-4\beta-\gamma-\theta-\omega$	$=\Sigma$
+ 2	$n+\kappa-\sigma-2C-4B-2j-3\chi-3\omega$	$=\Sigma$
− 3	$2q-2\rho+\beta+j$	$=\Sigma$
− 2	$3r+c-5\sigma-\beta-4\theta+\chi-\omega$	$=\Sigma$

and multiplying these equations by the numbers set opposite to them respectively, and adding, we find

$$-2n^3+12n^2+4n+b(12n-36)+c(12n-48)$$
$$-6q-6r-4C-10B-41\beta-30\gamma-24t-7j-8\chi+2\theta-4\omega=\Sigma,$$

and adding thereto (26) we have the equation (27); and from this (28), or by a like process, (29), is obtained without much difficulty. As to the 8 Σ-equations or symmetries, observe that the first, third, fourth, and fifth are in fact included among the original equations (for an expression which vanishes is in fact $=\Sigma$); we have from them moreover $3n-c=3a'-\kappa'$, and thence $3n-c-\kappa=3a'-\kappa-\kappa'$, which is $=\Sigma$, or we have thus the second equation; but the sixth, seventh, and eighth equations have yet to be obtained.

630. The equations (15), (16), (17) give

$$n' = a(a-1)-2\delta-3\kappa,$$
$$c' = 3a(a-2)-6\delta-8\kappa,$$
$$b' = \tfrac{1}{2}a(a-2)(a^2-9)-(a^2-a-6)(2\delta+3\kappa)+2\delta(\delta-1)+6\delta\kappa+\tfrac{9}{2}\kappa(\kappa-1);$$

from (7), (8), (9) we have

$$(a-b-c)(n-2) = \kappa-B - 6\beta-4\gamma-3t-\theta+2\omega,$$
$$(a-2b-3c)(n-2)(n-3) = 2(\delta-C)-8k-18h-6bc+18\beta+12\gamma+6i-6\omega,$$

and substituting these values for κ and δ, and for a its value $=n(n-1)-2b-3c$ we obtain the values of n', c', b'; viz. the value of n' is

$$\begin{aligned} n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 8b + 9c^2 + 15c \\ - 8k - 18h + 18\beta + 12\gamma + 12i - 9t \\ - 2C - 3B - 3\theta. \end{aligned}$$

Observe that the effect of a cnicnode C is to reduce the class by 2, and that of a binode B to reduce it by 3.

631. We have

$$(n-2)(n-3) = n^2 - n + (-4n+6) = a + 2b + 3c + (-4n+6),$$

and making this substitution in the equations (10), (11), (12), which contain $(n-2)(n-3)$, these become

$$\begin{aligned} a(-4n+6) &= 2(\delta - C) - a^2 - 4\rho - 9\sigma - 2j - 3\chi - 15\omega, \\ b(-4n+6) &= 4k - 2b^2 - 9\beta - 6\gamma - 3i - 2\rho - j, \\ c(-4n+6) &= 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma - \chi - 3\omega, \end{aligned}$$

(the foregoing equations (C) Salmon p. 586); and adding to each equation four times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2 - 2a &= 2(\delta - C) + 4(\kappa - B) - \sigma - 2j - 3\chi - 3\omega, \\ 2b^2 - 2b &= 4k - \beta + 6\gamma + 12t - 3i + 2\rho - j, \\ 3c^2 - 2c &= 6h + 10\beta + 4\theta - 2i + 5\sigma - \chi + \omega. \end{aligned}$$

Writing in the first of these $a^2 - 2a = n' + 2\delta + 3\kappa - a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n' - a &= -2C - 4B + \kappa - \sigma - 2j - 3\chi - 3\omega, \\ 2q + \beta + 3i + j &= 2\rho, \\ 3r + c + 2i + \chi &= 5\sigma + \beta + 4\theta + \omega, \end{aligned}$$

which give at once the last three of the 8 Σ-equations.

The reciprocal of the first of these is

$$\sigma' = a - n + \kappa' - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

or writing herein $a = n(n-1) - 2b - 3c$ and $\kappa' = 3n(n-2) - 6b - 8c$, this is

$$\sigma' = 4n(n-2) - 8b - 11c - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

giving the order of the spinode curve; viz. for a surface of the order n without singularities this is $= 4n(n-2)$, the product of the orders of the surface and its Hessian.

632. Instead of obtaining the second and third equations as above, we may to the value of $b(-4n+6)$ add twice the value of $b(n-2)$; and to twice the value of $c(-4n+6)$ add three times the value of $c(n-2)$, thus obtaining equations free from ρ and σ respectively; these equations are

$$\begin{aligned} b(-2n+2) &= 4k-2b^2-5\beta-3i+6t-j, \\ c(-5n+6) &= 12h-6c^2-5\gamma-4i-2\chi+3\theta-3\omega, \end{aligned}$$

equations which, introducing therein the values of q and r, may also be written

$$\begin{aligned} b(2n-4) &= 2q+5\beta+6\gamma+6t+3i+j+4f, \\ c(5n-12)+3\theta &= 6r+18\beta+5\gamma+4i+2\chi+3\omega. \end{aligned}$$

Considering as given, n the order of the surface; the nodal curve with its singularities b, k, f, t; the cuspidal curve and its singularities c, h; and the quantities β, γ, i which relate to the intersections of the nodal and cuspidal curves; the first of the two equations gives j, the number of pinch-points, being singularities of the nodal curve quoad the surface; and the second equation establishes a relation between θ, χ, ω, the numbers of singular points of the cuspidal curve quoad the surface.

In the case of a nodal curve only, if this be a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2=0$, and the first equation is

$$b(-2n+2)=4k-2b^2+6t-j;$$

or, assuming $t=0$, say $j=2(n-1)b-2b^2+4k$, which may be verified; and so in the case of a cuspidal curve only, when this is a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2=0$, where $AC-B^2=MP+NQ$; and the second equation is

$$c(-5n+6)=12h-6c^2-2\chi+3\theta-3\omega,$$

or, say

$$2\chi+3\omega=(5n-6)c-6c^2+12h+3\theta,$$

which may also be verified.

633. We may in the first instance out of the 46 quantities consider as given the 14 quantities

$$n; \qquad b, k, f, t; c, h, \theta, \chi \qquad ; \beta, \gamma, i; C, B,$$

then of the 26 relations, 17 determine the 17 quantities

$$\begin{array}{llllll} & a, \delta, \kappa, \rho, \sigma; j, q & ; r, \omega & ; & \\ n'; a', \delta', \kappa' & ; b', f' & ; c' & ; & i', \end{array}$$

and there remain the 9 equations

$$(18), (19), (20), (21), (22), (23), (24), (25), (28),$$

connecting the 15 quantities

$$\rho', \sigma'; k', t', j', q'; h', \theta', \chi', \omega', r'; \beta', \gamma'; C', B'.$$

Taking then further as given the 5 quantities j', χ', ω', C', B',

equations (18) and (21)	give	ρ', σ',
equation (19)	gives	$2\beta' + 3\gamma' + 3t'$,
" (20)	"	$4\beta' + \gamma' + \theta'$,
" (28)	"	$\beta' + \frac{1}{2}\theta'$,

so that taking also t' as given, these last three equations determine β', γ', θ'; and finally

equation (22)	gives	k',
" (23)	"	h',
" (24)	"	q',
" (25)	"	r',

viz. taking as given in all 20 quantities, the remaining 26 will be determined.

634. In the case of the general surface of the order n, without singularities, we have as follows:

$n = n$,

$a = n(n-1)$,

$\delta = \frac{1}{2}n(n-1)(n-2)(n-3)$,

$\kappa = n(n-1)(n-2)$,

$n' = n(n-1)^2$,

$a' = n(n-1)$,

$\delta' = \frac{1}{2}n(n-2)(n^2-9)$,

$\kappa' = 3n(n-2)$,

$b' = \frac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12)$,

$k' = \frac{1}{3}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464)$,

$t' = \frac{1}{3}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960)$,

$q' = n(n-2)(n-3)(n^2+2n-4)$,

$\rho' = n(n-2)(n^3-n^2+n-12)$,

$c' = 4n(n-1)(n-2)$,

$h' = \frac{1}{2}n(n-2)(16n^4-64n^3+80n^2-108n+156)$,

$r' = 2n(n-2)(3n-4)$,

$\sigma' = 4n(n-2)$,

$\beta' = 2n(n-2)(11n-24)$,

$\gamma' = 4n(n-2)(n-3)(n^3-3n+16)$,

the remaining quantities vanishing.

635. The question of singularities has been considered under a more general point of view by Zeuthen, in the memoir "Recherche des singularités qui ont rapport à une droite multiple d'une surface," *Math. Annalen*, t. IV. pp. 1—20, 1871. He attributes to the surface:

A number of singular points, viz. points at any one of which the tangents form a cone of the order μ, and class ν, with $y+\eta$ double lines, of which y are tangents to branches of the nodal curve through the point, and $z+\zeta$ stationary lines, whereof z are tangents to branches of the cuspidal curve through the point, and with u double planes and v stationary planes; moreover, these points have only the properties which are the most general in the case of a surface regarded as a locus of points; and Σ denotes a sum extending to all such points. {The foregoing general definition includes the cnicnodes ($\mu=\nu=2$, $y=\eta=z=\zeta=u=v=0$), and [also, but not properly] the binodes ($\mu=2$, $\eta=1$, $\nu=y=$ &c. $=0$), [it includes also the off-points ($\mu=\nu=3$, $z=v=1$, $y=\eta=(=0)$].}

And, further, a number of singular planes, viz. planes any one of which touches along a curve of the class μ' and order ν', with $y'+\eta'$ double tangents, of which y' are generating lines of the node-couple torse, $z'+\zeta'$ stationary tangents, of which z' are generating lines of the spinode torse, u' double points and v' cusps; it is, moreover, supposed that these planes have only the properties which are the most general in the case of a surface regarded as an envelope of its tangent planes; and Σ' denotes a sum extending to all such planes. {The definition includes the cnictropes ($\mu'=\nu'=2$, $y'=\eta'=z'=\zeta'=u'=v'=0$), and [also, but not properly] the bitropes ($\mu'=2$, $\eta'=1$, $\nu'=y'=$ &c. $=0$), [it includes also the off-planes ($\mu'=\nu'=3$, $z'=v'=1$, $y'=\eta'=\zeta'=0$)].}

636. This being so, and writing

$$x=\nu+2\eta+3\zeta,\quad x'=\nu'+2\eta'+3\zeta',$$

the equations (7), (8), (9), (10), (11), (12), contain in respect of the new singularities additional terms, viz. these are

$$\begin{aligned}
a(n-2)&=\ldots+\Sigma\,[x(\mu-2)-\eta-2\zeta],\\
b(n-2)&=\ldots+\Sigma\,[y(\mu-2)],\\
c(n-2)&=\ldots+\Sigma\,[z(\mu-2)],\\
a(n-2)(n-3)&=\ldots+\Sigma\,[x(-4\mu+7)+2\eta+4\zeta],\\
b(n-2)(n-3)&=\ldots+\Sigma\,[y(-4\mu+8)]-\Sigma'(4u'+3v'),\\
c(n-2)(n-3)&=\ldots+\Sigma\,[z(-4\mu+9)]-\Sigma'(2v'),
\end{aligned}$$

and there are of course the reciprocal terms in the reciprocal equations (18), (19), (20), (21), (22), (23). These formulæ are given without demonstration in the memoir just referred to: the principal object of the memoir, as shown by its title, is the consideration not of such singular points and planes, but of the multiple right lines of a surface; and in regard to these, the memoir should be consulted.

NOTES AND REFERENCES.

384. THE conclusion arrived at Nos. 27—30 that the transformed curve of the order $D+1$ depends upon $4D-6$ parameters is at variance with Riemann's theorem according to which the number of parameters is $3p-3$, (p Riemann $=D$ Cayley), $=3D-3$, and this last is the correct value. My erroneous conclusion is referred to in the preface to Clebsch and Gordan's *Theorie der Abel'schen Functionen* (Leipzig, 1866), "Unter den von Riemann behandelten Theilen der Theorie haben wir die Frage nach der Anzahl der Moduln einer Klasse von Abel'schen Functionen ausschliessen zu müssen geglaubt. Diese Frage ist durch die scharfsinnigen Betrachtungen des Herrn Cayley Gegenstand der Controverse geworden: sie ist überhaupt wohl zunächst nur durch tiefe algebraische Untersuchungen endgültig zu entscheiden, für deren Schwierigkeiten die gegenwärtig bekannten Methoden nicht mehr auszureichen scheinen." In the case D (or p) $=3$, my value is 10, Riemann's is 9: that the latter is correct was shown by a direct proof in the paper Brill, "Note bezüglich der Zahl der Moduln einer Klasse von algebraischen Gleichungen," *Math. Ann.*, t. I. (1869), pp. 401—406: the explanation of my error is given in the paper, Cayley, "Note on the Theory of Invariants," *Math. Ann.*, t. III. (1871), pp. 268—271.

400. The question here considered, viz., the expression of a binary sextic f in the form v^2-u^3, v and u a cubic and a quadric respectively, forms the basis of the very interesting investigations contained in the Memoir, Clebsch "Zur Theorie der binären Formen sechster Ordnung und zur Dreitheilung der hyperelliptischen Functionen," *Gött. Abh.*, t. XIV. (1869), pp. 1—59. Considering f as a given sextic it is remarked that the number of solutions, or what is the same thing the number of the functions u or v, although at first sight $=45$, is really $=40$; supposing that there is a given solution u, v, or that the sextic function is in the first instance given in the form v^2-u^3, then if any other solution is u', v', we have $v^2-u^3=v'^2-u'^3$, where v', u' are functions to be determined: there are in all 39 solutions, a set of 27 and a set of 12 solutions: viz. writing the equation in the form $(v+v')(v-v')=(u-u')(u-\epsilon u')(u-\epsilon^2u')$, ϵ an imaginary cube root of unity, then either the $v+v'$ and the $v-v'$ contain each of them as a factor one of the quadric functions $u-u'$, $u-\epsilon u'$, $u-\epsilon^2u'$ (which gives the set of 27 solutions) or else the $v+v'$ and the $v-v'$ are each of them the product of three linear factors of the quadric functions respectively (which gives the set of 12

solutions). It may be added that the 27 solutions form 9 groups of 3 each and that these 9 groups depend upon Hesse's equation of the order 9 for the determination of the inflexions of a cubic curve; and that the 12 solutions are determined by an equation of the order 12 which is the known resolvent of this order arising from Hesse's equation and is solved by means of a quartic equation with a quadrinvariant $=0$. As appears by the title of the memoir, the question is connected with that of the trisection of the hyperelliptic functions.

401, 403. On the subject of Pascal's theorem, see Veronese, "Nuove teoremi sull' hexagrammum mysticum," *R. Accad. dei Lincei* (1876—77), pp. 7—61; Miss Christine Ladd (Mrs Franklin), "The Pascal Hexagram," *Amer. Math. Jour.*, t. II. (1879), pp. 1—12, and Veronese, "Interprétations géométriques de la théorie des substitutions de n lettres, particulièrement pour $n=3, 4, 5$, en relation avec les groupes de l'Hexagramme Mystique," *Ann. di Matem.*, t. XI. 1882—83, pp. 93—236. See also Richmond, "A Symmetrical System of Equations of the Lines on a Cubic Surface which has a Conical Point," *Quart. Math. Jour.*, t. XXII. (1889), pp. 170—179, where the author discusses a perfectly symmetrical system of the lines on the cubic surface and deduces from them equations of the lines relating to a Pascal's hexagon: there are of course through the conical point 6 lines lying on a quadric cone and these by their intersections with the plane give the six points of the hexagon: the interest of the paper consists as well in the connexion established between the two theories as in the perfectly symmetrical form given to the equations.

406, 407. A correction was made by Halphen to the fundamental theorem of Chasles that the number of the conics $(X, 4Z)$ is $=\alpha\mu+\beta\nu$, he finds that a diminution is in some cases required, and thus that the general form is, Number of conics $(X, 4Z)=\alpha\mu+\beta\nu-\Gamma$: see Halphen's two Notes, *Comptes Rendus*, 4 Sep. and 13 Nov., 1876, t. LXXXIII. pp. 537 and 886, and his papers "Sur la théorie des caractéristiques pour les coniques," *Proc. Lond. Math. Soc.*, t. IX. (1877—1878), pp. 149—170, and "Sur les nombres des coniques qui dans un plan satisfont à cinq conditions projectives et indépendantes entre elles," *Proc. Lond. Math. Soc.*, t. X. (1878—79), pp. 76—87: also Zeuthen's paper "Sur la revision de la théorie des caractéristiques de M. Study," *Math. Ann.*, t. XXXVII. (1890), pp. 461—464, where the point is brought out very clearly and tersely.

The correction rests upon a more complete development of the notion of the line-pair-point, viz. this degenerate form of conic seems at first sight to depend upon three parameters only, the two parameters which determine the position of the coincident lines, and a third parameter which determines the position therein of the coincident points: but there is really a fourth parameter. {Compare herewith the point-pair, or indefinitely thin conic, which working with point-coordinates presents itself in the first instance as a coincident line-pair depending on two parameters only, but which really depends also on the two parameters which determine the position therein of the vertices.} As to the fourth parameter of the line-pair-point the most simple definition is a metrical one; taking the semiaxes of the degenerate conic to be a and b $(a=0,\ b=0)$ then we have two positive integers p and q prime to each other such that the ratio

$a^p : b^q$ is finite; and this being so the fractional or it may be integer number $p : q$ is the fourth parameter in question. But it is preferable to adopt Halphen's purely descriptive definition, viz. we consider a conic 1° in reference to three given points y, z, t on a given line, and take x, x' for the intersections of the conic with the line: we take $a=(y, z, t, x)-(y, z, t, x')$ for the difference of the corresponding anharmonic ratios of the three points with the points x, x' respectively; and 2° we consider the conic in reference to three given lines Y, Z, T through a given point and take X, X' for the tangents from the given point to the conic; we take $b=(Y, Z, T, X)-(Y, Z, T, X')$ for the difference of the corresponding anharmonic ratios of the three lines with the lines X, X' respectively $\left(\text{observe that these values are } a=\dfrac{x-x'}{z-x\,.\,z-x'}\div\dfrac{y-t}{z-y\,.\,z-t}, \text{ and } b=\dfrac{X-X'}{Z-X\,.\,Z-X'}\div\dfrac{Y-T}{Z-Y\,.\,Z-T}\right)$. Here when the conic is a line-pair-point, $x=x'$ and $X=X'$, where $a=0$ and $b=0$, but we have as before the integers p and q such that $a^p : b^q$ is finite, and we have thus the fourth parameter $p : q$.

Halphen's correction is now as follows, starting from the formula number of conics $(X, 4Z)=\alpha\mu+\beta\nu$, we may have among the $\alpha\mu+\beta\nu$ conics line-pair-points any one of which if we disregard altogether the fourth parameter is a conic satisfying the five conditions, but which unless the fourth parameter thereof has its proper value is an improper solution of the problem and as such it has to be rejected: if the number of such solutions is $=\Gamma$, then there is this number to be subtracted, and the formula becomes, Number of conics $(X, 4Z)=\alpha\mu+\beta\nu-\Gamma$.

It may be asked in what way the fourth parameter comes into the question at all: as an illustration suppose that a, b denoting the semiaxes of a conic, or else the above mentioned descriptively defined quantities, then p, q, k denoting given quantities (p and q positive integers prime to each other) the condition X may be that the conic shall be such that $a^p \div b^q = k$; this implies $a^p : b^q$ finite, and hence clearly if the system of conics $(X, 4Z)$ contains line-pair-points, no such line-pair-point can be a proper solution unless this relation $a^p \div b^q = k$ is satisfied.

412. Zeuthen's Memoir of 1876 presently referred to contains applications to the theory of Cubic Surfaces, the numerical results given in the table p. 539 agree for the most part with those of the Memoir 412, see p. 363, but for the surfaces III, VI, IX and XII discrepancies occur in the values of r' and h' relating to the spinode develope. As to this observe that Zeuthen's h', or say $\bar{h}'$ includes actual as well as apparent double planes, and we have $r'=c'^2-c'-2\bar{h}'-3\beta'$, my h' relates to apparent double planes only, but as I assume that there are no actual double planes the formula is $r'=c'^2-c'-2h'-3\beta'$, and as the values of c' and β' agree we have in fact in each of the four cases $r'+2h'$ (Cayley) $=r'+2h'$ (Zeuthen). The values found are

		III	VI	IX	XII		III	VI	IX	XII
Cayley	n'	72	24	12	6	r'	42	24	32	9
Zeuthen	n'	84	30	24	7	r'	18	12	8	7

and assuming the correctness of Zeuthen's values it would seem to follow that the four forms of surface have

$$12,\quad 6,\quad 12,\quad 1$$

actual double planes respectively.

413. In the equation No. 36, $\Omega = AP + BQ + CR + .. = 0$, it is implicitly assumed that the number of terms $P, Q, R, ..$ is finite, viz. the implied theorem is that any given k-fold relation whatever (k of course a finite number) there is always a *finite* number of functions $P, Q, R, ...$ such that every onefold relation included in the k-fold relation is of the form in question $\Omega, = AP + BQ + CR + ..., = 0$: this seems self-evident enough, but I never succeeded in finding a proof: a proof of the theorem has however been obtained by Hilbert, see his papers "Zur Theorie der algebraischen Gebilden (Erste Note)," *Gött. Nachr.* No. 16, (1888), pp. 450—457.

411, 415, 416. The first and second of these papers precede in date Zeuthen's Memoir of 1871 referred to in 416, but I ought in that paper to have referred also to his later Memoir, "Revision et extension des formules numériques de la théorie des surfaces réciproques," *Math. Ann.* t. x. (1876), pp. 446—546. I compare the notations as follows, viz. for the unaccented letters we have

Cayley.	Zeuthen.
$n, a, \delta, \kappa, \rho, \sigma$	$n, a, \delta, \kappa, \rho, \sigma$
b, q, k, t, γ	$b, q, \bar{k}, t, \gamma; s$
$c, r, h, \beta, \theta, \omega$	$c, r, \bar{h}, \beta \;; m$
j, χ	j, χ
C, B	B, U, O
f, i	f, i, d, g, e
23 letters in all.	27 letters in all.

Here for Zeuthen's k, h, I have written $\bar{k}$, $\bar{h}$, viz. these numbers represent the Plückerian equivalents of the number of double points for the nodal and cuspidal curves respectively. Zeuthen considers also the general node, say $\mathfrak{C}(\mu, \nu, y+\eta, z+\zeta, u, v)$, see 416, this includes the cnicnode C and off-point ω, and accordingly he includes under it and takes no special notice of these singularities, but it does not properly include, and he takes special notice of, the binode B; it does not extend to the case where the tangent cone breaks up into cones each or any of them more than once repeated, and accordingly not to the case of a unode U where the tangent cone is a pair of coincident planes. He introduces this singularity, and also the singularity of the osculating point O which is understood rather more easily by means of the reciprocal singularity of the osculating plane O', this is a tangent plane meeting the surface in a curve having the point of contact for a triple point; and he disregards my unexplained singularity θ. The letters s, m do not denote singularities; s is the class of the envelope of the osculating planes of the nodal curve, m the

class of the envelope of the osculating planes of the cuspidal curve. Finally d denotes the number of stationary points (cusps) of the nodal curve, exclusive of the points γ which lie on the cuspidal curve; and g and e denote, g the number of ordinary actual double points of the cuspidal curve, e the number of stationary points (cusps) of the same curve, exclusive of the points β which lie on the nodal curve.

Moreover with Zeuthen, the nodal curve has

$$3t+f+3O'+\Sigma' \text{ double points}$$

($\bar{k}=k+3t+f+3O'+\Sigma'$, if k denotes, as with me, the number of apparent double points of the curve), and it has

$$\gamma+d+\Sigma' \text{ stationary points.}$$

The cuspidal curve has

$$g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma' \text{ double points}$$

($\bar{h}=h+g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma'$, if h denotes, as with me, the number of apparent double points of the curve), and it has

$$\beta+e+2O' \text{ stationary points}$$

and the nodal and cuspidal curves intersect in

$$3\beta+2\gamma+i+12O'+\Sigma+\Sigma' \text{ points;}$$

where I have written Σ and Σ' to denote sums (different in the different equations) determined by Zeuthen, and depending on the singularities $\mathfrak{C}$ and $\mathfrak{C}'$ respectively.

For comparison of my formulæ with Zeuthen's it is thus proper in my formulæ to write $C=0$, $\omega=0$, $\theta=0$ (but in the first instance I retain θ) and in his formulæ to write $U=0$, $O=0$, $d=0$, $g=0$, $e=0$, $\Sigma=0$, $\Sigma'=0$. Doing this the last mentioned formulæ give as with me $3t+f$ double points and γ stationary points for the nodal curve, but they give for the cuspidal curve $6\chi'+12\beta'$ (instead of 0) double points and β stationary points; and the two curves intersect (as with me) in $3\beta+2\gamma+i$ points. There is a real discrepancy in the number $6\chi'+12\beta'$ of double points on the cuspidal curve.

I compare his $(6+26+1=)\,33$ relations:

(1) $a=a'.\qquad d=d'.$

$f=f'.\qquad g=g'.$

$i=i'.\qquad h=h'.$

(6) $n(n-1)=a+2b+3c.$

(7) $a(a-1)=n+2\delta'+3\kappa'.$

(8) $c-\kappa'=3(n-a).$

(9) $b(b-1)=q+2\bar{k}+3\gamma+3d+\Sigma'.$

(10) $[3(b-q)=\gamma+d-s+\Sigma'$, determines $s]$.

(11) $c(c-1)=r+2\bar{h}+3\beta+6O'+3e.$

(12) $[3(c-r)=\beta+e-m+2O'+\Sigma'$, determines $m]$.

(13) $a(n-2)=\kappa-B+\rho+2\sigma\qquad\qquad+\Sigma.$

(14) $b(n-2)=\rho+2\beta+3\gamma+3t\;+9O'\;+\Sigma.$

(15) $c(n-2)=2\sigma+4\beta+\gamma+8\chi'+16B'+12O'+\Sigma.$

(16) $a(n-2)(n-3)=2(\delta-3U)+3(ac-3\sigma-\chi)+2(ab-2\rho-j).$

(17) $b(n-2)(n-3)=4(\bar{k}-3t-f)+(ab-2\rho-j)+3(bc-3\beta-2\gamma-i)+39O'+\Sigma+\Sigma'.$

(18) $c(n-2)(n-3)=6(\bar{h}-6\chi'-12B'-U'-4O'-g)+(ac-3\sigma-\chi)+2(bc-3\beta-2\gamma-i).$
$-30O'+\Sigma+\Sigma',$

with the like reciprocal equations (6) to (18);

(19) $\sigma+m-r-\beta-4j'-3\chi'-14U'+\Sigma'$
$=\sigma'+m'-r'-\beta'-4j-3\chi-14U+\Sigma.$

where $\bar{k}=$

$\bar{h}=$

and my $(3+22+1=)\,26$ relations as follows:

(1) $a=a'$.

(2) $f=f'$.

(3) $i=i'$.

(4) $a=n(n-1)-2b-3c$.

(5) $\kappa'=3n(n-2)-6b-8c$.

(6) $\delta'=\frac{1}{2}n(n-2)(n^2-9)-$ &c.

(A) (13) $q=b^2-b-2k-2f-3\gamma-6t$.

(B) (14) $r=c^2-c-2h-3\beta$.

(C) (7) $a(n-2)=k-B+\rho+2\sigma+3\omega$.

(D) (8) $b(n-2)=\rho+2\beta+3\gamma+3t$.

(E) (9) $c(n-2)=2\sigma+4\beta+\gamma+\theta+\omega$.

(F) (10) $a(n-2)(n-3)=2(\delta-C-3\omega)+3(ac-3\sigma-\chi-3\omega)+2(ab-2\rho-j\quad)$.

(G) (11) $b(n-2)(n-3)=\quad 4k\quad+\quad(ab-2\rho-j\quad)+3(bc-2\beta-2\gamma-i)$.

(H) (12) $c(n-2)(n-3)=\quad 6h\quad+\quad(ac-3\sigma-\chi-3\omega)+2(bc-3\beta-2\gamma-i)$,

with the like reciprocal equations (4) to (14);

(I) (26) $2(n-1)(n-2)(n-3)-12(n-3)(b+c)+6q+6r+24t+42\beta+30\gamma-\frac{3}{2}\theta$.

$=2(n'-1)(n'-2)(n'-3)-12(n'-3)(b'+c')+6q'+6r'+24t'+42\beta'+30\gamma'-\frac{3}{2}\theta'$.

$k+3f+3t+3O'+\Sigma$.

$h+g+6\chi'+12B'+U'+4O'+\Sigma+\Sigma'$.

Substituting for $\bar{k}$, $\bar{h}$ their values we have instead of (A), (B), (C), (D) the equations

$$(A')\quad b^2-b \qquad\qquad = q+2k+2f+6t+6O'+3\gamma+3\delta+\Sigma+\Sigma'.$$

$$(B')\quad c^2-c \qquad\qquad = r+2h+2g+12\chi'+24B'+2U'+14O'+3\beta+3e+\Sigma+\Sigma'.$$

$$(G')\quad b(n-2)(n-3)=4k+27O'+(ab-2\rho-j)+3(bc-3\beta-2\gamma-i)+\Sigma+\Sigma'.$$

$$(H')\quad c(n-2)(n-3)=6h-30O'+(ac-3\sigma-\chi)+2(bc-3\beta-2\gamma-i)+\Sigma+\Sigma'.$$

Writing as before $C=0$, $\omega=0$; $U=0$, $O=0$, $d=0$, $g=0$, $e=0$, and neglecting the terms in Σ, Σ', the two equations (E) become

$$\text{Zeuthen}\qquad c(n-2)=2\sigma+4\beta+\gamma+8\chi'+16B',$$

$$\text{Cayley}\qquad c(n-2)=2\sigma+4\beta+\gamma+\theta,$$

which can be made to agree by writing $\theta=8\chi'+16B'$. But we have

$$\text{Zeuthen }(B')\quad c^2-c \qquad =r+2h+3\beta+12\chi'+24B',$$

$$\text{Cayley }(B)\quad c^2-c \qquad =r+2h+3\beta,$$

values which differ by the terms $12\chi'+24B'$, or if θ has the value just written down, the term $\frac{3}{2}\theta$.

I refrain from a comparison of the two equations (I.), and of the expressions for the deficiency given by these two equations respectively—but I notice here the expression for the deficiency obtained by Zeuthen in the last section (XIV.) of his Memoir, viz. this is

$$\begin{aligned}24(D+1)= \quad & c'-12a+24m+\beta+3r-15c+2\sigma+6\chi+12\chi'+6g+9e\\ &+8B+24B'+18U+6U'+6O'\\ &+\Sigma(3\nu+3z+8\eta+13\zeta)+\Sigma'(6\zeta).\end{aligned}$$

The problem is a very difficult one, and it cannot be held that as yet a complete solution has been obtained. Take in plane geometry the question of reciprocal curves: here, using throughout point-coordinates, we start with a curve represented by the general equation $(x, y, z)^n=0$, such a curve has only isolated singularities, viz. the line-singularities of the inflexion and the double tangent, we know the expression in point-coordinates of any such singularity (inflexion or double tangent as the case may be), viz. we can at once write down the equation of a curve of the order n having a given stationary tangent and point of contact therewith, or a given double tangent and two points of contact therewith. Returning to the general curve $(x, y, z)^n=0$, we know that the reciprocal curve has other isolated singularities, viz. the point-singularities which correspond to these, the double point (or node) and the stationary point (or cusp), and we know the expression of any such singularity (node or cusp as the case may be), viz. we can at once write down the equation of a curve of the order n having at a given point a node with given tangents, or a cusp with given tangent. And then starting afresh with a curve of the order n having a node or a cusp we obtain the effect

thereof as regards the line-singularities of the inflexion and the double tangent. We are thus led to consider as ordinary singularities in the theory the above-mentioned four singularities of the inflexion, the double tangent, the node and the cusp: and we know further that any other singularity whatever of a plane curve is compounded in a definite manner of a certain number of some or all of these singularities.

But in the theory of surfaces, starting in like manner with the general equation $(x, y, z, w)^n = 0$, such a surface has torse-singularities, the node-couple torse, and the spinode-torse; each of these is in general an indecomposable torse of a certain kind (but there is the new cause of complication that it may break into two or more separate torses), but we do not know the analytical expression of these singularities, nor consequently the analytical expression of the curve-singularities which correspond to them, the nodal curve and the cuspidal curve. Thus if we attempt to start with a surface $(x, y, z, w)^n = 0$ having a nodal curve, we can indeed write down the equation in its most general form, viz. if the nodal curve has for its complete expression the k equations $P = 0$, $Q = 0$, $R = 0$, &c. (viz. if the curve is such that every surface whatever through the curve is of the form $\Omega, = AP + BQ + CR + \ldots, = 0$) then the most general equation of the surface having this curve for a nodal curve is $(A, B, C, \ldots \between P, Q, R, \ldots)^2 = 0$, but this form is far too complicated to be worked with; and if for simplicity we take the nodal curve to be a complete intersection $P = 0$, $Q = 0$, and consequently the equation of the surface to be $(A, B, C \between P, Q)^2 = 0$, then it is by no means clear that we do not in this way introduce limitations extraneous to the general theory. The same difficulty applies of course, and with yet greater force, to the cuspidal curve; and even if we could deal separately with the cases of a surface having a given nodal curve, and a given cuspidal curve, this would in no wise solve the problem for the more general case of a surface having a given nodal curve and a given cuspidal curve. It is to be added that the general surface of the order n has no plane- or point-singularities, and thus that such singularities (which correspond most nearly to the singularities considered in the theory of reciprocal curves) present themselves in the theory of reciprocal surfaces as extraordinary singularities.

END OF VOL. VI.

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 4to. 10 vols. Vols. I. II. III. IV. V. and VI. 25*s.* each. [Vol. VII. *In the Press.*

The Scientific Papers of the late Prof. J. Clerk Maxwell. Edited by W. D. NIVEN, M.A., formerly Fellow of Trinity College. In 2 vols. Royal 4to. £3. 3*s.* (net).

Mathematical and Physical Papers, by Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Demy 8vo. Vol. I. 18*s.* Vol. II. 15*s.* Vol. III. 18*s.*

Mathematical and Physical Papers, by Sir G. G. STOKES, Sc.D., LL.D., F.R.S., Lucasian Professor of Mathematics in the University of Cambridge. Reprinted from the Original Journals and Transactions, with Additional Notes by the Author. Demy 8vo. Vol. I. 15*s.* Vol. II. 15*s.* [Vol. III. *In the Press.*

A Treatise on the Mathematical Theory of Elasticity by A. E. H. LOVE, M.A., Fellow of St John's College, Cambridge, in Two Volumes. Vol. I. Demy 8vo. 12*s.* Vol. II. 12*s.*

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.*
Vol. II. Saint-Venant to Lord KELVIN (Sir WILLIAM THOMSON). In 2 parts. By the same Editor. 30*s.*

A Treatise on Natural Philosophy. By Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Lord KELVIN (Sir W. THOMSON) and Professor P. G. TAIT. Demy 8vo. 9*s.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of Peterhouse, Cambridge. Demy 8vo. Vol. I. 14*s.* Vol. II. 10*s.*

An Elementary Treatise on Quaternions. By P. G. TAIT, M.A. *3rd Edition.* Demy 8vo. 18*s.*

Theory of Differential Equations. Part I. Exact Equations and Pfaff's Problem. By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, Cambridge. Demy 8vo. 12*s.*

A Treatise on the Theory of Functions of a Complex Variable, by A. R. FORSYTH, Sc.D., F.R.S. Royal 8vo. 21*s.* Net.

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow and Lecturer of Christ's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on the Theory of Determinants and their applications in Analysis and Geometry, by R. F. SCOTT, M.A., Fellow of St John's College. Demy 8vo. 12*s.*

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A. Demy 8vo. 12*s.*

Elementary Thermodynamics. By J. PARKER, M.A., Fellow of St John's College, Cambridge. Crown 8vo. 9*s.*

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Demy 4to. cloth (Vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half-morocco 28*s.* (net). Vol. 9. New Series for the years 1874—1883. Cloth 25*s.* (net), half morocco, 32*s.* (net). [Vol. X. *In the Press.*

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

CAMBRIDGE UNIVERSITY PRESS.

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. New and Enlarged Edition. Crown 8vo. 7*s.* 6*d.*

Solutions to the Examples in a Treatise on Elementary Dynamics. By the same Author. Crown 8vo. 7*s.* 6*d.*

Plane Trigonometry. By the same Author. Crown 8vo. 7*s.* 6*d.* Part I. up to and including the Solution of Triangles is published separately. 5*s.*

A History of the Study of Mathematics at Cambridge. By W. W. ROUSE BALL, M.A., Fellow and Lecturer on Mathematics in Trinity College, Cambridge. Crown 8vo. 6*s.*

The Electrical Researches of the Hon. H. Cavendish, F.R.S. Written between 1771 and 1781. Edited from the original MSS. in the possession of the Duke of Devonshire, K.G., by the late J. CLERK MAXWELL, F.R.S. Demy 8vo. 18*s.*

A Catalogue of the Portsmouth Collection of Books and Papers written by or belonging to Sir ISAAC NEWTON. Demy 8vo. 5*s.*

The Analytical Theory of Heat, by JOSEPH FOURIER. Translated, with Notes, by A. FREEMAN, M.A., formerly Fellow of St John's College, Cambridge. Demy 8vo. 12*s.*

A Short History of Greek Mathematics. By J. GOW, Litt.D., Fellow of Trinity College. Demy 8vo. 10*s.* 6*d.*

Diophantos of Alexandria; a Study in the History of Greek Algebra. By T. L. HEATH, M.A., Fellow of Trinity College, Cambridge. Demy 8vo. 7*s.* 6*d.*

A Treatise on the General Principles of Chemistry. By M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A., and D. J. CARNEGIE, M.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

Lectures on the Physiology of Plants. By S. H. VINES, Sc.D., Professor of Botany in the University of Oxford. Demy 8vo. With Illustrations. 21*s.*

PITT PRESS MATHEMATICAL SERIES.

Arithmetic for Schools. By C. SMITH, M.A., Master of Sidney Sussex College, Cambridge. Extra Fcap. 8vo. 3*s.* 6*d.*

Elementary Algebra. By W. W. ROUSE BALL, M.A., Fellow and Mathematical Lecturer of Trinity College, Cambridge. Extra Fcap. 8vo. 4*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Extra Fcap. 8vo. Books I.—IV. 3*s.* Books I.—VI. 4*s.* Books I. and II. 1*s.* 6*d.* Books III. and IV. 1*s.* 6*d.* Books V. and VI. 1*s.* 6*d.*

Solutions to the Exercises in Taylor's Euclid. Books I.—IV. By W. W. TAYLOR, M.A. 6*s.*

The Elements of Statics and Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. Third Edition. Extra Fcap. 8vo. 7*s.* 6*d.*

Part I. ELEMENTS OF STATICS. Extra Fcap. 8vo. 4*s.* 6*d.*

Part II. ELEMENTS OF DYNAMICS. Extra Fcap. 8vo. 3*s.* 6*d.*

Solutions to the Examples in the Elements of Statics and Dynamics. By S. L. LONEY, M.A. 7*s.* 6*d.*

Mechanics and Hydrostatics for Beginners. By S. L. LONEY, M.A. 4*s.* 6*d.*

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow of Christ's College, Cambridge, and University Lecturer in Mathematics, and C. M. JESSOP, M.A., Fellow of Clare College, and Assistant Master at Reading School. Extra Fcap. 8vo. 4*s.* 6*d.*

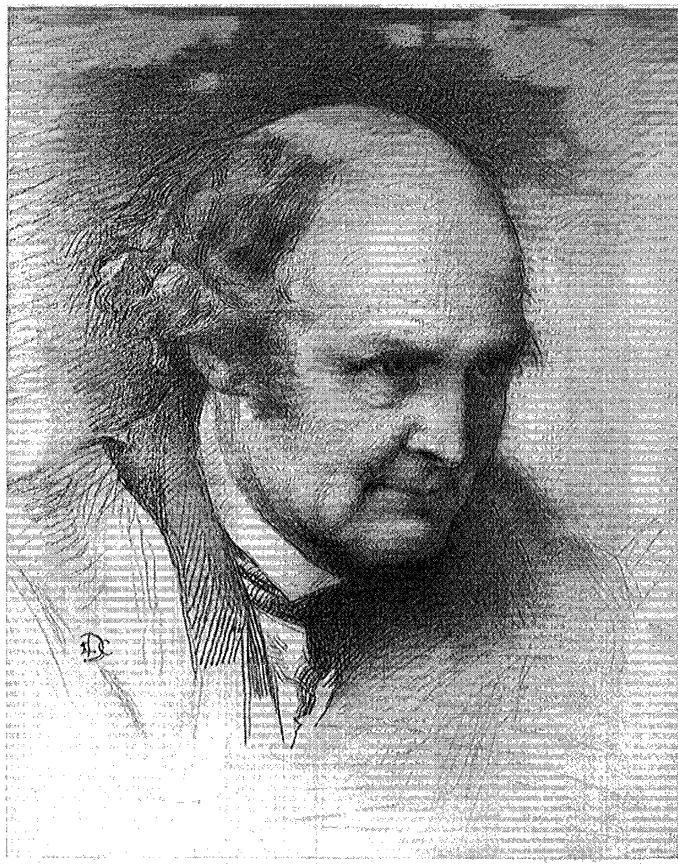
London: C. J. CLAY AND SONS,

CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.



THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. VII.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1894

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 69 papers numbered 417 to 485 published for the most part in the years 1866 to 1872; they include a series of astronomical papers published in the *Memoirs* and *Monthly Notices* of the Royal Astronomical Society.

The Portrait in Volume VI. is from the Painting in Oil by Mr Lowes Dickenson in the year 1874, presented by the Subscribers to Trinity College, Cambridge, and now in the Hall of the College: the portrait in the present volume is a photograph of a pencil sketch by Mr Lowes Dickenson in the year 1893.

The Table for the seven volumes is

Vol.	I.	Numbers	1	to	100.
,,	II.	,,	101	,,	158.
,,	III.	,,	159	,,	222.
,,	IV.	,,	223	,,	299.
,,	V.	,,	300	,,	383.
,,	VI.	,,	384	,,	416.
,,	VII.	,,	417	,,	485.

CONTENTS.

[An Asterisk denotes that the paper is not printed in full.]

PAGE

417. *On the Locus of the Foci of the Conics which pass through Four Given Points* 1
Phil. Mag. t. XXXII. (1866), pp. 362—365

418. *A Remark on Differential Equations*. 5
Phil. Mag. t. XXXII. (1866), pp. 379—381

419. *A Theorem on Differential Operators* 8
Phil. Mag. t. XXXII. (1866), pp. 461—472

420. *On Riccati's Equation* 9
Phil. Mag. t. XXXVI. (1868), pp. 348—351

421. *Note on the Solvibility of Equations by means of Radicals* . 13
Phil. Mag. t. XXXVI. (1868), pp. 386, 387

422. *On the Geodesic Lines on an Oblate Spheroid* 15
Phil. Mag. t. XL. (1870), pp. 329—340

423. *On the Plane Representation of a Solid Figure* . . . 26
Phil. Mag. t. XLI. (1871), pp. 286—290

424. *On the Attraction of a Terminated Straight Line* . . . 31
Phil. Mag. t. XLI. (1871), pp. 358—360

425. *Note on the Geodesic Lines on an Ellipsoid* 34
Phil. Mag. t. XLI. (1871), pp. 534, 535

*426. *On a supposed New Integration of Differential Equations of the Second Order* 36
Phil. Mag. t. XLII. (1871), pp. 197—199

PAGE

427. *On Gauss' Pentagramma Mirificum* 37

Phil. Mag. t. XLII. (1871), pp. 311, 312

*428. *Note sur la correspondance de deux points sur une courbe* . 39

Comptes Rendus, t. LXII. (1866), pp. 586—590

*429. *Sur les Coniques déterminées par cinq conditions de contact avec une courbe donnée* 40

Comptes Rendus, t. LXIII. (1866), pp. 9—12

430. *Note sur quelques formules de M. E. de Jonquières, relatives aux Courbes qui satisfont à des conditions données* . . 41

Comptes Rendus, t. LXIII. (1866), pp. 666—670

431. *Sur la transformation cubique d'une fonction elliptique* . . 44

Comptes Rendus, t. LXIV. (1867), pp. 560—563

432. *Théorème relatif à la théorie des substitutions* 47

Comptes Rendus, t. LXVII. (1868), pp. 784, 785

433. *Sur les surfaces tétraédrales* 48

Notes to De la Gournerie, *Recherches sur les Surfaces réglées tétraédrales symétriques*, 8vo. Paris, 1867

434. *On Certain Skew Surfaces, otherwise Scrolls* 54

Camb. Phil. Trans. t. XI. Part II. (1869), pp. 277—289

435. *On the Six Coordinates of a Line* 66

Camb. Phil. Trans. t. XI. Part II. (1869), pp. 290—323

436. *On a Certain Sextic Torse* 99

Camb. Phil. Trans. t. XI. Part III. (1871), pp. 507—523

*437. *Démonstration nouvelle du théorème de M. Casey par rapport aux cercles qui touchent à trois cercles donnés* . . . 115

Annali di Matematica, t. I. (1867), pp. 132—134

438. *Note sur quelques torses sextiques* 116

Annali di Matematica, t. II. (1868), pp. 99, 100

439. *Addition à la Note sur quelques torses sextiques* . . . 118

Annali di Matematica, t. II. (1868), pp. 219—221

PAGE

440. *Note sur une transformation géométrique* 121
Jour. der Mathem. (Crelle), t. LXVII. (1867), pp. 95, 96

441. *Note sur l'algorithme des tangentes doubles d'une courbe du quatrième ordre* 123
Jour. der Mathem. (Crelle), t. LXVIII. (1868), pp. 176—179

442. *Note sur la surface du quatrième ordre douée de seize points singuliers et de seize plans singuliers* 126
Jour. der Mathem. (Crelle), t. LXXIII. (1871), pp. 292, 293

443. *Note on the solution of the Quartic Equation* $\alpha U + 6\beta H = 0$. 128
Math. Ann. t. I. (1869), pp. 54, 55

444. *On the Centro-surface of an Ellipsoid* 130
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 16—18

445. *A Memoir on Quartic Surfaces* 133
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 19—69

446. *On the Mechanical Description of a Nodal Bicircular Quartic* 182
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 100—106

447. *On the Rational Transformation between Two Spaces* . . 189
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 127—180

448. *Note on the Cartesian with Two Imaginary Axial Foci* . . 241
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 181, 182

449. *Sketch of recent researches upon Quartic and Quintic Surfaces* 244
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 186—195

450. *Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points* . . . 253
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 196—198

451. *A Second Memoir on Quartic Surfaces* 256
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 198—202

452. *On an Analytical Theorem from a new point of view* . . 261
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 220, 221

PAGE

453. *On a Problem in the Calculus of Variations* 263
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 221, 222

454. *A Third Memoir on Quartic Surfaces* 264
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 234—266

455. *On Plücker's Models of certain Quartic Surfaces* . . . 298
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 281—285

456. *Note on the Discriminant of a Binary Quartic* . . . 303
Quart. Math. Jour. t. x. (1870), p. 23

457. *On the Quartic Surfaces* $(*\between U, V, W)^2=0$ 304
Quart. Math. Jour. t. x. (1870), pp. 24—34

458. *On the Anharmonic-Ratio Sextic* 314
Quart. Math. Jour. t. x. (1870), pp. 56, 57

459. *On the Double-Sixers of a Cubic Surface* 316
Quart. Math. Jour. t. x. (1870), pp. 58—71

460. *Note on Mr Frost's paper On the direction of the Lines of Curvature in the neighbourhood of an Umbilicus* . . 330
Quart. Math. Jour. t. x. (1870), pp. 111—113

461. *On the Geometrical Interpretation of the Covariants of a Binary Cubic* 332
Quart. Math. Jour. t. x. (1870), pp. 148, 149

462. *A Ninth Memoir on Quantics*. 334
Phil. Trans. t. CLXI. (for 1871), pp. 17—50

463. *Note on a Differential Equation* 354
Mem. Manchester Society, t. II. (1865), pp. 111—114

464. *Note on Plana's Lunar Theory* 357
Monthly Notices R. Ast. Society, t. XXIII. (1862—1863), pp. 211—215

465. *Note on the Lunar Theory* 361
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 182—189

PAGE

466. *Second Note on the Lunar Theory* 367
Monthly Notices R. Ast. Society, t. xxv. (1864—1865), pp. 203—207

467. *Expressions for Plana's e, γ in terms of the Elliptic e, γ* . 371
Monthly Notices R. Ast. Society, t. xxv. (1864—1865), pp. 265—271

468. *Addition to Second Note on the Lunar Theory* . . . 375
Monthly Notices R. Ast. Society, t. xxvii. (1866—1867), pp. 267—269

469. *On an Expression for the Angular Distance of Two Planets* . 377
Monthly Notices R. Ast. Society, t. xxvii. (1866—1867), pp. 312—315

470. *Note on the Attraction of Ellipsoids* 380
Monthly Notices R. Ast. Society, t. xxix. (1868—1869), pp. 254—257

471. *Note on the Problem of the Determination of a Planet's Orbit from three observations* 384
Monthly Notices R. Ast. Society, t. xxix. (1868—1869), pp. 257—259

472. *Note on Lambert's Theorem for Elliptic Motion* . . . 387
Monthly Notices R. Ast. Society, t. xxix. (1868—1869), pp. 318—320

473. *On the Graphical Construction of the Umbral or Penumbral Curve at any instant during a Solar Eclipse* . . . 390
Monthly Notices R. Ast. Society, t. xxx. (1869—1870), pp. 162—164

474. *On the Geometrical Theory of Solar Eclipses* 392
Monthly Notices R. Ast. Society, t. xxx. (1869—1870), pp. 164—168

475. *On a property of the Stereographic Projection* . . . 397
Monthly Notices R. Ast. Society, t. xxx. (1869—1870), pp. 205—207

476. *On the Determination of the Orbit of a Planet from three observations* 400
Mem. R. Ast. Society, t. xxxviii. (1870), pp. 17—111

477. *On the Graphical Construction of a Solar Eclipse* . . . 479
Mem. R. Ast. Society, t. xxxix. (1872), pp. 1—17

478. *On the Geodesic Lines on an Ellipsoid* 493
Mem. R. Ast. Society, t. xxxix. (1872), pp. 31—53

PAGE

479. *The Second Part of a Memoir On the Development of the Disturbing Function in the Lunar and Planetary Theories* 511

Mem. R. Ast. Society, t. XXXIX. (1872), pp. 55—74

480. *On the Expression of Delaunay's l, g, h in terms of his finally adopted Constants* 528

Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 8—16

481. *On the Expression of M. Delaunay's h+g in terms of his finally adopted Constants* 534

Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), p. 74

482. *Note on a pair of Differential Equations in the Lunar Theory* 535

Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 31, 32

483. *On a pair of Differential Equations in the Lunar Theory* . 537

Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 201—206

484. *On the variations of the position of the Orbit in the Planetary Theory* 541

Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 206—211

485. *Problems and Solutions* 546

Mathematical Questions with their Solutions from the Educational Times, vols. V. to XII. (1866—1869): for contents, see p. 607.

Notes and References 609

Plates . . . . . *to face pp.* 478, 492, 510

Portrait . . . . . *to face Title.*

CLASSIFICATION.

GEOMETRY :

Rational Transformation between Two Spaces, 447; Ditto Two Planes, and Special Systems of Points, 450
Quartic Surfaces, 445, 451, 454; Ditto and Quintic Surfaces, Recent Researches on, 449
Quartic Surfaces $(*\between U, V, W)^2 = 0$, 457
Recent Researches on Quartic and Quintic Surfaces, 449
Foci of Conics, 417
Geodesic Lines of Oblate Spheroid, 422; Ellipsoid, 425, 478
Plane Representation of Solid Figure, 423
Attraction of Terminated Line, 424
Gauss' Pentagramma Mirificum, 427
Correspondence of Two Points on a Curve, 428*
Conics determined by Five Conditions of Contact, 429*
Formulæ of M. de Jonquières, 430
Tetrahedral Surfaces, 433
Skew Surfaces otherwise Scrolls, 434
Six Coordinates of a Line, 435
Sextic Torse, 436, 438, 439
Tactions, Casey's Theorem, 437*
Geometrical Transformation, 440
Bitangents of Quartic Curve, 441
16 Nodal Quartic Surface, 442
Centro-surface of Ellipsoid, 444
Mechanical Description of Nodal Bicircular Quartic, 446
Cartesian with Two Imaginary Axial Foci, 448
Plücker's Models, 455
Attraction of Ellipsoids, 470
Double Sixers of Cubic Surface, 459
Curves of Curvature near Umbilicus, 460
Geometrical Interpretation of Covariants of Binary Cubic, 461

ANALYSIS :

Differential Equations, 418, 463
Differential Operators, 419
Riccati's Equation, 420

Solvibility of Equations by Radicals, 421
Supposed New Integration, 426*
Cubic Transformation of Elliptic Functions, 431
Substitutions, 432
Quartic Equation $\alpha U + 6\beta H = 0$, 443
Analytical Theorem from new point of view, 452
Problem in Calculus of Variations, 453
Discriminant of Binary Quartic, 456
Anharmonic-Ratio Sextic, 458
Ninth Memoir on Quantics, 462

ASTRONOMY:

Plana's Lunar Theory, 464, 467
Note on Lunar Theory, 465, 466, 468
Angular Distance of two Planets, 469
Planet's Orbit from three Observations, 471, 476
Lambert's Theorem for Elliptic Motion, 472
Solar Eclipses, 473, 474, 477
Stereographic Projection, 475
Disturbing Function in Planetary Theory, 479
Delaunay's l, g, h and $h+g$, 480, 481
Pair of Differential Equations in Lunar Theory, 482, 483
Position of Orbit in Planetary Theory, 484
Problems and Solutions, 485

417.

ON THE LOCUS OF THE FOCI OF THE CONICS WHICH PASS THROUGH FOUR GIVEN POINTS.

[From the *Philosophical Magazine*, vol. XXXII. (1866), pp. 362—365.]

THE curve which is the locus of the foci of the conics which pass through four given points is, as appears from a general theorem of M. Chasles, a sextic curve having a double point at each of the circular points at infinity; and Professor Sylvester, in his "Supplemental Note on the Analogues in Space to the Cartesian Ovals *in plano*" (*Phil. Mag.*, May 1866), has further remarked that the lines (eight in all) joining the circular points at infinity with any one of the four points are all of them double tangents of the curve; whence each of these points is a focus (more accurately a quadruple focus) of the curve. It is to be added that, besides the circular points at infinity, the curve has 6 double points (3 of these are the centres of the quadrangles formed by the 4 points), in all 8 double points; the class is therefore $=14$. Hence also the number of tangents to the curve from a circular point at infinity is $=10$; viz. these are the 4 double tangents each reckoned twice, and 2 single tangents; and the theoretical number of foci is $=100$; viz. we have

16 quadruple foci, or intersections of a double tangent by a double tangent . .	$16\times 4=64$
16 double foci, or intersections of a double tangent by a single tangent . .	$16\times 2=32$
4 single foci, or intersections of a single tangent by a single tangent . . .	$4\times 1=\ \ 4$
	$\overline{100}$

To verify the foregoing results, consider any two given points I, J, and the series of conics which pass through four given points A, B, C, D; we have thus a curve

the locus of the intersections of the tangents from I and the tangents from J to any conic of the series; which curve, if I, J are the circular points at infinity, is the required curve of foci. Taking $U+\lambda V=0$ for the equation of a conic of the series, the pair of tangents from I is given by an equation of the form

$$(\lambda, 1)^2 (x, y, z)^2 = 0,$$

and the pair of tangents from J by an equation of the like form

$$(\lambda, 1)^2 (x, y, z)^2 = 0;$$

and by eliminating λ from these equations, we obtain the equation of the required curve. This in the first instance presents itself as an equation of the eighth order; but it is to be observed that in the series of conics there are two conics each of them touching the line IJ, and that, considering the tangents drawn to either of these conics, the line IJ presents itself as part of the locus; that is, the line IJ twice repeated is part of the locus; and the residual curve is thus of the order $8-2, =6$; that is, the required curve is of the order 6. The consideration of the same two conics shows that each of the points I, J is a double point on the curve. Moreover, by taking for the conic any one of the line-pairs through the four points, it appears that each of the points $(AB.CD)$, $(AC.BD)$, $(AD.BC)$ is a double point on the curve: this establishes the existence of five double points. The two conics of the series which touch the line IA are a single conic taken twice, and the consideration of this conic shows that the line IA is a double tangent to the curve; similarly each of the eight lines $I(A, B, C, D)$ and $J(A, B, C, D)$ is a double tangent to the curve. Instead of seeking to establish directly the existence of the remaining three double points, the easier course is to show that, besides the four double tangents from I, the number of tangents from I to the curve is $=2$; for, this being so, the total number of tangents from I to the curve will be $(2\times 4+2=)\,10$; that is, I being a double point, the class of the curve is $=14$; and assuming that the depression $(6\times 5-14=)\,16$ in the class of the curve is caused by double points, the number of double points will be $=8$. But observing that in the series of conics there is one conic which passes through J, so that the tangents from J to this conic are the tangent at J twice repeated, then it is easy to see that the tangents from I to this conic, at the points where they meet the tangent at J, *touch* the required curve, and that these two tangents are in fact (besides the double tangents) the only tangents from I to the curve; that is, the number of tangents from I to the curve is $=2$.

Considering I, J as the circular points at infinity, and writing A, B, C, D to denote the squared distances of a point P from the four points A, B, C, D respectively, then, as remarked by Professor Sylvester, the equation

$$\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$$

(where λ, μ, ν, π are constants) is in general a curve of the order 8; but the ratios $\lambda:\mu:\nu:\pi$ may be so determined that the order of the curve in question shall be

$=6$; the resulting curve of the order 6 is (not one of a group of curves, but the very curve) the locus of the foci of the conics through the four points. And the determination of the ratios $\lambda : \mu : \nu : \pi$ is in fact quite simple; for writing

$$\begin{aligned} A &= (x-a)^2 + (y-a_1)^2 \\ &= \rho^2 - 2(ax + a_1 y) + \&c. \\ &\quad (\text{if } \rho^2 = x^2 + y^2), \end{aligned}$$

and therefore

$$\sqrt{A} = \rho - \frac{ax + a_1 y}{\rho} + \&c.,$$

with similar values for $\sqrt{B}$, $\sqrt{C}$, $\sqrt{D}$, it is easy to see that, considering λ, μ, ν, π as standing for $\pm\lambda$, $\pm\mu$, $\pm\nu$, $\pm\pi$ respectively, the conditions for the reduction to the order 6 are

$$\begin{aligned} \lambda + \mu + \nu + \pi &= 0, \\ \lambda a + \mu b + \nu c + \pi d &= 0, \\ \lambda a_1 + \mu b_1 + \nu c_1 + \pi d_1 &= 0, \end{aligned}$$

and hence that the required equation of the curve of foci is

$$\Sigma \left\{ \begin{vmatrix} 1, & 1, & 1 \\ b, & c, & d \\ b_1, & c_1, & d_1 \end{vmatrix} \sqrt{(x-a)^2 + (y-a_1)^2} \right\} = 0,$$

or, as this may also be written,

$$\Sigma \pm (B, \quad C, \quad D) \sqrt{A} = 0,$$

where (B, C, D), &c. are the areas of the triangles B, C, D, &c.

I remark, in conclusion, that the number of conditions to be satisfied in order that a curve may have for double points two given points I, J, may have besides six double points, and may have for double tangents eight given lines, is $(3+3+6+16=)\,28$; the number of constants contained in the general equation of the order 6 is $=27$. The conditions that a curve of the order 6 shall have for double points two given points I, J, shall besides have six double points, and shall have for double tangents four given lines through I and four given lines through J, are more than sufficient for the determination of the sextic curve; and the existence of a sextic curve satisfying these conditions is therefore a theorem.

In the case where the points I, J lie on a conic of the series, the consideration of this conic shows that the curve has a ninth double point, the pole of the line IJ in regard to the conic in question: in this case the sextic curve, as is known, breaks up into two cubic curves. [It need not do so, for a proper sextic curve may have nine (or indeed ten) double points.]

P.S. In general the curve $\lambda\sqrt{A}+\mu\sqrt{B}+\nu\sqrt{C}+\pi\sqrt{D}=0$ has (exclusively of multiple points at infinity) six double points; viz. these are situate at the intersections of the pairs of circles,

$$(\lambda\sqrt{A}+\mu\sqrt{B}=0,\quad \nu\sqrt{C}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\nu\sqrt{C}=0,\quad \mu\sqrt{B}+\pi\sqrt{D}=0),$$

$$(\lambda\sqrt{A}+\pi\sqrt{D}=0,\quad \mu\sqrt{B}+\nu\sqrt{C}=0).$$

In the case of the curve of foci, the first, second, and third pairs of circles intersect respectively in the points $(AB.CD)$, $(AC.BD)$, $(AD.BC)$, which, as mentioned above, are double points on the curve; and they besides intersect in three other points, which are the other three double points mentioned above.

Professor Sylvester reminds me that he mentioned to me in conversation that he had himself obtained the foregoing equation $\Sigma\pm(B, C, D)\sqrt{A}=0$, for the locus of the foci of the conics which pass through the four points A, B, C, D.

Cambridge, October 10, 1866.

418.

A REMARK ON DIFFERENTIAL EQUATIONS.

[From the *Philosophical Magazine*, vol. XXXII. (1866), pp. 379—381.]

CONSIDER a differential equation $f(x, y, p) = 0$, of the first order, but of the degree n, where f is a rational and integral function of (x, y, p) not rationally decomposable into factors: the integral equation contains an arbitrary constant c, and represents therefore a system of curves, for any one of which curves the differential equation is satisfied: the differential equation is assumed to be such that the curves are algebraical curves. The curves in question may be considered as undecomposable curves; in fact, if the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ (composed of the undecomposable curves $U=0$, $V=0$, $W=0, \ldots$) satisfies the differential equation, then either the curves $U=0$, $V=0$, $W=0, \ldots$ each satisfy the differential equation, and instead of the curve $U^\alpha V^\beta W^\gamma \ldots = 0$ we have thus the undecomposable curves $U=0$, $V=0$, $W=0, \ldots$ each satisfying the differential equation; or if any of these curves, for instance $W=0$, &c., do not satisfy the differential equation, then W^γ, &c. are mere extraneous factors which may and ought to be rejected, and instead of the original curve $U^\alpha V^\beta W^\gamma \ldots = 0$, we have the undecomposable curves $U=0$, $V=0$ satisfying the differential equation. Assuming, as above, the existence of an algebraical solution, this may always be expressed in the form $\phi(x, y, c) = 0$, where ϕ is a rational and integral function of (x, y, c), of the degree n as regards the arbitrary constant c: this appears by the consideration that for given values (x_0, y_0) of (x, y) the differential equation and the integral equation must each of them give the same number of values of p. It is to be observed that ϕ regarded as a function of (x, y, c) cannot be rationally decomposable into factors; for if the equation were $\phi = \Phi\Psi \ldots = 0$, Φ, Ψ, &c. being each of them rational and integral functions of (x, y, c), then the differential equation would be satisfied by at least one of the equations $\Phi = 0$, $\Psi = 0, \ldots$ that is, by an equation of a degree less than n in the arbitrary constant c.

But the equation $\phi(x, y, c)=0$ is not of necessity the equation of an undecomposable curve, and the undecomposable curve which constitutes the proper solution of the differential equation cannot always be represented by an equation of the form in question. For although ϕ regarded as a function of (x, y, c) is not rationally decomposable into factors, yet it may very well happen that ϕ regarded as a function of (x, y) is rationally decomposable into factors (geometrically the sections by the planes $z=c$ of the undecomposable surface $\phi(x, y, z)=0$ may each of them be composed of two or more distinct curves); and assuming that the function ϕ is thus decomposed into its prime factors, then each factor equated to 0 gives an undecomposable curve satisfying the differential equation, and constituting the proper solution thereof.

It may be observed that, by the foregoing process of decomposition, we sometimes reduce the original equation $\phi(x, y, c)=0$ into a like equation $\phi_1(x, y, c_1)=0$ of a more simple form. Thus, for instance, if we have $\phi(x, y, c)=U^2-c=0$, U being a rational and integral function of (x, y), then instead of $\phi=U^2-c=0$ we have the equations $U+\sqrt{c}=0$, $U-\sqrt{c}=0$, each of which is an equation of the form $U-c_1=0$, or we pass from the original equation $\phi(x, y, c)=U^2-c=0$ to the simplified equation

$$\phi_1(x, y, c_1)=U-c_1=0.$$

Again, to take a somewhat more complicated instance, if the given integral equation be

$$\phi(x, y, c)=U^4+c^2V^4+(c+1)^2W^2-2cU^2V^2-2(c+1)U^2W^2-2c(c+1)V^2W^2=0,$$

then the equation $U+V\sqrt{c}+W\sqrt{c+1}=0$, writing therein $\sqrt{c}=\dfrac{2c_1}{c_1^2-1}$, and therefore $\sqrt{c+1}=\dfrac{c_1^2+1}{c_1^2-1}$, becomes

$$U(c_1^2-1)+V.2c_1+W(c_1^2+1)=0;$$

so that we pass from the original equation $\phi(x, y, c)=0$ to the simplified equation

$$\phi_1(x, y, c_1)=U(c_1^2-1)+V.2c_1+W(c_1^2+1)=0.$$

But observe that the possibility of the rationalization depends on the form of the radicals $\sqrt{c}$ and $\sqrt{c+1}$; if we had had $\sqrt{c}$ and $\sqrt{c^2+1}$ (or c and $\sqrt{c^4+1}$), the rationalization could not have been effected.

Returning to the case of an integral equation $\phi(x, y, c)=0$, where ϕ regarded as a function of (x, y) is decomposable into factors, then equating to zero any one of the prime factors of ϕ, we obtain an integral equation $\psi(x, y, c_1, c_2, \ldots c_k)=0$, where $c_1, c_2 \ldots c_k$ are irrational functions (not of necessity representable by radicals, and without any superior limit to the number of these functions) of c: here ψ regarded as a function of (x, y) is of course undecomposable, and the equation $\psi(x, y, c_1, c_2, \ldots c_k)=0$ belongs to the undecomposable curve which is the proper solution of the differential equation. The result may be stated under a quasi-geometrical form; viz. regarding $c_1, c_2, \ldots c_k$ as the coordinates of a point in k-dimensional space, then as these are

functions of the single parameter c, the point to which they belong is an arbitrary point on a certain curve or $(k-1)$fold locus C in the k-dimensional space. And this curve must be such that to given values of (x, y) there shall correspond n points on the curve; that is, treating (x, y) as constants, the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)=0$, and the curve or $(k-1)$fold locus C, shall meet in n points. The conclusion stated in the foregoing quasi-geometrical form is, that the solution of the differential equation may be exhibited in the form $\psi(x, y, c_1, c_2 \ldots c_k)=0$; viz. ψ is a rational and integral function of $(x, y, c_1, c_2 \ldots c_k)$, where $(c_1, c_2 \ldots c_k)$ are the coordinates of an arbitrary or variable point on a curve or $(k-1)$fold locus C in a k-dimensional space, which curve meets the surface or onefold locus $\psi(x, y, c_1, c_2 \ldots c_k)$ in n points, and where ψ regarded as a function of (x, y) is not rationally decomposable into factors.

Cambridge, October 13, 1866.

419.

A THEOREM ON DIFFERENTIAL OPERATORS.

[From a *paper by* PROF. SYLVESTER, "*Note on the Test Operators which occur in the Calculus of Invariants, &c.*," *Philosophical Magazine*, vol. XXXII. (1866), pp. 461—472, see p. 471.]

THE paper concludes with an Observation from Professor Cayley as follows:

"In the case of two variables, if

$$P_1 = (ax + by)\frac{d}{dx} + (cx + dy)\frac{d}{dy},$$

then in the notation of matrices,

$$P_1 = \begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix} (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_2 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^2 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_3 = \tfrac{1}{6}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right);$$

whence also

$$P * P_2 = P_2 * P_1 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right) = 3P_3,$$

which accords with your theorem,

$$E_1 * E_2 * = E_2 * E_1 * = E_1 E_2 * + 3E_3 *."$$

I have taken the liberty of writing in the above $\frac{d}{dx}$, $\frac{d}{dy}$ for δ_x, δ_y, and P for δ in the original. It will be useful to bear in mind that in any operator such as E_1* or E_2*, *the asterisk forms an integral part* of the symbol. Thus $E_1 * E_2 *$, if we choose, may be written under the form of E_1* multiplied by E_2*, i.e. $(E_1*) \times (E_2*)$, where the cross is the sign of ordinary algebraical multiplication.

420.

ON RICCATI'S EQUATION.

[From the *Philosophical Magazine*, vol. XXXVI. (1868), pp. 348—351.]

THE following is, it appears to me, the proper form in which to present the solution of Riccati's equation.

The equation may be written

$$\frac{dy}{dx} + y^2 = x^{2q-2},$$

which is integrable by algebraic and exponential functions if $(2i+1)q = \pm 1$, i being zero, or a positive integer. To effect the integration, writing $y = \frac{1}{u}\frac{du}{dx}$, we have

$$\frac{d^2u}{dx^2} = x^{2q-2}u.$$

The peculiar advantage of this well-known transformation has not (so far as I am aware) been explicitly stated; it puts in evidence the form under which the sought-for function y contains the constant of integration. In fact if $u = P$, $u = Q$ be two particular solutions of the equation in u, then the general solution is $u = CP + DQ$; and denoting by P', Q' the derived functions, the value of y is

$$y = \frac{CP' + DQ'}{CP + DQ},$$

showing the form under which the constant of integration $C \div D$ is contained in y. To complete the solution, assume

$$u = ze^{\frac{1}{q}x^q};$$

we find

$$\frac{d^2z}{dx^2} + 2x^{q-1}\frac{dz}{dx} + (q-1)\,x^{q-2}\,z = 0:$$

considering first the particular integral of the form

$$z = A + Bx^q + Cx^{2q} + Dx^{3q} + \&c.,$$

we find that the equation will be satisfied if

$$\begin{aligned}
&(q-1)\,A + q(q-1)\,B = 0,\\
&(3q-1)\,B + 2q(2q-1)\,C = 0,\\
&(5q-1)\,C + 3q(3q-1)\,D = 0,\\
&(7q-1)\,D + 4q(4q-1)\,E = 0,\\
&\quad \&c.
\end{aligned}$$

If $A = 1$, this is

$$\begin{aligned}
A &= 1,\\
B &= -\frac{q-1}{q(q-1)},\\
C &= +\frac{(q-1)(3q-1)}{q(q-1)\,2q(2q-1)},\\
D &= -\frac{(q-1)(3q-1)(5q-1)}{q(q-1)\,2q(2q-1)\,3q(3q-1)},\\
&\&c.,
\end{aligned}$$

where it is to be noticed that the series may be considered to stop so soon as there is in the numerator a factor $= 0$. For instance, if $5q-1=0$, then if the particular integral had been assumed to be $z = A + Bx^q + Cx^{2q}$, the only conditions to be satisfied by the coefficients are the first and second equations giving the foregoing values of A, B, C. It is immaterial that the analytical expressions of F and the subsequent coefficients contain in the denominators the evanescent factor $5q-1$; the coefficients after C do not ever come into consideration.

Thus if $(2i+1)\,q = +1$, the series terminates, and we have for u the finite particular solution

$$u = P = \left(1 - \frac{q-1}{q(q-1)}x^q + \frac{(q-1)(3q-1)}{q(q-1)\,2q(2q-1)}x^{2q} - \&c.\right)e^{\frac{1}{q}x^q}:$$

and it is easy to see that we may herein change the sign of x^q, thereby obtaining another finite particular solution,

$$u = Q = \left(1 + \frac{q-1}{q(q-1)}x^q + \frac{(q-1)(3q-1)}{q(q-1)\,2q(2q-1)}x^{2q} + \&c.\right)e^{-\frac{1}{q}x^q}.$$

Reverting to the equation in z, we have next a particular solution of the form

$$z = Ax + Bx^{q+1} + Cx^{2q+1} + Dx^{3q+1} + \&c.,$$

giving between the coefficients the relation

$$\begin{aligned}
&(q+1)A + (q+1)\,q\,B = 0,\\
&(3q+1)B + (2q+1)\,2q\,C = 0,\\
&(5q+1)C + (3q+1)\,3q\,D = 0,\\
&(7q+1)D + (4q+1)\,4q\,E = 0,\\
&\quad \&c.
\end{aligned}$$

If $A = 1$, we have

$$\begin{aligned}
A &= 1,\\
B &= -\frac{(q+1)}{(q+1)\,q},\\
C &= +\frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q},\\
D &= -\frac{(q+1)(3q+1)(5q+1)}{(q+1)\,q\,(2q+1)\,2q\,(3q+1)\,3q},\\
&\&c.,
\end{aligned}$$

where, as in the former case, the series is considered to terminate as soon as there is an evanescent factor in the numerator, without any regard to the subsequent coefficients which contain in the denominators the same evanescent factor. [In particular, $q = -1$, we have the solution $z = x$.]

Hence if we have $(2i+1)q = -1$, the series terminates, and we have for u the finite particular solution,

$$u = P = x\left(1 - \frac{q+1}{(q+1)\,q}\,x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}\,x^{2q} - \&c.\right)e^{\frac{1}{q}x^q},$$

from which, changing the sign of x^q, we deduce the other finite particular solution,

$$u = Q = x\left(1 + \frac{q+1}{(q+1)\,q}\,x^q + \frac{(q+1)(3q+1)}{(q+1)\,q\,(2q+1)\,2q}\,x^{2q} + \&c.\right)e^{-\frac{1}{q}x^q}.$$

Hence, in the equation

$$\frac{dy}{dx} + y^2 = x^{2q-2},$$

where $q(2i+1) = \pm 1$, we have (writing $D = 1$)

$$y = \frac{CP' + Q'}{CP + Q},$$

where C is the constant of integration, P, Q are finite series as above, and P', Q' are the derived functions of P and Q. Writing successively $i=0$, $i=1$, $i=2$, &c., we may tabulate the solutions

$$\frac{dy}{dx}+y^2=1, \qquad P=e^x, \qquad Q=e^{-x},$$

$$\frac{dy}{dx}+y^2=x^{-4}, \qquad P=xe^{-\frac{1}{x}}, \qquad Q=xe^{\frac{1}{x}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{4}{3}}, \qquad P=(1-3x^{\frac{1}{3}})\,e^{3x^{\frac{1}{3}}}, \qquad Q=(1+3x^{\frac{1}{3}})\,e^{-3x^{\frac{1}{3}}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{8}{3}}, \qquad P=x\,(1+3x^{-\frac{1}{3}})\,e^{-3x^{-\frac{1}{3}}}, \qquad Q=x\,(1-3x^{-\frac{1}{3}})\,e^{3x^{-\frac{1}{3}}},$$

$$\frac{dy}{dx}+y^2=x^{-\frac{8}{3}}, \qquad P=(1-5x^{\frac{1}{3}}+\tfrac{25}{3}x^{\frac{2}{3}})\,e^{5x^{\frac{1}{3}}}, \qquad Q=(1+5x^{\frac{1}{3}}+\tfrac{25}{3}x^{\frac{2}{3}})\,e^{-5x^{\frac{1}{3}}},$$

&c.

It is hardly necessary to make the final step of calculating P' and Q' and substituting in y; but, as an example, take the above equation $\frac{dy}{dx}+y^2=x^{-\frac{4}{3}}$: we have

$$y=\frac{-3x^{-\frac{1}{3}}\,(Ce^{3x^{\frac{1}{3}}}+e^{-3x^{\frac{1}{3}}})}{C\,(1-3x^{\frac{1}{3}})\,e^{3x^{\frac{1}{3}}}+(1+3x^{\frac{1}{3}})\,e^{-3x^{\frac{1}{3}}}},$$

which is readily identified with the solution, p. 98 of Boole's *Differential Equations* (Cambridge, 1859).

Cambridge, September 29, 1868.

421.

NOTE ON THE SOLVIBILITY OF EQUATIONS BY MEANS OF RADICALS.

[From the *Philosophical Magazine*, vol. XXXVI. (1868), pp. 386, 387.]

IN regard to the theorem that the general quintic equation of the nth order is not solvible by radicals, I believe that the proofs which have been given depend, or at any rate that a proof may be given that shall depend, on the following two lemmas:

I. A one-valued (or symmetrical) function of n letters is a perfect kth power, only when the kth root is a one-valued function of the n letters.

There is an exception in the case $k=2$, whatever be the value of n: viz. the product of the squares of the differences is a one-valued function, a perfect square; but its square root, or the product of the simple differences, is a two-valued function. It is in virtue of this exception that a quadric equation is solvible by radicals; we have the one-valued function $(x_1-x_2)^2$, the square of a two-valued function x_1-x_2, and thence the two roots are each expressible in the form

$$\tfrac{1}{2}\{x_1+x_2+\sqrt{(x_1-x_2)^2}\}.$$

II. A two-valued function of n letters is a perfect kth power, only when the kth root is a two-valued function of the n letters.

There is an exception in the case $k=3$, when $n=3$ or 4: viz. for $n=3$ we have $(x_1+\omega x_2+\omega^2 x_2)^3$ (ω an imaginary cube root of unity) a two-valued function, and a perfect cube; whereas its cube root is the six-valued function $x_1+\omega x_2+\omega^2 x_3$. And similarly for $n=4$ we have, for instance,

$$\{x_1x_2+x_3x_4+\omega(x_1x_3+x_2x_4)+\omega^2(x_1x_4+x_2x_3)\}^3$$

a two-valued function, and a perfect cube, whereas its cube root is a six-valued function. And it is in virtue of this exception that a cubic or a quartic equation is solvible by radicals. But I assume that for $n > 4$ the lemma is true without exception.

The course of demonstration would be something as follows: Imagine, if possible, the root of an equation expressed, by means of radicals, in terms of the coefficients; the expression cannot contain any radical such as $\sqrt[p]{X}$, $p > 2$, where X is a one-valued (or rational) function of the coefficients, not a perfect pth power, for the reason that, expressing the coefficients in terms of the roots, such function $\sqrt[p]{X}$ is not a rational function of the roots; if it were so, by lemma I. it would be a one-valued (that is, a symmetrical) function of the roots; consequently a rational function of the coefficients, or X expressed in terms of the coefficients, would be a perfect pth power.

The expression may however contain a radical $\sqrt{X}$, X a one-valued (or rational) function of the coefficients, not a perfect square: viz. X may be any square function multiplied into that function of the coefficients which is equal to the product of the squared differences of the roots, or, say, multiplied into the discriminant; that is, we may have $X = Q^2\nabla$, or $\sqrt{X} = Q\sqrt{\nabla}$.

We have next to consider whether the expression can contain any radical $\sqrt[p]{X}$, where X, not being a rational function of the coefficients, is a function expressible by radicals. But the foregoing reasoning shows that if this be so, X cannot contain any radical other than the radical $\sqrt{Q^2\nabla}$ or $Q\sqrt{\nabla}$, as above; that is, X must be $= P + Q\sqrt{\nabla}$, where P and Q are rational functions of the coefficients, and where we may assume that $P + Q\sqrt{\nabla}$ is not a perfect pth power of a function of the like form $P' + Q'\sqrt{\nabla}$. But then, expressing the coefficients in terms of the roots, we have $P + Q\sqrt{\nabla}$, a (rational) two-valued function of the roots; and there is no radical $\sqrt[p]{P + Q\sqrt{\nabla}}$, which is a rational function of the roots; for by lemma II., if such radical existed we should have $\sqrt[p]{P + Q\sqrt{\nabla}}$ a (rational) two-valued function of the roots; that is, it would be $= P' + Q'\sqrt{\nabla}$, P' and Q' one-valued (symmetrical) functions of the roots, consequently rational functions of the coefficients; or $P + Q\sqrt{\nabla}$ would be a perfect pth power $(P' + Q'\sqrt[p]{\nabla})^p$.

The conclusion is that for $n > 4$ there is not (besides the function $P + Q\sqrt{\nabla}$) any function of the coefficients, expressible by means of radicals, which, when the coefficients are expressed in terms of the roots, will be a rational function of the roots, and consequently there is no possibility of expressing the roots in terms of the coefficients by means of radicals.

Cambridge, October 1, 1868.

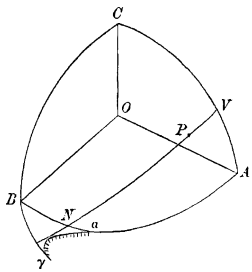
422.

ON THE GEODESIC LINES ON AN OBLATE SPHEROID.

[From the *Philosophical Magazine*, vol. XL. (1870), pp. 329—340.]

THE theory of the geodesic lines on an oblate spheroid of any excentricity whatever was investigated by Legendre (1); and the general course of them is well known, viz. each geodesic line undulates between two parallels equidistant from the equator (being thus either a closed curve, or a curve of indefinite length, according to the distance between the two parallels): at a point of contact with the parallel the curve is, of course, at right angles to the meridian; say this is V, a vertex of the geodesic

FIG. 1.

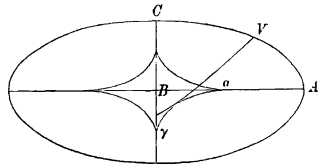


line, and let the meridian through V meet the equator in A; the geodesic line proceeds from V to meet the equator in a point N, the node, where AN is at most $= 90°$; and the undulations are obtained by the repetition of this portion VN of the geodesic line alternately on each side of the equator and of the meridian.

[1] *Mém. de l'Inst.* 1806; see also the *Exer. de Calcul Intégral*, t. I. (1811), p. 178, and the *Traité des Fonctions Elliptiques*, t. I. (1825), p. 360.

I consider in the present paper the series of geodesic lines which cut at right angles a given meridian AC, or, say, a series of geodesic normals. It may be remarked that as V passes from the position A on the equator to the pole C, the angular distance AN increases from a certain determinate value (equal, as will appear, to $\frac{C}{A}90°$, if C, A are the polar and equatorial axes respectively) up to the value $90°$; and it thus appears that, attending only to their course after they first meet the equator, the geodesic normals have an envelope resembling in its general appearance the evolute of an ellipse (see fig. 1 and also fig. 2), the centre hereof being the point

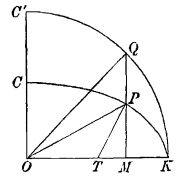
Fig. 2.



B at the distance $BA = 90°$, and the axes coinciding in direction with the equator BA and meridian BC: this is in fact a real geodesic evolute of the meridian CA. The point α is, it is clear, the intersection of the equator by the geodesic line for which V is consecutive to the point A (so that $\angle BOA = \left(1 - \frac{C}{A}\right)90°$); and the point γ is the intersection of the meridian CB by the geodesic line for which V is consecutive to the point C; and its position will be in this way presently determined. I was anxious, with a view to the construction of a drawing and a model, to obtain some numerical results in relation to a spheroid of considerable excentricity, and I selected that for which $\frac{C}{A} = \frac{1}{2}$ (polar axis $= \frac{1}{2}$ equatorial).

Before proceeding further, I remark that Legendre's expression "reduced latitude" is used in what is not, I think, the ordinary sense; and I propose to substitute the

Fig. 3.



term "parametric latitude": viz., in fig. 3, referring the point P on the ellipse by means

of the ordinate MPQ to a point Q on the circle, radius $OK (= OA$, fig. 1), and drawing the normal PT, then we have for the point P the three latitudes,

$$\lambda = \angle PTK, \text{ normal latitude,}$$
$$\lambda'' = \angle POK, \text{ central latitude,}$$
$$\lambda' = \angle QOK, \text{ parametric latitude;}$$

viz. λ' is the parameter most convenient for the expression of the values of the coordinates x, y ($x = A\cos\lambda'$, $y = C\sin\lambda'$) of a point P on the ellipse. The relations between the three latitudes are

$$\tan\lambda'' = \frac{C}{A}\tan\lambda' = \frac{C^2}{A^2}\tan\lambda,$$

so that λ'', λ', λ are in the order of increasing magnitude. I use in like manner l, l', l'' in regard to the vertex V. The course of a geodesic line is determined by the equation

$$\cos\lambda'\sin\alpha = \text{const.},$$

where λ' is the reduced latitude of any point P on the geodesic line, and α is at this point the azimuth of the geodesic line, or its inclination to the meridian. Hence, if l' be the parametric latitude of the vertex V, the equation is

$$\cos\lambda'\sin\alpha = \cos l'$$

(whence also, when $\lambda' = 0$, $\alpha = 90^\circ - l'$; that is, the geodesic line cuts the equator at an angle $= l'$, the parametric latitude of the vertex). The equation in question, $\cos\lambda'\sin\alpha = \cos l'$, leads at once to Legendre's other equations: viz. taking, as above, A, C for the equatorial and polar semiaxes respectively, and δ for the excentricity, $\delta = \sqrt{1 - \frac{C^2}{A^2}}$; and to determine the position of P on the meridian, using (instead of the parametric latitude λ') the angle ϕ determined by the equation

$$\cos\phi = \frac{\sin\lambda'}{\sin l'},$$

and writing, moreover, s to denote the geodesic distance VP, and Λ to denote the longitude of P measured from the meridian CA which passes through the vertex V, these are

$$ds = \quad d\phi\sqrt{C^2 + A^2\delta^2\sin^2 l'\cos^2\phi},$$

$$d\Lambda = \frac{\cos l'}{A}\cdot\frac{d\phi\sqrt{C^2 + A^2\delta^2\sin^2 l'\cos^2\phi}}{1 - \sin^2 l'\cos^2\phi};$$

which differential expressions are to be integrated from $\phi = 0$; and the equations then determine λ', s, and Λ, all in terms of the angle ϕ,—that is, virtually s and Λ, the length and longitude, in terms of the parametric latitude λ'.

Writing, with Legendre,

$$c^2 = \frac{A^2\delta^2 \sin^2 l'}{C^2 + A^2\delta^2 \sin^2 l'}, \quad = \quad \delta^2 \sin^2 l,$$

$$b^2 = 1 - c^2, \quad = \frac{C^2}{C^2 + A^2\delta^2 \sin^2 l'}, \quad = 1 - \delta^2 \sin^2 l;$$

also

$$n = \tan^2 l', \quad M = \frac{C}{bA \cos l'} = \frac{C}{A \cos l},$$

then the formulæ become

$$ds = \frac{C}{b} d\phi \sqrt{1 - c^2 \sin^2 \phi},$$

$$d\Lambda = M \frac{d\phi \sqrt{1 - c^2 \sin^2 \phi}}{1 + n \sin^2 \phi}.$$

Hence integrating from $\phi = 0$, and using the notations F, E, Π of elliptic functions, we have

$$s = \frac{C}{b} E(c, \phi),$$

$$\Lambda = \frac{M}{n} \{(n + c^2) \Pi(n, c, \phi) - c^2 F(c, \phi)\};$$

viz. these belong to any point P whatever on the geodesic line, parametric latitude of vertex $= l'$; and if we write herein $\phi = 90°$, then they will refer to the node N, or point of intersection with the equator.

The position of the point α is at once obtained by writing $l' = 0$: viz. this gives $c = 0$, $b = 1$, $M = \frac{C}{A}$, $n = 0$: the differential expressions are $ds = C d\phi$, $d\Lambda = \frac{C}{A} d\phi$. Or integrating from $\phi = 0$ to $\phi = \frac{1}{2}\pi$, we have $s = A \,.\, \frac{C}{A} \cdot \frac{1}{2}\pi$, $\Lambda = \frac{C}{A} \cdot \frac{1}{2}\pi$, agreeing with each other, and giving longitude of $\alpha = \frac{C}{A} \cdot \frac{1}{2}\pi$; or, what is the same thing, $\angle \alpha OB = \frac{1}{2}\pi \left(1 - \frac{C}{A}\right)$.

Writing in the formulæ $l' = 90°$, we have $c = \delta$, $b = \frac{C}{A}$, $\frac{M}{n} = 0$; whence $d\Lambda = 0$, or $\Lambda = \text{const.}$, $= \frac{1}{2}\pi$, since the geodesic line here coincides with the meridian CB; and moreover $s = AE(\delta, \phi)$; viz. this is merely the expression of the distance from C of a point P on the meridian CB. But we do not thus obtain the position of the point γ.

To find it we must consider a position of V consecutive to C, say, $l' = \frac{1}{2}\pi - \epsilon$, where ϵ is indefinitely small; n is thus indefinitely large, and the integral $\Pi(n, c, \phi)$ is not conveniently dealt with. But it may be replaced by an expression depending on

$\Pi\left(\frac{c^2}{n}, c, \phi\right)$, where $\frac{c^2}{n}$ is indefinitely small; viz. (Legendre, *Fonct. Ellipt.* vol. I. p. 69) we have

$$\Pi(n, c, \phi) = F(c, \phi) + \frac{1}{\sqrt{\alpha}} \tan^{-1} \frac{\sqrt{\alpha} \tan \phi}{\sqrt{1 - c^2 \sin^2 \phi}} - \Pi\left(\frac{c^2}{n}, c, \phi\right),$$

where

$$\alpha = (1+n)\left(1 + \frac{c^2}{n}\right).$$

We thus have

$$\Lambda = \frac{M}{n}\left\{nF(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}} \tan^{-1} \frac{\sqrt{\alpha} \tan \phi}{\sqrt{1 - c^2 \sin^2 \phi}} - (c^2+n)\,\Pi\left(\frac{c^2}{n}, c, \phi\right)\right\},$$

where, $\frac{c^2}{n}$ being small,

$$\Pi\left(\frac{c^2}{n}, c, \phi\right) = \int \frac{d\phi}{\left(1 + \frac{c^2}{n}\sin^2\phi\right)\sqrt{1 - c^2 \sin^2 \phi}},$$

$$= \int \frac{\left(1 - \frac{c^2}{n}\sin^2\phi\right) d\phi}{\sqrt{1 - c^2 \sin^2 \phi}}, \quad = \left(1 - \frac{1}{n}\right) F(c, \phi) + \frac{1}{n} E(c, \phi).$$

And expanding also the $\tan^{-1}$ term, we thus have

$$\Lambda = \frac{M}{n}\left\{nF(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}}\left[\tfrac{1}{2}\pi - \frac{\sqrt{1 - c^2 \sin^2 \phi}}{\tan \phi} \frac{\sqrt{n}}{\sqrt{(1+n)(c^2+n)}}\right]\right.$$

$$\left. - (c^2+n)\left[\left(1 - \frac{1}{n}\right) F(c, \phi) + \frac{1}{n} E(c, \phi)\right]\right\},$$

$$= \frac{M}{n}\left\{\left(b^2 + \frac{c^2}{n}\right) F(c, \phi) - \left(1 + \frac{c^2}{n}\right) E(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}} \cdot \tfrac{1}{2}\pi - \frac{n}{n+1} \cot \phi \sqrt{1 - c^2 \sin^2 \phi}\right\},$$

which, in the term in { } neglecting negative powers of n, becomes

$$\Lambda = \frac{M}{n}\left\{\sqrt{n} \cdot \tfrac{1}{2}\pi + b^2 F(c, \phi) - E(c, \phi) - \cot \phi \sqrt{1 - c^2 \sin^2 \phi}\right\}.$$

We may moreover write $c = \delta$, $b = \frac{C}{A}$, $\phi = 90^\circ - \lambda'$, $n = \frac{1}{\epsilon^2}$, $M = \epsilon$, and therefore $\frac{M}{n} = \epsilon$, so that the formula is

$$\Lambda = \epsilon\left\{\frac{1}{\epsilon} \cdot \tfrac{1}{2}\pi + b^2 F(c, 90^\circ - \lambda') - E(c, 90^\circ - \lambda') - \tan \lambda' \sqrt{1 - c^2 \cos^2 \lambda'}\right\},$$

$$= \tfrac{1}{2}\pi - \epsilon\left\{\tan \lambda' \sqrt{1 - c^2 \cos^2 \lambda'} + E(c, 90^\circ - \lambda') - b^2 F(c, 90^\circ - \lambda')\right\},$$

where I retain c, b as standing for $\sqrt{1 - \frac{C^2}{A^2}}$, $\frac{C}{A}$ respectively.

Writing herein $\lambda' = 0$, we have

$$\Lambda = \tfrac{1}{2}\pi - \epsilon\,(E, c - b^2 F, c),$$

where the coefficient $E, c - b^2 F, c$ is

$$= \int_0^{\frac{1}{2}\pi} d\theta \left(\sqrt{1 - c^2 \sin^2\theta} - \frac{1 - c^2}{\sqrt{1 - c^2 \sin^2\theta}}\right) = c^2 \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta\, d\theta}{\sqrt{1 - c^2 \sin^2\theta}}$$

consequently positive; that is, Λ, the longitude of the node, is less than $90°$, as it should be. Hence in order that Λ may be $= 90°$, we must have λ' negative, say, $\lambda' = -\mu'$, where μ' is positive; and, observing that we may under the signs E, F write $90° - \mu'$ instead of $90° + \mu'$, we thus have

$$\tfrac{1}{2}\pi = \tfrac{1}{2}\pi + \epsilon\,\{\sqrt{1 - c^2 \cos^2\mu'}\,\tan\mu' - E\,(c,\ 90° - \mu') + b^2 F\,(c,\ 90° - \mu')\};$$

that is, we must have

$$\tan\mu'\,\sqrt{1 - c^2 \cos^2\mu'} = E\,(c,\ 90° - \mu') - b^2 F\,(c,\ 90° - \mu');$$

viz. μ' is here the parametric latitude (south) of the intersection of the meridian CB with the consecutive geodesic line—that is, of the point γ. As μ' increases from 0 to $90°$, the left-hand side increases from 0 to ∞; and the right-hand side, beginning from a positive value and either attaining a maximum or not, ultimately decreases to 0; there is consequently a real root, which is easily found by trial.

Thus $\dfrac{C}{A} = \tfrac{1}{2}$, $C = \tfrac{1}{2}\sqrt{3}$ (angle of modulus $= 60°$), $b = \tfrac{1}{2}$; or the equation is

$$\tan\mu'\,\sqrt{1 - \tfrac{3}{4}\cos^2\mu'} = E\,(90° - \mu') - \tfrac{1}{4}F\,(90° - \mu').$$

Using Legendre's Table IX., we have

μ'.	$90° - \mu'$.	E.	F.	$E - \frac{1}{4}F$.	$\tan\mu'\sqrt{1 - \frac{3}{4}\cos^2\mu'}$.
0°	90°	1·21105	2·15651	·6719	·0
10	80	1·12248	1·81252	·6693	
20	70	1·02663	1·49441	·6530	
30	60	·91839	1·21253	·6153	·3819
40	50	·79538	·96465	·5542	·6278

so that we see the required value is between $30°$ and $40°$; and a rough interpolation gives the value $\mu' = 37° 40'$. But repeating the calculation with the values $37°$ and $38°$, we have

μ'.	$90° - \mu'$.	E.	F.	$E - \frac{1}{4}F$.	$\tan\mu'\sqrt{1 - \frac{3}{4}\cos^2\mu'}$.
37°	53°	·833879	1·035870	·57419	·54425
38	52	·821197	1·011849	·56823	·57108

whence, interpolating, $\mu' = 37° 55'$.

The semiaxes of the geodesic evolute, measured according to their longitude and parametric latitude respectively, are thus $B\alpha$, long. of $\alpha = 45°$; $B\gamma$, param. lat. $= 37° 55'$. But measuring them according to their geodesic distance, the equatorial radius A being taken $= 1$, we have

$$B\alpha = \tfrac{1}{4}\pi \qquad\qquad = \cdot 78540,$$

$$B\gamma = \left(\frac{C}{b} - 1\right)\{E_{,} - E(52° 5')\} = 1\cdot 21106 - \cdot 82225 = \cdot 38881.$$

Reverting to the general formulæ for s, Λ, but writing therein $A = 1$, and therefore $C = \sqrt{1-\delta^2}$; writing also $\phi = 90°$ (that is, making the formulæ to refer to the node N of the geodesic line), we have

$$s = \frac{\sqrt{1-\delta^2}}{\sqrt{1-\delta^2 \sin^2 l}} E_{,} c,$$

$$\Lambda = \frac{\sqrt{1-\delta^2}}{n \cos l} \{(n + c^2)\, \Pi_{,}(n,\ c) - c^2 F_{,} c\};$$

but for the calculation of the second of these formulæ by means of Legendre's Tables it is necessary to express $\Pi_{,}(n,\ c)$ in terms of the functions E, F.

The proper formula is given in *Fonct. Ellipt.* vol. I. p. 137; viz. this is

$$\frac{\Delta(b,\ \theta)}{\sin\theta\cos\theta}\Pi_{,}(n,\ c) = \tfrac{1}{2}\pi + \frac{\sin\theta}{\cos\theta}\Delta(b,\ \theta) F_{,} c + F_{,} c F(b,\ \theta) - F_{,} c E(b,\ \theta) - E_{,} c F(b,\ \theta),$$

where $\Delta(b,\ \theta) = \sqrt{1 - b^2 \sin^2\theta}$. θ is an angle given by the equation $\cot\theta = \sqrt{n}$; we have $n = \tan^2 l'$; consequently $\theta = 90° - l'$. Substituting this value, except that for shortness I retain $E(b,\ \theta)$, $F(b,\ \theta)$ in place of $E(b,\ 90° - l')$, $F(b,\ 90° - l')$, we have

$$\Delta(b,\ \theta) = \sqrt{1 - b^2 \cos^2 l'},$$

$$= \sqrt{1 - (1 - \delta^2 \sin^2 l)\cos^2 l'}\ , = \sin l;$$

and thence

$$\tan\theta\Delta(b,\ \theta) = \cot l \sin l = \frac{\cos l}{\sqrt{1-\delta^2}};$$

whence

$$\Pi_{,}(n,\ c) = \frac{\sin l' \cos l'}{\sin l}\left\{\tfrac{1}{2}\pi + F_{,} c\left[\frac{\cos l}{\sqrt{1-\delta^2}} + F(b,\ \theta) - E(b,\ \theta)\right] - E_{,} c F(b,\ \theta)\right\}.$$

But

$$n + c^2 = \tan^2 l' + \delta^2 \sin^2 l = \sin^2 l \sec^2 l.$$

Hence

$$(n + c^2)\, \Pi_{,}(n,\ c) - c^2 F_{,} c = \sin^2 l\, [\sec^2 l' \Pi_{,}(n,\ c) - \delta^2 F_{,} c];$$

and multiplying this by

$$\frac{\sqrt{1-\delta^2}}{n\cos l}, \quad = \frac{\sqrt{1-\delta^2}\cos l}{\tan^2 l' \cos^2 l},$$

the exterior factor is

$$\frac{\sqrt{1-\delta^2}\cos l \tan^2 l}{\tan^2 l'}, \quad = \frac{\cos l}{\sqrt{1-\delta^2}},$$

and we have

$$\Lambda = \frac{\cos l}{\sqrt{1-\delta^2}}\{\sec^2 l'\Pi, (n, c) - \delta^2 F, c\},$$

which is the formula I used in the calculations. It would, however, have been better to reduce a step further; viz. we have

$$\sec^2 l'\Pi, (n, c) = \frac{\tan l'}{\tan l \cos l}\{\ \},$$

$$= \frac{\sqrt{1-\delta^2}}{\cos l}\left\{\tfrac{1}{2}\pi + F, c\left[\frac{\cos l}{\sqrt{1-\delta^2}} + F(b, \theta) - E(b, \theta)\right] - E, c\, F(b, \theta)\right\},$$

$$= \frac{\sqrt{1-\delta^2}}{\cos l}\{\tfrac{1}{2}\pi + F, c[F(b, \theta) - E(b, \theta)] - E, c\, F(b, \theta)\} + F, c,$$

and thence

$$\sec^2 l'\Pi, (n, c) - \delta^2 F, c = \frac{\sqrt{1-\delta^2}}{\cos l}\{\tfrac{1}{2}\pi + F, c[\sqrt{1-\delta^2}\cos l + F(b, \theta) - E(b, \theta)] - E, cF(b, \theta)\};$$

or, finally,

$$\Lambda = \tfrac{1}{2}\pi + F, c\, F(b, \theta) - F, cE(b, \theta) - E, cF(b, \theta) + \sqrt{1-\delta^2}\cos l F, c.$$

It is easy with this expression of Λ to obtain the results already found for the extreme values $l' = 0°$, $l' = 90°$.

As Legendre's Tables have for argument, not the modulus c, but the angle of the modulus, say χ (that is, $\sin\chi = c = \delta \sin l$), it is convenient to replace $\sqrt{1-\delta^2\sin^2 l}$ by its value $\cos\chi$; and the formulæ thus are

$$s = \frac{\sqrt{1-\delta^2}}{\cos\chi} E, c,$$

$$\Lambda = \tfrac{1}{2}\pi + F, c[\sqrt{1-\delta^2}\cos l + F(b, \theta) - E(b, \theta)] - E, c\, F(b, \theta),$$

where

$$C = \sin\chi = \delta \sin l, \quad \tan l' = \sqrt{1-\delta^2}\tan l, \quad \theta = 90° - l';$$

and in the case intended to be numerically discussed, $\delta = \frac{1}{2}\sqrt{3}$, $\sqrt{1-\delta^2} = \frac{1}{2}$. I take l' as the argument, giving it the values $0°, 10°, \ldots 90°$, and perform the calculation as shown in the Table.

l'.	θ.	l.	$\chi = \angle c$.	$90° - \chi = \angle b$.	$\frac{1}{2}\cos l$.	F, c, log.	E, c, log.	F, c, nat.	E, c, nat.	$F(b, \theta)$, nat.	$E(b, \theta)$, nat.
0°											
10	80°	19° 26′	16°·75	73°·25	·47151	20548	18686	1·6050	1·5376	2·08962	1·03762
20	70	36 3	30·63	59·37	·40425	22814	16532	1·6910	1·4633	1·48840	1·02962
30	60	49 6	40·90	49·10	·32737	25478	14167	1·7980	1·3857	1·16024	·95214
40	50	59 13	48·07	41·93	·25589	27626	12163	1·8891	1·3232	·92141	·82827
50	40	67 14	52·98	37·02	·19349	29935	10645	1·9923	1·2777	·71820	·67903
60	30	73 54	56·32	33·68	·13866	31479	09557	2·0644	1·2461	·53083	·51655
70	20	79 41	58·43	31·57	·08954	32540	08850	2·1154	1·2260	·35099	·34716
80	10	84 58	59·62	30·38	·04387	33169	08446	2·1463	1·2147	·17475	·17431
90			60·0				08316				

l'.	$\frac{1}{2}\cos l + F(b, \theta) - E(b, \theta)$, nat.	Do. log.	Add log F, c.	(1) Nat.	Log $F(b, \theta)$.	Add log E, c.	(2) Nat.	$\frac{1}{2}\pi + (1) - (2)$.	* Do. in (°).	Log E, c $-$ log $\cos\chi$.	Nat.	* $\frac{1}{2}$ do., = length.
0°								·7854	45°			·7854
10	1·52308	·18272	·38820	2·4446	·32007	·50693	3·2132	·8022	45 58′	20569	1·6058	·8029
20	·86318	$\bar{1}$·93610	·16424	1·4596	·17272	·33804	2·1779	·8525	48 51	23075	1·7012	·8506
30	·53547	$\bar{1}$·72874	$\bar{1}$·98352	·9627	·06455	·20622	1·6078	·9257	53 2	26323	1·8333	·9166
40	·34903	$\bar{1}$·54286	$\bar{1}$·81912	·6594	$\bar{1}$·96445	·08608	1·2192	1·0110	57 56	29668	1·9801	·9900
50	·23266	$\bar{1}$·36672	$\bar{1}$·66607	·4635	$\bar{1}$·85625	$\bar{1}$·96270	·9177	1·1166	63 59	32682	2·1224	1·0612
60	·15292	$\bar{1}$·18446	$\bar{1}$·49925	·3157	$\bar{1}$·72496	$\bar{1}$·82053	·6615	1·2250	70 11	35178	2·2479	1·1239
70	·09327	$\bar{2}$·96974	$\bar{1}$·29514	·1973	$\bar{1}$·54529	$\bar{1}$·63379	·4303	1·3378	76 39	36939	2·3410	1·1705
80	·04431	$\bar{2}$·64650	$\bar{2}$·97819	·0951	$\bar{1}$·24242	$\bar{1}$·32688	·2123	1·4536	83 17	38454	2·4240	1·2120
90								1·5708	90			1·2111

χ, $90° - \chi$ in degrees and decimals of a degree, to correspond with Legendre's Tables.

where the columns marked with an * show respectively the longitude of the node, and the length (or distance of node from vertex), for the geodesic lines belonging to the different values of the argument l'.

The remarks which follow have reference to the stereographic projection of the figure on the plane of the equator, the centre of projection being the pole (say the South Pole) of the spheroid. It is to be remarked that if a point P of the spheroid is projected as above, by means of an ordinate into the point Q of the sphere radius $OK (= OA)$, then projecting stereographically as to the spheroid and the sphere from the south poles thereof respectively, the points P and Q have the same projection. And it is hence easy to show that an azimuth α at a point of the meridian $\left(\text{parametric latitude } \lambda', \text{ normal latitude } \lambda, \text{ and therefore } \tan\lambda' = \frac{C}{A}\tan\lambda\right)$ is projected into an angle (α) such that

$$\tan(\alpha) = \frac{\sin\lambda'}{\sin\lambda}\tan\alpha.$$

In fact in fig. 3, if we take therein OK, OC for the axes of x, z respectively, and the axis of y at right angles to the plane of the paper, and if we have at P on the surface of the spheroid an element of length PR at the inclination α to the meridian PK, then if x, y, z are the coordinates of P, and $x+\delta x$, $y+\delta y$, $z+\delta z$ those of R, we have

$$\begin{aligned} \delta x &= \rho\cos\alpha\sin\lambda, \\ \delta z &= -\rho\cos\alpha\cos\lambda, \\ \delta y &= \rho\sin\alpha, \end{aligned}$$

and thence

$$\tan\alpha = \frac{\delta y}{\sqrt{\delta x^2 + \delta z^2}}.$$

Now, if the meridian and the points P, R are referred by lines parallel to Oz to the surface of the sphere radius OA, the only difference is that the ordinates z are increased in the ratio $C : A$; so that if the projected angle be (α), we have

$$\tan(\alpha) = \frac{\delta y}{\sqrt{\delta x^2 + \frac{A^2}{C^2}\delta z^2}}:$$

and then projecting the sphere stereographically from its south pole, the angle in the projection is $=(\alpha)$. And according to the foregoing remark, the angle (α) thus obtained is also the projection of α from the south pole of the spheroid. We have thus

$$\frac{\tan(\alpha)}{\tan\alpha} = \frac{\sqrt{\delta x^2 + \delta z^2}}{\sqrt{\delta x^2 + \frac{A^2}{C^2}\delta z^2}}, \quad = \frac{\sqrt{\sin^2\lambda + \cos^2\lambda}}{\sqrt{\sin^2\lambda + \frac{A^2}{C^2}\cos^2\lambda}}, \quad = \sqrt{\frac{1+\cot^2\lambda}{1+\cot^2\lambda'}}, \quad = \frac{\sin\lambda'}{\sin\lambda},$$

which is the required relation.

The foregoing equations,

$$\cos\lambda'\sin\alpha = \cos l', \quad \tan\lambda' = \frac{C}{A}\tan\lambda,$$

$$\tan(\alpha) = \frac{\sin\lambda'}{\sin\lambda}\tan\alpha,$$

determine in the stereographic projection the inclination (α) to the radius, or projection of the meridian, of the geodesic line (parametric latitude of vertex $=l'$) at the point the parametric latitude of which is $=\lambda'$; viz. they enable the construction (in the projection) of the direction of the successive elements of the geodesic line. There would be no difficulty in performing the construction geometrically; but it would, I think, be more convenient to calculate (α) numerically for a given value of l' and for the successive values of λ'. Observe that for $\lambda'=0$ we have (as above) $90°-\alpha=l'$, and then $\frac{\sin\lambda'}{\sin\lambda}=\frac{\tan\lambda'}{\tan\lambda}=\frac{C}{A}$, consequently $\tan(\alpha)=\frac{C}{A}\cot l'$: but we have also $\cot l'=\frac{A}{C}\cot l$, so that this equation becomes $\tan(\alpha)=\cot l$, or we have $90°-(\alpha)=l$; viz. in the projection, the geodesic line cuts the equator at an angle $l=$ the normal latitude of the vertex of the geodesic line.

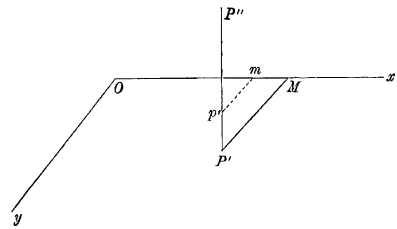
The preceding formulæ and results have enabled me to construct a drawing, on a large scale, of the stereographic projection of the geodesic lines for the spheroid, polar axis $=\frac{1}{2}$ equatorial axis.

423.

ON THE PLANE REPRESENTATION OF A SOLID FIGURE.

[From the *Philosophical Magazine*, vol. XLI. (1871), pp. 286—290.]

WE represent *in plano* the position of a point P whose coordinates in space are (x, y, z) by drawing these coordinates, on the same scale or on different scales, and in given directions from a fixed origin in the plane; $OM = x$, $MP' = y$, $P'P'' = z$. But observe that the point P'' *alone* does not completely represent the point P; in fact P'' represents a whole series of points lying in a line; any one such point is the

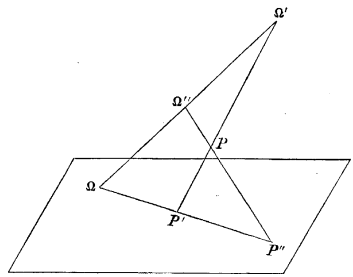


point whose coordinates are Om, mp', $p'P''$. For the complete representation of P we require the *two points* P', P'': these might be distinguished as the projection P'', and the foot-point P'. The two points P', P'' are obviously such that the line joining them is in a given direction.

The preceding is, of course, the ordinary method of orthogonal projection, or geometrical delineation of a solid figure: it may be used under various forms; for example, the coordinates x, y, z may be taken on the same scale and in directions inclined to each other at angles of 120° (isometrical projection); or the coordinates x, y may be drawn on the same scale and at their actual inclination, 90°, to each other;

and the coordinate z on the same or an altered scale in any given direction; the points P' then give a true ground-plan of the solid figure, and the lengths of the lines $P'P''$ give the altitudes of the several points P: this is also a method in ordinary use.

But it is to be observed that the points P', P'' are both of them *projections*, and that the general theory is as follows: we represent the position of the point P



by means of its projections P', P'', from two fixed points Ω', Ω'' respectively; the line joining these points passes, it is clear, through a fixed point Ω which is the intersection of the plane of projection by the line which joins the two points Ω', Ω''.

Hence we say that a point P in space is represented *in plano* by any two points P', P'' which are such that the line joining them passes through a fixed point Ω. And we have thus a *system of constructive geometry* which is the more simple on account of the generality of its basis, and which is at once applicable to any of the special projections above referred to. I establish the fundamental notions of such a geometry, and by way of illustration apply it to the solution of the well-known problem of finding the lines which meet four given lines in space.

A point P (as already mentioned) is given by its projections P', P'', which are points such that the line joining them passes through the fixed point Ω.

A line L is given by its projections L', L'', which are any two lines in the plane. We speak of the point (P', P''), meaning the point P whose projections are P', P''; and similarly of the line (L', L''), meaning the line whose projections are L', L''.

If P', P'' coincide, then the point P is in the plane of projection; and so if L', L'' coincide, then the line L is in the plane of projection.

If through Ω we draw a line meeting L', L'' in the points P', P'' respectively, these are the projections of a point P on the line L. In particular the intersection of L', L'' (considered as two coincident points) represents the intersection of the line L with the plane of projection.

The line through the points (P', P'') and (Q', Q'') has for its projections the lines $P'Q'$ and $P''Q''$.

Two lines (L', L'') and (M', M'') intersect each other if only the intersections $L'M'$ and $L''M''$ are the projections of a point P—that is, if the line through the points $L'M'$ and $L''M''$ passes through Ω. And then clearly P is the intersection of the two lines.

A plane Π is conveniently given by means of its trace Θ on the plane of projection, and of the projections (P', P'') of a point on the plane; or, say, by means of the trace Θ, and of a point P on the plane.

Suppose, however, that a plane is given by means of a line L and a point P on the plane. The trace Θ passes through the point of intersection of the line L with the plane of projection—that is, through the point of intersection of the projections L', L''. To find another point on the trace, we have only to imagine on the line L a point Q, and, joining this with P, to suppose the line PQ produced to meet the plane of projection. The construction is obvious; but by way of illustration I give it in full. Through Ω draw a line meeting L', L'' in Q', Q'' respectively (then these are the projections of a point Q on the line L); the lines $P'Q'$ and $P''Q''$ are the projections of the line PQ, and the intersection of $P'Q'$ and $P''Q''$ is therefore the required point on the trace Θ.

The line of intersection of two planes passes through the point of intersection of their traces Θ_1, Θ_2; whence, if the planes have in common a point P, the line of intersection is the line joining P with the intersection of the traces Θ_1, Θ_2.

In what precedes we have the solution of the following problem:—"Given a point P, and two lines L_1, L_2, to find a line through P meeting the two lines L_1, L_2." The required line is in fact the line of intersection of the planes (P, L_1) and (P, L_2); we have seen how to construct the traces Θ_1 and Θ_2 of these planes respectively; and the required line is the line joining P with the intersection of Θ_1 and Θ_2.

I proceed now to the problem to find the two lines, each of them meeting four given lines, L_1, L_2, L_3, L_4 (these being, of course, given by means of their projections (L_1', L_1'') &c.). The question is in effect to find on the line L_1 a point P such that, drawing from it a line to meet L_2, L_3, and also a line to meet L_2, L_4, these shall be one and the same line.

Now, considering in the first instance P as an arbitrary point on the line L_1, the line from P to meet L_2, L_3 is any line whatever meeting the lines L_1, L_2, L_3: say it is a generating line of the hyperboloid whose directrices are L_1, L_2, L_3, or of the hyperboloid $L_1L_2L_3$. Hence projecting from any point Ω' whatever, the generating lines and directrices are projected into tangents of one and the same conic. We know the projections L_1', L_2', L_3' of the directrices; to find two other tangents of the conic, we take two arbitrary positions of P on the line L_1, and construct as above the projections M', N' of the lines from these to meet the lines L_2, L_3. The conic is then

given as the conic touching the five lines L_1', L_2', L_3', M', N': say this is the conic Σ'. Similarly, instead of Ω', considering the point Ω'', we have the lines L_1'', L_2'', L_3'' and the lines M'', N'', which are the other projections of the lines through the two positions of P; and touching these five lines we have a conic Σ''. Each tangent T' of Σ', combined with the *corresponding* tangent T'' of Σ'', represents a line T meeting L_1, L_2, L_3; to establish the correspondence, observe that, inasmuch as the line T meets L_1, the intersections of T', L_1' and of T'', L_1'' must lie in a line with Ω; if T' be given, the point (T'', L_1'') is thus uniquely determined, and therefore also T'' (since L_1'' is a tangent of Σ''); and similarly if T'' be given, T' is uniquely determined; the correspondence T', T'' is thus, as it should be, a (1, 1) correspondence.

Considering in like manner the lines which meet L_1, L_2, L_4, we have touching L_1', L_2', L_4', $\overline{M}'$, $\overline{N}'$ a conic $\overline{\Sigma}'$; and touching L_1'', L_2'', L_4'', $\overline{M}''$, $\overline{N}''$ a conic $\overline{\Sigma}''$; each tangent T' of $\overline{\Sigma}'$, combined with the corresponding tangent $\overline{T}''$ of $\overline{\Sigma}''$, represents a line meeting L_1, L_2, L_4, the correspondence being a (1, 1) correspondence such as in the former case.

The conics Σ', $\overline{\Sigma}'$ both touch L_1', L_2'; hence they have in common two tangents. Say one of these is $T' = \overline{T}'$, the corresponding tangents T'' and $\overline{T}''$ will coincide with each other and be a common tangent of Σ'', $\overline{\Sigma}''$ (these conics both touch L_1'', L_2'', and have thus in common two tangents). We have thus $T' = \overline{T}'$, and $T'' = \overline{T}''$, as the projections of a line meeting L_1, L_2, L_3, L_4; and taking the other common tangents of Σ', $\overline{\Sigma}'$ and of Σ'', $\overline{\Sigma}''$, we have the projections of the other line meeting L_1, L_2, L_3, L_4.

The whole process is:—Construct M', M'' and N', N'' each of them the projections of a line through a point P of L_1, which meets L_2, L_3; and $\overline{M}'$, $\overline{M}''$ and $\overline{N}'$, $\overline{N}''$ each of them the projections of a line through a point P of L_1, which meets L_2, L_4; we have then the conics

Σ', Σ'' touching L_1', L_2', L_3', M', N', and L_1'', L_2'', L_3'', M'', N'' respectively,

$\overline{\Sigma}'$, $\overline{\Sigma}''$ „ L_1', L_2', L_4', $\overline{M}'$, $\overline{N}'$, „ L_1'', L_2'', L_4'', $\overline{M}''$, $\overline{N}''$ „ ;

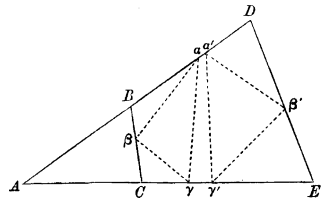
and then the projections of each of the required lines are $T' = \overline{T}'$, a common tangent of Σ', $\overline{\Sigma}'$, and $T'' = \overline{T}''$, the corresponding common tangent of Σ'', $\overline{\Sigma}''$.

It is material to remark how the construction is simplified when there is given one of the lines, say, M, which meets L_1, L_2, L_3, L_4. Here M is a common directrix of the two hyperboloids; we may for the hyperbolas Σ' and Σ'' consider, instead of L_1, L_2, L_3 and two new generating lines, the lines L_1, L_2, L_3, M, and a single new generating line N; and similarly for the hyperbolas $\overline{\Sigma}'$, $\overline{\Sigma}''$ the lines L_1, L_2, L_4, M and a single new generating line $\overline{N}$. Σ', $\overline{\Sigma}'$ have thus in common the three tangents L_1', L_2', M', and therefore only a single other common tangent, $T' = \overline{T}'$; and similarly Σ'', $\overline{\Sigma}''$ have in common the three tangents L_1'', L_2'', M'', and therefore only a single other common tangent, $T'' = \overline{T}''$; and we have thus the other line cutting the four given lines.

I take the opportunity of mentioning the following theorem:

"If in a given triangle we inscribe a variable triangle of given form, the envelope of each side of the variable triangle is a conic touching the two sides (of the given triangle) which contain the extremities of the variable side in question."

We have thence a solution of the problem (*Principia*, Book I. Sect. V. Lemma XXVII.), in a given quadrilateral to inscribe a quadrangle of given form. The question in effect is: in the triangle ABC to inscribe a triangle $\alpha\beta\gamma$ of given form; and in the triangle ADE a triangle $\alpha'\beta'\gamma'$ of given form, in such wise that the sides $\alpha\gamma$, $\alpha'\gamma'$



may be coincident. The envelope of $\alpha\gamma$ is a conic touching AD, AE, and the envelope of $\alpha'\gamma'$ a conic also touching AD, AE: there are thus two other common tangents, either of which may be taken for the position of the side $\alpha\gamma = \alpha'\gamma'$; and the problem admits accordingly of two solutions.

424.

ON THE ATTRACTION OF A TERMINATED STRAIGHT LINE.

[From the *Philosophical Magazine*, vol. XLI. (1871), pp. 358—360.]

WRITE for shortness $(a, b, c; \epsilon)$ to denote the shell included between the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(1+\epsilon)^2$$

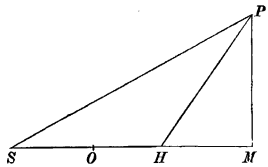
(where ϵ is indefinitely small); then, if the ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1$$

are confocal, the attractions of the shells $(a, b, c; \epsilon)$ and $(a', b', c'; \epsilon)$ upon any exterior point P are proportional to their masses. Hence, considering a prolate spheroid of revolution, $c=b$, the attractions of the shell $(a, b, b; \epsilon)$ will be proportional to those of the shell $(\sqrt{a^2-h}, \sqrt{b^2-h}, \sqrt{b^2-h}; \epsilon)$; or if, as usual, $b^2=a^2(1-e^2)$, then, if h increases and becomes ultimately equal to b^2, to those of the shell $(ae, 0, 0; \epsilon)$; viz. this last is the portion of the axis of x included between the limits $x=-ae$, $x=+ae$; or say it is the terminated line $x=\pm ae$; and I say that the mass is distributed over this line *uniformly*.

To see that this is so, observe in general that, in the spheroid $\frac{x^2}{a'^2}+\frac{y^2+z^2}{b'^2}=1$, the volume included between the planes $x=\alpha$, $x=\alpha+d\alpha$, is $=(y^2+z^2)\,d\alpha$, $=\pi\left(b'^2-\frac{b'^2}{a'^2}\alpha^2\right)d\alpha$; and thence, writing $a'(1+\epsilon)$, $b'(1+\epsilon)$ for a', b', in the shell $(a', b', b'; \epsilon)$ the volume included between the planes $x=\alpha$, $x=\alpha+d\alpha$ is $=\pi b'^2 . 2\epsilon' d\alpha$; viz. this is independent of α, and simply proportional to $d\alpha$. Hence, writing $b'=0$, when the shell shrinks up into a line, the mass must be disturbed uniformly over the line. It follows that for a line of uniform density the equipotential surfaces are each of them a prolate

spheroid of revolution having the extremities of the line for its foci, and that, if we have a shell bounded by any such surface and the consecutive *similar* surface, with its mass equal to that of the line, then such shell and the line will exert the same



attractions upon any point P exterior to the shell. The attractions of the line are obtained most easily by means of its potential; viz. taking S, H for the extremities of the line, and, as above, the origin at the middle point, and the axis of x in the direction of the line, and writing $2ae$ for the length of the line, x, y, z for the coordinates of P, and r, s for the values of HP, SP (that is, $r = \sqrt{(x-ae)^2+y^2+z^2}$, $s = \sqrt{(x+ae)^2+y^2+z^2}$), then the potential is at once found to be

$$V = \log \frac{x+ae+s}{x-ae+r};$$

and we can hereby verify that the equipotential surface is in fact a spheroid of revolution having the foci S, H; for, taking the equation of such a spheroid to be

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{a^2(1-e^2)} = 1,$$

(a is an arbitrary parameter, since only the value of ae has been defined), we have

$$s = a + ex, \quad r = a - ex$$

and thence

$$x + ae + s = (1+e)(x+a),$$
$$x - ae + r = (1-e)(x+a),$$

and the quotient is $= \dfrac{1+e}{1-e}$, a constant value, as it should be. The equation $V = \text{const.}$ may in fact be written

$$\frac{1+e}{1-e} = \frac{x+ae+s}{x-ae+r};$$

viz. this equation, apparently of the fourth order, breaks up into the twofold plane $y^2 = 0$, and the spheroid $\dfrac{x^2}{a^2} + \dfrac{y^2+z^2}{a^2(1-e^2)} = 1$.

The foregoing results in regard to the attraction of a line are not new. See Green's *Essay on Electricity*, 1828, and *Collected Works*, Cambridge, 1871, p. 68; also

Joachimsthal, "On the Attraction of a Straight Line," with Sir W. Thomson's Note, *Camb. and Dubl. Math. Journ.*, vol. III. (1848), p. 93; but it does not appear to have been noticed that they are, in fact, included in the theory of the attraction of ellipsoids.

The like considerations show that the attractions of the ellipsoidal shell $(a, b, c; \epsilon)$ upon an exterior point are equal to those of an elliptic disk $z=0$, $\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1$, the mass of which is equal to that of the shell, and which has the density at the point (x, y) proportional to $\left(1-\frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}\right)^{-\frac{1}{2}}$.

Sir W. Thomson informs me that the foregoing results have long been familiar to him.

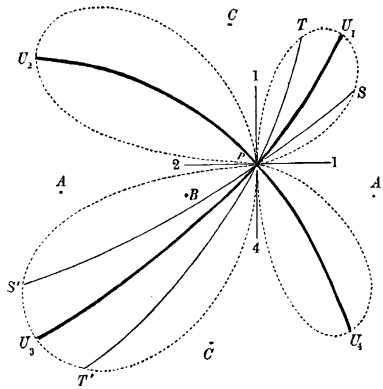
425.

NOTE ON THE GEODESIC LINES ON AN ELLIPSOID.

[From the *Philosophical Magazine*, vol. XLI. (1871), pp. 534, 535.]

THE general configuration of the geodesic lines on an ellipsoid is established by means of the known theorem (an immediate consequence of Jacobi's fundamental formulæ, but which was first given by Mr Michael Roberts, *Comptes Rendus*, vol. XXI. p. 1470, Dec. 1845) that every geodesic line touches a curve of curvature; that is, attending to the two opposite ovals which constitute the curve of curvature, the geodesic line is in general an infinite curve undulating between these opposite ovals, and so touching each of them an infinite number of times (but possibly in particular cases it is a reentrant curve touching each oval a finite number of times). The geodesic lines thus divide themselves into two kinds, accordingly as they touch a curve of curvature of the one or the other kind; and there is besides a third limiting kind, the lines which pass through an umbilicus: any such geodesic line passes through the opposite umbilicus, and is in general an infinite curve passing an infinite number of times alternately through the two umbilici; but possibly it is in particular cases a reentrant curve passing a finite number of times through the two umbilici. I annex a figure giving a general idea of the configuration of the geodesic lines drawn in different directions from a given point P on the surface of the ellipsoid: this is drawn (as it were) on the plane of the greatest and least axes; but it is not a perspective or geometrical representation of any kind, but a mere diagram for the purpose in question. We have A, A, B, C, C the extremities of the axes; U_1, U_2, U_3, U_4 the umbilici; P the point on the surface; $1P2$ and $1P4$ the curves of curvature through P, viz. these are ovals

containing the umbilici U_1, U_2 and U_1, U_4 respectively. Then U_1PU_3 and U_2PU_4 are the limiting geodesics passing through the umbilici; the line TPT' represents a



geodesic line of the one kind, viz. this at T touches an oval (curve of curvature) U_1U_4, and at T' the conjugate oval U_2U_3. Similarly SPS' is a geodesic line of the other kind, viz. this at S touches an oval (curve of curvature) U_1U_2, and at S' the conjugate oval U_3U_4; the dotted figure-of-eight curves are the loci of the points of contact T, T', S, S'.

426.

ON A SUPPOSED NEW INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

[From the *Philosophical Magazine*, vol. XLII. (1871), pp. 197—199.]

THIS refers to a paper, Challis, "On the Application of a new Integration of Differential Equations of the Second Order to some unsolved Problems in the Calculus of Variations," *Phil. Mag.* same volume, pp. 28—40.

427.

ON GAUSS'S PENTAGRAMMA MIRIFICUM.

[From the *Philosophical Magazine*, vol. XLII. (1871), pp. 311, 312.]

TAKE on a sphere (in the northern hemisphere) two points, A, B, whose longitudes differ by 90°, and refer them to the equator by the meridians AE and BC respectively; join A, B by an arc of great circle, and take in the southern hemisphere the pole D of this circle; and join D with E and C respectively by arcs of great circle. We have a spherical pentagon $ABCDE$, which is in fact the "Pentagramma mirificum," considered by Gauss, as appearing vol. III. pp. 481—490 of the *Collected Works*. Among its properties we have

$$\left.\begin{array}{l}\text{the distance of any two non-adjacent summits}\\ \text{the inclination of any two non-adjacent sides}\end{array}\right\} = 90^\circ;$$

so that each summit is the pole of the opposite side, or the pentagon is its own reciprocal.

Each angle is the supplement of the opposite side.

If the squared tangents of the sides (or angles) taken in order are α, β, γ, δ, ϵ, then

$$1+\alpha=\gamma\delta,\quad 1+\beta=\delta\epsilon,\quad 1+\gamma=\epsilon\alpha,\quad 1+\delta=\alpha\beta,\quad 1+\epsilon=\beta\gamma,$$

equivalent to three independent equations, so that any three of the quantities may be expressed in terms of the remaining two. (This agrees with the foregoing construction, where the arbitrary quantities are the latitudes of A, B respectively.)

Projecting from the centre of the sphere upon any plane, we have a plane pentagon which is such that the perpendiculars let fall from the summits upon the opposite sides respectively meet in a point. This (as easily seen) implies that the two portions into which each perpendicular is divided by the point in question have the same product.

Conversely, starting from the plane pentagon, and erecting from the point of intersection a perpendicular to the plane, the length of this perpendicular being equal to the square root of the product in question, we have the centre of a sphere such that the projection upon it of the plane polygon is the pentagramma mirificum.

I remark as to the analytical theory, that, taking the origin at the intersection of the perpendiculars, and for the coordinates of the summits $(\alpha_1, \beta_1), \ldots (\alpha_5, \beta_5)$ respectively, then we have

$$\alpha_1\alpha_4 + \beta_1\beta_4 = \alpha_5\alpha_5 + \beta_2\beta_5 = \alpha_3\alpha_1 + \beta_3\beta_1 = \alpha_4\alpha_2 + \beta_4\beta_2 = \alpha_5\alpha_3 + \beta_5\beta_3, \quad = -\gamma^2,$$

where γ^2 is the above-mentioned product, or γ is the radius of the sphere.

Cambridge, September 14, 1871.

428.

NOTE SUR LA CORRESPONDANCE DE DEUX POINTS SUR UNE COURBE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXII. (*Janvier—Juin*, 1866), pp. 586—590.]

THIS is to the same effect with the paper 385, "On the Correspondence of two Points on a Curve"; four examples of the theory are given, the first, second, and third of them the same as in this paper—the fourth example is as follows:

4°. *Recherche du nombre des points* sextactiques, c'est-à-dire des points qui sont tels que par chacun passe une conique qui a dans ce point un contact du cinquième ordre avec la courbe. Il faut prendre pour les points P les intersections avec la courbe de la conique qui a au point P' un contact du quatrième ordre: les points unis seront ceux dont il s'agit. La courbe $\Theta = 0$ est la conique qui a au point P' un contact du quatrième ordre. On a ainsi parmi les intersections le point P' 5 fois; donc $k = 5$. A chaque point P' correspondent $2m - 5$ points P; à chaque point P $10m^2 - 20m - 5 - 20\delta$ points P' (j'emprunte le terme -20δ d'une formule que vient de donner M. Zeuthen); donc la formule donne pour le nombre des points unis

$$10\,m^2 - 18\,m - 10 - 20\,\delta + 10\,D,$$

c'est-à-dire

$$15\,m^2 - 33\,m - 30\,\delta.$$

Mais cette expression comprend le nombre $3m\,(m-2) - 6\,\delta$ des inflexions; en effet pour un point d'inflexion la conique avec contact du quatrième ordre se réduit à la tangente prise deux fois, ce qui est une conique avec contact du cinquième ordre. Donc enfin le nombre des points sextactiques sera

$$m\,(12\,m - 27) - 24\,\delta,$$

ou, pour une courbe sans points doubles

$$m\,(12\,m - 27),$$

ce qui s'accorde avec la valeur que j'ai trouvée par d'autres moyens.

429.

SUR LES CONIQUES DÉTERMINÉES PAR CINQ CONDITIONS DE CONTACT AVEC UNE COURBE DONNÉE.

[From the *Comptes Rendus de l'Académie des Sciences à Paris*, tom. LXIII. (*Juillet—Décembre*, 1866), pp. 9—12.]

This paper (dated Cambridge, 26 June 1866), contains the expressions for the numbers (5), (4, 1), (3, 2) (3, 1, 1), (2, 2, 1), (2, 1, 1, 1) and (1, 1, 1, 1, 1), of the conics which satisfy five conditions of contact with a given curve, as obtained in the paper 406 "On the Curves which satisfy given conditions," see p. 214, and which expressions were found by the same process, viz. by consideration of functional equations obtained by supposing the given curve to break up into two curves of the orders m and m' respectively; there was in the expression for (1, 1, 1, 1, 1) a numerical error as mentioned in the footnote of the same page. The paper contains also the formula $\mu'' - \frac{3}{2}\nu'' + \frac{8}{3}\rho'' - \sigma'' = 0$, and the expression for $(2X, 3Z)$ given, pp. 203, 204.

430.

NOTE SUR QUELQUES FORMULES DE M. E. DE JONQUIÈRES, RELATIVES AUX COURBES QUI SATISFONT À DES CONDITIONS DONNÉES.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXIII. (*Juillet—Décembre*, 1866), pp. 666—670.]

LES formules dont il s'agit sont publiées dans les *Comptes Rendus*, séances du 3 et du 17 septembre 1866. En faisant une simple transformation algébrique pour y introduire la classe $M(=m^2-m)$ de la courbe donnée U^m, et en changeant un peu la forme, les théorèmes de M. de Jonquières peuvent s'énoncer comme il suit:

1°. Le nombre des contacts des courbes C^r qui ont un contact de l'ordre n avec une courbe fixe U^m, et qui passent en outre par $\frac{1}{2}r(r+3)-n$ points donnés, est

$$=\tfrac{1}{2}(n+1)[nM+(2r-2n)m].$$

OBSERVATION. Énoncé de cette manière, le théorème s'applique même au cas $n=0$. En effet, pour $n=0$, le nombre donné par le théorème est $=mr$, qui est le nombre des contacts de l'ordre 0 (intersections simples) de la courbe donnée U^m avec une courbe déterminée de l'ordre r.

2°. Le nombre des contacts de l'ordre n' (= ou $<n$) des courbes C^r qui ont deux contacts des ordres n et n' respectivement avec une courbe fixe U^m, et qui passent en outre par $\frac{1}{2}r(r+3)-n-n'$ points donnés est

$$\begin{aligned}=\tfrac{1}{4}(n+1)(n'+1)\{&[nM+(2r-2n)m][n'M+(2r-2n')m]\\&-2(n^2+nn'+n'^2+n+n')M\\&+\ [-4r(n+n'+1)+4(n^2+nn'+n'^2+n+n')]m\}.\end{aligned}$$

OBSERVATION. Énoncé de cette manière, le théorème s'applique même aux cas $n' = 0$, et $n' = n$. En effet, pour $n' = 0$, le nombre donné par le théorème est $= (rm - n - 1) . \frac{1}{2}(n+1)[nM + (2r - 2n)m]$, ce qui est égal au nombre des courbes C^r qui ont avec la courbe donnée U^m un contact de l'ordre n, multiplié par $rm - n - 1$, nombre des contacts de l'ordre 0 (intersections simples) de chacune de ces courbes avec la courbe U^m. Et pour $n' = n$, le nombre des contacts est le double du nombre des courbes C^r.

Je remarque que les deux théorèmes peuvent se démontrer de la manière dont je me suis servi en cherchant le nombre des coniques qui satisfont à cinq conditions données; car, en remplaçant la courbe m par l'ensemble de deux courbes m et m', on trouve que pour le théorème 1° le nombre cherché est

$$= \alpha M + \beta m,$$

où les coefficients (α, β) ne dépendent que de (r, n); et puis, en supposant que ce théorème soit connu, on trouve que pour le théorème 2° le nombre cherché est

$$= \tfrac{1}{4}(n+1)(n'+1)[nM + (2r - 2n)m][n'M + (2r - 2n')m] + \alpha M + \beta m,$$

où de même les coefficients (α, β) ne dépendent que de (r, n).

Or voici comment on peut déterminer les coefficients dans les deux théorèmes:

Pour le théorème 1°, on démontre que pour U^m une droite, le nombre cherché est $= (n+1)(r-n)$; et que pour U^m une conique, le nombre cherché se déduit de là en écrivant $2r$ au lieu de r; c'est-à-dire, que pour la conique, le nombre est $= (n+1)(2r-n)$. On a donc

$$\beta = (n+1)(r-n), \quad = \tfrac{1}{2}(n+1)(2r-2n),$$

$$2\alpha + 2\beta = (n+1)(2r-n),$$

et de là

$$\alpha = \tfrac{1}{2}(n+1)n;$$

ce qui achève la démonstration.

Pour le théorème 2°, on démontre que pour U^m une droite, le nombre cherché est $= (n+1)(n'+1)(r-n-n')(r-n-n'-1)$, et que pour U^m une conique, le nombre cherché se déduit de là en écrivant $2r$ au lieu de r; c'est-à-dire, pour la conique, le nombre est

$$= (n+1)(n'+1)(2r-n-n')(2r-n-n'-1).$$

On a donc

$$(n+1)(n'+1)(r-n-n')(r-n-n'-1) = (n+1)(n'+1)(r-n)(r-n') + \beta,$$

$$(n+1)(n'+1)(2r-n-n')(2r-n-n'-1) = (n+1)(n'+1)(2r-n)(2r-n') + 2\alpha + 2\beta;$$

cela donne pour α et β les valeurs

$$\alpha = \tfrac{1}{4}(n+1)(n'+1)[-2(n^2 + nn' + n'^2 + n + n')],$$

$$\beta = \tfrac{1}{4}(n+1)(n'+1)[-4r(n+n'+1) + 4(n^2 + nn' + n'^2 + n + n')];$$

et la démonstration est ainsi achevée.

Je remarque que sous les formes ici données les deux théorèmes s'appliquent à une courbe U^m avec des points doubles, mais sans point de rebroussement.

Le théorème dont je me suis servi pour la détermination des coefficients peut s'énoncer sous la forme plus générale que voici, savoir:

En dénotant par $\phi(r, n, n', \ldots)$ le nombre des courbes C^r qui ont avec une droite donnée des contacts des ordres $n, n', \ldots$, et qui passent en outre par $\frac{1}{2}r(r+3)-n-n'\ldots$ points donnés, alors si, au lieu de la droite donnée, on a une conique donnée, le nombre des courbes C^r sera $=\phi(2r, n, n', \ldots)$.

En effet, l'équation de la courbe cherchée C^r contient des coefficients indéterminés, lesquels, par les conditions de passer par les points donnés, se réduisent linéairement à $n+n'\ldots+1$ coefficients; en dénotant par $(A, B, \ldots)$ ces coefficients, l'équation de la courbe contiendra linéairement $(A, B, \ldots)$ et sera ainsi de la forme $(A, B, \ldots \between x, y, z)^r=0$. L'équation de la droite donnée est satisfaite en prenant pour (x, y, z) des fonctions linéaires déterminées d'un paramètre variable θ; donc, en coupant la courbe C^r par la droite donnée, on obtient une équation $(A, B, \ldots \between \theta, 1)^r=0$, et en exprimant que cette équation ait n racines égales, n' racines égales, etc., on obtient entre $(A, B, C, \ldots)$ des équations, lesquelles, en éliminant tous les coefficients, excepté deux quelconques (A, B), conduisent à une équation finale $(A, B)^p=0$, et le degré p de cette équation est ce qu'il s'agissait de trouver, le nombre des courbes C^r. Si au lieu d'une droite donnée on a une conique donnée, il n'y a rien à changer, sinon que les coordonnées (x, y, z) doivent être remplacées par des fonctions quadratiques de θ; on a ainsi une équation $(A, B, \ldots \between \theta, 1)^{2r}=0$, qui conduit à une équation finale $(A, B)^{p'}=0$, où p' est la même fonction de $(2r, n, n', \ldots)$ qu'est p de $(r, n, n', \ldots)$; et le nombre des courbes C^r est $=p'$. Le théorème est donc démontré. Et, précisément de la même manière, on démontre le théorème encore plus général:

En dénotant par $\phi(r, n, n', \ldots)$ le nombre des courbes C^r qui ont avec une droite donnée des contacts des ordres $n, n', \ldots$, et qui passent en outre par $\frac{1}{2}r(r+3)-n-n'\ldots$ points donnés, alors si, au lieu de la droite donnée, on a une courbe *unicursale* donnée de l'ordre m, le nombre des courbes C^r est $=\phi(mr, n, n', \ldots)$.

On aurait pu se servir directement de cela pour démontrer les théorèmes 1° et 2°. Par exemple, pour le théorème 1°, la considération de la courbe unicursale U^m donne

$$\alpha M+\beta m=\alpha(2m-2)+\beta m=(n+1)(mr-n);$$

c'est-à-dire

$$\alpha=\tfrac{1}{2}(n+1)n, \quad \beta=\tfrac{1}{2}(n+1)(2r-2n),$$

comme auparavant.

431.

SUR LA TRANSFORMATION CUBIQUE D'UNE FONCTION ELLIPTIQUE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXIV. *Janvier—Juin*, 1867, pp. 560—563.]

SOIT $U=(a, b, c, d, e \between x, 1)^4$ une fonction quartique quelconque de x: I, J les deux invariants:

$$(I = ae - 4bd + 3c^2, \quad J = ace - ad^2 - b^2e + 2bcd - c^3),$$

et prenons $\Omega = \dfrac{I^3 - 27J^2}{I^3}$ pour l'invariant absolu de U. Soient de même $U' = (a', b', \ldots \between x, 1)^4$ et $\Omega' = \dfrac{I'^3 - 27J'^2}{I'^3}$ l'invariant absolu de U'. En supposant que $\sqrt{U}$, $\sqrt{U'}$ soient les radicaux des deux fonctions elliptiques liées par la transformation du troisième ordre ou *cubique*, on peut se proposer la question quelle est la relation entre les deux invariants absolus Ω, Ω'? J'ai trouvé cette relation d'abord par des considérations géométriques qui me furent suggérées par une lettre de M. Sylvestre; puis je l'ai déduite des formules pour la transformation cubique données par M. Hermite, *Crelle*, t. LX., 1862, p. 304), et enfin, à l'aide d'une considération tirée de ces formules, j'ai réussi à l'obtenir à moyen des formules des *Fundamenta Nova*. Je vais donner ici cette dernière investigation de la relation dont il s'agit.

En supposant que les fonctions U, U' soient transformées linéairement en $(1-x^2)(1-k^2x^2)$, $(1-y^2)(1-\lambda^2y^2)$ respectivement, pour exprimer la liaison entre les modules k^2, λ^2, au lieu de l'équation explicite entre $\sqrt[4]{k}$, $\sqrt[4]{\lambda}$ (*Fund. Nova*, p. 23), je me sers des formules, p. 25, lesquelles en y écrivant

$$-\beta = \frac{\alpha+2}{2\alpha+1},$$

c'est-à-dire

$$2\alpha\beta + \alpha + \beta = 2,$$

deviennent

$$k^2 = -\alpha^2\beta, \quad \lambda^2 = -\alpha\beta^2.$$

Les transformations linéaires donnent sans peine

$$\Omega = \frac{108k^2(k^2-1)^4}{(k^4+14k^2+1)^3}, \quad \Omega' = \frac{108\lambda^2(\lambda^2-1)^4}{(\lambda^4+14\lambda^2+1)^3},$$

et il s'agit entre ces équations d'éliminer α, β, k, λ de manière à obtenir une équation entre Ω, Ω'.

J'écris

$$\alpha' = \frac{\frac{1}{2}(2\alpha+1)(\alpha+2)(\alpha-1)^4}{(\alpha^2+4\alpha+1)^3}, \quad \beta' = \frac{\frac{1}{2}(2\beta+1)(\beta+2)(\beta-1)^4}{(\beta^2+4\beta+1)^3}.$$

L'équation entre α, β donne

$$2\beta+1 = \frac{-3}{2\alpha+1}, \quad \beta+2 = \frac{3\alpha}{3\alpha+1}, \quad \beta-1 = \frac{-3(\alpha+1)}{3\alpha+1}, \quad \beta^2+4\beta+1 = -\frac{3(\alpha^2+4\alpha+1)}{(2\alpha+1)^2},$$

et on a de là

$$\beta' = \frac{\frac{27}{2}\alpha(\alpha+1)^4}{(\alpha^2+4\alpha+1)^3};$$

puis, en faisant attention à l'identité

$$(2\alpha+1)(\alpha+2)(\alpha-1)^4 + 27\alpha(\alpha+1)^4 = 2(\alpha^2+4\alpha+1)^3,$$

on obtient entre α', β', la relation très simple $\alpha' + \beta' = 1$.

L'expression de k^2 donne

$$k^2 = \frac{\alpha^2(\alpha+2)}{2\alpha+1},$$

$$k^2 - 1 = \frac{(\alpha-1)(\alpha+1)^3}{2\alpha+1},$$

$$k^4 + 14k^2 + 1 = \frac{1}{(2\alpha+1)^2}\{\alpha^6(\alpha+2)^2 + 14\alpha^3(\alpha+2)(2\alpha+1) + (2\alpha+1)^2\},$$

$$= \frac{1}{(2\alpha+1)^2}\{(\alpha^2+4\alpha+1)(\alpha^6+3\alpha^4+16\alpha^3+3\alpha^2+1)\},$$

et on a de là

$$\Omega = \frac{108\alpha^3(2\alpha+1)(\alpha+2)(\alpha-1)^4(\alpha+1)^{12}}{(\alpha^2+4\alpha+1)^3.(\alpha^6+3\alpha^4+16\alpha^3+3\alpha^2+1)^3}.$$

Or on a

$$\alpha' - 1 = \frac{\frac{1}{2}(2\alpha+1)(\alpha+2)(\alpha-1)^4 - (\alpha^2+4\alpha+1)^3}{(\alpha^2+4\alpha+1)^3}, \quad = \frac{-\frac{27}{2}\alpha(\alpha+1)^4}{(\alpha^2+4\alpha+1)^3}$$

$$8\alpha' + 1 = \frac{4(2\alpha+1)(\alpha+2)(\alpha-1)^4 + (\alpha^2+4\alpha+1)^3}{(\alpha^2+4\alpha+1)^3}, \quad = \frac{9(\alpha^6+3\alpha^4+16\alpha^3+3\alpha^2+1)}{(\alpha^2+4\alpha+1)^3},$$

et de là, en formant l'expression de la fonction $\frac{-64\alpha'(\alpha'-1)^4}{(8\alpha'+1)^3}$, on la trouve égale à la valeur qui vient d'être donnée pour Ω en termes de α: on a donc

$$\Omega = -\frac{64\alpha'(\alpha'-1)^3}{(8\alpha'+1)^3}$$

et de même

$$\Omega' = -\frac{64\beta'(\beta'-1)^3}{(8\beta'+1)^3}.$$

Avec la relation $\alpha' + \beta' = 1$, l'élimination de α', β' entre ces équations ne présente pas de difficulté.

432.

THÉORÈME RELATIF À LA THÉORIE DES SUBSTITUTIONS.

Extrait d'une lettre adressée à M. J. A. SERRET.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXVII. (*Juillet—Décembre*, 1868), pp. 784, 785.]

ON peut énoncer par rapport aux substitutions un théorème qui comprend les trois théorèmes III. IV. V., t. II. pp. 260—263 de votre *Cours d'Algèbre Supérieure.*

Pour un nombre quelconque μ on peut former avec les $\phi(\mu)$ nombres inférieurs et premiers à μ plusieurs systèmes de nombres lesquels sont chacun un système conjugué (mod. μ); c'est-à-dire que le produit de deux nombres quelconques d'un tel système est congru suivant le module μ, à un nombre du système. Comme cas extrêmes, l'unité est un tel système, et les $\phi(\mu)$ nombres forment aussi un système conjugué.

Pour μ premier, en dénotant par α une racine primitive de μ et par f un diviseur quelconque de $f-1$, les nombres $\equiv a^{f\alpha}$ (mod. μ), a étant un entier quelconque, forment un système conjugué. Et généralement pour un nombre μ quelconque ce nombre a des racines quasi-primitives $\alpha, \beta, \gamma, \ldots$, aux exposants $A, B, C, \ldots$, tels que $\alpha^A \equiv 1$ (mod. μ), $\beta^B \equiv 1$ (mod. μ), ... et $ABC \ldots = \phi(\mu)$. En choisissant une combinaison quelconque, par exemple α, β de ces racines, soient f, g des diviseurs quelconques de A, B respectivement, les nombres $\equiv a^{f\alpha} b^{g\beta}$ (mod. μ) forment un système conjugué, l'ordre du système ou nombre des termes étant $= \dfrac{AB}{fg}$.

Cela étant, on a ce théorème :

Une substitution T quelconque de l'ordre μ étant formée avec n lettres, l'on forme toutes les substitutions S telles que le produit STS^{-1} se réduise à une puissance de T dont l'exposant soit un nombre quelconque appartenant à un système conjugué (mod. μ), les substitutions S constitueront un système conjugué de l'ordre θM, où θ dénote l'ordre du système conjugué (mod. μ) et M le nombre des substitutions échangeables avec T.

La démonstration est tout à fait la même que celle que vous donnez p. 62 pour votre théorème IV, en y ajoutant seulement que les nombres i, j qui appartiennent au système conjugué (mod. μ) auront leur produit ij congru à un nombre de ce même système conjugué.

433.

SUR LES SURFACES TÉTRAÉDRALES.

[*Notes to the work* De la Gournerie, "*Recherches sur les surfaces réglées tétraédrales symétriques,*" 8vo. Paris, 1867.]

Premier Mémoire. Notes pp. 190—193.

ÉQUATIONS DE CERTAINES

1°. L'équation

$$
\begin{aligned}
0 = {} & + b^2c^2f^2x^8 && + c^2a^2g^2y^8 \\
& \ldots && - 2c^2bf(af-bg)\,x^6y^2 \\
& + 2c^2ag\,(af-bg)\,y^6x^2 && \ldots \\
& - 2b^2ah\,(ch-af)\,z^6x^2 && + 2a^2bh\,(bg-ch)\,z^6y^2 \\
& + 2f^2gh\,(bg-ch)\,w^6x^2 && + 2g^2hf\,(ch-af)\,w^6y^2
\end{aligned}
$$

$$
\begin{aligned}
& + a^2(b^2g^2+c^2h^2-4bgch)\,y^4z^4 && + b^2(c^2h^2+a^2f^2 \\
& + f^2(b^2g^2+c^2h^2-4bgch)\,w^4x^4 && + g^2(c^2h^2+a^2f^2
\end{aligned}
$$

$$
\begin{aligned}
& \ldots && + 2bf\,(afbg+c^2h^2-2ch\chi)\,x^4z^2w^2 \\
& - 2ag\,(afbg+c^2h^2-2ch\chi)\,y^4w^2z^2 && \ldots \\
& + 2ah\,(chaf+b^2g^2-2bg\chi)\,z^4y^2w^2 && - 2bh\,(bgch+a^2f^2-2af\chi)\,z^4w^2x^2 \\
& - 2gh\,(bcgh+a^2f^2-2af\chi)\,w^4y^2z^2 && - 2hf\,(cahf+b^2g^2-2bg\chi)\,w^4z^2x^2 \\
& + 2\Omega x^2y^2z^2w^2
\end{aligned}
$$

SURFACES DU HUITIÈME ORDRE.

$$+ a^2b^2h^2z^8 \qquad + f^2g^2h^2w^8$$

$$+ 2b^3cf\,(ch - af)\,x^6z^2 \qquad - 2bcf^2\,(bg - ch)\,x^6w^2$$

$$- 2a^2cg\,(bg - ch)\,y^6z^2 \qquad - 2cag^3\,(ch - af)\,y^6w^2$$

$$\ldots \qquad - 2abh^2\,(af - bg)\,z^6w^2$$

$$+ 2h^2fg\,(af - bg)\,w^6z^2 \qquad \ldots$$

$$- 4chaf)\,z^4x^4 \qquad + c^2\,(a^2f^2 + b^2g^2 - 4afbg)\,x^4y^4$$

$$- 4chaf)\,w^4y^4 \qquad + h^2\,(a^2f^2 + b^2g^2 - 4afbg)\,w^4z^4$$

$$- 2cf\,(chaf + b^2g^2 - 2bg\chi)\,x^4w^2y^2 + 2bc\,(bcgh + a^2f^2 - 2af\chi)\,x^4y^2z^2$$

$$+ 2cg\,(bgch + a^2f^2 - 2af\chi)\,y^4x^2w^2 + 2ca\,(cahf + b^2g^2 - 2bg\chi)\,y^4z^2x^2$$

$$\ldots \qquad + 2ab\,(abfg + c^2h^2 - 2ch\chi)\,z^4x^2y^2$$

$$- 2fg\,(abfg + c^2h^2 - 2ch\chi)\,w^4x^2y^2 \qquad \ldots$$

(où, pour abréger, on a écrit $\chi = af + bg + ch$, et Ω est une quantité quelconque) est celle d'une surface du huitième ordre qui a pour courbes doubles les quatre coniques

$$\begin{aligned}
x &= 0, & &+ cy^2 - bz^2 + fw^2 = 0;\\
y &= 0, & - cx^2 &\quad + az^2 + gw^2 = 0;\\
z &= 0, & + bx^2 - ay^2 &\quad + hw^2 = 0;\\
w &= 0, & - fx^2 - gy^2 - hz^2 &\quad = 0.
\end{aligned}$$

En déterminant Ω, savoir, en écrivant $\lambda + \mu + \nu = 0$, $af\lambda^2 + bg\mu^2 + ch\nu^2 = 0$ (ce qui donne deux systèmes de valeurs de $\lambda : \mu : \nu$), et puis

$$6\Omega = 4\Sigma \frac{\mu - \lambda}{\nu}(af + bg)^2 ch - 2\Sigma \frac{\mu^2 bg - \lambda^2 af}{\nu^2}(af + bg)^2 - 4(af - bg)(bg - ch)(ch - af),$$

la surface devient une surface réglée, savoir, la *quadrispinale* de M. de la Gournerie; et, en particulier, en supposant $\frac{1}{af} + \frac{1}{bg} + \frac{1}{ch} = 0$, on obtient

$$\lambda : \mu : \nu = \frac{1}{af} : \frac{1}{bg} : \frac{1}{ch},$$

et de là

$$6\Omega = (-2 - 4 =) - 6(af - bg)(bg - ch)(ch - af),$$

et la surface sera développable.

2°. L'équation

$$\begin{aligned}
0 = &+ a^2y^4z^4 + b^2z^4x^4 + c^3x^4y^4 + f^2x^4w^4 + g^2y^4w^4 + h^2z^4w^4\\
&\quad \ldots + 2bfx^4z^2w^2 - 2cfx^4y^2w^2 + 2bcx^4y^2z^2\\
&- 2agy^4z^2w^2 \quad \ldots + 2cgy^4x^2w^2 + 2cax^2y^4z^2\\
&+ 2ahz^4y^2w^2 - 2bhz^4x^2w^2 \quad \ldots + 2abx^2y^2z^4\\
&- 2ghw^4y^2z^2 - 2hfw^4z^2x^2 - 2fgw^4x^2y^2 \quad \ldots\\
&+ 2\Omega x^2y^2z^2w^2
\end{aligned}$$

(où Ω est une quantité quelconque) est celle d'une surface du huitième ordre ayant pour courbes doubles les quatre courbes du quatrième ordre

$$\begin{aligned}
x &= 0, & &hz^2w^2 - gw^2y^2 + ay^2z^2 = 0;\\
y &= 0, & - hz^2w^2 &\quad + fw^2x^2 + bz^2x^2 = 0;\\
z &= 0, & + gy^2w^2 - fw^2x^2 &\quad + cx^2y^2 = 0;\\
w &= 0, & - ay^2z^2 - bz^2x^2 - cx^2y^2 &\quad = 0.
\end{aligned}$$

En écrivant

$$\lambda + \mu + \nu = 0, \quad \frac{af}{\lambda^2} + \frac{bg}{\mu^2} + \frac{ch}{\nu^2} = 0$$

(ce qui donne quatre systèmes de valeurs de $\lambda : \mu : \nu$), et puis

$$\Omega = af\frac{\nu-\mu}{\lambda} + bg\frac{\lambda-\nu}{\mu} + ch\frac{\mu-\lambda}{\nu}:$$

la surface devient une surface réglée, savoir, la *quadricuspidale* de M. de la Gournerie; et, en supposant

$$(af)^{\frac{1}{3}} + (bg)^{\frac{1}{3}} + (ch)^{\frac{1}{3}} = 0,$$

et

$$\lambda : \mu : \nu = (af)^{\frac{1}{3}} : (bg)^{\frac{1}{3}} : (ch)^{\frac{1}{3}},$$

ce qui donne

$$\Omega = \left[(af)^{\frac{1}{3}} - (bg)^{\frac{1}{3}}\right]\left[(bg)^{\frac{1}{3}} - (ch)^{\frac{1}{3}}\right]\left[(ch)^{\frac{1}{3}} - (af)^{\frac{1}{3}}\right]:$$

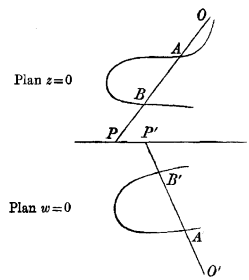
la surface devient développable.

Cambridge, 18 *Octobre* 1866.

Deuxième Mémoire. Notes pp. 279—283.

NOTE I. SUR LA DÉCOMPOSITION DU LIEU DES GÉNÉRATRICES EN SURFACES TÉTRAÉDRALES DISTINCTES.

Il me semble qu'une de vos conclusions a besoin d'être modifiée. Ainsi la surface tétraédrale dérivée de deux courbes triangulaires à exposant $\frac{1}{m}$ (m étant un entier positif), laquelle, selon un de vos théorèmes, serait de l'ordre $2m^2$, paraît se décomposer en m surfaces chacune de l'ordre $2m$. Il y a pour cela une raison *à priori*; en effet,



pour deux triangulaires de la forme en question, en employant votre construction, on peut établir une correspondance non-seulement entre les deux systèmes de points

$A, B, \ldots$, et $A', B', \ldots$, mais aussi entre chaque point A et un seul point correspondant A', car l'équation de la première courbe étant de la forme

$$fx^{\frac{1}{m}} + gy^{\frac{1}{m}} + hz^{\frac{1}{m}} = 0,$$

on satisfait à cette équation en écrivant

$$x : y : z = a(\theta + \alpha)^m : b(\theta + \beta)^m : c(\theta + \gamma)^m,$$

où θ est un paramètre variable. De même, l'équation de la seconde courbe étant

$$f'x^{\frac{1}{m}} + g'y^{\frac{1}{m}} + k'w^{\frac{1}{m}} = 0,$$

on satisfait à cette équation en écrivant

$$x : y : w = a'(\theta' + \alpha')^m : b'(\theta' + \beta')^m : d'(\theta' + \delta')^m,$$

où θ' est aussi un paramètre variable.

Pour la droite OP, on a

$$\frac{x}{y} = \frac{a(\theta + \alpha)^m}{b(\theta + \beta)^m},$$

et pour la droite $O'P'$

$$\frac{x}{y} = \frac{a'(\theta' + \alpha')^m}{b'(\theta' + \beta')^m};$$

donc, la condition pour la correspondance des droites est

$$\frac{a(\theta + \alpha)^m}{b(\theta + \beta)^m} = \lambda \frac{a'(\theta' + \alpha')^m}{b'(\theta' + \beta')^m},$$

ce qui donne m valeurs différentes pour θ' en termes de θ. Mais chacune de ces valeurs est de la forme

$$\theta' = \frac{A\theta + B}{C\theta + D},$$

et, en ne faisant attention qu'à une seule valeur de θ', on a le point

$$x : y : z = a(\theta + \alpha)^m : b(\theta + \beta)^m : c(\theta + \gamma)^m,$$

qui correspond à un point unique

$$x : y : w = a'(\theta' + \alpha')^m : b'(\theta' + \beta')^m : d'(\theta' + \delta')^m.$$

Pour le cas de l'exposant $\frac{1}{3}$, on a, de cette manière, une surface de l'ordre 6. J'ai vérifié cela dans le cas particulier de la surface développable. Il est très-singulier (c'est M. Salmon qui m'a fait cette remarque) qu'en écrivant dans cette équation (x^2, y^2, z^2, w^2) au lieu de (x, y, z, w), on obtient l'équation d'une surface du douzième ordre, lieu des centres de courbure d'un ellipsoïde.

Cambridge, 15 *Février* 1866.

NOTE II. À L'OCCASION DE L'ORDRE DES SURFACES TÉTRAÉDRALES.

Je crois que j'ai négligé de vous faire connaître un théorème assez général au sujet de l'ordre de ces surfaces. En considérant dans l'espace deux courbes (planes ou à double courbure) des ordres m et m' respectivement, et en supposant qu'il y ait entre les points de ces deux courbes une correspondance (α, α'), (c'est-à-dire qu'à un point donné de la courbe m correspondent α' points sur la courbe m', et à un point donné de la courbe m' correspondent α points sur la courbe m), alors la surface réglée que l'on obtient en unissant par des droites les points correspondants des courbes m et m' sera de l'ordre $m\alpha' + m'\alpha$.

Cambridge, 18 *Octobre*, 1866.

NOTE III. SUR LA SURFACE COMPLÉMENTAIRE.

Je puis reconnaître, par mes propres formules, que, des pq^2 surfaces de l'ordre $2p^2q$, il n'y en a que pq qui passent par la troisième directrice. En effet, le rapport anharmonique k est donné en termes des paramètres de la troisième directrice, au moyen d'une équation qui contient la quantité irrationnelle $k^{\frac{p}{q}}$. En rationalisant cette équation, on obtient pour k une équation de l'ordre pq; à chaque racine k_1 correspondent q surfaces, savoir celles qui appartiennent aux q valeurs de $k_1^{\frac{p}{q}}$; mais l'équation irrationnelle n'est satisfaite que par une seule valeur de $k_1^{\frac{p}{q}}$, à savoir la valeur de $k_1^{\frac{p}{q}}$ donnée par l'équation irrationnelle, en y substituant pour k, en tant que k y entre rationnellement, la valeur $k = k_1$. Donc, à chaque racine k_1 correspond une seule surface qui passe par la troisième directrice. La question à laquelle donne lieu cette circonstance paraît très-intéressante. La surface déterminée par les trois directrices est composée de pq surfaces chacune de l'ordre $2p^2q$, et d'une surface résiduale de l'ordre $2p^3(q^3 - q^2)$. Quelles sont la nature et les propriétés de cette surface résiduale? Je serais bien aise de savoir si vous avez fait des recherches à ce sujet.

Cambridge, 29 *Mars* 1866.

434.

ON CERTAIN SKEW SURFACES, OTHERWISE SCROLLS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part II. (1869), pp. 277—289. Read Nov. 11, 1867.]

THE investigations contained in the present Memoir were suggested to me by the Memoirs of M. De la Gournerie, presented by him to the Academy of Sciences in 1865 and 1866, published in extract in the *Comptes Rendus*, and reproduced in his work "*Recherches sur les surfaces réglées tétraédrales symétriques*, par Jules De la Gournerie, avec des notes par Arthur Cayley," 8vo. Paris, 1867. Although the results or the greater part of them, agree with those in the work just referred to, the mode of treatment is different, and more general, the orders, &c. of the different scrolls being obtained by considerations founded on the theory of Correspondence, and I have thought it not improper to submit to geometers in this altered form the theory of the very interesting class of Scrolls for which they are indebted to M. De la Gournerie's researches.

Article Nos. 1 to 10. *Geometrical Construction of a Class of Scrolls.*

1. Consider any two curves (plane or of double curvature) U, U', of the orders m, m' respectively, and let the points of U have with those of U' an (α, α') correspondence; viz. let the points of the two curves be so related that to each point of U correspond α' points of U', and to each point of U' correspond α points of U: then the lines joining the corresponding points of U, U' form a scroll the order of which is $= m\alpha' + m'\alpha$.

2. In particular let U, U' be plane curves in the planes Π, Π' respectively; and let the correspondence between the points of the two curves be established as follows; viz. consider in the plane Π a curve Ω of the class μ, and in the plane Π' a curve Ω' of the class μ'; and (to avoid useless generality) let the tangents of these two curves Ω, Ω' have to each other a (1, 1) correspondence; that is, to each tangent of

Ω there corresponds a single tangent of Ω', and to each tangent of Ω' a single tangent of Ω (this assumes that the curves Ω, Ω' are rational transformations one of the other, and that they have consequently the same Deficiency). This being so, let the points of U which lie on any tangent of Ω and the points of U' which lie on the corresponding tangent of Ω' be taken to be corresponding points of U, U'. The correspondence is then $(\mu m', \mu' m)$: in fact through a given point of U there pass μ tangents of Ω, and the corresponding μ tangents of Ω' meet U' in $\mu m'$ points, that is, to a given point of U correspond $\mu m'$ points of U'; and similarly to a given point of U' correspond $\mu' m$ points of U. And hence the order of the scroll formed by the lines joining the corresponding points of U, U' is $=(\mu+\mu')\,mm'$.

3. This conclusion may be otherwise established as follows; let K, K' be any two corresponding points of U, U', so that the scroll we are concerned with is that generated by the series of lines KK'; and let I denote the line of intersection of the planes Π, Π'. The line I meets the curve U in m points, and taking one of these points for a point K we may from this point draw μ tangents to the curve Ω, that is, the point in question is a point K in respect of μ different tangents of the curve Ω; to each of these tangents there corresponds a single tangent of Ω', and such tangent of Ω' meets the curve U' in m' points, that is, to the point K in question there correspond $\mu m'$ points K' and consequently $\mu m'$ lines KK' in the plane Π'; hence to each of the m points K on the line I there correspond $\mu m'$ lines KK' in the plane Π'; and we have thus $\mu mm'$ generating lines in the plane Π'; there are in like manner $\mu' mm'$ generating lines in the plane Π.

Take K an arbitrary point on the curve U; there are $\mu m'$ corresponding points K', and consequently $\mu m'$ generating lines through K, that is, through each point of the curve U; or the curve U (which is of the order m) is a $\mu m'$-tuple line on the scroll; similarly the curve U' (which is of the order m') is a $\mu' m$-tuple line on the scroll.

The complete section of the scroll by the plane Π consists of the curve U taken $\mu m'$ times (order $\mu mm'$) and of the $\mu' mm'$ generating lines in the plane Π; that is, the order of the section is $=(\mu+\mu')\,mm'$; and we thus see that the order of the scroll is $=(\mu+\mu')\,mm'$. Of course in like manner the complete section of the scroll by the plane Π' consists of the curve U' taken $\mu' m$ times (order $\mu' mm'$) and of the $\mu mm'$ generating lines in the plane Π', the order of the section being thus $=(\mu+\mu')mm'$.

4. There are on the scroll certain singular tangent planes; viz. if we have two corresponding tangents of Ω, Ω' meeting the line I in the same point, then we have m points K and m' points K' all lying in the plane of the two tangents; and of course the mm' lines KK' will all lie in the plane of the two tangents; that is, the intersection of the scroll by the plane in question will be made up of the mm' lines, and of a curve of the order $(\mu+\mu'-1)\,mm'$; and the plane in question is thus a singular tangent plane.

5. The number of these singular tangent planes is $=\mu+\mu'$; in fact considering as corresponding points on the line I, the intersection of this line by any tangent of Ω and the intersection by the corresponding tangent of Ω', the correspondence is

obviously (μ, μ'); viz. through a given point P considered as belonging to the first system there pass μ tangents of Ω, and corresponding thereto we have μ tangents of Ω' each intersecting I in a single point P'; that is, to a given point P correspond μ points P'; and similarly to a given point P' correspond μ' points P'. And this being so, the number of united points, that is, points of I through which pass corresponding tangents of Ω, Ω', is $=\mu+\mu'$.

6. In particular the curves Ω, Ω' may reduce themselves each to a point: the tangents to the two curves are here the lines passing through the points Ω, Ω' respectively: and the condition for the (1, 1) correspondence of the two tangents is that the pencils of lines shall be homographically related; or, what is the same thing, that these two pencils shall determine on the line I two ranges which are homographically related; the entire construction is then as follows:

Given in the plane Π a curve U and a point Ω, and in the plane Π' a curve U' and a point Ω'; and taking in the plane Π a pencil of lines through Ω, and in the plane Π' a pencil of lines through Ω', in such wise that the two pencils correspond homographically; then if a line of the first pencil meets the curve U in the m points K, and the corresponding line of the second pencil meets the curve U' in the m' points K', the scroll in question is that generated by the mm' lines KK'.

7. By what precedes, the scroll is of the order $2mm'$; the curve U is a m'-tuple line, and the complete section by the plane Π is made up of this curve taken m' times and of mm' generating lines; similarly the curve U' is a m-tuple line, and the complete section by the plane Π' is made up of this curve taken m times and of mm' generating lines; there are two singular tangent planes such that the section by each of them is made up of mm' generating lines and of a curve of the order mm'; the planes in question are obviously those through the lines $\Omega\Omega'$ and the coincident points of the two ranges on the line I, say the points A, B respectively.

8. The foregoing results will be modified in special cases. Suppose, for instance, that the curve U passes ω times, α times, and β times through the points Ω, A, B, respectively, and that the curve U' passes ω' times, α' times, and β' times through the points Ω', A, B respectively. Then to each point on the curve U there correspond the $m'-\omega'$ intersections (other than the point Ω') on a line through Ω', so that U' is a $(m'-\omega')$tuple line on the surface. The curve U' meets the line I in m' points and corresponding to each of them we have a line through Ω meeting the curve U in $(m-\omega)$ points, exclusive of the point Ω; this would give $m'(m-\omega)$ generating lines in the plane Π; but among the m' points are included the point $A\alpha'$ times, and the point $B\beta'$ times; the $(m-\omega)$ points corresponding to A include the point $A\alpha$ times, and we have thus the point A corresponding to itself $\alpha\alpha'$ times, and giving a reduction $=\alpha\alpha'$ in the number $m'(m-\omega)$ of generating lines: similarly the $m-\omega$ points corresponding to B include the point $B\beta$ times, and we have thus the point B corresponding to itself $\beta\beta'$ times and giving a reduction $=\beta\beta'$ in the number $m'(m-\omega)$ of generating lines; the number of generating lines in the plane Π is thus $=m'(m-\omega)-\alpha\alpha'-\beta\beta'$. The complete section by the plane Π is made up of the

curve $U(m-\omega')$ times (order $m(m'-\omega')$) and of the $m'(m-\omega)-\alpha\alpha'-\beta\beta'$ generating lines; the order of the section, and consequently also the order of the scroll, is thus $=2mm'-m\omega'-m'\omega-\alpha\alpha'-\beta\beta'$. It is clear that in like manner the curve U' is a $(m-\omega)$tuple line on the surface, and that the complete section by the plane Π' is made up of this curve taken $(m-\omega)$ times, order $m'(m-\omega)$, and of $m(m'-\omega')-\alpha\alpha'-\beta\beta'$ generating lines.

9. The section by the tangent plane through A is made up of $(m-\omega)(m'-\omega')-\alpha\alpha'$ generating lines (viz. these are, the line ΩA $\alpha'(m-\omega-\alpha)$ times, the line $\Omega' A$ $\alpha(m'-\omega'-\alpha')$ times, and $(m-\omega-\alpha)(m'-\omega'-\alpha')$ other generating lines) and of a curve of the order $mm'-\omega\omega'-\beta\beta'$: similarly the section by the tangent plane through B is made up of $(m-\omega)(m'-\omega')-\beta\beta'$ generating lines (viz. these are, the line ΩB $\beta'(m-\omega-\beta)$ times, the line $\Omega' B$ $\beta(m'-\omega'-\beta')$ times, and $(m-\omega-\beta)(m'-\omega'-\beta')$ other generating lines), and of a curve of the order $mm'-\omega\omega'-\alpha\alpha'$.

10. A very interesting case is when (m, m' being each even) we have

$$\omega=\alpha=\beta=\frac{1}{2m},\quad \omega'=\alpha'=\beta'=\tfrac{1}{2}m'.$$

Here the curve U is a $\frac{1}{2}m'$-tuple line on the scroll, and the complete section by the plane Π is this curve taken $\frac{1}{2}m'$ times; the order of the section, and therefore of the scroll is thus $=\frac{1}{2}mm'$; of course in like manner the curve U' is a $\frac{1}{2}m$-tuple line on the scroll, and the complete section by this plane is the curve U' taken $\frac{1}{2}m$ times: the section by each of the planes $\Omega\Omega' A$, $\Omega\Omega' B$ is a curve of the order $\frac{1}{2}mm'$, the planes in question being in the present case no longer singular tangent planes, or even tangent planes at all, of the scroll.

Article Nos. 11 to 14. *Analytical Theory.*

11. It will be convenient to denote by D, C respectively the points heretofore called Ω, Ω' respectively: this being so, we have a tetrahedron $ABCD$, of which the faces ABD, ABC are the planes heretofore called Π, Π' respectively, and the other two faces CDA, CDB are the singular tangent planes $\Omega\Omega' A$, $\Omega\Omega' B$ respectively. And then, taking

$$x=0,\quad y=0,\quad z=0,\quad w=0$$

for the equations of the faces BCD, CDA, DAB, ABC of the tetrahedron, we may write for the equations of the curve U, $z=0$, $f_3(x, y, w)=0$, for those of the curve U', $w=0$, $f_4(x, y, z)=0$; and take the homographic ranges on the line I $(z=0,\ w=0)$ to be given as the intersections of this line with the pencils of planes $x-\theta y$, $x-k\theta y=0$ respectively (θ a variable parameter, k a constant). The points K are therefore given by

$$x-\theta y=0,\quad z=0,\quad f_3(x, y, w)=0,$$

the points K' by

$$x-k\theta y=0,\quad w=0,\quad f_4(x, y, z)=0;$$

and then the lines KK' belonging to the different values of the parameter θ generate the scroll.

12. Or, what is the same thing, taking (X, Y, Z, W) as the coordinates of K, (X', Y', Z', W') as the coordinates of K', we have

$$\begin{aligned} X - \theta Y = 0, \quad Z = 0, \quad f_3(X, Y, W) = 0, \\ X' - k\theta Y = 0, \quad W' = 0, \quad f_4(X', Y', Z') = 0, \end{aligned}$$

and then the equations of the line KK' are

$$\left\| \begin{matrix} x, & y, & z, & w \\ X, & Y, & 0, & W \\ X', & Y', & Z', & 0 \end{matrix} \right\| = 0:$$

or, as these may be written,

$$\begin{aligned} & - WZ'y + & WY'z + & YZ'w = 0, \\ WZ'x + & \quad . \quad - & WX'z - & XZ'w = 0, \\ - W'Yx + & WX'y & . \quad + (XY' - X'Y)w & = 0, \\ - YZ'x + & XZ'y - (XY' - X'Y)z & . & = 0, \end{aligned}$$

equivalent of course to two equations. The elimination of $X, Y, W, X', Y', Z', \theta$ from all the equations gives the equation of the scroll.

13. Substituting the values $X = \theta Y$, $X' = k\theta Y'$, we have

$$f_3(\theta Y, Y, W) = 0, \quad f_4(k\theta Y', Y', Z') = 0,$$

$$\begin{aligned} . \quad & - WZ'y + & WY'z + & YZ'w = 0, \\ WZ'x & \quad . \quad - & k\theta WY'z - & \theta YZ'w = 0, \\ - WY'x + & k\theta WY'y & . \quad + \theta(1-k)YY'w & = 0, \\ - YZ'x + & \theta YZ'y - \theta(1-k)YY'z & . & = 0; \end{aligned}$$

or, what is the same thing, writing $\frac{W}{Y} = \omega$, and $\frac{Z'}{Y'} = \zeta$, we have

$$f_3(\theta, 1, \omega) = 0, \quad f_4(k\theta, 1, \zeta) = 0,$$

$$\begin{aligned} . \quad & - \omega\zeta y + & \omega z + & \zeta w = 0, \\ \omega\zeta x & \quad . \quad - & k\theta\omega z - & \theta\zeta w = 0, \\ - \omega x + & k\theta\omega y & . \quad + & \theta(1-k)w = 0, \\ - \zeta x + & \theta\zeta y - \theta(1-k)z & . & = 0. \end{aligned}$$

Recollecting that the last four equations are equivalent to two equations only, and substituting for ω, ζ their values in terms of θ, we have in effect two equations, which by the elimination of θ lead to a relation in (x, y, z, w), the equation of the scroll.

14. We may find the sections of the scroll by the planes $x = 0$, $y = 0$ respectively. Writing first $x = 0$, we have

$$\omega y = \frac{k-1}{k} w, \quad \zeta y = -(k-1)z.$$

Hence taking the other two equations in the form

$$f_3(\theta y,\ y,\ \omega y)=0,\quad f_4\left(\theta y,\ \frac{y}{k},\ \frac{\zeta y}{k}\right)=0,$$

and putting $\theta y = u$, we have

$$f_3\left(u,\ y,\ \frac{(k-1)\,w}{k}\right)=0,\quad f_4\left(u,\ \frac{y}{k},\ \frac{-(k-1)\,z}{k}\right)=0,$$

from which eliminating u we obtain an equation $f_1(y,\ z,\ w)=0$, the equation of the section by the plane $x=0$.

Similarly, writing $y=0$, we have

$$\frac{x\omega}{\theta}=-(k-1)\,w,\quad \frac{x\zeta}{\theta}=(k-1)\,z,$$

whence taking the equations in the form

$$f_3\left(x,\ \frac{x}{\theta},\ \frac{x\omega}{\theta}\right)=0,\quad f_4\left(kx,\ \frac{x}{\theta},\ \frac{x\zeta}{\theta}\right)=0,$$

and writing $\frac{x}{\theta}=v$, we have

$$f_3(x,\ v,\ -(k-1)\,w)=0,\quad f_4(kx,\ v,\ (k-1)\,z)=0,$$

from which, eliminating v, we obtain an equation $f_2(x,\ z,\ w)=0$, the equation of the section by the plane $y=0$.

Article Nos. 15 to 29. *The Curves U, U' are henceforward "triangular" curves.*

15. Let $r=\pm\frac{p}{q}$, where p, q are positive integers prime to each other, and let the given sections be

$$z=0,\quad A\,x^r+B\,y^r\quad .\quad +Dw^r=0,$$
$$w=0,\quad A'x^r+B'y^r+C'z^r\quad .\quad =0,$$

where it is to be observed that r being $=+\frac{p}{q}$, the two given sections are of the order pq, the order of the scroll is $=2p^2q^2$, each of the given sections is a pq-tuple line on the scroll, and the plane thereof meets the scroll in the section taken pq times, and in the pq generating lines: but r being $=-\frac{p}{q}$, the two given sections are each of the order $2pq$, with three pq-tuple points ($\omega=\alpha=\beta=pq$, $\omega'=\alpha'=\beta'=pq$), and thence the order of the scroll is $\frac{1}{2}(2pq)^2$, $=2p^2q^2$; each of the sections is a pq-tuple line on the scroll, and the plane meets the scroll only in the section taken pq times. But in either case, if q be >1, that is, if r be fractional, it will presently appear that the scroll of the order $2p^2q^2$ breaks up into q scrolls each of the order $2p^2q$.

16. To find the section by the plane $x=0$, we have

$$A\,u^r + B\quad y^r \qquad\qquad . \qquad + D\left(\frac{k-1}{k}\,w\right)^r = 0,$$

$$A'u^r + B'\left(\frac{y}{k}\right)^r + C'\left(-\frac{k-1}{k}\,z\right)^r \qquad\qquad . \qquad = 0,$$

and eliminating u we obtain

$$\left(AB'\frac{1}{k^r} - A'B\right)y^r + AC'\left(-\frac{k-1}{k}\right)^r z^r - A'D\left(\frac{k-1}{k}\right)^r w^r = 0:$$

writing $\left(\frac{k-1}{k}\right)^r = (-)^r\left(\frac{1-k}{k}\right)^r$, this is

$$\left(AB'\frac{1}{k^r} - A'B\right)y^r + AC'\quad\left(\frac{1-k}{k}\right)^r z^r - (-)^r\left(\frac{1-k}{k}\right)^r w^r = 0,$$

or, what is the same thing, it is

$$-(-)^r\frac{AB' - A'Bk^r}{(1-k)^r}y^r - (-)^{-r}AC'\qquad z^r + \qquad A'Dw^r = 0.$$

And in regard to this and the other equations which contain $(-)^{-r}$, it is to be observed that r being integral we have $(-)^{-r} = (-)^r$, and that r being fractional, every value of $(-)^{-r}$ is also a value of $(-)^r$; so that we may in every case write $(-)^r$ in place of $(-)^{-r}$.

Similarly for the section by the plane $y=0$, we have

$$Ax^r + Bv^r \qquad\qquad . \qquad + D\left(-(k-1)\;w\right)^r = 0,$$

$$A'(kx)^r + B'v^r + C\left((k-1)\,z\right)^r \qquad\qquad . \qquad = 0,$$

and eliminating v, we have

$$(AB' - A'Bk^r)\,x^r \qquad - BC'(k-1)^r z^r + B'D\left(-(k-1)\right)^r w^r = 0\,;$$

or, what is the same thing,

$$\frac{AB' - A'Bk^r}{(1-k)^r}x^r \qquad - (-)^r BC'z^r + B'D \qquad w^r = 0.$$

17. The four sections thus are

$$x=0, \qquad . \qquad -(-)^r\frac{AB' - A'Bk^r}{(1-k)^r}y^r - (-)^r AC'z^r + A'Dw^r = 0,$$

$$y=0, \quad \frac{AB' - A'Bk^r}{(1-k)^r}x^r \qquad\qquad . \quad - (-)^r BC'z^r + B'Dw^r = 0,$$

$$z=0, \qquad Ax^r \qquad + By^r \qquad . \qquad + Dw^r = 0,$$

$$w=0, \qquad A'x^r \qquad + B'y^r + \qquad C'z^r \quad . \quad = 0.$$

It will be convenient to speak of these four curves as *directrices* of the scroll.

18. Suppose for a moment that r is integral; as either of the given equations may be multiplied by a constant, we may assume that $D = -(-)^r C'$; substituting this value and dividing the first and second equations each by C', we have

$$\begin{array}{llllll}
x = 0, & \cdot & -(-)^r \dfrac{AB' - A'Bk^r}{(1-k)^r C'} y^r & -(-)^r A z^r & -(-)^r A' w^r & = 0, \\
y = 0, & \dfrac{AB' - A'Bk^r}{(1-k)^r C'} x^r & \cdot & -(-)^r B z^r & -(-)^r B' w^r & = 0, \\
z = 0, & A x^r + & B y^r & \cdot & -(-)^r C' w^r & = 0, \\
w = 0, & A' x^r + & B' y^r + & C' z^r & \cdot & = 0,
\end{array}$$

so that the diagonally opposite coefficients differ only by the factor $-(-)^r$; viz. the matrix is symmetrical or skew symmetrical according as r is odd or even.

19. If r be fractional, it is to be observed that, although the three symbols $(-)^r$ and the two symbols $(1-k)^r$ which enter into the first and second equations of No. 17, do not in the first instance represent of necessity the same values of $(-)^r$ and $(1-k)^r$ respectively, yet there is no loss of generality in assuming that they do so—the irrational equations are mere symbols for the rational equations to which they respectively give rise—and the irrationalities $(-)^r$ and $(1-k)^r$ will on the rationalisation of the equations disappear along with the irrationalities x^r, y^r, z^r, to which they are attached. But the case is otherwise with the irrationality k^r involved in the expression $AB' - A'Bk^r$; writing as before $r = \pm \frac{p}{q}$ (p and q positive integers prime to each other), the symbol k^r has q different values; and there is not in the first instance any relation between the k^r of the first equation and the k^r of the second equation: for each of these equations the rationalised equation (that is, the equation rationalised in regard to the coordinates) will contain the irrationality k^r, and will thus for each of the q values of k^r represent a distinct curve. The given equations (viz. the first and second equations) represent each of them a single curve of the order pq or $2pq$, according as r is positive or negative; the first and second equations represent each of them q such curves.

20. Hence, starting from the two given curves in the planes $z = 0$ and $w = 0$, respectively, and with a given value of k, the section of the scroll by the plane $y = 0$ is made up of q curves, viz. the curves obtained from the second equation of No. 17, by assigning to the radical k^r each of its q different values; the scroll consequently breaks up into q different scrolls, viz. the lines passing through the two given curves, and any one of the q curves in the plane $y = 0$, constitute a distinct scroll. The lines in question meet the plane $x = 0$, not indifferently in any one of the q curves in that plane, but in a certain one of these curves, viz. in that curve for which the radical k^r has the same value as for the curve in question in the plane $y = 0$. Hence we may in the first and second equations regard the radicals k^r as having the same meaning, and the system of four equations in effect breaks up into q systems, viz. the systems obtained by giving to the radical k^r its q different values; each of these systems gives a scroll, and the scroll derived from the two given curves with a given

value of k is made up of these q scrolls. And hence, attaching a unique value to each of the symbols $(-)^r$, $(1-k)^r$, and k^r, we may, as before, write $D=(-)^r C'$, and so reduce the original equations as in the case r integral, to the form No. 18, in which the diagonally opposite coefficients differ only by the factor $-(-)^r$.

21. Let the two given equations be taken to be

$$\begin{aligned} z=0, &\quad bx^r-(-)^r ay^r \quad . \quad +hw^r=0, \\ w=0, &\quad -(-)^r fx^r-(-)^r gy^r-(-)^r hz^r \quad . \quad =0; \end{aligned}$$

we have then

$$\frac{AB'-A'Bk^r}{(1-k)^r C'}=\frac{(-)^r bg+afk^r}{(-)^r h(1-k)^r},$$

or, putting this $=-(-)^r c$, that is, writing

$$afk^r+bg(-1)^r+ch(1-k)^r=0,$$

the four equations become

$$\begin{aligned} x=0, &\quad . \quad cy^r-(-)^r bz^r+fw^r=0, \\ y=0, &\quad -(-)^r cx^r \quad . \quad + \quad az^r+gw^r=0, \\ z=0, &\quad bx^r-(-)^r ay^r \quad . \quad +hw^r=0, \\ w=0, &\quad -(-)^r fx^r-(-)^r gy^r-(-)^r hz^r \quad . \quad =0; \end{aligned}$$

where c being considered as given, k is determined as mentioned above, or, what is the same thing, $k:-1:1-k=\lambda:\mu:\nu$, we have $\lambda:\mu:\nu$, and thence k, determined by the equations

$$\begin{aligned} \lambda+\mu+\nu&=0, \\ af\lambda^r+bg\mu^r+ch\nu^r&=0. \end{aligned}$$

22. Consider for a moment λ, μ, ν, as the coordinates of a point in a plane, then $\left(r=\pm\frac{p}{q}\text{ as before}\right)$, the equation $af\lambda^r+bg\mu^r+ch\nu^r=0$, is that of a curve of the order pq or $2pq$, according as r is positive or negative: and this curve is met by the line $\lambda+\mu+\nu=0$, in pq or $2pq$ points, that is, k has this number pq or $2pq$, of values: but to each of these values of k there corresponds (not q values but) only a single value of k^r, viz. that value for which $afk^r+bg(-1)^r+ch(1-k)^r=0$; that is, starting from the two directrices in the planes $z=0$, $w=0$, respectively, and a given third directrix in the plane $y=0$ (or in the plane $x=0$), we may by means of each of the pq or $2pq$ values of k construct a scroll passing through the three directrices, and which will also pass through the fourth directrix in the plane $x=0$ (or in the plane $y=0$), but such scroll is only one (not each) of the q scrolls which can be constructed from the two given sections in the planes $z=0$, $w=0$, respectively, and from the assumed value of k. It has been mentioned that whether r is $=+\frac{p}{q}$, or $=-\frac{p}{q}$, the total scroll constructed from the two given directrices in the planes $z=0$, $w=0$, and from a given value of k is of the order

$2p^2q^2$, and that such scroll breaks up into q distinct scrolls, hence the order of each of the distinct scrolls is $=2p^2q$. Whence, starting with the given directrices in the planes $z=0$, $w=0$, and a given third directrix in the plane $y=0$ (or in the plane $x=0$), we have pq or $2pq$ scrolls each of the order $2p^2q$, and passing through these three directrices, and through the given fourth directrix in the plane $x=0$ (or in the plane $y=0$).

23. It is to be observed that when q is >1, then considering the three directrices as given, the pq or $2pq$ scrolls each of the order $2p^2q$, do not make up the total scroll generated by the lines which pass through the three given directrices. I call to mind that for three given directrices the orders of which are m, n, p, respectively, and which meet, the second and third, the third and first, and the first and second, in α points, β points, and γ points respectively, the order of the scroll generated by the lines which meet the three directrices is $=2mnp-\alpha m-\beta n-\gamma p$. Suppose first, that $r=+\frac{p}{q}$, then the directrices are each of the order pq, and they do not any two of them meet; the order of the scroll is $=2p^3q^2$. Suppose secondly, $r=-\frac{p}{q}$, then the directrices are each of the order $2pq$, but each two of them have in common two pq-tuple points counting as $2p^2q^2$ intersections; the order of the scroll is thus $(16-3.4)p^3q^3=4p^3q^3$. In the first case the lines which meet the three directrices generate a residuary scroll of the order $2p^3(q^2-q^2)$, and the pq scrolls each of the order $2p^2q$; in the second case they generate a residuary scroll of the order $4p^3(q^2-q^2)$, and the $2pq$ scrolls each of the order $2pq$.

24. In the case $r=+\frac{p}{q}$, by way of illustration of the origin of the pq scrolls each of the order $2p^2q$, I consider the particular case $p=1$, that is, $r=\frac{1}{q}$, the reciprocal of a positive integer q, and where it is to be shown that we have q scrolls each of the order $2q$. The given directrices are here

$$z=0,\quad Ax^{\frac{1}{q}}+By^{\frac{1}{q}}\quad .\ +Dw^{\frac{1}{q}}=0,$$

$$w=0,\quad A'x^{\frac{1}{q}}+B'y^{\frac{1}{q}}+C'z^{\frac{1}{q}}\quad .\quad =0,$$

each of them a unicursal curve; we may in fact satisfy the two equations respectively, by writing in the first of them

$$x:y:w=a(\phi+\alpha)^q:b(\phi+\beta)^q:d(\phi+\delta)^q;$$

and in the second

$$x:y:z=a'(\phi'+\alpha')^q:b'(\phi'+\beta')^q:c'(\phi'+\gamma')^q,$$

where a, b, d, α, β, δ, a', b', c', α', β', γ' are properly determined constants, ϕ, ϕ' are variable parameters. It follows that, considering the points K, K' which are the intersections of the first curve by the line $x-\theta y=0$, and of the second curve by the corresponding line $x-k\theta y=0$, we have not only a correspondence of q points K with

k points K', but we may establish, and that in q different manners, a correspondence between single points K and K'. For, substituting the foregoing values of $x : y$ in the equations $x - \theta y = 0$ and $x - k\theta y = 0$ respectively, we have

$$\theta = \frac{a}{b}\frac{(\phi+\alpha)^q}{(\phi+\beta)^q}, \quad k\theta = \frac{a'}{b'}\frac{(\phi'+\alpha')^q}{(\phi'+\beta')^q},$$

and thence

$$\frac{a}{b}\frac{(\phi+\alpha)^q}{(\phi+\beta)^q} = \frac{1}{k}\frac{a'}{b'}\frac{(\phi'+\alpha')^q}{(\phi'+\beta')^q};$$

so that, extracting the qth root of each side, we have, in q different ways corresponding to the q values of the radical $\left(\frac{1}{k}\frac{ba'}{b'a}\right)^{\frac{1}{q}}$, a relation of the form $\phi' = \frac{l\phi+m}{n\phi+r}$; and considering ϕ' as having this value, the points K, K' as given by the equations

$$z = 0, \quad x : y : w = a(\phi+\alpha)^q : b(\phi+\beta)^q : d(\phi+\delta)^q,$$

and

$$w = 0, \quad x : y : z = a'(\phi'+\alpha')^q : b'(\phi'+\beta')^q : c'(\phi'+\gamma')^q,$$

respectively, correspond as single points to each other. We have thus in q different ways a series of corresponding points K, K', and consequently q series of lines KK' each of them generating a scroll which (as the order of the scroll generated by all the q series is $= 2q^2$), must be each of them of the order $2q$; and the decomposition in question is thus explained.

25. In the scroll of the order $2q^2$, each directrix is a q-tuple line, and the complete section by the plane of the directrix is made up of the directrix q times (order q^2), and of q^2 generating lines, in fact, of q q-fold generating lines: to show that this is so, consider the directrix in the plane $z = 0$, viz. the equation of this is $Ax^{\frac{1}{q}} + By^{\frac{1}{q}} + Dw^{\frac{1}{q}} = 0$. Writing herein $w = 0$, we have $Ax^{\frac{1}{q}} + By^{\frac{1}{q}} = 0$, that is, $A^q x - (-)^q B^q y = 0$; it is clear that the rationalised equation must reduce itself to $\{A^q x - (-)^q B^q y\}^q = 0$, and that the line $w = 0$, is thus a tangent of q-pointic intersection at the point $w = 0$, $A^q x - (-)^q B^q y = 0$. Taking K at this point we have, in each of the scrolls of the order $2q$, q coincident positions of K', that is, a q-fold line KK' in the plane $w = 0$; and the like for the plane $z = 0$, so that the total section by the plane $z = 0$ is made up of the directrix q times and of q q-fold generating lines; and it follows that for each of the scrolls of the order $2q$, the section by the plane $z = 0$ is made up of the directrix once, and of a q-fold generating line.

26. It is easy to see that in the general case $r = +\frac{p}{q}$, the like conclusion holds; for the scroll of the order $2p^2q^2$, the section by the plane of the directrix consists of the directrix pq times (order p^2q^2), and of p^2q q-fold generating lines; whence for each of the q component scrolls of the order $2p^2q$, the section is made up of the directrix p times (that is, the directrix is a p-tuple line on the scroll) and of p^2 q-fold generating lines.

27. In the case $r = -\frac{p}{q}$, where the order of the directrix is $= 2pq$, then in the scroll of the order $2p^2q^2$, the directrix is a pq-tuple line on the scroll, and taken pq times it constitutes the complete section by the plane of the directrix; whence in each of the q component scrolls of the order $2p^2q$, the directrix is a p-tuple line; and taken p times it constitutes the complete section by the plane of the directrix.

28. It is convenient to exhibit the foregoing results in a tabular form as follows:

$r = +\frac{p}{q}$.	$r = -\frac{p}{q}$.
Each directrix is of order pq.	Each directrix is of order $2pq$, with three pq-tuple points.
Scroll belonging to two directrices, and a given value of k, is of the order	
$2p^2q^2$,	$2p^2q^2$,
breaking up into q scrolls each of order $2p^2q$, each which scroll of the order $2p^2q$ has each directrix for a p-tuple line and has besides p^2 q-fold generating lines in the plane of the directrix.	breaking up into q scrolls each of the order $2p^2q$, each which scroll of the order $2p^2q$ has each directrix for a p-tuple line, and consequently no generating line in the plane of the directrix.
Considering two directrices and a given third directrix,	
k has pq values.	k has $2pq$ values.
Total scroll for the three directrices is made up of	
pq scrolls each of order $2p^2q$ (viz. one for each value of k), and residuary scroll of order $2p^2(q^3 - q^2)$.	$2pq$ scrolls each of order $2p^2q$ (viz. one for each value of k), and residuary scroll of order $4p^2(q^3 - q^2)$.

29. The following are noticeable cases; $r = 1$ gives the hyperboloid as derived from three directrix lines; $r = -1$ the hyperboloid as derived from three plane sections thereof; $r = 2$, an octic surface, M. De la Gournerie's Quadrispinal; $r = -2$, an octic surface, his Quadricuspidal; $r = \frac{1}{2}$, a sextic surface which (as remarked by Dr Salmon), on writing therein (x^2, y^2, z^2, w^2), in place of (x, y, z, w), is converted into a surface of the twelfth order, locus of the centres of curvature of an ellipsoid.

435.

ON THE SIX COORDINATES OF A LINE.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part II. (1869), pp. 290—323. Read November 11, 1867.]

THE notion of the six coordinates of a line was, so far as I am aware, first established in my paper "On a new analytical representation of Curves in Space," *Quart. Math. Jour.* t. III. (1860), pp. 225—236, [284]; see p. 226, where writing p, q, r, s, t, u for the six determinants of the matrix $\begin{Bmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{Bmatrix}$, I remark that these values give identically $ps+qt+ru=0$; and I consider a cone as represented by a homogeneous equation $V=0$ between the six coordinates (p, q, r, s, t, u); and many of the investigations of the present memoir, in which these coordinates are employed, have been in my possession for some years past. But these coordinates presented themselves independently to Prof. Plücker, and the theory of them is set forth in his most interesting and valuable memoir, "On a new Geometry of Space," *Phil. Trans.* t. CLV. (1865), pp. 725—791; the course of development there given to the theory is however altogether different from that in the present memoir. They have also more recently been made use of in a paper by Herr Lüroth, "Zur Theorie der windschiefen Flächen," *Crelle*, t. LXII. (1867), pp. 130—152.

I have in the present memoir applied these coordinates to the question of the Involution of six lines; the notion of this relation of six lines is due to Prof. Sylvester, to whom it presented itself in the year 1861, in connexion with a theorem in the *Lehrbuch der Statik*, by Möbius (Leipzig, 1837), that if four forces acting on a solid body are in equilibrium the lines along which the forces act are the generating lines of a hyperboloid. Prof. Sylvester was thereby led to consider six lines such that (regarding them as lines in a solid body) there exist along them forces which are in equilibrium; and he thence obtained, by the statical considerations reproduced in the present memoir, the construction (when five of the lines are given) of a sixth line to pass through a given point or to be situate in a given plane.

Article, Nos. 1 to 8. *The Six Coordinates of a Line; definition and general notions.*

1. Using any quadriplanar coordinates (x, y, z, w) whatever, consider a line; on the line two points the coordinates of which are $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ respectively; and through the line two planes, the equations whereof are $(A, B, C, D \between x, y, z, w) = 0$, and $(A', B', C', D' \between x, y, z, w) = 0$ respectively; we have

$$(A, B, C, D \between \alpha, \beta, \gamma, \delta) = 0,$$
$$(A, B, C, D \between \alpha', \beta', \gamma', \delta') = 0,$$
$$(A', B', C', D' \between \alpha, \beta, \gamma, \delta) = 0,$$
$$(A', B', C', D' \between \alpha', \beta', \gamma', \delta') = 0.$$

2. From the first and second equations, eliminating successively A, B, C, D, we find

$$\begin{vmatrix} 0, & \alpha\beta' - \alpha'\beta, & -(\gamma\alpha' - \gamma'\alpha), & \alpha\delta' - \alpha'\delta \\ -(\alpha\beta' - \alpha'\beta), & 0, & \beta\gamma' - \beta'\gamma, & \beta\delta' - \beta'\delta \\ \gamma\alpha' - \gamma'\alpha, & -(\beta\gamma' - \beta'\gamma), & 0, & \gamma\delta' - \gamma'\delta \\ -(\alpha\delta' - \alpha'\delta), & -(\beta\delta' - \beta'\delta), & -(\gamma\delta' - \gamma'\delta), & 0 \end{vmatrix} (A, B, C, D) = 0,$$

and from the third and fourth equations we find the like system with (A', B', C', D') in place of (A, B, C, D). Comparing the corresponding equations of the two systems, we find an equality of ratios, as will presently be mentioned.

3. From the first and third equations, eliminating successively $\alpha, \beta, \gamma, \delta$, we find

$$\begin{vmatrix} 0, & AB' - A'B, & -(CA' - C'A), & AD' - A'D \\ -(AB' - A'B), & 0, & BC' - B'C, & BD' - B'D \\ CA' - C'A, & -(BC' - B'C), & 0, & CD' - C'D \\ -(AD' - A'D), & -(BD' - B'D), & -(CD' - C'D), & 0 \end{vmatrix} (\alpha, \beta, \gamma, \delta) = 0,$$

and from the second and fourth equations we find the like system with $(\alpha', \beta', \gamma', \delta')$ in place of $(\alpha, \beta, \gamma, \delta)$: comparing the corresponding equations of the two systems, we find the same equality of ratios as before, viz.

4. This is

$$\beta\gamma' - \beta'\gamma : \gamma\alpha' - \gamma'\alpha : \alpha\beta' - \alpha'\beta : \alpha\delta' - \alpha'\delta : \beta\delta' - \beta'\delta : \gamma\delta' - \gamma'\delta$$
$$= AD' - A'D : BD' - B'D : CD' - C'D : BC' - B'C : CA' - C'A : AB' - A'B,$$

and putting each of these two equal sets of ratios

$$= a \quad : b \quad : c \quad : f \quad : g \quad : h,$$

then the quantities (a, b, c, f, g, h), which it is easy to see satisfy the condition

$$af + bg + ch = 0,$$

are said to be the 'six coordinates' of the line: as only the ratios of the six quantities are material, and as the last-mentioned equation establishes a single relation between these ratios, the system of the six coordinates contain four arbitrary ratios or parameters, for the determination of the particular line.

5. A line is thus determined by its six coordinates (a, b, c, f, g, h), which are such that $af+bg+ch=0$; and conversely any six quantities (a, b, c, f, g, h) satisfying this relation may be taken to be the six coordinates of a line.

6. It is proper to show that the ratios $a : b : c : f : g : h$ are independent of the particular two points on the line, or two planes through the line, used for their determination. In fact, if instead of the points

$$\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \end{array}$$

we have any other two points on the line, say the points

$$\begin{array}{cccc} \lambda\alpha+\mu\alpha', & \lambda\beta+\mu\beta', & \lambda\gamma+\mu\gamma', & \lambda\delta+\mu\delta', \\ \nu\alpha+\rho\alpha', & \nu\beta+\rho\beta', & \nu\gamma+\rho\gamma', & \nu\delta+\rho\delta', \end{array}$$

then the six determinants have their original values each multiplied by $\lambda\rho-\mu\nu$; and the ratios are unaltered.

And the like is the case, if instead of the planes

$$\begin{array}{cccc} A, & B, & C, & D, \\ A', & B', & C', & D', \end{array}$$

we have any other two planes through the line, say the planes

$$\begin{array}{cccc} \lambda A+\mu A', & \lambda B+\mu B', & \lambda C+\mu C', & \lambda D+\mu D', \\ \nu A+\rho A', & \nu B+\rho B', & \nu C+\rho C', & \nu D+\rho D', \end{array}$$

the determinants have their original values each multiplied by $\lambda\rho-\mu\nu$; and the ratios are unaltered.

7. It may be remarked, that the theory of the six coordinates considered as derived from the two points $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, and as derived from the two planes (A, B, C, D), (A', B', C', D'), is precisely the same in each case; and we may confine ourselves to the first point of view, regarding therefore the six coordinates as derived from the two points $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$. I further remark, that I do not at present in anywise fix the absolute magnitudes of the coordinates $(a, b\ c, f, g, h)$: it is only the ratios that we are concerned with.

8. The values of the ratios $a : b : c : f : g : h$ of the six coordinates do however depend on the particular coordinate planes $x=0$, $y=0$, $z=0$, $w=0$, made use of for their determination; and in the sequel it will be necessary to investigate the

formulæ of transformation to a new set of coordinate planes $x_0=0$, $y_0=0$, $z_0=0$, $w_0=0$. And I shall also show in what manner the absolute magnitudes of the coordinates may be fixed. But deferring the consideration of these questions, I consider the planes $x=0$, $y=0$, $z=0$, $w=0$ as given planes, and take the six coordinates (a, b, c, f, g, h) of a line to be determined as above in reference to these given planes, the absolute values of these coordinates remaining indeterminate, and their ratios only being attended to. And I proceed to consider the various questions which present themselves in the geometry of the line, considered as thus determined by means of its six coordinates (a, b, c, f, g, h).

Article, Nos. 9 to 18. (*Various Sub-headings.*) *Elementary Theorems.*

Condition that a line may be in a given plane.

9. Taking the line to be (a, b, c, f, g, h), the equation of the given plane to be

$$(A, B, C, D \between x, y, z, w)=0;$$

then if $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ are the coordinates of any two points on the line, we have the system of equations *ante*, No. 2, and substituting therein for $\beta\gamma'-\beta'\gamma$, &c. the values (a, b, c, f, g, h), we find

$$\left|\begin{matrix} 0, & c, & -b, & f \\ -c, & 0, & a, & g \\ b, & -a, & 0, & h \\ -f, & -g, & -h, & 0 \end{matrix}\right| (A, B, C, D)=0;$$

which equations, equivalent to a twofold relation, are the required condition. It may be remarked that, treating (A, B, C, D) as current plane coordinates, each equation of the system is that of a point lying in the line.

Condition that a line may pass through a given point.

10. The coordinates of the given point are taken to be $(\alpha, \beta, \gamma, \delta)$. If

$$(A, B, C, D \between x, y, z, w)=0, \quad (A', B', C', D' \between x, y, z, w)=0,$$

are the equations of any two planes through the line, then we have the system of equations *ante* No. 3, and substituting therein for $AB'-A'B$, &c. their values in terms of the coordinates (a, b, c, f, g, h) of the line, we have

$$\left|\begin{matrix} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{matrix}\right| (\alpha, \beta, \gamma, \delta)=0;$$

which equations, equivalent to a twofold relation, are the required condition. It is obvious that, treating $(\alpha, \beta, \gamma, \delta)$ as current point coordinates, each equation of the system is the equation of a plane through the given line.

Condition for the intersection of two lines.

11. The coordinates of the lines are taken to be (a, b, c, f, g, h), and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$, respectively. If $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, are the coordinates of any two points in the first line, and $(\alpha_{,}, \beta_{,}, \gamma_{,}, \delta_{,})$, $(\alpha_{,}', \beta_{,}', \gamma_{,}', \delta_{,}')$, are the coordinates of any two points on the second line, then the four points are in a plane, that is, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ \alpha_{,}, & \beta_{,}, & \gamma_{,}, & \delta_{,} \\ \alpha_{,}', & \beta_{,}', & \gamma_{,}', & \delta_{,}' \end{vmatrix} = 0,$$

that is, expanding the determinant and substituting for $\beta\gamma' - \beta'\gamma$, &c. and $\beta_{,}\gamma_{,}' - \beta_{,}'\gamma_{,}$, &c. their values in terms of the coordinates of the two lines respectively, we have

$$af_{,} + bg_{,} + ch_{,} + fa_{,} + gb_{,} + hc_{,} = 0,$$

or, as this may also be written,

$$(f_{,}, g_{,}, h_{,}, a_{,}, b_{,}, c_{,} \between a, b, c, f, g, h) = 0,$$

for the condition that the two lines may intersect.

12. The same result will be obtained if we take

$$(A, B, C, D \between x, y, z, w) = 0, \quad (A', B', C', D' \between x, y, z, w) = 0,$$

for the equations of any two planes through the first line, and

$$(A_{,}, B_{,}, C_{,}, D_{,} \between x, y, z, w) = 0, \quad (A_{,}', B_{,}', C_{,}', D_{,}' \between x, y, z, w) = 0,$$

for the equations of any two planes through the second line. The four planes will meet in a point, that is, we have

$$\begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A_{,}, & B_{,}, & C_{,}, & D_{,} \\ A_{,}', & B_{,}', & C_{,}', & D_{,}' \end{vmatrix} = 0,$$

or, expanding and substituting, we have the same condition as before.

13. In the case of any two lines (a, b, c, f, g, h), and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$, we may define the 'moment' of the two lines to be the function

$$af_{,} + bg_{,} + ch_{,} + fa_{,} + gb_{,} + hc_{,},$$

it being understood that we have not as yet any complete quantitative definition of the moment; this being so, we have, in what precedes, the theorem that the moment of two intersecting lines is $= 0$.

Plane through two intersecting lines.

14. Let $(A, B, C, D \between x, y, z, w) = 0$ be the equation of the plane through the two intersecting lines (a, b, c, f, g, h) and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$. We have two systems of equations, as in No. 9, and comparing the corresponding equations of the two systems, we find in the first instance

$$\begin{aligned} A : B : C : D = \quad \lambda \quad &: bf_{,} - b_{,}f : cf_{,} - c_{,}f : -(bc_{,} - b_{,}c) \\ = ag_{,} - a_{,}g : \quad \mu \quad &: cg_{,} - c_{,}g : -(ca_{,} - c_{,}a) \\ = ah_{,} - a_{,}h : bh_{,} - b_{,}h : \quad \nu \quad &: -(ab_{,} - a_{,}b) \\ = gh_{,} - g_{,}h : hf_{,} - h_{,}f : fg_{,} - f_{,}g : \quad &\rho \quad , \end{aligned}$$

where λ, μ, ν, ρ, are in the first instance unknown; the different sets of ratios are of course identical in virtue of the relation

$$(f_{,}, g_{,}, h_{,}, a_{,}, b_{,}, c_{,} \between a, b, c, f, g, h) = 0,$$

and comparing them we have equations which lead to the values of λ, μ, ν, ρ; and we thus obtain more completely,

$$\begin{aligned} A : B : C : D = f_{,}a + b_{,}g + c_{,}h &: bf_{,} + b_{,}f \quad : cf_{,} - c_{,}f \quad : -(bc_{,} - b_{,}c) \\ = ag_{,} - a_{,}g \quad &: a_{,}f + g_{,}b + c_{,}h : cg_{,} - c_{,}g \quad : -(ca_{,} - c_{,}a) \\ = ah_{,} - a_{,}h \quad &: bh_{,} - b_{,}h \quad : a_{,}f + b_{,}g + h_{,}c : -(ab_{,} - a_{,}b) \\ = gh_{,} - g_{,}h \quad &: hf_{,} - h_{,}f \quad : fg_{,} - f_{,}g \quad : af_{,} + bg_{,} + ch_{,}. \end{aligned}$$

15. It is in these equations easy to verify the identity of the different sets of values: we ought, for instance, to have

$$\frac{a_{,}f + b_{,}g + h_{,}c}{fg_{,} - f_{,}g} = -\frac{ab_{,} - a_{,}b}{af_{,} + bg_{,} + ch_{,}};$$

that is,

$$(h_{,}c + a_{,}f + b_{,}g)(h_{,}c + af_{,} + bg_{,}) + (ab_{,} - a_{,}b)(fg_{,} - f_{,}g) = 0,$$

and, observing that

$$\begin{aligned} &(a_{,}f + b_{,}g)(af_{,} + bg_{,}) + (ab_{,} - a_{,}b)(fg_{,} - f_{,}g) \\ &\quad = (af + bg)(a_{,}f_{,} + b_{,}g_{,}), = ch \,.\, c_{,}h_{,}, \end{aligned}$$

the left-hand side is

$$\begin{aligned} &= ch_{,}(ch_{,} + af_{,} + bg_{,} + a_{,}f + b_{,}g + c_{,}h), \\ &= ch_{,}(af_{,} + bg_{,} + ch_{,} + fa_{,} + gb_{,} + hc_{,}), = 0. \end{aligned}$$

Point on two intersecting lines.

16. Let $(\alpha, \beta, \gamma, \delta)$ be the coordinates of the point of intersection of the two intersecting lines (a, b, c, f, g, h) and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$. We have two systems of

equations such as in No. 10, and comparing the corresponding equations of the two systems, we find

$$\begin{aligned}
\alpha : \beta : \gamma : \delta = \quad & L & : \; & ag_{,} - a_{,}g & : \; & ah_{,} - a_{,}h & : \; & gh_{,} - g_{,}h \\
= \; & bf_{,} - b_{,}f & : \; & M & : \; & bh_{,} - b_{,}h & : \; & hf_{,} - h_{,}f \\
= \; & cf_{,} - c_{,}f & : \; & cg_{,} - c_{,}g & : \; & N & : \; & fg_{,} - f_{,}g \\
= \; & -(bc_{,} - b_{,}c) & : \; & -(ca_{,} - c_{,}a) & : \; & -(ab_{,} - a_{,}b) & : \; & P \quad ,
\end{aligned}$$

where L, M, N, P, are in the first instance unknown; but, comparing the different sets of values, we have equations for finding the values of these quantities, and we thus obtain the more complete system

$$\begin{aligned}
\alpha : \beta : \gamma : \delta = \quad & f_{,}a + b_{,}g + c_{,}h & : \; & ag_{,} - a_{,}g & : \; & ah_{,} - a_{,}h & : \; & gh_{,} - g_{,}h \\
= \; & bf_{,} - b_{,}f & : \; & a_{,}f + g_{,}b + c_{,}h & : \; & bh_{,} - b_{,}h & : \; & hf_{,} - h_{,}f \\
= \; & cf_{,} - c_{,}f & : \; & cg_{,} - c_{,}g & : \; & a_{,}f + b_{,}g + h_{,}c & : \; & fg_{,} - f_{,}g \\
= \; & -(bc_{,} - b_{,}c) & : \; & -(ca_{,} - c_{,}a) & : \; & -(ab_{,} - a_{,}b) & : \; & f_{,}a + g_{,}b + h_{,}c,
\end{aligned}$$

where it is to be observed that the right-hand side considered as a matrix is the transposed matrix of that which occurs in No. 13, in the formula for $A : B : C : D$. The verification of the identity of the different sets of values can of course be effected as in No. 15.

Expression for an arbitrary plane through a line.

17. The condition in order that the plane $(A, B, C, D \between x, y, z, w) = 0$, may pass through the line (a, b, c, f, g, h), is the twofold relation given, No. 9; it is satisfied by any one of the four systems

$$\begin{aligned}
A : B : C : D \quad &= \; 0 : \; h : \; g : a, \\
\text{or } &= -h : \; 0 : \; f : b, \\
\text{or } &= \; g : -f : \; 0 : c, \\
\text{or } &= -a : -b : -c : 0;
\end{aligned}$$

and consequently also by

$$\begin{aligned}
A : B : C : D = \; & (\; 0, & -h, & \; g, & -a \between \xi, \eta, \zeta, \omega) \\
& : (\; h, & 0, & -f, & -b \between \xi, \eta, \zeta, \omega) \\
& : (-g, & f, & \; 0, & -c \between \xi, \eta, \zeta, \omega) \\
& : (\; a, & b, & \; c, & 0 \between \xi, \eta, \zeta, \omega);
\end{aligned}$$

or, what is the same thing, by

$$\begin{aligned}
A : B : C : D = \; & (\; 0, & h, & -g, & a \between \xi, \eta, \zeta, \omega) \\
& : (-h, & 0, & \; f, & b \between \xi, \eta, \zeta, \omega) \\
& : (\; g, & -f, & \; 0, & c \between \xi, \eta, \zeta, \omega) \\
& : (-a, & -b, & -c, & 0 \between \xi, \eta, \zeta, \omega)
\end{aligned}$$

where $(\xi, \eta, \zeta, \omega)$ are arbitrary: there is, however, no loss of generality in putting any two of these quantities $= 0$.

Expression for an arbitrary point in a line.

18. The condition in order that the point $(\alpha, \beta, \gamma, \delta)$, may lie in the line (a, b, c, f, g, h), is the twofold relation given, No. 10; it is satisfied by any one of the four systems

$$\begin{aligned} \alpha : \beta : \gamma : \delta &= 0 : c : -b : f, \\ \text{or } &= -c : 0 : a : g, \\ \text{or } &= b : -a : 0 : h, \\ \text{or } &= -f : -g : -h : 0; \end{aligned}$$

and consequently, also by

$$\begin{aligned} \alpha : \beta : \gamma : \delta = &(0, -c, b, -f \between x, y, z, w) \\ &: (c, 0, -a, -g \between x, y, z, w) \\ &: (-b, a, 0, -h \between x, y, z, w) \\ &: (f, g, h, 0 \between x, y, z, w); \end{aligned}$$

or, what is the same thing, by

$$\begin{aligned} \alpha : \beta : \gamma : \delta = &(0, c, -b, f \between x, y, z, w) \\ &: (-c, 0, a, g \between x, y, z, w) \\ &: (b, -a, 0, h \between x, y, z, w) \\ &: (-f, -g, -h, 0 \between x, y, z, w) \end{aligned}$$

where (x, y, z, w) are arbitrary: there is, however, no loss of generality in putting two of these quantities $= 0$.

Article Nos. 19 to 25. *Geometrical considerations in regard to three, four, five, and six lines.*

Before proceeding further, I will establish certain geometrical notions in regard to three, four, five, and six lines. I use the term 'tractor' to denote a line which meets any given lines.

19. Three given lines have an infinity of tractors; viz. these are the generating lines of a hyperboloid having the three given lines for directrices.

20. Four given lines may be directrices (generating lines) of the same hyperboloid, viz. every tractor of any three of the four lines is then a tractor of all the four lines. But in general, four given lines have a pair of tractors; viz. considering the tractors of any three of the four lines, these form a hyperboloid having the three lines for directrices; the fourth line meets this hyperboloid in two points, and the generating line through either of these points is a line meeting each of the four given lines, that is, it is a tractor of the four given lines.

21. The fourth line may however touch the hyperboloid; and in this case, instead of a pair of tractors, the four lines have a twofold tractor. The relation of the four lines to each other is a symmetrical one; and we have thence the theorem, that if any one of four given lines touch the hyperboloid through the other three lines, then will each of the four given lines touch the hyperboloid through the other three lines. But the relation to each other of four lines having a twofold tractor may be otherwise expressed as follows; viz. considering a tractor of the four given lines, each line determines with the tractor a point, the intersection of the line and tractor; and it also determines a plane, viz. the plane containing the line and tractor; we have therefore a range of four points on the tractor, and a pencil of four planes through the tractor; and if the tractor be a two-fold tractor, the range and pencil will be homographic; and conversely, if the range and pencil are homographic, the tractor will be a twofold tractor. This is easily obtained as a limiting case from the general one where the four lines have a pair of tractors; each line determines with the one tractor a point and a plane as above, and this plane intersects the second tractor in a point; we have thus through the first tractor a pencil of planes, and on the second tractor a range of points, and these two are homographic. But, in the case of a twofold tractor, the range on the second tractor coincides with that on the first tractor; that is, the range of points on the tractor is homographic with the pencil of planes through the tractor.

22. Given any four lines, and a point O, then either in the general case where the four lines have a pair of tractors, or in the special case where they have a twofold tractor, there exists and can be found through the point O a single fifth line such that the five lines have (as the case may be) a pair of tractors, or a twofold tractor. And similarly, given the four lines and a plane Ω, there exists and can be found in the plane Ω a single fifth line such that the five lines have (as the case may be) a pair of tractors, or a twofold tractor.

23. Five given lines have not in general any tractor; the five lines may be directrices (generating lines) of the same hyperboloid, and they have then an infinity of tractors; or they may have a pair of tractors, viz. the fifth line may be a line meeting the tractors of the other four lines; or (as a particular case of the last relation) the five lines may have a twofold tractor; or the five lines may have a single tractor.

24. Given any five lines and a point O; then, selecting any four of the given lines, we may through O draw a line having with the four lines a pair of tractors. Treating in this manner each of the five sets of four lines, we obtain through the point O five lines constructed as above; we have the theorem which will be proved in the sequel, that these five lines lie in a plane Ω. And similarly, given the five lines, and a plane Ω, then selecting any four of the five lines, we may in the plane Ω draw a line having with the four lines a pair of tractors; treating in this manner each of the five sets of four lines, we obtain in the plane Ω five lines; and we have then the theorem that these five lines meet in a point O.

25. In the case of six given lines, we may have between the lines the like relations to those for the case of five given lines; or we may have the more general relation of the involution of six lines, depending on the last-mentioned theorems, viz. given any five lines, and the point O or the plane Ω, then determining in the one case the plane Ω and in the other case the point O, and taking as a sixth line any line whatever through the point O and in the plane Ω, the six lines are said to be in involution, or to form an involution of six lines. I now revert to the analytical theory of the line.

Article Nos. 26 to 51. (*Various sub-headings.*) *Cases of a linear relation or linear relations between the six Coordinates.*

26. If the coordinates (a, b, c, f, g, h) of a line are regarded as variable quantities connected by a single equation or by two or three equations, we have a system of lines with three or two arbitrary parameters or with a single arbitrary parameter; and so if there are four equations the system consists of a determinate number of lines. For a linear relation, the coefficients may be either (F, G, H, A, B, C), not the coordinates of a line, that is, not satisfying the relation $AF+BG+CH=0$, or they may be the coordinates of a line, satisfying the relation in question. I consider the several cases in order as follows:

Linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h)=0$, *where* (A, B, C, F, G, H) *are not the coordinates of a line.*

27. Considering any six lines which satisfy the relation in question, we may eliminate the coefficients F, G, H, A, B, C, and thus obtain an equation $\nabla=0$, where ∇ is the determinant formed with the coordinates of the six lines; this equation, regarding therein the coordinates of five of the six lines as given, is in regard to the coordinates of the remaining line, say the original line (a, b, c, f, g, h), a linear relation equivalent to the original linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h)=0$. The equation in its new form, viz. the equation $\nabla=0$, establishes between the six lines a relation which is in fact the relation of involution already referred to; viz. it will be shown in the sequel that, starting from the equation $\nabla=0$ as the definition of the relation of involution, we are led to a construction for a line in involution with five given lines the same as the construction explained *ante* No. 25.

Linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h)=0$, *where* (A, B, C, F, G, H) *are the coordinates of a line.*

28. The linear relation expresses that the two lines $(a, b, c, f, g, h \between A, B, C, F, G, H)$ intersect, or what is the same thing, that the line (a, b, c, f, g, h) is any line whatever meeting the line (A, B, C, F, G, H).

Two linear relations $(F, G, H, A, B, C \between a, b, c, f, g, h)=0$,

$$(F_1, G_1, H_1, A_1, B_1, C_1 \between a, b, c, f, g, h)=0,$$

where the two sets of coefficients respectively are or are not the coordinates of a line.

29. If the two sets of coefficients are each of them the coordinates of a line, then the two equations express that the line (a, b, c, f, g, h) is any line whatever cutting each of the two given lines. And the general case is in fact reducible to this particular one; for suppose that neither set of coefficients belongs to a line, then we may from the two given linear relations form the relation

$$(\lambda F+\lambda_1 F_1,\ \lambda G+\lambda_1 G_1,\ \lambda H+\lambda_1 H_1,\ \lambda A+\lambda_1 A_1,\ \lambda B+\lambda_1 B_1,\ \lambda C+\lambda_1 C_1 \between a,\ b,\ c,\ f,\ g,\ h)=0,$$

and if the ratio $\lambda : \lambda_1$ be properly determined, then $(\lambda A+\lambda_1 A_1, \ldots)$ will be the coordinates of a line. This will in fact be the case if

$$(\lambda A+\lambda_1 A)(\lambda F+\lambda_1 F_1)+(\lambda B+\lambda_1 B_1)(\lambda G+\lambda_1 G_1)+(\lambda C+\lambda_1 C_1)(\lambda H+\lambda_1 H_1)=0,$$

that is, if

$$(AF+BG+CH,\ AF_1+BG_1+CH_1+FA_1+GB_1+CH_1,\ A_1F_1+B_1G_1+C_1H_1 \between \lambda,\ \lambda_1)^2=0,$$

a quadric equation giving two values of the ratio $\lambda : \lambda_1$, that is, two linear relations in each of which the coefficients are the coordinates of a line: we have thus two derived lines, and the line (a, b, c, f, g, h) meets each of these derived lines.

There is no real difference if one or the other of the given systems of coefficients, say the system (A, B, C, F, G, H), are the coordinates of a line. We have then $AF+BG+CH=0$; the quadric equation in $\lambda : \lambda_1$ has a root $\lambda_1 : \lambda=0$, and rejecting it, the other root is determined by a simple equation: this only means that the line (A, B, C, F, G, H) is itself one of the two derived lines.

But there is a real difference in the case where the equation in $\lambda : \lambda_1$ has equal roots; to explain this special case, observe that if in the general case we consider the two derived lines as a pair of tractors of any four lines, then the linear relations express that the line (a, b, c, f, g, h) has with these four lines a pair of tractors; and in the special case under consideration the linear relations express that the line (a, b, c, f, g, h) has with the four lines, or (what is the same thing) with any three of them, that is with some three lines, a twofold tractor. According to what precedes (No. 21), the construction of the line (a, b, c, f, g, h) is in fact as follows, viz. if on the twofold tractor considered as given, we take a series of points p, and through the tractor, homographic with the range, a pencil of planes P, then the sought-for line will be any line through a point p, in the corresponding plane P. But it is proper to give an analytical proof of the construction.

30. I observe that we may without loss of generality assume $A_1F_1+B_1G_1+C_1H_1=0$, and this being so, the condition for the equality of the roots of the quadric equation is

$$AF_1+BG_1+CH_1+FA_1+BG_1+CH_1=0,$$

that is, writing $(a_1, b_1, c_1, f_1, g_1, h_1)$ in place of $(A_1, B_1, C_1, F_1, G_1, H_1)$, the case in question may be taken to be that of

Two linear relations

$$(f_1,\ g_1,\ h_1,\ a_1,\ b_1,\ c_1 \between a,\ b,\ c,\ f,\ g,\ h)=0,$$
$$(F,\ G,\ H,\ A,\ B,\ C \between a,\ b,\ c,\ f,\ g,\ h)=0,$$

where $(a_1, b_1, c_1, f_1, g_1, h_1)$ *are,* (A, B, C, F, G, H) *are not, the coordinates of a point, and where*

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between A, B, C, F, G, H) = 0\,;$$

that is, where the twofold derived line is in fact the original line

$$(a_1, b_1, c_1, f_1, g_1, h_1).$$

31. To simplify, we may take $x=0$, $y=0$ for the equations of the line; the coordinates of the line then are $(a_1, b_1, c_1, f_1, g_1, h_1) = (0, 0, 0, 0, 0, 1)$. Taking moreover $x=0$, $y=0$, $\frac{z}{\gamma} = \frac{w}{\delta}$ for the coordinates of the point p, and $\frac{x}{\alpha} = \frac{y}{\beta}$ for the equation of the plane P, the homographic relation of the point and plane is given by an equation of the form

$$-F\beta\gamma + G\alpha\gamma - A\alpha\delta - B\beta\delta = 0,$$

or, as this may be written,

$$(F, G, H, A, B, 0 \between -\beta\gamma, \alpha\gamma, 0, -\alpha\delta, -\beta\delta, \omega) = 0,$$

where H and ω, being each multiplied by 0, do not really enter into the equation.

The equations of any line whatever through the point p and in the plane P may be written $\beta x - \alpha y = 0$, $A'x + B'y + \delta z - \gamma\omega = 0$, where A', B' are arbitrary: hence arranging the coefficients in the order

$$\begin{array}{llll} \beta\,, & -\alpha\,, & 0, & 0, \\ A', & B', & \delta, & -\gamma, \end{array}$$

the coordinates (a, b, c, f, g, h) of the line in question are

$$(-\beta\gamma, \alpha\gamma, 0, -\alpha\delta, -\beta\delta, A'\alpha + B'\beta)\,;$$

so that we have

$$\begin{aligned} &(f_1, g_1, h_1, a_1, b_1, c_1 \between a, b, c, f, g, h) \\ &= (0\,, 0\,, 1\,, 0\,, 0\,, 0 \between a, b, c, f, g, h), = c, = 0\,; \end{aligned}$$

and morever the homographic relation, replacing therein the arbitrary quantity ω by $A'\alpha + B'\beta$, becomes

$$(F, G, H, A, B, 0 \between a, b, c, f, g, h) = 0.$$

Hence the linear relations satisfied by the coordinates (a, b, c, f, g, h) of the line in question are

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between a, b, c, f, g, h) = 0,$$
$$(F, G, H, A, B, C \between a, b, c, f, g, h) = 0,$$

with the coefficients

$$(f_1, g_1, h_1, a_1, b_1, c_1) = (0\,, 0\,, 1, 0\,, 0\,, 0\,),$$
$$(A, B, C, F, G, H) = (A, B, 0, F, G, H),$$

values which satisfy the condition

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between A, B, C, F, G, H) = 0.$$

Hence the line (a, b, c, f, g, h) through the point p and in the plane P is a line the coordinates of which satisfy two linear relations as mentioned in the heading; and the theorem is thus proved. The demonstration would be simplified by taking, as is allowable, the homographic relation to be $\frac{\alpha}{\beta} = k\frac{\gamma}{\delta}$.

32. It appears from the foregoing examination of the case of two linear relations that in the following cases of three or more linear relations there is no real loss of generality in assuming that the coefficients of each set are the coordinates of a line; for if originally this be not so, we have only to replace the given relations by linear functions of these relations, and to assign such values to the multipliers $\lambda, \lambda_1, \lambda_2 \ldots$ as in each case to make the new coefficients to be the coordinates of a line; and as there are two or more arbitrary ratios $\lambda : \lambda_1 : \lambda_2 \ldots$ to be assigned at pleasure and only a single condition to be satisfied, no cases of failure can arise. The remaining cases may consequently be stated in a more simple form.

Three linear relations, the coefficients of each set being the coordinates of a line.

33. The three relations express that the line (a, b, c, f, g, h) meets each of the three given lines; that is, that the line is any generating line of a hyperboloid having the three given lines for directrices.

Four linear relations, the coefficients of each set being the coordinates of a line.

34. The four relations express that the line (a, b, c, f, g, h) meets each of four given lines; or what is the same thing, that the line is a tractor of four given lines. It is to be noticed that the four linear relations serve to express the ratios $a : b : c : f : g : h$ linearly in terms of any one of these ratios, or what is the same thing, to express the several ratios in terms of an arbitrary ratio $u : v$. Substituting the resulting values in the equation

$$af + bg + ch = 0,$$

we have a quadric equation for the determination of the remaining ratio, or of the ratio $u : v$; and then each of the ratios of the coordinates can be expressed rationally in terms of either root of the quadric equation; we thus obtain the coordinates of each of the two tractors of the four given lines; or we have a complete analytical solution of the problem, to find the tractors of four given lines. The quadric equation may have equal roots; that is, the four given lines may have a twofold tractor, which is then determined linearly.

35. The theory of the linear relations of the coordinates (a, b, c, f, g, h) of a line may be considered in a different manner. It will be convenient to take the different cases in a reverse order, beginning with the extreme case (not before mentioned) of a fivefold relation and ascending to the case of a onefold or single relation.

Case of the fivefold relation.

36. The fivefold relation

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \end{matrix} \right\| = 0,$$

expresses that the quantities (a, b, c, f, g, h) are proportional to $(a_1, b_1, c_1, f_1, g_1, h_1)$. As the former set are by hypothesis the coordinates of a line, the given set $(a_1, b_1, c_1, f_1, g_1, h_1)$ must, it is clear, also be the coordinates of a line, and the relation then expresses that the line (a, b, c, f, g, h) coincides with the given line.

Case of the fourfold relation.

37. The fourfold relation is

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \end{matrix} \right\| = 0,$$

or what is the same thing, we have the six equations $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0$, &c., involving the indeterminate quantities λ, λ_1, λ_2. If the coefficients

$$(a_1, b_1, c_1, f_1, g_1, h_1), \quad (a_2, b_2, c_2, f_2, g_2, h_2)$$

are not either set the coordinates of a line; then substituting the foregoing values $-\lambda a = \lambda_1 a_1 + \lambda_2 a_2$, &c. in the equation $af + bg + ch = 0$, we have a quadric equation in $(\lambda_1 : \lambda_2)$: and for each root of this equation, the coefficients $\lambda_1 a_1 + \lambda_2 a_2$, &c. will be the coordinates of a line. There are thus in general two derived lines; and the fourfold relation expresses that the line (a, b, c, f, g, h) coincides with one or other of these derived lines. There is no real difference if one or the other of the two sets $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, or if each set, are the coordinates of a line; one of the derived lines or both of them will in these cases coincide with one or both of the given lines. And if the quadric equation has equal roots, then instead of two derived lines there is a twofold derived line, and the line (a, b, c, f, g, h) must coincide with this twofold line.

38. A case presenting peculiarity is however that in which the coefficients of the quadric equation vanish identically; this is only so when the coefficients $(a_1, b_1, c_1, f_1, g_1, h_1)$ and $(a_2, b_2, c_2, f_2, g_2, h_2)$ are the coordinates of two intersecting lines. The equations $-\lambda a = \lambda_1 a_1 + \lambda_2 a_2$, &c. here show that every line whatever which meets each of the two lines $(a_1, b_1, c_1, f_1, g_1, h_1)$ and $(a_2, b_2, c_2, f_2, g_2, h_2)$ meets also the line (a, b, c, f, g, h); that is, the line (a, b, c, f, g, h) is any line whatever in the plane and through the point of intersection of the two intersecting lines. We see moreover that not only

$$a_1 f_2 + b_1 g_2 + c_1 h_2 + a_2 f_1 + b_2 g_1 + c_2 h_1 = 0,$$

but also that $af_1 + bg_1 + ch_1 + fa_1 + gb_1 + ch_1 = 0$ and $af_2 + bg_2 + ch_2 + fa_2 + gb_2 + ch_2 = 0$; that is, the moment of each pair of lines is $= 0$. It may be remarked that the ratios $\lambda : \lambda_1 : \lambda_2$ may be determined from any two of the six equations

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0, \ldots \lambda h + \lambda_1 h_1 + \lambda_2 h_2 = 0;$$

but that in consequence of the moments being each $= 0$, there is not for the determination of these ratios any such set of equations as occur in the cases subsequently considered of a threefold relation, &c.

39. In what follows we have three or more sets $(a_1, b_1, c_1, f_1, g_1, h_1)$, &c.; and we may without loss of generality assume that each of these are the coordinates of a line: for replacing the several coefficients $a_1, \ldots$ by linear functions $\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 +$ &c., &c., the multipliers may be determined so that these are the coordinates of a point: and since for each set there is only a single condition to be satisfied by the two or more ratios $\mu_1 : \mu_2 : \mu_3 \ldots$, it is easy to see that no cases of failure will arise.

Case of the threefold relation.

40. The threefold relation is

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \end{matrix} \right\| = 0,$$

where $(a_1, \ldots)$, $(a_2, \ldots)(a_3, \ldots)$ are each the coordinates of a line. Here writing

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0 \ldots,$$

it is clear that every line which meets each of the lines $(a_1, \ldots)$, $(a_2, \ldots)$, $(a_3, \ldots)$ will also meet the line (a, b, c, f, g, h); the lines which meet the first-mentioned three lines are the generating lines of a hyperboloid having these three lines for directrices, and it hence appears that the line (a, b, c, f, g, h) is any directrix line whatever of the hyperboloid in question.

41. Using the notations 01, 02, 12, &c. to denote the moments of the several pairs of lines, viz.

$$01 = a f_1 + b g_1 + c h_1 + f a_1 + g b_1 + h c_1,$$
$$12 = a_1 f_2 + b_1 g_2 + c_1 h_2 + f_1 a_2 + g_1 b_2 + h_1 c_2,$$
&c.,

then from the equations $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$, &c., we deduce

$$\begin{array}{rcrcrcrcl} . & & \lambda_1 01 & + & \lambda_2 02 & + & \lambda_3 03 & = & 0, \\ \lambda 10 & + & . & + & \lambda_2 12 & + & \lambda_3 13 & = & 0, \\ \lambda 20 & + & \lambda_1 21 & & . & + & \lambda_3 23 & = & 0, \\ \lambda 30 & + & \lambda_1 31 & + & \lambda_2 32 & & . & = & 0, \end{array}$$

and hence eliminating λ, λ_1, λ_2, λ_3, we find

$$\begin{vmatrix} . & 01, & 02, & 03 \\ 10, & . & 12, & 13 \\ 20, & 21, & . & 23 \\ 30, & 31, & 32, & . \end{vmatrix} = 0,$$

a relation between the moments satisfied in virtue of the given threefold relation; but which as a mere onefold relation is of course not equivalent to the threefold relation. It will subsequently appear that the equation expresses that any one of the four lines, say the line (a, b, c, f, g, h) touches the hyperboloid having the other three lines for generatrices; this condition is satisfied in virtue of the threefold relation which, as we have seen, expresses that the line (a, b, c, f, g, h) lies wholly in the hyperboloid in question.

42. The last mentioned determinant is the Norm of

$$\sqrt{01 \,.\, 23} + \sqrt{02 \,.\, 31} + \sqrt{03 \,.\, 12};$$

so that the equation may be written

$$\sqrt{01 \,.\, 23} + \sqrt{02 \,.\, 31} + \sqrt{03 \,.\, 12} = 0,$$

or, what is the same thing,

$$\sqrt{01}\sqrt{23} + \sqrt{02}\sqrt{31} + \sqrt{03}\sqrt{12} = 0,$$

it being of course understood that the signs of the radicals must be determined in accordance with this equation; we then find

$$\lambda : \lambda_1 : \lambda_2 : \lambda_3 = \sqrt{23 \,.\, 31 \,.\, 12} : \sqrt{02 \,.\, 03 \,.\, 23} : \sqrt{03 \,.\, 01 \,.\, 31} : \sqrt{01 \,.\, 02 \,.\, 12},$$

or say

$$= \sqrt{23}\sqrt{31}\sqrt{12} : \sqrt{02}\sqrt{03}\sqrt{23} : \sqrt{03}\sqrt{01}\sqrt{31} : \sqrt{01}\sqrt{02}\sqrt{12};$$

in fact, substituting these last values in the linear equations for λ, λ_1, λ_2, λ_3, we find that the equations are all satisfied in virtue of the single equation

$$\sqrt{01}\sqrt{23} + \sqrt{02}\sqrt{31} + \sqrt{03}\sqrt{12} = 0.$$

Case of the twofold relation.

43. We have here

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \\ a_4, & b_4, & c_4, & f_4, & g_4, & h_4 \end{matrix} \right\| = 0,$$

where $(a_1, \ldots)(a_2, \ldots)(a_3, \ldots)(a_4, \ldots)$, are each the coordinates of a line. Here, writing

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0;$$

it is clear that every line which meets each of the four given lines, will also meet the line (a, b, c, f, g, h); but the only lines meeting the four given lines are two determinate lines, the tractors of the four given lines; and the conclusion is, that the line (a, b, c, f, g, h) is any line whatever which meets the two tractors.

44. If, however, the four given lines have a twofold tractor, then the line (a, b, c, f, g, h) is still a line having two conditions imposed upon it; it is in fact a line determined as in No. 21, viz. if on the tractor we take a series of points p, and through the tractor a series of planes P, corresponding homographically to the points, then the line (a, b, c, f, g, h) is any line through a point p, in the corresponding plane P.

45. Using as before 01, 02, ... 12, &c. to denote the moments of the several pairs of lines, we have

$$\begin{aligned}
&\quad . \quad + \lambda_1 01 + \lambda_2 02 + \lambda_3 03 + \lambda_4 04 = 0,\\
&\lambda 10 \quad . \quad + \lambda_2 12 + \lambda_3 13 + \lambda_4 14 = 0,\\
&\lambda 20 + \lambda_1 21 \quad . \quad + \lambda_3 23 + \lambda_4 24 = 0,\\
&\lambda 30 + \lambda_1 31 + \lambda_2 32 \quad . \quad + \lambda_4 34 = 0,\\
&\lambda 40 + \lambda_1 41 + \lambda_2 42 + \lambda_3 43 \quad . \quad = 0,
\end{aligned}$$

and thence also

$$\begin{vmatrix}
. & 01, & 02, & 03, & 04\\
10, & . & 12, & 13, & 14\\
20, & 21, & . & 23, & 24\\
30, & 31, & 32, & . & 34\\
40, & 41, & 42, & 43, & .
\end{vmatrix} = 0,$$

a relation between the moments satisfied in virtue of the original twofold relation; but which, as a single equation, is of course not equivalent to the twofold relation. It is in fact easy to see that this equation expresses that the five lines have a common tractor; this is true, since in virtue of the twofold relation there are really two common tractors.

I have not obtained from the linear equations any symmetrical expressions for the ratios $\lambda : \lambda_1 : \lambda_2 : \lambda_3$.

Case of a onefold relation.

46. The onefold relation is

$$\begin{vmatrix}
a, & b, & c, & f, & g, & h\\
a_1, & b_1, & c_1, & f_1, & g_1, & h_1\\
a_2, & b_2, & c_2, & f_2, & g_2, & h_2\\
a_3, & b_3, & c_3, & f_3, & g_3, & h_3\\
a_4, & b_4, & c_4, & f_4, & g_4, & h_4\\
a_5, & b_5, & c_5, & f_5, & g_5, & h_5
\end{vmatrix} = 0,$$

where $(a_1, \ldots)$, $(a_2, \ldots)$, $(a_3, \ldots)$, $(a_4, \ldots)$, $(a_5, \ldots)$, are each the coordinates of points in a line. The preceding mode of dealing with the question is inapplicable, since there is not in general any line which meets the five given lines; in the particular case, however, where the five given lines are met by a single line, say when they have a common tractor, then the line (a, b, c, f, g, h) is any line meeting this common tractor. The general case is that of the involution of six lines, mentioned No. 25, and the consideration of which was deferred.

47. The onefold relation implies that we can find multipliers $\lambda, \mu, \nu, \rho, \sigma, \tau$, such that

$$\begin{aligned} &\lambda a + \mu b + \nu c + \rho f + \sigma g + \tau h = 0, \\ &\lambda a_1 + \mu b_1 + \nu c_1 + \rho f_1 + \sigma g_1 + \tau h_1 = 0, \\ &\quad\vdots \\ &\lambda a_5 + \mu b_5 + \nu c_5 + \rho f_5 + \sigma g_5 + \tau h_5 = 0, \end{aligned}$$

we may by means of the last five equations determine the ratios of $\lambda, \mu, \nu, \rho, \sigma, \tau$, viz. these quantities will be proportional to the determinants formed out of the matrix

$$\left| \begin{matrix} a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \\ a_4, & b_4, & c_4, & f_4, & g_4, & h_4 \\ a_5, & b_5, & c_5, & f_5, & g_5, & h_5 \end{matrix} \right|$$

and the first equation is then a linear relation in (a, b, c, f, g, h), expressing the relation that exists between these coordinates.

48. Consider an arbitrary point O on the line (a, b, c, f, g, h); taking this point as origin, the coordinates of O are 0, 0, 0, 1; and if x, y, z, w, are the coordinates of any other point on the line, then writing

$$\begin{matrix} x, & y, & z, & w, \\ 0, & 0, & 0, & 1, \end{matrix}$$

we find

$$a : b : c : f : g : h = 0 : 0 : 0 : x : y : z;$$

and the equation

$$\lambda a + \mu b + \nu c + \rho f + \sigma g + \tau h = 0$$

becomes simply $\rho x + \sigma y + \tau z = 0$; viz. this equation expresses that the line (a, b, c, f, g, h), assumed to pass through a given point O, lies in a determinate plane Ω through this point.

49. To construct this plane Ω, I consider any four of the five given lines, say the lines 2, 3, 4, 5, and I endeavour to find the line OQ_1 through O, which has with these lines a pair of tractors; *quà* line through O, the coordinates of the line in question may be taken to be 0, 0, 0, F_1, G_1, H_1 (where F_1, G_1, H_1, are in fact the

coordinates x, y, z, of any point on the line OQ_1); and then the condition for the pair of tractors may be written

$$\begin{aligned}
p_2a_2 + p_3a_3 + p_4a_4 + p_5a_5 &= 0,\\
p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5 &= 0,\\
p_2c_2 + p_3c_3 + p_4c_4 + p_5c_5 &= 0,\\
p_2f_2 + p_3f_3 + p_4f_4 + p_5f_5 &= F_1,\\
p_2g_2 + p_3g_3 + p_4g_4 + p_5g_5 &= G_1,\\
p_2h_2 + p_3h_3 + p_4h_4 + p_5h_5 &= H_1,
\end{aligned}$$

where p_2, p_3... are arbitrary coefficients; and we hence deduce

$$\rho F_1 + \sigma G_1 + \tau H_1 = 0;$$

but in precisely the same way, if the line OQ_2 have with the lines 1, 3, 4, 5, a pair of tractors, and if F_2, G_2, H_2, be the coordinates of a point on the line OQ_2, and similarly for the lines OQ_3, OQ_4, OQ_5, and the coordinates (F_3, G_3, H_3), (F_4, G_4, H_4) (F_5, G_5, H_5), we have

$$\begin{aligned}
\rho F_2 + \sigma G_2 + \tau H_2 &= 0,\\
\rho F_3 + \sigma G_3 + \tau H_3 &= 0,\\
\rho F_4 + \sigma G_4 + \tau H_4 &= 0,\\
\rho F_5 + \sigma G_5 + \tau H_5 &= 0,
\end{aligned}$$

and these equations show that the five lines OQ_1, OQ_2, OQ_3, OQ_4, OQ_5, lie in the plane

$$\rho x + \sigma y + \tau z = 0;$$

so that this plane is given as the plane through the lines OQ_1, OQ_2, OQ_3, OQ_4, OQ_5; and we have thus (given the lines 1, 2, 3, 4, 5, and the arbitrary point O) the construction of the line (a, b, c, f, g, h) through O in involution with the given lines.

50. The original onefold relation may be replaced by the six equations

$$\begin{aligned}
\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 + \lambda_5 a_5 &= 0,\\
\lambda b + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 + \lambda_5 b_5 &= 0,\\
&\vdots\\
\lambda h + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 + \lambda_4 h_4 + \lambda_5 h_5 &= 0,
\end{aligned}$$

and hence denoting as before the moments by 01, 02, 12, &c. we have

$$\begin{aligned}
\cdot \quad \lambda_1 01 + \lambda_2 02 + \lambda_3 03 + \lambda_4 04 + \lambda_5 05 &= 0,\\
\lambda 10 \quad \cdot \quad + \lambda_2 12 + \lambda_3 13 + \lambda_4 14 + \lambda_5 15 &= 0,\\
\lambda 20 + \lambda_1 21 \quad \cdot \quad + \lambda_3 23 + \lambda_4 24 + \lambda_5 25 &= 0,\\
\lambda 30 + \lambda_1 31 + \lambda_2 32 \quad \cdot \quad + \lambda_4 34 + \lambda_5 35 &= 0,\\
\lambda 40 + \lambda_1 41 + \lambda_2 42 + \lambda_3 43 \quad \cdot \quad + \lambda_5 45 &= 0,\\
\lambda 50 + \lambda_1 51 + \lambda_2 52 + \lambda_3 53 + \lambda_4 54 \quad \cdot \quad &= 0,
\end{aligned}$$

which lead to

$$\begin{vmatrix} . & 01, & 02, & 03, & 04, & 05 \\ 10, & . & 12, & 13, & 14, & 15 \\ 20, & 21, & . & 23, & 24, & 25 \\ 30, & 31, & 32, & . & 34, & 35 \\ 40, & 41, & 42, & 43, & . & 45 \\ 50, & 51, & 52, & 53, & 54, & . \end{vmatrix} = 0,$$

a relation between the moments equivalent to the original onefold relation, and consequently expressing that the six lines are in involution. I have not obtained a symmetrical system of values for the ratios $\lambda : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5$.

51. Reverting to the relation which exists between the point O and the plane Ω, it is proper to remark that, since to any given point O there corresponds a single plane Ω, and to any given plane Ω a single point O, it follows that the point O and plane Ω are reciprocal figures; viz. they are reciprocals of the particular kind treated of by Möbius, wherein the reciprocal of a point is a plane through the point, and the reciprocal of a plane a point in the plane; and of which the analytical character is that the reciprocal of the point $(\alpha, \beta, \gamma, \delta)$ is the plane

$$\begin{aligned} &(\quad . \quad h\beta - g\gamma + l\delta)\,x \\ &+(-h\alpha \quad . \quad + f\gamma + m\delta)\,y \\ &+(\ g\alpha - f\beta \quad . \quad + n\delta)\,z \\ &+(-l\alpha - m\beta - n\gamma \quad . \quad)\,w = 0. \end{aligned}$$

Article No. 52. *A geometrical property of an involution of six lines.*

52. The figure of six lines in involution is connected in various ways with the theory of cubic curves in space, for instance, considering a point A of the curve, this determines with any given line l a plane meeting the curve in two other points, and the line λ which joins these two points may be called the projection of the line l. This being so, if in any osculating plane of the cubic we have six lines, $l, l_1, l_2, l_3, l_4, l_5$, tangents of a conic in that plane, the six projections $\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ of these tangents will be a set of lines in involution. I do not stop to prove this theorem or to develope any of its consequences.

Article No. 53. *To find the condition that four given lines may have a twofold tractor.*

53. Taking the coordinates of the given lines to be

(a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$

then if (A, B, C, F, G, H) be the coordinates of a tractor of these lines, we have

$$\begin{aligned}
&(F, G, H, A, B, C \between a, b, c, f, g, h) = 0,\\
&(F, G, H, A, B, C \between a_1, b_1, c_1, f_1, g_1, h_1) = 0,\\
&(F, G, H, A, B, C \between a_2, b_2, c_2, f_2, g_2, h_2) = 0,\\
&(F, G, H, A, B, C \between a_3, b_3, c_3, f_3, g_3, h_3) = 0.
\end{aligned}$$

In virtue of these relations the ratios $A : B : C : F : G : H$ are given linear functions of any one of these ratios or of an arbitrary ratio $u : v$; and we then have $AF + BG + CH = 0$, a quadric equation for determining the unknown ratio. In the case of a twofold tractor, this equation must have equal roots; whence employing as usual the method of indeterminate multipliers, we find

$$\begin{aligned}
A + \lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 &= 0,\\
B + \lambda b + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 &= 0,\\
C + \lambda c + \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 &= 0,\\
F + \lambda f + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 &= 0,\\
G + \lambda g + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 &= 0,\\
H + \lambda h + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 &= 0.
\end{aligned}$$

Hence representing as before the moments of the pairs of lines by 01, 02, &c., we deduce

$$\begin{aligned}
\cdot \quad \lambda_1 01 + \lambda_2 02 + \lambda_3 03 &= 0,\\
\lambda 10 + \quad \cdot \quad + \lambda_2 12 + \lambda_3 13 &= 0,\\
\lambda 20 + \lambda_1 21 \quad \cdot \quad + \lambda_3 23 &= 0,\\
\lambda 30 + \lambda_1 31 + \lambda_2 32 \quad \cdot \quad &= 0,
\end{aligned}$$

so that, as already mentioned, we have

$$\begin{vmatrix}
\cdot, & 01, & 02, & 03\\
10, & \cdot, & 12, & 13\\
20, & 21, & \cdot, & 23\\
30, & 31, & 32, & \cdot
\end{vmatrix} = 0,$$

as the condition that the four given lines may have a twofold tractor.

Article Nos. 54 to 56. *Hyperboloid passing through three given lines.*

54. The direct investigation is somewhat tedious; but I write down, and will afterwards verify, the equation of the hyperboloid passing through the three given lines

$$(a_1, b_1, c_1, f_1, g_1, h_1),\ (a_2, b_2, c_2, f_2, g_2, h_2),\ (a_3, b_3, c_3, f_3, g_3, h_3).$$

Writing for shortness (agh), &c. to denote the determinants

$$\begin{vmatrix}
a_1, & g_1, & h_1\\
a_2, & g_2, & h_2\\
a_3, & g_3, & h_3
\end{vmatrix}, \text{ \&c.}$$

the equation of the hyperboloid is

$$\begin{aligned}
&(agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 \\
&+ [(abg) - (cah)]\,xw + [(bfg) + (chf)]\,yz \\
&+ [(bch) - (abf)]\,yw + [(cgh) + (afg)]\,zx \\
&+ [(caf) - (bcg)]\,zw + [(ahf) + (bgh)]\,xy = 0.
\end{aligned}$$

In fact, we have

$$\begin{aligned}
(agh) &= a_1(g_2h_3 - g_3h_2) + g_1(h_2a_3 - h_3a_2) + h_1(a_2g_3 - a_3g_2) \\
&= a\,.\,gh + g\,.\,ha + h\,.\,ag,
\end{aligned}$$

where a, &c. stand for a_1, &c. and gh, &c. for $g_2h_3 - g_3h_2$, &c. Hence the foregoing equation may be written

$$\begin{aligned}
&x^2\,(a\,.\,gh + g\,.\,ha + h\,.\,ag) \\
&+ y^2\,(b\,.\,hf + h\,.\,fb + f\,.\,bh) \\
&+ z^2\,(c\,.\,fg + f\,.\,gc + g\,.\,cf) \\
&+ w^2\,(a\,.\,bc + b\,.\,ca + c\,.\,ab) \\
&+ xw\begin{pmatrix} a\,.\,bg + b\,.\,ga + g\,.\,ab \\ -c\,.\,ah - a\,.\,hc - h\,.\,ca \end{pmatrix} + yz\begin{pmatrix} b\,.\,fg + f\,.\,gb + g\,.\,bf \\ +c\,.\,hf + h\,.\,fc + f\,.\,ch \end{pmatrix} \\
&+ yw\begin{pmatrix} b\,.\,ch + c\,.\,hb - h\,.\,bc \\ -a\,.\,bf - b\,.\,fa - f\,.\,ab \end{pmatrix} + zx\begin{pmatrix} c\,.\,gh + g\,.\,hc + h\,.\,cg \\ +a\,.\,fg + f\,.\,ga + g\,.\,af \end{pmatrix} \\
&+ zw\begin{pmatrix} c\,.\,af + a\,.\,fc + f\,.\,ca \\ -b\,.\,cg - c\,.\,gb - g\,.\,bc \end{pmatrix} + xy\begin{pmatrix} a\,.\,hf + h\,.\,fa + f\,.\,ch \\ +b\,.\,gh + g\,.\,hb + h\,.\,bg \end{pmatrix} = 0.
\end{aligned}$$

55. This is

$$\begin{aligned}
&bc\,.\,w\,(hy - gz + aw) \\
&+ ca\,.\,w\,(-hx + fz + bw) \\
&+ ab\,.\,w\,(gx - fy + cw) \\
&+ gh\,.\,x\,(ax + by + cz) \\
&+ hf\,.\,y\,(ax + by + cz) \\
&+ fg\,.\,z\,(ax + by + cz) \\
&+ af\,\{w\,(ax + by + cz) - x\,(hy - gz + aw)\} \\
&+ bg\,\{w\,(ax + by + cz) - y\,(-hx + fz + bw)\} \\
&+ ch\,\{w\,(ax + by + cz) - z\,(gx - fy + cw)\} \\
&- bf\,.\,y\,(hy - gz + aw) \\
&- cf\,.\,z\,(hy - gz + aw) \\
&- cg\,.\,z\,(-hx + fz + bw) \\
&- ag\,.\,x\,(-hx + fz + bw) \\
&- ah\,.\,x\,(gx - fy + cw) \\
&- bh\,.\,y\,(gx - fy + cw) = 0.
\end{aligned}$$

56. Hence writing

$$(X,\ Y,\ Z,\ W)=\left|\begin{array}{cccc} .\ , & h, & -g, & a, \\ -h, & .\ , & f, & b, \\ g, & -f, & .\ , & c, \\ -a, & -b, & -c, & . \end{array}\right|(x,\ y,\ z,\ w),$$

the foregoing equation is

$$\begin{aligned} & bc\,.\,wX + ca\,.\,wY + ab\,.\,wZ - gh\,.\,xW - hf\,.\,yW - fg\,.\,zW \\ & - af(wW+xX) \quad - bg\,(wW+yY) \quad - ch\,(wW+zZ) \\ & - bf\,.\,yX - cg\,.\,zY - ah\,.\,xZ - cf\,.\,zX - ag\,.\,xY - bh\,.\,yZ = 0\,; \end{aligned}$$

or, collecting and arranging, this is

$$\begin{aligned} & X\,\{-af\,.\,x - bf\,.\,y - cf\,.\,z + bc\,.\,w\} \\ & + Y\,\{-ag\,.\,x - bg\,.\,y - cg\,.\,z + ca\,.\,w\} \\ & + Z\,\{-ah\,.\,x - bh\,.\,y - ch\,.\,z + ab\,.\,w\} \\ & + W\,\{-gh\,.\,x - hf\,.\,y - fg\,.\,z + (af.+bg.+ch.)\,w\} = 0, \end{aligned}$$

which is satisfied by $X=0$, $Y=0$, $Z=0$, $W=0$; that is, since (a, b, c, f, g, h) have been written in place of $(a_1, b_1, c_1, f_1, g_1, h_1)$, by $X_1=0$, $Y_1=0$, $Z_1=0$, $W_1=0$ (if we thus denote the corresponding functions of $(a_1, b_1, c_1, f_1, g_1, h_1)$), that is, the hyperboloid passes through the line $(a_1, b_1, c_1, f_1, g_1, h_1)$; and similarly it passes through the other two lines.

Article Nos. 57 and 58. *The six coordinates defined as to their absolute magnitudes.*

57. In all that precedes, the absolute magnitudes of the coordinates have been left indeterminate, only the ratios being attended to. But the magnitudes of the six coordinates may be fixed in a very simple manner as follows; viz. using ordinary rectangular coordinates, then for any line, if x_0, y_0, z_0 are the coordinates of a particular point on this line, and α, β, γ the inclinations of the line to the axes, the coordinates of another point on the line are

$$x_0 + r\cos\alpha,\ y_0 + r\cos\beta,\ z_0 + r\cos\gamma\,;$$

and hence writing

$$\begin{array}{lll} x_0 + r\cos\alpha, & y_0 + r\cos\beta, & z_0 + r\cos\gamma,\ 1, \\ x_0 \qquad\quad , & y_0 \qquad\quad , & z_0 \qquad\quad ,\ 1, \end{array}$$

we have

$$a : b : c : f : g : h = z_0\cos\beta - y_0\cos\gamma : x_0\cos\gamma - z_0\cos\alpha : y_0\cos\beta - x_0\cos\alpha : \cos\alpha : \cos\beta : \cos\gamma.$$

Or we may take

$$\begin{aligned} a &= z_0\cos\beta - y_0\cos\gamma, & f &= \cos\alpha, \\ b &= x_0\cos\gamma - z_0\cos\alpha, & g &= \cos\beta, \\ c &= y_0\cos\alpha - x_0\cos\beta, & h &= \cos\gamma, \end{aligned}$$

values which of course satisfy, as they should do, the relation $af+bg+ch=0$. It is hardly necessary to remark, that the values of a, b, c are not altered on substituting for x_0, y_0, z_0 the coordinates $x_0+s\cos\alpha$, $y_0+s\cos\beta$, $z_0+s\cos\gamma$ of any other point on the line.

58. Considering any two lines (a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$, if we define the moment of the two lines to be the product of the perpendicular distance into the sine of the inclination of the two lines, then we have, Moment

$$=af_1+bg_1+ch_1+fa_1+gb_1+hc_1,$$

viz. we have now a quantitative definition of the function of the coordinates previously called the moment of the two lines.

For the demonstration of this formula it is to be remarked, that taking on the first line a segment of the length r, the coordinates of its extremities being (x_0, y_0, z_0) and $(x_0+r\cos\alpha,\ y_0+r\cos\beta,\ z_0+r\cos\gamma)$,

and on the second line a segment of the length r_1 the coordinates of its extremities being (x_0', y_0', z_0') and $(x_0'+r_1\cos\alpha_1,\ y_0'+r_1\cos\beta_1,\ z_0'+r_1\cos\gamma_1)$ and joining the extremities of these segments so as to form a tetrahedron, the volume of the tetrahedron is

$$=\tfrac{1}{6}rr_1(af_1+bg_1+ch_1+fa_1+gb_1+hc_1).$$

But the volume of the tetrahedron is also equal to $\frac{1}{6}$ of the product of the opposite edges into their perpendicular distance into the sine of the inclination of the two edges [1]; that is, it is $=\frac{1}{6}rr_1$ into the moment of the two lines, and we have thus the formula in question.

Article Nos. 59 to 75. *Statical and Kinematical Applications.*

The coordinates (a, b, c, f, g, h), as last defined, are peculiarly convenient in kinematical and mechanical questions, as will appear from the following investigations.

59. Using the term rotation to denote an infinitesimal rotation, I say first that a rotation λ round the line (a, b, c, f, g, h) produces in the point (x, y, z) rigidly connected with this line the displacements

$$\begin{aligned}\delta x&=\lambda(\ \ .\ \ -hy+gz-a),\\ \delta y&=\lambda(\ \ hx\ \ .\ \ -fz-b),\\ \delta z&=\lambda(-gx+fy\ \ .\ \ -c).\end{aligned}$$

[1] I take the opportunity of mentioning a very simple demonstration of this formula: taking the opposite edges to be r, r_1, their inclination $=\theta$, and perpendicular distance $=h$; the section of the tetrahedron by a plane parallel to the two edges at the distances z, $h-z$ from the two edges respectively is a parallelogram, the sides of which are $\frac{r(h-z)}{h}$ and $\frac{r_1 z}{h}$ respectively, and their inclination is $=\theta$; the area of the section is therefore $\frac{rr_1}{h^2}\sin\theta\,.\,z(h-z)$ and the volume of the tetrahedron is $=\frac{rr_1}{h^2}\sin\theta\int_0^h z(h-z)\,dz$, $=\frac{1}{6}rr_1 h\sin\theta$. The same result is however obtained still more simply by drawing a plane through one of the two edges perpendicular to the other edge; the volume is then equal to the sum or the difference of the volumes of two tetrahedra standing on a common triangular base; and the required result at once follows.

In fact assuming for a moment that the axis of rotation passes through the origin, then for the point P coordinates x, y, z, the square of the perpendicular distance from the axis is

$$\begin{aligned} &(\ .\qquad -y\cos\gamma + z\cos\beta)^2 \\ +&(\ x\cos\gamma\quad .\qquad -z\cos\alpha\ )^2 \\ +&(-x\cos\beta + y\cos\alpha\quad .\qquad)^2, \end{aligned}$$

and the expressions which enter into this formula denote as follows; viz. if through the point P at right angles to the plane through P and the axis of rotation we draw a line PQ, $=$perpendicular distance of P from the axis of rotation, then the coordinates of Q referred to P as origin are

$$\begin{aligned} &.\qquad -y\cos\gamma + z\cos\beta, \\ &x\cos\gamma\quad .\qquad -z\cos\alpha, \\ -&x\cos\beta + y\cos\alpha\quad .\qquad , \end{aligned}$$

respectively. Hence the foregoing quantities each multiplied by λ are the displacements of the point P in the directions of the axes, produced by the rotation λ.

60. Suppose that the axis of rotation (instead of passing through the origin) pass through the point (x_0, y_0, z_0); the only difference is that we must in the formula write $(x-x_0, y-y_0, z-z_0)$ in place of (x, y, z): and attending to the significations of the six coordinates, it thus appears that the displacements produced by the rotation are equal to λ into the expressions

$$\begin{aligned} &.\quad -hy + gz - a, \\ &hx\quad .\quad -fz - b, \\ -&gx + fy\quad .\quad -c, \end{aligned}$$

respectively; which is the theorem in question.

61. I say secondly that considering in a solid body the point (x, y, z) situate in the line (a, b, c, f, g, h), and writing

$$a,\ b,\ c,\ f,\ g,\ h = z\cos\beta - y\cos\gamma,\ x\cos\gamma - z\cos\alpha,\ y\cos\alpha - x\cos\beta,\ \cos\alpha,\ \cos\beta,\ \cos\gamma,$$

then for any infinitesimal motion of the solid body the displacement of the point in the direction of the line is

$$= ap + bq + cr + fl + gm + hn,$$

where p, q, r, l, m, n are constants depending on the infinitesimal motion.

In fact for any infinitesimal motion of a solid body the displacements of the point (x, y, z) are

$$\begin{aligned} \delta x &= l\quad .\quad + ry - qz, \\ \delta y &= m - rx\quad .\quad + pz, \\ \delta z &= n + qx - py\quad .\quad , \end{aligned}$$

and hence the displacement in the direction of the line is

$$= \cos\alpha\,\delta x + \cos\beta\,\delta y + \cos\gamma\,\delta z,$$

which attending to the significations of (a, b, c, f, g, h) is

$$= ap + bq + cr + fl + gm + hn,$$

and we have thus the theorem in question.

62. It thus appears that for a system of rotations

λ_1 about the line $(a_1, b_1, c_1, f_1, g_1, h_1)$,
λ_2 „ $(a_2, b_2, c_2, f_2, g_2, h_2)$,
&c. „ &c.

the displacements of the point (x, y, z) rigidly connected with the several lines are

$$\begin{aligned} \delta x &= \quad . \quad - y\Sigma h\lambda + z\Sigma g\lambda - \Sigma a\lambda, \\ \delta y &= x\Sigma h\lambda \quad . \quad - z\Sigma f\lambda - \Sigma b\Sigma, \\ \delta z &= - x\Sigma g\lambda + y\Sigma f\lambda \quad . \quad - \Sigma c\lambda, \end{aligned}$$

and when the rotations are in equilibrium then the displacements $(\delta x, \delta y, \delta z)$ of any point (x, y, z) whatever must each of them vanish; that is, we must have

$$\Sigma\lambda a = 0, \quad \Sigma\lambda b = 0, \quad \Sigma\lambda c = 0, \quad \Sigma\lambda f = 0, \quad \Sigma\lambda g = 0, \quad \Sigma\lambda h = 0,$$

which are therefore the conditions for the equilibrium of the system of rotations λ_1, λ_2, &c.

63. And it further appears that for a system of forces acting on a rigid body,

λ_1 along the line $(a_1, b_1, c_1, f_1, g_1, h_1)$,
λ_2 „ $(a_2, b_2, c_2, f_2, g_2, h_2)$,
&c.

the conditions of equilibrium as given by the Principle of Virtual Velocities is

$$\Sigma\lambda\,(ap + bq + cr + fl + gm + hn) = 0,$$

or what is the same thing, that we have

$$\Sigma\lambda a = 0, \quad \Sigma\lambda b = 0, \quad \Sigma\lambda c = 0, \quad \Sigma\lambda f = 0, \quad \Sigma\lambda g = 0, \quad \Sigma\lambda h = 0,$$

for the conditions of equilibrium of the system of forces λ_1, λ_2, &c. The conditions of equilibrium are thus precisely the same in the case of a system of rotations (infinitesimal rotations) and in that of a system of forces.

64. It now appears that the greater portion of the investigations in the first part of the present paper are applicable, and may be considered as relating, to the equilibrium of forces (or of rotations; but as the two theories are identical, it is sufficient to attend to one of them), and that we have in effect solved the following

question, "Given any system of two, three, four, five or six lines considered as belonging to a solid body, to determine the relations between these lines in order that there may exist along them forces which are in equilibrium;" but for greater clearness I will consider the several cases in order; it is hardly necessary to remark that when the forces exist the equilibrium will depend on the ratios only, and that the absolute magnitude of any one of the forces may be assumed at pleasure.

65. The condition in the case of two lines is of course that these shall coincide together, or form one and the same line; and the forces are then equal and opposite forces.

66. In the case of three lines, these must meet in a point and lie in a plane; and the force along each line must then be as the sine of the angle between the other two lines.

67. Supposing that the forces are λ along the line (a, b, c, f, g, h), λ_1 along the line $(a_1, b_1, c_1, f_1, g_1, h_1)$, and λ_2 along the line $(a_2, b_2, c_2, f_2, g_2, h_2)$, the conditions of equilibrium are $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0, \ldots\ldots \lambda h + \lambda_1 h_1 + \lambda_2 h_2 = 0$, any two of which determine the ratios $\lambda : \lambda_1 : \lambda_2$; these ratios were not worked out *ante* No. 38 for the reason that with the coordinates there made use of, a symmetrical solution was not obtainable; but in the present case, selecting the last three equations, these are

$$\begin{aligned}
\lambda \cos\alpha + \lambda_1 \cos\alpha_1 + \lambda_2 \cos\alpha_2 &= 0,\\
\lambda \cos\beta + \lambda_1 \cos\beta_1 + \lambda_2 \cos\beta_2 &= 0,\\
\lambda \cos\gamma + \lambda_1 \cos\gamma_1 + \lambda_2 \cos\gamma_2 &= 0,
\end{aligned}$$

giving in the first instance an equation which expresses that the three lines (assumed to meet in a point) lie in the same plane: and then if 01, 02, 12 be the angles between the pairs of lines respectively, giving by an easy transformation

$$\begin{aligned}
\lambda \qquad\qquad + \lambda_1 \cos 01 + \lambda_2 \cos 02 &= 0,\\
\lambda \cos 10 + \lambda_1 \qquad\qquad + \lambda_2 \cos 12 &= 0,\\
\lambda \cos 20 + \lambda_1 \cos 21 + \lambda_2 \qquad &= 0.
\end{aligned}$$

68. Putting for shortness A, B, C in the place of 12, 20, 01 respectively, we thence find

$$\begin{vmatrix} 1 & , & \cos C & , & \cos B \\ \cos C & , & 1 & , & \cos A \\ \cos B & , & \cos A & , & 1 \end{vmatrix} = 0,$$

that is

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C = 0,$$

equivalent to $A + B + C = 2\pi$; and then from the first and second equations

$$\begin{aligned}
\lambda : \lambda_1 : \lambda_2 &= \cos A \cos C - \cos B &&: \cos B \cos C - \cos A &&: 1 - \cos^2 C,\\
&= \sin A \sin C &&: \sin B \sin C &&: \sin^2 C,\\
&= \sin A &&: \sin B &&: \sin C,
\end{aligned}$$

which is the required formula.

69. In the case of four given lines the condition (as noticed by Möbius) is that the four lines shall be generating lines of the same hyperboloid. In fact every line which meets three of the four lines must also meet the fourth line; for otherwise the moment of the system about such line would not be $=0$. Calling the lines 0, 1, 2, 3 and writing as before 01, 02, &c. for the moments of the several pairs of lines, then taking the moments of the system about the four lines respectively, we obtain directly the before-mentioned system of equations

$$\begin{aligned} &. \quad \lambda_1 01 + \lambda_2 02 + \lambda_3 03 = 0, \\ &\lambda 10 \quad . \quad + \lambda_2 12 + \lambda_3 13 = 0, \\ &\lambda 20 + \lambda_1 21 \quad . \quad + \lambda_3 23 = 0, \\ &\lambda 30 + \lambda_1 31 + \lambda_1 41 \quad . \quad = 0, \end{aligned}$$

leading as before to the relation

$$\sqrt{01}\sqrt{23} + \sqrt{02}\sqrt{31} + \sqrt{03}\sqrt{12} = 0,$$

and to the values

$$\lambda : \lambda_1 : \lambda_2 : \lambda_3 = \sqrt{12}\sqrt{23}\sqrt{31} : \sqrt{23}\sqrt{30}\sqrt{02} : \sqrt{30}\sqrt{01}\sqrt{13} : \sqrt{01}\sqrt{12}\sqrt{20}$$

for the proportional magnitudes of the forces. These last equations give

$$\lambda\lambda_1\, 01 = \lambda_2\lambda_3\, 23,$$

which, representing each force by a segment on the line along which the force acts, denotes that the tetrahedron of any two of the forces is equal to the tetrahedron of the other two forces; this is in fact equivalent to the theorem of M. Chasles, that if a system of forces be in any manner whatever reduced to two forces, the tetrahedron formed by these two forces has a constant volume.

70. In the case of five given lines, the lines must have a pair of tractors. Any four of the lines have in fact two tractors; and each of these tractors must meet the fifth line, for otherwise the moment of the system about the tractor would not be $=0$. In the case where the four lines have a twofold tractor, the foregoing consideration shows only that the fifth line meets the twofold tractor, but it fails to show that the twofold tractor is a twofold tractor in regard to the fifth line.

71. I stop to consider this particular case under the present statical point of view. Taking the twofold tractor for the axis of z; let the line 0 meet this line in the point $(0, 0, c)$, the coordinates (a, b, c, f, g, h) of this line being consequently

$$(c\cos\beta, \quad -c\cos\alpha, \quad 0, \quad \cos\alpha, \quad \cos\beta, \quad \cos\gamma)$$

and the like for the other four lines 1, 2, 3, 4. Using the sign Σ to refer to the last-mentioned four lines the equations of equilibrium become

$$\begin{aligned} &\lambda c\cos\beta + \Sigma\lambda_1 c_1 \cos\beta_1 = 0, \\ &\lambda c\cos\alpha + \Sigma\lambda_1 c_1 \cos\alpha_1 = 0, \\ &\lambda \ \cos\alpha + \Sigma\lambda_1 \ \cos\alpha_1 = 0, \\ &\lambda \ \cos\beta + \Sigma\lambda_1 \ \cos\beta_1 = 0, \\ &\lambda \ \cos\gamma + \Sigma\lambda_1 \ \cos\gamma_1 = 0. \end{aligned}$$

These equations give

$$\frac{\Sigma\lambda_1 c_1 \cos\beta_1}{\Sigma\lambda_1 \cos\alpha_1} = \frac{c\cos\beta}{\cos\alpha};$$

we may without loss of generality take the homographic conditions which express that the axis of z is a twofold tractor of the four lines to be

$$\frac{c_1\cos\beta_1}{\cos\alpha_1} = \frac{c_2\cos\beta_2}{\cos\alpha_2} = \frac{c_3\cos\beta_3}{\cos\alpha_3} = \frac{c_4\cos\beta_4}{\cos\alpha_4} = k,$$

and this being so, the last-mentioned equation becomes

$$\frac{c\cos\beta}{\cos\alpha} = k;$$

and it thus appears that the axis of z is a twofold tractor in regard also to the line 0.

72. In the case of six lines such that there exist along them forces which are in equilibrium, taking this as a definition of the involution of six lines, we may very readily obtain from statical considerations the before-mentioned construction of the sixth line; viz. it may be shown that given any five of the lines, say the lines 1, 2, 3, 4, 5 and a point O, we can through the point O determine a plane Ω, such that any line whatever through the point O and in the plane Ω is in involution with the five given lines. Consider the tractors of any four of the lines, say the lines 2, 3, 4, 5; we may through the point O draw a line OA meeting the two tractors; that is, the lines 2, 3, 4, 5 and the line OA will have a pair of common tractors. There consequently exist along these lines forces which are in equilibrium; and since only the ratios are material, the absolute magnitude of the force along the line OA may be anything whatever. Similarly, considering the tractors of the lines 1, 3, 4, 5, and through O a line OB meeting these tractors, then there exist along the lines 1, 3, 4, 5 and the line OB forces which are in equilibrium, and the absolute magnitude of the force along the line OB may be anything whatever. Hence, combining the two sets of forces, we have, along a line through O in the plane OA, OB, but otherwise indeterminate in its direction, a force in equilibrium with forces along the lines 1, 2, 3, 4, 5; that is, the line found as above is a line in involution with the lines 1, 2, 3, 4, 5.

73. It is to be added, that through O we cannot, out of the plane OA, OB, draw a line in involution with the lines 1, 2, 3, 4, 5; for if any such line OK existed, then we should have along each of the lines OA, OB, OK forces in equilibrium with forces along the lines 1, 2, 3, 4, 5; and the magnitudes of the three forces being each of them anything whatever, it would follow that along any line whatever through the point O there would exist a force in equilibrium with forces along the lines 1, 2, 3, 4, 5; that is, any line whatever through the point O would be a line in involution with these lines.

74. It hence appears, that drawing OA to meet the tractors of 2, 3, 4, 5; OB to meet those of 3, 4, 5, 1; OC to meet those of 4, 5, 1, 2; OD to meet those of 5, 1, 2, 3; and OE to meet those of 1, 2, 3, 4; the lines OA, OB, OC, OD, OE will be in one plane, say the plane Ω: and that any line through O in the plane Ω will be a line in involution with the lines 1, 2, 3, 4, 5.

75. There is another statical representation of the involution of six lines. If a system of forces act on a solid body, then taking six lines at random, the system will be in equilibrium if the sum of the moments be $=0$ in regard to each of the six lines. But if the six lines be in involution; then, for the very reason that a rotation about one of these lines is resolvable into rotations about the other five lines, if the sum of the moments be $=0$ for each of the five lines, it will also be $=0$ for the sixth line: that is, it is *not* sufficient for the equilibrium of the forces that the sum of the moments shall be $=0$ for each of the six lines. And we thus see that six lines in involution are lines such that the equilibrium of a system of forces about each of the six lines as axes does *not* insure the equilibrium of the system.

Article Nos. 76 and 77. *Transformation of Coordinates.*

76. Reverting to the general definition of the six coordinates (a, b, c, f, g, h) of a line by means of the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ on the line; suppose that instead of the original coordinate planes $x=0$, $y=0$, $z=0$, $w=0$ (forming a tetrahedron $ABCD$) we have new coordinate planes $x_0=0$, $y_0=0$, $z_0=0$, $w_0=0$ (forming a tetrahedron $A_0B_0C_0D_0$); and that the relations between the two sets of current coordinates are given by the equations

$$\begin{aligned} x : y : z : w = \ & (\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0) \\ & : (\lambda_2, \mu_2, \nu_2, \rho_2 \between x_0, y_0, z_0, w_0) \\ & : (\lambda_3, \mu_3, \nu_3, \rho_3 \between x_0, y_0, z_0, w_0) \\ & : (\lambda_4, \mu_4, \nu_4, \rho_4 \between x_0, y_0, z_0, w_0), \end{aligned}$$

with, of course, the like relations between the original coordinates $(\alpha, \beta, \gamma, \delta)$ and new coordinates $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, and between the original coordinates $(\alpha', \beta', \gamma', \delta')$ and the new coordinates $(\alpha_0', \beta_0', \gamma_0', \delta_0')$, of the two points on the line (a, b, c, f, g, h); then taking $(a_0, b_0, c_0, f_0, g_0, h_0)$ as the new values of the six coordinates of the line, viz. writing

$$\begin{aligned} & a_0 : b_0 : c_0 : f_0 : g_0 : h_0 \\ = \ & \beta_0\gamma_0' - \beta_0'\gamma_0 : \gamma_0\alpha_0' - \gamma_0'\alpha_0 : \alpha_0\beta_0' - \alpha_0'\beta_0 : \alpha_0\delta_0' - \alpha_0'\delta_0 : \beta_0\delta_0' - \beta_0'\delta_0 : \gamma_0\delta_0' - \gamma_0'\delta_0, \end{aligned}$$

we obtain a system of formulæ which may be conveniently written as follows:

$$\begin{aligned} & a : b : c : f : g : h \\ & = \frac{\mu\nu}{23} a_0 + \frac{\nu\lambda}{23} b_0 + \frac{\lambda\mu}{23} c_0 + \frac{\lambda\rho}{23} f_0 + \frac{\mu\rho}{23} g_0 + \frac{\nu\rho}{23} h_0 \\ & : 31 \\ & : 12 \\ & : 14 \\ & : 24 \\ & : 34 \end{aligned}$$

viz. the top line stands for $(\mu_2\nu_3 - \mu_3\nu_2) a_0 + (\nu_2\lambda_3 - \nu_3\lambda_2) b_0 +$ &c., and the other lines are obtained from this by mere alterations of the suffixes.

77. As to the interpretation of these formulæ, taking

$ABCD$ as the fundamental tetrahedron for (x, y, z, w),
$A_0B_0C_0D_0$ " " (x_0, y_0, z_0, w_0),

then

$(\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0) = 0$ is the equation of plane BCD,
$(\lambda_2, \mu_2, \nu_2, \rho_2 \between$ " $) = 0$ " CDA,
$(\lambda_3, \mu_3, \nu_3, \rho_3 \between$ " $) = 0$ " DAB,
$(\lambda_4, \mu_4, \nu_4, \rho_4 \between$ " $) = 0$ " ABC,

whence, observing that the second and third equations belong to two planes each passing through the line DA, it appears that the coefficients

$$\frac{\mu\nu}{23}, \frac{\nu\lambda}{23}, \frac{\lambda\mu}{23}, \frac{\lambda\rho}{23}, \frac{\mu\rho}{23}, \frac{\nu\rho}{23},$$

are the six coordinates of the line DA, expressed in regard to the tetrahedron $(A_0B_0C_0D_0)$; and similarly that the coefficients in the six expressions of the transformation formula are the six coordinates of the lines AD, BD, CD, BC, CA, AB respectively in regard to the tetrahedron $(A_0B_0C_0D_0)$.

In the preceding formulæ for the transformation of coordinates the ratios only have been attended to, no determinate absolute magnitudes have been assigned to the coordinates (a, b, c, f, g, h). But I will nevertheless show how we may attribute absolute magnitudes to these coordinates.

Article Nos. 78 to 80. *New definition of the six coordinates as to their absolute magnitudes.*

78. I assume (x, y, z, w) to be "volume" coordinates; viz. taking as before $ABCD$ for the fundamental tetrahedron, and denoting the point (x, y, z, w) by P, I assume that we have

$$x : y : z : w : 1 = PBCD : APCD : ABPD : ABCP : ABCD,$$

where $PBCD$, &c. denote the volumes of the several tetrahedra $PBCD$, &c. It is to be noticed that the volume is in every case taken with a determinate sign: analytically the sign may be fixed by taking (x_a, y_a, z_a), &c., as the Cartesian coordinates of the points A, &c. and writing

$$PBCD = \begin{vmatrix} x_p, & x_b, & x_c, & x_d \\ y_p, & y_b, & y_c, & y_d \\ z_p, & z_b, & z_c, & z_d \\ 1, & 1, & 1, & 1 \end{vmatrix}, \text{ \&c.}$$

(whence of course $PBCD = PCDA = -PCBD$, &c. according to the rule of signs): or we may in an equivalent manner, but less easily, determine the sign, by considering the sense of the rotation about CD (considered as an axis drawn from C to D) which would be produced by a force along PB (from P to B).

79. It is to be observed that the foregoing values give identically $x+y+z+w=1$, so that the equation of the plane infinity is $x+y+z+w=0$. The values of the coordinates (x, y, z, w) may be written

$$x : y : z : w : 1 = PBCD : PCAD : PABD : PCBA : ABCD;$$

or in the original form

$$x : y : z : w : 1 = PBCD : APCD : ABPD : ABCP : ABCD,$$

as may be most convenient.

80. Denoting the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ by Q, Q' respectively, we have

$$\alpha : \beta : \gamma : \delta : 1 = QBCD : AQCD : ABQD : ABCQ : ABCD$$

and

$$\alpha' : \beta' : \gamma' : \delta' : 1 = Q'BCD : AQ'CD : ABQ'D : ABCQ' : ABCD,$$

and writing

$$(a, b, c, f, g, h) = (\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\delta' - \alpha'\delta,\ \beta\delta' - \beta'\delta,\ \gamma\delta' - \gamma'\delta),$$

viz. the two sets being taken to be *equal*, $a = \beta\gamma' - \beta'\gamma$, &c. instead of merely proportional, then it is easily seen that we obtain

$$\begin{aligned} &a : b : c : f : g : h : 1 \\ &= AQQ'D : Q'BQD : QQ'CD : QBCQ' : AQCQ' : ABQQ' : ABCD, \end{aligned}$$

that is, in order to form the first six combinations we successively replace

$$(B, C), (C, A), (A, B), (A, D), (B, D), (C, D)$$

in $ABCD$ by (Q, Q').

Article No. 81. *Resulting formulæ of Transformation.*

81. For the transformation of coordinates if we assume

$$\begin{aligned} x &= (\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0), \\ y &= (\lambda_2, \mu_2, \nu_2, \rho_2 \between \quad ,, \quad), \\ z &= (\lambda_3, \mu_3, \nu_3, \rho_3 \between \quad ,, \quad), \\ w &= (\lambda_4, \mu_4, \nu_4, \rho_4 \between \quad ,, \quad), \end{aligned}$$

and take also (a, b, c, f, g, h), $(a_0, b_0, c_0, f_0, g_0, h_0)$ respectively *equal*, instead of merely proportional, to the foregoing values, then, observing that for the point A_0 we have $(x_0, y_0, z_0, w_0) = (1, 0, 0, 0)$ we see that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the $ABCD$ – coordinates of A_0; and the like as to the other sets of coefficients; viz. we have

$$\begin{aligned} \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : 1 &= A_0BCD : AA_0CD : ABA_0D : ABCA_0 : ABCD \\ \mu_1 : \mu_2 : \mu_3 : \mu_4 : 1 &= B_0 \quad ,, \quad : ,, B_0 \quad ,, \quad : \quad ,, B_0 ,, : \quad ,, \quad B_0 : \quad ,, \\ \nu_1 : \nu_2 : \nu_3 : \nu_4 : 1 &= C_0 \quad ,, \quad : ,, C_0 \quad ,, \quad : \quad ,, C_0 ,, : \quad ,, \quad C_0 : \quad ,, \\ \rho_1 : \rho_2 : \rho_3 : \rho_4 : 1 &= D_0 \quad ,, \quad : ,, D_0 \quad ,, \quad : \quad ,, D_0 ,, : \quad ,, \quad D_0 : \quad ,, \end{aligned}$$

and we hence find

$$\frac{\mu\nu}{23} : \frac{\nu\lambda}{23} : \frac{\lambda\mu}{23} : \frac{\lambda\rho}{23} : \frac{\mu\rho}{23} : \frac{\nu\rho}{23} : 1$$

$$= AB_0C_0D : AC_0A_0D : AA_0B_0D : AA_0D_0D : AB_0D_0D : AC_0D_0D : ABCD$$

viz. multiplying the last-mentioned set of terms by $A_0B_0C_0D_0 \div ABCD$, in order to make the last term equal to unity, we see that the coefficients $\frac{\mu\nu}{23}$, $\frac{\nu\lambda}{23}$ &c. are equal to $\frac{A_0B_0C_0D_0}{A\,B\,C\,D}$ into the six $(ABCD)_0$ – coordinates respectively of the line AD by means of the points A, D thereof. And similarly in the six expressions which enter into the formula of transformation, the coefficients are $= \frac{A_0B_0C_0D_0}{ABCD}$ into the six $(ABCD)_0$ – coordinates of the

line AD	in regard to	points A, D	thereof
„ BD	„	B, D	„
„ CD	„	C, D	„
„ BC	„	B, C	„
„ CA	„	C, A	„
„ AB	„	A, B	„

The foregoing theory of the transformation of coordinates seemed to me interesting for its own sake, and I have developed it in preference to the more simple theory which might easily be established of the case in which the coordinates are quantitatively defined as being equal to

$$(z_0 \cos\beta - y_0 \cos\gamma,\ x_0 \cos\gamma - z_0 \cos\alpha,\ y_0 \cos\beta - x_0 \cos\alpha,\ \cos\alpha,\ \cos\beta,\ \cos\gamma)$$

respectively.

436.

ON A CERTAIN SEXTIC TORSE.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XI. Part III. (1871), pp. 507—523. Read Nov. 8, 1869.]

THE torse (developable surface) intended to be considered is that which has for its edge of regression an excubo-quartic curve, or say a *unicursal* quartic curve. I call to mind that (excluding the plane quartic) a quartic curve is either a quadriquadric, viz. it is the complete intersection of two quadric surfaces; or else it is an excubo-quartic, viz. there is through the curve only one quadric surface, and the curve is the partial intersection of this quadric surface with a cubic surface through two generating lines (of the same kind) of the quadric surface. Returning to the quadriquadric curve, this may be general, nodal, or cuspidal; viz. if the two quadric surfaces have an ordinary contact, the curve of intersection is a nodal quadriquadric; if they have a stationary contact, the curve is a cuspidal quadriquadric.

The unicursal quartic is a curve such that the coordinates (x, y, z, w) of any point thereof are proportional to rational and integral quartic functions $(*\between\theta, 1)^4$ of a variable parameter θ; and the general unicursal quartic is in fact the excubo-quartic; but included as particular cases of the unicursal curve (although not as cases of the excubo-quartic as above defined) we have the nodal quadriquadric and the cuspidal quadriquadric. The torse having for its edge of regression a unicursal curve is a sextic torse; and this is in fact the order of the torse derived from the excubo-quartic, and from the nodal quadriquadric; but for the cuspidal quadriquadric, there is a depression of *one*, and the torse becomes a quintic torse. The equations have been obtained of (1) the sextic torse derived from the nodal quadriquadric, (2) the quintic torse derived from the cuspidal quadriquadric, (3) the sextic torse derived from a certain special excubo-quartic; but the equation of the torse derived from the general unicursal quartic has not yet been found. To show at the outset what the analytical problem is, I

anticipate the remark that the coordinates (x, y, z, w) of a point on the curve may by an obvious reduction be rendered proportional to the fourth powers $(\theta+\alpha)^4$, $(\theta+\beta)^4$, $(\theta+\gamma)^4$, $(\theta+\delta)^4$ in the parameter θ; this leads to an equation

$$\frac{x}{(\theta+\alpha)^2}+\frac{y}{(\theta+\beta)^2}+\frac{z}{(\theta+\gamma)^2}+\frac{w}{(\theta+\delta)^2}=0,$$

for the osculating plane at the point (x, y, z, w); or observing that this equation, when integralised, is of the form $(x, y, z, w \between \theta, 1)^6=0$, we see that the equation is obtained by equating to zero the discriminant of a certain sextic function in θ; the discriminant is of the order 10 in the coordinates (x, y, z, w), but it obviously contains the factor $xyzw$, or throwing this out we have an equation of the order 6, so that the torse is (as above stated) a sextic torse.

Theorem relating to Four Binary Quartics.

1. Consider the four quartics:

$$(a_1, b_1, c_1, d_1, e_1 \between x, y)^4,$$
$$(a_2, b_2, c_2, d_2, e_2 \between x, y)^4,$$
$$(a_3, b_3, c_3, d_3, e_3 \between x, y)^4,$$
$$(a_4, b_4, c_4, d_4, e_4 \between x, y)^4,$$

then if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are any four quantities, these may be determined, and that in four different ways, so that

$$\lambda_1(a_1, \ldots \between x, y)^4+\lambda_2(a_2, \ldots \between x, y)^4+\lambda_3(a_3, \ldots \between x, y)^4+\lambda_4(a_4, \ldots \between x, y)^4=(\beta x+\alpha y)^4,$$

a perfect fourth power; in fact, equating the coefficients of the different powers of $(x, y)^4$, we have five equations, which determine the ratios of the unknown quantities $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; α, β: eliminating $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we find the equation

$$\begin{vmatrix} \beta^4, & \beta^3\alpha, & \beta^2\alpha^2, & \beta\alpha^3, & \alpha^4 \\ a_1, & b_1, & c_1, & d_1, & e_1 \\ a_2, & b_2, & c_2, & d_2, & e_2 \\ a_3, & b_3, & c_3, & d_3, & e_3 \\ a_4, & b_4, & c_4, & d_4, & e_4 \end{vmatrix}=0,$$

giving four different values of the ratio $\alpha:\beta$; or, assigning at pleasure a value to α or β (say $\beta=1$), then to each of the four sets of values of (α, β) there correspond a determinate set of values of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$; that is, we have as stated four sets of values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; α, β.

Standard Equation of the Unicursal Quartic.

2. The coordinates (x, y, z, w) being originally taken to be proportional to any four given quartic functions $(*\between\theta, 1)^4$ of the parameter θ, then forming a linear function of the coordinates, we have four sets of values of the multipliers, each reducing the function of θ to a perfect fourth power; that is, writing (X, Y, Z, W) for the linear functions of the original coordinates, and taking (X, Y, Z, W) as coordinates, it appears that the unicursal quartic may be represented by the equations

$$X : Y : Z : W = (\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4.$$

Tangent Line, and Osculating Plane of the Unicursal Quartic.

3. The equations of the tangent line at the point (θ) (that is, the point the coordinates whereof are as $(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4$) are at once seen to be

$$\left\| \begin{matrix} X, & Y, & Z, & W \\ (\theta+\alpha)^4, & (\theta+\beta)^4, & (\theta+\gamma)^4, & (\theta+\delta)^4 \\ (\theta+\alpha)^3, & (\theta+\beta)^3, & (\theta+\gamma)^3, & (\theta+\delta)^3 \end{matrix} \right\| = 0,$$

and that of the osculating plane to be

$$\left| \begin{matrix} X, & Y, & Z, & W \\ (\theta+\alpha)^4, & (\theta+\beta)^4, & (\theta+\gamma)^4, & (\theta+\delta)^4 \\ (\theta+\alpha)^3, & (\theta+\beta)^3, & (\theta+\gamma)^3, & (\theta+\delta)^3 \\ (\theta+\alpha)^2, & (\theta+\beta)^2, & (\theta+\gamma)^2, & (\theta+\delta)^2 \end{matrix} \right| = 0.$$

Writing as in the sequel

$$\begin{aligned} a &= \beta-\gamma, & f &= \alpha-\delta, \\ b &= \gamma-\alpha, & g &= \beta-\delta, \\ c &= \alpha-\beta, & h &= \gamma-\delta, \end{aligned}$$

the equations of the tangent line become

$$\begin{aligned} \cdot \quad & \frac{hY}{(\theta+\beta)^3} - \frac{gZ}{(\theta+\gamma)^3} + \frac{aZ}{(\theta+\delta)^3} = 0, \\ -\frac{hX}{(\theta+\alpha)^3} \quad & \cdot \quad + \frac{fZ}{(\theta+\gamma)^3} + \frac{bZ}{(\theta+\delta)^3} = 0, \\ \frac{gX}{(\theta+\alpha)^3} - \frac{fY}{(\theta+\beta)^3} \quad & \cdot \quad + \frac{cZ}{(\theta+\delta)^3} = 0, \\ -\frac{aX}{(\theta+\alpha)^3} - \frac{bY}{(\theta+\beta)^3} - \frac{cZ}{(\theta+\gamma)^3} \quad & \cdot \quad = 0, \end{aligned}$$

(equivalent of course to two equations), and the equation of the osculating plane becomes

$$\frac{ahgX}{(\theta+\alpha)^2} + \frac{hbfY}{(\theta+\beta)^2} + \frac{gfcZ}{(\theta+\gamma)^2} + \frac{abcW}{(\theta+\delta)^2} = 0.$$

Modification of the foregoing notation, and final form for the Unicursal Quartic.

4. If instead of the coordinates (X, Y, Z, W) we introduce the coordinates (x, y, z, w) connected therewith by the relations

$$ahgX : hbfY : gfcZ : abcW = \quad x : \quad y : \quad z : \quad w,$$

or, what is the same thing,

$$X : \quad Y : \quad Z : \quad W = bcfx : cagy : abhz : fghw,$$

then the curve is given by the equations

$$x : y : z : w = ahg(\theta+\alpha)^4 : hbf(\theta+\beta)^4 : cfg(\theta+\gamma)^4 : abc(\theta+\delta)^4.$$

The equations of the tangent line are

$$\begin{aligned}
\cdot \qquad & \frac{cy}{(\theta+\beta)^3} - \frac{bz}{(\theta+\gamma)^3} + \frac{fw}{(\theta+\delta)^3} = 0,\\
-\frac{cx}{(\theta+\alpha)^3} \qquad & \cdot \qquad + \frac{az}{(\theta+\gamma)^3} + \frac{gw}{(\theta+\delta)^3} = 0,\\
\frac{bx}{(\theta+\alpha)^3} - \frac{ay}{(\theta+\beta)^3} \qquad & \cdot \qquad + \frac{hw}{(\theta+\delta)^3} = 0,\\
-\frac{fx}{(\theta+\alpha)^3} - \frac{gy}{(\theta+\beta)^3} - \frac{hz}{(\theta+\gamma)^3} \qquad & \cdot \qquad = 0,
\end{aligned}$$

and the equation of the osculating plane is

$$\frac{x}{(\theta+\alpha)^2} + \frac{y}{(\theta+\beta)^2} + \frac{z}{(\theta+\gamma)^2} + \frac{w}{(\theta+\delta)^2} = 0.$$

Determination of the Sextic Torse.

5. Starting from the equation of the osculating plane written under the form

$$\begin{aligned}
& x(\theta+\beta)^2(\theta+\gamma)^2(\theta+\delta)^2\\
+\, & y(\theta+\gamma)^2(\theta+\delta)^2(\theta+\alpha)^2\\
+\, & z(\theta+\delta)^2(\theta+\alpha)^2(\theta+\beta)^2\\
+\, & w(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2 = 0,
\end{aligned}$$

the equation of the torse is obtained by equating to zero the discriminant of the sextic function. Writing as before

$$\begin{aligned}
a &= \beta-\gamma, \quad f = \alpha-\delta,\\
b &= \gamma-\alpha, \quad g = \beta-\delta,\\
c &= \alpha-\beta, \quad h = \gamma-\delta,
\end{aligned}$$

equations which give

$$\begin{array}{rrrrl} . & h & -g & +a & =0, \\ -h & . & +f & +b & =0, \\ g & -f & . & +c & =0, \\ -a & -b & -c & . & =0, \end{array}$$

$$\begin{array}{rrrrl} . & h\beta & -g\gamma & +a\delta & =0, \\ -h\alpha & . & +f\gamma & +b\delta & =0, \\ g\alpha & -f\beta & . & +c\delta & =0, \\ -a\alpha & -b\beta & -c\gamma & . & =0, \end{array}$$

and also

$$af+bg+ch=0,$$

then the discriminant is a function of (x, y, z, w), (a, b, c, f, g, h) of the degree 10 in (x, y, z, w) and the degree 30 in (a, b, c, f, g, h). But the equation in θ has two equal roots, or the discriminant vanishes, if any one of the quantities (x, y, z, w) is $=0$; and again, if any one of the differences $\alpha-\beta$, &c. (that is any one of the quantities a, b, c, f, g, h) is $=0$: the discriminant thus contains the factors $xyzw$ and $(abcfgh)^2$, and throwing these out, we have an equation of the form

$$\Delta=(a, b, c, f, g, h)^{18}(x, y, z, w)^6=0,$$

which is the equation of the sextic torse.

Principal Sections of the Torse.

6. Consider for instance the section by the plane $w=0$. Writing $w=0$, the equation of the osculating plane is

$$(\theta+\delta)^2[x(\theta+\beta)^2(\theta+\gamma)^2+y(\theta+\gamma)^2(\theta+\alpha)^2+z(\theta+\alpha)^2(\theta+\beta)^2]=0.$$

The discriminant of the sextic function vanishes identically in virtue of the double factor $(\theta+\delta)^2$. But omitting this factor, the equation becomes

$$x(\theta+\beta)^2(\theta+\gamma)^2+y(\theta+\gamma)^2(\theta+\alpha)^2+z(\theta+\alpha)^2(\theta+\beta)^2=0.$$

The discriminant of this quartic function of θ is a function of x, y, z, a, b, c of the degree 6 in (x, y, z) and 12 in (a, b, c); it contains however the factors xyz, $a^2b^2c^2$, and the remaining factor is of the degree 3 in (x, y, z) and 6 in (a, b, c); this remaining factor is as will presently be seen

$$=(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz.$$

The last mentioned sextic equation in θ will have a triple root $\theta=-\delta$, if only the value $\theta=-\delta$ makes to vanish the factor in [ ], that is if we have

$$0=g^2h^2x+h^2f^2y+f^2g^2z.$$

The foregoing results lead to the conclusion that for $w=0$, we have

$$\Delta=(g^2h^2x+h^2f^2y+f^2g^2z)^3\left[(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz\right];$$

but this will appear more distinctly as follows.

7. First, as to the factor $(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz$: writing in the equation of the osculating plane $w=0$, the equation becomes

$$\frac{x}{(\theta+\alpha)^3}+\frac{y}{(\theta+\beta)^3}+\frac{z}{(\theta+\gamma)^3}=0,$$

which equation is therefore that of the trace of the osculating plane on the plane $w=0$; the envelope of the trace in question is *a part* of the section of the torse by the plane $w=0$. To find the equation of this envelope we must eliminate θ from the foregoing, and its derived equation

$$\frac{x}{(\theta+\alpha)^2}+\frac{y}{(\theta+\beta)^2}+\frac{z}{(\theta+\gamma)^2}=0;$$

the two equations give

$$x:y:z=a(\theta+\alpha)^2:b(\theta+\beta)^2:c(\theta+\gamma)^2,$$

and thence

$$(a^2x)^{\frac{1}{3}}+(b^2y)^{\frac{1}{3}}+(c^2z)^{\frac{1}{3}}=a(\theta+\alpha)+b(\theta+\beta)+c(\theta+\gamma)=0,$$

that is, we have

$$(a^2x)^{\frac{1}{3}}+(b^2y)^{\frac{1}{3}}+(c^2z)^{\frac{1}{3}}=0,$$

or, what is the same thing,

$$(a^2x+b^2y+c^2z)^3-27a^2b^2c^2\,xyz=0$$

for a part of the section in question.

8. I have said that the foregoing cubic is *a part* of the section; the equations

$$x:y:z:w=ahg(\theta+\alpha)^4:bhf(\theta+\beta)^4:cfg(\theta+\gamma)^4:abc(\theta+\delta)^4,$$

which for $w=0$ give $\theta=-\delta$, and thence $x:y:z=af^3:bg^3:ch^3$, show that the last mentioned point is a four-pointic intersection of the curve with the plane $w=0$. But the curve, having four consecutive points, will have three consecutive tangents in the plane $w=0$; that is, the tangent at the point in question will present itself as a threefold factor in the equation of the torse. Writing in the equations of the tangent $w=0$, $\theta=-\delta$, we find for the equation of the tangent in question

$$\frac{x}{f^2}+\frac{y}{g^2}+\frac{z}{h^2}=0,$$

or, what is the same thing,

$$g^2h^2x+h^2f^2y+f^2g^2z=0.$$

Hence the section by the plane $w=0$ is made up of this line taken three times, and of the last mentioned cubic curve.

9. By symmetry, we conclude that the sections by the principal planes $x=0$, $y=0$, $z=0$, $w=0$, are each made up of a line taken three times, and of a cubic curve: viz. these are

$$\begin{array}{llll|l}
x=0, & . & b^2f^2y+c^2f^2z+b^2c^2w & =0, & . \quad (h^2y)^{\frac{1}{3}}+(g^2z)^{\frac{1}{3}}+(a^2w)^{\frac{1}{3}}=0, \\
y=0, & a^2g^2x & . \quad c^2g^2z+c^2a^2w & =0, & (h^2x)^{\frac{1}{3}} \quad . \quad +(f^2z)^{\frac{1}{3}}+(b^2w)^{\frac{1}{3}}=0, \\
z=0, & a^2h^2x+b^2h^2y & . \quad +a^2b^2w & =0, & (g^2x)^{\frac{1}{3}}+(f^2y)^{\frac{1}{3}} \quad . \quad +(c^2w)^{\frac{1}{3}}=0, \\
w=0, & g^2h^2x+h^2f^2y+f^2g^2z & . & =0, & (a^2x)^{\frac{1}{3}}+(b^2y)^{\frac{1}{3}}+(c^2z)^{\frac{1}{3}} \quad =0,
\end{array}$$

where for shortness I have written the equations of the four cubics in their irrational forms respectively.

Partial Determination of the Equation.

10. As the value of Δ is known when any one of the coordinates x, y, z, w is put $=0$, we in fact know all the terms of Δ, except those which contain the factor $xyzw$, which unknown terms, as Δ is of the degree 6, are of the form $(*\between x, y, z, w)^2$.

I remark that if $(xyzw)$ is any homogeneous function $(*\between x, y, z, w)^2$, and (xyz), (xy), (x) are what $(xyzw)$ become on putting therein $(w=0)$, $(z=0, w=0)$, $(y=0, z=0, w=0)$ respectively, and the like for the other similar symbols, then that

$$\begin{aligned}
(xyzw) = \; & (x)+(y)+(z)+(w) \\
& -(xy)-(xz)-(xw)-(yz)-(yw)-(zw) \\
& +(xyz)+(xyw)+(xzw)+(yzw) \\
& +\text{terms multiplied by } xyzw;
\end{aligned}$$

in fact, omitting the last line, this equation on writing therein $x=0$ or $y=0$ or $z=0$ or $w=0$, becomes an identity, that is, the difference of the two sides vanishes when any one of these equations is satisfied, and such difference contains therefore the factor $xyzw$; which proves the theorem. It hence appears that the equation $\Delta=0$ of the torse is

$$\begin{aligned}
\Delta = \; & a^6g^6h^6x^6+b^6h^6f^6y^6+c^6f^6g^6z^6+a^6b^6c^6w^6 \\
& -(g^2h^2x+h^2f^2y)^3(a^2x+b^2y)^3 \\
& -(h^2f^2y+f^2g^2z)^3(b^2y+c^2z)^3 \\
& -(g^2h^2x+f^2g^2z)^3(a^2x+c^2z)^3 \\
& -(a^2h^2x+a^2b^2w)^3(g^2x+c^2w)^3 \\
& -(b^2h^2y+a^2b^2w)^3(f^2y+c^2w)^3 \\
& -(c^2g^2z+c^2a^2w)^3(f^2z+b^2w)^3 \\
& +(b^2f^2y+c^2f^2z+b^2c^2w)^3\,[(h^2y+g^2z+a^2w)^3-27a^2g^2h^2yzw] \\
& +(a^2g^2x+c^2g^2z+c^2a^2w)^3\,[(h^2x+f^2z+b^2w)^3-27b^2h^2f^2zxw] \\
& +(a^2h^2x+b^2h^2y+a^2b^2w)^3\,[(g^2x+f^2y+c^2w)^3-27c^2f^2g^2xyw] \\
& +(g^2h^2x+h^2f^2y+f^2g^2z)^3\,[(a^2x+b^2y+c^2z)^3-27a^2b^2c^2xyz] \\
& +xyzw)\,(*\between x, y, z, w)^2,
\end{aligned}$$

where the ten coefficients of $(*\between x, y, z, w)^2$ remain to be found.

Process for the Determination of the Unknown Coefficients.

11. At a point of the cubic curve in the plane $w=0$, we have

$$x : y : z = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3;$$

and the tangent plane at this point is the osculating plane of the curve; that is, it is the plane

$$\frac{x'}{(\theta+\alpha)^2}+\frac{y'}{(\theta+\beta)^2}+\frac{z'}{(\theta+\gamma)^2}+\frac{w'}{(\theta+\delta)^2}=0$$

if for a moment (x', y', z', w') are the current coordinates of a point in the tangent plane. But the equation of the tangent plane as deduced from the equation $\Delta=0$ is

$$x'\frac{d\Delta}{dx}+y'\frac{d\Delta}{dy}+z'\frac{d\Delta}{dz}+w'\frac{d\Delta}{dw}=0,$$

where in the differential coefficients of Δ, the coordinates (x, y, z, w) are considered as having the values

$$x : y : z : w = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3 : 0.$$

Hence, with these values of (x, y, z, w), we have

$$\frac{d\Delta}{dx} : \frac{d\Delta}{dy} : \frac{d\Delta}{dz} : \frac{d\Delta}{dw} = \frac{1}{(\theta+\alpha)^2} : \frac{1}{(\theta+\beta)^2} : \frac{1}{(\theta+\gamma)^2} : \frac{1}{(\theta+\delta)^2};$$

conditions which determine the values of certain of the coefficients of $(*\between x, y, z, w)^2$, viz. the six coefficients of the terms independent of w; and when these are known the values of the remaining four coefficients are at once obtained by symmetry.

12. To develope this process, disregarding the higher powers of w, we may write

$$\Delta = \Theta + 3w\Phi + xyzw(*\between x, y, z)^2,$$

where Θ denotes the terms independent of w, $3w\Phi$ the known terms which contain the factor w, and $xyzw\ (*\between x, y, z)^2$ the unknown terms which contain this same factor; the value of $(*\between x, y, z)^2$ being clearly $=(*\between x, y, z, 0)^2$.

We have, moreover,

$$\Theta = (g^2h^2x+h^2f^2y+f^2g^2z)^3[(a^2x+b^2y+c^2z)^3-27a^2b^2c^2xyz],$$

and

$$\begin{aligned}\Phi = {} & f^4(b^2y+c^2z)^2[a^2f^2(h^4y^2-7h^2g^2yz+g^4z^2)(b^2y+c^2z)+b^2c^2(h^2y+g^2z)^3] \\ & + g^4(a^2x+c^2z)^2[b^2g^2(h^4x^2-7h^2f^2xz+f^4z^2)(a^2x+c^2z)+c^2a^2(h^2x+f^2z)^3] \\ & + h^4(a^2x+b^2y)^2[c^2h^2(g^4x^2-7g^2h^2xy+f^4y^2)(a^2x+b^2y)+a^2b^2(g^2x+f^2y)^3] \\ & - a^6h^4g^4(b^2g^2+c^2h^2)x^5 \\ & - b^6h^4f^4(c^2h^2+a^2f^2)y^5 \\ & - c^6f^4g^4(a^2f^2+b^2g^2)z^5.\end{aligned}$$

13. The equations, putting after the differentiations $w=0$, and writing for shortness $(*)$ in place of $(*\between x, y, z)^2$, become

$$\frac{d\Theta}{dx} : \frac{d\Theta}{dy} : \frac{d\Theta}{dz} : 3\Phi + xyz\,(*) = \frac{1}{(\theta+\alpha)^2} : \frac{1}{(\theta+\beta)^2} : \frac{1}{(\theta+\gamma)^2} : \frac{1}{(\theta+\delta)^2}.$$

Now, observing that the second factor of Θ vanishes for the values

$$a(\theta+\alpha)^3,\quad b(\theta+\beta)^3,\quad c(\theta+\gamma)^3 \quad \text{of} \quad (x, y, z),$$

we have simply

$$\frac{d\Theta}{dx} = (g^2h^2x + h^2f^2y + f^2g^2z)^3 \,.\, 3a^2\left[(a^2x+b^2y+c^2z)^2 - 9b^2c^2yz\right].$$

But

$$\begin{aligned} a^2x+b^2y+c^2z &= a^3(\theta+\alpha)^3 + b^3(\theta+\beta)^3 + c^3(\theta+\gamma)^3, \\ &= 3abc\,(\theta+\alpha)(\theta+\beta)(\theta+\gamma), \end{aligned}$$

in virtue of the relation $a(\theta+\alpha)+b(\theta+\beta)+c(\theta+\gamma)=0$ and hence

$$\begin{aligned} \left[(a^2x+b^2y+c^2z)^2 - 9b^2c^2yz\right] &= 9b^2c^2(\theta+\beta)^2(\theta+\gamma)^2 \,.\, \left[a^2(\theta+\alpha)^2 - bc(\theta+\beta)(\theta+\gamma)\right], \\ &= 9b^2c^2(\theta+\beta)^2(\theta+\gamma)^2\, Q, \end{aligned}$$

where

$$\begin{aligned} Q &= a^2(\theta+\alpha)^2 - bc(\theta+\beta)(\theta+\gamma), \\ &= b^2(\theta+\beta)^2 - ca(\theta+\gamma)(\theta+\alpha), \\ &= c^2(\theta+\gamma)^2 - ab(\theta+\alpha)(\theta+\beta). \end{aligned}$$

Hence

$$\frac{d\Theta}{dx} = 27a^2b^2c^2(g^2h^2x + h^2f^2y + f^2g^2z)^3(\theta+\beta)^2(\theta+\gamma)^2\, Q,$$

and similarly

$$\frac{d\Theta}{dy} = 27a^2b^2c^2(g^2h^2x + h^2f^2y + f^2g^2z)^3(\theta+\gamma)^2(\theta+\alpha)^2\, Q,$$

$$\frac{d\Theta}{dz} = 27a^2b^2c^2(g^2h^2x + h^2f^2y + f^2g^2z)^3(\theta+\alpha)^2(\theta+\beta)^2\, Q;$$

whence the above-mentioned conditions reduce themselves to the single condition

$$(\theta+\delta)^2\{3\Phi + xyz\,(*)\} = 27a^2b^2c^2(g^2h^2x + h^2f^2y + f^2g^2z)^3(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2\, Q.$$

14. But we have

$$\begin{aligned} & g^2h^2x + h^2f^2y + f^2g^2z \\ &\quad = g^2h^2a(\theta+\delta+f)^3 + h^2f^2b(\theta+\delta+g)^3 + f^2g^2c(\theta+\delta+h)^3, \\ &\quad = (\theta+\delta)^2\left[(g^2h^2a + h^2f^2b + f^2g^2c)(\theta+\delta) + 3(gha+hfb+fgc)fgh\right], \\ &\quad = -abc(\theta+\delta)^2\left[(gh+hf+fg)(\theta+\delta) + 3fgh\right], \\ &\quad = -abc(\theta+\delta)^2\left[gh(\theta+\alpha) + hf(\theta+\beta) + fg(\theta+\gamma)\right], \\ &\quad = -abc(\theta+\delta)^2 P, \end{aligned}$$

if for shortness

$$P = gh(\theta+\alpha) + hf(\theta+\beta) + fg(\theta+\gamma).$$

Hence, substituting,

$$3\Phi + abc(\theta+\alpha)^3(\theta+\beta)^3(\theta+\gamma)^3(*) = -27(abc)^5(\theta+\delta)^4(\theta+\alpha)^2(\theta+\beta)^2(\theta+\gamma)^2 P^3 Q;$$

which when the values $a(\theta+\alpha)^2$, $b(\theta+\beta)^3$, $c(\theta+\gamma)^3$ for (x, y, z) are substituted in the functions Φ and $(*)$, will be an identical equation in θ.

15. It is right to remark that what we require is the expression of $(*)$, $=(*\between x, y, z)^2$; the foregoing equation leads to the value of $(*)$ expressed in terms of θ; and it is necessary to show that this leads back to the expression for $(*)$ as a function of (x, y, z); in fact, that the function of θ is transformable in a *definite* manner into a function of (x, y, z). Suppose that the function of θ could be expressed in two different manners as a function of (x, y, z); then we should have two different functions $(x, y, z)^2$ each equivalent to the same function of θ; and the difference of these functions would be identically $=0$; that is, we should have a function $(x, y, z)^2$ vanishing identically by the substitution

$$x : y : z = a(\theta+\alpha)^3 : b(\theta+\beta)^3 : c(\theta+\gamma)^3;$$

but these relations are equivalent to the single relation

$$(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\,xyz = 0,$$

which, *quà* cubic equation, is not equivalent to any equation whatever of the form

$$(x, y, z)^2 = 0;$$

that is, the function of θ is equivalent to a definite functi n $(x, y, z)^2$.

16. To proceed with the reduction, I remark that we have

$$\Phi = (a^2x + b^2y + c^2z)^2 \left\{ \begin{array}{l} f^4[a^2f^2(h^4y^2 - 7h^2g^2yz + g^4z^2)(b^2y + c^2z) + b^2c^2(h^2y + g^2z)^3] \\ + g^4[b^2g^2(h^4x^2 - 7h^2f^2xz + f^4z^2)(a^2x + c^2z) + c^2a^2(h^2x + f^2z)^3] \\ + h^4[c^2h^2(g^4x^2 - 7g^2f^2xy + f^4y^2)(a^2x + b^2y) + a^2b^2(g^2x + f^2y)^3] \\ - a^2h^4g^4(b^2g^2 + c^2h^2)x^3 \\ - b^2h^4f^4(c^2h^2 + a^2f^2)y^3 \\ - c^2f^4g^4(a^2f^2 + b^2g^2)z^3 \end{array} \right\}$$

$$+ xyz\Omega$$

where

$$\begin{aligned} -\Omega = \quad & 2(b^2c^2Ax^2 + c^2a^2By^2 + a^2b^2Cz^2) \\ & + (My + Nz)(a^4x + 2a^2b^2y + 2a^2c^2z) \\ & + (Oz + Px)(2a^2b^2x + b^4y + 2b^2c^2z) \\ & + (Qx + Ry)(2a^2c^2x + 2b^2c^2y + cz), \end{aligned}$$

if for shortness

$$A = a^2g^4h^4\,(b^2g^2 + c^2h^2),$$
$$B = b^2h^4f^4\,(c^2h^2 + a^2f^2),$$
$$C = c^2f^4g^4\,(a^2f^2 + b^2g^2),$$

$$M = f^4h^2\,(a^2f^2c^2h^2 - 7a^2f^2b^2g^2 + 3b^2g^2c^2h^2), \qquad N = f^4g^2\,(-7a^2f^2c^2h^2 + a^2f^2b^2g^2 + 3b^2g^2c^2h^2),$$
$$O = g^4f^2\,(b^2g^2a^2f^2 - 7b^2g^2c^2h^2 + 3c^2h^2a^2f^2), \qquad P = g^4h^2\,(-7b^2g^2a^2f^2 + b^2g^2c^2h^2 + 3c^2h^2a^2f^2),$$
$$Q = h^4g^2\,(c^2h^2b^2g^2 - 7c^2h^2a^2b^2 + 3a^2f^2b^2g^2), \qquad R = h^4f^2\,(-7c^2h^2b^2g^2 + c^2h^2a^2f^2 + 3a^2f^2b^2g^2);$$

and I represent the foregoing equation by

$$\Phi = (a^2x + b^2y + c^2z)^2\,U + xyz\Omega.$$

Hence, writing for x, y, z the foregoing values, we have

$$\Phi = 9a^2b^2c^2\,(\theta+\alpha)^2\,(\theta+\beta)^2\,(\theta+\gamma)^2\,U + abc\,(\theta+\alpha)^2\,(\theta+\beta)^2\,(\theta+\gamma)^2\,\Omega;$$

and thence

$$27\,U + \frac{1}{abc}\,(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,\big(3\Omega + (*)\big) = -27\,(abc)^3\,(\theta+\delta)^4\,P^3Q;$$

that is

$$27\,[U + (abc)^3\,(\theta+\delta)^4\,P^3Q] + \frac{1}{abc}\,(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,\big(3\Omega + (*)\big) = 0.$$

In order that this may be the case, it is clear that we must have

$$U + (abc)^3\,(\theta+\delta)^4\,P^3Q = (\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,M,$$

viz. the left-hand side expressed as a function of θ must be divisible by the product $(\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)$. Assuming for a moment that this is so, the quotient M will be a function $(\theta, 1)^6$ expressible in a unique manner in the form $(x, y, z)^2$, and assuming it to be so expressed, we have

$$27Mabc + 3\Omega + (*) = 0;$$

which equation, without any further substitution of the θ-values of (x, y, z), gives $(*)$ in its proper form as a function of (x, y, z).

Reduction of the Equation, $U + (abc)^3\,(\theta+\delta)^4\,P^3\,Q = (\theta+\alpha)\,(\theta+\beta)\,(\theta+\gamma)\,M$.

17. We have by an easy transformation

$$\begin{aligned} U = \quad & (a^2x + b^2y + c^2z)\left\{\begin{array}{l} a^2f^6\,(h^4y^2 - 7h^2g^2yz + g^4z^2) \\ + b^2g^6\,(f^4z^2 - 7f^2h^2zx + h^4x^2) \\ + c^2h^6\,(g^4x^2 - 7g^2f^2xy + f^4y^2) \end{array}\right\} \\ & + 7\,(a^4f^4 + b^4g^4 + c^4h^4)\,f^2g^2h^2\,xyz \\ & + U', \end{aligned}$$

if for shortness

$$\begin{aligned} U' = \quad & y^2z \,.\, c^2h^4f^4\,(3b^2g^2 - c^2h^2) \\ & + yz^2 \,.\, b^2g^4f^4\,(3c^2h^2 - b^2g^2) \\ & + z^2x \,.\, a^2f^4g^4\,(3c^2h^2 - a^2f^2) \\ & + zx^2 \,.\, c^2h^4g^4\,(3a^2f^2 - c^2h^2) \\ & + x^2y \,.\, b^2g^4h^4\,(3a^2f^2 - b^2g^2) \\ & + xy^2 \,.\, a^2f^4h^4\,(3b^2g^2 - a^2f^2). \end{aligned}$$

Substituting the θ-values, the terms of U, other than U', are at once seen to contain the factor $(\theta+\alpha)(\theta+\beta)(\theta+\gamma)$, and we have

$$\begin{aligned} M = \quad & 3abc \left\{ \begin{array}{l} a^2f^6\,(h^4y^2 - 7h^2g^2yz + g^4z^2) \\ + b^2g^6\,(f^4z^2 - 7f^2h^2zx + h^4x^2) \\ + c^2h^6\,(g^4x^2 - 7g^2f^2xy + f^4y^2) \end{array} \right\} \\ & + 7\,(a^4f^4 + b^4g^4 + c^4h^4)\,f^2g^2h^2\,abc\,(\theta+\alpha)^3(\theta+\beta)^3(\theta+\gamma)^3 \\ & + M', \end{aligned}$$

where

$$U' + (abc)^3(\theta+\delta)^4\,P^3Q = (\theta+\alpha)(\theta+\beta)(\theta+\gamma)\,M'.$$

18. Write for shortness $p,\ q,\ r = (af,\ bg,\ ch)$; after a complicated reduction, I obtain

$$\begin{aligned} 3abc\,M' = \quad & a^2g^2h^2\,(r-p)(p-q)(-2p^4 + 5p^3qr - 6q^2r^2)\,x^2 \\ & + b^2h^2f^2\,(p-q)(q-r)(-2q^4 + 5q^2rp - 6q^2p^2)\,y^2 \\ & + c^2f^2g^2\,(q-r)(r-p)(-2r^4 + 5r^2pq - 6p^2q^2)\,z^2 \\ & + 2f^2g^2h^2b^2c^2\,(7p^4 - 20p^2qr + 4q^2r^2)\,yz \\ & + 2f^2g^2h^2c^2a^2\,(7q^4 - 20pq^2r + 4r^2p^4)\,zx \\ & + 2f^2g^2h^2a^2b^2\,(7r^4 - 20pqr^2 + 4p^2q^2)\,xy \\ & - 2f^2g^2h^2\,(p^4 + q^4 + r^4)\,(a^2x + b^2y + c^2z)^2. \end{aligned}$$

We then have

$$9abc\,M = \text{terms } (x,\ y,\ z)^2 + 9abc\,M', \quad \Omega = \text{terms } (x,\ y,\ z)$$

as above; and

$$27M\,abc + 3\Omega + (*) = 0,$$

which gives $(*)$.

19. After all reductions we find:

$$\begin{aligned} -\tfrac{1}{3}(*) = \quad & a^2g^2h^2\,(28p^6 - 84p^4qr + 62p^2q^2r^2 - 28q^3r^3)\,x^2 \\ & + b^2h^2f^2\,(28q^6 - 84pq^4r + 62p^3q^2r^2 - 28r^3p^3)\,y^2 \\ & + c^2f^2g^2\,(28r^6 - 84pqr^4 + 62p^2q^2r^4 - 28p^3q^3)\,z^2 \\ & + f^2\,(-3p^8 + 14p^6qr - 130p^4q^2r^2 - 136p^2q^3r^3 + 42q^4r^4)\,yz \\ & + g^2\,(-3q^8 + 14pq^6r - 130p^2q^4r^2 - 136p^3q^2r^3 + 42r^4p^4)\,zx \\ & + h^2\,(-3r^8 + 14pqr^6 - 130p^2q^2r^4 - 136p^3q^3r^2 + 42p^4q^4)\,xy\,; \end{aligned}$$

or observing that the coefficients of $a^2g^2h^2x^2$, $b^2h^2f^2y^2$ and $c^2f^2g^2z^2$ are equal to each other and to

$$62p^2q^2r^2 - 28\,(q^3r^3 + r^3p^3 + p^3q^3),$$

the equation becomes

$$\begin{aligned}(*) = \quad & -3\,(62p^2q^2r^2 - 28\,(q^3r^3 + r^3p^3 + p^3q^3)\,(a^2g^2h^2x^2 + b^2h^2f^2y^2 + c^2f^2g^2z^2)\\ & + 3\,(3p^8 - 14p^6qr + 130p^4q^2r^2 + 136p^2q^3r^3 - 42q^4r^4)\,f^2yz\\ & + 3\,(3q^8 - 14q^6pr + 130q^4p^2r^2 + 136q^2p^3r^3 - 42r^4p^4)\,g^2zx\\ & + 3\,(3r^8 - 14r^6pq + 130r^4p^2q^2 + 136r^2p^3q^3 - 42p^4q^4)\,h^2xy\,;\end{aligned}$$

and we thence obtain by symmetry the complete value of $(*\between x, y, z, w)^2$, viz. we have only to complete the literal parts of the foregoing expression into the forms

$$\begin{aligned}& a^2g^2h^2x^2 + b^2h^2f^2y^2 + c^2f^2g^2z^2 + a^2b^2c^2w^2,\\ & f^2yz + a^2xw,\\ & g^2zx + b^2yw,\\ & c^2xy + a^2zw,\end{aligned}$$

respectively.

20. The equation of the torse thus is

$$\begin{aligned}\Delta = a^6g^6h^6x^6 + b^6h^6f^6y^6 + c^6f^6g^6z^6 + a^6b^6c^6w^6\\ -f^6\,(h^2y + g^2z)^3\,(b^2y + c^2z)^3\\ -g^6\,(f^2z + h^2x)^3\,(c^2z + a^2x)^3\\ -h^6\,(g^2x + f^2y)^3\,(a^2x + b^2y)^3\\ -a^6\,(g^2x + c^2w)^3\,(h^2x + b^2w)^3\\ -b^6\,(h^2y + a^2w)^3\,(f^2y + c^2w)^3\\ -c^6\,(f^2z + b^2w)^3\,(g^2z + a^2w)^3\end{aligned}$$

$$\begin{aligned}& + (b^2f^2y + c^2f^2z + b^2c^2w)^3\,[(h^2y + g^2z + a^2w)^3 - 27a^2g^2h^2\,yzw]\\ & + (a^2g^2x + c^2g^2z + c^2a^2w)^3\,[(f^2z + h^2x + b^2w)^3 - 27b^2h^2f^2\,zxw]\\ & + (a^2h^2x + b^2h^2y + a^2b^2w)^3\,[(g^2x + f^2y + c^2w)^3 - 27c^2f^2g^2\,xyw]\\ & + (g^2h^2x + h^2f^2y + f^2g^2z)^3\,[(a^2x + b^2y + c^2z)^3 - 27a^2b^2c^2\,xyz]\\ & + xyzw\,(*\between x, y, z, w)^2 = 0.\end{aligned}$$

I recall that

$$\begin{aligned}a &= \beta - \gamma, & f &= \alpha - \delta, & p &= af = (\alpha - \delta)(\beta - \gamma),\\ b &= \gamma - \alpha, & g &= \beta - \delta, & q &= bg = (\beta - \delta)(\gamma - \alpha),\\ c &= \alpha - \beta, & h &= \gamma - \delta, & r &= ch = (\gamma - \delta)(\alpha - \beta).\end{aligned}$$

Developing, we have finally the equation of the torse in the form following.

Equation of the Sextic Torse.

21. The equation is

$$
\begin{aligned}
0 = \quad & (a^6g^6h^6,\ b^6h^6f^6,\ c^6f^6g^6,\ a^6b^6c^6 \between x^6,\ y^6,\ z^6,\ w^6) \\
+ 3(q^2+r^2) \quad & (b^4h^4f^6,\ c^4g^4f^6,\ g^4h^4a^6,\ b^4c^4a^6 \between y^5z,\ yz^5,\ x^5w,\ xw^5) \\
+ 3(r^2+p^2) \quad & (c^4f^4g^6,\ a^4h^4g^6,\ h^4f^4b^6,\ c^4a^4b^6 \between z^5x,\ zx^5,\ y^5w,\ yw^5) \\
+ 3(p^2+q^2) \quad & (a^4g^4h^6,\ b^4f^4h^6,\ f^4g^4c^6,\ a^4b^4c^6 \between x^5y,\ xy^5,\ z^5w,\ zx^5) \\
+ 3(q^4+3q^2r^2+r^4) & (b^2h^2f^6,\ c^2g^2f^6,\ g^2h^2a^6,\ b^2c^2a^6 \between y^4z^2,\ y^2z^4,\ x^4w^2,\ x^2w^4) \\
+ 3(r^4+3r^2p^2+p^4) & (c^2f^2g^6,\ a^2h^2g^6,\ h^2f^2b^6,\ c^2a^2b^6 \between z^4x^2,\ z^2x^4,\ y^4w^2,\ y^2w^4) \\
+ 3(p^4+3p^2q^2+q^4) & (a^2g^2h^6,\ b^2f^2h^6,\ f^2g^2c^6,\ a^2b^2c^6 \between x^4y^2,\ x^2y^4,\ z^4w^2,\ z^2w^4) \\
+ (6p^4+9p^2q^2+9p^2r^2-21q^2r^2) & (a^2g^4h^4,\ a^2b^4c^4,\ f^2h^4b^4,\ f^2g^4c^4 \between x^4yz,\ w^4yz,\ y^4wx,\ z^4wy) \\
+ (6q^4+9q^2r^2+9q^2p^2-21r^2p^2) & (b^2h^4f^4,\ b^2c^4a^4,\ g^2f^4c^4,\ g^2h^4a^4 \between y^4zx,\ w^4zx,\ z^4wy,\ x^4wz) \\
+ (6r^4+9r^2p^2+9r^2q^2-21p^2q^2) & (c^2f^4g^4,\ c^2a^4b^4,\ h^2g^4a^4,\ h^2f^4b^4 \between z^4xy,\ w^4xy,\ x^4wz,\ y^4wx) \\
+ (q^6+9q^4r^2+9q^2r^4+r^6) \quad & (f^6,\ a^6 \between y^2z^2,\ x^2w^2) \\
+ (r^6+9r^4p^2+9r^2p^4+p^6) \quad & (g^6,\ b^6 \between z^2x^2,\ y^2w^2) \\
+ (p^6+9p^4q^2+9p^2q^4+q^6) \quad & (h^6,\ c^6 \between x^2y^2,\ z^2w^2) \\
+ 9(q^4r^2+q^2r^4+r^4p^2+r^2p^4+p^4q^2+p^2q^4-14p^2q^2r^2) & \\
\times \quad & (f^2g^2h^2,\ f^2b^2c^2,\ g^2c^2a^2,\ h^2a^2b^2 \between x^2y^2z^2,\ y^2z^2w^2,\ z^2x^2w^2,\ x^2y^2w^2) \\
+ 3\{p^6+3p^4(2q^2+r^2)+3p^2(q^4-7q^2r^2)+q^4r^2\} & \\
\times \quad & (g^2h^4,\ h^4b^2,\ g^2c^4,\ c^4b^2 \between x^2y^2z,\ y^2wx^2,\ z^2w^2x,\ w^2yz^2) \\
+ 3\{q^6+3q^4(2r^2+p^2)+3q^2(r^4-7r^2p^2)+r^4p^2\} & \\
\times \quad & (h^2f^4,\ f^4c^2,\ h^2a^4,\ a^4c^2 \between xy^2z^2,\ z^2wy^2,\ x^2w^2y,\ w^2zx^2) \\
+ 3\{r^6+3r^4(2p^2+q^2)+3r^2(p^4-7p^2q^2)+p^4q^2\} & \\
\times \quad & (f^2g^4,\ g^4a^2,\ f^2b,\ b^4a^2 \between x^2yz^2,\ x^2wz^2,\ y^2w^2z,\ w^2xy^2) \\
+ 3\{p^6+3p^4(2r^2+q^2)+3p^2(r^4-7q^2r^2)+q^2r^4\} & \\
\times \quad & (g^4h^2,\ h^2b^4,\ g^4c^2,\ c^2b^4 \between x^2yz^2,\ y^2w^2x,\ z^2wx^2,\ w^2y^2z) \\
+ 3\{q^6+3q^4(2p^2+r^2)+3q^2(p^4-7r^2p^2)+r^2p^4\} & \\
\times \quad & (h^4f^2,\ f^2c^4,\ h^4a^2,\ a^2c^4 \between x^2y^2z,\ z^2wy^2,\ x^2wy^2,\ w^2z^2x) \\
+ 3\{r^6+3r^4(2q^2+p^2)+3r^2(q^4-7p^2q^2)+p^2q^4\} & \\
\times \quad & (f^4g^2,\ g^2a^4,\ f^4b^2,\ b^2a^4 \between xy^2z^2,\ x^2w^2z,\ y^2wz^2,\ w^2x^2y) \\
+ xyzw \left\{ \begin{array}{ll}
-3\{62p^2q^2r^2-28(q^2r^3+r^2p^3+p^2q^3)\} & (a^2g^2h^2,\ b^2h^2f^2,\ c^2f^2g^2,\ a^2b^2c^2 \between x^2,\ y^2,\ z^2,\ w^2) \\
+3(3p^8-14p^6qr+130p^4q^2r^3+136p^2q^3r^3-42q^4r^4) & (f^2,\ a^2 \between yz,\ xw) \\
+3(3q^8-14q^6rp+130q^4r^2p^2+136q^2r^3p^3-42r^4p^4) & (g^2,\ b^2 \between zx,\ yw) \\
+3(3r^8-14r^6pq+130r^4p^2q^2+136r^2p^3q^3-42p^4q^4) & (h^2,\ c^2 \between xy,\ zw)
\end{array} \right\}.
\end{aligned}
$$

Comparison with the Equation of the Centro-surface of an Ellipsoid.

22. In the Equation

$$\frac{x}{(\theta+\alpha)^2}+\frac{y}{(\theta+\beta)^2}+\frac{z}{(\theta+\gamma)^2}+\frac{w}{(\theta+\delta)^2}=0,$$

for x, y, z, w write ξ^2, η^2, ζ^2, ω^2, and then $\delta=\infty$, the equation is converted into

$$\frac{\xi^2}{(\theta+\alpha)^2}+\frac{\eta^2}{(\theta+\beta)^2}+\frac{\zeta^2}{(\theta+\gamma)^2}+\omega^2=0;$$

or writing a^2, b^2, c^2 for α, β, γ, and understanding ξ^2, η^2, ζ^2, ω^2 to mean a^2x^2, b^2y^2, c^2z^2, -1, this is

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}+\frac{c^2z^2}{(\theta+c^2)^2}-1=0.$$

This is an equation, the envelope of which in regard to the variable parameter θ, gives the surface which is the locus of the centres of curvature of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, or say the Centro-surface of the Ellipsoid. (Salmon's *Solid Geometry*, Ed. 2, p. 400, [Ed. 4, p. 465].)

Making the same substitution in the foregoing equation $(*\between x, y, z, w)^6=0$, the quantities f, g, h become equal to $-\delta$, and p, q, r to $-a\delta$, $-b\delta$, $-c\delta$ respectively, and the whole equation divides by δ^{12}; throwing out this factor, we have a result which is obtained more simply by changing

$$x,\ y,\ z,\ w,\quad a,\ b,\ c,\ f,\ g,\ h,\quad p,\ q,\ r,$$

into

$$\xi^2,\ \eta^2,\ \zeta^2,\ \omega^2,\quad \alpha,\ \beta,\ \gamma,\ 1,\ 1,\ 1,\quad \alpha,\ \beta,\ \gamma,$$

where α, β, γ now signify b^2-c^2, c^2-a^2, a^2-b^2 respectively, and ξ^2, η^2, ζ^2, ω^2 are retained as standing for a^2x^2, b^2y^2, c^2z^2, -1 respectively; viz. the equation of the centro-surface is found to be

$$\begin{aligned}
0= \quad & (\alpha^6,\ \beta^6,\ \gamma^6,\ \alpha^6\beta^6\gamma^6\between\xi^{12},\ \eta^{12},\ \zeta^{12},\ \omega^{12})\\
+3(\beta^2+\gamma^2)\quad & (\beta^4,\ \gamma^2,\ \alpha^6,\ \alpha^6\beta^4\gamma^4\between\eta^{10}\zeta^2,\ \eta^8\zeta^{10},\ \xi^{10}\omega^2,\ \xi^2\omega^{10})\\
+3(\gamma^2+\alpha^2)\quad & (\gamma^4,\ \alpha^2,\ \beta^6,\ \beta^6\gamma^4\alpha^4\between\zeta^{10}\xi^2,\ \zeta^8\xi^{10},\ \eta^{10}\omega^2,\ \eta^2\omega^{10})\\
+3(\alpha^2+\beta^2)\quad & (\alpha^4,\ \beta^2,\ \gamma^6,\ \gamma^6\alpha^4\beta^4\between\xi^{10}\eta^2,\ \xi^8\eta^{10},\ \zeta^{10}\omega^2,\ \zeta^2\omega^{10})\\
+3(\beta^4+3\beta^2\gamma^2+\gamma^4)\quad & (\beta^2,\ \gamma^2,\ \alpha^6,\ \alpha^6\beta^2\gamma^2\between\eta^8\zeta^4,\ \eta^4\zeta^8,\ \xi^8\omega^4,\ \xi^4\omega^8)\\
+3(\gamma^4+3\gamma^2\alpha^2+\alpha^4)\quad & (\gamma^2,\ \alpha^2,\ \beta^6,\ \beta^6\gamma^2\alpha^2\between\zeta^8\xi^4,\ \zeta^4\xi^8,\ \eta^8\omega^4,\ \eta^4\omega^8)\\
+3(\alpha^4+3\alpha^2\beta+\beta^4)\quad & (\alpha^2,\ \beta^2,\ \gamma^6,\ \gamma^6\alpha^2\beta^2\between\xi^8\eta^4,\ \xi^4\eta^8,\ \zeta^8\omega^4,\ \zeta^4\omega^8)\\
+3(2\alpha^4+3\alpha^2\beta^2+3\alpha^2\gamma^2-7\beta^2\gamma^2)\ & (\alpha^2,\ \alpha^2\beta^4\gamma^4,\ \beta^4,\ \gamma^4\between\xi^8\eta^2\zeta^2,\ \omega^8\eta^2\zeta^2,\ \eta^8\omega^2\xi^2,\ \zeta^8\omega^2\eta^2)\\
+3(2\beta^4+3\beta^2\gamma^2+3\beta^2\alpha^2-7\gamma^2\alpha^2)\ & (\beta^2,\ \beta^2\gamma^4\alpha^4,\ \gamma^4,\ \alpha^4\between\eta^8\zeta^2\xi^2,\ \omega^8\zeta^2\xi^2,\ \zeta^8\omega^2\eta^2,\ \xi^8\omega^2\zeta^2)\\
+3(2\gamma^4+3\gamma^2\alpha^2+3\gamma^2\beta^2-7\alpha^2\beta^2)\ & (\gamma^2,\ \gamma^2\alpha^4\beta^4,\ \alpha^4,\ \beta^4\between\zeta^8\xi^2\eta^2,\ \omega^8\xi^2\eta^2,\ \xi^8\omega^2\zeta^2,\ \eta^8\omega^2\xi^2)
\end{aligned}$$

$$
\begin{aligned}
&+ (\beta^6 + 9\beta^4\gamma^2 + 9\beta^2\gamma^4 + \gamma^6) && (1,\ \alpha^6 \between \eta^3\zeta^3,\ \xi^3\omega^3)\\
&+ (\gamma^6 + 9\gamma^4\alpha^2 + 9\gamma^2\alpha^4 + \alpha^6) && (1,\ \beta^6 \between \zeta^3\xi^3,\ \eta^3\omega^3)\\
&+ (\alpha^6 + 9\alpha^4\beta^2 + 9\alpha^2\beta^4 + \beta^6) && (1,\ \gamma^6 \between \xi^3\eta^3,\ \zeta^3\omega^3)
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\,\{\alpha^6 + 3\alpha^4\,(2\beta^2 + \gamma^2) + 3\alpha^2\,(\beta^4 - 7\beta^2\gamma^2) + \beta^4\gamma^2\}\,(1,\ \beta^2,\ \gamma^4,\ \beta^2\gamma^4 \between \xi^6\eta^4\zeta^3,\ \eta^6\omega^3\xi^4,\ \zeta^6\omega^4\xi^3,\ \omega^6\eta^3\zeta^4)\\
&+ 3\,\{\beta^6 + 3\beta^4\,(2\gamma^2 + \alpha^2) + 3\beta^2\,(\gamma^4 - 7\gamma^2\alpha^2) + \gamma^4\alpha^2\}\,(1,\ \gamma^2,\ \alpha^4,\ \gamma^2\alpha^4 \between \eta^6\zeta^4\xi^3,\ \zeta^6\omega^3\eta^4,\ \xi^6\omega^4\eta^3,\ \omega^6\zeta^3\xi^4)\\
&+ 3\,\{\gamma^6 + 3\gamma^4\,(2\alpha^2 + \beta^2) + 3\gamma^2\,(\alpha^4 - 7\alpha^2\beta^2) + \alpha^4\beta^2\}\,(1,\ \alpha^2,\ \beta^4,\ \alpha^2\beta^4 \between \zeta^6\xi^4\eta^3,\ \xi^6\omega^3\zeta^4,\ \eta^6\omega^4\zeta^3,\ \omega^6\xi^3\eta^4)\\
&+ 3\,\{\alpha^6 + 3\alpha^4\,(2\gamma^2 + \beta^2) + 3\alpha^2\,(\gamma^4 - 7\beta^2\gamma^2) + \beta^2\gamma^4\}\,(1,\ \gamma^2,\ \beta^4,\ \beta^4\gamma^2 \between \xi^6\eta^3\zeta^4,\ \zeta^6\omega^3\xi^4,\ \eta^6\omega^4\xi^3,\ \omega^6\eta^4\zeta^3)\\
&+ 3\,\{\beta^6 + 3\beta^4\,(2\alpha^2 + \gamma^2) + 3\beta^2\,(\alpha^4 - 7\gamma^2\alpha^2) + \gamma^2\alpha^4\}\,(1,\ \alpha^2,\ \gamma^4,\ \gamma^4\alpha^2 \between \eta^6\zeta^3\xi^4,\ \xi^6\omega^3\eta^4,\ \zeta^6\omega^4\eta^3,\ \omega^6\zeta^4\xi^3)\\
&+ 3\,\{\gamma^6 + 3\gamma^4\,(2\beta^2 + \alpha^2) + 3\gamma^2\,(\beta^4 - 7\alpha^2\beta^2) + \alpha^2\beta^4\}\,(1,\ \beta^2,\ \alpha^4,\ \alpha^4\beta^2 \between \zeta^6\xi^3\eta^4,\ \eta^6\omega^3\zeta^4,\ \xi^6\omega^4\zeta^3,\ \omega^6\xi^4\eta^3)\\
&+ 9\,(\beta^4\gamma^2 + \beta^2\gamma^4 + \gamma^4\alpha^2 + \gamma^2\alpha^4 + \alpha^4\beta^2 + \alpha^2\beta^4 - 14\alpha^2\beta^2\gamma^2)\\
&\qquad\qquad (1,\ \beta^2\gamma^2,\ \gamma^2\alpha^2,\ \alpha^2\beta^2 \between \xi^4\eta^4\zeta^4,\ \eta^4\zeta^4\omega^4,\ \zeta^4\xi^4\omega^4,\ \xi^4\eta^4\omega^4)
\end{aligned}
$$

$$
+ \xi^3\eta^3\zeta^3\omega^3 \left\{
\begin{aligned}
&- 3\,\{62\alpha^2\beta^2\gamma^2 - 28\,(\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3)\}\,(\alpha^2,\ \beta^2,\ \gamma^2,\ \alpha^2\beta^2\gamma^2 \between \xi^4,\ \eta^4,\ \zeta^4,\ \omega^4)\\
&+ 3\,(3\alpha^8 - 14\alpha^6\beta\gamma + 130\alpha^4\beta^2\gamma^2 + 136\alpha^2\beta^3\gamma^3 - 42\beta^4\gamma^4)\ \ (1,\ \alpha^2 \between \eta^3\zeta^3,\ \xi^3\omega^3)\\
&+ 3\,(3\beta^8 - 14\beta^6\gamma\alpha + 130\beta^4\gamma^2\alpha^2 + 136\beta^2\gamma^3\alpha^3 - 42\gamma^4\alpha^4)\ \ (1,\ \beta^2 \between \zeta^3\xi^3,\ \eta^3\omega^3)\\
&+ 3\,(3\gamma^8 - 14\gamma^6\alpha\beta + 130\gamma^4\alpha^2\beta^2 + 136\gamma^3\alpha^3\beta^3 - 42\alpha^4\beta^4)\ \ (1,\ \gamma^2 \between \xi^3\eta^3,\ \zeta^3\omega^3)
\end{aligned}
\right\}.
$$

This agrees with the result given in Salmon's *Solid Geometry*, Ed. 2, p. 151, [Ed. 4, p. 178], and *Quarterly Mathematical Journal*, vol. II. p. 220 (1858); in the latter place, however, the term

$$\beta^2\eta^8\zeta^4 + \gamma^2\eta^4\zeta^8 + \alpha^6\xi^8\omega^4 + \alpha^6\beta^2\gamma^2\xi^4\omega^8$$

is by mistake written

$$\beta^2\eta^8\zeta^4 + \gamma^2\eta^4\zeta^8 + \alpha^6\xi^8\omega^4 + \quad \beta^2\gamma^2\xi^4\omega^8;$$

viz. a factor α^6 is omitted in one of the coefficients.

Some of the coefficients are presented under slightly different forms; viz. instead of

$$62\alpha^2\beta^2\gamma^2 - 28\,(\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3)$$

Salmon has

$$14\,(\beta^4\gamma^2 + \beta^2\gamma^4 + \gamma^4\alpha^2 + \gamma^2\alpha^4 + \alpha^4\beta^2 + \alpha^2\beta^4) + 20\alpha^2\beta^2\gamma^2;$$

and instead of

$$3\alpha^8 - 14\alpha^6\beta\gamma + 130\alpha^4\beta^2\gamma^2 + 136\alpha^2\beta^3\gamma^3 - 42\beta^4\gamma^4,$$

he has

$$-4\alpha^8 + 7\alpha^6\,(\beta^2 + \gamma^2) + 196\alpha^4\beta^2\gamma^2 - 68\alpha^2\beta^2\gamma^2\,(\beta^2 + \gamma^2) - 42\beta^4\gamma^4,$$

but these different forms are respectively equivalent in virtue of the relation

$$\alpha + \beta + \gamma = 0.$$

437.

DÉMONSTRATION NOUVELLE DU THÉORÈME DE M. CASEY PAR RAPPORT AUX CERCLES QUI TOUCHENT À TROIS CERCLES DONNÉS.

[From the *Annali di Matematica pura ed applicata*, tom. I. (1867), pp. 132—134.]

THIS is in fact the investigation contained in the paper 414, "On Polyzomal Curves otherwise the curves $\sqrt{U}+\sqrt{V}+\&c.=0$," Annex II. pp. 568—573, "On Casey's theorem for the circle which touches three given circles," viz. it is based on the identity of the two problems 1° to find a circle touching three given circles, 2° to find a cone-sphere (sphere of radius zero) passing through three given points in space.

438.

NOTE SUR QUELQUES TORSES SEXTIQUES.

[From the *Annali di Matematica pura ed applicata*, tom. II. (1868), pp. 99, 100.]

Je désire d'appeler attention aux surfaces développables, ou torses, données par l'équation

$$(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e + 2bcd - c^3)^2 = 0.$$

Dans cette équation (a, b, c, d, e) sont des fonctions linéaires quelconques des quatre coordonnées (x, y, z, t); ces quantités sont donc liées par une équation linéaire

$$Aa + 4Bb + 6Cc + 4Dd + Ee = 0,$$

et je remarque que la classification des torses comprises sous l'équation mentionnée dépend des propriétés invariantives de la fonction $(A, B, C, D, E \between \tau, 1)^4$.

En effet la torse a une courbe cuspidale, ou arête de rebroussement, donnée par les équations

$$ae - 4bd + 3c^2 = 0, \quad ace - ad^2 - b^2e + 2bcd - c^3 = 0,$$

et une courbe nodale donnée par les équations

$$\frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e}:$$

ces deux courbes se rencontrent dans les points donnés par les équations

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e},$$

lesquels sont des points stationnaires de la courbe cuspidale. Pour trouver ces points, en écrivant $a : b : c : d : e = \tau^4 : \tau^3 : \tau^2 : \tau : 1$, on obtient pour le paramètre τ l'équation

$$(A, B, C, D, E \between \tau, 1)^4 = 0$$

et l'on voit ainsi qu'il y a un rapport entre la théorie de la surface et cette équation; à toute particularité invariantive de l'équation, il y correspondra quelque particularité de la torse.

Les cas à considérer sont:

1°. Racines inégales, sans aucune relation invariantive. C'est le cas général; je l'ai considéré dans le Mémoire, "On a certain Sextic Developable," *Quart. Math. Journ.*, t. IX. (1868), pp. 129—142, [398].

2°. Deux racines égales. Ce cas n'a pas été considéré; je remarque que la courbe cuspidale est du cinquième ordre. En effet on peut supposer que les racines égales soient $=0$, ce qui revient à prendre $D=0$, $E=0$. On a donc $Aa+4Bb+6Cc=0$, c'est-à-dire les équations $a=0$, $b=0$ impliquent l'équation $c=0$; et on voit de là que les surfaces $ae-4bd+3c^2=0$, $ace-ad^2-b^2e+2bcd-c^3=0$ se coupent selon la droite $a=0$, $b=0$; il reste ainsi une courbe du cinquième ordre pour la courbe cuspidale.

3°. Trois racines égales. On peut supposer que ces racines sont $=0$, ce qui donne $C=0$, $D=0$, $E=0$; et l'on a ainsi $Aa+4Bb=0$, c'est-à-dire les plans $a=0$, $b=0$ sont ici un seul plan. L'équation de la torse contient le facteur a, et en l'écartant elle se réduit au cinquième ordre; on obtient ainsi la torse générale du cinquième ordre.

4°. Deux paires de racines égales. On peut supposer que ces racines sont $=\infty$, 0; cela donne $A=0$, $B=0$, $D=0$, $E=0$, et l'on a ainsi identiquement $c=0$. L'équation de la torse est $(ae-4bd)^3-27(-ad^2-b^2e)^2=0$. J'ai considéré ce cas dans le Mémoire, "On a Special Sextic Developable," *Quart. Math. Journ.*, t. VII. (1866), pp. 105—113, [373]; la courbe cuspidale est du quatrième ordre, une courbe excubo-quartique d'une forme particulière.

5°. Quatre racines égales; en prenant ces racines $=0$, on a $B=C=D=E=0$, donc identiquement $a=0$; l'équation de la torse contient le facteur b^2, et en l'écartant elle se réduit au quatrième ordre: on a dans ce cas la torse générale du quatrième ordre. Il y a encore deux cas à considérer.

6°. L'invariant I de la fonction $(A, B, C, D, E \between \tau, 1)^4$ est $=0$;

7°. L'invariant J de cette fonction est $=0$;

Mais je n'ai pas encore examiné ce que cela veut dire [1]. Il n'y a pas le cas à considérer où l'on a à la fois $I=0$, $J=0$; car cela revient au cas 3° de trois racines égales.

Cambridge, le 19 *mai* 1868.

[1] La courbe cuspidale étant du genre 0 (unicursale), on peut considérer la série des points de la courbe comme correspondant anharmoniquement aux points d'une droite. Si le système des quatre points stationnaires est harmonique on a $J=0$; si ce système est équi-anharmonique, on a $I=0$.

439.

ADDITION À LA NOTE SUR QUELQUES TORSES SEXTIQUES.

[From the *Annali di Matematica pura ed applicata*, tom. II. (1868), pp. 219—221.]

JE viens de trouver ce que signifie la condition $J=0$. Considérons deux surfaces quadriques qui se touchent (d'un contact ordinaire). Les équations tangentielles peuvent s'écrire sous la forme

$$a\,\xi^2+b\,\eta^2+c\,\zeta^2+2n\,\zeta\omega=0,$$

$$a'\xi^2+b'\eta^2+c'\zeta^2+2n'\zeta\omega=0,$$

et l'on satisfait à ces équations par des valeurs de ξ, η, ζ, ω qui contiennent un paramètre arbitraire θ, en écrivant

$$\xi:\eta:\zeta:\omega=\alpha\left(\theta+\frac{1}{\theta}\right):\beta\left(\theta-\frac{1}{\theta}\right):\epsilon:\gamma\left(\theta^2+\frac{1}{\theta^2}\right)+\delta;$$

en effet cela donne

$$a\,\alpha^2+b\,\beta^2+2n\,\gamma\epsilon=0,\qquad 2a\,\alpha^2-2b\,\beta^2+c\,\epsilon^2+2n\,\delta\epsilon=0,$$

$$a'\alpha^2+b'\beta^2+2n'\gamma\epsilon=0,\qquad 2a'\alpha^2-2b'\beta^2+c'\epsilon^2+2n'\delta\epsilon=0,$$

ce qui détermine les valeurs de $\alpha:\beta:\gamma:\delta:\epsilon$. L'équation du plan tangent commun sera donc

$$x\alpha\left(\theta+\frac{1}{\theta}\right)+y\beta\left(\theta-\frac{1}{\theta}\right)+z\epsilon+w\left[\gamma\left(\theta^2+\frac{1}{\theta^2}\right)+\delta\right]=0,$$

ou, en multipliant par $12\theta^2$, cette équation sera

$$(a,\ b,\ c,\ d,\ e\between\theta,\ 1)^4=0,$$

les valeurs des coefficients étant

$$\begin{aligned} a &= 12\ \gamma w, \\ b &= \ 3\,(\alpha x + \beta y), \\ c &= \ 2\,(\delta w + \epsilon z\ ), \\ d &= \ 3\,(\alpha x - \beta y), \\ e &= 12\ \gamma w. \end{aligned}$$

En représentant par $Aa + 4Bb + 6Cc + 4Dd + Ee = 0$ l'équation qui lie les fonctions linéaires a, b, c, d, e, cette équation sera $a - e = 0$; on a donc $A = -E = 1$, $B = C = D = 0$; l'invariant J de la fonction $(A, B, C, D, E \between \tau, 1)^4$ est donc $= 0$.

Nous arrivons ainsi à la conclusion que la torse sextique

$$(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e - c^3 + 2bcd)^2 = 0,$$

où les fonctions linéaires a, b, c, d, e sont liées par une équation

$$Aa + 4Bb + 6Cc + 4Dd + Ee = 0,$$

telle que l'invariant

$$J = ACE - AD^2 - EB^2 - C^3 - 2BCD$$

de la fonction $(A, B, C, D, E \between \tau, 1)^4$ est $= 0$ (cas 7° de la Note), est la torse enveloppée par le plan tangent commun de deux surfaces quadriques qui se touchent d'un contact ordinaire. J'ai trouvé l'équation de cette torse dans le Mémoire, "On the Developable Surfaces which arise from two Surfaces of the Second Order," *Camb. and Dubl. Math. Jour.*, t. v. (1850), pp. 46—57, voir p. 56, [85]: $a, b, c, n, a', b', c', n'$ y dénotent les mêmes coefficients comme à présent, et en écrivant

$$\begin{aligned} bc' - b'c &= f, \quad an' - a'n = p, \\ ca' - c'a &= g, \quad bn' - b'n = q, \\ ab' - a'b &= h, \quad cn' - c'n = r, \end{aligned}$$

(et de là $pf + qg + rh = 0$), l'équation trouvée est

$$f^2g^2h^2w^6 \ldots + 4pqr\,(qx^2 + py^2)^2 = 0.$$

Je vais vérifier ces termes. Partant de l'équation $(a, b, c, d, e \between \theta, 1)^4 = 1$, l'équation de la torse, en y introduisant pour commodité le facteur $-\frac{1}{216}\,pqr$, sera

$$-\tfrac{1}{216}\,pqr\,\{4\,[(\delta w + \epsilon z)^2 + 12\gamma^2w^2 - 3\,(\alpha^2x^2 - \beta^2y^2)]^3$$
$$-\,[2\,(\delta w + \epsilon z)^3 - 9\,(\delta w + \epsilon z)\,(\alpha^2x^2 - \beta^2y^2) - 27\,(\delta w + \epsilon z)\,\gamma^2w^2 + 54\,(\alpha^2x^2 + \beta^2y^2)\,\gamma w]^2\} = 0.$$

En prenant $\gamma\epsilon = ab' - a'b = h$, on obtient pour α, β, γ, δ, ϵ les valeurs

$$\alpha^2 = 2q, \quad \beta^2 = -2p, \quad \gamma\epsilon = h, \quad \epsilon^2 = -\frac{8pq}{r}, \quad \delta\epsilon = \frac{2(fp-gq)}{r},$$

et de là

$$\gamma^2 = -\frac{h^2 r}{8pq}, \quad \delta^2 = -\frac{(fp-gq)^2}{2pqr} = -\frac{h^2r^2 + 4fgpq}{2pqr} = 4\gamma^2 + \frac{2fg}{r}, \quad \delta^2 - 4\gamma^2 = \frac{2fg}{r}.$$

Les termes en w^6 et $(x^2, y^2)^3$ sont

$$\begin{aligned} &-\tfrac{1}{216}\, pqr\, \{[4(\delta^2 + 12\gamma^2)^3 - (2\delta^2 - 72\gamma^2\delta)^2]\, w^6 - 108\,(\alpha^2x^2 - \beta^2y^2)^3\}, \\ &= -\tfrac{1}{216}\, pqr\, \{432\gamma^2(\delta^2 - 4\gamma^2)^2\, w^6 - 108\,(\alpha^2x^2 - \beta^2y^2)^3\}\,; \end{aligned}$$

ces termes sont donc

$$\begin{aligned} &= -\tfrac{1}{216}\, pqr \left\{-432\,\frac{h^2r}{8pq}\cdot\frac{4f^2g^2}{r^2}\, w^2 + 864\,(qx^2 + pq^2)^2\right\} \\ &= f^2g^2h^2w^6 + 4pqr\,(qx^2 + py^2)^3, \end{aligned}$$

comme cela doit être.

Cambridge, le 22 *septembre* 1868.

440.

NOTE SUR UNE TRANSFORMATION GÉOMÉTRIQUE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXVII. (1867), pp. 95, 96.]

LA lecture de la Note de M. Hesse, "Ein Uebertragungsprincip" (t. LXVI. p. 15 de ce Journal) m'a suggéré les remarques suivantes:

Soient (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), (a_3, b_3, c_3, d_3) des constantes données, on peut supposer que les coordonnées (x, y) d'un point quelconque dans un plan soient exprimées en fonctions des paramètres variables (u, v) par les équations

$$x = \frac{a_1 + b_1u + c_1v + d_1uv}{a_3 + b_3u + c_3v + d_3uv}, \qquad y = \frac{a_2 + b_2u + c_2v + d_2uv}{a_3 + b_3u + c_3v + d_3uv}.$$

En introduisant une nouvelle indéterminée s, ces équations peuvent être écrites dans la forme

$$\begin{aligned} sx &= a_1 + b_1u + c_1v + d_1uv, \\ sy &= a_2 + b_2u + c_2v + d_2uv, \\ s &= a_3 + b_3u + c_3v + d_3uv\,; \end{aligned}$$

pour des valeurs données des coordonnées (x, y) la quantité s est en général déterminée par une équation quadratique, et les paramètres u et v sont des fonctions linéaires données de s; il y a cependant deux cas particuliers qu'il convient de distinguer.

1°. L'équation quadratique en s peut avoir la racine $s = 0$ et, débarrassée de ce facteur, se réduire par conséquent à une équation linéaire; ce cas particulier a lieu si

la condition $(abc)(bcd) = (abd)(acd)$ est remplie, où la notation (abc) désigne le déterminant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Dans ce cas u et v sont des fonctions rationnelles de (x, y) et la transformation a la signification géométrique suivante :

En considérant deux droites quelconques L, M dans l'espace et en menant par le point donné (x, y) la droite unique G qui rencontre ces deux droites, on peut supposer que u et v soient des paramètres qui déterminent les positions des points de rencontre sur les deux droites respectivement ; c. à. d. que u soit la distance d'un point fixe sur la droite L au point de rencontre avec la droite G, et de même que v soit la distance d'un point fixe sur la droite M au point de rencontre avec la droite G.

2°. Supposons $b_1 : c_1 = b_2 : c_2 = b_3 : c_3$, ou ce qui est au fond la même chose $b_1 - c_1 = 0$, $b_2 - c_2 = 0$, $b_3 - c_3 = 0$; alors s est déterminée par une équation simple, mais u et v ne sont plus des fonctions rationnelles de s ; on voit que dans ce cas $u + v$ et uv sont des fonctions rationnelles de (x, y), et que par conséquent u et v sont les racines d'une équation quadratique qui contient (x, y) linéairement. On peut supposer que u et v soient les paramètres de deux points sur une droite donnée, c. à. d. que u et v soient les distances de ces deux points respectivement à un point fixe situé sur la droite donnée ; on a ainsi la transformation de M. Hesse.

Je n'ai pas cherché la signification géométrique des formules générales.

Cambridge, 10 *octobre* 1866.

441.

NOTE SUR L'ALGORITHME DES TANGENTES DOUBLES D'UNE COURBE DU QUATRIÈME ORDRE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXVIII. (1868), pp. 176—179.]

On n'a pas, je crois, assez fait attention à l'algorithme (tiré de la considération d'une figure dans l'espace) qu'a trouvé M. Hesse (dans le mémoire "Ueber die Doppeltangenten der Curven vierter Ordnung," t. XLIX de ce Journal, 1855) pour dénoter les tangentes doubles (ou bitangentes) d'une courbe du quatrième ordre. Voici en quoi cet algorithme consiste. En employant les huit symboles 1, 2, 3, ... 8, les 28 bitangentes sont représentées par les combinaisons binaires 12, 13, 14, ... 78. Cela posé, considerons une expression quelconque 12 . 13 . 14, ou 12 . 34, ... ou disons un "terme" qui représente un système d'une seule ou de plusieurs des bitangentes. On peut opérer sur ce terme avec deux espèces de substitutions; la substitution ordinaire qui consiste à changer l'arrangement 12345678 des huit symboles en un autre arrangement quelconque; et la substitution "bifide" représentée par un symbole tel que 1234.5678, lequel dénote qu'il faut entrechanger les combinaisons 12 et 34, 13 et 24, 14 et 23, 56 et 78, 57 et 68, 58 et 67, en ne changeant pas les autres combinaisons. Par exemple en opérant avec 1234.5678 sur 34.45.56.17 on obtient 12.45.78.17. Le nombre de ces substitutions bifides est 35, ou en comptant la substitution, unité, qui ne change aucune des combinaisons, ce nombre est 36.

Appelons "homotypiques" deux termes qui se dérivent l'un de l'autre par une substitution ordinaire; "syntypiques" qui se dérivent l'un de l'autre par une substitution ordinaire ou bifide; "sous-groupe" le système entier des termes homotypiques à un terme donné: "groupe" le système entier des termes syntypiques à un terme donné. Un groupe peut contenir un seul sous-groupe, ou plusieurs sous-groupes; mais il importe de remarquer que la notion du sous-groupe n'a pas de signification géométrique, et ne

sert que comme moyen de former les termes du groupe. Cela étant, le théorème géométrique est celui-ci; "les systèmes de bitangentes représentées par des termes syntypiques (ou autrement dit, par des termes qui appartiennent au même groupe) ont les mêmes propriétés géométriques."

Par exemple, en considérant les bitangentes deux à deux, on a deux sous-groupes, l'un composé de termes homotypiques à 12.13; l'autre, de termes homotypiques à 12.34—ou disons, le sous-groupe 12.13 de 168 termes et le sous-groupe 12.34 de 210 termes; mais ces deux sous-groupes ne forment qu'un seul groupe: pour montrer cela il suffit d'opérer sur 12.13, par exemple avec la substitution 1245.3678, ce qui donne 45.13, terme homotypique à 12.34. Cela veut dire qu'il n'y a pas de combinaison de deux bitangentes qui se distingue d'une manière quelconque de toute autre combinaison de deux bitangentes.

Mais en combinant les bitangentes trois à trois, on a les deux sous-groupes 12.34.56 (420 termes) et 12.23.34 (840 termes) qui forment un groupe de 1260 termes; les trois bitangentes représentées par un quelconque des 1260 termes ont leurs six points de contact sur une même conique. Les trois autres sous-groupes 12.23.31 (56 termes), 12.23.45 (1680 termes) et 12.13.14 (280 termes) forment un groupe de 2016 termes, et pour trois bitangentes représentées par un terme quelconque de ce groupe, les six points de contact ne sont pas situés sur une même conique.

Comme un autre exemple j'explique la constitution des 63 "groupes" de Steiner (voir le mémoire de Steiner, "Eigenschaften der Curven vierten Grades rücksichtlich ihrer Doppeltangenten," t. XLIX. de ce journal, 1855) ou (pour éviter l'emploi de ce mot *groupe* dans une nouvelle signification) disons les 63 termes G de Steiner, chaque terme composé de 6 paires de bitangentes. On a ici un sous-groupe de 35 termes G_1 de la forme

$$12.34;\quad 13.42;\quad 14.23;\quad 56.78;\quad 57.86;\quad 58.67$$

(pour abréger on peut dénoter ce terme par 1234.5678), et un sous-groupe de 28 termes G_2 de la forme

$$13.32;\quad 14.42;\quad 15.52;\quad 16.62;\quad 17.72;\quad 18.82$$

(pour abréger on peut de même dénoter ce terme par 12.345678), les deux sous-groupes forment le groupe des 63 termes G.

Steiner a de plus considéré les "systèmes" ou disons les termes S_1, S_2, composés chacun de trois termes G; savoir 315 termes S_1 et 336 termes S_2. Les 315 termes S_1 sont ici un groupe composé d'un sous-groupe de 105 termes $3G_1$ de la forme

$$1234.5678;\quad 1256.3478;\quad 1278.3456$$

et un sous-groupe de 210 termes $2G_2 + G_1$ de la forme

$$12.345678;\quad 34.125678 \text{ et } 1234.5678.$$

Et de même les 336 termes S_2 sont un groupe composé d'un sous-groupe de 280 termes $2G_1 + G_2$ de la forme

$$1234.5678;\quad 5234.1678 \text{ et } 15.234678$$

et un sous-groupe de 56 termes $3G_2$ de la forme

$$12.345678;\quad 13.245678;\quad 31.245678.$$

Il va sans dire que je me suis servi de l'abréviation 1234.5678 pour dénoter le terme 12.34; 13.42; 14.23; 56.78; 57.86; 58.67; et de même pour les autres termes G_1 ou G_2.

M. Aronhold (dans le mémoire "Ueber den gegenseitigen Zusammenhang der 28 Doppeltangenten einer allgemeinen Curve vierten Grades," *Berl. Monatsber.* Juli 1864), partant de 7 bitangentes données, a trouvé une construction pour les autres 21 bitangentes. Les bitangentes données doivent être indépendantes; savoir pour trois quelconques de ces 7 bitangentes, les six points de contact ne sont pas situés sur une même conique. Les bitangentes représentées par les termes 12, 13, 14, 15, 16, 17, 18 sont un tel système de bitangentes indépendantes; et en dénotant de cette manière les 7 bitangentes données, la bitangente construite par le moyen de la conique qui touche cinq de ces droites, par exemple les droites 38, 48, 58, 68, 78, (ou conique 34567) peut être dénotée par 12, et de même pour les autres bitangentes cherchées; on a ainsi le système entier des bitangentes dénotées comme auparavant par 12, 13, 14, ... 78.

J'ajoute que le groupe qui contient 18, 28, 38, 48, 58, 68, 78 est composé d'un sous-groupe 18, 28, 38, 48, 58, 68, 78 de 8 termes, et d'un sous-groupe 12, 23, 31, 48, 58, 68, 78 de 280 termes; le groupe contient donc 288 termes; savoir il y a ce nombre 288 de systèmes de sept bitangentes indépendantes qui peuvent chacun servir à trouver par la construction d'Aronhold les autres 21 bitangentes.

P.S. J'ai trouvé à propos de la méthode de M. Aronhold une forme commode pour l'équation de la conique qui touche cinq droites données; en supposant que l'on ait identiquement $x+y+z+w=0$, et que les droites données soient $x=0$, $y=0$, $z=0$, $w=0$, et $ax+by+cz+dw=0$, l'équation de la conique est

$$(a-d)^2(b-c)^2(xw+yz)+(b-d)^2(c-a)^2(yw+zx)+(c-d)^2(a-b)^2(zw+xy)=0.$$

J'ajoute qu'en écrivant pour abréger

$$\alpha : \beta : \gamma = (a-d)(b-c) : (b-d)(c-a) : (c-d)(a-b)$$

(d'où $\alpha+\beta+\gamma=0$) les coordonnées (x, y, z, w) des points de contact avec les droites

$$x=0,\quad y=0,\quad z=0,\quad w=0 \text{ sont } (0, \gamma, \beta, \alpha),\ (\gamma, 0, \alpha, \beta),\ (\beta, \alpha, 0, \gamma),\ (\alpha, \beta, \gamma, 0)$$

respectivement; et que les coordonnées du point de contact avec la droite $ax+by+cz+dw=0$ sont

$$x : y : z : w = (bcd) : -(cda) : (dab) : -(abc)$$

où, pour abréger, (bcd) dénote $(b-c)(c-d)(d-b)$, et de même pour (cda), (dab), (abc).

Cambridge, le 23 *septembre* 1867.

442.

NOTE SUR LA SURFACE DU QUATRIÈME ORDRE DOUÉE DE SEIZE POINTS SINGULIERS ET DE SEIZE PLANS SINGULIERS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), tom. LXXIII. (1871), pp. 292—293.]

L'ÉQUATION de M. Kummer se transforme sans difficulté en celle-ci

$$\sqrt{\alpha x\left(\gamma'\gamma''y-\beta'\beta''z-\frac{w}{\alpha}\right)}+\sqrt{\beta y\left(\alpha'\alpha''z-\gamma'\gamma''x-\frac{w}{\beta}\right)}+\sqrt{\gamma z\left(\beta'\beta''x-\alpha'\alpha''y-\frac{w}{\gamma}\right)}=0,$$

où

$$\alpha+\beta+\gamma=0,\quad \alpha'+\beta'+\gamma'=0,\quad \alpha''+\beta''+\gamma''=0.$$

Or cette équation rendue rationnelle prend, après toutes les réductions nécessaires, la forme suivante:

$$\begin{aligned}&w^2(x^2+y^2+z^2-2yz-2zx-2xy)\\&+2w(\alpha\alpha'\alpha''(y^2z-yz^2)+\beta\beta'\beta''(z^2x-zx^2)+\gamma\gamma'\gamma''(x^2y-xy^2)+\theta xyz)\\&+\quad(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)^2=0,\end{aligned}$$

où, pour abréger, l'on a écrit

$$\begin{aligned}\theta&=(\beta-\gamma)\alpha'\alpha''+(\gamma-\alpha)\beta'\beta''+(\alpha-\beta)\gamma'\gamma'',\\&=(\beta'-\gamma')\alpha''\alpha+(\gamma'-\alpha')\beta''\beta+(\alpha'-\beta')\gamma''\gamma,\\&=(\beta''-\gamma'')\alpha\alpha'+(\gamma''-\alpha'')\beta\beta'+(\alpha''-\beta'')\gamma\gamma',\\&=-\tfrac{1}{3}\{(\beta-\gamma)(\beta'-\gamma')(\beta''-\gamma'')+(\gamma-\alpha)(\gamma'-\alpha')(\gamma''-\alpha'')+(\alpha-\beta)(\alpha'-\beta')(\alpha''-\beta'')\},\end{aligned}$$

l'identité de ces différentes valeurs de θ étant facile à vérifier.

En représentant par $Aw^2+2Bw+C=0$ la forme rationnelle de l'équation de la surface, on trouve pour le discriminant $AC-B^2$ de cette équation du second degré en w la valeur

$$AC-B^2=4\alpha\alpha'\alpha''\beta\beta'\beta''\gamma\gamma'\gamma''\,xyz\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}\right)\left(\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\left(\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}\right).$$

L'équation de la surface rendue rationnelle est symétrique par rapport aux trois systèmes de quantités (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; la forme irrationnelle de la même équation peut donc être présentée de trois manières différentes, savoir:

$$\sqrt{\alpha x\left(\gamma'\gamma''y-\beta'\beta''z-\frac{w}{\alpha}\right)}+\sqrt{\beta y\left(\alpha'\alpha''z-\gamma'\gamma''x-\frac{w}{\beta}\right)}+\sqrt{\gamma z\left(\beta'\beta''x-\alpha'\alpha''y-\frac{w}{\gamma}\right)}=0,$$

$$\sqrt{\alpha' x\left(\gamma''\gamma y-\beta''\beta z-\frac{w}{\alpha'}\right)}+\sqrt{\beta' y\left(\alpha''\alpha z-\gamma''\gamma x-\frac{w}{\beta'}\right)}+\sqrt{\gamma' z\left(\beta''\beta x-\alpha''\alpha y-\frac{w}{\gamma'}\right)}=0,$$

$$\sqrt{\alpha'' x\left(\gamma\gamma' y-\beta\beta' z-\frac{w}{\alpha''}\right)}+\sqrt{\beta'' y\left(\alpha\alpha' z-\gamma\gamma' x-\frac{w}{\beta''}\right)}+\sqrt{\gamma'' z\left(\beta\beta' x-\alpha\alpha' y-\frac{w}{\gamma''}\right)}=0$$

et l'on voit de plus que les équations des seize plans singuliers sont

$$x=0,\quad y=0,\quad z=0,\quad w=0,$$

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0,\qquad \frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}=0,\qquad \frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}=0,$$

$$\gamma'\gamma''y-\beta'\beta''z-\frac{w}{\alpha}=0,\quad \alpha'\alpha''z-\gamma'\gamma''x-\frac{w}{\beta}=0,\quad \beta'\beta''x-\alpha'\alpha''y-\frac{w}{\gamma}=0,$$

$$\gamma''\gamma y-\beta''\beta z-\frac{w}{\alpha'}=0,\quad \alpha''\alpha z-\gamma''\gamma x-\frac{w}{\beta'}=0,\quad \beta''\beta x-\alpha''\alpha y-\frac{w}{\gamma'}=0,$$

$$\gamma\gamma' y-\beta\beta' z-\frac{w}{\alpha''}=0,\quad \alpha\alpha' z-\gamma\gamma' x-\frac{w}{\beta''}=0,\quad \beta\beta' x-\alpha\alpha' y-\frac{w}{\gamma''}=0,$$

les quantités α, β, γ, etc. étant liées entre elles par les trois équations

$$\alpha+\beta+\gamma=0,\quad \alpha'+\beta'+\gamma'=0,\quad \alpha''+\beta''+\gamma''=0.$$

Voilà ce me semble la forme la plus simple pour l'équation de cette surface.

Cambridge, le 23 *février* 1871.

443.

NOTE ON THE SOLUTION OF THE QUARTIC EQUATION $\alpha U + 6\beta H = 0$.

[From the *Mathematische Annalen*, vol. I. (1869), pp. 54, 55.]

If U denote the quartic function $(a, b, c, d, e \between x, y)^4$, H its Hessian

$$= (ac - b^2,\ 2(ad - bc),\ ae + 2bd - 3c^2,\ 2(be - cd),\ ce - d^2 \between x, y)^4,$$

α and β constants, then we may find the linear factors of the function $\alpha U + 6\beta H$ (or what is the same thing solve the equation $\alpha U + 6\beta H = 0$) by a formula almost identical with that given by me (Fifth Memoir on Quantics, *Phil. Trans.* vol. CXLVIII. (1858), see p. 446, [156]) in regard to the original quartic function U.

In fact (reproducing the investigation) if I, J are the two invariants, $M = \frac{I^3}{4J^2}$, Φ the cubicovariant

$$= (-a^2d + 3abc - 2b^3,\ \&c \between x, y)^6,$$

then the identical equation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, may be written $(1, 0, -M, M \between IH, JU)^3 = -\frac{1}{4}I^3\Phi^2$, whence if ω_1, ω_2, ω_3 are the roots of the equation $(1, 0, -M, M \between \omega, 1)^3 = 0$, or what is the same thing $\omega^3 - M(\omega - 1) = 0$; then the functions

$$IH - \omega_1 JU,\ IH - \omega_2 JU,\ IH - \omega_3 JU$$

are each of them a square: writing

$$(\omega_2 - \omega_3)(IH - \omega_1 JU) = X^2,$$
$$(\omega_3 - \omega_1)(IH - \omega_2 JU) = Y^2,$$
$$(\omega_1 - \omega_2)(IH - \omega_3 JU) = Z^2,$$

so that identically $X^2+Y^2+Z^2=0$, the expression $\alpha X+\beta Y+\gamma Z$ will be a square if only $\alpha^2+\beta^2+\gamma^2=0$. (To see this observe that in virtue of the equation $X^2+Y^2+Z^2=0$, we have $X+iY$, $X-iY$ each of them a square, and thence

$$\alpha X+\beta Y+\gamma Z,\ =\tfrac{1}{2}(\alpha+i\beta)(X-iY)+\tfrac{1}{2}(\alpha-i\beta)(X-iY)-\gamma i\sqrt{X^2+Y^2},$$

is a square if the condition in question be satisfied.)

Hence in particular writing

$$\sqrt{\omega_2-\omega_3}\sqrt{\alpha I+6\beta\omega_1 J},\ldots,\quad \sqrt{\omega_1-\omega_2}\sqrt{\alpha I+6\beta\omega_3 J},$$

for α, β, γ, we have

$$(\omega_2-\omega_3)\sqrt{\alpha I+6\beta\omega_1 J}\sqrt{IH+\omega_1 JU}+\ldots+(\omega_1-\omega_2)\sqrt{\alpha I+6\beta\omega_3 J}\sqrt{IH+\omega_3 JU}$$

a perfect square, and since the product of the four different values is a multiple of $(\alpha U+6\beta H)^2$ (this is most readily seen by observing that for $\alpha U+6\beta H=0$, the irrational expression omitting a factor is $(\omega_2-\omega_3)(\alpha I+6\beta\omega_1 J)+\ldots+(\omega_1-\omega_2)(\alpha I+6\beta\omega_3 J)$, which vanishes identically) it follows that the expression in question is the square of a linear factor of $\alpha U+6\beta H$.

It thus appears that the radicals (other than those arising from the solution of $U=0$) contained in the solution of the equation $\alpha U+6\beta H=0$ are the three roots

$$\sqrt{\alpha I+6\beta\omega_1 J},\quad \sqrt{\alpha I+6\beta\omega_2 J},\quad \sqrt{\alpha I+6\beta\omega_3 J}.$$

Cambridge, September 2, 1868.

444.

ON THE CENTRO-SURFACE OF AN ELLIPSOID.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 16—18.]

THE President [Prof. Cayley] gave an account of his investigations on the centro-surface of an ellipsoid (locus of the centres of curvature of the ellipsoid). The surface has been studied by Dr Salmon, and also by Prof. Clebsch, but in particular the theory of the nodal curve on the surface admits of further development. The position of a point on the ellipsoid is determined by means of the parameters, or elliptic coordinates, h, k; viz., if as usual a, b, c are the semi-axes, and if X, Y, Z are the coordinates of the point in question, then

$$\frac{X^2}{a^2+h}+\frac{Y^2}{b^2+h}+\frac{Z^2}{c^2+h}=1,$$

$$\frac{X^2}{a^2+k}+\frac{Y^2}{b^2+k}+\frac{Z^2}{c^2+k}=1;$$

and hence

$$-\beta\gamma X^2=a^2(a^2+h)(a^2+k),$$

$$-\gamma\alpha Y^2=b^2(b^2+h)(b^2+k),$$

$$-\alpha\beta Z^2=c^2(c^2+h)(c^2+k),$$

if for shortness

$$\alpha=b^2-c^2,\quad \beta=c^2-a^2,\quad \gamma=a^2-b^2,\quad (\alpha+\beta+\gamma=0).$$

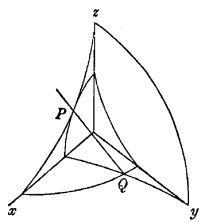
This being so, the coordinates of the point of intersection of the normal at (X, Y, Z) by the normal at the consecutive point of the curve of curvature

$$\frac{X^2}{a^2+k}+\frac{Y^2}{b^2+k}+\frac{Z^2}{c^2+k}=1$$

are given by the formulæ

$$-\beta\gamma a^2x^2 = (a^2+h)^3(a^2+k),$$
$$-\gamma\alpha b^2y^2 = (b^2+h)^3(b^2+k),$$
$$-\alpha\beta c^2z^2 = (c^2+h)^3(c^2+k);$$

viz., these equations, considering therein (h, k) as arbitrary parameters, determine the coordinates (x, y, z) of a point on the centro-surface. The principal sections (as is known) consist each of them of an ellipse counting three times, and of an evolute of an ellipse; the evolute and ellipse have four contacts (two-fold intersections) and four simple intersections, but the contacts and intersections respectively are in the different sections real and imaginary; and if (as we may without loss of generality assume) $a^2+c^2>2b^2$, then the form of the principal sections is as shown in the figure (which



represents only an octant of the surface); viz., there is a real contact at P in the plane of xz, and a real intersection at Q in the plane of xy. The surface has thus an exterior and an interior sheet, but (instead of meeting in a conical point, as in the wave surface) these intersect in a nodal curve QP. The curve has a cusp at Q, and a node at P; viz., the curve extends beyond P, but from that point is acnodal, or without any real sheet of the surface passing through it. For the nodal curve there must be two values (h, k), (h_1, k_1), giving the same values of (x, y, z); viz., there must exist the relations

$$(a^2+h)^3(a^2+k) = (a^2+h_1)^3(a^2+k_1),$$
$$(b^2+h)^3(b^2+k) = (b^2+h_1)^3(b^2+k_1),$$
$$(c^2+h)^3(c^2+k) = (c^2+h_1)^3(c^2+k_1);$$

from which equations eliminating h_1 and k_1, we should have between h, k a relation which, combined with the expressions of x, y, z in terms of (h, k), determines the nodal curve. But the better course is to eliminate k, k_1, thus obtaining a relation between h and h_1, in virtue whereof h_1 may be regarded as a known function of h; k and k_1 can then be readily expressed in terms of h, h_1; that is, we have k as a function of h, h_1, or in effect as a function of h. The relation between h, h_1 (after a

reduction of some complexity) assumes ultimately a form which is very simple and remarkable; viz., writing

$$P = a^2 + b^2 + c^2, \quad Q = b^2c^2 + c^2a^2 + a^2b^2, \quad R = a^2b^2c^2,$$

the relation is

$$\begin{aligned} &(6R + 3Qh + Ph^2) \\ + h_1 &(3Q + 4Ph + 3h^2) \\ + h_1^2 &(P \quad + 3h \qquad\quad) = 0; \end{aligned}$$

this is a (2, 2) correspondence between the two parameters h, h_1; the united values $h_1 = h$, are given by the equation $6(R + Qh + Ph^2 + h^3) = 0$, that is

$$(a^2 + h)(b^2 + h)(c^2 + h) = 0;$$

viz., the two points on the ellipsoid which have their common centre of curvature on the nodal curve are only situate on the same curve of curvature when this curve is a principal section of the ellipsoid.

{Since the date of the foregoing communication, Prof. Cayley has found that the squared coordinates x^2, y^2, z^2 of a point on the nodal curve can be expressed as rational functions of a single variable parameter σ.}

445.

A MEMOIR ON QUARTIC SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 19—69. Read February 10, 1870.]

THE present Memoir is intended as a commencement of the theory of the quartic surfaces which have nodes (conical points). A quartic surface may be without nodes, or it may have any number of nodes up to 16. I show that this is so, and I consider how many of the nodes may be given points. Although it would at first sight appear that the number is 8, it is in fact 7; viz., we can, with 7 given points as nodes (but not in a proper sense with 8 or more given points), find a quartic surface; such surface contains in its equation 6 constants, which may be such that the surface has an additional node or nodes. Suppose that the surface has an 8th node:—there are two distinct cases; viz., (1) the 8 nodes are the points of intersection of 3 quadric surfaces, or say they are an octad, and the surface is said to be octadic; (2) the 8th node is any point whatever on a certain sextic surface determined by means of the 7 given nodes, and called the dianodal surface of these 7 points; the quartic surface is said to be a dianome. The two cases are in general exclusive of each other; viz., the 7 given points being any points whatever, the dianodal surface does not pass through the 8th point of the octad; and thus the quartic surface with the 8 nodes is either octadic or else a dianome. Assuming it to be a dianome, the constants may be further determined so that there shall be a 9th node; it is necessary to examine whether this forms with 7 of the 8 nodes an octad. Supposing that it does not (viz., that there are not any 8 nodes in regard to which the surface is octadic), the 9th node is then any point whatever on a certain curve of the order 18, determined by means of the 8 nodes, and called the dianodal curve of these 8 points. And, finally, the constants may be further determined so that there shall be a 10th node; supposing, as before, that this does not form an octad with any 7 of the 9 nodes (viz., that

there are not any 8 nodes in regard to which the surface is octadic), the 10th node is then any one of a system of 22 [should be 13] points determined by means of the 9 nodes, and called the dianodal system of these 9 points. But the quartic surface is now completely determined; viz., starting with any 7 given points as nodes, we have a dianome with 8 nodes, 9 nodes, or 10 nodes, say, an octodianome, enneadianome, or decadianome, but not with any greater number of nodes; these can only present themselves when particular conditions are satisfied in regard to the 7 given nodes, and to the 8th and 9th node; and the consideration of the quartic surfaces with more than 10 nodes would thus form a separate branch of the subject.

The case of the decadianome (or quartic surface with 10 nodes formed as above with 7 given points as nodes) is peculiarly interesting. I identify this with the surface which I call a symmetroid; viz., the surface represented by an equation $\Delta = 0$, where Δ is a *symmetrical* determinant of the 4th order the several terms whereof are linear functions of the coordinates (x, y, z, w); this surface is related to the Jacobian surface of 4 quadric surfaces (itself a very remarkable surface), and this theory of the symmetroid and the Jacobian, and of questions connected therewith, forms a large portion of the present Memoir.

The theory of the Jacobian is connected also with the researches in regard to nodal quartic surfaces in general; and, for greater clearness, it has seemed to me proper to commence the Memoir with certain definitions, &c., in regard to this theory. It will be seen in what manner I extend the notion of the Jacobian.

I remark that the present researches on Quartic Surfaces were suggested to me by Professor Kummer's most interesting Memoir "Ueber die algebraischen Strahlensysteme u.s.w.," *Berl. Abh.* 1866, in which, without entering upon the general theory, he is led to consider the quartic surfaces, or certain quartic surfaces, with 16, 15, 14, 13, 12, or 11 nodes; the last of these, or surface with 11 nodes, being in fact a particular case of the symmetroid.

Considerations in regard to the Jacobian of four, or more or less than four, Surfaces.

1. In the case of any four surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, the differential coefficients of P, Q, R, S in regard to the coordinates (x, y, z, w) may be arranged as a square matrix in either of the ways

$$\begin{array}{c|cccc|c} & P, & Q, & R, & S & \\ \hline \delta_x & & & & & P \\ \delta_y & & & & & Q \\ \delta_z & & & & & R \\ \delta_w & & & & & S \end{array} \quad ; \quad \begin{array}{c|cccc|c} & \delta_x, & \delta_y, & \delta_z, & \delta_w & \\ \hline & & & & & P \\ & & & & & Q \\ & & & & & R \\ & & & & & S \end{array}$$

and with either arrangement we may form one and the same determinant, the Jacobian determinant $J(P, Q, R, S)$, or, equating it to zero, the Jacobian surface $J(P, Q, R, S)=0$, of the four surfaces.

2. In the case of more than four surfaces, adopting the arrangement

$$\begin{array}{c|cccc} & P, & Q, & R, & S, & T, \ldots \\ \hline \delta_x & & & & & \\ \delta_y & & & & & \\ \delta_z & & & & & \\ \delta_w & & & & & \end{array}$$

and considering the several determinants which can be formed with any four columns of the matrix, these equated to zero establish a more than one-fold relation between the coordinates; viz., in the case of five surfaces, we have $J(P, Q, R, S, T)\equiv 0$, a twofold relation representing a curve; and in the case of six surfaces, $J(P, Q, R, S, T, U)\equiv 0$, a threefold relation representing a point-system; and (since with four coordinates a relation is at most threefold) these are the only cases to be considered.

3. In the case of fewer than four surfaces, adopting the arrangement

$$\begin{array}{cccc|c} \delta_x, & \delta_y, & \delta_z, & \delta_w & \\ \hline & & & & P \\ & & & & Q \\ & & & & \vdots \end{array}$$

and considering the several determinants which can be formed with any 3 or 2 columns of the matrix, and equating these to zero, we have in like manner a more than one-fold relation between the coordinates; viz., in the case of three surfaces, we have $J(P, Q, R)\equiv 0$, a twofold relation representing a curve; and in the case of two surfaces $J(P, Q)\equiv 0$, a threefold equation representing a point-system, (viz., this denotes the points $\delta_x P : \delta_y P : \delta_z P : \delta_w P = \delta_x Q : \delta_y Q : \delta_z Q : \delta_w Q$); for a single surface we should have a fourfold relation, and the case is not considered. But observe that if the notation were used, $J(P)\equiv 0$ would denote $\delta_x P=0$, $\delta_y P=0$, $\delta_z P=0$, $\delta_w P=0$, equations which are satisfied simultaneously by the coordinates (x, y, z, w) of any node of the surface $P=0$. Although in what precedes I have used the sign $\equiv$, there is no objection to using, and I shall in the sequel use, the ordinary sign $=$, it being understood that while $J(P, Q, R, S)=0$ denotes a single equation or onefold relation, $J(P, Q, R, S, T)=0$ or $J(P, Q, R)=0$ will each denote a twofold relation, and $J(P, Q, R, S, T, U)=0$ or $J(P, Q)=0$ each of them a threefold relation.

4. It is not asserted that $\ldots J(P, Q, R)=0$, $J(P, Q, R, S)=0$, $J(P, Q, R, S, T)=0, \ldots$ form a continuous series of analogous relations; and there might even be a propriety in using, in regard to four or more surfaces, J, and in regard to four or fewer surfaces an inverted J (viz., in regard to four surfaces, either symbol indifferently); but there is no ambiguity in, and I have preferred to adopt, the use of the single symbol J.

5. Suppose that the orders of the surfaces $P=0$, $Q=0, \ldots$ are $a+1$, $b+1, \ldots$ (so that the orders of the differential coefficients of P, $Q \ldots$ are a, $b, \ldots$), then we have for the orders of the several loci,

$$\begin{array}{rll} J(P, Q)=0, & \text{point-system, order} & a^2+a^2b+ab^2+b^3; \\ J(P, Q, R)=0, & \text{curve,} \quad \text{,,} & a^2+b^2+c^2+bc+ca+ab; \\ J(P, Q, R, S)=0, & \text{surface,} \quad \text{,,} & a+b+c+d; \\ J(P, Q, R, S, T)=0, & \text{curve,} \quad \text{,,} & ab+ac \ldots + de; \\ J(P, Q, R, S, T, U)=0, & \text{point-system,} \quad \text{,,} & abc+abd \ldots + def; \end{array}$$

see, as to this, Salmon's *Solid Geometry*, Ed. 2, (1865), Appendix IV., "On the Order of Systems of Equations" [not reproduced in the later editions]. In particular, if $a=b=c \ldots =1$, then the orders are 4, 6, 4, 10, 20.

As to the Surface obtained by equating to zero a Symmetrical Determinant.

6. It is also shown (Salmon, Ed. 2, p. 495) that the surface obtained by equating to zero any symmetrical determinant has a determinate number of nodes; viz., if the orders of the terms in the diagonal be a, b, c, &c., then the number of nodes is $=\frac{1}{2}(\Sigma a \,.\, \Sigma ab - \Sigma abc)$, or, as this may also be written, $\frac{1}{2}(\Sigma a^2b + 2\Sigma abc)$. In particular, the formula applies to the case of the surface

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

(a, b, c, d) being here the orders of A, B, C, D respectively, and the orders of F, G, &c., being $\frac{1}{2}(b+c)$, $\frac{1}{2}(a+c)$, &c. If the terms are all of them linear functions of the coordinates, or $a=b=c=d=1$, then the number of nodes is $=10$.

7. That the surface has nodes is, in fact, clear from the consideration that any point for which the minors of the determinant all vanish will be a node; and that (for the symmetrical determinant), by making the minors all of them vanish, we establish only a threefold relation between the coordinates. The expression for the number of the nodes is, I think, obtained most readily as follows:

The nodes will be points of intersection of the curve and surface

$$\left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0, \quad \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} = 0,$$

these, however, contain in common the points

$$\left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0;$$

and not only so, but they touch at the points in question; so that, multiplying together the orders of the curve and surface, and subtracting *twice* the order of the point-system, we obtain the expression for the number of nodes. In the particular case where the functions are all linear, we have a sextic curve and cubic surface intersecting in 18 points; but the curve and surface touch in 4 points, and the number of nodes is $(18-2.4)=10$. And in the same way the formula may be established for the general case.

8. The subsidiary theorem of the contact of the curve and surface requires, however, to be proved. Seeking for the equation of the tangent plane of the surface at any one of the points in question, we have first

$$\begin{vmatrix} \delta B, & \delta F, & \delta M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ \delta F, & \delta C, & \delta N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ \delta M, & \delta N, & \delta D \end{vmatrix} = 0,$$

where, in virtue of the equations

$$\left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0,$$

the last term vanishes. Expanding the other two terms, the equation becomes

$$D(C\delta B + B\delta C - 2F\delta F) - (N^2\delta B - 2MN\delta F + M^2\delta C) + \delta M(FN - CM) + \delta N(BN - MF) = 0;$$

but, in virtue of the same equations, the coefficients of δM and δN each of them vanish, and we have also

$$N^2\delta B + M^2\delta C - 2MN\delta F = \frac{N^2}{C}(C\delta B + B\delta C - 2F\delta F);$$

so that the equation becomes finally $C\delta B + B\delta C - 2F\delta F = 0$. Investigating by a like process the equation of the tangent of the curve

$$\left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0,$$

we find between the differentials δA, δB, &c., a twofold linear relation, expressible by means of the foregoing equation $C\delta B + B\delta C - 2F\delta F = 0$, and one other equation; that is, at each of the points in question the tangent of the curve lies in the tangent plane of the surface, or, what is the same thing, the curve and surface touch at these points.

Surfaces represented by an equation $F(P, Q) = 0$, &c.

9. In the remarks which follow as to the surfaces $F(P, Q) = 0$, $F(P, Q, R) = 0$, &c., the function F is a rational and integral function of (P, Q), (P, Q, R), &c., not in general homogeneous in regard to $P, Q, R, \ldots$. but of such degrees in regard to these functions respectively as to be homogeneous in regard to the coordinates (x, y, z, w).

The surface $F(P, Q) = 0$ has in general a nodal curve $\delta_P F = 0$, $\delta_Q F = 0$; and if it has besides any nodes, these are points of the point-system $J(P, Q) = 0$.

The surface $F(P, Q, R) = 0$ has in general nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$; and if it has besides any nodes, these are points on the curve $J(P, Q, R) = 0$.

The surface $F(P, Q, R, S) = 0$ has not in general, but it may have, nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$, $\delta_S F = 0$; if it has any other nodes, these are points on the surface $J(P, Q, R, S) = 0$.

Nodes of a Quartic Surface; Circumscribed Cone having its vertex at a Node.

10. A quartic surface may be without nodes; or it may have any number of nodes up to 16. Consider a quartic surface having a node or nodes; and take the single node, or (if more nodes than one) any one of the nodes, as the vertex of a circumscribed cone; then, considering any plane through the vertex, the section will be a quartic curve having a node at the vertex, and the generating lines in the plane will be the tangents from the node to the quartic curve; the number of them is therefore 6, and the order of the circumscribed cone is thus $= 6$. Each tangent intersects the quartic curve in the node counting as two intersections, and in the point of contact counting as two intersections; there are consequently no singular tangents; and therefore in the circumscribed cone no singular lines arising from a singular tangency of the generating line. Hence, in the case of a single node on the surface, the circumscribed cone is a cone of the order 6 without nodal or stationary lines; and the class is $= 30$. But in the case of more than one node, say k nodes, the circumscribed cone passes through the remaining $k-1$ nodes, and the generating line through each of these nodes is a nodal line of the cone; that is, the cone has $k-1$ nodal lines, and its class is $= 30 - 2k + 2$. The cone is not of necessity a proper cone; the maximum number of nodal lines is when it breaks up into 6 planes, and we have then $k - 1 = 15$; that is, the number of nodes of the surface is at most $= 16$.

11. It is easy to form a table of the different *primâ facie* possible forms of the sextic cone, according to the number of nodes of the surface; viz., writing 6 for a proper sextic cone without nodal lines, 6_1, $6_2 \ldots 6_{10}$ for the proper sextic cone with 1, 2, ... or 10 nodal lines; and so 5, $5_1 \ldots 5_6$ for the proper quintic cones, 4, 4_1, 4_2, 4_3, 3, 3_1, 2 for the quartic, cubic, and quadric cones, and 1 for the plane, the table is

CIRCUMSCRIBED SEXTIC CONE.

Nodes of Surface.								
1	6							
2	6_1							
3	6_2							
4	6_3							
5	6_4							
6	6_5 ;	5 , 1						
7	6_6 ;	5_1, 1						
8	6_7 ;	5_2, 1						
9	6_8 ;	5_3, 1 ;	4 , 2					
10	6_9 ;	5_4, 1 ;	4_1, 2 ;	4 , 1, 1 ;	3 , 3			
11	6_{10};	5_5, 1 ;	4_2, 2 ;	4_1, 1, 1 ;	3_1, 3			
12	...	5_6, 1 ;	4_3, 2 ;	4_2, 1, 1 ;	3_1, 3_1 ;	3 , 2, 1		
13	...	...	...	4_3, 1, 1 ;	...	3_1, 2, 1 ;	3 , 1, 1, 1 ;	2 , 2, 2
14	...	...	...	...	...	...	3_1, 1, 1, 1 ;	2 , 2, 1, 1
15	...	...	...	...	...	...	...	2_1, 1, 1, 1, 1
16	...	...	...	...	...	...	...	1 , 1, 1, 1, 1, 1 ;

and moreover, in the cases where there are two or more forms of the sextic cone, then the k sextic cones may be of the different forms in various combinations. The total number of cases *primâ facie* possible is thus very great; but only a comparatively small number of them actually exist.

12. In the case where there is a plane 1, the sextic cone breaks up into this plane, and into a (proper or improper) quintic cone intersecting the plane in 5 lines; that is, there will be in the plane 6 nodes; the plane is, in fact, a singular tangent plane meeting the surface in a conic twice repeated; and the 6 nodes lie on this conic. Taking any one of these nodes as vertex, the corresponding sextic cone breaks up into the plane, and into a (proper or improper) quintic cone.

13. In the cases $k=1$, 2, 3, 4, 5, and $k=15$, 16, there is only one form of sextic cone; so that each node (at least so far as appears) stands in the same relation to the surface. Considering the last mentioned two cases; $k=16$,—each of the 16 nodes gives 6 singular tangent planes, but each of these passes through 6 nodes; therefore the number of planes is $=16$: similarly, $k=15$, the number of singular tangent planes is $15\times 4\div 6$, $=10$.

For $k=14$, the cones are $3_1, 1, 1, 1$, or $2, 2, 1, 1$: it is easy to see that we have only the three cases

Cones $3_1, 1, 1, 1 : 2, 2, 1, 1$			Singular tangent planes
No. may be	14 ,	0	gives $(14.3+\ 0.2)\div 6, =7$
"	8 ,	6	" $(\ 8.3+\ 6.2)\div 6, =6$
"	2 ,	12	" $(\ 2.3+12.2)\div 6, =5$

and we may in the like manner limit the number of possible cases, for other values of k. But I do not at present further pursue the inquiry.

As to the Number of Constants contained in a Surface.

14. We say that a surface $P=0$ contains or depends upon a certain number of constants; viz., this is the number of constants contained in the equation $P=0$ of the surface, taking the coefficient of any one term to be equal to unity; thus the general quadric surface contains 9 constants; the surface can in fact be determined so as to satisfy 9 conditions; or, as we might express it, the *Postulation* of the surface is $=9$. [I have elsewhere said *Postulandum* and *Capacity*: I prefer this last expression.] And if, in the general equation so containing 9 constants, k of these are given, or, what is the same thing, if the quadric surface be made to satisfy any k conditions, then the number of constants, or postulation of the surface, is $=9-k$.

15. But a different form of expression is sometimes convenient; the conditions to be satisfied are frequently such that, being satisfied by the surfaces $P=0$, $Q=0, \ldots$, they will be satisfied by the surface $\alpha P+\beta Q+\ldots=0$, where $\alpha, \beta, \ldots$ are any constant multipliers whatever. When this is so, there will be a certain number of solutions $P=0$, $Q=0, \ldots$ not connected by any such relation, or say of asyzygetic solutions, such that the general surface satisfying the conditions in question is $\alpha P+\beta Q+\ldots=0$; and hence, taking one of these coefficients as unity, the number of constants, or postulation of the surface, is equal to the number of the remaining coefficients, or, what is the same thing, it is less by unity than the number of the asyzygetic solutions $P=0$, $Q=0 \ldots$. Instead of considering the number of constants, or postulation, we may consider the number of solutions (that is, asyzygetic solutions) or surfaces $P=0$, $Q=0, \ldots$ which satisfy the conditions in question.

16. Thus, for the quadric not subjected to any conditions, there are 10 surfaces (for example, these may be taken to be the surfaces $x^2=0$, $y^2=0$, $z^2=0$, $w^2=0$, $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$); and the general quadric surface is by means of these expressed linearly in the form $(a, \ldots \between x, y, z, w)^2=0$. So for the quadric surfaces through k given points, the number of these is $=10-k$; thus for the surfaces through 4 given points, say the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, the 6 given surfaces may be taken to be $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$, and every other quadric surface through the 4 points is by means of these expressed linearly in the form $(a, \ldots \between yz, zx, xy, xw, yw, zw)=0$; for the quadric surfaces through 8 points there are two surfaces $P=0$, $Q=0$; and every quadric surface through the

8 points is by means of these expressed linearly in the form $\alpha P+\beta Q=0$; and (as the extreme case) if the quadric surface passes through 9 given points, then there is the single quadric surface $P=0$.

17. In the questions in which such quadric surfaces present themselves, it is in general quite immaterial what particular surfaces are selected as the surfaces $P=0$, $Q=0, \ldots$; the selection may be made at pleasure and, being so made, the surfaces are to be regarded as completely determinate; viz., there would be no gain of generality if these were replaced by any other surfaces $\alpha P+\beta Q \ldots =0$. For instance, in the theory of the quartic surfaces with 6 given points as nodes, we have through the 6 given points the 4 quartic surfaces $P=0$, $Q=0$, $R=0$, $S=0$, and we consider the quartic functions $(a, \ldots \between P, Q, R, S)^2$ and $J(P, Q, R, S)$: each of these is unaltered as to its form when P, Q, R, S are replaced each of them by any linear function of these quantities; viz., $(a, \ldots \between P, Q, R, S)^2$ is changed into a new quadric function $(a', \ldots \between P, Q, R, S)^2$, and $J(P, Q, R, S)$ into a mere constant multiple of its original value. We have herein a justification of the expressions in question, through 6 given points there are 4 quadric surfaces, &c.

General theory of the Quartic Surface with a given Node or Nodes.

18. A quartic surface contains 34 constants; and the number of conditions to be satisfied in order that a given point may be a node is $=4$. Hence, if the surface has k given points as nodes, the number of constants is $=34-4k$; and it would at first sight appear that k might be $=8$, and that with the 8 given points as nodes we should have a quartic surface containing 2 constants. But this is not so in a proper sense; for through the 8 given points we have 2 quadric surfaces $P=0$, $Q=0$; and we can by means of these form a quartic surface $(a, b, c \between P, Q)^2=0$, containing 2 constants, and having in a sense the 8 points as nodes. This, however, is no proper quartic surface, but is a system of 2 quadric surfaces, each of them passing through the 8 points, and the two quadric surfaces therefore intersecting in a quadri-quadric curve through the 8 points; which curve is therefore a nodal curve on the compound surface; and it is only as points on this nodal curve, and not in a proper sense, that the 8 given points are nodes of the quartic surface. The greatest value of k is thus $k=7$.

19. Of course, if $k=0$, we have the general quartic surface $U=0$, containing 34 constants. The cases $k=1$, $k=2$, $k=3$ (viz., a single given node, 2 given nodes, 3 given nodes), may be at once disposed of; taking for instance the 1st node to be the point (1, 0, 0, 0), the 2nd node the point (0, 1, 0, 0), the 3rd node the point (0, 0, 1, 0), we find at once an equation $U=0$, with 30, 26, or 22 constants, having the given node or nodes.

Four given Nodes.

20. The case of 4 given nodes is just as easy; but in reference to what follows, it is proper to consider it more in detail. The equation should contain 18 constants; we have through the 4 given points 6 quadric surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$,

$U=0$, and we can by means of them form a quartic equation $(a, \ldots \between P, Q, R, S, T, U)^2=0$, having the 4 given points as nodes; this contains, however, $(21-1=)$ 20 constants; the reduction to the right number 18 occurs by reason that the functions (P, Q, R, S, T, U), although linearly independent, are connected by two quadric equations

$$(*\between P, Q, R, S, T, U)^2=0, \quad (*'\between P, Q, R, S, T, U)^2=0;$$

hence writing the equation of the quartic surface in the form

$$(a, \ldots \between_{\prime\prime})^2-\lambda(*\between_{\prime\prime})^2-\mu(*\between_{\prime\prime})^2=0,$$

the coefficients λ, μ may be so determined as to reduce to zero the coefficients of any two terms of the equation, and the number of constants really is $20-2=18$, as it should be.

21. In proof, observe that, taking the 4 given nodes to be the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 0, 0, 0), the quadric surfaces may be taken to be $yz=0$, $zx=0$, $xy=0$, $xw=0$, $yw=0$, $zw=0$; the equation of the quartic surface will thus be

$$(a, \ldots \between yz, zx, xy, xw, yw, zw)^2=0;$$

but we have between the functions xy, &c., the two identical relations

$$xy\,.\,zw-xz\,.\,yw=0, \quad xy\,.\,zw-xw\,.\,yz=0;$$

and the number of constants is thus $=18$.

Five given Nodes.

22. In the case of 5 given nodes, the number of constants should be $=14$. We have through the 5 given points, 5 quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$, and we form herewith the quartic equation $(a, \ldots \between P, Q, R, S, T)^2=0$, containing the right number 14 of arbitrary constants. The functions P, Q, &c. are in this case not connected by any quadric relation, and the equation just written down is in fact the general equation of the quartic surface with the 5 given nodes.

23. In verification, take the first 4 nodes to be as above, and the 5th node to be the point (1, 1, 1, 1); we may write

$$(P, Q, R, S, T)=\{x(y-z), x(y-w), y(x-z), y(x-w), xy-zw\};$$

and if from the 5 equations $P=x(y-z)$, &c., we eliminate (x, y, z, w), we obtain one, and only one, relation between the functions P, Q, R, S, T; this is found to be

$$PS(Q+R-T)-QR(P+S-T)=0,$$

or, what is the same thing,

$$R(P-Q)(S-T)-P(R-S)(Q-T)=0;$$

viz., it is a cubic relation, and there is consequently no quadric relation between the 5 functions.

Six given Nodes.

24. In the case of 6 given nodes, the quartic surface should contain 10 constants. We have through the 6 given points 4 quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$; but if we form herewith the quartic surface $(a, \ldots \between P, Q, R, S)^2=0$, this contains only 9 constants. It is to be shown that the Jacobian surface $J(P, Q, R, S)=0$ of the 4 quadric surfaces (or say of the 6 points) is a quartic surface having the 6 given points as nodes, and not included in the foregoing form $(a, \ldots \between P, Q, R, S)^2=0$; this being so, we have the quartic surface

$$(a, \ldots \between P, Q, R, S)^2+\theta J(P, Q, R, S)=0,$$

having the 6 given points as nodes, and containing the complete number of constants, viz., 10.

25. The 6 given nodes being any points whatever, their coordinates may be taken to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), and $(\alpha, \beta, \gamma, \delta)$. I proceed to find the Jacobian of these 6 points. For this purpose, let (a, b, c, f, g, h) be the 6 coordinates of the line through the points (1, 1, 1, 1) and $(\alpha, \beta, \gamma, \delta)$, viz.,

$$\begin{aligned} a&=\beta-\gamma, & f&=\alpha-\delta,\\ b&=\gamma-\alpha, & g&=\beta-\delta,\\ c&=\alpha-\beta, & h&=\gamma-\delta, \end{aligned}$$

whence $af+bg+ch=0$, and also

$$\begin{aligned} .\quad\quad h-g+a&=0,\\ -h\quad .+f+b&=0,\\ g-f\quad .+c&=0,\\ -a-b-c\quad .&=0, \end{aligned}$$

we have through the 6 points the plane pairs

$$\begin{aligned} x\,(\quad .-hx-gz+aw)&=0,\\ y\,(-hx\quad .+fz+bw)&=0,\\ z\,(\quad gx-fy\quad .+cw)&=0,\\ w\,(-ax-by-cz\quad .)&=0, \end{aligned}$$

where, adding the four equations, we have identically $0=0$. For this reason, we cannot take these to be the equations of the 4 quadric surfaces, but we may take the first 3 of them for the surfaces $P=0$, $Q=0$, $R=0$; and for the 4th surface $S=0$, I take the quadric cone having its vertex at the point (0, 0, 0, 1); viz., the equation is

$$a\alpha yz+b\beta zx+c\gamma xy=0;$$

that is, I write

$$(P, Q, R, S)=\{x(hy-gz+aw),\ y(-hx+fz+bw),\ z(gx-fy+cw),\ (a\alpha yz+b\beta zx+c\gamma xy)\}.$$

26. The Jacobian is then easily found to be

$$\begin{aligned}&(b\beta zx + c\gamma xy)(-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -gB,\ hC,\ aA,\ b^2g,\ -c^2h \between x,\ y,\ z,\ w)^2\\ &+ (c\gamma xy + a\alpha yz)(agh,\ -bhf,\ cfg,\ abc,\ fA,\ -bg^2,\ -hC,\ -a^2f,\ bB,\ c^2h \between x,\ y,\ z,\ w)^2\\ &+ (a\alpha yz + b\beta zx)(agh,\ bhf,\ -cfg,\ abc,\ -fA,\ gB,\ -ch^2,\ a^2f,\ -b^2g,\ cC \between x,\ y,\ z,\ w)^2 = 0;\end{aligned}$$

where for the moment A, B, C denote $bg - ch$, $ch - af$, $af - bg$ respectively. Collecting and reducing, the whole divides by $2abc$; and if finally we replace a, b, c, f, g, h by their values, the result is

$$J = \left\{\begin{array}{l}(\beta-\gamma)\,yz\,(\alpha w^2 - \delta x^2) + (\alpha-\delta)\,xw\,(\beta z^2 - \gamma y^2)\\ + (\gamma-\alpha)\,zx\,(\beta w^2 - \delta y^2) + (\beta-\delta)\,yw\,(\gamma x^2 - \alpha z^2)\\ + (\alpha-\beta)\,xy\,(\gamma w^2 - \delta z^2) + (\gamma-\delta)\,zw\,(\alpha y^2 - \beta x^2)\end{array}\right\} = 0.$$

27. It may be shown *à posteriori* that J is not a quadric function of P, Q, R, S. For, attempting to express it in this form, J does not contain the terms x^2w^2, y^2w^2, z^2w^2, and it thence at once appears that the coefficients of P^2, Q^2, R^2 each of them vanish. Hence, introducing for convenience the factor 2, I assume $(0, 0, 0, D, F, G, H, L, M, N \between P, Q, R, S)^2 = 2J$. Comparing the terms in $w^2(yz,\ zx,\ xy)$, we obtain

$$bcF = a\alpha, \quad caG = b\beta, \quad abH = c\gamma;$$

and comparing the coefficients of $w(y^2z,\ z^2x,\ x^2y,\ yz^2,\ zx^2,\ xy^2)$, we obtain

$$\begin{aligned}-Ff + a\alpha M &= \frac{h\alpha}{b}, & Ff + a\alpha N &= -\frac{g\alpha}{c},\\ -Gg + b\beta N &= \frac{f\beta}{c}, & Gg + b\beta L &= -\frac{h\beta}{a},\\ -Hh + c\gamma L &= \frac{g\gamma}{a}, & Hh + c\gamma M &= -\frac{f\gamma}{b};\end{aligned}$$

substituting for F, G, H their values, we obtain from the first 3 equations $L, M, N = \frac{-f}{bc}, \frac{-g}{ca}, \frac{-h}{ab}$, and from the second 3 equations, $L, M, N = \frac{f}{bc}, \frac{g}{ca}, \frac{h}{ab}$; that is, the equations are inconsistent, and the function J is not expressible in the form in question.

Jacobian Surface of Six given Points.

28. The equation $J = 0$ is the locus of the vertices of the quadric cones which pass through the given 6 points; calling these 1, 2, 3, 4, 5, 6, we see at once that the surface passes through the 15 lines 12, 13, ... 56, and also through the ten lines 123 . 456 (viz., line of intersection of the planes through 1, 2, 3, and through 4, 5, 6), &c. In fact, taking the vertex at any point O in the line 1, 2, the lines drawn to the six points are $O1 = O2$, $O3$, $O4$, $O5$, $O6$; viz., there are only five lines, so that these lie in a quadric cone. And taking the vertex at any point in the line 123 . 456,

the lines to the 6 points lie in these planes 123 and 456 respectively, and the quadric cone is in fact this plane-pair. Moreover, the surface containing the lines 12, 13, 14, 15, 16, must have the point 1 for a node; and similarly, the points 2, 3, 4, 5, 6 are each of them a node on the surface. It is to be added that the surface contains the skew cubic through the 6 points, or say the skew cubic 123456. See, as to this, *post* No. 108.

29. The surface in question (the Jacobian of the 6 points) is a particular case of the Jacobian of any 4 quadric surfaces. This more general surface will be considered in the sequel; I only remark here that it contains 10 lines, corresponding to the 10 lines 123.456, &c., but it has not any other lines, or any nodes.

Jacobian Curve of Seven given Points, or of an Octad of Points.

30. In connexion with what precedes, we may here consider a curve which presents itself in the sequel; viz., the curve which is the locus of the vertices of the quadric cones which pass through seven given points. The general case is when no one of the points is the vertex of a quadric cone through the other 6 points. We have through the 7 points the three quadric surfaces $P=0$, $Q=0$, $R=0$; hence, forming the equation $\alpha P+\beta Q+\gamma R=0$ of the general quadric surface through the 7 points, and making this a cone, we find as the locus of the vertex $J(P, Q, R)=0$; the analytical form shows that this is a sextic curve. It appears, moreover, that the curve is symmetrically related to all the 8 points $P=0$, $Q=0$, $R=0$; and instead of calling it the Jacobian of the 7 points, we may call it the Jacobian of the octad. But in further explanation, take the points to be 1, 2, 3, 4, 5, 6, 7; the vertex will lie on each of the Jacobian surfaces 123456 and 123457; and it is at present assumed that 7 is not a point on the first surface, nor 6 a point on the second surface. The two surfaces have in common the lines 12, 13, ... 45, and they consequently besides intersect in a curve of the 6th order, or sextic curve, which is the locus in question. At the point 1 there is on the first surface a tangent cone through the lines 12, 13, 14, 15, 16, and on the second surface a tangent cone through the lines 12, 13, 14, 15, 17; these two cones have for their complete intersection the lines 12, 13, 14, 15, which lines belong to the complete intersection of the two surfaces, but not to the sextic curve. It thus appears, *à posteriori*, that the sextic curve does not pass through the point 1; and similarly, that it does not pass through any of the points 2, 3, 4, or 5. As to the points 6 and 7, each of these is on only one of the quartic surfaces, and therefore the curve of intersection does not pass through either of these points.

31. Suppose, however, that one of the seven points is the vertex of a cone through the other six; it is of course the same thing whether we take this to be one of the points 1, 2, 3, 4, 5, or one of the points 6 and 7, but the result comes out more easily in the latter case; viz., in the former case, taking 1 to be the point in question, the two tangent cones at 1 are one and the same cone, and all that appears is that there is nothing to hinder a branch or branches of the sextic curve from passing through the point 1. But in the latter case, taking 7 for the point in question, then 7 lies on the surface 123456, being a simple point on this surface, but a node on the surface 123457; and it thus appears that there are through 7 two

branches of the sextic curve; so that any one of the seven points, being the vertex of a cone through the other six, is an actual double point on the sextic curve.

32. In the case where two of the points are each of them the vertex of a cone through the other six points, then the seven points lie on a skew cubic; and the sextic curve of the general case becomes this skew cubic twice repeated.

Seven given Nodes.

33. In the case of 7 given nodes, the number of constants should be $=6$; the 7 given points determine 3 quadric surfaces $P=0$, $Q=0$, $R=0$; and we have hence the quartic surface $(a, \ldots \between P, Q, R)^2=0$, containing 5 constants only. That this is not the general quartic surface with the 7 given nodes, is also clear from the consideration that the surface in question has 8 nodes; viz., the 8 points of intersection of the three quadric surfaces. Suppose that a particular quartic surface, having the 7 given nodes, but not of the last mentioned form, is $\Delta=0$; then a quartic surface having the 7 given nodes is

$$(a, \ldots \between P, Q, R)^2+\theta\Delta=0;$$

and this, as containing 6 constants, will be the general quartic surface with the 7 given nodes.

34. It follows that, if $\Delta'=0$ be another quartic surface having the 7 given nodes, we must have identically $\Delta'-p\Delta=(*\between P, Q, R)^2$, where p is a determinate constant and $(*\between P, Q, R)^2$ a determinate quadric function of (P, Q, R). The formula extends to the case where $\Delta'=0$ has the 8 nodes $(P=0, Q=0, R=0)$, but we have then $p=0$, and the meaning is simply that the general quartic surface having the 8 nodes is $(*\between P, Q, R)^2=0$.

35. A particular quartic surface having (in an improper sense) the 7 given nodes, but not having the 8th node, is $M\Omega=0$, where $M=0$ in the plane through any 3 of the 7 points and $\Omega=0$ is the cubic surface through these same 3 points, and having the remaining 4 points as nodes. The equation of the cubic surface, if the 4 points are taken to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), is obviously of the form

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}=0, \text{ (that is, } ayzw+bzxw+cxyw+dxyz=0),$$

and by making the surface pass through the 3 points we determine linearly the coefficients (a, b, c, d), that is, their ratios. The equation of the quartic surface thus is

$$(a, \ldots \between P, Q, R)^2+\theta M\Omega=0,$$

the 7 given points being here proper nodes; and the formula being precisely equivalent to the preceding one containing Δ.

36. We can with the 7 given points form 35 such combinations $M\Omega=0$ of a plane and a cubic surface, and so present the equation of the quartic surface under 35 different forms; these are of course equivalent in virtue of the before mentioned formula for $\Delta'-p\Delta$; viz., we must have identically $M\Omega-pM'\Omega'=(*\between P, Q, R)^2$: a theorem of some interest, which it might be difficult to verify *à posteriori*.

Investigation of the cases of 8 Nodes.

37. It has already been shown that a quartic surface cannot in a proper sense have 8 given nodes. In regard to the quartic surfaces with 8 nodes, we start from the surface with 7 given nodes; viz.,

$$(a, \ldots \between P, Q, R)^2 + \theta\nabla = 0,$$

or, what is the same thing,

$$(a, \ldots \between P, Q, R)^2 + \theta M\Omega = 0;$$

and we inquire in what cases this surface has an 8th node. Obviously if $\theta = 0$. that is, if the surface is $(a, \ldots \between P, Q, R)^2 = 0$, the surface will have an 8th node, the remaining intersection of the quadric surfaces $P = 0$, $Q = 0$, $R = 0$ (observe that this is a point in no wise depending on the particular quadric surfaces, but uniquely determined by means of the 7 given points); and we have thus one kind, say the "octadic" surface, of the quartic surfaces with 8 nodes; viz., the nodes are the 8 points of intersection of any 3 quadric surfaces, or they are an octad of points. By what precedes, 7 of the nodes may be given points, and the remaining node is then a uniquely determinate point, the 8th point of the octad.

38. But if θ be not $= 0$, there may still be an 8th node; viz., this must then be a point on the Jacobian surface $J(P, Q, R, \nabla) = 0$, which is of the order 6. It is clear *à priori* that this must be a surface depending only on the 7 points, but independent of the particular surfaces $P = 0$, $Q = 0$, $R = 0$, $\nabla = 0$; to verify this, observe that, substituting for ∇ the function ∇', $= p\nabla + (*\between P, Q, R)^2$, we in fact leave the Jacobian unaltered; I call it the dianodal surface of the 7 points.

39. I say that the 8th node may be any point whatever on the dianodal surface; in fact, regarding for a moment the coordinates of the node as given, and expressing that the point is a node on the quartic surface, we have 4 equations containing

$$aP_0 + hQ_0 + gR_0, \quad hP_0 + bQ_0 + fR_0, \quad gP_0 + fQ_0 + cR_0,$$

(P_0, Q_0, R_0 the values of P, Q, R at the node,) but which, if only the point be on the dianodal surface, reduce themselves to three equations; viz., we have between the coefficients (a, b, c, f, g, h) and θ three equations which being satisfied, the point in question will be a node. And it thus appears that, taking the 8th node to be a given point on the dianodal surface, the equation $(a, \ldots \between P, Q, R)^2 + \theta\nabla = 0$ of the quartic surface will contain 3 constants. Observe that we may through the 8 nodes draw 2 quadric surfaces $P = 0$, $Q = 0$; and this being so if $\Delta = 0$ be a particular quartic surface with the 8 nodes, then the general quartic surface will be

$$(a, b, c \between P, Q)^2 + \theta\Delta = 0,$$

containing the right number 3 of constants. But there is not here any simple form of the surface $\Delta = 0$, such as the form $M\Omega = 0$ for the surface through 7 given points.

40. It is clear *à priori* that the relation between the 8 nodes is a symmetrical one; so that the 8th point being situate anywhere on the dianodal surface of the 7 points, each of the points will be situate on the dianodal surface of the remaining 7 points. This is a remarkable property of the dianodal surface, which will have to be again considered.

41. In what precedes, we have the second kind of quartic surfaces with 8 nodes, say the "dianome"; viz., each node is a point on the dianodal surface of the remaining 7 nodes; any 7 of the nodes may be taken to be given points, and the remaining node to be any point whatever on the dianodal surface of the 7 points.

The Dianodal Surface.

42. Consider the seven points 1, 2, 3, 4, 5, 6, 7. As already mentioned, through three of these, say 1, 2, 3, we may draw a plane $M=0$; and through the same three points, with the remaining points 4, 5, 6, 7 as nodes ($3+4.4=19$ conditions), a cubic surface $\Omega=0$; this surface passing through the six lines, 45, 46, ... 67. Hence we have $\Delta, =M\Omega, =0$, a quartic surface with the seven points as nodes. And using this form of Δ, it may be shown that the dianodal $J(P, Q, R, \Delta)=0$ passes through the 21 lines 12, 13, ... 67, and through 35 plane cubics such as $M=0$, $\Omega=0$; viz., this is a cubic in the plane 123 passing through the points 1, 2, 3, and through the intersections of the plane with each of the six lines 45, 46, ... 67 (nine points determining the cubic); the complete intersection by the plane 123 being therefore composed of this cubic and of the three lines 12, 13, 23. For the passage through the cubic, we have only to observe that

$$J(P, Q, R, M\Omega)=J(P, Q, R, \Omega)M+J(P, Q, R, M)\Omega=0$$

is satisfied by $M=0$, $\Omega=0$; and for the passage through the lines, taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the planes 567, 674, 745, and 456 respectively, each of the functions P, Q, R is of the form $ayz+bzx+cxy+fxw+gyw+hzw$, and the function Ω is of the form $Ayzw+Bzwx+Cwxy+Dxyz$. Hence, writing in the derived functions for instance $z=0$, $w=0$, the first and second lines of the determinant $J(P, Q, R, \Omega)$ will be of the form

$$\begin{vmatrix} cy, & c'y, & c''y, & 0 \\ cx, & c'x, & c''x, & 0 \end{vmatrix},$$

or the determinant vanishes for $z=0$, $w=0$; that is, for any point of the line 45 we have $\Omega=0$ and also $J(P, Q, R, \Omega)=0$; consequently $J(P, Q, R, M\Omega)=0$, and the like for the other lines. The theorem is thus proved.

43. I say that the dianodal surface passes through each of the 7 skew cubics, such as 123456. To prove this, it is only necessary to show that the skew cubic

123456 lies on the dianodal surface. For this purpose it will be enough to show that the skew cubic meets the plane 712 in a point of the surface; for then it will, in like manner, meet each of the 15 planes 712, 713, ... 756 in a point of the surface; that is, we shall have 15 intersections of the curve and surface, and there are, besides, the intersections 1, 2, 3, 4, 5, 6, in all 21 intersections; that is, the skew cubic must lie on the surface.

44. The plane 712 meets the surface in three lines and in a plane cubic determined by the points 7, 1, 2 and the six intersections of the plane with the lines 34, 35, ... 56. We have therefore to show that this plane cubic meets the skew cubic 123456. Consider for a moment the points 1, 2, 3, 4, 5, 6 and another point 7′. As seen above, we have in general, through the points 1, 2, 7′ and with the points 3, 4, 5, 6 as nodes, a determinate cubic surface, which surface passes through the lines 34, 35, ... 56. But the cubic surface becomes indeterminate if the points 1, 2, 7′, 3, 4, 5, 6 are on the same skew cubic; that is, if 7′ is any point whatever on the skew cubic 123456 (the proof presently). Taking, then, 7′ as the intersection of the skew cubic by the plane 712, we have in this plane the points 7′, 1, 2, and the intersections of the plane by the lines 34, 35, ... 56, nine points through which there pass an infinity of plane cubics; that is, the plane cubic determined by the points 7, 1, 2 and the six intersections will pass through the point 7′; viz., it meets the skew cubic 123456.

45. For the subsidiary theorem, taking X, Y, Z, W as current coordinates, viz., $X=0$, $Y=0$, $Z=0$, $W=0$ as the equations of the planes 456, 563, 634, 345 respectively, (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) as the coordinates of the points 1 and 2 respectively, and (x, y, z, w) for those of 7′; the equation of the cubic surface passing through 7′, 1, 2, and having the nodes 3, 4, 5, 6, is

$$\begin{vmatrix} \frac{1}{X}, & \frac{1}{Y}, & \frac{1}{Z}, & \frac{1}{W} \\ \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{vmatrix} = 0;$$

and this ceases to be a determinate function if only

$$\begin{vmatrix} \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{vmatrix} = 0;$$

viz., considering (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) as given, this is a twofold relation between the coordinates (x, y, z, w) of the point 7′. The relation may be represented by the four equations $(yzw)=0$, $(zwx)=0$, $(wxy)=0$, $(xyz)=0$, if for shortness

$$(yzw)=\begin{vmatrix} yz\ , & zw\ , & wy \\ y_1z_1, & z_1w_1, & w_1y_1 \\ y_2z_2, & z_2w_2, & w_2y_2 \end{vmatrix}$$

and the like as to the other symbols. The four equations represent quadric surfaces, each two intersecting in a line [e.g., $(yzw)=0$, $(zwx)=0$ in the line $z=0$, $w=0$], and the four surfaces besides intersecting in a skew cubic, which is the required locus of the point 7′, and which, as is seen at once, passes through the points 1, 2, 3, 4, 5, 6.

46. By what precedes, we have on the dianodal surface through the point 1 the lines 12, 13, 14, 15, 16, 17, and the skew cubics 123456, &c. The six lines are not on the same quadric cone, and it thus appears that the point 1 must be a cubic-node (point where, instead of the tangent plane, we have a cubic cone) on the surface. It is to be remarked that the lines 12, 13, 14, 15, 16, and the tangent at 1 to the skew cubic 123456, lie in a quadric cone; viz., this tangent is given as the sixth intersection of the cubic cone with the quadric cone through the lines 12, 13, 14, 15, 16.

47. I revert to the equation of the dianodal surface as given in the form $J=J(P, Q, R, M\Omega)=0$, where $M=0$ is the plane through the points 1, 2, 3, and $\Omega=0$ the cubic surface through these points, and having the points 4, 5, 6, 7, as nodes. We can find the orders of the several functions P, Q, R, M, Ω in the coordinates (x_1, y_1, z_1, w_1), &c., of the several points; viz., writing for shortness x_1^2 to denote the order 2 in regard to (x_1, y_1, z_1, w_1), and so in other cases, we have

$$P=Q=R=x^2(x_5, x_6, x_7)^2(x_1, x_2, x_3, x_4)^2,$$
$$M=x\,(x_5, x_6, x_7),$$
$$\Omega=x^3(x_5, x_6, x_7)^3(x_1, x_2, x_3, x_4)^9;$$

{where, of course, the x^2, x, x^3 show in like manner the orders in regard to the current coordinates (x, y, z, w); the proof in regard to Ω is easily supplied.} The order of J is equal that of $PQRM\Omega$, less 4 as regards the current coordinates, by reason of the differentiations; that is, we have $J=x^6(x_1x_2x_3)^{10}(x_4x_5x_6x_7)^{15}$; and we thus see that the equation of the dianodal surface as above obtained is encumbered with a constant factor of the form $(x_1x_2x_3)^4(x_4x_5x_6x_7)^9$. In fact, the relation between the 7 points and the current point (x, y, z, w), or say the point 8, as expressing that the 8 points are the nodes of a dianome, should be a symmetrical one in regard to the coordinates of the several points; and being of the order 6 in regard to the coordinates (x, y, z, w), it should be of the same order in regard to the other coordinates; that is, the true form would be $J=(xx_1x_2x_3x_4x_5x_6x_7)^6=0$.

48. It is possible that taking the 4 points, say 1, 2, 3, 4, to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), and the 3 points, say 5, 6, 7, to be (1, 1, 1, 1), $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, the extraneous factor might exhibit itself, and that the

equation divested of this factor might be of a tolerably simple form. I have not, however, worked this out, but I have, by an independent process, obtained in regard to the dianodal surface of the 7 points a result which may be interesting.

49. The dianodal surface, *quà* surface having the first-mentioned 4 points for cubic nodes, has its equation of the form

$$yzw(y, z, w)^2 + zxw(z, x, w)^2 + xyw(x, y, w)^2 + xyz(x, y, z)^2 + xyzw(x, y, z, w)^2 = 0;$$

where in the cubic functions the terms x^3, y^3, z^3, w^3 none of them appear. If for instance $w=0$, the equation becomes $(x, y, z)^2 = 0$, which, by what precedes, is a known cubic curve, viz., the curve through the points 1, 2, 3 and the intersections of the plane 123 by the lines 45, 46, 47, 56, 57, 67; and we can by this consideration find the cubic function $(x, y, z)^3$, and thence by symmetry the other cubic functions. I take

$$\left.\begin{array}{l}(a, b, c, f, g, h)\\(a', b', c', f', g', h')\\(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})\end{array}\right\}\text{ for coordinates of line through }\left\{\begin{array}{ll}(1, 1, 1, 1), & (\alpha, \beta, \gamma, \delta)\\(1, 1, 1, 1), & (\alpha', \beta', \gamma', \delta')\\(\alpha, \beta, \gamma, \delta), & (\alpha', \beta', \gamma', \delta')\end{array}\right.$$

respectively; viz., I write

$$\begin{array}{l|l|l}
a=\beta-\gamma, \quad f=\alpha-\delta & a'=\beta'-\gamma', \quad f'=\alpha'-\delta' & \mathrm{a}=\beta\gamma'-\beta'\gamma, \quad \mathrm{f}=\alpha\delta'-\alpha'\delta\\
b=\gamma-\alpha, \quad g=\beta-\delta & b'=\gamma'-\alpha', \quad g'=\beta'-\delta' & \mathrm{b}=\gamma\alpha'-\gamma'\alpha, \quad \mathrm{g}=\beta\delta'-\beta'\delta\\
c=\alpha-\beta, \quad h=\gamma-\delta & c'=\alpha'-\beta', \quad h'=\gamma'-\delta' & \mathrm{c}=\alpha\beta'-\alpha'\beta, \quad \mathrm{h}=\gamma\delta'-\gamma'\delta
\end{array}$$

and I write moreover

$$\begin{aligned}
\lambda &= \ .\ \ \mathrm{h}-\mathrm{g}+\mathrm{a},\\
\mu &= -\mathrm{h}\ \ .\ +\mathrm{f}+\mathrm{b},\\
\nu &= \mathrm{g}-\mathrm{f}\ \ .\ +\mathrm{c},\\
\varpi &= -\mathrm{a}-\mathrm{b}-\mathrm{c}\ \ .
\end{aligned}$$

50. This being so, the cubic curve through the last-mentioned six points has its equation of the form

$$\frac{A}{ax+by+cz}+\frac{B}{a'x+b'y+c'z}+\frac{C}{\mathrm{a}x+\mathrm{h}y+\mathrm{g}z}+\frac{D}{\lambda x+\mu y+\nu z}=0;$$

and to make this pass through the points 1, 2, 3, we write therein successively $(y=0, z=0)$, $(z=0, x=0)$, $(x=0, y=0)$; viz., we have for the ratios $A : B : C : D$ the three equations

$$\frac{A}{a}+\frac{B}{a'}+\frac{C}{\mathrm{a}}+\frac{D}{\lambda}=0,$$

$$\frac{A}{b}+\frac{B}{b'}+\frac{C}{\mathrm{h}}+\frac{D}{\mu}=0,$$

$$\frac{A}{c}+\frac{B}{c'}+\frac{C}{\mathrm{g}}+\frac{D}{\nu}=0.$$

In eliminating, for instance, B for the first and second equations, the resulting equation divides by $ab' - a'b$, $= \mathrm{a}+\mathrm{b}+\mathrm{c}$, and we thus obtain, between A, C, D, the three equations (equivalent to two)

$$\frac{A}{bc} + \frac{C\alpha'}{\mathrm{bc}} + \frac{Df'}{\mu\nu} = 0,$$

$$\frac{A}{ca} + \frac{C\beta'}{\mathrm{ca}} + \frac{Dg'}{\nu\lambda} = 0,$$

$$\frac{A}{ab} + \frac{C\gamma'}{\mathrm{ab}} + \frac{Dh'}{\lambda\mu} = 0,$$

from which the ratios $A : C : D$ may be obtained by actual calculation. After all reductions, we have

$$\begin{aligned}
A &= a\,b\,c\,\{(\alpha'\delta' + \beta'\gamma')\,\mathrm{af} + (\beta'\delta' + \gamma'\alpha')\,\mathrm{bg} + (\gamma'\delta' + \alpha'\beta')\,\mathrm{ch}\},\\
B &= -\,a'b'c'\,\{(\alpha\,\delta + \beta\,\gamma)\,\mathrm{af} + (\beta\,\delta + \gamma\,\alpha)\,\mathrm{bg} + (\gamma\,\delta + \alpha\,\beta)\,\mathrm{ch}\},\\
C &= \mathrm{a\,b\,c}\,\{(\alpha\alpha'\lambda + \beta\beta'\mu + \gamma\gamma'\nu + \delta\delta'\varpi\},\\
D &= -\,\lambda\,\mu\,\nu\,\{(\alpha\alpha'\mathrm{a} + \beta\beta'\mathrm{b} + \gamma\gamma'\mathrm{c}\};
\end{aligned}$$

viz., A, B, C, D are proportional to these values respectively. Multiplying by the product of the denominators, I find without much difficulty that the resulting cubic function is divisible by $\mathrm{a}+\mathrm{b}+\mathrm{c}$; hence, introducing the factor xyz, and an indeterminate multiplier l, I write

$$xyz\,(x,\ y,\ z)^3 = \frac{l}{\mathrm{a}+\mathrm{b}+\mathrm{c}}\,xyz\,(ax+by+cz)\,(a'x+b'y+c'z)\,(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\,(\lambda x+\mu y+\nu z)$$
$$\times\left\{\frac{A}{ax+by+cz} + \frac{B}{a'x+b'y+c'z} + \frac{C}{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z} + \frac{D}{\lambda x+\mu y+\nu z}\right\},$$

where A, B, C, D have the values above written down.

51. Considering the orders in regard to $(\alpha,\ \beta,\ \gamma,\ \delta)$, $(\alpha',\ \beta',\ \gamma',\ \delta')$, and observing that a, b, c and a', b', c' are linear functions of the two sets respectively, but that $\mathrm{a}, \mathrm{b} \ldots \mathrm{h}, \lambda \ldots \varpi$, are linear in the two sets conjointly, or say

$$a, \ldots = \alpha,\ a', \ldots = \alpha'\,;\ \mathrm{a}, \ldots = \alpha\alpha'\,;$$

we have

$$Aa'\mathrm{a}\lambda = \alpha^5\alpha'^4 \,.\, \alpha^2\alpha'^3 = \alpha^7\alpha'^7,$$

so that after the division by $\mathrm{a}+\mathrm{b}+\mathrm{c}$, $= \alpha\alpha'$, the order will be $\alpha^6\alpha'^6$. Hence l will be a mere numerical factor, and the last-mentioned equation gives, without any extraneous factor, the terms $xyz\,(x,\ y,\ z)^3$ in the equation of the dianodal surface of the seven points.

Octadic Surfaces with 9 *or* 10 *Nodes.*

52. In regard to the surfaces with 9 and 10 nodes, I consider first the octadic surfaces. Starting as before with the given points 1, 2, 3, 4, 5, 6, 7, we have a determinate point 8 completing the octad, and the surface with the 8 nodes is

$$(a, \ldots \between P,\ Q,\ R)^2 = 0,$$

(5 constants). Suppose that there is another node 9; this must be a point on the Jacobian curve $J(P, Q, R)=0$, which (as was seen) is a sextic curve not passing through any of the 8 points; the node 9 may be *any* point on this curve, viz, taking its coordinates as given, the condition of its being a node gives 4 equations, and these for the very reason that the point is on the Jacobian curve, reduce themselves to 2 equations, which can be satisfied by means of the constants $(a, \ldots)$; the resulting equation should therefore contain 3 constants.

53. In order to find it, taking as above 9 a given point on the Jacobian curve, this will be the vertex of a quadric cone, say $P=0$, through the 8 points; we may draw through the 9 points another quadric surface $Q=0$, and through the 8 points a quadric surface $R=0$; this being so, we have the quartic surface $(a, b, 0, 0, g, h \between P, Q, R)^2=0$, having the 9 nodes, and containing, as it should do, 3 constants; this may be written

$$(aP+2hQ+2gR)P+bQ^2=0;$$

viz., if $bR'=aP+2hQ+2gR$, that is, if $R'=0$ be the general quadric surface through the 8 points, then the equation is $Q^2-PR'=0$, where observe that R' is considered as containing implicitly 3 constants.

54. If there is a 10th node, say 10, this is also a point on the Jacobian curve $J(P, Q, R)=0$, and it may be any point whatever on the curve; taking it as a given point on the curve, the resulting equation should contain 1 constant. We may take $P=0$ to be the quadric cone, vertex 9, through the 8 points, $R=0$ the quadric cone, vertex 10, through the 8 points, $Q=0$ the quadric surface through the 8 points and the points 9 and 10 (viz., the surface through 9, 10 and any 7 of the 8 points will pass through the remaining 8th point). The equation of the quartic surface then is

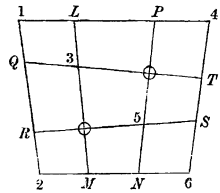
$$(0, b, 0, 0, g, 0 \between P, Q, R)^2=0;$$

that is, $bQ^2+2gPR=0$, containing 1 constant; we may reduce this to $Q^2-PR=0$, the constant being considered as contained implicitly in one of the functions. It is clear that the constant cannot be so determined as to give rise to an 11th node, nor indeed to any other singularity in the surface.

55. In the case of the surface with 9 nodes, it is clear that this is octadic in one way only; the node 9 cannot form an octad with any 7 of the remaining nodes. But in the case of the surface with 10 nodes, the question arises whether the nodes 9 and 10 may not be such as to form an octad with some six, say with the nodes 1, 2, 3, 4, 5, 6 of the remaining 8 nodes; that is, whether we can have 1, 2, 3, 4, 5, 6, 7, 8 forming an octad, and also 1, 2, 3, 4, 5, 6, 9, 10 forming an octad. I will show that this is impossible if only the points 1, 2, 3, 4, 5, 6 are given points, that is, points assumed at pleasure and not specially related to each other. For this purpose, assuming that the points form 2 octads as above, take through 1, 2, 3, 4, 5, 6, 7, 9 the quadric surfaces $P=0$, $Q=0$, then each of these passes through 8, 10; take $R=0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 7, 8, and $S=0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 9, 10. Then $P=0$, $Q=0$, $R=0$ intersect in the 1st octad,

and $P=0$, $Q=0$, $S=0$ intersect in the 2nd octad; the quartic surface (if it exists) must be simultaneously of the forms $(*\between P, Q, R)^2=0$, $(*\between P, Q, S)^2=0$; and this implies an identical equation $(*\between P, Q, R, S)^2=0$. The quadric surfaces are surfaces through the points 1, 2, 3, 4, 5, 6, and taking through these six points any other quadric surfaces $A=0$, $C=0$, $E=0$, $H=0$, we have P, Q, R, S each of them a linear function of A, C, E, H; and the relation between P, Q, R, S gives a like relation $(*\between A, C, E, H)^2=0$ between A, C, E, H. I assume $A=123.456$, $E=134.256$, $H=145.236$, $C=152.346$; viz., $A=0$ is the plane-pair formed by the planes through 1, 2, 3 and 4, 5, 6 respectively; and so for the others: we have to show that there is not any such identical relation $(*\between A, C, E, H)^2=0$.

56. We may through 3 draw the lines LM, QT to meet 14, 26 and 12, 46 respectively; and through 5 the lines RS, NP to meet 14, 26 and 12, 46 respectively. Observe that the points O in the figure are apparent intersections only; viz., NP does



not meet QT, *nor* LM meet RS. In fact, if NP met QT it would be a line in the series of lines meeting 14, QT, 26; or 5 would be situate in a hyperboloid, determined by means of the points 1, 2, 4, 6, 3; viz., 5 would not be an arbitrary point: and so LM does not meet RS. Now the quadrics E, H meet in the lines 14, 26, LM, NP, and the quadrics A, C in the lines 12, 46, QT, RS. Suppose that we had identically $(*\between A, C, E, H)^2=0$; putting therein $E=0$, $H=0$, we should have $(*\between A, C)^2=0$, viz., $(A+\lambda C)(A+\mu C)=0$; or there would exist quadrics of the forms $A+\lambda C=0$ containing the lines 14, 26, LM, NP. Now there is no quadric surface $A+\lambda C=0$ containing the line NP; for $A+\lambda C=0$ is a quadric containing the sides of the quadrilateral $QRST$; the generating lines of the one kind meet each of the lines RS, QT; those of the other kind neither. Hence NP, which meets RS but not QT, cannot be a generating line of either kind; and we have no identical relation $(A, C, E, H)^2=0$.

57. In the octadic surface with 9 nodes; starting with any 7 nodes of the octad, 9 is not the 8th point of the octad, and hence (by the theory of the dianome) it must lie in the dianodal surface of the 7 points; that is, the dianodal surface of the 7 points must pass through 9, viz., through any point whatever of the Jacobian curve of the 7 points, that is, of the octad; or (what is the same thing) the dianodal surface of the 7 points passes through the Jacobian curve of the octad. This is an obvious property of the dianodal surface, the surface $J(P, Q, R, \nabla)=0$ contains the Jacobian curve $J(P, Q, R)=0$. But it further appears that, starting with any 6 points of the octad and with the point 9 (that is, any point whatever of the Jacobian curve), the

dianodal surface of these 7 points must contain the remaining 2 points of the octad. And in the octadic surface with 10 nodes, starting with any 5 points of the octad and with the points 9 and 10 (that is, any two points on the Jacobian curve) the dianodal surface of these 7 points must contain the remaining three points of the octad. I have not attempted to verify these last properties of the dianodal surface.

Dianomes with 9 *or* 10 *Nodes.*

58. I now consider the dianomes with 9 and 10 nodes. Starting from the general form

$$(a,\ b,\ c \between P,\ Q)^2 + \theta\Delta = 0,$$

where $\Delta = 0$ is a particular quartic surface having the 8 nodes, it at once appears that if there is a 9th node, say 9, this must be a point on the Jacobian curve $J(P,\ Q,\ \Delta) = 0$, or say on the dianodal curve of the 8 points, viz. ($a = b = 1$, $c = 3$, in the formula No. 5), this is a curve of the order 18; the node may be any point whatever on this curve, and taking it to be a given point on the curve, the number of constants in the resulting equation should be 1. Hence if $P = 0$ be the quadric surface through the 9 points, and $\Delta = 0$ a particular quartic surface having the 9 points as nodes, the general equation is $aP^2 + \theta\Delta = 0$.

59. But we may consider the question somewhat differently. Starting with the 7 given points 1, 2, 3, 4, 5, 6, 7 and with 8 a given point on the dianodal surface of the 7 points; it is clear that 9 must be on the dianodal surface 1234567, and also on the dianodal surface 1234568; the complete intersection is of the order 36, and we have to consider how this breaks up so as to contain as part of itself the dianodal curve of the order 18.

Dianodal Curve of 8 *Points.*

60. Consider first any 8 points whatever 1, 2, 3, 4, 5, 6, 7, 8; where 8 is not on the dianodal surface 1234567, nor 7 on the dianodal surface 1234568. The two surfaces have in common the 15 lines 12, 13, ... 56 and the skew cubic 123456, they therefore besides intersect in a curve of the order 18. At the point 1 the tangent cubic cones of the two surfaces intersect in the lines 12, 13, 14, 15, 16 and the tangent to the skew cubic 123456, 6 lines lying in a quadric cone; they therefore besides intersect in 3 lines lying in a plane; that is, the point 1 is on the curve of the order 18 an actual triple point, the 3 tangents lying *in plano;* and the like of course in regard to each of the points 2, 3, 4, 5, 6. But as 7, 8 lie each of them on only one of the two surfaces, the curve of the order 18 does not pass through 7 or 8.

61. If, however, 8 lies on the dianodal surface 1234567, then each of the 8 points will lie on the dianodal surface of the other 7; and in particular 7 will lie on the dianodal surface 1234568. The surfaces intersect as before in a residual curve of the order 18; the only difference is that 7 and 8 are now points on each surface; viz., each of them is on one of the surfaces an ordinary point, and on the other a cubic node; the points 7 and 8 are thus each of them an actual triple point on the curve; and at each of them the 3 tangents are *in plano.* We thus see that the dianodal

curve 12345678 is a curve of the order 18, such that each of the 8 points is a triple point on the curve, the tangents at each of them being *in plano*.

Ten Nodes.

62. Suppose there is a 10th node, say 10; starting from the equation $aP^2 + \theta\Delta = 0$ ($P = 0$ the quadric surface through the 9 points, $\Delta = 0$ a particular quartic surface having the 9 points as nodes), it at once appears that the node must be one of the points $J(P, \Delta) = 0$; hence, taking it to be one of these points, we have 4 equations, which, in virtue of the node being one of the points in question, reduce themselves to a single equation determining the ratio $a : \theta$; we have thus a completely determinate surface, say $\square = 0$ having the 10 points as nodes. The number of points $J(P, \Delta)$, writing in the formula No. 5, $a = 1$, $b = 3$, is obtained as $1 + 3 + 9 + 27 = 40$, but it is to be observed that the surface $P = 0$ passes through each of the 9 nodes of the surface $\Delta = 0$; these count twice among the points $J(P, \Delta) = 0$, and the number of residual points (or say the dianodal centres of the 9 points) is $40 - 18 = 22$; viz., this is the number of positions of the node 10. [The nine points count each *three* times and the number of residual points, or positions of the node 10, is thus not $40 - 18 = 22$, but $40 - 27, = 13$.]

Dianodal Centres of 9 Points.

63. In further explanation, observe that 9 is any point on the dianodal curve 12345678; the node 10 must lie on this same curve, and also on the dianodal surface 1234569. Take $P = 0$ the quadric through all the 9 points, $Q = 0$ a quadric through all but the point 9, $R = 0$ through all but the point 8, $S = 0$ through all but the point 7. The dianodal curve 12345678 is $J(P, Q, \nabla) = 0$, and the dianodal surface 1234569 is $J(P, R, S, \nabla) = 0$; the total number of intersections is $6 \times 18 = 108$; these include the $4 \times 18 = 72$ points of intersection of the dianodal curve $J(P, Q, \Delta) = 0$ with the Jacobian surface $J(P, Q, R, S) = 0$, except the four points $J(P, Q) = 0$, which are the vertices of the 4 quadric cones through 1, 2, 3, 4, 5, 6, 7, 8 (which 4 points are not situate on the curve $J(P, R, S) = 0$), and there are besides 40 points $\{108 = (72 - 4) + 40\}$ which are the before mentioned points $J(P, \Delta) = 0$; viz., these are the 9 points each twice [three times], and the residual 22 [13] points which are the dianodal centres of the 9 points.

General result as to the Dianomes.

64. We have thus established the theory of the dianome quartic surfaces; viz., we have

The octodianome, 8 nodes, 7 of them arbitrary, and the 8th an arbitrary point on the dianodal surface (order 6) of the 7 points.

The enneadianome, 9 nodes, the 9th an arbitrary point on the dianodal curve (order 18) of the 8 points.

The decadianome, 10 nodes, the 10th any one of the 22 [13] dianodal centres of the 9 points.

And as already mentioned, so long as the first 7 nodes are arbitrary, there cannot be more than 10 nodes in all.

THE SYMMETROID.

The Lineolinear Correspondence of Quartic Surfaces.

65. I consider four equations $S=0$, $T=0$, $U=0$, $V=0$, lineolinear in regard to the two sets of coordinates (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$; viz., each of these equations is of the form

$$(*\between x, y, z, w \between \alpha, \beta, \gamma, \delta) = 0.$$

This implies that the point (x, y, z, w) lies on a certain quartic surface $\Theta=0$, and the point $(\alpha, \beta, \gamma, \delta)$ on a certain quartic surface $\Delta=0$, and that the two surfaces correspond point to point to each other. In fact, writing the four equations in the form

$$\begin{aligned} L\alpha + M\beta + N\gamma + P\delta &= 0, \\ L'\alpha + M'\beta + N'\gamma + P'\delta &= 0, \\ L''\alpha + M''\beta + N''\gamma + P''\delta &= 0, \\ L'''\alpha + M'''\beta + N'''\gamma + P'''\delta &= 0, \end{aligned}$$

where L, &c., are linear functions of (x, y, z, w), then eliminating $(\alpha, \beta, \gamma, \delta)$, we obtain the equation

$$\Theta = \begin{vmatrix} L, & M, & N, & P \\ L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix} = 0;$$

and similarly, writing the four equations in the form

$$\begin{aligned} Ax + By + Cz + Dw &= 0, \\ A'x + B'y + C'z + D'w &= 0, \\ A''x + B''y + C''z + D''w &= 0, \\ A'''x + B'''y + C'''z + D'''w &= 0, \end{aligned}$$

where A, &c., are linear functions of $(\alpha, \beta, \gamma, \delta)$, then eliminating (x, y, z, w), we obtain the equation

$$\Delta = \begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0.$$

Moreover, Θ being $=0$, the four linear equations in $(\alpha, \beta, \gamma, \delta)$ are equivalent to three equations, and give for instance $(\alpha, \beta, \gamma, \delta)$ proportional to the determinants formed with the matrix

$$\begin{vmatrix} L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix};$$

and similarly, Δ being $=0$, the four linear equations in (x, y, z, w) are equivalent to three equations, and give for instance (x, y, z, w) proportional to the determinants formed with the matrix

$$\begin{vmatrix} A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix};$$

which establishes the point-to-point correspondence of the two surfaces.

66. It would at first sight appear that any quartic surface $(*\between\alpha, \beta, \gamma, \delta)^4=0$ whatever might have its equation expressed in the foregoing determinant form $\Delta = 0$. This equation seems, in fact, to contain homogeneously as many as 64 constants. But if we multiply the determinant line into line by a constant determinant

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix}$$

and then column into column by another constant determinant, the coefficients, all but one of them, of these constant determinants may be used to specialize the form of the resulting equation, [say they are apoclastic constants]; this equation will really contain $64-(2.16-1)=33$ constants; and in order that the quartic surface $(*\between\alpha, \beta, \gamma, \delta)^4=0$ may have its equation expressible in the form $\Delta=0$, a single relation must hold good among the coefficients: but this in passing[1].

67. Returning to the quartic surface

$$\Delta = \begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0,$$

we may connect this not only with the foregoing surface $\Theta=0$, but in a similar manner with another quartic surface $\Phi=0$; viz., taking the current coordinates $(\xi, \eta, \zeta, \omega)$, we may form the lineolinear equations

$$\begin{aligned} A\xi + A'\eta + A''\zeta + A'''\omega &= 0, \\ B\xi + B'\eta + B''\zeta + B'''\omega &= 0, \\ C\xi + C'\eta + C''\zeta + C'''\omega &= 0, \\ D\xi + D'\eta + D''\zeta + D'''\omega &= 0, \end{aligned}$$

[1] Applying the same reasoning to a cubic determinant $\Delta=0$, the number of constants is $36-(2.9-1)=19$; so that a cubic surface is expressible in the form in question. And so for the quadric determinant $\Delta=0$, the number of constants is $16-(2.4-1)=9$; so that a quadric surface is expressible in the form in question, as is otherwise obvious.

which, by the elimination of $(\xi, \eta, \zeta, \omega)$, give $\Delta = 0$, and by the elimination of $(\alpha, \beta, \gamma, \delta)$ a determinant quartic equation $\Phi = 0$ between the coordinates $(\xi, \eta, \zeta, \omega)$; and of course the two surfaces $\Delta = 0$, $\Phi = 0$ have a point-to-point correspondence such as exists between the surfaces $\Theta = 0$, $\Delta = 0$. The relation of the point $(\alpha, \beta, \gamma, \delta)$ on the surface $\Delta = 0$ to the point (x, y, z, w) on the surface $\Theta = 0$, and to the point $(\xi, \eta, \zeta, \omega)$ on the surface $\Phi = 0$, may be conveniently indicated by means of the diagram

$$\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{x\ ,\ y\ ,\ z\ ,\ w}^{\Theta}} & \\ \hline A\ , & B\ , & C\ , & D & \xi \\ A'\ , & B'\ , & C'\ , & D' & \eta \\ A''\ , & B''\ , & C''\ , & D'' & \zeta \\ A'''\ , & B'''\ , & C'''\ , & D''' & \omega \end{array} \Bigg\} \Phi.$$

68. It is to be observed that, writing for $A, B, \ldots$ their values as linear functions of $(\alpha, \beta, \gamma, \delta)$, we have in all 64 constant coefficients, which we may conceive arranged in the form of a cube, thus:

$$\begin{array}{ll} a & b \\ \quad a' & \quad b' \\ a_1 & b_1 \\ \quad a_1' & \quad b_1' \end{array}$$

and taking these in fours height-wise, (a, a_1, a_2, a_3), &c., we compose with them the linear functions $a\alpha + a_1\beta + a_2\gamma + a_3\delta$, &c., which enter into the equation $\Delta = 0$; taking them in fours length-wise, (a, b, c, d), &c., we compose the linear functions $ax + by + cz + dw$, &c., which enter into the equation $\Theta = 0$; and taking them in fours breadth-wise (a, a', a'', a'''), &c., we compose the linear functions $a\xi + a'\eta + a''\zeta + a'''\omega$, &c., which enter into the equation $\Phi = 0$.

69. The process may be indefinitely repeated; we obtain always the same three surfaces over and over again, but on them an indefinite series of corresponding points; viz., we may write

$$\ldots \Theta\ ,\ \Delta\ ,\ \Phi\ ,\quad \Theta,\ \Delta,\ \Phi,\quad \Theta,\ \Delta,\ \Phi \ldots$$
$$\ldots P_1,\ Q_1,\ R_1,\quad P,\ Q,\ R,\quad P',\ Q',\ R' \ldots$$

viz., a point Q on Δ corresponds to a point P on Θ and to a point R on Φ; R corresponds to Q on Δ and to a new point P' on Θ; P' to R on Φ and to a new point Q' on Δ, and so on. And in the opposite direction P corresponds to Q on Δ, and to a new point R_1 on Φ; R_1 to P on Θ and to a new point Q_1 on Δ; and so on. And of course the correspondence of any two points of the series, whether belonging to the same surface or to different surfaces, is a one-to-one correspondence.

The Symmetrical Case; Symmetroid and Jacobian.

70. I have established the foregoing general theory; but it is only a particular case of it which connects itself with the theory of nodal quartics; viz., the cube of coefficients is a symmetrically arranged cube

$$\begin{array}{lllll} a & h & g & l & \\ h & b & f & m & \\ g & f & c & n & \\ l & m & n & d & \end{array}$$

$$\begin{array}{lll} a_1 & h_1 & \dots \\ h_1 & b_1 & \\ \vdots & & \ddots \end{array}$$

or say its upper face is the symmetrical square matrix

$$\left|\begin{array}{llll} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{array}\right|$$

and the other horizontal planes, the like squares with the several terms affected by suffixes.

The surface $\nabla = 0$ is here a surface of the form

$$\nabla = \left|\begin{array}{llll} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & P \end{array}\right| = 0$$

$\{A, B,$ &c. linear functions of $(\alpha, \beta, \gamma, \delta)\}$ viz., ∇ is a symmetrical determinant; I call this a symmetroid; the surfaces $\nabla = 0$, $\Phi = 0$ are one and the same surface, the Jacobian of 4 quadric surfaces; moreover the points P and R are one and the same point, and the correspondence R to P' is a reciprocal one; so that, instead of the indefinite series of points, we have only 2 points Q, Q' on the surface ∇, and 2 points P, P' on the surface $\Theta\ (= \Phi)$; viz., the diagram is

$$\dots \Delta,\ \Theta,\ \Theta,\ \Delta,\ \Theta,\ \Theta,\ \Delta \dots$$

$$\dots Q',\ P',\ P,\ Q,\ P,\ P',\ Q' \dots$$

moreover the symmetroid surface $\nabla = 0$ is a surface with 10 nodes, which is clearly not octadic, and which is therefore the decadianome.

71. Consider the quadric surfaces

$$\begin{aligned} S &= (a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0, \\ T &= (a_1, \ldots \qquad \between \quad ,, \quad)^2 = 0, \\ U &= (a_2, \ldots \qquad \between \quad ,, \quad)^2 = 0, \\ V &= (a_3, \ldots \qquad \between \quad ,, \quad)^2 = 0, \end{aligned}$$

and a point $(\alpha, \beta, \gamma, \delta)$ in the same or in a different space, such that the surface $\alpha S + \beta T + \gamma U + \delta V = 0$ is a cone, or say for shortness,

$$\alpha S + \beta T + \gamma U + \delta V = \text{cone};$$

$(\alpha, \beta, \gamma, \delta)$ is said to be the determining point, or determinator of the cone. And if we establish the equations

$$\begin{aligned} \delta_x\,(\alpha S + \beta T + \gamma U + \delta V) &= 0, \\ \delta_y\,(\qquad ,, \qquad) &= 0, \\ \delta_z\,(\qquad ,, \qquad) &= 0, \\ \delta_w(\qquad ,, \qquad) &= 0, \end{aligned}$$

which express that the surface is a cone, then the point (x, y, z, w) is the vertex of the cone. We have thus 4 equations lineolinear in (x, y, z, w) and also in $(\alpha, \beta, \gamma, \delta)$, so that the relation between the 2 points is of the nature of that above considered. The relation between (x, y, z, w) is given by the equation

$$J\,(S, T, U, V) = 0;$$

viz., the locus is the Jacobian of the 4 quadric surfaces. The relation between $(\alpha, \beta, \gamma, \delta)$ is given by the equation

$$\nabla = \begin{vmatrix} a\alpha + a_1\beta + a_2\gamma + a_3\delta, & h\alpha + \ldots, & g\alpha + \ldots, & l\alpha + \ldots \\ h\alpha + \ldots, & b\alpha + \ldots, & f\alpha + \ldots, & m\alpha + \ldots \\ g\alpha + \ldots, & f\alpha + \ldots, & c\alpha + \ldots, & n\alpha + \ldots \\ l\alpha + \ldots, & m\alpha + \ldots, & n\alpha + \ldots, & d\alpha + \ldots \end{vmatrix} = 0,$$

so that the locus is (by the foregoing definition) the symmetroid. And the determinator point on the symmetroid thus corresponds to the cone-vertex on the Jacobian.

72. But the Jacobian may be obtained in a different manner; viz., if we establish the equations

$$\begin{aligned} (\xi\delta_x + \eta\delta_y + \zeta\delta_z + \omega\delta_w)\,S &= 0, \\ (\qquad ,, \qquad)\,T &= 0, \\ (\qquad ,, \qquad)\,U &= 0, \\ (\qquad ,, \qquad)\,V &= 0, \end{aligned}$$

then the elimination of $(\xi, \eta, \zeta, \omega)$ leads to the equation $J\,(S, T, U, V) = 0$ of the Jacobian surface. And since each of the equations is symmetrical in regard to (x, y, z, w)

and $(\xi, \eta, \zeta, \omega)$, it appears that the point $(\xi, \eta, \zeta, \omega)$ is also a point on the Jacobian surface. We have on the symmetroid a point related to $(\xi, \eta, \zeta, \omega)$ in the same way that $(\alpha, \beta, \gamma, \delta)$ on the symmetroid is related to the point (x, y, z, w); and this completes the system of the 4 points, Q on the symmetroid, P and P' on the Jacobian, Q' on the symmetroid; but in what follows I make no use of this last point Q'.

73. The points (x, y, z, w), $(\xi, \eta, \zeta, \omega)$ on the Jacobian correspond in such wise that, taking the polar planes of either of them in regard to the quadrics $S=0$, $T=0$, $U=0$, $V=0$, these intersect in a single point, viz., in the other of the two corresponding points. Or, what is the same thing, the line joining the two points cuts each of the four quadrics harmonically, whence also it cuts harmonically any quadric surface whatever of the series $\alpha S+\beta T+\gamma U+\delta V=0$, ($\alpha$, β, γ, δ being here arbitrary multipliers); viz., this property is an immediate interpretation of the equation

$$(\xi\delta_x+\eta\delta_y+\zeta\delta_z+\omega\delta_w)(\alpha S+\beta T+\gamma U+\delta V)=0,$$

or, as this is more conveniently written,

$$(\alpha, \ldots \between \xi, \eta, \zeta, \omega \between x, y, z, w)=0,$$

if for a moment $(\alpha, \ldots)$ denote the coefficients of the quadric function $\alpha S+\beta T+\gamma U+\delta V$.

74. Consider any 6 pairs of points (x_1, y_1, z_1, w_1), $(\xi_1, \eta_1, \zeta_1, \omega_1)$, &c., related as above; the quartic surfaces $S=0$, $T=0$, $U=0$, $V=0$ are surfaces cutting harmonically the lines joining the two pairs of points respectively; or say they are quadrics cutting harmonically 6 given segments; and the general quadric surface which cuts harmonically the 6 given segments is $\alpha S+\beta T+\gamma U+\delta V=0$. We thus see that the Jacobian surface $J(S, T, U, V)=0$ is in fact the locus of the vertices of the quadric cones which cut harmonically 6 given segments. The surface so defined by M. Chasles (*Comptes Rendus*, tom. LII., 1861, pp. 1157—62), and shown by him to be a quartic surface, is thus identified with the Jacobian of any 4 quartic surfaces; and included herein we have the particular case, also considered by him, of the locus of the vertices of the quadric cones which pass through 6 given points, or Jacobian of the 6 given points.

75. It is to be shown that there are 10 systems of values $(\alpha, \beta, \gamma, \delta)$, or, what is the same thing, 10 points on the symmetroid, for each of which the quartic surface $\alpha S+\beta T+\gamma U+\delta V=0$ is a plane-pair. For any such system of values the plane-pair may be regarded as a cone, having its vertex at any point whatever on the line which is the axis of the plane-pair; that is, each point of this line is the vertex of a cone of the system of surfaces $\alpha S+\beta T+\gamma U+\delta V=0$; or, what is the same thing, the axis of the plane-pair lies on the Jacobian surface; viz., there will be on the Jacobian surface 10 lines. Moreover, to the point $(\alpha, \beta, \gamma, \delta)$ on the symmetroid there corresponds indifferently any point whatever on the axis of the plane-pair. The analytical expressions for (x, y, z, w) in terms of $(\alpha, \beta, \gamma, \delta)$ must therefore, for the values in question of $(\alpha, \beta, \gamma, \delta)$, become indeterminate; and this can only happen if for the values in question the first minors of the determinant ∇ all of them vanish. But a point $(\alpha, \beta, \gamma, \delta)$, for which the minors of ∇ all of them vanish, is obviously a node on the symmetroid; and it thus appears that there are on the symmetroid 10 nodes,

each corresponding to a line on the Jacobian, and that the condition for determining these is

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair};$$

viz., the values of $(\alpha, \beta, \gamma, \delta)$, which satisfy this condition, belong to a node of the symmetroid, and the line on the Jacobian is the axis of the plane-pair.

76. Reverting to the equation $\nabla = 0$ of the symmetroid, where ∇ is a symmetrical determinant the terms of which are linear functions of the coordinates $(\alpha, \beta, \gamma, \delta)$, it has already been shown, *ante* No. 7, that this is a surface with 10 nodes; but this may be also proved as follows. Writing as before

$$\alpha S + \beta T + \gamma U + \delta V = (A, B, C, D, F, G, H, L, M, N \between x, y, z, w)^2 = 0,$$

the condition that this shall be a plane-pair implies a threefold relation between the coefficients A, B, &c., and the required number of nodes is equal to the order of this threefold relation. Establishing between the coefficients A, B, &c., any 6 linear relations whatever, we should have a ninefold relation to determine the ratios of the 10 quantities; and the number of solutions would be equal to the order of the threefold relations. But taking the 6 linear relations to be of the form $(A, \ldots \between x_1, y_1, z_1, w_1)^2 = 0$, the question is in fact to find the number of the plane-pairs which pass through 6 given points; and this is clearly $= 10$.

77. Applying the conclusion to the system of quadric surfaces $\alpha S + \beta T + \gamma U + \delta V = 0$, we see that there are in the system 10 plane-pairs; and that the lines of intersection, or axes of the plane-pairs, are lines upon the Jacobian surface.

78. The equation $\nabla = 0$ of the symmetroid seems to contain homogeneously 40 constants. But starting with any given symmetrical determinant, we may multiply it line into line by a constant determinant, and then column into column by the same constant determinant, in such wise that the resulting product is still a symmetrical determinant; and the coefficients of the constant determinant may then be used to specialise the form of the equation. The equation $\nabla = 0$ of the symmetroid thus really contains $40 - 16 = 24$ constants; this is as it should be, for the symmetroid, *quà* quartic surface with 10 nodes, contains $34 - 10 = 24$ constants.

Symmetroid with given Nodes.

79. A symmetroid can be formed with 7 given points as nodes; but there is no proper symmetroid with 8 given points as nodes. If we endeavour to form such a symmetroid, we obtain a system of 2 quadric cones, each of them passing through the 8 points; viz., these are any 2 out of the 4 quadric cones which pass through the 8 points. This will be shown in a moment; for the complete *à posteriori* identification with the decadianome, it would be necessary to show that a symmetroid could be found having for nodes 7 given points, an 8th point anywhere on the dianodal surface, and a 9th point anywhere on the dianodal curve; but this I have not succeeded in effecting.

80. We have for any node $(\alpha, \beta, \gamma, \delta)$ of the symmetroid,

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair}.$$

If, then, 4 of the given nodes are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), we must have S, T, U, V each of them a plane-pair. We may without loss of generality assume $S = x^2 + y^2$, $T = z^2 + w^2$; this, however, does not determine the signification of the coordinates (x, y, z, w), for S will remain unaltered if we write therein

$$x \cos\theta + y \sin\theta, \quad x \sin\theta - y \cos\theta \text{ for } x, y;$$

and similarly T will remain unaltered if we write therein

$$z \cos\theta_1 + w \sin\theta_1, \quad z \sin\theta_1 - w \cos\theta_1 \text{ for } z, w.$$

Hence, if we go on to assume

$$U = k\,(x + m\,y + n\,z + p\,w)(x + m'\,y + n'\,z + p'\,w),$$
$$V = k_1\,(x + m_1 y + n_1 z + p_1 w)(x + m_1' y + n_1' z + p_1' w),$$

we may imagine the θ, θ_1 so determined that, for instance,

$$m + m' = 0, \quad p_1 + p_1' = 0;$$

we have thus

$$\begin{aligned} S &= x^2 + y^2, \\ T &= \qquad z^2 + w^2, \\ U &= k\,(x + m\,y + n\,z + p\,w)(x - m\,y + n'\,z + p'w), \\ V &= k_1 (x + m_1 y + n_1 z + p_1 w)(x + m_1' y + n_1' z - p_1 w); \end{aligned}$$

formulæ which contain the 12 constants

$$(k, m, n, p, n', p', k_1, m_1, n_1, p_1, m_1', n_1').$$

This is right, for the symmetroid containing 24 constants, the symmetroid with 4 given nodes should contain $(24 - 4.3 =)$ 12 constants. And each additional given node will determine 3 constants: hence for 4 new given nodes the expressions become determinate (not of necessity uniquely so).

81. But for any 4 new nodes, the equations may be satisfied by writing therein $n = n'$, $p = -p'$, $m = -m_1'$, $n_1 = n_1'$; viz., they then assume the form

$$\begin{aligned} S &= x^2 + \qquad y^2, \\ T &= \qquad z^2 \qquad + w^2, \\ U &= (a\,x + c\,z)^2 + (b\,y + d\,w)^2, \\ V &= (a'x + c'z)^2 + (b'y + d'w)^2, \end{aligned}$$

containing 8 constants, which may be determined so that the nodes shall be the 4 given points. If now with the last mentioned values we form the value of $\alpha S + \beta T + \gamma U + \delta V$, this will consist of two terms $(*\between x, z)^2$ and $(*\between y, w)^2$, the first of which will be a square if

$$(\alpha + \gamma a^2 + \delta a'^2)(\beta + \gamma c^2 + \delta c'^2) - (\gamma ac + \delta a'c')^2 = 0, \text{ say this is } \Lambda = 0,$$

and the second will be a square if

$$(\alpha+\gamma b^2+\delta b'^2)(\beta+\gamma d^2+\delta d'^2)-(\gamma bd+\delta b'd')^2=0, \text{ say this is } \Lambda'=0;$$

so that the condition

$$\alpha S+\beta T+\gamma U+\delta V=\text{cone}$$

will be satisfied if $\Lambda=0$, or if $\Lambda'=0$; that is, the equation of the symmetroid will be $\Lambda\Lambda'=0$, or the symmetroid breaks up into the 2 quadric surfaces $\Lambda=0$, $\Lambda'=0$, each of which is a cone.

82. It is to be further observed that, considering the first mentioned 4 points (1, 0, 0, 0), &c., and any other 4 given points whatever, the equation of any one of the 4 quadric cones through these 8 points will be of the form

$$(*\between\beta\gamma,\ \gamma\alpha,\ \alpha\beta,\ \alpha\delta,\ \beta\delta,\ \gamma\delta)=0;$$

viz., any equation of this form, being a cone, will admit of being expressed, and that in one way only, in the form $\Lambda=0$. Consider then any one of the 4 cones through the 8 points, and let its equation be thus expressed; we have the values of the coefficients a, c, a', c', which enter into the expressions of S, T, U, V; and similarly, considering any other of the 4 cones, and expressing its equation in the like form, we have the values of the coefficients b, d, b', d' which enter into the expressions of S, T, U, V.

83. If instead of taking 2 different cones through the 8 points, we take in each case the same cone, the expressions for S, T, U, V would be

$$\begin{aligned}
S &= x^2 \quad + y^2,\\
T &= \quad z^2 + \quad w^2,\\
U &= (ax+cz)^2+(ay+cw)\\
V &= (a'x+c'z)^2+(a'y+c'w)^2;
\end{aligned}$$

and we have identically

$$(ac'-a'c)(aa'S-cc'T)-a'c'U+acV=0.$$

This solution may be disregarded.

84. Instead of the assumption $S=x^2+y^2$, $T=z^2+w^2$, we may take $x=0$, $y=0$, $z=0$, $w=0$ to be planes of the plane-pairs S, T, U, V respectively; it is then easy to fix the remaining constants so that the 5th and 6th nodes of the symmetroid shall be given points. Suppose that the coordinates of the 5th node are (1, 1, 1, 1); to obtain the result in the most simple manner, I take for the moment Ω an arbitrary quadric function $(x, y, z, w)^2$, and I write

$$\begin{aligned}
S &= x\,(\delta_x\Omega \quad\quad + hy-gz+aw),\\
T &= y\,(\delta_y\Omega - hx \quad\quad + fz+bw),\\
U &= z\,(\delta_z\Omega + gx-fy \quad\quad + cw),\\
V &= w\,(\delta_w\Omega - ax-by-cz \quad\quad),
\end{aligned}$$

where the coefficients are arbitrary. We have identically $S+T+U+V=2\Omega$; wherefore the given point (1, 1, 1, 1) will be a node of the symmetroid if only $\Omega=0$ be a plane-pair; and it is easy to see that we may without loss of generality take one factor to be $x+y+z+w$, and write

$$\Omega=(x+y+z+w)(lx+my+nz+pw);$$

viz., Ω having this value, the symmetroid, $\alpha S+\beta T+\gamma U+\delta V=$ cone, will have the 5 given nodes; the equation contains, as it should do, 9 constants.

85. In order that the symmetroid may have a 6th given node $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, I observe that the constants may be determined so that $\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V$ shall be equal to an arbitrary quadric function, say

$$\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V=(\mathrm{a, b, c, d, f, g, h, l, m, n}\between x, y, z, w)^2;$$

this in fact gives

$$(\mathrm{l, m, n, p})=\left(\frac{\mathrm{a}}{\alpha_1}, \frac{\mathrm{b}}{\beta_1}, \frac{\mathrm{c}}{\gamma_1}, \frac{\mathrm{d}}{\delta_1}\right);$$

and then, completing the comparison,

$$\begin{aligned}
S=x\Bigg\{&\left[\frac{\mathrm{a}}{\alpha_1}\right]x+\left[\frac{2\mathrm{h}}{\alpha_1-\beta_1}-\frac{\beta_1}{\alpha_1-\beta_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{b}}{\gamma_1}\right)\right]y\\
&+\left[\frac{2\mathrm{g}}{\alpha_1-\gamma_1}-\frac{\gamma_1}{\alpha_1-\gamma_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{2\mathrm{l}}{\alpha_1-\delta_1}-\frac{\delta_1}{\alpha_1-\delta_1}\left(\frac{\mathrm{a}}{\alpha_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\Bigg\},\\
T=y\Bigg\{&\left[\frac{2\mathrm{h}}{\beta_1-\alpha_1}-\frac{\alpha_1}{\beta_1-\alpha_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{\mathrm{b}}{\beta_1}\right]y\\
&+\left[\frac{2\mathrm{f}}{\beta_1-\gamma_1}-\frac{\gamma_1}{\beta_1-\gamma_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{2\mathrm{m}}{\beta_1-\delta_1}-\frac{\delta_1}{\beta_1-\delta_1}\left(\frac{\mathrm{b}}{\beta_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\Bigg\},\\
U=z\Bigg\{&\left[\frac{2\mathrm{g}}{\gamma_1-\alpha_1}-\frac{\alpha_1}{\gamma_1-\alpha_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{2\mathrm{f}}{\gamma_1-\beta_1}-\frac{\beta_1}{\gamma_1-\beta_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{b}}{\beta_1}\right)\right]y\\
&+\left[\frac{\mathrm{c}}{\gamma_1}\right]z+\left[\frac{2\mathrm{n}}{\gamma_1-\delta_1}-\frac{\delta_1}{\gamma_1-\delta_1}\left(\frac{\mathrm{c}}{\gamma_1}+\frac{\mathrm{d}}{\delta_1}\right)\right]w\Bigg\},\\
V=w\Bigg\{&\left[\frac{2\mathrm{l}}{\delta_1-\alpha_1}-\frac{\alpha_1}{\delta_1-\alpha_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{a}}{\alpha_1}\right)\right]x+\left[\frac{2\mathrm{m}}{\delta_1-\beta_1}-\frac{\beta_1}{\delta_1-\beta_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{b}}{\beta_1}\right)\right]y\\
&+\left[\frac{2n}{\delta_1-\gamma_1}-\frac{\gamma_1}{\delta_1-\gamma_1}\left(\frac{\mathrm{d}}{\delta_1}+\frac{\mathrm{c}}{\gamma_1}\right)\right]z+\left[\frac{d}{\delta_1}\right]w\Bigg\};
\end{aligned}$$

viz., these values give

$$S+\quad T+\quad U+\quad V=(x+y+z+w)\left(\frac{\mathrm{a}}{\alpha_1}x+\frac{\mathrm{b}}{\beta_1}y+\frac{\mathrm{c}}{\gamma_1}z+\frac{\mathrm{d}}{\delta_1}w\right),$$

$$\alpha_1 S+\beta_1 T+\gamma_1 U+\delta_1 V=(\mathrm{a, b, c, d, f, g, h, l, m, n}\between x, y, z, w);$$

hence, taking the function $(\mathrm{a}, \ldots\between x, y, z, w)^2$ to be a plane-pair equal to $(x+iy+jz+kw)(x+i_1y+j_1z+k_1w)$ suppose, or considering the coefficients (a, ...) as given functions of (i, j, k, i_1, j_1, k_1), we have the symmetroid having the 6 given nodes and containing the last mentioned 6 constants.

The Jacobian with given Lines.

86. The Jacobian contains 24 constants; obviously it is uniquely determined if 4 of the plane-pairs thereof are given; and it is also determined, but not uniquely, if 6 of the lines thereof are given. We may enquire how many given nodes of the symmetroid may be considered as corresponding to given plane-pairs, or lines of the Jacobian. Take as given any 4 nodes of the symmetroid; the corresponding 4 plane-pairs may be taken to be given plane-pairs; and we may besides take as given a 5th node of the symmetroid. For let the first 4 nodes of the symmetroid be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1); the given plane-pairs $P_1Q_1=0$, $P_2Q_2=0$, $P_3Q_3=0$, $P_4Q_4=0$; (l_1, l_2, l_3, l_4) any system of values such that we have

$$l_1P_1Q_1+l_2P_2Q_2+l_3P_3Q_3+l_4P_4Q_4=\text{plane-pair};$$

and (1, 1, 1, 1) the 5th node of the symmetroid; we have only to assume

$$(S, T, U, V)=(l_1P_1Q_1, l_2P_2Q_2, l_3P_3Q_3, l_4P_4Q_4).$$

87. Suppose, however, that on the Jacobian we have given, not the 4 plane-pairs, but only the 4 axes of the plane-pairs; the plane-pairs may be taken to be

$$(1, b_1, c_1\between P_1, Q_1)^2=0, \ldots\ldots (1, b_4, c_4\between P_4, Q_4)^2=0,$$

where the 8 constants $(b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4)$ are in the first instance undetermined. If we attempt to find l_1, l_2, l_3, l_4, so that

$$l_1(1, b_1, c_1\between P_1, Q_1)^2 \ldots\ldots + l_4(1, b_4, c_4\between P_4, Q_4)^2=\text{plane-pair of given axis},$$

we have between the coefficients (b, c) 4 equations; and similarly, if we attempt to find m_1, m_2, m_3, m_4 such that

$$m_1(1, b_1, c_1\between P_1, Q_1)^2 \ldots\ldots + m_4(1, b_4, c_4\between P_4, Q_4)^2=\text{plane-pair of another given axis},$$

we have 4 more equations between the coefficients (b, c); viz., these will be determined by the 8 equations (this is in fact the before mentioned property that 6 lines of the Jacobian may be taken to be given lines). But considering only the first system of equations; in order that to the given axis may correspond a given node on the symmetroid, say the node (1, 1, 1, 1), we have only to write

$$S=l_1(1, b_1, c_1\between P_1, Q_1)^2, \ldots\ldots V=l_4(1, b_4, c_4\between P_4, Q_4)^2;$$

that is, we may take as given 5 nodes of the symmetroid, and the corresponding 5 lines of the Jacobian; the formulæ will contain 4 constants; we may by means of them make the Jacobian have a 6th given line, thus determining the constants; or we may make the symmetroid have a 6th given node, leaving in this case one constant arbitrary.

Correspondence on the Jacobian: Lines and Skew Cubics.

88. I consider the correspondence of two points on the Jacobian; it is to be shown that when one of the points is on a line of the Jacobian, the corresponding point will be on a skew cubic; that is, that corresponding to each line of the Jacobian we have (on the Jacobian) a skew cubic. Call the plane-pairs of the system of quadric surfaces 1, 2, 3, ... 10; selecting any 4 of these, say 1, 2, 3, 4, the polar planes of any point of the Jacobian in regard to these 4 plane-pairs will meet in a point which will be the required corresponding point. And observe that, in regard to any one of the plane-pairs, say 1, the polar plane of a point P is the plane through the axis harmonic to the plane through the axis and the point P. Hence, for a point on the axis of 1, the polar plane in regard to 1 is indeterminate; the polar planes in regard to the plane-pairs 2, 3, 4 respectively meet in a point which is the required corresponding point. We may for any point whatever take the polar planes in regard to the plane-pairs 2, 3, 4 respectively, and call the intersection of these planes the corresponding point; this being so, if the first mentioned point moves along a line, the corresponding point moves along a curve, which is easily shown to be a skew cubic cutting the axis of each plane-pair twice; that is, in regard to the plane-pairs 2, 3, 4, the locus corresponding to any line whatever is a skew cubic cutting the axis of each plane-pair twice. In particular, the corresponding curve of the axis of 1, is a skew cubic cutting the axis of the plane-pairs 2, 3, 4 each twice; but the axis of 1 does not stand in any special relation to the plane-pairs 2, 3, 4, as distinguished from the remaining plane-pairs 5, 6...10; we have therefore the more complete theorem, that the skew cubic cuts the axes of the plane-pairs 2, 3,...10 each twice; or, instead of the plane-pairs, speaking of the line 1, 2, 3,...10, we may say that corresponding to any one of the lines we have a skew cubic meeting the other 9 lines each of them twice.

89. I stop for a moment to prove the subsidiary theorem assumed in the foregoing demonstration. Let the 3 plane-pairs be $PQ=0$, $RS=0$, $TU=0$, and let the line be that joining the points (x_0, y_0, z_0, w_0) and (x_1, y_1, z_1, w_1); the coordinates of any point in the line may be taken to be $\lambda x_0+\mu x_1$, $\lambda y_0+\mu y_1$, $\lambda z_0+\mu z_1$, $\lambda w_0+\mu w_1$; and hence for the polar plane in regard to the plane-pair $PQ=0$ we have

$$\{(\lambda x_0+\mu x_1)\,\delta_x \ldots + (\lambda w_0+\mu w_1)\,\delta_w\}\,PQ=0\,;$$

viz., this equation may be written

$$\lambda\,(PQ_0+P_0Q)+\mu\,(PQ_1+P_1Q)=0\,;$$

forming the like equations in regard to the other 2 plane-pairs respectively, and eliminating λ, μ, we obtain for the required locus

$$\left\| \begin{matrix} PQ_0+P_0Q, & RS_0+R_0S, & TU_0+T_0U \\ PQ_1+P_1Q, & RS_1+R_1S, & TU_1+T_1U \end{matrix} \right\| = 0,$$

a skew cubic; and on writing herein $P=0$, $Q=0$, the equations become

$$\begin{vmatrix} RS_0+R_0S, & TU_0+T_0U \\ RS_1+R_1S, & TU_1+T_1U \end{vmatrix} = 0;$$

viz., the line $(P=0,\ Q=0)$ meets the skew cubic in the points where the line meets the quadric surface determined by this last equation, that is in 2 points.

90. We have thus on the Jacobian the 10 lines 1, 2, ... 9, 10, and corresponding thereto respectively the 10 skew cubics 1′, 2′, ... 9′, 10′, where each line meets twice each of the skew cubics except that denoted by the same number; a relation similar to that which exists between the lines 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′, which compose a double-sixer on a cubic surface.

Suppose that there are given on the Jacobian the lines 1, 2, 3, 4, 5, 6; meeting each of these twice, we have the skew cubics 7′, 8′, 9′, 10′; and then

$$\text{the lines } \begin{matrix} 7 \\ 8 \\ 9 \\ 10 \end{matrix} \text{ meet twice each of the cubics } \begin{matrix} 8', & 9', & 10' \\ 9', & 10', & 7' \\ 10', & 7', & 8' \\ 7', & 8', & 9' \end{matrix}$$

so that the determination of the remaining 4 lines depends upon that of the skew cubics 7′, 8′, 9′, 10′, which meet each of the given lines twice.

91. To determine a skew cubic cutting twice each of 6 given lines, I proceed as follows. Let the lines be 1, 2, 3, 4, 5, 6; take $U=0$ the general quadric surface through the lines 1 and 2, $V=0$ the general quadric surface through the lines 1, 3 (the equations contain each of them homogeneously 4 constants). The 2 surfaces intersect in the line 1, and in a skew cubic cutting twice each of the lines 1, 2, 3; we have therefore to determine the constants so that the 2 surfaces may meet the line 4 in the same 2 points, the line 5 in the same 2 points, the line 6 in the same two points. Imagine for a moment the equations of any one of the lines 4, 5, 6 to be $z=0$, $w=0$; the equations of the 2 surfaces, substituting therein these values, would assume the forms

$$(a,\ b,\ c \between x,\ y)^2=0,\quad (a',\ b',\ c' \between x,\ y)^2=0;$$

and the conditions for the intersection in the same 2 points would be $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}, =p$ suppose. This is in fact the form of the conditions, understanding a, b, c to be linear functions of the coefficients of U, and a', b', c' to be linear functions of the coefficients of V. We have in this manner 3 sets of equations involving respectively the indeterminate quantities p, q, r; viz., these may be represented by

$$a=pa',\ b=pb',\ c=pc';\quad d=qd',\ e=qe',\ f=qf';\quad g=rg',\ h=rh',\ i=ri';$$

where the unaccented letters $a, b, \ldots i$ are linear functions of the coefficients of U, and the accented letters $a', b', \ldots i'$ linear functions of the coefficients of V. Eliminating

the coefficients of U, V, we have between p, q, r a twofold relation, which may be represented as follows:

$$\left\|\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \end{matrix}\right\| = 0,$$

it being understood that the 1's represent constants, and the p's, q's, and r's linear functions of these variables respectively. The several equations of the system, regarding therein p, q, r as coordinates, represent each of them a quartic curve; any 2 of these intersect in 16 points; but the number of points common to all the curves is $=10$. But each of the curves passes through the 3 points (1, 0, 0), (0, 1, 0), (0, 0, 1); these are consequently included among the 10 points, but they do not give a proper solution of the question; and the number of solutions is thus reduced to $10-3=7$. There is yet another solution to be rejected; viz., $U=0$ being a quadric surface through the lines 1, 2, and $V=0$ the quadric surface through the lines 1, 3, it is possible to determine the coefficients of U, V so that each of these surfaces shall be the quadric surface through the lines 1, 2, 3; and if we then have identically $U=\theta V$, it is clear that corresponding values of p, q, r are $p=q=r\,(=\theta)$. We have thus the point $p=q=r$ common to all the curves of the system; this solution counts, I believe, once only, and the number of relevant solutions is $7-1=6$.

92. It may be observed, in regard to the foregoing solution, that if we take $123=0$ as the equation of the quadric surface through the lines 1, 2, 3, and so in other cases, then the equation of the surfaces $U=0$ and $V=0$ may be taken to be

$$\lambda\,.\,123+\mu\,.\,124+\nu\,.\,125+\rho\,.\,126=0,$$
$$\lambda'\,.\,132+\mu'\,.\,134+\nu'\,.\,135+\rho'\,.\,136=0,$$

respectively, the coefficients of the two surfaces being here put in evidence. And it is clear that for $\mu=\nu=\rho=0$, $\mu'=\nu'=\rho'=0$, the surfaces become each of them the surface through the lines 1, 2, 3.

93. The conclusion is, that touching twice each of the six lines 1, 2, 3, 4, 5, 6, we have six skew cubics; it would appear that any four of these may be taken for the skew cubics 7′, 8′, 9′, 10′ (so that there are 15 such tetrads of cubics). I am not, however, able to verify that we then have the remaining 4 lines each cutting twice 3 of the 4 skew cubics; assuming that for each system of 4 skew cubics there is one and only one, such system of lines, then of course to the given system of lines 1, 2, 3, 4, 5, 6, there will belong 15 systems of lines 7, 8, 9, 10, and therefore also 15 Jacobian surfaces.

Further Investigations as to the Jacobian, &c.

94. Taking $(\xi, \eta, \zeta, \omega)$ as plane-coordinates, two quadric surfaces

$$(a, b, c, d, f, g, h, l, m, n \between \xi, \eta, \zeta, \omega)^2 = 0$$

and

$$(A, B, C, D, F, G, H, L, M, N \between x, y, z, w)^2 = 0$$

are said to be interverts (or interverse) one of the other, when we have between the coefficients the relation

$$(a, b, c, d, f, g, h, l, m, n \between A, B, C, D, F, G, H, L, M, N) = 0,$$

that is

$$aA + \ldots + 2fF + \ldots = 0.$$

The condition that the two surfaces may be interverts of each other is linear in regard to the coefficients of each surface separately; hence, using a before explained locution, we may say—interverse to a given quadric surface we have 9 quadrics; interverse to two given quadrics 8 quadrics; or generally, that interverse to k given quadrics we have $10-k$ quadrics. And, moreover, if the quadrics of the two systems be $L=0$, $M=0$, &c., and $S=0$, $T=0$, $U=0$, &c., then every quadric $\lambda L + \mu M + \ldots = 0$ is interverse to each of the quadrics $\alpha S + \beta T + \gamma U + \ldots = 0$.

If the quadric $(a, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0$ be an intervert of the plane-pair

$$(lx + my + nz + pw \between l'x + m'y + n'z + p'w) = 0,$$

the condition is

$$(a, \ldots \between l, m, n, p \between l', m', n', p') = 0;$$

viz., this expresses that the two planes are harmonics in regard to the pair of planes drawn through the axis of the plane-pair to touch the quadric surface; or say, that the plane-pair is harmonic in regard to the quadric.

95. To apply this to the Jacobian surface, I recall that, starting with the given quadric surfaces $S=0$, $T=0$, $U=0$, $V=0$, and taking $(\alpha, \beta, \gamma, \delta)$ to be such that

$$\alpha S + \beta T + \gamma U + \delta V = \text{plane-pair},$$

there are 10 such plane-pairs, and that the axes of these are the lines of the Jacobian. If instead of the given quadric surfaces, we consider the six interverse surfaces $(a_1, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0, \ldots (a_6, \ldots \between \xi, \eta, \zeta, \omega)^2 = 0$, then the condition is that the plane-pair shall be harmonic in regard to each of these surfaces. Let the quadric surfaces be called 1, 2, 3, 4, 5, 6; then, attending to any three of these, say 1, 2, 3, the plane-pair is harmonic in regard to these three surfaces. Through the axis of the plane-pair draw tangent planes to 1, 2, and 3 respectively; each of these pairs of planes is harmonic in regard to the planes of the plane-pair; that is, the three pairs of tangent planes are in involution; or, as we may also express it, the axis is (*quoad* its planes) in involution in regard to the three quadric surfaces. Conversely, when the axis is thus in involution in regard to the surfaces 1, 2, and 3, we may by

means of the surfaces 1 and 2 determine the two planes of the plane-pair, and then these will be harmonics in regard to the surface 3. It thus appears that the axis is given as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, to the surfaces 1, 2, 4, the surfaces 1, 2, 5, and the surfaces 1, 2, 6, respectively; or, as we may express it, as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, 4, 5, 6.

96. It is substantially the same thing, but it is rather easier, to consider the whole question under the reciprocal form; viz., instead of a plane-pair and a quadric surface represented by an equation in plane-coordinates, to take a point-pair and a quadric surface represented by an equation in point-coordinates; we have thus a line which is (*quoad* its points) in involution in regard to three given quadric surfaces, or as we may more simply express it, which cuts in involution the three given surfaces; and we thus arrive at the problem of finding a line which cuts in involution six given quadric surfaces; viz., this is equivalent to the above problem where the line has to satisfy (*quoad* its planes) the like condition; and in each problem the number of solutions should be $=10$.

97. Consider a line which cuts in involution the three given surfaces $(a_1, \ldots \between x, y, z, w)=0$, $(a_2, \ldots \between x, y, z, w)^2=0$, $(a_3, \ldots \between x, y, z, w)^2=0$. I will presently show that this implies a cubic relation $(* \between a, b, c, f, g, h)^3$ between the six coordinates of the line. But assuming it for the moment, suppose that the line cuts in involution the three surfaces and a fourth quadric surface $(a_4, \ldots \between x, y, z, w)^2=0$. Considering the line as cutting in involution the surfaces 1, 2, 4, we have between the six coordinates a second cubic relation; there is, however, a reduction, and the order of the resulting twofold relation between the coordinates is $3 \,.\, 3-4=5$. To explain this, observe that every line which cuts in the same two points the surfaces 1 and 2 respectively (that is, which cuts the curve of intersection twice) will in an improper sense cut in involution the surfaces 1, 2, 3, and also the surfaces 1, 2, 4. There is thus a reduction equal to the order in the six coordinates of the twofold relation which expresses that the line cuts twice the curve of intersection of the surfaces 1 and 2. Join hereto the relations that the line meets each of two given lines; the coordinates of the line are determined by the twofold relation (say its order is $=\lambda$) two linear equations, and the universal equation $af+bg+ch=0$; the number of solutions is $=2\lambda$. But the number of solutions is equal to that of the lines which meet the quadri-quadric curve of intersection twice, and meet also each of two given lines; or what is the same thing, it is equal to the order of the scroll generated by the lines which meet the curve twice, and also a given line. We have for the curve of intersection (m the order, h the number of apparent double points) $m=4$, $h=2$; whence order of the scroll is $2+\frac{1}{2} \,.\, 4 \,.\, 3=8$; that is, $2\lambda=8$, or $\lambda=4$, which is the required reduction.

98. If the line cut in involution 5 given quadric surfaces {say the 5th surface is $(a_5, \ldots \between x, y, z, w)^2=0$}; then we have between the 6 coordinates a threefold relation, the order of which is $3.5-$reduction. This should be $=10$, and consequently the reduction $=5$; for admitting the value to be 10, the order (in the ordinary sense) of the scroll generated by the lines which cut in involution the 5 given quadrics should be $=20$;

and conversely. But the value 20 may be verified without difficulty. For the question may be transformed as follows:—If a point-pair be harmonic in regard to each of 5 given quadrics, how many of the axes (or lines through the 2 points of a point-pair) cut a given line. Take (x, y, z, w), (x', y', z', w') as the coordinates of the 2 points of a point-pair; the harmonic condition in regard to a quadric surface $U=0$ is $x'\delta_x U + y'\delta_y U + z'\delta_z U + w'\delta_w U = 0$ {where U is regarded as a function of the (x, y, z, w) belonging to a point of the point-pair}; the condition for the intersection with a given line is a lineolinear equation in the coordinates (x, y, z, w) and (x', y', z', w'), or say it is $Lx' + My' + Nz' + Pw' = 0$, where L, M, N, P are linear functions of the coordinates; we have thence for (x, y, z, w) the threefold relation

$$\left\| \begin{array}{lllllll} L, & \delta_x U_1, & \delta_x U_2, & \delta_x U_3, & \delta_x U_4, & \delta_x U_5 \\ M, & \delta_y U_1 & . & . & & \\ N, & \delta_z U_1 & . & & & \\ P, & \delta_w U_1 & & & & \end{array} \right\| = 0,$$

which denotes a system of $\frac{1}{6} . 6 . 5 . 4 = 20$ points.

It would seem that if the line cuts in involution 6 given quadrics, there should be between the 6 coordinates a fourfold relation of the order $\frac{1}{2} . 10 = 5$; this would imply a reduction 25, viz. we should have $5 = 3 . 10 - 25$. I do not understand this, and I drop the question.

99. I return to the question to find the relation between the coordinates (a, b, c, f, g, h) of a line which cuts in involution the 3 quadric surfaces

$$(a_1, b_1, c_1, d_1, f_1, g_1, h_1, l_1, m_1, n_1 \between x, y, z, w)^2 = 0, \quad (a_2, \ldots \between x, y, z, w)^2 = 0, \quad (a_3, \ldots \between x, y, z, w)^2 = 0.$$

Writing down any two of the equations of the line, for instance

$$\begin{aligned} \mathrm{h}y - \mathrm{g}z + \mathrm{a}w &= 0, \\ -\mathrm{h}x \quad + \mathrm{f}z + \mathrm{b}w &= 0, \end{aligned}$$

if we substitute the values of (x, y) in the equation of the first surface, it becomes

$$(a_1, \ldots \between \mathrm{f}z + \mathrm{b}w,\ \mathrm{g}z - \mathrm{a}w,\ \mathrm{h}z,\ \mathrm{h}w)^2 = 0;$$

or if we write for shortness

$$\Pi = (\mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{o}), \quad \Pi' = (\mathrm{b}, -\mathrm{a}, \mathrm{o}, \mathrm{h}),$$

then the equation is

$$(a_1, \ldots \between \Pi)^2 . z^2 + 2(a_1, \ldots \between \Pi \between \Pi') . zw + (a_1, \ldots \between \Pi')^2 . w^2 = 0,$$

and forming the like equations for the other two surfaces, the condition of involution is at once found to be

$$\left| \begin{array}{lll} (a_1, \ldots \between \Pi)^2, & (a_1, \ldots \between \Pi \between \Pi'), & (a_1, \ldots \between \Pi')^2 \\ (a_2, \ldots \between \Pi)^2, & (a_2, \ldots \between \Pi \between \Pi'), & (a_2, \ldots \between \Pi')^2 \\ (a_3, \ldots \between \Pi)^2, & (a_3, \ldots \between \Pi \between \Pi'), & (a_3, \ldots \between \Pi')^2 \end{array} \right| = 0.$$

100. It is convenient, in working this out, to consider Π, Π' as standing, in the first instance, for (x, y, z, w), (x', y', z', w'), these symbols being ultimately replaced by the above-mentioned values. Writing also, for shortness, (abc) to denote the determinant $a_1(b_2c_3 - b_3c_2) +$ &c., and so in other cases, it is at once seen that the function on the right-hand side is a sum of such determinants each into a proper factor, containing the coordinates (a, b, c, f, g, h), originally of the order 6, but where each term contains the factor h^3, which may be omitted; or finally the result is of the order 3 in the coordinates. Thus we have a term

$$(abc)\begin{vmatrix} x^2, & xx', & x'^2 \\ y^2, & yy', & y'^2 \\ z^2, & zz', & z'^2 \end{vmatrix}$$

where the second factor is

$$\begin{aligned} x^2y'z'\,(yz' - y'z) + y^2z'x'\,(zx' - z'x) + z^2x'y'\,(xy' - x'y), \quad &= z^2x'y'.(xy' - x'y), \\ &= h^2(-\mathrm{ab})(-\mathrm{af} - \mathrm{bg}), \quad = -\mathrm{abch}^3, \end{aligned}$$

or, omitting the factor $-h^3$, the term is $(abc)\,\mathrm{abc}$.

101. There are in all 120 terms, but 16 of these are found to vanish (viz., these are the terms in *agh, bhf, cfg*; *ahl, bfm, cgn*; *agl, bhm, cfn*; *dmn, dnl, dlm*; *fgn, ghl, hfm*). The final result contains therefore 104 terms; viz., as a further abbreviation writing *abc* &c., instead of (abc) &c., to denote the above-mentioned determinants, the equation is

$$\begin{aligned}
& abc\ .\,\mathrm{abc} - bcd\,.\,\mathrm{agh} - cad\,.\,\mathrm{bhf} - abd\,.\,\mathrm{cfg} \\
& + bcf\ .\,\mathrm{a}^3 + cag\,.\,\mathrm{b}^3 + abh\,.\,\mathrm{c}^3 + adl\,.\,\mathrm{f}^3 + bdm\,.\,\mathrm{g}^3 + cdn\,.\,\mathrm{h}^3 \\
& + abn\ .\,\mathrm{c}(\mathrm{bg} - \mathrm{af}) + adf\,.\,\mathrm{f}(\mathrm{ch} - \mathrm{bg}) \\
& + bcl\ .\,\mathrm{a}(\mathrm{ch} - \mathrm{bg}) + bdg\,.\,\mathrm{g}(\mathrm{af} - \mathrm{ch}) \\
& + cam\,.\,\mathrm{b}(\mathrm{af} - \mathrm{ch}) + cdh\,.\,\mathrm{h}(\mathrm{bg} - \mathrm{af}) \\
& - bcg\ .\,\mathrm{a}^2\mathrm{b} - bch\,.\,\mathrm{a}^2\mathrm{c} + bcm\,.\,\mathrm{a}^2\mathrm{g} - bcn\,.\,\mathrm{a}^2\mathrm{h} \\
& - cah\ .\,\mathrm{b}^2\mathrm{c} - caf\,.\,\mathrm{b}^2\mathrm{a} + can\,.\,\mathrm{b}^2\mathrm{h} - cal\,.\,\mathrm{b}^2\mathrm{f} \\
& - abf\ .\,\mathrm{c}^2\mathrm{a} - abg\,.\,\mathrm{c}^2\mathrm{b} + abl\,.\,\mathrm{c}^2\mathrm{f} - abm\,.\,\mathrm{c}^2\mathrm{g} \\
& - adg\,.\,\mathrm{bf}^2 + adh\,.\,\mathrm{cf}^2 + adm\,.\,\mathrm{f}^2g + adn\,.\,\mathrm{f}^2\mathrm{h} \\
& - bdh\,.\,\mathrm{cg}^2 + bdf\,.\,\mathrm{ag}^2 + bdn\,.\,\mathrm{g}^2\mathrm{h} + bdl\,.\,\mathrm{g}^2\mathrm{f} \\
& - cdf\,.\,\mathrm{ah}^2 + cdg\,.\,\mathrm{bh}^2 + cdl\,.\,\mathrm{h}^2\mathrm{f} + cdm\,.\,\mathrm{h}^2\mathrm{g} \\
& + 2\left\{\begin{array}{l} afg\,.\,\mathrm{b}^2\mathrm{c} - afh\,.\,\mathrm{bc}^2 + afl\,.\,\mathrm{bcf} - afm\,.\,\mathrm{c}^2\mathrm{h} - afn\,.\,\mathrm{b}^2\mathrm{g} \\ + bgh\,.\,\mathrm{c}^2\mathrm{a} - bgf\,.\,\mathrm{ca}^2 + bgm\,.\,\mathrm{cag} - bgl\,.\,\mathrm{a}^2\mathrm{f} - bgl\,.\,\mathrm{c}^2\mathrm{h} \\ + chf\,.\,\mathrm{a}^2\mathrm{b} - chg\,.\,\mathrm{ab}^2 + chn\,.\,\mathrm{abh} - chm\,.\,\mathrm{b}^2\mathrm{g} - chm\,.\,\mathrm{a}^2\mathrm{f} \end{array}\right\}
\end{aligned}$$

$$+2\left\{\begin{array}{l} agm\,.\,\mathrm{bcf}-agn\,.\,\mathrm{b^2f}-ahm\,.\,\mathrm{c^2f}+ahn\,.\,\mathrm{bcf} \\ +bhn\,.\,\mathrm{cag}-bfl\,.\,\mathrm{c^2g}-bfn\,.\,\mathrm{a^2g}+bfl\,.\,\mathrm{cag} \\ +cfl\,.\,\mathrm{abh}-chm\,.\,\mathrm{a^2h}-cgl\,.\,\mathrm{b^2h}+cgm\,.\,\mathrm{abh} \end{array}\right\}$$

$$+2\left\{\begin{array}{l} -amn\,.\,\mathrm{af^2}-anl\,.\,\mathrm{bf^2}-alm\,.\,\mathrm{cf^2}+dfg\,.\,\mathrm{ch^2} \\ -bnl\,.\,\mathrm{bg^2}-blm\,.\,\mathrm{cg^2}-bmn\,.\,\mathrm{ag^2}+dgh\,.\,\mathrm{af^2} \\ -clm\,.\,\mathrm{ch^2}-cmn\,.\,\mathrm{ah^2}-cnl\,.\,\mathrm{bh^2}+dhf\,.\,\mathrm{bg^2} \end{array}\right\}$$

$$+2\left\{\begin{array}{l} -dfl\,.\,\mathrm{fgh}-dfm\,.\,\mathrm{g^2h}-dfn\,.\,\mathrm{gh^2} \\ -dgm\,.\,\mathrm{fgh}-dgn\,.\,\mathrm{h^2f}-dgl\,.\,\mathrm{hf^2} \\ -dhn\,.\,\mathrm{fgh}-dhl\,.\,\mathrm{f^2g}-dhm\,.\,\mathrm{fg^2} \end{array}\right\}$$

$$+4\left\{\begin{array}{l} fgh\,.\,\mathrm{bch}-fgm\,.\,\mathrm{ach}-fmn\,.\,\mathrm{agh}-fnl\,.\,\mathrm{bgh}-flm\,.\,\mathrm{cgh} \\ +ghm\,.\,\mathrm{caf}-ghn\,.\,\mathrm{baf}-gnl\,.\,\mathrm{bhf}-glm\,.\,\mathrm{chf}-gmn\,.\,\mathrm{ahf} \\ +hfn\,.\,\mathrm{abg}-hfl\,.\,\mathrm{cbg}-hlm\,.\,\mathrm{cfg}-hmn\,.\,\mathrm{afg}-hnl\,.\,\mathrm{bfg} \end{array}\right\}$$

$$-4fgh\,.\,\mathrm{abc} \qquad\qquad =0.$$

And observe, by what precedes, this triple system of lines contains each of the following double systems: viz., the lines which meet the quadriquadric curve (2, 3) twice, those which meet the curve (3, 1) twice, those which meet the curve (1, 2) twice.

Persymmetrical Case: the Hessian of a Cubic.

102. Reverting to the general equation

$$\alpha S+\beta T+\gamma U+\delta V=\text{cone},$$

which connects the symmetroid and Jacobian, it is evident that if S, T, U, V are the derivatives, in regard to the coordinates, of a single cubic function $U, =(*\between x, y, z, w)^3$, then the symmetroid and the Jacobian become one and the same surface; viz., this is the Hessian surface $H=0$ derived from the given cubic surface. The two corresponding points on the symmetroid and the Jacobian respectively, and the two corresponding points on the Jacobian, become one and the same pair of corresponding points on the Hessian; viz., either of these points is such that its first polar surface in regard to the cubic is a quadric cone having for its vertex the other corresponding point. And the Hessian surface unites the properties of the Jacobian and the symmetroid, viz., it has 10 nodes and 10 lines. It is, in fact, known that there are five planes such that the intersection of every two of them is a line on the Hessian surface, and the intersection of every three of them a node on the surface; viz., if the equations of the five planes are $x=0$, $y=0$, $z=0$, $w=0$, $u=0$, then the equation of the Hessian surface is

$$xyzwu\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}+\frac{e}{u}\right)=0,$$

a form which puts in evidence the properties just referred to.

Quartics with 11 *or more Nodes.*

103. I mention two results which, although they relate to quadric surfaces with more than 10 nodes, present themselves in such immediate connexion with the present Memoir, that it is natural to speak of them. If, in the equation

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

of the symmetroid (A, B, ... linear functions of the coordinates), we have identically $A = 0$, then the surface has evidently a node $H = 0$, $G = 0$, $L = 0$; viz., this is a node in addition to the usual 10 nodes, or the surface has in all 11 nodes. And so also if (identically in every case) B is $= 0$, there are 12 nodes; if C is $= 0$, there are 13 nodes; and if D is $= 0$, there are 14 nodes. These are, in fact, quartic surfaces with 11, 12, 13, and 14 nodes respectively, mentioned in Kummer's Memoir.

104. We may consider the symmetroid derived from the quadric surfaces which pass through 6 given points; viz., taking as before (see No. 25) the coordinates of the 6 points to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), (α, β, γ, δ), and (a, b, c, f, g, h) as the coordinates of the line joining the last-mentioned two points; and, to avoid confusion, taking for the present purpose (X, Y, Z, W) instead of (α, β, γ, δ) for the coordinates of a point on the symmetroid, the equation is obtained by arranging in the form of a determinant the coefficients of the quadric form

$$\begin{aligned} & Xx\,(\qquad\quad hy - \quad gz + aw) \\ & + Yy\,(-hx \qquad\quad + \quad fz + bw) \\ & + Zz\,(\quad gx - \quad fy \qquad\quad + cw) \\ & + W\,(a\alpha\, yz + b\beta\, zx + c\gamma\, xy \qquad); \end{aligned}$$

viz., the equation in question is

$$\begin{vmatrix} \cdot & , h(X-Y)+c\gamma\, W, & g(Z-X)+b\beta\, W, & aX \\ h(X-Y)+c\gamma\, W, & \cdot & , f(Y-Z)+a\alpha\, W, & bY \\ g(Z-X)+b\beta\, W, & f(Y-Z)+a\alpha\, W, & \cdot & , cZ \\ aX \quad , & bY \quad , & cZ \quad , & \cdot \end{vmatrix} = 0;$$

or, as it may be more simply written,

$$\sqrt{aX\,\{f(Y-Z)+a\alpha W\}} + \sqrt{bY\,\{g(Z-X)+b\beta W\}} + \sqrt{cZ\,\{h(X-Y)+c\gamma W\}} = 0.$$

This is, in fact, a surface with 16 nodes. It would appear that additional nodes correspond to the six common intersections of the quadric surfaces, or nodes of the Jacobian; and it would seem that for four quadric surfaces having in common 1, 2, 3, 4, 5, or 6 points, the corresponding symmetroid would have 11, 12, 13, 14, 15, or 16 nodes. But I reserve this for future consideration.

I take the opportunity of mentioning some results which have a connexion, although not an immediate one, with the subject of the present Memoir.

Quadric Surface through three given Lines.

105. To find the equation to the quadric surface through the three lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$. Take on one of the lines the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$; then the equation of a quadric surface through this line will be of the form

$$\begin{vmatrix} x^2 & y^2 & z^2 & w^2 & yz & zx & xy & xw & yw & zw \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \beta\gamma & \gamma\alpha & \alpha\beta & \alpha\delta & \beta\delta & \gamma\delta \\ 2\alpha\alpha' & 2\beta\beta' & 2\gamma\gamma' & 2\delta\delta' & \beta\gamma'+\beta'\gamma & \gamma\alpha'+\gamma'\alpha & \alpha\beta'+\alpha'\beta & \alpha\delta'+\alpha'\delta & \beta\delta'+\beta'\delta & \gamma\delta'+\gamma'\delta \\ \alpha'^2 & \beta'^2 & \gamma'^2 & \delta'^2 & \beta'\gamma' & \gamma'\alpha' & \alpha'\beta' & \alpha'\delta' & \beta'\delta' & \gamma'\delta' \\ \vdots & & & & & & & & & \end{vmatrix} = 0;$$

and if we form thus a determinant with three of its lines relating to the line 1, three of them to the line 2, and three to the line 3, we have the equation of the quadric surface through the three lines. But considering in the determinant the three lines which refer to the line 1, it is clear that the determinant is a function of the order 3 of the coordinates $(a_1, b_1, c_1, f_1, g_1, h_1)$ of the line in question; and the like as regards the other two lines respectively. Now observe that if two of the lines intersect, the problem becomes indeterminate (in fact, the plane of the intersecting lines, and any plane whatever through the third line, constitute a solution); the condition for the intersection of the lines 1 and 2 is $a_1f_2+a_2f_1+b_1g_2+b_2g_1+c_2h_1+c_1h_2=0$; hence, if this condition be satisfied, the determinant must vanish; it therefore divides by the factor a_1f_2+ &c.; but, similarly, it divides by the factors a_2f_3+ &c. and a_3f_1+ &c.; and throwing out the three factors, the result should be of the order 1, that is linear, in regard to the three sets of coordinates respectively. I have obtained this reduced result in my "Memoir on the Six Coordinates of a Line" (*Camb. Phil. Trans.*, t. XI., 1869, p. 311 [435]); viz., writing (abc) to denote the determinant $a_1(b_2c_3-b_3c_2)+$ &c., and so for the other like determinants, the result is

$$\begin{aligned} & (agh)\,x^2+(bhf)\,y^2 + (cfg)\,z^2+(abc)\,w^2 \\ + & [(abg) - (cah)]\,xw+[(bfg) + (chf)]\,yz \\ + & [(bch) - (abf)]\,yw+[(cgh) + (afg)]\,zx \\ + & [(caf) - (bcg)]\,zw+[(ahf) + (bgh)]\,xy=0. \end{aligned}$$

Condition that five given lines may lie in a Cubic Surface.

106. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1), \ldots (a_5, b_5, c_5, f_5, g_5, h_5)$, and $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ the coordinates of any two points on one of the lines, the equation of a cubic surface through this line would be

$$\left|\begin{array}{llll} \alpha^3, \ \ldots\ldots & \alpha^2 y, & \ldots\ldots\ xyz, & \ldots\ldots \\ \alpha^2, & \alpha^2\beta, & \alpha\beta\gamma, & \\ 3\alpha^2\alpha', & 2\alpha\alpha'\beta + \alpha^2\beta', & \alpha'\beta\gamma + \alpha\beta'\gamma + \alpha\beta\gamma', & \\ 3\alpha\alpha'^2, & 2\alpha\alpha'\beta' + \alpha'^2\beta, & \alpha\beta'\gamma' + \alpha'\beta\gamma' + \alpha\beta'\gamma', & \\ \alpha'^3, & \alpha'^2\beta', & \alpha'\beta'\gamma', & \end{array}\right| = 0;$$

and hence it at once appears that, forming a determinant of 20 lines, wherein four lines relate to the line 1, four to the line 2, ..., four to the line 5, and equating this to zero, we have the required condition. But the condition so obtained is of the order $(\frac{1}{2}4.3=)6$ in regard to the coordinates of each line; and, as for the quadric, it is satisfied identically if we have any such equation as $a_1f_2 + \&c. = 0$; it consequently contains the several factors $a_1f_2 + \&c.$, which can be formed with the coordinates of any two of the five lines; and throwing out these factors, the condition should be of the order 2 in regard to the coordinates of each line. We in fact know that the required relation between the five lines is that they shall all of them be cut by a sixth line; and moreover that, writing $a_1f_2 + a_2f_1 + b_1g_2 + b_2g_1 + c_1h_2 + c_2h_1 = 12$, &c., then that the condition for this is

$$\left|\begin{array}{lllll} .\,, & 12, & 13, & 14, & 15 \\ 21, & .\,, & 23, & 24, & 25 \\ 31, & 32, & .\,, & 34, & 35 \\ 41, & 42, & 43, & .\,, & 45 \\ 51, & 52, & 53, & 54, & . \end{array}\right| = 0,$$

being, as it should be, of the order 2 in regard to the coordinates of each line.

Condition that 7 given lines shall lie on a Quartic Surface.

107. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1), \ldots (a_7, b_7, c_7, f_7, g_7, h_7)$, then in precisely the same way we form a determinant of the order $(\frac{1}{2}5.4=)$ 10 in regard to the coordinates of each line; this determinant however divides out by the several factors $a_1f_2 + \&c.$, which can be formed with the seven lines; or throwing these out and equating the quotient to zero, we have an equation of the order 4 in regard to the coordinates of each line. It would not be practicable to obtain the reduced equation in this manner, and I do not know how to obtain it otherwise, but the material conclusion is that the order is $= 4$.

The Jacobian of 6 points.

108. Any 6 points whatever may be regarded as points on a skew cubic; and the coordinates (x, y, z, w) may be taken so that the equations of the skew cubic shall be $\left\|\begin{array}{lll} x, & y, & z \\ y, & z, & w \end{array}\right\| = 0$. This being so, the coordinates of the 6 given points may be taken to be $(1, t_1, t_1^2, t_1^3), \ldots (1, t_6, t_6^2, t_6^3)$; and the equation of the Jacobian surface of

the 6 points can then be expressed in a very simple form, putting in evidence the passage of the surface through the skew cubic; viz. writing

$$p_1 = \Sigma t_1,\quad p_2 = \Sigma t_1 t_2,\quad p_3 = \Sigma t_1 t_2 t_3,\quad p_4 = \Sigma t_1 t_2 t_3 t_4,\quad p_5 = \Sigma t_1 t_2 t_3 t_4 t_5,\quad p_6 = t_1 t_2 t_3 t_4 t_5 t_6:$$

moreover,

$$\square = \tfrac{1}{2}(6xyzw - 4xz^3 - 4y^3w + 3y^2z^2 - x^2w^2),$$

and therefore

$$\begin{aligned}
\delta_x \square &= -\quad xw^2 - 2z^3 \quad + 3yzw,\\
\delta_y \square &= \quad 3yz^2 - 6y^2w + 3xzw,\\
\delta_z \square &= \quad 3y^2z - 6xz^2 + 3xyz,\\
\delta_w \square &= -\quad x^2w - 2y^3 \quad + 3xyz;
\end{aligned}$$

then the equation of the Jacobian surface is

$$\begin{aligned}
&3(\ xp_3 \qquad\qquad + \ zp_1 - 2w\ )\ \delta_x \square\\
&+\ (\qquad\qquad\qquad 2zp_2 - wp_1)\ \delta_y \square\\
&+\ (\ xp_5 - 2yp_4 \qquad\qquad\qquad)\ \delta_z \square\\
&+3(2xp_6 - \ yp_5 \qquad\qquad - wp_3)\ \delta_w \square = 0.
\end{aligned}$$

There is not much difficulty in the direct investigation; but a simple verification may be obtained by showing that the surface contains upon it the 15 lines 12, 13, ... 56. Write in the equation

$$(x,\ y,\ z,\ w) = (\lambda + \mu,\ \lambda s + \mu t,\ \lambda s^2 + \mu t^2,\ \lambda s^3 + \mu t^3),$$

the values $\delta_x \square$ &c. are found to contain the factor $\lambda\mu(s-t)^2$, and omitting this common factor the values are as

$$\tfrac{1}{3}(\lambda s^3 - \mu t^3),\quad -(\lambda s^2 - \mu t^2),\quad (\lambda s - \mu t),\quad -\tfrac{1}{3}(\lambda - \mu):$$

the equation thus becomes

$$\begin{aligned}
&\{\lambda(-2s^3 \ + \ s^2p_1 + \ p_3) + \mu(-2t^3 \ + \ t^2p_1 + \ p_3)\}(\lambda s^3 - \mu t^3)\\
&-\{\lambda(-\ s^3p_1 + 2s^2p_2 \qquad) + \mu(-\ t^3p_1 + 2t^2p_2 \qquad)\}(\lambda s^2 - \mu t^2)\\
&+\{\lambda(-2sp_4 + \ p_5 \qquad) + \mu(-2tp_4 + \ p_5 \qquad)\}(\lambda s - \mu t)\\
&-\{\lambda(-\ s^2p_3 - \ sp_5 + 2p_6) + \mu(-\ t^2p_3 - \ tp_5 + 2p_6)\}(\lambda \ - \mu \ ) = 0,
\end{aligned}$$

viz., collecting the terms, the coefficient of $\lambda\mu$ vanishes, and the whole is

$$\begin{aligned}
&-2\lambda^2(1,\ p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6 \between s,\ -1)^6\\
&+2\mu^2(1,\ p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6 \between t,\ -1)^6 = 0;
\end{aligned}$$

viz., this equation is satisfied if s denote any one of the quantities $(t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6)$, and t any one of the same 6 quantities; that is, the equation of the surface is satisfied when $(x,\ y,\ z,\ w)$ are the coordinates of a point on the line joining any 2 of the 6 points.

Locus of the vertex of a Quadric Cone which touches each of Six given Lines.

109. Representing as before each line by means of its six coordinates, let (x, y, z, w) be the coordinates of the vertex, and (X, Y, Z, W) current coordinates. Suppose that (a, b, c, f, g, h) are the coordinates of any one of the lines, the equation of the plane through this line and the vertex is

$$\begin{aligned} &a(xW - wX) + b(yW - wY) + c(zW - wZ) \\ &+ f(yZ - zY) + g(zX - xZ) + h(xY - yX) = 0; \end{aligned}$$

or, what is the same thing, writing for shortness

$$\begin{aligned} P &= \quad . \quad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \end{aligned}$$

the equation is

$$PX + QY + RZ + SW = 0.$$

The plane in question is a tangent plane to the cone touched by the 6 lines. Now when 6 planes touch a quadric cone, their traces on any plane whatever touch a conic the intersection of the cone by that plane. Hence taking the plane $W = 0$, the equation of the trace is

$$PX + QY + RZ = 0,$$

and forming in like manner tne equations belonging to each of the given lines, the condition that the 6 traces may touch a conic is

$$(P^2, Q^2, R^2, QR, RP, PQ) = 0,$$

where the left-hand side represents a determinant of 6 lines, the several lines being respectively P_1^2, Q_1^2, R_1^2, Q_1R_1, R_1P_1, P_1Q_1, P_2^2, &c..... Or more simply we may denote the equation by

$$[(P, Q, R)^2] = 0.$$

To ascertain the form of this, write for a moment $y = 0$, $z = 0$; the equation is

$$[(aw, -hx + bw, gx + cw)^2] = 0,$$

or attending only to the highest and lowest powers of w, this is

$$w^{12}[(a, b, c)^2] \ldots + w^4x^8[(a, -h, g)^2] = 0;$$

and it is thence easy to infer that the whole equation divides by w^4; so that, omitting this factor, the form of the equation is

$$((a, b, c, f, g, h)^2 \between x, y, z, w)^8 = 0;$$

viz., the equation is of the order 8 in the coordinates (x, y, z, w), and of the degree 2 in the coordinates (a, b, c, f, g, h) of each of the lines. It would not be very

difficult to actually develope the equation; in fact, starting from the term $w^8[(a, b, c)^2]$ the other terms are obtained therefrom by changing a, b, c into $a+\frac{1}{w}(hy-gz)$, $b+\frac{1}{w}(-hx+fz)$, $c+\frac{1}{w}(gx-fy)$ respectively; the equation may therefore be written in the symbolic form

$$w^8 . \exp. \frac{1}{w}\{(hy-gz)\,\delta_a+(-hx+fz)\,\delta_b+(gx-fy)\,\delta_c\} . [(a,\ b,\ c)^2]=0.$$

or, what is the same thing,

$$w^8 . \exp. \frac{1}{w}\{x(g\delta_c-h\delta_b)+y(h\delta_a-f\delta_c)+z(f\delta_b-g\delta_a)\} . [(a,\ b,\ c)^2]=0,$$

where exp. θ (read exponential) denotes e^θ, and $[(a, b, c)^2]$ represents a determinant as above explained. The equation contains, it is clear, the four terms

$$x^8[(a,\ -h,\ g)^2]+y^8[(-h,\ b,\ -f)^2]+z^8[(-g,\ f,\ c)^2]+w^8[(a,\ b,\ c)^2].$$

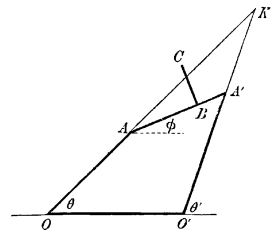
I am not sure whether this surface of the eighth order has been anywhere considered.

446.

ON THE MECHANICAL DESCRIPTION OF A NODAL BICIRCULAR QUARTIC.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 100—106.]

THE ingenious method, devised by Mr S. Roberts (*Proceedings*, vol. II. p. 133) for the description of a nodal bicircular quartic suggests a further investigation. We have a quadrilateral $OAA'O'$, in which the adjacent sides OA, AA' are equal to each other, and the other two adjacent sides OO', $O'A'$ are also equal to each other; O, O' are fixed points; and we have thus a link AA', the extremities of which are connected



with the radii OA, $O'A'$ respectively, and consequently describe circles about the centres O, O' respectively, the radius OA of the one circle being equal to the length AA' of the link, and the radius $O'A'$ of the other circle being equal to the distance OO' of the centres. The theorem is, that any point C, rigidly connected with the link AA', describes a nodal bicircular quartic, that is, a quartic curve with three nodes (or unicursal quartic), two of the nodes being the circular points at infinity. Any such curve is the inverse of a conic, and it is also the antipode of a conic; viz., if at each

point of the curve we draw a line at right angles to the radius vector from the node, these lines envelope a conic having for its pedal the curve in question. It is worth noticing at the outset that to a given position of A' there correspond two positions of A, viz., the broken line OAA' may occupy two positions situate symmetrically on the opposite sides of the line OA'. But to a given position of A, there corresponds only one position of A'; viz., the broken line $AA'O'$ is situate symmetrically with AOO' on the opposite side of the axis of symmetry $O'A$; the only other position would be A' coinciding with O, that is, AA' with AO, and the locus of C would then be a circle. If the equalities $OA = AA'$, $O'A' = OO'$ did not subsist, then to a given position of A' there would correspond two positions of A, and to a given position of A two positions of A', and the locus of C would be of a higher order than in the actual problem.

I have called AA' the link; OO' may be called the bar. OA is then the link-radius, $O'A'$ the bar-radius; moreover $AA'C$ may be called the constant triangle; and, producing OA, $O'A'$ to meet in K, then $AA'K$ may be called the variable triangle. Since at any instant the motion of A is normal to KA, and the motion of A' normal to KA', it is clear that the motion at that instant of the constant triangle is a motion of rotation about the point K.

Imagine any two positions of the link; say these are A_1A_1', and A_2A_2'. Join A_1A_2, and at its mid-point draw a perpendicular thereto; join in like manner $A_1'A_2'$, and at its mid-point draw a perpendicular thereto; and let these two perpendiculars meet in Γ; we have the two equal triangles $A_1A_1'\Gamma$, $A_2A_2'\Gamma$ (viz., $\Gamma A_1 = \Gamma A_2$, $\Gamma A_1' = \Gamma A_2'$, $A_1A_1' = A_2A_2$) with the common vertex Γ, and which may be brought to coincide with each other by a finite rotation about this point Γ. Considering any particular given position of Γ, if we take the constant triangle $AA'C$ equal to $A_1A_1'\Gamma$ or $A_2A_2'\Gamma$ (viz., $AC = A_1\Gamma$, $A'C = A_1'\Gamma$), then the constant triangle $AA'C$ will, in the course of its motion, come at two different times to coincide with the triangles $A_1A_1'\Gamma$ and $A_2A_2'\Gamma$ respectively; that is, Γ will be a node on the locus described by the point C; and moreover, if K_1 and K_2 be the corresponding positions of K, then by what precedes, the directions of the motion (or tangents at the node) will be normal to $K_1\Gamma$ and $K_2\Gamma$ respectively.

It is to be observed that the point Γ is determined by means of two arbitrary positions A_1A_1', A_2A_2' of the link; that is, the position of Γ depends upon two arbitrary parameters, and therefore Γ may be any point whatever in the plane; if, for an assumed position of Γ, the two positions A_1A_1', A_2A_2' of the link are real, then Γ is a crunode on the locus; but if imaginary, then Γ is an acnode on the locus. The transition case is when the two positions A_1A_1', A_2A_2', coincide with each other, Γ being in this case a cusp on the locus. But from the foregoing general construction for Γ, it appears that when A_1A_1' and A_2A_2' coincide, Γ is in fact the point K, the vertex of the variable triangle. I find that the locus of K is a nodal bicircular quartic, symmetrical in regard to the axis OO', and having the point O for a node; viz., when, as in the figure, AA' is $< OO'$, then the point O is an acnode, but when AA' is $> OO'$, then the point O is a crunode. The curve in question—say the "cuspidal locus"—

is a curve such that any point whatever thereof is a cusp on the curve described by some point C; it separates those points Γ, such that each of them is a crunode on the curve described by some point C, from the points Γ which are such that each of them is an acnode of the curve, described by some point C. If (as in the figure) AA' is $< OO'$, then the cuspidal curve is a closed curve (the inverse of an ellipse), the interior region being crunodal, and the exterior region acnodal. If AA' is $> OO'$, then the cuspidal curve is a figure of eight (inverse of a hyperbola), the two interior regions being crunodal, and the exterior region acnodal.

Passing now to the analytical investigation, I take the origin at O, the axis of x being in the direction from O to O', and the axis of y, at right angles thereto, upwards from O. The inclinations of OA, AA', $O'A'$ to the axis Ox, are taken to be θ, ϕ, θ' respectively. I write also $OA = AA' = a$, and $OO' = O'A' = a'$; and

$$m = \frac{a' - a}{a' + a},$$

or, what is the same thing,

$$m : 1 : 1 + m : 1 - m = a' - a : a' + a : 2a' : 2a;$$

and finally $AB = b$, $BC = c$.

Observing that the angle $AA'O'$ is $= \theta$, we have $\theta' = \theta + \phi$; and then, in the quadrilateral $OAA'O'$, the angles A, O' are $= \pi - \theta + \phi$, $\pi - \theta - \phi$ respectively; whence, projecting on the diagonal OA', we have

$$a \cos \tfrac{1}{2}(\theta - \phi) = a' \cos \tfrac{1}{2}(\theta + \phi),$$

which, attending to the value of m, is

$$\tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\phi = m;$$

whence, writing

$$\tan \tfrac{1}{2}\theta = u,$$

we have

$$\tan \tfrac{1}{2}\phi = \frac{m}{u},$$

and the sines and cosines of the angles θ, ϕ, θ' can be all of them expressed in terms of the single parameter u.

For the locus of C we have

$$x = a \cos \theta + b \cos \phi - c \sin \phi,$$
$$y = a \sin \theta + b \sin \phi + c \cos \phi,$$

or, instead of θ, ϕ introducing u, we have

$$\left.\begin{aligned} x &= -a\frac{u^2 - 1}{u^2 + 1} + b\frac{u^2 - m^2}{u^2 + m^2} - c\frac{2mu}{u^2 + m^2} \\ y &= a\frac{2u}{u^2 + 1} + b\frac{2mu}{u^2 + m^2} + c\frac{u^2 - m^2}{u^2 + m^2} \end{aligned}\right\},$$

which, in fact, show that the locus is a bicircular quartic. To put in evidence the third node, I assume that the values belonging thereto are $u=u_1$, $u=u_2$, and that the coordinates of the node are α, β; we have thus

$$\alpha = -a\frac{u_1^2-1}{u_1^2+1}+b\frac{u_1^2-m^2}{u_1^2+m^2}-c\frac{2mu_1}{u_1^2+m^2}, \quad = -a\frac{u_2^2-1}{u_2^2+1}+b\frac{u_2^2-m^2}{u_2^2+m^2}-c\frac{2mu_2}{u_2^2+m^2},$$

$$\beta = \quad a\frac{2u_1}{u_1^2+1}+b\frac{2mu_1}{u_1^2+m^2}+c\frac{u_1^2-m^2}{u_1^2+m^2}, \quad = -a\frac{2u_2}{u_2^2+1}+b\frac{2mu_2}{u_2^2+m^2}+c\frac{u_2^2+m^2}{u_2^2+m^2}.$$

These give b, c, α, β in terms of a, m, u_1, u_2; and we may then express the values of $x-\alpha$, $y-\beta$ in terms of a, m, u_1, u_2, u. I find

$$b=\frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[(u_1+u_2)^2+(1-m)(1-u_1u_2)\right]\right\},$$

$$c=\frac{a}{m}\left\{\qquad\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[\quad-(u_1+u_2)(m-u_1u_2)\quad\right]\right\};$$

and then

$$x=-a\frac{u^2-1}{u^2+1}+\frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[(u_1+u_2)^2+(1-m)(1-u_1u_2)\right]\right\}\frac{u^2-m^2}{u^2+m^2}$$
$$+\frac{a}{m}\left\{\qquad\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[\quad(u_1+u_2)(m-u_1u_2)\quad\right]\right\}\frac{2mu}{u^2+m^2},$$

$$y=\quad a\frac{2u}{u^2+1}+\frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[(u_1+u_2)^2+(1-m)(1-u_1u_2)\right]\right\}\frac{2mu}{u^2+m^2}$$
$$+\frac{a}{m}\left\{\qquad\frac{m+1}{(u_1^2+1)(u_2^2+1)}\left[\quad-(u_1+u_2)(m-u_1u_2)\quad\right]\right\}\frac{u^2-m^2}{u^2+m^2},$$

$$\alpha=\frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(1-u_1u_2)(m+u_1u_2),$$

$$\beta=\frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(u_1+u_2)(m+u_1u_2);$$

and then

$$(x-\alpha)=-\frac{2(m+1)a}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)(u-u_2)}{(u^2+1)(u^2+m^2)}\times\left[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u\right],$$

$$(y-\beta)=-\frac{2(m+1)a}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)(u-u_2)}{(u^2+1)(u^2+m^2)}\times\left[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u\right],$$

where, of course, the factors $(u-u_1)$, $(u-u_2)$ indicate the node (α, β). We have moreover

$$(x-\alpha)^2+(y-\beta)^2=\frac{4(m+1)^2a^2}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)^2(u-u_2)^2}{(u^2+1)(u^2+m^2)},$$

so that, writing

$$\frac{x-\alpha}{(x-\alpha)^2+(y-\beta)^2}=-\frac{1}{2(m+1)a}\frac{\left[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u\right]}{(u-u_1)(u-u_2)},$$

$$\frac{y-\beta}{(x-\alpha)^2+(y-\beta)^2}=-\frac{1}{2(m+1)a}\frac{\left[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u\right]}{(u-u_1)(u-u_2)},$$

we have the locus as the inverse of a conic. To exhibit it as the antipode of a conic, taking X, Y as current coordinates *measured from the node as origin*, the equation of the line through a point of the locus, at right angles to the radius vector from the node, is

$$X(x-\alpha)+Y(y-\beta)-(x-\alpha)^2-(y-\beta)^2=0;$$

or, substituting for $(x-\alpha)$, $(y-\beta)$ their values, this is

$$\begin{aligned}&X[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u]\\&+Y[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u]+2(m+1)a(u-u_1)(u-u_2)=0;\end{aligned}$$

and the antipodal conic is thus the envelope of the line represented by this equation. Putting for shortness

$$P=X(1-u_1u_2)+Y(u_1+u_2),\quad Q=X(u_1+u_2)-Y(1-u_1u_2),$$

the equation is

$$u^2\{P+2(m+1)a\}+u\{(1-m)Q-2(m+1)a(u_1+u_2)\}+mP+2(m+1)au_1u_2=0,$$

and the equation of the conic therefore is

$$4\{P+2(m+1)a\}\{mP+2(m+1)au_1u_2\}-\{(1-m)Q-2(m+1)a(u_1+u_2)\}^2=0,$$

so that the conic touches each of the lines $P+2(m+1)a=0$, $mP+2(m+1)au_1u_2=0$ at its intersection with the line $(1-m)Q-2(m+1)au_1=0$. If these lines were constructed, one other condition would suffice for the construction of the conic.

The before-mentioned equations

$$\alpha=\frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(1-u_1u_2)(m+u_1u_2),$$

$$\beta=\frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(u_1+u_2)(m+u_1u_2),$$

give

$$\alpha^2+\beta^2=\frac{a^2}{m^2}\frac{(m+1)^2}{(u^2+1)(u_2^2+1)}(m+u_1u_2)^2;$$

and thence

$$\frac{\alpha}{\alpha^2+\beta^2}=\frac{m}{(m+1)a}\frac{1-u_1u_2}{m+u_1u_2},$$

$$\frac{\beta}{\alpha^2+\beta^2}=\frac{m}{(m+1)a}\frac{u_1+u_2}{m+u_1u_2},$$

which determine u_1+u_2 and u_1u_2 rationally in terms of α, β. For the cuspidal curve, writing $u_1=u_2=v$, we have

$$\frac{\alpha}{\alpha^2+\beta^2}=\frac{m}{(m+1)a}\frac{1-v^2}{m+v^2},$$

$$\frac{\beta}{\alpha^2+\beta^2}=\frac{m}{(m+1)a}\frac{2v}{m+v^2},$$

which show that the cuspidal curve is the inverse of a conic (viz., of an ellipse, if, as in the figure, m is positive). The result in the very same form would be obtained by considering the curve as the locus of the vertex K of the variable triangle.

If we imagine a plane rigidly connected with the link AA', and carried along with it, then (b, c) are the coordinates of the point C in this moveable plane; and if, as above, (α, β) are the coordinates of the node, then (b, c) and also (α, β), are given functions of (u_1, u_2). We have thus (b, c) functions of (α, β), and reciprocally (α, β) functions of (b, c); that is, we have a correspondence between the points of the fixed plane and those of the variable plane. It is worth while to investigate the nature of this correspondence, although the result does not appear to be one of any elegance.

Writing

$$A = \frac{(m+1)a}{m} \frac{\alpha}{\alpha^2+\beta^2},$$

$$B = \frac{(m+1)a}{m} \frac{\beta}{\alpha^2+\beta^2},$$

we may, in place of (α, β), consider the point in the fixed plane as given by means of the inverse coordinates (A, B). And then, if $p = u_1 + u_2$, $q = 1 - u_1 u_2$, we have

$$A = \frac{q}{m+1-q}, \qquad B = \frac{q}{m+1-q};$$

whence

$$p = \frac{(m+1)B}{1+A}, \qquad q = \frac{(m+1)A}{1+A},$$

$$p^2 + q^2 = (1+u_1^2)(1+u_2^2) = \frac{(m+1)^2(A^2+B^2)}{(1+A)^2}.$$

Hence

$$\frac{m}{a}\left(b + \frac{a}{m}\right) = \frac{m+1}{p^2+q^2}[p^2 - (m-1)q],$$

$$-\frac{m}{a}c = \frac{m+1}{p^2+q^2}p(m-1+q),$$

which determine (b, c) in terms of (p, q); that is, of (A, B) or of (α, β).

In reference to some other constructions given in Mr Roberts' paper, it may be remarked that if we have a moveable plane Π_1 always coincident with a fixed plane Π, and if a condition of the motion is that a circle C_1, fixed in the plane Π_1 and carried along with it, always touches a fixed circle C in the plane Π, then this same condition may be expressed indifferently in either of the forms—(1) a circle C_1 in the plane Π_1 always passes through a fixed point of Π; (2) a point in the plane Π_1 is always situate on a fixed circle C in the plane Π. But if either of the circles C, C_1 reduce itself to a line, then we have two distinct forms of condition; viz., *first*, if a fixed line L_1 in the plane Π_1 always touches a fixed circle C in the plane Π, this is equivalent to the condition that a fixed line L_1 in the plane Π_1 always passes

through a fixed point of the plane Π. And secondly, if a fixed circle C_1 in the plane Π_1 always touches a fixed line L in the plane Π, this is equivalent to the condition that a fixed point in the plane Π_1 is always situate in a fixed line L_1 in the plane Π_1. The different forms of condition therefore are:

(α) A fixed circle C_1 in the plane Π_1 always touches a fixed circle C in the plane Π (where, as above, either circle indifferently may be reduced to a point).

(β) A fixed line L_1 in the plane Π_1 always passes through a fixed point C in the plane Π.

(γ) A fixed point C_1 in the plane Π_1 is always situate in a fixed line L of the plane Π.

Hence, if the motion of the plane Π_1 satisfy any two such conditions (of the same form or of different forms, viz., the conditions may be each α, or they may be α and β, &c.), then the motion of the plane Π_1 will depend on a single variable parameter, and the question arises as to the locus described by a given point, or enveloped by a given line, of the plane Π; and again of the locus traced out, or enveloped, on the moving plane Π_1 by a given point of the plane Π. The case considered in the present paper is of course a particular case of the two conditions being each of them of the form α.

It may be remarked, that if the two conditions be each of them β, then there will be in the plane Π_1 a fixed point C_1 which describes a circle; and similarly, if the two conditions be each of them γ, then there will be in the plane Π_1 a fixed point C_1 which describes a circle[1]; that is, the combination $\beta\beta$ is a particular case of $\alpha\beta$, and the combination $\gamma\gamma$ a particular case of $\alpha\gamma$.

[1] The theorem is, that if an isosceles triangle, on the base AA' and with angle $=2\omega$ at the vertex C, slide between two lines OA, OA' inclined to each other at an angle ω, in such manner that C is the centre of the circle circumscribed about OAA', then the locus of C is a circle having O for its centre.

447.

ON THE RATIONAL TRANSFORMATION BETWEEN TWO SPACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 127—180. Account of the Paper given at the Meeting 11 March 1869.]

Two figures are rationally transformable each into the other (or, say, there is a rational transformation between the two figures) when to a variable point of each of them there corresponds a single variable point of the other. The figures may be either loci in a space, or *locus in quo* of any number of dimensions; or they may be such spaces themselves. Thus the figures may be each a line (or space of one dimension), each a plane (or space of two dimensions), or each a space of three dimensions; these last are the cases intended to be considered in the present Memoir, which is accordingly entitled, "On the Rational Transformation between Two Spaces." I observe in explanation (to fix the ideas, attending to the case of two planes), that any rational transformation between two planes gives rise to a rational transformation between curves in these planes respectively (one of these curves being any curve whatever): but *non constat*, and it is not in fact the case, that every rational transformation between two plane curves thus arises out of a rational transformation between two planes. The problem of the rational transformation between two planes (or generally between two spaces) is thus a distinct problem from that of the rational transformation between two plane curves (or loci in the two spaces respectively).

I consider in the Memoir, (1) the rational transformation between two lines; this is simply the homographic transformation: (2) the rational transformation between two planes; and here there is little added to what has been done by Prof. Cremona in his memoirs, "Sulle Trasformazioni Geometriche delle Figure Piane," (*Mem. di Bologna*, t. II., 1863, and t. V., 1865; see also "On the Geometrical Transformation of Plane Curves," *British Assoc. Report*, 1864): (3) the rational transformation between two spaces; in regard hereto I examine the general theory, but attend mainly to what I call the lineo-linear transformation; viz., it is assumed that the coordinates

of a point in the one space, and the coordinates of the corresponding point in the other space are connected by three lineo-linear equations (that is, each equation is linear in the two sets of coordinates respectively). The lineo-linear transformation presents itself in the preceding two cases; viz., between two lines, the homographic transformation (which, as already mentioned, is the only rational transformation) is lineo-linear; and between two planes, the lineo-linear transformation is in fact the well-known inverse transformation $\left(x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. As regards two spaces, the lineo-linear transformation has not, I think, been discussed in a general manner, and it gives rise to a theory of some complexity, and of great interest.

The General Principle of the Rational Transformation between Two Spaces.

1. In all that follows, the two spaces (lines, planes, or three dimensional spaces, as the case may be), or any corresponding loci in the two spaces respectively, are referred to as the first and second figures respectively. The two figures are in general considered, not as superimposed or situate in a common space, but as existing, each independently of the other, as a separate *locus in quo* or figure in such locus. The unaccented coordinates (x, y), (x, y, z), or (x, y, z, w), as the case may be, refer throughout to a point of the first figure; the accented coordinates refer in like manner to the corresponding point of the second figure [1]. Moreover $X, Y, \ldots$ are used to denote functions of the same order, say n, of the coordinates $(x, y, \ldots)$; viz., (X, Y) are each of them of the form $(*\between x, y)^n$; (X, Y, Z) each of the form $(*\between x, y, z)^n$, (X, Y, Z, W) each of the form $(*\between x, y, z, w)^n$, as the case may be; and in like manner $X', Y', \ldots$ are used to denote functions of the same order, say n', of the coordinates $(x', y', \ldots)$. This being so:

The condition of a rational transformation is that we have simultaneously

$$x' : y', \ldots = X : Y, \ldots; \qquad x : y, \ldots = X' : Y', \ldots$$

viz., these equations must be such that either set shall imply the other set.

2. If, to fix the ideas, we attend to the case of two planes, or take the sets to be

$$x' : y' : z' = X : Y : Z; \qquad x : y : z = X' : Y' : Z',$$

[1] The coordinates (x, y) of a point in a line may be conceived as proportional to given multiples (α times, β times) of the distances of the point from two fixed points on the line; similarly the coordinates (x, y, z) of a point in a plane as proportional to given multiples (α times, β times, γ times) of the perpendicular distances of the point from three fixed lines in the plane; and the coordinates (x, y, z, w) of a point in a space as proportional to given multiples (α times, β times, γ times, δ times) of the perpendicular distances of the point from four fixed planes in the space. Observe that even if the coordinates (x, y) and (x', y') refer to the same line, and to the same two fixed points in this line, they are not of necessity the same coordinates; viz., the factors for x, y may be α, β, and those for x', y' may be α', β'. If these are proportional (viz., if $\alpha : \beta = \alpha' : \beta'$), then (x', y') will be the same coordinates of P' that (x, y) are of P; and in this case, but not otherwise, the equation $xy' - x'y = 0$ will imply the coincidence of the points P, P'. The like remarks apply to the coordinates (x, y, z) and (x, y, z, w).

then starting with the set $x' : y' : z' = X : Y : Z$, for any given point (x, y, z) whatever in the first figure, we have a single corresponding point (x', y', z') in the second figure; but for any given point (x', y', z') in the second figure, we have *primâ facie* a system of n^2 points in the second figure, viz., these are the common points of intersection of the curves $x' : y' : z' = X : Y : Z$ (in which equations x', y', z' are regarded as given parameters, x, y, z as current coordinates, and the equations therefore represent curves of the order n in the first figure). The curves may however have only a single variable point of intersection; viz., this will be the case if each of the curves passes through the same $n^2 - 1$ fixed points (points, that is, the positions of which are independent of x', y', z'); and in order that the curves in question may each pass through the $n^2 - 1$ points, it is necessary and sufficient that these shall be common points of intersection of the curves $X = 0$, $Y = 0$, $Z = 0$. {Observe that the condition thus imposed upon the curves $X = 0$, $Y = 0$, $Z = 0$ will in certain cases imply that the curves have n^2 common intersections; or, what is the same thing, that the functions X, Y, Z are connected by an identical equation, or syzygy, $\alpha X + \beta Y + \gamma Z = 0$. This must not happen; for if it did, not only there will be no variable point of intersection, and the transformation will on this account fail; but there would also arise a relation $\alpha x' + \beta y' + \gamma z' = 0$ between (x', y', z'), contrary to the hypothesis that (x', y', z') are the coordinates of any point whatever of the second figure. It thus becomes necessary to show that there exist curves $X = 0$, $Y = 0$, $Z = 0$, satisfying the required condition of the $n^2 - 1$ common intersections, but without a remaining common intersection, or, what is the same thing, without any syzygy $\alpha X + \beta Y + \gamma Z = 0$.}

3. The curves $x' : y' : z' = X : Y : Z$ having then a single variable point of intersection, if we take (x, y, z) to be the coordinates of this point, the ratios $x : y : z$ will be determined rationally; that is, as a consequence of the first set of equations, we obtain a second set $x : y : z = X' : Y' : Z'$, where X', Y', Z' will be rational and integral functions of the same order, say n', of the coordinates (x', y', z'); that is, we have a second set of equations, and consequently a rational transformation, as mentioned above.

4. It is easy to see that we have $n' = n$; in fact, consider in the first figure a curve $\alpha X + \beta Y + \gamma Z = 0$, and an arbitrary line $ax + by + cz = 0$; to these respectively correspond, in the second figure, the line $\alpha x' + \beta y' + \gamma z' = 0$, and the curve $aX' + bY' + cZ' = 0$; the curves are of the orders n, n' respectively, or the curve and line of the first figure intersect in n points, and the line and curve of the second figure intersect in n' points; which two systems of points must correspond point to point to each other; that is, we must have $n' = n$. It will presently appear how different the analogous relation is in the transformation between two spaces.

5. Ascending to the case of two spaces, we have here the two sets

$$x' : y' : z' : w' = X : Y : Z : W; \qquad x : y : z : w = X' : Y' : Z' : W',$$

the theory is analogous; the surfaces $x' : y' : z' : w' = X : Y : Z : W$ (surfaces of the order n in the first figure) must have a single variable point of intersection, and they must therefore have a common fixed intersection equivalent to $n^3 - 1$ points of inter-

section: I say *equivalent* to n^3-1 points, for this fixed intersection need not be n^3-1 points, but it may be or include a curve of intersection(1). The surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ must consequently have a common intersection equivalent to n^3-1 points; there is (as in the preceding case) a cause of failure to be guarded against, viz., the condition as to the intersection must not be such as to imply one more point of intersection, that is, to imply an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W=0$ between the functions X, Y, Z, W; but it is assumed that they are not thus connected. There is, then, a single variable point of intersection of the surfaces $x':y':z':w'=X:Y:Z:W$; or taking the coordinates of this point to be (x, y, z, w), we have the ratios $x:y:z:w$ rationally determined; that is, we have a second set of equations $x:y:z:w=X':Y':Z':W'$, where X', Y', Z', W' are rational and integral functions of the same order, say n', in the coordinates (x', y', z', w'); viz., we have the rational transformation, as above, between the two spaces.

6. Suppose that the common intersection of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ is or includes a curve of the order ν; and consider in the first figure the two surfaces

$$\alpha X+\beta Y+\gamma Z+\delta W=0,\quad \alpha_1 X+\beta_1 Y+\gamma_1 Z+\delta_1 W=0,$$

and the arbitrary plane $ax+by+cz+dw=0$. The two surfaces intersect in the fixed curve ν, and in a residual curve of the order $n^2-\nu$; hence the two surfaces and the plane meet in ν points on the fixed curve, and in $n^2-\nu$ other points. Corresponding to the surfaces and plane in the first figure, we have in the second figure the two planes

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0,\quad \alpha_1 x'+\beta_1 y'+\gamma_1 z'+\delta_1 w'=0,$$

and the surface $aX'+bY'+cZ'+dW'=0$ of the order n': these intersect in n' points, being a system corresponding point to point with the $n^2-\nu$ points of the first figure; that is, we must have $n'=n^2-\nu$. And conversely, it follows that in the second figure the common intersection of the surfaces $X'=0$, $Y'=0$, $Z'=0$, $W'=0$ will be or include a curve of the order ν'; and that we shall have $n=n'^2-\nu$. Hence also

$$\nu-\nu'=(n-n')(n+n'+1).$$

7. The principle of the rational transformation comes out more clearly in the foregoing two cases than in the case of two lines, which from its very simplicity fails to exhibit the principle so well; and I have accordingly postponed the consideration of it: but the theory is similar to that of the foregoing cases. We must have the two sets (each a single equation) $x':y'=X:Y$, and $x:y=X':Y'$. The equation $x':y'=X:Y$ must give for the ratio $x:y$ a single variable value; viz., there must be $n-1$ constant values (values, that is, independent of x', y'); this can only be the case by reason of the functions having a common factor M of the order $n-1$; but this being so, the common factor divides out, and the equation assumes the form $x':y'=X:Y$, where X, Y are linear functions of (x, y): and we have then reciprocally

[1] The curve of intersection may consist of distinct curves, each or any of which may be a singular curve of any kind in regard to the several surfaces.

$x : y = X' : Y'$, where X', Y' are linear functions of (x', y'). Thus in the present case, instead of an infinity of transformations for different values of n, n', we have only the well-known homographic transformation wherein $n = n' = 1$.

8. In the discussion of the foregoing cases of the transformation between two planes and two spaces, it was tacitly assumed that n was greater than 1, and the transformations considered were thus different from the homographic transformation; but it is hardly necessary to remark that the homographic transformation applies to these cases also; viz., for two planes we may have $x' : y' : z' = X : Y : Z$, and $x : y : z = X' : Y' : Z'$, where (X, Y, Z), (X', Y', Z') are linear functions of the two sets of coordinates respectively; and similarly for two spaces $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$, where (X, Y, Z, W), (X', Y', Z', W') are linear functions of the two sets of coordinates respectively. We may, if we please, separate off the homographic transformation (as between two lines, planes, and spaces respectively), and restrict the notion of the rational transformation to the higher or non-linear transformations; in this point of view, the case of two lines would not be considered at all, but the theory of the rational transformation would begin with the case of the two planes. Such severance of the theory is, however, somewhat arbitrary; and moreover the homographic transformation between two lines (being, as mentioned, the only rational transformation) is analogous not only to the homographic transformation between two planes, and to the homographic transformation between two spaces, but it is also analogous to the lineo-linear (or quadric) transformation between two planes, and to the lineo-linear (which is a cubic) transformation between two spaces.

9. For the sake of bringing out this analogy, I shall consider in some detail the homographic transformation between two lines; but as regards the homographic transformations between two planes and between two spaces respectively (although there is room for a like discussion) the theories may be considered as substantially known, and I do not propose to go into them.

The Homographic Transformation between Two Lines.

10. By what precedes, it appears that we have $x' : y' = X : Y$, where (X, Y) are linear functions of (x, y); and conversely, $x : y = X' : Y'$, where X', Y' are linear functions of (x', y'); or what is the same thing, the relation is expressed by a single equation

$$(ax + by)\,x' + (cx + dy)\,y' = 0\,;$$

or, as this may conveniently be written,

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} (x,\ y)\,(x',\ y') = 0,$$

or, when the expression of the actual values of the coefficients is unnecessary,

$$(*\between x,\ y)\,(x',\ y') = 0.$$

We thus see that the rational transformation between two lines is in fact the homographic transformation; and also that it is the lineo-linear transformation.

11. A special case is when

$$ad - bc = 0.$$

Writing here

$$\frac{c}{a} = \frac{d}{b} = m, \quad = \frac{b'}{a'},$$

the equation is

$$(ax + by)(\ \ x' + my') = 0,$$

that is

$$(ax + by)(a'x' + b'y') = 0;$$

viz., either $ax + by = 0$, without any relation between x', y'; or else $a'x' + b'y' = 0$, without any relation between x, y; that is, to the single point $ax + by = 0$ of the first figure there corresponds any point whatever of the second figure; and to the single point $a'x' + b'y' = 0$ of the second figure there corresponds any point whatever of the first figure.

12. In the general case where $ad - bc \neq 0$, we may either by a linear transformation ($ax + by$, $cx + dy$ into y, $-x$ or into x, $-y$) of the coordinates of a point of the first figure, or by a linear transformation ($ax' + cy'$, $bx' + dy'$ into y', $-x'$ or into x', $-y'$) of the coordinates of a point in the second figure (or in a variety of ways by simultaneous linear transformations of the two sets of coordinates) transform the relation indifferently into either of the forms $xy' - x'y = 0$, $xx' - yy' = 0$; the former of these, or $x' : y' = x : y$, is the most simple expression of the homographic transformation; the latter, or $x' : y' = \frac{1}{x} : \frac{1}{y}$, is its expression as an inverse transformation.

13. If, to fix the precise signification of the coordinates (x, y), we employ the distances from a fixed point O in the line; taking the distances of the two fixed points (say A, B) to be α, β, and that of the variable point P to be ρ, then we have x, y proportional to given multiples $p(\rho - \alpha)$, $q(\rho - \beta)$ of the distances from the two fixed points; or writing $\frac{p}{q} = n$, we may say that the coordinate $\frac{x}{y}$ of the point P is $= n\frac{\rho - \alpha}{\rho - \beta}$; or in particular, if $n = 1$, then the coordinate is $= \frac{\rho - \alpha}{\rho - \beta}$. If for $n\frac{\rho - \alpha}{\rho - \beta}$ we write $\frac{-\beta\rho + \lambda}{\rho - \beta}$, and then take $\beta = \infty$, we see that in a particular system of coordinates, A at O, B at ∞, the coordinate $\frac{x}{y}$ is $= \rho$. Proceeding in the same manner in regard to the coordinates (x', y'), for a particular system of coordinates, A' at O', B' at ∞, the coordinate $\frac{x'}{y'}$ of P' will be $= \rho'$. And the correspondence of the points P, P' will be given by an equation

$$a\rho\rho' + b\rho' + c\rho + d = 0.$$

14. The equation just mentioned is often convenient for obtaining a precise statement of theorems. Thus taking A, B at pleasure on the first line, A', B' the corresponding points on the second line, we have

$$\rho' = -\frac{c\rho + d}{a\rho + b},$$

and thence

$$\alpha' = -\frac{c\alpha + d}{a\alpha + b},$$

$$\beta' = -\frac{c\beta + d}{a\beta + b},$$

$$\rho' - \alpha' = \frac{(ad - bc)(\rho - \alpha)}{(a\alpha + b)(a\rho + b)},$$

$$\rho' - \beta' = \frac{(ad - bc)(\rho - \beta)}{(a\beta + b)(a\rho + b)};$$

and consequently

$$\frac{\rho' - \alpha'}{\rho' - \beta'} = \frac{a\beta + b}{a\alpha + b}\,\frac{\rho - \alpha}{\rho - \beta},$$

which is of the form

$$\frac{\rho' - \alpha'}{\rho' - \beta'} = m\,\frac{\rho - \alpha}{\rho - \beta},$$

where (the correspondence $a\rho\rho' + b\rho' + c\rho + d = 0$ being given, and also the fixed points A, B) m has a determinate value not assumable at pleasure. If, however, the fixed points A, B be not given, then we may determine a relation between them, such that m shall have any given value not being $= 1$; we have in fact only to write

$$a\beta + b = m(a\alpha + b),$$

that is

$$a(\beta - m\alpha) + b(1 - m) = 0,$$

($m = 1$ would give $\alpha = \beta$ and the transformation would fail). In particular we may write $m = -1$, we have then

$$a(\alpha + \beta) + 2b = 0;$$

or the sum of the two distances OA, OB has a given value $= -\dfrac{2b}{a}$ dependent on the transformation; one of these points being assumed at pleasure, the other is known: the points A', B' are also known, and the equation of correspondence is

$$\frac{\rho' - \alpha'}{\rho' - \beta'} + \frac{\rho - \alpha}{\rho - \beta} = 0;$$

it is moreover easy to show that we have

$$a(\alpha' + \beta') + 2c = 0.$$

15. In what precedes, the two lines are considered as distinct lines, not of necessity existing in a common space. But they may be considered, not only as existing in the common space, but as superimposed the one on the other. Suppose this is so, and moreover that the fixed points A', B' coincide with A, B respectively, and that the coordinates (x, y) and (x', y') are the same coordinates; so that the equation $xy' - x'y = 0$ will imply the coincidence of the points P, P'.

16. If $ad-bc=0$, the equation of correspondence becomes

$$(ax+by)(a'x'+b'y')=0,$$

and as before, to a single given point $ax+by=0$, considered as belonging to the first figure, there corresponds every point whatever of the line, or second figure: to a single given point $a'x'+b'y'=0$ (the same as, or different from, the first point), considered as belonging to the second figure, there corresponds every point whatever of the line, or first figure.

17. Excluding the foregoing case, or assuming $ad-bc\neq 0$, there are in general on the line two points such that to each of them considered as belonging to either figure there corresponds the same point considered as belonging to the other figure, or say there are two united points: in fact, writing $x':y'=x:y$, we find $ax^2+(b+c)xy+dy^2=0$, a quadric equation for the determination of the points in question. Unless $4ad-(b+c)^2=0$, this equation will have two unequal roots; and taking the two points so determined for the fixed points $A=A'$, $B=B'$, the equation of correspondence will assume the form $xy'-kx'y=0$. In this equation k cannot be $=1$; for if it were so, the equation would be $xy'-x'y=0$; that is, the points P, P' would be always one and the same point. The equation may, however, be $xy'+x'y=0$; the points P, P' are then harmonics in regard to the fixed points A, B. It is to be observed, that if the equation $xy'-kx'y=0$ be unaltered by the interchange of (x, y) and (x', y') we must have $k^2-1=0$, or since $=1$ is excluded, we must have $k=-1$.

18. The original equation $(ax+by)x'+(cx+dy)y'=0$ is unaltered by the interchange, only if $b-c=0$; the equation $4ad-(b+c)^2=0$ becomes in this case $ad-bc=0$, which by hypothesis is not satisfied; the two distinct points $A=A'$, $B=B'$ consequently exist. That is, if the correspondence between the two points P, P' is such that whether P be considered as belonging to the first figure or to the second figure, there corresponds to it in the other figure the same point P'—or say if the correspondence between the points P, P' is a symmetrical correspondence—then as united points in the superimposed figures we have the two distinct points A, B: and the correspondence of the points P, P' is given by the condition that these are harmonics in regard to the points A, B.

19. There is still the case to be considered where $4ad-(b+c)^2=0$; the equation $ax^2+(b+c)xy+dy^2=0$ has here equal roots, or the two united points coincide together, or form a single point. Taking this point to be the point A, the coordinate whereof is $x:y=0:1$, we must, it is clear, have $d=0$, and therefore also $b+c=0$: the relation between the coordinates (x, y) and (x', y') is then $axx'+b(xy'-x'y)=0$; viz., this is the form assumed by the equation of correspondence when instead of two united points there is a double united point, and this is taken to be the fixed point A.

20. It is to be observed, that we cannot have either $b=0$, for this would give $xx'=0$, which belongs to the excluded case $ad-bc=0$; nor $a=0$, for this would give $xy'-x'y=0$: excluding these cases, the equation is of necessity altered by the inter-

change of (x, y) and (x', y'); that is, in the case of a double united point, the transformation is essentially unsymmetrical.

By what precedes, if the other fixed point be taken to be at infinity, the coordinates $x : y$ and $x' : y'$ may be taken to be ρ, ρ' respectively; viz., ρ, ρ' will denote the distances of the points P, P' from the double united point A; and the equation of correspondence then becomes $\rho\rho' + b(\rho - \rho') = 0$; that is, $(\rho - b)(\rho' + b) + b^2 = 0$.

21. The original equation $axx' + byx' + cxy' + dyy' = 0$ can be reduced to the inverse form $xx' - yy' = 0$ only (it is clear) in the symmetrical case $b = c$; here, transforming to the united points, the equation is, by what precedes (*ante*, No. 17) $xy' + x'y = 0$. This equation can be written $(lx + my)(lx' + my') - (lx - my)(lx' - my') = 0$, where $l : m$ is arbitrary; viz., we have thus an equation of the required form.

22. In further explanation, start from the equation $a\rho\rho' + b(\rho + \rho') + d = 0$; that is, $(a\rho + b)(a\rho' + b) + ad - b^2 = 0$, or say $(\rho - \alpha)(\rho' - \alpha) - k^2 = 0$; this may be reduced to $\rho\rho' - 1 = 0$; viz., the point O from which are measured the distances ρ, ρ' is here the mid-point between the two united points A, B; and the unit of distance is $\frac{1}{2}AB$; the equation expresses that the points P, P', harmonics in regard to the two points A, B, are the images one of the other in regard to the circle described upon AB as diameter. Take any two corresponding points L, L'; if the distances of these be λ, λ', we have $\lambda\lambda' = 1$; and hence

$$(\rho - \lambda)(\rho' - \lambda) = 1 - \lambda(\rho + \rho') + \lambda^2 = \lambda(\lambda + \lambda' - \rho - \rho'),$$
$$(\rho - \lambda')(\rho' - \lambda') = 1 - \lambda'(\rho + \rho') + \lambda'^2 = \lambda'(\lambda + \lambda' - \rho - \rho');$$

and consequently

$$\frac{\rho - \lambda}{\rho - \lambda'} \cdot \frac{\rho' - \lambda}{\rho' - \lambda'} = \frac{\lambda}{\lambda'};$$

which, writing

$$\frac{x}{y} = \frac{k(\rho - \lambda)}{\rho - \lambda'}, \quad \frac{x'}{y'} = \frac{k(\rho' - \lambda)}{\rho' - \lambda'},$$

$$k^2 = \frac{\lambda}{\lambda'} \text{ (so that } k^2 \neq 1\text{); or, } k = \lambda = \frac{1}{\lambda'},$$

becomes $xx' - yy' = 0$; that is, the correspondence of the points P, P' being symmetrical, if the coordinate $\frac{x}{y}$ of P be taken to be a multiple of the ratio of the distances PL, PL' of P from any two corresponding points L, L' (and of course the coordinate $\frac{x'}{y'}$ of P' to be the same multiple of the ratio of the distances $P'L$, $P'L'$), the equation of correspondence is obtained in the inverse form $xx' - yy' = 0$.

The Rational Transformation between Two Planes.

23. Starting from the equations $x' : y' : z' = X : Y : Z$, where $X = 0$, $Y = 0$, $Z = 0$ are curves in the first plane, of the same order n, it has been seen that in order that we may thence have a rational transformation between the two planes, the curves

$X=0$, $Y=0$, $Z=0$ must have a common intersection of n^2-1 points, and no more; that is, they must not have a complete common intersection of n^2 points. In the case $n=2$, taking the n^2-1 points in the first plane to be any three points whatever, the condition that the curves shall be conics passing through the three points does not in anywise imply that the conics shall have a common fourth point of intersection; and we have thus a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are conics passing through the same three points of the first plane; and as it is easy to see (but which will be subsequently shown more in detail), the second set is the similar one $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are conics passing through the same three points in the second plane; this may be called the quadric transformation between the two planes.

24. But the like theory would not apply to the case $n=3$; if the n^2-1 points in the first plane were any eight points whatever, the cubics $X=0$, $Y=0$, $Z=0$, intersecting in these eight points, would have a common ninth point of intersection, and the transformation would fail; and so for any higher value of n, taking at pleasure any $\frac{1}{2}n(n+3)-1$ of the n^2-1 points of the first plane, the curves $X=0$, $Y=0$, $Z=0$ of the order n passing through these $\frac{1}{2}n(n+3)-1$ points, would have in common all their remaining points of intersection, and the transformation would fail. A transformation can only be obtained by taking the n^2-1 points in such wise that these can be made to be the common intersection of the curves, and at the same time that the number of conditions imposed upon each of the curves $X=0$, $Y=0$, $Z=0$ shall be at most $=\frac{1}{2}n(n+3)-1$.

25. And this requirement may be satisfied; viz., the number of conditions may be made to be $=\frac{1}{2}n(n+3)-1$, by assuming that certain of the n^2-1 points of intersections are multiple intersections of the curves. For if we have a given point which is an α-tuple point on each of the curves $X=0$, $Y=0$, $Z=0$, then this counts for α^2 points of intersection of any two of the curves, and thus for α^2 points of the n^2-1 points: but the condition that the given point shall be on any one of the curves, say the curve $X=0$, an α-tuple point, imposes on the curve, not α^2, but only $\frac{1}{2}\alpha(\alpha+1)$ conditions: and we have in this way a reduction whereby the number of conditions for passing through the n^2-1 points can be lowered from n^2-1 to the required number $\frac{1}{2}n(n+3)-1$.

26. In particular, for $n=3$, we may for the n^2-1 points of the first plane take a point as a double point on each of the cubic curves $X=0$, $Y=0$, $Z=0$ (which therefore reckons as four points), and take any other four points. Each of the curves is determined by the conditions of having a given point for double point, and of passing through the same four other given points; that is, by $3+4=7$ conditions; and the three cubic curves $X=0$, $Y=0$, $Z=0$ have for the common intersection the double point reckoning as four points, and the given other four points; that is, they have a common intersection of $4+4=8$ points; but this does not imply that they have a common ninth point of intersection; we have therefore a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$; where $X=0$, $Y=0$, $Z=0$ are cubics in the first plane having each of them a double point at the same given point and

also each passing through the same four given points; the second set of equations is $x : y : z = X' : Y' : Z'$, where $X' = 0$, $Y' = 0$, $Z' = 0$ are like cubics in the second plane.

27. Generally suppose that the $n^2 - 1$ points in the first plane are made up of α_1 points, which are simple points; α_2 points, which are double points; α_3 points, which are triple points, ... α_{n-1} points, which are $(n-1)$tuple points ($\alpha_{n-1} = 1$ or 0), on each of the three curves; these will represent a system of $n^2 - 1$ points if only

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 \ldots \quad + \quad (n-1)^2 \alpha_{n-1} = n^2 - 1.$$

The number of conditions imposed on each of the curves $X = 0$, $Y = 0$, $Z = 0$ will be $\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \frac{1}{2} n(n-1) \alpha_{n-1}$; for the reason presently appearing, I exclude the case of this being $< \frac{1}{2} n(n+3) - 2$; and therefore assume it to be $= \frac{1}{2} n(n+3) - 2$. In fact, writing

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots \quad + \tfrac{1}{2} n(n-1) \ \alpha_{n-1} = \tfrac{1}{2} n(n+3) - 2,$$

this combined with the former equation gives

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2) \ \alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2);$$

viz., the singularities are equivalent to $\frac{1}{2}(n-1)(n-2)$ double points, that is, to the maximum number of double points of a curve of the order n; or say each of the curves $X = 0$, $Y = 0$, $Z = 0$ is a curve of the order n having a deficiency $= 0$; that is, it is a unicursal curve of the order n. Hence also, taking (a, b, c) any constant factors whatever, the curve $aX + bY + cZ = 0$ is unicursal.

28. It is important to remark that the conclusion follows directly from the general notion of the rational transformation; in fact, the equation $aX + bY + cZ = 0$ is satisfied if $x : y : z = X' : Y' : Z'$; $ax' + by' + cz' = 0$. The last of these equations determines the ratios $x' : y' : z'$ in terms of a single parameter (e.g. the ratio $x' : y'$), and we have then $x : y : z$ expressed as rational functions of this parameter; that is, the curve is unicursal.

29. Suppose for a moment that it was possible to have

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \quad \tfrac{1}{2} n(n-1) \alpha_{n-1} < \tfrac{1}{2} n(n+3) - 2.$$

Combining in the same way with the first equation, it would follow that

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2) \ \alpha_{n-1} > \tfrac{1}{2}(n-1)(n-2),$$

which would imply that the curves $X = 0$, $Y = 0$, $Z = 0$ break up each of them into inferior curves: but more than this, the coefficients a, b, c being arbitrary, it would imply that the curve $aX + bY + cZ = 0$ breaks up into inferior curves; this can only be the case if X, Y, Z have a common factor, say M; that is, if X, Y, $Z = MX_1$, MY_1, MZ_1; but we could then omit the common factor, and in place of $x' : y' : z' = X : Y : Z$ write $x' : y' : z' = X_1 : Y_1 : Z_1$, where $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$, are proper curves, not breaking up; the above supposition may therefore be excluded from consideration.

30. We have thus a transformation in which the first set of equations is $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are curves in the first plane, of the same order n, having in common $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ points which are simple points, double points,... $(n-1)$tuple points respectively on each of the curves; these numbers satisfy the conditions

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 \ldots + \qquad (n-1)^2 \alpha_{n-1} = n^2 - 1,$$

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \qquad \tfrac{1}{2}n(n-1)\ \alpha_{n-1} = \tfrac{1}{2}(n^2+3n) - 2;$$

conditions which give, as above,

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2)\alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

and also

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 \ldots + \qquad (n-1)\alpha_{n-1} = 3n-3;$$

so that the relations between $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ are given by any two of these four equations.

31. The second set of equations then is $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are curves in the second plane, of the same order n; and it is clear that these must be curves such as those in the first plane; viz., they must have in common $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$ points, which are simple points, double points, ... $(n-1)$tuple points respectively on each of the curves, the relations between these numbers being expressed by any two of the four equations

$$\alpha_1' + 4\alpha_2' + 9\alpha_3' \ldots + \qquad (n-1)^2 \alpha'_{n-1} = n^2 - 1,$$

$$\alpha_1' + 3\alpha_2' + 6\alpha_3' \ldots + \qquad \tfrac{1}{2}n(n-1)\ \alpha'_{n-1} = \tfrac{1}{2}n(n+3) - 2,$$

$$\alpha'_2 + 3\alpha_3' \ldots + \tfrac{1}{2}(n-1)(n-2)\ \alpha'_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

$$\alpha_1' + 2\alpha_2' + 3\alpha_3' \ldots + \qquad (n-1)\ \alpha'_{n-1} = 3n-3.$$

32. To any line $ax' + by' + cz' = 0$ in the second plane there corresponds in the first plane a curve $aX + bY + cZ$ of the order n; and to any line $a'x + b'y + c'z = 0$ in the first plane there corresponds in the second plane a curve $a'X' + b'Y' + c'Z' = 0$ of the same order n; the curves $aX + bY + cZ = 0$ in the first plane are, it is clear, a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X=0$, $Y=0$, $Z=0$, and being as already mentioned unicursal; and similarly the curves $a'X' + b'Y' + c'Z' = 0$ in the second plane are a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X'=0$, $Y'=0$, $Z'=0$; and being also unicursal. We may say that to the lines of the second plane there corresponds in the first plane the *réseau* of curves $aX + bY + cZ = 0$; and to the lines of the first plane there corresponds in the second plane the *réseau* of curves $a'X' + b'Y' + c'Z' = 0$; these *réseau* being systems satisfying respectively the conditions just referred to.

33. We have next to enquire what are the curves in the second plane which correspond to the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points of the first plane. I remark that the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points are termed by Cremona the *principal points* of the first plane, and the corresponding curves the *principal curves* of the second plane. But it will be

more convenient to say that the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points are the *principal system* of the first plane, and the corresponding curves the *principal counter-system* of the second plane. And of course the $\alpha_1' + \alpha_2' \ldots + \alpha'_{n-1}$ points will be the principal system of the second plane, and the corresponding curves the principal counter-system of the first plane.

34. The Jacobian (curve) of the curves $X=0$, $Y=0$, $Z=0$ is, of course, the Jacobian of any three curves $aX+bY+cZ=0$ of the first plane, or it may be called the Jacobian of the reseau of the first plane; and similarly, the Jacobian of the curves $X'=0$, $Y'=0$, $Z'=0$ is the Jacobian of the reseau of the second plane.

35. I say that to each point α_1 of the first figure there corresponds in the second figure a line; to each point α_2 a conic; to each point α_3 a nodal cubic; ... and generally, to each point α_r a unicursal r-thic curve; the entire system of the curves corresponding to the $\alpha_1 + \alpha_2 + \alpha_3 \ldots + \alpha_{n-1}$ points, that is, the principal counter-system of the second plane, is thus made up of α_1 lines, α_2 conics, α_3 nodal cubics, ... α_r unicursal r-thics, ... α_{n-1} unicursal $(n-1)$thics. It is thus a curve of the aggregate order $\alpha_1 + 2\alpha_2 + 3\alpha_3 \ldots + (n-1)\alpha_{n-1}, = 3n-3$; and it is in fact the Jacobian of the reseau of the second plane; as such, it passes through each point α_1' two times, each point α_2' five times, ... each point α_r' $3r-1$ times, ... each point α'_{n-1} $3n-4$ times.

36. The reciprocal theorem is of course true. The Jacobian of the reseau of the first plane is thus made up of α_1' lines, α_2' conics, α_3' nodal cubics, ... α_r' unicursal r-thics, ... α'_{n-1} unicursal $(n-1)$thics. Calculating the Jacobian of the reseau of the first plane, we have thus the numbers α_1', α_2', ... α'_{n-1}, which determine the nature of the principal system of the second plane.

37. I indicate as follows the analytical proof of the theorem that to a principal point α_r of the first plane there corresponds in the second plane a unicursal r-thic. Consider the simplest case, $r=1$; if in the equations $x' : y' : z' = X : Y : Z$ the coordinates (x, y, z) are considered as belonging to a point α_1, these values give identically $X=0$, $Y=0$, $Z=0$; hence for the consecutive point $x+\delta x$, $y+\delta y$, $z+\delta z$, if (A, B, C) denote the derived functions of X, (A_1, B_1, C_1) those of Y, (A_2, B_2, C_2) those of Z, we have

$$\begin{aligned} x' : y' : z' = \quad & A\,\delta x + B\,\delta y + C\,\delta z \\ : \; & A_1\delta x + B_1\delta y + C_1\delta z \\ : \; & A_2\delta x + B_2\delta y + C_2\delta z. \end{aligned}$$

We have $\begin{vmatrix} A, & B, & C \\ A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \end{vmatrix} = 0$, for the determinant is the value, at the point α_1 in question, of the Jacobian of the reseau of the first plane; and the Jacobian curve passing through α_1 (in fact, having there a double point), the value is $=0$.

38. Hence x', y', z', considered as corresponding to a point indefinitely near to α_1, are connected by a linear equation. Corresponding to α_1 we have in the second figure

a line. But it is to be observed, further, that the equation of the line is that obtained by writing in the foregoing equations, say $\delta z = 0$, and eliminating the remaining quantities δx, δy; or, what is the same thing, we may consider the equation of the line as given by the equations

$$\begin{aligned} x' : y' : z' = \quad & A\,\delta x + B\,\delta y \\ : \; & A_1\delta x + B_1\delta y \\ : \; & A_2\delta x + B_2\delta y, \end{aligned}$$

where δx, δy are indeterminate parameters to be eliminated.

39. In the case of a point α_r we have in like manner

$$\begin{aligned} x' : y' : z' = \quad & (a\,, \ldots \between \delta x,\ \delta y,\ \delta z)^r \\ : \; & (a_1, \ldots \between \delta x,\ \delta y,\ \delta z)^r \\ : \; & (a_2, \ldots \between \delta x,\ \delta y,\ \delta z)^r, \end{aligned}$$

where $(a, \ldots)$, $(a_1, \ldots)$, $(a_2, \ldots)$ are the r-th derived functions of X, Y, Z respectively. In virtue of the relation of the point α_r to the curves $X = 0$, $Y = 0$, $Z = 0$, the coefficients will be such as to allow of the simultaneous elimination from these equations of the three quantities δx, δy, δz. The result of the elimination will be the same as if, writing say $\delta z = 0$, we eliminate δx, δy; or, what is the same thing, the relation of x', y', z' may be regarded as given by the equations

$$\begin{aligned} x' : y' : z' = \quad & (a\,, \ldots \between \delta x,\ \delta y)^r \\ : \; & (a_1, \ldots \between \delta x,\ \delta y)^r \\ : \; & (a_2, \ldots \between \delta x,\ \delta y)^r, \end{aligned}$$

where δx, δy are indeterminate parameters. These equations obviously express that the point (x', y', z') is situate on a unicursal curve of the order r.

40. It is further to be shown that the r-thic curve thus corresponding to α_r is part of the Jacobian of the reseau of the second plane. The Jacobian in question is the locus of the new double point of those curves of the reseau which have a new double point; that is, a double point not included among the $\alpha_2' + \alpha_3' \ldots + \alpha'_{n-1}$ singular points of the principal system of the second plane. But a curve of the reseau being unicursal, can only acquire a new double point by breaking up into inferior curves. Consider, in the first figure, any line through α_r, the corresponding curve in the second figure is made up of the unicursal r-thic curve, which corresponds to the point α_r, together with a residual curve variable with the line through α_r; this is a unicursal curve of the order $n-r$. The aggregate curve of the order $r+(n-r)$ has singular points equivalent to $\frac{1}{2}(n-1)(n-2)+1$ double points[1]; that is, the singularities are those belonging to the principal system of the second plane, together with a new double

[1] In general, if $r+r'=n$, and the curves r, r' are each unicursal, then the aggregate singularity arising from the singularities of the two curves and from their intersections, is equivalent to $\frac{1}{2}(r-1)(r-2)+\frac{1}{2}(r'-1)(r'-2)+rr'$, that is, to $\frac{1}{2}(r+r'-1)(r+r'-2)+1$, or $\frac{1}{2}(n-1)(n-2)+1$ double points.

point constituted by an intersection of the curves r, $n-r$. {Observe that the two curves have only this single intersection; viz., the remaining $r(n-r)-1$ intersections are at points $\alpha_2' + \alpha_3' \ldots + \alpha'_{n-1}$ of the principal system of the second plane.} We have thus, in the second plane, a series of curves, each of them having a new double point; viz., these are the several curves which correspond to the lines through α_r in the first figure. Each of the curves is a fixed curve r together with a variable curve $n-r$. The new double point is an intersection of the two curves; that is, it is a variable point on the curve r. The locus of the new double point is thus the curve r; therefore the curve r is part of the Jacobian of the reseau of the second plane. Since each point α_r gives a curve r, the curves in question form an aggregate curve of the order $\alpha_1 + 2\alpha_2 \ldots + (n-1)\alpha_{n-1}, = 3n-3$; viz., this is the order of the Jacobian; or, as stated, the curves r (that is, the principal counter-system of the second plane) constitute the Jacobian of the reseau of this plane.

41. The numerical systems $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are each of them a solution of the same two indeterminate equations

$$\Sigma r^2 \alpha_r = n^2 - 1, \quad \Sigma r \alpha_r = 3n - 3,$$

but not every solution of these equations is admissible; for instance, if $r > \frac{1}{2}n$, then α_r is $=0$ or 1, for $\alpha_r = 2$ would imply a curve of the order n with two r-tuple points, and the line joining these would meet the curve in more than r points; similarly, $r > \frac{2}{5}n$, α_r is $=4$ at most, for $\alpha_r = 5$ would imply a curve of the order n with five r-tuple points, and the conic through these would meet the curve in more than $2n$ points; and there are of course other like restrictions. The different admissible systems up to $n = 10$ are tabulated in Cremona's Memoir; and he has also given systems belonging to certain specified forms of n: these results are as follows:

n	2	3	4		5			6				7				
a_1	3	4	6	3	8	3	0	10	1	4	3	12	2	0	5	3
a_2		1	0	3	0	3	6	0	4	1	4	0	3	3	0	5
a_3			1	0	0	1	0	0	2	3	0	0	2	4	3	0
a_4					1	0	0	0	0	0	1	0	1	0	1	0
a_5								1	0	0	0	0	0	0	0	1
a_6												1	0	0	0	0
a_7																
a_8																
a_9																
	1	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5

n	8									9									
a_1	14	3	1	0	3	6	0	2	3	16	4	2	0	3	7	1	3	0	1
a_2	0	2	3	0	6	0	5	0	3	0	1	3	4	7	0	4	0	3	1
a_3	0	3	2	7	0	1	2	5	0	0	4	1	0	0	0	3	4	3	3
a_4	0	0	2	0	0	3	0	1	3	0	0	2	4	0	3	0	1	1	3
a_5	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	1	0
a_6	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
a_7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_8										1	0	0	0	0	0	0	0	0	0
a_9																			
	1	2	3	4	5	6	7	8	9(1)	1	2	3	4	5	6	7	8	9	10

n	10																
a_1	18	5	1	0	0	3	8	2	4	1	2	3	3	3	0	0	1
a_2	0	0	4	2	0	8	0	3	0	3	1	3	3	0	6	1	0
a_3	0	5	0	2	7	0	0	4	3	2	3	0	1	0	0	5	2
a_4	0	0	2	3	0	0	1	0	2	2	1	3	0	6	0	0	5
a_5	0	0	2	1	0	0	3	0	0	0	2	0	3	0	3	2	0
a_6	0	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0
a_7	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
a_8	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
a_9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

[1] Omitted by Cremona.

$n = p$

$a_1 =$	$2p-2$
$a_{p-1} =$	1

$n = 2p$

$a_1 =$	3
$a_2 =$	$2p-2$
$a_{p-1} =$	0
$a_p =$	0
$a_{2p-2} =$	1

$2p-2$
0
1
3
0

$n = 2p+1$

$a_1 =$	3
$a_2 =$	$2p-1$
$a_p =$	0
$a_{p+1} =$	0
$a_{2p-1} =$	1

$2p-1$
0
3
1
0

$n = 3p$

$a_1 =$	1
$a_2 =$	4
$a_3 =$	$2p-3$
$a_p =$	0
$a_{p+1} =$	0
$a_{2p-1} =$	0
$a_{3p-3} =$	1

$2p-3$
0
0
4
1
1
0

$a_1 =$	4
$a_2 =$	1
$a_3 =$	$2p-2$
$a_{p-1} =$	0
$a_p =$	0
$a_{2p} =$	0
$a_{3p-3} =$	1

$2p-2$
0
0
1
4
1
0

$n = 3p+1$

$a_1 =$	2
$a_2 =$	3
$a_3 =$	$2p-2$
$a_p =$	0
$a_{p+1} =$	0
$a_{2p} =$	0
$a_{3p-2} =$	1

$2p-2$
0
0
3
2
1
0

$a_1 =$	5
$a_3 =$	$2p-1$
$a_p =$	0
$a_{2p+1} =$	0
$a_{3p-2} =$	1

$2p-1$
0
5
1
0

$n = 3p+2$

$a_1 =$	3
$a_2 =$	2
$a_3 =$	$2p-1$
$a_p =$	0
$a_{p+1} =$	0
$a_{2p+1} =$	0
$a_{3p-1} =$	1

$2p-1$
0
0
2
3
1
0

$a_1 =$	0
$a_3 =$	5
$a_3 =$	$2p-2$
$a_{p+1} =$	0
$a_{2p} =$	0
$a_{3p-1} =$	1

$2p-2$
0
0
5
1
0

$n = 4p$

$a_1 =$	1
$a_2 =$	3
$a_3 =$	2
$a_4 =$	$2p-3$
$a_p =$	0
$a_{p+1} =$	0
$a_{2p-1} =$	0
$a_{2p} =$	0
$a_{4p-1} =$	1

$2p-3$
0
0
0
3
1
1
2
0

$a_1 =$	2
$a_3 =$	5
$a_4 =$	$2p-4$
$a_p =$	0
$a_{p+1} =$	0
$a_{3p-1} =$	0
$a_{4p-4} =$	1

$2p-4$
0
0
5
2
1
0

$a_1 =$	3
$a_2 =$	3
$a_4 =$	$2p-2$
$a_{p-1} =$	0
$a_p =$	0
$a_{2p} =$	0
$a_{4p-4} =$	1

$2p-2$
0
0
1
3
3
0

$a_1 =$	6
$a_3 =$	1
$a_4 =$	$2p-2$
$a_{p-1} =$	0
$a_p =$	0
$a_{3p} =$	0
$a_{4p-4} =$	1

$2p-2$
0
0
1
6
1
0

$n = 4p + 1$

a_1	$= 0$	$2p-3$	a_1	$= 2$	$2p-2$	a_1	$= 3$	$2p-3$	a_1	$= 7$	$2p-1$
a_2	$= 3$	0	a_2	$= 3$	0	a_3	$= 4$	0	a_4	$= 2p-1$	0
a_3	$= 3$	0	a_3	$= 1$	0	a_4	$= 2p-3$	0	a_p	$= 0$	7
a_4	$= 2p-3$	0	a_4	$= 2p-2$	0	a_p	$= 0$	4	a_{3p+1}	$= 0$	1
a_p	$= 0$	1	a_p	$= 0$	3	a_{p+1}	$= 0$	3	a_{4p-3}	$= 1$	0
a_{p+1}	$= 0$	3	a_{p+1}	$= 0$	1	a_{3p}	$= 0$	1			
a_{2p}	$= 0$	3	a_{2p}	$= 0$	2	a_{4p-3}	$= 1$	0			
a_{4p-3}	$= 1$	0	a_{2p+1}	$= 0$	1						
			a_{4p-3}	$= 1$	0						

$n = 4p + 2$

a_1	$= 0$	$2p-4$	a_1	$= 1$	$2p-2$	a_1	$= 3$	$2p-1$	a_1	$= 4$	$2p-2$
a_3	$= 7$	0	a_2	$= 3$	0	a_2	$= 3$	0	a_3	$= 3$	0
a_4	$= 2p-4$	0	a_3	$= 2$	0	a_4	$= 2p-1$	0	a_4	$= 2p-2$	0
a_{p+1}	$= 0$	7	a_4	$= 2p-2$	0	a_p	$= 0$	3	a_p	$= 0$	3
a_{3p}	$= 0$	1	a_p	$= 0$	1	a_{p+1}	$= 0$	1	a_{p+1}	$= 0$	4
a_{4p-2}	$= 1$	0	a_{p+1}	$= 0$	3	a_{2p+1}	$= 0$	3	a_{3p}	$= 0$	1
			a_{2p}	$= 0$	1	a_{4p-2}	$= 1$	0	a_{4p-2}	$= 1$	0
			a_{2p+1}	$= 0$	2						
			a_{4p-2}	$= 1$	0						

$n = 4p + 3$

a_1	$= 0$	$2p-2$	a_1	$= 9$	$2p-3$	a_1	$= 2$	$2p-1$	a_1	$= 5$	$2p-1$
a_2	$= 3$	0	a_3	$= 6$	0	a_2	$= 3$	0	a_3	$= 2$	0
a_3	$= 3$	0	a_4	$= 2p-3$	0	a_3	$= 1$	0	a_4	$= 2p-1$	0
a_4	$= 2p-2$	0	a_{p+1}	$= 0$	6	a_4	$= 2p-1$	0	a_p	$= 0$	2
a_{p+1}	$= 0$	3	a_{p+2}	$= 0$	1	a_p	$= 0$	1	a_{p+1}	$= 0$	5
a_{p+2}	$= 0$	1	a_{3p+1}	$= 0$	1	a_{p+1}	$= 0$	3	a_{3p+2}	$= 0$	1
a_{2p+1}	$= 0$	3	a_{4p-1}	$= 1$	0	a_{2p+1}	$= 0$	2	a_{4p-1}	$= 1$	0
a_{4p-1}	$= 1$	0				a_{2p+2}	$= 0$	1			
						a_{4p-1}	$= 1$	0			

42. The system $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ geometrically determines completely the system $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; it ought therefore to determine it arithmetically; that is, given the one series of numbers, we ought to be able to determine, or at least to select from the table, the other series of numbers. Cremona has shown that the two series *consist of the same numbers in the same or a different order.* By examination of the tables, it appears that there are certain columns which are single (that is, no other column contains in a different order the same numbers), others that occur in pairs, the two columns of a pair containing the same numbers in a different order. Where the column is single, it is clear that this must give as well the values of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ as of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$. Where there is a pair of columns, as far as Cremona has examined, if the one column is taken to be $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ the other column is $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; it appears, however, not to be shown that this is universally the case; viz., it is not shown but that the two columns, instead of being reckoned as a pair, might be reckoned as two separate columns, each by itself representing the values as well of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ as of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; neither is it shown that there are not, in any case, more than two columns having the same numbers in different orders. It seems, however, natural to suppose that the law, as exhibited in the tables, holds good generally; viz., that the tables contain only single columns, each giving the values as well of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ as of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; or else pairs of columns, one giving the values of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$, and the other those of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; or, say, that the partitions are either *sibi-reciprocal,* or else *conjugate.*

43. Assuming that the two systems $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are each known, there is still a question of grouping to be settled; viz., the Jacobian of the first plane consists of α_1' lines, α_2' conics, $\ldots \alpha'_{n-1}$ unicursal $(n-1)$-thics; each line, each conic, &c., passes a certain number of times through *certain* of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: but through which of them? For instance, each of the α_1' lines will pass through two of the points $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$: will these be points α_1, or points α_2, &c., or a point α_1 and a point α_2, &c.? The mere symmetry of the different groups of points determines certain conditions of the solution(1); for instance, if any particular one of the α_1' lines passes through two points α_r, then each of the α_1' lines must pass through two points α_r; and since the points α_r are symmetrical, we must in this way use all the pairs of points α_r; that is, if $\alpha_1' = \frac{1}{2}\alpha_r(\alpha_r + 1)$, but not otherwise, *it may be* that each of the α_1' lines passes through two of the points α_r. In the case of an equality $\alpha_r = \alpha_s$ we could not hereby decide whether the line passed through two points α_r or through two points α_s. So, again, if any one of the α_1' lines pass through a point α_r and a point α_s, then each of the α_1' lines must do so likewise, and we must hereby exhaust the combinations of a point α_r with a point α_s; viz., the assumed relation can only hold good if $\alpha_1' = \alpha_r\alpha_s$. Similarly, each of the α_2' conics will pass through five of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$; each of the α_2' nodal cubics will pass twice through one (have a double point there) and through six others of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: which are the points so passed through? I do not know how a general solution is to be obtained, but most of the cases within the limits of the foregoing table have

[1] It is by such considerations of symmetry that Cremona has demonstrated the before mentioned theorem of the identity of the numbers $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$.

been investigated by Cremona. The results may conveniently be stated in a tabular form; the tables exhibit in the outside upper line the values of $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$, and in the outside left-hand line the values of $\alpha_1', \alpha_2' \ldots \alpha'_{n-1}$, and they are to be read as follows: Each of the $\left\{\begin{matrix}\alpha_1' \text{ lines} \\ \alpha_2' \text{ conics} \\ \&c.\end{matrix}\right\}$ passes ( ) times through ( ) of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ respectively; the numbers in the table being those of the points passed through, and the indices in the table (index $=1$ when no index is expressed) showing the number of times of passage, that is, showing whether the point is a simple, double, triple, &c., point on the curve referred to.

44. Thus (in the tables which follow) the last of the tables $n=6$ gives the constitution of the Jacobian of the first plane, where the principal system is (3, 4, 0, 1, 0); and it is to be read:

Each of the 4 lines	passes through	1	of the points α_1 and through the point α_4;
The 1 conic	„ „	4	of the points α_2 and through the point α_4;
Each of the 3 cubics	„ „	2	of the points α_1, 4 of the points α_2, and twice through the point α_4 (that is, α_4 is a double point on each cubic).

It is hardly necessary to remark that the tables are sibi-reciprocal, or else conjugate, as appears by the outer lines of each table.

TABLE $n=2$.

	$\alpha_1 = 3$
$\alpha_1' = 3$	2

[was originally printed, 3.]

TABLE $n=3$.

	$\alpha_1 = 4$	$\alpha_2 = 1$
$\alpha_1' = 4$	1	1
$\alpha_2' = 1$	4	1

TABLES $n=4$.

	$\alpha_1 = 6$	$\alpha_2 = 0$	$\alpha_3 = 1$
$\alpha_1' = 6$	1		1
$\alpha_2' = 0$			
$\alpha_3' = 1$	6		1^2

	$\alpha_1 = 3$	$\alpha_2 = 3$	$\alpha_3 = 0$
3		2	
3	2	3	
0			

TABLES $n = 5$.

	$a_1 = 8$	$a_2 = 0$	$a_3 = 0$	$a_4 = 1$
$a_1' = 8$	1			1
$a_2' = 0$				
$a_3' = 0$				
$a_4' = 1$	8			1^3

	$a_1 = 3$	$a_2 = 3$	$a_3 = 1$	$a_4 = 0$
3		1	1	
3	1	3	1	
1	3	3	1^2	
0				

	$a_1 = 0$	$a_2 = 6$	$a_3 = 0$	$a_4 = 0$
0				
6		5		
0				
0				

TABLES $n = 6$.

	$a_1 = 10$	$a_2 = 0$	$a_3 = 0$	$a_4 = 0$	$a_5 = 1$
$a_1' = 10$	1				1
$a_2' = 0$					
$a_3' = 0$					
$a_4' = 0$					
$a_5' = 1$	10				1^4

	$a_1 = 1$	$a_2 = 4$	$a_3 = 2$	$a_4 = 0$	$a_5 = 0$
1			2		
4		3	2		
2	1	4	1^2,1*		
0					
0					

	$a_1 = 4$	$a_2 = 1$	$a_3 = 3$	$a_4 = 0$	$a_5 = 0$
$a_1' = 3$			2		
$a_2' = 4$	1	1	3		
$a_3' = 0$					
$a_4' = 1$	4	1	3^2		
$a_5' = 0$					

	$a_1 = 3$	$a_2 = 4$	$a_3 = 0$	$a_4 = 1$	$a_5 = 0$
4	1			1	
1		4		1	
3	2	4		1^2	
0					
0					

* Read, "Each of the two cubics passes through the point a_1, the four points a_2, and, (1^2, 1), twice through one and once through the other of the points a_2."

45. It is to be remarked upon the tables—first, as regards the lines: if we add the numbers in each line, reckoning m^p as mp, (that is, each multiple point, according to the number of branches through it,) the sums for the successive lines are 2, 5, 8, 11, 14, &c.; that is, each line passes through 2 points, each conic through 5 points, each cubic through 8 points, each quartic through 11 points, &c. But if we add the numbers reckoning m^p as $m \,.\, \frac{1}{2}p(p+1)$, (that is, each multiple point according to its effect in the determination of the curve,) then the sums are 2, 5, 9, 14, 20, &c., that is, all the curves are completely determined, viz., the line by 2 conditions, the conic by 5 conditions, the cubic by 9 conditions, &c. Secondly, as regards the columns, if for any column, reckoning m^p as mp, we multiply each number by the corresponding outside left-hand number, add, and divide the sum by the outside number at the head of the column, the successive results are 2, 5, 8, 11, 14, &c.; this merely expresses the known circumstance that the Jacobian passes $3r-1$ times through each point α_r.

46. The analogous tables showing the passage of the Jacobian through the principal system, in the solutions belonging to certain special forms of n, are

TABLE $n = p$.

	$\alpha_1 = 2p-2$	$\alpha_{p-1} = 1$
$\alpha_1' = 2p-2$	1	1
$\alpha'_{p-1} = 1$	$2p-2$	1^{p-2}

TABLES $n = 2p$.

	$\alpha_1 = 3$	$\alpha_2 = 2p-2$	$\alpha_{p-1} = 0$	$\alpha_p = 0$	$\alpha_{2p-2} = 1$
$\alpha_1' = 2p-2$	1				1
$\alpha_2' = 0$					
$\alpha'_{p-1} = 1$		$2p-2$			1^{p-2}
$\alpha_p' = 3$	2	$2p-2$			1^{p-1}
$\alpha'_{2p-2} = 0$					

	$\alpha_1 = 2p-2$	$\alpha_2 = 0$	$\alpha_{p-1} = 1$	$\alpha_p = 3$	$\alpha_{2p-2} = 0$
$\alpha_1' = 3$				2	
$\alpha_2' = 2p-2$	1		1	3	
$\alpha'_{p-1} = 0$					
$\alpha_p' = 0$					
$\alpha'_{2p-2} = 1$	$2p-2$		1^{p-2}	3^{p-1}	

TABLES $n = 2p+1$.

	$\alpha_1 = 3$	$\alpha_2 = 2p-1$	$\alpha_p = 0$	$\alpha_{p+1} = 0$	$\alpha_{2p-1} = 1$
$\alpha_1' = 2p-1$		1			1
$\alpha_2' = 0$					
$\alpha_p' = 3$	1	$2p-1$			1^{p-1}
$\alpha'_{p+1} = 1$	3	$2p-1$			1^p
$\alpha'_{2p-1} = 0$					

	$\alpha_1 = 2p-1$	$\alpha_2 = 0$	$\alpha_p = 3$	$\alpha_{p+1} = 1$	$\alpha_{2p-1} = 0$
$\alpha_1' = 3$	1			1	
$\alpha_2' = 2p-1$	1		3	1	
$\alpha_p' = 0$					
$\alpha'_{p+1} = 0$					
$\alpha'_{2p-1} = 1$	$2p-1$		3^{p-1}	1^p	

TABLES $n = 3p$.

	$\alpha_1 = 1$	$\alpha_2 = 4$	$\alpha_3 = 2p-3$	$\alpha_p = 0$	$\alpha_{p+1} = 0$	$\alpha_{2p-1} = 0$	$\alpha_{3p-3} = 1$
$\alpha_1' = 2p-3$			1				1
$\alpha_2' = 0$							
$\alpha_3' = 0$							
$\alpha_p' = 4$		3	$2p-3$				1^{p-1}
$\alpha'_{p+1} = 1$	1	4	$2p-3$				1^p
$\alpha'_{2p-1} = 1$	1	4	$(2p-3)^2$				1^{2p-3}
$\alpha'_{3p-3} = 0$							

	$a_1 = 2p-3$	$a_2 = 0$	$a_3 = 0$	$a_p = 4$	$a_{p+1} = 1$	$a_{2p-1} = 1$	$a_{3p-3} = 0$
$a_1' = 1$					1	1	
$a_2' = 4$				3	1	1	
$a_3' = 2p-3$	1			4	1	1^2	
$a_p' = 0$							
$a'_{p+1} = 0$							
$a'_{2p-1} = 0$							
$a'_{3p-3} = 1$	$2p-3$			4^{p-1}	1^p	1^{2p-3}	

	$a_1 = 4$	$a_2 = 1$	$a_3 = 2p-2$	$a_{p-1} = 0$	$a_p = 0$	$a_{2p} = 0$	$a_{3p-3} = 1$
$a_1' = 2p-2$			1				1
$a_2' = 0$							
$a_3' = 0$							
$a'_{p-1} = 1$			$2p-2$				1^{p-2}
$a_p' = 4$	1	1	$2p-2$				1^{p-1}
$a'_{2p} = 1$	4	1	$(2p-2)^2$				1^{2p-2}
$a'_{3p-3} = 0$							

	$\alpha_1 = 2p-2$	$\alpha_2 = 0$	$\alpha_3 = 0$	$\alpha_{p-1} = 1$	$\alpha_p = 4$	$\alpha_{2p} = 1$	$\alpha_{3p-3} = 0$
$\alpha_1' = 4$					1	1	
$\alpha_2' = 1$					4	1	
$\alpha_3' = 2p-2$	1			1	4	1^2	
$\alpha'_{p-1} = 0$							
$\alpha_p' = 0$							
$\alpha'_{2p} = 0$							
$\alpha'_{3p-3} = 1$	$2p-2$			1^{p-2}	4^{p-1}	1^{2p-2}	

47. The before mentioned theorem, that $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are the same series of numbers, of course implies $\Sigma\alpha_r = \Sigma\alpha_r'$; this relation Cremona demonstrates independently, by consideration of the pencil of curves $(aX + bY + cZ) + \theta(a_1X + b_1Y + c_1Z) = 0$, ($\theta$ a variable parameter,) which corresponds in the first plane to the pencil of lines $(ax' + by' + cz') + \theta(a_1x' + b_1y' + c_1z') = 0$, which pass through a fixed point $(ax' + by' + cz' = 0$, $a_1x' + b_1y' + c_1z' = 0)$ in the second figure. In general, in the pencil $U + \theta V = 0$ (U, V given functions of the order n) there are $3(n-1)^2$ values of θ, each giving a nodal curve. But in the present case each of the curves $U = 0$, $V = 0$ has multiple points at the principal points α_r of the first plane: the question is to obtain the number of values which give a curve having one new double point; and this is found to be $= 3(n-1)^2 - \Sigma(r-1)(3r+1)\alpha_r$. We have $\Sigma r^2\alpha_r = n^2 - 1$, $\Sigma r\alpha_r = 3n - 3$; or, substituting, the value of θ is $= \Sigma\alpha_r$. But the curves which have an additional double point are those which correspond to the lines which in the second figure pass through one of the principal points α_r'; viz., these are the lines drawn from the point $(ax' + by' + cz' = 0$, $a_1x' + b_1y' + c_1z' = 0)$ to the several principal points α_r'; and the number of them is $= \Sigma\alpha_r'$. We have thus the required relation $\Sigma\alpha_r = \Sigma\alpha_r'$.

The Quadric Transformation between Two Planes.

48. This is of course given by what precedes. The principal system in each plane is a set of three points; and the Jacobian of the same plane is the set of three lines joining each pair of points; that is, the three lines of either plane are the principal counter-system of the other plane. But to give the analytical investigation

directly: taking the coordinates (x, y, z) to refer to the principal system of the first plane (viz., taking the three points to be the vertices of the triangle formed by the lines $x=0$, $y=0$, $z=0$), then $X=0$, $Y=0$, $Z=0$ being conics through the three points, the functions X, Y, Z will be each of them of the form $fyz+gzx+hxy$; x', y', z' being proportional to three such functions, there will be linear functions of x', y', z' proportional to yz, zx, xy; or taking these linear functions of the original (x', y', z') for the coordinates (x', y', z') of a point in the second plane, the formulæ of transformation will be $x' : y' : z' = yz : zx : xy$, and we have then conversely $x : y : z = y'z' : z'x' : x'y'$; that is, the formulæ for the transformation in question are

$$x' : y' : z' = yz : zx : xy, \text{ and } x : y : z = y'z' : z'x' : x'y'.$$

We at once verify *à posteriori* that the Jacobian in the first plane is $xyz=0$, and that in the second plane is $x'y'z'=0$.

The equations may be written

$$x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}, \text{ and } x : y : z = \frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'},$$

or, if we please, $xx' = yy' = zz'$; the transformation is thus given as an inverse transformation.

{49. With respect to the metrical interpretation and actual construction of the transformation, it is to be observed that if x, y, z be taken to be proportional (not to given multiples of the perpendicular distances, but) to the perpendicular distances of P from the sides of the triangle in the first plane, and similarly x', y', z' to be proportional to the perpendicular distances of P' from the sides of the triangle in the second plane, then in general the equations of transformation must be written, not as above, but in the form $\frac{xx'}{f} = \frac{yy'}{g} = \frac{zz'}{h}$, involving arbitrary multipliers $f : g : h$. We may imagine in the second plane a point P'' determined by coordinates (x'', y'', z''),—the same coordinates as (x', y', z'), that is, proportional to the perpendicular distances of P'' from the sides of the triangle in the second plane,—which point P'' corresponds homographically to P in such wise that $\frac{x}{f} : \frac{y}{g} : \frac{z}{h} = x'' : y'' : z''$. We have then, in the second plane, the two points P', P'' corresponding to each other in such wise that $x'x'' = y'y'' = z'z''$; and either of these points being given, the other can at once be constructed; viz., it is obvious that, joining P', P'' with any vertex, say A', of the triangle $A'B'C'$, the lines $A'P'$, $A'P''$ are equally inclined to the bisectors of the angle A'; and consequently, P' being given, we have the three lines $A'P''$, $B'P''$, $C'P''$ intersecting in a common point P'', which is therefore determined by means of any two of these lines. We have thus a geometrical construction of the transformation between P and P'.}

50. The analysis assumes that the principal points A, B, C of the first figure are three distinct points; but they may two of them, or all three, coincide. In the first case, say if B, C coincide, the line BC is still to be regarded as having a definite direction; and taking $x=0$ for this line, $y=0$ for the line joining A with

(BC), and $z=0$ an arbitrary line through A, the functions X, Y, Z will be each of them of the form $by^2+2gzx+2hxy$; and replacing, as before, the original x', y', z' by linear functions of these quantities, these linear functions being taken for the coordinates (x', y', z'), we may write $x' : y' : z' = y^2 : xy : xz$. Forming the converse system, the equations for the transformation are

$$x' : y' : z' = y^2 : xy : xz, \text{ and } x : y : z = y'^2 : x'y' : x'z',$$

so that the points A', B', C' in the second plane are related as the points in the first plane; viz., B', C' coincide, the line $B'C'$ being definite.

It is easy to verify that the Jacobian in the first plane is $xy^2=0$, and the Jacobian in the second plane is $x'y'^2=0$.

51. Secondly, if A, B, C all coincide, these being however consecutive points on a curve of finite curvature, or say on a conic; then, taking $x=0$ for the tangent at (ABC), $z=0$ for any other tangent, and $y=0$ for the chord of contact, the functions X, Y, Z will be of the form $ax^2+b(y^2-zx)+2hxy$; whence we may write $x' : y' : z' = x^2 : xy : y^2-xz$. Forming the converse equations, the equations of transformation are

$$x' : y' : z' = x^2 : xy : y^2-xz, \text{ and } x : y : z = x'^2 : x'y' : y'^2-x'z';$$

so that the points A', B', C' in the second plane are related as those of the first plane; viz., they are the consecutive points of a curve of continuous curvature.

We may verify that the Jacobian of the first plane is $x^3=0$, and the Jacobian of the second plane $x'^3=0$.

The Lineo-linear Transformation between Two Planes.

52. We have two equations of the form

$$(a\ ,\ldots\between x,\ y,\ z\between x',\ y',\ z')=0,$$
$$(a_1,\ldots\between x,\ y,\ z\between x',\ y',\ z')=0;$$

writing these in the form

$$P\,x'+Q\,y'+R\,z'=0,$$
$$P_1x'+Q_1y'+R_1z'=0,$$

where $(P,\ Q,\ R,\ P_1,\ Q_1,\ R_1)$ are linear functions of $(x,\ y,\ z)$, we have

$$x',\ y',\ z' \text{ proportional to } \left\| \begin{matrix} P\ , & Q\ , & R \\ P_1, & Q_1, & R_1 \end{matrix} \right\|,$$

that is to $X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are conics each passing through the same three points in the first plane.

And conversely, writing the equations in the form

$$P'x+Q'y+R'z=0,$$
$$P_1'x+Q_1'y+R_1'z=0,$$

where $(P', Q', R', P_1', Q_1', R_1')$ are linear functions of (x', y', z'), we have

$$x, y, z \text{ proportional to } \left\| \begin{matrix} P', & Q', & R' \\ P_1', & Q_1', & R_1' \end{matrix} \right\|,$$

that is to X', Y', Z', where $X'=0$, $Y'=0$, $Z'=0$ are conics each passing through the same three points in the second plane.

53. The lineo-linear transformation is thus the same thing as the quadric transformation. It is, moreover, clear that the equations must, by linear transformations on the two sets of variables respectively, and by linear combination of the two equations, be reducible into forms giving the before-mentioned values of $x : y : z$ and $x' : y' : z'$ respectively. Thus, in the general case, where in each plane the three points are distinct points, the lineo-linear equations will be reducible to

$$xx' - yy' = 0, \quad xx' - zz' = 0;$$

in the case where B, C in the first plane, and B', C' in the second plane respectively coincide, the forms will be

$$xx' - yy' = 0, \quad yz' - y'z = 0;$$

and in the case where A, B, C in the first plane, and A', B', C' in the second plane respectively coincide, the forms will be

$$xy' - yx' = 0, \quad xz' - yy' + zx' = 0.$$

The determination of the actual formulæ for these reductions would, it is probable, give rise to investigations of considerable interest.

The General Rational Transformation between Two Planes (resumed).

54. Consider, as above, the first plane or figure with a principal system $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$, and the second plane or figure with a principal system $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$. To a line in the second plane there corresponds in the first plane a curve of the order n passing 1 time through each of the points α_1, 2 times through each of the points α_2, 3 times through each of the points α_3, &c.; or, as we may write this:

First figure.

Points	α_1	α_2	α_3	$\ldots$	α_{n-1}	
	1	2	3		$n-1$	curve order n
	1	2	3		$n-1$	
	1	⋮	3		$n-1$	
	⋮		⋮		⋮	

Second figure.

Points	α_1'	α_2'	α_3'	$\ldots$	α'_{n-1}	
	0	0	0		$n-1$	curve order 1
	0	0	0		$n-1$	
	⋮	⋮	0		⋮	
			⋮			

viz., the 1's denote the number of times which the curve of the order n passes through the several points α_1 respectively; the 2's the number of times which the curve passes through the several points α_2 respectively; and so on.

55. We may, in the second figure, in the place of a line consider a curve of the order k'. If the equation hereof is $(*\between x', y', z')^{k'}=0$, then the corresponding curve in the first figure is $(*\between X, Y, Z)^{k'}=0$; viz., this is a curve of the order $k=nk'$. If, however, the curve in the second figure passes once or more times through all or any of the points $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$, then there will be a depression in the order of the corresponding curve in the first figure; and, moreover, this curve will pass a certain number of times through all or some of the points $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{n-1}$. The diagram of the correspondence will be:

First figure.

$$\begin{array}{cccc} \alpha_1, & \alpha_2, & \alpha_3, \ldots & \alpha_{n-1} \\ \hline a_1 & a_2 & a_3 & a_{n-1} \\ b_1 & b_2 & b_3 & b_{n-1} \\ c_1 & c_2 & \vdots & \vdots \\ \vdots & \vdots & & \end{array} \Bigg\} \text{ curve order } k$$

Second figure.

$$\begin{array}{cccc} \alpha_1', & \alpha_2', & \alpha_3', \ldots & \alpha'_{n-1} \\ \hline a_1' & a_2' & a_3' & a'_{n-1} \\ b_1' & b_2' & b_3' & b'_{n-1} \\ c_1' & \vdots & \vdots & c'_{n-1} \\ \vdots & & & \vdots \end{array} \Bigg\} \text{ curve order } k'$$

where $a_1, b_1, c_1 \ldots$ denote the number of times that the curve of the order k passes through the several points α_1 respectively, (viz., the number of the letters $a_1, b_1, c_1 \ldots$ is $=\alpha_1$, any or all of them being zeros,) $a_2, b_2, c_2 \ldots$ the number of times that the curve passes through the several points α_2 respectively, (viz., the number of the letters $a_2, b_2, c_2 \ldots$ is $=\alpha_2$, any or all of them being zeros,) and so on; and the like for the curve in the second figure.

56. By what precedes, it is easy to see that, if the curve k' passes through a point α_1', then the curve k throws off a line, and the depression of order is $=1$; so, if the curve passes 2 times, 3 times, ... or α_1' times through the point in question, then the curve throws off the line repeated 2 times, 3 times, ... α_1' times, or the depression of order is $=2, 3, \ldots$ or α_1'; and the like for each of the points α_1'; so that, writing for shortness $\alpha_1'+b_1'+c_1'+\ldots=\Sigma\alpha_1'$, the depression of order on account of the passages through the several points α_1 is $=\Sigma\alpha_1'$. Similarly, for each time of passage through a point α_2', there is thrown off a conic; or if $\alpha_2'+b_2'+\ldots=\Sigma\alpha_2'$, then the depression of order is $=2\Sigma\alpha_2'$, and so on; and the like for the figure in the other plane; and we thus arrive at the equations

$$k=k'n-\Sigma(\alpha_1'+2\alpha_2'+3\alpha_3'\ldots+\overline{n-1}\,\alpha'_{n-1})$$
$$k'=kn-\Sigma(\alpha_1+2\alpha_2+3\alpha_3\ldots+\overline{n-1}\,\alpha_{n-1}).$$

57. The simplest case is when the curve k' does not pass through any of the points $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$. We have then

$$\alpha_1'=b_1'=c_1'\ldots=0,\ \alpha_2'=b_2'\ldots=0,\ \ldots\ldots\ \alpha'_{n-1}=b'_{n-1}\ldots=0;$$

consequently $k=k'n$. And, moreover, it is easy to see that

$$\alpha_1=b_1\ldots=k',\quad \alpha_2=b_2\ldots=2k',\ \ldots\ldots\ \alpha_{n-1}=b_{n-1}\ldots=(n-1)\,k';$$

so that the correspondence is:

First Figure.					Second Figure.				
α_1,	α_2,	α_3, ...	α_{n-1}		α_1',	α_2',	α_3', ...	α'_{n-1}	
k'	$2k'$	$3k'$	$(n-1)k'$	curve order $k = nk'$	0	0	0	0	curve order k'.
k'	$2k'$	$3k'$	$(n-1)k'$		0	0	0	0	
⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	

We have

$$\Sigma a_1 = k'\alpha_1, \quad \Sigma a_2 = 2k'\alpha_2, \ \ldots\ldots \ \Sigma a_1(a_1 - 1) = k'(k'-1)\alpha_1, \ \&c.,$$

and the formulæ for k, k' become

$$k = k'n, \quad k' = kn - k'\{\alpha_1 + 4\alpha_2 \ldots + (n-1)^2\alpha_{n-1}\};$$

viz., the second equation is here $k' = kn - k'(n^2-1)$; that is, $k'n^2 = kn$, agreeing, as it should do, with the first equation.

58. Moreover, the deficiency-relation is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\tfrac{1}{2}\{k'(k'-1) + 2k'(2k'-1)\ldots + \overline{n-1}\,k'(\overline{n-1}\,k'-1)\} = \tfrac{1}{2}(k'-1)(k'-2);$$

or, what is the same thing, this is

$$\begin{aligned}(nk'-1)(nk'-2) - (k'-1)(k'-2) = \ & k'^2\{\alpha_1 + 4\alpha_2 \ldots + (n-1)^2\alpha_{n-1}\}. \\ & - k'\{\alpha_1 + 2\alpha_2 \ldots + (n-1)\alpha_{n-1}\}.\end{aligned}$$

The right-hand side is

$$k'^2(n^2-1) - k'(3n-3) = (n-1)\{(\overline{n+1}\,k'^2 - 3k')\},$$

and we have thus the identical equation

$$(nk'-1)(nk'-2) - (k'-1)(k'-2) = (n-1)k'\{(\overline{n+1}\,k' - 3\}.$$

59. It should be possible, when the nature of the correspondence between the two planes is completely given, to express each of the numbers a_1, b_1, $c_1, \ldots a_{n-1}$, $b_{n-1}, \ldots$ in terms of k', a_1', b_1', $c_1', \ldots a'_{n-1}$, $b'_{n-1}, \ldots$; and reciprocally each of the numbers a_1', b_1', $c_1', \ldots a'_{n-1}$, $b'_{n-1}, \ldots$ in terms of k, a_1, b_1, $c_1, \ldots a_{n-1}$, $b_{n-1}, \ldots$; thus completing a system of relations between the two sets

$$(k, a_1, b_1, \ldots a_{n-1}, b_{n-1}, \ldots), \qquad (k', a_1', b_1', \ldots a'_{n-1}, b'_{n-1}, \ldots);$$

but even if the theory was known, there would be considerable difficulty in forming a proper algorithm for the expression of these relations.

60. The two curves must have each of them the same deficiency. It is to be noticed, that if the curve in the first plane passes any number of times through a point P, which is not one of the points α_1, α_2, $\alpha_3, \ldots$ or α_{n-1}, then the corresponding curve in the second plane will pass the same number of times through the corresponding point P', which point will not be one of the points α_1', $\alpha_2', \ldots \alpha'_{n-1}$. The

points P, P' will therefore contribute equal values to the deficiencies of the two curves respectively; so that, in equating the two deficiencies, we may disregard P, P', and attend only to the points $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ of the first plane, and $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$ of the second plane. The required relation thus is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\, \tfrac{1}{2}\{a_1(a_1-1) + a_2(a_2-1)\ldots + a_{n-1}(a_{n-1}-1)\}$$
$$= \tfrac{1}{2}(k'-1)(k'-2) - \Sigma\, \tfrac{1}{2}\{a_1'(a_1'-1) + a_2'(a_2'-1)\ldots + a'_{n-1}(a'_{n-1}-1)\}.$$

61. In the case of the quadric transformation $n=2$, we have in the first plane the three points α_1, say these are A, B, C; and in the second plane the three points α_1', say these are A', B', C'. And if in the first plane the curve of the order k passes a, b, c times through the three points respectively, and in the second plane the corresponding curve of the order k' passes a', b', c' times through the three points respectively, then it is easy to obtain

$k' = 2k - a - b - c,$	$k = 2k' - a' - b' - c',$
$a' = k - b - c,$	$a = k' - b' - c',$
$b' = k - c - a,$	$b = k' - c' - a',$
$c' = k - a - b.$	$c = k' - a - b'.$

The Quadric Transformation any number of times repeated.

62. We may successively repeat the quadric transformation according to the type:

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A', B', C'		
	D', E', F'	D'', E'', F''	
		G'', H'', I''	G''', H''', I'''

viz., in the transformation between the first and second figures, the principal systems are ABC and $A'B'C'$ respectively; in that between the second and third figures, they are $D'E'F'$ and $D''E''F''$ respectively; in that between the third and fourth figures, they are $G''H''I''$ and $G'''H'''I'''$; and so on. And it is then easy to see that between the first and any subsequent figure we have a rational transformation of the order 2 for the second figure, 4 for the third figure, 8 for the fourth figure, and so on.

63. But to further explain the relation, we may complete the diagram, by taking, in the transformation between the second and third figures, A'', B'', C'' to correspond to A', B', C'; similarly, in that between the third and fourth, A''', B''', C''' to correspond to A'', B'', C''; and D''', E''', F''' to correspond to D'', E'', F''. And so in the transformation between the second and third figure, we may make G', H', I'

correspond to G'', H'', I'', and between the first and second figures make D, E, F correspond to D', E', F', and G, H, I to G', H', I', the diagram being thus:

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A', B', C'	A'', B'', C''	A''', B''', C'''
D, E, F	D', E', F'	D'', E'', F''	D''', E''', F'''
G, H, I	G', H', I'	G'', H'', I''	G''', H''', I'''

Observe that in the principal systems (for instance, A, B, C and A', B', C') the points A, B, C correspond, not to the points A', B', C', but to the lines $B'C'$, $C'A'$, $A'B'$ respectively; and so in the other case.

64. Consider now a line in the first figure: there corresponds hereto in the second figure a conic through the points A', B', C'; and to this conic there corresponds in the third figure a quartic curve passing through each of the points A'', B'', C'' once, and through each of the points D'', E'', F'' twice. And conversely, to a line in the third figure corresponds in the second figure a conic through the points D', E', F'; and hereto in the first figure a quartic through the points D, E, F, once and through the points A, B, C twice; that is, we have between the first and third figures a quartic transformation wherein $\alpha_1 = \alpha_2 = 3$ and $\alpha_1' = \alpha_2' = 3$, or say a quartic transformation $3_1 3_2$ and $3_1 3_2$. In like manner, passing to the fourth figure, to a line in the first figure corresponds in the fourth figure an octic curve passing through A''', B''', C''' once, through D''', E''', F''' twice, and through G''', H''', I''' four times; and conversely, to a line in the fourth figure there corresponds in the first figure an octic curve passing through the points G, H, I once, the points D, E, F twice, and the points A, B, C four times; that is, between the first and fourth figures we have an octic transformation, wherein $\alpha_1 = \alpha_2 = \alpha_4 = 3$, $\alpha_1' = \alpha_2' = \alpha_4' = 3$, or say a transformation, order 8, of the form $3_1 3_2 3_4$ and $3_1 3_2 3_4$. And so between the first and fifth figures there is a transformation, order 16, of the form $3_1 3_2 3_4 3_8$ and $3_1 3_2 3_4 3_8$.

65. It is, moreover, easy to find the Jacobians or counter-systems in the several transformations respectively. Thus, in the transformation between the first and second figures, in the second figure the Jacobian consists of 3 lines such as $B'C'$ (viz., these are, of course, the lines $B'C'$, $C'A'$, $A'B'$). Hence, in the transformation between the first and third figures, the Jacobian in the third figure consists of

$$\begin{array}{ll} 3 \text{ conics} & B''C''(D''E''F''), \\ 3 \text{ lines} & D''E''; \end{array}$$

viz., one of the conics is that through the five points B'', C'', D'', E'', F'', one of the lines that through the two points D'', E''. And so in the fourth figure, the Jacobian consists of

$$\begin{array}{lll} 3 \text{ quartics} & B'''C'''(D'''E'''F''')_1 & (G'''H'''I''')_2, \\ 3 \text{ conics} & D'''E''' & (G'''H'''I''')_1, \\ 3 \text{ lines} & & G'''H'''; \end{array}$$

viz., one of the quartics passes through B''', C'''; through D''', E''', F'''' each once; and through G''', H''', I''' each twice. And so in the fifth figure the Jacobian consists of

3 octics	$B''''C''''$	$(D''''E''''F'''')_1$	$(G''''H''''I'''')_2$	$(J''''K''''L'''')_4$,
3 quartics		$D''''E''''$	$(G''''H''''I'''')_1$	$(J''''K''''L'''')_2$,
3 conics			$G''''H''''$	$(J''''K''''L'''')_1$,
3 lines				$J''''K''''$,

and so on.

66. The conditions are in each case sufficient for the determination of the curve. This depends on the numerical relation

$$4+3\{1\,.\,2+2\,.\,3+4\,.\,5+8\,.\,9\ldots+2^\theta(2^\theta+1)\}=2^{\theta+1}(2^{\theta+1}+3).$$

The term in { } is

$$\begin{array}{l} 1+4+16\ldots+2^{2\theta} \\ +1+2+\ \ 4\ldots+2^\theta, \end{array}$$

that is

$$\frac{2^{2\theta+2}-1}{2^2-1}+\frac{2^{\theta+1}-1}{2-1},$$

which is

$$\begin{aligned} &=\tfrac{1}{3}[2^{2\theta+2}-1+3(2^{\theta+1}-1)], \\ &=\tfrac{1}{3}[2^{2\theta+2}+3\,.\,2^{\theta+1}-4]; \end{aligned}$$

and the relation is thus identically true.

67. Conversely, in the transformation between the first figure and the several other figures respectively, the Jacobian of the first figure is

3 lines	AB; and so			for order 2, between first and second figures;
3 conics	DE	$(ABC)_1$		for order 4, between first and third figures;
3 lines	AB			
3 quartics	GH	$(DEF)_1$	$(ABC)_2$	for order 8, between first and fourth figures;
3 conics	DE	$(ABC)_1$		
3 lines	AB			
3 octics	JK	$(GHI)_1$	$(DEF)_2$	$(ABC)_4$
3 quartics	GH	$(DEF)_1$	$(ABC)_2$	for order 16, between first and fifth figures;
3 conics	DE	$(ABC)_1$		
3 lines	AB			

and so on.

Special Cases—Reduction of the General Rational Transformation to a Series of Quadric Transformations.

68. It was remarked by Mr Clifford that any Cremona-transformation whatever may be obtained by this method of repeated quadric transformations, if only the principal systems, instead of being completely arbitrary, are properly related to each other. To take the simplest instance; suppose that we have

First figure.	Second figure.	Third figure.
A, B, C	A', B', C'	B'', C''
	$\parallel$	
E, F	D', E', F'	D'', E'', F''

viz., in the transformation between the first and second figures, we have the principal systems ABC and $A'B'C'$ (arbitrary as before); but in the transformation between the second and third figures, the principal systems are $D'E'F'$ and $D''E''F''$, where D', instead of being arbitrary, coincides with A'. And we then have B'', C'' in the third figure corresponding to B', C' in the second figure, and E, F in the first figure corresponding to E', F' in the second figure. This being so, to a line in the first figure corresponds in the second figure a conic through A', B', C'. But $A' = D'$; viz., this conic passes through a point D' of the principal system of the second figure, in regard to the transformation between the second and third figures. That is, (k, a, b, c referring to the second figure, and k', a', b', c' to the third figure, $k=2$, $a=1$, $b=0$, $c=0$, and therefore $k'=3$, $a'=2$, $b'=1$, $c'=1$,) corresponding to the conic we have in the third figure a curve, order 3 (cubic curve), passing twice through D'', but once through E'' and F'' respectively; this cubic curve passes also through the points B'', C'' which correspond to B', C' respectively; that is,

cubic passes through E'', F'', B'', C'' each 1 time
" " D'' 2 times;

or, corresponding to a line in the first figure, we have in the third figure a curve, order 3, passing through four fixed points each 1 time, and through one fixed point 2 times. That is, we have $n=3$, $\alpha_1'=4$, $\alpha_2'=1$. And in the same manner, to a line in the third figure there corresponds in the first figure a cubic through four fixed points (viz., B, C, E, F) each 1 time, and through one fixed point, A, 2 times; so that also $\alpha_1=4$, $\alpha_2=1$. The transformation is thus of the order 3, and the form $4_1 1_2$ and $4_1 1_2$ (this is in fact the only cubic transformation; see the Tables, *ante*, No. 41).

69. Mr Clifford has also devised a very convenient algorithm for this decomposition of a transformation of any order into quadric transformations. The quadric transformation is denoted by [3], the cubic transformation by [41], the quartic transformations by [601], [330], the quintic ones by [8001], [3310], [0600], and so on; see the Tables just referred to. (This is substantially the same as a notation employed above, the zeros enabling the omission of the suffixes; viz., $[8001] = 8_1 1_4$; and so in other cases.)

70. The foregoing result is represented thus $[4, 1] = [3\mathfrak{C}0, 0, 1)$, which I proceed to explain. Consider in the first figure a line; the symbol [3] denotes that in the second figure we have a conic with three points (α_1'). We are about to apply to this a quadric transformation; (0, 0, 0) would denote that the three points of the principal system in the second figure were all of them arbitrary; (0, 0, 1) that one of these points was a point α_1'; (0, 1, 1) that two of them were points α_1'; (1, 1, 1) that all three of them were points α_1; (0, 0, 2) would denote that one of the points was a point α_2'; only in the present case we can have no such symbol, by reason that there are no points α_2'. Hence $[3\mathfrak{C}001)$ denotes that the conic has applied to it a quadric transformation such that, in the transformation thereof, one point of the principal system coincides with one of the points (α_1') on the conic. To [3], *quà* quadric transformation, belongs the number 2; and from 2, (001) we derive 3, (112), [in general k, (a, b, c) gives k', (a', b', c'), where $k' = 2k - a - b - c$, $a' = k - b - c$, $b' = k - c - a$, $c' = k - a - b$]. $k = 2$ corresponds to a symbol [3] of one number, $k' = 3$ to a symbol of two numbers; viz., we change [3] into [30]; we then, in the symbols (112) and (001), consider the frequencies of the several numbers 1, 2, ... taking those in the first symbol as positive, and those in the second symbol as negative; or, what is the same thing, representing the frequency as an index, we have $1^2 2^1$, 1^{-1}; or, combining, $1^{2-1} 2^1$; these indices are then added on to the numbers of [30]; viz., the index of 1 to the first number, the index of 2 to the second number (and, in the case of more numbers, so on): [30] is thus converted into [41], and we have the required equation

$$[41] = [3\mathfrak{C}001),$$

where the *rationale* of this algorithmic process appears by the explanation, *ante*, No. 68.

71. As another example take

$$[8001] = [601\mathfrak{C}003).$$

To [601], *quà* quartic transformation, belongs the number 4; and from 4, (003) we form 5, (114); where the 5 indicates that [601] is to be changed into [6010]; then (114), (003), writing them in the form $1^2 2^0 3^{-1} 4^1$, show that to the numbers of [6010] we are to add 2, 0, -1, 1; thus changing the symbol into [8001], so that we have the required relation.

72. Mr Clifford calculated in this way the following table, showing how any transformation of an order not exceeding 8 can be expressed by means of a series of quadric transformations; the symbols Cr. 3, Cr. 4 . 1; 4 . 2, &c., refer to the order and number of Cremona's tables, *ante*, No. 41.

Cr. 3 . $= [\ 41] = [3\mathfrak{C}001)$,

Cr. 4 . 1 $= [601] = [41\mathfrak{C}002) = [3\mathfrak{C}001\mathfrak{C}002)$,

4 . 2 $= [330] = [\ 3\mathfrak{C}000) = [41\mathfrak{C}011) = [3\mathfrak{C}001\mathfrak{C}011)$,

Cr. 5 . 1 = [8001]	= [601ʒ003)	= [3ʒ001ɣ002ɣ003),		
5 . 2 = [3310]	= [41ʒ001)	= [3ʒ001ɣ001),		
5 . 3 = [0600]	= [330ʒ111)	= [3ʒ000ɣ111)	= [3ʒ001ɣ011ɣ111),	
Cr. 6 . 1 = [10,0001]	= [8001ʒ004)	= [3ʒ001ɣ002ɣ003ɣ004),		
6 . 2 = [14200]	= [330ʒ011)	= [3ʒ000ɣ011)	= [3ʒ001ɣ011ɣ011),	
6 . 3 = [41300]	= [601ʒ011)	= [3ʒ001ɣ002ɣ011)	= [3ʒ001ɣ000),	
	= [3310ʒ022)	= [3ʒ001ɣ001ɣ022)		
6 . 4 = [34010]	= [330ʒ002)	= [3ʒ000ɣ002),		
	= [3310ʒ013)	= [3ʒ001ɣ001ɣ013),		
Cr. 7 . 1 = [12,00001]	= [10,0001ʒ005)	= [3ʒ001ɣ002ɣ003ɣ004ɣ005),		
7 . 2 = [330ʒ001)	= [3ʒ000ɣ001)	= [232100],		
7 . 3 = [034000]	= [3310ʒ111)	= [3ʒ001ɣ001ɣ111),		
7 . 4 = [503100]	= [601ʒ001)	= [3ʒ001ɣ002ɣ001),		
7 . 5 = [350010]	= [3310ʒ003)	= [3ʒ001ɣ001ɣ003),		
Cr. 8 . 1 = [14,000001]	= [3ʒ001ɣ002ɣ003ɣ004ɣ005ɣ006),			
8 . 2 = [3230100]	= [3310ʒ002)	= [3ʒ001ɣ001ɣ002),		
8 . 3 = [1322000]	= [3310ʒ011)	= [3ʒ001ɣ001ɣ011),		
8 . 4 = [0070000]	= [034000ʒ222)	= [3ʒ001ɣ001ɣ111ɣ222),		
8 . 5 = [3600010]	= [34010ʒ004)	= [330ʒ002ɣ004)	= [3ʒ000ɣ002ɣ004),	
8 . 6 = [6013000]	= [601ʒ000)	= [3ʒ001ɣ002ɣ000),		
8 . 7 = [0520100]	= [0600ʒ002)	= [3ʒ000ɣ111ɣ222),		
8 . 8 = [2051000]	= [41300ʒ112)	= [3ʒ001ɣ000ɣ112),		
8 . 9 = [3303000]	= [3ʒ000ɣ000ɣ000).			

73. The reduction as above of a transformation to a series of quadric transformations, enables the determination of the reciprocal transformation; or, what is the same thing, the determination of the Jacobian of the first figure; see the example, *ante*, No. 67, where it appears that the reciprocal transformation of [41] is [41]. But I do not see any easy algorithmic process for the determination of the reciprocal transformation, or still less any general form in which the result can be expressed; and I do not at present pursue the inquiry.

The Rational Transformation between Two Spaces.

74. The general principles have been already explained: the two systems $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$ must be derivable the one from the other; and starting with the first system, this will be the case if

only the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ have a common intersection equivalent to n^3-1 points of intersection, but not equivalent to a complete common intersection of n^3 points. The last-mentioned circumstance would arise, if the condition of the common intersection should impose upon the surface more than $\frac{1}{6}(n+1)(n+2)(n+3)-4$ conditions; viz., the surfaces would then be connected by an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W=0$. The common intersection is a figure composed of points and curves: say it is the principal system in the first space; the problem is, to determine a principal system equivalent to n^3-1 points of intersection but such that the number of conditions to be satisfied by a surface passing through it is not more than

$$\tfrac{1}{6}(n+1)(n+2)(n+3)-4.$$

75. The following locutions are convenient. We may say that the number of conditions imposed upon a surface of the order n which passes through the common intersection is the *Postulation* of this intersection; and that the number of points represented by the common intersection (in regard to the points of intersection of any three surfaces each of the order n which pass through it) is the *Equivalence* of this intersection. The conditions above referred to are thus

$$\text{Equivalence} = n^3-1,$$
$$\text{Postulation} \not> \tfrac{1}{6}(n+1)(n+2)(n+3)-4.$$

76. It would appear by the analogy of the rational transformation between two planes, that the only cases to be considered are those for which

$$\text{Postulation} = \tfrac{1}{6}(n+1)(n+2)(n+3)-4;$$

but I cannot say whether this is so.

77. In the transformation between two planes, the two conditions lead, as was seen, to the result that the curve $aX+bY+cZ=0$ is unicursal. I do not see that in the present case of two spaces, the two conditions lead to the corresponding result that the surface $aX+bY+cZ+dW=0$ is unicursal; that this is so, appears, however, at once from the general notion of the rational transformation. In fact, the equation in question $aX+bY+cZ+dW=0$ is satisfied by $x:y:z:w=X':Y':Z':W'$ and $ax'+by'+cz'+dw'=0$; the last equation determines the ratios $x':y':z':w'$ in terms of two arbitrary parameters (say these are $x':y'$ and $x':z'$), and we have then $x:y:z:w$ proportional to rational functions of these two parameters; that is, the surface $aX+bY+cZ+dW=0$ is unicursal. And similarly the surface $aX'+bY'+cZ'+dW'=0$ is unicursal.

78. In the most general point of view, the principal system will contain a given number of points which are simple points, a given number which are quadriconical points, a given number which are cubiconical points, &c. &c., on the surfaces; and similarly a given number of curves which are simple curves, a given number which are double curves, &c. &c., on the surfaces. But, to simplify, I will consider that it includes only points which are simple points, and a curve which is a simple curve

on the surfaces: this curve may, however, break up into separate curves, and we thus, in fact, admit the case where there are any number of separate curves each of them a simple curve on the surfaces. It is right to remark that we cannot assert *à priori*—and it is not in fact the case—that the principal system in the second space will be subject to the like restrictions: starting with such a principal system in the first space, we may be led in the second space to a principal system including a curve which is a double curve on the surfaces; an instance of this will in fact occur.

79. It is shown (Salmon's *Solid Geometry*, 2nd ed., p. 283, [Ed. 4, p. 321]), that in the intersection of three surfaces of the orders μ, ν, ρ respectively, a curve of intersection of the order m and class r counts as $m(\mu+\nu+\rho-2)-r$ points of intersection. For a curve without actual double points or stationary points, we have $r=m(m-1)-2h$, where h is the number of apparent double points; or, substituting, we have the curve counting for $m(\mu+\nu+\rho-2)-m(m-1)+2h$ points of intersection; this is in fact a more general form of the formula, inasmuch as it extends to the case of a curve with actual double points and stationary points. Or, what is the same thing, the three surfaces intersecting in the curve of the order m with h apparent double points, will besides intersect in $\mu\nu\rho-m(\mu+\nu+\rho-2)+m(m-1)-2h$ points; viz., the curve may, besides the apparent double points, have actual double points and stationary points; but these do not affect the formula.

80. Some caution is necessary in the application of the theorem. For instance, to consider cases that will present themselves in the sequel: let the surfaces be cubics ($\mu=\nu=\rho=3$); the number of remaining intersections is given as $=27-7m+m(m-1)-2h$. Suppose that the curve consists of four non-intersecting lines, $m=4$, $h=6$, the number is given as $=-1$. But observe in this case there are two lines each meeting the four given lines; that is, any cubic surface passing through the four given lines meets these two lines each of them in four points, that is, the cubic passes also through each of the two lines; the complete *curve*-intersection of the surfaces is made up of the six lines $m=6$, $h=7$ (since each of the two lines, as intersecting the four lines, gives actual double points, but the two lines together give one apparent double point), and the expression for the number of the remaining points of intersection becomes $=27-42+30-14=1$, which is correct.

81. Similarly, if the given curve of intersection be a conic and two non-intersecting lines, there is here in the plane of the conic a line meeting each of the two given lines, and therefore meeting the cubic surface, in four points, that is, lying wholly in the cubic surface: the complete *curve*-intersection consists of the conic, the two given lines, and the last-mentioned line, $m=5$, $h=5$, and the number of points of intersection is $=27-35+20-10$, $=2$, which is correct. Again, if the given curve of intersection be two conics, here the line of intersection of the planes of the conics lies in the cubic surface; or, for the complete *curve*-intersection we have $m=5$, $h=4$; and the number of points is $27-35+20-8$, $=4$. If in this last case each or either of the conics become a pair of intersecting lines, or if in the preceding case the conic becomes a pair of intersecting lines, the results remain unaltered.

82. If a surface of the order μ pass through a curve of the order m and class r without stationary points or actual double points, this imposes on the surface a number of conditions $=(\mu+1)m-\frac{1}{2}r$. In the case in question, the value of r is $=m(m-1)-2h$; or, substituting, the number of conditions is $=(\mu+1)m-\frac{1}{2}m(m-1)+h$; and the formula in this form holds good even in the case where the curve has stationary points and actual double points. Thus $\mu=3$, the number of conditions is $=4m-\frac{1}{2}m(m-1)+h$. If the curve be a line, $m=1$, $h=0$, number of conditions is $=4$; if the curve be a pair of non-intersecting lines, $m=2$, $h=1$, number of conditions is $=8$. And so in general, if the curve consist of k non-intersecting lines ($k=4$ at most), then $m=k$, $h=\frac{1}{2}k(k-1)$, and the number of conditions is $=4k$. If the curve be a conic, or a pair of intersecting lines, $m=2$, $h=1$, and the number of conditions is $=7$. If the curve consist of k lines, such that there are θ pairs of intersecting lines, then $m=k$, $h=\frac{1}{2}k(k-1)-\theta$, and the number of conditions is $=4k-\theta$. It is obvious that, the number of conditions for a line being $=4$, that for the k lines with θ intersecting pairs must have the foregoing value $4k-\theta$. In fact, when the lines do not intersect, we take on each line 4 points, and the cubic surface passing through any such 4 points will contain the line; but for two lines which intersect, taking this point, and on each of the intersecting lines 3 other points, the cubic surface through the 7 points will pass through the two lines; and so in other cases.

83. The formula must, in some instances, be applied with caution. Thus, given five non-intersecting lines $k=5$, $\theta=0$, and the number of conditions is $=20$; and a cubic surface cannot be, in general, made to pass through the lines. But if the five lines are met by any other line, then a cubic surface, if it pass through the five lines, will pass through this sixth line; for the six lines $k=6$, $\theta=5$, and the number of conditions is $24-5=19$; so that there is a determinate cubic surface through the six lines, and therefore through the five lines related in the manner just referred to.

84. Recurring to the problem of transformation, it appears by what precedes, that if the principal system in the first plane consists of α_1 points, and of a curve of the order m_1 with h_1 apparent double points (the α_1 points being simple points, and the curve a simple curve on the surfaces), then the conditions for a transformation are

$$(3n-2)m_1-\ \ m_1(m_1-1)+2h_1+\alpha_1=n^3-1,$$
$$(\ n+1)m_1-\tfrac{1}{2}m_1(m_1-1)+\ \ h_1+\alpha_1=\tfrac{1}{6}(n+1)(n+2)(n+3)-4,$$

where, in the second line, instead of $\not>$ I have written $=$. I remark, in passing, that I have ascertained that an actual triple point counts as an apparent double point; or, what is the same thing, that if the curve has t_1 actual triple points, then we may, instead of h_1, write h_1+t_1. The equations give

$$m_1(4n-5-m_1)=\tfrac{1}{3}(n-1)(5n^2-\ \ n-12)-2h_1,$$
$$(n-4)m_1-\alpha=\tfrac{1}{3}(n-1)(2n^2-4n-15),$$

to which may be joined

$$(3n+8)m_1-2m_1(m_1-1)+4h_1+5\alpha_1=(n-1)(6n+17).$$

The first two equations for the successive values of n give

$$\begin{array}{lll}
n=2, & m_1(3-m_1)= 2-2h_1, & 2m_1+\alpha_1= 5; \\
n=3, & m_1(7-m_1)= 20-2h_1, & m_1+\alpha_1= 6; \\
n=4, & m_1(11-m_1)= 64-2h_1, & \alpha_1=-1; \\
n=5, & m_1(15-m_1)=144-2h_1, & -m_1+\alpha_1=-20; \\
n=6, & m_1(19-m_1)=270-2h_1, & -2m_1+\alpha_1=-55; \\
\&c. & \&c. & \&c.
\end{array}$$

85. It is remarkable that for $n=4$ there is no solution of the geometrical problem; in fact, $\alpha_1=-1$, a negative value of α_1, shows that this is so. For the higher values of n, there seem to be solutions with large values of m_1, h_1, α_1; for example, $n=5$, we have $m_1=20+\alpha_1$, is $=20$ at least. Writing $m_1=20$, we have $-100=144-2h_1$, or $2h_1=244$. The highest value of $2h_1$ is $=(m_1-1)(m_1-2)$, which for $m_1=20$ is $=342$; or the foregoing value $2h_1=244$ is admissible. Thus $m_1=20$, $h_1=122$, $\alpha_1=0$ gives a solution; and, moreover, any larger value of m_1, say $m_1=20+\alpha$, gives an admissible solution, $m_1=20+\alpha$, $h_1=122+\frac{1}{2}\alpha(\alpha+25)$, $\alpha_1=\alpha$. And so for $n=6$, &c.; but I have not further examined any of these cases, and do not understand them.

There remain the cases $n=2$, $n=3$. For $n=2$, since $2m_1+\alpha_1=5$, we have $m_1=0$, 1, or 2; $m_1=0$ gives $h_1=0$, which is not admissible. The remaining solutions are $m_1=1$, $h_1=0$, $\alpha_1=3$; and $m_1=2$, $h_1=0$, $\alpha_1=1$.

For $n=3$, since $m_1+\alpha_1=6$, we have $m_1=0$, 1, 2, 3, 4, 5, or 6. $m_1=0$ gives $h_1=10$; $m_1=1$ gives $h_1=7$; $m_1=2$ gives $h_1=5$; $m_1=3$ gives $h_1=4$: these values are not geometrically admissible. The remaining cases are $m_1=4$, $h_1=4$, $\alpha_1=2$; $m_1=5$, $h_1=5$, $\alpha_1=1$; $m_1=6$, $h_1=7$, $\alpha_1=0$.

86. The reciprocal transformation is in every case of the order $n'=n^2-m_1$. Hence considering the quadric transformations:

First, the case $n=2$, $m_1=1$, $h_1=0$, $\alpha_1=3$; the reciprocal transformation is of the order $n'=3$. Suppose for a moment that the principal system in the second space is of the same nature as that above considered in the first space, consisting of α_1' points, and a curve of the order m_1' with h_1' apparent double points (the α_1' points each a simple point, and the curve a simple curve on the surfaces $X'=0$, &c.). Passing back to the original transformation, we should have $2=9-m_1'$, that is, $m_1'=7$. But it has just been seen that, for $n=3$, the only values of m_1 are 4, 5, 6; hence for $n'=3$ we cannot have $m_1'=7$. The explanation is, that the principal system in the second space *is not of the form in question;* it, in fact, consists (as will appear) of three lines each a simple line, and of another line which is a double line on the surfaces $X'=0$, &c. In the intersection of any two of these surfaces, the three lines count each once, the double line four times, and the order of the curve of intersection is thus $3+4=7$, as it should be. The principal system may be characterized $\alpha_1'=0$, $m_1'=3$, $h_1'=3$, $m_2'=1$, $h_2'=0$.

Next, the case $n=2$, $m_1=2$, $h_1=0$, $\alpha_1=1$: the reciprocal transformation is of the order $n'=2$; it is evidently not of the form above considered (for this would make the original transformation to be of the order 3). Hence, assuming (as it seems allowable to do) that the principal system does not contain any multiple point or curve, the reciprocal transformation will be of the same form as the original one; viz., we shall have $n'=2$, $m_1'=2$, $h_1'=0$, $\alpha_1'=1$.

87. Considering next the cubic transformations, or those belonging to $n=3$; in the case $m_1=4$, $h_1=4$, $\alpha_1=2$, the reciprocal transformation is of the order $9-4$, $=5$; and in the case $m_1=5$, $h_1=5$, $\alpha_1=1$, the reciprocal transformation is of the order $9-5$, $=4$: I do not consider these cases. But $m_1=6$, $h_1=7$, $\alpha_1=0$, the reciprocal transformation is of the order $9-6$, $=3$; and assuming (as seems allowable) that the principal system does not contain any multiple point or curve, it must be of the same form as the original transformation, that is, we must have $n'=3$, $m_1'=6$, $h_1'=7$, $\alpha_1'=0$.

88. The transformations to be studied are thus,—1° The quadri-quadric transformation $n=2$, $m_1=2$, $h_1=0$, $\alpha_1=1$, and $n'=2$, $m_1'=2$, $h_1'=0$, $\alpha_1'=1$; the principal system in each space consists of a point and of a conic (which may be a pair of intersecting lines); and the surfaces are quadrics. 2° The quadri-cubic transformation $n=2$, $m_1=1$, $h_1=0$, $\alpha_1=3$, and $n'=3$, $\alpha_1'=0$, $m_1'=3$, $h_1'=3$, $m_2'=1$, $h_2'=0$: in the first space the principal system consists of three points and a line, and the surfaces are quadrics: in the second figure, the principal system consists of three simple lines and a double line; and the surfaces are cubic surfaces passing through this principal system, that is, they are cubic scrolls. 3° The cubo-cubic transformation $n=3$, $\alpha_1=0$, $m_1=6$, $h_1=7$, and $n'=3$, $\alpha_1'=0$, $m_1'=6$, $h_1'=7$; in each space the principal system is a sextic curve with seven apparent double points (but there are different cases to be considered according as the sextic curve does or does not break up into inferior curves), and the surfaces are cubic surfaces through the sextic curve.

The Quadri-quadric Transformation between Two Spaces.

89. Starting from the equations $x':y':z':w'=X:Y:Z:W$, we have here $X=0$, &c., quadric surfaces passing through a given point and a given conic (which may be a pair of intersecting lines). Take $x=0$, $y=0$, $z=0$ for the coordinates of the given point; $w=0$ for the equation of the plane of the conic; the conic is then given as the intersection of this plane by a cone having the given point for its vertex; or say the equations of the conic are $w=0$, $(a,\ldots\between x,y,z)^2=0$; the general equation of a quadric through the point and conic is $w(\alpha x+\beta y+\gamma z)+\delta(a,\ldots\between x,y,z)^2=0$; and it hence appears that the equations of the transformation may be taken to be

$$x':y':z':w'=xw:yw:zw:(a,\ldots\between x,y,z)^2;$$

these give at once a reciprocal system of the same form; viz., the two sets are

$$x':y':z':w'=xw:yw:zw:(a,\ldots\between x,y,z)^2,$$

and

$$x:y:z:w=x'w':y'w':z'w':(a,\ldots\between x',y',z')^2.$$

90. The Jacobian of the first space is at once found to be

$$w^2(a, \ldots \between x, y, z)^2 = 0\,;$$

that of the second space is of course

$$w'^2(a, \ldots \between x', y', z')^2 = 0.$$

The two spaces are similar to each other; we may say that there is in each of them a principal point and a principal conic; that the plane of the conic is the principal plane, and the cone having its vertex at the point and passing through the conic is the principal cone. To the principal point of either space corresponds any point whatever in the principal plane of the other space; and conversely. More definitely, the points of the one principal plane and the infinitesimal elements of direction through the principal point of the other space correspond according to the equations $x : y : z = x' : y' : z'$. To any point on the principal conic of either space corresponds in the other space, not a mere element of direction through the principal point of the other space, but a line of the principal cone; that is, to the points of the principal conic of the one space correspond the lines of the principal cone of the other space. The Jacobian of either space, consisting of the principal plane twice, and of the principal cone, is thus the principal counter-system of the other space.

91. {Writing $(a, \ldots \between x, y, z)^2 = x^2 + y^2 + z^2$, $w = w' = 1$, the equations of transformation become

$$x' : y' : z' : 1 = x : y : z : x^2 + y^2 + z^2,$$

and

$$x : y : z : 1 = x' : y' : z' : x'^2 + y'^2 + z'^2,$$

or, what is the same thing, if for shortness

$$x^2 + y^2 + z^2 = r^2, \quad x'^2 + y'^2 + z'^2 = r'^2,$$

the equations are

$$x' = \frac{x}{r^2}, \quad y' = \frac{y}{r^2}, \quad z' = \frac{z}{r^2}; \text{ and } x = \frac{x'}{r'^2}, \quad y = \frac{y'}{r'^2}, \quad z = \frac{z'}{r'^2},$$

whence also $rr' = 1$; this is the well known transformation by reciprocal radius vectors.}

92. The principal conic may be a pair of intersecting lines; taking its equations to be $w = 0$, $xy = 0$, the equations of transformation here become

$$x' : y' : z' : w' = xw : yw : zw : xy,$$

and

$$x : y : z : w = x'w' : y'w' : z'w' : x'y'.$$

There is no difficulty in the further development of the theory.

The Quadri-cubic Transformation between Two Spaces.

93. It will be convenient to have the unaccented letters (x, y, z, w) referring to the cubic surfaces. I will therefore take the quadric surfaces in the second figure; viz., I will start from the equations $x : y : z : w = X' : Y' : Z' : W'$, where $X' = 0$,

$Y'=0$, $Z'=0$, $W'=0$ are quadric surfaces passing through three fixed points (say the principal points) and through a fixed line (say the principal line) in the second figure. Taking $x'=0$, $y'=0$ for the planes passing through the principal line and through two of the principal points respectively; $z'=0$ for the plane passing through the three principal points, $w'=0$ for an arbitrary plane passing through the first mentioned two principal points, the implicit factors of x', y', w' may be so determined that for the third principal point $x'=y'=-w'$. That is, we shall have

$$\begin{aligned} &\text{for principal line} && x'=0,\ y'=0, \\ &\text{for principal points} && (x'=0,\ z'=0,\ w'=0), \\ &\quad\text{,,} && (y'=0,\ z'=0,\ w'=0), \\ &\quad\text{,,} && (x'=y'=-w',\ z'=0), \end{aligned}$$

and this being so, the equation of a quadric surface through the principal points and line will be

$$(\alpha x'+\beta y')\,z'+\gamma x'\,(y'+w')+\delta y'\,(x'+w'),$$

and the equations of transformation may be taken to be

$$x : y : z : w \qquad = x'z' : y'z' : x'\,(y'+w') : y'\,(x'+w').$$

94. Writing these in the extended form

$$x : y : z : w : x-y : z-w = x'z' : y'z' : x'\,(y'+w') : y'\,(x'+w') : z'\,(x'-y') : w'\,(x'-y')$$

and forming also the equation

$$xy : (xw-yz) = z' : x'-y',$$

we at once derive the reciprocal system of equations

$$x' : y' : z' : w' \qquad = x\,(xw-yz) : y\,(xw-yz) : (x-y)\,xy : (z-w)\,xy,$$

so that this is a cubic transformation. And the cubic surface in the first space (corresponding to an arbitrary plane $ax'+by'+cz'+dw'=0$ of the second space) is $(ax+by)(xw-yz)+c\,(x-y)\,xy+d\,(z-w)\,xy=0$; viz., this is a cubic surface having the fixed double line $(x=0,\ y=0)$, the fixed simple lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, and $(x-y=0,\ z-w=0)$; it has also the variable simple line $(dz+cx=0,\ dw+cy=0)$. The principal figure of the first space thus consists of the three simple lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$, and of the line $(x=0,\ y=0)$, a double line counting four times in the intersection of two of the cubic surfaces.

95. The cubic surface as having the double line $(x=0,\ y=0)$ is a cubic scroll, and this line is the nodal directrix thereof; the line $(dz+cx=0,\ dw+cy=0)$ is the simple directrix; the lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$ are at once seen to be lines meeting each of these directrix lines; and they are generating lines of the scroll. To explain the generation of the scroll, observe that the section by any plane is a cubic curve having a given double point (viz., the intersection of the plane with the nodal directrix); and three other given points (viz., the intersections of

the plane with the three generating lines respectively); this cubic also passes through the intersection of the plane with the simple directrix. Conversely, if the plane be assumed at pleasure, and if, taking for the simple directrix any line which meets the given generating lines, we draw a cubic as above, then the scroll is the scroll generated by a line which meets each of the directrix lines, and also the cubic.

If the plane be taken to pass through any generating line, then the cubic section breaks up into this line, and a conic; the conic does not meet the simple directrix, but it meets the nodal directrix; and any such conic will serve as a directrix; viz., the scroll is generated by the lines which meet the two directrix lines and the conic.

96. Any two scrolls as above meet in the three fixed generating lines, and in the nodal directrix counting four times; they consequently meet besides in a curve of the second order, which is a conic (one of the conics just referred to). In order to further explain the theory, suppose for a moment that the two scrolls had only a common nodal directrix; they would besides meet in a quintic curve; this curve would meet the nodal directrix in four points, viz., the points at which the two scrolls have a common tangent plane. Now if at any point of the nodal directrix the two scrolls have a common generating line, then the plane through this line and the nodal line is one of the two tangent planes of each scroll; that is, the scrolls have this plane for a common tangent plane. Hence, in the case of the common three generating lines, the points where these meet the nodal line are three of the four points just referred to; there remains therefore one point, which is the point where the conic meets the nodal line; through this point there are for each of the scrolls two generating lines; one of these for the first scroll, and one for the second scroll, lie in a plane with the nodal line; the other two determine the plane of the conic; and the tangent to the conic at its intersection with the nodal line is the intersection of the plane of the conic with the plane of the first-mentioned two generating lines.

97. Analytically we have the two equations

$$c\,(x-y)\,xy+(ax+by)\,(xw-yz)+d\,(z-w)\,xy=0,$$
$$c'\,(x-y)\,xy+(a'x+b'y)\,(xw-yz)+d'\,(z-w)\,xy=0\,;$$

or, combining these equations so as to eliminate successively the terms in $x\,(xw-yz)$ and $y\,(xw-yz)$, and for this purpose writing

$$(bc'-b'c,\ ca'-c'a,\ ab'-a'b,\ ad'-a'd,\ bd'-b'd,\ cd'-c'd)=(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}),$$

and therefore

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0,$$

we have

$$\mathrm{b}\,(x-y)\,x-\mathrm{c}\,(xw-yz)-\mathrm{f}\,(z-w)\,x=0,$$
$$-\mathrm{a}\,(x-y)\,y+\mathrm{c}\,(xw-yz)-\mathrm{g}\,(z-w)\,y=0,$$

and multiplying the first of these by $\mathrm{c}+\mathrm{g}$ and the second by $\mathrm{c}-\mathrm{f}$, and adding, the whole divides by $x-y$, and the final result is

$$(\mathrm{c}+\mathrm{g})\,(\mathrm{b}x-\mathrm{f}z)-(\mathrm{c}-\mathrm{f})\,(\mathrm{a}y+\mathrm{g}w)=0\,;$$

viz., this is the equation of the plane of the conic.

98. Any two scrolls as above meeting in a conic, a third scroll will meet the conic in six points; but these include the point on the nodal directrix twice, and the points on the three fixed generating lines each once; there is left a single point of intersection, viz., this is the *one variable* point of intersection of the three scrolls; which is in accordance with the theory.

99. For the Jacobian of the second space, we have

$$\begin{vmatrix} z', & 0, & x', & 0 \\ 0, & z', & y', & 0 \\ y'+w', & x', & 0, & 0 \\ y', & x'+w', & 0, & y' \end{vmatrix} = 0;$$

that is, $2x'y'z'(x'-y')=0$; viz., $z'=0$ is the plane containing the three principal points; and $x'=0$, $y'=0$, $x'-y'=0$ are the planes which pass through the principal line and the three principal points respectively.

100. For the Jacobian of the first space, we have

$$\begin{vmatrix} 2xw-yz, & -xz, & -xy, & x^2 \\ yw, & xw-2yz, & -y^2, & xy \\ 2xy-y^2, & x^2-2xy, & 0, & 0 \\ (z-w)y, & (z-w)x, & xy, & -xy \end{vmatrix} = 0;$$

that is, $3x^2y^2(x-y)^2(xw-yz)=0$; viz., $x=0$, $y=0$, $x-y=0$ are the planes through the nodal directrix and the three fixed generators respectively (each plane therefore occurring twice); and $xw-yz=0$ is the quadric scroll generated by the lines which meet each of the three generators $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$; this scroll passing also through the nodal directrix $x=0$, $y=0$.

The Cubo-cubic Transformation between Two Spaces.

101. The principal system in the first space is a sextic curve with 7 apparent double points; but this curve may be either a single curve, or it may break up into inferior curves. I have not examined all the cases which may arise; but the two extreme cases are—(A) The sextic curve breaks up into six lines, viz., two non-intersecting lines, and four other lines each meeting each of the two lines (this implies that no two of the four lines meet each other): here the two lines give 1 apparent double point, and the four lines give 6 apparent double points; total number is $=7$, as it should be. (B) The curve is a proper sextic curve, with 7 apparent double points: this gives, as will be shown, the general lineo-linear transformation. The two cases are each of them symmetrical.

(A) *The Principal System consists of Six Lines.*

102. Taking in the first space, for the equations of the two lines, $(x=0,\ y=0)$ and $(z=0,\ w=0)$, and for the equations of the four lines, $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$, $(x-py=0,\ z-qw=0)$, then, if the equations of transformation are taken to be

$$\begin{aligned} x'-py' : x'-y' : z'-qw' : z'-w' = \ & (x-py)(xw-yz) \\ & : (x-y)(qxw-pyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(qxw-pyz); \end{aligned}$$

these lead conversely (see *post*, No. 104) to a like system,

$$\begin{aligned} x-py : x-y : z-qw : z-w = \ & (x'-py')M' \\ & : (x'-y')N' \\ & : (z'-qw')M' \\ & : (z'-w')N', \end{aligned}$$

where for shortness

$$\begin{aligned} M' &= p(q-1)^2 x'w' - q(p-1)^2 y'z' + (pq-1)(p-q)y'w', \\ N' &= \ \ (q-1)^2 x'w' - \ \ (p-1)^2 y'z' + (pq-1)(p-q)y'w'; \end{aligned}$$

or, as these are better written,

$$\begin{aligned} M' &= -q(p-1)y'\{(p-1)z'-(pq-1)w'\} + p(q-1)w'\{(q-1)x'-(pq-1)y'\}, \\ N' &= -\ \ (p-1)y'\{(p-1)z'-(pq-1)w'\} + \ \ (q-1)w'\{(q-1)x'-(pq-1)y'\}. \end{aligned}$$

Hence the principal system in the second plane is composed of the two non-intersecting lines $(x'=0,\ y'=0)$, $(z'=0,\ w'=0)$ and the four lines $\{(p-1)z'-(pq-1)w'=0,\ (q-1)x'-(pq-1)y'=0\}$, $(y'=0,\ w'=0)$, $(x'-y'=0,\ z'-w'=0)$, $(x'-py'=0,\ z'-qw'=0)$, each meeting each of the two lines.

103. The Jacobian of the first space is

$$\begin{vmatrix} 2xw-yz-pyw, & -xz-pxw+2pyz, & -y(x-py), & x(x-py) \\ 2qxw-pyz-qyw, & -pxz-qxw+2pyz, & -py(x-y), & qx(x-y) \\ w(z-qw), & -z(z-qw), & xw-2yz+qyw, & xz-2qxw+qyz \\ qw(z-w), & -pz(z-w), & qxw-2pyz+pyw, & qxz-2qxw+pyz \end{vmatrix} = 0,$$

viz., this is $xyzw(x-y)(x-py)(z-w)(z-qw)=0$, the equation of the planes each passing through one of the four lines and one of the two lines.

Similarly, the Jacobian of the second space is

$$\{(q-1)x'-(pq-1)y'\}\{(p-1)z'-(pq-1)w'\}(x'-y')(z'-w')(x'-py')(z'-qw')y'w'=0;$$

viz., this is the equation of the eight planes each passing through one of the four lines and one of the two lines.

104. To effect the foregoing transformation, writing

$$\begin{aligned} x' : y' : z' : w' = \ & (x-py)(xw-yz) \\ & : (x-y)(qxw-pyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(qxw-pyz); \end{aligned}$$

or what will ultimately be the same thing, but it is more convenient for working with,

$$\begin{aligned} x' &= (x-py)(xw-yz), \\ y' &= (x-y)(qxw-pyz), \\ z' &= (z-qw)(xw-yz), \\ w' &= (z-w)(qxw-pyz); \end{aligned}$$

these give

$$\begin{aligned} x-py &= M'x', \\ x-y &= N'y', \\ z-qw &= M'z', \\ z-w &= N'w', \end{aligned}$$

where M', N' are quantities which have to be determined; and thence

$$\begin{aligned} (1-p)x &= M'x' - pN'y', \\ (1-p)y &= M'x' - N'y', \\ (1-q)z &= M'z' - qN'w', \\ (1-q)w &= M'z' - N'w'; \end{aligned}$$

whence also

$$\begin{aligned} (1-p)(1-q)(xw-yz) &= N'[\{(q-1)x'w' - (p-1)y'z'\}M' + (p-q)y'w'N'], \\ (1-p)(1-q)(qxw-pyz) &= M'[-(p-q)x'z'M' + \{(pq-q)x'w' - (pq-p)y'z'\}N']; \end{aligned}$$

but we have

$$\frac{xw-yz}{qxw-pyz} = \frac{x'}{y'} \div \frac{x-py}{x-y} = \frac{x'}{y'} \div \frac{M'x'}{N'y'} = \frac{N'}{M'};$$

or, substituting,

$$\begin{aligned} & M'\{(q-1)x'w' - (p-1)y'z'\} + N'(p-q)y'w' \\ = \ & M'\{-(p-q)x'z'\} + N'\{(pq-q)x'w' - (pq-p)y'z'\}; \end{aligned}$$

that is

$$M'\{(q-1)x'w' - (p-1)y'z' + (p-q)x'z'\} = N'\{(pq-q)x'w' - (pq-p)y'z' - (p-q)y'w'\};$$

or, what is the same thing,

$$\begin{aligned} M' &= (pq-q)x'w' - (pq-p)y'z' - (p-q)y'w', \\ N' &= (q-1)x'w' - (p-1)y'z' + (p-q)x'z'; \end{aligned}$$

viz., M', N' having these values, the original equations

$$\begin{aligned} x' : y' : z' : w' = \ & (x-py)(xw-yz) \\ & : (x-y)(pxw-qyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(pxw-qyz), \end{aligned}$$

give

$$x-py : x-y : z-qw : z-w = M'x' : N'y' : M'z' : N'w'.$$

If, in these equations, in place of (x', y', z', w') we write $(x'-py', x'-y', z'-qw', z'-w')$, the new values of M', N' are found to be

$$\begin{aligned} M' &= p(q-1)^2 x'w' - q(p-1)^2 y'z' + (pq-1)(p-q) y'w', \\ N' &= \ \ (q-1)^2 x'w' - \ \ (p-1)^2 y'z' + (pq-1)(p-q) y'w', \end{aligned}$$

and we have the formulæ of No. 102.

(B) *The Principal System of a Proper Sextic Curve; the Lineo-linear Transformation between Two Spaces.*

105. I start with the lineo-linear transformation, and show that this is in fact a transformation such that the principal system in either space is a sextic curve with seven apparent double points. I do not attempt any formal proof, but assume that the lineo-linear transformation is the most general one which gives rise to such a principal system.

We have between (x, y, z, w), (x', y', z', w') three lineo-linear equations; writing these first under the form

$$\begin{aligned} (P_1, Q_1, R_1, S_1 \between x', y', z', w') &= 0, \\ (P_2, Q_2, R_2, S_2 \between x', y', z', w') &= 0, \\ (P_3, Q_3, R_3, S_3 \between x', y', z', w') &= 0, \end{aligned}$$

we have $x' : y' : z' : w' = X : Y : Z : W$, where X, Y, Z, W are the determinants (each with its proper sign) formed out of the matrix

$$\begin{vmatrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{vmatrix}.$$

106. Each of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$, or generally any surface $aX+bY+cZ+dW=0$, is thus a cubic surface passing through the curve

$$\left\| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right\| = 0,$$

which is at once seen to be of the order 6. In fact any two of these surfaces, for instance

$$\begin{vmatrix} P_1, & Q_1, & R_1 \\ P_2, & Q_2, & R_2 \\ P_3, & Q_3, & R_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} P_1, & Q_1, & S_1 \\ P_2, & Q_2, & S_2 \\ P_3, & Q_3, & S_3 \end{vmatrix} = 0,$$

have in common a curve

$$\left\| \begin{matrix} P_1, & P_2, & P_3 \\ Q_1, & Q_2, & Q_3 \end{matrix} \right\| = 0,$$

which is of the order 3; they consequently besides intersect in a curve of the order 6, which is the before mentioned curve of intersection of all the surfaces. And it further appears that the number of the apparent double points is $=7$; in fact the formula in the case of two surfaces of the orders μ, ν, the complete intersection of which consists of a curve of the order m with h apparent double points, and of a curve of the order m' with h' apparent double points, the numbers m, m', h, h' are connected by the equation $2(h-h')=(m-m')(\mu-1)(\nu-1)$. (Salmon's *Solid Geometry*, 2nd Ed., p. 273 [Ed. 4, p. 311]). Hence, in the case of the two cubic surfaces intersecting as above (since for the cubic curve we have $m'=3$, $h'=1$, and for the sextic $m=6$), the formula becomes $2(h-1)=12$, that is $h=1+6=7$; or the number of apparent double points is $=7$.

107. It thus appears that the principal system in the first plane is a curve of the order 6, with seven apparent double points: it is to be added that there are not in general any actual double points or stationary points, so that the class of the curve is $6.5-2.7, =16$, and its deficiency is $\frac{1}{2}5.4-7, =3$. For convenience I will refer to this as the curve Σ.

The transformation is obviously a symmetrical one; hence the principal system in the second space is in like manner a curve of the order 6, with seven apparent double points; say it is the curve Σ'.

108. Consider in the first space any point P on the curve Σ; for this point the three equations

$$\begin{aligned} (P_1, Q_1, R_1, S_1 \between x', y', z', w') &= 0, \\ (P_2, Q_2, R_2, S_2 \between \quad \text{,,} \quad) &= 0, \\ (P_3, Q_3, R_3, S_3 \between \quad \text{,,} \quad) &= 0, \end{aligned}$$

are not independent, but are equivalent to two linear equations in (x', y', z', w'); that is, to the point P on the curve Σ there corresponds in the second space, not a determinate point P', but any point whatever on a certain line L'; or say to the point P on Σ there corresponds a line L'; and as P describes the curve Σ, L' describes a scroll Π'; that is, to the curve Σ there corresponds a scroll Π', the principal counter-system in the second space. Similarly to the curve Σ' there corresponds a scroll Π, the principal counter-system of the first space.

109. The scroll Π is the Jacobian of the first space; and as such it is of the order 8, having the curve Σ for a triple line—and it thus appears that the Jacobian

of the first space is a scroll (a theorem the analytical verification of which seems by no means easy). But without assuming the identity of the scroll Π with this Jacobian, or taking the order of the scroll to be known, I proceed to show that the scroll Π is the scroll generated by the lines each of which meets the curve Σ three times; it will thereby appear that the order is $=8$, and that the curve is a triple line on the scroll.

Consider a point P' on Σ', and the corresponding line L of the first space: take Θ' a plane in the second space; corresponding to it the cubic surface Θ in the first space. By imposing a single relation on the coefficients (a, b, c, d) in the equation $ax'+by'+cz'+dw'=0$ of the plane Θ', we make it pass through the point P'; therefore by imposing this same *single* relation on the coefficients (a, b, c, d) of the cubic surface Θ, we make it pass through the line L; Θ is a cubic surface through Σ; and it is easy to see that the effect will be as above only if the line L cuts the curve Σ three times; this being so, the general cubic surface Θ meets L in three points (viz., the three intersections of L with Σ), and if Θ be made to pass through a fourth point on the line L, it will pass through the line L; it thus appears that the line L meets Σ three times, and consequently that the scroll Π is generated by the lines which meet Σ three times.

110. The theory of a scroll so generated is considered in my "Memoir on Skew Surfaces, otherwise Scrolls"(1). Writing $m=6$, $h=7$ and therefore $M[=-\frac{1}{2}m(m-1)+h], =-8$, the order of the scroll is $(\frac{1}{3}[m]^3+(m-2)M=40-32)=8$; but calculating the values of

$$NG(m^3)=\tfrac{1}{2}[m]^4+6m+M(3[m]^2-12m+33)+M^2.3,$$

$$NR(m^3)=\tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3-3m+M(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}m^2+8m-20)+M^2(\tfrac{1}{2}[m]^2-2m);$$

these are found to be respectively $=0$; viz., there are no nodal generators, and no nodal residue; the sextic curve Σ is a triple curve on the surface, and there is not any other multiple line.

111. It may be remarked that any plane Θ' meets the sextic curve Σ' in six points; hence the corresponding cubic surface Θ contains six lines, generatrices of Π, and, therefore, each meeting the curve Σ three times; say six lines L. Through one of these lines L, draw to the cubic surface Θ a triple tangent plane meeting it in the line L and in two other lines, say M, N; this plane must meet Σ in three new points which must lie on the lines M, N; viz., one of these lines must pass through two of the points, and the other line through the third point.

Addition—September, 1870.

[Some corrections have been made in accordance with the concluding paragraph of a paper "Note on the Rational Transformation and on Special Systems of Points," 450.]

The formulæ of No. 84 are included in the following more general formulæ; viz., if the principal system consist of α_1 points, each a simple point, α_2 points each a

(1) *Phil. Trans.* vol. CLIII. 1863, pp. 453—483, [329]. See the Table $S(m^3)$ &c., p. 457; in the value of $NR(m^3)$ instead of term $+3m$ read $-3m$. [This correction should have been made in the present Reprint.]

quadri-conical point, α_3 points each a cubi-conical point, &c., and of a simple curve order m_1 with h_1 apparent double points, a double curve order m_2 with h_2 apparent double points, and so on; and if moreover, the curves m_1, m_2 intersect in $k_{1,2}$ points, the curves m_1, m_3 in $k_{1,3}$ points, &c.; then writing in general $\rho = \frac{1}{2}m(m-1) - h$; that is, $\rho_1 = \frac{1}{2}m_1(m_1-1) - h_1$, $\rho_2 = \frac{1}{2}m_2(m_2-1) - h_2$, &c., I find that the general condition of equivalence is

$$\left.\begin{array}{l} \alpha_1 + (3n-2)\,m_1 - 2\rho_1 \\ + 8\alpha_2 + (12n-16)\,m_2 - 16\rho_2 \\ \quad\vdots \\ + r^3\alpha_r + (3r^2n - 2r^3)\,m_r - 2r^3\rho_r \\ - 5k_{1,2} - 8k_{1,3} \ldots - (3r-1)\,k_{1,r} \\ \qquad - 28k_{2,3} \\ \qquad\qquad \ldots - s^2(3r-s)\,k_{s,r}\,(s<r) \end{array}\right\} = n^3 - 1;$$

and that the general condition of postulation is

$$\left.\begin{array}{l} \alpha_1 + (n+1)\,m_1 - \rho_1 \\ + 4\alpha_2 + (3n+1)\,m_2 - 5\rho_2 \\ \quad\vdots \\ + \frac{1}{6}r(r+1)(r+2)\,\alpha_r \\ + [\frac{1}{2}r(r+1)\,n - \frac{1}{6}r(r+1)(2r-5)]\,m_r \\ \quad - \frac{1}{24}[(r-1)(r-2)(r-3)(r-4) \\ \qquad + 4r(r+1)(2r+1)]\,\rho_r \\ - 2k_{1,2} - 3k_{1,3} \ldots - rk_{1,r} \\ \quad - 8k_{2,3} \\ \quad\vdots \\ - \frac{1}{2}s(s+1)\{r+1-\frac{1}{3}(s+2)\}\,k_{s,r}\,(s<r) \end{array}\right\} = \frac{1}{6}(n+1)(n+2)(n+3) - 4:$$

in which formulæ it is however assumed that the curves have not any actual multiple points. This implies that if any one of the curves, say m_r, break up into two or more curves, the component curves do not intersect each other; for, of course, any such point of intersection would be an actual double point on the curve m_r. I believe, however, that the formulæ will extend to this case by admitting for s the value $s = r$; viz., if we suppose the curve m_r to be the aggregate of the two curves m_r', m_r'' intersecting in K_r points, then that the corresponding terms in the equivalence-equation are

$$(3r^2n - 2r^3)(m_r' + m_r'') - 2r^3(\rho_r' + \rho_r'') - 2r^3K_r,$$

and that those in the postulation-equation are

$$\begin{array}{l} [\frac{1}{2}r(r+1)\,n - \frac{1}{6}r(r+1)(2r-5)]\,(m_r' + m_r'') \\ \qquad - \frac{1}{24}[(r-1)(r-2)(r-3)(r-4) + 4r(r+1)(2r+1)]\,(\rho_r' + \rho_r'') \\ \qquad - \frac{1}{6}r(r+1)(2r+1)\,K_r. \end{array}$$

Let the r-tuple curve consist of three right lines meeting in a point: this is an actual triple point, and the formulæ do not apply. But calculating the postulation-terms by the formula, we have $m_r = 3$, $\rho_r = \frac{1}{2}3 \,.\, 2 - 0, = 3$; and the terms are

$$[\tfrac{1}{2}r(r+1)n - \tfrac{1}{6}r(r+1)(2r-5)]\,3 - \tfrac{1}{8}[(r-1)(r-2)(r-3)(r-4) + 4r(r+1)(2r+1)],$$

which are

$$= \tfrac{1}{2}r(r+1)(3n-4r+4) - \tfrac{1}{8}(r-1)(r-2)(r-3)(r-4),$$

or say

$$= \tfrac{1}{2}r(r+1)(3n-4r+4) + \tfrac{1}{8}(-r^4 + 10r^3 - 35r^2 + 50r - 24).$$

I have found by an independent investigation that this value requires the correction

$$+\tfrac{1}{8}[r^4 - 8r^3 + 30r^2 - 56r + 24 + \tfrac{1}{2}\{1 - (-)^r 1\}],$$

and that the true value of the postulation is

$$= \tfrac{1}{2}r(r+1)(3n-4r+6) + \tfrac{1}{8}[\quad 2r^3 - 5r^2 \;\; - 6r \qquad + \tfrac{1}{2}\{1 - (-)^r 1\}],$$

viz., that this is the number of the conditions to be satisfied that a surface of the order n may have for an r-tuple curve three given right lines meeting in a point.

448.

NOTE ON THE CARTESIAN WITH TWO IMAGINARY AXIAL FOCI.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 181, 182. Read June 9, 1870.]

LET A, A', B, B' be a pair of points and antipoints; viz.,

(A, A') the two imaginary points, coordinates $(\pm \beta i, 0)$,

(B, B') the two real points, coordinates $(0, \pm \beta)$;

and write $\rho, \rho', \sigma, \sigma'$ for the distances of a point (x, y) from the four points respectively; say

$$\rho = \sqrt{(x+\beta i)^2 + y^2}, \quad \sigma = \sqrt{x^2 + (y+\beta)^2},$$

$$\rho' = \sqrt{(x-\beta i)^2 + y^2}, \quad \sigma' = \sqrt{x^2 + (y-\beta)^2}.$$

We have

$$\rho^2 + \rho'^2 = 2x^2 + 2y^2 - 2\beta^2 = \sigma^2 + \sigma'^2 - 4\beta^2,$$

$$\rho\rho' = \sqrt{(x+\beta i+yi)(x+\beta i-yi)(x-\beta i+yi)(x-\beta i-yi)} = \sigma\sigma';$$

and thence

$$(\rho+\rho')^2 = (\sigma+\sigma')^2 - 4\beta^2,$$

$$(\rho-\rho')^2 = (\sigma-\sigma')^2 - 4\beta^2;$$

or say

$$\rho + \rho' = \sqrt{(\sigma+\sigma')^2 - 4\beta^2},$$

$$i(\rho-\rho') = \sqrt{4\beta^2 - (\sigma-\sigma')^2}.$$

The equation of a Cartesian having the two imaginary axial foci A, A' is

$$(p+qi)\rho + (p-qi)\rho' + 2k^2 = 0;$$

viz., this is

$$p(\rho+\rho')+qi(\rho-\rho')+2k^2=0;$$

or, what is the same thing, it is

$$p\sqrt{(\sigma+\sigma')^2+4\beta^2}+q\sqrt{4\beta^2-(\sigma-\sigma')^2}+2k^2=0,$$

which is the equation expressed in terms of the distances σ, σ' from the non-axial real foci B, B'. Of course, the radicals are to be taken with the signs $\pm$. This equation gives, however, the Cartesian in combination with an equal curve situate symmetrically therewith in regard to the axis of y.

The distances σ, σ' may conveniently be expressed in terms of a single variable parameter θ; in fact, we may write

$$\pm p\sqrt{(\sigma+\sigma')^2-4\beta^2}=-k^2-k\theta,$$

$$\pm q\sqrt{4\beta^2-(\sigma-\sigma')^2}=-k^2+k\theta;$$

that is

$$(\sigma+\sigma')^2-4\beta^2=\frac{k^2}{p^2}(k+\theta)^2,$$

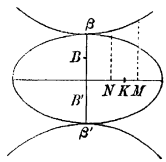
$$4\beta^2-(\sigma-\sigma')^2=\frac{k^2}{q^2}(k-\theta)^2;$$

and therefore

$$\sigma+\sigma'=\sqrt{4\beta^2+\frac{k^2}{p^2}(k+\theta)^2},$$

$$\sigma-\sigma'=\pm\sqrt{4\beta^2-\frac{k^2}{q^2}(k-\theta)^2}:$$

so that, assigning to θ any given value, we have σ, σ', and thence the position of the point on the curve. We may draw the hyperbola $y^2=4\beta^2+\frac{k^2}{p^2}x^2$, and the ellipse $y^2=4\beta^2-\frac{k^2}{q^2}x^2$; and then measuring off in these two curves respectively the ordinates which belong to the abscissæ $k+\theta$ for the hyperbola, $k-\theta$ for the ellipse, we have



the values $\sigma+\sigma'$ and $\sigma-\sigma'$, which determine the point on the curve. Considering k, p, q, β as disposable quantities, the conics may be any ellipse and hyperbola whatever, having a pair of vertices in common; and the complete construction is,—From the

fixed point K in the axis of x, measure off in opposite directions the equal distances KM, KN, and take

$$\sigma+\sigma' \text{ the ordinate at } M \text{ in the hyperbola,}$$

$$\pm(\sigma-\sigma') \quad \text{,,} \quad \text{,,} \quad N \quad \text{,,} \quad \text{ellipse;}$$

where σ, σ' denote the distances of the required point from the fixed points B and B' respectively, the distance of each of these from the origin being $=\frac{1}{2}$ the common semi-axis. We may imagine N travelling from one extremity of the x-axis of the ellipse to the other, the value of $\sigma+\sigma'$ will be real and greater than BB', that of $\sigma-\sigma'$ real and less than BB', and the point (σ, σ') will be real. The construction gives, it will be observed, the two symmetrically situated curves.

The x-semi-axis of the ellipse is $\frac{q}{k}2\beta$, and the form of the curve depends chiefly on the value of the ratio $k : \frac{q}{k}2\beta$; or, what is the same thing, $k^2 : 2\beta q$. We see, for instance, that, in order that the curve may meet the axis of y in two real points between the foci, the value $\theta=-k$ must give a real value of $\sigma-\sigma'$; viz., that we must have $4\beta^2 > \frac{4k^4}{q^2}$; that is, $\beta^2q^2 > k^4$, or $k^2 < \beta q$. If k has this value, viz., $k=\frac{1}{2}\frac{q}{k}2\beta=\frac{1}{2}$ semi-axis, the curve *touches* the axis of y at the origin; if $k<\frac{1}{2}$ semi-axis, the curve cuts the axis of y in two real points between the foci; if $k>\frac{1}{2}$ semi-axis, the curve does not cut the axis of y between the foci.

449.

SKETCH OF RECENT RESEARCHES UPON QUARTIC AND QUINTIC SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 186—195. Read Nov. 10, 1870.]

THE classification of quartic surfaces is even as to its highest divisions incomplete; and it is by no means easy to make it at once exhaustive and precise; an enumeration of all the *primâ facie* possible cases would include forms which do not really exist. Thus the singular curve (if any) is of the order 1, 2, or 3—but in the case where the order is $=3$, the curve, as is at once evident, cannot be a plane cubic, nor (among other excluded forms) a system of three non-intersecting lines. And certain forms of the singular curve, e.g. all but one of the admissible forms of a curve of the order 3, make the surface to be a scroll, so that, if (as is convenient) we wish to separate the scrolls, certain forms otherwise admissible must be excluded. The expression "singular" means double or cuspidal, or refers to a higher singularity, but the cases of higher singularity are very special. I will, at the cost of some inaccuracy, use the expression "nodal" as meaning, in general, double, but as including the signification "cuspidal"; and, if there are any cases of higher singularity, as extending to cases of higher singularity: and I provisionally arrange the non-scrolar quartic surfaces as follows:

1. Without a nodal curve.

2. With a nodal line.

3. With a nodal conic, or line-pair (pair of intersecting lines).

4. With three nodal lines (not in the same plane) meeting in a point.

(Observe that the omitted cases are cases which, as I believe, ought to be omitted; thus the case of a nodal skew cubic is omitted, because the surface is then of necessity a scroll.) And to these I join:

5. The quartic scrolls;

omitting altogether the torse and cones.

The references, by the name of the author, and number (if any) of his paper, are to the subjoined list of Memoirs.

As to the scrolls, we have Cayley (3) and (4), and Cremona; the division into 12 species is, I believe, complete: see *post*, the remarks upon Schwarz's paper on quintic scrolls.

As regards the non-scrolar surfaces:

1. Without a singular curve. The surface may be without a cnicnode (conical point), or it may have any number of cnicnodes up to 16, Cayley (7): the cases of singularity higher than a cnicnode are probably very numerous, but they have been scarcely at all examined. The memoir just referred to relates chiefly to the several cases of not more than 10 nodes; the cases of 11, 12, 13, 14, 15, 16 nodes are considered incidentally, Kummer (2), but it was not the object of his paper to make an enumeration, and there *may be* cases which are not considered; the discussion of the cases considered is very full and interesting. The case of 16 nodes is also considered, Kummer (1). As to the surface with 16 nodes, it is to be remarked that the wave-surface, or generally the surface obtained by the homographic deformation of the wave-surface—called, Cayley (1), the "tetrahedroid"—is a special form of surface with 16 nodes: its relation to the general surface is explained, Cayley (2).

2. Quartic surface with nodal line: considered incidentally, Clebsch (2) and (3). There are through the nodal line 8 planes, each meeting the surface in a line-pair: considering any 7 of these, and taking out of each of them a line, the 7 lines are met by a conic which also meets a determinate line out of the remaining line-pair; there are thus on the surface $2^7 = 128$, conics; viz., these form 64 pairs, each pair lying in a plane, and being the complete intersection of the surface by such plane; the number of these planes is of course $= 64$.

Although not properly included in the present case, I mention the quartic surface which is the reciprocal of the cubic surface $\mathrm{XIX} = 12 - B_6 - C_2$, Cayley (5): the nodal curve is here an oscnodal line counting as three nodal lines.

3. Quartic surfaces with nodal conic. Such a surface may be without cnicnodes, or it may have 1, 2, 3, or 4 cnicnodes; the cases, other than that of 3 cnicnodes, are mentioned, Kummer (3); but the question is examined, and the remaining case of 3 cnicnodes established, Cayley (6).

The general case of the nodal conic without cnicnodes is elaborately considered, Clebsch (1): it is shown that there are on the surface 16 lines, each meeting the conic, and which in their arrangement are strikingly analogous to the 27 lines on a cubic surface; viz., if on a cubic surface we select at pleasure any one of the 27 lines, and through this line draw a plane which besides meets the cubic surface in a conic; then, disregarding the line in question and the 10 lines which meet it, the remaining 16 lines each meet the conic, and are related to it and to each other in the same manner that the 16 lines of the quartic surface are related to the nodal

conic and to each other. And the ground hereof appears, Geiser; viz., it is shown that the quartic surface with the nodal conic, is rationally transformable into a cubic surface, the 16 lines and the nodal conic corresponding respectively to the 16 lines and the conic of the cubic surface.

The several cases of 1, 2, 3, and 4 cnicnodes are considered, Korndörfer.

In the case where the nodal conic is the circle at infinity, the surfaces have been termed "anallagmatic" (perhaps "bicircular" would be a more convenient name), and a great deal has been written upon these surfaces by Moutard, Clifford, and others. Such a surface may of course have 1, 2, 3, or 4 cnicnodes; these surfaces, viz. the cnicnodal anallagmatics, in fact arise from the inversion of a quadric surface by the method of reciprocal radius vectors $\left(\text{that is, by the change of } x, y, z \text{ into } \frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$: the centre of inversion is a node on the quartic surface. If the quadric surface is a cone, there is another node, the inverse point of the vertex; if the quadric surface is one of revolution, there are two other nodes; and if it is a cone of revolution, there are three other nodes—viz., in all, four nodes. The last-mentioned surface is, or includes, the Cyclide; viz., this is a quartic surface having the circle at infinity for a nodal curve, and having besides four nodes, which are a system of skew antipoints. The surface was first considered by Dupin (*Applications de Géométrie &c.*, 1822) as the envelope of a sphere touching three given spheres—its lines of curvature are thus circles; and the surface has been very frequently considered in reference to this property and otherwise: see Maxwell, where a classification (not quite complete) is made of the different forms of the surface, and also stereographic drawings given. It is to be observed, that one interesting form, the parabolic cyclide, is not a quartic but a cubic surface.

In the class of surfaces which have been under consideration, the cnicnodes have been points not on the nodal conic—in fact, a point on a nodal curve cannot be, properly speaking, a cnicnode, though it may be a point of higher singularity in the nature of a cnicnode; viz., there may be on the nodal curve points which, in the classification of the surfaces, must be counted as cnicnodes. Such a case presents itself in the "Conic Torus," or surface generated by the rotation of a conic about a line whether not in or in the plane of the conic. The surface has been considered, De la Gournerie (1), although more in reference to the constructions of descriptive geometry than as a theory of pure geometry, and Cayley (6). The surface has a nodal circle, and upon it two singular points, the circular points at infinity; so that it belongs to the case of a nodal conic with two cnicnodes. In the particular case where the axis of rotation is in the plane of the conic, then there are on the axis two cnicnodes; so that the case is that of a nodal conic and four cnicnodes; and when the generating conic is a circle, viz., when the surface is the ordinary torus, or anchor ring, generated by the rotation of a circle about a line in its own plane, then the nodal conic is the circle at infinity having upon it two cnicnodal points (its intersections by the planes at right angles to the axis) and the surface has also two cnicnodes on the axis: the surface, although presenting considerable peculiarity, may be

regarded as a particular case of the cyclide. In reference to the plane sections of the conic torus and its various particular cases, see De la Gournerie (2); the ordinary torus has been the subject of numerous papers by Darboux and others, and possesses very interesting properties.

In connexion with the foregoing, I speak of the surfaces having a cuspidal conic: the general case is briefly referred to, Cayley (6); viz., this is the surface (AA) the equation of which is $V^2 - x^3y = 0$, and which it is shown has a reciprocal of the order 6. A special case is the surface (AB) having a reciprocal of the order 3; viz., the quartic surface is here the reciprocal of the cubic surface $\mathrm{XX} = 12 - U_3$, Cayley (5). And it appears from the memoir last referred to, that there is another cubic surface, $\mathrm{XVII} = 12 - 2B_3 - C_3$, the reciprocal of which is a quartic surface having a cuspidal conic. But the theory of the quartic surfaces with a cuspidal conic has been hardly at all considered.

I do not know that anything has been done in regard to the quartic surfaces where the nodal conic becomes a line-pair; that is, where we have two intersecting nodal lines. Although not properly belonging to the case in question, I mention here the quartic surface which is the reciprocal of the cubic surface $\mathrm{XVIII} = 12 - B_4 - 2C_3$, Cayley (5); the nodal curve consists of two intersecting lines, but one of them is tacnodal, counting as two nodal lines.

4. Quartic surface with three nodal lines (not in the same plane) meeting in a point. This is, in fact, Steiner's quartic surface; and it has been the subject of numerous investigations.

The equation of the surface may be taken to be $\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 0$; and the surface thus presents itself as the reciprocal of the cubic surface $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} = 0$, $(\mathrm{XVI} = 12 - 4C_3,)$ with four cnicnodes [1].

It was convenient to make the foregoing enumeration before speaking of Kummer's paper (3), and of the several memoirs which relate to the Abbildung of certain quartic and quintic surfaces.

As regards Kummer's paper, the object appears by the title, viz., he considers in what cases a quartic surface has upon it a system of conics; or, what is the same thing, in what cases there is a system of planes each intersecting the surface in two conics. It is, in the first place, remarked that there is no proper quartic surface cut by every plane in a pair of conics, or even a proper quartic surface cut in a pair of conics by every plane through a fixed point. The cases considered are—I., where the planes are non-tangent planes; II., where they are single tangent planes; and III., where they are double tangent planes. The case I. is—(1) when there is a nodal conic and two cnicnodes; viz., any plane through the 2 cnicnodes gives a section with 4 nodes, therefore a pair of conics, (and the special case of 4 cnicnodes is noticed

[1] The Author exhibited, and pointed out some of the properties of, a model of Steiner's surface.

incidentally);—(2) when there is a nodal line; any plane through the nodal line besides meets the surface in a conic;—(3) when the surface has two "Selbstberührungspuncte"; viz., either of these is a point where the tangent plane is replaced by two coincident planes, and which, when a plane passes through it, gives in the section a tacnode, $=2$ nodes; the section by a plane through two such points, consists of two conics touching each other at the point in question: the equation of such a surface is $\phi^2=(*\between p, q)^4$, where ϕ is a quadric function, and p, q linear functions of the coordinates. In all the cases the planes pass through a fixed line, and the surface may be considered as the locus of a variable conic, the plane of which always passes through such line. II. is—(1) Steiner's surface, where every tangent plane meets the surface in a pair of conics; and (2) surface with a nodal conic and one cnicnode, where every tangent plane through the cnicnode meets the surface in a pair of conics. And III. is (1) the surface with a nodal conic, where every double tangent plane meets the surface in a pair of conics: it is shown that there are 5 quadric cones, such that a tangent plane of any one of these cones is always a double tangent plane of the surface. Or the surface is (2) a quartic scroll; any plane through two intersecting lines of the surface besides meets the surface in a conic.

It is in the paper, Cayley (6), remarked, that the quartic surface $(*\between U, V, W)^2=0$ can also be expressed in the form $UW-V^2=0$; under which form the surface is seen to be the envelope of the series of quadric surfaces $(U, V, W\between\theta, 1)^2=0$; and by reason of this property it is very easy to find the equations of the reciprocal surfaces, or plane-equations of the quartic surfaces in question. And, in the same paper, it is noticed that the surfaces of the form in question include the reciprocals of several interesting surfaces of the orders 6, 8, 9, 10, and 12; viz., order 6, parabolic ring: order 8, elliptic ring: order 9, centro-surface of paraboloid: order 10, parallel surface of paraboloid; envelope of planes through the points of an ellipsoid at right angles to the radius vectors from the centre: order 12, centro-surface of ellipsoid; parallel surface of ellipsoid.

It will be noticed that several of the papers by Clebsch and others refer in their titles to the "Abbildung" of a surface; viz., they show that a (1, 1) correspondence exists between the points of the surface and the points of a plane. The most simple instance is the quadric surface; here, taking any fixed point O on the surface, the line OP drawn to any point P on the surface meets a plane in a point P', and the points P, P' have, it is clear, a (1, 1) correspondence. And, of course, to any curve on the quadric surface there corresponds a curve on the plane, and the discussion of the nature of the plane curves which correspond to the different curves on the quadric surface would constitute a theory of the Abbildung of the quadric surface.

Similarly, as remarked, Clebsch (2), for a cubic surface, taking upon it any two lines which do not meet, if from a point P on the surface we draw, meeting each of the two lines, a line to meet the plane in P', then the point P on the cubic surface and the point P' on the plane will have a (1, 1) correspondence; and we have thus a like theory for the cubic surface. The Abbildung of a cubic surface had been, however, previously effected by Clebsch [in the paper "Die Geometrie auf den Flächen

dritter Ordnung," *Crelle*, t. LXV. (1866), pp. 359—380}, and by Cremona, in a different and really the most simple manner [1], but having a less obvious geometrical signification.

For surfaces of the higher orders, it is only certain surfaces which admit of an Abbildung, or (1, 1) correspondence of the points thereof with the points of a plane; viz. (in the same way as a plane curve, in order to its being unicursal, must have a sufficient number of nodes or cusps) a surface, in order that it may thus correspond with the plane (or say, in order that it may be unicursal), must have a sufficient singularity in the way of a nodal or cuspidal curve. The quartic and quintic surfaces considered in the enumerated memoirs are there considered for the sake of the Abbildung theory which they give rise to; whereas, in the present sketch, the Abbildung theory is considered only for the sake of the quartic and quintic surfaces to which the theory has been applied. But the methods of the theory furnish results in relation to these surfaces; and it is proper to give some account of them.

Clebsch's memoirs (2) and (3) relate to the same subject, which is elaborately treated in the latter of them: the former of them contains, however, some valuable remarks which are not reproduced in the other. In these memoirs (2) and (3), after explaining the above method of the transformation of a cubic surface by means of two of the lines thereof, the author goes on to notice that the like method is applicable to certain quartic and quintic surfaces; viz., (1) quartic surface with a nodal conic: there are here, as already mentioned, 16 lines, each meeting the conic; if, selecting any one of these, from a point P on the surface we draw, meeting the line and the conic, a line to cut the plane in P', then the points P on the surface and P' on the plane have a (1, 1) correspondence. (2) Quartic surface with a nodal line: as already mentioned, there are on the surface 128 conics, each meeting the nodal line; selecting any one of these, if from a point P of the surface we draw, meeting the nodal line and the conic, a line cutting the plane in P', then the point P on the surface, and the point P' on the plane, have a (1, 1) correspondence.

Similarly, (3), for a quintic surface having a nodal skew cubic; then if from a point P on the surface we draw, meeting the skew cubic twice, a line to cut the plane in P', the point P on the surface and the point P' on the plane have a (1, 1) correspondence. The nodal skew cubic may break up into a conic and line which meets it, or into three lines, two of them not meeting each other, but each met by the third line; and the like theory applies to these quintic surfaces.

It is to be noticed that (as for the cubic surface) the above methods of Abbildung, although they have the most obvious geometrical significations, are (as explained in the foregoing foot-note) not the most simple ones; but for each of the foregoing cases (1), (2), (3), the most simple transformation is established in the memoirs now under consideration. The memoir [Memoirs (1) and (2)] of Korndörfer, as indicated by its title, relates to the Abbildung of a quartic surface having a nodal conic and 1, 2, 3, or 4 cnicnodes.

[1] Any method of transformation leads to an expression of the coordinates of a point on the surface as proportional to rational and integral functions of a given degree ν of the coordinates (x, y, z) of a point on the plane, and that transformation is the more simple for which ν has the smaller value: for the method of the text, the value is $\nu=3$, but for the methods previously given by Clebsch and Cremona, it is $\nu=2$.

Clebsch's paper (4) relates to the Abbildung of a quartic scroll.

As regards quintic surfaces (not being scrolls), we have, so far as I am aware, only the before-mentioned paper, Clebsch (3), relating to quintic surfaces with a nodal skew cubic; and the paper, Clebsch (5), which relates to the Abbildung of a quintic surface having a nodal quadriquadric. The method employed is that of a preliminary Abbildung upon a twofold plane (2-blättrige Ebene); that is, it consists in establishing, in the first instance, a (1, 2) correspondence between the surface and the plane; and by means hereof it is shown that there exist on the surface the conics K and C presently referred to, and which give, ultimately, an ordinary Abbildung or (1, 1) correspondence of the points of the surface with those of the plane; viz., this final result is as follows:

There is on the surface a system of conics K, such that their planes pass through a point and envelope a quadricone; and also 64 conics C each meeting each of the conics K in a single point: we select one of these and call it the conic C.

Take now the plane B_1 of a conic K_1 of the series of conics K, which plane B_1 passes, of course, through the vertex V of the cone enveloped by the planes of the conics K; viz., these planes intersect the plane B_1 in a series of lines passing through the point V.

Take a point P on the surface; this lies on a conic K meeting the conic C in a point ξ; and if we draw the line ξP to meet the plane B_1 in P', then P on the quintic surface, and P' on the plane B_1, will have a (1, 1) correspondence; in fact, it appears that, given P, there exists a single position of P'; and conversely, given P', this lies on a line VP' through which there passes the plane B_1 and one other tangent plane of the cone: this tangent plane contains a conic K meeting the conic C in a point ξ; and joining $\xi P'$, this meets the conic K in one other point P; viz., given P', there is a single position of P; and there is thus a (1, 1) correspondence.

There are, as originally shown by Schläfli, and as further appears by my memoir on cubic surfaces, Cayley (5), 3 kinds of cubic surfaces of the class 5; viz., these are the surfaces $\text{XIII} = 12 - B_3 - 2C_2$, $\text{XIV} = 12 - B_5 - C_2$, and $\text{XV} = 12 - U_7$; for each of these the reciprocal surface is a quintic surface of the class 3, having a nodal line and a cuspidal quartic curve. For the reciprocal of XIII, the cuspidal curve is a quadriquadric; for that of XIV, the cuspidal curve breaks up into the nodal line (viz., this is a cuspnodal line) and into a skew cubic; for that of XV, the cuspidal curve is a cuspidal quadriquadric, or curve of intersection of two quadric surfaces with singular contact.

It only remains to speak of Schwarz's memoir on quintic scrolls: it is to be remarked that the theory of scrolls is allied more closely with that of plane curves than with that of surfaces; viz., considering any plane section of the scroll, the lines of the scroll have, in general, a (1, 1) correspondence with the points of the plane section, and the scrolls of any given order are properly arranged according to the deficiency of the plane section. This is what is done by Cremona in the memoir on quartic

scrolls above referred to; viz., for a quartic scroll the deficiency is either 0 or 1; and of the 12 species, there are 10 for which the deficiency is $=0$ (or which are unicursal), and 2 for which the deficiency is $=1$. And this is the principle of classification in Schwarz's memoir; viz., for a quintic scroll the deficiency is $=0$, 1, or 2; the number of species established being 10, 4, and 1 for these deficiencies respectively.

List of Memoirs.

Cayley. 1. Sur la surface des ondes. *Liouv.* t. XI. (1846), pp. 291—296, [47].

——— 2. Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers. *Crelle*, t. LXV. (1866), pp. 284—290, [356].

——— 3. Second Memoir on Skew Surfaces, otherwise Scrolls. *Phil. Trans.*, vol. CLIV. (1864), pp. 559—577, [340].

——— 4. Third Memoir on Skew Surfaces, otherwise Scrolls. *Phil. Trans.*, vol. CLIX. (1869), pp. 111—126, [410].

——— 5. Memoir on Cubic Surfaces. *Phil. Trans.*, vol. CLIX. (1869), pp. 231—326, [412].

——— 6. On the Quartic Surfaces $(*\between U, V, W)^2=0$. *Quart. Math. Journ.* vol. X. (1868), pp. 24—34; and vol. XI. (1870), pp. 15—25, and pp. 111—113.

——— 7. A Memoir on Quartic Surfaces. *Proc. Lond. Math. Soc.*, vol. III. (1870), pp. 19—69, [445].

Clebsch. 1. Ueber die Flächen vierter Ordnung welche eine Doppelcurve zweiten Grades besitzen. *Crelle*, t. LXIX. (1868), pp. 355—358.

——— 2. Intorno alla rappresentazione di superficie algebriche sopra un piano. *Atti del R. Ist. Lomb.* (12 Nov. 1868), 13 p.

——— 3. Ueber die Abbildung algebraischer Flächen, insbesondere der vierten und fünften Ordnung. *Math. Ann.*, t. I. (1869), pp. 253—316.

——— 4. Ueber die ebene Abbildung der geradlinigen Flächen vierter Ordnung, welche eine Doppelcurve dritten Grades besitzen. *Math. Ann.*, t. II. (1870), pp. 445—466.

——— 5. Ueber die Abbildung einer Classe von Flächen fünfter Ordnung. *Gött. Abh.*, t. XV. 64 p.

Cremona. Sulle superficie gobbe di quarto grado. *Mem. di Bologna*, t. VIII. (30 April, 1868), 15 p.

De la Gournerie. 1. Mémoire sur la surface engendrée par la révolution d'une conique autour d'une droite située d'une manière quelconque dans l'espace. *Jour. de l'Éc. Polyt.*, t. XXIII. (1863), pp. 1—74.

——— 2. Mémoire sur les lignes spiriques. *Liouv.*, t. XIV. (1869), 92 p.

Geiser. Ueber die Flächen vierten Grades welche eine Doppelcurve zweiten Grades haben. *Crelle*, t. LXX. (1869), pp. 249—257.

Korndörfer. [There was only a general reference; the several memoirs are

——— 1. Die Abbildung einer Fläche vierter Ordnung mit einer Doppelcurve zweiten Grades und einem oder mehreren Knotenpunkten. *Math. Ann.*, t. I. (1869), pp. 592—626.

——— 2. Fortsetzung dieses Aufsatzes. t. II. (1870), pp. 41—64.

——— 3. Ueber diejenigen Raumcurven deren Coordinaten sich als rationale Functionen eines Parameters darstellen. t. III. (1871), pp. 415—423.

——— 4. Die Abbildung einer Fläche vierter Ordnung mit zwei sich schneidenden Doppelgeraden. t. III. (1871), pp. 496—522.

——— 5. Die Abbildung einer Fläche vierter Ordnung mit einer Doppelcurve zweiten Grades welche aus zwei sich schneidenden unendlich nahen Geraden besteht. t. IV. (1871), pp. 117—134.]

Kummer. 1. {Surfaces of the fourth order with sixteen conical points.} *Berl. Monatsb.* (1864), pp. 246—260, and 495—499.

——— 2. Ueber die algebraischen Strahlensysteme, insbesondere über die der ersten und zweiten Ordnung. *Berl. Abh.* (1866), pp. 1—120.

——— 3. Ueber die Flächen vierten Grades auf welchen Schaaren von Kegelschnitten liegen. *Berl. Monatsb.* (July, 1863). *Crelle*, t. LXIV. (1864), pp. 66—76.

Maxwell. On the Cyclide. *Quart. Math. Journ.*, t. IX. (1867), pp. 111—126.

Schwarz. Ueber die geradlinigen Flächen fünften Grades. *Crelle*, t. LXVII. (1867), pp. 23—57.

450.

NOTE ON THE THEORY OF THE RATIONAL TRANSFORMATION BETWEEN TWO PLANES, AND ON SPECIAL SYSTEMS OF POINTS.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 196—198. Read December 8, 1870.]

In Prof. Cremona's theory of the transformation of plane curves, the fundamental equations are taken to be

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 + \ldots = n^2 - 1 \qquad (1),$$

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots = \tfrac{1}{2}(n^2 + 3n) - 2 \qquad (2);$$

and from these we have as a consequence

$$\alpha_2 + 3\alpha_3 + \ldots = \tfrac{1}{2}(n-1)(n-1) \qquad (3);$$

viz., the first equation expresses that any two curves of the system intersect in a single variable point; the second, that the curves form a *réseau*, or system containing two arbitrary parameters; and the third, that the curves are unicursal.

In the equivalent theory of the rational transformation between two planes, as given in my "Memoir on the Rational Transformation between Two Spaces," [447], we have the equation (1); but instead of the equation (2), it would *primâ facie* appear to be sufficient if we had the inequation

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots \overset{=}{<} \tfrac{1}{2}(n^2 + 3n) - 2;$$

but on the ground there explained, the case

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots < \tfrac{1}{2}(n^2 + 3n) - 2$$

is excluded, and we thus have the equation (2), giving with (1) the equation (3).

I believe the better course is to assume (1) and (3) as the fundamental equations, from them deducing (2); and we thus also get over a difficulty presently referred to, but which did not occur to me when the memoir was written.

In fact, starting with the equations $x' : y' : z' = X : Y : Z$ (which are to give $x : y : z = X' : Y' : Z'$), we have in the first instance the equation (1). Moreover, establishing for x', y', z' a linear equation $ax' + by' + cz' = 0$, we have corresponding hereto a curve $aX + bY + cZ = 0$, and the coordinates x, y, z of a point on this curve are proportional to $X' : Y' : Z'$; that is, substituting for z' the value $-\frac{1}{c}(ax' + by')$, they are proportional to rational and integral (homogeneous) functions of (x', y'), that is, to rational and integral functions of the single parameter $x' : y'$; wherefore the curve $aX + bY + cZ = 0$ is unicursal; whence the equation (3). The like change may be made in the theory of the rational transformation between two spaces; and it is in this case a more important one.

The difficulty is as follows: It is not self-evident that we are at liberty to assume

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots \overline{\overline{<}} \tfrac{1}{2}(n^2 + 3n) - 2;$$

for imagine that we had a system of $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ points, such that $\alpha_1 + 4\alpha_2 + \ldots$ being $= n^2 - 1$, and $\alpha_1 + 3\alpha_2 + \ldots$ being $> \frac{1}{2}(n^2 + 3n) - 2$, the points were such that the conditions in question (viz., the condition that the curve passes once through each of the points α_1, twice through each of the points $\alpha_2 \ldots$) should be *less* than $\alpha_1 + 3\alpha_2 + \ldots$, and in fact $=$ or $< \frac{1}{2}(n^2 + 3n) - 2$; then the functions X, Y, Z would not of necessity be connected by a linear relation $\lambda X + \mu Y + \nu Z = 0$, and the ground for the assumption in question, $\alpha_1 + 3\alpha_2 + \ldots \overline{\overline{<}} \frac{1}{2}(n^2 + 3n) - 2$, would no longer exist. And except by the process now adopted of deriving the equation (2) from the equations (1) and (3), I do not know how the impossibility of such a system is to be established; viz., I do not know how we are to prove the following theorem:—There is not any system of $(\alpha_1, \alpha_2, \alpha_3 \ldots)$ points, where

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 \ldots = n^2 - 1,$$

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots > \tfrac{1}{2}(n^2 + 3n) - 2,$$

such that (for a curve of the order n passing once through each point α_1, twice through each point $\alpha_2, \ldots$) the number of conditions actually imposed on the curve is $=$ or $< \frac{1}{2}(n^2 + 3n) - 2$.

A system of $(\alpha_1, \alpha_2 \ldots)$ points such that the number of actually imposed conditions is less than $\alpha_1 + 3\alpha_2 + \ldots$, may be termed a special system; we have, of course, the well-known case $(\alpha_1 = n^2)$ of a system of n^2 points, such that any curve of the order n passing through $\frac{1}{2}(n^2 + 3n) - 1$ of these passes through all the remaining points {or what is the same thing, where the number of conditions actually imposed is $= \frac{1}{2}(n^2 + 3n) - 1$}; and we have the following special system, which presented itself to Dr Clebsch, in his researches on the Abbildung of a quintic surface with two non-intersecting nodal

lines; viz., "$\alpha_1 = 12$, $\alpha_2 = 2$. We may have 12 points and 2 points such that, for a quintic curve passing once through each of the 12 points and twice through each of the 2 points, the number of conditions actually imposed (instead of being $12 + 3 . 2, = 18$) is $= 17$." The construction is as follows: viz., starting with the 2 points and any 10 points, we may draw a quartic passing twice through the first of the 2 points, once through the second of them, and through the 10 points; and another quartic passing twice through the second of the 2 points, once through the first of them, and through the 10 points: the two quartics intersect in the 2 points each twice, in the 10 points, and in 2 new points, forming, with the 10 points, a system of 12 points; and the first-mentioned 2 points and the 12 points form the system in question.

A more complicated case, $\alpha_1 = 10$, $\alpha_2 = 6$, $\alpha_3 = 1$, occurs in Dr Nöther's paper, "Ueber Flächen, welche Schaaren rationaler Curven besitzen," [*Math. Ann.*, t. III. (1871), pp. 161—227]. Except these two, I do not know any other case of a special system for which $\alpha_2, \alpha_3 \ldots$ are not all $= 0$; the investigation of such systems would, I think, be very interesting.

[A concluding paragraph of seven lines gave some corrections to the "Memoir on the Rational Transformation between Two Spaces," 447, which corrections are made in the present reprint of that paper.]

451.

A SECOND MEMOIR ON QUARTIC SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 198—202. Read December 8, 1870.]

In my Memoir on Quartic Surfaces, *ante* pp. 19—69, [445], although remarking (see No. 79) that the identification was not completely made out, I tacitly assumed that the symmetroid and the decadianome (each of them a quartic surface with 10 nodes) were in fact identical. There is yet a good deal which I cannot completely explain; but the truth appears to be, that the decadianome includes two cases of coordinate generality, say the sextic decadianome, and the bicubic decadianome = symmetroid: viz., in the first of these the circumscribed cone, having for vertex any one of the 10 nodes, is a proper sextic cone with 9 double lines; in the second it is a system of two cubic cones, intersecting, of course, in 9 lines, which are double lines of the aggregate sextic cone: or, in the notation of the Table No. 11, in the case of the sextic decadianome, the circumscribed cones are each of them 6_9; in that of the bicubic decadianome = symmetroid, they are each of them (3, 3). We thus arrive at a very remarkable system of 10 points in space, viz., giving the name "ennead" to any 9 points *in plano*, which are the intersections of two cubic curves, or to any 9 lines through a point which are the intersections of two cubic cones; the 10 points in space are such that, taking any one whatever of them as vertex, and joining it with the remaining points, the 9 lines form an ennead. I purpose in the present short Memoir to consider the theories in question: the paragraphs are numbered consecutively with those of the Memoir on Quartic Surfaces.

Plane Sextic Curve with 9 Nodes.

110. A sextic curve contains 27 constants; and the number of conditions to be satisfied in order that a given point may be a node is $=3$. Hence it would at first sight appear that the curve could be found so as to have 9 given nodes; this would

be $9 \times 3, = 27$ conditions, or the curve would be completely determinate. But observe that through the 9 given points we have a determinate cubic curve $U=0$; we have therefore $U^2=0$ a sextic curve, and the only sextic curve with the 9 given nodes; that is, there is not in a proper sense any sextic curve with the 9 given nodes. The number of given nodes is thus $=8$ at most.

111. The sextic curve with 8 given nodes should contain $27-3.8=3$ constants. We may through the 8 given points draw the two cubics $P=0$, $Q=0$; and we have then $(a, b, c \between P, Q)^2=0$, a bicubic, or improper sextic curve having the 8 nodes, and also a ninth node, viz., the remaining point of intersection of the two cubic curves, or say the remaining point of the ennead. Hence if $\nabla=0$ be any particular sextic curve having the 8 given nodes, we have

$$(a, b, c \between P, Q)^2+\theta\nabla=0$$

a proper sextic curve having the 8 given nodes; and this, as containing the right number $(=3)$ of constants, will be the general sextic curve having the 8 given nodes.

112. There will be a ninth node if $\theta=0$; viz., the curve is then $(a, b, c \between P, Q)^2=0$, a bicubic, or improper sextic curve, having for nodes the 9 points of the ennead. Observe that the ninth node is here a point completely and uniquely determined by means of the given 8 nodes. Moreover the number of constants is $=2$, so that we have here a general (improper) solution of the question of finding a sextic curve with 9 nodes, 8 of them given.

113. But if θ is not $=0$, then the ninth node must be a point on the curve $J(P, Q, \nabla)=0$; viz., this is a curve of the order 9, determined by means of the given 8 points; say it is the "dianodal curve" of these 8 points, and, as is easy to see, it has each of these 8 points for a node. The ninth node of the sextic may be any point whatever on the dianodal curve; and regarding it as a given point, the sextic will still contain 1 constant; that is, we have the general solution of the problem of finding a sextic curve with 9 nodes, 8 of them given, and the 9th a given point on the dianodal curve.

114. So long as the 8 points are arbitrary, the dianodal curve does not pass through the 9th point of the ennead, and the two cases above considered are mutually exclusive. It will be noticed how closely analogous this theory of the plane sextic with 9 nodes, is to that of the quartic surface with 8 nodes.

115. Of course, instead of the plane sextic curve, we may have a sextic cone; such a cone has at most 8 given double lines; and if there be a 9th double line, then there are the two cases of coordinate generality; viz., (1), the new double line is the ninth line of the ennead, the cone being in this case not a proper sextic cone, but a bicubic cone; (2), the new double line may be any line whatever on the dianodal cone, (cone of the order 9 determined by the 8 given lines, and having each of these for a double line,) and regarding it as a given line on the dianodal cone, the sextic cone contains 1 constant.

Each circumscribed cone of the Symmetroid is (3, 3).

116. Using (x, y, z, w) as current coordinates of a point of the symmetroid, I take S, T, U, V to be quadric functions of the coordinates $(\alpha, \beta, \gamma, \delta)$; the equation of the symmetroid is therefore given by

$$xS + yT + zU + wV = \text{cone},$$

and the nodes thereof are determined by

$$xS + yT + zU + wV = \text{plane-pair}.$$

Suppose that a node is $(x=0,\ y=0,\ w=0)$; the condition for this is $V = \text{plane-pair}$; and we may without loss of generality write $V = \gamma^2 + \delta^2$. Hence, putting for shortness $\Theta = xS + yT + zU$, that is Θ a quadric function

$$(a, b, c, d, f, g, h, l, m, n \between \alpha, \beta, \gamma, \delta)^2,$$

wherein the coefficients $a, b, \ldots$ are arbitrary linear functions of (x, y, z), but not containing w, the equation of the symmetroid is given by

$$\Theta + w(\gamma^2 + \delta^2) = \text{cone}.$$

117. It follows that the equation is

$$\begin{vmatrix} a, & h, & g & , & l \\ h, & b, & f & , & m \\ g, & f, & c+w, & & n \\ l, & m, & n & , & d+w \end{vmatrix} = 0;$$

viz., this is

$$\nabla + w(\delta_c + \delta_d)\nabla + \tfrac{1}{2}w^2(\delta_c + \delta_d)^2\nabla = 0,$$

where ∇ denotes the foregoing determinant, writing therein $w=0$. Or, observing that ∇ contains c and d each only linearly, the equation may be written

$$\nabla + w(\delta_c + \delta_d)\nabla + w^2\delta_c\delta_d\nabla = 0,$$

which is a quartic surface having, as it should have, the point (0, 0, 0) for one of its ten nodes.

118. The equation of the circumscribed cone is

$$(\delta_c\nabla + \delta_d\nabla)^2 - 4 \quad \nabla.\delta_c\delta_d\nabla = 0;$$

or, what is the same thing, it is

$$(\delta_c\nabla - \delta_d\nabla)^2 + 4(\delta_c\nabla.\delta_d\nabla - \nabla.\delta_c\delta_d\nabla) = 0.$$

But we have identically

$$\delta_c\nabla.\delta_d\nabla - (\tfrac{1}{2}\delta_n\nabla)^2 = \nabla.\delta_c\delta_d\nabla;$$

so that the equation is

$$(\delta_c\nabla - \delta_d\nabla)^2 + (\delta_n\nabla)^2 = 0\,;$$

a sextic cone breaking up into the two cubic cones

$$\delta_c\nabla - \delta_d\nabla \pm i\delta_n\nabla = 0,$$

so that the cone is (3, 3). And since clearly the point (0, 0, 0) may be regarded as representing any one whatever of the 10 nodes, it follows that for any node whatever of the symmetroid, the circumscribed cone is (3, 3), so that, as stated above, bicubic decadianome = symmetroid.

Deductions from the foregoing theory.

119. Referring to No. 85 of the original memoir, it appears that, with 6 given points as nodes, we can actually find for the symmetroid an equation containing 6 constants. I cannot discover any ground for doubting that 3 of these may be determined so as to give to the symmetroid a seventh given node; and I therefore assume that with 7 given points as nodes, an equation can be found with 3 constants. The symmetroid is certainly not octadic, hence the eighth node must lie on the dianodal surface of the 7 given points. I can discover no ground for doubting but that two of the constants may be determined so that the eighth node shall be any given point whatever on the dianodal surface of the 7 points; and (this being so) that further the remaining constant may be determined so that the ninth node shall be any given point whatever on the dianodal curve of the 8 points. But if all this be so, the consequence is very remarkable; the tenth node is not any one whatever of the 22 dianodal centres of the 9 points, but it is a uniquely determinate "enneadic centre," viz., we must have the following theorem:

120. "Take any 7 points; an eighth point at pleasure on the dianodal surface of the 7 points; a ninth point at pleasure on the dianodal curve of the 8 points. In the system of 9 points so determined, take any one as vertex, and joining it with the remaining 8, construct the ninth line of the ennead. Performing this construction with each of the 9 points successively as vertex, we obtain 9 lines passing through the 9 points respectively. These 9 lines meet in a point which is the 'enneadic centre' of the 9 points: and further, the 10 points form a completely symmetrical system, so that each one of them is the enneadic centre of the remaining 9."

121. Assuming that the 9 lines do intersect so as to give rise to an enneadic centre, there is no difficulty in conceiving that the loci, which by their intersection determine the dianodal centres, do each of them pass through the enneadic centre; so that this enneadic centre counts once or more among the dianodal centres, and the number of proper dianodal centres, instead of being $=22$, will be suppose $=22-\omega$, and if, further, the 9 points, together with the enneadic centre, are the nodes of a symmetroid, but the 9 points together with any one of the $22-\omega$ dianodal centres are the nodes of a sextic decadianome, then we must also have as follows:

122. "Considering any 9 points as above; taking any one as vertex, and joining it with the remaining 8, these 8 lines determine a dianodal ninthic cone. We have thus 9 dianodal cones, which cones pass all of them through the same $22-\omega$ points."

123. I am not able to verify these theorems *à posteriori*. It appears to me that the theorem in regard to the enneadic centre subsists for a system of 9 points such as referred to in the statement; but that if by possibility the statement be too general, the theorem must, at all events, subsist for a more special system of 9 points; and that there certainly exist systems of 10 points, such that each 9 of the points have as an enneadic centre the tenth point. {I have since ascertained that if a quartic surface with 10 nodes has a single node (3, 3), the surface is a symmetroid; whence, by what precedes, the remaining nine nodes are each of them (3, 3). Added 25 March, 1871.}

124. I notice, as a subject of investigation, the following system of correspondence viz., given any 8 points in space: then to every point in space corresponds a line through this point, viz., the ninth line of the ennead obtained by joining the point with the 8 given points respectively; and to each line in space a point or points on the line, viz., the point or points for each of which the line is the ninth line of the ennead obtained by joining the point with the eight given points respectively.

452.

ON AN ANALYTICAL THEOREM FROM A NEW POINT OF VIEW.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 220, 221. Read February 9, 1871.]

THE theorem is a well-known one, derived from the equation

$$(az^2+2bz+c)\,w^2+2\,(a'z^2+2b'z+c')\,w+a''z^2+2b''z+c''=0\,;$$

viz., considering this equation as establishing a relation between the variables z and w, and writing it in the forms

$$2u=Aw^2+2Bw+C=A'z^2+2B'z+C'=0,$$

(where, of course, A, B, C are quadric functions of z, and A', B', C' quadric functions of w,) we have

$$0=\frac{du}{dw}\,dw+\frac{du}{dz}\,dz,\ =(Aw+B)\,dw+(A'z+B')\,dz;$$

but in virtue of the equation $u=0$, we have $Aw+B=\sqrt{B^2-AC}$, and $A'z+B'=\sqrt{B'^2-A'C'}$, and the differential equation thus becomes

$$\frac{dw}{\sqrt{B'^2-A'C'}}+\frac{dz}{\sqrt{B^2-AC}}=0,$$

where $B'^2-A'C'$ and B^2-AC are quartic functions of w and z respectively. This is, of course, integrable (viz., the integral is the original equation $u=0$); and it follows, from the theory of elliptic functions, that the two quartic functions must be linearly transformable into each other; viz., they must have the same absolute invariant $I^3\div J^2$. It is, in fact, easy to verify, not only that this is so, but that the two functions have the same quadrinvariant I, and the same cubinvariant J.

The new point of view is, that we take the coefficients a, b, &c., to be homogeneous functions of (x, y), their degrees being such that the equation $u=0$ is a quartic equation $(*\between x, y, z, w)^4=0$: viz., this equation now represents a quartic surface having a node (conical point) at the point $(x=0, y=0, z=0)$, and also a node at the point $(x=0, y=0, w=0)$, say, these points are O, O' respectively. The equation $B'^2-A'C'=0$ gives the circumscribed sextic cone having O for its vertex, and the equation $B^2-AC=0$ the circumscribed sextic cone having O' for its vertex; each of these cones has the line OO' $(x=0, y=0)$ for a nodal line, as appears geometrically, and also by the equations containing z, w respectively in the degree 4. Considering $B'^2-A'C'$ as a quartic function of z, its quadrinvariant is a function $(x, y)^8$, and its cubinvariant a function $(x, y)^{12}$; and similarly, considering B^2-AC as a quartic function of w, its invariants are functions $(x, y)^8$ and $(x, y)^{12}$. We have thus, between the two cones, a geometrical relation answering to the analytical one of the identity of the invariants; but the nature of this geometrical relation is not obvious; and it presents itself as an interesting subject of investigation.

453.

ON A PROBLEM IN THE CALCULUS OF VARIATIONS.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 221, 222. Read February 9, 1871.]

THE problem is, $z = \frac{1}{3}(3x - y^2)y$, to find y a function of x such that $\int z dx = \text{max.}$ or min., subject to a given condition $\int y dx = c$ (the limits of each integral being x_1, x_0, where these quantities are each positive, and $x_1 > x_0$). The ordinary method of solution gives $y^2 = x + \lambda$, where $(x_1 + \lambda)^{\frac{3}{2}} - (x_0 + \lambda)^{\frac{3}{2}} = \frac{3}{2}c$; so long as c is not less than $(x_1 - x_0)^{\frac{3}{2}}$, there is a real value of λ, but for a smaller value of c there is no real value. The difficulty arising in this last case is somewhat illustrated by replacing the original problem by a like problem of ordinary maxima and minima; viz., $x_1, x_2 \ldots x_n$ being given positive values of x, in the order of increasing magnitude; and if, in general, $z_i = (3x_i - y_i^2)y_i$, then the problem is to find y_i a function of x_i, such that $\Sigma z_i = \text{max.}$ or min., subject to the condition $\Sigma y_i = c$. We have here $y_i^2 = x_i + \lambda$, where λ is to be determined by the condition $\Sigma y_i = c$; the remainder of the investigation turns on the question of the sign $y_i = +\sqrt{x_i + \lambda}$ or $y_i = -\sqrt{x_i + \lambda}$, to be taken for the several values of i respectively.

454.

A THIRD MEMOIR ON QUARTIC SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 234—266. Read April 13, 1871.]

THE present Memoir is a continuation of my former researches on Nodal Quartic Surfaces, [445, 451]. The leading idea is, that for a quartic surface with k-nodes, given the nature of the circumscribed, $(k-1)$ nodal, sextic cone belonging to any one node of the surface {for instance, $k=10$, that it is a cone (3, 3) composed of two cubic cones}, we thereby determine the equation of the quartic surface, and consequently the nature of the remaining $(k-1)$ nodes thereof. By means of this general theory I complete, in an essential point, the theory of the *Symmetroid;* viz., I show that a 10-nodal quartic surface having a single node (3, 3) is a Symmetroid; whence, as appears by my second Memoir, [451], each of the remaining nine nodes is also a node (3, 3); and we have the theory of the remarkable system of ten points in space such that, joining any one of them with the remaining nine, the nine lines thus obtained are the intersections of two cubic cones. A large part of the Memoir is devoted to the consideration of the surfaces with 16, 15, 14, and 13 nodes: this is substantially a reproduction of the results obtained by Kummer in the Memoir "Ueber die algebraischen Strahlensysteme, &c.," already referred to; but the results in question are brought into connexion with the theory of the present Memoir, and they are, by a change of the constants, exhibited in a form of much greater symmetry and elegance. I attach importance also to the square diagrams by means of which I have exhibited, in a compendious form, the relation between the several nodes and circumscribed sextic cones.

The paragraphs are numbered consecutively with those of the first and second Memoirs.

Preliminary Considerations and Classification.

125. I call to mind that if a quartic surface has a node (conical point), then there is for this node a tangent quadricone and a circumscribed sextic cone; viz., if the surface has $(k-1)$ other nodes, or in all k nodes, then the sextic cone has $(k-1)$

nodal lines (passing through the other nodes respectively), and we have thus for the different forms of the sextic the table No. 11; viz., this is

CIRCUMSCRIBED SEXTIC CONE.

Nodes of Surface.	Nodal Lines of Cones.								
1	0	6							
2	1	6_1							
3	2	6_2							
4	3	6_3							
5	4	6_4							
6	5	6_5	5 , 1						
7	6	6_6	5_1, 1						
8	7	6_7	5_2, 1						
9	8	6_8	5_3, 1	4 , 2					
10	9	6_9	5_4, 1	4_1, 2	4 , 1, 1	3 , 3			
11	10	6_{10}	5_5, 1	4_2, 2	4_1, 1, 1	3_1, 3			
12	11	...	5_6, 1	4_3, 2	4_2, 1, 1	3_1, 3_1	3 , 2, 1		
13	12	...	...	...	4_3, 1, 1	...	3_1, 2, 1	3 , 1, 1, 1	2 , 2, 2
14	13	...	...	...	...	...	...	3_1, 1, 1, 1	2 , 2, 1, 1
15	14	...	...	...	...	...	...	...	2_1, 1, 1, 1, 1
16	15	...	...	...	...	...	...	...	1 , 1, 1, 1, 1, 1

viz., 6 denotes a proper sextic cone without nodal lines; 6_1 a proper sextic cone with one nodal line; 5, 1 a proper quintic cone and a plane, &c.

We may distinguish the nodes according to the sextic cones; thus, a node 6 means a node for which the circumscribed cone is a proper sextic cone, (1, 1, 1, 1, 1, 1) a node where the circumscribed cone breaks up into six planes, &c.

126. A 16-nodal surface has 16 nodes (1, 1, 1, 1, 1, 1), and a 15-nodal surface has 15 nodes (2, 1, 1, 1, 1); but, for a 14-nodal surface, the question arises how many nodes are (3_1, 1, 1, 1), and how many (2, 2, 1, 1). It was remarked, No. 13, that the only possible cases were 14, 0; 8, 6; or 2, 12; and that we might, in like manner, limit the number of possible cases for other values of k; but that the inquiry was not then further pursued. I resume this inquiry, but without obtaining as yet a complete answer.

127. It is to be observed that a line joining any two nodes is not, in general, a line on the surface, but that it may be so; the surfaces for which this is so (viz., any surface which contains upon it a line through two nodes) form, however,

a class by themselves, which at present I *altogether exclude from consideration.* This being so, it will appear in the sequel that there is but one kind of surface having a node (2, 2, 1, 1), and but one kind of surface having a node $(3_1, 1, 1, 1)$. Now there is a surface, Kummer's 14-nodal, the nodes of which are $8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$; wherefore the two kinds are identical, and are each of them Kummer's 14-nodal surface. Similarly, for the 13-nodal surfaces, there is but one kind having a node $(4_3, 1, 1)$, but one kind having a node $(3, 1, 1, 1)$, and moreover but one kind having a node $(3_1, 2, 1)$; and we have Kummer's 13-nodal surface with the nodes $3(4_3, 1, 1)+1(3, 1, 1, 1)+9(3_1, 2, 1)$; hence the three kinds are identical with each other and with Kummer's. Moreover, there is but one kind having a node (2, 2, 2); hence all the other nodes must be (2, 2, 2), and we have a surface 13 (2, 2, 2) not given by Kummer. And in like manner for the 12-nodal surfaces, we have the two kinds given by Kummer, and a third kind $12(4_2, 1, 1)$ not given by him; the arrangement thus far being

No. of Nodes.	Character of Surface.
16	$16(1, 1, 1, 1, 1, 1)$,
15	$15(2, 1, 1, 1, 1)$,
14	$8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$,
13 (α)	$3(4_3, 1, 1)+1(3, 1, 1, 1)+9(3_1, 2, 1)$,
„ (β)	$13(2, 2, 2)$,
12 (α)	$12(4_3, 2)$,
„ (β)	$2(5_6, 1)+6(3_1, 3_1)+4(3, 2, 1)$,
„ (γ)	$12(4_2, 1, 1)$.

128. But in the next following case we have Kummer's surface, viz.

$$11\ (\alpha)\quad 1(6_{10})+10(3_1, 3),$$

and I do not know whether one, two, or three kinds of surface having nodes $(4_1, 1, 1)$, $(4_2, 2)$, and $(5_5, 1)$. And in the next case we have (as will appear) the Symmetroid, viz.,

$$10\ (\alpha)\quad 10(3, 3),$$

and I do not know how many kinds of surfaces having a node or nodes 6_9, $(5_4, 1)$, $(4_1, 2)$, $(4, 1, 1)$.

It will be observed that the present division has nothing to do with the octadic and dianodal division in the former Memoir.

129. I consider a conic $A=0$, and any six tangents thereof, $t_1=0$, $t_2=0$, $t_3=0$, $t_4=0$, $t_5=0$, $t_6=0$; we have an identical equation which might be written $AC-B^2=t_1t_2t_3t_4t_5t_6$, but it will be convenient, introducing a constant factor K, to write it

$$AC-B^2=Kt_1t_2t_3t_4t_5t_6,$$

B being a cubic function and C a quartic function of the coordinates.

Consider now the series of factors, such as

$$\begin{array}{ll} t_1, & \\ lA & +m\,t_1t_2, \\ sA & +m\,t_1t_2t_3, \\ UA & +m\,t_1t_2t_3t_4, \\ VA & +m\,t_1t_2t_3t_4t_5, \\ WA & +m\,t_1t_2t_3t_4t_5t_6, \end{array}$$

where m is a constant, l a constant, s a linear function, U, V, W functions of the degrees 2, 3, 4 respectively; and compose with one or more such factors an expression involving the term $Kt_1t_2t_3t_4t_5t_6$; for instance, such an expression is

$$\frac{K}{mm'}(sA+mt_1t_2t_3)(l'A+m't_4t_5)\,t_6;$$

this is, of the form $A\Omega+Kt_1t_2t_3t_4t_5t_6$, viz., $A\Omega+(AC-B^2)$, or $A(\Omega+C)-B^2$, say $A\Gamma-B^2$; or what is the same thing, introducing a new coordinate w, we have a quadric function

$$Aw^2+2Bw+\Gamma,$$

the discriminant of which, $A\Gamma-B^2$, is equal to the expression in question.

130. In the sequel (x, y, z, w) are considered as the coordinates of a point in space; $A=0$ is thus a quadric cone, $t_1=0$, $t_2=0\ldots t_6=0$, any six tangent planes thereof; and hence $Aw^2+2Bw+\Gamma=0$ a quartic surface, having the point $(x=0, y=0, z=0)$ for a node, whereof the circumscribed cone $A\Gamma-B^2=0$ breaks up in the assumed manner.

Thus, in order that the circumscribed cone may be as above

$$(sA+mt_1t_2t_3)(l'A+m't_4t_5)\,t_6,$$

we have only to assume

$$\Gamma=C+\frac{K}{mm'}(sl'A+sm't_4t_5+l'mt_1t_2t_3)\,t_6,$$

and so in other cases. Observe that $sA+mt_1t_2t_3=0$ is a cubic cone, which, so long as s, m are arbitrary, has no nodal line; but establishing a single relation (say s remains arbitrary, but a proper value is assigned to m) it will be a cubic cone having a nodal line. And so $UA+mt_1t_2t_3t_4=0$ is a quartic cone without any nodal line, but by particularising the constants it may be made to have one, two, or three nodal lines. Such nodal determinations are obviously required in order that the formula may extend to all the before-mentioned forms of the circumscribed cone. The foregoing analysis is the foundation of the whole theory: I have given it, as above, apart from the theory, in order that the nature of it may be the better perceived; but I have now to bring it into connexion with the theory.

On the Sextic Curves, $A_2B_4-C_3^2=0$.

131. I revert to the consideration of plane curves. The equation of a sextic curve $(*\between x, y, z)^6=0$ cannot be in general expressed in the form $AC-B^2=0$, where

the degrees of A, B, C are 2, 3, 4 respectively; in fact, the existence of such a form implies that there is a conic $A=0$ touching the sextic 6 times; and since a conic can only be made to satisfy 5 conditions, there is not in general any such conic.

132. Such conic, when it exists, is said to be inscribed in the sextic, and the sextic to be circumscribed about the conic, or to be an "amphigram;" and then, $A=0$ being the equation of the conic, that of the sextic is expressible in the form in question $AC-B^2=0$. It is clear that $B=0$ is a cubic curve passing through the 6 points of contact of the conic with the sextic, and that any such curve may be taken for the curve B; in fact if a particular cubic through the 6 points is $B'=0$, and the equation of the sextic is $AC'-B'^2=0$, then taking p an arbitrary linear function of the coordinates, the equation of the general cubic is $B=B'+pA=0$; and then writing

$$A=A,$$
$$B=B'+pA,$$
$$C=C'+2B'p+Ap^2,$$

we have $AC-B^2=AC'-B'^2$; so that the original form $AC'-B'^2=0$ becomes $AC-B^2=0$. But the cubic $B=0$ being assumed at pleasure, the quartic $C=0$ is a determinate curve.

133. It is to be observed that a sextic curve may be an amphigram in more than one way: certainly in two, three, or four, and possibly in a greater number of ways. For the equation of the curve contains 27 constants, and hence determining the sextic so as to touch 4 given conics each of them 6 times, there are still 3 constants; and the curve will be an amphigram in regard to each of the 4 conics; say it is a quadruple amphigram. But in the sequel we are only concerned with a sextic curve considered as an amphigram in regard to a given conic $A=0$ (no attention being paid to the other inscribed conics, if any); and then, by what precedes, taking $B=0$ any cubic whatever through the 6 points of contact, we have a determinate quartic curve $C=0$, and the equation of the sextic curve assumes the form $AC-B^2=0$.

134. The curves $A=0$, $B=0$, $C=0$ contain respectively 5, 9, 14 constants; whence considering the function B as containing an arbitrary constant factor, for the curve $AC-B^2=0$, the number of constants is *primâ facie* $5+9+14+1=29$; but on account of the arbitrary linear function p, the real number is $29-3=26$: this is right, for a sextic curve contains 27 constants; and the curve being an amphigram, there is one relation between the constants, $27-1=26$.

135. Suppose now that the sextic curve $AC-B^2=0$ breaks up into two or more separate curves, say into the two curves $P=0$, $Q=0$ of the orders f, g respectively; $f+g=6$. We have

$$AC-B^2=PQ=0;$$

and the conic $A=0$ touching the sextic six times, must, it is clear, touch the curves $P=0$, $Q=0$, f and g times respectively. And so when the sextic breaks up into any number of curves, each component curve $P=0$ of the order f must touch the sextic g times.

136. It follows that if the sextic break up into six lines, say $AC-B^2=t_1t_2t_3t_4t_5t_6=0$, then that each of the lines $t_1=0$, $t_2=0, \ldots t_6=0$ is a tangent to the conic. And conversely, starting with the conic $A=0$ and any six tangents thereof $t_1=0$, $t_2=0, \ldots t_6=0$, we have an identity of the form in question. In fact, taking any two of the tangents, say $t_1=0$ and $t_2=0$, then, if $p=0$ be the equation of the line joining their points of intersection, the equation of the conic will be of the form $t_1t_2+p^2=0$, that is, we may write $A=t_1t_2+p^2$, or what is the same thing, $t_1t_2=A-p^2$. (Considering A as a given quadric function of the coordinates, this of course implies that the implicit constant factors of t_1, t_2, p are properly determined.) Similarly, $q=0$ being the line through the points of contact of t_3, t_4, and $r=0$ that through the points of contact of t_5, t_6, we have $t_3t_4=A-q^2$ and $t_5t_6=A-r^2$; whence, to satisfy the equation

$$AC-B^2=t_1t_2t_3t_4t_5t_6,$$

we have only to assume $B=lA+pqr$, l an arbitrary linear function of the coordinates, and the equation then gives

$$C=A^2-A\,(p^2+q^2+r^2)+(q^2r^2+r^2p^2+p^2q^2)+l^2A+2lpqr.$$

137. It will be observed that the grouping of the six tangents into pairs is arbitrary. By altering this grouping, we merely alter the linear function l, but do not obtain any new solution. Thus, say that the new form is $B=l'A+p'q'r'$, then, by properly determining the linear function l, we can reduce this to the original form $B=lA+pqr$; viz., we can satisfy identically the equation $(l-l')\,A+pqr-p'q'r'=0$; or what is the same thing, $\lambda A+pqr-p'q'r'=0$, where λ is a linear function of the coordinates. We have, in fact, the conic $A=0$ and the cubic $pqr=0$ intersecting in the six points of contact any other cubic through these six points; and consequently the cubic $p'q'r'=0$ must be expressible in the form $\lambda A+pqr=0$, and we have thus the identity in question.

138. We have just seen that the value of B is necessarily of the form $B=lA+pqr$, but we are not concerned with its expression in this particular form. What we require in the sequel is a value of B, and thence one of C, satisfying the identity in question, $AC-B^2=t_1t_2t_3t_4t_5t_6$; or what is the same thing, introducing for convenience a constant factor K, the identity

$$AC-B^2=Kt_1t_2t_3t_4t_5t_6.$$

139. Instead of C, I write Γ, and consider the sextic amphigram $A\Gamma-B^2=0$ touched by the conic $A=0$ in the points of contact of the conic with the six tangents $t_1=0$, $t_2=0$, $\ldots t_6=0$. Suppose the sextic curve breaks up into factors; if one of these factors is a line, it is one of the six tangents, say the tangent $t_1=0$. If there is a conic factor, this is a conic touching the conic $A=0$ at its points of contact with two of the tangents, say the equation is $lA+mt_1t_2=0$. Similarly, if there is a cubic, quartic, or quintic factor, then the equation hereof is $sA+mt_1t_2t_3=0$, $UA+mt_1t_2t_3t_4=0$, or $VA+m_1t_2t_3t_4t_5=0$. Or going on to the next case of a sextic factor (being of course the whole curve), we may say that this is $WA+mt_1t_2t_3t_4t_5t_6=0$. (Observe that since $AC-B^2=Kt_1t_2t_3t_4t_5t_6$, this means only that the equation of the sextic amphigram is of the assumed form $A\Gamma-B^2=0$.)

140. By what precedes we can, for a sextic amphigram which breaks up in any assigned manner, determine the value of Γ. For instance, let the amphigram break up into two cubic curves; say these are $sA + mt_1t_2t_3 = 0$, $s'A + m't_4t_5t_6 = 0$. Assume

$$A\Gamma - B^2 = \frac{K}{mm'}(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6),$$

then this equation is

$$A\Gamma - B^2 = \frac{K}{mm'}\{ss'A^2 + (sm't_4t_5t_6 + s'mt_1t_2t_3)A\} + AC - B^2;$$

that is, we have

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3),$$

and so in any other case.

I have already adverted to the question of the "nodal determination" of the formulæ, and it might be properly here considered; viz., the question is as to the determination of the constants in such manner that, for instance, $sA + mt_1t_2t_3 = 0$ may be a nodal cubic, $UA + mt_1t_2t_3t_4 = 0$ a nodal, binodal, or trinodal quartic, &c.; but I defer it for the moment in order first to apply the theory to the quartic surfaces.

Application to Quartic Surfaces.

141. If a quartic surface has a node or nodes, we may take for a node the point $x = 0$, $y = 0$, $z = 0$; the equation of the quartic surface is then of the form

$$Aw^2 + 2Bw + \Gamma = 0,$$

where A, B, Γ are functions of x, y, z of the degrees 2, 3, 4 respectively. $A = 0$ is the tangent quadricone at the node in question; and the circumscribed cone is $A\Gamma - B^2 = 0$. By what precedes, this is an amphigram touching the quadricone along six generating lines thereof; say the tangent planes of the cone $A = 0$ along these six lines respectively are $t_1 = 0$, $t_2 = 0, \ldots t_6 = 0$. We have then an identical equation

$$AC - B^2 = Kt_1t_2t_3t_4t_5t_6,$$

viz., regarding for a moment this equation as an equation for the determination of B, and B' as any particular solution thereof, then its general solution is $B' + tA$, where t is an arbitrary linear function of (x, y, z), and the B in the equation of the surface is properly $= B' + tA$. But by the substituting $w - t$ in place of w, the B of the equation of the surface would then be made $= B'$; and it thus appears that we may, without loss of generality, take the B of this equation to be *any particular value* satisfying the identity in question; and then, B having such particular value, C is a quartic function of (x, y, z) completely determined by the same identity. And we then, by what precedes, at once determine Γ so that the circumscribed cone $A\Gamma - B^2 = 0$ may be a cone breaking up in any assigned manner; for instance, if it be a cone

(3, 3), then, as just mentioned, the two cubic cones are $sA + mt_1t_2t_3 = 0$, $s'A + m't_4t_5t_6 = 0$; and Γ has the value

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3),$$

above obtained.

On the Nodal Determination.

142. I am not able to discuss with much completeness the question of nodal determination. We have to consider a cubic curve $sA + mt_1t_2t_3 = 0$, a quartic curve $UA + mt_1t_2t_3t_4 = 0$, &c., as the case may be, and to determine the constants so that this shall have a node or nodes. Consider for a moment the form $PA + t_1Q = 0$, where Q denotes the product $mt_2t_3\ldots$ of all or any of the tangents $t_2, \ldots t_5$; the orders of PA, t_1Q are of course equal, that is, the order of P is less by unity than that of Q. I say that, by establishing a single relation between the constants, this may be made to have a node *at the point of contact* $A = 0$, $t_1 = 0$. In fact, writing $\Delta = \lambda\delta_x + \mu\delta_y + \nu\delta_z$, where λ, μ, ν are arbitrary, there will be a node at any point if for that point $\Delta(PA + t_1Q) = 0$. But for the point $A = 0$, $t_1 = 0$ this becomes $P\Delta A + Q\Delta t_1 = 0$; moreover, if $t = 0$ be any other tangent of the conic $A = 0$, and if $p = 0$ be the line joining the points of contact of the tangents t, t_1, then we may write $A = tt_1 - p^2$, and thence (since at the point in question, $A = 0$, $t_1 = 0$, we have also $p = 0$) we find $\Delta A = t\Delta t_1$, and the foregoing equation thus becomes $(tP + Q)\Delta t_1 = 0$; viz., this equation is satisfied irrespectively of the values of λ, μ, ν, if only at the point in question (that is, for the values of the coordinates which belong to the point $t_1 = 0$, $A = 0$) we have $tP + Q = 0$, which is a single relation between the constants.

143. In particular the cubic curve $sA + mt_1t_2t_3 = 0$ may be made to have a node at the point of contact of any one of the three tangents; the quartic curve $UA + mt_1t_2t_3t_4 = 0$, a node, or two or three nodes, at the point or points of contact of any one, two, or three of the four tangents; and so in other cases. These are not the only solutions, and they are in fact solutions which (as afterwards explained) I propose to reject, attending in each case only to the remaining or proper solutions of the problem.

144. To obtain in a different manner the foregoing result, consider again the cubic curve $sA + mt_1t_2t_3 = 0$; regarding this as a given curve, the conic $A = 0$ is a conic determined (not of course completely) as a conic having therewith 3 points of 2-pointic intersection; viz., if the cubic has a node, then the cone $A = 0$ is either a conic passing through the node and besides touching the curve twice, or else it is a conic touching the curve 3 times; the former is of course the above mentioned case where there is a node at one of the points of contact on the conic $A = 0$; the latter is regarded as the proper solution. So in the case of a quartic curve $UA + mt_1t_2t_3t_4 = 0$, regarding this as a given curve, the conic $A = 0$ is a conic having therewith four points of 2-pointic intersection; viz., if the quartic curve has one, two, or three nodes, then the conic is either a conic passing through one, two, or three nodes, and besides touching the quartic thrice, twice, or once; or else it is a conic touching the quartic

four times. The former is the above mentioned case where there is a node or nodes at a point or points of contact with the conic $A=0$; the latter is regarded as the proper solution.

145. To fix the ideas, and at the same time obtain a result which will be afterwards useful, I work out the formulæ for the cubic curve $sA+mt_1t_2t_3=0$, taking this equation under the form

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+m\,xyz=0.$$

This may have a node in two different ways; viz.,

1°. At the point of contact of one of the tangents $x=0$, $y=0$, $z=0$ with the conic $A=0$; say at the point of contact of $x=0$, that is, the point $x=0$, $y-z=0$. The value of m is $=\frac{4}{\mu}+\frac{4}{\nu}$; hence $\frac{1}{\nu}=-\frac{1}{\mu}+\frac{1}{4}m$; and, substituting, the equation of the curve becomes

$$\left(\frac{x}{\lambda}+\frac{y-z}{\lambda}\right)\{x^2-2x(y+z)+(y-z)^2\}+\tfrac{1}{4}mz(x+y-z)^2=0,$$

which has obviously a node at the point in question.

2°. The node may be at a point not on the conic $A=0$, viz. the value of m is $=\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}$, the equation is

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}xyz=0.$$

In fact, writing for shortness

$$\begin{aligned}-\lambda+\mu+\nu&=L,\\ \lambda-\mu+\nu&=M,\\ \lambda+\mu-\nu&=N\\ \lambda+\mu+\nu&=P,\end{aligned}$$

the node is at the point $x:y:z=L\lambda:M\mu:N\nu$; which is at once verified, if we remark that, writing for convenience x, y, $z=L\lambda$, $M\mu$, $N\nu$, then we have

$$\begin{aligned}-x+y+z&=MN,\\ x-y+z&=NL,\\ x+y-z&=LM,\end{aligned}$$

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}=P,\quad x^2+y^2+z^2-2yz-2zx-2xy=-LMNP\,(=A).$$

For, of the three equations for the coordinates of a node, the first is

$$\frac{1}{\lambda}A+\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)2(x-y-z)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}yz=0,$$

that is, for the values in question,

$$-\frac{1}{\lambda}LMNP+P(-2MN)+\frac{P^2}{\lambda\mu\nu}MN\mu\nu=0,$$

that is

$$-LMNP-2\lambda MNP+P^2MN=0,$$

or, finally, $-L-2\lambda+P=0$, which is satisfied; and similarly the other two equations are satisfied.

Quartic Surfaces resumed.

146. Passing now from curves to cones, and to the theory of the quartic surface, suppose that there is a component cone having a nodal line, say the cubic cone $sA+mt_1t_2t_3=0$: if the remaining factor is $t_4t_5t_6$, then we have

$$A\Gamma-B^2=\frac{K}{m}(sA+mt_1t_2t_3)\,t_4t_5t_6.$$

Suppose the nodal line is a line of contact with the cone $A=0$, say its equations are $t_1=0$, $p=0$ (p a linear function), then $sA+mt_2t_3t_3$ is a quadric function $(*\between t_1, p^2)$, (of course with variable coefficients); hence $A\Gamma-B^2$ is a quadric function; and A being a linear function $(*\between t_1, p)$, it follows that B is a linear function, and thence that Γ is also a linear function; that is, A, B, Γ are each of them a linear function $(*\between t_1, p)$, or the line in question (viz. the line of contact $A=0$, $t_1=0$) is a line on the quartic surface $Aw^2+2Bw+\Gamma=0$. As already mentioned, *I exclude from consideration* the surfaces which have upon them a line through two nodes; that is, I exclude from consideration the case in question where any component cone, or say where the sextic cone, has a nodal line which is a line on the tangent cone $A=0$.

147. Now, excluding the case just referred to, *I assume as a postulate* that there is but one way in which the cubic cone $sA+mt_1t_2t_3=0$ can be made to have a nodal line, or the quartic cone $UA+mt_1t_2t_3t_4=0$ one, two, or three nodal lines &c., as the case may be. It is to be understood that this does not mean that the constants are in any of these cases completely determined, but that there is between them a relation or relations constituting a general solution which includes in itself every particular solution whatever. I have no doubt that as regards the cubic cone at least the assumption is correct. This being so, the character of a single node determines the nature of the surface; for instance, if there is a node $(3_1, 3)$, then taking this as the point $(x=0, y=0, z=0)$ the equation of the surface is $Az^2+2Bw+\Gamma=0$, where

$$\Gamma=C+\frac{K}{mm'}(ss'A+l'st_4t_5t_6+ls't_1t_2t_3),$$

a surface of a determinate nature; so that the character of all the remaining nodes is completely determined.

148. The point to be attended to is, that if for instance there were two essentially distinct ways of giving the cubic cone $sA+mt_1t_2t_3=0$ a nodal line (such as there would be if the excluded case were considered admissible), then the foregoing equation

of the surface would or might include two distinct forms of equation applying to different kinds of surface. The conclusion is that there is but one kind of quartic surface having a node $(3_1, 3)$. Admitting this, and similarly that there is but one kind of quartic surface having a node 6_{10}, it follows that if (as the fact is) there is a surface having the nodes $1(6_{10}) + 10(3_1, 3)$ (Kummer's 11-nodal surface), then that the two first-mentioned kinds are in fact each of them this last-mentioned kind of surface; and it was in this manner that I arrived at the enumeration given near the beginning of the present Memoir.

149. The reasoning is, of course, in place of a direct demonstration which would consist in showing that a surface having a node $(3_1, 3)$ has 9 other like nodes, and also a node 6_{10}; and that a surface having a node 6_{10} has 10 other nodes $(3_1, 3)$; and that, starting from either form of equation, we could, by passing to a node of the other kind, obtain the other form of equation.

Enumeration of the Cases.

150. I collect the results as follows: I call to mind that we have always the identical equation $AC - B^2 = Kt_1t_2t_3t_4t_5t_6$, that the equation of the surface is $Aw^2 + 2Bw + \Gamma = 0$, and that the circumscribed cone is $A\Gamma - B^2 = 0$. The equation of a surface having different kinds of nodes will assume different forms according as the origin (or point $x = 0$, $y = 0$, $z = 0$) is taken to be at a node of one or other of these kinds; these forms of the equations are distinguished as "node-forms,"—viz., we speak of the node-form $(3_1, 3)$ when the origin is a node $(3_1, 3)$, and so in other cases.

The 16-nodal surface

$$16(1, 1, 1, 1, 1, 1),$$

node-form

$$(1, 1, 1, 1, 1, 1),$$

cone is

$$t_1t_2t_3t_4t_5t_6 = 0,$$

and

$$\Gamma = C;$$

viz., equation is

$$Aw^2 + 2Bw + C = 0.$$

The 15-nodal surface

$$15(2, 1, 1, 1, 1),$$

node-form

$$(2, 1, 1, 1, 1)$$

cone is

$$(lA + mt_1t_2)\, t_3t_4t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{Kl}{m}\, t_3t_4t_5t_6 = 0.$$

The 14-nodal surface

$$8\,(3_1,\ 1,\ 1,\ 1) + 6\,(2,\ 2,\ 1,\ 1),$$

node-form

$$(3_1,\ 1,\ 1,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,t_4t_5t_6 = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{Ks}{m}\,t_4t_5t_6\,;$$

node-form

$$(2,\ 2,\ 1,\ 1),$$

cone is

$$(lA + mt_1t_2)\,(l'A + m't_3t_4)\,t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(ll'A + lm't_3t_4 + l'mt_1t_2)\,t_5t_6.$$

The 13 (α)-nodal surface

$$3\,(4_3,\ 1,\ 1) + 1\,(3,\ 1,\ 1,\ 1) + 9\,(3_1,\ 2,\ 1),$$

node-form

$$(4_3,\ 1,\ 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a trinodal quartic } 4_3,$$

and

$$\Gamma = C + \frac{K}{m}\,Ut_5t_6\,;$$

node-form

$$(3,\ 1,\ 1,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,t_4t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{m}\,st_4t_5t_6\,;$$

node-form

$$(3_1,\ 2,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,(l'A + m't_4t_5)\,t_6 = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(sl'A + m'st_4t_5 + ml't_1t_2t_3)\,t_6.$$

The 13 (β)-nodal surface

$$13\,(2,\ 2,\ 2),$$

node-form

$$(2,\ 2,\ 2),$$

cone is

$$(lA + mt_1t_2)\,(l'A + m't_3t_4)\,(l''A + m''t_5t_6) = 0,$$

and

$$\Gamma = C + \frac{K}{mm'm''}\{ll'l''\,A^2 + (l'l''mt_1t_2 + l'lm't_3t_4 + ll'm''t_5t_6)\,A$$
$$+ lm'm''t_3t_4t_5t_6 + l'm''mt_5t_6t_1t_2 + l''mm't_1t_2t_3t_4\}.$$

The 12 (α)-nodal surface

$$12\,(4_3,\ 2),$$

node-form

$$(4_3,\ 2),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,(l'A + m't_5t_6) = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a trinodal quartic } 4_3,$$

and

$$\Gamma = C + \frac{K}{mm'}(l'UA + m'Ut_5t_6 + l'mt_1t_2t_3t_4).$$

The 12 (β)-nodal surface

$$2\,(5_6,\ 1) + 6\,(3_1,\ 3_1) + 4\,(3,\ 2,\ 1),$$

node-form

$$(5_6,\ 1),$$

cone is

$$(VA + mt_1t_2t_3t_4t_5)\,t_6 = 0,$$

where

$$VA + mt_1t_2t_3t_4t_5 = 0 \text{ is a 6-nodal quintic } 5_6,$$

and

$$\Gamma = C + \frac{K}{m}\,Vt_6\,;$$

node-form

$$(3_1,\ 3_1),$$

cone is

$$(sA + mt_1t_2t_3)\,(s'A + m't_4t_5t_6) = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ and } s'A + m't_4t_5t_6 = 0 \text{ are each of them a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'}(ss'A + m'st_4t_5t_6 + ms't_1t_2t_3)\,;$$

node-form

$$(3,\ 2,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,(l'A + m't_4t_5)\,t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{mm'}(l'sA + m'st_4t_5 + l'mt_1t_2t_3)\,t_6.$$

The 12 (γ)-nodal surface,

$$12\,(4_2,\ 1,\ 1),$$

node-form

$$(4_2,\ 1,\ 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a binodal quartic } 4_2,$$

and

$$\Gamma = C + \frac{K}{m}\,Ut_5t_6.$$

The 11 (α)-nodal surface,

$$1\,(6_{10}) + 10\,(3_1,\ 3)$$

node-form

$$(6_{10}),$$

cone is

$$WA + mt_1t_2t_3t_4t_5t_6 = 0,$$

where this is a 10-nodal sextic 6_{10},

and

$$\Gamma = C + \frac{K}{m}\,W;$$

node-form

$$(3_1,\ 3),$$

cone is

$$(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6) = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(ss'A + m'st_4t_5t_6 + ms't_1t_2t_3).$$

Other 11-nodal surfaces,

node-form

$$(5_5,\ 1),$$

cone is

$$(VA + mt_1t_2t_3t_4t_5)\,t_6 = 0,$$

where

$$VA + mt_1t_2t_3t_4t_5 = 0 \text{ is a 5-nodal quintic } 5_5,$$

and

$$C = \Gamma + \frac{K}{m}\,Vt_6 = 0;$$

node-form

$$(4_2,\ 2),$$

cone is

$$(UA + mt_1t_2t_3t_4)(l'A + m't_5t_6) = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a binodal quartic } 4_2,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(l'UA + m'Ut_5t_6 + l'mt_1t_2t_3t_4);$$

node-form

$$(4,\ 1,\ 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 \text{ is a nodal quartic } 4_1,$$

and

$$\Gamma = C + \frac{K}{m}\,Ut_5t_6;$$

but whether these node-forms belong to the same or to different surfaces is not ascertained.

The enumeration is not extended to the 10-nodal surfaces, but I consider one case of these surfaces.

The 10 (α)-*nodal surface* 10 (3, 3).

151. I assume only that there is a single node (3, 3): taking the cone to be

$$(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6) = 0;$$

then for the equation of the surface, in the node-form (3, 3) in question, we have

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3).$$

But I present this result under a different form, as follows: I write

$$A = p^2 + ft_1t_2 = q^2 + gt_3t_4 = r^2 + ht_5t_6,$$

where f, g, h are constants, and, as before, $p = 0$, $q = 0$, $r = 0$ are the lines joining the points of contact of t_1, t_2; t_3, t_4; and t_5, t_6 respectively: we have

$$sA + mt_1t_2t_3 = sA + mt_3\left(\frac{A - p^2}{f}\right), \text{ and } s'A + m't_4t_5t_6 = s'A + m't_4\left(\frac{A - r^2}{h}\right);$$

or in place of s, s' introducing new linear functions σ, σ', the cubic curves may be taken to be $\sigma A - \frac{m}{f}p^2t_3$, $\sigma'A - \frac{m'}{h}r^2t_4$, so that we have

$$\begin{aligned} A\Gamma - B^2 &= \frac{K}{mm'}\left(\sigma A - \frac{m}{f}p^2t_3\right)\left(\sigma'A - \frac{m'}{h}r^2t_4\right), \\ &= \frac{K}{mm'}\left(\sigma\sigma'A^2 - \sigma A\frac{m'}{h}r^2t_4 - \sigma'A\frac{m}{f}p^2t_3 + \frac{mm'}{fh}p^2r^2\frac{A - q^2}{g}\right), \\ &= \frac{K}{mm'}\left(\sigma\sigma'A^2 - \sigma A\frac{m'}{h}r^2t_4 - \sigma'A\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2A\right) - \frac{K}{fgh}p^2q^2r^2; \end{aligned}$$

whence $B = \left(\frac{K}{fgh}\right)^{\frac{1}{2}}(pqr + tA)$, where t is a linear function of the coordinates; and we then have

$$\Gamma = \frac{K}{mm'}\left(\sigma\sigma'A - \sigma\frac{m'}{h}r^2t_4 - \sigma'\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2\right) + \frac{K}{fgh}(t^2A + 2tpqr),$$

where A may be considered as standing for $q^2+gt_3t_4$. The equation $Aw^2+2Bw+\Gamma=0$ of the surface, substituting throughout for A its value, is therefore

$$(q^2+gt_3t_4)\,w^2+2\left(\frac{K}{fgh}\right)^{\frac{1}{2}}\{pqr+t\,(q^2+gt_3t_4)\}\,w$$

$$+\frac{K}{mm'}\left[\sigma\sigma'\,(q^2+gt_3t_4)-\sigma\frac{m'}{h}r^2t_4-\sigma'\frac{m}{f}p^2t_3+\frac{mm'}{fgh}p^2r^2\right]+\frac{K}{fgh}\left[t^2\,(q^2+gt_3t_4)+2tpqr\right]=0,$$

where the cone is

$$\left\{\sigma\,(q^2+gt_3t_4)-\frac{m}{f}p^2t_3\right\}\left\{\sigma'\,(q^2+gt_3t_4)-\frac{m'}{h}r^2t_4\right\}=0.$$

152. Writing in the equation of the surface $w\left(\frac{K}{fgh}\right)^{\frac{1}{2}}$ instead of w, it becomes

$$(q^2+gt_3t_4)\,w^2+2\left[pqr+t\,(q^2+gt_3t_4)\right]w$$

$$+\frac{fgh}{mm'}\left[\sigma\sigma'\,(q^2+gt_3t_4)-\sigma\frac{m'}{h}r^2t_4-\sigma'\frac{m}{f}p^2t_3+\frac{mm'}{fgh}p^2r^2\right]+t^2\,(q^2+gt_3t_4)+2tpqr=0\,;$$

and then writing $\frac{m}{f}\sigma$ and $\frac{m'}{h}\sigma'$ for σ and σ' respectively, this is

$$(q^2+gt_3t_4)\,w^2+2pqrw+2tw\,(q^2+gt_3t_4)$$

$$+g\left[\sigma\sigma'\,(q^2+gt_3t_4)-\sigma r^2t_4-\sigma'p^2t_3+\frac{1}{g}p^2r^2\right]+t^2\,(q^2+gt_3t_4)+2tpqr=0.$$

We may consider t_3, t_4 as denoting not the functions originally so represented, but these functions each multiplied by a suitable constant, and thereupon write $g=-1$; viz., $t_3=0$, $t_4=0$, will now denote any two tangents to the conic $A=0$, the implicit factors being so determined that $A=q^2-t_3t_4$. The equation of the surface is

$$(q^2-t_3t_4)\,w^2+2pqrw+2tw\,(q^2-t_3t_4)$$

$$-\sigma\sigma'\,(q^2-t_3t_4)+\sigma r^2t_4+\sigma'p^2t_3+p^2r^2+t^2\,(q^2-t_3t_4)+2pqrt=0\,;$$

viz., this is

$$(q^2-t_3t_4)\left[(w+t)^2-\sigma\sigma'\right]+2pqr\,(w+t)+\sigma r^2t_4+\sigma'p^2t_3+p^2r^2=0,$$

the sextic cone being

$$\{\sigma\,(q^2-t_3t_4)-p^2t_3\}\,\{\sigma'\,(q^2-t_3t_4)-r^2t_4\}=0.$$

153. But the foregoing equation of the surface is

$$\begin{vmatrix} -\sigma', & w+t, & . \;, & r \\ w+t, & -\sigma\,, & p\;, & . \\ .\;, & p\;, & t_4\;, & -q \\ r\;, & .\;, & -q, & t_3 \end{vmatrix}=0,$$

as is at once seen by developing the determinant; the functions $w+t$, σ, σ', p, q, r, t_3, t_4 are all of them linear; and the determinant is thus a symmetrical quartic determinant the terms whereof are linear functions of the coordinates; viz. the surface is a

symmetroid. That is, a surface having a single node (3, 3) is a symmetroid; but I have shown (*Second Memoir*, No. 116) that a symmetroid has each of its ten nodes (3, 3); wherefore the surface having a single node (3, 3) is the $10(\alpha)$-nodal surface, nodes $10(3, 3)$.

154. Start from two cubic cones $U=0$, $V=0$, having each the same vertex $(x=0,\ y=0,\ z=0)$; we may in a variety of ways determine the two cones $\alpha U+\beta V=0$, $\gamma U+\delta V=0$, having a common inscribed quadric cone $A=0$ (viz., $\alpha:\beta$ being assumed at pleasure, then $\gamma:\delta$ will be determined; not, I believe, uniquely, but I do not know what the multiplicity is). This being so, the quadric cone $A=0$ is uniquely determined; and then, assuming at pleasure the plane $w=0$, the $10(\alpha)$-nodal surface $Aw^2+2Bw+\Gamma=0$ is uniquely determined: consequently the remaining nine nodes are determinate points on the nine lines $U=0$, $V=0$ respectively. And we have thus a system of ten points in space such that, joining any one of them with the remaining nine, the nine lines so obtained are the intersections of two cubic cones, or say that they are an ennead of lines.

Notation for the Cases afterwards considered.

155. I proceed to further develope the theory of some of the different surfaces. The same node-form of equation will, of course, assume different shapes according to the actual expressions in terms of the coordinates (x, y, z) of the several functions A, &c., which enter into it. I have found it convenient to attribute to A and B certain specific values which are not in every case those of the coefficients of w^2, w in the equation of the surface: this means that we must, in the equation of the surface, substitute new symbols for these coefficients, and write the equation say in the form $A'w^2+2B'w+\Gamma=0$; the change of notation, when it occurs, will be duly explained.

156. It is in general (but not always) convenient to take the equation of the tangent cone to be $x^2+y^2+z^2-2yz-2zx-2xy=0$; for then any plane $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$, where $\alpha+\beta+\gamma=0$, will be a tangent plane; so that six tangent planes may be represented by $x=0$, $y=0$, $z=0$, and by three equations of the form just referred to. And in reference to this assumed form of the equation of the tangent cone, and to what follows, I write

$$\alpha+\beta+\gamma=0,$$
$$\alpha'+\beta'+\gamma'=0,$$
$$\alpha''+\beta''+\gamma''=0,$$

$$P=\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma},$$

$$P'=\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'},$$

$$P''=\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''},$$

$$\begin{aligned} X &= \alpha\ (\gamma'\gamma''\ y - \beta'\beta'' z), \\ Y &= \beta\ (\alpha'\alpha''\ z - \gamma'\gamma'' x), \\ Z &= \gamma\ (\beta'\beta'' x - \alpha'\alpha'' y), \end{aligned}$$

$$\begin{aligned} X' &= \alpha'\ (\gamma''\gamma\ y - \beta''\beta\ z), \\ Y' &= \beta'\ (\alpha''\alpha\ z - \gamma''\gamma\ x), \\ Z' &= \gamma'\ (\beta''\beta\ x - \alpha''\alpha\ y), \end{aligned}$$

$$\begin{aligned} X'' &= \alpha''\ (\gamma\ \gamma'\ y - \beta\ \beta'\ z), \\ Y'' &= \beta''\ (\alpha\ \alpha'\ z - \gamma\ \gamma'\ x), \\ Z'' &= \gamma''\ (\beta\ \beta'\ x - \alpha\ \alpha'\ y), \end{aligned}$$

$$A = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,$$

$$B = \alpha\alpha'\alpha''(y^2z - yz^2) + \beta\beta'\beta''(z^2x - zx^2) + \gamma\gamma'\gamma''(x^2y - xy^2) + Mxyz,$$

$$C = (\alpha\alpha'\alpha''yz + \beta\beta'\beta''zx + \gamma\gamma'\gamma''xy)^2;$$

where

$$\begin{aligned} M &= (\beta - \gamma)\alpha'\alpha'' + (\gamma - \alpha)\beta'\beta'' + (\alpha - \beta)\gamma'\gamma'', \\ &= (\beta' - \gamma')\alpha''\alpha + (\gamma' - \alpha')\beta''\beta + (\alpha' - \beta')\gamma''\gamma, \\ &= (\beta'' - \gamma'')\alpha\alpha' + (\gamma'' - \alpha'')\beta\beta' + (\alpha'' - \beta'')\gamma\gamma', \\ &= -\tfrac{1}{8}\{(\beta - \gamma)(\beta' - \gamma')(\beta'' - \gamma'') + (\gamma - \alpha)(\gamma' - \alpha')(\gamma'' - \alpha'') + (\alpha - \beta)(\alpha' - \beta')(\alpha'' - \beta'')\}; \end{aligned}$$

also

$$K = 4\alpha\alpha'\alpha''\beta\beta'\beta''\gamma\gamma'\gamma'':$$

and we have identically

$$AC - B^2 = Kxyz\left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}\right)\left(\frac{x}{\alpha'} + \frac{y}{\beta'} + \frac{z}{\gamma'}\right)\left(\frac{x}{\alpha''} + \frac{y}{\beta''} + \frac{z}{\gamma''}\right).$$

The 16-*nodal Surface* 16 (1, 1, 1, 1, 1, 1).

157. Kummer starts from an irrational equation, which is readily converted into the following

$$\sqrt{x(X - w)} + \sqrt{y(Y - w)} + \sqrt{z(Z - w)} = 0,$$

and then, rationalizing, we have

$$Aw^2 + 2Bw + C = 0,$$

where as above

$$AC - B^2 = Kxyz\,PP'P''.$$

This agrees with the foregoing theory; viz., the point ($x = 0$, $y = 0$, $z = 0$) being a node, the rationalized equation must, of course, be in the node-form (1, 1, 1, 1, 1, 1), (being the only node-form); and the symmetry of the formulæ enables us at once to write

down the equations of the 16 singular planes, and thence to deduce the coordinates of the 16 nodes; viz.,

the singular planes are	and the nodes are
(1) $x=0$,	(1) $(0, -\beta, \gamma, \alpha'\alpha''\beta\gamma)$,
(2) $y=0$,	(2) $(\alpha, 0, -\gamma, \beta'\beta''\gamma\alpha)$,
(3) $z=0$,	(3) $(-\alpha, \beta, 0, \gamma'\gamma''\alpha\beta)$,
(4) $w=0$,	(4) $(\alpha'\alpha'', \beta'\beta'', \gamma'\gamma'', 0)$,
(5) $X-w=0$,	(5) $(1, 0, 0, 0)$,
(6) $Y-w=0$,	(6) $(0, 1, 0, 0)$,
(7) $Z-w=0$,	(7) $(0, 0, 1, 0)$,
(8) $P=0$,	(8) $(0, 0, 0, 1)$,
(9) $X'-w=0$,	(9) $(0, -\beta', \gamma', \alpha''\alpha\beta'\gamma')$,
(10) $Y'-w=0$,	(10) $(\alpha', 0, -\gamma', \beta''\beta\gamma'\alpha')$,
(11) $Z'-w=0$,	(11) $(-\alpha', \beta', 0, \gamma''\gamma\alpha'\beta')$,
(12) $P'=0$,	(12) $(\alpha''\alpha, \beta''\beta, \gamma''\gamma, 0)$,
(13) $X''-w=0$,	(13) $(0, -\beta'', \gamma'', \alpha\alpha'\beta''\gamma'')$,
(14) $Y''-w=0$,	(14) $(\alpha'', 0, -\gamma'', \beta\beta'\gamma''\alpha'')$,
(15) $Z''-w=0$,	(15) $(-\alpha'', \beta'', 0, \gamma\gamma'\alpha''\beta'')$,
(16) $P''=0$,	(16) $(\alpha\alpha', \beta\beta', \gamma\gamma', 0)$,

where the nodes and planes are numbered as by Kummer; and by means of his (differently arranged) diagram of the relation between the several nodes and planes, I was enabled to form the following square diagram, which exhibits this relation in, I think, the most convenient form. To explain this, observe that in the upper and left-hand margins, the numbers refer to the nodes; in the body of the table, and in the right-hand margin to the planes, the table shows that for the node 1, the circumscribed cone is made up of the planes 1, 6, 7, 8, 9, 13; and that the remaining 15 nodes are situate on the nodal lines of this cone, the node 2 on the intersection of the planes 7, 8; the node 3 on the intersection of the planes 6, 8, and so on; and the like as regards the other lines of the table.

NODES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	7, 8	6, 8	6, 7	9, 13	1, 6	1, 7	1, 8	1, 9	6, 13	7, 13	8, 13	1, 13	6, 9	7, 9	8, 9	1, 9, 13, 8, 7, 6
2	7, 8	*	5, 8	5, 7	2, 5	10, 14	2, 7	2, 8	5, 14	2, 10	7, 14	8, 14	5, 10	2, 14	7, 10	8, 10	2, 10, 14, 7, 8, 5
3	6, 8	5, 8	*	5, 6	3, 5	3, 6	11, 15	3, 8	5, 15	6, 15	3, 11	8, 15	5, 11	6, 11	3, 15	8, 11	3, 11, 15, 6, 5, 8
4	6, 7	5, 7	5, 6	*	4, 5	4, 6	4, 7	12, 16	5, 16	6, 16	7, 16	4, 12	5, 12	6, 12	7, 12	4, 16	4, 12, 16, 5, 6, 7
5	9, 13	2, 5	3, 5	4, 5	*	3, 4	2, 4	2, 3	5, 9	2, 13	3, 13	4, 13	5, 13	2, 9	3, 9	4, 9	5, 13, 9, 4, 3, 2
6	1, 6	10, 14	3, 6	4, 6	3, 4	*	1, 4	1, 3	1, 14	6, 10	3, 14	4, 14	1, 10	6, 14	3, 10	4, 10	6, 14, 10, 3, 4, 1
7	1, 7	2, 7	11, 15	4, 7	2, 4	1, 4	*	1, 2	1, 15	2, 15	7, 11	4, 15	1, 11	2, 11	7, 15	4, 11	7, 15, 11, 2, 1, 4
8	1, 8	2, 8	3, 8	12, 16	2, 3	1, 3	1, 2	*	1, 16	2, 16	3, 16	8, 12	1, 12	2, 12	3, 12	8, 16	8, 16, 12, 1, 2, 3
9	1, 9	5, 14	5, 15	5, 16	5, 9	1, 14	1, 15	1, 16	*	15, 16	14, 16	14, 15	1, 5	9, 14	9, 15	9, 16	9, 1, 5, 16, 15, 4
10	6, 13	2, 10	6, 15	6, 16	2, 13	6, 10	2, 15	2, 16	15, 16	*	13, 16	15, 13	10, 13	2, 6	10, 15	10, 16	10, 2, 6, 15, 16, 13
11	7, 13	7, 14	3, 11	7, 16	3, 13	3, 14	7, 11	3, 16	14, 16	13, 16	*	13, 14	11, 13	11, 14	3, 7	11, 16	11, 3, 7, 14, 13, 16
12	8, 13	8, 14	8, 15	4, 12	4, 13	4, 14	4, 15	8, 12	14, 15	13, 15	13, 14	*	12, 13	12, 14	12, 15	4, 8	12, 4, 8, 13, 14, 15
13	1, 13	5, 10	5, 11	5, 12	5, 13	1, 10	1, 11	1, 12	1, 5	10, 13	11, 13	12, 13	*	11, 12	10, 12	10, 11	13, 5, 1, 12, 11, 10
14	6, 9	2, 14	6, 11	6, 12	2, 9	6, 14	2, 11	2, 12	9, 14	2, 6	11, 14	12, 14	11, 12	*	9, 12	9, 11	14, 6, 2, 11, 12, 9
15	7, 9	7, 10	3, 15	7, 12	3, 9	3, 10	7, 15	3, 12	9, 15	10, 15	3, 7	12, 15	10, 12	9, 12	*	9, 10	15, 7, 3, 10, 9, 12
16	8, 9	8, 10	8, 11	4, 16	4, 9	4, 10	4, 11	8, 16	9, 16	10, 16	11, 16	4, 8	10, 11	9, 11	9, 10	*	16, 8, 4, 9, 10, 11

158. The before mentioned irrational equation may be written

$$\sqrt{1.5} + \sqrt{2.6} + \sqrt{3.7} = 0,$$

and by symmetry we see that also

$$\sqrt{1.9} + \sqrt{2.10} + \sqrt{3.11} = 0,$$

$$\sqrt{1.13} + \sqrt{2.14} + \sqrt{3.15} = 0;$$

viz., these are three equations each containing the planes 1, 2, 3, which are three of the planes belonging to the node 1; the other three planes in any such equation (for instance, the planes 5, 6, 7, in the first equation) being three planes belonging to another node. Instead of the planes 1, 2, 3, we may have any other three planes belonging to the node 1; and instead of the node 1, any other node; but each equation belongs to two nodes: the number of equations is thus

$$\frac{6.5.4}{1.2.3} \times 16 \times 3 \div 2, = 480.$$

159. To obtain the planes belonging to any such equation, combine any two of the outside right-hand lines of the diagram, these contain in common two numbers the places of which are interchanged; striking these out, we have four columns, and taking out of these any three columns, we have the corresponding sets of planes. For instance, lines 1 and 2 contain 78 and 87 respectively; striking these out, the lines are

$$1, \quad 9, \quad 13, \quad 6;$$

$$2, \quad 10, \quad 14, \quad 5;$$

whence we have the sets (1, 9, 13) and (2, 10, 14); viz., there is an irrational equation of the form

$$\sqrt{1.2} + \sqrt{9.10} + \sqrt{13.14} = 0,$$

but it is probably necessary to introduce constant factors along with the products 1.2, 9.10, and 13.14 respectively. There are $\frac{1}{2}16.15, = 120$ pairs of lines, and each line gives 4 equations; in all $120 \times 4, = 480$ equations, as above.

160. I stop to remark that Kummer gives for his 13-nodal surface an equation containing three arbitrary constants, say λ, μ, ν, such that, putting one of these $= 0$, we have the 14-nodal surface; putting two of them each $= 0$, the 15-nodal surface; putting all three of them $= 0$, the 16-nodal surface. The equations for the 16-nodal surface and that for the 14-nodal surface, made use of by Kummer, are, in fact, those deduced as above from the equation of the 13 (α)-nodal surface; and the like form might have been used for the 15-nodal surface. But the form actually used by Kummer, as presently appearing, is an equivalent form not thus deducible from the equation of the 13 (α)-nodal surface.

The 15-*nodal Surface* 15 (2, 1, 1, 1, 1).

161. Kummer's equation is readily converted into the following:

$$Aw^2 + 2Bw + C + \frac{Kl}{m} xyz\, P = 0,$$

the circumscribed cone being thus

$$(lA + mP'P'')\, xyz\, P = 0,$$

and the equation being in the node-form (2, 1, 1, 1, 1).

The formulæ for the 15 nodes and the 10 singular planes depend upon a quadric equation, for the symmetrical expression of which I write

$$\alpha'\alpha'' - \beta''\gamma' = \beta'\beta'' - \gamma''\alpha' = \gamma'\gamma'' - \alpha''\beta' = \omega,$$
$$\alpha'\alpha'' - \beta'\gamma'' = \beta'\beta'' - \gamma'\alpha'' = \gamma'\gamma'' - \alpha'\beta'' = \varpi;$$

so that

$$\omega - \varpi = \beta'\gamma'' - \beta''\gamma' = \gamma'\alpha'' - \gamma''\alpha' = \alpha'\beta'' - \alpha''\beta',$$
$$\omega + \varpi = \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'':$$

the equation in question then is

$$(\rho - \omega)(\rho - \varpi) + \frac{Kl}{4\alpha\beta\gamma m} = 0;$$

so that, calling the roots of it ρ_1, ρ_2, we have

$$\rho_1 + \rho_2 = \omega + \varpi, \quad \rho_1\rho_2 = \omega\varpi + \frac{Kl}{4\alpha\beta\gamma m};$$

or we may write

$$\rho_1 = \tfrac{1}{2}(\omega + \varpi + \sqrt{\Omega}), \quad = \tfrac{1}{2}(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' + \sqrt{\Omega}),$$
$$\rho_2 = \tfrac{1}{2}(\omega + \varpi - \sqrt{\Omega}), \quad = \tfrac{1}{2}(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' - \sqrt{\Omega}),$$

if for shortness

$$\Omega = (\omega - \varpi)^2 - \frac{Kl}{\alpha\beta\gamma m}.$$

162. I write also for shortness

$$\mathrm{a} = (\beta - \gamma)\,\alpha'\alpha'' + \alpha(\beta'\beta'' - \gamma'\gamma''),$$
$$\mathrm{b} = (\gamma - \alpha)\,\beta'\beta'' + \beta(\gamma'\gamma'' - \alpha'\alpha''),$$
$$\mathrm{c} = (\alpha - \beta)\,\gamma'\gamma'' + \gamma(\alpha'\alpha'' - \beta'\beta'');$$

and I say that the singular planes are

(1) (1) $x = 0$,

(2) (2) $y = 0$,

(3) (3) $z = 0$,

(4) (9) $-\dfrac{1}{\beta\gamma}(X'-w)+(\rho_1-\omega)\left(\dfrac{y}{\beta}+\dfrac{z}{\gamma}\right)=0,$

(5) (10) $-\dfrac{1}{\gamma\alpha}(Y'-w)+(\rho_1-\omega)\left(\dfrac{z}{\gamma}+\dfrac{x}{\alpha}\right)=0,$

(6) (11) $-\dfrac{1}{\alpha\beta}(Z'-w)+(\rho_1-\omega)\left(\dfrac{x}{\alpha}+\dfrac{y}{\beta}\right)=0,$

(7) (13) $-\dfrac{1}{\beta\gamma}(X''-w)+(\rho_2-\varpi)\left(\dfrac{y}{\beta}+\dfrac{z}{\gamma}\right)=0,$

(8) (14) $-\dfrac{1}{\gamma\alpha}(Y''-w)+(\rho_2-\varpi)\left(\dfrac{z}{\gamma}+\dfrac{x}{\alpha}\right)=0,$

(9) (15) $-\dfrac{1}{\alpha\beta}(Z''-w)+(\rho_2-\varpi)\left(\dfrac{x}{\alpha}+\dfrac{y}{\beta}\right)=0,$

(10) (8) $P=0$;

and that the nodes are

(1) (1) $(0,\ -\beta,\ \gamma,\ \alpha'\alpha''\beta\gamma),$

(2) (2) $(\alpha,\ 0,\ -\gamma,\ \beta'\beta''\gamma\alpha),$

(3) (3) $(-\alpha,\ \beta,\ 0,\ \gamma'\gamma''\alpha\beta),$

(4) (9) $\{0,\ \rho_1-\gamma'\gamma'',\ -(\rho_2-\beta'\beta''),\ \alpha(\rho_1-\gamma'\gamma'')(\rho_2-\beta'\beta'')\},$

(5) (10) $\{-(\rho_2-\gamma'\gamma''),\ 0,\ \rho_1-\alpha'\alpha'',\ \beta(\rho_1-\alpha'\alpha'')(\rho_2-\gamma'\gamma'')\},$

(6) (11) $\{\rho_1-\beta'\beta'',\ -(\rho_2-\alpha'\alpha''),\ 0,\ \gamma(\rho_1-\beta'\beta'')(\rho_2-\alpha'\alpha'')\},$

(7) (13) $\{0,\ \rho_2-\gamma'\gamma'',\ -(\rho_1-\beta'\beta''),\ \alpha(\rho_2-\gamma'\gamma'')(\rho_1-\beta'\beta'')\},$

(8) (14) $\{-(\rho_1-\gamma'\gamma''),\ 0,\ \rho_2-\alpha'\alpha'',\ \beta(\rho_2-\alpha'\alpha'')(\rho_1-\gamma'\gamma'')\},$

(9) (15) $\{\rho_2-\beta'\beta'',\ -(\rho_1-\alpha'\alpha''),\ 0,\ \gamma(\rho_2-\beta'\beta'')(\rho_1-\alpha'\alpha'')\},$

(10) (8) $(0,\ 0,\ 0,\ 1),$

(11) (5) $(1,\ 0,\ 0,\ 0),$

(12) (6) $(0,\ 1,\ 0,\ 0),$

(13) (7) $(0,\ 0,\ 1,\ 0),$

(14) (16) $\left(\tfrac{1}{2}\alpha(\mathrm{a}-\alpha\sqrt{\Omega}),\ \tfrac{1}{2}\beta(\mathrm{b}-\beta\sqrt{\Omega}),\ \tfrac{1}{2}\gamma(\mathrm{c}-\gamma\sqrt{\Omega}),\ \dfrac{kl}{4m}\right),$

(15) (12) $\left(\tfrac{1}{2}\alpha(\mathrm{a}+\alpha\sqrt{\Omega}),\ \tfrac{1}{2}\beta(\mathrm{b}+\beta\sqrt{\Omega}),\ \tfrac{1}{2}\gamma(\mathrm{c}+\gamma\sqrt{\Omega}),\ \dfrac{kl}{4m}\right).$

163. The small reference numbers are those used by Kummer. It is, I think, better to retain the reference numbers belonging to the case of the 16-nodal surface; viz., there are here given, large, 1, 2, 3, 8, 9, 10, 11, 13, 14, 15 for the planes, and 1, 2, 3, 5, 6, ... to 16 for the nodes. Belonging to each node (that is, with the node as vertex) there is a quadric cone passing through 8 other nodes; and each node lies (exclusively of the cone whose vertex it is) in 8 such cones. We have thus the following square diagram:

NODES	1	2	3	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	C 8	C 8	9, 13	C 1	C 1	1, 8	1, 9	C 13	C 13	8, 13	1, 13	C 9	C 9	8, 9	1, 8, 9, 13 (6, 7)
2	C 8	*	C 8	C 2	10, 14	C 2	2, 8	C 14	2, 10	C 14	8, 14	C 10	2, 14	C 10	8, 10	2, 8, 13, 14 (5, 7)
3	C 8	C 8	*	C 3	C 3	11, 15	3, 8	C 15	C 15	3, 11	8, 15	C 11	C 11	3, 15	8, 11	3, 8, 11, 15 (5, 6)
5	9, 13	C 2	C 3	*	C 3	C 2	2, 3	C 9	2, 13	3, 13	C 13	C 13	2, 9	3, 9	C 9	2, 3, 9, 13 (4, 5)
6	C 1	10, 14	C 3	C 3	*	C 1	1, 3	1, 14	C 10	3, 14	C 14	1, 10	C 14	3, 10	C 10	1, 3, 10, 14 (4, 6)
7	C 1	C 2	11, 15	C 2	C 1	*	1, 2	1, 15	2, 15	C 11	C 15	1, 11	2, 11	C 15	C 11	1, 2, 11, 15 (4, 7)
8	1, 8	2, 8	3, 8	2, 3	1, 3	1, 2	*	C 1	C 2	C 3	C 8	C 1	C 2	C 3	C 8	1, 2, 3, 8 (12, 16)
9	1, 9	C 14	C 15	C 9	1, 14	1, 15	C 1	*	C 15	C 14	14, 15	C 1	9, 14	9, 15	C 9	1, 9, 14, 15 (5, 16)
10	C 13	2, 10	C 15	2, 13	C 10	2, 15	C 2	C 15	*	C 13	13, 15	10, 13	C 2	10, 15	C 10	2, 10, 13, 15 (6, 16)
11	C 13	C 14	3, 11	3, 13	3, 14	C 11	C 3	C 14	C 13	*	13, 14	11, 13	11, 14	C 3	C 11	3, 11, 13, 14 (7, 16)
12	8, 13	8, 14	8, 15	C 13	C 14	C 15	C 8	14, 15	13, 15	13, 14	*	C 13	C 14	C 15	C 8	8, 13, 14, 15 (4, 12)
13	1, 13	C 10	C 11	C 13	1, 10	1, 11	C 1	C 1	10, 13	11, 13	C 13	*	C 11	C 10	10, 11	1, 10, 11, 13 (5, 12)
14	C 9	2, 14	C 11	2, 9	C 14	2, 11	C 2	9, 14	C 2	11, 14	C 14	C 11	*	C 9	9, 11	2, 9, 11, 14 (6, 12)
15	C 9	C 10	3, 15	3, 9	3, 10	C 15	C 3	9, 15	10, 15	C 3	C 15	C 10	C 9	*	9, 10	3, 9, 10, 15 (7, 12)
16	8, 9	8, 10	8, 11	C 9	C 10	C 11	C 8	C 9	C 10	C 11	C 8	10, 11	9, 11	9, 10	*	8, 9, 10, 11 (4, 16)

164. The arrangement is the same as in the 16-nodal square diagram; only, in the right-hand margin, a bracket (6, 7) denotes that instead of the planes (6, 7) we have a quadric cone; which cones are, in the body of the table, denoted by C. Thus for the node 1 the sextic cone is made up of the planes 1, 8, 9, 13 and of a quadric cone (6, 7), $= C$: the remaining 14 nodes lie on the nodal lines of the sextic cone, viz., the node 2 on an intersection of the cone C with the plane 8, the node 3 on an intersection of the same cone and plane, the node 5 on the intersection of the planes 9, 13, and so on.

The Equation of the 15*-nodal Surface, as deduced from that of the* 13 (α)*-nodal.*

165. If, in the equation hereafter given for the 13-nodal surface, we write $\nu = 0$, $\mu = 0$, or (what is the same thing) in that of the 14-nodal surface we write $\mu = 0$, the form is

$$\begin{aligned} & w^2 (A + 4\lambda yz) \\ + & 2w \ (B - 2\lambda yz\, X) \\ + & \qquad C = 0. \end{aligned}$$

The circumscribed sextic cone is thus (2, 1, 1, 1, 1),

$$(\lambda\alpha^2 yz + \beta\gamma x^2 + \gamma\alpha xy + \alpha\beta xz)\, yz\, (P' - \rho_1 P'')(P' - \rho_2 P'') = 0,$$

where ρ_1, ρ_2 now denote the roots of the equation

$$\lambda(\rho\alpha - \alpha'')^2 - \frac{1}{\beta'\beta''\gamma'\gamma''}(\omega - \varpi)^2 \rho\alpha'\alpha'' = 0.$$

The singular planes are

(4)	$w = 0,$
(12)	$P' - \rho_1 P'' = 0,$
(16)	$P' - \rho_2 P'' = 0,$
(3)	$z = 0,$
(2)	$y = 0,$
(15)	$\dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1 (Z' - w) - (Z'' - w) = 0,$
(11)	$\dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2 (Z' - w) - (Z'' - w) = 0,$
(14)	$\dfrac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\rho_1 (Y' - w) - (Y'' - w) = 0,$
(10)	$\dfrac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\rho_2 (Y' - w) - (Y'' - w) = 0,$
(5)	$X - w = 0,$

and we have then the same square table as before: the coordinates of the 15 nodes may be obtained without difficulty.

166. The form is really equivalent to that first considered in regard to the 15-nodal surface. To show that this is so, we have only to arrange according to powers of x; viz., the equation thus becomes

$$\begin{aligned} & x^2\{(\gamma\gamma'\gamma'')^2 y^2 + (\beta\beta'\beta'')^2 z^2 + w^2 - 2\beta\beta'\beta'' zw + 2\gamma\gamma'\gamma'' wy + 2\beta\beta'\beta''\gamma\gamma'\gamma'' yz\} \\ + & 2x\{\gamma\gamma'\gamma''\alpha\alpha'\alpha'' y^2 z + \alpha\alpha'\alpha''\beta\beta'\beta'' yz^2 + \beta\beta'\beta'' z^2 w - zw^2 - w^2 y - \gamma\gamma'\gamma'' wy^2 + Myzw\} \\ + & \quad (\alpha\alpha'\alpha'' yz + wy - wz)^2 - 4\lambda yzw(X - w) = 0, \end{aligned}$$

where, if for a moment A denotes the coefficient of x^2, we have $y=0$, $z=0$, $w=0$, $X-w=0$, four tangent planes of the quadric cone $A=0$.

14-*nodal Surface* $8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$, *Node-form* $(3_1, 1, 1, 1)$.

167. In the equation hereafter given for the 13-nodal surface, writing $\nu=0$, the sextic cone becomes

$$4z(\lambda\alpha^2 y^2 z + \mu\beta^2 zx^2 + \beta\gamma x^2 y + \gamma\alpha xy^2 + \alpha\beta xyz)(P' - \rho_1 P'')(P' - \rho_2 P'') = 0;$$

viz., this is of the form in question $(3_1, 1, 1, 1)$; and the equation of the surface is

$$\begin{aligned} & w^2 A + 4(\lambda y + \mu x) z - 4\lambda\mu z^2 \\ + & 2w\{B - 2(\lambda yX + \mu xY) z\} \\ + & \quad C = 0. \end{aligned}$$

The singular planes are

$$w = 0, \qquad (4) \quad (4)$$

$$P' - \rho_1 P'' = 0, \qquad (12) \quad (2)$$

$$P' - \rho_2 P'' = 0, \qquad (16) \quad (3)$$

$$z = 0, \qquad (3) \quad (1)$$

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(Z' - w) - (Z'' - w) = 0, \qquad (15) \quad (6)$$

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(Z' - w) - (Z'' - w) = 0, \qquad (11) \quad (5)$$

and the nodes are

$$(0, 0, 0, 1), \qquad (8) \quad (1)$$

$$(1, 0, 0, 0), \qquad (5) \quad (13)$$

$$(0, 1, 0, 0), \qquad (6) \quad (14)$$

$$(0, 0, 1, 0), \qquad (7) \quad (5)$$

$$\left(-\frac{\beta'' - \beta'\rho_1}{\beta'\beta''}, \frac{\alpha'' - \alpha'\rho_1}{\alpha'\alpha''}, 0, \frac{\gamma\gamma'\gamma''(\beta'' - \beta'\rho_1)(\alpha'' - \alpha'\rho_1)}{\alpha''\beta''\gamma' - \alpha'\beta'\gamma''\rho_1}\right), \qquad (11) \quad (4)$$

$$\left(-\frac{\beta'' - \beta'\rho_2}{\beta'\beta''}, \frac{\alpha'' - \alpha'\rho_2}{\alpha'\alpha''}, 0, \frac{\gamma\gamma'\gamma''(\beta'' - \beta'\rho_2)(\alpha'' - \alpha'\rho_2)}{\alpha''\beta''\gamma' - \alpha'\beta'\gamma''\rho_2}\right), \qquad (15) \quad (3)$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_1},\ \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_1},\ \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_1},\ 0\right), \qquad (16)\ {\scriptstyle (7)}$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_2},\ \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_2},\ \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_2},\ 0\right), \qquad (12)\ {\scriptstyle (8)}$$

$$(\ \alpha'\alpha''\ ,\ \beta'\beta''\ ,\ \gamma'\gamma''\ ,\ 0\ ), \qquad (4)\ {\scriptstyle (6)}$$

$$(\ -\alpha\ ,\ \beta\ ,\ 0\ ,\ \gamma'\gamma''\alpha\beta), \qquad (3)\ {\scriptstyle (2)}$$

two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(Z'-w)-(Z''-w)=0,\quad P'-\rho_1P''=0, \qquad (13,\ 14)\ {\scriptstyle (9,\ 10)}$$

and two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(Z'-w)-(Z''-w)=0,\quad P'-\rho_2P''=0, \qquad (9,\ 10)\ {\scriptstyle (11,\ 12)}$$

where the large numbers are those for the 16-nodal surface, the small numbers are Kummer's.

168. In these formulæ, ρ_1, ρ_2 denote the roots of the equation

$$\begin{aligned}&(\lambda\beta''^2+\mu\alpha''^2)(\rho\alpha'\beta')^2\\ &+(\lambda\beta'^2+\mu\alpha'^2)(\alpha''\beta'')^2\\ &-2\left[\lambda\beta'\beta''+\mu\alpha'\alpha''+\frac{1}{2\gamma'\gamma''}(\omega-\varpi)^2\right]\rho\alpha'\beta'\alpha''\beta''=0.\end{aligned}$$

169. The relation of the nodes and sextic cones is given by means of the square diagram on the opposite page:

NODES	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
3	*	×	C 3	C 3	11, 15	C 3	C 15	C 15	3, 11	C 15	C 11	C 11	3, 15	C 11	3, 11, 15, (5, 6, 8)
4	×	*	C 4	C 4	C 4	12, 16	C 16	C 16	C 16	4, 12	C 12	C 12	C 12	4, 16	4, 12, 16, (5, 6, 7)
5	D 3	D 4	*	3, 4	D 4	D 3			D' 3	D' 4			D' 3	D' 4	3, 4, (2, 5), (9, 13)
6	D 3	D 4	3, 4	*	D 4	D 3			D' 3	D' 4			D' 3	D' 4	3, 4, (1, 6), (10, 14)
7	11, 15	C 4	C 4	C 4	*	×	C 15	C 15	C 11	4, 15	C 11	C 11	C 15	4, 11	4, 11, 15, (1, 2, 7)
8	C 3	12, 16	C 3	C 3	×	*	C 16	C 16	3, 16	C 12	C 12	C 12	3, 12	C 16	3, 12, 16, (1, 2, 8)
9	D 15	D 16			D' 15	D' 16	*	15, 16	D 16	D 15			D' 15	D' 16	15, 16, (5, 14), (1, 9)
10	D 15	D 16			D' 15	D' 16	15, 16	*	D 16	D 15			D' 15	D' 16	15, 16, (6, 13), (2, 10)
11	3, 11	C 16	C 3	C 3	C 11	3, 16	C 16	C 16	*	×	C 11	C 11	C 3	11, 16	3, 11, 16, (7, 13, 14)
12	C 15	4, 12	C 4	C 4	4, 15	C 12	C 15	C 15	×	*	C 12	C 12	12, 15	C 4	4, 12, 15, (8, 13, 14)
13	D 11	D 12			D' 11	D' 12			D' 11	D' 12	*	11, 12	D 12	D 11	11, 12, (5, 10), (1, 13)
14	D 11	D 12			D' 11	D' 12			D' 11	D' 12	11, 12	*	D 12	D 11	11, 12, (6, 9), (2, 14)
15	3, 15	C 12	C 3	C 3	C 15	3, 12	C 15	C 15	C 3	12, 15	C 12	C 12	*	×	3, 12, 15, (7, 9, 10)
16	C 11	4, 16	C 4	C 4	4, 11	C 16	C 16	C 16	11, 16	C 4	C 11	C 11	×	*	4, 11, 16, (8, 9, 10)

where the arrangement is the same as before; only in the right-hand margin, a bracket (5, 6, 8) denotes that, instead of the planes 5, 6, 8, we have a nodal cubic cone (5, 6, 8), which is in the body of the table referred to as C, and the nodal line thereof by $\times$; and the brackets (2, 5), (9, 13) denote that, instead of the planes 2, 5, we have a quadric cone (2, 5), and instead of the planes 9, 13, a quadric cone (9, 13); which cones are in the body of the table referred to as D, D' respectively. And it is to be understood that each vacant square of the table should contain the symbol (D, D'), this being omitted only for the purpose of better exhibiting the form of the table.

170. The equation of the surface may be presented in the irrational form

$$\begin{aligned}&\sqrt{(P'-\rho_1 P'')\left[\gamma''\alpha'\beta'\rho_2(Z'-w)-\gamma'\alpha''\beta''(Z''-w)\right]}\\ +&\sqrt{(P'-\rho_2 P'')\left[\gamma''\alpha'\beta'\rho_1(Z'-w)-\gamma'\alpha''\beta''(Z''-w)\right]}\\ +&\ (\rho_1-\rho_2)\sqrt{\alpha'\beta'\alpha''\beta''\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)}\,zw=0.\end{aligned}$$

In fact the norm of the left-hand side is

$$\begin{aligned}=(\rho_1-\rho_2)^2(\omega-\varpi)^2\{&w^2[A+4z(\lambda y+\mu x)-4\lambda\mu z^2]\\ &+2w[B-2z(\lambda yX+\mu xY)]\\ &+C\qquad\qquad\qquad\qquad\}.\end{aligned}$$

To partially verify this, observe that, writing the equation under the form $\sqrt{R}+\sqrt{S}+\sqrt{T}=0$, on writing therein $w=0$, we ought to have

$$(R-S)^2=(\rho_1-\rho_2)^2(\omega-\varpi)^2(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)^2.$$

But writing $w=0$, we have

$$\begin{aligned}R-S=&\ (P'-\rho_1P'')(\gamma''\alpha'\beta'\rho_2Z'-\gamma'\alpha''\beta''Z'')\\ &-(P'-\rho_2P'')(\gamma''\alpha'\beta'\rho_1Z'-\gamma'\alpha''\beta''Z''),\\ =&\ (\rho_2-\rho_1)(\gamma''\alpha'\beta'P'Z'-\gamma'\alpha''\beta''P''Z''),\\ =&\ (\rho_2-\rho_1)\{\gamma''(\beta'\gamma'x+\gamma'\alpha'y+\alpha'\beta'z)(\beta\beta''x-\alpha\alpha''y)\\ &\qquad-\gamma'(\beta''\gamma''x+\gamma''\alpha''y+\alpha''\beta''z)(\beta\beta'x-\alpha\alpha'y)\},\end{aligned}$$

which is easily found to be

$$=-(\rho_2-\rho_1)(\omega-\varpi)(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy);$$

and thus $(R-S)^2$ has the value in question.

171. In further verification, observe that, writing x, y, $z=\alpha'\alpha''$, $\beta'\beta''$, $\gamma'\gamma''$, and therefore $P'=0$, $P''=0$, we ought to have

$$(\rho_1-\rho_2)^2(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)z^2w^2=(\rho_1-\rho_2)^2(\omega-\varpi)^2w^2[A+4z(\lambda y+\mu x)-4\lambda\mu z^2],$$

observing that, for the values in question, B, X, Y, C all vanish.

This is

$$(\rho_1-\rho_2)^2(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)^2(\gamma'\gamma'')^2=(\omega-\varpi)^2\{(\omega-\varpi)^2+4\gamma'\gamma''(\lambda\beta'\beta''+\mu\alpha'\alpha''-\lambda\mu\gamma'\gamma'')\},$$

which is in fact the value of $(\rho_1-\rho_2)^2$ obtained from the equation in ρ.

The 13 (α)-*nodal Surface* $3(4_3,\ 1,\ 1)+1(3,\ 1,\ 1,\ 1)+9(3_1,\ 2,\ 1)$.

172. The equation, node-form $(4_3,\ 1,\ 1)$, is

$$\begin{aligned}&w^2\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}\\+&\ 2w\{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}\\+&\qquad C=0;\end{aligned}$$

viz., for the circumscribed cone we have

$$\begin{aligned}&\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}\,C\\-&\ \{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}^2\\&=4\{\lambda\alpha^2y^2z^2+\mu\beta^2z^2x^2+\nu\gamma^2x^2y^2+\beta\gamma x^2yz+\gamma\alpha y^2zx+\alpha\beta z^2xy\}\\&\qquad\times\left\{-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right\},\end{aligned}$$

where, on the right-hand side, the first factor, equated to zero, represents a trinodal quartic cone, the nodal lines whereof are $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$.

173. As regards the second factor, it is to be observed that, writing as above

$$\omega-\varpi=\beta'\gamma''-\beta''\gamma',\quad=\gamma'\alpha''-\gamma''\alpha',\quad=\alpha'\beta''-\alpha''\beta',$$

we have identically

$$\alpha'P'-\alpha''P''=\frac{\omega-\varpi}{\beta'\beta''\gamma'\gamma''}\,\frac{X}{\alpha},$$

$$\beta'P'-\beta''P''=\frac{\omega-\varpi}{\gamma'\gamma''\alpha'\alpha''}\,\frac{Y}{\beta},$$

$$\gamma'P'-\gamma''P''=\frac{\omega-\varpi}{\alpha'\alpha''\beta'\beta''}\,\frac{Z}{\gamma};$$

so that the second factor is

$$\begin{aligned}=\frac{1}{(\omega-\varpi)^2}\{&-\lambda(\beta'\beta''\gamma'\gamma'')^2(\alpha'P'-\alpha''P'')^2-\mu(\gamma'\gamma''\alpha'\alpha'')^2(\beta'P'-\beta''P'')^2\\&-\nu(\alpha'\alpha''\beta'\beta'')^2(\gamma'P'-\gamma''P'')^2+(\omega-\varpi)^2\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\};\end{aligned}$$

or, what is the same thing,

$$\frac{1}{(\omega-\varpi)^2}\{-\ [\lambda(\beta''\gamma'')^2+\mu(\gamma''\alpha'')^2+\nu(\alpha''\beta'')^2](\alpha'\beta'\gamma'P')^2$$
$$-\ [\lambda(\beta'\gamma')^2+\mu(\gamma'\alpha')^2+\nu(\alpha'\beta')^2](\alpha''\beta''\gamma''P'')^2$$
$$+2[\lambda(\beta'\gamma'\beta''\gamma'')+\mu(\gamma'\alpha'\gamma''\alpha'')+\nu(\alpha'\beta'\alpha''\beta'')+\tfrac{1}{2}(\omega-\varpi)^2]\times\alpha'\beta'\gamma'P'\alpha''\beta''\gamma''P''\},$$

so that, equating to zero, we have a pair of planes, each passing through the line $P'=0$, $P''=0$, and which it is clear must be the tangent planes from this line to the quadric cone $A'=0$. I will presently return to this, but I consider first the foregoing identical equation in regard to the circumscribed cone.

174. In verification hereof, observe first that, if $x=0$, the equation becomes

$$\{(y-z)^2+4\lambda yz-4\lambda(\nu y^2+\mu z^2)\}(\alpha\alpha'\alpha'')^2y^2z^2-\{2\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)\}^2$$
$$=4\lambda y^2z^2\{-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-(\alpha\alpha'\alpha'')^2(\nu y^2+\mu z^2)-\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)\},$$

viz., omitting terms which destroy each other, this is

$$[(y-z)^2+4\lambda yz](\alpha\alpha'\alpha'')^2y^2z^2-[\alpha\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)]^2$$
$$=4\lambda y^2z^2[-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)].$$

Or again, this is

$$4\lambda yz(\alpha\alpha'\alpha'')^2y^2z^2+4\lambda yz\,\alpha^2\alpha'\alpha''(y^2z-yz^2)(\gamma'\gamma''y-\beta'\beta''z)$$
$$=-4\lambda y^2z^2\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z);$$

viz., this is

$$\alpha'\alpha''yz+(y-z)(\gamma'\gamma''y-\beta'\beta''z)=(\gamma'y+\beta'z)(\gamma''y+\beta''z),$$

which is at once seen to be true.

175. Again, compare the terms which contain x^4yz. On the right-hand side, we have

$$4\beta\gamma x^2yz\times\text{term in }x^2\text{ of }\left(-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right);$$

viz., the coefficient is

$$=4\beta\gamma\{-\mu(\gamma'\gamma'')^2-\nu(\beta'\beta'')^2+\beta'\beta''\gamma'\gamma''\}.$$

On the left-hand side, that is in $A'C-B'^2$, the only terms which give rise to the terms in question are

$$\text{in } A',\ (1-4\mu\nu)x^2;\ \text{ in } C,\ 2\beta\beta'\beta''\gamma\gamma'\gamma''\,x^2yz;$$

and in B'

$$(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')x^2y,\quad -(\beta\beta'\beta''-2\mu\beta\gamma'\gamma'')x^2z;$$

whence the coefficient is

$$2\beta\beta'\beta''\gamma\gamma'\gamma''(1-4\mu\nu)+2(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')(\beta\beta'\beta''-2\mu\beta\gamma'\gamma''),$$

which is in fact

$$= 4\beta\gamma\,\{-\mu\,(\gamma'\gamma'')^2 - \nu\,(\beta'\beta'')^2 + \beta'\beta''\gamma'\gamma''\},$$

which is right; and the verification may be completed without difficulty.

176. The singular planes are

$$w = 0, \qquad (4)$$

$$P' - \rho_1 P'' = 0, \qquad (12)$$

$$P' - \rho_2 P'' = 0, \qquad (16);$$

and the nodes are

$$(0,\ 0,\ 0,\ 1), \qquad (8) \quad {\scriptstyle (1)}$$

$$(1,\ 0,\ 0,\ 0), \qquad (5) \quad {\scriptstyle (5)}$$

$$(0,\ 1,\ 0,\ 0), \qquad (6) \quad {\scriptstyle (6)}$$

$$(0,\ 0,\ 1,\ 0), \qquad (7) \quad {\scriptstyle (7)}$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha'' - \alpha'\rho_1},\ \frac{\beta\beta'\beta''}{\beta'' - \beta'\rho_1},\ \frac{\gamma\gamma'\gamma''}{\gamma'' - \gamma'\rho_1},\ 0\right), \qquad (16) \quad {\scriptstyle (2)}$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha'' - \alpha'\rho_2},\ \frac{\beta\beta'\beta''}{\beta'' - \beta'\rho_2},\ \frac{\gamma\gamma'\gamma''}{\gamma'' - \gamma'\rho_2},\ 0\right), \qquad (12) \quad {\scriptstyle (3)}$$

$$(\ \alpha'\alpha''\ ,\ \beta'\beta''\ ,\ \gamma'\gamma''\ ,\ 0), \qquad (4) \quad {\scriptstyle (4)}$$

three nodes in $P' - \rho_1 P'' = 0$ (13, 14, 15) (8, 9, 10)

" " $P' - \rho_2 P'' = 0$ (9, 10, 11) (11, 12, 13);

where the small numbers are those used by Kummer, the large ones are those referring to the 16-nodal surface, and are here adopted. In the foregoing formulæ ρ_1, ρ_2 are the roots of

$$\begin{aligned}
&[\lambda\,(\beta''\gamma'')^2 + \mu\,(\gamma''\alpha'')^2 + \nu\,(\alpha''\beta'')^2]\,(\rho\alpha'\beta'\gamma')^2 \\
&+ [\lambda\,(\beta'\gamma')^2 + \mu\,(\gamma'\alpha')^2 + \nu\,(\alpha'\beta')^2]\,(\alpha''\beta''\gamma'')^2 \\
&- 2\,[\lambda\beta'\gamma'\beta''\gamma'' + \mu\gamma'\alpha'\gamma''\alpha'' + \nu\alpha'\beta'\alpha''\beta'' + \tfrac{1}{2}\,(\omega - \varpi)^2]\,\rho\alpha'\beta'\gamma'\alpha''\beta''\gamma'' = 0.
\end{aligned}$$

177. We have the square diagram in the following page:

NODES	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
4	*	C 4	C 4	C 4	12, 16	C 16	C 16	C 16	4, 12	C 12	C 12	C 12	4, 16	4, 12, 16, (5, 6, 7)
5	C 4	*	C 4	C 4	×				C' 4				C' 4	4, (2, 3 ; 5) nodal, (9, 13)
6	C 4	C 4	*	C 4	×				C' 4				C' 4	4, (1, 3 ; 6) nodal, (10, 14)
7	C 4	C 4	C 4	*	×				C' 4				C' 4	4, (1, 2 ; 7) nodal, (11, 15)
8	12, 16	×	×	×	*	C 16	C 16	C 16	C 16	C 12	C 12	C 12	C 12	12, 16 (8 ; 1, 2, 3) trinodal
9	C 16				C' 16	*	C 16	C 16	×				C' 16	16, (5 ; 14, 15) nodal, (1, 9)
10	C 16				C' 16	C 16	*	C 16	×				C' 16	16, (6 ; 13, 15) nodal, (2, 10)
11	C 16				C' 16	C 16	C 16	*	×				C' 16	16, (7 ; 13, 14) nodal, (3, 11)
12	4, 12	C 4	C 4	C 4	C 12	×	×	×	*	C 12	C 12	C 12	C 4	4, 12 (8 ; 13, 14, 15) trinodal
13	C 12				C' 12				C' 12	*	C 12	C 12	×	12, (5 ; 10, 11) nodal, (1, 13)
14	C 12				C' 12				C' 12	C 12	*	C 12	×	12, (6 ; 9, 11) nodal, (2, 14)
15	C 12				C' 12				C' 12	C 12	C 12	*	×	12, (7 ; 9, 10) nodal, (3, 15)
16	4, 16	C 4	C 4	C 4	C 16	C 16	C 16	C 16	C 4	×	×	×	*	4, 16, (8 ; 9, 10, 11) trinodal

where for greater clearness I have omitted the symbol CC', which is to be understood as occupying each of the vacant squares.

The arrangement is the same as before; the right-hand margin shows the sextic cone; viz., for the node 4 this is made up of the singular planes 4, 12, 16 and of a cubic cone represented by (5, 6, 7) (as replacing the planes 5, 6, 7 in the case of the 16-nodal surface). Similarly for the node 5, the sextic cone is made up of the singular plane 4, the nodal cubic cone (2, 3; 5), and the quadric cone (9, 13) (the numbers in these last symbols indicating the planes in the case of the 16-nodal surface, which are here replaced by cones). So for the node 8, the sextic cone is made up of the singular planes 12, 16 and of the trinodal quartic cone (8; 1, 2, 3). As regards a nodal cubic cone, for example (2, 3; 5), the semicolon is used to indicate that the nodal line replaces the intersection of the planes 2, 3; the other intersections 2, 5 and 3, 5 having disappeared. And so for a trinodal quartic cone (8; 1, 2, 3), the semicolon is used to indicate that the nodal lines replace the intersection (1, 2), the intersection (1, 3), and the intersection (2, 3) respectively; the other intersections 1, 2; 2, 8; and 3, 8 having disappeared. Finally, in the body of the table, C is used to denote the cubic or the quartic cone (as the case may be); $\times$ to denote a nodal line of either of these cones; and C' the quadric cone; as already mentioned, the vacant places are considered to be CC'.

The reading of the table is then as follows; viz., for the node 4, the remaining twelve nodes lie on the nodal lines of the sextic cone 4, 12, 16, (5, 6, 7), as shown; viz., 5, 6, 7 are each of them on the intersection of the cubic cone with the plane 4; 8 is on the intersection of the planes 12 and 16; and so on.

I reserve for another Memoir the discussion of the 13 (β)-nodal surface, and the surfaces with less than 13 nodes.

455.

ON PLÜCKER'S MODELS OF CERTAIN QUARTIC SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 281—285. Read June 8, 1871.]

THE Society possesses a series of 14 wooden models of surfaces, constructed under the direction of the late Prof. Plücker, in illustration of the theory developed in his posthumous work, "Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelemente," Leipzig, 1869. These all of them represent, I believe, Equatorial Surfaces; viz., models 1 to 8 represent cases of the 78 forms of equatorial surfaces "deren Breiten-Curven eine feste Axenrichtung besitzen," vol. II. pp. 352—363, the remaining models, Nos. 9—14, I have not completely identified. I propose to go into the theory only so far as is required for the explanation of the models.

In a "Complex," or triply infinite system of lines, there is in any plane whatever a singly infinite system of lines enveloping a curve; and if we attend only to the curves the planes of which pass through a given fixed line, the locus of these curves is a "complex surface." Similarly, there is through any point whatever a single infinite series of lines generating a cone; and if we attend only to the cones which have their vertices in the given fixed line, then the envelope of these cones is the same complex surface. In the case considered of a complex of the second degree, the curves and cones are each of them of the second order; the fixed line is a double line on the surface, so that (attending to the first mode of generation) the complete section by any plane through the fixed line is made up of this line twice, and of a conic; the surface is thus of the order 4: it is also of the class 4; the surface has, in fact, the nodal line, and also 8 nodes (conical points), and we have thus a reduction $=32$ in the class of the surface.

In the particular case where the nodal line is at infinity, the complex surface becomes an equatorial surface; viz., (attending to the first mode of generation) we have

here a series of parallel planes each containing a conic, and the locus of these conics is the equatorial surface.

It is convenient to remark that, taking a, b, h to be homogeneous functions of (x, w) of the order 2; f, g of the order 1; and c of the order 0 (a constant); then the equation of a complex surface (see vol. I. p. 162) is

$$\begin{vmatrix} & y, & z, & 1 \\ y, & a, & h, & g \\ z, & h, & b, & f \\ 1, & g, & f, & c \end{vmatrix} = 0;$$

and that, writing $w=1$, or considering a, h, b; f, g; c as functions of x of the orders 2, 1, 0 respectively, we have an equatorial surface.

A particular form of equatorial surface is thus $bcy^2 + caz^2 + ab = 0$, or taking $c=1$, this is $by^2 + az^2 + ab = 0$, where a, b are quadric functions of x.

The surface is still, in general, of the fourth order: it may however degenerate into a cubic surface, or even into a quadric surface; the last case is however excluded from the enumeration. The section by any plane parallel to that of yz is a conic; the section by the plane $y=0$ is made up of the pair of lines $a=0$, and of the conic $z^2+b=0$; that by the plane $z=0$ is made up of the pair of lines $b=0$, and of the conic $y^2+a=0$; the last-mentioned planes may be called the principal planes, and the conics contained in them principal conics. The surface is thus the locus of a variable conic, the plane of which is parallel to that of yz, and which has for its vertices the intersections of its plane with the two principal conics respectively. And we have thus the particular equatorial surfaces considered by Plücker, vol. II. pp. 346—363 (as already mentioned), under the form

$$\frac{y^2}{Ex^2+2Ux+C} + \frac{z^2}{Fx^2-2Rx+B} + 1 = 0,$$

and of which he enumerates 78 kinds; viz., these are

1 to 17	Principal conics each proper.
18 „ 29	One of them a line-pair.
30 „ 32	Each a line-pair.
33 „ 39	Principal conics, each proper, but having a common point.
40 „ 43	One of them a line-pair, its centre on the other principal conic.
44 „ 61	One principal conic a parabola.
62 „ 73	One principal conic a pair of parallel lines.
74 „ 76	Principal conics each a parabola.
77 and 78	Principal conics, one of them a parabola, the other a pair of parallel lines.

The models 1 to 8 correspond hereto, as follows:

Mod.	1	is 13,	i.e. 2	is Mod.	5
„	2	9	3	„	6
„	3	40	4	„	7
„	4	34	9	„	2
„	5	2	13	„	1
„	6	3	32	„	8
„	7	4	34	„	4
„	8	32	40	„	3.

Mod. 5 is 2: the form of the equation is here

$$\frac{y^2}{l^2[(x-\alpha)^2+\beta^2]}-\frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]}=1;$$

viz., the principal conics are one of them a hyperbola, the other imaginary; hence the generating conic has always two, and only two, real vertices, viz., it is always a hyperbola: there are no real lines.

Mod. 6 is 3: the form of the equation is

$$\frac{y^2}{l^2[(x-\alpha)^2+\beta^2]}+\frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]}=1;$$

viz., the principal conics are each of them a hyperbola; the generating conic has four real vertices, viz., it is always an ellipse: there are no real lines.

Mod. 7 is 4: the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)}+\frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]}+1=0.$$

The principal conics are one of them an ellipse, the other imaginary; for values of x between γ and δ, the variable conic has two real vertices or it is a hyperbola; for any other values it is imaginary, so that the surface lies wholly between the planes $x=\gamma$, $x=\delta$: the surface contains the real lines $y=0$, $x=\gamma$ and $y=0$, $x=\delta$.

Mod. 2 is 9: the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)}+\frac{z^2}{l'^2(x-\gamma')(x-\delta')}+1=0,$$

where, say the values γ, δ lie between the values γ', δ': the principal conics are each of them an ellipse, the vertices (on the axis or line $y=0$, $z=0$) of the one ellipse lying between those of the other ellipse. The variable conic for values of x between γ, δ has four real vertices, or it is an ellipse; for values beyond these limits, but within the limits γ', δ'—say from γ to γ', and from δ to δ'—there are two real vertices, or the conic is a hyperbola; and for values beyond the limits γ', δ', the variable conic is imaginary.

There are four real lines $(y=0,\ x=\gamma)$, $(y=0,\ x=\delta)$, $(z=0,\ x=\gamma')$, $(z=0,\ x=\delta')$. The surface consists of a central pillow-like portion, joined on by two conical points to an upper portion, and by two conical points to an under portion, the whole being included between the planes $x=\gamma'$, $x=\delta'$.

Mod. 1 is 13: the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)}-\frac{z^2}{l'^2(x-\gamma')(x-\delta')}+1=0;$$

the values γ', δ' lying between γ, δ; the principal conics are one of them a hyperbola, the other an ellipse, the vertices (on the axis or line $y=0$, $z=0$) of the hyperbola lying between those of the ellipse.

The variable conic, for values of x between γ', δ', has two real vertices, or it is a hyperbola; for the values, say, from γ' to γ, and δ' to δ, there are four real vertices, or the conic is an ellipse; for values beyond the limits γ, δ, there are two real vertices, and the conic is a hyperbola. There are the four real lines $(y=0,\ x=\gamma)$, $(y=0,\ x=\delta)$, and $(z=0,\ x=\gamma')$, $(z=0,\ x=\delta')$. The surface consists of 8 portions joined to each other by 8 conical points, but the form can scarcely be explained by a description.

Mod. 8 is 32: the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)^2}+\frac{z^2}{l'^2(x-\gamma')^2}=1;$$

viz., the principal conics are each of them a line-pair, the variable conic is always an ellipse.

There are the two real nodal lines $(y=0,\ x=\gamma)$ and $(z=0,\ x=\gamma')$, each of these being in the neighbourhood of the axis crunodal, and beyond certain limits acnodal; the surface is a scroll, being, in fact, the well-known surface which is the boundary of a small circular pencil of rays obliquely reflected, and consequently passing through two focal lines.

Mod. 4 is 34: the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)}+\frac{z^2}{l'^2(x-\gamma')(x-\delta)}+1=0,$$

where $x=\delta$ is not intermediate between the values $x=\gamma$ and $x=\gamma'$; say the order is δ, γ, γ'. The surface is thus a *cubic* surface; the principal conics are ellipses having on the axis a common vertex at the point $x=\delta$, and the remaining two vertices on the same side of the last-mentioned one. The variable conic for values between δ and γ has four real vertices, or it is an ellipse; for values between γ and γ' two real vertices, or it is a hyperbola; and for values beyond the limits δ, γ' it is imaginary. There are on the surface the two real lines $(y=0,\ x=\gamma)$ and $(z=0,\ x=\gamma')$. The surface consists of a finite portion joined on by two conical points to the remaining portion.

Mod. 3 is 40: the form of equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)}+\frac{z^2}{l'^2(x-\delta)^2}+1=0.$$

The surface is thus a *cubic* surface; the principal conics are, one of them an ellipse, the other a pair of imaginary lines intersecting on the ellipse; for values of x between γ and δ, the variable conic has thus two real vertices, and it is a hyperbola; for values beyond these limits it is imaginary, and the whole surface is thus included between the planes $x=\gamma$ and $x=\delta$. There are the two real lines $(y=0, x=\gamma)$ and $(z=0, x=\delta)$.

Taking $l^2=l'^2=1$, the surface is

$$\frac{y^2}{(x-\gamma)(x-\delta)}+\frac{z^2}{(x-\delta)^2}+1=0,$$

which is a *particular* case of the parabolic cyclide.

As already mentioned, I have not completely identified the remaining models 9 to 14, but I will say a few words about them.

The equatorial surfaces, not included in the preceding 78 cases, Plücker distinguishes (vol. II. p. 363) as "gedrehte" or "tordirte," say as twisted equatorial surfaces; the equation of such a surface is

$$by^2+2hyz+az^2+ab-h^2=0,$$

where

$$b=Fx^2-2Rx+B,$$

$$a=Ex^2+2Ux+C,$$

$$h=Kx^2-\ Ox-G \text{ (or in particular } =-Ox-G).$$

Mod. 13 is such a surface, being a twisted form of model 2.

Mod. 9 and Mod. 14 belong, I think, to the case $a=0$; viz., the form of the equation is $by^2+2hyz-h^2=0$. The variable conic is a hyperbola, the direction of one of the asymptotes being constant (vol. II. p. 368).

There are moreover (p. 372) equatorial surfaces in which the variable conic is always a parabola, and where there are on the surface four real or imaginary lines.

Mods. 10, 11, and 12 seem to represent such surfaces.

456.

NOTE ON THE DISCRIMINANT OF A BINARY QUANTIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), p. 23.]

It is well known that the discriminant of a binary quantic $(a, b, c, d, \ldots \between t, 1)^n$ is of the form

$$Ma + Nb^2,$$

but it is further to be remarked that if $b=0$, then the form is

$$a\,(Ma + Nc^3),$$

if $b=0$, $c=0$, the form is

$$a^2\,(Ma + Nd^4),$$

and so on, until only the lowest two coefficients are not put $=0$. Or, what is the same thing, if in the discriminant of the original function we put $a=0$, then the discriminant divides by b^2; if $b=0$, the discriminant divides by a, and, omitting this factor, if we then write $a=0$, it divides by c^3; if $b=0$, $c=0$, the discriminant divides by a^2, and omitting this factor, if we then write $a=0$, it divides by d^4; and so on, until as before.

Thus if $b=0$, the discriminant of $(a, 0, c, d, e \between t, 1)^4$, divides by a, and omitting this factor it is

$$\begin{aligned} & a^2e^2 \\ & -18\,ac^2e^2 \\ & +54\,acd^2e \\ & -27\,ad^4 \\ & +81\,c^4e \\ & -54\,c^3d^2 \end{aligned}$$

which for $a=0$ has the factor c^3; if $b=0$, $c=0$, the discriminant of $(a, 0, 0, d, e \between t, 1)^4$ has the factor a^2, and omitting this factor it is

$$\begin{aligned} & ae^2 \\ & -27\,d^4, \end{aligned}$$

which for $a=0$ has the factor d^4; the series of theorems here terminates, since the lowest two coefficients d, e are not to be put $=0$.

457.

ON THE QUARTIC SURFACES $(*\between U, V, W)^2 = 0$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 24—34.]

I PROPOSE to myself for investigation the quartic surfaces represented by an equation

$$(*\between U, V, W)^2 = 0,$$

where U, V, W are quadric functions of the coordinates.

Such a surface has 8 nodes (conical points), viz., these are the points of intersection of the quadric surfaces $U=0$, $V=0$, $W=0$. It is to be observed, that not every quartic surface with 8 nodes is included under the above form; in fact the equation of a quartic surface contains (homogeneously) 35 coefficients, or say 34 arbitrary parameters; in order that a given point may be a node, 4 conditions must be satisfied, and it is consequently possible to find a quartic surface having 8 given points as nodes (and having in its equation $(34 - 8 . 4 =)$ 2 arbitrary parameters): but 8 given points are not in general the intersections of three quadric surfaces, and such a quartic surface is therefore not in general included under the above form. I think, however, that it may be assumed that the above form includes all the quartic surfaces having 8 nodes, points of intersection of three quadric surfaces. It will presently appear that, included in the form, we have surfaces where (instead of the 8 nodes) there is a nodal or cuspidal conic; and that these are the most general forms of such quartic surfaces. A quartic surface has at most 16 nodes, and the general form with 8 nodes must admit of being particularised so that the surface shall acquire any number not exceeding 8 of additional nodes. This does not show, but it is probable, that the above special form with 8 nodes can be particularised so that the surface shall in like manner acquire any number not exceeding 8 of additional nodes. Similarly, a quartic surface with a nodal conic may have besides 1, 2, 3, or 4 nodes; and it will be shown in the

sequel how the form, particularised so as to give a nodal conic, may be further particularised so as to give the 1, 2, 3, or 4 nodes. So a quartic surface with a cuspidal conic may besides have 1 node, and it will be shown how the form, particularised so as to give a cuspidal conic, may be further particularised so as to give 1 node.

Starting from the equation $(*\between U, V, W)^2 = 0$, we may, by substituting for U, V, W linear functions of these expressions, transform the equation precisely in the manner of a conic, and therefore into any of the forms under which the equation of a conic can be exhibited; for instance, in the forms $aU^2 + bV^2 + cW^2 = 0$, $fVW + gWU + hUV = 0$, $UW - V^2 = 0$, &c. I attend at present only to the last-mentioned form $UW - V^2 = 0$, which, it thus appears, is equally general with the original form $(*\between U, V, W)^2 = 0$.

The quartic surface

$$UW - V^2 = 0$$

where U, V, W are any quadric functions of the coordinates, may be considered as the envelope of the quadric surface

$$(U, V, W \between \theta, 1)^2 = 0,$$

where θ is an arbitrary parameter. And it thus appears that it is very easy to reciprocate (in regard to any given quadric surface) the quartic surface. For the reciprocal of the quartic surface is clearly the envelope of the reciprocal of the variable quadric surface; this reciprocal is itself a quadric surface, and the reciprocal of the quartic surface is thus given in the same form as the original surface, viz., as the envelope of a quadric surface the equation whereof contains rationally the variable parameter θ; the equation of the reciprocal surface is consequently obtained by equating to zero the discriminant in regard to θ, of the equation of the reciprocal quadric surface.

It is to be observed that, inasmuch as the equation of the reciprocal quadric surface is of the third degree in the coefficients of the original quadric, it is in general of the degree 6 in the parameter θ; we have thus a sextic function of θ, the coefficients whereof are quadric functions of the coordinates; and the discriminant is a function of the order 10 in these coefficients, that is, of the order 20 in the coordinates. The reciprocal of the quartic surface is thus a surface of the order 20; this is right, for in a general quartic surface the order of the reciprocal surface is $= 36$, and the 8 nodes reduce the order by 16; $36 - 16 = 20$.

In the equation $UW - V^2 = 0$, or say $V^2 - UW = 0$; if U reduce itself to the square of a linear function, $U = P^2$, the equation becomes $V^2 - P^2W = 0$, which is the general form of the quartic surface having the nodal conic $V = 0$, $P = 0$. And if, moreover, W be the product of this same linear function P by another linear function Q, $W = PQ$, then the form is $V^2 - P^3Q = 0$, which is the general form of the quartic surface having the cuspidal conic $V = 0$, $P = 0$.

Writing for greater convenience x, y in the place of P, Q respectively, we have the quartic

$$(AA) \qquad V^2 - x^3y = 0,$$

having the cuspidal conic $V=0$, $x=0$; and which has besides the conic of plane contact $V=0$, $y=0$. In virtue of the cuspidal conic the reciprocal surface should be of the order 6; and by the foregoing method of obtaining the equation of the reciprocal surface, I will verify that this is so. To effect this as simply as possible, I fix the remaining coordinates z, w as follows. The line $x=0$, $y=0$ is not in general a tangent to the surface $V=0$; it therefore meets this surface in two points, and we may take $z=0$, $w=0$ to be the equations of the tangent planes at these two points respectively; we have thus $V=ax^2+2hxy+by^2+2nzw$. Introducing for convenience the numerical factor 2, and taking the equation of the surface to be

$$(ax^2+2hxy+by^2+2nzw)^2-2x^3y=0,$$

this is the envelope of the quadric surface

$$\theta^2x^2+2\theta(ax^2+2hxy+by^2+2nzw)+2xy=0,$$

which is a surface $(a, b, c, d, f, g, h, l, m, n\between x, y, z, w)^2=0$, where $a=\theta^2+2a\theta$, $b=2\theta b$, $h=2\theta h+1$, $n=2\theta n$, and where all the other coefficients vanish. Assuming, as usual, that the reciprocation is effected in regard to the surface $x^2+y^2+z^2+w^2=0$, the general equation is

$$\begin{aligned}
& x^2 . d(bc-f^2)-cm^2-bn^2+2fmn \\
+\ & y^2 . d(ca-g^2)-an^2-cl^2+2gnl \\
+\ & z^2 . d(ab-h^2)-bl^2-am^2+2hlm \\
+\ & w^2 . abc-af^2-bg^2-ch^2+2fgh \\
& +2yz . d(gh-af)+l^2f+amn-hnl-glm \\
& +2zx . d(hf-bg)+m^2g+bnl-flm-hmn \\
& +2xy . d(fg-ch)+n^2h+clm-gmn-fnl \\
& +2xw . -l(bc-f^2)-n(hf-bg)-m(fg-ch) \\
& +2yw . -m(ca-g^2)-l(fg-ch)-n(gh-af) \\
& +2zw . -n(ab-h^2)-m(gh-af)-l(hf-bg),
\end{aligned}$$

(I write down this general result as it will be useful for reference in other cases); in the present case this becomes simply

$$x^2 . -bn^2+y^2 . -an^2+2xy . nh^2+2zw(-nab+nh^2)=0,$$

where a, b, n, h have the foregoing values; the equation is thus only of the order 4 in regard to θ; but it in fact divides by $n(=2\theta n)$ and thus reduces itself to the third order, viz. it becomes

$$n(bx^2+ay^2)-2h^2xy+2(ab-h^2)zw=0,$$

or, substituting for a, b, n, h their values, this is

$$x^2 . 4bn\theta^2+y^2(2n\theta^3+4an\theta^2)+2zw(2b\theta^3+4ab\theta^2)-2(xy+zw)(2\theta h+1)^2=0,$$

say this is

$$(A, B, C, D \between \theta, 1)^2 = 0,$$

where A, B, C, D are each of them a quadric function of the coordinates; it being observed that C and D are respectively numerical multiples of the same function $xy + zw$. Hence equating the discriminant to zero, we have

$$A^2D^2 + 4AC^3 + 4B^3D - 3B^2C^2 - 6ABCD = 0,$$

which equation, inasmuch as every term contains either C or D as a factor, divides by $xy + zw$, and thus becomes an equation of the order 6 in the coordinates: that is the order of the reciprocal surface is $= 6$. Multiplying by $\frac{3}{2}$ to avoid fractions, the actual values of A, B, C, D are

$$\begin{cases} A = 3\ (ny^2 + 2bzw), \\ B = 2\ \{bnx^2 + any^2 + 2abzw - 2h^2(xy + zw)\}, \\ C = -4h\ (xy + zw), \\ D = -3\ (xy + zw), \end{cases}$$

or say $A = 3\alpha$, $B = 2\beta$, $C = -4h\gamma$, $D = -3\gamma$; where $\gamma = xy + zw$; substituting these values and omitting the factor 3γ, the equation is

$$27\alpha^2\gamma - 256h^3\alpha\gamma^2 - 32\beta^3 - 64h^2\beta^2\gamma - 144h\alpha\beta\gamma = 0,$$

which is an equation of the form $(*\between \alpha, \beta, \gamma)^3 = 0$. The sextic surface has thus singular points $\alpha = 0$, $\beta = 0$, $\gamma = 0$, viz. these are the two points $(x = 0, y = 0, z = 0)$, $(x = 0, y = 0, w = 0)$ each four times. The further discussion of the sextic surface is reserved for another occasion.

I do not at present attempt to enumerate the particular cases of the surface $V^2 - x^3y = 0$, but content myself with the discussion of a particular case in which the order of the reciprocal surface is $= 3$. Suppose that $V = 0$ is a cone, $y = 0$ a tangent plane to the cone (so that the conic $y = 0$, $V = 0$ breaks up into a line twice repeated), $x = 0$ an arbitrary plane (so that we have still the proper cuspidal conic $x = 0$, $V = 0$). Any other tangent plane of the cone may be taken for the plane $z = 0$; the plane containing the lines of contact of the two tangent planes for the plane $w = 0$; the equation of the conic then is $V = dw^2 + 2fyz = 0$; and the equation of the surface is

$$(AB) \qquad (dw^2 + 2fyz)^2 - x^3y = 0.$$

For convenience of comparison, I change x, y, w, z into y, w, z, x, and assign numerical values to the coefficients, writing the equation under the form

$$(AB) \qquad 27(4xw + z^2)^2 - 64y^2w = 0.$$

The quartic is here the envelope of the quadric surface

$$\theta^2 . 4yw + \theta . 9(4xw + z^2) + 12y^2 = 0,$$

viz., comparing with the general form

$$(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0,$$

we have $b = 12$, $c = 9\theta$, $l = 18\theta$, $m = 2\theta^2$, all the other coefficients vanishing. The reciprocal equation is

$$x^2(-cm^2) + y^2(-cl^2) + z^2(-bl^2) + 2xy \,.\, clm + 2xw(-lbc) = 0,$$

or substituting for b, c, l, m their values, this is found to be

$$\theta(\theta x - 9y)^2 + 108(z^2 + xw) = 0.$$

Representing this by $(A, B, C, D \between \theta, 1)^3 = 0$, the discriminant in regard to θ would, in virtue of the values of A, B, C, contain D as a factor; the reason of this appears from the original form; in fact, forming the derived equation in regard to θ, this is found to be $(\theta x - 9y)(\theta x - 3y) = 0$; the value $\theta x - 9y = 0$ gives as a factor of the discriminant $z^2 + xw$; the value $\theta x - 3y = 0$ gives $3y(-6y)^2 + 108x(z^2 + xw)$, that is the factor $y^3 + x(z^2 + xw)$; the complete value of the discriminant as obtained by substitution of the values of A, B, C, D being $x^3(z^2 + xy)\{y^3 + x(z^2 + xw)\}$; the equation of the reciprocal surface is

$$y^3 + x(z^2 + xw) = 0,$$

viz. this is a cubic surface, Prof. Schläfli's Case XX., having a uniplanar point $x = 0$, $y = 0$, $z = 0$ reducing the class by 8, and so giving a reciprocal surface of the order $(12 - 8 =) 4$, viz. the surface $27(4xw + z^2)^2 - 64y^3w = 0$. See the Memoir, Schläfli, "On the distribution of surfaces of the third order into species in reference to the absence or presence of singular points and the reality of their lines," *Phil. Trans.*, vol. CLIII. (1853), pp. 193—241.

I pass to the case of a surface

$$V^2 - P^2U = 0$$

having a nodal conic $V = 0$, $P = 0$, but not having in general any nodes. And I propose to show how the constants may be determined so that the surface shall have 1, 2, 3, or 4 nodes. It is to be remarked that in the above equation the plane $P = 0$ is a determinate plane, but the quadric surface $V = 0$ is not a determinate quadric, we may in fact substitute for it the quadric $V + \lambda P^2 = 0$, writing the equation under the form

$$(V + \lambda P^2)^2 - P^2(U + 2\lambda V + \lambda^2 P^2),$$

so that we may without loss of generality, by means of the disposable constant λ, subject the surface $V = 0$ to any single condition; for instance, take it to be a cone, or to pass through a given point, &c.

Taking the planes $x = 0$, $y = 0$, $z = 0$, $w = 0$ to be arbitrary planes, the implicit constant factors in these equations may be determined in such wise that the equation of the given plane $P = 0$ shall be $x + y + z + w = 0$. The equation of the surface will then be

$$\{(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2\}^2$$
$$= (x + y + z + w)^2 . (a', b', c', d', f', g', h', l', m', n' \between x, y, z, w)^2,$$

and we may assume that the node or nodes (if any) lie at a vertex or vertices of the tetrahedron $x=0$, $y=0$, $z=0$, $w=0$, say at the points A, B, C, D. The conditions for a node at each of these points are at once found to be

Node at A,	Node at B,	Node at C,	Node at D,
$2a^2 = a' + a'$	$2bh = h' + b'$	$2cg = g' + c'$	$2dl = l' + d'$
$2ah = h' + a'$	$2b^2 = b' + b'$	$2cf = f' + c'$	$2dm = m' + d'$
$2ag = g' + a'$	$2bf = f' + b'$	$2c^2 = c' + c'$	$2dn = n' + d'$
$2al = l' + a'$	$2bm = m' + b'$	$2cn = n' + c'$	$2d^2 = d' + d'$

The first set of equations gives

$$a' = a^2, \quad h' = 2ah - a^2, \quad g' = 2ag - g^2, \quad l' = 2al - a^2.$$

If the first and second sets are satisfied simultaneously, we have $2(a-b)h = a^2 - b^2$, that is $a = b$, or else $h = \frac{1}{2}(a+b)$; that is, the two sets may be satisfied in two different ways according as a and b are equal or unequal. Similarly the first, second, and third sets may be satisfied in three different ways and the four sets in five different ways according as there are or are not any equalities between a, b, c, and between a, b, c and d respectively. The several solutions are shown in the annexed table, viz., in the line I no set is satisfied; in the line II only the first set; in the lines III and IV the first and second sets; in the lines V to VII the first, second, and third sets; and in the lines VIII to XII the four sets.

[See next page for this Table, which should come in here.]

I is the general case, $V^2 = P^2 U$, of a quartic and a nodal conic but without nodes. (AC).

II is the case of a single node; writing, as without loss of generality we may do, $a=0$, the equation is

$$[(0, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2]^2 = (x+y+z+w)^2 . (b, c, d, n, m, f \between y, z, w)^2,$$

viz. the quadric $U=0$ is here a cone having its vertex on the quadric $V=0$. (AD).

III and IV are two cases each of them with two nodes, viz. III, the equation is

$$\begin{aligned} &\{(ax+by)(x+y) + cz^2 + 2nzw + dw^2 + 2x(gz+lw) + 2y(fz+mw)\}^2 \\ &\qquad = (x+y+z+w)^2 [(ax+by)^2 + c'z^2 + 2n'zw + d'w^2 \qquad (AE) \\ &\qquad\qquad + 2x\{(2ag - a^2)z + (2al - a^2)w\} \\ &\qquad\qquad + 2y\{(2bf - b^2)z + (2bm - b^2)w\}], \end{aligned}$$

where it is to be observed that the line $z=0$, $w=0$ joining the two nodes $(y=0, z=0, w=0)$ and $(w=0, z=0, x=0)$ is a line on the surface. Writing, as we may do, $a=0$, the equation assumes the more simple form

$$\begin{aligned} &\{by(x+y) + cz^2 + 2nzw + dw^2 + 2x(gz+lw) + 2y(fz+mw)\}^2 \qquad (AE) \\ &\qquad = (x+y+z+w)^2 [b^2y^2 + c'z^2 + 2n'zw + d'w^2 + 2y\{(2bf - b^2)z + (2bm - b^2)w\}]. \end{aligned}$$

I	a	b	c	d	f	g	h	l	m	n	a'	b'	c'	d'	f'	g'	h'	l'	m'	n'
II	$\widetilde{a\ a}$	b	c	d	f	g	h	l	m	n	a^2	b'	c'	d'	f'	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	m'	n'
III	$\widetilde{a\ b}$		c	d	f	g	$\frac{1}{2}(a+b)$	l	m	n	a^2	b^2	c'	d'	$2bf-b^2$	$2ag-a^2$	ab	$2al-a^2$	$2bm-b^2$	n'
IV	a	a	c	d	f	g	h	l	m	n	a^2	a^2	c'	d'	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	n'
V	$\widetilde{a\ b\ c}$			d	$\frac{1}{2}(b+c)$	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+b)$	l	m	n	a^2	b^2	c^2	d'	bc	ac	ab	$2al-a^2$	$2bm-b^2$	$2cn-c^2$
VI	a	a	c	d	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	l	m	n	a^2	a^2	c^2	d'	ac	ac	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2cn-c^2$
VII	a	a	a	d	f	g	h	l	m	n	a^2	a^2	a^2	d'	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2an-a^2$
VIII	$\widetilde{a\ b\ c\ d}$				$\frac{1}{2}(b+c)$	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+b)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(b+d)$	$\frac{1}{2}(c+d)$	a^2	b^2	c^2	d^2	bc	ac	ab	ad	bd	cd
IX	a	a	c	d	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(c+d)$	a^2	a^2	c^2	d^2	ac	ac	$2ah-a^2$	ad	ad	cd
X	a	a	a	d	f	g	h	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	$\frac{1}{2}(a+d)$	a^2	a^2	a^2	d^2	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	ad	ad	ad
XI	a	a	c	c	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	h	$\frac{1}{2}(a+c)$	$\frac{1}{2}(a+c)$	n	a^2	a^2	c^2	c^2	ac	ac	$2ah-a^2$	ac	ac	$2cn-c^2$
XII	a	a	a	a	f	g	h	l	m	n	a^2	a^2	a^2	a^2	$2af-a^2$	$2ag-a^2$	$2ah-a^2$	$2al-a^2$	$2am-a^2$	$2an-a^2$

In IV the equation is

$$\begin{aligned}\{a(x^2+y^2)+2hxy+cz^2+dw^2+2nzw+2x(gz+lw)+2y(fz+mw)\}^2 \\ = (x+y+z+w)^2[a^2(x^2+y^2)+2(2af-a^2)xy+c'z^2+2n'zw+d'w^2 \\ +2x\{(2ag-a^2)z+(2al-a^2)w\} \\ +2y\{(2af-a^2)z+(2am-a^2)w\}],\end{aligned} \qquad (AF)$$

where it will be observed that the line $z=0$, $w=0$ joining the two nodes is *not* a line on the surface.

Writing, as we may do, $a=0$, the equation becomes

$$\begin{aligned}\{2hxy+cz^2+2nzw+dw^2+2x(gz+lw)+2y(fz+mw)\}^2 \\ = (x+y+z+w)^2(c'z^2+2n'zw+d'w^2),\end{aligned} \qquad (AF)$$

viz. the form is $V^2 = P^2QR$, the quadric surface $U=0$ breaking up into the two planes $Q=0$, $R=0$; and the nodes being situate at the intersections of the line $Q=0$, $R=0$ with the surface $V=0$.

V, VI, VII are apparently cases with three nodes, but in fact VI is the only case of a proper quartic surface with three nodes. For in V the equation is

$$\begin{aligned}\{(ax+by+cz)(x+y+z)+dw^2+2w(lx+my+nz)\}^2 \\ = (x+y+z+w)^2[(ax+by+cz)^2+d'w^2+2w\{(2al-a^2)x+(2bm-b^2)y+(2cn-c^2)z\}],\end{aligned}$$

which is satisfied by $w=0$, and the surface thus breaks up into the plane $w=0$ and a cubic surface.

And in VII the equation is

$$\begin{aligned}\{a(x^2+y^2+z^2)+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw\}^2 \\ = (x+y+z+w)^2[a^2(x^2+y^2+z^2)+d'w^2 \\ +2\{(2af-a^2)yz+(2ag-a^2)zx+(2ah-a^2)xy \\ + (2al-a^2)xw+(2am-a^2)yw+(2an-a^2)zw\}],\end{aligned}$$

which putting $a=0$ is

$$(dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw)^2 = (x+y+z+w)^2 d'w^2,$$

viz. this is a pair of quadric surfaces.

In the remaining case VI the equation is

$$\begin{aligned}\{a(x^2+y^2)+2hxy+cz^2+dw^2+(a+c)(yz+zx)+2w(lx+my+nz)\}^2 \\ = (x+y+z+w)^2[a^2(x^2+y^2)+c^2z^2+d'w^2+2acz(x+y)+2(2ah-a^2)xy \\ +2w\{(2al-a^2)x+(2am-a^2)y+(2an-a^2)z\}],\end{aligned} \qquad (AG)$$

which putting therein $a=0$ is

$$\begin{aligned}\{2hxy+dw^2+cz(x+y+z) \qquad +2w(lx+my+nz)\}^2 \\ = (x+y+z+w)^2\{c^2z^2+d'w^2+2(2cn-c^2)zw\},\end{aligned} \qquad (AG)$$

which is a surface having the nodes A, B, C; and it is to be observed that the lines CA, CB, but *not* the line AB, are lines on the surface.

IX, X, XI, XII are apparently cases with four nodes, but it is only XI which is a proper quartic with four nodes. In fact IX is

$$\{(ax+ay+cz+dw)(x+y+z+w)+2(h-a)xy\}^2$$
$$=(x+y+z+w)^2\{(ax+ay+cz+dw)^2+4a(h-a)xy\},$$

which is satisfied if $x=0$ or if $y=0$; that is, the surface breaks up into the two planes $x=0$, $y=0$, and a quadric surface.

X is

$$\{a(x^2+y^2+z^2)+dw^2+2fyz+2gzx+2hxy+(a+d)(xw+yw+zw)\}^2$$
$$=(x+y+z+w)^2\{a^2(x^2+y^2+z^2)+d^2w^2$$
$$+2(2af-a^2)yz+2(2ag-a^2)zx+2(2af-a^2)xy+2ad(xw+yw+zw)\},$$

which putting therein $a=0$ is

$$\{dw(x+y+z+w)+2fyz+2gzx+2hxy\}^2=(x+y+z+w)^2d^2w^2,$$

and thus breaks up into two quadrics.

And XII is

$$\{a^2(a^2+y^2+z^2+w^2)+2fyz+2gzx+2hxy+2lxw+2myw+2nw\}^2$$
$$=(x+y+z+w)^2\{a^2(x^2+y^2+z^2+w^2)$$
$$+2(2af-a^2)yz+2(2ag-a^2)zx+2(2ah-a^2)xy$$
$$+2(2al-a^2)xw+2(2am-a^2)yw+2(2ah-a^2)zw\},$$

which putting $a=0$ is

$$(2fyz+2gzx+2hxy+2lxw+2myw+2nzw)^2=0,$$

and is thus a quadric surface twice repeated.

There remains XI, and here the equation is

$$\{(ax+ay+cz+cw)(x+y+z+w)+2(h-a)xy+2(n-c)zw\}^2$$
$$=(x+y+z+w)^2\{(ax+ay+cz+cw)^2+4a(h-a)xy+4c(n-c)zw\},$$

or writing $h+a$, $n+c$ in place of a, c respectively, this is

$$\{(ax+ay+cz+cw)(x+y+z+w)+2hxy+2nw\}^2$$
$$=(x+y+z+w)^2\}(ax+ay+cz+cw)^2+4ahxy+4cnzw\}^2, \qquad (AH)$$

or putting herein $a=0$ it is

$$\{c(z+w)(x+y+z+w)+2hxy+2nzw\}^2=c(x+y+z+w)^2\{c(z+w)^2+4nzw\}, \quad (AH)$$

which may also be written

$$c(x+y+z+w)\{hxy(z+w)-nzw(x+y)\}+(hxy+nzw)^2=0, \qquad (AH)$$

the equation of a quartic surface with the four nodes A, B, C, D; it is to be observed that the lines AC, AD, BC, BD are, the lines AB and CD are *not*, lines on the surface.

A more simple form may be given to the equation as follows; using the second of the above forms, multiplying the equation by 4, and writing therein

$$p=\sqrt{(c)}(x+y+z+w),$$
$$qr=c(z+w)^2+4nzw,$$
$$st=c(x+y)^2-4hxw,$$

q, r, and s, t being the linear factors of the two quadric functions respectively, we have

$$qr-st=c(x+y+z+w)(-x-y+z+w)+4hxy+4nzw,$$

and thence

$$p^2+qr-st=2c(z+w)(x+y+z+w)+4hxy+4nzw,$$

wherefore the equation is

$$(p^2+qr-st)^2=4p^2qr, \qquad (AH)$$

or, what is the same thing,

$$p+\sqrt{(qr)}+\sqrt{(st)}=0, \qquad (AH)$$

where p, q, r, s, t are any linear functions of the coordinates; this is the equation of a quartic surface having the nodal conic $p=0$, $qr-st=0$; and the four nodes $(q=0, r=0, p^2-st=0)$ and $(s=0, t=0, p^2-qr=0)$. It includes the Cyclide, the equation of which may be written

$$b^2=\sqrt{\{(ax-ek)^2+b^2y^2\}}+\sqrt{\{(ex-ak)^2-b^2z^2\}}.$$

I remark that Prof. Kummer in his most valuable Memoir, "Ueber die Flächen vierten Grades auf welchen Schaaren von Kegelschnitten liegen," *Crelle*, t. LXVI. (1864), pp. 66—76, has considered several of the cases of a quartic surface with a nodal conic, viz. no node, (AC); a single node, (AD); two nodes (the case AF); and four nodes, (AH); but he has not considered two nodes, the case (AE); nor three nodes, (AG).

In reference to the general case of a quartic surface with a nodal conic, some most interesting properties have recently been obtained by Prof. Clebsch, see *Berl. Monatsb.*, April 30, 1868, where it is shown that there are on the surface 16 right lines forming 20 systems of double-fours, analogous in some respect to the 27 lines and 36 systems of double-sixes of a cubic surface.

458.

ON THE ANHARMONIC-RATIO SEXTIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 56, 57.]

MR WALKER'S equation is $\Delta(\lambda^2-\lambda+1)^3+I^3(\lambda^2-\lambda)^2=0$; changing the sign of λ, and also the numerical multipliers of I, Δ (so as to convert the discriminant equation into its standard form $\Delta=I^3-27J^2$), the equation is

$$4\Delta(\lambda^2+\lambda+1)^3-27I^3(\lambda^2+\lambda)^2=0.$$

I remark that this is most readily obtained as follows; writing

$$A=(a-d)(b-c),$$
$$B=(b-d)(c-a),$$
$$C=(c-d)(a-b),$$

then we have $A+B+C=0$,

$$I=\tfrac{1}{24}(A^2+B^2+C^2)=-\tfrac{1}{12}(BC+CA+AB),$$
$$J=\tfrac{1}{432}(B-C)(C-A)(A-B),$$
$$\sqrt{(\Delta)}=\tfrac{1}{16}ABC,$$

see my Fifth Memoir on Quantics, *Phil. Trans.*, vol. CXLVIII. (1858), pp. 429—460, [156]. And observe also, that in virtue of the relation $A+B+C=0$, we have

$$12I=A^2+AB+B^2=A^2+AC+C^2=B^2+BC+C^2.$$

Hence writing

$$u=\frac{4\sqrt{(\Delta)}}{3I}\left(\lambda+\frac{1}{\lambda}+1\right),$$

when λ has any one of the values $\frac{A}{B}, \frac{B}{A}, \frac{A}{C}, \frac{C}{A}, \frac{B}{C}, \frac{C}{B}$, we see that u assumes only the values A, B, C, and u is thus determined by the equation

$$u^3 - 12Iu - 16\sqrt{(\Delta)} = 0.$$

Eliminating u, we obtain

$$16\sqrt{(\Delta)}\left\{\frac{4\Delta}{27I^3}\left(\lambda + \frac{1}{\lambda} + 1\right)^3 - \left(\lambda + \frac{1}{\lambda} + 1\right) - 1\right\} = 0,$$

or, what is the same thing,

$$4\Delta\left(\lambda + \frac{1}{\lambda} + 1\right)^3 - 27I^3\left(\lambda + \frac{1}{\lambda} + 2\right) = 0,$$

that is

$$4\Delta(\lambda^2 + \lambda + 1)^3 - 27I^3\lambda^2(\lambda + 1)^2 = 0,$$

the required equation.

459.

ON THE DOUBLE-SIXERS OF A CUBIC SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 58—71.]

THE 27 lines on a cubic surface include, and that in 36 different ways, *a double-sixer;* viz. a system of two sets of six lines 1, 2, 3, 4, 5, 6; 1′, 2′, 3′, 4′, 5′, 6′, such that every line of the one set intersects all the non-corresponding lines of the other set, thus

	1	2	3	4	5	6
1′		.	.	.	.	.
2′	.		.	.	.	.
3′	.	.		.	.	.
4′	.	.	.		.	.
5′	.	.	.	.		.
6′	.	.	.	.	.	

there being in all 30 intersections.

Any line say 4, of the one set, intersects five lines 1′, 2′, 3′, 5′, 6′ of the other set; and these six lines being given the double-sixer may be constructed; viz. (besides the line 4) we have a line 1 meeting the lines 2′, 3′, 5′, 6′; a line 2 meeting the lines 3′, 5′, 6′, 1′; a line 3 meeting the lines 5′, 6′, 1′, 2′; a line 5 meeting the lines 6′, 1′, 2′, 3′; and a line 6′ meeting the lines 1′, 2′, 3′, 5′; and then the lines 1, 2, 3, 5, 6 are all of them met by a single line 4′, which completes the system.

We may, if we please, consider the lines 4, 2 as given, and then 1′, 3′, 5′, 6′ will be any four lines each of them meeting the two given lines 4, 2; 2′ will be any line meeting 4; and we have to determine a line 4′ meeting 2, such that there may exist the lines 1, 3, 5, 6, completing the system as above. Or what is the same thing, we have a skew quadrilateral 1′, 2, 3′, 4; 5′ and 6′ meet 2 and 4; 2′ meets

4, and 4′ meets 2: 5 and 6 meet 1′ and 3′; 1 meets 3′ and 3 meets 1′; and the two sets 2′, 4′, 5′, 6′ and 1, 3, 5, 6 meet thus

	1	3	5	6
2′	.	.	.	.
4′	.	.	.	.
5′	.	.		.
6′	.	.	.	

Hence, starting with the skew quadrilateral 1′23′4, and taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the four planes 41′, 1′2, 23′, 3′4 respectively; or what is the same thing $x=0$, $y=0$ for the equations of the line 1′; $y=0$, $z=0$ for those of the line 2; $z=0$, $w=0$ for those of the line 3′; and $w=0$, $x=0$ for those of the line 4; the several lines may be determined, each of them by means of its six coordinates, as follows:

	a	b	c	f	g	h
1′	0	0	0	0	0	1
2	0	0	0	1	0	0
3′	0	0	1	0	0	0
4	1	0	0	0	0	0
2′	A_2	B_2	C_2	0	G_2	H_2
4′	0	B_4	C_4	F_4	G_4	H_4
5′	0	B_5	C_5	0	G_5	H_5
6′	0	B_6	C_6	0	G_6	H_6
1	a_1	b_1	c_1	f_1	g_1	0
3	a_3	b_3	0	f_3	g_3	h_3
5	a_5	b_5	0	f_5	g_5	0
6	a_6	b_6	0	f_6	g_6	0

where

$$B_2G_2 + C_2H_2 = 0,$$
$$B_4G_4 + C_4H_4 = 0,$$
$$B_5G_5 + C_5H_5 = 0,$$
$$B_6G_6 + C_6H_6 = 0,$$

$$a_1f_1 + b_1g_1 = 0,$$
$$a_3f_3 + b_3g_3 = 0,$$
$$a_5f_5 + b_5g_5 = 0,$$
$$a_6f_6 + b_6g_6 = 0.$$

The conditions in regard to the intersections of the lines 2′, 4′, 5′, 6′ and 1, 3, 5, 6, are formed by means of the diagram

$$\begin{array}{c|cccccc||cccccc|c}
1 & f_1 & g_1 & 0 & a_1 & b_1 & c_1 & A_2 & B_2 & C_2 & 0 & G_2 & H_2 & 2', \\
3 & f_3 & g_3 & h_3 & a_3 & b_3 & 0 & 0 & B_4 & C_4 & F_1 & G_4 & H_4 & 4', \\
5 & f_5 & g_5 & 0 & a_5 & b_5 & 0 & 0 & B_5 & C_5 & 0 & G_5 & H_5 & 5', \\
6 & f_6 & g_6 & 0 & a_6 & b_6 & 0 & 0 & B_6 & C_6 & 0 & G_6 & H_6 & 6',
\end{array}$$

viz. we have the equations

first set,

$$\begin{aligned}
f_1A_2+g_1B_2 \qquad\qquad &+b_1G_2+c_1H_2=0,\\
g_1B_4+a_1F_4+b_1G_4 &+c_1H_4=0,\\
g_1B_5 \qquad\qquad &+b_1G_5+c_1H_5=0,\\
g_1B_6 \qquad\qquad &+b_1G_6+c_1H_6=0;
\end{aligned}$$

second set,

$$\begin{aligned}
f_3A_2+g_3B_2+h_3C_2 \qquad &+b_3G_2=0,\\
g_3B_4+h_3C_4+a_3F_3 &+b_3G_4=0,\\
g_3B_5+h_3C_5 \qquad &+b_3G_5=0,\\
g_3B_6+h_3C_6 \qquad &+b_3G_6=0;
\end{aligned}$$

third set,

$$\begin{aligned}
f_5A_2+g_5B_2 \qquad\qquad &+b_5G_2=0,\\
g_5B_4 \qquad &+a_5F_4+b_5G_4=0,\\
g_5B_6 \qquad\qquad &+b_5G_6=0;
\end{aligned}$$

fourth set,

$$\begin{aligned}
f_6A_2+g_6B_2 \qquad\qquad &+b_6G_2=0,\\
g_6B_4 \qquad &+a_6F_4+b_6G_4=0,\\
g_6B_5 \qquad\qquad &+b_6G_5=0;
\end{aligned}$$

and it is to be shown, that taking as given the coordinates of 2′, 5′, 6′, that is $(A_2, B_2, C_2, G_2, H_2)$, (B_5, C_5, G_5, H_5) and (B_6, C_6, G_6, H_6), we can find the coordinates of the remaining lines 4′, 1, 3, 5, 6.

The first set of equations gives

$$g_1,\ b_1,\ c_1=\left\|\begin{matrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{matrix}\right\|,$$

viz. g_1, b_1, c_1 are proportional, but as only the ratios are material, they may be taken equal, to the determinants $G_5H_6-G_6H_5$, $H_5B_6-H_6B_5$, $B_5G_6-B_6G_5$. And then retaining g_1, b_1, c_1 to signify these values respectively, the first equation gives f_1A_2, and the second equation gives a_1F_4; multiplying together these values, and writing $a_1f_1=-b_1g_1$, we find

$$(B_2B_4,\ G_2G_4,\ H_2H_4,\ G_2H_4+G_4H_2,\ H_2B_4+H_4B_2,\ B_2G_4+B_4G_2+A_2F_4\between g_1,\ b_1,\ c_1)^2=0.$$

Proceeding in a similar manner with the second set of equations, we have first

$$g_3,\ h_3,\ b_3 = \left\| \begin{matrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{matrix} \right\|,$$

$$(\text{observe } h_3 = G_5 B_6 - G_6 B_5,\ = -c_1),$$

and then

$$(B_2B_4,\ C_2C_4,\ G_2G_4,\ C_2G_4 + C_4G_2,\ G_2B_4 + G_4B_2 + A_2F_4,\ B_2C_4 + B_4C_2 \between g_3,\ h_3,\ b_3)^2 = 0.$$

The third set gives more simply

$$g_5^2 B_2B_4 + g_5 b_5 (B_2G_4 + B_4G_2 + A_2F_4) + b_5^2 G_2G_4 = 0,$$

or since

$$g_5 : b_5 = G_6 : -B_6,$$

this is

$$G_6^2 B_2B_4 - G_6B_6 (B_2G_4 + B_4G_2 + A_2F_4) + B_6^2 G_2G_4 = 0,$$

and similarly, the fourth set gives

$$g_6^2 B_2B_4 + g_6 b_6 (B_2G_4 + B_4G_2 + A_2F_4) + b_6^2 G_2G_4 = 0,$$

or since $g_6 : b_6 = G_5 : -B_5$, this is

$$G_5^2 B_2B_4 - G_5B_5 (B_2G_4 + B_4G_2 + A_2F_4) + B_5^2 G_2G_4 = 0:$$

and these last two results lead to the values of the ratios of B_2B_4, $B_2G_4 + B_4G_2 + A_2F_4$, G_2G_4: viz. these are proportional to expressions containing the common factor $B_5G_6 - B_6G_5$, and omitting this common factor, and taking them equal instead of merely proportional to the resulting expressions (which is allowable, since the absolute values are not material), we have

$$B_2B_4,\ B_2G_4 + B_4G_2 + A_2F_4,\ G_2G_4 = B_5B_6,\ B_5G_6 + B_6G_5,\ G_5G_6.$$

Returning to the result obtained from the first set of equations, this now becomes

$$(B_5B_6,\ G_5G_6,\ H_2H_4,\ G_2H_4 + G_4H_2,\ H_2B_4 + H_4B_2,\ B_5G_6 + B_6G_5 \between g_1,\ b_1,\ c_1)^2 = 0:$$

but the terms containing g_1, b_1 are $(B_5g_1 + G_5b_1)(B_6g_1 + G_6b_1)$, viz. this is $= -H_5c_1 . - H_6c_1$, that is $H_5H_6c_1^2$; the whole equation is thus divisible by c_1, and omitting this factor, it becomes

$$g_1 (H_2B_4 + H_4B_2) + b_1 (G_2H_4 + G_4H_2) + c_1 (H_2H_4 + H_5H_6) = 0.$$

Proceeding in like manner with the result obtained from the second set of equations, this becomes

$$(B_5B_6,\ C_2C_4,\ G_5G_6,\ C_2G_4 + C_4G_2,\ B_5G_6 + B_6G_5,\ B_2G_4 + B_4C_2 \between g_3,\ h_3,\ b_3)^2 = 0,$$

where the terms containing g_3, b_3 are $(Bg_5 + G_5b_3)(Bg_3 + G_3b)$, viz. this is $-h_3C_5 . - h_3C_6 = h_3^2C_5C_6$; the whole equation divides by h_3, and it then becomes

$$g_3 (B_2C_4 + B_4C_2) + h_3 (C_2C_4 + C_5C_6) + b_3 (C_2G_4 + C_4G_2) = 0.$$

Considering B_4, G_4, F_4 as given by the equations

$$B_2B_4 = B_5B_6,\ G_2G_4 = G_5G_6,\ B_2G_4 + G_2B_4 + A_2F_4 = B_5G_6 + B_6G_5,$$

the equations last obtained determine the values of H_4 and C_4, viz. these equations may be written

$$(g_1B_2+b_1G_2+c_1H_2)\,H_4+H_2\,(g_1B_4+b_1G_4)+c_1H_5H_6=0,$$
$$(g_3B_2+h_3C_2+b_3G_2)\,C_4+C_2\,(g_3B_4+b_3G_4)+h_3C_5\,C_6=0\,;$$

but in order that the values $(B_4,\ C_4,\ F_4,\ G_4,\ H_4)$ given by these five equations may belong to a line $(0,\ B_4,\ C_4,\ F_4,\ G_4,\ H_4)$, they must satisfy the equation

$$B_4G_4+C_4H_4=0,$$

viz. in order to the existence of the line 4, this equation must be satisfied identically by the foregoing values; and I proceed to show that it is in fact thus satisfied. Multiplying the values of C_4, H_4, and writing $C_4H_4=-B_4G_4$, the identity to be verified is

$$(g_1B_2+b_1G_2+c_1H_2)(g_3B_2+h_3C_2+b_3G_2)\,B_4G_4$$
$$+[H_2\,(g_1B_4+b_1G_4)+c_1H_5H_6]\,[C_2\,(g_3B_4+b_3G_4)+h_3C_5C_6]=0.$$

The first line includes the terms

$$\{g_1g_3B_2^2+b_1b_3G_2^2+(b_1g_3+b_3g_1)\,B_2G_4+c_1h_3C_2H_2\}\,B_4G_4,$$

which, writing $C_2H_2=-B_2G_2$ and $B_2B_4=B_5B_6$, $G_2G_4=G_5G_6$, are

$$=g_1g_3B_2G_4B_5B_6+b_1b_3G_2B_4G_5G_6+(b_1g_3+g_1b_3-c_1h_3)\,B_5B_6G_5G_6.$$

The second line includes the terms

$$C_2H_2\,(g_1B_4+b_1G_4)\,(g_3B_4+b_3G_4)+c_1h_3C_5H_5C_6H_6,$$

which, reducing in like manner, are

$$=-g_1g_3G_2B_4B_5B_6-b_1b_3B_2G_4G_5G_6-(b_1g_3+g_1b_3-c_1h_3)\,B_5B_6G_5G_6,$$

and these are together

$$=\ (g_1g_3B_5B_6-b_1b_3G_5G_6)\,(B_2G_4-B_4G_2).$$

The remaining terms from the first line are at once reduced to

$$(g_1h_3C_2G_4+c_1g_3H_2G_4)\,B_5B_6+(b_1h_3C_2B_4+c_1b_3H_2B_4)\,G_5G_6,$$

and those from the second line are

$$C_5C_6\,H_2\,(g_1h_3B_4+b_1h_3G_4)+H_5H_6C_5\,(c_1g_3B_4+c_1b_3G_4).$$

Hence, attending to the relation $c_1=-h_3$, and collecting and arranging, the equation to be verified is

$$\begin{aligned}&(g_1g_3B_5B_6-b_1b_3G_5G_6)\,(B_2G_4-B_4G_2)\\&\quad+h_3C_2\,G_4\,(g_1B_5B_6-b_3H_5H_6)\\&\quad+h_3H_2B_4\,(g_1C_5C_6-b_3G_5\,G_6)\\&\quad+h_3C_2\,B_4\,(b_1G_5G_6-g_3H_5H_6)\\&\quad+h_3H_2G_4\,(b_1C_5C_6-g_3B_5\,B_6)=0.\end{aligned}$$

But we have

$$\begin{aligned} g_1 B_5 B_6 - b_3 H_5 H_6 &= B_5 B_6 (G_5 H_6 - G_6 H_5) - H_5 H_6 (B_5 C_6 - B_6 C_5) \\ &= B_6 H_6 (B_5 G_5 + C_5 H_5) - B_5 H_5 (B_6 G_6 + C_6 H_6) = 0, \end{aligned}$$

and similarly

$$\begin{aligned} g_1 C_5 C_6 - b_3 G_5 G_6 &= 0, \\ b_1 G_5 G_6 - g_3 H_5 H_6 &= 0, \\ b_1 C_5 C_6 - g_3 B_5 B_6 &= 0. \end{aligned}$$

Moreover, writing $g_1 B_5 B_6 = b_3 H_5 H_6$, we have

$$\begin{aligned} & g_1 g_3 B_5 B_6 - b_1 b_3 G_5 G_6 \\ &= b_3 (g_3 H_5 H_6 - b_1 G_5 G_4) = 0, \end{aligned}$$

and the five terms of the equation in question thus separately vanish; and the equation is consequently verified.

We may collect the results as follows:

Data are lines 1′, 2, 3′, 4, 2′, 5′, 6′;

and then, for the remaining lines, 1, 3, 5, 6, 4′ the coordinates are as follows:

For 4′,

$$B_2 B_4 = B_5 B_6, \quad G_2 G_4 = G_5 G_6, \quad A_2 F_4 = B_5 G_6 + B_6 G_5 - B_2 G_4 - B_4 G_2, \quad A_4 = 0,$$

$$\begin{vmatrix} B_2, & G_2, & H_2 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix} H_4 + \begin{vmatrix} H_2 B_4, & H_2 G_4, & H_5 H_6 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix} = 0,$$

$$\begin{vmatrix} B_2, & C_2, & G_2 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix} C_4 + \begin{vmatrix} C_2 B_4, & C_5 C_6, & C_2 G_4 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix} = 0,$$

$$(B_4 G_4 + C_4 H_4 = 0, \text{ identity}).$$

For 1,

$$(g_1, b_1, c_1) = \left\| \begin{matrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{matrix} \right\|, \quad A_2 f_2 = -\begin{vmatrix} B_2, & G_2, & H_2 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix}, \quad F_4 a_1 = -\begin{vmatrix} B_4, & G_4, & H_4 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix}, \quad h_1 = 0,$$

$$(a_1 f_1 + b_1 g_1 = 0, \text{ identity}).$$

For 3,

$$(g_3, h_3, b_3) = \left\| \begin{matrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{matrix} \right\|, \quad A_2 f_3 = -\begin{vmatrix} B_2, & C_2, & G_2 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix}, \quad F_4 a_3 = -\begin{vmatrix} B_4, & C_4, & G_4 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix}, \quad c_3 = 0,$$

$$(a_3 f_3 + b_3 g_3 = 0, \text{ identity}).$$

For 5,

$$g_5 = G_6, \quad b_5 = -B_6, \quad A_2 f_5 = B_6 G_2 - B_2 G_6, \quad F_4 a_5 = B_6 G_4 - B_4 G_6, \quad c_5 = 0, \quad h_5 = 0,$$

$$(a_5 f_5 + b_5 g_5 = 0, \text{ identity}).$$

For 6,

$$g_6 = G_5, \quad b_6 = -B_5, \quad A_2 f_6 = B_5 G_2 - B_2 G_5, \quad F_4 a_6 = B_5 G_4 - B_4 G_5, \quad c_6 = 0, \quad h_6 = 0;$$

and, for actual calculation, it is convenient to remark that as only the ratios are material, a set of six coordinates may be multiplied or divided by any common number at pleasure.

But these results may be further reduced. Writing

$$(g_3, h_3, b_3) = \left\| \begin{matrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{matrix} \right\|,$$

we have

$$-(g_3 B_2 + h_3 C_2 + b_3 G_2) C_4 = C_2 \left(g_3 \frac{B_5 B_6}{B_2} + b_3 \frac{G_5 G_6}{C_2} \right) + h_3 C_5 C_6 = \frac{C_2}{B_2 G_2} (g_3 B_5 B_6 G_2 + b_3 G_5 G_6 B_2) + h_3 C_5 C_6.$$

But

$$g_3 B_5 B_6 G_2 + b_3 G_5 G_6 B_2 = \left\{ \begin{matrix} B_5 B_6 G_2 (C_5 G_6 - C_6 G_5) \\ + G_5 G_6 B_2 (B_5 C_6 - C_5 B_6) \end{matrix} \right\} = \left\{ \begin{matrix} B_6 G_6 C_5 (B_5 G_2 - B_2 G_5) \\ + B_5 G_5 C_6 (G_6 B_2 - G_2 B_6) \end{matrix} \right\}$$

$$= C_5 C_6 \left\{ \begin{matrix} - H_6 (B_5 G_2 - B_2 G_5) \\ - H_5 (G_6 B_2 - G_2 B_6) \end{matrix} \right\}$$

$$= C_5 C_6 \{ B_2 (G_5 H_6 - G_6 H_5) + G_2 (B_6 H_5 - B_5 H_6) \}$$

$$= C_5 C_6 (B_2 g_1 + G_2 b_1),$$

since

$$(g_1, b_1, c_1) = \left\| \begin{matrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{matrix} \right\|;$$

the equation obtained is thus

$$g_3 B_4 + b_3 G_4 = \frac{C_5 C_6}{B_2 G_2} (g_1 B_2 + b_1 G_2).$$

We then have

$$-(g_3 B_2 + h_3 C_2 + b_3 G_2) C_4 = \frac{C_5 C_6}{B_2 G_2} [C_2 (g_1 B_2 + b_1 G_2) + h_3 B_2 G_2]$$

$$= \frac{-C_5 C_6}{C_2 H_2} [C_2 (g_1 B_2 + b_1 G_2) + c_1 C_2 H_2]$$

$$= -\frac{C_5 C_6}{H_2} (g_1 B_2 + b_1 G_2 + c_1 H_2),$$

that is

$$C_4 = \frac{C_5 C_6}{H_2} \frac{g_1 B_2 + b_1 G_2 + c_1 H_2}{g_3 B_2 + h_3 C_2 + b_3 G_2};$$

and in like manner

$$-(g_1B_2+b_1G_2+c_1H_2)\,H_4=\frac{H_2}{B_2G_2}(g_1G_2B_5B_6+b_1B_2G_5G_6)+c_1H_5H_6,$$

$$g_1G_2B_5B_6+b_1B_2G_5G_6=\left\{\begin{matrix}G_2B_5B_6(G_5H_6-G_5H_5)\\+B_2G_5G_6(H_5B_6-H_6G_5)\end{matrix}\right\}=\left\{\begin{matrix}B_5G_5H_6(G_2B_6-G_6B_2)\\+B_6G_6H_5(B_2G_5-B_5G_2)\end{matrix}\right\}$$

$$=H_5H_6\left\{\begin{matrix}-C_5(G_2B_6-G_6B_2)\\-C_6(B_2G_5-B_5G_2)\end{matrix}\right\}$$

$$=H_5H_6\{B_2(C_5C_6=C_6G_5)+G_2(B_5C_6-B_6C_5)\}$$

$$=H_5H_3(B_2g_3+G_2b_3);$$

the equation obtained is thus

$$g_1B_4+b_1G_4=\frac{H_5H_6}{B_2G_2}(B_2g_3+G_2b_3);$$

and then

$$-(g_1B_2+b_1G_2+c_1H_2)\,H_4=\frac{H_5H_6}{B_2G_2}\{H_2(B_2g_3+G_2b_3)+c_1B_2G_2\}$$

$$=-\frac{H_5H_6}{C_2H_2}\{H_2(B_2g_3+G_2b_3)+h_3C_2H_2\}$$

$$=-\frac{H_5H_6}{C_2}(B_2g_3+C_2h_3+G_2b_3),$$

that is

$$H_4=\frac{H_5H_6}{C_2}\,\frac{g_3B_2+h_3C_2+b_3G_2}{g_1B_2+b_1G_2+c_1H_2},$$

which values of C_4, H_4 satisfy, as they should do, the relation

$$C_4H_4=-B_4G_4.$$

We have also

$$A_2F_4=B_5G_6+B_6G_5-\frac{B_2G_5G_6}{G_2}-\frac{G_2B_5C_6}{B_2}$$

$$=-\frac{1}{B_2G_2}(G_2B_5-B_2G_5)(G_2B_6-B_2G_6)$$

$$=\frac{1}{C_2H_2}(G_2B_5-B_2G_5)(G_2B_6-B_2G_6),$$

which gives F_4.

Moreover

$$-F_4a_1=g_1B_4+b_1G_4+c_1H_4=-\frac{H_5H_6}{C_2H_2}(g_3B_2+b_2G_3)-\frac{h_3H_5H_6}{C_2}\,\frac{g_3B_2+b_3G_2+h_3C_2}{g_1B_2+b_1G_2+c_1H_2}$$

$$=\frac{-H_5H_6}{C_2H_2(g_1B_2+b_1G_2+c_1H_2)}\left\{\begin{matrix}(g_3B_2+b_3G_2)(g_1B_2+b_1G_2+c_1H_2)\\+(g_3B_2+b_3G_2+h_3C_2)\,h_3H_2\end{matrix}\right\}$$

$$=\frac{-H_5H_6}{C_2H_2(g_1B_2+b_1G_2+c_1H_2)}\{(g_3B_2+b_3G_2)(g_1B_2+b_1G_2)+c_1h_3B_2G_2\},$$

which combined with the foregoing value of F_4, gives a_1.

Again,

$$-F_4 a_3 = g_3 B_4 + h_3 C_4 + b_3 G_4 = -\frac{C_5 C_6}{C_2 H_2}(g_1 B_2 + b_1 G_2) = \frac{c_1 C_5 C_6}{H_2}\,\frac{g_1 B_2 + b_1 G_2 + c_1 H_2}{g_3 B_2 + h_3 C_2 + b_3 G_2}$$

$$= -\frac{C_5 C_6}{C_2 H_2 (g_3 B_2 + h_3 C_2 + b_3 G_2)}\left\{\begin{array}{l}(g_1 B_2 + b_1 G_2)(g_3 B_2 + h_3 C_2 + b_3 G_2)\\ +(g_1 B_2 + b_1 G_2 + c_1 H_2)\, c_1 C_2\end{array}\right\}$$

$$= \frac{-C_5 C_6}{C_2 H_2 (g_3 B_2 + h_3 C_2 + b_3 G_2)}\{(g_1 B_2 + b_1 G_2)(g_1 B_2 + b_3 G_2) + c_1 h_3 B_2 C_2\},$$

which combined with the foregoing value of F_4, gives a_3.

Write

$$\omega = B_5 G_6 + B_6 G_5 + C_5 H_6 + C_3 H_5,$$

we have

$$\begin{aligned} b_1 g_1 &= (H_5 B_6 - H_6 B_5)(G_5 H_6 - G_6 H_5)\\ &= -H_5^2 B_6 G_6 - H_6^2 B_5 G_5 + H_5 H_6 (B_5 G_6 + B_6 G_5)\\ &= H_5 H_6 (C_6 H_5 + C_5 H_6 + B_5 G_6 + B_6 G_5), \end{aligned}$$

that is

$$b_1 g_1 = H_5 H_3 \omega,$$

and similarly

$$b_3 g_3 = C_5\, C_6\, \omega,$$

$$b_1 b_3 = B_5\, B_6\, \omega,$$

$$g_1 g_3 = G_5\, G_6\, \omega,$$

$$b_1 g_3 + b_3 g_1 + c_1 h_3 = -(B_5 G_6 + B_3 G_5)\,\omega,$$

$$\begin{aligned}(b_1 G_2 + g_1 B_2)(b_3 G_2 + g_3 B_2) + c_1 h_2 B_2 G_2 &= \{G_2^2 B_5 B_6 + B_2^2 G_5 G_6 - B_2 G_2 (B_5 G_6 + B_6 G_5)\}\,\omega\\ &= (G_2 B_5 - B_2 G_5)(G_2 B_6 - B_2 G_6)\,\omega,\end{aligned}$$

which last value is to be substituted for the left-hand function in the formulæ for a_1 and a_3 respectively.

Whence, finally recollecting that

$$B_2 G_2 + C_2 H_2 = 0, \quad B_5 G_5 + C_5 H_5 = 0, \quad B_6 G_6 + C_6 H_6 = 0,$$

and

$$\omega = C_6 H_5 + C_5 H_6 + B_6 G_5 + B_5 G_6,$$

we have

$$\text{For 1}\left\{\begin{array}{ll}(g_1,\ b_1,\ c_1) = \left\|\begin{array}{ccc} B_5, & G_5, & H_5\\ B_6, & G_6, & H_6\end{array}\right\|, & f_1 = -\dfrac{1}{A_2}(g_1 B_2 + b_1 G_2 + c_1 H_2),\\ b_1 g_1 = H_5 H_6 \omega. & a_1 = \dfrac{A_2 H_5 H_6 \omega}{g_1 B_2 + h_1 G_2 + c_1 H_2}, \quad h_1 = 0.\end{array}\right.$$

$$\text{For 3}\begin{cases}(g_3,\ h_3,\ b_3)=\left\|\begin{matrix}B_5, & C_5, & G_6\\ B_6, & C_6, & G_6\end{matrix}\right\|, & f_3=-\dfrac{1}{A_2}(g_3B_2+h_3C_2+b_3G_2),\\ b_3g_3=C_5C_6\omega. & a_3=\dfrac{A_2C_5C_6\omega}{g_3B_2+h_3C_2+b_3G_2},\quad h_3=0,\end{cases}$$

where observe that $c_1+h_3=0$.

$$\text{For 5}\begin{cases}g_5=G_6,\quad b_5=-B_6, & f_5=\dfrac{1}{A_2}(G_2B_6-B_2G_6),\\ c_5=0\ ,\quad h_5=\ \ 0, & a_5=\dfrac{A_2B_6G_6}{G_2B_6-B_2G_6}.\end{cases}$$

$$\text{For 6}\begin{cases}g_6=G_5,\quad b_6=-B_5, & f_6=\dfrac{1}{A_2}(G_2B_5-B_2G_5),\\ c_6=0\ ,\quad h_6=\ \ 0, & a_6=\dfrac{A_2B_5G_5}{G_2B_5-B_2G_5}.\end{cases}$$

$$\text{For 4}'\begin{cases}A_4=0, & B_4=\dfrac{B_5B_6}{B_2},\\ F_4=\dfrac{(G_2B_5-G_5B_2)(G_2B_6-G_6B_2)}{A_2C_2H_2}, & G_4=\dfrac{G_5G_6}{G_2},\\ C_4=\dfrac{C_5C_6}{H_2}\ \dfrac{g_1B_2+b_1G_2+c_1H_2}{g_3B_2+h_3C_2+b_3G_2}, & \\ H_4=\dfrac{H_5H_6}{C_2}\ \dfrac{g_3B_2+h_3C_2+b_3G_2}{g_1B_2+b_1G_2+c_1H_2}. & \end{cases}$$

I have thought it worth while to effect the numerical calculations for enabling the construction of a drawing or model. For this purpose taking X, Y, Z as ordinary rectangular coordinates, I write

$$\begin{aligned}x&=\ \ X+Y+Z-10,\\ y&=\qquad\quad Z,\\ z&=-X+Y+Z-10,\\ w&=\qquad\ \ Y,\end{aligned}$$

that is, I take 1′ and 2 to be lines in the plane of XY, defined by the equations $X+Y=10$ and $X-Y=-10$ respectively, and 3′ and 4 to be lines in the plane of XZ defined by the equations $X-Z=-10$, $X+Z=10$ respectively. And I take 5′ to be the line joining the points (2, 0, 8) and (−9, 1, 0); 6′ the line joining the points (3, 0, 7) and (−8, 2, 0); 2′ the line joining the points (9, 0, 1) and (−3, 7, 4). We

calculate then for each of the lines the $xyzw$ coordinates of two points thereof; and thence the six coordinates of the line, viz.:

	x	y	z	w	x	y	z	w	A	B	C	F	G	H	reduced A	B	C	F	G	H
5′	0,	8,	−4,	0	−18,	0,	0,	1	0,	72,	144,	0,	8,	−4	0,	18,	36,	0,	2,	−1
6′	0,	7,	−6,	0	−16,	0,	0,	2	0,	96,	112,	0,	14,	−12	0,	48,	56,	0,	7,	−6
2′	0,	1,	−18,	0	−2,	4,	4,	7	76,	36,	2,	0,	7,	−126	76,	36,	2,	0,	7,	−126

and effecting the calculations for the remaining lines, we have

	A	B	C	F	G	H
4′	0	24	−944	$-\frac{9}{38}$	2	$\frac{3}{59}$
1	$\frac{380}{59}$	60	30	$\frac{385}{19}$	−5	0
3	127680	−720	0	$\frac{15}{19}$	140	−30
5	304	−48	0	$\frac{21}{19}$	7	0
6	$\frac{152}{3}$	−18	0	$\frac{27}{38}$	0	0
	a	b	c	f	g	h

or reducing to integers, the values are

	A	B	C	F	G	H
5′	0	18	36	0	2	−1
6′	0	48	56	0	7	−6
2′	76	36	2	0	7	−126
4′	0	53808	−2116448	−531	4484	114
1	1444	13452	6726	10443	−1121	0
3	485184	−2736	0	3	532	−114
5	5776	−912	0	21	133	0
6	5776	−2052	0	81	228	0
	a	b	c	f	g	h

The line (a, b, c, f, g, h) is given as the intersection of any two of the four planes

$$\left(\begin{array}{cccc} & h, & -g, & a \\ -h, & & f, & b \\ g, & -f, & & c \\ -a, & -b, & -c, & \end{array}\right)(x, y, z, w) = 0,$$

or substituting for x, y, z, w the values $X+Y+Z-10,\ Z,\ -X+Y+Z-10,\ Y$, these become

$$\left(\begin{array}{cccc} g\ , & a-g\ , & h-g\ , & g \\ -f-h, & b+f-h, & f-h\ , & h-g \\ g\ , & c+g\ , & g-f\ , & -g \\ c-a, & -c-a\ , & -c-a-b, & c+a \end{array}\right)(X, Y, Z, 10) = 0,$$

or, what is the same thing,

$$\left(\begin{array}{cccc} .\ , & 2g-a+c\ , & 2g-f-h\ , & -2g \\ -2g+a-c, & .\ , & a+b+c+f-h, & -a-c \\ -2g+f+h, & -a-b-c-f+h, & .\ , & -f+h \\ 2g\ , & a+c\ , & -f+h\ , & . \end{array}\right)(X, Y, Z, 10) = 0.$$

And substituting, we have the equations of the several lines, viz.:

(1′) $X+Y=10,\quad Z=0,$

(2) $-X+Y=10,\quad Z=0,$

(3′) $-X+Z=10,\quad Y=0,$

(4) $X+Z=10,\quad Y=0,$

(5′) $$\left(\begin{array}{cccc} .\ , & 40, & 5, & -4 \\ -40, & .\ , & 55, & -36 \\ -5, & -55, & .\ , & 1 \\ 4, & 36, & -1, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(6′) $$\left(\begin{array}{cccc} .\ , & -35, & 10, & -7 \\ -35, & .\ , & 55, & -28 \\ -10, & -55, & .\ , & 3 \\ 7, & 28, & -3, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

$$(2')\quad \left(\begin{array}{cccc} . , & -30, & 70, & -7 \\ 30, & . , & 120, & -39 \\ -70, & -120, & . , & 63 \\ 7, & 39, & -63, & . \end{array}\right) \between X, Y, Z, 10) = 0,$$

$$(4')\quad \left(\begin{array}{cccc} . , & -2107480, & 9385, & -8968 \\ 2107480, & . , & -2063285, & 2116448 \\ -9385, & 2063285, & . , & -645 \\ 8968, & -2116448, & 645, & . \end{array}\right) \between X, Y, Z, 10) = 0,$$

$$(1)\quad \left(\begin{array}{cccc} . , & 3040, & -12685, & 2242 \\ -3040, & . , & 32065, & -8170 \\ 12685, & -32065, & . , & 10443 \\ -2242, & 8170, & -10443, & . \end{array}\right) \between X, Y, Z, 10) = 0,$$

$$(3)\quad \left(\begin{array}{cccc} . , & -484120, & 1175, & -1064 \\ 484120, & . , & 482565, & -485184 \\ -1175, & -482565, & . , & 117 \\ 1064, & 485184, & -117, & . \end{array}\right) \between X, Y, Z, 10) = 0,$$

$$(5)\quad \left(\begin{array}{cccc} . , & 5510, & 245, & -266 \\ -5510, & . , & 4885, & -5776 \\ -245, & -4885, & . , & 21 \\ 266, & 5776, & -21, & . \end{array}\right) \between X, Y, Z, 10) = 0,$$

$$(6)\quad \left(\begin{array}{cccc} . , & -5320, & 375, & -456 \\ 5320, & . , & 3805, & -5776 \\ -3751, & -3805, & . , & 81 \\ 456, & 5776, & -81, & . \end{array}\right) \between X, Y, Z, 10) = 0.$$

The several lines intersect as they should do, the coordinates of the points of intersection being as follows:

	1′	2′	3′	4′	5′	6′
1	*	$\left.\begin{matrix}621\\1239\\836\end{matrix}\right\}\div 305$	$\left.\begin{matrix}-354\\0\\76\end{matrix}\right\}\div 43$	$\left.\begin{matrix}42417\\-177\\8968\end{matrix}\right\}\div 5098$	$\left.\begin{matrix}-493\\59\\152\end{matrix}\right\}\div 78$	$\left.\begin{matrix}-1188\\354\\532\end{matrix}\right\}\div 253$
2	$\begin{matrix}0\\10\\0\end{matrix}$	*	$\begin{matrix}-10\\0\\0\end{matrix}$	$\left.\begin{matrix}-472\\-2\\0\end{matrix}\right\}\div 47$	$\begin{matrix}-9\\1\\0\end{matrix}$	$\begin{matrix}-8\\2\\0\end{matrix}$
3	$\left.\begin{matrix}912\\-2\\0\end{matrix}\right\}\div 91$	$\left.\begin{matrix}9693\\-21\\1064\end{matrix}\right\}\div 1073$	*	$\left.\begin{matrix}-2484\\66\\251464\end{matrix}\right\}\div 24727$	$\left.\begin{matrix}807\\-1\\3192\end{matrix}\right\}\div 398$	$\left.\begin{matrix}3672\\-6\\8512\end{matrix}\right\}\div 1213$
4	$\begin{matrix}10\\0\\0\end{matrix}$	$\begin{matrix}9\\0\\1\end{matrix}$	$\begin{matrix}0\\0\\10\end{matrix}$	*	$\begin{matrix}2\\0\\8\end{matrix}$	$\begin{matrix}3\\0\\7\end{matrix}$
5	$\left.\begin{matrix}304\\-14\\0\end{matrix}\right\}\div 29$	$\left.\begin{matrix}459\\-21\\38\end{matrix}\right\}\div 47$	$\left.\begin{matrix}6\\0\\76\end{matrix}\right\}\div 7$	$\left.\begin{matrix}4752\\42\\71744\end{matrix}\right\}\div 6521$	*	$\left.\begin{matrix}1080\\-42\\2128\end{matrix}\right\}\div 283$
6	$\left.\begin{matrix}76\\-6\\0\end{matrix}\right\}\div 7$	$\left.\begin{matrix}2421\\-189\\152\end{matrix}\right\}\div 233$	$\left.\begin{matrix}54\\0\\304\end{matrix}\right\}\div 25$	$\left.\begin{matrix}1479\\9\\8968\end{matrix}\right\}\div 727$	$\left.\begin{matrix}65\\-3\\154\end{matrix}\right\}\div 16$	*

viz. the coordinates of 12′ (intersection of lines 1 and 2′) are $(\frac{621}{305}, \frac{1239}{305}, \frac{836}{305})$, and so in other cases; where there is no divisor the coordinates are integers. I find however, on laying down the figure, that the lines 3 and 4, 3′ and 4′ come so close together, that the figure cannot be obtained with any accuracy.

460.

NOTE ON MR FROST'S PAPER ON THE DIRECTION OF LINES OF CURVATURE IN THE NEIGHBOURHOOD OF AN UMBILICUS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 111—113.]

I REMARK as follows:

1. In regard to a quadric surface, it is not, I think, correct to say that the generatrices through an umbilic are curves of curvature; notwithstanding that, as shown p. 80, the normals at every point of such generatrix lie in one plane and consequently intersect. The way in which these generatrices as *quasi*-curves-of-curvature present themselves is as follows:

The curves of curvature satisfy a certain differential equation, the complete integral of which gives these curves as the intersections of the given quadric surface by the series of confocal surfaces $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1$, h being the constant of integration of the differential equation. The singular solution of the differential equation, or envelope of the curves of curvature determined as above, gives the umbilicar generatrices.

2. In regard to a surface in general, I think it must be considered, not that there pass through the umbilic three distinct curves, but that the umbilicar curve of curvature is a curve having at the umbilic a triple point, or rather a point at which there are in general three distinct directions of the curve. The umbilicar curve of curvature in fact presents itself as the curve belonging to a certain value of the constant of integration h; in order that the curve of curvature may pass through a given point on the surface, h must satisfy a certain quadratic equation, that is for a given point of the surface there are two values of h, and therefore two curves of

curvature; but an umbilic is a point for which (as in effect shown, p. 81, for the particular case of a quadric surface) the two values of h become equal; that is, there is through the umbilic only a singular curve of curvature; but $\frac{dy}{dx}$ is determined by a cubic equation, and the umbilic is thus (as just mentioned) a point at which there are in general three distinct directions of the curve.

3. Some researches on the subject are contained in my paper "On Differential Equations and Umbilici," *Phil. Mag.*, vol. XXVI. (1863), pp. 373—379 and 441—452, [330]. It is noticeable that in the integral equations which I have there obtained for the differential equations $cy(p^2-1)+(a-c)xp=0$, and the more general form $(bx+cy)(p^2-1)+2(fx+gy)=0$, which belong to the neighbourhood of an umbilic, the curve through the umbilic *does* break up into three distinct curves; and the same is the case with the umbilic on the surface $xyz=1$ presently referred to.

4. In the paper "Mémoire sur les surfaces orthogonales," *Liouv.*, t. XII. (1847), pp. 241—254, M. Serret has given two very remarkable cases of three systems of surfaces intersecting each other at right angles, and consequently in the curves of curvature of the surfaces of each system. It was only on referring to this paper, in connexion with that of Mr Frost, that I perceived an obvious enough simplification of M. Serret's formulæ, whereby it appears that the curves of curvature on the surface $xyz=1$ are given as the intersection of this surface with the series of surfaces

$$h=(x^2+\omega y^2+\omega^2 z^2)^{\frac{3}{2}}+(x^2+\omega^2 y^2+\omega z^2)^{\frac{3}{2}},$$

where ω is an imaginary cube root of unity; the rationalised equation is of the twelfth order in (x, y, z), and for the particular value $h=0$, reduces itself as is easily seen to $0=(y^2-z^2)^2(z^2-x^2)^2(x^2-y^2)^2$. The point $x=y=z=1$ is obviously an umbilic on the surface $xyz=1$, and the corresponding value of h being $h=0$, the equation just obtained determines the umbilicar curves of curvature, viz. combining therewith the equation $xyz=1$ of the surface, we have the three hyperbolic curves

$$(y=z,\ xy^2=1),\quad (z=x,\ yz^2=1),\quad (x=y,\ zx^2=1).$$

461.

ON THE GEOMETRICAL INTERPRETATION OF THE COVARIANTS OF A BINARY CUBIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 148—149.]

CONSIDER the binary cubic $U=(a, b, c, d \between x, y)^3$, and its covariants, viz. the discriminant (invariant)

$$\nabla = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2,$$

the Hessian

$$H = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

and the cubicovariant

$$\begin{aligned}\Phi = \;& (\quad a^2d - 3abc + 2b^3\,)\, x^3 \\ & - (-3abd + 6ac^2 - 3b^2c)\, x^2y \\ & + (-3acd + 6bd^2 - 3bc^2)\, xy^2 \\ & - (\quad ad^2 - 3bcd + 2c^3\,)\, y^3,\end{aligned}$$

connected by the identical equation

$$\Phi^2 - \nabla U^2 = -4H^3.$$

Then if we regard (a, b, c, d) as the coordinates of a point in space, but (x, y) as variable parameters, the equation

$$\nabla = 0$$

represents a quartic torse, having for its cuspidal curve the skew cubic $ac - b^2 = 0$, $ad - bc = 0$, $bd - c^2 = 0$; the equation

$$U = 0$$

is that of the tangent plane to the torse along the line $ax^2+2bxy+cy^2=0$, $bx^2+2cxy+dy^2=0$: this line meets the cuspidal curve in the point whose coordinates are $a:b:c:d=y^3:-xy^2:x^2y:-y^3$. The equation

$$H=0$$

is that of a quadric cone having the last mentioned point for its vertex, and passing through the cuspidal curve: and the equation

$$\Phi=0$$

is that of the cubic surface which is the first polar of the same point in regard to the torse.

The equation $\Phi^2-\nabla U^2=-4H^3$, writing therein $U=0$, gives $\Phi^2=-4H^3$, a result which implies that $U=0$, $H=0$ is a certain curve repeated twice, and that $U=0$, $\Phi=0$ is the same curve repeated three times. The curve in question is at once seen to be the line of contact $\delta_x U=0$, $\delta_y U=0$; it thus appears that the tangent plane $U=0$ meets the cubic surface $\Phi=0$ in this line taken three times. This can only be the case if the equation $\Phi=0$ be expressible in the form $MU+(\delta_x U)^3=0$, or, what is the same thing,

$$MU+(\alpha\delta_x U+\beta\delta_y U)^3=0,$$

α and β constants, M a quadric function of (a, b, c, d); that is, Φ must be equal to a function of the form

$$MU+(\alpha\delta_x U+\beta\delta_y U)^2.$$

Seeking for this expression of Φ, and writing the symbols out at length, I find that the required identical equation is

$$-(\beta x-\alpha y)^3\left\{\begin{array}{l}(\quad a^2d-3abc+2b^3\ )\,x^3\\ -(-\ 3abd+6ac^2-3b^2c)\,x^2y\\ +(-\ 3acd+6bd^2-3bc^2)\,xy^2\\ -(-\quad ad^2-3bcd+2c^3\ )\,y^3\end{array}\right\}+2\,\{\alpha\,(ax^2+2bxy+cy^2)+\beta\,(bx^2+2cxy+dy^2)\}^3=$$

$$(a, b, c, d\between x, y)^3.\left(\begin{array}{cccc}2a^2, & 6ab, & 6b^2, & -ad+3bc\\ 6ab, & 12ac+6b^2, & 3ad+15bc, & 6c^2\\ 6b^2, & 3ad+15bc, & 12bd+\ 6c^2, & 6cd\\ -ad+3bc, & 6c^2, & 6cd, & 2d^2\end{array}\right)\dagger(x, y)^3(\alpha, \beta)^3,$$

(where the $\dagger$ indicates that the binomial coefficients are *not* to be inserted, viz. the function on the right hand is $\{2a^2x^3+6abx^2y+6b^2xy^3+(-ad+3bc)\,y^3\}\,\alpha^3+$ &c.). As a verification remark that for $x=\alpha$, $y=\beta$, the equation becomes simply $2U^3=U.2U^2$.

462.

A NINTH MEMOIR ON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXI. (for the year 1871), pp. 17—50. Received April 7,—Read May 19, 1870.]

It was shown not long ago by Professor Gordan that the number of the irreducible covariants of a binary quantic of any order is finite (see his memoir "Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist," *Crelle*, t. LXIX. (1869), Memoir dated 8 June 1868), and in particular that for a binary quintic the number of irreducible covariants (including the quintic and the invariants) is $=23$, and that for a binary sextic the number is $=26$. From the theory given in my "Second Memoir on Quantics," *Phil. Trans.*, 1856, [141], I derived the conclusion, which, as it now appears, was erroneous, that for a binary quintic the number of irreducible covariants was infinite. The theory requires, in fact, a modification, by reason that certain linear relations, which I had assumed to be independent, are really not independent, but, on the contrary, linearly connected together: the interconnexion in question does not occur in regard to the quadric, cubic, or quartic; and for these cases respectively the theory is true as it stands; for the quintic the interconnexion first presents itself in regard to the degree 8 in the coefficients and order 14 in the variables, viz. the theory gives correctly the number of covariants of any degree not exceeding 7, and also those of the degree 8 and order less than 14; but for the order 14 the theory as it stands gives a non-existent irreducible covariant $(a,..)^8(x, y)^{14}$, viz. we have, according to the theory, $5=(10-6)+1$, that is, of the form in question there are 10 composite covariants connected by 6 syzygies, and therefore equivalent to $10-6$, $=4$ asyzygetic covariants; but the number of asyzygetic covariants being $=5$, there is left, according to the theory, 1 irreducible covariant of the form in question. The fact is that the 6 syzygies being interconnected and equivalent to 5 independent syzygies only, the composite covariants are equivalent to $10-5$, $=5$, the full number of the asyzygetic covariants. And similarly the theory as it stands gives a non-existent

irreducible covariant $(a, ..)^8 (x, y)^{20}$. The theory being thus in error, by reason that it omits to take account of the interconnexion of the syzygies, there is no difficulty in conceiving that the effect is the introduction of an infinite series of non-existent irreducible covariants, which, when the error is corrected, will disappear, and there will be left only a finite series of irreducible covariants.

Although I am not able to make this correction in a general manner so as to show from the theory that the number of the irreducible covariants is finite, and so to present the theory in a complete form, it nevertheless appears that the theory can be made to accord with the facts; and I reproduce the theory, as well to show that this is so as to exhibit certain new formulæ which appear to me to place the theory in its true light. I remark that although I have in my Second Memoir considered the question of finding the number of irreducible covariants of a given degree θ in the coefficients but of any order whatever in the variables, the better course is to separate these according to their order in the variables, and so consider the question of finding the number of the irreducible covariants of a given degree θ in the coefficients, and of a given order μ in the variables. (This is, of course, what has to be done for the enumeration of the irreducible covariants of a given quantic; and what is done completely for the quadric, the cubic, and the quartic, and for the quintic up to the degree 6 in my Eighth Memoir, *Phil. Trans.* 1867, [405].) The new formulæ exhibit this separation; thus (Second Memoir, No. 49), writing a instead of x, we have for the quadric the expression $\frac{1}{(1-a)(1-a^2)}$, showing that we have irreducible covariants of the degrees 1 and 2 respectively, viz. the quadric itself and the discriminant: the new expression is $\frac{1}{(1-ax^2)(1-a^2)}$, showing that the covariants in question are of the actual forms $(a, ..\between x, y)^2$ and $(a, ..)^2$ respectively. Similarly for the cubic, instead of the expression No. 55, $\frac{1-a^8}{(1-a)(1-a^2)(1-a^3)(1-a^4)}$, we have $\frac{1-a^8x^6}{(1-ax^3)(1-a^2x^2)(1-a^3x^3)(1-a^4)}$, exhibiting the irreducible covariants of the forms $(a, ..\between x, y)^3$, $(a, ..)^2 (x, y)^2$, $(a\,..)^3 (x, y)^3$, and $(a, ..)^4$, connected by a syzygy of the form $(a, ..)^6 (x, y)^6$; and the like for quantics of a higher order.

In the present Ninth Memoir I give the last-mentioned formulæ; I carry on the theory of the quintic, extending the Table No. 82 of the Eighth Memoir up to the degree 8, calculating all the syzygies, and thus establishing the interconnexions in virtue of which it appears that there are really no irreducible covariants of the forms $(a, ..)^8 (x, y)^{14}$, and $(a, ..^8\between x, y)^{20}$. I reproduce in part Gordan's theory so far as it applies to the quintic, and I give the expressions of such of the 23 covariants as are not given in my former memoirs; these last were calculated for me by Mr W. Barrett Davis, by the aid of a grant from the Donation Fund at the disposal of the Royal Society. [The expressions referred to are in fact printed, 143.] The paragraphs of the present memoir are numbered consecutively with those of the former memoirs on Quantics.

Article Nos. 328 to 332. *Reproduction of my original Theory as to the Number of the Irreducible Covariants.*

328. I reproduce to some extent the considerations by which, in my Second Memoir on Quantics, I endeavoured to obtain the number of the irreducible covariants of a given binary quantic $(a, b, \ldots \between x, y)^n$.

Considering in the first instance the covariants as functions of the coefficients $(a, b, c, \ldots)$, without regarding the variables (x, y), and attending only to the following properties—1°, a covariant is a rational and integral homogeneous function of the coefficients; 2°, if $P, Q, R, \ldots$ are covariants, any rational and integral function $F(P, Q, R, \ldots)$, homogeneous in regard to the coefficients, is also a covariant,—we say that the covariants $X, Y, \ldots$ of the same degree in regard to the coefficients, and not connected by any identical equation $\alpha X + \beta Y \ldots = 0$ (where $\alpha, \beta, \ldots$ are quantities independent of the coefficients $(a, b, c, \ldots)$), are *asyzygetic* covariants, and that a covariant not expressible as a rational and integral function of covariants of lower degrees is an *irreducible* covariant; and it is assumed that we know the number of the asyzygetic covariants of the degrees 1, 2, 3, ...; say, these are $A_1, A_2, A_3, \ldots$, or, what is the same thing, that the number of the asyzygetic covariants of the degree θ, or form $(a, b, \ldots)^\theta$, is equal to the coefficient of a^θ in a given function

$$\phi(a) = 1 + A_1 a + A_2 a^2 \ldots + A_\theta a^\theta + \ldots,$$

where I have purposely written a, as a representative of the coefficients $(a, b, c, \ldots)$, in place of the x of my Second Memoir.

329. The theory was, that determining $\alpha_1, \alpha_2, \ldots$ by the conditions

$$\begin{aligned} A_1 &= \alpha_1, \\ A_2 &= \tfrac{1}{2}\alpha_1(\alpha_1+1) + \alpha_2, \\ A_3 &= \tfrac{1}{6}\alpha_1(\alpha_1+1)(\alpha_1+2) + \alpha_1\alpha_2 + \alpha_3, \\ &\vdots \end{aligned}$$

that is, throwing

$$1 + A_1 a + A_2 a^2 + A_3 a^3 + \ldots$$

into the form

$$(1-a)^{-\alpha_1}(1-a^2)^{-\alpha_2}(1-a^3)^{-\alpha_3}\ldots,$$

the index α_r would express the number of irreducible covariants of the degree r *less* the number of the (irreducible) linear relations, or syzygies, between the composite or non-irreducible covariants of the same degree. Thus $A_1 = \alpha_1$, there would be α_1 covariants of the degree 1[1]; these give rise to $\frac{1}{2}\alpha_1(\alpha_1+1)$ composite covariants of the degree 2; or, assuming that these are connected by k_2 syzygies, the number of asyzygetic composite covariants of the degree 2 would be $\frac{1}{2}\alpha_1(\alpha_1+1) - k_2$; and thence there would be $A_2 - \frac{1}{2}\alpha_1(\alpha_1+1) + k_2$, that is, $\alpha_2 + k_2$ irreducible covariants of the same degree; so that (irreducible invariants *less* syzygies) $(\alpha_2 + k_2) - k_2$ is $= \alpha_2$.

[1] For the case of covariants, α_1 is of course $= 1$; but in the investigation the term covariant properly stands for any function satisfying the conditions 1° and 2°.

330. The k_2 syzygies are here irreducible syzygies; for, calling $P, Q, R, \ldots$ the covariants of the degree 1, there is no identical relations between the terms $P^2, Q^2, PQ, \ldots$: imagine for a moment that we could have l_2 such identical relations (viz. this might very well be the case if instead of the $\frac{1}{2}\alpha_1(\alpha_1+1)$ functions $P^2, Q^2, PQ, \ldots$, we were dealing with the same number of other quadric functions of these quantities), that is, relations not establishing any relation between $P^2, Q^2, PQ, \ldots$, and besides these k_2 non-identical relations as above; then the number of irreducible invariants would be $\alpha_2+k_2+l_2$, and the number of irreducible syzygies being as before k_2, the difference would be not α_2 but α_2+l_2. The l_2 identical relations are here relations between composite covariants, and the effect (if any such relation could subsist) would, it appears, be to increase α_2; between syzygies such identical relations do actually exist, and the effect is contrariwise to diminish the α; we may, for instance, for the degree s have irreducible covariants *less* irreducible syzygies $=\alpha_s-l_s$.

331. Assume for a moment that, for a given value of s, α_s is positive; but for the term l_s it would of course follow that there was for the degree in question a certain number of irreducible covariants; and it was in this manner that I was led to infer that the number of the covariants of a quintic was infinite—viz. the transformed expression for the number of asyzygetic covariants is

$$= \text{coeff. } a^\theta \text{ in } (1-a^4)^{-1}(1-a^8)^{-3}(1-a^{12})^{-6}(1-a^{14})^{-4}\ldots,$$

a product which does not terminate, and as to which it is also assumed that the series of negative indices does not terminate.

332. The principle is the same, but the discussion as to the number of the irreducible covariants becomes more precise, if we attend to the covariants as involving not only the coefficients $(a, b, \ldots)$ but also the variables (x, y); we have then to consider the covariants of the form $(a, b, \ldots)^\theta(x, y)^\mu$, or, say, of the form $a^\theta x^\mu$ (degree θ and order μ), and the number of the asyzygetic covariants of this form is given as the coefficient of $a^\theta x^\mu$ in a given function of (a, x), (I write a instead of the z of my Second Memoir in the formulæ which contain x and z): by taking account of the composite covariants and syzygies, we successively determine, from the given number of asyzygetic covariants for each value of θ and μ, the number of the irreducible covariants for the same values of θ and μ. This is, in fact, done for the quintic in my Eighth Memoir up to the covariants and syzygies of the degree 6. But before resuming the discussion for the quintic, I will consider the preceding cases of the quadric, the cubic, and the quartic.

Article Nos. 333 to 336. *New formulæ for the number of Asyzygetic Covariants.*

333. For the quadric $(a, b, c \between x, y)^2$, the number of asyzygetic covariants $a^\theta x^\mu$

$$= \text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)},$$

(see Second Memoir, No. 35, observing that q is there $=\theta-\frac{1}{2}\mu$, and that the subtraction of successive coefficients is effected by means of the factor $1-x$ in the

numerator. See also Eighth Memoir, No. 251, where a like form is used for the quintic). Writing ax^2 for a, and $\frac{1}{x^2}$ for x, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^2)(1-a)\left(1-\frac{a}{x^2}\right)}.$$

The development is

$$\begin{array}{ll|l}
1 & -\frac{1}{x^2} & 1 \\
+ax^2 & & +a\left(\frac{1}{x^2}\right) \\
+a^2(x^4+1) & & +a^2\left(\frac{1}{x^4}+1\right) \\
+a^3(x^6+x^2) & & +a^3\left(\frac{1}{x^6}+\frac{1}{x^2}\right) \\
+a^4(x^8+x^4+1) & & +a^4\left(\frac{1}{x^8}+\frac{1}{x^4}+1\right) \\
\vdots & & \vdots
\end{array}$$

which is

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1}{(1-ax^2)(1-a^2)};$$

and, since $\frac{1}{x^2} A\left(\frac{1}{x}\right)$ contains only negative powers, the required number is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1}{(1-ax^2)(1-a^2)},$$

indicating that the covariants are powers and products of (ax^2 and a^2), the quadric itself, and the discriminant. Compare Second Memoir, No. 49, according to which, writing therein a for x, the number of asyzygetic covariants is

$$= \text{coeff. } a^\theta \text{ in } \frac{1}{(1-a)(1-a^2)}.$$

334. For the cubic $(a, b, c, d \between x, y)^3$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta-\frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^2)(1-ax)(1-ax^{-1})(1-ax^{-3})}:$$

the function is here

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1 - a^6x^6}{(1 - ax^3)(1 - a^2x^2)(1 - a^3x^3)(1 - a^4)}$$

(that this is so may be easily verified); and since the second term contains only negative powers, the required number is $=$ coeff. $a^\theta x^\mu$ in $A(x)$. The formula, in fact, indicates that the covariants are made up of $(ax^3, a^2x^2, a^3x^3, a^4)$, the cubic itself, the Hessian, the cubicovariant, and the discriminant, these being connected by a syzygy (a^6x^6) of the degree 6 and order 6. Compare Second Memoir, No. 50, according to which the number of covariants of degree θ is

$$= \text{coeff. } a^\theta \text{ in } \frac{1 - a^6}{(1 - a)(1 - a^2)(1 - a^3)(1 - a^4)}.$$

335. For the quartic $(a, b, c, d, e \between x, y)^4$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1 - x}{(1 - a)(1 - ax)(1 - ax^2)(1 - ax^3)(1 - ax^4)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1 - x^{-2}}{(1 - ax^4)(1 - ax^2)(1 - a)(1 - ax^{-2})(1 - ax^{-4})}:$$

the function is here

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1 - a^6x^{12}}{(1 - ax^4)(1 - a^2x^4)(1 - a^2)(1 - a^3)(1 - a^3x^6)};$$

and the second term containing only negative powers, the required number is $=$ coeff. $a^\theta x^\mu$ in $A(x)$. The formula indicates that the covariants are made up of $(ax^4, a^2x^4, a^2, a^3, a^3x^6)$, the quartic itself, the Hessian, the quadrinvariant, the cubinvariant, and the cubicovariant, these being connected by a syzygy (a^6x^{12}) of the degree 6 and order 12. Compare Second Memoir, No. 51, according to which the number of covariants of degree θ is

$$= \text{coeff. } a^\theta \text{ in } \frac{1 - a^6}{(1 - a)(1 - a^2)^2(1 - a^3)^2}.$$

336. For the quintic $(a, b, c, d, e, f \between x, y)^5$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1 - x}{(1 - a)(1 - ax)(1 - ax^2)(1 - ax^3)(1 - ax^4)(1 - ax^5)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-x^{-2}}{(1-ax^5)(1-ax^3)(1-ax)(1-ax^{-1})(1-ax^{-3})(1-ax^{-5})}.$$

The developed expression is

$$\begin{array}{ll|l} 1 & -\dfrac{1}{x^2} & 1 \\ +ax^5 & & +ax^{-5} \\ +a^2(x^{10}+x^6+x^2) & & +a^2(x^{-10}+x^{-6}+x^{-2}) \\ \vdots & & \vdots \end{array}$$

but here there is not any *finite* function $A(x)$ such that this development is

$$= A(x) \qquad -\frac{1}{x^2}A\left(\frac{1}{x}\right).$$

The numerical coefficients are of course the same as those in the development of the untransformed function; viz. they are the numbers given in the third column of Table No. 82 (Eighth Memoir), and also (carried further) in the third column of the following Table, No. 87. And we can, from the discussion of these coefficients, deduce the form of $A(x)$, viz. this is

$$= \frac{\begin{array}{c|c|c|

Article Nos. 337 to 346. *The* 23 *Fundamental Covariants.*

337. Gordan's result is that the entire number of the irreducible covariants of the binary quintic is $=23$. I represent these by the letters $A, B, C, \ldots, W$, identifying such of them as were given in my former Memoirs on Quantics with the Tables of these Memoirs, and the new ones, O, P, R, S, T, V, with the Tables Nos. 90, 91, 92, 93, 94, 95 of the present Memoir.

Table No. 87. Identification of the 23 irreducible covariants of the binary quintic.

			Table No.
A	$(a, b, c, d, e, f \between x, y)^5$	f	13
$B = \frac{1}{28000}(A, A)^4$	$(\quad)^2(\quad)^2$	$\iota = (ff)^4$	14
$C = \frac{1}{800}(A, A)^2$	$(\quad)^2(\quad)^6$	$\phi = (ff)^2$	15
$D = -\frac{1}{3}(A, B)^2$	$(\quad)^3(\quad)^3$	$j = (f\iota)^2$	16
$E = \frac{1}{5}(A, B)$	$(\quad)^3(\quad)^5$	$(f\iota)$	17
$F = \frac{1}{15}(A, C)$	$(\quad)^3(\quad)^9$	$(f\phi)$	18
$G = -\frac{1}{2}(B, B)^2$	$(\quad)^4(\quad)^0$	$(\iota\iota)^2$	19
$H = -\frac{1}{5}(B, C)^2 + \frac{2}{5}B^2$	$(\quad)^4(\quad)^4$	$p = (\phi\iota)^2$	20
$I = -\frac{1}{6}(B, C)$	$(\quad)^4(\quad)^6$	$(\phi\iota)$	21
$J = -\frac{1}{4}(B, D)^2$	$(\quad)^5(\quad)^1$	$\alpha = (j\iota)^2$	22
$K = -(B, D)$	$(\quad)^5(\quad)^3$	$(j\iota)$	23
$L = -\frac{1}{20}(A, H) + \frac{1}{2}BE$	$(\quad)^5(\quad)^7$	(fp)	24
$M = -\frac{1}{48}(B, H)^2 - \frac{1}{6}BG$	$(\quad)^6(\quad)^2$	$\tau = (p\iota)^2$	83
$N = \frac{1}{4}(B, H)$	$(\quad)^6(\quad)^4$	$(p\iota)$	84
$O = -(B, J)$	$(\quad)^7(\quad)^1$	$(\iota\alpha)$	*90
$P = -\frac{1}{5}(A, M) - BK$	$(\quad)^7(\quad)^5$	$(f\tau)$	*91
$Q = \frac{1}{2}(B, M)^2$	$(\quad)^8(\quad)^0$	$(\iota\tau)^2$	25
$R = -\frac{1}{2}(B, M)$	$(\quad)^8(\quad)^2$	$(\tau\iota)$	*92
$S = -96(D, M) + 16BO - 7GK$	$(\quad)^9(\quad)^3$	$(j\tau)$	*93
$T = -(J, M)$	$(\quad)^{11}(\quad)^1$	$\gamma = (\tau\alpha)$	*94
$U = \frac{1}{18}(J, O) + \frac{1}{9}GQ$	$(\quad)^{12}(\quad)^0$	$((\iota\alpha), \alpha)$	29
$V = -(B, T)$	$(\quad)^{13}(\quad)^1$	$(\iota\gamma)$	*95
$W = -\frac{1}{6}(O, T)$	$(\quad)^{13}(\quad)^0$	$((\iota\alpha), \gamma)$	29A

338. The Table exhibits the generation of the several covariants; viz. (A, B) denotes $\partial_x A \,.\, \partial_y B - \partial_y A \,.\, \partial_x B$, $(A, B)^2$ denotes $\partial_x^2 A \,.\, \partial_y^2 B - 2\partial_x\partial_y A \,.\, \partial_x\partial_y B + \partial_y^2 A \,.\, \partial_x^2 B$, &c. (see *post*, No. 348). The column f, $\iota = (ff)^4$, &c. shows Gordan's notation, and the generation of his 23 forms $\big((ff)^4$ written as with him for $(f, f)^4$, &c.$\big)$: it will be observed that the forms are not identical; if the calculations had been made *de novo*, I should have adopted his values, simply omitting numerical factors of the several forms (thus every term of ι, $=(ff)^4$ contains the factor $2\,.\,(120)^2$, $= 28800$): of course the presence of these numerical factors renders the f, ι, ϕ, &c. as they stand inconvenient for the expression of results; and the numerical fixation of the values was no part of Gordan's object. But by reason of the existing Tables the change of notation is in fact more than this; thus H instead of being a submultiple of $(B, C)^2$, that is, of p, is in fact $= -\frac{1}{5}(B, C)^2 + \frac{2}{5}B^2$; and so in other cases. If the occasion for it arises, there is no difficulty in expressing any one of the forms f, ι, ϕ, &c. in terms of the $(A, B, C \,..\, V, W)$; thus in the instance just referred to, $p = (\phi\iota)^2$, we have

$$\phi = (ff)^2 = (A, A)^2 = 800\, C,$$

and

$$\iota = (ff)^4 = (A, A)^4 = 28800\, B,$$

whence $p = 2304000\,(B, C)^2$; also $(B, C)^2 = -5H + 2B^2$; and therefore, finally,

$$p = -11520000\, H + 4608000\, B^2.$$

339. I remark upon the value $S = -96\,(D, M) + 16BO - 7GK$, that S is the complete value of a covariant $(\ \ )^9\,(\ \ )^3$, the leading coefficient of which is given in Table No. 86 of my Eighth Memoir; the form (D, M), omitting a numerical factor (if any), would have had smaller numerical coefficients, but there is in the form actually adopted the advantage that it vanishes for $a = 0$, $b = 0$, that is, when the quintic has two equal roots, [see *post*, No. 346].

340. I now form the following Table No. 88, viz. this is the Table No. 82 of my Eighth Memoir, carried as far as a^8, but with the composite covariants expressed by means of the foregoing letters $A, B, C, \ldots, W$; instead of giving the syzygies as in Table No. 82, I transfer them to a separate Table, No. 89. In all other respects the arrangement is as explained, Eighth Memoir, No. 253; but in place of N, S, S' I have written $*$, Σ, Σ' to denote new covariant, new syzygy, derived syzygy, respectively; and I have, as to the terms a^8x^{14}, a^8x^{20} respectively, introduced the new symbol σ to denote an interconnexion of syzygies, as appearing by the Table No. 89, and as will be further explained.

Table No. 88.

[In this Table and the subsequent Table 89, I have for convenience used, instead of capitals, the small italic letters $a, b, c, \ldots w$ to denote the 23 irreducible covariants of the quintic.]

Ind. a.	Ind. x.	Coeff.		
1	5	1	a	*
	3	0		
	1	0		
2	10	1	a^2	
	8	0		
	6	1	c	*
	4	0		
	2	1	b	*
	0	0		
3	15	1	a^3	
	13	0		
	11	1	ac	
	9	1	f	*
	7	1	ab	
	5	1	e	*
	3	1	d	*
	1	0		
4	20	1	a^5	
	18	0		
	16	1	a^2c	
	14	1	af	
	12	2	a^2b, c^2	
	10	1	ae	
	8	2	ad, bc	
	6	1	i	*
	4	2	b^2, h	*
	2	0		
	0	1	g	*

Table No. 88 (continued).

Ind. a.	Ind. x.	Coeff.		
5	25	1	a^5 .	
	23	0	.	
	21	1	a^3c .	
	19	1	a^2f .	
	17	2	a^3b, ac^2 .	
	15	2	a^2e, cf .	
	13	2	a^2d, abc .	
	11	2	ai, bf, ce .	Σ
	9	3	ab^2, ah, cd .	
	7	2	be, l .	*
	5	2	ag, bd .	
	3	1	k .	*
	1	1	j .	*
6	30	1	a^6 .	
	28	0	.	
	26	1	a^4c .	
	24	1	a^3f .	
	22	2	a^4b, a^2c^2 .	
	20	2	a^3e, acf .	
	18	3	a^3d, a^2bc, c^3, f^2 .	Σ
	16	2	a^2i, abf, ace .	Σ'
	14	4	a^2b^2, a^2h, acd, bc^2, ef .	Σ
	12	3	abe, al, ci, df .	Σ
	10	4	a^2g, abd, b^2c, ch, e^2 .	Σ
	8	2	ak, bi, de .	Σ
	6	4	aj, b^3, bh, cg, d^2 .	Σ
	4	1	n .	*
	2	2	bg, m .	*
	0	0	.	
7	35	1	a^7 .	
	33	0	.	
	31	1	a^5c .	
	29	1	a^4f .	
	27	2	a^5b, a^3c^2 .	
	25	2	a^4e, a^3cf .	
	23	3	a^4d, a^3bc, ac^3, af^2 .	Σ'
	21	3	a^3i, a^2bf, a^2ce, c^2f .	Σ'
	19	4	a^3b^2, a^3h, a^2cd, abc^2, aef .	Σ'
	17	4	a^2be, a^2l, aci, adf, bcf, c^2e .	$2\Sigma'$
	15	5	a^2g, a^2bd, ab^2c, ach, ae^2, c^2d, fi .	Σ', Σ
	13	4	a^2k, abi, ade, b^2f, bce, cl, fh .	Σ
	11	5	a^2j, ab^3, abh, acg, ad^2, bcd, ei .	Σ
	9	4	an, b^2e, bl, ck, di, eh, fg .	3Σ
	7	4	abg, am, b^2d, cj, dh .	Σ
	5	3	bk, p, eg .	*
	3	2	bj, dg .	
	1	1	o .	*

Table No. 88 (concluded).

Ind. a.	Ind. x.	Coeff.		
8	40	1	a^8 .	
	38	0	.	
	36	1	a^6c .	
	34	1	a^5f .	
	32	2	$a^6b,\ a^4c^2$.	
	30	2	$a^5e,\ a^3cf$.	
	28	3	$a^5d,\ a^4bc,\ a^2c^3,\ a^2f^2$.	Σ'
	26	3	$a^4i,\ a^3bf,\ a^3ce,\ ac^2f$.	Σ'
	24	5	$a^4b^2,\ a^4h,\ a^3cd,\ a^2bc^2,\ a^2ef,\ cf^2,\ c^4$.	Σ'
	22	4	$a^3be,\ a^3l,\ a^2ci,\ a^2df,\ abcf,\ ac^2e$.	$2\Sigma'$
	20	6	$a^4g,\ a^3bd,\ a^2b^2c,\ a^2ch,\ a^2e^2,\ ac^2d,\ afi,\ bc^3,\ bf^2,\ cef$.	σ
	18	5	$a^3k,\ a^2bi,\ a^2de,\ ab^2f,\ abce,\ acl,\ afh,\ c^2i,\ cdf$.	$4\Sigma'$
	16	7	$a^3j,\ a^2b^3,\ a^2bh,\ a^2cg,\ a^2d^2,\ abcd,\ aei,\ b^2c^2,\ bef,\ c^2h,\ ce^2,\ fl$.	$5\Sigma'$
	14	5	$a^2n,\ ab^2e,\ abl,\ ack,\ adi,\ aeh,\ afg,\ bci,\ bdf,\ cde$.	σ
	12	7	$a^2bg,\ a^2m,\ ab^2d,\ acj,\ adh,\ b^3c,\ bch,\ be^2,\ c^2g,\ cd^2,\ el,\ fk,\ i^2$.	3Σ
	10	5	$abk,\ aeg,\ ap,\ b^2i,\ bde,\ cn,\ dl,\ fj,\ hi$.	3Σ
	8	6	$abj,\ adg,\ b^4,\ b^2h,\ bcg,\ bd^2,\ cm,\ ek,\ h^2$.	2Σ
	6	3	$ao,\ bn,\ dk,\ ej,\ gi$.	2Σ
	4	4	$b^2g,\ bm,\ dj,\ gh$.	
	2	1	r .	*
	0	2	$g^2,\ q$.	*

341. The syzygies and interconnexions of syzygies are given in

Table No. 89.

[See *ante* Table No. 88.]

(5, 11)	$ai + bf - ce = 0$
(6, 18)	$a^2d - a^2bc + 4c^3 + f^2 = 0$
(6, 14)	$a^2h - 6acd - 4bc^2 - ef = 0$
(6, 12)	$al - 2ci + 3df = 0$
(6, 10)	$a^2g - 12abd - 4b^3c - e^2 = 0$
(6, 8)	$ak + 2bi - 3de = 0$
(6, 6)	$aj - b^3 + 2bh - cg - 9d^2 = 0$
(7, 15)	$a^2bd - ab^2c + ach - 6c^2d - fi = 0$
(7, 13)	$a^2k - abi - 3b^2f + 6cl + 3fh = 0$
(7, 11)	$a^2j - ab^3 + abh - 9ad^2 - 6bcd - ei = 0$
(7, 9)	$an - b^2e - 6di + 2eh - fg = 0$
	$2bl + 6di - eh + fg = 0$
	$2ck - 12di + eh - fg = 0$
(7, 7)	$am + 2b^2d + cj - 3dh = 0$

Table No. 89 (continued).

σ, (8, 20)	$0 \,.\, a^2(a^2g - 12abd - 4b^3c - e^2)$ $-a(a^2bd - abc^2 + ach - 6c^2d - fi)$ $+b(a^3d - a^2bc + 4c^3 + f^2)$ $+c(a^2h - 6acd - 4bc^2 - ef)$ $-f(ai + bf - ce) \quad = 0$	*suprà* (6, 10) „ (7, 15) „ (6, 18) „ (6, 14) „ (5, 11)
σ, (8, 14)	$0 \,.\, a(an - b^2e - 6di + 2eh - fg)$ $+ a(2bl + 6di - eh + fg)$ $+ a(2ck - 12di + eh - fg)$ $- 2b(al - 2ci + 3df)$ $- 2c(ak + 2bi - 3de)$ $+ 6d(ai + bf - ce) \quad = 0$	*suprà* (7, 9) „ („) „ („) „ (6, 12) „ (6, 8) „ (5, 11)
(8, 12)	$ab^2d - b^3c + 2bch - c^2g + i^2 = 0$ $-3adh - 2bch + 2c^2g + 18cd^2 + fk - 2i^2 = 0$ $el + fk - 2i^2 = 0$	
(8, 10)	$abk - cn - 6dl - 2fj + hi = 0$ $ap + 2cn + fj = 0$ $b^2i + cn + 3dl + fj - 2hi = 0$	
(8, 8)	$abj - b^4 + 4b^2h - 9bd^2 + 12cm - ek - 3h^2 = 0$ $adg + 2b^2h - 12bd^2 + 8cm - ek - 2h^2 = 0$	
(8, 6)	$ao + 6dk - 3ej + 2gi = 0$ $bn + 3dk - ej + gi = 0$	

342. In illustration take any one of the lines of Table No. 88, for instance [resuming the notation by capital letters] the line

$$(7, 17) \mid 4 \mid \quad A^2BE,\ A^2L,\ ACI,\ ADF,\ BCF,\ C^2E \quad \mid 2\Sigma' \mid$$

there are here 6 composite covariants, but the number of asyzygetic covariants is $=4$; there must therefore be $6-4, =2$ syzygies; we have however (see Table No. 89) two derived syzygies of the right form, viz. these are

$$A(AL - 2CI + 3DF) = 0,$$
$$C(AI + BF - CE) = 0,$$

which are designated as $2\Sigma'$, and there is consequently no new syzygy Σ.

But in the line

$$(7, 15) \mid 5 \mid \quad A^2G,\ A^2BD,\ AB^2C,\ ACH,\ AE^2,\ C^2D,\ FI \quad \mid \Sigma', \Sigma \mid$$

there are 7 composite covariants, but the number of asyzygetic covariants is $=5$; there must therefore be $7-5, =2$ syzygies. One of these is the derived syzygy

$$A(A^2G - E^2 - 12ABD - 4B^2C) = 0,$$

which is designated by Σ'; the other is a new syzygy (see Table No. 89),

$$A^2BD - ABC^2 + ACH - 6C^2D - FI = 0,$$

designated by Σ.

343. Take now the line

$$(8, 20) \mid 6 \mid \quad A^4G,\ A^3BD,\ A^2B^2C,\ A^3CH,\ A^2E^2,\ AC^2D,\ AFI,\ BC^3,\ BF^2,\ CEF \quad \mid 5\Sigma',\ \sigma \mid ;$$

there are here 10 composite covariants, but the number of irreducible covariants is $= 6$; there should therefore be $10 - 6, = 4$ syzygies. There are, however, the 5 derived syzygies

$$A^2(A^2G - 12ABD - 4B^3C - E^2) = 0, \text{ \&c. (see Table No. 89)}$$

designated by $5\Sigma'$; since these are equivalent to 4 syzygies only there must be 1 identical relation between them (designated by σ), viz. this is the equation $0 = 0$ obtained by adding the several syzygies, multiplied each by the proper numerical factor as shown Table No. 89.

344. Again, for the line

$$(8, 14) \mid 5 \mid \quad A^2N,\ AB^2E,\ ABL,\ ACK,\ ADI,\ AEH,\ AFG,\ BCI,\ BDF,\ CDE \quad \mid 6\Sigma',\ \sigma \mid$$

there are here 10 composite covariants, but only 5 irreducible covariants; there should therefore be $10 - 5, = 5$ syzygies; we have in fact the 6 derived syzygies

$$A(AN - B^2E - 6DI + 2EH - FG) = 0 \text{ \&c. (see Table No. 89)}$$

designated by $6\Sigma'$; these must therefore be connected by 1 identical relation (designated by σ), viz. this is the equation $0 = 0$ obtained by adding the several syzygies, each multiplied by the proper numerical factor as shown Table No. 89.

345. These two cases (σ) are in fact the instances which present themselves where a correction is required to my original theory. The two identical relations in question were disregarded in my original theory, and this accordingly gave the two non-existent irreducible covariants $(a, \ldots)^8(x, y)^{14}$ and $(a, \ldots)^8(x, y)^{20}$. And reverting to No. 336, these give in the denominator of $A(x)$ the factors $(1 - a^8x^{20})(1 - a^8x^{14})$. In virtue hereof, writing $x = 1$, we have in $A(x)$ the factor $\dfrac{(1 - a^8)^{10}}{(1 - a^8)^4}$, $= (1 - a^8)^6$, agreeing with the function $(1 -)^{-1}(1 - a)^{-2} \ldots (1 - a^8)^6 \ldots$. And we thus see that the denominator factors of $A(x)$ do not all of them refer to irreducible covariants; viz. we have

$$ax^5,\ a^2x^6,\ a^2x^8,\ a^3x^9,\ a^3x^5,\ a^3x^3,\ a^4x^6,\ a^4x^4,\ a^4,\ a^5x^7,\ a^5x^3,\ a^5x,\ a^6x^4,\ a^6x^2,\ a^7x^3,\ a^7x,\ a^8x^2,\ a^8,$$

each referring to an irreducible covariant, but a^8x^{20} and a^8x^{14} each referring to an identical relation (σ) or interconnexion of syzygies. And we thus understand how, consistently with the number of the irreducible covariants being finite, the expression for $A(x)$ may be as above the quotient of two infinite products; viz. there will be in the denominator a finite number of factors each referring to an irreducible covariant, but the remaining infinite series of denominator factors will refer each factor to an

identical relation or interconnexion of syzygies. But I do not see how we can by the theory distinguish between the two classes of factors, so as to determine the number of the irreducible covariants, or even to make out affirmatively that the number of them is finite.

346. The new covariants O, P, R, S, T, V are as follows:

[Table No. 90 (Covariant O),
Table No. 91 (Covariant P),
Table No. 92 (Covariant R),
Table No. 93 (Covariant S),
Table No. 94 (Covariant V),

printed in the paper 143, "Tables of the Covariants M to W of the Binary Quintic; from the second, third, fifth, eighth, ninth and tenth Memoirs on Quantics" with the insertion as therein mentioned of the terms with zero coefficients. The covariant $S, = -96(D, M) + 16BO - 7GK$, of the present Memoir is there called S', and there is given the more simple form $S = (D, M)$, of this covariant.]

Article Nos. 347 to 365. *Sketch of Professor* GORDAN'S *proof for the finite Number,* $= 23$, *of the Covariants of a Binary Quintic.*

347. I propose to reproduce the leading points of Professor Gordan's proof that the binary quintic $(a, b, c, d, e, f \between x, y)^5$ has a finite system of 23 covariants, viz. a system such that every other covariant whatever is a rational and integral function of these 23 covariants.

348. DERIVATION. Consider for a moment any two binary quantics ϕ, ψ of the same or different orders, and which may be either independent quantics, or they may be both or one of them covariants, or a covariant, of a binary quantic f. We may form the series of *derivatives*

$$(\phi, \psi)^0 = \phi\psi,$$

$$(\phi, \psi)^1 = \overline{12}\ \phi_1\psi_2 = \partial_x\phi\ .\ \partial_y\psi\ - \partial_y\phi\ .\ \partial_x\psi,$$

$$(\phi, \psi)^2 = \overline{12}^2\ \phi_1\psi_2 = \partial_x^2\phi\ .\ \partial_y^2\psi - 2\partial_x\partial_y\phi\ .\ \partial_x\partial_y\psi + \partial_y^2\phi\ .\ \partial_x^2\psi,$$

$$\vdots$$

where, however, there is no occasion to use the notation $(\phi, \psi)^0$ (as this is simply the product $\phi\psi$), and the succeeding derivatives may (when there is no risk of ambiguity) be written more shortly $(\phi\psi)$, $(\phi\psi)^2$, $(\phi\psi)^3$, &c.; in all that follows the word "derivative" (Gordan's *Uebereinanderschiebung*) is to be understood in this special sense.

349. The degree of the derivative $(\phi\psi)^k$ is the sum of the degrees of the constituents ϕ, ψ; the order of the derivative is the sum of the orders *less* $2k$; it being understood throughout that the word degree refers to the coefficients, and the

word order to the variables. In speaking generally of the covariants or of all the covariants of a quantic f, or of the covariants or all the covariants of a given degree or order, we of course exclude from consideration covariants linearly connected with other covariants (for otherwise the number of terms would be infinite); but unless it is expressly so stated, we do not carry this out rigorously so as to make the system to consist of asyzygetic covariants; viz. it is assumed that the system is complete, but not that it is divested of superfluous terms.

350. THEOREM A. The covariants of a quantic f of a given degree m can be all of them obtained by derivation from f and the covariants of the next inferior degree $(m-1)$.

In particular for the degree 1 the only covariant is the quantic f itself; for the degree 2 the covariants are $(ff)^0$, $(ff)^2$, $(ff)^4, \ldots$: using for a moment β to denote each of these in succession, the covariants of the third degree are $(\beta f)^0$, $(\beta f)^1$, $(\beta f)^2, \ldots$; and so on.

351. Suppose that the covariants of the second degree $(ff)^0$, $(ff)^2$, $(ff)^4 \ldots$ are in this order represented by β_1, β_2, $\beta_3 \ldots$, then the covariants of the third degree written in the order

$$(\beta_1 f)^0,\ (\beta_1 f),\ (\beta_1 f)^2,\ \ldots (\beta_2 f)^0,\ (\beta_2 f),\ (\beta_2 f)^2,\ \ldots (\beta_3 f)^0,\ (\beta_3 f),\ (\beta_3 f)^2 \ldots$$

may be represented by γ_1, γ_2, $\gamma_3, \ldots$, the covariants of the fourth degree written in the order

$$(\gamma_1 f)^0,\ (\gamma_1 f),\ (\gamma_1 f)^2,\ \ldots (\gamma_2 f)^0,\ (\gamma_2 f),\ (\gamma_2 f)^2,\ \ldots (\gamma_3 f)^0,\ (\gamma_3 f),\ (\gamma_3 f)^2 \ldots$$

may be represented by δ_1, δ_2, $\delta_3 \ldots$, and so on: we thus obtain in a definite order the covariants of a given degree m; say, these are μ_1, μ_2, μ_3, $\mu_4, \ldots$: any term μ_3 is said to be a *later* term than the preceding terms μ_1, μ_2, and an *earlier* term than the following ones, μ_5, μ_6, &c.

Observe that each term μ_r is a derivative $(\lambda_q f)^k$, the derivatives of an earlier λ are earlier than those of a later λ; and as regards the derivatives of the same λ, the derivative with a less index of derivation is earlier than that with a greater index of derivation, or, what is the same thing, those are earlier which are of the higher order.

352. The series μ_1, μ_2, μ_3, $\mu_4 \ldots$ is not asyzygetic; we make it so, by considering in succession whether the several terms μ_2, $\mu_3, \ldots$ respectively are expressible as linear functions of the earlier terms, and by omitting every term which is so expressible. The reduced series thus obtained is called T_1, T_2, $T_3, \ldots$. Observe that not every μ is a T, but that every T is a μ; every T therefore arises from a derivation upon f and a certain term λ; which term λ (supposing the λ series reduced in like manner to S_1, S_2, $S_3, \ldots$) is a linear function of certain of the S's. Each later T is derived from later S's, or it may be from the same S's as an earlier T; viz. if the later T is derived from $(S_1, S_2, \ldots S_\theta)$, then the earlier T is derived, it may be, from $(S_1, S_2, \ldots S_\theta)$, or from $(S_1, S_2, \ldots S_{\theta-k})$, but so that there is not in the series any term later than S_θ.

And if, considering any T as thus derived from certain of the S's, and in like manner each of these S's as derived from certain of the R's, and so on, we descend to any preceding series,

$$M_1,\ M_2,\ M_3 \ldots$$

it will appear that the T is derived from a certain number $(M_1,\ M_2, \ldots M_\phi)$ of the terms of this series.

353. The quadricovariants $(ff)^0$, $(ff)^2$, $(ff)^4, \ldots$ are of different orders, and consequently asyzygetic. They form therefore a series such as the T-series, and they may be represented by

$$B_1,\ B_2,\ B_3, \ldots.$$

Supposing f to be of the order n, B_1 is of the order $2n$, B_2 of the order $2n-4$, B_3 of the order $2n-8$, and so on. Those terms which are of an order greater than n, are said to be of the form W (agreeing with a subsequent more general definition of W); those which are of an order equal to or less than n, are said to be of the form χ; so that the earlier terms of the B series are W, and the later terms are χ; viz. the χ terms taken in order, beginning with the earliest, are $\chi_1,\ \chi_2,\ \chi_3, \ldots.$

354. By what precedes any particular T is derived from certain terms $B_1,\ B_2, \ldots B_\theta$, of the B series. This series, $B_1,\ B_2, \ldots B_\theta$, may stop short of the terms χ, or it may include a certain number of them, say $\chi_1,\ \chi_2, \ldots \chi_r$. The terms derived from the χ's are in the sequel denoted by P_χ.

355. Every covariant whatever is a form or sum of forms such as

$$\overline{12}^\alpha\, \overline{13}^\beta\, \overline{23}^\gamma \ldots f_1 f_2, \ldots f_m;$$

writing in regard to any such expression

$$\Sigma \text{ ind. } 1 = i,\ \Sigma \text{ ind. } 2 = j, \ldots$$

(viz. i is the sum of all those indices α, β, &c. which belong to a term containing the symbolic number 1, j the sum of all the indices α, γ, &c. which belong to a term containing the symbolic number 2, and so on) then each of the numbers $i,\ j, \ldots$ is at most $= n$, that is $n-i,\ n-j, \ldots$ may be any of them $= 0$, but they cannot be any of them negative; the degree of the function is $= m$, and its order is $= mn - i - j \ldots$ It is to be further observed that the form is a function of the differential coefficients of f of the orders $n-i,\ n-j$, &c. respectively. It follows that if $n-i,\ n-j, \ldots$ are none of them $= 0$, the form in question may be obtained from a like form belonging to a quantic f' of the next inferior order $n-1$ by replacing therein the coefficients $a',\ b', \ldots$ by $ax+by,\ bx+cy$, &c. respectively: for example, if f denote the cubic function $(a,\ b,\ c,\ d \between x,\ y)^3$, then the Hessian hereof is $\overline{12}^2 f_1 f_2$; the like form in regard to the quadric $f' = (a',\ b',\ c' \between x,\ y)^2$ is $\overline{12}^2 f_1' f_2'$, which is $= a'c' - b'^2$; and substituting herein $ax+by,\ bx+cy,\ cx+dy$ for $a',\ b',\ c'$ respectively, we have the Hessian $\overline{12}^2 f_1 f_2$ of the cubic. A covariant of f derivable in this manner from a covariant of the next inferior quantic f' is said to be a special covariant.

356. Reverting to the form

$$\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots f_1 f_2 \ldots f_m;$$

if, as before, $n-1$, $n-j$, &c. are each of them >0; if there is at least one index i which is $=$ or $<\frac{1}{2}n$ (that is, for which $n-i>\frac{1}{2}n$), and if the order $mn-i-j\ldots$ be $>n$, then the form, or any sum of such forms, is said to be a form or covariant W. Every covariant W is thus a special covariant, but not conversely. In the particular case $m=2$, the form is

$$\overline{12}^{\alpha} f_1 f_2,$$

which will be a form W if $n-\alpha>\frac{1}{2}n$, or, what is the same thing, $2n-2\alpha>n$, that is if the order be $>n$. Hence, as already mentioned, the covariants T of the degree 2 are W, or else χ, according as the order is greater than n, or as it is equal to or less than n.

357. *Theorem B.* If any covariant T be expressible as the sum of a form W and of earlier T's than itself, then forming the derivative $(Tf)^k$, either this is not a form T, or being a form T, it is expressible as the sum of a form W and of earlier T's than itself; or, what is the same thing, $(Tf)^k$, if it be a form T, is (like the original T) the sum of a form W and of earlier T's than itself.

Hence also every form T is the sum of a form W, and of forms derived from the functions $\chi_1, \chi_2, \ldots$, say

$$T = W + P_\chi,$$

or, what is the same thing, every covariant whatever is of the form $W+P_\chi$.

358. The proof that for a form f of the order n the number of covariants is finite, depends on the assumption that the number is finite for a form f' of the next inferior order $n-1$: this being so, the number of the special covariants of f will be finite; say these are $A_1, A_2, A_3, \ldots$ (f is itself one of the series, but we may separate it, and speak of the form f and its special covariants): the forms W are functions of the special covariants, and hence every covariant whatever of f is of the form $F(A)+P_\chi$; but it requires still a long investigation to pass from this to the theorem of the existence of a finite number of forms V such that every covariant whatever is $F(V)$. I pass this over, and reproduce only the investigation for the case of the quintic.

359. Starting from the assumed system of forms,

$$\begin{gathered}
f,\ \phi=(ff)^2,\ i=(ff)^4,\ j=(fi)^2,\ \alpha=(ji)^2,\ p=(\phi i)^2,\ \tau=(pi)^2,\ \gamma=(\tau\alpha),\\
(f\phi),\ (fp),\ (f\tau),\ (j\tau),\\
(fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),\\
(i\alpha),\ (i\gamma),\ (ii)^2,\ ((i\alpha),\alpha),\ (i\tau)^2,\ ((i\alpha),\gamma),
\end{gathered}$$

say, the 23 forms U, it is to be shown that every other covariant whatever of the quintic is of the form $F(U)$.

The special covariants are f, ϕ, $(f\phi)$, i, j, which are forms U; the only form χ is i, so that instead of P_χ writing P_i, every covariant whatever of f is

$$=F(U)+P_i;$$

and it remains to show that every form P_i is $F(U)$; or, what is the same thing, that if H be any form $F(U)$ whatever, then that (Hi) and $(Hi)^2$ are each of them $F(U)$.

360. In order to show that every covariant of a degree not exceeding m is $F(U)$, it will be sufficient to show that the several forms (Hi) and $(Hi)^2$ of a degree not exceeding m are each of them $F(U)$; and if for this purpose we assume that it is shown that every covariant of a degree not exceeding $m-1$ is $F(U)$, then in regard to the forms (Hi) and $(Hi)^2$ of the degree m, it will be sufficient to show that any such form is a function of covariants of a degree inferior to m.

361. First for the form (Hi): we have $(PQ, i)=P(Qi)+Q(Pi)$; and hence we see that (Hi) will be $F(U)$ if only (Ui) is always $F(U)$.

In forming the derivative of i with the several covariants U, we may omit i itself, and also the four invariants $(ii)^2$, $(i\tau)^2$, $((i\alpha), \alpha)$, $((i\alpha), \gamma)$, since in each of these cases the derivative is $=0$. We have therefore to consider the derivative of i with

$$f,\ \phi,\ j,\ \alpha,\ p,\ \tau,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),\ (i\alpha),\ (i\gamma),$$

respectively: the first seven of these are each of them U; the remaining eleven are each of them of the form $((PQ), i)$. Now $((PQ), i)$ is a linear function of $P(Qi)^2$, $Q(Pi)^2$, and $i(PQ)^2$, that is $((PQ), i)$ is a function of covariants of a lower degree than itself.

362. Next for the form $(Hi)^2$, we have $(PQ, i)^2$, a linear function of $P(Qi)^2$, $Q(Pi)^2$, $i(PQ)^2$; and we hence see that $(Hi)^2$ will be $F(U)$ if only $(Ui)^2$ is always $F(U)$.

In forming the second derivative of i with the several covariants U, we may omit as before the four invariants, and also omit the four linear covariants α, $i\alpha$, γ, $i\gamma$; we have therefore to consider the second derivatives of i with

$$f,\ \phi,\ i,\ j,\ p,\ \tau,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),$$

respectively: the first six of these are each of them U; the remaining nine are each of the form $((PQ), i)^2$. Now $((PQ), i)^2$ is a linear function of $((Pi)^2, Q)$, $((Qi)^2, P)$, $P(Qi)^2$, and $Q(Pi)^2$. The first two of these are terms of the same form; $(Pi)^2$, as a covariant of a lower degree than $((PQ), i)^2$, is $F(U)$, and hence $((Pi)^2, Q)$ will be $F(U)$ if only (U, Q) is $F(U)$; Q being here any one of the functions f, ϕ, i, j, p, τ, and U being any one of the functions

$$f,\ \phi,\ i,\ j,\ p,\ \tau,\ \alpha,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i)\,(ji)\,(pi)\,(\tau i)\,(i\alpha)\,(i\gamma).$$

363. For U equal to any one of the last eleven values, the form is $(Q, S)R$ which is $=R(QS)+S(QR)$, and is thus a function of covariants of a lower degree; there remains only the derivatives formed with two of the functions f, ϕ, i, j, p, τ, or of one of these with α or γ. But these are all U other than the derivatives

$$(fj),\ (\phi j),\ (\phi p),\ (\phi\tau),\ (p\tau);\ (f\alpha),\ (\phi\alpha),\ (j\alpha),\ (p\alpha);\ (f\gamma),\ (\phi\gamma),\ (j\gamma),\ (p\gamma),\ (\tau\gamma),$$

and since $\gamma=(\tau\alpha)$, the derivatives containing γ will depend upon covariants of a lower degree; there remain therefore only (fj), (ϕj), (ϕp), $(\phi\tau)$, $(p\tau)$; $(f\alpha)$, $(\phi\alpha)$, $(j\alpha)$, $(p\alpha)$: each of these can be actually calculated in the form $F(U)$.

Hence finally, assuming that every covariant of a degree inferior to m is $F(U)$, it follows that every coyariant of the degree m is $F(U)$; whence every covariant whatever is $F(U)$, viz. it is a rational and integral function of the 23 covariants U.

364. It will be observed that, writing A, B, C for P, Q, i, the proof depends on the theorems

$$\begin{array}{llll}
((AB), C), & \text{a linear function of} & A(BC)^2,\ B(CA)^2,\ C(AB)^2, & \\
(AB, C)^2 & \quad\text{"} \quad\quad\text{"} & \text{do.} \quad\text{do.} \quad\text{do.} & \\
((AB), C)^2 & \quad\text{"} \quad\quad\text{"} & ((AC)^2, B),\ ((BC)^2, A),\ B(AC)^2,\ C(AB)^2, &
\end{array}$$

which are theorems relating to any three functions A, B, C whatever.

365. I remark upon the proof that the really fundamental theorem seems to be that which I have called theorem A. As to the forms W it is difficult to see *à priori* why such forms are to be considered, or what the essential property involved in their definition is; and in fact in a more recent paper, "Die simultanen Systeme binären Formen" (*Math. Annalen*, t. II. (1869), see p. 256), Professor Gordan has modified the definition of the forms W by omitting the condition that the order of the function shall exceed n; if it were possible further to omit the condition of at least one index being $=$ or $<\frac{1}{2}n$, and so only retain the conditions $n-i$, $n-j$, &c., each of them >0, then the essential property of the forms W would be that any such form was a rational and integral function of the special covariants formed, as above, by means of the quantic of the next inferior order. And moreover, as regards the theorem B, there seems something indirect and artificial in the employment of such a property; one sees no reason why, when a system of irreducible covariants is once written down, it should not be possible to show that the derivatives of $F(U)$ with the original quantic f are each of them $F(U)$, instead of having to show this in regard to the derivatives of $F(U)$ with the several covariants χ: as regards the quintic, where there is a single covariant χ, the *quadric* function i, there is obviously a great abbreviation in this employment of i in place of f; but for the higher orders, assuming that the proof could be conducted by means of the quantic f itself, it does not appear that there would be even an abbreviation in the employment in its stead of the several covariants χ. The like remarks apply to the proof in the last-mentioned paper. I cannot but hope that a more simple proof of Professor Gordan's theorem will be obtained—a theorem the importance of which, in reference to the whole theory of forms, it is impossible to estimate too highly.

463.

NOTE ON A DIFFERENTIAL EQUATION.

[From the *Memoirs of the Literary and Philosophical Society of Manchester*, vol. II. (1865), pp. 111—114. Read February 18, 1862.]

THE following investigation was suggested to me by Mr Harley's "Remarks on the Theory of the Transcendental Solution of Algebraic Equations," communicated to the Society at the Meeting of the 4th of February.

Mr Harley's equation

$$y^n - ny + (n-1)\,x = 0$$

may be written

$$y = \frac{n-1}{n}\,x + \frac{1}{n}\,y^n;$$

or putting

$$\frac{n-1}{n}\,x = u,\ \frac{1}{n} = a,$$

it becomes

$$y = u + ay^n,$$

which equation may be considered instead of the original equation; and it is to be shown that y, regarded as a function of u, satisfies a certain linear differential equation of the order $n-1$. In fact, expanding y by Lagrange's theorem, we have

$$y = u + au^n + \frac{a^2}{1\,.\,2}\,(u^{2n})' + \frac{a^3}{1\,.\,2\,.\,3}\,(u^{3n})'' + \&c.,$$

$$= u + au^n + \frac{a^2}{1\,.\,2}\,2n\,.\,u^{2n-1} + \frac{a^3}{1\,.\,2\,.\,3}\,3n\,(3n-1)\,u^{3n-2} + \&c.,$$

the law whereof is obvious, and using the ordinary notation of factorials, viz. $[n]^r = n(n-1)\dots(n-r+1)$, we may write

$$y = S_\theta . \frac{[n\theta]^{\theta-1}}{[\theta]^\theta} a^\theta u^{(n-1)\theta+1},$$

where θ extends from 0 to ∞.

It is now very easy to show that y satisfies the differential equation

$$\left[u\frac{d}{du}\right]^{n-1} y = na\left[\frac{n}{n-1}u\frac{d}{du} - \frac{2n-1}{n-1}\right]^{n-1} u^{n-1} y.$$

In fact, using on the left-hand side the foregoing value of y, and on the right-hand side the following value of $u^{n-1}y$, obtained from that of y by writing $\theta - 1$ in the place of θ, viz.

$$u^{n-1} y = S_\theta \frac{[n\theta - n]^{\theta-2}}{[\theta-1]^{\theta-1}} a^{\theta-1} u^{(n-1)\theta+1},$$

and observing that in general the symbol $u\frac{d}{du}$, as regards u^m, is $= m$, the equation in question will be satisfied, if only

$$\frac{[n\theta]^{\theta-1}}{[\theta]^\theta}[(n-1)\theta+1]^{n-1} = \frac{n[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}\left[\frac{n}{n-1}((n-1)\theta+1) - \frac{2n-1}{n-1}\right]^{n-1},$$

where the right-hand side is

$$= \frac{n[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}[n\theta-1]^{n-1};$$

and the equation may be written

$$\frac{n\theta[n\theta-1]^{\theta-2}}{\theta[\theta-1]^{\theta-1}}[(n-1)\theta+1]^{n-1} = \frac{n[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}[n\theta-1]^{n-1},$$

that is,

$$[n\theta-1]^{\theta-2}[(n-1)\theta+1]^{n-1} = [n\theta-1]^{n-1}[n\theta-n]^{\theta-2},$$

which, since each side of the equation is $= [n\theta-1]^{\theta+n-3}$, is obviously true.

The foregoing differential equation is developable in the form

$$\left\{\alpha_0 + \alpha_1 u\frac{d}{du} + \alpha_2 u^2\left(\frac{d}{du}\right)^2 \dots + \alpha_{n-1}u^{n-1}\left(\frac{d}{du}\right)^{n-1}\right\} y = \frac{1}{na}\left(\frac{d}{du}\right)^{n-1} y;$$

but to find the coefficients $\alpha_0, \alpha_1, \dots \alpha_{n-1}$, I start from this form, and proceed to substitute in the equation the value of y, which on the left-hand side I use in the original form, and on the right-hand side in the form obtained by writing $\theta+1$ in the place of θ, viz.

$$y = S_\theta \frac{[n(\theta+1)]^\theta}{[\theta+1]^{\theta+1}} \alpha^{\theta+1} u^{(n-1)\theta+n}.$$

The equation to be satisfied is

$$\frac{[n\theta]^{\theta-1}}{[\theta]^{\theta}}\left(\alpha_0+\alpha_1[(n-1)\theta+1]^1+\alpha_2[(n-1)\theta+1]^2\ldots+\alpha_{n-1}[(n-1)\theta+1]^{n-1}\right)$$
$$=\frac{1}{n}\frac{[n(\theta+1)]^{\theta}}{[\theta+1]^{\theta+1}}[(n-1)\theta+n]^{n-1},$$

or, what is the same thing,

$$\frac{1}{[\theta]^{\theta}}\left(\alpha_0[n\theta]^{\theta-1}+\alpha_1[n\theta]^{\theta}+\alpha_2[n\theta]^{\theta+1}\ldots+\alpha_{n-1}[n\theta]^{\theta+n-2}\right)=\frac{1}{n}\frac{[n(\theta+1)]^{\theta+n-1}}{[\theta+1]^{\theta+1}}.$$

Observing that the right-hand side may be written

$$\frac{1}{n}\cdot\frac{n(\theta+1)[n\theta+n-1]^{\theta+n-2}}{(\theta+1)[\theta]^{\theta}},$$

the equation becomes

$$\alpha_0[n\theta]^{\theta-1}+\alpha_1[n\theta]^{\theta}+\alpha_2[n\theta]^{\theta+1}\ldots+\alpha_{n-1}[n\theta]^{\theta+n-2}=[n\theta+n-1]^{\theta+n-2},$$

or, what is the same thing,

$$\alpha_0+\alpha_1[(n-1)\theta+1]^1+\alpha_2[(n-1)\theta+1]^2\ldots+\alpha_{n-1}[(n-1)\theta+1]^{n-1}=[n\theta+n-1]^{n-1};$$

so that $\alpha_0, \alpha_1, \ldots \alpha_{n-1}$ are the coefficients of the expansion of $[n\theta+n-1]^{n-1}$ (which is a rational and integral function of θ, of the degree $n-1$) in a factorial series, as shown by the left-hand side of the equation.

To determine the actual values, write

$$(n-1)\theta+1=\phi,$$

this gives

$$n\theta+n-1=\frac{n\phi+n^2-3n+1}{n-1};$$

and we have therefore

$$\left[\frac{n\phi+n^2-3n+1}{n-1}\right]^{n-1}=\alpha_0+\alpha_1[\phi]^1+\alpha_2[\phi]^2\ldots+\alpha_{n-1}[\phi]^{n-1};$$

and thus the general expression is

$$\alpha_s=\frac{1}{[s]^s}\Delta^s\left(\frac{n\phi+n^2-3n+1}{n-1}\right),$$

where Δ denotes the difference in regard to ϕ ($\Delta U_\phi=U_{\phi+1}-U_\phi$), and, after the operation Δ^s is performed, ϕ is to be put equal to zero.

464.

NOTE ON PLANA'S LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXIII. (1862—1863), pp. 211—215.]

I HAVE been much surprised to find that there is an error of the order $m^2\gamma^4$, arising from the omission of a factor $(1+\gamma^2)^{-1}$, in the expression for $\frac{d^2\delta u}{dv^2}+\delta u$, as given by the equation (II.)′ (*Théorie de la Lune*, t. I., p. 267), being the equation made use of in the theory for the determination of δu, the perturbation of the reciprocal of the radius vector. This error may probably be the cause of some of the discrepancies in the terms of the fourth and higher orders, between Plana's results and those of Pontécoulant and Delaunay.

Plana's equation (6), t. I., p. 260, is

$$\begin{aligned}\frac{d^2\delta u}{dv^2}+\delta u = \quad & \mathrm{a}R'' \\ & +f(e,\gamma)\,Q'e\cos\left(cv-\int\varpi dv\right) \\ & -\left\{f(e,\gamma)(1+\gamma^2)(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}(1+s^2)^{-\frac{3}{2}}}{\mathrm{a}_{,}\psi(e,\gamma)}\right\} \\ & +f(e,\gamma)\,P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta,\end{aligned}$$

if for shortness

$$\Theta=\tfrac{3}{8}\gamma^2-(1+\tfrac{1}{2}\gamma^2)\cos\left(2gv-2\int\theta dv\right)+\tfrac{1}{8}\gamma^2\cos\left(4gv-4\int\theta dv\right).$$

R'' (p. 256) should be

$$R''=\frac{1}{\sigma\mathrm{a}_{,}\psi(e,\gamma)}\,\frac{\Omega_2-\frac{du}{dv}\frac{u^2dv}{d\Omega}-\sigma(1+s^2)^{-\frac{3}{2}}\,2\int Udv}{1+2\int Udv},$$

but, by an error which is implicitly corrected, the σ which multiplies $(1+s^2)^{-\frac{3}{2}}\,2\int Udv$ is omitted. Hence the equation (6) becomes

$$\begin{aligned}(1+2\int Udv)\left(\frac{d^2\delta u}{dv^2}+\delta u\right)=&\frac{\mathrm{a}}{\sigma\mathrm{a}_{,}\,\psi(e,\gamma)}\left\{\Omega_2-\frac{du}{dv}\frac{d\Omega}{u^2dv}-\sigma(1+s^2)^{-\frac{3}{2}}\,2\int Udv\right\}\\&+(1+2\int Udv)f(e,\gamma)\,Q'e\cos\left(cv-\int\varpi dv\right)\\&-(1+2\int Udv)\left\{f(e,\gamma)(1+\gamma^2)(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}(1+s^2)^{-\frac{3}{2}}}{\mathrm{a}_{,}\psi(e,\gamma)}\right\}\\&+(1+2\int Udv)f(e,\gamma)\,P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta,\end{aligned}$$

in which equation

$$\Omega_2=\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds},\ \text{pp. 26, 245,}\quad \frac{d\Omega}{u^2dv}=\frac{\sigma\mathrm{a}_{,}}{\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}}\,U,\ \text{p. 265,}$$

$$\psi(e,\gamma)=\lambda^{-\frac{2}{3}}(1+\gamma^2)^{-\frac{1}{3}},\quad f(e,\gamma)=\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{2}{3}},\ \text{p. 261.}$$

But retaining for greater convenience the function $f(e,\gamma)$ in two of the terms, we have

$$\begin{aligned}(1+2\int Udv)\left(\frac{d^2\delta u}{dv^2}+\delta u\right)=&\\&\frac{\mathrm{a}}{\sigma\mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left\{\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds}-\frac{\sigma\mathrm{a}_{,}}{\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}}U\frac{du}{dv}-\sigma(1+s^2)^{-\frac{3}{2}}\,2\int Udv\right\}\\&+(1+2\int Udv)f(e,\gamma)\,Q'e\cos\left(cv-\int\varpi dv\right)\\&-(1+2\int Udv)\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\}\\&+(1+2\int Udv)f(e,\gamma)\,P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta\\=&\frac{\mathrm{a}}{\sigma\mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left(\frac{d\Omega}{du}+\frac{s}{u}\frac{d\Omega}{ds}\right)\\&-\mathrm{a}U\frac{du}{dv}\\&-\frac{\mathrm{a}}{\mathrm{a}_{,}}\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}(1+s^2)^{-\frac{3}{2}}\,2\int Udv\\&+(1+2\int Udv)f(e,\gamma)\,Q'e\cos\left(cv-\int\varpi dv\right)\\&-(1+2\int Udv)\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{1}{3}}\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\}\\&+(1+2\int Udv)f(e,\gamma)\,P\gamma^2(1+s_{,}^2)^{-\frac{3}{2}}\,\Theta,\end{aligned}$$

which is

$$
\begin{aligned}
= \quad & \frac{\mathrm{a}}{\sigma \mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left(\frac{d\Omega}{du} + \frac{s}{u}\frac{d\Omega}{dv}\right) \\
& - \mathrm{a} U \frac{du}{dv} \\
& - \frac{\mathrm{a}}{\mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s^2)^{-\frac{3}{2}} 2 \int U dv \text{ (destroyed by term } \textit{infrà}) \\
& + (1 + 2\int U dv) f(e, \gamma) Q' e \cos\left(cv - \int \varpi dv\right) \\
& - \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left\{(1+s_{,}^2)^{-\frac{3}{2}} - \frac{\mathrm{a}}{\mathrm{a}_{,}} (1+s^2)^{-\frac{3}{2}}\right\} \\
& - \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s_{,}^2)^{-\frac{3}{2}} 2 \int U dv \\
& + \frac{\mathrm{a}}{\mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} (1+s^2)^{-\frac{3}{2}} 2 \int U dv \text{ (destroyed by term } \textit{suprà}) \\
& + (1 + 2\int U dv) f(e, \gamma) P \gamma^2 (1+s_{,}^2)^{-\frac{3}{2}} \Theta;
\end{aligned}
$$

or, putting $u = \frac{1}{\mathrm{a}}(u_{,} + \delta u)$, this becomes

$$
\begin{aligned}
(1 + 2\int U dv)\left(\frac{d^2\delta u}{dv^2} + \delta u\right) = \quad & \frac{\mathrm{a}}{\sigma \mathrm{a}_{,}} \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left(\frac{d\Omega}{du} + \frac{s}{u}\frac{d\Omega}{dv}\right) \\
& - U \left(\frac{du_{,}}{dv} + \frac{d\delta u}{dv}\right) \\
& - \lambda^{\frac{2}{3}} (1+\gamma)^{\frac{4}{3}} (1+s_{,}^2)^{-\frac{3}{2}} 2 \int U dv \\
& + (1 + 2\int U dv) f(e, \gamma) Q' e \cos\left(cv - \int \varpi dv\right) \\
& - \lambda^{\frac{2}{3}} (1+\gamma^2)^{\frac{4}{3}} \left\{(1+s_{,}^2)^{-\frac{3}{2}} - \frac{\mathrm{a}}{\mathrm{a}_{,}} (1+s^2)^{-\frac{3}{2}}\right\} \\
& + (1 + 2\int U dv) f(e, \gamma) P \gamma^2 (1+s_{,}^2)^{-\frac{3}{2}} \Theta,
\end{aligned}
$$

agreeing with the Formula II. p. 265, except that in Plana's last term, instead of the factor $f(e, \gamma)\left(= \lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}\right)$, we have the factor $\lambda^{\frac{2}{3}}(1+\gamma^2)^{\frac{4}{3}}$. That is, the last term, as given by Plana, should be divided by $1+\gamma^2$. And this error is introduced

from the formula (II.) into the formula (II.)′, p. 267, viz. the incorrect factor $\lambda^{\frac{3}{2}}(1+\gamma^2)^{\frac{1}{2}}$ is there replaced by its value q; whereas, the true value being $\lambda^{\frac{3}{2}}(1+\gamma^2)^{\frac{1}{4}}$, the factor in (II.)′ should be $=\dfrac{q}{1+\gamma^2}$.

The corrected formula (II.)′ is

$$
\begin{aligned}
-\frac{d^2\delta u}{dv^2}-\delta u = & -Q'\frac{qe}{1+\gamma^2}\cos\left(cv-\int\varpi dv\right) \\
& +q\left\{(1+s_{,}^2)^{-\frac{3}{2}}-\frac{\mathrm{a}}{\mathrm{a}_{,}}(1+s^2)^{-\frac{3}{2}}\right\} \\
& +\mu^2(R_4+R_5)-\mu^2\left(\frac{du_{,}}{dv}+\frac{d\delta u}{dv}\right)R_{,} \\
& -2\mu^2\left\{\frac{d^2\delta u}{dv^2}+\delta u+q(1+s_{,}^2)^{-\frac{3}{2}}-\frac{Q'qe}{1+\gamma^2}\cos\left(cv-\int\varpi dv\right)\right\}\int R_{,}dv \\
& -Pq\gamma^3(1+\gamma^2)^{-1}(1+s_{,}^2)^{-\frac{3}{2}}\left(1-2\mu^2\int R_{,}dv\right)\times \\
& \left\{\tfrac{3}{8}\gamma^2-(1+\tfrac{1}{2}\gamma^2)\cos\left(2gv-2\int\theta dv\right)+\tfrac{1}{8}\gamma^2\cos\left(4gv-4\int\theta dv\right)\right\}.
\end{aligned}
$$

Observing that P is of the order m^2, and that q is approximately equal to unity, the error in $\dfrac{d^2\delta u}{dv^2}+\delta u$ is of the order $m^2\gamma^4$, as noticed above. It may be right to mention that I obtained the correction in the first instance by starting from the fundamental equations, and not as here from the intermediate equation (6), so that there is not in that equation any error afterwards implicitly corrected in the transformation to (II.)′.

465.

NOTE ON THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXV. (1864—1865), pp. 182—189.]

I ATTEND, in the expressions for the lunar coordinates, only to the coefficients independent of m. Plana's values, taken to the fourth order only, are as follows; for greater simplicity I write $a=1$; and, instead of $nt+$ constant, $cnt+$ constant, $gnt+$ constant, I write l, c, g respectively; viz., l is the mean longitude, c the mean anomaly, g the mean distance from node: this being so, then r, v, y, denoting the radius vector longitude and latitude respectively, we have

$$\frac{1}{r}\,(\text{Plana}) =$$

$$\begin{array}{rlll}
 & e - \frac{1}{8}e^3 - \frac{1}{4}\gamma^2 e & \cos & c \\
+ & e^2 - \frac{1}{3}e^4 - \frac{1}{2}\gamma^2 e^2 & ,, & 2c \\
+ & \frac{9}{8}e^3 & ,, & 3c \\
+ & \frac{4}{3}e^4 & ,, & 4c \\
- & \frac{5}{4}\gamma^2 e^2 & ,, & 2g \\
- & \frac{5}{8}\gamma^2 e & ,, & c-2g
\end{array}$$

(but I omit Plana's term $+\frac{1}{8}\gamma^2 e^2$ $\cos$ $2c+2g$ which should be $=0$).

$$v\ (\text{Plana}) = l +$$

$$\begin{array}{rlll}
+ & 2\ e - \frac{1}{4}e^3 - \frac{1}{2}\gamma^2 e & \sin & c \\
+ & \frac{5}{4}e^3 - \frac{11}{24}e^4 - \frac{15}{16}\gamma^2 e^2 & ,, & 2c \\
+ & \frac{13}{12}e^3 & ,, & 3c \\
+ & \frac{103}{96}e^4 & ,, & 4c \\
- & \frac{1}{4}\gamma^2 - \frac{9}{16}\gamma^2 e^2 + \frac{1}{8}\gamma^4 & ,, & 2g \\
+ & \frac{3}{4}\gamma^2 e & ,, & c-2g
\end{array}$$

$-\frac{1}{2}\gamma^2 e$	sin	$c+2g$
$-\frac{1}{8}\gamma^2 e^2$	„	$2c-2g$
$-\frac{13}{16}\gamma^2 e^2$	„	$2c+2g$
$+\frac{1}{32}\gamma^4$	„	$4g.$

y (Plana) =

$\gamma - \gamma e^2 - \frac{3}{8}\gamma^3$	sin	g
$+\gamma e - \frac{5}{8}\gamma e^3$	„	$c-g$
$+\gamma e - \frac{5}{4}\gamma e^2 - \frac{5}{8}\gamma^3 e$	„	$c+g$
$+\frac{3}{4}\gamma e^2$	„	$2c-g$
$+\frac{9}{8}\gamma e^2$	„	$2c+g$
$+\frac{17}{24}\gamma e^3$	„	$3c-g$
$+\frac{4}{3}\gamma e^3$	„	$3c+g$
$-\frac{1}{24}\gamma^3$	„	$3g$
$+\frac{1}{2}\gamma^3 e$	„	$c-3g$
$-\frac{1}{8}\gamma^3 e$	„	$c+3g.$

To compare these with the elliptic values, it is necessary to write $e(1+\frac{1}{4}\gamma^2)$ in place of e. Making this change, or say reducing Plana's (e, γ) to the elliptic (e, γ), I write down in a first column the transformed coefficients, and in a second column the elliptic coefficients, as follows:

Plana, with Elliptic e, γ	Elliptic		
$\frac{1}{r}=$	$\frac{1}{r}=$		
1	1	cos	c
$+e-\frac{1}{8}e^3$	$+e-\frac{1}{8}e^3$	„	$2c$
$+e^2-\frac{1}{3}e^4$	$+e^2-\frac{1}{3}e^4$	„	$3c$
$+\frac{9}{8}e^3$	$+\frac{9}{8}e^3$	„	$4c$
$+\frac{4}{3}e^4$	$+\frac{4}{3}e^4$	„	$2g$
$-\frac{5}{4}\gamma^2 e^2$	0	„	$c-2g.$
$-\frac{5}{8}\gamma^2 e$	0		

Plana, with Elliptic e, γ	Elliptic		
$v=$	$v=$		
l	l	sin	c
$+2e-\frac{1}{4}e^3$	$+2e-\frac{1}{4}e^3$	„	$2c$
$+\frac{5}{4}e^2-\frac{11}{24}e^4-\frac{5}{16}\gamma^2 e^2$	$+\frac{5}{4}e^2-\frac{11}{24}e^4$	„	$3c$
$+\frac{13}{12}e^3$	$+\frac{13}{12}e^3$	„	$4c$
$+\frac{103}{96}e^4$	$+\frac{103}{96}e^4$	„	$2g$
$-\frac{1}{4}\gamma^2-\frac{9}{16}\gamma^2 e^2+\frac{1}{8}\gamma^4$	$-\frac{1}{4}\gamma^2+\gamma^2 e^2+\frac{1}{8}\gamma^4$	„	$c-2g$
$+\frac{3}{4}\gamma^2 e$	$-\frac{1}{2}\gamma^2 e$	„	$c+2g$
$-\frac{1}{2}\gamma^2 e$	$-\frac{1}{2}\gamma^2 e$	„	$2c-2g$
$-\frac{1}{8}\gamma^2 e^2$	$+\frac{3}{16}\gamma^2 e^2$	„	$2c+2g$
$-\frac{13}{16}\gamma^2 e^2$	$-\frac{13}{16}\gamma^2 e^2$	„	$4g.$
$+\frac{1}{32}\gamma^4$	$+\frac{1}{32}\gamma^4$		

Plana, with Elliptic e, γ	Elliptic		
$y =$	$y =$		
$\gamma - \gamma e^2 - \frac{3}{8}\gamma^3$	$+ \gamma e - \gamma e^2 - \frac{3}{8}\gamma^3$	sin	g
$+ \gamma e - \frac{5}{4}\gamma e^3 - \frac{3}{8}\gamma^3 e$	$+ \gamma e - \frac{5}{4}\gamma e^3 - \frac{3}{8}\gamma^3 e$	,,	$c + g$
$+ \gamma e - \frac{5}{8}\gamma e^3 + \frac{1}{4}\gamma^3 e$	$+ \gamma e \quad - \frac{3}{8}\gamma^3 e$	,,	$c - g$
$+ \frac{3}{4}\gamma e^2$	$+ \frac{1}{8}\gamma e^2$	,,	$2c - g$
$+ \frac{9}{8}\gamma e^2$	$+ \frac{9}{8}\gamma e^2$	,,	$2c + g$
$+ \frac{17}{24}\gamma e^3$	$+ \frac{1}{12}\gamma e^3$	,,	$3c - g$
$+ \frac{4}{3}\gamma e^3$	$+ \frac{4}{3}\gamma e^3$	,,	$3c + g$
$- \frac{1}{24}\gamma^3$	$- \frac{1}{24}\gamma^3$	,,	$3g$
$+ \frac{1}{2}\gamma^3 e$	$- \frac{1}{8}\gamma^3 e$	,,	$c - 3g$
$- \frac{1}{8}\gamma^3 e$	$- \frac{1}{8}\gamma^3 e$	,,	$c + 3g$,

where, for greater clearness, I remark that the values called "elliptic" of e, γ, c, g, refer to an ellipse, such that the longitude of the node, and the longitude (in orbit) of the pericentre, vary uniformly with the time,—viz., we have mean distance $= 1$, excentricity $= e$, tangent of inclination $= \gamma$, mean longitude $= l$, mean anomaly $= c$, distance from node $= g$.

We have therefore

$$\begin{array}{lll}
\delta\dfrac{1}{r} = & -\frac{5}{4}\gamma^2 e^2 & \cos \quad 2g \\
 & -\frac{5}{8}\gamma^2 e & ,, \quad c - 2g \\
\delta v = & -\frac{5}{16}\gamma^2 e^2 & \sin \quad 2c \\
 & -\frac{25}{16}\gamma^2 e^2 & ,, \quad 2g \\
 & +\frac{5}{4}\gamma^2 e & ,, \quad c - 2g \\
 & -\frac{5}{16}\gamma^2 e^2 & ,, \quad 2c - 2g \\
\delta y = & -\frac{5}{8}\gamma e^3 + \frac{5}{8}\gamma^3 e & ,, \quad c - g \\
 & +\frac{5}{8}\gamma e^2 & ,, \quad 2c - g \\
 & +\frac{5}{8}\gamma e^3 & ,, \quad 3c - g \\
 & +\frac{5}{8}\gamma^3 e & ,, \quad c - 3g,
\end{array}$$

viz., these are the increments to be added to the elliptic values of $\frac{1}{r}$, v, y, respectively, in order to obtain the disturbed values of $\frac{1}{r}$, v, y, attending only to the coefficients independent of m; *they represent, in fact, the lunar inequalities which rise two orders by integration.*

The elliptic values of $\frac{1}{r}$ and y are functions, and that of v, is equal $l +$, a function, of e, γ, c, g, and the foregoing disturbed values may be obtained by affecting each of

the quantities e, γ, c, g, and l, with an inequality depending on the argument $2c-2g$, viz., these inequalities are

$$\begin{aligned}
\delta e &= -\tfrac{5}{8}\,\gamma^2 e \cos 2c-2g \\
\delta c &= \tfrac{5}{8}\,\gamma^2 \sin 2c-2g \\
\delta\gamma &= \tfrac{5}{8}\,\gamma e^2 \cos 2c-2g \\
\delta g &= \tfrac{5}{8}\,e^2 \sin 2c-2g \\
\delta l &= -\tfrac{5}{16}\,\gamma^2 e^2 \sin 2c-2g.
\end{aligned}$$

The verification may be effected without difficulty; thus, for instance, starting from the elliptic value of $\frac{1}{r}$, we have to the fourth order

$$\begin{aligned}
\delta\frac{1}{r}=\delta\begin{pmatrix} e^2\cos c \\ +e\cos 2c\end{pmatrix} &= \begin{pmatrix} -e\sin c \\ -2e^2\sin 2c\end{pmatrix}\delta c+\begin{pmatrix}\cos c \\ +2e\cos 2c\end{pmatrix}\delta e \\
&= \tfrac{5}{8}\gamma^2 e\,(-\sin c\sin 2c-2g-\cos c\cos 2c-2g) \\
&\quad +\tfrac{5}{4}\gamma^2 e^2(-\sin 2c\sin 2c-2g-\cos 2c\cos 2c-2g) \\
&= -\tfrac{5}{8}\gamma^2 e\cos c-g \\
&\quad -\tfrac{5}{4}\gamma^2 e^2\cos 2g,
\end{aligned}$$

which is right; and the verification of the values of δv, δy, may be effected in a similar manner.

I have, in order to fix the ideas, preferred to give in the first instance the foregoing *à posteriori* proof; but I now inquire generally as to the form of the values of $\frac{1}{r}$, v, y, or say of r, v, y, taking account only of coefficients independent of m; and I proceed to show that these may be obtained from the elliptic values expressed as above in terms of l, e, γ, c, g, by affecting l, e, γ, c, g, each with an inequality depending on the multiple sines or cosines of $c-g$.

Writing for greater simplicity $n=1$, we have $l=t+L$, $c=\mathrm{c}t+C$, $g=\mathrm{g}t+G$, where $\mathrm{c}=1-\frac{3}{4}m^2+\&\text{c.}$, $\mathrm{g}=1+\frac{3}{4}m^2+\&\text{c.}$; viz., c, g, are constants which differ from unity by terms involving m^2.

The required values of r, v, y, satisfy the *undisturbed* equations of motion, if after the differentiations we write in the coefficients (which coefficients are functions of m through c, g) $m=0$; that is, if we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$. In fact, the required values of r, v, y, are what the complete values become, upon writing in the coefficients of the complete values $m=0$; that is, the required values of r, v, y, differ from the complete values by terms the coefficients whereof contain m as a factor; and the disturbed equations differ from the undisturbed equations in that they contain the differential coefficients of the disturbing function; that is, terms the coefficients whereof have the factor m^2. Imagine the complete values of r, v, y, substituted in the disturbed equations of motion; the resulting equations are satisfied identically; and, therefore, whatever be the value of m; that is, they are satisfied if in these equations respectively

we write $m=0$: it requires a little consideration to see that this is so, if *in the coefficients only* we write $m=0$; but recollecting that c, g, stand for functions $\mathrm{c}t+C$, $\mathrm{g}t+G$, so that, for example, $c-g$, $=(\mathrm{c}-\mathrm{g})t+C-G$, upon writing therein $m=0$, becomes equal, not to zero, but to the constant value $C-G$, the identity subsists in regard to the coefficient of the sine or cosine of each separate argument $\alpha c+\beta g$, and, consequently, it subsists notwithstanding that in the arguments c and g, instead of being each put $=1$, are left indeterminate. And granting this (viz. that the equations are satisfied if *in the coefficients only* we write $m=0$), then it is clear that, as above stated, the required values of r, v, y, satisfy the undisturbed equations of motion, if after the differentiations we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$.

The required values of r, v, y, are of the form $r=\phi(c, g)$, $y=\psi(c, g)$, $v=l+\chi(c, g)$, but writing $w=v+c-l$, $=c+\chi(c, g)$, the last mentioned property will equally subsist in regard to the functions r, w, y: in fact, v enters into the differential equations only through its differential coefficient $\frac{dv}{dt}$, and the differential coefficients of v and w, that is, of $l+\chi(c, g)$ and $c+\chi(c, g)$, differ only by the quantity $\mathrm{c}-1$, which becomes $=0$, in virtue of the assumed relations $\mathrm{c}=1$, $\mathrm{g}=1$.

Hence the undisturbed equations are satisfied by the values $r=\phi(c, g)$, $y=\psi(c, g)$, $w=c+\chi(c, g)$, when after the differentiations we write in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$; the foregoing values contain t through the quantities c, g, only; and we have, therefore, $\frac{d}{dt}=\mathrm{c}\frac{d}{dc}+\mathrm{g}\frac{d}{dg}$.

Hence, writing in the coefficients $\mathrm{c}=1$, $\mathrm{g}=1$, we have $\frac{d}{dt}=\frac{d}{dc}+\frac{d}{dg}$; that is, the values $r=\phi(c, g)$, $y=\psi(c, g)$, $w=\chi(c, g)$, regarding r, v, y, as functions of c, g, satisfy the partial differential equations obtained from the undisturbed equations of motion by writing therein $\frac{d}{dc}+\frac{d}{dg}$ in place of $\frac{d}{dt}$. Hence also, considering r, w, y, as functions of c and $c-g$, then observing that $\left(\frac{d}{dc}+\frac{d}{dg}\right)(c-g)$ is $=0$, the values of r, v, y, satisfy the partial differential equations obtained by writing $\frac{d}{dc}$ in place of $\frac{d}{dt}$; and inasmuch as these partial differential equations do not contain $\frac{d}{dg}$, they are to be integrated as ordinary differential equations in regard to c as the independent variable, the constants of integration being replaced by arbitrary functions of $c-g$.

Consider the pure elliptic values of r, v, y, in an elliptic orbit with the following elements: A, the mean distance; N, the mean motion ($N^2A^3=1$ and therefore $A=N^{-\frac{2}{3}}$); E, the excentricity; $Nt+D$, the mean anomaly; $Nt+H$, the mean distance from node; $Nt+K$, the mean longitude; then writing c in place of t, we have

$$\begin{aligned} r &= N^{-\frac{2}{3}}\,\mathrm{elqr}\,(E,\ Nc+D),\\ v\,(=l-c+w) &= l-c+Nc+K+P\,(E,\ \Gamma,\ Nc+D,\ Nc+H),\\ y &= \qquad\qquad Q\,(E,\ \Gamma,\ Nc+D,\ Nc+H), \end{aligned}$$

where N, E, Γ, D, H, K, are arbitrary functions of $c-g$: P and Q denote given functional expressions. But, in order that r, v, y, considered as functions of c and g may be of the proper form, it is necessary as regards N to write simply $N=1$; we have then

$$\begin{aligned} r &= \text{elqr}\,(E,\ c+D), \\ v &= l+K+P\,(E,\ \Gamma,\ c+D,\ c+H), \\ y &= \qquad\quad Q\,(E,\ \Gamma,\ c+D,\ c+H), \end{aligned}$$

where E, Γ, D, H, K, are arbitrary functions of $c-g$; or, what is the same thing, writing for these quantities respectively $e+\delta e$, $\gamma+\delta\gamma$, δc, $g-c+\delta g$, δl, where δe, $\delta\gamma$, δc, δg, δl are arbitrary functions of $c-g$, we have

$$\begin{aligned} r &= \text{elqr}\,(e+\delta e,\ c+\delta c), \\ v &= l+\delta l+P\,(e+\delta e,\ \gamma+\delta\gamma,\ c+\delta c,\ g+\delta g), \\ y &= \qquad\quad Q\,(e+\delta e,\ \gamma+\delta\gamma,\ c+\delta c,\ g+\delta g), \end{aligned}$$

that is, the values of r, v, y, are obtained from the elliptic values

$$\begin{aligned} r &= \text{elqr}\,(e,\ c), \\ v &= l+P\,(e,\ \gamma,\ c,\ g), \\ y &= \quad Q\,(e,\ \gamma,\ c,\ g), \end{aligned}$$

by affecting each of the quantities e, γ, c, g, l, with an inequality which is a function of $c-g$.

466.

SECOND NOTE ON THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXV. (1864—1865), pp. 203—207.]

THE elliptic values of

r, the radius vector,
v, the longitude,
y, the latitude,

are functions of

a, the mean distance,
e, the excentricity,
γ, the tangent of the inclination,
l, the mean longitude,
c, the mean anomaly,
g, the mean distance from node;

see my Note in the last *Monthly Notice*, p. 182, [465], where, for the present purpose, $\frac{a}{r}$ should be written instead of $\frac{1}{r}$; and it is there shown that the disturbed values, attending only to the coefficients independent of m, are obtained by affecting a, e, γ, c, g, l, with the inequalities

$$\begin{aligned}
\delta a &= 0 \\
\delta e &= -\tfrac{5}{8}\gamma^2 e && \cos\ 2c-2g \\
\delta\gamma &= +\tfrac{5}{8}\gamma e^2 && \text{,,}\ 2c-2g \\
\delta c &= +\tfrac{5}{8}\gamma^2 && \sin\ 2c-2g \\
\delta g &= +\tfrac{5}{8}e^2 && \text{,,}\ 2c-2g \\
\delta l &= -\tfrac{5}{16}\gamma^2 e^2 && \text{,,}\ 2c-2g,
\end{aligned}$$

or, what is the same thing, adding to the elliptic values the inequalities

$$
\begin{array}{lll}
\delta\dfrac{a}{r} = -\tfrac{5}{4}\gamma^2e^2 & \cos & 2g \\
\qquad\quad -\tfrac{5}{8}\gamma^2e & \text{,,} & c-2g, \\
\delta v = -\tfrac{5}{16}\gamma^2e^2 & \sin & 2c \\
\qquad\quad -\tfrac{25}{16}\gamma^2e^2 & \text{,,} & 2g \\
\qquad\quad +\tfrac{5}{4}\gamma^2e & \text{,,} & c-2g \\
\qquad\quad -\tfrac{5}{16}\gamma^2e^2 & \text{,,} & 2c-2g, \\
\delta y = -\tfrac{5}{8}\gamma e^3+\tfrac{5}{8}\gamma^3e & \sin & c-g \\
\qquad\quad +\tfrac{5}{8}\gamma e^2 & \text{,,} & 2c-g \\
\qquad\quad +\tfrac{5}{8}\gamma e^3 & \text{,,} & 3c-g \\
\qquad\quad +\tfrac{5}{8}\gamma^3e & \text{,,} & c-3g.
\end{array}
$$

I propose to show how these results may be obtained by the method of the variation of the elements. For this purpose, treating a, e, γ, c, g, l, as elements, the proper formulæ are obtained very readily from those given in my "Memoir on the Problem of Disturbed Elliptic Motion," *Mem. R. Ast. Soc.*, vol. XXVII. (1859), pp. 1—29, [212]; viz., writing c in place of g, the formulæ, p. 25, give the variations of a, e, c, ϖ, θ, ϕ; we have then

$$
\begin{aligned}
g &= c+\varpi \\
l &= c+\varpi+\theta \\
\gamma &= \tan\phi,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
dg &= dc+d\varpi \\
dl &= dc+d\varpi+d\theta \\
d\gamma &= (1+\gamma^2)\,d\phi,
\end{aligned}
$$

which give for the transformation of the differential coefficients of Ω,

$$
\begin{aligned}
\frac{d\Omega}{dc} &= \frac{d\Omega}{dc}+\frac{d\Omega}{dg}+\frac{d\Omega}{dl} \\
\frac{d\Omega}{d\varpi} &= \qquad\frac{d\Omega}{dg}+\frac{d\Omega}{dl} \\
\frac{d\Omega}{d\theta} &= \qquad\qquad\frac{d\Omega}{dl} \\
\frac{d\Omega}{d\phi} &= \qquad(1+\gamma^2)\frac{d\Omega}{d\gamma}
\end{aligned}
$$

and the formulæ finally become

$$
\frac{da}{dt} = \frac{2}{na}\frac{d\Omega}{dc}+\qquad\frac{2}{na}\frac{d\Omega}{dg}+\qquad\frac{2}{na}\frac{d\Omega}{dl},
$$

$$
\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{d\Omega}{dc}+\frac{1-e^2-\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{dg}+\frac{1-e^2-\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{dl},
$$

$$\frac{d\gamma}{dt} = \frac{1+\gamma^2}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{dg} + \frac{(1+\gamma^2)(1-\sqrt{1+\gamma^2})}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{dl},$$

$$\frac{dc}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2}{na^2e}\frac{d\Omega}{de},$$

$$\frac{dg}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2-\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de} - \frac{1+\gamma^2}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{d\gamma},$$

$$\frac{dl}{dt} = -\frac{2}{na}\frac{d\Omega}{da} - \frac{1-e^2-\sqrt{1+e^2}}{na^2e}\frac{d\Omega}{de} - \frac{(1+\gamma^2)(1-\sqrt{1+\gamma^2})}{na^2\sqrt{1-e^2}\,\gamma}\frac{d\Omega}{d\gamma}.$$

The disturbing function contains the term

$$m^2n^2a^2\left(+\tfrac{15}{16}e^2\gamma^2\right)\quad\cos\quad 2c-2g.$$

If after the differentiations we write for greater simplicity $a=1$, $n=1$, we have

$$\frac{d\Omega}{da} = +\tfrac{15}{8}m^2e^2\gamma^2 \quad\cos\quad 2c-2g,$$

$$\frac{d\Omega}{de} = +\tfrac{15}{8}m^2e\,\gamma^2 \quad\text{,,}\quad 2c-2g,$$

$$\frac{d\Omega}{d\gamma} = +\tfrac{15}{8}m^2e^2\gamma \quad\text{,,}\quad 2c-2g,$$

$$\frac{d\Omega}{dc} = -\tfrac{15}{8}m^2e^2\gamma^2 \quad\sin\quad 2c-2g,$$

$$\frac{d\Omega}{dg} = -\tfrac{15}{8}m^2e^2\gamma^2 \quad\text{,,}\quad 2c-2g,$$

$$\frac{d\Omega}{dl} = 0,$$

and the formulæ for the variations give

$$\frac{da}{dt} = 2\left(\frac{d\Omega}{dc}+\frac{d\Omega}{dg}\right) = 0$$

$$\frac{de}{dt} = \frac{1}{e}\frac{d\Omega}{dc} = -\tfrac{15}{8}m^2e\gamma^2 \quad\sin\quad 2c-2g,$$

$$\frac{d\gamma}{dt} = \frac{1}{\gamma}\frac{d\Omega}{dg} = -\tfrac{15}{8}m^2e^2\gamma \quad\text{,,}\quad 2c-2g,$$

$$\frac{dc}{dt} = -\frac{1}{e}\frac{d\Omega}{de} = -\tfrac{15}{8}m^2\gamma^2 \quad\cos\quad 2c-2g,$$

$$\frac{dg}{dt} = -\frac{1}{\gamma}\frac{d\Omega}{d\gamma} = -\tfrac{15}{8}m^2e^2 \quad\text{,,}\quad 2c-2g,$$

$$\frac{dl}{dt} = -2\frac{d\Omega}{da} + \tfrac{1}{2}e\frac{d\Omega}{de} + \tfrac{1}{2}\gamma\frac{d\Omega}{d\gamma} = \left(-\tfrac{15}{4}+\tfrac{15}{16}+\tfrac{15}{16}=\right) -\tfrac{15}{8}m^2e^2\gamma^2 \quad\text{,,}\quad 2c-2g,$$

but this value of $\frac{dl}{dt}$ is, as will presently be seen, incomplete.

Writing $a+\delta a$, $e+\delta e$, &c., in place of a, e, &c., and observing that the divisor for the integration of the term in $2c-2g$ is $2(c-g)$, $=-3m^2$, the first five equations give respectively

$$\begin{aligned}
\delta a &= \quad 0, \\
\delta e &= -\tfrac{5}{8}\gamma^2 e \quad \cos \quad 2c-2g, \\
\delta\gamma &= +\tfrac{5}{8}\gamma e^2 \quad \text{,,} \quad 2c-2g, \\
\delta c &= +\tfrac{5}{8}\gamma^2 \quad \sin \quad 2c-2g, \\
\delta g &= +\tfrac{5}{8}c^2 \quad \text{,,} \quad 2c-2g.
\end{aligned}$$

The constant term in Ω is

$$= m^2n^2a^2\left(\tfrac{1}{4}+\tfrac{3}{8}e^2-\tfrac{3}{8}\gamma^2\right),$$

and this gives in

$$\frac{dl}{dt}, = -2\frac{d\Omega}{da}+\tfrac{1}{2}e\frac{d\Omega}{de}+\tfrac{1}{2}\gamma\frac{d\Omega}{d\gamma},$$

a term

$$\begin{aligned}
m^2(&-1-\tfrac{3}{2}e^2+\tfrac{3}{2}\gamma^2 \\
&+\tfrac{3}{8}e^2-\tfrac{3}{8}\gamma^2),
\end{aligned}$$

which is

$$= m^2\left(-1-\tfrac{9}{8}e^2+\tfrac{9}{8}\gamma^2\right).$$

Substituting for e, γ, their correct values $e+\delta e$, $\gamma+\delta\gamma$, it appears that $\frac{dl}{dt}$ contains the term

$$m^2\left(-\tfrac{9}{4}e\delta e+\tfrac{9}{4}\gamma\delta\gamma\right),$$

which is

$$\begin{aligned}
&= m^2\left(\tfrac{45}{32}+\tfrac{45}{32}=\right)\tfrac{45}{16} \quad e^2\gamma^2 \quad \cos \quad 2c-2g, \\
&= \qquad\qquad \tfrac{45}{16}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,
\end{aligned}$$

and joining to this the before-mentioned term

$$= \qquad -\tfrac{15}{8}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,$$

we find

$$\frac{dl}{dt} = \quad \left(\tfrac{45}{16}-\tfrac{15}{8}=\right)\tfrac{15}{16}m^2e^2\gamma^2 \quad \text{,,} \quad 2c-2g,$$

whence, writing as above $l+\delta l$ for l, and integrating, we have

$$\delta l = \qquad -\tfrac{5}{16} \quad e^2\gamma^2 \quad \sin \quad 2c-2g,$$

and it thus appears that the values of δa, δe, $\delta\gamma$, δc, δg, δl, agree with those obtained in my former Note.

467.

EXPRESSIONS FOR PLANA'S e, γ IN TERMS OF THE ELLIPTIC e, γ.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXV. (1864—1865), pp. 265—271.]

THE coefficient of $\sin cnt$ in Plana's expression for the true longitude v (see Plana, t. I. p. 574), putting therein $E' = \epsilon' = e'$, that is, neglecting the terms which depend on the variation of the solar excentricity, is

$$
\begin{aligned}
= \; & e \left(2 + \tfrac{3}{2} m^2 - \tfrac{75}{64} m^3 - \tfrac{6659}{256} m^4 - \tfrac{48843375}{36864} m^5 - \tfrac{657568819}{147456} m^6 \right) \\
& + e^3 \left(-\tfrac{1}{4} - 17 m^2 - \tfrac{3195}{32} m^3 - \tfrac{463359}{7680} m^4 \right) \\
& + e^5 \left(\tfrac{5}{96} + \tfrac{66863}{6144} m^2 \right) \\
& + e^7 \left(\tfrac{5921}{161280} \right) \\
& + e\gamma^2 \left(-\tfrac{1}{2} - \tfrac{63}{32} m^2 + \tfrac{1467}{256} m^3 + \tfrac{22857}{512} m^4 \right) \\
& + e^3\gamma^2 \left(\tfrac{23}{16} - \tfrac{405}{128} m + \tfrac{16029}{512} m^2 \right) \\
& + e^5\gamma^2 \left(-\tfrac{195}{128} \right) \\
& + e\gamma^4 \left(-\tfrac{3}{8} + \tfrac{135}{256} m + \tfrac{3749}{2048} m^2 \right) \\
& + e^3\gamma^4 \left(\tfrac{1761}{1280} \right) \\
& + e\gamma^6 \left(-\tfrac{5}{16} \right) \\
& + ee'^2 \left(\left(-\tfrac{45}{4} + \tfrac{9}{4} = \right) - 9m^2 + \left(-\tfrac{6455}{64} + \tfrac{165}{42} = \right) - \tfrac{6125}{64} m^3 \right. \\
& \qquad \left. + \left(-\tfrac{281095}{512} - \tfrac{147}{256} = \right) - \tfrac{281382}{512} m^4 \right) \\
& + e^3e'^2 \left(\left(-\tfrac{12831}{480} - \tfrac{33}{2} = \right) - \tfrac{20751}{480} m^2 \right) \\
& + ee^2\gamma^2 \left(\left(\tfrac{5171}{128} - \tfrac{453}{128} = \right) + \tfrac{2359}{64} m^2 \right) \\
& + ee'^4 \left(\left(\tfrac{45}{16} - \tfrac{2025}{64} = \right) - \tfrac{1845}{64} m^2 \right) \\
& + eb^4 \left(-\tfrac{135}{32} m^2 \right) \\
& + ee'^2b^4 \left(-\tfrac{75}{16} \right).
\end{aligned}
$$

Taking this to the fifth order only, and comparing it with the coefficient in the elliptic theory, we have

Plana.		Elliptic.
$= e\ (2 - \frac{3}{2}m^2 - \frac{75}{64}m^3 - \frac{6659}{256}m^4)$	$=$	$e\ (2)$
$+ e^3\ (-\frac{1}{4} - 17m^2)$		$+ e^3 (-\frac{1}{4})$
$+ e^5\ (\frac{5}{96})$		$+ e^5 (\frac{5}{96})$
$+ e\gamma^2\ (-\frac{1}{2} - \frac{63}{32}m^2)$		
$+ e^3\gamma^2\ (\frac{23}{16})$		
$+ e\gamma^4\ (-\frac{3}{8})$		
$+ ee'^2\ (-9m^2)$.		

The coefficient of $\sin gnt$ in Plana's expression for the latitude (see t. I. p. 704) is

$$
\begin{aligned}
= \ & \gamma\ (1 + \tfrac{33}{128}m^3 + \tfrac{241}{512}m^4 - \tfrac{82495}{24576}m^5) \\
& + \gamma e^2\ (-1 - \tfrac{31}{512}m^2 - \tfrac{7977}{256}m^3) \\
& + \gamma e^4\ (\tfrac{5}{64} + \tfrac{245}{512}m) \\
& + \gamma^3\ (-\tfrac{3}{8} + \tfrac{5}{128}m^2 + \tfrac{69}{256}m^3) \\
& + \gamma^3 e^2\ (\tfrac{7}{32} - \tfrac{405}{256}m) \\
& + \gamma^5\ (\tfrac{1}{4}) \\
& + \gamma e'^2\ (\tfrac{27}{8}m^2 - \tfrac{113}{128}m^3).
\end{aligned}
$$

But according to the calculation of Prof. Adams (quoted by M. Delaunay, *Comptes Rendus*, t. LIV. (1862), this should be

$$
\begin{aligned}
= \ & \gamma\ (1 + \tfrac{33}{128}m^3 - \tfrac{1}{512}m^4 - \tfrac{82497}{24576}m^5 - \tfrac{4801697}{294012}m^6) \\
& + \gamma e^2\ (-1 - \tfrac{1111}{256}m^2 - \tfrac{7977}{256}m^3) \\
& + \gamma e^4\ (\tfrac{3}{16} - \tfrac{135}{512}m) \\
& + \gamma^3\ (-\tfrac{3}{8} + \tfrac{5}{128}m^2 - \tfrac{15}{128}m^3) \\
& + \gamma^3 e^2\ (\tfrac{23}{32} + \tfrac{135}{256}m) \\
& + \gamma^5\ (\tfrac{15}{64}) \\
& + \gamma e'^2\ (\tfrac{9}{8}m^2 - \tfrac{113}{128}m^3 + \tfrac{3521}{1024}m^4).
\end{aligned}
$$

Adopting this as the true expression according to Plana's theory, taking it to the fifth order only, and comparing with the elliptic value of the same coefficient, we have

Plana.		Elliptic.
$\gamma\ (1 + \frac{33}{128}m^3 - \frac{1}{512}m^4)$	$=$	$\gamma\ (1)$
$+ \gamma e^2\ (-1 - \frac{1111}{256}m^2)$		$+ \gamma e^2\ (-1)$
$+ \gamma e^4\ (\frac{3}{16})$		$+ \gamma e^4\ (\frac{7}{64})$
$+ \gamma^3\ (-\frac{3}{8} + \frac{5}{128}m^2)$		$+ \gamma^3\ (-\frac{3}{8})$
$+ \gamma^3 e^2\ (\frac{23}{32})$		$+ \gamma^3 e^2\ (\frac{3}{8})$
$+ \gamma^5\ (\frac{15}{64})$		$+ \gamma^5\ (\frac{55}{64})$.
$+ \gamma e'^2\ (\frac{9}{8}m^2)$		

We have thus two equations for the determination of Plana's e, γ in terms of the elliptic e, γ. And the solution of these equations give

Elliptic.

$$\begin{aligned}
e\ (\text{Plana}) = \ & e\ \left(1 - \tfrac{3}{4}\ m^2 + \tfrac{75}{128} m^3 + \tfrac{6947}{512} m^4\right) \\
& + e^5\ \left(\tfrac{263}{32} m^2\right) \\
& + \gamma^2 e\ \left(\tfrac{1}{4} + \tfrac{39}{64}\ m^2\right) \\
& + \gamma^2 e^3 \left(-\tfrac{5}{8}\right) \\
& + \gamma^4 e\ \left(\tfrac{1}{4}\right) \\
& + ee'^2\ \left(\tfrac{9}{2} m^2\right),
\end{aligned}$$

$$\begin{aligned}
\gamma\ (\text{Plana}) = \ & \gamma\ \left(1 - \tfrac{33}{128} m^2 + \tfrac{1}{512} m^4\right) \\
& + \gamma e^2\ \left(\tfrac{727}{256} m^2\right) \\
& + \gamma e^4\ \left(-\tfrac{5}{64}\right) \\
& + \gamma^3\ \left(-\tfrac{5}{128} m^2\right) \\
& + \gamma^3 e^2 \left(\tfrac{5}{32}\right) \\
& + \gamma^3\ \left(\tfrac{5}{8}\right) \\
& + \gamma e'^2 \left(-\tfrac{9}{8}\ m^2\right).
\end{aligned}$$

I annex the verification of these expressions; we have

Plana.		Elliptic.
$e\ \left(2 + \tfrac{3}{2} m^2 - \tfrac{75}{64} m^3 - \tfrac{6659}{256} m^4\right) =$		$e\ \left(2 - \tfrac{3}{2} m^2 + \tfrac{75}{64} m^3 + \tfrac{6947}{256} m^4\right.$
		$+ \tfrac{3}{2} m^2 \quad - \tfrac{9}{8}\ m^4$
		$\left. - \tfrac{75}{64} m^3 - \tfrac{6659}{256} m^4\right)$
		$+ e^3\ \left(\tfrac{263}{32} m^2\right)$
		$+ e\gamma^2\ \left(\tfrac{1}{2} + \tfrac{39}{32}\ m^2\right.$
		$\left. + \tfrac{3}{8}\ m^2\right)$
		$+ e^3\gamma^2 \left(-\tfrac{5}{4}\right)$
		$+ e\gamma^4\ \left(\tfrac{1}{2}\right)$
		$+ ee'^2\ (9m^2),$
$e^3\ \left(-\tfrac{1}{4} - 17m^2\right)$	$=$	$e^3\ \left(-\tfrac{1}{4} + \tfrac{9}{16}\ m^2\right.$
		$\left. - 17\ m^2\right)$
		$+ e^3\gamma^2 \left(-\tfrac{3}{16}\right),$
$e^5\ \left(\tfrac{5}{96}\right)$	$=$	$e^5\ \left(\tfrac{5}{96}\right),$
$e\gamma^2\ \left(-\tfrac{1}{2} - \tfrac{63}{32} m^2\right)$	$=$	$e\gamma^2\ \left(-\tfrac{1}{2} - \tfrac{63}{32} m^2\right.$
		$\left. + \tfrac{3}{8}\ m^2\right)$
		$+ e\gamma^4\ \left(-\tfrac{1}{8}\right)$
$e^3\gamma^2 \left(\tfrac{23}{16}\right)$	$=$	$e^3\gamma^2 \left(\tfrac{25}{16}\right)$
$e\gamma^4\ \left(-\tfrac{3}{8}\right)$	$=$	$e\gamma^4\ \left(-\tfrac{3}{8}\right)$
$ee'^2\ (-9m^2)$	$=$	$ee'^2\ (-9m^3),$

whence, adding, we have the first equation.

And, moreover,

$$\begin{aligned}
\gamma\ (1+\tfrac{33}{128}m^3-\tfrac{1}{512}m^4) &= \gamma\ (\ 1-\tfrac{33}{128}m^3+\tfrac{1}{512}m^4 \\
&\qquad\qquad +\tfrac{33}{128}m^3-\tfrac{1}{512}m^4) \\
&\quad +\gamma e^2\ (\qquad \tfrac{727}{256}m^2) \\
&\quad +\gamma^4 e\ (-\tfrac{5}{64}) \\
&\quad +\gamma^3\ (\qquad -\tfrac{5}{128}m^2) \\
&\quad +\gamma^3 e^2\ (\tfrac{5}{32}) \\
&\quad +\gamma^5\ (\tfrac{5}{8}) \\
&\quad +\gamma e'^2\ (\qquad -\tfrac{9}{8}m^2), \\
\gamma e^2\ (-1-\tfrac{1111}{256}m^2) &= \gamma e^2\ (-1+\tfrac{3}{2}m^2 \\
&\qquad\qquad -\tfrac{1111}{256}m^2) \\
&\quad +\gamma^3 e^2\ (\qquad -\tfrac{1}{2}), \\
\gamma e^4\ (\tfrac{3}{16}) &= \gamma e^4\ (\tfrac{3}{16}), \\
\gamma^3\ (-\tfrac{3}{8}+\tfrac{5}{128}m^2) &= \gamma^3\ (-\tfrac{3}{8}+\tfrac{5}{128}m^2) \\
\gamma^3 e^2\ (\ \tfrac{23}{32}) &= \gamma^3 e^2\ (\ \tfrac{23}{32}) \\
\gamma^5\ (\ \tfrac{15}{64}) &= \gamma^5\ (\ \tfrac{15}{64}) \\
\gamma e'^2\ (\qquad \tfrac{9}{8}m^2) &= \gamma e'^2\ (\qquad \tfrac{9}{8}m^2),
\end{aligned}$$

whence, adding, we have the second equation.

It may be noticed that, taking the foregoing expressions only as far as the third order, we have

Plana.		Elliptic.
e	$=$	$e\ (1+\frac{1}{4}\gamma^2-\frac{3}{4}m^2)$,
γ	$=$	γ.

And moreover that, attending only to the terms which are independent of m, we have

$$\begin{aligned}
e &= e\ (1+\tfrac{1}{4}\gamma^2-\tfrac{5}{8}\gamma^2+\tfrac{1}{4}\gamma^4), \\
\gamma &= \gamma\,(1-\tfrac{5}{64}e^4+\tfrac{5}{32}e^2\gamma^2-\tfrac{5}{8}\gamma^4).
\end{aligned}$$

which are formulæ that may be found useful.

468.

ADDITION TO SECOND NOTE ON THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXVII. (1866—1867), pp. 267—269.]

WRITING as in my Second Note, *Monthly Notices*, Vol. XXV., pp. 203—207 (May 1865), [466], for the Moon,

a, the mean distance,

e, the excentricity,

γ, the tangent of the inclination,

l, the mean longitude,

c, the mean anomaly,

g, the mean distance from node,

I obtained by the ordinary method of the variation of the elements, from the constant term of R and the term involving $\cos(2c-2g)$, the following expressions of the variations,

$$\begin{aligned}
\delta a &= 0, \\
\delta e &= -\tfrac{5}{8}\gamma^2 e \quad \cos \quad 2c-2g, \\
\delta \gamma &= +\tfrac{5}{8}\gamma e^2 \quad \text{,,} \quad 2c-2g, \\
\delta c &= +\tfrac{5}{8}\gamma^2 \quad \sin \quad 2c-2g, \\
\delta g &= +\tfrac{5}{8}e^2 \quad \text{,,} \quad 2c-2g, \\
\delta l &= +\tfrac{5}{16}\gamma^2 e^2 \quad \text{,,} \quad 2c-2g,
\end{aligned}$$

viz. if in the elliptic expressions of the radius vector, longitude, and latitude, we apply to a, e, γ, c, g, l, the foregoing increments, we obtain to the fourth order in (e, γ) the portions independent of m in the expressions of the radius vector, latitude,

and longitude. I wish to notice that the results, to the very limited extent to which they go, agree with those obtained by M. Delaunay in his "Théorie du Mouvement de la Lune," from his 49th operation, the object of which is to take away the term (63) of R, that is the term involving $\cos(2c-2g)$. The formulæ (see vol. I. p. 788), taken only to the necessary degree of approximation are

$$\begin{array}{rcrcl}
a & \text{replaced by} & a, & & \\
e^2 & \text{,,} & e^2 - 5\gamma^2 e^2 & \cos & 2g, \\
\gamma^2 & \text{,,} & \gamma^2 + \frac{5}{4}\gamma^2 e^2 & \text{,,} & 2g, \\
l & \text{,,} & l \ - \frac{5}{2}\gamma^2 & \sin & 2g, \\
h+g+l & \text{,,} & h+g+l \ + \frac{5}{4}\gamma^2 e^2 & \text{,,} & 2g, \\
h & \text{,,} & h \ + \frac{5}{8}e^2 & \text{,,} & 2g,
\end{array}$$

which, observing that

$$\begin{array}{ll}
\gamma \text{ (Del.)} & = \frac{1}{2}\gamma \text{ (for present purpose)}, \\
l & = c, \\
g+l & = g, \\
h+g+l & = l,
\end{array}$$

and therefore $\quad g \quad = -(c-g),$

become

$$\begin{array}{rcrcl}
a & \text{replaced by} & a, & & \\
e^2 & \text{,,} & e^2 - \frac{5}{4}\gamma^2 e^2 & \cos & 2c-2g, \\
\gamma^2 & \text{,,} & \gamma^2 + \frac{5}{4}\gamma^2 e^2 & \text{,,} & 2c-2g, \\
c & \text{,,} & c \ + \frac{5}{8}\gamma^2 & \sin & 2c-2g, \\
l & \text{,,} & l \ - \frac{5}{16}\gamma^2 e^2 & \text{,,} & 2c-2g, \\
l-g & \text{,,} & l-g \ - \frac{5}{8}e^2 & \text{,,} & 2c-2g,
\end{array}$$

the last of which may be changed into

$$g \quad \text{,,} \quad g + \tfrac{5}{8}e^2 \quad \text{,,} \quad 2c-2g,$$

or if the new values of a, e, γ, c, g, l, are called $a+\delta a$, $e+\delta e$, $\gamma+\delta\gamma$, $c+\delta c$, $g+\delta g$, $l+\delta l$, then the increments δa, δe, $\delta\gamma$, δc, δg, δl, have the values given above. The process of my Second Note, taken as a first transformation, has in fact the object of removing the term $\cos(2c-2g)$, and to the degree of approximation regarded, the result is not affected by the previous transformations, or by the substitution, t. II. p. 800, introducing for a, e, γ, their standard elliptic values.

469.

ON AN EXPRESSION FOR THE ANGULAR DISTANCE OF TWO PLANETS.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXVII. (1866—1867), pp. 312—315.]

IF for the planet m, referred to any fixed plane and origin of longitudes, we have

v, the longitude in orbit,

θ, the longitude of node,

ϕ, the inclination,

and similarly for the planet m' referred to the same fixed plane and origin of longitudes, if the corresponding quantities are v', θ', ϕ'; then the angular distance of the two planets will of course be expressible in terms of v, θ, ϕ, v', θ', ϕ', but I am not aware that the actual expression has been given. To obtain it in the most simple manner, I write further for the planet m:

$\theta + x$, the reduced longitude,

y, the latitude,

z, the distance from node,

so that z $(= v - \theta)$, x, y, are the hypothenuse, base, and perpendicular of a right-angled spherical triangle, the base angle of which is $= \phi$. And similarly $\theta' + x'$, y', z', have the like significations for the planet m'. I write also r, r', for the distances of the two planets respectively.

This being so, the rectangular coordinates of the planet m are

$$r \cos y \cos (\theta + x),$$
$$r \cos y \sin (\theta + x),$$
$$r \sin y.$$

But observing that from the right-angled triangle we have

$$\begin{aligned}\cos z &= \cos x \cos y,\\ \cos \phi &= \tan x \cot z,\\ \sin x &= \cot \phi \tan y,\\ \sin y &= \sin \phi \sin z,\end{aligned}$$

and therefore also

$$\sin x \cos y = \cot \phi \sin y, = \cos \phi \sin z,$$

the expressions for the coordinates become

$$\begin{aligned}&r(\cos z \cos \theta - \sin z \sin \theta \cos \phi),\\ &r(\cos z \sin \theta + \sin z \cos \theta \cos \phi),\\ &r(\qquad\qquad \sin z \sin \phi).\end{aligned}$$

Forming the analogous expressions for the coordinates of m', then if H be the angular distance of the two planets, we deduce at once the expression for $\cos H$, viz. this is

$$\begin{aligned}\operatorname{Cos} H = \;& (\cos z \cos \theta - \sin z \sin \theta \cos \phi)(\cos z' \cos \theta' - \sin z' \sin \theta' \cos \phi')\\ &+ (\cos z \sin \theta + \sin z \cos \theta \cos \phi)(\cos z' \sin \theta' + \sin z' \cos \theta' \cos \phi')\\ &+ (\qquad \sin z \sin \phi \qquad)(\qquad \sin z' \sin \phi' \qquad),\end{aligned}$$

or, multiplying out, this is

$$\begin{aligned}\operatorname{Cos} H = \;& \cos z \cos z' \cos(\theta - \theta')\\ &+ \cos z \sin z' \sin(\theta - \theta') \cos \phi'\\ &- \sin z \cos z' \sin(\theta - \theta') \cos \phi\\ &+ \sin z \sin z' \left(\cos(\theta - \theta') \cos \phi \cos \phi' + \sin \phi \sin \phi'\right),\end{aligned}$$

say this is

$$\begin{aligned}= \;& A \cos z \cos z'\\ &+ B \cos z \sin z'\\ &+ C \sin z \cos z'\\ &+ D \sin z \sin z',\end{aligned}$$

viz. it is

$$\begin{aligned}= \;& \cos(z - z').\ \tfrac{1}{2} A + \tfrac{1}{2} D\\ &+ \sin(z - z').\ - \tfrac{1}{2} B + \tfrac{1}{2} C\\ &+ \cos(z + z').\ \tfrac{1}{2} A - \tfrac{1}{2} D\\ &+ \sin(z + z').\ \tfrac{1}{2} B + \tfrac{1}{2} C.\end{aligned}$$

But we have

$$z - z' = v - v' - \theta + \theta', \quad z + z' = v + v' - \theta - \theta',$$

whence the expression becomes

$$\begin{aligned}\operatorname{Cos} H = \quad & \cos(v-v') \,.\, (\tfrac{1}{2}A+\tfrac{1}{2}D)\cos(\theta-\theta') - (-\tfrac{1}{2}B+\tfrac{1}{2}C)\sin(\theta-\theta') \\ & + \sin(v-v') \,.\, (\tfrac{1}{2}A+\tfrac{1}{2}D)\sin(\theta-\theta') + (-\tfrac{1}{2}B+\tfrac{1}{2}C)\cos(\theta-\theta') \\ & + \cos(v+v') \,.\, (\tfrac{1}{2}A-\tfrac{1}{2}D)\cos(\theta+\theta') - (\ \tfrac{1}{2}B+\tfrac{1}{2}C)\sin(\theta+\theta') \\ & + \sin(v+v') \,.\, (\tfrac{1}{2}A-\tfrac{1}{2}D)\sin(\theta+\theta') + (\ \tfrac{1}{2}B+\tfrac{1}{2}C)\cos(\theta+\theta'),\end{aligned}$$

or substituting for A, B, C, D, their values, and after a few easy reductions, we find

$$\begin{aligned}\operatorname{Cos} H = \quad & \cos(v-v') \left\{ \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\cos\phi\cos\phi' - \tfrac{1}{2}(1-\cos\phi)(1-\cos\phi')\sin^2(\theta-\theta') \\ +\tfrac{1}{2}\sin\phi\sin\phi' \qquad\qquad \cos(\theta-\theta') \end{matrix} \right\} \\ & + \sin(v-v') \left\{ \begin{matrix} \tfrac{1}{2}(1-\cos\phi)(1-\cos\phi')\sin(\theta-\theta')\cos(\theta-\theta') \\ +\tfrac{1}{2}\sin\phi\sin\phi' \qquad\qquad \sin(\theta-\theta') \end{matrix} \right\} \\ & + \cos(v+v') \left\{ \begin{matrix} \tfrac{1}{2}(1-\cos\phi\cos\phi')\cos(\theta-\theta')\cos(\theta+\theta') \\ +\tfrac{1}{2}(\cos\phi-\cos\phi') \quad \sin(\theta-\theta')\sin(\theta+\theta') \\ -\tfrac{1}{2}\sin\phi\sin\phi' \qquad\qquad \cos(\theta+\theta') \end{matrix} \right\} \\ & + \sin(v+v') \left\{ \begin{matrix} \tfrac{1}{2}(1-\cos\phi\cos\phi')\cos(\theta-\theta')\sin(\theta+\theta') \\ -\tfrac{1}{2}(\cos\phi-\cos\phi') \quad \sin(\theta-\theta')\cos(\theta+\theta') \\ -\tfrac{1}{2}\sin\phi\sin\phi' \qquad\qquad \sin(\theta+\theta') \end{matrix} \right\}\end{aligned}$$

For $\phi=\phi'=0$, the formula becomes, as of course it should do,

$$\operatorname{Cos} H = \cos(v-v').$$

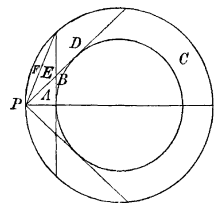
It may be added, that if f, f' are the true anomalies, ω, ω' the longitudes of pericentre in orbit, then $v=\omega+f$, $v'=\omega'+f'$; and we thence have for $\cos H$, formulæ of the like form, containing $\cos f\cos f'$, $\cos f\sin f'$, $\sin f\cos f'$, $\sin f\sin f'$, or containing $\cos(f-f')$, $\sin(f-f')$, $\cos(f+f')$, $\sin(f+f')$, respectively, in place of the like functions of z, z', but with of course altered values of the coefficients.

470.

NOTE ON THE ATTRACTION OF ELLIPSOIDS.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXIX. (1868—1869), pp. 254—257.]

If an indefinitely thin shell of uniform density, bounded by two similar and similarly-situated ellipsoids, attracts a point P on its outer surface, it has been shown geometrically by M. Chasles that the attraction is in the direction of the normal at P, and is equal to twice the attraction of an infinite plate, the thickness of which is equal to the normal thickness at P. Assuming that the attraction is in the direction of the normal, the proof is in fact as follows:—with P as vertex, circumscribe to the interior surface a cone; this divides the shell into three parts; the one, $D+E+F$, exterior to the cone, the other two, $A+B$ and C, interior to the cone. It is shown that in the direction of the normal the attraction of C is equal to that of $A+B$;



and it is assumed that in comparison with these the attraction of $D+E+F$ may be neglected; the whole attraction is thus equal to twice that of the portion $A+B$. At the point where the normal at P meets the internal surface draw the tangent plane to the internal surface, thus dividing the portion $A+B$ into the solid cone A and

a remaining portion B; it is assumed that in comparison with that of A the attraction of B may be neglected; the whole attraction is thus equal to twice the attraction of the solid cone A; and the attraction of this solid cone is in the limit (the aperture or solid angle then becoming $=2\pi$) equal to the attraction of an infinite plate whose thickness is equal to the altitude of the solid cone, that is, to the normal thickness at P. And the attraction of the whole ellipsoidal shell is thus ultimately (that is, when the shell is indefinitely thin) equal to twice the attraction of the infinite plate.

It is interesting to ascertain the orders of magnitude of the attractions of the several portions of the shell, which attractions are compared in the foregoing investigation; and this can be done very easily, when, instead of the ellipsoidal shell, we have a spherical shell (bounded by two concentric spherical surfaces). The tangent plane to the inner surface divides the portion $D+E+F$ into two portions D and $E+F$; and if with P as vertex we describe a cone standing on the circle in which the tangent plane meets the outer surface, the last-mentioned portion is hereby divided into the portions E and F; the whole shell is thus divided into the portions A, B, C, D, E, F, each of them symmetrical in regard to the normal or radius at P, and consequently attracting in the direction of this radius. I proceed to find the attractions of each of these portions; it will appear, in accordance with the assumptions of the foregoing investigation, that, taking the radii to be 1 and $1+\alpha$, that is, α the thickness of the shell, and supposing ultimately α to become indefinitely small, the attractions of A and C are each ultimately $=2\pi\alpha$, that is $=$ to the attraction of the infinite plate, while the attractions of the other portions are of the order $\alpha^{\frac{3}{2}}$, and thus vanish in comparison with that of A or C.

The attraction of an indefinitely thin cone or frustum of a cone, length r and solid angle $d\omega$ is $=rd\omega$; considering any such cone having P for its vertex, if the inclination of r to the radius through P is $=\theta$, and if the azimuth of the plane through r and the radius is $=\phi$, then we have $d\omega=\sin\theta\, d\theta\, d\phi$, the attraction $rd\omega$ is $=r\sin\theta d\theta d\phi$, and this attraction resolved in the direction of the radius is $=r\sin\theta\cos\theta d\theta d\phi$. For the several cases which have to be considered, the value of r is independent of ϕ, and the integration in regard to ϕ is always from $\phi=0$ to $\phi=2\pi$;—the attraction is thus in each case $=2\pi\int r\sin\theta\cos\theta\, d\theta$, the expression of r in the terms of θ, and the limits of θ being known for each of the several portions of the shell. Taking θ_1 for the semi-angle of the tangent cone, we have it is clear

$$\sin\theta_1=\frac{1}{1+\alpha},\quad \cos\theta_1=\frac{\sqrt{2\alpha+\alpha^2}}{1+\alpha};$$

and taking θ_2 for the semi-angle of the cone which divides the portions E, F,

$$\tan\theta_2=\sqrt{\frac{2+\alpha}{\alpha}},\quad \sin\theta_2=\frac{\sqrt{2+\alpha}}{\sqrt{2}\,(1+\alpha)},\quad \cos\theta_2=\frac{\sqrt{\alpha}}{\sqrt{2(1+\alpha)}}.$$

For F we have

$$r = 2(1+\alpha)\cos\theta, \quad \theta = \theta_2 \text{ to } \theta = \tfrac{1}{2}\pi,$$

Integral is

$$= 2(1+\alpha)\int \sin\theta\cos^2\theta\, d\theta, = \tfrac{2}{3}(1+\alpha)\cos^3\theta_2.$$

For $D+E$ we have

$$r = 2(1+\alpha)\cos\theta, \quad \theta = \theta_1 \text{ to } \theta = \theta_2,$$

Integral is

$$= 2(1+\alpha)\int \sin\theta\cos^2\theta\, d\theta, = \tfrac{2}{3}(1+\alpha)(\cos^3\theta_1 - \cos^3\theta_2).$$

For E we have

$$r = \frac{\alpha}{\cos\theta}, \quad \theta = \theta_1 \text{ to } \theta = \theta_2,$$

Integral is

$$= \alpha\int \sin\theta\, d\theta, = \alpha(\cos\theta_1 - \cos\theta_2).$$

For A we have

$$r = \frac{\alpha}{\cos\theta}, \quad \theta = 0 \text{ to } \theta = \theta_1,$$

Integral is

$$= \alpha\int \sin\theta\, d\theta, = \alpha(1 - \cos\theta_1).$$

For $A+B$ we have

$$r = (1+\alpha)\cos\theta - \sqrt{1-(1+\alpha)^2\sin^2\theta}, \quad \theta = 0 \text{ to } \theta = \theta_1,$$

Integral is

$$= \int \{(1+\alpha)\cos\theta - \sqrt{1-(1+\alpha)^2\sin^2\theta}\}\sin\theta\cos\theta\, d\theta,$$

$$= (1+\alpha)(-\tfrac{1}{3}\cos^3\theta) + \frac{1}{3(1+\alpha)^2}\{1-(1+\alpha)^2\sin^2\theta\}^{\frac{3}{2}}, \text{ between the limits,}$$

$$= \tfrac{1}{3}\left\{(1+\alpha)(1-\cos^3\theta_1) - \frac{1}{(1+\alpha)^2}\right\},$$

and subtracting the above value of the integral for A, it at once appears that, for B, the integral is

$$= 2\pi\left\{\alpha(-1+\cos\theta_1) + \tfrac{1}{3}(1+\alpha)(1-\cos^2\theta_1) - \tfrac{1}{3}\frac{1}{(1+\alpha)^2}\right\}.$$

Hence, calculating the approximate values, and restoring in each case the omitted factor, 2π, we have

$$\begin{aligned}
&\text{Attraction} && A = 2\pi\alpha - 2\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && B = \tfrac{2}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && C = 2\pi\alpha - \tfrac{4}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && D = \tfrac{4}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && E = 1\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && F = \tfrac{1}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}};
\end{aligned}$$

or, if we please,

$$\begin{aligned}
&\text{Attraction} && A + B = 2\pi\alpha - \tfrac{4}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && C = 2\pi\alpha - \tfrac{4}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}},\\
&\quad\text{,,} && D + E + F = \tfrac{8}{3}\sqrt{2}\,\pi\alpha^{\frac{3}{2}};
\end{aligned}$$

so that ultimately the attraction of the portion $D+E+F$ vanishes in comparison with those of the portions $A+B$ and C; and the attraction of these last, that is, of the whole shell, is $=4\pi\alpha$, twice the attraction of an infinite plate of the thickness α.

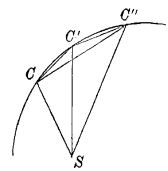
471.

NOTE ON THE PROBLEM OF THE DETERMINATION OF A PLANET'S ORBIT FROM THREE OBSERVATIONS.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXIX. (1868—1869), pp. 257—259.]

THE principle of the solution given in the *Theoria Motus* may be explained very simply as follows:

Consider three successive positions of C, C', C'', of a planet revolving about the focus S; let n, n', n'', denote the doubles of the triangular areas $C'SC''$, CSC', and CSC'' respectively (viz. the triangular area means the area of the triangle included between the two radius vectors and the chord joining their extremities), r' the radius



vector SC'; θ'', θ, the times of describing the arcs CC' and $C'C''$ respectively, the units of time and distance being such that the time is equal to the double area divided by the square root of the half latus rectum ($t = 2\pi a^{\frac{3}{2}}$ for the Period in a circular or elliptic orbit).

Then writing

$$P = \frac{n''}{n}, \quad Q = 2\left(\frac{n+n''}{n'} - 1\right) r'^3,$$

(observe that $n+n''-n'$ is $=$twice the triangle $CC'C''$), for neighbouring positions of the planet, the values of P and Q are *approximately* $=\frac{\theta''}{\theta}$ and $\theta\theta''$ respectively: the solution consists in the determination of an orbit for which P and Q have these approximate values; then, by means of such approximate orbit, the values of P and Q are more accurately determined, and by means of these new values of P and Q, a new determination is effected of the orbit: and so on, to the requisite accuracy of approximation.

The foregoing approximate values of P and Q respectively are deduced from the accurate values

$$P=\frac{\theta''\eta}{\theta\eta''},\quad Q=\frac{\theta\theta''}{\eta\eta''}\,\frac{r'^2}{r'r''}\,\frac{1}{\cos f\cos f'\cos f''};$$

where r, r', r'' are the radius vectors SC, SC', SC''; $2f$, $2f'$, $2f''$ are the angular distances $C'SC''$, CSC'', CSC' $(f'=f+f'')$ and η, η', η'' are the ratios of the sectorial areas $C'SC''$, CSC'', $C'SC''$, to the triangular areas represented by the same letters respectively: the doubles of the sectorial areas are thus $n\eta$, $n'\eta'$, and $n''\eta''$, and if the half latus rectum be denoted by p, then we have

$$\sqrt{p}=\frac{n\eta}{\theta}=\frac{n'\eta'}{\theta'}=\frac{n''\eta''}{\theta''};$$

and it thus at once appears that the accurate value of P is $=\frac{\theta''\eta}{\theta\eta''}$, as above. To obtain the expression for Q, taking ϕ, ϕ', ϕ'' for the true anomalies (and, for greater symmetry, writing for the moment ν, $-\nu'$, ν'', g, $-g'$, g'' in place of n, n', n'', f, f', f'' respectively), we have

$$r=\frac{p}{1+e\cos\phi},\quad 2g=\phi''-\phi',$$

$$r'=\frac{p}{1+e\cos\phi'},\quad 2g'=\phi-\phi'',$$

$$r''=\frac{p}{1+e\cos\phi''},\quad 2g''=\phi'-\phi,$$

$$(g+g'+g''=0);$$

whence identically

$$\frac{\sin 2g}{r}+\frac{\sin 2g'}{r'}+\frac{\sin 2g''}{r''}=-\frac{4\sin g\sin g'\sin g''}{p};$$

or writing

$$\nu=r'r''\sin 2g,\quad \nu'=r''r\sin 2g',\quad \nu''=rr'\sin 2g'',$$

this is

$$\nu+\nu'+\nu''=-\frac{4rr'r''\sin g\sin g'\sin g''}{p},$$

$$=-\frac{(rr'r'')^2\sin 2g\sin 2g'\sin 2g''}{2prr'r''\cos g\cos g'\cos g''}$$

$$=-\frac{\nu\nu'\nu''}{2prr'r''\cos g\cos g'\cos g''}.$$

This is, in fact,

$$n - n' + n'' = \frac{nn'n''}{2prr'r''\cos f\cos f'\cos f''},$$

or since

$$\frac{nn''}{p} = \frac{\theta\theta''}{\eta\eta''},$$

it is

$$2\left(\frac{n+n''}{n'} - 1\right) = \frac{\theta\theta''}{\eta\eta''rr'r''\cos f\cos f'\cos f''},$$

viz. multiplying by r'^3, it is

$$Q \quad = \frac{\theta\theta''}{\eta\eta''}\,\frac{r'^2}{rr''}\,\frac{1}{\cos f\cos f'\cos f''},$$

the above-mentioned value of Q.

472.

NOTE ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXIX. (1868—1869), pp. 318—320.]

CONSIDER any two positions, A, B, in an elliptic orbit, focus S, and semi-axis major $= a$; then if ρ, ρ', c denote the radius vectors SA, SB, and the chord AB respectively, and if $P, = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$, be the periodic time, the time of passage from A to B is given by the formula

$$\text{Time } AB = \frac{P}{2\pi}(\chi - \chi' - \sin\chi + \sin\chi')$$

where

$$2a\cos\chi = 2a - \rho - \rho' - c, \quad 2a\cos\chi' = 2a - \rho - \rho' + c.$$

To fix the ideas we may consider the time of passage as being in every case positive; and, for Time AB, the motion from A as being towards the apocentre; Time BA will, of course, in like manner denote that the motion from B is towards the apocentre; and we thus have according to the positions of A, B, either Time $AB =$ Time BA; or else Time $AB +$ Time $BA = P$.

This being so (see the *Theoria Motus*, p. 120), χ will be always a positive arc between 0 and 360°; χ' a positive or negative arc between 0 and $\pm 180°$; and moreover χ' will be positive or negative according as the described focal angle is $< 180°$ or $> 180°$; whence, $\cos\chi'$ being known, the arc χ' is determined without ambiguity.

But as noticed in the place referred to, there is when only ρ, ρ', c, a, are known, a real ambiguity as regards the arc χ; viz. χ may be either the arc $> 180°$ or the arc $< 180°$, having for its cosine the given value of $\cos\chi$. For, given the points S, A, B, and the semi-axis major a, there exist two elliptic orbits determined by these data; and the two values of χ correspond to the times of passage between A and B, in these two orbits respectively. If, however, the actual orbit be given,

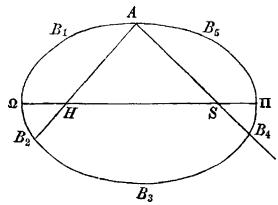
there is no longer any real ambiguity; and it must be possible to decide between the two values of χ: the criterion is, in fact, a very simple one, viz. drawing a chord from A through the other focus H of the ellipse, this either separates, or it does not separate, B from the force-focus S; and I say that in the expression of Time AB, in the former case (viz. when chord A is a separator) we have $\chi < 180°$; in the latter case (viz. when chord A is not a separator) we have $\chi > 180°$.

It of course follows that, in the case of transition, when the line AB passes through H, we must have $\chi = 180°$: this is at once seen to be so; for $\chi = 180°$ gives the condition $4a = \rho + \rho' + c$; but if σ, σ', are the distances of AB from the focus H, then $2a = \rho + \sigma$, $2a = \rho' + \sigma'$, and the condition becomes $\sigma + \sigma' = c$; that is AB must pass through H.

As a verification of the new criterion, I consider the point A as having a fixed position on the orbit, but the point B as having successively different positions; and writing down the two formulæ

$$\text{Time } AB = \chi - \chi' - \sin\chi + \sin\chi',$$
$$\text{Time } BA = \omega - \omega' - \sin\omega + \sin\omega',$$

(where for simplicity the constant factor $P \div 2\pi$ is omitted) I proceed to compare these for different positions of the point B. We have, in every case, $\cos\omega = \cos\chi$, and $\cos\omega' = \cos\chi'$; whence (χ, ω being each positive and less than 360°) $\omega = \chi$ or else $\omega + \chi = 360°$, viz. the former equation subsists if ω, χ, are each less or each greater than 180°, the latter if the one is greater, the other less than 180°. And again (χ', ω' being each less than $\pm 180°$) we have $\omega' = \chi'$, or else $\omega' = -\chi'$, according as ω', χ' have the same or opposite signs.



Now in the figure, suppose that B occupies successively the different positions B_1, B_2, ... B_5, the criteria for χ, χ' (or ω, ω') give as follows,

	Ch. A.	Ch. B.			therefore	$\angle AS$	$\angle BSA$			
1	sep.	not	$\chi < 180°$	$\omega > 180°$	or $\omega + \chi = 2\pi$	$< 180°$	$> 180°$	$\chi' = +$	$\omega' = -$	or $\omega' = -\chi'$,
2	sep.	sep.	$<$	$<$	„ $\omega = \chi$	$<$	$<$	$+$	$+$	„ $\omega' = \chi'$,
3	not	not	$>$	$>$	„ $\omega = \chi$	$>$	$>$	$+$	$+$	„ $\omega' = \chi'$,
4	not	not	$>$	$>$	„ $\omega = \chi$	$>$	$>$	$-$	$-$	„ $\omega' = \chi'$,
5	not	sep.	$>$	$<$	„ $\omega + \chi = 2\pi$	$>$	$<$	$-$	$-$	„ $\omega' = -\chi'$.

Hence substituting for ω, ω' their values in terms of χ, χ', we have

$$\begin{aligned}
\text{Time } AB_1 \ldots_5 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_1 &= 2\pi - \chi + \chi' + \sin\chi - \sin\chi', \\
\text{,, } BA_2 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_3 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_4 &= \chi - \chi' - \sin\chi + \sin\chi', \\
\text{,, } BA_5 &= 2\pi - \chi + \chi' + \sin\chi - \sin\chi';
\end{aligned}$$

and thence (restoring the omitted factor $P \div 2\pi$)

$$\begin{aligned}
\text{Time } AB_1 + \text{Time } BA_1 &= P, \\
\text{,, } AB_2 - \text{ ,, } BA_2 &= 0, \\
\text{,, } AB_3 - \text{ ,, } BA_3 &= 0, \\
\text{,, } AB_4 - \text{ ,, } BA_4 &= 0, \\
\text{,, } AB_5 + \text{ ,, } BA_5 &= P,
\end{aligned}$$

which are the relations which in fact subsist between the times AB_1 and BA_1 &c.

473.

ON THE GRAPHICAL CONSTRUCTION OF THE UMBRAL OR PENUMBRAL CURVE AT ANY INSTANT DURING A SOLAR ECLIPSE.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXX. (1869—1870), pp. 162—164.]

THE curve in question, say the penumbral curve, is the intersection of a sphere by a right cone,—I wish to show that the stereographic projection of this curve may be constructed as the envelope of a variable circle, having its centre on a given conic, and cutting at right angles a fixed circle; this fixed circle being in fact the projection of the circle which is the section of the sphere by the plane through the centre and the axis of the cone, or say by the axial plane. The construction thus arrived at is Mr Casey's construction for a bicircular quartic; and it would not be difficult to show that the stereographic projection of the penumbral curve is in fact a bicircular quartic.

The construction depends on the remark that a right cone is the envelope of a variable sphere, having its centre on a given line and its radius proportional to the distance of the centre from a given point on this line; and on the following theorem of plane geometry:

Imagine a fixed circle, and a variable circle having its centre on a given line and its radius proportional to the distance of the centre from a given point on the line (or, what is the same thing, the variable circle always touches a given line); then the locus of the pole in regard to the fixed circle, of the common chord of the two circles (or, what is the same thing, the locus of the centre of a new variable circle which cuts the fixed circle at right angles in the points where it is met by the first-mentioned variable circle) is a conic.

To fix the ideas, say that P is the centre of the first variable circle; AB its common chord with the fixed circle; Q the centre of the circle which cuts the fixed circle at right angles in the points A and B; then the locus of Q is a conic.

To prove this, take $x^2+y^2=1$ for the equation of the fixed circle, $(x-\alpha)^2+(y-\beta)^2=\gamma^2$ for that of the variable circle; the foregoing law of variation being in fact such that α, β, γ, are linear functions of a variable parameter θ; the equation of the common chord AB is $-2\alpha x-2\beta y+1+\alpha^2+\beta^2-\gamma^2=0$; viz., this equation contains θ quadratically; hence the envelope of the common chord is a conic; and thence (reciprocating in regard to the fixed circle) the locus of the pole of AB, that is, of the point Q, is also a conic.

Consider now a solid figure in which the circles are replaced by spheres; viz. we have a fixed sphere, and a variable sphere having its centre on a given line and its radius proportional to the distance of the centre from a given point on the line. The envelope of the variable sphere is a right cone; the intersection of the cone with the fixed sphere is the envelope of the small circle of the sphere, say the circle AB, which is the intersection of the fixed sphere by the variable sphere. This circle AB is also the intersection of the fixed sphere by a sphere, centre Q, which cuts the fixed sphere at right angles; and by what precedes the locus of Q is a conic. Hence the penumbral curve is given as the envelope of the circle AB which is the intersection of the fixed sphere by a sphere which has its centre Q on a conic, and which cuts the fixed sphere at right angles. It is obvious that the circle AB always cuts at right angles the great circle which is the section of the fixed sphere by the axial plane, or say the axial circle. Project the whole figure stereographically; the projection of the circle AB is a variable circle which cuts at right angles the circle which is the projection of the axial circle, and which has for its centre the point Q' which is the projection of Q. But the locus of Q being a conic, the locus of its projection Q' is also a conic; and we have thus the projection of the penumbral curve as the envelope of a variable circle which has its centre on a conic, and which cuts at right angles a fixed circle.

We may in the axial plane construct five points of the conic which is the locus of Q, by means of any five assumed positions of the variable circle, and somewhat simplify the construction by a proper choice of the five positions of the variable circle. This is not a convenient construction, and even if it were accomplished we should still have to construct the projection of the conic so obtained, in order to find, in the figure of the stereographic projection, the conic which is the locus of Q'. I do not at present perceive any direct construction for the last-mentioned conic; but assuming that a tolerably simple construction can be obtained, the construction of the projection of the penumbral curve as the envelope of the variable circle is as easy and rapid as possible. Probably the easiest course would be (without using the conic at all) to calculate numerically, for a given position of the variable sphere, the terrestrial latitude and longitude of the two points of intersection of the variable sphere by the axial circle; laying these down on the projection, we have then a position of the variable circle; and a small number of properly selected positions would give the penumbral curve with tolerable accuracy.

I have throughout spoken of the penumbral curve, as it is in regard hereto that a graphical construction is most needed; but the theory is applicable, without any alteration, to the umbral curve.

474.

ON THE GEOMETRICAL THEORY OF SOLAR ECLIPSES.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXX. (1869—1870), pp. 164—168.]

THE fundamental equation in a solar eclipse is, I think, most readily established as follows:

Take the centre of the Earth for origin, and consider a set of axes fixed in the Earth and moveable with it; viz., the axis of z directed towards the North Pole; those of x, y, in the plane of the Equator; the axis of x directed towards the point longitude 0°; that of y towards the point longitude 90° W. of Greenwich. Take a, b, c, for the coordinates of the Moon; k for its radius (assuming it to be spherical); a', b', c', for the coordinates of the Sun; k' for its radius (assuming it to be spherical); then, writing $\theta + \phi = 1$, the equation

$$\{\theta(x-a)+\phi(x-a')\}^2+\{\theta(y-b)+\phi(y-b')\}^2+\{\theta(z-c)+\phi(z-c')\}^2=(\theta k \pm \phi k')^2$$

is the equation of the surface of the Sun or Moon, according as θ, $\phi = 1$, 0 or $= 0$, 1: and for any values whatever of θ, ϕ, it is that of a variable sphere, such that the whole series of spheres have a common tangent cone. Writing the equation in the form

$$\begin{aligned} &\theta^2\{(x-a)^2+(y-b)^2+(z-c)^2-k^2\} \\ +\ &2\theta\phi\{(x-a)(x-a')+(y-b)(y-b')+(z-c)(z-c')\mp kk'\} \\ +\ &\phi^2\{(x-a')^2+(y-b')^2+(z-c')^2-k'^2\}=0, \end{aligned}$$

or, putting for shortness,

$$\begin{aligned} \rho &= a^2 + b^2 + c^2 - k^2 \\ \rho' &= a'^2 + b'^2 + c'^2 - k'^2 \\ \sigma &= aa' + bb' + cc' \mp kk' \\ P &= ax + by + cz \\ P' &= a'x + b'y + c'z, \end{aligned}$$

the equation is

$$\begin{aligned} &\theta^2\ (x^2+y^2+z^2-2P+\rho) \\ &+2\theta\phi\ (x^2+y^2+z^2-P-P'+\sigma) \\ &+\ \ \phi^2(x^2+y^2+z^2-2P'+\rho')=0, \end{aligned}$$

and the equation of the envelope consequently is

$$(x^2+y^2+z^2-2P+\rho)(x^2+y^2+z^2-2P'+\rho')-(x^2+y^2+z^2-P-P'+\sigma)^2=0,$$

that is

$$(x^2+y^2+z^2)(\rho+\rho'-2\sigma)-(P-P')^2-2(\rho'-\sigma)P-2(\rho-\sigma)P'+\rho\rho'-\sigma^2=0,$$

which is the equation of the cone in question.

Observe that one sphere of the series is a *point*, viz., taking first the upper signs if we have $\theta k+\phi k'=0$, that is

$$\theta=\frac{k'}{k'-k},\quad \phi=\frac{-k}{k'-k},$$

then the sphere in question is the point the coordinates whereof are

$$x=\frac{k'a-ka'}{k'-k},\quad y=\frac{k'b-kb'}{k'-k},\quad z=\frac{k'c-kc'}{k'-k},$$

which point is the vertex of the cone: it hence appears that, taking the upper signs, the cone is the *umbral* cone, having its vertex on this side of the Moon; and similarly taking the lower signs, then if we have $\theta k-\phi k'=0$, that is

$$\theta=\frac{k'}{k'+k},\quad \phi=\frac{k}{k'+k},$$

then the variable sphere will be the point the coordinates of which are

$$\frac{k'a+ka'}{k'+k},\quad \frac{k'b+kb'}{k'+k},\quad \frac{k'c+kc'}{k'+k},$$

which point is the vertex of the cone; viz. the cone is here the penumbral cone having its vertex between the Sun and Moon.

Taking as unity the Earth's equatorial radius, if p, p' are the parallaxes, κ, κ' the angular semi-diameters of the Moon and Sun respectively, then the distances are $\frac{1}{\sin p}$, $\frac{1}{\sin p'}$ and the radii are $\frac{\sin\kappa}{\sin p}$, $\frac{\sin\kappa'}{\sin p'}$ respectively; hence, if h, h' are the hour-angles west from Greenwich, Δ, Δ' the N.P.D.'s of the Moon and Sun respectively, we have

$$\begin{aligned} a&=\frac{1}{\sin p}\sin\Delta\cos h, & a'&=\frac{1}{\sin p'}\sin\Delta'\cos h', \\ b&=\frac{1}{\sin p}\sin\Delta\sin h, & b'&=\frac{1}{\sin p'}\sin\Delta'\sin h', \\ c&=\frac{1}{\sin p}\cos\Delta\quad, & c'&=\frac{1}{\sin p'}\cos\Delta', \\ k&=\frac{\sin\kappa}{\sin p}\quad, & k'&=\frac{\sin\kappa'}{\sin p'}; \end{aligned}$$

and thence

$$\rho = \frac{1}{\sin^2 p}(1-\sin^2\kappa'),$$

$$\rho' = \frac{1}{\sin^2 p'}(1-\sin^2\kappa'),$$

$$\sigma = \frac{1}{\sin p \sin p'}[\cos\Delta\cos\Delta' + \sin\Delta\sin\Delta'\cos(h'-h) \mp \sin\kappa\sin\kappa'],$$

$$P = \frac{1}{\sin p}\quad\{\sin\Delta\ (x\cos h + y\sin h) + z\cos\Delta\},$$

$$P' = \frac{1}{\sin p'}\quad\{\sin\Delta'(x\cos h' + y\sin h') + z\cos\Delta'\}.$$

Moreover, if the right ascensions of the Moon and Sun are α, α' respectively, and if the R.A. of the meridian of Greenwich (or sidereal time in angular measure) be $=\Sigma$, then we have

$$h = \Sigma - \alpha, \quad h' = \Sigma - \alpha'.$$

It is to be observed that $h-h'$, Δ, Δ' are slowly varying quantities, viz., their variation depends upon the variation of the celestial positions of the Sun and Moon; but h and h' depend on the diurnal motion, thus varying about 15° per hour; to put in evidence the rate of variation of the several angles h, h', Δ, Δ' during the continuance of the eclipse, instead of the foregoing values of h, h', I write

$$h' = \left\{E + \left(1 + \frac{E_1 - E}{24}\right)t\right\}15°,$$

where t is the Greenwich mean time, E, E_1 are the values (reckoned in parts of an hour) of the Equation of Time at the preceding and following mean noons respectively, taken positively or negatively, so that E, E_1 are the mean times of the two successive apparent noons respectively; whence also

$$h = \left\{E + \left(1 + \frac{E_1 - E}{24}\right)t\right\}15° - \alpha + \alpha';$$

and moreover

$$\alpha = A + m\ (t-T),$$
$$\alpha' = A + m'(t-T),$$
$$\Delta = D + n\ (t-T),$$
$$\Delta' = D' + n'\ (t-T),$$

if T be the time of conjunction, A, A, D, D' the values at that instant of the R.A.'s and N.P.D.'s; m, m' and n, n' the horary motions in R.A. and N.P.D. respectively.

It appears to me not impossible but that the foregoing form of equation,

$$(x^2+y^2+z^2)(\rho+\rho'-2\sigma)-(P-P')^2-2(\rho'-\sigma)P-2(\rho-\sigma)P'+\rho\rho'-\sigma^2=0,$$

for the umbral or penumbral cone might present some advantage in reference to the calculation of the phenomena of an eclipse over the Earth generally: but in order to obtain in the most simple manner the equation of the same cone referred to a set of principal axes, I proceed as follows:

Writing

$$\mathrm{a}=bc'-b'c, \qquad \mathrm{f}=a-a',$$

$$\mathrm{b}=ca'-c'a, \qquad \mathrm{g}=b-b',$$

$$\mathrm{c}=ab'-a'b, \qquad \mathrm{h}=c-c',$$

$$(\text{and therefore} \quad \mathrm{af}+\mathrm{bg}+\mathrm{ch}=0).$$

Then, if

$$X=\frac{(\mathrm{bh}-\mathrm{cg})x+(\mathrm{cf}-\mathrm{ah})y+(\mathrm{ag}-\mathrm{bf})z}{\sqrt{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}\sqrt{\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}},$$

$$Y=\frac{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z}{\sqrt{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}},$$

$$Z=\frac{\mathrm{f}x+\mathrm{g}y+\mathrm{h}z}{\sqrt{\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}},$$

X, Y, Z, will be coordinates referring to a new set of rectangular axes; viz., the origin is, as before, at the centre of the Earth, the axis of Z is parallel to the line joining the centres of the Sun and Moon; the axis of X cuts at right angles the last-mentioned line; and the axis of Y is perpendicular to the plane of the other two axes; or, what is the same thing, to the plane through the centres of the Earth, Sun, and Moon.

The coordinates of the vertex of the cone are therefore X_0, Y_0, Z_0, where these denote what the foregoing values of X, Y, Z, become on substituting therein for x, y, z, the values

$$\frac{k'a\mp ka'}{k'\mp k}, \quad \frac{k'b\mp kb'}{k'\mp k}, \quad \frac{k'c\mp kc'}{k'\mp k},$$

and the equation of the cone therefore is

$$(X-X_0)^2+(Y-Y_0)^2=\tan^2\lambda\,(Z-Z_0)^2,$$

where

$$\sin\lambda=\frac{k'\mp k}{G},$$

if for a moment G denotes the distance between the centres of the Sun and Moon. We have therefore

$$\tan\lambda = \frac{k' \mp k}{\sqrt{G^2 - (k' \mp k)^2}},$$

or since

$$G^2 = (a' - a)^2 + (b' - b)^2 + (c' - c)^2,$$

this is in fact

$$\tan\lambda = \frac{k' \mp k}{\sqrt{\rho + \rho' - 2\sigma}},$$

where ρ, ρ', σ signify as before; and thus X_0, Y_0, Z_0, $\tan\lambda$ are all of them given functions of a, b, c, k, a', b', c', k', and consequently of the before-mentioned astronomical data of the problem. The form is substantially the same as Bessel's equation (3), *Ast. Nach.* No. 321 (1837), (but the direction of the axes of X, Y is not identical with those of his x, y); and it is therefore unnecessary to consider here the application of it to the calculation of the eclipse for a given point on the Earth.

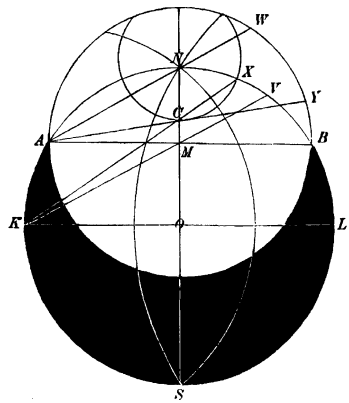
475.

ON A PROPERTY OF THE STEREOGRAPHIC PROJECTION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXX. (1869—1870), pp. 205—207.]

I AM not aware whether it has been noticed that the very same circles which in the direct stereographic projection of a hemisphere (viz., that wherein the projection is on the plane of a meridian) represent the meridians and parallels respectively,—represent also in the oblique projection of the hemisphere meridians and parallels respectively. In fact, in the direct projection where the poles N, S, are in the horizon-meridian, or bounding circle of the projection, if we take a chord AB at right angles to NS, and on AB as diameter describe a circle, the original (meridian and parallel) circles will, as the appearance of the figure at once suggests, represent meridians and parallels in the oblique projection in which the horizon or bounding circle of the projection is the circle diameter AB, and where consequently the North Pole N is brought into view, the South Pole S being beyond the limits of the projection. That this really is so, is clear from the consideration that in any stereographic projection whatever, the meridians will be circles passing through two fixed points N, S, and the parallels be circles cutting the meridians at right angles. (Or, what is the same thing, the parallels also pass each of them through two fixed imaginary points, the antipoints of N, S, but this in passing.) And moreover since in the oblique, as well as in the direct, projection, the longitude of any meridian, as reckoned from the central meridian NS, is the angle at N between the two meridians, the longitude for a given meridian is the same in the two projections respectively. But the co-latitudes are not the same in the two projections respectively; viz., a circle which in the direct projection represents the parallel co-latitude c, will in the oblique projection represent the parallel of a different co-latitude c'. The relation between the values of c, c', will of course depend upon the position of the bounding

circle AB of the oblique direction: to define this position, we may use either the arc NM which in the direct projection determines the co-latitude of the centre M of the oblique projection (say $NM = \Delta$, that is, $NV = \Delta$), or by the arc NM which in the oblique projection determines the distance of N from the centre, or co-latitude of the



centre (say $NM = \Delta'$, that is, $BW = \Delta'$). The obliquity in the oblique projection is thus $90° - \Delta'$, viz., this is the inclination of the plane of projection to that of the horizon-meridian in the direct projection. We have also $c = NX$, $c' = WY$. The relation between the angles Δ, Δ', is easily found to be

$$\tan \tfrac{1}{2}\Delta = \tan^2 \tfrac{1}{2}\Delta',$$

viz., taking the radius in the direct projection to be $= 1$, we have

$$\begin{aligned} OM &= \tan \tfrac{1}{2}(90° - \Delta), \\ MA &= \sqrt{1 - \tan^2 \tfrac{1}{2}(90° - \Delta)}, \\ MN &= 1 - \tan \tfrac{1}{2}(90° - \Delta); \end{aligned}$$

wherefore

$$\sqrt{1 - \tan^2 \tfrac{1}{2}(90° - \Delta)} \,.\, \tan \tfrac{1}{2}\Delta' = 1 - \tan \tfrac{1}{2}(90° - \Delta),$$

and thence

$$\tan^2 \tfrac{1}{2}\Delta' = \frac{1 - \tan \frac{1}{2}(90° - \Delta)}{1 + \tan \frac{1}{2}(90° - \Delta)} = \tan \tfrac{1}{2}\Delta,$$

the required relation.

We have moreover

$$\begin{aligned} NC = 1 - \tan\tfrac{1}{2}(90^\circ - c) &= AM \quad \{\tan\tfrac{1}{2}\Delta' - \tan\tfrac{1}{2}(\Delta' - c')\}, \\ &= \sin\Delta' \{\tan\tfrac{1}{2}\Delta' - \tan\tfrac{1}{2}(\Delta' - c')\}, \\ &= 2\sin^2\tfrac{1}{2}\Delta' - \sin\Delta' \tan\tfrac{1}{2}(\Delta' - c'), \end{aligned}$$

that is

$$\begin{aligned} \tan\tfrac{1}{2}(90^\circ - c) &= \cos\Delta' + \sin\Delta' \tan\tfrac{1}{2}(\Delta' - c'), \\ &= \frac{\cos\frac{1}{2}(\Delta' + c')}{\cos\frac{1}{2}(\Delta' - c')}, \end{aligned}$$

or, what is the same thing,

$$\frac{1 - \tan\frac{1}{2}c}{1 + \tan\frac{1}{2}c} = \frac{1 + \delta\tan\frac{1}{2}c' \tan\frac{1}{2}\Delta'}{1 + \tan\frac{1}{2}c' \tan\frac{1}{2}\Delta'},$$

that is

$$\tan\tfrac{1}{2}c = \tan\tfrac{1}{2}\Delta' \tan\tfrac{1}{2}c',$$

which is the required relation between c and c'. In the particular case $\Delta = \Delta' = 90^\circ$, the two projections coincide, and we have, as we should do, $c' = c$.

476.

ON THE DETERMINATION OF THE ORBIT OF A PLANET FROM THREE OBSERVATIONS.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXXVIII. (1870), pp. 17—111. Read December 10, 1869.]

I PROPOSE to consider from a geometrical point of view the problem of the determination of the orbit of a planet from three observations. The orbit is a conic, having the Sun for a focus; and each observation shows that the planet is at the date thereof in a given line. We have thus a given point or focus S, and three given lines, say the "rays." The orbit-plane, if known, would, by its intersections with the three rays, determine the three positions of the planet; that is, we should have the focus and three points on the orbit; or (what is the same thing) three radius vectors from the focus, say a "trivector." Geometrically, through three given points, and with a given focus, there may be described four conics; but (as will be explained) there is only one of these which can be the orbit; we may therefore say that the orbit will be determined, and that uniquely, by means of a given trivector. The problem is therefore to find the orbit-plane, such that in the orbit determined by means of the trivector the times of passage between the three positions on the orbit may have the observed values; or (what is the same thing) that the orbital areas, each divided by the square root of the latus rectum, may have given values. If, instead of the orbit-plane, we consider the orbit-axis (that is, the line normal to the orbit-plane at the point S), or, what is more convenient, the orbit-pole, or intersection of the axis with a sphere about the centre S; then to a given position of the orbit-pole, there corresponds, as above, a determinate orbit; and the problem is to find the position of the orbit-pole, so that in the orbit belonging thereto the times of passage may have given values as already mentioned; and it is clear that the required position of the orbit-pole may be obtained as the intersection of two spherical curves; the one of them, the locus of those positions of the orbit-pole for which the time of passage

between the first and second points on the orbit has its proper given value; the other of them, the locus of those positions for which the time of passage between the second and third points on the orbit has its proper given value: and in connexion therewith we may consider other isoparametric loci of the orbit-pole; for instance, the iseccentric lines, or loci of the orbit-pole such that along each of them the eccentricity of the orbit has a given value. It is in this point of view that the problem is considered in the present memoir, viz., the object proposed is the discussion of the configuration, &c. of these loci. I consider, in the first instance, any three given rays whatever; but in the ulterior discussion of the spherical curves, which it is difficult to carry out otherwise than numerically, I have confined myself to the case of a particular symmetrical position of the three rays; viz., these are taken to be lines each of them at an inclination of 60° to a fixed plane through S, and such that their projections on this plane form an equilateral triangle having S for its centre, and that each ray cuts the plane in the mid-point of the corresponding side of the triangle.

The general theory as above explained is further developed in the memoir; and I consider the formulæ for the determination of the orbit, &c. by means of a given trivector; those relating to the determination of the trivector obtained as above by means of a variable plane passing through a given point and intersecting three given rays; and lastly, the application to the particular system of three rays already referred to. The Plates refer to this particular system; they are as follow:

Plate 1.	General Planogram for a single ray,	See Nos. 8—10 for explanation of the terms Planogram and Spherogram.
„ 2.	Planogram for Meridian 90°—270°,	
„ 3.	Planogram for Meridian 0°—180°,	
„ 4.	Spherogram for the Eccentricity,	
„ 5.	Spherogram for the Time.	

Article Nos. 1 to 14. *Considerations on the General Theory.*

1. As explained in the introduction, we have a point or focus S, and three lines called the "rays." The orbit-plane is any plane through S; it meets the rays in three points, which are points on the orbit; and joining these with S, we have a "trivector." The orbit is for the present considered as in general uniquely determined by means of the trivector.

2. There are certain critical positions of the orbit-plane.

First, the orbit-plane may be parallel to one of the rays; or (what is the same thing) it may pass through the line through S parallel to the ray: the point on the ray is at infinity; or say that it is at an indefinitely great distance in one direction or in the other direction along the ray; and (from the particular way in which the orbit is selected as one of four conics) there is, as will appear (see *post,* No. 20), a discontinuity of orbit as the point passes from the one to the other of these positions.

3. *Secondly*, the orbit-plane may be parallel to two of the rays; or (what is the same thing) it may pass through the lines through S parallel to these two rays; the points on the two rays are each at infinity; viz. each of them is at an indefinitely great distance in one or the other direction along the ray; and there is a discontinuity of the orbit as each point passes from the one to the other of its two positions.

4. *Thirdly*, the orbit-plane may be such that the orbit is a right line. To see how this arises, observe that we may consider a system of lines meeting each of the three rays, and of course generating a hyperboloid; say these are the generating lines: there is on the hyperboloid another system of lines, say the directrix lines, in which are included the three rays; the point S is not on the hyperboloid. Then, if the orbit-plane pass through a generating line, it will meet the three rays in the points in which these are met by the generating line: and the orbit is, consequently, the generating line (described, as being a right line not passing through S, with a velocity $=\infty$). Any plane through S and a generating line also meets the hyperboloid in a directrix line; and consequently touches it at the intersection of the two lines, viz. it is a tangent plane of the hyperboloid. The planes in question thus envelope the circumscribed cone whose vertex is S; or (what is the same thing) when the orbit-plane is any tangent-plane of this cone, the orbit is a right line.

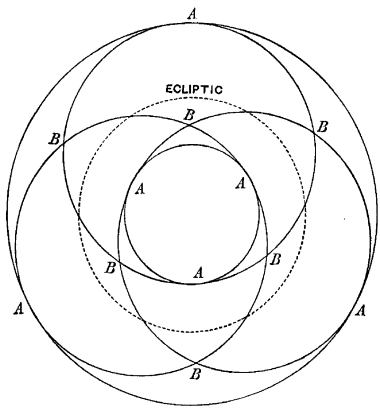
5. The only exception is, *fourthly*, when the orbit-plane passes through one of the rays. Observe that the plane then meets the hyperboloid in another line, that is, a generating line, or the case under consideration is included in the third case; it is also included in the first case. The point on the ray in question is here not a determinate point, but any point whatever of the ray; the points on the other two rays being (as in general) determinate: the orbit is consequently indeterminate; viz. to any point selected at pleasure as the intersection of the orbit-plane with the ray contained therein, there corresponds a determinate orbit (in particular, the selected point may be such that the orbit is, as in the third case, a right line); and, corresponding to the position in question of the orbit-plane, we have the entire system of such orbits.

6. Consider now the corresponding positions of the orbit-pole on a sphere described about the centre S. It will be convenient for the moment to attend to the two opposite positions of the orbit-pole belonging to any position of the orbit-plane, and thus to regard the orbit-pole as moving over the entire spherical surface. The parallel through S to a ray meets the sphere in two points, poles of a great circle which I call a "separator;" we have thus three separators, each two meeting in a pair of opposite points which I call the points B; viz., these are the intersections with the sphere of a line through S perpendicular to the plane containing the parallels of the two rays. A line through S perpendicular to the plane through a ray meets the sphere in a pair of opposite points which I call the points A; these lying on the corresponding separator; there are thus three pairs of points A. The cone reciprocal to the circumscribed cone (that is, generated by a line through S at right angles to any tangent plane of the circumscribed cone) meets the sphere in a spherical conic which I call the "regulator;" this touches each of the separators at the pair of points A on such separator.

7. I say that in the *first* of the cases above considered the locus of the orbit-pole is a separator; in the *second* case the orbit-pole is a point B; in the *third* case the locus is the regulator; and in the *fourth* case the orbit-pole is a point A.

8. In the absence of models, the spherical figure must be represented by a projection; the stereographic projection is convenient for facility of description; and it has the very great advantage that we can by means of it exhibit, no matter how large a portion of the spherical surface. In the figures called "spherograms," afterwards referred to, the representation of a hemisphere is all that is required; but, to give a more distinct general idea, I annex a figure representing a larger portion of the surface; the data are those belonging to the particular symmetrical case referred to as intended to be specially considered: and the regulator conic is accordingly a pair of opposite small circles, the points A and B being related to it symmetrically; but, disregarding these specialities, the figure is adapted to the illustration of the general

Fig. 1.



case (at least if the point S be situate *within* the hyperboloid), and it is here given for that purpose. The circle marked "Ecliptic" does not properly belong to the figure: it is added as showing the boundary of a hemisphere, so that, by omitting all that lies outside this circle, the figure would be limited to the representation of a hemisphere; and the orbit-pole be in every case represented, no longer as a pair of opposite points, but as a single point; we should have the separators each as a half circle, and the regulator as a single small circle; the separators would intersect in pairs, in the *three* points B, and would touch the regulator in the *three* points A, &c.

9. The figure constructed as above, but omitting so much of it as lies outside the ecliptic circle, is the representation of a hemisphere—say of the northern hemi-

sphere. It is readily seen that the central triangle *BBB* and the three circumjacent triangles *BBB*, represent also the half-surface of the sphere, viz., instead of the omitted portions of the northern hemisphere we have the equal opposite portions of the southern hemisphere. The adoption of this figure as the representation of the half-surface of the sphere has the great advantage that the spherical curves can be delineated without the apparent breaks which would otherwise occur at their intersections with the ecliptic circle: I accordingly adopt it, and call the figure in question (viz., that composed of the four triangles) a blank "spherogram." We wish for any given position thereon of the orbit-pole to determine the values of certain parameters (eccentricity, latus rectum, time of passage between two rays, &c., as the case may be) belonging to the orbit, with a view to the subsequent delineation of the corresponding isoparametric (iseccentric, isochronic, &c.) lines, so constructing a "spherogram" for any such parameter, or system of lines.

10. It is for this purpose convenient to consider the values of the parameter corresponding to a single series of positions of the orbit-pole, viz., we consider the orbit-pole as describing on the sphere a curve selected at pleasure. Consider for a moment the orbit-plane as a material plane rigidly connected with the orbit-axis; the motion of the orbit-pole does not absolutely determine the motion of the orbit-plane, inasmuch as the orbit-plane, occupying the same position in space, might rotate about the orbit-axis; but if we exclude any such motion by the assumption that the motion of the orbit-plane is always about an axis in the orbit-plane, then the motion of the orbit-pole determines that of the orbit-plane, viz., the orbit-plane envelopes a cone, the reciprocal to that described by the orbit-axis. If then on the orbit-plane in each position thereof we mark, as well its line of contact with the enveloped cone, as also its intersections with the three rays, we obtain a figure (which may, if we please, be regarded as drawn on the orbit-plane in some particular position thereof), such figure consisting of a series of trivectors, and (belonging to each of them) a line through S serving to fix the position of the trivector in space. The locus of each extremity of the trivector is a certain curve, and the construction establishes a point-to-point correspondence between these three curves; viz., to any point on one of them there corresponds on each of the other two a single point, the three points being the extremities of a trivector. The figure would be rendered more complete by drawing the orbit belonging to each trivector thereof. Such a figure (with or without the orbits) is termed a "planogram."

11. The most simple case is when the orbit-pole describes a great circle; the orbit-plane here rotates about a fixed line, the axis of the circle, or (what is the same thing) the enveloped cone reduces itself to this axis of rotation; and the line of contact is thus a fixed line in the orbit-plane; or (what is the same thing) the lines through S in the planogram are here a single fixed line, the axis of rotation. I say that, for each extremity of the trivector, the locus is a hyperbola, having the axis of rotation for its conjugate axis. In fact, attending to any one ray, it is the same thing whether the orbit-plane be made to revolve round the axis of rotation, so as continually to intersect the ray, or whether, considering the orbit-plane as fixed,

and the ray as rigidly connected with the axis, we make the ray to rotate about this axis, so as continually to intersect the orbit-plane. But in this last case the ray describes about the axis a hyperboloid of revolution, and the orbit-plane, as an axial plane, meets this surface in a hyperbola having the axis for its conjugate axis; which hyperbola is the required locus of the trivector-extremity. It is moreover easy to see that if the angle of position of the variable orbit-plane, or (what is the same thing) the angle of position of the orbit-pole in the great circle which it describes be $=q$ (where q is measured from any fixed plane or point), and if the coordinates x' and y' be measured from S in the direction of and perpendicular to the axis of rotation, then the coordinates of the point on the hyperbola are expressed in the form $x' = \alpha + a \tan(q+\beta)$, $y' = b \sec(q+\beta)$, where α, a, b, β, are constants depending on the position of the ray in regard to the axis of rotation: see as to this *post*, No. 49.

12. Considering the orbit-pole as describing a given curve, the value for the several positions thereof of any parameter of the orbit may be exhibited by means of a "diagram," viz., we may take for abscissa any quantity serving to fix the position of the orbit-pole on the described curve, and for ordinate the value of the parameter in question. In the particular case where the orbit-pole describes a great circle passing through the axis of the stereographic projection, and which is consequently in the spherogram represented by a diameter of the ecliptic or bounding circle, it is natural to take for the abscissa the distance (from the centre) of the representation of the orbit-pole; the diagram will then fit on to the diameter, and for any position of the orbit-pole on such diameter give at once the value of the parameter to which the diagram relates.

13. It is right to remark that the construction of planograms and diagrams is merely subsidiary to that of the spherograms; the information given by any number of planograms or diagrams would be all of it embodied in a spherogram for the same parameter. And theoretically the construction of a spherogram is a mere matter of geometry; for a given position of the orbit-pole we construct the trivector, thence the orbit, and in relation thereto any parameters which it is desired to consider; and so, for a sufficient number of points on the spherogram, determine the value of the parameter, or parameters; and lay down the isoparametric lines. The construction of the orbit from a given trivector, and in particular the selection of the orbit as one of the four conics given by the trivector, has not yet been explained: in connexion herewith we have the discontinuity of orbit which arises when the orbit-pole is upon a separator, and which is a leading circumstance in the theory; until it is gone into, there is little more to be said in the way of general explanation as to the spherogram, or the isoparametric lines thereof.

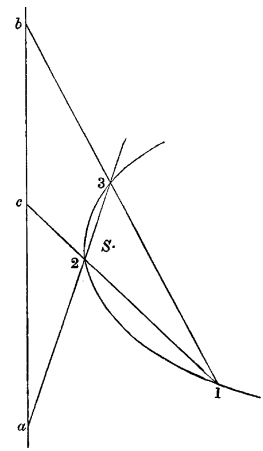
14. It may however be noticed that for any parameter whatever, the points A of the spherogram are common points, through which pass in general the lines belonging to any value whatever of the parameter; the reason of course is that the orbit-plane then passing through the ray, and the orbit itself being indeterminate, the value of any parameter belonging to the orbit is also indeterminate. Moreover, for some parameters the curve belonging to any particular value of the parameter not only

passes through the points A, but passes through each point twice, or (what is the same thing) has each of the points A for a nodal point; when this is so, then it is to be further observed that, for certain values of the parameters, they will be acnodal points, properly belonging to the curve, although there is not any real branch of the curve passing through the points A; for others they will be crunodal points, with two real branches through each; and in the transition between the two cases they will be cuspidal points on the isoparametric curve; it will appear in the sequel that this is really the case in regard to the iseccentric lines.

Article Nos. 15 to 30. *Determination of the Orbit from a given Trivector.*

15. With a given point S as focus, and through three given points, that is with a given trivector, there may be described *four* conics. This appears from the general theory according to which a given focus is equivalent to two given tangents; and also

Fig. 2.



from the geometrical construction, *Principia*, book I. sect. 4; Scholium to Prop. XXI.: viz. given the focus S and the points 1, 2, 3, then if

On 23 we find a so that $a\,2 : a\,3 = S\,2 : S\,3$,

" 31 " b " $b\,3 : b\,1 = S\,3 : S\,1$,

" 12 " c " $c\,1 : c\,2 = S\,1 : S\,2$,

the points a, b, c, are each of them on the directrix, so that any two of them determine the directrix. In the figure (as in Newton's) the distances $S\,1$, $S\,2$, $S\,3$, are

each regarded as positive, but the very same construction, taking two of the distances each as positive and negative successively, would lead to three other positions of the directrix; or the construction would give in all four conics.

16. In the figure the directrix lies on the same side of the three points; and the conic is thus an ellipse or parabola, or, if a hyperbola, then the three points lie in the same branch thereof; and it is consequently an orbit such that along it a body can pass through the three points successively. The construction as varied would give in each case a directrix having on one side of it one, and on the other side two, of the three points; so that the conic would be a hyperbola having the three points not on the same branch thereof; consequently it would not be an orbit such that along it a body could pass through the three points successively.

And it thus appears that though the trivector really determines four conics, yet it is only one of these in which the directrix lies on the same side of the three points; and this conic I call the "orbit:" the given trivector thus determines a single orbit.

17. It is to be noticed however that the orbit constructed as above *may* be a hyperbolic branch separated by the directrix from the focus S, and consequently convex to the focus S; viz., the three points lie here in a hyperbolic branch convex to S, and which is therefore not an orbit which can be described under the action of an *attractive* force at S: say we have a "convex orbit." I regard this as a real orbit, but the times of passage therein as imaginary, or rather as non-existent, and the case is thus excluded from consideration in the formulæ and figures which relate to the times of passage.

18. The same results are established analytically in a very similar manner, viz., taking the focus for origin and starting from the focal equation

$$r = Ax + By + C;$$

then if we take (x_1, y_1), (x_2, y_2), (x_3, y_3), as the coordinates of the three given points and write

$$r_1 = \sqrt{x_1^2 + y_1^2}, \quad r_2 = \sqrt{x_2^2 + y_2^2}, \quad r_3 = \sqrt{x_3^2 + y_3^2},$$

we have for the determination of the constants

$$\begin{aligned} r_1 &= Ax_1 + By_1 + C, \\ r_2 &= Ax_2 + By_2 + C, \\ r_3 &= Ax_3 + By_3 + C, \end{aligned}$$

and the equation therefore is

$$\begin{vmatrix} r, & x, & y, & 1 \\ r_1, & x_1, & y_1, & 1 \\ r_2, & x_2, & y_2, & 1 \\ r_3, & x_3, & y_3, & 1 \end{vmatrix} = 0,$$

which, attributing therein to r_1, r_2, r_3, the signs $+$, $-$ at pleasure, represents eight different equations: these however give only four conics, viz., we have the same conic whether we attribute to r_1, r_2, r_3, any particular combination of signs, or reverse all the signs simultaneously.

19. But the focal equation $r = Ax + By + C$ is precisely equivalent to the equation

$$r = \frac{p}{1 + e\cos(\theta - \varpi)},$$

and in this equation (taking as is allowable p as positive) then if $\pm e$ be $=$ or < 1, that is for an ellipse or parabola whatever be the value of $\theta - \varpi$, r is always positive; but if $\pm e$ be > 1, that is for a hyperbola, r is positive for those values of $\theta - \varpi$ which belong to one branch, negative for those which belong to the other branch, of the curve. Hence in the determinant equation, unless r_1, r_2, r_3, have the same sign, the curve will be a hyperbola with the points two of them on one branch, the third on the other branch thereof. But in the remaining case, when r_1, r_2, r_3, have all the same sign, or say when they are all positive, then the conic is an ellipse or parabola, or else it is a hyperbola with the three points on the same branch thereof; that is, the foregoing determinant equation, regarding therein r_1, r_2, r_3, as all of them positive, gives the orbit.

20. When one of the points is at infinity on a given line there is a discontinuity of orbit. To explain this, suppose that the point (x_1, y_1) is situate on the line $y = x\tan\alpha$, at an indefinitely great distance r_1 in one or the other direction along the line; viz., r_1 is an indefinitely large positive quantity, and we have in the one case x_1, $y_1 = r_1\cos\alpha$, $r_1\sin\alpha$; and in the other case x_1, $y_1 = -r_1\cos\alpha$, $-r_1\sin\alpha$: the corresponding equations of the orbit, putting therein ultimately $r_1 = +\infty$, are

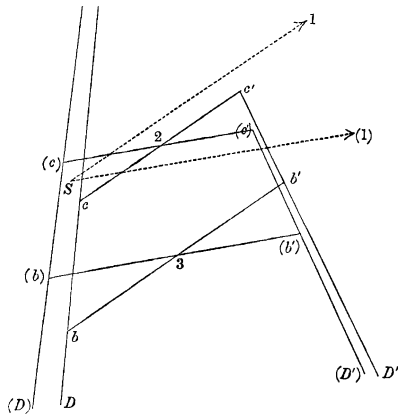
$$\begin{vmatrix} r, & x, & y, & 1 \\ 1, & \cos\alpha, & \sin\alpha, & 0 \\ r_2, & x_2, & y_2, & 1 \\ r_3, & x_3, & y_3, & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} r, & x, & y, & 1 \\ 1, & -\cos\alpha, & -\sin\alpha, & 0 \\ r_2, & x_2, & y_3, & 1 \\ r_3, & x_3, & y_3, & 1 \end{vmatrix} = 0,$$

which equations belong, it is clear, to two distinct conics; or as the point (x_1, y_1) passes from a positive to a negative infinity along the given line, there is an abrupt change of orbit. It is proper to remark that the two orbits are the very same as would be obtained by writing x_1, $y_1 = r_1\cos\alpha$, $r_1\sin\alpha$, $r_1 = +\infty$ and $r_1 = -\infty$ in the determinant equation: that is, the orbit passes abruptly from one to another of the four conics which belong to the position (x_1, y_1), and we thus understand how the transition from $+\infty$ to $-\infty$, which is geometrically no breach of continuity, occasions in the actual problem a discontinuity.

21. The same thing appears from the geometrical construction; and we derive a further result which will be useful. Suppose first that the point 1 is at infinity in the direction shown by the arrow; then drawing $2c = 2S$ and $3b = 3S$ each in the direction opposite to $S1$, we have the points b, c on the directrix, which is thus the

line D joining these points. But if 1 is at infinity on the same line in the opposite direction, then instead of c, b we have the points c', b', and the directrix is the line D' joining these points.

Fig. 3.



22. Observe that in the first case the focus S and the three points are on opposite sides of the directrix D, or the orbit is convex; but in the second case the focus S and the three points are on the same side of the directrix D', and the orbit is concave. That is, the line S_1 does not separate the two points 2, 3, and the orbits are the one convex, the other concave.

23. But if 1 be at infinity along the line $S(1)$ first in the direction shown by the arrow, and then in the opposite direction; in the first case the directrix is (D) not separating the focus S from the three points, and the orbit is concave; in the second case the orbit is (D'), not separating S from the three points, and the orbit is still concave; here the line $S(1)$ does separate the points 2, 3, and the orbits are both concave.

24. And we thus see in general that as the point 1 passes from a positive to a negative infinity along a line passing through S; then, according as the line through S does not or does separate the remaining two points 2, 3, the orbits corresponding to the two positions of 1 are the one convex, the other concave, or they are both concave.

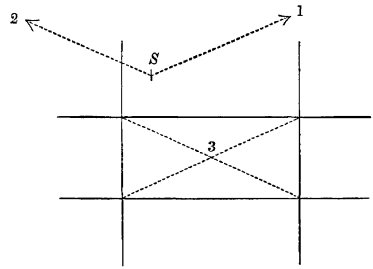
25. The points 1 and 2 may be each of them at infinity along a given ray; we have here in a similar manner x_1, $y_1 = r_1 \cos \alpha_1$, $r_1 \sin \alpha_1$, or else $= -r_1 \cos \alpha_1$, $-r_1 \sin \alpha_1$, where r_1 is an indefinitely large positive quantity; and x_2, $y_2 = r_2 \cos \alpha_2$, $r_2 \sin \alpha_2$, or else

$= -r_2 \cos \alpha_2$, $-r_2 \sin \alpha_2$, where r_2 is an indefinitely large positive quantity. And writing ultimately $r_1 = +\infty$, $r_2 = +\infty$, the equation of the orbit is obtained in the form

$$\begin{vmatrix} r, & x, & y, & 1 \\ 1, & \pm\cos\alpha_1, & \pm\sin\alpha_1, & 0 \\ 1, & \pm\cos\alpha_2, & \pm\sin\alpha_2, & 0 \\ r_3, & x_3, & y_3, & 1 \end{vmatrix} = 0,$$

where the $\pm$ of the second line and the $\pm$ of the third line have each of them the value $+$ or $-$ at pleasure. There are consequently four distinct orbits, corresponding to the combinations of each of the two directions of the point 1 with each of the two directions of the point 2. And it is moreover clear that these are the very conics which are obtained from the determinant equation by writing therein x_1, $y_1 = r_1 \cos \alpha_1$, $r_1 \sin \alpha_1$; x_2, $y_2 = r_2 \cos \alpha_2$, $r_2 \sin \alpha_2$ and $r_1 = +\infty, -\infty$; $r_2 = +\infty, -\infty$ successively; viz., the orbit changes abruptly between the four conics which correspond to the given position of the points 1, 2, 3.

Fig. 4.



26. The geometrical construction is very simple indeed; viz., measuring off from 3 in the directions $S1$, $S2$, and in the opposite directions respectively, a distance $= S3$, we have four points, the angles of a rectangle; and joining these in pairs, we have the four positions of the directrix: the figure shows at once that the orbits are three of them concave, the remaining one convex.

27. The determinant equation obtained for the orbit is an equation of the form

$$r = Ax + By + C;$$

and it is clear that the equation of the directrix is $Ax + By + C = 0$. By what precedes, this line will lie on the same side of the three points; viz., either it does not separate them from the focus, and the orbit is then concave, or it does separate them from the focus, and the orbit is then convex. Although in general the sign of C is no criterion (for the equations $r = Ax + By + C$ and $r = -Ax - By - C$ represent

the same curve) yet in the present case it is so; for, observe that, in taking r_1, r_2, r_3 each of them positive, we make r to be positive for the orbit, that is, for the entire curve if an ellipse or parabola, but for the branch containing the three points if the curve is a hyperbola. Hence, considering the radius vector through S parallel to the directrix, this is positive for a concave, negative for a convex orbit; or writing $Ax+By=0$, we have $r=C$ positive for a concave, negative for a convex orbit; wherefore the orbit is concave or convex according as C is positive or negative.

28. Comparing the equation with

$$r=e(x\cos\varpi+y\sin\varpi)\pm a(1-e^2),$$

we see that the eccentricity and semiaxis major, taken to be each of them positive, are

$$e=\sqrt{A^2+B^2},\quad a=\frac{\pm C}{1-A^2-B^2},$$

($+C$ or $-C$, according as $e<1$ or $e>1$); and inasmuch as the focus and directrix are known, there is no ambiguity as to the position of the orbit: it may be added that the coordinates of the centre are given by

$$\begin{aligned}(A^2-1)\,x+AB\quad\;\; y+AC&=0,\\ AB\quad x+(B^2-1)\,y+BC&=0,\end{aligned}$$

that is, we have for the coordinates of the centre

$$x=\frac{AC}{1-A^2-B^2},\quad y=\frac{BC}{1-A^2-B^2};$$

and thence also

$$x=\frac{2AC}{1-A^2-B^2},\quad y=\frac{2BC}{1-A^2-B^2}$$

for the coordinates of the other focus.

29. But to effect the comparison rather more precisely it is to be observed that a, e being positive, then for a concave orbit, if X be measured from the focus in the direction *away from* the directrix, we should have

$$r=eX\pm a(1-e^2)$$

($+$ for the ellipse, $-$ for the hyperbola, so that $\pm a(1-e^2)$ is positive): whence

$$e=\sqrt{A^2+B^2},\quad X=\frac{Ax+By}{\sqrt{A^2+B^2}},\quad a=\frac{\pm C}{1-A^2-B^2}$$

(by what precedes, C is $=+$, so that the formula gives as it should do $a=+$).

And similarly for a convex orbit, if X be measured in the direction *towards* the directrix, we should have

$$r=eX-a(e^2-1);$$

whence

$$e=\sqrt{A^2+B^2},\quad X=\frac{Ax+By}{\sqrt{A^2+B^2}},\quad a=\frac{-C}{A^2+B^2-1},$$

where by what precedes C is $=-$, and the formula gives as it should do $a=+$.

30. It is not necessary for the purpose of the present memoir, but I notice an elegant form of the polar equation of the orbit belonging to a given trivector; viz., taking (r, θ) as polar coordinates, and therefore (r_1, θ_1), (r_2, θ_2), (r_3, θ_3), as the coordinates of the given points, the equation of the orbit is

$$\frac{1}{r}=\Sigma\frac{1}{r_1}\cdot\frac{\sin\frac{1}{2}(\theta-\theta_2)\sin\frac{1}{2}(\theta-\theta_3)}{\sin\frac{1}{2}(\theta_1-\theta_2)\sin\frac{1}{2}(\theta_1-\theta_3)}.$$

In fact, it is clear that this is an equation of the form

$$\frac{1}{r}=(\alpha, \beta, \gamma)(\sin\tfrac{1}{2}\theta, \cos\tfrac{1}{2}\theta)^2;$$

that is of the form

$$\frac{1}{r}=\lambda\cos\theta+\mu\sin\theta+\nu;$$

and that it thus represents a conic with the given focus; and moreover that the equation is satisfied by writing therein (r_1, θ_1), (r_2, θ_2), or (r_3, θ_3), in place of (r, θ); that is, the conic passes through the three given points. The foregoing remarks as to the signs of r_1, r_2, r_3, apply without alteration to this polar equation.

Article Nos. 31 to 41. *Time Formulæ;* LAMBERT'S *Equation.*

31. Suppose for a moment that the orbit is an ellipse; as the ellipse may be described in either direction, the time of passage between any two points, 1 to 2, or 2 to 1, indifferently, may be regarded as positive. With only two points 1, 2, we might pass, say from 1 to 2, in either direction along the ellipse, and the time of passage would have ambiguously either of two positive values. In the case however where we have on the ellipse three points, 1, 2, 3, this ambiguity is avoided; viz., it is assumed that the passage between any two of the points is along the elliptic arc which does not contain the third point; the three times of passage are thus all of them positive, and their sum is equal to the periodic time, or time of describing the entire ellipse.

32. But if the orbit be a parabola or concave hyperbolic branch, then, if the points taken in their order of position along the orbit be 1, 2, 3, we have in like manner a positive time of passage between 1 and 2, and also a positive time of passage between 2 and 3; but, inasmuch as there is no passage between 1 and 3 except through 2 (which mode is excluded from consideration), I say that there is no time of passage between 1 and 3; and so consider only two times of passage; viz., between 1 and 2, and between 2 and 3.

33. In the case of a convex hyperbolic branch, since this cannot be described under the action of an attractive force, there is not any time of passage to be considered.

In the transition case of a right line not passing through the focus, since, as mentioned, the velocity is infinite, if the order of the points on the line is 1, 2, 3, the times of passage from 1 to 2 and from 2 to 3 are each $=0$; and these are the only times of passage which are to be considered.

34. The preceding conventions are of course to be attended to in the application of any formula to the calculation of the times of passage between given points of the orbit; in the case of a parabolic or hyperbolic orbit we have only to ascertain which are the two times of passage to be calculated; but, in the case of an ellipse, we must take care that the time of passage between each two of the three points is calculated along the arc not containing the third point; viz., it is in some cases to be calculated through the angle $<\pi$ between the two radius vectors, and in other cases through the angle $>\pi$ between the two radius vectors; or, more simply, the time to be calculated is sometimes the longer, and at other times the shorter time of passage.

35. For the purpose of the present memoir the unit of time is so fixed that the periodic time in a circle radius 1 shall be equal 3. The period in a circle or ellipse, radius or semiaxis major $=a$, is thus $=3a^{\frac{3}{2}}$, and generally

$$\text{Time} = \frac{3}{\pi} \cdot \frac{\text{Area}}{\sqrt{\frac{1}{2}\ \text{latus rectum}}}.$$

The time formulæ are first the ordinary ones in which the time from pericentre is expressed in terms of an angle (the eccentric anomaly for an ellipse or hyperbola, true anomaly for the parabola); secondly, Lambert's formulæ, in which the time between any two points on the orbit is expressed by means of the two radius vectors and the chord.

36. The first set of formulæ may be written:

Ellipse. u, the eccentric anomaly from pericentre, viz. $x=a(\cos u-e)$, $y=a\sqrt{1-e^2}\sin u$, if x, y, are the coordinates from the focus, x measured in the direction *towards* the directrix.

$$\text{Time from pericentre} = \frac{3}{\pi} a^{\frac{3}{2}} (u - e \sin u).$$

Parabola. θ, the true anomaly, viz., $r = p \sec^2 \frac{1}{2}\theta$, if p be the pericentric distance or $\frac{1}{4}$-latus rectum.

$$\text{Time from pericentre} = \frac{3}{\pi} \frac{p^{\frac{3}{2}}}{\sqrt{2}} (\tan \tfrac{1}{2}\theta + \tfrac{1}{3} \tan^3 \tfrac{1}{2}\theta).$$

Hyperbola; concave branch. u, the eccentric anomaly from pericentre, viz., $x = a(\sec u - e)$, $y = a\sqrt{e^2-1}\tan u$, if x, y are the coordinates from the focus, x measured in the direction *away* from the directrix.

$$\text{Time from pericentre} = \frac{3a^{\frac{3}{2}}}{2\pi} \{e \tan u - \text{hyp. log} \tan (\tfrac{1}{4}\pi + \tfrac{1}{2} u)\},$$

and by taking the sum or the difference of two of these expressions, we obtain the time of passage between two given points of the orbit.

37. I remark that as to the elliptic and parabolic orbits, I have preferred using Lambert's equations, and I should have done the same for the hyperbolic orbits, but for the absence of a table (see *post*, No. 39). As it is, for the few hyperbolic orbits which it was necessary to calculate, I have used the foregoing formula[1]: a table of hyp. log tan $(\frac{1}{4}\pi + \frac{1}{2}u)$, $u = 0°$ to $u = 90°$, at intervals of 30′ to 12 places of decimals, with fifth differences is given, Table IV. Legendre, *Traité des Fonctions Elliptiques*, t. II. pp. 256—259.

38. The other set of formulæ may be written:

Ellipse. r, r' the radius vectors, γ the chord.

$$2a\cos\chi = 2a - r - r' - \gamma, \qquad 2a\cos\chi' = 2a - r - r' + \gamma.$$

$$\text{Time} = \frac{3}{2\pi}a^{\frac{3}{2}}(\chi - \chi' - \sin\chi + \sin\chi').$$

Parabola. r, r', γ, *ut suprà;*

$$\text{Time} = \frac{1}{4\pi}\{(r + r' + \gamma)^{\frac{3}{2}} - (r + r' - \gamma)^{\frac{3}{2}}\}.$$

Hyperbola.

$$2a\cosh\chi = 2a + r + r' + \gamma, \qquad 2a\cosh\chi' = 2a + r + r' - \gamma.$$

$$\text{Time} = \frac{3}{2\pi}a^{\frac{3}{2}}(-\chi + \chi' + \sinh\chi - \sinh\chi'),$$

where cosh, sinh, denote the hyperbolic cosine and sine of χ, viz.:

$$\cosh\chi = \tfrac{1}{2}(e^{\chi} + e^{-\chi}), \qquad \sinh\chi = \tfrac{1}{2}(e^{\chi} - e^{-\chi}).$$

39. The logarithms (ordinary) of the functions $\cosh\chi$, $\sinh\chi$, and of $\tanh\chi$ are tabulated by Gudermann, *Crelle*, tt. VIII. and IX. from $\chi = 2{\cdot}000$ to $\chi = 8{\cdot}00$ at intervals of ·001 and subsequently of ·01 to eight places of decimals. I do not know why the tabulation was not commenced from $\chi = 0$, but the omission from them of the values 0 to 2 rendered the tables unavailing for the present purpose, and I therefore, for the hyperbolic orbits, resorted to the first set of formulæ.

40. As regards the elliptic formulæ it remains to be explained how the values of χ, χ' are to be selected from those which satisfy the required conditions

$$2a\cos\chi = 2a - r - r' - \gamma, \qquad 2a\cos\chi' = 2a - r - r' + \gamma.$$

It is remarked in Gauss' *Theoria Motûs*, p. 120, that χ is a positive angle between 0° and 360°; χ' a positive or negative angle between $+180°$, $-180°$, viz. χ' is positive

[1] I rather regret that I did not use the foregoing formulæ in all cases.

or negative according as the angle between the two radius vectors is $< 180°$ or $> 180°$. This determines χ', but it is said that χ is really indeterminate; viz. it is so if only the values r, r', γ, a, are given, for there are then two orbits in which these quantities have their given values, and the times in these have different values. But when, as in the case here considered the orbit is known, χ will of course have a determinate signification, and it is easy to explain how this is to be fixed. I observe, in the first place, that if $\chi = \pi$ we have $\gamma = (2a - r) + (2a - r')$, that is, the chord γ passes through the other focus of the ellipse. The criterion thus depends on the position of the two points on the ellipse in relation to the other focus, and it is easy to see that it is as follows: viz. let the time between the points 1, 2, on the ellipse be understood to mean the time of passage from 1 through apocentre to 2; then I say that, in the preceding formula

$$\text{Time} = \frac{3}{2\pi} a^{\frac{3}{2}} (\chi - \chi' - \sin\chi + \sin\chi'),$$

χ will be $< 180°$ or $> 180°$ according as the chord from 1 through the other focus H does not or does separate the point 2 from the focus S.

41. It is hardly necessary to remark that in the application of the formulæ, χ, χ' must be reckoned according to their lengths as circular arcs to the radius unity: a table for the conversion of degrees and minutes to such circular measure, is given in most collections of Trigonometrical Tables.

Article Nos. 42 to 45. *Formulæ for the Transformation between two sets of Rectangular Axes.*

42. Consider an arbitrary set of fixed rectangular axes, Sx, Sy, Sz, which are considered as intersecting the sphere, centre S, in the points X, Y, Z, and so the axes Sx', Sy', Sz', afterwards defined are considered as intersecting the sphere in the points X', Y', Z'. For convenience Sx is considered as an origin of longitudes, which are measured in the plane of xy in the direction towards y; and an angular distance from Sz is termed a polar distance or colatitude; so that the position of any line through S, or point on the sphere, will be determined by its longitude b and colatitude c.

43. It is wished in the sequel to made the orbit-pole revolve about an arbitrary line Sx', and for this purpose I take the new set of rectangular axes, Sx', Sy', Sz', or points on the sphere X', Y', Z', as follows,

$$X', \text{ longitude } G, \text{ colatitude } 90° + N.$$

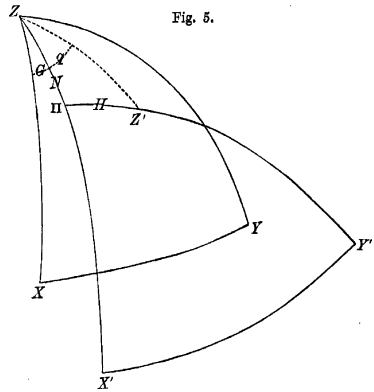
$Y'Z'$, is then a great circle, pole X', meeting ZX' in a point Π, longitude G, colatitude N, and the position of Z' in this great circle is fixed by its distance from Π, $\Pi Z' = H$, the distance of Y' being $\Pi Y' = 90° + H$, and these being each of them reckoned from Π in the direction of longitude X to Y. The position of the new axes Sx', Sy', Sz', or points X', Y', Z', is thus fixed by means of the three angles G, N, H.

It is to be added that if the angle $X'ZZ'$ is called q, and if b, c, are the longitude and colatitude of Z', then we have $\sin N = \cot q \tan H$, which gives q, and then

$$b = G + q$$

$$\cos c = \cos N \cos H.$$

Fig. 5.



44. The transformation-formulæ between the two sets of axes are at once found to be

	X	Y	Z
X'	$\cos G \cos N$	$\sin G \cos N$	$-\sin N$
Y'	$-\sin G \cos H - \cos G \sin H \sin N$	$\cos G \cos H - \sin G \sin H \sin N$	$-\sin H \cos N$
Z'	$-\sin G \sin H + \cos G \cos H \sin N$	$\cos G \sin H + \sin G \cos H \sin N$	$\cos H \cos N$

which are for shortness represented by

	X	Y	Z
X'	α	β	γ
Y'	α'	β'	γ'
Z'	α''	β''	γ''

45 In the particular case where Sx' is in the plane of xy, $N=0$; Π coincides with Z, and the longitude and colatitude of Z' are $b=G+90°$, $c=H$. Writing accordingly in the formula $N=0$, and introducing b, c in the place of G, H, the formulæ become

	X	Y	Z
X'	$\sin b$	$-\cos b$	0
Y'	$\cos b \cos c$	$\sin b \cos c$	$-\sin c$
Z'	$\cos b \sin c$	$\sin b \sin c$	$\cos c$

and in particular if $c=0$, (Sz' here coincides with Sz, and the axes Sx', Sy', are in the plane of xy) then we have simply

	X	Y	Z
X'	$\sin b$	$-\cos b$	0
Y'	$\cos b$	$\sin b$	0
Z'	0	0	1

Article Nos. 46 to 60. *Application to finding the Intersection of the Orbit-plane by a Single Ray.*

46. The equations of the ray referred to the fixed axes are taken to be

$$\frac{x-A}{\mathrm{f}}=\frac{y-B}{\mathrm{g}}=\frac{z-C}{\mathrm{h}}, \quad =R \text{ suppose,}$$

or, what is the same thing,

$$x=A+R\mathrm{f},$$
$$y=B+R\mathrm{g},$$
$$z=C+R\mathrm{h},$$

and if in the foregoing formulæ the point Z' is taken to be the orbit-pole (longitude $b=G+90°$, and colatitude $c=\cos^{-1}\cos N\cos H$ as above) then the equation of the

orbit-plane is $z'=0$. We have therefore merely to transform the equations of the ray to the new axes by writing for x, y, z, the values

$$\alpha x'+\alpha' y'+\alpha'' z',$$
$$\beta x'+\beta' y'+\beta'' z',$$
$$\gamma x'+\gamma' y'+\gamma'' z',$$

and then putting $z'=0$, we find x', y', the coordinates in the orbit-plane of its intersections with the ray.

47. The equations thus become,

$$\alpha x'+\alpha' y'-A-R\mathrm{f}=0,$$
$$\beta x'+\beta' y'-B-R\mathrm{g}=0,$$
$$\gamma x'+\gamma' y'-C-R\mathrm{h}=0,$$

or, what is the same thing, we have

$$x' : y' : R : 1$$

$$= \begin{vmatrix} 1 & & & \\ \alpha, & \alpha', & \mathrm{f}, & A \\ \beta, & \beta', & \mathrm{g}, & B \\ \gamma, & \gamma', & \mathrm{h}, & C \end{vmatrix} : \begin{vmatrix} & 1 & & \\ \alpha, & \alpha', & \mathrm{f}, & A \\ \beta, & \beta', & \mathrm{g}, & B \\ \gamma, & \gamma', & \mathrm{h}, & C \end{vmatrix} : \begin{vmatrix} & & -1 & \\ \alpha, & \alpha', & \mathrm{f}, & A \\ \beta, & \beta', & \mathrm{g}, & B \\ \gamma, & \gamma', & \mathrm{h}, & C \end{vmatrix} : \begin{vmatrix} & & & -1 \\ \alpha, & \alpha', & \mathrm{f}, & A \\ \beta, & \beta', & \mathrm{g}, & B \\ \gamma, & \gamma', & \mathrm{h}, & C \end{vmatrix}$$

$$= \begin{vmatrix} \alpha', & \mathrm{f}, & A \\ \beta', & \mathrm{g}, & B \\ \gamma', & \mathrm{h}, & C \end{vmatrix} : - \begin{vmatrix} \alpha, & \mathrm{f}, & A \\ \beta, & \mathrm{g}, & B \\ \gamma, & \mathrm{h}, & C \end{vmatrix} : - \begin{vmatrix} A, & \alpha, & \alpha' \\ B, & \beta, & \beta' \\ C, & \gamma, & \gamma' \end{vmatrix} : \begin{vmatrix} \mathrm{f}, & \alpha, & \alpha' \\ \mathrm{g}, & \beta, & \beta' \\ \mathrm{h}, & \gamma, & \gamma' \end{vmatrix}.$$

In these formulæ we have identically

$$\beta\gamma'-\beta'\gamma,\quad \gamma\alpha'-\gamma'\alpha,\quad \alpha\beta'-\alpha'\beta=\alpha'',\ \beta'',\ \gamma'',$$

and if we write moreover

$$\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ =C\mathrm{g}-B\mathrm{h},\quad A\mathrm{h}-C\mathrm{f},\quad B\mathrm{f}-A\mathrm{g},$$

(whence identically $\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0$, and where (a, b, c, f, g, h) are the "six coordinates" of the ray), then we have the very simple formulæ

$$x' : y' : R : 1$$
$$=(\mathrm{a},\ \mathrm{b},\ \mathrm{c} \between \alpha',\ \beta',\ \gamma') : -(\mathrm{a},\ \mathrm{b},\ \mathrm{c} \between \alpha,\ \beta,\ \gamma) : (A,\ B,\ C \between \alpha'',\ \beta'',\ \gamma'') : (\mathrm{f},\ \mathrm{g},\ \mathrm{h} \between \alpha'',\ \beta'',\ \gamma''),$$

or omitting (as not required for the present purpose) one of the proportional terms, we have

$$x' : y' : 1=(\mathrm{a},\ \mathrm{b},\ \mathrm{c} \between \alpha',\ \beta',\ \gamma') : -(\mathrm{a},\ \mathrm{b},\ \mathrm{c} \between \alpha,\ \beta,\ \gamma) : (\mathrm{f},\ \mathrm{g},\ \mathrm{h} \between \alpha'',\ \beta'',\ \gamma''),$$

which are the required expressions for the coordinates.

48. Consider in the equations just obtained the axis of x' as fixed but H as variable; that is, let the orbit-pole Z' describe a great circle about the fixed pole X' (longitude G, colatitude $90° + N$). We have x', y', 1, proportional to linear functions of $\sin H$, $\cos H$; viz., writing for shortness

$$\begin{aligned}
X_c &= -\mathrm{a}\sin G + \mathrm{b}\cos G,\\
X_s &= (-\mathrm{a}\cos G - \mathrm{b}\sin G)\sin N - \mathrm{c}\cos N,\\
Y_o &= (-\mathrm{a}\cos G - \mathrm{b}\sin G)\cos N + \mathrm{c}\sin N,\\
W_c &= (\ \ \mathrm{f}\cos G + \mathrm{g}\sin G)\sin N + \mathrm{h}\cos N,\\
W_s &= (-\mathrm{f}\sin G + \mathrm{g}\cos G),
\end{aligned}$$

we have

$$x' = \frac{X_c\cos H + X_s\sin H}{W_c\cos H + W_s\sin H},$$

$$y' = \frac{Y_o}{W_c\cos H + W_s\sin H}.$$

49. I write

$$\left\{\begin{aligned}
&\frac{W_c}{Y_o} = \frac{1}{m}\cos\Delta, \qquad \frac{W_s}{Y_o} = \frac{1}{m}\sin\Delta,\\
&\frac{X_c}{Y_o} = \frac{l}{m}\cos\Delta - \cot\delta\sin\Delta,\\
&\frac{X_s}{Y_o} = \frac{l}{m}\sin\Delta + \cot\delta\cos\Delta,
\end{aligned}\right.$$

equations which determine m, Δ, l, δ, viz., we have

$$\tan\Delta = \frac{W_s}{W_c}, \qquad m = \frac{Y_o}{\sqrt{W_c^2 + W_s^2}},$$

$$l = m\,\frac{X_c\cos\Delta + X_s\sin\Delta}{Y_o} = \frac{1}{W_c^2 + W_s^2}(X_cW_c + X_sW_s),$$

$$\cot\delta = \frac{-X_c\sin\Delta + X_s\cos\Delta}{Y_o} = \frac{1}{Y_o\sqrt{W_c^2 + W_s^2}}(X_sW_c - X_cW_s),$$

and we then very easily find

$$\begin{aligned}
x' &= l + m\cot\delta\ \tan(H - \Delta),\\
y' &= \qquad\qquad m\sec(H - \Delta),
\end{aligned}$$

and thence also

$$y'^2 - (x' - l)^2\tan^2\delta = m^2;$$

viz. the orbit-plane revolving about the fixed axis SX', meets the ray in a series of points forming in the orbit-plane a hyperbola having the line SX' for its conjugate axis.

50. As already remarked (*ante*, No. 11), this hyperbola is nothing else than the intersection of the orbit-plane regarded as fixed, by the hyperboloid generated by the rotation of the ray about the axis SX'. And we thus see the interpretation of the constants, viz.

l is the distance from S along the axis SX' of the "arm," or shortest distance of SX' and the ray.

m is the length of this arm.

δ is the inclination of the ray to the axis SX';

and for the remaining quantity Δ, imagine parallel to the ray a line through S meeting the sphere in L (L is the pole of the separator), I say that $\Delta - H$ is the angle $LX'Z'$: or (what is the same thing) drawing $X'L$ to meet $\Pi Z'Y'$ in Λ, we have $\Pi\Lambda = \Delta = H + Z'\Lambda$, or (what is the same thing) $Z'\Lambda = \Delta - H$.

51. To verify this, observe that the cosine distances of L from X, Y, Z, are as $\mathrm{f} : \mathrm{g} : \mathrm{h}$; and thence its cosine distances from X', Y', Z', are as $(\mathrm{f}, \mathrm{g}, \mathrm{h} \between \alpha, \beta, \gamma) : (\mathrm{f}, \mathrm{g}, \mathrm{h} \between \alpha', \beta', \gamma') : (\mathrm{f}, \mathrm{g}, \mathrm{h} \between \alpha'', \beta'', \gamma'')$; say, for a moment, as $\mathrm{f}' : \mathrm{g}' : \mathrm{h}'$.

Now $L\Lambda$ is the perpendicular from L on the side $Y'Z'$ of the quadrantal spherical triangle $LY'Z'$, and we thence have

$$\frac{\mathrm{h}'}{\mathrm{g}'} = \frac{\cos \Lambda Y'}{\cos \Lambda Z'} = \tan \Lambda Z' = \tan(\Delta - H),$$

if Δ has the geometrical signification just assigned to it. But this equation is

$$\mathrm{g}' \cos(H - \Delta) + \mathrm{h}' \sin(H - \Delta) = 0,$$

that is

$$\tan \Delta = \frac{\mathrm{g}' \cos H + \mathrm{h}' \sin H}{-\mathrm{g}' \sin H + \mathrm{h}' \cos H},$$

or substituting for g', h' their values, the numerator is

$$\mathrm{f}(\alpha' \cos H + \alpha'' \sin H) + \mathrm{g}(\beta' \cos H + \beta'' \sin H) + \mathrm{h}(\gamma' \cos H + \gamma'' \sin H),$$

which is

$$= -\mathrm{f} \sin G + \mathrm{g} \cos G, \; = W_s,$$

and the denominator is

$$\mathrm{f}(-\alpha' \sin H + \alpha'' \cos H) + \mathrm{g}(-\beta' \sin H + \beta'' \cos H) + \mathrm{h}(-\gamma' \sin H + \gamma'' \cos H),$$

which is

$$= (\mathrm{f} \cos G + \mathrm{g} \sin G) \sin N + \mathrm{h} \cos N, \; = W_c,$$

so that the formula becomes

$$\tan \Delta = \frac{W_s}{W_c},$$

which is the original expression of $\tan \Delta$.

52. We might in the equations

$$x' : y' : 1 = (\mathrm{a}, \mathrm{b}, \mathrm{c} \between \alpha', \beta', \gamma') : -(\mathrm{a}, \mathrm{b}, \mathrm{c} \between \alpha, \beta, \gamma) : (\mathrm{f}, \mathrm{g}, \mathrm{h} \between \alpha'', \beta'', \gamma'')$$

consider for instance G or N as alone variable, and then eliminate the variable parameter so as to obtain a locus; but the results would be complicated and the geometrical interpretations not very obvious.

53. I assume (as was done before) $N=0$, $G=b-90°$, $H=c$, that is, the position of the orbit-pole Z' is longitude b, colatitude c, and the axis SX' is the line of nodes or intersection of the orbit-plane with the ecliptic, viz., the longitude of this line is $=b-90°$.

The formulæ become

$$\begin{aligned} x' : y' : 1 = \quad & (\mathrm{a}\cos b + \mathrm{b}\sin b)\cos c - \mathrm{c}\sin c \\ : \; & -\mathrm{a}\sin b + \mathrm{b}\cos b \\ : \; & (\mathrm{f}\cos b + \mathrm{g}\sin b)\sin c + \mathrm{h}\cos c. \end{aligned}$$

or if these are

$$x' = \frac{X_c \cos c + X_s \sin c}{W_c \cos c + W_s \sin c},$$

$$y' = \frac{Y_o}{W_c \cos c + W_s \sin c},$$

the values now are

$$\begin{aligned} X_c &= \mathrm{a}\cos b + \mathrm{b}\sin b, \\ X_s &= -\mathrm{c}, \\ Y_o &= -\mathrm{a}\sin b + \mathrm{b}\cos b, \\ W_c &= \mathrm{h}, \\ W_s &= \mathrm{f}\cos b + \mathrm{g}\sin b, \end{aligned}$$

and thence forming as before the values of $\tan\Delta$, l, m, $\cot\delta$, and putting for shortness

$$\sqrt{W_c^2 + W_s^2}, = \sqrt{\mathrm{h}^2 + (\mathrm{f}\cos b + \mathrm{g}\sin b)^2}, = \Omega$$

we find after some easy reductions

$$\tan\Delta = \frac{\mathrm{f}}{\mathrm{h}}\cos b + \frac{\mathrm{g}}{\mathrm{h}}\sin b,$$

$$m = \frac{1}{\Omega}(-\mathrm{a}\sin b + \mathrm{b}\cos b),$$

$$l = \frac{1}{\Omega^2}[(\mathrm{ah} - \mathrm{cf})\cos b + (\mathrm{bh} - \mathrm{cg})\sin b],$$

$$\cot\delta = \frac{1}{\Omega Y_o}(-\mathrm{a}\sin b + \mathrm{b}\cos b)(-\mathrm{f}\sin b + \mathrm{g}\cos b),$$

$$= \frac{1}{\Omega}(-\mathrm{f}\sin b + \mathrm{g}\cos b),$$

and with these values

$$x' = l + m \cot \delta \ \tan (c - \Delta),$$

$$y' = \qquad m \sec (c - \Delta),$$

and thence

$$y'^2 - (x' - l)^2 \tan^2 \delta = m^2,$$

viz., this is the hyperbola obtained by rotating the orbit-plane about the line of nodes, longitude $b - 90°$.

54. Imagine the orbit-plane (having upon it the hyperbola) brought by such rotation into the plane $z = 0$, or plane of the ecliptic, so that the hyperbola will be a curve in this plane, the inclination to Sx, or longitude of the axis Sx', being of course $= b - 90°$. Transforming the equation to axes Sx, Sy, we must write in the equation

$$x' = x \sin b - y \cos b,$$

$$y' = x \cos b + y \sin b,$$

and the equation thus becomes

$$(x \cos b + y \sin b)^2 - (x \sin b - y \cos b - l)^2 \tan^2 \delta = m^2.$$

55. It will be recollected that the equations of the ray were

$$\frac{x - A}{\mathrm{f}} = \frac{y - B}{\mathrm{g}} = \frac{z - C}{\mathrm{h}};$$

writing herein $z = 0$ we find

$$x = A - \frac{\mathrm{f}}{\mathrm{h}} C, = \frac{\mathrm{b}}{\mathrm{h}},$$

$$y = B - \frac{\mathrm{f}}{\mathrm{g}} C, = -\frac{\mathrm{a}}{\mathrm{h}},$$

and it is clear that this point $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$ should lie on the hyperbola.

Substituting for (x, y) the values in question, we have first

$$\mathrm{b} \sin b + \mathrm{a} \cos b - \mathrm{h} l$$

$$= \frac{1}{\Omega^2} \left\{\left(\mathrm{h}^2 + (\mathrm{f} \cos b + \mathrm{g} \sin b)^2\right)(\mathrm{b} \sin b + \mathrm{a} \cos b) - \mathrm{h}\,(\mathrm{ah} - \mathrm{cf})\left(\cos b + (\mathrm{bh} - \mathrm{cg}) \sin b\right)\right\}$$

$$= \frac{1}{\Omega^2} \left\{(\mathrm{f} \cos b + \mathrm{g} \sin b)^2 (\mathrm{b} \sin b + \mathrm{a} \cos b) + (\mathrm{f} \cos b + \mathrm{g} \sin b)\,\mathrm{ch}\right\}$$

$$= \frac{1}{\Omega^2} (\mathrm{f} \cos b + \mathrm{g} \sin b) \left\{(\mathrm{f} \cos b + \mathrm{g} \sin b)(\mathrm{b} \sin b + \mathrm{a} \cos b) + \mathrm{ch}\,(\cos^2 b + \sin^2 b)\right\}$$

$$= \frac{1}{\Omega^2} (\mathrm{f} \cos b + \mathrm{g} \sin b)(-\mathrm{a} \sin b + \mathrm{b} \cos b)(\mathrm{f} \sin b - \mathrm{g} \cos b);$$

or observing that

$$\tan\delta = \frac{-\Omega}{\mathrm{f}\sin b - \mathrm{g}\cos b},$$

we have

$$(\mathrm{b}\sin b + \mathrm{a}\cos b - \mathrm{h}l)\tan\delta = -\frac{1}{\Omega}(\mathrm{f}\cos b + \mathrm{g}\sin b)(-\mathrm{a}\sin b + \mathrm{b}\cos b);$$

and hence the result of the substitution is at once found to be

$$(-\mathrm{a}\sin b + \mathrm{b}\cos b)^2 - \frac{1}{\Omega^2}(-\mathrm{a}\sin b + \mathrm{b}\cos b)^2(\mathrm{g}\sin b + \mathrm{f}\cos b)^2$$
$$= m^2\mathrm{h}^2, = \frac{\mathrm{h}^2(-\mathrm{a}\sin b + \mathrm{b}\cos b)^2}{\Omega^2};$$

viz., the factor $(-\mathrm{a}\sin b + \mathrm{b}\cos b)^2$ divides out, and the equation then becomes

$$1 - \frac{1}{\Omega^2}(\mathrm{g}\sin b + \mathrm{f}\cos b)^2 = \frac{\mathrm{h}^2}{\Omega^2},$$

that is

$$\Omega^2 = \mathrm{h}^2 + (\mathrm{g}\sin b + \mathrm{f}\cos b)^2,$$

which is in fact the value of Ω^2.

56. I seek for the direction of the hyperbola at the point $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$ in question. We have

$$dx : dy = \quad (\mathrm{b}\cos b - \mathrm{a}\sin b)\sin b + \cos b\tan^2\delta(\mathrm{b}\sin b + \mathrm{a}\cos b - \mathrm{h}l)$$
$$: -(\mathrm{b}\cos b - \mathrm{a}\sin b)\cos b + \sin b\tan^2\delta(\mathrm{b}\sin b + \mathrm{a}\cos b - \mathrm{h}l),$$

and from the above values of $(\mathrm{b}\sin b + \mathrm{a}\cos b - \mathrm{h}l)$ and $\tan\delta$, we have

$$\tan^2\delta(\mathrm{b}\sin b + \mathrm{a}\cos b - \mathrm{h}l) = \frac{\mathrm{g}\sin b + \mathrm{f}\cos b}{\mathrm{f}\sin b - \mathrm{g}\cos b}(-\mathrm{a}\sin b + \mathrm{b}\cos b);$$

whence

$$dx : dy = \quad (\mathrm{b}\cos b - \mathrm{a}\sin b)\sin b(\mathrm{f}\sin b - \mathrm{g}\cos b) + (\mathrm{g}\sin b + \mathrm{f}\cos b)\cos b(-\mathrm{a}\sin b + \mathrm{b}\cos b)$$
$$: -(\mathrm{b}\cos b - \mathrm{a}\sin b)\cos b(\mathrm{f}\sin b - \mathrm{g}\cos b) + (\mathrm{g}\sin b + \mathrm{f}\cos b)\sin b(-\mathrm{a}\sin b + \mathrm{b}\cos b),$$

which, multiplying out and reducing by means of the relation $\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0$, becomes

$$dx : dy = (-\mathrm{a}\sin b + \mathrm{b}\cos b)(\sin^2 b + \cos^2 b)\mathrm{f} : (-\mathrm{a}\sin b + \mathrm{b}\cos b)(\sin^2 b + \cos^2 b)\mathrm{g};$$

that is

$$dx : dy = \mathrm{f} : \mathrm{g}, \quad \text{or } \frac{dy}{dx} = \frac{\mathrm{g}}{\mathrm{f}},$$

which shows that the hyperbola, at the point $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$ where it meets the ray, touches the projection

$$\frac{x - A}{\mathrm{f}} = \frac{y - B}{\mathrm{g}}$$

of the ray on the plane of xy, which contains the hyperbola.

57. We may consider various particular forms of the hyperbola $y'^2-(x'-l)^2\tan^2\delta=m^2$.

1°. If $\tan\delta=0$, the hyperbola is the pair of parallel lines $y'^2=m^2$.

This can only happen if $\mathrm{h}=0$, $\mathrm{f}\cos b+\mathrm{g}\sin b=0$. The first equation gives $\mathrm{af}+\mathrm{bg}=0$, whence $\tan b=-\frac{\mathrm{f}}{\mathrm{g}}=\frac{\mathrm{b}}{\mathrm{a}}$; we have thus $m=\frac{-\mathrm{a}\sin b+\mathrm{b}\cos b}{\Omega}=\frac{0}{0}$, which is consistent with m finite. The equations show that the ray is parallel to the line of nodes.

2°. If $\tan\delta=\infty$, the hyperbola is $(x'-l)^2=0$, viz., the line $x'=l$ twice: the condition is $-\mathrm{f}\sin b+\mathrm{g}\cos b=0$; viz., the ray (not in general cutting the line of nodes) is at right angles to the line of nodes.

3°. If $m=0$, the hyperbola is the pair of intersecting lines $y'^2=(x'-l)^2\tan^2\delta$. The condition is $-\mathrm{a}\sin b+\mathrm{b}\cos b=0$, signifying that the ray cuts the line of nodes.

4°. We may have simultaneously $\tan\delta=\infty$, $m=0$. The hyperbola (as in 2°) is $(x'-l)^2=0$. The conditions are $-\mathrm{f}\sin b+\mathrm{g}\cos b=0$, $-\mathrm{a}\sin b+\mathrm{b}\cos b=0$, whence $\tan b=\frac{\mathrm{g}}{\mathrm{f}}=\frac{\mathrm{b}}{\mathrm{a}}$, and therefore also $\mathrm{ag}-\mathrm{bf}=0$; these signify that the ray cuts at right angles the line of nodes.

The line $x'=l$ passes through the point $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$, that is, we ought to have $\mathrm{h}^2l^2=\mathrm{a}^2+\mathrm{b}^2$. The value of l is in the first instance given in the form

$$l=\frac{1}{\Omega^2}\{(\mathrm{ah}-\mathrm{cf})\cos b+(\mathrm{bh}-\mathrm{cg})\sin b\},$$

where

$$\Omega^2=\mathrm{h}^2+(\mathrm{f}\cos b+\mathrm{g}\sin b)^2=\mathrm{h}^2+\mathrm{f}^2+\mathrm{g}^2-(-\mathrm{f}\sin b+\mathrm{g}\cos b)^2=\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2.$$

But observe that the equations

$$\mathrm{ag}-\mathrm{bf}=0,$$
$$\mathrm{bg}+\mathrm{af}=-\mathrm{ch},$$

give

$$\mathrm{f}=\frac{-\mathrm{ch}}{\mathrm{a}^2+\mathrm{b}^2}\mathrm{a},\quad \mathrm{g}=\frac{-\mathrm{ch}}{\mathrm{a}^2+\mathrm{b}^2}\mathrm{b},$$

and thence

$$\Omega^2=\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2 \quad =\mathrm{h}^2\left(1+\frac{\mathrm{c}^2}{\mathrm{a}^2+\mathrm{b}^2}\right) \quad =\frac{\mathrm{h}^2(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2)}{\mathrm{a}^2+\mathrm{b}^2},$$

$$\mathrm{ah}-\mathrm{cf}=\mathrm{ah}\frac{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}{\mathrm{a}^2+\mathrm{b}^2} \quad =\frac{\mathrm{a}}{\mathrm{h}}\Omega,$$

$$\mathrm{bh}-\mathrm{cg}=\mathrm{bh}\frac{\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2}{\mathrm{a}^2+\mathrm{b}^2} \quad =\frac{\mathrm{b}}{\mathrm{h}}\Omega;$$

consequently

$$l=\frac{1}{\Omega^2\mathrm{h}}(\mathrm{a}\cos b+\mathrm{b}\sin b)\,\Omega^2=\frac{1}{\mathrm{h}}(\mathrm{a}\cos b+\mathrm{b}\sin b)=\frac{1}{\mathrm{h}}\sqrt{\mathrm{a}^2+\mathrm{b}^2},$$

which is right.

58. I return to the equation of the hyperbola written in the form

$$(x\cos b + y\sin b)^2 - (x\sin b - y\cos b - l)^2\tan^2\delta = m^2;$$

being (as was shown) a hyperbola passing through the point $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$ where its plane is met by the ray, and touching at this point the projection $\frac{x-A}{\mathrm{f}} = \frac{y-B}{\mathrm{g}}$.

If in the equation we consider b as variable, we have a series of hyperbolas, viz., these are the intersections of the plane of xy with the hyperboloids of revolution obtained by making the ray rotate successively round the several lines $x\cos b + y\sin b = 0$ through the focus S. And, as just seen, these hyperbolas all of them touch at $\left(\frac{\mathrm{b}}{\mathrm{h}}, -\frac{\mathrm{a}}{\mathrm{h}}\right)$ the projection of the ray.

59. The hyperbola to any particular angle b is the hyperbola belonging to the ray, in the planogram for an orbit-plane rotating about the axis $x\cos b + y\sin b = 0$; so that the system of hyperbolas would be useful for the construction of any such planogram. And there is another series of curves which, if they could be constructed with moderate facility, would be very useful for the same purpose; viz., reverting to the equations

$$\begin{aligned} x' : y' : 1 = \quad & (\mathrm{a}\cos b + \mathrm{b}\sin b)\cos c - \mathrm{c}\sin c \\ : \; & -\mathrm{a}\sin b + \mathrm{b}\cos b \\ : \; & (\mathrm{f}\cos b + \mathrm{g}\sin b)\sin c + \mathrm{h}\cos c, \end{aligned}$$

which determine in the orbit-plane the coordinates x', y' of the intersection thereof with the ray: imagine as before that the point is marked on the orbit-plane, and let it by a rotation of the orbit-plane be brought into the plane of xy; so that x', y', will be the coordinates in the direction of and perpendicular to the line of nodes of a point on the hyperbola $y'^2 - (x'-l)^2\tan^2\delta = m^2$, or $(x\cos b + y\sin b)^2 - (x\sin b - y\cos b - l)^2\tan^2\delta = m^2$, viz., of the point corresponding to an orbit-pole, colatitude c. Suppose that x, y, are the coordinates of this same point referred to the fixed axes, we have

$$\begin{aligned} x = \quad & x'\sin b + y'\cos b, \\ x = \; & -x'\cos b + y'\sin b, \end{aligned}$$

and thence

$$\begin{aligned} x : y : 1 = \quad & (\mathrm{a}\cos b + \mathrm{b}\sin b)\sin b\cos c - \mathrm{c}\sin b\sin c + (-\mathrm{a}\sin b + \mathrm{b}\cos b)\cos b \\ : \; & -(\mathrm{a}\cos b + \mathrm{b}\sin b)\cos b\cos c + \mathrm{c}\cos b\sin c + (-\mathrm{a}\sin b + \mathrm{b}\cos b)\sin b \\ : \; & (\mathrm{f}\cos b + \mathrm{g}\sin b)\sin c + \mathrm{h}\cos c, \end{aligned}$$

the coordinates of the point just referred to. Now, if from these equations we could eliminate b, we should have a series of curves containing the variable parameter c, intersecting the series of hyperbolas; and thus marking out on each of these hyperbolas the points which belong to the successive values of the parameter c; we should thus have in the plane of xy the point corresponding to an orbit-pole longitude b and

colatitude c. The series of curves in question may be called "graduation curves," viz., they would serve for the graduation of the hyperbola in the planogram for an orbit-plane rotating round any line $x\cos b + y\sin b = 0$ in the plane of xy. But the elimination cannot be easily effected, and I am not in possession of any method of tracing the series of curves.

60. I remark that from the equations

$$\begin{aligned} x' : y' : 1 = & \quad (\mathrm{a}\cos b + \mathrm{b}\sin b)\cos c - \mathrm{c}\sin c \\ : & - \mathrm{a}\sin b + \mathrm{b}\cos b \\ : & \quad (\mathrm{f}\cos b + \mathrm{g}\sin b)\sin c + \mathrm{h}\cos c, \end{aligned}$$

we may without difficulty eliminate b; the result is, in fact,

$$\begin{aligned} & [x'(-\mathrm{ah}\cos c) + y'(-\mathrm{bh}\cos^2 c - \mathrm{cg}\sin^2 c) - \mathrm{ac}\sin c]^2 \\ + & [x'(\ \mathrm{bh}\cos c) + y'(-\mathrm{ah}\cos^2 c + \mathrm{cf}\sin^2 c) + \mathrm{bc}\sin c]^2 \\ = & [x'(\ \mathrm{ch}\sin c) + y'(\ \mathrm{ag} - \mathrm{bf})\sin c\cos c + (\mathrm{a}^2 - \mathrm{b}^2)\cos c]^2, \end{aligned}$$

a conic; but the geometrical signification of this result is not obvious, and I do not make any use of it.

Article Nos. 61 to 63. *The Trivector and the Orbit.*

61. Considering now the three rays, these are determined by their six coordinates,

$$\begin{aligned} & (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{f}_1, \mathrm{g}_1, \mathrm{h}_1), \\ & (\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2, \mathrm{f}_2, \mathrm{g}_2, \mathrm{h}_2), \\ & (\mathrm{a}_3, \mathrm{b}_3, \mathrm{c}_3, \mathrm{f}_3, \mathrm{g}_3, \mathrm{h}_3), \end{aligned}$$

respectively; and the intersections with the orbit-plane are given by

$$\begin{aligned} x_1' : y_1' : 1 &= (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \between \alpha, \beta, \gamma) : -(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \between \alpha', \beta', \gamma') : (\mathrm{f}_1, \mathrm{g}_1, \mathrm{h}_1 \between \alpha'', \beta'', \gamma''), \\ x_2' : y_2' : 1 &= (\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2 \between \quad ,, \quad) : -(\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2 \between \quad ,, \quad) : (\mathrm{f}_2, \mathrm{g}_2, \mathrm{h}_2 \between \quad ,, \quad), \\ x_3' : y_3' : 1 &= (\mathrm{a}_3, \mathrm{b}_3, \mathrm{c}_3 \between \quad ,, \quad) : -(\mathrm{a}_3, \mathrm{b}_3, \mathrm{c}_3 \between \quad ,, \quad) : (\mathrm{f}_3, \mathrm{g}_3, \mathrm{h}_3 \between \quad ,, \quad), \end{aligned}$$

where the axes Sx', Sy', are an arbitrary set of rectangular axes in the orbit-plane; or where, as before, the axis Sx' may be taken to be the line of nodes.

There is no difficulty in finding the equation of the orbit. Writing $r_1 = \sqrt{x_1'^2 + y_1'^2}$, we have

$$r_1 = \frac{u_1}{(\mathrm{f}_1, \mathrm{g}_1, \mathrm{h}_1 \between \alpha'', \beta'', \gamma'')},$$

if

$$u_1 = \pm\sqrt{[(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \between \alpha', \beta', \gamma')]^2 + [(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \between \alpha, \beta, \gamma)]^2},$$

the sign being taken in such manner that r_1 shall be positive; viz., the sign must be the same as that of $(f_1, g_1, h_1 \between \alpha'', \beta'', \gamma'')$. And we have the like formulæ for r_2 and r_3. Substituting these values, the equation of the orbit becomes

$$\begin{vmatrix} r, & x', & y', & 1 \\ u_1, & (a_1, b_1, c_1 \between \alpha', \beta', \gamma'), & -(a_1, b_1, c_1 \between \alpha, \beta, \gamma), & (f_1, g_1, h_1 \between \alpha'', \beta'', \gamma'') \\ u_2, & (a_2, b_2, c_2 \between \quad ,, \quad), & -(a_2, b_2, c_2 \between \quad ,, \quad), & (f_2, g_2, h_2 \between \quad ,, \quad) \\ u_3, & (a_3, b_3, c_3 \between \quad ,, \quad), & -(a_3, b_3, c_3 \between \quad ,, \quad), & (f_3, g_3, h_3 \between \quad ,, \quad) \end{vmatrix} = 0.$$

62. Considering the minor determinants formed with the terms under the x' and y', for instance

$$(a_1, b_1, c_1 \between \alpha', \beta', \gamma') . -(a_2, b_2, c_2 \between \alpha, \beta, \gamma)$$
$$+ (a_1, b_1, c_1 \between \alpha, \beta, \gamma) . \quad (a_2, b_2, c_2 \between \alpha', \beta', \gamma')$$

this is

$$= (b_1c_2 - b_2c_1)(\beta\gamma' - \beta'\gamma) + (c_1a_2 - c_2a_1)(\gamma\alpha' - \gamma'\alpha) + (a_1b_2 - a_2b_1)(\alpha\beta' - \alpha'\beta)$$
$$= \alpha''(b_1c_2 - b_2c_1) + \beta''(c_1a_2 - c_2a_1) + \gamma''(a_1b_2 - a_2b_1),$$

or, what is the same thing,

$$= (b_1c_2 - b_2c_1, \; c_1a_2 - c_2a_1, \; a_1b_2 - a_2b_1 \between \alpha'', \beta'', \gamma'');$$

with the like expressions for the other two minors. And we thus obtain the following developed form of the equation, viz.

$$\begin{aligned} &\{x'(a_1, b_1, c_1 \between \alpha, \beta, \gamma) + y'(a_1, b_1, c_1 \between \alpha', \beta', \gamma')\}[-u_2(f_3, g_3, h_3 \between \alpha'', \beta'', \gamma'') \\ &\qquad\qquad + u_3(f_2, g_2, h_2 \between \alpha'', \beta'', \gamma'')] \\ &+ \{x'(a_2, b_2, c_2 \between \quad ,, \quad) + y'(a_2, b_2, c_2 \between \quad ,, \quad)\}[-u_3(f_1, g_1, h_1 \between \quad ,, \quad) \\ &\qquad\qquad + u_1(f_3, g_3, h_3 \between \quad ,, \quad)] \\ &+ \{x'(a_3, b_3, c_3 \between \quad ,, \quad) + y'(a_3, b_3, c_3 \between \quad ,, \quad)\}[-u_1(f_2, g_2, h_2 \between \quad ,, \quad) \\ &\qquad\qquad + u_2(f_1, g_1, h_1 \between \quad ,, \quad)] \\ &+ (b_2c_3 - b_3c_2, \; c_2a_3 - c_3a_2, \; a_2b_3 - a_3b_2 \between \alpha'', \beta'', \gamma'')[r(f_1, g_1, h_1 \between \alpha'', \beta'', \gamma'') - u_1] \\ &+ (b_3c_1 - b_1c_3, \; c_3a_1 - c_1a_3, \; a_3b_1 - a_1b_3 \between \quad ,, \quad)[r(f_2, g_2, h_2 \between \quad ,, \quad) - u_2] \\ &+ (b_1c_2 - b_2c_1, \; c_1a_2 - c_2a_1, \; a_1b_2 - a_2b_1 \between \quad ,, \quad)[r(f_3, g_3, h_3 \between \quad ,, \quad) - u_3] = 0, \end{aligned}$$

being an equation of the form $\Omega r = Ax' + By' + C$.

63. The coefficient of r is a quadric function of $(\alpha'', \beta'', \gamma'')$, and if this vanish the orbit is a right line. It thus appears that the orbit will be a right line provided only the orbit-axis be situate in a certain quadric cone, or (what is the same thing) the orbit-pole be situate in a certain spherical conic: agreeing with a preceding result, viz. the cone is that reciprocal to the cone, vertex S, circumscribed about the hyper-

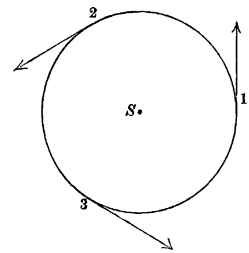
boloid which contains the three rays. And we see that the equation of this reciprocal cone is

$$\begin{vmatrix} & & & \alpha'', \beta'', \gamma'' & \\ (f_1, & g_1, & h_1 \between \alpha'', \beta'', \gamma''), & a_1, & b_1, & c_1 \\ (f_2, & g_2, & h_2 \between \quad \text{,,} \quad), & a_2, & b_2, & c_2 \\ (f_3, & g_3, & h_3 \between \quad \text{,,} \quad), & a_3, & b_3, & c_3 \end{vmatrix} = 0.$$

Article Nos. 64 and 65. *The Special Symmetrical System of three Rays.*

64. In what follows I consider the three rays forming a symmetrical system as already referred to: viz. the three rays intersect the plane of the ecliptic at points equidistant from S at longitudes 0°, 120°, 240°; each of them is at right angles to

Fig. 6.



the line joining S with the intersection with the plane of the ecliptic, and at an inclination $=60°$ to this plane: the figure shows the projection on the plane of the ecliptic of the portions which lie above this plane of the three rays respectively.

The three rays lie on a hyperboloid of revolution having the line Sz for its axis; the circumscribed or asymptotic cone vertex S, is a right cone of the semi-aperture $=30°$; the reciprocal cone is therefore a right cone semi-aperture 60°, or (what is the same thing) the regulator is a small circle, angular radius 60°, and the regulator and separators have the positions shown in fig. 1, see No. 8.

Taking $S1 = S2 = S3 = 1$, and writing down the equations of the three rays in the forms

$$\frac{x-1}{0} = \frac{y}{1} = \frac{z}{\tan 60°},$$

$$\frac{x+\cos 60°}{-\sin 60°} = \frac{y-\sin 60°}{-\cos 60°} = \frac{z}{\tan 60°},$$

$$\frac{x+\cos 60°}{\sin 60°} = \frac{y+\sin 60°}{-\cos 60°} = \frac{z}{\tan 60°},$$

we obtain the six coordinates of the three rays respectively

$$\begin{aligned}
(a_1, b_1, c_1, f_1, g_1, h_1) &= (\ 0, \quad \sqrt{3}, \quad -1, \quad 0, \quad 1, \quad \sqrt{3}),\\
(a_2, b_2, c_2, f_2, g_2, h_2) &= (\ 3, \quad \sqrt{3}, \quad 2, \quad \sqrt{3}, \quad 1, \quad -2\sqrt{3}),\\
(a_3, b_3, c_3, f_3, g_3, h_3) &= (-3, \quad \sqrt{3}, \quad 2, \quad -\sqrt{3}, \quad 1, \quad -2\sqrt{3}),
\end{aligned}$$

whence the intersections with the orbit-plane are given by

$$\begin{aligned}
x_1' : y_1' : 1 &= \beta'\sqrt{3} - \gamma' : -\beta\sqrt{3} + \gamma : \beta'' + \gamma''\sqrt{3},\\
x_2' : y_2' : 1 &= 3\alpha' + \beta'\sqrt{3} + 2\gamma' : -3\alpha - \beta\sqrt{3} - 2\gamma : \alpha''\sqrt{3} + \beta'' - 2\sqrt{3}\gamma'',\\
x_3' : y_3' : 1 &= -3\alpha' + \beta'\sqrt{3} + 2\gamma' : 3\alpha - \beta\sqrt{3} - 2\gamma : -\alpha''\sqrt{3} + \beta'' - 2\sqrt{3}\gamma'',
\end{aligned}$$

where if (as before) the position of the orbit-plane be determined by means of the longitude b and colatitude c of the orbit-pole, we have

$$\begin{aligned}
\alpha, \beta, \gamma &= \sin b, & -\cos b, & \quad 0,\\
\alpha', \beta', \gamma' &= \cos b \cos c, & \sin b \cos c, & -\sin c,\\
\alpha'', \beta'', \gamma'' &= \cos b \sin c, & \sin b \sin c, & \cos c,
\end{aligned}$$

and the passage from the coordinates x', y', to x, y, is given by

$$\begin{aligned}
x' &= x \sin b - y \cos b,\\
y' &= x \cos b + y \sin b,
\end{aligned}$$

or conversely

$$\begin{aligned}
x &= x' \sin b + y' \cos b,\\
y &= -x' \cos b + y' \sin b.
\end{aligned}$$

65. To develope the results, I consider the orbit-pole as passing through certain series of positions. The locus may be a meridian circle: by reason of the symmetry of the system, the results are not altered by a change of 120° in the longitude of the meridian; so that, by considering the two meridians 0°—180° and 90°—270°, we, in fact, consider twelve half meridians at the intervals of 30°. An illustration is afforded by Plate I.; the orbit-pole describes successively the meridians 0°, 30°, 60°, 90°, and the line 1, by its intersection with the orbit-plane, traces out on this plane a series of hyperbolas shown in the figure; the hyperbola for the meridian 90° is a right line, but (except for the position where the orbit-plane passes through the line 1) the locus is a determinate point on this line. Planogram No. 1 (Plate II.) refers to the meridian 90°—270°, and Planogram No. 2 (Plate III.) to the meridian 0°—180°. Next, if the orbit-pole be at one of the points A, that is, if the orbit-plane pass through a ray—though the position of the orbit-pole be here determinate, yet as there is a series of orbits, this also will give rise to a planogram: I call it Planogram No. 3. The orbit-pole may pass along a separator circle (viz. the orbit-plane be parallel to a ray), this is Planogram No. 4. And, lastly, the orbit-pole may pass along the ecliptic (or the orbit-plane may pass through the axis SZ), I call this Planogram No. 5. But the last three planograms are not considered in the like detail as the first two, and I have not, in regard to them, tabulated the results, nor given any Plates.

Article Nos. 66 to 82. *Planogram No.* 1, *the Meridian* 90°—270° (see Plate II.).

66. Supposing that the orbit-plane rotates about the axis $S1$ (fig. 6, see No. 64) in the plane of the ecliptic, the orbit-pole will describe the meridian 90°—270°, the position of the orbit-pole being $b=90°$, $c=0°$ to 90°, or else $b=270°$, $c=0°$ to 90°. But the same analytical formula extends to the two half meridians, viz., we may take $b=90°$, and extend c over 180°, in the final results making c an arc between 0° and 90°, and $b=90°$, or $=270°$, as the case requires.

67. Assuming then $b=90°$, we have

$$\begin{array}{llll} \alpha\,,\ \beta\,,\ \gamma & =1, & 0\,, & 0\,, \\ \alpha',\ \beta',\ \gamma' & =0, & \cos c, & -\sin c, \\ \alpha'',\ \beta'',\ \gamma'' & =0, & \sin c, & \cos c, \end{array}$$

and, moreover, x', $y'=x$, y: so that instead of (x_1', y_1'), &c., we may write at once (x_1, y_1), &c. The formulæ become

$$\begin{array}{l} x_1 : y_1 : 1 = \sqrt{3}\cos c + \sin c : \quad 0 : \sin c + \sqrt{3}\cos c, \\ x_2 : y_2 : 1 = \sqrt{3}\cos c - 2\sin c : -3 : \sin c - 2\sqrt{3}\cos c, \\ x_3 : y_3 : 1 = \sqrt{3}\cos c - 2\sin c : \ \ 3 : \sin c - 2\sqrt{3}\cos c, \end{array}$$

that is

$$x_1=1, \quad y_1=0,$$

(viz. the orbit-plane, as is evident, meets the ray 1 in a fixed point, its intersection with the plane of xy);

$$x_2=\frac{\sqrt{3}\cos c-2\sin c}{\sin c-2\sqrt{3}\cos c}, \quad x_3=x_2,$$

$$y_2=\frac{-3}{\sin c-2\sqrt{3}\cos c}, \quad y_3=-y_2,$$

and writing

$$\frac{2\sqrt{3}}{\sqrt{13}}=\cos\omega, \qquad \frac{1}{\sqrt{13}}=\sin\omega, \qquad \frac{1}{2\sqrt{3}}=\tan\omega,$$

(whence $\omega=16°\,6'$) we find

$$x_2=-\tfrac{8}{13}+\frac{3\sqrt{3}}{13}\tan(c+\omega),$$

$$y_2=\qquad\frac{3}{\sqrt{13}}\sec(c+\omega),$$

and we thence have for the hyperbola, the locus of (x_2, y_2) and (x_3, y_3)

$$(x+\tfrac{8}{13})^2=\tfrac{3}{13}(y^2-\tfrac{9}{13}),$$

viz. the points (x_2, y_2) and (x_3, y_3) are situate on the hyperbola, symmetrically on opposite sides of the axis of x. For $c=0$, we have $x_2=-\frac{1}{2}$, $y_2=\frac{1}{2}\sqrt{3}$, $(x_2^2+y_2^2=1)$, and the hyperbola at this point touches the circle $x^2+y^2=1$; and similarly for x_3, y_3. The inclination of the asymptotes to the axis of y is given by $\tan\eta=\sqrt{\frac{3}{13}}$, $\eta=22^\circ\,56'$.

68. The orbits are conics, focus S and vertex 1. It will be convenient to consider c as passing from 0° to $90^\circ-\omega$, and from 0° to $-(90^\circ+\omega)$; that is, from 0° to $73^\circ\,54'-\epsilon$, and from 0° to $-116^\circ\,6'+\epsilon$, if ϵ be indefinitely small: the point 2 will thus traverse the upper branch (alone shown in the Plate) of the guide-hyperbola, viz., for $c=0^\circ$ it will be at the point of contact with the circle; for $c=73^\circ\,54'-\epsilon$ it will be at ∞, and for $c=-106^\circ\,6'+\epsilon$ at ∞'. For $c=0^\circ$ the orbit is the circle; as c increases positively, it becomes an ellipse of increasing eccentricity and major axis, until for a certain value ($c=46^\circ\,48'$ as will appear) it becomes a parabola; it then becomes a hyperbola (concave branch); for $c=52^\circ\,45'$ it becomes the hyperbola Σ' subsequently referred to; and for $c=60^\circ$ (the point 2 being then on the line shown in the figure) the orbit becomes this right line. As c continues to increase, the orbit becomes a hyperbola (convex branch); and ultimately for $c=73^\circ\,54'-\epsilon$, the point 2 goes to ∞, and the orbit becomes a hyperbola (convex) Σ, having an asymptote parallel to that of the guide-hyperbola: the inclination to the axis of x being thus $90^\circ-22^\circ\,56'$, $=67^\circ\,4'$.

69. Next as c increases negatively, the point 2 moves from the point of contact in the other direction to ∞': for $c=0^\circ$ the orbit is of course the circle, and as c increases negatively the orbits are at first the very same series of orbits as those belonging to the positive values [1], viz., they are first ellipses, of increasing eccentricity and major axis; then for $c=-92^\circ\,54'$ the orbit is the parabola; the orbits are then hyperbolas (concave), and finally for $c=-106^\circ\,6'+\epsilon$, when 2 is at ∞', the orbit is a hyperbola Σ', the asymptote of which is parallel to that of the guide-hyperbola, viz., the inclination to the axis of x is $=67^\circ\,4'$.

70. It will be observed that the orbits from the circle to the hyperbola Σ' each intersect the guide-hyperbola (that is, the branch shown in the figure) in two points, the one corresponding to a positive, the other to a negative value of c; in the positive series, the remaining orbits from the hyperbola Σ', through the right line to the convex hyperbola Σ, each intersect the guide-hyperbola (same branch) in a single point only, for which c is positive.

71. There is, in the passage of the orbit-pole from $c=-106^\circ\,6'+\epsilon$ to $c=73^\circ\,54'-\epsilon$, say at $c=73^\circ\,54'$, a discontinuity of orbit, viz., an abrupt change from the concave hyperbola Σ' to the concave hyperbola Σ; observe that the direction of the asymptotes being the same in each, the eccentricity e has the same value.

[1] Of course, as corresponding to different values of c, they are not the same orbits in space, but they are only the same curves in the planogram.

The point in question ($b=90°$, $c=73°\,54'$) is one of the points B of the spherogram, and the hyperbolas Σ, Σ' are two of the four orbits belonging to this point. And, by what precedes, it appears that as the orbit-pole passes through this point along a meridian downwards to the ecliptic the change is from a concave to a convex orbit.

72. On account of the symmetry in regard to the axis of x, the equation of the orbit will be of the form $r=Ax+B$; viz., the equation is at once found to be

$$r-1=\frac{r_2-1}{x_2-1}(x-1).$$

73. The eccentricity is the coefficient A taken positively ($e=\pm A$): it is in the present case proper to attend to the value of the coefficient itself,

$$A=\frac{r_2-1}{x_2-1},$$

the sign of A will then indicate the position of the centre of the orbit, viz., according as A is positive or negative the centre will be on the negative or the positive side of the focus S. To investigate the variation of A, we may express it as a function of $\tan c$, $=\lambda$ suppose. We have

$$x_2=\frac{\sqrt{3-2\lambda}}{\lambda-2\sqrt{3}},\qquad y_2=\frac{-3}{\lambda-2\sqrt{3}},$$

and thence

$$r_2=\frac{1}{\lambda-2\sqrt{3}}R_2,\qquad R_2=\pm\sqrt{12-4\sqrt{3}\lambda+13\lambda^2};$$

viz., r_2 must be positive, that is, R_2 is positive or negative according to the sign of $\lambda-2\sqrt{3}$; negative if $\lambda<2\sqrt{3}$ or $c<73°\,54'$, positive if $\lambda>2\sqrt{3}$ or $c>73°\,54'$. And we have then

$$A=\frac{\lambda-2\sqrt{3}-R_2}{3(\lambda-\sqrt{3})}.$$

But a more convenient formula is obtained by writing

$$\theta=-\cot c+\frac{1}{2\sqrt{3}},$$

$$a=\qquad\frac{1}{2\sqrt{3}},$$

we then have

$$\sqrt{1+\theta^2}=\theta r_2,$$

which determines the sign of the radical, viz., this must have the same sign as θ; and then for the coefficient

$$A=\frac{2}{3(\theta+a)}(-\sqrt{1+\theta^2}+\theta).$$

74. For c a small arc $=\epsilon$, θ is large and negative, and $\sqrt{1+\theta^2}$, having the same sign as θ, is $=\theta+\frac{1}{2\theta}$ nearly; we have therefore

$$A=\frac{2}{3\theta}\cdot\frac{-1}{2\theta}=-\frac{1}{3\theta^2} \text{ approximately.}$$

For c nearly $=60°$, say $c=60°\pm\epsilon$,

$$\cot c=\cot 60°\pm\epsilon\operatorname{cosec}^2 60°=\frac{1}{\sqrt{3}}\pm\frac{4\epsilon}{3},$$

$$\theta=-\frac{1}{2\sqrt{3}}\pm\frac{4\epsilon}{3}\qquad \theta+a=\pm\frac{4\epsilon}{3},\qquad \sqrt{1+\theta^2}=-\frac{\sqrt{13}}{2\sqrt{3}},$$

and thence

$$A=\frac{-1+\sqrt{13}}{2\sqrt{3}}\div\pm\frac{4\epsilon}{3}=\pm\frac{\sqrt{3}}{8}\cdot\frac{-1+\sqrt{13}}{\epsilon},$$

viz., this is $-\infty$ for $c=60°-\epsilon$, and $+\infty$ for $c=60°+\epsilon$.

For c nearly $=90°-\omega$, say first $c=73°\,54'-\epsilon$, we have

$$\cot c=\frac{1}{2\sqrt{3}}+\tfrac{13}{12}\epsilon,\qquad \theta=-\tfrac{13}{12}\epsilon,\qquad \theta+a=\frac{1}{2\sqrt{3}},\qquad \sqrt{1+\theta^2}=-1,$$

whence

$$A=\frac{4}{\sqrt{3}}=2{\cdot}30940;$$

but if $c=73°\,54'+\epsilon$, then $\theta=\frac{13}{12}\epsilon$, $\theta+a=\frac{1}{2\sqrt{3}}$, $\sqrt{1+\theta^2}=+1$, and

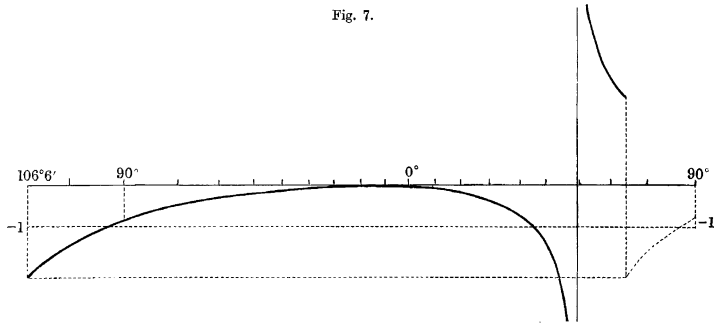
$$A=-\frac{4}{\sqrt{3}}=-2{\cdot}30940,$$

viz., there is an abrupt change from $A=+\frac{4}{\sqrt{3}}$ to $A=-\frac{4}{\sqrt{3}}$; corresponding to the discontinuity of orbit already referred to. We may diminish c by 180°, and consider the last-mentioned value, $A=-\frac{4}{\sqrt{3}}$, as belonging to $c=-90°-\omega+\epsilon=-(106°\,6'-\epsilon)$.

75. Consider next that c passes from 0 to $-(106°\,6'-\epsilon)$. First if c is a small negative quantity $c=-\epsilon$, θ is large and positive, and $\sqrt{1+\theta^2}$ having the same sign as θ (positive) is $=\theta+\frac{1}{2\theta}$ nearly, we have therefore $A=\frac{2}{3\theta}\cdot\frac{-1}{2\theta}=-\frac{1}{3\theta^2}$ (same as for $c=+\epsilon$). And it is easy to see that as c increases negatively, A is always increasing negatively, its value for $c=-90°$ being $A=\frac{1-\sqrt{13}}{3}=-{\cdot}8685$, and for $c=-106°\,6'+\epsilon$ being $=-2{\cdot}30940$ as above. We have a diagram of A (see next page).

76. It thus appears that from $A=0$ to $A=-\frac{4}{\sqrt{3}}$, there are always to any given value of A two values of c, or positions of the orbit-pole. In particular if A be $=-1$, the curve will be a parabola; the values of c lying between 0°, 60° and between 73° 54′, 90° respectively.

Fig. 7.



To find them, writing $A=-1$, we have

$$-3\theta-3a=2\theta-2\sqrt{1+\theta^2}, \text{ that is, } 5\theta+3a=2\sqrt{1+\theta^2},$$

or

$$21\theta^2+30a\theta+9a^2-4=0,$$

that is, substituting for a its value $=\frac{1}{2\sqrt{3}}$,

$$21\,\theta^2+5\sqrt{3}\,\theta-\tfrac{13}{4}=0, \qquad (14\theta\sqrt{3}+5)^2=116,$$

or

$$\theta=\frac{-5\pm\sqrt{116}}{14\sqrt{3}},$$

that is

$$\theta=-\cdot65034, \qquad \theta=\cdot23797,$$

giving

$$\cot c=+\cdot93902, \qquad \cot c=+\cdot05071,$$

or

$$c=46^\circ\,48', \qquad c=87^\circ\,6'.$$

77. It has been seen that $c=73^\circ\,54'+\epsilon$ gives $A=-\frac{4}{\sqrt{3}}=-2\cdot30940$; there will be between 0° and 60° another value of c, giving for A this same value; to find this value write $A=-\frac{4}{\sqrt{3}}$, then we have

$$-4\sqrt{3}\left(\theta+\frac{1}{2\sqrt{3}}\right)=2\theta-2\sqrt{1+\theta^2},$$

that is

$$(1+2\sqrt{3})\,\theta+1=\sqrt{1+\theta^2},$$

or

$$(12+4\sqrt{3})\,\theta^2+(2+4\sqrt{3})\,\theta=0,$$

satisfied as it should be by $\theta=0$, and also by

$$\theta=-\frac{1+2\sqrt{3}}{2\,(3+\sqrt{3})}=-\cdot 47170,$$

giving

$$\cot c=\cdot 76038 \text{ or } c=52^\circ\,45'.$$

78. Representing the equation of the orbit by

$$r=Ax\pm a\,(1-A^2),$$

we have for the point 1,

$$1=A\quad\pm a\,(1-A^2),$$

that is

$$a=\frac{\pm 1}{1+A},$$

where the sign is to be taken so that a shall be positive.

79. With a view to the calculation of the times of passage, I calculate a series of values of x_2, y_2, r_2, A, a, for values of c at the intervals of 5° and for a few intermediate values; we have x_3, y_3, $r_3=x_2$, y_2, r_2, so that these are known; so long as the orbit is an ellipse, the time of passage between the points 2 and 3, say T_{23}, may be calculated by Lambert's equation, the length of the chord y_2-y_3, $=2y_2$ being known without any fresh calculation. And then the times T_{12} and T_{31} being equal, and the sum $T_{12}+T_{23}+T_{31}$ being equal to the whole periodic time (reckoned as $=3a^{\frac{3}{2}}$) the times T_{12} and T_{31} are also known. But when the orbit is a concave hyperbola there is no time T_{23}, and the other two times T_{12}, $=T_{31}$ must be calculated. For the reason referred to (*ante*, No. 39) I did not use Lambert's equation,—and it was less necessary to do so, by reason that, the transverse axis coinciding with the axis of x, the other formula could be employed without difficulty.

80. The formulæ for x_2, y_2 adapted to logarithmic calculation are

$$\log\,(x_2+\cdot 61539)=\overline{11}\cdot 60174+\log\tan\,(c+16^\circ\,6'),$$

$$\log\,y_2\qquad\qquad=\overline{11}\cdot 92015+\log\sec\,(c+16^\circ\,6'),$$

where y_2 is always positive, but the sign of x_2 must be attended to. The values of r_2 and its inclination ϕ_2 to the axis of x are then to be calculated from

$$\tan\phi_2=\frac{y_2}{x_2};\quad r_2=x_2\sec\phi_2 \text{ or } =y_2\operatorname{cosec}\phi_2,$$

(viz. for r_2 it is proper to use the first or the second value, according as x_2 is greater or less than y_2).

We have then $e = (\pm A)$ and a from the foregoing formulæ

$$A = \frac{r_2 - 1}{x_2 - 1}, \quad a = \frac{\pm 1}{1 + A},$$

where a, e are each of them positive.

And then for the Times

$$T_{23} = \frac{3}{2\pi}(\chi - \chi' - \sin\chi + \sin\chi'); \quad \left(\log\frac{3}{2\pi} = \bar{1}{\cdot}67894\right),$$

where

$$a\cos\chi = a - r_2 - y_2,$$
$$a\cos\chi' = a - r_2 + y_2,$$

and attention is necessary in order to the selection of the proper values of the angles χ, χ'.

And finally

$$T_{12} = T_{31} = \tfrac{1}{2}\left(3a^{\frac{3}{2}} - T_{23}\right).$$

81. I subjoin a specimen; the characteristics of the logarithms are (as in the actual calculations) omitted.

$b = 90°$ $\quad c = 20°$,

$c + 16°\,6' \quad = 36°\,6'$

log sec	09259	log tan	86285
	92015		60174
	01274		46459
$y_2 =$	1·0297		61539
			29147
		$x_2 =$	$-$ ·32392
		log =	51044
	01274		02046
	51044		01274
	50230		03320
$\phi_2 =$	72° 33′	$r_2 =$	1·0794
·0794 log =	89982		·94003 log = 97314
1·3239 log =	12185		comp = 02686
	77797		

$A = -\cdot 059975$ $\qquad a = 1\cdot 0638$

$$
\begin{array}{rr}
 & 1\cdot 0638 \\
 & 1\cdot 0794 \\
\hline
A - r_2 = - & \cdot 0156 \\
y_2 = + & 1\cdot 0297 \\
\hline
 & 1\cdot 0141 = a \cos \chi' \\
 & -1\cdot 0453 = a \cos \chi
\end{array}
$$

$$
\begin{array}{rr}
00608 & 01924 \\
02686 & 02686 \\
\hline
97922 & 99238
\end{array}
$$

$\chi' = 17^\circ\, 35'$ $\qquad \chi\,(= \text{Supp. } 10^\circ\, 42') = 169^\circ\, 18'$ $\qquad \chi - \chi' = 151^\circ\, 43'$

$$
\begin{array}{rrrr}
151^\circ & 2\cdot 63544 & & \\
43' & \cdot 01250 & & \\
-\sin\chi & & -\cdot 18566 & \\
\sin\chi' & \cdot 30209 & & 02686 \\
\hline
 & 2\cdot 95003 & & 01343 \\
 & \cdot 18566 & & 47712 \\
\hline
 & 2\cdot 76437 & \log = 44160 & 51741 \\
 & & 02686 & 3a^{\frac{3}{2}} = 3\cdot 2916 \\
 & & 01343 & 1\cdot 4482 \\
 & & 67894 & \\
\hline
 & T_{23} = 1\cdot 4482 & 16083 & T_{12} = T_{31} = \cdot 9217
\end{array}
$$

82. For the Time in a hyperbola, we have

$$T_{12} = T_{31} = \frac{3}{2\pi}\, a^{\frac{3}{2}} \left\{ e \tan u_2 - h\,.\,l.\, \tan\left(45^\circ + \tfrac{1}{2} u_2\right) \right\},$$

where

$$\tan u_2 = \frac{y_2}{a\sqrt{e^2 - 1}}\,.$$

Taking as a specimen the case $c = 75°$, we have here

$a = \cdot 9004$	$e = 2\cdot 1106$	$y_2 = 43\cdot 341$
$\log = \cdot 95444$	$\log = \cdot 32441$	$\log = 1\cdot 63690$
$a(e^2 - 1) = 3\cdot 1106$		
$\log \quad ,, \quad = \cdot 49284$		

and then the calculation is

$$\begin{array}{lr}
\log a & = 95444 \\
,, \; a(e^2-1) & = \underline{49284} \\
 & 44728 \\
,, \; a\sqrt{e^2-1} & = 22364 \\
\log \quad y_2 & = \underline{63690} \\
 & \\
\log \tan u & = 41326 \\
 & \underline{32441} \\
 & 73767
\end{array}$$

$u = 87° \; 47'$

$h \,.\, l \tan (45° + \frac{1}{2} u) = 3\cdot 95140$

$$\begin{array}{rr}
e \tan u = & 54\cdot 660 \\
 & \underline{3\cdot 951} \\
 & 50\cdot 709 \\
 & \\
\log = & 70508 \\
 & 95444 \\
 & 47722 \\
 & \underline{67894} \\
 & 31568 \\
T_{12} = T_{31} = & 20\cdot 686
\end{array}$$

83. In the case of the parabola $p = 1$, and the expression for the Times is

$$T_{13} = T_{21} = \frac{1}{4\pi} \{(\rho + \rho' + \gamma)^{\frac{3}{2}} - (\rho + \rho' - \gamma)^{\frac{3}{2}}\},$$

where for

$$c = 46° \; 48'$$
$$c = 87° \; 6'$$

we have

$$T_{12} = T_{31} = \cdot 787,$$
$$T_{12} = T_{31} = 2\cdot 588.$$

Planogram No. 1, *first part,* $b = 90°$.

	c	x_2	y_2	ϕ_2	r_2	A	a	T_{12}	T_{23}	T_{31}
Circle	0°	− ·500	+ ·866	60°	1·000	0	1·000	1·000	1·000	1·000
Ellipses	5	·461	·892	62° 39′	1·004	− ·003	1·003	..		
	10	·420	·927	65 38	1·017	·012	1·012	·960	1·135	·960
	15	·374	·972	68 56	1·041	·030	1·031	..		
	20	·324	1·030	72 33	1·079	·060	1·064	·922	1·448	·922
	25	·267	1·104	76 26	1·136	·107	1·120			
	30	·200	1·200	80 32	1·216	·180	1·220	·887	2·275	·887
	35	·120	1·325	84 49	1·330	·295	1·418			
	40	− ·021	1·492	90 48	1·492	·482	1·931	·838	6·371	·838
	40° 54′	·000	1·515	90	1·515	·515	2·061			
	45	+ ·109	1·722	93 37	1·725	·814	5·362			
Parab.	46° 48′	·166	1·826	95 11	1·834	1·000	∞	·787	∞	·787
Hyperbs.	50	·287	2·054	97 57	2·074	1·505	1·981	·750	~	·750
	52° 45′	·418	2·306	100 16	2·344	2·309	·764		~	
	55	·552	2·569	102 8	2·627	− 3·632	·380	·628	~	·628
Line	60	1·000	3·464	196 6	3·605	− ∞ + ∞	·000	·000	~	·000
Convex	65	1·937	5·378	109 48	5·716	+ 5·032	·166	Convex orbit.		
	70	+ 5·248	12·233	113 13	13·311	2·898	·257			
	73° 54′	+ ∞ − ∞	∞	115 39 64 21	∞	+ 2·309 − 2·309	·302 ·764	∞	~	∞
Hyperbs.	75	− 21·432	43·341	63 42	48·346	2·111	·900	20·68	~	20·68
	80	4·356	7·830	60 55	8·960	1·486	2·056	3·856	~	3·856
	85	2·653	4·322	58 27	5·072	1·115	8·718	2·912	~	2·912
Parab.	87° 6′	2·320	3·644	57 31	4·320	1·000	∞	2·588	∞	2·588
Ellipse	90	− 2·000	3·000	56 18	3·606	− ·869	7·622	2·255	58·62	2·255

The mark ~ in the T_{23} column shows that there is no Time T_{23}.

Planogram No. 1, *second part*, $b = 270°$.

	c	x_2	y_2	ϕ_2	r_2	A	a	T_{12}	T_{23}	T_{31}
Circ.	0°	− ·500	+ ·866	60° ′	1·000	0	1·000	1·000	1·000	1·000
	5	·537	·848	57 39	1·003	− ·002	1·002			
	10	·573	·837	55 37	1·014	·009	1·009	1·044	·951	1·044
	15	·608	·832	53 52	1·030	·019	1·019			
	20	·643	·834	52 23	1·053	·032	1·033	1·091	·969	1·091
	25	·678	·842	51 10	1·081	·048	1·051			
	30	·714	·857	50 11	1·116	·068	1·073	1·145	1·043	1·145
	35	·752	·879	49 27	1·157	·090	1·098			
All Ellipses	40	·793	·910	48 51	1·207	·115	1·130	1·207	1·192	1·207
	45	·836	·950	48 40	1·266	·145	1·169			
	50	·884	1·002	48 36	1·336	·179	1·217	1·283	1·464	1·283
	55	·938	1·069	48 45	1·442	·218	1·278			
	60	1·000	1·154	49 7	1·527	·264	1·358	1·377	1·983	1·377
	65	1·074	1·266	49 42	1·660	·318	1·466			
	70	1·164	1·412	50 31	1·830	·383	1·622	1·506	3·036	1·506
	75	1·280	1·611	51 35	2·056	·464	1·864			
	80	1·431	1·891	52 53	2·372	·564	2·295	1·771	6·888	1·771
	85	1·651	2·311	54 28	2·840	·694	3·269			
	90	− 2·000	+ 3·000	56 18	3·606	− ·869	7·622	2·255	58·62	2·255

Article Nos. 84 to 94. *Planogram No.* 2, *the Meridian* 0°—180° (see Plate III.).

84. The orbit-plane here rotates about an axis in the plane of the ecliptic at right angles to $S1$ (Fig. 6). The entire meridian is given by $b=0°$, $c=0°$ to $90°$, and $b=180°$, $c=0°$ to $90°$, but it is sufficient to consider one of these half meridians, say the latter of them, as the series of values is the same for each of them, with only an interchange of the points 2, 3. I write therefore, $b=180°$, so that we have

$$\begin{aligned} \alpha\,,\ \beta\,,\ \gamma &= 0\,,\ 1,\ 0\,, \\ \alpha'\,,\ \beta'\,,\ \gamma' &= -\cos c,\ 0,\ -\sin c, \\ \alpha''\,,\ \beta''\,,\ \gamma'' &= -\sin c,\ 0,\ \cos c, \end{aligned}$$

consequently

$$\begin{aligned} x_1' : y_1' : 1 &= \sin c : -\sqrt{3} : \sqrt{3}\cos c, \\ x_2' : y_2' : 1 &= -3\cos c - 2\sin c : -\sqrt{3} : -\sqrt{3}\sin c - 2\sqrt{3}\cos c, \\ x_3' : y_3' : 1 &= 3\cos c - 2\sin c : -\sqrt{3} : \sqrt{3}\sin c - 2\sqrt{3}\cos c, \end{aligned}$$

and moreover $x'=y$, $y'=-x$; so that, introducing into the formulæ (x_1, y_1), &c., in place of the (x_1', y_1'), &c., we have

$$x_1 = \sec c, \qquad y_1 = \frac{1}{\sqrt{3}}\tan c,$$

$$x_2 = \frac{1}{\sin c + 2\cos c}, \quad y_2 = \frac{1}{\sqrt{3}}\,\frac{2\sin c + 3\cos c}{\sin c + 2\cos c},$$

$$x_3 = \frac{-1}{\sin c - 2\cos c}, \quad y_3 = \frac{1}{\sqrt{3}}\,\frac{2\sin c - 3\cos c}{\sin c - 2\cos c},$$

which, putting

$$\cos 8 = \frac{2}{\sqrt{5}}, \quad \sin 8 = \frac{1}{\sqrt{5}}, \quad \tan 8 = \tfrac{1}{2}, \quad 8 = 26°\,34',$$

become

$$x_1 = \sec c, \qquad y_1 = \frac{1}{\sqrt{3}}\tan c,$$

$$x_2 = -\frac{1}{\sqrt{5}}\sec(c-8), \quad y_2 = \frac{1}{\sqrt{3}}\{\tfrac{8}{5} + \tfrac{1}{5}\tan(c-8)\},$$

$$x_3 = -\frac{1}{\sqrt{5}}\sec(c+8), \quad y_3 = \frac{1}{\sqrt{3}}\{-\tfrac{8}{5} + \tfrac{1}{5}\tan(c+8)\};$$

so that the guide-hyperbolas are

$$\begin{aligned} x_1^2 - 3y_1^2 &= 1, && \tfrac{1}{2}\text{ angle of asymptotes} = 30° \\ x_2^2 = 15y_2^2 - 16\sqrt{3}\,y_2 + 13, &&\ \text{,,}\quad \text{,,}\quad \tan^{-1}\frac{1}{\sqrt{15}} = 14°\,28' \\ x_3^2 = 15y_3^2 + 16\sqrt{3}\,y_2 + 13, &&\ \text{,,}\quad \text{,,}\quad \text{,,}\quad \text{,,} \end{aligned}$$

It is easy to verify that

Hyperbola 2 passes through $x_2 = -\frac{1}{2}$, $y_2 = \frac{1}{2}\sqrt{3}$, and touches there circle $x^2 + y^2 = 1$,

" 3 " $x_3 = -\frac{1}{2}$, $y_3 = -\frac{1}{2}\sqrt{3}$ " " "

and we thus have the figure in the Plate.

85. The figure shows the motion of the points 1, 2, 3, along their respective hyperbolas, viz. $c = 0°$ to $90°$, the point 1 moves from contact with the circle, along a half branch to infinity: 2 moves from contact along a small portion of the half branch; 3 moves from contact, along the half branch to infinity for $c = \tan^{-1} 2 = 63° 26'$, and then reappearing at the opposite infinity, as c increases to $90°$ describes a portion of the opposite half branch.

86. For $c = 0$, the orbit is the circle; as c increases the orbit becomes elliptic; then parabolic, $c = 51°$, and afterwards hyperbolic (concave); until for $c = 60°$, the three points are on the horizontal line of the figure, and the orbit is this right line; it is to be noticed that the arrangement of the points on these orbits is 1, 2, 3; so that for the parabola, T_{31} is $= \infty$, and for the hyperbolas and right line T_{31} does not exist.

87. For $c < 60°$ until $c = 63° 26'$ the orbit is a convex hyperbola, the arrangement of the points being still 1, 2, 3: say for $c = 63° 26' - \epsilon$, the orbit is the convex hyperbola Ω. At $c = 63° 26'$ there is an abrupt change of orbit; say for $c = 63° 26' + \epsilon$ the orbit is a concave hyperbola Ω_1; and for $c = 65° 52'$ the orbit is a parabola; the arrangement of the points on these orbits is 2, 1, 3; so that for the hyperbolas T_{23} does not exist, and T_{23} is $= \infty$ for the parabola. Observe also that for the hyperbola Ω_1, the point 3 is at infinity, or we have $T_{31} = \infty$. As c continues to increase, the orbit becomes an ellipse, the eccentricity having a minimum value $= \cdot 628$ (about), for $c = 69°$ (about). For $c = 89° 20'$ the orbit is again a parabola, and then until $c = 90°$ it is a hyperbola; the order of the points on the last-mentioned parabola and hyperbolas being 1, 3, 2; so that for the parabola T_{12} is $= \infty$, and for the hyperbolas T_{12} does not exist. In the hyperbola for $c = 90°$, say the hyperbola Ω', the point 1 is at infinity, or we have $T_{12} = \infty$. The foregoing results, obtained (except as to the numerical values) by consideration of the figure, will be confirmed by means of the calculated values of e.

88. The equation of the orbit may be written

$$\begin{vmatrix} \frac{r}{\sqrt{3}}, & \frac{x}{\sqrt{3}}, & y, & \frac{1}{\sqrt{3}} \\ r_1 \cos c, & 1, & \sin c, & \cos c \\ r_2 (\sin c + 2 \cos c), & -1, & 2 \sin c + 3 \cos c, & \sin c + 2 \cos c \\ r_3 (\sin c - 2 \cos c), & 1, & -2 \sin c + 3 \cos c, & \sin c - 2 \cos c \end{vmatrix} = 0,$$

or developing, this is

$$\frac{r}{\sqrt{3}}\,6\,(\sin^2 c - 3\cos^2 c),$$

$$\begin{aligned}
-\frac{x}{\sqrt{3}}\{\;& 4r_1(\sin^2 c - 3\cos^2 c)\cos c \\
&- r_2(\sin c + 2\cos c)(\sin^2 c - 3\cos^2 c) \\
&+ r_3(\sin c - 2\cos c)(\sin^2 c - 3\cos^2 c)\} \\
+y\;\{&- 2r_1 \sin c \cos c \\
&+ r_2(-\sin^2 c + \sin c\cos c + 6\cos^2 c) \\
&+ r_3(\;\sin^2 c + \sin c\cos c - 6\cos^2 c)\} \\
-\frac{1}{\sqrt{3}}\{\;& r_1 . - 6\cos^2 c \\
&+ 3r_2(\sin^2 c + \sin c\cos c - 2\cos^2 c) \\
&+ 3r_3(\sin^2 c - \sin c\cos c - 2\cos^2 c)\} = 0;
\end{aligned}$$

(observe that the orbit will be a right line if $\sin^2 c - 3\cos^2 c = 0$, that is if $c = 60°$, which is right, since 60° is the angular radius of the regulator circle).

89. Putting in the equation $\tan c = \lambda$, and therefore $\cos c = \frac{1}{\sqrt{1+\lambda^2}}$, the equation becomes

$$\begin{aligned}
r = \;& \frac{1}{6\sqrt{1+\lambda^2}}\left(4r_1 - (\lambda+2)\,r_2 + (\lambda-2)\,r_3\right)x \\
&+ \frac{1}{2\sqrt{3}\,(\lambda^2-3)}\left(2\lambda\,r_1 + (\lambda+2)(\lambda-3)\,r_2 - (\lambda+3)(\lambda-2)\,r_3\right)y \\
&+ \frac{1}{2\,(\lambda^2-3)}\left(-2r_1 + (\lambda-1)(\lambda+2)\,r_2 + (\lambda+1)(\lambda-2)\,r_3\right).
\end{aligned}$$

We have

$$x_1 = \sqrt{1+\lambda^2},\quad x_2 = \frac{-\sqrt{1+\lambda^2}}{\lambda+2},\quad x_3 = \frac{\sqrt{1+\lambda^2}}{\lambda-2},$$

$$y_1 = \frac{1}{\sqrt{3}}\lambda,\quad y_2 = \frac{1}{\sqrt{3}}\,\frac{2\lambda+3}{\lambda+2},\quad y_3 = -\frac{1}{\sqrt{3}}\,\frac{2\lambda-3}{\lambda-2},$$

and thence, writing for shortness

$$R_1 = \sqrt{1+\tfrac{4}{3}\lambda^2},$$

$$R_2 = \frac{1}{\sqrt{3}}\sqrt{7\lambda^2+12\lambda+12},$$

$$R_3 = \frac{1}{\sqrt{3}}\sqrt{7\lambda^2-12\lambda+12},$$

we have

$$r_1 = R_1,$$
$$r_2(\lambda+2) = R_2,$$
$$r_3(\lambda-2) = R_3,$$

where r_1, r_2, r_3 are positive, and the signs of R_1, R_2, R_3 must be determined accordingly; viz., R_1 is always positive, and ($c=0°$ to $c=90°$, as here supposed) R_2 is also positive; but R_3 has the same sign as $\lambda-2$; viz., $c=0°$ to $c=63°\,26'$, R_3 is negative; and $c=63°\,26'$ to $c=90°$, R_3 is positive. It is to be observed that this position, $c=\tan^{-1}2=63°\,26'$, of the pole is the intersection of the meridian $b=180°$ by a separator circle, and corresponds to an intersection at infinity on the ray 3.

90. Substituting the foregoing values of r_1, r_2, r_3, the equation of the orbit becomes

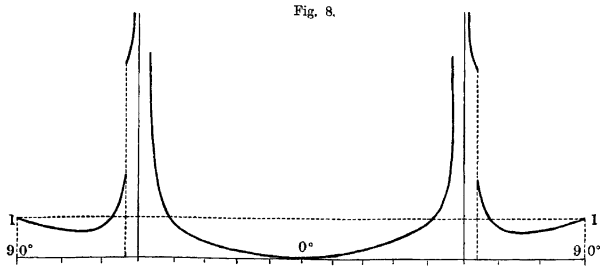
$$r = \frac{1}{6\sqrt{1+\lambda^2}}(4R_1 - R_2 + R_3)x$$
$$+ \frac{1}{2\sqrt{3}(\lambda^2-3)}\{\lambda(2R_1+R_2-R_3) - 3(R_2+R_3)\}y$$
$$+ \frac{1}{2(\lambda^2-3)}\{\lambda(R_2+R_3) - 2R_1 - R_2 + R_3\},$$

where $\lambda = \tan c$; and the equation of the orbit may thence be calculated for any given value of c.

91. The analytical expression for the eccentricity is

$$e = \sqrt{A^2+B^2},$$

Fig. 8.



where, as above,

$$A = \frac{1}{6\sqrt{1+\lambda^2}}(4R_1 - R_2 + R_3),$$

$$B = \frac{1}{2\sqrt{3}(\lambda^2-3)}\{\lambda(2R_1+R_2-R_3) - 3(R_2+R_3)\};$$

but this expression is too complicated to allow of an analytical discussion of the series of values of e (such as was given for A, $=\pm e$, in planogram No. 1). The numerical calculation gives the results mentioned *ante* No. 87, viz., $c=0$, $e=0$; $c=51°$, $e=1$; $c=60°$, $e=\infty$; $c=63°\,26'-\epsilon$, $e=4{\cdot}912$; $c=63°\,26'+\epsilon$, $e=1{\cdot}853$; $c=69°$, $e={\cdot}628$ (min.); $c=89°\,20'$, (viz. $\lambda=86{\cdot}176$), $e=1$; $c=90°$, $e=1{\cdot}018$; values which are exhibited in the diagram in the preceding page.

92. It may be further remarked, in reference to the formula

$$r=Ax+By+C,$$

that for $c=60°$, that is $\lambda=\sqrt{3}$, we have A finite, B and C each infinite, but equal and of opposite signs; viz., the equation becomes $r={\cdot}2242\,x\pm\infty\,(y-1)$, that is $y=1$, orbit a right line as above.

The abrupt change at $c=63°\,26'$, $\lambda=2$, arises from the change of sign of R_3; viz., $c=63°\,26'-\epsilon$, $R_3=-\dfrac{4}{\sqrt{3}}=-2{\cdot}309$, but $c=63°\,26'+\epsilon$, $R_3=\dfrac{4}{\sqrt{3}}=+2{\cdot}309$; the two orbits are

$$c=63°\,26'-\epsilon,\qquad r={\cdot}234\,x+4{\cdot}906\,y-3{\cdot}671,\qquad e=4{\cdot}912,\qquad a={\cdot}159,$$

$$c=63°\,26'+\epsilon,\qquad r={\cdot}578\,x-1{\cdot}761\,y+3{\cdot}257,\qquad e=1{\cdot}853,\qquad a=1{\cdot}338$$

For $c=90°$ the equation is

$$r=\frac{4}{3\sqrt{3}}x+\tfrac{2}{3}\;y+\sqrt{\tfrac{7}{3}}$$

$$={\cdot}770\,x+{\cdot}666\,y+1{\cdot}527$$

and therefore $e=\sqrt{\frac{28}{27}}=1{\cdot}018$ as above; $a=9\sqrt{21}=41{\cdot}243$.

It is to be added that for c nearly $=90°$, or λ very large, we have

$$R_1=\frac{2}{\sqrt{3}}\lambda,\quad R_2=\sqrt{\tfrac{7}{3}}\lambda+2\sqrt{\tfrac{3}{7}},\quad R_3=\sqrt{\tfrac{7}{3}}\lambda-2\sqrt{\tfrac{3}{7}},$$

and thence

$$A=\frac{4}{3\sqrt{3}}-\frac{2}{\sqrt{21}}\frac{1}{\lambda}={\cdot}770-{\cdot}430\frac{1}{\lambda},$$

$$B=\tfrac{2}{3}-\frac{5}{\sqrt{7}}\frac{1}{\lambda}={\cdot}666-1{\cdot}890\frac{1}{\lambda},$$

$$C=\sqrt{\tfrac{7}{3}}-\frac{1}{\sqrt{3}}\frac{2}{\lambda}=1{\cdot}527-1{\cdot}555\frac{1}{\lambda}.$$

It was, in fact, by means of these expressions that the value $\lambda=86{\cdot}176$ ($c=89°\,20'$) corresponding to the last-mentioned parabolic orbit was obtained.

93. For the calculation of the table we have

$$\log x_1 = \overline{10} \qquad + \log \sec c,$$

$$\log y_1 = \overline{10}{\cdot}76144 + \log \tan c,$$

$$\log x_2 = \overline{10}{\cdot}65052 + \log \sec\ (c - 26^\circ\ 34'),$$

$$\log (y_2 - {\cdot}92376) = \overline{10}{\cdot}06247 + \log \tan\ (c - 26^\circ\ 34'),$$

$$\log x_3 = \overline{10}{\cdot}65052 + \log \sec\ (c + 26^\circ\ 34'),$$

$$\log (y_3 + {\cdot}92376) = \overline{10}{\cdot}06247 + \log \sec\ (c + 26^\circ\ 34'),$$

the values of r_1, r_2, r_3, are then calculated from

$$x_1 = r_1 \cos \phi_1, \quad y_1 = r_1 \sin \phi_1,$$

or say

$$\frac{y_1}{x_1} = \tan \phi_1, \qquad r_1 = x_1 \sec \phi_1, \text{ \&c.}$$

and those of the chords γ_{12}, γ_{23}, γ_{31}, from

$$x_1 - x_2 = \gamma_{12} \cos \theta_{12}, \quad y_1 - y_2 = \gamma_{12} \sin \theta_{12},$$

or say

$$\tan \theta_{12} = \frac{x_1 - x_2}{y_1 - y_2}, \quad \gamma_{12} = (x_1 - x_2) \sec \theta_{12}.$$

We have then to find the equation of the orbit $r = Ax + By + C$; this might be done by substituting in the determinant expression the numerical values of x_1, y_1, r_1, x_2, y_2, r_2, x_3, y_3, r_3, and so calculating the result, but I have preferred to employ the formula of No. 90, using only the calculated values of r_1, r_2, r_3; viz. we have

$$r_1 = R_1,$$

$$r_2 (\lambda + 2) = R_2,$$

$$r_3 (\lambda - 2) = R_3,$$

which gives the values of R_1, R_2, R_3. And then we have e, ϖ, a, from the equations

$$A = e \cos \varpi, \quad B = e \sin \varpi, \quad a = \frac{\pm C}{1 - e^2},$$

e and a being each regarded as positive. The times in the elliptic, and parabolic orbits are then calculated from Lambert's equation, as explained in regard to Planogram No. 1, but for the hyperbolic orbits, the other formulæ were made use of.

94. I annex a specimen; the characteristics of the logarithms are omitted.

$c = 20°$.

20°		$-6° 34'$		$+46° 34'$	
02701	56107	00286	06113	16272	02376
	76144	65052	06247	65052	06247
	32251	65338	12360	81324	08623
$x_1 = 1{\cdot}06418$	$y_1 = {\cdot}21014$	$x_2 = -{\cdot}45017$	·92367	$x_3 = -{\cdot}65049$	·92367
32251	02701		·01329		·12196
02701	00830	$y_2 = +$	·91038	$y_3 =$	·80171
29950	03531	log =	·95922	log $y_3 =$	90402
$\phi_1 = 11° 10'$	$r_1 = 1{\cdot}0847$	95922	95922	90402	90402
		65338	04752	81324	10980
		30584	00674	09078	01382

$\phi_2 (= 63° 41') = 116° 19'$, $r_2 = 1{\cdot}0157$ $\phi_3 (= 50° 57') = 230° 57'$, $r_3 = 1{\cdot}0323$.

The calculation of the equation of the orbit is then as follows:

$\lambda =$ ·36397 $\log R_1 = 03531$

log = 56107 $R_1 = 1{\cdot}0847$

12214 $\lambda + 2 = 2{\cdot}36397$

$\lambda^2 =$ ·13248 log = 37364

$\lambda^2 - 3 = -2{\cdot}86752$ $\log r_2 =$ 00674

log = 45750 38038

$\log \sqrt{1+\lambda^2} = 02701$ $R_2 = +2{\cdot}4010$

77815

$\log 6\sqrt{1+\lambda^2} = 80516$ (a) $\lambda - 2 = -1{\cdot}63603$

log = 21378

45750 $\log r_3 =$ 01382

30103 22760

$\log 2(\lambda^2 - 3) = 75853$ (c) $R_3 = -1{\cdot}6889$

23856

$\log 2\sqrt{3}(\lambda^2 - 3) = 99709$ (b)

$4R_1 = 4{\cdot}3388$		
$-R_2$	$-2{\cdot}4010$	
$+R_3$	$-1{\cdot}6889$	
	$4{\cdot}0899$	
	${\cdot}2489$	
$\log = 39602$		
$(a) = 80516$		
	59086	
$A = {\cdot}038982$		

$2R_1 = 2{\cdot}1694$		
$+R_2$	$2{\cdot}4010$	
$-R_3$	$1{\cdot}6889$	
	$6{\cdot}2593^*$	
$\log = 79653$		
$\lambda = 56107$		
	35760	
	$+2{\cdot}2782$	
$-3R_2$		$-7{\cdot}2032$
$-3R_3$	$5{\cdot}0667$	
	$7{\cdot}3449$	
	$-7{\cdot}2032$	
	$0{\cdot}1417$	
$\log = 15137$		
$(b) = 99709$		
	${\cdot}15428$	
$B = -{\cdot}014265$		

$R_2 =$	$2{\cdot}4010$
$R_3 = -$	$1{\cdot}6889$
$+$	${\cdot}7121$
$\log =$	85254
$\lambda =$	56107
	41361
$+$	${\cdot}25919$
$-$	$6{\cdot}2593^*$
$-$	$6{\cdot}00011$
$\log =$	77818
$(c) =$	75853
	${\cdot}01965$
$C = +$	$1{\cdot}0464$

$\log B = 15428$	02729
$\log A = 59086$	59086
56342	61815
$\varpi\ (= 20^\circ\ 6') = 160^\circ\ 52'$	$e = {\cdot}04151$
	23630
	$e^2 = {\cdot}001723$
	$1 - e^2 = {\cdot}998277$
	$a = 1{\cdot}0481$

$\log C = 01965$	
$\log\ = 99925$	
	02040

The calculation of the Times is similar to that for the first planogram, and requires no further illustration.

The Table for Planogram No. 2 is as follows:

Planogram No

	c	x_1 all +	y_1 all +	x_2 all −	y_2 all +	x_3	y_3	r_1	ϕ_1	r_2	ϕ_2	r_3	ϕ
Circle	0°	1·000	·000	·500	·866	− ·500	− ·866	1·000	0° 0′	1·000	120° 0′	1·000	240
Ellipses	5	1·004	·051	·481	·878	·525	·853	1·005	2 52	1·001	118 42	1·001	238
	10	1·015	·102	·467	·889	·557	·838	1·020	5 44	1·004	117 41	1·006	236
	15	1·035	·155	·456	·900	·598	·821	1·047	8 30	1·009	116 54	1·016	233
	20	1·064	·210	·450	·910	·650	·802	1·085	11 10	1·016	116 19	1·032	230
	25	1·103	·269	·447	·921	·719	·778	1·136	13 43	1·024	115 55	1·060	227
	30	1·155	·333	·448	·931	·812	·749	1·202	16 6	1·033	115 42	1·104	222
	35	1·221	·404	·452	·941	·939	·710	1·286	18 19	1·044	115 40	1·178	217
	40	1·305	·484	·460	·951	1·125	·657	1·392	20 21	1·056	115 48	1·302	210
	45	1·414	·577	·471	·962	1·414	·577	1·527	22 12	1·071	116 6	1·528	202
	50	1·556	·688	·487	·974	1·925	·440	1·701	23 51	1·088	116 35	1·975	192
Parab.	51° 0′	1·589	·713	·491	·976	2·077	·400	1·741	24 9	1·093	116 43	2·115	190
Hyperbs.	52	1·624	·739	·495	·978	2·256	·353	1·787	24 28	1·097	116 51	2·283	188
	54	1·701	·795	·504	·984	2·729	·229	1·878	25 2	1·105	117 7	2·738	184
	55	1·743	·824	·509	·986	3·049	− ·145	1·928	25 18	1·109	117 16	3·053	182
	56° 18′	1·802	·866	·515	·990	3·601	·000	1·999	25 39	1·116	117 30	3·601	180
	59	1·942	·961	·530	·997	5·786	+ ·566	2·166	26 20	1·129	118 59	5·813	174
Line	60	2·000	1·000	·536	1·000	7·468	1·000	2·236	26 34	1·134	118 11	7·534	172
Convex	61	2·063	1·042	·542	1·003	− 10·53	+ 1·793	2·311	26 48	1·140	118 24	10·68	170
	63° 26′ − ε 63° 26′ + ε	2·236	1·155	·559	1·010	− ∞ + ∞	+ ∞ − ∞	2·517	27 19	1·155	118 57	∞	165 345
Hyperbs.	64	2·281	1·184	·563	1·012	+ 45·22	− 12·60	2·570	27 26	1·157	119 6	46·94	344
	65	2·366	1·238	·571	1·015	16·36	5·146	2·670	27 37	1·165	119 21	17·15	342
Parab.	65° 52′	2·446	1·289	·578	1·019	10·12	3·552	2·765	27 47	1·171	119 35	10·80	340
Ellipses	66	2·459	1·297	·579	1·019	9·987	3·500	2·779	27 48	1·172	119 37	10·59	340
	68	2·669	1·429	·596	1·026	5·617	2·369	3·028	28 10	1·186	120 11	6·090	337
	70	2·924	1·586	·616	1·033	3·912	1·927	3·326	28 29	1·202	120 48	4·360	333
	72	3·237	1·777	·638	1·041	3·008	1·694	3·693	28 47	1·221	121 29	3·455	330
	75	3·864	2·155	·674	1·054	2·230	1·488	4·424	29 9	1·251	122 36	2·681	326
	80	5·759	3·274	·751	1·079	1·568	1·312	6·624	29 37	1·315	124 49	2·045	320
	85	11·47	6·599	·854	1·112	1·217	1·216	13·25	29 54	1·402	127 32	1·720	315
Parab.	89° 20′	86·41	49·79	·979	1·148	1·024	1·161	99·5	29 56	1·508	130 24	1·548	311
Hyperbs.	90 − ε	∞	∞	1·000	1·155	+ 1·000	− 1·155	∞	30 0	1·527	130 54	1·527	310

$b = 180°$, $c = 0°$ to $90°$.

Equation of Orbit. $r = Ax$	$+By$	$+C$	e	ϖ	a	γ_{23}	γ_{31}	γ_{12}	T_{23}	T_{31}	T_{12}	
·000	·000	+ 1·000	·000	ind.° ′	1·000	1·732	1·732	1·732	1·000	1·000	1·000	0°
												5
+ ·0101	− ·0015	1·0104	·010	171 19	1·010	1·729	1·832	1·678	·987	1·106	·953	10
												15
·039	·014	1·046	·041	160 52	1·048	1·724	1·991	1·668	·956	1·316	·946	20
												25
·083	·061	1·126	·103	143 48	1·138	1·718	2·244	1·710	·924	1·777	·962	30
												35
·135	·209	1·317	·248	122 54	1·404	1·740	2·684	1·826	·878	3·238	·966	40
·161	·395	1·527	·426	112 12	1·867	1·805	3·055	1·925				45
·186	·815	1·972	·836	102 50	6·554	2·016	3·659	2·063	·878	48·60	·849	50
·191	·982	2·140	1·000	101 14	∞	2·100	3·831	2·097	·879	∞	·820	51° 0′
·196	1·150	2·319	1·166	99 39	6·434							52
·203	1·719	2·898	1·720	96 44	1·481							54
·207	2·182	3·366	2·192	95 25	·885	2·781	4·890	2·258	·895	~	·665	55
·212	3·074	4·227	3·081	93 56	·498							56° 18′
·221	− 14·15	+ 15·42	14·15	90 30	·077							59
·224	± ∞	$(y-1)$	∞	90 0	·000	6·932	9·468	2·536	·000	~	·000	60
									Convex Orbits.			61
·234	+ 4·906	− 3·671	4·912	87 17	·159							63° 26′ − ε
·578	− 1·761	+ 3·257	1·853	108 11	1·338	∞	∞	2·799	~	∞	·909	63° 26′ + ε
												64
·587	·979	2·494	1·134	120 21	8·666	18·014	15·380	2·945				65
·591	·805	2·257	1·000	126 19	∞	11·633	9·072	3·036	∞	7·746	1·386	65° 52′
·593	·779	2·221	·979	127 15	53·83							66
·606	·338	1·894	·693	150 53	3·645							68
·619	− ·120	1·708	·630	169 0	2·834	5·409	3·649	3·584	5·735	6·343	2·685	70
·635	+ ·027	1·599	·636	182 25	2·674							72
·654	·185	1·497	·680	195 47	2·783	3·859	3·981	4·644	2·62	6·68	4·64	75
·692	·366	1·439	·783	207 52	3·716	3·327	6·212	6·870	1·97	10·21	9·343	80
·740	·514	1·455	·892	214 47	7·721	3·115	12·895	13·480				85
·764	·645	1·505	1·000	219 51	∞	3·055	99·43	102·7	1·200	225·4	∞	89° 20′
+ ·770	+ ·666	+ 1·527	1·018	220 6	41·000	3·055	∞	∞	1·148	∞	~	90 − ε

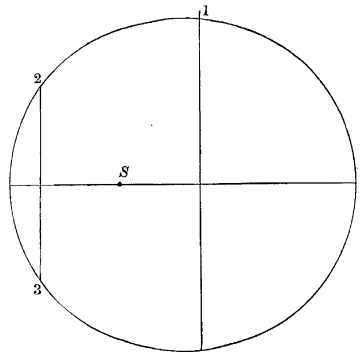
The mark ~ in any of the T columns shows that the Time does not exist.

Article Nos. 95 to 98. *Planogram No. 3, the Orbit-pole at one of the points A.*

95. When the orbit-pole is at one of the points A, the orbit-plane passes through one of the rays, and as there is no longer on this ray any determinate point of intersection, the orbit (as was seen) becomes indeterminate. Thus consider the point A for which $b = 270°$, $c = 60°$: we have

$$\begin{array}{llll} \alpha\,,\ \beta\,,\ \gamma\ = -1, & 0\ , & 0\ , \\ \alpha',\ \beta',\ \gamma' = \ \ 0, & -\frac{1}{2}\ , & -\frac{1}{2}\sqrt{3}, \\ \alpha'',\ \beta'',\ \gamma'' = \ \ 0, & -\frac{1}{2}\sqrt{3}, & \frac{1}{2}\ , \end{array}$$

Fig. 9.



and consequently the formula gives

$$\begin{array}{llll} x_1' : y_1' : 1 = & 0 & : 0 : & 0\ , \\ x_2' : y_2' : 1 = & -\frac{1}{2}\sqrt{3} - \sqrt{3} & : \ 3 : & -\frac{1}{2}\sqrt{3} - \sqrt{3}, \\ x_3' : y_3' : 1 = & -\frac{1}{2}\sqrt{3} - \sqrt{3} & : -3 : & -\frac{1}{2}\sqrt{3} - \sqrt{3}, \end{array}$$

and, moreover, $x = -x'$, $y = -y'$. From the formula the value of x_1' or x_1 is given as $\frac{0}{0}$, but the true value is obviously $x_1 = 1$; the value of y_1 is actually indeterminate. The formulæ give the values of (x_2, y_2), (x_3, y_3), viz. the system is

$$\begin{array}{ll} x_1 = \ \ 1, & y_1 = \text{ind.} \\ x_2 = -1, & y_2 = \dfrac{2}{\sqrt{3}}, \quad \text{whence } r_2 = r_3 = \sqrt{\tfrac{7}{3}}, \\ x_3 = -1, & y_3 = -\dfrac{2}{\sqrt{3}}, \end{array}$$

so that the orbits in the planogram are the whole series of conics having a given focus, S, and passing through two fixed points, 2, 3, having the common abscissa $x=-1$, and at equal distances $\frac{2}{\sqrt{3}}$ $(=1{\cdot}15470)$ on opposite sides of the axis. The axis of x is obviously the common transverse axis for all the orbits; that is, the equation of the orbit will be of the form $r=Ax+B$; and writing $x=-1$, we have $\sqrt{\frac{7}{3}}=-A+B$, viz. the equation is $r-\sqrt{\frac{7}{3}}=A(x+1)$; the value of A will be determined if we assume for the point 1 a determinate position on the line $x=1$, say its ordinate is $=y_1$; for then if $r_2=\sqrt{1+y_1^2}$ we have $r_1-\sqrt{\frac{7}{3}}=2A$, and the equation is $r-\sqrt{\frac{7}{3}}=\frac{1}{2}(r_1-\sqrt{\frac{7}{3}})(x+1)$. In particular if $y_1=0$, we have $r_1=1$, and the equation of the orbit is $r-\sqrt{\frac{7}{3}}=\frac{1}{2}(1-\sqrt{\frac{7}{3}})(x+1)$: this is the orbit, eccentricity $\frac{1}{2}(\sqrt{\frac{7}{3}}-1)$, $=\cdot264$, belonging to the point A as a point in planogram No. 1: for the value of y, being in that planogram originally assumed $=0$, is of course $=0$ when the orbit-pole comes to be the point A.

96. We may conversely take the equation of the orbit, or say the value of $A\,(=\pm e)$ in the equation $r-\sqrt{\frac{7}{3}}=A(x+1)$, to be given; and then writing $x=x_1=1$, we have

$$r_1=\sqrt{\tfrac{7}{3}}+2A, \qquad \text{that is } y_1^2-(\sqrt{\tfrac{7}{3}}+2A)^2-1\,;$$

for

$$r_1=1 \text{ or } y_1=0, \qquad A=\tfrac{1}{2}(1-\sqrt{\tfrac{7}{3}})=-\cdot264,$$

and as r_1 increases to $r_1=\sqrt{\frac{7}{3}}$, or y_1 increases to $\pm\frac{2}{\sqrt{3}}$, A diminishes from $-\cdot264$ to 0; viz., for $r_1=\sqrt{\frac{7}{3}}$, or $y_1=+\frac{2}{\sqrt{3}}$, the orbit is a circle; as r_1 increases from $\sqrt{\frac{7}{3}}$, or y_1 from $\pm\frac{2}{\sqrt{3}}$, A increases from 0 positively; for $r_1=\sqrt{\frac{7}{3}}+2$, $=3{\cdot}527$, or $y_1=\pm\sqrt{\frac{16+2\sqrt{21}}{3}}$, $=\pm2{\cdot}896$, A becomes $=1$; that is, the orbit is a parabola; and for larger positive values of r_1, or positive or negative values of y_1, the orbit is a hyperbola (concave); and ultimately for $r_1=\infty$ or $y_1=\pm\infty$, the orbit is the right line $x+1=0$. Thus A extends from $-\cdot264$ to 0, and thence from 0 positively to $+\infty$.

97. In further illustration, suppose that the orbit-pole, instead of being at A, is a point in the immediate neighbourhood of A, say that the rectangular spherical coordinates, measured from A in the direction of the meridian and perpendicular thereto, are ξ and η; the colatitude and longitude of the orbit-pole being thus $c=60^\circ+\xi$, and $b=270^\circ+\frac{2}{\sqrt{3}}\eta$; we have then, ξ, η being indefinitely small,

$$\begin{aligned}
\alpha\,,\ \beta\,,\ \gamma &= -1\,,\ -\frac{2}{\sqrt{3}}\eta, \qquad 0,\\
\alpha',\ \beta',\ \gamma' &= \frac{1}{\sqrt{3}}\eta,\ -\tfrac{1}{2}+\frac{\sqrt{3}}{2}\xi,\ \frac{\sqrt{3}}{2}-\tfrac{1}{2}\xi,\\
\alpha'',\ \beta'',\ \gamma'' &= \eta\ ,\ -\frac{\sqrt{3}}{2}-\tfrac{1}{2}\xi,\ \ \tfrac{1}{2}-\frac{\sqrt{3}}{2}\eta\,;
\end{aligned}$$

and thence

$$x_1' = \left(-\tfrac{1}{2}+\frac{\sqrt{3}}{2}\,\xi\right)\sqrt{3}+\frac{\sqrt{3}}{2}+\tfrac{1}{2}\xi = \quad 2\xi$$

$$: y_1' \quad : -\frac{2}{\sqrt{3}}\eta. \quad \sqrt{3} \qquad : -2\eta$$

$$: 1 \quad : -\frac{\sqrt{3}}{2}-\tfrac{1}{2}\xi+\left(\tfrac{1}{2}-\frac{\sqrt{3}}{2}\right)\xi \quad : -2\xi;$$

that is, $x_1' = -1$, $y_1' = \frac{\eta}{\xi}$, or what is the same thing, $x_1 = -1$, $y_1 = \frac{\eta}{\xi}$; the values of x_2, y_2, and x_3, y_3, differ from their former values only by terms in ξ, η, which may be neglected; that is, we have as before $x_2 = -1$, $y_2 = \frac{2}{\sqrt{3}}$ and $x_3 = -1$, $y_3 = -\frac{2}{\sqrt{3}}$; and we thus see that the foregoing determination of the orbit for an arbitrary value of y_1, writing therein $y_1 = -\frac{\eta}{\xi}$ $\left(\text{or what would be the same thing } y_1 = \frac{\eta}{\xi}\right)$ gives the orbit for the neighbouring position $c = 60° + \xi$, and $b = 270° + \frac{2}{\sqrt{3}}\eta$ of the orbit-pole. Writing for greater convenience $\xi = \rho\cos\psi$, $\eta = \rho\sin\psi$, the indefinitely small quantity ρ will denote the distance of the orbit-pole from A, and its azimuth measured from the meridian will be $=\psi$. We then have $y_1 = -\tan\psi$, and $r_1 = \sqrt{1+y_1^2} = \pm\sec\psi$, or, if to fix the ideas, ψ be considered as $< \pm 90°$, then $r_1 = \sec\psi$: we have thus ($A \doteq \pm e$ as before) $A = \frac{1}{2}(-\sqrt{\tfrac{7}{3}}+\sec\psi)$; viz., observing that $\sqrt{\tfrac{7}{3}} = 1{\cdot}527$, we obtain

$\psi = 0,$	$A = -\frac{1}{2}(\sqrt{\tfrac{7}{3}}-1) = -{\cdot}264$
$\psi = \sec^{-1}\sqrt{\tfrac{7}{3}} = \pm 49°\ 6',$	$A = 0$
$\psi = \sec^{-1}(2\sqrt{\tfrac{7}{3}}-1) = \pm 60°\ 52',$	$A = \frac{1}{2}(\sqrt{\tfrac{7}{3}}-1) = +{\cdot}264$
$\psi = \sec^{-1}(\sqrt{\tfrac{7}{3}}+2) = \pm 73°\ 32',$	$A = 1$
$\psi = \pm(90° - \epsilon),$	$A = +\infty.$

98. These results will have to be further considered in reference to the course of the iseccentric curves through the point A. I remark here that, although it appears that although for eccentricities less than ·264, and in particular for the eccentricity $=0$, there are real directions of passage from A to a neighbouring point, yet there are not through A any real branches of the corresponding iseccentric curves; viz., A is in regard to these curves, an isolated point with *real* tangents; that is a point in the nature of an evanescent lemniscate. As regards the eccentricity $=0$, it is obvious that this must be so; viz., there can be no real branch through A. In fact, the orbit can only be a circle when the intersection by the orbit-plane of the hyperboloid which contains the three rays is also a circle; that is, the orbit is a circle only when the orbit-plane coincides with the plane of the ecliptic.

Article Nos. 99 to 103. *Planogram No. 4, the Orbit-pole in the Ecliptic.*

99. When the orbit-pole describes the circle of the ecliptic, the orbit-plane passes through the axis of z, or polar axis. We have $c = 90°$, and consequently

$$\begin{aligned} \alpha\,,\ \beta\,,\ \gamma &= \sin b, & -\cos b, & & 0, \\ \alpha',\ \beta',\ \gamma' &= 0\ , & 0\ , & & -1, \\ \alpha'',\ \beta'',\ \gamma'' &= \cos b, & \sin b, & & 0. \end{aligned}$$

Reverting for a moment to the general case where the six coordinates of the ray are (a, b, c, f, g, h), the formulæ for the intersection by the orbit-plane are

$$\begin{aligned} x' : y' : 1 = \ & (\mathrm{a,\ b,\ c} \between \alpha',\ \beta',\ \gamma') = -\mathrm{c} \\ : & -(\mathrm{a,\ b,\ c} \between \alpha\ ,\ \beta\ ,\ \gamma\ ) : -\mathrm{a} \sin b + \mathrm{b} \cos b \\ : & \ (\mathrm{f,\ g,\ h} \between \alpha'',\ \beta'',\ \gamma'') : \ \mathrm{f} \cos b + \mathrm{g} \sin b, \end{aligned}$$

that is

$$\frac{1}{x'} + \frac{\mathrm{f}}{\mathrm{c}} \cos b + \frac{\mathrm{g}}{\mathrm{c}} \sin b = 0,$$

$$\frac{y'}{x'} + \frac{\mathrm{b}}{\mathrm{c}} \cos b - \frac{\mathrm{a}}{\mathrm{c}} \sin b = 0\,;$$

and thence

$$\begin{aligned} 1 : \cos b : \sin b &= \frac{-\mathrm{af} - \mathrm{bg}}{\mathrm{c}^2} : \frac{\mathrm{g}y' + \mathrm{a}}{\mathrm{c}x'} : \frac{\mathrm{b} - \mathrm{f}y'}{\mathrm{c}x'} \\ &= \quad \mathrm{h}x' \quad : \mathrm{g}y' + \mathrm{a} : -\mathrm{f}y' + \mathrm{b}\,; \end{aligned}$$

consequently

$$\mathrm{h}x'^2 = (\mathrm{g}y' + \mathrm{a})^2 + (\mathrm{f}y' - \mathrm{b})^2,$$

or, what is the same thing,

$$\mathrm{h}x'^2 = (\mathrm{f}^2 + \mathrm{g}^2)\, y'^2 + 2\,(\mathrm{ag} - \mathrm{bf})\, y' + \mathrm{a}^2 + \mathrm{b}^2,$$

or, in particular, if (as in the special symmetrical case) $\mathrm{ag} - \mathrm{bf} = 0$, then

$$\mathrm{h}x'^2 = (\mathrm{f}^2 + \mathrm{g}^2)\, y'^2 + \mathrm{a}^2 + \mathrm{b}^2.$$

100. For the symmetrical system of rays we have as before

$$\begin{aligned} \mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{f}_1,\ \mathrm{g}_1,\ \mathrm{h}_1 &= \ 0,\ \sqrt{3},\ -1,\ 0\ ,\ 1,\ \sqrt{3}, \\ \mathrm{a}_2,\ \mathrm{b}_2,\ \mathrm{c}_2,\ \mathrm{f}_2,\ \mathrm{g}_2,\ \mathrm{h}_2 &= \ 3,\ \sqrt{3},\ 2,\ \sqrt{3},\ 1,\ -2\sqrt{3}, \\ \mathrm{a}_3,\ \mathrm{b}_3,\ \mathrm{c}_3,\ \mathrm{f}_3,\ \mathrm{g}_3,\ \mathrm{h}_3 &= -3,\ \sqrt{3},\ 2,\ \sqrt{3},\ 1,\ -2\sqrt{3}, \end{aligned}$$

and thence

$$\begin{aligned} x_1' : y_1' : 1 &= \ 1 : \qquad \sqrt{3} \cos b : \sin b\ , \\ x_2' : y_2' : 1 &= -2 : -3 \sin b + \sqrt{3} \cos b : \sin b + \sqrt{3} \cos b, \\ x_3' : y_3' : 1 &= -2 : \ 3 \sin b + \sqrt{3} \cos b : \sin b - \sqrt{3} \cos b, \end{aligned}$$

or, what is the same thing,

$$x_1' = \operatorname{cosec} b \quad , \quad y_1' = \sqrt{3}\cot b,$$

$$x_2' = \frac{-1}{\sin b + \sqrt{3}\cos b}, \quad y_2' = \frac{\sqrt{3}(\cos b - \sqrt{3}\sin b)}{\sin b + \sqrt{3}\cos b},$$

$$x_3' = \frac{-2}{\sin b - \sqrt{3}\cos b}, \quad y_3' = \frac{\sqrt{3}(\cos b + \sqrt{3}\sin b)}{\sin b - \sqrt{3}\cos b};$$

or as these may also be written

$$x_1' = \operatorname{cosec} b \quad , \quad y_1' = \sqrt{3}\cot b \quad ,$$

$$x_2' = -\operatorname{cosec}(b + 60^\circ), \quad y_2' = \sqrt{3}\cot(b + 60^\circ),$$

$$x_3' = -\operatorname{cosec}(b - 60^\circ), \quad y_3' = \sqrt{3}\cot(b - 60^\circ),$$

so that for each of these sets we have

$$x'^2 - \tfrac{1}{3}y'^2 = 1.$$

(The curve is in fact a section of the hyperboloid of revolution, $x^2 + y^2 - \frac{1}{3}z^2 = 1$, which passes through the three rays.)

101. As regards the equation of the orbit I will first consider the particular cases $b = 90^\circ$, $b = 0^\circ$, which should agree with the orbits for $c = 90^\circ$ in the planograms 1 and 2 respectively.

For $b = 90^\circ$ we have $x' = x$, $y' = y$ and

$$x_1' = 1, \quad y_1' = 0,$$

$$x_2' = -2, \quad y_2' = -3,$$

$$x_3' = -2, \quad y_3' = 3,$$

and the orbit is at once found to be

$$r = \tfrac{1}{3}(1 - \sqrt{13})(x' - 1),$$

the eccentricity (regarded as positive) being thus $\frac{1}{3}(\sqrt{13} - 1)$, $= \cdot 7685$ as before. For $b = 0^\circ$ there is a discontinuity, and I write successively $b = +\epsilon$, and $b = -\epsilon$. For $b = +\epsilon$ we have $x' = -y$, $y' = x$, and

$$x_1' = \infty \quad , \quad y_1' = \infty\sqrt{3},$$

$$x_2' = -\frac{2}{\sqrt{3}}, \quad y_2' = -1,$$

$$x_3' = \frac{2}{\sqrt{3}}, \quad y_3' = -1,$$

and the orbit is found to be

$$r = \tfrac{2}{3}x' + \frac{4}{3\sqrt{3}}y' + \frac{\sqrt{7}}{\sqrt{3}} = \cdot 666\,x + \cdot 770\,y + 1\cdot 527\,;$$

and similarly for $b = -\epsilon$ the equation is

$$r = -\tfrac{2}{3}x' + \frac{4}{\sqrt{3}}y' + \frac{\sqrt{7}}{\sqrt{3}} = -\cdot 666\,x + \cdot 770\,y + 1\cdot 527\,;$$

hence the eccentricity is

$$e = \sqrt{\tfrac{28}{27}}, \quad = 1\cdot 018, \text{ as before.}$$

102. Considering now the general case where b has any value whatever, the equation of the orbit is

$$\begin{vmatrix} r & , & x', & y' & , & 1 \\ r_1 \sin b & , & 1, & \sqrt{3}\cos b & , & \sin b \\ r_2(\sin b + \sqrt{3}\cos b), & & -2, & -3\sin b + \sqrt{3}\cos b, & & \sin b + \sqrt{3}\cos b \\ r_3(\sin b - \sqrt{3}\cos b), & & -2, & 3\sin b + \sqrt{3}\cos b, & & \sin b - \sqrt{3}\cos b \end{vmatrix} = 0,$$

($x' = x\sin b - y\cos b$, $y' = x\cos b + y\sin b$, as before).

The coefficient of r is readily found to be $-6\sqrt{3}(\sin^2 b + \cos^2 b)$, $= -6\sqrt{3}$; hence completing the development, dividing by $6\sqrt{3}$, and transposing, the equation of the orbit is

$$\begin{aligned} r = \ & \tfrac{1}{6}\,[2r_1 \sin b \quad - r_2(\sin b + \sqrt{3}\cos b) - r_3(\sin b - \sqrt{3}\cos b)]\,x' \\ & + \frac{1}{6\sqrt{3}}\big[4r_1 \sin b\cos b + r_2\big(-2\sin b\cos b + \sqrt{3}(\cos^2 b - \sin^2 b)\big) \\ & \qquad\qquad + r_3\big(-2\sin b\cos b - \sqrt{3}(\cos^2 b - \sin^2 b)\big)\big]\,y' \\ & + \tfrac{1}{6}\,[4r_1 \sin^2 b \quad + r_2(\sin^2 b + 3\cos^2 b + 2\sqrt{3}\sin b\cos b) \\ & \qquad\qquad + r_3(\ \sin^2 b + 3\cos^2 b - 2\sqrt{3}\sin b\cos b)], \end{aligned}$$

where

$$r_1 = \frac{\sqrt{\sin^2 b + 4\cos^2 b}}{\sin b},$$

$$r_2 = \frac{\sqrt{13\sin^2 b + 7\cos^2 b - 6\sqrt{3}\sin b\cos b}}{\sin b + \sqrt{3}\cos b},$$

$$r_3 = \frac{\sqrt{13\sin^2 b + 7\cos^2 b + 6\sqrt{3}\sin b\cos b}}{\sin b - \sqrt{3}\cos b};$$

in which expressions the signs of the radicals must be such that r_1, r_2, r_3 shall be positive. Hence writing $\tan b = \eta$, ($\sec b = \sqrt{1+\eta^2}$, which determines the sign of $\sqrt{1+\eta^2}$), also

$$R_1 = \sqrt{\eta^2 + 4}, \quad R_2 = \sqrt{13\eta^2 - 6\sqrt{3}\eta + 7}, \quad R_3 = \sqrt{13\eta^2 + 6\sqrt{3}\eta + 7},$$

and therefore

$$\eta r_1 = R_1, \qquad (\eta + \sqrt{3})\, r_2 = R_2, \qquad (\eta - \sqrt{3})\, r_3 = R_3,$$

which last equations determine the signs of R_1, R_2, R_3 respectively, the equation of the orbit is

$$\begin{aligned} r = \; & \frac{1}{6\sqrt{1+\eta^2}} \quad (2R_1 - R_2 - R_3)\, x' \\ & + \frac{1}{6(1+\eta^2)\sqrt{3}} \left(4R_1 + R_2(1-\eta\sqrt{3}) + R_3(1+\eta\sqrt{3})\right) y' \\ & + \frac{1}{6(1+\eta^2)} \quad \left(4R_1\eta + R_2(\eta+\sqrt{3}) + R_3(\eta-\sqrt{3})\right). \end{aligned}$$

Thus if $b = +\epsilon$, then also $\eta = +\epsilon$,

$$\sqrt{1+\eta^2} = 1, \quad R_1 = 2, \quad R_2 = \sqrt{7}, \quad R_3 = -\sqrt{7},$$

and the equation is

$$r = \tfrac{1}{6}\, 4x' + \frac{1}{6\sqrt{3}}\, 8y' + \tfrac{1}{6}\, 2\sqrt{7}\sqrt{3}, \; = \tfrac{2}{3}\, x' + \frac{4}{3\sqrt{3}}\, y' + \frac{\sqrt{7}}{\sqrt{3}}, \; = \cdot 666\, x' + \cdot 770\, y' + 1\cdot 527,$$

as before; and similarly if $b = 90°$.

And moreover, if $b = 30°$, then

$$\eta = \frac{1}{\sqrt{3}}, \quad R_1 = \sqrt{\tfrac{13}{3}}, \quad R_2 = \frac{4}{\sqrt{3}}, \quad R_3 = -\frac{2\sqrt{13}}{3},$$

whence the equation of the orbit is

$$\begin{aligned} r &= \tfrac{1}{3}(\sqrt{13} - 1)\, x' + 0\, y' + \tfrac{1}{3}(\sqrt{13} + 2), \\ &= \quad \cdot 868\, x' + 0\, y' + \; 1\cdot 868. \end{aligned}$$

103. The equation of the orbit should be tabulated from $b = 0$ to $b = 30°$, the equations for the remainder of the circumference will be then found by successive repetition of this interval in direct and reverse order, with however a change of sign, in the manner about to be explained,

$$\begin{aligned} & b = \epsilon, && r = +\cdot 666\, x' + \cdot 770\, y' + 1\cdot 527, \\ & b = 30°, && r = +\cdot 868\, x' + \;\; 0 \;\; y' + 1\cdot 868, \\ & b = 60° - \epsilon, && r = +\cdot 666\, x' - \cdot 770\, y' + 1\cdot 527, \\ & b = 60° + \epsilon, && r = -\cdot 666\, x' + \cdot 770\, y' + 1\cdot 527, \\ & b = 90°, && r = -\cdot 868\, x' + \;\; 0 \;\; y' + 1\cdot 868, \\ & b = 120° - \epsilon, && r = -\cdot 666\, x' - \cdot 770\, y' + 1\cdot 527, \end{aligned}$$

$30° + \beta$ same as $30° - \beta$, reversing sign of the y' coefficient.

$90° + \beta$ same as $90° - \beta$, reversing sign of the y' coefficient, and whole interval 60° to 120° same as interval 0° to 60°, except that the signs of the x' coefficient are reversed, and the remaining two intervals, 120° to 240° and 240° to 360°, are merely repetitions of the interval 0° to 120°.

As regards the interval 0° to 30° the only intermediate value that I have calculated is $b=15°$, viz., we then have

$$b=15°,\quad r=\cdot 811\,x'+\cdot 403\,y'+1\cdot 787.$$

Calculating for the foregoing values $b=0°$, $b=15°$, $b=30°$, the values of e, ϖ, a, these are found to be

$b=\ 0°,$	$e=1\cdot 018$	$\varpi=220°\ \ 6'$	$a=41\cdot 24$
$b=15°,$	$e=\ \cdot 906$	$\varpi=206°\ 27'$	$a=10\cdot 008$
$b=30°,$	$e=\ \cdot 868$	$\varpi=180°$	$a=\ 7\cdot 604$

Article Nos. 104 to 113. *Planogram No. 5. The Orbit-pole on a Separator.*

104. If the orbit-plane rotate round a line parallel to one of the rays, the orbit-pole will describe a separator circle, and conversely. I consider the general case of a ray the six coordinates of which are (a, b, c, f, g, h), and for which the intersections with the orbit-plane are given by

$$x' : y' : 1=(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between\alpha',\ \beta',\ \gamma') : -(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between\alpha,\ \beta,\ \gamma) : (\mathrm{f},\ \mathrm{g},\ \mathrm{h}\between\alpha'',\ \beta'',\ \gamma'').$$

The axis of x' is parallel to the ray

$$\frac{x-A}{\mathrm{f}}=\frac{y-B}{\mathrm{g}}=\frac{z-C}{\mathrm{h}},$$

that is, we have

$$\alpha : \beta : \gamma=\mathrm{f} : \mathrm{g} : \mathrm{h},$$

whence, putting for shortness

$$\Omega=\sqrt{\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}\ \text{ and }\ \Pi=\sqrt{\mathrm{f}^2+\mathrm{g}^2},$$

we have

$$\alpha=\frac{\mathrm{f}}{\Omega}=\cos N\cos G,\quad \beta=\frac{\mathrm{g}}{\Omega}=\cos N\sin G,\quad \gamma=\frac{\mathrm{h}}{\Omega}=-\sin N,$$

and thence

$$\tan G=\frac{\mathrm{g}}{\mathrm{f}},\quad \sin G=\frac{\mathrm{g}}{\Pi},\quad \cos G=\frac{\mathrm{f}}{\Pi},\quad \cos N=\frac{\Pi}{\Omega},$$

and we thus obtain the values of α', β', γ'; α'', β'', γ'' in terms of f, g, h and the variable angle H, viz., these are

$$\alpha'=-\frac{\mathrm{g}\cos H}{\Pi}+\frac{\mathrm{hf}\sin H}{\Pi\Omega},\qquad \alpha''=\frac{-\mathrm{g}\sin H}{\Pi}-\frac{\mathrm{hf}\cos H}{\Pi\Omega},$$

$$\beta'=\frac{\mathrm{f}\cos H}{\Pi}+\frac{\mathrm{gh}\sin H}{\Pi\Omega},\qquad \beta''=\frac{\mathrm{f}\sin H}{\Pi}-\frac{\mathrm{hg}\cos H}{\Pi\Omega},$$

$$\gamma'=-\frac{\Pi^2\sin H}{\Pi\Omega},\qquad \gamma''=\frac{\Pi^2\cos H}{\Pi\Omega},$$

where H is the angular distance of the orbit-pole, along the separator, from the point A. The foregoing values give

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between\alpha\ ,\ \beta\ ,\ \gamma\ )=0,$$

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between\alpha'\ ,\ \beta'\ ,\ \gamma'\ )=-\frac{1}{\Pi}\{(\mathrm{ag}-\mathrm{bf})\cos H+\mathrm{c}\Omega\sin H\},$$

$$(\mathrm{f},\ \mathrm{g},\ \mathrm{h}\between\alpha''\ \ \beta'',\ \gamma'')=0,$$

so that the coordinates x', y' of the intersection with the ray are given in the form

$$x' : y' : 1 = M : 0 : 0,$$

that is

$$x'=\frac{M}{0}=\infty,\quad y'=\frac{0}{0},$$

but the value of y' is determinate, viz., this is equal to the perpendicular distance of the ray from the point S.

105. In particular when the rays are the special symmetrical system before considered, then if (a, b, c, f, g, h) refer to the ray 1, we have $\mathrm{f}=0$, $\mathrm{g}=1$, $\mathrm{h}=\sqrt{3}$, $\Pi=1$, $\Omega=2$, and thence

$$\begin{array}{llll}
\alpha\ ,\ \beta\ ,\ \gamma\ = & 0\ , & \frac{1}{2}\ , & \frac{1}{2}\sqrt{3}\ , \\
\alpha',\ \beta',\ \gamma' = & -\cos H, & \frac{1}{2}\sqrt{3}\sin H, & -\frac{1}{2}\sin H, \\
\alpha'',\ \beta'',\ \gamma'' = & -\sin H, & \frac{1}{2}\sqrt{3}\cos H, & \frac{1}{2}\cos H.
\end{array}$$

For the intersection with the ray 1 we have

$$x_1'=\pm\infty,\quad y_1'=1,$$

and for the intersections with the other two lines

$$\begin{array}{l}
x_2' : y_2' : 1 = \\
\quad(\ 3,\ \sqrt{3},\ \ 2\ \ )(-\cos H,\ \ \frac{1}{2}\sqrt{3}\sin H,\ \ -\frac{1}{2}\sin H) = -3\cos H+\frac{1}{2}\sin H \\
:-(\ 3,\ \sqrt{3},\ \ 2\ \ )(\ \ 0,\ 1,\ \ \frac{1}{2}\sqrt{3}) \qquad\qquad\qquad\qquad : \frac{3}{2}\sqrt{3} \\
:\ \ (\sqrt{3},\ \ 1,\ -2\sqrt{3})(-\sin H,\ -\frac{1}{2}\sqrt{3}\cos H,\ \ \frac{1}{2}\cos H) \ \ : -\sqrt{3}\sin H-\frac{3}{2}\sqrt{3}\cos H,
\end{array}$$

and

$$\begin{array}{l}
x_3' : y_3' : 1 = \\
\quad(-\ 3,\ \sqrt{3},\ \ 2\ \ )(-\cos H,\ \ \frac{1}{2}\sqrt{3}\sin H,\ \ -\frac{1}{2}\sin H) = \ \ 3\cos H+\frac{1}{2}\sin H \\
:-(-\ 3,\ \sqrt{3},\ \ 2\ \ )(\ \ 0,\ \frac{1}{2},\ \ \frac{1}{2}\sqrt{3}) \qquad\qquad\qquad\qquad : \frac{3}{2}\sqrt{3} \\
:\ \ (-\sqrt{3},\ \ 1,\ -2\sqrt{3})(-\sin H,\ -\frac{1}{2}\sqrt{3}\cos H,\ \ \frac{1}{2}\cos H) \ \ : \ \sqrt{3}\sin H-\frac{3}{2}\sqrt{3}\cos H,
\end{array}$$

that is, we have

$$x_2' = \frac{1}{\sqrt{3}}\,\frac{6\cos H - \sin H}{3\cos H + 2\sin H}, \qquad x_3' = -\frac{1}{\sqrt{3}}\,\frac{6\cos H + \sin H}{3\cos H - 2\sin H},$$

$$y_2' = \frac{3}{3\cos H + 2\sin H}, \qquad y_3' = \frac{3}{3\cos H - 2\sin H}.$$

106. Writing herein

$$\cos\omega = \frac{3}{\sqrt{13}}, \quad \sin\omega = \frac{2}{\sqrt{13}}, \quad \tan\omega = \tfrac{2}{3}, \quad \omega = 33^\circ\,41'$$

the formulæ are readily converted into

$$x_2' = \frac{1}{13\sqrt{3}}\{16 - 15\tan(H-\omega)\}, \qquad x_3' = \frac{1}{13\sqrt{3}}\{-16 - 15\tan(H+\omega)\},$$

$$y_2' = \frac{3}{\sqrt{13}}\sec(H-\omega), \qquad y_3' = \frac{3}{\sqrt{13}}\sec(H+\omega),$$

where, in regard to this angle ω, it is to be observed that it represents the angular distance from the ecliptic along the separator to a point B, or what is the same thing, the complement of the angular distance on the separator, of the points A and B. We have, in fact, a right-angled spherical triangle ZAB, $\angle Z = 60^\circ$, $\angle A = 90^\circ$, $ZA = 60^\circ$ whence $\sin 60^\circ = \tan AB \cot 60^\circ$, that is, $\tan AB = \sin 60^\circ \tan 60^\circ = \frac{3}{2}$, or $AB = 90^\circ - \omega$.

Hence, $H = \pm 90^\circ$, the orbit-pole is on the ecliptic, $H = \pm(90^\circ - \omega)$, it is at a point B (the intersection of the separator by one of the other two separators), and $H = 0$, it is at the point A on the separator.

The foregoing values of (x_2', y_2') satisfy the equation

$$25y^2 = 39x^2 - 32x\sqrt{3} + 37,$$

and similarly the values of (x_3', y_3') satisfy

$$25y^2 = 39x^2 + 32x\sqrt{3} + 37,$$

results which would be useful for the delineation of the planogram.

107. As regards the equation of the orbit we have $x_1' = \pm\infty$, and consequently $x_1' = \pm r_1 = \theta r_1$ if for convenience θ be written to stand for ± 1. The equation of the orbit then is

$$0 = \begin{vmatrix} r, & x', & y', & 1 \\ 1, & \theta, & 0, & 0 \\ r_2(3\cos H + 2\sin H), & \frac{1}{\sqrt{3}}(6\cos H - \sin H), & 3, & 3\cos H + 2\sin H \\ r_3(3\cos H - 2\sin H), & \frac{1}{\sqrt{3}}(-6\cos H - \sin H), & 3, & 3\cos H - 2\sin H \end{vmatrix},$$

that is

$$-(r\theta - x')\,12\sin H =$$

$$y'\left\{\frac{1}{\sqrt{3}}(36\cos^2 H + 4\sin^2 H) - \theta r_2(9\cos^2 H - 4\sin^2 H) + \theta r_3(9\sin^2 H - 4\cos^2 H)\right\}$$

$$-12\sqrt{3}\cos H + 3\theta(3\cos H + 2\sin H)\,r_2 - 3\theta(3\cos H - 2\sin H)\,r_3,$$

where

$$r_2 = \frac{\sqrt{21\cos^2 H - 4\cos H\sin H + \frac{28}{3}\sin^2 H}}{3\cos H + 2\sin H},$$

$$r_3 = \frac{\sqrt{21\cos^2 H + 4\cos H\sin H + \frac{28}{3}\sin^2 H}}{3\cos H - 2\sin H}.$$

Hence, writing $\tan H = \lambda$, and therefore $\sec H = \sqrt{1+\lambda^2}$, which determines the sign of $\sqrt{1+\lambda^2}$, and moreover

$$R_2 = \sqrt{21 - 4\lambda + \tfrac{28}{3}\lambda^2}, \quad R_3 = \sqrt{21 + 4\lambda + \tfrac{28}{3}\lambda^2},$$

and thence also

$$(3+2\lambda)\,r_2 = R_2, \quad (3-2\lambda)\,r_3 = R_3,$$

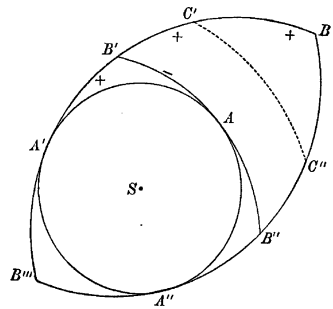
which last equations, since r_2, r_3 must be positive, determine the signs of the radicals R_2, R_3; the equation of the orbit is

$$r = \theta x' + \frac{y'}{12\lambda\sqrt{1+\lambda^2}}\left\{\frac{\theta}{\sqrt{3}}(36+4\lambda^2) - (3-2\lambda)\,R_2 + (3+2\lambda)\,R_3\right\} + \frac{-4\theta\sqrt{3} + R_2 - R_3}{4\lambda},$$

where θ it will be recollected denotes $+1$ or -1 at pleasure.

108. I remark that $\theta = +1$ and $\theta = -1$ may be considered as belonging to positions of the orbit-pole indefinitely near the separator on the opposite sides thereof

Fig. 10.



respectively; the annexed figure represents a portion of the blank spherogram, and the two sides of the half-separator $A'C'$ will be traversed by the orbit-pole, if H

extend from $0°$ to $90° - \omega$ ($= 56° 19'$, value at B') and thence to $90°$, $\theta = +1$ belonging to the side marked + in the figure, and $\theta = -1$ to the opposite side. But the same result may be stated, more conveniently, in reference to the blank spherogram, as follows:

$H = 0°$ to $H = 56° 19'$, $\theta = +1$ belongs to the outside of AB', viz. to positions within the region of convex orbits,

$\theta = -1$, to inside of AB',

$H = 56° 19'$ to $H = 90°$, $\theta = +1$ belongs to inside of $B'C'$,

$H = 90°$ to $H = 123° 41'$, $\theta = +1$ belongs to inside of $C'B$,

the last-mentioned values being identical with those for $H = 90°$ to $H = 56° 19'$, $\theta = -1$: viz. the formula for $H = 90° + K$, $\theta = +1$ is equivalent to that for $H = 90° - K$, $\theta = -1$.

109. I consider some particular cases.

Orbit-pole at A: here $H = 0$ and therefore $\lambda = 0$, $R_2 = R_3 = \sqrt{21}$; the orbit is $r = \theta x' + \frac{\theta\sqrt{3}}{\lambda}(y' - 1)$, viz. it is the right line $y' - 1 = 0$.

Orbit-pole in the neighbourhood of B. Suppose first $H = 90° - \omega - \epsilon$, $\lambda = \cot\omega - \epsilon \operatorname{cosec}^2\omega = \frac{3}{2} - \frac{13}{4}\epsilon$, $3 - 2\lambda$, $= \frac{13}{2}\epsilon$, is positive, and therefore R_3 is positive, and we have $R_2 = 6$, $R_3 = 4\sqrt{3}$; whence the equation is

$$r = \theta x' + \frac{y'}{9\sqrt{13}}(15\theta + 24) + 1 - \frac{2}{\sqrt{3}}(\theta + 1),$$

viz. $\theta = -1$, this is

$$r = +x' + \sqrt{\tfrac{3}{13}}\, y' + 1,$$

and $\theta = +1$, it is

$$r = x' + \sqrt{\tfrac{13}{3}}\, y' + 1 - \frac{4}{\sqrt{3}},$$

and so secondly, if $H = 90° - \omega + \epsilon$, $\lambda = \frac{3}{2} + \frac{13}{4}\epsilon$, $3 - 2\lambda$, $= -\frac{13}{2}\epsilon$, is negative, or R_3 is also negative, viz. $R_2 = 6$, $R_3 = -4\sqrt{3}$, and the equation is

$$r = \theta x' - \frac{y'}{9\sqrt{13}}(15\theta - 24) + 1 - \frac{2}{\sqrt{3}}(\theta - 1),$$

viz. $\theta = +1$, this is

$$r = x' - \sqrt{\tfrac{3}{13}}\, y' + 1,$$

and $\theta = -1$, it is

$$r = -x' - \sqrt{\tfrac{13}{3}}\, y' + 1 + \frac{4}{\sqrt{3}}.$$

At the point B there are thus four orbits: viz. $H = -90° - \omega - \epsilon$, $\theta = +1$, and $H = 90° - \omega + \epsilon$, $\theta = -1$, these are orbits wherein the eccentricity is $= \sqrt{\frac{16}{3}}$, $= 2{\cdot}309$, agreeing with that found for the point B in planogram No. 1, or say for an orbit-pole near B in the direction of the meridian; whereas for $H = 90° - \omega - \epsilon$, $\theta = -1$ and $H = 90° - \omega + \epsilon$, $\theta = +1$ the eccentricity is $\sqrt{\frac{16}{13}} = 1{\cdot}101$.

Suppose again that the orbit-pole is on the ecliptic, or say $H = 90° - \epsilon$, $\lambda = +\infty$, $R_2 = 2\sqrt{\tfrac{7}{3}}\lambda$, $R_3 = -2\sqrt{\tfrac{7}{3}}\lambda$, and $\sqrt{1+\lambda^2} = \lambda$, and the equation is

$$r = \theta\left(x' + \frac{y'}{3\sqrt{3}}\right) + \sqrt{\tfrac{7}{3}},$$

and similarly for $H = 90° + \epsilon$, $\lambda = -\infty$, $R_2 = 2\sqrt{\tfrac{7}{3}}\lambda$, $R_3 = -2\sqrt{\tfrac{7}{3}}\lambda$, $\sqrt{1+\lambda^2} = \lambda$, and the equation still is

$$r = \theta\left(x' + \frac{y'}{3\sqrt{3}}\right) + \sqrt{\tfrac{7}{3}},$$

viz. θ retaining the same sign, there is no discontinuity in the passage through 90°.

The eccentricity, whether $\theta = +1$ or $= -1$, is $\sqrt{\tfrac{28}{27}}$, $= 1{\cdot}018$, agreeing with Planogram No. 2.

110. For the more complete discussion of the eccentricity, we have

$$e^2 = 1 + \frac{1}{144\lambda^2(1+\lambda^2)}\left(\frac{4\theta}{\sqrt{3}}(9+\lambda^2) - (3-2\lambda)R_2 + (3+2\lambda)R_3\right)^2.$$

The eccentricity cannot be less than 1, which is evidently right, for the point 3 being at infinity, the orbit cannot be an ellipse. We may have $e = 1$ (or the orbit a parabola), viz. this will be the case if

$$\frac{4\theta}{\sqrt{3}}(9+\lambda^2) - (3-2\lambda)R_2 + (3+2\lambda)R_3 = 0.$$

Proceeding to rationalize this equation, we have first

$$(3-2\lambda)^2 R_2^2 + (3+2\lambda)^2 R_3^2 - \tfrac{16}{3}(9+\lambda^2)^2 = 2(9-4\lambda^2)R_2R_3,$$

viz. substituting for R_2, R_3 their values $\sqrt{21 - 4\lambda + \tfrac{28}{3}\lambda^2}$ and $\sqrt{21 + 4\lambda + \tfrac{28}{3}\lambda^2}$, this is found to be

$$2(9-4\lambda^2)\sqrt{\left(21 + \frac{28}{3\lambda^2}\right)^2 - 16\lambda^2} = -54 + 336\lambda^2 + \tfrac{208}{3}\lambda^4;$$

or, what is the same thing,

$$(9-4\lambda^2)\sqrt{3969 + 3384\lambda^2 + 784\lambda^4} = -81 + 504\lambda^2 + 104\lambda^4$$

whence, squaring and reducing, we have

$$432(4\lambda^8 - 248\lambda^6 - 819\lambda^4 + 162\lambda^2 + 729) = 0;$$

or, what is the same thing,

$$432(\lambda^2+1)(4\lambda^6 - 252\lambda^4 - 567\lambda^2 + 729) = 0,$$

or, finally, the condition for a parabola is

$$4\lambda^6 - 252\lambda^4 - 567\lambda^2 + 729 = 0.$$

111. I stop to remark that this equation may be obtained differently, as follows. Since the point 1 is at infinity on the axis of x, this line will be the axis of the parabola; or the equation of the parabola will be

$$-y^2 + 4ax + 4a^2 = 0,$$

and we have therefore

$$-y_2^2 + 4ax_2 + 4a^2 = 0,$$
$$-y_3^2 + 4ax_3 + 4a^2 = 0,$$

that is

$$1 : 4a : 4a^2 = x_2 - x_3 : y_2^2 - y_3^2 : -y_2^2 x_3 + y_3^2 x_2,$$

and therefore

$$(y_2^2 - y_3^2)^2 = -4(x_2 - x_3)(y_2^2 x_3 - y_3^2 x_2)$$

as the condition for a parabola.

But the values of x_2, y_2; x_3, y_3, *ante* No. 104, introducing λ in the place of H, are

$$x_2 = \frac{1}{\sqrt{3}}\frac{6-\lambda}{3+2\lambda}, \quad x_3 = \frac{1}{\sqrt{3}}\frac{6+\lambda}{3-2\lambda},$$

$$y_2 = \frac{3\sqrt{1+\lambda^2}}{3+2\lambda}, \quad y_3 = \frac{3\sqrt{1+\lambda^2}}{3-2\lambda},$$

and thence

$$x_2 - x_3 = \frac{4}{\sqrt{3}}\frac{9+\lambda^2}{9-4\lambda^2},$$

$$y_2^2 - y_3^2 = -\frac{216\lambda(1+\lambda^2)}{(9-4\lambda^2)^2},$$

$$y_2^2 x_3 - y_3^2 x_2 = -\frac{36}{\sqrt{3}}\frac{(1+\lambda^2)(9-\lambda^2)}{(9-4\lambda^2)^2},$$

and substituting these values and omitting a factor $\frac{1+\lambda^2}{(9-4\lambda^2)^2}$, the result is

$$\frac{243\lambda^2(1+\lambda^2)}{9-4\lambda^2} = (9+\lambda^2)(9-\lambda^2),$$

viz. this is

$$(4\lambda^2 - 9)(\lambda^4 - 81) - 243\lambda^2(\lambda^2 + 1) = 0,$$

that is

$$4\lambda^6 - 252\lambda^4 - 567\lambda^2 + 729 = 0,$$

as before.

112. The equation considered as a cubic equation in λ^2 has its three roots real, but only two of them are positive; viz. there is a root not very different from 1, and which is easily approximated to by writing $\lambda^2 = 1 - x$, this gives

$$4x^3 + 240x^2 - 1068x + 86 = 0,$$

or nearly $x = \frac{86}{1088} = \cdot 08$; a second approximation gives $x = \cdot 0802$; or we have $\lambda^2 = \cdot 9198$, $\lambda = \cdot 9592$, whence $H = 43° 49'$. Substituting in the equation

$$\frac{4\theta}{\sqrt{3}}(9 + 4\lambda^2) - (3 - 2\lambda) R_2 + (3 + 2\lambda) R_3 = 0,$$

this will be satisfied by $\theta = -1$, viz. the parabola belongs (as it obviously should do) to a point of AB' within the triangle $BB'B''$.

To obtain the other positive root we may write the equation in the form

$$\lambda^2 = 63 + \frac{141 \cdot 75}{\lambda^2} - \frac{182 \cdot 25}{\lambda^4},$$

the approximate value $\lambda^2 = 63$, gives more nearly $\lambda^2 = 65$ and then

$$\lambda^2 = 63 + \frac{141 \cdot 75}{65} - \frac{128 \cdot 24}{4225}, = 65 \cdot 177,$$

whence $\lambda^2 = 8 \cdot 073$ or $H = 82° 56'$. Substituting in the equation

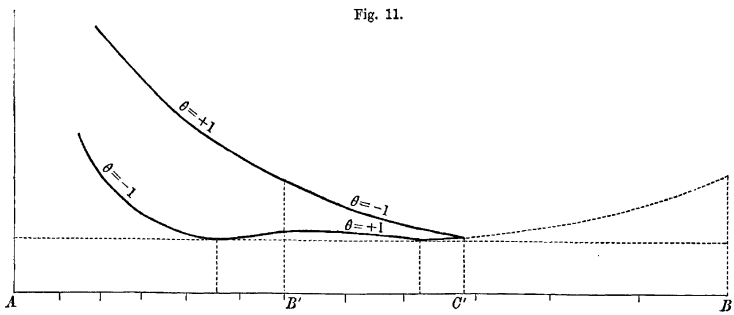
$$\frac{4\theta}{\sqrt{3}}(9 + 4\lambda^2) - (3 - 2\lambda) R_2 + (3 + 2\lambda) R_3 = 0,$$

we have $\theta = +1$, viz. this parabola belongs to a point of $B'C'$ within the triangle $BB'B''$.

The two values of e for $\theta = +1$ and $\theta = -1$, are each infinite for $\lambda = 0$, and they become equal for $\lambda = \infty$ (viz. when the orbit-pole is on the ecliptic), but not in any other case; in fact they can only do so for $9 + \lambda^2 = 0$, or else for $(3 - 2\lambda) R_2 = (3 + 2\lambda) R_3$, that is, $\lambda (288 + 128 \lambda^2) = 0$, viz., $\lambda (9 + 4\lambda^2) = 0$.

113. In further explanation I give a diagram of the eccentricity.

Fig. 11.



The base $AB'C'B$ is here the broken line $AB'C'B'$ of figure 10: the ordinates along the base AC' (= 90°) of the two continuous curves exhibit the values of e, as

given by $\theta = +1$ and $\theta = -1$ respectively; the dotted curve on the base $C'B\,(=C'B')$ is merely the upper curve on the base $C'B'$ transferred to the base $C'B$; and the curve composed of the lower curve on the base AC' and of the dotted curve gives by its ordinates the value of the eccentricity as the orbit-pole moves along $AB'B$ within the triangle $B'BB''$: the upper curve on the base AB' gives by its ordinates the value of the eccentricity as the orbit-pole moves along AB' on the other side thereof, that is, within the convex region.

The base of the diagram is graduated not for the value of H, but for that of the angular distance (or distance in longitude) of the orbit-pole from the point A (or A'); viz. this is the angle opposite H in a right-angled spherical triangle, the sides and hypothenuse of which are 60°, H, c; writing β for the angle in question we have

$$\cos c = \tfrac{1}{2}\cos H, \quad \tan\beta = \frac{2}{\sqrt{3}}\tan H\left(=\frac{2\lambda}{\sqrt{3}}\right),$$

and any position of the orbit-pole on the separator may be conveniently laid down by means of this angle β. The values of β corresponding to the before-mentioned values $\lambda = \cdot 9592$ and $\lambda = 8\cdot 073$ are $\beta = 47°\ 54'$ and $\beta = 83°\ 53'$ respectively.

Article Nos. 114 and 115. *The Spherogram and Isoparametric Lines—General Considerations.*

114. We first construct a blank spherogram, as already explained (and see also Plates IV. and V.), viz., we draw on the stereographic projection a hemisphere—say the northern hemisphere: the meridians being radii and the parallels of colatitude circles with the pole as centre; the parallel of 60° is the regulator circle, and the separators are great circles touching this at the points A, A, A, in longitudes 30°, 150°, 270° respectively; the separators intersect in the points B, B, B, in the northern hemisphere, and they are produced to meet again in the points B, B, B, of the southern hemisphere; but instead of taking the whole northern hemisphere, we omit portions thereof, and take in the opposite portions of the southern hemisphere; the spherogram being thus bounded by portions of the separator circles, and consisting of the inner spherical triangle B, B, B, and three surrounding triangles B, B, B. The inner triangle contains the regulator-circle, touching its sides at the points A, A, A respectively, and dividing it into an inner circular region and three surrounding regions A, B, A; these last are the *loci in quibus* of the orbit-poles which correspond to convex orbits; and to mark them off from the other regions, it is proper to shade them in the spherogram. Excluding them from consideration, we have the inner circular region and the outer triangular regions separated off from each other by the shaded regions, except at the points A, where these are thinned away to nothing. The points A are positions of the orbit-pole for which the orbit is indeterminate; and consequently any parameter belonging to the orbit is also indeterminate. Hence the isoparametric line for any given value of the parameter will always pass through the points A; that is, all the isoparametric lines will pass through these points, which are thus points of connexion

between the inner circular region and the three outer regions, but it must be recollected that for certain given values of the parameter, the points A may be isolated points on the isoparametric line.

115. It is sometimes necessary (more particularly as regards the Time-spherogram and isochronic lines) to distinguish from each other the several points A and B; and for this purpose I consider the several points, as situated in the spherogram, to be accented in the following manner:

$$\begin{array}{ccccc} B^{\text{IV}} & & B' & & B \\ & A' & & A & \\ B''' & & A'' & & B'' \\ & & B^{\text{V}} & & \end{array}$$

so that the inner triangle is $B'B''B'''$ and the outer triangles are $BB'B''$, $B'B^{\text{IV}}B'''$ and $B'''B^{\text{V}}B''$ respectively; this distinction has been already partially made in Fig. 10.

Article Nos. 116 to 122. *The e-spherogram and Iseccentric Lines. See Plate IV.*

116. Constructing a blank spherogram as above, we may from the tables for planograms Nos. 1 and 2 lay down numerically the values of the eccentricity at the several points of each meridian for the longitudes 0°, 30°,..330°, viz.

LONGITUDES 0°, 60°, 120°, 180°, 240°, 300°.	Planogram No. 2 shows that e increases from 0 at the centre to ∞ at 60°, then, 60° to 63° 26′ (shaded region), it diminishes from ∞ to 4·912; on passing 63° 26′ it changes abruptly to 1·853; thence diminishes to a minimum = ·628 at 59°, and again increases to 1·018 at 90°.
LONGITUDES 10°, 210°, 330°.	Planogram No. 1, part 1, shows that e increases from 0 at the centre to ∞ at 60°, then, 60° to 73° 54′ (shaded region), it diminishes from ∞ to 2·309, this last value being at a point B, the termination of the spherogram.
LONGITUDES 30°, 150°, 210°.	Planogram No. 1, part 2, and for values over 90°, part 1, shows that e increases from 0 at the centre to ·264 at 60° (point A), ·869 at 90°, and 2·309 at 100° 6′, point B.

It will be recollected that, although e has the same value, 2·309 at the two opposite points B, yet there is an abrupt change of orbit, indicated by the change of sign of A $(= \pm e)$.

117. Planogram No. 3 shows the directions at the points A of the several iseccentric lines. Planogram No. 4, if the calculations were completed, would give the

value of the eccentricity at the several points of the ecliptic, but besides the already-mentioned values 1·018 at 0°, 60°, &c., and ·868 at 30°, 90°, &c., the only value calculated is ·906 at 15°, 45°, &c. It thus appears that the eccentricity $=1·018$ for longitude 0° diminishes through ·906 at 15° to ·868 at 30°, and then again increases through ·906 at 45° to 1·018 at 60°, and so on through successive intervals of 60°.

118. Planogram No. 5, if the calculations were completed, would give the value of e for the arc AB within the shaded region (but no values have been found except those given by Planograms 1 and 2, viz. $e=\infty$ at A, $=4·912$ at longitude 30° from A, and $=2·309$ at B); and it would also give the value of e for the whole bounding arc ABB within the exterior triangular region. We have $e=\infty$ at A, $=1·853$ at longitude 30° from A, $=1$ at distance $H=43°\ 49'$ from A, $=1·101$ at B, and then proceeding along the arc BB, $=1$ at distance $H=82°\ 56'$ from A, $=1·018$ on the ecliptic, and, finally, $=2·309$ at B. The two values $e=1$ are very important, as will presently appear, with regard to the parabolic curve.

119. It is now easy to trace the form of the iseccentric lines.

$e=0$, the curve is a point at the centre, and for any value less than ·264 it is a trigonoid form surrounding the centre, the maxima radii being directed towards the points A. The points A belong as isolated points to all these curves.

$e=·264$, the curve is tricuspidal, having a cusp at each of the points A. The numerical values seem to show a singularly blunt form of cusp (the points A are, in fact, not ordinary cusps, but singular points of a higher order); but the data do not enable me to draw with certainty the precise forms of the arcs between the three cusps: the wavy form was drawn purposely, but there is no sufficient evidence for its correctness.

120. It is convenient to pass at once to the case $e=1$, or say the parabolic curve, locus of the orbit-pole when the orbit is a parabola. This is a three-looped curve cutting itself (having a node) at each of the points A; and it appears from planogram No. 5 that each loop touches at four points (two points, $H=43°\ 49'$, and two points, $H=82°\ 56'$), the sides of the bounding triangle BBB. The loop thus divides the triangle BBB into six regions, viz. one within the loop, two subjacent, two lateral, and one superjacent.

For any value between $e=·264$ and $e=1$, the curve is a three-looped curve intersecting itself at the points A, and such that the loops lie wholly within those of the parabolic curve, and the remaining portions between the parabolic and cuspidal curves.

121. For any value of $e>1$, we must imagine a three-looped curve intersecting itself at the points A, the loops respectively containing those of the parabolic curve, and the remaining portions within the regulator-circle lying between the regulator-circle and the parabolic curve; and we must then obliterate so much of each loop as lies in the shaded regions, or outside the spherogram; viz. instead of a continuous loop there will be thus a broken loop with detached portions thereof in the subjacent regions, the lateral regions, and the superjacent region respectively. More precisely

this is the form for any value of e from $e=1$ to $e=1{\cdot}101$, but for this last value the unobliterated portion for each lateral region evanesces; for any value of e between $e=1{\cdot}101$ and $e=2{\cdot}309$, the unobliterated portions lie wholly within the subjacent regions and the superjacent region; for $e=2{\cdot}309$ the portion within the superjacent region evanesces; and for any greater value of e the unobliterated portion lies wholly within the subjacent regions, the loop being thus a mere fragment.

122. The iseccentric curves within the shaded regions form a distinct system: such curves belong to the values $e=2{\cdot}309$ to $e=\infty$, and any one of them is a fragment of a three-looped curve intersecting itself at the points A, obtained by obliterating so much of the complete curve as lies outside the shaded regions. But it is perhaps better to disregard these curves altogether, thus in effect excluding the shaded regions from the spherogram.

Article Nos. 123 to 143. *The Time-spherogram and Isochronic Lines. See Plate V.*

123. We construct a blank spherogram, and lay down upon it the parabolic curve; we may then lay down (as will be explained) the numerical values, say of the times T_{13}, but in order to gain some idea of the form of the T_{13}-lines I will first consider the question in a more general manner.

124. When the orbit is a line, parabola, or hyperbola, we may distinguish it by the letters L, P, H accordingly; and by the numbers 1, 2, 3, written in the proper order, show the arrangement of the three points on the orbit; observe that if 1 be the middle point on the orbit, we may write indifferently 213 or 312, and so in other cases, the fixation of the middle number is alone material. When the orbit is a line the distances of the points are always finite; and if the orbit be, for example, $L\,123$, then T_{12} and T_{13} are each $=0$, but T_{13} is non-existent. For the parabola and hyperbola the distances are in general finite; but it is necessary to distinguish for the parabola, e.g. the case $P\,\dot{1}23$ where an extreme point, and for the hyperbola, e.g. the cases $H\,\dot{1}23$ and $H\,\dot{1}2\dot{3}$ where one or each of the extreme points, is at infinity. We have in these cases respectively

$P\,123$,	T_{12} finite,	T_{23} finite,	$T_{31}=\infty$
$P\,\dot{1}23$,	$T_{12}=\infty$,	T_{23} finite,	$T_{31}=\infty$

and it may be added, as regards $P\,1\dot{2}3$, that, by a continuous change of the parabolic orbit the point 1 may change over to infinity on the other half-branch of the parabola, or the arrangement become $P\,23\dot{1}$. And, moreover

$H\,123$,	T_{12} finite,	T_{23} finite,	T_{31} non-existent.
$H\,\dot{1}23$,	$T_{12}=\infty$,	T_{23} finite,	T_{31} non-existent.
$H\,\dot{1}2\dot{3}$,	$T_{12}=\infty$,	$T_{23}=\infty$,	T_{31} non-existent.

Thus the proper symbol $L\,123$, $P\,\dot{1}23$, &c. as the case may be, will always at once indicate as to each of the times T_{12}, T_{23}, T_{31}, whether this is $=0$, finite, infinite, or non-existent.

125. We may without difficulty attach to the several portions of the regulator, the separators and the parabolic curve, to each portion its proper symbol L, P, H and 123, $\dot{1}23$, &c. as the case may be.

First, as to the regulator, it is obvious that this is separated by the points A into the three portions $L\,213$, $L\,321$, $L\,132$, respectively. And inside the regulator, adjacent to these, we have portions of the parabolic curve $P\,213$, $P\,321$, $P\,132$, respectively.

Again, for one of the separators, say $B^{\text{IV}}B'AB''B^{\text{V}}$ (see here and in all that follows the notation-diagram, No. 115); since the point 2 is here at infinity this must be at every portion thereof either $H\,13\dot{2}$ or else $H\,31\dot{2}$. The point B^{IV} is $H\,\dot{1}3\dot{2}$ and the point B' is $H\,\dot{3}1\dot{2}$; consequently, as the orbit-pole passes along the separator from B^{IV} to B', the symbol is at first $H\,13\dot{2}$ and at last $H\,31\dot{2}$; the transition takes place at the point of contact of the parabolic curve which is indifferently $P\,13\dot{2}$ or $P\,\dot{2}13$. (In further explanation of the transition, consider the orbit-pole as passing from B^{IV} to B, not on the separator, but indefinitely near it; it can only do so by twice crossing the parabolic curve near the point of contact; the orbit is first $H\,13\dot{2}$, or say $H\,132$, then $P\,132$, then an ellipse, which when the orbit-pole again arrives at the parabolic curve changes into $P\,312$; and it finally becomes $H\,312$ or $H\,31\dot{2}$.)

126. Again, since, on the two separators through B^{IV}, in the portions adjacent to B^{IV}, the symbols are $H\,13\dot{2}$ and $H\,\dot{1}32$, it is clear that in the adjacent portion of the parabolic curve (terminated each way by a point of contact with these separators respectively) the symbol must be $P\,132$; at the point of contact with the first-mentioned separator $B^{\text{IV}}B'AB''B^{\text{V}}$, this becomes $P\,13\dot{2}$, $= P\,\dot{2}13$; and beyond the point of contact it becomes $P\,213$, continuing so until it arrives at the next point of contact with the separator $B'A'B''$: there is always in the symbol for the parabolic curve this change of form as we pass through a point of contact with a separator; and there is the same change, when *travelling along the loop* (that is without going inside the regulator) we pass through a point A. The foregoing considerations fully explain how the proper symbol is to be attached to each portion of the regulator, the separators, and the parabolic curve: to avoid confusion, I have abstained from attaching them in the Plate.

127. Imagining the symbols attached as above, it at once appears that, for the two portions $A'A$ and AA'' of the regulator curve, we have $T_{13} = 0$; while, for the arc $A''A'$ of the parabolic curve we have $T_{13} = \infty$. Moreover, T_{13} can only be infinite on one of the separators through B''' and on the parabolic curve; and the symbols show that the curve T_{13} is made up, in a peculiar discontinuous manner, of portions of these two separators and of the parabolic curve, as shown by the strongly marked line of the figure; we have thus the boundary of certain *lightly* shaded regions within which (as well as within the shaded regions) T_{13} is non-existent; excluding these, the remaining regions (instead of a trilateral symmetry) have a symmetry about the axis BB'''; there are still four regions which may be distinguished as the inner region, the axial outer region, and the lateral outer regions; or, more shortly, as the inner, axial, and lateral regions.

128. The times T_{12}, T_{23}, T_{31} are calculated, Planogram 1, part 1, for the meridian long. 90°, and ditto part 2 for the meridian long. 270°; and in Planogram 2 for the meridian long. 180°. As regards these last values, it is easy to see that, in order to pass to the meridian long. 0°, the numbers 2, 3 must be interchanged; that is, long. 0°, the T_{12}, T_{13}, T_{23} are respectively equal to the values, long. 180°, T_{13}, T_{12}, T_{23}. Moreover, the numbers 1, 2, 3 may be changed into 2, 3, 1, or into 3, 1, 2, provided the longitude is increased by 120° and 240° in the two cases respectively; that is,

$$\begin{aligned} T_{31} \text{ long. } \alpha &= T_{31} \text{ long. } \alpha \\ &= T_{12} \text{ long. } (\alpha + 120^\circ) \\ &= T_{23} \text{ long. } (\alpha + 240^\circ). \end{aligned}$$

129. By means of the foregoing two relations, T_{13} for the several longitudes 0°, 30°, 60°, ... 330°, is given as equal to the T_{12}, T_{23}, or T_{31}, for long. 90°, 270°, or 180°, that is, to the T_{12}, T_{23}, or T_{31}, of Planogram No. 1, part 1 or 2, or of Planogram No. 2. For example, T_{31} long. 240° = T_{12} long. 0° = T_{13} long. 180°, that is, it is equal to the T_{31} of Planogram No. 2. We thus find

Long.	T_{13} is =		
0°	T_{12} of Plan. No. 2		
30°		T_{23} of Plan. No. 1, pt. 2	
60°	T_{12} „		
90°			T_{12} of Plan. No. 1, pt. 1
120°	T_{23} „		
150°		T_{31} „	
180°	T_{31} „		
210°			T_{23} „
240°	T_{12} „		
270°		T_{12} „	
300°	T_{23} „		
330°			T_{31} „

and observing that for Planogram No. 1, part 1 or 2, we have $T_{12} = T_{31}$, it hence appears as above, that the meridian 30°—210° is an axis of symmetry of the spherogram. In what precedes it has been assumed that the colatitudes only extend from 0° to 90°, but in the spherogram they extend for the meridians 30°, 150°, 270°, to the colatitude 106° 6′, the values for the colatitudes above 90° are those for the omitted portions 90° to 73° 54′ of the opposite meridian.

N.B. A meridian extends from the pole *in one direction only*, unless the contrary is expressed or implied, as in speaking of a meridian 0°—180°.

130. I attend, in the first instance, to the axis of symmetry or meridian 30°—210°. Proceeding along the meridian long. 30° or towards the point A, the value of T_{13} decreases from 1 at the centre to a minimum $= \cdot 950$ at colatitude 11° (call this the point X), and it then increases to 1·983 at A, and thence to 58·62 at 90° and ∞ at the parabolic boundary of the axial region. In the opposite direction it increases from 1 at the centre to ∞ at the parabolic boundary of the inner region. The minimum value ·950 on the axis of symmetry indicates a node on the isochronic curve; that is, the point X is a node on the isochronic $T_{13} = \cdot 950$. This will consist of two branches, proceeding from A', A'', respectively, cutting the axis and each other at X, then again cutting at A, and thence passing on into the axial region, and respectively terminating on the separator boundary $B'AB''$ thereof.

131. This curve, which I call the nodal isochronic, divides the inner region into a loop, antiloop, and two side regions. On each of the meridians 0°, 60°, the value of T_{13} diminishes from 1 at the centre to a minimum which is less, and then increases to a maximum which is greater, than ·950; the value then diminishes to 0 on the regulator: on emergence of the meridian from the shaded into the axial region, the value is $= \cdot 909$, and it thence increases to ∞ at the parabolic boundary of the axial region; these data further determine the form of the nodal isochronic, viz., each of the two half meridians cuts the loop twice, and again cuts the curve in the axial region.

The nodal isochronic, at each of the points A', A'', continues its course into the lateral region, returning to the same point A or A', so as to form in each of the lateral regions a loop. Considering the loop as formed of two branches, each proceeding from A' or A'', the one which is the continuation of the course within the inner region I call the lower branch; the other, the upper branch; and I say that the upper branch *touches* the separator at A' or A''. The two branches and the entire loop lie on the left-hand side (or side away from A) of the meridians through A' or A''. As to the contact of the upper branch of this and other isochronics at A' or A'' with the separator, see *post* No. 142.

132. It is convenient at this point to consider the form of the isochronic curves within the axial region. The parabolic boundary thereof is an isochronic $T_{13} = \infty$, and it thence appears that for any large value of T_{13} the isochronic curve (portion of the curve) is a curve not meeting the parabolic boundary, and terminated each way in the separator boundary $B'AB''$. As the value of T_{13} diminishes, the curve (which is of course always symmetrical in regard to the axis) bends inwards towards the point A and for $T_{13} = 1 \cdot 983$ (value on the axis at A) the curve acquires a cusp at A. I call this the cuspidal isochronic; I remark that it intersects in the axial region each of the meridians 0° and 60°.

As T_{13} further diminishes to any value between 1·983 and ·950, the curve, commencing in the separator boundary, passes through A into the inner region, and, forming a loop within the loop of the nodal isochronic, emerges through A into the axial region, terminating again in the separator boundary.

133. On the meridians 90°, 330°, through the points B', B'', respectively, the value of T_{13} diminishes from 1 at the centre to 0 at the regulator, where these meridians are considered as terminating.

On the meridians 120°, 300° (meridian at right angles to the axis of symmetry), the value of T_{13} diminishes from 1 at the centre to a minimum less than ·878, and then increasing to a maximum of over ·895 diminishes to 0 at the regulator. On emergence of the meridian from the shaded and half-shaded region on the parabolic boundary of the lateral region the value is $=\infty$, and it thence diminishes to 1·148 on the separator boundary $B^{\text{IV}}B'$ or $B^{\text{V}}B''$.

On the meridians 150°, 270°, which pass through A', A'', respectively, the value of T_{13} increases from 1 at the centre to 1·377 at the regulator, and thence through 2·255 at 90° to ∞ at B^{IV} or B^{V}.

And finally, on the meridians 180°, 240°, the value of T_{13} increases from 1 at the centre to ∞ at the parabolic inner boundary, and then on emergence from the half-shaded and shaded region at the separator boundary $B'''A'$ or $B'''A''$, the value is $=\infty$, and it thence diminishes to a minimum under 6·343, and again increases to ∞ at the separator boundary $B'''B^{\text{IV}}$ or $B'''B^{\text{V}}$.

134. By what precedes, it appears that on the separator boundary $B^{\text{IV}}B'$ or $B^{\text{V}}B''$ of either of the lateral regions, the values of T_{13} is at each extremity $=\infty$, and at an intermediate point $=1{\cdot}148$; there is consequently a minimum value less than 1·148, and therefore two points at each of which the value is $=1{\cdot}983$.

Now resuming the consideration of the cuspidal isochronic ($T_{13}=1{\cdot}983$) as regards the remaining portions thereof, viz., those in the lateral and inner regions; and considering first the lateral region $B'''B^{\text{IV}}B'$, there will be from each of the points just referred to on the boundary $B^{\text{IV}}B'$ a branch; one (which I call the lower branch) from the point nearer B', passes, on the right-hand side of the meridian through A', to A'; the other (which I call the upper branch) proceeding from the point nearer B^{IV}, cuts the same meridian, and then on the left-hand side thereof arrives at A', touching there the separator: at A'' in the other lateral region there are in like manner an upper and a lower branch (situate symmetrically, in regard to the axis, with the upper and lower branches at A'); and continuous with the two lower branches there is a branch from A' to A'', through the antiloop of the inner region.

135. Imagine the given value of T_{13} as continuously increasing from the value ·950, which belongs to the nodal isochronic; and attend in the first instance to the form within the lateral regions. There will be a loop of continually increasing magnitude (viz., the loop for a larger value of T_{13} will always wholly include that for a smaller value); each loop formed by an upper branch, which at A' touches the separator, and a lower branch the direction of which from A' is variable. So long as T_{13} is less than 1·377 (value at A' along the meridian) the lower branch, and consequently the whole loop, will lie on the left hand of the meridian; but when T_{13} is $=1{\cdot}377$, the lower branch touches the meridian, and for any greater value of T_{13} lies on the right of the meridian; and in either of the last-mentioned cases the loop

is cut by the meridian, and thus lies partly on the left, and partly on the right thereof.

136. Now by what precedes there is on the separator boundary $B'B^{\text{IV}}$ of the lateral region a point where T_{13} has a minimum value less than 1·148, and consequently, for any given value, say for a value between this minimum and 1·377, there are on $B'B^{\text{IV}}$ two points where T_{13} has the given value. These points cannot lie on the loop of the curve belonging to the given value (for this loop is wholly on the left hand of the meridian); hence the complete curve for the given value of T_{13} will include (within the lateral region) besides the loop, a branch uniting the two points in question; say a link branch.

137. It follows that there is between $T_{13} = 1{\cdot}377$ and 1·983, a value (to fix the ideas, say $= 1{\cdot}80$?, it being understood that I do not attempt to determine this value) for which the loop and link branch will unite themselves together, the point of junction becoming as usual a node; viz., there will be a curve $T_{13} = 1{\cdot}80$? having in the two lateral regions respectively the nodes Y, Y'; or say the curve has in each lateral region a self-intersecting loop. For any greater value of T_{13} (as for example the value 1·983 belonging to the cuspidal curve) there are two branches inclosing the self-intersecting loop; for a less value, as has been seen, instead of the self-intersecting loop, there is a loop and link branch; at least this is the case until for the minimum value $< 1{\cdot}148$ of T_{13} on the separator boundary $B^{\text{IV}}B'$ the link branch disappears. For smaller values down to $T_{13} = {\cdot}950$, which belongs to the nodal isochronic, there is no link branch, but only the loop; and as T_{13} diminishes below this value, there is still a continually diminishing loop, lying wholly on the left hand of the meridian, and with its upper branch always touching the separator; and ultimately for $T_{13} = 0$ the loop vanishes.

138. We have attended wholly to the lateral regions; but the consideration of the axial and inner regions is very easy: for any value between the values 1·983 and ·950, there are in the axial region (between the nodal and cuspidal curves) two branches each proceeding from the separator to A, where they unite, and, crossing each other, pass into the inner region, forming a loop within the loop of the nodal isochronic; and, moreover, there is in the inner region a branch, the continuation of the lower branches of the lateral loops, uniting the points A', A'', and lying between the nodal and cuspidal isochronic. And for T_{13} less than ·950 there are in the axial region, between the nodal curve and the separator, two branches, each proceeding from the separator to A, where, crossing each other, they enter the inner region passing outside the nodal curve (or in the side regions of the inner region) to the points A', A'', where they respectively join on to the lower branches of the lateral loops. Ultimately, for $T_{13} = 0$, the curve coincides with the finite portions AA', AA'' of the regulator circle.

139. We have finally to consider the case T_{13} greater than 1·983: there is in the axial region a branch lying outside the cuspidal curve, and extending from separator to separator; in each lateral region two branches (lying outside those of the cuspidal curve) each proceeding from A' (or A'') to the separator boundary $B'B^{\text{IV}}$

or $B''B^{\text{v}}$, an upper branch touching the separator at A' or A'', and a lower branch; and in the inner region, a branch (continuation of the lower branches) lying between the cuspidal curve and the parabolic boundary of the antiloop, and uniting the points A', A''. In the ultimate case $T_{13} = \infty$, the curve coincides with the before-mentioned discontinuous curve composed of portions of the parabolic curve and of two separators.

140. To obtain a comprehensive statement of the foregoing results, we may (as in the case of the iseccentric lines) imagine the curves completed and rendered continuous by the insertion of portions lying outside the spherogram, or within the half-shaded and shaded regions; which inserted portions are to be ultimately obliterated. The upper and under branches terminating in the separator boundary of a lateral region are thus completed into a loop; the link branch into a closed curve or oval; the vanishing of the link branch happens when the oval, on the point of passing outside the separator boundary of the lateral region, just touches this boundary; as T_{13} diminishes to the value for which this happens, and continues still further to diminish, I think it may be assumed that there is some value (to fix the ideas, say $T_{13} = 1{\cdot}10$?, but I do not attempt to determine it) for which the oval becomes a conjugate point, viz., for this value $T_{13} = 1{\cdot}10$? the curve will have two conjugate points (nodes) Z', Z'', outside the two lateral regions respectively.

141. We may now state the forms of the curve. The points A, A', A'', are always nodes, viz., A', A'', nodes with real branches, but A is either a conjugate point, a cusp, or a node with real branches.

$T_{13} > 1{\cdot}983$: two-looped curve, containing within it A as a conjugate point:

as T_{13} diminishes, the curve bends inwards towards A, and

$T_{13} = 1{\cdot}983$: cuspidal isochronic; A, a cusp.

$T_{13} < 1{\cdot}983$, the curve cuts itself at A, having thus acquired an internal loop: as T_{13} diminishes, changes occur first as regards the lateral loops, and afterwards as regards the internal loop; viz., each of the lateral loops is gradually pinched together until

$T_{13} = 1{\cdot}80$? there are two new nodes Y', Y'', each lateral loop being a figure of 8.

As T_{13} diminishes the figure-of-8-loop breaks up into a loop and oval, which oval continually diminishes until for

$T_{13} = 1{\cdot}10$? the ovals have each become conjugate points, or there is a curve with two conjugate points Z', Z''. As T_{13} diminishes the conjugate points have disappeared, and we have again a curve with an internal and two lateral loops; but in the meantime the internal loop and the branch $A'A''$ are continually approaching each other; and, $T_{13} = {\cdot}950$, nodal isochronic, there is a node X on the axis. The curve consists of two figures of 8, each crossing itself at one of the points A', A'', and the two crossing each other at the points A, X.

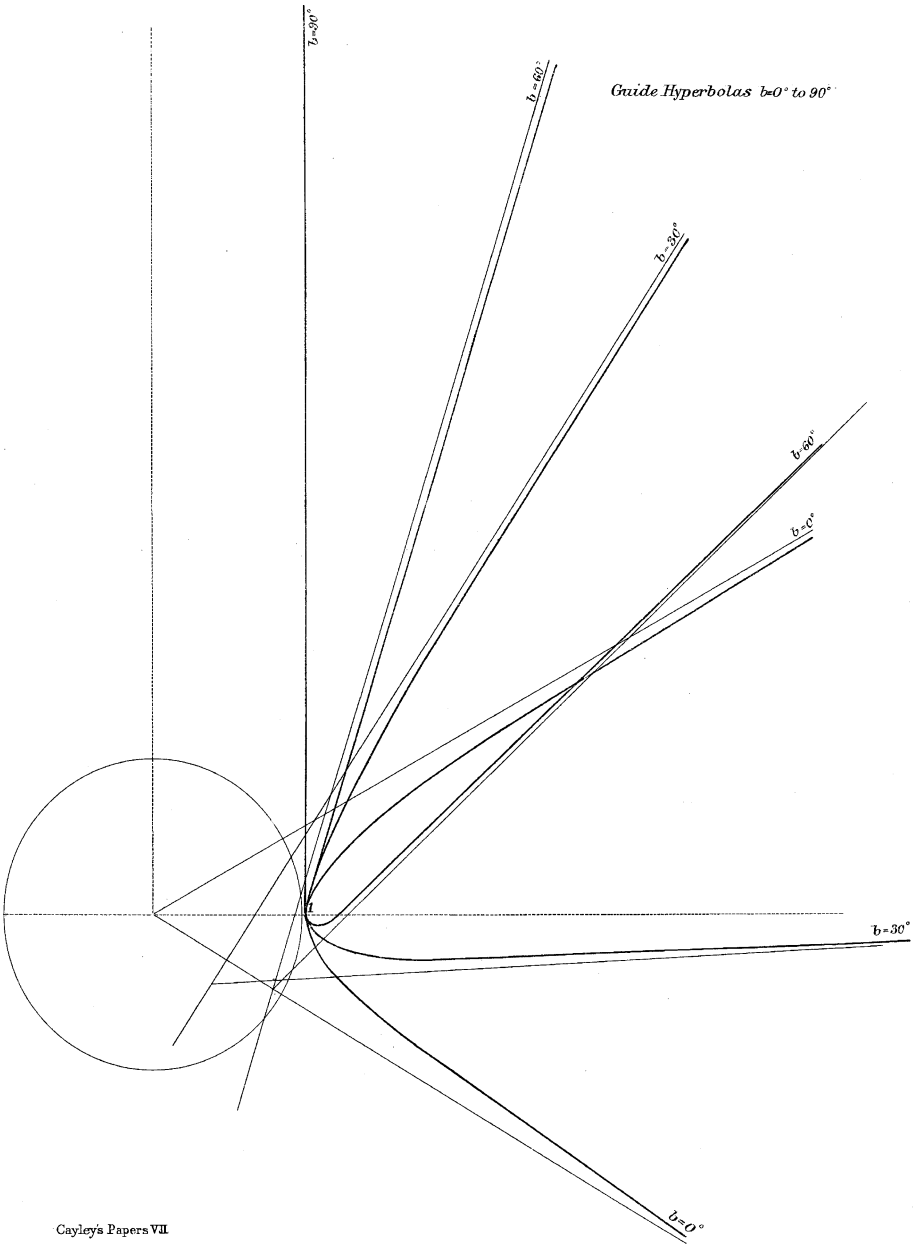
As T_{13} diminishes, the curve breaks off from X on each side of the axis so as not any longer to cross the axis (except at A), that is

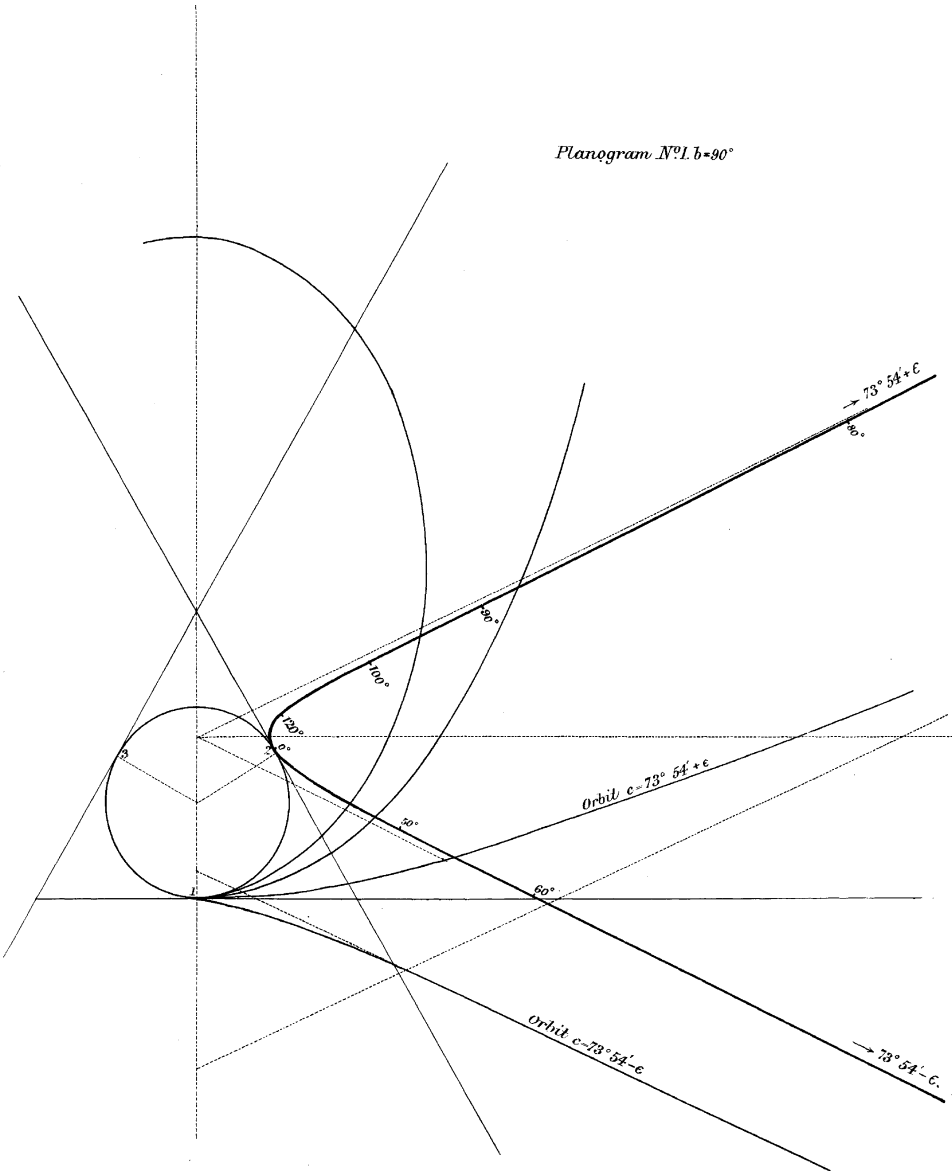
$T_{13} < \cdot 950$; curve is a chain intersecting itself at A', A, A''; viz., from each loop there pass two branches, one inside, the other outside, the regulator, uniting themselves at A with the branches from the other loop outside and inside the regulator respectively; and finally

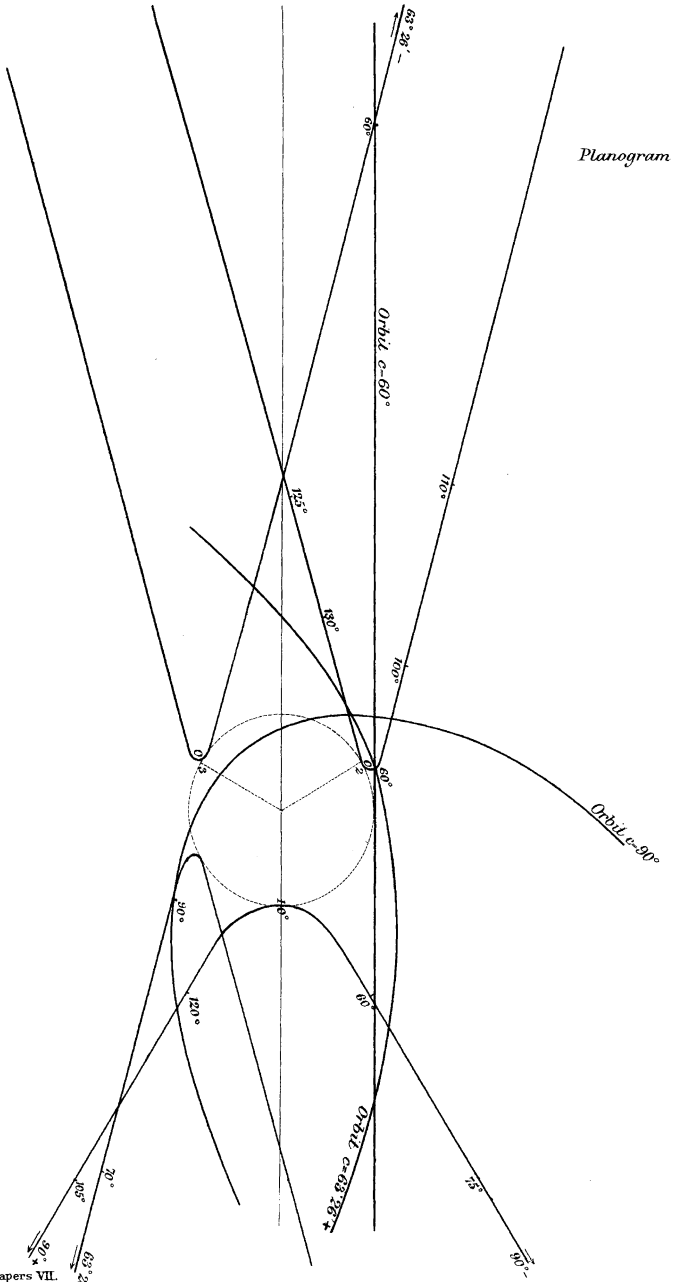
$T_{13} = 0$, the curve is the arc $A'AA''$ of the regulator circle.

142. There is not in the several curves any discontinuity of direction at the point A' or A'': the branch from A within the shaded or half-shaded region, emerges at A' or A'' into the lateral region, uniting itself with the upper branch of the loop; it can only do this in virtue of its being at A' or A'' a tangent to the separator (for otherwise it would cross the separator and regulator into the inner region); that is, the continuation thereof, or upper branch of the loop, must at the point A' or A'' *touch the separator;* it has been previously throughout assumed that this is so.

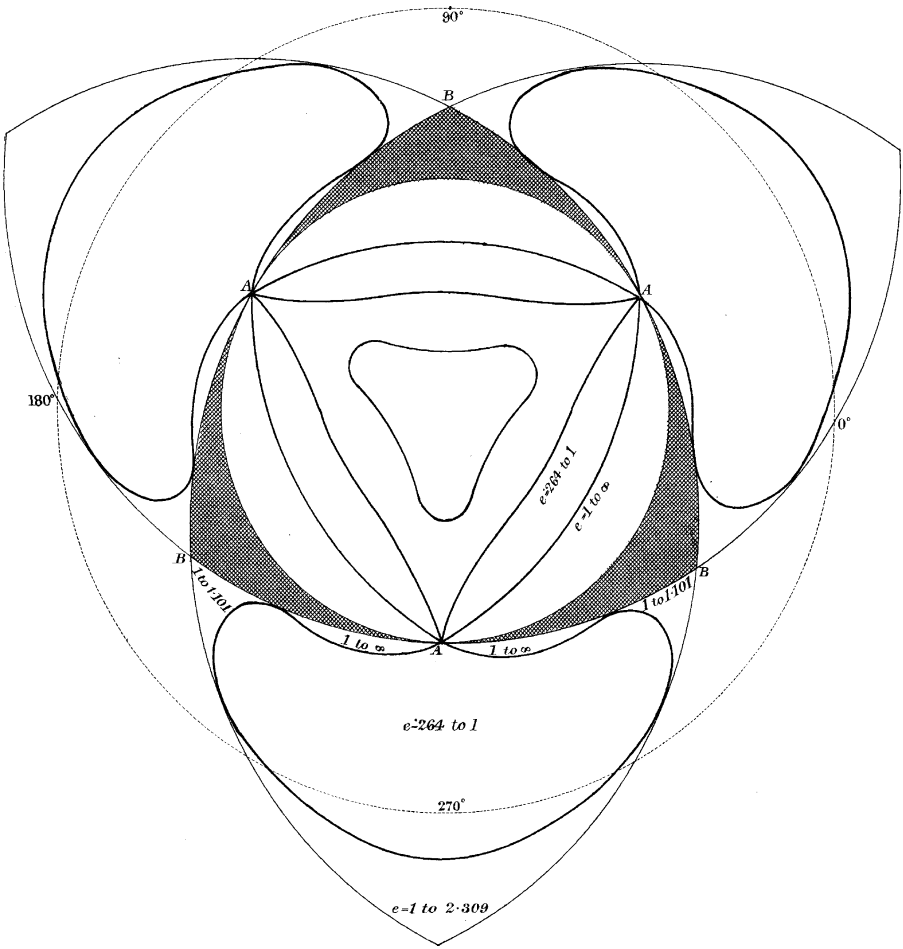
143. It is to be observed, both as regards the iseccentric and the isochronic curves, that there is a real meaning in the obliterated portions; viz., to any position of the orbit-pole on such obliterated portion of the curve there corresponds a conic determined by means of a given trivector, but which, by reason of its being a convex hyperbola, or hyperbola such that the three points do not lie on the same branch thereof, is not regarded as an orbit. The obliterated portions have been in the present Memoir considered only so far as they present themselves in continuity with the curves which are the loci of the pole of a proper orbit, and for the purpose of explaining the course of these curves; and the curves completed as above are not the complete loci which would be obtained if, instead of the selected conic called the orbit, we had considered simultaneously the four conics determined by means of any given trivector; such extension of the theory would, it is probable, be interesting geometrically; but it would be devoid of all astronomical significance.


Guide Hyperbolas b=0° to 90°
b=90°
b=60°
b=30°
b=60°
b=0°
1
b=30°
b=0°

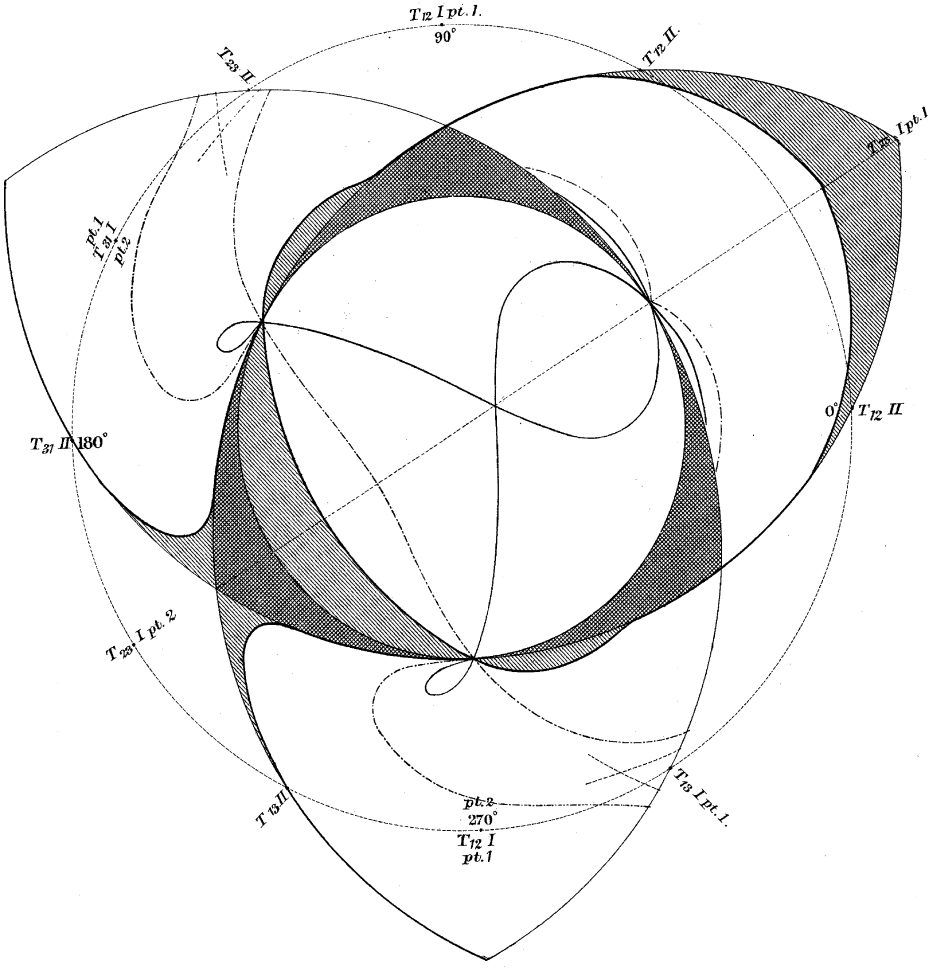

Planogram No. I. b=90°
→ 73° 54′ + ε
80°
90°
100°
120°
0°
3
2
1
Orbit c=73° 54′ + ε
30°
60°
Orbit c=73° 54′ − ε
→ 73° 54′ − ε.


Planogram No 2. b=180°.
Orbit c=60°
Orbit c=90°
Orbit c=63°26′
63°26′
60°
125°
180°
110°
100°
90°
120°
70°
105°
75°
90°
63°26′
90°

e-Spherogram.



T_{13} - *Spherogram.*



477.

ON THE GRAPHICAL CONSTRUCTION OF A SOLAR ECLIPSE.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXXIX. (1870—1871), pp. 1—17. Read January 13, 1871.]

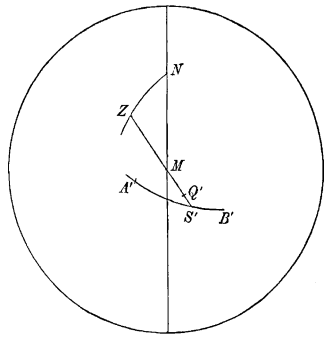
THE present Memoir contains the explanation of a Graphical Construction of a Solar Eclipse, which (it appears to me) is at once easy, and susceptible of considerable accuracy: I think that if made on the suggested scale (radius = 12 inches) we might by means of it construct a diagram such as the eclipse-diagrams of the *Nautical Almanac*, with at least as much accuracy as could be exhibited in a diagram on that scale.

Article Nos. 1 to 9. *General Explanation of the Construction.*

1. We may imagine the celestial sphere as seen from the centre of the Earth stereographically projected at each instant during the eclipse—the radius of the bounding circle of the projected hemisphere being a given length, say twelve inches, which is taken as unity—in such wise that the centre of the Moon is always at the centre of the projection, say M, and the pole (suppose the north pole, say N) of the Earth on a given radius: its position on this radius will in strictness be variable, viz. distance from centre = projection of Moon's N. P. D. = $\tan \frac{1}{2}\Delta$. Suppose, for a moment, that the position at each instant of the Sun's centre were also laid down on the projection, so as to obtain the projection of the Sun's relative orbit; this will be a terminated short line $A'B'$ (fig. 1), nearly straight, and lying near the centre of the projection (this relative orbit is not to be actually laid down, but it is replaced, as will presently be explained, by a relative orbit on a very enlarged scale); if at any instant the position of the Sun on the relative orbit be denoted by S', then the straight line MS' is the projection of the arc of great circle through the centres of the Moon and Sun, so that E being the angular distance of the centres, the length of the line MS' is $= \tan \frac{1}{2}E$, or (E being small) it is $= \frac{1}{2}E$.

2. Produce $S'M$ through the centre M to a point Z, and consider Z as representing a point on the Earth's surface: to determine the geographical position of Z, we must consider the projected meridian NZ which passes through Z: the arc NZ,

Fig. 1.



regarded as a projection, represents the N.P.D. or colatitude of Z, and the actual angle at N which the tangent of NZ makes with the line NM is equal to the celestial angle ZNM which is = Moon's hour-angle from Z, or what is the same thing = difference of Moon's hour-angle from Greenwich and of the longitude of Z (as the figure is drawn, $\angle ZNM$ = Moon's hour-angle E. of Greenwich, *less* E. longitude of Z).

3. Now, considering the Moon and Sun as seen from Z, we may disregard the parallactic depression of the Sun, and attribute to the Moon a displacement equal to the difference of the parallactic displacements of the Moon and Sun; that is, regarding the zenith distances ZM, ZS' as equal, we may consider the Moon's centre as depressed by parallax in the direction of the arc MS' through an arc MQ', $= \sin^{-1} P' \sin ZM$, where $P' = \cdot 99837\,(\sigma' - \pi')$ is the quantity thus designated in the Appendix to the *Nautical Almanac* for 1836, viz. it is $= \sigma' - \pi'$, the difference of the equatoreal horizontal parallaxes at the time of the eclipse, multiplied by a factor ·99837, which answers to a distance of Z from the Earth's centre = Earth's radius for latitude 45°. And if we take Q' such that its angular distance from S' = sum of angular semidiameters of the Sun and Moon, the locus of Q' is very nearly a circle about the centre S', and the corresponding positions of Z give the positions on the Earth where the limbs are in exterior contact, or, what is the same thing, give the penumbral curve on the Earth's surface for the position S' of the Sun.

4. Instead of

$$\text{Arc } MQ' = \sin^{-1} P' \sin ZM,$$

we may write

$$\text{Arc } MQ' = \quad P' \sin ZM,$$

or, using ρ to denote the linear distance ZM in the projection, we have $\rho = \tan \frac{1}{2} ZM$, and therefore $\sin ZM = \frac{2\rho}{\rho^2+1}$, hence

$$\text{Arc } MQ' = P' \frac{2\rho}{\rho^2+1},$$

and the linear distance MQ' in the projection is $= \tan \frac{1}{2} \text{arc } MQ'$, say this is $= \frac{1}{2} \text{arc } MQ'$, or calling this linear distance r' we have

$$r' = P' \frac{\rho}{\rho^2+1}.$$

5. Hence, if instead of the original representation of the Sun's relative orbit we consider an enlarged representation thereof and of the depressed positions Q' of the Moon, obtained by increasing the several distances from the centre of the projection in the ratio $\frac{1}{2}P'$ to 1, and if instead of A', B', S', Q', we use A, B, S, Q, as referring to this enlarged representation, then representing by r the linear distance MQ, we have $r = \frac{2}{P'} r'$, and consequently

$$r = \frac{2\rho}{\rho^2+1}.$$

We have here r representing the parallactic depression corresponding to the zenith distance ZM, where $\rho = \tan \frac{1}{2} ZM$; that is, $ZM = 90°$, $\rho = 1$, and therefore $r = 1$; but for $ZM = 90°$ the parallactic depression is $= P'$; that is, the scale of the enlarged representation of the Sun's relative orbit, or say simply the scale of the relative orbit (for on the original scale it was never actually constructed at all) is such that we have P' (= about 60′) represented by the radius of the bounding circle of the projected hemisphere, = 12 inches.

6. The process is, construct the relative orbit on the scale P' = radius of bounding circle: take S for the position at any given instant of the Sun in the relative orbit, and with centre S and radius $= s + \sigma$ (sum of the angular semidiameters, of course on the same scale) describe a circle. The positions A and B of the Sun at the beginning and end of the eclipse respectively are such that this circle just touches the bounding circle externally, viz. the distances of A and B from the centre of the projection are each = radius of bounding circle $+ s + \sigma$. At any intermediate instant the circle, radius $s + \sigma$, lies wholly or partly within the bounding circle; in the latter case we attend only to the arc thereof which lies within the bounding circle. Taking then Q any point whatever on the circle or arc in question, we join Q with the centre M of the projection, and produce this line through M to a point Z, such that the distances MQ, MZ, being r, ρ respectively, we have as above

$$r = \frac{2\rho}{\rho^2+1},$$

or, what is the same thing, writing θ in place of z, and regarding this angle θ as a variable parameter, the relation between r, ρ, may be expressed by means of the two equations, $\rho = \tan \frac{1}{2} \theta$, $r = \sin \theta$.

7. Practically the construction may be performed very easily by means of a straight edge twenty-four inches long, graduated from the centre, one half of it for the values of r, and the other half for the corresponding values of ρ (that is, the first half is graduated for $\sin\theta$, and the second half for $\tan\frac{1}{2}\theta$): we have thus, corresponding to the circle or arc of circle which is the locus of Q, a closed curve, or arc thereof terminated each way at the bounding circle, for the locus of Z: which curve or arc of a curve is the penumbral curve on the Earth's surface for the position S of the Sun in the relative orbit.

8. The north pole of the Earth occupies in the projection a given position, viz. it is situate on a given radius at a distance $=\tan\frac{1}{2}$ Moon's N.P.D.; which N.P.D. may be considered as being throughout the eclipse constant, and equal to its value at the middle of the eclipse. But in order to arrive at the geographical signification of the figure it is necessary to lay down on the projection the position of the meridian of Greenwich; which position, it will be remembered, varies according to the position of S. Supposing this done, we could of course (at least theoretically) draw the whole series of meridians and parallels, and thereby determine the latitudes and longitudes of the several points of the penumbral curve, or (if need is) transfer it to a different projection of the Earth's surface. The actual description of the meridians and parallels would, however, be very laborious, and fortunately it can be avoided by means of a single blank projection and a slight modification of the foregoing process, as will be explained.

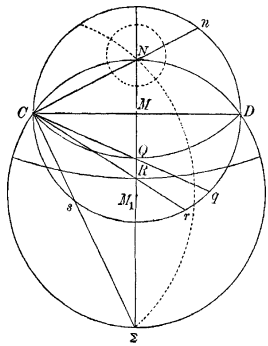
9. But before considering how this is, it is proper to remark that constructing as above a figure of the penumbral curves corresponding to the several positions of the Sun: by what precedes these different curves may indeed be considered as belonging to the same position of the north pole in the projection, but they belong to different positions of the meridian of Greenwich; and thus they do not constitute a representation of the penumbral curves each in its proper terrestrial position, but only a representation in which the penumbral curves are affected each of them by a different displacement in longitude.

Article Nos. 10 to 13. *Modification in order to the Applicability of a Single Blank Projection.*

10. Imagine a stereographic projection of the meridians and parallels *on the plane of a meridian*, radius of this meridian, that is of the bounding circle of the projected hemisphere, being $=12$ inches as before; and the poles N, Σ being of course opposite points on the circumference of the bounding circle—the meridians and parallels are, however, to be produced outside the bounding circle; say this is the "blank projection," and let its centre be denoted by M_1. Then, if at any point M on the radius MN, we draw the chord CD at right angles to M_1N, and on CD as diameter describe a circle, this will cut out from the blank projection a new projection having the last-mentioned circle for its bounding circle, and in which N is the north pole; viz. the meridians of the blank projection will be meridians, and the parallels of the blank projection will be parallels, in this new projection. And, moreover, if the longitudes are reckoned from the meridian NMM_1, then the meridian of a given longitude in the blank projection will in the new projection be the meridian of the *same*

longitude—but the parallel of a given colatitude c in the blank projection will, in the new projection, be the parallel of a different colatitude c',—the relation of c, c' being, however, a very simple one, as presently explained.

Fig. 2.

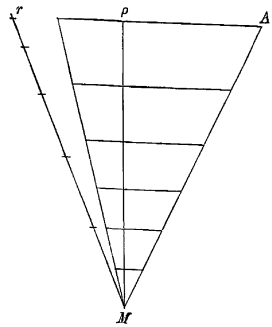


11. The blank projection thus at once gives a projection in which the north pole N has any assumed position whatever; and it is easy to see that in order that its distance MN from the centre of the projection may represent a given angle Δ, we have only to take $M_1M = \cos\Delta$ (that is $= 12$ inches $\times \cos\Delta$), the corresponding value of MC being $MC = \sin\Delta$ (that is $= 12$ inches $\times \sin\Delta$). Hence Δ denoting the Moon's N.P.D. at the middle of the eclipse, we can by means of the blank projection construct a projection such as that above referred to, only the radius of its bounding circle, instead of being unity (12 inches), is in the reduced ratio of $1 : \sin\Delta$.

12. The figure of the penumbral curves as originally constructed requires, therefore, to be reduced in the ratio $1 : \sin\Delta$, viz. each of the distances from the centre M should be reduced in this ratio; this could of course be done easily enough with a pair of proportional compasses; but by means of a different graduation of the straight edge we may, *in the first instance*, construct the penumbral curves on the proper reduced scale; viz. assuming that we have on the proper scale a proportional-scale figure such as is here shown, the line Mr ($= 12$ inches) being graduated for $\sin\theta$, and the line MA (also $= 12$ inches) for $\tan\frac{1}{2}\theta$, and a set of parallel lines being drawn through the last-mentioned graduations—then taking the distance $M\rho = \sin\Delta$, that is $= 12$ inches $\times \sin\Delta$, and drawing the line $M\rho$, this line will, it is clear, be graduated for $\sin\Delta \tan\frac{1}{2}\theta$: so that we may from the figure graduate the straight edge, the one half of it by means of the line Mr, and the other half of it by means of the line $M\rho$; and with the straight edge thus graduated, at once lay down the penumbral curve on the scale now in question. And we thus obtain a figure containing as well

the penumbral curves, as the meridians and parallels which serve to fix their terrestrial position.

Fig. 3.



13. It remains in the new projection to find the colatitude belonging to any given parallel. Supposing that the colatitude in the blank projection is $=c'$, then it may be shown that the colatitude c of the same parallel in the reduced projection is given by means of the equation

$$\tan \tfrac{1}{2} c = \cot \tfrac{1}{2} \Delta \tan \tfrac{1}{2} c',$$

from which c might be calculated numerically: but the required value may also be obtained graphically. In fact, considering the parallel which cuts $N\Sigma$ (see fig. 2) in a point R, then, if by lines drawn from C as a centre we project N, R, Σ, on the circumference of the bounding circle of the new projection—say the projections of these points are n, r, s, respectively, the arc ns is a semicircle, and the arcs nr, sr, are respectively the N.P.D. and the S.P.D. of the parallel in question. It may be added that in the new projection the equator is represented by the parallel through the points C, D; so that if this cuts $N\Sigma$ in Q, and the point Q be in like manner projected on the bounding circle—say its projection is q, then the arcs nq, sq, will be each of them a quadrant, and the arc qr will be the latitude of the parallel in question.

Article Nos. 14 to 18. *As to the Construction of the Relative Orbits.*

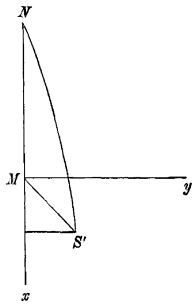
14. It is convenient to notice that if e, e′, be the values of the equation of time at the preceding and following Greenwich Mean Noons (viz. e or e′ = G.M.T. of apparent Noon) then that the Sun's hour-angle E. of Greenwich at the Greenwich mean time t is

$$h' = \mathrm{e} + t\left(1 + \frac{\mathrm{e}' - \mathrm{e}}{24^{\mathrm{h}}}\right)$$

and that if a, a', are the R.A.'s of the Moon and Sun respectively, then $h'-h=a'-a$, which is also of the form $A+Bt$. In the reduced projection, the Moon is always at the centre M; by means of the values of $h'-h$ we lay down at any instant the Sun's position in R.A. and then by means of the values of h', the position of the meridian of Greenwich; and we thus at any instant read off the terrestrial longitude of any point of the reduced projection, or say, of a point on the penumbral curve.

15. With regard to the construction of the relative orbit, it is to be observed that if at any instant the hour-angle and N.P.D. of the Moon are h, Δ, and those of the Sun, h', Δ', then taking M as origin, and the axes Mx, My, in the direction

Fig. 4.



of NM produced, and perpendicular hereto to the right (or eastwards), then the rectangular coordinates of S' are approximately $x=\frac{1}{2}(\Delta'-\Delta)$, $y=\frac{1}{2}(h'-h)\sin\Delta$, where $h'-h$ is equal to the difference of R.A. of the Sun and Moon. Hence, in the adopted relative orbit, the coordinates of S would be

$$x=\frac{\Delta'-\Delta}{P'}\,12\text{ in.}\qquad y=\frac{h'-h}{P'}\sin\Delta\,.\,12\text{ in.}$$

where, P' being reckoned in minutes, $\Delta'-\Delta$ and $h'-h$ are also reckoned in minutes.

16. Moreover, Δ may be considered as constant during the eclipse: and the relative orbit, assumed to be a straight line, will be determined by means of two points thereof; viz. knowing the values of $\Delta'-\Delta$, and $h'-h$ at about the time of the beginning and at about the time of the end of the eclipse, we construct by these formulæ two points of the orbit, and joining them by a straight line, we have the orbit. Also the position at any instant of the Sun in this relative orbit will be obtained by considering its motion therein as being uniform. I think there is no advantage in the adoption of a more accurate construction: for although we may for any given instant use the accurate values of h, Δ, h', Δ', and so construct the position in the relative orbit, and the corresponding penumbral curve, yet if in the deter-

mination of the geographical significance thereof, we were to use for each curve a different value of Δ, the simplicity of the construction would disappear; and it is, moreover, doubtful whether the trifling corrections would not be within the limits of the necessary errors of the drawing.

17. But if MS' be $=E$, and $\angle xMS' = \theta$, the accurate values for the coordinates of S' are $x = \tan\frac{1}{2}E.\cos\theta$, $y = \tan\frac{1}{2}E.\sin\theta$, and the values for the coordinates of S are $x = \frac{2}{P'.\text{arc } 1'}\tan\frac{1}{2}E.\cos\theta.12$ in., $y = \frac{2}{P'.\text{arc } 1'}\tan\frac{1}{2}E.\sin\theta.12$ in., where P' is still reckoned in minutes, and of course arc $1' = \frac{\pi}{10800}$. As the scale is considerable, it is worth while to inquire whether the employment of the accurate formulæ would produce an appreciable difference in the position of S.

We have $\sin\theta \div \sin\Delta' = \sin(h'-h) \div \sin E$, that is, $\sin E \sin\theta = \sin(h'-h)\sin\Delta'$, and $\cos E = \cos\Delta\cos\Delta' + \sin\Delta\sin\Delta'\cos(h'-h)$; or putting for shortness $\Delta'-\Delta = \alpha$, $h'-h = \beta$, we have $\sin E\sin\theta = \sin\beta\sin\Delta'$, and $\cos E = \cos\alpha - \sin\Delta\sin\Delta'.2\sin^2\frac{1}{2}\beta$. Hence, attending to the equations $\cos^2\frac{1}{2}E = 2(\cos^2\frac{1}{2}\alpha - \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)$ and $\sin^2\frac{1}{2}E = 2(\sin^2\frac{1}{2}\alpha + \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)$, we find

$$\tan\tfrac{1}{2}E\sin\theta = \frac{\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\Delta'}{\cos^2\frac{1}{2}\alpha - \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta},$$

and

$$\tan\tfrac{1}{2}E\cos\theta = \sqrt{\frac{\sin^2\frac{1}{2}\alpha + \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta}{\cos^2\frac{1}{2}\alpha - \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta} - \frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{(\cos^2\frac{1}{2}\alpha - \sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)^2}},$$

whence, considering α, β as small quantities of the same order, and neglecting terms of the third order, we have $\tan\frac{1}{2}E\sin\theta = \sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\Delta'$, or what is the same thing, $=\sin\frac{1}{2}\beta\sin\Delta$, or finally, $=\frac{1}{2}\beta\sin\Delta$, that is $\frac{1}{2}(h'-h)\sin\Delta$, which is the foregoing approximate value, and thus in the adopted orbit $y = \frac{h'-h}{P'}12$ in. = approx. value. As regards the expression for $\tan\frac{1}{2}E\cos\theta$, writing for a moment $\Omega = \sec^2\frac{1}{2}\alpha\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta$, the quantity under the radical sign is

$$\frac{\tan^2\frac{1}{2}\alpha + \Omega}{1-\Omega} - \frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{\cos^4\frac{1}{2}\alpha.(1-\Omega)^2},$$

and, taking this to the third order, it is

$$= \tan^2\alpha + \Omega(1+\tan^2\tfrac{1}{2}\alpha) - \frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{\cos^4\frac{1}{2}\alpha},$$

which, substituting for Ω its value, is

$$= \tan^2\tfrac{1}{2}\alpha + \frac{\sin^2\frac{1}{2}\beta\sin\Delta'}{\cos^4\frac{1}{2}\alpha}(\sin\Delta - \sin\Delta'\cos^2\tfrac{1}{2}\beta),$$

where $\sin\Delta - \sin\Delta'\cos^2\frac{1}{2}\beta = \sin\Delta - \sin(\Delta+\alpha)\cos^2\frac{1}{2}\beta$,

or neglecting herein terms of the second order, this is

$$= \sin\Delta - (\sin\Delta + \sin\alpha\cos\Delta)\cos^2\tfrac{1}{2}\beta,$$
$$= -\sin\alpha\cos\Delta, \; = -2\tan\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\alpha\cos\Delta,$$

so that to the third order the quantity under the radical sign is

$$= \tan^2\tfrac{1}{2}\alpha - \frac{2\tan\frac{1}{2}\alpha\sin^2\frac{1}{2}\beta\sin\Delta\cos\Delta}{\cos^2\frac{1}{2}\alpha};$$

and to the second order, that is finally neglecting terms of the third order,

$$\tan\tfrac{1}{2}E\cos\theta = \tan\tfrac{1}{2}\alpha - \frac{\sin^2\frac{1}{2}\beta\sin\Delta\cos\Delta}{\cos^2\frac{1}{2}\alpha},$$

or, what is the same thing,

$$= \tfrac{1}{2}\alpha - \sin\Delta\cos\Delta\,.\,\tfrac{1}{4}\beta^2.$$

18. Hence, writing $\alpha = (\Delta' - \Delta)$ arc 1′, $\beta = (h' - h)$ arc 1′, and passing to the adopted orbit, we have

$$x = \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - \tfrac{1}{2}\sin\Delta\cos\Delta\,\frac{h' - h}{P'}\,12\text{ in.} \times (h' - h)\text{ arc }1',$$

viz. putting

$$y = \frac{h' - h}{P'}\sin\Delta\,.\,12\text{ in.}$$

we have

$$x = \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - y\,.\,\tfrac{1}{2}\cos\Delta \times (h' - h)\text{ arc }1',$$

or say

$$= \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - y\,.\,\tfrac{1}{2}\cos\Delta\,(h' - h)\,\frac{\pi}{10800}.$$

The value of the second term may amount to about $\frac{1}{10}$ of an inch, and thus be sensible, but there is no difficulty in taking account of it.

Article No. 19. *As to the Equation* $r = \dfrac{2\rho}{\rho^2 + 1}$.

19. It may be remarked that the equation $r = \dfrac{2\rho}{\rho^2 + 1}$, which served for the graduation of the straight edge, was in effect obtained from the equations

$$r = \frac{2}{P'}\tan\tfrac{1}{2}E, \quad P'\sin z = \sin E, \quad \rho = \tan\tfrac{1}{2}z$$

by assuming therein $\tan\frac{1}{2}E = \frac{1}{2}E$ and $\sin E = E$ respectively. But the elimination of E and z can be effected without this assumption, viz. we have $\sin E = \dfrac{2\tan\frac{1}{2}E}{1 + \tan^2\frac{1}{2}E} = \dfrac{P'r}{1 + \frac{1}{4}P'^2 r^2}$,

and then as before, $\sin z = \frac{2\tan\frac{1}{2}z}{1+\tan^2\frac{1}{2}z} = \frac{2\rho}{1+\rho^2}$, whence the relation between r and ρ is found to be

$$\frac{r}{1+\frac{1}{4}P'^2 r^2} = \frac{2\rho}{1+\rho^2},$$

which however assumes that P' is reckoned in parts of the radius; reckoning it as before in minutes, we must, instead of P', write $P' \operatorname{arc} 1' = \frac{P'\pi}{10800}$, viz. the numerical value is about $\frac{1}{60}$, and taking it to be this number, the formula is

$$\frac{r}{1+\frac{1}{14400}r^2} = \frac{2\rho}{1+\rho^2},$$

where r, ρ are reckoned in parts of the radius (= 12 inches). Supposing that r_1 is calculated from the formula $r_1 = \frac{2\rho}{1+\rho^2}$, then we have very nearly $r = r_1\left(1+\frac{r_1^2}{14400}\right)$, and r_1 being $=1$ at most, the correction is inappreciable: if however this were not the case, the more accurate formula might have been used; the only difference being that the making of the graduation would have been more laborious.

Article No. 20. *Remark as to the Geometrical Theory of the Projection of the Penumbral Curve.*

20. The stereographic projection of the penumbral curve on the Earth's surface (assumed to be spherical) is, as I have elsewhere shown, a bicircular quartic. It may be shown that the stereographic projection, as given by the foregoing approximate method, is a bicircular quartic: we have, in fact a circle, the equation of which in the polar coordinates r, θ is

$$(r\cos\theta - \alpha)^2 + r^2\sin^2\theta = \beta^2,$$

and where (θ being unaltered) r is changed into ρ, where $r = \frac{2\rho}{\rho^2+1}$, that is $\frac{1}{r} = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$. The equation of the circle is

$$r^2 - 2\alpha r\cos\theta + \alpha^2 - \beta^2 = 0,$$

or say

$$1 - \frac{2\alpha\cos\theta}{r} + \frac{\alpha^2-\beta^2}{r^2} = 0,$$

and the transformed equation is therefore

$$1 - \alpha\cos\theta\left(\rho+\frac{1}{\rho}\right) + \frac{1}{4}(\alpha^2-\beta^2)\left(\rho+\frac{1}{\rho}\right)^2 = 0,$$

that is

$$(\alpha^2-\beta^2)(\rho^2+1)^2 - 4\alpha\cos\theta\rho(\rho^2+1) + 4\rho^2 = 0,$$

or in rectangular coordinates

$$\rho^4 + 2\rho^2 + 1 - \frac{4\alpha}{\alpha^2-\beta^2}x(\rho^2+1) + \frac{4}{\alpha^2-\beta^2}\rho^2 = 0,$$

that is

$$\rho^4 - \frac{4\alpha}{\alpha^2-\beta^2}x\rho^2 + \left(2 + \frac{4}{\alpha^2-\beta^2}\right)\rho^2 - \frac{4\alpha}{\alpha^2-\beta^2}x + 1 = 0,$$

where $\rho^2 = x^2 + y^2$; the form of the equation shows that the curve is a bicircular quartic. Writing for shortness $\frac{2}{\alpha^2-\beta^2} = m$, the equation is

$$\rho^4 - 2m\alpha x\rho^2 + (2+2m)\rho^2 + 1 - 2m\alpha x = 0,$$

that is

$$\{\rho^2 - m(\alpha x - 1) + 1\}^2 - m^2(\alpha x - 1)^2 - 2m = 0,$$

or, what is the same thing,

$$\{(x - \tfrac{1}{2}m\alpha)^2 + y^2 - \tfrac{1}{4}m^2\alpha^2 + m + 1\}^2 - m^2(\alpha x - 1)^2 - 2m = 0,$$

which putting $x + \frac{1}{2}m\alpha$ for x is

$$(x^2 + y^2 - \tfrac{1}{4}m^2\alpha^2 + m + 1)^2 - m^2(\alpha x + \tfrac{1}{2}m\alpha^2 - 1)^2 - 2m = 0,$$

viz. the terms of the fourth order being $(x^2+y^2)^2$, and there being no terms of the third order, the curve represented by this equation is a bicircular quartic.

Article Nos. 21 to 30. *Practical Details and Application to Eclipse of December* 21–22, 1870.

21. There are some practical details which it is proper to explain, using to fix the ideas the eclipse of December 21–22, 1870: the constant value of Δ (see *infrà*) is taken to be $+90° + 22° 35'$(1).

I have a blank projection (radius 12 in. as above) with the meridians and parallels each at intervals of 5°. And also another blank form which has on it merely a circle, radius 12 in., graduated as to one quadrant thereof with lines about $1\frac{1}{2}$ in. long, inwards towards the centre. It contains also in a corner the foregoing proportional-scale figure.

22. On the blank projection I measure off, downwards from the centre, a distance $M_1M = 12 \sin 22° 35' (= 4{\cdot}61)$, distances all in inches; and then with the centre M and radius $MC = 12 \cos 22° 35' (= 11{\cdot}08)$, describe a circle which is the bounding circle of the reduced projection. With this same radius I describe on the second form, concentric with the 12-inch circle and above the horizontal diameter thereof, a semicircle: and, cutting out the included area, replace it with tracing cloth. The form thus prepared is placed over the blank projection, so that the semicircle shall coincide with the corresponding semicircle on the blank projection, and the two sheets are fixed together by their lower edges, and by folding down the remaining sides. We have thus the upper half of the reduced projection, represented by the semicircle, with the

[1] See Plate, which exhibits in dotted lines the blank projection under the other blank form; the part within the red semicircle, as seen through the tracing cloth, the rest really hidden.

meridians and parallels marked out thereon by lines seen through the tracing cloth. See the Plate; the dotted line shows a paper scale afterwards affixed to the second form or upper sheet. Observe that so far the only eclipse-datum made use of is the value $\Delta = 90° + 22° 35'$.

23. We have for the eclipse in question, taking t for the G.M.T. in hours, positive or negative according as the time is after or before G.M. Noon, Dec. 22, and h' also in hours,

$$h' = 0^h{\cdot}02 + t\,{\cdot}9996,$$

and then taking the values of α, α', Δ, Δ' from the N.A. we have as follows:

G.M.T. 1870, Dec.	$h' + 2^h =$	$\alpha =$	$\alpha' =$	$\Delta = 90° +$	$\Delta' = 90° +$	$h + 2^h = h' + 2^h + \alpha' - \alpha$	$\Delta' - \Delta$ in Minutes of Arc.	$h' - h$ in ditto.
d h	h m s	h m s	h m s	° ′ ″	° ′ ″	h m s	′	′
21 22	0 1 13·42	17 56 1·84	18 1 48·72	22 27 8·5	23 27 17·1	0 7 0·30	60·143	− 86·620
22 3	5 1 7·37	18 9 26·74	18 2 44·27	22 43 12·5	23 27 13·9	4 54 24·90	44·023	+ 100·632

and moreover

Moon's Parallax $\sigma' = 60' 38''{\cdot}6$

Sun's ditto $\pi' = 9{\cdot}1$

$\sigma' - \pi' = 60\ 29{\cdot}5 = 60'{\cdot}49$

$P' = 60{\cdot}39$

Moon's Semidiam. $s = 16' 33''{\cdot}2$

Sun's ditto $s' = 16\ 17{\cdot}9$

$s + s' = 32\ 51{\cdot}2 = 32'{\cdot}85$

whence

$$\frac{s + s'}{P'}\ 12 \text{ in.} = 6{\cdot}53$$

viz. this is the radius of the circles used in the construction of the penumbral curves.

24. We have for x, y the formulæ

$$x = \frac{\Delta' - \Delta}{P'}\ 12 \text{ in.} + y\,(h' - h)\ 0{\cdot}00006,$$

$$y = \frac{h' - h}{P'}\ 12 \text{ in.} \times \sin \Delta,$$

viz. I find Dec. 21, 22^h,

$$x = 11{\cdot}95 - {\cdot}06 = 11{\cdot}89,$$

$$y \qquad -15{\cdot}80,$$

and Dec. 22—3^h,

$$x = 8{\cdot}75 + {\cdot}11 = 8{\cdot}85,$$
$$y = +18{\cdot}45,$$

where I have taken account of the small corrections to the approximate values of x: it may be added that, using the conjunction-value $52'\,9''{\cdot}4$ of $\Delta' - \Delta$, we have at conjunction,

$$x = 10{\cdot}36, \quad y = 0.$$

25. We thus lay down on the relative orbit the two points 22^h and 3^h, and the point of conjunction or intersection with the axis of x; the three points are found to be sensibly in a straight line: the distance between the extreme points is about 34 inches, representing 5 hours, so that the scale is nearly 7 inches to an hour: the line is then graduated to quarters of an hour. We then, by means of the distance $12 + 6{\cdot}53 = 18{\cdot}53$, mark off on the relative orbit, the points B, E, which correspond to the beginning and end of the eclipse respectively: the times as read off from my figure, and compared with the true times given in the N. A. are

	from figure	N. A.
Beginning	$22^h\,12^m{\cdot}5$	22 13·6
End	2 40 ·5	2 41·1

26. With centre B describing a circle radius 6·53 this will of course just touch the 12-inch circle, and the penumbral curve will be a mere point, viz., this is the point B' on the bounding circle, opposite to the point of contact. And so with centre E describing a circle of the same radius 6·53, that will just touch the 12-inch circle, and the penumbral curve will be a mere point, viz., the point E' on the bounding circle, opposite the point of contact.

27. I draw the circles corresponding to the times $22^h\,30^m$, 45^m, $23^h\,0^m$, viz., so much of each as lies within the 12-inch circle. Each of these is then transformed into a penumbral curve, drawn in the upper semicircle on the tracing cloth. For this purpose we construct a straight edge of paper, the one half graduated for $12 \sin\theta$, the other half for $11{\cdot}08 \tan\frac{1}{2}\theta$, by means of the proportional-scale figure, as already explained: $\theta = 0°$ to $90°$ at intervals of $5°$, is quite sufficient; the points on any particular penumbral curve are laid down in pairs with the utmost facility, and the curve is traced by hand from 4 or 5 pairs of points.

28. We then graduate for latitude; viz., we see through the tracing cloth, the equator cutting the vertical radius in Q, and a parallel cutting the same radius, say in R; drawing lines from C, we refer these to the points q, r on the bounding circle, viz., on the quadrant thereof which is graduated by means of the graduation-lines of the 12-inch circle; and we thus read off the latitude of the parallel in question; this latitude is then marked for each parallel on the vertical radius from Q up to the bounding circle, viz., not on the tracing cloth, but on the paper affix; and we then on this same radius (on the paper affix) interpolate the positions where this would be intersected by the parallels for the colatitudes, 5°, 10°, 15°, &c. Or

(what is perhaps better) we may without marking the latitudes of the parallels of the blank form, construct directly the last-mentioned graduations; viz., marking off on the bounding circle from the point q, equal intervals of 5°, and from any such mark drawing to C, a line to meet the vertical radius, the point of intersection is the point belonging to the parallel, latitude equal to the corresponding multiple of 5°.

29. Finally, we must (not on the tracing cloth but on the paper affix) graduate an arc of the equator for the position of the meridian of Greenwich, that is for h. We have

$$\text{At } 22^h \quad h = -2^h + \overset{h}{0}\ \overset{m}{7}\ \overset{s}{0{\cdot}30} = -\overset{h}{1}\ \overset{m}{52}\ \overset{s}{52{\cdot}30} = -\overset{\circ}{28}\ \overset{\prime}{13{\cdot}08}$$

$$\text{At } 3^h \quad h = -2^h + 4\ 54\ 24{\cdot}90 = +2\ 54\ 24{\cdot}90 = +43\ 36{\cdot}20$$

The equator is already graduated in longitude by means of the meridians of the blank projection: hence we lay down the marks for 22^h and 3^h in the positions belonging to $-28°\,13'$, and $+43°\,36'$ respectively. And then since the interval of 5 hours answers to $71°\,49'$, that of 1 hour will answer to $14°\,22'$, so that, measuring off these intervals of longitude, we have the marks for the intermediate times 23^h, 0^h, 1^h, 2^h; or it might be proper to find in this way the marks corresponding to each interval of 20^m of time, answering to about 5° of longitude; the further subdivisions would be proportional to the intervals of time.

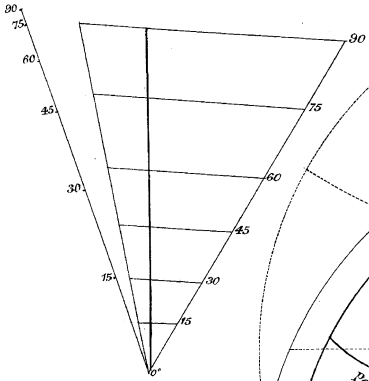
30. I have in this way read off the positions of the points B' and E' belonging to the beginning and end of the eclipse; the values, as compared with the true ones, are

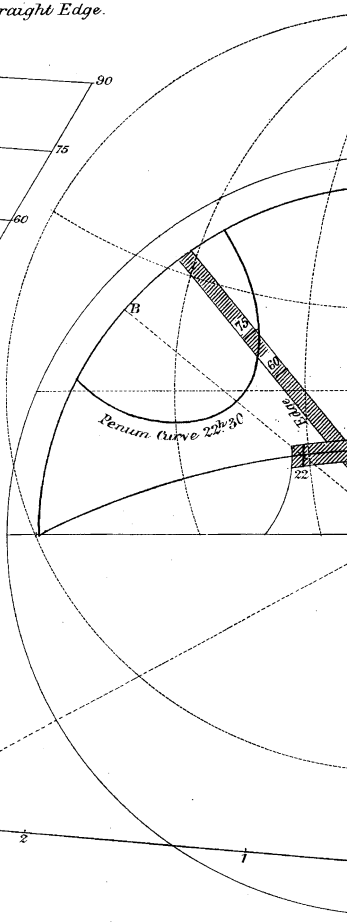
			From Figure	N. A.
			°	° ′
B′	latitude	N.	34	35 37
	longitude	W.	47	45 44
E′	latitude	N.	26	26 5
	longitude	W.	38½	37 16

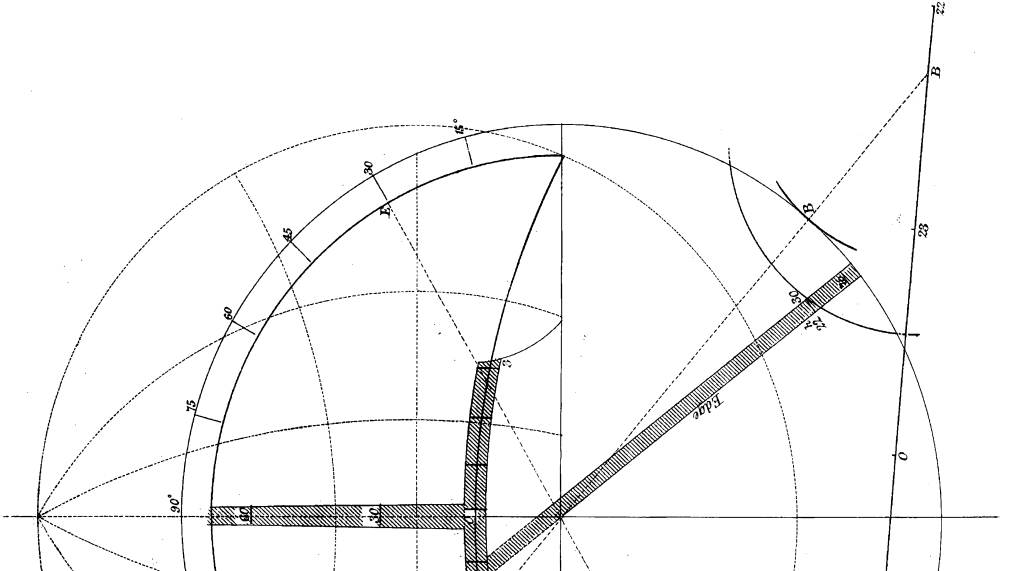
I remark that my figure, although drawn carefully, is not drawn with anything like the degree of accuracy which would be easily attainable; and I think that far better results might be obtained. I merely from a scale lay down tenths and estimate hundredths of an inch, but certainly fiftieths might be laid down from a scale.

Cayley's Paper VII.

Scale for graduation of straight Edge.






90°
75
60
45
30
15
45°
90
Face
B
B
22
23
24
O
C

478.

ON THE GEODESIC LINES ON AN ELLIPSOID.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXXIX. (1872), pp. 31—53. Read January 13, 1871.]

THE fundamental equations, in regard to the geodesic lines on an ellipsoid, were established by Jacobi, viz., representing by a, b, c, the squares of the semiaxes, that is, taking the ellipsoid to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$$

(where $a > b > c$), if we introduce the elliptic coordinates h, k, and write

$$\frac{x^2}{a+h} + \frac{y^2}{b+h} + \frac{z^2}{c+h} = 1,$$

$$\frac{x^2}{a+k} + \frac{y^2}{b+k} + \frac{z^2}{c+k} = 1,$$

or, what is the same thing,

$$x^2 = \frac{a(a+h)(a+k)}{(a-b)(a-c)},$$

$$y^2 = \frac{b(b+h)(b+k)}{(b-c)(b-a)},$$

$$z^2 = \frac{c(c+h)(c+k)}{(c-a)(c-b)};$$

then, if β be an arbitrary constant, the differential equation of a geodesic line is

$$(1) \quad \text{const.} = \int dh \sqrt{\frac{h}{(a+h)(b+h)(c+h)(\beta+h)}} + \int dk \sqrt{\frac{h}{(a+k)(b+k)(c+k)(\beta+k)}},$$

and the expression for the length of any arc of the curve is given by

$$(2)\quad s=\int dh\sqrt{\frac{h(\beta+h)}{(a+h)(b+h)(c+h)}}+\int dk\sqrt{\frac{k(\beta+k)}{(a+k)(b+k)(c+k)}}.$$

I propose in the present Memoir to develope the theory to the extent of showing how we can, by means of the first of these equations, explain the course of the geodesic lines; and for given numerical values of a, b, c, calculate, construct, and exhibit in a drawing the course of these lines: I attend more particularly to the series of geodesic lines through an umbilicus (which lines pass also through the opposite umbilicus), and to the case where the semiaxes are connected by the equation $ac-b^2=0$, a relation which simplifies the formulæ.

General Considerations as to the Course of the Lines.

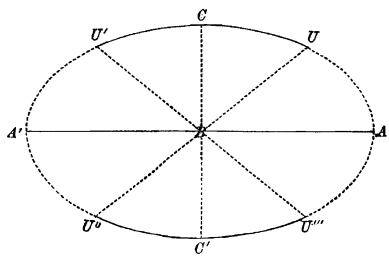
1. It will be observed that h and k enter into the formulæ symmetrically: it will be convenient to distinguish between these coordinates by considering h as extending between the values $-a$, $-b$; and k as extending between the values $-b$, $-c$. Thus:

$h=$ const. denotes a curve of curvature of the one kind, viz.:

$h=-a$, the principal section ABA' (or major-mean section), $h=-b$, the curves UU' and $U''U'''$ (or portions of the umbilicar section $ACA'C'$); similarly,

$k=$ const. denotes a curve of curvature of the other kind, viz.:

$k=-c$, the principal section CBC' (or minor-mean section), $k=-b$, the curves UU''' and $U'U''$ (remaining portions of the umbilicar section $ACA'C'$).



2. To any given (admissible) values of h, k there correspond eight points, situate in the eight octants of the surface respectively; but, unless the contrary is expressed, it is assumed that the coordinates x, y, z, are positive, and that the point is situate in the octant ABC.

3. The constant β may have any value from $+a$ to $+c$; viz., if it has a value between a and b, or say, if $-\beta$ has an h-value, then the geodesic lines wholly

between the two ovals of the curve of curvature $h=-\beta$ (being in general an indefinite undulating curve touching each oval an indefinite number of times). Similarly, if β has any value between b and c, or say, if $-\beta$ has a k-value, then the geodesic line lies wholly between the two ovals of the curve of curvature $k=-\beta$ (being in general an indefinite undulating curve touching each oval an indefinite number of times). The intermediate case is when $\beta=b$, or say when $-\beta$ has the umbilicar value: here the geodesic line is in general an indefinite undulating curve passing an infinite number of times through the opposite umbilici U, U'', or U', U'''; to fix the ideas, say through U, U''.

Lines through an Umbilicus.

4. I attend in particular to the last-mentioned case, and thus write $\beta=b$. We may in the formula (1) fix at pleasure a limit of each integral; and writing for convenience

$$\Pi(h)=\int_{-a}^{h}\frac{-dh}{b+h}\sqrt{\frac{h}{(a+h)(c+h)}},$$

$$\Psi(k)=\int_{k}^{-e}\frac{dk}{b+k}\sqrt{\frac{k}{(a+k)(c+k)}},$$

the equation (1) becomes

$$\text{Const.}=\Pi(h)+\Psi(k).$$

5. It is to be observed, in regard to these integrals, that writing $h=-a+u$, we have

$$\Pi(h)=\int_{o}^{u}\frac{du}{a-b-u}\sqrt{\frac{a-u}{u(a-c-u)}},$$

which, for u small, is

$$=\frac{1}{a-b}\sqrt{\frac{a}{a-c}}\int_{o}^{u}\frac{du}{\sqrt{u}},\quad =\frac{2\sqrt{u}}{a-b}\sqrt{\frac{a}{a-c}}.$$

By the assistance of this formula the value of the integral may be calculated by quadratures; viz., the formula gives the integral for any small value of u, and we can then proceed by the method of quadratures. The integral becomes infinite for $h=-b$: suppose that we have by quadratures calculated it up to $h=-b-m$ (m small), then to calculate it up to any value $-b-m+u$ nearer to $-b$, we have

$$\Pi(h)=\Pi(-b-m)+\int_{o}^{u}\frac{du}{m-u}\sqrt{\frac{b+m-u}{(a-b-m+u)(b-c+m-u)}}$$

$$=\Pi(-b-m)+\sqrt{\frac{b}{(a-b)(b-c)}}\int_{o}^{u}\frac{du}{m-u}$$

$$=\Pi(-b-m)-\sqrt{\frac{b}{(a-b)(b-c)}}\log\left(1-\frac{u}{m}\right),\text{[1]}$$

where the second term is positive, and the value thus increases slowly with u, becoming as it should do $=\infty$ for $u=m$ or $h=-b$.

[1] Except when the contrary is stated, the symbol "log" denotes throughout the hyperbolic logarithm.

6. Similarly in the second integral writing $k=-c-v$, we have

$$\Psi(k)=\int_0^v \frac{dv}{b-c-v}\sqrt{\frac{c+v}{(a-c-v)\,v}},$$

which, for v small, is

$$=\frac{1}{b-c}\sqrt{\frac{c}{a-c}}\int\frac{dv}{\sqrt{v}},\quad =\frac{2\sqrt{v}}{b-c}\sqrt{\frac{c}{a-c}},$$

which is of the like assistance in regard to the calculation by quadratures. And if we have by quadratures calculated the integral up to $h=-b+n$ (n small), then, to calculate it up to any value $-b+n-v$ nearer to $-b$, we have

$$\begin{aligned}\Psi(k)&=\Psi(-b+n)+\int_0^v\frac{dv}{n-v}\sqrt{\frac{b-n+v}{(a-b+n-v)(b-c-n+v)}}\\&=\Psi(-b+n)+\sqrt{\frac{b}{(a-b)(b-c)}}\int_0^v\frac{dv}{n-v}\\&=\Psi(-b+n)-\sqrt{\frac{b}{(a-b)(b-c)}}\log\left(1-\frac{v}{n}\right),\end{aligned}$$

where the second term is positive, and the value thus increases slowly with v, becoming as it should do $=\infty$ for $v=n$, or $k=-b$.

7. It may be remarked that in $\Pi(h)$ and $\Psi(k)$ respectively the coefficient of the logarithmic term has in each case the same value $=\sqrt{\frac{b}{(a-b)(b-c)}}$. As regards the initial terms $\sqrt{u}$ and $\sqrt{v}$, the coefficients are $\frac{1}{a-b}\sqrt{\frac{a}{a-c}}$ and $\frac{1}{b-c}\sqrt{\frac{c}{a-c}}$ respectively, which are equal if $\frac{a}{(a-b)^2}=\frac{c}{(b-c)^2}$, or $ac-b^2=0$.

8. We may consider the two geodesic lines $\Pi(h)\pm\Psi(k)=\text{const.}$; suppose that these each of them pass through the point P, coordinates (h_0, k_0) in the ABC octant of the ellipsoid; then for one of them we have $\Pi(h)-\Psi(k)=\Pi(h_0)-\Psi(k_0)$, and for the other of them we have $\Pi(h)+\Psi(k)=\Pi(h_0)+\Psi(k_0)$: I attend first to the former of these, say $\Pi(h)-\Psi(k)=C$ (where C is $=\Pi(h_0)-\Psi(k_0)$); and I say that this denotes the curve UPU''. In fact, by reason of the equation $\Pi(h)$ and $\Psi(k)$ must both increase or both diminish; they both increase as h passes from h_0 to $-b$, and as k passes from k_0 to $-b$: we may have $h=-b+u$, $k=-b+v$ where u and v are both indefinitely small, the functions Π and Ψ being then indefinitely large, but $\Pi-\Psi=C$; and we have thus a series of points nearer and nearer to the umbilicus U; that is, we have the portion PU of the curve. Tracing the curve in the opposite direction, or considering h as passing from h_0 to $-a$, and k as passing from k_0 to $-c$, then if C be positive, k will attain the value $-c$, before h attains the value $-a$, say that we have simultaneously $h=h_1$, $k=-c$; the equation is $\Pi(h_1)-\Psi(-c)=C$, that is, $\Pi(h_1)=C$; and the geodesic line then arrives at a point P, on the arc CB of the minor-mean

principal section. The function Ψ then changes its sign, viz., considering it as always positive, the equation is now $\Pi(h)+\Psi(k)=C$, k passing from the value $-c$ towards $-b$, that is, $\Psi(k)$ increasing, and therefore $\Pi(h)$ diminishing, or h passing from h_1 towards the value $-a$; until at last, say for $k=k_2$, we have $h=-a$, that is, $C=\Pi(-a)+\Psi(k_2)$, or $C=\Psi(k_2)$; the geodesic line here arrives at a point P_2 on the arc BA' of the major mean principal section. The function Π then changes its sign, viz., Π denoting a positive function as before, the equation is $-\Pi(h)+\Psi(k)=C$; h passes from $-a$ towards $-b$, that is $\Pi(h)$ increases, and therefore $\Psi(k)$ must also increase, or k pass from k_2 towards $-b$: we have at length $h=-b-u$, $k=-b+v$, u and v being each indefinitely small; and therefore Π and Ψ each indefinitely large (but $-\Pi+\Psi=C$); that is, we arrive at the umbilicus U'', completing the geodesic line UPU''.

9. If instead of $C=+$ we have $C=-$, everything is similar, but the geodesic line proceeding from U in the direction UP will first cut the arc BA of the major mean section at a point P_1; then the arc BC' of the minor mean section at a point P_2; and, finally, arrive as before at the umbilicus U'''.

10. The intermediate case is when $C=0$, viz., we have here $\Pi(h)-\Psi(k)=0$; the geodesic line here passes from U in the direction UP to B (extremity of the mean axis, $h=-a$, $k=-c$); Π and Ψ then each change their sign, so that, considering them as positive, the equation still is $\Pi(h)-\Psi(k)=0$, and the geodesic line at last arrives at the umbilicus U''. It will be easily understood how in the like manner $\Pi(h)+\Psi(k)=C$ refers to the line $U'PU'''$.

11. Reverting to the equation $\Pi(h)-\Psi(k)=C$, or as I will now write it

$$\Pi(h)-\Psi(k)=\Pi(h_0)-\Psi(k_0),$$

which belongs to the portion UP of the geodesic line UPU'', we require when h is $=-b-u$, and $k=-b+v$ (u and v indefinitely small) to know the ratio of the increments u, v; this in fact serves to determine the direction at U of the geodesic line through the given point (h_0, k_0).

12. For this purpose writing $h=-b-u$, we find

$$\Pi(h)=\int_u^{a-b}\frac{du}{u}\sqrt{\frac{b+u}{(a-b-u)(b-c+u)}},$$

which is

$$=\int_u^{a-b}\frac{du}{u}\left\{\sqrt{\frac{b+u}{(a-b-u)(b-c+u)}}-\sqrt{\frac{b}{(a-b)(b-c)}}\right\}$$
$$+\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{a-b}{u},$$

and, when u is indefinitely small, this is

$$\Pi(h)=\int_0^{a-b}\frac{du}{u}\left\{\sqrt{\frac{b+u}{(a-b-u)(b-c+u)}}-\sqrt{\frac{b}{(a-b)(b-c)}}\right\}+\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{a-b}{u}.$$

Similarly, when $k = -b + v$, where v is indefinitely small

$$\Psi(k) = \int_0^{b-c} \frac{dv}{v} \left\{ \sqrt{\frac{b-v}{(a-b+v)(b-c-v)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\} + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{b-c}{v}.$$

13. Each of the integrals is of the dimension $-\frac{1}{2}$ in a, b, c, and the difference of the integrals may be represented by

$$M \sqrt{\frac{b}{(a-b)(b-c)}};$$

we have therefore

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}} \left\{ M + \log \frac{a-b}{b-c} \frac{v}{u} \right\},$$

where

$$\sqrt{\frac{b}{(a-b)(b-c)}} M = \int_0^{a-b} \frac{du}{u} \left\{ \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\}$$

$$- \int_0^{b-c} \frac{dv}{v} \left\{ \sqrt{\frac{b-v}{(a-b+v)(b-c-v)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\}.$$

14. Suppose the inferior limits replaced by the indefinitely small positive quantities ϵ, ϵ' respectively; and for the variable in the second integral write $-u$; then

$$M = \int_{-(b-c)}^{a-b} \left\{ \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \right\},$$

it being understood that the values $u = -\epsilon'$ to $u = +\epsilon$ are omitted from the integration: this is

$$= \int_{-(b-c)}^{a-b} \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} - \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{\epsilon} \frac{\epsilon'}{b-c}$$

with the same convention as to the integral; or if $\epsilon' = \epsilon$, then

$$M = M' - \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{a-b}{b-c},$$

where

$$\sqrt{\frac{b}{(a-b)(b-c)}} M' = \int_{-(b-c)}^{a-b} \frac{du}{u} \sqrt{\frac{b+u}{(a-b-u)(b-c+u)}} = \int_{-a}^{c} \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}},$$

the omitted elements being from $u = -\epsilon$ to $u = +\epsilon$; that is (in the language of Cauchy) we take for the integral its *principal* value. And hence

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}} \left\{ M' + \log \frac{v}{u} \right\}.$$

15. By what precedes this is $= \Pi(h_0) - \Psi(k_0)$; or if we write simply (h, k) instead of (h_0, k_0), that is, consider the geodesic line UP, which is drawn from the

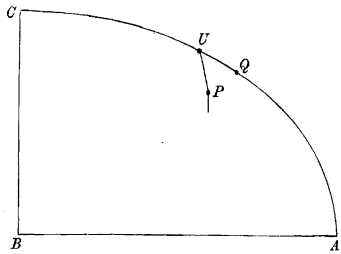
point P, coordinates (h, k), to the umbilicus U, the coordinates of a point consecutive to the umbilicus are $-b-u$, $-b+v$, where u, v are connected by the last-mentioned equation, in which M' is a transcendental function depending on (a, b, c) but independent of the particular geodesic line.

16. If for the geodesic line through the point B, or say for the B-geodesic $\frac{v}{u}=\frac{v_0}{u_0}$, then $M'=-\log\frac{v_0}{u_0}$, and we have in general

$$\Pi(h)-\Psi(k)=\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{v\,u_0}{u\,v_0},$$

a result which I proceed to further transform as follows:

If x_0, y_0, z_0 refer to the umbilicus U, then considering first the consecutive point P on the geodesic line (coordinates $-b-u$, $-b+v$) and next the consecutive



point Q on the umbilicar section, we have for these two points respectively,

$$dx_0=\frac{\frac{1}{2}\sqrt{a}\,(v-u)}{\sqrt{(a-b)(a-c)}},$$

$$dy_0=\frac{\sqrt{b}\,\sqrt{uv}}{\sqrt{(a-b)(b-c)}},$$

$$dz_0=\frac{\frac{1}{2}\sqrt{c}\,(u-v)}{\sqrt{(b-c)(a-c)}},$$

$$\delta x_0=\frac{\frac{1}{2}\sqrt{a}}{\sqrt{(a-b)(a-c)}},$$

$$\delta y_0=0,$$

$$\delta z_0=\frac{-\frac{1}{2}\sqrt{c}}{\sqrt{(b-c)(a-c)}},$$

say these are α, β, γ, and α', β', γ'; and then

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = \tfrac{1}{4}\left\{\frac{a}{(a-b)(a-c)} + \frac{c}{(b-c)(a-c)}\right\}(v-u),$$

$$= \frac{\frac{1}{4}(v-u)}{a-c}\left\{\frac{a}{a-b} + \frac{c}{b-c}\right\}, \quad = \frac{\frac{1}{4}b(v-u)}{(a-b)(b-c)},$$

$$\alpha^2 + \beta^2 + \gamma^2 = \left\{\frac{a}{(a-b)(a-c)} + \frac{c}{(b-c)(a-c)}\right\}\cdot\tfrac{1}{4}(u-v)^2 + \frac{buv}{(a-b)(b-c)}$$

$$= \frac{b}{(a-b)(b-c)}\cdot\tfrac{1}{4}(u+v)^2,$$

$$\alpha'^2 + \beta'^2 + \gamma'^2 = \frac{b}{(a-b)(b-c)}\cdot\tfrac{1}{4},$$

whence

$$\sqrt{\alpha^2+\beta^2+\gamma^2}\sqrt{\alpha'^2+\beta'^2+\gamma'^2} = \frac{\frac{1}{4}(u+v)b}{(a-b)(b-c)},$$

and hence

$$\cos\phi = \frac{\alpha\alpha' + \beta\beta' + \gamma\gamma'}{\sqrt{\alpha^2+\beta^2+\gamma^2}\sqrt{\alpha'^2+\beta'^2+\gamma'^2}} = \frac{v-u}{v+u},$$

that is,

$$\cos(180^\circ - \phi) = \frac{u-v}{u+v}, \quad \text{or} \quad \tan^2\tfrac{1}{2}\phi = \frac{u}{v},$$

where if U is the umbilicus, P the consecutive point $-b-u$, $-b+v$, and UQ the element of the umbilicar principal section, $\phi = \angle PUQ$, $180^\circ - \phi = \angle PUQ'$. For the B-geodesic we have

$$2\log\tan\tfrac{1}{2}\phi_0 = \log\frac{u_0}{v_0} = M'.$$

17. The foregoing equation for $\Pi(h) - \Psi(k)$ now becomes

$$\Pi(h) - \Psi(k) = \sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{\tan^2\frac{1}{2}\phi_0}{\tan^2\frac{1}{2}\phi};$$

viz., ϕ_0 is the south azimuth of the B-geodesic at the umbilicus, a mere function of (a, b, c) and ϕ is the south azimuth at the umbilicus, of the geodesic line under consideration, so that we may consider the geodesic line to be determined by the south azimuth ϕ as its parameter.

Formulæ for the case $ac - b^2 = 0$.

18. I annex the following investigation in regard to the case $ac - b^2 = 0$.

We have in general

$$\frac{1}{\sqrt{b(a-b)(b-c)}}\frac{d}{dx}\log\frac{\sqrt{-x(a-b)(b-c)}+\sqrt{-b(a+x)(c+x)}}{\sqrt{-x(a-b)(b-c)}-\sqrt{-b(a+x)(c+x)}}$$

$$= -\frac{1}{b}\frac{1}{\sqrt{x(a+x)(c+x)}}$$

$$+\frac{1}{b}\frac{1}{b+x}\sqrt{\frac{x}{(a+x)(c+x)}}$$

$$+\frac{1}{bx+ac}\sqrt{\frac{x}{(a+x)(c+x)}}.$$

In fact, denoting the logarithm by $\log\frac{P+Q}{P-Q}$, we have

$$\frac{d}{dx}\log\frac{P+Q}{P-Q} = \frac{2(PQ'-P'Q)}{P^2-Q^2}$$

where

$$2(PQ'-P'Q) = 2PQ\left(\frac{Q'}{Q}-\frac{P'}{P}\right) = \sqrt{x(a+x)(c+x)b(a-b)(b-c)}\left\{\frac{1}{a+x}+\frac{1}{c+x}-\frac{1}{x}\right\}$$

$$= \frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}}(x^2-ac);$$

$$P^2-Q^2 = -x(a-b)(b-c)+b(a+x)(c+x)$$

$$= (bx+ac)(b+x);$$

that is

$$\frac{2(PQ'-P'Q)}{P^2-Q^2} = -\frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}}\frac{x^2-ac}{(bx+ac)(b+x)},$$

$$= \frac{\sqrt{b(a-b)(b-c)}}{\sqrt{x(a+x)(c+x)}}\left\{-\frac{1}{b}+\frac{x}{b(b+x)}+\frac{x}{bx+ac}\right\},$$

which proves the theorem.

19. Hence in the particular case $ac = b^2$ we have

$$\frac{1}{\sqrt{b(a-b)(b-c)}}\log\frac{\sqrt{-h(a-b)(b-c)}+\sqrt{-b(a+h)(c+h)}}{\sqrt{-h(a-b)(b-c)}-\sqrt{-b(a+h)(c+h)}},$$

$$= -\frac{1}{b}\int_{-a}^{h}\frac{dh}{\sqrt{h(a+h)(c+h)}}$$

$$-\frac{2}{b}\int_{-a}^{h}\frac{dh}{b+h}\sqrt{\frac{h}{(a+h)(c+h)}}\left(= +\frac{2}{b}\Pi(h)\right),$$

that is

$$\Pi(h)=\tfrac{1}{2}\int_{-a}^{h}\frac{dh}{\sqrt{h(a+h)(c+h)}}+\tfrac{1}{2}\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{\sqrt{-h(a-b)(b-c)}+\sqrt{-b(a+h)(c+h)}}{\sqrt{-h(a-b)(b-c)}-\sqrt{-b(a+h)(c+h)}},$$

or say

$$=\tfrac{1}{2}\int_{-a}^{h}\frac{dh}{\sqrt{h(a+h)(c+h)}}+\tfrac{1}{2}\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{1+H}{1-H},$$

where

$$H^2=\frac{b}{(a-b)(b-c)}\,\frac{(a+h)(c+h)}{h},$$

viz. we see that $\Pi(h)$ depends on the more simple integral

$$\int_{-a}^{h}\frac{dh}{\sqrt{h(a+h)(c+h)}}.$$

20. Similarly

$$\frac{1}{\sqrt{b(a-b)(b-c)}}\log\frac{\sqrt{-k(a-b)(b-c)}+\sqrt{-b(a+k)(c+k)}}{\sqrt{-k(a-b)(b-c)}-\sqrt{-b(a+k)(c+k)}}$$

$$=\frac{1}{b}\int_{k}^{-c}\frac{dk}{\sqrt{k(a+k)(c+k)}}$$

$$+\frac{2}{b}\int_{k}^{-c}\frac{dk}{b+k}\sqrt{\frac{k}{(a+k)(c+k)}}\left(=+\frac{2}{b}\Psi(k)\right),$$

that is

$$\Psi(k)=-\tfrac{1}{2}\int_{k}^{-c}\frac{dk}{\sqrt{k(a+k)(c+k)}}+\tfrac{1}{2}\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{\sqrt{-k(a-b)(b-c)}+\sqrt{-b(a+k)(c+k)}}{\sqrt{-k(a-b)(b-c)}-\sqrt{-b(a+k)(c+k)}},$$

or say

$$\Psi(k)=-\tfrac{1}{2}\int_{k}^{-c}\frac{dk}{\sqrt{k(a+k)(c+k)}}+\tfrac{1}{2}\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{1+K}{1-K},$$

where

$$K^2=\frac{b}{(a-b)(b-c)}\,\frac{(a+k)(c+k)}{k},$$

that is, $\Psi(k)$ depends on the more simple integral,

$$\int_{k}^{-c}\frac{dk}{\sqrt{k(a+k)(c+k)}}.$$

Write $h=-b-u$, $k=-b+v$, where u and v are indefinitely small, then

$$\Pi(h)-\Psi(k)=\tfrac{1}{2}\int_{-a}^{-c}\frac{dh}{\sqrt{h(a+h)(c+h)}}+\tfrac{1}{2}\sqrt{\frac{b}{(a-b)(b-c)}}\log\frac{1+U}{1-U}\,\frac{1-V}{1+V},$$

where

$$U^2 = \frac{\left(1 - \frac{u}{a-b}\right)\left(1 + \frac{u}{b-c}\right)}{1 + \frac{u}{b}} = 1 - \frac{bu^2}{(a-b)(b-c)(b+u)} \text{ (attending to } ac = b^2\text{)},$$

and

$$V^2 = \frac{\left(1 + \frac{v}{a-b}\right)\left(1 - \frac{v}{b-c}\right)}{1 - \frac{v}{b}} = 1 - \frac{bv^2}{(a-b)(b-c)(b-v)},$$

$$\Pi(h) - \Psi(k) = \tfrac{1}{2}\int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}} + \sqrt{\frac{b}{(a-b)(b-c)}} \log \frac{v}{u}.$$

21. Comparing with the result obtained for the general case the two agree, if only

$$\int_{-a}^{-c} \frac{dh}{b+h} \sqrt{\frac{h}{(a+h)(c+h)}} = \tfrac{1}{2}\int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}},$$

where on the left-hand side the integral has its principal value: a result which must therefore hold good when $ac = b^2$.

Calculation of the Umbilicar Geodesics for Ellipsoid $a : b : c = 4 : 2 : 1$.

22. As a specimen of the way in which we may, on a given ellipsoid, calculate the course of a geodesic line, I take the semiaxes to be as $2 : \sqrt{2} : 1$, or, for convenience, $a = 1000$, $b = 500$, $c = 250$; and, considering the geodesic lines through the umbilicus, I calculate by quadratures the functions

$$\Pi(h) = 100{,}000 \int_{-1000}^{h} \frac{-dh}{500+h} \sqrt{\frac{h}{(1000+h)(250+h)}},$$

$$\Psi(k) = 100{,}000 \int_{k}^{-250} \frac{dk}{500+k} \sqrt{\frac{k}{(1000+k)(250+k)}}.$$

The results do not pretend to minute accuracy: I have not attempted to estimate or correct for any error occasioned by the intervals (10 units) being too large; and there may possibly be accidental errors.

TABLE I.

$-h=$	Π'	$\Pi(h)$	$-h$	Π'	$\Pi(h)$	$-h$	Π'	$\Pi(h)$
1000	∞	0	840	27·6	6746	630	51·5	13972
999	231·4	462	830	27·8	7023	620	54·1	14499
998	164·0	659	820	27·9	7301	610	59·2	15066
997	134·2	809	810	28·1	7582	600	65·5	15689
996	116·5	934	800	28·4	7865	590	72·1	16377
995	104·4	1044	790	28·7	8151	580	80·9	17142
990	74·6	1492	780	29·2	8440	570	93·0	18011
980	54·0	2135	770	29·7	8735	560	106·8	19010
970	45·1	2630	760	30·3	9035	550	127·6	20183
960	40·1	3056	750	31·0	9341	540	159·1	21616
950	36·6	3439	740	31·8	9655	530	211·5	23469
940	34·2	3794	730	32·6	9977	520	316·7	26111
930	32·5	4127	720	33·6	10308	510	632·7	30858
920	31·2	4446	710	34·7	10650	505	1265·0	35602
910	30·2	4753	700	36·0	11004	504	1581·2	37014
900	29·4	5051	690	37·4	11371	503	2107·7	38834
890	28·8	5342	680	39·0	11754	502	3162·3	41398
880	28·4	5628	670	40·8	12153	501	6324·1	45792
870	28·1	5911	660	42·0	12567	500	∞	∞
860	27·9	6190	650	45·4	13005			
850	27·8	6469	640	48·2	13473			

TABLE II.

$-k$	Ψ'	$\Psi(k)$	$-k$	Ψ'	$\Psi(k)$	$-k$	Ψ'	$\Psi(k)$
250	∞	0	320	45·5	4655	440	107·1	12207
251	232·2	462	330	46·1	5114	450	127·9	13383
252	165·5	661	340	47·3	5581	460	159·2	14818
253	136·0	811	350	48·9	6062	470	211·6	16673
254	118·6	939	360	51·1	6562	480	316·7	19314
255	106·8	1051	370	53·8	7086	490	632·7	24062
260	78·1	1514	380	57·2	7641	495	1265·0	28806
270	60·5	2207	390	61·4	8235	496	1581·2	30218
280	51·7	2768	400	66·7	8875	497	2108·1	32037
290	48·2	3268	410	73·2	9575	498	3162·3	34602
300	46·3	3741	420	81·6	10349	499	6234·1	38995
310	45·5	4200	430	91·4	11214	500	∞	∞

23. But it is obviously convenient to revert these Tables so as to have for the common arguments a series of uniformly increasing values of Π or Ψ, viz., we obtain by interpolation the values of h and k belonging to the given values of Π or Ψ, and thus obtain the following Table. Here, in any line of the Table the values of h, k are such that $\Pi(h) - \Psi(k) = 0$, viz., the values in question belong to successive points of the B-geodesic. And to obtain the values for any other geodesic line $\Pi(h) - \Psi(k) = \pm 500\,m$, we have only to take each value of k from the line m lines above or below the line from which h is taken; and similarly the table gives at once the values belonging to a geodesic line $\Pi(h) + \Psi(k) = 500\,m$.

TABLE III.

$\Pi = \Psi =$	h	D.	k	D.	$\Pi = \Psi =$	h	D.	k	D.
0	1000		250		13000	650·1		446·7	
		1·2		1·4			20·6		7·8
500	998·8		251·4		14000	629·5		454·5	
		3·4		3·1			18·3		6·5
1000	995·4		254·5		15000	611·2		461·0	
		5·5		5·2			15·7		5·3
1500	989·9		259·7		16000	595·5		466·3	
		7·8		7·3			13·6		4·9
2000	982·1		267·0		17000	581·9		471·2	
		9·5		8·2			11·8		3·8
2500	972·6		275·2		18000	570·1		475·0	
		11·5		9·4			10·0		3·8
3000	961·1		284·6		19000	560·1		478·8	
		12·8		10·3			8·5		2·6
3500	948·3		294·9		20000	551·6		481·4	
		14·5		10·7			7·3		2·2
4000	933·8		305·6		21000	544·3		483·6	
		15·6		11·0			6·2		2·0
4500	918·2		316·6		22000	538·1		485·6	
		16·5		10·9			4·9		2·2
5000	901·7		327·5		23000	533·2		487·8	
		17·2		10·8			4·6		2·1
5500	884·5		338·3		24000	528·6		489·9	
		17·7		10·4			8·1		2·1
6000	866·8		348·7		26000	520·5		492·0	
		17·9		10·1			4·5		2·1
6500	848·9		358·8		28000	516·0		494·1	
		18·1		9·5			4·2		1·7
7000	830·8		368·3		30000	511·8		495·8	
		17·9		9·2			3·0		1·2
7500	812·9		377·5		32000	508·8		497·0	
		17·6		8·6			2·1		0·8
8000	795·3		386·1		34000	506·7		497·8	
		17·3		8·0			2·0		0·5
8500	778·0		394·1		36000	504·7		498·3	
		16·8		7·7			1·2		0·5
9000	761·2		401·8		38000	503·5		498·8	
		16·3		7·1			1·0		
9500	744·9		408·9		39000	..		499·0	
		15·6		6·6					
10000	729·3		415·5		40000	502·5			
		14·9		6·2			0·6		
10500	714·4		421·7		42000	501·9			
		14·3		5·9			0·5		
11000	700·1		427·6		44000	501·4			
		13·5		5·3			0·4		
11500	686·6		432·9		45800	501·0			
		12·8		5·0					
12000	673·8		437·9		∞	500		500	
		23·7		8·8					

Graphical Construction: Projection on the Umbilicar Plane.

24. The most convenient mode of delineation of the geodesic lines is obtained by projecting them orthogonally on the umbilicar plane: the contour of the figure is here the umbilicar section, or ellipse $\frac{x^2}{a}+\frac{z^2}{c}=1$; and the curves of curvature of each series are projected into elliptic arcs lying within the ellipse in question, the one set cutting at right angles the axes AA', the other cutting at right angles the axes CC'; the equations of the complete ellipses being

$$x^2\frac{a-b}{a(a+h)}+z^2\frac{c-b}{c(c+h)}-1=0$$

and

$$x^2\frac{a-b}{a(a+k)}+z^2\frac{c-b}{c(c+k)}-1=0.$$

25. I constructed, by means of the table, a drawing of this kind for the ellipsoid a, b, $c=1000$, 500, 250, the lengths $\sqrt{a}$ and $\sqrt{c}$ being taken to be 12 inches and 6 inches respectively: the process consists in taking from the table for a series of values $\Pi=\Psi$ (say $\Pi=\Psi=1000$, $=2000$ &c.), the values of h and k, laying down for such values the elliptic arcs which represent the two curves of curvature respectively, thus dividing the bounding ellipse into a series of curvilinear rectangles, and then obtaining the geodesic lines by drawing the diagonals of these rectangles, and of course rounding off the corners so as to form continuous curves. The Plate shows on a reduced scale so much of the drawing as is comprised within a quadrant of the bounding ellipse (viz. it is a representation of an octant of the ellipsoid).

Elliptic-Function Formulæ.

26. I have in all that precedes abstained from the use of elliptic functions, since obviously the form $\sqrt{1-k^2\sin^2\phi}$ of the radical of an elliptic function is in nowise specially appropriate to the present question. But (more particularly in the above-mentioned case $ac-b^2=0$, where the radical is $\sqrt{h(a+h)(c+h)}$ without any exterior factor $b+h$ in the denominator) the formulæ are expressible easily and elegantly by elliptic functions, and it is desirable to make the transformation. Reverting to the formulæ which, in the case in question (viz. when $ac-b^2=0$), give the values of $\Pi(h)$ and $\Psi(k)$; and writing therein $h=-a+(a-c)\sin^2\phi$, $k=-a+(a-c)\sin^2\psi$, also

$$\kappa=\sqrt{1-\frac{c}{a}},\ \text{or}\ \frac{c}{a}=1-\kappa^2,\ =\kappa'^2,$$

we have

$$\int_{-a}^{h}\frac{dh}{\sqrt{h(a+h)(c+h)}}=2\int_0^{\phi}\frac{d\phi}{\sqrt{a-(a-c)\sin^2\phi}}=\frac{2}{\sqrt{a}}F(\kappa,\ \phi),$$

$$\int_{k}^{-c}\frac{dk}{\sqrt{k(a+k)(c+k)}}=2\int_{\psi}^{\frac{1}{2}\pi}\frac{d\psi}{\sqrt{a-(a-c)\sin^2\psi}}=\frac{2}{\sqrt{a}}\left\{F_{,}(\kappa)-F(\kappa,\ \psi)\right\}.$$

27. Hence

$$\Pi(h) = \frac{1}{\sqrt{a}} F(\kappa, \phi) + \frac{1}{2\sqrt{a(1-\kappa')}} \log \frac{1+H}{1-H},$$

where

$$H = \frac{(1-\kappa')\sin\phi\cos\phi}{\sqrt{1-\kappa^2\sin^2\phi}};$$

(observe, as h passes from $-a$ to $-b$, ϕ passes from $\phi=0$ to $\sin^2\phi = \frac{1}{1+\kappa}$ and H from $H=0$ to $H=1$).

Similarly

$$\Psi(k) = \frac{1}{\sqrt{a}}\left\{F_{,}(\kappa) - F(\kappa, \psi)\right\} + \frac{1}{2\sqrt{a(1-\kappa')}} \log \frac{1+K}{1-K},$$

where

$$K = \frac{(1+\kappa')\sin\psi\cos\psi}{\sqrt{1-\kappa^2\sin^2\psi}},$$

and as k passes from $-c$ to $-b$, ψ passes from $\frac{1}{2}\pi$ to $\sin^2\psi = \frac{1}{1+\kappa}$, and K from 0 to 1.

28. The before-mentioned identical equation

$$\int_{-a}^{-c} \frac{dh}{b+h}\sqrt{\frac{h}{(a+h)(c+h)}} = \tfrac{1}{2}\int_{-a}^{-c} \frac{dh}{\sqrt{h(a+h)(c+h)}}$$

is by the same transformation converted into

$$\int_0^{\frac{1}{2}\pi} \frac{1-(1-\kappa')\sin^2\phi}{1-(1+\kappa')\sin^2\phi} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = 0.$$

To prove this, I remark that the equation is

$$0 = \int_{-0}^{\frac{1}{2}\pi} d\phi \frac{\frac{1-\kappa'}{1+\kappa'}\{1-(1+\kappa')\sin^2\phi\} + \frac{2\kappa'}{1+\kappa'}}{1-(1+\kappa')\sin^2\phi} \frac{1}{\Delta\phi},$$

viz. this is

$$0 = \frac{1-\kappa'}{1+\kappa'} F_{,} + \frac{2\kappa}{1+\kappa'} \Pi_{,}(-1-\kappa'),$$

or, what is the same thing,

$$\Pi_{,}(-1-\kappa') = -\frac{1-\kappa'}{2\kappa} F_{,},$$

where $\Pi_{,}(-1-\kappa')$ denotes the principal value of the integral

$$\int_0^{\frac{1}{2}\pi} d\phi \frac{1}{1-(1+\kappa')\sin^2\phi} \frac{1}{\Delta\phi}.$$

Now (Leg. *Fonct. Ellip.*, t. I., p. 71), we have

$$\Pi_{,}(-\kappa^2 \sin^2\theta) + \Pi_{,}\left(-\frac{1}{\sin^2\theta}\right) = F_{,},$$

where, upon examination, it will appear that $\Pi_{,}\left(-\frac{1}{\sin^2\theta}\right)$ in fact represents the principal value of the integral.

Writing herein $\sin^2\theta = \frac{1}{1-\kappa'}$, and therefore $\cos^2\theta = \frac{\kappa'}{1+\kappa'}$, or $\tan^2\theta = \kappa'$, this is

$$\Pi_{,}(-1+\kappa') + \Pi_{,}(-1-\kappa'), = F_{,},$$

and the formula (p'), p. 141, attributing therein to θ the foregoing value, becomes

$$\Pi_{,}(-1-\kappa') = E_{,} + \frac{1}{\kappa'}\left\{F_{,}E(\theta) - E_{,}F(\theta)\right\}.$$

But θ is the value for the bisection of the function $F_{,}$, viz., we have

$$2F(\theta) = F_{,},$$

$$2E(\theta) = E_{,} + 1 - \kappa',$$

whence

$$F_{,}E(\theta) - E_{,}F(\theta) = \tfrac{1}{2}(1-\kappa')F_{,},$$

or the formula in question gives

$$\Pi_{,}(-1+\kappa') = \frac{1+\kappa'}{2\kappa'}E_{,},$$

whence

$$\Pi_{,}(-1-\kappa') = -\frac{1-\kappa'}{2\kappa'}F_{,},$$

the result which was to be proved.

29. The value of M' $\left(\text{observing that } \frac{b}{(a-b)(b-c)} = \frac{1}{(\sqrt{a}-\sqrt{c})^2} = \frac{1}{a(1-\kappa')^2}\right)$ is

$$\frac{1}{\sqrt{a}(1-\kappa')}M' = \tfrac{1}{2}\int_{-a}^{-c}\frac{dh}{\sqrt{h(a+h)(c+h)}},$$

which is

$$= \tfrac{1}{2}\frac{2}{\sqrt{2}}F_{,}(\kappa),$$

that is we have

$$M' = (1-\kappa')F_{,}(\kappa),$$

or, what is the same thing,

$$\log\tan\tfrac{1}{2}\phi_0 = \frac{1-\kappa'}{2}F_{,}(\kappa),$$

that is

$$\tan\tfrac{1}{2}\phi_0 = \left(\frac{1-\kappa'}{2}F_{,}(\kappa)\right)$$

(ϕ_0 the South azimuth of the B-geodesic at the umbilicus).

30. I purposely calculated the Table by quadratures as being a method available where the equation $ac - b^2 = 0$ is not satisfied; but in the present case, where this

equation is satisfied, the table might have been calculated from Legendre's Tables of Elliptic Integrals. Observe that $a = 1000$, $b = 500$, $c = 250$, gives $\kappa = \frac{1}{2}\sqrt{3}$ or angle of modulus $= 60°$. As an instance of the comparison[1], suppose $h = -800$, then $\sin^2\phi = \frac{200}{750} = \frac{4}{15}$, $\log\sin\phi = 9{\cdot}71298$, $\phi = 31°\,5'$.

$$\sqrt{\frac{b}{(a-b)(b-c)}} = \sqrt{\frac{500}{500\,.\,250}} = \frac{\sqrt{10}}{50} = {\cdot}06326,$$

$$H^2 = \frac{500\,.\,200\,.\,550}{800\,.\,500\,.\,250} = \frac{110}{200},\ \log = \bar{1}{\cdot}87018,$$

$$H = {\cdot}7416\,\frac{1+H}{1-H} = \frac{1{\cdot}7416}{2{\cdot}584} = 6{\cdot}7582,$$

$$\Pi(h) = {\cdot}03163\,F(31°\,5') + {\cdot}03163\ \text{h. l. } 6{\cdot}7582,$$

$$\begin{array}{rcr}
F\,31° & = & {\cdot}56166 \\
 & & 163 \\
\hline
F\,31°\,5' & = & {\cdot}56329 \\
\text{h. l. } 6{\cdot}7582 & = & 1{\cdot}91075 \\
\hline
 & & 2{\cdot}47404 \\
\times \text{ by} & & {\cdot}03163 \\
\hline
 & & {\cdot}0782043
\end{array}$$

or multiplying by 100,000 (factor introduced into my Table) this is $= 7820{\cdot}43$. The value $\Pi(-800) = 7864$ given by my Table agrees sufficiently well with this, the correct value.

31. I calculate also the angle ϕ_0, viz. we have

$$\text{h. l. } \tan\tfrac{1}{2}\phi_0 = \frac{1-\kappa'}{2}\,F_{,}\kappa, \quad = \tfrac{1}{4}\,F_{,}(60°). \qquad \text{Leg. vol. III. Table VIII.}$$

$$= \tfrac{1}{4}\,2{\cdot}15651 = {\cdot}53913,$$

whence by Leg. Table IV.

$$\tfrac{1}{2}\phi_0 = 45° + \tfrac{1}{2}\,.\,29°\,29'{\cdot}64$$

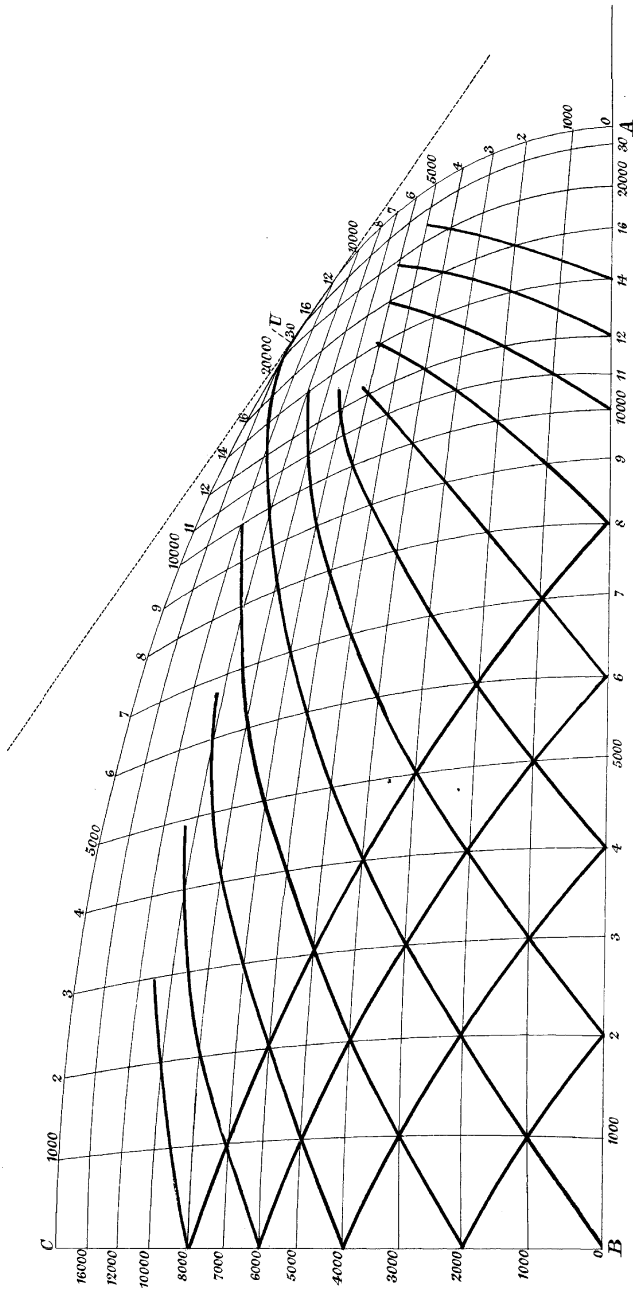
$$= 59°\,44'{\cdot}82$$

or

$$\phi_0 = 119°\,29'{\cdot}64.$$

This exceeds 90°, and since at the umbilicus the tangent plane is at right angles to the plane of projection, the B-geodesic should in the drawing proceed (as it in fact does) from U in the sense UC, touching the bounding ellipse at the point U.

[1] In the present calculation, log denotes an ordinary logarithm, the hyperbolic logarithm being distinguished as h. l.


A
B
C
U
0
1000
2000
3000
4000
5000
6000
7000
8000
10000
12000
16000
20000
30000
40000
50000
1
2
3
4
5
6
7
8
9
11
12
14
16
18
20
30
40

479.

THE SECOND PART OF A MEMOIR ON THE DEVELOPMENT OF THE DISTURBING FUNCTION IN THE LUNAR AND PLANETARY THEORIES.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXXIX. (1872), pp. 55—74. Read January 12, 1872.]

THE present communication is a sequel to my paper, "The First Part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories," *Memoirs R.A.S.*, vol. XXVIII. (1859), pp. 187—215, [214], and I have therefore entitled it as above, but it, in fact, relates only to the Planetary Theory. In the First Part, I gave in effect, but not explicitly, an expression for the general coefficient $D(j, j')$ in terms of the coefficients of the multiple cosines of θ in the expansions of the several powers $(r^2+r'^2-2rr'\cos\theta)^{-s-\frac{1}{2}}$, or say $(a^2+a'^2-2aa'\cos\theta)^{-s-\frac{1}{2}}$; viz., at the foot of page 208 I speak of the term involving $\cos(jU+j'U')$ as having a certain given value; the term in question is $D(j, j')\cos(jU+j'U')$; and consequently the expression for $D(j, j')$ is

$$D(j, j') = \Sigma \frac{\Pi_1(x-\frac{1}{2})}{\Pi x} \eta^{2x} \Sigma M_x^2 R_x^2;$$

the omission was, however, a material one, inasmuch as this expression for the general coefficient serves to connect my formulæ with Leverrier's development, *Annales de l'Observ. de Paris*, t. I. (1855), pp. 275—330 and 358—383, and I resume the question for the purpose of applying it.

Formula for the general Coefficient $D(j, j')$.

In the First Part, the reciprocal of the distance of the two planets, or function

$$\{r^2+r'^2-2rr'(\cos U\cos U'+\sin U\sin U'\cos\Phi)\}^{-\frac{1}{2}}$$

is taken to be developed in multiple cosines of U, U', the general term being

$$D(j, j')\cos(jU+j'U'),$$

where j, j' have each of them any integer value from $-\infty$ to $+\infty$ (zero not excluded), but so that j, j' are simultaneously even or simultaneously odd. We have $D(-j, -j') = D(j, j')$ and $D(j', j) = D(j, j')$; and it hence appears that the really distinct values of the coefficient may be taken to be those for which j is not negative, and as regards absolute magnitude is not less than j'; and for such values of j, j' we have the above-mentioned expression

$$D(j, j') = \Sigma \frac{\Pi_1(x-\frac{1}{2})}{\Pi x} \eta^{2x} \Sigma M_x^{\mathfrak{s}} R_x^{\mathfrak{s}},$$

which I proceed to explain and develope.

$\Pi_1(x-\frac{1}{2})$ and Πx (x being a positive integer) denote respectively $\frac{1}{2}.\frac{3}{2}\ldots(x-\frac{1}{2})$, and $1.2.3\ldots x$; in particular for $x=0$, the value of each factorial is $=1$.

η denotes $\sin\frac{1}{2}\Phi$.

The coefficients $R_x^{\mathfrak{s}}$ are those of the multiple cosines in certain developments, viz. we have

$$r^x r'^x \{r^2 + r'^2 - 2rr'\cos(U-U')\}^{-x-\frac{1}{2}} = \Sigma R_x^i \cos i(U-U'),$$

where, as usual, i extends from $-\infty$ to ∞ and $R_x^{-i} = R_x^i$. Writing with Leverrier

$$(a^2 + a'^2 - 2aa'\cos H)^{-\frac{1}{2}} = \tfrac{1}{2}\Sigma A^i \cos iH,$$

$$aa'(a^2 + a'^2 - 2aa'\cos H)^{-\frac{3}{2}} = \tfrac{1}{2}\Sigma B^i \cos iH,$$

$$a^2a'^2(a^2 + a'^2 - 2aa'\cos H)^{-\frac{5}{2}} = \tfrac{1}{2}\Sigma C^i \cos iH,$$

$$a^3a'^3(a^2 + a'^2 - 2aa'\cos H)^{-\frac{7}{2}} = \tfrac{1}{2}\Sigma D^i \cos iH,$$

then $2R_0^i$, $2R_1^i$, $2R_2^i$, $2R_3^i$ are the same functions of r, r' that A^i, B^i, C^i, D^i respectively are of a, a'.

The expression of $M_x^{\mathfrak{s}}$ is

$$M_x^{\mathfrak{s}} = (-)^{x-\frac{1}{2}(j+j')} \frac{\Pi x}{\Pi\frac{1}{2}(x-j-\mathfrak{S})\,\Pi\frac{1}{2}(x+j'+\mathfrak{S})} \; \frac{\Pi x}{\Pi\frac{1}{2}(x-j+\mathfrak{S})\,\Pi\frac{1}{2}(x+j'-\mathfrak{S})};$$

and, finally, in the expression for $D(j, j')$, x has every integer value from 0 to ∞, and, for any given value of x, $\mathfrak{S}$ extends by steps of two units from the inferior value $-(x-j')$ to the superior value $x-j$.

It is convenient to write $x = \frac{1}{2}(j+j') + s$; we have then $\mathfrak{S}$ extending from $-\frac{1}{2}(j-j') - s$ to $-\frac{1}{2}(j-j') + s$, or writing $\mathfrak{S} = -\frac{1}{2}(j-j') + \theta$, θ has the $s+1$ values s, $s-2$, $s-4, \ldots -s$, viz. for $s = 2p+1$ the values are ± 1, $\pm 3, \ldots \pm(2p+1)$, and for $s = 2p$ they are 0, ± 2, $\pm 4, \ldots \pm 2p$.

Making these changes we have

$$D(j, j') = \Sigma \frac{\Pi_1 \{\frac{1}{2}(j+j')+s-\frac{1}{2}\}}{\Pi \{\frac{1}{2}(j+j')+s\}} \eta^{j+j'+2s} \Sigma M^{-\frac{1}{2}(j-j')+\theta}_{\frac{1}{2}(j+j')+s} R^{-\frac{1}{2}(j-j')+\theta}_{\frac{1}{2}(j+j')+s},$$

where

$$M^{-\frac{1}{2}(j-j')+\theta}_{\frac{1}{2}(j+j')+s} = (-)^s \frac{\Pi \{\frac{1}{2}(j+j')+s\}}{\Pi \frac{1}{2}(s-\theta) \Pi \frac{1}{2}(j+j'+s+\theta)} \frac{\Pi \{\frac{1}{2}(j+j')+s\}}{\Pi \frac{1}{2}(s+\theta) \Pi \frac{1}{2}(j+j'+s-\theta)},$$

viz. this is $(-)^s$ into the product of two binomial coefficients, each belonging to the exponent $\frac{1}{2}(j+j')+s$.

Particular Cases, $j+j'=0, 2, 4, 6$, being those required in the Planetary Theory.

Considering successively the cases $j+j'=0, 2, 4, 6$, we have, first,

$$D(j, -j) = \Sigma \frac{\Pi_1 (s-\frac{1}{2})}{\Pi s} \eta^{2s} \Sigma (-)^s \left\{ \frac{\Pi s}{\Pi \frac{1}{2}(s-\theta) \Pi \frac{1}{2}(s+\theta)} \right\}^2 R_s^{-j+\theta}$$

which, developed as far as η^6, is

$$(*) \qquad \begin{aligned} D(j, -j) = & \quad \tfrac{1}{2} A^{-j} \\ & - \frac{1}{2} \eta^2 \tfrac{1}{2}(B^{-j+1} + B^{-j-1}) \\ & + \frac{1.3}{2.4} \eta^4 \tfrac{1}{2}(C^{-j+2} + 4C^{-j} + C^{-j-2}) \\ & - \frac{1.3.5}{2.4.6} \eta^6 \tfrac{1}{2}(D^{-j+3} + 9D^{-j+1} + 9D^{-j-1} + D^{-j-3}), \end{aligned}$$

where, and in what immediately follows, A, B, C, D are used to denote functions (not of (a, a'), but) of r, r'.

Secondly,

$$D(j, -j+2) = \Sigma \frac{\Pi_1 (s+\frac{1}{2})}{\Pi (s+1)} \eta^2 \Sigma \eta^{2s} \left\{ (-)^s \frac{\Pi (s+1)}{\Pi \frac{1}{2}(s-\theta) \Pi \frac{1}{2}(s+\theta)+1} \times \frac{\Pi (s+1)}{\Pi \frac{1}{2}(s+\theta) \Pi \frac{1}{2}(s-\theta)+1} R_{s+1}^{-j+1+\theta} \right\},$$

which, developed to η^6, is

$$(*) \qquad \begin{aligned} D(j, -j+2) = \eta^2 \Big\{ & \frac{1}{2} \cdot \tfrac{1}{2} B^{-j+1} \\ & - \frac{1.3}{2.4} \eta^2 \cdot \tfrac{1}{2}(2C^{-j+2} + 2C^{-j}), \\ & + \frac{1.3.5}{2.4.6} \eta^4 \cdot \tfrac{1}{2}(3D^{-j+3} + 9D^{-j+1} + 3D^{-j-1}) \Big\}. \end{aligned}$$

Thirdly,

$$D(j,\ -j+4)=\Sigma\frac{\Pi_1(s+\frac{3}{2})}{\Pi(s+2)}\eta^4\,.\,\Sigma\eta^{2s}(-)^s\left\{\frac{\Pi(s+2)}{\Pi\frac{1}{2}(s-\theta)\,\Pi\frac{1}{2}(s+\theta)+2}\right.$$
$$\left.\times\frac{\Pi(s+2)}{\Pi\frac{1}{2}(s+\theta)\,\Pi\frac{1}{2}(s-\theta)+2}R^{-j+2\theta}_{s+2}\right\},$$

which, developed to η^6, is

$$(*)\qquad D(j,\ -j+4)=\eta^4\left\{\frac{1\,.\,3}{2\,.\,4}\cdot\tfrac{1}{2}C^{-j+2}\right.$$
$$\left.-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\eta^2\cdot\tfrac{1}{2}(3D^{-j+3}+3D^{-j+1})\right\}:$$

and, fourthly,

$$D(j,\ -j+6)=\Sigma\frac{\Pi_1(s+\frac{5}{2})}{\Pi(s+3)}\eta^6\,\Sigma\eta^{2s}(-)^s\left\{\frac{\Pi(s+3)}{\Pi\frac{1}{2}(s-\theta)\,\Pi\frac{1}{2}(s+\theta)+3}\right.$$
$$\left.\times\frac{\Pi(s+3)}{\Pi\frac{1}{2}(s+\theta)\,\Pi\frac{1}{2}(s-\theta)+3}R^{-j+3+\theta}_{s+3}\right\},$$

which, developed to η^6, is simply

$$(*)\qquad D(j,\ -j+6)=\eta^6\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\tfrac{1}{2}D^{-j+3}.$$

The foregoing formulæ, although obtained on the supposition $j=0$, or positive, apply without alteration to the case $j=$ negative, and the entire series of terms of an order not exceeding 6 as regards η may be written,

$$\begin{aligned}
&D(j,\ -j) && \cos(jU-jU')\\
&+2D(j,\ -j+2) && \cos\big(jU+(-j+2)\,U'\big)\\
&+2D(j,\ -j+4) && \cos\big(jU+(-j+4)\,U'\big)\\
&+2D(j,\ -j+6) && \cos\big(jU+(-j+6)\,U'\big),
\end{aligned}$$

where j has every integer value from $-\infty$ to $+\infty$.

Comparison with LEVERRIER.

This is in fact what Leverrier's expression becomes on putting therein $e=e'=0$. To verify this, observe that Leverrier having defined his A^i, B^i, C^i, D^i, as above, writes further

$$\begin{aligned}
E^i &= \tfrac{1}{2}(B^{i-1}+B^{i+1}),\\
G^i &= \tfrac{3}{8}(C^{i-2}+4C^i+C^{i+2}),\\
H^i &= \tfrac{5}{16}(D^{i-3}+9D^{i+1}+9D^{i+1}+D^{i+3}),\\
L^i &= \tfrac{3}{4}(C^{i-2}+C^i),\\
S^i &= \tfrac{15}{16}(D^{i-3}+3D^{i-1}+D^{i+1}),\\
T^i &= \tfrac{15}{16}(D^{i-3}+D^{i-1}),
\end{aligned}$$

(consequently $E^{-i}=E^i$, $G^{-i}=G^i$, $H^{-i}=H^i$, $L^{-i+2}=L^i$, $S^{-i+2}=S^i$, $T^{-i+4}=T^i$), and that the terms in question, putting in the coefficients $e=e'=0$, are with him

$$\begin{aligned}
&\{(1)^i+(11)^i\eta^2+(17\ \ )^i\eta^4+(20\ \ )^i\eta^6\} && \cos(il'-i\lambda),\\
&\{(212)^i\eta^2+(218)^i\eta^4+(221)^i\eta^6\} && \cos[il'-(i-2)\lambda-2\tau],\\
&\{(372)^i\eta^4+(375)^i\eta^6\} && \cos[il'-(i-4)\lambda-4\tau],\\
&\{(449)^i\eta^6\} && \cos[il'-(i-6)\lambda-6\tau],
\end{aligned}$$

where, substituting for $(1)^i$, $(11)^i$, &c., their values, the coefficients are

$$\begin{aligned}
&\tfrac{1}{2}A^i-\eta^2\tfrac{1}{2}E^i+\eta^4\cdot\tfrac{1}{2}G^i-\eta^6\tfrac{1}{2}H^i,\\
&=\tfrac{1}{2}A^i-\eta^2\cdot\tfrac{1}{4}(B^{i-1}+B^{i+1})+\eta^4\cdot\tfrac{3}{16}(C^{i-2}+4C^i+C^{i+2})-\eta^6\cdot\tfrac{5}{32}(D^{i-3}+9D^{i-1}+9D^{i+1}+D^{i+3});\\
&\eta^2\cdot\tfrac{1}{2}B^{i-1}-\eta^4\cdot L^i+\eta^6S^i,\ =\eta^2\cdot\tfrac{1}{2}B^{i+1}-\eta^4(\tfrac{3}{4}C^{i-2}+C^i)+\eta^6\cdot\tfrac{15}{16}(D^{i-3}+3D^{i-1}+D^{i+1});\\
&\eta^4\cdot\tfrac{3}{8}C^{i-2}-\eta^6T^i,\ =\eta^4\cdot\tfrac{3}{8}C^{i-2}-\eta^6\cdot\tfrac{15}{16}(D^{i-3}+D^{i-1});
\end{aligned}$$

and

$$\eta^6\cdot\tfrac{5}{16}D^{i-3}.$$

Writing herein j in place of i, and for A^j, B^{j-1}, &c., the equal values A^{-j}, B^{-j+1}, &c., we have precisely the foregoing coefficients $D(j,\ -j),\ldots D(j,\ -j+6)$.

The Development in Powers of e, e'.

The complete expression of the reciprocal of the distance is obtained from

$$\begin{aligned}
&D(j,\ -j) && \cos(jU-jU')\\
&+2D(j,\ -j+2) && \cos\big(jU+(-j+2)U'\big)\\
&+2D(j,\ -j+4) && \cos\big(jU+(-j+4)U'\big)\\
&+2D(j,\ -j+6) && \cos\big(jU+(-j+6)U'\big),
\end{aligned}$$

by writing therein for r, r', U, U', instead of the circular, the elliptic values, that is the values

$$\begin{aligned}
r &= a\ \mathrm{elqr}(e,\ L-\Pi) &&, =a(1+x),\\
r' &= a'\ \mathrm{elqr}(e',\ L'-\Pi') &&, =a'(1+x'),\\
U &= \Pi-\Theta+\mathrm{elta}(e,\ L-\Pi), && =\Pi-\Theta+f,\\
U' &= \Pi'-\Theta'+\mathrm{elta}(e',\ L'-\Pi'), && =\Pi'-\Theta'+f';
\end{aligned}$$

L, Π, Θ the mean longitude in orbit, longitude of perihelion in orbit, and longitude of node; and the like for L', Π', Θ'; "elqr" = elliptic quotient radius, "elta" = elliptic true anomaly; or, what is the same thing, if we write $\mathrm{elta}(e,\ L-\Pi)=L-\Pi+\mathrm{eltt}(e,\ L-\Pi)$, and the like for $\mathrm{elta}(e',\ L'-\Pi')$, then

$$\begin{aligned}
U &= L-\Theta+\mathrm{eltt}(e,\ L-\Pi), && =L-\Theta+y,\\
U' &= L'-\Theta'+\mathrm{eltt}(e',\ L'-\Pi'), && =L'-\Theta'+y'.
\end{aligned}$$

The process for doing this is explained, First Part, pp. 205—207, [214], viz., writing $r = a(1+x)$, $r' = a'(1+x')$, and restoring j' (instead of its value $-j, \ldots -j+6$, as the case may be), we have a general term

$$\frac{1}{\Pi\alpha\Pi\alpha'} a^\alpha a'^{\alpha'} \left(\frac{d}{da}\right)^\alpha \left(\frac{d}{da}\right)^{\alpha'} . D(j, j') . x^\alpha x'^{\alpha'} \cos[j(\Pi - \Theta + f) + j'(\Pi' - \Theta' + f')],$$

where $D(j, j')$ now denotes the value obtained by writing a, a' in place of r, r' and f, f' are the true anomalies elta $(e, L - \Pi)$ and elta $(e', L' - \Pi')$. And the second factor, $x^\alpha x'^{\alpha'}$ into the cosine, is given as a series

$$\Sigma\Sigma([\cos]^i + [\sin]^i)([\cos]^{i'} + [\sin]^{i'}) \cos[i(L - \Pi) + i'(L' - \Pi') + j(\Pi - \Theta) - j'(\Pi' - \Theta')],$$

where $[\cos]^i$, $[\sin]^i$ are functions of e, $[\cos]^{i'}$, $[\sin]^{i'}$ functions of e'. Or, what is better, the term $x^\alpha x'^{\alpha'}$ into the cosine may be written $x^\alpha x'^{\alpha'} \cos[j(L - \Theta + y) + j'(L' - \Theta' + y')]$, and the expansion then is

$$\Sigma\Sigma([\cos]^i + [\sin]^i)([\cos]^{i'} + [\sin]^{i'}) \cos[i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta')],$$

where as before $[\cos]^i$, $[\sin]^i$ are functions of e, $[\cos]^{i'}$, $[\sin]^{i'}$ are the same functions of e', viz. the e-functions are those given in the two "datum-tables" $(x^0 \ldots x^7) \cos jy$ and $(x^0 \ldots x^7) \sin jy$, taken from Leverrier, which I have given in my "Tables of the Developments of Functions in the Theory of Elliptic Motion," *Memoirs R.A.S.* vol. XXIX. (1861), pp. 191—306, [216]. In order to better show which are the symbols referred to, we may, instead of $[\cos]^i$, &c., write $[x^\alpha \cos jy]^i$, &c., the formula will then be

$$\begin{aligned} x^\alpha x'^{\alpha'} \cos[j(L - \Theta + y) + j'(L' - \Theta' + y')] = \\ \Sigma\Sigma([x^\alpha \cos jy]^i + [x^\alpha \sin jy]^i)([x'^{\alpha'} \cos j'y']^{i'} + [x'^{\alpha'} \sin j'y']^{i'}) \\ \times \cos[i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta')]; \end{aligned}$$

and if we attribute to i, i' any given values, that is, attend to any particular multiple cosine,

$$\cos[i(L - \Pi) + i'(L' - \Pi') + j(L - \Theta) + j'(L' - \Theta')],$$

the coefficient hereof will be

$$\Sigma \frac{1}{\Pi\alpha\Pi\alpha'} a^\alpha \left(\frac{d}{da}\right)^\alpha a'^{\alpha'} \left(\frac{d}{da}\right)^{\alpha'} D(j, j') . ([x^\alpha \cos jy]^i + [x^\alpha \sin jy]^i)([x'^{\alpha'} \cos j'y']^{i'} + [x'^{\alpha'} \sin j'y']^{i'}),$$

where α, α' each extend from zero to infinity, but to obtain the expression up to a given order p in e, e', we take only the values up to $\alpha + \alpha' = p$.

Particular Case.

Thus, for instance, in $\cos[j(L - \Theta) - j'(L' - \Theta')]$ the terms independent of e' are

$$\begin{aligned} & D(j, -j)\{[x^0 \cos jy]^0 + [x^0 \sin jy]^0\} \\ + \frac{1}{1} a \left(\frac{d}{da}\right) & D(j, -j')\{[x' \cos jy]^0 + [x' \sin jy]^0\} \\ + \frac{1}{1.2} a^2 \left(\frac{d}{da}\right)^2 & D(j, -j)\{[x^2 \cos jy]^0 + [x^2 \sin jy]^0\}, \\ & + \&c. \end{aligned}$$

which, observing that in the present case the sine terms vanish, is

1	$\frac{e^2}{8}$	$\frac{e^4}{384}$	$\frac{e^6}{46080}$			
1	$-8j^2$	$+96j^4$	$-1280j^6$			
		$-54j^2$	$+3920j^4$	1	.	$D(j, -j)$
			$-3440j^2$			
	$+4$	$-48j^2$	$-360j^2$	$\frac{1}{1}\ a\frac{d}{da}$	.	,,
	$+4$	$-96j^2$	$+1920j^4$			
			$-1320j^2$	$\frac{1}{1.2}\ a^2\left(\frac{d}{da}\right)^2$	.	,,
		$+144$	$-2880j^2$	$\frac{1}{1.2.3}\ a^3\left(\frac{d}{da}\right)^3$	.	,,
		$+144$	$-5760j^2$	$\frac{1}{1.2.3.4}\ a^4\left(\frac{d}{da}\right)^4$	.	,,
			$+14400$	$\frac{1}{1.2..5}\ a^5\left(\frac{d}{da}\right)^5$	.	,,
			$+14400$	$\frac{1}{1.2..6}\ a^6\left(\frac{d}{da}\right)^6$	.	,,
			0	$\frac{1}{1.2..7}\ a^7\left(\frac{d}{da}\right)^7$	.	,,

viz. the term in e^2 is

$$e^2\left\{-j^2+\tfrac{1}{2}\,a\frac{d}{da}+\tfrac{1}{4}\,a^2\left(\frac{d}{da}\right)^2\right\}D(j,\ -j):$$

viz. writing $\eta=0$, and therefore $D(j,\ -j)=\frac{1}{2}A^{-j}$, the term in e^2 is

$$e^2\left\{-j^2+\tfrac{1}{2}\,a\frac{d}{da}+\tfrac{1}{4}\,a^2\left(\frac{d}{da}\right)^2\right\}\tfrac{1}{2}A^{-j},$$

which, conformably with Leverrier's subscript notation

$$A_1^{\,i}=\frac{1}{1}\,a\frac{d}{da}A^i,\ A_2^{\,i}=\frac{1}{1.2}\,a^2\left(\frac{d}{da}\right)^2A^i,\ \&c.,$$

I write

$$e^2\{-j^2+\tfrac{1}{2}(\quad)_1+\tfrac{1}{4}2(\quad)_2\}\tfrac{1}{2}A^{-j}=e^2\{-\tfrac{1}{2}j^2A^{-j}+\tfrac{1}{4}A_1^{-j}+\tfrac{1}{4}A_2^{-j}\}.$$

The term in question is given by Leverrier as $(\frac{1}{2}e)^2(2)^i$, $=e^2.\frac{1}{4}(2)$, $h=i$ and $K^i=A^i$, $=e^2.\frac{1}{4}(-2i^2A^i+A_1^i+A_2^i)$, which agrees.

Similarly the term in e^4 is

$$\frac{e^4}{384}\{96j^4-54j^2-48j^2(\quad)_1-96j^2(\quad)_2+144(\quad)_3+144(\quad)_4\}\tfrac{1}{2}A^{-j},$$

$$=\frac{e^4}{768}\{(96j^4-54j^2)A^{-j}-48j^2A_1^{-j}-96j^2A_2^{-j}+144A_3^{-j}+144A_4^{-j}\},$$

and the term in question is given by Leverrier as $(\frac{1}{2}e)^4(4)^i=e^4.\frac{1}{16}(4)$, $h=i$ and $K^i=A^i$,

$$=e^4\tfrac{1}{16}\{\tfrac{1}{8}(-9i^2+16i^4)A^i-i^2A_1^i-2i^2A_2^i+3A_3^i+3A_4^i\},$$

which agrees. I have not made the comparison of any more terms.

LEVERRIER'S *Results expressed in terms of the Arguments*, $L'-\Theta'$, $L'-\Pi'$, $L-\Theta$, $L-\Pi$.

The angles which Leverrier uses in his arguments are l', λ, ω, ϖ', and τ', viz. we have,

$$\begin{aligned}
l' &= \Theta'+(L'-\Theta'),\\
\lambda &= \Theta'+(L-\Theta),\\
\varpi' &= \Theta'+(\Pi'-\Theta'),\\
\omega &= \Theta'+(\Pi-\Theta),\\
\tau' &= \Theta',
\end{aligned}$$

where L, Π, Θ are the mean longitude of the planet m, its perihelion and the mutual node, all in the orbit of m; and similarly L', Π', Θ' are the mean longitude of the planet m', of its perihelion and of the mutual node, all in the orbit of m'. On substituting the foregoing values of l', λ, &c., Θ', as it should do, disappears, and the arguments are all of them linear functions of $L'-\Theta'$, $\Pi'-\Theta'$, $L-\Theta$, $\Pi-\Theta$; or, if we please, of $L'-\Theta'$, $L'-\Pi'$, $L-\Theta$, $L-\Pi$, that is of the distances of each planet from its own perihelion and from the mutual node. It is, I think, convenient to use these last angular distances, and accordingly in Leverrier's arguments, I write,

$$\begin{aligned}
l' &= \Theta'+(L'-\Theta'),\\
\lambda &= \Theta' \quad . \quad . \quad . \quad . \quad +(L-\Theta),\\
\varpi' &= \Theta'+(L'-\Theta')-(L'-\Pi'),\\
\omega &= \Theta' \quad . \quad . \quad . \quad . \quad . \quad +(L-\Theta)-(L-\Pi),\\
\tau' &= \Theta',
\end{aligned}$$

and for the purpose of reference form as it were an Index to his result as follows:

Reciprocal of Distance = as follows:

Terms of order zero: terms of orders 2, 4, 6, *having the same arguments.*

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(1)^i$	(1 .. 20) ..	cos	i	0	$-i$	0
$(21)^i(\frac{1}{2}e)(\frac{1}{2}e')$	(21 .. 30) ..	,,	i	$+1$	$-i$	-1
$(31)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^2$	(31 .. 34) ..	,,	i	$+2$	$-i$	-2
$(35)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^3$	(35 .. 35) ..	,,	i	$+3$	$-i$	-3
$(36)^i(\frac{1}{2}e)^2\eta^2$	(36 .. 39) ..	,,	i	0	$-i+2$	-2
$(40)^i(\frac{1}{2}e)(\frac{1}{2}e')\eta^2$	(40 .. 43) ..	,,	i	-1	$-i+2$	-1
$(44)^i(\frac{1}{2}e')^2\eta^2$	(44 .. 47) ..	,,	i	-2	$-i+2$	0
$(48)^i(\frac{1}{2}e)^3(\frac{1}{2}e')\eta^2$	(48 .. 48) ..	,,	i	$+1$	$-i+2$	-3
$(49)^i(\frac{1}{2}e)(\frac{1}{2}e')^3\eta^2$	(49 .. 49) ..	,,	i	-3	$-i+2$	$+1$

Terms of the first order: terms of orders 3, 5, 7, *having the same arguments.*

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(50)^i\ \frac{1}{2}e$	(50 .. 69) ..	cos	i	0	$-i$	$+1$
$(70)^i\ \frac{1}{2}e'$	(70 .. 89) ..	,,	i	$+1$	$-i$	0
$(90)^i(\frac{1}{2}e)^2(\frac{1}{2}e')$	(90 .. 99) ..	,,	i	$+1$	$-i$	-2
$(100)^i(\frac{1}{2}e)(\frac{1}{2}e')^2$	(100 .. 109) ..	,,	i	$+2$	$-i$	-1
$(110)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2$	(110 .. 113) ..	,,	i	$+2$	$-i$	-3
$(114)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3$	(114 .. 117) ..	,,	i	$+3$	$-i$	-2
$(118)^i(\frac{1}{2}e)^4(\frac{1}{2}e')^3$	(118 .. 118) ..	,,	i	$+3$	$-i$	-4
$(119)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^4$	(119 .. 119) ..	,,	i	$+4$	$-i$	-3
$(120)^i(\frac{1}{2}e)\eta^2$	(120 .. 129) ..	,,	i	0	$-i+2$	-1
$(130)^i(\frac{1}{2}e')\eta^2$	(130 .. 139) ..	,,	i	-1	$-i+2$	0
$(140)^i(\frac{1}{2}e)^3\eta^2$	(140 .. 143) ..	,,	i	0	$-i+2$	-3
$(144)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\eta^2$	(144 .. 147) ..	,,	i	$+1$	$-i+2$	-2
$(148)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\eta^2$	(148 .. 151) ..	,,	i	-1	$-i+2$	-2
$(152)^i(\frac{1}{2}e)(\frac{1}{2}e')^2\eta^2$	(152 .. 155) ..	,,	i	-2	$-i+2$	-1
$(156)^i\ \frac{1}{2}e\ (\frac{1}{2}e')^2\eta^2$	(156 .. 159) ..	,,	i	-2	$-i+2$	$+1$
$(160)^i(\frac{1}{2}e')^3\eta^2$	(160 .. 163) ..	,,	i	-3	$-i+2$	0
$(164)^i(\frac{1}{2}e)^4(\frac{1}{2}e')\eta^2$	(164 .. 164) ..	,,	i	$+1$	$-i+2$	-4
$(165)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2\eta^2$	(165 .. 165) ..	,,	i	$+2$	$-i+2$	-3
$(166)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3\eta^2$	(166 .. 166) ..	,,	i	-3	$-i+2$	$+2$
$(167)^i(\frac{1}{2}e)(\frac{1}{2}e')^4\eta^2$	(167 .. 167) ..	,,	i	-4	$-i+2$	$+1$
$(168)^i(\frac{1}{2}e)^3\eta^4$	(168 .. 168) ..	,,	i	0	$-i+2$	-3
$(169)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\eta^4$	(169 .. 169) ..	,,	i	-1	$-i+4$	-2
$(170)^i(\frac{1}{2}e)(\frac{1}{2}e')^2\eta^4$	(170 .. 170) ..	,,	i	-2	$-i+4$	-1
$(171)^i(\frac{1}{2}e')^3\eta^4$	(171 .. 171) ..	,,	i	-3	$-i+4$	0

Terms of second order: terms of orders 4, 6, having the same arguments.

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(172)^i(\frac{1}{2}e)^2$	(172 .. 181) ..	cos	i	0	$-i$	$+2$
$(182)^i(\frac{1}{2}e)(\frac{1}{2}e')$	(182 .. 191) ..	,,	i	$+1$	$-i$	$+1$
$(192)^i(\frac{1}{2}e')^2$	(192 .. 201) ..	,,	i	$+2$	$-i$	0
$(202)^i(\frac{1}{2}e)^3(\frac{1}{2}e')$	(202 .. 205) ..	,,	i	$+1$	$-i$	-3
$(206)^i(\frac{1}{2}e)(\frac{1}{2}e')^3$	(206 .. 209) ..	,,	i	$+3$	$-i$	-1
$(210)^i(\frac{1}{2}e)^4(\frac{1}{2}e')^2$	(210 .. 210) ..	,,	i	$+2$	$-i$	-4
$(211)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^4$	(211 .. 211) ..	,,	i	$+4$	$-i$	-2
$(212)^i\eta^2$	(212 .. 221) ..	,,	i	0	$-i+2$	0
$(222)^i(\frac{1}{2}e)(\frac{1}{2}e')\eta^2$	(222 .. 225) ..	,,	i	$+1$	$-i+2$	-1
$(226)^i(\frac{1}{2}e)(\frac{1}{2}e')\eta^2$	(226 .. 229) ..	,,	i	-1	$-i+2$	$+1$
$(230)^i(\frac{1}{2}e)^4\eta^2$	(230 .. 230) ..	,,	i	0	$-i+2$	-4
$(231)^i(\frac{1}{2}e)^3(\frac{1}{2}e')\eta^2$	(231 .. 231) ..	,,	i	-1	$-i+2$	-3
$(232)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^2\eta^2$	(232 .. 232) ..	,,	i	-2	$-i+2$	-2
$(233)^i(\frac{1}{2}e)(\frac{1}{2}e')^3\eta^2$	(233 .. 233) ..	,,	i	-3	$-i+2$	-1
$(234)^i(\frac{1}{2}e')^4\eta^2$	(234 .. 234) ..	,,	i	-4	$-i+2$	0
$(235)^i(\frac{1}{2}e')^2(\frac{1}{2}e')^2\eta^2$	(235 .. 235) ..	,,	i	$+2$	$-i+2$	-2
$(236)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^2\eta^2$	(236 .. 236) ..	,,	i	-2	$-i+2$	$+2$
$(237)^i(\frac{1}{2}e)^2\eta^4$	(237 .. 237) ..	,,	i	0	$-i+4$	-2
$(238)^i(\frac{1}{2}e)(\frac{1}{2}e')\eta^4$	(238 .. 238) ..	,,	i	-1	$-i+4$	-1
$(239)^i(\frac{1}{2}e')^2\eta^4$	(239 .. 239) ..	,,	i	-2	$-i+4$	0

Terms of third order: terms of orders 5, 7, having the same arguments.

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(240)^i(\frac{1}{2}e)^3$	(240 .. 249) ..	cos	i	0	$-i$	$+3$
$(250)^i(\frac{1}{2}e)^2(\frac{1}{2}e')$	(250 .. 259) ..	,,	i	$+1$	$-i$	$+2$
$(260)^i(\frac{1}{2}e)(\frac{1}{2}e')^2$	(260 .. 269) ..	,,	i	$+2$	$-i$	$+1$
$(270)^i(\frac{1}{2}e')^3$	(270 .. 279) ..	,,	i	$+3$	$-i$	0

Terms of third order (concluded):

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(280)^i(\frac{1}{2}e)^4(\frac{1}{2}e')$	(280 .. 283) ..	cos	i	$+1$	$-i$	-4
$(284)^i(\frac{1}{2}e)(\frac{1}{2}e')^4$	(284 .. 287) ..	,,	i	$+4$	$-i$	-1
$(288)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2$	(288 .. 289) ..	,,	i	$+2$	$-i$	-5
$(290)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3$	(290 .. 299) ..	,,	i	0	$-i+2$	$+1$
$(300)^i(\frac{1}{2}e')\eta^2$	(300 .. 309) ..	,,	i	$+1$	$-i+2$	0
$(310)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\eta^2$	(310 .. 313) ..	,,	i	-1	$-i+2$	$+2$
$(314)^i(\frac{1}{2}e)(\frac{1}{2}e')^2\eta^2$	(314 .. 317) ..	,,	i	$+2$	$-i+2$	-1
$(318)^i(\frac{1}{2}e)^3\eta^2$	(318 .. 318) ..	,,	i	0	$-i+2$	-5
$(319)^i(\frac{1}{2}e)^4(\frac{1}{2}e')\eta^2$	(319 .. 319) ..	,,	i	-1	$-i+2$	-4
$(320)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2\eta^2$	(320 .. 320) ..	,,	i	-2	$-i+2$	-3
$(321)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3\eta^2$	(321 .. 321) ..	,,	i	-3	$-i+2$	-2
$(322)^i(\frac{1}{2}e)(\frac{1}{2}e')^4\eta^2$	(322 .. 322) ..	,,	i	-4	$-i+2$	-1
$(323)^i(\frac{1}{2}e')^5\eta^2$	(323 .. 323) ..	,,	i	-5	$-i+2$	0
$(324)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2\eta^2$	(324 .. 324) ..	,,	i	-2	$-i+2$	$+3$
$(325)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3\eta^2$	(325 .. 325) ..	,,	i	$+3$	$-i+2$	-2
$(326)^i(\frac{1}{2}e)\eta^4$	(326 .. 329) ..	,,	i	0	$-i+4$	-1
$(330)^i(\frac{1}{2}e')\eta^4$	(330 .. 333) ..	,,	i	-1	$-i+4$	0
$(334)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\eta^4$	(334 .. 334) ..	,,	i	$+1$	$-i+4$	-2
$(335)^i(\frac{1}{2}e)(\frac{1}{2}e')^2\eta^4$	(335 .. 335) ..	,,	i	-2	$-i+4$	$+1$

Terms of fourth order: terms of order 6, *and of same argument.*

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(336)^i(\frac{1}{2}e)^4$	(336 .. 339) ..	cos	i	0	$-i$	$+4$
$(340)^i(\frac{1}{2}e)^3(\frac{1}{2}e')$	(340 .. 343) ..	,,	i	$+1$	$-i$	$+3$
$(344)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^2$	(344 .. 347) ..	,,	i	$+2$	$-i$	$+2$
$(348)^i(\frac{1}{2}e)(\frac{1}{2}e')^3$	(348 .. 351) ..	,,	i	$+3$	$-i$	$+1$
$(352)^i(\frac{1}{2}e')^4$	(352 .. 355) ..	,,	i	$+4$	$-i$	0
$(356)^i(\frac{1}{2}e)^5(\frac{1}{2}e')$	(356 .. 356) ..	,,	i	$+1$	$-i$	-5
$(357)^i(\frac{1}{2}e)(\frac{1}{2}e')^5$	(357 .. 357) ..	,,	i	$+5$	$-i-1$	0

Terms of fourth order (concluded):

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(358)^i\,(\frac{1}{2}e)^2\,\eta^2$	(358 .. 358) ..	cos	i	0	$-i+2$	$+2$
$(362)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')\,\eta^2$	(362 .. 364) ..	,,	i	$+1$	$-i+2$	$+1$
$(366)^i\,(\frac{1}{2}e')^2\,\eta^2$	(366 .. 369) ..	,,	i	$+2$	$-i+2$	0
$(370)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')\,\eta^2$	(370 .. 370) ..	,,	i	-1	$-i+2$	$+3$
$(371)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^3\,\eta^2$	(371 .. 371) ..	,,	i	$+3$	$-i+2$	-1
$(372)^i\,\eta^4$	(372 .. 375) ..	,,	i	0	$-i+4$	0
$(376)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')\,\eta^4$	(376 .. 376) ..	,,	i	$+1$	$-i+4$	-1
$(377)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')\,\eta^4$	(377 .. 377) ..	,,	i	-1	$-i+4$	$+1$

Terms of fifth order: terms of order 7 having the same arguments.

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(378)^i\,(\frac{1}{2}e)^5$	(378 .. 381) ..	cos	i	0	$-i$	$+5$
$(382)^i\,(\frac{1}{2}e)^3\,(\frac{1}{2}e')$	(382 .. 385) ..	,,	i	$+1$	$-i$	$+4$
$(386)^i\,(\frac{1}{2}e)^3\,(\frac{1}{2}e')^2$	(386 .. 389) ..	,,	i	$+2$	$-i$	$+3$
$(390)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')^3$	(390 .. 393) ..	,,	i	$+3$	$-i$	$+2$
$(394)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^4$	(394 .. 397) ..	,,	i	$+4$	$-i$	$+1$
$(398)^i\,(\frac{1}{2}e')^5$	(398 .. 401) ..	,,	i	$+5$	$-i$	0
$(402)^i\,(\frac{1}{2}e)^6\,(\frac{1}{2}e')$	(402 .. 402) ..	,,	i	$+1$	$-i$	-6
$(403)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^6$	(403 .. 403) ..	,,	i	$+6$	$-i$	-1
$(404)^i\,(\frac{1}{2}e)^3\,\eta^2$	(404 .. 407) ..	,,	i	0	$-i+2$	$+3$
$(408)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')\,\eta^2$	(408 .. 411) ..	,,	i	$+1$	$-i+2$	$+2$
$(412)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^2\,\eta^2$	(412 .. 415) ..	,,	i	$+2$	$-i+2$	$+1$
$(416)^i\,(\frac{1}{2}e')^3\,\eta^2$	(416 .. 419) ..	,,	i	$+3$	$-i+2$	0
$(420)^i\,(\frac{1}{2}e)^4\,(\frac{1}{2}e')\,\eta^2$	(420 .. 420) ..	,,	i	-1	$-i+2$	$+4$
$(421)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^4\,\eta^2$	(421 .. 421) ..	,,	i	$+4$	$-i+2$	-1
$(422)^i\,(\frac{1}{2}e)\,\eta^4$	(422 .. 425) ..	,,	i	0	$-i+4$	$+1$
$(426)^i\,(\frac{1}{2}e')\,\eta^4$	(426 .. 429) ..	,,	i	$+1$	$-i+4$	0
$(430)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')\,\eta^2$	(430 .. 430) ..	,,	i	-1	$-i+4$	$+2$
$(431)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^2\,\eta^2$	(431 .. 431) ..	,,	i	$+2$	$-i+4$	-1
$(432)^i\,(\frac{1}{2}e)\,\eta^6$	(432 .. 432) ..	,,	i	0	$-i+6$	-1
$(433)^i\,(\frac{1}{2}e')\,\eta^4$	(433 .. 433) ..	,,	i	-1	$-i+6$	0

Terms of sixth order.

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(434)^i\,(\frac{1}{2}e)^6$	(434 .. 434) ..	cos	i	0	$-i$	$+6$
$(435)^i\,(\frac{1}{2}e)^5\,(\frac{1}{2}e')$	(435 .. 435) ..	,,	i	$+1$	$-i$	$+5$
$(436)^i\,(\frac{1}{2}e)^4\,(\frac{1}{2}e')^2$	(436 .. 436) ..	,,	i	$+2$	$-i$	$+4$
$(437)^i\,(\frac{1}{2}e)^3\,(\frac{1}{2}e')^3$	(437 .. 437) ..	,,	i	$+3$	$-i$	$+3$
$(438)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')^4$	(438 .. 438) ..	,,	i	$+4$	$-i$	$+2$
$(439)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^5$	(439 .. 439) ..	,,	i	$+5$	$-i$	$+1$
$(440)^i\,(\frac{1}{2}e')^6$	(440 .. 440) ..	,,	i	$+6$	$-i$	0
$(441)^i\,(\frac{1}{2}e)^4\,\eta^2$	(441 .. 441) ..	,,	i	0	$-i+2$	$+4$
$(442)^i\,(\frac{1}{2}e)^3\,(\frac{1}{2}e')\,\eta^2$	(442 .. 442) ..	,,	i	$+1$	$-i+2$	$+3$
$(443)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')^2\,\eta^2$	(443 .. 443) ..	,,	i	$+2$	$-i+2$	$+2$
$(444)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^3\,\eta^2$	(444 .. 444) ..	,,	i	$+3$	$-i+2$	$+1$
$(445)^i\,(\frac{1}{2}e')^4\,\eta^2$	(445 .. 445) ..	,,	i	$+4$	$-i+2$	0
$(446)^i\,(\frac{1}{2}e)^2\,\eta^4$	(446 .. 446) ..	,,	i	0	$-i+4$	$+2$
$(447)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')\,\eta^4$	(447 .. 447) ..	,,	i	$+1$	$-i+4$	$+1$
$(448)^i\,(\frac{1}{2}e')^2\,\eta^4$	(448 .. 448) ..	,,	i	$+2$	$-i+4$	0
$(449)^i\,\eta^6$	(449 .. 449) ..	,,	i	0	$-i+6$	0

Terms of seventh order.

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(450)^i\,(\frac{1}{2}e)^7$	(450 .. 450) ..	cos	i	0	$-i$	$+7$
$(451)^i\,(\frac{1}{2}e)^6\,(\frac{1}{2}e')^2$	(451 .. 451) ..	,,	i	1	$-i$	$+6$
$(452)^i\,(\frac{1}{2}e)^5\,(\frac{1}{2}e')^2$	(452 .. 452) ..	,,	i	2	$-i$	$+5$
$(453)^i\,(\frac{1}{2}e)^4\,(\frac{1}{2}e')^3$	(453 .. 453) ..	,,	i	3	$-i$	$+4$
$(454)^i\,(\frac{1}{2}e)^3\,(\frac{1}{2}e')^4$	(454 .. 454) ..	,,	i	4	$-i$	$+3$
$(455)^i\,(\frac{1}{2}e)^2\,(\frac{1}{2}e')^5$	(455 .. 455) ..	,,	i	5	$-i$	$+2$
$(456)^i\,(\frac{1}{2}e)\,(\frac{1}{2}e')^6$	(456 .. 456) ..	,,	i	6	$-i$	$+1$
$(457)^i\,(\frac{1}{2}e')^7$	(457 .. 457) ..	,,	i	7	$-i$	0
$(458)^i\,(\frac{1}{2}e)^5\,\eta^2$	(458 .. 458) ..	,,	i	0	$-i+2$	$+5$

Terms of seventh order (concluded):

			$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$(459)^i(\frac{1}{2}e)^4(\frac{1}{2}e')\,\eta^2$	(459 .. 459) ..	cos	i	$+1$	$-i+2$	$+4$
$(460)^i(\frac{1}{2}e)^3(\frac{1}{2}e')^2\eta^2$	(460 .. 460) ..	,,	i	$+2$	$-i+2$	$+3$
$(461)^i(\frac{1}{2}e)^2(\frac{1}{2}e')^3\eta^2$	(461 .. 461) ..	,,	i	$+3$	$-i+2$	$+2$
$(462)^i(\frac{1}{2}e)\,(\frac{1}{2}e')^4\eta^2$	(462 .. 462) ..	,,	i	$+4$	$-i+2$	$+1$
$(463)^i(\frac{1}{2}e')^5\eta^2$	(463 .. 463) ..	,,	i	$+5$	$-i+2$	0
$(464)^i(\frac{1}{2}e)^3\eta^4$	(464 .. 464) ..	,,	i	0	$-i+4$	$+3$
$(465)^i(\frac{1}{2}e)^2(\frac{1}{2}e')\,\eta^4$	(465 .. 465) ..	,,	i	$+1$	$-i+4$	$+2$
$(466)^i(\frac{1}{2}e)\,(\frac{1}{2}e')^2\eta^4$	(466 .. 466) ..	,,	i	$+2$	$-i+4$	$+1$
$(467)^i(\frac{1}{2}e')^3\eta^4$	(467 .. 467) ..	,,	i	$+3$	$-i+4$	0
$(468)^i(\frac{1}{2}e)\,\eta^6$	(468 .. 468) ..	,,	i	0	$-i+6$	$+1$
$(469)^i(\frac{1}{2}e')\,\eta^6$	(469 .. 469) ..	,,	i	$+1$	$-i+6$	0

Here the several coefficients are ultimately given in terms of the before-mentioned quantities A^i, B^i, C^i, D^i, E^i, G^i, H^i, L^i, S^i, T^i (functions of a, a'), and their differential coefficients in regard to a

$$\left(A_1{}^i=\frac{1}{1}\,a\,\frac{d}{da}A^i,\quad A_2{}^i=\frac{1}{1\,.\,2}\,a\,\frac{d^2}{da^2}A^i,\ \&c.\right),$$

as follows:—we have Leverrier, pp. 299—330, a list of functions (1), (2), ... (154) of the form $(1)=\frac{1}{2}K^i$, $(2)=-2h^2K^i+K_1{}^i+K_2{}^i$, $(3)=-2i^2K^i+K_1{}^i+K_2{}^i$, &c., involving i, h, and K^i and its derived functions $K_1{}^i$, $K_2{}^i$, &c. The coefficients of the several cosines are given by means of the functions in question, thus, first coefficient, above denoted as $(1)^i(1\ldots 20)$, is

$$=(1)^i+(2)^i(\tfrac{1}{2}e)+(3)^i(\tfrac{1}{2}e')\ldots+(20)^i\eta^6$$

where $(1)^i=(1)$, $(2)^i=(2)\ldots$ writing in the functions (1), (2) ... (10), $h=i$, and $K^i=A^i$;

$(11)^i=(1)$, $(12)^i=(2)$, &c., writing $h=i$ and $K^i=-E^i$,

$(20)^i=(1)$, writing $h=i$ and $K^i=-H^i$,

and so on for the various component coefficients $(1)^i$, $(2)^i\ldots(469)^i$.

But the resulting expressions, for the several integer values $i=-10$ to $+10$, are worked out in the Addition II. (*Numerical Tables for the Calculation of the Coefficients of the Development of the Disturbing Function*), pp. 358—383. And this Addition contains also, indicated by the letters δ and Δ respectively, the expressions of the

terms which experience an alteration in passing from the development of the reciprocal of the distance to those of the disturbing functions m' upon m, and m upon m' respectively.

We have

Disturbing Function m' upon m

$$= m' \left\{ -\frac{r \cos H}{r'^2} + \frac{1}{\rho} \right\}.$$

Disturbing Function m upon m'

$$= m \left\{ -\frac{r' \cos H}{r^2} + \frac{1}{\rho} \right\}.$$

The expressions of $-\frac{r \cos H}{r'}$ and $-\frac{r' \cos H}{r^2}$, developed to the third order in the eccentricities and inclination, are given, Leverrier, pp. 272 and 274. Expressed in the terms of the foregoing arguments $L' - \Theta'$, &c., and in terms of a, a' in place of a and α, these are as follows:

$-\frac{r \cos H}{r'^2} = \frac{a}{a'^2}$ into		$L' - \Theta'$	$L' - \Pi'$	$L - \Theta$	$L - \Pi$
$-1 + \frac{1}{2}(e^2 + e'^2) + \eta^2$	cos	1	0	-1	0
$-ee'$	,,	$+1$	$+1$	-1	-1
$+\frac{3}{2}e - \frac{3}{4}ee'^2 - \frac{3}{2}e\eta^2$	,,	$+1$	0	-1	$+1$
$-\frac{1}{2}e + \frac{1}{4}ee'^2 + \frac{3}{8}e^3 + \frac{1}{2}e\eta^2$	,,	$+1$	0	-1	-1
$-2e' + e^2e' + \frac{3}{2}e'^3 + 2e'\eta^2$	,,	$+1$	$+1$	-1	0
$-\frac{3}{4}e^2e'$	,,	$+1$	$+1$	-1	-2
$+\frac{3}{16}ee'^2$	,,	-1	$+2$	$+1$	-1
$-\frac{27}{16}ee'^2$	,,	$+1$	$+2$	-1	-1
$+\frac{3}{2}e\eta^2$	,,	$+1$	0	$+1$	-1

$-\frac{r\cos H}{r'^2}=\frac{a}{a'^2}$ into		$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$-\frac{1}{8}e^2$	cos	+1	0	−1	+2
$-\frac{3}{8}e^2$	,,	+1	0	−1	−2
$+3\,ee'$	,,	+1	+1	−1	+1
$-\frac{1}{8}e'^2$	,,	−1	+2	+1	0
$-\frac{27}{8}e'^2$	,,	+1	−2	−1	0
$-\eta^2$	,,	+1	0	+1	0
$-\frac{1}{24}e^3$	,,	+1	0	−1	+3
$-\frac{1}{3}e^3$	,,	+1	0	−1	−3
$-\frac{1}{4}e^2e'$	,,	+1	+1	−1	+2
$-\frac{1}{16}ee'^2$	,,	−1	+2	+1	+1
$+\frac{81}{16}ee'^2$	,,	+1	+2	−1	+1
$-\frac{1}{6}e'^3$	,,	−1	+3	+1	0
$-\frac{16}{3}e'^3$	,,	+1	+3	−1	0
$-\frac{1}{2}e\eta^2$	,,	+1	0	+1	+1
$-2\,e'\eta^2$	,,	+1	+1	0	+1

$-\frac{r'\cos H}{r^2}=\frac{a'}{a^2}$ into		$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$-1+\frac{1}{2}(e^2+e'^2)+\eta^2$	cos	1	0	−1	0
$-ee'$	,,	+1	+1	−1	−1
$-2\,e+e\,e'^2+\frac{3}{2}e^3+2\,e\eta^2$	,,	+1	0	−1	−1
$+\frac{3}{2}e'-\frac{3}{4}e^2e'-\frac{3}{2}e'\eta^2$	,,	−1	+1	+1	0
$-\frac{1}{2}e'+\frac{1}{4}e^2e'+\frac{3}{8}e'^3+\frac{1}{2}e'\eta^2$	,,	+1	+1	−1	0
$+\frac{5}{16}e^2e'$	,,	+2	−1	−2	+2
$-\frac{27}{16}e^2e'$	,,	+1	+1	−1	−2
$-\frac{3}{4}ee'^2$	,,	+1	+2	−1	−1
$+\frac{3}{2}e'\eta^2$	,,	+1	−1	+1	0
$-\frac{1}{8}e^2$	,,	+1	0	−1	+2
$-\frac{27}{8}e^2$	,,	+1	0	−1	−2
$+3\,ee'$	,,	−1	+1	+1	+1
$-\frac{1}{8}e'^2$	,,	−1	+2	+1	0

$-\frac{r'\cos H}{r^2} = \frac{a}{a^2}$ into		$L'-\Theta'$	$L'-\Pi'$	$L-\Theta$	$L-\Pi$
$-\frac{3}{8}e'^2$	cos	+1	+2	−1	0
$-\eta^2$	,,	+1	0	+1	0
$-\frac{1}{6}e^3$	,,	+1	0	−1	+3
$-\frac{16}{3}e^3$	,,	+1	0	−1	−3
$+\frac{81}{16}e^2e'$	,,	−1	+1	+1	+2
$-\frac{1}{16}e^2e'$	,,	+1	+1	−1	+2
$-\frac{1}{4}ee'^2$	,,	−1	+2	+1	+1
$-\frac{1}{24}e'^3$	,,	−1	+3	+1	0
$-\frac{1}{3}e'^3$	,,	+1	+3	−1	0
$-2e\eta^2$	,,	+1	0	+1	+1
$-\frac{1}{2}e'\eta^2$	,,	+1	+1	+1	0

It is hardly necessary to observe that, to obtain the expressions of the Disturbing Functions, these additional terms are to be combined with the corresponding terms in the expression of the reciprocal of the distance: thus, in the Disturbing Function Ω (m' upon m), the entire term depending on $\cos[L'-\Theta'-(L-\Theta)]$ is

$$= m'\left\{2(1, \ldots 20)_{i=1} + \frac{a}{a'^2}\left(-1+\tfrac{1}{2}(e^2+e'^2)+\eta^2\right)\right\}\cos[(L'-\Theta')-(L-\Theta)],$$

where, however, the supplemental term is taken to the third order only.

480.

ON THE EXPRESSION OF DELAUNAY'S l, g, h, IN TERMS OF HIS FINALLY ADOPTED CONSTANTS.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXII. (1871—72), pp. 8—16.]

WE have in Delaunay's lunar theory,

l, the mean anomaly of the Moon,

g, the mean distance of perigee from ascending node,

h, the mean longitude of ascending node,

quantities which vary directly as the time, the coefficients of t, or values of $\frac{dl}{dt}$, $\frac{dg}{dt}$, $\frac{dh}{dt}$, being given in his *Théorie du Mouvement de la Lune*, vol. II. pp. 237, 238. But these values are not expressed in terms of his constants a (or n), e, γ, finally adopted as explained p. 800, and it seems very desirable to obtain the expressions of l, g, h, in terms of these finally adopted constants: I have accordingly effected this transformation (which I found less laborious than I had anticipated). It will be convenient to imagine the a, n, e, γ of pp. 237, 238 replaced by A, N, E, Γ respectively. This being so, and writing m for the $\frac{n}{n'}$ of p. 800 we have, p. 800,

$$\begin{aligned}
A = a \Big\{ 1 &+ \left[-\tfrac{3}{4} m^2 - \tfrac{825}{256} m^3 \right] \frac{a^2}{a'^2} \\
&+ \left(-\tfrac{2}{3} + 3\gamma^2 - \tfrac{3}{4} e^2 - e'^2 - 2\gamma^2 + \tfrac{5}{2} \gamma^2 e^2 + \tfrac{9}{2} \gamma^2 e'^2 - \tfrac{1}{16} e^4 - \tfrac{9}{8} e^2 e'^2 - \tfrac{5}{4} e'^4 \right) m^2 \\
&+ \left(-\tfrac{9}{4} \gamma^2 - \tfrac{225}{16} e^2 + \tfrac{45}{8} \gamma^4 + \tfrac{81}{2} \gamma^2 e^2 - \tfrac{23}{4} \gamma^2 e'^2 + \tfrac{675}{128} e^4 - \tfrac{825}{16} e^2 e'^2 \right) m^3 \\
&+ \left(\tfrac{1705}{288} - \tfrac{1529}{64} \gamma^2 - \tfrac{14639}{256} e^2 + \tfrac{7469}{192} e'^2 \right) m^4 \\
&+ \left(\tfrac{787}{48} - \tfrac{9323}{256} \gamma^2 - \tfrac{227555}{1024} e^2 + \tfrac{7083}{32} e'^2 \right) m^5 \\
&+ \tfrac{5887}{162} m^6 \\
&+ \tfrac{29809}{432} m^7,
\end{aligned}$$

and hence calculating N from the formula $N^2A^3 = n^2a^3$, we find

$$\begin{aligned}
N = n\Big\{1 &+ \left[\tfrac{9}{8}m^2 + \tfrac{2475}{512}m^3\right]\frac{a^2}{a'^2} \\
&+ (1 - \tfrac{9}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{3}{8}e'^2 + 3\gamma^4 - \tfrac{15}{4}\gamma^2e^2 - \tfrac{27}{4}\gamma^2e'^2 + \tfrac{3}{32}e^4 + \tfrac{27}{16}e^2e'^2 + \tfrac{15}{8}e'^4)\,m \\
&+ (\tfrac{27}{8}\gamma^2 + \tfrac{675}{32}e^2 - \tfrac{135}{16}\gamma^4 - \tfrac{243}{4}\gamma^2e^2 + \tfrac{69}{8}\gamma^2e'^2 - \tfrac{2025}{256}e^4 + \tfrac{2475}{32}e^2e'^2)\,m^3 \\
&+ (-\tfrac{515}{64} + \tfrac{3627}{128}\gamma^2 + \tfrac{44877}{512}e^2 - \tfrac{7149}{128}e'^2)\,m^4 \\
&+ (-\tfrac{787}{32} + \tfrac{30849}{512}\gamma^2 + \tfrac{754665}{2048}e^2 - \tfrac{21249}{64}e'^2)\,m^5 \\
&+ (-\tfrac{13183}{192})\,m^6 \\
&+ (-\tfrac{20807}{144})\,m^7,
\end{aligned}$$

$= n(1 + Q)$ suppose.

The values of E, Γ are given p. 800, but for the present purpose we only require E^2, and Γ^2 to the fifth order, viz. the values of these are at once found to be

$$E^2 = e^2\left(1 + \tfrac{81}{64}m^2 - \tfrac{2595}{128}m^3\right),$$
$$\Gamma^2 = \gamma^2\left(1 + \tfrac{57}{64}m^2 - \tfrac{129}{128}m^3\right),$$

whence also $E^4 = e^4$ and $\Gamma^4 = \gamma^4$.

The formulæ of pp. 237—238 now give

$$\begin{aligned}
l = nt\Big\{1 &+ \left[-\tfrac{81}{32}m^2 - \tfrac{2475}{512}m^3\right]\frac{a^2}{a'^2} + Q \\
&+ \left\{\begin{array}{l}
(-\tfrac{7}{4} + \tfrac{21}{2}\gamma^2 - \tfrac{3}{4}e^2 - \tfrac{21}{8}e'^2 + \tfrac{33}{4}\gamma^4 - \tfrac{39}{8}\gamma^2e^2 + \tfrac{63}{4}\gamma^2e'^2 - \tfrac{9}{8}e^2e'^2 - \tfrac{105}{32}e'^4)\,m^2 \\
+(\tfrac{1197}{128}\gamma^2 - \tfrac{243}{256}e^2)\,m^4 \\
+(-\tfrac{2709}{256}\gamma^2 + \tfrac{7785}{512}e^2)\,m^5
\end{array}\right\}(1+Q)^{-1} \\
&+ \left\{\begin{array}{l}
(-\tfrac{225}{32} + \tfrac{81}{4}\gamma^2 - \tfrac{675}{64}e^2 - \tfrac{825}{32}e'^2 - \tfrac{243}{4}\gamma^4 + \tfrac{1863}{32}\gamma^2e^2 + \tfrac{639}{8}\gamma^2e'^2 + \tfrac{2025}{256}e^4 - \tfrac{2475}{64}e^2e'^2)\,m^3 \\
\qquad + (\tfrac{4617}{256}\gamma^2 - \tfrac{54675}{4096}e^2)\,m^5
\end{array}\right\}(1+Q)^{-2} \\
&+ (-\tfrac{3265}{128} + \tfrac{3345}{32}\gamma^2 - \tfrac{7059}{256}e^2 - \tfrac{48225}{256}e'^2)\,m^4(1+Q)^{-3} \\
&+ (-\tfrac{243225}{2048} + \tfrac{17545425}{256}\gamma^2 - \tfrac{1675335}{2048}e^2 - \tfrac{15032265}{1024}e'^2)\,m^5(1+Q)^{-4} \\
&+ (-\tfrac{126267759}{24576})\,m^6(1+Q)^{-5} \\
&+ (-\tfrac{13651131021}{589824})\,m^7(1+Q)^{-6}
\end{aligned}$$

(Observe that writing herein $Q = 0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 237)

$$\begin{aligned}
g = nt\Big\{&\left[\tfrac{45}{16}m^2 + \tfrac{585}{32}m^3\right]\frac{a^2}{a'^2} \\
&+ \left\{\begin{array}{l}
(\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{9}{4}e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2e^2 - \tfrac{45}{4}\gamma^2e'^2 - \tfrac{27}{64}e^4 + \tfrac{27}{16}e^2e'^2 + \tfrac{45}{16}e'^4)\,m^2 \\
\qquad + (-\tfrac{855}{128}\gamma^2 + \tfrac{729}{512}e^2)\,m^4 \\
\qquad + (\tfrac{1935}{256}\gamma^2 - \tfrac{23355}{1024}e^2)\,m^5
\end{array}\right\}(1+Q)^{-1}
\end{aligned}$$

$$+\left\{\begin{array}{l}\left(\frac{27}{4}-\frac{351}{16}\gamma^2-\frac{297}{64}e^2+\frac{401}{16}e'^2+\frac{135}{2}\gamma^4-\frac{1053}{32}\gamma^2e^2-\frac{1297}{16}\gamma^2e'^2+\frac{675}{256}e^4-\frac{1079}{64}e^2e'^2\right)m^3\\ \quad+\left(-\frac{20007}{1024}\gamma^2-\frac{24057}{4096}e^2\right)m^5\end{array}\right\}(1+Q)^{-2}$$

$$+\left(\frac{1995}{64}-\frac{7989}{64}\gamma^2-\frac{9969}{256}e^2+\frac{29535}{128}e'^2\right)m^4(1+Q)^{-3}$$

$$+\left(\frac{17709}{128}-\frac{376653}{512}\gamma^2-\frac{440787}{2048}e^2+\frac{883245}{512}e'^2\right)m^5(1+Q)^{-4}$$

$$+\frac{2481349}{4096}m^6(1+Q)^{-5}$$

$$+\frac{62329307}{24576}m^7(1+Q)^{-6}$$

(where writing $Q=0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 237). And

$$h=nt\left\{\left[-\frac{45}{32}m^2-\frac{1935}{512}m^3\right]\frac{a^2}{a'^2}\right.$$

$$+\left\{\begin{array}{l}\left(-\frac{3}{4}+\frac{3}{2}\gamma^2-\frac{3}{2}e^2-\frac{9}{8}e'^2-\frac{51}{8}\gamma^2e^2+\frac{9}{4}\gamma^2e'^2+\frac{21}{64}e^2-\frac{9}{4}e^2e'^2-\frac{45}{32}e'^4\right)m^2\\ \quad+\left(\frac{171}{128}\gamma^2-\frac{243}{128}e^2\right)m^4\\ \quad+\left(-\frac{387}{256}\gamma^2+\frac{7785}{256}e^2\right)m^5\end{array}\right\}(1+Q)^{-1}$$

$$+\left\{\begin{array}{l}\left(\frac{9}{32}-\frac{27}{16}e^2-\frac{189}{32}\gamma^2+\frac{23}{32}e'^2+\frac{27}{16}\gamma^4+\frac{567}{16}\gamma^2e^2-\frac{99}{16}\gamma^2e'^2-\frac{675}{256}e^4-\frac{349}{16}e^2e'^2\right)m^3\\ \quad+\left(-\frac{1539}{1024}\gamma^2-\frac{15309}{2048}e^2\right)m^5\end{array}\right\}(1+Q)^{-2}$$

$$+\left(\frac{177}{128}-\frac{195}{64}\gamma^2-\frac{699}{32}e^2+\frac{2685}{256}e'^2\right)m^4(1+Q)^{-3}$$

$$+\left(\frac{10949}{2048}-\frac{6369}{512}\gamma^2-\frac{133839}{1024}e^2+\frac{767159}{1024}e'^2\right)m^5(1+Q)^{-4}$$

$$+\frac{467977}{24576}m^6(1+Q)^{-5}$$

$$+\frac{26983045}{589824}m^7(1+Q)^{-6}$$

(where writing $Q=0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 238). We hence have

$$\begin{aligned}l&=nt\{A+(1+B)Q+CQ^2\},\\&=nt\{A+Q+BQ+CQ^2\},\\g&=nt\{A'+B'Q+C'Q^2\},\\h&=nt\{A''+B''Q+C''Q^2\},\end{aligned}$$

where $\left(\text{omitting the terms in } \frac{a^2}{a'^2}\right)$

$$\begin{aligned}A=1&+\left(-\frac{7}{4}+\frac{21}{2}\gamma^2-\frac{3}{4}e^2-\frac{21}{8}e'^2+\frac{33}{4}\gamma^4-\frac{39}{8}\gamma^2e^2+\frac{63}{4}\gamma^2e'^2-\frac{9}{8}e^2e'^2-\frac{109}{32}e'^4\right)m^2\\&+\left(-\frac{225}{32}+\frac{81}{4}\gamma^2-\frac{675}{64}e^2-\frac{895}{32}e'^2-\frac{243}{4}\gamma^4+\frac{1863}{32}\gamma^2e^2+\frac{629}{8}\gamma^2e'^2+\frac{2025}{256}e^4-\frac{105}{32}e'^4\right)m^3\\&+\left(-\frac{3265}{128}+\frac{14577}{128}\gamma^2-\frac{1833}{64}e^2-\frac{48225}{256}\right)m^4\\&+\left(-\frac{243925}{2048}+\frac{177333}{256}\gamma^2-\frac{328065}{4096}e^2-\frac{1502265}{1024}e'^2\right)m^5\\&+\left(-\frac{12626759}{24576}\right)m^6\\&+\left(-\frac{136513021}{589824}\right)m^7.\end{aligned}$$

$$
\begin{aligned}
B = \; & \left(\tfrac{7}{4} - \tfrac{21}{2}\gamma^2 + \tfrac{3}{4}e^2 + \tfrac{21}{8}e'^2\right)m^2 \\
& + \left(\tfrac{225}{16} - \tfrac{81}{2}\gamma^2 + \tfrac{675}{32}e^2 + \tfrac{825}{16}e'^2\right)m^3 \\
& + \tfrac{2795}{128}m^4 \\
& + \tfrac{243925}{512}m^5.
\end{aligned}
$$

$$
\begin{aligned}
C = \; & - \tfrac{7}{4}m^2 \\
& - \tfrac{675}{32}m^3.
\end{aligned}
$$

$$
\begin{aligned}
A' = \; & \left(\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{9}{4}e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2e^2 - \tfrac{45}{4}\gamma^2e'^2 - \tfrac{27}{16}e^4 + \tfrac{27}{16}e^2e'^2 + \tfrac{45}{16}e'^4\right)m^2 \\
& + \left(\tfrac{27}{4} - \tfrac{351}{16}\gamma^2 - \tfrac{297}{64}e^2 + \tfrac{401}{16}e'^2 + \tfrac{135}{2}\gamma^4 - \tfrac{1053}{32}\gamma^2e^2 - \tfrac{1297}{16}\gamma^2e'^2 + \tfrac{675}{256}e^4 - \tfrac{1079}{64}e^2e'^2\right)m^3 \\
& + \left(\tfrac{1995}{64} - \tfrac{16833}{128}\gamma^2 - \tfrac{19209}{512}e^2 + \tfrac{29535}{128}e'^2\right)m^4 \\
& + \left(\tfrac{17709}{128} - \tfrac{705573}{1024}\gamma^2 - \tfrac{999051}{4096}e^2 + \tfrac{8833245}{512}e'^2\right)m^5 \\
& + \tfrac{2431349}{4096}m^6 \\
& + \tfrac{623293307}{24576}m^7.
\end{aligned}
$$

$$
\begin{aligned}
B' = \; & \left(-\tfrac{3}{2} + \tfrac{15}{2}\gamma^2 - \tfrac{9}{8}e^2 - \tfrac{9}{4}e'^2\right)m^2 \\
& + \left(-\tfrac{27}{2} + \tfrac{351}{8}\gamma^2 + \tfrac{297}{32}e^2 - \tfrac{401}{8}e'^2\right)m^3 \\
& - \tfrac{5985}{64}m^4 \\
& - \tfrac{17709}{32}m^5.
\end{aligned}
$$

$$
\begin{aligned}
C' = \; & \tfrac{3}{2}m^2 \\
& + \tfrac{81}{4}m^3.
\end{aligned}
$$

$$
\begin{aligned}
A'' = \; & \left(-\tfrac{3}{4} + \tfrac{3}{2}\gamma^2 - \tfrac{3}{2}e^2 - \tfrac{9}{8}e'^2 - \tfrac{51}{8}\gamma^2e^2 + \tfrac{9}{4}\gamma^2e'^2 + \tfrac{21}{64}e^4 - \tfrac{9}{4}e^2e'^2 - \tfrac{45}{32}e'^4\right)m^2 \\
& + \left(\tfrac{9}{32} - \tfrac{27}{16}\gamma^2 - \tfrac{189}{32}e^2 + \tfrac{33}{32}e'^2 + \tfrac{27}{16}\gamma^4 + \tfrac{567}{16}\gamma^2e^2 - \tfrac{99}{16}\gamma^2e'^2 - \tfrac{675}{256}e^4 - \tfrac{349}{16}e^2e'^2\right)m^3 \\
& + \left(\tfrac{177}{128} - \tfrac{219}{128}\gamma^2 - \tfrac{3039}{128}e^2 + \tfrac{2685}{256}e'^2\right)m^4 \\
& + \left(\tfrac{10949}{2048} - \tfrac{15825}{1024}\gamma^2 - \tfrac{220707}{2048}e^2 + \tfrac{75759}{1024}e'^2\right)m^5 \\
& + \tfrac{467977}{24576}m^6 \\
& + \tfrac{26983045}{589824}m^7.
\end{aligned}
$$

$$
\begin{aligned}
B'' = \; & + \left(\tfrac{3}{4} - \tfrac{3}{2}\gamma^2 + \tfrac{3}{2}e^2 + \tfrac{9}{8}e'^2\right)m^2 \\
& + \left(-\tfrac{9}{16} + \tfrac{27}{8}\gamma^2 + \tfrac{189}{16}e^2 - \tfrac{23}{16}e'^2\right)m^3 \\
& - \tfrac{531}{128}m^4 \\
& - \tfrac{10949}{512}m^5.
\end{aligned}
$$

$$
\begin{aligned}
C'' = \; & - \tfrac{3}{4}m^2 \\
& + \tfrac{27}{32}m^3.
\end{aligned}
$$

And in terms BQ, $B'Q$, $B''Q$, we have

$$\begin{aligned} Q = \quad & (1 - \tfrac{9}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{3}{2}e'^2)\, m^2 \\ & + (\tfrac{27}{8}\gamma^2 + \tfrac{675}{32}e^2)\, m^3 \\ & - \tfrac{515}{64}\, m^4 \\ & - \tfrac{787}{32}\, m^5, \end{aligned}$$

and in the terms CQ^2, $C'Q^2$, $C''Q^2$, simply $Q^2 = m^4$. Hence finally the required values of l, g, h, are

$$\begin{aligned} l = nt \Big\{ & 1 + [-\tfrac{45}{32}m^2 - \tfrac{7425}{512}m^3]\, \frac{a^2}{a'^2} \\ & + (-\tfrac{3}{4} + 6\gamma^2 + \tfrac{3}{8}e^2 - \tfrac{9}{8}e'^2 + \tfrac{45}{4}\gamma^4 - \tfrac{69}{8}\gamma^2e^2 + 9\gamma^2e'^2 + \tfrac{3}{32}e^4 + \tfrac{9}{16}e^2e'^2 - \tfrac{45}{32}e'^4)\, m^2 \\ & + (-\tfrac{225}{32} + \tfrac{189}{8}\gamma^2 + \tfrac{675}{64}e^2 - \tfrac{825}{32}e'^2 - \tfrac{1107}{16}\gamma^4 - \tfrac{81}{32}\gamma e^2 + \tfrac{349}{4}\gamma^2e'^2 + \tfrac{2475}{64}e^2e'^2)\, m^3 \\ & + (-\tfrac{4071}{128} + \tfrac{15852}{128}\gamma^2 + \tfrac{31605}{512}e^2 - \tfrac{61179}{256}e'^2)\, m^4 \\ & + (-\tfrac{2654293}{2048} + \tfrac{3354403}{512}\gamma^2 + \tfrac{14836865}{4096}e^2 - \tfrac{1767840}{1024}e'^2)\, m^5 \\ & + (-\tfrac{12822631}{24576})\, m^6 \\ & + (-\tfrac{1273325965}{589824})\, m^7, \end{aligned}$$

$$\begin{aligned} g = nt \Big\{ & [\tfrac{45}{16}m^2 + \tfrac{585}{32}m^3]\, \frac{a^2}{a'^2} \\ & + (\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{9}{4}e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2e^2 - \tfrac{45}{4}\gamma^2e'^2 - \tfrac{27}{64}e^4 + \tfrac{27}{16}e^2e'^2 + \tfrac{45}{16}e'^4)\, m^2 \\ & + (\tfrac{27}{4} - \tfrac{351}{16}\gamma^2 - \tfrac{297}{64}e^2 + \tfrac{401}{16}e'^2 + \tfrac{135}{2}\gamma^4 - \tfrac{1053}{32}\gamma^2e^2 - \tfrac{1297}{16}\gamma^2e'^2 + \tfrac{675}{256}e^4 - \tfrac{1079}{64}e^2e'^2)\, m^3 \\ & + (\tfrac{1899}{64} - \tfrac{15009}{128}\gamma^2 - \tfrac{20649}{512}e^2 - \tfrac{28959}{128}e'^2)\, m^4 \\ & + (\tfrac{15981}{128} - \tfrac{663621}{1024}\gamma^2 - \tfrac{1152843}{4096}e^2 + \tfrac{847213}{512}e'^2)\, m^5 \\ & + \tfrac{2103893}{4096}\, m^6 \\ & + \tfrac{52802843}{24576}\, m^7, \end{aligned}$$

$$\begin{aligned} h = nt \Big\{ & [-\tfrac{45}{32}m^2 - \tfrac{1935}{512}m^3)\, \frac{a^2}{a'^2} \\ & + (-\tfrac{3}{4} + \tfrac{3}{2}\gamma^2 - \tfrac{3}{2}e^2 - \tfrac{9}{8}e'^2 - \tfrac{51}{8}\gamma^2e^2 + \tfrac{9}{4}\gamma^2e'^2 + \tfrac{21}{64}e^4 - \tfrac{9}{4}e^2e'^2 - \tfrac{45}{32}e'^4)\, m^2 \\ & + (\tfrac{9}{32} - \tfrac{27}{16}\gamma^2 - \tfrac{189}{32}e^2 + \tfrac{33}{32}e'^2 + \tfrac{27}{16}\gamma^4 + \tfrac{567}{16}\gamma^2e^2 - \tfrac{99}{16}\gamma^2e'^2 - \tfrac{675}{256}e^4 - \tfrac{349}{16}e^2e'^2)\, m^3 \\ & + (\tfrac{273}{128} - \tfrac{843}{128}\gamma^2 - \tfrac{2739}{128}e^2 + \tfrac{3261}{256}e'^2)\, m^4 \\ & + (\tfrac{9797}{2048} - \tfrac{7185}{1024}\gamma^2 - \tfrac{165411}{2048}e^2 + \tfrac{73423}{1024}e'^2)\, m^5 \\ & + \tfrac{199273}{24576}\, m^6 \\ & + \tfrac{66577333}{589824}\, m^7, \end{aligned}$$

which values satisfy, as they should do, the equation $l+g+h=nt$. I recall that the precise signification of the constants is as follows: n is the coefficient of t in the expression of the Moon's longitude in terms of the time, a the corresponding elliptic value of the mean distance ($n^2a^3=$ sum of masses), e the eccentricity, such that in the expression of the longitude the coefficient of the leading term of the equation of the centre has its elliptic value

$$=2e-\tfrac{1}{4}e^3+\tfrac{5}{96}e^5$$

and γ the sine of the half-inclination, such that in the expression of the latitude the coefficient of the leading term has its elliptic value

$$=2\gamma-2\gamma e^2-\tfrac{1}{4}\gamma^5+\tfrac{7}{32}\gamma e^4+\tfrac{1}{4}\gamma^5e^2-\tfrac{5}{144}\gamma e^6$$

n', a' are the mean motion and mean distance of the Sun, $m=\dfrac{n'}{n}$, and e' is the eccentricity of the Sun's orbit, considered as constant.

481.

ON THE EXPRESSION OF M. DELAUNAY'S $h+g$ IN TERMS OF HIS FINALLY ADOPTED CONSTANTS.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXII. (1871—72), p. 74.]

I HAD the pleasure of receiving from M. Delaunay a letter dated Paris, 17th Dec. 1871, in which he informs me that, on referring to his papers, he had found there expressions for l, g, h, identical with those given by me in the November Number of the *Monthly Notices*,—with only a single typographical error, $\frac{23}{33} e'^2 m^3$ instead of $\frac{23}{32} e'^2 m^3$ [*ante* p. 532, corrected] in my expression of h.

M. Delaunay mentions also that he had obtained four additional terms in the expression for $h+g$ (longitude of the Moon's perigee), and that the complete expression in terms of the finally adopted constants is

$$
\begin{aligned}
h+g = \\
nt \Big\{ & \left(\tfrac{3}{4} - 6\gamma^2 - \tfrac{3}{8} e^2 + \tfrac{9}{8} e'^2 - \tfrac{45}{4}\gamma^4 + \tfrac{69}{8}\gamma^2 e^2 - 9\gamma^2 e'^2 - \tfrac{3}{32} e^4 - \tfrac{9}{16} e^2 e'^2 + \tfrac{45}{32} e'^4\right) m^2 \\
& + \left(\tfrac{235}{32} - \tfrac{189}{8}\gamma^2 - \tfrac{675}{64} e^2 + \tfrac{825}{32} e'^2 + \tfrac{1107}{16}\gamma^4 + \tfrac{81}{32}\gamma^2 e^2 - \tfrac{349}{4}\gamma^2 e'^2 - \tfrac{2475}{64} e^2 e'^2\right) m^3 \\
& + \left(\tfrac{4071}{128} - \tfrac{3963}{32}\gamma^2 - \tfrac{31605}{512} e^2 + \tfrac{61179}{256} e'^2\right) m^4 \\
& + \left(\tfrac{265493}{2048} - \tfrac{335403}{512}\gamma^2 - \tfrac{1483665}{4096} e^2 + \tfrac{1767849}{1024} e'^2\right) m^5 \\
& + \left(\tfrac{12822631}{24576} - \tfrac{25291729}{16384} e^2\right) m^6 \\
& + \left(\tfrac{1273925965}{589824} + \tfrac{352038855}{1179648} e^2\right) m^7 \\
& + \tfrac{11028685589}{7077888} m^8 \\
& + \tfrac{2214591470774 1}{67947248} m^9 \\
& + \left[\tfrac{45}{32} m^2 + \tfrac{7425}{512} m^3\right] \frac{a^2}{a'^2} \Big\}.
\end{aligned}
$$

[Observe that $h+g$ is $= nt - l$, and compare with the expression for l, *ante* p. 532.]

482.

NOTE ON A PAIR OF DIFFERENTIAL EQUATIONS IN THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXII. (1871—72), pp. 31—32.]

THE equations

$$\frac{d}{dt}\frac{d\rho}{dt} - \rho\left(\frac{dv}{dt}\right)^2 + \frac{1}{\rho^2} = km^2\rho\left\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v - 2mt)\right\},$$

$$\frac{d}{dt}\rho^2\frac{dv}{dt} \qquad\qquad = jm^2\rho^2\left\{\ -\tfrac{3}{2}\sin(2v - 2mt)\right\},$$

taking therein $j = k = 1$ in effect present themselves in the Lunar Theory, and particular integrals in series have been obtained, the development being carried to a great extent; but I give the results only as far as m^4, viz., writing

$$t - mt = D,$$

we have

$$v = t + \left(\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4\right)\sin 2D$$
$$+ \tfrac{201}{256}m^4 \sin 4D,$$

$$\frac{1}{\rho} = 1 + \tfrac{1}{6}m^2 \qquad - \tfrac{179}{288}m^4$$
$$+ \left(\ m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4\right)\cos 2D$$
$$+ \tfrac{7}{8}m^4 \cos 4D.$$

In the Lunar Theory j and k are properly each $= \dfrac{1}{1 + \dfrac{E}{m'}}$ (E the mass of the Earth, m' that of the Sun), but they are taken to be $= 1$; the numerical difference is inappreciable; but there would be a considerable theoretical advantage in retaining

in the equations the coefficients j, k [regarded as each of them $=k$]: in fact, the developments could then be arranged according to the powers of k, that is according to the powers of the disturbing force; whereas, when k is taken $=1$, we have only a development in powers of m, and since m also presents itself through the coefficient $2-2m$ of t in $2v-2mt$, terms which are really of different orders in regard to the disturbing force, are united together into a single term: so that, instead of a term of the form $(Ak+Bk^2+\&c.)\,m^p$, where A, B, are numerical, we have the term $(A+B+..)\,m^p$, where of course $A+B..$ is given as a single numerical coefficient. There is no equal advantage in retaining the two coefficients k, j, as this only serves to show how a term arises from the central and tangential forces respectively; thus retaining these coefficients, the integrals as far as m^2 are

$$v = \quad t \quad + (\tfrac{1}{2}k + \tfrac{7}{8}j)\, m^2 \sin 2D,$$

$$\frac{1}{\rho} = 1 + \tfrac{1}{6}\, m^2 k + (\tfrac{1}{2}k + \tfrac{1}{2}j)\, m^2 \cos 2D,$$

agreeing with the former result when $k=j=1$; but there is, nevertheless, some interest in retaining the two coefficients. I hope to develope the results somewhat further, and to communicate them to the Society.

483.

ON A PAIR OF DIFFERENTIAL EQUATIONS IN THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXII. (1871—72), pp. 201—206.]

I CONSIDER the differential equations

$$\frac{d}{dt}\frac{d\rho}{dt} - \rho\left(\frac{dv}{dt}\right)^2 + \frac{1}{\rho^2} = km^2\rho\left\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v - 2mt)\right\},$$

$$\frac{d}{dt}\left(\rho^2\frac{dv}{dt}\right) \qquad\qquad = jm^2\rho^2\left\{\ -\tfrac{3}{2}\sin(2v - 2mt)\right\},$$

which when $j = k = 1$ give the following equations in the lunar theory ($D = t - mt$):

$$\begin{aligned}\frac{1}{\rho} = 1 &+ \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4 - \tfrac{97}{48}m^5 - \tfrac{757}{162}m^6 - \tfrac{4039}{432}m^7 - \tfrac{347511189}{1990656}m^8 - \tfrac{1550676635}{4976640}m^9 \\ &+ \cos 2D\left[m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4 + \tfrac{383}{27}m^5 + \tfrac{5105665}{20736}m^6 + \tfrac{231407781}{622080}m^7\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. + \tfrac{3550221217}{9331200}m^8 + \tfrac{27888590059}{34992000}m^9\right] \\ &+ \cos 4D\left[\tfrac{7}{8}m^4 + \tfrac{2737}{480}m^5 + \tfrac{162869}{7200}m^6 + \tfrac{7554833}{108000}m^7 + \tfrac{2389416723}{12960000}m^8 + \tfrac{233523012528 3}{5443200000}m^9\right] \\ &+ \cos 6D\left[\tfrac{219}{256}m^6 + \tfrac{151339}{17920}m^7 + \tfrac{29887443}{627200}m^8 + \tfrac{989786223957}{4443328000}m^9\right] \\ &+ \cos 8D\left[\tfrac{2701}{3072}m^8 + \tfrac{700336 33}{6021120}m^9\right],\end{aligned}$$

or as far as m^7,

$$\begin{aligned}\rho = 1 &- \tfrac{1}{6}m^2 + \tfrac{331}{288}m^4 + \tfrac{83}{16}m^5 + \tfrac{42775}{2592}m^6 + \tfrac{4787}{108}m^7 \\ &+ \cos 2D\left[-\ m^2 - \tfrac{19}{6}m^3 - \tfrac{135}{18}m^4 - \tfrac{709}{54}m^5 - \tfrac{485173}{20736}m^6 - \tfrac{24487949}{622080}m^7\right] \\ &+ \cos 4D\left[-\tfrac{3}{8}m^4 - \tfrac{1217}{480}m^5 - \tfrac{74069}{7200}m^6 - \tfrac{1749779}{54000}m^7\right] \\ &+ \cos 6D\left[-\tfrac{59}{256}m^6 - \tfrac{126193}{53760}m^7\right],\end{aligned}$$

$\left(\frac{1}{\rho}\right.$ is given by M. Delaunay only as far as m^5, the additional terms of $\frac{1}{\rho}$ and expression for ρ were kindly communicated to me by Prof. Adams$\left.\right)$; and

$$\begin{aligned} v = t & \\ & + \sin 2D \left(\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{833}{72} m^4 + \tfrac{2855}{108} m^5 + \tfrac{83044449}{165888} m^6 \right. \\ & \qquad \left. + \tfrac{102859902}{1244160} m^7 + \tfrac{7596606727}{74649600} m^8 - \tfrac{805141816 1}{1119744000} m^9 \right) \\ & + \sin 4D \left(\tfrac{201}{256} m^4 + \tfrac{649}{120} m^5 + \tfrac{6476293}{28800} m^6 + \tfrac{31363361}{432000} m^7 + \tfrac{12303037 7303}{4147200000} m^8 \right) \\ & + \sin 6D \left(\tfrac{3715}{6144} m^6 + \tfrac{6664571}{107520} m^7 \right) \end{aligned}$$

(Delaunay, t. II. pp. 815, 836, 845).

To integrate the original equations write

$$\rho = 1 + \rho_1 + \rho_2 + \dots,$$
$$v = t + v_1 + v_2 + \dots,$$

where the suffixes indicate the degrees in the coefficients k, j conjointly: the equations for ρ_n, v_n take the form

$$\frac{d}{dt}\frac{d\rho_n}{dt} - 3\rho_n - 2\frac{dv_n}{dt} + V_n = Q_n,$$
$$\frac{d}{dt}\left(\frac{dv_n}{dt} + 2\rho_n + U_n\right) \qquad = P_n,$$

where V_n, U_n, P_n, Q_n do not contain ρ_n or v_n. From the second equation we have

$$\frac{dv_n}{dt} + 2\rho_n + U_n = \Omega_n + \int P_n\, dt,$$

where Ω_n is a constant of integration, the integral $\int P_n\, dt$ containing only periodic terms; and then adding twice this to the first equation we have

$$\frac{d}{dt}\frac{d\rho_n}{dt} + \rho_n + V_n + 2U_n = 2\Omega_n + Q_n + 2\int P_n\, dt$$

which determines ρ_n; and substituting its value in the other equation we have $\frac{dv_n}{dt}$, and thence v_n; the constant Ω_n is determined so that $\frac{dv_n}{dt}$ may contain no constant term. We have

$V_1 = 0,$	$U_1 = 0,$
$V_2 = -\left(\frac{dv_1}{dt}\right)^2 - 2\rho_1\frac{dv_1}{dt} + 3\rho_1^2,$	$U_2 = 2\rho_1\frac{dv_1}{dt} + \rho_1^2,$
$V_3 = -2\frac{dv_1}{dt}\frac{dv_2}{dt} - 2\rho_1\frac{dv_2}{dt} - \rho_1\left(\frac{dv_1}{dt}\right)^2$	$U_3 = 2\rho_1\frac{dv_2}{dt} + (2\rho_2 + \rho_1^2)\frac{dv_1}{dt} + 2\rho_1\rho_2,$
$\quad - 2\rho_2\frac{dv_2}{dt} + 6\rho_1\rho_2 - 4\rho_1^3,$	
&c.	&c.

$$Q_1 = km^2(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D),$$
$$Q_2 = km^2\{3v_1 \sin 2D + \rho_1(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D)\},$$
$$Q_3 = km^2\{-3v_2 \sin 2D - 3v_1^2 \cos 2D + \rho_1 v_1 . 3 \sin 2D + \rho_2(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D)\},$$

&c.

$$P_1 = jm^2(-\tfrac{3}{2}\sin 2D),$$
$$P_2 = jm^2(-3v_1 \cos D - 3\rho_1 \sin 2D),$$
$$P_3 = jm^2\{-3v_2 \cos 2D + 3v_1^2 \sin 2D - 6\rho_1 v_1 \cos 2D + (2\rho_2 + \rho_1^2) . -\tfrac{3}{2}\sin 2D\},$$

&c.

In particular attending to the values of P_1, Q_1 the equations for ρ_1, v_1 are in their original form

$$\frac{d}{dt}\frac{d\rho_1}{dt} - 3\rho_1 + 2\frac{dv_1}{dt} = km^2(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D),$$

$$\frac{d}{dt}\left(\frac{dv_1}{dt} + 2\rho_1\right) \quad = jm^2(\ -\tfrac{3}{2}\sin 2D),$$

whence in the transformed form they are

$$\frac{dv_1}{dt} + 2\rho_1 = \Omega_1 + \frac{3jm^2}{4(1-m)}\cos 2D,$$

and

$$\frac{d^2\rho_1}{dt^2} + \rho_1 = 2\Omega_1 + km^2(\tfrac{1}{2} + \tfrac{3}{2}\cos 2D) + \frac{\frac{3}{2}jm^2}{1-m}\cos 2D.$$

Thus the constant term of ρ_1 is $2\Omega_1 + \frac{1}{2}km^2$, giving in $\frac{dv_1}{dt}$ a constant term $-3\Omega_1 - km^2$ this must vanish, or we have $\Omega_1 = -\frac{1}{3}km^2$; and the equations thus become

$$\frac{dv_1}{dt} + 2\rho_1 = -\tfrac{1}{3}km^2 + \frac{3jm^2}{4(1-m)}\cos 2D,$$

$$\frac{d^2\rho_1}{dt^2} + \rho_1 = -\tfrac{1}{6}km^2 + \left(\tfrac{3}{2}km^2 + \frac{\frac{3}{2}jm^2}{1-m}\right)\cos 2D,$$

and then completing the integration

$$\rho_1 = -\tfrac{1}{6}km^2 + \left\{\frac{-\frac{3}{2}km^2}{3 - 8m + 4m^2} + \frac{-\frac{3}{2}jm^2}{(1-m)(3-8m+4m^2)}\right\}\cos 2D,$$

$$v_1 = \left\{\frac{\frac{3}{2}km^2}{(1-m)(3-8m+4m^2)} + \frac{\frac{3}{8}jm^2(7-8m+4m^2)}{(1-m)^2(3-8m+4m^2)}\right\}\sin 2D,$$

which are the accurate values of ρ_1 and v_1.

Expanding as far as m^6 we have

$$\rho_1 = k(-\tfrac{1}{6}m^2) + \cos 2D\{\ k(-\tfrac{1}{2}m^2 - \tfrac{4}{3}m^3 - \tfrac{26}{9}m^4 - \tfrac{160}{27}m^5 - \tfrac{968}{81}m^6) + j(-\tfrac{1}{2}m^2 - \tfrac{11}{6}m^3 - \tfrac{85}{18}m^4 - \tfrac{575}{54}m^5 - \tfrac{3661}{162}m^6)\},$$

which for $j = k$ is

$$= k(-\ m^2 - \tfrac{19}{6}m^3 - \tfrac{137}{18}m^4 - \tfrac{895}{54}m^5 - \tfrac{5597}{162}m^6),$$

and

$$\begin{aligned} v_1 = \quad & \sin 2D \{ \; k(\tfrac{1}{2} m^2 + \tfrac{11}{6} m^3 + \tfrac{85}{18} m^4 + \tfrac{575}{54} m^5 + \tfrac{3661}{162} m^6) \\ & + j(\tfrac{7}{8} m^2 + \tfrac{37}{12} m^3 + \tfrac{589}{72} m^4 + \tfrac{1037}{54} m^5 + \tfrac{27331}{648} m^6)\}, \\ \text{which for } j=k \text{ is} \quad & = k(\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{929}{72} m^4 + \tfrac{896}{27} m^5 + \tfrac{41975}{648} m^6) . \end{aligned}$$

I have, not in general, but for the value $j=k$, calculated ρ_2 and v_2 as far as m^6: I have not made the calculation for ρ_3 and v_3, but their values may be deduced from the foregoing values of ρ, v; the final expressions (when $j=k$) of ρ, $=1+\rho_1+\rho_2+\rho_3+\ldots$ and v, $=t+v_1+v_2+v_3\ldots$ are

$$\begin{aligned} \rho = 1 & \\ & + k\,(-\tfrac{1}{6} m^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad) \\ & + k^2 (\qquad\qquad\qquad\qquad \tfrac{331}{288} m^4 + \tfrac{83}{16} m^5 + \tfrac{5113}{288} m^6) \\ & + k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{1021}{1296} m^6) \\ + \cos 2D \{ & \; k\,(- \quad m^2 - \tfrac{10}{6} m^3 - \tfrac{137}{18} m^4 - \tfrac{895}{54} m^5 - \tfrac{5597}{162} m^6) \\ & + k^2 (\qquad\qquad\qquad\qquad \tfrac{2}{3} m^4 + \tfrac{31}{9} m^5 + \tfrac{329}{27} m^6) \\ & + k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{2381}{2304} m^6)\} \\ + \cos 4D \{ & \; k^2 (\qquad\qquad\qquad\qquad - \tfrac{3}{8} m^4 - \tfrac{1217}{480} m^5 - \tfrac{70583}{7200} m^6) \\ & + k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{7}{20} m^6)\} \\ + \cos 6D \{ & \; k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{59}{250} m^6)\}, \end{aligned}$$

and

$$\begin{aligned} v = t & \\ + \sin 2D \{ & \; k\,(\tfrac{11}{8} m^2 + \tfrac{59}{12} m^3 + \tfrac{929}{72} m^4 + \tfrac{896}{27} m^5 + \tfrac{41975}{648} m^6) \\ & + k^2 (\qquad\qquad - \tfrac{1}{2} m^4 - \tfrac{41}{12} m^5 - \tfrac{43}{3} m^6) \\ & + k^3 (\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{783}{2048} m^6)\} \\ + \sin 4D \{ & \; k^2 (\qquad\qquad\qquad\qquad \tfrac{201}{256} m^4 + \tfrac{649}{120} m^5 + \tfrac{665263}{28800} m^6) \\ & + k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \tfrac{49}{80} m^6)\} \\ + \sin 6D \{ & \; k^3 (\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{3715}{6144} m^6)\}; \end{aligned}$$

which for $k=1$ agree with the foregoing formulæ (verifying them as far as m^5); the present formulæ exhibit the manner in which the expressions depend on the several powers of the disturbing force.

484.

ON THE VARIATIONS OF THE POSITION OF THE ORBIT IN THE PLANETARY THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXII. (1871—72), pp. 206—211.]

It has always appeared to me that in the Planetary Theory, more especially when the method of the variation of the elements is made use of, there is a difficulty as to the proper mode of dealing with the inclinations and longitudes of the nodes, hindering the ulterior development of the theory. Considering the case of two planets m, m', and referring their orbits to any fixed plane and fixed origin of longitudes therein, let θ, θ' be the longitudes of the nodes, ϕ, ϕ' the inclinations ($p = \tan\phi \sin\theta$, $q = \tan\phi \cos\theta$, &c., as usual); then the disturbing functions for m, m' respectively are developed, not explicitly in terms of ϕ, ϕ', θ, θ', but in terms of Φ, the mutual inclination of the two orbits, and of Θ, Θ' the longitudes in the two orbits respectively of the mutual node of the two orbits; Φ and Θ, Θ' being functions (and complicated ones) of ϕ, ϕ', θ, θ'. Moreover, although in the general theory of the secular variations of the orbits of the planetary system, θ, ϕ, &c., are, as above, referred to one fixed plane (the ecliptic of a certain date), yet in the theory of each particular planet it is the practice, and obviously the convenient one, to refer for such planet the θ, ϕ to its own fixed plane (the orbit of the planet at a certain date), the effect of course being that ϕ, and consequently p, q, instead of being of the order of the inclinations to the ecliptic, are only of the order of the disturbing forces. It has occurred to me that the last-mentioned plan should be adhered to *throughout;* viz., that for each planet m, the position of its variable orbit should be determined by θ, the longitude of its node, and ϕ, the inclination in reference to the appropriate fixed plane (orbit of the planet at a certain date) and origin of longitude therein. The disturbing functions for the planets m and m' will of course depend not only on θ, θ', ϕ, ϕ', but on the quantities Φ, Θ, Θ' which determine the mutual positions of the two fixed

planes of reference and origins of longitude therein, *these last being however absolute constants not affected by any variation of the elements;* so that as regards the variation of the elements the disturbing functions are in fact given as *explicit* functions of the variable elements θ, θ', ϕ, ϕ'; and where ϕ, ϕ' and therefore also p, q, p', q' are only of the order of the disturbing forces.

I proceed to work out this idea, for the present considering the development of the Disturbing Function only as far as the first powers of p, q, &c. For comparison with the ordinary theory, observe that in this theory the disturbing function contains only the *second* powers of the p, q, &c., made use of therein; these are in fact of a form such as $P+p$, $Q+q$, ... where P, Q are absolute constants and p, q, ... are the p, q, ... of the present theory; the ordinary theory gives therefore in the disturbing function a series of terms involving $(P+p)^2$, $(P+p)(Q+q)$, ... which I now take account of only as far as the first powers of p, q, ... viz., they are in effect reduced to P^2+2Pp, $PQ+Pq+Qp$, &c. ... The present theory is thus not now developed to the extent of giving the p, q, ... of the ordinary theory in the more complete form as the solutions of a system of simultaneous linear differential equations, but only to the extent of obtaining for these p, q, ... respectively the terms which are proportional to the time.

I commence with the following subsidiary problem. Consider a spherical triangle ABC (sides a, b, c, angles A, B, C, as usual), and taking the side c as constant, but the angles A and B as variable, let it be required to find the variations of C, a, b in terms of variations dA, dB and the variable elements C, a, b themselves. Although the geometrical proof would be more simple, I give the analytical one, as it may be useful.

We have

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c,$$

and thence

$$\begin{aligned} -\sin C dC = & \ (\sin A \cos B + \cos A \sin B \cos c)\, dA \\ & + (\sin B \cos A + \sin A \cos B \cos c)\, dB \\ = & \ \frac{\sin B \sin c}{\tan b}\, dA + \frac{\sin A \sin c}{\tan a}\, dB, \end{aligned}$$

that is

$$-\frac{\sin C}{\sin c}\, dC = \frac{\sin B \cos b}{\sin b}\, dA + \frac{\sin A \cos a}{\sin a}\, dB,$$

or finally

$$-dC = \cos b dA + \cos a dB.$$

Next

$$\sin a = \sin c \frac{\sin A}{\sin C},$$

or, differentiating,

$$\cos a\, da = \frac{\sin c}{\sin^2 C}(\sin C \cos A dA - \cos C \sin A dC)$$

or, substituting for dC its value,

$$= \frac{\sin c}{\sin^2 C}\left\{dA\,(\sin C\cos A + \cos C\sin A\cos b) + dB\cos C\sin A\cos a\right\},$$

$$= \frac{\sin c}{\sin^2 C}\left\{dA\,\frac{\sin A\sin b\cos a}{\sin a} + dB\cos C\sin A\cos a\right\},$$

that is

$$da = \frac{1}{\sin C}\left\{dA\,\frac{\sin A}{\sin a}\sin b + dB\cos C\sin A\right\} \div \frac{\sin C}{\sin c},$$

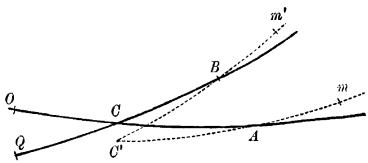
or, on the right-hand writing $\frac{\sin A}{\sin a}$ instead of $\frac{\sin C}{\sin c}$, this is

$$da = \frac{1}{\sin C}(dA\sin b + dB\cos C\sin a);$$

and similarly

$$db = \frac{1}{\sin C}(dB\sin a + dA\cos C\sin b).$$

Now let the continuous lines represent the orbits of m, m' at certain dates, O, Q the origins of longitude therein; and the dotted lines the variable orbits of the planets respectively.



Write

$$OC = \Theta, \quad CA = \theta, \quad \angle CAC' = \phi,$$
$$QC = \Theta', \quad CB = \theta', \quad \angle CBC = \phi',$$
$$\angle C = \Phi.$$

Then, answering to the notation of the lemma, we have

$$a = \theta', \quad b = \theta, \quad C = \Phi, \qquad dA = \phi, \qquad dB = -\phi',$$
$$\text{or say} \qquad = \tan\phi, \qquad = -\tan\phi',$$

whence

$$C'B = a + da,$$

$$= \theta' + \frac{1}{\sin\Phi}(\tan\phi\sin\theta - \tan\phi'\cos\Phi\sin\theta'),$$

$$= \theta' + \frac{1}{\sin\Phi}(p - p'\cos\Phi),$$

$$C'A = b + db,$$
$$= \theta + \frac{1}{\sin\Phi}(-\tan\theta'\sin\phi' + \tan\phi\cos\Phi\sin\theta),$$
$$= \theta - \frac{1}{\sin\Phi}(p' - p\cos\Phi),$$
$$\angle C' = C + dC = \Phi - \cos\theta\tan\phi + \cos\theta'\tan\phi', = \Phi - q + q'.$$

Suppose v, v' are the longitudes of the planets in their two orbits respectively; that is

$$v = OA + Am = \Theta + \theta + Am,$$
$$v' = QB + Bm' = \Theta + \theta' + Bm',$$

whence

$$C'm = C'A + Am, = v - \Theta - \frac{1}{\sin\Phi}(p' - p\cos\Phi),$$
$$C'm' = C'B + Bm, = v' - \Theta' + \frac{1}{\sin\Phi}(p - p'\cos\Phi),$$
$$\angle C' = \Phi - q + q';$$

say these values are $v - \Theta + x$, $v' - \Theta' + x'$, $\Phi + y$. Then if $\bar{H}$ is the angular distance mm' of the two planets,

$$\cos\bar{H} = \cos(v - \Theta + x)\cos(v' - \Theta' + x') + \sin(v - \Theta + x)(\sin v' - \Theta' + x')\cos(\Phi + y),$$
$$= \cos(v - \Theta)\cos(v' - \Theta') + \sin(v - \Theta)\sin(v' - \Theta')\cos\Phi$$
$$+ x\,[-\sin(v - \Theta)\cos(v' - \Theta') + \cos(v - \Theta)\sin(v' - \Theta')\cos\Phi]$$
$$+ x'\,[-\cos(v - \Theta)\sin(v' - \Theta') + \sin(v - \Theta)\cos(v' - \Theta')\cos\Phi]$$
$$+ y\,[-\sin(v - \Theta)\sin(v' - \Theta')\sin\Phi],$$
$$= \cos H + \nabla \text{ suppose.}$$

The disturbing function for the planet m disturbed by m' is

$$\Omega = m'\left\{\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\bar{H}}} - \frac{r\cos\bar{H}}{r'^2}\right\},$$

($\Omega = -R$, if R is the disturbing function of the *Mécanique Céleste*); and the term hereof which involves ∇ is

$$= \nabla\frac{d\Omega}{d\,.\cos\bar{H}}$$

where after the differentiation $\cos\bar{H}$ is replaced by $\cos H$,

$$= m'\left\{\frac{rr'}{(r^2 + r'^2 - 2rr'\cos H)^{\frac{3}{2}}} + \frac{r}{r'^2}\right\}\nabla,$$

viz., this is a linear function of x, x', y, that is of p, q, p', q', with coefficients which of course involve the other variable elements and the time; but it will be remembered that Θ, Θ', Φ are not variable elements, but are absolute constants. The variations of p depend upon $\frac{d\Omega}{dq}$ and those of q on $\frac{d\Omega}{dp}$, and the quantities p, q, p', q', ... disappear from these differential coefficients $\frac{d\Omega}{dq}$, $\frac{d\Omega}{dp}$; that is, disregarding periodic terms, and the variations of the elements, we obtain $\frac{dp}{dt}$, $\frac{dq}{dt}$ as absolute constants, or reckoning the time from the epoch belonging to the fixed orbit of m, we have p, q as mere multiples of the time ($p = At$, $q = Bt$, where A and B are constants); agreeing with the statement preceding the investigation.

Observe that the p, q, as used above, have reference not only to the fixed orbit of m, but also to the node thereon of the fixed orbit of m': we may, if we please, write $\mathrm{p} = \tan\phi \sin(\Theta + \theta)$, $\mathrm{q} = \tan\phi \cos(\Theta + \theta)$, that is, $\mathrm{p} = q \sin\Theta + p \cos\Theta$, $Q = q \cos\Theta - p \sin\Theta$ (or $p = \mathrm{p} \cos\Theta - \mathrm{q} \sin\Theta$, $q = P \sin\Theta + \mathrm{q} \cos\Theta$), and in place of p, q introduce into the formulæ p and q, which have reference only to the fixed orbit of m, and similarly writing $\mathrm{p}' - \tan\phi' \sin(\Theta' + \theta')$, $\mathrm{q}' = \tan\phi' \cos(\Theta + \theta')$, instead of p', q' introduce p′, q′ which have reference only to the fixed orbit of m'.

I remark that a table for the relative positions of the orbits of the eight Planets for the Epoch 1st January, 1850, is given in Leverrier's *Annales de l'Observ. de Paris*, t. II. (1856), pp. 64—66.

485.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. V. to XII. (1866—1869).]

[Vol. V., January to July, 1866, p. 17.]

1791. (PROPOSED by Professor CAYLEY.)—Given a quartic curve $U=0$, to find three cubic curves $P=0$, $Q=0$, $R=0$, each meeting the quartic in the same six points 1, 2, 3, 4, 5, 6, and such that $P=0$, $R=0$ may besides meet the quartic in the same three points a, b, c, and that $Q=0$, $R=0$ may besides meet the quartic in the same three points α, β, γ.

[Vol. V. pp. 25, 26.]

Note on the Problems in regard to a Conic defined by five Conditions of Intersection.

I use the word "intersection" rather than "contact" because it extends to the case of a 1-pointic intersection, which cannot be termed a contact. The conditions referred to are that the conic shall have with a given curve, at a point given or not given, a 1-pointic intersection, a 2-pointic intersection (= ordinary contact), a 3-pointic intersection, &c., as the case may be. It may be noticed that when the point on the curve is a given point, the condition of a k-pointic intersection is really only the condition that the conic shall pass through k given points; though from the circumstance that these are consecutive points on a conic, the formulæ for a conic passing through k discrete points require material alteration; for instance, in the two questions to find the equation of a conic passing through five given points, and to find the equation of a conic having at a given point of a given curve 5-pointic intersection with the curve, the forms of the solutions are very different from each other.

The foregoing remark shows, however, that it is proper to detach the conditions which relate to intersections at given points; and consequently attending only to the

conditions which relate to intersection at an unascertained point (of course the intersections referred to must be at least 2-pointic, for otherwise there is no condition at all) we may consider the conics which pass through four points and satisfy one condition; or which pass through three points and satisfy two conditions; or which pass through two points and satisfy three conditions; or which pass through one point and satisfy four conditions; or which satisfy five conditions. Considering in particular the last case, let 1 denote that the conic has 2-pointic intersection, 2 that it has 3-pointic intersection, ... 5 that it has 6-pointic intersection with a given curve at an unascertained point.

Then the problems are in the first instance

$$5;\ 4, 1;\ 3, 2;\ 3, 1, 1;\ 2, 2, 1;\ 2, 1, 1, 1;\ 1, 1, 1, 1, 1.$$

But the intersections may be intersections with the same given curve or with different given curves; and we have thus in all 27 problems, viz. these are as given in the following table, where the colons (:) separate those conditions which refer to different curves:

No. of Prob.	Conditions.	No. of Prob.	Conditions.	No. of Prob.	Conditions.
1	5	10	3, 1 : 1	19	3 : 1 : 1
2	4, 1	11	3 : 1, 1	20	2 : 2 : 1
3	3, 2	12	2, 2 : 1	21	2, 1 : 1 : 1
4	3, 1, 1	13	2, 1 : 2	22	2 : 1, 1 : 1
5	2, 2, 1	14	2, 1, 1 : 1	23	1, 1, 1 : 1 : 1
6	2, 1, 1, 1	15	2, 1 : 1, 1	24	1, 1 : 1, 1 : 1
7	1, 1, 1, 1, 1	16	2 : 1, 1, 1	25	2 : 1 : 1 : 1
8	4 : 1	17	1, 1, 1, 1 : 1	26	1, 1 : 1 : 1 : 1
9	3 : 2	18	1, 1, 1 : 1, 1	27	1 : 1 : 1 : 1 : 1

Thus Problem 1 is to find a conic having 6-pointic intersection with a given curve; Problem 2 a conic having 5-pointic intersection and also 2-pointic intersection with a given curve... Problem 7 is to find a conic having five 2-pointic intersections with (touching at five distinct points) a given curve....Problem 27 is to find a conic having 2-pointic intersection with (touching) each of five given curves. Or we may in each case take the problem to be merely to find the number of the conics which satisfy the required conditions. This number is known in Prob. 1, for the case of a curve of the order m without singularities, viz. the number is $= m(12m - 27)$. It is also known in Problems 25 and 26 in the case where the first curve (that to which the symbol 2, or 1, 1 relates) is a curve without singularities; and it is known in Prob. 27, viz. if m, n, p, q, r be the orders and M, N, P, Q, R the classes of the five curves

respectively, then the number is $=(M, m)(N, n)(P, p)(Q, q)(R, r)\{1, 2, 4, 4, 2, 1\}$, that is, $1\,MNPQR + 2\,\Sigma MNPQr +$ &c. The number is not, I believe, known in any other of the problems. In particular, (Prob. 7) we do not as yet know the number of the conics which touch a given curve at five points. It would be interesting to obtain this number; but (judging from the analogous question of finding the double tangents of a curve) the problem is probably a very difficult one.

[Vol. v. p. 37.]

1857. (Proposed by Professor CAYLEY.)—If for shortness we put

$$P = x^3 + y^3 + z^3, \quad Q = yz^2 + y^2z + zx^2 + z^2x + xy^2 + x^2y, \quad R = xyz,$$
$$P_0 = a^3 + b^3 + c^3, \quad Q_0 = bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b, \quad R_0 = abc;$$

then (α, β, γ) being arbitrary, show that the cubic curves $\begin{vmatrix} \alpha, & \beta, & \gamma \\ P, & Q, & R \\ P_0, & Q_0, & R_0 \end{vmatrix} = 0$ pass all of them through the same nine points, lying six of them upon a conic and three of them upon a line; and find the equations of the conic and line, and the coordinates of the nine points of intersection; find also the values of $(\alpha : \beta : \gamma)$ in order that the cubic curve may break up into the conic and line.

[Vol. v. p. 37.]

1730. (Proposed by Professor CAYLEY.)—Show that (I) the condition in order that the roots k_1, k_2, k_3 of the equation

$$\gamma k^3 + (-g - \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{3}{2}\gamma)\,k^2 + (-g - \tfrac{3}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)\,k - \alpha = 0 \qquad \text{(A)}$$

may be connected by a relation of the form

$$k_3(k_1 - k_2) - (k_2 - k_3) = 0, \qquad (1)$$

and (II) the result of the elimination of a, b, c from the equations

$$a^2(b + c) = -2\alpha, \qquad (2)$$
$$b^2(c + a) = 2\beta, \qquad (3)$$
$$c^2(a + b) = -2\gamma, \qquad (4)$$
$$(b - c)(c - a)(a - b) = -4g, \qquad (5)$$

are each

$$4(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)g^3 + 4(-\Sigma\alpha^3\beta + 4\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma)\,g^2$$
$$+ (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)\,g + 2(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = 0. \qquad \text{(B)}$$

[Vol. v. pp. 38, 39.]

1834. (Proposed by Professor CAYLEY.)—1. It is required to find on a given cubic curve three points A, B, C, such that, writing $x=0$, $y=0$, $z=0$ for the equations of the lines BC, CA, AB respectively, the cubic curve may be transformable into itself by the inverse substitution $(\alpha x^{-1}, \beta y^{-1}, \gamma z^{-1})$ in place of x, y, z respectively, α, β, γ being disposable constants.

2. In the cubic curve $ax(y^2+z^2)+by(z^2+x^2)+cz(x^2+y^2)+2lxyz=0$ the inverse points (x, y, z) and (x^{-1}, y^{-1}, z^{-1}) are corresponding points (that is, the tangents at these two points meet on the curve).

Solution by the PROPOSER, S. ROBERTS, M.A., *and others.*

Since the points A, B, C are on the curve, the equation is of the form

$$fy^2z+\quad gz^2x+\quad hx^2y+\quad iyz^2+\quad jzx^2+\quad hxy^2+2lxyz=0;$$

hence this equation must be equivalent to

$$\frac{f\beta^2\gamma}{y^2z}+\frac{g\gamma^2\alpha}{z^2x}+\frac{h\alpha^2\beta}{x^2y}+\frac{i\beta\gamma^2}{yz^2}+\frac{j\gamma\alpha^2}{zx^2}+\frac{k\alpha\beta^2}{xy^2}+\frac{2l\alpha\beta\gamma}{xyz}=0,$$

or,

$$j\frac{\alpha}{\beta}y^2z+k\frac{\beta}{\gamma}z^2x+i\frac{\gamma}{\alpha}x^2y+h\frac{\alpha}{\gamma}yz^2+f\frac{\beta}{\alpha}zx^2+g\frac{\gamma}{\beta}xy^2+2lxyz=0,$$

which will be the case if

$$f=j\frac{\alpha}{\beta},\quad g=k\frac{\beta}{\gamma},\quad h=i\frac{\gamma}{\alpha},\quad i=h\frac{\alpha}{\gamma},\quad j=f\frac{\beta}{\alpha},\quad k=g\frac{\gamma}{\beta}.$$

This implies $fgh=ijk$; and if this condition be satisfied, then $\alpha:\beta:\gamma$ can be determined, viz. we have $\alpha:\beta:\gamma=if:ij:hf$, which satisfy the remaining equations, so that the only condition is $fgh=ijk$.

Writing in the equation of the curve $x=0$, we find $fy^2z+iyz^2=0$, that is, the line $x=0$ meets the curve in the points $(x=0, y=0)$, $(x=0, z=0)$, and $(x=0, fy+iz=0)$. We have thus on the curve the three points

$$(x=0, fy+iz=0),\quad (y=0, gz+jx=0),\quad (z=0, hx+ky=0),$$

and in virtue of the assumed relation $fgh=ijk$, these three points lie in a line. Hence the points A, B, C must be such that BC, CA, AB respectively meet the curve in points A', B', C', which three points lie in a line; that is, we have a quadrilateral whereof the six angles A, B, C, A', B', C' all lie on the curve. It is well known that the opposite angles A and A', B and B', C and C' must be *corresponding points*, that is, points the tangents at which meet on the curve. And conversely taking A, C any two points on the curve, A' a corresponding point to A (any one of the four

corresponding points), then AC, $A'C$ will meet the curve in the corresponding points B', B; and AB, $A'B'$ will meet on the curve in a point C' corresponding to C, giving the inscribed quadrilateral (A, B, C, A', B', C'); the triangle ABC is therefore constructed.

It is to be remarked that the equation $fgh = ijk$ being satisfied, we may without any real loss of generality write $f = j$, $g = k$, $h = i$, and therefore $\alpha = \beta = \gamma$; hence changing the constants we have the theorem: the inverse points (x, y, z), (x^{-1}, y^{-1}, z^{-1}) are corresponding points on the curve

$$ax(y^2 + z^2) + by(z^2 + x^2) + cx(x^2 + y^2) + 2lxyz = 0.$$

[Vol. v. pp. 57, 58.]

Addition to the Note on the Problems in regard to a Conic defined by five Conditions of Intersection.

Since writing the Note in question, I have found that a solution of Problem 7 has been given by M. De Jonquières in the paper "Du Contact des Courbes Planes, &c.," *Nouvelles Annales de Mathématiques*, vol. III. (1864), pp. 218—222: viz. the number of conics which touch a curve of the order n in five distinct points is stated to be

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(n^5 + 15n^4 - 55n^3 - 495n^2 + 1584n + 15).$$

There are given also the following results; the number of conics which pass through two given points and touch a curve of the order n in three distinct points is

$$\frac{n(n-1)(n-2)}{2}(n^3 + 6n^2 - 19n - 12),$$

and the number of conics which pass through a given point and touch a curve of the order n in four distinct points is

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(n^4 + 10n^3 - 37n^2 - 118n + 282).$$

These formulæ are given without demonstration, and with an expression of doubt as regards their exactness—("elles sont exactes, je crois"); they apply, of course, to a curve of the order n without singularities; but assuming them to be accurate, the means exist for adapting them to the case of a curve with singularities.

[There is also a paper on the same subject in the *Annales* for January, 1866 (pp. 17—20), from the Editor's *Note* to which we have introduced a correction ($+15$ instead of -35) in the formula given above.]

[Vol. v. pp. 58, 59.]

1876. (Proposed by R. BALL, M.A.)—If three of the roots of the equation $(a, b, c, d, e \between x, 1)^4 = 0$ be in arithmetical progression, show that

$$55296\, H^3J - 2304\, aH^2I^2 - 16632\, a^2HIJ + 625\, a^3I^3 - 9261\, a^3J^2 = 0,$$

where

$$H = ac - b^2, \quad I = ae - 4bd + 3c^2, \quad J = ace + 2bcd - ad^2 - b^2e - c^3.$$

Solution by PROFESSOR CAYLEY.

Write $(a, b, c, d, e \between x, 1)^4 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$; then putting for a moment $\beta+\gamma+\delta = p$, $\beta\gamma+\beta\delta+\gamma\delta = q$, $\beta\gamma\delta = r$, and forming the equation

$$(\beta+\gamma-2\delta)(\beta+\delta-2\gamma)(\gamma+\delta-2\beta) = 0,$$

this is easily reduced to

$$-2p^3 + 9pq - 27r = 0.$$

But we have

$$a(x^3 - px^2 + qx - r)(x-\alpha) = (a, b, c, d, e \between x, 1)^4,$$

and hence

$$p = -\frac{4b}{a} - \alpha, \quad q = \frac{6c}{a} + \frac{4b}{a}\alpha + \alpha^2, \quad r = -\frac{4d}{a} - \frac{6c}{a}\alpha - \frac{4b}{a}\alpha^2 - \alpha^3.$$

Substituting these values of p, q, r, the foregoing equation becomes, after all reductions,

$$(20\, a^3,\ 20\, a^2b,\ -16\, ab^2 + 36\, a^2c,\ 128\, b^3 - 216\, abc + 108\, a^2d \between \alpha, 1)^3 = 0,$$

and from this and the equation $(a, b, c, d, e \between \alpha, 1)^4 = 0$, eliminating α, we should find the condition for three roots in arithmetical progression. But it appears from the theory of invariants that the result of the elimination may be obtained by writing $b = 0$, and expressing the result so obtained in terms of a, H, I, J. Hence, writing in the two equations $b = 0$, the first equation contains the factor $4a^2$, and throwing this out, the equations become

$$5a\alpha^4 + 27c\alpha + 27d = 0, \quad a\alpha^4 + 6c\alpha^2 + 4d\alpha + e = 0;$$

or multiplying the first by α and reducing by means of the second, the two equations become

$$5a\alpha^3 + 27c\alpha + 27d = 0, \quad 3c\alpha^2 - 7d\alpha + 5e = 0.$$

The result is of the degree 5 in the coefficients, but in order to avoid fractions in the final result it is proper to multiply it by a^4; it then becomes

$$625\, a^6e^3 - 4050\, a^5c^2e^2 + 6561\, a^4c^4e - 1890\, a^5ced^2 + 13122\, a^4c^3d^2 + 9261\, a^5d^4 = 0.$$

But writing as above $b = 0$, we have

$$a = a, \quad c = \frac{H}{a}, \quad e = \frac{I}{a} - \frac{3H^2}{a^3}, \quad d^2 = -\frac{J}{a} + \frac{HI}{a^2} - \frac{4H^3}{a^4};$$

and substituting these values, the result is found to contain the terms $\frac{IH^4}{a}$, $\frac{H^6}{a^3}$ with coefficients which vanish; viz. the coefficient of the first of these terms is

$$+16875+24300+\ 6561+\ 7560+18792-74088,\ =0;$$

and the coefficient of the second of the two terms is

$$-16875-36450-19683-75168+148176,\ =0.$$

The remaining terms give

$$\left.\begin{array}{ll} +\ 625 & =+\ 625\,a^3I^3 \\ -\ 5625-4050-1890+9261 & =-\ 2304\,aH^2I^2 \\ +\ 1890-18522 & =-\ 16632\,a^2HIJ \\ -\ 18792+74088 & =+\ 55296\,H^3J \\ +\ 9261 & =+\ 9261\,a^3J^2 \end{array}\right\}=0,$$

which is the required result; a more convenient form of writing it is

$$(55296\,J,\ \ -768\,I^2,\ \ -5544\,IJ,\ \ 625\,I^3+9261\,J^2 \between H,\ a)^3=0.$$

REMARK. If I and J denote as above the two invariants of the form $U=(a, b, c, d, e \between x, 1)^4$, and if we now use H to denote the Hessian of the form, viz.

$$H=(ac-b^2,\ \ \tfrac{1}{2}(ad-bc),\ \ \tfrac{1}{6}(ae+2bd-3c^2),\ \ \tfrac{1}{2}(be-cd),\ \ ce-d^2 \between x,\ 1)^4,$$

then it appears by the theory of invariants that the equation of the twelfth order

$$(55296\,J,\ \ -768\,I^2,\ \ -5544\,IJ,\ \ 625\,I^3+9261\,J^2 \between H,\ U)^3=0,$$

is such that each of its roots forms with some three of the roots of the equation $U=0$ a harmonic progression; viz. if the three roots are β, γ, δ, then we have

$$\frac{2}{x-\gamma}=\frac{1}{x-\beta}+\frac{1}{x-\delta},\ \text{ or }\ x=\frac{2\beta\delta-(\beta+\delta)\gamma}{\beta+\delta-2\gamma};$$

so that the roots of the equation of the twelfth order are the twelve values of the last-mentioned function of three roots.

[Vol. v. pp. 65, 66.]

On the Problems in regard to a Conic defined by five Conditions of Intersection.

There has been recently published in the *Comptes Rendus* (t. LXII. pp. 177—183, January, 1866) an extract of a memoir "Additions to the Theory of Conics," by M. H. G. Zeuthen (of Copenhagen). The extract gives the solutions of fourteen problems, with a brief indication of the method employed for obtaining them. Of these problems, four relate to intersections at given points, the remaining ten are included

among the twenty-seven problems enumerated in my *Note* on this subject in the January Number of the *Educational Times* (*Reprint*, vol. v., p. 25); but two of these ten are the problems 25 and 26 which are in my *Note* stated to have been solved; there are, consequently, of the twenty-seven problems, in all twelve which are solved: viz. these are where it is to be observed that Zeuthen's solutions apply to the case

No. of Prob.	1, 8, 10, 12, 14, 17, 19, 21, 23, 25, 26, 27
Zeuthen's No.	—, 14, 13, 11, 8, 3, 12, 7, 2, 6, 1, —

of a curve of a given order with given numbers of double points and cusps. The problems 25 and 26 had been previously solved only in the case of a curve without singularities. As to Prob. 27, the solution mentioned in my former *Note* is in fact applicable to the general case. The solution for Prob. 1 may also be extended to this general case, viz. for a curve of the order m with δ double points and κ cusps the required number is $=m(12m-27)-24\delta-27\kappa$; or, if n be the class, then this number is $=12n-15m+9\kappa$; so that all the twelve problems are solved in the general case.

The results obtained by M. de Jonquieres, as stated in my *Note* in the March Number (*Reprint*, vol. v., p. 57), seem to be all of them erroneous. In fact, for the number of conics passing through two given points and touching a curve of the order m in three distinct points (which is a particular case of Prob. 23), Zeuthen's formula applied to a curve without singularities gives this

$$=\tfrac{1}{6}m(m-2)(m^4+5m^3-17m^2-49m+108)$$

instead of the value

$$\tfrac{1}{2}m(m-1)(m-2)(m^3+6m^2-19m-12)$$

which is

$$=\tfrac{1}{6}m(m-2)(m^4+5m^3-25m^2+7m+12);$$

and I have by my own investigation verified Zeuthen's Number. So for the number of conics through a given point and touching a curve of the order m in four distinct points (which is a particular case of Prob. 17), Zeuthen's formula applied to a curve without singularities gives this

$$=\tfrac{1}{24}m(m-2)(m-3)(m^5+9m^4-15m^3-225m^2+140m+1050)$$

instead of the value

$$\tfrac{1}{24}m(m-1)(m-2)(m-3)(m^4+10m^3-37m^2-118m+282)$$

which is

$$=\tfrac{1}{24}m(m-2)(m-3)(m^3+9m^4-47m^3-81m^2+400m-282),$$

and it may I think be inferred that the expression obtained for the number of conics which touch a given curve in five distinct points (Prob. 7), containing as it does the factor $(m-1)$, is also erroneous.

I have obtained for Prob. 2 a solution which I believe to be accurate; viz. the number of the conics (4, 1), (that is, the conics which have with a given curve a 5-pointic intersection and also a 2-pointic intersection, or ordinary contact), is

$$=10n^2+10nm-20m^2-130n+140m+10\kappa(m+n-9)-4[(n-3)\kappa+(m-3)\iota]$$

where ι (the number of inflexions) is $=3n-3m+\kappa$, but I prefer to retain the foregoing form, without effecting the substitution.

[Vol. v. pp. 88, 89.]

1890. (Proposed by Professor CAYLEY.)—Find the equation of a conic passing through three given points and having double contact with a given conic.

Solution by the PROPOSER.

Let the given points be the angles of the triangle $(x=0,\ y=0,\ z=0)$, and let the equation of the given conic be $U=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0$; then the equation of the required conic is

$$U-(x\sqrt{a}+y\sqrt{b}+z\sqrt{c})^2=0,$$

for this is a conic having double contact with the conic $U=0$, and, since the terms in $(x^2,\ y^2,\ z^2)$ each vanish, it is also a conic passing through the given points.

It is clear that there are four conics satisfying the conditions of the Problem, viz. putting for shortness

$$P=x\sqrt{a}+y\sqrt{b}+z\sqrt{c},\qquad P_1=-x\sqrt{a}+y\sqrt{b}+z\sqrt{c},$$
$$P_2=x\sqrt{a}-y\sqrt{b}+z\sqrt{c},\qquad P_3=\ \ x\sqrt{a}+y\sqrt{b}-z\sqrt{c},$$

the four conics are

$$U-P^2=0,\quad U-P_1^2=0,\quad U-P_2^2=0,\quad U-P_3^2=0.$$

It may be remarked that the conics P, P_1 have a fourth intersection lying on the line $y\sqrt{b}+z\sqrt{c}=0$, and the conics P_2, P_3 a fourth intersection lying on the line $y\sqrt{b}-z\sqrt{c}$; which two lines are harmonics in regard to the lines $y=0$, $z=0$.

Similarly the conics P_1, P_2 have a fourth intersection on the line $x\sqrt{a}+z\sqrt{c}=0$, and the conics P, P_3 a fourth intersection on the line $x\sqrt{a}-z\sqrt{c}=0$; which two lines are harmonics in regard to the lines $z=0$, $x=0$. And the conics P_1, P_3 have a fourth intersection on the line $x\sqrt{a}+y\sqrt{b}=0$, and the conics P, P_2 a fourth intersection on the line $x\sqrt{a}-y\sqrt{b}=0$; which two lines are harmonics in regard to the lines $x=0$, $y=0$. It may further be remarked that the equations of any two of the four conics may be taken to be

$$\alpha yz+\beta zx+\gamma xy=0,\quad \alpha' yz+\beta' zx+\gamma' xy=0.$$

The general equation of a conic having double contact with each of these conics then is

$$n^2z^2 - 2n(\gamma\alpha' + \gamma'\alpha)yz - 2n(\gamma\beta' + \gamma'\beta)zx - 4n\gamma\gamma'xy + [(\beta\gamma' - \beta'\gamma)x - (\gamma\alpha' - \gamma'\alpha)y]^2 = 0,$$

where n is arbitrary: and, having double contact with this conic, we have (besides the above-mentioned two conics) two new conics each passing through the angles of the triangle; viz. writing for greater convenience

$$n = \frac{(\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)}{K - \gamma\gamma'}, \quad \text{or } K = \gamma\gamma' + \frac{(\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)}{n},$$

then the equations of the two new conics are

$$\gamma'\alpha\, yz + \gamma\beta'\, zx + Kxy = 0, \quad \gamma\alpha'\, yz + \gamma'\beta\, zx + Kxy = 0.$$

In fact, writing the equation under the form

$$\begin{aligned} &[xz + (\beta\gamma' - \beta'\gamma)x + (\gamma\alpha' - \gamma'\alpha)y]^2 \\ &- 4\ (\beta\gamma' - \beta'\gamma)(\gamma\alpha' - \gamma'\alpha)xy - 4n\gamma\gamma'xy \\ &- 2n(\beta\gamma' - \beta'\gamma)xz - 2n(\beta\gamma' + \beta'\gamma)\ xz \\ &- 2n(\gamma\alpha' - \gamma'\alpha)yz - 2n(\gamma\alpha' + \gamma'\alpha)yz = 0, \end{aligned}$$

we at once see that this is a conic having double contact with the conic $\gamma'\alpha yz + \gamma\beta' zx + Kxy = 0$, the equation of the chord of contact being $nz + (\beta\gamma' - \beta'\gamma)x + (\gamma\alpha' - \gamma'\alpha)y = 0$: and similarly it has double contact with the conic $\gamma\alpha'\, yz + \gamma'\beta\, zx + Kxy = 0$, the equation of the chord of contact being $nz - (\beta\gamma' - \beta'\gamma)x - (\gamma\alpha' - \gamma'\alpha)y = 0$.

[Vol. v. pp. 99, 100.]

1554. (Proposed by Professor CAYLEY.)—Show that, in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are

For the circle of maximum curvature	$y_1 = a(1-e)(1+2e)^{\frac{1}{2}}$,
for the ellipse	$y_2 = a(1-e^2)$,
for the circle of minimum curvature	$y_3 = \dfrac{a\{(1-e^2+e^4)^{\frac{1}{2}} - e^2\}}{(1-e^2)^{\frac{1}{2}}}$,

and that these values are in the order of increasing magnitude.

[Vol. VI., July to December, 1866, pp. 18, 19.]

1931. (Proposed by Professor CAYLEY.)—Find the stationary tangents (or tangents at the inflexions) of the nodal cubic

$$x(y-z)^2 + y(z-x)^2 + z(x-y)^2 = 0.$$

Solution by the PROPOSER.

The equation may be transformed into the form

$$(-8x+y+z)^{\frac{1}{3}}+\ (x-8y+z)^{\frac{1}{3}}+\ (x+y-8z)^{\frac{1}{3}}=0,$$

and it thence follows immediately that the stationary tangents are the lines

$$-8x+y+z=0,\quad x-8y+z=0,\quad x+y-8z\ =0,$$

respectively, and that the three points of contact, or inflexions, are the intersections of these lines with the line $x+y+z=0$.

In fact, writing

$$X=kx+y+z,\quad Y=x+ky+z,\quad Z=x+y+kz,$$

we have identically

$$\begin{aligned}
&(X+Y+Z)^3-27XYZ\\
&=(k+2)^3(x+y+z)^3-27(kx+y+z)(x+ky+z)(x+y+kz),\\
&=\quad (x^3+y^3+z^3)\{(k+2)^3-27k\}\\
&\quad+3(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)\{(k+2)^3-9(k^2+k+1)\}\\
&\quad+3\,xyz\,\{2(k+2)^3-9(k^3+3k+2)\}\\
&=(k-1)^2(k+8)(x^3+y^3+z^3)+3(k-1)^2(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)-3(k-1)^2(7k+2)xyz.
\end{aligned}$$

Hence, writing $k=-8$, we have

$$\begin{aligned}
(X+Y+Z)^3-27XYZ&=-2187\,\{yz^2+y^2z+zx^2+z^2x+xy^2+x^2y-6xyz\},\\
&=-2187\,\{x(y-z)^2+y(z-x)^2+z(x-y)^2\}.
\end{aligned}$$

The equation of the given curve is therefore

$$(X+Y+Z)^3-27XYZ=0,\quad \text{or}\ X^{\frac{1}{3}}+Y^{\frac{1}{3}}+Z^{\frac{1}{3}}=0,$$

where of course X, Y, Z have the values

$$X=-8x+y+z,\quad Y=x-8y+z,\quad Z=x+y-8z.$$

[Vol. VI. pp. 35—39.]

1990. (Proposed by Professor SYLVESTER.)—Prove that the three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through the four foci of the cubic.

Solution by PROFESSOR CAYLEY.

Some preliminary explanations are required in regard to this remarkable theorem.

1. I call to mind that a circular cubic (or cubic through the two circular points at infinity) has 16 foci, which lie 4 together on 4 different circles; and that the

property of 4 concyclic foci is that taking any three of them A, B, C, the distances of a point P of the curve from these three foci are connected by a linear relation $\lambda . AP + \mu . BP + \nu . CP = 0$, where $\lambda \pm \mu \pm \nu = 0$, or if as is more convenient the distances are considered as $\pm$, then where $\lambda + \mu + \nu = 0$. A circular cubic may be determined so as to satisfy 7 conditions; having a focus at a given point is 2 conditions; hence a circular cubic may be determined so as to pass through three given points, and to have as foci two given points.

2. A Cartesian, or bicircular cuspidal quartic (that is a quartic having a cusp at each of the circular points at infinity) has nine foci, but of these there are three which lie in a line with the centre of the Cartesian (or intersection of the cuspidal tangents), and which are preeminently the foci of the Cartesian. We may, therefore, say that the Cartesian has three foci, which foci lie in a line, the axis of the Cartesian. A Cartesian may be determined to satisfy 6 conditions; having a focus at a given point is 2 conditions; but having for foci three given points on a line is 5 conditions; and hence a Cartesian may be found having for foci three given points on a line, and passing through a given point; there are in fact two such Cartesians, intersecting at right angles at the given point.

3. The theorem at first sight appears impossible; for take any three points F, G, H in a line and any other point A; then, as just remarked, there are, having F, G, H for foci and passing through A, two Cartesians. And we may draw through F, G, H, and with A for focus, a circular cubic depending upon two arbitrary parameters; the position of a second focus of the circular cubic is (on account of the two arbitrary parameters) *primâ facie* indeterminate; and this is confirmed by the remark that the circular cubic can actually be so determined as to have for focus an arbitrary point B; and yet the theorem in effect asserts that the foci concyclic with A, of the circular cubic, lie on one or other of the two Cartesians.

4. To explain this, it is to be remarked that the arbitrary point B is a focus which is either concyclic with A or else not concyclic with A. In the latter case, although B is arbitrary, yet the foci concyclic with A may and in fact do lie on one of the Cartesians; the difficulty is in the former case if it arises; viz., if we can describe a cubic through the points F, G, H in a line, and with A and B as *concyclic* foci; that is, if we can find a third focus C, such that the distances from A, B, C of a point P on the curve are connected by a relation $\lambda . AP + \mu . BP + \nu . CP = 0$, where $\lambda + \mu + \nu = 0$. It may be shown that this is in a sense possible, but that the resulting cubic is not a proper circular cubic, but is the cubic made up of the line FGH taken twice, and of the line infinity. To show this, since the required cubic passes through the points F, G, H we have

$$\begin{array}{l} \lambda . AF + \mu . BF + \nu . CF = 0 \\ \lambda . AG + \mu . BG + \nu . CG = 0 \\ \lambda . AH + \mu . BH + \nu . CH = 0 \\ \lambda \quad + \mu \quad + \nu \quad = 0 \end{array} \text{ and thence } \left\| \begin{array}{cccc} AF, & AG, & AH, & 1 \\ BF, & BG, & BH, & 1 \\ CF, & CG, & CH, & 1 \end{array} \right\| = 0,$$

being two conditions for the determination of the position of the point C; these give CG, CH as linear functions of CF; the distances CF, CG, CH of the point C from the points F, G, H in the line FGH are connected by a quadratic equation, and hence substituting for CG, CH their values in terms of CF, we have a quadratic equation for CF; as the given conditions are satisfied when C coincides with A or with B, the roots of this equation are $CF=AF$ and $CF=BF$. But if $CF=AF$, then the linear relations give $CG=AG$ and $CH=AH$, that is, C is a point opposite to A in regard to the line FGH. And similarly if $CF=BF$, then C is a point opposite to B in regard to the line FGH. But C being opposite to A or B, the fourth concyclic focus D will be opposite to B or A; that is, the pairs A, B and C, D of concyclic foci lie symmetrically on opposite sides of the line FGH; this of course implies that the four points lie on a circle.

5. Taking $Y=0$ as the equation of the line FGH, $x^2+y^2-1=0$ as the equation of the circle through the four points A, B, C, D, then these lie on a proper cubic

$$(x^2+y^2+1)x+lx^2+ny^2=0$$

(not passing through the points F, G, H) and the four foci are given as the intersections with the circle $x^2+y^2-1=0$ of the pair of lines

$$x^2-2nx-nl=0.$$

But if we attempt to describe with the same four foci a cubic

$$(x^2+y^2+1)y+l'x^2+2m'xy+n'y^2=0,$$

then the foci are given as the intersections with the circle $x^2+y^2-1=0$ of the conic

$$y^2+2m'x-2l'y+m'^2-n'l'=0.$$

In order that these may coincide with the points (A, B, C, D) we must have

$$(x^2-2nx-nl)+(y^2+2m'x-2l'y+m'^2-n'l')=x^2+y^2-1;$$

that is

$$m'=n, \quad l'=0, \quad -nl+n^2-n'l'=-1.$$

The last equation is $n'l'=n^2+1-nl$, which, assuming that nl is not equal to n^2+1, {in this case the cubic $(x^2+y^2+1)x+lx^2+ny^2=0$ would reduce itself to the line and conic $(x+n)\left(x^2+y^2+\frac{x}{n}\right)=0$}, since $l'=0$, gives $n'=\infty$, and therefore the cubic

$$(x^2+y^2+1)y+l'x^2+2m'xy+n'y^2=0,$$

reduces itself to $y^2=0$, that is, the cubic in question reduces itself to the line FGH twice repeated, and the line infinity.

6. The conclusion is that F, G, H being given points on a line, and A and B being any other given points, there is not any proper cubic passing through F, G, H and having A, B for concyclic foci: and the *primâ facie* objection to the truth of the theorem is thus removed.

7. Considering the points F, G, H on a line and the point A as given, it has been seen that there are *two* Cartesians through A with the foci F, G, H; and the theorem asserts that in the circular cubics through F, G, H with the focus A, the foci concyclic with A lie on one or other of the two Cartesians: there are consequently through F, G, H with the focus A two systems of circular cubics corresponding to the two Cartesians respectively, each system depending upon two arbitrary parameters. But if we attend only to one of the two Cartesians and to the corresponding system of cubics, then the Cartesian passes through the four foci of each cubic, and if (instead of taking as given the points F, G, H and the focus A) we take as given the four concyclic foci A, B, C, D of a cubic, the theorem asserts that we have through A, B, C, D a Cartesian depending on two arbitrary parameters (or having for its axis an arbitrary line), and such that the foci of the Cartesian are the points of intersection F, G, H of its axis with the cubic. And I proceed to the proof of the theorem in this form.

8. The equation of a circular cubic having four foci on the circle $x^2+y^2-1=0$ is

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0;$$

and this being so, the four foci are the intersections of the circle with the conic

$$(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl=0.$$

9. The general equation of a Cartesian is

$$(x^2+y^2+2Ax+2By+C)^2+2Dx+2Ey+F=0,$$

and by assuming for A, B, C, D, E, F, the following values which contain the two arbitrary parameters α and θ, viz. by writing

$$2A=\theta Q,\ 2B=-\theta P,\ C=\alpha-1,\ D=-n\theta^2P+(m\theta^2-\alpha\theta)Q,$$
$$E=(m\theta^2+\alpha\theta)P-l\theta^2Q,\ F=-\alpha^2+\theta^2(m^2-nl),$$

we have the equation of a system (the selected one out of two systems) of Cartesians through the four foci; in fact, substituting the foregoing values, the equation of the Cartesian is

$$\{x^2+y^2+\theta(Qx-Py)+\alpha-1\}^2-2\alpha\theta(Qx-Py)$$
$$+2\theta^2(-nP+mQ)x+2\theta^2(mP-lQ)y-\alpha^2+\theta^2(m^2-nl)=0,$$

and writing herein $x^2+y^2-1=0$, the equation reduces itself to

$$\theta^2\{(Qx-Py)^2+2(-nP+mQ)x+2(mP-lQ)y+m^2-nl\}=0,$$

verifying that the Cartesian passes through the four foci.

The coordinates of the centre of the Cartesian are $x=-A$, $y=-B$, and the equation of its axis is $E(x+A)-D(y+B)=0$; we have therefore to show that the points of intersection of this line with the cubic are the foci of the Cartesian.

10. To find where the line in question meets the cubic

$$(x^2+y^2+1)(Px+Qy)+lx^2+2mxy+ny^2=0,$$

writing in this equation

$$x=-A+D\Omega,\quad y=-B+E\Omega,$$

we have for the determination of Ω the equation

$$\{A^2+B^2+1-2(AD+BE)\Omega+(D^2+E^2)\Omega^2\}\times$$
$$\{-AP-BQ+(DP+EQ)\Omega\}+(l,\ m,\ n\between -A+D\Omega,\ -B+E\Omega)^2=0,$$

or observing that we have $AP+BQ=0$, this equation becomes

$$\begin{aligned}&(D^2+E^2)(DP+EQ)\Omega^3\\ &+\{-2(AD+BE)(DP+EQ)\quad +\ lD^2+2mDE+nE^2\}\Omega^2\\ &+\{\ (A^2+B^2+1)(DP+EQ)\quad -2lAD-2m(AE+BD)-2nBE\}\Omega\\ &+\{\qquad\qquad\qquad\qquad lA^2+2mAB+nB^2\}=0.\end{aligned}$$

11. Substituting for A, B, D, E their values in terms of $(P,\ Q,\ \alpha,\ \theta)$, we find

$$\begin{aligned}DP+\ EQ&=-\theta^2(nP^2-2mPQ+lQ^2),\\ lA^2+2mAB+nB^2&=\tfrac{1}{4}\theta^2(nP^2-2mPQ+lQ^2),\\ lAD+m(AE+BD)+nBE&=-\tfrac{1}{2}\alpha\theta^2(nP^2-2mPQ+lQ^2),\\ lD^2+2mDE+nE^2&=\left((nl-m^2)\theta^4+\alpha^2\theta^2\right)(nP^2-2mPQ+lQ^2),\end{aligned}$$

and substituting these values in the equation for Ω, the whole equation divides by $\theta^2(nP^2-2mPQ+lQ^2)$, and it then becomes

$$4(D^2+E^2)\Omega^3+4\{-2(AD+BE)-(nl-m^2)\theta^2-\alpha^2\}\Omega^2+4\{A^2+B^2+1-\alpha\}\Omega-1=0,$$

or, putting for shortness

$$\begin{aligned}C'&=C-A^2-B^2, &&=\alpha-1-A^2-B^2,\\ F'&=F-2(AD+BE), &&=-\alpha^2-\theta^2(nl-m^2)-2(AD+BE),\end{aligned}$$

the equation in Ω is

$$4(D^2+E^2)\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

so that, Ω satisfying this equation, the intersections of the axis with the cubic are given by $x=-A+D\Omega$, $y=-B+E\Omega$.

12. The equation of the Cartesian, writing therein $x+A=u$ and $y+B=v$, and attending to the values of C' and F', is

$$(u^2+v^2+C')^2+2Du+2Ev+F'=0.$$

And to find the foci, writing in this equation $u+\rho$, $v+i\rho$ in place of u, v, we find

$$\{u^2+v^2+C'+2(u+vi)\rho\}^2+2(D+Ei)\rho+2Du+2Ev+F'=0,$$

that is

$$(u^2+v^2+C')^2+2Du+2Ev+F'+\{2(u+vi)(u^2+v^2+C')+D+Ei\}\,2\rho+4(u+vi)^2\rho^2=0.$$

Expressing that this equation in ρ has two equal roots, we find

$$4(u+vi)^2\{(u^2+v^2+C')^2+2Du+2Ev+F'\}-\{2(u+vi)(u^2+v^2+C')+D+Ei\}^2=0,$$

that is

$$4(2Du+2Ev+F')(u+vi)^2-4(u^2+v^2+C')(u+vi)(D+Ei)-(D-Ei)^2=0,$$

which equation is in fact the equation of the three tangents from one of the circular points at infinity. Writing it under the form $U+Vi=0$, the nine foci of the Cartesian are given as the intersections of the two cubics $U=0$, $V=0$. But of these nine points, three, the foci that we are concerned with, lie on the axis, or line $Eu-Dv=0$; in fact, we have

$$\begin{aligned} U=4(u^2-v^2)(2Du+2Ev+F')\\ -4(uD-vE)(u^2+v^2+C')\\ -(D^2-E^2), \end{aligned} \qquad \begin{aligned} V=8uv(2Du+2Ev+F')\\ -4(uE+vD)(u^2+v^2+C')\\ -2DE; \end{aligned}$$

and hence

$$2DEU-(D^2-E^2)V=(Eu-Dv)\{8(Du+Ev)(2Du+2Ev+F')-4(D^2+E^2)(u^2+v^2+C')\}=0,$$

which shows that the nine points lie three of them on the line $Eu-Dv=0$, and the remaining six on the conic

$$2(Du+Ev)(2Du+2Ev+F')-(D^2+E^2)(u^2+v^2+C')=0.$$

13. We have thus the three foci given as the intersections of the axis $Eu-Dv=0$, with the cubic

$$U=4(u^2-v^2)(2Du+2Ev+F')-4(uD-vE)(u^2+v^2+C')-(D^2-E^2)=0;$$

or, writing in this last equation $u=D\Omega$, $v=E\Omega$, that is $x=-A+D\Omega$, $y=-B+E\Omega$, we have

$$u^2-v^2=(D^2-E^2)\Omega^2,\quad uD-vE=(D^2-E^2)\Omega.$$

The whole equation divides by (D^2-E^2), and omitting this factor, it is

$$4\Omega^2\{2(D^2+E^2)\Omega+F'\}-4\Omega\{(D^2+E^2)\Omega^2+C'\}-1=0,$$

that is

$$4(D^2+E^2)\Omega^3+4F'\Omega^2-4C'\Omega-1=0,$$

the same equation as the equation in Ω before obtained; that is the intersections of the cubic with the axis are the three foci of the Cartesian.

[Vol. VI. pp. 57—59.]

1949. (Proposed by Professor CAYLEY.)—Find the conic of five-pointic intersection at any point of the cuspidal cubic $y^3=x^2z$.

Solution by the PROPOSER.

The equation $y^3 = x^2 z$, is satisfied by the values $x : y : z = 1 : \theta : \theta^3$; and conversely, to any given value of the parameter θ there corresponds a point on the cubic $y^3 = x^2 z$. Consider the five points corresponding to the values $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ respectively; the equation of the conic through these five points is

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0,$$

where the remaining four lines of the determinant are obtained from the second line by writing therein $\theta_2, \theta_3, \theta_4, \theta_5$ successively in place of θ_1. Writing for shortness $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ to denote the product of the differences of the quantities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, the equation contains the factor $\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, and we may therefore write it in the simplified form

$$\frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix} = 0.$$

Hence putting in this equation $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \phi$, we have the equation of the conic of five-pointic intersection at the point (ϕ). The result in its reduced form may be obtained directly without much difficulty, but it is obtained most easily as follows: let the function on the left hand of the foregoing equation be represented by

$$(a, b, c, f, g, h \between x, y, z)^2,$$

then writing $x : y : z = 1 : \theta : \theta^3$, we have

$$(a, b, c, f, g, h \between 1, \theta, \theta^3)^2$$

$$= \frac{1}{\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix},$$

$$= \frac{(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)}{\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)} \begin{vmatrix} 1, & \theta^2, & \theta^6, & \theta^4, & \theta^3, & \theta \\ 1, & \theta_1^2, & \theta_1^6, & \theta_1^4, & \theta_1^3, & \theta_1 \\ \vdots & & & & & \end{vmatrix},$$

$$= (\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)(\theta-\theta_4)(\theta-\theta_5)(\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5);$$

for the determinant, which is a function of the order 16 in the quantities $(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ conjointly, divides by $\zeta^{\frac{1}{2}}(\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, which is a function of the order 15; and as the quotient is a symmetrical function of $\theta, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, it must be equal, save to a numerical factor which may be disregarded, to $\theta+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5$.

Hence if ϕ be the parameter of the given point, writing $\theta_1=\theta_2=\theta_3=\theta_4=\theta_5=\phi$, we have

$$(a,\ b,\ c,\ f,\ g,\ h\between 1,\ \theta,\ \theta^3)^2=(\theta-\phi)^5(\theta+5\phi)$$
$$=(1,\ 0,\ -15,\ +40,\ -45,\ +24,\ -5\between\theta,\ \phi)^6,$$

where the left-hand side is

$$a+b\theta^2+c\theta^6+f\theta^4+g\theta^3+h\theta,=\{c,\ 0,\ f,\ g,\ b,\ h,\ a\}(\theta,\ 1)^6,$$

that is we have

$$c=1,\quad f=-15\phi^2,\quad g=40\phi^3,\quad b=-45\phi^4,\quad h=24\phi^5,\quad a=-5\phi^6,$$

and the equation of the conic of five-pointic intersection therefore is

$$(-5\phi^6,\ -45\phi^4,\ 1,\ -15\phi^2,\ 40\phi^3,\ 24\phi^5\between x,\ y,\ z)^2=0,$$

or, what is the same thing,

$$-5\phi^6x^2-45\phi^4y^2+z^2-15\phi^2yz+40\phi^3zx+24\phi^5xy=0,$$

which is the required result.

NOTE. The condition in order that any six points $(\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6)$ of the cubic $y^3=x^2z$ may lie on a conic, is

$$\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6=0.$$

[Vol. VI. p. 65.]

1872. (Proposed by Professor CAYLEY.)—Show that the surfaces

$$xyz=1,\ yz+zx+xy+x+y+z+3=0,$$

intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

[Vol. VI. pp. 67—69.]

1969. (Proposed by Professor SYLVESTER.)—In two given great circles of a sphere intersecting at O are taken respectively two points P and Q, the arc joining which is of given length: prove that S, H two fixed points, and M a fixed line, in a plane may be found such that, for all positions of the arc PQ, a point M in the fixed line may be found satisfying the equations

$$SM\pm HM=\sin OP,\quad SM\mp HM=\sin OQ.$$

Solution by PROFESSOR CAYLEY.

1. In the spherical triangle OPQ, whereof the sides OP, OQ, PQ are θ, ϕ, β and the angle O is $=\alpha$, the relation between these quantities is $\cos\alpha = \dfrac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi}$; hence treating α, β as constants, and θ, ϕ as variable angles connected by the foregoing equation, it is required to show that we can find two fixed points S, H and a fixed line, such that taking M a variable point in this line and writing $SM=r$, $HM=s$, the relation between r and s (or equation of the fixed line in terms of r, s as coordinates of a point thereof) is obtained by substituting in the foregoing equation for θ and ϕ the values given by the two equations

$$\sin\theta = (r+s), \quad \sin\phi = (r-s),$$

or as, for the sake of homogeneity, it will be more convenient to write these equations,

$$m\sin\theta = (r+s), \quad m\sin\phi = (r-s).$$

2. Suppose that the perpendicular distances of S, H from the fixed line are a and b, and that the distance between the feet of the two perpendiculars is $2c$, then if x denote the distance of the point M from the midway point between the feet of the two perpendiculars, we have

$$r = \sqrt{\{(c+x)^2 + a^2\}}, \quad s = \sqrt{\{(c-x)^2 + b^2\}},$$

and (a, b, c) being properly determined, the elimination of x from these equations should give between (r, s) a relation equivalent to that obtained by the elimination of (θ, ϕ) from the before-mentioned equations. Or, what is the same thing, the elimination of (r, s, x) from the equations

$$m\sin\theta = r+s, \quad m\sin\phi = r-s, \quad r = \sqrt{\{(c+x)^2 + a^2\}}, \quad s = \sqrt{\{(c-x)^2 + b^2\}}$$

should give between (θ, ϕ) the relation

$$\cos\alpha = \frac{\cos\beta - \cos\theta\cos\phi}{\sin\theta\sin\phi};$$

that is, the last-mentioned equation should be obtained by the elimination of x from the equations

$$m(\sin\theta + \sin\phi) = 2\sqrt{\{(c+x)^2 + a^2\}}, \quad m(\sin\theta - \sin\phi) = 2\sqrt{\{(c-x)^2 + b^2\}}.$$

3. The equation in (θ, ϕ) may be written

$$\cos\beta - \cos\alpha\sin\theta\sin\phi = \cos\theta\cos\phi,$$

or, squaring and reducing,

$$\sin^2\theta + \sin^2\phi = \sin^2\beta + 2\cos\alpha\cos\beta\sin\theta\sin\phi + \sin^2\alpha\sin^2\theta\sin^2\phi,$$

that is

$$\sin^2\theta + \sin^2\phi = \frac{1 - \cos^2\alpha - \cos^2\beta}{\sin^2\alpha} + \left(\sin\alpha\sin\theta\sin\phi + \frac{\cos\alpha\cos\beta}{\sin\alpha}\right)^2.$$

But from the two equations in x, we have

$$m^2(\sin^2\theta+\sin^2\phi)=4c^2+2a^2+2b^2+4x^2,\quad m^2\sin\theta\sin\phi=4cx+a^2-b^2,$$

whence

$$2x=\frac{b^2-a^2+m^2\sin\theta\sin\phi}{2c},$$

therefore

$$\sin^2\theta+\sin^2\phi=\frac{4c^2+2b^2+2a^2}{m^2}+\left(\frac{b^2-a^2+m^2\sin\theta\sin\phi}{2cm}\right)^2.$$

Hence, comparing the two results, we have

$$\frac{1-\cos^2\alpha-\cos^2\beta}{\sin^2\alpha}=\frac{4c^2+2b^2+2a^2}{m^2},\quad \frac{\cos\alpha\cos\beta}{\sin\alpha}=\frac{b^2-a^2}{2cm},\quad \sin\alpha=\frac{m}{2c};$$

or, as these may also be written,

$$\sin\alpha=\frac{m}{2c},\quad \cos^2\alpha+\cos^2\beta=\frac{-b^2-a^2}{2c^2},\quad 2\cos\alpha\cos\beta=\frac{b^2-a^2}{2c^2};$$

whence

$$(\cos\alpha+\cos\beta)^2=\frac{-a^2}{c^2},\quad (\cos\alpha-\cos\beta)^2=\frac{-b^2}{c^2},\quad \sin\alpha=\frac{m}{2c};$$

so that m being put equal to unity, or otherwise assumed at pleasure, a, b, c are given functions of α, β. Or conversely, if a, b, c are assumed at pleasure, then α, β, m are given functions of these quantities.

5. It is to be remarked that (α, β) being real, a and b will be imaginary, and consequently the points S, H of Professor Sylvester's theorem are imaginary [1]; if, however, we write $-a^2$, $-b^2$ in place of a^2, b^2 respectively, then the radicals $\sqrt{\{(c+x)^2-a^2\}}$, $\sqrt{\{(c-x)^2-b^2\}}$ have a real geometrical interpretation. The system of relations between $(\alpha, \beta, a, b, c, m)$ becomes

$$(\cos\alpha+\cos\beta)^2=\frac{a^2}{c^2},\quad (\cos\alpha-\cos\beta)^2=\frac{b^2}{c^2},\quad \sin\alpha=\frac{m}{2c};$$

and considering (a, b, c) as given, we may write

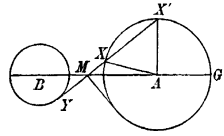
$$\cos\alpha=\frac{a+b}{2c},\quad \cos\beta=\frac{a-b}{2c},\quad m=\sqrt{\{4c^2-(a+b)^2\}},$$

viz. we have either this system or the similar system obtained by writing $-b$ in place of b.

6. Consider two circles with the radii a, b and having the distance of their centres $=2c$, and to fix the ideas assume that $2c>a+b$, that is, that the circles are

[1] Prof. Sylvester remarks that according as β is less or greater than α, we may find real values of θ, ϕ equal to one another in the one case and supplementary in the other. Hence we must in any case be able to make $r=0$ and $s=0$ indifferently, which shows *à priori* that the line being supposed real, each point S, H must be imaginary, but so that the squared distance of either from the line must be a *real negative quantity*, conformably to Prof. Cayley's important observation in the text. W. J. M.

exterior to each other. The foregoing equations signify that $90^\circ - \alpha$, $90^\circ - \beta$ are the inclinations to the line of centres of the inverse and the direct common tangents respectively, and that m is the length of the inverse common tangent. And the theorem is, that considering two circles as above, and taking M a variable point in



the line of centres, if r, s denote the tangential distances of M from the two circles respectively, and if m be the length of the inverse common tangent of the two circles, then the angles θ, ϕ determined by the equations

$$m \sin \theta = r + s, \quad m \sin \phi = r - s,$$

are connected by the relation

$$\cos \beta = \cos \theta \cos \phi + \sin \theta \sin \phi \cos \alpha,$$

(α, β) being constant angles, determined as above.

7. It is to be remarked that, assuming

$$k = \frac{\sin \alpha}{\sin \beta} = \frac{\sqrt{\{4c^2 - (a+b)^2\}}}{\sqrt{\{4c^2 - (a-b)^2\}}},$$

that is, $k =$ inverse common tangent $\div$ direct common tangent, then we have

$$\cos \alpha = \sqrt{(1 - k^2 \sin^2 \beta)} = \Delta\beta,$$

or the equation in θ, ϕ becomes

$$\cos \beta = \cos \theta \cos \phi + \sin \theta \sin \phi \, \Delta\beta,$$

which is the algebraical equation connecting the amplitudes of the elliptic functions in the relation $F(\theta) + F(\phi) = F(\beta)$.

8. It is very noticeable that the above figure leads to another relation in elliptic functions, viz. it is the very figure employed for that purpose by Jacobi; in fact, considering therein YM as a variable tangent meeting the circle A in the two points X and X', then if 2ψ, $2\psi'$ denote the angles GAX, GAX' respectively, it is easy to see geometrically that we have $d\psi : d\psi' = YX : YX'$; where

$$(YX)^2 = (BX)^2 - b^2, \; = 4c^2 + a^2 + 4ac \cos 2\psi - b^2, \; = (2c+a)^2 - b^2 - 8ac \sin^2 \psi,$$

and similarly $(YX')^2 = (2c+a)^2 - b^2 - 8ac \sin^2 \psi'$, that is, writing $l^2 = \dfrac{8ac}{(2c+a)^2 - b^2}$, the differential equation is

$$\frac{d\psi}{\sqrt{(1 - l^2 \sin^2 \psi)}} - \frac{d\psi'}{\sqrt{(1 - l^2 \sin^2 \psi')}} = 0,$$

corresponding to an integral equation

$$F(\psi) - F(\psi') = F(\mu),$$

the modulus of the elliptic functions being

$$l, = \frac{\sqrt{8ac}}{\sqrt{\{(2c+a)^2 - b^2\}}}.$$

In the problem above considered the modulus is

$$k, = \frac{\sqrt{\{4c^2 - (a+b)^2\}}}{\sqrt{\{4c^2 - (a-b)^2\}}},$$

and it is not very easy to see the connexion between the two results.

[Vol. VI. p. 81.]

Theorem: by PROFESSOR CAYLEY.

If (A, A'), (B, B') are four points (two real and the other two imaginary) related to each other as foci and antifoci (that is, if the lines AA', BB' intersect at right angles in a point O in such wise that $OA = OA' = i \,.\, OB = i \,.\, OB'$), then the product of the distances of any point P from the points A, A' is equal to the product of the distances of the same point P from the points B, B'.

In fact, the coordinates of A, A' may be taken to be $(\alpha, 0)$, $(-\alpha, 0)$, and those of B, B' to be $(0, \alpha i)$, $(0, -\alpha i)$; whence, if (x, y) are the coordinates of P, we have

$$\begin{aligned}
(AP)^2 &= (x-\alpha)^2 + y^2 = (x - \alpha + iy)(x - \alpha - iy),\\
(A'P)^2 &= (x+\alpha)^2 + y^2 = (x + \alpha + iy)(x + \alpha - iy),\\
(BP)^2 &= x^2 + (y - i\alpha)^2 = (x + iy + \alpha)(x - iy - \alpha),\\
(B'P)^2 &= x^2 + (y + i\alpha)^2 = (x + iy - \alpha)(x - iy + \alpha),
\end{aligned}$$

from which the theorem is at once seen to be true.

An important application of the theorem consists in the means which it affords of passing from the foci (A, B, C, D) of a bicircular quartic, to the antifoci (A, B) and (C, D); viz. if these are (A', B', C', D'), then the equation $l\sqrt{(A)} + m\sqrt{(B)} + n\sqrt{(C)} = 0$ must be transformable into $l'\sqrt{(A')} + m'\sqrt{(B')} + n'\sqrt{(C')} = 0$. Writing these respectively under the forms

$$l^2A + m^2B - n^2C + 2lm\sqrt{(AB)} = 0, \quad l'^2A' + m'^2B' - n'^2C' + 2l'm'\sqrt{(A'B')} = 0,$$

the two radicals $\sqrt{(AB)}$, $\sqrt{(A'B')}$ are identical; and the remaining terms in the two equations respectively are rational functions, which when the ratios $l' : m' : n'$ are properly determined will be to each other in the ratio $lm : l'm'$; the two equations being thus identical.

[Vol. VI. p. 99.]

1970. (Proposed by Professor CAYLEY.)—Find the conditions in order that the conics

$$U=(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0,\quad U'=(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2=0,$$

may have double contact.

Solution by the PROPOSER.

The coefficients of the two conics must be so related that for a properly determined value of θ we shall have identically $U-\theta U'=(\lambda x+\mu y+\nu z)^2$; but when this is so, the inverse coefficients of the quadric function $U-\theta U'$ are each $=0$; that is, writing

$$\begin{aligned}
(A,\ B,\ C,\ F,\ G,\ H) &= (bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch)\\
(A',\ B',\ C',\ F',\ G',\ H') &= (b'c'-f'^2,\ldots \qquad\qquad g'h'-a'f',\ldots)\\
(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}) &= (bc'+b'c-2ff',\ldots \qquad gh'+g'h-af'-a'f,\ldots),
\end{aligned}$$

then we have the six equations $A-\theta\mathfrak{A}+\theta^2A'=0$, &c.

Or, eliminating θ, the required conditions are

$$\left\|\begin{matrix} A, & B, & C, & F, & G, & H\\ A', & B', & C', & F', & G', & H'\\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H}\end{matrix}\right\|=0,$$

equivalent to three relations between the two sets of coefficients.

[Vol. VII., January to July, 1867, pp. 17—19.]

2110. (Proposed by Professor CAYLEY.)—Prove that the locus of the foci of the parabolas which pass through three given points is a unicursal quintic curve passing through the two circular points at infinity.

Solution by the PROPOSER.

More generally it may be shown that for the conics which pass through three given points and touch a given line, the locus of the intersection of the tangents drawn from two fixed points Q, Q' on this line to each conic of the series is a unicursal quintic passing through the two points Q and Q'.

Taking the three given points to be the angles of the triangle $(x=0,\ y=0,\ z=0)$, and the points Q, Q' to be the points $(\alpha,\ \beta,\ \gamma)$ and $(\alpha',\ \beta',\ \gamma')$ respectively, the equation of a conic through the three points is

$$fyz+gzx+hxy=0,$$

which conic will touch the line through the points $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma')$, if

$$\sqrt{\{f(\beta\gamma' - \beta'\gamma)\}} + \sqrt{\{g(\gamma\alpha' - \gamma'\alpha)\}} + \sqrt{\{h(\alpha\beta' - \alpha'\beta)\}} = 0.$$

The equation of the pair of tangents from (α, β, γ) to the conic is

$$(f^2, g^2, h^2, -gh, -hf, -bg \between \gamma y - \beta z, \alpha z - \gamma x, \beta x - \alpha y)^2 = 0,$$

that is

$$\begin{aligned} & x^2(g\gamma + h\beta)^2 + y^2(h\alpha + f\gamma)^2 + z^2(f\beta + g\alpha)^2 \\ & + 2yz\{2gh\alpha^2 - (h\alpha + f\gamma)(f\beta + g\alpha)\} \\ & + 2zx\{2hg\beta^2 - (f\beta + g\alpha)(g\gamma + h\beta)\} \\ & + 2xy\{2fg\gamma^2 - (g\gamma + h\beta)(h\alpha + f\gamma)\} = 0, \end{aligned}$$

but one of the tangents through (α, β, γ) being

$$x(\beta\gamma' - \beta'\gamma) + y(\gamma\alpha' - \gamma'\alpha) + z(\alpha\beta' - \alpha'\beta) = 0,$$

it follows that the other tangent is

$$x\frac{(g\gamma + h\beta)^2}{\beta\gamma' - \beta'\gamma} + y\frac{(h\alpha + f\gamma)^2}{\gamma\alpha' - \gamma'\alpha} + z\frac{(f\beta + g\alpha)^2}{\alpha\beta' - \alpha'\beta} = 0.$$

Hence, writing for shortness

$$\begin{aligned} A &= g\gamma + h\beta, & B &= h\alpha + f\gamma, & C &= f\beta + g\alpha, \\ A' &= g\gamma' + h\beta', & B' &= h\alpha' + f\gamma', & C' &= f\beta' + g\alpha', \end{aligned}$$

the equations of the tangents from Q, Q' respectively are

$$A^2\frac{x}{\beta\gamma' - \beta'\gamma} + B^2\frac{y}{\gamma\alpha' - \gamma'\alpha} + C^2\frac{z}{\alpha\beta' - \alpha'\beta} = 0,$$

$$A'^2\frac{x}{\beta\gamma' - \beta'\gamma} + B'^2\frac{y}{\gamma\alpha' - \gamma'\alpha} + C'^2\frac{z}{\alpha\beta' - \alpha'\beta} = 0,$$

and for the coordinates of the intersection of these tangents, we have

$$\frac{x}{\beta\gamma' - \beta'\gamma} : \frac{y}{\gamma\alpha' - \gamma'\alpha} : \frac{z}{\alpha\beta' - \alpha'\beta} = B^2C'^2 - B'^2C^2 : C^2A'^2 - C'^2A^2 : A^2B'^2 - A'^2B^2.$$

$$\begin{aligned} BC' - B'C &= f\{-f(\beta\gamma' - \beta'\gamma) + g(\gamma\alpha' - \gamma'\alpha) + h(\alpha\beta' - \alpha'\beta)\} \\ BC' + B'C &= 2gh\alpha\alpha' + f\{f(\beta\gamma' + \beta'\gamma) + g(\gamma\alpha' + \gamma'\alpha) + h(\alpha\beta' + \alpha'\beta)\}. \end{aligned}$$

To satisfy the equation

$$\sqrt{\{f(\beta\gamma' - \beta'\gamma)\}} + \sqrt{\{g(\gamma\alpha' - \gamma'\alpha)\}} + \sqrt{\{h(\alpha\beta' - \alpha'\beta)\}},$$

write

$$f = \frac{a^2}{\beta\gamma' - \beta'\gamma}, \qquad g = \frac{b^2}{\gamma\alpha' - \gamma'\alpha}, \qquad h = \frac{c^2}{\alpha\beta' - \alpha'\beta},$$

and therefore $a + b + c = 0$; we then have

$$-f(\beta\gamma' - \beta'\gamma) + g(\gamma\alpha' - \gamma'\alpha) + h(\alpha\beta' - \alpha'\beta), = -a^2 + b^2 + c^2, = -2bc;$$

and thence

$$f\{-f(\beta\gamma'-\beta'\gamma)+g(\gamma\alpha'-\gamma'\alpha)+h(\alpha\beta'-\alpha'\beta)\}=\frac{a^2}{\beta\gamma'-\beta'\gamma}(-2bc),$$

and the equations become

$$x : y : z = a(BC'+B'C) : b(CA'+C'A) : c(AB'+A'B),$$

where $BC'+B'C$, $CA'+C'A$, $AB'+A'B$, substituting therein for f, g, h the values $\frac{a^2}{\beta\gamma'-\beta'\gamma}$, $\frac{b^2}{\gamma\alpha'-\gamma'\alpha}$, $\frac{c^2}{\alpha\beta'-\alpha'\beta}$, are respectively functions of the fourth degree in a, b, c; hence (a, b, c) being connected by the relation $a+b+c=0$, x, y, z are proportional to quintic functions of (a, b, c), or what is the same thing, writing $a, b, c=1, \theta, -1-\theta$, then x, y, z are proportional to quintic functions of θ, that is, the locus is a unicursal quintic curve.

That the curve passes through the points $(\alpha', \beta', \gamma')$ and (α, β, γ) appears by considering the conics $fyz+gzx+hxy=0$, which pass through these points respectively.

For the first of these conics we have $f : g : h = \alpha(\beta\gamma'-\beta'\gamma) : \beta(\gamma\alpha'-\gamma'\alpha) : \alpha(\beta\gamma'-\beta'\gamma)$; the equation

$$A^2\frac{x}{\beta\gamma'-\beta'\gamma}+B^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C^2\frac{z}{\alpha\beta'-\alpha'\beta}=0,$$

reduces itself to $x(\beta\gamma'-\beta'\gamma)+y(\gamma\alpha'-\gamma'\alpha)+z(\alpha\beta'-\alpha'\beta)=0$, and as the other equation

$$A'^2\frac{x}{\beta\gamma'-\beta'\gamma}+B'^2\frac{y}{\gamma\alpha'-\gamma'\alpha}+C'^2\frac{z}{\gamma\alpha'-\gamma'\alpha}=0,$$

is that of a line through $(\alpha', \beta', \gamma')$ the two lines meet of course in the point $(\alpha', \beta', \gamma')$. And the like for the conic

$$f : g : h = \alpha'(\beta\gamma'-\beta'\gamma) : \beta'(\gamma\alpha'-\gamma'\alpha) : \gamma'(\alpha\beta'-\alpha'\beta).$$

If the triangle is equilateral, and (x, y, z) are respectively proportional to the perpendicular distances from the three sides, then we have for the circular points at infinity

$$(\alpha, \beta, \gamma)=(1, \omega, \omega^2), \quad (\alpha', \beta', \gamma')=(1, \omega^2, \omega),$$

where ω is an imaginary cube root of unity. These values give

$$\beta\gamma'-\beta'\gamma=\gamma\alpha'-\gamma'\alpha=\alpha\beta'-\alpha'\beta=\omega^2-\omega$$
$$\alpha\alpha'=\beta\beta'=\gamma\gamma'=1, \quad \beta\gamma'+\beta'\gamma=\gamma\alpha'+\gamma'\alpha=\alpha\beta'+\alpha'\beta=-1;$$

and the expressions for (x, y, z) take the form

$$\begin{aligned} x : y : z = a&\{2b^2c^2-a^2(a^2+b^2+c^2)\} \\ : b&\{2c^2a^2-b^2(a^2+b^2+c^2)\} \\ : c&\{2a^2b^2-c^2(a^2+b^2+c^2)\}, \end{aligned}$$

or, what is the same thing, reducing by means of the relation $a+b+c=0$,

$$x : y : z = a(a^4-2a^2bc-2b^2c^2) : b(b^4-2b^2ca-2c^2a^2) : c(c^4-2c^2ab-2a^2b^2),$$

and the equation of the curve is obtained by eliminating (a, b, c) from these equations and the before mentioned equation $a+b+c=0$.

N.B. The above is a particular case of the following general theorem of M. Chasles: If the conics of a system (μ, ν) all of them touch the line QQ', the locus of the intersection of the tangents through Q, Q' to each conic of the series is a curve of the order $\frac{1}{2}\mu+\nu$, having a $(\frac{1}{2}\mu)$-tuple point at the points Q, Q' respectively.

[Vol. VII. pp. 26, 27.]

2250. (Proposed by Professor CAYLEY.)—From the focal equation $x^2+y^2=(lx+n)^2$ of a conic, deduce the remaining three focal equations.

Solution by the PROPOSER.

We are to find α, β, L, M, N such that the equation

$$(x-\alpha)^2+(y-\beta)^2=(Lx+My+N)^2$$

may be identical with the given equation. It is at once seen that we must have $M=0$ or else $L=0$; the first supposition gives two solutions, one of which is the given equation itself, the other is

$$\left(x-\frac{2ln}{1-l^2}\right)^2+y^2=\left\{lx-n\frac{1+l^2}{1-l^2}\right\}^2.$$

The second supposition, $L=0$, gives two solutions, which only differ by the sign of i $(=\sqrt{-1})$, viz. these are

$$\left(x-\frac{ln}{1-l^2}\right)^2+\left(y\mp\frac{lni}{1-l^2}\right)^2=\frac{-(ly\pm ni)^2}{1-l^2}.$$

There is, of course, no difficulty in *verifying* the identity of each of the three forms with the given form $x^2+y^2=(lx+n)^2$.

[Vol. VII. pp. 33, 34.]

1991. (Proposed by Professor CAYLEY.)—Given a point and three lines; it is required to draw through the point a plane meeting the three lines in three points equidistant from the given point.

Solution by the PROPOSER.

Let O be the given point, $OA'=a$, $OB'=b$, $OC'=c$ the perpendiculars let fall from O on the given lines respectively. Take θ an arbitrary line, and from the points

A', B', C' measure off on the three lines respectively the distances $A'A = \pm\sqrt{(\theta^2 - a^2)}$, $B'B = \pm\sqrt{(\theta^2 - b^2)}$, $C'C = \pm\sqrt{(\theta^2 - c^2)}$, or, considering each radical as containing implicitly the sign $\pm$, what is the same thing, the distances $A'A = \sqrt{(\theta^2 - a^2)}$, $B'B = \sqrt{(\theta^2 - b^2)}$, $C'C = \sqrt{(\theta^2 - c^2)}$, then we have $OA = OB = OC (= \theta)$; and consequently the problem is to determine θ in such wise that the plane ABC may pass through the given point O: for we shall then have through O a plane meeting the three given lines in the points A, B, C equidistant from O.

The coordinates of A, B, C are linear functions of the radicals $\sqrt{(\theta^2 - a^2)}$, $\sqrt{(\theta^2 - b^2)}$, $\sqrt{(\theta^2 - c^2)}$ respectively. Taking O as origin, the condition in order that the plane ABC may pass through O is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

and substituting for the coordinates their values in terms of θ, this is an equation linear in each of the three radicals, or say, an equation of the form

$$\left(\sqrt{(\theta^2 - a^2)},\ 1\right)\left(\sqrt{(\theta^2 - b^2)},\ 1\right)\left(\sqrt{(\theta^2 - c^2)},\ 1\right) = 0.$$

But we may represent any one of the three radicals, say $\sqrt{(\theta^2 - c^2)}$ by a single letter s; and this being so, we have $\sqrt{(\theta^2 - a^2)} = \sqrt{(s^2 + c^2 - a^2)} = \sqrt{P}$ suppose, and $\sqrt{(\theta^2 - b^2)} = \sqrt{(s^2 + c^2 - b^2)} = \sqrt{Q}$ suppose; and it is to be observed that there is no loss of generality in assuming that the distance $C'c = s$ is measured off from C' in a determinate sense, for as s passes from $-\infty$ to $+\infty$, we thus obtain for c every position whatever on the line in question; whereas the other two distances $A'A$, $B'B$, represented by the radicals $\sqrt{P}$ and $\sqrt{Q}$ respectively, remain each of them with the double sense $\pm$. The equation in s is of the form

$$(s,\ 1)(\sqrt{P},\ 1)(\sqrt{Q},\ 1) = 0,$$

or, what is the same thing, it is of the form

$$\alpha\sqrt{(PQ)} + \beta\sqrt{P} + \gamma\sqrt{Q} + \delta = 0,$$

where $(\alpha, \beta, \gamma, \delta)$ are respectively linear functions of s.

Proceeding to rationalise the equation, we have first

$$\alpha^2 PQ + 2\alpha\delta\sqrt{(PQ)} + \delta^2 = \beta^2 P + \gamma^2 Q + 2\beta\gamma\sqrt{(PQ)},$$

and then finally

$$(\alpha^2 PQ - \beta^2 P - \gamma^2 Q + \delta^2)^2 = 4(\beta\gamma - \alpha\delta)^2 PQ,$$

which, observing that P, Q are each of them of the second order in s, is an equation of the twelfth order in s; that is, the number of solutions is $= 12$.

The solution of the problem is greatly simplified when $a = b = c$, that is, when the three given lines are tangents to a sphere having its centre at the given point. We have in this case $\sqrt{P} = \pm s$, $\sqrt{Q} = \pm s$, or the equation in s is

$$(s,\ 1)(\pm s,\ 1)(\pm s,\ 1) = 0;$$

that is, the equation of the twelfth order breaks up into four equations each of the third order. The geometrical theory may also be further developed. In fact, assuming on each of the three lines respectively a certain sense as positive (and thus isolating a set of three solutions) the construction is, on the three lines, from the points A', B', C' respectively, measure off the distances $A'A = B'B = C'C = s$. Then the points A, B, C form on the three lines respectively three homographic series; that is, the lines BC, CA, AB are respectively generating lines of three hyperboloids, viz. hyperboloids which pass respectively through the second and third lines, the third and first lines, and the first and second lines. Taking the given point O as the centre of projection, and projecting the whole figure on any plane whatever, the projections of the lines BC are the tangents of a conic which is the projection of the visible contour of the hyperboloid generated by the lines BC; and the like for the lines CA and AB. Hence in the projection, or plane figure, we have a triangle whereof the sides A', B', C' are the projections of the three given lines respectively; inscribed in this triangle we have a variable triangle ABC, such that the side

BC envelopes a conic, say (A), which touches B' and C',

CA envelopes a conic, say (B), which touches C' and A',

AB envelopes a conic, say (C), which touches A' and B'.

The conics $(A)(B)(C)$ have three common tangents, say L, M, N; the conics

(B) and (C) having besides the common tangent A',

(C) and (A) having besides the common tangent B',

(A) and (B) having besides the common tangent C',

so that the common tangents of the conics (B) and (C), (C) and (A), (A) and (B) are the lines A', B', C' each once, and the lines L, M, N each three times. In the entire series of triangles ABC there are three triangles which degenerate into the lines L, M, N respectively, these being in fact the projections of the triangles ABC of the solid figure which lie in a plane with O. Or, what is the same thing, the planes of the required triangles ABC of the solid figure are the planes through O and the three lines L, M, and N, respectively.

[Vol. VII. pp. 34—36.]

1993. (Proposed by T. COTTERILL, M.A.)—If P is a point on a circle, in which A and B are fixed points on a diameter at equal distances from its centre, the curve envelope of lines cutting harmonically the two circles whose centres are A and B and radii AP, BP respectively, is independent of the position of P on the circle.

Solution by PROFESSOR CAYLEY.

1. More generally, the problem may be thus stated: If two conics touch at I, J the lines OI, OJ respectively; if P be a variable point on the first conic, and OAB

a fixed line through O meeting the second conic in the points A and B; then considering the conic which passes through P and touches at I, J the lines AI, AJ respectively, and also the conic which passes through P and touches at I, J the lines BI, BJ respectively; the envelope of the lines which cut harmonically the last-mentioned two conics is a conic independent of the position of P.

2. Taking $x=0$, $y=0$, $z=0$ for the equations of the lines OI, JI, and OJ respectively, the equations of the two given conics are

$$xz-y^2=0, \quad kxz-y^2=0;$$

hence the coordinates of P may be taken to be

$$x : y : z = 1 : \theta : \theta^2,$$

and the coordinates of the points A and B may be taken to be

$$x : y : z = 1 : k\alpha : k\alpha^2, \text{ and } x : y : z = 1 : -k\alpha : k\alpha^2.$$

The equations of the lines AI, AJ are

$$k\alpha x - y = 0, \quad z - \alpha y = 0;$$

hence the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(k\alpha x - y)(z-\alpha y)}{(k\alpha-\theta)(\theta-\alpha)} = \frac{y^2}{\theta},$$

and similarly the equations of the lines BI, BJ being

$$k\alpha x + y = 0, \quad z + \alpha y = 0,$$

the equation of the conic touching these lines at the points I, J respectively, and also passing through the point P, is

$$\frac{(k\alpha x + y)(z+\alpha y)}{(k\alpha+\theta)(\theta+\alpha)} = \frac{y^2}{\theta},$$

or multiplying out and reducing, if the equations of the two conics are represented by

$$(a, b, c, f, g, h \between x, y, z)^2 = 0, \quad (a', b', c', f', g', h' \between x, y, z)^2 = 0,$$

respectively, then the values of the coefficients are

$$\begin{aligned}
a &= 0, & a' &= 0,\\
b &= 2(k\alpha + \theta^2 - k\alpha\theta), & b' &= 2(-k\alpha^2 - \theta^2 - k\alpha\theta),\\
c &= 0, & c' &= 0,\\
f &= -\theta, & f' &= \theta,\\
g &= \theta k\alpha, & g' &= \theta k\alpha,\\
h &= -\theta k\alpha^2, & h' &= \theta k\alpha^2.
\end{aligned}$$

Now the tangential equation of the envelope of the line which cuts harmonically the last-mentioned two conics, is

$$(bc'+b'c-2ff', \ldots gh'+g'h-af'-a'f, \ldots \between \xi, \eta, \zeta)^2;$$

or substituting for a &c. a' &c., their values, it is found that the coefficients of this equation have all of them the common factor $2\theta^2$, and that omitting this factor the equation is independent of θ, viz. the tangential equation of the envelope in question is

$$(1, -k^2\alpha^2, k^2\alpha^4, 0, k(2k-1)\alpha^2, 0 \between \xi, \eta, \zeta)^2 = 0,$$

which proves the theorem.

3. *In particular*, if $k=1$, that is if the points A, B lie on the conic $xz-y^2=0$, then the tangential equation of the envelope is

$$(1, -\alpha^2, \alpha^4, 0, \alpha^2, 0 \between \xi, \eta, \zeta)^2 = 0,$$

that is

$$\xi^2 - \alpha^2\eta^2 + \alpha^4\zeta^2 + 2\alpha^2\xi\zeta = 0;$$

or, what is the same thing, the equation is

$$(\xi - \alpha\eta + \alpha^2\zeta)(\xi + \alpha\eta + \alpha^2\zeta) = 0,$$

and thus the envelope breaks up into the two points

$$\xi - \alpha\eta + \alpha^2\zeta = 0, \quad \xi + \alpha\eta + \alpha^2\zeta = 0;$$

that is, the points $(1, -\alpha, \alpha^2)$ and $(1, \alpha, \alpha^2)$, which are the points A and B respectively. That is, in the problem in its original form, if the points A and B are the extremities of a diameter of a given circle, then the two constructed circles are a pair of orthotomic circles with the centres A and B respectively; and the theorem is the very obvious one, that any line through the centre of either circle cuts the two circles harmonically.

[Vol. VII. pp. 52, 53.]

2270. (Proposed by Professor CAYLEY.)—To reduce the equation of a bicircular quartic into the form $SS'-k^2L=0$, where $S=0$, $S'=0$ are the equations of two circles, $L=0$ the equation of a line. (See Salmon's *Higher Plane Curves*, p. 128.)

Solution by the PROPOSER.

The equation of a bicircular quartic may be taken to be

$$(x^2+y^2)^2 + (u_1+u_0)(x^2+y^2) + v_2 + v_1 + v_0 = 0,$$

where, and in what follows, the subscript numbers denote the degrees in the coordinates (x, y) of the several functions to which they are attached.

Introducing an arbitrary constant θ_0, and putting the equation under the form

$$(x^2+y^2)^2+(u_1+u_0-\theta_0)(x^2+y^2)+\theta_0(x^2+y^2)+v_2+v_1+v_0=0,$$

this may be identified with

$$(x^2+y^2+p_1+p_0)(x^2+y^2+q_1+q_0)+r_1+r_0=0\ ;$$

viz. the conditions in order to this identity are

$$p_1+p_0+q_1+q_0=u_1+u_0-\theta_0,$$
$$(p_1+p_0)(q_1+q_0)+r_1+r_0=\theta_0(x^2+y^2)+v_2+v_1+v_0,$$

that is

$$p_1+q_1=u_1,\quad p_0+q_0=u_0-\theta_0,$$
$$p_1q_1=\theta_0(x^2+y^2)+v_2,\quad p_1q_0+p_0q_1+r_1=v_1,\quad p_0q_0+r_0=v_0.$$

Hence

$$(p_1-q_1)^2=u_1^2-4v_2-4\theta_0(x^2+y^2),$$

where the right-hand side is a quadric function $(x, y)^2$, which, when the discriminant thereof is put $=0$, (that is, when θ_0 is determined as the root of a quadric equation,) is a perfect square, p_1-q_1 is then a known linear function, and p_1+q_1 being equal to the linear function u_1, we have p_1 and q_1 as linear functions of (x, y). We may take for the constants p_0 and q_0 any values satisfying the equation $p_0+q_0=u_0-\theta_0$; and we then have

$$r_1=v_1-p_1q_0-p_0q_1,\quad r_0=v_0-p_0q_0,$$

which completes the determination; the form

$$(x^2+y^2+p_1+p_0)(x^2+y^2+q_1+q_0)+r_1+r_0=0$$

is of course the same as the proposed form $SS'-k^3L=0$.

COR. A somewhat more convenient form is $UU'-k^2V=0$, where $U=0$, $U'=0$ are the equations of two evanescent circles (pairs of imaginary lines), $V=0$ the equation of a circle; in fact the original form $SS'-k^3L=0$ may be written $(S-\alpha)(S'-\alpha')+(\alpha S'+\alpha' S-\alpha\alpha'-k^3L)=0$, which, when α, α' are so determined that $S-\alpha=0$, $S'-\alpha'=0$ may be evanescent circles, is of the required form $UU'-k^2V=0$. The equation $UU'=0$ is that of the two pairs of tangents to the curve at the circular points at infinity respectively; in fact, writing $U=pq$, $U'=p'q'$, each of the lines $p=0$, $q=0$, $p'=0$, $q'=0$ meets the circle $V=0$ in one or other of the circular points at infinity, and therefore only in a single point not at infinity; hence each of these lines meets the curve $UU'-k^2V=0$ three times in one of the circular points at infinity, that is, the line in question is a tangent to one of the two branches through the circular point at infinity.

[Vol. VII. pp. 87, 88.]

2309. (Proposed by Professor CAYLEY.)—Show that for n things

$$1-(\text{no. of partitions into 2 parts})+1\,.\,2\,(\text{no. of partitions into 3 parts})\ldots.$$
$$\pm 1\,.\,2\,.\,3\,..\,(n-1)(\text{no. of partitions into } n \text{ parts})=0.$$

For instance, $n=4$; partitions of (a, b, c, d) into two parts are (a, bcd), (b, cda), (c, dab), (d, abc), (ab, cd), (ac, db), (ad, bc); no. is $=7$. Partitions into three parts are (ab, c, d), (ac, b, d), (ad, b, c), (bc, a, d), (b, d, ac), (cd, a, b); no. is $=6$. Partition into 4 parts is (a, b, c, d); no. is $=1$. And we have

$$1-1.7+2.6-6.1=13-13=0.$$

Solution by the PROPOSER.

Write $n=a\alpha+b\beta+c\gamma+\dots$, where α, β, $\gamma\dots$ are positive integers all of them different, and α, β, $\gamma\dots$ are positive integers; and consider the partitions wherein we have a parts each of α things, b parts each of β things, &c. Writing as usual $\Pi(n)=1.2.3\dots n$, the number of partitions of the form in question is

$$=\frac{\Pi n}{\Pi a.\Pi b\dots(\Pi\alpha)^a(\Pi\beta)^b\dots};$$

whence, putting for shortness $\alpha+\beta+\dots=p$, the theorem may be written

$$\Sigma(-)^{p-1}\frac{\Pi(p-1)\Pi n}{\Pi a.\Pi b\dots(\Pi\alpha)^a(\Pi\beta)^b\dots}=0,$$

the summation extending to all the partitions $n=a\alpha+b\beta+\dots$, as explained above.

Now if the n quantities $x, y, z,\dots$ are the nth roots of unity, we have $x+y+z\dots=0$, and therefore also $(x+y+z\dots)^n=0$, and the general term of the left-hand is

$$\frac{\Pi n}{(\Pi\alpha)^a(\Pi\beta)^b\dots}[\alpha^a\beta^b\dots],$$

where $[\alpha^a\beta^b\dots]$ denotes the symmetrical function $\Sigma x^\alpha y^\alpha\dots(a \text{ factors})\, u^\beta v^\beta\dots(b \text{ factors})\dots$ of the roots $x, y, z, u, v\dots$ of the equation $\theta^n-1=0$; where, as above, $n=a\alpha+b\beta+\dots$. Now by a formula not, I believe, generally known, but which is given on p. 175 of the translation of Hirsch's Algebra (Hirsch's *Collection of Examples &c. on the Literal Calculus and Algebra*, translated by the Rev. J. A. Ross, London, 1827), the value of the sum in question is $=(-)^{p-1}\frac{\Pi(p-1)}{\Pi a.\Pi b\dots}n$, where $p=a+b+\dots$, (the sign $\pm$, given in the formula as quoted, is at once seen to be $(-)^{p-1}$); whence, substituting and omitting the factor n, we have

$$\Sigma(-)^{p-1}\frac{\Pi(p-1)\Pi n}{\Pi a.\Pi b\dots(\Pi\alpha)^a(\Pi\beta)^b\dots}=0,$$

which is the required theorem.

OBSERVATION. In Cauchy's *Exercices d'Analyse &c.*, t. III., p. 173, is given a formula relating to the same mode of partition of the number n, viz. this is

$$\Sigma\frac{\Pi n}{\Pi a.\Pi b\dots\alpha^a\beta^b\dots}=\Pi n.$$

I have somewhere made the remark that, on the left-hand side, the terms which belong to the odd and the even values of $a+b+\ldots(=p)$ are equal, and that we have therefore

$$\Sigma(-)^{p-1}\frac{\Pi n}{\Pi a\,.\,\Pi b\ldots\alpha^a\beta^b\ldots}=0,$$

which is a theorem having a curious analogy with that demonstrated above.

[Vol. VII. pp. 99—102.]

2286. (Proposed by W. H. LAVERTY.)—If we have $(n-2)$ sets of n quantities each, $(\alpha_1, \alpha_2\ldots\alpha_n)$, $(\beta_1, \beta_2\ldots\beta_n)$, $\ldots(\lambda_1, \lambda_2\ldots\lambda_n)$, connected with the n quantities $(r_1, r_2\ldots r_n)$ by $\frac{1}{2}n(n-1)$ equations of which the type form is

$$(\alpha_k-\alpha_l)^2+(\beta_k-\beta_l)^2+\ldots(\lambda_k-\lambda_l)^2=r_k^2+r_l^2;$$

then show that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\ldots+\frac{1}{r_n^2}=0 \text{ and } \frac{P_1}{r_1^2}+\frac{P_2}{r_2^2}+\ldots\frac{P_n}{r_n^2}=0,$$

where P is any one of the quantities $\alpha, \beta, \gamma\ldots\lambda$.

Solution by PROFESSOR CAYLEY.

Consider the case $n=4$; we have between $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$,$\ldots(r_1, r_2, r_3, r_4)$ six equations, such as the equation

$$(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2=r_1^2+r_2^2; \tag{12}$$

and it is in effect required to show that these equations give

$$\frac{1}{r_1^2}:\frac{1}{r_2^2}:\frac{1}{r_3^2}:\frac{1}{r_4^2}=(234):-(341):(412):-(123),$$

where

$$(123)=\begin{vmatrix}\alpha_1, & \beta_1, & 1\\ \alpha_2, & \beta_2, & 1\\ \alpha_3, & \beta_3, & 1\end{vmatrix}, \text{ \&c.},$$

viz. considering (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4) as the rectangular coordinates of four points in a plane, then (123) is the area (taken with a proper sign) of the triangle formed by the points 1, 2, 3; and the like for (234) &c.

Combining the equations as follows,

$$(12)+(34)-(13)-(24),$$

the r's disappear, and we have an equation

$$(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)+(\beta_1-\beta_4)(\beta_2-\beta_3)=0,$$

which shows that the lines 14 and 23 intersect at right angles; similarly the lines 12 and 34, and also the lines 13 and 24, intersect at right angles; or starting from the given points 1, 2, 3, the point 4 is the intersection of the perpendiculars let fall from the angles 1, 2, 3 of the triangle 123 on the opposite sides respectively.

Again combining the equations as follows,

$$(12)+(13)-(23),$$

we obtain

$$r_1^2=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)+(\beta_1-\beta_2)(\beta_1-\beta_3).$$

The entire system of equations will remain unaltered if we pass from the original axes to any other system of rectangular axes; hence taking the axes of x in the sense from 1 to 2 along the line 12, $\beta_1-\beta_2$ becomes $=0$, and we have

$$\alpha_2-\alpha_1=12,\quad \alpha_3-\alpha_1=1\,(12,\ 34);$$

viz. $\alpha_2-\alpha_1$ is the distance 12 of the points 1 and 2, $\alpha_3-\alpha_1$ is the distance $1\,(12,\ 34)$ of the point 1 from the point (12, 34) which is the intersection of the lines 12 and 34; we have therefore

$$r_1^2=12\,.\,1\,(12,\ 34).$$

But similarly

$$r_2^2=21\,.\,2\,(12,\ 34),\ =12\,.\,(12,\ 34)\,2,$$

(since $21=-12$ and $2\,(12,\ 34)=-(12,\ 34)\,2$). And we have therefore

$$r_1^2:r_2^2=1\,(12,\ 34):(12,\ 34)\,2,\ \text{or}\ \frac{1}{r_1^2}:\frac{1}{r_2^2}=(12,\ 34)\,2:1\,(12,\ 34).$$

Write

$$\lambda=\frac{(12,\ 34)\,2}{12},\qquad \mu=\frac{1\,(12,\ 34)}{12}$$

where $1\,(12,\ 34)$ and $(12,\ 34)\,2$ are as above the distances from 1 to (12, 34) and from (12, 34) to 2; and, in the denominators, 12 is the distance from 1 to 2; we have $\lambda+\mu=1$; the coordinates of (12, 34) are $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, and the values of λ, μ are obtained by writing $\lambda\alpha_1+\mu\alpha_2$, $\lambda\beta_1+\mu\beta_2$, $\lambda+\mu$ for x, y, 1 in the equations

$$\begin{vmatrix} x, & y, & 1 \\ \alpha_3, & \beta_3, & 1 \\ \alpha_4, & \beta_4, & 1 \end{vmatrix}=0$$

of the line 34. Making this substitution, we find

$$\lambda\,(134)+\mu\,(234)=0,$$

where as above

$$(134)=\begin{vmatrix} \alpha_1, & \beta_1, & 1 \\ \alpha_3, & \beta_3, & 1 \\ \alpha_4, & \beta_4, & 1 \end{vmatrix},\ \&c.,$$

we have therefore

$$\lambda : \mu = (234) : -(134) = (234) : -(341),$$

or, what is the same thing,

$$(12, 34)\,2 : 1\,(12, 34) = (234) : -(341);$$

and consequently

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} \qquad = (234) : -(341);$$

or completing the system by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} = (234) : -(341) : (412) : -(123),$$

which is the required result.

In the case $n = 5$, we have between

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5), \quad (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5), \quad (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5), \quad (r_1, r_2, r_3, r_4, r_5)$$

ten equations such as the equation

$$(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 = r_1^2 + r_2^2. \tag{12}$$

We obtain as before the equation

$$(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3) + (\beta_1 - \beta_4)(\beta_2 - \beta_3) + (\gamma_1 - \gamma_4)(\gamma_2 - \gamma_3) = 0,$$

which, considering $(\alpha_1, \beta_1, \gamma_1)$ &c. as the rectangular coordinates of five points 1, 2, 3, 4, 5 in space, signifies that the line 14 is at right angles to the line 23; the five points are therefore such that the line joining any two of them is at right angles to the line joining any other two of them, whence also the line joining any two is at right angles to the plane through the remaining three points. (The points 1, 2, 3, 4 form a tetrahedron such that that 12 and 34, also 13 and 42, also 14 and 23 are at right angles to each other, two of these conditions imply the third; and this being so, if a further condition be satisfied, the perpendiculars from 1, 2, 3, and 4 on the opposite faces respectively, will meet in a point 5, and we shall have the system of points 1, 2, 3, 4, 5 related as above.)

We further obtain as before

$$r_1^2 = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) + (\beta_1 - \beta_2)(\beta_1 - \beta_3) + (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3),$$

or taking the axis of x in the sense from 1 to 2 along the line 12, we have $\beta_1 - \beta_2 = 0$, $\gamma_1 - \gamma_2 = 0$, and the equation becomes

$$r_1^2 = 12 \,.\, 1\,(12, 345),$$

and similarly

$$r_2^2 = 12 \,.\, (12, 345)\,2;$$

whence

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} = (12, 345)\,2 : (12, 345)\,1.$$

Writing them $\lambda = \frac{(12.345)\,2}{12}$, $\mu = \frac{1\,(12,\ 345)}{12}$ (and therefore $\lambda + \mu = 1$) we find (λ, μ) by substituting $\lambda\alpha_1 + \mu\alpha_2$, $\lambda\beta_1 + \mu\beta_2$, $\lambda\gamma_1 + \mu\gamma_2$, $\lambda + \mu$ for x, y, z, 1 in the equation

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \\ \alpha_5, & \beta_5, & \gamma_5, & 1 \end{vmatrix} = 0$$

of the plane 345; we have thus

$$\lambda\,(1345) + \mu\,(2345) = 0,$$

that is

$$\lambda : \mu = (2345) : -(1345) = (2345) : (3451),$$

whence

$$(12,\ 345)\,2 : 1\,(12,\ 345) = (2345) : (3451),$$

that is

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} \qquad = (2345) : (3451),$$

or completing by symmetry

$$\frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r_3^2} : \frac{1}{r_4^2} : \frac{1}{r_5^2} = (2345) : (3451) : (4512) : (5123) : (1234),$$

which is the theorem for the case $n = 5$. The general case depends, it is clear, upon similar reasoning in a $(n-2)$dimensional geometry; leading to the conception in this geometry of a figure of $(n-1)$ points such that the line joining any two of them is at right angles to the line joining any other two of them.

[Vol. VII. p. 106.]

2331. (Proposed by Professor CAYLEY.)—Show that it is possible to find (X, Y, Z) linear functions of the trilinear coordinates (x, y, z) such that the equations $xX = yY = zZ$ may determine four given points.

[Vol. VIII., July to December, 1867, p. 26.]

2321. (Proposed by Professor CAYLEY.)—Given a conic, to find four points such that all the conics through the four points may have their centres in the given conic.

[Vol. VIII. p. 36.]

2371. (Proposed by Professor CAYLEY.)—(4). If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

[Vol. VIII. pp. 51, 52.]

Note on Question 1990. *By* PROFESSOR CAYLEY.

The theorem of paragraph 4 (*Reprint*, vol. VI. p. 88), (ascribed by Professor Sylvester to Mr Crofton), that "if a circle and a straight line be cut by any transversal in three points, these will be the foci of a system of Cartesian ovals having double contact with one another at two fixed points," may be enunciated under a more complete form, as follows:

If in a given circle the chords PP_1, BC meet in A, then each of the two Cartesians, foci A, B, C, which pass through P, will also pass through P_1; and moreover, if α, α' be the diametrals of the chord PP_1 (that is, the extremities of the diameter at right angles to PP_1) then the tangents at P, P_1 to one of the Cartesians will be αP, αP_1 respectively, and to the other of them $\alpha' P$, $\alpha' P_1$ respectively, these tangents being thus independent of the position of the chord BC; and thence also thus;

Given the points A, B, C *in lineâ*, and the point P;

through	P, B, C	draw a circle	(A)	and let	PA	meet this in	P_1,
„	P, C, A	„	(B)	„	PB	„	P_2,
„	P, A, B	„	(C)	„	PC	„	P_3,

then each of the Cartesians, foci A, B, C, which pass through P will also pass through P_1, P_2, P_3; and if

α, α'	are the diametrals of	PP_1	in circle	(A),
β, β'	„	PP_2	„	(B),
γ, γ'	„	PP_3	„	(C),

then (the points of the several pairs being properly selected) the points (α, β, γ) and the points $(\alpha', \beta', \gamma')$ will each lie in a line through P, viz. the lines $P\alpha\beta\gamma$ and $P\alpha'\beta'\gamma'$ will be the tangents at P to the two Cartesians respectively.

The two Cartesians meet in the points P, P_1, P_2, P_3, and in the symmetrically situated points in regard to the axis ABC; the theorem contains as part of itself the well-known property that the two Cartesians cut at right angles at each of their points of intersection; it gives moreover the construction of the following problem:—given the foci A, B, C, and one intersection P of a pair of triconfocal Cartesians, to find the remaining intersections, and the tangents at each of the intersections.

[Vol. VIII. pp. 70—72.]

1911. (Proposed by Professor CAYLEY.)—Given four points, and also the "conic of centres"—viz. the conic which is the locus of the centres of the several conics which pass through the four given points; then if a conic through the four given points has for its centre a given point on the conic of centres, it is required to find a construction for the asymptotes of this conic.

Solution by the PROPOSER.

1. Consider four given points, and in connection therewith a given line IJ; the locus of the poles of IJ, in regard to the several conics which pass through the four points, is a conic, the "conic of poles." Consider a particular conic Θ, through the four points; the pole of IJ in regard to the conic Θ is a point C on the conic of poles, and the tangents from C to the conic Θ meet the conic of poles in two points H, K; the chord of intersection HK passes through the point Π which is the pole of IJ in regard to the conic of poles. Moreover, the polars of a point C', in regard to the several conics through the four points, meet in a point Ω', the "common pole" of C', and in particular if C' be the point C on the conic of poles, then the common pole is a point Ω on the line IJ; this being so, the line HK passes (as already mentioned) through Π, and the lines HK and $\Pi\Omega$ are harmonics in regard to the conic of poles.

2. Assuming the foregoing properties, then, given the four points, the line IJ, the conic of poles, and the point C on this conic; we may construct Π the pole of IJ in regard to the conic of poles; and also Ω the common pole of C; the line HK is then given as a line passing through Π, and harmonic to $\Pi\Omega$ in regard to the conic of poles; this line meets the conic of poles in the points H, K; and then CH, CK are the tangents from C to a conic Θ which passes through the four points.

3. In particular if IJ be the line infinity, then the conic of poles is the conic of centres; Π is the centre of this conic; Ω is as before the common pole of C; HK is given as the diameter of the conic of centres, conjugate to $\Pi\Omega$; H, K are the extremities of this diameter; and then CH, CK are the asymptotes of the conic through the four points, which has the point C for its centre; and the asymptotes are therefore constructed as required. If the points H, K are imaginary, the asymptotes will be also imaginary; the conic Θ is in this case an ellipse.

4. It is hardly necessary to remark, in regard to the construction of the point Ω, that we have among the conics through the four points, three pairs of lines meeting in points P, Q, R respectively (it is clear that the conic of poles passes through these three points); the harmonics of CP, CQ, CR in regard to the three pairs of lines respectively meet in a point, which is the required point Ω. In the particular case where the point C is on the conic of centres, the three harmonics are parallel; it is therefore sufficient to construct *one* of them; and the line HK is then the diameter of the conic of poles, conjugate to the harmonic so constructed.

5. It remains to prove the properties assumed in (1). We may take $z=0$ for the equation of the line IJ, $x=0$, $y=0$ for the equations of the tangents to the conic Θ at its intersections with the line IJ, so that we have $(x=0,\ y=0)$ for the coordinates of the point C; the equation of the conic Θ will be of the form $z^2-xy=0$, and the four points may then be taken to be the intersections of the conic $z^2-xy=0$, and the arbitrary conic

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0.$$

The equation of the conic of centres is found to be

$$x(ax+hy+gz)-y(hx+by+fz)=0, \text{ or } ax^2-by^2+gzx-hxy=0\,;$$

or, as it may also be written,

$$(2a,\ -2b,\ 0,\ -f,\ g,\ 0 \between x,\ y,\ z)^2 = 0;$$

and it is convenient to remark that the equation in line coordinates (or condition that this conic may be touched by the line $\xi x + \eta y + \zeta z = 0$) is

$$(-f^2,\ -g^2,\ -4ab,\ 2af,\ 2bg,\ -fg \between \xi,\ \eta,\ \zeta)^2 = 0.$$

The line $x=0$ meets the conic of poles in the point $x=0$, $by+fz=0$, and the line $y=0$ meets the same conic in the point $y=0$, $ax+gz=0$; hence the line HK, which is the line joining these two points, has for its equation

$$afx + bgy + fgz = 0;$$

and it only remains to be shown that this line passes through the point Π, and is the harmonic of the line $\Pi\Omega$ in regard to the conic of centres. The point Π is the pole of the line $z=0$ in regard to the conic of centres, its coordinates are at once found to be

$$x : y : z = bg : af : -2ab;$$

and we thence see that Π is a point on the line HK. The point Ω is given as the intersection of the polars of C in regard to the conics $z^2 - xy = 0$, and $(a, b, c, f, g, h \between x, y, z)^2 = 0$ respectively; that is, as the intersection of the lines $z=0$, and $gx+fy+cz=0$; its coordinates therefore are

$$x : y : z = -f : g : 0.$$

Hence the equation of the line $\Pi\Omega$ is

$$2abg\,x + 2abf\,y + (af^2 + bg^2)\,z = 0.$$

Now, in general, if we have a conic the line-equation whereof is $(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0$, then the condition in order that, in regard thereto, the lines $\lambda x + \mu y + \nu z = 0$ and $\lambda' x + \mu' y + \nu' z = 0$ may be harmonics, is

$$(A,\ B,\ C,\ F,\ G,\ H \between \lambda,\ \mu,\ \nu \between \lambda',\ \mu',\ \nu') = 0;$$

that is

$$A\lambda\lambda' + B\mu\mu' + C\nu\nu' + F(\mu\nu' + \mu'\nu) + G(\nu\lambda' + \nu'\lambda) + H(\lambda\mu' + \lambda'\mu) = 0.$$

Hence, in order that the two lines HK and $\Pi\Omega$ may be harmonics in regard to the conic of centres, we should have

$$(-f^2,\ -g^2,\ -4ab,\ 2af,\ 2bg,\ -fg \between af,\ bg,\ fg \between 2abg,\ 2abf,\ af^2 + bg^2) = 0.$$

But developing, and omitting the common factor $abfg$, which enters into all the terms, this equation is

$$-(2af^2) - (2bg^2) - 4(af^2 + bg^2) + \{4af^2 + 2(af^2 + bg^2)\} + \{4bg^2 + 2(af^2 + bg^2)\} - 2(af^2 + bg^2) = 0,$$

which is identically true; and the lines HK and $\Pi\Omega$ are therefore harmonics in regard to the conic of centres.

[Vol VIII. p. 74.]

2371. (Proposed by Professor CAYLEY.)—If P, Q be two points taken at random within the triangle ABC, what is the chance that the points A, B, P, Q may form a convex quadrangle?

[Vol. VIII. pp. 86, 87.]

2466. (Proposed by H. MURPHY.)—If four points A, B, C, D be either in the same plane or not, and if the three rectangles $AB.CD$, $AC.DB$, $AD.BC$ be taken; the sum of any two of them is greater than the third, except when the points lie on the circumference of a circle.

Solution by PROFESSOR CAYLEY.

Write for shortness $BC=f$, $CA=g$, $AB=h$; $AD=a$, $BD=b$, $CD=c$; then, Lemma, if r be the radius of the sphere circumscribed about the tetrahedron $ABCD$, we have

$$4r^2\left\{\begin{array}{l}-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2\\+(a^2f^2+b^2c^2)(g^2+h^2-f^2)\\+(b^2g^2+c^2a^2)(h^2+f^2-g^2)\\+(c^2h^2+a^2b^2)(f^2+g^2-h^2)\end{array}\right\}=2b^2c^2g^2h^2+2c^2a^2h^2f^2+2a^2b^2f^2g^2-a^4f^4-b^4g^4-c^4h^4,$$

where the left-hand side is $=576V^2r^2$, if V be the volume of the tetrahedron.

Suppose first that the points are not in the same plane, then the left-hand side $(=576\,V^2r^2)$ is positive; therefore the right-hand side is also positive, or putting for shortness $af=\alpha$, $bg=\beta$, $ch=\gamma$, we have

$$2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4=+,\text{ that is, }4\beta^2\gamma^2-(\alpha^2-\beta^2-\gamma^2)^2=+,$$

and thence $\alpha<\beta+\gamma$; for if α were equal to or greater than $\beta+\gamma$, say $\alpha=\beta+\gamma+x$, the left-hand side would be $4\beta^2\gamma^2-\{2\beta\gamma+2(\beta+\gamma)x+x^2\}^2$, which vanishes if $x=0$, and is negative for x positive. Similarly $\beta<\gamma+\alpha$, $\gamma<\alpha+\beta$; and the theorem is thus proved for the case where the four points are not in a plane.

Starting from this general case, if we imagine the point D continually to approach and ultimately to coincide with the plane ABC, but so as not to be in the circle ABC, then the expression $2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4$, which does not vanish in the limit, is throughout equal to the positive quantity $576\,V^2r^2$ (in the limit V is $=0$ and $r=\infty$, but Vr is finite, and of course V^2r^2 is positive), that is, the expression in question is $=+$, and the theorem follows as before. Of course when the four points are in a circle, then the expression is $=0$, and consequently one of the quantities α, β, γ is equal to the sum of the other two.

The lemma is at once proved by means of my theorem for the relation between the distances of five points in space, {*Cambridge Mathematical Journal*, vol. II. (1841), p. 269, [1],} viz. if the point 1 is the centre of the circumscribed sphere, and the points 2, 3, 4, 5 are the points A, B, C, D respectively, then the relation in question, viz.

$$\begin{vmatrix} 0, & (12)^2, & (13)^2, & (14)^2, & (15)^2, & 1 \\ (21)^2, & 0, & (23)^2, & (24)^2, & (25)^2, & 1 \\ (31)^2, & (32)^2, & 0, & (34)^2, & (35)^2, & 1 \\ (41)^2, & (42)^2, & (43)^2, & 0, & (45)^2, & 1 \\ (51)^2, & (52)^2, & (53)^2, & (54)^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0$$

becomes

$$\begin{vmatrix} 0, & r^2, & r^2, & r^2, & r^2, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Multiplying the last line by $-r^2$ and adding it to the first line, this is

$$\begin{vmatrix} -r^2, & 0, & 0, & 0, & 0, & 1 \\ r^2, & 0, & h^2, & g^2, & a^2, & 1 \\ r^2, & h^2, & 0, & f^2, & b^2, & 1 \\ r^2, & g^2, & f^2, & 0, & c^2, & 1 \\ r^2, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

and then proceeding in the same way with the first and last columns the equation is

$$\begin{vmatrix} -2r^2, & 0, & 0, & 0, & 0, & 1 \\ 0, & 0, & h^2, & g^2, & a^2, & 1 \\ 0, & h^2, & 0, & f^2, & b^2, & 1 \\ 0, & g^2, & f^2, & 0, & c^2, & 1 \\ 0, & a^2, & b^2, & c^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is in fact the equation of the Lemma. See my papers in the *Quarterly Journal of Mathematics*, vol. III. (1859), pp. 275—277, [286], and vol. V. (1861), pp. 381—384, [297].

COR.—It appears by the demonstration that for any four points not in the same plane, the expression

$$-a^4f^2-b^4g^2-c^4h^2-f^2g^2h^2$$
$$+(a^2f^2+b^2c^2)(g^2+h^2-f^2)+(b^2g^2+c^2a^2)(h^2+f^2-g^2)+(c^2h^2+a^2b^2)(f^2+g^2-h^2)$$

is always positive.

[Vol. VIII. pp. 105, 106.]

2472. (Proposed by Professor CAYLEY.)—Through four points on a circle to draw a conic such that an axis may pass through the centre of the circle.

Solution by the PROPOSER.

Let the equation of the conic be $(a, b, c, f, g, h \between x, y, 1)^2=0$, then if as usual the inverse coefficients are represented by (A, B, C, F, G, H), the equation of the two axes is

$$(a-b)(Cx-G)(Cy-F)+h[(Cx-G)^2-(Cy-F)^2]=0,$$

whence if an axis pass through the origin

$$(a-b)FG+h(G^2-F^2)=0.$$

Consider now the circle $x^2+y^2-1=0$ and on it the four points in which it is intersected by the conic $(a, b, c, f, g, h \between x, y, 1)^2=0$; then for any conics through the four points we have

$$(a, b, c, f, g, h \between x, y, 1)^2+\lambda(x^2+y^2-1)=0;$$

so that, taking this for the equation of the required conic, and representing it by

$$(a', b', c', f', g', h' \between x, y, 1)^2=0,$$

the values of the coefficients are

$$a'=a+\lambda,\quad b'=b+\lambda,\quad c'=c+\lambda,\quad f'=f,\quad g'=g,\quad h'=h,$$

and we thence have

$$F'=F-\lambda f,\quad G'=G-\lambda g,\quad a'-b'=a-b,\quad h'=h.$$

The required relation is

$$(a'-b')F'G'+h'(G'^2-F'^2)=0,$$

that is

$$(a-b)(F-\lambda f)(G-\lambda g)+h\{(G-\lambda g)^2-(F-\lambda f)^2\}=0,$$

a quadric equation in λ; and substituting for λ each of its two values, we have the two required conics

$$(a, b, c, f, g, h \between x, y, 1)^2+\lambda(x^2+y^2-1)=0,$$

for each of which an axis passes through the centre of the circle.

[Vol. IX., January to June, 1868, pp. 20, 21.]

Note on Question 2471. *By* PROFESSOR CAYLEY.

In the singularly beautiful solution which Mr Woolhouse has given of this question (see *Reprint*, vol. VIII. p. 100), it is important to note what is the analytical problem solved, and how the solution is obtained. Considering a plane area bounded by any closed convex curve, and in it three points P, P', P'', Mr Woolhouse investigates the average area of the triangle $PP'P''$, viz. this depends on the sextuple integral

$$\int \pm \{x'y'' - x''y' + x''y - xy'' + xy' - x'y\}\, dx\, dy\, dx'\, dy'\, dx''\, dy'',$$

where the sign $\pm$ has to be taken so that $\pm\{\ \}$ shall be positive, and where the integration in respect to each set of coordinates extends over the entire closed area; the difficulty is as to the mode of dealing with the discontinuous sign. It is remarked that the integral is

$$= 6 \int \pm \{x'y'' - x''y' + x''y - xy'' + xy' - x'y\}\, dx\, dy\, dx'\, dy'\, dx''\, dy'';$$

the variables in this last expression being restricted in such wise that x, x'', x' are in the order of increasing magnitude; the term $\pm\{\ \}$ is of the form $\pm(x'-x)(y''-\beta)$, where β is independent of y, and where (as is easily seen) if v'', u'' be the upper and lower ordinate corresponding to the abscissa x'', then β lies between the values u'' and v''. But $x'-x$ is positive, hence the sign $\pm$ must be so taken that $\pm(y''-\beta)$ shall be positive, that is, from $y''=u''$ to $y''=\beta$ the sign is $-$, and from $y''=\beta$ to $y''=v''$ the sign is $+$.

Hence for the integration in regard of y'' we have

$$\int \pm (y''-\beta)\, dy'' = \int_{\beta}^{v''} +(y''-\beta)\, dy'' + \int_{u'}^{\beta} -(y''-\beta)\, dy'', \ = \tfrac{1}{2}(v''-\beta)^2 + \tfrac{1}{2}(\beta - u'')^2;$$

and the discontinuous sign $\pm$ is thus got rid of. The remaining integrations are then effected in the order x'', y', y, x', x, the limits being for x'' from x to x', for y' from u' to v', and for y from u to v (if the upper and lower ordinates corresponding to the abscissa x and x' are v, u and v', u' respectively) and finally for x' from x to the maximum abscissa, and for x from the minimum to the maximum abscissa. The final result involves only single definite integrals between the extreme values of x, the functions under the integral sign containing indefinite integrations from the same arbitrary inferior limit, say $x=0$; the form of the result (previous to its simplification by taking the axes to be principal axes through the centre of gravity of the area) is however somewhat complicated; and it would not be easy to show *a posteriori*, that the value is invariantive, that is, independent of the position of the axes: that this is so is of course apparent from the original form of the integral.

[Vol. IX. pp. 38, 39.]

2530. (Proposed by Professor CAYLEY.)—Trace the curve

$$\frac{1}{\sqrt{z}}+\frac{1}{\sqrt{(x+iy)}}+\frac{1}{\sqrt{(x-iy)}}=0,$$

where the coordinates x, y, z are the perpendicular distances of the current point P from the sides of an equilateral triangle, the coordinates being positive for a point within the triangle.

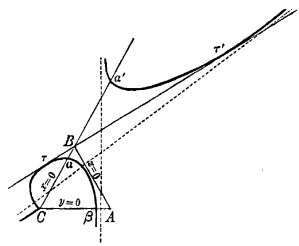
Solution by the PROPOSER.

The form of the equation shows that the curve is a tricuspidal quartic, having a real cusp at the point $(x=0, y=0)$, and two imaginary cusps at the points $(z=0, x+iy=0)$ and $(z=0, x-iy=0)$. The rationalised form of the equation is

$$(x^2+y^2)^2-4zx(x^2+y^2)-4z^2y^2=0.$$

$x=0$ gives $y^2(y^2-4z^2)=0$, the point C twice, and two other real points α, α' on the line BC.

$y=0$ gives $x^3(x-4z)=0$, the point C three times, and a real point β on the line CA.



It is easy to find that there is a double tangent $z+2x=0$, viz. $z+2x=0$ gives $(3x^2-y^2)^2=0$, two points τ, τ' (each twice) on the line in question.

Laying down these points, it appears that the curve must have two real asymptotes, and that the form is as shown in the figure.

[Vol. IX. pp. 55, 56.]

2553. (Proposed by Professor CAYLEY.)—Show that the surface $y^2z^2+z^2x^2+x^2y^2-2xyz=0$ meets the sphere $x^2+y^2+z^2=1$ in four circles; and explain in a general manner the

form of the curve of intersection of the surface by any other sphere having the same centre, and thence the form of the surface itself (being a particular case of Steiner's surface, and which by the homographic transformations $w^{-1}x$, $w^{-1}y$, $w^{-1}z$ for x, y, z gives $y^2z^2+z^2x^2+x^2y^2-2wxyz=0$, the general equation of Steiner's surface).

Solution by the PROPOSER.

Take X, X', Y, Y', Z, Z' the intersections of the sphere $x^2+y^2+z^2=1$ by the three axes respectively; then we have $x^2+y^2+z^2=1$, $x+y+z=-1$, the equations of the circle through the points X', Y', Z'; and from these two equations we deduce $yz+zx+xy=0$, and thence

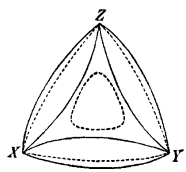
$$y^2z^2+z^2x^2+x^2y^2+2xyz\,(x+y+z)=0,$$

that is

$$y^2z^2+z^2x^2+x^2y^2-2xyz=0\,;$$

so that the circle lies on the quartic surface; and by changing successively the signs of each two of the three coordinates, we have three other circles lying on the sphere and also on the quartic surface; viz. we have in all four circles, the above-mentioned circle through (X', Y', Z'), and three other circles through (X', Y, Z), (X, Y', Z), (X, Y, Z') respectively, making together a curve of the order 8, the complete intersection of the quartic surface by the sphere.

The quartic surface lies entirely in the four octants of space xyz, $xy'z'$, $x'yz'$, $x'y'z$; and as to the portion of the surface which lies in the octant xyz, this meets the sphere $x^2+y^2+z^2=1$ in portions of the three circles (X', Y, Z) (X, Y', Z) (X, Y, Z') constituting a tricuspidal form lying within the octant XYZ as shown in the figure. The intersection by a sphere, radius <1, projected on the octant XYZ, is a trinodal form, lying outside the tricuspidal one, as shown by a dotted line in the figure; the intersection by a sphere radius >1, projected in the same way, is a trigonoid form lying inside the tricuspidal one, as also shown by a dotted line in the figure; as the radius approaches to and ultimately becomes $=\frac{2}{\sqrt{3}}$, this diminishes, and becomes ultimately a mere point, and when the radius is greater than this value the intersection is imaginary.



Imagine on the solid sphere, radius $=1$, the four tricuspidal forms lying in alternate octants as above; cut away down to the centre the portions lying without

these tricuspidal forms; and build up on the tricuspidal forms, until the greatest distance from the centre becomes $=\frac{2}{\sqrt{3}}$; we have a solid figure with four prominences situate as the summits of a tetrahedron, the bounding surface whereof is the surface in question: it is to be added that the axes are nodal lines on the surface, viz. the portions which lie within the solid figure are the intersections of two real sheets of the surface, the portions which lie without the solid figure are isolated, or acnodal, lines on the surface.

[Vol. IX. pp. 73, 74.]

2573. (Proposed by Professor CAYLEY.)—The envelope of a variable circle having for its diameter the double ordinate of a rectangular cubic is a Cartesian.

{DEFINITION. The expression "a rectangular cubic" is used to express a cubic with three real asymptotes, having a diameter at right angles to one of the asymptotes and at an angle of 45° to each of the other two asymptotes, viz. the equation of such a cubic is $xy^2 = x^3 + bx^2 + cx + d$.}

Solution by the PROPOSER.

The equation of the variable circle may be taken to be

$$(x-\theta)^2 + y^2 = \theta^2 - 2m\theta + \alpha + \frac{2A}{\theta},$$

viz. θ being the abscissa of the rectangular cubic, the squared ordinate is taken to be $=\frac{1}{\theta}(\theta^3 - 2m\theta^2 + \alpha\theta + 2A)$, or, what is the same thing, the equation of the variable circle is

$$x^2 + y^2 - \alpha - 2(x-m)\theta - \frac{2A}{\theta} = 0.$$

Hence, taking the derived equation in regard to θ, we have

$$x - m - \frac{A}{\theta^2} = 0,$$

and thence

$$x^2 + y^2 - \alpha = \frac{4A}{\theta};$$

therefore

$$(x^2 + y^2 - \alpha)^2 = \frac{16A^2}{\theta^2} = 16A(x-m);$$

that is, the equation of the envelope is

$$(x^2 + y^2 - \alpha)^2 - 16A(x-m) = 0,$$

which is a known form of the equation of a Cartesian.

[Vol. IX. pp. 82, 83.]

2493. (Proposed by Professor CAYLEY.)—1. Given the conic $U=0$ (but observe that the *function* U contains implicitly an arbitrary constant factor which is *not* given) and also the conic $U+1=0$, to construct the conic $U+l=0$, where l is a given constant.

2. Given the conics $U=0$, $U+1=0$, $V=0$, $V+1=0$, and the constants θ, k, to construct the conic $\theta U+\theta^{-1}V+2k=0$.

Solution by the PROPOSER.

1. The conics $U=0$, $U+1=0$, $U+l=0$ are obviously concentric similar and similarly situated conics, and if drawing a line in any direction from the centre, the radius-vectors for the three conics respectively are r, r', R, then it is easy to see that we have

$$R^2=lr'^2+(1-l)r^2.$$

There is no difficulty in constructing geometrically the radius R, and then the conic $U+l=0$ is given as the concentric similar and similarly situated conic passing through the extremity of this radius.

2. To construct the conic $\theta U+\theta^{-1}V+2k=0$. By what precedes, we may construct the two conics $\theta U+k=0$, $\theta^{-1}V+k=0$; the four points of intersection of these lie on the required conic $\theta U+\theta^{-1}V+2k=0$, and also on the conic $\theta U-\theta^{-1}V=0$; which last conic is consequently given as a conic passing through the four points in question, and also through the four points of intersection of the given conics $U=0$, $V=0$. But the conic $\theta U-\theta^{-1}V=0$ being constructed, the conic $\theta U+\theta^{-1}V=0$ can also be constructed; viz. the tangents of these two conics and of the conics $U=0$, $V=0$, at each of the four intersections $U=0$, $V=0$, form a harmonic pencil; and we have thus the conic $\theta U+\theta^{-1}V=0$ a conic passing through four given points, and having at each of these a given tangent. And then finally the required conic $\theta U+\theta^{-1}V+2k=0$ is given as a conic concentric similar and similarly situated with the conic $\theta U+\theta^{-1}V=0$, and passing through the four given points

$$\theta U+k=0,\quad \theta^{-1}V+k=0.$$

3. Treating k as an absolute constant but θ as a variable parameter, the envelope of the conic $\theta U+\theta^{-1}V+2k=0$ is the quartic curve $UV-k^2=0$. This is a curve used by Plücker (in the *Theorie der algebraischen Curven*) for the purpose of showing that the 28 double tangents of a quartic curve may be all of them real. In fact, if $U=0$, $V=0$ be ellipses intersecting in four real points; and if, moreover, the implicit constants be such that U is positive for points *without* the first ellipse, V positive for points *within* the second ellipse, then since $UV, =k^2$, is positive for all points of the curve in question, the curve must be wholly situate in the four closed spaces which lie outside the one and inside the other of the two ellipses; consisting therefore of four detached portions. And when k is sufficiently small, then the figure of each portion is that of a concavo-convex lens with its angles rounded off: viz. each such portion has a real double tangent of its own. Any two portions have obviously four real double tangents, and hence the total number of real double tangents is $4+6\times 4, =28$.

4. A construction has been given by Aronhold (*Berl. Monatsber.*, July, 1864) by which, taking any 7 given lines as double tangents of a quartic curve, the remaining 21 double tangents can be constructed, and which, when the seven given lines are real, leads to a system of 28 real double tangents; but wishing to construct the figure of the 28 real double tangents, it occurred to me that the easier manner might be to construct Plücker's curve $UV - k^2 = 0$, as the envelope of the conic $\theta U + \theta^{-1} V + 2k = 0$, and then to draw the tangents of this curve: the construction is, however, practically one of considerable difficulty, and I have not yet accomplished it.

[Vol. IX. p. 87.]

2451. (Proposed by Professor CAYLEY.)—If A, B, C, D are the intersections of a conic by a circle, then the antipoints of A, B, and the antipoints of C, D, lie on a confocal conic.

N.B. If AB, $A'B'$ intersect at right angles in a point O in such wise that $OA' = OB' = i \,.\, OA = i \,.\, OB$ {where $i = \sqrt{(-1)}$ as usual}, then A', B' are the antipoints of A, B, and conversely.

[Vol. IX. pp. 101—103.]

2590. (Proposed by Professor CAYLEY.)—It is required to verify Professor Kummer's theorem that "if a quartic surface is such that every section by a plane through a certain fixed point is a pair of conics, the surface is a pair of quadric surfaces (except only in the case where it is a quartic cone having its vertex at the fixed point)."

Solution by the PROPOSER.

The theorem may be more generally stated as follows; if a surface is such that every section through a certain fixed point (is or) includes a proper conic, then the surface (is or) includes a proper quadric surface. In order to the demonstration, I premise the following *Lemma:* If a surface is such that every section through a certain fixed line includes a conic, then the line meets each of these conics in the same two points.

In fact, if the line meet the surface in any n points, then it is clear that each of the conics will meet the line in some two of these n points; and as the plane of the section passes continuously from any one to any other position, the two points of intersection with the conic cannot pass abruptly from being some two to being some other two of the n points, that is, they are always the same two points.

Consider now a surface which is such that every section through a fixed point O includes a conic; and consider three lines xx', yy', zz' meeting in the point O. Let the conics in the planes yz, zx, xy be A, B, C respectively; then since the conics

through the line xx' all pass through the same two points, and since B, C are two of these conics, B and C meet xx' in the same two points X, X'; similarly C and A meet yy' in the same two points Y, Y'; and A, B meet zz' in the same two points Z, Z'; that is, we have the conics A, B, C intersecting

$$\begin{array}{llll} B,\ C & \text{in the two points} & X,\ X', \\ C,\ A & \quad\text{,,}\quad\quad\text{,,} & Y,\ Y', \\ A,\ B & \quad\text{,,}\quad\quad\text{,,} & Z,\ Z'; \end{array}$$

hence taking on the conics A, B, C the points α, β, γ respectively, and drawing a quadric surface Σ through the nine points X, X', Y, Y', Z, Z', α, β, γ, this meets the conic A in the five points Y, Y', Z, Z', α; that is, it passes through the conic A, and similarly it passes through the conic B, and through the conic C.

Consider how any plane whatever through O intersecting the conics A, B, C in the points L and L', M and M', N and N' respectively; the section of the quadric surface Σ by the plane in question is a conic through the six points L, L', M, M', N, N'. But the section of the surface includes a conic through these same six points, and which is consequently the same conic; in fact, the section of the surface by the plane in question includes a conic, and since every section through the line LL' includes a conic through the same two points, and one of these conics is the conic A which passes through the points L and L', the conic in question passes through the points L and L'; and similarly it passes through the points M and M', and through the points N and N'. That is, for any plane whatever through O, the section of the surface includes the conic which is the section of the quadric surface Σ, and the surface thus includes as part of itself the quartic surface Σ.

The foregoing demonstration ceases, however, to be applicable if O is a point on the surface, and the conic included in the section through O is always a conic passing through the point O. In the case where O is a non-singular point of the surface (that is, where there is at O a unique tangent plane) a like demonstration applies. Take through O a section, and let this include the conic A; on A take any point O' and through OO' a section including the conic B; we have thus the conics A, B intersecting in the points O, O'. Take through O any plane meeting the conics A, B in the points X, Y respectively—the section by this plane includes a conic C passing through the points O, X, Y; and each of the conics A, B, C touches at O the same plane, viz. the tangent plane of the surface. Hence, taking on the conic A the point α, consecutive to O, and any other point α'; on the conic B the point β, consecutive to O, and any other point β'; and on the conic C a point γ'; we may, through the nine points O, α, β, O', α', β', X, Y, γ' describe a quadric surface Σ; this will touch at O the tangent plane of the surface, that is, it will touch the conic C, or (what is the same thing) pass through a point γ of this conic consecutive to O. Hence the quadric surface meets the conic A in the five points O, O', α, α', X, that is, it entirely contains the conic A; similarly it meets the conic B in five points O, O', B, B', Y, that is, it entirely contains the conic B; and it meets the conic C in the five points O, γ, X, Y, γ', that is, it entirely contains this conic. And it may then be shown as

before that the surface will include the quadric surface Σ. But there still remains for consideration the case where O is a conical point on the surface, and I do not at present see how this is to be treated.

I remark that, taking three lines xx', yy', zz' which meet in a point O, then if a surface be such that every section through xx' includes a conic, every section through yy' includes a conic, and every section through zz' includes a conic; and if besides the two points, say X, X', on the conics through the line xx' are ordinary points on the surface, then the surface will include a quadric surface. In fact, if the surface has at each of the points X, X' an ordinary tangent plane, then the conics through xx', and (as conics of the series) the two conics B, C all of them touch the two tangent planes; hence, constructing as before the quadric surface Σ, this also touches the two tangent planes: and taking through xx' a plane meeting the conic A in the points L, L', the section of the surface includes a conic which touches the section of the quadric surface Σ at the points X, X', and besides meets it in the points L, L'; such conic coincides therefore with the section of the quadric surface Σ; that is, every section of the surface through the line xx' includes the conic which is the section of the quadric surface Σ; and the surface thus includes as part of itself the quadric surface Σ.

[Vol. X., July to December, 1868, pp. 17—19.]

2609. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A', on the second a point B', and on the third a point C', such that the intersections of the lines

$A'B'$ and AC, $A'C'$ and AB, lie on the first conic;

$B'C'$ and BA, $B'A'$ and BC, lie on the second conic;

$C'A'$ and CB, $C'B'$ and CA, lie on the third conic.

Solution by the PROPOSER.

Let the six intersections in question be called α, α'; β, β'; γ, γ', respectively; then BC intersects second conic in β', third conic in γ; CA intersects third conic in γ', first conic in α; AB intersects first conic in α', second conic in β; and we have

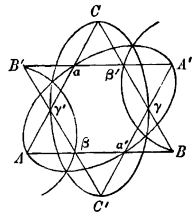
A' the intersection of $\alpha\beta'$, $\gamma\alpha'$,

B' the intersection of $\beta\gamma'$, $\alpha\beta'$,

C' the intersection of $\gamma\alpha'$, $\beta\gamma'$;

and it has to be shown that the points A', B', C' so determined lie—A' on the first conic, B' on the second conic, C' on the third conic.

Taking $x=0$, $y=0$, $z=0$ for the equations of the sides of the triangle ABC, the equations of the three conics may be taken to be $U=0$, $V=0$, $W=0$, where the functions U, V, W are such that identically $U+V+W=0$; and then observing that



the conics pass through the points $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$, respectively, we see that the equations may be taken to be

$$\begin{aligned}
&(\ 0,\ -b,\ c,\ f_1,\ g_1,\ h_1 \between x,\ y,\ z)^2=0,\\
&(\ a,\ 0,\ -c,\ f_2,\ g_2,\ h_2 \between x,\ y,\ z)^2=0,\\
&(-a,\ b,\ 0,\ f_3,\ g_3,\ h_3 \between x,\ y,\ z)^2=0,
\end{aligned}$$

where

$$f_1+f_2+f_3=0,\quad g_1+g_2+g_3=0,\quad h_1+h_2+h_3=0.$$

The coordinates of the points α, β, γ, α', β', γ' are at once found to be

$$\begin{aligned}
&\alpha,\ (\ c\ ,\ 0,\ -2g_1);\qquad \alpha',\ (\ b\ ,\ 2h_1,\ 0\ )\\
&\beta,\ (-2h_2,\ a\ ,\ 0\ );\qquad \beta',\ (\ 0\ ,\ c\ ,\ 2f_2)\\
&\gamma,\ (\ 0\ ,\ -2f_3,\ b\ );\qquad \gamma',\ (2g_3,\ 0\ ,\ a\ );
\end{aligned}$$

and hence the equations of $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ are

$$\begin{aligned}
&\beta\gamma';\qquad ax+2h_2y-2g_3z=0,\\
&\gamma\alpha';\qquad -2h_1x+by+2f_3z=0,\\
&\alpha\beta';\qquad 2g_1x-2f_2y+cz=0.
\end{aligned}$$

Hence for the point A', which is the intersection of $\gamma\alpha'$, $\alpha\beta'$, coordinates are

$$bc+4f_2f_3,\quad 4f_3g_1+2ch_1,\quad 4h_1f_2-2bg_1;$$

and A' will be on the first conic if only

$$(0,\ -b,\ c,\ f_1,\ g_1,\ h_1 \between bc+4f_2f_3,\quad 4f_3g_1+2ch_1,\quad 4h_1f_2-2bg_1)^2=0,$$

viz. this equation is

$$\begin{aligned}
&-\ b\ (\ 16f_3^2g_1^2 \quad +16f_3g_1h_1c\ +4h_1^2c^2)\\
&+\ c\ (\ 16h_1^2f_2^2 \quad -16f_2g_1h_1b\ +4g_1^2b^2)\\
&+2f_1(\ 16g_1h_1f_2f_3-\ 8g_1^2f_3b\ \ +8h_1^2f_2c-4g_1h_1bc)\\
&+2g_1(+16h_1f_2^2f_3\ -\ 8g_1f_2f_3b+4h_1f_2bc-2g_1b^2c\ )\\
&+2h_1(+16g_1f_2f_3^2\ +\ 8h_1f_2f_3c+4g_1f_3bc+2h_1bc^2\ )=0,
\end{aligned}$$

viz. this is easily found to be

$$8\,(2g_1f_3+ch_1)\,(2h_1f_2-bg_1)\,(f_1+f_2+f_3)=0,$$

which is satisfied in virtue of $f_1+f_2+f_3=0$; that is, A' is on the first conic; and similarly, in virtue of $g_1+g_2+g_3=0$, B' is on the second conic; and in virtue of $h_1+h_2+h_3=0$, C' is on the third conic. But the same thing appears at once by the remark that the equations of the three conics are

$$\begin{aligned}
-y\,(-2h_1x+\ \ by+2f_3z)+z\,(\ \ 2g_1x-2f_2y+\ \ cz)&=0,\\
-z\,(\ \ 2g_1x-2f_2y+\ \ cz)+x\,(\ \ \ ax+2h_2y-2g_3z)&=0,\\
-x\,(\ \ \ ax+2h_2y-2g_3z)+y\,(-2h_1x+\ \ by+2f_3z)&=0.
\end{aligned}$$

It may be added that, taking $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$, $(x_3,\ y_3,\ z_3)$, $(x_4,\ y_4,\ z_4)$ as the coordinates of the four points of intersection of the three conics, the first conic is given by means of these four points and the fifth point $(y=0,\ z=0)$; and similarly for the other two conics; whence, denoting the determinants formed with any four columns out of the matrix

$$\left\|\begin{matrix}
x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1\\
x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2\\
x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3\\
x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4
\end{matrix}\right\|$$

by 1234, 1235, &c., we easily find the equations of the three conics, viz. these may be written

$$\begin{array}{rcccccccl}
 & x^2 & , & y^2 & , & z^2 & , & yz & , & zx & , & xy & \\
1456\,(& 0 & , & +3456, & & -2456, & & +2356, & & +2364, & & +2345) & =0,\\
2356\,(& -3456, & & 0 & , & +1456, & & +3156, & & +3164, & & +3145) & =0,\\
3456\,(& 2456, & & -1456, & & 0 & , & +1256, & & +1264, & & +1245) & =0,
\end{array}$$

the exterior factors 1456, 2356, 3456 being introduced in order to bring the equations into the above-mentioned form, wherein the sum of the quadric functions is $=0$.

[Vol. x. pp. 88, 89.]

2743. (Proposed by M. JENKINS, M.A.)—Show that if p be a prime number and m and n any positive integers, the highest power of p contained in $\dfrac{\Pi\,(m+n)}{\Pi\,(m)\,\Pi\,(n)}$ may be obtained by expressing $m+n$ and either m or n in the scale of p; the number of times that it would be necessary to borrow in subtracting the latter number from the former being the index of the power of p required.

Solution by PROFESSOR CAYLEY.

1. In adding any two numbers, we carry a certain number of times; and it is easy to see that the sum of the digits of the two components, less the sum of the digits of the sum, is equal to nine times the number of carryings; moreover, that the number of carryings is equal to the number of borrowings, if either of the components be subtracted from the sum.

2. The same thing is true in any scale of notation, only, instead of nine, we have the radix of the scale, less unity: say the theorem is

$$S(m)+S(n)-S(m+n)=(p-1)x.$$

3. If p be a prime number, the number of times that the factor p occurs in $\Pi(m)$ is

$$E\left(\frac{m}{p}\right)+E\left(\frac{m}{p^2}\right)+E\left(\frac{m}{p^3}\right)+\&c.,$$

where $E\left(\frac{m}{p}\right)$ denotes the integer part of $\frac{m}{p}$, and similarly $E\left(\frac{m}{p^2}\right)$ &c. the integer part of $\frac{m}{p^2}$, &c.; the series is, of course, finite.

Hence the number of times that the factor p occurs in $\frac{\Pi(m+n)}{\Pi(m)\,\Pi(n)}$ is

$$N=E\left(\frac{m+n}{p}\right)+E\left(\frac{m+n}{p^2}\right)+\&c.-E\left(\frac{m}{p}\right)-E\left(\frac{m}{p^2}\right)-\&c.-E\left(\frac{n}{p}\right)-E\left(\frac{n}{p^2}\right)-\&c.$$

4. Hence, expressing m, n, $m+n$ in the scale to the radix p, suppose

$$m=a+bp+cp^2+dp^3,\quad n=a'+b'p+c'p^2+d'p^3,\quad m+n=\alpha+\beta p+\gamma p^2+\delta p^3,$$

we have

$$E\left(\frac{m}{p}\right)+E\left(\frac{m}{p^2}\right)+\&c.=b+cp+dp^2+c+dp+d=d(p^2+p+1)+c(p+1)+b;$$

and similarly for $\qquad E\left(\frac{n}{p}\right)+\&c.,\ E\left(\frac{m+n}{p}\right)+\&c.\ldots;$

whence

$$\begin{aligned}(p-1)N=\ & \delta(p^3-1)+\gamma(p^2-1)+\beta(p-1)\\ & -d(p^3-1)-c(p^2-1)-b(p-1)\\ & -d'(p^3-1)-c'(p^2-1)-b'(p-1)\\ =\ & \{m+n-S(m+n)\}-\{m-S(m)\}-\{n-S(n)\}\\ =\ & S(m)+S(n)-S(m+n),\ =(p-1)x,\end{aligned}$$

if x be the number of times of carrying for the sum $m+n$, or of borrowing for the difference $(m+n)-m$ or $(m+n)-n$; that is, $N=x$, the required theorem. I remark that although the foregoing expression of the number N is a very elegant and ingenious one, yet the original form of N, as given at the end of (3), is the natural and proper expression of the number of times that the factor p occurs in the binomial coefficient $\frac{\Pi(m+n)}{\Pi(m)\Pi(n)}$.

[Vol. x. p. 98.]

2756. (Proposed by J. GRIFFITHS, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle.

Solution by PROFESSOR CAYLEY.

It is a known theorem that if two triads of points, say A, B, C and A', B', C', are self-conjugate in regard to a conic S, they lie in a conic Σ. Take the conic S and the points A, B, C as given; then Σ will be a conic passing through A, B, C; and if on this conic we take any point A', and then take B' to be either of the intersections of the conic Σ by the polar of A in regard to S, and finally construct C' as the pole of $A'B'$ in regard to S, then, by what precedes, C' will be on a conic through A, B, C, A', B', that is, on the conic Σ. Or given the conic S, the triangle ABC, and the conic Σ through A, B, C, we obtain an infinity of triangles $A'B'C'$, self-conjugate in regard to S and inscribed in Σ. That is, if S, Σ are circles, we have an infinity of triangles self-conjugate in regard to the circle S and inscribed in the circle Σ; and inasmuch as the nine-points circle can be constructed by means of the two circles S, Σ alone, the triangles have all of them the same nine-points circle.

[Vol. x. p. 108.]

2737. (Proposed by Professor CAYLEY.)—Find *in solido* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

[Vol. XI., January to June, 1869, pp. 33—38.]

2718. (Proposed by Professor CAYLEY.)—Find *in plano* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

2757. (Proposed by Professor CAYLEY.)—If

$$x_0^2 + y_0^2 = 1, \quad x_1^2 + y_1^2 = 1, \quad \text{and} \quad \begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix} = L;$$

show that each of the equations

$$\frac{a^2(x-x_0)^2 + b^2(y-y_0)^2}{(xx_0 + yy_0 - 1)^2} = \frac{a^2(x-x_1)^2 + b^2(y-y_1)^2}{(xx_1 + yy_1 - 1)^2}, \tag{1}$$

$$\frac{a^2(x-x_0)^2 + b^2(y-y_0)^2}{(xy_0 - x_0y)^2 - (x-x_0)^2 - (y-y_0)^2} = \frac{a^2(x-x_1)^2 + b^2(y-y_1)^2}{(xy_1 - x_1y)^2 - (x-x_1)^2 - (y-y_1)^2}, \tag{2}$$

represents the right line $L = 0$ and a cubic curve.

1819. (Proposed by C. TAYLOR, M.A.)—From two fixed points on a given conic pairs of tangents are drawn to a variable confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

Solution of the above Problems by PROFESSOR CAYLEY.

1. It is easy to see that drawing through the points A, C a circle, and through B, D a circle, such that the radii of the two circles are proportional to the lengths AC, BD, then that the required locus is that of the intersections of the two variable circles.

Take $AC = 2l$, MO perpendicular to it at its middle point M, and $= p$; a, b the coordinates of M, and λ the inclination of p to the axis of x; then

$$\text{coordinates of } O \text{ are } \quad a + p\cos\lambda, \quad b + p\sin\lambda,$$
$$\text{coordinates of } A, C \text{ are } \quad a \pm l\sin\lambda, \quad b \mp l\cos\lambda,$$

and hence the equation of a circle, centre O and passing through A, C, is

$$(x - a - p\cos\lambda)^2 + (y - b - p\sin\lambda)^2 = l^2 + p^2;$$

or, what is the same thing,

$$(x-a)^2 + (y-b)^2 - l^2 = 2p\,[(x-a)\cos\lambda + (y-b)\sin\lambda].$$

If $2m$, q, c, d, μ refer in like manner to the points B, D, then the equation of a circle, centre say Q, and passing through B, D, is

$$(x-c)^2+(y-d)^2-m^2=2q[(x-c)\cos\mu+(y-d)\sin\mu];$$

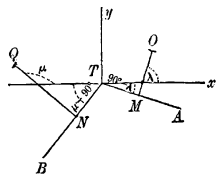
and the condition as to the radii is $l^2+p^2 : m^2+q^2=l^2 : m^2$, that is, $p^2 : q^2=l^2 : m^2$, or $p : q=\pm l : m$. And we thus have for the equation of the required locus

$$\frac{(x-a)^2+(y-b)^2-l^2}{(x-a)\cos\lambda+(y-b)\sin\lambda}=\pm\frac{l}{m}\,\frac{(x-c)^2+(y-d)^2-m^2}{(x-c)\cos\mu+(y-d)\sin\mu},$$

viz. the locus is composed of two cubics, which are at once seen to be *circular* cubics. One of these will however belong (at least for some positions of the four points) to the case of the subtended angles being *equal*, the other to that of the subtended angles being *supplementary;* and we may say that the required locus is a circular cubic.

2. If two of the points coincide, suppose C, D at T; then, taking T as the origin, we may write

$$a=\ \ l\ \sin\lambda,\quad b=-l\ \cos\lambda,$$
$$c=-m\sin\mu,\quad d=\ \ m\cos\mu,$$



and the equation becomes

$$\frac{x^2+y^2+2l(x\sin\lambda-y\cos\lambda)}{x\cos\lambda+y\sin\lambda}=\pm\frac{l}{m}\,\frac{x^2+y^2+2m(x\sin\mu-y\cos\mu)}{x\cos\mu+y\sin\mu},$$

viz. this is

$$(x^2+y^2)[m(x\cos\mu+y\sin\mu)\mp l(x\cos\lambda+y\sin\lambda)]-2lm\{(x\sin\lambda-y\cos\lambda)(x\cos\mu+y\sin\mu)$$
$$\pm(x\sin\mu-y\cos\mu)(x\cos\lambda+y\sin\lambda)\}=0.$$

Taking the lower signs, the term in { } is $(x^2+y^2)\sin(\lambda-\mu)$, and the equation is

$$(x^2+y^2)\{m(x\cos\mu+y\sin\mu)+l(x\cos\lambda+y\sin\lambda)-2lm\sin(\lambda-\mu)\}=0,$$

viz. this is $x^2+y^2=0$, and a line which is readily seen to be the line AB; and in fact from any point whatever of this line the points A, T and the points B, T subtend *supplementary* angles.

Taking the upper signs, the equation is

$$(x^2+y^2)\left[m(x\cos\mu+y\sin\mu)-l(x\cos\lambda+y\sin\lambda)\right]$$
$$-2lm\left\{(x^2-y^2)\sin(\lambda+\mu)-xy\cos(\lambda+\mu)\right\}=0,$$

which is the locus for *equal* angles, a circular cubic as in the case of the four distinct points.

3. The question is connected with Question 1819, which is given above. In fact, taking A, B for the fixed points on the given conic, and P for the intersection of any two of the tangents, if in the conic (foci A, B) which passes through P, the tangent or normal at P passes through a fixed point T, then it is clear that at P the points A, T and B, T subtend equal angles, and consequently the locus of P should be a circular cubic as above. The theorem will therefore be proved if it be shown that the locus of P considered as the intersection of tangents from A, B to the variable confocal conic is in fact the foregoing circular cubic. I remark that the fixed point T is in fact the intersection of the tangents AT, BT to the given conic at the points A, B respectively.

4. Consider the points A, B, (which we may in the first instance take to be arbitrary points, but we shall afterwards suppose them to be situate on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,) and from each of them draw a pair of tangents to the confocal conic $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$. Take (x_0, y_0) for the coordinates of A, and (x_1, y_1) for those of B; then the equation of the pair of tangents from A is

$$\left(\frac{x_0^2}{a^2+h}+\frac{y_0^2}{b^2+h}-1\right)\left(\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}-1\right)-\left(\frac{xx_0}{a^2+h}+\frac{yy_0}{b^2+h}-1\right)^2=0,$$

or, what is the same thing,

$$\frac{(xy_0-x_0y)^2}{(a^2+h)(b^2+h)}-\frac{(x-x_0)^2}{a^2+h}-\frac{(y-y_0)^2}{b^2+h}=0,$$

that is

$$(xy_0-x_0y)^2-(b^2+h)(x-x_0)^2-(a^2+h)(y-y_0)^2=0,$$

or as this may also be written

$$(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2=h\left[(x-x_0)^2+(y-y_0)^2\right];$$

and similarly for the tangents from B we have

$$(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2=h\left[(x-x_1)^2+(y-y_1)^2\right];$$

in which equations the points (x_0, y_0), (x_1, y_1) are in fact any two points whatever.

5. Eliminating h, we have as the locus of the intersection of the tangents

$$\frac{(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2}{(x-x_1)^2+(y-y_1)^2},$$

which is apparently a quartic curve; but it is obvious *à priori* that the locus includes as part of itself the line AB which joins the two given points. In fact, there is in the series of confocal conics one conic which touches the line in question, and since for this conic one of the tangents from A and also one of the tangents from B is the line AB, we see that every point of the line AB belongs to the required locus. The locus is thus made up of the line in question and of the cubic curve.

6. To effect the reduction it will be convenient to write ax, by in the place of x, y, (ax_0, by_0, ax_1, by_1 in place of x_0, y_0, x_1, y_1,) and thus consider the equation under the form

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2};$$

it is to be shown that this equation represents the line $L=0$, and a cubic curve.

Writing for a moment $x_0=x+\xi_0$, $y_0=y+\eta_0$, and $x_1=x+\xi_1$, $y_1=y+\eta_1$, the equation becomes

$$\frac{a^2\xi_0^2+b^2\eta_0^2}{(x\eta_0-y\xi_0)^2-\xi_0^2-\eta_0^2}=\frac{a^2\xi_1^2+b^2\eta_1^2}{(x\eta_1-y\xi_1)^2-\xi_1^2-\eta_1^2},$$

and hence, multiplying out, the equation is at once seen to contain the factor $\xi_0\eta_1-\xi_1\eta_0$ (which is in fact the determinant just mentioned), and when divested of this factor the equation is

$$a^2[(x^2-1)(\xi_0\eta_1+\xi_1\eta_0)-2xy\xi_0\xi_1]=b^2[(y^2-1)(\xi_0\eta_1+\xi_1\eta_0)-2xy\eta_0\eta_1].$$

Writing herein for ξ_0, η_0, ξ_1, η_1 their values, and consequently

$$\xi_0\xi_1=x^2-x(x_0+x_1)+x_0x_1,$$
$$\eta_0\eta_1=y^2-y(y_0+y_1)+y_0y_1,$$
$$\xi_0\eta_1+\xi_1\eta_0=2xy-x(y_0+y_1)-y(x_0+x_1)+x_0y_1+x_1y_0,$$

and arranging the terms, the equation is found to be

$$(a^2x^2+b^2y^2)[-x(y_1+y_0)-y(x_1+x_0)]+(a^2x^2+b^2y^2)(x_0y_1+x_1y_0)-2xy[a^2(1+x_0x_1)-b^2(1+y_0y_1)]$$
$$+(a^2-b^2)[x(y_1+y_0)+y(x_1+x_0)-(x_0y_1+x_1y_0)]=0,$$

which is the required cubic curve.

7. Restoring the original coordinates, or writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c. in place of x, y, x_0, &c., we have

$$(x^2+y^2)[-x(y_1+y_0)+y(x_1+x_0)]+(x^2-y^2)(x_0y_1+x_1y_0)-2xy(a^2-b^2+x_0x_1-y_0y_1)$$
$$+(a^2-b^2)[x(y_1+y_0)+y(x_1+x_0)-(x_0y_1+x_1y_0)]=0,$$

which is a circular cubic the locus of the intersections of the tangents from the arbitrary points (x_0, y_0), (x_1, y_1) to the series of confocal conics $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$; the origin of the coordinates is at the centre of the conics.

8. Supposing that the points (x_0, y_0), (x_1, y_1) are on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and that we have consequently $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, the equations of the tangents at these points respectively are $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$; and hence, writing for shortness $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, we find $x=-\frac{a^2\alpha}{\gamma}$, $y=-\frac{b^2\beta}{\gamma}$ as the coordinates of the point of intersection T, of the two tangents; and in order to transform to this point as origin, we must in place of x, y write $x-\frac{a^2\alpha}{\gamma}$, $y-\frac{b^2\beta}{\gamma}$ respectively. Or what is more convenient, we may in the equation at the end of (6), in which it is to be now assumed that $x_0^2+y_0^2=1$, $x_1^2+y_1^2=1$, write $x-\frac{\alpha}{\gamma}$, $y-\frac{\beta}{\gamma}$ for x, y, and then restore the original coordinates by writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c., for x, y, x_0, &c., and $\frac{\alpha}{b}$, $\frac{\beta}{a}$, $\frac{\gamma}{ab}$ for α, β, γ, these quantities throughout signifying $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$. I however obtained the equation referred to the point T as origin by a different process, as follows:

9. Starting from the equation at the commencement of (5), I found that the points (x_0, y_0), (x_1, y_1) being on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation could be transformed into the form

$$\frac{\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2}{(x-x_1)^2+(y-y_1)^2},$$

an equation which (not, as the original one, for all values of (x_0, y_0), (x_1, y_1), but) for values of (x_0, y_0), (x_1, y_1) such that $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, breaks up into the line AB and a cubic curve.

10. To simplify the transformation, write as before ax, by, ax_0, &c., for x, y, x_0, &c. We have thus to consider the equation

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2},$$

where $x_0^2+y_0^2=1$, $x_1^2+y_1^2=1$, and which equation, I say, breaks up into the line $L=0$, and into a cubic.

Write for shortness $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, so that the equation of the last-mentioned line is $\alpha x+\beta y+\gamma=0$. Then it may be verified that, in virtue of the relations between (x_0, y_0), (x_1, y_1), we have identically

$$(x-x_0)(xx_1+yy_1-1)+(x-x_1)(xx_0+yy_0-1)=(\alpha x+\beta y+\gamma)\frac{x_0+x_1}{\alpha\gamma}(\gamma x+\alpha),$$

$$(x-x_0)(xx_1+yy_1-1)-(x-x_1)(xx_0+yy_0-1)=\beta x^2-\alpha xy-\gamma y-\beta;$$

and, similarly,

$$(y-y_0)(xx_1+yy_1-1)+(y-y_1)(xx_0+yy_0-1)=(\alpha x+\beta y+\gamma)\frac{y_0+y_1}{\beta\gamma}(\gamma y+\beta),$$

$$(y-y_0)(xx_1+yy_1-1)-(y-y_1)(xx_0+yy_0-1)=\beta xy-\alpha y^2+\gamma x+\alpha.$$

11. The equation in question may be written $a^2P+b^2Q=0$, where

$$P=(x-x_0)^2(xx_1+yy_1-1)^2-(x-x_1)^2(xx_0+yy_0-1)^2,$$
$$Q=(y-y_0)^2(xx_1+yy_1-1)^2-(y-y_1)^2(xx_0+yy_0-1)^2,$$

values which are given by means of the formulæ just obtained; there is a common factor $\alpha x+\beta y+\gamma$ which is to be thrown out; and we have also, as is at once verified, $\frac{y_0+y_1}{\beta}=\frac{x_0+x_1}{\alpha}$, so that these equal factors may be thrown out. We thus obtain the cubic equation

$$a^2(\gamma x+\alpha)(\beta x^2-\alpha xy-\gamma y-\beta)+b^2(\gamma y+\beta)(\beta xy-\alpha y^2+\gamma x+\alpha)=0.$$

This is simplified by writing $x-\frac{\alpha}{\gamma}$ for x, $y-\frac{\beta}{\gamma}$ for y. It thus becomes

$$a^2x[(\gamma x-\alpha)(\beta x-\alpha y)-\gamma^2y]+b^2y[(\gamma y-\beta)(\beta x-\alpha y)+\gamma^2x]=0;$$

or, what is the same thing,

$$a^2x[\gamma x(\beta x-\alpha y)-\alpha\beta x+(\alpha^2-\gamma^2)y]+b^2y[\gamma y(\beta x-\alpha y)-(\beta^2-\gamma^2)x+\alpha\beta y]=0;$$

that is

$$\gamma(a^2x^2+b^2y^2)(\beta x-\alpha y)+a^2[-\alpha\beta x^2+(\alpha^2-\gamma^2)xy]+b^2[-(\beta^2-\gamma^2)xy+\alpha\beta y^2]=0.$$

12. Restoring $\frac{x}{a}$, $\frac{x_0}{a}$, $\frac{x_1}{a}$ for x, x_0, x_1, and $\frac{y}{a}$, $\frac{y_0}{a}$, $\frac{y_1}{a}$ for y, y_0, y_1; writing consequently $\frac{\alpha}{b}$, $\frac{\beta}{a}$, $\frac{\gamma}{ab}$ in place of α, β, γ, if α, β, γ are still used to denote $\alpha=y_0-y_1$, $\beta=x_1-x_0$, $\gamma=x_0y_1-x_1y_0$, the equation becomes

$$\gamma(x^2+y^2)[b^2\beta x-a^2\alpha y]+a^2[-b^2\alpha\beta x^2+(a^2\alpha^2-\gamma^2)xy]+b^2[-(b^2\beta^2-\gamma^2)xy+a^2\alpha\beta y^2]=0,$$

where now, as originally, $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$; viz. this is the equation, referred to the point T as origin, of the locus of the point P considered as the intersection of tangents from A, B to the variable confocal conic; and it is consequently the equation which would be obtained as indicated in (8). The locus is thus a circular cubic; the equation is identical in form with that obtained (2) for the locus of the point at which A, T and B, T subtend equal angles, and the complete identification of the two equations may be effected without difficulty.

13. I may remark that M. Chasles has given (*Comptes Rendus*, tom. 58, February, 1864) the theorem that the locus of the intersections of the tangents drawn from a fixed conic to the conics of a system (μ, ν) is a curve of the order 3ν. The confocal

conics, *quà* conics touching four fixed lines, are a system (0, 1); hence, taking for the fixed conic the two points A, B, we have, as a very particular case, the foregoing theorem, that the locus of the intersections of the tangents drawn from two fixed points to a variable confocal conic is a cubic curve.

[Vol. XI. p. 49.]

Note on Question 2740. *By* PROFESSOR CAYLEY.

The envelope of the curve

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

(where A, B, C, D, E are any functions of the coordinates, and θ is a variable parameter,) is obtained in the particular case $E=0$ (Salmon's *Higher Plane Curves*, p. 116), and the same process is applicable in the general case where E is not $=0$. From the great variety of the problems which depend upon the determination of such an envelope, the result is an important one, *and ought to be familiarly known* to students of analytical geometry. We have only to write $z=\cos\theta + i\sin\theta$, the trigonometrical functions are then given as rational functions of z, and the equation is converted into a quartic equation in z; the result is therefore obtained by equating to zero the discriminant of a quartic function. The equation, in fact, becomes

$$A\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)+B\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)+C\frac{1}{2}\left(z+\frac{1}{z}\right)+D\frac{1}{2i}\left(z-\frac{1}{z}\right)+E=0,$$

that is

$$A(z^4+1)-Bi(z^4-1)+C(z^3+z)-Di(z^3-z)+2Ez^2=0;$$

or, multiplying by 12 to avoid fractions, this is

$$(a,\ b,\ c,\ d,\ e \between z,\ 1)^4=0,$$

where

$$a=12(A-Bi),\quad b=3(C-Di),\quad c=4E,$$
$$e=12(A+Bi),\quad d=3(C+Di);$$

and substituting in

$$(ae-4bd+3c^2)^3-27(ace-ad^2-b^2e+2bcd-c^3)^2=0,$$

the equation divides by 1728, and the final result is

$$\{12(A^2+B^2)-3(C^2+D^2)+4E^2\}^3$$
$$-\{27A(C^2-D^2)+54BCD-[72(A^2+B^2)+9(C^2+D^2)]E+8E^3\}^2=0.$$

It is to be noticed, that in developing the equation according to the powers of E, the terms in E^6, E^4 each disappear, so that the highest power is E^3; the degree in the coordinates, or order of the curve, is on this account sometimes lower than it would otherwise have been.

[Vol. XII., July to December, 1869, p. 69.]

2920. (Proposed by Professor CAYLEY.)—Imagine a tetrahedron $BB'CC'$ in which the opposite sides BB', CC' are at right angles to each other and to the line joining their middle points M, N; and in which moreover $\overline{CN}^2+\overline{NM}^2+\overline{MB}^2=0$, (or, what is the same thing, the sides CB, CB', $C'B$, $C'B'$ are each $=0$; the tetrahedron is of course imaginary; viz. the lines CC', BB' and points M, N may be real; but the distances $MB=MB'$ and $NC=NC'$ may be one real and the other imaginary, or both imaginary, but they cannot be both real) the points B, B' and C, C' are said to be "skew antipoints." Then it is required to prove that

1. A given system of skew antipoints may be taken to be the nodes (conical points) of a tetranodal cubic surface, passing through the circle at infinity, and which is in fact a Parabolic Cyclide.

2. The equation of the surface may be expressed in the form

$$x(x+\beta)(x+\gamma)+(x+\beta)y^2+(x+\gamma)z^2=0.$$

3. The section through either of the lines $(y=0,\ x+\gamma=0)$ and $(z=0,\ x+\beta=0)$ is made up of this line and a circle; the two systems of circles being the curves of curvature of the surface; it is required to verify this *à posteriori;* viz. by means of the equation of the surface to transform the differential equation of the curves of curvature in such manner that the transformed equation shall have the integrals

$$y=C(x+\gamma),\quad z=C'(x+\beta).$$

In the following Contents, the Problems are referred to each by its No. and the Proposer's name, and the Subject is briefly indicated. An asterisk shows that no solution was given. A line —— shows that there is no No.

			PAGE
*1791	Cayley	Quartic and three Cubics	546
——	,,	Conic defined by five Conditions	ib.
*1857	,,	Ternary Cubic Form	548
*1730	,,	Relation between roots of Cubic Equation	ib.
1834	,,	Points on Cubic Curve	549
——	,,	Conic defined by five Conditions	550
1876	Ball	Roots of Quartic Equation	551
——	Cayley	Conic defined by five Conditions	552
1890	,,	Conic through three Points and with double Contact	554
*1554	,,	Ellipse and Circles of maximum and minimum Curvature	555
1931	,,	Nodal Cubic	ib.
1990	Sylvester	Cartesian Curves and Cubic Curve	556
1949	Cayley	Cuspidal Cubic	561

			PAGE
*1872	Cayley	Intersection of two Surfaces	563
1969	Sylvester	Spherical Problem	ib.
——	Cayley	Foci and Antifoci	567
1970	,,	Conics of double Contact	568
2110	,,	Locus of Foci	ib.
2250	,,	Foci of Conic	571
1991	,,	Plane determined by Point and three Lines	ib.
1993	Cotterill	Envelope depending on two Circles	573
2270	Cayley	Bicircular Quartic	575
2309	,,	Partitions	576
2286	Laverty	System of Equations	578
2331	Cayley	System of Equations	581
2321	,,	Conic and four Points	ib.
2371	,,	Random Points within Triangle	ib.
1990	Crofton	Cartesian Curves	582
1911	Cayley	Four Points and Conic of Centres	ib.
*2371	,,	Random Points	585
2466	Murphy	Four Points in Plane or Space	ib.
2472	Cayley	Four Points and Conic	587
2471	Woolhouse	Theorem of Integration	588
2530	Cayley	Tricuspidal Quartic	589
2553	,,	Quartic Surface and Sphere	ib.
2573	,,	Envelope of Circle	591
2493	,,	Construction of Conic	592
2451	,,	Antipoints	593
2590	Cayley	Sections of Quartic Surface	ib.
2609	,,	Three Conics	595
2743	Jenkins	Factorials	597
2756	Griffiths	Series of Triangles	599
2737	Cayley	Locus *in solido*	ib.
2718	,,		
2757	,,	Locus *in plano*	600
1819	Taylor		
2740	Cayley	Envelope of Plane Curve	606
*2920	,,	Tetrahedron and Cubic Surface	607

NOTES AND REFERENCES.

445, 451, 454. We have the two papers by K. Rohn, "Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte und ihrer Gestalten," *Preisschr. der F. J. Gesell. zu Leipzig* (Leipzig, 1886, pp. 1—58), and same title *Math. Ann.* t. XXIX. (1887), pp. 81—97. I have not been able to examine the conclusions arrived at in these papers with as much care as would have been desirable.

I call to mind that for a k-nodal quartic surface the tangent cone from any node is a sextic cone with $(k-1)$ nodal lines, breaking up it may be into cones of lower orders—see table p. 265: and that we distinguish the quartic surfaces according to the forms of the sextic cones corresponding to the k nodes respectively. It will be recollected that (6) denotes a sextic cone, (6_1) a sextic cone with one nodal line, $(5_1, 1)$ a sextic cone breaking up into a quintic cone with one nodal line and a plane; and so in other cases.

There is a sort of break in the theory; in fact when the number of nodes is not greater than 7 these may be any given points whatever, and taking the 7 points at pleasure we have surfaces with 8 nodes, and 9 nodes, but not with any greater number of nodes, viz. for a surface with 10 or more nodes, it is not permissible to take 7 of these as points at pleasure, so that the theory of the surfaces with 10 or more nodes is so to speak separated off from that of the surfaces with a smaller number of nodes. For the case of 10 nodes we have the symmetroid 10 (3, 3) and other forms, for 11 nodes Rohn finds 3 or ? 4 forms; for 12 nodes he has four forms, viz. my 3 forms and a fourth form 12_d; for 13 nodes he has two forms, 13_b agreeing with my 13_a, and 13_a which replaces my non-existent form 13_β; for 14 nodes, 15 nodes and 16 nodes he has in each case a single form, agreeing with my results. Without endeavouring to complete the theory, I write down a table as follows:

No. of Nodes		Form of Cones	Remarks
16		16 (1, 1, 1, 1, 1, 1)	
15		15 (2, 1, 1, 1, 1)	
14		$8(3_1, 1, 1, 1) + 6(2, 2, 1, 1)$	
13	$13_b = 13_a$	$3(4_3, 1, 1) + 1(3, 1, 1, 1) + 9(3_1, 2, 1)$	
"	13_a	$1(2, 2, 2) + 12(4_3, 1, 1)$	13_a replaces my non-existent 13_β, $= 13(2, 2, 2)$
12	$12_b = 12_a$	$12(4_3, 2)$	
"	$12_a = 12_\beta$	$2(5_6, 1) + 6(3_1, 3_1) + 4(3, 2, 1)$	

Table continued.

No. of Nodes		Form of Cones	Remarks
12	$12_c = 12_\gamma$	$12\,(4_2,\ 1,\ 1)$	$12_c = 12_\gamma$ is a peculiarly simple and elegant form; the equation is $A^2 - xyzw = 0$, where A is a quadric function of the coordinates.
"	12_d	$2\,(4_2,\ 1,\ 1) + 8\,(5_5,\ 1) + 2\,(4_3,\ 2)$	
11	$11_a = 11_\alpha$	$1\,(6_{10}) + 10\,(3_1,\ 3_1)$	
"	11_b	$8\,(6_{10}) + 3\,(4_2,\ 2)$	
"	11_c	$6\,(5_5,\ 1) + 5\,(6_{10})$	
"	11_d	?	
10		$10\,(3,\ 3)$	The quartic surface is here the symmetroid.
9			
8			
7			
6			
5		$5\,(6_4)$	
4		$4\,(6_3)$	
3		$3\,(6_2)$	
2		$2\,(6_1)$	
1		$1\,(6)$	

The suffixes a, b, c, d refer to Rohn's forms, the suffixes α, β, γ to my forms. The form 11_d is given in the first but not in the second of Rohn's two memoirs, and I am not sure as to the intended character of the sextic cones. I have not attempted to fill up the third column of the table for the Nos. of nodes 9, 8, 7, 6, as there may be particular cases which I have not considered. For the Nos. 5, 4, 3, 2, 1, the cone is a sextic cone with at most 4 nodal lines, and consequently in each case a proper sextic cone not breaking up into cones of inferior orders.

END OF VOL. VII.

CAMBRIDGE:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 4to. 10 vols. Vols. I. II. III. IV. V. VI. and VII. 25*s.* each. [Vol. VIII. *In the Press.*

The Scientific Papers of the late Prof. J. Clerk Maxwell. Edited by W. D. NIVEN, M.A., formerly Fellow of Trinity College. In 2 vols. Royal 4to. £3. 3*s.* (net).

Mathematical and Physical Papers, by Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., Pres.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Demy 8vo. Vol. I. 18*s.* Vol. II. 15*s.* Vol. III. 18*s.*

The Scientific Papers of John Couch Adams, M.A., D.Sc., D.C.L., LL.D., F.R.S., F.R.A.S., F.C.P.S., &c., late Fellow of Pembroke College, Honorary Fellow of St John's College, and Lowndean Professor of Astronomy and Geometry in the University of Cambridge. Vol. I. Edited by WILLIAM GRYLLS ADAMS, M.A., D.Sc., F.R.S., &c., Professor of Natural Philosophy in King's College, London, late Fellow of St John's College, Cambridge. With a Memoir by J. W. L. GLAISHER, Sc.D., F.R.S., &c., Fellow of Trinity College, Cambridge. Demy 4to. [*In the Press.*

A Treatise on the Mathematical Theory of Elasticity, by A. E. H. LOVE, M.A., Fellow of St John's College, Cambridge, in Two Volumes. Demy 8vo. Vol. I. 12*s.* Vol. II. 12*s.*

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.*
Vol. II. Saint-Venant to Lord KELVIN (Sir WILLIAM THOMSON). In 2 parts. By the same Editor. 30*s.*

The Steam Engine and other Heat Engines. By J. A. EWING, M.A., F.R.S., M.Inst. C.E., Professor of Mechanism and Applied Mathematics in the University of Cambridge. Demy 8vo. 15*s.*

A Treatise on Natural Philosophy. By Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Lord KELVIN (Sir W. THOMSON) and Professor P. G. TAIT. Demy 8vo. 9*s.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of Peterhouse, Cambridge. Demy 8vo. Vol. I. 14*s.* Vol. II. 10*s.*

Theory of Differential Equations. Part I. Exact Equations and Pfaff's Problem. By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, Cambridge. Demy 8vo. 12*s.*

A Treatise on the Theory of Functions of a Complex Variable, by A. R. FORSYTH, Sc.D., F.R.S. Royal 8vo. 21*s.* Net.

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A., F.R.S., late Fellow of Trinity College, Cambridge, Professor of Mathematics in the Owens College, Manchester. Demy 8vo. New Edition. [*In the Press.*

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow and Tutor of Christ's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on Spherical Astronomy. By Sir ROBERT S. BALL, M.A., F.R.S., Lowndean Professor of Astronomy and Geometry. Demy 8vo. [*Preparing.*

A Treatise on Elementary Hydrostatics. By J. GREAVES, M.A., Fellow and Lecturer of Christ's College. Crown 8vo. 5*s.*

Elementary Thermodynamics. By J. PARKER, M.A., Fellow of St John's College, Cambridge. Crown 8vo. 9*s.*

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

Electricity and Magnetism. By J. J. THOMSON, M.A., Cavendish Professor of Experimental Physics. Crown 8vo. [*In the Press.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Demy 4to. cloth (Vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half morocco 28*s.* (net). New Series for the years 1874—1883. Vols. 9—10, cloth 25*s.* (net) each, half morocco 32*s.* (net) each. [Vol. XI. *In the Press.*

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

CAMBRIDGE UNIVERSITY PRESS.

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. New and Enlarged Edition. Crown 8vo. 7*s.* 6*d.*

Solutions to the Examples in a Treatise on Elementary Dynamics. By the same Author. Crown 8vo. 7*s.* 6*d.*

Plane Trigonometry. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. 7*s.* 6*d.* Or in separate Parts. Part I. up to and including the Solution of Triangles. 5*s.* Part II. Analytical Trigonometry. 3*s.* 6*d.*

A Treatise on the General Principles of Chemistry. By M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A., and D. J. CARNEGIE, M.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

Lectures on the Physiology of Plants. By S. H. VINES, Sc.D., Professor of Botany in the University of Oxford. Demy 8vo. With Illustrations. 21*s.*

CAMBRIDGE NATURAL SCIENCE MANUALS.

Elementary Palaeontology for Geological Students. By HENRY WOODS, B.A., F.G.S. Crown 8vo. 6*s.*

Heat and Light. By R. T. GLAZEBROOK, M.A., F.R.S., Assistant Director of the Cavendish Laboratory. Crown 8vo. 5*s.*

The Two Parts are also published separately.

Heat. 3*s.* **Light.** 3*s.*

Other Volumes preparing.

PITT PRESS MATHEMATICAL SERIES.

Arithmetic for Schools. By C. SMITH, M.A., Master of Sidney Sussex College, Cambridge. Extra Fcap. 8vo. With or without Answers. Second edition, 3*s.* 6*d.* Or in two Parts 2*s.* each.

Key to Smith's Arithmetic. [*In the Press.*

Elementary Algebra. By W. W. ROUSE BALL, M.A., Fellow and Mathematical Lecturer of Trinity College, Cambridge. Extra Fcap. 8vo. 4*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Extra Fcap. 8vo. Books I.—IV. 3*s.* Books I.—VI. 4*s.* Books I. and II. 1*s.* 6*d.* Books III. and IV. 1*s.* 6*d.* Books V. and VI. 1*s.* 6*d.* Books XI. and XII. [*In the Press.*

Solutions to the Exercises in Taylor's Euclid. Books I.—IV. By W. W. TAYLOR, M.A. 6*s.*

The Elements of Statics and Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. Third Edition. Extra Fcap. 8vo. 7*s.* 6*d.*
Also published separately STATICS, 4*s.* 6*d.* DYNAMICS, 3*s.* 6*d.*

Solutions to the Examples in the Elements of Statics and Dynamics. By S. L. LONEY, M.A. 7*s.* 6*d.*

Mechanics and Hydrostatics for Beginners. By S. L. LONEY, M.A. 4*s.* 6*d.*

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow of Christ's College, Cambridge, and University Lecturer in Mathematics, and C. M. JESSOP, M.A., Fellow of Clare College. Extra Fcap. 8vo. 4*s.* 6*d.*

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

Vol. 8. Notes & References.

Ribaucour C.R. t. 75-1872 pp. 533-536. Referring to my note remarks that the condition can be ~~expressed~~ (by means of the imaginary coordinates of M. Ossian Bonnet) expressed in a simple form. Communicated by him to the Philomathic Society, May 1870. ~~The proof is not given, but it is not difficult to prove it in a quite intelligible form. The result is as follows: he takes~~ $p = f(x, y, z)$ to represent a family of surfaces belonging to a triply orthotomic system. Considering two neighbouring surfaces (A) & (A') corresponding to the values z & $z+dz$; A & A' the two points where they meet the trajectories of the surfaces; AT, A'T' the tangents to the curves of curvature of the same system at A, A' respectively. Then according to the remark of M. Léroy, it is to be expressed that these lines meet, and this is done by expressing that along the trajectory AA', the variation of the angle of AT with the osculating plane at A is equal to the angle of the osculating planes at A, A' respectively.

Let B' be the spherical image of A', the plane OBB' is parallel to the osculating plane at A of the trajectory, and the angle of the two osculating planes measures the geodesic curvature of BB': denote this by $d\gamma$

Let β be the angle of BB' with BX, θ the angle of AT with BX; $\beta - \theta$ is the angle of AT with the osculating plane at A of the trajectory: $d\beta - d\theta = d\gamma$

But dx & dy being the increments of x, y corresponding to dz in the passage from A to A', then by a theorem of M. Liouville

$$d\gamma = d\beta - i\left(\frac{d\lambda}{\lambda\, dx}\, dx - \frac{d\lambda}{\lambda\, dy}\, dy\right);$$

the condition thus is

$$d\theta = i\left(\frac{d\lambda}{\lambda\, dx}\, dx - \frac{d\lambda}{\lambda\, dy}\, dy\right)$$

introducing the symmetric imaginary coordinates x & y we write

$$a = \frac{dp}{\lambda^2 dx},\quad b = \frac{dp}{\lambda^2 dy},\quad c = \frac{1}{\lambda^2}\frac{dp}{dx\,dy}$$

$$ds^2 = 4\lambda^2 \frac{da}{dx}\frac{db}{dy}\, dx\, dy$$

and the formula $e^{-2i\theta} = \pm\sqrt{\frac{da}{dx}} \div \sqrt{\frac{db}{dx}}$

enables this to be written in the definitive form

$$dx \frac{d}{dx}\left\{\left(\frac{db}{dy} \div \lambda^4 \frac{da}{dx}\right) - dy \frac{d}{dy}\left\{\left(\frac{da}{dx} \div \lambda^4 \frac{db}{dy}\right) + dz \frac{d}{dz}\left(\left\{\frac{db}{dy} - \frac{d}{dz}\left\{\frac{da}{dx}\right)\right) = 0.\right.\right.$$

We have

$$dx\left(\tfrac{1}{2}p + c\right) + dy \frac{db}{dy} + dz \frac{db}{dz} = 0$$

$$dx \frac{da}{dx} + dy\left(\tfrac{1}{2}p + c\right) + dz \frac{da}{dz} = 0$$

and hence eliminating dx, dy, dz we have

$$\begin{vmatrix} \frac{d}{dx}\left\{\left(\frac{db}{dy} \div \lambda^4 \frac{da}{dx}\right), & \frac{d}{dy}\left\{\left(\frac{db}{dy} \div \lambda^4 \frac{da}{dx}\right), & \frac{d}{dz}\left\{\left(\frac{db}{dy} \div \frac{da}{dx}\right) \\ \frac{1}{2}p + c, & \frac{db}{dy}, & \frac{db}{dz} \\ \frac{da}{dx} & \frac{1}{2}p + c & \frac{da}{dz} \end{vmatrix} = 0$$

which defines the triply orthotomic system.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. VIII.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1895

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 70 papers, numbered 486 to 555, published for the most part in the years 1871 to 1873.

The Table for the eight volumes is

Vol.	I.	Numbers	1	to	100.
,,	II.	,,	101	,,	158.
,,	III.	,,	159	,,	222.
,,	IV.	,,	223	,,	299.
,,	V.	,,	300	,,	383.
,,	VI.	,,	384	,,	416.
,,	VII.	,,	417	,,	485.
,,	VIII.	,,	486	,,	555.

PREFATORY NOTE.

THE death of Professor Cayley, which occurred on the 26th of January, 1895, has deprived the later part of this volume, as it will deprive the succeeding volumes, of the advantage of his supervision. The Syndics of the Press desired that the collection of the papers should be completed; and on the 15th of February, they asked me to undertake the duty of editing the remaining volumes. I willingly acceded to their request.

Professor Cayley had himself passed the first thirty-eight sheets of this volume for press; his illness prevented him from even revising any succeeding sheets. He had prepared one Note for the volume: it is printed at the end. The remaining volumes must appear without Notes and References: the reason being that he did not prepare these Notes in advance but only when the corresponding papers came before him in the proof-sheets.

The actual manuscript of the Note has been reproduced in facsimile upon the kind of paper which he regularly used during his mathematical investigations. As it refers to the memoir that ends only in the last sheet but one which he passed for press, it is one of the last pieces of his writing.

He left no instructions as to the Collection of his Mathematical Papers; the statement, prefixed to the first volume, is the only account of his method of arrangement. A comparison of the contents of the first seven volumes with the list of his papers in the Royal Society's *Catalogue of Scientific Papers* has enabled me to make out the detailed course of the method which will, of course, be followed in the remaining volumes.

The Syndics expressed their desire that I should insert some biographical notice of Professor Cayley in a volume of the series. Accordingly, one is inserted in the present volume; it is a reprint (with only slight verbal changes) of the notice which was written for the *Proceedings of the Royal Society*. And I have ventured to add a complete list of the lectures which he announced from year to year after his return to Cambridge in 1863 as Sadlerian Professor.

A. R. FORSYTH.

8 *June*, 1895.

ARTHUR CAYLEY.

[From the Obituary Notices in the *Proceedings of the Royal Society*, vol. LVIII. 1895.]

ARTHUR CAYLEY was the second son of Henry Cayley and Maria Antonia Doughty; he was born at Richmond, in Surrey, on 16 August, 1821.

The family, to whose fame so much honour has been added by one of the greatest mathematicians of all time, is of old origin and illustrious descent. Its name, like not a few English names, is derived from a locality in Normandy; there was a Castellum Cailleii, near Rouen, held by baronial tenure. The head of the house appears to have come to England with William the Conqueror and to have settled in Norfolk, becoming Lord of Massingham, Cranwich, Brodercross, and Hiburgh in that county. The influence of the family increased and, by the time of Edward II., Sir Thomas de Cailli possessed estates also in Yorkshire. On his decease without issue, the Yorkshire property was transferred to a younger branch of the family and was inherited by a long succession of Cayleys who made their home at Thormanby. One of these was knighted, as Sir William Cayley, in 1641; in 1661 he was created a baronet in recognition of his services during the Civil Wars, the title surviving to the present day. The fourth son of Sir William, Cornelius, settled at York; and the eldest son of the latter, also Cornelius, born in 1692, was a barrister and in 1725 was appointed Recorder of Kingston-upon-Hull, an office which he held until a few years before his death in 1779. Probably the advantages offered by Hull, then, as now, the greatest port on the northern coast of England, suggested commerce as an occupation for some members of the Recorder's large family; two of his sons became Russia merchants, settling in St Petersburg. The younger of these, being the fifth son of the Recorder, was Henry Cayley, born in 1768; he married, in 1814, Maria Antonia Doughty, a daughter of William Doughty. The eldest son of this marriage died in infancy. The youngest son, Charles Bagot, was a scholar, possessed of linguistic genius; he was particularly interested in the Romance Languages and he made verse-translations of Homer's *Iliad*, Dante's *Divine Comedy*, and the *Sonnets* of Petrarch. The second son was Arthur, the subject of the present sketch; he was born during a visit of his parents to England. Before passing to the details of his life, it may be added that the second of his father's sisters married Edward Moberly—also a Russia merchant living in St Petersburg—and was the mother of the late Dr. George Moberly, Bishop of Salisbury.

Mr. Henry Cayley took his young family to Russia and remained there for a few years. On retiring from business in 1829, he returned to England and settled into residence at Blackheath. Arthur was sent soon afterwards to a private school there, kept by the Rev. G. B. F. Potticary; and when he was fourteen he was transferred to King's College School, London. At a very early age he had begun to show some of those preferences by which the existence of mathematical ability is wont to reveal itself; he had a great liking for numerical calculations and he developed a great aptitude for them.

In his new school the boy showed himself to be possessed of remarkable ability: his power of grasping a new subject very rapidly and of seizing its central principles was certainly unusual. An old friend tells of an examination in chemistry: the subject had not been studied by Cayley before, but he soon acquired sufficient knowledge to carry off the medal from the professedly chemical students, to their surprise and mortification *. But it was most of all by the indications of mathematical genius that he astonished his teachers. It had been Mr. Cayley's intention to educate his son with the view of placing him in his former business—an intention not abandoned without reluctance. The impression, however, produced upon his teachers could not lightly be set aside; and the advice of the Principal to send him to Cambridge, where his abilities promised to secure brilliant distinction, was adopted.

Accordingly, he went to Cambridge. He was entered at Trinity College on 2nd May, 1838, as a pensioner, and he began residence in the succeeding October at the unusually early age of seventeen. He passed through the ordinary stages in the career of a successful student of mathematics. Like the other able undergraduates of his period, he "coached" with William Hopkins of Peterhouse, who has been described as a great and stimulating teacher—a description justified by the high achievements of a long line of distinguished and grateful pupils.

Cayley's fame grew rapidly: and, as is the way of Cambridge undergraduates, he soon was pointed out as the future Senior Wrangler of the year. It is interesting to find a record of him written about this time and published not long afterwards by an acquaintance†, who says that:—

> "As an undergraduate he had generally the reputation of a mere mathematician, which did him great injustice, for he was really a man of much varied information, and that on some subjects the very opposite of scientific—for instance, he was well up in all the current novels, an uncommon thing at Cambridge where novel-reading is not one of the popular weaknesses."

* It may be added that he maintained his interest in chemistry throughout his life, and acquired a considerable knowledge of it. When he was at Baltimore, in 1882, lecturing at the Johns Hopkins University by special invitation, he attended Professor Remsen's lectures with a pleasure which found expression in his letters home to his children in England. And on one occasion, at Professor Remsen's request, he lectured to the chemistry class on the hydrocarbon "trees" (*Brit. Assoc. Report*, 1875, pp. 257—305).

† Bristed, *Five Years in an English University* (second edition, 1852), p. 95.

It may be added that Cayley declared the story about him in the tripos, recorded by Bristed, to be quite apocryphal.

So also was another story, belonging to a later part of his life, according to which he is reported to have said that "the object of law was to say a thing in the greatest number of words, and of mathematics to say it in the fewest": this view, and the possibility of his ever having held it, he repudiated entirely.

Novel-readers are more frequent in Cambridge now than they appear to have been in 1842, and Cayley in his later days avoided reading some of the modern novels; but it is worth noting, as will subsequently be seen more in detail, that he had this "popular weakness" all his life.

He was admitted a scholar of the College on 1st May, 1840, winning his scholarship at the earliest time when it was possible to do so: and he secured a first class in each of the annual examinations of the College. No record of marks for the first and the second years is given in the Trinity Head Examiner's Book; but in the third year the marks are given and, as he then scored more than twice the marks of the second candidate, the Head Examiner separated him from the rest of the first class by drawing a line under his name. This presage of his powers was confirmed in the following year, 1842, when he graduated as Senior Wrangler; the Examiners were so definitely satisfied that he was first as to dispense in his case with the viva voce tests which at that time were a customary part of the Tripos. And in due course the first Smith's Prize was awarded to him in the succeeding examination.

Cayley's own "year" at Trinity was a distinguished one; for, in addition to himself, it contained Mr. (now the Right Honourable) George Denman, for many years a Judge of the High Court of Justice, and Mr. Hugh Andrew Johnstone Munro, one of the foremost of Latin Scholars of any period. And the distinction of Cayley's contemporaries in neighbouring years is marked: it is impossible to avoid noticing the names of some of the graduates in the Mathematical Tripos about that time. Sylvester and Green (second and fourth wranglers respectively in 1837), Leslie Ellis (senior in 1840), Stokes (senior in 1841), Cayley (senior in 1842), Adams (senior in 1843), Thomson—now Lord Kelvin—(second in 1845), constitute an extraordinary succession of mathematicians of whom England is justly proud. Their achievements in mathematical science have done much to render their University one of the acknowledged chief mathematical schools of the world.

Cayley was elected a Fellow of Trinity and admitted to fellowship on 3rd October, 1842, at an age younger than any other Fellow of the College, at least in the present century; and he was promoted from the position of Minor Fellow to that of Major Fellow on 2nd July, 1845, the year in which he proceeded to his M.A. degree. He was an Assistant Tutor of the College for three years; but such a post was then of an almost nominal character, and there appears to be no indication that any of the mathematical teaching of the College fell to him. He did, indeed, accept some private pupils: his lifelong friend, Canon Venables, has given a pleasant account* of a reading-party which Cayley took to Aberfeldie in 1842.

His pupils, however, did not tie him strictly to Cambridge, for it appears that the latter half of the year 1843 was devoted to continental rambles. The summer was spent in Switzerland, where his zest for walking and for mountain-climbing, a pleasure that never failed while his health lasted, found an active outlet: he had become a member of the Alpine Club in its comparatively early days. The last four months of the year were spent in Italy, partly in the North and in Florence, partly in Rome and Naples. It may have been on this tour that he acquired his love for

* *Guardian*, 6th Feb. 1895, p. 201.

both painting and architecture. The works of painters such as Masaccio, Giovanni Bellini, Perugino, and Luini, then first became known to him; they proved a delight at the time and remained a happy remembrance with him.

These and other continental journeys from time to time, while he remained in residence as a Fellow of his College, were his relaxations. He had no formal lecturing and he did not attempt to obtain a large number of private pupils. The leisure that he thus secured was turned to the best, and to him the most pleasant, of uses, in carrying out mathematical researches. It was, indeed, as an undergraduate that Cayley began the marvellous series of publications which, extending over more than fifty years of his life, have been concerned with practically every branch of pure mathematics as well as with theoretical dynamics and physical astronomy.

The time seemed ripe for the outburst of some mathematical activity. By the efforts of Herschel, Peacock, and Whewell, Cambridge teaching had been set free from the bonds that restricted methods of procedure to those which had proved effective in Newton's days; and the struggle to secure the admittance of analytical methods had been successfully completed. One sign of the new freedom was the foundation of the *Cambridge Mathematical Journal*, in 1837, by D. F. Gregory and Leslie Ellis. Before that time, practically the only English means of publication open to mathematicians was in the *Philosophical Transactions of the Royal Society;* and young writers, whether modest or not about the value of their researches, might well have hesitated before seeking publication in a quarter that exacts so high a standard. The new journal then founded was open to young students and gave them an opportunity, previously difficult to obtain, of making their researches known; and it proved a great stimulus to the intellectual activity of those members of the University. Only four volumes of the journal appeared; but it was continued, first under the name of the *Cambridge and Dublin Mathematical Journal*, and, subsequently down to the present time, under that of the *Quarterly Journal of Pure and Applied Mathematics*. Though the opportunities of publication, which now are afforded to mathematicians both in England and abroad, are vastly more numerous than they were half a century ago, the undoubted service rendered to English mathematics by the initial venture of the two young Cambridge men should not be forgotten.

It was in the second volume of this journal that Cayley's earliest paper, written in 1841, was printed: and two other papers bearing the same date—it was the year before his degree—are included in the third volume. Though the results are not remarkable, the freshness and the independence of these early investigations are worthy of notice. Cayley had evidently read with enquiring and critical care the *Mécanique Analytique* of Lagrange, some of the work of Laplace, and several memoirs in the two continental journals of the time, those of Liouville and Crelle. These achievements of an undergraduate of nineteen or twenty, which are rarely accomplished now and were still rarer in his day, recall Abel's dictum*:—

> "Si l'on veut faire des progrès dans les mathématiques, il faut étudier les maîtres et non pas les écoliers."

* *Niels-Henrik Abel* (par Bjerknes, Paris, 1885), p. 173.

It was as certainly one of the characteristics of Cayley to find a stimulus to new developments in the main ideas of other writers as it was one of his characteristics to be able to follow out his own ideas with the insistent unwearying patience of an investigator creating a new work complete. And it is interesting to see how this faculty of receiving inspiration reveals itself from the beginning of his career.

Once free from the necessity of preparing for his Tripos and his Fellowship examination, he was able to throw himself into the work of production. His activity may be estimated from the fact that he produced three papers in 1842, eight in 1843, four in 1844, and thirteen in 1845. Moreover, these papers deal with a great variety of subjects. Thus he makes his first investigations in the numerative calculus of plane curves: he initiates his discussions about geometry of n dimensions: he founds the theory of invariants and covariants: and he elucidates the connexion between doubly-infinite products and elliptic functions. Some of these early papers are now classical; and the briefest inspection of them is sufficient to reveal the suggestiveness and the easy strength of the young mathematician who was not yet in his twenty-fifth year.

Even by this date the opportunities of publication in England had become inadequate to his needs. Curiously enough, he does not appear to have sent any paper to the Royal Society until the year 1852, when Sylvester communicated the "Analytical Researches connected with Steiner's Extension of Malfatti's Problem*" to that Society. Later in the same year, Cayley was elected a Fellow of the Society, and thereafter many of his papers appear in its *Philosophical Transactions*. Before 1852, there were few journals either at home or abroad which did not receive communications from him: and even in the quite early years of his researches, several of his papers, written in French, appeared in Liouville's journal and in Crelle's journal. As societies and journals grew in number, so the area over which his papers spread became ever wider.

At first, after winning his Trinity Fellowship, he remained at Cambridge, and his time must then have been largely at his own disposal. This freedom, in his circumstances, could last for only a limited time because, unless he either entered holy orders or devoted himself to teaching in some permanent post (if obtainable) in the College, the Fellowship could be held for not more than seven years after his M.A. degree—a period that would expire in 1852. He was unwilling to take holy orders—not that there was any religious obstacle in his way, for he was not harassed either by philosophical doubts or critical difficulties. His simple reason for remaining a layman was that, though devout in spirit and an active Churchman, he felt no vocation for the sacred office.

In consequence, it became necessary to choose some profession. Cayley selected the law, left Cambridge in 1846, entered at Lincoln's Inn, and became a pupil of the famous conveyancer, Mr. Christie. A story of their first interview, that Mr. Christie used to tell in after years, is an illustration of the modesty and the lack of self-

* Cayley's *Collected Mathematical Papers*, vol. II. No. 114. Subsequent references to this series will be made in the form *C. M. P.*

assertiveness which were leading features of Cayley's character: and this impression is confirmed by the recollections of a fellow-pupil, Mr. T. C. Wright, who says:—

> "... We fellow-pupils knew that Arthur Cayley had been the Senior Wrangler of his year, and that he possessed extraordinary abilities; but they were not indicated by his personal bearing, and the retiring modesty of his disposition prevented him from ever alluding to the honours he had won at Cambridge. He had one of the most unsophisticated minds I have ever known; jokes, and the badinage of the pupil-room, seemed to be delightful novelties to him, and his face beamed with amusement as he listened to them without taking much part in the conversation, being content to devote his time assiduously to work which I suspect was not altogether congenial to his taste...."

But if the modest, almost shy, man did not display his honours, he could not conceal his powers; and very soon his clearness of head, his almost intuitive grasp of the principles of any subject that came before him, his capacity for work and his power of concentration, made him a favourite pupil. He was called to the Bar on 3rd May, 1849, and thereafter he had no occasion to wait for business. Mr. Christie was always ready to supply him with at least as much conveyancing work as he was willing to undertake: but no advice, no encouragement, no opening however favourable, least of all any wish for fame or fortune, could tempt him to subside into a large practice. He restricted himself to "devilling" for Mr. Christie, and he limited the amount of work he would undertake in this way, always refusing work that came to him at first hand. There is no doubt that, had he remained at the Bar and devoted himself to its business, he could have made a great legal reputation and a substantial fortune: even as it was, some of his drafts* have been made to serve as models. But the spirit of research possessed him; it was not merely will but an irresistible impulse that made the pursuit of mathematics, not the practice of law, his chief desire. To achieve this desire, he reserved with jealous care a due portion of his time; and he regarded his legal occupations mainly as the means of providing a livelihood.

He remained at the Bar for fourteen years. Between two and three hundred papers are the mathematical outcome of that period; and they include some of the most brilliant of his discoveries. Among these papers are to be found the majority of his famous memoirs on quantics (particularly the sixth memoir, in which he develops his theory of geometry, and shows that all geometry can be made entirely descriptive), his work upon matrices, numerous contributions to the theory of symmetric functions of the roots of an equation, the elaborate calculations connected with the development of functions arising in the planetary and the lunar theories, and his valuable reports on theoretical dynamics. The enormous range over which his papers of these fourteen years extend is not more remarkable than the vigour of his contributions to knowledge; and a reference to them will show that he frequently recurs to some given problem, always adding something to the development.

* In Davidson's *Precedents and Forms in Conveyancing* (third edition, 1873), vol. III. Part II. p. 1067, the author adds a footnote, calling "attention to the remarkable skill exhibited in [a] settlement, the work of Mr. Arthur Cayley."

In judging of this persistent and unflagging activity, some account ought to be taken of his surroundings. It can hardly be that 2, Stone Court, from which many of his papers are dated, proved an inspiration to mathematical research. For part of the time, his friend Sylvester was in London—then as an actuary; and I have heard Cayley describe how Sylvester and he walked round the Courts of Lincoln's Inn discussing the theory of invariants and covariants which occupied (and occasionally absorbed) the attention of both of them during the fifties. And on matters which related to analytical geometry he was in frequent (but formal) correspondence with Salmon; indeed, the relation that existed between the two men developed ultimately into one of warm friendship and deep mutual regard: its sincerity can be gathered from the spirit animating Salmon's notice of Cayley, published in *Nature* in 1883, at the time when the latter was President of the British Association. But, with special exceptions of the types indicated, his work was so largely of the kind that is called path-breaking that he was bound to do it alone: he did it with a simple unconscious courage and with unfailing resolution.

It may easily be imagined that his links with life at Cambridge had now become slight. During the earliest of the years spent at the bar, he had returned on a few occasions. In 1848, the year before his call, he was the junior mathematical examiner in the regular annual examinations of Trinity; in 1849, and also in 1850, he was the senior mathematical examiner in the same examinations. In 1851 he was Senior Moderator for the Mathematical Tripos; one of the wranglers, Lightfoot, becoming subsequently his friend, and his colleague in the University, before going to his great work in the diocese of Durham as Bishop. In 1852 he was Senior Examiner for the Tripos, the senior wrangler of the year being Tait (also afterwards one of his intimate friends), now Professor of Natural Philosophy at Edinburgh. These seem to have been the only occasions when he was recalled to Cambridge; and they did not require any permanent connexion with the College or the University. He was settled in London, his allegiance divided between law and mathematics.

A change, however, in the statutes of the University offered an opportunity for his return to Cambridge; a professorship of pure mathematics was established upon an old foundation. Lady Mary Sadleir (who endowed the Croonian Lecture Fund of the Royal College of Physicians of London and also that of the Royal Society in memory of her first husband, Dr. William Croone, a physician and one of the earliest Fellows of the Royal Society) had, by her will, dated 25th September, 1701, and proved 6th November, 1706, given to the University an estate, which was to be used as an endowment of lectureships in algebra at nine of the colleges in Cambridge. These posts were duly established. The great developments of analysis, which took place at the end of the last century and during the first half of the present century, gradually proved that the restriction to algebra prevented the lectureships from being as adequate an encouragement to the advancement of mathematics as they were designed to be at the time of their establishment. Moreover, the lecturers had ceased to attract undergraduates to their lectures: so that the purpose of the foundation was not being fulfilled. Consequently, in 1857, a proposal was made by the Council of the Senate of the University that a new direction should be given to the endowment by the establishment

of a professorship, to be called the Sadlerian Professorship of Pure Mathematics: the duty of the professor was "to explain and teach the principles of pure mathematics, and to apply himself to the advancement of that science." The proposal was approved by the Senate on 3rd December, 1857, and the new statute was sanctioned by an Order of the Queen in Council on 7th March, 1860. Some time had to elapse before certain provisional arrangements could be completed, and it was not until after three years that the University was in a position to act.

On 10th June, 1863, Cayley was elected Sadlerian professor: he held the chair for the rest of his life. The stipend attached to the professorship was modest, though it was improved in the course of subsequent legislation; these changes, however, could not have been foreseen at the time when Cayley was elected. Yet he had no hesitation about returning to Cambridge: for the post enabled him to devote his life to the pursuit he liked best. He never showed the slightest regret at having neglected the prospects of distinction at the bar, or at having chosen to return to his University; and he always expressed perfect satisfaction and content with his life in Cambridge, which was one of great happiness.

His appointment as Sadlerian professor marks a turning point in his life. Henceforward he lived, for the most part, in the quiet of the University; yet it was by no means in seclusion, for he took his share in administration, which claims a part (often too large a part) of the leisure of men fitted for this necessary duty. But he was not burdened by heavy claims arising out of his official position: and he was directed by the statutes governing him to do what was, as a matter of fact, his ideal in life. No man could have been better suited than Cayley was to fulfil the charge of the statutes: his knowledge and his power of research pointed him out as the obvious choice of the electors.

He settled in Cambridge at once. On 8th September, 1863, he married Susan, daughter of Robert Moline, of Greenwich. This is not the place to dwell upon his domestic life; but it is impossible to omit in silence all reference to its singular happiness, based upon the affection felt by its members for one another. Friends and visitors who have been in that home will not soon forget the kindness and the gracious courtesy of the welcome they received, or the atmosphere of peace into which they were raised. Sometimes in the old garden by the river-side, more often in the drawing-room, the talk went on; the professor himself listening, attentive and watchful, frequently taking only a slight share, but ever ready to join in. No cynicism or paradox in speech was ventured upon in his presence; no harshness of judgment was tolerated without a quiet protest; no sense of bustle or ambition was felt there; in all things the charm of an old-world home, centred round him. His widow and their two children, Mary and Henry, remain to mourn their loss.

His teaching duty was limited to the delivery of one course of lectures in the academic year, and he usually chose the Michaelmas term. This practice was maintained for twenty-three years until he was placed under the new statutes, which in 1882 had come into operation so far as concerned all future appointments. After that change, he delivered two courses of lectures, one in the Michaelmas term, the other in the Lent term. An inspection of the list of his lectures* shows that he chose his

* The list is given on pp. xlv, xlvi.

subjects by preference from analytical geometry, dynamics (in his view, theoretical dynamics is a portion of pure mathematics), differential equations, theory of equations, Abelian functions, elliptic functions, and modern algebra. The titles of the lectures, as announced, were sometimes vague, nor were they intended to limit his range; in all cases he went far beyond the boundary that so frequently limits Cambridge studies. Thus a course of lectures on differential equations, announced for the Michaelmas term in 1879, was chiefly concerned with conformal representation, polyhedral functions, and Schwarz's investigations on the hypergeometric series.

For many years he dispensed with the use of blackboard and chalk in his class-room; this was possible because his class usually was small. He brought his work written out upon the blue draft-paper,* which was regularly used by him in all his writing of mathematics; the exposition consisted partly of verbal explanations made as he showed the manuscript, partly of details written out at the moment. A change came in 1881, when his class amounted to fifteen or sixteen: he was then obliged to use the blackboard, and he subsequently maintained the new practice. Occasionally his older habit of explaining his manuscript recurred—he then placed it upon the board. This was especially the case when he brought carefully prepared diagrams, such as those used in the modular-function division of the plane: these diagrams were made much clearer by the use of water-colours to distinguish different sets of regions, and their preparation evidently gave him pleasure.

But, as may be surmised, his influence as a teacher was overshadowed by his influence as an investigator. Those whom he affected by his lectures belonged for the most part to the mathematical teachers in Cambridge: the number of undergraduates whom he influenced was small, though, when any one of them did come under his influence, the effect was well marked. His starting point in any subject was usually beyond the range of all other than quite advanced students; but to any able undergraduate who was willing to devote time, not merely to the comprehension of the matter in the lectures but also to collateral reading, the lectures were stimulating and inspiring. This effect was partly due to the easy strength with which he worked, partly to the spirit in which he approached old and new subjects alike; an independent suggestiveness and a singular freshness marked his views, and gave an added interest to his exposition even of a well-known theory. One reason of this freshness may be found in the fact that his lectures consisted of the current researches upon which he was engaged at the time; sometimes, even, a lecture would be devoted to results which he had obtained since the preceding lecture. Though the titles of the courses occasionally recur from one year to another, the same course was never given twice. The new matter in any course, once given, was usually incorporated in a paper or memoir; and when the same subject was nominally lectured upon again, it was a distinct part of the subject—old notes were never used a second time.

It was not alone by his lectures that he acted as professor. Students, seeking help or desiring to interest him in their work, found him always willing to give them the benefit of his advice, his criticism, and his knowledge. Nor was it merely mathematicians in Cambridge whom he helped in this way. He was continually consulted by

* It was the customary "scribbling paper" of his undergraduate days.

foreigners, who appreciated the promptness no less than the fulness of information in his replies.

It frequently happens that a man of genius, great enough to leave a distinct impress of his originality upon his science, finds it irksome to study what others have written. With the growth of all sciences during the last fifty years, especially—it may be said—with the growth of pure mathematics in that time, the tendency of workers is to become specialists in their own subject and, perhaps, in subjects immediately cognate with it, and to acquire only a slight acquaintance with what is being done outside the circle of their limited interests. Not so was Cayley: he was singularly learned in the work of other men, and catholic in his range of knowledge. Yet he did not read a memoir completely through: his custom was to read only so much as would enable him to grasp the meaning of the symbols and understand its scope. The main result would then become to him a subject of investigation: he would establish it (or test it) by algebraical analysis and, not infrequently, develop it so as to obtain other results. This faculty of grasping and testing rapidly the work of others, together with his great knowledge, made him an invaluable referee; his services in this capacity were used through a long series of years by a number of societies to which he almost was in the position of standing mathematical adviser.

Concurrently with his teaching, he continued his investigations. He wrote only one book—a *Treatise on Elliptic Functions,* published in 1876, which was intended to bridge over the gap from Legendre's *Traité des Fonctions Elliptiques* to Jacobi's *Fundamenta Nova*; it contains a considerable amount of new matter. But paper after paper was published in a long unfailing succession almost until his death; their tale amounts to more than 800. Happily for the convenience of mathematicians, the republication of his papers in collected form was undertaken by the Cambridge University Press,—perhaps the most enduring, certainly not the least fitting, monument of his fame. The request was made to him in 1889 by the Syndics of the Press; he willingly acceded to it and deeply appreciated, both then and afterwards, what he regarded as a great compliment to himself. Seven large quarto volumes, under his own editorship, have already appeared. The preparation of them was always a great happiness to him; and, especially in the later years of his life, it gave him an occupation in his science which was still within the range of his failing strength. At the time when the collection was begun it was estimated that ten volumes would suffice for the purpose, but it is now evident that ten will be certainly insufficient. The Syndics of the Press intend to complete the series of volumes; it is a matter of regret that the illustrious author of the papers has not lived to complete it himself.

Even his teaching and investigations did not fully occupy his time. For the first few years after his return he was left comparatively free from a large share in administration, but gradually it was assigned to him. As he became better known for his effective business capacity, his share in administration grew until he came to be regarded as an indispensable member of the Council of the Senate. He was elected a member of that body on 7th November, 1876, and with the exception of some six months when he was absent in America, he continued a member of it until 1892, when failing health compelled him to resign. During this period of service he was

re-elected three times. Party feeling ran rather strongly at times during the discussions that led to the new statutes; but both parties included his name among their lists of nominations—an adequate proof that he possessed the confidence of the Senate. He was free from party bias, and he became established in his position of strength by his fairmindedness, his sound judgment, and his calm temperament. He would listen to a discussion, speaking only when he had something of importance to add; when speaking he was listened to with full attention. More frequently he would take no part in the discussion until his opinion was asked, as was usually the case in difficult questions; his opinion was always valued and sometimes final. Similarly, on syndicates, his co-operation was much sought, and in particular the services which he rendered to the Library Syndicate and the Press Syndicate were of substantial importance. He also took great interest in the movement for the higher education of women. In the early days of Girton College he gave direct help in teaching, and for some years he was Chairman of the Council of Newnham College, in the progress of which he took the keenest interest even to the last.

But, with all his general aptitude for business, he was perhaps most specially helpful by his legal knowledge. The training he had undergone and the knowledge he had acquired at the bar ultimately proved invaluable. His opinion on legal matters was sought by the University, by his own college, and by the scientific societies with which he was connected; when given, it frequently had the effect of a judicial decision. His powers of drafting were constantly being called into requisition; he responded to the calls upon him and, with unstinted generosity, placed his time and skill at the disposal of these bodies, so that the new statutes of Trinity College, and not a few of the statutes and ordinances of the University, owe much to him.

One other illustration, at once of his general business capacity and of the confidence reposed in him, may be given. The elections for representatives of the Universities in the House of Commons are still conducted openly and by means of voting papers, delivered either by the elector himself or by another elector whom he has nominated; objections may be raised against any voting paper, but they must be decided at once. In Cambridge the Vice-Chancellor, being the returning officer, nominates a number of assessors to act with him in the case of a contested election. At a bye-election in 1882, when the candidates were Mr. H. C. Raikes and Professor James Stuart, Cayley was nominated as presiding officer at one of the polling places. His imperturbable firmness, his calm courtesy, and the justice of his decisions secured for his effectiveness in this capacity the admiration of the University.

This brief account of his participation in business affairs is necessary; without some such indication a proper estimate of his position in Cambridge cannot be framed. And it also may help to show that his supremacy in the subjects of his investigations neither made him a recluse, nor limited his other interests, nor restricted his practical usefulness.

The merits of such a man were recognised by the only means at the disposal of a grateful and appreciative University. He was elected an honorary Fellow of Trinity College on 22nd May, 1872, at the same time as Dr. Lightfoot, Mr. James

Spedding, and Professor Clerk Maxwell; and on 11th October, 1875, he was made an ordinary Fellow, a position which he retained for the rest of his life. His friends subscribed for a presentation portrait,* painted by Lowes Dickenson in 1874; it now hangs in the College Hall. The simplest of inscriptions is on its frame, but the humorous lines which Clerk Maxwell† wrote at the time should not readily be forgotten. The graver element, seldom absent from his verses, is not entirely repressed even by his wit, and the lines were based upon a deep admiration of the man

> "Whose soul, too large for vulgar space,
> In *n* dimensions flourished unrestricted."

His bust, by Mr. Henry Wiles, was given to Trinity College by a donor who wished to remain anonymous. It was placed in the beautiful library of the College on 3rd December, 1888, an honour that has been conferred during life in only two other cases—Tennyson and Sedgwick.

After the new statutes came into operation, the Senate on 27th May, 1886, decided that the Sadlerian Professorship should at once be made subject to the improved provisions, a decision which, though it increased the amount of lecturing required, gave him the benefit of the full stipend. At the same time the Lucasian Professorship, held by Professor Stokes, was also made subject to the new statutes; and it was currently believed that the Lowndean Professorship would have been included in the proposal had Professor Adams been willing to have the change made. There was a wish on the part of members of the University to give some recognition to the glory conferred upon the mathematical school by Stokes, Adams, and Cayley; one possibility remained. The opportunity came in 1888 when Prince Edward (as he was known in Cambridge), afterwards Duke of Clarence, received the degree of LL.D. Such an occasion is customarily marked by the conferment of a number of honorary degrees upon distinguished men; among them, on this particular occasion, were the three professors who had been colleagues for a quarter of a century. On the 9th of June in that year a great assembly gathered to see these degrees conferred upon the recipients. It need hardly be said that the men singled out for honour received ovations on being presented; among the most enthusiastic ovations were those accorded to the three professors.

Nor were external bodies and learned societies, both at home and abroad, backward in recognising the merits of his work; the honours he received were numerous and came from all quarters. Honorary degrees were conferred upon him by several universities as well as his own, among them being Oxford, Dublin, Edinburgh, Göttingen, Heidelberg, Leyden, and Bologna. President Carnot nominated him an Officer of the Legion of Honour. He was either a Fellow or a foreign corresponding member of most of the scientific societies of the Continent, among them being the French Institute, the Academies of Berlin, Göttingen, St. Petersburg, Milan, Rome, Leyden, Upsala, and Hungary. He was also a Fellow of the Royal Society of Edinburgh, of the Royal Irish

* A photographic reproduction of the portrait is prefixed to vol. VI. of the *C. M. P.*

† See Campbell and Garnett's *Life of James Clerk Maxwell*, p. 636.

Academy, and of the Royal Astronomical Society. He had been President of the Cambridge Philosophical Society, and he sat on its Council for many years; also President of the London Mathematical Society and of the Royal Astronomical Society. He was elected a Fellow of the Royal Society on 3rd June, 1852, and he served as a member of its Council for six periods of office. In 1859 he received from the Royal Society a Royal medal, and in 1882 the Copley Medal, the highest scientific distinction it is in its power to bestow. When the De Morgan Medal was instituted in connexion with the London Mathematical Society, the first award was fitly made to Cayley. And from Leyden he received the Huyghens Medal.

Mention should be made of one other honour which he received: it is of a kind seldom conferred. The high opinion of his work which was held in America was indicated by an invitation in 1881 to deliver a course of lectures in the Johns Hopkins University, Baltimore, where his friend and fellow investigator, Sylvester, was then professor. He accepted the invitation, and left England in December of that year. During the next five months he lectured on Abelian and Theta Functions; the substance of these lectures was incorporated in a memoir subsequently published in the *American Journal of Mathematics**. He returned to England in June, 1882, bringing back pleasant remembrances of kindnesses and friendships.

His life, spent in mathematical research and in the quiet round of activity in the University, offered little of either interest or incident to make his name known by the outside world to the same extent or in the same way as the names of many scientific men, engaged in other lines of enquiry, are known. Once, however, in his life circumstances brought him prominently into notice. In 1883 he was President of the British Association for the Advancement of Science, the meeting being held at Southport; and, in that capacity at the opening of the meeting, he had to deliver a formal address, an abstract of which appeared as usual in the leading newspapers of the country.

In the early days of the Association, the President's address frequently reviewed the whole field of science; but as knowledge has developed, a tendency has set in, according to which each later President has confined himself more particularly to those matters within whose range he is an authority. And, subject to this restriction, it is hoped that the address may be legitimately popular. There have been critics of presidential addresses prepared to assert that science was sacrificed to popularity; there have been immense audiences convinced that popularity was sacrificed to science. Taken together, the presidential addresses, some severe and others popular, form an interesting series of reviews of the successive stages in scientific achievements.

Cayley's address belonged to the severely scientific class. From the nature of his subject—the progress of mathematics, more particularly of pure mathematics—it was bound to have this character. Few of the members of a regular Association audience have more than a slight acquaintance with pure mathematics; and, consequently, it is impossible to deliver to such a gathering an address which, in a reasonable time, can give them any real idea of the condition or the progress of the science. Cayley felt

* Vol. v. (1883), pp. 137—179; vol. VII. (1885), pp. 101—167.

this and confessed to the feeling in a passage which is perhaps the best known in the address:—

> "It is difficult to give an idea of the vast extent of modern mathematics. The word 'extent' is not the right one: I mean extent crowded with beautiful detail—not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for everything else, so for a mathematical theory —beauty can be perceived but not explained."

But he also felt that the respect due to the Association requires its President to deal with that branch of science about which, as he knows it best, he is best fitted to tell them, so that different subjects may thus in turn be brought before successive meetings.

> "So much the worse," he added, "it may be, for a particular meeting; but the meeting is the individual which on evolution principles must be sacrificed for the development of the race."

Granting then the inevitably stern character (as popularly estimated) that must mark any proper exposition of his subject, the address is one of singular interest. It undoubtedly made a great impression. Parts of it were incomprehensible to all but mathematicians: still, there was much which others could understand and, understanding, found excellent. Even leader-writers at the time recognised its lucidity, its finish, its native elegance, and its instructive and stimulating essence. To mathematicians it counts for much. Not merely is it a valuable historical review of various mathematical theories; but the exposition possesses all the freshness, the independence of view, the suggestiveness and the amazing knowledge that were so characteristic of Cayley. And, consequently, it can often be recurred to with unfailing profit.

After this event, his life pursued the unbroken tenor of its scientific course. Ever thinking, working, writing, he maintained the flow of his papers with the same unslackening vigour, and he showed the same sympathetic encouragement of others, as had marked him before the scientific world had tried to acknowledge his genius by showering its honours upon him.

It is now some years since the painful internal malady, which ultimately was the cause of death, began to show itself. At first, its action was slow; and there was reasonable hope that his naturally strong constitution would enable him to throw it off. Unfortunately these hopes were not realised; its growth was steady, its undermining influence persistent. Change of scene was tried once or twice, but without good effect; and it soon appeared that Cambridge itself troubled him least. Three years ago his friends saw that his health began to fail: he had occasional attacks of severe illness which confined him to his bed for weeks together, each of them leaving him gravely frailer than before. Gradually he became confined to his house and his garden; he could see only very few friends, and usually even them only for a short time. When

they did see him, they found only too clearly how rare and brief were his intervals of relief from pain, though occasionally his gentleness and his patience would almost delude them into hope.

The last of the severe attacks began on the 8th of January; he seemed to be getting better when, on the 21st, his strength suddenly began to collapse. He died about six o'clock on the evening of Saturday, 26th January, 1895. The funeral took place on the succeeding Friday when, in Trinity Chapel, a great assemblage, composed of members of the University, of representatives of the embassies of Russia and America, as well as of various learned societies and of personal friends, gathered to pay him their last homage of respect and reverence.

Sufficient has been said to show that Cayley was a man of general activities; but his scientific work and his public duties by no means exhausted or limited his general interests.

It has already been stated that, as an undergraduate, he was fond of reading novels; this practice remained with him all his days. He preferred a novel of the old orthodox type with a "happy ending"; and though his greatest delight was in the older novels, a modern book, such as *Beside the Bonnie Briar Bush* (which he read quite late in 1894), met with words of warm praise. He had a good memory, and used to discuss plots and characters with considerable animation. The two novelists, by whose works many English people are divided into one or other of two classes, did not affect him much; Thackeray he read but did not like, and he would not read Dickens. His favourite authors were Scott and Jane Austen; all their works had been read by him many times, and they were read aloud to him during the long period of his illness. *Guy Mannering* and *The Heart of Midlothian*, among Scott's, and *Persuasion*, among Jane Austen's, were the books he liked the best. He also was fond of George Eliot's novels, particularly of *Romola*. Indeed, though he had aversions, his taste was somewhat general. Commendation of a book was enough to make him willing to try it; and there was only one limitation to his range of novel-reading—he had an instinctive abhorrence of anything that suggested either coarseness or vulgarity.

His English reading was not confined to novels. He had a keen liking for many of Shakespeare's plays, notably *Much Ado About Nothing*, and some of the historical dramas. He delighted in Milton's shorter poems, though he would not tolerate *Paradise Lost*. Scott's poems were frequently read; and he had a great appreciation of Byron's *Tales* and of Coleridge's *Ancient Mariner*. Grote's *History of Greece* and Macaulay's *History of England* he read repeatedly and with zest; and he never seemed tired of Lockhart's *Life of Scott*.

He was also a good linguist. He knew French well; it was a second writing-language to him, as will be seen from the large number of papers, written in French, which occur in his collected mathematical papers. He read (but he did not talk) German and Italian with ease, and his Greek remained fresh throughout his life. This last power may have been due to the admiration he felt for Plato; he referred to the *Republic* and the *Theætetus* in his Presidential Address; and, on the afternoon of the

day of the "Greek division"* in the Senate House, I remember finding him at home reading the *Gorgias*.

He had the keenest interest, amounting almost to a passionate delight, in travelling; cities of historic or artistic fame delighted him equally with beautiful scenery. Long after he had become an invalid, he found a fascination in guide-books and maps; and all his younger friends will recall the sympathetic zeal with which he entered into their projected journeys, and the happy pleasure he took in hearing them speak of recent journeyings and in recalling, with a wonderful vivid memory, his own experiences and ideas about places they had visited.

Reference has been made to his early pleasure in the old Italian masters. Yet, if any inferences can be drawn from the likings of his later years, architecture attracted him even as much as pictures. He had a true feeling and a clear judgment as to genuine excellence: he sketched well, and had a quick eye for proportions, perspective, light and shade. One of his relaxations was to make coloured sketches of buildings that he liked, notably sepia drawings of some of the great Gothic cathedrals and churches of northern France. He kept up his practice of water-colour painting all his life, and in his closing years it proved a great solace to him at times when his strength was so far reduced that he could not work. He had great happiness in looking at architectural pictures and at books on architecture, one of his favourites among the latter being Street's *Brick and Marble in the Middle Ages*.

Financial matters and accounts also interested him; and only a few months before his death he published a brief pamphlet on book-keeping by double entry, which he has been known to declare one of the two perfect sciences. He could not resist some reference to the subject in his Presidential Address, making the remark that the notion of a negative magnitude "is used in a very refined manner in book-keeping by double-entry."

His bearing was gentle, and it was marked by a courtesy that was unfailing. On questions of administration and in discussions, his opinions were stated clearly and quietly. Not that he did not hold decided views or that he would abate one jot of his firm, even chivalrous, defence of what he held to be right; but there was a judicial temper in his mind which prevented the subjective element in a discussion from disturbing his equanimity. The even balance of his mind enabled him to recognise and appreciate the position of one who differed from him, and his quiet "I do not think so" was all the more effective because its very calmness excluded the slightest suggestion of hostile spirit.

His figure was spare: until his illness, he could easily endure the fatigue of long walks, in which he delighted, especially in hill country. In later years it became rather bent, and he had the appearance of being frail. His head was very impressive,

* In 1891 a proposal was made by the Council of the Senate for the appointment of a Syndicate to enquire, among other things, into the expediency of allowing alternatives for one of the two classical languages in the Previous Examination. Many members of the Senate were convinced that the adoption of an alternative would lead to the extinction of the study of Greek except in the greater public schools; they consequently opposed the proposal, which, on 29th October, 1891, was rejected by a great majority (525 to 185).

It may be added that Cayley was in the minority. He allowed his signature to be added to a letter which was sent to the London newspapers as an appeal for assistance in defeating the attempt to resist enquiry.

as may be seen from his portrait and from photographs. In repose, and when his attention was not concentrated upon what was passing, his face had a grave air and the blue-grey eyes suggested that he was far away in thought; but when attentive or amused, and when expressing pleasure, the eyes became singularly keen and a peculiar charm lightened up the whole face.

He was absolutely modest. The honours conferred on him in full profusion never injured in the least degree the grand simplicity of his character, never gave rise to the slightest trace of vanity, which was alien to his nature. He rarely spoke of them, and, when he did, it never was as of honours: they pleased him, but, perhaps, rather as recognition of his work than as tributes to the worker. If any one expressed appreciation of any of his papers, owing to the help it had given, he would reply very quietly: but he did not stint the expression of his pleasure at advances beyond his own results when they were made by others. Public appearances were rather distressing to him at first, for his disposition was retiring and he could be reserved; but as time wore on, duty often compelled him to take part in them. In such cases he accepted the claim and discharged it with a straightforward simplicity that was entirely devoid of self-consciousness; but he gladly avoided demonstrations whenever it was possible.

In the spirit of his work one great quality was his generosity to others, particularly to young men, whose work he was always willing to recognise. He ignored the fact that he was a great mathematician—probably it never occurred to him to think of his doings: but it may be doubted whether this unconsciousness of his greatness ever proved at once more fascinating or more bewildering than when he was discussing scientific results with young men. He so evidently had his wishes centred on a single-hearted desire for the right result that it was difficult to conceive him approaching a question merely as a learner: yet he was ever a learner. There are few men, if any, with not even a tithe of his scientific achievements, who have had less of controversy or have had such immunity from questions as to priority of discovery. This arose not merely from the indisputable priority of his results: it was partly owing to his nature. Salmon says of him:—

> "His motto has always been 'esse quam videri,' and I do not know any one to whom it would be more repulsive to engage in a personal contest by claiming for himself a particle of honour or of money more than was spontaneously conceded. He would be apt to take for his model the patriarch Isaac, who, when the Philistines claimed a well which he had dug, went on and dug another, and when they claimed that, too, went on and dug a third":

an exceedingly happy description of the man the tide of whose genius was

> "Too full for sound or foam."

Some account of his work, some estimate of its character, some indication of the original contributions made by him to his science, may not improperly be given here. It is, of course, impossible to predict what his permanent influence will be upon mathematics, or what opinion coming generations of workers will hold of him: certainly, by his own contemporaries, he was deemed one of the greatest mathematicians the

world has seen. Bertrand, Darboux, and Glaisher have compared him to Euler, alike for his range, his analytical power, and, not least, for his prolific production of new views and fertile theories. There is hardly a subject in the whole of pure mathematics at which he has not worked. Some new subjects owe their existence to him; to others he has made very definite contributions, so that their boundaries have been enlarged often to an enormous extent; there are few upon which he has not left the mark of his genius.

In several of the notices that appeared at his death he was described as a great explorer. Such he undoubtedly was, but he was more. He not merely discovered new countries but he also opened them up, so that others were able to enter into some possession of those regions without undergoing the difficulties that he had overcome. And if the metaphor may be carried further, he had the restlessness of the explorer: he could not long remain satisfied with an achievement concluded, but must try his fortune again and elsewhere.

Varying opinions have been expressed as to Cayley's style; the variations are largely due to preconceived views of what a mathematical paper should be. It certainly is not easy to skim one of his papers; any attempt to do so leads to an inadequate estimate of what it usually establishes. It is not difficult to read one of his papers, even to grasp the contents well, provided proper care be devoted to it, because difficulties that occur are completely solved, and nothing lies in the background to cause doubt or suggest incompleteness. He has been well described by Glaisher as an unequalled master of analytical processes; it is especially in algebraical manipulation that his strength and his facility stand out in clear view. His success in this direction was achieved by a skill that cannot be explained by describing it as due to acquired knowledge, or to practice, or to long consideration and patient selection. It was rather an instinct for the management of the most complicated processes, and the way in which he controls the most elaborate calculations is sometimes little short of extraordinary.

As regards his methods, he does not seem to have cast about so as to choose one rather than another. As soon as he had thought of any method the possible effectiveness of which he could settle almost intuitively ("one's best things are done in five minutes," he once said to me, in confirmation of the satisfaction I was expressing at the fruitfulness of an idea that had occurred to me unexpectedly), the rest was the exercise of his powers. Among the methods he preferred, especially during the last twenty-five years of his life, was that of verification; in his hands it proved a weapon of great force. Indeed, only less remarkable than his algebraical skill, was the insight which enabled him to preserve the exact equivalence of all the equations in any particular process, so that he could have reversed each process merely by reversing the steps as they were made, and could have proceeded to the required theorem from the initial expression of an algebraical fact. Numerous instances of this quality in his work could be adduced; it will be sufficient to refer to some parts of his paper * "On the centro-surface of an ellipsoid."

But though Cayley was specially happy in the treatment of algebraical developments, an inadequate estimate of his genius would be obtained by supposing that he

* *C. M. P.* vol. VIII. No. 520: *Camb. Phil. Trans.* vol. XII. (1873), pp. 319—365.

was almost entirely an analyst. Much of his thinking, not a little of his writings, is completely geometrical; and his contributions to line geometry, his introduction of the Absolute into geometry, his continued recurrence to the methods in pure geometry invented by Poncelet and Chasles, should be sufficient to range him among geometricians.

Moreover, even in strictly analytical work, the synthetic element is often not far away though it does not always appear on the surface. In this connexion an acute suggestion, made by Salmon and perhaps based upon his remembrance of their mathematical correspondence that lasted through many years, is confirmed by one of the Notes Cayley himself added at the end of the second volume of his *Collected Mathematical Papers*. An enquiry sometimes begins by a comparatively easy problem which, when solved, leads to wider inferences; so that, ultimately in the development, considerable generalisations are effected. Now the usual writer, in publishing the results of such an enquiry, draws them up in a sequence that partly marks the order of their connected discovery: and, in doing so, he makes his work easier for his readers. But Cayley was not the usual writer. When he had reached his most advanced generalisations he proceeded to establish them directly by some method or other, though he seldom gave the clue by which they had first been obtained: a proceeding which does not tend to make his papers easy reading. An instance of the fact occurs* in his "Memoir on the Theory of Matrices," where he proves that a matrix satisfies an algebraical equation of his own order; he proves it by verification in simple cases, but he gives no clue as to his line of discovery. An instance of the method occurs in a note† added to one of his papers, where he says that the general equations

$$\{yd_x\} - yd_x = 0, \quad \{xd_y\} - xd_y = 0,$$

characteristic of covariants and invariants of binary quantics, were initially suggested by considering the relation of the quadratic $ax^2 + 2bxy + cy^2$ and its discriminant $ac - b^2$ to these equations. In the paper he drops linear transformation as connected with the covariantive property and defines a covariant as a function satisfying these two equations.

His literary style is direct, simple and clear. His legal training had an influence, not merely upon his mode of arrangement but also upon his expression; the result is that his papers are severe and present a curious contrast to the luxuriant enthusiasm which pervades so many of Sylvester's papers. He used to prepare his work for publication as soon as he had carried his investigations in any subject far enough for his immediate purpose. He found it an easy matter to do this part of his work, and thus differed widely in experience from those to whom the preparation of a paper is laborious even when the results to be incorporated have been obtained. As a matter of fact, he took the straightforward course of saying what he had to say in a clear and simple manner, fixing his mind upon the substance and never going out of his way in order to secure beautiful form for the presentation of results. Yet not infrequently his papers are so admirably written that they satisfy the exacting critics; thus it is perhaps not too much to affirm that his "Sixth Memoir on Quantics‡" could not be presented in more attractive form—a character due, however, to the tendency of

* *C. M. P.* vol. II. No. 152, pp. 482, 483; *Phil. Trans.* (1858), pp. 24, 25.

† *C. M. P.* vol. II. p. 600.

‡ *C. M. P.* vol. II. No. 158; *Phil. Trans.* (1859), pp. 61—90.

his method and to his results, but not acquired by any effort specially devoted to elaboration of clear expression. Again, a paper once written out was promptly sent for publication; this practice he maintained throughout his life. He undoubtedly formed projects for the immediate future; thus to the second edition* of his *Treatise on Elliptic Functions* he intended to add a couple of chapters, which, however, remained unwritten solely for the reason that all such projects were carried into effect only about the time when the need arose. The consequence is that he has left few arrears of unfinished or unpublished papers; his work has been given by himself to the world.

Only one other remark as to the form of his papers need be made. Readers must be struck with the number of exact references he makes to other writers. It was a practice about which he had very decided opinions: he wished not merely to make honourable acknowledgment of indebtedness but also to give indications of the history of the subject. In the latter particular he was always careful to insert in the reference the year in which the book or the paper had appeared; and he steadily urged others to insert dates in their references.

Cayley made additions to every important subject that lies within the range of pure mathematics. Their importance and their amount have varied in different subjects; thus on geometry his writings have a dominating influence: while on the general theory of functions, though he knew the subject well, he has left little mark, for he concerned himself chiefly with details such as the solution of more or less special problems in conformal representation. His papers in general have such value that he is the author most frequently quoted by the great body of current mathematicians. A full record of what he has done in pure mathematics could be made only by writing its history during the last half century; all that is attempted here consists of some brief indications of a selection among his more obviously important contributions to mathematical knowledge.

One of the subjects with which Cayley's name will probably be most closely associated is the theory of invariance. It is easy to cite simple cases of what is implied by an invariantive function: two will suffice.

It is known that, in solving an ordinary algebraical equation with literal coefficients, a certain functional combination of these coefficients (called the discriminant) must vanish in order that two roots of the equation may be equal; for example, the equation $ax^2+2bx+c=0$, has equal roots if (and only if) the quantity $ac-b^2$ vanishes. When the variable is transformed from x to y by a relation $(l'x+m')y=lx+m$, where l, m, l', m' are constants, then evidently two values of y, corresponding to the two equal values of x, are equal. When x is eliminated from the equation by means of the assumed relation, a new quadratic arises having y for its variable; let it be $a'y^2+2b'y+c'=0$, where a', b', c' depend upon a, b, c and l, m, l', m'. The two values of y determined by this equation are equal if (and only if) the quantity $a'c'-b'^2$ vanishes. But the equality of the two values of y depends upon and is determined by the equality of the two values of x, the latter equality being secured if the quantity $ac-b^2$ vanishes. It follows that the vanishing of either of the quantities $a'c'-b'^2$ and $ac-b^2$ requires

* It was published four months after his death; only the earlier sheets had the benefit of his revision.

the vanishing of the other; and it is therefore inferred that, when neither of them vanishes, one of them contains the other as a factor. When the actual calculation is made, it is found that $a'c'-b'^2$ is the product of $ac-b^2$ and $(lm'-l'm)^2$, the latter being a quantity that depends solely upon the transforming relation. Consequently it appears that a combination of the coefficients in the original equation exists, such that when the equation is transformed by any relation of the type indicated and exactly the same combination of the new coefficients is constructed, the two combinations are equal to one another save as to a factor depending solely upon the transforming relation. Such a combination of the coefficients is called an invariant.

Again, it is known that every curve (of degree higher than two) possesses a number of points where a tangent to the curve not merely touches it but, having contact of one degree closer, crosses it; and it is found that all these points, called points of inflexion, also lie upon another curve uniquely derived from the first. When the curves are represented by means of equations, the statement is that the points of inflexion of a curve $U=0$ are given as the intersections of this curve with a curve $H=0$, the latter equation being uniquely derived from $U=0$. Now suppose that the axes, to which the curves have been referred, are changed to another system, so that new co-ordinates x', y' are connected with the former co-ordinates by relations

$$\frac{x'}{a_1x+b_1y+c_1}=\frac{y'}{a_2x+b_2y+c_2}=\frac{1}{ax+by+c}.$$

A new equation $U'=0$, obtained by eliminating x and y between these relations and $U=0$, will now represent the curve. The change thus made does not affect the geometrical properties of the curve; its points of inflexion are still given as its intersections with the curve $H=0$. But the points of inflexion of the curve represented by $U'=0$ are the intersections of this curve with another curve represented by $H'=0$, an equation derived from $U'=0$ in exactly the same way as $H=0$ is derived from $U=0$. It therefore appears that the associated curve $H'=0$ cuts the given curve in precisely the same points as the associated curve $H=0$, a result which suggests that the associated curves are the same. Now $H'=0$ has been derived from $U'=0$; but actual calculation shows that, if the relations between x', y' and x, y be used to eliminate x, y from $H=0$, the resulting equation is $H'=0$; in other words, the relations between x', y' and x, y transform the equation $H=0$, derived from $U=0$, into the equation $H'=0$, derived in the same way from $U'=0$. Moreover, as in the case of the invariant, it is found that H', a specially constructed function of x', y' and the coefficients in U', is divisible by H, the same function of x, y and the coefficients in U; the quotient being a quantity depending again only upon the constants in the transforming relations. Consequently it appears that a combination of the coefficients and the variables in the original equation exists such that, when the equation is transformed by means of relations of the type indicated, and exactly the same combination of the new coefficients and the new variables is constructed, the two combinations are equal to one another save as to a factor dependent solely upon the transforming relations. Such a combination of the coefficients and the variables is called a covariant.

The first notice of such a property appears to have been made by Lagrange. And Gauss discussed the invariance of the discriminants of certain expressions when the latter are subjected to linear transformations. Again, Boole in 1841 had shown that this invariantive property belongs to all discriminants, and he gave a method of deducing some other functions of this kind. Boole's paper suggested to Cayley a much more general subject—the permanence of invariantive form—so that he set himself the question of finding "all the derivatives of any number of functions which have the property of preserving their form unaltered after any linear transformation of the variables." The first set of results obtained by his investigations related to invariants; they appeared in his famous paper,* "On the Theory of Linear Transformations," published half a century ago. The second set of results related to covariants; they appeared in the paper,† "On Linear Transformations," published in the succeeding year. In these two papers Cayley demonstrated the general existence of a number of functions, both invariants and covariants (at first he called them hyperdeterminants), which preserve their form under linear transformation.

These discoveries of Cayley establish him as the founder of what is called sometimes modern algebra, sometimes invariants and covariants, sometimes theory of forms; the origination of the theory is incontestably his, and it is universally ascribed to him.

A discovery of this general importance and complete novelty soon attracted the attention of other workers. It is not too much to say that the subsequent investigations long absorbed the active interest of many mathematicians, and, as a result, the theory has influenced all that domain of mathematical science which is in any way connected with algebraical form. Among the first to enter the field was Sylvester, then living in London; he and Cayley were in constant communication, alike oral and written, and carried on their work in the most friendly relations with one another. Boole also resumed his investigations, and both he and Salmon made substantial additions to the theory. The continental mathematicians also had begun their important contributions, chief among them being Aronhold, Hesse, and, at a later date, Hermite. Aronhold, indeed, devised the so-called symbolical method, now the favourite method with German workers; in its origin it is nearly the same as the symbolical method introduced by Cayley, but the subsequent developments—due largely also to Clebsch and to Gordan—run on lines entirely different from Cayley's.

After a time, Cayley began his series of ten memoirs on quantics; they must rank among the most wonderful combinations of original researches and papers upon a single theory ever produced. They contain a splendid exposition of the theory as already established; they are full of original contributions to the subject, and as they take account of the work done by other authors, they have the further interest of showing how the subject grew between the appearance of the "Introductory Memoir" in 1854 and the appearance of the "Tenth Memoir" in 1878. This is hardly the

* *C. M. P.* vol. I. No. 13; *Camb. Math. Jour.* vol. IV. (1845), pp. 193—209.

† *C. M. P.* vol. I. No. 14; *Camb. and Dubl. Math. Jour.* vol. I. (1846), pp. 104—122. The two papers were rewritten, and appeared in *Crelle*, vol. xxx. (1846), pp. 1—37, under the title "Mémoire sur les Hyperdéterminants."

opportunity to write a history of the subject by apportioning among the various investigators the sections which they respectively originated;* yet reference should be made to two matters.

First, one of the problems that greatly interested Cayley was the determination of the complete asyzygetic system of irreducible invariants and covariants appertaining to a binary form, that is, the system such that every invariant and every covariant of the form can be expressed as a rational integral algebraical function of the members of the system, the coefficients in the function being numerical only. In his "Second Memoir on Quantics"† he had accurately determined the number (and their degrees) of the asyzygetic invariants for binary forms of orders 2, 3, 4, 5, 6; he had also accurately inferred the number (together with their degrees and their orders) of the asyzygetic covariants for binary forms of orders 2, 3, 4, all these concomitants being subsequently tabulated. But, in regard to the invariants of forms of order higher than 6 and the covariants of forms of order higher than 4, he came to the erroneous conclusion that the respective numbers are infinite. The error was not corrected until Gordan in his memoir‡, dated 8th June, 1868, and entitled "Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist," showed that the complete system for a binary quantic of any order contains only a limited number of members. Cayley at once returned to the question, and having found a source of error (it was the neglected interdependence of certain syzygies, reducing the numbers of invariants and covariants; the interdependence had not previously been suspected), he dedicated his "Ninth Memoir on Quantics§" (dated 7th April, 1870), to the correction of the error and a further development of the theory in the light of Gordan's results. His promptness in recognising and giving immediate prominence to the work of the younger author possibly prevented some controversy among unwise partisans; it was characteristic of the man.

And, secondly, though his series of memoirs was brought to an end with the tenth, his interest in the subject did not cease, and he frequently wrote upon parts of it under other titles. In particular, Captain P. A. MacMahon's discovery of a relation of a new character between seminvariants and symmetric functions (viz., that the leading coefficients of the covariants of a binary quantic are the same as the non-unitary partition symmetric functions of the roots of an equation connected with a modified quantic) proved of the keenest satisfaction to him. From time to time he wrote in the *American Journal of Mathematics* upon this subject and upon symmetric functions generally in this connexion, always sympathetic and appreciative of the advances made by others, able to grasp and assimilate

* Some information will be found in an appendix to Salmon's *Lessons on Higher Algebra*; also in the notes and references at the end of the second volume (pp. 598—601) of the *Collected Mathematical Papers*. A valuable and exhaustive report, containing a full history of the subject, was drawn up by Prof. Dr. Franz Meyer, and published under the title "Bericht über den gegenwärtigen Stand der Invariantentheorie" (*Jahresber. d. Deutschen Mathem.-Vereinigung*, I. 1892).

† *C. M. P.* vol. II. No. 250; *Phil. Trans.* (1856), pp. 101—126.

‡ *Crelle*, vol. LXIX. (1869), pp. 323—354.

§ *C. M. P.* vol. VII. No. 462; *Phil. Trans.* (1871), pp. 17—50.

their ideas, but using them as a master and not as a follower. It was not alone, however, to symmetric functions, upon which he had written long and important memoirs as early as 1857, but to many other cognate subjects that he extended his researches upon invariants and covariants. The theory of equations of the fifth and higher degrees, Sturm's functions, Tschirnhausen's transformation, partition of numbers, Arbogast's method of derivation, skew determinants*—to quote no others—are titles and subjects of papers, all of which contain investigations of great value. The reason that they are less known (if such be the case) than his other work in the same line of ideas is perhaps due to the fact that the direct theory of invariants and covariants was rapidly brought within the range of students through Salmon's *Lessons on Higher Algebra*, dedicated by the author to Cayley and Sylvester.

Another subject, of which he must be regarded as the creator, is the theory of matrices. His first memoir† upon this theory, "wherein," to quote Sylvester,‡ "he may be said to have laid the foundation-stone of multiple quantity," was published in 1858. A couple of isolated results had been obtained by Hamilton in 1852 through the methods of quaternions; but they were unknown to Cayley at the time of his memoir, and, owing to the connexion in which they occur, they have an entirely detached aspect.

A matrix may initially be defined as a symbol of linear operation; thus, when the equations

$$X = ax + by + cz, \quad Y = a'x + b'y + c'z, \quad Z = a''z + b''y + c''z$$

are expressed in the form

$$(X, Y, Z) = \left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array} \right) (x, y, z) = M(x, y, z),$$

the symbol M is a matrix. Cayley was the first to discuss the theory of such symbols as subjects of functional operation and to dispense with the hitherto regular return at each stage to the equations of substitution in which the symbol first arises; in fact, he replaces the notion of substitutional operation by the notion of a new class of quantity.

Matrices (being of the same order or dimension) can be added like ordinary algebraical quantities; as regards multiplication, they are subject to the associative law, but not to the commutative law. Hence powers of a matrix (positive and negative, integral and fractional) can be obtained, and likewise algebraical functions of a matrix. It also follows that two general matrices are not convertible, that is, LM is not the same as ML save under special conditions; and it is a part of the theory to find the most general matrix convertible with a given matrix. The expression of this convertible

* His discoveries in this subject alone, have done much to simplify the analytical investigations connected with Pfaff's problem and the allied theory.

† *C. M. P.* vol. II. No. 152; *Phil. Trans.* (1858), pp. 17—37.

‡ *Amer. Jour. Math.* vol. VI. (1884), p. 271.

matrix can be deduced by means of the fundamental equation which every matrix satisfies, viz., an algebraical equation of its own order, the coefficient of the highest term being unity, and the last term being the determinant of the constants in the matrix. All these results were given by Cayley in his initial memoir; and, at the same time, they were applied by him to obtain the most general automorphic linear transformation of a bipartite quadric function, an extension of the problem which requires the most general (orthogonal) substitution transforming the function $x^2+y^2+z^2+\ldots\ldots$ into the function $x'^2+y'^2+z'^2+\ldots\ldots$

How fruitful the subject has proved may be inferred by noting the subsequent investigations of Sylvester, who has developed it on Cayley's lines, and has added to it many new ideas; of Tait, who developed the theory of quaternions on parallel lines: of the Peirces, father and son, whose researches on linear associate algebra* gave rise to the notion of matrices from a different source; of Clifford and Buchheim, who connected the theory with Grassmann's methods; of Laguerre, in whose memoir† the treatment of a "linear system" (the same as a Cayley matrix) is similar to Cayley's; and of many other writers, among whom Taber should be mentioned.

Connected with non-commutative algebraical quantities, Cayley's researches on the theory of groups require a passing notice. He devoted several papers to questions in this theory. Some of them relate to those groups of substitutions, the introduction of which by Galois made an epoch in the theory of equations, others of them relate to groups of homographic transformations, particularly those related to the polyhedral functions. But, so far as can be seen, he limited his published investigations to those groups which are finite and discontinuous.

Abstract geometry—the ideal geometry of n dimensions—is a subject that he may almost be said to have created; no other name than his has been associated with its origin. More than anything else, it marks the line of difference between the kinds of homage accorded to him. Experts regard it as an illustration of his imaginative power: the unlearned regard it as an incomprehensible mystery.

It finds a place among his earliest investigations,‡ it was steadily present to his mind, illuminating many of his researches; and occasionally it found explicit treatment, e.g., in his "Memoir on Abstract Geometry,"§ and in his Presidential Address at Southport. The theory presents itself in two connexions: one, as a need in analysis, the other as a generalisation of the ordinary geometries of two dimensions and of three dimensions.

The former origin can be indicated in a brief statement. When an occasion arises for dealing with a number of variables, connected in any manner and regarded as either variable or determinate (wholly or partially), the nature of the relations among them is frequently indicated, and often is made more easily intelligible, by associating some geometrical interpretation with the given system of relations. Thus

* *Amer. Jour. Math.* vol. IV. (1881), pp. 97—229.

† "Sur le Calcul des Systèmes Linéaires (*Journal de l'Éc. Poly.* vol. XXV. 1867, pp. 215—264).

‡ *C. M. P.* vol. I. No, 11; *Camb. Math. Jour.* vol. IV. (1845), pp. 119—127.

§ *C. M. P.* vol. VI. No. 413; *Phil. Trans.* (1870), pp. 51—63.

the momental ellipsoid is of great use in the discussion of moments of inertia, in representing the motion of a body round a fixed point when there are no impressed forces, and in other questions in dynamics. Again, two non-homogeneous (or three homogeneous) variables can be regarded as the co-ordinates of a point in a two-dimensional geometry, such as that of a plane or the surface of a sphere or any analytical surface; and any equation among the co-ordinates is then interpreted as representing a curve (or curves, or portion of a curve or curves) upon the surface. Similarly, when there are three non-homogeneous (or four homogeneous) variables, they can be regarded as the co-ordinates of a point in a three-dimensional geometry, such as that of ordinary space; corresponding to an equation among the variables, there is a surface (or surfaces) in space; corresponding to two independent equations among the variables, there is a curve (or curves) in space; and corresponding to three independent equations, there is a point (or points) in space. In such cases the analytical relations can often, with great advantage, be exhibited as geometrical properties. When the number of non-homogeneous co-ordinates is greater than three (or the number of homogeneous co-ordinates is greater than four), the circumstances have greater need of such a representation, while there is a greater difficulty in constructing some geometrical illustration; and then it can be obtained in a corresponding form only by the idea of a space of the proper number of dimensions. To secure the possibility of such a representation, it is necessary to evolve the geometry of multiple space.

For example, there are four single theta-functions, and their squares are connected by linear homogeneous relations. In order to obtain other properties of the functions themselves, it is convenient to regard them as homogeneous co-ordinates of a point in (ordinary) space; the amplitude in space that then is to be selected is the quadri-quadric tortuous curve represented by those linear relations, viz., the curve which is common to two quadric cylinders with intersecting axes. Similarly there are sixteen double theta-functions, with corresponding linear relations among their squares. The associated geometry is fifteen-dimensional; the manifoldness in this space to be selected for the discussion of the properties is the quadri-quadric two-dimensional amplitude common to thirteen quadric hyper-cylinders.

An initial difficulty in the construction of an analytical geometry of n dimensions is the expression of an amplitude of less than $n-1$ dimensions by means of equations that shall represent the complete amplitude, and nothing besides the amplitude. It occurs in ordinary solid geometry, the difficulty there being to obtain the expression of a tortuous curve in space by means of equations that represent it alone. For instance, a twisted cubic is frequently taken as the intersection of two quadrics having one common generator; but the equations of the quadrics taken together represent not the cubic curve alone but also the common generator. And the like for other cases.

Cayley's purpose in his "Memoir on Abstract Geometry," already referred to, was the exposition of some of the elementary principles of the subject. The paper is a remarkable instance of his power of presentation of abstract ideas, and of his clear precision of statement. Moreover, he makes it an explanatory paper; and, in view of the prevailing estimate of him as an analyst, it is worthy of notice that the paper does not contain a single equation, and contains only a few symbols. It is

unnecessary to summarise its contents; the furthest stage reached is the establishment of the notion that underlies the principle of duality in geometry.

But though the necessity for hyperdimensional geometry can thus be met so far as it arises in connexion with analysis, it is a different matter when the geometry is to be regarded as the generalisation of the geometries of two-dimensional space and of three-dimensional space. Cayley's reply to his own question as to the meaning to be attached to hyperdimensional space is* that

> "It may be at once admitted that we cannot conceive of a fourth dimension of space; that space as we conceive of it, and the physical space of our experience, are alike three-dimensional; but we can, I think, conceive of space as being two- or even one-dimensional; we can imagine rational beings living in a one-dimensional space (a line) or in a two-dimensional space (a surface), and conceiving of space accordingly, and to whom, therefore, a two-dimensional space, or (as the case may be) a three-dimensional space would be as inconceivable as a four-dimensional space is to us."

By not a few people the first clause in this passage has been neglected and the later clauses have not always been read rightly; and his further remark, "I need hardly say that the first step is the difficulty, and that granting a fourth dimension we may assume as many more dimensions as we please," has left some readers rather puzzled as to whether Cayley had not, after all, some mysterious incommunicable conception of a fourth dimension. His position is stated in the first clause of the former passage; his conclusion is that hypergeometry is, and is only, a branch of mathematics.

Before passing from the consideration of his larger contributions to hypergeometry, it is proper to mention his introduction of the six co-ordinates of a line. These are six quantities connected by a homogeneous equation $af+bg+ch=0$; and as only their ratios are used, they are thus equivalent to only four independent magnitudes, sufficient for the unique specification of a right line. They were first established, and primarily used by him, in connexion with his new analytical representation of curves in space;† and he often recurred to the subject, devoting in particular one paper‡ to the calculus of the six co-ordinates and to a discussion of Sylvester's involution of six lines. It should, however, be stated that these co-ordinates presented themselves independently to Plücker; the development of Plücker's theory as set forth in his memoir§ *On a New Geometry of Space*, and in his book‖ *Neue Geometrie des Raumes*, is entirely different from that obtained by Cayley, and it ought to be regarded as a separate creation. And it need hardly be remarked that while the introduction of a line, as an entity represented by a set of co-ordinates, leads to a new geometry of space, it is also clear that line-geometry can be regarded as a geometry of four dimensions.

* *Brit. Assoc. Report*, 1883, President's Address, p. 9.
† *C. M. P.* vol. IV. Nos. 284, 294.
‡ *C. M. P.* vol. VII. No. 435.
§ *Phil. Trans.* 1865, pp. 725—791.
‖ Leipzig, *Teubner*, 1868.

Another notion, entirely due to Cayley in its first form, is that of the Absolute; it was first introduced in his *Sixth Memoir on Quantics,** which was devoted chiefly to his investigations on the generalised theory of metrical geometry.

It is a known property that the angle between two lines AB, AC, when multiplied by $2\sqrt{-1}$, is equal to the logarithm of the cross-ratio of the pencil made up of the lines AB, AC and (conjugate imaginary) lines joining A to the circular points at infinity; and the measure of the angle between two lines can thus be replaced by the consideration of a projective property of an extended system of lines. Other examples of similar changes could easily be quoted. The purpose of Cayley's theory was to replace metrical properties of a figure or figures by projective properties of an extended system composed of a given figure or figures and of an added figure.

But it is not solely owing to the generalisation of distance that the memoir is famous. It has revolutionised the theory of the so-called non-Euclidian geometry; and it has important bearings on the logical and philosophical analysis of the axioms of space-intuition. The independence and the importance of the ideas, originated by Cayley in this memoir, have never been questioned; but, as is often (and naturally) the case with the discoverer of a fertile subject, Cayley himself did not explain or foresee the full range of application of his new ideas. He did not recognise, at the time when his memoir was first published, the beautiful identification of his generalised theory of metrical geometry with the non-Euclidian geometry of Lobatchewsky and Bolyai. This fundamental step was taken by Klein in his admirable memoir†, *Ueber die sogenannte Nicht-Euklidische Geometrie,* which contains a considerable simplification in statement of Cayley's original point of view, and contributes one of the most important results of the whole theory. The work of the two mathematicians now being an organic whole, there is no advantage—at least here—in attempting to subdivide the subject for the purpose of specifying the exact share of each in its construction.

The scope of the Cayley-Klein ideas may briefly be gathered from the following sketch. Let A_1 and A_2 be two points, often called a point-pair; they are to be either both real or, if not both real, then conjugate imaginaries so far as their coordinates are concerned. Let P, Q, R be three other points on the line $A_1 A_2$; and let the symbol (PQ) denote

$$2\gamma \log \frac{A_1P \,.\, A_2Q}{A_1Q \,.\, A_2P} \text{ or } 2i\gamma \log \frac{A_1P \,.\, A_2Q}{A_1Q \,.\, A_2P},$$

according as A_1 and A_2 are a real point-pair, or an imaginary point-pair. Then it is manifest that

$$(PQ) + (QR) = (PR),$$

so that the functions (PQ), (QR), (PR) satisfy the fundamental property of the distances between P and Q, Q and R, and P and R. Consequently (PQ) may be taken as a generalised conception of the distance between the points P and Q.

* *C. M. P.* vol. II. No. 158; *Phil. Trans.* (1859), pp. 61—90.

† *Math. Ann.* vol. IV. (1871), pp. 573—625.

Now let a conic be described in a plane, either imaginary, say, of the form $x^2+y^2+z^2=0$ or real, say, of the form $x^2+y^2-z^2=0$. Choosing the latter case, let attention be confined to points lying within the conic, so that every straight line through a point cuts the conic in a real point-pair. Take two points, P and Q; and let the line joining them cut the conic in two points, A_1 and A_2. Then (PQ), as defined above (the constant γ being the same for all such lines), is the generalised distance between P and Q. This conic, which has been arbitrarily assumed, and upon which the generalised conception of distance depends, is termed by Cayley the Absolute.

Cayley, however, avoided the unsatisfactory procedure of using one conception of distance to define a more general conception. As he himself explains more fully,* he regarded the co-ordinates of points as some quantities which define the relative properties of points, considered without any reference to the idea of distance but conceived as ordered elements of a manifold. Thus if α_1, β_1, γ_1 and α_2, β_2, γ_2 be the co-ordinates of the point-pair A_1 and A_2, the co-ordinates of the points P and Q on the line A_1A_2 can be taken as $\lambda_1\alpha_1+\lambda_2\alpha_2$, $\lambda_1\beta_1+\lambda_2\beta_2$, $\lambda_1\gamma_1+\lambda_2\gamma_2$ and $\mu_1\alpha_1+\mu_2\alpha_2$, $\mu_1\beta_1+\mu_2\beta_2$, $\mu_1\gamma_1+\mu_2\gamma_2$ respectively. The function (PQ) can then be defined as

$$2\gamma\log\frac{\lambda_2\mu_1}{\lambda_1\mu_2} \text{ or } 2i\gamma\log\frac{\lambda_2\mu_1}{\lambda_1\mu_2};$$

the generalised idea of distance thus finds its definition without any antecedent use of the conception in its ordinary form. Cayley's view is summed up in his sentence†:— ".....the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the absolute."

The metrical formulæ obtained when the absolute is real are identical with those of Lobatchewsky's and Bolyai's "hyperbolic" geometry: when the absolute is imaginary the formulæ are identical with those of Riemann's "elliptic" geometry; the limiting case between the two being that of ordinary Euclidian ("parabolic") geometry.

Cayley's memoir leads inevitably to the question, as to how far projective geometry can be defined in terms of space perception without the introduction of distance. This has been discussed by von Staudt‡ (in 1847, previous to Cayley's memoir), by Klein§ and by Lindemann‖. The memoir thus points to a division of our space intuitions into two distinct parts: one, the more fundamental as not involving the idea of distance, the other, the more artificial as adding the idea of distance to the former. The consideration of the relation of these ideas to the philosophical account of space has not yet been brought to its ultimate issue.

* See the note which he added, *C. M. P.* vol. II. p. 604, to the Sixth Memoir; it contains some interesting historical and critical remarks.

† *Loc. cit.* § 230.

‡ *Geometrie der Lage*; also in his later *Beiträge zur Geometrie der Lage*, 1857.

§ *Math. Ann.* vol. VI. (1873), pp. 112—145.

‖ *Vorlesungen über Geometrie* (Clebsch-Lindemann), vol. II. part I.; the third section is devoted to the subject.

It is in analytical geometry, both of curves and of surfaces, that the greatest variety of Cayley's contributions is to be found. There is hardly an important question in the whole range of either subject in the solution of which he has not had some share; and there are many properties our acquaintance with which is due chiefly, if not entirely, to him. How widely he has advanced the boundaries of knowledge in analytical geometry can be inferred even from the amount of his researches already incorporated in treatises such as those by Salmon, Clebsch and Frost; and yet they represent only a portion of what he has done. In these circumstances only a selection among his contributions can be indicated: it must be understood that, here as elsewhere, the statement does not pretend to be a complete account.

It is an old-established property that two curves of degrees m and n cut in mn points, but that it is not possible to draw a curve of degree n through any mn arbitrarily selected points on a curve of degree m. As early as 1843, Cayley extended the property and showed that when a curve of degree r higher than either m or n is to be drawn through the mn points common to the two curves, they do not count for mn conditions in its determination, but only for a number of conditions smaller than mn by $\frac{1}{2}(m+n-r-1)(m+n-r-2)$. A single addition was made to the theorem by Bacharach* in 1886—taking account of the case when the undetermining points lie on a curve of degree $m+n-r-3$; with this exception the algebraical problem was completely solved by Cayley in his original paper†. The result is often called Cayley's intersection-theorem.

Another geometrical research of fundamental importance was embodied by him in a memoir‡, "On the higher singularities of a plane curve," published in 1866: it is there proved that any singularity whatever on a plane algebraical curve can be reckoned as equivalent to a definite number of the simple singularities constituted by the node, the ordinary cusp, the double tangent and the ordinary inflexional tangent. The theory has, since that date, been developed on lines different from Cayley's—owing to its importance in other theories, such as Abelian functions, variety in its development has proved both necessary and useful; but it was Cayley's investigations in continuation of Plücker's theory that have cleared the path for the later work of others.

The classification of cubic curves had been effected by Newton in his tract "Enumeratio linearum tertii ordinis," published in 1704: and six species had been added by Stirling and Cramer, the total then being 78. Plücker effected a new classification in his "System der analytischen Geometrie," published in 1835: his total number of species is 219, the division into species being more detailed than Newton's. Cayley re-examined the subject in his memoir§, "On the classification of cubic curves," expounding the principles of the two classifications and bringing them into comparison with one another; and entering into the discussion with full minuteness, he obtains the exact relation of the two classifications to one another—a result of great value in the theory.

* *Math. Ann.* vol. XXVI. (1886), pp. 275—299.

† *C. M. P.* vol. I. No. 5; *Camb. Math. Jour.* vol. III. (1843), pp. 211—213.

‡ *C. M. P.* vol. V. No. 374; *Quart. Math. Jour.* vol. VII. (1866), pp. 212—223.

§ *C. M. P.* vol. V. No. 350; *Camb. Phil. Trans.* vol. XI. (1864), pp. 81—128.

To the theories of rational transformation and correspondence he made considerable additions. Two figures are said to be rationally transformable into one another when to a variable point of one of them corresponds reciprocally one (and only one) variable point of the other. The figure may be a space or it may be a locus in a space. Rational transformations between two spaces give rational transformations between loci in those spaces; but it is not in general true that rational transformations between two loci necessarily give rational transformations between the spaces in which those loci exist. There is thus a distinction between the theory of transformation of spaces and the theory of correspondence of loci. Both theories have occupied many investigators, the latter in particular; and Cayley's work may fairly be claimed to have added much to the knowledge of the theory as due* to Riemann, Cremona and others.

Further, there may be singled out for special mention, his investigations on the bitangents of plane curves, and, in particular, on the 28 bitangents of a non-singular quartic; his developments of Plücker's conception of foci; his discussion of the osculating conics of curves, and of the sextactic points on a plane curve (these are the places where a conic can be drawn through six consecutive points); his contributions to the geometrical theory of the invariants and covariants of plane curves; and his memoirs on systems of curves subjected to specified conditions. Moreover, he was fond of making models and of constructing apparatus intended for the mechanical description of curves. The latter finds record in various of his papers; even so lately as 1893 he exhibited, at a meeting of the Cambridge Philosophical Society, a curve-tracing mechanism connected with three-bar motion.

All the preceding results belong to plane geometry; no less important or less numerous were the results he contributed to solid geometry. The twenty-seven lines that lie upon a cubic surface were first announced in his memoir†, "On the triple tangent planes of surfaces of the third order," published in 1849, after a correspondence between Salmon and himself. Cayley devised a new method for the analytical expression of curves in space by introducing into the representation the cone passing through the curve and having its vertex at an arbitrary point. Again, by using Plücker's equations that connect the ordinary (simple) singularities of plane curves, he deduced equations connecting the ordinary (simple) singularities of the developable surface that is generated by the osculating plane of a given tortuous curve, and, therefore, also of any developable surface. He greatly extended Salmon's theory of reciprocal surfaces; and resuming a subject already discussed by Schläfli he produced‡ in 1869 his "Memoir on cubic surfaces," in which he dealt with their complete classification. Many of his memoirs are devoted to the theory of skew ruled surfaces, or scrolls as he called them. Our knowledge of geodesics, of orthogonal systems of surfaces, of the centro-surface of an ellipsoid, of the wave-surface, of the 16-nodal quartic surface, not to mention more,

* In this connexion a report by Brill and Noether, "Bericht über die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit" (*Jahresber. d. Deutschen Mathem.-Vereinigung*, vol. III. 1894) will be found—particularly the sixth and the tenth sections—to give a very valuable *résumé* of the theory and its history.

† *C. M. P.* vol. I. No. 76; *Camb. and Dubl. Math. Jour.* vol. IV. (1849), pp. 118—132. See also Salmon's *Solid Geometry* (third edition, 1874), p. 464, note.

‡ *C. M. P.* vol. VI. p. 412; *Phil. Trans.* (1869), pp. 231—326.

is due in part to the extensions he achieved. It is difficult to indicate parts of the general theory of surfaces and of twisted curves that do not owe at least something and frequently much to his labours; a mere reference to the index of a book like Salmon's *Solid Geometry* will show how vast has been his influence.

One group of subjects interested him throughout his life, the theory of periodic functions, in particular, of elliptic functions: it was to the latter that his only book was devoted. But in a subject, the main lines of which were established so definitely before he began to write*, it is impossible, without entering into great detail, to mark out the contributions that are directly due to him. When a theory is in such a stage as was that of elliptic functions about 1842, the work of one writer sometimes helps to fill the gaps left by that of another, sometimes develops another writer's results from a different point of view; the composite theory depends, in part, upon the coordination of complementary results.

Abel's famous paper†, "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendantes," presented to the French Academy of Sciences in 1826, and unfortunately delayed in publication‡ for nearly fifteen years, attracted Cayley's attention quite early in his scientific career. In 1845 Cayley published his "Mémoire sur les fonctions doublement périodiques,"§ in which he considered Abel's doubly-infinite products of the form

$$u(x) = x\Pi\Pi\left(1 + \frac{x}{w}\right),$$

where $w = (m, n) = m\Omega + n\Upsilon$, the ratio $\Omega : \Upsilon$ is not real, and the product is taken for all positive and all negative integer values of m and of n between positive and negative infinity, except simultaneous zero values. He showed that such products can be used to obtain Jacobi's elliptic functions by constructing fractions such as

$$u(x + \tfrac{1}{2}\Omega) \div u(x);$$

and he also showed that the actual value of any product involves an exponential factor $e^{\frac{1}{2}Bx^2}$, where the value of the constant B depends upon the relation‖ between the infinities of m and of n. The results were of definite importance at the time of their discovery, and they still hold their place. But the form of the doubly-infinite product has been modified¶ by Weierstrass, who takes

$$\sigma(x) = x\Pi\Pi\left\{\left(1 + \frac{x}{w}\right) e^{-\frac{x}{w} + \frac{1}{2}\frac{x^2}{w^2}}\right\},$$

* The history will be found in Casorati, *Teorica delle funzioni di variabili complesse*, 1868, and in Enneper, *Elliptische Functionen, Theorie und Geschichte*, second edition, 1890, where other references are given.

† *Œuvres complètes d'Abel* (Christiania, 1881), vol. I. pp. 145—211.

‡ The circumstances are recited in § 9 of the appendix to the volume, by Bjerknes, *Niels Henrik Abel, Tableau de sa vie et de son action scientifique* (Gauthier-Villars, Paris, 1885).

§ *C. M. P.* vol. I. No. 25; *Liouville*, vol. x. (1845), pp. 385—420.

‖ This is sometimes expressed differently, as follows. Points are taken having m and n for their Cartesian co-ordinates; those which occur for infinite values of m and of n lie at infinity, and may be considered to lie upon a curve altogether at infinity, the shape of which is determined by the relation between the infinities of m and of n.

The value of the constant B is said to depend upon the shape of this bounding curve.

¶ Weierstrass's investigations on infinite products are contained in his memoir "Zur Theorie der eindeutigen analytischen Functionen" (*Abh. d. K. Akad. d. Wiss. zu Berlin*, 1876); also in his book *Abhandlungen aus der Functionenlehre*, 1886.

a function the value of which is independent of any particular form of relation between the infinities of m and of n. Owing to the latter simplification, Cayley's results are, as he himself remarked*, partly superseded by those of Weierstrass.

Cayley had great admiration for the works of both Abel and Jacobi; he had begun to read the latter's *Fundamenta Nova* immediately after his degree. The prominent position occupied in that work by the theory of transformation naturally attracted his interest; and, even as early as 1844 and 1846, he wrote short memoirs upon the subject, obtaining in one of them a function, due to Abel and now known as the octahedral function. Further memoirs of a similar tenor appeared occasionally; they deal chiefly with transformation as concerned with the known differential relation of the form

$$\{(1-x^2)(1-k^2x^2)\}^{-\frac{1}{2}}\,dx = M\{(1-y^2)(1-\lambda^2y^2)\}^{-\frac{1}{2}}\,dy.$$

The contributions made to the transformation theory by Sohnke, Joubert, and Hermite, as well as Jacobi's original investigations, all depend upon the use of transcendental functions of the quantity $q\left(=e^{-\pi\frac{K'}{K}}\right)$: yet the results are such that they ought to be deducible by ordinary algebraical processes. It was Cayley's wish to deal with this theory by pure algebra; two simple cases had already thus been discussed by Jacobi, but the extension to the less simple cases proved difficult. Cayley's "Memoir on the transformation of elliptic functions†," carries on the algebraical theory and places it in a clearer light than before. But though he made a distinct advance in dealing with particular cases, he still found it necessary to use the q-transcendents for making any definite advance in the general case. And the same compulsion occurs in the chapters of his *Treatise on Elliptic Functions*, where transformation is discussed at considerable length.

He resumed his investigations in 1886, still dealing with the algebraical method, but applying it to a simplified form of elliptic integral due to Brioschi. Though the problem is not solved‡ completely for the general case, he has devised a method which is effective at least in part; it easily leads to new results connected with the modular equations in the known simpler cases previously solved.

The theta-functions are the subject of several of his papers. He began§ with a direct establishment of Jacobi's relation

$$\sqrt{k}\,snu = \mathrm{H}(u) \div \Theta(u),$$

obtained in the *Fundamenta Nova* by a long and cumbrous process; and he proceeded to the construction of the linear differential equations satisfied by the theta-functions. Except, however, in so far as they arise in the transformation theory, they do not appear to have occupied him until about 1877. In that year and in the succeeding

* *C. M. P.* vol. I. p. 586.

† *C. M. P.* vol. IX. No. 577; *Phil. Trans.* 1874, pp. 397—456.

‡ The memoirs of this period belonging to the transformation of elliptic functions were published in the *American Journal of Mathematics*, vol. IX. (1887), pp. 193—224; vol. X. (1888), pp. 71—93.

§ "On the Theory of Elliptic Functions," *C. M. P.* vol. I. No. 45; *Camb. and Dubl. Math. Jour.* vol. II. (1847), pp. 256—266.

years he wrote a number of papers dealing with the theta-functions as on an independent basis and not as a detail in elliptic functions. Though the investigations are concerned with p-tuple functions, yet, partly for simplicity, and partly in order to secure the greater detailed development of the theory, the papers deal chiefly with the cases $p=1$, $p=2$.

Previous to Cayley's investigations, the most valuable algebraical results in this subject were those of Rosenhain* and Göpel† which had connected the double theta-functions with the theory of the Abelian functions of two variables, and those of Weierstrass, developed by Königsberger‡ to give the "addition-theorem." Proceeding in his "Memoir on the single and double theta-functions"§ more by Göpel's method than by Rosenhain's, Cayley resumes the whole theory. He pays special attention to the relations among the squares of the functions and to the derivation of the biquadratic relation among four of the functions, which is the same as the equation of Kummer's sixteen-nodal quartic surface. To this relation and to the geometry of this associated surface he frequently recurred, both specifically in isolated papers and generally in researches upon quartic surfaces.

As connected, in part, with elliptic functions, his investigations on the porism of the in- and circumscribed polygon should be mentioned. The porismatic property of two conics, viz. that they may be related to each other so that one polygon (and, if one polygon, then an infinite number of polygons) can be inscribed in one and circumscribed about the other, is due to the geometrician Poncelet. The special case when the conics are two circles had been discussed analytically by Jacobi‖, using elliptic functions for the purpose. Cayley undertook, first in 1853, the analytical discussion of the most general case of two conics, also using elliptic functions; and he obtained¶ the relations, necessary for the porism, for the several polygons as far as the enneagon. And it may be remarked, as a characteristic instance of Cayley's habit of proceeding to general cases, that he did not leave the matter at this stage. In a memoir** "On the problem of the in- and circumscribed triangle" he raises the question as to the number of polygons which are such that their angular points lie on a given curve or given curves of any order and their sides touch another given curve or given curves of any class. Using the theory of correspondence, he solves the question completely in the case of a triangle—taking account of the fifty-two cases that arise through the possibility of two curves, or more than two curves, being one and the same curve.

From time to time Cayley turned his attention to questions in theoretical dynamics, choosing them as subjects of his lectures during his earlier years as professor. Among them may be mentioned his investigations on attractions, specially those on the attraction of ellipsoids, to which he devotes five memoirs††, discussing the methods of Legendre,

* *Mém. des Sav. Étr.* vol. XI. (1851), pp. 361—468; the paper is dated 1846.

† *Crelle*, vol. XXXV. (1847), pp. 277—312.

‡ *Crelle*, vol. LXIV. (1865), pp. 17—42.

§ *Phil. Trans.* 1880, pp. 897—1002.

‖ *Ges. Werke*, vol. I. pp. 277—293; this paper was published first in *Crelle*, vol. III. (1828), pp. 376—389.

¶ In a set of five papers, *C. M. P.* vol. II. Nos. 113, 115, 116, 128; *ibid.*, vol. IV. No. 267.

** *C. M. P.* vol. VIII. No. 514; *Phil. Trans.* (1871), pp. 369—412.

†† *C. M. P.* vol. I. Nos. 75, 89; vol. II. Nos. 164, 173, 193.

Jacobi, Gauss, Laplace, and Rodrigues; and his evaluations or reductions of multiple definite integrals connected with attractions and potentials in general, particularly his "Memoir on Prepotentials*," in which he discusses the reduction of the most general integral of the type that can occur in dealing with the potential-problem related to hyperspace. He also frequently recurred at intervals, before drawing up his reports about to be quoted, to the consideration of the motion of rotation of a solid body about a fixed point under no forces. By introducing Rodrigues's co-ordinates into the equations of motion he was able to reduce the solution of the problem to quadratures; but the final solution of this case, in the most elegant form, is due to Jacobi himself; it involves single theta-functions. It may be remarked that the next substantial advance made in the theory of motion of a body under the action of forces is due to the late Madame Sophie Kovalewsky, who, in a memoir†, to which the Bordin Prize of 1888 was awarded by the Paris Academy of Sciences, has shown that the motion can, in a particular case, be determined in terms of double theta-functions when the body rotating round a fixed point is subject to the force of gravity.

Sometimes, after reading widely upon a subject, Cayley would draw up a report recounting the chief researches in it made by the great writers. It occasionally happens in the development of a theory that periods come when the incorporation and the marshalling of created ideas seem almost necessary preliminaries to further progress. Cayley was admirably fitted for work of this kind, owing not only to his faculty of clear and concise exposition, but also to his wide and accurate knowledge. Among such reports, two are of particular importance; his "Report on the recent progress of theoretical dynamics‡" and his "Report on the progress of the solution of certain special problems of dynamics§" have proved of signal service to other writers and to students. His knowledge and his power of summarising are shown also in some interesting articles on mathematical topics, written by him for the *Encyclopædia Britannica*.

Cayley also had a great enthusiasm for some of the branches of physical astronomy. Some idea of the value and importance of his labours in this subject, particularly in connexion with the development of the disturbing function in both the lunar theory and the planetary theory, and with the general developments of the functions that arise in elliptic motion, may be gathered by consulting the series of memoirs‖ which he communicated to the Royal Astronomical Society.

Special reference should be made to one of Cayley's astronomical papers. In 1853 Adams had made a new investigation of the value of the secular acceleration of the moon's mean motion, and, taking account of the variation in the eccentricity of the earth's orbit, had obtained a value which differed from that given by Laplace. Unfortunately, Adams's result was disputed by some of the great school of French physical

* *Phil. Trans.* 1875, pp. 675—774.

† *Mém. des Sav. Étr.*, vol. XXXI. (1894), No. 1.

‡ *C. M. P.* vol. III. No. 195; *Brit. Assoc. Report* (1857), pp. 1—42.

§ *C. M. P.* vol. IV. No. 298; *Brit. Assoc. Report* (1862), pp. 184—252.

‖ They are included, with very few exceptions, in the third and the seventh volumes of the *Collected Mathematical Papers*.

astronomers, notably by Pontécoulant, and, in consequence, some hesitation about acceptance was felt by some English astronomers, perhaps not unnaturally in view of the severe criticisms expressed. Cayley made an independent investigation of the necessary approximations, and devised a new method for introducing the variation of the eccentricity in question—a method effective perhaps chiefly owing to the instinct and power with which he carried out the laborious analysis required. The memoir, in which he embodied his results and which was entitled "On the secular acceleration of the moon's mean motion*," completely confirmed the value obtained by Adams, and was of substantial help in settling the controversy.

And, in the last place, the preceding sketch of Cayley's contributions to mathematical science seems to refer, for the most part, only to long memoirs. Yet it must not therefore be supposed that his shorter papers (which are very numerous) can safely be neglected. Sometimes he wrote a simple note not so much to convey new results as to set out his view of some particular theorem; these notes were always fresh and often suggestive. He was specially gratified when he had obtained a brief solution of some question, and his quite short papers frequently contain most important results. For instance, in the brief paper†, "On the theory of the singular solutions of differential equations of the first order," he was the first to give a clear exposition of the theory which in Boole's book had been left in an imperfect state. He there obtained the broad essential results of the theory, and it is particularly on his work, and on the work of Darboux published very soon after Cayley's, that ulterior researches are based.

What has been said may be sufficient to point out Cayley's place among the mathematicians of his time, and to indicate the services he rendered to the science which he loved so well. But he was more than a mathematician. With a singleness of aim, which Wordsworth could have chosen for his "Happy Warrior," he persevered to the last in his nobly lived ideal. His life had a significant influence on those who knew him: they admired his character as much as they respected his genius: and they felt that, at his death, a great man had passed from the world.

A. R. F.

1 *June*, 1895.

* *C. M. P.* vol. III. No. 221; *Monthly Not. R. A. S.* vol. XXII. (1862), pp. 171—231.
† *C. M. P.* vol. VIII. No. 545; *Messenger of Math.* vol. II. (1873), pp. 6—12.

COURSES OF LECTURES DELIVERED BY PROFESSOR CAYLEY.

(M. denotes Michaelmas Term; L. denotes Lent Term.)

1863. M. Analytical Geometry.
1864. M. Analytical Geometry.
1865. M. Analytical Geometry and Mechanics.
1866. M. Dynamics.
1867. M. Miscellaneous Analysis.
1868. M. Dynamics and Differential Equations.
1869. M. Analytical Geometry.
1870. M. Theories of correspondence and transformation in analytical geometry.
1871. M. Graphical Geometry.
1872. M. Elliptic Functions.
1873. M. Theory of Equations and Miscellaneous Analysis.
1874. M. Integral Calculus.
1875. M. On a course of pure mathematics.
1876. M. Differential Equations.
1877. M. Algebra.
1878. M. Solid Geometry.
1879. M. Differential Equations.
1880. M. Theory of Equations.
1881. M. Abel's Theorem and the Theta Functions.
1882. M. Abelian and Theta Functions.
1883. M. Higher Algebra and the Theory of Numbers.
1884. M. Some recent developments in Analysis and Geometry.
1885. M. Higher Algebra.
1886. M. Differential Equations and Analytical Geometry.

1887.	L.	Differential Equations.
	M.	Quaternions and other non-commutative algebras.
1888.	L.	Analytical Geometry.
	M.	Elliptic Functions.
1889.	L.	Analytical Geometry.
	M.	Solid Geometry.
1890.	L.	Theory of Equations.
	M.	Elliptic Functions, in particular Transformation and the modular equations.
1891.	L.	Analytical Geometry.
	M.	Higher Algebra and Analytical Geometry.
1892.	L.	Higher Algebra and Analytical Geometry.
	M.	On a course of pure mathematics.
1893.	L.	On a course of pure mathematics.
	M.	Analytical Geometry.
1894.	L.	The known transcendental functions.
	M.	Analytical Geometry.
1895.	L.	Theory of Equations (announced).

CONTENTS.

[An Asterisk means that the paper is not printed in full.]

PAGE

486. *Note on Dr. Glaisher's paper on a theorem in definite integration* 1
Quart. Math. Jour. t. x. (1870), pp. 355, 356

487. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 2
Quart. Math. Jour. t. xi. (1871), pp. 15—25

488. *Note on a relation between two circles* 12
Quart. Math. Jour. t. xi. (1871), pp. 82, 83

489. *On the porism of the in-and-circumscribed polygon, and the* (2, 2) *correspondence of points on a conic* 14
Quart. Math. Jour. t. xi. (1871), pp. 83—91

490. *On a problem of elimination* 22
Quart. Math. Jour. t. xi. (1871), pp. 99—101

491. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 25
Quart. Math. Jour. t. xi. (1871), pp. 111—113

492. *Note on a system of algebraical equations* 29
Quart. Math. Jour. t. xi. (1871), pp. 132, 133

493. *On evolutes and parallel curves* 31
Quart. Math. Jour. t. xi. (1871), pp. 183—200

494. *Example of a special discriminant* 46
Quart. Math. Jour. t. xi. (1871), pp. 211—213

495. *On the envelope of a certain quadric surface* 48
Quart. Math. Jour. t. xi. (1871), pp. 244—246

PAGE

496. *Tables of the binary cubic forms for the negative determinants* $\equiv 0 \pmod{4}$ *from* -4 *to* -400; *and* $\equiv 1 \pmod{4}$ *from* -3 *to* -99; *and for five irregular negative determinants* . 51
Quart. Math. Jour. t. XI. (1871), pp. 246—261

497. *Note on the calculus of logic* 65
Quart. Math. Jour. t. XI. (1871), pp. 282, 283

498. *On the inversion of a quadric surface* 67
Quart. Math. Jour. t. XI. (1871), pp. 283—288

499. *On the theory of the curve and torse* 72
Quart. Math. Jour. t. XI. (1871), pp. 294—317

500. *On a theorem relating to eight points on a conic* . . . 92
Quart. Math. Jour. t. XI. (1871), pp. 344—346

501. *Review. Pineto's tables of logarithms* 95
Quart. Math. Jour. t. XI. (1871), pp. 375, 376

502. *On the surfaces divisible into squares by their curves of curvature* 97
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 8, 9

503. *On the surfaces each the locus of the vertex of a cone which passes through m given points and touches* $6-m$ *given lines* 99
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 11—47

504. *On the mechanical description of certain sextic curves* . . 138
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 105—111

505. *On the surfaces divisible into squares by their curves of curvature* 145
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 120, 121

506. *On the mechanical description of a cubic curve* . . . 147
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 175—178

507. *On the mechanical description of certain quartic curves by a modified oval chuck* 151
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 186—190

508. *On geodesic lines, in particular those of a quadric surface* . 156
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 191—211

PAGE

509. *Plan of a curve-tracing apparatus* 179
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 345—347

510. *On bicursal curves* 181
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 347—352

511. *Addition to the memoir on geodesic lines, in particular those of a quadric surface* 188
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 368—380

512. *On a correspondence of points in relation to two tetrahedra* . 200
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 396—404

513. *On a bicyclic chuck* 209
Phil. Mag. t. XLIII. (1872), pp. 365—367

514. *On the problem of the in-and-circumscribed triangle* . . 212
Phil. Trans. t. CLXI. (for 1871), pp. 369—412

515. *Sur les courbes aplaties* 258
Comptes Rendus, t. LXXIV. (1872), pp. 708—712

516. *Sur une surface quartique aplatie* 262
Comptes Rendus, t. LXXIV. (1872), pp. 1393—1395

517. *Sur les surfaces divisibles en carrés par leurs courbes de courbure et sur la théorie de Dupin* 264
Comptes Rendus, t. LXXIV. (1872), pp. 1445—1449

518. *Sur la condition pour qu'une famille de surfaces données puisse faire partie d'un système orthogonal* 269
Comptes Rendus, t. LXXV. (1872), pp. 177—185, 246—250, 324—330, 381—385, 1800—1803

519. *On curvature and orthogonal surfaces* 292
Phil. Trans. t. CLXIII. (for 1873), pp. 229—251

520. *On the centro-surface of an ellipsoid* 316
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 319—365

521. *On Dr. Wiener's model of a cubic surface with 27 real lines; and on the construction of a double-sixer* 366
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 366—383

PAGE

522. *Note on the theory of invariants* 385
Math. Ann. t. III. (1871), pp. 268—271

523. *On the transformation of unicursal surfaces* 388
Math. Ann. t. III. (1871), pp. 469—474

524. *On the deficiency of certain surfaces* 394
Math. Ann. t. III. (1871), pp. 526—529

525. *An example of the higher transformation of a binary form* . 398
Math. Ann. t. IV. (1871), pp. 359—361

526. *On a surface of the eighth order* 401
Math. Ann. t. IV. (1871), pp. 558—560

527. *On a theorem in covariants* 404
Math. Ann. t. V. (1872), pp. 625—629

528. *On the non-Euclidian geometry* 409
Math. Ann. t. V. (1872), pp. 630—634

529. *A "Smith's Prize" paper* [1868]; *solutions by Prof. Cayley* . 414
Oxford, Camb. and Dubl. Messenger of Mathematics, t. IV. (1868), pp. 201—226

530. *Solution of a Senate-House problem* 436
Oxford, Camb. and Dubl. Messenger of Mathematics, t. V. (1871), pp. 24—27

531. *A "Smith's Prize" paper* [1869]; *solutions by Prof. Cayley* . 439
Oxford, Camb. and Dubl. Messenger of Mathematics, t. V. (1871), pp. 41—64

532. *Note on the integration of certain differential equations by series* 458
Oxford, Camb. and Dubl. Messenger of Mathematics, t. V. (1871), pp. 77—82

533. *On the binomial theorem, factorials, and derivations* . . 463
Oxford, Camb. and Dubl. Messenger of Mathematics, t. V. (1871), pp. 102—114

534. *A "Smith's Prize" paper* [1870]; *solutions by Prof. Cayley*. 474
Oxford, Camb. and Dubl. Messenger of Mathematics, t. V. (1871), pp. 182—203

PAGE

535. *Note on the problem of envelopes* 491
Messenger of Mathematics, t. I. (1872), pp. 3, 4

536. *Note on Lagrange's demonstration of Taylor's theorem* . . 493
Messenger of Mathematics, t. I. (1872), pp. 22—24

537. *Solutions of a Smith's Prize paper for* 1871 496
Messenger of Mathematics, t. I. (1872), pp. 37—47, 71—77, 89—95

538. *Extract from a letter from Prof. Cayley to Mr. C. W. Merrifield* 517
Messenger of Mathematics, t. I. (1872), pp. 87, 88

*539. *Further note on Lagrange's demonstration of Taylor's theorem* 519
Messenger of Mathematics, t. I. (1872), pp. 105, 106

540. *On a property of the torse circumscribed about two quadric surfaces* 520
Messenger of Mathematics, t. I. (1872), pp. 111, 112

541. *On the reciprocal of a certain equation of a conic* . . . 522
Messenger of Mathematics, t. I. (1872), pp. 120, 121

*542. *Further note on Taylor's theorem* 524
Messenger of Mathematics, t. I. (1872), p. 137

543. *On an identity in spherical trigonometry*. 525
Messenger of Mathematics, t. I. (1872), p. 145

544. *On a penultimate quartic curve* 526
Messenger of Mathematics, t. I. (1872), pp. 178—180

545. *On the theory of the singular solutions of differential equations of the first order* 529
Messenger of Mathematics, t. II. (1873), pp. 6—12

546. *Theorems in relation to certain sign-symbols* 535
Messenger of Mathematics, t. II. (1873), pp. 17—20

547. *On the representation of a spherical or other surface on a plane: a Smith's Prize dissertation* 538
Messenger of Mathematics, t. II. (1873), pp. 36, 37

PAGE

548. *On Listing's theorem.* 540

Messenger of Mathematics, t. II. (1873), pp. 81—89

549. *Note on the maxima of certain factorial functions* . . . 548

Messenger of Mathematics, t. II. (1873), pp. 129, 130

550. *Problem and hypothetical theorems in regard to two quadric surfaces* 550

Messenger of Mathematics, t. II. (1873), p. 137

551. *Two Smith's Prize dissertations* [1872] 551

Messenger of Mathematics, t. II. (1873), pp. 145—149

552. *On a differential formula connected with the theory of confocal conics* 556

Messenger of Mathematics, t. II. (1873), pp. 157, 158

553. *Two Smith's Prize dissertations* [1873] 558

Messenger of Mathematics, t. II. (1873), pp. 161—166

554. *An elliptic-transcendent identity* 564

Messenger of Mathematics, t. II. (1873), p. 179

555. *Notices of Communications to the British Association for the Advancement of Science* 565

Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1870, 1871, 1873).

Notes and References 569
Prefatory Note vii
Arthur Cayley: biographical notice by the Editor . . . ix
List of courses of lectures delivered by Professor Cayley . . xlv
Facsimile of the manuscript of his note on p. 569 . . *Frontispiece*

CLASSIFICATION.

GEOMETRY :

Conics, 500, 541, 552

Mechanical description of curves, 504, 506, 507, 509, 513

Bicursal curves, 510

Penultimate forms of curves, 515, 516, 544

In-and-circumscribed figures, 489, 514

Evolutes and parallel curves, 493

Quadric surfaces, 495, 498, 550

Centro-surface of ellipsoid, 520

Cubic surfaces, 521

Quartic surfaces, 487, 491

Envelopes, 535

Curve and torse, 499, 540

Curves of curvature on surfaces, 502, 505, 517

Geodesics on quadrics, 508, 511

Correspondence, transformation, and deficiency of surfaces, 512, 523, 524

Cones satisfying six conditions, 503

Orthogonal surfaces, 518, 519

Surface of eighth order, 526

Non-Euclidian geometry, 528

Listing's theorem, 548

Miscellaneous geometry, 488, 555

ANALYSIS:

Pineto's table of logarithms, 501

Tables of binary cubic forms, 496

Calculus of logic, 497

Algebraical equations, 492

Elimination, 490

Discriminant, 494

Invariants and covariants, 522, 525, 527

Definite integrals, 486

Differential equations, 532, 545

Miscellaneous analysis, 533, 536, 538, 539, 542, 543, 546, 549, 554, 555

Smith's-Prize papers and solutions, 529, 530, 531, 534, 537, 547, 551, 553

486.

NOTE ON DR GLAISHER'S PAPER ON A THEOREM IN DEFINITE INTEGRATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 355, 356.]

It is worth noticing how easily the case when $\phi = 1$ may be proved independently of the general formula with Θ; for (1) the equation

$$v = ax - \frac{a_1}{x-\lambda_1} - \frac{a_2}{x-\lambda_2} \dots - \frac{a_n}{x-\lambda_n}$$

is

$$(ax - v)(x-\lambda_1)(x-\lambda_2)\dots - a_1(x-\lambda_2)\dots - \dots = 0,$$

and has $n+1$ roots, say $x_1, x_2 \dots x_{n+1}$ where

$$x_1 + x_2 \dots + x_{n+1} = \lambda_1 + \lambda_2 \dots + \lambda_n + \frac{v}{a},$$

and (2) the equation

$$v = -\frac{a_1}{x-\lambda_1} - \frac{a_2}{x-\lambda_2} \dots - \frac{a_n}{x-\lambda_n}$$

is

$$-v(x-\lambda_1)(x-\lambda_2)\dots - a_1(x-\lambda_2)\dots - \dots = 0,$$

and has n roots $x_1, x_2 \dots x_n$ where

$$x_1 + x_2 \dots + x_n = \lambda_1 + \lambda_2 \dots + \lambda_n - \frac{a_1 + a_2 \dots + a_n}{v};$$

wherefore

$$fv\,dx_1 + fv\,dx_2 \dots = fv\,(dx_1 + dx_2 \dots)$$

$$= fv\frac{dv}{a} \qquad \text{in the first case}$$

and

$$= fv\frac{(a_1 + a_2 \dots + a_n)\,dv}{v^2} \quad \text{in the second}$$

[which are the two formulæ in question].

487.

ON THE QUARTIC SURFACES $(*\between U, V, W)^2 = 0$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 15—25.]

AMONG the surfaces of the form in question are included the reciprocals of several interesting surfaces of the orders 6, 8, 9, 10, and 12, viz.

Order 6, parabolic ring.
" 8, elliptic ring.
" 9, centro-surface of a paraboloid.
" 10, parallel surface of a paraboloid.
" " envelope of planes through the points of an ellipsoid at right angles to the radius vectors from the centre.
" 12, centro-surface of an ellipsoid.
" " parallel surface of an ellipsoid.

I propose to consider these surfaces, not at present in any detail, but merely for the purpose of presenting them in connexion with each other and with the present theory. It will be convenient to use homogeneous equations, but for the metrical interpretation the coordinate W or w may be considered as equal to unity: I have not thought it necessary so to adjust the constants that the equations shall be homogeneous in regard to the constants; this can of course be done without difficulty, and in many cases it would be analytically advantageous to make the change.

I take throughout (X, Y, Z, W) for the coordinates of a point on the quartic surface (so that (U, V, W) in the equation $(*\between U, V, W)^2 = 0$ are to be considered as quadric functions of (X, Y, Z, W)), reserving (x, y, z, w) for the coordinates of a point on the reciprocal surface of the order 6, 8, 9, 10, or 12. The reciprocation is performed in regard to the imaginary sphere $x^2 + y^2 + z^2 + w^2 = 0$: the relation between

the coordinates (X, Y, Z, W) and (x, y, z, w) is then $Xx + Yy + Zz + Ww = 0$, and the equation $(X, Y, Z, W)^4 = 0$ is the equation in point-coordinates of the quartic surface, or in line-coordinates of the reciprocal surface: and similarly the equation $(x, y, z, w)^2 = 0$ is the equation in point-coordinates of the reciprocal surface, or in line-coordinates of the quartic surface.

Parabolic ring, or envelope of a sphere of constant radius having its centre on a parabola.

Taking k for the radius of the sphere, and $z = 0$, $y^2 = 4ax$ for the equations of the parabola, then the coordinates of a point on the parabola are $a\theta^2$, $2a\theta$, 0; where θ is a variable parameter. The equation of the sphere therefore is

$$(x - a\theta^2 w)^2 + (y - 2a\theta w)^2 + z^2 - k^2 w^2 = 0,$$

and the ring is the envelope of this sphere.

The reciprocal of the sphere is

$$k^2 (X^2 + Y^2 + Z^2) - (a\theta^2 X + 2a\theta Y + W)^2 = 0;$$

writing this in the form

$$a\theta^2 X + 2a\theta Y + W + k\sqrt{(X^2 + Y^2 + Z^2)} = 0,$$

and taking the envelope in regard to θ, we have

$$X\{W + k\sqrt{(X^2 + Y^2 + Z^2)}\} - aY^2 = 0,$$

or, what is the same thing,

$$(aY^2 - XW)^2 - k^2 X^2 (X^2 + Y^2 + Z^2) = 0,$$

for the equation of the quartic surface. This has the line $X = 0$, $Y = 0$ for a tacnodal line, but I am not in possession of a theory enabling me thence to infer that the parabolic ring is of the order 6.

To show that it is so, I revert to the equation of the variable sphere

$$(x - a\theta^2 w)^2 + (y - 2a\theta w)^2 + z^2 - k^2 w^2 = 0,$$

or, what is the same thing,

$$(A, B, C, D, E \between \theta, 1)^4 = 0,$$

where

$$\begin{aligned} A &= 3a^2 w^2, \\ B &= 0, \\ C &= a(2aw^2 - xw), \\ D &= -3ayw, \\ E &= 3(x^2 + y^2 + z^2 - k^2 w^2). \end{aligned}$$

Then $I = 3a^2w^2I'$, $J = a^3w^3J'$, and the equation is $I'^3 - J'^2 = 0$, viz. this is

$$\{4x^2 + 3y^2 - 4axw + 4a^2w^2 + 3(z^2 - k^2w^2)\}^3$$
$$- \{(2aw - x)[8x^2 + 9y^2 + 4axw - 4a^2w^2 + 9(z^2 - k^2w^2)] - 27ay^2w\}^2 = 0,$$

or, as this may also be written,

$$\{4x^2 + 3y^2 - 4axw + 4a^2w^2 + 3(z^2 - k^2w^2)\}^3$$
$$- \{-8x^3 - 9xy^2 + 12ax^2w + 12a^2xw^2 - 8a^3w^3 - 9(x - 2aw)(z^2 - h^2w^2)\} = 0.$$

Developing, the whole divides by 27, and the equation of the ring finally is

$$(y^2 - 4axw)^2\{y^2 + (x - aw)^2\}$$
$$+ \{3y^4 + y^2(2x^2 - 2axw + 20a^2w^2) + 8ax^3w + 8a^2x^2w^2 - 32a^3xw^3 + 16a^4w^4\}(z^2 - k^2w^2)$$
$$+ (3y^2 + x^2 + 8axw - 8a^2w^2)(z^2 - k^2w^2)^2$$
$$+ (z^2 - k^2w^2)^3 = 0.$$

Elliptic ring, or envelope of a sphere of constant radius having its centre on an ellipse.

Taking k for the radius of the sphere, and $z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for the equations of the ellipse, the coordinates of a point on the ellipse are $a\cos\theta$, $b\sin\theta$; hence the equation of the variable sphere is

$$(x - aw\cos\theta)^2 + (y - bw\sin\theta)^2 + z^2 - k^2w^2 = 0.$$

The reciprocal of this is

$$k^2(X^2 + Y^2 + Z^2) - (aX\cos\theta + bY\sin\theta + W)^2 = 0,$$

viz. writing this under the form

$$aX\cos\theta + bY\sin\theta + W + k\sqrt{(X^2 + Y^2 + Z^2)} = 0,$$

and taking the envelope in regard to θ, the equation of the reciprocal surface is

$$a^2X^2 + b^2Y^2 = \{W + k\sqrt{(X^2 + Y^2 + Z^2)}\}^2,$$

viz. this is

$$(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 - W^2 = 2kW\sqrt{(X^2 + Y^2 + Z^2)},$$

or

$$\{(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 - W^2\}^2 - 4k^2W^2(X^2 + Y^2 + Z^2) = 0,$$

that is

$$\{(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2\}^2 - 2W^2\{(a^2 + k^2)X^2 + (b^2 + k^2)Y^2 + k^2Z^2\} + W^4 = 0,$$

which is a quartic surface having the nodal conic $W = 0$, $(a^2 - k^2)X^2 + (b^2 - k^2)Y^2 - k^2Z^2 = 0$. This singularity alone would only reduce the order of the reciprocal surface to 12; the reciprocal surface or elliptic ring is in fact (as I proceed to show) of the order 8.

For this purpose reverting to the equation

$$(x - aw\cos\theta)^2 + (y - bw\sin\theta)^2 + z^2 - k^2w^2 = 0,$$

this may be written

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

where

$$\begin{aligned} A &= (a^2 - b^2)\,w^2, \\ B &= 0, \\ C &= -4axw, \\ D &= -4byw, \\ E &= (a^2 + b^2)\,w^2 + 2\,(x^2 + y^2 + z^2 - k^2w^2), \end{aligned}$$

and the equation is

$$\begin{aligned} &\{12\,(A^2 + B^2) - 3\,(C^2 + D^2) + 4E^2\}^3 \\ &\qquad - \{27A\,(C^2 - D^2) + 54BCD - [72\,(A^2 + B^2) + 9\,(C^2 + D^2)]\,E + 8E^3\}^2 = 0, \end{aligned}$$

or say

$$\{12A^2 - 3\,(C^2 + D^2) + 4E^2\}^3 - \{27A\,(C^2 - D^2) - [72A^2 + 9\,(C^2 + D^2)]\,E + 8E^3\}^2 = 0.$$

This is of the order 12, but it is easy to see that the terms in E^6 and $E^4\,(C^2 + D^2)$ disappear from the equation, all the other terms divide by w^4; and the equation is thus of the order 8.

The equation may be obtained somewhat differently as follows. The equation of the variable sphere is

$$(x - \alpha w)^2 + (y - \beta w)^2 + z^2 - k^2w^2 = 0,$$

where (α, β) vary subject to the condition $\dfrac{\alpha^2}{a^2} + \dfrac{\beta^2}{b^2} = 1$. We have therefore

$$x - \alpha w - \lambda\,\frac{\alpha w}{a^2} = 0,$$

$$y - \beta w - \lambda\,\frac{\beta w}{b^2} = 0,$$

and thence

$$\alpha w = \frac{a^2x}{a^2 + \lambda}, \qquad x - \alpha w = -\frac{\lambda x}{a^2 + \lambda},$$

$$\beta w = \frac{b^2y}{b^2 + \lambda}, \qquad y - \beta w = -\frac{\lambda y}{b^2 + \lambda}.$$

Consequently

$$\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2 - k^2w^2}{\lambda^2} = 0,$$

$$\frac{a^2x^2}{(a^2 + \lambda)^2} + \frac{b^2y^2}{(b^2 + \lambda)^2} - \qquad w^2 \quad = 0,$$

from which λ is to be eliminated. The second equation may be replaced by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2-k^2w^2}{\lambda}-w^2=0,$$

which has the first for its derived equation in regard to λ. Hence, writing this last equation in the form

$$w^2(a^2+\lambda)(b^2+\lambda)\lambda-(b^2+\lambda)\lambda x^2-(a^2+\lambda)\lambda y^2-(a^2+\lambda)(b^2+\lambda)(z^2-k^2w^2)=0,$$

we have to equate to zero the discriminant of this cubic function of λ. Calling the equation

$$(A,\ B,\ C,\ D\between\lambda,\ 1)^3=0,$$

we have

$$\begin{aligned} A &= 3w^2,\\ B &= (a^2+b^2)w^2-x^2-y^2-(z^2-k^2w^2),\\ C &= a^2b^2w^2-b^2x^2-a^2y^2-(a^2+b^2)(z^2-k^2w^2),\\ D &= -3a^2b^2(z^2-k^2w^2). \end{aligned}$$

The required equation then is

$$A^2D^2+4AC^3+4B^3D-3B^2C^2-6ABCD=0.$$

The developed equation (Salmon's *Conic Sections*, Ed. v., p. 325) is

$$\begin{aligned} &(b^2x^2+a^2y^2-a^2b^2w^2)^2\{(x-cw)^2+y^2\}\{(x+cw)^2+y^2\}\\ &-\left\{\begin{array}{l} 2b^2(a^2-2b^2)x^6-2(a^4-a^2b^2+3b^4)x^4y^2-2(3a^4-a^2b^2+b^4)x^2y^4+2a^2(b^2-2a^2)y^6\\ -b^2(6a^4-10a^2b^2+6b^4)x^4w^2+(4a^6-6a^4b^2-6a^2b^4+4b^6)x^2y^2w^2\\ \qquad\qquad -a^2(6a^4-10a^2b^2+6b^4)y^4w^2\\ +2c^2(3a^4-a^2b^2+b^4)x^2w^4-2c^2(a^4-3a^2b^2+3b^4)y^2w^4-2a^2b^2c^4(a^2+b^2)w^6 \end{array}\right\}(z^2-k^2w^2)\\ &+\left\{\begin{array}{l} (a^4-6a^2b^2+6b^4)x^4+(6a^4-10a^2b^2+6b^4)x^2y^2+(6a^4-6a^2b^2+b^4)y^4\\ -2c^2(a^4-a^2b^2+3b^4)x^2w^2+2c^2(3a^4-a^2b^2+b^4)y^2w^2+c^4(a^4+4a^2b^2+b^4)w^4 \end{array}\right\}(z^2-k^2w^2)^2\\ &+\left\{\begin{array}{l} (a^2-2b^2)x^2+(2a^2-b^2)y^2\\ \qquad +c^2(a^2+b^2)w^2 \end{array}\right\}2c^2(z^2-k^2w^2)^3\\ &+c^4(z^2-k^2w^2)^4=0. \end{aligned}$$

I remark that the before-mentioned nodal conic $W=0$, $(a^2-k^2)X^2+(b^2-k^2)Y^2-k^2Z^2=0$ is the reciprocal of a quadric cone, which is a bitangent cone of the ring: this is a cone, vertex at the centre of the ring, and which is the envelope of the right cone, vertex the same point, circumscribed about the variable sphere which generates the ring.

Centro-surface of a paraboloid.

For the paraboloid $\dfrac{X^2}{a}+\dfrac{Y^2}{b}-2ZW=0$, it may be shown that the centro-surface is the envelope of the quadric

$$\frac{ax^2}{(a+\theta)^2}+\frac{by^2}{(b+\theta)^2}-2zw-2\theta w^2=0.$$

The quartic surface is consequently the envelope of the quadric

$$\frac{(a+\theta)^2}{a}X^2+\frac{(b+\theta)^2}{b}Y^2+2\theta Z^2-2ZW=0,$$

viz. this is

$$\theta^2\left(\frac{X^2}{a}+\frac{Y^2}{b}\right)+2\theta(X^2+Y^2+Z^2)+aX^2+bY^2-2ZW=0.$$

Hence the quartic surface is

$$\left(\frac{X^2}{a}+\frac{Y^2}{b}\right)(aX^2+bY^2-2ZW)-(X^2+Y^2+Z^2)^2=0,$$

or, what is the same thing,

$$X^2Y^2(a-b)^2-2ZW(bX^2+aY^2)-2abZ^2(X^2+Y^2)-abZ^4=0.$$

This has four conic nodes; viz. considering the equations

$$\frac{X^2}{a}+\frac{Y^2}{b}=0,\quad aX^2+bY^2-2ZW=0,\quad X^2+Y^2+Z^2=0,$$

these give the point $X=0$, $Y=0$, $Z=0$ four times, and four other points which are the nodes in question; the point $(X=0, Y=0, Z=0)$ is a singular point of a higher order; the reduction caused by these singularities should be $=8+19$, so as to make the order of the surface of centres $=9$; that is the reduction on account of the point $(X=0, Y=0, Z=0)$ must be $=19$; but it is not by any means obvious how this is so.

Parallel surface of the paraboloid.

This is given, Salmon's *Solid Geometry*, 2nd Edit., pp. 146 and 148, [Ed. 4, p. 180], for the paraboloid $aX^2+bY^2+2rZW=0$, as the envelope of the quadric surface

$$\frac{\theta ax^2}{\theta a+1}+\frac{\theta by^2}{\theta b+1}+2\theta rzw-(\theta^2r^2+k^2)w^2\quad=0.$$

The reciprocal quartic is thus the envelope of

$$\frac{\theta a+1}{\theta a}X^2+\frac{\theta b+1}{\theta b}Y^2+\frac{\theta^2r^2+k^2}{\theta^2r^2}Z^2+\frac{2}{\theta r}ZW=0,$$

that is

$$(X^2+Y^2+Z^2)+\frac{1}{\theta}\left(\frac{X^2}{a}+\frac{Y^2}{b}+\frac{2}{r}ZW\right)+\frac{1}{\theta^2}\frac{k^2}{r^2}Z^2=0,$$

whence the equation is

$$4\frac{k^2}{r^2}Z^2(X^2+Y^2+Z^2)-\left(\frac{X^2}{a}+\frac{Y^2}{b}+\frac{2}{r}ZW\right)^2=0,$$

viz. this is a quartic having the nodal line-pair $Z=0$, $\frac{X^2}{a}+\frac{Y^2}{b}=0$; and a further singularity at the point $X=0$, $Y=0$, $Z=0$. It would require some consideration to show that the order of the parallel surface is thence $=10$, as it should be.

Envelope of the planes through the points of an ellipsoid at right angles to the radius vectors from the centre.

This is given in my paper "Sur la surface &c." in the *Annali di Matematica*, t. II. (1859), [250], as the envelope of the quadric surface

$$\frac{x^2}{2-\frac{\theta}{a^2}}+\frac{y^2}{2-\frac{\theta}{b^2}}+\frac{z^2}{2-\frac{\theta}{c^2}}-\theta w^2=0.$$

The reciprocal quartic surface is thus the envelope of

$$\left(2-\frac{\theta}{a^2}\right)X^2+\left(2-\frac{\theta}{b^2}\right)Y^2+\left(2-\frac{\theta}{c^2}\right)Z^2-\frac{1}{\theta}W^2=0,$$

or, what is the same thing,

$$\theta\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-2(X^2+Y^2+Z^2)+\frac{1}{\theta}W^2=0,$$

viz. this is

$$W^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

which is in fact the inverse surface

$$\left(\frac{X}{X^2+Y^2+Z^2},\quad \frac{Y}{X^2+Y^2+Z^2},\quad \frac{Z}{X^2+Y^2+Z^2}\text{ for } X, Y, Z\right)$$

of the ellipsoid $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$; this is obvious geometrically inasmuch as the reciprocal of the variable plane is the inverse of the point on the ellipsoid.

The quartic surface has the nodal conic

$$W=0,\quad X^2+Y^2+Z^2=0;$$

and also the node $X=0$, $Y=0$, $Z=0$; there is consequently in the order of the reciprocal surface a reduction $24+2=26$, or the order of the reciprocal surface is $=10$.

Centro-surface of the ellipsoid.

Writing the equation of the ellipsoid in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-w^2=0$, the centro-surface is given as the envelope of the quadric surface

$$\frac{a^2x^2}{(\theta+a^2)^2}+\frac{b^2y^2}{(\theta+b^2)^2}+\frac{c^2z^2}{(\theta+c^2)^2}-w^2=0,$$

(Salmon, [Ed. 2], p. 400, [Ed. 4, p. 179]), and hence the reciprocal quartic surface is the envelope of

$$\left(a+\frac{\theta}{a}\right)^2X^2+\left(b+\frac{\theta}{b}\right)^2Y^2+\left(c+\frac{\theta}{c}\right)^2Z^2-W^2=0,$$

in regard to the variable parameter θ, viz. the equation is

$$\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)(a^2X^2+b^2Y^2+c^2Z^2-W^2)-(X^2+Y^2+Z^2)^2=0,$$

(see Salmon, [Ed. 2], p. 144 [Ed. 4, p. 172]). It hence at once appears, that the quartic surface has 12 nodes, viz. these are the four angles of the fundamental tetrahedron $(XYZW)$, and the eight points

$$\begin{cases} \dfrac{X^2}{a^2}+\dfrac{Y^2}{b^2}+\dfrac{Z^2}{c^2}=0, \\ X^2+Y^2+Z^2=0, \\ a^2X^2+b^2Y^2+c^2Z^2-W^2=0, \end{cases}$$

or writing as it is convenient to do

$$(\alpha, \beta, \gamma)=(b^2-c^2,\ c^2-a^2,\ a^2-b^2);$$

and therefore

$$\alpha+\beta+\gamma=0,\quad a^2\alpha+b^2\beta+c^2\gamma=0,\quad a^4\alpha+b^4\beta+c^4\gamma=-\alpha\beta\gamma;$$

these are the eight points

$$\frac{X^2}{W^2}=-\frac{a^2}{\beta\gamma},\quad \frac{Y^2}{W^2}=-\frac{b^2}{\gamma\alpha},\quad \frac{Z^2}{W^2}=-\frac{c^2}{\alpha\beta};$$

the order of the reciprocal of the quartic surface is thus $36-2.12, =12$, which is in fact the order of the surface of centres.

The equation of the centro-surface is given, Salmon, [Ed. 2], p. 151, and *Quart. Math. Jour.*, t. II. (1858), p. 220, in the form

$$(\alpha, \beta, \gamma)^6(\xi, \eta, \zeta, \omega)^{12}=0,$$

where ξ, η, ζ, ω stand for ax, by, cz, iw; it is therefore of the degree 18 in regard to a, b, c.

Parallel surface of the ellipsoid.

This is given, Salmon, [Ed. 2], p. 148 [Ed. 4, p. 176], as the envelope of the quadric surface

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-\left(1+\frac{k^2}{\theta}\right)w^2=0.$$

The reciprocal quartic is thus the envelope of

$$(a^2+\theta)X^2+(b^2+\theta)Y^2+(c^2+\theta)Z^2-\frac{\theta W^2}{k^2+\theta}=0,$$

or writing $k^2+\theta=\lambda$, this is

$$(a^2-k^2+\lambda)X^2+(b^2-k^2+\lambda)Y^2+(c^2-k^2+\lambda)Z^2-\left(1-\frac{k^2}{\lambda}\right)W^2=0,$$

or, what is the same thing,

$$\lambda^2(X^2+Y^2+Z^2)+\lambda[(a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2-W^2]+k^2W^2=0,$$

whence the equation is

$$\{(a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2-W^2\}^2-4k^2W^2(X^2+Y^2+Z^2)=0,$$

viz. this is a quartic having the nodal conic

$$W=0,\quad (a^2-k^2)X^2+(b^2-k^2)Y^2+(c^2-k^2)Z^2=0.$$

The order of the reciprocal or parallel surface is thus $36-24, =12$, as it should be. The nodal conic of the quartic surface is the reciprocal of a bitangent or node-couple quadric cone, vertex the centre, in the parallel surface: this cone is imaginary for the ellipsoid, but real for either of the hyperboloids, and its existence in the case of the hyperboloid is readily perceived.

Reverting to the equation

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-\left(1+\frac{k^2}{\theta}\right)w^2=0,$$

or say

$$(a^2+\theta)(b^2+\theta)(c^2+\theta)(k^2+\theta)w^2$$
$$-x^2(b^2+\theta)(c^2+\theta)\theta-y^2(c^2+\theta)(a^2+\theta)\theta-z^2(a^2+\theta)(b^2+\theta)\theta=0,$$

this is

$$(A, B, C, D, E\between\theta, 1)^4=0,$$

where putting for shortness

$$\begin{aligned}
\alpha &= a^2+b^2+c^2+k^2,\\
\beta &= b^2c^2+c^2a^2+a^2b^2+k^2(a^2+b^2+c^2),\\
\gamma &= a^2b^2c^2+k^2(b^2c^2+c^2a^2+a^2b^2),\\
\delta &= a^2b^2c^2k^2,
\end{aligned}$$

and

$$\begin{aligned}
p &= x^2+y^2+z^2,\\
q &= (b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2,\\
r &= b^2c^2x^2+c^2a^2y^2+a^2b^2z^2,
\end{aligned}$$

we have

$$\begin{aligned}
A &= 12w^2,\\
B &= 3\alpha w^2-3p,\\
C &= 2\beta w^2-2q,\\
D &= 3\gamma w^2-3r,\\
E &= 12\delta w^2.
\end{aligned}$$

The equation of the parallel surface is of course

$$(AE-4BD+3C^2)^3-27(ACE-AD^2-B^2E+2BCD-C^3)^2=0.$$

It is remarked (Salmon, [Ed. 2], p. 148 [Ed. 4, p. 176]) that there is in the plane $z=0$, a nodal conic $\frac{x^2}{a-c}+\frac{y^2}{b-c}-\left(1+\frac{k^2}{c}\right)w^2=0$, the complete section being made up of this conic twice, and of the curve of the eighth order which is the parallel curve of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-w^2=0$; the like is of course the case as to the sections by the other two principal planes $x=0$ and $y=0$. For the section by the plane $w=0$ (or plane infinity) we have at once $p^2r^2(4pr-q^2)=0$, where observe that

$$\begin{aligned} q^2-4pr &= \{(b^2+c^2)x^2+(c^2+a^2)y^2+(a^2+b^2)z^2\}^2-4(x^2+y^2+z^2)(b^2c^2x^2+c^2a^2y^2+a^2b^2z^2), \\ &= (1,\ 1,\ 1,\ -1,\ -1,\ -1\between(b^2-c^2)x^2,\ (c^2-a^2)y^2,\ (a^2-b^2)z^2)^2 \\ &= \text{norm. }\{x\sqrt{(b^2-c^2)}+y\sqrt{(c^2-a^2)}+z\sqrt{(a^2-b^2)}\}. \end{aligned}$$

The section is thus made up of two conics, each twice, and of four right lines: viz. the conics are $x^2+y^2+z^2=0$, the circle at infinity and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$, the section at infinity of the ellipsoid; and the lines are

$$x\sqrt{(b^2-c^2)}\pm y\sqrt{(c^2-a^2)}\pm z\sqrt{(a^2-b^2)}=0,$$

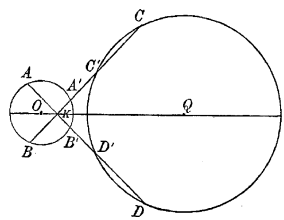
viz. these are the common tangents of the two conics. The circle at infinity is a nodal conic on the surface, which has thus 4 nodal conics.

488.

NOTE ON A RELATION BETWEEN TWO CIRCLES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 82, 83.]

CONSIDER any two circles O, Q; and let AC, BD, $A'D'$, $B'C'$ be the common tangents touching the circles in the points A, A', B, B', C, C', D, D': the locus of a point P such that the pairs of tangents from it to the two circles respectively form a harmonic pencil, is a conic through the 8 points A, A', B, B', C, C', D, D'; but this conic may break up into two lines, viz. if (as in the figure) the points A, B', D', D are in a line, then the points C, C', A', B will be in a symmetrically situated line, and the conic breaks up into this pair of lines, meeting suppose in K. The condition



for this, if a, a' are the distances of the centres from a fixed point in the line of centres, and if the radii are c, c', is readily found to be

$$(a-a')^2 = 2(c^2+c'^2).$$

Suppose in general, that (given any two conics) the point P' is the intersection of the polars of P in regard to the two given conics respectively; then if P describes

a line, the locus of P' is a conic passing through the three conjugate points of the given conics; if, however, the line which is the locus of P pass through one of the conjugate points, then the conic the locus of P' breaks up into a pair of lines, one of them a fixed line through the other two conjugate points, the other of them a line through the first-mentioned conjugate point. That is, if the locus of P be a line through a conjugate point, the locus of P' is a line through the same conjugate point; but in every other case the locus of P' is a conic.

Reverting to the figure of the two circles, in order that it may be possible that the two lines AD and BC may be loci of points P, P', related as above, it is necessary that K shall be a conjugate point of the two circles; that is, if the two circles intersect in points Λ, Λ' lying symmetrically in the radical axis, which meets, suppose, the line of centres in M, then it is necessary that K shall be one of the anti-points of Λ, Λ'; or, what is the same thing, the distance KM must be $= i$ into $M\Lambda$ or $M\Lambda'$; this condition, if as above $(a-a')^2 = 2(c^2+c'^2)$, implies $c^2 = c'^2$, and we have then $(a-a')^2 = 4c^2$, that is, the circles must be equal, and the distance of the centres must be twice the radius, or, what is the same thing, the circles must be equal circles touching each other; when this is so, the two lines AD, BC (being then lines at right angles to each other intersecting in the point of contact), have, in fact, the above-mentioned relation. And it thus appears that given two circles, the necessary and sufficient conditions for the coexistence of the properties mentioned in the theorem are that they shall be equal circles touching each other.

489.

ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON, AND THE (2, 2) CORRESPONDENCE OF POINTS ON A CONIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 83—91.]

THE present paper includes, as will at once be seen, much that is perfectly well known; but the separate theories required, it seemed to me, to be put together; and there are, particularly as regards the unsymmetrical case afterwards referred to, some results which I believe to be new.

The porism of the in-and-circumscribed polygon has its foundation in the theory of the symmetrical (2, 2) correspondence of points on a conic; viz. a (2, 2) correspondence is such that to any given position of either point there correspond two positions of the other point; and in a symmetrical (2, 2) correspondence either point indifferently may be considered as the first point and the other of them will then be the second point of the correspondence. Or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form $(*\between x, 1)^2(y, 1)^2 = 0$, which is symmetrical in regard to the two parameters (x, y). In the case of such symmetrical relation it is easy to show that the line joining the two points envelopes a conic. For the relation may be expressed in the form $(*\between 1, x+y, xy)^2 = 0$; we may imagine the coordinates (P, Q, R) fixed in such manner that for the point (x) on the first conic we have $P : Q : R = 1 : x : x^2$, and for the point (y), $P : Q : R = 1 : y : y^2$; the equation of the line joining the two points is then

$$\begin{vmatrix} P, & Q, & R \\ 1, & x, & x^2 \\ 1, & y, & y^2 \end{vmatrix} = 0;$$

that is

$$Pxy - Q(x+y) + R = 0,$$

or representing this by

$$P\xi + Q\eta + R\zeta = 0,$$

we have $\xi : \eta : \zeta = xy : -x-y : 1$; and consequently (ξ, η, ζ) are connected by a quadric equation; that is, the envelope is a conic.

The relation $(*\between x, 1)^2 (y, 1)^2 = 0$, whether symmetrical or not, leads as will be presently shown to a differential equation of the form

$$\frac{dx}{\sqrt{(X)}} \pm \frac{dy}{\sqrt{(Y)}} = 0,$$

where X, Y are quartic functions of x, y respectively; viz. these are unlike or like functions of the two variables according as the integral equation is not or is symmetrical in regard to the two variables. In the former case, however, the functions X, Y are so related to each other, that the two can be by a linear transformation converted into like functions of the variables: for instance, if y be changed into $ay_1 + b \div cy_1 + d$, then the constants may be determined in suchwise that Y is the same function of y_1, that X is of x; the original integral equation being hereby converted into a symmetrical equation $(*\between x, 1)^2 (y_1, 1)^2 = 0$ between x and y_1, so that in one point of view the unsymmetrical case is not really more general than the symmetrical one. It is to be added that the integral equation contains really one more constant than the differential equation (this is most readily seen in the symmetrical case, the differential equation depends only on the ratio of five constants a, b, c, d, e, whereas the integral equation depends on the ratio of six constants), so that the integral equation is really the complete integral of the differential equation.

Attending now to the symmetrical case; if A and B are corresponding points, then the corresponding points of B are A and a new point C; those of C are B and a new point D, and so on; so that the points form a series $A, B, C, D, \ldots$; and the porismatic property is that, if for a given position of A this series closes at a certain term, for instance, if $D = A$, then it will always thus close, whatever be the position of A. And this follows at once from the consideration of the differential equation $\frac{dx}{\sqrt{(X)}} = \frac{dy}{\sqrt{(Y)}}$; viz. as this is at once integrable *per se* in the form

$$\Pi(y) - \Pi(x) = \Pi(k),$$

this equation must be a transformation of the original equation $(*\between x, 1)^2 (y, 1)^2 = 0$, and equally with it represent the relation between the parameters x, y of the two points A, B; the constant of integration k is of course completely determined in terms of the coefficients of the last-mentioned equation, assumed to be given.

Hence forming the equations for the correspondences, B, C; C, D; ... and assuming that the series closes F, A; we have

$$\Pi(z) - \Pi(y) = \Pi(k),$$
$$\vdots$$
$$\Pi(x) - \Pi(u) = \Pi(k);$$

where, however, the $\Pi(x)$ of the last equation must be regarded as differing from that of the first equation by a period, say Ω, of the integral; hence adding, we have

$$\Omega = n\Pi(k),$$

or

$$\Pi(k) = \frac{1}{n}\Omega,$$

which gives between the constants of the integral equation $(*\between x, 1)^2(y, 1)^2 = 0$, a relation which must be satisfied when the series closes at the nth term (viz. when the term after this coincides with the first term); and this relation is independent of x, that is, of the position of the point A.

The analysis in regard to the differential equation is as follows:

Consider the equation

$$\begin{aligned} U = \quad & y^2(ax^2 + 2bx + c) \\ & + 2y\,(a'x^2 + 2b'x + c') \\ & + \quad (a''x^2 + 2b''x + c'') = 0, \end{aligned}$$

say

$$U = (P, Q, R\between y, 1)^2 = (L, M, N\between x, 1)^2 = 0,$$

we have

$$dU = 0 = (Py + Q)\,dy + (Lx + M)\,dx.$$

But the equation $U = 0$ gives $(Py + Q)^2 = Q^2 - PR$, $(Lx + M)^2 = M^2 - NL$, and the differential equation therefore becomes

$$dy\,\sqrt{(Q^2 - PR)} \pm dx\,\sqrt{(M^2 - NL)} = 0,$$

viz. it is

$$\frac{dy}{\sqrt{\{(ay^2 + 2a'y + a'')(cy^2 + 2c'y + c'') - (bx^2 + 2b'y + b'')^2\}}}$$

$$\pm \frac{dx}{\sqrt{\{(ax^2 + 2bx + c)(a''x^2 + 2b''x + c'') - (a'x^2 + 2b'x + c')^2\}}} = 0.$$

Suppose the equation is

$$\begin{aligned} & y^2(ax^2 + 2hx + g) \\ & + 2y\,(hx^2 + 2bx + f) \\ & + \quad (gx^2 + 2fx + c) = 0, \end{aligned}$$

then the differential equation is

$$\frac{dy}{\sqrt{\{(ay^2 + 2hy + g)(gy^2 + 2fy + c)^2 - (hy^2 + 2by + f)^2\}}}$$

$$\pm \frac{dx}{\sqrt{\{(ax^2 + 2hx + g)(gx^2 + 2fx + c) - (hx^2 + 2bx + f)^2\}}} = 0,$$

say

$$\frac{dy}{\sqrt{(Y)}} = \frac{\pm\,dx}{\sqrt{(X)}}.$$

Now starting from the differential equation

$$\frac{dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4\}}} = \pm \frac{dy}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between y,\ 1)^4\}}},$$

the integral equation is known to be

$$\left[\frac{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4\}} - \sqrt{\{(a,\ b,\ c,\ d,\ e \between y,\ 1)^4\}}}{x-y}\right] = a\,(x+y)^2 + 4b\,(x+y) + 6\,\theta,$$

where θ is the constant of integration. Writing, for shortness, $X = (a,\ b,\ c,\ d,\ e \between x,\ 1)^4$, $Y = (a,\ b,\ c,\ d,\ e \between y,\ 1)^4$, this is

$$X + Y - 2\sqrt{(XY)} = a\,(x^2-y^2)^2 + 4b\,(x-y)(x^2-y^2) + 6\,\theta\,(x-y)^2;$$

or, what is the same thing,

$$\begin{aligned} a\,(x^4+y^4) - 2\sqrt{(XY)} = a\,(x^2-y^2)^2 &+ 4b\,(x-y)(x^2-y^2) + 6\,\theta\,(x-y)^2, \\ &+ 4b\,(x^3+y^3) \\ &+ 6c\,(x^2+y^2) \\ &+ 4d\,(x+y) \\ &+ 2e, \end{aligned}$$

viz. this gives

$$\begin{aligned} \sqrt{(XY)} = \quad & ax^2y^2 \\ &+ 2b\,(x^2y + xy^2) \\ &+ 3c\,(x^2 + y^2) \\ &+ 3\theta\,(x - y)^2 \\ &+ 2d\,(x + y) \\ &+ e, \end{aligned}$$

and, rationalising, the integral equation becomes

$$\begin{aligned} &- 6a\theta x^2y^2 \\ &- 4adxy\,(x+y) \\ &- ae\,(x+y)^2 \\ &+ 4b^2x^2y^2 \\ &+ 12bcxy\,(x+y) - 12b\theta xy\,(x+y) \\ &- 8bdxy \\ &- 4be\,(x+y) \\ &+ 9c^2\,(x+y)^2 - 18c\theta\,(x^2+y^2) \\ &- 12cd\,(x+y) \\ &+ 9\theta^2\,(x-y)^2 - 12d\theta\,(x+y) - 6e\theta + 4d^2 = 0; \end{aligned}$$

or, as it may be written,

$$\begin{aligned} & x^2y^2(4b^2-6a\theta) \\ + & (x^2y+xy^2)(-4ad+12bc-12b\theta) \\ + & (x^2+y^2)(-ae+9c^2-18c\theta+9\theta^2) \\ + & \quad xy(-2ae-8bd+18c^2-18\theta^2) \\ + & (x+y)(-4be+12cd-12d\theta) \\ + & \quad 4d^2-6e\theta=0. \end{aligned}$$

Comparing this with the original integral equation $V=0$, and the form of differential equation deduced therefrom, we ought to have identically

$$\begin{aligned} & [(4b^2-6a\theta)x^2+(-2ad+6bc-6b\theta)x+(-ae+9c^2-18c\theta+9\theta^2)] \\ & \qquad \times[(-ae+9c^2-18c\theta+9\theta^2)x^2+(-2be+6cd-6d\theta)x+(4d^2-6e\theta)] \\ & -[(-2ad+6bc-6b\theta)x^2+(-ae-4bd+9c^2-9\theta^2)x+(-2be+6cd-6d\theta)]^2 \\ & = \text{multiple of } X, \\ & = \{(-4ad^2-4b^2e+24bcd)+(6ae-24bd-54c^2)\theta+108c\theta^2-54\theta^3\}(a,\ b,\ c,\ d,\ e\between x,\ 1)^4, \end{aligned}$$

by comparing the coefficients of x^4.

I obtain this otherwise:

Write

$$V=\alpha U+6\beta H,$$

then, forming the Hessian of V, we have

$$\begin{aligned} \tilde{H}V &= (\alpha^2-3I\beta^2)H+(I\alpha\beta+9J\beta^3)U, \\ &= \frac{(\alpha^2-3I\beta^2)}{6\beta}(V-\alpha U)+(I\alpha\beta+9J\beta^3)U, \\ &= \frac{\alpha^2-3I\beta^2}{6\beta}V+\frac{1}{6\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U, \end{aligned}$$

that is

$$d_x^2Vd_y^2V-(d_xd_yV)^2-\frac{2(\alpha^2-3I\beta^2)}{\beta}(x^2d_x^2V+2xyd_xd_yV+y^2d_y^2V)=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

or writing

$$K=-\frac{2(\alpha^2-3I\beta^2)}{\beta},$$

this is

$$(d_x^2V+Ky^2)(d_y^2V+Kx^2)-(d_xd_yV-Kxy)^2=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

so that the components are

$$d_x^2V+Ky^2,\quad d_xd_yV-Kxy,\quad d_y^2V+Kx^2,$$

$$\begin{aligned} V=\alpha U+6\beta H= & \\ & \alpha(a,\ b,\ c,\ d,\ e\between x,\ 1)^4+6\beta(ac-b^2,\ 2ad-2bc,\ ae+2bd-3c^2,\ 2be-2cd,\ ce-d^2\between x,\ 1)^4, \end{aligned}$$

viz. the components are

$$\begin{array}{lll}
(\alpha a + 6\beta(ac - b^2), & \alpha b + 3\beta(ad - bc), & \alpha c + \beta(ae + 2bd - 3c^2) + \tfrac{1}{12}K \between x, 1)^2, \\
(\alpha b + 3\beta(ad - bc), & \alpha c + \beta(ae + 2bd - 3c^2) + \tfrac{1}{24}K, & \alpha d + 3\beta(be - cd) \between x, 1)^2, \\
(\alpha c + \beta(ae + 2bd - 3c^2) + \tfrac{1}{12}K, & \alpha d + 3\beta(be - cd), & \alpha e + 6\beta(ce - d^2) \between x, 1)^2,
\end{array}$$

where as before

$$K = -\frac{2(\alpha^2 - 3I\beta^2)}{\beta}.$$

I assume

$$\beta = -\tfrac{2}{3}, \quad \alpha = 4c - 6\theta, \quad K = 3\{(4c - 6\theta)^2 - \tfrac{4}{3}I\}.$$

$$\begin{aligned}
\alpha a + 6\beta(ac - b^2) &= a(4c - 6\theta) - 4(ac - b^2) = 4b^2 - 6a\theta, \\
\alpha b + 3\beta(ad - bc) &= b(4c - 6\theta) - 2(ad - bc) = -2ad + 6bc - 6b\theta, \\
\alpha d + 3\beta(be - cd) &= d(4c - 6\theta) - 2(be - cd) = -2be + 6cd - 6d\theta, \\
\alpha e + 6\beta(ce - d^2) &= e(4c - 6\theta) - 4(ce - d^2) = 4d^2 - 6e\theta, \\
\alpha c + \beta(ae + 2bd - 3c^2) - \tfrac{1}{24}K &= c(4c - 6\theta) - \tfrac{2}{3}(ae + 2bd - 3c^2) - \tfrac{1}{8}\{(4c - 6\theta)^2 - \tfrac{4}{3}I\} \\
&= -\tfrac{1}{2}ae - 2bd + \tfrac{9}{2}c^2 - \tfrac{9}{2}\theta^2, \\
\alpha c + \beta(ae + 2bd - 3c^2) + \tfrac{1}{12}K & \\
&= c(4c - 6\theta) - \tfrac{2}{3}(ae + 2bd - 3c^2) + \tfrac{1}{4}\{(4c - 6\theta)^2 - \tfrac{4}{3}I\} \\
&= -ae + 9c^2 - 18c\theta + 9\theta^2,
\end{aligned}$$

agreeing with the former result.

I return to the general form

$$\begin{aligned}
& y^2(a, \ b, \ c \between x, 1)^2 \\
+ & 2y(a', \ b', \ c' \between x, 1)^2 \\
+ & \quad (a'', \ b'', \ c'' \between x, 1)^2 = 0,
\end{aligned}$$

giving

$$\frac{dx}{\sqrt{[(a, b, c \between x, 1)^2 (a'', b'', c'' \between x, 1)^2 - \{(a', b', c' \between x, 1)^2\}^2]}}$$

$$= \frac{dy}{\sqrt{[(a, a', a'' \between y, 1)^2 (c, c', c'' \between y, 1)^2 - \{(b, b', b'' \between y, 1)^2\}^2]}}.$$

Operate a linear transformation on the x, say

$$x = \frac{\lambda x' + \mu}{\nu x' + \rho};$$

the new coefficients are

$$\begin{array}{lll}(a\,,\,b\,,\,c\between\lambda,\,\nu)^2, & (a\,,\,b\,,\,c\between\lambda,\,\nu\between\mu,\,\rho), & (a\,,\,b\,,\,c\between\mu,\,\rho)^2,\\ (a',\,b',\,c'\between\lambda,\,\nu)^2, & (a',\,b',\,c'\between\lambda,\,\nu\between\mu,\,\rho), & (a',\,b',\,c'\between\mu,\,\rho)^2,\\ (a'',\,b'',\,c''\between\lambda,\,\nu)^2, & (a'',\,b'',\,c''\between\lambda,\,\nu\between\mu,\,\rho), & (a'',\,b'',\,c''\between\mu,\,\rho)^2:\end{array}$$

assume now

$$\begin{array}{rrrr} a'\lambda+(b'-\theta)\,\nu- & a\mu- & b\rho=0,\\ (b'+\theta)\,\lambda+ & c'\nu- & b\mu- & c\rho=0,\\ a''\lambda+ & b''\nu- & a'\mu-(b'+\theta)\,\rho=0,\\ b''\lambda+ & c''\nu-(b'-\theta)\,\mu- & c'\rho=0,\end{array}$$

then it is easy to show that

$$\begin{aligned}(a\,,\,b\,,\,c\between\lambda,\,\nu\between\mu,\,\rho)&=(a'\,,\,b'\,,\,c'\between\lambda,\,\nu)^2,\\ (a',\,b',\,c'\between\mu,\,\rho)^2&=(a'',\,b'',\,c''\between\lambda,\,\nu\between\mu,\,\rho),\\ (a\,,\,b\,,\,c\between\mu,\,\rho)^2&=(a'',\,b'',\,c''\between\lambda,\,\nu)^2\\ &[=(a'\,,\,b'\,,\,c'\between\lambda,\,\nu\between\mu,\,\rho)+\theta\,(\lambda\rho-\mu\nu)],\end{aligned}$$

and the equations give

$$\begin{vmatrix} a' & , & b'-\theta, & a & , & b\\ b'+\theta, & c' & , & b & , & c\\ a'' & , & b'' & , & a' & , & b'+\theta\\ b'' & , & c'' & , & b'-\theta, & c' \end{vmatrix}=0,$$

that is

$$\begin{aligned}&(a'c'-b'^2+\theta^2)^2+(a''c''-b''^2)\,(ac-b^2)\\ &-(a'b''-a''b'+a''\theta)\,(bc'-b'c+c\theta)\\ &+(a'c''-b''b'+b''\theta)\,(bb'-a'c+b\theta)\\ &+(b'b''-a''c'+b''\theta)\,(ac'-b'b+b\theta)\\ &-(b'c''-b''c'+c''\theta)\,(ab'-a'b+a\theta)=0,\end{aligned}$$

which is

$$\begin{array}{ll}(a'c'-b'^2)^2+(a''c''-b''^2)\,(ac-b^2) & -2\theta\begin{vmatrix} a\,, & b\,, & c\\ a'\,, & b'\,, & c'\\ a''\,, & b''\,, & c'' \end{vmatrix}\\ +a'^2\,(-cc'') & \\ +b'^2\,(-ac''-2bb''-a''c) & \\ +c'^2\,(-aa'') & \\ +2b'c'\,(ab''+a''b) & +\theta^2\big(2\,(a'c'-b'^2)-ac''+2bb''-a''c\big)\\ +2c'a'\,(-bb'') & \\ +2a'b'\,(bc''+b''c) & +\theta^4=0.\end{array}$$

If the original matrix be symmetrical $= \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, this is

$$\begin{aligned}
&(fh-b^2)^2+(ag-h^2)(cg-f^2) && -2\theta \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \\
&+h^2(-cg) && \\
&+b^2(-ac-g^2-2fh) && \\
&+f^2(-ag) && \\
&+2bf(af+gh) && +\theta^2[2(fh-b^2)-ac-g^2+2hf] \\
&+2fh(-fh) && \\
&+2bh(fg+ch) && +\theta^4=0,
\end{aligned}$$

that is

$$\begin{aligned}
(b-g)\{(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\} \\
-2\theta(abc-af^2-bg^2-ch^2+2fgh)+\theta^2(4fh-2b^2-ac-g^2)+\theta^4=0,
\end{aligned}$$

satisfied by

$$\theta+b-g=0,$$

viz. the equation in θ is

$$(\theta+b-g)\{\theta^3-(b-g)\theta^2+(4fh-ac-2bg-b^2)\theta+(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\}=0.$$

490.

ON A PROBLEM OF ELIMINATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 99—101.]

I WRITE

$$P = (\alpha, \ldots \between x, y, z)^k, \quad Q = (\alpha', \ldots \between x, y, z)^k,$$
$$U = (a, \ldots \between x, y, z)^m, \quad V = (b, \ldots \between x, y, z)^n,$$

and I seek for the form of the relation between the coefficients $(\alpha, \ldots)$, $(\alpha', \ldots)$, $(a, \ldots)$, $(b, \ldots)$, in order that there may exist in the pencil

$$\lambda P + \mu Q = 0$$

a curve passing through *two* of the intersections of the curves $U = 0$, $V = 0$.

The ratio $\lambda : \mu$ may be determined so as that the curve $\lambda P + \mu Q = 0$ shall pass through one of the intersections of the curves $U = 0$, $V = 0$; or, what is the same thing, so as that the three curves shall have a common point; the condition for this is

$$\text{Reslt. } (\lambda P + \mu Q, U, V) = 0,$$

a condition of the form

$$(\lambda\alpha + \mu\alpha', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km} = 0;$$

or, what is the same thing,

$$(\alpha, \ldots, \alpha', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km} (\lambda, \mu)^{mn} = 0,$$

which, for shortness, may be written

$$(A, \ldots \between \lambda, \mu)^{mn} = 0.$$

Suppose this equation has equal roots, then we have

$$\text{Disct. Reslt. } (\lambda P + \mu Q,\ U,\ V) = 0,$$

the discriminant being taken in regard to λ, μ. This is of the form

$$(A, \ldots)^{2\,(mn-1)} = 0\,;$$

that is

$$(a, \ldots, \alpha', \ldots)^{2mn\,(mn-1)}\,(a, \ldots)^{2kn\,(mn-1)}\,(b, \ldots)^{2km\,(mn-1)} = 0.$$

It is moreover clear that the nilfactum is a combinant of the functions P, Q; and the form of the equation is therefore

$$\left(\left\| \begin{matrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \ldots \end{matrix} \right\|\right)^{mn\,(mn-1)} (a, \ldots)^{2kn\,(mn-1)}\,(b, \ldots)^{2km\,(mn-1)} = 0.$$

Now the equation in question will be satisfied, 1°. if the curves $U=0$, $V=0$ touch each other; let the condition for this be $\nabla = 0$. 2°. If there exists a curve $\lambda P + \mu Q = 0$ passing through two of the intersections of the curves $U=0$, $V=0$; let the condition be $\Omega = 0$. There is reason to think that the equation contains the factor Ω^2, and that the form thereof is $\Omega^2 \nabla = 0$.

Assuming that this is so, and observing that ∇, the osculant or discriminant of the functions U, V, is of the form

$$\nabla = (a, \ldots)^{n\,(n+2m-3)}\,(b, \ldots)^{m\,(m+2n-3)},$$

we have

$$\Omega^2 = \left(\left\| \begin{matrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \end{matrix} \right\|\right)^{mn\,(mn-1)} (a, \ldots)^{kn\,(n-1)\,(2m-1)+(k-1)\,n\,(n+2m-3)} \times$$
$$(b, \ldots)^{km\,(m-1)\,(2n-1)+(k-1)\,m\,(m+2n-3)},$$

and consequently

$$\Omega = \left(\left\| \begin{matrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \end{matrix} \right\|\right)^{\frac{1}{2}mn\,(mn-1)}$$
$$(a, \ldots)^{\frac{1}{2}n\,(n-1)\,k\,(2m-1)+\frac{1}{2}(k-1)\,n\,(n+2m-3)} \times$$
$$(b, \ldots)^{\frac{1}{2}m\,(m-1)\,k\,(2n-1)+\frac{1}{2}(k-1)\,m\,(m+2n-3)},$$

which is the solution of the proposed question. Suppose for instance $n=1$, then

$$\Omega = \left(\left\| \begin{matrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \end{matrix} \right\|\right)^{\frac{1}{2}m\,(m-1)} (a, \ldots)^{(k-1)\,(m-1)}\,(b, \ldots)^{\frac{1}{2}m\,(m-1)\,k+\frac{1}{2}(k-1)\,(m-1)}.$$

If moreover $k=1$, then

$$\Omega = \left(\left\| \begin{matrix} \alpha, & \beta, \ldots \\ \alpha', & \beta', \end{matrix} \right\|\right)^{\frac{1}{2}m\,(m-1)} (b, \ldots)^{\frac{1}{2}m\,(m-1)}\,;$$

this is right, for writing $P=\alpha x+\beta y+\gamma z$, $Q=\alpha' x+\beta' y+\gamma' z$, $V=bx+b'y+b''z$, then if two of the intersections of the curve $U=0$ with the line $V=0$ lie in a line with the point $P=0$, $Q=0$, then the point in question, that is the point $(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)$, must lie in the line $V=0$; and the condition reduces itself to

$$\{(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta\between b,\ b',\ b'')\}^{\frac{1}{2}m(m-1)}=0,$$

where the index $\frac{1}{2}m(m-1)$ is accounted for as denoting the number of pairs of points out of the m intersections of the curve $U=0$ with the line $V=0$.

If in general $k=1$, then writing as before $P=\alpha x+\beta y+\gamma z$, $Q=\alpha' x+\beta' y+\gamma' z$, we have

$$\Omega=(\beta\gamma'-\beta'\gamma,\ldots)^{\frac{1}{2}mn(mn-1)}(a,\ldots)^{\frac{1}{2}n(n-1)(2m-1)}(b,\ldots)^{\frac{1}{2}m(m-1)(2n-1)},$$

where $\Omega=0$ is the condition in order that the point $(\beta\gamma'-\beta'\gamma,\ldots)$ may lie *in lined* with two of the intersections of the curves $U=0$, $V=0$. Or writing $(X,\ Y,\ Z)$ for the coordinates of the given point, the condition is

$$\Omega=(a,\ldots)^{\frac{1}{2}n(n-1)(2m-1)}(b,\ldots)^{\frac{1}{2}m(m-1)(2n-1)}(X,\ Y,\ Z)^{\frac{1}{2}mn(mn-1)}=0.$$

I have found that if

$$U=(a,\ldots\between x,\ y,\ z)^m,\quad V=(b,\ldots\between x,\ y,\ z)^n,$$
$$W=(c,\ldots\between x,\ y,\ z)^p,\quad T=(d,\ldots\between x,\ y,\ z)^q,$$

the condition in order that the point $(X,\ Y,\ Z)$ may lie *in lined* with one of the intersections of the curves $U=0$, $V=0$, and one of the intersections of the curves $W=0$, $T=0$, is

$$\Omega=(a,\ldots)^{npq}(b,\ldots)^{mpq}(c,\ldots)^{mnq}(d,\ldots)^{mnp}(X,\ Y,\ Z)^{mnpq}=0.$$

Supposing that the curves $W=0$, $T=0$ become identical with the curves $U=0$, $V=0$ respectively, this becomes

$$\Omega=(a,\ldots)^{n^2\cdot 2m}(b,\ldots)^{m^2\cdot 2n}(X,\ Y,\ Z)^{mn\cdot mn}=0,$$

and the variation from the correct form given above is what might have been expected.

491.

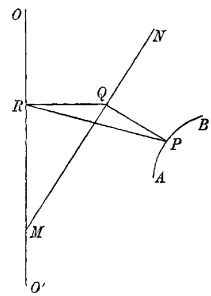
ON THE QUARTIC SURFACES $(*\between U, V, W)^2 = 0$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 111—113.]

THE general Torus, or surface generated by the rotation of a conic about a fixed axis anywise situate, has been investigated by M. De La Gournerie, *Jour. de l'École Polyt.*, t. XXIII. (1863), pp. 1—74. The surface is one of the fourth order, having a nodal circle; and with its equation of the form $V^2 - UW = 0$, consequently of the form in question. The leading points of the theory are as follows:

Consider (fig. 1) the plane of the conic in any particular position thereof; let this

Fig. 1.



meet the axis of rotation OO' in the point M, and let the projection of OO' on the plane of the conic be MN. Take P any point of the conic; draw PQ in the plane of the

conic, perpendicular to MN, and QR perpendicular to OO', and join PR: the point P of the conic describes a circle radius RP, $=\sqrt{(\overline{RQ}^2+\overline{QP}^2)}$. Hence if $\angle OMN=\alpha$, and if $MQ=\mathrm{x}$, $QP=\mathrm{y}$ are the coordinates in the plane of the conic of the point P; and if the coordinates x, y, z are measured, z upwards from M in the direction MO, and x, y in the plane at right angles to the axis OO': we have

$$z=\mathrm{x}\cos\alpha,\quad \sqrt{(x^2+y^2)}=\sqrt{(\mathrm{x}^2\sin^2\alpha+\mathrm{y}^2)};$$

or, what is the same thing,

$$\mathrm{x}=z\sec\alpha,\quad \mathrm{y}=\sqrt{(x^2+y^2-z^2\tan^2\alpha)}.$$

Hence the equation of the conic being $F(\mathrm{x},\mathrm{y})=0$, that of the torus is

$$F\left(z\sec\alpha,\quad \sqrt{(x^2+y^2-z^2\tan^2\alpha)}\right)=0.$$

Thus taking the equation of the conic to be

$$(a, b, c, f, g, h\between \mathrm{x}, \mathrm{y}, 1)^2=0;$$

or, as this may be written,

$$(a\mathrm{x}^2+2g\mathrm{x}+c+b\mathrm{y}^2)^2 \qquad =4\mathrm{y}^2(h\mathrm{x}+f)^2,$$

we have at once the equation of the torus in the form

$$\{az^2\sec^2\alpha+2gz\sec\alpha+c+b(x^2+y^2-z^2\tan^2\alpha)\}^2=4(x^2+y^2-z^2\tan^2\alpha)(hz\sec\alpha+f)^2,$$

which is of the form $V^2-4UW=0$; or, as it is better to write it, $V^2-4UL^2=0$, where

$$V=az^2\sec^2\alpha+2gz\sec\alpha+c+b(x^2+y^2-z^2\tan^2\alpha),$$
$$U=x^2+y^2-z^2\tan^2\alpha,$$
$$L=hz\sec\alpha+f,\quad W=L^2.$$

There is thus a nodal circle $V=0$, $L=0$, that is

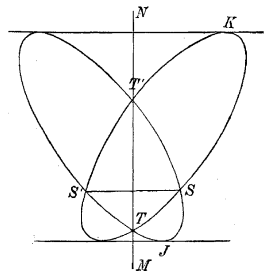
$$z=-\frac{f}{h}\cos\alpha,$$

$$bh^2(x^2+y^2)-bf^2\sin^2\alpha+af^2-2gfh+ch^2=0.$$

But the origin of this nodal circle is better seen geometrically. For observe that the radius of the circle described by the point P of the conic depends only on the *square* of the ordinate PQ: hence if we have on the conic two points S, S' situate symmetrically in regard to the line MN, these points S, S' will describe one and the same circle, which will be a nodal circle on the surface. And there is in fact one such pair of points S, S'; for (see fig. 2) considering in the plane of the conic the equal conic situate symmetrically thereto on the other side of the line MN, the two conics intersect in two points T, T' (real or imaginary) on the line MN, and in two

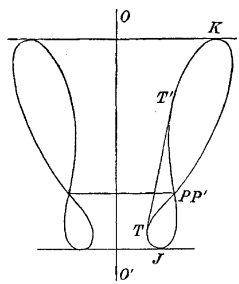
other points S, S' (real or imaginary) situate symmetrically in regard to MN; we have thus the required pair of points which generate the nodal circle.

Fig. 2.



A meridian section of the torus (or section through the axis OO') is a quartic curve symmetrical in regard to this axis, and having two (real or imaginary) nodes the intersections of the plane by the nodal circle: see fig. 3, which shows the section for the surface generated by a conic such as in fig. 2. The quartic curve has 8 double tangents, 2 of them at right angles to the axis OO', the remaining 6 forming 3 pairs

Fig. 3.



of tangents situate symmetrically in regard to this axis; so that attending only to one tangent of each pair, we may say that there are 3 oblique bitangents: one of these is the line TT'; and the section of the torus by a plane through this line at right angles to the plane of the meridian section is in fact the two conics of fig. 2, either of which by its rotation about OO' generates the torus. But taking either of the other two oblique bitangents, the section by a plane through the bitangent at right angles to the meridian plane is in like manner a pair of conics situate symmetrically in regard to the bitangent, and such that either of them by its rotation

about the axis OO' generates the torus. It thus appears that the same torus may be generated in three different ways by the rotation of a conic about the axis OO'.

In the particular case where the plane of the conic passes through the axis, the meridian section consists it is clear of two symmetrically situate conics, intersecting the axis in the points T, T', which are nodes of the surface, the surface having as before a nodal circle generated by the rotation of the two symmetrically situate intersections S, S' of the two conics. The equation is included under the foregoing form, but it is at once obtained from that of the conic,

$$(a\mathrm{x}^2 + 2g\mathrm{x} + c + b\mathrm{y}^2)^2 = 4\mathrm{y}^2 (h\mathrm{x} + f)^2,$$

by writing therein z for x and $\sqrt{(x^2+y^2)}$ for y; viz. the equation of the torus here is

$$\{az^2 + 2gz + c + b(x^2 + y^2)\}^2 = 4(x^2 + y^2)(hz + f)^2,$$

and the two nodes thus are $x = 0$, $y = 0$, $az^2 + 2gz + c = 0$.

492.

NOTE ON A SYSTEM OF ALGEBRAICAL EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 132, 133.]

CONSIDER the system of equations

$$a + b(y+z)^2 + cy^2z^2 = 0,$$
$$a + b(z+x)^2 + cz^2x^2 = 0,$$
$$a + b(x+y)^2 + cx^2y^2 = 0,$$

which is a particular case of that belonging to the porism of the in-and-circumscribed triangle. We have y and z the roots of

$$a + bx^2 + 2u \,.\, bx + u^2(b + cx^2) = 0;$$

consequently

$$y + z = \frac{-2bx}{b + cx^2},$$

$$yz = \frac{a + bx^2}{b + cx^2},$$

or substituting in the equation between y and z, this becomes

$$(ac + b^2)(a + 4bx^2 + cx^4) = 0,$$

so that if $ac + b^2$ is not $= 0$, we have

$$a + 4bx^2 + cx^4 = 0,$$

and moreover

$$(x-y)(x-z) = x^2 + \frac{2bx^2}{b+cx^2} + \frac{a+bx^2}{b+cx^2}, \; = \frac{1}{b+cx^2}(a+4bx^2+cx^4) = 0,$$

so that $x=y$ or else $x=z$. If $x=z$, the three equations reduce themselves to the two

$$\begin{aligned} a + bx^2 + 2y\,.\,bx + y^2(b+cx^2) &= 0, \\ a \;+\; 4bx^2 \;+\; \quad cx^4 \quad &= 0, \end{aligned}$$

giving $y=x$, or else $y = -\dfrac{3bx+cx^3}{b+cx^2}$; and it hence appears that if from this last equation and $a+4bx^2+cx^4=0$ we eliminate x, the result must be $a+4by^2+cy^4=0$. For in the same way that the elimination of y, z from the original three equations gives $a+4bx^2+cx^4=0$, the elimination of x, z from the same three equations will give $a+4by^2+cy^4=0$, so that in any case y is a root of this equation.

493.

ON EVOLUTES AND PARALLEL CURVES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 183—200.]

IN abstract geometry we have a conic called the Absolute; lines which are harmonics of each other in regard to the absolute, or, what is the same thing, which are such that each contains the pole of the other in regard to the absolute, are said to be at right angles. Similarly, points which are harmonics of each other in regard to the absolute, or, what is the same thing, which are such that each lies in the polar of the other, are said to be quadrantal.

A conic having double contact with the absolute is said to be a circle; the intersection of the two common tangents is the centre of the circle; the line joining the two points of contact, or chord of contact, is the axis of the circle.

Taking as a definition of equidistance that the points of a circle are equidistant from the centre, we arrive at the notion of distance generally, and we can thence pass down to that of equal circles; but the notion of equal circles may be established descriptively in a more simple manner:

Any two circles have an axis of symmetry, viz. this is the line joining their centres; and they have a centre of homology, viz. this is the intersection of their axes. They intersect in four points, lying in pairs on two lines through the centre of homology: they have also four tangents meeting in pairs in two points on the axis of symmetry. Now if the two lines through the centre of homology are harmonically related to the two axes, or, what is the same thing, if the two points on the axis of symmetry are harmonically related to the two centres, then the circles are *equal*.

Circles which are equal to the same circle are equal to each other, and the entire series of circles which are equal to a given circle, are said to be a system of circles of constant magnitude.

Starting from these general considerations, I pass to the question of evolutes and parallel curves: it will be understood that everything—lines at right angles, circles, poles, polars, reciprocal curves, &c.—refers to the absolute.

At any point of a curve we have a normal, viz. this is a line at right angles to the tangent; or, what is the same thing, it is the line joining the point with the pole of the tangent. The locus of the pole of the tangent is the reciprocal curve, and for any point of a given curve, the pole of the tangent at that point is the corresponding point of the reciprocal curve. Hence, also the normal is the line from the point to the corresponding point of the reciprocal curve. And the curve and its reciprocal have at corresponding points the same normal.

The envelope of the normals is the evolute; any curve having with the given curve the same normals (and therefore the same evolute) is a parallel curve; in other words, the parallel curve is any orthogonal trajectory of the normals of the given curve.

The parallel curve is also the envelope of a circle of constant radius having its centre on the given curve; or, again, it is the envelope of a circle of constant radius touching the given curve.

The theory in the above form is directly applicable to spherical, or rather conical, geometry; but in ordinary plane geometry the absolute degenerates into a point-pair, the two circular points at infinity, or say the points I, J; and this is a case that requires to be separately treated. The theory in the general case, the absolute a conic, is the more symmetrical and elegant, and it might appear advantageous to commence with this; but upon the whole I prefer the opposite course, and will commence with the case of plane geometry, the absolute a point-pair.

The subject connects itself with that of foci: I call to mind that a common tangent of the curve and the absolute is a focal tangent, and the intersection of two focal tangents a focus. In the case where the absolute is a point-pair, the focal tangents are the tangents from I to the curve, and the tangents from J to the curve, or say these are the I-tangents and the J-tangents; a focus is the intersection of an I-tangent and a J-tangent; the line IJ, when it touches the curve, and (when the curve passes through I and J or either of them) the tangents at I or J to the curve are usually not reckoned as focal tangents; and other singular tangents, for instance a double or stationary tangent through I or J, are also excluded from the focal tangents; and the number of foci is of course reckoned accordingly, viz. it is the product of the number of the I-tangents into that of the J-tangents. So when the absolute is a conic; if this is touched by the curve, the common tangent at the point of contact is not reckoned as a focal tangent; and we may also exclude any singular tangents which touch the absolute; and the number of foci is reckoned accordingly, viz. it is equal to the number of pairs of focal tangents.

Let the Plückerian numbers for the given curve be $(m, n, \delta, \kappa, \tau, \iota)$, viz. m the order, n the class, δ the number of nodes, κ of cusps, τ of bitangents, ι of inflexions; and suppose moreover that D is the deficiency, and α the statitude; viz.

$$\alpha = 3m + \iota, \quad = 3n + \kappa;$$

$$2D = (m-1)(n-2) - 2\delta - 2\kappa, \quad = (n-1)(n-2) - 2\tau - 2\iota, \quad = -2m - 2n + 2 + \alpha$$

$$= n - 2m + 2 + \kappa, \quad = m - 2n + 2 + \iota,$$

And let the corresponding numbers for the evolute be

$$(m'', n'', \delta'', \kappa'', \tau'', \iota''; D'', \alpha'').$$

These are most readily obtained, as in Clebsch's paper, "Ueber die Singularitäten algebraischer Curven," *Crelle*, t. LXIV. (1864), pp. 98—100, viz. it being assumed that we have

$$n'' = m + n, \ \iota'' = 0,$$

then by reason that the evolute has a (1, 1) correspondence with the original curve, the two curves have the same deficiency, or writing this relation under the form

$$m'' - 2n'' + \iota'' = m - 2n + \iota,$$

we have $m'' = 3m + \iota, = \alpha$; and the Plückerian relations then give the values of $\kappa'', \delta'', \tau''$.

In regard to these equations $n'' = m + n$, $\iota'' = 0$, I remark that if we have two curves of the orders m, m', and on these points P, Q having an (α, α') correspondence, the line PQ envelopes a curve of the class $m\alpha' + m'\alpha$, and the number of inflexions is in general $= 0$. Now in the present case, taking P on the given curve and Q the point of intersection with IJ of the normal (or harmonic of the tangent), the orders of the curves are $(m, 1)$, and the correspondence is $(n, 1)$; whence as stated $m'' = m + n$, $\iota'' = 0$.

The formulæ thus are

$$\begin{aligned} m'' &= \alpha, \\ n'' &= m + n, \\ \iota'' &= 0, \\ \kappa'' &= -3m - 3n + 3\alpha, \\ \alpha'' &= 3\alpha, \\ D'' &= D, \end{aligned}$$

in which formulæ it is assumed that the curve has no special relations to the points I, J; or, what is the same thing, that the line IJ intersects the curve in m points distinct from each other, and from the points I, J.

It is to be added (see Salmon's *Higher Plane Curves*, [Ed. 2], (1852), pp. 109 et seq.) that m of the κ'' cusps arise from the intersections of the curve with IJ, these cusps being situate on the line IJ, and each of them the harmonic of one of the intersections in question, and the cuspidal tangent being for each of them the line IJ. The intersections of the evolute by the line IJ are these m cusps each 3 times, and besides ι points arising from the ι inflexions of the curve; viz. at any inflexion the two consecutive normals intersect in a point on the line IJ, being in fact the harmonic of the intersection of IJ with the tangent at the inflexion. It was in this manner that Salmon obtained the number $3m + \iota$ of the points at infinity of the evolute, that is the expression $m'' = 3m + \iota$ $(= \alpha)$ for the order of the evolute.

The remaining $-4m - 3n + 3\alpha$ cusps arise from the points on the curve where there is a circle of 4-pointic intersection, or contact of the third order, and in this

manner the number of them was found, Salmon, [Ed. 2], p. 113, in the particular case of a curve without nodes or cusps, and generally in Zeuthen's *Nyt Bidrag* &c., p. 91; the number of the points in question, in the foregoing form $-4m-3n+3\alpha$, is also obtained in my Memoir, "On the curves which satisfy given conditions," *Phil. Trans.* (1868), pp. 75—143, see p. 97, [406].

It is further to be noticed that the $m+n$ tangents to the evolute from either of the points I, J are made up of the line IJ counting m times (in respect that it is a tangent at each of the above-mentioned m cusps) and of the n tangents from the points in question to the original curve. Or taking the two points I, J conjointly, say the $2m+2n$ common tangents of the absolute and the evolute are made up of the line IJ (or axis of the absolute) counting m times, and of the $2n$ focal tangents of the original curve. The focal tangents of the original curve and of the evolute are thus the same $2n$ lines; and the two curves have the same foci.

The above are the ordinary values of m'', n'', ι'', κ'', but if the given curve touch the line IJ, then the evolute has at the point of contact an inflexion, the stationary tangent being the line IJ; and if the given curve pass through one or other of the points I, J, the evolute has in this case an inflexion on the tangent at the point in question, this tangent being the stationary tangent of the evolute: but observe that the inflexion is not at the point I or J in question: and for each inflexion there is a diminution $=1$ in the class, 3 in the order, and 5 in the number of cusps. Suppose that the point I is a f_2-tuple point on the given curve; then the evolute has f_1 inflexions; and similarly if the point J is a f_2-tuple point on the given curve, then the evolute has f_2 inflexions. Hence writing $f_1+f_2=f$, we have thus f inflexions; and if moreover the number of contacts with the line IJ be $=g$, then we have on this account g inflexions; or in all $f+g$ inflexions, and the formulæ become

$$\begin{aligned} m'' &= \alpha - 3f - 3g, \\ n'' &= m+n - f - g, \\ \iota'' &= f + g, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f - 5g. \end{aligned}$$

It is to be noticed here that the number of the intersections of the given curve with the line IJ (other than the points I, J and the points of contact) is $=m-f_1-f_2-2g$, that is $m-f-2g$: each of these gives as before a cusp on the evolute, the cuspidal tangent being IJ; we have besides on the line IJ (in respect of the g contacts) g inflexions, the stationary tangent being the line IJ; and each of the ι inflexions gives for the evolute a point on the line IJ; hence the whole number of intersections with the line IJ is $3(m-f-2g)+3g+\iota$, $=3m+\iota-3f-3g$, which is thus the order of the evolute.

The tangents from the point I or J to the evolute are the line IJ counting $m-f-2g$ times in respect of the cusps on this line and $2g$ times in respect to the inflexions, that is $m-f$ times; the tangents at the point in question to the given curve each twice as touching the evolute at an inflexion, $2f_1$ or $2f_2$: and the remaining

$n-2f_1-g$, or $n-2f_2-g$ tangents from the point in question to the given curve; the whole number is thus $(m-f)+2f_1+(n-2f_1-g)$ or $(m-f)+2f_2+(n-2f_2-g)$, $=m+n-f-g$, the class of the evolute. The two values of n'' give

$$2n''=2m+(n-2f_1-g)+(n-2f_2-g),$$

viz. twice the class of the evolute = twice the order of the curve + the number of the focal I-tangents + that of the focal J-tangents; but this is not true for all relations whatever of the curve to the absolute.

The tangents from I to the given curve (excluding the line IJ and the tangents at I) are $n-2f_1-g$ tangents; and similarly the tangents from I to the evolute (excluding the line IJ and the stationary tangents through I) are the same $n-2f_1-g$ tangents; say the curve and the evolute have the same $n-2f_1-g$ I-tangents. Similarly they have the same $n-2f_2-g$ J-tangents; or together the same $2(n-f_1-f_2-g)$, $=2(n-f-g)$ focal tangents. And the curve and evolute have the same $(n-2f_1-g)(n-2f_2-g)$ foci.

The foregoing specialities f and g refer, g to the ordinary contacts of the line IJ with the curve, viz. the curve is supposed to have with the curve at an ordinary or non-singular point thereof a contact or 2-pointic intersection, and f, that is f_1 or f_2, to the multiple points having f_1 or f_2 ordinary branches, none of them touching the line IJ. Thus the formulæ do not apply to the cases of IJ passing through a node or a cusp of the given curve, or touching it at an inflexion; nor to the cases where at I or J the curve touches IJ, or where there is at I or J an ordinary double point with one of its branches touching IJ, or where there is at I or J a cusp, where the cuspidal tangent is or is not IJ.

It is easy to see that in the case of a multiple point of any kind whether situate on IJ or at I or J, each branch of the curve produces its own separate effect on the singularities of the evolute: thus if we have on IJ a double point neither branch touching IJ, then the separate effect of each branch is *nil*, therefore the effect of the double point is also *nil*: but if one branch touch the line IJ, then the whole effect is the same as if we had this branch only; viz. we have here the case $g=1$. And so if there is at I or J a double point with one branch touching the line IJ, then the effect of this branch is as if we had this branch only (a case not yet investigated) but the other branch is the case $f=1$. And so if we have at I or J a double point with two ordinary branches touching each other (tacnode or, if the two branches have a contact higher than the first order, oscnode), then if the branches do not touch the line IJ the case is $f=2$, but if they do, then the effect is twice that of an ordinary branch touching IJ. In support of these conclusions, observe that such multiple points, with ordinary branches, present themselves in the case of two or more curves which intersect or touch each other in any manner; and that the evolute of a system of two or more curves is simply the system of the evolutes of the several curves.

It follows as regards the relations of the given curve to the points I, J, and the effect thereby produced on the evolute, we only need to consider the case of a single branch; viz. the cases are

the given curve intersects the line IJ at a point other than I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ;

and the given curve passes through the point I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ.

I have succeeded in determining the effect, not for a singular branch of any kind whatever, but for branches of the form $y = x^k$, $y^{k-1} = x^k$; viz. $k = 2$, each of these is an ordinary branch, $k = 3$, the first $y = x^3$ is an inflexional branch and the second $y = x^{\frac{3}{2}}$ a cuspidal branch; and so $k > 3$ the two branches are respectively inflexional and cuspidal of a higher order. I do this very simply by consideration of the curve $x^{k-1} z = y^k$.

The curve in question $x^{k-1} z = y^k$, is a unicursal curve, and it has a reciprocal of the same form $X^{k-1} Z = Y^k$, hence

$$m = n = k\,; \quad 0 = n - 2m + 2 + \kappa,$$

whence

$$\iota = \kappa = k - 2, \quad \tau = \delta = \tfrac{1}{2}(k-2)(k-3)\,;$$

viz. the point $x = 0$, $y = 0$ is a cusp equivalent to $k - 2$ cusps and $\frac{1}{2}(k-2)(k-3)$ nodes; and the point $z = 0$, $y = 0$ is an inflexion equivalent to $k - 2$ inflexions and $\frac{1}{2}(k-2)(k-3)$ bitangents.

The equation $U = x^{k-1} z - y^k = 0$ of the curve is satisfied by writing therein $x : y : z = 1 : \theta : \theta^k$; and these values give

$$d_x U : d_y U : d_z U = (k-1) x^{k-2} z : -k y^{k-1} : x^{k-1}, = (k-1)\theta^k : -k\theta^{k-1} : 1.$$

Taking the coordinates of I, J to be (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively, and X, Y, Z as current coordinates, the equation of the normal at the point (x, y, z) of the curve $U = 0$ is readily found to be

$$(\alpha' d_x U + \beta' d_y U + \gamma' d_z U) \begin{vmatrix} X, & Y, & Z \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix}$$

$$+ (\alpha\, d_x U + \beta\, d_y U + \gamma\, d_z U) \begin{vmatrix} X, & Y, & Z \\ \alpha', & \beta', & \gamma' \\ x, & y, & z \end{vmatrix} = 0.$$

Hence for the curve in question the equation of the normal is

$$\{(k-1)\alpha'\theta^k - k\beta'\theta^{k-1} + \gamma'\}\{(\beta X - \alpha Y)\theta^k + (\alpha Z - \gamma X)\theta + (\gamma Y - \beta Z)\}$$
$$+ \{(k-1)\alpha\theta^k - k\beta\theta^{k-1} + \gamma\}.\{(\beta' X - \alpha' Y)\theta^k + (\alpha' Z - \gamma' X)\theta + (\gamma' Y - \beta' Z)\} = 0,$$

or, expanding and reducing, this equation is

$$\begin{aligned}
&\theta^{2k}.(k-1)\ \{(\alpha\beta' + \alpha'\beta)X - 2\alpha\alpha' Y\}\\
&+\theta^{2k-1}.\quad -k\{2\beta\beta' X - (\alpha\beta' + \alpha'\beta)Y\}\\
&+\theta^{k+1}.(k-1)\{2\alpha\alpha' Z - (\alpha\gamma' + \alpha'\gamma)X\}\\
&+\theta^{k}.\{(k+1)(\beta\gamma' + \beta'\gamma)X + (k-2)(\gamma\alpha' + \gamma'\alpha)Y - (2k-1)(\alpha\beta' + \alpha'\beta)Z\}\\
&+\theta^{k-1}.\quad -k\{(\beta\gamma' + \beta'\gamma)Y - 2\beta\beta' Z\}\\
&+\theta\quad .\qquad \{(\gamma\alpha' + \gamma'\alpha)Z - 2\gamma\gamma' X\}\\
&+\qquad\qquad \{2\gamma\gamma' Y - (\beta\gamma' + \beta'\gamma)Z\} = 0,
\end{aligned}$$

where k is a positive integer not less than 2; hence except in the case $k=2$, all the terms θ^{2k}, $\theta^{2k-1}, \ldots \theta$, θ^0, have different indices, and the coefficients $k-1$, k, &c. none of them vanish; if however $k=2$, then the terms θ^{2k-1}, θ^{k+1} coalesce into a single term, as do also the terms θ^{k-1} and θ; moreover the coefficient $k-2$ is $=0$.

The evolute is the envelope of the line represented by the foregoing equation, considering therein θ as an arbitrary parameter; viz. the equation is obtained by equating to zero the discriminant of the foregoing equation in θ. Hence in general the class of the evolute is $=2k$, and its order is $=2(2k-1)$; results which agree with the formulæ for n'', m'', since in the present case $m+n$, $=k+k$, $=2k$, $\alpha = 3n+\kappa$, $=3k+(k-2)$, $=4k-2$. And moreover there are not any inflexions, $\iota''=0$ as before.

The equation may however contain a factor in θ independent of (X, Y, Z), and throwing out this factor, say its order is $=s$, the expression for the class is $2k-s$, $=m+n-s$, and that for the order is $4k-2-2s$, $=\alpha-2s$. Moreover, in the original equation or in the equation thus reduced, it may happen that the equation will on writing therein $\Omega=0$ (Ω a linear function of X, Y, Z) acquire a factor of the order ω, independent of (X, Y, Z); the line $\Omega=0$ is in this case a stationary tangent, $=\omega-1$ inflexions; and the discriminant contains the factor $\Omega^{\omega-1}$, which may be thrown out; that is we have here $n''=2k-s$, $\iota''=\omega-1$, $m''=4k-2-2s-(\omega-1)$; agreeing with the relation $m''-2n''+2+\iota''=0$ which holds good in virtue of the evolute being a unicursal curve. It is in this manner that the values of m'', n'', ι'' are obtained in the several cases to be considered, viz.:

A_k Inflexion situate on IJ, which is not a tangent.
B_k Inflexion situate on IJ, which is a tangent.
C_k Cusp situate on IJ, which is not a tangent.
D_k Cusp situate on IJ, which is a tangent.
P_k Inflexion at J, IJ not a tangent.
Q_k Inflexion at J, IJ a tangent.
R_k Cusp at J, IJ not a tangent.
S_k Cusp at J, IJ a tangent.

The results are respectively as follows:

	A_k	B_k	C_k	D_k	P_k	Q_k	R_k	S_k
$m'' = \alpha -$	0	$3k-3$	$k-2$	$k+1$	k	$3k-2$	$2k-2$	$2k$
$n'' = m+n -$	0	$k-1$	0	1	1	k	$k-1$	k
$\iota'' = 0 +$	0	$k-1$	$k-2$	$k-1$	$k-2$	$k-2$	0	0
$\kappa'' = -3m-3n+3\alpha -$	0	$5k-5$	$2k-4$	$2k+1$	$2k-1$	$5k-4$	$3k-3$	$3k$

	A_2C_2	B_2D_2	A_3	B_3	C_3	D_3	$\bar{P}_2\bar{R}_2$	Q_2S_2	P_3	Q_3	R_3	S_3
$m'' = \alpha -$	0	3	0	6	0	4	3	4	3	7	4	6
$n'' = m+n -$	0	1	0	2	0	1	1	2	1	3	2	3
$\iota'' = 0 +$	0	1	0	2	0	2	1	0	1	1	0	0
$\kappa'' = -3m-3n+3\alpha -$	0	5	0	10	0	7	5	6	5	11	6	9

read for instance in B_k, $m'' = \alpha - (3k-3)$, $n'' = m+n-(k-1)$, $\iota'' = 0+(k-1)$, and $\kappa'' = -3m-3n+3\alpha+(5k-5)$; and so in other cases.

A_2C_2 (that is indifferently A_2 or C_2) is when there is on IJ an ordinary point, IJ not a tangent; and so B_2D_2 when there is on IJ an ordinary point, IJ a tangent. Similarly P_2R_2 when there is at J an ordinary point, IJ not a tangent; only instead thereof I have written $\bar{P}_2\bar{R}_2$ to indicate that (for a reason which will appear) the numbers are *not* deducible from those for P_2 or R_2 by writing therein $k=2$; and Q_2S_2 is when there is at J an ordinary point, IJ a tangent.

Case A_k. We have to take the line IJ passing through the inflexion; the condition for this is $\beta\gamma' - \beta'\gamma = 0$: there is no speciality, or we have $n'' = 2k$, $m'' = 4k-2$, $\iota'' = 0$; whence also $\kappa'' = 0$; the value of κ'' being in every case deduced from those of m'', n'', ι'' by the formula

$$3m'' + \iota'' = 3n'' + \kappa''.$$

Case B_k. I write $\gamma = \gamma' = 0$, the equation of the normal is

$$\begin{array}{lll|ll}
 & \theta^{2k} & .(k-1)\{(\alpha\beta'+\alpha'\beta)X - 2\alpha\alpha'Y\} & & \theta^{k+1}(X,\ Y) \\
+ & \theta^{2k-1} & .-k\{2\beta\beta'X - (\alpha\beta'+\alpha'\beta)Y\} & + & \theta^{k}\ (X,\ Y) \\
+ & \theta^{k+1} & .(k-1)\,2\alpha\alpha'Z & + & \theta^{2}\quad Z \\
+ & \theta^{k} & .-(2k-1)(\alpha\beta'+\alpha'\beta)Z & + & \theta\quad Z \\
+ & \theta^{k-1} & .\ k.\quad 2\beta\beta'Z \qquad = 0, & + & \quad Z = 0,
\end{array}$$

where throwing out the factor θ^{k-1}, the form is as shown on the right hand. Writing $Z=0$, we have a factor θ^k, whence $\iota''=k-1$, and then $n''=k+1$, $m''=2k-(k-1)=k+1$, agreeing with the table. The process holds good for $k=2$.

Case C_k. I write $\beta=\beta'=0$; this brings as well the inflexion as the cusp upon the line IJ; but it has been seen (Case A_k) that there is not any reduction on account of this position of the inflexion, hence the whole effect will be due to the cusp. The equation is

$$\begin{array}{ll|ll}
 & \theta^{2k}\,.(k-1)\{-2\alpha\alpha'Y\} & & \theta^{2k}\quad Y \\
+ & \theta^{k+1}\,.(k-1)\{2\alpha\alpha'Z-(\alpha\gamma'+\alpha'\gamma)X\} & + & \theta^{k+1}(Z,\ X) \\
+ & \theta^{k}\,.(k-2)(\gamma\alpha'+\gamma'\alpha)Y & + & \theta^{k}(k-2)Y \\
+ & \theta\quad .\qquad\{(\gamma\alpha'+\gamma'\alpha)Z-2\gamma\gamma'X\} & + & \theta\ (Z,\ X) \\
+ & \qquad 2\gamma\gamma'Y=0, & + & \qquad Y=0,
\end{array}$$

so that here $n''=2k$. On writing $Y=0$, there is a factor $\left(1-\dfrac{\theta}{\infty}\right)^{k-1}$ thrown out (indicated by the reduction of the order from $2k$ to $k+1$), whence

$$\iota''=k-2,\quad m''=2(2k-1)-(k-2),\quad =3k.$$

The process holds good for $k=2$.

Case D_k. We may write $\alpha=\alpha'=0$; the equation is

$$\begin{array}{ll|ll}
 & \theta^{2k-1}\,.-k\,.\,2\beta\beta'X & & \theta^{2k-1}\ X \\
+ & \theta^{k}\quad .\quad (k+1)(\beta\gamma'+\beta'\gamma)X & + & \theta^{k}\quad X \\
+ & \theta^{k-1}\,.-k\,[(\beta\gamma'+\beta'\gamma)Y-2\beta\beta'Z] & + & \theta^{k-1}(Y,\ Z) \\
+ & \theta\quad .-2\gamma\gamma'X & + & \theta\quad X \\
 & \qquad +2\gamma\gamma'Y-(\beta\gamma'+\beta'\gamma)Z=0, & + & \qquad (Y,\ Z)=0,
\end{array}$$

so that $n''=2k-1$. Writing $X=0$, we have the factor $\left(1-\dfrac{\theta}{\infty}\right)^{k}$ (indicated by the reduction of order from $2k-1$ to $k-1$), whence $\iota''=k-1$, and then $m''=(4k-2)-2-(k-1)$, $=3k-3$, agreeing with the table. The process holds good for $k=2$.

Case P_k. We have $\beta'=\gamma'=0$; the equation is

$$\begin{array}{ll|ll}
 & \theta^{2k}\,.(k-1)\{\alpha'\beta X-2\alpha\alpha'Y\} & & \theta^{2k-1}(X,\ Y) \\
+ & \theta^{2k-1}.-k\qquad\{-\alpha'\beta Y\} & + & \theta^{2k-2}Y \\
+ & \theta^{k+1}\,.(k-1)\{-\alpha'\gamma X\} & + & \theta^{k}\quad X \\
+ & \theta^{k}\quad .(k-2)\gamma\alpha'Y-(2k-1)\alpha'\beta Z & + & \theta^{k-1}(k-2)Y+Z \\
+ & \theta\quad .\qquad \gamma\alpha'Z=0, & + & \qquad Z=0,
\end{array}$$

so that here $n''=2k-1$. Writing $Z=0$, we have the factor θ^{k-1}, whence $\iota''=k-2$, $m''=4k-4-(k-2)=3k-2$, agreeing with the table. If, however, $k=2$, then on writing $Z=0$ the equation (instead of the factor θ^{k-1}) acquires the factor $\theta^k\,(=\theta^2)$; so that here $n''=3$, $\iota''=1$, $m''=3$, agreeing with the column $\bar{P}_2\bar{R}_2$ of the table.

Case Q_k. We have $\gamma=0$, $\beta'=\gamma'=0$, viz. $\beta'=0$ in the formulæ of B_k. The equation is

$$\begin{array}{l|l}
\theta^{2k} \;.\; (k-1)\,\alpha'\beta Y & \theta^{k}\; X \\
+\,\theta^{2k-1}\,.-k\,(-\alpha'\beta Y) & +\,\theta^{k-1} Y \\
+\,\theta^{k+1}\;.\;(k-1)\,2\alpha\alpha' Z & +\,\theta\;\; Z \\
+\,\theta^{k}\;\;.-(2k-1)\,\alpha'\beta Z=0, & +\;\;\; Z=0,
\end{array}$$

so that $n''=k$. For $Z=0$ there is the factor θ^{k-1}, hence $\iota''=k-2$, $m''=2(k-1)-(k-2)$, $=k$. The process holds good for $k=2$.

Case R_k. We have $\beta=0$, $\alpha'=0$, $\beta'=0$, viz. $\alpha'=0$ in the formulæ of C_k. The equation is

$$\begin{array}{l|l}
\theta^{k+1}\,.\,(k-1)\,(-\alpha\gamma' X) & \theta^{k+1} X \\
+\,\theta^{k}\;\;.\,(k-2)\,(\alpha\gamma' Y) & +\,\theta^{k}\;\; Y(k-2) \\
+\,\theta\;\;\;.\qquad (\alpha\gamma' Z-2\gamma\gamma' X) & +\,\theta\;\;(Z,\;X) \\
+\qquad\qquad 2\gamma\gamma' Y=0, & +\;\;\; Y=0,
\end{array}$$

so that $n''=k+1$, $\iota''=0$, $m''=2k$. But observe that in the particular case $k=2$, the form is $\theta^3 X+\theta(Z,\;X)+Y=0$, the term $\theta^k Y$ disappearing on account of the factor $k-2$. Here on writing $X=0$, there is a factor $\left(1-\dfrac{\theta}{\infty}\right)^2$ (indicated by the reduction of order from 3 to 1), hence $\iota''=1$, $n''=3$, $m''=2\,.\,2-1=3$, agreeing with the column $\overline{P}_2\overline{R}_2$.

Case S_k. We have $\alpha=0$, $\alpha'=\beta'=0$, viz. $\beta'=0$ in the formulæ of D_k. The equation is

$$\begin{array}{l|l}
\theta^{k}\;\;.\,(k+1)\,\beta\gamma' X & \theta^{k}\;\; X \\
+\,\theta^{k-1}.\qquad -k\beta\gamma' Y & +\,\theta^{k-1}\, Y \\
+\,\theta\;\;\;.\qquad -2\gamma\gamma' X & +\,\theta\;\;\; X \\
+\qquad\qquad 2\gamma\gamma' Y-\beta\gamma' Z=0, & +\;\;\; Y+Z=0,
\end{array}$$

so that $n''=k$, $\iota''=0$, $m''=2k-2$. The process applies to the case $k=2$.

As to the formula for A_3, B_3, ... S_3, there is nothing special in these; they are simply deduced from those for A_k, B_k, ... S_k by writing therein $k=3$. And we have thus the foregoing series of formulæ, which will apply to the greater part of the cases which ordinarily arise. For instance suppose there is at I or J a triple point = cusp + 2 nodes; there is here an ordinary branch and a cuspidal (ordinary cuspidal) branch and according as IJ touches neither branch, the ordinary branch, or the cuspidal branch, the corrections to m'', n'', ι'', κ'' are $\overline{R}_2+R_3$, S_2+R_3, $\overline{R}_2+S_3$ respectively. Observe moreover that A_2C_2 is no speciality, B_2D_2 is the speciality $g=1$, $\overline{P}_2\overline{R}_2$ the speciality $f=1$.

There is a remarkable case in which the fundamental assumption of the (1, 1) correspondence of the evolute with the original curve ceases to be correct. In fact,

in the case about to be considered of a parallel curve; the parallel to any given curve is *in general* a curve not breaking up into two distinct curves of the same order with such given curve, and when this is so (viz. when the parallel curve does not break up) each normal of the parallel curve is a normal at two distinct points thereof: the evolute of the parallel curve is thus the evolute of the given curve taken twice; and the parallel curve and its evolute have not a (1, 1) but (1, 2) correspondence. Hence, (m, n, δ, κ, τ, ι) the unaccented letters referring to the parallel curve, or say rather to a curve which has a (1, 2) correspondence with its evolute, and, as before, the twice accented letters to the evolute, it is *not true* that $m'' - 2n'' + \iota'' = m - 2n + \iota$; it will subsequently appear that the values of m'', n'' are correct, those of ι'', κ'' suffering a modification; viz. the formulæ are

$$\begin{aligned} m'' &= \qquad\qquad\quad \alpha - 3f - 3g, \\ n'' &= \quad m + n \quad\quad - f - g, \\ \iota'' &= \qquad\qquad\qquad\quad f + g - \Theta, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f - 5g - \Theta, \end{aligned}$$

where, for the present, I leave Θ undetermined.

Coming now to the parallel curve, let the numbers in regard to it be m', n', δ', κ', τ', ι'; α', D'. Supposing in the first instance that the given curve does not stand in any special relation to I, J, the formulæ are

$$\begin{aligned} m' &= 2m + 2n, & \alpha &= 6n + 2\alpha, \\ n' &= 2n, & 2D' &= -4m + 2 + 2\alpha, = -2m + 2n + \alpha. \\ \iota' &= 2\alpha - 6m, \\ \kappa' &= 2\alpha. \end{aligned}$$

Considering the parallel curve as the envelope of a circle of constant radius having its centre on the given curve, it appears (e.g. by consideration of the case of the ellipse) that when the radius of the circle is $=0$, there is not any depression in the order of the parallel curve, but that the parallel curve reduces itself to the given curve twice, together with the system of tangents from the points I, J to the given curve: the order of the parallel curve is thus $m' = 2m + 2n$.

To find the class, consider the tangents from a given point to the parallel curve; about the point as centre describe a circle, radius k; then the tangents in question are respectively parallel to, and correspond each to each with, the common tangents of the circle and the given curve, and the number of these is $= 2n$, that is $n' = 2n$.

Each inflexion of the given curve gives rise to two inflexions of the parallel curve; and the inflexions of the parallel curve arise in this way only: that is $\iota' = 2\iota$, $= 2\alpha - 6m$. And the Plückerian relations then give $\kappa' = 2\alpha$; a value which may be investigated independently.

Attending now to the singularities f and g; the values of n', ι' are unaltered: to obtain m' we as before consider the particular case where the radius of the variable

circle is $=0$: the parallel curve here breaks up into the original curve, together with the focal tangents from the points I, J; viz. we have

$$m' = 2n + (n - 2f_1 - g) + (n - 2f_2 - g), \quad = 2m + 2n - 2f - 2g;$$

and knowing m', n', ι' we have κ'.

The points on IJ of the original curve are I, J counting as f_1 and f_2 respectively; or together as f points: the points of contact counting as $2g$: and besides $m-f-2g$ points. As regards the parallel curve, we have the same points on IJ; but here I is $(n-g)$ tuple point, having in respect of each branch of the f_1-tuple point on the original curve a pair of branches touching each other, and in respect of each of the tangents from I to the given curve a single branch, together $2f_1 + (n - 2f_1 - g)$, $= n - g$ branches; and thus counting $n-g$ times: similarly J counts $n-g$ times. Hence also for the parallel curve $f_1' = f_2' = n - g$. In respect of each of the points g, we have a point where there are two branches touching each other and the line IJ; and thus counting 4 times, or together as $4g$: moreover, on account of the two branches at each of these points, $g' = 2g$. Lastly, each of the $m-f-2g$ points is a node on the parallel curve; and as such counts twice; $m' = 2(n-g) + 4g + 2(m-f-2g)$, $= 2m + 2n - 2f - 2g$ as above.

And we have thus the formulæ

$$\begin{aligned}
m' &= 2m + 2n - 2f - 2g, \\
n' &= 2n, \\
\iota' &= -6m + 2\alpha, = 2\iota, \\
\kappa' &= 2\alpha - 6f - 6g, = 6n + 2\kappa - 6f - 6g, \\
f' &= 2n - 2g, \\
g' &= 2g,
\end{aligned}$$

where observe that m' is $=2n''$; that is, twice the class of the evolute (which relation however is not in all cases true for a curve with singularities); and further that $n' - f' - g'$ is $=0$.

The case of a curve for which $n-f-g=0$ is very interesting and remarkable. Recurring to the formulæ for the evolute, we have here $m'' = \kappa$, $n'' = m$, $\iota'' = n$, $\kappa'' = n - 3m + 3\kappa$. And for the parallel curve $m' = 2m$, $n' = 2n$, $\iota' = 2\iota$, $\kappa' = 2\kappa$, $f' = 2f$, $g' = 2g$; formulæ which lead to the assumption that the parallel curve here breaks up into two distinct curves, each such as the given curve.

Observe further that for a curve possessing the singularities f and g, but where $n-f-g$ is not $=0$; then for the parallel curve we have as above $n'-f'-g'=0$; or the parallel of the parallel curve should, according to the assumption, break up into two distinct curves such as the parallel curve; this is of course correct.

Consider the evolute of the parallel curve: since for the parallel curve $n'-f'-g'=0$, the formulæ for the evolute thereof (viz. those containing the undetermined quantity Θ)

are $m''' = \kappa'$, $n''' = m'$, $\iota''' = n' - \Theta$, $\kappa''' = n' - 3m' + 3\kappa' - \Theta$, or substituting for m', n', κ' their values, and comparing with the formulæ in regard to the evolute, we have

$$\begin{aligned}
m''' &= 2\alpha - 6f - 6g, && = 2m'',\\
n''' &= 2m + 2n - 2f - 2g, && = 2n'',\\
\iota''' &= 2n - \Theta, && = 2\iota'' + 2(n - f - g) - \Theta,\\
\kappa''' &= -6m - 4n + 6\alpha - 12f - 12g - \Theta, && = 2\kappa'' + 2(n - f - g) - \Theta,
\end{aligned}$$

where m'', n'', ι'', κ'' refer to the evolute. Hence by assuming $\Theta = 2(n - f - g)$, the values of m''', n''', ι''', κ''' become $2m''$, $2n''$, $2\iota''$, $2\kappa''$, viz. the evolute of the parallel curve is the evolute of the original curve taken twice. Observe that in the foregoing value of Θ, the letters n, f, g refer not to the parallel curve, the evolute whereof is under consideration, but to the curve from which such parallel curve was derived; this value $\Theta = 2(n - f - g)$ is not a value of Θ applicable to be substituted in the evolute-formulæ for the case of a curve which has with its evolute a (1, 2) correspondence.

Instead of the foregoing case of the f, viz. f- and g-singularities, we may, as regards the parallel curve, consider the original curve as having any I- and J-singularities whatever. Suppose in this case (excluding always the line IJ and the tangents at I or J) the number of tangents from I to the curve is $= n - I$, and the number of tangents from J to the curve is $= n - J$, then when the radius of the variable curve is $= 0$, the parallel curve becomes the original curve twice together with the $(n - I) + (n - J)$, $= 2n - I - J$ tangents; so that the order is $m' = 2m + 2n - I - J$[1]; we have, as before, $n' = 2n$ and $\iota' = 2\iota$, and these values give κ', so that the equations are

$$\begin{aligned}
m' &= 2m + 2n - I - J,\\
n' &= 2n,\\
\iota' &= -6m + 2\alpha, && = 2\iota,\\
\kappa' &= 2\alpha - I - J, && = 6n + 2\kappa - 3I - 3J.
\end{aligned}$$

Suppose $2n - I - J = 0$; this implies $n - I = 0$, $n - J = 0$ since neither $n - I$ nor $n - J$ can be negative; viz. that there are no I- or J-tangents; and conversely, when this is the case $2n - I - J = 0$: and we have then m', n', ι', $\kappa' = 2m$, $2n$, 2ι, 2κ; viz. it is assumed, as before, that the parallel curve breaks up into two distinct curves such as the original curve; that is, the condition in order that the parallel curve should break up, is that the original curve has no focal tangents. Observe that the number of foci is $= (n - I)(n - J)$ which is $= 0$ if only $n - I = 0$ or $n - J = 0$; but as regards real curves $I = J$, so that the equations $n - I = 0$ and $n - J = 0$ are one and the same equation, satisfied if $(n - I)(n - J) = 0$; so that for a real curve without foci (real or imaginary) the parallel curve will break up. An instance given to me by Dr Salmon is the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} - c^{\frac{2}{3}} = 0$ or $(x^2 + y^2 - c^2)^3 + 27c^2x^2y^2 = 0$, here $m = 6$, $n = 4$,

[1] That the order of the evolute is not (in every case of a curve with singularities) one-half this, or $= m + n - \frac{1}{2}(I + J)$, is at once seen by remarking that there is no reason why $I + J$ should be even.

$\delta = 4$, $\kappa = 6$, $\iota = 0$, $\tau = 3$: the points I, J are each of them a cusp, the tangents being the line IJ; the number of tangents from a cusp is $n - 3$, $= 1$, but for the cusp I or J, this tangent is the line IJ itself, so that we have $I = J = 4$.

Theory when the Absolute is a conic.

When the Absolute is a conic the formulæ for the evolute are essentially the same as those in the former case, but the formulæ for the parallel curve are modified essentially and in a very remarkable manner. I observe that corresponding to a passage of the given curve through I or J we have a contact with the Absolute, so that in the present case f will properly denote the number of contacts of the given curve with the Absolute, and attending to this singularity only, viz. considering a given curve $(m, n, \delta, \kappa, \iota, \tau; \alpha, D)$ having f contacts with the Absolute, the formulæ for the evolute are

$$\begin{aligned} m'' &= \qquad\qquad \alpha - 3f, \\ n'' &= m + n \qquad - f, \\ \iota'' &= \qquad\qquad\quad f, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f. \end{aligned}$$

In the case $f = 0$, these at once follow from the two equations $n'' = m + n$, and $\iota'' = 0$. The normal is the line joining a point of the given curve with the pole of the tangent; or, what is the same thing, it is the line joining the point of the given curve with the corresponding point of the reciprocal curve: the degrees of the two curves are m, n, and the correspondence is a (1, 1) correspondence. Hence, by the general theorem previously referred to, it follows that we have $n'' = m + n$, and $\iota'' = 0$. Compare herewith the demonstration of the theorem in the case where the Absolute is a point-pair.

The formulæ for the parallel curve are

$$\begin{aligned} m' &= 2m + 2n - 2f, \\ n' &= 2m + 2n - 2f, \\ \iota' &= \qquad 2\alpha - 6f, \\ \kappa' &= \qquad 2\alpha - 6f, \\ f' &= 2m + 2n - 2f, \end{aligned}$$

(so that $m' = n'$, $\iota' = \kappa'$). The intersections of the curve and Absolute are in this case the points f each twice, and besides $2m - 2f$ points; similarly the common tangents are the tangents at f each twice and besides $2n - 2f$ tangents. Now I remark that the parallel curve, when the radius of the variable circle is $= 0$, reduces itself to the original curve twice, together with the $2n - 2f$ common tangents, and the $2m - 2f$ common points; the order is thus $= 2m + (2n - 2f)$, and the class $= 2n + (2m - 2f)$: and these are the values in the general case where the radius of the variable circle is not $= 0$.

But in the remarkable case where the curve and its evolute have a (1, 2) correspondence, then I correct the formulæ by adding $-\Theta$ to the expressions for ι', κ' respectively. We have for the evolute of the parallel curve

$$\begin{aligned} m''' &= 2m'', \\ n''' &= 2n'', \\ \iota''' &= 2\iota'' + (2m + 2n - 4f) - \Theta, \\ \kappa''' &= 2\kappa'' + (2m + 2n - 4f) - \Theta, \end{aligned}$$

viz. assuming $\Theta = 2m + 2n - 4f$, this means that the evolute is the evolute of the original curve taken twice.

A very interesting case is when $m = n = f$: observe that neither $m - f$ nor $n - f$ can be negative, so that the assumed relation $m + n - 2f = 0$ would imply these two relations. We have here for the parallel curve $m' = 2m$, $n' = 2n$, $\iota' = 2\iota$, $\kappa' = 2\kappa$; the parallel curve in fact breaking up into two curves such as the given curve. And in this case the formulæ for the evolute assume the very simple form $m'' = \kappa$, $n'' = f$, $\iota'' = f$, $\kappa'' = -2f + 3\kappa$.

Whatever the original curve may be, we have for the parallel curve $m' = n' = f'$, so that the formulæ for the evolute of the parallel curve are of the foregoing form $m''' = \kappa'$, $n''' = f'$, $\iota''' = f' - \Theta$, $\kappa''' = -2f' + 3\kappa' - \Theta$, which agree with the above values of m''', n''', ι''', κ'''. In the particular case $m = n = f$, we have $\Theta = 0$, so that the evolute-formulæ, if originally written down without the terms in Θ, would still be $m''' = 2m''$, $n''' = 2n''$, $\iota''' = 2\iota''$, $\kappa''' = 2\kappa''$; viz. the evolute is here the original evolute taken twice; as already seen, the parallel curve consisted of two curves such as the original curve, and each of these has for its evolute the evolute of the original curve.

494.

EXAMPLE OF A SPECIAL DISCRIMINANT.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 211—213.]

If we have a function $(a, \ldots \between x, y, z)^n$, where the coefficients $(a, \ldots)$ are such that the curve $(a, \ldots \between x, y, z)^n = 0$ has a node, and *à fortiori* if this curve has any number of nodes or cusps, the discriminant of the function (that is, the discriminant of the general function $(* \between x, y, z)^n$, substituting in such discriminant for the coefficients their values for the particular function in question) vanishes *identically*. But the particular function has nevertheless a *special discriminant*, viz. this is a function of the coefficients which, equated to zero, gives the condition that the curve may have (besides the nodes or cusps which it originally possesses) one more node; and the determination of this special discriminant (which, observe, is not deducible from the expression of the discriminant of the general function $(* \between x, y, z)^n$) is an interesting problem. I have, elsewhere, shown that if the curve in question $(a, \ldots \between x, y, z)^n = 0$ has δ nodes and κ cusps, then the degree of the special discriminant in regard to the coefficients a, &c., of the function is $= 3(n-1)^2 - 7\delta - 11\kappa$: and I propose to verify this in the case of a quartic curve with two cusps.

Consider the curve

$$6nx^2y^2 + 12rz^2xy + (4gx + 4iy + cz)\, z^3 = 0,$$

where $x = 0$ is the tangent at a cusp; $y = 0$ the tangent at a cusp; and $z = 0$ the line joining the two cusps.

For the special discriminant we have

$$3nxy^2 + 3ryz^2 + gz^3 = 0,$$

$$3nx^2y + 3rxz^2 + iz^3 = 0,$$

$$z\,\{6rxy + (3gx + 3iy + 4cz)\, z\} = 0\,;$$

the last of which may be replaced by the equation of the curve.

Assume $x=\lambda z$, $y=\mu z$, the first two equations give

$$3(n\lambda\mu+r)\mu+g=0,$$
$$3(n\lambda\mu+r)\lambda+i=0,$$

whence also

$$6n\lambda^2\mu^2+6r\lambda\mu+g\lambda+i\mu=0,$$

and the equation of the curve gives

$$6n\lambda^2\mu^2+12r\lambda\mu+4g\lambda+4i\mu+c=0,$$

whence eliminating $g\lambda+i\mu$ we find

$$18n\lambda^2\mu^2+12r\lambda\mu-c=0.$$

Moreover the first two equations give

$$9(n\lambda\mu-r)^2\lambda\mu-ig=0,$$

or putting $\lambda\mu=\theta$ we have

$$18n\theta^2+12r\theta-c=0,$$
$$9(n\theta+r)^2\theta-ig=0,$$

from which θ is to be eliminated.

The equations are

$$18n\theta^2+12r\theta-c=0,$$
$$9n^2\theta^3+18nr\theta^2+9r^2\theta-ig=0,$$

and thence

$$\begin{aligned}
18n^2\theta^3+36nr\theta^2+18r^2\theta-2ig&=0,\\
18n^2\theta^3+12nr\theta^2-cn\theta\qquad\quad&=0,\\
24nr\theta^2+(18r^2+cn)\theta-2ig&=0,\\
18nr\theta^2\quad+\quad 12r^2\theta-cr&=0,\\
(6r^2+3cn)\theta-6ig+4cr&=0,
\end{aligned}$$

$$\theta=\frac{6ig-4cr}{6r^2+3cn}=\tfrac{2}{3}\,\frac{3ig-2cr}{2r^2+cn};$$

or substituting in $18n\theta^2+12r\theta-c=0$, this is

$$8n(3ig-2cr)^2+8r(3ig-2cr)(2r^2+cn)-c(2r^2+cn)^2=0.$$

Hence, developing, the special discriminant is

$$\begin{aligned}
\square=-\ &1\,c^3n^2\\
&+12\,c^2nr^2\\
&-72\,cginr\\
&-36\,cr^4\\
&+72\,g^2i^2n\\
&+48\,gir^3,
\end{aligned}$$

which is as it should be of the degree 5, $=3\,.\,3^2-11\,.\,2$.

495.

ON THE ENVELOPE OF A CERTAIN QUADRIC SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 244—246.]

To find the envelope of the quadric surface

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

where the coefficients vary subject to the conditions

$$\begin{cases} a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0, \\ \dfrac{p^2}{a} + \dfrac{q^2}{b} + \dfrac{r^2}{c} + \dfrac{s^2}{d} = 0, \end{cases}$$

$(\alpha, \beta, \gamma, \delta)$ and (p, q, r, s) being respectively constant.

We have in the usual manner

$$x^2 + \lambda\alpha^2 + \mu\frac{p^2}{a^2} = 0,$$

$$y^2 + \lambda\beta^2 + \mu\frac{q^2}{b^2} = 0,$$

$$z^2 + \lambda\gamma^2 + \mu\frac{r^2}{c^2} = 0,$$

$$w^2 + \lambda\delta^2 + \mu\frac{s^2}{d^2} = 0$$

and thence $\alpha^2 = \dfrac{-\mu p^2}{x^2 + \lambda\alpha^2}$, &c., and substituting these values μ disappears and we have

$$p\sqrt{(x^2+\lambda\alpha^2)} + q\sqrt{(y^2+\lambda\beta^2)} + r\sqrt{(z^2+\lambda\gamma^2)} + s\sqrt{(w^2+\lambda\delta^2)} = 0,$$

$$\frac{\alpha^2 p}{\sqrt{(x^2+\lambda\alpha^2)}} + \frac{\beta^2 q}{\sqrt{(y^2+\lambda\beta^2)}} + \frac{\gamma^2 r}{\sqrt{(z^2+\lambda\gamma^2)}} + \frac{\delta^2 s}{\sqrt{(w^2+\lambda\delta^2)}} = 0,$$

from which λ is to be eliminated; the second equation is here the derived function of the first in regard to λ, so that rationalising the first equation, the result is, as will be shown, of the form $(*\between\lambda, 1)^4 = 0$, and the result is obtained by equating to zero the discriminant of the quartic function.

Denoting for shortness the first equation by

$$A + B + C + D = 0,$$

the rationalised form is

$$(A^4 + B^4 + C^4 + D^4 - 2A^2B^2 - 2A^2C^2 - 2A^2D^2 - 2B^2C^2 - 2B^2D^2 - 2C^2D^2)^2 - 64A^2B^2C^2D^2 = 0,$$

which is of the form

$$-(\mathfrak{A} + 2\mathfrak{B}\lambda + \mathfrak{C}\lambda^2)^2 + (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e}\between 1,\ \lambda)^4 = 0,$$

where

$$\begin{aligned}
\mathfrak{A} &= p^4x^4 \ldots - 2p^2q^2x^2y^2 \ldots,\\
\mathfrak{B} &= p^4\alpha^2x^2 \ldots - p^2q^2(\alpha^2y^2 + \beta^2x^2) \ldots,\\
\mathfrak{C} &= p^4\alpha^4 \ldots - 2p^2q^2\alpha^2\beta^2 \ldots,\\
\mathrm{a} &= 8\,.\,x^2y^2z^2w^2,\\
4\mathrm{b} &= 8\,.\,\alpha^2y^2z^2w^2 + \ldots,\\
6\mathrm{c} &= 8\,.\,\alpha^2\beta^2z^2w^2 + \ldots,\\
4\mathrm{d} &= 8\,.\,\alpha^2\beta^2\gamma^2w^2 + \ldots,\\
\mathrm{e} &= 8\,.\,\alpha^2\beta^2\gamma^2\delta^2.
\end{aligned}$$

Writing I', J' for the two invariants we find without difficulty

$$\begin{aligned}
I' &= I - \tfrac{4}{3}P + \Delta^2,\\
J' &= J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3,
\end{aligned}$$

where

$$\begin{aligned}
I &= \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2,\\
J &= \mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} - \mathrm{c}^3 + 2\mathrm{bcd},\\
\Delta &= \mathfrak{A}\mathfrak{C} - \mathfrak{B}^2,\\
P &= \mathrm{a}\mathfrak{C}^2 - 4\mathrm{b}\mathfrak{B}\mathfrak{C} + 2\mathrm{c}\,(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) - 4\mathrm{d}\mathfrak{A}\mathfrak{B} + \mathrm{e}\mathfrak{A}^2,\\
Q &= (\mathrm{ce} - \mathrm{d}^2)\,\mathfrak{A}^2 + (\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2)\,.\,\tfrac{1}{3}(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) + (\mathrm{ac} - \mathrm{b}^2)\,\mathfrak{C}^2\\
&\qquad - 2\,(\mathrm{ad} - \mathrm{bc})\,\mathfrak{B}\mathfrak{C} \qquad\qquad - 2\,(\mathrm{be} - \mathrm{cd})\,\mathfrak{A}\mathfrak{B}.
\end{aligned}$$

The result thus is

$$(I - P + \tfrac{4}{3}\Delta^2)^3 - 27\,(J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3)^2 = 0,$$

or, what is the same thing, it is

$$\begin{aligned}(I - P)^3 - 27\,(J - Q)^2 - 9\Delta P\,(J - 2Q)\\ + \;\Delta^2\,(4I^2 - 8IP + P^2)\\ + 8\Delta^3\,(J - 2Q)\\ + \;\Delta^4 \cdot \tfrac{16}{3} I = 0,\end{aligned}$$

where the left-hand side is of the order 24 in (x, y, z, w). I apprehend that the order should be $=12$ only; for writing (x, y, z, w) in place of (x^2, y^2, z^2, w^2), the equations which connect (a, b, c, d) express that these quantities are the coordinates of a point on a plane cubic; and the problem is in fact that of finding the reciprocal of the plane cubic: this is a sextic cone, or restoring (x^2, y^2, z^2, w^2) instead of (x, y, z, w), we should have a surface of the order 12. I cannot explain how the reduction is effected.

496.

TABLES OF THE BINARY CUBIC FORMS FOR THE NEGATIVE DETERMINANTS, $\equiv 0$ (MOD. 4), FROM -4 to -400; AND $\equiv 1$ (MOD. 4), FROM -3 TO -99; AND FOR FIVE IRREGULAR NEGATIVE DETERMINANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 246—261.]

THE theory of binary cubic forms for determinants, as well positive as negative, has been studied by M. Arndt in the memoir "Versuch einer Theorie der homogenen Functionen des dritten Grades mit zwei Variabeln," *Grunert's Archiv*, t. XVII. (1851, pp. 1—54); and in the later memoir, "Tabellarische Berechnung der reducirten binären cubischen Formen und Klassification derselben für alle negativen Determinanten $(-D)$ von $D=3$ bis $D\equiv 2000$," *ditto*, t. XXXI. (1858), pp. 335—445, he has given a very valuable Table of the forms for a Negative Determinant. It has appeared to me suitable to arrange this Table in the manner made use of for Quadratic Forms in my memoir "Tables des formes quadratiques binaires pour les déterminants négatifs $D\equiv -1$ jusqu'à $D\equiv -100$, pour les déterminants positifs non carrés depuis $D\equiv 2$ jusqu'à $D\equiv 99$, et pour les treize déterminants négatifs du premier millier," *Crelle*, t. LX. (1862), pp. 357—372, [335]; and confining myself to the limits of the last-mentioned tables I deduce from that of M. Arndt the three Tables which follow.

To explain the arrangement, I give in the first instance the following extract from M. Arndt's Table:

D.	*Reducirte Formen mit Charakteristik.*	*Klassen.*
3	(0, 1, 1, 0) (1, 0, −1, −1) (1, 1, 0, −1) (2, 1, 2) (2, 1, 2) (2, 1, 2)	(0, 1, 1, 0), (1, 0, −1, ±1)
4	(0, 1, 0, −1) (1, 0, −1, 0) (2, 0, 2) (2, 0, 2)	(0, −1, 0, 1)

D.	*Reducirte Formen mit Charakteristik.*	*Klassen.*
7	(0, 1, 1, − 1) (2, 1, 4)	(0, − 1, − 1, 1)
8	(0, 1, 0, − 2) (2, 0, 4)	(0, − 1, 0, 2)
11	(0, 1, 1, − 2) (2, 1, 6)	(0, − 1, − 1, 2)
12	(0, 1, 0, − 3) (2, 0, 6)	(0, − 1, 0, 3)
15 ⋮	(0, 1, 1, − 3) (2, 1, 8)	(0, − 1, − 1, 3)
44 ⋮	(0, 1, 0, − 11) (1, − 1, − 2, 0) (2, 0, 22) (6, 2, 8)	(0, − 1, 0, 11), (0, − 2, ± 1, 1)
112 ⋮	(0, 1, 0, − 28) (0, 2, 2, − 2) (1, 2, 0, − 4) (2, 0, 56) (8, 4, 16) (8, 4, 16) (1, − 1, − 3, − 1) (8, 4, 16)	(0, − 1, 0, 28), (0, 2, 2, − 2), (1, ± 1, − 3, ± 1)
144	(0, 1, 0, − 36) (0, 2, 2, − 3) (2, 0, 72) (8, 4, 20)	(0, − 1, 0, 36), (0, − 2, − 2, 3)
156	(0, 1, 0, − 39) (1, − 1, − 3, 1) (2, 0, 78) (8, 2, 20)	(0, − 1, 0, 39), (1, ∓ 1, − 3, ± 1)
216	(0, 1, 0, − 54) (1, − 2, − 3, 0) (2, 0, − 3, 0) (2, 0, 108) (14, 6, 18) (12, 0, 18)	(0, − 1, 0, 54), (0, ∓ 3, 0, ± 2), (0, ∓ 3, 2, ± 1)

The first column contains the value of the determinant, the second column contains the *reduced* forms, omitting the *contrary* and *opposite* forms; viz. for the cubic form (a, b, c, d), the contrary form (equal, that is, properly equivalent to the given form) is $(-a, -b, -c, -d)$; and the opposite form (improperly equivalent to the given form) is $(a, -b, c, -d)$ or $(-a, b, -c, d)$; this second column contains also the *characteristic* of each cubic form, viz. the cubic form (a, b, c, d) has for its characteristic the quadratic form

$$\{2(b^2 - ac),\ bc - ad,\ 2(c^2 - bd)\},$$

(so that the cubic form and its characteristic have the same determinant

$$-D = (bc - ad)^2 - 4(b^2 - ac)(c^2 - bd),\ \equiv 1 \text{ or } 0 \pmod{4}),$$

and a cubic form which corresponds to a reduced characteristic is itself a reduced form. The third column contains for each determinant the entire series of *unequal* cubic forms (that is of the forms whereof no two are properly equivalent to each other), the representatives of the *classes* for this determinant. M. Arndt has included in his table the non-primitive classes (for example Det. = − 112, the form (0, 2, 2, − 2)), for which the terms (a, b, c, d) have a common divisor μ, but as these are at once

deducible from the classes which belong to the determinant $= -\frac{D}{\mu^4}$, it seems better to omit the non-primitive classes.

The two opposite forms included in a single expression by means of the sign $\pm$ have opposite characteristics which are for the most part unequal to each other, for instance

Det. -44; $(0, -2, 1, 1)$ has the characteristic $(6, -2, 8)$,
$(0, 2, 1, -1)$ " " $(6, 2, 8)$,

where $(6, -2, 8)$, $(6, 2, 8)$ are unequal forms, but this is not always the case, for instance

Det. -112; $(1, -1, -3, -1)$ has the characteristic $(8, -4, 16)$,
$(1, 1, -3, 1)$ " " $(8, 4, 16)$,

where $(8, -4, 16) = (8, 4, 16)$, since each is an ambiguous form. Instead of the two unequal forms $(1, -1, -3, -1)$, $(1, 1, -3, 1)$ which correspond to the opposite (though equal) characteristics $(8, -4, 16)$, $(8, 4, 16)$, M. Arndt might have given the two forms $(1, 2, 0, -4)$ and $(1, -1, -3, -1)$ corresponding to the *same* characteristic $(8, 4, 16)$; but then it would not have appeared at a glance that the two classes were opposite to each other; and I presume that it is for this reason that he has selected the two representative forms $(1, -1, -3, -1)$ and $(1, 1, -3, 1)$. It must not, however, be imagined that the opposite cubic forms which correspond to opposite characteristics, which are ambiguous (and therefore equal to each other), are always, as in the last preceding example, unequal: for example Det. -144, there is only the form $(0, -2, -2, 3)$ given as corresponding to the ambiguous characteristic $(8, 4, 20)$; the opposite form $(0, 2, -2, -3)$ corresponding to the opposite but equal characteristic $(8, -4, 20)$ is equal to $(0, -2, -2, 3)$, and so does not give rise to a distinct opposite class. In the new tables, the sign $\pm$ is only employed in regard to opposite ambiguous characteristics; for instance, Det. -4×28 there are given (not included in a single expression by means of the sign $\pm$) the two forms $(1, -1, -3, 1)$, $(1, 1, -3, 1)$ corresponding to the characteristic $2(2, \pm 1, 4)$.

I remark that, in a few instances M. Arndt, in passing from the second to the third column, has modified the expression for a cubic form in such manner that the characteristic has ceased to be a reduced form; for instance, Det. -216, he has given in the third column the two forms $(0, \mp 3, 2, \pm 1)$ belonging to the characteristic $(18, \mp 6, 14)$; it would have been better, it appears to me, to preserve the expression of the second column $(1, -2, -3, 0)$, and adopt the two representative forms $(1, \mp 2, -3, 0)$; I have accordingly made this change.

I divide M. Arndt's table into two tables; the first of them corresponding to the determinants $\equiv 0$ (mod. 4), the second to the determinants $\equiv 1$ (mod. 4). In the first table I take for the characteristic the form

$$\{b^2 - ac,\ \tfrac{1}{2}(bc - ad),\ bd - c^2\},$$

which belongs to the determinant $-\frac{1}{4}D$, and I arrange the cubic classes according to their *order*; viz. we have the properly primitive order (pp) when the terms $(a, 3b, 3c, d)$ have no common divisor; and the improperly primitive order (ip) when the terms $(a, 3b, 3c, d)$ have no common divisor other than 3, or what is the same thing when a and d being each of them divisible by 3, the terms (a, b, c, d) have no common divisor. But, moreover, the characteristic $\{b^2-ac, \frac{1}{2}(bc-ad), bd-c^2\}$, may be of the properly primitive order pp; or of the improperly primitive order ip; or it may be of a derived order $\mu(A', B', C')$, $=\mu.pp$ or $\mu.ip$, according as (A', B', C') considered as a form belonging to the determinant $B'^2-A'C'$, $=-\frac{1}{4\mu^2}D$, is of the properly primitive or the improperly primitive order. And in these different cases, the cubic class is said to be of the order pp on pp, pp on ip, pp on $\mu.pp$, pp on $\mu.ip$, ip on pp, &c., as the case may be.

For the determinants $\equiv 1$ (mod. 4), I retain the characteristic

$$\{2(b^2-ac), bc-ad, 2(c^2-bd)\},$$

and this being so, the division into orders is the same as in the former case; only as the characteristic, when primitive, is of necessity improperly primitive, the orders pp on pp and ip on pp no longer exist.

To every characteristic I annex in the tables the symbol of its composition; viz. 1 denotes the principal form, c a form which by its duplication, d a form which by its triplication, &c., produces the principal form, σ denotes the most simple form of order ip, σc, σd, &c., the forms obtained by combining σ with the forms c, d, &c., of the order pp. Similarly to a characteristic $\mu(A', B', C')$ I annex the symbol of composition of the form (A', B', C'), $\left(\text{considered as belonging to the determinant } B'^2-A'C', =-\frac{D}{4\mu^2}\right)$ multiplying this symbol by the number μ; for instance, $\mu.1$ denotes that (A', B', C') is the principal form, and similarly in other cases.

I have given a third table for the determinants

$$-4\times 243,\ -4\times 307,\ -4\times 339,\ -4\times 459,\ -4\times 675,$$

where -243, -307, -339, -459, -675 are those of the thirteen irregular negative determinants in the first thousand for which the number of classes is divisible by 3. The number -4×675, $=-2700$, is beyond the limits of M. Arndt's Table, but the calculation (at least for the order pp on pp) presents no difficulty.

I remark that, according to M. Arndt (*Grunert*, t. XVII. p. 19), the number of cubic forms corresponding to a given characteristic (A, B, C) is equal to the number of proper transformations of $(A, -B, C)$, Det. D, into $(\frac{1}{2}A^2, B^2-\frac{1}{2}AC, \frac{1}{2}C^2)$, Det. DB^2, so that when there is no such transformation, there exists no cubic form corresponding to the characteristic (A, B, C). This includes, I believe, the theorem in a letter of mine to M. Hermite, *Quarterly Mathematical Journal*, t. I. (1857), p. 85, [162], viz. that

for a *pp* form (A, B, C) of negative determinant, there is either no corresponding cubic form, or else a single corresponding cubic form, according as (A, B, C) does not, or does, produce by its triplication the principal form; but the particular theorem, in the cases to which it applies, is the more convenient one: it shows at once that for a regular negative determinant the number of cubic forms corresponding to a properly primitive characteristic (or, what is the same thing, number of cubic classes of the order (*pp* or *ip*) on *pp*) is 1 or 3, according as the number of quadratic classes is not, or is, divisible by 3.

The inspection of the tables gives rise to other remarks, but at present I abstain from pursuing the subject further; I will only notice that in some instances, for example Det. -224, the classes which correspond to characteristics of the principal genus are partly of the order *pp* on *pp* and partly of the order *ip* on *pp*.

Table I. *of the binary cubic forms, the determinants of which are the negative numbers* $\equiv 0$ (*mod.* 4) *from* -4 *to* -400.

Det. 4 ×	Classes	Order	on	Charact.	Compn.
1	0, − 1, 0, 1	*pp*	*pp*	1, 0, 1	1
2	0, − 1, 0, 2	*pp*	*pp*	1, 0, 2	1
3	0, − 1, 0, 3	*ip*	*pp*	1, 0, 3	1
4	0, − 1, 0, 4	*pp*	*pp*	1, 0, 4	1
	1, − 1, − 1, 1	*pp*	2*pp*	2 (1, 0, 1)	2 . 1
5	0, − 1, 0, 5	*pp*	*pp*	1, 0, 5	1
6	0, − 1, 0, 6	*ip*	*pp*	1, 0, 6	1
7	0, − 1, 0, 7	*pp*	*pp*	1, 0, 7	1
	1, 0, − 2, 2 1, 0, − 2, − 2	*pp*	2*ip*	2 (2, ± 1, 4)	2σ
8	0, − 1, 0, 8	*pp*	*pp*	1, 0, 8	1
	0, − 2, 0, 1	*pp*	2*pp*	2 (1, 0, 2)	2 . 1
9	0, − 1, 0, 9	*ip*	*pp*	1, 0, 9	1
10	0, − 1, 0, 10	*pp*	*pp*	1, 0, 10	1
11	0, − 1, 0, 11	*pp*	*pp*	1, 0, 11	1
	0, − 2, − 1, 1			3, 1, 4	d
	0, − 2, 1, 1			3, − 1, 4	d^2
12	0, − 1, 0, 12	*ip*	*pp*	1, 0, 12	1
13	0, − 1, 0, 13	*pp*	*pp*	1, 0, 13	1
14	0, − 1, 0, 14	*pp*	*pp*	1, 0, 14	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
15	0, − 1, 0, 15	*ip*	*pp*	1, 0, 15	1
	1, − 2, 0, 2	*pp*	*ip*	4, ± 1, 4	σc
	1, 2, 0, − 2				
16	0, − 1, 0, 16	*pp*	*pp*	1, 0, 16	1
17	0, − 1, 0, 17	*pp*	*pp*	1, 0, 17	1
18	0, − 1, 0, 18	*ip*	*pp*	1, 0, 18	1
	1, 1, − 2, − 2	*pp*	3*pp*	3 (1, 0, 2)	3 . 1
	1, − 1, − 2, 2				
19	0, − 1, 0, 19	*pp*	*pp*	1, 0, 19	1
	0, 2, 1, − 2			4, 1, 5	d
	0, − 2, 1, 2			4, − 1, 5	d^2
20	0, − 1, 0, 20	*pp*	*pp*	1, 0, 20	1
	0, − 2, − 2, 1	*pp*	2*pp*	2 (2, 1, 3)	$2c$
21	0, − 1, 0, 21	*ip*	*pp*	1, 0, 21	1
22	0, − 1, 0, 22	*pp*	*pp*	1, 0, 22	1
23	0, − 1, 0, 23	*pp*	*pp*	1, 0, 23	1
	1, − 1, − 2, 4			3, − 1, 8	d^2
	1, 1, − 2, − 4			3, 1, 8	d
24	0, − 1, 0, 24	*ip*	*pp*	1, 0, 24	1
	0, − 2, 0, 3	*ip*	2*pp*	2 (2, 0, 3)	$2c$
25	0, − 1, 0, 25	*pp*	*pp*	1, 0, 25	1
	1, − 2, − 1, 2	*pp*	5*pp*	5 (1, 0, 1)	5 . 1
	1, 2, − 1, − 2				
26	0, − 1, 0, 26	*pp*	*pp*	1, 0, 26	1
	1, 0, − 3, 2			3, − 1, 9	g^2
	1, 0, − 3, − 2			3, 1, 9	g^4
27	0, − 1, 0, 27	*ip*	*pp*	1, 0, 27	1
	0, − 2, 1, 3			4, − 1, 7	d^2
	0, 2, 1, − 3			4, 1, 7	d
	0, − 3, 0, 1	*pp*	3*pp*	3 (1, 0, 3)	3 . 1
28	0, − 1, 0, 28	*pp*	*pp*	1, 0, 28	1
	1, 1, − 3, 1	*pp*	2*ip*	2 (2, ± 1, 4)	2σ
	1, − 1, − 3, − 1				

Det. 4 ×	Classes	Order	on	Charact.	Compn.
29	0, − 1, 0, 29			1, 0, 29	1
	2, 1, − 2, − 2	*pp*	*pp*	5, 1, 6	g^2
	2, − 1, − 2, 2			5, − 1, 6	g^4
30	0, − 1, 0, 30	*ip*	*pp*	1, 0, 30	1
31	0, − 1, 0, 31			1, 0, 31	1
	2, 1, − 2, − 3	*pp*	*pp*	5, 2, 7	d
	2, − 1, − 2, 3			3, − 2, 7	d^2
32	0, − 1, 0, 32	*pp*	*pp*	1, 0, 32	1
33	0, − 1, 0, 33	*ip*	*pp*	1, 0, 33	1
34	0, − 1, 0, 34	*pp*	*pp*	1, 0, 34	1
35	0, − 1, 0, 35			1, 0, 35	1
	0, 2, 1, − 4	*pp*	*pp*	4, 1, 9	g^2
	0, − 2, 1, 4			4, − 1, 9	g^4
36	0, − 1, 0, 36	*ip*	*pp*	1, 0, 36	1
	0, − 2, − 2, 3	*ip*	2*pp*	2 (2, 1, 5)	2c
37	0, − 1, 0, 37	*pp*	*pp*	1, 0, 37	1
38	0, − 1, 0, 38			1, 0, 38	1
	2, − 2, − 1, 3	*ip*	*pp*	6, 2, 7	g^2
	2, 2, − 1, − 3			6, − 2, 7	g^4
39	0, − 1, 0, 39	*ip*	*pp*	1, 0, 39	1
	1, 1, − 3, − 1	*pp*	*ip*	4, − 1, 10	σe
	1, − 1, − 3, 1			4, 1, 10	σe^3
40	0, − 1, 0, 40	*pp*	*pp*	1, 0, 40	1
	0, − 2, 0, 5	*pp*	2*pp*	2 (2, 0, 5)	2 . 1
41	0, − 1, 0, 41	*pp*	*pp*	1, 0, 41	1
42	0, − 1, 0, 42	*ip*	*pp*	1, 0, 42	1
43	0, − 1, 0, 43			1, 0, 43	1
	0, − 2, 1, 5	*pp*	*pp*	4, − 1, 11	d^2
	0, 2, 1, − 5			4, 1, 11	d
44	0, − 1, 0, 44	*pp*	*pp*	1, 0, 44	1
	1, 2, − 1, − 4	*pp*	2*ip*	2 (2, ± 1, 6)	2σ
	1, − 2, − 1, 4				

Det. 4×	Classes	Order	on	Charact.	Compn.
45	0, − 1, 0, 45	*ip*	*pp*	1, 0, 45	1
	2, 0, − 3, 3	*pp*	3*pp*	3 (2, ± 1, 3)	3*c*
	2, 0, − 3, − 3				
46	0, − 1, 0, 46	*pp*	*pp*	1, 0, 46	1
47	0, − 1, 0, 47	*pp*	*pp*	1, 0, 47	1
	1, − 2, − 2, 2	*pp*	*ip*	6, 1, 8	σf
	1, 2, − 2, − 2			6, − 1, 8	σf^4
48	0, − 1, 0, 48	*ip*	*pp*	1, 0, 48	1
	1, − 1, − 3, 3	*pp*	4*pp*	4 (1, 0, 3)	4 . 1
	1, 1, − 3, − 3				
49	0, − 1, 0, 49	*pp*	*pp*	1, 0, 49	1
50	0, − 1, 0, 50	*pp*	*pp*	1, 0, 50	1
	2, 0, − 3, 2			6, − 2, 9	g^4
	2, 0, − 3, − 2			6, 2, 9	g^2
51	0, − 1, 0, 51	*ip*	*pp*	1, 0, 51	1
	0, − 2, 1, 6			4, − 1, 13	g^4
	0, 2, 1, − 6			4, 1, 13	g^2
52	0, − 1, 0, 52	*pp*	*pp*	1, 0, 52	1
	0, − 2, − 2, 5	*pp*	2*pp*	2 (2, 1, 7)	2*c*
53	0, − 1, 0, 53	*pp*	*pp*	1, 0, 53	1
	1, − 3, 0, 2			6, − 1, 9	g^2
	1, 3, 0, − 2			6, 1, 9	g^4
54	0, − 1, 0, 54	*ip*	*pp*	1, 0, 54	1
	1, 2, − 3, 0	*pp*		7, − 3, 9	g^4
	1, − 2, − 3, 0	*pp*		7, 3, 9	g^2
	0, − 3, 0, 2	*pp*	3*pp*	3 (2, 0, 3)	3*c*
	0, 3, 0, − 2				
55	0, − 1, 0, 55	*pp*	*pp*	1, 0, 55	1
	1, − 1, − 3, 5	*pp*	*ip*	4, − 1, 14	σe^3
	1, 1, − 3, − 5			4, 1, 14	σe
56	0, − 1, 0, 56	*pp*	*pp*	1, 0, 56	1
	0, − 2, 0, 7	*pp*	2*pp*	2 (2, 0, 7)	$2e^2$
57	0, − 1, 0, 57	*ip*	*pp*	1, 0, 57	1
58	0, − 1, 0, 58	*pp*	*pp*	1, 0, 58	1

Det. 4×	Classes	Order	on	Charact.	Compn.
59	0, − 1, 0, 59			1, 0, 59	1
	0, − 2, 1, 7	*pp*	*pp*	4, − 1, 15	j^6
	0, 2, 1, − 7			4, 1, 15	j^3
60	0, − 1, 0, 60	*ip*	*pp*	1, 0, 60	1
	1, 0, − 4, 4	*pp*	2*ip*	2 (2, ± 1, 8)	2σ
	1, 0, − 4, − 4				
61	0, − 1, 0, 61			1, 0, 61	1
	1, − 2, − 1, 6	*pp*	*pp*	5, − 2, 13	g^2
	1, 2, − 1, − 6			5, 2, 13	g^4
62	0, − 1, 0, 62	*pp*	*pp*	1, 0, 62	1
63	0, − 1, 0, 63	*ip*	*pp*	1, 0, 63	1
	1, 0, − 4, 2	*pp*	*ip*	4, − 1, 16	σe
	1, 0, − 4, − 2			4, 1, 16	σe^3
64	0, − 1, 0, 64	*pp*	*pp*	1, 0, 64	1
	1, 0, − 4, 0	*pp*	4*pp*	4 (1, 0, 4)	4 . 1
65	0, − 1, 0, 65	*pp*	*pp*	1, 0, 65	1
66	0, − 1, 0, 66	*ip*	*pp*	1, 0, 66	1
67	0, − 1, 0, 67			1, 0, 67	1
	0, − 2, 1, 8	*pp*	*pp*	4, − 1, 17	d^2
	0, 2, 1, − 8			4, 1, 17	d
68	0, − 1, 0, 68	*pp*	*pp*	1, 0, 68	1
	0, − 2, − 2, 7	*pp*	2*pp*	2 (2, 1, 9)	$2e^2$
69	0, − 1, 0, 69	*ip*	*pp*	1, 0, 69	1
70	0, − 1, 0, 70	*pp*	*pp*	1, 0, 70	1
71	0, − 1, 0, 71	*pp*	*pp*	1, 0, 71	1
	1, − 3, 1, 3	*pp*	*ip*	8, − 3, 10	σh^3
	1, 3, 1, − 3			8, 3, 10	σh^4
72	0, − 1, 0, 72	*ip*	*pp*	1, 0, 72	1
	0, − 2, 0, 9	*ip*	2*pp*	2 (2, 0, 9)	$2c$
	1, − 2, − 2, 4	*pp*	6*pp*	6 (1, 0, 2)	6 . 1
	1, 2, − 2, − 4				
	2, − 3, 0, 3	*pp*	3*pp*	3 (3, ± 1, 3)	$3c$
	2, 3, 0, − 3				
73	0, − 1, 0, 73	*pp*	*pp*	1, 0, 73	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
74	0, − 1, 0, 74	*pp*	*pp*	1, 0, 74	1
75	0, − 1, 0, 75			1, 0, 75	1
	0, − 2, 1, 9	*ip*	*pp*	4, − 1, 19	g^2
	0, 2, 1, − 9			4, 1, 19	g^4
76	0, − 1, 0, 76			1, 0, 76	1
	0, − 4, 1, 1	*pp*	*pp*	5, − 2, 16	g^2
	0, 4, 1, − 1			5, 2, 16	g^4
77	0, − 1, 0, 77	*pp*	*pp*	1, 0, 77	1
78	0, − 1, 0, 78	*ip*	*pp*	1, 0, 78	1
79	0, − 1, 0, 79	*pp*	*pp*	1, 0, 79	1
	2, − 2, − 2, 3	*pp*	*ip*	8, − 1, 10	σf
	2, 2, − 2, − 3			8, 1, 10	σf^4
80	0, − 1, 0, 80	*pp*	*pp*	1, 0, 80	1
81	0, − 1, 0, 81	*pp*		1, 0, 81	1
	0, − 3, 2, 2	*ip*	*pp*	9, − 3, 10	g^2
	0, 3, 2, − 2	*ip*		9, 3, 10	g^4
82	0, − 1, 0, 82	*pp*	*pp*	1, 0, 82	1
83	0, − 1, 0, 83			1, 0, 83	1
	0, − 2, 1, 10	*pp*	*pp*	4, − 1, 21	j^6
	0, 2, 1, − 10			4, 1, 21	j^3
84	0, − 1, 0, 84	*ip*	*pp*	1, 0, 84	1
	0, − 2, − 2, 9	*ip*	2*pp*	2 (2, 1, 11)	2σ
85	0, − 1, 0, 85	*pp*	*pp*	1, 0, 85	1
86	0, − 1, 0, 86	*pp*	*pp*	1, 0, 86	1
87	0, − 1, 0, 87	*ip*		1, 0, 87	1
	1, − 2, − 3, 2	*pp*	*pp*	7, − 2, 13	g^4
	1, 2, − 3, − 2	*pp*		7, 2, 13	g^2
88	0, − 1, 0, 88	*pp*	*pp*	1, 0, 88	1
	0, − 2, 0, 11	*pp*	2*pp*	2 (2, 0, 11)	2σ
89	0, − 1, 0, 89			1, 0, 89	1
	1, − 1, − 4, 2	*pp*	*pp*	5, − 1, 18	m^8
	1, 1, − 4, − 2			5, 1, 18	m^4
90	0, − 1, 0, 90	*ip*	*pp*	1, 0, 90	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
91	0, − 1, 0, 91			1, 0, 91	1
	0, − 2, 1, 11	*pp*	*pp*	4, − 1, 23	g^4
	0, 2, 1, − 11			4, 1, 23	g^2
92	0, − 1, 0, 92			1, 0, 92	1
	2, − 3, 0, 4	*pp*	*pp*	9, − 4, 12	g^4
	2, 3, 0, − 4			9, 4, 12	g^2
93	0, − 1, 0, 93	*ip*	*pp*	1, 0, 93	1
94	0, − 1, 0, 94	*ip*	*pp*	1, 0, 94	1
95	0, − 1, 0, 95	*pp*	*pp*	1, 0, 95	1
	1, − 2, − 2, 6	*pp*	*ip*	6, − 1, 16	σi^7
	1, 2, − 2, − 6			6, 1, 16	σi
96	0, − 1, 0, 96	*ip*	*pp*	1, 0, 96	1
97	0, − 1, 0, 97	*pp*	*pp*	1, 0, 97	1
98	0, − 1, 0, 98	*pp*	*pp*	1, 0, 98	1
99	0, − 1, 0, 99			1, 0, 99	1
	0, − 2, 1, 12	*ip*	*pp*	4, − 1, 25	g^2
	0, 2, 1, − 12			4, 1, 25	g^4
100	0, − 1, 0, 100	*pp*	*pp*	1, 0, 100	1
	0, − 2, − 2, 11	*pp*	2*pp*	2 (2, 1, 13)	2 . 1
	1, − 1, − 4, 4	*pp*	5*pp*	5 (1, 0, 4)	5 . 1
	1, 1, − 4, − 4				
	1, − 3, − 1, 3	*pp*	10*pp*	10 (1, 0, 1)	10 . 1
	1, 3, − 1, − 3				

Table II. *of the binary cubic forms the determinants of which are the positive numbers* ≡ 1 (mod. 4) *from* − 3 to − 99.

Det. 4 ×	Classes	Order	on	Charact.	Compn.
3	0, 1, 1, 0	*ip*			
	1, 0, − 1, 1	*pp*	*ip*	2, ± 1, 2	σ
	1, 0, − 1, − 1	*pp*			
7	0, − 1, − 1, 1	*pp*	*ip*	2, 1, 4	σ
11	0, − 1, − 1, 2	*pp*	*ip*	2, 1, 6	σ
15	0, − 1, − 1, 3	*ip*	*ip*	2, 1, 8	σ
19	0, − 1, − 1, 4	*pp*	*ip*	2, 1, 10	σ

Det. 4 ×	Classes	Order	on	Charact.	Compn.
23	0, − 1, − 1, 5			2, 1, 12	σ
	1, − 1, − 1, 2	*pp*	*ip*	4, − 1, 6	σd
	1, 1, − 1, − 2			4, 1, 6	σd^2
27	0, − 1, − 1, 6	*ip*	*ip*	2, 1, 14	σ
	1, 1, − 2, − 1				
	1, − 1, − 2, 1	*pp*	3*ip*	3 (2, ± 1, 2)	3σ
	2, − 1, − 1, 2				
31	0, − 1, − 1, 7			2, 1, 16	σ
	1, 0, − 2, 1	*pp*	*ip*	4, − 1, 8	σd
	1, 0, − 2, − 1			4, 1, 8	σd^2
35	0, − 1, − 1, 8	*pp*	*ip*	2, 1, 18	σ
39	0, − 1, − 1, 9	*ip*	*ip*	2, 1, 20	σ
43	0, − 1, − 1, 10	*pp*	*ip*	2, 1, 22	σ
47	0, − 1, − 1, 11	*pp*	*ip*	2, 1, 24	σ
51	0, − 1, − 1, 12	*ip*	*ip*	2, 1, 26	σ
55	0, − 1, − 1, 13	*pp*	*ip*	2, 1, 28	σ
59	0, − 1, − 1, 14			2, 1, 30	σ
	1, − 1, − 2, 1	*pp*	*ip*	6, 1, 10	σj
	1, 1, − 2, − 1			6, − 1, 10	σj^2
63	0, − 1, − 1, 15	*ip*	*ip*	2, 1, 32	σ
67	0, − 1, − 1, 16	*pp*	*ip*	2, 1, 34	σ
71	0, − 1, − 1, 17	*pp*	*ip*	2, 1, 36	σ
75	0, − 1, − 1, 18	*ip*	*ip*	2, 1, 38	σ
79	0, − 1, − 1, 19	*pp*	*ip*	2, 1, 40	σ
83	0, − 1, − 1, 20			2, 1, 42	σ
	1, − 1, − 2, 3	*pp*	*ip*	6, − 1, 14	σj^2
	1, 1, − 2, − 3			6, 1, 14	σj
87	0, − 1, − 1, 21	*ip*		2, 1, 44	σ
	1, − 2, 0, 3	*pp*	*ip*	8, − 3, 12	σg^2
	1, 2, 0, − 3	*pp*		8, 3, 12	σg^4
91	0, − 1, − 1, 22	*pp*	*ip*	2, 1, 46	σ
95	0, − 1, − 1, 23	*pp*	*ip*	2, 1, 48	σ
99	0, − 1, − 1, 24	*ip*	*ip*	2, 1, 50	σ
	1, 0, − 3, 3	*pp*	3*ip*	3 (2, ± 1, 6)	3σ
	1, 0, − 3, − 3				

Table III. *of the binary cubic forms the determinants of which are the negative numbers* -972, -1228, 1336, -1836 *et* -2700. ($-4 \times 675 = -2700$ is beyond Arndt's Tables.)

Det. $4\times$	Classes	Order	on	Charact.	Compn.
243	0, − 1, 0, 243	*ip*		1, 0, 243	1
	1, − 1, − 6, 0	*pp*		7, 3, 36	d
	1, 1, − 6, 0	*pp*		7, − 3, 36	d^2
	0, 2, 1, − 30	*ip*		4, 1, 61	d_1
	2, 3, − 2, − 5	*pp*	*pp*	13, − 2, 19	dd_1
	0, 3, 2, − 8	*pp*		9, 3, 28	d^2d_1
	0, − 2, 1, 30	*ip*		4, − 1, 61	d_1^2
	0, − 3, 2, 8	*pp*		9, − 3, 28	dd_1^2
	2, − 3, − 2, 5	*pp*		13, 2, 19	$d^2d_1^2$
307	0, − 1, 0, 307			1, 0, 307	1
	1, 1, − 6, − 8			7, 1, 44	d
	1, − 1, − 6, 8			7, − 1, 44	d^2
	0, 2, 1, − 38			4, 1, 77	d_1
	1, − 3, − 2, 8	*pp*	*pp*	11, − 1, 28	dd_1
	4, 1, − 4, − 3			17, 4, 19	d^2d_1
	0, − 2, 1, 38			4, − 1, 77	d_1^2
	4, − 1, − 4, 3			17, − 4, 19	dd_1^2
	1, 3, − 2, 8			11, 1, 28	$d^2d_1^2$
339	0, − 1, 0, 339	*ip*		1, 0, 339	1
	1, 0, − 7, − 4	*pp*		7, 2, 49	d
	1, 0, − 7, 4	*pp*		7, − 2, 49	d^2
	0, 2, 1, − 42	*ip*		4, 1, 85	d_1
	3, 0, − 5, − 4	*pp*	*pp*	15, 6, 25	dd_1
	2, − 3, − 2, 8	*pp*		13, − 5, 28	d^2d_1
	0, − 2, 1, 42	*ip*		4, − 1, 85	d_1^2
	2, 3, − 2, − 8	*pp*		13, 5, 28	dd_1^2
	3, 0, − 5, 4	*pp*		15, − 6, 25	$d^2d_1^2$
459	0, − 1, 0, 459	*ip*		1, 0, 459	1
	0, 3, 2, − 16	*pp*		9, 3, 52	d
	0, − 3, 2, 16	*pp*		9, − 3, 52	d^2
	0, 2, 1, − 57	*ip*		4, 1, 115	d_1
	1, 4, − 3, − 4	*pp*	*pp*	19, − 4, 25	dd_1
	2, − 1, 6, 0	*pp*		13, 3, 36	d^2d_1
	0, − 2, 1, 57	*ip*		4, − 1, 115	d_1^2
	2, 1, − 6, 0	*pp*		13, − 3, 36	dd_1^2
	1, − 4, − 3, 4	*pp*		19, 4, 25	$d^2d_1^2$
	0, − 3, 0, 17	*pp*	3*pp*	3 (3, 0, 17)	$3g^3$

Det. 4 ×	Classes	Order	on	Charact.	Compn.
675	0, − 1, 0, 675			1, 0, 675	1
	0, 3, 2, − 24			9, 3, 76	d
	0, − 3, 2, 24			9, − 3, 76	d^2
	0, 2, 1, − 168			4, 1, 169	d_1
	0, 5, − 4, − 3	*ip*	*pp*	25, − 10, 31	dd_1
	3, − 1, − 6, 0			19, 3, 36	d^2d_1
	0, − 1, 1, 168			4, − 1, 169	d_1^2
	3, 1, − 6, 0			19, − 3, 36	dd_1^2
	0, − 5, − 4, 3			25, 10, 31	$d^2d_1^2$

N.B. For this last determinant -4×675, there may possibly be other cubic classes based on a non-primitive characteristic; I have not ascertained whether such forms do or do not exist.

497.

NOTE ON THE CALCULUS OF LOGIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 282, 283.]

IT appears to me that the theory of the Syllogism, as given in Boole's paper, "The Calculus of Logic," *Camb. and Dubl. Math. Jour.*, t. III. (1848), pp. 183—198, may be presented in a more concise and compendious form as follows:

We are concerned with complementary classes, X, X'; viz. these together make up the universe (of things under consideration), $X + X' = 1$; viz. X' is the class not-X, and X the class not-X'.

Any kind whatever of simple relation between two classes (if we attend also to the complementary classes) can be expressed as a relation of total exclusion, $XY = 0$, or as a relation of partial (it may be total) inclusion, YX not $= 0$; viz. the relation $XY = 0$ may be read in any of the forms

No X's are X's,
No Y's are X's,
All X's are not-Y's,
All Y's are not-X's,

and the relation XY not $= 0$ in either of the forms

Some X's are Y's,
Some Y's are X's.

I say the above are the *only* kinds of simple relations; it being understood that X' may be substituted for X, or Y' for Y; so that the example $X'Y = 0$ (all Y's are X's) is the same kind of relation as $XY = 0$; and $X'Y$ not $= 0$ (some Y's are not-X's) the same kind of relation as XY not $= 0$.

Now taking X or X' and Z or Z' for the extreme terms, and Y or Y' for the middle term, of a syllogism; the only combinations of premises are

(1) $XY=0$, $ZY=0$.

(2) $XY=0$, ZY not $=0$, therefore $X'Z$ not $=0$.

(3) XY not $=0$, ZY not $=0$.

(4) $XY=0$, $ZY'=0$, therefore $XZ=0$.

(5) $XY=0$, ZY' not $=0$.

(6) XY not $=0$, ZY' not $=0$.

And of these, there are (as shown by the third column) only two which give rise to a conclusion (or relation between the extreme terms). As regards the negative cases, this is at once seen to be so; thus $XY=0$, $ZY=0$ (no X's are Y's, no Z's are Y's) leads to no conclusion in regard to X, Z. As regards the positive cases, it is also at once seen that the conclusions do follow; but we may obtain the conclusions by symbolical reasoning, thus

(2) $Y = YX + YX', = YX'$;

therefore $ZY = ZYX'$, not $=0$; therefore ZX' not $=0$.

(4) $XZ = XZY + XZY'$, where on the right-hand side each term (the first as containing XY, the second as containing ZX') is $=0$; that is, $XZ=0$; where the logical signification of each step is obvious.

498.

ON THE INVERSION OF A QUADRIC SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 283—288.]

THE inversion intended to be considered is that by reciprocal radius vectors, viz. if x, y, z are rectangular coordinates, and $r^2 = x^2 + y^2 + z^2$, then x, y, z are to be changed into $\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}$. But it is convenient to introduce for homogeneity a fourth coordinate $w, = 1$; and the change then is x, y, z into $\frac{xw^2}{r^2}, \frac{yw^2}{r^2}, \frac{zw^2}{r^2}$.

Starting from the quadric surface

$$(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0,$$

or, what is the same thing,

$$\begin{aligned} &(a, b, c, f, g, h \between x, y, z)^2 \\ &+ 2w\,(lx + my + nz) \\ &+ dw^2 \qquad\qquad = 0, \end{aligned}$$

the equation of the inverse surface is

$$\begin{aligned} &w^2\,(a, b, c, f, g, h \between x, y, z)^2 \\ &+ 2w\,(lx + my + nz)\,r^2 \\ &+ dr^4 \qquad\qquad = 0, \end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$. The inverse surface is thus a quartic having the nodal conic $w = 0$, $x^2 + y^2 + z^2 = 0$ (circle at infinity); and having the node $x = 0$, $y = 0$, $z = 0$ (the centre of inversion); or say it is a nodal bicircular quartic surface, or nodal anallagmatic.

For x, y, z write $x-\frac{1}{2}\frac{l}{d}w$, $y-\frac{1}{2}\frac{m}{d}w$, $z-\frac{1}{2}\frac{n}{d}w$, and put for shortness

$$\begin{aligned}
&lx+my+nz=u,\ l^2+m^2+n^2=\alpha,\\
&al+hm+gn=\mathrm{a}, \qquad (a,\ b,\ c,\ f,\ g,\ h \between l,\ m,\ n)^2=A,\\
&hl+bm+fn=\mathrm{b},\\
&gl+fm+cn=\mathrm{c},
\end{aligned}$$

then

$$\begin{aligned}
r^2 \qquad &\text{becomes}\ r^2-\frac{uw}{d}+\tfrac{1}{4}\frac{\alpha}{d^2}w^2,\\
lx+my+nz \qquad &\text{,,}\quad u-\tfrac{1}{2}\frac{\alpha}{d}w,\\
(a,\ldots \between x,\ y,\ z)^2 \qquad &\text{,,}\quad (a,\ldots \between x,\ y,\ z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\tfrac{1}{4}A\frac{w^2}{d^2}.
\end{aligned}$$

Hence the equation is

$$\begin{aligned}
&d\left\{r^4-2r^2\frac{uw}{d}+w^2\left(\tfrac{1}{2}\frac{\alpha}{d^2}r^2+\frac{u^2}{d^2}\right)-\tfrac{1}{2}\frac{\alpha uw^2}{d^3}+\tfrac{1}{16}\frac{\alpha^2}{d^4}w^4\right\}\\
&+2\left(wr^2-\frac{uw^2}{d}+\tfrac{1}{4}\frac{\alpha}{d^2}w^3\right)\left(u-\tfrac{1}{2}\frac{\alpha}{d}w\right)\\
&+w^2\left\{(a,\ldots \between x,\ y,\ z)^2-(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\frac{w}{d}+\tfrac{1}{4}A\frac{w^2}{d^2}\right\}=0;
\end{aligned}$$

viz. arranging and reducing, this is

$$\begin{aligned}
&dr^4\\
&+w^2\left\{-\tfrac{1}{2}\frac{\alpha}{d}r^2-\frac{u^2}{d}+(a,\ldots \between x,\ y,\ z)^2\right\}\\
&+w^3\left\{\quad\frac{\alpha u}{d^2}\ -\frac{1}{d}(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\right\}\\
&+w^4\left\{-\tfrac{3}{16}\frac{\alpha^2}{d^3}+\tfrac{1}{4}A\frac{1}{d^2}\right\} \qquad =0;
\end{aligned}$$

and we may without loss of generality assume

$$\begin{aligned}
&-\frac{mn}{d}+f=0, \text{ that is } \quad df-mn=0,\\
&-\frac{nl}{d}+g=0, \quad \text{,,} \quad dg-nl=0,\\
&-\frac{lm}{d}+h=0, \quad \text{,,} \quad dh-lm=0.
\end{aligned}$$

The equation then is

$$
\begin{aligned}
& r^4 \\
& + w^2 \left\{ -\tfrac{1}{2}\frac{\alpha}{d^2}(x^2+y^2+z^2) + \left(\frac{a}{d} - \frac{l^2}{d^2}\right)x^2 + \left(\frac{b}{d} - \frac{m^2}{d^2}\right)y^2 + \left(\frac{c}{d} - \frac{n^2}{d^2}\right)z^2 \right\} \\
& + w^3 \left\{ \frac{\alpha u}{d^3} - \frac{1}{d^2}(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) \right\} \\
& + w^4 \left\{ -\tfrac{3}{16}\frac{\alpha^2}{d^4} + \tfrac{1}{4}A\frac{1}{d^3} \right\} = 0.
\end{aligned}
$$

Write

$$
\begin{aligned}
ad - l^2 &= a'd, \\
bd - m^2 &= b'd, \\
cd - n^2 &= c'd.
\end{aligned}
$$

We have

$$\mathrm{a} = al + hm + gn = gn = al + \frac{lm^2}{d} + \frac{ln^2}{d} = \frac{l}{d}(ad - l^2 + \alpha),$$

that is

$$\mathrm{a} = la' + \frac{l\alpha}{d},$$

and similarly

$$\mathrm{b} = mb' + \frac{m\alpha}{d},$$

$$\mathrm{c} = nc' + \frac{n\alpha}{d}.$$

Hence also

$$A = l^3a' + m^3b' + n^3c' + \frac{\alpha^2}{d},$$

and the equation is

$$
\begin{aligned}
& r^4 \\
& + w^2 \left\{ \left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d}\right)x^2 + \left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d}\right)y^2 + \left(-\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d}\right)z^2 \right\} \\
& + w^3 \left\{ -\frac{la'}{d^2}x - \frac{mb'}{d^2}y - \frac{nc'}{d^2}z \right\} \\
& + w^4 \left\{ \frac{1}{4d^3}(l^3a' + m^3b' + n^3c') + \tfrac{1}{16}\frac{\alpha^2}{d^4} \right\} = 0.
\end{aligned}
$$

This is Kummer's form, say

$$r^4 = 4w^2\{\alpha_1x^2 + \beta_1y^2 + \gamma_1z^2 + \delta_1w^2 + 2w(a_1x + b_1y + c_1z)\},$$

where

$$-4\alpha_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{a'}{d},$$

$$-4\beta_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{b'}{d},$$

$$-4\gamma_1 = -\tfrac{1}{2}\frac{\alpha}{d^2} + \frac{c'}{d},$$

$$-4\delta_1 = \frac{1}{4d^3}(l^2a' + m^2b' + n^2c') + \tfrac{1}{16}\frac{\alpha^2}{d^4},$$

$$-8a_1 = -\frac{la'}{d^2},$$

$$-8b_1 = -\frac{mb'}{d^2},$$

$$-8c_1 = -\frac{nc'}{d^2}.$$

Hence Kummer's equation

$$\delta_1 + \lambda^2 = \frac{a_1^2}{\lambda + \alpha_1} + \frac{b_1^2}{\lambda + \beta_1} + \frac{c_1^2}{\lambda + \gamma_1},$$

or say

$$64\delta_1 + 64\lambda^2 = \frac{256a_1^2}{4\lambda + 4\alpha_1} + \frac{256b_1^2}{4\lambda + 4\beta_1} + \frac{256c_1^2}{4\lambda + 4\gamma_1},$$

becomes

$$64\lambda^2 - \frac{4}{d^3}(l^2a' + m^2b' + n^2c') - \frac{\alpha^2}{d^4} = \frac{4l^2a'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{a'}{d} + 4\lambda\right)} + \frac{4m^2b'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{b'}{d} + 4\lambda\right)} + \frac{4n^2c'^2}{d^4\left(\frac{1}{2}\frac{\alpha}{d^2} - \frac{c'}{d} + 4\lambda\right)},$$

which is satisfied by $4\lambda = -\frac{1}{2}\frac{\alpha}{d^2}$. Writing therefore

$$4\lambda + \tfrac{1}{2}\frac{\alpha}{d^2} = -\frac{\theta}{d},$$

that is

$$8\lambda = -\frac{2\theta}{d} - \frac{\alpha}{d^2},$$

$$64\lambda^2 = \frac{4\theta^2}{d^2} + \frac{4\theta\alpha}{d^3} + \frac{\alpha^2}{d^4};$$

the equation is

$$\frac{4\theta^2}{d^3} + \frac{4\theta\alpha}{d^3} - \frac{4}{d^3}(l^2a' + m^2b' + n^2c') = \frac{4l^2a'^2}{d^4\left(-\frac{\theta}{d} - \frac{a'}{d}\right)} + \frac{4m^2b'^2}{d^4\left(-\frac{\theta}{d} - \frac{b'}{d}\right)} + \frac{4n^2c'^2}{d^4\left(-\frac{\theta}{d} - \frac{c'}{d}\right)},$$

viz. this is

$$l^2a' + m^2b' + n^2c' - \theta\alpha - \theta^2 d = \frac{l^2a'^2}{\theta + a'} + \frac{m^2b'^2}{\theta + b'} + \frac{n^2c'^2}{\theta + c'},$$

which is of course satisfied by $\theta=0$. Moreover the derived equation

$$-\alpha-2\theta d=-\frac{l^2a'^2}{(\theta+a')^2}-\frac{m^2b'^2}{(\theta+b')^2}-\frac{n^2c'^2}{(\theta+c')^2}$$

is also satisfied by $\theta=0$, so that this is a double root. The equation in fact is

$$\begin{aligned}&\{\theta^2d+\theta\alpha-(l^2a'+m^2b'+n^2c')\}(\theta+a')(\theta+b')(\theta+c')\\&\quad+\{l^2a'^2(\theta+b')(\theta+c')+m^2b'^2(\theta+c')(\theta+a')+n^2c'^2(\theta+a')(\theta+b')\}=0,\end{aligned}$$

or, expanding and dividing by θ^2, this is

$$\begin{aligned}&d(\theta+a')(\theta+b')(\theta+c')\\&+\alpha\{\theta^2+\theta(a'+b'+c')+b'c'+c'a'+a'b'\}\\&-(l^2a'+m^2b'+n^2c')(\theta+a'+b'+c')\\&+l^2a'^2+m^2b'^2+n^2c'^2=0,\end{aligned}$$

which gives the remaining three roots.

If $a'=b'=c'$ the equation is

$$(\theta+a'+\alpha)(\theta+a')^2=0.$$

I recall that we have

$$a,\ b,\ c,\ d,\quad f=\frac{mn}{d},\quad g=\frac{nl}{d},\quad h=\frac{lm}{d},\quad l,\ m,\ n,$$

$$a'=a-\frac{l^2}{d},\quad b'=b-\frac{m^2}{d},\quad c'=c-\frac{n^2}{d},\quad \alpha=l^2+m^2+n^2,$$

so that the quadric surface is

$$d(a'x^2+b'y^2+c'z^2)+(lx+my+nz+dw)^2=0,$$

and that, α_1, β_1, γ_1, δ_1, a_1, b_1, c_1 denoting as before, the equation of the inverse surface (referred to a different origin) is

$$r^4=4w^2\{\alpha_1x^2+\beta_1y^2+\gamma_1z^2+\delta_1w^2+2w(a_1x+b_1y+c_1z)\}.$$

499.

ON THE THEORY OF THE CURVE AND TORSE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 294—317.]

THE fundamental relations in the theory of the Curve and Torse were first established in my "Mémoire sur les Courbes à double courbure et sur les Surfaces developpables," *Liouv.* t. X. (1845), [30] see also *Camb. and Dubl. Math. Jour.*, t. IV. (1850), [83], viz. I showed that the systems (m, r, β, h, n, y) and (r, n, m, x, α, g), (the notation is subsequently explained), each of them satisfied the Plückerian relations. An additional set of equations giving the values of $(\gamma, t, k, q, \gamma', t', k', q')$ was furnished by Dr Salmon's "Theory of Reciprocal Surfaces," *Trans. R. I. Acad.*, t. XXIII. (1857), see also the *Solid Geometry*. The theory as thus established is complete in itself, but it does not take account of certain singularities υ, ω, H, G; the singularity υ was first considered in my paper "On a special sextic Developable," *Quart. Math. Jour.* t. VII. (1865), [373], (there called θ), and I afterwards endeavoured to take account of the remaining singularities ω, H, G. I was in correspondence on the subject with Prof. Cremona, and the discovery of the complete forms of several of the formulæ is due to him.

There has recently appeared a very valuable memoir by M. Zeuthen, "Sur les singularités ordinaires d'une courbe gauche et d'une surface developpable," *Annali di Matem.* t. III. (1869); he excludes, however, from consideration the singularity ω, and does not throughout attend to H, G.

I propose in the present memoir to reproduce and develope the whole theory.

Explanations and Notation.

1. We have a singly infinite series of points, lines, and planes; viz. each line passes through two consecutive points and lies in two consecutive planes; each plane passes through three consecutive points and contains two consecutive lines; each point

is the intersection of three consecutive planes and of two consecutive lines. The points describe and the lines envelope a *curve*; the lines describe and the planes envelope a *torse*; the entire system of points, lines, and planes is thus the system of the curve and torse. The curve is the edge of regression or cuspidal curve of the torse; in regard to the curve the points are points (or ineunts), the lines tangents, and the planes osculating planes: in regard to the torse the points are points on the edge of regression, the lines generating lines, and the planes tangent planes.

2. Each line of the system is met by a certain number of non-consecutive lines, and the locus of the points of intersection (or say the locus of the intersection of two intersecting non-consecutive lines) is a nodal curve on the torse, or say simply it is the *nodal curve*. The plane containing the two intersecting non-consecutive lines envelopes a torse which is called the *nexal torse*.

3. There is occasion to consider

m, order of the system; this is the number o points of the system which lie in a given plane; or it is the order of the curve.

r, rank of the system; this is the number of lines of the system which meet a given line. It is thus the class of the curve; and the order of the torse.

n, class of the system; this is the number of planes of the system which pass through a given point. It is thus the class of the torse.

α, number of stationary planes; that is, planes each passing through four consecutive points of the system.

β, number of stationary points, that is, points each of them the intersection of four consecutive planes of the system.

g, number of *lines in two planes* (that is, lines each of them the intersection of two non-consecutive planes of the system) contained in a given plane; or say, number of apparent double planes of the torse.

G, number of double planes, or *tropes*, of the torse; viz. considering the torse as the envelope of a variable plane, if the plane in the course of its motion comes twice into the same position, we have then a double plane or trope.

h, number of *lines through two points* (that is, lines each through two non-consecutive points of the system) passing through a given point; or say, number of apparent double points of the curve.

H, number of double points, or *nodes* of the curve; viz. considering the curve as described by a variable point, if the point in the course of its motion comes twice into the same position we have then a double point or node of the curve.

x, number of *points in two lines* (that is, points each of them the intersection of two non-consecutive lines of the system) contained in a given plane; or what is the same thing, order of nodal curve.

y, number of *planes through two lines* (that is, planes each containing two non-consecutive lines of the system) passing through a given point; or what is the same thing, class of the nexal torse.

υ, number of stationary lines of the system, that is, lines each containing three consecutive points of the system.

ω, number of double lines of the system, that is, lines each containing two pairs of consecutive points of the system.

t, number of *points on three lines* (that is, points each of them the common intersection of three non-consecutive lines of the system): these are also triple points on the curve.

γ, number of points of the system, through each of which passes a non-consecutive line of the system: these are intersections of the curve with the nodal curve, stationary points on the latter curve.

k, number of apparent double points of nodal curve.

q, class of nodal curve.

t', number of *planes through three lines* (that is, planes each of them through three non-consecutive lines of the system): these are also triple tangent planes of the torse.

γ', number of planes of the system each of them passing through a non-consecutive line of the system: these are common tangent planes of the torse and nexal torse, stationary planes of the latter torse.

k', number of apparent double planes of nexal torse.

q', order of nexal torse.

4. The formulæ thus contains in all the 21 quantities

$$m,\ r,\ n,\ \alpha,\ \beta,\ g,\ G,\ h,\ H,\ x,\ y,\ \upsilon,\ \omega \parallel t,\ \gamma,\ k,\ q \mid t',\ \gamma',\ k',\ q'.$$

My own Plückerian equations, or, say, the Plücker-Cayley equations, establish in all 6 relations between the first 13 quantities, and thus enable the expression of them in terms of any seven, say of

$$m,\ r,\ n,\ G,\ H,\ \upsilon,\ \omega,$$

and the Salmon-Cremona equations then lead to the expressions in terms of these, of the remaining eight quantities $t,\ \gamma,\ k,\ q,\ t',\ \gamma',\ k',\ q'$.

I also consider

D_m, the deficiency of the curve,

D_x, the deficiency of the nodal curve.

I will first consider the equations themselves, and the mere algebraical transformations thereof; and afterwards the geometrical theory.

The Plücker-Cayley Equations.

5. These are found (as will be further explained) by considering first the cone, vertex an arbitrary point, which passes through the curve; and secondly, the section of the torse by an arbitrary plane. We have in the two figures

Cone.	*Section.*
m, order,	n, class,
r, class,	r, order,
$h+H$, double lines,	$g+G$, double tangents,
β, stationary lines,	α, stationary tangents,
$y+\omega$, double planes,	$x+\omega$, double points,
$n+v$, stationary planes.	$m+v$, stationary points.

And hence the two sets of quantities respectively are connected by the Plückerian relations, viz. these are

$$\begin{aligned}
r &= m(m-1)-2(h+H)-3\beta,\\
n+v &= 3m(m-2)-6(h+H)-8\beta,\\
y+\omega &= \tfrac{1}{2}m(m-2)(m^2-9)\\
&\quad -(m^2-m-6)\{2(h+H)+3\beta\}\\
&\quad +2(h+H)(h+H-1)\\
&\quad +6(h+H)\beta\\
&\quad +\tfrac{9}{2}\beta(\beta-1),\\
m &= r(r-1)-2(y+\omega)-3(n+v),\\
\beta &= 3r(r-2)-6(y+\omega)-8(n+v),\\
h+H &= \tfrac{1}{2}r(r-2)(r^2-9)\\
&\quad -(r^2-r-6)\{2(y+\omega)+3(n+v)\}\\
&\quad +2(y+\omega)(y+\omega-1)\\
&\quad +6(y+\omega)(n+v)\\
&\quad +\tfrac{9}{2}(n+v)(n+v-1),\\
n+v-\beta &= 3(r-m),\\
y+\omega-h-H &= \tfrac{1}{2}(r-m)(r+m-9),\\
\tfrac{1}{2}(r-1)(r-2)-(y+\omega)-(n+v)&\\
&= \tfrac{1}{2}(m-1)(m-2)\\
&\quad -(h+H)-\beta,
\end{aligned}$$

$$\begin{aligned}
n &= r(r-1)-2(x+\omega)-3(m+v),\\
\alpha &= 3r(r-2)-6(x+\omega)-8(m+v),\\
g+G &= \tfrac{1}{2}r(r-2)(r^2-9)\\
&\quad -(r^2-r-6)\{2(x+\omega)+3(m+v)\}\\
&\quad +2(x+\omega)(x+\omega-1)\\
&\quad +6(x+\omega)(m+v)\\
&\quad +\tfrac{9}{2}(m+v)(m+v-1),\\
r &= n(n-1)-2(g+G)-3\alpha,\\
m+v &= 3n(n-2)-6(g+G)-8\alpha,\\
x+\omega &= \tfrac{1}{2}n(n-2)(n^2-9)\\
&\quad -(n^2-n-6)\{2(g+G)+3\alpha\}\\
&\quad +2(g+G)(g+G-1)\\
&\quad +6(g+G)\alpha\\
&\quad +\tfrac{9}{2}\alpha(\alpha-1),\\
\alpha-(m+v) &= 3(n-r),\\
g+G-(x+\omega) &= \tfrac{1}{2}(n-r)(n+r-9),\\
\tfrac{1}{2}(r-1)(r-2)-(x+\omega)-(m+v)&\\
&= \tfrac{1}{2}(n-1)(n-2)\\
&\quad -(g+G)-\alpha,
\end{aligned}$$

and combining the two systems

$$\beta = \alpha + 2(m-n),$$
$$y = x + (m-n),$$
$$h + H = g + G + \tfrac{1}{2}(m-n)(m+n-7),$$
$$\tfrac{1}{2}(m-1)(m-2) - (h+H) - \beta = \tfrac{1}{2}(n-1)(n-2) - (g+G) - \alpha,$$
$$\tfrac{1}{2}(r-1)(r-2) - (x+\omega) - (m+v) = \tfrac{1}{2}(r-1)(r-2) - (y+\omega) - (n+v).$$

6. Taking as data r, m, n, G, H, v, ω, we find very easily

$$h = \tfrac{1}{2}(m^2 - 10m - 3n + 8r - 3v - 2H),$$
$$g = \tfrac{1}{2}(n^2 - 10n - 3m + 8r - 3v - 2G),$$
$$x = \tfrac{1}{2}(r^2 - r - n - 3m - 3v - 2\omega),$$
$$y = \tfrac{1}{2}(y^2 - r - m - 3n - 3v - 2\omega),$$
$$\alpha = m - 3r + 3n + v,$$
$$\beta = n - 3r + 3m + v.$$

The Salmon-Cremona Equations.

7. These are

$$m(r-2) = 2n + 4\beta + \gamma + 4v + 4\omega + 4H,$$
$$x(r-2) = n(r-4) + 2\beta + 3\gamma + 3t + v(3r-14) + \omega(2r-10) + 12H,$$
$$x(r-2)(r-3) = n(x-2r+8) + 3mx + 4k - 3\alpha - 9\beta - 6\gamma + v(3x-6r+18) + \omega(2x-4r+8) - 12H,$$
$$q = x^2 - x - 2k - 3\gamma - 6t - 3v(r-6) - 2\omega(r-8) - 2G - 18H,$$
$$\{= r(n-3) - 3\alpha - 2G\},$$

and

$$n(r-2) = 2m + 4\alpha + \gamma' + 4v + 4\omega + 4G,$$
$$y(r-2) = m(r-4) + 2\alpha + 3\gamma' + 3t' + v(3r-14) + \omega(2r-10) + 12G,$$
$$y(r-2)(r-3) = m(y-2r+8) + 3ny + 4k' - 3\beta - 9\alpha - 6\gamma' + v(3y-6r+18) + \omega(2y-4r+8) - 12G,$$
$$q' = y^2 - y - 2k' - 3\gamma' - 6t' - 3v(r-6) - 2\omega(r-8) - 2H - 18G,$$
$$\{= r(m-3) - 3\beta - 2H\},$$

where the second values of q, q' respectively are reduced forms obtained by the aid of the foregoing Plückerian relations.

8. Expressing these in terms of r, m, n, G, H, v, ω, we obtain

$$\gamma = rm + 12r - 14m - 6n - 8v - 4\omega - 4H,$$
$$t = \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3r(n + 3m + 3v + 2\omega) + 42n + 78m + 78v + 48\omega],$$
$$k = \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)(n + 3m + 3v + 2\omega) + (n + 3m + 3v + 2\omega)^2 - 58n - 126m - 126v - 76\omega - 24H],$$
$$q = rn + 6r - 3m - 9n - 3v - 2G,$$

and

$$\begin{aligned}
\gamma' &= rn + 12r - 14n - 6m - 8\upsilon - 4\omega - 4G,\\
t' &= \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3r(m + 3n + 3\upsilon + 2\omega) + 42m + 78n + 78\upsilon + 48\omega],\\
k' &= \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)(m + 3n + 3\upsilon + 2\omega)\\
&\qquad + (m + 3n + 3\upsilon + 2\omega)^2 - 58m - 126n - 126\upsilon - 76\omega - 24G],\\
q' &= rm + 6r - 3n - 9m - 3\upsilon - 2H.
\end{aligned}$$

9. We have thence

$$\begin{aligned}
\gamma' - \gamma &= -(r-8)(m-n) - 4(G-H),\\
t' - t &= (r-6)(m-n),\\
q' - q &= (r-6)(m-n) + 2(G-H),\\
k' - k &= \tfrac{1}{2}(m-n)(r^2 - 5r - 2m - 2n - 3\upsilon - 2\omega + 17) - 3(G-H)\\
&= \tfrac{1}{2}(m-n)(x + y - 4r + 17) - 3(G-H).
\end{aligned}$$

10. Instead of obtaining the above values of γ, t, k, q directly it is convenient to verify them by substitution in the equations from which they were obtained; viz. writing for shortness $n + 3m + 3\upsilon + 2\omega = P$, these may be written

$$\begin{aligned}
&-m(r-2) + 2n + 4\beta + 4\upsilon + 4\omega + 4H + \gamma &&= 0,\\
&-2x(r-2) + 2r(P - 3m) - 8n + 4\beta - 28\upsilon - 20\omega + 6\gamma + 24H + 6t &&= 0,\\
&-2x(r^2 - 5r + 6 - P) - 4(r-4)n - 18\beta - 12\gamma - 6\alpha + 36\upsilon + 16\omega - 24H\\
&\qquad - 4r(-n - 3m + P) + 8k &&= 0,\\
&-4q + 2x(2x-2) - 8k - 8G - 4r(-n - 3m + P) + 72\upsilon + 64\omega - 12\gamma - 24t - 72H &&= 0,
\end{aligned}$$

which are to be satisfied by

$$\begin{aligned}
\gamma &= rm + 12r - 14m - 6n - 8\upsilon - 4\omega - 4H,\\
t &= \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3rP + 42n + 78m + 78\upsilon + 48\omega],\\
k &= \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)P + P^2 - 58n - 126m - 126\upsilon - 76\omega - 24H],\\
q &= rn + 6r - 3m - 9n - 3\upsilon - 2G,\\
x &= \tfrac{1}{2}(r^2 - r - P),\\
\alpha &= m - 3r + 3n + \upsilon,\\
\beta &= n - 3r + 3m + \upsilon,\\
P &= n + 3m + 3\upsilon + 2\omega.
\end{aligned}$$

11. We have, in fact, first

$$\left.\begin{aligned}
&-mr + 2m\\
&\qquad\qquad + 2n\\
&\qquad + 12m + 4n - 12r + 4\upsilon\\
&\qquad\qquad\qquad\qquad + 4\upsilon\\
&\qquad\qquad\qquad\qquad\qquad + 4\omega\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 4H\\
&+ mr - 14m - 6n + 12r - 8\upsilon - 4\omega - 4H
\end{aligned}\right\} = 0,$$

secondly

$$
\begin{array}{l}
-r^3+r^2+rP \\
+2r^2-2r-2P \\
+2rP-6mr \\
-8n \\
-12r+4n+12m+4\upsilon \\
-28\upsilon \\
-20\omega \\
+72r+6mr-36n-84m-48\upsilon-24\omega-24H \\
+24H \\
+r^3-3r^2-58r-3rP+42n+78m+78\upsilon+48\omega \\
\hline
-2P+2n+6m+6\upsilon \qquad =0,
\end{array}
$$

that is

thirdly

$$
\begin{array}{l}
-r^4+5r^3-6r^2 \\
+r^3-5r^2+6r+P(2r^2-6r+6)-P^2 \\
-4rn+16n \\
+54r-18n-54m-18\upsilon \\
-144r-12rm+72n+168m+96\upsilon+48\omega+48H \\
+18r-18n-6m-6\upsilon \\
+36\upsilon \\
+16\omega \\
-24H \\
+P(-4r)+12rm+4rn \\
+r^4-6r^3+11r^2+66r+P(-4r^2+10r)+P^2-58n-126m-126\upsilon-76\omega-24H \\
\hline
6P-6n-18m-18\upsilon-12\omega-24H=0,
\end{array}
$$

that is

and lastly

$$
\begin{array}{l}
-24r-4rn+12m+36n+12\upsilon+8G \\
+r^4-2r^3-r^2+2r+P(-2r^2+2r+2)+P^2+126m+58n+126\upsilon+76\omega+24H \\
-r^4+6r^3-11r^2-66r+P(2r^2-10r)-P^2-8G \\
+P(-4r)+4rn+12rm+72\upsilon+64\omega \\
-144r-12rm+168m+72n+96\upsilon+48\omega+48H \\
-4r^3+12r^2+232r+P(12r)-312m-168n-312\upsilon-192\omega-72H \\
\hline
2P-6m-2n-6\upsilon-4\omega \qquad =0,
\end{array}
$$

that is

which completes the verification.

12. The deficiency of the curve m is given by the equation

$$2D_m = r - 2m + 2 + \beta,$$

or substituting for β its value $= n - 3r + 3m + v$, this is

$$2D_m = m + n - 2r + v + 2.$$

The deficiency of the nodal curve x is given by

$$2D_x = q - 2x + 2 + \gamma + v(r-6) + 2H,$$

which, substituting for q, x, and γ, the values

$$\begin{aligned} rn + 6r - 3m - 9n - 3v - 2G, \\ \tfrac{1}{2}(r^2 - r - n - 3m - 3v - 2\omega), \\ rm + 12r - 14m - 6n - 8v - 4\omega - 4H, \end{aligned}$$

respectively, becomes

$$2D_x = (r-14)(m+n+v) - r^2 + 19r + 2 - 2\omega - 2G - 2H;$$

whence, also, writing herein $m + n + v = 2D_m + 2r - 2$, we have

$$2D_x - (r-14).2D_m = (r-5)(r-6) - 2\omega - 2G - 2H,$$

a relation between the two deficiencies.

Geometrical Theory of the foregoing Relations.

13. In considering the geometrical theory, we have to speak of the original curve, or curve of the system, and also of the nodal curve; it will be convenient to call them the curve m and the curve x respectively. I speak of the torse absolutely, to signify the torse of the system, as in what follows there is not the like occasion to speak of the nexal torse. I speak also of a plane α, meaning thereby any one of the stationary planes, the number of which is $= \alpha$; and so of a line α, meaning the line in the stationary plane α; and a point α, meaning the point of contact of such line with the curve m; or in the plural, the planes α, lines α, &c. And so in other cases; thus we have the stationary tangents v, and the points v, which are the points of contact hereof with the curve m. As regards a double tangent ω, we have here two points of contact; one of these separately would be a point ω; and we may speak of the points (or pair) 2ω, meaning thereby the two points of contact of the same tangent ω; or of the points 2ω, meaning the system of the 2ω points of contact of the tangents ω.

14. Observe that the expressions, the planes α, lines α, &c., have an absolute signification; there are other such expressions which have only a relative signification, in regard to the system considered in connexion with a given point, line, or plane, as the case may require. Thus the expression, *the lines g*, must be understood of the system in connexion with *a given plane*, to signify the lines in two planes contained in the *given plane*; the planes g, points g, would of course mean the planes or points of the system belonging to the lines g.

In particular the points m are the points of the system which lie in a given plane, the lines r are the lines which meet a given line; the planes n are the planes

which pass through a given point. There will be occasion to speak, not only of the planes n, but also of the lines n and points n; these are of course the lines and points in the planes n.

15. It is to be remarked that, considering the torse as a surface, the nodal curve thereof is made up of the curve x and of the double lines ω (or its order is $=x+\omega$); the cuspidal curve is made up of the curve m, and of the stationary lines υ (or its order is $=m+\upsilon$).

The Plücker-Cayley Equations.

16. The mode of obtaining these equations has already been indicated. We in fact consider the system in connexion with an arbitrary point, and with an arbitrary plane. The point is the vertex of a cone passing through the curve m, and this cone is of the order m, the class r, with $h+H$ double lines, β stationary lines, $y+\omega$ double tangent planes, and $n+\upsilon$ stationary tangent planes; viz. the order of the cone is equal to the number of lines in which this is intersected by a plane through the vertex; but each of these is determined as the line joining the vertex with an intersection of the plane by the curve m, and the number of them is thus $=m$. The class of the cone is equal to the number of the tangent planes which can be drawn through an arbitrary line through the vertex; but this is in fact the number of lines of the system which meet the arbitrary line, viz. it is $=r$. Again, any line drawn from the vertex to meet the curve twice, and also any line drawn to one of the points H, is a double line of the cone, that is, the whole number of double lines is $=h+H$. A line from the vertex to one of the points β is a stationary line of the cone; the number of these is $=\beta$. A plane through the vertex, and containing two tangents of the curve m, or containing a double tangent ω, is a double tangent plane of the cone, the number of these is thus $=y+\omega$. A plane through the vertex, which is also a plane of the system, is a stationary tangent plane; in fact, we have here on the curve m three consecutive points lying in a plane with the vertex, the tangent plane of the cone is the plane through the vertex, and the first and second of the points on the curve; but this is also the plane through the vertex, and the second and third points, or the plane is a stationary tangent plane. But the plane through the vertex and the tangent υ is also a stationary tangent plane of the cone; and the number of stationary tangent planes is thus $=n+\upsilon$.

17. Similarly for the section by the arbitrary plane; this is a curve of the order r and class n with $x+\omega$ double points, $m+\upsilon$ stationary points, $g+G$ double tangents, and α stationary tangents. In fact, the order of the curve is equal to that of the torse; that is, to the number of lines which meet an arbitrary line, or $=r$. The class of the curve is equal to the number of tangents which pass through an arbitrary point of the plane; or, what is the same thing, the number of planes of the system which pass through this same arbitrary point, viz. it is $=n$. Each point of the plane which is the intersection of two lines of the system, and also each intersection of the plane by a line ω, is a double point of the curve; viz. the number of these is $=x+\omega$. Each intersection with the curve m, and also each intersection with the tangent υ, is a stationary point of the curve; viz. the number is $=m+\upsilon$. Each line

in the plane, which is the intersection of two planes of the system, and also each intersection of the plane with a double tangent plane G, is a double tangent of the curve; viz. the number is $=g+G$. And, finally, each intersection of the plane with a stationary plane α is a stationary tangent of the curve; viz. the number of these is $=\alpha$.

We have thus the Plückerian relations as above between

$$m,\ r,\ h+H,\ \beta,\ y+\omega,\ n+v,$$

and between

$$n,\ r,\ g+G,\ \alpha,\ x+\omega,\ n+v.$$

Zeuthen's Tables.

18. The vertex of the cone and the plane of the section may occupy special positions. We have a table given by Zeuthen, but which I have completed so far as regards the double lines ω as follows:

Cone through the Curve.

	Vertex	Order	Class	Double lines	Stationary lines	Double tangent planes	Stationary tangent planes
1	Arbitrary	m	r	h $+H$	β	y $+\omega$	n $+v$
2	On a tangent	m	$r-1$	$h-1$ $+H$	β $+1$	$y-r+4$ $+\omega$	$n-2$ $+v$
3	On the curve	$m-1$	$r-2$	$h-m+2$ $+H$	β	$y-2r+8$ $+\omega$	$n-3$ $+v$
4	At a point H	$m-2$	$r-4$	$h-2m+6$ $+H-1$	β	$y-4r+20$ $+\omega$	$n-6$ $+v$
5	At a point β	$m-2$	$r-3$	$h-2m+6$ $+H$	$\beta-1$	$y-3r+13$ $+\omega$	$n-4$ $+v$
6	On stationary tangent v	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+9$ $+\omega$	$n-3$ $+v-1$
7	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+14$ $+\omega$	$n-4$ $+v-1$
8	On double tangent ω	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+10$ $+\omega-1$	$n-4$ $+v$
9	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+15$ $+\omega-1$	$n-5$ $+v$

Plane Section of the Torse.

	Plane	Class	Order	Double tangents	Stationary tangents	Double points	Stationary points
1	Arbitrary	n	r	g $+G$	α	x $+\omega$	m $+v$
2	Through a tangent	n	$r-1$	$g-1$ $+G$	α $+1$	$x-r+4$ $+\omega$	$m-2$ $+v$
3	A tangent plane	$n-1$	$r-2$	$g-n+2$ $+G$	α	$x-2r+8$ $+\omega$	$m-3$ $+v$
4	A double tangent plane G	$n-2$	$r-4$	$g-2n+6$ $+G-1$	α	$x-4r+20$ $+\omega$	$m-6$ $+v$
5	A stationary tangent plane α	$n-2$	$r-3$	$g-2n+6$ $+G$	$\alpha-1$	$x-3r+13$ $+\omega$	$m-4$ $+v$
6	Through stationary tangent v	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+9$ $+\omega$	$m-3$ $+v-1$
7	Tangent plane at contact of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+14$ $+\omega$	$m-4$ $+v-1$
8	Through double tangent ω	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+10$ $+\omega-1$	$m-4$ $+v$
9	A tangent plane at one of contacts of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+15$ $+\omega-1$	$m-5$ $+v$

19. To avoid confusion with the geometrical term line, I will speak (not of the lines, but) of the cases of these tables; the numbers in each case satisfy the foregoing Plückerian relations. To fix the ideas, I attend to the second table. We require to know in each case, say the numbers in the (n, r, α) columns; these being known, the other three numbers will be determined.

20. First for the n column; for the Cases 2, 6, 8, the plane is not a tangent plane; the number of tangent planes which pass through a fixed point in the plane is still $=n$. For Cases 3, 7, 9, the plane is a tangent plane, it therefore counts 1 among the tangent planes which pass through a fixed point thereof; and the number of the remaining tangent planes is $=n-1$. And so for Cases 4 and 5, the plane counts for 2 among the tangent planes which pass through a fixed point thereof, and the number of the remaining tangent planes is $=n-2$.

21. Next as to the r and α columns.

Case (2). The plane passes through a generating line, and therefore besides cuts the torse in a curve of the order $r-1$; this generating line is a stationary tangent cutting the curve $r-1$ in the point of contact with the curve m, counting 3 times (in all $\alpha+1$ stationary tangents), and in $r-4$ other points.

Case (3). The plane cuts the torse in the generating line counting twice, and in a curve of the order $r-2$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m (so that the number of stationary tangents remains $=\alpha$), and besides cuts the curve in $r-4$ points as in Case 2.

Case (4). The plane cuts the torse in two generating lines, each counting twice, and in a curve of the order $r-4$; each of the generating lines is in regard to this curve an ordinary tangent at the point of contact with the curve m (number of stationary tangents remains $=\alpha$), and besides cuts the curve in $r-6$ other points.

Case (5). The plane cuts the torse in a generating line counting 3 times, and in a curve of the order $r-3$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m, and besides cuts it in $r-5$ points. The plane being in the present case a plane α, its intersections with the remaining planes α, give the $\alpha-1$ stationary tangents.

Case (6). The plane meets the torse in a generating line counting twice, and in a curve of the order $r-2$; the generating line is in regard to the curve a singular tangent meeting it in the point of contact with the curve m, counting 4 times, and besides meeting it in $r-6$ points. The generating line in respect of this four-pointic intersection counts as a stationary tangent twice; and the number of stationary tangents is $=\alpha+2$.

Case (7). The plane meets the torse in the generating line counting 3 times, and in a curve of the order $r-3$; the generating line is in regard to this curve a stationary tangent at the point of contact with the curve m, counting 3 times, and besides meeting it in $r-6$ points as in Case 6. The whole number of stationary tangents is thus $=\alpha+1$.

Case (8). The plane meets the torse in a generating line counting twice and in a curve of the order $r-2$. The generating line is in respect of the curve a stationary tangent at each of the points of contact with the curve m, viz. each of these points counts 3 times, and there are besides $r-8$ intersections. Moreover, the generating line counting as 2 stationary tangents, the whole number of stationary tangents is $=\alpha+2$.

Case (9). The plane meets the torse in a generating line counting 3 times, and in a curve of the order $r-3$. The generating line is in respect of the curve an ordinary tangent at one of the points of contact with the curve m (viz. the point at which the plane is a tangent plane of the torse), so that we have here two intersections; and it is a stationary tangent at the other of the points of contact with the curve m (viz. the point at which the plane is not a tangent plane of the torse),

so that there are here 3 intersections; there are therefore $r-8$ other intersections. The generating line reckons once as a stationary tangent, and the whole number of stationary tangents is $=\alpha+1$.

22. The $r-6$ points in Cases 6 and 7 are the points of intersection of the stationary tangent υ by other tangents of the curve m; and so the $r-8$ points in Cases 8 and 9 are the points of intersection of the double tangent ω by other tangents of the curve m; these numbers $r-6$ and $r-8$ will present themselves in the sequel.

We may as to the α-column sum up by saying, that in Cases 2, 7, 9 there is a generating line which reckons as a stationary tangent; in Case 6 a generating line, which, in respect of 4 consecutive intersections, reckons as two stationary tangents; and in Case 8 a generating line, which, in respect of two pairs of 3 consecutive intersections, reckons as two stationary tangents.

In the $(x+\omega)$ and $(m+\upsilon)$-columns, observe that in Cases 6, 7, 8, 9, we have in the first two ω, $\upsilon-1$, and in the last two $\omega-1$, υ; viz. these numbers refer to the intersections of the plane with the tangent ω, υ respectively; the actual numbers $\begin{pmatrix} x-2r+9 \\ +\omega \end{pmatrix}$ and $\begin{pmatrix} x-2r+10 \\ +\omega-1 \end{pmatrix}$, &c. are equal for the Cases 6 and 8, and also for the Cases 7 and 9. So in the $g+G$ column in Case 4, we have $G-1$ for G.

The Nodal Curve x; Intersections with the Curve m; and Singularities.

23. The intersections of the curve m with the nodal curve x, are points α, β, γ, H, υ or ω.

24. At a point α, four consecutive points of the curve m lie in a plane; the point may be considered as the intersection of two consecutive tangents, viz. of the line through two consecutive points with that through the next two consecutive points; and it is thus a point on the curve x. We may imagine the points A, A' starting from α in opposite directions along the curve m, and moving in such manner that the tangents at these two points respectively continually intersect; we have thus a portion of the curve x, proceeding apparently in one direction only from the point α; and being, as regards the portion in question, an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from α, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve x thus meets the curve m in the several points α, the curve x, each time that it passes through such a point of intersection, changing its character from crunodal to acnodal. The two *half-sheets*[1] of the torse cross each other in the crunodal portion, extending

[1] In a curve (plane or twisted), the portions extending each way from a cusp (and considered without reference to a termination) are called half-branches; and so in a surface which has a cuspidal curve, or in particular, in a torse, the portions extending each way from the cuspidal curve (and considered without reference to a termination) are called half-sheets. In the case of higher singularities of a like nature, we may speak of a partial branch or partial sheet (as the case may be).

in one direction from the point α of the nodal curve; and the acnodal portion extends in the opposite direction from the same point.

25. A point β, or stationary point (cusp) on the curve m, is the intersection of four consecutive osculating planes; and it is thus a point of intersection of two consecutive tangents (viz. of the line of intersection of two consecutive osculating planes, and of that of the next two consecutive osculating planes), consequently a point on the curve x. We may imagine the points B, B' starting from β, and moving along the two half-branches of the curve m, in such manner that the tangents at the two points respectively continually intersect. We have thus a portion of the curve x, proceeding apparently in one direction only from the point β (and having at this point a common tangent with the curve m), and being as regards this portion thereof an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from the point β, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve thus meets the curve m in each of the points β, the curve x, each time that it passes through such a point, changing its character from crunodal to acnodal. There are at β three partial sheets of the torse; viz. if we imagine at this point the half-tangent b in the sense of the two half-branches of the curve m, and the half-tangent b' in the opposite sense, then we have through b' and one of the half-branches a partial sheet, and through b' and the other half-branch a partial sheet; these two partial sheets touching along b', and intersecting in the crunodal portion of the curve x; and a third partial sheet through b and the two half-branches of the curve m.

26. At a point γ on the cuspidal curve m, we have, traversing the curve and the two half-sheets which meet along it, another sheet of the torse, meeting the two half-sheets respectively in two half-branches, which are a portion of the nodal curve x, and which unite together (as at a cusp), in the point γ, which is thus a cusp or stationary point on the curve x.

27. A point H is the intersection of two branches of the cuspidal curve m. There are for each branch two half-sheets; and we have thus at the point H four (say) quarter-branches of the curve x, touching each other at the point (viz. the common tangent is the intersection of the osculating planes belonging to the two branches of the curve m respectively). I find in a special manner that in regard to the curve x a point H is equivalent to six double points, plus two stationary points; it thus causes a reduction $2\cdot 6 + 3\cdot 2 = 18$ in the class of the curve.

28. The nodal curve x has in each of the double tangent planes G an actual double point. In fact the plane is an osculating plane of the curve m at two points thereof; that is, it contains two consecutive tangents R, R', and two other consecutive tangents S, S'; hence RS is a point on the nodal curve x; and not only so, but this is an actual double point, the two tangents being R and S; for since R is met by S and S', there is a consecutive point on the line R; that is, R is a tangent; and similarly since S is met by R and R' there is a consecutive point on the line S; that is, S is also a tangent.

29. At a point v the tangent has with the curve x a 3-pointic intersection (whence also the tangent is a stationary tangent v in regard to the curve x), the curves m and x have also a 3-pointic intersection at v.

30. At each of the two points ω the tangent has with the curve x a 3-pointic intersection (viz. the tangent is in regard to the curve x more than a double tangent ω, instead of two 2-pointic intersections, or ordinary contacts, there are two 3-pointic intersections; I am unable to perceive this directly, but accept it on other grounds). But as at each of the points ω the intersection of the tangent with the curve m is 2-pointic, the intersection of the curves x and m is only 2-pointic.

31. The curves m and x meet in the points α each once, β each 3 times, γ each twice, v each 3 times, 2ω each twice, and H each 8 times: we have thus

$$\alpha + 3\beta + 2\gamma + 3v + 4\omega + 8H$$

for the number of actual intersections of the two curves; and the number of apparent intersections is therefore

$$= mx - \alpha - 3\beta - 2\gamma - 3v - 4\omega - 8H,$$

a result which is required in the sequel.

32. Consider the cone (vertex an arbitrary point) through the curve x, or say simply the cone x.

Each line n (*quà* ordinary line of the system) is met by $r-4$ other lines, the $r-4$ points being situate on the curve x; the line n at each of these points *touches* the cone x, and it therefore besides intersects it in $x-2r+8$ points.

33. A line v meets the curve x in the point v counting 3 times, and in $r-6$ points each a stationary point of the curve; it consequently meets the cone x, in the point v counting 3 times, in each of the $r-6$ points counting twice, and besides in $x-2r+9$ points.

34. A line ω meets the curve x in the two points ω each counting 3 times, and in $r-8$ points each an actual double point of the curve; it consequently meets the cone x in the two points ω each counting 3 times, in the $r-8$ points each counting twice, and besides in $x-2r+10$ points.

35. Each of the $r-8$ points in which the double tangent ω meets another tangent of m is an actual double point of the curve x; and each of the $r-6$ points in which a stationary tangent v meets another tangent of m is a stationary point or cusp on the curve x. We have thus as singularities of the curve x, the k apparent double points, G actual double points, $\omega(r-8)$ ditto, t triple points, H 4-branch cuspidal points, γ stationary or cuspidal points, $v(r-6)$ ditto. In regard to the effect upon the class of the curve x, each of the points t is equivalent to 3 double points and produces a reduction $=6$; each of the points H is equivalent to 6 double points $+2$ cusps, and produces a reduction $2\cdot 6+3\cdot 2=18$, as already mentioned. We have thus the relation

$$q = x(x-1) - 2k - 2G - 3v(r-6) - 2\omega(r-8) - 3\gamma - 6t - 18H,$$

which is one of the Salmon-Cremona equations.

36. The stationary points of the curve x are the points γ, the $\upsilon(r-6)$ points, and the points H, each counting as 2 stationary points; that is, we have

$$2D_x = q - 2x + 2 + \gamma + \upsilon(r-6) + 2H,$$

used above for finding the value of the deficiency D_x.

The remaining Salmon-Cremona Equations.

37. The formulæ for $m(r-2)$, $x(r-2)$ and $x(r-2)(r-3)$ correspond to those for $c(n-2)$, $b(n-2)$ and $b(n-2)(n-3)$ in the case of a general surface (*Solid Geometry*, 2nd Ed., Nos. 522 and 525 [4th Ed., Nos. 610, 613]), being obtained as follows; considering as before the cone, vertex an arbitrary point, which passes through the curve m, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve m is $=m(r-2)$; and, similarly, considering the cone, vertex the same arbitrary point, which passes through the curve x, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve x is $=x(r-2)$; viz. these numbers $m(r-2)$ and $x(r-2)$ are the numbers of the intersections of the two curves respectively with the surface of the order $(r-2)$, which is the second polar of the arbitrary point in regard to the torse. And so also in the cone, vertex the arbitrary point, which passes through the curve x, the total number of lines which touch the torse at a point not on the curve x is $=x(r-2)(r-3)$; viz. this is obtained by Salmon, the number of intersections of the curve x with a certain surface of the order $(r-2)(r-3)$ (Salmon, Nos. 269 and 273, writing r for n), but in a preferable manner by Cremona thus; the cone through the curve x meets the torse in the curve x counting twice and in a residual curve of the order $x(r-2)$; the lines in question are the tangents from the vertex to this curve, which is a curve meeting each line of the cone $r-2$ times; we may on the surface of the cone metrically, as in the plane, construct the polar of the vertex in regard to the curve of the order $x(r-2)$; viz. we have thus on the cone a curve meeting each generating line $(r-3)$ times; and this polar curve meets the curve of the order $x(r-2)$ in $x(r-2)(r-3)$ points (this would be clearly the case if only the curve of the order $x(r-2)$ were the complete intersection of the cone by a surface of the order $r-2$, for then the polar curve would be the intersection of the cone by the polar surface of the order $r-3$; but in the case in hand, where this is not so, some additional considerations would be required in order to sustain the result), and we have thus the number $x(r-2)(r-3)$ of the lines in question.

38. I consider the second polar surface, order $r-2$, which belongs to a given arbitrary point. Any point on the torse, such that the line joining it with the arbitrary point cuts the torse 3-pointically at the point on the torse, is a point on the second polar $r-2$. Such points are, as will be shewn, the points n, $n(r-4)$, β, γ, υ, 2ω, $\upsilon(r-6)$, $\omega(r-8)$, t, H.

39. We may consider through any such point and the arbitrary point either a particular section of the torse or any section whatever. If the line joining the two points has at the point in question a 3-pointic intersection with the section, then it has a 3-pointic intersection with the torse, viz. the point possesses the required

property. For the points n and $n(r-8)$, the plane may be taken to be the osculating plane of the curve m. This meets the torse in the line n twice, and in a curve of the order $r-2$ touching this line at the point n and meeting it in the $r-4$ points. Hence, considering the complete section made up of the line twice and the curve of the order $r-2$, the line from the arbitrary point to the point n or to any one of the $r-4$ points meets the section 3-pointically; viz. the lines to the points n and to the points $n(r-4)$ meet the torse 3-pointically.

40. For the points β; we may consider any section through the arbitrary point and β; in effect any section through the point β. A plane section through a point β has at this point an invisible triple point, that is a point not in appearance differing from an ordinary point of the curve, but which by considering a consecutive position of the plane of section is seen to be equivalent to a double point and two cusps; viz. the node is a point of intersection of the plane with the curve x, the cusps are the two intersections with the curve m, in the neighbourhood of the point β. The line from β to the arbitrary point has thus with the torse a 3-pointic intersection at β.

41. Similarly for the points γ; we take any section through the arbitrary point and γ; in effect any section through the point γ. The section through a point γ has at this point a triple point, at which an ordinary branch passes through a cusp, and thus equivalent to a cusp $+2$ nodes; in fact, for a consecutive position of the cutting plane, the section has actually a cusp and two nodes; the cusp at the intersection of the plane with the curve m, the nodes at the two intersections of the plane with the curve x in the neighbourhood of the cusp γ. The line through the point γ has thus a 3-pointic intersection with the torse.

42. For a point υ I consider the section through the tangent υ; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 4-pointic intersection at the point υ, and besides meeting it in $r-6$ points. Hence in the plane, a line through υ, or through one of the $r-6$ points has at such point a 3-pointic intersection with the curve. And thus the lines through the points υ and $\upsilon(r-6)$ respectively have a 3-pointic intersection with the torse.

43. Similarly for a point ω, I consider the section through a tangent ω; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 3-pointic intersection at each of the points ω, and besides meeting it in $r-8$ points. Hence in the plane a line through either of the points ω or through one of the $r-8$ points has with the curve a 3-pointic intersection, and thus the lines through the points 2ω and $\omega(r-8)$ respectively have a 3-pointic intersection with the torse.

44. For a point H I consider any section through the arbitrary point and H; in effect any section through H. There is at H a singularity $= 6$ nodes $+ 2$ cusps. But a line through H in the plane of the section cuts the section in 4 points only; that is, the line from the arbitrary point to H has with the curve a 4-pointic intersection at H; and *à fortiori* it may be regarded as a line of 3-pointic intersection. The lines to the points H have thus a (4-pointic, that is, more than) 3-pointic intersection with the torse.

45. For a point t I consider any section through the arbitrary point and t; in effect any section through t; there is at t a triple point, and the line through t has thus a 3-pointic intersection at t. Hence a line through t has 3-pointic intersections with the torse.

46. The several points above referred to lie on the curve m or on the curve x, or on both of these curves; and each curve has at these points respectively a simple or multiple intersection with the polar surface $r-2$, as shown in the table.

	Intersection of second polar $r-2$	
	with curve m	curve x
Points n	2	0
,, $n(r-4)$	0	1
,, β	4	2
,, γ	1	3
,, v	4	4
,, 2ω	2	3
,, $v(r-6)$	0	3
,, $\omega(r-8)$	0	2
,, H	4	12
,, t	0	3

where the figures 1, 2, &c. denote a simple intersection, 2-pointic intersection, &c. of the curve and surface; 0 denotes of course that there is not any intersection, viz. that the curve does not pass through the point referred to.

47. Several of the foregoing numbers are obtained without difficulty; thus we see that the points n, $n(r-4)$ are ordinary points on the second polar $r-2$, the surface at each of the points n *touching* the curve m, but at each of the points $n(r-4)$ simply cutting the curve x. So also the points β, γ, v, 2ω, $v(r-6)$, $\omega(r-8)$ are ordinary points on the second polar; at a point β the two half-branches of the curve m touch the surface in a special manner so as to give a 4-pointic intersection; whereas the curve x simply touches the surface. At γ the curve m cuts the surface, but the two half-branches of x *touch* the surface. At v, each of the curves m, x has a 4-pointic intersection with the surface; at each of the two points ω, the curve m touches the surface, but the curve x has with it a 3-pointic intersection, and at $\omega(r-8)$ it simply touches the surface.

48. The point H is in the nature of a biplanar point on the polar surface; this appears, or is at least indicated, by the circumstance that the line to the arbitrary point has with the torse (not a 3-pointic but) a 4-pointic intersection; the two branches of the curve m each simply cut the two coincident sheets of the polar surface, giving 2×2, $=4$ intersections; but for the curve x, the four partial branches each touch the

two coincident sheets; for a single sheet the number of intersections would be 6, but for the two coincident sheets it is twice this, or $=12$. Finally, a point t is an ordinary point on the second polar, each of the three branches of x simply cuts the surface, or the number of intersections is $=3$.

49. The last table gives at once

$$m(r-2) = \quad 2n+4\beta+\gamma+4\upsilon+2.2\omega+4H,$$
$$x(r-2) = (r-4)+2\beta+3\gamma+4\upsilon+3.2\omega+3\upsilon(r-6)+2\omega(r-8)+12H+3t,$$

which are the true theoretical forms of the equations for $m(r-2)$ and $x(r-2)$, in which these were obtained by Cremona.

50. The $x(r-2)(r-3)$ points are those points in which the Cremona $x(r-2)$ curve is met 2-pointically by the line from the arbitrary point (I recall that taking the arbitrary point as the vertex of a cone through the curve x, this cone, say the cone x, meets the torse in the curve x twice, and in the $x(r-2)$ curve in question); viz. these points are either points of contact of tangents from the vertex to the $x(r-2)$ curve; or they are double points, or else cusps of the $x(r-2)$ curve; in which several cases respectively they count 1, 2 or 3 times, among the $x(r-2)(r-3)$ points.

51. The points of contact are the $n(x-2r+8)$ points of intersection of the lines n with the cone x. We have in fact a plane n through the vertex of the cone, and in this plane two consecutive lines of the system; hence at each of the $x-2r+8$ points the generating line of the cone meets the two consecutive lines of the system; that is, there is with the curve $x(r-2)$ a 2-pointic intersection, not arising out of any singularity of the curve, and consequently a contact of this curve with the generating line of the cone.

52. The actual double points of the curve $x(r-2)$ are first the $2k$ apparently coincident points of the curve x, and secondly the $\omega(x-2r+10)$ points on the lines ω. For first if we consider through the vertex a line meeting the curve x in two points, say A, B, this meets the torse in these points each twice and in $r-4$ other points. Now imagine a line from the vertex to the point P in the vicinity of A, this meets the torse in the point P twice and in $r-2$ points, which are points on the $x(r-2)$ curve; hence as P travels through A, 2 of the $r-2$ points come together at B, and again separate, that is B is an actual double point on the $x(r-2)$ curve; and similarly A is an actual double point on the curve; and we have thus the $2k$ double points. Secondly, since the line ω is a nodal line on the torse, a generating line of the cone, in the neighbourhood of and considered as travelling through one of the $x-2r+10$ points, meets the torse in two points which come to coincide and then again separate; that is each of the $x-2r+10$ points is an actual double point on the curve $x(r-2)$; and the whole number of these is $=\omega(x-2r+10)$.

53. The stationary points of the curve $x(r-2)$ are first the points on the curve m which apparently coincide with the curve x; viz. the number of these, as was seen, is $=mx-\alpha-3\beta-2\gamma-3\upsilon-4\omega-8H$; secondly, the $\upsilon(x-2r+9)$ points on the lines υ;

thirdly, the points H each counting as 4 cusps. For first consider a generating line meeting the curve x in B and the curve x in A; if we imagine on the curve x a point Q which approaches and ultimately coincides with B, the generating line through Q meets the torse in the neighbourhood of its cuspidal edge in two points which come ultimately to coincide with the point A, and we thus see that A is a stationary point on the $x(r-2)$ curve.

54. Secondly, observing that the line v is a cuspidal line on the torse, and considering in like manner a generating line of the x cone, which approaches and comes ultimately to coincide with one of the $x-2r+9$ points, we see that this is a stationary point on the $x(r-2)$ curve. And thirdly, any line through a point H meets the torse in this point counting 4 times, and in $r-4$ other points. Hence considering the generating line of the x cone, which travelling along any one of the four partial branches of the x curve comes ultimately to coincide with H, 2 of the $r-2$ points on such generating line come to coincide at the point H; and we have thus the point H as a singular point on the $x(r-2)$ curve; viz. it reckons as a stationary point once in respect of each of the four partial branches of the curve x (it must be assumed that this is so, but a further proof is required), that is as 4 cusps on the $x(r-2)$ curve.

55. By what precedes we have

$$\begin{aligned} x(r-2)(r-3) = \quad & n(x-2r+8) \\ & + 2\{2k+\omega(x-2r+10)\} \\ & + 3\{(mx-\alpha-3\beta-2\gamma-3v-4\omega-8H)+v(x-2r+9)+4H\}, \end{aligned}$$

which is the true theoretical form in which the equation for $x(r-2)(r-3)$ was obtained by Cremona.

500.

ON A THEOREM RELATING TO EIGHT POINTS ON A CONIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 344—346.]

THE following is a known theorem:

"In any octagon inscribed in a conic, the two sets of alternate sides intersect in the 8 points of the octagon and in 8 other points lying in a conic."

In fact the two sets of sides are each of them a quartic curve, hence any quartic curve through 13 of the $8+8$ points passes through the remaining 3 points: but the original conic together with a conic through 5 of the 8 new points form together such a quartic curve; and hence the remaining 3 of the new points (inasmuch as obviously they are not situate on the original conic) must be situate on the conic through the 5 new points, that is the 8 new points must lie on a conic.

We may without loss of generality take $(\alpha_1^2, \alpha_1, 1)$, $(\alpha_2^2, \alpha_2, 1)$, ... $(\alpha_8^2, \alpha_8, 1)$, as the coordinates (x, y, z) of the 8 points of the octagon; and obtain hereby an *à posteriori* verification of the theorem, by finding the equation of the conic through the 8 new points: the result contains cyclical expressions of an interesting form.

Calling the points of the octagon 1, 2, 3, 4, 5, 6, 7, 8, the 8 new points are

$$12.45,\quad 23.56,\quad 34.67,\quad 45.78,\quad 56.81,\quad 67.12,\quad 78.23,\quad 81.34,$$

viz. 12.45 is the intersection of the lines 12 and 45; and so on. The 8 points lie on a conic, the equation of which is to be found.

The equation of the line 12 is

$$x-(\alpha_1+\alpha_2)\,y+\alpha_1\alpha_2 z=0,$$

or as it is convenient to write it

$$x-(1+2)\,y+12.z=0,$$

viz. 1, 2, &c., are for shortness written in place of α_1, α_2, &c. respectively.

The coordinates of the 12.45 are consequently proportional to the terms of

$$1,\ -(1+2),\ 12,$$
$$1,\ -(4+5),\ 45,$$

or say they are as

$$12(4+5)-45(1+2) : 12-45 : 1+2-(4+5).$$

The equation of the line (12.45)(23.56) which joins the points 12.45 and 23.56 thus is

$$\begin{vmatrix} x, & y, & z \\ 12(4+5)-45(1+2), & 12+45, & (1+2)-(4+5) \\ 23(5+6)-56(2+3), & 23-56, & (2+3)-(5+6) \end{vmatrix} = 0,$$

where the determinant vanishes identically if $2-5=0\ (\alpha_2-\alpha_5=0)$; it in fact thereby becomes

$$\begin{vmatrix} x, & y, & z \\ 2^2(1-4), & 2(1-4), & (1-4) \\ 2^2(3-6), & 3(3-6), & (3-6) \end{vmatrix},$$

which is $=0$; the determinant divides therefore by $2-5$; the coefficient of x is easily found to be

$$=(2-5)(12-23+34-45+56-61),$$

and so for the other terms; and omitting the factor $2-5$ the equation is

$$\begin{aligned} & x\{12-23+34-45+56-61\} \\ & -y\{12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)\} \\ & +z\{1234-2345+3456-4561+5612-6123\}=0. \end{aligned}$$

There is now not much difficulty in forming the equation of the required conic; viz. this is

$$\begin{aligned} & (2-8)\{x-(6+7)y+67z\}\times \\ & \left\{\begin{aligned} & x[12-23+34-45+56-61] \\ & -y[12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)] \\ & +z[1234-2345+3456-4561+5612-6123] \end{aligned}\right\} \\ & +(6-8)\{x-(1+2)y+12z\}\times \\ & \left\{\begin{aligned} & x[23-34+45-56+67-72] \\ & -y[23(5+6)-34(6+7)+45(7+2)-56(2+3)+67(3+4)-72(4+5)] \\ & +z[2345-3456+4567-5672+6723-7234] \end{aligned}\right\} \\ & =0. \end{aligned}$$

In fact this equation written with an indeterminate coefficient λ, say, for shortness, thus

$$67\,[(12\,.\,45)(23\,.\,56)] = \lambda 12\,.\,[(23\,.\,56)(34\,.\,67)] = 0,$$

is the general equation of the conic through the 4 points 12.67, 34.67, 12.45, and 23.56; and by making the conic pass through 1 of the remaining 4 of the 8 points, I succeeded in finding the value $\lambda = \frac{6-8}{2-8}$, so that the conic in question passes through 5 of the 8 points, and is therefore by the theorem the conic through the 8 points. But as thus written down the equation contains the extraneous factor $2-6$, as appears at once by the observation that the left-hand side on writing therein $6 = 2\,(\alpha_6 = \alpha_2)$ becomes identically $=0$; the value in fact is

$$\begin{aligned}&-(2-8)\,[x-(2+7)\,y+27z]\,(23-34+45-52)\,[x-(1+2)\,y+12z]\\&+(2-8)\,[x-(1+2)\,y+12z]\,(23-34+45-52)\,[x-(2+7)\,y+27z]\end{aligned}$$

which is $=0$; there is consequently the factor $2-6$ to be rejected, and throwing this out the equation assumes a symmetrical form in regard to the 8 symbols 1, 2, 3, 4, 5, 6, 7, 8. The coefficient of x^2 is very easily found to be

$$=(2-6)\,(12-23+34-45+56-67+78-81),$$

and similarly that of z^2 to be

$$=(2-6)\,\{123456-234567+345678-456781+567812-678123+781234-812345\}:$$

those of the other terms are somewhat more difficult to calculate; but the final result, throwing out the factor $(2-6)$, and introducing an abbreviated notation

$$\Sigma 12 = (12-23+34-45+56-67+78-81),$$

and the like in other cases, is found to be

$$\begin{aligned}&x^2\,.\,\Sigma 12\\&+y^2\,.\,[\Sigma 12\,(4+5)\,(6+7)-\tfrac{1}{2}\,\Sigma 1256]\\&+z^2\,.\,\Sigma 123456\\&-yz\,.\,\Sigma 16\,(234+235+245+345)\\&+zx\,.\,[\Sigma 1234 \qquad\qquad +\tfrac{1}{2}\,\Sigma 1256]\\&-xy\,.\,\Sigma 12\,(4+5+6+7) \qquad\qquad =0,\end{aligned}$$

where it is to be observed that $\Sigma 1256$ consists of 4 distinct terms each twice repeated: $\frac{1}{2}\,\Sigma 1256$ consists therefore of these 4 terms; and in the coefficient of y^2 they destroy 4 of the 32 terms of $\Sigma 12\,(4+5)\,(6+7)$ so that the coefficient of y^2 contains $32-4$, $=28$ terms. In the coefficient of zx there is no destruction, and this contains therefore $12+4$, $=16$ terms.

501.

REVIEW. *Tables de Logarithmes vulgaires à dix decimales construites d'après un nouveau mode* par S. Pineto, approuvées par l'Académie des Sciences de S. Pétersbourg. S. Pétersbourg, 1871.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 375—376.]

THE tables occupy 56 pages—the principal one being a table in 44 pages, the 22 left-hand pages containing the 10 figure logarithms of the numbers from 1,000,000 to 1,010,999, and the 22 right-hand pages the proportional parts ·01, ·02, ... ·99 of the differences. A like table 100,000 to 999,999 would occupy 3600 pages. By means of an auxiliary table of 3 pages, and of a slight increase of the numerical calculation, *the table of* 44 *pages does the work of the table of* 3600 *pages*. To explain how this is: the auxiliary table gives for any number A the initial four digits of which are equal to or exceed 1011, a multiplier M, such that in the product MA the initial four digits are between 1000 and 1011; this multiplier M contains only 1, 2 or 3 figures, and when there are 3 figures, then in general either the middle figure is 0, or two of the figures are equal; the table gives also $\log \frac{1}{M}$ to 12 decimals; and there is a third column, as will be explained. Hence A being as above, the auxiliary table gives M, we form the product MA, obtain the logarithm thereof from the principal table, and adding thereto $\log \frac{1}{M}$, we have the required $\log A$. Conversely, when there is given a logarithm B the first five digits in the mantissa of which are not included between 00000 and 00474 (being the limits of the first five digits of the logarithms in the principal table), the auxiliary table by means of its third column gives M; adding $\log \frac{1}{M}$ to B, we have a logarithm included in the limits of

the principal table, and seeking for the corresponding number, this is $=\frac{1}{M}$ number having B for its logarithm: that is, the required number is $=M$ times the number obtained as above. Of course as regards the principal Table, the proportional parts are employed in the usual manner; the tabulation of them to hundredths (instead of tenths) facilitates the interpolation; for better securing the accuracy of the last figure, directions are given in regard to the 11th and 12th figures. An example of the determination of a logarithm is as follows:

$$\begin{array}{rr}
\pi = & 3{\cdot}14159\ 26536 \\
M = & 32 \\
\hline
M\pi = & 100{\cdot}53096\ 49152 \\
\log\frac{1}{M} = & 8{\cdot}49485\ 00216.80 - 10 \\
\log = & 2{\cdot}00229\ 95705.75 \\
64 & 2764.80 \\
91 & 39.31 \\
52 & 22 \\
\hline
\log\pi = & 0{\cdot}49714\ 98726.88
\end{array}
\quad \left.\begin{array}{r} \\ \\ \\ \\ \end{array}\right\} D = 4320$$

(correct value of last two figures $= 94$).

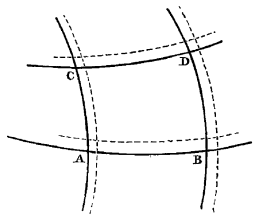
The labour saved by the small bulk of the Tables goes far to balance that occasioned by the additional steps in the calculation.

502.

ON THE SURFACES DIVISIBLE INTO SQUARES BY THEIR CURVES OF CURVATURE.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 8, 9. Read December 14, 1871.]

GEOMETRICALLY, the question is as follows:—Consider any two curves of curvature AB, CD of one set, and any two AC, BD of the other set, as shown by the continuous lines of the figure: drawing the consecutive curves as shown by dotted lines, the curve consecutive to AB at an arbitrary (infinitesimal) distance from AB, the other three curves may be drawn at such distances that the elements at A, B, and C shall be each of them a square; but this being so, the element at D will not be in general a square, and it is only for certain surfaces that it is so. But if (whatever the curves



of curvature AB, CD, AD, BC may be) the element at D is a square, then it is clear that the whole surface can be, by means of its curves of curvature, divided into infinitesimal squares.

Analytically, if for a given surface the equations of its curves of curvature are expressed in the form $h=f(x, y, z)$, $k=\phi(x, y, z)$; then the coordinates x, y, z can

be expressed each of them as a function of the parameters h, k, and we have for the element of distance between two consecutive points on the surface

$$dx^2 + dy^2 + dz^2 = A\,dh^2 + C\,dk^2,$$

where A, C are in general each of them a function of h and k. The condition for the divisibility into squares is that the quotient $A \div C$ shall be of the form function h $\div$ function k.

It was shown by M. Bertrand that, in a triple system of orthotomic isothermal surfaces, each surface possesses the property in question of divisibility into squares by means of its curves of curvature. But in such a triple system, each surface of the system is necessarily a quadric; so that the theorem comes to this, that a quadric surface is, by means of its curves of curvature, divisible into squares. The analytical verification is at once effected: taking the equation of the surface to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

then the expressions for the coordinates in terms of the parameters h, k of a curve of curvature are

$$x^2 = \frac{a\,(a+h)\,(a+k)}{(a-b)\,(a-c)},$$

$$y^2 = \frac{b\,(b+h)\,(b+k)}{(b-c)\,(b-a)},$$

$$z^2 = \frac{c\,(c+h)\,(c+k)}{(c-a)\,(c-b)},$$

and we have

$$4\,(dx^2 + dy^2 + dz^2) = (h-k)\left\{\frac{h\,dh^2}{(a+h)\,(b+h)\,(c+h)} - \frac{k\,dk^2}{(a+k)\,(b+k)\,(c+k)}\right\};$$

so that $A \div C$ is of the required form.

But there is nothing to show that the property is confined to quadric surfaces; and the question of the determination of the surfaces possessing the property appears to be one of considerable difficulty, and which has not hitherto been examined.

503.

ON THE SURFACES EACH THE LOCUS OF THE VERTEX OF A CONE WHICH PASSES THROUGH m GIVEN POINTS AND TOUCHES $6-m$ GIVEN LINES.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 11—47. Read January 11, 1872.]

I CONSIDER the surfaces, each of them the locus of the vertex of a (quadri-)cone which passes through m given points and touches $6-m$ given lines; viz. calling the given points $a, b, c, \ldots$ and the given lines $\alpha, \beta, \gamma, \ldots$, the surfaces in question are:

	Order
$abcdef$	4
$abcde\alpha$	8
$abcd\alpha\beta$	16
$abc\alpha\beta\gamma$	24
$ab\alpha\beta\gamma\delta$	24
$a\alpha\beta\gamma\delta\epsilon$	14
$\alpha\beta\gamma\delta\epsilon\zeta$	8

I remark that the orders of these several surfaces are in effect determined by the investigations of M. Chasles in regard to the conics in space which satisfy seven conditions. The surface $abcdef$ was long ago considered by M. Chasles, and it is treated of in my "Memoir on Quartic Surfaces," [445], and in the same Memoir the surface $\alpha\beta\gamma\delta\epsilon\zeta$ is also referred to: these two surfaces, and also the surfaces $a\alpha\beta\gamma\delta\epsilon$ and $ab\alpha\beta\gamma\delta$ are considered by Dr Hierholzer (1) in his excellent paper "Ueber Kegelschnitte im Raume," *Math. Annalen*, t. II. (1870), pp. 563—586, and to him are due the equations given in the sequel for the surfaces $abcdef$ and $\alpha\beta\gamma\delta\epsilon\zeta$: the researches of the present Memoir are in fact a continuation and development of those in the Memoir last referred to.

[1] I was grieved to hear of Dr Hierholzer's death last autumn, at Carlsruhe, at the early age of 30.

Table of Singularities, and Explanations in regard thereto.

1. We have on the before-mentioned surfaces respectively certain simple or multiple points, right lines, and curves, as shown in the following Table:

	$abcdef$, 4	$abcdea$, 8	$abcda\beta$, 16	$abca\beta\gamma$, 24	$aba\beta\gamma\delta$, 24	$aa\beta\gamma\delta\epsilon$, 14	$a\beta\gamma\delta\epsilon\zeta$, 8	
Points a	$6 \times (2)$	$5 \times (4)$	$4 \times (8)$	$3 \times (8)$	$2 \times (4)$	$1 \times (2)$	0	(0)
Lines ab	$15 \times (1)$, C	$10 \times (2)$, C	$6 \times (4)$, C	$3 \times (4)$, C	$1 \times (2)$, C	0	0	(1)
α	0	$1 \times (2)$, C	$2 \times (4)$, C	$3 \times (8)$, C	$4 \times (8)$, C	$5 \times (4)$, C	$6 \times (2)$, C	(2)
$[ab, \alpha, \beta, \gamma]$	0	0	0	$6 \times (4)$, P	(3) $8 \times (2+2)$, L	0	0	(3)
$[\alpha, \beta, \gamma, \delta]$	0	0	0	0	$2 \times (8)$, P	$10 \times (2)$, L	$30 \times (1)$, L	(4)
$[ab, cd, \alpha, \beta]$	0	0	$6 \times (2)$, P	0	0	0	0	(5)
abc, def	$10 \times (1)$, P	0	0	0	0	0	0	(6)
abc, de, α	0	(1) $10 \times (1)$, P	0	0	0	0	0	(7)
abc, α, β	0	0	(2) $4 \times (2+2)$, P	$3 \times (4)$, P	0	0	0	(8)
Cubic $abcdef$	$1 \times (1)$, C	0	0	0	0	0	0	(9)
Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$	0	0	0	0	0	0	$(10) \times (1)$, L	(10)
Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, α	0	0	0	0	0	(4) $10 \times (1)$, L	0	(11)

[1] Tangent plane along line is plane abc.
[2] Tacnodal line, each sheet touched along line by plane abc.
[3] Tacnodal line, each sheet of surface touched along line by hyperboloid $\alpha\beta\gamma$.
[4] Surface touched along line by hyperboloid $\alpha\beta\gamma$.

2. In the Table, the upper margin refers to the surfaces, and the left-hand margin to the points, lines, and curves situate on these surfaces respectively; the body of the Table showing the number, and in ( ) the multiplicity, of these points, lines, and curves in regard to the several surfaces respectively. Thus, points a; for the surface $abcdef$, $6 \times (2)$, there are 6 such points, each of them a 2-conical (ordinary conical) point on the surface: so $abcde\alpha$, $5 \times (4)$, there are 5 such points, each a 4-conical point on the surface (viz. instead of the tangent plane there is a quartic cone); and so on. Similarly, lines ab (viz. these are the lines joining two points a, b); for the surface $abcdef$, $15 \times (1)$, there are 15 such lines, each a simple line on the surface; surface $abcde\alpha$, $10 \times (2)$, there are 10 such lines, each a double (ordinary nodal) line on the surface; and so on. We have in two places the multiplicity $(2+2)$, which refers to a tacnodal line, as presently explained. The corner letters C, P, L denote respectively proper cone, plane-pair, and line-pair, as afterwards explained.

3. The lines and curves referred to in the left-hand margin are:

(1) ab, line joining the points a and b.

(2) α, line α.

(3) $[ab, \alpha, \beta, \gamma]$, pair of lines meeting each of the four lines, or say the tractors of the four lines ab, α, β, γ. As regards the surface $ab\alpha\beta\gamma\delta$, the multiplicity is given as $(2+2)$, viz. the line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the hyperboloid $\alpha\beta\gamma$.

(4) $[\alpha, \beta, \gamma, \delta]$, tractors of the four lines α, β, γ, δ.

(5) $[ab, cd, \alpha, \beta]$ tractors of the four lines ab, cd, α, β.

(6) abc, def, line of intersection of the planes abc and def.

(7) abc, de, α, line in the plane abc joining the intersections of this plane by the lines de and α respectively.

(8) abc, α, β, line in the plane abc joining the intersections of this plane by the lines α and β respectively. As regards the surface $abcd\alpha\beta$, the multiplicity is given as $(2+2)$, viz. each line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the plane abc.

(9) Cubic $abcdef$, cubic curve through the six points a, b, c, d, e, f, common intersection of the cones each having its vertex at one of the points and passing through the other five.

(10) Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$, intersection of the quadric surfaces $\alpha\beta\gamma$ and $\delta\epsilon\zeta$, that is, the quadric surfaces through the lines α, β, γ and δ, ϵ, ζ respectively.

(11) Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, a, quartic curve generated as follows: viz. taking any line whatever which meets the lines α, β, γ (or say any generating line of the quadric $\alpha\beta\gamma$), the plane through this line and the point a meets the lines δ, ϵ in two points respectively; and the line joining these meets the generating line in a point having for its locus the excuboquartic curve in question (theory further considered in the sequel).

Special forms of (Quadri-)Cones.

4. We have to consider the special forms of (quadri-)cones; these are: 1°. The sharp-cone, or plane-pair; that is, a pair of two planes, intersecting in a line called the axis, the vertex being in this case an indeterminate point on the axis. Observe that a plane-pair passes through a given point when either of its planes passes through such point; it touches a given line when its axis meets the given line. 2°. The flat-cone, or line-pair; viz. this is a pair of intersecting lines, their point of intersection being the vertex of the line-pair, and the plane of the two lines being the *diametral* of the line-pair. Observe that the line-pair passes through a given point when its diametral passes through such point; it touches a given line when either of its lines meets the given line. 3°. There is a third kind, the line-pair-plane; viz. the two planes of the plane-pair may come to coincide, retaining, however, a definite line of intersection, or axis: or again, the two lines of a line-pair may come to coincide, retaining a definite plane or diametral; that is, in either case we have a plane passing through a line; and which is to be considered indifferently as two coincident planes intersecting in the line, or as two coincident lines lying in the plane. But there is not, in the present Memoir, any occasion to consider this third kind of special cone.

The letters C, P, L in the Table denote that the cone is a (proper) cone, plane-pair, or line-pair, as the case may be.

Singular Lines and Curves on the Surfaces.

5. We may establish *à priori* the existence, and even to some extent the multiplicity, of the several lines and curves on the surfaces $abcdef, \ldots \alpha\beta\gamma\delta\epsilon\zeta$. Thus:

1°. Lines ab: take for the vertex of the cone a point at pleasure on the line ab; the cone passing through b will *ipso facto* pass through a; and the conditions are thus that the cone shall pass through b and satisfy four other conditions—in all, five conditions: and there is thus a cone with the point in question as vertex; that is, the line ab is situate on the surface. Moreover, for the surfaces $abcdef$, $abcde\alpha$, $abcd\alpha\beta$, $abc\alpha\beta\gamma$, $ab\alpha\beta\gamma\delta$ respectively, for a given position of the vertex on the line ab, the number of cones is 1, 2, 4, 4, 2 respectively: and these are the multiplicities of the line ab on the several surfaces respectively.

2°. Lines α: take for the vertex of the cone a point at pleasure on the line α; then the cone *ipso facto* touches the line α, and there are only five other conditions to be satisfied; that is, we have a cone with the vertex in question; or the line α is situate on the surface. Moreover, for the surfaces $abcde\alpha$, $abcd\alpha\beta$, $abc\alpha\beta\gamma$, $ab\alpha\beta\gamma\delta$, $a\alpha\beta\gamma\delta\epsilon$, $\alpha\beta\gamma\delta\epsilon\zeta$ respectively, the number of cones is 1, 2, 4, 4, 2, 1 respectively: and it may be seen that the multiplicities of the line α are the doubles of these numbers, or are $=2$, 4, 8, 8, 4, 2 for the several surfaces respectively.

3°. Lines $[ab, \alpha, \beta, \gamma]$: taking the vertex in one of these tractors, the cone cannot be a proper cone, but (if it exist) it must be either a line-pair having the tractor for one of its lines, or else a plane-pair having the tractor for its axis. The two cases are:

Surface $abc\alpha\beta\gamma$. Cone is a plane-pair, the two planes intersecting in the tractor, and passing, the one of them through the points a, b, the other through the point c. The vertex being an indeterminate point on the tractor, the tractor is situate on the surface.

Surface $ab\alpha\beta\gamma\delta$. Cone is a line-pair, one line being the tractor, the other a line drawn in the plane of the tractor and ab to meet δ, and which meets the tractor in an arbitrary point thereof: the tractor is thus a line on the surface.

4°. Lines $[\alpha, \beta, \gamma, \delta]$: taking the vertex in one of these tractors, then, as in the last case, the cone is either a line-pair having the tractor for one of its lines or a plane-pair having the tractor for its axis. The three cases are:

Surface $ab\alpha\beta\gamma\delta$. Cone is a plane-pair, the two planes intersecting in the tractor and passing through the points a, b respectively.

Surface $a\alpha\beta\gamma\delta\epsilon$. Cone is a line-pair, one line being the tractor, the other a line in the plane of the tractor and a, meeting the line ϵ and meeting the tractor in an indeterminate point.

Surface $\alpha\beta\gamma\delta\epsilon\zeta$. Cone is a line-pair, one line being the tractor, the other a line drawn from an indeterminate point of the tractor to meet the lines ϵ and ζ.

5°. Lines $[ab, cd, \alpha, \beta]$. Cone is a plane-pair, the two planes intersecting in the tractor, and passing through the points a, b and the points c, d respectively.

6°. Line abc, def. Cone is a plane-pair, consisting of the two planes abc and def.

7°. Line abc, de, α. Cone is a plane-pair, the two planes intersecting in the line; one plane being abc, the other a plane through the line de.

8°. Line abc, α, β. There are two cases:

Surface $abcd\alpha\beta$. Cone is a plane-pair, the two planes intersecting in the line; the one being abc, and the other passing through the point d.

Surface $abc\alpha\beta\gamma$. Cone is a line-pair; one line being abc, α, β, the other a line in the plane abc meeting the line δ, and meeting the line abc, α, β in an indeterminate point.

9°. Cubic $abcdef$. Each point of the cubic is the vertex of a proper cone passing through the cubic, and therefore through the six points; that is, the cubic is a line on the surface $abcdef$.

10°. Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$. Cone is a line-pair; viz. it is composed of the lines drawn from any point of the curve, one of them to meet the lines α, β, γ, and the other to meet the lines δ, ϵ, ζ.

11°. Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, a. Cone is a line-pair; the two lines being, one of them a line at pleasure meeting α, β, γ, the other the line which, in the plane of the other line and the point a, meets the lines δ, ϵ.

Mode of obtaining the several Equations: Notations and Formulæ.

6. The equations of the several surfaces are obtained by taking as centre of projection an assumed position of the vertex, and projecting everything upon an arbitrary plane; the projections of the given points and lines are points and lines in the arbitrary plane, and the section of the cone by this plane is a conic; the equation of the surface is thus obtained as the condition that there shall be a conic passing through m given points and touching $6-m$ given lines.

7. We take as current coordinates (X, Y, Z, W), or when plane-coordinates are employed $(\xi, \eta, \zeta, \omega)$: the coordinates of the vertex are throughout represented by (x, y, z, w); but in explanations &c., these are also used as current coordinates. The plane of projection is taken to be $W=0$. The coordinates of the given points a, &c., are taken to be (x_a, y_a, z_a, w_a), &c. There is no confusion occasioned by so doing, and I retain the ordinary letters (a, b, c, f, g, h) for the six coordinates of a line, it being understood that these letters so used have no reference whatever to the given points a, b, &c.; viz. the coordinates of the given lines α, &c., are $(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, &c.; there is sometimes occasion to consider the coordinates of other lines ab, &c., but the notation will always be explained.

8. I write l, m, n, p, q, r for the coordinates of the line joining the vertex (x, y, z, w) with a point (x', y', z', w'); viz.

$$\begin{aligned} l &= yz'-y'z, & p &= xw'-x'w, \\ m &= zx'-z'x, & q &= yw'-y'w, \\ n &= xy'-x'y, & r &= zw'-z'w, \end{aligned}$$

($l_a = yz_a - y_a z$, &c., this being explained when necessary); and also

$$\begin{aligned} P &= \quad . \quad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \ , \end{aligned}$$

($P_a = h_a y - g_a z + a_a w$, &c., this being explained when necessary).

This being so, then projecting from the vertex (x, y, z, w), say on the plane $W=0$, the x, y, z coordinates of the projection of a point a are as $p_a : q_a : r_a$ ($p_a = xw_a - x_a w$, &c.); and the equation of the projection of a line α is

$$P_\alpha X + Q_\alpha Y + R_\alpha Z = 0,$$

($P_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c.). We thus have, in the projection on the plane $W=0$, the m points and $6-m$ lines situate in and touched by the conic.

The following notations and formulæ are convenient:

9. $pabc=0$ is the equation of the plane through the points a, b, c; viz.

$$pabc=\begin{vmatrix} x\,, & y\,, & z\,, & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \\ x_c, & y_c, & z_c, & w_c \end{vmatrix}.$$

Of course $pbac=-pabc$, &c. Observe that here, and in the notations which follow, the letter p is used as referring to the coordinates (x, y, z, w), and that the index of $p\,(=1$ when no index is expressed) shows the degree in these coordinates.

10. $pa\alpha=0$ is the equation of the plane through the point a and the line α; viz. $pa\alpha$ is the foregoing determinant, if for a moment b, c are any two points on the line α; or, what is the same thing,

$$pa\alpha=P_a x+Q_a y+R_a z+S_a w,$$

where

$$\begin{aligned} P_a &= \quad . \quad hy_a-gz_a+aw_a, \\ Q_a &= -hx_a \quad . \quad +fz_a+bw_a, \\ R_a &= \quad gx_a-fy_a \quad . \quad +cw_a, \\ S_a &= -ax_a-by_a-cz_a \quad . \ ; \end{aligned}$$

and (a, b, c, f, g, h) are the coordinates of the line α: observe that $pa\alpha=pa\alpha$.

11. $p^2\alpha\beta\gamma=0$ is the equation of the quadric surface through the lines α, β, γ; viz. we have

$$\begin{aligned} p^2\alpha\beta\gamma=(agh)\,x^2+(bhf)\,y^2+(cfg)\,z^2+(abc)\,w^2 \\ +[(abg)-(cah)]\,xw \\ +[(bch)-(abf)]\,yw \\ +[(caf)-(bcg)]\,zw \\ +[(bfg)+(chf)]\,yz \\ +[(cgh)+(afg)]\,zx \\ +[(ahf)+(bgh)]\,xy, \end{aligned}$$

where

$$agh=\begin{vmatrix} a_\alpha, & g_\alpha, & h_\alpha \\ a_\beta, & g_\beta, & h_\beta \\ a_\gamma, & g_\gamma, & h_\gamma \end{vmatrix} \text{ \&c.}$$

$(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, $(a_\beta, \ldots)$, $(a_\gamma, \ldots)$ being the coordinates of the given lines α, β, γ. Observe that $p^2\beta\alpha\gamma=-p^2\alpha\beta\gamma$, &c.

12. It is to be noticed that, writing

$$\begin{aligned} P &= \quad . \quad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \quad , \end{aligned}$$

viz. $P_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c., then that we have identically

$$\begin{vmatrix} \lambda\,, & \mu\,, & \nu\,, & \rho \\ P_\alpha\,, & Q_\alpha\,, & R_\alpha\,, & S_\alpha \\ P_\beta\,, & Q_\beta\,, & R_\beta\,, & S_\beta \\ P_\gamma\,, & Q_\gamma\,, & R_\gamma\,, & S_\gamma \end{vmatrix} = -(\lambda x + \mu y + \nu z + \rho w) \,.\, p^2\alpha\beta\gamma,$$

and further that we have identically

$$-p^2\alpha\beta\gamma = L_{\alpha\beta}P_\gamma + M_{\alpha\beta}Q_\gamma + N_{\alpha\beta}R_\gamma + \Omega_{\alpha\beta}S_\gamma,$$

where

$$\begin{aligned} L &= (af' - a'f)\,x + (bf' - b'f)\,y + (cf' - c'f)\,z - (bc' - b'c)\,w, \\ M &= (ag' - a'g)\,x + (bg' - b'g)\,y + (cg' - c'g)\,z - (ca' - c'a)\,w, \\ N &= (ah' - a'h)\,x + (bh' - b'h)\,y + (ch' - c'h)\,z - (ab' - a'b)\,w, \\ \Omega &= (gh' - g'h)\,x + (hf' - h'f)\,y + (fg' - f'g)\,z + (af' - a'f + bg' - b'g + ch' - c'h)\,w\,; \end{aligned}$$

and $L_{\alpha\beta}$, &c. are the values of L, &c. on substituting therein $(a_\alpha, \ldots)$ and $(a_\beta, \ldots)$ for the unaccented and accented letters respectively.

13. Observe that we have

$$\begin{aligned} L + (a'f + b'g + c'h)\,x &= \quad . \quad - c'Q + b'R - f'S, \\ M + (\quad \text{,,} \quad)\,y &= \quad c'P \quad . \quad - a'R - g'S, \\ N + (\quad \text{,,} \quad)\,z &= - b'P + a'Q \quad . \quad - h'S, \\ \Omega + (\quad \text{,,} \quad)\,w &= \quad f'P + g'Q + h'R \quad . \quad ; \end{aligned}$$

and similarly

$$\begin{aligned} -L + (af' + bg' + ch')\,x &= \quad . \quad - cQ' + bR' - fS', \\ -M + (\quad \text{,,} \quad)\,y &= \quad cP' \quad . \quad - aR' - gS', \\ -N + (\quad \text{,,} \quad)\,z &= - bP' + aQ' \quad . \quad - hS', \\ -\Omega + (\quad \text{,,} \quad)\,w &= \quad fP' + gQ' + hR' \quad . \quad ; \end{aligned}$$

whence also

$$\begin{aligned} \quad . \quad h'M - g'N + a'\Omega &= -(a'f + b'g + c'h)\,P', \\ -h'L \quad . \quad + f'N + b'\Omega &= -(\quad \text{,,} \quad)\,Q', \\ g'L - f'M \quad . \quad + c'\Omega &= -(\quad \text{,,} \quad)\,R', \\ -a'L - b'M - c'N \quad . \quad &= -(\quad \text{,,} \quad)\,S'\,; \end{aligned}$$

and

$$\begin{array}{rrrrl}
. & hM & -gN & +a\Omega & = (af'+bg'+ch')\,P, \\
-hL & . & +fN & +b\Omega & = (\quad\text{"}\quad)\,Q, \\
gL & -fM & . & +c\Omega & = (\quad\text{"}\quad)\,R, \\
-aL & -bM & -cN & . & = (\quad\text{"}\quad)\,S.
\end{array}$$

14. $p^3a\,.\,\alpha\beta\,.\,\gamma\delta=0$ is the equation of the cubic surface through the lines α, β, γ, δ and $a\alpha\beta$, $a\gamma\delta$ (viz. $a\alpha\beta$ is the line from a to meet α, β, and so $a\gamma\delta$ is the line from a to meet γ, δ). Observe that the conditions which determine this cubic surface thus are that the cubic shall pass through

a; the points of $a\alpha\beta$ on α and β respectively, 3 other points on α, 3 on β, and 1 on $a\alpha\beta$;

also the points of $a\gamma\delta$ on γ and δ respectively, 3 other points on γ, 3 on δ, and 1 on $a\gamma\delta$; in all, $1+9+9=19$ points;

viz. the conditions completely determine the surface.

15. We have

$$p^3a\,.\,\alpha\beta\,.\,\gamma\delta=\begin{vmatrix} x\,, & y\,, & z\,, & w \\ x_a\,, & y_a\,, & z_a\,, & w_a \\ L_{\alpha\beta}, & M_{\alpha\beta}, & N_{\alpha\beta}, & \Omega_{\alpha\beta} \\ L_{\gamma\delta}, & M_{\gamma\delta}, & N_{\gamma\delta}, & \Omega_{\gamma\delta} \end{vmatrix},$$

viz. this determinant, equated to zero, gives the equation of the surface.

To prove this, take as before the unaccented letters (a, b, c, f, g, h) to refer to the line α, and the letters with one, two, and three accents to refer to the lines β, γ, δ respectively; write also L, M, N, Ω and L', M', N', Ω' for $L_{\alpha\beta}$, &c., and $L_{\gamma\delta}$, &c., respectively. Referring to the foregoing expressions for L, M, N, Ω, and observing that for a point on the line α, the values of P, Q, R, S are each $=0$, then for such a point we have $L+(a'f+b'g+c'h)\,x=0$, &c., that is, $L:M:N:\Omega=x:y:z:w$, and these values satisfy the equation of the surface, which is thus a surface passing through the line α; and similarly it passes through the lines β, γ, δ.

To show that the surface passes through the line $a\alpha\beta$, take the coordinates of the point a to be 0, 0, 0, 1; then the line $a\alpha\beta$ is given as the intersection of the planes $ax+by+cz=0$ and $a'x+b'y+c'z=0$, that is, $S=0$ and $S'=0$. And the equation of the surface, writing therein x_a, y_a, z_a, $w_a=0$, 0, 0, 1, becomes

$$\begin{vmatrix} x\,, & y\,, & z \\ L\,, & M\,, & N \\ L', & M', & N' \end{vmatrix}=0,$$

or, as this may be written,

$$\begin{vmatrix} S & , & S' & , & z \\ aL+bM+cN, & & a'L+b'M+c'N, & & N \\ aL'+bM'+cN', & & a'L'+b'M'+c'N', & & N' \end{vmatrix} = 0;$$

and, for a point on the line $a\alpha\beta$, this is

$$\begin{vmatrix} aL+bM+cN, & a'L+b'M+c'N \\ aL'+bM'+cN', & a'L'+b'M'+c'N' \end{vmatrix} = 0.$$

But in the equations $-a'L-b'M-c'N = -(a'f+b'g+c'h)\,S'$, and $-aL-bM-cN = (af'+bg'+ch')\,S$, writing $S=0$ and $S'=0$, we have $aL+bM+cN=0$ and $a'L+b'M+c'N=0$, and the equation is satisfied; that is, the surface passes through the line $a\alpha\beta$, and similarly it passes through the line $a\gamma\delta$.

Surface abcdef.

16. The equation may be written

$$pabe\,.\,pcde\,.\,pacf\,.\,pdbf - pabf\,.\,pcdf\,.\,pace\,.\,pdbe = 0,$$

where $pabe=0$ is the equation of the plane through the points a, b, e; and the like for the other symbols. The form is one out of 45 like forms, depending on the partitionment

$$\left\{\begin{matrix} ab\,.\,cd \\ ac\,.\,db \\ ad\,.\,bc \end{matrix}\right\}(ef),$$

of the six letters.

17. *Investigation.* In the projection, the six points (p_a, q_a, r_a) are situate on a conic; the condition for this is

$$(p, q, r)^2 = 0,$$

where the left-hand side represents the determinant obtained by writing successively (p_a, q_a, r_a), &c., for (p, q, r). The equation in question may be written

$$abe\,.\,cde\,.\,acf\,.\,dbf - abf\,.\,cdf\,.\,ace\,.\,dbe = 0;$$

where

$$abe = \begin{vmatrix} p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \\ p_e, & q_e, & r_e \end{vmatrix}, \text{ \&c.};$$

and substituting for $p_a, \ldots$, their values, we have $abe = w^2\,.\,pabe$, whence the foregoing result.

{Surface *abcdef.*}

18. *Singularities.* The form of the equation shows at once that [1]

(0)[2] The point a is a 2-conical point; in fact, for this point we have $pabe=0$, $pacf=0$, $pabf=0$, $pace=0$.

(1) The line ab a simple line; in fact, for any point of this line we have $pabe=0$, $pabf=0$.

(2) The line $abe\,.\,cdf$ a simple line; in fact, for any point of this line we have $pabe=0$, $pcdf=0$.

(9) To show analytically that the cubic curve $abcdef$ is a line on the surface, observe that the equation of the surface is satisfied if we have simultaneously (λ being arbitrary)

$$pabe\,.\,pacf-\lambda\,.\,pabf\,.\,pace=0,$$
$$\lambda\,.\,pcde\,.\,pdbf-\quad pcdf\,.\,pdbe=0.$$

The first of these equations is a cone, vertex a, which passes through the points $b, e\ c, f$, and which, if λ is properly determined, will pass through the point d; the second is a cone, vertex d, which passes through the points b, e, c, f, and which, if λ is properly determined, will pass through the point a; the two determinations of λ are

$$dabe\,.\,dacf-\lambda\,.\,dabf\,.\,dace=0,$$
$$\lambda\,.\,acde\,.\,adbf-\quad acdf\,.\,adbe=0\,;$$

giving the same value of λ; and the equations then represent cones, the first having a for its vertex, and passing through d, b, e, c, f; the second having d for its vertex, and passing through a, b, e, c, f; the two intersect in the line ad, and in the cubic curve $abcdef$, which is thus a curve on the surface.

Surface abcdeα.

19. The equation may be written

$$(pabe\,.\,pcde\,.\,p^2\alpha ac\,.\,db-pace\,.\,pdbe\,.\,p^2\alpha ab\,.\,cd)^2$$
$$+\,4pabe\,.\,pcde\,.\,pace\,.\,pdbe\,.\,pabc\,.\,pdbc\,.\,pa\alpha\,.\,pd\alpha=0,$$

or, what is the same thing,

$$(pabe\,.\,pcde\,.\,p^2\alpha ac\,.\,db+pace\,.\,pdbe\,.\,p^2\alpha ab\,.\,cd)^2$$
$$+\,4pabe\,.\,pcde\,.\,pace\,.\,pdbe\,.\,pbad\,.\,pcad\,.\,pb\alpha\,.\,pc\alpha=0,$$

(the equivalence of the two depending on the identity

$$-\,p^2\alpha ab\,.\,cd\,.\,p^2\alpha ac\,.\,db+pabc\,.\,pdbc\,.\,pa\alpha\,.\,pd\alpha-pbad\,.\,pcad\,.\,pb\alpha\,.\,pc\alpha=0)$$

[1] Or course, as regards the present surface and the other surfaces for which the equation is given in an unsymmetrical form, the conclusion obtained in regard to any point or line of the surface applies to every point or line of the same kind. Thus ab being a simple line, we have also ad a simple line, although the equation, as written down, does not put this in evidence.

[2] The bracketed numbers refer to the lines of the Table.

where, as before, $pabe=0$ is the equation of the plane through the points a, b, e; $p^2\alpha acdb=0$ the equation of the quadric surface through the lines α, ac, db; and $pa\alpha=0$ the equation of the plane through the point a and the line α.

The above forms are 2 out of 30 like forms, as appears by the partitionment

$$\left\{\begin{matrix} ab, & cd \\ ac, & db \\ ad, & bc \end{matrix}\right\} e\alpha.$$

20. *Investigation.* In the projection, the equation of the conic through the five points may be written

$$\begin{vmatrix} (X, & Y, & Z)^2 \\ (p, & q, & r)^2 \end{vmatrix} = 0,$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to (p, q, r) the suffixes a, b, c, d, e respectively. This is at once transformed into

$$abe\,.\;cde\,.\,ac\Delta\,.\,db\Delta - ace\,.\;dbe\,.\,ab\Delta\,.\,cd\Delta = 0,$$

or, what is the same thing,

$$pabe\,.\,pcde\,.\,ac\Delta\,.\,db\Delta - pace\,.\,pdbe\,.\,ab\Delta\,.\,cd\Delta = 0,$$

or say,

$$pabe\,.\,pcde\,(A''X + B'Y + C''Z)(A'''X + B'''Y + C'''Z)$$
$$- pace\,.\,pdbe\,(AX + BY + CZ)(A'X + B'Y + C'Z),$$

where $pabe$, &c. signify as before, and

$$AX + BY + CZ = \begin{vmatrix} X, & Y, & Z \\ p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \end{vmatrix}$$

and so for $A'X + B'Y + C'Z$, &c., the suffixes for A', B', C' being (c, d), and those for A'', B'', C'' and A''', B''', C''' being (a, c) and (d, b) respectively.

21. Passing to the reciprocal equation, and making the conic touch the line α, we obtain the equation of the surface in the form

$$\left\{pabe\,.\,pcde \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix} - pace\,.\,pdbe \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A', & B', & C' \end{vmatrix}\right\}^2$$
$$+ 4\,pace\,.\,pdbe\,.\,pabe\,.\,pcde \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A'', & B'', & C'' \end{vmatrix} \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix} = 0,$$

{Surface *abcdea*.}

(where $P_a = h_a y - g_a z + a_a w = 0$) or in the equivalent form wherein we have in the first term + instead of −, and in the second term the determinants

$$\begin{vmatrix} P_a, & Q_a, & R_a \\ A\ , & B\ , & C \\ A''', & B''', & C''' \end{vmatrix}, \quad \begin{vmatrix} P_a, & Q_a, & R_a \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}.$$

22. {The question, in fact, is to find the reciprocal of the form

$$\lambda(ax+by+cz)(a'x+b'y+c'z) - \mu(a''x+b''y+c''z)(a'''x+b'''y+c'''z) = 0;$$

taking ξ, η, ζ for the reciprocal variables, the coefficient of ξ^2 is

$$\{\lambda(bc'+b'c) - \mu(b''c'''+b'''c'')\}^2 - (2\lambda bb' - 2\mu b''b''')(2\lambda cc' - 2\mu c''c'''),$$

viz. this is

$$\lambda^2(bc'-b'c)^2 + \mu^2(b''c'''-b'''c'')^2 + 2\lambda\mu\{2bb'c''c''' + 2b''b'''cc' - (bc'+b'c)(b''c''+b'''c'')\},$$

or, as it may be written,

$$\{\lambda(bc'-b'c) \pm \mu(b''c'''-b'''c'')\}^2 + 2\lambda\mu \left\{\begin{matrix} 2bb'c''c''' + 2b''b'''cc' \\ \mp(bc-b'c)(b''c''-b'''c'') \\ -(bc'+b'c)(b''c'''+b'''c'') \end{matrix}\right\}.$$

Taking the upper signs, this is

$$\{\lambda(bc'-b'c) + \mu(b''c'''-b'''c'')\}^2 + 4\lambda\mu \left(\begin{matrix} bb'c''c''' + b''b'''cc' \\ -bc'b''c''' - b'cb'''c'' \end{matrix}\right);$$

viz. the term in $\lambda\mu$ is

$$= +4\lambda\mu(bc'''-b'''c)(b'c''-b''c').$$

Taking the lower signs, it is

$$\{\lambda(bc'-b'c) - \mu(b''c'''-b'''c'')\}^2 + 4\lambda\mu \left(\begin{matrix} bb'c''c''' + b''b'''cc' \\ -bc'b'''c'' - b'cb''c''' \end{matrix}\right);$$

viz. the term in $\lambda\mu$ is

$$4\lambda\mu(bc''-b''c)(b'c'''-b'''c');$$

and it is thence easy to infer the forms of the other coefficients, and to obtain the reciprocal equation in the two equivalent forms

$$\{\lambda \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} + \mu \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\}^2 + 4\lambda\mu \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a\ , & b\ , & c \\ a''', & b''', & c''' \end{vmatrix} \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0,$$

$$\{\lambda \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} - \mu \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\}^2 + 4\lambda\mu \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a\ , & b\ , & c \\ a'', & b'', & c'' \end{vmatrix} \begin{vmatrix} \xi\ , & \eta\ , & \zeta \\ a', & b', & c' \\ a''', & b''', & c''' \end{vmatrix} = 0,$$

which are the required auxiliary formulæ.}

{Surface *abcdea*.}

23. To reduce the foregoing result, we have

$$A,\ B,\ C = \left\| \begin{matrix} xw_a - wx_a, & yw_a - wy_a, & zw_a - wz_a \\ xw_b - wx_b, & yw_b - wy_b, & zw_b - wz_b \end{matrix} \right\|$$

proportional to the three determinants which contain w, of the set

$$\left\| \begin{matrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \end{matrix} \right\|, \quad \text{viz. } A = w \begin{vmatrix} y, & z, & w \\ y_a, & z_a, & w_a \\ y_b, & z_b, & w_b \end{vmatrix}, \text{ \&c.};$$

and similarly A', B', C' are proportional to the three determinants which contain w, of the set

$$\left\| \begin{matrix} x, & y, & z, & w \\ x_c, & y_c, & z_c, & w_c \\ x_d, & y_d, & z_d, & w_d \end{matrix} \right\|, \quad \text{viz. } A' = w \begin{vmatrix} y, & z, & w \\ y_c, & z_c, & w_c \\ y_d, & z_d, & w_d \end{vmatrix}, \text{ \&c.}$$

Hence, omitting the factor w, and writing (a, b, c, f, g, h) and (a′, b′, c′, f′, g′, h′) for the coordinates of the lines ab and cd respectively, we have

$$\begin{aligned} A &= \quad\quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w, & A' &= \quad\quad \mathrm{h}'y - \mathrm{g}'z + \mathrm{a}'w, \\ B &= -\mathrm{h}x \quad + \mathrm{f}z + \mathrm{b}w, & B' &= -\mathrm{h}'x \quad + \mathrm{f}'z + \mathrm{b}'w, \\ C &= \mathrm{g}x - \mathrm{f}y \quad + \mathrm{c}w, & C' &= \mathrm{g}'x - \mathrm{f}'y \quad + \mathrm{c}'w; \end{aligned}$$

and thence

$$\begin{aligned} BC' - B'C &= \Omega x - Lw, \\ CA' - C'A &= \Omega y - Mw, \\ AB' - A'B &= \Omega z - Nw, \end{aligned}$$

where

$$\begin{aligned} L &= (\mathrm{af}' - \mathrm{a}'\mathrm{f})\,x + (\mathrm{bf}' - \mathrm{b}'\mathrm{f})\,y + (\mathrm{cf}' - \mathrm{c}'\mathrm{f})\,z - (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,w, \\ M &= (\mathrm{ag}' - \mathrm{a}'\mathrm{g})\,x + (\mathrm{bg}' - \mathrm{b}'\mathrm{g})\,y + (\mathrm{cg}' - \mathrm{c}'\mathrm{g})\,z - (\mathrm{ca}' - \mathrm{c}'\mathrm{a})\,w, \\ N &= (\mathrm{ah}' - \mathrm{a}'\mathrm{h})\,x + (\mathrm{bh}' - \mathrm{b}'\mathrm{h})\,y + (\mathrm{ch}' - \mathrm{c}'\mathrm{h})\,z - (\mathrm{ab}' - \mathrm{a}'\mathrm{b})\,w, \\ \Omega &= (\mathrm{gh}' - \mathrm{g}'\mathrm{h})\,x + (\mathrm{hf}' - \mathrm{h}'\mathrm{f})\,y + (\mathrm{fg}' - \mathrm{f}'\mathrm{g})\,z - (\mathrm{af}' - \mathrm{a}'\mathrm{f} + \mathrm{bg}' - \mathrm{b}'\mathrm{g} + \mathrm{ch}' - \mathrm{c}'\mathrm{h})\,w; \end{aligned}$$

and consequently

$$\begin{aligned} \begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} &= \Omega\,(xP_a + yQ_a + zR_a) - w\,(LP_a + MQ_a + NR_a) \\ &= -w\,(LP_a + MQ_a + NR_a + \Omega S_a); \end{aligned}$$

or omitting the factor $-w$, say it is $= LP_a + MQ_a + NR_a + \Omega S_a$, viz. this is $= p^2\alpha\, ab\,.\,cd$.

{Surface *abcdea*.}

We have similarly

$$\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix}$$

taken to be $=p^2\alpha\, ac\,.\,db$.

24. We have in like manner the other two determinants

$$\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A\ , & B\ , & C \\ A'', & B'', & C'' \end{vmatrix} \text{ and } \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix}$$

taken to be $=p^2\alpha\, ab\,.\,ac$ and $p^2\alpha\, cd\,.\,db$ respectively.

But we have

$$p^2\alpha\, ab\,.\,ac = p\alpha a\,.\,pabc,$$

(viz. geometrically the hyperboloid through the lines α, ab, ac breaks up into the plane $p\alpha a$ through the line α and point a, and the plane $pabc$ through the points a, b, c).

And similarly

$$p^2\alpha\, cd\,.\,db = -p^2\alpha\, dc\,.\,db = +p^2\alpha\, db\,.\,dc = p\alpha d\,.\,pdbc;$$

whence, substituting for the several determinants, we have the foregoing equation of the surface.

25. *Singularities.* The form of the equation shows that

(0) The point a is a 4-conical point: in fact, for this point we have $pabe=0$, $p^2\alpha\, ac\,.\,db=0$, $pace=0$, $p^2\alpha\, ab\,.\,cd=0$.

(1) The line ab is a double line: in fact, for any point of the line we have $pabe=0$, $p^2\alpha\, ab\,.\,cd=0$, $pabc=0$.

(2) The line α is a double line: in fact, for any point of the line we have $p^2\alpha\, ac\,.\,db=0$, $p^2\alpha\, ab\,.\,cd=0$, $pa\alpha=0$, $pd\alpha=0$.

(7) The line $abe\,.\,cd\,.\,\alpha$ is a simple line: in fact, for any point of the line we have $pabe=0$, $p^2\alpha\, ab\,.\,cd=0$. Observe that, on writing in the equation $pabe=0$ the equation becomes $(p^2\alpha\, ab\,.\,cd)^2=0$; so that the surface along the line in question touches the plane $pabe$.

Surface $abcd\alpha\beta$.

26. The equation of the surface is

$$\text{Norm}\ \{\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd - \sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda + \sqrt{pc\alpha\,.\,pc\beta}\,.\,pdab - \sqrt{pd\alpha\,.\,pd\beta}\,.\,pabc\} = 0,$$

where the norm is the product of 8 factors.

As before, $pa\alpha=0$ is the equation of the plane through the point a and the line α; and $pbcd=0$ the equation of the plane through the points b, c, d. The form is unique.

{Surface $abcd\alpha\beta$.}

27. *Investigation.* In the projection, the equation of the conic touching the projections of the lines α, β is

$$\sqrt{(P_\alpha X+Q_\alpha Y+R_\alpha Z)(P_\beta X+Q_\beta Y+R_\beta Z)}+AX+BY+CZ=0,$$

where A, B, C are arbitrary coefficients. To make this pass through the projection of the point a, we must write $X : Y : Z = p_a : q_a : r_a$; viz. we thus have

$$\begin{aligned} P_\alpha X+Q_\alpha Y+R_\alpha Z = \; & w_a(x\,P_\alpha+y\,Q_\alpha+z\,R_\alpha) \\ & -w\,(x_\alpha P_\alpha+y_\alpha Q_\alpha+z_\alpha R_\alpha), \\ = & -w\,(x_\alpha P_\alpha+y_\alpha Q_\alpha+z_\alpha R_\alpha+w_\alpha S_\alpha), \\ = & -w\,.\,pa\alpha\,; \end{aligned}$$

and similarly

$$P_\beta X+Q_\beta Y+R_\beta Z=-w\,.\,pa\beta.$$

We thus have

$$w\sqrt{pa\alpha\,.\,pa\beta}+Ap_a+Bq_a+Cr_a=0.$$

Or, forming the like equations for the points b, c, d respectively and eliminating, the equation is

$$\begin{vmatrix} \sqrt{pa\alpha\,.\,pa\beta}, & p_a, & q_a, & r_a \\ \sqrt{pb\alpha\,.\,pb\beta}, & p_b, & q_b, & r_b \\ \sqrt{pc\alpha\,.\,pc\beta}, & p_c, & q_c, & r_c \\ \sqrt{pd\alpha\,.\,pd\beta}, & p_d, & q_d, & r_d \end{vmatrix}=0\,;$$

which, substituting for (p_a, q_a, r_a), &c., their values, viz. $p_a=xw_a-x_aw$, &c., is readily converted into

$$\begin{vmatrix} & x\,, & y\,, & z\,, & w \\ \sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & w_a \\ \sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & w_b \\ \sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & w_c \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}=0.$$

or, what is the same thing,

$$\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd-\sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda+\sqrt{pc\alpha\,.\,pc\beta}\,.\,pdab-\sqrt{pd\alpha\,.\,pd\beta}\,.\,pabc=0\,;$$

viz. taking the norm, we have the form mentioned above.

28. *Singularities.* The equation shows that

(0) The point a is an 8-conical point; in fact, for the point in question $pa\alpha=0$, $pa\beta=0$, $pcda=0$, $pdab=0$, $pabc=0$; each factor is of the form 0^1, and the norm is 0^8.

{Surface $abcda\beta$.}

(1) The line ab is a 4-tuple line. To show this, observe in the first instance, that we may obtain the 8 factors of the norm by giving to the radical $\sqrt{pa\alpha . pa\beta}$ the sign $+$, and to the other three radicals the signs $+$, $-$, at pleasure. For a point on the line in question, we have $pdab=0$, $pabc=0$; hence the norm is the product of the four equal factors

$$\sqrt{pa\alpha . pa\beta} . pbcd - \sqrt{pb\alpha . pb\beta} . pcda,$$

and the other four equal factors obtained by writing herein $+$ instead of $-$.

Now for a point on the line ab, we may write for x, y, z, w the values $ux_a + vx_b$, $uy_a + vy_b$, $uz_a + vz_b$, $uw_a + vw_b$, where u, v are arbitrary coefficients. We have

$$\begin{aligned}
pa\alpha &= u . aa\alpha + v . ba\alpha = v . ba\alpha = -v . ab\alpha,\\
pa\beta &= v . ba\beta = -v . ab\beta,\\
pb\alpha &= u . ab\alpha + v . bb\alpha = u . ab\alpha,\\
pb\beta &= u . ab\beta,\\
pbcd &= u . abcd + v . bbcd = u . abcd,\\
pcda &= u . acda + v . bcda = v . bcda = -v . abcd,
\end{aligned}$$

where $ab\alpha=0$ is the condition that the points a, b and the line α may be in the same plane (or, what is the same thing, that the lines ab and α may intersect), viz. $ba\alpha$ is $=P_a x_b + Q_a y_b + R_a z_b + S_a w_b$. And similarly $abcd=0$ is the condition that the four points a, b, c, d may be in a plane; viz. we have

$$abcd = \begin{vmatrix} x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \\ x_c, & y_c, & z_c, & w_c \\ x_d, & y_d, & z_d, & w_d \end{vmatrix}.$$

Substituting, we have $\sqrt{pa\alpha . pa\beta} . pbcd$ and $\sqrt{pb\alpha . pb\beta} . pcda$, each equal (save as to sign) to $uv\sqrt{ab\alpha . ab\beta} . abcd$; that is, the four equal factors of one set will vanish. The vanishing factors are of the form $0^{\frac{1}{2}}$, and the norm is 0^4, that is, the line in question, ab, is a 4-tuple line.

(2) The line α is a 4-tuple line; in fact, for any point of the line we have $pa\alpha=0$, $pb\alpha=0$, $pc\alpha=0$, $pd\alpha=0$; each factor of the norm is therefore evanescent, of the form $0^{\frac{1}{2}}$, and the norm itself is thus $=0^4$.

29. (5) The line (ab, cd, α, β) is a double line. To show this, take $z=0$, $w=0$ as the equations of the line in question; then we have $h_\alpha=0$, $h_\beta=0$, $z_a w_b - z_b w_a = 0$; or say $w_a=\lambda z_a$, $w_b=\lambda z_b$: and $z_c w_d - z_d w_c = 0$; or say $w_c=\mu z_c$, $w_d=\mu z_d$ (λ and μ arbitrary coefficients). Putting for shortness

$$I=(g-\lambda a)x-(f+\lambda b)y, \quad J=(g-\mu a)x-(f+\mu b)y;$$

{Surface $abcd\alpha\beta$.}

viz. $I_\alpha = (g_a - \lambda a_\alpha)x - (f_a + \lambda b_a)y$, &c., and writing $z=0$, $w=0$, we have $pa\alpha \,.\, pa\beta = z_a^3 I_\alpha I_\beta$, $pb\alpha \,.\, pb\beta = z_b^3 I_\alpha I_\beta$, $pc\alpha \,.\, pc\beta = z_c^3 J_\alpha J_\beta$, $pd\alpha \,.\, pd\beta = z_d^3 J_\alpha J_\beta$; and the factor of the norm (reverting to the expression thereof as a determinant) is

$$\begin{vmatrix} & x, & y & & \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & \lambda z_a \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b, & z_b, & \lambda z_b \\ z_c\sqrt{J_\alpha J_\beta}, & x_c, & y_c, & z_c, & \mu z_c \\ z_d\sqrt{J_\alpha J_\beta}, & x_d, & y_d, & z_d, & \mu z_d \end{vmatrix}$$

which vanishes. In fact, resolving the determinant into a set of products of the form $\pm 2 \,.\, 13 \,.\, 45$, where the single symbol denotes a term of the top line, and the binary symbols refer to the second and third lines, and the fourth and fifth lines respectively (denoting minors composed with the terms in these pairs of lines respectively); then each product will contain a term 14, 15, or 45, and the minor so designated (to whichever of the two pairs of lines it belongs) is $=0$. The factor is thus evanescent, being, as it is easy to see, $=0^1$. There are two factors which vanish; viz. taking the first radical to be $+$, the second radical must be also $+$, but the third and fourth radicals may be either both $+$ or both $-$; the norm is thus $=0^2$, viz. the line (ab, cd, α, β) is a double line.

30. (8) The line abc, α, β is a double line. To prove this, take $w=0$ for the equation of the plane abc, and $(z=0, w=0)$ for those of the line in question; we have $h_a=0$, $h_\beta=0$, $w_a=0$, $w_b=0$, $w_c=0$; and writing $I_\alpha = -g_\alpha x + f_\alpha y$, $I_\beta = -g_\beta x + f_\alpha y$, then for $z=0$, $w=0$, the factor expressed as a determinant is

$$\begin{vmatrix} & x, & y & \cdot & \cdot \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & \cdot \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b, & z_b, & \cdot \\ z_c\sqrt{I_\alpha I_\beta}, & x_c, & y_c, & z_c, & \cdot \\ \sqrt{pd\alpha \,.\, pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}$$

which is

$$= w_d\sqrt{I_\alpha I_\beta} \begin{vmatrix} \cdot & x, & y & \cdot \\ z_a, & x_a, & y_a, & z_a \\ z_b, & x_b, & y_b, & z_b \\ z_c, & x_c, & y_c, & z_c \end{vmatrix}$$

and consequently vanishes, the form being 0^1. There are two such factors, viz. the radical $\sqrt{pd\alpha \,.\, pd\beta}$ may be either $+$ or $-$, hence the norm is $=0^2$.

31. But it is to be further shown that the line is tacnodal, each sheet of the surface being touched along the line by the plane $w=0$: we have to show that the

factor operated upon by $\Delta = X\delta_x + Y\delta_y + Z\delta_z + W\delta_w$, reduces itself for $z=0$, $w=0$ to a multiple of W. Considering the factor in the form of a determinant, the result of the operation is

$$\begin{vmatrix} & X, & Y, & Z, & W \\ \sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & . \\ \sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & . \\ \sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix} + \begin{vmatrix} & x, & y, & . & . \\ \Delta\sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & . \\ \Delta\sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & . \\ \Delta\sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix};$$

the first term is

$$\begin{vmatrix} & X, & Y, & Z, & W \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & . \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b, & z_b, & . \\ z_c\sqrt{I_\alpha I_\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix},$$

where the first column may be replaced by

$$\begin{matrix} -Z\sqrt{I_\alpha I_\beta} \\ \vdots \\ \sqrt{pd\alpha\,.\,pd\beta} - z_d\sqrt{I_\alpha I_\beta}, \end{matrix}$$

and the term in question thus becomes

$$\{w_d Z\sqrt{I_\alpha I_\beta} + W(-z_d\sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha\,.\,pd\beta})\}\,.\,abc,$$

if for shortness

$$\begin{vmatrix} x_a, & y_a, & z_a \\ x_b, & y_b, & z_b \\ x_c, & y_c, & z_c \end{vmatrix} = abc.$$

As regards the second term, we have

$$\Delta\sqrt{pa\alpha\,.\,pa\beta} = \frac{pa\alpha\,.\,\Delta pa\beta + pa\beta\,.\,\Delta pa\alpha}{2\sqrt{pa\alpha\,.\,pa\beta}},$$

which is

$$= \frac{I_\alpha \Delta pa\beta + I_\beta \Delta pa\alpha}{2\sqrt{I_\alpha I_\beta}}.$$

But

$$pa\alpha = x\,(-g_a z_a) + y\,(f_a z_a) + z\,(g_a x_a - f_a y_a) + w\,(a_a x_a - b_a y_a - c_a z_a),$$
$$= x_a\,(g_a z - a_a w) + y_a\,(-f_a z - b_a w) + z_a\,(-g_a x + f_a y - c_a w);$$

and thence

$$\Delta pa\alpha = x_a\,(g_a Z - a_a W) + y_a\,(-f_a Z - b_a W) + z_a\,(-g_a X + f_a Y - c_a W),$$

{Surface $abcd\alpha\beta$.}

with the like formula for $\Delta pa\beta$; hence

$$\frac{I_\alpha \Delta pa\beta + I_\beta \Delta pa\alpha}{2\sqrt{I_\alpha I_\beta}} = Ax_a + By_a + Cz_a,$$

where

$$A = \frac{1}{2\sqrt{I_\alpha I_\beta}}\{I_\beta(\ g_\alpha Z - a_\alpha W) + I_\alpha(\ g_\beta Z - a_\beta W)\},$$

$$B = \frac{1}{2\sqrt{I_\alpha I_\beta}}\{I_\beta(-f_\alpha Z - b_\alpha W) + I_\alpha(-f_\beta Z - b_\beta W)\},$$

$$C = \frac{1}{2\sqrt{I_\alpha I_\beta}}\{I_\beta(-g_\alpha X + f_\alpha Y - c_\alpha W) + I_\alpha(-g_\beta X + f_\beta Y - c_\beta W)\}.$$

The term in question is thus

$$\begin{vmatrix} & x, & y, & . & . \\ Ax_a + By_a + Cz_a, & x_a, & y_a, & z_a & . \\ Ax_b + By_b + Cz_b, & x_b, & y_b, & z_b, & . \\ Ax_c + By_c + Cz_c, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha \,.\, pd\beta}\ , & x_d, & y_d, & z_d, & w_d \end{vmatrix}$$

viz. replacing the first column by

$$\begin{matrix} -\ Ax - By \\ \vdots \\ \Delta\sqrt{pd\alpha \,.\, pd\beta} - Ax_d - By_d - Cz_d; \end{matrix}$$

this is

$$= (Ax + By)\, w_d \,.\, abc;$$

and we have

$$Ax + By = \frac{1}{2\sqrt{I_\alpha I_\beta}}\begin{matrix}[\ \ I_\beta(\ \ g_\alpha x - f_\alpha y) + I_\alpha(\ \ g_\beta x - f_\beta y)]\, Z \\ [+\ I_\beta(-\ a_\alpha x - b_\alpha y) + I_\alpha(-\ a_\beta x - b_\beta y)]\, W,\end{matrix}$$

$$= \frac{1}{2\sqrt{I_\alpha I_\beta}}(-2I_\alpha I_\beta Z - MW),$$

if for shortness

$$M = (-\ g_\beta x + f_\beta y)(a_\alpha x + b_\alpha y) + (-\ g_\alpha x + f_\alpha y)(a_\beta x + b_\beta y);$$

viz. the whole term is

$$w_d\left\{-\sqrt{I_\alpha I_\beta}\, Z - \frac{\frac{1}{2}M}{\sqrt{I_\alpha I_\beta}}\, W\right\} abc.$$

Hence the first and second terms together are

$$= W\left\{-z_d\sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha \,.\, pd\beta} - \frac{\frac{1}{2}M}{\sqrt{I_\alpha I_\beta}}\, w_d\right\} abc;$$

viz. this is a multiple of W, which was the theorem to be proved.

{Surface $abcd\alpha\beta$.}

Surface abcαβγ.

32. The equation is

$$\text{Norm}\begin{vmatrix} \sqrt{pa\alpha}, & \sqrt{pb\alpha}, & \sqrt{pc\alpha} \\ \sqrt{pa\beta}, & \sqrt{pb\beta}, & \sqrt{pc\beta} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{vmatrix} = 0,$$

where the norm is a product of 16 factors, each of the order $\frac{3}{2}$. As before, $pa\alpha = 0$ is the equation of the plane through the point a and the line α; viz. $pa\alpha$ has the value already mentioned.

33. *Investigation.* In the projection, the equation of the conic touching the projections of the lines α, β, γ is

$$A\sqrt{P_\alpha X + Q_\alpha Y + R_\alpha Z} + B\sqrt{P_\beta X + Q_\beta Y + R_\beta Z} + C\sqrt{P_\gamma X + Q_\gamma Y + R_\gamma Z} = 0;$$

and to make this pass through the projection of the point a, we must write herein $X : Y : Z = p_a : q_a : r_a$. As before, we have

$$\begin{aligned} P_\alpha X + Q_\alpha Y + R_\alpha Z = \; & w_a(x\,P_\alpha + y\,Q_\alpha + z\,R_\alpha) \\ & - w\,(x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha), \\ = & - w\,(x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha + w_a S_\alpha), \\ = & - w\,.\,pa\alpha; \end{aligned}$$

and so for the other terms; the equation thus is

$$A\sqrt{pa\alpha} + B\sqrt{pa\beta} + C\sqrt{pa\gamma} = 0;$$

or forming the like equations in regard to the points b, c respectively, and eliminating we have a determinant $=0$, and then, taking the norm, we obtain the above-written equation of the surface.

34. *Singularities.* The equation of the surface shows that

(0) The point a is 8-conical: in fact, for the point in question we have $pa\alpha = 0$, $pa\beta = 0$, $pa\gamma = 0$; each factor is $0^{\frac{1}{2}}$, and the norm is 0^8.

(1) The line ab is 4-tuple. To prove this, observe that the sixteen factors are obtained by attributing at pleasure the signs $+$, $-$ to the radicals $\sqrt{pb\beta}$, $\sqrt{pc\beta}$, $\sqrt{pb\gamma}$, $\sqrt{pc\gamma}$; hence there are four factors in which $\sqrt{pb\beta}$, $\sqrt{pb\gamma}$ have determinate signs, but in which we attribute to the radicals $\sqrt{pc\beta}$, $\sqrt{pc\gamma}$ the signs $+$ or $-$ at pleasure. It is to be shown that the four factors each vanish for a point on the line ab; that is, on writing therein for x, y, z, w the values $ux_a + vx_b$, $uy_a + vy_b$, &c. But we thus

have, as before, $pa\alpha = -v\,.\,ab\alpha$ and $pb\alpha = u\,.\,ab\alpha$, with the like formulæ with β and γ in place of α. The factor thus becomes

$$\sqrt{-uv}\begin{vmatrix} \sqrt{ab\alpha}, & \sqrt{ab\alpha}, & \sqrt{pc\alpha} \\ \sqrt{ab\beta}, & \sqrt{ab\beta}, & \sqrt{pc\beta} \\ \sqrt{ab\gamma}, & \sqrt{ab\gamma}, & \sqrt{pc\gamma} \end{vmatrix},$$

which vanishes, being $=0^1$; and the norm is thus $=0^4$, viz. the line is 4-tuple.

(2) The line α is 8-tuple: in fact, for a point on the line we have $pa\alpha=0$, $pb\alpha=0$, $pc\alpha=0$, whence each factor vanishes, being $=0^{\frac{1}{2}}$, and the norm is therefore 0^8.

(3) The line $(ab,\ \alpha,\ \beta,\ \gamma)$ is 4-tuple: in fact, writing $z=0$, $w=0$ for the equations of the line, we have $h_\alpha=0$, $h_\beta=0$, $h_\gamma=0$, and $z_a w_b - z_b w_a = 0$, or say $w_a = \lambda z_a$, $w_b = \lambda z_b$. Hence, writing

$$I = (g-\lambda a)\,x - (f+\lambda b)\,y,$$

viz. $I_\alpha = (g_\alpha - \lambda a_\alpha)\,x - (f_\alpha + \lambda b_\alpha)\,y$, &c., for $z=0$, $w=0$, we have $pa\alpha = z_a I_\alpha$, $pb\alpha = z_b I_\alpha$; and similarly $pa\beta = z_a I_\beta$, $pb\beta = z_b I_\beta$, and $pa\gamma = z_a I_\gamma$, $pb\gamma = z_b I_\gamma$. The factor thus is

$$\sqrt{z_a z_b}\begin{vmatrix} \sqrt{I_\alpha}, & \sqrt{I_\alpha}, & \sqrt{pc\alpha} \\ \sqrt{I_\beta}, & \sqrt{I_\beta}, & \sqrt{pc\beta} \\ \sqrt{I_\gamma}, & \sqrt{I_\gamma}, & \sqrt{pc\gamma} \end{vmatrix},$$

which vanishes, being $=0^1$; there are four such factors, or the norm is 0^4; whence the line is 4-tuple.

(8) The line $abc\,.\,\alpha\,.\,\beta$ is a 4-tuple line. To prove it, take as before $w=0$ for the equation of the plane abc, and $(z=0,\ w=0)$ for the equations of the line in question. We have $h_\alpha=0$, $h_\beta=0$, $w_a=0$, $w_b=0$, $w_c=0$; whence (if $z=0$, $w=0$), writing for shortness $I=gx-fy$ (viz. $I_\alpha = g_\alpha x - f_\alpha y$, $I_\beta = g_\beta x - f_\beta y$), we have paa, $pb\alpha$, $pc\alpha = I_\alpha z_a$, $I_\alpha z_b$, $I_\alpha z_c$, and similarly $pa\beta$, $pb\beta$, $pc\beta = I_\beta z_a$, $I_\beta z_b$, $I_\beta z_c$: the factor thus is

$$\begin{vmatrix} \sqrt{I_\alpha z_a}, & \sqrt{I_\alpha z_b}, & \sqrt{I_\alpha z_c} \\ \sqrt{I_\beta z_a}, & \sqrt{I_\beta z_b}, & \sqrt{I_\beta z_c} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{vmatrix}$$

which vanishes, being $=0^1$: and there are four such factors, obtained by giving to the radicals the signs $+$, $-$ at pleasure: hence the norm is $=0^4$.

{Surface $abca\beta\gamma$.}

Surface $ab\alpha\beta\gamma\delta$.

35. The equation is

$$\text{Norm}\ \{\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta - \sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha + \sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta - \sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma\} = 0,$$

where the norm is the product of 8 factors each of the order 3. As before, $pa\alpha = 0$ is the equation of the plane through the point a and the line α; viz. $pa\alpha$ has the value previously mentioned: and $p^2\beta\gamma\delta = 0$ is the equation of the quadric surface through the lines β, γ, δ.

36. *Investigation.* In the projection, taking ξ, η, ζ as current line-coordinates, the equation of the conic passing through the projections of the points a, b is

$$\sqrt{(p_a\xi + q_a\eta + r_a\zeta)(p_b\xi + q_b\eta + r_b\zeta)} + A\xi + B\eta + C\zeta = 0,$$

where A, B, C are arbitrary coefficients. To make this touch the projection of the line α, we must write $\xi : \eta : \zeta = P_\alpha : Q_\alpha : R_\alpha$; and then

$$\begin{aligned} p_a\xi + q_a\eta + r_a\zeta = &\quad p_aP_\alpha + q_aQ_\alpha + r_aR_\alpha, \\ = &\quad w_a(x\ P_\alpha + y\ Q_\alpha + z\ R_\alpha) \\ &\quad - w\ (x_aP_\alpha + y_aQ_\alpha + z_aR_\alpha), \\ = &\quad - w\ (x_aP_\alpha + y_aQ_\alpha + z_aR_\alpha + w_aS_\alpha), \\ = &\quad - w\,.\,pa\alpha, \end{aligned}$$

and similarly

$$p_b\xi + q_b\eta + r_b\zeta = - w\,.\,pb\alpha.$$

Hence the equation is

$$w\sqrt{pa\alpha\,.\,pb\alpha} + AP_\alpha + BQ_\alpha + CR_\alpha = 0;$$

and forming the like equations for the lines β, γ, δ respectively, and eliminating, we have

$$\begin{vmatrix} \sqrt{pa\alpha\,.\,pb\alpha}, & P_\alpha, & Q_\alpha, & R_\alpha \\ \sqrt{pa\beta\,.\,pb\beta}, & P_\beta, & Q_\beta, & R_\beta \\ \sqrt{pa\gamma\,.\,pb\gamma}, & P_\gamma, & Q_\gamma, & R_\gamma \\ \sqrt{pa\delta\,.\,pb\delta}, & P_\delta, & Q_\delta, & R_\delta \end{vmatrix} = 0;$$

which, throwing out a factor w, becomes

$$\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta - \sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha + \sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta - \sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma = 0;$$

or, taking the norm, we have the above written equation.

37. *Singularities.* The equation shows that

(0) The point a is a 4-conical point; in fact, for the point in question we have $pa\alpha = 0$, $pa\beta = 0$, $pa\gamma = 0$, $pa\delta = 0$; each factor is $= 0^{\frac{1}{2}}$, and the norm is $= 0^4$.

(1) The line ab is a 2-tuple line. To prove this, we have for the coordinates of a point on the line in question $ux_a + vx_b$, $uy_a + vy_b$, &c.; the values of $pa\alpha$, $pb\alpha$ become as before $-v\,.\,ab\alpha$, $+u\,.\,ab\alpha$, and similarly for $pa\beta$, $pb\beta$, &c.; so that, omitting the constant factor $\sqrt{-uv}$, the value of the factor is

$$ab\alpha\,.\,p^2\beta\gamma\delta - ab\beta\,.\,p^2\gamma\delta\alpha + ab\gamma\,.\,p^2\delta\alpha\beta - ab\delta\,.\,p^2\alpha\beta\gamma.$$

Taking (a, b, c, f, g, h) for the coordinates of the line ab, we have

$$ab\alpha = \mathrm{a}f_\alpha + \mathrm{b}g_\alpha + ch_\alpha + \mathrm{f}a_\alpha + \mathrm{g}b_\alpha + \mathrm{h}c_\alpha,$$

with the like expressions for $ab\beta$, &c.; and substituting for $p^2\beta\gamma\delta$, &c., their values, the factor is

	x^2	y^2	z^2	w^2	xw	yw	zw	yz	zx	xy
a	$fagh$			$fabc$	$fabg - fach$	$fbch$	$-fbcg$		$fcgh$	$fbgh$
b		$gbhf$		$gabc$	$-gach$	$gbch - gabf$	$gcaf$	$gchf$	$hafg$	$gahf$
c			$hcfg$	$habc$	$habg$	$-habf$	$hcaf - hbcg$	$hbfg$		
f		$abhf$	$acfg$			$abch$	$-abcg$	$abfg + achf$	$acgh$	$abgh$
g	$bagh$		$bcfg$		$-bach$		$bcaf$	$bchf$	$bcgh + bafg$	$bahf$
h	$cagh$	$cbhf$			$cabg$	$-cabf$		$cbfg$	$cafg$	$cahf + cbgh$

viz. the value of the factor is $\{\mathrm{a}\,(fagh) + \mathrm{g}\,(bagh) + \mathrm{h}\,(cagh)\}\,x^2 + \&c.$, where $fagh = f_\alpha a_\beta g_\gamma h_\delta$ is the determinant

$$\begin{vmatrix} f, & a, & g, & h \\ \vdots & & & \end{vmatrix},$$

the suffixes in the four lines being α, β, γ, δ respectively.

Collecting, this is

$$\begin{aligned}
&(\quad . \quad cbhfy - bcfgz + fabcw)(\quad . \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w) \\
&(-cagh x \quad . \quad + acfgz + gabcw)(-\mathrm{h}x \quad . \quad + \mathrm{f}z + \mathrm{b}w) \\
&(+bagh x - abhfy \quad . \quad + habcw)(\quad \mathrm{g}x - \mathrm{f}y \quad . \quad + cw) \\
&(-afgh x - bfghy - cfghz \quad . \quad)(\quad \mathrm{a}x + \mathrm{b}y + \mathrm{c}z \quad . \quad) \\
&\quad + bcgh\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - x\,(\quad . \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w)] \\
&\quad + cahf\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - y\,(-\mathrm{h}x \quad . \quad + \mathrm{f}z + \mathrm{b}w)] \\
&\quad + abfg\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - z\,(\quad \mathrm{g}x - \mathrm{f}y \quad . \quad + cw)] = 0;
\end{aligned}$$

or, what is the same thing,

$$AP + BQ + CR + DS = 0,$$

{Surface $abcda\beta$.}

where

$$\begin{aligned}
A &= \{-bcghx + cbhfy - bcfgz + fabcw\},\\
B &= \{-caghx + cahfy + acfgz + gabcw\},\\
C &= \{+baghx - abhfy + abfgz + habcw\},\\
D &= \{-afghx - bfghy - cfghz + (bcgh + cahf + abfg)\,w\},\\
P &= (\quad . \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w),\\
Q &= (-\mathrm{h}x \quad . \quad + \mathrm{f}z + \mathrm{b}w),\\
R &= (\quad \mathrm{g}x - \mathrm{f}y \quad . \quad + \mathrm{c}w),\\
S &= (\quad \mathrm{a}x + \mathrm{b}y + \mathrm{c}z \quad . \quad) = 0,
\end{aligned}$$

the right-hand factors vanishing for the values $ux_a + vx_b$ of the coordinates.

38. It thus appears so far that the factor is $= 0^1$; it is, in fact, $= 0^2$, viz. we can show that, operating upon it with

$$\Delta = Xd_x + Yd_y + Zd_z + Wd_w,$$

the value (for any point of the line ab) is $= 0$. We have

$$\Delta\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta = \frac{pa\alpha\,.\,lb\alpha + pb\alpha\,.\,la\alpha}{2\sqrt{pa\alpha\,.\,pb\alpha}}\,p^2\beta\gamma\delta + \sqrt{pa\alpha\,.\,pb\alpha}\,.\,\Delta\,.\,p^3\beta\gamma\delta,$$

where $lb\alpha\,(=\Delta pb\alpha)$ is what $pb\alpha$ becomes on writing therein (X, Y, Z, W) in place of (x, y, z, w). Writing, as before, for x, y, z, w the values $ux_a + vx_b$, &c., we have $pa\alpha = -v\,.\,ab\alpha$, $pb\alpha = u\,.\,ab\alpha$; and putting for shortness

$$-v\,.\,lb\alpha + u\,.\,la\alpha = lk\alpha, \text{ \&c.},$$

the expression in question, divided by $\sqrt{-uv}$, is

$$\begin{aligned}
&= -2vu\,\{ab\alpha\,.\,\Delta p^3\beta\gamma\delta - \text{\&c.}\}\\
&\qquad + \{lk\alpha\,.\,p^3\beta\gamma\delta - \text{\&c.}\},
\end{aligned}$$

where, denoting the determinants

$$\begin{vmatrix} X & Y & Z & W \\ ux_a - vx_b, & uy_a - vy_b, & uz_a - vz_b, & uw_a - vw_b \end{vmatrix}$$

by (a′, b′, c′, f′, g′, h′), we have

$$lk\alpha = \mathrm{a}'\mathrm{f}_a + \mathrm{b}'\mathrm{f}_a + \mathrm{c}'\mathrm{g}_a + \mathrm{f}'a_a + \mathrm{g}'b_a + \mathrm{h}'c_a.$$

But $ab\alpha\,.\,\Delta p^3\beta\gamma\delta = \Delta ab\alpha\,.\,p^3\beta\gamma\delta$, since $ab\alpha$ is independent of (x, y, z, w); and the expression is

$$\begin{aligned}
&= -2vu\Delta\,(AP + BQ + CR + DS)\\
&\qquad + AP' + BQ' + CR' + DS',
\end{aligned}$$

{Surface $abcd\alpha\beta$.}

where P', Q', R', S' denote $\mathrm{h}'y - \mathrm{g}'z + \mathrm{a}'w$, &c., and where, finally, x, y, z, w are to be replaced by $ux_a + vx_b$, &c. Since for these values P, Q, R, S vanish, the expression becomes

$$\begin{aligned} &= -2vu\,(A\Delta P + B\Delta Q + C\Delta R + D\Delta S) \\ &\qquad\quad + AP' + BQ' + CR' + DS'; \end{aligned}$$

that is

$$= A\,(P' - 2uv\Delta P) + B\,(Q' - 2uv\Delta Q) + C\,(R' - 2uv\Delta R) + D\,(S' - 2uv\Delta S)$$

and we have, in fact, $P' - 2uv\Delta P = 0$, &c. For, writing for a moment

$$\begin{aligned} x,\ y,\ z,\ w &= ux_a + vx_b,\ uy_a + vy_b,\ uz_a + vz_b,\ uw_a + vw_b, \\ x',\ y',\ z',\ w' &= ux_a - vx_b,\ uy_a - vy_b,\ uz_a - vz_b,\ uw_a - vw_b; \end{aligned}$$

then, for instance,

$$S' = \mathrm{a}'x + \mathrm{b}'y + \mathrm{c}'z,$$

where

$$\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}' = Yz' - Zy',\ Zx' - Xz',\ Xy' - Yx';$$

and thence

$$\begin{aligned} S' &= -\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ x', & y', & z' \end{vmatrix} \\ &= 2uv\,(\mathrm{a}X + \mathrm{b}Y + \mathrm{c}Z) \\ &= 2uv\Delta S; \end{aligned}$$

and similarly for the other equations. The factor is thus $= 0^2$; there is only one such factor, and the line ab is double.

(2) The line α is an 8-tuple line: in fact, for a point on the line we have $p\alpha\alpha = 0$, $pb\alpha = 0$, $p^2\gamma\delta\alpha = 0$, $p^2\delta\alpha\beta = 0$, $p^2\alpha\beta\gamma = 0$; and the factor vanishes, being $= 0^1$. Each of the factors is 0^1, and the norm is $= 0^8$.

39. (3) The line $[ab, \alpha, \beta, \gamma]$ is a double line. To prove this, observe first that for a point on this line we have $p^2\alpha\beta\gamma = 0$.

Taking as before $z = 0$, $w = 0$ for the equation of the line ab, α, β, γ, we have $h_\alpha = 0$, $h_\beta = 0$, $h_\gamma = 0$, and $z_a w_b - z_b w_a = 0$; or say $w_a = \lambda z_a$, $w_b = \lambda z_b$; whence, writing for shortness $I = -(g - \lambda a)\,x + (f + \lambda b)\,y$, viz. $I_\alpha = -(g_\alpha - \lambda a_\alpha)\,x + (f_\alpha + \lambda b_\alpha)\,y$, we have (when $z = 0$, $w = 0$) $pa\alpha = z_a I_\alpha$, $pb\alpha = z_b I_\alpha$, or omitting the factor $\sqrt{z_a z_b}$, $\sqrt{pa\alpha\,.\,pb\alpha} = I_\alpha$; and so for $\sqrt{pa\beta\,.\,pb\beta}$ and $\sqrt{pa\gamma\,.\,pb\gamma}$. The factor thus is

$$I_\alpha\,.\,p^2\beta\gamma\delta - I_\beta\,.\,p^2\gamma\delta\alpha + I_\gamma\,.\,p^2\delta\alpha\beta;$$

viz. writing $z = 0$, $w = 0$ in the expressions of $p^2\beta\gamma\delta$, &c., this may be written

$$\Sigma\,[(g - \lambda a)\,x - (f + \lambda b)\,y]\,\{(agh)\,x^2 + [(ahf) + (bgh)]\,xy + (bhf)\,y^2\},$$

where observe that Σ denotes a sum of three terms of the form

$$\alpha\,.\,\beta\gamma\delta - \beta\,.\,\gamma\delta\alpha + \gamma\,.\,\delta\alpha\beta.$$

{Surface $abcd\alpha\beta$.}

Adding thereto a fourth term $-\delta\,.\,\alpha\beta\gamma$, the value of the sum would be $=\alpha\beta\gamma\delta$, or the sum of the three terms is $=\alpha\beta\gamma\delta+\delta\,.\,\alpha\beta\gamma$, where the symbols represent determinants. But in each case the determinant $\alpha\beta\gamma$ is $=0$, as containing the column h_α, h_β, h_γ, the terms of which are each $=0$: thus $\Sigma g\,.\,agh$ is $=gagh-g_\delta\,.\,agh$, where in $gagh$ the suffixes are α, β, γ, δ, and in agh they are α, β, γ: that is, we have $\Sigma g\,.\,agh=gagh$. And the whole expression thus is

$$\begin{aligned}
= \quad & x^3\ (gagh-\lambda aagh) \\
& + x^2y\ (gahf-\lambda aahf+gbgh-\lambda abgh-fagh-\lambda bagh) \\
& + xy^2\ (gbhf-\lambda abhf \qquad\qquad -fahf-\lambda bahf-fbgh-\lambda bbgh) \\
& + y^3\ (\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -fhbf-\lambda bbhf),
\end{aligned}$$

where $gahf$ denotes the determinant $\begin{vmatrix} g, & a, & h, & f \\ \vdots & & & \end{vmatrix}$, with the suffixes α, β, γ, δ, in the four lines respectively, and so in other cases: the terms, such as $gagh$, which contain a twice-repeated letter, vanish of themselves; and in the coefficients of x^2y and xy^2, the terms which do not separately vanish destroy each other in pairs, $gahf-fagh=0$, &c.; whence the factor vanishes, being $=0^1$; there are two such factors (viz. the zero term $\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma$ may be taken with the sign $+$ or $-$ at pleasure), and the norm is thus $=0^2$.

40. But the line is tacnodal, each sheet of the surface touching along the line in question the hyperboloid $p^2\alpha\beta\gamma$. To prove this, write

$$\Delta=X\delta_x+Y\delta_y+Z\delta_z+W\delta_w;$$

we have for the hyperboloid, writing $z=0$, $w=0$,

$$\Delta p^2\alpha\beta\gamma=(afg\,.\,x+bfg\,.\,y)\,Z+(abg\,.\,x-abf\,.\,y)\,W;$$

and it is to be shown that

$$\Delta\,(\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta\mp\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma)$$

each contain the factor $\Delta p^2\alpha\beta\gamma$; or, what is the same thing, that

$$\Delta\Sigma\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta$$

contains the factor in question, Σ denoting the sum of the first three terms of the original expression. The value is

$$=\Sigma\left(\frac{pa\alpha\,.\,Pb\alpha+pb\alpha\,.\,Pa\alpha}{2\sqrt{pa\alpha\,.\,pb\alpha}}\,p^2\beta\gamma\delta+\sqrt{pa\alpha\,.\,pb\alpha}\,.\,\Delta p^2\beta\gamma\delta\right);$$

where $Pa\alpha$, $=\Delta pa\alpha$, denotes what $pa\alpha$ becomes on writing therein X, Y, Z, W for x, y, z, w; and the like as to $Pb\alpha$. Substituting for $pa\alpha$ and $pb\alpha$ their values z_aI_α and z_bI_α, and multiplying by $\sqrt{z_az_b}$, the expression is

$$=\Sigma\,\{(z_aPb\alpha+z_bPa\alpha)\,p^2\beta\gamma\delta+2z_az_bI_\alpha\,\Delta p^2\beta\gamma\delta\},$$

where we have

$$\begin{aligned}
&z_a P b\alpha + z_b P a\alpha \\
&\quad = z_b\{x_a(Zg_a - Wa_a) + y_a(-Zf_a - Wb_a) + z_a[X(-g_a + \lambda a_a) + Y(f_a + \lambda b_a) + (\lambda Z - W)c_a]\} \\
&\quad + z_a\{x_b(Zg_a - Wa_a) + y_b(-Zf_a - Wb_a) + z_b[X(-g_a + \lambda a_a) + Y(f_a + \lambda b_a) + (\lambda Z - W)c_a]\}, \\
&\quad = (z_b x_a + z_a x_b)(\quad Zg_a - Wa_a) \\
&\quad + (z_b y_a + z_a y_b)(-Zf_a - Wb_a) \\
&\quad + 2z_a z_b\{X(-g_a + \lambda a_a) + Y(f_a + \lambda b_a) + (\lambda Z - W)c_a\}.
\end{aligned}$$

Also

$$\begin{aligned}
z_a z_b I_a &= z_a z_b\{(-g_a + \lambda a_a)x + (f_a + \lambda b_a)y\}, \\
p^2\beta\gamma\delta &= x^2 . agh + xy(ahf + bgh) + y^2 . hbf, \\
\Delta p^2\beta\gamma\delta &= X . 2x . agh + \quad y(ahf + bgh) \\
&\quad + Y . x(ahf + bgh) + 2y . hbf \\
&\quad + Z . x(cgh + afg) + \quad y(bfg + chf) \\
&\quad + W . x(abg - cah) + \quad y(bch - abf).
\end{aligned}$$

41. The whole expression is a linear function of X, Y, Z, W, and it is easy to see *à priori*, or to verify, that the coefficients of X, Y, each of them vanish. The coefficient of Z is

$$\begin{aligned}
&= \Sigma\{(z_b x_a + z_a x_b)g_a - (z_b y_a + z_a y_b)f_a + 2\lambda z_a z_b c_a\}p^2\beta\gamma\delta \\
&\qquad + \Sigma z_a z_b[(-g_a + \lambda a_a)x + (f_a + \lambda b_a)y][x(cgh + afg) + y(bfg + chf)],
\end{aligned}$$

with a like expression for the coefficient of W.

The foregoing expression may be written

$$\begin{aligned}
&(z_b x_a + z_a x_b)\Sigma g[agh . x^2 + (ahf + bgh)xy + bhf . y^2] \\
&-(z_b y_a + z_a y_b)\Sigma f[agh . x^2 + (ahf + bgh)xy + bhf . y^2] \\
&+ 2\lambda z_a z_b \Sigma\{c[agh . x^2 + (ahf + bgh)xy + bhf . y^2] \\
&\qquad\qquad + (ax + by)[(cgh + afg)x + (bfg + chf)y]\} \\
&+ 2z_a z_b \Sigma(-gx + fy)[(cgh + afg)x + (bfg + chf)y].
\end{aligned}$$

The first sum is

$$\begin{aligned}
&x^2 . gagh + xy(gahf + gbgh) + y^2 . gbhf, \\
&\quad = -xy . afgh - y^2 . bfgh, \\
&\quad = -h_\delta y(afg . x + bfg . y);
\end{aligned}$$

where afg, bfg denote determinants with the suffixes α, β, γ. Similarly the second sum is

$$= -h_\delta x(afg . x + bfg . y);$$

the third sum is

$$(a_\delta x + b_\delta y)(afg . x + bfg . y),$$

and the fourth sum is

$$(-g_\delta x + f_\delta y)(afg . x + bfg . y).$$

The whole coefficient of Z thus contains the factor $(afg\,.\,x+bfg\,.\,y)$; and similarly it would appear that the whole coefficient of W contains the factor $(abg\,.\,x-abf\,.\,y)$, the other factor being the same in each case; viz. the two terms together are

$$\left\{\begin{array}{l} -(z_b x_a + z_a x_b)\,h_\delta y \\ +(z_b y_a + z_a y_b)\,h^\delta x \\ +2\lambda z_a z_b (a_\delta x + b_\delta y) \\ +2 z_a z_b (-g_\delta x + f_\delta y) \end{array}\right\} \{Z\,(afg\,.\,x+bfg\,.\,y) + W\,(abg\,.\,x-abf\,.\,y)\};$$

where the second factor is $\Delta p^3\alpha\beta\gamma$, which is the required result. See *post*, Nos. 59 *et seq.*

42. (4) The line $[\alpha, \beta, \gamma, \delta]$ is an 8-tuple line; in fact, for any point of the line in question we have $p^2\beta\gamma\delta=0$, $p^2\gamma\delta\alpha=0$, $p^2\delta\alpha\beta=0$, $p^2\alpha\beta\gamma=0$; whence each factor is 0^1, or the norm is 0^8.

I notice that the surface meets the quadric $p^2\alpha\beta\gamma$ in

lines	α, β, γ	each 8	times	24
,,	$(\alpha, \beta, \gamma, \delta)$	,,	,,	16
,,	$(ab, \alpha, \beta, \gamma)$	,, 4	,,	8
			$24\times 2=$	48

Surface $a\alpha\beta\gamma\delta\epsilon$.

43. The equation is

$$(p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3a\alpha\gamma\,.\,\delta\beta + p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3a\alpha\beta\,.\,\gamma\delta)^2$$
$$-4p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\alpha\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,p\alpha a\,.\,p\delta a=0;$$

or, what is the same thing,

$$(p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3a\alpha\gamma\,.\,\delta\beta - p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3a\alpha\beta\,.\,\gamma\delta)^2$$
$$-4p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\beta a\delta\,.\,p^2\gamma a\delta\,.\,p\beta a\,.\,p\gamma a=0;$$

the equivalence of the two depending on the identity

$$p^3a\alpha\beta\,.\,\gamma\delta\,.\,p^3a\alpha\gamma\,.\,\delta\beta$$
$$-p^2\alpha\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,p\alpha a\,.\,p\delta a$$
$$+p^2\beta a\delta\,.\,p^2\gamma a\delta\,.\,p\beta a\,.\,p\gamma a=0;$$

where, as before, $p^2\alpha\beta\epsilon=0$ is the equation of the quadric through the lines α, β, ϵ, and $p\alpha a=0$ is the equation of the plane through the line α and the point a; viz. $p^2\alpha\beta\epsilon$, &c., and $p\alpha a$, &c., have the values already mentioned: $p^3a\alpha\beta\,.\,\gamma\delta=0$ as already mentioned is the cubic surface through the lines α, β, γ, δ and $a\alpha\beta$, $a\gamma\delta$.

{Surface $a\alpha\beta\gamma\delta\epsilon$.}

44. *Investigation.* In the projection, using line-coordinates, the equation of the conic touching the five lines may be written

$$\begin{vmatrix} (\xi, & \eta, & \zeta)^2 \\ (P, & Q, & R)^2 \end{vmatrix} = 0;$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to (P, Q, R) the suffixes α, β, γ, δ, ϵ respectively. This is at once transformed into

$$\alpha\beta\epsilon\,.\,\gamma\delta\epsilon\,.\,\alpha\gamma\Delta\,.\,\delta\beta\Delta - \alpha\gamma\epsilon\,.\,\delta\beta\epsilon\,.\,\alpha\beta\Delta\,.\,\gamma\delta\Delta = 0,$$

or, what is the same thing,

$$p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,\alpha\gamma\Delta\,.\,\delta\beta\Delta - p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,\alpha\beta\Delta\,.\,\gamma\delta\Delta = 0;$$

or say

$$p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,(A''\xi + B''\eta + C''\zeta)\,(A'''\xi + B'''\eta + C'''\zeta)$$
$$- p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,(A\xi + B\eta + C\zeta)\,(A'\xi + B'\eta + C'\zeta) = 0;$$

where $p^2\alpha\beta\epsilon$, &c., signify as before; and

$$A\xi + B\eta + C\zeta = \begin{vmatrix} \xi\,, & \eta\,, & \zeta \\ P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \end{vmatrix},$$

and so for $A'\xi + B'\eta + C'\zeta$, &c., the suffixes for A', B', C' being (γ, δ); and those for $A''\xi + B''\eta + C''\zeta$ and $A'''\xi + B'''\eta + C'''\zeta$ being (α, γ) and (δ, β) respectively.

45. Passing to the reciprocal equation, and making the conic pass through the point a, we obtain the equation of the surface in the form

$$\left\{p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon \begin{vmatrix} p_a, & q_a, & r_a \\ A\,, & B\,, & C \\ A', & B', & C' \end{vmatrix} - p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \begin{vmatrix} p_a\,, & q_a\,, & r_a \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix}\right\}^2$$
$$+ 4p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \begin{vmatrix} p_a\,, & q_a\,, & r_a \\ A\,, & B\,, & C \\ A'', & B'', & C'' \end{vmatrix} \begin{vmatrix} p_a\,, & q_a\,, & r_a \\ A'\,, & B'\,, & C' \\ A''', & B''', & C''' \end{vmatrix} = 0;$$

or in the equivalent form, where in the first term we have + instead of −, and in the second term the determinants are

$$\begin{vmatrix} p_a\,, & q_a\,, & r_a \\ A\,, & B\,, & C \\ A''', & B''', & C''' \end{vmatrix}, \quad \begin{vmatrix} p_a\,, & q_a\,, & r_a \\ A'\,, & B'\,, & C' \\ A'', & B'', & C'' \end{vmatrix}.$$

46. To reduce this result, observe that we have

$$A, B, C = \left\| \begin{matrix} hy - gz + aw, & -hx + fz + bw, & gx - fy + cw \\ h'y - g'z + a'w, & -h'x + f'z + b'w, & g'x - f'y + c'w \end{matrix} \right\|$$

{Surface $aa\beta\gamma\delta\epsilon$.}

where, for convenience, I retain the unaccented and accented letters $(a, \ldots)$, $(a', \ldots)$ instead of these letters with the suffixes α and β respectively. Writing as before

$$\begin{aligned} L &= (af' - a'f)\,x + \ldots \\ M &= (ag' - a'g)\,x + \ldots \\ N &= (ah' - a'h)\,x + \ldots \\ \Omega &= (gh' - g'h)\,x + \ldots \end{aligned}$$

then

$$\begin{aligned} A &= \Omega x - Lw, \\ B &= \Omega y - Mw, \\ C &= \Omega z - Nw, \end{aligned}$$

and similarly

$$\begin{aligned} A' &= \Omega' x - L'w, \\ B' &= \Omega' y - M'w, \\ C' &= \Omega' z - N'w; \end{aligned}$$

where for L', M', N', Ω' we have $(a'', \ldots)$ and $(a''', \ldots)$. Hence

$$BC' - B'C = w \begin{vmatrix} y, & z, & w \\ M, & N, & \Omega \\ M', & N', & \Omega' \end{vmatrix}$$

with like expressions for $CA' - C'A$ and $AB' - A'B$; and substituting, we have

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = w \begin{vmatrix} p_a, & q_a, & r_a, & \\ x, & y, & z, & w \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

or substituting for p_a, q_a, r_a their values $xw_a - wx_a$, $yw_a - wy_a$, $zw_a - wz_a$, this is

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = -w^2 \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

whence, omitting the factors w^2, the equation is

$$\begin{aligned} &\{p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix} - p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L'', & M'', & N'', & \Omega'' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix}\}^2 \\ &+ 4p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon \,.\, p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L'', & M'', & N'', & \Omega'' \end{vmatrix} \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L', & M', & N', & \Omega' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix} = 0, \end{aligned}$$

{Surface $aa\beta\gamma\delta\epsilon$.}

where I recall that for $(L, \ldots)$, $(L', \ldots)$, $(L'', \ldots)$, $(L''', \ldots)$ the suffixes are (α, β), (γ, δ), (α, γ), and (δ, β) respectively. The values of the first two determinants thus are $p^3 a\alpha\beta \,.\, \gamma\delta$ and $p^3 a\alpha\gamma \,.\, \delta\beta$ respectively: that of the third is $p^3 a\alpha\beta \,.\, \alpha\gamma$; viz. this is $= p^2 \alpha\beta\gamma \,.\, pa\alpha$; similarly, that of the fourth is $p^3 a\gamma\delta \,.\, \delta\beta$, which is $= -p^3 a\delta\gamma \,.\, \delta\beta = +p^3 a\delta\beta \,.\, \delta\gamma$; or finally this is $= p^2 \delta\beta\gamma \,.\, pa\delta$. And we have thus the before-mentioned equation of the surface.

47. *Singularities.* The equation of the surface shows that

(0) The point a is a 2-conical point: in fact, we have for this point $p^3 a\alpha\beta \,.\, \gamma\delta = 0$, $p^3 a\alpha\gamma \,.\, \delta\beta = 0$, $pa\alpha = 0$, $pa\delta = 0$.

(2) The line α is a 4-tuple line: in fact, for any point on this line $p^2 \alpha\beta\epsilon = 0$, $p^3 a\alpha\beta \,.\, \gamma\delta = 0$, $p^2 \alpha\gamma\epsilon = 0$, $p^3 a\alpha\gamma \,.\, \delta\beta = 0$, $p^2 \alpha\beta\gamma = 0$, $p^2 a\alpha = 0$.

(4) The line $(\alpha, \beta, \gamma, \epsilon)$ is a 2-tuple line: in fact, for any point on the line we have $p^2 \alpha\beta\epsilon = 0$, $p^2 \alpha\gamma\epsilon = 0$.

(10) The excuboquartic $\alpha\beta\epsilon \,.\, \gamma\delta \,.\, a$ is a simple curve: in fact, for any point of this curve we have $p^2 \alpha\beta\epsilon = 0$, $p^3 a\alpha\beta \,.\, \gamma\delta = 0$, these two surfaces intersecting in the lines α, β and the curve. It is, moreover, obvious that the surface is touched along the curve by the hyperboloid $p^2 \alpha\beta\epsilon$.

I notice that the surface meets the quadric $p^2 \alpha\beta\gamma$ in

lines (α, β, γ)	each 4 times,		12
" $(\alpha, \beta, \gamma, \delta)$	"	twice,	4
" $(\alpha, \beta, \gamma, \epsilon)$	"	"	4
curve $a\alpha\beta\gamma \,.\, \delta\epsilon$	"	"	8
		$14 \times 2 =$	28

Surface $\alpha\beta\gamma\delta\epsilon\zeta$.

48. The equation of the surface may be written

$$p^2 \alpha\beta\epsilon \,.\, p^2 \gamma\delta\epsilon \,.\, p^2 \alpha\gamma\zeta \,.\, p^2 \delta\beta\zeta - p^2 \alpha\beta\zeta \,.\, p^2 \gamma\delta\zeta \,.\, p^2 \alpha\gamma\epsilon \,.\, p^2 \delta\beta\epsilon = 0,$$

where $p^2 \alpha\beta\epsilon = 0$ is the equation of the quadric through the lines α, β, ϵ; viz. $p^2 \alpha\beta\epsilon$ has the value already mentioned.

The form is one of 45 like forms depending on the partitionment

$$\left\{\begin{matrix} \alpha\beta \,.\, \gamma\delta \\ \alpha\gamma \,.\, \delta\beta \\ \alpha\delta \,.\, \beta\gamma \end{matrix}\right\} (\epsilon, \zeta)$$

of the six letters.

{Surface $\alpha\beta\gamma\delta\epsilon\zeta$.}

49. *Investigation.* The projections of the six lines are tangents to a conic: the condition for this is $(P, Q, R)^2 = 0$, where the left-hand side represents the determinant obtained by writing successively $(P_\alpha, Q_\alpha, R_\alpha)$, &c. for (P, Q, R). The equation may be written

$$\alpha\beta\epsilon \,.\, \gamma\delta\epsilon \,.\, \alpha\gamma\zeta \,.\, \delta\beta\zeta - \alpha\beta\zeta \,.\, \gamma\delta\zeta \,.\, \alpha\gamma\epsilon \,.\, \delta\beta\gamma = 0,$$

where

$$\alpha\beta\epsilon = \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \\ P_\epsilon, & Q_\epsilon, & R_\epsilon \end{vmatrix}$$

and substituting for P_α, &c., their values, we have $\alpha\beta\epsilon = w \,.\, p^2\alpha\beta\epsilon$; whence the foregoing result.

50. *Singularities.* The equation shows that

(2) The line α is a 2-tuple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\gamma\zeta = 0$, $p^2\alpha\beta\zeta = 0$, $p^2\alpha\gamma\epsilon = 0$.

(4) The line $(\alpha, \beta, \epsilon, \zeta)$ is a simple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\beta\zeta = 0$.

(9) The quadriquadric $\alpha\beta\epsilon \,.\, \gamma\delta\zeta = 0$ is a simple curve on the surface: in fact, for each point of the curve we have $p^2\alpha\beta\epsilon = 0$, $p^2\gamma\delta\zeta = 0$.

It may be remarked that the surface meets the hyperboloid $p^2\alpha\beta\epsilon$ in

lines $(\alpha, \beta, \epsilon)$	each	twice,	6
„ $(\alpha, \beta, \epsilon, \gamma)$	„	once,	2
„ $(\alpha, \beta, \epsilon, \delta)$	„	„	2
„ $(\alpha, \beta, \epsilon, \zeta)$	„	„	2
curve $\alpha\beta\epsilon \,.\, \gamma\delta\zeta$	„	„	4
		$2 \times 8 =$	16

51. It might be thought that there should be on the surface some curve $\alpha\beta\gamma\delta\epsilon\zeta$, such as the cubic *abcdef* on the surface *abcdef*; but I cannot find that this is so. The equation of the surface is satisfied if we have simultaneously (λ being arbitrary)

$$p^2\alpha\beta\epsilon \,.\, p^2\alpha\gamma\zeta - \lambda p^2\alpha\beta\zeta \,.\, p^2\alpha\gamma\epsilon = 0,$$
$$\lambda p^2\gamma\delta\epsilon \,.\, p^2\delta\beta\zeta - p^2\gamma\delta\zeta \,.\, p^2\delta\beta\epsilon = 0;$$

which equations represent quartic surfaces, the first of them having α for a double line, and passing through the lines $\beta, \gamma, \epsilon, \zeta$ ($13 + 4 \times 5 = 33$ conditions, so that the equation of such a surface contains only an arbitrary parameter λ); and the second having δ for a double line, and passing through the lines $\beta, \gamma, \epsilon, \zeta$. But I see no condition by which λ can be determined so as to have the same value in the two equations respectively. Of course, leaving it arbitrary, the two quartic surfaces intersect in the lines $\beta, \gamma, \epsilon, \zeta$ and in a curve of the order 12 depending on the arbitrary value of λ, which curve lies on the surface $\alpha\beta\gamma\delta\epsilon\zeta$.

{Surface $\alpha\beta\gamma\delta\epsilon\zeta$.}

The Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, a.

52. The notion is, that we have a fixed point a, two fixed lines δ, ϵ, and a singly infinite series of lines, or say the generating lines of a skew surface: each generating line determines, with the point a, a plane; and if in this plane we draw, meeting the lines δ, ϵ, a line to meet the generating line in a point P, then the locus of this point P is the curve about to be considered.

53. In the case in question, the singly infinite series of lines is that of the lines which meet each of the lines α, β, γ, or say these are the generatrices of the hyperboloid $\alpha\beta\gamma$: the locus, or curve $\alpha\beta\gamma$, $\delta\epsilon$, a, is (as mentioned above) an excuboquartic. It is not necessary for the purpose of the memoir, but it is interesting to consider in conjunction therewith the excuboquartic arising in like manner from the directrices of the hyperboloid; it will appear that the two curves are the complete intersection of the quadric $\alpha\beta\gamma$ by a quartic surface. Observe that the two curves are given as follows: viz. considering for the quadric $\alpha\beta\gamma$ any tangent-plane through the point a, and drawing in this plane, to meet the lines δ and ϵ, a line, this meets the section of the quadric surface by the tangent-plane in two points, the locus of which is the aggregate of the two curves: viz. the section being a line-pair, the two points belong, one of them to a generatrix and the other to a directrix of the quadric surface.

54. It is convenient to take $x=0$, $y=0$ for the equations of the line δ; $z=0$, $w=0$ for those of the line ϵ: for then, for any plane $Ax+By+Cz+Dw=0$, the line in this plane and meeting the lines δ and ϵ, has for its equations $Ax+By=0$, $Cz+Dw=0$; or, what is the same thing, for the plane $P=0$ the equations of the line are $P_{xy}=0$, $P_{zw}=0$, where P_{xy}, P_{zw} denote the terms in x, y and in z, w respectively.

I take also x_0, y_0, z_0, w_0 for the coordinates of the point a, and $PS-QR=0$ for the equation of the quadric surface, P, Q, R, S being given linear functions of (x, y, z, w): we have then say $P-\theta R=0$, $Q-\theta S=0$ for the equations of any generatrix, and $P-\phi Q=0$, $R-\phi S=0$ for the equations of any directrix of the hyperboloid.

The equation of the plane through the point a and the generatrix $P-\theta R=0$, $Q-\theta S=0$, is clearly

$$(Q_0-\theta S_0)(P\ \ -\theta R\ \ )-(P_0-\theta R_0)(Q\ \ -\theta S\ \ )=0;$$

so that for the line in this plane, meeting the lines δ and ϵ, we have

$$(Q_0-\theta S_0)(P_{xy}-\theta R_{xy})-(P_0-\theta R_0)(Q_{xy}-\theta S_{xy})=0,$$
$$(Q_0-\theta S_0)(P_{zw}-\theta R_{zw})-(P_0-\theta R_0)(Q_{zw}-\theta S_{zw})=0;$$

and joining thereto the equations

$$\theta=\frac{P}{R}=\frac{Q}{S}=\frac{P_{xy}+P_{zw}}{R_{xy}+R_{zw}}=\frac{Q_{xy}+Q_{zw}}{R_{xy}+R_{zw}},$$

(equivalent in all to three equations,) the elimination of θ gives the required curve: the equations thus are

$$PS-QR=0,$$
$$(Q_0S-QS_0)(P_{xy}R-PR_{xy})-(P_0R-PR_0)(Q_{xy}S-QS_{xy})=0,$$

or, as the second equation may also be written,

$$(Q_0S - QS_0)(P_{xy}R_{zw} - P_{zw}R_{xy}) - (P_0R - PR_0)(Q_{xy}S_{zw} - Q_{zw}S_{xy}) = 0;$$

viz. the second equation represents a cubic surface having upon it the lines $(P=0,\ R=0)$ and $(Q=0,\ S=0)$: it therefore intersects the quadric $PS - QR = 0$ in these two lines, and besides in an excuboquartic curve, which is the required locus.

55. Representing the determinants

$$\begin{vmatrix} P\ , & Q\ , & R\ , & S \\ P_0\ , & Q_0\ , & R_0\ , & S_0 \end{vmatrix} \text{ by } (\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}'), \text{ viz. } \begin{aligned} \mathrm{a}' &= QR_0 - Q_0R, \ldots \\ \mathrm{f}' &= PS_0 - P_0S, \ldots; \end{aligned}$$

$$\begin{vmatrix} P_{xy}, & Q_{xy}, & R_{xy}, & S_{xy} \\ P_{zw}, & Q_{zw}, & R_{zw}, & S_{zw} \end{vmatrix} \text{ by } (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}), \text{ viz. } \mathrm{a} = Q_{xy}R_{zw} - Q_{zw}R_{xy}, \ldots;$$

so that $(\mathrm{a}', \ldots)$ are linear functions, $(\mathrm{a}, \ldots)$ quadric functions, of the coordinates; the equation of the cubic surface is $\mathrm{gb}' - \mathrm{bg}' = 0$, viz. the excuboquartic arising from the generatrices is the partial intersection of the quadric $PS - QR = 0$ and the cubic $\mathrm{gb}' - \mathrm{g}'\mathrm{b} = 0$; the two surfaces besides intersecting in the lines $(P=0,\ R=0)$ and $(Q=0,\ S=0)$.

It appears, in the same manner, that the excuboquartic arising from the directrices is the partial intersection of the quadric $PS - QR = 0$ and the cubic $\mathrm{hc}' - \mathrm{ch}' = 0$; the two surfaces besides intersecting in the lines $(P=0,\ Q=0)$ and $(R=0,\ S=0)$.

56. But the elimination may be performed in a different manner, as follows: from the first two equations in θ, multiplying by P_{zw}, $-P_{xy}$ and adding, and so with Q_{zw}, $-Q_{xy}$, &c., we obtain

$$\begin{aligned}
(Q_0 - \theta S_0)(\quad -\theta\mathrm{b}) &- (P_0 - \theta R_0)(-\mathrm{c} + \theta\mathrm{f}) = 0, \\
(Q_0 - \theta S_0)(\ \mathrm{c} + \theta\mathrm{a}) &- (P_0 - \theta R_0)(\qquad \theta\mathrm{g}) = 0, \\
(Q_0 - \theta S_0)(-\mathrm{b}\qquad) &- (P_0 - \theta R_0)(\ \mathrm{a} + \theta\mathrm{h}) = 0, \\
(Q_0 - \theta S_0)(\ \mathrm{f} - \theta\mathrm{h}) &- (P_0 - \theta R_0)(\ \mathrm{g}\qquad) = 0.
\end{aligned}$$

We then have

$$\theta = \frac{-\mathrm{c} + \theta\mathrm{f}}{\mathrm{a} + \theta\mathrm{h}} = \frac{\mathrm{c} + \theta\mathrm{a}}{\mathrm{f} - \theta\mathrm{h}},$$

or, what is the same thing,

$$\mathrm{h}\theta^2 + (\mathrm{a} - \mathrm{f})\theta + \mathrm{c} = 0.$$

Using this equation, written in the form $(\mathrm{a} + \theta\mathrm{h})\theta = -\mathrm{c} + \theta\mathrm{f}$, to transform the first or third of the four equations in θ, we obtain

$$-\mathrm{a}P_0 - \mathrm{b}Q_0 - \mathrm{c}R_0 + \theta(-\mathrm{h}P_0\ \ . + \mathrm{f}R_0 + \mathrm{b}S_0) = 0;$$

and using the same equation, written in the form $(\mathrm{f} - \theta\mathrm{h})\theta = \mathrm{c} + \theta\mathrm{a}$, to transform the second or fourth equation, we obtain

$$\mathrm{g}P_0 - \mathrm{f}Q_0 + \mathrm{c}S_0 + \theta(\qquad \mathrm{h}Q_0 - \mathrm{g}R_0 + \mathrm{a}S_0) = 0;$$

and hence, eliminating θ, we obtain

$$(\mathrm{h}Q_0 - \mathrm{g}R_0 + \mathrm{a}S_0)(-\mathrm{a}P_0 - \mathrm{b}Q_0 - \mathrm{c}R_0) - (-\mathrm{h}P_0 + \mathrm{f}R_0 + \mathrm{b}S_0)(\mathrm{g}P_0 - \mathrm{f}Q_0 . + \mathrm{c}S_0) = 0,$$

which, as being of the second order in (a, ...), represents a quartic surface. The equation remains unaltered by the interchange of Q, R, and the consequent interchanges among (a, b, c, f, g, h): hence the quartic surface contains not only the excuboquartic arising from the generatrices, but also that arising from the directrices; and these two curves are the complete intersection of the quartic by the quadric $PS - QR = 0$.

57. I obtain this same result also as follows. Consider a point (P_1, Q_1, R_1, S_1) on the quadric surface; $P_1S_1 - Q_1R_1 = 0$; the tangent plane at the point is

$$PS_1 - QR_1 - RQ_1 + SP_1 = 0;$$

and if this passes through the point a, then

$$P_0S_1 - Q_0R_1 - R_0Q_1 + S_0P_1 = 0.$$

The line which in the tangent-plane meets the lines δ, ϵ is given, as before, by the equations

$$P_{xy}S_1 - Q_{xy}R_1 - R_{xy}Q_1 + S_{xy}P_1 = 0,$$
$$P_{zw}S_1 - Q_{zw}R_1 - R_{zw}Q_1 + S_{zw}P_1 = 0.$$

Remembering the significations of (a, ...), the last three equations give

$$\begin{aligned} S_1 : R_1 : -Q_1 : -P_1 = \quad & . \quad \mathrm{h}Q_0 - \mathrm{g}R_0 + \mathrm{a}S_0 \\ : & -\mathrm{h}P_0 \quad . \quad + \mathrm{f}R_0 + \mathrm{b}S_0 \\ : & \mathrm{g}P_0 - \mathrm{f}Q_0 \quad . \quad + \mathrm{c}S_0 \\ : & -\mathrm{a}P_0 - \mathrm{b}Q_0 - \mathrm{c}R_0 \quad . \quad ; \end{aligned}$$

and substituting these values in $S_1P_1 - Q_1R_1 = 0$, we have the above equation of the quadric surface.

58. Or again, changing the notation, I take the equation of the quadric surface to be

$$(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0.$$

A tangent-plane hereof is

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

where ξ, η, ζ, ω are any quantities satisfying the relation

$$(A, B, C, D, F, G, H, L, M, N \between \xi, \eta, \zeta, \omega)^2 = 0,$$

the capitals denoting the inverse coefficients.

Supposing that the tangent-plane passes through a fixed point a, coordinates $(\alpha, \beta, \gamma, \delta)$, we have

$$\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega = 0;$$

and if the equations of the lines δ, ϵ are as before $(x=0,\ y=0)$ and $(z=0,\ w=0)$; then for the line in the tangent-plane meeting the lines δ, ϵ, we have

$$\xi x+\eta y=0,\quad \zeta z+\omega w=0.$$

These last equations may be represented by

$$\xi=ly,\quad \eta=-lx,\quad \zeta=mw,\quad \omega=-mz;$$

and, substituting these values, we have

$$(A,\ldots\between ly,\ -lx,\ mw,\ -mz)^2=0,$$
$$(\alpha,\ldots\between ly,\ -lx,\ mw,\ -mz)^1=0,$$

that is

$$(Ay^2-2Hxy+Bx^2,\ -Fxw+Gyw-Lyz+Mxz,\ Cw^2-2Nwz+Dz^2\between l,\ m)^2=0,$$

and

$$(\alpha y-\beta x,\ \gamma w-\delta z\between l,\ m)=0.$$

Whence, eliminating l, m, we have the quartic equation

$$(Ay^2-2Hxy+Bx^2,\ -Fxw+Gyw-Lyz+Mxz,\ Cw^2-2Nzw+Dz^2\between \gamma w-\delta z,\ \beta x-\alpha y)^2=0.$$

Further Investigation as to the Surface $ab\alpha\beta\gamma\delta$.

59. The theorem that in the surface $ab\alpha\beta\gamma\delta$, the equation of which is

$$\text{Norm}\ \{\sqrt{p\alpha\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta-\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma\}=0;$$

the lines $(ab,\ \alpha,\ \beta,\ \gamma)$ are tacnodal, each sheet touching along the line the quadric $p^2\alpha\beta\gamma$, may be proved in a different manner by investigating the intersection of the surface with the quadric $p^2\alpha\beta\gamma$.

For this purpose take the equation of the quadric to be $yz-xw=0$; the equations of the lines α, β, γ will be

$$\begin{pmatrix} z-\lambda_\alpha w=0\\ x-\lambda_\alpha y=0\end{pmatrix},\quad \begin{pmatrix} z-\lambda_\beta w=0\\ x-\lambda_\beta y=0\end{pmatrix},\quad \begin{pmatrix} z-\lambda_\gamma w=0\\ x-\lambda_\gamma y=0\end{pmatrix};$$

and we may write $(a,\ b,\ c,\ f,\ g,\ h)$ for the coordinates of the line δ. The equation of the surface will be

$$\text{Norm}\left\{\Sigma\left[\pm\sqrt{p\alpha\alpha\,.\,pb\alpha}\,(\lambda_\beta-\lambda_\gamma)\left\{\begin{array}{l}(a-f)\,xz-(\lambda_\beta+\lambda_\gamma)\,yz+\lambda_\beta\lambda_\gamma yw\\ +(b-g)\,\lambda_\beta\lambda_\gamma\,(yz-xw)\\ \qquad +c\,(z-\lambda_\beta w)\,(z-\lambda_\gamma w)\\ \qquad +h\,(x-\lambda_\beta y)\,(x-\lambda_\gamma z)\end{array}\right\}\right]\right.$$
$$\left.-\sqrt{pa\delta\,.\,pb\delta}\,(\lambda_\beta-\lambda_\gamma)\,(\lambda_\gamma-\lambda_\alpha)\,(\lambda_\alpha-\lambda_\beta)\,(yz-xw)\right\};$$

where Σ denotes the sum of the three terms obtained by the cyclical interchange of α, β, γ; and

$$pa\alpha = (z_a - \lambda w_a)(x - \lambda y) - (x_a - \lambda y_a)(z - \lambda w),$$
$$pb\alpha = (z_b - \lambda w_b)(x - \lambda y) - (x_b - \lambda y_b)(z - \lambda w);$$

λ here standing for λ_α; and similarly for $pa\beta$, &c.

60. To obtain the intersection with $xw - yz = 0$, writing $w = \frac{yz}{x}$, then

$$pa\alpha = [z_a - \lambda w_a - \frac{z}{x}(x_a - \lambda y_a)](x - \lambda y), \quad (\lambda = \lambda_\alpha),$$

$$pb\alpha = [z_b - \lambda w_b - \frac{z}{x}(x_b - \lambda y_b)](x - \lambda y);$$

or say

$$\sqrt{pa\alpha \,.\, pb\alpha} = \sqrt{M_\alpha}\,(x - \lambda_\alpha y);$$

also the expression in { } becomes

$$= \{(a - f)\frac{z}{x} + c\frac{z^2}{x^2} + h\}(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

so that the norm in question is

$$\text{Norm } \Sigma \sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma)\{(a - f)\frac{z}{x} + c\frac{z^2}{x^2} + h\}(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

or say

$$\text{Norm } \Sigma \sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma)\{hx^2 + (a - f)zx + cz^2\}(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

where M_α is now considered to stand for

$$\{(z_a x - z x_a) - \lambda(w_a x - y_a z)\}\{(z_b x - z x_b) - \lambda(w_b x - y_b z)\}.$$

Observing that the norm was originally the product of 8 factors, this breaks up into

$$\{hx^2 + (a - f)zx + cz^2\}^8\{(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y)\}^8 = 0,$$

and

$$\text{Norm}^2 \sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma) = 0,$$

where the new norm is the product of 4 factors.

61. Writing for greater convenience λ, μ, ν in place of λ_α, λ_β, λ_γ, and observing that M_α is a quadric function of λ_α, that is of λ, the last-mentioned norm is

$$\text{Norm } \sqrt{A + B\lambda + C\lambda^2}\,(\mu - \nu),$$

which is easily seen to be

$$= (4AC - B^2)(\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2;$$

or writing for a moment

$$(A + B\lambda + C\lambda^2) = (P - Q\lambda)(P' - Q'\lambda),$$

whence

$$A = PP', \quad B = -(PQ' + P'Q), \quad C = QQ';$$

then

$$4AC - B^2 = -(PQ' - P'Q)^2;$$

and we have

$$\begin{aligned} P,\ Q &= z_a x - z x_a, \quad w_a x - y_a z, \\ P',\ Q' &= z_b x - z x_b, \quad w_b x - y_b z, \end{aligned}$$

whence

$$\begin{aligned} PQ' - P'Q = \ & (z_a w_b - z_b w_a) x^2 \\ & + [y_a z_b - y_b z_a - (x_a w_b - x_b w_a)] xz \\ & + (x_a y_b - x_b y_a) z^2; \end{aligned}$$

viz. if (a, b, c, f, g, h) are the coordinates of the line ab, this is

$$= \mathrm{h}x^2 + (\mathrm{a} - \mathrm{f}) xz + \mathrm{c}z^2.$$

Hence, omitting the constant factor $(\mu - \nu)^4 (\nu - \lambda)^4 (\lambda - \mu)^4$ {that is $(\lambda_\beta - \lambda_\gamma)^4 (\lambda_\gamma - \lambda_\alpha)^4 (\lambda_\alpha - \lambda_\beta)^4$}, the foregoing equation $\mathrm{norm}^2 = 0$ becomes

$$[\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f}) xz + \mathrm{c}z^2]^4 = 0,$$

and the intersections of the quadric with the surface are obtained by combining the equation $xw - yz = 0$ with the several equations

$$\begin{aligned} &\{\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f}) zx + \mathrm{c}z^2\}^8 = 0, \\ &\{(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y)\}^8 = 0, \\ &\{\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f}) zx + \mathrm{c}z^2\}^4 = 0; \end{aligned}$$

viz. these are

lines $(\alpha, \beta, \gamma, \delta)$	each	8 times	16	
line $(x = 0,\ z = 0)$		16 „		16
lines α, β, γ	each	8 „	24	
line $(x = 0,\ y = 0)$		24 „		24
lines $[ab, \alpha, \beta, \gamma]$	each	4 „	8	
line $(x = 0,\ z = 0)$		8 „		8
		$(16 + 24 + 8) \times 2 =$	48	+ 48

But it is clear that the lines $(x = 0,\ y = 0)$ and $(x = 0,\ z = 0)$ are introduced by the process of elimination, and are no part of the intersection. The complete intersection consists of the lines $(\alpha, \beta, \gamma, \delta)$ each 8 times, the lines (α, β, γ) each 8 times, and the lines $[ab, \alpha, \beta, \gamma]$ each 4 times. But the last-mentioned lines being only double lines on the surface, this means that the two sheets each touch the quadric surface, or that the lines are tacnodal.

504.

ON THE MECHANICAL DESCRIPTION OF CERTAIN SEXTIC CURVES.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 105—111. Read April 11, 1872.]

THE curves in question might be taken to be those described by a point C rigidly connected with points A and B, each of which describes a circle: but the construction is considered under a somewhat more general form. I consider a quadrilateral, the sides of which are a, b, c, d, and the inclinations of these to a fixed line $\alpha, \beta, \gamma, \delta$. This being so, if a, b, c, d, and one of the angles, say δ, are constant, then we have between the three variable angles the relations

$$a \cos \alpha + b \cos \beta + c \cos \gamma + d \cos \delta = 0,$$

$$a \sin \alpha + b \sin \beta + c \sin \gamma + d \sin \delta = 0,$$

giving rise to a single relation between any two of the variable angles; and we consider a curve such that the coordinates x, y of any point thereof are given linear functions of the sines and cosines of the three variable angles, or, what is the same thing, of the sines and cosines of any two of these angles. We thus unite together what would otherwise be distinct cases; for everything is symmetrical in regard to the sides a, b, c and the corresponding variable angles α, β, γ, irrespectively of the order of succession of these sides: and we can thus, in the discussion of the curve, employ any two at pleasure, say α, β, of the variable angles, without determining whether the sides a, b are contiguous or opposite.

Eliminating, then, the variable angle γ, we obtain between α, β a relation which, if we write therein $\tan \frac{1}{2}\alpha = u$, $\tan \frac{1}{2}\beta = v$, takes the form $(*\between u, 1)^2 (v, 1)^2 = 0$; viz. either of the variables u, v is expressible rationally in terms of the other of them and of the root of a quartic function thereof; say v is a rational function of u and $\sqrt{U}$. And

hence a curve for which the coordinates x, y are rational functions of u, v, is a curve having a deficiency $D=1$, or, what is the same thing, having a number of dps. less by unity than the maximum number $\{=\frac{1}{2}(n-1)(n-2)$, if n be the order of the curve$\}$.

It will further appear that the relation $(*\between u,\ 1)^2(v,\ 1)^2=0$ is satisfied by the values $u=v=i$ and $u=v=-i$ (if, as usual, $i=\sqrt{-1}$).

In the curve in question, the coordinates (x, y) are given linear functions of the sines and cosines of α, β; and if we make the curve meet an arbitrary line $Ax+By+C=0$, we obtain between the sines and cosines of α, β a linear relation which, substituting therein the expressions in terms of u, v, takes the form

$$(*\between u,\ 1)^2.(1+v^2)+(*\between v,\ 1)^2.(1+u^2)=0,$$

viz. this is a relation of the form $(\dagger\between u,\ 1)^2(v,\ 1)^2=0$, such that it is satisfied by the four sets of values $u=\pm i$, $v=\pm i$, and therefore in particular by the values $u=v=i$ and $u=v=-i$.

Hence, considering the intersections of the curve by the arbitrary line, the values of (u, v) are given by the two equations $(*\between u,\ 1)^2(v,\ 1)^2=0$, $(\dagger\between u,\ 1)^2(v,\ 1)^2=0$; these, regarding for a moment u, v as ordinary rectangular coordinates, represent each of them a quartic curve having two dps. at infinity on the axes $u=0$, $v=0$ respectively: each of these points reckons therefore as 4 intersections, and the number of the remaining intersections therefore is $4.4-2.4$, $=8$. But, by what precedes, the two quartic curves have also in common the points $u=v=i$ and $u=v=-i$; and rejecting these, there remain $8-2$, $=6$ intersections.

The conclusion is, that the curve is a sextic curve of deficiency 1, that is, having 9 dps. The reasoning may be presented under a slightly different form as follows: regarding u, v as coordinates, we have the curve $(*\between u,\ 1)^2(v,\ 1)^2=0$, a binodal quartic curve, and having therefore the deficiency 1; the curve passes, as above-mentioned, through the points $u=v=i$ and $u=v=-i$. The required curve is obtained as a transformation of the quartic curve by formulæ of the form $x:y:z(=1)=P:Q:R$, where P, Q, and R $\{=(1+u^2)(1+v^2)\}$ are quartic functions of the coordinates u, v, such that $P=0$, $Q=0$, $R=0$ are each of them a quartic curve passing twice through each of the nodes and once through each of the before-mentioned points, $(u=v=i)$ and $(u=v=-i)$, of the binodal quartic curve. Hence the curve in question is a curve of the order $4.4-2.4-2.1$, $=6$, and having the same deficiency as the binodal quartic, that is, the deficiency is $=1$.

I observe that the sextic curve does not, in general, pass through the circular points at infinity, but it intersects the line at infinity in three distinct pairs of points; one of these, or all three of them, (but not two pairs only,) may coincide with the circular points at infinity, the circular points at infinity being, in the latter case, triple points, or the curve being tricircular: this will appear presently.

To obtain the foregoing equation $(*\between u,\ 1)^2(v,\ 1)^2=0$, the elimination of γ gives

$$(a\cos\alpha+b\cos\beta+d\cos\delta)^2+(a\sin\alpha+b\sin\beta+d\sin\delta)^2=c^2,$$

that is

$$a^2+b^2-c^2+d^2+2ab\cos(\alpha-\beta)+2ad\cos(\alpha-\delta)+2bd\cos(\beta-\delta)=0\,;$$

or, substituting herein the values

$$\cos\alpha=\frac{1-u^2}{1+u^2},\quad \sin\alpha=\frac{2u}{1+u^2},\quad \cos\beta=\frac{1-v^2}{1+v^2},\quad \sin\beta=\frac{2v}{1+v^2},$$

this is easily found to be $0=$

	1	$2u$	u^2
1	$(a+b)^2-2(a+b)\,d\cos\delta+d^2-c^2$	$2ad\sin\delta$	$(a-b)^2+2(a-b)\,d\cos\delta+d^2-c^2$
$2v$	$2bd\sin\delta$	$2ab$	$2bd\sin\delta$
v^2	$(a-b)^2-2(a-b)\,d\cos\delta+d^2-c^2$	$2ad\sin\delta$	$(a+b)^2+2(a+b)\,d\cos\delta+d^2-c^2$

or writing, for greater convenience, $\delta=0$, that is, taking α, β, γ to be the inclinations of the sides a, b, c to the fixed side d, this is $0=$

	1	$2u$	u^2
1	$(a+b-d)^2-c^2$	0	$(a-b+d)^2-c^2$
$2v$	0	$2ab$	0
v^2	$(a-b-d)^2-c^2$	0	$(a+b+d)^2-c^2$

,

or say this is

$$(A+Bu^2)+8abuv+v^2(C+Du^2)=0,$$

where, writing for shortness

$$\begin{aligned}
a+d-b-c&=\lambda, & -a+b+c+d&=\lambda',\\
b+d-c-a&=\mu, & -b+c+d+a&=\mu',\\
c+d-a-b&=\nu, & -c+d+a+b&=\nu',\\
a+b+c+d&=\sigma, & -d+a+b+c&=\rho';
\end{aligned}$$

then

$$A=-\nu\rho',\quad B=\lambda\mu',\quad C=\lambda'\mu,\quad D=\nu'\sigma.$$

We at once verify that the equation is satisfied by $u=v=\pm i$, viz. this will be so if only

$$A-B-C+D-8ab=0,$$

and we have

$$A + D = 2(a+b)^2 + 2d^2 - 2c^2,$$
$$B + C = 2(a-b)^2 + 2d^2 - 2c^2;$$

whence the relation in question.

Writing $u = i$, we have

$$\{v(C-D) + 4abi\}^2 = -16a^2b^2 - (A-B)(C-D)$$
$$= -\tfrac{1}{4}(A - B + C - D)^2,$$

whence

$$v(C-D) = \pm\tfrac{1}{2}i\{\mp 8ab + A - B + C - D\}$$
$$= i(C-D) \text{ or } -i(A-B);$$

so that, corresponding to $u = i$, we have the values $v = i$ and $v = -i\dfrac{A-B}{C-D}$; and similarly, corresponding to $u = -i$, the values $v = -i$ and $v = i\dfrac{A-B}{C-D}$.

And in like manner, corresponding to $v = i$, we have the values $u = i$ and $u = -i\dfrac{A-C}{B-D}$; and to $v = -i$, the values $u = -i$ and $u = i\dfrac{A-C}{B-D}$.

It is easy to show that, if $u = i + \epsilon$, $v = i + \zeta$ are the values consecutive to $u = v = i$, then $a\epsilon + b\zeta = 0$: in fact, substituting the foregoing values in the relation between u, v, and writing for $8ab$ its value, $= A - B - C + D$, we have

$$A + B(-1 + 2i\epsilon) + (A - B - C + D)\{-1 + i(\epsilon + \zeta)\} + C(-1 + 2i\zeta) + D\{1 - 2i(\epsilon + \zeta)\} = 0,$$

which is

$$= \epsilon(A + B - C - D) + \zeta(A - B + C - D) = 0;$$

or finally $a\epsilon + b\zeta = 0$. And similarly, if $u = -i + \epsilon$, $v = -i + \zeta$ are the values consecutive to $u = v = -i$, then we have the same relation $a\epsilon + b\zeta = 0$.

The points at infinity on the sextic curve are those for which $1 + u^2$ or $1 + v^2$, or each of these, is $= 0$; viz. the values of u, v for the six points are

$$u = i + \epsilon, \quad -i - \epsilon, \quad i, \quad -i, \quad -i\left(\frac{A-C}{B-D}\right), \quad i\left(\frac{A-C}{B-D}\right);$$

$$v = i + \zeta, \quad -i - \zeta, \quad -i\frac{A-B}{C-D}, \quad i\frac{A-B}{C-D}, \quad i, \quad -i,$$

where, instead of $u = v = i$ and $u = v = -i$, I have written down the consecutive values of u, v, and as before ϵ, ζ are infinitesimals such that $a\epsilon + b\zeta = 0$.

Suppose that the coordinates x, y of a point on the sextic curve are

$$x = \mathrm{a}\cos\alpha + \mathrm{c}\sin\alpha + \mathrm{b}\cos\beta + \mathrm{d}\sin\beta,$$
$$y = \mathrm{a}'\cos\alpha + \mathrm{c}'\sin\alpha + \mathrm{b}'\cos\beta + \mathrm{d}'\sin\beta;$$

then, if $u = \pm(i+\epsilon)$, $\cos\alpha = \frac{-i}{\epsilon}$, $\sin\alpha = \frac{\pm 1}{\epsilon}$, and similarly if $v = \pm(i+\zeta)$, then $\cos\beta = -\frac{i}{\zeta}$, $\sin\beta = \frac{\pm 1}{\zeta}$. Hence the points at infinity of the sextic curve are as follows:

1°. $u = \pm(i+\epsilon)$, v not $= \pm(i+\zeta)$,

$$x = \frac{-\mathrm{a}i \pm \mathrm{c}}{\epsilon}, \quad y = \frac{-\mathrm{a}'i \pm \mathrm{c}'}{\epsilon}, \text{ first pair of points;}$$

2°. $v = \pm(i+\zeta)$, u not $= \pm(i+\epsilon)$,

$$x = \frac{-\mathrm{b}i \pm \mathrm{d}}{\zeta}, \quad y = \frac{-\mathrm{b}'i \pm \mathrm{d}'}{\zeta}, \text{ second pair of points;}$$

3°. $u = \pm(i+\epsilon)$, $v = \pm(i+\zeta)$,

$$x = \frac{-\mathrm{a}i \pm \mathrm{c}}{\epsilon} + \frac{-\mathrm{b}i \pm \mathrm{d}}{\zeta}, \quad y = -\frac{\mathrm{a}'i \pm \mathrm{c}'}{\epsilon} + \frac{-\mathrm{b}'i \pm \mathrm{d}}{\zeta};$$

$a\epsilon + b\zeta = 0$, as above, third pair of points;

which six points are in general distinct from each other, and from the circular points at infinity.

The foregoing values of x, y may be said to be "circular *quoad* α," if $\mathrm{a} = \mathrm{c}'$, $\mathrm{a}' = -\mathrm{c}$; and similarly to be "circular *quoad* β," if $\mathrm{b} = \mathrm{d}'$, $\mathrm{b}' = -\mathrm{d}$.

And we see at once that if the values are circular *quoad* α, then the first pair of points coincide with the circular points at infinity; and that, in like manner, if the values are circular *quoad* β, then the second pair of points coincide with the circular points at infinity; but if the values are circular *quoad* α and β respectively, then each of the three pairs of points coincides with the circular points at infinity: so that these are then triple points on the curve; or the curve is tricircular, having besides the two triple points, 3 dps.

The relation between u, v gives

$$\{v(C+Du^2)+4abu\}^2 = 16a^2b^2 \,.\, u^2 - (A+Bu^2)(C+Du^2),$$

and it thus appears that if any one of the functions A, B, C, D is $=0$, the function under the radical sign is a mere linear function of u^2, say it is $L+Mu^2$; introducing a new parameter θ such that $u = \sqrt{\left(\frac{L}{M}\right)}\frac{2\theta}{1+\theta^2}$, we have $\sqrt{L+Mu^2} = \sqrt{L}\,\frac{1+\theta^2}{1-\theta^2}$, and consequently u, v are each of them a rational function of θ. Hence, when any one of the relations in question is satisfied, or say, when $a+d=b+c$, $b+d=c+a$, or $c+d=a+b$, the curve becomes unicursal: there is no diminution of the order, and the curve is consequently a unicursal sextic, or sextic with 10 dps.

It would at first sight appear that the curve might become unicursal in a different manner; viz. it would be unicursal if

$$16a^2b^2u^2 - (A + Bu^2)(C + Du^2)$$

was a perfect square; but this is only the case when one of the four sides a, b, c, d is $=0$. The condition in fact is

$$AD + BC - 16a^2b^2 = 2\sqrt{ABCD};$$

that is

$$\lambda\lambda'\mu\mu' - \nu\nu'\rho'\sigma - 16a^2b^2 = 2\sqrt{-\lambda\lambda'\mu\mu'\rho'\sigma};$$

where, putting for shortness $M = d^2 - a^2 - b^2 - c^2$, we have

$$\begin{aligned}
\lambda\lambda' &= M - 2bc + 2ca + 2ab,\\
\mu\mu' &= M + 2bc - 2ca + 2ab,\\
\nu\nu' &= M + 2bc + 2ca - 2ab,\\
-\rho'\sigma &= M - 2bc - 2ca - 2ab;
\end{aligned}$$

and thence

$$\begin{aligned}
\lambda\lambda'\mu\mu' &= M^2 - 4b^2c^2 - 4c^2a^2 + 4a^2b^2 + 4abM + 8c^2ab,\\
-\nu\nu'\rho'\sigma &= M^2 - 4b^2c^2 - 4c^2a^2 + 4a^2b^2 - 4abM - 8c^2ab,
\end{aligned}$$

and the equation thus becomes

$$M^2 - 4b^2c^2 - 4c^2a^2 - 4a^2b^2 = \sqrt{(M^2 - 4b^2c^2 - 4c^2a^2 + 4a^2b^2)^2 - 16a^2b^2(M + 2c^2)^2},$$

viz. putting for a moment $X = M^2 - 4c^2(a^2 + b^2)$, this is

$$(X - 4a^2b^2)^2 = (X + 4a^2b^2)^2 - 16a^2b^2(M + 2c^2)^2,$$

that is

$$16a^2b^2\{X - (M + 2c^2)^2\} = 0;$$

or, substituting for X its value, the equation is

$$64a^2b^2c^2(M + a^2 + b^2 + c^2) = 0,$$

that is $a^2b^2c^2d^2 = 0$.

We may have simultaneously 1°, $a = d$, $b = c$; 2°, $b = d$, $a = c$; 3°, $c = d$, $a = b$; the three cases are really equivalent, but the results present themselves in different forms.

1°. Here $A = 0$, $B = 4a(a - b)$, $C = 0$, $D = 4a(a + b)$; the relation between u, v contains the factor u, and throwing this out, and also the constant factor $4a$, it is

$$u[(a - b) + (a + b)v^2] + 2bv = 0,$$

viz. u is given as a rational function of v.

2°. Here $A=0$, $B=0$, $C=4b(b-a)$, $D=4b(b+a)$; the equation contains the factor v, and throwing out this and also the constant factor $4b$, the equation is

$$v\,[(b-a)+(b+a)\,u^2]+2au=0,$$

viz. v is given as a rational function of u.

3°. Here $A=4a(a-c)$, $B=0$, $C=0$, $D=4a(a+c)$; or, dividing by $4a$, the equation is

$$(a-c)+2a\,uv+(a+c)\,u^2v^2=0;$$

viz. this is

$$(uv+1)\,[(a+c)\,uv+a-c]=0,$$

which may be reduced to

$$(a+c)\,uv+a-c\;=0,$$

giving u or v each a rational function of the other.

I do not discuss the theory in detail, but only remark that in each case there is a conic thrown off, and that in place of the sextic we have a unicursal (or trinodal) quartic curve.

505.

ON THE SURFACES DIVISIBLE INTO SQUARES BY THEIR CURVES OF CURVATURE.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 120, 121. Read June 13, 1872.]

PROFESSOR CAYLEY gave an account of an investigation recently communicated by him to the Academy of Sciences at Paris. The fundamental theorem is that, if the coordinates x, y, z of a point on a surface are expressed as functions of two parameters p, q (such expressions, of course replacing the equation of the surface); and if these parameters are such that $p = \text{const.}$, $q = \text{const.}$ are the equations of the two sets of curves of curvature respectively; then (writing for shortness

$$\frac{dx}{dp} = x_1, \quad \frac{dx}{dq} = x_2, \quad \frac{d^2x}{dp^2} = x_3, \quad \frac{d^2x}{dp\,dq} = x_4, \quad \frac{d^2x}{dq^2} = x_5,$$

and the like for y, z), the coordinates x, y, z, considered always as functions of p, q, satisfy the equations

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_4, & y_4, & z_4 \end{vmatrix} = 0.$$

The last equation is equivalent to

$$x_4 + Ax_1 + Bx_2 = 0,$$

$$y_4 + Ay_1 + By_2 = 0,$$

$$z_4 + Az_1 + Bz_2 = 0;$$

and if in the notation of Gauss we write

$$x_1^2 + y_1^2 + z_1^2 = E,$$
$$x_2^2 + y_2^2 + z_2^2 = G,$$

then adding the equations multiplied by x_1, y_1, z_1 respectively, and also adding the equations multiplied by x_2, y_2, z_2 respectively, we find

$$A = -\tfrac{1}{2}\frac{1}{E}\frac{dE}{dq}, \quad B = -\tfrac{1}{2}\frac{1}{G}\frac{dG}{dq}$$

and the equations thus become

$$2x_4 - \frac{1}{E}\frac{dE}{dq}x_1 - \frac{1}{G}\frac{dG}{dq}x_2 = 0,$$
$$\&c. \qquad \&c. \qquad \&c.,$$

which, in fact, agree with the equations (10 bis) in Lamé's "Leçons sur les coordonnées curvilignes," Paris (1859), p. 89. The surface will be divisible into squares if only $E : G$ is the quotient of a function of p by a function of q, or say if

$$E = \Theta P, \quad G = \Theta Q,$$

where Θ is any function of (p, q), but P and Q are functions of p and q respectively; we then have

$$\frac{1}{E}\frac{dE}{dq} = \frac{1}{\Theta}\frac{d\Theta}{dq}, \quad \frac{1}{G}\frac{dG}{dp} = \frac{1}{\Theta}\frac{d\Theta}{dp},$$

and the equations for x, y, z are

$$2x_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}x_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}x_2 = 0,$$
$$\&c. \qquad \&c. \qquad \&c.,$$

viz. x, y, z being functions of p, q such that $x_1x_2 + y_1y_2 + z_1z_2 = 0$, and which besides satisfy these equations, or say which each of them satisfy the equation

$$2u_4 - \frac{1}{\Theta}\frac{d\Theta}{dq}u_1 - \frac{1}{\Theta}\frac{d\Theta}{dp}u_2 = 0,$$

then the values of x, y, z in terms of (p, q) determine a surface which has the property in question.

506.

ON THE MECHANICAL DESCRIPTION OF A CUBIC CURVE.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 175—178. Read November 14, 1872.]

If the coordinates x, y of a point on a curve are rational functions of $\sin\phi$, $\cos\phi$, $\sqrt{1-k^2\sin^2\phi}$, the curve has the deficiency 1, and conversely in any curve of deficiency 1 the coordinates x, y can be thus expressed in terms of the parameter ϕ. Hence writing $\sin\theta = k\sin\phi$, the coordinates will be rational functions of $\sin\phi$, $\cos\phi$, $\cos\theta$, or say of $\sin\phi$, $\cos\phi$, $\sin\theta$, $\cos\theta$; and for the mechanical representation of the relation $k\sin\phi = \sin\theta$, we require only a rod OA rotating about the fixed point O, and connected with it by a pin at A, a rod AB, the other extremity of which, B, moves in a fixed line Ox. The curve most readily obtained by such an arrangement is that described by a point C rigidly connected with the rod AB; this is however a quartic curve (with two dps., since its deficiency is $=1$). I first considered the cubic curve

$$xy - 1 = \sqrt{(1-x^2)(1-k^2x^2)},$$

or say

$$xy - 1 = -\sqrt{(1-x^2)(1-k^2x^2)};$$

writing herein $x = \sin\phi$, and as before $k\sin\phi = \sin\theta$, we have then $y\sin\phi = 1 - \cos\theta\cos\phi$; which values may be written

$$x = \sin\phi,$$

$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi} - \sin\theta.$$

I found, however, that this was *not* the cubic curve most easily constructed; and I ultimately devised a mechanical arrangement consisting of

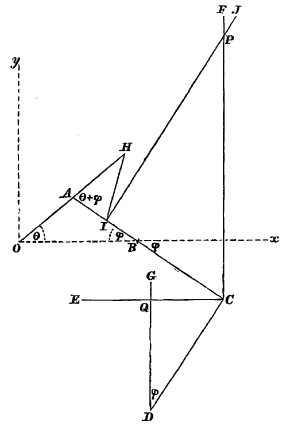
1. Rod OH, and connected with it by a pin at H, rod HI [1].

[1] There was a mechanical convenience in this, but observe that producing OH to meet IP in I', the single straight rod OHI' might have been made use of.

2 Square ACD, and connected with it by a pin at D, rod DG.

3. Square ECF; the two squares being connected by a pin at C.

4. Rod IJ.



The rod OH rotates about a pin at O; taking $HA = HI$, there is a pin at A connecting a fixed point of this rod with the extremity A of the square ACD: the fixed point B of this square moves along the line Ox. There is a pin at I connecting the extremities of the rods HI, IJ; and this slides along the leg AC of the square ACD, the rod IJ being always at right angles thereto: finally the legs of the square ECF are always parallel to Ox, Oy, and the rod DG at right angles to EC. I have omitted from the description the parallel-motion rods or other arrangements necessary for giving these fixed directions to the rod IJ, the square ECF, and the rod DG. It will be seen that the angles AOB, ABO are variable angles connected by an equation of the form above referred to; and that the lines IJ, CF determine by their intersection the point P; and the lines CE, DG determine by their intersection the point Q; the curve about to be considered is that determined by the relative motion of P in regard to Q; or say the curve the coordinates of a point of which are

$$x = QC, \quad y = CP.$$

I write

$$\angle AOB = \theta, \quad \angle ABO = \phi,$$

$$OA = a, \quad AB = b, \quad AC = c, \quad CD = d,$$

$$AH = HI = \tfrac{1}{2}h.$$

We then have $a \sin\theta = b \sin\phi$; and moreover the length AI being $= h\cos(\theta+\phi)$, and therefore $IC = c - h\cos(\theta+\phi)$, we have

$$y = CP = \frac{c - h\cos(\theta+\phi)}{\sin\phi},$$

$$x = QC = d\sin\phi;$$

whence also

$$xy = d\{c - h\cos(\theta+\phi)\};$$

or we have

$$xy = d\left\{c - h\sqrt{\left(1 - \frac{x^2}{d^2}\right)}\sqrt{\left(1 - \frac{b^2}{a^2}\frac{x^2}{d^2}\right)} + h\frac{b}{a}\frac{x^2}{d^2}\right\},$$

that is

$$x\left(y - \frac{bh}{ad}x\right) - cd = -dh\sqrt{\left(1 - \frac{x^2}{d^2}\right)}\sqrt{\left(1 - \frac{b^2}{a^2}\frac{x^2}{d^2}\right)}:$$

or rationalising and reducing, this is

$$x^2y^2 - \frac{2bh}{ad}x^2y - 2cdxy + \left\{2\frac{b}{a}ch + h^2\left(1 + \frac{b^2}{a^2}\right)\right\}x^2 + d^2(c^2 - h^2) = 0,$$

a quartic curve with two dps.

In the particular case $a = b$, the relation between θ, ϕ is simply $\theta = \phi$; the curve should become unicursal.

Writing in the equation $\frac{b}{a} = 1$, the equation takes the form

$$\left\{x\left(y - \frac{2h}{d}x\right) - d(c-h)\right\}\{xy - d(c+h)\} = 0;$$

the second factor is extraneous, and the curve is the hyperbola

$$x\left(y - \frac{2h}{d}x\right) - d(c-h) = 0,$$

as at once appears from the foregoing irrational form of the equation.

In the particular case $h = c$, the equation contains the factor x, and omitting this it becomes

$$x\left(y^2 - \frac{2bc}{ad}x^2\right) - 2cdy + c\left(1 + \frac{b}{a}\right)^2 x = 0;$$

viz. we have here a cubic curve with three real asymptotes meeting in a point which is also the centre of the curve.

If simultaneously $a = b$ and $h = c$, then the equation is

$$x\left(y - \frac{2c}{d}x\right)(xy - 2cd) = 0,$$

the actual locus being in this case the line $y - \frac{2c}{d}x = 0$.

Writing $h=c$, and for greater convenience $h=c=d=1$; also to fix the ideas supposing $b<a$, and writing $\frac{b}{a}=k$, $=\sin\lambda$, then we have

$$\sin\theta = k\sin\phi,$$
$$x = \sin\phi,$$
$$y = \frac{1-\cos(\theta+\phi)}{\sin\phi};$$

that is

$$xy = 1-\sqrt{1-x^2}\sqrt{1-k^2x^2}+kx^2,$$

giving the rationalised equation

$$x(y^2-2kx^2)-2y+4x=0;$$

the angle ϕ may be anything whatever, but θ varies between the limits $\pm\lambda$, the simultaneous values of these angles and of the coordinates being

$\phi = 0$	$\theta=0$	$x=0$	$y=0$
$\phi = 90°$	$\theta=\lambda$	$x=1$	$y=1+\sin\lambda$
$\phi = 180°$	$\theta=0$	$x=0$	$y=\pm\infty$
$\phi = 270°$	$\theta=-\lambda$	$x=-1$	$y=-(1+\sin\lambda)$
$\phi = 360°$	$\theta=0$	$x=0$	$y=0$;

and it thus appears that the mechanism gives the continuous branch which belongs to the asymptote $x=0$ of the cubic curve; the other two branches belong to $x=\sin\phi$, $y=\frac{1+\cos(\theta+\phi)}{\sin\phi}$, which would require a slight alteration in the arrangement of the mechanism.

I remark that if AH, HI had been unequal, then writing $\angle HIA=\chi$, this would be connected with $\theta+\phi$ by an equation of the form

$$\sin(\theta+\phi)=m\sin\chi,$$

and the coordinates x, y would be rational functions of the sines and cosines of θ, ϕ, χ; the deficiency is in this case >1.

507.

ON THE MECHANICAL DESCRIPTION OF CERTAIN QUARTIC CURVES BY A MODIFIED OVAL CHUCK.

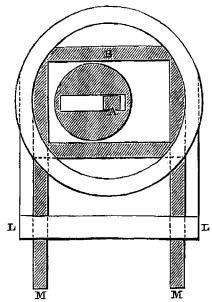
[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 186—190. Read December 12, 1872.]

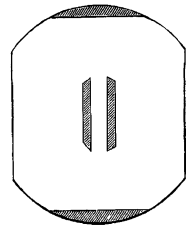
THE geometrical principle of the oval chuck is the well-known one that if a plane move in such manner that two lines Ox, Oy, fixed in the plane and moveable with it, pass through two fixed points A, B, respectively, then any fixed point P traces out on the plane an ellipse. The point A is on the (geometrical) axis of the mandril; there is connected with the head a guide-ring moving horizontally; the point B is the centre of the guide-ring, this being a ring connected with the head, moveable horizontally at right angles to the axis in such wise that the distance AB of the two centres is adjustable to any given value; the fixed point P is the tool, which practically is held on the level of the axis, that is, at a point in the line AB. The guide-ring remains fixed during the motion of the lathe.

It occurred to me that a chuck applicable to ornamental turning might be constructed by giving to the guide-ring a reciprocating motion synchronous with the rotation of the mandril; viz. for this purpose it is only necessary to affix to the axis of the mandril an eccentric, working in a frame attached to the guide-ring so as to move the centre B of the guide-ring backwards and forwards along the line AB: the curve is thus that described by the fixed point P upon a plane moving in such manner that the lines Ox, Oy pass always through the points A, B respectively; the former of these being a fixed point, the latter of them a point moving according to determinate law backwards and forwards along a fixed line through A.

The plan is carried out in a drawing apparatus which I have had constructed in wood, the axis being here vertical instead of horizontal, and the details of course different from what they would be for a lathe.

The apparatus consists of a piece of inch-board, about 10 inches long by 7 inches broad, pierced with a circular hole of 1 inch diameter for a vertical axis: the edges of the board serve as guides for the frame L, which carries the guide-ring, and resting on the board we have the frame M, itself guided by the frame L: the two frames move independently of each other, but they can be clamped together; the axis has



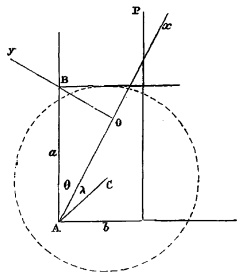


upon it a square nut, the sides of which work in the slot of an eccentric, the throw of this being adjustable by means of a screw passing through the nut and axis, and there is above the eccentric a square nut shown in the figure. This is capable of rotation round the axis, so that two of its sides may be placed either parallel with or inclined to the sides of the slot; but I fix it with two sides parallel to those of the slot by means of a screw run into the axis. The upper surfaces of the last-mentioned nut and of the guide-ring are flush with each other; and we then have a table or bed having, on its under-surface, guides which work on the outer edges of the guide-ring and on two edges of the nut. It will be observed that the bed may be placed in two different positions, viz. the guides may work on either pair of edges of the nut, those which are parallel to the sides of the slot, or those which are at right angles to it.

Supposing the bed placed as above upon the guide-ring and nut, then if the frames L and M are disconnected, and the former of them is fixed, the frame M will, on rotation of the axis, be carried backwards and forwards by the eccentric, but this will in no wise affect the motion of the bed; the arrangement is then equivalent to the oval chuck, and a pencil fixed above the bed in any given position will trace out upon it an ellipse. If, however, the frame L, instead of being fixed, is clamped to the frame M, then the two frames, and therefore the guide-ring, are carried backwards and forwards by the eccentric, and the curve traced out by the pencil is no longer an ellipse; it is, as I proceed to show, a special form of trinodal quartic; viz. there is a tacnode (= two nodes) at infinity, and a third node, which may be a crunode, cusp, or acnode. In the last-mentioned case, the acnode or conjugate point is, as usual, not exhibited by the mechanical description, and the curve has no visible singularity.

Let the coordinates of the fixed point P, referred to axes through A, the first of them perpendicular to, and the second coincident with, AB, be b, c; let the distance AB be $=a$; and let θ denote the angle BAO: then, if x, y are the coordinates of P referred to the origin O and axes Ox, Oy, we have

$$\begin{aligned} x + a\cos\theta &= \quad b\sin\theta + c\cos\theta, \\ y \qquad\quad &= -b\cos\theta + c\sin\theta, \end{aligned}$$



which, if a be constant, gives a quadric equation, or the curve is an ellipse; and, in particular, if $b=0$, that is if the point P is on the line AB, then we have

$$x = (c-a)\cos\theta, \quad y = c\sin\theta,$$

or the curve is

$$\frac{x^2}{(c-a)^2} + \frac{y^2}{c^2} = 1.$$

But if a is a given function of θ, then the equation is still found by eliminating θ between the two equations for x and y. In particular, if the distance AB is given as the perpendicular upon the tangent of a circle, as shown in the figure, then if k be the radius AC of this circle, and λ the inclination of AC to Ax (k and λ being taken to be constants), we have

$$a = k\cos(\theta+\lambda),$$

and the equations are

$$\begin{aligned} x &= \quad b\sin\theta + \{c - k\cos(\theta+\lambda)\}\cos\theta, \\ y &= -b\cos\theta + c\sin\theta. \end{aligned}$$

The elimination is nearly the same as if b were $=0$; viz. we may determine γ, α in such wise that

$$\begin{aligned} b\sin\theta + c\cos\theta &= \gamma\cos(\theta+\alpha), \quad &= \gamma\cos\phi \quad \text{suppose}, \\ -b\cos\theta + c\sin\theta &= \gamma\sin(\theta+\alpha), \quad &= \gamma\sin\phi; \end{aligned}$$

and then

$$x = \gamma \cos \phi - k \cos (\phi + \lambda - \alpha) \cos (\phi - \alpha),$$
$$y = \gamma \sin \phi.$$

I will, for greater simplicity, at once write $b = 0$: the equations thus are

$$x = (c - k \cos \theta \cos \lambda + k \sin \theta \sin \lambda) \cos \theta,$$
$$y = c \sin \theta;$$

or say the first is

$$x = (c + k \sin \theta \sin \lambda) \cos \theta - k \cos^2 \theta \cos \lambda;$$

whence we have

$$x + \frac{k \cos \lambda}{c^2} (c^2 - y^2) = \left(c + \frac{ky}{c} \sin \lambda\right) \frac{1}{c} \sqrt{c^2 - y^2},$$

that is

$$c^2 x + k \cos \lambda (c^2 - y^2) = (c^2 + ky \sin \lambda) \sqrt{c^2 - y^2},$$

an equation which will assume a more simple form if either $\lambda = 0$ or $\lambda = 90°$; that is, if in the apparatus the nut-sides which guide the bed are either at right angles, or parallel, to the sides of the slot.

Taking the general case, and writing for convenience $\cos \theta = \xi$, $\sin \theta = \eta$, the curve is given by equations of the form

$$x = (\xi, \eta, 1)^2,$$
$$y = (\xi, \eta, 1)^2,$$
$$\xi^2 + \eta^2 = 1;$$

viz. the elimination of ξ, η from these equations leads to the equation of the curve. The points of the curve have thus a (1, 1) correspondence with those of the circle $\xi^2 + \eta^2 = 1$; or, the circle being unicursal, the curve is also unicursal. Moreover, considering the intersections of the curve with an arbitrary line $ax + by + c = 0$, the points of intersection correspond to the points of intersection of the circle by the quadric $a(\xi, \eta, 1)^2 + b(\xi, \eta, 1)^2 + c = 0$; viz. there are four points of intersection, or the curve is a quartic, and hence it is a binodal quartic. But it is a binodal quartic of a special form: to show this more clearly, I introduce for homogeneity the coordinates z, ζ, so that the foregoing equations become

$$x : y : z = (\xi, \eta, \zeta)^2 : (\xi, \eta, \zeta)^2 : \zeta^2, \text{ where } \xi^2 + \eta^2 - \zeta^2 = 0;$$

the curve corresponding to these equations is, as just seen, a binodal quartic. But in the case in hand the form is the more special one,

$$x : y : z = (\xi, \eta, \zeta)^2 : \xi\zeta \quad : \zeta^2, \text{ where } \xi^2 + \eta^2 - \zeta^2 = 0.$$

The intersections by the arbitrary line $ax + by + cz = 0$ are the points corresponding to the intersections of the circle $\xi^2 + \eta^2 - \zeta^2 = 0$ by the quadric $a(\xi, \eta, \zeta)^2 + b\xi\zeta + c\zeta^2 = 0$,

giving four intersections. But the intersections by the line $by+cz=0$ (that is, by any line through the point $y=0$, $z=0$) are obtained from the equation $b\xi\zeta+c\zeta^2=0$; viz. this breaks up into $b\xi+c\zeta=0$, $\zeta=0$, and the last factor combined with the equation of the circle gives $\zeta=0$, $\xi^2+\eta^2=0$, the two circular points at infinity, corresponding each to the point $y=0$, $z=0$: the other factor gives points corresponding to two variable points on the curve; that is, a line through the point $y=0$, $z=0$ meets the curve in this point twice and in two other points. Again, making $b=0$, or taking the line to be the line at infinity $z=0$, the equations then are $\zeta^2=0$, $\xi^2+\eta^2=0$; viz. we then have the circular points at infinity each twice, corresponding to the point $y=0$, $z=0$ four times, and no other point; that is, the line $z=0$ meets the curve in the point $y=0$, $z=0$ four times. We thus see that the curve has at $y=0$, $z=0$, that is at infinity on the line $y=0$, a tacnode (counting as two nodes), the tangent at this point being the line at infinity $z=0$. The curve being trinodal has of course one other node.

508.

ON GEODESIC LINES, IN PARTICULAR THOSE OF A QUADRIC SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 191—211. Read December 12, 1872.]

THE present Memoir contains an investigation of the differential equation (of the second order) of the geodesic lines on a surface, the coordinates of a point on the surface being regarded as given functions of two parameters p, q, and researches in connection therewith; a deduction of Jacobi's differential equation of the first order in the case of a quadric surface, the parameters p, q being those which determine the two sets of curves of curvature; formulæ where the parameters are those which determine the two right lines through the surface; and a discussion of the forms of the geodesic lines in the two cases of an ellipsoid and a skew hyperboloid respectively.

Preliminary Formulæ.

1. I call to mind the fundamental formulæ in the Memoir by Gauss, "Disquisitiones generales circa superficies curvas," *Comm. Gott. recent.* t. VI., 1827, (reprinted as an Appendix in Liouville's edition of Monge,) together with some that I have added to them. The coordinates x, y, z of a point on a surface are regarded as given functions of two parameters p, q, these expressions of x, y, z in effect determining the equation of the surface, and we have

$$dx + \tfrac{1}{2}d^2x = a\,dp + a'dq + \tfrac{1}{2}(\alpha\,dp^2 + 2\alpha'\,dp\,dq + \alpha''\,dq^2),$$
$$dy + \tfrac{1}{2}d^2y = b\,dp + b'dq + \tfrac{1}{2}(\beta\,dp^2 + 2\beta'dp\,dq + \beta''dq^2),$$
$$dz + \tfrac{1}{2}d^2z = c\,dp + c'dq + \tfrac{1}{2}(\gamma\,dp^2 + 2\gamma'dp\,dq + \gamma''dq^2),$$
$$A,\ B,\ C = bc' - b'c,\ ca' - c'a,\ ab' - a'b;$$

whence differential equation of surface is

$$Adx + Bdy + Cdz = 0.$$

Also

$$E,\ F,\ G = a^2 + b^2 + c^2,\quad aa' + bb' + cc',\quad a'^2 + b'^2 + c'^2;$$

so that element of length on the surface is given by

$$dx^2 + dy^2 + dz^2 = Edp^2 + 2Fdp\,dq + Gdq^2;$$

or, as I write it,

$$= (E,\ F,\ G \between dp,\ dq)^2;$$

and moreover

$$V^2 = A^2 + B^2 + C^2 = EG - F^2.$$

The equation $(E,\ F,\ G \between dp,\ dq)^2 = 0$ determines at each point on the surface two directions (necessarily imaginary) which are called the "circular" directions. Passing on the surface from point to point along the circular directions, we obtain two series of curves (always imaginary) which are the "circular" curves; the equation $(E,\ F,\ G \between dp,\ dq)^2 = 0$ is the differential equation of these curves; and if we have $E = 0$, $G = 0$, then this becomes $dp\,dq = 0$; viz. we have in this case $p = \text{const.}$ and $q = \text{const.}$ as the equations of the two sets of circular curves respectively. It is clear *à priori*, and will be shown analytically in the sequel, that the circular curves are geodesic lines.

I write also

$$E',\ F',\ G' = A\alpha + B\beta + C\gamma,\quad A\alpha' + B\beta' + C\gamma',\quad A\alpha'' + B\beta'' + C\gamma'',$$

or, what is the same thing, E', F', G' represent the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ respectively,}$$

(these last symbols do not occur in Gauss). [They are the D, D', D'' of Gauss.]

2. The radius of curvature of normal section corresponding to direction $dp : dq$ is given by

$$\frac{\rho}{V} = \frac{(E,\ F,\ G \between dp,\ dq)^2}{(E',\ F',\ G' \between dp,\ dq)^2};$$

whence it appears that the directions of the inflexional or chief tangents (the *Haupt-tangenten*) are determined by the equation

$$(E',\ F',\ G' \between dp,\ dq)^2 = 0.$$

The directions in question are imaginary on a surface such as the ellipsoid where the curvatures are in the same direction, but on a concavo-convex surface they are real; and in particular on the hyperboloid they coincide with the directions of the generating lines. We may on any surface pass from point to point along the chief directions; we have thus on the surface two sets of curves which are the chief curves;

the differential equation of these is $(E', F', G' \between dp, dq)^2 = 0$; and in particular if $E' = 0$, $G' = 0$, then this becomes $dp\,dq = 0$, or we have $p = \text{const.}$, $q = \text{const.}$ for the equations of the two sets of chief curves respectively. On the hyperboloid the chief curves are the two sets of generating lines. The chief curves are not in general geodesic lines, but on the hyperboloid, *quà* straight lines, they are, it is clear, geodesic lines.

3. The directions of the curves of curvature, or the principal tangents, and the corresponding values of the radius of curvature are determined by

$$\frac{\rho}{V} = \frac{Edp + Fdq}{E'dp + F'dq} = \frac{Fdp + Gdq}{F'dp + G'dq};$$

or, what is the same thing, these directions are determined by the equation

$$\begin{vmatrix} dq^2, & -dq\,dp, & dp^2 \\ E\,, & F\,, & G \\ E', & F', & G' \end{vmatrix} = 0.$$

The same equations may be written

$$-dq : dp = \frac{\rho E' - VE}{\rho F' - VF} = \frac{\rho F' - VF}{\rho G' - VG};$$

that is the principal radii of curvature are determined by the equation

$$\rho^2(E'G' - F'^2) - \rho V(EG' + E'G - 2FF') + V^2(EG - F^2) = 0,$$

(last term is $= V^4$, but it is better to retain the original form): and then, ρ being either root, the last preceding equations give the direction of the curve of curvature corresponding to the given value of the radius of curvature.

If $p = \text{const.}$, $q = \text{const.}$ are the equations of the two systems of curves of curvature respectively, then the quadric equation in (dp, dq) must become $dp\,dq = 0$; this will be so if $F = 0$, $F' = 0$; and we thus have these equations, viz. written at full length they are

$$d_p x\, d_q x + d_p y\, d_q y + d_p z\, d_q z = 0,$$

$$\begin{vmatrix} d_p x\,, & d_p y\,, & d_p z \\ d_q x\,, & d_q y\,, & d_q z \\ d_p d_q x, & d_p d_q y, & d_p d_q z \end{vmatrix} = 0,$$

as the conditions in order that $p = \text{const.}$, $q = \text{const.}$ may be the two systems of curves of curvature. The former of these equations merely expresses that the two sets of curves always intersect at right angles.

General Theory of the Geodesic Lines on a Surface.

4. I now proceed to investigate the theory of geodesic lines on a surface, the surface being determined as above by means of given expressions of the coordinates x, y, z in terms of the parameters p, q.

The differential equation obtained by Gauss for the geodesic lines is in a form not symmetrical in regard to the two variables; viz. his equation is

$$\frac{dE}{dq}dp^2 + 2\frac{dF}{dp}dp\,dq + \frac{dG}{dp}dp^2 = 2ds\,d\frac{Edp + Fdq}{ds},$$

where, as above,

$$ds^2 = (E,\ F,\ G \between dp,\ dq)^2.$$

If we herein consider p, q as functions of a parameter θ, and write for shortness

$$d_\theta p,\ d_\theta q,\ d_\theta^2 p,\ \&c. = p',\ q',\ p'',\ \&c.,$$

also

$$\Omega = (E,\ F,\ G \between p',\ q')^2,$$

and

$$d_p E = E_1,\quad d_q E = E_2,\ \&c.,$$

then the equation is

$$(E_1,\ F_1,\ G_1 \between p',\ q')^2 - 2\sqrt{\Omega}\left(\frac{Ep' + Fq'}{\sqrt{\Omega}}\right)' = 0.$$

We have

$$\left(\frac{Ep' + Fq'}{\sqrt{\Omega}}\right)' = \frac{1}{\Omega\sqrt{\Omega}}(M + N),$$

where N is the part containing p'', q'', which I will first calculate; viz. we have

$$\begin{aligned} N &= \Omega\,(Ep'' + Fq'') - (Ep' + Fq')\,\tfrac{1}{2}\Omega', \\ &= \Omega\,(Ep'' + Fq'') - (Ep' + Fq')\,\{(Ep' + Fq')\,p'' + (Fp' + Gq')\,q''\}, \\ &= p''\,\{E\Omega - (Ep' + Fq')^2\} + q''\,\{F\Omega - (Ep' + Fq')\,(Fp' + Gq')\}; \end{aligned}$$

or substituting for Ω its value, this is

$$= p''\,(EG - F^2)\,q'^2 - q''\,(EG - F^2)\,p'q',\quad = -q'\,(EG - F^2)(p'q'' - p''q');$$

wherefore

$$\left(\frac{Ep' + Fq'}{\sqrt{\Omega}}\right)' = \frac{1}{\Omega\sqrt{\Omega}}\{M - q'\,(EG - F^2)\,(p'q'' - p''q')\},$$

and the equation becomes

$$\Omega\,(E_1 p'^2 + 2F_1 p'q' + G_1 q'^2) - 2M + 2q'\,(EG - F^2)\,(p'q'' - p''q') = 0;$$

whence we foresee that the whole equation must divide by q'.

5. We have

$$\begin{aligned} M = & \ (p'dE + q'dF)\,\Omega - (Ep' + Fq')\,(\tfrac{1}{2}p'^2dE + p'q'dF + \tfrac{1}{2}q'^2dG), \\ = & \ dE\,\{p'\Omega - \tfrac{1}{2}p'^2\,(Ep' + Fq')\} \\ & + dF\,\{q'\Omega - p'q'\,(Ep' + Fq')\} \\ & + dG\,\{\qquad - \tfrac{1}{2}q'^2\,(Ep' + Fq')\}, \end{aligned}$$

or say

$$\begin{aligned} 2M = & \ dE\,[\ \ p'^2\,(p'E + q'F) + 2p'q'\,(p'F + q'G)] \\ & + dF\,[\qquad\qquad\qquad\quad 2q'^2\,(p'F + q'G)] \\ & + dG\,[-q'^2\,(p'E + q'F)\qquad\qquad\qquad\quad], \\ = & \ (E_1p' + E_2q')\,[\ \ p'^2\,(Ep' + Fq') + 2p'q'\ (Fp' + Gq')] \\ & + (F_1p' + F_2q')\,[\qquad\qquad\qquad\quad 2q'^2\,(Fp' + Gq')] \\ & + (G_1p' + G_2q')\,[-q'^2\,(Ep' + Fq')\qquad\qquad\qquad\quad]. \end{aligned}$$

The term in p'^4 is $EE_1p'^4$, which is also the term in p'^4 of $\Omega\,(E_1p'^2 + 2F_1p'q' + G_1q'^2)$; whence $\Omega\,(E_1p'^2 + 2F_1p'q' + G_1q'^2) - 2M$ divides by q'.

Proceeding to the reduction:

Term in E_1 is

$$E_1 . p'^2\,\Omega - p'^2\,(Ep' + Fq') - 2p'^2q'\,(Fp' + Gq'),\ = E_1 . - p'^2q'\,(Fp' + Gq');$$

term in F_1 is

$$F_1 . 2p'q'\Omega - 2p'q'^2\,(Fp' + Gq'),\ = F_1 . 2p'^2q'\,(Ep' + Fq');$$

term in G_1 is

$$G_1 . q'^2\Omega + p'q'^2\,(Ep' + Fq'),\ = G_1\,\{2p'q'^2\,(Ep' + Fq') + q'^3\,(Fp' + Gq')\}.$$

6. The remaining terms in E_2, F_2, G_2 require no reduction, and the result is

$$\begin{aligned} & E_1\,\{-p'^2\,(Fp' + Gq')\} - E_2\,\{\ p'^2\,(Ep' + Fq') + 2p'q'\,(Fp' + Gq')\} \\ & + F_1\,\{2\ p'^2\,(Ep' + Fq')\} - F_2\,\{2q'^2\,(Fp' + Gq')\} \\ & + G_1\,\{2p'q'\,(Ep' + Fq') + q'^2\,(Fp' + Gq')\}\ - G_2\,\{-q'^2\,(Ep' + Fq')\} \\ & \quad + 2\,(EG - F^2)\,(p'q'' - p''q') = 0, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} & (Ep' + Fq')\,\{(2F_1 - E_2)\,p'^2 + 2G_1p'q' + G_2q'^2\} \\ & - (Fp' + Gq')\,\{E_1p'^2 + 2E_2p'q' + (2F_2 - G_1)\,q'^2\} \\ & \quad + 2\,(EG - F^2)\,(p'q'' - p''q') = 0, \end{aligned}$$

which is the required differential equation of the second order: the independent variable has been taken to be the arbitrary quantity θ; but taking $\theta = p$, or q, say $\theta = p$, we have $p' = 1$, $p'' = 0$, and the equation then contains (besides p and q) only q'' and q', that is, d_p^2q and d_pq, and is therefore a differential relation between p and q.

7. Instead of starting, as above, from the equation given by Gauss, we may use the geometrical property that at each point of a geodesic line the osculating plane passes through the normal of the surface.

Considering, as above, p, q as functions of a variable θ, then, θ becoming $\theta + d\theta$, the new values of p, q are

$$p + p'd\theta + \tfrac{1}{2}p''d\theta^2, \quad q + q'd\theta + \tfrac{1}{2}q''d\theta^2;$$

and that of x is

$$x + a(p'd\theta + \tfrac{1}{2}p''d\theta^2) + a'(q'd\theta + \tfrac{1}{2}q''d\theta^2) + \tfrac{1}{2}(\alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2)\,d\theta^2;$$

or calling this $x + x'd\theta + \tfrac{1}{2}x''d\theta^2$, we have

$$x' = ap' + a'q', \quad x'' = ap'' + a'q'' + \alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2,$$

and so

$$y' = bp' + b'q', \quad y'' = bp'' + b'q'' + \beta p'^2 + 2\beta' p'q' + \beta'' q'^2,$$

$$z' = cp' + c'q', \quad z'' = cp'' + c'q'' + \gamma p'^2 + 2\gamma' p'q' + \gamma'' q'^2.$$

The condition in question is expressed by the equation

$$\begin{vmatrix} A, & B, & C \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} A, & B, & C \\ ap' + a'q', & bp' + b'q', & cp' + c'q' \\ ap'' + a'q'', & bp'' + b'q'', & cp'' + c'q'' \end{vmatrix}$$

$$+ \begin{vmatrix} A, & B, & C \\ ap' + a'q', & bp' + b'q', & cp' + c'q' \\ \alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2, & \beta p'^2 + 2\beta' p'q' + \beta'' q'^2, & \gamma p'^2 + \gamma' p'q' + \gamma'' q'^2 \end{vmatrix} = 0.$$

8. The first determinant is the sum of three terms such as $A(bc' - b'c)(p'q'' - p''q')$; viz. this is $A^2(p'q'' - p''q')$, or the determinant is

$$(A^2 + B^2 + C^2)(p'q'' - p''q'), = (EG - F^2)(p'q'' - p''q').$$

The second determinant is the sum of three terms such as

$$(\alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2)[B(cp' + c'q') - C(bp' + b'q')],$$

where the factor in [ ] is

$$p'[c(ca' - c'a) - b(ab' - a'b)] + q'[c'(ca' - c'a) - b'(ab' - a'b)],$$

which is $=p'(a'E-aF)+q'(a'F-aG)$. We have thus the second determinant, and the equation becomes

$$\begin{aligned}(EG-F^2)(p'q''-p''q')\\ &+(\alpha p'^2+2\alpha' p'q'+\alpha'' q'^2)\{p'(a'E-aF)+q'(a'F-aG)\}\\ &+(\beta p'^2+2\beta' p'q'+\beta'' q'^2)\{p'(b'E-bF)+q'(b'F-bG)\}\\ &+(\gamma p'^2+2\gamma' p'q'+\gamma'' q'^2)\{p'(c'E-cF)+q'(c'F-cG)\}=0,\end{aligned}$$

an equation of the same form as that previously obtained, and which can be without difficulty identified therewith.

The Circular Curves are Geodesics.

9. I proceed to show that the circular curves are geodesics; viz. that an integral of the geodesic equation is

$$(E,\ F,\ G \between p',\ q')^2=0.$$

Starting from this equation, we have

$$2\{(Ep'+Fq')p''+(Fp'+Gq')q''\}+(E_1p'+E_2q')p'^2+2(F_1p'+F_2q')p'q'+(G_1p'+G_2q')q'^2=0.$$

Now the equation, writing therein $Ep'+Fq'=\lambda q'$, gives $Fp'+Gq'=-\lambda p'$: these equations may be written

$$\begin{aligned}Ep'+(F-\lambda)q'&=0,\\ (F+\lambda)p'+\ \ G\qquad q'&=0,\end{aligned}$$

the value of λ being therefore $-\lambda^2=EG-F^2$. The result just obtained thus becomes

$$\begin{aligned}2\lambda(p''q'-p'q'')\\ &+[(E_1p'+E_2q')p'+(F_1p'+F_2q')q']\,.\,-\frac{1}{\lambda}(Fp'+Gq')\\ &+[(F_1p'+F_2q')p'+(G_1p'+G_2q')q']\,.\ \ \frac{1}{\lambda}(Ep'+Fq'),\end{aligned}$$

that is

$$\begin{aligned}2(EG-F^2)(p'q''-p''q')\\ &-(Fp'+Gq')[E_1p'^2+(E_2+F_1)p'q'+F_2q'^2]\\ &+(Ep'+Fq')[F_1p'^2+(F_2+G_1)p'q'+G_2q'^2]=0;\end{aligned}$$

or what is the same thing, adding hereto the zero value

$$\Delta(E,\ F,\ G \between p',\ q')^2,\ =(Fp'+Gq')\Delta q'+(Ep'+Fq')\Delta p',$$

where Δ is arbitrary, the equation is

$$\begin{aligned}2(EG-F^2)(p'q''-p''q')\\ &+(Ep'+Fq')[F_1p'^2+(F_2+G_1)p'q'+G_2q'^2+\Delta p']\\ &-(Fp'+Gq')[E_1p'^2+(E_2+F_1)p'q'+F_2q'^2-\Delta q']=0;\end{aligned}$$

viz. taking $\Delta=(F_1-E_2)p'+(G_1-F_2)q'$, this agrees with the geodesic equation.

The foregoing integral, $(E,\ F,\ G \between p',\ q')^2=0$, is, I believe, a particular, not a singular, solution of the geodesic equation.

The Chief Lines are not in general Geodesics.

10. That the chief lines are not in general geodesics appears most readily as follows:

To find the condition in order that $p=\text{const.}$ may be a geodesic, we write in the geodesic equation $p'=0$, $p''=0$: the equation thus becomes

$$Fq' \,.\, G_2 q'^2 - Gq' \,(2F_2 - G_1)\, q'^2 = 0\,;$$

that is we have

$$FG_2 - 2F_2 G + GG_1 = 0$$

as the condition in order that $p=\text{const.}$ may be a geodesic: the condition that it may be a chief curve is $G'=0$, which is a different condition.

We have of course, in like manner,

$$2EF_1 - E_1 F - EE_2 = 0$$

as the condition in order that $q=\text{const.}$ may be a geodesic; and $E'=0$ as the condition that this may be a chief curve. If $p=\text{const.}$, $q=\text{const.}$ are each of them at once a geodesic and a chief curve, then the four equations must all be satisfied, viz. we must have

$$\begin{aligned} FG_2 - 2F_2 G + GG_1 = 0, \quad & 2EF_1 - E_1 F - EE_2 = 0, \\ G' = 0, \quad & E' = 0. \end{aligned}$$

Special Form of the Geodesic Equation.

11. In the case where the curves $p=\text{const.}$, $q=\text{const.}$ intersect at right angles (and in particular when these are the curves of curvature), we have $F=0$; whence also $F_1=0$, $F_2=0$; and the geodesic equation assumes the more simple form,

$$\begin{aligned} & Ep' \;\; (-E_2 p'^2 + 2G_1 p'q' + G_2 q'^2) \\ & - Gq' \;\; (\;\; E_1 p'^2 + 2E_2 p'q' - G_1 q'^2) \\ & + 2EG\,(p'q'' - p''q') = 0. \end{aligned}$$

[11[a]. In the case of a surface of revolution we have

$$x = p\cos q, \quad y = p\sin q, \quad z = p\,;$$

E is of the form $1+P'^2$, $P' = d_p P$, where P is a function of p only, and we have

$$ds^2 = (1+P'^2)\,dp^2 + p^2 dq^2,$$

that is

$$\begin{aligned} & E = 1 + P'^2, \quad E_1 = 2P'P'', \quad E_2 = 0, \\ & F = 0, \\ & G = p^2 \quad\;\; , \quad G_1 = 2p \quad\;\; , \quad G_2 = 0\,; \end{aligned}$$

hence the differential equation is

$$(1+P'^2)\{p^2(p'q''-p''q')+2pp'^2q'\}-p^2p'^2q'P'P''+p^2q'^3=0;$$

this has an integral

$$\frac{(1+P'^2)p'^2}{p^4q'^2}+\frac{1}{p^2}=\frac{1}{C^2},$$

or say

$$C^2s'^2=p^4q'^2$$

where

$$s'^2=(1+P'^2)p'^2+p^2q'^2.$$

Writing here ρ, ψ for p, q, where ρ is the distance of the point from the axis, and ψ is the longitude reckoned from an arbitrary meridian, then the equation is

$$Cds=\rho^2d\psi,$$

which is the equation given by Legendre, *Théorie des fonctions elliptiques*, t. I. p. 361. This may also be written

$$\frac{C}{\rho}=\cos\gamma$$

if γ be the inclination of the geodesic line to the parallel of latitude.]

Geodesics on a Quadric Surface.

12. In the case of a quadric surface $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$, writing for shortness α, β, $\gamma=b-c$, $c-a$, $a-b$, we may express the coordinates x, y, z in terms of two parameters p, q as follows:

$$-\beta\gamma\, x^2=a\,(a+p)\,(a+q),$$
$$-\gamma\alpha\, y^2=b\,(b+p)\,(b+q),$$
$$-\alpha\beta\, z^2=c\,(c+p)\,(c+q),$$

where, in fact, $p=$ const., $q=$ const. are the equations of the two sets of curves of curvature respectively. Writing moreover

$$P=\frac{p}{(a+p)(b+p)(c+p)},\quad Q=\frac{q}{(a+q)(b+q)(c+q)},$$

we have

$$ds^2=\tfrac{1}{4}(p-q)(Pdp^2-Qdq^2),$$

that is

$$E,\ F,\ G=\tfrac{1}{4}(p-q)P,\ 0,\ \tfrac{1}{4}(q-p)Q;$$

and the geodesic equation becomes

$$\begin{aligned}&Pp'\{Pp'^2-2Qp'q'+(Q+\overline{q-p}\,Q')q'^2\}\\+\;&Qq'\{(P+\overline{p-q}\,P')p'^2-2Pp'q'+Qq'^2\}\\-\;&2(p-q)PQ(p'q''-p''q')=0,\end{aligned}$$

where P', Q' stand for d_pP and d_qQ respectively; viz. this is

$$\begin{aligned} & p'^3 \;.\; P^2 \\ & + p'^2q' \,.\, -PQ + (p-q)\,P'Q \\ & + p'q'^2 \,.\, -PQ + (q-p)\,PQ' \\ & + \quad q'^3 \,.\, \quad Q^2 \\ & - 2\,(p-q)\,PQ\,(p'q''-p''q') = 0. \end{aligned}$$

13. This has a first integral,

$$p'\sqrt{\left(\frac{P}{\theta+p}\right)} + q'\sqrt{\left(\frac{Q}{\theta+q}\right)} = 0,$$

where θ is the constant of integration; or say this is

$$\theta\,(p'^2P - q'^2Q) + p'^2qP - q'^2pQ = 0\,;$$

viz. differentiating logarithmically, this gives

$$\frac{2p'p''P - 2q'q''Q + p'^3P' - q'^3Q'}{p'^2P - q'^2Q} = \frac{2p'p''qP - 2q'q''pQ + p'^2\,(qp'P' + q'P) - q'^2\,(pq'Q' + p'Q)}{p'^2qP - q'^2pQ},$$

which, multiplying out the denominators, is in fact the foregoing geodesic equation. To verify, consider first the part involving p'', q'': this is

$$(2p'p''P - 2q'q''Q)\,(p'^2qP - q'^2pQ) - (2p'p''qP - 2q'q''pQ)\,(p'^2P - q'^2Q),$$

which is

$$= 2p'p''P \,.\, q'^2Q\,(q-p) - 2q'q''Q \,.\, p'^2P\,(q-p),$$

that is

$$= 2\,(q-p)\,PQp'q'\,(p''q' - p'q''),$$

or say

$$= 2\,(p-q)\,PQp'q'\,(p'q'' - p''q').$$

We have next the part

$$(p'^3P' - q'^3Q')\,(p'^2qP - q'^2pQ) - \{p'^2\,(qp'P' + q'P) + q'^2\,(pq'Q' + p''Q)\}\,(p'^2P - q'^2Q),$$

which is readily found to be

$$= -p'q'\,\{p'^3P^2 + p'^2q'\,(-PQ + \overline{p-q}\,P'Q) + p'q'^2\,(-PQ + \overline{q-p}\,PQ') + q'^3Q^2\}$$

and the equation is thus verified.

14. We have consequently

$$dp\sqrt{\left(\frac{p}{(a+p)\,(b+p)\,(c+p)\,(\theta+p)}\right)} + dq\sqrt{\left(\frac{q}{(a+q)\,(b+q)\,(c+q)\,(\theta+q)}\right)} = 0,$$

involving the arbitrary constant θ as the differential equation of the first order of the geodesic lines on the quadric surface $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$: the geodesic lines in question

all touch the curve of curvature determined by the parameter θ, that is the curve which is the intersection of the surface by the confocal surface

$$\frac{x^2}{a+\theta}+\frac{y^2}{b+\theta}+\frac{z^2}{c+\theta}=1.$$

15. In the particular case $\theta=\infty$, the equation becomes

$$Pdp^2-Qdq^2=0,$$

that is

$$\frac{pdp^2}{(a+p)(b+p)(c+p)}-\frac{qdq^2}{(a+q)(b+q)(c+q)}=0,$$

which is the differential equation of the circular curves on the surface.

16. The signification of the case $\theta=0$ is not at first sight so obvious. Supposing that θ is first indefinitely small, and writing the equation in the form

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1-\theta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\&c.=0,$$

we have the series of geodesics touching the (imaginary) curve of curvature, the intersection of the surface by the imaginary cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$. *These are, in fact, the right lines on the surface:* I apprehend that the intersection in question is not a proper envelope, but is the locus of nodes of the geodesics, viz. each geodesic is to be considered as a pair of lines belonging to the two sets: I do not, however, quite understand this.

17. I say that the geodesics in question are the right lines on the surface; viz. writing in the differential equation $\theta=0$, it is to be shown that the differential equation of the right lines is

$$\frac{dp}{\sqrt{(a+p)(b+p)(c+p)}}+\frac{dq}{\sqrt{(a+q)(b+q)(c+q)}}=0,$$

or what is the same thing, that the integral of this equation represents the right lines on the surface.

Writing the equation of the surface in the form

$$\frac{x^2}{a}+\frac{y^2}{b}=1-\frac{z^2}{c},$$

we have at once

$$\frac{x}{\sqrt{a}}+\frac{iy}{\sqrt{b}}=\sigma\left(1+\frac{z}{\sqrt{c}}\right),$$

$$\frac{x}{\sqrt{a}}-\frac{iy}{\sqrt{b}}=\frac{1}{\sigma}\left(1-\frac{z}{\sqrt{c}}\right),$$

(σ an arbitrary parameter) as the equations of a right line on the surface; viz. considering x, y, z as denoting the foregoing functions of p and q, these two equations are forms of a single relation between p, q, σ, which relation expresses that the point (p, q) is situate on the right line determined by the parameter σ. We may from this integral equation deduce without difficulty the foregoing differential equation; viz. we have

$$\frac{dx}{\sqrt{a}} + \frac{idy}{\sqrt{b}} = \sigma \frac{dz}{\sqrt{c}},$$

$$\frac{dx}{\sqrt{a}} - \frac{idy}{\sqrt{b}} = -\frac{1}{\sigma}\frac{dz}{\sqrt{c}},$$

or multiplying these equations,

$$\frac{dx^2}{a} + \frac{dy^2}{b} + \frac{dz^2}{c} = 0;$$

and substituting herein for dx, dy, dz their values in terms of dp and dq, we find the required equation

$$\frac{dp^2}{(a+p)(b+p)(c+p)} - \frac{dq^2}{(a+q)(b+q)(c+q)} = 0.$$

18. I return to the integral equation involving σ: we have to rationalise this equation, that is, obtain from it an equation containing x^2, y^2, z^2, and then substituting for these their values in terms of p, q, we have the required relation between p, q, σ. We at once obtain

$$\left\{2\left(\frac{x^2}{a} - \frac{y^2}{b}\right) - \left(\sigma^2 + \frac{1}{\sigma^2}\right)\left(1 + \frac{z^2}{c}\right)\right\}^2 - 4\left(\sigma^2 - \frac{1}{\sigma^2}\right)^2 \frac{z^2}{c} = 0;$$

or if for greater convenience we introduce in place of σ a new parameter ϕ, determined by the equation $\sigma^2 + \frac{1}{\sigma^2} = \frac{2\phi}{\gamma}$, the equation is

$$\left\{\gamma\left(\frac{x^2}{a} - \frac{y^2}{b}\right) - \phi\left(1 + \frac{z^2}{c}\right)\right\}^2 - 4(\phi^2 - \gamma^2)\frac{z^2}{c} = 0.$$

Writing for shortness $p + q = X$, $pq = Y$, we have

$$-\beta\gamma\frac{x^2}{a} = a^2 + aX + Y, \quad -\gamma\alpha\frac{y^2}{b} = b^2 + bX + Y, \quad -\alpha\beta\frac{z^2}{c} = c^2 + cX + Y;$$

and substituting these values, the equation becomes

$$\{\beta(b^2 + bX + Y) - \alpha(a^2 + aX + Y) - \phi\gamma(\alpha\beta - c^2 - cX - Y)\}^2 + 4\alpha\beta(\phi^2 - \gamma^2)(c^2 + cX + Y) = 0,$$

or, what is the same thing,

$$\{\beta b^2 - \alpha a^2 - \phi\gamma(\alpha\beta - c^2) + X(\beta b - \alpha a + \phi\gamma c) + Y(\beta - \alpha + \phi\gamma)\}^2 + 4\alpha\beta(\phi^2 - \gamma^2)(c^2 + cX + Y) = 0.$$

19. This is an equation, quadric as regards p, and also as regards q, viz. it is of the form

$$\begin{aligned} &(\mathrm{a} + 2\mathrm{h}p + \mathrm{g}p^2) \\ &+ 2q\,(\mathrm{h} + 2\mathrm{b}p + \mathrm{f}p^2) \\ &+ q^2(\mathrm{g} + 2\mathrm{f}p + \mathrm{c}p^2) = 0, \end{aligned}$$

and it leads to a differential equation

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0,$$

where

$$P=(\mathrm{h}+2\mathrm{b}p+\mathrm{f}p^2)^2-(\mathrm{a}+2\mathrm{h}p+\mathrm{g}p^2)(\mathrm{g}+2\mathrm{f}p+\mathrm{c}p^2),$$
$$Q=(\mathrm{h}+2\mathrm{b}q+\mathrm{f}q^2)^2-(\mathrm{a}+2\mathrm{h}q+\mathrm{g}q^2)(\mathrm{g}+2\mathrm{f}q+\mathrm{c}q^2);$$

and upon effecting the calculation, it is found that we have

$$P=-8\alpha^2\beta^2(\phi^2-\gamma^2)(a+b-2c-\phi)(a+p)(b+p)(c+p),$$
$$Q=-8\alpha^2\beta^2(\phi^2-\gamma^2)(a+b-2c-\phi)(a+q)(b+q)(c+q),$$

viz. P, Q are the same multiples of $(a+p)(b+p)(c+p)$, $(a+q)(b+q)(c+q)$ respectively; so that, omitting the common factor, or taking P, Q to represent the last-mentioned functions respectively, we have

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0;$$

and since the parameter ϕ has disappeared, we see that the original equation involving ϕ is the general integral of this differential equation; viz. that the differential equation belongs to the right lines on the surface.

20. The form of the integral equation may be simplified by introducing instead of ϕ a new parameter K, connected with it by the equation

$$K=\frac{\beta b-\alpha a+\phi c}{\beta-\alpha+\phi},\quad\text{or}\quad \phi=\frac{(\beta-\alpha)K-\beta b+\alpha a}{c-K},$$

viz. we thence deduce

$$\begin{aligned}
\beta-\alpha+\phi&=-2\alpha\beta && (\div),\\
\beta b-\alpha a+\phi c&=-2\alpha\beta K && (\div),\\
\beta b^2-\alpha a^2-\phi(\alpha\beta-c^2)&=\ \ 2\alpha\beta(ac+bc-ab+2cK) && (\div),\\
\phi+\gamma&=\ \ 2\beta(K-b) && (\div),\\
\phi-\gamma&=-2\alpha(K-a) && (\div),
\end{aligned}$$

where the sign $(\div)$ is used to signify that the functions preceding it have to be divided by a denominator which in fact is $=c-K$. The equation thus becomes

$$(ac+bc-ab-2cK-KX-Y)^2-4(K-a)(K-b)(c^2+cX+Y)=0;$$

and if we moreover write

$$\nu,\ \mu,\ \lambda=abc,\ bc+ca+ab,\ a+b+c,$$

and instead of K introduce the parameter C, $=\lambda-K$, the equation becomes

$$\{-\mu-2c^2+2cC-(\lambda-C)X-Y\}^2+(b+c-C)(c+a-C)(c^2+cX+Y)=0;$$

or expanding and reducing, this is

$$\begin{aligned}&\{Y+(\lambda-C)X\}^2\\&\quad+Y(-2\mu+4\lambda C-4C^2)\\&\quad+X(2\mu\lambda-4\nu-2\mu C)\\&\quad+\mu^2-4\nu C=0,\end{aligned}$$

or say

$$\begin{aligned}&\mu^2-4\nu C\\&+(2\mu\lambda-4\nu-2\mu C)(p+q)\\&+(-2\mu+4\lambda C-4C^2)pq\\&+(\lambda-C)^2(p^2+q^2+2pq)\\&+2(\lambda-C)pq(p+q)\\&+\qquad p^2q^2\qquad=0,\end{aligned}$$

viz. this, containing the constant C, is the general integral of the differential equation

$$\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0,$$

where

$$\begin{aligned}P&=(a+p)(b+p)(c+p),\quad=\nu+\mu p+\lambda p^2+p^3,\\Q&=(a+q)(b+q)(c+q),\quad=\nu+\mu q+\lambda q^2+q^3.\end{aligned}$$

21. The constant C is connected with the parameter σ, which originally served to determine the right line, by the equation

$$\sigma+\frac{1}{\sigma}=\frac{2}{\gamma}\,\frac{(\beta-\alpha)(\lambda-C)-\beta b+\alpha a}{c-(\lambda-C)},$$

or, what is the same thing,

$$\sigma+\frac{1}{\sigma}=\frac{2}{b-c}\,\frac{2c^2-a^2-b^2-C(2c-a-b)}{C-a-b}.$$

Reverting to the equation between p, q, ϕ, I remark that if ϕ be therein considered as variable, we have the differential equation

$$\sqrt{Q}\,dp+\sqrt{P}\,dq+\sqrt{\Phi}\,d\phi=0,$$

where P, Q have the foregoing values

$$P=-8\alpha^2\beta^2(\phi^2-\gamma^2)(a+b-c-\phi)(a+p)(b+p)(c+p),\quad Q=\&c.;$$

and where, if the integral equation be written in the form

$$L+2M\phi+N\phi^2=0,$$

then we have $\Phi=M^2-NL$, viz. we thus find

$$\Phi=16\alpha^2\beta^2(a+p)(b+p)(c+p)(a+q)(b+q)(c+q).$$

22. Changing the notation, and writing

$$P = (a+p)(b+p)(c+p),$$
$$Q = (a+q)(b+q)(c+q),$$
$$\Phi = (\phi^2 - \gamma^2)(a+b-2c-\phi),$$

the equation is

$$\frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} + \frac{\sqrt{2}\, d\phi}{\sqrt{\Phi}} = 0;$$

or if, instead of ϕ, we introduce the original parameter σ, then, observing that

$$\frac{2d\sigma}{\sigma} = \frac{d\phi}{\sqrt{\phi^2 - \gamma^2}},$$

we at once find

$$\frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} + \frac{4d\sigma}{\sqrt{\Sigma}} = 0,$$

where

$$\Sigma = \gamma(1+\sigma^4) - 2(a+b-2c)\sigma^2,$$

or, what is the same thing,

$$\Sigma = a(\sigma^2-1)^2 - b(\sigma^2+1)^2 + c\,.\,4\sigma^2;$$

viz. passing from a point (p, q) on the line σ to a consecutive point $(p+dp, q+dq)$ on the line $\sigma + d\sigma$, the above is the relation between the variations dp, dq, $d\sigma$. If τ be the parameter of the other line through the same point, then we have in like manner, say

$$\frac{dp}{\sqrt{P}} - \frac{dq}{\sqrt{Q}} + \frac{4d\tau}{\sqrt{\mathrm{T}}} = 0,$$

(viz. one of the radicals $\sqrt{P}$, $\sqrt{Q}$ must present itself with a reversed sign): and we thus have dp, dq each expressed in terms of $d\sigma$, $d\tau$; viz. we have the increments dp, dq when a point passes from (σ, τ) to $(\sigma + d\sigma, \tau + d\tau)$. These results will be presently obtained in a more simple manner.

Formulæ where the position of a Point on the Surface is determined by means of the two Lines through the Point.

23. We may determine the position of a point by means of the parameters σ, τ of the two lines through the point. The equations of these are

$$\frac{x}{\sqrt{a}} + \frac{iy}{\sqrt{b}} = \sigma\left(1 + \frac{z}{\sqrt{c}}\right), \qquad \frac{x}{\sqrt{a}} + \frac{iy}{\sqrt{b}} = \tau\left(1 - \frac{z}{\sqrt{c}}\right),$$

$$\frac{x}{\sqrt{a}} - \frac{iy}{\sqrt{b}} = \frac{1}{\sigma}\left(1 - \frac{z}{\sqrt{c}}\right), \qquad \frac{x}{\sqrt{a}} - \frac{iy}{\sqrt{b}} = \frac{1}{\tau}\left(1 + \frac{z}{\sqrt{c}}\right),$$

and from these equations we deduce

$$\frac{x}{\sqrt{a}}=\frac{\sigma\tau+1}{\tau+\sigma},\quad \frac{iy}{\sqrt{b}}=\frac{\sigma\tau-1}{\tau+\sigma},\quad \frac{z}{\sqrt{c}}=\frac{\tau-\sigma}{\tau+\sigma}.$$

We have thence

$$\frac{dx}{\sqrt{a}}=(\tau^2-1)\,d\sigma+(\sigma^2-1)\,d\tau\quad(\div),$$

$$\frac{idy}{\sqrt{b}}=(\tau^2+1)\,d\sigma+(\sigma^2+1)\,d\tau\quad(\div),$$

$$\frac{dz}{\sqrt{c}}=\quad-2\tau d\sigma+\qquad 2\sigma d\tau\quad(\div),$$

where denom. $=(\tau+\sigma)^2$: regarding σ, τ as the parameters in place of p, q, these show the values of the first differential coefficients a, a'; b, b'; c, c'. We deduce

$$A=-2i\sqrt{bc}\,(\sigma\tau+1)\quad\div,\quad B=-2i\sqrt{ca}\,(\sigma\tau-1)\quad\div,\quad C=-2i\sqrt{ba}\,(\tau-\sigma)\quad\div,$$

where denom. $=(\tau+\sigma)^2$. We have, moreover,

$$\frac{d^2x}{\sqrt{a}}=-2\,(\tau^2-1)\,d\sigma^2+2\,(\sigma\tau+1)\,2d\sigma\,d\tau-2\,(\sigma^2-1)\,d\tau^2\quad(\div),$$

$$\frac{2d^2y}{\sqrt{b}}=-2\,(\tau^2+1)\,d\sigma^2+2\,(\sigma\tau-1)\,2d\sigma\,d\tau-2\,(\sigma^2+1)\,d\tau^2\quad(\div),$$

$$\frac{d^2z}{\sqrt{c}}=+\qquad 4\tau\,d\sigma^2+2\,(\tau-\sigma)\,2d\sigma\,d\tau+\qquad 4\sigma\,d\tau^2\quad(\div),$$

where denom. $=(\tau+\sigma)^2$: giving α, α', α''; β, β', β''; γ, γ', γ''. We deduce as the numerators of $E'\,(=A\alpha+B\beta+C\gamma)$ and $G'\,(=A\alpha''+B\beta''+C\gamma'')$,

$$4i\sqrt{abc}\,\{(\sigma\tau+1)(\tau^2-1)-(\sigma\tau-1)(\tau^2+1)-2\tau(\tau-\sigma)\},=0,$$

and

$$4i\sqrt{abc}\,\{(\sigma\tau+1)(\sigma^2-1)-(\sigma\tau-1)(\sigma^2+1)+2\sigma(\tau-\sigma)\},=0;$$

that is, $E'=0$ and $G'=0$; or the differential equation of the chief lines is $d\sigma\,d\tau=0$, which is right. The value of $F'\,(=A\alpha'+B\beta'+C\gamma')$ is hardly required, but it is readily found to

$$=4i\sqrt{abc}\,\{-(\sigma\tau+1)^2+(\sigma\tau-1)^2-(\tau-\sigma)\}^2\div(\tau+\sigma)^6,$$

or since the term in $\{\quad\}$ is $=-(\tau+\sigma)^2$, we have

$$F'=\frac{-4i\sqrt{abc}}{(\tau+\sigma)^4}.$$

24. The values of E, F, G $(ds^2=Ed\sigma^2+2Fd\sigma\,d\tau+Gd\tau^2)$ are

$$E=a\,(\tau^2-1)^2\qquad-b\,(\tau^2+1)^2\qquad+c\,.\,4\tau^2\quad(\div),$$
$$F=a\,(\tau^2-1)(\sigma^2-1)-b\,(\tau^2+1)(\sigma^2+1)-c\,.\,4\tau\sigma\quad(\div),$$
$$G=a\,(\sigma^2-1)^2\qquad-b\,(\sigma^2+1)^2\qquad+c\,.\,4\sigma^2\quad(\div),$$

where denom. $=(\tau+\sigma)^4$.

We have, it is clear, ($E_1 = d_\sigma E$, $E_2 = d_\tau E$, &c.)

$$E_1 = -\frac{4}{\sigma+\tau}E, \quad G_2 = -\frac{4}{\sigma+\tau}G.$$

Hence the condition $FG_2 - 2F_2G + GG_1 = 0$, in order that $\sigma = \text{const.}$ may be a geodesic, reduces itself to

$$-\frac{4}{\sigma+\tau}F - 2F_2 + G_1 = 0;$$

and similarly, the condition $-2EF_1 + E_1F + EE_2 = 0$, in order that $\tau = \text{const.}$ may be a geodesic, reduces itself to

$$-\frac{4}{\sigma+\tau}F - 2F_1 + E_2 = 0.$$

We have at once

$$\begin{aligned}
E_2 &= 4a(\tau^2-1)(\tau\sigma+1) && -4b(\tau^2+1)(\tau\sigma-1) && +4c.2\tau(\sigma-\tau) && (\div),\\
G_1 &= 4a(\sigma^2-1)(\tau\sigma+1) && -4b(\sigma^2+1)(\tau\sigma-1) && -4c.2\sigma(\sigma-\tau) && (\div),\\
F_1 &= 2a(\tau^2-1)(\tau\sigma-\sigma^2+2) && -2b(\tau^2+1)(\tau\sigma-\sigma^2-2) && -2c.2\tau(\tau-3\sigma) && (\div),\\
F_2 &= 2a(\sigma^2-1)(\tau\sigma-\tau^2+2) && -2b(\sigma^2+1)(\tau\sigma-\tau^2-2) && -2c.2\sigma(\sigma-3\tau) && (\div),
\end{aligned}$$

where denom. $= (\sigma+\tau)^5$; and substituting these values, the conditions are verified: we thus again see *à posteriori* that the right lines $\sigma = \text{const.}$ and $\tau = \text{const.}$ are geodesics.

25. The last-mentioned values of E, G are $E = \mathrm{T} \div (\tau+\sigma)^4$, $G = \Sigma \div (\tau+\sigma)^4$; and writing for a moment

$$A = a(\tau^2-1)(\sigma^2-1) - b(\tau^2+1)(\sigma^2+1) - c.4\sigma\tau,$$

we have $F = A \div (\tau+\sigma)^4$, the value of ds^2 is thus

$$= \mathrm{T}d\sigma^2 + 2Ad\sigma d\tau + \Sigma d\tau^2 \div (\tau+\sigma)^4,$$

which should be

$$= \tfrac{1}{4}(p-q)\left(p\frac{dp^2}{P} - q\frac{dq^2}{Q}\right),$$

where, as before,

$$P,\ Q = (a+p)(b+p)(c+p),\quad (a+q)(b+q)(c+q),$$

respectively. We have already found

$$\frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} + \frac{4d\sigma}{\sqrt{\Sigma}} = 0,$$

$$\frac{dp}{\sqrt{P}} - \frac{dq}{\sqrt{Q}} + \frac{4d\tau}{\sqrt{\mathrm{T}}} = 0;$$

or, what is the same thing,

$$\frac{dp}{\sqrt{P}} = -2\left(\frac{d\sigma}{\sqrt{\Sigma}} + \frac{d\tau}{\sqrt{\mathrm{T}}}\right),$$

$$\frac{dq}{\sqrt{Q}} = -2\left(\frac{d\sigma}{\sqrt{\Sigma}} - \frac{d\tau}{\sqrt{\mathrm{T}}}\right);$$

and we ought therefore to have identically

$$(p-q)\left\{p\left(\frac{d\sigma}{\sqrt{\Sigma}}+\frac{d\tau}{\sqrt{\mathrm{T}}}\right)^2-q\left(\frac{d\sigma}{\sqrt{\Sigma}}-\frac{d\tau}{\sqrt{\mathrm{T}}}\right)^2\right\}=\mathrm{T}d\sigma^2+2Ad\sigma d\tau+\Sigma d\tau^2\div(\tau+\sigma)^4;$$

that is

$$\begin{aligned}(p-q)^2 &= \mathrm{T}\Sigma\div(\tau+\sigma)^4,\\ p^2-q^2 &= A\sqrt{\mathrm{T}\Sigma}\div(\tau+\sigma)^4;\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}(p-q)^2 &= \mathrm{T}\Sigma\div(\tau+\sigma)^4,\\ p+q &= A\div(\tau+\sigma)^2,\end{aligned}$$

which are easily verified.

26. In fact, the equation

$$\frac{x^2}{a+u}+\frac{y^2}{b+u}+\frac{z^2}{c+u}=1$$

gives for u a quadric equation, the roots of which are $u=p$, $u=q$; that is, we have

$$\begin{aligned}p+q &= x^2+y^2+z^2-a-b-c,\\ pq &= -(b+c)x^2-(c+a)y^2-(a+b)z^2+bc+ca+ab;\end{aligned}$$

and substituting herein for x^2, y^2, z^2 their values in terms of σ, τ, we find

$$\begin{aligned}p+q &= a(\sigma^2-1)(\tau^2-1)-b(\sigma^2+1)(\tau^2+1)-4c\tau\sigma \div(\tau+\sigma)^2,\\ pq &= bc(\sigma\tau+1)^2-ca(\sigma\tau-1)^2+ab(\sigma-\tau)^2 \div(\tau+\sigma)^4,\end{aligned}$$

the first of which is, in fact, $p+q=A\div(\tau+\sigma)^2$. And from the two equations, forming the combination $(p+q)^2-4pq$, we at once obtain the other equation

$$(p-q)^2=\Sigma\mathrm{T}\div(\tau+\sigma)^4.$$

27. The most ready way of obtaining the relations between the differentials of p, q, σ, τ, is from the foregoing expressions of $p+q$, pq. Writing for a moment $p+q=A\div(\sigma+\tau)^2$, $pq=B\div(\sigma+\tau)^2$, we have $p^2(\sigma+\tau)^2-Ap+B=0$, $q^2(\sigma+\tau)^2-Aq+B=0$; viz. the first of these equations is

$$\begin{aligned}p^2(\sigma+\tau)^2-p\,[a(\sigma^2-1)(\tau^2-1)-b(\sigma^2+1)(\tau^2+1)-4c\sigma\tau]\\ +bc(\sigma\tau+1)^2-ca(\sigma\tau-1)^2+ab(\sigma-\tau)^2=0,\end{aligned}$$

which is quadric in p, σ, τ. The negative discriminants in regard to these variables respectively are $\Sigma\mathrm{T}$, $4P\mathrm{T}$, $4P\Sigma$ respectively, and we have thus the equation

$$\frac{dp}{\sqrt{P}}+2\left(\frac{d\sigma}{\sqrt{\Sigma}}+\frac{d\tau}{\sqrt{\mathrm{T}}}\right)=0;$$

and the like equation for dq.

28. In the first integral of the geodesic lines, introducing σ, τ instead of p, q, the equation becomes

$$\sqrt{\left(\frac{p}{p+\theta}\right)}\left(\frac{d\sigma}{\sqrt{\Sigma}}+\frac{d\tau}{\sqrt{\mathrm{T}}}\right)-\sqrt{\left(\frac{q}{q+\theta}\right)}\left(\frac{d\sigma}{\sqrt{\Sigma}}-\frac{d\tau}{\sqrt{\mathrm{T}}}\right)=0;$$

or, what is the same thing,

$$p(q+\theta)\left(\frac{d\sigma}{\sqrt{\Sigma}}+\frac{d\tau}{\sqrt{\mathrm{T}}}\right)^2-q(p+\theta)\left(\frac{d\sigma}{\sqrt{\Sigma}}-\frac{d\tau}{\sqrt{\mathrm{T}}}\right)^2=0;$$

that is,

$$(p-q)\,\theta\left(\frac{d\sigma^2}{\Sigma}+\frac{d\tau^2}{\mathrm{T}}\right)+2\,[2pq+\theta(p+q)]\frac{d\sigma\,d\tau}{\sqrt{\Sigma\mathrm{T}}}=0;$$

or substituting herein for $p-q$, pq, $p+q$ the values $\sqrt{\Sigma\mathrm{T}}$, B, A, each divided by $(\sigma+\tau)^2$, this is

$$\theta\,(\mathrm{T}d\sigma^2+\Sigma d\tau^2)+2\,(2B+\theta A)\,d\sigma\,d\tau=0,$$

or say

$$\theta\,(\mathrm{T}d\sigma^2+2Ad\sigma\,d\tau+\Sigma d\tau^2)+4Bd\sigma\,d\tau=0;$$

viz. writing herein $\theta=0$, the equation is $d\sigma\,d\tau=0$, giving the right lines on the surface; and writing $\theta=\infty$, it is $\mathrm{T}d\sigma^2+2Ad\sigma\,d\tau+\Sigma d\tau^2=0$, giving the circular lines.

29. The equation $ds^2=\mathrm{T}d\sigma^2+2Ad\sigma d\tau+\Sigma d\tau^2\div(\tau+\sigma)^4$ shows that the right lines σ, $\sigma+d\sigma$, τ, $\tau+d\tau$ form on the surface an indefinitely small parallelogram, the sides whereof are $\sqrt{\mathrm{T}}\,d\sigma\div(\tau+\sigma)^2$ and $\sqrt{\Sigma}\,d\tau\div(\tau+\sigma)^2$, viz. the ratio of the coefficients of $d\sigma$, $d\tau$ is of the form function $\sigma\div$function τ; and it thus appears that it is possible to draw on the surface the two sets of right lines, the lines of each set being at such intervals that the surface is divided into parallelograms, the sides of which have to each other any given ratio (the angles being variable); viz. if this ratio be as $m:1$, then, to determine the relation between σ, τ, we must have $\sqrt{\mathrm{T}}\,d\sigma=\pm m\sqrt{\Sigma}\,d\tau$, or what is the same thing, $\frac{d\sigma}{\sqrt{\Sigma}}=\pm m\frac{d\tau}{\sqrt{\mathrm{T}}}$. In particular, if $m=1$, the parallelograms will be rhombs; and we must then have

$$\frac{d\sigma}{\sqrt{\Sigma}}=\pm\frac{d\tau}{\sqrt{\mathrm{T}}};$$

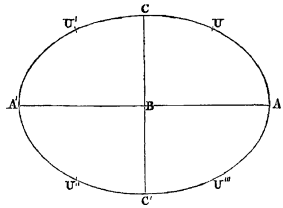
viz. this being in terms of σ, τ, the differential equation of the curves of curvature, it appears that the two sets of lines may be taken so as to divide the surface into indefinitely small rhombs, such that, drawing the diagonals of these, we have the two sets of curves of curvature.

The Ellipsoid and the Skew Hyperboloid.

30. I have thus far considered a quadric surface in general, the various theorems being applicable as well to the ellipsoid and the hyperboloid of two sheets as to the skew hyperboloid, the right lines being of course imaginary for the first-mentioned surfaces; but I will now consider the ellipsoid and the skew hyperboloid separately.

31. First the ellipsoid. We have here a, b, c all positive, and I assume as usual $a>b>c$. The principal sections are all ellipses, viz. $\frac{x^2}{a}+\frac{y^2}{b}=1$ is the major-

mean, or say the minor section, $\frac{y^2}{b}+\frac{z^2}{c}=1$ the minor-mean, or say the major section, and $\frac{x^2}{a}+\frac{z^2}{c}=1$ the mean, or umbilicar section. The elliptic coordinates p, q enter into the equations symmetrically, but we distinguish them by taking p to extend from $-c$ to $-b$, and q to extend from $-b$ to $-a$. Thus $p=$const. denotes the curves of

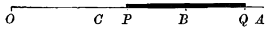


curvature of the one kind; viz. $p=-c$ denotes the major-mean section $\frac{x^2}{a}+\frac{y^2}{b}=1$, $p=-b$ the portions UU' and $U''U'''$ of the umbilicar section; and $q=$const. denotes the curves of curvature of the other kind, viz. $q=-b$ the remaining portions $U'U''$ and $U'''U$ of the umbilicar section, $q=-a$ the minor-mean section $\frac{y^2}{b}+\frac{z^2}{c}=1$; say $p=$const. the major-mean curves, and $q=$const. the minor-mean curves.

32. Hence, in order that the equation

$$dp\sqrt{\left(\frac{p}{(a+p)(b+p)(c+p)(\theta+p)}\right)}\pm dq\sqrt{\left(\frac{q}{(a+q)(b+q)(c+q)(\theta+q)}\right)}=0$$

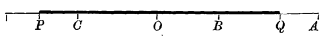
of the geodesic lines may be real {observing that we have $a+p$, $b+p=+$, $c+p$, $p=-$, and $a+q=+$, $b+q$, $c+q$, $q=-$, consequently $p\div(a+p)(b+p)(c+p)=+$, but $q\div(a+q)(b+q)(c+q)=-$}, we must have $\theta+p$, $\theta+q$ of opposite signs, that is $\theta+p=+$ and $\theta+q=-$; or θ included between the limits a, c. Or, what is the same thing, $-\theta$ is included between the limits $-c$, $-b$, say $-\theta$ has a p-value; or else between the limits $-b$, $-a$, say $-\theta$ has a q-value. This is conveniently shown



in the annexed diagram of the values of $-p$, $-q$, $-\theta$. Hence on the ellipsoid we have two kinds of geodesic lines, each of them touching a real curve of curvature; viz. those which touch a major-mean curve and those which touch a minor-mean curve: the transition case, answering to the value $\theta=b$, is that of the geodesic lines which pass through an umbilicus. I have considered the theory more in detail in my memoir "On the Geodesic Lines of an Ellipsoid," *Mem. R. Ast. Soc.*, t. xxx., pp. 31—53, 1872, [478].

33. Next, for the skew hyperboloid, we have a and $b=+$, $c=-$, and I assume for convenience $a>b$. Attending to the signs, we still have therefore $a>b>c$. The principal sections are one of them an ellipse, and the other two hyperbolas, viz. the minor section is the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$, the major section is the hyperbola $\frac{y^2}{b}+\frac{z^2}{c}=1$, and the mean section is the hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$: there are no umbilici. The elliptic coordinates enter symmetrically; but, as before, we distinguish them, viz. we take p to extend from $-c$ (a positive value) to infinity, and q from $-b$ to $-a$. Thus $p=\text{const.}$ denotes the curves of curvature of the one kind, viz. $p=-c$ the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$, and every other value of p an oval curve surrounding the hyperboloid; and $q=\text{const.}$ the curves of curvature of the other kind, viz. $q=-c$ the major hyperbola $\frac{y^2}{b}+\frac{z^2}{c}=1$, $q=-b$ the mean hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$, and each intermediate value gives a curve of curvature of a hyperbolic form: we may say that $p=\text{const.}$ determines the oval curves of curvature, and $q=\text{const.}$ the hyperbolic curves of curvature.

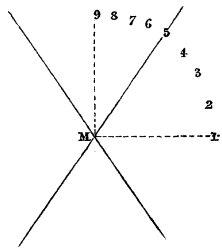
34. In the equation of the geodesic lines we have $a+p$, $b+p$, $c+p$, p all positive; but $a+q=+$, $b+q$, $c+q$, q each $=-$; hence $p\div(a+p)(b+p)(c+p)=+$, but $q\div(a+q)(b+q)(c+q)=-$; therefore $\theta+p$ and $\theta+q$ must be of opposite signs, or we must have $\theta+p=+$ and $\theta+q=-$; or what is the same thing, θ may have any value from $-p$ to $-q$, or say $-\theta$ any value from p to q; that is, the value of $-\theta$ may be positive and greater than $-c$, positive and less than $-c$, negative and less than $-b$, negative and between $-b$ and $-a$; viz. in the first case $-\theta$ has a p-value,



and in the fourth case it has a q-value, but in the second and third cases it has neither a p- nor a q-value. This is better seen from the diagram. It follows that we have, on the hyperboloid, geodesic lines of four different kinds: those which touch a real curve of curvature, oval or hyperbolic, and those which touch no real curve of curvature, but for which $-\theta$ has a positive value from 0 to $-c$, or a negative value from 0 to $-b$. And there are the transitional cases $-\theta=-c$, where the geodesic touches the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$; $\theta=0$, where the geodesic becomes a right line; and $-\theta=-b$, where the geodesic touches the mean hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$.

35. To explain this more in detail, consider the geodesics which start from a point M of the hyperboloid. To fix the ideas, consider the axis of z as vertical, and take the point M in the positive octant of the hyperboloid; and let $M1$ represent the direction of the oval curve of curvature, $M9$ that of the hyperbolic curve of curvature, $M5$ that of one of the right lines.

The geodesic of initial direction $M1$ touches at M the oval curve of curvature $M1$, and lies wholly above this curve; it makes an infinity of convolutions round the upper part of the hyperboloid, cutting all the oval curves of curvature for which p has a (positive) value greater than p_1 (if p_1 is the value of p corresponding to the oval curve through M), and ascending to infinity: or considering the curve as described in the opposite sense, it descends from infinity to touch the oval curve through M, after which it again ascends to infinity.



Next, if the initial direction is $M2$, we have a geodesic of the same kind, only descending below M to touch a certain oval curve having for its parameter p_2 ($p_2 > -c < p_1$).

We come next to a critical direction $M3$, for which the geodesic descends below M to touch the oval curve of parameter $p_3 = -c$, that is, the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$. But it is to be observed that, whatever the initial point M may be, the geodesic makes below M an infinity of convolutions round the hyperboloid, so that it does not in fact ever actually touch the ellipse, but has this ellipse for an asymptote. That this is so appears from the consideration that the ellipse, *quà* plane curve of curvature, is a geodesic; so that, starting from a point of the ellipse in the direction of the ellipse, the geodesic coincides with the ellipse, or, besides the ellipse itself, there is not any geodesic which touches the ellipse.

Next, if the initial direction be $M4$, the geodesic does not here touch any oval curve; it descends through M below the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$, lying in the upper and lower portions of the hyperboloid, and making round it an infinity of convolutions.

36. We come, then, to the initial direction $M5$, which is that of the right line; the geodesic here coincides with the right line.

In the cases which follow, the geodesic lies in the upper and lower portions of the hyperboloid, cutting all the oval curves of curvature.

Initial direction $M6$: the geodesic does not touch any hyperbolic curve of curvature, but makes round the hyperboloid an infinity of convolutions.

Initial direction $M7$: the geodesic touches at opposite infinities the mean hyperbola $\frac{x^2}{a}+\frac{z^2}{c}=1$, it lies wholly in front of the plane $y=0$ of this hyperbola.

Initial direction $M8$: the geodesic touches a hyperbolic curve of curvature parameter q_8 where q_8 (negative) is between $-b$ and q_9 the parameter of the hyperbolic curve of curvature through M; viz. it cuts all the hyperbolic curves the parameters of which are between $-b$ and q_8, but does not cut the remaining curves the parameters of which extend from q_8 to $-a$.

Lastly, initial direction is $M9$, that of the hyperbolic curve of curvature through M; the geodesic touches this curve, cutting all the hyperbolic curves the parameters of which are between $-b$ and q_9, but not any of those the parameters of which are between q_9 and $-a$.

37. If in the differential equation of the geodesic line we consider p, q as the elliptic coordinates of a given point M of the curve, the equation for a given value of θ determines the direction of the curve; or conversely, if the direction be given, the equation determines the value of the parameter θ. Writing

$$\frac{dp\sqrt{p}}{\sqrt{(a+p)(b+p)(c+p)}}=P,\quad \frac{dq\sqrt{q}}{\sqrt{(a+q)(b+q)(c+q)}}=Q,$$

then P, Q are proportional to the rectangular coordinates of a consecutive point M', measured from M in the directions of the hyperbolic and oval curves of curvature respectively; and the differential equation of the geodesic lines gives

$$\frac{P}{\sqrt{p+\theta}}\pm\frac{Q}{\sqrt{q+\theta}}=0;$$

viz. if ϕ be the inclination of the geodesic to the hyperbolic curve of curvature, then $Q=P\tan\phi$, or we have $\frac{1}{p+\theta}=\frac{\tan^2\phi}{q+\theta}$, that is, $p\tan^2\phi-q=\theta(1-\tan^2\phi)$; hence, if for the right line $\phi=\lambda$, then $p\tan^2\lambda-q=0$; and therefore $\theta=\frac{p(\tan^2\phi-\tan^2\lambda)}{1-\tan^2\phi}$; viz. $\phi=0$, $\theta=-q$, $=\infty$, $\theta=-p$, as it should be.

509.

PLAN OF A CURVE-TRACING APPARATUS.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 345—347. Read May 8, 1873.]

I HAVE devised a curve-tracing apparatus on the following plan:

Imagine two planes Π, Π' moving in the same horizontal plane, and above the two planes respectively the two points P, P' moving in the same or a parallel plane.

To fix the ideas, suppose that the two planes each move according to a law (that is, let the motion of each of them depend on a single variable parameter; for instance, they may each of them rotate about an axis); but let the motion of the two points be free.

Suppose, next, that the planes are connected together in such manner that the motion of one of them determines the motion of the other (e.g. by a train of wheel-work); and that the two points are also connected together in such manner that the motion of one of them determines the motion of the other (e.g. by a pentagraph; or by a slotted rod, the slot of which works on an axle, so as to allow the rod to move lengthways as well as rotate).

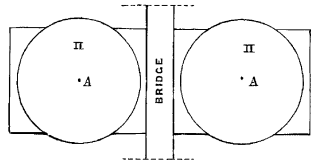
Suppose, finally, that one of the points, say P', is attached to a point of the plane Π'; then the plane Π being set in motion, this determines the motion of Π', consequently of P', consequently of P; and the moving point P, or say the pencil P, will describe on the moving plane Π a curve, the nature of which will of course depend on the nature of the motions of Π, Π', and on that of the connection between these planes and of the connections between the points P and P'.

I propose to describe the apparatus as nearly as I have actually constructed it. (See sketch-plan Fig. 1, and side-elevation Fig. 2.)

The framework of the instrument consists of two longitudinal bars (B) each about three feet long, one inch thick, and three inches broad, supported edgewise at a distance of about eighteen inches on the cross-pieces C, C; and half-way between them, supported by the same cross-pieces, is an axis carrying at each extremity two mitre wheels. The bars B support three cross-pieces D, D, D, and between these are

the moveable cross-pieces E, E carrying the axes A, A with the attached mitre wheels, and circular disks X, X. Each of these wheels separately may be placed (as in the figure) out of gear with the two vertical wheels, or it can (by moving the

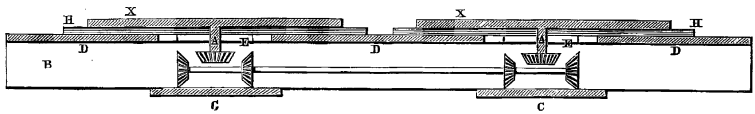
Fig. 1.



cross-piece E) be brought into gear with either of these wheels. Each axis A passes through a circular disk H, capable of rotating about it, so that it may be fixed in any position, and serving as the bearing for the plane X.

The disks X, X may be regarded as being themselves the planes Π, Π (say these planes are rigidly attached to X, X respectively); or we may, in any other manner, move the plane Π by means of the disk X; for instance, X may carry a spur-wheel gearing into a spur-wheel on the under surface of Π, and thereby communicating a

Fig. 2.



rotation of different velocity to the plane Π; or the connection may be as in the ordinary oval chuck. In any such case, the disk H (which, for this reason, was made to project beyond X) serves as a support for the plane Π, and the apparatus connected therewith; and observe that the angular position of such apparatus, and therefore of the path of any point of Π, may be varied at pleasure by moving the disk H through any angle.

Detached altogether from the rest of the instrument, or what is better, supported on longitudinal pieces carried by the cross-pieces C, C we have a bridge (see fig. 1) capable of adjustment as regards height, and serving as a support for the pentagraph-apparatus, or other connection of the one plane Π with the pencil which works upon the other plane Π.

It is hardly necessary to remark that in the simple form of the instrument where the disks X, X are themselves the planes Π, Π, then putting the mitre wheel of one plane X in gear with either of the corresponding vertical wheels, and making the plane rotate, the other plane X will rotate with the same angular velocity, in the same or the opposite direction, or it will remain at rest, according as its mitre wheel is in gear with one or the other of the corresponding vertical wheels, or is out of gear with each of them.

510.

ON BICURSAL CURVES.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 347—352. Read May 8, 1873.]

A CURVE of deficiency 1 may be termed bicursal: there is some distinction according as the order is even or odd, and to fix the ideas I take it to be even.

A bicursal curve of the order n contains

$$\tfrac{1}{2}n(n+3)-\{\tfrac{1}{2}(n-1)(n-2)-1\},\ =3n \text{ constants};$$

hence, if the order is $=2n$, the number of constants is $=6n$; such a curve is normally represented by a system of equations

$$(x,\ y,\ z)=(1,\ \theta)^n+(1,\ \theta)^{n-2}\sqrt{\Theta},$$

where Θ is a quartic function, which may be taken to be of the form $(1-\theta^2)(1-k^2\theta^2)$, or otherwise to depend on a single constant; viz. $(x,\ y,\ z)$ are proportional to n-thic functions involving such a radical: since in the values of $(x,\ y,\ z)$ one constant divides out, the number of constants is $3\{(n+1)+(n-1)\}-1+1,\ =6n$, as it should be.

But the curve of the order $2n$ may be abnormally or improperly represented by a system of equations

$$(x,\ y,\ z)=(1,\ \theta)^{n+k}+(1,\ \theta)^{n+k-2}\sqrt{\Theta},$$

viz. these equations, instead of representing a curve of the order $2n+2k$, will represent a curve of the order $2n$, provided only there exist $2k$ common values of θ for which each of the three functions vanish. The passage to a normal representation is effected by finding θ' a function of θ, $\sqrt{\Theta}$ (viz. an irrational function of θ) such that the foregoing equations become

$$(x,\ y,\ z)=(1,\ \theta')^n+(1,\ \theta')^{n-2}\sqrt{\Theta'};$$

it is shown that such a transformation is possible, and a mode of effecting it, derived from a theorem of Hermite's in relation to the Jacobian H, Θ functions, is given in Clebsch's Memoir "Ueber diejenigen Curven, deren Coordinaten sich als elliptische Functionen eines Parameters darstellen lassen," *Crelle*, t. LXIV. (1865), pp. 210—270. The demonstration is a very interesting one, and I reproduce it at the end of this paper. I remark, in passing, that the analogous reduction in the case of unicursal curves is self-evident; the equations

$$(x,\ y,\ z) = (1,\ \theta)^{n+k}$$

will represent a curve, not of the order $n+k$, but of the order n, provided there exist k common values of θ for which the three functions vanish; in fact, the three functions have then a common factor of the order k, and omitting this, the form is $(x,\ y,\ z) = (1,\ \theta)^n$.

Returning to the curves of deficiency 1, we see that a curve of the order $2m+2n$ contains $6(m+n)$ constants, and is normally represented by a system of equations

$$(x,\ y,\ z) = (1,\ \theta)^{m+n} + (1,\ \theta)^{m+n-2}\sqrt{\Theta}.$$

Such a curve may be otherwise represented: we may derive it by a rational transformation from the curve $(D=1)$ of the order 4 (binodal quartic), the equation of which is

$$(1,\ u)^2(1,\ v)^2 = 0;$$

viz. the coordinates are here connected by a quadriquadric equation; and we then express x, y, z in terms of these by a system of equations

$$(x,\ y,\ z) = (1,\ u)^m(1,\ v)^n.$$

It is, however, to be observed that the form of these functions is not determinate: each of them may be altered by adding to it a term $\{(1,\ u)^{m-2}(1,\ v)^{n-2}\}\{(1,\ u)^2(1,\ v)^2\}$, where the second factor is that belonging to the quadriquadric transformation, and the first factor is arbitrary. Using the arbitrary function to simplify the form, the real number of constants is reduced to $(m+1)(n+1)-(m-1)(n-1),\ =2(m+n)$; or the three functions contain together $6(m+n)$ constants, one of which divides out. The quadriquadric equation, dividing out one constant, contains eight constants; but reducing by linear transformations on u, v respectively, the number of constants is $8-6,\ =2$. Hence, in the system of equations, the whole number of constants is $6(m+n)-1+2,\ =6(m+n)+1$; viz. this is greater by unity than the number of constants in the curve $(D=1)$ of the order $2(m+n)$. The explanation of the excess is that the same curve of the order $2(m+n)$ may be derived from the different quartic curves $(1,\ u)^2(1,\ v)^2=0$; this will be further examined.

The transition from the one form to the other is not immediately obvious; in fact, if from the quadriquadric equation $(1,\ u)^2(1,\ v)^2=0$ (say this is $A+2Bv+Cv^2=0$, where A, B, C are quadric functions of u) we determine v; this gives $Cv = -B+\sqrt{B^2-AC}$,

$= -B + \sqrt{\Omega}$ suppose; and then substituting in the equations $(x, y, z) = (1, u)^m (1, v)^n$, we find

$$(x, y, z) = (1, u)^m (C, -B + \sqrt{\Omega})^n;$$

viz. we have (x, y, z) proportional to functions of u, involving the quartic radical $\sqrt{\Omega}$; but these functions are of the order (not $m+n$, but) $m+2n$.

In particular, if $n = m$, then, instead of functions of the order $2n$, we have functions of the order $3n$. The reduction in this last case to the form where the order is $2n$ can be effected without difficulty, but in the general case where m and n are unequal, I do not know how it is to be effected except by the general process explained in Clebsch's Memoir.

We may, in fact, by linear transformations on u, v, reduce the quadriquadric relation to

$$\begin{array}{lll} 1 & & + u^2 \\ & + 2buv & \\ + v^2 & & + cu^2v^2 = 0, \end{array}$$

that is

$$1 + u^2 + 2buv + v^2 + cu^2v^2 = 0;$$

or putting herein $u + v = p$, $uv = q$, the relation is

$$1 + p^2 - 2q + 2bq + cq^2 = 0,$$

that is

$$\begin{aligned} p^2 &= -1 + (\ \ 2 - 2b)\, q - cq^2, \\ p^2 - 4q &= -1 + (-2 - 2b)\, q - cq^2; \end{aligned}$$

viz. extracting the square roots,

$$u + v = \sqrt{Q}, \quad u - v = \sqrt{Q'},$$

if for shortness

$$\begin{aligned} Q &= -1 + (\ \ 2 - 2b)\, q - cq^2, \\ Q' &= -1 + (-2 - 2b)\, q - cq^2; \end{aligned}$$

we may then rationalise one of the radicals, for instance, Q; viz. writing

$$-c\{-1 + (2 - 2b)\, q - cq^2\} = \{cq - (1-b)\}^2 + c - (1-b)^2,$$

then, if

$$cq - (1-b) = \sqrt{c - (1-b)^2} \cdot \tfrac{1}{2}\left(\theta - \frac{1}{\theta}\right),$$

this becomes

$$\begin{aligned} -cQ &= \left\{c - (1-b)^2\right\} \left\{\tfrac{1}{4}\left(\theta - \frac{1}{\theta}\right)^2 + 1\right\} \\ &= \left\{c - (1-b)^2\right\} \cdot \tfrac{1}{4}\left(\theta + \frac{1}{\theta}\right)^2, \end{aligned}$$

that is

$$\sqrt{Q} = \tfrac{1}{2}\sqrt{\left(\frac{c-(1-b)^2}{-c}\right)\left(\theta+\frac{1}{\theta}\right)};$$

and the corresponding value of $\sqrt{Q'}$ is

$$\sqrt{Q'} = \sqrt{\left\{\tfrac{1}{4}\frac{c-(1-b)^2}{-c}\left(\theta+\frac{1}{\theta}\right)^2 - 4q\right\}},$$

where q stands for its value

$$\frac{1}{c}\left\{1-b+\sqrt{c-(1-b)^2}\,.\,\tfrac{1}{2}\left(\theta-\frac{1}{\theta}\right)\right\}.$$

We may write these in the form

$$1 : \tfrac{1}{2}\sqrt{Q} : \tfrac{1}{2}\sqrt{Q'} = M\theta : 1+\theta^2 : \sqrt{\Theta},$$

where M is a constant, and Θ is a quartic function of θ, such that $(1+\theta^2)^2 - \Theta$ is a quadric function only of θ.

The equations

$$(x,\ y,\ z) = (1,\ u)^m\,(1,\ v)^n$$

thus assume the form

$$(x,\ y,\ z) = (M\theta,\ 1+\theta^2+\sqrt{\Theta})^m\,(M\theta,\ 1+\theta^2-\sqrt{\Theta})^n;$$

and on the right-hand side the term of the highest order in θ is

$$(1+\theta^2+\sqrt{\Theta})^m\,(1+\theta^2-\sqrt{\Theta})^n,$$

viz. if $n =$ or $> m$, then this is

$$\{(1+\theta^2)^2-\Theta\}^m\,(1+\theta^2-\sqrt{\Theta})^{n-m}.$$

This is

$$= (1,\ \theta)^{2m}\,(1+\theta^2-\sqrt{\Theta})^{n-m},$$

which is of the order $2m+2(n-m), = 2n$ (which, in virtue of $n =$ or $> m$, is $=$ or $> m+n$). In particular, if $n = m$, then the highest order is $= 2n$; or the curve of the order $2n$, as represented by the equations

$$(x,\ y,\ z) = (1,\ u)^n\,(1,\ v)^n,$$

where $(u,\ v)$ are connected by a quadriquadric equation, is also represented by the equations

$$(x,\ y,\ z) = (1,\ \theta)^n + (1,\ \theta)^{n-2}\sqrt{\Theta},$$

which is the required transformation of the original equations.

It is to be noticed that the foregoing form,

$$\begin{array}{lll} 1 & & +u^2 \\ & +2buv & \\ +v^2 & & +cu^2v^2 = 0, \end{array}$$

is the most special form to which the quadriquadric relation can be reduced by the linear transformations of u, v; in fact, by mere division, the equation is made to have the constant term 1; the number of the coefficients of transformation is then $3+3, =6$; and to reduce the relation to the foregoing form, we have the six conditions, coeff. $u=0$, coeff. $v=0$, coeff. $vu^2=0$, coeff. $v^2u=0$, coeff. $u^2=1$, coeff. $v^2=1$; but in this form, expressing v in terms of u, the radical is $\sqrt{\Omega}$, where

$$\Omega=(1+u^2)(1+cu^2)-b^2u^2,$$

which is not more general than if we had

$$\Omega=(1+u^2)(1+cu^2),$$

viz. there is a superfluous constant b. And we thus see how it is that the system of equations

$$(x,\ y,\ z)=(1,\ u)^n(1,\ v)^n,$$

$(u,\ v)$ connected by a quadriquadric relation, contains, not $6n$, but $6n+1$ constants, one of these being superfluous.

Clebsch's transformation, above referred to, is as follows:

Starting with the equations

$$(x,\ y,\ z)=(1,\ \theta)^{n+k}+(1,\ \theta)^{n+k-2}\sqrt{\Theta},$$

if the function Θ is not originally of the standard form, we may, by a linear substitution, reduce it to this form, viz. we may write

$$\Theta=\theta(1-\theta)(1-k^2\theta);$$

and then writing $\theta=\operatorname{sn}^2 u$, $(\sin^2 \operatorname{am} u)$, we have

$$\begin{aligned}\sqrt{\Theta}&=\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\\&=\operatorname{sn} u \operatorname{sn}' u;\end{aligned}$$

so that the formulæ become

$$(x,\ y,\ z)=(1,\ \operatorname{sn}^2 u)^{n+k}+(1,\ \operatorname{sn}^2 u)^{n+k-2}\operatorname{sn} u \operatorname{sn}' u.$$

Hermite's theorem, used in the demonstration, is that any such function of $\operatorname{sn} u$ is expressible in the form

$$C\frac{H(u-\alpha_1)H(u-\alpha_2)\ldots H(u-\alpha_{2n+2k})}{\Theta^{2n+2k}(u)}$$

(H, Θ denoting here the two Jacobian functions), where

$$\alpha_1+\alpha_2\ldots+\alpha_{2n+2k}=0.$$

{Observe, in passing, that the equation

$$0=(1,\ \operatorname{sn}^2 u)^{n+k}+(1,\ \operatorname{sn}^2 u)^{n+k-2}\operatorname{sn} u \operatorname{sn}' u$$

gives, when rationalised, an equation in $\mathrm{sn}^2 u$ of the order $2n+2k$; the roots of this equation are $\mathrm{sn}^2 \alpha_1$, $\mathrm{sn}^2 \alpha_2 \ldots \mathrm{sn}^2 \alpha_{2n+2k}$. Considering the functions $(1, \mathrm{sn}^2 u)^{n+k}$ and $(1, \mathrm{sn}^2 u)^{n+k-2}$ as indeterminate, the coefficients can be found so that all but one of the roots of the equation in $\mathrm{sn}^2 u$ shall have any given values whatever, $\mathrm{sn}^2 \alpha_1$, $\mathrm{sn}^2 \alpha_2, \ldots \mathrm{sn}^2 \alpha_{2n+2k-1}$; the theorem then shows that the remaining root is $\mathrm{sn}^2 \alpha_{2n+2k}$, where

$$-\alpha_{2n+2k} = \alpha_1 + \alpha_2 \ldots + \alpha_{2n+2k-1},$$

which is, in fact, Abel's theorem.}

Now, supposing that the three functions of θ all vanish for $2k$ common values of θ, each of the functions of u will contain the same $2k$ H functions, say these are $H(u-\alpha_{2n+1}) \ldots H(u-\alpha_{2n+2k})$. Omitting these and also the denominator factor $\Theta^{2k}(u)$, we have the set of equations

$$(x, y, z) = C \frac{H(u-\alpha_1) H(u-\alpha_2) \ldots H(u-\alpha_{2n})}{\Theta^{2n}(u)},$$

where, however,

$$\alpha_1 + \alpha_2 \ldots + \alpha_{2n} \neq 0,$$

{the values $\alpha_1, \alpha_2 \ldots \alpha_{2n}$ are of course different for the three coordinates x, a, z respectively}; viz. we have

$$\begin{aligned} \alpha_1 + \alpha_2 \ldots + \alpha_{2n} &= -(\alpha_{2n+1} + \ldots + \alpha_{2n+2k}) \\ &= 2ns \text{ suppose.} \end{aligned}$$

Writing then

$$u = u' + s, \quad \alpha_1 = \alpha_1' + s, \quad \alpha_2 = \alpha_2' + s, \ldots \alpha'_{2n} = \alpha'_{2n} + s,$$

and consequently

$$\alpha_1' + \alpha_2' + \ldots + \alpha'_{2n} = 0,$$

we have

$$(x, y, z) = C \frac{H(u'-\alpha_1') \ldots H(u'-\alpha'_{2n})}{\Theta^{2n}(u'+s)};$$

or, changing the common denominator,

$$(x, y, z) = C \frac{H(u'-\alpha_1') \ldots H(u'-\alpha'_{2n})}{\Theta^{2n}(u')},$$

where

$$\alpha_1' + \alpha_2' \ldots + \alpha'_{2n} = 0;$$

or, what is the same thing,

$$(x, y, z) = (1, \mathrm{sn}^2 u')^n + (1, \mathrm{sn}^2 u')^{n-2} \mathrm{sn}\, u' \,\mathrm{sn}'.\, u';$$

viz. writing $\mathrm{sn}\, u' = \theta'$, and $\Theta' = \theta'(1-\theta')(1-k^2\theta')$, this is

$$(x, y, z) = (1, \theta')^n + (1, \theta')^{n-2} \sqrt{\Theta'},$$

a normal representation of the curve of the order $2n$.

The relation between the parameters θ, θ' is given by $\theta' = \mathrm{sn}^2(u-s)$, $\mathrm{sn}^2 u = \theta$, that is, we have

$$\sqrt{\theta'} = \frac{\sqrt{\theta}\,\mathrm{cn}\,s\,\mathrm{dn}\,s - \mathrm{sn}\,s\,\sqrt{(1-\theta)(1-k^2\theta)}}{1-k^2\,\mathrm{sn}^2 s\,.\,\theta};$$

or, writing $\mathrm{sn}^2 s = \sigma$, this is

$$\theta' = \frac{\sqrt{\theta}\,\sqrt{(1-\sigma)(1-k^2\sigma)} - \sqrt{\sigma}\,\sqrt{(1-\theta)(1-k^2\theta)}}{1-k^2\sigma\theta};$$

and, conversely, we have

$$\theta = \frac{\sqrt{\theta'}\,\sqrt{(1-\sigma)(1-k^2\sigma)} + \sqrt{\sigma}\,\sqrt{(1-\theta')(1-k^2\theta')}}{1-k^2\sigma\theta'},$$

and the theorem, in fact, shows that, substituting this value of θ in the functions of θ which serve to express x, y, z, these become *proportional* to the functions of θ'; viz. they become equal to these functions, each multiplied by an irrational function $A' + B'\sqrt{\Theta'}$, (A', B' rational functions of θ').

511.

ADDITION TO THE MEMOIR ON GEODESIC LINES, IN PARTICULAR THOSE OF A QUADRIC SURFACE (509). [1]

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 368—380. Read June 12, 1873.]

38. IN the Memoir above referred to, speaking of the geodesic lines on the skew hyperboloid, I say (No. 35), "The geodesic of initial direction $M1$ touches at M the oval curve of curvature $M1$, and lies wholly above this curve; *it makes an infinity of convolutions round the upper part of the hyperboloid*, cutting all the oval curves of curvature for which p has a positive value greater than p_1 (if p_1 is the value of p corresponding to the oval curve through M), and ascending to infinity." The statement as to the infinity of convolutions is incorrect; I was led to it by the assumption that the geodesic could not touch any hyperbolic curve of curvature. The fact is, that it touches at infinity (has for asymptotes) in general two hyperbolic curves of curvature; viz. the geodesic descending from infinity in the direction of a hyperbolic curve of curvature, so as to touch the oval curve through M, again ascends to infinity in the direction of a hyperbolic curve of curvature (the same as the first-mentioned one, or a different curve), making in its whole course say k convolutions, where m is a positive finite number; if $k < 1$, there is no complete convolution, and when $k = 1$ or any integer number, then the two hyperbolic curves are one and the same curve; k is infinite only in the special case afterwards referred to in the same No. 35, where the oval curve of curvature is the ellipse which is a principal section of the hyperboloid, and does not even attain to the value 1 except for an oval curve exceedingly close to this ellipse. The error was on consideration obvious enough, though I was in fact led to perceive it by the numerical calculations about to be referred to, which gave me geodesics not making a complete convolution.

[1] The articles are numbered consecutively with those of the Memoir, (509).

39. I have effected, for a particular skew hyperboloid and oval curve of curvature thereof, the numerical calculations for laying down the geodesic lines which touch this curve of curvature. Taking in general the equation of the hyperboloid to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

and the θ-curve of curvature to be the intersection by the confocal surface

$$\frac{x^2}{a-\theta} + \frac{y^2}{b-\theta} + \frac{z^2}{c-\theta} = 1,$$

then the selected values for the hyperboloid, and oval curve of curvature touched by the geodesics, are

$$\begin{aligned} a &= 900, \\ b &= 400, \\ c &= -c' = -1600, \\ \theta &= -\theta' = -1650; \end{aligned}$$

so that a, b, c', θ' are the positive values 900, 400, 1600, and 1650 respectively. I recall that $p = c'$ to $p = \infty$ gives the oval curves of curvature, viz. $p = c'$, the elliptic principal section; $p = \theta'$, the given oval curve: and that we are in the sequel concerned only with the oval curves above this, for which p extends from θ' to ∞. Moreover, $q = -b$ to $-a$ gives the hyperbolic curves of curvature, viz. $q = -b$ the xz-principal section; and $q = -a$ the yz-principal section of the hyperboloid. We have, in fact, to deal with the integrals

$$\Pi(p) = \int_{\theta'} dp \sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}},$$

and

$$\Psi(q) = \int_{-b} dq \sqrt{\frac{q}{(q+a)(q+b)(q-c')(q-\theta')}};$$

or if $p = \theta' + u$, u extending from 0 towards ∞, then

$$\Pi(p) = \int_0^u du \sqrt{\frac{u+\theta'}{(u+a+\theta')(u+b+\theta')(u+\theta'-c')u}},$$

and so if $q = -b - v$, v extending from 0 towards $(a-b)$, which is its limit,

$$\Psi(q) = \int_0^v dv \sqrt{\frac{v+b}{(a-b-v)v(v+b+c')(v+b+\theta')}};$$

the relation for any particular geodesic of the series being

$$\pm \Pi(p) \pm \Psi(q) = C.$$

40. To avoid discontinuity as to sign, it is convenient to take the integral $\Psi(q)$ in a particular manner. The hyperboloid is by the xz- and yz-principal planes divided into four quadrants; or since we attend only to the upper half of the hyperboloid, say this upper half is thus divided into four quadrants, x to y, y to x', x' to y', and y' to x; or call them the first, second, third, and fourth quadrants. But we may consider the quadrants as forming an infinite succession, first, second, third, fourth, fifth,

and so on; or we may take them in the reverse order, -1, -2, -3, &c. For a hyperbolic curve $q=-b-v$ in the first quadrant the integral is to be taken $v=0$ to $v=v$; for a curve in the second quadrant $v=0$ to $a-b$, and thence positively $a-b$ to v; for a curve in the third quadrant $v=0$ to $a-b$, thence $a-b$ to 0, and thence 0 to v; and so on: and so for a point in the quadrant -1, the integral is from 0 to v, taken negatively, &c.; that is, as the hyperbolic curve travels from the xz-position in the positive direction, the integral $\Psi(q)$ continually increases from zero; and if the curve travels from the xz-position in the negative direction, then the integral $\Psi(q)$ continually decreases from zero; that is, it increases negatively. It is to be remarked that the integral $v=0$ to $a-b$ is finite, say it is $=K'$; and of course it is only thus far that the integral requires to be calculated, the subsequent values differing from the preceding ones only by multiples of this complete integral.

The integral $\Pi(p)$ requires no explanation; it is taken from $u=0$, giving a certain oval curve, up to any positive value of u, giving the oval curves above this one; and, in particular, taking the integral to $u=\infty$ (or, what is the same thing, to $p=\infty$) its value is finite, $=K$, suppose.

41. Consider the geodesic which touches the given oval curve at a point for which $\Psi(q)$ has a given value Q; at this point $p=\theta'$, or $\Pi(p)=0$; so that, taking for the equation of the geodesic $\Psi(q)\pm\Pi(p)=C$, we have $Q=C$, and consequently

$$\Psi(q)=Q\mp\Pi(p).$$

Taking the positive sign, then as p increases from θ', $\Psi(q)$ increases, or the describing point of the geodesic moves upwards from the point of contact in the direction of positive rotation; and taking the negative sign, then $\Psi(q)$ decreases, or the describing point of the geodesic moves upwards from the point of contact in the direction of negative rotation; and, in particular, p becoming infinite, then the first-mentioned branch touches at infinity the hyperbolic curve, for which q is such that $\Psi(q)=Q+K$, and the second branch that for which q is such that $\Psi(q)=Q-K$.

42. The graphical process is as follows: we describe, on the hyperboloid, a series of hyperbolic curves of curvature, numbering them according to the values of $\Psi(q)$; viz. considering the hyperbolic branches which form the xz, yz, $x'z$ and $y'z$ sections respectively, these are 0, K', $2K'$, $3K'$, and on going round a second time they would be $4K'$, $5K'$, $6K'$, $7K'$, and so on respectively. We similarly describe, say on the upper half of the hyperboloid, the oval curves of curvature, numbering them according to the values of $\Pi(p)$, viz. beginning with that for which $p=\theta'$, which is 0, we go successively up to the oval curve at infinity, which is K.

In the example, and drawing belonging thereto [1], where, for convenience, the values of the integrals have been multiplied by 100,000, we have, as will appear, $K=12490$, $K'=34726$: the two sets of curves are drawn at intervals of 2000; viz. we have the hyperbolic curves 0, 2000, 4000, ... 34000, 34726; and the oval curves 0, 2000, 4000, 6000, 8000, 10000; the distance of the successive oval curves increases very rapidly, since the curve at infinity would be K, $=12490$, and the curve $K=10000$ is the last which comes into the limits of the figure.

[1] This drawing was exhibited at the meeting of the Society.

43. The two sets of curves of curvature being thus drawn at equal intervals of $\Pi(p)$ and $\Psi(q)$ respectively, dividing the hyperboloid into quadrilateral spaces (which of course should theoretically be indefinitely small), the diagonals of these quadrilateral spaces are the elements of the geodesic lines; and by a series of such elements we have a particular geodesic line. The general character comes out in the drawing very distinctly; viz. the geodesic is a hyperbola-like curve descending from infinity to touch the oval curve, and again ascending to infinity; by reason of the small value of K in comparison of K', there is nothing like a complete convolution, but the whole curve is included within a quadrant of the hyperboloid.

44. I remark that the calculations were performed roughly. I made no attempt to estimate or allow for errors arising from the intervals being too great; and there are very probably accidental errors of calculation. But starting with the value 10411 of $\Pi(p)$ for $p=10000$, I found, with some care, superior and inferior limits of the remainder of the integral, $p=10000$ to $p=\infty$; and the process is, I think, an interesting one. Consider in general the integral

$$\begin{aligned} I = & \int_{\theta'}^{\infty} dp \sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}} \\ = & I_1 + \int_{p_1}^{\infty} dp \sqrt{\frac{p}{(p+a)(p+b)(p-c')(p-\theta')}}, \end{aligned}$$

where I_1 is the integral calculated up to a somewhat large value $p=p_1$.

Writing

$$\begin{aligned} \alpha &= \tfrac{1}{2}(a+b), \\ \beta &= \tfrac{1}{2}(c'+\theta'), \\ \gamma &= \tfrac{1}{2}(a+b)-m, \\ \delta &= \tfrac{1}{2}(c'+\theta')+n, \end{aligned}$$

where m and n are as yet undetermined, we have

$$\begin{aligned} (p+a)(p+b) &= (p+\alpha)^2 - \tfrac{1}{4}(a-b)^2, \quad > (p+\alpha)^2, \\ (p-c')(p-\theta') &= (p-\beta)^2 - \tfrac{1}{4}(\theta'-c')^2, \quad > (p-\beta)^2; \end{aligned}$$

and the integral is thus

$$> I_1 + \int_{p_1}^{\infty} dp \frac{\sqrt{p}}{(p+\alpha)(p-\beta)}.$$

But we may determine m and n, so that for $p=$ or $>p_1$,

$$\begin{aligned} (p+a)(p+b) &< (p+\gamma)^2, \\ (p-c')(p-\theta') &< (p-\delta)^2; \end{aligned}$$

and the integral is then

$$< I_1 + \int_{p_1}^{\infty} dp \frac{\sqrt{p}}{(p+\gamma)(p-\delta)}.$$

The determination thus depends on the last-mentioned integrals, the values of which are at once obtainable by writing therein $\sqrt{p}=x$; viz. we have

$$\int dp \frac{\sqrt{p}}{(p+\alpha)(p-\beta)} = 2\int \frac{x^2 dx}{(x^2+\alpha)(x^2-\beta)} = C + \frac{2}{\alpha+\beta}\left\{\tfrac{1}{2}\beta h.l\frac{x-\sqrt{\beta}}{x+\sqrt{\beta}}\right\} + \frac{2\sqrt{\alpha}}{\alpha+\beta}\tan^{-1}\frac{x}{\sqrt{\alpha}};$$

and hence, substituting in the formula, for $h.l.x$ its value $\frac{\log x}{\log e}$, the superior limit is

$$I_1 + \left\{\frac{\sqrt{\delta}}{\gamma+\delta}\frac{1}{\log e}\log\frac{\sqrt{p_1}+\sqrt{\delta}}{\sqrt{p_1}-\sqrt{\delta}} + \frac{2\sqrt{\gamma}}{\gamma+\delta}\cot^{-1}\frac{\sqrt{p_1}}{\sqrt{\gamma_1}}\right\},$$

and the inferior limit is

$$I_1 + \left\{\frac{\sqrt{\beta}}{\alpha+\beta}\frac{1}{\log e}\log\frac{\sqrt{p_1}+\sqrt{\beta}}{\sqrt{p_1}-\sqrt{\beta}} + \frac{2\sqrt{\alpha}}{\alpha+\beta}\cot^{-1}\frac{\sqrt{p_1}}{\sqrt{\alpha_1}}\right\}.$$

45. The numerical values are $p_1 = 10{,}000$; a, b, c', $\theta' = 900$, 400, 1600, 1650; and thence determining by trial values of m and n,

$$\alpha = 650, \qquad \gamma = 650 - 4 = 646,$$
$$\beta = 1625, \qquad \delta = 1625 + 160 = 1785,$$

I obtained for the logarithmic and circular terms of the two limits respectively

	Superior	Inferior
Logarithmic	·015668	·015144
Circular	·005202	·005593
	·020870	·020737

The value of I_1 was $10411 \div 100{,}000 = \cdot 104110$, and the two limits thus are ·124980 and ·124850; or restoring the factor 100,000, they are 12498 and 12485; the mean of these, say 12490, was taken for the value $\Pi(p)$, $p = \infty$; that is $K = 12490$.

46. As regards the calculation of the integrals $\Pi(p)$ and $\Psi(q)$, introducing the numerical values, and multiplying by the before-mentioned factor 100,000, we have $(q = -400 - v)$,

$$\Psi(q) = 100{,}000\int_0^v dv\sqrt{\frac{v+400}{(500-v)\,v\,(v+2000)\,(v+2050)}},$$

which for any small value of v is

$$= 100{,}000\sqrt{\frac{400}{500\,.\,2000\,.\,2050}}\left(\int\frac{dv}{\sqrt{v}}, = 2\sqrt{v}\right);$$

viz. this is

$$= 883{\cdot}45\,(\log = 2{\cdot}9461830)\sqrt{v},$$

which was used for the values $v = 1, 2, \ldots 10$, that is, to $q = -410$; after which the calculation was continued by quadratures giving to v the values 10, 20, 30, ... up to $v = 490$, or $q = -890$. For the remainder of the integral, writing $500 - v = w$ (that is, $q = -900 + w$), we have

$$\Psi(q) - \Psi(-890) = 100{,}000\int dw\sqrt{\frac{900-w}{w\,(500-w)\,(2500-w)\,(2550-w)}},$$

$$= 100{,}000\sqrt{\frac{900}{500\,.\,2500\,.\,2550}}\left\{\int\frac{dw}{\sqrt{w}}, = 2\,(\sqrt{10}-\sqrt{w})\right\}$$

$$= 1062{\cdot}7\,(\log = 3{\cdot}0264261)\,(\sqrt{10}-\sqrt{w}),$$

which was used for the values $w = 9, 8, 7, \ldots 1, 0$, to complete the calculation up to $q = -900$.

47. We have in like manner, $p = 1650 + u$,

$$\Pi(p) = 100000 \int_0^u du \sqrt{\frac{u + 1650}{(u + 2550)(u + 2050)(u + 50)u}},$$

which for small values of u is

$$= 100000 \sqrt{\frac{1650}{2550 \,.\, 2050 \,.\, 50}} \left(\int \frac{du}{\sqrt{u}}, = 2\sqrt{u} \right);$$

viz. this is

$$502{\cdot}5\,(\log = 2{\cdot}7011399)\sqrt{u},$$

used for $u = 1, 2, \ldots 10$, that is to $p = 1660$. The calculation was afterwards continued by quadratures, by giving to u a succession of values, at intervals at first of 10, and afterwards of 20, 50, 100, 200, and 500, up to $p = 10{,}000$, giving for the integral the value 10411; and thence, as appearing above, the value for $p = \infty$ was found to be $= 12490$.

48. After the calculation of the values of $\Pi(p)$ and $\Psi(q)$, it was easy by interpolation to revert these tables, so as to obtain a table which, for Π or Ψ as argument, gives the values of p and q. The arguments are taken at intervals of 500; up to 10000 as regards p, since the original table was only calculated thus far; and up to 34726 as regards q. I had thus calculated the annexed Table III., when it occurred to me that there was a convenience in taking the arguments to be submultiples of the complete integral 34726; say we divide this into 90 parts, or, as it were, graduate the quadrant of the hyperboloid by means of hyperbolic curves of curvature adapted for the geodesics in question. Taking every fifth part, or in fact dividing the quadrant into 18 parts, we have the Table IV.

49. It will be remembered that the foregoing results apply only to the geodesics which touch the oval curve of curvature $p = +1650$; for the geodesics touching any other oval curve of curvature, the values of the integrals, and the mutual distances of the curves of curvature used for tracing the geodesics, would be completely altered. But it is possible to derive some general conclusions as to the geodesics that touch a given oval curve of curvature.

Observe that the integral K' ($= 34726$ in the case considered) measures the quadrant of the hyperboloid; viz. $\Psi(q) = 0$, $\Psi(q) = K'$ determine two hyperbolic curves of curvature (principal sections), the mutual distance whereof is a quadrant. Each geodesic touches the given oval curve of curvature, and it touches at infinity the two hyperbolic curves $\Psi(q) = Q \pm K$ ($K = 12490$ in the case considered); viz. the distance of these in regard to the circuit of four quadrants, or say the amplitude of the geodesic, is measured by the ratio $\dfrac{K}{2K'}$.

50. Now it is easy to see that as the oval curve of curvature approaches the principal elliptic section, that is, as θ' approaches c' (or writing $\theta' = c' + m$, as m

diminishes towards zero), the integral K' alters its value only slowly, increasing towards a certain constant limit; but, contrariwise, K increases without limit, its value for any small value of m being of the form $A - B \log m$, $= \infty$ in the limit; wherefore, as m diminishes, the value of $\frac{K}{2K'}$, the amplitude of the geodesic, continually increases. If this is $= 1$, the geodesic touching at infinity a certain hyperbolic curve of curvature, in descending to touch the oval curve, makes round the hyperboloid a half-convolution, and then again ascends through another half-convolution to touch at infinity the same hyperbolic curve of curvature; viz. it makes in all one entire convolution, or say in descending it makes a half-convolution. But if $K \div 2K' = 2$, then the curve makes in descending a complete convolution; and so, if $K \div 2K' = 2s$, then the geodesic makes in descending s convolutions; and, as already mentioned, ultimately when $m = 0$ the geodesic makes an infinity of convolutions; that is, it never actually reaches the elliptic principal section, but has this line for an asymptote.

51. To sustain the foregoing statements, I write $\theta' (= c' + m) = 1600 + m$, and I consider the integral

$$K'_m = 100000 \int_0^\infty du \sqrt{\frac{u + 1600 + m}{(u + 2500 + m)(u + 2000 + m)(u + m) u}},$$

say for a moment this is

$$K'_m = \int_0^\infty U_m \, du.$$

Supposing m to be small, we divide the integral into two parts, say from 0 to α [where α, $=$ for example 50 or 100, is large in comparison with m, but small in comparison with the numbers (c', &c.), 1600, &c.], and from α to ∞. In the second part, the expression under the integral sign and the value of the integral varies slowly with m, and we may, as an approximation, write $m = 0$. We have thus

$$K_m = \int_0^\alpha U_m \, du + \int_\alpha^\infty U_0 \, du,$$

and the first part hereof is

$$= 100000 \sqrt{\left(\frac{1600}{2500 \,.\, 2000}\right)} \int_0^\alpha \frac{du}{\sqrt{u(u+m)}};$$

viz. the integral is here

$$h.l. \left\{u + \tfrac{1}{2} m + \sqrt{u(u+m)}\right\} = h.l. \frac{\alpha + \frac{1}{2} m + \sqrt{\alpha(\alpha + m)}}{\frac{1}{2} m} = h.l. \frac{4\alpha}{m} \text{ approximately};$$

or say this is

$$= \frac{1}{\log e} \log \frac{4\alpha}{m}.$$

The first term is thus

$$= 100000 \sqrt{\frac{1600}{2500 \,.\, 2000}} \frac{1}{\log e} \log \frac{4\alpha}{m},$$

which is

$$= 4119 \log \frac{4\alpha}{m},$$

or we have

$$K'_m = 4119 \log \frac{4\alpha}{m} + \int_\alpha^\infty U_0\, du.$$

We ought to have the same value of the integral, whatever, within proper limits, the assumed value of α may be. Taking, for instance, $\alpha = 50$ and $\alpha = 100$, we ought to have

$$K'_m = 4119 \log \frac{400}{m} + \int_{100}^\infty U_0\, du$$

$$= 4119 \log \frac{200}{m} + \int_{50}^\infty U_0\, du\,;$$

that is,

$$4119 \log 2 = \int_{50}^{100} U_0\, du.$$

In verification, I calculated the second side by quadratures; viz. for the values 50, 60, 70, 80, 90, 100, the values of U_0 are 35·532, 29·570, 25·311, 22·373, 19·632, 17·645; whence, adding the half sum of the extreme terms to the sum of the mean terms, and multiplying by 10, the value of the integral is $= 1234{\cdot}74$. The value of the left-hand side is $= 1239{\cdot}94$, which is a sufficient agreement.

52. Returning to the formula for K'_m, this may be written

$$K'_m = \left(4119 \log 400 + \int_{100}^\infty U_0\, du\right) - 4119 \log m.$$

I did not calculate the value of the integral in this formula, but determined the term in ( ) in such wise that the formula should be correct for the foregoing value $m = 50$; viz. the term thus is

$$= 12490 + 4119 \log 50 = 12490 + 6998,\ = 19488, \text{ or say } 19500\,;$$

we thus have

$$K_m = 19500 - 4119 \log m\,;$$

and we may roughly assume that, for any small value of m, K'_m has the same value as for $m = 50$; viz. we may write

$$K'_m = 34726, \text{ or say } = 35000.$$

We thus see how to give to m such a value that the quantity $\frac{K}{2K'}$, which is the number of convolutions of the geodesic, may have any given value; and, in particular, we see how exceedingly small m must be for any moderately large number of convolutions; for instance, $m = \frac{1}{100{,}000000}$ or $\log m = -8$, $K = 19500 + 32952$, = say 52500, or the number is $= \frac{525}{100}$, about five-sevenths of a convolution.

CORRECTION. Instead of speaking, as above, of a geodesic as touching at infinity a hyperbolic curve of curvature, the accurate expression is that the geodesic at infinity is parallel to a certain hyperbolic curve of curvature. The geodesic has, in fact, for asymptote the right line on the surface parallel at infinity to such curve of curvature. Added Dec. 1873.

TABLE I.

p	$\Pi(p)$
1650	000
1	502
2	711
3	870
4	1005
5	1124
6	1231
7	1329
8	1421
9	1507
1660	1589
70	2188
80	2605
90	2933
1700	3205
10	3438
20	3642
30	3824
40	3989
50	4130
60	4287
70	4414
80	4532
90	4642
1800	4745
20	4945
40	5106
60	5261
80	5403
1900	5533
20	5654
40	5766
60	5871
80	5970
2000	6063
50	6275
100	6462
150	6629
200	6779
250	6916
300	7041
350	7156
400	7262
450	7362

TABLE II.

q	$\Psi(q)$
− 400	000
1	883
2	1249
3	1530
4	1767
5	1975
6	2164
7	2337
8	2499
9	2650
410	2794
20	4016
30	4952
40	5752
50	6467
60	7124
70	7739
80	8321
90	8877
500	9413
10	9931
20	10436
30	10923
40	11406
50	11880
60	12347
70	12809
80	13266
90	13720
600	14171
10	14619
20	15067
30	15513
40	15960
50	16408
60	16857
70	17308
80	17762
90	18220
700	18682
10	19149
20	19623
30	20101
− 40	20588

TABLE I. (*continued*).

p	$\Pi(p)$
2500	7454
600	7625
700	7777
800	7913
900	8037
3000	8151
200	8352
400	8526
600	8679
800	8815
4000	8936
500	9216
5000	9423
500	9593
6000	9737
500	9861
7000	9970
500	10066
8000	10152
500	10229
9000	10299
500	10363
10000	10411
⋮	
∞	12490

TABLE II. (*continued*).

q	$\Psi(q)$
− 750	21086
60	21595
70	22117
80	22653
90	23206
800	23778
10	24374
20	24998
30	25655
40	26355
50	27105
60	27928
70	28861
80	29936
890	31365
1	31538
2	31720
3	31914
4	32123
5	32350
6	32600
7	32886
8	33224
9	33663
− 900	34726

TABLE III.

Π or Ψ	p	Diff.	q	Diff.
0	1650·0	1	− 400·0	·3
500	1651·0	3	400·3	1·0
1000	1654·0	5	401·3	1·6
1500	1659·0	7·8	402·9	2·2
2000	1666·8	10·7	405·1	2·9
2500	1677·5	14·9	408·0	3·7
3000	1692·4	20·6	411·7	4·0
3500	1713·0	27·8	415·7	4·3
4000	1740·8	36·6	420·0	5·2
4500	1777·4	49·4	425·2	5·4
5000	1826·8	68·1	430·6	6·2
5500	1894·9	91·5	436·8	6·6
6000	1986·4	124·6	− 443·4	7·1

TABLE III. (*continued*).

Π or Ψ	p	Diff.	q	Diff.
6500	2111	124·6	−450·5	7·1
7000	2283	172	458·1	7·6
7500	2527	244	466·1	8·0
8000	2870	353	474·5	8·4
8500	3285	415	483·2	8·7
9000	4114	829	492·3	9·1
9500	5226	1112	501·7	9·4
10000	7156	1930	511·3	9·6
500	.		521·1	9·8
11000	.		531·5	10·1
500	.		542·0	10·5
12000	.		552·5	10·8
500	∞		563·3	10·8
13000			574·2	10·9
500			585·1	10·9
14000			596·2	11·1
500			607·3	11·1
15000			618·5	11·2
500			629·7	11·2
16000			641·0	11·3
500			652·1	11·1
17000			663·2	11·1
500			674·2	11·0
18000			685·2	11·0
500			696·1	10·9
19000			706·8	10·7
500			717·4	10·6
20000			728·0	10·6
500			738·4	10·4
21000			748·3	9·9
500			758·1	9·8
22000			767·7	9·6
500			777·4	9·4
23000			786·4	9·0
500			795·1	8·7
24000			803·7	8·6
500			812·0	8·3
25000			820·0	8·0
500			827·6	7·6
26000			834·9	7·3
500			−841·9	7·0
				6·7

TABLE III. (*continued*).

Π or Ψ	p	Diff.	q	Diff.
27000			−848·6	6·7
500			854·8	6·2
28000			860·8	6·0
500			866·1	5·3
29000			871·3	5·2
500			876·0	4·7
30000			880·6	4·6
500			884·0	3·4
31000			887·5	3·5
500			890·8	3·3
32000			893·4	3·6
500			895·6	2·2
33000			897·5	1·7
500			898·6	1·1
34000			899·3	0·7
500			899·8	0·5
34726			−900	0·2

TABLE IV.

	Π or Ψ	p	Diff.	q	Diff.
0	0	1650		−400	
1	1929	1665·7	15·7	404·8	4·8
2	3858	1732·1	66·4	418·7	13·9
3	5788	1944·2	212·1	440·5	21·8
4	7717	2657·7	713·5	469·6	29·1
5	9646	5697·7	3040·0	504·5	34·9
6	11575			543·6	39·1
7	13504			585·2	41·6
8	15434			628·0	42·8
9	17363			671·2	43·2
10	19292			713·0	41·8
11	21221			752·7	39·7
12	23150			789·0	36·3
13	25079			821·2	32·2
14	27008			848·6	27·4
15	28938			870·7	22·1
16	30868			886·5	15·8
17	32797			896·7	10·2
18	34726			−900	3·3

512.

ON A CORRESPONDENCE OF POINTS IN RELATION TO TWO TETRAHEDRA.

[From the *Proceedings of the London Mathematical Society*, vol. IV. (1871—1873), pp. 396—404. Read June 12, 1873.]

THE following question has been considered by R. Sturm in an interesting paper, "Das Problem der Projectivität und seine Anwendung auf die Flächen zweiten Grades," *Math. Ann.*, t. I. (1870), pp. 533—574: Given *in plano* two groups of the same number (5, 6, or 7) of points, to find points P, P' homographically related to these two groups respectively; viz. the lines from P to the points of the first group and those from P' to the points of the second group are to be homographic pencils. In the present paper I require only a particular form of these results; viz. in each group two of the points are the circular points at infinity; or, disregarding these, we have two groups of 3, 4, or 5 points such that the points of the first group at P, and those of the second group at P', subtend equal angles. I give for this particular case an independent analytical investigation; but I will first state the results included in the more general ones obtained by Sturm.

If the points A, B, C at P and the points A', B', C' at P' subtend equal angles, then to any given position of the one point corresponds a single position of the other point; viz. the two points have a (1, 1) correspondence; the nature of this being, that to any line in the one figure corresponds in the other figure a quintic curve, having 6 dps.; viz. the three points, the two circular points at infinity I, J, and one other fixed point of that figure {say for the first figure this fixed point is (ABC)}.

If the points A, B, C, D at P and the points A', B', C', D' at P' subtend equal angles, then the locus of each point is a cubic curve; viz. the locus of P passes through A, B, C, D, I, J and the four fixed points (ABC), (ABD), (ACD), (BCD); and the like for the locus of P'.

Finally (although this is a theorem which I do not require), if the points A, B, C, D, E at P and the points A', B', C', D', E' at P' subtend equal angles, then there are three positions of each point.

The problem I propose to consider is: Given the tetrahedra $ABCD$ and $A'B'C'D'$, it is required in the planes ABC and $A'B'C'$ respectively to find the points P, P' such that A, B, C, D at P, and A', B', C', D' at P', subtend equal angles. I was led to this by the more general problem, which I do not at present discuss: Given the two tetrahedra, it is required to find the loci of the points P, P' such that A, B, C, D at P, and A', B', C', D' at P', subtend equal angles.

Here, drawing from D, D' the perpendiculars DK, $D'K'$ on the planes ABC and $A'B'C'$ respectively, we have A, B, C, K at P, and A', B', C', K' at P', subtending equal angles, and such that the distances PK and $P'K'$ are proportional to the heights of the tetrahedra (for the triangles PDK and $P'D'K'$ are obviously similar). The required points P, P' are each the intersection of two loci, viz.:

1. P is such that A, B, C, K at P, and A', B', C', K' at P', subtend equal angles; locus is a cubic through A, B, C, K, I, J, (ABC), (ABK), (ACK), (BCK).
2. P is such that A, B, K at P, and A', B', K' at P', subtend equal angles, and that PK and $P'K'$ are in a given ratio; locus is a certain octic curve Ω;

and the required positions of P are obtained as the intersections of the two loci.

I proceed to the analytical investigation.

Preliminary Formulæ.

1. Consider a triangle ABC, and let the position of a point P be determined by means of its coordinates x, y, z, which are equal to the perpendicular distances of P from the sides, each divided by the perpendicular distance of the opposite vertex (as usual, x, y, z are positive for a point within the triangle); or what is the same thing, x, y, $z = PBC$, PCA, PAB, divided each by ABC, whence identically $x+y+z=1$.

Suppose for a moment the rectangular coordinates of A, B, C are (α_1, β_1), (α_2, β_2), (α_3, β_3) respectively; and that those of P are X, Y. Also let the sides BC, CA, AB be $= a$, b, c respectively.

We have

$$\begin{aligned} X &= \alpha_1 x + \alpha_2 y + \alpha_3 z, \\ Y &= \beta_1 x + \beta_2 y + \beta_3 z, \\ 1 &= x + y + z; \end{aligned}$$

and if we consider a second point P', the coordinates of which are x', y', z' and X', Y', we have the like relations between these quantities. Calling δ the distance of the two points P, P', we may write

$$\begin{aligned}\delta^2 = \quad & \{\alpha_1 (x-x') + \alpha_2 (y-y') + \alpha_3 (z-z')\}^2 \\ & + \{\beta_1 (x-x') + \beta_2 (y-y') + \beta_3 (z-z')\}^2 \\ & - \{(\alpha_1^2+\beta_1^2)(x-x') + (\alpha_2^2+\beta_2^2)(y-y') + (\alpha_3^2+\beta_3^2)(z-z')\} \\ & \qquad \times \{(x-x') + \qquad (y-y') + \qquad (z-z')\},\end{aligned}$$

the last term being in fact $=0$; viz. this is

$$\begin{aligned}\delta^2 = & -\{(\alpha_2-\alpha_3)^2 + (\beta_2-\beta_3)^2\}(y-y')(z-z') \\ & -\{(\alpha_3-\alpha_1)^2 + (\beta_3-\beta_1)^2\}(z-z')(x-x') \\ & -\{(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\}(x-x')(y-y');\end{aligned}$$

or what is the same thing, the expression for the distance δ^2 of the two points P, P' is

$$\delta^2 = -a^2(y-y')(z-z') - b^2(z-z')(x-x') - c^2(x-x')(y-y'),$$

which expression may be modified by means of the identical equations

$$1 = x+y+z, \quad 1 = x'+y'+z';$$

viz. writing

$$yz'-y'z,\ zx'-z'x,\ xy'-x'y = \xi,\ \eta,\ \zeta,$$

we have

$$\begin{aligned} x-x' &= x(x'+y'+z') - x'(x+y+z) = \zeta - \eta, \\ y-y' & \qquad\qquad\qquad\qquad\quad = \xi - \zeta, \\ z-z' & \qquad\qquad\qquad\qquad\quad = \eta - \xi;\end{aligned}$$

and consequently

$$\begin{aligned}\delta^2 = \quad & a^2(-\xi^2 - \eta\zeta + \zeta\xi + \xi\eta) \\ & + b^2(-\eta^2 + \eta\zeta - \zeta\xi + \xi\eta) \\ & + c^2(-\zeta^2 + \eta\zeta + \zeta\xi - \xi\eta).\end{aligned}$$

2. Treating x', y', z' as constants and x, y, z as current coordinates, the formula for δ^2 is of course the equation of a circle, centre x', y', z' and radius δ. It thus also appears that the general equation of a circle is

$$-a^2yz - b^2zx - c^2xy + (Lx+My+Nz)(x+y+z) = 0;$$

viz. writing $-a^2yz - b^2zx - c^2xy = U$, and $x+y+z = \Omega$, this is

$$U + (Lx+My+Nz)\,\Omega = 0,$$

where $U=0$ is the circle circumscribed about the triangle ABC, and $\Omega = 0$ is the line infinity. Of course the general equation of a circle passing through the points

(B, C) is $U + Lx\,\Omega = 0$, and similarly those of circles through (C, A) and through (A, B) are $U + My\,\Omega = 0$ and $U + Nz\,\Omega = 0$ respectively. But we require the interpretation of the coefficients L, M, N which enter into these equations.

3. Considering the triangle ABC, if through B, C we have a circle, this is by the side BC divided into two segments, and I consider that lying on the same side with A as the positive segment, and define the angle of the circle to be the angle in this positive segment. It is clear that if we have *within* the triangle a point P, and, through this point and (B, C), (C, A), (A, B) respectively, three circles, then if α, β, γ be the angles of these circles, we have $\alpha + \beta + \gamma = 2\pi$; and conversely, if the circles through (B, C), (C, A), (A, B) are such that their angles α, β, γ satisfy the relation $\alpha + \beta + \gamma = 2\pi$, then the three circles meet in a point. But it is further to be noticed, that if, producing the sides of the triangle so as to divide the plane into seven spaces, the triangle, three trilaterals, and three bilaterals, we take the point P within one of the bilaterals, we still have $\alpha + \beta + \gamma = 2\pi$; but taking it within one of the trilaterals, we have $\alpha + \beta + \gamma = \pi$. And the converse theorem is, that if the three circles (B, C), (C, A), (A, B) are such that $\alpha + \beta + \gamma = \pi$ or 2π, then the circles meet in a point; viz. if the sum is 2π, then this point lies in the triangle or one of the bilaterals; but if the sum is $= \pi$, then this point lies in a trilateral.

4. I seek for the equation of a circle through the points B, C, and containing the angle L. The equation in rectangular coordinates is easily seen to be

$$(X - \alpha_2)(X - \alpha_3) + (Y - \beta_2)(Y - \beta_3) - \cot L\,\{(\beta_2 - \beta_3)\,X - (\alpha_2 - \alpha_3)\,Y + \alpha_2\beta_3 - \alpha_3\beta_2\} = 0.$$

In fact this is the equation of a circle through (B, C); and taking for a moment the origin at B, and axis of X to coincide with BC, or writing α_2, $\beta_2 = 0$, 0; α_3, $\beta_3 = a$, 0, the equation is

$$X(X - a) + Y^2 - aY \cot L = 0,$$

viz. the equation of the tangent at B is $-aX - aY \cot L = 0$, that is, $Y = -X \tan L$, or the angle in the positive segment is $= L$.

If for a moment λ, μ, ν are the inclinations of the sides of the triangle ABC to the axis of X, then A, B, C being the angles, we may write

$$\begin{aligned} \mu - \nu &= \pi - A, \\ \nu - \lambda &= \pi - B, \\ \lambda - \mu &= -\pi - C, \end{aligned}$$

and

$$\begin{aligned} X - \alpha_2 &= (\alpha_1 x + \alpha_2 y + \alpha_3 z) - \alpha_2 (x + y + z) = c \cos \nu \,.\, x - a \cos \lambda \,.\, z, \\ X - \alpha_3 &= (\alpha_1 x + \alpha_2 y + \alpha_3 z) - \alpha_3 (x + y + z) = -b \cos \mu \,.\, x + a \cos \lambda \,.\, y, \\ Y - \beta_2 &= \beta_1 x + \beta_2 y + \beta_3 z - \beta_2 (x + y + z) = c \sin \nu \,.\, x - a \sin \lambda \,.\, z, \\ Y - \beta_3 &= \beta_1 x + \beta_2 y + \beta_3 z - \beta_3 (x + y + z) = -b \sin \nu \,.\, x + a \sin \lambda \,.\, z; \end{aligned}$$

whence

$$\begin{aligned} &(X - \alpha_2)(X - \alpha_3) + (Y - \beta_2)(Y - \beta_3) \\ &= -a^2 yz - b^2 zx - c^2 xy + bc \cos A \,.\, x^2 + (b^2 - ab \cos C)\, zx + (c^2 - ac \cos B)\, xy; \end{aligned}$$

viz. this is

$$= -a^2yz - b^2zx - c^2xy + bc\cos A \,.\, x(x+y+z).$$

Moreover, if $\Delta =$ twice the area of the triangle, then

$$(\beta_2 - \beta_3)X - (\alpha_2 - \alpha_3)Y + \alpha_2\beta_3 - \alpha_3\beta_2 = \Delta x(x+y+z) = bc\sin A \,.\, x(x+y+z);$$

so that the equation becomes

$$-a^2yz - b^2zx - c^2xy + bc\sin A\,(\cot A - \cot L)\,x(x+y+z) = 0,$$

or, what is the same thing,

$$-a^2yz - b^2zx - c^2xy + \Delta\,(\cot A - \cot L)\,x(x+y+z) = 0,$$

or, if we please,

$$-a^2yz - b^2zx - c^2xy + \Delta\,(\cot A - \cot L)\,x = 0.$$

Writing as before,

$$-a^2yz - b^2zx - c^2xy = U, \qquad x+y+z = \Omega,$$

the equation is

$$U + \Delta\,(\cot A - \cot L)\,\Omega x = 0;$$

or forming the like equations of two other similar circles, we have the circles (B, C), (C, A), (A, B) containing the angles L, M, N respectively; and the equations are

$$U + \Delta\,(\cot A - \cot L)\,\Omega x = 0,$$
$$U + \Delta\,(\cot B - \cot M)\,\Omega y = 0,$$
$$U + \Delta\,(\cot C - \cot N)\,\Omega z = 0.$$

Correspondence, A, B, C at P, and A', B', C' at P', subtending equal angles.

5. Consider now the two figures A', B', C', subtending at P' the same angles L, M, N which A, B, C subtend at P; then we have

$$\frac{U}{\Omega\Delta x} + \cot A - \cot L = 0, \quad \frac{U'}{\Omega'\Delta' x'} + \cot A' - \cot L = 0,$$

$$\frac{U}{\Omega\Delta y} + \cot B - \cot M = 0, \quad \frac{U'}{\Omega'\Delta' y'} + \cot B' - \cot M = 0,$$

$$\frac{U}{\Omega\Delta z} + \cot C - \cot N = 0, \quad \frac{U'}{\Omega'\Delta' z'} + \cot C' - \cot N = 0;$$

and thence

$$\frac{U}{\Omega\Delta x} + \cot A = \frac{U'}{\Omega'\Delta' x'} + \cot A',$$

$$\frac{U}{\Omega\Delta y} + \cot B = \frac{U'}{\Omega'\Delta' y'} + \cot B',$$

$$\frac{U}{\Omega\Delta z} + \cot C = \frac{U'}{\Omega'\Delta' z'} + \cot C';$$

and consequently

$$\frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'} = \frac{U}{\Omega\Delta x} + \cot A - \cot A'$$

$$: \frac{U}{\Omega\Delta y} + \cot B - \cot B'$$

$$: \frac{U}{\Omega\Delta z} + \cot C - \cot C',$$

or, what is the same thing,

$$x' : y' : z' = \frac{x}{U + \Delta(\cot A - \cot A')\,\Omega x}$$

$$: \frac{y}{U + \Delta(\cot B - \cot B')\,\Omega y}$$

$$: \frac{z}{U + \Delta(\cot C - \cot C')\,\Omega z},$$

where observe that the equations

$$U + \Delta(\cot A - \cot A')\,\Omega x = 0,$$
$$U + \Delta(\cot B - \cot B')\,\Omega y = 0,$$
$$U + \Delta(\cot C - \cot C')\,\Omega z = 0,$$

represent circles (B, C), (C, A), (A, B) containing the angles A', B', C'; and since $A' + B' + C' = \pi$, these meet in a point O. We may for convenience write

$$x' : y' : z' = \frac{BC}{BCO} : \frac{CA}{CAO} : \frac{AB}{ABO},$$

where $BC = 0$ denotes $(x = 0)$ the line BC; $BCO = 0$ the circle through B, C, O. And of course, in like manner,

$$x : y : z = \frac{B'C'}{B'C'O'} : \frac{C'A'}{C'A'O'} : \frac{A'B'}{A'B'O'};$$

so that the points P, P' have a rational, or (1, 1), correspondence.

Writing

$$\begin{array}{ccccccc} x' : y' : z' = & BC.CAO.ABO & : & CA.ABO.BCO & : & AB.BCO.CAO \\ = & X & : & Y & : & Z \end{array}$$

suppose, X, Y, Z are quintic functions of x, y, z, and the curve in the first figure corresponding to the line $\alpha x' + \beta y' + \gamma z' = 0$ of the second figure is

$$\alpha X + \beta Y + \gamma Z = 0;$$

viz. this is a quintic curve having dps. at each of the points A, B, C, O, I, J. In fact, if for BCO we write $BCOIJ$, and so for the other two circles respectively, we have in an algorithm which will be at once understood $X = BC.CAOIJ.ABOIJ, = (ABCOIJ)^2$, and similarly $Y = Z, = (ABCOIJ)^2$, or the curve is $(ABCOIJ)^2 = 0$.

Correspondence, A, B, C, D at P and A', B', C', D' at P' subtending equal angles.

6. Consider now *in plano* the points A, B, C, D which at P, and the points A', B', C', D' which at P', subtend equal angles. Let a, b, c, f, g, h denote the perpendicular distances of P from the lines BC, CA, AB, AD, BD, CD respectively; and the like as to a', b', c', f', g', h'. Observe that, neglecting constant factors, a, b, c are what were before represented by x, y, z; we may consider the coordinates of P in regard to the triangles ABC, BCD, CAD, ABD to be (a, b, c), (a, h, g), (b, f, h), (c, g, f) respectively. We have in regard to ABC the point O as before, and in regard to BCD, CAD, ABD the points O_1, O_2, O_3 respectively. Then A, B, C at P and A', B', C' at P' subtending equal angles, we may write

$$a' : b' : c' = \frac{a}{BCO} : \frac{b}{CAO} : \frac{c}{ABO};$$

viz. $BCO = 0$ is here the circle through B, C, O, and the like for CAO and ABO, the expressions being multiplied into the proper constant factors to take account of the constant factors whereby a, b, c and a', b', c' differ from x, y, z and x', y', z' respectively.

We have in like manner

$$a' : h' : g' = \frac{a}{BCO_1} : \frac{h}{CDO_1} : \frac{g}{BDO_1},$$

$$b' : f' : h' = \frac{b}{CAO_2} : \frac{f}{ADO_2} : \frac{h}{CDO_2},$$

$$c' : g' : f' = \frac{c}{ABO_3} : \frac{g}{BDO_3} : \frac{f}{ADO_3}.$$

From the ratios of (f', g', h'), (b', c', f'), (c', a', g'), (a', b', h') respectively we deduce

$$CDO_1 \,.\, ADO_2 \,.\, BDO_3 - BDO_1 \,.\, CDO_2 \,.\, ADO_3 = 0,$$

$$CAO \,.\, ABO_3 \,.\, ADO_2 - ABO \,.\, ADO_3 \,.\, CAO_2 = 0,$$

$$ABO \,.\, BCO_1 \,.\, BDO_3 - BCO \,.\, BDO_1 \,.\, ABO_3 = 0,$$

$$BCO \,.\, CAO_3 \,.\, CDO_1 - CAO \,.\, CDO_2 \,.\, BCO_1 = 0,$$

each of which equations represents a sextic curve; and admitting that it can be shown that these pass through O, O_1, O_2, O_3 respectively, the forms are

$$D^3 A\, B\, C\, O\, O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A^3 B\, C\, O\, O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A\, B^3 C\, O\, O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A\, B\, C^3 O\, O_1 O_2 O_3 I^3 J^3 = 0.$$

7. Now the locus of P is evidently a curve, and this can only happen by reason that the four left-hand functions contain a common factor, and the form of them suggests that this common factor is $ABCDOO_1O_2O_3IJ$, the four extraneous factors being $D^2I^2J^2$, $A^2I^2J^2$, $B^2I^2J^2$, $C^2I^2J^2$; viz. $ABCDOO_1O_2O_3IJ=0$ is a cubic curve passing through the ten points; and $D^2I^2J^2=0$ a cubic curve through each of the points D, I, J twice; viz. it is the triad of lines IJ, DI, DJ; and the like as to the other extraneous factors $A^2I^2J^2$, $B^2I^2J^2$, and $C^2I^2J^2$. I have not worked out the analysis to verify this *à posteriori*; but, the conclusion agreeing with Sturm, I accept it without further investigation, viz. the result is that A, B, C, D at P and A', B', C', D' at P' subtending equal angles, the locus of P is a cubic curve $ABCDOO_1O_2O_3O_4IJ=0$ through the ten points thus represented; and of course the locus of P' is in like manner a cubic curve $A'B'C'D'O'O_1'O_2'O_3'O_4'IJ=0$ through the ten points thus represented.

Correspondence, A, B, C, D, E at P and A', B', C', D', E' at P' subtending equal angles.

8. We may go a step further, and consider A, B, C, D, E at P and A', B', C', D', E' at P' subtending equal angles. Attending only to the points A, B, C, D and A', B', C', D', the locus of P is a cubic curve

$$ABCDOO_1O_2O_3O_4IJ=0;$$

and similarly attending to the points A, B, C, E and A', B', C', E', the locus of P is a cubic curve

$$ABCEOQ_1Q_2Q_3IJ=0.$$

(Observe that O, as depending only on A, B, C, is the same point as before; but that Q_1, Q_2, Q_3, as depending on E instead of D, are not the same as O_1, O_2, O_3.) The two cubic curves have in common the points A, B, C, I, J, O, and they consequently intersect in three other points; that is, there are three positions of the point P, and of course three corresponding positions of P'.

Correspondence, A, B, C at P and A', B', C' at P' subtending equal angles, and AP, $A'P'$ in a given ratio.

9. Consider, as before, A, B, C at P and A', B', C' at P' subtending equal angles, and the points P, P' being moreover such that the distances AP, $A'P'$ are in a given ratio. I write for shortness

$$x' : y' : z' = \frac{x}{L} : \frac{y}{M} : \frac{z}{N},$$

where L, M, N denote

$$U+\Delta(\cot A-\cot A')\,\Omega x,\quad U+\Delta(\cot B-\cot B')\,\Omega y,\quad U+\Delta(\cot C-\cot C')\,\Omega z,$$

respectively. We have

$$\begin{aligned}(AP)^2 &= -a^2yz-b^2z(x-1)-c^2(x-1)y,\\ &= -a^2yz-b^2zx-c^2xy+(b^2z+c^2y)(x+y+z),\\ &= \quad c^2y^2+b^2z^2+(b^2+c^2-a^2)yz;\end{aligned}$$

or, what is the same thing,

$$(A\,P\,)^2 = c^2y^2 + b^2z^2 + 2bc\ \cos A\ .\ yz;$$

and similarly

$$(A'P')^2 = c'^2y'^2 + b'^2z'^2 + 2b'c'\cos A'\,.\,y'z'.$$

The required relation therefore is

$$\frac{c'^2y'^2 + b'^2z'^2 + 2b'c'\cos A'\,.\,y'z'}{(x'+y'+z')^2} = \theta^2\,.\,\frac{c^2y^2 + b^2z^2 + 2bc\cos A\,.\,yz}{(x+y+z)^2};$$

viz. substituting for x', y', z' their values, this is

$$L^2(x+y+z)^2(b'^2z^2M^2 + c'^2y^2N^2 + 2b'c'yzMN\cos A)$$
$$= \theta^2(b^2z^2 + c^2y^2 + 2bcyz\cos A)(xMN + yNL + zLM)^2,$$

which is an equation of the 12th order. I say that the points A, B, C, O, I, J are each quadruple. In fact, according to the foregoing algorithm, we may write

$$x+y+z = IJ,\quad zM = AB\,.\,CAOIJ,\ \&c.,$$
$$xL = yM = zN = A^2BCOIJ,$$
$$y = z = A,\quad xMN = BC\,.\,CAOIJ\,.\,ABOIJ,\ \&c.,$$
$$xMN = yNL = zLM = (ABCOIJ)^2;$$

and the equation is

$$(BCOIJ)^2(IJ)^2(A^2BCOIJ)^2 = \theta^2\,.\,A^2(ABCOIJ)^4,$$

that is

$$(IJ)^2(ABCOIJ)^4 = \theta^2\,.\,A^2(ABCOIJ)^4;$$

so that the points are each quadruple.

The two Tetrahedra; A, B, C, D at P in ABC and A', B', C', D' at P' in $A'B'C'$ subtending equal angles.

10. I consider now the before-mentioned problem of the two tetrahedra; viz. on the two bases ABC and $A'B'C'$ respectively, letting fall the perpendiculars DK and $D'K'$, then first A, B, C, K at P and A', B', C', K' at P' subtend equal angles; the locus of P is a cubic curve $ABCKOO_1O_2O_3IJ = 0$ through these ten points. ($O = ABC$ is derived from the points A, B, C; and in like manner $O_1 = BCK$, $O_2 = CAK$, $O_3 = ABK$.)

Next, B, C, K at P and B', C', K' at P' subtend equal angles, and moreover the distances KP and $K'P'$ are in a given ratio; the locus of P is a 12-thic curve

$$(BCKO_1IJ)^4 = 0,$$

having each of these six points as a quadruple point. Hence among the 36 intersections of the two curves we have the points B, C, K, O_1, I, J each 4 times, and there remain $36 - 24$, $= 12$ intersections.

The conclusion is that A, B, C, D at a point P of ABC, and A', B', C', D' at a point P' of $A'B'C'$, subtending equal angles, there are 12 positions of P, and of course 12 corresponding positions of P'.

513.

ON A BICYCLIC CHUCK.

[From the *Philosophical Magazine*, vol. XLIII. (1872), pp. 365—367.]

THE apparatus, although I have called it a chuck, is constructed not for turning, but for drawing; viz. it rotates horizontally on a table (being moved, not from the inside by the axle of the lathe, but from the outside by a handle-frame), carrying a drawing-board which works under a fixed pencil supported by a bridge. Two points of the drawing-board describe circles; and the curve traced out on the drawing-board is consequently that described by a fixed point upon a moving plane two points of which describe circles; or, what is really the same thing, it is the curve described on a fixed plane by a point rigidly connected with two points each of which describes a circle. The apparatus is at once convertible into an oval chuck of nearly the ordinary construction; viz. it may be arranged so that the curve described on the drawing-board shall be an ellipse.

Bottom plane is a rectangular board (1) (see figure) about 30 inches by 24 inches, having in the middle a sliding-piece (2) carrying a block (3).

Second plane contains two circular segments (4) fixed to the bottom plane, serving as an axle for the moving piece (5) next referred to, and allowing the block (3) to move between them. And in the same plane we have a moving piece (5) in the form of a rectangle with a circle cut out thereof, rotating about the segments (4), and having upon it a groove in which works a sliding-piece (6) carrying a block (7); there is in this block a circular hole, *D*. The second plane includes also two sides (8) of a handle-frame, which two sides slide along two of the sides of the piece (5).

Third plane consists of a rectangular piece (9) rotating about an axle fixed to the block (3), and having a sliding-piece (10) in which is a circular hole, *C*. The third plane includes also the before-mentioned block (7), having upon it the hole *D*;

and it includes also the remaining two sides (11) of the handle-frame, and, let into the same so as to be flush therewith on the upper surface, two slips (12) completing, in this plane, the handle-frame.

We have thus on a level the sides (11), (12) of the handle-frame and the holes C, D, where C rotates about the point B, which is the centre of the block (3); and D rotates about the point A, which is the centre of the segments (4), each hole being capable of describing a complete circle; and the distances AB, BC, CD, and DA are (within limits) adjustable to any given values: the distance of the holes C, D is made equal to that of the two pegs next referred to.

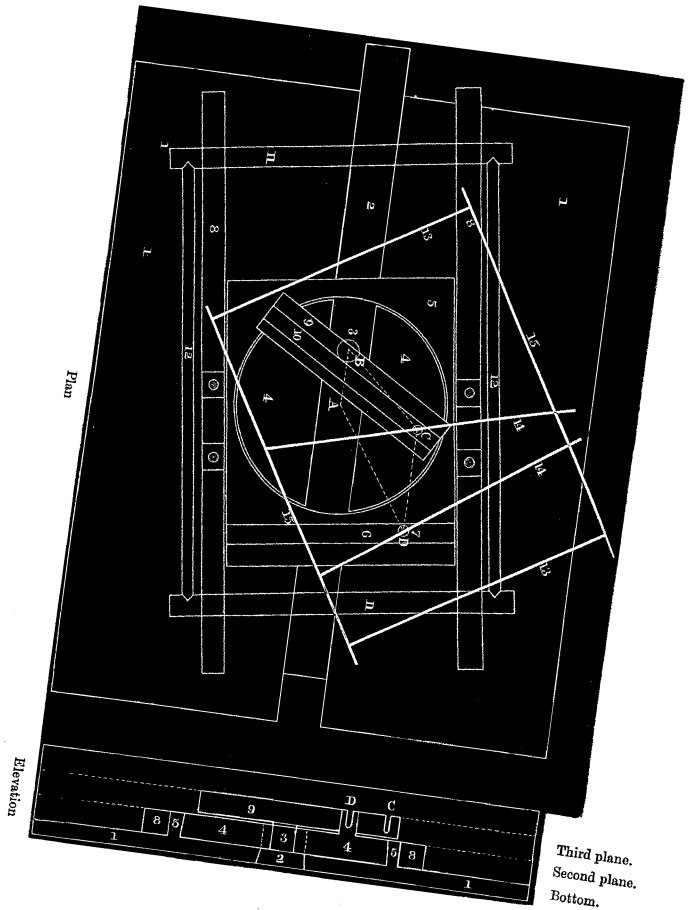
Connected herewith by means of cylindrical pegs working in the holes C, D respectively, we have a carrying-frame; viz. the fourth plane contains two sides (13) of this carrying-frame, and two moveable bars (14), attached to the remaining two sides (15) of the carrying-frame, and having on their lower surfaces the pegs which work in the holes C and D respectively—each bar being free to rotate about one extremity, and being clampable at the other extremity so as to allow the two pegs to be adjusted at a given distance from each other. And then in the fifth plane we have the remaining two sides (15) of the carrying-frame.

Rigidly connected with the carrying-frame we have the drawing-board; or, to make the whole more complete, this should be adjustable to any given position in regard to the carrying-frame by giving it two sliding motions crosswise, and a rotating motion, in the manner of an eccentric chuck.

To convert the apparatus into an oval chuck, we remove altogether the carrying-frame; and in the third plane we fix to the sides (8) of the handle-frame two bars at right angles to these sides, by means of pegs on the lower surfaces of these bars fitting tightly into holes on the sides (8) (which holes and the ends of the bars are shown in the figure), in such wise that these bars include between them the piece (9), which is thereby kept in a direction at right angles to the sides (8), and thus slides between the two bars. There are thus in the handle-frame two lines at right angles to each other, which pass through the fixed points A and B respectively; so that, now connecting the drawing-board directly with the handle-frame, the apparatus has become an oval chuck, viz. the curve traced out on the drawing-board will be an ellipse. The drawing-board should be adjustable to any given position in regard to the handle-frame, in like manner as it was to any given position in regard to the carrying-frame; it is easy to arrange as to this.

It is hardly necessary to remark that the pencil should have two sliding motions crosswise, so as to allow it to be adjusted to any given position; and a small up-and-down motion, so that it may be loaded to press with the proper force upon the drawing-board.

The variety of forms, even with a fixed adjustment of the chuck, only the position of the pencil being altered, is very considerable: among them we have bent ovals and pear-shapes, passing through cuspidal forms into bent figures-of-eight.


Plan
Elevation
Third plane.
Second plane.
Bottom.

514.

ON THE PROBLEM OF THE IN-AND-CIRCUMSCRIBED TRIANGLE.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXI. (for the year 1871), pp. 369—412. Received December 30, 1870,—Read February 9, 1871.]

THE problem of the In-and-Circumscribed Triangle is a particular case of that of the In-and-Circumscribed Polygon: the last-mentioned problem may be thus stated—to find a polygon such that the angles are situate in and the sides touch a given curve or curves. And we may in the first instance inquire as to the number of such polygons. In the case where the curves containing the angles and touched by the sides respectively are all of them distinct curves, the number of polygons is obtained very easily and has a simple expression: it is equal to twice the product of the *orders* of the curves containing the several angles respectively into the product of the *classes* of the curves touched by the several sides respectively; or, say, it is equal to twice the product of the orders of the angle-curves into the product of the classes of the side-curves. But when several of the curves become one and the same curve, and in particular when the angles are all of them situate in and the sides all touch one and the same curve, it is a much more difficult problem to find the number of polygons. The solution of this problem when the polygon is a triangle, and for all the different relations of identity between the different curves, is the object of the present memoir, which is accordingly entitled "On the Problem of the In-and-Circumscribed Triangle;" the methods and principles, however, are applicable to the case of a polygon of any number of sides, the method chiefly made use of being that furnished by the theory of correspondence, as will be explained. The results (for the triangle) are given in the following Table; for the explanation of which I remark that the triangle is taken to be $aBcDeF$; viz. a, c, e are the angles, B, D, F the sides; that is, B, D, F are the sides ac, ce, ea respectively, and a, c, e are the angles FB, BD, DF

respectively. And I use the same letters a, c, e, B, D, F to denote the curves containing the angles and touched by the sides respectively; viz. the angle a is situate in the curve a, the side B touches the curve B, and so for the other angles and sides respectively. An equation such as $a=c$ or $a=B$ denotes that the curves a, c or, as the case may be, the curves a, B are one and the same curve: it is in general convenient to use a new letter for denoting these identical curves; viz. I write, for instance, $a=c=x$ or $a=B=x$, to denote that the curves a, c or, as the case may be, the curves a, B are one and the same curve x; the new letters thus introduced are x, y, z, there being in regard to them no distinction of small letters and capitals. The expression "no identities" denotes that the curves are all distinct. But I use also the letters $a, c, e, b, d, f, x, y, z$, and $A, C, E, B, D, F, X, Y, Z$ quantitatively, to denote the orders and classes of the curves $a, c, e, B, D, F, x, y, z$ respectively; thus, in the Table, for the case 1 "no identities" the number of triangles is given as $=2aceBDF$, which agrees with the before-mentioned result for the polygon: for the case 2 the several separate identities $a=c$, $a=e$, $c=e$ are of course equivalent to each other; and selecting one of them, $a=c=x$, the number of triangles is given as $=2x(x-1)eBDF$. There is a convenience in thus writing down the several forms $a=c$, $a=e$, $c=e$ of the identity or identities which constitute the 52 distinct cases of the Table; and I have accordingly done so throughout the Table, the expression for the number of triangles being however in each case given under one form only. It only remains to mention that for the curve x the Greek letter ξ denotes what may be termed the "stativity" of the curve, viz. this is = number of cusps + 3 times the class, or, what is the same thing, = number of inflections + 3 times the order; the curve being determined by its order x, class X, and ξ; and similarly for η and ζ.

Observe that, in the column "Specification," each line is to be read separately from the others, and, where the word "or" occurs, the two parts of the line are to be read separately; thus case 5, the six forms are $a=B$, $a=F$, $c=D$, $c=B$, $e=F$, $e=D$: the letter x (or, as the case may be, x, y, or x, y, z) accompanies the first of the given forms; in the present instance $a=B=x$, and it is to this first form that the number of triangles, here $2(Xx-X-x)ceDF$, applies.

I remark that what is primarily determined is the number of positions of a particular angle of the triangle, and that in some cases, on account of the symmetry of the figure, the number of triangles is a submultiple of this number; viz. the number of positions of the angle is to be divided by 2 or 6; this is expressly shown, by means of a separate column, in the Table.

No. of Case	Specification	No. of forms	Totals	No. of triangles	Divided by
1	No identities	1	1	$2aceBDF$	
2	$a = c = x$ $c = e$ $e = a$	3		$2x(x-1)\,eBDF$	
3	$D = F = x$ $F = B$ $B = D$	3		$2X(X-1)\,Bace$	
4	$a = D = x$ $c = F$ $e = B$	3		$2Xxce\,BF$	
5	$a = B = x$, or $a = F$ $c = D$ „ $c = B$ $e = F$ „ $e = D$	6	15	$2(Xx - X - x)\,ceDF$	
6	$a = c = e = x$	1		$\{2x(x-1)(x-2) + X\}\,BDF$	
7	$B = D = F = x$	1		$\{2X(X-1)(X-2) + x\}\,ace$	
8	$a = c = B = x$ $c = e = D$ $e = a = F$	3		$2x(x-3)(X-2)\,eDF$	

9	$D=F=e=x$ $F=B=a$ $B=D=c$	3		$2X(X-3)(x-2)acB$	
10	$a=c=D=x$, or $a=c=F$ $c=e=F$,, $c=e=B$ $e=a=B$,, $e=a=D$	6		$2(x-1)(Xx-X-x)eBF$	
11	$D=F=a=x$, or $D=F=c$ $F=B=c$,, $F=B=e$ $B=D=e$,, $B=D=a$	6		$2(X-1)(Xx-X-x)ceB$	
12	$c=e=x$, $a=D=y$ $e=a$ $c=F$ $a=c$ $e=B$	3	20	$2x(x-1)yYBF$	
13	$F=B=x$, $a=D=y$ $B=D$ $c=F$ $D=F$ $e=B$	3		$2X(X-1)Yyce$	
14	$c=e=x$, $a=B=y$, or $c=e$, $a=F$ $e=a$ $c=D$,, $e=a$, $c=B$ $a=c$ $e=F$,, $a=c$, $e=D$	6		$2x(x-1)(Yy-Y-y)DF$	
15	$F=B=x$, $D=e=y$, or $F=B$, $D=c$ $B=D$ $F=a$,, $B=D$, $F=e$ $D=F$ $B=c$,, $D=F$, $B=a$	6		$2X(X-1)(Yy-Y-y)ac$	
	Carried over	(18)	(36)		

TABLE (*continued*).

No. of Case	Specification	No. of forms	Totals	No. of triangles	Divided by
	Brought over	(18)	(36)		
16	$c=e=x,\ D=F=y,$ or $c=e,\ D=B$ $e=a\quad F=B$ „ $e=a,\ F=D$ $a=c\quad B=D$ „ $a=c,\ B=F$	6		$2x(x-1)\,Y(Y-1)\,aB$	
17	$c=e=x,\ B=F=y$ $e=a\quad D=B$ $a=c\quad F=D$	3		$2x(x-1)\,Y(Y-1)\,aD$	2
18	$a=D=x,\ c=B=y,$ or $a=D,\ e=F$ $c=F\quad e=D$ „ $c=F,\ a=B$ $e=B\quad a=F$ „ $e=B,\ c=D$	6		$2xX(Yy-Y-y)\,eF$	
19	$c=F=x,\ e=B=y$ $e=B\quad a=D$ $a=D\quad c=F$	3		$2xyXYaD$	
20	$c=D=x,\ e=F=y,$ or $c=B,\ e=D$ $e=F\quad a=B$ „ $e=D,\ a=F$ $a=B\quad c=D$ „ $a=F,\ c=B$	6		$2\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}\,aB$	
21	$c=B=x,\ e=F=y$ $e=D\quad a=B$ $a=F\quad c=D$	3		$2\{xyXY-xy(X+Y)-XY(x+y)+2xY+2yX\}\,aD$	
22	$a=D=x,\ c=F=y,\ e=B=z$	1	45	$2xyzXYZ$	
23	$a=B=x,\ c=D=y,\ e=F=z$ $a=F\quad c=B\quad e=D$	2		$2\{xyzXYZ-xyz(YZ+ZX+XY)-XYZ(yz+zx+xy)$ $+2xyz(X+Y+Z)+2XYZ(x+y+z)-4xyz-4XYZ\}$	

24	$a=D=x,\ c=B=y,\ e=F=z$ $c=F \quad e=D \quad a=B$ $e=B \quad a=F \quad c=D$	3		$2xX\{yzYZ-yz(Y+Z)-YZ(y+z)+2yZ+2zY\}$ [Originally printed $2yz+2YZ\}$ but the correction was made in a footnote to Case 24.]	
25	$a=c=x,\ D=F=y,\ e=B=z$ $c=e \quad F=B \quad a=D$ $e=a \quad B=D \quad c=F$	3		$2x(x-1)\,Y(Y-1)\,zZ$	2
26	$a=c=x,\ B=D=y,\ e=F=z$ or $a=c,\ B=F,\ e=D$ $c=e \quad D=F \quad a=B$,, $c=e,\ D=B,\ a=F$ $e=a \quad F=B \quad c=D$,, $e=a,\ F=D,\ c=B$	6	15	$2x(x-1)\,Y(Y-1)(zZ-z-Z)$	
27	$a=c=e=x,\ B=F=y$ $a=c=e \quad D=B$ $a=c=e \quad F=D$	3		$\{2x(x-1)(x-2)+X\}\,Y(Y-1)\,D$	2
28	$B=D=F=x,\ c=e=y$ $B=D=F \quad e=a$ $B=D=F \quad a=c$	3		$\{2X(X-1)(X-2)+x\}\,y(y-1)\,a$	2
29	$a=c=B=x,\ D=F=y$ $c=e=D \quad F=B$ $e=a=F \quad B=D$	3		$2x(x-3)(X-2)\,Y(Y-1)\,e$	2
30	$e=D=F=x,\ a=c=y$ $a=F=B \quad c=e$ $c=B=D \quad e=a$	3		$2X(X-3)(x-2)\,y(y-1)\,B$	2
	Carried over	(12)	(96)		

TABLE (*continued*).

No. of Case	Specification	No. of forms	Totals	No. of triangles	Divided by
	Brought over	(12)	(96)		
31	$c=e=D=x,\ a=B=y,$ or $c=e=D,\ a=F$			$2x(x-3)(X-2)(yY-y-Y)F$	
	$e=a=F\quad c=D$ „ $e=a=F,\ c=B$				
	$a=c=B\quad e=F$ „ $a=c=B,\ e=D$	6			
32	$F=B=a=x,\ D=e=y,$ or $F=B=a,\ D=c$			$2X(X-3)(x-2)(yY-y-Y)c$	
	$B=D=c\quad F=a$ „ $B=D=c,\ F=e$				
	$D=F=e\quad B=c$ „ $D=F=e,\ B=a$	6			
33	$B=F=y,\ a=e=D=x,$ or $B=F,\ a=c=D$			$2(x-1)(xX-x-X)Y(Y-1)c$	
	$D=B\quad c=a=F$ „ $D=B,\ c=e=F$				
	$F=D\quad e=c=B$ „ $F=D,\ e=a=B$	6			
34	$c=e=y,\ B=D=a=x,$[1] or $c=e,\ D=F=a$			$2(X-1)(xX-x-X)y(y-1)F$	
	$e=a\quad D=F=a$ „ $e=a,\ F=B=c$				
	$a=c\quad F=B=c$ „ $a=c,\ B=D=e$	6			
35	$a=D=y,\ c=e=B=x,$ or $a=D,\ c=e=F$			$2(x-1)(xX-x-X)yYF$	
	$c=F\quad e=a=D$ „ $c=F,\ e=a=B$				
	$e=B\quad a=c=F$ „ $e=B,\ a=c=D$	6			
36	$a=D=y,\ B=F=e=x,$ or $a=D,\ B=F=c$			$2(X-1)(xX-x-X)yYc$	
	$c=F\quad D=B=a$ „ $c=F,\ D=B=e$				
	$e=B\quad F=D=c$ „ $e=B,\ F=D=a$	6			
37	$a=e=D=x,\ c=B=y,$ or $a=e=B,\ c=D$			$2(x-1)\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}F$	
	$c=a=F\quad e=D$ „ $c=a=D,\ e=F$				
	$e=c=B\quad a=F$ „ $e=c=F,\ a=B$	6			

[1] Originally printed $B=D=e=x$. The correction was made *post*, Case 34.

38	$B=D=a=x,\ F=e=y,$ or $B=D=e,\ F=a$ $D=F=c \quad B=a$ „ $D=F=a,\ B=c$ $F=B=e \quad D=c$ „ $F=B=c,\ D=e$	6	60	$2(X-1)\{xyXY-xy(X+Y)-XY(x+y)+2xy+2XY\}c$	
39	$a=c=e=B=x$ „ $=D$ „ $=F$	3		$\{X^2+X(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}DF$	
40	$B=D=F=e=x$ „ $=a$ „ $=c$	3		$\{x^2+x(2X^3-10X^2+12X-1)-4X^3+20X^2-16X-3\xi\}ac$	
41	$c=e=D=F=x,$ or $c=e=D=B$ $e=a=B=B$ „ $e=a=F=D$ $a=c=F=D$ „ $a=c=B=F$	6		$2(x-3)(X-3)(xX-x-X)aB$	
42	$a=c=D=F=x$ $c=e=F=B$ $e=a=B=D$	3		$\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}eB$	2
43	$a=c=e=x,\ B=D=F=y$	1	15	$2x(x-1)(x-2)Y(Y-1)(Y-2)+yx(x-1)(x-2)$ $+XY(Y-1)(Y-2)$	6
44	$e=D=F=x,\ a=c=B=y$ $a=F=B \quad c=e=D$ $c=B=D \quad e=a=F$	3		$2(x-2)X(X-3)(Y-2)y(y-3)$	2
	Carried over	(4)	(171)		

TABLE (*continued*).

No. of Case	Specification	No. of forms	Totals	No. of triangles	Divided by
	Brought over	(4)	(171)		
45	$a=D=B=x,\ c=e=F=y,$ or $a=D=F,\ c=e=B$ $c=F=D \quad e=a=B$,, $c=F=B,\ e=a=D$ $e=B=F \quad a=c=D$,, $e=B=D,\ a=c=F$	6	10	$2(X-1)(y-1)\{XYxy-XY(x+y)$ $-xy(X+Y)+2xy+2XY\}$	
46	$a=c=y,\ B=D=F=e=x$ $c=e \quad B=D=F=a$ $e=a \quad B=D=F=c$	3		$y(y-1)\{x^2+x(2X^3-10X^2+12X-1)$ $-4X^3+20X^2-16X-3\xi\}$	2
47	$D=F=y,\ a=c=e=B=x$ $F=B \quad a=c=e=D$ $B=D \quad a=c=e=F$	3		$Y(Y-1)\{X^2+X(2x^2-10x^2+12x-1)$ $-4x^3+20x^2-16x-3\xi\}$	2
48	$a=c=D=F=x,\ e=B=y$ $c=e=F=B \quad a=D$ $e=a=B=D \quad c=F$	3		$\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}yY$	2
49	$a=B=y,\ c=e=D=F=x,$ or $a=F,\ c=e=B=D$ $c=D \quad e=a=F=B$,, $c=B,\ e=a=D=F$ $e=F \quad a=c=B=D$,, $e=D,\ a=c=F=B$	6	15	$2(x-3)(X-3)\{xyXY-(x+y)XY$ $-(X+Y)xy+2xy+2XY\}$	
50	$c=e=B=D=F=x$ $e=a=B=D=F$ $a=c=B=D=F$	3		a into $x^3(\qquad\qquad\qquad +1)$ $+x^2(\quad 2X^3-14X^2+28X-11)$ $+x(-10X^3+70X^2-116X-8)$ $+12X^3-76X^2+64X$ $+\xi(-6x-4X+42)$	2

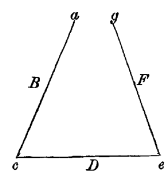
51	$F = B = a = c = e = x$ $B = D = a = c = e$ $D = F = a = c = e$	3	6	D into $X^3(\quad + 1)$ $+X^2(\quad 2x^3 - 14x^2 + 28x - 11)$ $+X(-10x^3 + 70x^2 - 116x - 8)$ $+12x^3 - 76x^2 + 64x$ $+\xi(-6X - 4x + 42)$	2
52	$a = c = e = B = D = F = x$	1	1 203	$X^4(\quad + 1)$ $+X^3(\quad 2x^3 - 18x^2 + 52x - 46)$ $+X^2(\quad -18x^3 + 162x^2 - 420x + 221)$ $+X(\quad 52x^3 - 420x^2 + 704x + 172)$ $+1x^4 - 46x^3 + 221x^2 + 172x$ $+\xi\left\{\begin{array}{l} X^2(\quad - 9) \\ +X(\quad - 12x + 135) \\ -9x^2 + 135x - 600 \end{array}\right\}$	6

The foregoing results are chiefly obtained by means of the theory of correspondence; viz. if instead of the triangle $aBcDeF$ we consider the unclosed trilateral $aBcDeFg$, where the points a and g are situate on one and the same curve, say the curve $a=g$, then the points a and g have a certain correspondence, say a (χ, χ') correspondence with each other; and when a, g are a "united point" of the correspondence, the trilateral in question becomes an in-and-circumscribed triangle $aBcDeF$; that is, the number of triangles is equal to that of the united points of the correspondence, subject however (in many of the cases) to a reduction on account of special solutions. It may be remarked that by the theory of correspondence the number of the united points is, in several of the cases, but not in all of them, $=\chi+\chi'$. But in some instances I employ a functional method, by assuming that the identical curves are each of them the aggregate of the two curves x, x': we here obtain for the number ϕx of the triangles belonging to the curve x a functional equation $\phi(x+x')-\phi x-\phi x'=$ given function; viz. the expression on the right-hand side depends on the solution of the preceding cases, wherein the number of identities between the several curves is less than in the case under consideration; and taking it to be known, the functional equation gives $\phi x=$ particular solution + linear function of (x, X, ξ). The particular solution is always easily obtainable, and the constants of the linear function can be determined by means of particular forms of the curve x.

Article Nos. 1 to 6. *The Principle of Correspondence as applied to the present Problem.*

1. Consider the unclosed trilateral $aBcDeFg$, where the points a and g are on one and the same curve, $a=g$. Starting from an arbitrary point a on the curve a, we have aBc any one of the tangents from a to the curve B, touching this curve, say at the point B, and intersecting the curve c in a point c; viz. c is any one of the intersections of aBc with the curve c; we have then similarly cDe any one of

Fig. 1.



the tangents from c to the curve D, touching it, say at D, and intersecting the curve e in a point e; viz. the point e is any one of the intersections in question; and then in like manner we have eFg any one of the tangents from e to the curve F, touching it, say at F, and intersecting the curve g $(=a)$ in a point g; viz. g is any one of the intersections in question. Suppose that to a given position of a there correspond χ positions of g; it is easy to find the value of χ; viz. if (as above tacitly supposed)

the curves a, B, c, D, e, F are all of them distinct curves, then the number of the tangents aBc is $=B$; there are on each of them c points c; through each of these we have D tangents cDe; on each of these e points e; through each of these F tangents eFg; and on each of these a points g; that is, $\chi = BcDeFa$. But if some of the curves become one and the same curve—if, for instance, $a=B=c$,—the line aBc is here a tangent from a point a on the curve, we exclude the tangent at the point a, and the number of the remaining tangents is $=(A-2)$; each tangent meets the curve in the point a counting once, the point B counting twice, and in $(a-3)$ other points; that is, the number of the points c is $=(A-2)(a-3)$, and so in other cases; the calculation is always immediate, and the only difference is that, instead of a factor a or A, we have such factor in its original form or diminished by 1, 2, or 3, as the case may be. Similarly starting from g, considered as a given point on the curve g $(=a)$, we find χ' the number of the corresponding points a; thus in the case where the curves are all distinct curves, we have $\chi' = FeDcBa$ $(=\chi)$; and so in other cases we find the value of χ'. The points (a, g) have thus a (χ, χ') correspondence, where the values of χ, χ' are found as above.

2. There will be occasion to consider the case where in the triangle $aBcDeF$ (or say the triangle $aBcDeFa$) the point a is not subjected to any condition whatever, but is a free point. There is in this case a "locus of a," which is at once constructed as follows: viz. starting with an arbitrary tangent aBc of the curve B, touching it at B and intersecting the curve c in a point c; through c we draw to the curve D the tangent cDe, touching it at D and intersecting the curve e in a point e; and finally from e to the curve F the tangent eFa, touching it at F and intersecting the original arbitrary tangent aBc in a point a, which is a point on the locus in question. We can, it is clear, at once determine how many points of the locus lie on an arbitrary tangent of the curve B (or of the curve F).

3. The general form of the equation of correspondence is

$$p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\ldots=k\Delta\ (^1);$$

viz. if on a curve for which twice the deficiency is $=\Delta$ we have a point P corresponding to certain other points P', Q', ... in such wise that P, P' have an (α, α') correspondence, P, Q' a (β, β') correspondence, &c.; and if (a) be the number of the united points (P, P'), (b) the number of the united points (P, Q'), &c.; and if moreover for a given position of P on the curve the points P', Q', ... are obtained as the intersections of the curve with a curve Θ (depending on the point P) which meets the curve k times at P, p times at each of the points P', q times at each of the

[1] To avoid confusion with the notation of the present memoir, I abstain in the text from the use of D as denoting the deficiency, and there is a convenience in the use of a single symbol for twice the deficiency; but writing for the moment D to denote the deficiency, I remark, in passing, that perhaps the true theoretical form of the equation is

$$k(0-D-D)+p(\mathrm{a}-\alpha-\alpha')+q(\mathrm{b}-\beta-\beta')+\ldots=0;$$

viz. the point P is here considered as having with itself a (D, D) correspondence, the number of the united points therein being $=0$.

points Q', &c.; then the relation between the several quantities is as stated above: see my "Second Memoir on the Curves which satisfy given conditions," *Philosophical Transactions*, vol. 159 (1868), pp. 145—172, [407]. I omit for the present purpose the term "Supp.," treating it as included in the other terms.

4. In the present case we consider, as already mentioned, the unclosed trilateral $aBcDeFg$, where the angles a, g are on one and the same curve a $(=g)$ (the curve in the general theorem); and the curve Θ is the system of lines eFg which by their intersection with the curve a determine the points g. Considering these as the points (P, P') of the general theorem we have $p=1$: I change the notation, and instead of $\mathrm{a}-\alpha-\alpha'$ write $\mathrm{g}-\chi-\chi'$; viz. I take (g) for the number of the united points (a, g), and suppose that the points (a, g) have a (χ, χ') correspondence. The most simple case is when the curve a is distinct from each of the curves e, F; here all the intersections of the line-system eFg with the curve a are points g, that is we have *only* the correspondence (a, g); and since the line-system eFg does not pass through the point a, we have simply

$$\mathrm{g}-\chi-\chi' \doteq 0.$$

5. But suppose that the curves a, e, F are one and the same curve, say that $a=e=F$; understanding by the point F the point of contact of a line eFg with the curve a, then the intersections of the line-system eFg with the curve a are the points g each once, the points F each twice, and the points e each as many times as there are lines eFg through the point e, say each M times. (In the present case, where the curves e, F are identical, we have $M=F-2$ or $F-3$ according as the curve D is or is not distinct from the curve F; in the cases afterwards referred to, the values may be F or $F-1$; that is, we have always $M=F$, $F-1$, $F-2$, $F-3$, as the case may be.) We have to consider the several correspondences (a, g), (a, F), (a, e); k is as before $=0$; and the form of the theorem is

$$(\mathrm{g}-\chi-\chi')+2(\mathrm{f}-\phi-\phi')+M(\mathrm{e}-\epsilon-\epsilon')=0,$$

where the symbols denote as follows, viz.

(a, g)	have a	(χ, χ')	correspondence, and	No. of united points		$=\mathrm{g}$,
(a, F)	,,	(ϕ, ϕ')	,,	,,	,,	$=\mathrm{f}$,
(a, e)	,,	(ϵ, ϵ')	,,	,,	,,	$=\mathrm{e}$,

so that the determination of g here depends upon that of $\mathrm{f}-\phi-\phi'$ and $\mathrm{e}-\epsilon-\epsilon'$.

6. The curve a might however have been identical with only one of the curves e, F; viz. if $a=F$, but e is a distinct curve, then the equation will contain the term $2(\mathrm{f}-\phi-\phi')$, but not the term $M(\mathrm{e}-\epsilon-\epsilon')$; and so if $a=e$, but F is a distinct curve, then the equation will not contain $2(\mathrm{f}-\phi-\phi')$, but will contain $M(\mathrm{e}-\epsilon-\epsilon')$: it is to be noticed that in this last case we have $M=F$ or $M=F-1$, according as the curve D is not, or is, one and the same curve with F. The determination of (g) here depends upon that of $\mathrm{f}-\phi-\phi'$ or $\mathrm{e}-\epsilon-\epsilon'$, as the case may be. These sub-

sidiary values $f-\phi-\phi'$ and $e-\epsilon-\epsilon'$ are obtained by means of a more simple application of the principle of correspondence, as will appear in the sequel [1], but for the moment I do not pursue the question.

Article Nos. 7 to 14. *Locus of a free angle* (a).

7. I consider the case where a is a distinct curve $\neq e, \neq F$, and where, as was seen, the equation is simply

$$g-\chi-\chi'=0.$$

I suppose further that a is distinct from all the other curves, or say, *simpliciter*, that a is a distinct curve. The values of χ, χ' will here each of them contain the factor a, say we have $\chi=a\omega$, $\chi'=a\omega'$; and therefore the equation gives $g=a(\omega+\omega')$. It is obvious that ω, ω' are the values assumed by χ, χ' respectively in the particular case where the curve a is an arbitrary line ($a=1$); and $\omega+\omega'$ is the number of the united points on this line.

8. Suppose now that in the triangle $aBcDeFa$ the point a is a free point, we have, as above-mentioned, a locus of a, and the united points on the arbitrary line are the intersections of the line with this locus; that is, the locus meets the arbitrary line in $\omega+\omega'$ points; or, what is the same thing, the order of the locus is $=\omega+\omega'$.

9. I stop for a moment to remark that in the particular case where the curve B is a point ($B=1$), then in the construction of the locus of a the arbitrary tangent aBc is an arbitrary line through B, and the construction gives on this line ω positions of the point a. But drawing from B a tangent to the curve F, and thus constructing in order the points F, e, D, c, a, the construction shows that B is an ω'-tuple point on the locus; and (by what precedes) an arbitrary line through B meets the locus in ω other points; that is, in the particular case where the curve B is a point, the order of the locus of a is $=\omega+\omega'$, which agrees with the foregoing result.

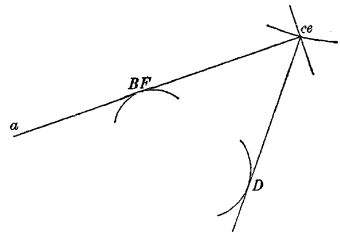
10. The construction for the locus of a may be presented in the following form: viz. drawing to the curve D a tangent cDe, meeting the curves c, e in the points c, e respectively; then if from any point c we draw to the curve B a tangent cBa, and from any point e to the curve F a tangent eFa, the tangents cBa, eFa intersect in a point on the required locus. Hence if in any particular case (that is for any particular position of the tangent cDe) the lines cBa, eFa become one and the same line, the point a will be an indeterminate point on this line; that is, the line in question will be part of the locus of a.

11. The case cannot in general arise so long as the curves B, F are distinct from each other; but when these are one and the same curve, say when $B=F$, it will arise, and that in two distinct ways. To show how this is, suppose, to fix the ideas, that the curves c, D, e are distinct from each other and from the curve $B=F$. Then the first mode is that shown in the annexed "first-mode figure," viz. we have

[1] See *post*, Nos. 24 *et seq.*

here a tangent at D passing through a point ce of the intersection of the curves c, e, and from this point a tangent drawn to the curve $B=F$. For the position in question of the tangent of D, the points c, e coincide with each other, and we have thus the coincident tangents cBa and eFa to the identical curves $B=F$. It is further

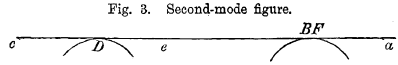
Fig. 2. First-mode figure.



to be remarked that the number of the points of intersection is $=ce$; from each of these there are B tangents to the curve $B=F$ (in all $ce.B$ tangents), and each of these counts once in respect of each of the D tangents to the curve D, that is, it counts D times. We have thus, as part of the locus of a, $ce.B$ lines each D times, or, say, first-mode reduction $=ce.B.D$.

12. The second mode is that shown in the annexed "second-mode figure." The tangent from D is here a common tangent of the curves D, and $B=F$. This meets the curve c in c points, and the curve e in e points; and attending to any pair of points c, e, these give the tangents cBa, eFa, coinciding with the common tangent in

Fig. 3. Second-mode figure.



question, and forming part of the locus of a. The number of the common tangents is $=BD$; but each of these counts once in respect of each combination of the points c, e, that is in all ce times. And we have thus as part of the locus BD lines each $c.e$ times, or, say, second-mode reduction $=BD.c.e$. This is (as it happens) the same number as for the first mode; but to distinguish the different origins I have written as above $ce.B.D$ and $BD.c.e$ respectively.

13. It is important to remark that each of the two modes arises whatever relations of identity subsist between the curves c, e, D, and $B=F$, but with considerable modification of form. Thus if the curves c, e are identical ($c=e$) but distinct from D, then in the first-mode figure ce may be a node or a cusp of the curve $c=e$, or it may be a point of contact of a common tangent of the curves D, and $c=e$. As regards the node, remark that if we consider a tangent of D meeting the curve $c=e$ in the neighbourhood of the node, then of the two points of intersection each

in succession may be taken for the point c, and the other of them will be the point e; so that the node counts twice. It requires more consideration to perceive, but it will be readily accepted that the cusp counts three times. Hence if for the curve $c=e$ the number of nodes be $=\delta$ and that of cusps $=\kappa$, the value of the first-mode reduction is $=(2\delta+3\kappa+C)\,BD$, or, what is the same thing, it is $=(c^2-c)\,BD$.

As regards the second-mode figure, the only difference is that c, e will be here any pair of intersections (each pair twice) of the tangent with the curve $c=e$; the value is thus $=(c^2-c)\,BD$.

It would be by no means uninteresting to enumerate the different cases, and indeed there might be a propriety in doing so here; but I have (instead of this) considered the several cases, when and as they arise in connexion with any of the cases of the in-and-circumscribed triangle.

14. Observe that the general result is, that in the case $B=F$ of the identity of the curves B and F, but not otherwise, the locus of a includes as part of itself a system of lines; or, say, that it is made up of these lines, and of a residual curve of the order $\omega+\omega'-\text{Red.}$, which is the proper locus.

Article Nos. 15 to 17. *Application of the foregoing Theory as to the locus of* (a).

15. Reverting now to the case where the angle a is not a free angle but is situate on a given curve a, then if the curve a is distinct from the curves e, F, the number of positions of a is, as was seen, $g=\chi+\chi'$. But the points in question are the intersections of the curve a with the locus of a considered as a free angle; and hence in the case $B=F$, but not otherwise, they are made up of the intersections of the curve a with the system of lines, and of its intersections with the proper locus of a. But the intersections with the system of lines are improper solutions of the problem (or, to use a locution which may be convenient, they are "heterotypic" solutions): the true solutions are the intersections with the proper locus of a; and the number of these is not $\chi+\chi'$, $=a(\omega+\omega')$, but it is $=a(\omega+\omega'-\text{Red.})$; say it is $=\chi+\chi'-\text{Red.}$, where the symbol "Red." is now used to signify a times the number of lines, or reduction in the expression $\omega+\omega'-\text{Red.}$ of the order of the proper locus of a.

16. It is however to be noticed that if the curve a, being as is assumed distinct from the curves e, and $F=B$, is identical with one or both of the remaining curves c, D, the foregoing expression $\chi+\chi'-\text{Red.}$ *may* include positions which are not true solutions of the problem, viz. the curve a may pass through special points on the proper locus of a, giving intersections which are a new kind of heterotypic solutions[1].

[1] More generally, if the curve a be a curve identical with any of the other curves, then if treating in the first instance the angle a as free we find in any manner the locus of a, the required positions of the angle a are the intersections of this locus and of the curve a; but these intersections will in general include intersections which give heterotypic solutions. The determination of these is a matter of some delicacy, and I have in general treated the problems in such manner that the question does not arise; but as an example see *post*, Case 43.

17. But this cannot happen if the curve a is distinct also from the curves c, D; or, say, simply when a is a distinct curve. The conclusion is, that in the case where a is a distinct curve we have

$$g = \chi + \chi' - \text{Red.},$$

where the term "Red." vanishes except in the case of the identity $B = F$ of the curves B, F; and that when this identity subsists it is $= a$ times the reduction in the order of the locus of a considered as a free angle; viz. this consists of a first-mode and a second-mode reduction as above explained.

Article Nos. 18 to 21. *Remarks in regard to the Solutions for the* 52 *Cases.*

18. Before going further I remark that the principle of correspondence applies to corresponding and united tangents in like manner as to corresponding and united points, and that all the investigations in regard to the in-and-circumscribed triangle might thus be presented in the reciprocal form, where, instead of points and lines, we have lines and points respectively. But there is no occasion to employ any such reciprocal process; the result to which it would lead is the reciprocal of a result given by the original process, and as such it can always be obtained by reciprocation of the original result, without any performance of the reciprocal process.

19. It is hardly necessary to remark that although reciprocal results would, by the employment of the two processes respectively, be obtained in a precisely similar manner, yet that this is not so when only one of the reciprocal processes is made use of; so that, using one process only, it may be and in general is easier and more convenient to obtain directly one than the other of two reciprocal results; for instance, to consider the case $B = D = F$ rather than $a = c = e$, or *vice versâ*; and that it is sufficient to do this, and having obtained the one result, directly to deduce from it the other by reciprocity; but that it may nevertheless be interesting to obtain each of the two results directly.

20. It is moreover obvious that although the several forms of the same case, for instance Case 2, $a = c$, $a = e$, or $c = e$, are absolutely equivalent to each other, yet that, when as above we select a vertex a, and seek for the number of the united points (a, g), the process of obtaining the result will be altogether different according to the different form which we employ. For instance, in the case just referred to, if the form is taken to be $a = c$ or $c = e$, then the equation $g = \chi + \chi'$ is applicable to it; but not so if the form is taken to be $a = e$. It would be by no means uninteresting in every case to consider the several forms successively and get out the result from each of them; I shall not, however, do this, but only consider two or more forms of the same case when for comparison, illustration, verification, or otherwise it appears proper so to do. The translation of a result, for instance, of a form $a = e$ or $c = e$ into that for the form $a = c = x$ is so easy and obvious, that it is not even necessary formally to make it.

21. I do not at present further consider the general theory, but proceed to consider in order the 52 cases, interpolating in regard to the general theory such further discussion or explanation as may appear necessary. In the several instances in which the equation $g = \chi + \chi'$ is applicable, it is sufficient to write down the values of χ, χ', the mode of obtaining these being already explained.

The 52 Cases for the in-and-circumscribed triangles.

Case 1. No identities.

$$\chi = BcDeFa, \quad \chi' = FeDcBa\,(=\chi),$$
$$g = 2aceBDF.$$

Case 2. $a = c = x$.

$$\chi = B(x-1)\,DeFx, \quad \chi' = FeDxB\,(x-1)\,(=\chi),$$
$$g = 2x\,(x-1)\,eBDF.$$

Second process, for form $a = e = x$. The equation of correspondence is here

$$g - \chi - \chi' + F(e - \epsilon - \epsilon') = 0\,;$$

but the points e being given as all the intersections of the curve $a\,(=e)$ by the line-system cDe which does not pass through a, we have $e - \epsilon - \epsilon' = 0$; so that $g = \chi + \chi'$; and then

$$\chi = BcDxF\,(x-1), \quad \chi' = F\,(x-1)\,DcBx,$$

giving the former result[1].

Case 3. $D = F = x$. Reciprocation from 2; or else, *second process*,

$$\chi = BcXe\,(X-1)\,a, \quad \chi' = Xe\,(X-1)\,cBa,$$
$$g = 2X\,(X-1)\,Bace.$$

Third process: form $F = B = x$. We have here $g = \chi + \chi' - \text{Red.}$

$$\chi = XcDeXa, \quad \chi' = XeDcXa\,(=\chi),$$
$$\chi + \chi' = 2X^2Dace\,;$$

and the reductions are those of the first and second mode, as explained *ante*, Nos. 11, 12, viz. each of these is $= XDace$, and together they are $= 2XDace$; whence the foregoing result.

Case 4. $a = D = x$.

$$\chi = BcXeFx, \quad \chi' = FeXBx\,(=\chi),$$
$$g = 2Xx\,ceBF.$$

[1] Of course, the result is obtained in the form belonging to the new form of specification, viz. here it is $= 2x\,(x-1)\,cBDF$; and so in other instances; but it is unnecessary to refer to this change.

Observe this is what the result for Case 1 becomes on writing therein $a = D = x$, viz. the opposite curves a, D may become one and the same curve without any alteration in the form of the result.

Case 5. $a = B = x$.

$$\chi = (X-2)\,cDeFx, \quad \chi' = FeDcX\,(x-2),$$

where

$$(X-2)\,x + X\,(x-2) = 2\,(Xx - X - x);$$

therefore

$$g = 2\,(Xx - X - x)\,ceDF.$$

Case 6. $a = c = e = x$: perhaps most easily by reciprocation of Case 7; or

Second process, functionally by taking the curve $a = c = e$ to be the aggregate curve $x + x'$. The triangle $aBcDeF$ is here in succession each of the eight triangles:

$x\,B\,x\,D\,x\,F$	$x'\,B\,x'\,D\,x'\,F$
x' „ x „ x „	x „ x' „ x' „
x „ x' „ x „	x' „ x „ x' „
x' „ x „ x „	x „ x' „ x' „

where the two top triangles give ϕx and $\phi x'$ respectively; the remaining triangles all belong to Case 2, and those of the first column give each $2\,(x^2 - x)\,x'BDF$, and those of the second column each $2\,(x'^2 - x')\,xBDF$. We have thus

$$\phi\,(x + x') - \phi x - \phi x' = \{6\,(x^2x' + xx'^2) - 12xx'\}\,BDF.$$

Hence obtaining a particular solution and adding the constants, we have

$$\phi x = (2x^3 - 6x^2 + \alpha x + \beta X + \gamma\xi)\,BDF;$$

it is easy to see that α, β, γ are independent of the curves B, D, F; and taking each of these to be a point, and the curve $a = c = e$ to be a conic, then it is known that $\phi x = 2$; we have therefore $2 = 16 - 24 + 2\alpha + 2\beta + 6\gamma$, that is $\alpha + \beta + 3\gamma = 5$.

The case where the curve $a = c = e$ is a line gives $0 = 2 - 6 + \alpha + 3\gamma$, that is $\alpha + 3\gamma = 4$; but it is not easy to find another condition; assuming however $\gamma = 0$, we have $\alpha = 4$, $\beta = 1$, and thence

$$\phi x = (2x^3 - 6x^2 + 4x + X)\,BDF,$$

or say

$$g = \{2x\,(x-1)\,(x-2) + X\}\,BDF:$$

this is a good easy example of the functional process, the use of which begins to exhibit itself; and I have therefore given it, notwithstanding the difficulty as to the complete determination of the constants.

Third process. The equation of correspondence is

$$\mathrm{g}-\chi-\chi'+F(\mathrm{e}-\epsilon-\epsilon')=0,$$

but for the correspondence (a, e) we have

$$\mathrm{e}-\epsilon-\epsilon'+D(\mathrm{c}-\gamma-\gamma')=0,$$

and for the correspondence (a, c) we have

$$\mathrm{c}-\gamma-\gamma'=B\Delta,$$

whence

$$\mathrm{g}=\chi+\chi'+BDF\,.\,\Delta\,;$$

and then

$$\chi=B(x-1)\,D(x-1)\,F(x-1),\quad \chi'=F(x-1)\,D(x-1)\,B(x-1)(=\chi)\,;$$

that is

$$\chi+\chi'=BDF\,.\,2(x-1)^3.$$

Moreover

$$\Delta=X-2x+2+\kappa$$

(if κ be the number of cusps of the curve $a=c=e$), and the resulting value is

$$\mathrm{g}=\{2(x-1)^3+X-2x+2+\kappa\}\,BDF\,;$$

that is

$$=\{2x(x-1)(x-2)+X+\kappa\}\,BDF,$$

where, however, the term κBDF is to be rejected. I cannot quite explain this; I should rather have expected a rejection $=2\kappa BDF$, introducing the term $-\kappa$. For consider a tangent from the curve D from a cusp of the curve $a=c=e$: there are D such tangents; each gives in the neighbourhood of the cusp two points, say c, e; and from these we draw B tangents cBa to the curve B, and F tangents eFa to the curve F; we have thus in respect of the given tangent of D, BF positions of a, or in all BDF positions of a which will ultimately coincide with the cusp; that is, BDF infinitesimal triangles of which the angles a, c, e coincide together at the cusp; and for all the cusps together κBDF such triangles: this would be what is wanted; the difficulty is that as (of the two intersections at the cusp) each in succession might be taken for c, and the other of them for e, it would seem that the foregoing number κBDF should be multiplied by 2.

Case 7. $B=D=F=x$. Here $\mathrm{g}=\chi+\chi'-$ Red. and

$$\chi=Xc(X-1)\,e(X-1)\,a,\quad \chi'=Xe(X-1)\,c(X-1)\,a(=\chi)\,;$$

that is,

$$\chi+\chi'=2X(X-1)^2\,ace.$$

The reductions of the two modes are as above, with only the variation that in the present case D is the same curve with the two curves $B=F$. That of the first mode is $=X(X-1)\,ace$, and that of the second mode is $(2\tau+3\iota)\,ace$, which is $=\{X(X-1)-x\}\,ace$; together they are $=\{2X(X-1)-x\}\,ace$, or subtracting, we have

$$\mathrm{g}=\{2X(X-1)(X-2)+x\}\,ace.$$

Case 8. $a=c=B=x$.

$$\chi=(X-2)(x-3)\,DeFx,\quad \chi'=FeDx\,(X-2)(x-3)\,(=\chi),$$
$$g=2x(x-3)(X-2)\,eDF.$$

Case 9. $D=F=e=x$. By reciprocation of 8.

$$\text{No.}=2X(X-3)(x-2)\,acB.$$

Case 10. $a=c=D=x$.

$$\chi=B(x-1)(X-2)\,eFx,\quad \chi'=FeX(x-2)\,B(x-1),$$
$$g=2(x-1)(Xx-X-x)\,eBF.$$

Case 11. $D=F=a=x$. By reciprocation of 10.

$$\text{No.}=2(X-1)(Xx-X-x)\,ceB.$$

Second process: form $a=B=D=x$.

$$\chi=(X-2)\,c\,(X-1)\,eFx,\quad \chi'=FeXc\,(X-1)(x-2),$$

giving the former result.

Case 12. $c=e=x,\ a=D=y$.

$$\chi=BxY(x-1)\,Fy,\quad \chi'=FxY(x-1)\,By\,(=\chi),$$
$$g=2x(x-1)\,y\,YBF.$$

Case 13. $F=B=x,\ a=D=y$. By reciprocation of 12.

$$\text{No.}=2X(X-1)\,Yyce.$$

Case 14. $c=e=x,\ a=B=y$.

$$\chi=(Y-2)\,xD\,(x-1)\,Fy,\quad \chi'=FxD\,(x-1)\,Y(y-2),$$
$$g=2x(x-1)(Yy-Y-y)\,DF.$$

Case 15. $F=B=x,\ D=e=y$. By reciprocation of 14.

$$\text{No.}=2X(X-1)(Yy-Y-y)\,ac.$$

Case 16. $c=e=x,\ D=F=y$.

$$\chi=BxY(x-1)(Y-1)\,a,\quad \chi'=Yx(Y-1)(x-1)\,Ba\,(=\chi),$$
$$g=2x(x-1)\,Y(Y-1)\,aB.$$

Case 17. $c=e=x,\ B=F=y$.

$$\chi=D(x-1)\,Ya\,(Y-1)\,x,\quad \chi'=Ya\,(Y-1)\,xD\,(x-1)\,(=\chi),$$
$$g=2x(x-1)\,Y(Y-1)\,aD.$$

But we have here aD as an axis of symmetry, so that each triangle is counted twice, or the number of distinct triangles is $=\frac{1}{2}g$.

Case 18. $a = D = x,\ c = B = y.$

$$\chi = Y(y-2)\,XeFx, \quad \chi' = FeXy\,(Y-2)\,x\,(=\chi),$$
$$g = 2xX\,(Yy - Y - y)\,eF.$$

Case 19. $c = F = x,\ e = B = y.$

$$\chi = YxDyXa, \quad \chi' = XyDxYa\,(=\chi),$$
$$g = 2xyXYaD.$$

Case 20. $c = D = x,\ e = F = y.$

$$\chi = Bx\,(X-2)\,y\,(Y-2)\,a, \quad \chi' = Y(y-2)\,X(x-2)\,Ba,$$
$$g = \{xy\,(X-2)(Y-2) + XY(x-2)(y-2)\}\,aB$$
$$= 2\,\{xyXY - xy\,(X+Y) - XY(x+y) + 2xy + 2XY\}\,aB.$$

Case 21. $c = B = x,\ e = F = y.$

$$\chi = X(x-2)\,Dy\,(Y-2)\,a, \quad \chi' = Y(y-2)\,Dx\,(X-2)\,a,$$
$$g = \{X\,(Y-2)\,y\,(x-2) + Y(X-2)\,x\,(y-2)\}\,aD$$
$$= 2\,\{xyXY - xy\,(X+Y) - XY(x+y) + 2xY + 2yX\}\,aD.$$

Case 22. $a = D = x,\ c = F = y,\ e = B = z.$

$$\chi = ZyXzYx, \quad \chi' = YzXyZx\,(=\chi),$$
$$g = 2xyzXYZ.$$

Case 23. $a = B = x,\ c = D = y,\ e = F = z.$

$$\chi = (X-2)\,y\,(Y-2)\,z\,(Z-2)\,x, \quad \chi' = Z(z-2)\,Y(y-2)\,X(x-2),$$
$$g = xyz\,(X-2)(Y-2)(Z-2) + XYZ(x-2)(y-2)(z-2)$$
$$= 2\,\{xyzXYZ - xyz\,(YZ + ZX + XY) - XYZ(yz + zx + xy)$$
$$+ 2xyz\,(X+Y+Z) + 2XYZ(x+y+z) - 4xyz - 4XYZ\}.$$

Case 24. $a = D = x,\ c = B = y,\ e = F = z.$

$$\chi = Y(y-2)\,Xz\,(Z-2)\,x, \quad \chi' = Z(z-2)\,Xy\,(Y-2)\,x,$$
$$g = xX\,\{Y(Z-2)\,z\,(y-2) + Z(Y-2)\,y\,(z-2)\}$$
$$= 2xX\,\{yzYZ - yz\,(Y+Z) - YZ(y+z) + 2yZ + 2zY\}.$$

Case 25. $a = c = x,\ D = F = y,\ e = B = z.$

$$\chi = Z(x-1)\,Yz\,(Y-1)\,x, \quad \chi' = Yz\,(Y-1)\,xZ\,(x-1)\,(=\chi),$$
$$g = 2x\,(x-1)\,Y(Y-1)\,zZ.$$

But we have here eB as an axis of symmetry, so that each triangle is counted twice, or the number of distinct triangles is $=\frac{1}{2}g$.

Case 26. $a=c=x,\ B=D=y,\ e=F=z.$

$$\chi = Y(x-1)(Y-1)\,z(Z-2)\,x,\quad \chi' = Z(z-2)\,Yx(Y-1)(x-1),$$
$$\mathrm{g} = x(x-1)\,Y(Y-1)\{z(Z-2)+Z(z-2)\}$$
$$= 2x(x-1)\,Y(Y-1)(zZ-z-Z).$$

Case 27. $a=c=e=x,\ B=F=y.$ By reciprocation of 28.

$$\text{No.} = \{2x(x-1)(x-2)+X\}\,Y(Y-1)\,D,$$

where each triangle is counted twice, so that the number is really one half of this.

Case 28. $B=D=F=x,\ c=e=y.$

Here

$$\mathrm{g} = \chi+\chi' - \text{Red.}$$
$$\chi = Xy(X-1)(y-1)(X-1)\,a,\quad \chi' = Xy(X-1)(y-1)(X-1)\,a\,(=\chi),$$
$$\chi+\chi' = ay(y-1)\,.\,2X(X-1)^2.$$

The reductions are those of the first and second mode as explained above, with the variation that the curves c and e are here identical, $c=e$, and that the curve D is identical with the curves $B=F$.

First-mode reduction is

$$a(C+2\delta+3\kappa)\,B(B-1)$$

(where δ, κ refer to the curve $c=e$), which is

$$= a\,c(c-1)\,B(B-1);$$

that is, the reduction is $= a\,y(y-1)\,X(X-1)$.

Second-mode reduction is

$$a(2\tau+3\iota)\,c(c-1)$$

(where τ, ι refer to the curve $B=D=F$), which is

$$= a\{B(B-1)-b\}\,c(c-1);$$

that is, the reduction is $= a\,y(y-1)\{X(X-1)-x\}$.

Hence the two together are $= a\,y(y-1)\{2X(X-1)-x\}$; and subtracting from $\chi+\chi'$ we have

$$\mathrm{g} = a\,y(y-1)\,.\,\{2X(X-1)(X-2)+x\};$$

but on account of the symmetry each triangle is reckoned twice, and the number of triangles is $=\frac{1}{2}\mathrm{g}$.

Case 29. $a=c=B=x,\ D=F=y.$

$$\chi = (X-2)(x-3)\,Ye(Y-1)\,x,\quad \chi' = Ye(Y-1)\,x(X-2)(X-3)\,(=\chi),$$
$$\mathrm{g} = 2x(x-3)(X-2)\,Y(Y-1)\,e.$$

Second process. Taking the form

$$C = D = e = x, \quad B = F = y;$$

here

$$\text{No.} = \chi + \chi' - \text{Red.},$$
$$\chi = Yx(X-2)(x-3)\,Ya, = \gamma',$$

and

$$\chi + \chi' = 2Y^2 x(x-3)(X-2)\,a.$$

There is a first-mode reduction,

$$aY\{2\tau + 2\delta(X-4) + 3\kappa(X-3)\},$$

viz. this is

$$\begin{aligned} aY\{ \quad & X^2 - X + 8x - 3\xi \\ & + (X-4)(x^2 - x + 8X - 3\xi) \\ & + (X-3)(\qquad - 9X + 3\xi)\}, \end{aligned}$$

which is

$$= aY\{X(x^2 - x - 6) - 4x^2 + 12x\};$$

and a second-mode reduction

$$= aYX(x-2)(x-3).$$

Hence the two together are

$$= aY\{X(2x^2 - 6x) - 4x^2 + 12x\}$$
$$= 2Yx(x-3)(X-2)\,a,$$

whence the result is

$$= 2(Y^2 - Y)\,x(x-3)(X-2)\,a,$$

which agrees with that obtained above.

On account of the symmetry we must divide by 2.

Case 30. $e = D = F = x,\ a = c = y.$ By reciprocation of 29.

$$\text{No.} = 2X(X-3)(x-2)\,y(y-1)\,B.$$

On account of the symmetry we must divide by 2.

Case 31. $c = e = D = x,\ a = B = y.$

$$\chi = (Y-2)\,x(X-2)(x-3)\,Fy, \quad \chi' = Fx(X-2)(x-3)\,Y(y-2),$$
$$g = x(x-3)(X-2)\,F\{(Y-2)\,y + Y(y-2)\}$$
$$= 2x(x-3)(X-2)(yY - y - Y)\,F.$$

Case 32. $F = B = a = x,\ D = e = y.$ By reciprocation of 31.

$$\text{No.} = 2X(X-3)(x-2)(yY - y - Y)\,c.$$

Case 33. $B=F=y,\ a=e=D=x.$ By reciprocation of 34.

$$\text{No.}=2(x-1)(xX-x-X)\,Y(Y-1)\,c.$$

Case 34. $c=e=y,\ B=D=a=x.$

$$\chi=(X-2)\,y\,(X-1)(y-1)\,Fx,\quad \chi'=FyX\,(y-1)\,(X-1)\,(x-2),$$
$$g=y\,(y-1)\,(X-1)\,\{(X-2)\,x+X\,(X-2)\}\,F$$
$$=2\,(X-1)\,(xX-x-X)\,y\,(y-1)\,F.$$

Case 35. $a=D=y,\ c=e=B=x.$

$$\chi=X\,(x-2)\,Y\,(x-1)\,Fy,\quad \chi'=FxY\,(x-1)\,(X-2)\,y,$$
$$g=yY(x-1)\,\{X\,(x-2)+(X-2)\,x\}\,F$$
$$=2\,(x-1)\,(xX-x-X)\,yYF.$$

Case 36. $a=D=y,\ B=F=e=x.$ By reciprocation of 35.

$$\text{No.}=2\,(X-1)\,(Xx-x-X)\,yYc.$$

Case 37. $a=e=D=x,\ c=B=y.$ By reciprocation of 38.

$$\text{No.}=2\,(x-1)\,\{xyXY-xy\,(X+Y)-XY\,(x+y)+2xy+2XY\}\,F.$$

Case 38. $B=D=a=x,\ F=e=y.$

$$\chi=(X-2)\,c\,(X-1)\,y\,(Y-2)\,x,\quad \chi'=Y\,(y-2)\,Xc\,(X-1)\,(x-2),$$
$$g=(X-1)\,c\,\{xy\,(X-2)\,(Y-2)+XY\,(x-2)\,(y-2)\}$$
$$=2\,(X-1)\,\{xyXY-xy\,(X+Y)-XY\,(x+y)+2xy+2XY\}\,c.$$

Case 39. $a=c=e=B=x.$

Functional process; the curve is assumed to be the aggregate of two curves, say $a=c=e=B=x+x'$. Forming the enumeration

		Case
$x\,X\,x\,DxF$	$x'X'x'Dx'F$	39
$x'X\,x\,.\,x\,.$	&c.	10
$x\,X'x\,.\,x\,.$	.	6
$x'X'x\,.\,x\,.$	.	14
$x\,X\,x'\,.\,x\,.$	.	10
$x'X\,x'\,.\,x\,.$	.	12
$x\,X'x'\,.\,x\,.$	.	14
$x'X'x'\,.\,x\,.$	.	8

(where the second column is derived from the first by a mere interchange of the accented and unaccented letters), I annex to each line the number of the case to

which it belongs; thus $x'XxDxF$ is $B=c=e=x$, which is Case 10, and so in the other instances. Observing that cases 10 and 14 occur each twice, we have thus

$$\begin{aligned}
\phi(x+x')-\phi x-\phi x' = DF \text{ multiplied into}\\
&4(x-1)(Xx-X-x)x' \quad + \;..\quad (10)\times 2\\
&+\{2x(x-1)(x-2)+X\}X' + \;..\quad (6)\\
&+4x(x-1)(X'x'-X'-x') + \;..\quad (14)\times 2\\
&+2x(x-1)x'X' \qquad + \;..\quad (12)\\
&+2x(x-3)(X-2)x' \qquad + \;..\quad (8)
\end{aligned}$$

where the (. .)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is

$$\begin{aligned}
\phi(x+x')-\phi x-\phi x' = DF \text{ multiplied into}\\
&2XX'\\
&+2X(3x^2x'+3xx'^2+x'^3-10xx'-5x'^2+6x')\\
&+2X'(x^3+3x^2x'+3xx'^2-5x^2-10xx'+6x)\\
&-12(x^2x'+xx'^2)+40xx',
\end{aligned}$$

and thence

$$\begin{aligned}
\phi x = \qquad DF \text{ multiplied into}\\
&X^2\\
&+X(2x^3-10x^2+12x)-LX\\
&-4x^3+20x^2 \qquad -lx-\lambda\xi,
\end{aligned}$$

where the constants L, l, λ have to be determined. Now for a cubic curve the number of triangles vanishes; that is, we have $\phi x=0$ in each of the three cases,

$$\begin{aligned}
x=3,\quad & X=6,\quad \xi=18,\\
\text{,,}\quad & X=4,\quad \xi=12,\\
\text{,,}\quad & X=3,\quad \xi=10,
\end{aligned}$$

and we thus obtain the three equations

$$\begin{aligned}
0 &= 108-6L-3l-18\lambda,\\
0 &= 88-4L-3l-12\lambda,\\
0 &= 81-3L-3l-10\lambda,
\end{aligned}$$

giving $L=1$, $l=16$, $\lambda=3$. Whence, finally,

$$\phi x=\{X^2+X(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}\,DF.$$

Second process, by correspondence. We have

$$\begin{aligned}
g-\chi-\chi'+F(\mathrm{e}-\epsilon-\epsilon')&=0,\\
\mathrm{e}-\epsilon-\epsilon'+D(\mathrm{c}-\gamma-\gamma')&=0,
\end{aligned}$$

and thence

$$\mathrm{g}-\chi-\chi'=DF(\mathrm{c}-\gamma-\gamma').$$

Moreover

$$\chi=(X-2)(x-3)\,D\,(x-1)\,F(x-1),$$
$$\chi'=F(x-1)\,D\,(x-1)\,(X-2)\,(x-1),\ =\chi,$$
$$\chi+\chi'=DF(X-2)\,2\,(x-3)\,(x-1)^2,$$

and

$$\mathrm{c}-\gamma-\gamma'=2\tau+(X-3)\,\kappa-2\,(X-2)\,(x-3),$$

as is easily obtained, but see also *post*, No. 29; hence

$$\begin{aligned}\mathrm{g}=DF\ &\text{multiplied into}\\ &(X-2)\,.\,2\,(x-3)\,(x-1)^2\\ &+(X-2)\,.-2\,(x-3)\\ &+2\tau+(X-3)\,\kappa;\end{aligned}$$

but I reject the term $DF\,.\,(X-3)\,\kappa$ as in fact giving a heterotypic solution; I do not go into the explanation of this. And then substituting for 2τ its value, we have

$$\begin{aligned}\mathrm{g}=DF\ &\text{multiplied into}\\ &(X-2)\,.\,2x\,(x-1)\,(x-2)\\ &+X^2-X+8x-3\xi,\end{aligned}$$

where the second factor is

$$=X^2+X\,(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi,$$

which is the foregoing result.

Case 40. $B=D=F=e=x$. By reciprocation of 39.

$$\text{No.}=\{x^2+x\,(2X^3-10X^2+12X-1)-4X^3+20X^2-16X-3\xi\}\,ac.$$

Case 41. $c=e=D=F=x$.

$$\begin{aligned}\chi&=Bx\,(X-2)\,(x-3)\,(X-3)\,a,\\ \chi'&=X\,(x-2)\,(X-3)\,(x-3)\,Ba,\\ \mathrm{g}&=\quad(x-3)\,(X-3)\,aB\,\{x\,(X-2)+X\,(x-2)\},\\ &=2\quad(x-3)\,(X-3)\,(xX-x-X)\,aB.\end{aligned}$$

Case 42. $a=c=D=F=x$.

Functional Process; the curve is supposed to be the aggregate of two curves, say $a=c=D=F=x+x'$.

The enumeration is

		Case
$x\ Bx\ X\ e\ X$	$x'Bx'X'eX'$,	(42)
$x'\ .\ x\ X\ .\ X$	&c.	(11)
$x\ .\ x'X\ .\ X$		(11)
$x'\ .\ x'X\ .\ X$		(17)
$x\ .\ x\ X'\ .\ X$		(10)
$x'\ .\ x\ X'\ .\ X$		(19)
$x\ .\ x'X'\ .\ X$		(21)
$x'\ .\ x'X'\ .\ X$		(10)

whence

$$\begin{aligned}
&\phi(x+x')-\phi x-\phi x'=eB \text{ multiplied into}\\
&\quad 4(X-1)(Xx-X-x)\,x' && +\,..\ (11)\times 2\\
&\quad +2x(x-1)X'(X'-1) && +\,..\ (17)\\
&\quad +4(x-1)(Xx-x-x)\,X' && +\,..\ (10)\times 2\\
&\quad +2xXx'X' && +\,..\ (19)\\
&\quad +2xx'XX'-2(x+x')XX'-2(X+X')xx'+4(Xx'+X'x)+\,.. && (21)
\end{aligned}$$

where the (. .)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, we have

$$\begin{aligned}
&\phi(x+x')-\phi x-\phi x'=eB \text{ multiplied into}\\
&\quad X^2\ (4xx'+2x'^2-6x')\\
&\quad +XX'(4x^2+8xx'+4x'^2-12x-12x'+8)\\
&\quad +X'^2\ (2x^2+4xx'-6x)\\
&\quad +X\ (-12xx'-6x'^2+18x')\\
&\quad +X'\ (-6x^2-12xx'+18x)\\
&\quad +\ 8xx',
\end{aligned}$$

and consequently

$$\begin{aligned}
\phi x = \quad & eB \text{ multiplied into}\\
&X^2(2x^2-6x+4)\\
&+X\ (-6x^2+18x+L)\\
&+\ 4x^2+lx\ +\lambda\xi,
\end{aligned}$$

where the constants L, l, λ have to be determined. The number of triangles vanishes when the curve is a line or a conic, that is $\phi x=0$ for $x=1$, $X=0$, $\xi=0$, and for $x=X=2$, $\xi=6$; we thus have

$$\begin{aligned}
0 &= 4+l,\\
0 &= 40+2L+2l+6\lambda.
\end{aligned}$$

Moreover, the data being sibireciprocal, the result must be so likewise; we must therefore have $L=l$. We thus obtain $L=l=\lambda=-4$; so that finally

$$\phi x=\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}\,eB.$$

Second process, by correspondence: form $a=c=D=F=x$. We have

$$\mathrm{c}-\chi-\chi'+2(\mathrm{f}-\phi-\phi')=0;$$

also from the special consideration that the points D, F are given as the intersections of the curve x, by the first polar of the point e, which first polar does not pass through a, we have

$$(\mathrm{f}-\phi-\phi')+e(\mathrm{d}-\delta-\delta')=0,$$

and by the consideration that c, D are given as intersections, c a double intersection, of the curve with the first polar of the point c, which first polar does not pass through a,

$$\mathrm{d}-\delta-\delta'+2(\mathrm{c}-\gamma-\gamma')=0,$$

whence

$$\mathrm{g}-\chi-\chi'=-4e(\mathrm{c}-\gamma-\gamma')$$

and

$$\mathrm{c}-\gamma-\gamma'=B\Delta,$$

so that this is

$$\begin{aligned}\mathrm{g}-\chi-\chi'&=-4Be\Delta\\&=-4Be(-2X-2x+2+\xi).\end{aligned}$$

Also

$$\begin{aligned}\chi&=B(x-1)(X-2)\,e(X-1)(x-2),\\ \chi'&=(X-2)\,e(X-1)(x-2)\,B(x-1),\ =\chi,\end{aligned}$$

so that

$\mathrm{g}=Be$ multiplied into

$$2(X-1)(X-2)(x-1)(x-2)-4(-2X-2x+2+\xi),$$

viz. this is

$$Be\,\{X^2(2x^2-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi\}.$$

Third process: form $c=e=F=B=x$.

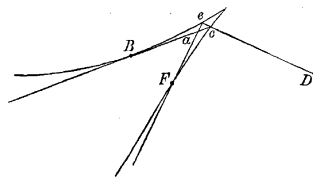
$$\begin{aligned}\mathrm{g}&=\chi+\chi'-\text{Red.},\\ \chi&=X(x-2)\,D(x-1)(X-2)\,a,\\ \chi'&=X(x-2)\,D(x-1)(X-2)\,a,\ =\chi,\\ \chi+\chi'&=aD\,.\,2X(X-2)(x-1)(x-2).\end{aligned}$$

The first-mode reduction is here

$$aD\,[(X-2)\,X+(X-4)\,2\delta+(X-3)\,3\kappa+\kappa];$$

where the last term $aD\kappa$ arises from the tangents cBa and eFa, each coinciding with a cuspidal tangent, as shown in the figure.

Fig. 4.



The second-mode reduction is

$$= aD \,.\, X(x-2)(x-3),$$

so that the two reductions together are

$$= aD\,\{(X-2)\,X + (X-4)\,2\delta + (X-3)\,3\kappa + \kappa + X(x-2)(x-3)\},$$

viz. this is

$$= aD\,\{(X-2)\,X + (X-4)(2\delta + 3\kappa) + 4\kappa + X(x-2)(x-3)\}\,;$$

or substituting for $2\delta + 3\kappa$ and κ the values $x^2 - x - X$ and $-3X + \xi$ respectively, and reducing, it is

$$aD\,\{X(2x^2 - 6x - 4) - 4x^2 + 4x + 4\xi\}.$$

Hence subtracting from $\chi + \chi'$, written in the form

$$aD\,\{X^2(2x^2 - 6x + 4) + X(-4x^2 + 12x - 8)\},$$

the result is

$$= aD\,\{X^2(2x^2 - 6x + 4) + X(-6x^2 + 18x - 4) + 4x^2 - 4x - 4\xi\}.$$

On account of the symmetry we must divide by 2.

Case 43. $a = c = e = x,\ B = D = F = y.$

Suppose for a moment that the angle a is a free point; the locus of a is a curve the order of which is obtained from Case 28, by writing $c = e = x,\ B = D = F = y$; the locus in question meets a curve order a in $\{2Y(Y-1)(Y-2) + y\}\,x(x-1)\,a$ points; wherefore the order of the locus is

$$= \{2Y(Y-1)(Y-2) + y\}\,x\,(x-1),$$

and this locus meets the curve $a = c = e = x$ in a number of points

$$= \{2Y(Y-1)(Y-2) + y\}\,x^2(x-1),$$

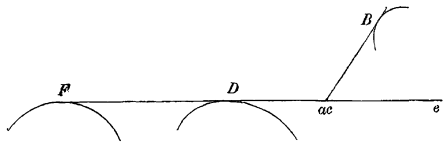
viz. this is the number of positions of the angle a; but several of these belong to special forms of the triangle $aBcDeF$, giving heterotypic solutions, which are to be rejected; the required number is thus

$$\{2Y(Y-1)(Y-2) + y\}\,x^2(x-1) - \text{Red.}$$

The reduction is due first and secondly to triangles wherein the angle a coincides with an angle c or e, and thirdly to triangles wherein the angles a, c, e all coincide.

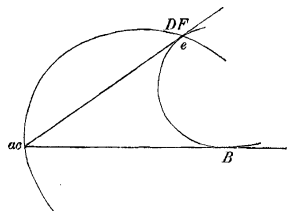
1°. Take for the side cDe a double tangent of the curve $B=D=F$, this meets the curve $a=c=e$ in x points, and selecting any one of them for e and any other for c,

Fig. 5.



we have from the last-mentioned point $Y-2$ tangents to the curve $B=D=F$; and in respect of each of these a position of a coincident with c. The reduction on this account is $2\tau x(x-1)(Y-2)$; but since we may in the figure interchange c and e, B and F, we have the same number belonging to the coincidence of the angles a, e, or together the reduction is $=4\tau x(x-1)(Y-2)$.

Fig. 6.



But instead of a double tangent we may have cDe a stationary tangent; we have thus reductions $3\iota x(x-1)(Y-2)$ and $3\iota x(x-1)(Y-2)$, together $6\iota x(x-1)(Y-2)$; and for the double and stationary tangents together we have

$$(4\tau+6\iota)x(x-1)(Y-2),$$

$$=2\{Y(Y-1)-y\}x(x-1)(Y-2),$$

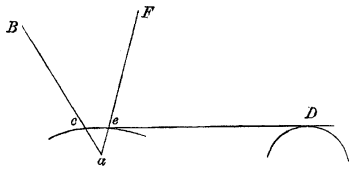
that is

$$=2x(x-1)Y(Y-1)(Y-2)-2x(x-1)y(Y-2).$$

2°. The side cDe may be taken to be a tangent to the curve $B=D=F$ at any one of its intersections with the curve $a=c=e$. Taking then the point e at the intersection in question, and the point c at any other of the intersections of the tangent with the curve $a=c=e$, and from c drawing any other tangent to the curve $B=D=F$, there is in respect of each of these tangents a position of a at c; and

the reduction on this account is $= xy(x-1)(Y-1)$. But interchanging in the figure the letters c, e, B, F, there is an equal reduction belonging to the coincidence of a, e; and the whole reduction in this manner is $= 2x(x-1)y(Y-1)$.

Fig. 7.



3°. If the side cDe intersects the curve $a = c = e$ in two coincident points, then taking these in either order for the points c, e, and from the two points respectively drawing two other tangents to the curve $D = B = F$, we have a triangle wherein the angles a, c, e all coincide. The side cDe may be a proper tangent to the curve $a = c = e$, or it may pass through a node or a cusp of this curve, viz. it is either a common tangent of the curves $B = D = F$ and $a = c = e$ (as in the figure, except that for greater distinctness the points c and e are there drawn nearly instead of actually coincident), or it may be a tangent to the curve $B = D = F$ from a node or a cusp of the curve $a = c = e$; we have thus the numbers

Common tangent	$XY(Y-1)(Y-2)$,
Tangent from node	$2\delta Y(Y-1)(Y-2)$,
Tangent from cusp	$2\kappa Y(Y-1)(Y-2)$;

but (as we are counting intersections with the curve $a = c = e$) the second of these, as being at a node of this curve, is to be taken 2 times; and the third, as being at a cusp, 3 times; and the three together are thus

$$\begin{aligned}&(X + 4\delta + 6\kappa\ \ )\,Y(Y-1)(Y-2),\\ &= \{2x(x-1) - X\}\,Y(Y-1)(Y-2).\end{aligned}$$

The reductions 1°, 2°, 3° altogether are

$$\begin{aligned}&2x(x-1)\,Y(Y-1)(Y-2)\\ &-2x(x-1)\,y(Y-2)\\ &+2x(x-1)\,y(Y-1)\\ &+2x(x-1)\,Y(Y-1)(Y-2)\\ &-XY(Y-1)(Y-2),\end{aligned}$$

which is

$$\begin{aligned}&= 4x(x-1)\,Y(Y-1)(Y-2)\\ &\quad +2x(x-1)\,y\\ &-XY(Y-1)(Y-2);\end{aligned}$$

and subtracting from the before-mentioned number

$$2x^2(x-1)\,Y(Y-1)(Y-2)$$
$$+\,x^2(x-1)\,y,$$

the required number of positions of the angle a is

$$=2x(x-1)(x-2)\,Y(Y-1)(Y-2)$$
$$+\,yx(x-1)(x-2)\,y+XY(Y-1)(Y-2).$$

The number of triangles is on account of the symmetry equal to one-sixth of this number.

Case 44. $e=D=F=x,\ a=c=B=y.$

$$\chi=(Y-2)(y-3)\,X(x-2)(X-3)\,y,$$
$$\chi'=X(x-2)(X-3)\,y(Y-2)(y-3)\,(=\chi),$$
$$g=2(x-2)\,X(X-3)(Y-2)\,y(y-3):$$

there is a division by 2 on account of the symmetry.

Case 45. $a=D=B=x,\ c=e=F=y.$

$$\chi=(X-2)\,y(X-1)(y-1)(Y-2)\,x,$$
$$\chi'=Y(y-2)\,X(y-1)(X-1)(x-2),$$
$$g=(X-1)(y-1)\{xy(X-2)(Y-2)+XY(x-2)(y-2)\}$$
$$=2(X-1)(y-1)\{XYxy-XY(x+y)-xy(X+Y)+2xy+2XY\}.$$

Case 46. $a=c=y,\ B=D=F=e=x.$ By reciprocation of 47,

$$\text{No.}=y(y-1)\{x^2+x(2X^3-10X^2+12X-1)-4X^3+20X^2-16X-3\xi\}:$$

there is a division by 2 on account of the symmetry.

Case 47. $D=F=y,\ a=c=e=B=x.$

The functional process is exactly the same as for No. 39 ($a=c=e=B=x$), with only $Y(Y-1)$ written instead of DF; hence

$$\text{No.}=Y(Y-1)\{X^2+X(2x^3-10x^2+12x-1)-4x^3+20x^2-16x-3\xi\}:$$

there is a division by 2 on account of the symmetry.

Case 48. $a=c=D=F=x,\ e=B=y.$

The functional process, writing $a=c=D=F=x+x'$, would be precisely the same as for Case 42, with only the factor yY written instead of eB; and we have thus the like result, viz.

$$\text{No.}=\{X^2(2x^3-6x+4)+X(-6x^2+18x-4)+4x^2-4x-4\xi)\,yY,$$

which on account of the symmetry must be divided by 2.

Case 49. $a = B = y,\ c = e = D = F = x.$

$$\chi = (Y-2)\,x\,(X-2)\,(x-3)\,(X-3)\,y,$$
$$\chi' = X\,(x-2)\,(X-3)\,(x-3)\,Y\,(y-2),$$
$$\begin{aligned} g &= (x-3)\,(X-3)\,\{xy\,(X-2)\,(Y-2) + XY\,(x-2)\,(y-2)\} \\ &= 2\,(x-3)\,(X-3)\,\{xyXY - (x+y)\,XY - (X+Y)\,xy + 2xy + 2XY\}. \end{aligned}$$

Case 50. $c = e = B = D = F = x.$

Functional process; by taking the curve $c = e = B = D = F$ as the aggregate of two curves, say $= x + x'$. The cases are

		Case
$aX\,x\,X\,x\,X$	$aX'x'X'x'X'$	50
$.\,X'x\,X\,x\,.$	&c.	41
$.\,X\,x'X\,x\,.$	.	40
$.\,X'x'X\,x\,.$	.	32
$.\,X\,x\,X'x\,.$	.	42
$.\,X'x\,X'x\,.$	.	33
$.\,X\,x'X'x\,.$	.	38
$.\,X'x'X'x\,.$	.	32
$.\,X\,x\,X\,x'.$	.	40
$.\,X'x\,X\,x'.$	.	36
$.\,X\,x'X\,x'.$	.	28
$.\,X'x'X\,x'.$	.	33
$.\,X\,x\,X'x'.$	.	38
$.\,X'x\,X'x'.$	.	36
$.\,X\,x'X'x'.$	.	29
$.\,X'x'X'x'X$	.	41

and we thus have

$\phi\,(x + x') - \phi x - \phi x' = a$ multiplied into

$$\begin{aligned}
= \quad & 4\,(x-3)\,(X-3)\,(xX - x - X)\,X' + \ldots && 2\,(41) \\
& + 2x'\,[x^2 + x\,(2X^3 - 10X^2 + 12X - 1) - 4X^3 + 20X^2 - 16X - 3\xi] && + \ldots\ 2\,(40) \\
& + 4X\,(X-3)\,(x-2)\,(x'X' - X' - x') && + \ldots\ 2\,(32) \\
& + [X^2\,(2x^2 - 6x + 4) + X\,(-6x^2 + 18x - 4) + 4x^2 - 4x - 4\xi]\,X' && + \ldots\ (42) \\
& + 4\,(x-1)\,(xX - x - X)\,(X'^2 - X') && + \ldots\ 2\,(33) \\
& + 4\,(X-1)\,[XX'xx' - XX'\,(x + x') - xx'\,(X + X') + 2XX' + 2xx'] && + \ldots\ 2\,(38) \\
& + 4xX\,(X'-1)\,(X'x' - X' - x') && + \ldots\ 2\,(36) \\
& + (x^2 - x)\,(2X'^3 - 6X'^2 + 4X' + x') && + \ldots\ (28) \\
& + 2x\,(x-3)\,(X-2)\,(X'^2 - X') && + \ldots\ (29)
\end{aligned}$$

where as before the (. .)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is

$$
\begin{aligned}
\phi(x+x') - \phi x - \phi x' = a \text{ multiplied into}\\
3x^2x' + 3xx'^2\\
+ x^2 . 6X^2X' + 6XX'^2 + 2X'^3\\
- 28XX' - 14X'^2\\
+ 28X'\\
+ xx' . 4X^3 + 12X^2X' + 12XX'^2 + 4X'^3\\
- 28X^2 - 56XX' - 28X'^2\\
+ 56X + 56X'\\
- 22\\
+ x'^2 . 2X^3 + 6X^2X' + 6XX'^2\\
- 14X^2 - 28XX'\\
+ 28X\\
+ x . - 30X^2X' - 30XX'^2 - 10X'^3\\
+ 140XX' + 70X'^2\\
- 116X' - 6\xi'\\
+ x' . - 10X^3 - 30X^2X' - 30XX'^2\\
+ 70X^2 + 140XX'\\
- 116X - 6\xi\\
+ 36X^2X' + 36XX'^2\\
- 152XX'\\
- 4(X\xi' + X'\xi);
\end{aligned}
$$

whence

$$
\begin{aligned}
\phi x = a \text{ multiplied into}\\
x^3 (\qquad\qquad\qquad\qquad + 1)\\
+ x^2 (\quad 2X^3 - 14X^2 + 28X - 11)\\
+ x\ (- 10X^3 + 70X^2 - 116X + l)\\
+ \qquad 12X^3 - 76X^2 \qquad + LX\\
+ \xi\ (- \ 6x - 4X \qquad + \lambda),
\end{aligned}
$$

where the constants l, L, λ have to be determined. We should have $\phi x = 0$ for a cubic curve; viz. $x = 3$: $X = 6$, $\xi = 18$; $X = 4$, $\xi = 12$; or $X = 3$, $\xi = 10$. Writing first $x = 3$, the equation is

$$8X^2 - 96X - 72 - \xi(18 + 4X) + 3l + XL + \xi\lambda = 0,$$

giving in the three cases respectively

$$\begin{aligned} 3l + 6L + 18\lambda &= 1116, \\ 3l + 4L + 12\lambda &= 736, \\ 3l + 3L + 10\lambda &= 588\,; \end{aligned}$$

and we have then $l = -8$, $L = 64$, $\lambda = 42$, so that the required number is

$$\begin{aligned} = \; & x^3 (\qquad\qquad\qquad + 1) \\ & + x^2 (\quad 2X^3 - 14X^2 + 28X - 11) \\ & + x\,(-10X^3 + 70X^2 - 116X - 8) \\ & + \qquad 12X^3 - 76X^2 + 64X \\ & + \xi\,(-\quad 6x - 4X \quad + 42 \qquad). \end{aligned}$$

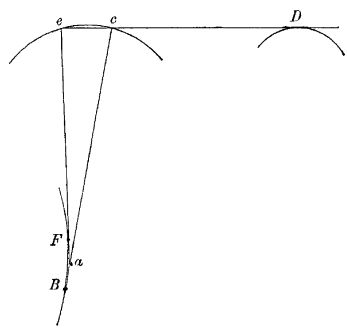
As a verification, observe that for a conic, $x = X = 2$, $\xi = 6$, this is $= 0$.

Second process, by correspondence: form $c = e = B = D = F = x$.

We have

$$\begin{aligned} g &= \chi + \chi' - \text{Red.}, \\ \chi &= X(x-2)(X-3)(x-3)(X-3)\,a, \\ \chi' &= X(x-2)(X-3)(x-3)(X-3)\,a, = \chi, \\ \chi + \chi' &= a \text{ into} \\ & 2(x-2)(x-3)\,X(X-3)^2. \end{aligned}$$

Fig. 8.



There is a first-mode reduction, which is

$$= a\,\{2\delta\,(X-4)(X-5) + 3\kappa\,(X-3)(X-4) + \kappa\,(X-3) + 2\tau\,(X-3)\},$$

where the term $a \,.\, 2\tau\,(X-3)$ arises, as shown in the figure, and a second-mode reduction, which is

$$= a\,\{2\tau\,(x-4)\,(x-5) + 3\iota\,(x-3)\,(x-4)\}\,;$$

and the two together are $= a$ into

$$\begin{aligned}
&(X-4)\,(X-5)\,(x^2 - x + 8X \qquad\quad - 3\xi)\\
&+ (X-3)\,(X-4)\,(\qquad\quad 9X \qquad\quad + 3\xi)\\
&+ \qquad (X-3)\left(\begin{array}{l} \qquad\quad - 3X \qquad\quad + \xi \\ + X^2 - X + 8x - 3\xi \end{array}\right)\\
&+ (x-4)\,(x-5)\,(X^2 - X + 8x \qquad\quad - 3\xi)\\
&+ (x-3)\,(x-4)\,(\qquad\qquad\qquad - 9x + 3\xi)\,;
\end{aligned}$$

that is, $= a$ into

$$\begin{aligned}
&- x^3\\
&+ x^2 \,.\, 2X^2 - 10X + 11\\
&+ x \,.\, - 10X^2 + 26X + 8\\
&+ \;\; 4X^2 + 44X\\
&+ \xi\,(6x + 4X - 42)\,;
\end{aligned}$$

and subtracting this from the foregoing value of $\chi + \chi'$, which is $= a$ into

$$\begin{aligned}
&x^2\,(\quad 2X^3 - 12X^2 + \;\; 18X)\\
&+ x\,(- 10X^3 + 60X^2 - \;\; 90X)\\
&+ \qquad 12X^3 - 72X^2 + 108X,
\end{aligned}$$

the result is as before.

There is a division by 2 on account of the symmetry.

Case 51. $a = c = e = B = D = x$. By reciprocation of 50,

$$\begin{aligned}
\text{No. is} = \;\; &X^3\,(\qquad\qquad\qquad\qquad + \;\; 1)\\
&+ X^2\,(\quad 2x^3 - 14x^2 + \;\; 28x - 11)\\
&+ X\;(-10x^3 + 70x^2 - 116x - \;\; 8)\\
&+ \qquad 12x^3 - 76x^2 + \;\; 64x\\
&+ \xi\,(- 6X - 4x + 42).
\end{aligned}$$

There is a division by 2 on account of the symmetry.

Case 52. $a = c = e = B = D = F = x$.

Functional process, by taking the curve to be the aggregate of two curves, say $= x + x'$. The enumeration of the cases is conveniently made in a somewhat different manner from that heretofore employed, viz. we may write

x or x'	x' or x	Case	times
$\parallel$	$\parallel$		
all	none	(52)	1
a	residue	(50)	3
B	"	(51)	3
a, c	"	(46)	3
B, D	"	(47)	3
a, D	"	(48)	3
a, B	"	(49)	6
a, c, e	B, D, F	(43)	1
a, B, F	c, e, D	(44)	3
a, B, D	c, e, F	(45)	6
			32;

and the functional equation then is

$$\phi(x+x') - \phi x - \phi x'$$

$$= 3x' \left\{\begin{array}{l} x^2(\qquad\qquad\qquad + 1) \\ x^2(\quad 2X^3 - 14X^2 + 28X - 11) \\ x\,(-10X^3 + 70X^2 - 116X - 8) \\ \quad + 12X^3 - 76X^2 + 64X \\ \quad + \xi(-6x - 4X \qquad + 42) \end{array}\right\} + \ldots \qquad (50)\times 3$$

$$+ 3X' \left\{\begin{array}{l} X^2(\qquad\qquad\qquad + 1) \\ X^2(\quad 2x^3 - 14x^2 + 28x - 11) \\ X\,(-10x^3 + 70x^2 - 116x - 8) \\ \quad + 12x^3 - 76x^2 + 64x \\ \quad + \xi(-6X - 4x \qquad + 42) \end{array}\right\} \qquad (51)\times 3$$

$$+ 3(x'^2 - x') \left\{\begin{array}{l} x^2 \\ + x(2X^3 - 10X^2 + 12X - 1) \\ - \quad 4X^3 + 20X^2 - 16X - 3\xi \end{array}\right\} + \ldots \qquad (46)\times 3$$

$$+3(X'^2-X')\left\{\begin{array}{l} X^2 \\ +X\ (2x^3-10x^2+12x-1) \\ \quad -4x^3+20x^2-16x-3\xi \end{array}\right\} \quad +\,.. \qquad (47)\times 3$$

$$+3x'X' \left\{\begin{array}{l} X^2(\ \ 2x^2-\ 6x+4) \\ +X\ (-6x^2+18x-4) \\ \quad +4x^2-\ 4x-4\xi \end{array}\right\} \quad +\,.. \qquad (48)\times 3$$

$$+12(x'-3)(X'-3)\{xx'XX'-xx'(X+X')-XX'(x+x')+2xx'+2XX'\} \quad +\,.. \quad (49)\times 6$$

$$+\{2x'(x'-1)(x'-2)X(X-1)(X-2)+xx'(x'-1)(x'-2)+X'X(X-1)(X-2)\} \quad +\,.. \quad (43)$$

$$+6(x'-2)X'(X'-3)(X-2)(x-3) \quad +\,.. \quad (44)\times 3$$

$$+12(X'-1)(x-1)\{xx'XX'-xx'(X+X')-XX'(x+x')+2xx'+2XX'\} \quad +\,.. \quad (45)\times 6$$

where as before the (..)'s refer to the like functions with the two sets of letters interchanged. Developing and collecting, this is found to be

$$\begin{array}{ll} = & 4X^3X'+6X^2X'^2+4XX'^3 \\ & +X^3\left\{\begin{array}{l} 6x^2x'+6xx'^2+2x'^3 \\ -36xx'-18x'^2 \\ +52x' \end{array}\right\} \\ & +(X^2X'+XX'^2)\left\{\begin{array}{l} 6x^3+\ 18x^2x'+18xx'^2+6x'^3 \\ -\ 54x^2-108xx'-54x'^2 \\ +156x+156x' \\ -138 \end{array}\right\} \\ & +X'^3\left\{\begin{array}{l} 2x^3+6x^2x'+6xx'^2 \\ -18x^2-36xx' \\ +52x \end{array}\right\} \\ & +\ \&c.\ \&c. \end{array}$$

I abstain from writing down the remaining terms, as they can at once be obtained backwards from the value of ϕx; they were in fact found directly, and the integration of the functional equation then gives

$$\begin{array}{ll} \phi x = & X^4(\qquad\qquad\qquad +\ 1) \\ & +X^3(\qquad 2x^3-\ 18x^2+\ 52x-\ 46) \\ & +X^2(\ -18x^3+162x^2-420x+221) \\ & +X\ (\qquad 52x^3-420x^2+704x+l\quad) \\ & +\qquad x^4-46x^3+221x^2+\quad lx \\ & +\xi\left\{\begin{array}{l} X^2(\qquad\qquad -\ 9) \\ +X\ (\ -\ 12x+135) \\ \quad -9x^2+135x+\quad \lambda \end{array}\right\} \end{array}$$

where the constants l, λ have to be determined; I have in the first instance written $l(X+x)+\lambda\xi$, instead of $LX+lx+\lambda\xi$, thus introducing two constants only, since it is clear from the symmetry in regard to x, X that we must have $l=L$. We must have $\phi x=0$, when the curve is a conic or cubic. Writing $x=2$, we have

$$\phi x = X^4+2X^3-115X^2+144X+532+\xi(-9X^2+111X+234)+l(2+X)+\xi\lambda,$$

and then for the conic, $X=2$, $\xi=6$.

Writing $x=3$, we have

$$\phi x = X^4+2X^3-67X^2-264X+828+\xi(-9X^2+99X+324)+l(3+X)+\lambda\xi,$$

and then for the three cases of the cubic $X=6$, $\xi=18$; $X=4$, $\xi=12$; and $X=3$, $\xi=10$. We have thus the four equations

$$\begin{aligned} 2912+4l+\ \ 6\lambda&=0,\\ 9252+9l+18\lambda&=0,\\ 5796+7l+12\lambda&=0,\\ 4968+6l+10\lambda&=0, \end{aligned}$$

all satisfied by $l=+172$, $\lambda=-600$. Hence finally

$$\begin{aligned} \phi x = \quad & X^4(\qquad\qquad\qquad\qquad +1)\\ & +X^3(\qquad 2x^3-18x^2+52x-46)\\ & +X^2(\quad -18x^3+162x^2-420x+221)\\ & +X\ (\qquad 52x^3-420x^2+704x+172)\\ & +\qquad x^4-46x^3+221x^2+172x\\ & +\xi\left\{\begin{array}{l} X^2(\qquad\qquad -9)\\ +X\ (\qquad -12x+135)\\ \qquad -9x^2+135x-600 \end{array}\right\} \end{aligned}$$

but on account of the symmetry the number of triangles is = one-sixth of this expression.

Article Nos. 22 to 36. *The Case* 52, *as belonging to a different series of Problems.*

22. In the foregoing Case 52, where all the curves are one and the same curve, we have the unclosed trilateral $aBcDeFg$, and we seek for the number of the united points (a, g). But we may consider this as belonging to a series of questions, viz. we may seek for the number of the united points (a, B), (a, c), (a, D), (a, e), (a, F), (the last four of these giving by reciprocity the numbers of the united points (B, D), (B, e), (B, F), (B, g)), and finally the number of the united points (a, g). It is very instructive to consider this series of questions, and the more so that in those which precede (a, F) there are only special solutions having reference to the singular points and tangents of the curve, and that the solutions thus explain themselves.

23. Thus the first case is that of the united points (a, B), viz. we have here a point a on the curve, and from it we draw to the curve a tangent aB touching it at B; the points a and B are to coincide together. Observe that from a point in general a of the curve we have $X-2$ tangents (X the class as heretofore), viz. we disregard altogether the tangent *at* the point, counting as 2 of the X tangents from a point not on the curve, and attend exclusively to the $X-2$ tangents *from* the point. Now if the point a is an inflection, or if it is a cusp, there are only $X-3$ tangents, or, to speak more accurately, one of the $X-2$ tangents has come to coincide with the tangent *at* the point; such tangent is a tangent of three-pointic intersection, viz. we have the point a and the point B (counting, as a point of contact, twice) all three coinciding; that is, we have a position of the united point (a, B); and the number of these united points is $=\iota+\kappa$.

24. It is important to notice that neither a point of contact of a double tangent, nor a double point, is a united point. In the case of the point of contact of a double tangent, one of the tangents from the point coincides with the double tangent; but the point B is here the other point of contact of this tangent, so that the points a, B are not coincident. In the case of a double point, regarding the assumed position of a at the double point as belonging to one of the two branches, then of the $X-2$ tangents there are two, each coinciding with the tangent to the other branch; hence, attending to either of these, the point B belongs to the other branch, and thus, though a and B are each of them at the double point, the two do not constitute a united point. (In illustration remark that for a unicursal curve, the position of a answers to a value $=\lambda$, and that of B to a value $=\mu$ of the parameter θ, viz. λ, μ are the two values of θ at the double point; contrariwise in the foregoing case of a cusp, where there is a single value $\lambda=\mu$. Hence the whole number of the united points (a, B) is $=\iota+\kappa$, and this is in fact the value given, as will presently appear, by the theory of correspondence.)

I recall that I use Δ, $=2D$, to denote twice the deficiency of the curve, viz. that we have $\Delta=X-2x+2+\kappa$, $=-2x-2X+2+\xi$.

25. The several cases are

United points.	
(a, B)	$\mathrm{b}-\beta-\beta'=2\Delta$,
(a, c)	$\mathrm{c}-\gamma-\gamma'+2(\mathrm{b}-\beta-\beta')=(X-2)\Delta$,
(B, D)	$\mathrm{c}_0-\gamma_0-\gamma_0'$ by reciprocity,
(a, D)	$\mathrm{d}-\delta-\delta'+2(\mathrm{c}_0-\gamma_0-\gamma_0')+(X-3)(\mathrm{b}-\beta-\beta')=0$,
(a, e)	$\mathrm{e}-\epsilon-\epsilon'+2(\mathrm{d}-\delta-\delta')+(X-3)(\mathrm{c}-\gamma-\gamma')=0$,
(B, F)	$\mathrm{e}_0-\epsilon_0-\epsilon_0'$ by reciprocity,
(a, F)	$\mathrm{f}-\phi-\phi'+2(\mathrm{e}_0-\epsilon_0-\epsilon_0')+(X-3)(\mathrm{d}-\delta-\delta')=0$,
(a, g)	$\mathrm{g}-\chi-\chi'+2(\mathrm{f}-\phi-\phi')+(X-3)(\mathrm{e}-\epsilon-\epsilon')=0$,
(B, H)	$\mathrm{g}_0-\chi_0-\chi_0'$ by reciprocity,

and so on.

26. The mode of obtaining these equations appears *ante*, Nos. 5 and 6, but for greater clearness I will explain it in regard to a pair of the equations, say those for (a, e), (a, D). Regarding a as given, we draw from a the tangents aBc, touching at B and besides intersecting at c (viz. the number of tangents is $=X-2$, and the number of the points c is $=(X-2)(x-3)$); from each of the positions of c we draw to the curve the $(X-3)$ tangents cDe touching at D and intersecting at e; the whole number of these tangents is $=(X-2)(x-3)(X-3)$; and this is also the number of the points D, but the number of the points e is $=(X-2)(x-3)(X-3)(x-3)$. Now this system of the $(X-2)(x-3)(X-3)$ tangents is the curve Θ of the general theory (*ante*, Nos. 3, 4), viz. the curve Θ (which does not pass through a) intersects the given curve in the three classes of points c, D, e, the number of intersections at a point e being $=1$, at a point D being $=2$, and at a point c being $=X-3$. And we have thus the equation

$$\mathrm{e}-\epsilon-\epsilon'+2(\mathrm{d}-\delta-\delta')+(X-3)(\mathrm{c}-\gamma-\gamma')=0,$$

where e, d, c are the numbers of united points and (ϵ, ϵ'), (δ, δ'), (γ, γ') the correspondences in the three cases respectively.

27. Observe that we cannot, starting from a, obtain in this manner the equation for the number of the united points (a, D); for we introduce per force the points e, and thus obtain the foregoing equation for (a, e). But starting from D, the tangent at this point besides intersects the curve in $(x-2)$ points, each of which is a position of c; and from each of these drawing a tangent cBa to the curve, we have the curve Θ consisting of these $(x-2)(X-3)$ tangents, not passing through D, but intersecting the given curve in the three classes of points c, B, a, viz. the number of intersections at each point c is $=X-3$, at each point B it is $=2$, and at each point a it is $=1$; and we have thus the equation

$$(\mathrm{d}-\delta-\delta')+2(\mathrm{c}_0-\gamma_0-\gamma_0')+(X-3)(\mathrm{b}-\beta-\beta')=0,$$

where the numbers $(\mathrm{d}, \delta, \delta')$, $(\mathrm{c}_0, \gamma_0, \gamma_0')$, $(\mathrm{b}, \beta, \beta')$ refer to the correspondences (D, a), (D, B), and (D, c) (or what is the same thing (a, B)) respectively.

28. Correspondence (a, B).

We have

$$\beta=X-2, \quad \beta'=x-2,$$

and thence

$$\begin{aligned}\mathrm{b}&=x+X-4+2\Delta\\&=-3x-3X+2\xi,\end{aligned}$$

which is the solution: the value obtained above was $\mathrm{b}=\iota+\kappa$, and we in fact have identically

$$\iota+\kappa=-3x-3X+2\xi.$$

It was in this manner that I originally applied the principle of correspondence to investigating the number of inflections of a curve, regarding, however, the term κ as a special solution; it is better to put the cusp and inflection on the same footing as above.

29. Correspondence (a, c).

Since $\mathrm{b}-\beta-\beta'=2\Delta$, we have here

$$\mathrm{c}-\gamma-\gamma'=(X-6)\Delta,$$

and

$$\gamma=\gamma'=(X-2)(x-3),$$

whence

$$\begin{aligned}\mathrm{c}&=2(X-2)(x-3)+(X-6)(-2x-2X+2+\xi)\\&=-2X^2+8X+8x+(X-6)\xi;\end{aligned}$$

this is in fact $=2\tau+(X-3)\kappa$, viz. we have

$$2\tau=X^2-X+8x-3\xi$$

$$(X-3)\kappa=(X-3)(-3X+\xi)=-3X^2+9X+(X-3)\xi,$$

and therefore

$$2\tau+(X-3)\kappa=\text{as above},$$

viz. the united points (a, c) are the 2τ points of contact of the double tangents, and the κ cusps each $(X-3)$ times in respect of the $(X-3)$ tangents from it to the curve. This is the way in which I originally applied the principle to finding the number of double tangents of a curve.

30. Correspondence (B, D). By reciprocation

$$\mathrm{c}_0-\gamma_0-\gamma_0'=(x-6)\Delta,$$

$$\begin{aligned}\mathrm{c}_0&=-2x^2+8x+8X+(x-6)\xi\\&=\quad 2\delta+(x-3)\iota.\end{aligned}$$

31. It may be remarked, as regards the cases which follow, that although the result in terms of $(\delta, \kappa, \iota, \tau)$ when once known can be explained and verified easily enough, there is great risk of oversight if we endeavour to find it in the first instance; while on the other hand the transformation from the form in terms of (x, X, ξ), as given by the principle of correspondence, to the required form in terms of $(\delta, \kappa, \iota, \tau)$ is by no means easy. I in fact first obtained the expression in (x, X, ξ), and then, knowing in some measure the form of the other expression, was able to find it by the actual transformation of the expression in (x, X, ξ).

32. Correspondence (a, D).

From the values of $\mathrm{c}_0-\gamma_0-\gamma_0'$ and $\mathrm{b}-\beta-\beta'$ we have

$$\mathrm{d}-\delta-\delta'=-(2X+2x-18)\Delta,$$

and then

$$\delta=(X-2)(x-3)(X-3),\quad \delta'=(x-2)(X-3)(x-3),$$

whence

$$\begin{aligned}&=(x-3)(X-3)(X+x-4)\\&\quad+(-2X-2x+18)(-2X-2x+2+\xi)\end{aligned}$$

which is

$$\begin{aligned} &= X^2(\quad x+1) \\ &\quad + X\,(x^2-2x-19) \\ &\quad + \quad x^2-19x \\ &\quad + \xi(-2X-2x+18). \end{aligned}$$

And then, by means of the equations

$$\begin{aligned} (x-4)\,2\tau &= (x-4)(X^2-X+8x-3\xi), \\ (X-4)\,2\delta &= (X-4)(x^2-x+8X-3\xi), \\ (x-3)\,\iota &= (x-3)(\quad -3x \quad +\xi), \\ (X-3)\,\kappa &= (X-3)(\quad -3X \quad +\xi), \end{aligned}$$

we verify that

$$\mathrm{d} = (x-4)\,2\tau + (X-4)\,2\delta + (x-3)\,\iota + (X-3)\,\kappa.$$

33. Correspondence (a, e).

From the values of $\mathrm{d}-\delta-\delta'$, $\mathrm{c}-\gamma-\gamma'$ we have

$$\mathrm{e}-\epsilon-\epsilon' = (-X^2+13X+4x-54)\,\Delta,$$

and then

$$\epsilon = \epsilon' = (X-2)(x-3)(X-3)(x-3);$$

that is

$$\begin{aligned} \mathrm{e} &= 2\,(x-3)^2(X-2)(X-3) \\ &\quad + (-X^2+13X+4x-54)(-2X-2x+2+\xi), \end{aligned}$$

which is

$$\begin{aligned} &= X^3(\qquad\qquad 2) \\ &\quad + X^2(\quad 2x^2-10x \quad -10) \\ &\quad + X\,(-10x^2+26x \quad +44) \\ &\quad + \qquad 4x^2+44x \\ &\quad + \xi\,(-X^2+13X+4x-54), \end{aligned}$$

and then

$$\begin{aligned} (x-4)(x-5)\,2\tau &= (x-4)(x-5)(X^2-X+8x-3\xi), \\ \{(X-4)(X-5)+x-3\}\,2\delta &= \{(X-4)(X-5)+x-3\}(x^2-x+8X-3\xi), \\ \{3\,(x-3)(x-4)+x-3\}\,\iota &= (x-3)(3x-11)(-3x+\xi), \\ 2\,(X-3)(X-4)\,\kappa &= 2\,(X-3)(X-4)(-3X+\xi); \end{aligned}$$

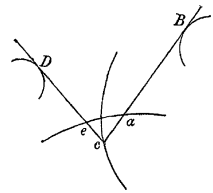
and summing these values and comparing,

$$\begin{aligned} \mathrm{c} &= (x-4)(x-5)\,2\tau + 2\,(X-3)(X-4)\,\kappa \\ &\quad + [(X-4)(X-5)+x-3]\,2\delta + [3\,(x-3)(x-4)+x-3]\,\iota. \end{aligned}$$

The united points (a, e) are in fact, 1°, each of the $x-4$ intersections of a double tangent with the curve, in respect of the two contacts and of the remaining $x-5$ intersections; 2°, each double point in respect of the two branches and of the pairs of tangents from it to the curve; 3°, each of the $x-3$ intersections of each of the

tangents at a double point with the curve; 4°, each of the $x-3$ intersections of a tangent at an inflection (stationary tangent) with the curve, in respect of the $(x-4)$ remaining intersections; 5°, each inflection in respect of the $x-3$ intersections of the

Fig. 9.



tangent with the curve; and 6°, each cusp in respect of the pairs of tangents from it to the curve. Thus (2°), the double point in respect of the branch which contains c, and of the two tangents from it to the curve, is a position of the united point (a, e), as appearing in the figure.

34. Correspondence (B, F). By reciprocation of (a, e)

$$\mathrm{e}_0 - \epsilon_0 - \epsilon_0' = (-x^2 + 13x + 4X - 54)\,\Delta,$$

$$\mathrm{e}_0 = (X-4)(X-5)\,2\delta + 2(x-3)(x-4)\,\iota$$
$$+ [(x-4)(x-5) + X - 3]\,2\tau + [3(X-3)(X-4) + (X-3)]\,\kappa.$$

35. Correspondence (a, F). By means of the values of $\mathrm{e}_0 - \epsilon - \epsilon'$ and $\mathrm{d} - \delta - \delta'$, we have

$$\mathrm{f} - \phi - \phi' = (2X^2 + 2Xx + 2x^2 - 32X - 32x + 162)\,\Delta,$$

and then

$$\phi = (X-2)(x-3)(X-3)(x-3)(X-3),$$
$$\phi' = (x-2)(X-3)(x-3)(X-3)(x-3),$$

whence

$$\mathrm{f} = (X + x - 4)(x-3)^2(X-3)^2$$
$$+ (2X^2 + 2Xx + 2x^2 - 32X - 32x + 162)(-2X - 2x + 2 + \xi)$$

which is

$$\begin{aligned}
= \; & X^3(\qquad\qquad x^3 - 6x + 5) \\
& + X^2(\quad x^3 - 16x^2 + 61x - 22) \\
& + X\,(-6x^3 + 61x^2 - 120x - 91) \\
& + \qquad 5x^3 - 22x^2 - 91x \\
& + \xi \left\{ \begin{array}{l} X^2(\qquad\qquad 2) \\ + X\,(\qquad 2x - 32) \\ + \quad 2x^2 - 32x + 132 \end{array} \right\}
\end{aligned}$$

This result includes proper solutions of the problem of finding the number of the triangles $aBcDeF$, which are such that the side ea touches the curve at a; and also heterotypic solutions having reference to the singular points of the curve; but I have not determined the number of solutions of each kind.

36. Correspondence (a, g): from the values of $\mathrm{f}-\phi-\phi'$ and $\mathrm{e}-\epsilon-\epsilon'$, we have

$$\mathrm{g}-\chi-\chi' = (X^3-20X^2-8Xx-4x^2+125X+44x-486)\,\Delta,$$

and then

$$\chi=\chi' = (X-2)(x-3)(X-3)(x-3)(X-3)(x-3),$$

wherefore

$$\begin{aligned} g = {} & 2(X-2)(X-3)^2(x-3)^3 \\ & +(X^3-20X^2-8Xx+125X+44x-486)(-2X-2x+2+\xi), \end{aligned}$$

viz. this is

$$\begin{aligned} g = {} & X^4(\qquad\qquad\qquad\qquad -2) \\ & +X^3(\quad 2x^3-18x^2+52x-12) \\ & +X^2(-16x^3+144x^2-376x+142) \\ & +X\,(\quad 42x^3-362x^2+780x+88) \\ & \qquad -36x^3+236x^2+88x \\ & +\xi\left\{\begin{array}{l} X^3(\qquad\qquad\qquad 1) \\ +X^2(\qquad\qquad -20) \\ +X\,(\quad -8x+125) \\ \qquad +44x-486. \end{array}\right\} \end{aligned}$$

Comparing with the expression of ϕx, Case 52, we have

$$\begin{aligned} g-\phi x = {} & X^4(\qquad\qquad\qquad\qquad -3) \\ & +X^3(\qquad\qquad\qquad\qquad +34) \\ & +X^2(\qquad 2x^3-18x^2+44x-79) \\ & +X\,(\quad -10x^3+58x^2+76x-84) \\ & \qquad -x^4+10x^3+15x^2-84x \\ & +\xi\left\{\begin{array}{l} X^3(\qquad\qquad\qquad 1) \\ +X^2(\qquad\qquad -11) \\ +X\,(\qquad 4x-10) \\ \qquad +9x^2-91x+114, \end{array}\right\} \end{aligned}$$

which difference must be the number of heterotypic solutions having relation to the singularities of the curve; but I have not further considered this.

515.

SUR LES COURBES APLATIES.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXXIV. (*Janvier—Juin*, 1872), pp. 708—712.]

EN lisant la thèse de M. S. Maillard, *Recherches des caractéristiques des systèmes élémentaires des courbes planes du troisième ordre* (Paris, 1871), j'ai été conduit à quelques réflexions sur la théorie générale des courbes aplaties de M. Chasles [1].

Je considère une courbe représentée par une équation de l'ordre n, $f(x, y, k) = 0$, laquelle pour $k = 0$ se réduit à la forme $P^\alpha Q^\beta \ldots = 0$. Pour k un infiniment petit, ou disons pour $k = 0^1$, cette courbe sera ce que je nomme la pénultième de $P^\alpha Q^\beta \ldots = 0$; la courbe $P^\alpha Q^\beta \ldots = 0$ elle-même sera la courbe finale; et les courbes $P = 0$, $Q = 0, \ldots$, les facteurs. Or en menant par un point donné quelconque les tangentes à la courbe pénultième, ces tangentes approchent continuellement aux droites que voici: 1° les tangentes aux courbes $P = 0$, $Q = 0, \ldots$, respectivement; 2° les droites par les points singuliers de ces mêmes courbes respectivement; 3° les droites par les intersections de deux quelconques de ces mêmes courbes $P = 0$, $Q = 0, \ldots$, respectivement; 4° les droites par certains points situés sur l'une quelconque des mêmes courbes $P = 0$, $Q = 0, \ldots$. En ne faisant aucune supposition particulière par rapport à la courbe pénultième, cette courbe sera une courbe sans points singuliers, et ainsi de la classe $n^2 - n$: le nombre des droites 1°, 2°, 3°, 4° (en faisant attention à la multiplicité de quelques-unes de ces droites) sera donc égal à $n^2 - n$. Les droites 3° sont comptées chacune un certain nombre de fois; en supposant que pour un point d'intersection $P = 0$, $Q = 0$ quelconque ce nombre soit θ, nous dirons qu'il y a à ce point un nombre θ de *sommets fixes*. Les droites 4° sont comptées en général chacune une seule fois; les points par lesquels passent ces droites (points sur l'une quelconque des courbes $P = 0$, $Q = 0, \ldots$) seront

[1] *Comptes Rendus*, t. LXIV. p. 799—805 et 1079—1081; séances des 22 avril et 27 mai 1867.

nommés *sommets libres*. Cela étant, on peut considérer la courbe pénultième comme équivalente à la courbe finale $P^\alpha Q^\beta \ldots = 0$ *plus* les sommets: il s'agit, pour un cas donné quelconque, de trouver le nombre et la distribution de ces sommets.

Je considère d'abord le cas le plus simple, celui d'une conique aplatie, pénultième de $x^2 = 0$; l'équation d'une telle conique est

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0,$$

où, en prenant $a = 1$, tous les autres coefficients seront des infiniment petits, pas en général du même ordre. Les tangentes menées à la courbe par un point donné $(\alpha,\ \beta,\ \gamma)$ seront déterminées par l'équation

$$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch) \times (\gamma y - \beta z,\ \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0;$$

ou disons

$$(bc - f^2,\ c\ - g^2,\ \ b - h^2,\ gh -\ f,\ hf - bg,\ fg - ch) \times (\gamma y - \beta z,\ \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0.$$

En considérant pour un moment tous les coefficients comme étant des infiniment petits du même ordre, $= 0^1$, cette équation se réduit à

$$(0,\ c,\ b,\ -f,\ 0,\ 0 \between \gamma y - \beta z,\ \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0,$$

ou, ce qui est la même chose,

$$(c,\ -f,\ b \between \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0;$$

et ces tangentes coupent la droite $x = 0$ dans les deux points donnés par l'équation $(c,\ -f,\ b \between \alpha z,\ -\alpha y)^2 = 0$, c'est-à-dire $by^2 + 2fyz + cz^2 = 0$, points indépendants de la position du point donné $(\alpha,\ \beta,\ \gamma)$; ces points sont en effet *les intersections de la pénultième par la droite* $x = 0$.

Mais il y a là une restriction qu'on évite au moyen d'une supposition plus générale, savoir: en prenant $g,\ h$ du premier, $b,\ c,\ f$ du second ordre, ou disons $g,\ h = 0^1$, $b,\ c,\ f = 0^2$, l'équation des tangentes devient

$$(0,\ c - g^2,\ b - h^2,\ gh - f,\ 0,\ 0 \between \gamma y - \beta z,\ \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0,$$

ou

$$(c - g^2,\ gh - f,\ b - h^2 \between \alpha z - \gamma x,\ \beta x - \alpha y)^2 = 0.$$

Or, en écrivant $x = 0$, cette équation devient

$$(c - g^2,\ gh - af,\ b - h^2 \between \alpha z,\ -\alpha y)^2 = 0,$$

c'est-à-dire

$$by^2 + 2fyz + cz^2 - (hy + gz)^2 = 0;$$

nous avons ainsi, sur la droite $x = 0$, deux points indépendants de la position du point donné $(\alpha,\ \beta,\ \gamma)$, et qui ne sont plus les intersections de la conique par cette droite (autrement dit, ces points ne sont pas situés sur la conique); ces points sont en effet deux points quelconques sur cette droite. Il y a ainsi pour la conique aplatie pénultième de $x^2 = 0$ deux sommets situés à volonté sur la droite $x = 0$ (et qui ainsi ne sont pas situés sur la conique pénultième).

Je passe à un cas nouveau, celui de la courbe quartique pénultième de $x^2y^2=0$; mais pour simplifier l'analyse, au lieu d'un point quelconque (α, β, γ) je prends successivement les points $(y=0, z=0)$ et $(x=0, y=0)$. On conçoit, en effet, que s'il y a p sommets libres sur la droite $x=0$, q sommets libres sur la droite $y=0$, et r sommets fixes au point $(x=0, y=0)$, alors les droites par le point donné $(y=0, z=0)$ seront les droites par les p points, *plus* la droite $y=0$, $q+r$ fois; et de même les droites par le point donné $(x=0, z=0)$ seront les droites par les q points, *plus* la droite $x=0$, $p+r$ fois: de manière que le procédé donnera les nombres cherchés p, q, r.

J'écris l'équation de la pénultième sous les deux formes

$$\begin{aligned}
& x^4 . a \\
+\ & 4x^3 (h, j \between y, z) \\
+\ & 6x^2 (1, p, m \between y, z)^2 \\
+\ & 4x\ (k, q, r, g \between y, z)^3 \\
+\ & \quad (b, f, l, i, c \between y, z)^4 = 0, \\
& y^4 . b \\
+\ & 4y^3 (k, f \between x, z) \\
+\ & 6y^2 (1, q, l \between x, z)^2 \\
+\ & 4y\ (k, p, r, l \between x, z)^3 \\
+\ & \quad (a, j, m, g, c \between x, z)^4 = 0,
\end{aligned}$$

où le coefficient de x^2y^2 est $=6$, et tous les autres coefficients sont des infiniment petits, pas nécessairement du même ordre. Je représente ces deux équations par

$$(A, B, y^2+C, D, E \between x, 1)^4 = 0, \quad (A', B', x^2+C', D', E' \between y, 1)^4 = 0$$

respectivement.

Cela étant, on obtient l'équation des tangentes par le point $(y=0, z=0)$ en égalant à zéro le discriminant de la fonction quartique de x; et de même pour les tangentes par le point $(x=0, z=0)$: les deux équations seront

$$\begin{aligned}
0 = \ & (y^2+C)^4 . 81AE \\
& + (y^2+C)^3 (-54AD^2 - 54B^2E) \\
& + (y^2+C)^2 (-18A^2E^2 - 180ABDE + 36B^2D^2) \\
& + \ldots . \\
0 = \ & (x^2+C')^4 . 81A'E' \\
& + \ldots .
\end{aligned}$$

En prenant pour le moment tous les coefficients $=0^1$, chaque équation contiendra un seul terme de l'ordre le plus bas 0^2, et en négligeant les autres termes, les équations deviendront simplement

$$y^8 . AE = 0, \qquad x^8 . A'E' = 0;$$

il y a ainsi sur la droite $x=0$ quatre sommets libres donnés par l'équation $E=0$; et de même sur la droite $y=0$, quatre sommets libres donnés par l'équation $E'=0$; donc quatre sommets fixes au point $x=0$, $y=0$. Les sommets libres sur les droites $x=0$ et $y=0$ sont *les intersections de la quartique par ces deux droites respectivement.*

Mais, au contraire, prenons $b, f, l, i, c = 0^2$, les autres coefficients étant $= 0^1$. On a d'abord $A, B, D = 0^1$, $E = 0^2$; la première équation se réduit à

$$27Ay^6(3Ey^2 - 2D^2) = 0,$$

ce qui donne, sur la droite $x = 0$, six sommets libres déterminés par l'équation

$$3Ey^2 - 2D^2 = 0.$$

On a depuis $A' = 0^2$, $B', D', E' = 0^1$; la seconde équation est donc

$$27E'x^6(3A'x^2 - 2B'^2) = 0;$$

mais ici

$$E' = (a, j, m, g, c \between x, z)^4, = x(ax^3 + 4jx^2z + 6mxz^2 + 4gz^3),$$

à cause de $c = 0^2$; et, de plus,

$$3A'x^2 - 2B'^2 = 3bx^2 - 2(kx + fz)^2, = (3b - 2k^2)x^2,$$

à cause de $f = 0^2$; donc l'équation se réduit à

$$x^8(ax^3 + 4jx^2z + 6mxz^2 + 4gz^3) = 0,$$

et il y a sur la droite $y = 0$, trois sommets libres déterminés par l'équation

$$ax^3 + 4jx^2z + 6mxz^2 + 4gz^3 = 0.$$

Remarquons que la droite $y = 0$ rencontre la quartique dans les quatre points donnés par l'équation $E' = 0$, c'est-à-dire un point infiniment près de $(x = 0, y = 0)$ et trois autres points, lesquels sont précisément les trois sommets libres sur la droite $y = 0$. Il y a de plus trois sommets fixes au point $(x = 0, y = 0)$.

Conclusion. Il y a ainsi une courbe quartique pénultième de $x^2y^2 = 0$, avec neuf sommets libres, trois sur l'une des deux droites (disons la droite $y = 0$) et qui sont trois des intersections de la quartique par cette même droite (la quatrième intersection étant infiniment près du point $x = 0$, $y = 0$), six situés à volonté sur l'autre droite $x = 0$, et trois sommets fixes à l'intersection des deux droites.

On peut se figurer une telle courbe quartique: elle peut consister en trois ovales aplaties *plus* une trigonoïde (savoir, figure fermée avec trois angles saillants et trois angles réentrants) rétrécie; l'une des ovales coïncide à peu près avec la droite $y = 0$, les deux autres à peu près avec la droite $x = 0$; la trigonoïde entoure le point $x = 0$, $y = 0$, de manière que les angles réentrants, très-approchés de ce point, soient les trois sommets fixes: mais il n'est pas facile d'en faire un dessin.

Je considère le système des courbes quartiques, qui satisfont chacune aux $(14 - 1 =) 13$ conditions que voici: toucher deux droites données 1, 2 en des points donnés A, B; passer par deux points donnés C, D; toucher sept droites données 3, 4, ..., 9. Prenons $y = 0$ pour la droite AB, et $x = 0$ pour la droite CD: il y aura dans le système une courbe quartique pénultième de $x^2y^2 = 0$, laquelle compte sept fois au moins; cette courbe pénultième est censée toucher les droites 1, 2 dans les points donnés A, B, et l'une quelconque des sept droites à son intersection avec la droite $y = 0\,(AB)$; les autres six droites à leurs intersections avec la droite $x = 0\,(CD)$. Cette courbe pénultième entre donc dans la théorie des caractéristiques d'un tel système de courbes quartiques.

516.

SUR UNE SURFACE QUARTIQUE APLATIE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXXIV. (*Janvier—Juin*, 1872), pp. 1393—1395.]

IL y a évidemment pour les surfaces une théorie analogue à celle des courbes aplaties: la pénultième d'une surface $P^\alpha Q^\beta \ldots = 0$ est, pour ainsi dire, composée des surfaces $P=0$, $Q=0$, &c., *plus* des lignes courbes ou *arêtes*, lesquelles correspondent aux sommets d'une courbe aplatie[1]; par exemple une surface quadrique peut se réduire à $P^2=0$, un plan deux fois, *plus* une conique qui est l'arête de la surface aplatie. Pour les surfaces quartiques, un exemple assez intéressant se rencontre dans le beau Mémoire de M. Casey, "On cyclides and spheroquartics," (*Phil. Trans.*, vol. CLXI. pp. 585—721, 1871). L'auteur, d'après M. Darboux, nomme *cyclide* la surface quartique générale qui a pour ligne double le cercle à l'infini (*surface quartique anallagmatique* de M. Moutard), et *spheroquartic* la courbe d'intersection d'une sphère par une surface quadrique quelconque; et il est conduit à considérer la sphéroquartique comme cas particulier de la cyclide. J'aime mieux dire qu'il y a une cyclide aplatie ayant pour arête une courbe sphéroquartique.

Voici comment on y arrive: la cyclide est l'enveloppe des sphères dont chacune a son centre sur une surface quadrique nommée *focale*, et coupe orthogonalement une sphère fixe, nommée *sphère d'inversion*, disons la sphère S. Cela étant, en envisageant la focale comme une surface réglée, chaque droite sur la surface donne lieu à une infinité de sphères, qui passent toutes par un même cercle. En supposant que la droite coupe la sphère S aux points O, O', ce cercle est ce que j'appelle l'*anticircle* des points O, O', savoir, le plan du cercle est perpendiculaire à la corde OO' au point central M, et le rayon en est égal à $i\,OM\,(=i\,O'M)$, de manière que le cercle est réel ou imaginaire, selon que les points O, O' sont imaginaires ou réels: toute

[1] Voir *Comptes Rendus*, t. LXXIV. p. 708, [515].

sphère ayant son centre sur la droite OO', et coupant orthogonalement la sphère S, passe par le cercle dont il s'agit, disons le cercle L. On voit sans peine que chaque point du cercle L est situé sur la cyclide. Il y a donc sur la cyclide une série infinie single de cercles L qui correspondent un à une aux directrices de la surface focale; il y a de même une série infinie single de cercles L' qui correspondent un à une aux génératrices de la surface focale. La cyclide est le lieu des cercles de l'une ou l'autre série; chaque cercle de la première série coupe en deux points opposés chaque cercle de l'autre série, mais deux cercles de la même série ne se rencontrent pas, &c.

Or, en supposant avec M. Casey que la surface focale se réduise à un cône, les deux séries de cercles se réduisent à une seule série de cercles L, dont chacun est situé sur la sphère, centre le sommet du cône, qui coupe orthogonalement la sphère S, disons la sphère T. On a sur la sphère T la série des cercles L, lesquels ont pour enveloppe une courbe sphérique, la sphéroquartique de M. Casey. Les points des différents cercles L ne remplissent pas la surface sphérique entière, mais seulement une partie de cette surface, limitée par la courbe sphéroquartique. Cela étant, on pourrait dire que la surface cyclide se réduit à la sphère T deux fois, mais il vaut mieux la considérer comme une cyclide aplatie ayant pour arête la courbe sphéroquartique.

La sphéroquartique, considérée comme courbe sur une sphère T, est donnée (comme le remarque M. Casey) par une construction tout à fait analogue à celle pour la cyclide comme surface dans l'espace, savoir (en considérant toujours les courbes sphériques sur une même sphère), la sphéroquartique est l'enveloppe des cercles qui ont leurs centres sur une sphéro-conique et qui coupent orthogonalement un cercle fixe. Le cône, sommet le centre de la sphère, qui passe par la sphéroquartique, est de l'ordre 4, avec deux droites doubles (la classe est donc $=8$); j'ajoute qu'il touche quatre fois la sphère-cône $x^2+y^2+z^2=0$, ayant le même sommet [1].

M. Casey dit que le cône quartique a 16 droites focales: cela a besoin d'explication. Le cône quartique et le sphère-cône ont en commun $8\times 2=16$ plans tangents, y compris les plans tangents selon les 4 droites de contact, chacun deux fois; hormis ceux-ci, il y a donc 8 plans tangents communs. L'intersection de deux quelconques de ces 8 plans est droite focale du cône quartique: donc $\frac{1}{2}(8\times 7)$, $=28$ droites focales. Mais je trouve que les 8 plans tangents forment deux systèmes de 4 plans chacun: les 4 points de l'un de ces systèmes coupent les 4 plans de l'autre système dans 16 droites, lesquelles sont les droites focales de M. Casey; il y a de plus $6+6$ droites, dont chacune est l'intersection de deux plans du même système. Je n'ai pas cherché les distinctions qui doivent exister entre ces différents systèmes de droites focales.

[1] En général, en considérant une courbe quelconque sur une surface S, et un point O quelconque, les deux cônes, sommet O, dont l'un passe par la courbe et l'autre est circonscrit à la surface, se touchent partout où ils se rencontrent: autrement dit, ils n'ont que des droites d'intersection doubles ou de contact.

517.

SUR LES SURFACES DIVISIBLES EN CARRÉS PAR LEURS COURBES DE COURBURE ET SUR LA THÉORIE DE DUPIN.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXXIV. (*Janvier—Juin*, 1872), pp. 1445—1449.]

SOIENT Θ une fonction arbitraire de h, k; x, y, z des fonctions de h, k telles que

$$2\Theta \frac{d^2x}{dh\,dk} - \frac{d\Theta}{dh}\frac{dx}{dk} - \frac{d\Theta}{dk}\frac{dx}{dh} = 0,$$

$$2\Theta \frac{d^2y}{dh\,dk} - \frac{d\Theta}{dh}\frac{dy}{dk} - \frac{d\Theta}{dk}\frac{dy}{dh} = 0,$$

$$2\Theta \frac{d^2z}{dh\,dk} - \frac{d\Theta}{dh}\frac{dz}{dk} - \frac{d\Theta}{dk}\frac{dz}{dh} = 0,$$

et que, de plus,

$$\frac{dx}{dh}\frac{dx}{dk} + \frac{dy}{dh}\frac{dy}{dk} + \frac{dz}{dh}\frac{dz}{dk} = 0;$$

en éliminant h, k, on a, entre x, y, z, l'équation $V=0$ d'une surface. Je dis que les équations $h=\text{const.}$, $k=\text{const.}$ déterminent les deux systèmes des courbes de courbure de cette surface, et, de plus, que cette surface est divisible en carrés par ses courbes de courbure.

En effet, les équations donnent

$$\Theta \frac{d}{dk}\left[\left(\frac{dx}{dh}\right)^2 + \left(\frac{dy}{dh}\right)^2 + \left(\frac{dz}{dh}\right)^2\right] - \frac{d\Theta}{dk}\left[\left(\frac{dx}{dh}\right)^2 + \left(\frac{dy}{dh}\right)^2 + \left(\frac{dz}{dh}\right)^2\right] = 0,$$

ce qui implique

$$\left(\frac{dx}{dh}\right)^2 + \left(\frac{dy}{dh}\right)^2 + \left(\frac{dz}{dh}\right)^2 = \Theta H,$$

où H est fonction de h seulement; et l'on trouve de même

$$\left(\frac{dx}{dk}\right)^2 + \left(\frac{dy}{dk}\right)^2 + \left(\frac{dz}{dk}\right)^2 = \Theta K,$$

où K est fonction de k seulement; donc en écrivant, comme à l'ordinaire,

$$dx^2 + dy^2 + dz^2 = E\,dh^2 + 2F\,dh\,dk + G\,dk^2,$$

cette expression se réduit à

$$dx^2 + dy^2 + dz^2 = \Theta\,(H\,dh^2 + K\,dk^2),$$

ce qui fait voir que la surface est divisible en carrés par les courbes

$$h = \text{const.}, \qquad k = \text{const.}$$

Les équations donnent aussi

$$\begin{vmatrix} \dfrac{dx}{dh}, & \dfrac{dy}{dh}, & \dfrac{dz}{dh} \\ \dfrac{dx}{dk}, & \dfrac{dy}{dk}, & \dfrac{dz}{dk} \\ \dfrac{d^2x}{dh\,dk}, & \dfrac{d^2y}{dh\,dk}, & \dfrac{d^2z}{dh\,dk} \end{vmatrix} = 0;$$

et, cela étant, l'équation différentielle des courbes de courbure se réduit, comme je vais le montrer, à $dh\,dk = 0$; on a donc $h = \text{const.}$, $k = \text{const.}$, pour les équations des courbes de courbure de la surface.

Pour cela, en considérant x, y, z comme des fonctions données de h, k, j'écris, comme à l'ordinaire,

$$\frac{dx}{dh} = a, \quad \frac{dx}{dk} = a', \quad \frac{d^2x}{dh^2} = \alpha, \quad \frac{d^2x}{dh\,dk} = \alpha', \quad \frac{d^2x}{dk^2} = \alpha'',$$

et de même b, b', β, β', β'', et c, c', γ, γ', γ'' pour les coefficients différentiels de y et z respectivement. J'écris aussi

$$A = bc' - b'c, \qquad B = c\alpha' - c'\alpha, \qquad C = ab' - a'b,$$
$$E = a^2 + a'^2 + a''^2, \quad F = a\alpha' + bb' + cc', \quad G = \alpha'^2 + b'^2 + c'^2.$$

L'équation différentielle des courbes de courbure est

$$\begin{vmatrix} dx, & dy, & dz \\ A, & B, & C \\ dA, & dB, & dC \end{vmatrix} = 0.$$

Le premier terme de ce déterminant est $dx\,(B\,dC - C\,dB)$, savoir:

$$\begin{aligned}(a\,dh + a'dk)\,\{ \quad & B\,[(a\beta' - b\alpha' + b'\alpha - a'\beta)\,dh + (a\beta'' - b\alpha'' + b'\alpha' - a'\beta')\,dk] \\ & - C\,[(c\alpha' - a\gamma' + a'\gamma - c'\alpha)\,dh + (c\alpha'' - a\gamma'' + a'\gamma' - c'\alpha')\,dk]\},\end{aligned}$$

ce qui se réduit tout de suite à

$$\begin{aligned}(adh + a'dk)\{ \quad & [a(A\alpha' + B\beta' + C\gamma') - a'(A\alpha + B\beta + C\gamma)]\,dh \\ & - [a(A\alpha'' + B\beta'' + C\gamma'') - a'(A\alpha' + B\beta' + C\gamma')]\,dk\};\end{aligned}$$

en formant les expressions analogues du second et du troisième terme, et en prenant la somme, l'équation devient

$$\begin{aligned}&[E(A\alpha' + B\beta' + C\gamma') - F(A\alpha + B\beta + C\gamma)]\,dh^2 \\ &+ [E(A\alpha'' + B\beta'' + C\gamma'') - G(A\alpha + B\beta + C\gamma)]\,dh\,dk \\ &+ [F(A\alpha'' + B\beta'' + C\gamma'') - G(A\alpha' + B\beta' + C\gamma')]\,dk^2 = 0,\end{aligned}$$

ou, ce qui est la même chose,

$$\begin{vmatrix} dk^2, & -dh\,dk, & dh^2 \\ E, & F, & G \\ A\alpha + B\beta + C\gamma, & A\alpha' + B\beta' + C\gamma', & A\alpha'' + B\beta'' + C\gamma'' \end{vmatrix} = 0;$$

celle-ci est l'équation différentielle des courbes de courbure d'une surface quand les coordonnées x, y, z d'un point de la surface sont données comme fonctions de deux paramètres h, k.

En supposant $F = 0$, l'équation se réduit à

$$\begin{aligned}&(A\alpha' + B\beta' + C\gamma')(Edh^2 - Gdk^2) \\ &+ [(A\alpha'' + B\beta'' + C\gamma'')E - (A\alpha + B\beta + C\gamma)G]\,dh\,dk = 0;\end{aligned}$$

et en supposant de plus $A\alpha' + B\beta' + C\gamma' = 0$, l'équation se réduit simplement à $dh\,dk = 0$; mais cette équation $A\alpha' + B\beta' + C\gamma' = 0$, savoir

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix} = 0,$$

ou

$$\begin{vmatrix} \dfrac{dx}{dh}, & \dfrac{dy}{dh}, & \dfrac{dz}{dh} \\ \dfrac{dx}{dk}, & \dfrac{dy}{dk}, & \dfrac{dz}{dk} \\ \dfrac{d^2x}{dh\,dk}, & \dfrac{d^2y}{dh\,dk}, & \dfrac{d^2z}{dh\,dk} \end{vmatrix} = 0,$$

et aussi $F = 0$, subsistent dans le cas actuel; et nous avons ainsi $dk\,dh = 0$ pour équation différentielle des courbes de courbure.

On vérifie sans peine les équations fondamentales, en prenant $\Theta = h - k$,

$$-(c-a)(a-b)x^2 = a(a+h)(a+k),$$
$$-(a-b)(b-c)y^2 = b(b+h)(b+k),$$
$$-(b-c)(c-a)z^2 = c(c+h)(c+k);$$

ce qui donne les courbes de courbure de l'ellipsoïde $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$; l'ellipsoïde étant, comme on sait, une surface divisible en carrés par des courbes de courbure; mais je n'ai pas encore cherché d'autres solutions.

Je remarque que l'équation pour x peut s'écrire sous la forme

$$\frac{d}{dh}\left(\frac{1}{\Theta}\frac{dx}{dk}\right)+\frac{d}{dk}\left(\frac{1}{\Theta}\frac{dx}{dh}\right)=0;$$

donc, en posant

$$-\frac{d}{dh}\left(\frac{1}{\Theta}\frac{dx}{dk}\right)=\frac{d}{dk}\left(\frac{1}{\Theta}\frac{dx}{dh}\right)=\frac{d^2\Omega}{dh\,dk},$$

on trouve

$$\frac{dx}{dh}=\Theta\frac{d\Omega}{dh},\quad \frac{dx}{dk}=-\Theta\frac{d\Omega}{dk},$$

ce qui donne

$$\frac{d}{dk}\left(\Theta\frac{d\Omega}{dh}\right)+\frac{d}{dh}\left(\Theta\frac{d\Omega}{dk}\right)=0,$$

équation pour Ω de la même forme que celle pour x.

On déduit une démonstration très-simple du théorème de Dupin. En considérant comme auparavant (x, y, z) comme des fonctions données de (h, k), le point (x, y, z) sera situé sur une surface, et les conditions pour que les courbes de courbure soient $h = \text{const.}$, $k = \text{const.}$ seront

$$\frac{dx}{dh}\frac{dx}{dk}+\frac{dy}{dh}\frac{dy}{dk}+\frac{dz}{dh}\frac{dz}{dk}=0,$$

$$\begin{vmatrix} \frac{dx}{dh}, & \frac{dy}{dh}, & \frac{dz}{dh} \\ \frac{dx}{dk}, & \frac{dy}{dk}, & \frac{dz}{dk} \\ \frac{d^2x}{dh\,dk}, & \frac{d^2y}{dh\,dk}, & \frac{d^2z}{dh\,dk} \end{vmatrix}=0.$$

Cela étant, en introduisant un troisième paramètre l, soient h, k, l des fonctions données de (x, y, z), ou réciproquement (x, y, z) des fonctions données de (h, k, l). On

a ici les trois systèmes de surfaces $h = \text{const.}$, $k = \text{const.}$, $l = \text{const.}$, et les conditions pour que ces surfaces se coupent orthogonalement peuvent s'écrire sous la forme

$$\frac{dx}{dk}\frac{dx}{dl} + \frac{dy}{dk}\frac{dy}{dl} + \frac{dz}{dk}\frac{dz}{dl} = 0,$$

$$\frac{dx}{dl}\frac{dx}{dh} + \frac{dy}{dl}\frac{dy}{dh} + \frac{dz}{dl}\frac{dz}{dh} = 0,$$

$$\frac{dx}{dh}\frac{dx}{dk} + \frac{dy}{dh}\frac{dy}{dk} + \frac{dz}{dh}\frac{dz}{dk} = 0.$$

On a donc

$$\frac{dx}{dl} : \frac{dy}{dl} : \frac{dz}{dl} = \frac{dy}{dh}\frac{dz}{dk} - \frac{dz}{dh}\frac{dy}{dk} : \frac{dz}{dh}\frac{dx}{dk} - \frac{dx}{dh}\frac{dy}{dk} : \frac{dx}{dh}\frac{dy}{dk} - \frac{dy}{dh} : \frac{dz}{dh}.$$

Pour abréger, j'écris

$$\frac{dx}{dh}\frac{dx}{dk} + \frac{dy}{dh}\frac{dy}{dk} + \frac{dz}{dh}\frac{dz}{dk} = [h\,.\,k], \ldots,$$

et de même

$$\frac{dx}{dh}\frac{d^2x}{dk\,dl} + \frac{dy}{dh}\frac{d^2y}{dk\,dl} + \frac{dz}{dh}\frac{d^2z}{dk\,dl} = [h\,.\,kl], \ldots.$$

Les conditions données sont ainsi

$$[k\,.\,l] = 0, \quad [l\,.\,h] = 0, \quad [h\,.\,k] = 0\,;$$

en différentiant ces équations par rapport à h, k, l respectivement, on obtient

$$[k\,.\,lh] + [l\,.\,hk] = 0,$$
$$[l\,.\,kh] + [h\,.\,kl] = 0,$$
$$[h\,.\,kl] + [k\,.\,lh] = 0\,;$$

donc

$$[h\,.\,kl] = 0, \quad [k\,.\,lh] = 0, \quad [l\,.\,kh] = 0.$$

Mais l'équation $[h\,.\,k] = 0$ et l'équation $[l\,.\,hk] = 0$, en substituant dans celle-ci les valeurs de $\frac{dx}{dl}$, $\frac{dy}{dl}$, $\frac{dz}{dl}$, sont précisément les conditions pour que la surface $l = \text{const.}$ soit coupée par les autres surfaces selon ses courbes de courbure: donc le théorème.

518.

SUR LA CONDITION POUR QU'UNE FAMILLE DE SURFACES DONNÉES PUISSE FAIRE PARTIE D'UN SYSTÈME ORTHOGONAL.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXXV. (*Juillet—Décembre*, 1872), pp. 177—185, 246—250, 324—330, 381—385, 1800—1803.]

[Pp. 177—185.]

1. SOIT $\rho = f(x, y, z)$ l'équation d'une famille de surfaces qui fait partie d'un système orthogonal. On sait que ρ satisfait à une équation à différences partielles du troisième ordre, et en suivant la route tracée par M. Levy, dans son excellent "Mémoire sur les coordonnées curvilignes orthogonales" (*Journal de l'École Polytechnique*, t. XXVI., pp. 157—200, 1870), je suis parvenu à trouver cette équation.

2. Je remarque que le théorème fondamental de M. Levy est, en effet, assez évident. Considérons une surface de la famille ρ: soit P un point quelconque de cette surface, et PT, PT_1, PT_2 la normale et les tangentes aux deux courbes de courbure par le point P. Passons, suivant la normale au point P' de la surface consécutive $\rho + d\rho$, et soient $P'T'$, $P'T_1'$, $P'T_2'$ la normale et les tangentes aux deux courbes de courbure par le point P'. Or, si les surfaces ρ forment partie d'un système orthogonal, évidemment PP' sera élément d'une courbe de courbure d'une surface ρ_1 et aussi d'une surface ρ_2 des deux autres familles du système orthogonal, et PT_1, $P'T_1'$ seront les normales à deux points consécutifs de cette courbe de courbure de la surface ρ_1: et de même PT_2 et $P'T_2'$ seront les normales à deux points consécutifs de cette courbe de courbure de la surface ρ_2. Donc PT_1 et $P'T_1'$ se rencontrent; et de même PT_2 et $P'T_2'$ se rencontrent. En se souvenant que PT_1, PT_2 sont perpendiculaires l'une à l'autre, et de même $P'T_1'$, $P'T_2'$, on voit sans peine que les deux conditions se réduisent à une seule. Réciproquement, si PT_1, $P'T_1'$ se rencontrent (ou, ce qui est la même chose, PT_2 et $P'T_2'$), la famille ρ fera partie d'un système orthogonal; ce qui est le théorème de M. Levy.

3. Soient (X, Y, Z) les fonctions dérivées de ρ du premier ordre; (a, b, c, f, g, h) celles du second ordre; $(a, b, c, f, g, h, i, j, k, l)$ celles du troisième ordre, savoir:

$$(X, Y, Z) = (\partial_x, \partial_y, \partial_z)\,\rho,$$

$$(\mathrm{a, b, c, f, g, h}) = (\partial_x^2, \partial_y^2, \partial_z^2, \partial_y\partial_z, \partial_z\partial_x, \partial_x\partial_y)\,\rho,$$

$$(a, b, c, f, g, h, i, j, k, l) = (\partial_x^3, \partial_y^3, \partial_z^3, \partial_y^2\partial_z, \partial_z^2\partial_x, \partial_x^2\partial_y, \partial_y\partial_z^2, \partial_z\partial_x^2, \partial_x\partial_y^2, \partial_x\partial_y\partial_z)\,\rho;$$

soient de plus

$$A = 2(Z\mathrm{h} - Y\mathrm{g}), \quad F = X(\mathrm{c} - \mathrm{h}) + Y\mathrm{h} - Z\mathrm{g},$$

$$B = 2(X\mathrm{f} - Z\mathrm{h}), \quad G = Y(\mathrm{a} - \mathrm{c}) + Z\mathrm{f} - X\mathrm{h},$$

$$C = 2(Y\mathrm{g} - X\mathrm{f}), \quad H = Z(\mathrm{h} - \mathrm{a}) + X\mathrm{g} - Y\mathrm{f},$$

valeurs qui satisfont aux équations

$$A + B + C = 0 \text{ et } (A, B, C, F, G, H \between X, Y, Z)^2 = 0.$$

Alors les tangentes PT_1, PT_2 sont données par les équations

$$(A, B, C, F, G, H \between x, y, z)^2 = 0,$$

$$Xx + Yy + Zz = 0,$$

et en partant de ces équations, mais en supposant que pour le point P les valeurs de X, Y soient $X = 0$, $Y = 0$, M. Levy obtient comme condition de l'intersection dont il s'agit

$$\left(\frac{dX}{dx} - \frac{dY}{dy}\right)\frac{d^2}{dxdy}\frac{1}{Z} + \frac{dY}{dx}\left(\frac{d^2}{dy^2} - \frac{d^2}{dx^2}\right)\frac{1}{Z} = 0,$$

ou, ce qui est la même chose

$$2\mathrm{fg}(\mathrm{a} - \mathrm{b}) + 2\mathrm{h}(\mathrm{f}^2 - \mathrm{g}^2) - Z[(f - j)\mathrm{h} + l(\mathrm{a} - \mathrm{b})] = 0;$$

savoir: cette équation est ce que devient l'équation cherchée du troisième ordre en y écrivant $X = 0$, $Y = 0$.

4. Je passe à la recherche de l'équation générale; pour cela (X, Y, Z) dénotant comme auparavant, nous pouvons considérer ces quantités comme les coordonnées (mesurées du point P comme origine) d'un point sur la normale PT; soient de même X_1, Y_1, Z_1 les coordonnées d'un point sur la tangente PT_1 et X_2, Y_2, Z_2 les coordonnées d'un point sur la tangente PT_2. Il s'agit seulement des valeurs relatives de ces coordonnées; et celles de X_1, Y_1, Z_1 et X_2, Y_2, Z_2 sont les valeurs de (x, y, z), données par les équations

$$(A, B, C, F, G, H \between x, y, z)^2 = 0,$$

$$Xx + Yy + Zz = 0.$$

Ces équations impliquent $X_1X_2+Y_1Y_2+Z_1Z_2=0$, et en se rappelant une équation déjà mentionnée, on a le système

$$\begin{aligned}
(A, \ldots \between X\,,\ Y\,,\ Z\,)^2 &= 0,\\
(A, \ldots \between X_1,\ Y_1,\ Z_1)^2 &= 0,\\
(A, \ldots \between X_2,\ Y_2,\ Z_2)^2 &= 0,\\
X_1X_2+Y_1Y_2+Z_1Z_2 &= 0,\\
XX_1+YY_1+ZZ_1 &= 0,\\
XX_2+YY_2+ZZ_2 &= 0.
\end{aligned}$$

L'origine étant quelconque, prenons (x, y, z) pour coordonnées de P, et $x+\delta x$, $y+\delta y$, $z+\delta z$ pour coordonnées de P'; nous avons $\delta x : \delta y : \delta z = X : Y : Z$; et comme il ne s'agit que des valeurs relatives, nous pouvons omettre un facteur infinitésimal commun, et écrire simplement δx, δy, $\delta z = X$, Y, Z. De même, en supposant qu'une fonction quelconque u de (x, y, z) devient $u+\delta u$, en passant du point P au point P', la valeur de δu sera $X\dfrac{du}{dx}+Y\dfrac{du}{dy}+Z\dfrac{du}{dz}$, ou, ce qui est la même chose, nous aurons $\delta = X\dfrac{d}{dx}+Y\dfrac{d}{dy}+Z\dfrac{d}{dz}$. Dans tout ce qui suit, δ aura cette signification.

5. Cela étant, si pour un moment nous prenons ξ, η, ζ pour coordonnées courantes, et θ pour un paramètre arbitraire, les équations de PT seront

$$\xi = x+\theta X_1, \quad \eta = y+\theta Y_1, \quad \zeta = z+\theta Z_1,$$

et si cette droite rencontre PT_1, alors en prenant ξ, η, ζ pour les coordonnées du point d'intersection, nous aurons $0 = \delta x + X_1\delta\theta + \theta\delta X_1, \ldots$: ou en éliminant $\delta\theta$ et θ,

$$0 = \begin{vmatrix} \delta x, & X_1, & \delta X_1 \\ \delta y, & Y_1, & \delta Y_1 \\ \delta z, & Z_1, & \delta Z_1 \end{vmatrix},$$

ou, ce qui est la même chose,

$$0 = \begin{vmatrix} X, & X_1, & \delta X_1 \\ Y, & Y_1, & \delta Y_1 \\ Z, & Z_1, & \delta Z_1 \end{vmatrix}.$$

Mais nous avons $X_2 : Y_2 : Z_2 = YZ_1 - ZY_1 : ZX_1 - XZ_1 : XY_1 - YX_1$: donc cette équation devient $X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 = 0$. Or nous avons $\delta(X_1X_2+Y_1Y_2+Z_1Z_2) = 0$; l'équation trouvée peut donc s'écrire sous la forme plus symétrique

$$X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 - (X_1\delta X_2 + Y_1\delta Y_2 + Z_1\delta Z_2) = 0,$$

équation qui exprime la condition pour l'intersection des tangentes PT_1, $P'T_1'$ (ou PT_2, $P'T_2'$).

6. Dans la démonstration précédente, je me suis servi du théorème de Dupin; mais il convient de remarquer qu'en partant du système orthogonal, et dénotant par X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2 les dérivées de ρ, ρ_1, ρ_2, respectivement, il serait possible de déduire cette même équation des seules équations

$$X X_1 + Y Y_1 + Z Z_1 = 0,$$
$$X X_2 + Y Y_2 + Z Z_2 = 0,$$
$$X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 = 0.$$

En effet, l'équation fut démontrée de cette manière par R. L. Ellis, dans une démonstration du théorème de Dupin, publiée dans l'ouvrage de Gregory (*Examples of the processes of the differential and integral calculus;* Cambridge, 1841). Les premières deux équations donnent $X : Y : Z = Y_1Z_2 - Y_2Z_1 : Z_1X_2 - Z_2X_1 : X_1Y_2 - X_2Y_1$; on a donc l'expression

$$(Y_1Z_2 - Y_2Z_1)\,dx + (Z_1X_2 - Z_2X_1)\,dy + (X_1Y_2 - X_2Y_1)\,dz,$$

intégrable par un facteur; ce qui donne

$$(Y_1Z_2 - Y_2Z_1)\left\{\frac{d}{dy}(X_1Y_2 - X_2Y_1) - \frac{d}{dz}(Z_1X_2 - Z_2X_1)\right\}\ldots = 0.$$

Le terme en { } est égal à

$$\left(X_2\frac{dX_1}{dx} + Y_2\frac{dX_1}{dy} + Z_2\frac{dX_1}{dz}\right) - \left(X_1\frac{dX_2}{dx} + Y_1\frac{dX_2}{dy} + Z_1\frac{dX_2}{dz}\right)$$
$$- X_2\left(\frac{dX_1}{dx} + \frac{dY_1}{dy} + \frac{dZ_1}{dz}\right) + X_1\left(\frac{dX_2}{dx} + \frac{dY_2}{dy} + \frac{dZ_2}{dz}\right),$$

et la somme qui correspond à la deuxième ligne de cette expression s'évanouit identiquement; la première ligne peut s'écrire sous la forme $\delta_2X_1 - \delta_1X_2$; donc, en rétablissant X, Y, Z au lieu de $Y_1Z_2 - Y_2Z_1, \ldots$, la condition devient simplement

$$X(\delta_2X_1 - \delta_1X_2) + Y(\delta_2Y_1 - \delta_1Y_2) + Z(\delta_2Z_1 - \delta_1Z_2) = 0.$$

Mais nous avons

$$\delta_2X_1 = X_2\frac{dX_1}{dx} + Y_2\frac{dX_1}{dy} + Z_2\frac{dX_1}{dz}, \quad = X_2\frac{dX_1}{dx} + Y_2\frac{dY_1}{dx} + Z_2\frac{dZ_1}{dx},$$

$$\delta_1X_2 = X_1\frac{dX_2}{dx} + Y_1\frac{dX_2}{dy} + Z_1\frac{dX_2}{dz}, \quad = X_1\frac{dX_2}{dx} + Y_1\frac{dY_2}{dx} + Z_1\frac{dZ_2}{dx},$$

et ainsi $\delta_2X_1 + \delta_1X_2 = \frac{d}{dx}(X_1X_2 + Y_1Y_2 + Z_1Z_2) = 0$; c'est-à-dire $\delta_1X_2 = -\delta_2X_1$, et de même $\delta_1Y_2 = -\delta_2Y_1$, $\delta_1Z_2 = -\delta_2Z_1$, et l'équation trouvée se réduit à

$$X\delta_2X_1 + Y\delta_2Y_1 + Z\delta_2Z_1 = 0, \quad \text{ou} \quad X\delta_1X_2 + Y\delta_1Y_2 + Z\delta_1Z_2 = 0;$$

on a de même

$$X_1\delta X_2 + Y_1\delta Y_2 + Z_1\delta Z_2 = 0, \quad \text{ou} \quad X_1\delta_2X + Y_1\delta_2Y + Z_1\delta_2Z = 0,$$

et

$$X_2\delta_1X + Y_2\delta_1Y + Z_2\delta_1Z = 0, \quad \text{ou} \quad X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 = 0,$$

et ainsi l'équation dont il s'agit

$$X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 - (X_1\delta X_2 + Y_1\delta Y_2 + Z_1\delta Z_2) = 0.$$

On ne savait pas auparavant la signification géométrique de cette équation.

7. Dans la question actuelle, partant de cette équation, je rappelle que les valeurs de X, Y, Z, X_1, Y_1, Z_1 sont celles de (x, y, z) données par les équations

$$(A, B, C, F, G, H \between x, y, z)^2 = 0, \quad Xx + Yy + Zz = 0.$$

En supposant que ces équations donnent

$$X_1 : Y_1 : Z_1 = U + U' : V + V' : W + W',$$
$$X_2 : Y_2 : Z_2 = U - U' : V - V' : W - W',$$

la condition devient

$$U\delta U' + V\delta V' + W\delta W' - (U'\delta U + V'\delta V + W'\delta W) = 0.$$

8. Pour effectuer la réduction de cette formule, nous avons besoin de plusieurs formules subsidiaires. J'écris

$$(BC - F^2, CA - G^2, AB - H^2, GH - AF, HF - BG, FG - CH \between X, Y, Z)^2$$
$$= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between X, Y, Z)^2 = -\phi,$$

et je dénote par (a), (b), (c), (f), (g), (h) les coefficients de $\lambda^2, \ldots$ dans la fonction

$$(A, B, C, F, G, H \between \nu Y - \mu Z, \lambda Z - \nu X, \mu X - \lambda Y)^2,$$

savoir, j'écris

$$\begin{aligned}
(\mathrm{a}) &= BZ^2 + CY^2 - 2FYZ,\\
(\mathrm{b}) &= CX^2 + AZ^2 - 2GZX,\\
(\mathrm{c}) &= AY^2 + BX^2 - 2HXY,\\
(\mathrm{f}) &= -AYZ - FX^2 + GXY + HXZ,\\
(\mathrm{g}) &= -BZX + FXY - GY^2 + HYZ,\\
(\mathrm{h}) &= -CXY + FYZ + GYZ - HZ^2,
\end{aligned}$$

où je remarque qu'en vertu des valeurs de $A, \ldots$ nous avons

$$(\mathrm{a}) + (\mathrm{b}) + (\mathrm{c}) = 0.$$

Cela étant, nous avons les identités

$$[(\mathrm{a}), (\mathrm{h}), (\mathrm{g})](X, Y, Z) = 0,$$
$$[(\mathrm{h}), (\mathrm{b}), (\mathrm{f})](X, Y, Z) = 0,$$
$$[(\mathrm{g}), (\mathrm{f}), (\mathrm{c})](X, Y, Z) = 0,$$
$$[(\mathrm{b})(\mathrm{c}) - (\mathrm{f})^2, (\mathrm{c})(\mathrm{a}) - (\mathrm{g})^2, (\mathrm{a})(\mathrm{b}) - (\mathrm{h})^2, (\mathrm{g})(\mathrm{h}) - (\mathrm{a})(\mathrm{f}), (\mathrm{h})(\mathrm{f}) - (\mathrm{b})(\mathrm{g}), (\mathrm{f})(\mathrm{g}) - (\mathrm{c})(\mathrm{h})]$$
$$= -(X^2, Y^2, Z^2, YZ, ZX, XY)\phi,$$

savoir, $(\mathrm{b})(\mathrm{c})-(\mathrm{f})^2=-X^2\phi, \ldots$. De plus

$$\begin{aligned}
(A, H, G)[(\mathrm{a}), (\mathrm{h}), (\mathrm{g})] &= -X(\mathfrak{A}X+\mathfrak{H}Y+\mathfrak{G}Z)-\phi,\\
(H, B, F)[(\mathrm{a}), (\mathrm{h}), (\mathrm{g})] &= -Y(\mathfrak{A}X+\mathfrak{H}Y+\mathfrak{G}Z),\\
(G, F, C)[(\mathrm{a}), (\mathrm{h}), (\mathrm{g})] &= -Z(\mathfrak{A}X+\mathfrak{H}Y+\mathfrak{G}Z),\\
(A, H, G)[(\mathrm{h}), (\mathrm{b}), (\mathrm{f})] &= -X(\mathfrak{H}X+\mathfrak{B}Y+\mathfrak{F}Z),\\
(H, B, F)[(\mathrm{h}), (\mathrm{b}), (\mathrm{f})] &= -Y(\mathfrak{H}X+\mathfrak{B}Y+\mathfrak{F}Z)-\phi,\\
(G, F, C)[(\mathrm{h}), (\mathrm{b}), (\mathrm{f})] &= -Z(\mathfrak{H}X+\mathfrak{B}Y+\mathfrak{F}Z),\\
(A, H, G)[(\mathrm{g}), (\mathrm{f}), (\mathrm{c})] &= -X(\mathfrak{G}X+\mathfrak{F}Y+\mathfrak{C}Z),\\
(H, B, F)[(\mathrm{g}), (\mathrm{f}), (\mathrm{c})] &= -Y(\mathfrak{G}X+\mathfrak{F}Y+\mathfrak{C}Z),\\
(G, F, C)[(\mathrm{g}), (\mathrm{f}), (\mathrm{c})] &= -Z(\mathfrak{G}X+\mathfrak{F}Y+\mathfrak{C}Z)-\phi;
\end{aligned}$$

aussi

$$A(\mathrm{a})+B(\mathrm{b})+C(\mathrm{c})+2F(\mathrm{f})+2G(\mathrm{g})+2H(\mathrm{h})+2\phi=0.$$

Multipliant cette dernière équation par l'un quelconque des coefficients (a), ..., et réduisant, on obtient six équations; mais je forme seulement celle qui se dérive de (g), savoir, nous avons

$$\begin{aligned}
(\mathrm{g})[A(\mathrm{a})+B(\mathrm{b})+C(\mathrm{c})+2F(\mathrm{f})+2G(\mathrm{g})+2H(\mathrm{h})]&\\
+2\phi(-BZX+FXY-GY^2+HYZ)&=0.
\end{aligned}$$

Ici la seconde ligne est égale à

$$2B[(\mathrm{f})(\mathrm{h})-(\mathrm{b})(\mathrm{g})]-2F[(\mathrm{f})(\mathrm{g})-(\mathrm{c})(\mathrm{h})]+2G[(\mathrm{c})(\mathrm{a})-(\mathrm{g})^2]-2H[(\mathrm{g})(\mathrm{h})-(\mathrm{a})(\mathrm{f})],$$

et l'équation est

$$A(\mathrm{a})(\mathrm{g})+B[2(\mathrm{h})(\mathrm{f})-(\mathrm{b})(\mathrm{g})]+C(\mathrm{c})(\mathrm{g})+2F(\mathrm{c})(\mathrm{h})+2G(\mathrm{c})(\mathrm{a})+2H(\mathrm{a})(\mathrm{f})=0.$$

Des équations $(\mathrm{g})(\mathrm{h})-(\mathrm{a})(\mathrm{f})=-YZ\phi$, $(\mathrm{h})(\mathrm{f})-(\mathrm{b})(\mathrm{g})=-ZX\phi$, multipliant par $-X$, $-Y$ et ajoutant, nous obtenons $-(\mathrm{h})[(\mathrm{g})X+(\mathrm{f})Y]+(\mathrm{a})(\mathrm{f})X+(\mathrm{b})(\mathrm{g})Y=2XYZ$, c'est-à-dire

$$(\mathrm{a})(\mathrm{f})X+(\mathrm{b})(\mathrm{g})Y+(\mathrm{c})(\mathrm{f})Z=2XYZ.$$

9. Je reviens à la question principale. A moins de se servir de quantités arbitraires qui rendraient les formules plus complexes, il n'y a pas d'expression symétrique pour les valeurs de $X_1 : Y_1 : Z_1$ et $X_2 : Y_2 : Z_2$: et ainsi j'écris

$$\begin{aligned}
X_1 : Y_1 : Z_1 &= (\mathrm{a}) : (\mathrm{h})+Z\sqrt{\phi} : (\mathrm{g})-Y\sqrt{\phi},\\
X_2 : Y_2 : Z_2 &= (\mathrm{a}) : (\mathrm{h})-Z\sqrt{\phi} : (\mathrm{g})+Y\sqrt{\phi},
\end{aligned}$$

et la condition devient

$$[(\mathrm{h})\,\delta Z-Z\delta(\mathrm{h})-(\mathrm{g})\,\delta Y+Y\delta(\mathrm{g})]\sqrt{\phi}+[(\mathrm{h})Z-(\mathrm{g})Y]\,\delta\sqrt{\phi}=0,$$

ou, puisque $\delta\sqrt{\phi}=\dfrac{1}{2\sqrt{\phi}}\delta\phi$, ceci est

$$2[(\mathrm{h})\,\delta Z-Z\delta(\mathrm{h})-(\mathrm{g})\,\delta Y+Y\delta(\mathrm{g})]\,\phi+[(\mathrm{h})Z-(\mathrm{g})Y]\,\delta\phi=0,$$

équation qui contient, comme nous le verrons, le facteur (a); et, en omettant ce facteur, l'équation deviendra symétrique.

J'écris

$$\delta(\mathrm{g}) = \Delta(\mathrm{g}) + \delta'(\mathrm{g}), \quad \delta(\mathrm{h}) = \Delta(\mathrm{h}) + \delta'(\mathrm{h}), \quad \delta\phi = \Delta\phi + \delta'\phi,$$

en dénotant par Δ les parties qui dépendent de δX, δY, δZ, et par δ' celles qui dépendent de $\delta A, \ldots$. La fonction à droite est ainsi la somme des deux parties

$$\Omega_1 = 2\,[(\mathrm{h})\,\delta Z - Z\Delta(\mathrm{h}) - (\mathrm{g})\,\delta Y + Y\Delta(\mathrm{g})]\,\phi + [(\mathrm{h})\,Z - (\mathrm{g})\,Y]\,\Delta\phi,$$

$$\Omega_2 = 2\,[\qquad - Z\delta'(\mathrm{h}) \qquad + Y\delta'(\mathrm{g})]\,\phi + [(\mathrm{h})\,Z - (\mathrm{g})\,Y]\,\delta'\phi,$$

où cette seconde partie Ω_2 est la seule qui contient les dérivées de ρ du troisième ordre.

10. Je réduis l'expression de Ω_1. Nous avons

$$\Delta(\mathrm{h}) = (-CY + FZ)\,\delta X + (-CX + GZ \qquad)\,\delta Y + (FX + GY - 2HZ)\,\delta Z,$$

$$\Delta(\mathrm{g}) = (-BZ + FY)\,\delta X + (FX - 2GY + HZ)\,\delta Y + (-BX + HY \qquad)\,\delta Z,$$

et de là

$$\tfrac{1}{2}\Omega_1 = \phi\,\{[(C-B)\,YZ + F(Y^2 - Z^2)]\,\delta X + [-AXZ + G(Y^2 + Z^2)]\,\delta Y + [AXZ + H(Y^2 + Z^2)]\,\delta Z\}$$
$$+ \tfrac{1}{2}\,[(\mathrm{h})\,Z - (\mathrm{g})\,Y]\,\Delta\phi,$$

où la dernière ligne est égale à

$$[(\mathrm{g})\,Y - (\mathrm{h})\,Z]\,[(\mathfrak{A}X + \mathfrak{H}Y + \mathfrak{G}Z)\,\delta Z + (\mathfrak{H}X + \mathfrak{B}Y + \mathfrak{F}Z)\,\delta Z + (\mathfrak{G}X + \mathfrak{F}Y + \mathfrak{C}Z)\,\delta Z].$$

Ici le coefficient de δX est égal à

$$(C-B)\,[(\mathrm{a})\,(\mathrm{f}) - (\mathrm{g})\,(\mathrm{h})] + F(\mathrm{g})^2 - (\mathrm{a})\,(\mathrm{c}) - (\mathrm{h})^2 + (\mathrm{a})\,(\mathrm{b})]$$
$$+ [(\mathrm{g})\,Y - (\mathrm{h})\,Z]\,(\mathfrak{A}X + \mathfrak{H}Y + \mathfrak{G}Z),$$

où la seconde ligne est égale à

$$-(\mathrm{g})\,(H,\ B,\ F)\,[(\mathrm{a}),\ (\mathrm{b}),\ (\mathrm{g})] + (\mathrm{h})\,(G,\ F,\ C)\,[(\mathrm{a}),\ (\mathrm{h}),\ (\mathrm{g})],$$

et ainsi l'expression entière se réduit à

$$(\mathrm{a})\,\{-(B-C)\,\mathfrak{O} + F[(\mathrm{b}) - (\mathrm{c})] - H(\mathrm{g}) + G(\mathrm{h})\}$$

c'est-à-dire le coefficient de δX contient le facteur (a).

Le coefficient de δY est

$$[-AXZ - F(Y^2 + Z^2)]\,\phi + [(\mathrm{g})\,Y - (\mathrm{h})\,Z]\,(\mathfrak{H}X + \mathfrak{B}Y + \mathfrak{F}Z),$$

où la seconde partie est

$$(\mathrm{g})\,\{-\phi - (H,\ B,\ F)\,[(\mathrm{h}),\ (\mathrm{b}),\ (\mathrm{f})]\} + (\mathrm{h})\,(G,\ F,\ C)\,[(\mathrm{h}),\ (\mathrm{b}),\ (\mathrm{f})]\,;$$

on a donc les termes

$$-\phi\,[(\mathrm{g}) + AZX + G(Y^2 + Z^2)],$$

c'est-à-dire

$$-\phi\,[(A-B)\,ZX + FXY + GZ^2 + HYZ],$$

et l'expression entière est ainsi égale à

$$\begin{aligned}(A-B)[(\mathrm{h})(\mathrm{f})-(\mathrm{b})(\mathrm{g})]+F[(\mathrm{f})(\mathrm{g})-(\mathrm{c})(\mathrm{h})]+G[(\mathrm{a})(\mathrm{b})-(\mathrm{h})^2]\\ +H[(\mathrm{g})(\mathrm{h})-(\mathrm{a})(\mathrm{f})]-(\mathrm{g})(H,\ B,\ F)[(\mathrm{h}),\ (\mathrm{b}),\ (\mathrm{f})]\\ +(\mathrm{h})(G,\ F,\ C)[(\mathrm{h}),\ (\mathrm{b}),\ (\mathrm{f})],\end{aligned}$$

c'est-à-dire à

$$A[(\mathrm{h})(\mathrm{f})]-(\mathrm{b})(\mathrm{g})-B(\mathrm{h})(\mathrm{f})+C(\mathrm{h})(\mathrm{f})+F[(\mathrm{b})-(\mathrm{c})](\mathrm{h})+G(\mathrm{a})(\mathrm{b})-H(\mathrm{a})(\mathrm{f}),$$

ou enfin à

$$A-(\mathrm{b})(\mathrm{g})+B-2(\mathrm{h})(\mathrm{f})+F[(\mathrm{b})(\mathrm{c})](\mathrm{h})+G(\mathrm{a})(\mathrm{b})+H-(\mathrm{a})(\mathrm{f}).$$

J'ajoute la quantité nulle

$$\begin{aligned}A(\mathrm{a})(\mathrm{g})+B[2(\mathrm{h})(\mathrm{f})-(\mathrm{b})(\mathrm{g})]+C(\mathrm{c})(\mathrm{g})\\ +F\,2(\mathrm{c})(\mathrm{h})+G\,2(\mathrm{c})(\mathrm{a})\qquad\qquad +H\,2(\mathrm{a})(\mathrm{f}),\end{aligned}$$

et, en réduisant au moyen de $(\mathrm{a})+(\mathrm{b})+(\mathrm{c})=0$ et $A+B+C=0$, le coefficient de δY devient

$$=(\mathrm{a})\{-(C-A)(\mathrm{g})+H(\mathrm{f})+G[(\mathrm{c})-(\mathrm{a})]-F(\mathrm{h})\},$$

et, de même, le coefficient de δZ est

$$=(\mathrm{a})\{-(A-B)(\mathrm{h})-G(\mathrm{f})+F(\mathrm{g})+H[(\mathrm{a})-(\mathrm{b})]\},$$

ce qui achève le calcul de Ω_1.

[Pp. 246—250.]

11. Pour trouver Ω_2, nous avons

$$\begin{aligned}\delta'(\mathrm{g})&=-ZX\delta B+YX\delta F-Y^2\delta G+YZ\delta H,\\ \delta'(\mathrm{h})&=-XY\delta C+XZ\delta F+YZ\delta G-Z^2\delta H,\\ \delta'\phi\ &=-[(\mathrm{a})\,\delta A+(\mathrm{b})\,\delta B-(\mathrm{c})\,\delta C+2(\mathrm{f})\,\delta F+2(\mathrm{g})\,\delta G+2(\mathrm{h})\,\delta H],\end{aligned}$$

et de là

$$\begin{aligned}\Omega_2=2\phi\,[-XYZ(\delta B-\delta C)+X(Y^2-Z^2)\,\delta F-Y(Y^2+Z^2)\,\delta G+Z(Y^2+Z^2)\,\delta H]\\ +[(\mathrm{g})\,Y-(\mathrm{h})\,Z]\,[(\mathrm{a})\,\delta A+(\mathrm{b})\,\delta B+(\mathrm{c})\,\delta C+2(\mathrm{f})\,\delta F+2(\mathrm{g})\,\delta G+2(\mathrm{h})\,\delta H],\end{aligned}$$

ce qui se réduit tout de suite à

$$\begin{aligned}&-[X(\mathrm{a})(\mathrm{f})+Y(\mathrm{b})(\mathrm{g})+Z(\mathrm{c})(\mathrm{h})]\,(\delta B-\delta C)\\ &+[(\mathrm{g})\,Y-(\mathrm{h})\,Z][(\mathrm{a})\,\delta A+(\mathrm{b})\,\delta B+(\mathrm{c})\,\delta C]\\ &+2\,[Y(\mathrm{c})(\mathrm{h})-Z(\mathrm{b})(\mathrm{g})]\,\delta F\\ &+2(\mathrm{a})\,[Y(\mathrm{c})-Z(\mathrm{f})\qquad]\,\delta G\\ &+2(\mathrm{g})\,[Y(\mathrm{f})-Z(\mathrm{b})\qquad]\,\delta H.\end{aligned}$$

Les premières deux lignes se réduisent facilement à

$$(a)\,[-X(f)+Z(h)]\,(\delta B-\delta C)+(a)\,[(g)\,Y-(h)\,Z]\,(\delta A-\delta C),$$

et la troisième ligne à $2(\mathrm{a})[Z(\mathrm{g})-X(\mathrm{c})]\,\delta F$. Donc l'expression entière contient le facteur (a), et nous aurons

$$\begin{aligned}\Omega_2 : (\mathrm{a}) = &-\ [X(\mathrm{f})+Z(\mathrm{h})](\delta B-\delta C)\\ &+\ [Y(\mathrm{g})-Z(\mathrm{h})](\delta A-\delta C)\\ &+2[Z(\mathrm{g})-X(\mathrm{c})]\,\delta F\\ &+2[Y(\mathrm{c})-Z(\mathrm{f})]\,\delta G\\ &+2[Y(\mathrm{f})-Z(\mathrm{b})]\,\delta H,\end{aligned}$$

expression qui se réduit sans peine à la forme symétrique sous laquelle je la présente dans l'équation finale.

12. Cette équation est $\Omega_1+\Omega_2=0$; savoir, en omettant le facteur (a), nous avons

$$\begin{aligned}&2[\{F[(\mathrm{b})-(\mathrm{c})]-(B-C)(\mathrm{f})-H(\mathrm{g})+G(\mathrm{h})\quad\}\,\delta X\\ &\quad+\{G[(\mathrm{c})-(\mathrm{a})]+H(\mathrm{f})-(C-A)(\mathrm{g})-F(\mathrm{h})\ \}\,\delta Y\\ &\quad+\{H[(\mathrm{a})-(\mathrm{b})]-G(\mathrm{h})+F(\mathrm{g})-(A-B)(\mathrm{h})\}\,\delta Z]\\ &\quad-X(\mathrm{f})(\delta B-\delta C)\\ &\quad-Y(\mathrm{g})(\delta C-\delta A)\\ &\quad-Z(\mathrm{h})(\delta A-\delta B)\\ &\quad+\{X[(\mathrm{b})-(\mathrm{c})]-Y(\mathrm{h})+Z(\mathrm{g})\}\,\delta F\\ &\quad+\{X(\mathrm{h})+Y[(\mathrm{c})-(\mathrm{a})]-Z(\mathrm{f})\}\,\delta G\\ &\quad+\{-X(\mathrm{g})+Y(\mathrm{f})+Z[(\mathrm{a})-(\mathrm{b})]\}\,\delta H=0.\end{aligned}$$

On se rappelle que δ signifie

$$X\frac{d}{dx}+Y\frac{d}{dy}+Z\frac{d}{dz}.$$

13. Pour déduire de là le résultat de M. Levy, j'écris d'abord $X=0$, $Y=0$; nous avons alors

$$[(\mathrm{a}),\ (\mathrm{b}),\ (\mathrm{c}),\ (\mathrm{f}),\ (\mathrm{g}),\ (\mathrm{h})]=(BZ^2,\ AZ^2,\ 0,\ 0,\ 0,\ -HZ^2),$$

et l'équation devient

$$2[(AF-GH)\,\delta X+(-BG+FH)\,\delta Y]+HZ(\delta A-\delta B)-Z(A-B)\,\delta\Pi=0;$$

mais ici

$$(A,\ B,\ C,\ F,\ G,\ H)=[2Z\mathrm{h},\ -2Z\mathrm{h},\ 2\mathrm{h},\ 0,\ -Z\mathrm{g},\ Z\mathrm{f},\ -Z(\mathrm{a}-\mathrm{b})],$$

et l'équation devient

$$2\{[\mathrm{f}(\mathrm{a}-\mathrm{b})-2\mathrm{gh}]\,\delta X+[\mathrm{g}(\mathrm{a}-\mathrm{b})+2\mathrm{fh}]\,\delta Y\}-(\mathrm{a}-\mathrm{b})(\delta A-\delta B)-4\mathrm{h}\delta H=0.$$

Mais nous avons $\delta X = gZ$, $\delta Y = fZ$, $\delta Z = cZ$, et, de plus,

$$\begin{aligned}
\delta A &= 2\delta Z h - 2g\delta Y = \quad 2lZ^2 + 2(ch - fg)Z,\\
\delta B &= 2f\delta X - 2\delta Z h = -2lZ^2 - 2(ch - fg)Z,\\
\delta H &= -\delta Z(a-b) + g\delta X - f\delta Y = (f-j)Z^2 + (-ac + bc - f^2 + g^2)Z;
\end{aligned}$$

l'équation est donc

$$4fg(a-b) + 4(f^2-g^2)h - (a-b)[4lZ + 4(ch-fg)] - 4h[-c(a-b) - (f^2-g^2) + (f-j)Z] = 0,$$

ou enfin

$$2fg(a-b) + 2h(f^2-g^2) - Z[(f-j)h + l(a-b)] = 0,$$

ce qui s'accorde avec le résultat cité.

14. En changeant la signification de X, Y, Z, écrivons $\rho = X + Y + Z$, où X, Y, Z dénotent à présent des fonctions de x, y, z respectivement; en dénotant par X', Y', Z' les fonctions dérivées de celles-ci, les fonctions premièrement représentées par X, Y, Z seront X', Y', Z'. Je cherche, au moyen de l'équation générale, la condition pour que la famille $\rho = X + Y + Z$ puisse faire partie d'un système orthogonal.

Dénotons par X', X'', X''' les dérivées de X, et de même celles de Y et Z, et écrivons, pour abréger, α, β, $\gamma = Y'' - Z''$, $Z'' - X''$, $X'' - Y''$, nous avons

$$(a, b, c, f, g, h) = (X'', Y'', Z'', 0, 0, 0),$$

et de là

$$(A, B, C, F, G, H) = (0, 0, 0, -\alpha X', -\beta Y', -\gamma Z'),$$

et, de plus,

$$\begin{aligned}
[(a), (b), (c), (f), (g), (h)] = [&2\alpha X'Y'Z', 2\beta X'Y'Z', 2\gamma X'Y'Z',\\
&X'(\alpha X'^2 - \beta Y'^2 - \gamma Z'^2),\\
&Y'(-\alpha X'^2 + \beta Y'^2 - \gamma Z'^2),\\
&Z'(-\alpha X'^2 - \beta Y'^2 + \gamma Z'^2)].
\end{aligned}$$

Nous avons aussi

$$\begin{aligned}
(\delta X', \delta Y', \delta Z') &= (X'X'', Y'Y'', Z'Z''),\\
(\delta A, \delta B, \delta C) &= (0, 0, 0),\\
(\delta F, \delta G, \delta H) &= [X'(-\alpha X'' + Z'Z''' - Y'Y'''),\\
&\quad Y'(-\beta Y'' + X'X''' - Z'Z'''),\\
&\quad Z'(-\gamma Z'' + Y'Y''' - X'X''')].
\end{aligned}$$

15. Donc, dans l'équation générale, la première ligne est

$$\begin{aligned}
2[-\alpha X'.2X'Y'Z'(\beta-\gamma) + \gamma Y'Z'(-\alpha X'^2 + \beta Y'^2 - \gamma Z'^2)\\
- \beta Y'Z'(-\alpha X'^2 - \beta Y'^2 + \gamma Z'^2)]X'X'',
\end{aligned}$$

c'est-à-dire

$$\begin{aligned}
2X'Y'Z'.X''[-2\alpha(\beta-\gamma)X'^2 + \gamma(-\alpha X'^2 + \beta Y'^2 - \gamma Z'^2)\\
- \beta(-\alpha X'^2 - \beta Y'^2 + \gamma Z'^2)]
\end{aligned}$$

ou, ce qui est la même chose,

$$2X'Y'Z' \,.\, \alpha X'' \left[(\gamma - \beta) X'^2 - \beta Y'^2 + \gamma Z'^2\right],$$

et la somme des premières trois lignes sera aussi $= 2X'Y'Z'$ multiplié par

$$\begin{aligned} &\alpha X'' \left[(\gamma - \beta) X'^2 - \beta Y'^2 + \gamma Z'^2\right] \\ &+ \beta Y'' \left[\alpha X'^2 + (\alpha - \gamma) Y'^2 - \gamma Z'^2\right] \\ &+ \gamma Z'' \left[-\alpha X'^2 + \beta Y'^2 + (\beta - \alpha) Z'^2\right], \end{aligned}$$

savoir dans ce second facteur le coefficient de $\alpha X'^2$ est $X''(\gamma - \beta) + \beta Y'' - \gamma Z''$, $= -2\beta\gamma$, et de même les coefficients de $\beta Y'^2$ et $\gamma Z'^2$ sont $-2\gamma\alpha$, $-2\alpha\beta$ respectivement, donc le terme entier, ou première partie de l'équation est

$$4X'Y'Z' (X'^2 + Y'^2 + Z'^2)(-\alpha\beta\gamma).$$

Les termes en δA, δB, δC s'évanouissent, et il ne reste que les termes en δF, δG, δH qui forment la seconde partie de l'équation. Le premier de ceux-ci est

$$\begin{aligned} &[X' \,.\, 2(\beta - \gamma) X'Y'Z' - Y'Z' (-\alpha X'^2 - \beta Y'^2 + \gamma Z'^2) \\ &\qquad + Y'Z' (-\alpha X'^2 + \beta Y'^2 - \gamma Z'^2)] \times X' (-X''\alpha + Z'Z''' - Y'Y'''), \end{aligned}$$

c'est-à-dire

$$2X'Y'Z' \left[(\beta - \gamma) X'^2 + \beta Y'^2 - \gamma Z'^2\right] (-X''\alpha + Z'Z''' - Y'Y''').$$

On a donc $2X'Y'Z'$ multiplié par

$$\begin{aligned} &\left[(\beta - \gamma) X'^2 + \beta Y'^2 - \gamma Z'^2\right] (-X''\alpha + Z'Z''' - Y'Y''') \\ &\quad + \left[-\alpha X'^2 + (\gamma - \alpha) Y'^2 + \gamma Z'^2\right] (-Y''\beta + X'X''' - Z'Z''') \\ &\quad + \left[\alpha X'^2 - \beta Y'^2 + (\alpha - \beta) Z'^2\right] (-Z''\gamma + Y'Y''' - X'X'''), \end{aligned}$$

où dans le second facteur nous avons d'abord le terme $-2\alpha\beta\gamma \times (X'^2 + Y'^2 + Z'^2)$ et puis le terme $-2(\alpha X'X''' + \beta Y'Y''' + \gamma Z'Z''') \times (X'^2 + Y'^2 + Z'^2)$.

La seconde partie est donc

$$4X'Y'Z' (X'^2 + Y'^2 + Z'^2) \left[-\alpha\beta\gamma - (\alpha X'X''' + \beta Y'Y''' + \gamma Z'Z''')\right]$$

et en réunissant les deux parties et en omettant le facteur $-4X'Y'Z' \times (X'^2 + Y'^2 + Z'^2)$, l'équation devient

$$2\alpha\beta\gamma + \alpha X'X''' + \beta Y'Y''' + \gamma Z'Z''' = 0,$$

savoir :

$$2(Y''' - Z'')(Z'' - X'')(X'' - Y'') + (Y'' - Z'') X'X''' + (Z'' - X'') Y'Y''' + (X'' - Y'') Z'Z''' = 0,$$

équation trouvée par M. Bouquet dans sa "Note sur les surfaces orthogonales" (*Journal de M. Liouville*, t. XI., pp. 446—450, 1846), et reproduite par M. Serret dans son "Mémoire sur les surfaces orthogonales" (*Journal de M. Liouville*, t. XII., pp. 241—254, 1847).

[Pp. 324—330.]

En considérant une famille orthogonale (savoir: une famille de surfaces qui fait partie d'un système orthogonal), on peut se proposer la question: *Étant donnée une surface de la famille, trouver de la manière la plus générale la famille.* J'essaye de résoudre cette question en développant les trois coordonnées selon les puissances d'un paramètre; et, quoique je n'aie encore calculé que les trois premiers termes des trois développements, les résultats me paraissent assez intéressants pour les soumettre aux géomètres.

On peut, pour la surface donnée, considérer les coordonnées x, y, z d'un point quelconque de la surface comme des fonctions déterminées de deux paramètres p, q. Si, de plus, ces paramètres sont tels, que les équations des deux systèmes de courbes de courbure soient $p = \text{const.}$, $q = \text{const.}$ respectivement, alors (en écrivant pour abréger

$$\frac{dx}{dp} = x_1, \quad \frac{dx}{dq} = x_2, \quad \frac{d^2x}{dp^2} = x_3, \quad \frac{d^2x}{dp\,dq} = x_4, \quad \frac{d^2x}{dq^2} = x_5,$$

et de même pour y et z) ces coordonnées x, y, z, considérées toujours comme des fonctions de p, q, seront telles, que

$$x_1x_2 + y_1y_2 + z_1z_2 = 0, \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0.$$

J'écris ici et dans la suite X, Y, $Z = y_1z_2 - y_2z_1$, $z_1x_2 - z_2x_1$, $x_1y_2 - x_2y_1$. On a donc identiquement

$$Xx_1 + Yy_1 + Zz_1 = 0,$$
$$Xx_2 + Yy_2 + Zz_2 = 0,$$

et les deux équations mentionnées sont

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$
$$Xx_4 + Yy_4 + Zz_4 = 0.$$

Je m'arrête pour remarquer que la dernière équation, dans sa forme originale, peut être remplacée par trois équations de la forme $x_4 + Ax_1 + Bx_2 = 0$, et qu'en ajoutant les trois équations multipliées par x_1, y_1, z_1 respectivement, et aussi multipliées par x_2, y_2, z_2 respectivement, on obtient les valeurs de A, B, exprimées en termes de

$$E = x_1^2 + y_1^2 + z_1^2 \text{ et } G = x_2^2 + y_2^2 + z_2^2$$

(E, G de Gauss), et que l'on trouve de là

$$2\frac{d^2x}{dp\,dq} - \frac{1}{E}\frac{dE}{dq}\frac{dx}{dp} - \frac{1}{G}\frac{dG}{dp}\frac{dx}{dq} = 0,$$

avec les équations semblables en y et z. Ces équations sont, en effet, les équations (10 *bis*) de Lamé, "Mémoire sur les coordonnées curvilignes" (*Liouville*, t. v. 1840, p. 322).

Je suppose que les surfaces de la famille dépendent du paramètre r, lequel pour la surface donnée se réduit à $r=0$. Par le point (p, q) de la surface donnée on peut mener une trajectoire orthogonale aux différentes surfaces de la famille; les coordonnées ξ, η, ζ d'un point quelconque sur cette courbe seront des fonctions de p, q, r, lesquelles, pour $r=0$, se réduisent à x, y, z respectivement; et j'écris

$$\xi = x + ar + dr^2 + \ldots,$$
$$\eta = y + br + er^2 + \ldots,$$
$$\zeta = z + cr + fr^2 + \ldots,$$

où a, b, c, d, e, f, ... sont des fonctions inconnues de p et q.

Pour exprimer que la courbe coupe orthogonalement les différentes surfaces de la famille, écrivons pour abréger

$$\eta_1\zeta_2 - \eta_2\zeta_1 = X + Ar + Dr^2 + \ldots, \quad X = y_1z_2 - y_2z_1,$$
$$\zeta_1\xi_2 - \zeta_2\xi_1 = Y + Br + Er^2 + \ldots, \quad A = y_1c_2 - y_2c_1 + b_1z_2 - b_2z_1,$$
$$\xi_1\eta_2 - \xi_2\eta_1 = Z + Cr + Fr^2 + \ldots, \quad \ldots\ldots$$

$\left(\text{où } \xi_1 = \dfrac{d\xi}{dp} \ldots, \text{ comme pour } x, y, z\right)$. La condition cherchée est

$$\frac{X + Ar + Dr^2 + \ldots}{a + 2dr + \ldots} = \frac{Y + Br + Er^2 + \ldots}{b + 2er + \ldots} = \frac{Z + Cr + Fr^2 + \ldots}{c + 2fr + \ldots},$$

laquelle doit être satisfaite pour une valeur quelconque de r; on a donc

$$(1) \qquad \frac{X}{a} = \frac{Y}{b} = \frac{Z}{c},$$

$$(2) \qquad \frac{A}{a} - \frac{2dX}{a^2} = \frac{B}{b} - \frac{2eY}{b^2} = \frac{C}{c} - \frac{2fZ}{c^2};$$

savoir, les équations (1) contiennent (a, b, c), les équations (2) contiennent de plus (d, e, f), et ainsi de suite.

Pour qu'il y ait un système orthogonal, il faut et il suffit que l'on ait

$$\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2 = 0,$$

pour toute valeur de r; on aura donc

$$[0] \qquad x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

$$[1] \qquad x_1a_2 + x_2a_1 + y_1b_2 + y_2b_1 + z_1c_2 + z_2c_1 = 0,$$

$$[2] \qquad x_1d_2 + x_2d_1 + y_1e_2 + y_2e_1 + z_1f_2 + z_2f_1 + a_1a_2 + b_1b_2 + c_1c_2 = 0,$$

savoir, l'équation [0] est satisfaite d'elle-même; l'équation [1] contient (a, b, c), l'équation [2] contient de plus (d, e, f), et ainsi de suite.

Il paraît donc qu'il y a les trois équations (1), [1] pour déterminer (a, b, c); les trois équations (2), [2] pour déterminer (d, e, f), et ainsi de suite. Mais les choses ne se comportent pas ainsi. On satisfait à (1), [1] par des valeurs de (a, b, c) qui contiennent une fonction arbitraire λ, fonction qui est ensuite déterminée au moyen d'une équation à différences partielles du second ordre, obtenue au moyen des équations (2), [2]; on satisfait alors à (2), [2] par des valeurs de (d, e, f) qui contiennent une fonction arbitraire θ; je présume que cette fonction serait ensuite déterminée au moyen des équations (3), [3], et ainsi de suite; mais je n'ai pas encore fait les calculs ultérieurs.

Par rapport à λ, en remplaçant cette fonction par $\rho = \lambda\sqrt{X^2 + Y^2 + Z^2}$, l'équation pour ρ est

$$2\frac{d^2\rho}{dp\,dq} - \frac{1}{E}\frac{dE}{dq}\frac{d\rho}{dp} - \frac{1}{G}\frac{dG}{dp}\frac{d\rho}{dq} = 0,$$

savoir, c'est la même équation que pour x, y, z: ainsi l'on y satisfait en prenant ρ égal à une fonction linéaire (avec terme constant) quelconque de x, y, z.

Pour obtenir ces conclusions, partant des équations (1), [1], les équations (1) donnent

$$a, b, c = \lambda X, \lambda Y, \lambda Z,$$

où λ est une fonction de p, q: ces valeurs satisfont d'elles-mêmes à l'équation [1]. La vérification se fait sans peine; j'écris pour abréger x_1x_2 pour dénoter $x_1x_2 + y_1y_2 + z_1z_2$, et ainsi dans les cas semblables: l'équation à vérifier est donc

$$x_1(\lambda X)_2 + x_2(\lambda X)_1 = 0,$$

c'est-à-dire

$$\lambda(x_1X_2 + x_2X_1) + \lambda_2x_1X + \lambda_1x_2X = 0,$$

où nous avons

$$x_1X = 0, \quad x_2X = 0;$$

reste à trouver le coefficient $x_1X_2 + x_2X_1$. Nous avons

$$X = y_1z_2 - y_2z_1,$$

et de là

$$X_1 = y_1z_4 - y_4z_1 + y_3z_2 - y_2z_3,$$
$$X_2 = y_1z_5 - y_5z_1 + y_4z_2 - y_2z_4,$$

et de là, en faisant la somme des trois termes de x_1X_2 et x_2X_1 respectivement, on trouve

$$x_1X_2 = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_4, & y_4, & z_4 \end{vmatrix} = x_2X_1,$$

savoir: $x_1X_2 + x_2X_1$ est égal à -2 multiplié par ce déterminant, $= -2Xx_4$, c'est-à-dire $x_1X_2 + x_2X_1 = 0$. Donc la fonction λ est jusqu'ici indéterminée.

Passons aux équations (2), [2]. Substituant dans (2) les valeurs de (a, b, c), ces équations deviennent

$$\frac{A}{\lambda X}-\frac{2d}{\lambda^2 X}=\frac{B}{\lambda Y}-\frac{2e}{\lambda^2 Y}=\frac{C}{\lambda Z}-\frac{2f}{\lambda^2 Z}.$$

On y satisfait en écrivant

$$2d,\ 2e,\ 2f=\lambda(\theta X+A),\quad \lambda(\theta Y+B),\quad \lambda(\theta Z+C),$$

où θ est fonction de (p, q); en substituant ces valeurs dans l'équation [2], la fonction θ disparaît d'elle-même; mais on obtient pour λ une équation linéaire entre λ, λ_1, λ_2 et λ_4, laquelle est ainsi une équation à différences partielles du second ordre, et, cela étant, on a pour d, e, f les expressions mentionnées, qui contiennent la fonction θ, fonction qui n'est pas déterminée par les équations (2), [2].

L'équation [2], sous la forme abrégée, est

$$x_1 d_2 + x_2 d_1 + a_1 a_2 = 0,$$

c'est-à-dire

$$x_1[\lambda(\theta X+A)]_2 + x_2[\lambda(\theta X+A)]_1 + 2a_1 a_2 = 0,$$

ou, ce qui est la même chose,

$$\lambda[x_1(\theta X+A)_2 + x_2(\theta X+A)_1] + \lambda_2 x_1(\theta X+A) + \lambda_1 x_2(\theta X+A) + 2a_1 a_2 = 0.$$

Les termes en θ sont

$$\lambda[x_1(\theta X_2+\theta_2 X) + x_2(\theta X_1+\theta_1 X)] + \lambda_2\theta x_1 X + \lambda_1\theta x_2 X.$$

qui s'évanouissent d'eux-mêmes; l'équation se réduit donc à

$$\lambda(A_2 x_1 + A_1 x_2) + \lambda_2 A x_1 + \lambda_1 A x_2 + 2a_1 a_2 = 0,$$

ou, en substituant la valeur de $a_1 a_2$,

$$\lambda(A_2 x_1 + A_1 x_2) + \lambda_2 A x_1 + \lambda_1 A x_2 + 2(\lambda X)_1(\lambda X)_2 = 0;$$

on a

$$(\lambda X)_1(\lambda X)_2 = (\lambda X_1+\lambda_1 X)(\lambda X_2+\lambda_2 X),\ = \lambda_2 X_1 X_2 + \lambda\lambda_2 X X_1 + \lambda\lambda_1 X X_2 + \lambda_1\lambda_2 X^2,$$

et l'on trouve sans peine $Ax_1 = -a_1 X$, $Ax_2 = -a_2 X$, et de là

$$Ax_1 = -(\lambda X)_1 X,\ = -\lambda_1 X^2 - \lambda X X_1,$$
$$Ax_2 = -(\lambda X)_2 X,\ = -\lambda_2 X^2 - \lambda X X_2.$$

Substituant ces valeurs, l'équation entière contiendra le facteur λ, et en l'écartant, elle devient

$$A_2 x_1 + A_1 x_2 + \lambda_2 X X_1 + \lambda_1 X X_2 + 2\lambda X_1 X_2 = 0.$$

Pour abréger encore la notation, au lieu de x_1^2, $(=x_1^2+y_1^2+z_1^2)$, j'écris simplement 11, et ainsi dans les cas semblables: savoir, je me sers des abréviations

$$\begin{aligned} 11 &= x_1^2 + y_1^2 + z_1^2, \\ 12 &= x_1x_2 + y_1y_2 + z_1z_2 \,(=0), \\ &\vdots \end{aligned}$$

et je remarque que l'équation $12=0$, en prenant les dérivées par rapport à p, q respectivement, donne $15+24=0$, $23+14=0$, équations qui servent pour éliminer des formules les expressions 15 et 23. Si pour un moment nous dénotons ainsi par 124 le déterminant $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}$, alors, en multipliant par les déterminants analogues 123 et 125 respectivement, l'équation $124=0$ donne

$$\begin{vmatrix} 11 & . & 14 \\ . & 22 & 24 \\ 51 & 52 & 54 \end{vmatrix} = 0, \quad \begin{vmatrix} 11 & . & 14 \\ . & 22 & 24 \\ 31 & 32 & 34 \end{vmatrix} = 0,$$

dont chacune est une équation à trois termes entre les quantités 11, 22, ...

Nous avons

$$\begin{aligned} A &= y_1(\lambda Z)_2 - y_2(\lambda Z)_1 + z_2(\lambda Y)_1 - z_1(\lambda Y)_2, \\ &= \lambda(y_1Z_2 - y_2Z_1 + z_2Y_1 - z_1Y_2) + \lambda_1(z_2Y - y_2Z) + \lambda_2(y_1Z - z_1Y); \end{aligned}$$

or nous avons

$$Y,\ Z = z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1,$$

et en formant de là les valeurs de Y_1, Y_2, Z_1, Z_2 on obtient sans peine

$$\begin{aligned} A = \lambda\,[&x_1(15-24) + x_2(23-14) - x_3 . 22 + 2x_4 . 12 - x_5 . 11] \\ &+ \lambda_1(x_2 . 12 - x_1 . 22) + \lambda_2(x_1 . 12 - x_2 . 11), \end{aligned}$$

ou, ce qui est la même chose,

$$A = \lambda\,[-2x_1 . 24 - 2x_2 . 14 - x_3 . 22 - x_5 . 11] - \lambda_1 x_1 . 22 - \lambda_2 x_2 . 11.$$

Écrivons pour un moment

$$A = \lambda P + \lambda_1 P' + \lambda_2 P'';$$

nous avons

$$\begin{aligned} A_1 &= P_1\lambda + (P + P_1')\lambda_1 + P_1''\lambda_2 + P'\lambda_3 + P''\lambda_4, \\ A_2 &= P_2\lambda + P_2'\lambda_1 + (P + P_2'')\lambda_2 + P'\lambda_4 + P''\lambda_5, \end{aligned}$$

et de là

$$\begin{aligned} A_1x_2 + A_2x_1 = \ &\lambda\,(P_1x_2 + P_2x_1) + \lambda_1[(P + P_1')x_2 + P_2'x_1] \\ &+ \lambda_2[P_1''x_2 + (P + P_2'')x_1] + \lambda_3 P'x_2 \\ &+ \lambda_4(P''x_2 + P'x_1) + \lambda_5 P''x_1. \end{aligned}$$

[Pp. 381—385.]

Les expressions de P_1, P_2,... contiennent les dérivées du troisième ordre

$$\frac{d^3x}{dp^3} = x_6, \quad \frac{d^3x}{dp^2dq} = x_7, \quad \frac{d^3x}{dpdq^2} = x_8, \quad \frac{d^3x}{dq^3} = x_9, \ldots$$

En formant les dérivées des équations $23 + 14 = 0$, $15 + 24 = 0$ par rapport à p et q respectivement, on obtient

$$\begin{aligned} 17 + 26 + \ \ 2 \,.\, 34 \ &= 0, \\ 18 + 27 + 35 + 44 &= 0, \\ 19 + 28 + \ \ 2 \,.\, 45 \ &= 0. \end{aligned}$$

On obtient alors

$$\begin{aligned} P_1 = &- 2x_1(44 + 17) + 2x_2(34 + 17) \\ &- 4x_3 \,.\, 24 - 2x_4 \,.\, 14 - 2x_5 \,.\, 13 - x_6 \,.\, 22 - x_8 \,.\, 11, \end{aligned}$$

et de là la somme P_1x_2 est

$$= -2 \,.\, 22(34 + 17) - 4 \,.\, 23 \,.\, 24 - 2 \,.\, 24 \,.\, 14 - 2 \,.\, 25 \,.\, 13 - 26 \,.\, 22 - 28 \,.\, 11,$$

ou, ce qui est la même chose,

$$P_1x_2 = -22 \,.\, 17 - 11 \,.\, 28 + 2 \,.\, 14 \,.\, 24 - 2 \,.\, 25 \,.\, 13.$$

On a de même

$$\begin{aligned} P_2 = &- 2 \,.\, x_1(45 + 28) - 2 \,.\, x_2(44 + 18) \\ &- 2x_3 \,.\, 25 - 2x_4 \,.\, 24 - 4x_5 \,.\, 14 - x_7 \,.\, 22 - x_9 \,.\, 11, \end{aligned}$$

et de là la somme P_2x_1 est

$$= -2 \,.\, 11(45 + 28) - 2 \,.\, 13 \,.\, 25 - 2 \,.\, 14 \,.\, 24 - 4 \,.\, 15 \,.\, 14 - 17 \,.\, 22 - 19 \,.\, 11,$$

ou, ce qui est la même chose,

$$P_2x_1 \qquad = -22 \,.\, 17 - 11 \,.\, 28 + 2 \,.\, 14 \,.\, 24 - 2 \,.\, 25 \,.\, 13 \,(= P_1x_2);$$

on a donc

$$P_1x_2 + P_2x_1 = -2 \,.\, 22 \,.\, 17 - 2 \,.\, 11 \,.\, 28 + 4 \,.\, 14 \,.\, 24 - 4 \,.\, 25 \,.\, 13.$$

On obtient sans peine les autres sommes

$$\begin{aligned} &P'x_2 = 0, \quad P''x_2 + P'x_1 = -2 \,.\, 11 \,.\, 22, \quad P''x_1 = 0, \\ &Px_1 = -11 \,.\, 24 - 22 \,.\, 13, \quad Px_2 = -11 \,.\, 25 - 22 \,.\, 14, \end{aligned}$$

et l'on a ainsi

$$\begin{aligned} A_1x_2 + A_2x_1 = \ &\lambda\,(-2 \,.\, 11 \,.\, 28 - 2 \,.\, 22 \,.\, 17 + 4 \,.\, 14 \,.\, 24 - 4 \,.\, 25 \,.\, 13) \\ &+ \lambda_1(-3 \,.\, 11 \,.\, 25 - 2 \,.\, 22 \,.\, 14 - 22 \,.\, 23) \\ &+ \lambda_2(-2 \,.\, 11 \,.\, 24 - 11 \,.\, 15 - 3 \,.\, 22 \,.\, 13) \\ &+ \lambda_4(-2 \,.\, 11 \,.\, 22). \end{aligned}$$

L'équation en λ est

$$A_1x_2 + A_2x_1 + 2\lambda X_1X_2 + \lambda_1 XX_2 + \lambda_2 XX_1 = 0,$$

et l'on obtient sans peine

$$\begin{aligned} X_1X_2 &= 11 \,.\, 45 + 22 \,.\, 34 + 3 \,.\, 14 \,.\, 24 + 25 \,.\, 13, \\ XX_2 &= 11 \,.\, 25 + 22 \,.\, 14, \\ XX_1 &= 11 \,.\, 24 + 22 \,.\, 13. \end{aligned}$$

Donc enfin l'équation en λ est

$$\lambda[11(-28+45) + 22(-17+34) + 3 \,.\, 14 \,.\, 24 - 25 \,.\, 13] - \lambda_1 \,.\, 11 \,.\, 25 - \lambda_2 \,.\, 22 \,.\, 13 - \lambda_4 \,.\, 11 \,.\, 22 = 0.$$

Cette équation est vérifiée par la valeur $R = \frac{1}{V}(V = \sqrt{X^2 + Y^2 + Z^2})$; en effet, en dénotant pour un moment le premier coefficient par Λ, l'équation à vérifier est

$$\Lambda V^2 + 11 \,.\, 25 \,.\, XX_1 + 22 \,.\, 13 \,.\, XX_2 + (X^2 \,.\, X_1X_2 + X^2 \,.\, XX_4 - 3 \,.\, XX_1 \,.\, XX_2) = 0,$$

c'est-à-dire

$$\begin{aligned} 11 \,.\, 22\Lambda &+ 11 \,.\, 25(22 \,.\, 13 + 11 \,.\, 24) + 22 \,.\, 13(22 \,.\, 14 + 11 \,.\, 25) \\ &+ 11 \,.\, 22(11 \,.\, 45 + 22 \,.\, 34 + 3 \,.\, 14 \,.\, 24 + 24 \,.\, 13 + XX_4) \\ &- 3(22 \,.\, 13 + 11 \,.\, 24)(22 \,.\, 14 + 11 \,.\, 25) = 0, \end{aligned}$$

et l'on remarque qu'il n'y a ici que les termes $-2(11^2 \,.\, 24 \,.\, 25 + 22^2 \,.\, 13 \,.\, 24)$ qui ne contiennent pas le facteur $11 \,.\, 22$.

Savoir, l'équation est de la forme

$$11 \,.\, 22\,\Omega - 2(11^2 \,.\, 24 \,.\, 25 + 22^2 \,.\, 13 \,.\, 14) = 0;$$

mais, des équations mentionnées $123 \,.\, 124 = 0$ et $125 \,.\, 124 = 0$, on obtient

$$\begin{aligned} 22^2 \,.\, 13 \,.\, 14 &= 11 \,.\, 22(22 \,.\, 34 + 14 \,.\, 24), \\ 11^2 \,.\, 24 \,.\, 25 &= 11 \,.\, 22(11 \,.\, 45 + 14 \,.\, 24). \end{aligned}$$

Donc l'équation entière contient le facteur $11 \,.\, 22$ et, en l'écartant, elle devient

$$\Omega - 2(11 \,.\, 45 + 22 \,.\, 34 + 2 \,.\, 14 \,.\, 24) = 0.$$

On a

$$\Omega = 11 \,.\, (-28 + 2 \,.\, 45) + 22(-17 + 2 \,.\, 34) - 25 \,.\, 13 + 3 \,.\, 14 \,.\, 24 + XX_4;$$

l'équation est donc

$$-11 \,.\, 28 - 22 \,.\, 17 - 25 \,.\, 13 - 14 \,.\, 25 + XX_4 = 0;$$

et l'on vérifie sans peine que la valeur de XX_4 est actuellement

$$XX_4 = 11 \,.\, 28 + 22 \,.\, 17 + 25 \,.\, 13 + 14 \,.\, 24.$$

Donc, en écrivant $\lambda = \frac{\rho}{V}$, l'équation en ρ ne contiendra que les termes en ρ_1, ρ_2, ρ_4. En effet, l'équation devient

$$-11.25\frac{\rho_1}{V} - 22.13\frac{\rho_2}{V} - V^2\left(\frac{\rho_4}{V} - \frac{\rho_1}{V^3}XX_2 - \frac{\rho_2}{V^3}XX_1\right),$$

où, comme auparavant, XX_1 dénote $XX_1 + YY_1 + ZZ_1$, et de même XX_2 dénote $XX_2 + YY_2 + ZZ_2$. Nous avons déjà trouvé

$$XX_1 = 22.13 + 11.24, \quad XX_2 = 22.14 + 11.25;$$

l'équation devient ainsi

$$11.22\rho_4 - 14.22\rho_1 - 24.11\rho_2 = 0.$$

Savoir, cette équation est

$$2\frac{d^2\rho}{dp\,dq} - \frac{1}{E}\frac{dE}{dq}\frac{d\rho}{dp} - \frac{1}{G}\frac{dG}{dp}\frac{d\rho}{dq} = 0.$$

Pour compléter la solution, il convient d'exprimer A, B, C en termes de ρ. Nous avons

$$A = \lambda(-2x_1.24 - 2x_2.14 - x_3.22 - x_5.11) - \lambda_1 x_1.22 - \lambda_2 x_2.11.$$

Substituant la valeur $\lambda = \frac{\rho}{V}$, le coefficient de ρ est

$$\frac{1}{V}(-2x_2.24 - 2x_2.14 - x_3.22 - x_5.11) + \frac{1}{V^3}(x_1.22XX_1 + x_2.11.XX_2)$$

$$= \frac{1}{V^3}[11.22(-2x_1.24 - 2x_2.14 - x_3.22 - x_5.11)$$

$$+ x_1.22(13.22 + 24.11) + x_2.11(14.22 + 25.11)],$$

ou, ce qui est la même chose

$$\frac{1}{V^3}[x_1.22(22.13 + 11.15) + x_2.11(11.25 + 22.23) - 11.22(x_3.22 + x_5.11)].$$

Le terme entre [ ] est fonction linéaire de x_3, y_3, z_3, x_5, y_5, z_5, et en réunissant les termes qui contiennent ces quantités respectivement, on le réduit sans peine à la forme

$$X[11(Xx_5 + Yy_5 + Zz_5) + 22(Xx_3 + Yy_3 + Zz_3)],$$

ou, ce qui est la même chose

$$X(11.125 + 22.123).$$

Nous avons donc

$$A = -\frac{\rho X}{V^3}(11.125 + 22.123) - \frac{1}{V}(x_1\rho_1.22 + x_2\rho_2.11).$$

Nous avons

$$11 = E, \quad 22 = G, \quad V = \sqrt{EG};$$

donc, en écrivant, pour abréger,

$$-\frac{\rho}{EG\sqrt{EG}}(11 \,.\, 125 + 22 \,.\, 123) = \theta',$$

la valeur est

$$A = \theta' X - \frac{1}{\sqrt{EG}}\left(G\frac{dx}{dp}\frac{d\rho}{dp} + E\frac{dx}{dq}\frac{d\rho}{dq}\right),$$

avec des expressions semblables pour B et C. Dans les expressions $2d = \frac{\rho}{\sqrt{EG}}(\theta X + A)\ldots$, la fonction θ' se combine avec la fonction arbitraire θ, de manière qu'il serait permis de remplacer $\theta + \theta'$ par un seul symbole θ, mais je retiens $\theta + \theta'$.

Donc, enfin, les expressions de ξ, η, ζ deviennent

$$\xi = x + \frac{\rho X}{\sqrt{EG}}r + \tfrac{1}{2}\left[\frac{(\theta+\theta')\rho X}{\sqrt{EG}} - \frac{\rho}{EG}\left(G\frac{dx}{dp}\frac{d\rho}{dp} + E\frac{dx}{dq}\frac{d\rho}{dq}\right)\right]r^2 + \ldots,$$

$$\eta = y + \frac{\rho Y}{\sqrt{EG}}r + \tfrac{1}{2}\left[\frac{(\theta+\theta')\rho Y}{\sqrt{EG}} - \frac{\rho}{EG}\left(G\frac{dy}{dp}\frac{d\rho}{dp} + E\frac{dy}{dq}\frac{d\rho}{dq}\right)\right]r^2 + \ldots,$$

$$\zeta = z + \frac{\rho Z}{\sqrt{EG}}r + \tfrac{1}{2}\left[\frac{(\theta+\theta')\rho Z}{\sqrt{EG}} - \frac{\rho}{EG}\left(G\frac{dz}{dp}\frac{d\rho}{dp} + E\frac{dz}{dq}\frac{d\rho}{dq}\right)\right]r^2 + \ldots.$$

Je remarque que l'on satisfait à toutes les conditions en prenant $\rho = \text{const.}$ (ou, ce qui est la même chose, $\rho = 1$), $\theta + \theta' = 0$: cela donne

$$\xi,\ \eta,\ \zeta = x + \frac{rX}{\sqrt{EG}}, \quad y = \frac{rY}{\sqrt{EG}}, \quad z = \frac{rZ}{\sqrt{EG}};$$

savoir, la famille est ici celle des surfaces parallèles à la surface donnée.

[Pp. 1800—1803.]

J'ai trouvé que l'équation différentielle du troisième ordre, sous la forme [ci-dessus trouvée], contient le facteur étranger $X^2 + Y^2 + Z^2$, et que l'équation débarrassée de ce facteur devient beaucoup plus simple. La réduction et aussi la nouvelle méthode dont je me suis servi pour obtenir l'équation réduite sont toutes les deux assez pénibles; mais cette nouvelle méthode a l'avantage d'établir un résultat intermédiaire qui a quelque valeur. J'ai changé un peu la notation; aussi je commence en l'expliquant.

Je prends $U = 0$ l'équation d'une surface; X, Y, Z les coefficients différentiels du premier ordre; a, b, c, f, g, h les coefficients du second ordre;

$$(A,\ B,\ C,\ F,\ G,\ H \between dx,\ dy,\ dz)^2 = 0$$

l'équation différentielle des courbes de courbure; savoir:

$$\begin{aligned}
A &= 2(hZ - gY),\\
B &= 2(fX - hZ),\\
C &= 2(gY - fX),\\
F &= hY - gZ - (b-c)X,\\
G &= fZ - hX - (c-a)Y,\\
H &= gX - fY - (a-b)Z;
\end{aligned}$$

et

$$[(A), (B), (C), (F), (G), (H)](dx, dy, dz)^2$$
$$= (A, B, C, F, G, H \between Ydz - Zdy, Zdx - Xdz, Xdy - Ydx)^2;$$

savoir:

$$\begin{aligned}
(A) &= BZ^2 + CY^2 - 2FYZ,\\
(B) &= CX^2 + AZ^2 - 2GZX,\\
(C) &= AY^2 + BX^2 - 2HXY,\\
(F) &= -aYZ - fX^2 + gXY + hXZ,\\
(G) &= -bZX + fYX - gY^2 + hYZ,\\
(H) &= -cXY + fZX + gZY - hZ^2,
\end{aligned}$$

ce qui explique la signification des symboles (A), (B), ...; aussi

$$V^2 = X^2 + Y^2 + Z^2,$$

et

$$(\bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{g}, \bar{h}) = (bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch).$$

Cela étant, à chaque point P de la surface $U = 0$, je prends sur la normale une distance infinitésimale $PP' = \rho$, où ρ est une fonction quelconque des coordonnées x, y, z du point P; on obtient ainsi une surface, lieu des points P', laquelle se nomme la surface voisine, et les points P et P' sont des points correspondants sur les deux surfaces.

Pour que les deux surfaces forment partie d'un système orthogonal, je trouve que la distance ρ, considérée comme fonction des coordonnées (x, y, z), doit satisfaire à cette équation différentielle du second ordre

$$[(A), (B), (C), (F), (G), (H)](d_x, d_y, d_z)^2 \rho = 0.$$

Or, si la surface donnée $U = 0$ et la surface voisine sont des surfaces consécutives d'une famille $r - f(x, y, z) = 0$; savoir, si l'équation de la surface donnée est $r - f(x, y, z) = 0$, et celle de la surface voisine $r + dr - f(x, y, z) = 0$, on a

$$\rho = \frac{dr}{V}, \quad V = \sqrt{X^2 + Y^2 + Z^2},$$

où dr est une constante; l'équation devient ainsi

$$[(A), \ldots](d_x, d_y, d_z)^2 \frac{1}{V} = 0,$$

où, à présent, X, Y, Z, a, b, c, f, g, h dénotent les coefficients différentiels de $f(x, y, z)$, ou (ce qui est la même chose) du paramètre r, considéré comme fonction des coordonnées x, y, z. Cette équation est donc une équation du troisième ordre, à laquelle doit satisfaire la fonction ρ; multipliée par V^5, elle est en effet l'équation [dont il s'agit], laquelle, comme j'ai déjà dit, contient le facteur V^2: donc pour écarter le dénominateur, il suffira de multiplier par V^3.

J'écris $\delta = Xd_x + Yd_y + Zd_z$,

et de là

$$\delta X,\ \delta Y,\ \delta Z = aX + hY + gZ,\quad hX + bY + fZ,\quad gX + fY + cZ,$$

respectivement. On trouve

$$d_x^2 \frac{1}{V} = -\frac{1}{V^3}(a^2 + h^2 + g^2 + \delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_x \frac{1}{V} = -\frac{1}{V^3}(gh + bf + cf + \delta f) + \frac{3}{V^5}\delta Y\,\delta Z,$$

ou, en écrivant $\omega = a + b + c$, $\bar{\omega} = \bar{a} + \bar{b} + \bar{c}$, ces valeurs deviennent

$$d_x^2 \frac{1}{V} = -\frac{1}{V^3}(a\omega - \bar{\omega} + \bar{a} + \delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_x \frac{1}{V} = -\frac{1}{V^3}(f\omega \qquad + \bar{f} + \delta f) + \frac{3}{V^5}\delta Y\,\delta Z.$$

et, en substituant ces valeurs, les termes en ω, $\bar{\omega}$ disparaissent d'eux-mêmes, et l'équation, multipliée seulement par $-V^2$, se réduit à

$$[(A), \ldots](\bar{a}, \ldots) + [(A), \ldots](\delta a, \ldots) - \frac{3}{V^2}[(A), \ldots](\delta X,\ \delta Y,\ \delta Z)^2 = 0,$$

où le premier terme est

$$(A)\bar{a} + (B)\bar{b} + (C)\bar{c} + 2(F)\bar{f} + 2(G)\bar{g} + 2(H)\bar{h},$$

et de même pour le second terme.

Or je trouve que l'on a identiquement

$$[(A), \ldots](\delta X,\ \delta Y,\ \delta Z)^2 = -V^2(A, \ldots)(\delta X,\ \delta Y,\ \delta Z)^2,$$

de manière que l'équation est

$$[(A), \ldots](\bar{a}, \ldots) + [(A), \ldots](\delta a, \ldots) + 3(A, \ldots)(\delta X,\ \delta Y,\ \delta Z)^2 = 0,$$

et, de plus, que l'on a identiquement

$$[(A), \ldots](\bar{a}, \ldots) = -(A, \ldots)(\delta X,\ \delta Y,\ \delta Z)^2 = 2\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix},$$

où $\bar{\delta}X$, $\bar{\delta}Y$, $\bar{\delta}Z$ dénotent respectivement $\bar{a}X+\bar{h}Y+\bar{g}Z$, $\bar{h}X+\bar{b}Y+\bar{f}Z$, $\bar{g}X+\bar{f}Y+\bar{c}Z$; l'équation se réduit donc à

$$[(A),\ldots](\delta a,\ldots)+\Omega=0,$$

où Ω peut être exprimé à volonté sous l'une quelconque des trois formes

$$\begin{aligned}
&=+2\,[(A),\ldots](\bar{a},\ldots),\\
&=+2\,(A,\ldots\between\delta X,\ \delta Y,\ \delta Z)^2,\\
&=-4\begin{vmatrix}\delta X, & \delta Y, & \delta Z\\ X, & Y, & Z\\ \bar{\delta}X, & \bar{\delta}Y, & \bar{\delta}Z\end{vmatrix}.
\end{aligned}$$

Prenant la première forme, l'équation est

$$[(A),\ldots](\delta a,\ldots)-2\,[(A),\ldots](\bar{a},\ldots)=0,$$

ou, ce qui est la même chose,

$$\begin{aligned}
(A)\,\delta a+(B)\,\delta b+(C)\,\delta c+2\,(F)\,\delta f+2\,(G)\,\delta g+2\,(H)\,\delta h\\
-2\,[(A)\,\bar{a}+(B)\,\bar{b}+(C)\,\bar{c}+2\,(F)\,\bar{f}+2\,(G)\,\bar{g}+2\,(H)\,\bar{h}]=0,
\end{aligned}$$

où les coefficients sont des fonctions données de X, Y, Z, a, b, c, f, g, h, les coefficients différentiels de r du premier et du second ordre, et δ dénote $Xd_x+Yd_y+Zd_z$.

519.

ON CURVATURE AND ORTHOGONAL SURFACES.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXIII. (for the year 1873), pp. 229—251. Received December 27, 1872,—Read February 13, 1873.]

THE principal object of the present Memoir is the establishment of the partial differential equation of the third order satisfied by the parameter of a family of surfaces belonging to a triple orthogonal system. It was first remarked by Bouquet that a given family of surfaces does not in general belong to an orthogonal system, but that (in order to its doing so) a condition must be satisfied; it was afterwards shown by Serret that the condition is that the parameter, considered as a function of the coordinates, must satisfy a partial differential equation of the third order: this equation was not obtained by him or the other French geometers engaged on the subject, although methods of obtaining it, essentially equivalent but differing in form, were given by Darboux and Levy; the last-named writer even found a particular form of the equation, viz. what the general equation becomes on writing therein $X=0$, $Y=0$ (X, Y, Z the first derived functions, or quantities proportional to the cosine-inclinations of the normal). Using Levy's method, I obtained the general equation, and communicated it to the French Academy, [518]. My result was, however, of a very complicated form, owing, as I afterwards discovered, to its being encumbered with the extraneous factor $X^2+Y^2+Z^2$; I succeeded, by some difficult reductions, in getting rid of this factor, and so obtaining the equation in the form given in the present memoir, viz.

$$\begin{aligned}&\big((A), (B), (C), (F), (G), (H)\between \delta a,\ \delta b,\ \delta c,\ 2\delta f,\ 2\delta g,\ 2\delta h\big)\\ &-2\big((A), (B), (C), (F), (G), (H)\between \bar{a},\ \bar{b},\ \bar{c},\ 2\bar{f},\ 2\bar{g},\ 2\bar{h}\big)=0:\end{aligned}$$

but the method was an inconvenient one, and I was led to reconsider the question. The present investigation, although the analytical transformations are very long, is in theory extremely simple: I consider a given surface, and at each point thereof take along the normal an infinitesimal length ρ (not a constant, but an arbitrary function

of the coordinates), the extremities of these distances forming a new surface, say the vicinal surface; and the points on the same normal being considered as corresponding points, say this is the conormal correspondence of vicinal surfaces. In order that the two surfaces may belong to an orthogonal system, it is necessary and sufficient that at each point of the given surface the principal tangents (tangents to the curves of curvature) shall correspond to the principal tangents at the corresponding point of the vicinal surface; and the condition for this is that ρ shall satisfy a partial differential equation of the second order,

$$((A), (B), (C), (F), (G), (H) \between d_x, d_y, d_z)^2 \rho = 0,$$

where the coefficients depend on the first and second differential coefficients of U, if $U=0$ is the equation of the given surface. Now, considering the given surface as belonging to a family, or writing its equation in the form $r - r(x, y, z) = 0$ (the last r a functional symbol), the condition in order that the vicinal surface shall belong to this family, or say that it shall coincide with the surface $r + \delta r - r(x, y, z) = 0$, is $\rho = \frac{\delta r}{V}$, where $V = \sqrt{X^2 + Y^2 + Z^2}$, if X, Y, Z are the first differential coefficients of $r(x, y, z)$, that is, of the parameter r considered as a function of the coordinates; we have thus the equation

$$((A), (B), (C), (F), (G), (H) \between d_x, d_y, d_z)^2 \frac{1}{V} = 0,$$

viz. the coefficients being functions of the first and second differential coefficients of r, and V being a function of the first differential coefficients of r, this is in fact a relation involving the first, second, and third differential coefficients of r, or it is the partial differential equation to be satisfied by the parameter r considered as a function of the coordinates. After all reductions, this equation assumes the form previously mentioned.

Article Nos. 1 to 21. *On the Curvature of Surfaces.*

1. Curvature is a metrical theory having reference to the circle at infinity; each point in space may be regarded as the vertex of a cone passing through this circle, say the circular cone; a line and plane through the vertex are at right angles to each other when they are polar line and polar plane in regard to the cone; and so two lines or two planes are at right angles when they are harmonics in regard to the cone, that is, when each line lies in the polar plane, or each plane passes through the polar line of the other. A plane through the vertex meets the cone in two lines, which are the "circular lines" in the plane and through the point; a line through the vertex has through it two tangent planes, which might be called the "circular planes" of the point and through the line; but the term is hardly required. Lines in the plane and through the point, at right angles to each other, are also harmonics (polar lines) in regard to the two circular lines.

2. Consider now a surface, and any point thereof; we have at this point a tangent plane and a normal. The tangent plane meets the surface in a curve having

at the point a node, and the tangents to the two branches of the curve (being of course lines in the tangent plane) are the "chief tangents" of the surface at the point.

3. The chief tangents are the intersections of the tangent plane by a quadric cone, which may be called the chief cone; but it is important to observe that this cone is not independent of the particular form under which the equation of the surface is presented. To explain this, suppose that the rational equation of the surface is $U=0$; taking ξ, η, ζ as current coordinates measured from the point as origin, the equation of the chief cone is $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2\,U=0$, where x, y, z denote the coordinates of the point. But it is in the sequel necessary to present the equation of the surface in a different manner; say we have an equation between the coordinates (x, y, z) and a parameter r (r being therefore in general an irrational function of x, y, z), which, when $r=r_1$, reduces itself to $U=0$: we have then $r=r_1$ as the equation of the surface; and the corresponding equation of the chief cone is $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2\,r=0$; this is not the same as the cone $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2\,U=0$, although of course it intersects the tangent plane in the same two lines, viz. the chief lines; and so in general there is a distinct chief cone corresponding to each form of the equation of the surface. But adopting a definite form of equation, we have a definite chief cone intersecting the tangent plane in the chief tangents.

4. Observe that the equations $U=0$, $r=r_1$, although each relating to one and the same surface, serve to represent this surface, and that in different ways, as belonging to a family of surfaces, viz. one of these is the family $U=\text{const.}$, and the other the family $r=\text{const.}$ In order to represent a given surface as belonging to a certain family, we need the irrational form of equation; thus r denoting the irrational function of x, y, z determined by the equation $\frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}=1$, we have $r=0$ as the equation of the ellipsoid $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$, considered as belonging to a family of confocal quadrics.

5. Although at first sight presenting some difficulty, it is convenient to use the same letter r to denote the parameter considered as a function of the coordinates, and the special value of the parameter; thus in general the equation of a surface may be written $r(x, y, z)-r=0$ (in which form the first r may be regarded as a functional symbol), or simply $r-r=0$, viz. the first r here denotes the given function of (x, y, z), and the second r the particular value of the parameter.

6. By what precedes we have through the point and in the tangent plane two circular lines, the intersections of the tangent plane by the circular cone having the point for its vertex.

We have also through the point and in the tangent plane two other lines, termed the principal tangents, viz. the definition of these is that they are the double (or sibiconjugate) lines of the involution formed by the circular lines and the chief tangents, or, what is the same thing, they are the bisectors (and as such at right angles to each other) of the angles formed by the chief tangents.

7. The principal tangents may also be considered as the intersections of the tangent plane by a quadric cone, called the principal cone; this being a cone constructed by means of the circular cone and the chief cone, and thus depending on the particular chief cone, that is, on the form of the equation of the surface. The definition is that the principal cone is the locus of a line (through the point), such that the line itself, the perpendicular (or harmonic in regard to the circular cone) of the polar plane of the line in regard to the chief cone, and the normal of the surface are *in plano*.

8. Analytically, taking, as before, (x, y, z) for the coordinates of the point, and u, v, w as current coordinates measured from the point as origin, then the equation of the circular cone is $u^2+v^2+w^2=0$; and taking $Xu+Yv+Zw=0$ for the equation of the tangent plane, and $(a, b, c, f, g, h \between u, v, w)^2=0$ for that of the chief cone, then, if the line be $u : v : w = \xi : \eta : \zeta$, we have

$$(a, \ldots \between \xi, \eta, \zeta \between u, v, w)=0$$

for the equation of the polar plane, and thence

$$u : v : w = a\xi+h\eta+g\zeta : h\xi+b\eta+f\zeta : g\xi+f\eta+c\zeta$$

for those of the perpendicular, or harmonic in regard to the circular cone; also for the normal $u, v, w = X : Y : Z$; whence, if the three lines are *in plano*, we have

$$\begin{vmatrix} \xi \;, & \eta \;, & \zeta \\ a\xi+h\eta+g\zeta, & h\xi+b\eta+f\zeta, & g\xi+f\eta+c\zeta \\ X \;, & Y \;, & Z \end{vmatrix} = 0$$

as the equation of the principal cone. This is in the sequel written, for shortness, as

$$\begin{vmatrix} \xi\,, & \eta\,, & \zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} = 0.$$

9. Consider any point P', not in general on the surface, in the neighbourhood of the point on the surface, say P; then the point P' has in regard to the surface a polar plane, which plane, however, is dependent on the particular form of equation—viz. x', y', z' being the coordinates of P', and U' the same function of these that U is of x, y, z, then the form $U=0$ of the equation of the surface gives for P' the polar plane $(ud_{x'}+vd_{y'}+wd_{z'})\,U'=0$; and we may through P' draw hereto a perpendicular (or harmonic in regard to the circular cone), say this is the normal line of P'. Then for points P' in the neighbourhood of P, when these are such that their normal lines meet the normal at P, the locus of P' is the before-mentioned principal cone. The analytical investigation presents no difficulty.

10. Taking P' on the surface, the normal line of P' becomes the normal at a consecutive point P' of the surface (being now a line independent of the particular form of equation), and this normal meets the normal at P; that is, we have the

principal cone meeting the tangent plane in two lines, the principal tangents, such that at a consecutive point P' on either of these the normal meets the normal at P; viz. we have the principal tangents at the tangents of the two curves of curvature through the point P.

The plane through the normal and a principal tangent is termed a principal plane; we have thus at the point of the surface two principal planes, forming with the tangent plane an orthogonal triad of planes.

11. I proceed to further develop the theory, commencing with the following lemma:

Lemma. Given the line $Xu + Yv + Zw = 0$, and conic

$$(a,\ b,\ c,\ f,\ g,\ h \between u,\ v,\ w)^2 = 0,$$

then, to determine the coordinates $(u_1,\ v_1,\ w_1)$, $(u_2,\ v_2,\ w_2)$ of the points of intersection of the line and conic, we have

$$\begin{aligned}&(a, \ldots \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2\\ &= (\xi u_1 + \eta v_1 + \zeta w_1)(\xi u_2 + \eta v_2 + \zeta w_2),\end{aligned}$$

or, what is the same thing, we have

$$(a, \ldots \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2 = 0$$

as the equation, in line coordinates, of the two points of intersection. The proof is obvious.

12. Making the equations refer to a plane and a cone, and writing throughout $\xi,\ \eta,\ \zeta$ as current point coordinates, the theorem is:

Given the plane $X\xi + Y\eta + Z\zeta = 0$, and cone

$$(a,\ b,\ c,\ f,\ g,\ h \between \xi,\ \eta,\ \zeta)^2 = 0;$$

then, to determine the lines of intersection of the plane and cone, we have

$$(a, \ldots \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2 = 0$$

as the equation of the pair of planes at right angles to the two lines respectively.

13. Denoting the coefficients by (a), (b), &c., that is, writing

$$\begin{aligned}&(a, \ldots \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2\\ &= \big((a),\ (b),\ (c),\ (f),\ (g),\ (h) \between \xi,\ \eta,\ \zeta\big)^2,\end{aligned}$$

the values of these are

$$\begin{aligned}(a) &= bZ^2 + cY^2 - 2fYZ,\\ (b) &= cX^2 + aZ^2 - 2gZX,\\ (c) &= aY^2 + bX^2 - 2hXY,\\ (f) &= -aYZ - fX^2 + gXY + hXZ,\\ (g) &= -bZX + fYX - gY^2 + hYZ,\\ (h) &= -cXY + fZX + gZY - hZ^2.\end{aligned}$$

We have the following identities:

$$(a)\,X+(h)\;Y+(g)\,Z=0,$$
$$(h)\,X+(b)\;Y+(f)\,Z=0,$$
$$(g)\,X+(f)\,Y+(c)\;Z=0,$$
$$((b)(c)-(f)^2,\ldots,\ (g)(h)-(a)(f),\ldots)=-(X^2,\ Y^2,\ Z^2,\ YZ,\ ZX,\ XY)\,\phi,$$

that is, $(b)(c)-(f)^2=-X^2\phi$ &c., where

$$\phi=(bc-f^2,\ldots gh-af,\ldots \between X,\ Y,\ Z)^2.$$

Writing also

$$aX+hY+gZ,\ hX+bY+fZ,\ gX+fY+cZ=\delta X,\ \delta Y,\ \delta Z,$$

and $X^2+Y^2+Z^2=V^2$; also $a+b+c=\omega$, then

$$\begin{aligned}
(a)&=(b+c)\,V^2-\omega X^2\ +X\delta X-Y\delta Y-Z\delta Z,\\
(b)&=(c+a)\,V^2-\omega Y^2\ -X\delta X+Y\delta Y-Z\delta Z,\\
(c)&=(a+b)\,V^2-\omega Z^2\ -X\delta X-Y\delta Y+Z\delta Z,\\
(f)&=-fV^2\qquad-\omega YZ\ +Y\delta Z\ +Z\delta Y,\\
(g)&=-gV^2\qquad-\omega ZX\ +Z\delta X\ +X\delta Z,\\
(h)&=-hV^2\qquad-\omega XY+X\delta Y+Y\delta X.
\end{aligned}$$

14. I give also the following lemma:

Lemma. The condition in order that the plane $X\xi+Y\eta+Z\zeta=0$ may meet the cones

$$(A,\ B,\ C,\ F,\ G,\ H \between \xi,\ \eta,\ \zeta)^2=0,$$
$$(A',\ B',\ C',\ F',\ G',\ H' \between \xi,\ \eta,\ \zeta)^2=0$$

in two pairs of lines harmonically related to each other, is

$$(BC'+B'C-2FF',\ldots,\ GH'+G'H-AF'-A'F,\ldots \between X,\ Y,\ Z)^2=0.$$

Writing here

$$\begin{aligned}
&(A,\ldots \between Y\zeta-Z\eta,\ Z\xi-X\zeta,\ X\eta-Y\xi)^2\\
&\quad=((A),\ (B),\ (C),\ (F),\ (G),\ (H) \between \xi,\ \eta,\ \zeta)^2,
\end{aligned}$$

that is, $(A)=BZ^2+CY^2-2FYZ$, &c., the condition may be written

$$(A)\,A'+(B)\,B'+(C)\,C'+2\,(F)\,F'+2\,(G)\,G'+2\,(H)\,H'=0,$$

or say

$$((A),\ldots \between A',\ldots)=0;$$

and we may, it is clear, interchange the accented and unaccented letters respectively.

15. I take $r-r=0$ for the equation of a surface, X, Y, Z for the first derived functions of r, (a, b, c, f, g, h) for the second derived functions. The equation of the tangent plane at the point (x, y, z), taking ξ, η, ζ as current coordinates measured from this point, is

$$X\xi + Y\eta + Z\zeta = 0;$$

the equation of the chief cone in regard to this form of the equation of the surface is

$$(a, b, c, f, g, h \between \xi, \eta, \zeta)^2 = 0,$$

and the equation of the circular cone is $\xi^2+\eta^2+\zeta^2=0$, or, what is the same thing,

$$(1, 1, 1, 0, 0, 0 \between \xi, \eta, \zeta)^2 = 0.$$

Imagine a quadric cone

$$(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0,$$

such that it meets the tangent plane in the sibiconjugate lines of the involution formed by the intersections of the tangent plane by the chief cone and the circular cone respectively; that is, in a pair of lines harmonically related to the intersections with the chief cone, and also to the intersections with the circular cone; the conditions are

$$((A), \ldots \between a, \ldots) = 0,$$

and

$$(A)+(B)+(C)=0,$$

viz. if only these two conditions are satisfied the cone will intersect the tangent plane in the two principal tangents.

16. The principal cone, writing, for shortness,

$$a\xi+h\eta+g\zeta,\ h\xi+b\eta+f\zeta,\ g\xi+f\eta+c\zeta=\delta\xi,\ \delta\eta,\ \delta\zeta,$$

was before taken to be the cone

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} = 0.$$

Representing this equation by

$$\tfrac{1}{2}(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0,$$

the expressions of the coefficients are

$$\begin{aligned} A &= 2hZ - 2gY, \\ B &= 2fX - 2hZ, \\ C &= 2gY - 2fX, \\ F &= hY - gZ - (b-c)X, \\ G &= fZ - hX - (c-a)Y, \\ H &= gX - fY - (a-b)Z. \end{aligned}$$

These values give

$$AX + HY + GZ = Z\delta Y - Y\delta Z,$$
$$HX + BY + FZ = X\delta Z - Z\delta X,$$
$$GX + FY + CZ = Y\delta X - X\delta Y;$$

whence also

$$(A, \ldots \between X, Y, Z)^2 = 0,$$

as is, in fact, at once obvious from the determinant-form; and also

$$A + B + C = 0.$$

17. Writing for shortness

$$(\bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{g}, \bar{h}) = (bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch),$$

we find

$$Aa + Hh + Gg = \omega(hZ - gY) + \bar{h}Z - \bar{g}Y,$$
$$Hh + Bb + Ff = \omega(fX - hZ) + \bar{f}X - \bar{h}Z,$$
$$Gg + Ff + Cc = \omega(gY - fX) + \bar{g}Y - \bar{f}X;$$

whence

$$(A, \ldots \between a, \ldots) = 0.$$

18. By what precedes, we have

$$((A), \ldots \between \xi, \eta, \zeta)^2 = 0$$

for the equation of the two principal planes, where the coefficients (A), (B), &c. are functions of A, B, &c. and of X, Y, Z, as mentioned above. These coefficients satisfy of course the several relations similar to those satisfied by (a), (b), &c., and other relations dependent on the expressions of A, B, &c. in terms of a, b, &c. and X, Y, Z.

19. Proceeding to consider the coefficients (A), (B), &c., we have then

$$(A) + (B) + (C) = (A + B + C)\, V^2 - (A, \ldots \between X, Y, Z)^2,$$

that is

$$(A) + (B) + (C) = 0.$$

Observing the relation $A + B + C = 0$, the equations analogous to

$$(a) = (b + c)\, V^2 - (a + b + c)\, X^2 + \&c., \text{ are } (A) = -A\, V^2 + X\delta' X - Y\delta' Y - Z\delta' Z, \ \&c.$$

if for a moment we write $\delta' X$, $\delta' Y$, $\delta' Z$ to denote the functions

$$AX + HY + GZ,\ HX + BY + FZ,\ GX + FY + CZ.$$

But, from the above values, $X\delta' X + Y\delta' Y + Z\delta' Z = 0$, or the equation is $(A) = -A\, V^2 + 2X\delta' X$, that is $= -A\, V^2 + 2X\,(Z\delta Y - Y\delta Z)$. The equation for (F) is $(F) = -F V^2 + Y\delta' Z + Z\delta' Y$, where $Y\delta' Z + Z\delta' Y$ is $= Y(Y\delta X - X\delta Y) + Z(X\delta Z - Z\delta X)$, viz. this is

$$= (Y^2 - Z^2)\,\delta X - XY\delta Y + XZ\delta Z.$$

We have thus the system of equations

$$\begin{aligned}
(A) &= -AV^2 \quad . \quad + 2XZ\delta Y \quad - 2XY\delta Z,\\
(B) &= -BV^2 - 2YZ\delta X \quad . \quad + 2XY\delta Z,\\
(C) &= -CV^2 + 2YZ\delta X \quad - 2XZ\delta Y \quad .\\
(F) &= -FV^2 + (Y^2 - Z^2)\,\delta X - XY\delta Y \quad + XZ\delta Z,\\
(G) &= -GV^2 + XY\delta X \quad + (Z^2 - X^2)\,\delta Y - YZ\delta Z,\\
(H) &= -HV^2 - XZ\delta X \quad + YZ\delta Y \quad + (X^2 - Y^2)\,\delta Z.
\end{aligned}$$

20. We hence find

$$\begin{aligned}
(A)\,a + (H)\,h + (G)\,g &= -(Aa + Hh + Gg)\,V^2 + (Z\delta Y - Y\delta Z)\,\delta X + XP,\\
(H)\,h + (B)\,b + (F)\,f &= -(Hh + Bb + Ff)\,V^2 + (X\delta Z - Z\delta X)\,\delta Y + YQ,\\
(G)\,g + (F)\,f + (C)\,c &= -(Gg + Ff + Cc)\,V^2 + (Y\delta X - X\delta Y)\,\delta Z + ZR,
\end{aligned}$$

if for shortness

$$\begin{aligned}
P &= (gY - hZ)\,\delta X + (aZ - gX)\,\delta Y + (hX - aY)\,\delta Z,\\
Q &= (fY - bZ)\,\delta X + (hZ - fX)\,\delta Y + (bX - hY)\,\delta Z,\\
R &= (cY - fZ)\,\delta X + (gZ - cX)\,\delta Y + (fX - gY)\,\delta Z.
\end{aligned}$$

Forming the sum $PX + QY + RZ$, the coefficient of δX is found to be

$$= -Z(hX + bY + fZ) + Y(gX + fY + cZ),\ = -Z\delta Y + Y\delta Z;$$

hence the whole is

$$= \delta X\,(Y\delta Z - Z\delta Y) + \delta Y\,(Z\delta X - X\delta Z) + \delta Z\,(X\delta Y - Y\delta X),\ \text{which is } = 0, \text{ that is,}$$

$$PX + QY + RZ = 0.$$

21. Hence, adding, we find

$$((A), \ldots \between a, \ldots) = 0;$$

viz. in this and the before-mentioned equation

$$(A) + (B) + (C) = 0$$

we have the *à posteriori* verification that the cone $(A, \ldots \between \xi, \eta, \zeta)^2 = 0$ cuts the tangent plane in the double lines of the involution.

In what precedes I have given only those relations between the several sets of quantities a, $\bar{a}$, (a), A, (A), &c. which have been required for establishing the results last obtained; but there are various other relations required in the sequel, and which will be obtained as they are wanted.

The Conormal Correspondence of Vicinal Surfaces.
Art. Nos. 22 to 35.

22. We consider a surface $U=0$ (or $r=r$), and at each point P thereof measure along the normal an infinitesimal length ρ, dependent on the position of the point P (that is, ρ is a function of x, y, z). We have thus a point P', the coordinates of which are

$$x',\ y',\ z'=x+\rho\alpha,\ y+\rho\beta,\ z+\rho\gamma,$$

where α, β, γ are the cosine-inclinations of the normal, that is,

$$\alpha,\ \beta,\ \gamma=\frac{X}{V},\ \frac{Y}{V},\ \frac{Z}{V},\ \text{if } V=\sqrt{X^2+Y^2+Z^2};$$

the locus of P' is of course a surface, say the vicinal surface, and we require to find the direction of the normal at P', or, what is the same thing, the differential equation $X'dx'+Y'dy'+Z'dz'$ of the surface. We have

$$\begin{aligned}
dx' &= (1+d_x\rho\alpha)\ dx + \quad d_y\rho\alpha\ .\ dy + \quad d_z\rho\alpha\ .\ dz,\\
dy' &= \quad d_x\rho\beta\ .\ dx + (1+d_y\rho\beta)\ dy + \quad d_z\rho\beta\ .\ dz,\\
dz' &= \quad d_x\rho\gamma\ .\ dx + \quad d_y\rho\gamma\ .\ dy + (1+d_z\rho\gamma)\ dz,\\
0 &= \quad X\ dx + \quad Y\ dy + \quad Z\ dz;
\end{aligned}$$

whence, eliminating dx, dy, dz, we have between dx', dy', dz' a linear equation, the coefficients of which may be taken to be X', Y', Z'. Taking these only as far as the first power of ρ, we have

$$X'=X(1+d_y\rho\beta+d_z\rho\gamma)-Yd_x\rho\beta-Zd_x\rho\gamma,$$

or, what is the same thing,

$$X'=X(1+d_x\rho\alpha+d_y\rho\beta+d_z\rho\gamma)-Xd_x\rho\alpha-Yd_x\rho\beta-Zd_x\rho\gamma,$$

with the like expressions for Y' and Z'. I proceed to reduce these. The formula for X' is

$$\begin{aligned}
X'=X\{1+\rho(d_x\alpha+d_y\beta+d_z\gamma)+\alpha d_x\rho+\beta d_y\rho+\gamma d_z\rho\}\\
-\rho(Xd_x\alpha+Yd_x\beta+Zd_x\gamma)-(\alpha X+\beta Y+\gamma Z)\,d_x\rho.
\end{aligned}$$

23. I write, for shortness, $\delta=Xd_x+Yd_y+Zd_z$, whence δX, δY, $\delta Z=aX+hY+gZ$, $hX+bY+fZ$, $gX+fY+cZ$, agreeing with the former significations of δX, δY, δZ; also Vd_xV, Vd_yV, $Vd_zV=\delta X$, δY, δZ, and $V\delta V=X\delta X+Y\delta Y+Z\delta Z$. It is now easy to form the values of

$$\begin{aligned}
&d_x\alpha,\ d_x\beta,\ d_x\gamma, \quad \text{viz. these are } \frac{a}{V}-\frac{X\delta X}{V^3},\quad \frac{h}{V}-\frac{Y\delta X}{V^3},\quad \frac{g}{V}-\frac{Z\delta X}{V^3},\\
&d_y\alpha,\ d_y\beta,\ d_y\gamma, \qquad \frac{h}{V}-\frac{X\delta Y}{V^3},\quad \frac{b}{V}-\frac{Y\delta Y}{V^3},\quad \frac{f}{V}-\frac{Z\delta Y}{V^3},\\
&d_z\alpha,\ d_z\beta,\ d_z\gamma, \qquad \frac{g}{V}-\frac{X\delta Z}{V^3},\quad \frac{f}{V}-\frac{Y\delta Z}{V^3},\quad \frac{c}{V}-\frac{Z\delta Z}{V^3};
\end{aligned}$$

and hence

$$d_x\alpha + d_y\beta + d_z\gamma = \frac{a+b+c}{V} - \frac{\delta V}{V^2},$$

$$Xd_x\alpha + Yd_x\beta + Zd_x\gamma = \frac{\delta X}{V} - \frac{V^2}{V^3}\delta X, = 0,$$

$$\alpha d_x\rho + \beta d_y\rho + \gamma d_z\rho = \frac{1}{V}\delta\rho,$$

$$\alpha X + \beta Y + \gamma Z \quad = V;$$

and we have

$$X' = X\left\{1 + \rho\left(\frac{a+b+c}{V} - \frac{\delta V}{V^2}\right) + \frac{1}{V}\delta\rho\right\} - Vd_x\rho,$$

with the like values of Y' and Z'. But we are only concerned with the ratios $X' : Y' : Z'$; whence, dividing the foregoing values by the coefficient in { }, and taking the second terms only to the first order in ρ, we have simply

$$X',\ Y',\ Z' = X - Vd_x\rho,\ Y - Vd_y\rho,\ Z - Vd_z\rho.$$

24. We may investigate the condition in order that the surface x', y', z' may be the consecutive surface $r + dr = r(x, y, z)$. This will be the case of

$$r + dr = r\left(x + \rho\frac{X}{V},\quad y + \rho\frac{Y}{V},\quad z + \rho\frac{Z}{V}\right),$$

that is, $r + dr = r + \rho V$, or $\rho = \frac{dr}{V}$. This value of ρ gives $d_x\rho = -\frac{dr}{V^2}d_xV = -\frac{\rho}{V^3}\delta X$, and similarly $d_y\rho = -\frac{\rho}{V^2}\delta Y$, $d_z\rho = -\frac{\rho}{V^2}\delta Z$; whence

$$X',\ Y',\ Z' = X + \frac{\rho}{V}\delta X,\ Y + \frac{\rho}{V}\delta Y,\ Z + \frac{\rho}{V}\delta Z,$$

which is as it should be, viz. these are what X, Y, Z become on substituting therein for x, y, z the values $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$.

25. I return to the case where ρ is arbitrary, and I investigate the values of a, b, ... for the point P' on the vicinal surface; say these are a', b', &c., then we have $a' = d_xX'$ &c. The relation between the differentials may be written

$$\begin{aligned}
dx &= (1 - d_x\rho\alpha)\,dx' - \quad d_y\rho\alpha\ \ dy' - \quad d_z\rho\alpha\ dz',\\
dy &= \quad -d_x\rho\beta\ dx' + (1 - d_y\rho\beta)\,dy' - \quad d_z\rho\beta\ dz',\\
dz &= \quad -d_x\rho\gamma\ dx' - \quad d_y\rho\gamma\ \ dy' + (1 - d_z\rho\gamma)\,dz',
\end{aligned}$$

and we thence have $d_{x'} = (1 - d_x\rho\alpha)\, d_x - d_x\rho\beta d_y - d_x\rho\gamma d_z$ &c.; hence

$$\begin{aligned} a' &= \{(1 - d_x\rho\alpha)\, d_x - d_x\rho\beta d_y - d_x\rho\gamma d_z\}\,(X - Vd_x\rho) \\ &= (1 - d_x\rho\alpha)\, a - d_x\rho\beta\,.\,h - d_x\rho\gamma\,.\,g - d_x\,(Vd_x\rho) \\ &= a - \rho\,(ad_x\alpha + hd_x\beta + gd_x\gamma) \\ &\qquad - (a\alpha + h\beta + g\gamma)\, d_x\rho \\ &\qquad - \frac{1}{V}\,\delta X d_x\rho - Vd_x^2\rho\,; \end{aligned}$$

and similarly, $f' = d_{y'}Z'$ (or $d_{z'}Y'$), that is

$$\begin{aligned} f' &= f - \rho\,(gd_y\alpha + fd_y\beta + cd_y\gamma) \\ &\qquad - (g\alpha + f\beta + c\gamma)\, d_y\rho \\ &\qquad - \frac{1}{V}\,\delta Y d_z\rho - Vd_yd_z\rho. \end{aligned}$$

26. Completing the reduction, we find

$$a' = a - \rho\left(\frac{a\omega - \bar{b} - \bar{c}}{V} - \frac{(\delta X)^2}{V^3}\right) - \frac{2}{V}\,\delta X d_x\rho - Vd_x^2\rho,$$

$$b' = b - \rho\left(\frac{b\omega - \bar{c} - \bar{a}}{V} - \frac{(\delta Y)^2}{V^3}\right) - \frac{2}{V}\,\delta Y d_y\rho - Vd_y^2\rho,$$

$$c' = c - \rho\left(\frac{c\omega - \bar{a} - \bar{b}}{V} - \frac{(\delta Z)^2}{V^3}\right) - \frac{2}{V}\,\delta Z d_z\rho - Vd_z^2\rho,$$

$$f' = f - \rho\left(\frac{\omega f + \bar{f}}{V} - \frac{\delta Y\delta Z}{V^3}\right) - \frac{1}{V}\,(\delta Y d_z\rho + \delta Z d_x\rho\,) - Vd_yd_z\rho,$$

$$g' = g - \rho\left(\frac{\omega g + \bar{g}}{V} - \frac{\delta Z\delta X}{V^3}\right) - \frac{1}{V}\,(\delta Z d_x\rho + \delta X d_y\rho) - Vd_zd_x\rho,$$

$$h' = h - \rho\left(\frac{\omega h + \bar{h}}{V} - \frac{\delta X\delta Y}{V^3}\right) - \frac{1}{V}\,(\delta X d_y\rho + \delta Y d_z\rho) - Vd_xd_y\rho\,;$$

say these expressions are $a' = a + \Delta a$, &c.

27. Taking ξ, η, ζ for the coordinates, referred to P as origin, of a point on the given surface near to P, and ξ', η', ζ' for the coordinates, referred to P' as origin, of the corresponding point on the vicinal surface, the relation between ξ', η', ζ' and ξ, η, ζ is the same as that between dx', dy', dz' and dx, dy, dz; viz. we have

$$\begin{aligned} \xi &= (1 - d_x\rho\alpha)\,\xi' - d_y\rho\alpha\,.\,\eta' \qquad - d_z\rho\alpha\,.\,\zeta', \\ \eta &= - d_x\rho\beta\,.\,\xi' \quad + (1 - d_y\rho\beta)\,\eta' - d_z\rho\beta\,.\,\zeta', \\ \zeta &= - d_x\rho\gamma\,.\,\xi' \quad - d_y\rho\gamma\,.\,\eta' \qquad + (1 - d_z\rho\gamma)\,\zeta'\,; \end{aligned}$$

or conversely

$$\begin{aligned}
\xi' &= (1 + d_x\rho\alpha)\,\xi + \quad d_y\rho\alpha\,.\,\eta + \quad d_z\rho\alpha\,.\,\zeta,\\
\eta' &= \quad d_x\rho\beta\,.\,\xi + (1 + d_y\rho\beta)\,\eta + \quad d_z\rho\beta\,.\,\zeta,\\
\zeta' &= \quad d_x\rho\gamma\,.\,\xi + \quad d_y\rho\gamma\,.\,\eta + (1 + d_z\rho\gamma)\,\zeta,
\end{aligned}$$

say $\xi',\ \eta',\ \zeta' = \xi + \Delta\xi,\ \eta + \Delta\eta,\ \zeta + \Delta\zeta$; hence

$$\begin{aligned}
X'\xi' + Y'\eta' + Z'\zeta' &= (X - Vd_x\rho)\,(\xi + \Delta\xi) + \&c.\\
&= \quad X\xi + Y\eta + Z\zeta\\
&\quad + X\Delta\xi + Y\Delta\eta + Z\Delta\zeta\\
&\quad - V\,(\xi d_x\rho + \eta d_y\rho + \zeta d_z\rho),
\end{aligned}$$

where second line is

$$\begin{aligned}
&(X\alpha + Y\beta + Z\gamma)\,(\xi d_x\rho + \eta d_y\rho + \zeta d_z\rho)\\
&+ \rho\,\{(Xd_x\alpha + Yd_x\beta + Zd_x\gamma)\,\xi + (Xd_y\alpha + Yd_y\beta + Zd_y\gamma)\,\eta + (Xd_z\alpha + Yd_z\beta + Zd_z\gamma)\,\zeta\}.
\end{aligned}$$

But

$$\begin{aligned}
Xd_x\alpha + Yd_x\beta + Zd_x\gamma &= \frac{\delta X}{V} - \frac{V^2}{V^3}\delta X = 0,\\
Xd_y\alpha + Yd_y\beta + Zd_y\gamma &= 0,\\
Xd_z\alpha + Yd_z\beta + Zd_z\gamma &= 0,
\end{aligned}$$

or second line is $= V\,(\xi d_x\rho + \eta d_y\rho + \zeta d_z\rho)$; and we have therefore

$$X'\xi' + Y'\eta' + Z'\zeta' = X\xi + Y\eta + Z\zeta.$$

We require

$$(A',\ B',\ C',\ F',\ G',\ H'\between\xi',\ \eta',\ \zeta')^2;$$

viz., to the first order in ρ, this is

$$\begin{aligned}
&= (A',\ \ldots\between\xi,\ \eta,\ \zeta)^2\\
&\quad + 2\,(A,\ \ldots\between\Delta\xi,\ \Delta\eta,\ \Delta\zeta\between\xi,\ \eta,\ \zeta).
\end{aligned}$$

28. Here second line is

$$2\,\{(A\xi + H\eta + G\zeta)\,\Delta\xi + (H\xi + B\eta + F\zeta)\,\Delta\eta + (G\xi + F\eta + C\zeta)\,\Delta\zeta\}:$$

but

$$A\xi + H\eta + G\zeta = Z\delta\eta - Y\delta\zeta + \begin{vmatrix} a, & h, & g\\ X, & Y, & Z\\ \xi, & \eta, & \zeta \end{vmatrix},$$

$$H\xi + B\eta + F\zeta = X\delta\zeta - Z\delta\xi + \begin{vmatrix} h, & b, & f\\ X, & Y, & Z\\ \xi, & \eta, & \zeta \end{vmatrix},$$

$$G\xi + F\eta + C\zeta = Y\delta\xi + X\delta\eta + \begin{vmatrix} g, & f, & c\\ X, & Y, & Z\\ \xi, & \eta, & \zeta \end{vmatrix},$$

whence term in { } is

$$\begin{vmatrix} \Delta\xi, & \Delta\eta, & \Delta\zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} + \begin{vmatrix} a\Delta\xi + h\Delta\eta + g\Delta\zeta, & h\Delta\xi + b\Delta\eta + f\Delta\zeta, & g\Delta\xi + f\Delta\eta + c\Delta\zeta \\ X, & Y, & Z \\ \xi, & \eta, & \zeta \end{vmatrix},$$

which might be written

$$\begin{vmatrix} \Delta\xi, & \Delta\eta, & \Delta\zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} - \begin{vmatrix} \delta\Delta\xi, & \delta\Delta\eta, & \delta\Delta\zeta \\ \xi, & \eta, & \zeta \\ X, & Y, & Z \end{vmatrix}$$

but it is perhaps more convenient to retain the second term in its original form.

29. As regards the first line, we have

$$\begin{aligned} A' &= 2h'Z' - 2g'Y' \\ &= 2(h+\Delta h)(Z - Vd_z\rho) - 2(g+\Delta g)(Y - Vd_y\rho) \\ &= A + 2(Z\Delta h - Y\Delta g) - 2V(hd_z\rho - gd_y\rho), \end{aligned}$$

with similar expressions for the other coefficients. Attending only to the terms of the first order, we thus obtain

$$\begin{aligned} A' &= A + 2(Z\Delta h - Y\Delta g) - 2V(hd_z - gd_y)\rho, \\ B' &= B + 2(X\Delta f - Z\Delta h) - 2V(fd_x - hd_z)\rho, \\ C' &= C + 2(Y\Delta g - X\Delta f) - 2V(gd_y - fd_x)\rho, \\ F' &= F + Y\Delta h - Z\Delta g - X(\Delta b - \Delta c) - V(hd_y - gd_z - (b-c)d_x)\rho, \\ G' &= G + Z\Delta f - X\Delta h - Y(\Delta c - \Delta a) - V(fd_z - hd_x - (c-a)d_y)\rho, \\ H' &= H + X\Delta g - Y\Delta f - Z(\Delta a - \Delta b) - V(gd_x - fd_y - (a-b)d_z)\rho, \end{aligned}$$

say these are $A' = A + \theta A$, &c., where θ is a functional symbol; we thus have

$$(A', \ldots \between \xi', \eta', \zeta')^2 = (A, \ldots \between \xi, \eta, \zeta)^2 + (\theta A, \ldots \between \xi, \eta, \zeta)^2 + 2(A, \ldots \between \xi, \eta, \zeta \between \Delta\xi, \Delta\eta, \Delta\zeta),$$

which, for shortness, I represent by

$$= (A, \ldots \between \xi, \eta, \zeta)^2 + (A'', \ldots \between \xi, \eta, \zeta)^2;$$

and I proceed to complete the calculation of the coefficients A'', B'', &c.

30. We have

$$\begin{aligned} A'' &= \theta A + \text{coeff. } \xi^2 \text{ in} \\ &\quad 2[(A\xi + H\eta + G\zeta)\Delta\xi + (H\xi + B\eta + F\zeta)\Delta\eta + (G\xi + F\eta + C\zeta)\Delta\zeta] \\ &= \theta A + 2(Ad_x\rho\alpha + Hd_x\rho\beta + Gd_x\rho\gamma), \end{aligned}$$

that is,

$$\begin{aligned} A'' = \theta A &+ 2\frac{1}{V}(AX + HY + GZ)d_x\rho \\ &+ 2\rho(Ad_x\alpha + Hd_x\beta + Gd_x\gamma), \end{aligned}$$

where coeff. 2ρ is

$$= \frac{Aa + Hh + Gg}{V} - \frac{(AX + HY + GZ)\,\delta X}{V^3}$$

$$= \frac{1}{V}\left\{\omega\,(hZ - gY) + \bar{h}Z - \bar{g}Y\right\} - \frac{\delta X}{V^3}\,(Z\delta Y - Y\delta Z).$$

31. And similarly,

$$\begin{aligned} F'' &= \theta F + (H\alpha + B\beta + F\gamma)\,d_z\rho + (G\alpha + F\beta + C\gamma)\,d_y\rho \\ &\quad + \rho\,\{(Hd_z\alpha + Bd_z\beta + Fd_z\gamma) + (Gd_y\alpha + Fd_y\beta + Cd_y\gamma)\} \\ &= \theta F + \frac{1}{V}\{(HX + BY + FZ)\,d_z\rho + (GX + FY + CZ)\,d_y\rho\} \\ &\quad + \rho\left\{\frac{Hg + Bf + Fc}{V} - \frac{(HX + BY + FZ)\,\delta Z}{V^3}\right. \\ &\quad \left. + \frac{Gh + Fb + Cf}{V} - \frac{(GX + FY + CZ)\,\delta Y}{V^3}\right\}, \end{aligned}$$

$$\begin{aligned} Gh + Fb + Cf &= \omega\,(hY - bX) + \bar{h}Y - \bar{b}X + \bar{\omega}X + \bar{a}X + \bar{h}Y + \bar{g}Z, \\ Hg + Bf + Fc &= -\omega\,(gZ - cX) - \bar{g}Y + \bar{c}X - \bar{\omega}X - \bar{a}X - \bar{h}Y - \bar{g}Z. \end{aligned}$$

Sum is $\omega\,\{hY - gZ - (b - c)X\} + \bar{h}Y - \bar{g}Z - (\bar{b} - \bar{c})\,X$, which is $= \omega F + \bar{h}Y - \bar{g}Z - (\bar{b} - \bar{c})\,X$: hence

$$\begin{aligned} F'' &= \theta F + (X\delta Z - Z\delta X)\left(\frac{1}{V}\,d_z\rho - \frac{\rho\delta Z}{V^3}\right) + (Y\delta X - X\delta Y)\left(\frac{1}{V}\,d_y\rho - \frac{\rho\delta Y}{V^3}\right) \\ &\quad + \frac{\rho}{V}\,\{\omega F + \bar{h}Y - \bar{g}Z - (\bar{b} - \bar{c})\,X\}. \end{aligned}$$

32. We may write

$$A'' = \theta A + 2\left(\frac{1}{V}\,d_x\rho - \frac{\rho\delta X}{V^3}\right)(Z\delta Y - Y\delta Z) + \frac{\rho}{V}\,\{\omega A + \bar{A}\},$$

$$B'' = \theta B + 2\left(\frac{1}{V}\,d_y\rho - \frac{\rho\delta Y}{V^3}\right)(X\delta Z - Z\delta X) + \frac{\rho}{V}\,\{\omega B + \bar{B}\},$$

$$C'' = \theta C + 2\left(\frac{1}{V}\,d_z\rho - \frac{\rho\delta Z}{V^3}\right)(Y\delta X - X\delta Y) + \frac{\rho}{V}\,\{\omega C + \bar{C}\},$$

$$F'' = \theta F + \left(\frac{1}{V}\,d_z\rho - \frac{\rho\delta Z}{V^3}\right)(X\delta Z - Z\delta X) + \left(\frac{1}{V}\,d_y\rho - \frac{\rho\delta Y}{V^3}\right)(Y\delta X - X\delta Y) + \frac{\rho}{V}\,\{\omega F + \bar{F}\},$$

$$G'' = \theta G + \left(\frac{1}{V}\,d_x\rho - \frac{\rho\delta X}{V^3}\right)(Y\delta X - X\delta Y) + \left(\frac{1}{V}\,d_z\rho - \frac{\rho\delta Z}{V^3}\right)(Z\delta Y - Y\delta Z) + \frac{\rho}{V}\,\{\omega G + \bar{G}\},$$

$$H'' = \theta H + \left(\frac{1}{V}\,d_y\rho - \frac{\rho\delta Y}{V^3}\right)(Z\delta Y - Y\delta Z) + \left(\frac{1}{V}\,d_x\rho - \frac{\rho\delta X}{V^3}\right)(X\delta Z - Z\delta X) + \frac{\rho}{V}\,\{\omega H + \bar{H}\},$$

in which equations $\bar{A}$, $\bar{B}$, &c. are the like functions of $\bar{a}$, $\bar{b}$, &c. that A, B, &c. are of a, b, &c.; viz. $\bar{A} = 2\bar{h}Z - 2\bar{g}Y$, &c.

The value of θA is

$$\theta A = 2Z\left(\left\{-\frac{\rho}{V}(h\omega+\bar{h})+\frac{\rho}{V^3}\delta X\delta Y\right\}-\frac{1}{V}(\delta Y d_x\rho+\delta X d_y\rho)-Vd_xd_y\rho\right)$$
$$-2Y\left(\left\{-\frac{\rho}{V}(g\omega+\bar{g})+\frac{\rho}{V^3}\delta Z\delta X\right\}-\frac{1}{V}(\delta Z d_x\rho+\delta X d_z\rho)-Vd_xd_z\rho\right)$$
$$-2V(hd_z-gd_y)\rho,$$

which is

$$=-\frac{\rho}{V}(\omega A+\bar{A}) \qquad +\frac{2\rho}{V^3}\delta X(Z\delta Y-Y\delta Z)$$
$$-\frac{2\delta X}{V}(Zd_y-Yd_z)\rho \qquad -2V(hd_z-gd_y)\rho$$
$$-\frac{2}{V}(Z\delta Y-Y\delta Z)d_x\rho-2V(Zd_y-Yd_z)d_x\rho.$$

Hence the value of A'' is equal to the last-mentioned expression, together with the following terms:—

$$+\frac{\rho}{V}(\omega A+\bar{A})-\frac{2\rho}{V^3}\delta X(Z\delta Y-Y\delta Z)+\frac{2}{V}(Z\delta Y-Y\delta Z)d_x\rho,$$

which destroy certain of the foregoing ones; viz. we have

$$A''=\left(2Vg-\frac{2Z\delta X}{V}\right)d_y\rho-2\left(Vh-\frac{2Y\delta X}{V}\right)d_z\rho-2V(Zd_y-Yd_z)d_x\rho.$$

33. Similarly, the value of θF is

$$\theta F = Y\left(-\frac{\rho}{V}(h\omega+\bar{h})+\frac{\rho}{V^3}\delta X\delta Y-\frac{1}{V}(\delta Y d_x\rho+\delta X d_y\rho)-Vd_xd_y\rho\right)$$
$$-Z\left(-\frac{\rho}{V}(g\omega+\bar{g})+\frac{\rho}{V^3}\delta Z\delta X-\frac{1}{V}(\delta Z d_x\rho+\delta X d_z\rho)-Vd_xd_z\rho\right)$$
$$-X\left(-\frac{\rho}{V}\{(b-c)\omega+\bar{b}-\bar{c}\}+\frac{\rho\delta Y^2}{V^3}-\frac{\rho\delta Z^2}{V^3}-\frac{2}{V}\delta Y d_y\rho+\frac{2}{V}\delta Z d_z\rho-Vd_y^2\rho+Vd_z^2\rho\right)$$
$$-V\left(hd_y-gd_z-(b-c)d_x\right)\rho,$$

which is

$$=\frac{\rho}{V}(-F\omega-\bar{F})+\frac{\rho\delta Z}{V^3}(X\delta Z-Z\delta X)+\frac{\rho\delta Y}{V^3}(Y\delta X-X\delta Y)$$
$$+\left\{-\frac{1}{V}(Y\delta Y-Z\delta Z)+V(b-c)\right\}d_x\rho$$
$$+\left\{-\frac{Y}{V}\delta X+\frac{2X}{V}\delta Y-Vh\right\}d_y\rho$$
$$+\left\{\frac{Z}{V}\delta X+\frac{2X}{V}\delta Y+Vg\right\}d_z\rho$$
$$+(-VYd_xd_y+VZd_xd_z+VXd_y^2-VXd_z^2)\rho.$$

Hence F'' is equal to the foregoing expression, together with the following terms:—

$$+\frac{\rho}{V}(F\omega+\overline{F})-\frac{\rho\delta Z}{V^3}(X\delta Z-Z\delta X)-\frac{\rho\delta Y}{V^3}(Y\delta X-X\delta Y)$$

$$+\frac{1}{V}(Y\delta X-X\delta Y)\,d_y\rho+\frac{1}{V}(X\delta Z-Z\delta X)\,d_z\rho,$$

which destroy certain of the foregoing terms; viz. we thus have

$$F''=\left\{-\frac{1}{V}(Y\delta Y-Z\delta Z)+V(b-c)\right\}d_x\rho+\left\{\frac{X}{V}\delta Y-Vh\right\}d_y\rho+\left\{-\frac{X}{V}\delta Z-Vg\right\}d_z\rho$$
$$+V(-Yd_xd_y+Zd_zd_x+Xd_y^2-Xd_z^2)\,\rho.$$

34. We thus have

$$A''=\qquad\qquad 2\left(Vg-\frac{Z\delta X}{V}\right)d_y\rho-2\left(Vh-\frac{Y\delta X}{V}\right)d_z\rho+2V(-Zd_yd_x+Yd_zd_x)\,\rho,$$

$$B''=-2\left(Vf-\frac{Z\delta Y}{V}\right)d_x\rho\qquad\qquad +2\left(Vh-\frac{X\delta Y}{V}\right)d_z\rho+2V(-Xd_zd_y+Zd_xd_y)\,\rho,$$

$$C''=+2\left(Vf-\frac{Y\delta Z}{V}\right)d_x\rho-2\left(Vg-\frac{X\delta Z}{V}\right)d_y\rho\qquad\qquad +2V(-Yd_xd_z+Xd_yd_z)\,\rho,$$

$$F''=\left\{V(b-c)-\frac{1}{V}(Y\delta Y-Z\delta Z)\right\}d_x\rho-\left(Vh-\frac{X\delta Y}{V}\right)d_y\rho+\left(Vg-\frac{X\delta Z}{V}\right)d_z\rho$$
$$+V(-Yd_xd_y+Zd_xd_z+Xd_y^2-Xd_z^2)\,\rho,$$

$$G''=\left(Vh-\frac{Y\delta X}{V}\right)d_x\rho+\left\{V(c-a)-\frac{1}{V}(Z\delta Z-X\delta X)\right\}d_y\rho-\left(Vf-\frac{Y\delta Z}{V}\right)d_z\rho$$
$$+V(-Zd_yd_z+Xd_yd_x+Yd_z^2-Yd_x^2)\,\rho,$$

$$H''=-\left(Vg-\frac{Z\delta X}{V}\right)d_x\rho-\left(Vf-\frac{Z\delta Y}{V}\right)d_y\rho+\left\{V(a-b)-\frac{1}{V}(X\delta X-Y\delta Y)\right\}d_z\rho$$
$$+V(-Xd_zd_x+Yd_zd_y+Zd_x^2-Zd_y^2)\,\rho.$$

35. It will be recollected that we have $X'\xi'+Y'\eta'+Z'\zeta'=X\xi+Y\eta+Z\zeta$; by what precedes it appears that for the given surface the principal tangents are determined by the equations

$$(A,\ldots\between\xi,\ \eta,\ \zeta)^2=0,$$
$$X\xi+Y\eta+Z\zeta=0,$$

and that the lines which (in the tangent plane of the given surface) correspond to the principal tangents of the corresponding point of the vicinal surface are determined by the equations

$$(A,\ldots\between\xi,\ \eta,\ \zeta)^2+(A'',\ldots\between\xi,\ \eta,\ \zeta)^2=0,$$
$$X\xi+Y\eta+Z\zeta=0.$$

Condition that the two surfaces may belong to an Orthogonal System.
Art. Nos. 36 to 41.

36. The condition in order that the two surfaces may belong to an orthogonal system is that the principal tangents shall correspond, or, what is the same thing, the lines which (in the tangent plane of the given surface) correspond to the principal tangents of the vicinal surface must be the principal tangents of the given surface. When this is the case, the plane and cone $X\xi + Y\eta + Z\zeta = 0$, $(A'', \ldots \between \xi, \eta, \zeta)^2 = 0$ intersect in the principal tangents, and this is therefore the required condition.

The plane $X\xi + Y\eta + Z\zeta = 0$ meets the cone $(A'', \ldots \between \xi, \eta, \zeta)^2 = 0$ in the principal tangents, that is, in a pair of lines harmonically related to the circular lines and also to the chief tangents. Forming then the coefficients (A''), (B''), (C''), (F''), (G''), (H'') from A'', &c. in the same way as (A) &c. are formed from A, &c., that is, writing $(A'') = B''Z^2 + C''Y^2 - 2F''YZ$, &c., the conditions are

$$(A'') + (B'') + (C'') = 0,$$
$$((A''), \ldots \between a, \ldots) = 0,$$

or, what is the same thing,

$$(A'', \ldots \between (a), \ldots) = 0.$$

The former of these, as about to be shown, is satisfied identically; we have therefore the second of them, say $(A'', \ldots \between (a), \ldots) = 0$ as the required condition.

37. We have

$$(A'') + (B'') + (C'') = (A'' + B'' + C'')\, V^2 - (A'', \ldots \between X, Y, Z)^2,$$

$$A'' + B'' + C'' = \frac{2}{V}\{(Z\delta Y - Y\delta Z)\, d_x\rho + (X\delta Z - Z\delta X)\, d_y\rho + (Y\delta X - X\delta Y)\, d_z\rho\}.$$

Forming next the expressions of $A''X + H''Y + G''Z$ &c., and, for convenience, writing down separately the terms which involve the second differential coefficients of ρ, we have

$$\begin{aligned}
&A''X + H''Y + G''Z = \\
&\quad d_x\rho\,.\,V(hZ - gY) + d_y\rho\,[V\delta Z - Z\delta V + V(gX - aZ)] + d_z\rho\,[-(V\delta Y - Y\delta V) - V(hX - aY)], \\
&H''X + B''Y + F''Z = \\
&\quad d_x\rho\,[-(V\delta Z - Z\delta V) - V(fY - bZ)] + d_y\rho\,.\,V(fX - hZ) + d_z\rho\,[(V\delta X - X\delta V) + V(hY - bX)], \\
&G''X + F''Y + C''Z = \\
&\quad d_x\rho\,[V\delta Y - Y\delta V + V(fZ - cY)] + d_y\rho\,[-(V\delta X - X\delta V) - V(gZ - cX)] + d_z\rho\,.\,V(gY - fX),
\end{aligned}$$

where δV stands for $\frac{1}{V}(X\delta X + Y\delta Y + Z\delta Z)$, and where the three expressions contain also the following terms respectively:

$$\begin{aligned}
&\{\quad . \quad - YZd_y^2 + YZd_z^2 + (Y^2 - Z^2)\, d_yd_z + \quad XYd_zd_x - \quad XZd_xd_y\}\,\rho, \\
&\{\; ZXd_x^2 \quad . \quad - ZXd_z^2 - \quad XYd_yd_z + (Z^2 - X^2)\, d_zd_x + \quad YZd_xd_y\}\,\rho, \\
&\{- XYd_x^2 + XYd_y^2 \quad . \quad + \quad XZd_yd_z - \quad YZd_zd_x + (X^2 - Y^2)\, d_xd_y\}\,\rho.
\end{aligned}$$

Multiplying by X, Y, Z, and adding, the terms which contain the second differential coefficients disappear, and we obtain

$$(A'', \ldots \between X,\ Y,\ Z)^2 = 2V[(Z\delta Y - Y\delta Z)\, d_x\rho + (X\delta Z - Z\delta X)\, d_y\rho + (Y\delta X - X\delta Y)\, d_z\rho]\,;$$

so that, attending to the above value of $A'' + B'' + C''$, we have the required equation

$$(A'') + (B'') + (C'') = 0.$$

38. Proceeding now to form the value of $(A'', \ldots \between (a), \ldots)$, that is,

$$A''(a) + B''(b) + C''(c) + 2F''(f) + 2G''(g) + 2H''(h),$$

it will be shown that the terms involving the first differential coefficients of ρ vanish of themselves; as regards those containing the second differential coefficients, forming the auxiliary equations

$$\begin{aligned}
(A) &= 2(h)Z - 2(g)Y\,,\\
(B) &= 2(f)X - 2(h)Z\,,\\
(C) &= 2(g)Y - 2(f)X,\\
(F) &= (h)Y - (g)Z - ((b)-(c))X,\\
(G) &= (f)Z - (h)X - ((c)-(a))Y,\\
(H) &= (g)X - (f)Y - ((a)-(b))Z,
\end{aligned}$$

we find without difficulty that the terms in question (being, in fact, the complete value of the expression) are

$$= V((A), \ldots \between d_x,\ d_y,\ d_z)^2\rho.$$

39. As regards the terms involving the first differential coefficients, observe that the whole coefficient of $d_x\rho$ is

$$\begin{aligned}
&-2(b)\left(Vf - \frac{Z\delta Y}{V}\right)\\
&+2(c)\left(Vg - \frac{Y\delta Z}{V}\right)\\
&+2(f)\left(V(b-c) - \frac{1}{V}(Y\delta Y - Z\delta Z)\right)\\
&+2(g)\left(Vh - \frac{Y\delta X}{V}\right)\\
&-2(h)\left(Vg - \frac{Z\delta X}{V}\right),
\end{aligned}$$

which is

$$\begin{aligned}
&= 2V\{(g)h + (f)b + (c)g - ((h)g + (b)f + (f)c)\}\\
&+ \frac{2}{V}\{Z((h)\delta X + (b)\delta Y + (f)\delta Z) - Y((g)\delta X + (f)\delta Y + (c)\delta Z)\}.
\end{aligned}$$

40. The reduction depends on the following auxiliary formulæ:

$$\begin{array}{l|l|l}
a(a)+h(h)+g(g)=V\bar{\delta}V-X\bar{\delta}X, & a(h)+h(b)+g(f)= \quad -X\bar{\delta}Y, & a(g)+h(f)+g(c)= \quad -X\bar{\delta}Z, \\
h \;\,,, +b \;,, +f \;,, = \quad -Y\bar{\delta}X, & h \;,, +b \;,, +f \;,, = V\bar{\delta}V-Y\bar{\delta}Y, & h \;,, +b \;,, +f \;,, = \quad -Y\bar{\delta}Z, \\
g \;,, +f \;,, +c \;,, = \quad -Z\bar{\delta}X, & g \;,, +f \;,, +c \;,, = \quad -Z\bar{\delta}Y, & g \;,, +f \;,, +c \;,, = V\bar{\delta}V-Z\bar{\delta}Z,
\end{array}$$

where, for shortness, I have written $\bar{\delta}X$, $\bar{\delta}Y$, $\bar{\delta}Z$ to stand for $\bar{a}X+\bar{h}Y+\bar{g}Z$, $\bar{h}X+\bar{b}Y+\bar{f}Z$, $\bar{g}X+\bar{f}Y+\bar{c}Z$ respectively, and $V\bar{\delta}V$ for $X\bar{\delta}X+Y\bar{\delta}Y+Z\bar{\delta}Z$, $(=\bar{a},\ldots\between X,\ Y,\ Z)^2$.

From these we immediately have

$$\begin{aligned}
(a)\,\delta X+(h)\ \delta Y+(g)\ \delta Z &= V(X\bar{\delta}V-V\bar{\delta}X),\\
(h)\,\delta X+(b)\ \delta Y+(f)\,\delta Z &= V(Y\bar{\delta}V-V\bar{\delta}Y),\\
(g)\,\delta X+(f)\,\delta Y+(c)\ \delta Z &= V(Z\bar{\delta}V-V\bar{\delta}Z).
\end{aligned}$$

Hence, in the coefficient of $d_x\rho$, the first line is

$$=2V(-Y\bar{\delta}Z+Z\bar{\delta}Y),$$

and the second line is

$$=\frac{2}{V}\{VZ(Y\bar{\delta}V-V\bar{\delta}Y)-VY(Z\bar{\delta}V-V\bar{\delta}Z)\},\ =2V(Y\bar{\delta}Z-Z\bar{\delta}Y);$$

so that the sum, or whole coefficient of $d_x\rho$, is $=0$. Similarly, the coefficients of $d_y\rho$ and $d_z\rho$ are each $=0$.

41. We have thus arrived at the equation

$$((A),\ldots\between d_x,\ d_y,\ d_z)^2\,\rho=0$$

as the condition to be satisfied by the normal distance ρ in order that the given surface and the vicinal surface may belong to an orthogonal system, viz. this is a partial differential equation of the second order, its coefficients being given functions of X, Y, Z, a, b, c, f, g, h, the first and second differential coefficients of r (where $r=r(x,\ y,\ z)$ is the equation of the given surface).

The equation, it is clear, may also be written in the two forms

$$(A,\ldots\between Zd_y-Yd_z,\ Xd_z-Zd_x,\ Yd_x-Xd_y)^2\,\rho=0,$$

and

$$\begin{vmatrix}
P\ , & Q\ , & R\\
aP+hQ+gR, & hP+bQ+fR, & gP+fQ+cR\\
X\ , & Y\ , & Z
\end{vmatrix}\rho=0,$$

if, for shortness, P, Q, R are written to denote Zd_y-Yd_z, Xd_z-Zd_x, Yd_x-Xd_y respectively, it being understood that in each of these forms the d_x, d_y, d_z operate on the ρ only.

Condition that a family of surfaces may belong to an Orthogonal System.
Art. Nos. 42 to 49.

42. We pass at once to the condition in order that the family of surfaces

$$r - r(x, y, z) = 0$$

may belong to an orthogonal system, viz. when the vicinal surface belongs to the family, we have ρ proportional to $\frac{1}{V}\left(= \frac{1}{\sqrt{X^2 + Y^2 + Z^2}}\right)$, and the condition is

$$\left((A), \ldots \between d_x, d_y, d_z\right)^2 \frac{1}{V} = 0,$$

where r is a function of (x, y, z), the first and second differential coefficients of which are $X, Y, Z, a, b, c, f, g, h$; and the equation is thus a partial differential equation of the third order satisfied by r. The form is by no means an inconvenient one, but it admits of further reduction.

43. We have $d_x \frac{1}{V}$, $d_y \frac{1}{V}$, $d_z \frac{1}{V}$ equal to $-\frac{1}{V^3}\delta X$, $-\frac{1}{V^3}\delta Y$, $-\frac{1}{V^3}\delta Z$ respectively, and thence

$$d_x^2 \frac{1}{V} = -\frac{1}{V^3}(a^2 + h^2 + g^2 + \delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_z \frac{1}{V} = -\frac{1}{V^3}(gh + bf + cf + \delta f) + \frac{3}{V^5}\delta Y \delta Z,$$

or, as these may be written,

$$d_x^2 \frac{1}{V} = -\frac{1}{V^3}(a\omega - \bar{\omega} + \bar{a} + \delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_z \frac{1}{V} = -\frac{1}{V^3}(f\omega \qquad + \bar{f} + \delta f) + \frac{3}{V^5}\delta Y \delta Z,$$

with the like values for $d_y^2 \frac{1}{V}$, &c. Substituting, the equation contains a term multiplied by ω, viz. this is

$$-\frac{1}{V^3}\omega\left((A), \ldots \between a, \ldots\right),$$

which vanishes; and a term multiplied by $\bar{\omega}$, viz. this is

$$\frac{1}{V^3}\bar{\omega}\left((A) + (B) + (C)\right),$$

which also vanishes. Writing down the remaining terms, and multiplying the whole by $-V^3$, the equation becomes

$$\left((A), \ldots \between \bar{a}, \ldots\right) + \left((A), \ldots \between \delta a, \ldots\right) - \frac{3}{V^2}\left((A), \ldots \between \delta X, \delta Y, \delta Z\right)^2 = 0.$$

44. The last term admits of reduction; from the equations

$$(A) = -AV^2 + 2XZ\delta Y - 2XY\delta Z, \text{ \&c., we find}$$

$$(A)\,\delta X + (H)\,\delta Y + (G)\,\delta Z = -V^2(A\delta X + H\delta Y + G\delta Z) + V\delta V(Z\delta Y - Y\delta Z),$$

$$(H)\,\delta X + (B)\,\delta Y + (F)\,\delta Z = -V^2(H\delta X + B\delta Y + F\delta Z) + V\delta V(X\delta Z - Z\delta X),$$

$$(G)\,\delta X + (F)\,\delta Y + (C)\,\delta Z = -V^2(G\delta X + F\delta Y + C\delta Z) + V\delta V(Y\delta X - X\delta Y),$$

and hence

$$\big((A), \ldots \between \delta X,\ \delta Y,\ \delta Z\big)^2 = -V^2(A, \ldots \between \delta X,\ \delta Y,\ \delta Z)^2;$$

wherefore the equation becomes

$$\big((A), \ldots \between \bar{a}, \ldots\big) + \big((A), \ldots \between \delta a \ldots\big) + 3(A, \ldots \between \delta X,\ \delta Y,\ \delta Z)^2 = 0.$$

45. It will be shown that we have identically

$$\big((A), \ldots \between \bar{a}, \ldots\big) = -(A, \ldots \between \delta X,\ \delta Y,\ \delta Z)^2 = 2\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix}.$$

The partial differential equation thus assumes the form

$$\big((A), \ldots \between \delta a, \ldots\big) + \Omega = 0,$$

where Ω may be expressed indifferently in the three forms,

$$= +2(A, \ldots \between \bar{a}, \ldots),$$

$$= +2(A, \ldots \between \delta X,\ \delta Y,\ \delta Z)^2,$$

$$= -4\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix}.$$

46. Taking the first of these, the partial differential equation is

$$\big((A), \ldots \between \delta a, \ldots\big) - 2\big((A), \ldots \between \bar{a}, \ldots\big) = 0;$$

or, written at full length, it is

$$(A)\,\delta a + (B)\,\delta b + (C)\,\delta c + 2(F)\,\delta f + 2(G)\,\delta g + 2(H)\,\delta h$$
$$-2\{(A)\ \bar{a} + (B)\ \bar{b} + (C)\ \bar{c} + 2(F)\ \bar{f} + 2(G)\ \bar{g} + 2(H)\,\bar{h}\} = 0,$$

where the coefficients are given functions of X, Y, Z, a, b, c, f, g, h, the first and second differential coefficients of r; and δ is written to denote $Xd_x + Yd_y + Zd_z$.

47. It remains to prove the above-mentioned identities.

To reduce the term $(A, .. \between \delta X, \delta Y, \delta Z)^2$, we have

$$\begin{aligned}
&A\delta X + H\delta Y + G\delta Z \\
&\quad = A(aX + hY + gZ) + H(hX + bY + fZ) + G(gX + fY + cZ) \\
&\quad = X\{\ \omega(hZ - gY) + \bar{h}Z - \bar{g}Y\} \\
&\qquad + Y\{-\omega(fY - bZ) - (\bar{f}Y - \bar{b}Z) - \omega Z - \bar{\delta}Z\} \\
&\qquad + Z\{\ \omega(fZ - cY) + fZ - \bar{c}Y + \bar{\omega}Y + \bar{\delta}Z\} \\
&\quad = \omega(Z\delta Y - Y\delta Z) + (Z\bar{\delta}Y - Y\bar{\delta}Z) + (Z\bar{\delta}Y - Y\bar{\delta}Z),
\end{aligned}$$

that is,

$$A\delta X + H\delta Y + G\delta Z = \omega(Z\delta Y - Y\delta Z) + 2(Z\bar{\delta}Y - Y\bar{\delta}Z);$$

and similarly

$$H\delta X + B\delta Y + F\delta Z = \omega(X\delta Z - Z\delta X) + 2(X\bar{\delta}Z - Z\bar{\delta}X),$$

$$G\delta X + F\delta Y + C\delta Z = \omega(Y\delta X - X\delta Y) + 2(Y\bar{\delta}X - X\bar{\delta}Y),$$

whence

$$(A, .. \between \delta X, \delta Y, \delta Z)^2 = -2 \begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta}X, & \bar{\delta}Y, & \bar{\delta}Z \end{vmatrix}.$$

48. Now, from the equations $AX + HY + GZ = Z\delta Y - Y\delta Z$, &c. we have for the value of twice the foregoing determinant

$$\begin{aligned}
2\,\text{det.} = 2\{&(\bar{a}X + \bar{h}Y + \bar{g}Z)(AX + HY + GZ) \\
&+ (\bar{h}X + \bar{b}Y + \bar{f}Z)(HX + BY + FZ) \\
&+ (\bar{g}X + \bar{f}Y + \bar{c}Z)(GX + FY + CZ)\};
\end{aligned}$$

and subtracting herefrom the function $((A), .. \between \bar{a}, ..)$, which is

$$\begin{aligned}
= \quad & (BZ^2 + CY^2 - 2FYZ)\bar{a} \\
& + (CX^2 + AZ^2 - 2GZX)\bar{b} \\
& + (AY^2 + BX^2 - 2HYZ)\bar{c} \\
& + 2(-AYZ - FX^2 + GXY + HXZ)\bar{f} \\
& + 2(-BZX + FXY - GY^2 + HYZ)\bar{g} \\
& + 2(-CXY + FXZ + GYZ - HZ^2)\bar{h},
\end{aligned}$$

the difference is found to be

$$\begin{aligned}
= \quad & \bar{a}\{(A, .. \between X, Y, Z)^2 + AV^2\} \\
& + \bar{b}\{(A, .. \between X, Y, Z)^2 + BV^2\} \\
& + \bar{c}\{(A, .. \between X, Y, Z)^2 + CV^2\} \\
& + 2\bar{f}\{(A + B + C)YZ + FV^2\} \\
& + 2\bar{g}\{(A + B + C)ZX + GV^2\} \\
& + 2\bar{h}\{(A + B + C)XY + HV^2\},
\end{aligned}$$

which, on account of $(A, .. \between X, Y, Z)^2 = 0$, and $A + B + C = 0$, reduces itself to

$$(A, .. \between \bar{a}, \ldots) . V^2.$$

49. We have

$$\begin{aligned} A\bar{a} + H\bar{h} + G\bar{g} = \; & \bar{a}\,(2hZ - 2gY) \\ & + \bar{h}\,(\,gX - fY - (a-b)\,Z) \\ & + g\,(\,fZ - hX - (c-a)\,Y) \\ = \; & X\,(g\bar{h} - h\bar{g}) \\ & + Y\,(a\bar{g} - g\bar{a} - (g\bar{a} + f\bar{h} + c\bar{g})) \\ & + Z\,(h\bar{a} - a\bar{h} + (h\bar{a} + b\bar{h} + f\bar{g})); \end{aligned}$$

or, observing that in the coefficients of Y and Z the second terms each vanish, this is

$$A\bar{a} + H\bar{h} + G\bar{g} = X\,(\bar{h}g - \bar{g}h) + Y\,(\bar{g}a - \bar{a}g) + Z\,(\bar{a}h - \bar{h}a);$$

and similarly

$$H\bar{h} + B\bar{b} + F\bar{f} = X\,(\bar{b}f - \bar{f}b) + Y\,(\bar{f}h - \bar{h}f) + Z\,(\bar{h}b - \bar{b}h),$$

$$G\bar{g} + H\bar{f} + C\bar{c} = X\,(\bar{f}c - \bar{c}f) + Y\,(\bar{c}g - \bar{g}c) + Z\,(\bar{g}f - \bar{f}g).$$

Adding these equations, the coefficient of X is the difference of two expressions each of which vanishes; and the like as regards the coefficients of Y and Z; that is, we have

$$(A, .. \between \bar{a}, ..) = 0;$$

and consequently

$$2 \begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix} = ((A), .. \between \bar{a}, \ldots) = -(A, \ldots \between \delta X, \delta Y, \delta Z)^2,$$

the required relation.

520.

ON THE CENTRO-SURFACE OF AN ELLIPSOID.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XII. Part I. (1873), pp. 319—365. Read March 7, 1870.]

THE Centro-surface of any given surface is the locus of the centres of curvature of the given surface, or say it is the locus of the intersections of consecutive normals, (the normals which intersect the normal at any particular point of the surface being those at the consecutive points along the two curves of curvature respectively which pass through the point on the surface). The terms, *normal, centre of curvature, curve of curvature*, may be understood in their ordinary sense, or in the generalised sense referring to the case where the Absolute (instead of being the imaginary circle at infinity) is any quadric surface whatever; viz. the normal at any point of a surface is here the line joining that point with the pole of the tangent plane in respect of the quadric surface called the Absolute: and of course the centre of curvature and curve of curvature refer to the normal as just defined.

The question of the centro-surface of a quadric surface has been considered in the two points of view, viz. 1°, when the terms "normal," &c. are used in the ordinary sense, and the equation of the quadric surface (assumed to be an ellipsoid) is taken to be $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$; 2°, when the Absolute is the surface $X^2+Y^2+Z^2+W^2=0$, and the equation of the quadric surface is taken to be $\alpha X^2+\beta Y^2+\gamma Z^2+\delta W^2=0$: in the first of them by Salmon, *Quart. Math. Jour.* t. II. pp. 217—222 (1858), and in the second by Clebsch, *Crelle*, t. LXII. pp. 64—107 (1863): see also Salmon's *Solid Geometry*, 2nd Ed. 1865, pp. 143, 402, &c. In the present Memoir, as shown by the title, the quadric surface is taken to be an Ellipsoid; and the question is considered exclusively from the first point of view: the theory is further developed in various respects, and in particular as regards the nodal curve upon the centro-surface: the distinction of real and

imaginary is of course attended to. The new results suitably modified would be applicable to the theory treated from the second point of view; but I do not on the present occasion attempt so to present them.

The Ellipsoid; Parameters ξ, η, &c. Art. Nos. 1—6.

1. The position of a point (X, Y, Z) on the ellipsoid

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$$

may be determined by means of the parameters, or elliptic coordinates, ξ, η; viz. these are such that we have

$$\frac{X^2}{a^2+\xi}+\frac{Y^2}{b^2+\xi}+\frac{Z^2}{c^2+\xi}=1,$$

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1;$$

or, what is the same thing, ξ, η are the roots of the quadric equation

$$\frac{X^2}{a^2+v}+\frac{Y^2}{b^2+v}+\frac{Z^2}{c^2+v}=1.$$

(In its actual form this is a cubic equation, but there is a root $v=0$, which is to be thrown out, and the quadric equation is thus

$$\begin{aligned}&v^2\\&+v\,(a^2+b^2+c^2-X^2-Y^2-Z^2)\\&+\{b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)\,X^2-(c^2+a^2)\,Y^2-(a^2+b^2)\,Z^2\}=0,\end{aligned}$$

or putting

$$\begin{aligned}P&=a^2+b^2+c^2,\\Q&=b^2c^2+c^2a^2+a^2b^2,\\R&=a^2b^2c^2,\end{aligned}$$

the equation is

$$v^2+v\,(P-X^2-Y^2-Z^2)+Q-(b^2+c^2)\,X^2-(c^2+a^2)\,Y^2-(a^2+b^2)\,Z^2=0.)$$

2. It is convenient to write throughout

$$\begin{aligned}b^2-c^2&=\alpha,\\c^2-a^2&=\beta,\\a^2-b^2&=\gamma,\end{aligned}$$

(whence $\alpha+\beta+\gamma=0$).

As usual, a is taken to be the greatest and c the least of the semi-axes; we have thus α, γ each of them positive, and β negative, $=-\beta'$ where β' is a positive quantity $=\alpha+\gamma$. A distinction arises in the sequel between the two cases $a^2+c^2>2b^2$ and $a^2+c^2<2b^2$, but the two cases are not essentially different, and it is convenient to assume $a^2+c^2>2b^2$, that is, $a^2-b^2>b^2-c^2$ or $\gamma>\alpha$, say $\gamma-\alpha$ positive. The limiting case $a^2+c^2=2b^2$ or $\gamma=\alpha$ requires special consideration.

3. We have

$$-\beta\gamma\, X^2=a^2(a^2+\xi)(a^2+\eta),$$
$$-\gamma\alpha\, Y^2=b^2(b^2+\xi)(b^2+\eta),$$
$$-\alpha\beta\, Z^2=c^2(c^2+\xi)(c^2+\eta).$$

It is in fact easy to verify that these values satisfy as well the equation of the ellipsoid as the assumed equations defining the elliptic coordinates ξ, η. We may also obtain the relations

$$X^2+Y^2+Z^2=a^2+b^2+c^2+\xi+\eta,$$
$$a^2X^2+b^2Y^2+c^2Z^2=a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2+(a^2+b^2+c^2)(\xi+\eta)+\xi\eta.$$

These, however, are obtained more readily from the equation in v, viz. the roots thereof being ξ, η, we have

$$-\xi-\eta=a^2+b^2+c^2-X^2-Y^2-Z^2,$$
$$\xi\eta=b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)X^2-(c^2+a^2)Y^2-(a^2+b^2)Z^2,$$

which lead at once to the relations in question.

4. Considering ξ as constant, the locus of the point (X, Y, Z) is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2+\xi}+\frac{Y^2}{b^2+\xi}+\frac{Z^2}{c^2+\xi}=1;$$

viz. this is one of the curves of curvature through the point; and similarly considering η as constant, the locus of the point is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1;$$

viz. this is the other of the curves of curvature through the point.

5. If instead of ξ and η we write h and k, we may consider h as extending between the values $-a^2$, $-b^2$, and k as extending between the values $-b^2$, $-c^2$.

$h=$ const. will thus give the series of curves of curvature one of which is the section by the plane $X=0$, or ellipse semi-axes b, c; say this is the *minor-mean* series. In particular $h=-a^2$ gives the ellipse just referred to; and $h=-b^2$, or say $h=-b^2-\epsilon$, gives two detached portions of the ellipse semi-axes a, c; viz. each of these portions extends from an umbilicus above the plane of xy, through the extremity of the semi-axis a, to an umbilicus below the plane of xy.

And in like manner $k=$ const. gives the series of curves of curvature one of which is the section by the plane $Z=0$, or ellipse semi-axes a, b; say this is the *major-mean* series. In particular $k=-c^2$ gives the ellipse just referred to; and $k=-b^2$, or say $k=-b^2+\epsilon$, gives the remaining portions of the ellipse semi-axes a, c; viz. these are two portions each extending from an umbilicus above the plane of xy, through the extremity of the semi-axis c, to an umbilicus above the plane of xy.

The ellipse last referred to may be called the umbilicar section, the other two principal sections being the major-mean section and the minor-mean section respectively.

In the limiting case $h=k=-b^2$, we have the umbilici, viz. these are given by

$$\frac{X^2}{a^2}=-\frac{\gamma}{\beta},\quad Y=0,\quad \frac{Z^2}{c^2}=-\frac{\alpha}{\beta}.$$

The two series of curves of curvature cover the whole real surface of the ellipsoid; so that at any real point thereof we have $\xi=h$, $\eta=k$, or else $\xi=k$, $\eta=h$, where h, k are negative real values lying within the foregoing limits $-a^2$, $-b^2$ and $-b^2$, $-c^2$ respectively. But observe that ξ, η taken separately may each extend between the limits $-a^2$, $-c^2$.

6. Suppose $\xi=\eta$, the equation in v will have equal roots, or the condition is

$$(P-X^2-Y^2-Z^2)^2=4\{Q-(b^2+c^2)X^2-(c^2+a^2)Y^2-(a^2+b^2)Z^2\},$$

viz. this surface by its intersection with the ellipsoid determines the envelope of the curves of curvature. This envelope is in fact a system of eight imaginary lines, four of them belonging to one of the systems of right lines on the ellipsoid, the other four to the other of the systems of right lines. For in the values of X^2, Y^2, Z^2 writing $\eta=\xi$, we find

$$\pm\sqrt{-\beta\gamma}\,\frac{X}{a}=a^2+\xi,$$

$$\pm\sqrt{-\gamma\alpha}\,\frac{Y}{b}=b^2+\xi,$$

$$\pm\sqrt{-\alpha\beta}\,\frac{Z}{c}=c^2+\xi,$$

or representing for shortness the left-hand functions by $\pm X'$, $\pm Y'$, $\pm Z'$, the eight lines are

$$\begin{array}{l|l|l|l}
a^2+\xi= X' & =X' & =-X' & =-X' \\
b^2+\xi= Y' & =-Y' & =Y' & =-Y' \\
c^2+\xi= Z' & =-Z' & =-Z' & =Z' \\
\hline
a^2+\xi=-X' & =X' & =X' & =-X' \\
b^2+\xi= Y' & =-Y' & =Y' & =-Y' \\
c^2+\xi= Z' & =Z' & =-Z' & =-Z',
\end{array}$$

so that in the two tetrads each line intersects the four lines of the other tetrad, but it does not intersect the remaining three lines of its own tetrad. The intersections are four points corresponding to $\xi = -a^2$, being the imaginary umbilici in the plane $X=0$: four to $\xi = -b^2$, being the real umbilici in the plane $Y=0$: four to $\xi = -c^2$, being the imaginary umbilici in the plane $Z=0$: and four corresponding to $\xi = \infty$, which may be called the umbilici at infinity [1].

Sequential and Concomitant Centro-curves. Art. No. 7.

7. Consider any particular curve of curvature; the normals at the several points thereof successively intersect each other in a series of points forming a curve; and we have thus, corresponding to the particular curve of curvature, a curve on the centro-surface, which curve may be called the *sequential centro-curve.* Again the same normals, viz. those at the several points of the particular curve of curvature, are intersected, the normal at each point by the consecutive normal belonging to the other curve of curvature through that point; and we have thus, corresponding to the particular curve of curvature, a curve on the centro-surface, which curve may be called the *concomitant centro-curve.* If instead of a single curve of curvature we consider the whole series, say of the major-mean curves of curvature, we have a series of major-mean sequential centro-curves, and also a series of major-mean concomitant centro-curves; and similarly considering the series of the minor-mean curves of curvature we have a series of minor-mean sequential centro-curves and also a series of minor-mean concomitant curves; the configuration of the several curves will be discussed further on, but it may be convenient to remark here that the centro-surface may be considered as consisting of two portions, say,

(A) locus of the major-mean sequential centro-curves; and also of the minor-mean concomitant centro-curves;

(B) locus of the minor-mean sequential centro-curves, and also of the major-mean concomitant centro-curves.

Investigation of expressions for the Coordinates of a point on the Centro-surface. Art. Nos. 8 to 13.

8. Consider the normal at the point (X, Y, Z). Taking in the first instance (x, y, z) as current coordinates, the equations are

$$\frac{x-X}{\frac{X}{a^2}} = \frac{y-Y}{\frac{Y}{b^2}} = \frac{z-Z}{\frac{Z}{c^2}}, = \lambda \text{ suppose,}$$

[1] According to Salmon, *Solid Geometry*, [2nd Ed. 1865], p. 229, the number of umbilici for a surface of the nth order is $= n(10n^2 - 25n + 16)$; viz. for $n=2$, this is $=12$, as in the ordinary theory, not recognizing the umbilici at infinity. But whether properly umbilici or not, the 4 points which I call the umbilici at infinity do in the present theory present themselves in like manner with the 12 umbilici.

or, what is the same thing,

$$x = X\left(1 + \frac{\lambda}{a^2}\right), \quad y = Y\left(1 + \frac{\lambda}{b^2}\right), \quad z = Z\left(1 + \frac{\lambda}{c^2}\right).$$

Suppose now that the normal meets the consecutive normal, or normal at the point $X + dX$, $Y + dY$, $Z + dZ$; and let x, y, z belong to the point of intersection of the two normals; we must have

$$0 = dX\left(1 + \frac{\lambda}{a^2}\right) + \frac{X}{a^2}d\lambda,$$

$$0 = dY\left(1 + \frac{\lambda}{b^2}\right) + \frac{Y}{b^2}d\lambda,$$

$$0 = dZ\left(1 + \frac{\lambda}{c^2}\right) + \frac{Z}{c^2}d\lambda,$$

which determine the direction of the consecutive point; the equations in fact give

$$0 = \begin{vmatrix} dX, & \frac{dX}{a^2}, & \frac{X}{a^2} \\ dY, & \frac{dY}{b^2}, & \frac{Y}{b^2} \\ dZ, & \frac{dZ}{c^2}, & \frac{Z}{c^2} \end{vmatrix},$$

or, what is the same thing,

$$0 = \begin{vmatrix} a^2dX, & dX, & X \\ b^2dY, & dY, & Y \\ c^2dZ, & dZ, & Z \end{vmatrix},$$

which is the differential equation of the curve of curvature. This equation must therefore be satisfied by taking for $X + dX$, $Y + dY$, $Z + dZ$, the coordinates of the consecutive point along either of the curves of curvature,—say along that which is the intersection with the surface

$$\frac{X^2}{a^2 + \eta} + \frac{Y^2}{b^2 + \eta} + \frac{Z^2}{c^2 + \eta} = 1.$$

9. To verify this, observe that we then have

$$\frac{XdX}{a^2} + \frac{YdY}{b^2} + \frac{ZdZ}{c^2} = 0,$$

$$\frac{XdX}{a^2 + \eta} + \frac{YdY}{b^2 + \eta} + \frac{ZdZ}{c^2 + \eta} = 0;$$

or, what is the same thing,

$$XdX : YdY : ZdZ = a^2(a^2 + \eta)\alpha : b^2(b^2 + \eta)\beta : c^2(c^2 + \eta)\gamma.$$

But from the equations $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$ &c., these become

$$XdX : YdY : ZdZ = \frac{X^2}{a^2+\xi} : \frac{Y^2}{b^2+\xi} : \frac{Z^2}{c^2+\xi},$$

or, what is the same thing,

$$dX : dY : dZ = \frac{X}{a^2+\xi} : \frac{Y}{b^2+\xi} : \frac{Z}{c^2+\xi};$$

and, substituting these values in the determinant equation, it becomes

$$0 = \frac{XYZ}{(a^2+\xi)(b^2+\xi)(c^2+\xi)} \begin{vmatrix} a^2, & 1, & a^2+\xi \\ b^2, & 1, & b^2+\xi \\ c^2, & 1, & c^2+\xi \end{vmatrix},$$

which is identically true, since evidently the determinant vanishes.

10. Proceeding with the solution, we have from the three equations

$$XdX + YdY + ZdZ + \lambda\left(\frac{XdX}{a^2} + \frac{YdY}{b^2} + \frac{ZdZ}{c^2}\right) + d\lambda\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2}\right) = 0,$$

and observing that from the equation

$$X^2 + Y^2 + Z^2 = a^2 + b^2 + c^2 + \xi + \eta,$$

considering therein η as constant, we have

$$XdX + YdY + ZdZ = \tfrac{1}{2}d\xi,$$

the equation becomes

$$\tfrac{1}{2}d\xi + d\lambda = 0;$$

and the three equations then are

$$0 = dX\left(1 + \frac{\lambda}{a^2}\right) - \tfrac{1}{2}\frac{X}{a^2}d\xi, \text{ \&c.},$$

or say

$$0 = dX(a^2+\lambda) - \tfrac{1}{2}Xd\xi, \text{ \&c.}$$

But from the equation $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$, considering therein η as a constant, we have

$$\frac{dX}{X} = \frac{\frac{1}{2}d\xi}{a^2+\xi},$$

and the equations thus become

$$0 = \frac{a^2+\lambda}{a^2+\xi} - 1, \text{ \&c.},$$

viz. these are all satisfied if only $\lambda = \xi$.

11. The coordinates of the point of intersection of the two normals thus are

$$x = X\left(1 + \frac{\xi}{a^2}\right), \quad y = Y\left(1 + \frac{\xi}{b^2}\right), \quad z = Z\left(1 + \frac{\xi}{c^2}\right),$$

or squaring, and substituting for X^2, &c., their values as given by

$$-\beta\gamma X^2 = a^3(a^2+\xi)(a^2+\eta), \text{ \&c.},$$

the equations become

$$-\beta\gamma\, a^2x^2 = (a^2+\xi)^3(a^2+\eta),$$
$$-\gamma\alpha\, b^2y^2 = (b^2+\xi)^3(b^2+\eta),$$
$$-\alpha\beta\, c^2z^2 = (c^2+\xi)^3(c^2+\eta),$$

viz. these equations give (x, y, z) the coordinates of a point on the centro-surface, the intersection of the normal at the point (X, Y, Z) of the ellipsoid, (determined by the parameters ξ, η), by the normal at the consecutive point along the curve of curvature

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1,$$

or say η is the sequential parameter [1].

Of course by interchanging ξ and η we should obtain the coordinates of the point of intersection of the normal at the same point (X, Y, Z) by the normal at the consecutive point along the other curve of curvature: ξ being in this case the sequential parameter.

12. I stop for a moment to consider the foregoing two equations

$$\lambda=\xi, \quad d\lambda=-\tfrac{1}{2}d\xi,$$

which at first sight appear inconsistent. But observe that in the foregoing solution λ is the parameter of the point (x, y, z) of the centro-surface considered as a point on the normal at (X, Y, Z); $\lambda+d\lambda$ is the parameter of the *same point* considered as a point on the normal at the consecutive point $(X+dX, Y+dY, Z+dZ)$: the value $\lambda+d\lambda=\xi+d\xi$ would belong to a different point, viz. the consecutive point of the centro-surface considered as a point on the consecutive normal—wherefore the $d\lambda$ of the solution ought not to be $=d\xi$. In further explanation, observe that the equations

$$x=X\left(1+\frac{\lambda}{a^2}\right), \text{ \&c. where } \lambda=\xi,$$

if we pass from (x, y, z) to the consecutive point on the centro-surface, give

$$dx=dX\left(1+\frac{\lambda}{a^2}\right)+\frac{X}{a^2}d\xi;$$

but since by what precedes,

$$0=dX\left(1+\frac{\lambda}{a^2}\right)-\tfrac{1}{2}\frac{X}{a^2}d\xi,$$

this is

$$dx=\tfrac{3}{2}\frac{X}{a^2}d\xi.$$

[1] The expressions are given in effect, but not explicitly, Salmon, p. 143.

Or since

$$a^2x = X(a^2+\xi),$$

this is

$$\frac{dx}{x} = \tfrac{3}{2}\frac{d\xi}{a^2+\xi};$$

and similarly

$$\frac{dy}{y} = \tfrac{3}{2}\frac{d\xi}{b^2+\xi},$$

$$\frac{dz}{z} = \tfrac{3}{2}\frac{d\xi}{c^2+\xi},$$

which are the correct values of dx, dy, dz as derived from the equations

$$-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta), \text{ \&c.}$$

13. The equations $-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta)$, &c. give expressions for the coordinates (x, y, z) of a point on the centro-surface in terms of the two parameters (ξ, η): the elimination of (ξ, η) from these equations will therefore lead to the equation of the surface; but the discussion of the surface may also be effected by means of these expressions for the coordinates in terms of the two parameters.

Discussion by means of the equations $-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta)$, &c.; *Principal Sections, &c.* Art. Nos. 14 to 24 (*several subheadings*).

14. To fix the ideas consider the section of the surface by the plane $z=0$; we have in the surface $z=0$, that is, $\xi = -c^2$, or else $\eta = -c^2$, values which give respectively

$$\begin{array}{l|l} -\beta\gamma a^2x^2 = -\beta^3(a^2+\eta), & -\beta\gamma a^2x^2 = -\beta(a^2+\xi)^3, \\ -\gamma\alpha b^2y^2 = \quad\alpha^3(b^2+\eta); & -\gamma\alpha b^2y^2 = \quad\alpha(b^2+\xi)^3; \end{array}$$

or, what is the same thing,

$$\begin{array}{l|l} \frac{\gamma}{\beta^2}a^2x^2 = \quad a^2+\eta, & \gamma a^2x^2 = \quad (a^2+\xi)^3, \\ \frac{\gamma}{\alpha^2}b^2y^2 = -b^2-\eta; & \gamma b^2y^2 = -(b^2+\xi)^3. \end{array}$$

The first set of equations gives

$$\frac{a^2x^2}{\beta^2} + \frac{b^2y^2}{\alpha^2} = 1,$$

which is the equation of an ellipse.

The second set gives

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \gamma^{\frac{2}{3}},$$

or in a rationalised form

$$(a^2x^2+b^2y^2-\gamma^2)^3 + 27a^2b^2\gamma^2x^2y^2 = 0,$$

which is the equation of an evolute of an ellipse.

15. The ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$ is a cuspidal curve on the surface, and the section by the plane $z=0$ is consequently made up of this ellipse counting three times, and of the evolute; it is therefore of the twelfth order; and the order of the surface is in fact $=12$.

It is clear that the section of the centro-surface arises from the section $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, viz. the normal at any point of this ellipse lies in the plane $Z=0$, and its intersection by a normal at the consecutive point of the ellipse gives a point of the evolute; the evolute being thus the sequential centro-curve of this section: the intersection by the normal at the consecutive point on the other curve of curvature gives a point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$, which ellipse is therefore the concomitant centro-curve. Observe that this other curve of curvature cuts the ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$ at right angles, and that the normals at the consecutive points above and below the point on the ellipse will meet each other and also the normal at the point of the same ellipse at the point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$: this shows that the last-mentioned ellipse is a cuspidal curve on the centro-surface.

16. The three principal sections of the centro-surface are consequently

$$x=0,\quad \frac{b^2y^2}{\gamma^2}+\frac{c^2z^2}{\beta^2}=1,\quad \text{and } (by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=\alpha^{\frac{2}{3}};$$

$$y=0,\quad \frac{c^2z^2}{\alpha^2}+\frac{a^2x^2}{\gamma^2}=1,\quad \text{and } (cz)^{\frac{2}{3}}+(ax)^{\frac{2}{3}}=\beta^{\frac{2}{3}};$$

$$z=0,\quad \frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad \text{and } (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\gamma^{\frac{2}{3}};$$

viz. each section is made up of an ellipse counting three times and of an evolute (of an ellipse). I have for shortness represented the three evolutes by their irrational equations. It will presently appear that the section (imaginary) by the plane infinity is of the like character.

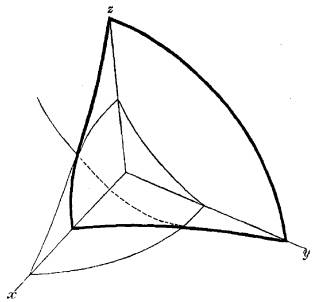
17. Considering only the positive directions of the axes, we have on each axis two points, viz.

$$\text{axis of } x,\quad x=\frac{\gamma}{a},\quad x=-\frac{\beta}{a};$$

$$\text{axis of } y,\quad y=\frac{\alpha}{b},\quad y=\frac{\gamma}{b};$$

$$\text{axis of } z,\quad z=-\frac{\beta}{c},\quad z=\frac{\alpha}{c};$$

through each of which, in the two different planes through the axis respectively, there passes an ellipse and an evolute. In the assumed case $a^2+c^2>2b^2$, the disposition of the points is as shown in the figure.



Plane of xz, evolute is outside ellipse,
yz, „ inside „
xy, „ cuts „ ;

but in the contrary case $a^2+c^2<2b^2$, the disposition is

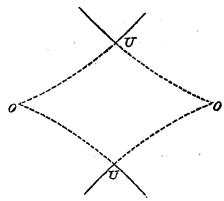
Plane of xz, evolute is outside ellipse,
yz, „ cuts „
xy, „ is inside „ ;

there is no real difference, and to fix the ideas I attend exclusively to the first-mentioned case

$$a^2+c^2>2b^2.$$

18. In each of the principal planes, the evolute and ellipse, quà curves of the orders 6 and 2 respectively, intersect in twelve points, 3 in each quadrant; viz. of the 3 points two unite together into a twofold point or point of contact, and the third is a point of simple intersection; assuming for the moment that this is so, the figure at once shows that in the plane of xz or umbilicar plane the contact is real, the intersection imaginary; in the plane of xy, or major-mean plane, the contact is imaginary, the intersection real; but in the plane of yz or minor-mean plane the contact and intersection are each imaginary. The contacts arise, as will appear, from the umbilici of the ellipsoid, and may be termed "umbilicar centres," or "omphaloi;" the simple intersections "points of outcrop," or simply "outcrops." By what precedes there are in the umbilicar plane, four real umbilicar centres (in each quadrant one); and in the major-mean plane four real outcrops (in each quadrant one); the other umbilicar centres and outcrops are respectively imaginary.

19. The surface consists of two sheets intersecting in a *nodal curve* connecting the outcrop with the umbilicar centre. As to the form of this curve there is a cusp at the outcrop; and the curve does not terminate at the umbilicar centre but, on passing it, from crunodal becomes acnodal (viz. there is no longer through the curve any real sheet of the surface): moreover the curve is not at the umbilicar centre



perpendicular to the plane of xz, and there is consequently on the opposite side of the plane a symmetrically situate branch of the curve, viz. the umbilicar centre is a node on the nodal curve. Completing the curve, the nodal curve consists of two distinct portions, one on the positive side of the plane of yz or minor-mean plane consisting of two cuspidal branches as shown in the figure; the other a symmetrically situate portion on the negative side of the minor-mean plane.

Intersections of Evolute and Ellipse.

20. Consider in the plane of xy the ellipse and evolute,

$$\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad (a^2x^2+b^2y^2-\gamma^2)^3+27\gamma^2a^2b^2x^2y^2=0.$$

First, these are satisfied by

$$\left.\begin{aligned} a^2x^2&=-\frac{\beta^3}{\gamma},\\ b^2y^2&=-\frac{\alpha^3}{\gamma},\end{aligned}\right\}\ \text{Coordinates of Umbilicar centres in plane of } xy \text{ (imaginary),}$$

viz. the equations respectively become

$$-\frac{\beta}{\gamma}-\frac{\alpha}{\gamma}=1,\quad \left(-\frac{\beta^3+\alpha^3}{\gamma}-\gamma^2\right)^3+27\alpha^2\beta^3=0,$$

the first of which is $\alpha+\beta+\gamma=0$, and the second is $(\alpha^3+\beta^3+\gamma^3)^3-27\alpha^2\beta^3\gamma^3=0$. But the equation $\alpha+\beta+\gamma=0$ gives $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, and the two equations are thus identically satisfied. Moreover the condition for a contact is at once found to be

$$\beta^2[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2]=\alpha^2[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2],$$

or, what is the same thing,

$$(\alpha^2-\beta^2)(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2(\alpha^2a^2x^2-\beta^2b^2y^2)=0;$$

and substituting the foregoing values, this is

$$(\alpha^2-\beta^2)\left(-\frac{\alpha^3+\beta^3}{\gamma}-\gamma^2\right)^2+9\gamma^2\frac{-\alpha^2\beta^3+\alpha^3\beta^2}{\gamma}=0,$$

that is,

$$\frac{\alpha^2-\beta^2}{\gamma^2}(\alpha^3+\beta^3+\gamma^3)^2+9\gamma\alpha^2\beta^2(\alpha-\beta)=0,$$

which, putting therein $\alpha+\beta=-\gamma$, and $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, is also satisfied; that is, the points in question are points of contact of the ellipse and evolute.

21. Secondly, consider the values

$$\left.\begin{aligned} a^2x^2&=-\frac{\beta^3}{\gamma}\left(\frac{\gamma-\alpha}{\alpha-\beta}\right)^3,\\ b^2y^2&=-\frac{\alpha^3}{\gamma}\left(\frac{\beta-\gamma}{\alpha-\beta}\right)^3,\end{aligned}\right\}\text{ Coordinates of outcrops in plane of } xy \text{ (real).}$$

Substituting in the equation of the ellipse, we have

$$\alpha(\beta-\gamma)^3+\beta(\gamma-\alpha)^3+\gamma(\alpha-\beta)^3=0,$$

which is

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)=0,$$

or the equation is satisfied identically: and substituting in the equation of the evolute, we have first

$$a^2x^2+b^2y^2-\gamma^2=-\frac{\alpha^3(\beta-\gamma)^3+\beta^3(\gamma-\alpha)^3+\gamma^3(\alpha-\beta)^3}{\gamma(\alpha-\beta)^3};$$

which in virtue of $\alpha(\beta-\gamma)+\beta(\gamma-\alpha)+\gamma(\alpha-\beta)=0$ becomes

$$\begin{aligned} a^2x^2+b^2y^2-\gamma^2&=-\frac{3\alpha\beta\gamma(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{\gamma(\alpha-\beta)^3},\\ &=-\frac{3\alpha\beta(\beta-\gamma)(\gamma-\alpha)}{(\alpha-\beta)^2},\end{aligned}$$

and then, completing the substitution, it is seen that the equation of the evolute is also satisfied. The points last considered are simple intersections, and we have thus the complete number $(8+4,=12)$ of the intersections of the evolute and ellipse.

22. We have α, γ positive, β negative; whence $\alpha-\beta$ is positive, $\beta-\gamma$ negative; $\gamma-\alpha\,(=a^2+c^2-2b^2)$ is positive, and hence, the outcrops in the plane of xy are real; the umbilicar centres are imaginary for this plane, but real for the plane of zx, the coordinates being

$$\left.\begin{aligned} c^2z^2&=-\frac{\alpha^3}{\beta},\\ a^2x^2&=-\frac{\gamma^3}{\beta},\end{aligned}\right\}\text{ Coordinates of Umbilicar centres in plane of } xz \text{ (real).}$$

Nodes of the Evolute.

23. The Evolute is a curve with four nodes, all of them imaginary; viz. for the evolute in the plane of xy, the equation of which is

$$(a^2x^2+b^2y^2-\gamma^2)^2+27\gamma^2a^2b^2x^2y^2=0,$$

these are

$$\left.\begin{aligned} a^2x^2&=-\gamma^2,\\ b^2y^2&=-\gamma^2,\end{aligned}\right\}\text{ Coordinates of Nodes of evolute in plane of } xy \text{ (imaginary)},$$

in fact these values satisfy as well the equation of the evolute, as the two derived equations

$$6a^2x\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2]=0,$$
$$6b^2y\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2]=0,$$

or the points in question are nodes of the evolute.

The evolute has the four cusps on the axes and two cusps at infinity, in all 6 cusps as just mentioned; it has 4 nodes: and the order being 6, the class is

$$30-2\,.\,4-3\,.\,6,\ =4.$$

Section by the plane infinity.

24. The surface itself is finite, and the section by the plane infinity is therefore imaginary; but by what precedes the nodal curve must have real points at infinity, viz. there must be real acnodal points on this imaginary section. The section by the plane infinity resembles in fact the principal sections; viz. writing successively $\xi=\infty$, and $\eta=\infty$, we have

$$-\beta\gamma a^2x^2\ :\ -\gamma\alpha b^2y^2\ :\ -\alpha\beta c^2z^2=a^2+\eta\ :\ b^2+\eta\ :\ c^2+\eta$$

or

$$=(a^2+\xi)^3\ :\ (b^2+\xi)^3\ :\ (c^2+\xi)^3,$$

giving respectively

$$a^2x^2+b^2y^2+c^2z^2=0,\quad\text{and}\quad(\alpha x)^{\frac{2}{3}}+(b\beta y)^{\frac{2}{3}}+(c\gamma z)^{\frac{2}{3}}=0,$$

where the first equation represents an imaginary conic which counts three times; and the second equation, the rationalised form of which is

$$(a^2\alpha^2x^2+b^2\beta^2y^2+c^2\gamma^2z^2)^3-27a^2b^2c^2\alpha^2\beta^2\gamma^2x^2y^2z^2=0,$$

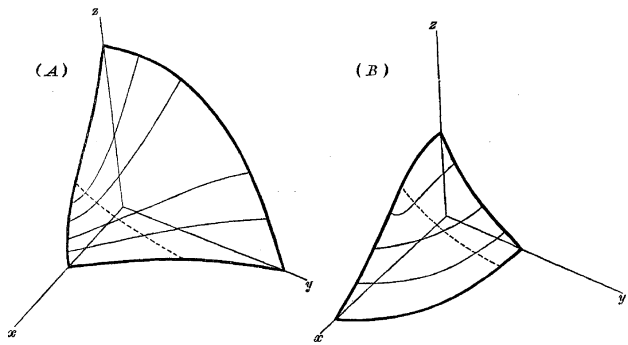
an imaginary evolute. The conic and evolute have four contacts and four simple intersections (in all $4\,.\,2+4=12$ intersections) which are all of them imaginary. But the evolute has four real nodes (acnodes) $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$; or, what is the same thing, there are four real lines $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$, which are respectively asymptotes of the nodal curve: viz. inasmuch as the equation of the surface contains only the squares

x^2, y^2, z^2, the lines in question will be not merely parallel to, but will be, the asymptotes of the nodal curve.

The plane infinity may be reckoned as a principal plane, and we may say that in each of the four principal planes there are four umbilicar centres, four outcrops, and four evolute-nodes.

The generation of the surface considered geometrically. Art. Nos. 25 to 28.

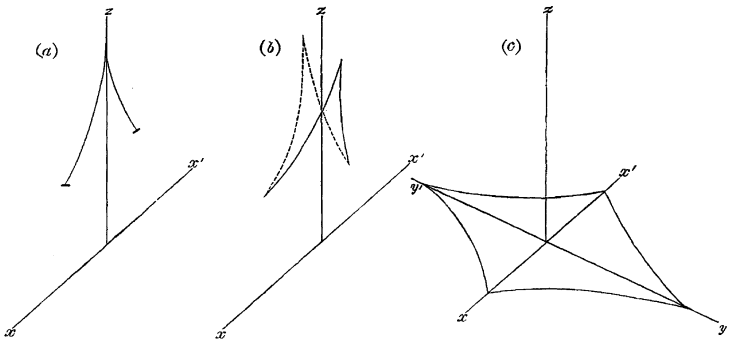
25. I have deferred until this point the discussion of the generation of the centro-surface by means of the centro-curves, for the reason that it can be carried on more precisely now that we know the forms of the principal sections and of the nodal



curve. The two figures exhibit (as regards one octant of the surface) the portions already distinguished as (A), and (B): they intersect each other in the nodal curve, shown in each of the figures.

26. Consider first the generation of the portion (A) by means of the major-mean sequential centro-curves. The major-mean curves of curvature (attending to those below the plane of xy) commence with a portion (extending from umbilicus to umbilicus) of the ellipse $\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1$, this may be termed the vertical curve, and they end with the whole ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, which may be termed the horizontal curve. The normals at the several points of the vertical curve successively intersect along a portion (terminated each way at an umbilicar centre) of the evolute in the plane of xz or umbilicar plane; viz. this portion of the evolute, shown fig. (a), is the sequential centro-curve belonging to the vertical curve of curvature. The curve of curvature is at first a narrow oval surrounding the vertical curve; the corresponding form of the sequential centro-curve is at once seen to be a four-cusped curve as in fig. (b), and which we may imagine as derived from the curve (a) by first doubling this curve and then

opening out the two component parts thereof: the two upper cusps of the curve (*b*) are situate on the *yz*-ellipse of the centro-surface, and the two lower cusps upon two detached portions respectively of the *xz*-ellipse of the centro-surface. And as the curve



of curvature gradually broadens out and ultimately coincides with the XY-section of the ellipsoid, the four-cusped curve continues to open itself out, and ultimately coincides as shown figure (*c*) with the *xy*-evolute of the centro-surface, viz. this evolute is the sequential centro-curve belonging to the horizontal curve of curvature or XY-section of the ellipsoid. The successive sequential curves are also shown (so far as regards an octant of the surface) in the figure (A).

27. We consider next the generation of the portion (B) by means of the major-mean concomitant centro-curves. Starting as before with the vertical curve of curvature, the concomitant centro-curve is a finite portion (terminated each way at an umbilicar centre) of the *xz*-ellipse of the centro-surface. As the curve of curvature opens itself out into an oval, the concomitant centro-curve in like manner opens itself out into an oval, the two further vertices thereof situate on two detached portions of the *xz*-evolute of the centro-surface, and the two nearer vertices on the *yz*-evolute of the central surface. And as the curve of curvature continues to open itself out, and ultimately coincides with the horizontal curve or XY-section of the ellipsoid, so the concomitant centro-curve continues to open itself out and ultimately coincides with the *xy*-ellipse of the centro-surface. The successive forms (so far as relates to an octant of the surface) are shown in the figure (B). We have in each case attended only to the curves of curvature below the plane of *xy*, and the corresponding centro-curves above the plane of *xy*, but of course everything is symmetrical as regards the two sides of the plane.

28. There is a precisely similar generation of the portion (A) by the minor-mean concomitant centro-curves, and of the portion (B) by means of the minor-mean sequential centro-curves.

The Nodal Curve. Art. Nos. 29 to 60.

29. If two different points on the ellipsoid correspond to the same point on the centro-surface, this will be a point on the Nodal Curve: the conditions for this if (ξ, η), (ξ_1, η_1) are the parameters for the two points on the ellipsoid, are obviously

$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1),$$
$$(b^2+\xi)^3(b^2+\eta) = (b^2+\xi_1)^3(b^2+\eta_1),$$
$$(c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1);$$

these equations in effect determine η as a function of ξ, so that the equations

$$-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta), \text{ \&c.}$$

then determine the coordinates (x, y, z) of a point on the Nodal Curve in terms of the single parameter ξ.

The relation between ξ and η would be obtained by eliminating ξ_1, η_1 from the foregoing equation: but it is easier to eliminate η and η_1, thus obtaining between ξ_1 and ξ a relation in virtue of which ξ_1 may be regarded as a known function of ξ; η and η_1 can then be expressed in terms of ξ, ξ_1, so that each of these quantities will be in effect a known function of ξ[1].

30. The relation between ξ, ξ_1 is in the first instance given in the form

$$\begin{vmatrix} a^2[(a^2+\xi)^3-(a^2+\xi_1)^3], & (a^2+\xi)^3, & (a^2+\xi_1)^3 \\ b^2[(b^2+\xi)^3-(b^2+\xi_1)^3], & (b^2+\xi)^3, & (b^2+\xi_1)^3 \\ c^2[(c^2+\xi)^3-(c^2+\xi_1)^3], & (c^2+\xi)^3, & (c^2+\xi_1)^3 \end{vmatrix} = 0.$$

Throwing out a factor $(\xi-\xi_1)^2$, this becomes

$$\Sigma[a^2\{3a^4+3a^2(\xi+\xi_1)+\xi^2+\xi\xi_1+\xi_1^2\}$$
$$\times(b^2-c^2).(1, 1, 1)\{(b^2+\xi)(c^2+\xi_1), (b^2+\xi_1)(c^2+\xi)\}^2] = 0,$$

where the left-hand side is a symmetrical function of ξ, ξ_1 vanishing for $\xi=\xi_1$, and therefore divisible by $(\xi-\xi_1)^2$; it is also divisible by Δ, $=(b^2-c^2)(c^2-a^2)(a^2-b^2)(=\alpha\beta\gamma)$. To work this out, write $\xi+\xi_1=p$, $\xi\xi_1=q$, the equation may be written

$$\Sigma\left\{(b^2-c^2)a^2 \left|\begin{array}{l} 3a^4 \\ +3a^2p \\ +p^2-q \end{array}\right| \left|\begin{array}{l} 3b^4c^4 \\ +3b^2c^2(b^2+c^2)p \\ +(b^4+c^4)(p^2-q) \\ +b^2c^2(p^2+8q) \\ +3(b^2+c^2)pq \\ +3q^2 \end{array}\right|\right\} = 0,$$

where the left-hand side divides by $\Delta(p^2-4q)$.

[1] This was my first method of solution; and I have thought the results quite interesting enough to retain them—but it will appear in the sequel that I have succeeded in expressing ξ, η, ξ_1, η_1, in terms of a single parameter σ.

31. Developing and reducing, and omitting this factor, the final result is

$$6R + 3Qp + P(p^2 + 2q) + 3pq = 0,$$

where as before P, Q, R denote $a^2 + b^2 + c^2$, $b^2c^2 + c^2a^2 + a^2b^2$, $a^2b^2c^2$, respectively; that is,

$$6R + 3Q(\xi + \xi_1) + P(\xi^2 + 4\xi\xi_1 + \xi_1^2) + 3(\xi + \xi_1)\xi\xi_1 = 0,$$

or, as this may be written,

$$\begin{aligned} &6R + 3Q\xi + P\xi^2 \\ &+ \xi_1\,(3Q + 4P\xi + 3\xi^2) \\ &+ \xi_1^2(P \;+\; 3\xi \qquad) = 0, \end{aligned}$$

viz. the parameters ξ, ξ_1 have a symmetrical (2, 2) correspondence.

32. From the equations $(a^2 + \xi)^3(a^2 + \eta) = (a^2 + \xi_1)^3(a^2 + \eta_1)$, &c., we have

$$\Sigma(b^2 - c^2)\{(a^2 + \xi)^3(a^2 + \eta) - (a^2 + \xi_1)^3(a^2 + \eta_1)\} = 0,$$
$$\Sigma b^2c^2(b^2 - c^2)\{(a^2 + \xi)^3(a^2 + \eta) - (a^2 + \xi_1)^3(a^2 + \eta_1)\} = 0;$$

and observing that the term in { } is

$$\begin{aligned} &a^6(3\xi + \quad \eta - 3\xi_1 - \quad \eta_1) \\ &+ a^4(3\xi^2 + 3\xi\eta - 3\xi_1^2 - 3\xi_1\eta_1) \\ &+ a^2(\;\xi^3 + 3\xi^2\eta - \;\xi_1^3 - 3\xi_1^2\eta_1) \\ &+ \quad(\;\xi^3\eta \qquad - \xi_1^3\eta_1), \end{aligned}$$

these are readily reduced to

$$\begin{aligned} &(3\xi + \eta - 3\xi_1 - \eta_1)P + (3\xi^2 + 3\xi\eta - 3\xi_1^2 - 3\xi_1\eta_1) = 0, \\ &(3\xi + \eta - 3\xi_1 - \eta_1)R + \xi^3\eta - \xi_1^3\eta_1 \qquad\qquad = 0, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} &3(\xi - \xi_1)(P + \xi + \xi_1) + \eta(P + 3\xi) - \eta_1(P + 3\xi_1) = 0, \\ &3(\xi - \xi_1)\,R \qquad\qquad + \eta(R + \;\xi^3) - \eta_1(R + \;\xi_1^3) = 0, \end{aligned}$$

and if we hence determine the ratios $3(\xi - \xi_1) : \eta : \eta_1$, the first of the resulting terms divides by $\xi - \xi_1$, and we have

$$\begin{aligned} 3 : \eta : \eta_1 = &-P(\xi^2 + \xi\xi_1 + \xi_1^2) + 3R - 3\xi\xi_1(\xi + \xi_1) \\ &: R(2\xi_1 - \xi\;) - \xi_1^3(P + \xi + \xi_1) \\ &: R(2\xi\; - \xi_1) - \xi^3\;(P + \xi + \xi_1). \end{aligned}$$

Hence observing that by the relation between ξ, ξ_1 the first term is

$$= 3\{P\xi\xi_1 + Q(\xi + \xi_1) + 3R\},$$

the equations become

$$\begin{aligned} 1 : \eta : \eta_1 = &P\xi\xi_1 + Q(\xi + \xi_1) + 3R \\ &: R(2\xi_1 - \xi\;) - \xi_1^3(P + \xi + \xi_1) \\ &: R(2\xi\; - \xi_1) - \xi^3\;(P + \xi + \xi_1); \end{aligned}$$

and we thus have

$$\eta = \frac{R(2\xi_1 - \xi) - \xi_1^3(P + \xi + \xi_1)}{P\xi\xi_1 + Q(\xi + \xi_1) + 3R},$$

which, considering ξ_1 as a given function of ξ, gives η as a function of ξ.

33. I write $\xi + \xi_1 = 2\mathrm{x}$, $\xi - \xi_1 = 2\mathrm{y}$, so that $p = 2\mathrm{x}$, $q = \mathrm{x}^2 - \mathrm{y}^2$. The relation between ξ, ξ_1 takes the form

$$6(R + Q\mathrm{x} + P\mathrm{x}^2 - \mathrm{x}^3) - (6\mathrm{x} + 2P)\mathrm{y}^2 = 0,$$

or, what is the same thing,

$$\mathrm{y}^2 = \frac{(\mathrm{x} + a^2)(\mathrm{x} + b^2)(\mathrm{x} + c^2)}{\mathrm{x} + \frac{1}{3}(a^2 + b^2 + c^2)};$$

so that taking x at pleasure and considering y as denoting this function of x, the values of ξ, ξ_1 belonging to a point on the nodal curve are $\xi = (\mathrm{x} + \mathrm{y})$, $\xi_1 = (\mathrm{x} - \mathrm{y})$; and the value of η is then given as before.

34. The form just given is analytically the most convenient, but there is some advantage in writing $\frac{1}{\sqrt{2}}\mathrm{x}$, $\frac{1}{\sqrt{2}}\mathrm{y}$, in the place of x, y respectively; viz. we then have

$$\mathrm{y}^2 = \frac{(\mathrm{x} + a^2\sqrt{2})(\mathrm{x} + b^2\sqrt{2})(\mathrm{x} + c^2\sqrt{2})}{\mathrm{x} + \frac{1}{3}\sqrt{2}(a^2 + b^2 + c^2)},$$

where $\xi = \frac{1}{\sqrt{2}}(\mathrm{x} + \mathrm{y})$, $\xi_1 = \frac{1}{\sqrt{2}}(\mathrm{x} - \mathrm{y})$, so that if (ξ, ξ_1) be taken as rectangular coordinates of a point in a plane, (x, y) will be the rectangular coordinates of the same point referred to axes inclined at angles of 45° to the first-mentioned axes respectively.

35. The curve is a cubic curve symmetrical in regard to the axis of x, and having the three asymptotes,

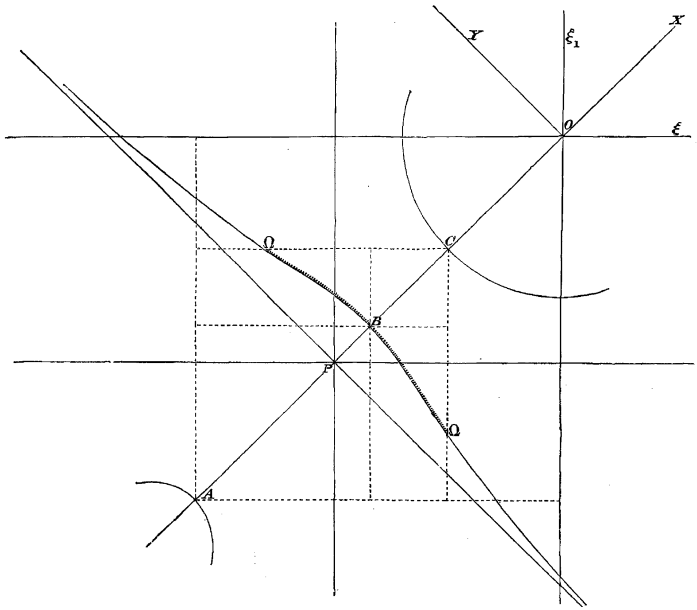
$$\mathrm{x} = -\tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}, \quad \mathrm{y} = \pm\{\mathrm{x} + \tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}\},$$

viz. these all meet in the point P the coordinates of which are

$$\mathrm{x} = -\tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}, \quad \mathrm{y} = 0:$$

moreover we have $\mathrm{y} = 0$ for the values $\mathrm{x} = -a^2\sqrt{2}$, $-b^2\sqrt{2}$, $-c^2\sqrt{2}$, that is, the curve meets the axis of x in the points A, B, C; the order in the direction of $-\mathrm{x}$ being C, B, P, A as shown in the figure: and with these data it is easy to draw the curve: the portion which gives the crunodal part of the nodal curve is that extending from B to the points Ω; viz. at B we have $\xi = \xi_1 = -b^2$, corresponding to the umbilicar centre; and at Ω, Ω we have ξ or $\xi_1 = -c^2$, ξ_1 or $\xi = -c^2 + \frac{3\alpha\beta}{\alpha - \beta}$, corresponding to the outcrop.

36. The nodal curve passes through (I) the umbilicar centres, (II) the outcrops, (III) the nodes of the evolute. The geometrical construction led to the conclusion that the real umbilicar centre was a node on the nodal curve, and that the real outcrop was a cusp (the tangent lying in the principal plane). It will presently appear generally, as regards the several points real or imaginary, that the umbilicar centre is a node on the nodal curve, and the outcrop a cusp—the tangent at the outcrop being in the principal plane: as regards the node on the evolute this is a simple point on the nodal curve, and by reason of the symmetry in regard to the principal



plane, the nodal curve will at this (imaginary) point cut the principal planes at right angles. Hence considering the intersections of the nodal curve by a principal plane, the umbilicar centre, outcrop and node of the evolute count respectively as 2 points, 3 points and 1 point, and as for each kind the number is 4, the whole number of intersections is $4(2+3+1)$, $=24$. It may be shown that these are the only intersections of the nodal curve with the principal plane; and this being so, it follows that the order of the nodal curve is $=24$; which agrees with the result of a subsequent analytical investigation.

37. The umbilicar centres or points (I) belong to values such as $\xi = \xi_1 = -a^2$ which are the *united values* in the equation between (ξ, ξ_1), viz. writing herein $\xi_1 = \xi$ the equation becomes

$$(\xi + a^2)(\xi + b^2)(\xi + c^2) = 0,$$

so that the united values are $\xi = \xi_1 = -a^2$, $-b^2$ or $-c^2$. (It may be remarked, that treating this cubic as a degenerate quartic, a united value would be $\xi = \xi_1 = \infty$, corresponding to the umbilicar centres at infinity.)

To a value such as $\xi = -a^2$ there corresponds (not only the value $\xi_1 = -a^2$, but also) a value $\xi_1 = -a^2 + \dfrac{3\beta\gamma}{\beta - \gamma}$, as it is easy to verify. And the outcrops or points (II) belong to such values $\xi = -a^2$, $\xi_1 = -a^2 + \dfrac{3\beta\gamma}{\beta - \gamma}$.

And the nodes of the evolute or points (III) belong to values such as $\xi = \omega b^2 + \omega^2 c^2$, $\xi_1 = \omega^2 b^2 + \omega c^2$ (ω an imaginary cube root of unity) which, as it is easy to see, satisfy the relation between (ξ, ξ_1). But to complete the theory we require to have the values of η, η_1 and also the coordinates of the points on the centro-surface, and of the two points on the ellipsoid.

38. I exhibit the results first for the umbilicar centres (imaginary), outcrops (imaginary), and nodes of the evolute (imaginary), in the plane $x = 0$; secondly for the real umbilicar centres in the plane $y = 0$ and for the real outcrops in the plane $z = 0$. The formulæ contain an expression Ω which is a symmetrical function of α, β, γ (or a, b, c), viz. it is

$$\Omega = \alpha^2 - \beta\gamma = \beta^2 - \gamma\alpha = \gamma^2 - \alpha\beta = \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = -(\beta\gamma + \gamma\alpha + \alpha\beta).$$

We have

I. $\xi = -a^2$, $\eta = -a^2$; $\xi_1 = -a^2$, $\eta_1 = -a^2$.

$$\left.\begin{array}{ll} X = 0, & X_1 = 0, \\ Y^2 = -b^2\dfrac{\gamma}{\alpha}, & Y_1^2 = -b^2\dfrac{\gamma}{\alpha}, \\ Z^2 = -c^2\dfrac{\beta}{\alpha}, & Z_1^2 = -c^2\dfrac{\beta}{\alpha}, \end{array}\right\} \text{(Umbilicus).}$$

$$\left.\begin{array}{l} x = 0, \\ b^2y^2 = -\dfrac{\gamma^3}{\alpha}, \\ c^2z^2 = -\dfrac{\beta^3}{\alpha}, \end{array}\right\} \text{(Umbilicar centre).}$$

II. $\xi = -a^2, \qquad \eta = -a^2 + \dfrac{9\beta\gamma\Omega}{(\beta-\gamma)^3},$

$$\left(\text{or } \eta + b^2 = -\gamma\frac{(\alpha-\beta)^3}{(\beta-\gamma)^3}, \quad \eta + c^2 = \frac{\beta(\gamma-\alpha)^3}{(\beta-\gamma)^3}\right).$$

$$\xi_1 = -a^2 + \frac{3\beta\gamma}{\beta-\gamma}, \quad \eta_1 = -a^2.$$

$$\left.\begin{aligned} X &= 0, & X_1 &= 0, \\ Y^2 &= -b^2\frac{\gamma}{\alpha}\frac{(\alpha-\beta)^3}{(\beta-\gamma)^3}, & Y_1^2 &= -b^2\frac{\gamma}{\alpha}\frac{\alpha-\beta}{\beta-\gamma}, \\ Z^2 &= -c^2\frac{\beta}{\alpha}\frac{(\gamma-\alpha)^3}{(\beta-\gamma)^3}, & Z_1^2 &= -c^2\frac{\beta}{\alpha}\frac{\gamma-\alpha}{\beta-\gamma}, \end{aligned}\right\}$$

Ellipse, concomitant. Ellipse, sequential.

$$\left.\begin{aligned} x &= 0, \\ b^2y^2 &= -\frac{\gamma^3(\alpha-\beta)^3}{\alpha(\beta-\gamma)^3}, \\ c^2z^2 &= -\frac{\beta^3(\gamma-\alpha)^3}{\alpha(\beta-\gamma)^3}, \end{aligned}\right\} \text{ (Outcrop).}$$

(Observe that at point Y_1, Z_1 of ellipse $\dfrac{Y^2}{b^2} + \dfrac{Z^2}{c^2} = 1$, the coordinates of the centre of curvature are $y = \dfrac{\alpha Y_1^3}{b^4}$, $z = -\dfrac{\alpha Z_1^3}{c^4}$, and it thence appears that this is the point in regard to which the ellipse is sequential.)

III. $\xi = \omega b^2 + \omega^2 c^2, \quad \eta = -a^2; \qquad \xi_1 = \omega^2 b^2 + \omega c^2, \quad \eta_1 = -a^2.$

$$\begin{aligned} X &= 0, & X_1 &= 0, \\ Y^2 &= -b^2\omega^2, & Y_1^2 &= -b^2\omega, \\ Z^2 &= -c^2\omega, & Z_1^2 &= -c^2\omega^2, \end{aligned}$$

$$\left.\begin{aligned} x &= 0, \\ b^2y^2 &= -a^2, \\ c^2z^2 &= -a^2, \end{aligned}\right\} \text{ (Node of evolute).}$$

39. Observe that these are the only ways in which it is possible to satisfy the equations

$$0 = (a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1),$$

viz. starting from this equation we have

I. $\qquad a^2 + \xi = 0, \quad a^2 + \xi_1 = 0,$

whence in the equations for η, η_1, substituting the values $\xi = \xi_1 = -a^2$, we have

$$\begin{aligned} 1 : \eta : \eta_1 &= a^4 P - 2a^2 Q + 3R, \\ &: -a^2 R + a^6 (P - 2a^2), \\ &: -a^2 R + a^6 (P - 2a^2), \end{aligned}$$

that is,

$$1 : \eta : \eta_1 = -a^2\beta\gamma : a^4\beta\gamma : a^4\beta\gamma,$$

or

$$\eta = \eta_1 = -a^2.$$

40. II. $a^2 + \xi = 0$ without $a^2 + \xi_1 = 0$, consequently $a^2 + \eta_1 = 0$; writing $\xi = -a^2$, in the relation between (ξ, ξ_1), this is

$$6R + 3Q(\xi_1 - a^2) + P(\xi_1^2 - 4a^2\xi_1 + a^4) - 3a^2\xi_1(\xi_1 + a^2) = 0,$$

viz. this is

$$\xi_1^2(b^2 + c^2 - 2a^2) + \xi_1(-a^4 - a^2b^2 - a^2c^2 + 3b^2c^2) + a^2(a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2) = 0,$$

where the left-hand side should divide by $\xi_1 + a^2$; the equation in fact is

$$(\xi_1 + a^2)\{\xi_1(b^2 + c^2 - 2a^2) + a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2\} = 0;$$

or, what is the same thing,

$$(\xi_1 + a^2)\{(\xi_1 + a^2)(\beta - \gamma) - 3\beta\gamma\} = 0,$$

whence

$$\xi_1 = -a^2 + \frac{3\beta\gamma}{\beta - \gamma}.$$

41. Considering these values of ξ, ξ_1 as given, the verification of the value $\eta_1 = -a^2$, and determination of $\eta = -a^2 + \dfrac{9\beta\gamma\Omega}{(\beta - \gamma)^3}$ is somewhat complex.

Writing for a moment $\Lambda = -\dfrac{3\beta\gamma}{\beta - \gamma}$, we have

$$\begin{aligned} 1 : \eta : \eta_1 = &\quad P(a^4 + a^2\Lambda) - Q(2a^2 + \Lambda) + 3R \\ &: -R(a^2 + 2\Lambda) - (a^2 + \Lambda)^3(2a^2 - P + \Lambda \\ &: -R(a^2 - \Lambda) - a^6(2a^2 - P + \Lambda). \end{aligned}$$

The first term is

$$a^4 P - 2a^2 Q + 3R + \Lambda(a^2 P - Q),$$

which is

$$= -a^2\beta\gamma + \Lambda(a^4 - b^2c^2);$$

and for the value of η_1, proceeding to the third term, this is

$$-a^2 R - a^6(2a^2 - P) + \Lambda(R - a^6),$$

which is

$$= a^4\beta\gamma - a^2\Lambda(a^4 - b^2c^2),$$

so that without any further reduction $\eta_1 = -a^2$.

42. We have then

$$\eta = \frac{-R(a^2+2\Lambda)-(a^2+\Lambda)^3(a^2-b^2-c^2+\Lambda)}{-a^2\beta\gamma+\Lambda(a^4-b^2c^2)},$$

and I assume

$$\eta = -a^2 + \frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega,$$

and investigate the value of Ω.

We have

$$-R(a^2+2\Lambda)-(a^2+\Lambda)^3(a^2-b^2-c^2+\Lambda), = a^4\beta\gamma+\Lambda\odot, \text{ suppose.}$$

The equation therefore is

$$\frac{a^4\beta\gamma+\Lambda\odot}{-a^2\beta\gamma+\Lambda(a^4-b^2c^2)} = -a^2 + \frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega,$$

that is,

$$\Lambda\odot = -a^2\Lambda(a^4-b^2c^2) + \frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega\{-a^2\beta\gamma+\Lambda(a^4-b^2c^2)\} = 0,$$

or writing $\frac{9\beta\gamma}{(\beta-\gamma)^3} = -\frac{3\Lambda}{(\beta-\gamma)^2}$, omitting the factor Λ, and multiplying by $(\beta-\gamma)^2$, this is

$$(\beta-\gamma)^2\{\odot + a^2(a^4-b^2c^2)\} + 3\Omega\{-a^2\beta\gamma+\Lambda(a^4b^2c^2)\} = 0,$$

in which equation

$$\odot = -2R - a^6 - (3a^4+3a^2\Lambda+\Lambda^2)(a^2-b^2-c^2+\Lambda),$$

and thence

$$\begin{aligned}\odot + a^2(a^4-b^2c^2) &= \text{same } + a^2(a^4-b^2c^2),\\ &= -3a^6+3a^4(b^2+c^2)-3a^2b^2c^2\\ &\quad +\Lambda\{-6a^4+3a^2(b^2+c^2)\}\\ &\quad +\Lambda^2(-4a^2+\quad b^2+c^2)\\ &\quad -\Lambda^3\\ &= 3a^2\beta\gamma+3a^2\Lambda(\beta-\gamma)+\Lambda^2(\beta-\gamma-2a^2)-\Lambda^3.\end{aligned}$$

43. Hence, substituting for Λ its value and multiplying by $(\beta-\gamma)^3$, we have

$$\begin{aligned}&(\beta-\gamma)^3\{\odot + a^2(a^4-b^2c^2)\}\\ &= 3a^2\beta\gamma(\beta-\gamma)^3 - 9a^2\beta\gamma(\beta-\gamma)^3 + 9\beta^2\gamma^2(\beta-\gamma-2a^2)(\beta-\gamma) + 27\beta^3\gamma^3,\end{aligned}$$

which is

$$= -6a^2\beta\gamma(\beta-\gamma)^3 + 9\beta^2\gamma^2(\beta-\gamma)^2 - 18a^2\beta^2\gamma^2(\beta-\gamma) + 27\beta^3\gamma^3;$$

viz. this is

$$\begin{aligned}&= \{-6a^2(\beta-\gamma)+9\beta\gamma\}\{(\beta-\gamma)^2+3\beta\gamma\}\beta\gamma,\\ &= \{-6a^2(\beta-\gamma)+9\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma,\end{aligned}$$

and the equation thus is

$$\{-2a^2(\beta-\gamma)+3\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma+\Omega\left\{-a^2\beta\gamma-\frac{3\beta\gamma}{\beta-\gamma}(a^4-b^2c^2)\right\}(\beta-\gamma)=0,$$

or finally

$$\Omega\{a^2(\beta-\gamma)+3(a^4-b^2c^2)\}=(-2a^2(\beta-\gamma)+3\beta\gamma)(\beta^2+\beta\gamma+\gamma^2).$$

But $c^2=a^2+\beta$, $b^2=a^2-\gamma$, and hence $a^4-b^2c^2=-a^2(\beta-\gamma)+\beta\gamma$, and therefore

$$a^2(\beta-\gamma)+3(a^4-b^2c^2)=-2a^2(\beta-\gamma)+3\beta\gamma;$$

the equation thus divides by $-2a^2(\beta-\gamma)+3\beta\gamma$ and we have

$$\Omega=\beta^2+\beta\gamma+\gamma^2,$$

or, as this may also be written, $\Omega=\alpha^2-\beta\gamma$, $=\beta^2-\gamma\alpha$, $=\gamma^2-\alpha\beta$. So that Ω has the value originally so denoted, and we have then

$$\eta=-a^2+\frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega.$$

44. III. Lastly the equation $0=(a^2+\xi)^3(a^2+\eta)=(a^2+\xi_1)^3(a^2+\eta_1)$ is satisfied if $a^2+\eta=0$, $a^2+\eta_1=0$: the equations

$$(b^2+\xi)^3(b^2+\eta)=(b^2+\xi_1)^3(b^2+\eta_1),$$
$$(c^2+\xi)^3(c^2+\eta)=(c^2+\xi_1)^3(c^2+\eta_1),$$

then give

$$(b^2+\xi)^3=(b^2+\xi_1)^3,$$
$$(c^2+\xi)^3=(c^2+\xi_1)^3,$$

which can be satisfied by $\xi=\xi_1$, leading to $\xi=\xi_1=-a^2$, which is the case I., or else by

$$b^2+\xi=\omega\,(b^2+\xi_1),$$
$$c^2+\xi=\omega^2(c^2+\xi_1),$$

that is,

$$\xi=\omega b^2+\omega^2c^2,\quad \xi_1=\omega^2b^2+\omega c^2.$$

To show that these values satisfy the relation between ξ, ξ_1, observe that they give

$$\xi+\xi_1=-b^2-c^2,\quad \xi\xi_1=b^4-b^2c^2+c^4,$$

whence also

$$\xi^2+4\xi\xi_1+\xi_1^2=3(b^4+c^4),$$

and the relation becomes

$$6a^2b^2c^2-3[a^2(b^2+c^2)+b^2c^2](b^2+c^2)$$
$$+[a^2+(b^2+c^2)].3(b^4+c^4)-3(b^2+c^2)(b^4-b^2c^2+c^4)=0,$$

which is an identity

45. I will show that these values of ξ, ξ_1 give the foregoing values $\eta = \eta_1 = -a^2$. We have

$$\begin{aligned} 1 : \eta - \eta_1 : \eta + \eta_1 &= P\xi\xi_1 + Q(\xi + \xi_1) + 3R \\ &: (\xi_1 - \xi)\{3R - (\xi^2 + \xi\xi_1 + \xi_1^2)(P + \xi + \xi_1)\} \\ &: (\xi_1 + \xi)\{\ R - (\xi^2 - \xi\xi_1 + \xi_1^2)(P + \xi + \xi_1)\}, \end{aligned}$$

this is

$$1 : \eta - \eta_1 : \eta + \eta_1 = a^2(b^2 + c^2) : 0(\xi_1 - \xi) : -2a^2\alpha^2(b^2 + c^2),$$

or

$$\eta - \eta_1 = 0,\ \eta + \eta_1 = -2a^2;\ \text{that is, } \eta = \eta_1 = -a^2.$$

46. For the real umbilicar centres and outcrops we have

I. $$\xi = -b^2,\ \eta = -b^2,\ \xi_1 = -b^2,\ \eta_1 = -b^2.$$

$$X^2 = -a^2\frac{\gamma}{\beta}, \qquad X_1^2 = -a^2\frac{\gamma}{\beta},$$

$$Y = 0, \qquad Y_1 = 0,$$

$$Z^2 = -c^2\frac{\alpha}{\beta}, \qquad Z_1^2 = -c^2\frac{\alpha}{\beta},$$

$$\left.\begin{aligned} a^2x^2 &= -\frac{\gamma^3}{\beta}, \\ y &= 0, \\ c^2z^2 &= -\frac{\alpha^3}{\beta}, \end{aligned}\right\} \text{(real umbilicar centre).}$$

II. $$\xi = -c^2, \qquad \eta = -c^2 + \frac{9\alpha\beta}{(\alpha - \beta)^2}$$

$$\left(\text{or } \eta + a^2 = -\beta\frac{(\gamma - \alpha)^2}{(\alpha - \beta)^2},\ \eta + b^2 = \frac{\alpha(\beta - \gamma)^2}{(\alpha - \beta)^2}\right).$$

$$\xi_1 = -c^2 + \frac{3\alpha\beta}{\alpha - \beta},\ \eta_1 = -c^2,$$

$$\left.\begin{aligned} X^2 &= -a^2\frac{\beta}{\gamma}\frac{(\gamma - \alpha)^3}{(\alpha - \beta)^3}, & X_1^2 &= -a^2\frac{\beta}{\gamma}\frac{\gamma - \alpha}{\alpha - \beta}, \\ Y^2 &= -b^2\frac{\alpha}{\gamma}\frac{(\beta - \gamma)^3}{(\alpha - \beta)^3}, & Y_1^2 &= -b^2\frac{\alpha}{\gamma}\frac{\beta - \gamma}{\alpha - \beta}, \\ Z &= 0, & Z_1 &= 0, \end{aligned}\right\}$$

ellipse concomitant. ellipse sequential.

$$\left.\begin{aligned} a^2x^2 &= -\frac{\beta^3}{\gamma}\frac{(\gamma - \alpha)^3}{(\alpha - \beta)^3} \\ b^2y^2 &= -\frac{\alpha^3}{\gamma}\frac{(\beta - \gamma)^3}{(\alpha - \beta)^3} \\ z &= 0. \end{aligned}\right\} \text{(real outcrop).}$$

Nodal curve in vicinity of umbilicar centre, $a^2x^2 = -\frac{\gamma^3}{\beta}$, $y=0$, $c^2z^2 = -\frac{\alpha^3}{\beta}$. Art. Nos. 47 to 49.

47. Write

$$\xi = -b^2 + q\,, \quad \eta = -b^2 + r\,,$$
$$\xi_1 = -b^2 + q_1, \quad \eta_1 = -b^2 + r_1,$$

we have to find the relation between q, q_1, r, r_1; first for q, q_1, the equation of correspondence gives

$$\begin{aligned} & 6R \\ + & \ 3Q(-2b^2 + q + q_1) \\ + & \ P\{6b^4 - 6b^2(q+q_1) + q^2 + 4qq_1 + q_1^2\} \\ + & \ 3\ \{-2b^6 + 3b^4(q+q_1) - b^2(q^2 + 4qq_1 + q_1^2) + qq_1(q+q_1)\} = 0, \end{aligned}$$

that is,

$$\begin{aligned} & 3(q+q_1)(3b^4 - 2b^2P + Q) \\ + & \ (q^2 + qq_1 + q_1^2)(-3b^2 + P) \\ + & \ 3qq_1(q+q_1) = 0, \end{aligned}$$

viz. this is

$$\begin{aligned} & -3(q+q_1)\alpha\gamma \\ + & \ (q^2 + 4qq_1 + q_1^2)(\gamma - \alpha) \\ + & \ 3qq_1(q+q_1) = 0, \end{aligned}$$

whence approximately $q + q_1 = 0$; but it will appear that the value is required to the second order; we have therefore

$$\begin{aligned} q + q_1 &= \tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(q^2 + 4qq_1 + q_1^2) \\ &= -\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}q^2. \end{aligned}$$

48. Now the equations

$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1), \text{ and } (c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1),$$

putting therein for ξ, η, ξ_1, η_1, their values, give the first of them

$$\log\left(1+\frac{r}{\gamma}\right) + 3\log\left(1+\frac{q}{\gamma}\right) = \log\left(1+\frac{r_1}{\gamma}\right) + 3\log\left(1+\frac{q_1}{\gamma}\right),$$

that is,

$$r + 3q - \frac{1}{2\gamma}(r^2+3q^2) + \frac{1}{3\gamma^2}(r^3+3q^3) = r_1 + 3q_1 - \frac{1}{2\gamma}(r_1^2+3q_1^2) + \frac{1}{3\gamma^2}(r_1^3+3q_1^3);$$

and similarly the second equation

$$r + 3q + \frac{1}{2\alpha}(r^2+3q^2) + \frac{1}{3\alpha^2}(r^3+3q^3) = r_1 + 3q_1 + \frac{1}{2\alpha}(r_1^2+3q_1^2) + \frac{1}{3\alpha^2}(r_1^3+3q_1^3);$$

whence multiplying by γ, α, and adding,

$$(\gamma+\alpha)\left\{r+3q+\frac{1}{3\alpha\gamma}(r^3+3q^3)\right\}=(\gamma+\alpha)\left\{r_1+3q_1+\frac{1}{3\alpha\gamma}(r_1^3+3q_1^3)\right\},$$

which, neglecting terms of the third order, is

$$r+3q=r_1+3q_1.$$

Subtracting the two equations we have

$$\tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)(r^2+3q^2)+\tfrac{1}{3}\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)(r^3+3q^3)=\tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)(r_1^2+3q_1^2)+\tfrac{1}{3}\left(\frac{1}{\alpha^3}-\frac{1}{\gamma^3}\right)(r_1^3+3q_1^3),$$

viz. this is

$$r^2+3q^2+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(r^3+3q^3)=r_1^2+3q_1^2+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(r_1^3+3q_1^3),$$

or, what is the same thing,

$$r^2-r_1^2+3(q^2-q_1^2)+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}\{r^3-r_1^3+3(q^3-q_1^3)\}=0,$$

which, putting therein $r-r_1=-3(q-q_1)$, is

$$-r-r_1+q+q_1+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-r^2-rr_1-r_1^2+q^2+qq_1+q_1^2)=0,$$

say this is

$$-r-r_1+q+q_1+2\Delta=0;$$

combining herewith

$$r-r_1+3q-3q_1\quad=0,$$

we have

$$r+q-2q_1-\Delta=0,$$

and

$$r_1-2q+q_1-\Delta=0,$$

where

$$\Delta=\tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-r^2-rr_1-r_1^2+q^2+qq_1+q_1^2).$$

But substituting herein the values $r=-q+2q_1$, $r_1=2q-q_1$, this becomes

$$\Delta=\tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-2q^2+4qq_1-2q_1^2),\ =-\tfrac{8}{3}\frac{\gamma-\alpha}{\gamma\alpha}q^2,$$

and then

$$r=-q+2q_1+\Delta,$$

that is,

$$r+3q=2(q+q_1)+\Delta,\ =-\frac{4(\gamma-\alpha)}{\gamma\alpha}q^2.$$

49. We have then

$$a^2x^2 = -\frac{\gamma^3}{\beta}\left(1+\frac{r}{\gamma}\right)\left(1+\frac{q}{\gamma}\right)^3,$$

$$= -\frac{\gamma^3}{\beta}\left(1+\frac{r+3q}{\gamma}+\frac{3q(r+q)}{\gamma^2}\right),$$

$$= -\frac{\gamma^3}{\beta}\left(1+\frac{r+3q}{\gamma^2}-\frac{6}{\gamma^2}q^2\right)$$

$$= -\frac{\gamma^3}{\beta}\left\{1+q^2\left(-\frac{4(\gamma-\alpha)}{\gamma^2\alpha}-\frac{6\alpha}{\gamma^2\alpha}\right)\right\}$$

$$= -\frac{\gamma^3}{\beta}\left\{1+q^2\frac{2(\beta-\gamma)}{\gamma^2\alpha}\right\};$$

and in the same way from $c^2z^2 = -\frac{\alpha^3}{\beta}\left(1-\frac{r}{\alpha}\right)\left(1-\frac{r}{\alpha}\right)^3$, we have

$$c^2z^2 = -\frac{\alpha^3}{\beta}\left\{1-q^2\frac{2(\alpha-\beta)}{\gamma\alpha^2}\right\}:$$

moreover we have at once

$$b^2y^2 = -\frac{q^3r}{\gamma\alpha} = \frac{3q^4}{\gamma\alpha}.$$

Hence, writing $x+\delta x$, $0+\delta y$, $z+\delta z$ for x, y, z, we find

$$\delta x = \tfrac{1}{2}x\,.\,\frac{2(\beta-\gamma)}{\gamma^2\alpha}\,.\,q^2,$$

$$\delta y = \pm\frac{1}{b}\sqrt{\frac{3}{\gamma\alpha}}\,.\,q^2,$$

$$\delta z = \tfrac{1}{2}z\,.\,\frac{-2(\alpha-\beta)}{\gamma\alpha^2}\,.\,q^2,$$

or, what is the same thing,

$$\delta x : \delta y : \delta z = x\frac{2(\beta-\gamma)}{\gamma^2\alpha} : \pm\frac{2}{b}\sqrt{\frac{3}{\gamma\alpha}} : z\frac{-2(\alpha-\beta)}{\gamma\alpha^2},$$

where x, z denote the values at the umbilicar centre.

Nodal curve in vicinity of real outcrop, viz.

$$a^2x^2 = -\frac{\beta^3(\gamma-\alpha)^3}{\gamma(\alpha-\beta)^3},\quad b^2y^2 = -\frac{\alpha^3(\alpha-\gamma)^3}{\gamma(\alpha-\beta)^3},\quad z=0.$$ Art. Nos. 50 to 52.

50. Write

$$\xi = -c^2+q, \qquad \eta = -c^2+\frac{q\alpha\beta\Omega}{(\alpha-\beta)^3}+\theta,$$

$$\xi_1 = -c^2+\frac{3\alpha\beta}{\alpha-\beta}+q_1, \qquad \eta_1 = -c^2+\theta_1;$$

and first for the relation between q and q_1, writing for a moment $\frac{3\alpha\beta}{\alpha-\beta}+q_1=Q_1$, and therefore $\xi_1=-c^2+Q_1$, the equation of correspondence gives

$$-3\alpha\beta(q+Q_1)+(q^2+4qQ_1+Q_1^2)(\alpha-\beta)+3qQ_1(q+Q_1)=0,$$

which, putting for Q_1 its value, is

$$\begin{aligned}&-3\alpha\beta\left(q+q_1+\frac{3\alpha\beta}{\alpha-\beta}\right)\\&+(\alpha-\beta)\left(q^2+4qq_1+q_1^2+(4q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right)\\&+3q\left\{q(q+q_1)+(q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right\}=0;\end{aligned}$$

that is,

$$\begin{aligned}&-3\alpha\beta(q+q_1)\\&+3\alpha\beta(4q+2q_1)+(\alpha-\beta)(q^2+4qq_1+q_1^2)\\&+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}q+\frac{9\alpha\beta}{\alpha-\beta}(q^2+2qq_1)+3qq_1(q+q_1)=0,\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}&\left(9\alpha\beta+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}\right)q+3\alpha\beta q_1\\&+q^2\left\{\frac{\alpha^2+7\alpha\beta+\beta^2}{\alpha-\beta}+qq_1\frac{\alpha^2+16\alpha\beta+\beta^2}{\alpha-\beta}+q_1^2(\alpha-\beta)\right\}\\&+3qq_1(q+q_1)=0,\end{aligned}$$

or, for small values,

$$\left(3+\frac{9\alpha\beta}{(\alpha-\beta)^2}\right)q+q_1=0,\text{ that is, }\frac{3\Omega}{(\alpha-\beta)^2}q+q_1=0.$$

51. Moreover, from the equation $(c^2+\xi)^3(c^2+\eta)=(c^2+\xi_1)^3(c^2+\eta_1)$, we have

$$q^3\frac{9\alpha\beta\Omega}{(\alpha-\beta)^2}=\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3.\theta_1,\text{ that is, }\theta_1=\tfrac{1}{3}\frac{\Omega}{\alpha^2\beta^2}.q^3,$$

or, since q and q_1 are of the same order, θ_1 is of the order q_1^3. Hence, starting from the equations $-\beta\gamma a^2x^2=(a^2+\xi_1)^3(a^2+\eta_1)$ &c., the terms of x, y arising from the variation of η_1 are indefinitely small in regard to those arising from the variation of ξ_1; and we have

$$\frac{2\delta x}{x}=\frac{3q_1}{-\beta+\frac{3\alpha\beta}{\alpha-\beta}},=-3q_1\frac{(\alpha-\beta)}{\beta(\gamma-\alpha)},$$

$$\frac{2\delta y}{y}=\frac{3q_1}{\alpha+\frac{3\alpha\beta}{\alpha-\beta}},=3q_1\frac{\alpha-\beta}{\alpha(\beta-\gamma)},$$

and for δz $(=z)$ we have

$$c^2(\delta z)^2 = -\frac{1}{\alpha\beta}\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3\theta_1, \quad = \frac{-9\Omega q^3}{(\alpha-\beta)^3}, \quad = \frac{(\alpha-\beta)^3}{3\Omega^2}\,q_1^3,$$

so that writing for greater simplicity, $(\alpha-\beta)\,q_1 = -\alpha\beta\varpi$, the formulæ become

$$\frac{2\delta x}{x} = \frac{3\alpha}{\gamma-\alpha}\,\varpi,$$

$$\frac{2\delta y}{y} = -\frac{3\beta}{\beta-\gamma}\,\varpi,$$

$$c\delta z = \frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{\Omega\sqrt{3}}.$$

52. This shows that there is at the outcrop a cusp, the cuspidal tangent being in the plane of xy. It appears moreover that this tangent coincides with the tangent of the evolute. In fact, from the equation $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} - \gamma^{\frac{2}{3}} = 0$ of the evolute we have

$$\frac{(ax)^{\frac{2}{3}}.\,dx}{x} + \frac{(by)^{\frac{2}{3}}.\,dy}{y} = 0,$$

or substituting for (x, y) their values at the outcrop,

$$\frac{\beta(\gamma-\alpha)}{\gamma^{\frac{1}{3}}(\alpha-\beta)}\,\frac{dx}{x} + \frac{\alpha(\beta-\gamma)}{\gamma^{\frac{1}{3}}(\alpha-\beta)}\,\frac{dy}{y} = 0;$$

that is,

$$\beta(\gamma-\alpha)\frac{dx}{x} + \alpha(\beta-\gamma)\frac{dy}{y} = 0,$$

which is satisfied by the foregoing values of $\frac{\delta x}{x}$, and $\frac{\delta y}{y}$, and the two tangents therefore coincide.

We have

$$4\{(\delta x)^2 + (\delta y)^2\} = \frac{-9\varpi^2\alpha^2\beta^2}{\gamma(\alpha-\beta)^3}\cdot\left\{\frac{\beta(\gamma-\alpha)}{a^2} + \frac{\alpha(\beta-\gamma)}{b^2}\right\},$$

which in virtue of

$$a^2\alpha(\beta-\gamma) + b^2\beta(\gamma-\alpha) + c^2\gamma(\alpha-\beta) = 3\alpha\beta\gamma,$$

is

$$4\{(\delta x)^2 + (\delta y)^2\} = \frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2(\alpha-\beta)^3}\{3\alpha\beta - c^2(\alpha-\beta)\}$$

(observe $3\alpha\beta - c^2(\alpha-\beta)$, $= -c^2(\gamma-\alpha) - 3a^2\alpha$, is negative)

$$= \frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2(\alpha-\beta)^2}\,\xi_1,$$

if ξ_1 be the value at the outcrop. Writing δs for the element of the arc we have

$$\delta s = -\tfrac{3}{2}\frac{\alpha\beta}{ab(\alpha-\beta)}\sqrt{-\xi_1}\,.\,\varpi,$$

$$\delta z = \frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{c\,\Omega\sqrt{3}},$$

which exhibit the form at the outcrop.

The Nodal Curve; expressions for the coordinates in terms of a single parameter σ.
Art. Nos. 53 to 60.

53. After the foregoing investigation of the nodal curve, I was led to perceive that it is possible to express ξ, η, ξ_1, η_1 in terms of a single variable σ, and thus to obtain expressions for the coordinates of a point of the nodal curve in terms of the single variable σ. The result was obtained by the consideration that the acnodal portion of the nodal curve could only arise from imaginary values of ξ, η; the question thus was, what imaginary values of these quantities give real values for the coordinates x, y, z. To make y real we may assume

$$\xi = -b^2 - p(\theta - \phi i),$$
$$\eta = -b^2 + p(\theta + \phi i)^2,$$

($i = \sqrt{-1}$ as usual): this being so, if Δ denote one or other of the quantities

$$\gamma,\ -\alpha\,(= a^2 - b^2,\ c^2 - b^2),$$

the expressions for $-\beta\gamma a^2x^2$, $-\gamma\alpha b^2y^2$ will be

$$= \{\Delta - p(\theta - \phi i)\}^2\,\{\Delta + p(\theta + \phi i)^2\},$$

and we have therefore the condition that this shall be real (for the two values $\Delta = \gamma$, $\Delta = -\alpha$): being real, it will in certain cases be positive, and we shall then have real values for the remaining coordinates x, z.

54. The condition of reality is easily found to be

$$\Delta^2(3\theta^2 - \phi^2 + 3) - 6\theta p\Delta(\theta^2 + \phi^2 + 1) + p^2\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\} = 0,$$

viz. this equation in Δ must have the roots γ, $-\alpha$, or the expression on the left hand must be

$$= (3\theta^2 - \phi^2 + 3)\{\Delta^2 - (\gamma - \alpha)\Delta - \alpha\gamma\}:$$

we have therefore

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha} = \frac{36(\theta^2 + \phi^2 + 1)^2}{(3\theta^2 - \phi^2 + 3)\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\}},$$

$$\gamma - \alpha = \frac{6\theta p(\theta^2 + \phi^2 + 1)}{3\theta^2 - \phi^2 + 3};$$

and writing $\theta^2+\phi^2=X$, $3\theta^2-\phi^2=Y$, the first of these is

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha}=\frac{9(X+Y)(X+1)^2}{(Y+3)\{3(X^2-1)+Y+3\}},$$

which regarding X, Y as the coordinates of a point in a plane is a cubic curve having the point $X+1=0$, $Y+3=0$ as a node: hence writing $Y+3=3\sigma(X+1)$, X and Y will be each of them a rational function of σ. The second equation is

$$\frac{6\theta p(X+1)}{Y+3}=\gamma-\alpha, \text{ that is, } p=\frac{(\gamma-\alpha)\sigma}{2\theta}, \; =\frac{(\gamma-\alpha)\sigma}{\sqrt{X+Y}};$$

and we have also

$$2\theta=\sqrt{X+Y}, \quad 2\phi=\sqrt{3X-Y};$$

the equations thus become

$$\xi=-b^2-\frac{(\gamma-\alpha)\sigma}{2}\left\{1-i\sqrt{\frac{3X-Y}{X+Y}}\right\},$$

$$\eta=-b^2+\frac{(\gamma-\alpha)\sigma(X+Y)}{8}\left\{1+i\sqrt{\frac{3X-Y}{X+Y}}\right\}^3,$$

which are better written in the form

$$\xi=-b^2-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{-3X+Y}{X+Y}}\right\},$$

$$\eta=-b^2+\tfrac{1}{8}(\gamma-\alpha)\sigma(X+Y)\left\{1+\sqrt{\frac{-3X+Y}{X+Y}}\right\}^3,$$

where X, Y are given functions of σ. We in fact thus obtain an analytical expression of the nodal curve, quite independent of the considerations as to real and imaginary which suggested the process: the foregoing values substituted for ξ, η will give $-\beta\gamma a^2x^2$, &c. equal to rational functions of σ, so that taking for ξ_1, η_1 the same expressions, only changing therein the sign of the radical $\sqrt{\frac{-3X+Y}{X+Y}}$, these values of ξ_1, η_1 give the very same values of $-\beta\gamma a^2x^2$, &c., or the values of ξ, η, ξ_1, η_1 satisfy the conditions

$$(a^2+\xi)^2(a^2+\eta)=(a^2+\xi_1)^2(a^2+\eta_1), \text{ \&c.}$$

for a point on the nodal curve.

55. To complete the investigation, writing as above $Y+3=3\sigma(X+1)$, we obtain

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha}=\frac{(3\sigma+1)X+3\sigma-3}{\sigma(X+\sigma-1)};$$

or putting for a moment

$$\frac{(\gamma-\alpha)^2\sigma}{-\gamma\alpha}=K,$$

we have

$$X = \frac{(K-3)(\sigma-1)}{3\sigma+1-K}, \qquad X+1 = \frac{K(\sigma-2)+4}{3\sigma+1-K};$$

$$Y+3 = \frac{3K\sigma(\sigma-2)+12\sigma}{3\sigma+1-K}, \qquad Y = \frac{3(\sigma-1)\{K(\sigma-1)+1\}}{3\sigma+1-K};$$

$$X+Y = \frac{(\sigma-1)(3\sigma-2)K}{3\sigma+1-K}, \quad -3X+Y = \frac{3(\sigma-1)\{K(\sigma-2)+4\}}{3\sigma+1-K};$$

or substituting for K its value we have

$$K(\sigma-2)+4 = -\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left\{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}$$

$$= -\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right),$$

$$3\sigma+1-K = \frac{1}{\gamma\alpha}\{(3\sigma+1)\gamma\alpha+(\gamma-\alpha)^2\sigma\}, \quad = \frac{1}{\gamma\alpha}(\Omega\sigma+\gamma\alpha),$$

if as before $\Omega = \beta^2-\gamma\alpha$; and the result is

$$\xi = -b^2-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\},$$

$$\eta = -b^2-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3,$$

and changing the sign of the radical we have the values of ξ_1, η_1.

56. Write for a moment

$$\left[\Delta-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}\right] = (\Delta-\mathrm{a}+\mathrm{a}\sqrt{S})^3 = A+B\sqrt{S},$$

$$\Delta-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3 = A'+B'\sqrt{S};$$

then in the product of these two expressions the rational part is $= AA'+BB'S$; but from the manner in which they were arrived at we have $0 = AB'+A'B$, and the rational part is thus

$$= -\frac{B'}{B}(A^2-B^2S).$$

We have

$$A^2 - B^2 S = \{(\Delta - \mathrm{a})^2 - \mathrm{a}^2 S\}^3,$$

$$B' = -\tfrac{1}{2}(\gamma - \alpha)^3 \frac{\sigma^2(\sigma - 1)(3\sigma - 2)}{\Omega\sigma + \gamma\alpha}(3 + S),$$

$$B = \tfrac{1}{2}(\gamma - \alpha)\{3(\Delta - \mathrm{a})^2 + \mathrm{a}^2 S\};$$

hence the rational part in question is

$$= \tfrac{1}{4}\frac{(\gamma - \alpha)^2}{\Omega\sigma + \gamma\alpha}\,\frac{\sigma(\sigma - 1)(3\sigma - 2)(3 + S)}{3(\Delta - \mathrm{a})^2 + \mathrm{a}^2 S}\{(\Delta - \mathrm{a})^2 + \mathrm{a}^2 S\}^3,$$

which putting therein $\Delta = 0$ gives the value of $-\gamma\alpha b^2 y^2$; and putting $\Delta = \gamma$, or $\Delta = -\alpha$, gives the value of $-\beta\gamma a^2 x^2$ or $-\alpha\beta c^2 z^2$.

57. We have

$$1 - S = \frac{1}{\sigma(3\sigma - 2)}\left[3\sigma^2 - 2\sigma - 3\left\{\sigma^2 - 2\sigma - \frac{4\gamma\alpha}{(\gamma - \alpha)^2}\right\}\right]$$

$$= \frac{4\left\{\sigma + \dfrac{3\alpha\gamma}{(\gamma - \alpha)^2}\right\}}{\sigma(3\sigma - 2)},$$

$$3 + S = \frac{3}{\sigma(3\sigma - 2)}\left[3\sigma^2 - 2\sigma + \left\{\sigma^2 - 2\sigma - \frac{4\alpha\gamma}{(\gamma - \alpha)^2}\right\}\right]$$

$$= \frac{12}{\sigma(3\sigma - 2)}\left(\sigma + \frac{\alpha}{\gamma - \alpha}\right)\left(\sigma - \frac{\gamma}{\gamma - \alpha}\right).$$

Hence we have at once the value of

$$-\gamma\alpha b^2 y^2, \quad = \tfrac{1}{4}\frac{(\gamma - \alpha)^2}{\Omega\sigma + \gamma\alpha}\cdot\frac{\sigma(\sigma - 1)(3\sigma - 2)}{\mathrm{a}^2}\mathrm{a}^6(1 - S)^3,$$

where

$$\mathrm{a} = \tfrac{1}{2}(\beta - \gamma)\sigma.$$

58. Moreover

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2 S = \Delta^2 - 2\mathrm{a}\Delta + \mathrm{a}^2(1 - S)$$

$$= \frac{1}{3\sigma - 2}[(3\sigma - 2)\{-(\gamma - \alpha)\Delta\sigma + \Delta^2\} + (\gamma - \alpha)^2\sigma^2 + 3\alpha\gamma],$$

where the term in [ ] is

$$\sigma^2(\gamma - \alpha)(\gamma - \alpha - 3\Delta) + \sigma\{3\Delta^2 + 2(\gamma - \alpha)\Delta + 3\alpha\gamma\} - 2\Delta^2,$$

and since $\Delta = \gamma$ or $-\alpha$, that is, $\Delta^2 - (\gamma - \alpha)\Delta - \alpha\gamma = 0$, the coefficient of σ is

$$= \Delta\{6\Delta - (\gamma - \alpha)\},$$

or the term is the product of two linear functions of σ; and we have

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2 S = \frac{1}{3\sigma - 2}\{\sigma(\gamma - \alpha) - 2\Delta\}\{\sigma(\gamma - \alpha - 3\Delta) + \Delta\}.$$

Similarly

$$3(\Delta - \mathrm{a})^2 + \mathrm{a}^2 S = 3(\Delta^2 - 2\mathrm{a}\Delta) + \mathrm{a}^2(3 + S)$$

$$= \frac{3}{3\sigma - 2}\left[(3\sigma - 2)\{-(\gamma - \alpha)\sigma\Delta + \Delta^2\} + \sigma\{(\gamma - \alpha)\sigma + \alpha\}\{(\gamma - \alpha)\sigma - \gamma\}\right],$$

where the term in [ ] is

$$(\gamma - \alpha)^2\sigma^3 - (\gamma - \alpha)(\gamma - \alpha + 3\Delta)\sigma^2 + \{3\Delta^2 + 2(\gamma - \alpha)\Delta - \alpha\gamma\}\sigma - 2\Delta^2,$$

in which the coefficient of σ is $= \Delta\{2\Delta + 3(\gamma - \alpha)\}$, and the term is a product of three linear functions: hence

$$3(\Delta - \mathrm{a})^2 + \mathrm{a}^2 S = \frac{3(\sigma - 1)}{3\sigma - 2}\{(\gamma - \alpha)\sigma - \Delta\}\{(\gamma - \alpha)\sigma - 2\Delta\}.$$

59. Substituting these values we have the expression

$$\frac{1}{\Omega\sigma + \gamma\alpha}\,\frac{\{(\gamma - \alpha)\sigma + \alpha\}\{(\gamma - \alpha)\sigma - \gamma\}\{(\gamma - \alpha)\sigma - 2\Delta\}^2\{(\gamma - \alpha - 3\Delta)\sigma + \Delta\}^3}{\{(\gamma - \alpha)\sigma - \Delta\}(3\sigma - 2)^2};$$

which writing therein $\Delta = \gamma$ gives $-\beta\gamma\alpha^2x^2$, and writing $\Delta = -\alpha$ gives $-\alpha\beta c^2z^2$; we have above an expression for $-\gamma\alpha b^2y^2$ requiring only a simple reduction, and the final results are

$$-\beta\gamma\alpha^2x^2 = \frac{\{(\gamma - \alpha)\sigma + \alpha\}\{(\gamma - \alpha)\sigma - 2\gamma\}^2\{(\beta - \gamma)\sigma + \gamma\}^3}{(\Omega\sigma + \gamma\alpha)(3\sigma - 2)^2},$$

$$-\gamma\alpha b^2y^2 = \frac{(\sigma - 1)\sigma^2\{(\gamma - \alpha)^2\sigma + 3\alpha\gamma\}^3}{(\Omega\sigma + \gamma\alpha)(3\sigma - 2)^2},$$

$$-\alpha\beta c^2z^2 = \frac{\{(\gamma - \alpha)\sigma - \gamma\}\{(\gamma - \alpha)\sigma + 2\alpha\}^2\{(\alpha - \beta)\sigma - \alpha\}^3}{(\Omega\sigma + \gamma\alpha)(3\sigma - 2)^2},$$

where it is to be observed that, equating the denominator to 0, we have a triple root $\sigma = \infty$; to indicate this, we may insert in the denominator the factor $(1 - 0\sigma)^3$.

60. We see here the meaning of all the factors, viz.

Planes.

	$x = 0$	$y = 0$	$z = 0$	∞
Evolute nodes	$\sigma = -\dfrac{\alpha}{\gamma - \alpha}$	$\sigma = 1$	$\sigma = \dfrac{\gamma}{\gamma - \alpha}$	$\sigma = \dfrac{-\gamma\alpha}{\Omega}$
Umbilicar centres	$\sigma = \dfrac{2\gamma}{\gamma - \alpha}$	$\sigma = 0$	$\sigma = \dfrac{-2\alpha}{\gamma - \alpha}$	$\sigma = \frac{2}{3}$
Outcrops	$\sigma = \dfrac{-\gamma}{\beta - \gamma}$	$\sigma = \dfrac{-3\gamma\alpha}{(\gamma - \alpha)^2}$	$\sigma = \dfrac{\alpha}{\alpha - \beta}$	$\sigma = \infty$

For the real curve σ extends from $\frac{\alpha}{\alpha-\beta}$ through 0 to $-\frac{\gamma\alpha}{\Omega}$, viz.

$$\sigma = \frac{\alpha}{\alpha-\beta} \text{ gives outcrop in plane } z=0,$$

$$\sigma = \quad 0 \quad \text{,, umbilicar centre in plane } y=0,$$

$$\sigma = \frac{-\gamma\alpha}{\Omega} \quad \text{,, evolute-node in plane } \infty.$$

It is to be noticed that the order of magnitude of the terms in the table is

$$\infty,\ \frac{2\gamma}{\gamma-\alpha},\ \frac{\gamma}{\gamma-\alpha},\ 1,\ \tfrac{2}{3},\ \frac{-\gamma}{\beta-\gamma},\ \frac{\alpha}{\alpha-\beta},\ 0,\ \frac{-\gamma\alpha}{\Omega},\ \frac{-\alpha}{\gamma-\alpha},\ \frac{-2\alpha}{\gamma-\alpha},\ \frac{-3\gamma\alpha}{(\gamma-\alpha)^2},\ -\infty,$$

so that the values $\frac{\alpha}{\alpha-\beta}$, 0, $\frac{-\gamma\alpha}{\Omega}$ which belong to the real curve are contiguous; this is as it should be. Several of the preceding investigations conducted by means of the quantities ξ, η, ξ_1, η_1 might have been conducted more simply by means of the formulæ involving σ.

The Eight Cuspidal Conics. Art. Nos. 61 to 71.

61. The centro-surface is the envelope of the quadric

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0.$$

Hence it has a cuspidal curve given by means of this equation and the first and second derived equations

$$\frac{a^2x^2}{(a^2+\xi)^3}+\frac{b^2y^2}{(b^2+\xi)^3}+\frac{c^2z^2}{(c^2+\xi)^3}=0,$$

$$\frac{a^2x^2}{(a^2+\xi)^4}+\frac{b^2y^2}{(b^2+\xi)^4}+\frac{c^2z^2}{(c^2+\xi)^4}=0,$$

which equations determine a^2x^2, b^2y^2, c^2z^2 in terms of ξ, viz. we have

$$-\beta\gamma\, a^2x^2=(a^2+\xi)^4,$$

$$-\gamma\alpha\, b^2y^2=(b^2+\xi)^4,$$

$$-\alpha\beta\, c^2z^2=(c^2+\xi)^4;$$

so that, comparing with the equations $-\beta\gamma\, a^2x^2=(a^2+\xi)^3(a^2+\eta)$ &c. which give the centro-surface, we see that for the cuspidal curve $\xi=\eta$; or the cuspidal curve now in question arises from the eight lines on the ellipsoid, which lines are the envelope of the curves of curvature: it is clear that the curve is imaginary.

62. From the foregoing equations we have:

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \sqrt{-\alpha\beta\gamma},$$

$$\alpha^{\frac{3}{2}} \sqrt{ax} + \beta^{\frac{3}{2}} \sqrt{by} + \gamma^{\frac{3}{2}} \sqrt{cz} = 0,$$

the second of which is best written in the rationalised form

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1)\,(\alpha\sqrt{\alpha}\, ax,\ \beta\sqrt{\beta}\, by,\ \gamma\sqrt{\gamma}\, cz)^2 = 0,$$

and combining herewith the equation

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \sqrt{-\alpha\beta\gamma},$$

then for any given signs of $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{-\alpha\beta\gamma}$ the first of these equations represents a quadric surface, the second a plane, or the two equations together represent a conic.

By changing the signs of the radicals (observing that when all the signs are changed simultaneously the curve is unaltered) we obtain in all 8 conics, but only four quadric surfaces; viz. the two conics

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \pm\sqrt{-\alpha\beta\gamma}$$

lie on the same quadric surface.

63. The conics form two sets of four, corresponding to the two sets of four lines on the ellipsoid. The analysis seems to establish a correspondence of each conic of the one set to a single conic of the other set; viz. the conics have been obtained in pairs as the intersections of the same quadric surface by a pair of planes: there is a like correspondence of each line of the one set to a single line of the other set, viz. the lines meet in pairs on the umbilici at infinity, but this correspondence is included in a more general property: in fact each line of the one set meets each line of the other set in an umbilicus; and the corresponding conics (not only meet but) touch at the corresponding umbilicar centre; and *quà* touching conics they have two points of intersection, and consequently lie on the same quadric surface. It is to be added that the two conics touch also, at the umbilicar centre, the cuspidal conic of the principal section.

64. The 8 conics form two tetrads, and the principal conics (reckoning as one of them the conic at infinity) another tetrad: the complete cuspidal curve consists therefore of three tetrads of conics: with these we may form (one conic out of each tetrad) 16 triads; viz. each conic of one tetrad is combined with each conic of either of the other tetrads, and with a determinate conic of the third tetrad, to form a triad. And the conics of each triad, not only meet but touch at an umbilicar centre, the common tangent being also by what precedes, the tangent of the evolute at that point, which point is also a node of the nodal curve.

65. In fact consider the two conics

$$\sqrt{\alpha}\,ax \pm \sqrt{\beta}\,by + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1)\,(\alpha\sqrt{\alpha}\,ax,\ \pm\beta\sqrt{\beta}\,by,\ \gamma\sqrt{\gamma}\,cz)^2 = 0;$$

for the intersections with the plane $y=0$ we have

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz)^2 = 0;$$

so that the two conics each meet the plane in question in the same two coincident points, that is, they each touch the plane $y=0$ at the same point, viz. the point given by the equations

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz = 0;$$

viz. this is the point, $ax - \dfrac{\gamma\sqrt{\gamma}}{\sqrt{-\beta}}$, $cz = \dfrac{\alpha\sqrt{\alpha}}{\sqrt{-\beta}}$, which is one of the umbilicar centres

$$\left(a^2x^2 = -\frac{\gamma^3}{\beta},\quad c^2z^2 = -\frac{\alpha^3}{\beta}\right);$$

and the common tangent at this point is

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

which is also the common tangent of the ellipse and evolute in the plane $y=0$.

66. It has been seen that the nodal curve meets each principal conic at four outcrops, which points are cusps of the nodal curve: it is to be further shown that the nodal curve meets each of the 8 cuspidal conics in four points (giving 32 new points, which may be called 'outcrops,' the 16 points heretofore so called being distinguished as the principal outcrops or 16 outcrops, and the points now in question as the 32 outcrops), which points are cusps of the nodal curve.

In fact to obtain the intersections of the nodal curve with the 8 cuspidal conics, we must in the equation of the nodal curve, or (what is the same thing) in the expressions of ξ, η in terms of σ, write $\eta = \xi$.

67. Putting for shortness,

$$\Theta = \tfrac{1}{4}\frac{(\gamma-\alpha)^2\,\sigma\,(\sigma-1)\,(3\sigma-2)}{\Omega\sigma+\gamma\alpha},$$

and as before

$$S = \frac{3\left(\sigma + \dfrac{2\alpha}{\gamma-\alpha}\right)\left(\sigma - \dfrac{2\gamma}{\gamma-\alpha}\right)}{\sigma\,(3\sigma-2)},$$

we have thus

$$\Theta\,(1+\sqrt{S})^3 = 1 - \sqrt{S},$$

or, what is the same thing,

$$\Theta(1+3S)-1+\sqrt{S}\{\Theta(3+S)+1\}=0:$$

we have without difficulty

$$\Theta(3+S)+1=\frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\},$$

$$\Theta(1+3\sqrt{S})-1=\frac{(\gamma-\alpha)^2(3\sigma-2)\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\Omega\sigma+\gamma\alpha},$$

so that the resulting equation contains the factor

$$\sqrt{\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)},\quad =\sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}}.$$

Omitting it, the equation becomes

$$\sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}}(3\sigma-2)^{\frac{3}{2}}\sqrt{\sigma}+\sqrt{3}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}^2=0,$$

or putting for shortness $\frac{4\gamma\alpha}{(\gamma-\alpha)^2}=M$, and rationalising, this is

$$-(\sigma^2-2\sigma-M)(3\sigma-2)^3\sigma+3(3\sigma^3-6\sigma^2+4\sigma+M)^2=0,$$

and, working this out, the terms in σ^6, σ^5 disappear, and the result is

$$(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2=0,$$

or, as this may also be written,

$$3M^2+M(27\sigma^4-36\sigma^3+16\sigma)+4(9\sigma^4-16\sigma^3+8\sigma^2)=0,$$

a quartic equation in σ: to each of the 4 roots there correspond 8 intersections, viz. there will be in all 32 intersections, lying in 4's upon the 8 cuspidal conics.

68. To show that these points are cusps, or stationary points on the nodal curve, starting from the expressions of $-\beta\gamma a^2x^2$ &c. in terms of σ we have, first for dy,

$$-\frac{2dy}{y}=d\sigma\left\{\frac{1}{\sigma-1}+\frac{2}{\sigma}+\frac{3(\gamma-\alpha)^2}{(\gamma-\alpha)^2\sigma+3\alpha\gamma}-\frac{\Omega}{\Omega\sigma+\gamma\alpha}-\frac{6}{3\sigma-2}\right\},$$

or, as this may be written,

$$-\frac{2dy}{y}=d\sigma\left\{\frac{1}{\sigma-1}+\frac{2}{\sigma}+\frac{12}{4\sigma+3M}-\frac{4+3M}{(4+3M)\sigma+M}-\frac{6}{3\sigma-2}\right\},$$

$$=d\sigma\left\{\frac{3\sigma^2-6\sigma+4}{3\sigma^3-5\sigma^2+2\sigma}+\frac{(32+24M)\sigma-9M^2}{(16+12M)\sigma^2+(16M+9M^2)\sigma+3M^2}\right\},$$

$$=d\sigma\,\frac{4\{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2\}}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}},$$

viz. the numerator vanishes when σ is a root of the quartic equation.

69. We have next

$$-\frac{2dx}{x}=d\sigma\left\{\frac{\gamma-\alpha}{(\gamma-\alpha)\sigma+\alpha}+\frac{2(\gamma-\alpha)}{(\gamma-\alpha)\sigma-2\gamma}+\frac{3(\beta-\gamma)}{(\beta-\gamma)\sigma+\gamma}-\frac{\Omega}{\Omega\sigma+\gamma\alpha}-\frac{6}{3\sigma-2}\right\},$$

which, putting $\frac{\gamma}{\gamma-\alpha}=B$, and therefore $\frac{\alpha}{\gamma-\alpha}=B-1$, and $\frac{\gamma}{\beta-\gamma}=C$, is

$$=d\sigma\left\{\frac{1}{\sigma+B-1}+\frac{2}{\sigma-2B}+\frac{3}{\sigma+C}-\frac{4+3M}{(4+3M)\sigma+M}-\frac{6}{3\sigma-2}\right\},$$

and adding the fractions except $\frac{3}{\sigma+C}$, the numerator is

$$\begin{aligned}&\sigma^2(27MB+36B-4)\\&+\sigma\{54(B^2-B)M+72B^3-80B+8\}\\&+\quad 4M-16B^3+16B,\end{aligned}$$

which, observing that $B^2-B=\frac{1}{4}M$, is

$$\begin{aligned}=\quad&\sigma^2(27MB+36B-4)\\&+\sigma(\tfrac{27}{2}M^2+18M-8B+8),\end{aligned}$$

and, substituting for M and B their values, this is found to be

$$=\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma^2+\frac{8(2\gamma+\alpha)^3\alpha}{(\gamma-\alpha)^4}\sigma,$$

$$=\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right).$$

70. Hence observing that $C=\frac{\gamma}{\beta-\gamma}=\frac{-\gamma}{2\gamma+\alpha}$, the whole coefficient of $d\sigma$ is

$$=\frac{\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\left(\sigma^2+\frac{2\alpha}{\gamma-\alpha}\sigma\right)}{(3\sigma-2)(\sigma+B-1)(\sigma-2B)[(3M+4)\sigma+M]}+\frac{3}{\sigma+C},$$

and the numerator of this expressed as a single fraction is

$$\begin{aligned}&=\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)[(2\gamma+\alpha)\sigma-\gamma]\\&+3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M-B\sigma)\{(3M+4)\sigma+M\},\end{aligned}$$

which is

$$\begin{aligned}&=3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M)\{(3M+4)\sigma+M\}\\&+\sigma\Bigg[-3B(3\sigma-2)\{(3M+4)\sigma+M\}\\&\quad+\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^2}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\{(2\gamma+\alpha)\sigma-\gamma\}\Bigg]:\end{aligned}$$

the term in [ ] is

$$\begin{aligned} &= \sigma^3\left[-(27M+36)B+\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\right] \\ &+\sigma\left[3B(3M+8)+\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^3}\left(-\gamma+\frac{2\alpha(2\gamma+\alpha)}{\gamma-\alpha}\right)\right] \\ &+\left[6BM-\frac{8(2\gamma+\alpha)^2\gamma\alpha}{(\gamma-\alpha)^4}\right], \end{aligned}$$

which is found to be

$$=-4\sigma^2+\sigma(8+15M+\tfrac{27}{2}M^2)-2M-\tfrac{9}{2}M^2,$$

and the whole numerator is thus

$$\begin{aligned} &3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M)[(3M+4)\sigma+M] \\ &-4\sigma^3+\sigma^2(8+15M+\tfrac{27}{2}M^2)+\sigma(-2M-\tfrac{9}{7}M^2), \end{aligned}$$

which is

$$=(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2.$$

71. We have thus

$$-\frac{2dx}{x}=d\sigma\,\frac{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma+\dfrac{\gamma}{\beta-\gamma}\right)},$$

and thence also

$$-\frac{2dz}{z}=d\sigma\,\frac{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma-\dfrac{\alpha}{\alpha-\beta}\right)},$$

so that dx and dz also vanish when σ is a root of the quartic equation: the points in question are therefore cusps of the nodal curve.

Centro-surface as the envelope of the quadric $\Sigma a^2x^2(a^2+\xi)^{-2}=1$. Art. Nos. 72 to 76.

72. The equations $-\beta\gamma a^2x^2=(a^2+\xi)^3(a^2+\eta)$, &c. considering therein ξ, η as variable give the centro-surface; considering η as a given constant but ξ as variable they give the sequential centro-curve; and considering ξ as a given constant but η as variable they give the concomitant centro-curve.

73. Suppose first that η is a given constant; to eliminate ξ we may write the equations in the form

$$-(\beta\gamma)^{\frac{1}{3}}(ax)^{\frac{2}{3}}(a^2+\eta)^{-\frac{1}{3}}=(a^2+\xi),\ \&c.,$$

and then multiplying first by $\alpha(a^2+\eta)$, &c. and adding, and secondly by α, &c., and adding (observing that $\Sigma\alpha(a^2+\xi)(a^2+\eta)=-\alpha\beta\gamma$, $\Sigma\alpha(a^2+\xi)=0$); we have

$$\begin{aligned} \Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{\frac{2}{3}} &= (\alpha\beta\gamma)^{\frac{2}{3}}, \\ \Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{-\frac{1}{3}} &= 0, \end{aligned}$$

which equations, considering therein η as a given constant, are the equations of a sequential centro-curve.

If from the two equations we eliminate η we should have the equation of the centro-surface; the second equation is the derivative of the first in regard to η; and it thus appears that the equation of the centro-surface might be obtained by equating to zero the discriminant of the rationalised function

$$\text{norm. } [\{\Sigma\,(\alpha a x)^{\frac{2}{3}}\,(a^2+\eta)^{\frac{2}{3}}\} - (\alpha\beta\gamma)^{\frac{2}{3}}];$$

but the form is too inconvenient to be of any use.

74. Taking next ξ as a given constant, and writing the equations in the form

$$-\beta\gamma a^2 x^2\,(a^2+\xi)^{-3} = (a^2+\eta),\ \&c.;$$

then multiplying by $\alpha\,(a^2+\xi)$, &c. and adding, and again multiplying by α, &c. and adding, we have

$$\Sigma a^2 x^2\,(a^2+\xi)^{-2} = 1,$$
$$\Sigma a^2 x^2\,(a^2+\xi)^{-3} = 0;$$

or writing these equations at full length,

$$\frac{a^2x^2}{(a^2+\xi)^2} + \frac{b^2y^2}{(b^2+\xi)^2} + \frac{c^2z^2}{(c^2+\xi)^2} - 1 = 0,$$
$$\frac{a^2x^2}{(a^2+\xi)^3} + \frac{b^2y^2}{(b^2+\xi)^3} + \frac{c^2z^2}{(c^2+\xi)^3} \qquad = 0,$$

which equations, considering therein ξ as a constant, are the equations of any concomitant centro-curve: since the equations are each of the second order it thus appears that the concomitant centro-curves are quadriquadrics.

75. If from the two equations we eliminate ξ, we have the equation of the centro-surface; the second equation is the derivative of the first in regard to ξ; and it thus appears that the equation of the centro-surface is obtained by equating to zero the discriminant in regard to ξ of the integralised function

$$(a^2+\xi)^2\,(b^2+\xi)^2\,(c^2+\xi)^2\,\{(\Sigma a^2x^2\,(a^2+\xi)^{-2} - 1\},$$

or, what is the same thing, the discriminant of the sextic function

$$(a^2+\xi)^2\,(b^2+\xi)^2\,(c^2+\xi)^2 - \Sigma a^2x^2\,(b^2+\xi)^2\,(c^2+\xi)^2.$$

76. If instead hereof we consider the homogeneous function

$$w^2\,(a^2+\xi)^2\,(b^2+\xi)^2\,(c^2+\xi)^2 - \Sigma a^2x^2\,(b^2+\xi)^2\,(c^2+\xi)^2,$$

then the coefficients are of the second order in (x, y, z, w), and the discriminant, being of the tenth order in the coefficients, is of the order 20 in (x, y, z, w). But the sextic function has a twofold factor $\left(1+\frac{\xi}{\infty}\right)^2$ if $w^2=0$, and it has evidently a twofold factor if $x^2=0$ or $y^2=0$ or $z^2=0$, that is, the discriminant contains the factor $x^2y^2z^2w^2$; or, omitting this factor, it will be of the order 12 in (x, y, z, w); whence writing $w=1$, the centro-surface is of the order 12. I have in this manner actually obtained the equation of the centro-surface: see the memoir "On a certain Sextic Torse," *Camb. Phil. Trans.* t. XI. (1871), pp. 507—523, [436].

Another generation of the Centro-surface. Art. Nos. 77 to 83.

77. By what precedes, the equation of the centro-surface is obtained as the condition in order that the equation

$$\{\Sigma a^2x^2(a^2+\xi)^{-2}\}-1=0$$

may have two equal roots. But taking m an arbitrary constant, this is the derived equation of

$$\{\Sigma a^2x^2(a^2+\xi)^{-1}\}+\xi+m=0,$$

and as such it will have two equal roots if the last-mentioned equation has three equal roots; and conversely, we have thus the equation of the centro-surface by expressing that the last-mentioned equation, or, what is the same thing, the quartic equation

$$(\xi+m)(\xi+a^2)(\xi+b^2)(\xi+c^2)-\Sigma a^2x^2(\xi+b^2)(\xi+c^2)=0$$

has three equal roots. The conditions for this are that the quadrinvariant and the cubinvariant shall each of them vanish; the two invariants are respectively a quadric and a cubic function of m; viz. the equations are

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c})(m,\ 1)^2=0,\quad (\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{d}')(m,\ 1)^3=0;$$

where the degrees in $(x,\ y,\ z)$ of a, b, c are 0, 2, 4 and those of a′, b′, c′, d′ are 0, 2, 4, 6 respectively: the equation of the centro-surface then is

$$\begin{vmatrix} & & \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ & \mathrm{a}, & \mathrm{b}, & \mathrm{c} & \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} & & \\ & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' & \end{vmatrix}=0,$$

which is of the right order 12; but it would be difficult to obtain thereby the developed equation.

78. For the nodal curve the cubic equation must be satisfied by each root of the quadric equation, or, what is the same thing, the quadric function must completely divide the cubic function; the conditions are

$$\left\|\begin{matrix} & & \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} & & \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' & \end{matrix}\right\|=0,$$

where the degrees may be taken to be

$$\begin{vmatrix} 0, & 0, & 2, & 4 \\ 0, & 2, & 4, & 0 \\ 0, & 2, & 4, & 6 \end{vmatrix},$$

and the order of the nodal curve is thus $=24$: two of the equations in fact are

$$\begin{vmatrix} & \mathrm{a}, & \mathrm{b} \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}' \end{vmatrix} = 0, \qquad \begin{vmatrix} & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}, & \mathrm{c} & \\ \mathrm{a}', & \mathrm{c}', & \mathrm{d}' \end{vmatrix} = 0,$$

which are surfaces of the orders 4, 6; or the nodal curve is a complete intersection 4×6. By the results above obtained as to the nodal curve, it appears that the two surfaces must have an ordinary contact at each of the 16 umbilicar centres, and a stationary or singular contact at each of 48 outcrops.

79. The derivation of the centro-surface from the surface

$$\frac{a^2x^2}{a^2+\xi} + \frac{b^2y^2}{b^2+\xi} + \frac{c^2z^2}{c^2+\xi} + \xi - m = 0$$

requires to be further explained. The surface, say $V=0$, is a quadric surface depending on the two parameters ξ, m; the axes coincide in direction with those of the ellipsoid, and their relative magnitudes are as

$$\frac{1}{a}\sqrt{a^2+\xi} : \frac{1}{b}\sqrt{b^2+\xi} : \frac{1}{c}\sqrt{c^2+\xi},$$

viz. these are as the axes of the confocal surface

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1 = 0$$

divided by a, b, c respectively; to fix the absolute magnitudes observe that the equation may be written

$$x^2+y^2+z^2-m-\xi\left(\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1\right) = 0,$$

viz. the surface $V=0$ is a surface through the spheroconic which is the intersection of the confocal surface by the arbitrary sphere $x^2+y^2+z^2-m=0$; but, while the surface is hereby and by the preceding condition as to the axes completely determined, the geometrical significance is anything but clear.

80. Considering then the quadric surface $V=0$, depending on the parameters ξ, m; suppose that m remains constant while ξ alone varies; we have thus three consecutive surfaces $V=0$, $V'=0$, $V''=0$; and these I say intersect in a point of the centro-surface; the point in question will depend on the two parameters (ξ, m), and if these vary simultaneously we have the whole system of points on the centro-surface; but if only one of them varies, the other being constant, we have a curve on the centro-surface.

The three equations may be replaced by $V=0$, $\delta_\xi V=0$, $\delta_\xi^2 V=0$; of which the first alone contains m; and it thus appears that if m be the variable parameter, the equations of the curve are $\delta_\xi V=0$, $\delta_\xi^2 V=0$, viz. the curve is then the quadriquadric curve which is the concomitant centro-curve of the curve of curvature for the parameter ξ. But if

the variable parameter be ξ, then this is a curve on the 12-thic surface $\Omega = 0$ obtained by the elimination of ξ from the equations $V = 0$, $\delta_\xi V = 0$; viz. we have $\Omega = S^3 - T^2 = 0$, where $S = (\mathrm{a, b, c})(m, 1)^2$, $T = (\mathrm{a', b', c', d'})(m, 1)^3$, and the curve in question is the curve $S = 0$, $T = 0$, which is the cuspidal curve on the surface $\Omega = 0$; the elimination of m from the two equations $S = 0$, $T = 0$ gives as above the equation of the centro-surface.

81. The surface $\Omega = S^3 - T^2 = 0$ obtained as above by the elimination of ξ from the equations $V = 0$, $\delta_\xi V = 0$, (or, what is the same thing, by equating to zero the discriminant of V in regard to ξ,) may be termed the sociate-surface: we have then the quartic and sextic surfaces $S = 0$, $T = 0$ intersecting in the before-mentioned curve, which may be called the sociate-edge; and the locus of these sociate-edges is the centro-surface.

82. We may if we please, changing the parameter in one of the functions, consider the two series of surfaces $S = 0$, $T = 0$ depending on the parameters m, m' respectively; a surface of the first series will correspond to one of the second series when the parameters are equal, $m = m'$, and we have then a sociate-edge. Taking a point anywhere in space, through this point there pass two surfaces $S = 0$, and three surfaces $T = 0$; but there is no pair of corresponding surfaces, or sociate-edge. If however the point be taken anywhere on the centro-surface, then there is a pair of corresponding surfaces $S = 0$, $T = 0$; that is, through each point of the centro-surface there passes a single sociate-edge; and if the point be taken anywhere on the nodal curve of the centro-surface, then there are two pairs of corresponding surfaces; that is, through each point of the nodal curve there are two sociate-edges: this explains the method above made use of for finding the equations of the nodal curve, by giving to the equations $S = 0$, $T = 0$, considered as equations in m, two equal roots.

83. The *à posteriori* verification that the surfaces $V = 0$, $V' = 0$, $V'' = 0$ intersect in a point of the centro-surface, is not without interest; the parameters ξ_1, η_1 of the point of intersection are found to be $\xi_1 = \xi$, $\eta_1 = m - a^2 - b^2 - c^2 - 3\xi$; whence in the equation $V = 0$, writing $-\beta\gamma a^2 x^2 = (a^2 + \xi_1)^3 (a^2 + \eta_1)$ and $m = a^2 + b^2 + c^2 + 3\xi_1 + \eta_1$, the resulting equation considered as an equation in ξ should have three roots $\xi = \xi_1$: the fourth root is at once seen to be $\xi = \eta_1$, and we ought therefore to have identically

$$\frac{-\alpha(a^2+\xi_1)^3(a^2+\eta_1)}{a^2+\xi} - \&\text{c.} + \alpha\beta\gamma(\xi - 3\xi_1 - \eta_1 - a^2 - b^2 - c^2)$$
$$= \frac{\alpha\beta\gamma(\xi-\xi_1)^3(\xi-\eta_1)}{(a^2+\xi)(b^2+\xi)(c^2+\xi)};$$

and by decomposing the right-hand side into its component fractions this is at once seen to be true.

Third generation of the Centro-surface. Art. Nos. 84 and 85.

84. Instead of the foregoing equation $V = 0$, consider the equation

$$W = \left(\frac{a^2x^2}{a^2+\xi} + \frac{b^2y^2}{b^2+\xi} + \frac{c^2z^2}{c^2+\xi} + \xi\right) - \left(\frac{a^2x^2}{a^2+\eta} + \frac{b^2y^2}{b^2+\eta} + \frac{c^2z^2}{c^2+\eta} + \eta\right) = 0.$$

The equations $d_\xi W=0$, $d_\xi^2 W=0$ contain only ξ, and are in fact identically the same as the equations $d_\xi V=0$, $d_\xi^2 V=0$; the elimination of ξ from the equations $d_\xi V=0$, $d_\xi^2 W=0$ would therefore lead to the equation of the centro-surface: and the centro-surface is connected with the surfaces $W=0$, $d_\xi W=0$, $d_\xi^2 W=0$ and the parameters ξ, η in the same way as it is with the surfaces $V=0$, $d_\xi V=0$, $d_\xi^2 V=0$ and the parameters ξ, m. That is, if from the equations $W=0$, $d_\xi W=0$ we eliminate ξ we have a surface $\Omega=0$, depending upon η and having a cuspidal curve; and the locus of the cuspidal curve (as η varies) is the centro-surface. But the equation $W=0$ divides by $\xi-\eta$, and throwing out this factor it becomes

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}-1=0,$$

so that the surface $\Omega=0$ is obtained by eliminating ξ from this equation and the derived equation in regard to ξ; or, what is the same thing, by equating to zero the discriminant in regard to ξ of the cubic function

$$(a^2+\xi)(b^2+\xi)(c^2+\xi)-\Sigma\frac{a^2x^2}{a^2+\eta}(b^2+\xi)(c^2+\xi).$$

This surface is in fact the torse generated by the normals at the several points of the curve of curvature belonging to the parameter η; the cuspidal curve is the edge of regression of this torse, that is, it is the sequential centro-curve of the curve of curvature; and we thus fall back upon the original investigation for the centro-surface.

85. In verification I remark that if X, Y, Z be the coordinates of a point on the curve of curvature in question, and (x, y, z) current coordinates, then the tangent plane of the torse, or plane through the normal and the tangent of the curve of curvature, has for its equation

$$\frac{Xx}{a^2+\eta}+\frac{Yy}{b^2+\eta}+\frac{Zz}{c^2+\eta}-1=0,$$

and if in this equation we consider the point (X, Y, Z) to be the point belonging to the parameters (η, ξ), viz. if we have $-\beta\gamma X^2=a^2(a^2+\xi)(a^2+\eta)$, &c., then this plane will be always touched by the before-mentioned ellipsoid,

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}=1.$$

The condition for the contact in fact is

$$\Sigma\frac{X^2}{(a^2+\eta)^2}\,\frac{(a^2+\xi)(a^2+\eta)}{a^2}=1,$$

viz. substituting for (X, Y, Z) their values, this is

$$-\frac{1}{\alpha\beta\gamma}\Sigma\alpha(a^2+\xi)^2=1,$$

which is true. And this being so, the ellipsoid and the plane have each the same envelope, viz. this is the torse in question.

Reciprocal Surface. Art. No. 86.

86. The centro-surface is the envelope of

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0;$$

hence the reciprocal surface in regard to the sphere $x^2+y^2+z^2-k^2=0$ is the envelope of

$$\frac{(a^2+\xi)^2}{a^2}X^2+\frac{(b^2+\xi)^2}{b^2}Y^2+\frac{(c^2+\xi)^2}{c^2}Z^2-k^2=0,$$

that is,

$$a^2X^2+b^2Y^2+c^2Z^2-k^2+2\xi(X^2+Y^2+Z^2)+\xi^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)=0,$$

viz. the envelope is

$$(a^2X^2+b^2Y^2+c^2Z^2-k^4)\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

or, expanding and multiplying by $a^2b^2c^2$, this is

$$a^2(b^2-c^2)^2Y^2Z^2+b^2(c^2-a^2)^2Z^2X^2+c^2(a^2-b^2)^2X^2Y^2$$
$$-k^4(b^2c^2X^2+c^2a^2Y^2+a^2b^2Z^2)=0,$$

or, what is the same thing,

$$a^2\alpha^2Y^2Z^2+b^2\beta^2Z^2X^2+c^2\gamma^2X^2Y^2-k^4(b^2c^2X^2+c^2a^2Y^2+a^2b^2Z^2)=0,$$

which may be written

$$\mathrm{a}^2Y^2Z^2+\mathrm{b}^2Z^2X^2+\mathrm{c}^2X^2Y^2+\mathrm{f}^2X^2+\mathrm{g}^2Y^2+\mathrm{h}^2Z^2=0,$$

where

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h})=(a\alpha,\ b\beta,\ c\gamma,\ 2k^2bc,\ 2k^2ca,\ 2k^2ab),$$

and consequently,

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=2k^2abc(\alpha+\beta+\gamma)=0.$$

It would doubtless be interesting to discuss this surface as it here presents itself, and with reference to its geometrical signification as the locus of the pole, in regard to the sphere, of the plane through two intersecting consecutive normals of the ellipsoid: but I abstain from any consideration of the question.

Delineation of the centro-surface for given numerical values of the semiaxes. Art. Nos. 87 and 88.

87. I constructed on a large scale a drawing of the centro-surface for the values

$$a^2=50,\quad b^2=25,\quad c^2=15.$$

(These were chosen so that a, b, c should have approximately the integer values 7, 5, 4, and that a^2+c^2 should be well greater than $2b^2$; they give a good form of surface,

though perhaps a better selection might have been made; there is a slight objection to the existence of the relation $a^2 = 2b^2$, as in the xy-section it brings a cusp of the evolute on the ellipse.) We have therefore

$$\alpha = 10, \quad \beta = -35, \quad \gamma = 25;$$

the ellipses in the principal planes of the centro-surface are

$$\frac{y^2}{(5)^2} + \frac{z^2}{(8{\cdot}937)^2} = 1,$$

$$\frac{z^2}{(2{\cdot}582)^2} + \frac{x^2}{(3{\cdot}535)^2} = 1,$$

$$\frac{x^2}{(4{\cdot}950)^2} + \frac{y^2}{(2)^2} = 1,$$

and these determine on each axis the two points which are the cusps of the evolutes. We have moreover for the umbilicar centre $x = 2{\cdot}988$, $y = 0$, $z = 1{\cdot}380$, and for the outcrop $x = 1{\cdot}127$, $y = 1{\cdot}947$, $z = 0$.

88. For the delineation of the nodal curve (crunodal portion) we have first to find the values of ξ, ξ_1; these are given in terms of x, y *ante* No. 33 [p. 334], where y is a given function of x, and x extends between the values $\{-b^2$ and $-\frac{1}{3}(a^2+b^2+c^2)\} - 25$ and $-26\frac{2}{3}$. It was thought sufficient to divide the interval into 6 equal parts, that is, the values of x were taken to be -25, $-25{\cdot}\dot{3}$, ... $-26{\cdot}\dot{6}$. The values of ξ, ξ_1 being found, those of η, η_1 were obtained from them by means of the original equations $(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)(a^2+\eta)$ &c. viz. we have thus for the determination of η, η_1 three simple equations, affording a verification of each other.

For the performance of these calculations (viz. of the values of y, ξ, ξ_1, η, η_1) I am indebted to the kindness of Mr J. W. L. Glaisher, of Trinity College. The results being obtained it is then easy to calculate as well the coordinates (x, y, z) of the point on the nodal curve as also the coordinates (X, Y, Z) and (X_1, Y_1, Z_1) of the corresponding two points on the ellipsoid (these last are of course not required for the delineation of the nodal curve, but it was interesting to obtain them). The whole series of the results is given in the annexed Table, and from them the drawing was constructed.

I find also in the neighbourhood of the umbilicar centre (if $\xi = -25 + q$),

$$\delta x = {\cdot}02868\, q^2,$$

$$\delta y = \pm\, {\cdot}02484\, q^2,$$

$$\delta z = {\cdot}02191\, q^2,$$

and in the neighbourhood of the outcrop (if $\xi_1 = -38{\cdot}333 + \frac{70}{9}\varpi$),

$$\delta x = 1{\cdot}127\, \varpi,$$

$$\delta y = -1{\cdot}704\, \varpi,$$

$$\delta z = \pm\, 4{\cdot}582\, \varpi^{\frac{3}{2}}.$$

x	y	ξ	ξ_1	η	η_1	x	y	z	X	Y	Z	X_1	Y_1	Z_1
−25	0·	−25·	−25·	−25·	−25·	2·988	0·	1·380	5·326	0·	2·070	5·326	0·	2·070
−25·$\dot{3}$	4·2669	−21·0664	−29·6002	−38·3193	−16·6728	2·543	0·360	0·996	4·394	2·289	2·491	6·233	1·957	1·023
−25·$\dot{6}$	6·3191	−19·3475	−31·9858	−42·9911	−15·4693	2·148	0·721	0·662	3·504	3·189	2·283	5·956	2·580	0·584
−26·	8·1240	−17·8760	−34·1240	−45·7879	−15·1047	1·786	1·175	0·373	2·780	3·847	1·948	5·626	3·005	0·293
−26·$\dot{3}$	9·8760	−16·4573	−36·2094	−47·5684	−15·0106	1·448	1·500	0·139	2·159	4·391	1·426	5·251	3·346	0·098
−26·$\dot{6}$	11·6667	−15·0000	−38·3333	−48·7037	−15·0000	1·127	1·947	0·	1·610	4·869	0·	4·829	3·651	0·

The calculations were performed before I had obtained the formulæ in σ, which would have given the results more easily.

521.

ON DR. WIENER'S MODEL OF A CUBIC SURFACE WITH 27 REAL LINES; AND ON THE CONSTRUCTION OF A DOUBLE-SIXER.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XII. Part I. (1873), pp. 366—383. Read May 15, 1871.]

I.

I CALL to mind that a cubic surface has upon it in general 27 lines which may be all of them real. We may out of the 27 lines (and that in 36 different ways) select 12 lines forming a "double-sixer," viz. denoting such a system of lines by

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,$$
$$b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6;$$

then no two lines a meet each other, nor any two lines b, but each line a meets each line b, except that the two lines of a pair $(a_1,\ b_1)$, $(a_2,\ b_2)$, ... $(a_6,\ b_6)$ do *not* meet each other. And such a system of twelve lines leads at once to the remaining fifteen lines; viz. we have a line c_{12}, the intersection of the planes which contain the pairs of lines $(a_1,\ b_2)$ and $(a_2,\ b_1)$ respectively.

The model is formed of plaster, and is contained within a cube, the edge of which is $= 18{\cdot}2$ inches: the lines a, b, c are coloured blue, yellow, and red respectively; the lines a_1, b_2, b_5 being at right angles to each other, in such wise that taking the origin at the centre of the cube, the axes parallel to the edges, and the unit of length $= 1{\cdot}6$ inches, the equations of these three lines are

$$\begin{array}{lll} a_1, & x=0,\ y=0, & \\ b_2, & x=0, & z=\ \ 1, \\ b_5, & & y=0,\ z=-1. \end{array}$$

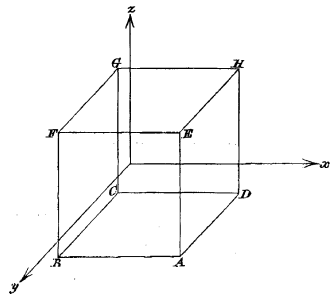
The model is a solid figure bounded by portions of the faces of the cube, and by a portion of the cubic surface, being a surface with three apertures, the collocation of which is not easily explained.

To determine the construction I measured, on the faces of the cube, the coordinates of the two extremities of each of the twelve lines; these were measured in tenths of an inch (taking account of the half division, or twentieth of an inch), and the resulting numbers divided by 16 to reduce them to the before-mentioned unit of 1·6 inches. These reduced values are shewn in the table: knowing then the coordinates of two points on each line, the equations of the several lines became calculable; the true theoretical form of these results—(viz. the form which, but for errors of the model, or of the measurement, they would have assumed)—is

b_1,	$x = B_1 z + D$,	$y = B_1' z + D'$,	
b_2,	$x = 0$,		$z = 1$,
b_3,	$x = B_3 (z + \beta_3)$,	$y = B_3' (z + \beta_3)$,	
b_4,	$x = B_4 (z + \beta_4)$,	$y = B_4' (z + \beta_4)$,	
b_5,		$y = 0$,	$z = -1$,
b_6,	$x = B_6 (z + \beta_6)$,	$y = B_6' (z + \beta_6)$.	
a_1,	$x = 0$,	$y = 0$,	
a_2,	$x = A_2 z + C_2$,	$y = A_2' (z - 1)$,	
a_3,	$x = A_3 (z + 1)$,	$y = A_3' (z - 1)$,	
a_4,	$x = A_4 (z + 1)$,	$y = A_4' (z - 1)$,	
a_5,	$x = A_5 (z + 1)$,	$y = A_5' z + C_5'$,	
a_6,	$x = A_6 (z + 1)$,	$y = A_6' (z - 1)$;	

but in consequence of such errors, the results are not accurately of the form in question.

The faces of the cube being as in the diagram:



the Table is

Equations calculated from the measurements of the model.		*ABCD* $z=-5{\cdot}688$	*EFGH* $z=+5{\cdot}688$	*AEBF* $y=+5{\cdot}688$	*BCFG* $x=-5{\cdot}688$	*CDGH* $y=-5{\cdot}688$	*AEDH* $x=+5{\cdot}688$
$x=0$ $y=0$	a_1	$x=0$ $y=0$	$x=0$ $y=0$				
$x=-\ {\cdot}780z-\ {\cdot}187$ $y=-\ {\cdot}423z+\ {\cdot}406$	a_2	$x=4{\cdot}250$ $y=2{\cdot}812$	$x=-4{\cdot}625$ $y=-2{\cdot}000$				
$x=-\ {\cdot}654z-\ {\cdot}656$ $y=-\ {\cdot}588z+\ {\cdot}531$	a_3	$x=3{\cdot}062$ $y=3{\cdot}875$	$x=-4{\cdot}375$ $y=-2{\cdot}812$				
$x=-2{\cdot}912z-2{\cdot}959$ $y=-\ {\cdot}736z+\ {\cdot}752$	a_4	.....	.....	.....	$y={\cdot}0625$ $z={\cdot}9375$	.....	$y=\ 2{\cdot}937$ $z=-2{\cdot}969$
$x=\ 1{\cdot}024z+1{\cdot}014$ $y=-1{\cdot}049z-\ {\cdot}277$	a_5	$x=-4{\cdot}812$ $y=\ 5{\cdot}688$	.....	.....	.....	.....	$y=-5{\cdot}063$ $z=\ 4{\cdot}562$
$x=\ {\cdot}264z+\ {\cdot}187$ $y=-\ {\cdot}104z+\ {\cdot}219$	a_6	$x=-1{\cdot}313$ $y=\ {\cdot}8125$	$x=\ 1{\cdot}687$ $y=-\ {\cdot}375$				
$x=-1{\cdot}611z+\ {\cdot}151$ $y=-1{\cdot}438z+\ {\cdot}288$	b_1	.....	.....	.....	$y=\ 5{\cdot}500$ $z=-3{\cdot}625$	.....	$y=-4{\cdot}656$ $z=\ 3{\cdot}437$
$x=\ 0$ $z=-1$	b_2	.....	.....	$x=0$ $z=-1$	.....	$x=0$ $z=-1$	
$x=-1{\cdot}352z-\ {\cdot}685$ $y=-2{\cdot}034z-\ {\cdot}984$	b_3	.....	.....	$x=\ 3{\cdot}750$ $z=-3{\cdot}281$	.....	$x=-3{\cdot}812$ $z=\ 2{\cdot}313$	
$x=-\ {\cdot}753z-\ {\cdot}0315$ $y=-\ {\cdot}500z-\ {\cdot}0315$	b_4	$x=\ 4{\cdot}250$ $y=\ 2{\cdot}812$	$x=-4{\cdot}313$ $y=-2{\cdot}875$				
$y=\ 0$ $z=+1$	b_5	.....	.....	.....	$y=0$ $z=1$	.....	$y=0$ $z=1$
$x=\ 1{\cdot}123z-\ {\cdot}702$ $y=-1{\cdot}123z+\ {\cdot}702$	b_6	.....	.....	$x=-5{\cdot}688$ $z=-4{\cdot}438$	.....	.....	$y=-5{\cdot}688$ $z=\ 5{\cdot}688$

I hence calculate the intersections: considering any two lines which ought to intersect, then projecting on the horizontal plane and calculating x, y the coordinates

of the point of intersection of the two projections, these values of x, y substituted in the equations should give the same value of z; but if the lines do not accurately intersect, then the values of z will be different.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	*	0 0 − 1	0 0 − ·495 ± ·011	0 0 − ·052 ∓ ·010	0 0 + 1	0 0 + ·625
a_2	− ·077 + ·455 − ·129 ∓ ·013	*	+ ·381 + ·771 − ·796 ± ·067	+ 4·292 + 2·803 − 5·704 ± ·036	− ·967 − ·008 ∓ ·008 + 1	− ·398 + ·227 + ·346 ± ·076
a_3	− ·423 + ·699 − ·264 ± ·021	− ·001 ∓ ·001 + 1·119 − 1	*	− 4·782 − 3·227 + 6·350 ± ·042	− 1·310 − ·028 ∓ ·028 + 1	− ·673 + ·344 + ·162 ± ·146
a_4	− ·957 + 1·238 − ·674 ± ·014	− ·023 ∓ ·023 + 1·488 − 1	+ 1·286 + 1·736 − 1·398 ± ·060	*	− 5·871 + ·008 ± ·008 + 1	− 1·330 + ·847 − ·344 ± ·215
a_5	+ 2·519 − 1·801 + 1·462 ± ·008	− ·005 ∓ ·005 + ·772 − 1	+ ·282 + ·476 − ·717 ± ·001	+ ·412 + ·194 − ·518 ± ·070	*	+ 18·764 − 14·155 + 15·282 ± 4·052
a_6	+ ·194 + ·214 + ·040 ± ·022	− ·038 ∓ ·038 + ·323 − 1	+ ·045 + ·283 − ·582 ± ·042	− ·131 + ·284 − ·423 ± ·208	+ ·451 + ·057 ± ·057 − 1	*

Starting from the assumed equations of b_2, b_3, b_4, b_5, b_6, a_1, and calculating by the theory the remaining lines, the equations of the b-lines (those of b_1 being calculated) are

$$
\begin{aligned}
b_1, \quad & x = 1{\cdot}321\, z - {\cdot}310, \\
& y = -1{\cdot}295\, z + {\cdot}581\,; \\
b_2, \quad & x = 0,\ z = -1\,; \\
b_3, \quad & x = -1{\cdot}352\,(z + {\cdot}510), \\
& y = -2{\cdot}034\,(z + {\cdot}510)\,; \\
b_4, \quad & x = -\ {\cdot}753\,(z + {\cdot}052), \\
& y = -\ {\cdot}500\,(z + {\cdot}052)\,; \\
b_5, \quad & y = 0,\ z = +1\,; \\
b_6, \quad & x = \ 1{\cdot}123\,(z - {\cdot}624), \\
& y = -1{\cdot}123\,(z - {\cdot}624)\,;
\end{aligned}
$$

and the equations of the a-lines (those of all but a_1 being calculated) are

$$
\begin{aligned}
&a_1, && x = 0,\ y = 0\,; \\
&a_2, && x = -\ \cdot 753\, z - \cdot 091, \\
& && y = -\ \cdot 498\,(z-1)\,; \\
&a_3, && x = -\ \cdot 609\,(z+1), \\
& && y = -\ \cdot 677\,(z-1)\,; \\
&a_4, && x = -\ 2\cdot 506\,(z+1), \\
& && y = -\ \cdot 841\,(z-1)\,; \\
&a_5, && x = \ \cdot 874\,(z+1), \\
& && y = -\ \cdot 967\, z - \cdot 288\,; \\
&a_6, && x = \ \cdot 170\,(z+1), \\
& && y = -\ \cdot 071\,(z-1)\,;
\end{aligned}
$$

and thence for the points of intersection the coordinates are

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	*	0 0 −1	0 0 − ·510	0 0 − ·052	0 0 +1	0 0 + ·624
a_2	− ·170 + ·446 + ·105	*	+ ·662 + ·996 − 1	+ 197· + 131· − 262· i.e. lines a_2, b_4 nearly parallel.	− ·844 0 +1	− ·336 + ·336 + ·325
a_3	− ·515 + ·782 − ·155	0 +1·354 −1	*	− 1·805 − 2·007 + 3·964	−1·218 0 +1	− ·641 − ·641 + ·053
a_4	−1·071 +1·323 − ·573	0 +1·682 −1	+1·438 +2·164 −1·574	*	−5·012 0 +1	−1·259 +1·259 − ·497
a_5	+3·189 −2·849 +2·649	0 + ·679 −1	+ ·259 + ·291 − ·702	+ ·383 + ·255 − ·561	*	+6·410 −6·410 +6·333
a_6	+ ·241 + ·041 + ·417	0 + ·142 − 1	− ·074 + ·112 − ·565	+ ·131 + ·087 − ·226	+ ·340 0 +1	*

II.

I have in a paper "On the double-sixers of a cubic surface," *Quart. Math. Journal*, t. x. (1870), pp. 58—71, [459], obtained analytical expressions for the twelve lines of a double-sixer, and also calculated numerical values, which however (as there remarked) did not come out convenient ones for the construction of a figure. A different mode of treatment since occurred to me, by means of the following equation of the cubic surface

$$\left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\left(\frac{xz}{\alpha\gamma}-\frac{yw}{\beta\delta}\right)-k\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\left(\frac{xz}{\alpha'\gamma'}-\frac{yw}{\beta'\delta'}\right)=0,$$

which as will appear is a very convenient one for the purpose. We in fact obtain at once eight lines of the double-sixer; viz. these are

1. $x=0,\ w=0,$	2′. $x=0,\ y=0,$
3. $y=0,\ z=0,$	4′. $z=0,\ w=0,$
5. $\frac{x}{\alpha}-\frac{y}{\beta}=0,\ \frac{z}{\gamma}-\frac{w}{\delta}=0,$	5′. $\frac{x}{\alpha'}-\frac{w}{\delta'}=0,\ \frac{y}{\beta'}-\frac{z}{\gamma'}=0,$
6. $\frac{x}{\alpha'}-\frac{y}{\beta'}=0,\ \frac{z}{\gamma'}-\frac{w}{\delta'}=0,$	6′. $\frac{x}{\alpha}-\frac{w}{\delta}=0,\ \frac{y}{\beta}-\frac{z}{\gamma}=0;$

and also five lines not belonging to the double-sixer, viz.

12. $$x=0,\quad \left(-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

23. $$y=0,\quad \left(\frac{x}{\alpha'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$$

34. $$z=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

41. $$w=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\right)\frac{1}{\alpha'\gamma'}=0,$$

56. $$\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}=0,\quad \frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}=0.$$

The remaining lines of the double-sixer are then easily determined; viz. the lines 3, 5, 6, and 12 are met by the line 2′, and by a second line 1′; this, as a line meeting 3, 5, 6, will be given by equations of the form

$$x-\frac{\alpha}{\beta}y=\phi\left(\frac{\delta}{\gamma}z-w\right),\quad x-\frac{\alpha'}{\beta'}y=\phi\left(\frac{\delta'}{\gamma'}z-w\right),$$

and observing that these equations, writing therein $x=0$, give

$$\frac{z}{\gamma}-\frac{w}{\delta}=-\frac{\alpha}{\beta\delta\phi}y,\quad \frac{z}{\gamma'}-\frac{w}{\delta'}=-\frac{\alpha'}{\beta'\delta'\phi}y,$$

the condition of intersection with the line 12 gives

$$\phi = -\frac{\alpha' - k\alpha}{\delta' - k\delta},$$

which is the value of ϕ in the foregoing equations: and to these we may join the resulting equation

$$y\gamma\gamma'(\alpha\beta' - \alpha'\beta) = z\phi\beta\beta'(\gamma\delta' - \gamma'\delta).$$

Proceeding in like manner for the lines 3′, 2, 4, the equations for the remaining four lines of the double-sixer are

2.
$$\phi = \frac{\alpha' - k\alpha}{\beta' - k\beta},$$
$$x - w\frac{\alpha'}{\delta'} = \phi\left(y - z\frac{\beta'}{\gamma'}\right),$$
$$x - w\frac{\alpha}{\delta} = \phi\left(y - z\frac{\beta}{\gamma}\right),$$
$$w\gamma\gamma'(\alpha\delta' - \alpha'\delta) = z\phi\delta\delta'(\beta\gamma' - \beta'\gamma).$$

4.
$$\phi = \frac{\gamma' - k\gamma}{\delta' - k\delta},$$
$$\phi\left(x\frac{\delta'}{\alpha'} - w\right) = y\frac{\gamma'}{\beta'} - z,$$
$$\phi\left(x\frac{\delta}{\alpha} - w\right) = y\frac{\gamma}{\beta} - z,$$
$$x\phi\beta\beta'(\alpha\delta' - \alpha'\delta) = y\alpha\alpha'(\beta\gamma' - \beta'\gamma).$$

1′.
$$\phi = \frac{\alpha' - k\alpha}{\delta' - k\delta},$$
$$x - y\frac{\alpha'}{\beta'} = \phi\left(z\frac{\delta'}{\gamma'} - w\right),$$
$$x - y\frac{\alpha}{\beta} = \phi\left(z\frac{\delta}{\gamma} - w\right),$$
$$y\gamma\gamma'(\alpha\beta' - \alpha'\beta) = z\phi\beta\beta'(\gamma\delta' - \gamma'\delta).$$

3′.
$$\phi = -\frac{\gamma' - k\gamma}{\beta' - k\beta},$$
$$\phi\left(x\frac{\beta'}{\alpha'} - y\right) = z - w\frac{\gamma'}{\delta'},$$
$$\phi\left(x\frac{\beta}{\alpha} - y\right) = z - w\frac{\gamma}{\delta},$$
$$x\phi\delta\delta'(\alpha\beta' - \alpha'\beta) = w\alpha\alpha'(\gamma\delta' - \gamma'\delta).$$

It may be added that:—

In plane $x = 0$,

intersection of 1′ lies on line $z : w = (\alpha\beta' - \alpha'\beta)\gamma\gamma' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

" 2 " $y : z = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\delta' - \alpha'\delta)\gamma\gamma'$,

and that the line joining these intersections is the line 12.

In plane $y = 0$,

intersection of 2 lies on line $x : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : -(\beta\gamma' - \beta'\gamma)\delta\delta'$,

" 3′ " $z : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\beta' - \alpha'\beta)\delta\delta'$,

and that the line joining these intersections is the line 23.

In plane $z=0$,

$$\text{intersection of } 3' \text{ lies on line } x : y = (\gamma\delta' - \gamma'\delta)\,\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$
$$\text{,, } \quad 4 \quad \text{,, } \quad x : w = -(\beta\gamma' - \beta'\gamma)\,\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$

and that the line joining these intersections is the line 34.

And in plane $w=0$,

$$\text{intersection of } 4 \text{ is on line } y : z = (\alpha\delta' - \alpha'\delta)\,\beta\beta' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$
$$\text{,, } \quad 1' \quad \text{,, } \quad x : y = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\gamma\delta' - \gamma'\delta)\,\beta\beta',$$

and that the line joining these intersections is the line 14.

The equations of the remaining ten lines of the surface may be obtained without difficulty, and also the forty-five triple planes, but I do not stop to effect this; the planes $x=0$, $y=0$, $z=0$, $w=0$, are, it is clear, triple planes, containing the lines 1, 2′, 12; 2′, 3, 23; 3, 4′, 34; and 4′, 1, 41 respectively.

If, to fix the ideas, the planes $x=0$, $y=0$, $z=0$, $w=0$ are taken to be those of the tetrahedron $ABCD$ ($x=BCD$ &c., as usual), then the edges AB, BC, CD, DA (but not the remaining opposite edges AC, BD) will be lines on the surface. Each plane of the tetrahedron, for instance ABC ($w=0$), is met by the ten lines not contained therein in two vertices A, C, three points on the edge BA, three points on the edge BC, and two other points, viz. these are the intersections of the plane ABC by the lines 4 and 1′. For the construction of a model it is sufficient to determine the three points on each edge, and the two points, say in the plane ABC and in the plane DBC ($x=0$) respectively; for then each of the remaining eight lines will be determined as a line joining two points in these two planes respectively. If in the first instance k is considered as a variable parameter, then the two points in the plane $w=0$ are given as the intersections of two fixed lines by a variable line (14) rotating round the fixed point $\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}=0$, $\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}=0$; and the like as regards the two points in the plane $x=0$. By making (with assumed values of the other parameters) the proper drawings for the two planes $w=0$, $x=0$, it is easy to fix upon a convenient value of the parameter k; and I have in this manner succeeded in making a string model of the double-sixer; viz. the coordinates x, y, z, w are taken to be as the perpendicular distances of the current point from the faces of a regular tetrahedron (the coordinates being positive for an interior point); the values of α, β, γ, δ were put $=3, 4, 5, 6$ and those of α', β', γ', $\delta'=1, 1, 1, 1$; the value of k fixed upon as above was $k=-\frac{1}{8}$; this however brings the lines 2 and 4 too close together (viz. the shortest distance between them is not great enough), and also their apparent intersection too close to their intersections with the line 6′; and it is probable that a slightly different value of k would be better.

The results just obtained may be exhibited in a compendious form as follows:

	x :	y :	z :	w
1 is line BC	0			0
$2'$ „ CD	0	0		
3 „ DA		0	0	
$4'$ „ AB			0	0
5 meets CD			γ	δ
„ AB	α	β		
6 „ CD			γ'	δ'
„ AB	α'	β'		
$6'$ „ BC		β	γ	
„ AD	α			δ
$5'$ „ BC		β'	γ'	
„ AD	α'			δ'
$1'$ „ AD	$\alpha'-k\alpha$			$\delta'-k\delta$
„ BCD		$-(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
„ ABC	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$-(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	
$3'$ „ BC		$\beta'-k\beta$	$\gamma'-k\gamma$	
„ ACD	$-(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$		$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta'$
„ ABD	$(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$	$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta'$
2 „ AB	$\alpha'-k\alpha$	$\beta'-k\beta$		
„ BCD		$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
„ ACD	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$-(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
4 „ CD			$\gamma'-k\gamma$	$\delta'-k\delta$
„ ABD	$-(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$-(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$		$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
„ ABC	$(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$	$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	

$$1' \text{ and } 2 \text{ meet } BCD \text{ on line } \left(-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$2 \text{ and } 3' \text{ ,, } CDA \text{ ,, } \left(\frac{x}{\alpha'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$$

$$3' \text{ and } 4 \text{ ,, } DAB \text{ ,, } \left(\frac{x}{\alpha'}-\frac{y}{\beta'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$4 \text{ and } 1' \text{ ,, } ABC \text{ ,, } \left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\right)\frac{1}{\alpha'\gamma'}=0;$$

or calculating the numerical values from the foregoing assumed data, say

	x	$:y$	$:z$	$:w$	
1 is line BC	0			0	
2′ ... CD	0	0			
3 ... DA		0	0		
4′ ... AB			0	0	
5 meets CD			5	6	$z=45{\cdot}5$, $w=54{\cdot}5$.
... AB	3	4			$x=42{\cdot}9$, $y=57{\cdot}1$.
6 meets CD			1	1	$z=50$, $w=50$.
... AB	1	1			$x=50$, $y=50$.
6′ meets BC		4	5		$y=44{\cdot}4$, $z=55{\cdot}6$.
... AD	3			6	$x=33{\cdot}3$, $w=66{\cdot}7$.
5′ meets BC		1	1		$y=50$, $z=50$.
... AD	1			1	$x=50$, $w=50$.
1′ meets AD	11			14	$x=44$, $z=56$.
... BCD		-44	70	126	$y=-25{\cdot}1$, $z=39{\cdot}9$, $w=71{\cdot}8$.
... ABC	99	44	-70		Not required.
3′ meets BC		12	13		$y=48$, $z=52$.
... ACD	-36		117	78	Not required.
... ABD	36	108		-78	$x=47{\cdot}2$, $y=141{\cdot}7$, $w=-102{\cdot}3$.
2 meets AB	11	12			$x=47{\cdot}8$, $y=52{\cdot}2$.
... BCD		100	180	66	$y=26{\cdot}4$, $z=44$, $w=16{\cdot}2$.
... ACD	-99		189	66	Not required.
4 meets CD			13	14	$z=48{\cdot}1$, $w=51{\cdot}9$.
... ABD	42	156		-126	$x=50{\cdot}5$, $y=187{\cdot}6$, $w=-151{\cdot}5$.
... ABC	42	156	117		Not required.

$$\begin{array}{llll} 1' \text{ and } 2 & \text{meet } BCD & \text{on line} & 35y - 32z + 30w = 0, \\ 2 \text{ and } 3' & \ldots\ CDA & \ldots\ldots & 26x + 22z - 21w = 0, \\ 3' \text{ and } 4 & \ldots\ DAB & \ldots\ldots & 8x - 7y - 6w = 0, \\ 4' \text{ and } 1 & \ldots\ ABC & \ldots\ldots & 52x - 47y - 44z = 0, \end{array}$$

which last four equations serve as a verification.

The outside numerical values are given in the manner most convenient for the construction of a drawing; viz. when the coordinates refer to a point on an edge of the tetrahedron, or say on the side of an equilateral triangle, then taking the length of this edge (or side) to be $=100$, the numerical values are fixed so that the sum of the two coordinates may be $=100$, and the two coordinates thus denote the distances from the extremities of the edge or side: but when the three coordinates belong to a point in the face of the tetrahedron, or say in the plane of an equilateral triangle, then the sum of the coordinates is made $=86{\cdot}6$, and the three coordinates thus denote the perpendicular distances from the sides of the triangle.

III.

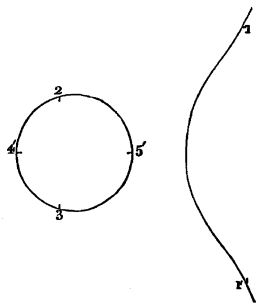
It is possible to find on a cubic curve a double-sixer of points 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′ such that any six points such as 1, 2, 3, 4′, 5′, 6′ lie in a conic. In fact considering a cubic surface having upon it the double-sixer of lines 1, 2, 3, 4, 5, 6 and 1′, 2′, 3′, 4′, 5′, 6′, the section by any plane is a cubic curve meeting the lines, say in the points 1, 2, 3, 4, 5, 6, 1′, 2′, 3′, 4′, 5′, 6′: each of the lines 1, 2, 3 meets each of the lines 4′, 5′, 6′, and consequently the six lines lie in a quadric surface: therefore the points 1, 2, 3, 4′, 5′, 6′ lie in a conic: and so in the other cases; the number of the conics is of course $=60$.

The cubic curve may be a given curve, and six of the points upon it (not being points on a conic) may also be taken to be given; for instance the points 1, 2, 3, 1′, 4′, 5′. For take through the points 2, 3 respectively any two lines 1, 2; through 1′, 4′, 5′ respectively the lines 1′, 4′, 5′ each meeting each of the lines 2, 3: and through 1 a line meeting each of the lines 4′, 5′. It is easy to see that a cubic surface may be drawn through the cubic curve and the lines 1, 2, 3, 1′, 4′, 5′: for the passage through the cubic curve requires 9 conditions; the surface then passes through the point 2 and to make it pass through the line 2 requires 3 conditions; similarly the surface passes through the point 3, and to make it pass through the line 3 requires 3 conditions. The surface now passes through 1′ and through the points of intersection of the line 1′ with the lines 2, 3: to make it pass through the line 1′ requires 1 condition; similarly to make it pass through the lines 4′, 5′, 1 requires in each case 1 condition; or there are in all 19 conditions, so that the cubic surface is completely determined. Take now through the points 1, 2, 3, 4′, 5′, a conic meeting the cubic in the point 6′: then through the lines 1, 2, 3, 4′, 5′ we have a quadric surface passing through this

conic, and therefore through 6′: hence through 6′ we may draw a line 6′ meeting each of the lines 1, 2, 3; and since the cubic surface passes through the point 6′ and also through the intersections of the line 6′ with the lines 1, 2, 3, it passes through the line 6′. We complete in this manner by constructions in the plane of the cubic the system of the twelve points, viz. each new point is given as the intersection of the cubic curve by a conic drawn through five points of the cubic curve. It is then shown as for the point 6′ and the line 6′ through it, that through each new point there can be drawn a line denoted by the same number and meeting each of the lines which it ought to meet, and hence lying on the cubic surface: the twelve points are thus the intersections of the plane of the cubic curve by the twelve lines of the double-sixer; and it follows that the six points which ought to lie in a conic (in every case where such conic has not been used in the plane construction) do actually lie in a conic.

I was anxious to construct such a double-sixer of points on a cubic curve; for this purpose I take the equation of the curve to be $y^2 = \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right)\left(1 - \frac{x}{c}\right)$, or say for shortness $y^2 = X$; where, to fix the ideas, a, b are supposed to be positive, a greater than b; and c to be negative.

The cubic curve is thus a parabola symmetrical in regard to the axis of x, and consisting of a loop and infinite branch; and I take upon it the points 1, 2, 3, 1′, 4′, 5′



as shown in the figure, viz. the coordinates of these points are as stated in the Table, where m is the x coordinate, and $\sqrt{M} = \sqrt{\left(1 - \frac{m}{a}\right)\left(1 - \frac{m}{b}\right)\left(1 - \frac{m}{c}\right)}$ and so in other cases, $\sqrt{14} = 3{\cdot}74165$.

	x	y	x	y
1	m	$\sqrt{M}$	6	$\sqrt{14}=3{\cdot}742$
2	0	1	0	1
3	0	-1	0	-1
4	θ	$\sqrt{\Theta}$	$2-\frac{75}{1369}=1{\cdot}945$	$-\frac{2280}{(37)^3}\sqrt{14}=-{\cdot}168$
5	ϕ	$\sqrt{\Phi}$	$-1+\frac{75}{361}=-0{\cdot}792$	$\frac{1110}{(19)^3}\sqrt{14}={\cdot}606$
6	m_1	$-\sqrt{M_1}$	$\frac{13}{3}=4{\cdot}333$	$-\frac{4}{9}\sqrt{14}=-1{\cdot}641$
1′	m	$-\sqrt{M}$	6	$-\sqrt{14}=-3{\cdot}742$
2′	σ	$\sqrt{\Sigma}$	$-\frac{1560\,(14-\sqrt{14})}{(31\sqrt{14}-5)^2}=1{\cdot}299$	.. $=+{\cdot}676$
3′	τ	$\sqrt{\mathrm{T}}$	$-\frac{1560\,(14+\sqrt{14})}{(31\sqrt{14}+5)^2}=1{\cdot}887$	.. $=+{\cdot}247$
4′	c	0	-1	0
5′	b	0	2	0
6′	m_1	$\sqrt{M_1}$	$\frac{13}{3}=4{\cdot}333$	$\frac{4}{9}\sqrt{14}=1{\cdot}641.$

The numerical values belong to the curve $y^2=\left(1-\frac{x}{3}\right)\left(1-\frac{x}{2}\right)\left(1+x\right)$ and to $m=6$.

Starting with the points 1, 2, 3, 1′, 4′, 5′ we have to find the remaining points 6′, 6, 4, 5, 2′, 3′.

Point 6′ by means of the conic 1234′5′6′, as follows.

The equation of the conic is

$$(x-b)(x-c)-bc\,y^2+k\,xy=0,\ (2,\ 3,\ 4',\ 5'),$$

and making this pass through the point 1 $(x=m,\ y=\sqrt{M})$ we find

$$(m-b)(m-c)+ka\sqrt{M}=0.\quad (1).$$

Hence taking the coordinates of 6′ to be m_1, $\sqrt{M_1}$, we have

$$(m_1-b)(m_1-c)+ka\sqrt{M_1}=0,\quad (6'),$$

and thence

$$\frac{\sqrt{M_1}}{\sqrt{M}}=\frac{(m_1-b)(m_1-c)}{(m-b)(m-c)}=\frac{M_1}{M}\,\frac{(m-a)}{(m_1-a)},$$

that is,

$$\frac{\sqrt{M_1}}{\sqrt{M}} = \frac{(m_1 - b)(m_1 - c)}{(m - b)(m - c)} = \frac{m_1 - a}{m - a}.$$

We have thus for m, a quadric equation satisfied by $m = m_1$, so that throwing out the factor $m - m_1$, the equation is a linear one, viz. we find

$$m_1 = \frac{ma - ab - ac + bc}{m - a},$$

or, what is the same thing,

$$m_1 - a = \frac{(a - b)(a - c)}{m - a},$$

and thence also

$$\sqrt{M_1} = \frac{(a - b)(a - c)}{(m - a)^2}\sqrt{M},$$

viz. $\sqrt{M_1}$ is determined rationally in terms of m, $\sqrt{M}$; this is of course as it should be, since the point 6′ is uniquely determinate.

Point 6 by means of the conic 2361′4′5′.

In precisely the same manner the coordinates are m_1, $-\sqrt{M_1}$, where m_1, $\sqrt{M_1}$, denote the same quantities as before.

Point 4 by means of the conic 2341′5′6′.

The equation of the conic is

$$Fx + Gy + H = \frac{1 - y^2}{x}, \qquad (2, 3),$$

where

$$Fb \qquad + H = \frac{1}{b}, \qquad (5')$$

$$Fm_1 + G\sqrt{M_1} + H = \frac{1 - M_1}{m_1}, \qquad (6')$$

$$Fm - G\sqrt{M} + H = \frac{1 - M}{m}, \qquad (1')$$

which give without difficulty

$$abc\,F = -a - c + P,$$
$$\sqrt{M}\,abc\,G = (m - b)(-m + P),$$
$$abc\,H = ab + ac + bc - bP,$$

where $P = 2a - c - \dfrac{2(a - c)(b - c)}{m + m_1 - 2c}$, a quantity which will presently be expressed in terms of m only.

And then

$$F\theta + G\sqrt{\Theta} + H = \frac{1 - \Theta}{\theta},$$

or say

$$F(\theta-b)+G\sqrt{\Theta}=\frac{1-\Theta}{\theta}-\frac{1}{b}$$

$$=-(\theta-b)\left(\frac{1}{ab}+\frac{1}{bc}-\frac{\theta}{abc}\right),$$

that is,

$$(\theta-b)(abc\,F+a+c-\theta)+G\,abc\sqrt{\Theta}=0,$$

viz. that is,

$$(\theta-b)(P-\theta)+(m-b)\frac{\sqrt{\Theta}}{\sqrt{M}}(P-m)=0$$

or, rationalising and throwing out the factor $\theta-b$, this is

$$(\theta-b)(\theta-P)^2-(m-b)(m-P)^2\frac{(\theta-a)(\theta-b)}{(m-a)(m-b)}=0,$$

which is a cubic equation satisfied by $\theta=m$ and $\theta=m_1$; so that throwing out the factors $\theta-m$, $\theta-m_1$ we have for θ a linear equation.

Putting for shortness

$$A=(m-a)^2-(a-b)(a-c),$$
$$B=(m-b)^2-(b-c)(b-a),$$
$$C=(m-c)^2-(c-a)(c-b),$$

the value of θ may be expressed in the forms

$$\theta-a=\frac{B^2}{C^2}(c-a),\quad \theta-b=\frac{A^2}{C^2}(c-b),\quad \theta-c=\frac{4(m-a)(m-b)(m-c)(b-c)(a-c)}{C^2}.$$

We have moreover

$$P-c=\frac{2(a-c)(m-b)(m-c)}{C},\quad P-m=-\frac{(m-c)A}{C},$$

equations which express P in terms of m only; also

$$\theta-P=\frac{-2(a-c)(m-b)(m-c)B}{C^2},$$

and then

$$\sqrt{\Theta}=-\sqrt{M}\,\frac{\theta-b}{m-b}\,\frac{P-\theta}{P-m},$$

whence

$$\sqrt{\Theta}=2\sqrt{M}(b-c)(c-a)\frac{AB}{C^3},$$

so that θ, $\sqrt{\Theta}$ are now determined.

Point 5 by means of the conic 2351′4′6′.

The conic is

$$Fx + Gy + H = \frac{1 - y^2}{x}, \qquad (2,\ 3)$$

where

$$Fc. \qquad + H = \frac{1}{c}, \qquad (4')$$

$$Fm_1 + G\sqrt{M_1} + H = \frac{1 - M_1}{m_1}, \qquad (6')$$

$$Fm - G\sqrt{M} + H = \frac{1 - M}{m}, \qquad (1').$$

Everything is the same as for the point 4 except that b, c are interchanged: hence writing Q instead of P, and using A, B, C to denote as before, we have

$$abc\,F = -a - b + Q,$$

$$\sqrt{M}\,abc\,G = (m - c)(-m + Q),$$

$$abc\,H = ab + ac + bc - cQ,$$

and

$$\phi - a = \frac{C^2(b - a)}{B^2},$$

$$\phi - b = \frac{4(m - a)(m - b)(m - c)(c - b)(a - b)}{B^2},$$

$$\phi - c = \frac{A^2(b - c)}{B^2},$$

$$Q - b = \frac{2(m - b)(m - c)(a - b)}{B},$$

$$Q - m = -\frac{A(m - b)}{B},$$

$$\phi - Q = -\frac{2(m - b)(m - c)C(a - b)}{B^2},$$

and

$$\sqrt{\Phi} = 2\sqrt{M}(c - b)(b - a)\frac{AC}{B^3},$$

which determine ϕ, $\sqrt{\Phi}$.

Point 3′ by means of the conic 1263′4′5′.

The conic is

$$Fx + Gy + H = \frac{(x - b)(x - c)}{y}, \qquad (4',\ 5')$$

and we have

$$G + H = bc, \qquad (2)$$

$$Fm + G\sqrt{M} + H = \frac{(m-b)(m-c)}{\sqrt{M}}, \qquad (1)$$

$$Fm_1 - G\sqrt{M_1} + H = \frac{(m_1-b)(m_1-c)}{-\sqrt{M_1}}, \qquad (6).$$

Eliminating F, we have

$$G(m_1\sqrt{M} + m\sqrt{M_1}) + H(m - m_1) = \frac{m_1(m-b)(m-c)}{\sqrt{M}} + \frac{m(m_1-b)(m_1-c)}{\sqrt{M_1}},$$

which is easily reduced first to

$$G\frac{2mm_1 - a(m+m_1)}{(m-a)\sqrt{M}} + H(m - m_1) = (m + m_1)\frac{(m-a)(m-b)}{\sqrt{M}},$$

and then to

$$G\{aA + 2m(a-b)(a-c)\} - H\frac{(m-a)A}{\sqrt{M}} + abc\{-A + 2m(m-a)\} = 0;$$

and combining herewith $G + H = bc$, we have

$$H = \frac{2bc\,m\,[a(m-a) + (a-b)(a-c)]}{aA + 2m(a-b)(a-c) + \dfrac{(m-a)A}{\sqrt{M}}};$$

$$G = bc - H;$$

and we have then

$$F(m + m_1) + G(\sqrt{M} - \sqrt{M_1}) + 2H = 0,$$

that is,

$$F\{2m(m-a) - A\} + G\frac{A\sqrt{M}}{m-a} + 2H(m-a) = 0,$$

or, what is the same thing,

$$F\{2m(m-a) - A\} = -bc\frac{A\sqrt{M}}{m-a} - H\left\{2(m-a) - \frac{A\sqrt{M}}{m-a}\right\}.$$

We then have

$$Fx + H = y\left(-G + \frac{(x-b)(x-c)}{y^2}\right)$$

$$= y\left(-G - \frac{abc}{x-a}\right) = -\frac{y(Ha + Gx)}{x-a},$$

that is,

$$(Fx + H)^2 = -\frac{(x-b)(x-c)}{abc(x-a)}(Ha + Gx)^2,$$

or

$$abc\,(x-a)\,(Fx+H)^2+(x-b)\,(x-c)\,(Gx+Ha)^2=0.$$

Developing and throwing out the factor x, this is

$$\begin{aligned}&G^2x^3\\&+\{2a\,GH-(b+c)\,G^2+abc\,F^2\}\,x^2\\&+\{a^2H^2-2a\,(b+c)\,GH+bc\,G^2+abc\,(2FH-aF^2)\}\,x\\&+\{-(b+c)\,a^2H^2+2abc\,GH+abc\,(H^2-2aFH)\}=0.\end{aligned}$$

This must be satisfied by $x=m$, $x=m_1$; hence the left hand must be $=G^2(x-m)(x-m_1)(x-\sigma)$, or equating the constant terms we have

$$G^2\,mm_1\,\sigma=aH\,\{-2abc\,F+2bc\,G+(bc-ab-ac)\,H\},$$

which gives σ; and we then have

$$\sqrt{\Sigma}=-\frac{\sigma-a}{G\sigma+Ha}(F\sigma+H),$$

but I have not attempted the further reduction of these expressions.

The numerical values for the example are

$$3F=\frac{-140+62\sqrt{14}}{5+21\sqrt{14}},\quad G=\frac{-10+62\sqrt{14}}{5+21\sqrt{14}},\quad H=\frac{-104\sqrt{14}}{5+\sqrt{14}},$$

whence σ as in the Table.

Point 2′ by means of the conic 1362′4′5′.

The equation of the conic is

$$Fx+Gy+H=\frac{(x-b)\,(x-c)}{y},\qquad (4',\ 5')$$

where

$$-G+H=-bc,\qquad (3)$$

$$Fm+G\sqrt{M}+H=\frac{(m-b)\,(m-c)}{\sqrt{M}},\qquad (1)$$

$$Fm_1-G\sqrt{M_1}+H=\frac{(m_1-b)\,(m_1-c)}{-\sqrt{M_1}},\qquad (6),$$

which are the same as for point 3′, if only we reverse the signs of F, H and $\sqrt{M}$, $\sqrt{M_1}$.

Hence the formulæ are

$$H = -\frac{2bc\,m\,[a(m-a)+(a-b)(a-c)]}{aA+2m(a-b)(a-c)-\dfrac{(m-a)A}{\sqrt{M}}},$$

$$G = \quad bc + H,$$

$$F\{2m(m-a)-A\} = -bc\frac{A\sqrt{M}}{m-a} - H\left\{2(m-a)+\frac{A\sqrt{M}}{m-a}\right\},$$

$$G^2 mm_1\,\tau = aH\{-2abc\,F - 2bc\,G + (bc-ab-ac)\,H\},$$

which gives τ; and then

$$\sqrt{T} = \frac{(\tau-a)}{G\tau - Ha}(F\tau + H),$$

which are also unreduced.

The numerical values are

$$3F = \frac{140+62\sqrt{14}}{5-21\sqrt{14}}, \quad G = \frac{-10-62\sqrt{14}}{5-21\sqrt{14}}, \quad H = \frac{-104\sqrt{14}}{5-21\sqrt{14}},$$

whence τ as in the Table.

522.

NOTE ON THE THEORY OF INVARIANTS.

[From the *Mathematische Annalen*, vol. III. (1871), pp. 268—271.]

If two binary quantics $(a,..)(x, y)^n$, $(a',..)(x', y')^n$ are linearly transformable the one into the other, and if for the first of them P, Q are any two invariants whatever of the same degree, and P', Q' are the like invariants for the second of them, then we have

$$P : Q = P' : Q',$$

(or, what is the same thing, the absolute invariants have the same values for the two functions respectively); and the entire system of these equations constitutes only a $(n-3)$ fold relation between the two sets of coefficients. But the converse theorem, viz. that if the entire system of equations is satisfied, the two functions are linearly transformable the one into the other, is only true *sub modo*.

For instance, considering the two binary sextics

$$(0, 0, 0, d, e, f, g)(x, y)^6 \text{ and } (0, 0, 0, d', e', f', g')(x', y')^6,$$

or, what is the same thing,

$$(20d, 5e, 2f, g)(x, y)^3 y^3 \text{ and } (20d', 5e', 2f', g')(x', y')^3 y'^3,$$

the invariants of the two functions respectively are each and all of them $=0$, and yet the two functions are not in general linearly transformable the one into the other.

For they can be transformable only by the substitution

$$x = \lambda x' + \mu y', \quad y = \rho y';$$

or, what is the same thing, only if the cubic functions are transformable by the substitution $x = x' + ay'$, $y = y'$; and forming for these the seminvariants $ac - b^2$ and $a^2d - 3abc + 2b^3$ for the cubic $(a, b, c, d)(x, y)^3$, we have as the necessary condition for the transformability

$$(8df - 5e^2)^3 : (8d^2g - 12def + 5e^3)^2 = (8d'f' - 5e'^2)^3 : (8d'^2g' - 12d'e'f' + 5e'^3)^2.$$

To deduce this result from the theory of the sextic function, I observe that denoting by A, B, C, Δ, the values of the quadrinvariant, the sextinvariant, and the discriminant, as given in Salmon's *Higher Algebra*, Ed. 2, pp. 202—211, then in the particular case $a=0$, $b=0$, we have

$$\begin{array}{llll} A = -10\,d^2 & B = d^4 & C = -8\,d^6 & \Delta = 0, \\ +15\,ce, & -3\,cd^3e & +36\,cd^4e & \\ & +c^2e^2, & -39\,c^2d^2e^2 & \\ & & -8\,c^3e^3, & \end{array}$$

and hence forming the new invariants

$$\begin{aligned} \mathrm{B} &= 100\,B - A^2, \\ \Gamma &= 1000\,C - 1200\,AB + 4A^3, \end{aligned}$$

the values of these in the same particular case $a=b=0$ are

$$\begin{array}{ll} \mathrm{B} = 25\,c^2\,(8\,df - 5e^2) & \Gamma = -2500\,c^3\,(\,8\,d^2g - 12\,def + 5e^3) \\ \phantom{\mathrm{B} = 25\,c^2\,(8\,df}-100\,c^3g, & +3000\,c^4\,(10\,eg\ - 9f^2). \end{array}$$

Taking now A, B, C, Δ as the invariants of the sextic, one of the conditions for the transformation is $B^3 : C^2 = B'^3 : C'^2$.

In the particular case $a=b=c=0$ and $a'=b'=c'=0$, the invariants vanish and the equation is satisfied identically. But if we assume in the first instance only $a=b=0$, $a'=b'=0$, then the terms contain the common factors c^6 and c'^6 respectively; and throwing these out, and then writing $c=0$, $c'=0$, we obtain the condition previously found in a different manner.

It will be observed that the condition is of the original form $P : Q = P' : Q'$, but with the difference that P, Q and the corresponding functions P', Q', are not invariants. As possessing the foregoing property these functions may however be called "imperfect invariants," it being understood that an imperfect invariant is not an invariant, and is not in any case included in the term "invariant" used without qualification.

And we may now establish the general theory as follows: Consider the similarly constituted special forms $(a,\,..)(x,\,y,\,z,\,..)^n$ and $(a',\,..)(x',\,y',\,z',\,..)^n$: to fix the ideas the coefficients $(a,\,..)$ may be regarded as homogeneous functions of the elements $(\alpha,\,\beta,\,..)$ which are either independent, or homogeneously connected together in any manner; and then the coefficients $(a',\,..)$ will be the like functions of the elements $(\alpha',\,\beta',\,..)$ which are either independent or (as the case may be) homogeneously connected in the like manner.

The entire series of functions $P, Q, \ldots$ of $(\alpha,\,\beta,..)$, which are such that P, Q being of the same degree, and P', Q' being the like functions of $(\alpha',\,\beta')$, we have for the linearly transformable functions $(a,..)(x,\,y,\,z,..)^n$ and $(a',\ldots)(x',\,y',\,z',..)^n$ the relation

$$P : Q = P' : Q',$$

may be called the "perfect and imperfect invariants" of $(a, ..)(x, y, z, ..)^n$; and the relation in question be briefly referred to by the expression that the perfect and imperfect invariants are proportional.

We have then the theorem that if the two functions $(a, ..)(x, y, z, ..)^n$ and $(a', ..)(x', y', z', ..)^n$ are linearly transformable the one into the other, the two functions have their perfect and imperfect invariants proportional; and *conversely* the theorem, that two functions which have their perfect and imperfect invariants proportional, are linearly transformable the one into the other.

There is thus a wide field of inquiry in regard to the imperfect invariants, even of a binary function, but still more so as to those of a ternary or quaternary function representing a curve or surface possessed of singularities.

We have in what precedes the explanation of an error into which I fell in my paper "On the transformation of plane curves," *Proc. Lond. Math. Soc.*, vol. I. No. 3, Oct. 1865, [384], see Arts. Nos. 27—30. Considering a given curve of deficiency D and, by means of a system of $D-3$ points chosen at pleasure on the curve, transforming this into a curve of the order $D+1$ with deficiency D; then for any two of the transformed curves (that is, two curves obtained by means of different systems of the $D-3$ points) I showed that these had the same absolute invariants—or in the language of the present paper, that they had their invariants proportional, and I thence inferred that the two transformed curves were linearly transformable the one into the other—whereas, to sustain this conclusion, it is necessary that the two curves should have their perfect and imperfect invariants proportional; and this was in no wise proved. That the two transformed curves are not in fact linearly transformable the one into the other has since been shown *a posteriori* by Dr Brill in the particular case $D=4$. Riemann's conclusions, with which my own were at variance, are thus correct.

I remark that if a binary function of an odd or even degree $n=2p+1$ or $=2p$, has $p+1$ equal factors, then the invariants all of them vanish; but the equality of the $p+1$ factors implies only a p-fold relation between the coefficients; that is, the vanishing of all the invariants gives only a p-fold relation between the coefficients, viz. the relation is $\frac{1}{2}(n-1)$ fold or $\frac{1}{2}n$-fold according as n is odd or even. Thus for a sextic function the equations $A=0$, $B=0$, $C=0$, $\Delta=0$ constitute only a 3-fold relation between the coefficients.

Similarly if the function has p equal factors, then every invariant is a mere numerical multiple of a power of one and the same function Θ; so that the vanishing invariants can be at once formed. And we have thus only a $(p-1)$ fold relation between the coefficients, viz. the relation is $\frac{1}{2}(n-3)$ fold or $\frac{1}{2}(n-2)$ fold according as n is odd or even.

Cambridge, 4 August, 1870.

523.

ON THE TRANSFORMATION OF UNICURSAL SURFACES.

[From the *Mathematische Annalen*, vol. III. (1871), pp. 469—474.]

I CONSIDER the question of the transformation (Abbildung auf einer Ebene) of unicursal surfaces. Taking (x, y, z, w) for the coordinates of a point on the surface, (x', y', z') for those of the corresponding point on the plane; then if X', Y', Z', W' denote each of them a function $(x', y', z')^{n'}$, the equations of transformation are

$$x : y : z : w = X' : Y' : Z' : W':$$

and assuming that each of the curves

$$X' = 0, \quad Y' = 0, \quad Z' = 0, \quad W' = 0$$

(or, what is the same thing, the general curve

$$aX' + bY' + cZ' + dW' = 0)$$

passes once through each of α_1 points, twice through each of α_2 points, ... , r times through each of α_r points (for convenience I write α_r instead of α_r'); and writing also

$$n = n'^2 - \Sigma r^2 \alpha_r,$$

$$0 = \tfrac{1}{2} n'(n'+3) - 3 - \Theta - \Sigma \tfrac{1}{2} r(r+1)\alpha_r,$$

(where Θ is $= 0$ or positive except in the case of special relations between the positions of the fixed points $\alpha_1, \alpha_2, \ldots, \alpha_r$), which equations give

$$-n = 3n' - 6 - 2\Theta - \Sigma r\alpha_r;$$

then the order of the surface is $= n$, and the order of the nodal curve is $b = \tfrac{1}{2}(n-2)(n-3) + \Theta$. I assume that the nodal curve has h apparent double points and t actual triple points, but no stationary points, so that q being the class, we have $q = b^2 - b - 2h - 6t$; and I endeavour to find these numbers q, t, h.

For this purpose, imagine through the nodal curve a surface of the order k, which therefore meets the surface besides in a curve of the order $nk-2b$; this curve I call the k-thic residue of the nodal curve, or simply the "residue." The projection (Abbildung) of the complete intersection kn is a curve of the order kn' passing kr times through each of the points α_r: this is made up of the projection of the nodal curve *once*, and of the projection of the residue. But as shown by Dr Clebsch the projection of the nodal curve is of the order $(n-4)n'+3$, and it passes $(n-4)r+1$ times through each of the points α_r; hence the projection of the residue is of the order $(k-n+4)n'-3$, and it passes $(k-n+4)r-1$ times through each of the points α_r. I assume that the projection of the residue is the *general* curve which satisfies the foregoing conditions, viz. that the residue, and its projection as defined by the foregoing conditions, *depend each of them on the same number of constants.* The necessity for this is I confess by no means obvious: but take as an illustration Steiner's quartic surface as transformed by the equations $x:y:z:w=x'^2:y'^2:z'^2:(x'+y'+z')^2$: the nodal curve consists of three lines meeting in a point, the quadric residue is the remaining intersection of the surface by a quadric cone passing through the three lines; and the projection thereof is a line; the quadric cone, and therefore the conic, each depend upon 2 constants; and the line which is the projection of the conic depends upon the same number (2) of constants: at all events I make the assumption provisionally.

Now in the projection of the residue, we have twice the number of constants

$$=[(k-n+4)n'-3](k-n+4)n'-\Sigma[(k-n+4)r-1](k-n+4)r\alpha_r,$$

viz. this is

$$=(k-n+4)^2(n'^2-\Sigma r^2\alpha_r)+(k-n+4)(-3n'+\Sigma r\alpha_r),$$

or, what is the same thing, it is

$$=(k-n+4)^2n+(k-n+4)(n-6-2\Theta),$$

viz. reducing, and replacing Θ by its value $=-\frac{1}{2}(n-2)(n-3)+b$, the number in question is

$$=k^2n+k(-n^2+4n-2b)+2(n-4)b.$$

Now k being $=$ or $>n-3$, the curve of intersection of a given surface n by a surface k depends on

$$\tfrac{1}{6}(k+1)(k+2)(k+3)-\tfrac{1}{6}(k-n+1)(k-n+2)(k-n+3)-1$$

constants; and making the surface k to pass through the curve b we have to subtract herefrom $(k+1)b-\frac{1}{2}g-2t$; that is, for the residue, twice the number of constants is

$$=\tfrac{1}{3}(k+1)(k+2)(k+3)-\tfrac{1}{3}(k-n+1)(k-n+2)(k-n+3)-2-2(k+1)b+g+4t,$$

viz. this is

$$=k^2n+k(-n^2+4b-2b)+\tfrac{1}{3}(n-1)(n-2)(n-3)-2b+g+4t.$$

Hence comparing the two expressions in question we have

$$2(n-4)b = \tfrac{1}{3}(n-1)(n-2)(n-3) - 2b + q + 4t,$$

that is,

$$0 = \tfrac{1}{3}(n-1)(n-2)(n-3) - 2(n-3)b + q + 4t,$$

or, as I prefer to write it,

$$0 = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)b + \tfrac{1}{2}q + 2t;$$

which agrees with a more general formula in my "Memoir on the theory of Reciprocal Surfaces," *Phil. Trans.* vol. CLIX. (1869), [411], see p. 227, [*Coll. Math. Papers*, vol. VI. p. 356]. I consider any two residues, a k-thic residue and a l-thic residue; to each intersection of these there corresponds an intersection of their projections: or the number of intersections of the two residues must be equal to that of the two projections. Now the projections being (as above)

order $(k-n+4)n'-3$ passing $(k-n+4)r-1$ times through each point α_r,
" $(l-n+4)n'-3$ " $(l-n+4)r-1$ " " "

the number of the intersections in question is

$$= [(k-n+4)n'-3][(l-n+4)n'-3] - \Sigma[(k-n+4)r-1][(l-n+4)r-1]\alpha_r + \omega,$$

where for a reason which will be afterwards explained I have added the term ω: this is

$$= (k-n+4)(l-n+4)(n'^2 - \Sigma r^2\alpha_r) + (k+l-2n+8)(-3n' + \Sigma r\alpha_r) + 9 - (\Sigma\alpha_r - \omega),$$

viz. it is

$$= (k-n+4)(l-n+4)n + (k+l-2n+8)(n-6-2\Theta) + 9 - (\Sigma\alpha_r - \omega),$$

viz. substituting for Θ its value, $= -\tfrac{1}{2}(n-2)(n-3) + b$, and reducing, the number is

$$= kln - 2(k+l)b - n^3 + 8n^2 - 16n + 9 + 4(n-4)b - (\Sigma\alpha_r - \omega).$$

But the surfaces n, k, l, having in common the curve b which is a nodal curve on n, besides intersect in

$$kln - b(n+2k+2l-4) + 2q + qt$$

points (Salmon's *Geometry of three Dimensions*, 2nd Ed. p. 283, except that in the formula as there given the singularity t is not taken account of); that is, the number of intersections of the two residues is

$$= kln - 2(k+l)b - (n-4)b + 2q + 9t,$$

which is equal to the number of intersections of the two projections[1]: or comparing the numbers in question we have

$$-n^3 + 8n^2 - 16n + 9 + 4(n-4)b - (\Sigma\alpha_r - \omega) = -(n-4)b + 2q + 9t,$$

that is,

$$2q + 9t = 5(n-4)b - n^3 + 8n^2 - 16n + 9 - (\Sigma\alpha_r - \omega).$$

[1] I remark that $n+\lambda$ being positive or not less than $n-3$, two $(n+\lambda)$-thic residues meet in $n(\lambda+4)(\lambda+6) - 12\lambda - 39 - 4(\lambda+4) - (\Sigma\alpha_r - \omega)$ points: in particular, two $(n-3)$-thic residues meet in $3n-3-4\Theta-(\Sigma\alpha_r-\omega)$ points; and two $(n-2)$-thic residues meet in $8n-15-8\Theta-(\Sigma\alpha_r-\omega)$ points.

But we have already found

$$2q + 8t = 4(n-3)b - \tfrac{2}{3}n^3 + 4n^2 - \tfrac{22}{3}n + 4,$$

and we have therefore

$$t = (n-8)b - \tfrac{1}{3}n^3 + 4n^2 - \tfrac{26}{3}n + 5 - (\Sigma\alpha_r - \omega),$$

and

$$q = -2(n-13)b + n^3 - 14n^2 + 31n - 18 + 4(\Sigma\alpha_r - \omega).$$

I obtain these results in a different manner by investigating expressions for the deficiency (Geschlecht) of the nodal residue $nk - 2b$ and for that of its projection.

First for the projection, we have

$$\begin{aligned}\text{Twice Deficiency} \quad &= [(k-n+4)n' - 4][(k-n+4)n' - 5] \\ &\quad - \Sigma[(k-n+4)r - 1][(k-n+4)r - 2]\alpha_r + 2\omega,\end{aligned}$$

where I have added the term 2ω, as afterwards explained: this is

$$= (k-n+4)^2(n'^2 - \Sigma r^2\alpha_r) + (k-n+4)(-9n' + 3\Sigma r\alpha_r) + 20 - 2(\Sigma\alpha_r - \omega),$$

viz. it is

$$= (k-n+4)^2 n + (k-n+4)(3n - 18 - 6\Theta) + 20 - 2(\Sigma\alpha_r - \omega),$$

or substituting for Θ its value $-\frac{1}{2}(n-2)(n-3) + b$ and reducing, it is

$$= k^2n + k(n^2 - 4n - 6b) - 2n^3 + 16n^2 - 32n + 20 + 6(n-4)b - 2(\Sigma\alpha_r - \omega).$$

Next as regards the residue, the number h' of its apparent double points is obtained in terms of h and t by the formula

$$8h + 6t - 2h' = (kn - 4b)(k-1)(n-1) - 2b(k-1),$$

(Salmon, l. c., p. 284, except that the singularity t is not there taken account of); and we thence have

$$\begin{aligned}\text{Twice Deficiency} \quad &= (kn-1)(kn-2) - 2h' \\ &= kn(k+n-4) + 4b^2 + b(-4n - 6k + 12) + 2 - 8h - 6t,\end{aligned}$$

or introducing q instead of h by the formula $4b^2 - 8h = 4q + 4b + 24t$, this is

$$= kn(k+n-4) + b(-4n - 6k + 12) + 4q + 4b + 2 + 18t,$$

viz. it is

$$= k^2n + k(n^2 - 4n - 6b) - 4(n-4)b + 4q + 18t.$$

So that comparing with the deficiency of the projection we have

$$-2n^3 + 16n^2 - 32n + 20 + 6(n-4)b - 2(\Sigma\alpha_r - \omega) = -4(n-4)b + 4q + 2 + 18t,$$

that is,

$$2q + 9t = 5(n-4)b - n^3 + 8n^2 - 16n + 9 - (\Sigma\alpha_r - \omega),$$

the same result as before.

The necessity for the term ω appears by the consideration that if we apply to the plane figure a Cremona-transformation, thus obtaining a new transformation of the surface, the value of $\Sigma\alpha_r$ will in general be altered; whereas the expressions for q, t should it is clear remain unaltered; and it arises as follows, viz. for certain transformations of the surface the curve of the order $(k-n+4)n'-3$, passing $(k-n+4)r-1$ times through each point α_r and assumed to be the projection of the residue, is not an indecomposable curve but contains a certain number ω of factors (each belonging to a unicursal curve definable by means of the number of its passages through the several points α_r), which factors are to be rejected in order to obtain the equation of the proper residue. Thus reverting to the transformation

$$x : y : z : w = x'^2 : y'^2 : z'^2 : (x'+y'+z')^2$$

of Steiner's surface, the projection of the quadric residue was (as already remarked) a line; applying to the plane figure the ordinary quadric (or inverse) transformation we introduce three fixed points, $(\alpha_2 = 3)$, say these are A, B, C; viz. in the new transformation of the surface the projection of any plane section is a quartic curve having a node at each of the fixed points: the projection of the residue ought clearly to be a conic through the three points; but according to the general formula it is a quintic having at each of these points a triple point: the quintic is in fact made up of the lines BC, CA, AB and of the conic which is the proper residue; viz. in the case in question there are 3 factors thrown out, or we have $\omega = 3$. To apply this to the second investigation of $2q+9t$, by comparison of the two deficiencies, observe that in general if a curve is made up of $\omega+1$ indecomposable curves, the deficiency of the compound curve is equal to the sum of the deficiencies of the component curves $-\omega$; hence if ω of the curves are unicursal, the deficiency of the compound curve is equal to that of the remaining curve $-\omega$; or, what is the same thing, the deficiency of the remaining curve is $=$ that of the compound curve $+\omega$; and the addition of the term $+\omega$ to the expression for the deficiency is thus accounted for. It is easy to see that a like explanation applies to the first investigation of $2q+9t$.

I further remark, reverting to the equations

$$x : y : z : w = X' : Y' : Z' : W'$$

of the transformation, that the product of the ω factors is given as the common factor (if any) of the Jacobians

$$J(Y', Z', W'),\quad J(Z', W', X'),\quad J(W', X', Y') \text{ and } J(X', Y', Z').$$

Such common factor exists whenever we can by a Cremona-transformation of the plane figure reduce the number of the points α_r upon which the transformation of the surface depends; viz. for any given transformation of the surface, ω is equal to the excess of $\Sigma\alpha_r$ above the minimum value of $\Sigma\alpha_r$, or, what is the same thing, $\Sigma\alpha_r - \omega$ is equal to the minimum value of $\Sigma\alpha_r$, and is thus independent of the particular transformation. And of course if $\Sigma\alpha_r$ has this minimum value, viz. if the transformation is such that the number of the points α_r cannot be reduced by any Cremona-trans-

formation of the plane figure, then we have $\omega = 0$. I presume that for the most simple transformation, that is, when n' has its least value, $\Sigma\alpha_r$ has also its least value, and consequently that ω is $= 0$.

Recapitulating, the results obtained are

$$q = -2(n-13)b + n^3 - 14n^2 + 31n - 18 + 4(\Sigma\alpha_r - \omega),$$
$$t = \quad (n-8)b - \tfrac{1}{3}n^3 + 4n^2 - \tfrac{26}{3}n + 5 - (\Sigma\alpha_r - \omega),$$

where it will be recollected that

$$b = \tfrac{1}{2}(n-2)(n-3) + \Theta;$$

the formulæ are verified in the several cases:

n'	a_1	a_2'	n	Θ	ω	b	q	t	
2	2	0	2	0	1	0	0	0	Quadric surface
2	1	0	3	1	0	1	0	0	Cubic scroll
2	0	0	4	2	0	3	0	1	Steiner's quartic surface
3	6	0	3	0	0	0	0	0	Cubic surface
3	5	0	4	1	0	2	2	0	Quartic with nodal conic
2	8	1	4	0	0	1	0	0	Do. with nodal line
2	7	1	5	1	0	4	8	0	Quintic with nodal quadriquadric
2	11	0	5	0	0	3	4	0	Do. with nodal skew cubic
2	12	2	5	-1	0	2	0	0	Do. with two non-intersecting nodal lines

which are the transformations chiefly as yet examined: but the first-mentioned case (quadric surface, generalised stereographic projection), although as stated the formulæ are verified with the value $\omega = 1$, does not really come under the foregoing theory. It is interesting to see that they are verified in the last-mentioned case, belonging to a negative value of Θ, that is, to a special system of fixed points.

Cambridge, Dec. 5, 1870.

524.

ON THE DEFICIENCY OF CERTAIN SURFACES.

[From the *Mathematische Annalen*, vol. III. (1871), pp. 526—529.]

If a given point or curve is to be an ordinary or singular point or curve on a surface of the order n, this imposes on the surface a certain number of conditions, which number may be termed the "Postulation"; thus "Postulation of a given curve *quà* i-tuple curve on a surface n" will denote the number of conditions to be satisfied by the surface in order that the given curve may be an i-tuple curve on the surface.

The "deficiency" (Flächengeschlecht) of a given surface of the order n is

$=\frac{1}{6}(n-1)(n-2)(n-3)$ *less* deficiency-value of the several singularities; viz. as shown by Dr Noether, if the surface has a given i-tuple curve, the deficiency-value hereof is

$=$ Postulation of the curve *quà* $(i-1)$ tuple curve on a surface $n-4$;

and if the surface has an i-conical point, the deficiency-value hereof is

$=$ Postulation of the point *quà* $(i-2)$ conical point on a surface $n-4$; viz. this is $=\frac{1}{6}i(i-1)(i-2)$, and is thus independent of the order of the surface.

I remark that if the tangent-cone at the i-conical point has δ double lines and κ cuspidal lines, then the deficiency-value is

$$=\tfrac{1}{6}i(i-1)(i-2)+(i-2)(n-i-1)(\delta+\kappa).$$

In the case of a double or cuspidal curve i is $=2$, and the deficiency-value is

$=$ Postulation of given curve *quà* simple curve on a surface $n-4$;

and so for an ordinary conical point i is $=2$, and the deficiency-value is $=0$: results which were first obtained by Dr Clebsch.

I found in this manner the expression for the deficiency of a surface n having a double and cuspidal curve and the other singularities considered in my "Memoir on the Theory of Reciprocal Surfaces," *Phil. Trans.* vol. CLIX. (1869), [411, *Coll. Math. Papers*, vol. VI. p. 356]; viz. this was

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}y + i - \tfrac{1}{8}\theta,$$

where we have

b, order of double curve,

q, class of Do.,

c, order of cuspidal curve,

r, class of Do.,

β, number of intersections of the two curves, stationary points on b,

y, number of intersections, stationary points on c,

i, number of intersections, not stationary points on either curve,

θ, number of certain singular points on c, the nature of which I do not completely understand; it is here taken to be $=0$.

Before going further I remark that

Postulation of right line *quà* i-tuple on surface n

$$= \tfrac{1}{2}i(i+1)n - \tfrac{1}{6}i(i+1)(2i-5),$$
$$= \tfrac{1}{6}i(i+1)(3n-2i+5).$$

Whence if a surface n has an i-tuple right line, the deficiency-value hereof is

$$= \tfrac{1}{6}i(i-1)(3n-2i-5),$$

or we have

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - \tfrac{1}{6}i(i-1)(3n-2i-5)$$
$$= \tfrac{1}{6}(i-n+1)(i-n+2)(2i+n-3);$$

so that $D=0$ if either $i=n-1$ or $i=n-2$; the former case is that of a scroll (skew surface) with a $(n-1)$ tuple right line, the latter that of a surface with a $(n-2)$ tuple line: whence (as shown by Dr Noether) such surface is rationally transformable into a plane.

For a surface of the order n with an i-conical point where the tangent cone has δ double lines and κ cuspidal lines, we have

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - \{\tfrac{1}{6}i(i-1)(i-2) + (i-2)(n-i-1)(\delta+\kappa)\}$$
$$= \tfrac{1}{6}(n-i-1)\{n^2 + n(i-5) + i^2 - 4i + 6 - 6(i-2)(\delta+\kappa)\};$$

viz. for $i=n-1$ this is $D=0$ (in fact, a surface n with a $(n-1)$ conical point is at once seen to be rationally transformable into a plane): and for $i=n$, that is, for a cone of the order n, we have

$$D = -\tfrac{1}{2}(n-1)(n-2) + (n-2)(\delta+\kappa) - (n-3)(\delta+\kappa),$$

where the last term $-(n-3)(\delta+\kappa)$ is added because in the present case the surface has the δ double lines and the κ cuspidal lines.

The formula therefore gives

$$D = -\tfrac{1}{2}(n-1)(n-2) + \delta + \kappa,$$

viz. this is equal to the deficiency of the plane sections *taken negatively.*

I find that the same property exists *first* in the case of a scroll (skew surface) having only a double curve; and *secondly* in the case of a torse (developable surface) having a cuspidal curve with the ordinary singularities; and this being so there can I think be no doubt but that it is true for any scroll or torse whatever—viz. that for any ruled surface whatever the deficiency is equal to that of the plane section taken negatively.

First, for the scroll, we have

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)b + \tfrac{1}{2}q + 2t,$$

which should be

$$= -\tfrac{1}{2}(n-1)(n-2) + b.$$

Salmon's equations give in the case of a scroll

$$3t = (n-4)\{3b - n(n-2)\},$$
$$q = n(n-2)(n-5) - 2(n-6)b,$$

and with these values the relation is at once verified.

Secondly, for the torse; changing the notation into that used for the singularities of the curve and torse, we have

$$D = \tfrac{1}{6}(r-1)(r-2)(r-3) - (r-3)(x+m) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + \alpha,$$

which should be

$$= -\tfrac{1}{2}(m-1)(m-2) + h + \beta.$$

We have $q = r(n-3) - 3\alpha$, and substituting this value and expressing everything in terms of r, m, n by means of the formulæ

$$x = \tfrac{1}{2}(r^2 - r - n - 3m),$$
$$\alpha = m - 3r + 3n,$$
$$\beta = n - 3r + 3m,$$
$$t = \tfrac{1}{6}\{r^3 - 3r^2 - 58r - 3r(n+3m) + 42n + 78m\},$$
$$\gamma = rm + 12r - 14m - 6n,$$
$$h = \tfrac{1}{2}(m^2 - 10m - 3n + 8r),$$

we have after all reductions

$$D = -\tfrac{1}{2}(m+n) + r - 1 = -\tfrac{1}{2}(m-1)(m-2) + h + \beta.$$

We have thus a class of surfaces of *negative deficiency;* viz. any rational transformation of a cone for which the plane section has a given (positive) deficiency produces such a surface: and I think it may be assumed conversely that a surface of negative deficiency is always the rational transformation of a cone for which the deficiency is equal to that of the surface taken with the reverse sign. As an instance, take a quintic surface having a nodal conic and two 3-conical (cubiconical) points (this of course implies that the line joining the two cubiconical points is a line on the surface); the formula for the deficiency is ($n=5$, $b=2$, $q=2$, $r=0$, $t=0$)

$$D=\tfrac{1}{6}(n-1)(n-2)(n-3)-(n-3)b+\tfrac{1}{2}q+2t-2,$$

(viz. a term -1 for each of the cubiconical points)

$$=4-4+1\quad-2,\quad=-1.$$

Such a surface can be obtained as the quadric inverse of a cubic cone; viz. taking for the vertex the point $x:y:z:w=\alpha:\beta:\gamma:\delta$ and the cone to pass through the point $x=0$, $y=0$, $z=0$, the equation of the cone is

$$(\delta x-\alpha w,\quad \delta y-\beta w,\quad \delta z-\gamma w)^3=0,$$

where $(\alpha, \beta, \gamma)^3=0$.

Taking Q a quadric function $(x, y, z)^2$, the transformation in question consists in the change of x, y, z, w into xw, yw, zw, Q; viz. the new equation, rejecting the factor w which divides out, is

$$\frac{1}{w}(\delta xw-\alpha Q,\quad \delta yw-\beta Q,\quad \delta zw-\gamma Q)^3=0,$$

which is a quintic surface, having the two cubiconical points $x=0$, $y=0$, $z=0$ and $x:y:z:w=\alpha:\beta:\gamma:\frac{1}{\delta}Q_0$ (where Q_0 is the value of Q on writing therein α, β, γ in place of x, y, z): and having the nodal conic $w=0$, $Q=0$.

Cambridge, 5 *Jan.* 1871.

525.

AN EXAMPLE OF THE HIGHER TRANSFORMATION OF A BINARY FORM.

[From the *Mathematische Annalen*, vol. IV. (1871), pp. 359—361.]

THE quartic

$$(1) \qquad (a,\ b,\ c,\ d,\ e)(x,\ y)^4$$

is by means of the two quadrics

$$(2) \qquad (\alpha,\ \beta,\ \gamma)(x,\ y)^2 \quad \text{and} \quad (\alpha',\ \beta',\ \gamma')(x,\ y)^2$$

transformed into

$$(3) \qquad (a_1,\ b_1,\ c_1,\ d_1,\ e_1)(x_1,\ y_1)^4,$$

that is, eliminating x, y from

$$(a,\ b,\ c,\ d,\ e)(x,\ y)^4 = 0,$$

$$x_1(\alpha,\ \beta,\ \gamma)(x,\ y)^2 + y_1(\alpha',\ \beta',\ \gamma')(x,\ y)^2 = 0,$$

we obtain

$$(a_1,\ b_1,\ c_1,\ d_1,\ e_1)(x_1,\ y_1)^4 = 0.$$

It is required to express the invariants of (3) in terms of the simultaneous invariants of (1) and (2).

Write

$$P,\ Q,\ R = \alpha x_1 + \alpha' y_1,\quad \beta x_1 + \beta' y_1,\quad \gamma x_1 + \gamma' y_1\,;$$

the equations from which $(x,\ y)$ have to be eliminated are

$$(a,\ b,\ c,\ d,\ e)(x,\ y)^4 = 0,\quad (P,\ Q,\ R)(x,\ y)^2 = 0,$$

and the result of the elimination therefore is

$$\begin{vmatrix} & a, & 4b, & 6c, & 4d, & e \\ a, & 4b, & 6c, & 4d, & e & \\ & & & P, & 2Q, & R \\ & & P, & 2Q, & R & \\ & P, & 2Q, & R & & \\ P, & 2Q, & R & & & \end{vmatrix} = 0,$$

viz. this determinant is the transformed quartic $(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4$.

The developed expression of the determinant is

$$\begin{aligned} & a^2R^4 - 8abQR^3 \\ & + \begin{pmatrix} -12\,ac \\ +16\,b^2 \end{pmatrix} PR^3 - 24\,acQ^2R^2 + \begin{pmatrix} 24\,ad \\ -48\,bc \end{pmatrix} PQR^2 - 32\,adQ^3R \\ & + \begin{pmatrix} 2\,ae \\ -32\,bd \\ +36\,c^2 \end{pmatrix} P^2R^2 + \begin{pmatrix} -16\,ae \\ +64\,bd \end{pmatrix} PQ^2R + 16\,aeQ^4 \\ & + \begin{pmatrix} 24\,be \\ -48\,cd \end{pmatrix} P^2QR - 32\,bePQ^3 + \begin{pmatrix} -12\,ce \\ +16\,d^2 \end{pmatrix} P^3R \quad + 24\,ceP^2Q^2 \\ & - 8\,deP^3Q + e^2P^4, \end{aligned}$$

so that writing for P, Q, R their values, we have the transformed function $(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4$, the coefficients being of the forms

$$\begin{aligned} a_1 &= (a, b, c, d, e)^2 (\alpha, \beta, \gamma)^4 \\ b_1 &= (\quad \text{,,} \quad)^2 (\alpha, \beta, \gamma)^3 (\alpha', \beta', \gamma') \\ &\vdots \\ e_1 &= (\quad \text{,,} \quad)^2 \quad . \quad . \quad . \quad (\alpha', \beta', \gamma')^4. \end{aligned}$$

Writing I, J for the invariants of the quartic (1), and

$$\begin{aligned} A &= 4(\alpha\beta' - \alpha'\beta)(\beta\gamma' - \beta'\gamma) - (\gamma\alpha' - \gamma'\alpha)^2, \\ B &= (e, c, a, b, c, d)(\alpha\beta' - \alpha'\beta, \gamma\alpha' - \gamma'\alpha, \beta\gamma' - \beta'\gamma)^2, \end{aligned}$$

we have I, J, A, B simultaneous invariants of the forms (1) and (2). Putting moreover $\nabla = I^3 - 27J^2$, and writing I_1, J_1, ∇_1, for the like invariants of the form (3), I find

$$\begin{aligned} I_1 &= 4(4\,IB^2 + 12\,JAB + \tfrac{1}{3}I^2A^2), \\ J_1 &= 8\{8\,JB^3 + \tfrac{4}{3}I^2AB^2 + 2\,IJA^2B + (2J^2 - \tfrac{1}{27}I^3)A^3\}, \end{aligned}$$

and thence

$$\nabla_1 = 256\,(4\,B^3 - IA^2B - JA^3)\,\nabla.$$

As a verification, suppose $(a, b, c, d, e)(x, y)^4 = x^4 + y^4$ (whence $I = 1$, $J = 0$). And take $x_1(x+y)^2 - y_1(x-y)^2 = 0$ for the transforming equation, that is, $(\alpha, \beta, \gamma) = (1, 1, 1)$ and $(\alpha', \beta', \gamma') = (-1, 1, -1)$. We have $P = R = x_1 - y_1$ and $Q = x_1 + y_1$, and thence

$$\begin{aligned}\text{Det.} &= (P^2 + R^2)^2 - 16\,PQ^2R + 16\,Q^4 \\ &= (2P^2 - 4Q^2)^2 = (-2x_1^2 - 12\,x_1y_1 - 2y_1^2)^2,\end{aligned}$$

that is,

$$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4 = 4\,(x_1^2 + 6x_1y_1 + y_1^2)^2,$$

whence

$$I_1 = \frac{4096}{3}, \quad = \frac{2^{12}}{3}; \quad J_1 = -\frac{262144}{27}, \quad = -\frac{2^{18}}{27};$$

also

$$A = -16, \quad B = 8,$$

and the equations for I_1, J_1 become

$$\frac{4096}{3} = 4\,(4\,.\,64 + \tfrac{1}{3}\,256),$$

$$-\frac{262144}{27} = 8\,(\tfrac{4}{3}\,.\,-16\,.\,64 + \tfrac{1}{27}\,4096),$$

which are true. More generally, assuming

$$(a, b, c, d, e)(x, y)^4 = x^4 + 6\Theta x^2y^2 + y^4,$$

(whence $I = 1 + 3\Theta^2$, $J = \Theta - \Theta^3$), and the same transforming equation, we have

$$(a_1, b_1, c_1, d_1, e_1)(x_1, y_1)^4 = 4\,\{(1 + 3\Theta)\,x_1^2 + (3 - 3\Theta)\,2x_1y_1 + (1 + 3\Theta)\,y_1^2\}^2,$$

whence

$$I_1 = \frac{2^{12}}{3}(1 - 3\Theta)^2, \quad J_1 = -\frac{2^{18}}{27}(1 - 3\Theta)^3;$$

also

$$A = -16, \quad B = 8\,(1 - \Theta).$$

Substituting these different values in the equations for I_1, J_1, we obtain

$$16\,(1 - 3\Theta)^2 = 12\,(1 + 3\Theta^2)(1 - \Theta)^2 - 72\,(\Theta - \Theta^3)(1 - \Theta) + 4\,(1 + 3\Theta^2)^2,$$

and

$$\begin{aligned}-8\,(1 - 3\Theta)^3 = {} & 27\,(\Theta - \Theta^3)(1 - \Theta)^3 - 9\,(1 + 3\Theta^2)^2(1 - \Theta)^2 \\ & + 27\,(1 + 3\Theta^2)(\Theta - \Theta^3)(1 - \Theta) - 54\,(\Theta - \Theta^3)^2 + (1 + 3\Theta^2)^3,\end{aligned}$$

which are in fact satisfied identically.

Cambridge, 26 *July,* 1871.

526.

ON A SURFACE OF THE EIGHTH ORDER.

[From the *Mathematische Annalen*, vol. IV. (1871), pp. 558—560.]

I REPRODUCE in an altered form, so as to exhibit the application thereto of the theory of the six coordinates of a line, the analysis by which Dr Hierholzer obtained the equation of the surface of the eighth order, the locus of the vertex of a quadricone which touches six given lines.

I call to mind that if $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ are the coordinates of any two points on a line, then the quantities (a, b, c, f, g, h), which denote respectively

$$(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\delta' - \alpha'\delta,\ \beta\delta' - \beta'\delta,\ \gamma\delta' - \gamma'\delta),$$

and which are such that $af + bg + ch = 0$, are the six coordinates of the line[1].

Consider the given point (x, y, z, w) and the given line (a, b, c, f, g, h), and write for shortness

$$\begin{aligned} P &= \quad hy - gz + aw, \\ Q &= -hx \quad + fz + bw, \\ R &= gx - fy \quad + cw, \\ S &= -ax - by - cz \quad , \end{aligned}$$

then taking (X, Y, Z, W) as current coordinates, the equation of the plane through the given point and line is

$$PX + QY + RZ + SW = 0.$$

Considering in like manner the given point (x, y, z, w) and the three given lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, \ldots)$, $(a_3, \ldots)$, then we have the three planes

$$\begin{aligned} P_1X + Q_1Y + R_1Z + S_1W &= 0, \\ P_2X + Q_2Y + R_2Z + S_2W &= 0, \\ P_3X + Q_3Y + R_3Z + S_3W &= 0, \end{aligned}$$

[1] Cayley, "On the six coordinates of a line," *Camb. Phil. Trans.* vol. XI. (1869), [435], pp. 290—323.

and if these planes have a common line, the point (x, y, z, w) is in a line meeting each of the three given lines; that is, the locus of the point is the hyperboloid through the three given lines. It follows that the equations

$$\left\| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right\| = 0$$

reduce themselves to a single equation, that of the hyperboloid in question.

I write for shortness

$$\begin{aligned}(000) = (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 \\ + [(abg) - (cah)]\,xw \\ + [(bch) - (abf)]\,yw \\ + [(caf) - (bcg)]\,zw \\ + [(bfg) + (chf)]\,yz \\ + [(cgh) + (afg)]\,zx \\ + [(ahf) + (bgh)]\,xy,\end{aligned}$$

viz. (123) will mean $(a_1g_2h_3)\,x^2 + \text{etc.}$ where $(a_1g_2h_3)$ etc. denote as usual the determinants

$$\left| \begin{matrix} a_1, & g_1, & h_1 \\ a_2, & g_2, & h_2 \\ a_3, & g_3, & h_3 \end{matrix} \right| \text{ etc.};$$

then the equations in question are found to be $x(123) = 0$, $y(123) = 0$, $z(123) = 0$, $w(123) = 0$, reducing themselves to the single equation $(123) = 0$, which is accordingly that of the hyperboloid through the three lines[1].

Proceeding now to the above-mentioned problem, we have the point (x, y, z, w), and the six lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, \ldots)$ etc., say the lines 1, 2, 3, 4, 5, 6: the six planes

$$P_1X + Q_1Y + R_1Z + S_1W = 0, \text{ etc.}$$

must be tangents to the same quadricone; that is, considering the sections by the plane $W = 0$, the six lines

$$P_1X + Q_1Y + R_1Z = 0, \text{ etc.}$$

must be tangents to the same conic, and the condition for this is

$$[1\,2\,3\,4\,5\,6] = 0,$$

[1] This equation is given in the paper above referred to, § 54.

where the symbol stands for the determinant

$$\begin{vmatrix} P_1^2, & Q_1^2, & R_1^2, & Q_1R_1, & R_1P_1, & P_1Q_1 \\ P_2^2 \ldots & & & & & \\ \vdots & & & & & \end{vmatrix}.$$

But as is well known this equation may be written

$$(*) \qquad (126)(346)(145)(235) - (146)(236)(125)(345) = 0,$$

where (126) etc. denote the determinants

$$\begin{vmatrix} P_1, & Q_1, & R_1 \\ P_2, & Q_2, & R_2 \\ P_3, & Q_3, & R_3 \end{vmatrix} \text{ etc.};$$

or, what is the same thing, they denote the functions above represented by the like symbols $(126) = (a_1 g_2 h_3) x^2 + \text{etc.}$ The equation $(*)$ just obtained is Hierholzer's equation for the surface of the eighth order, the locus of the vertex of a quadricone which touches six given lines.

I remark that in my "Memoir on Quartic Surfaces," *Proc. Lond. Math. Soc.* vol. III. (1870), [445], pp. 19—69, I obtained the equation of the surface under the foregoing form $[1\,2\,3\,4\,5\,6] = 0$ or say $[(P, Q, R)^2] = 0$, noticing that there was a factor w^4, so that the order of the surface is $= 8$; and further that the equation might be written

$$w^8 \exp. \frac{1}{w} \{x(g\partial_c - h\partial_b) + y(h\partial_a - f\partial_c) + z(f\partial_b - g\partial_a)\} [(a, b, c)^2] = 0,$$

where exp. Θ (read exponential) denotes e^Θ, and $[(a, b, c)^2]$ denotes

$$\begin{vmatrix} a_1^2, & b_1^2, & c_1^2, & b_1c_1, & c_1a_1, & a_1b_1 \\ a_2^2, \ldots & & & & & \\ \vdots & & & & & \end{vmatrix}.$$

Also that the equation contains the four terms

$$x^8 [(a, -h, g)^2] + y^8 [(h, b, -f)^2] + z^8 [(-g, f, c)^2] + w^8 [(a, b, c)^2] = 0.$$

Cambridge, 12 *September*, 1871.

527.

ON A THEOREM IN COVARIANTS.

[From the *Mathematische Annalen*, vol. v. (1872), pp. 625—629.]

THE proof given in Clebsch "Theorie der binären algebraischen Formen" (Leipzig, 1872) of the finite number of the covariants of a binary form depends upon a subsidiary proposition which is deserving of attention for its own sake.

I use my own hyperdeterminant notation, which is as follows: Considering a function $U=(a,..)(x,y)^n$, (viz. $U_1=(a,..)(x_1,y_1)^n$ &c.), and writing $\overline{12}=\partial_{x_1}\partial_{y_2}-\partial_{y_1}\partial_{x_2}$ &c., then the general form of a covariant of the degree m is

$$k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots)\,U_1U_2..\,U_m,$$

where k is a merely numerical factor, the indices $\alpha, \beta, \gamma,..$ are positive integers, and after the differentiations each set of variables $(x_1, y_1), .., (x_m, y_m)$ is replaced by (x, y). I say that the general form of a covariant is as above; viz. a covariant is equal to a single term of the above form, or a sum of such terms.

Attending to a single term: the sum of the indices of all the duads which contain a particular number 1, 2,.. as the case may be is called an index-sum; each index-sum is at most $=n$; so that, calling the index-sums $\sigma_1, \sigma_2, \ldots, \sigma_m$ respectively, we have $n-\sigma_1, n-\sigma_2, \ldots, n-\sigma_m$ each of them zero or positive: the term, before the several sets of variables are each replaced by (x, y), is of the orders $n-\sigma_1, n-\sigma_2, \ldots, n-\sigma_m$ in the several sets of variables respectively.

The term may be expressed somewhat differently: for writing $\nabla_1=x\partial_{x_1}+y\partial_{y_1}$, $\nabla_2=x\partial_{x_2}+y\partial_{y_2}$ &c.—then (except as to a numerical factor) it is for a function $(*)\,(x_1, y_1)^p$ the same thing whether we change (x_1, y_1) into (x, y), or operate on this function with $\nabla_1{}^p$, and so for the other sets: the term may therefore be written

$$\nabla_1{}^{n-\sigma_1}\ldots\nabla_m{}^{n-\sigma_m}k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots)\,U_1U_2\ldots U_m,$$

being now in the first instance a function of the single set (x, y) of variables.

We may omit the operand $U_1U_2 \dots U_m$, and consider only the symbol

$$k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\dots) \text{ or } \nabla_1^{n-\sigma_1}\dots\nabla_m^{n-\sigma_m}k(\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\dots),$$

which, under either of the two forms, I represent for shortness by $[12\dots m]$: observe that this is considered as a symbol involving the m symbolic numbers 1, 2, $3\dots m$, even although in particular cases one or more of these numbers may be wanting from the actual expression of the symbol: thus [123] may denote $\overline{12}^{\alpha}$, but the operand to be supplied thereto is always $U_1U_2U_3$.

A sum of symbols is not in general equal to a single symbol: but a single symbol can be expressed in a variety of ways as a sum of symbols: the most simple transformation-formulæ relate to three or four symbolic numbers; viz. for three such numbers, say 1, 2, 3, we have

$$\nabla_1 . 23 + \nabla_2 . 31 + \nabla_3 . 12 = 0,$$

showing that in a symbol, which written with the ∇'s involves $\nabla_1 . 23$, this may be replaced by its value $\nabla_2 . 13 - \nabla_3 . 12$; and so in other cases.

For the four numbers 1, 2, 3, 4 we have a group of the like formulae

$$\begin{aligned}
\quad . \quad - \nabla_2 . 34 + \nabla_3 . 24 - \nabla_4 . 23 &= 0,\\
\nabla_1 . 34 \quad . \quad - \nabla_3 . 14 - \nabla_4 . 31 &= 0,\\
- \nabla_1 . 24 + \nabla_2 . 14 \quad . \quad - \nabla_4 . 12 &= 0,\\
\nabla_1 . 23 + \nabla_2 . 23 + \nabla_3 . 31 \quad . \quad &= 0,
\end{aligned}$$

leading to

$$23 . 14 + 31 . 24 + 12 . 34 \qquad = 0,$$

which is a form not involving the ∇'s and consequently is applicable to the transformation of invariant-symbols where the numbers

$$n - \sigma_1,\ n - \sigma_2,\ \dots,\ n - \sigma_m$$

are all $= 0$.

I establish the following definitions:

A symbol $[12\dots m]$ is *proximate* when each index-sum is $< n$; otherwise it is *ultimate;* viz. this is the case when any one or more of the index-sums is or are $= n$. We may say that the symbol is ultimate as to 1 if $\sigma_1 = n$; and that it is ultimate as to 1, 2 if σ_1 and σ_2 are each $= n$: and so in other cases.

A proximate symbol which has any one index-sum thereof $< \frac{1}{2}n$ is said to be *inferior:* thus if $\sigma_1 < \frac{1}{2}n$ the symbol is inferior in regard to 1; and so if σ_1 and σ_2 are each $< \frac{1}{2}n$, it is inferior in regard to 1 and 2: and the like in other cases.

Observe that if a symbol is inferior then in the covariant the order exceeds the

degree by a number which is greater than $\frac{1}{2}n-1$: in fact, suppose it inferior in regard to 1, then the order is

$$(n-\sigma_1)+(n-\sigma_2)+\ldots+(n-\sigma_m),$$

where each term after the first is at least $=1$, that is, the order is at least $=n-\sigma_1+m-1$; hence order − degree is at least $=n-\sigma_1-1$; viz. σ_1 being less than $\frac{1}{2}n$, this is greater than $\frac{1}{2}n-1$.

Conversely, if for any symbol order-degree is $\overline{\overline{>}}\,\frac{1}{2}n-1$, then the symbol is not inferior.

A symbol $[12\ldots m]$ is *sharp* when any index is $\overline{\overline{>}}\,\frac{1}{2}n$; otherwise it is *flat;* viz. this is so when each index is $<\frac{1}{2}n$. The symbol is sharp as to any particular duad or duads when the index or indices thereof is or are each of them $\overline{\overline{>}}\,\frac{1}{2}n$.

The subsidiary theorem is now as follows: "A symbol is inferior or sharp: or it can be expressed as a sum of symbols each of which is inferior or sharp"—or what is the same thing, the only symbols which need to be considered are those which are either inferior or sharp.

Thus for the degree 1 the symbol is [1] (which is simply unity) $\sigma_1=0$, and the symbol is inferior.

For the degree 2 the symbol is [2], $=\overline{12}^k$; if $k<\frac{1}{2}n$ the symbol is inferior, if $k\,\overline{\overline{>}}\,\frac{1}{2}n$ then it is sharp.

A proof is first required for the degree 3, here $[123]=\overline{12}^\alpha\,\overline{13}^\beta\,\overline{23}^\gamma$ ($\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ each $=$ or $<n$) which may very well be neither inferior nor sharp; for instance, if $n=5$, we have $\overline{12}^2\,\overline{13}^2\,\overline{23}^2$, where each index being $=2$, the symbol is not sharp; and each index-sum being $=4$ the symbol is not inferior. But writing the symbol in the form $\nabla_1\nabla_2\nabla_3\,\overline{12}^2\,\overline{13}^2\,\overline{23}^2$, then by means of the relation

$$\nabla_1.23+\nabla_2.31+\nabla_3.12=0,$$

(or, what is the same thing, $\nabla_1.23=\nabla_2.13-\nabla_3.12$), the symbol becomes

$$\nabla_2\,\nabla_3\,\overline{12}^2\,\overline{13}^2\,\overline{23}^2\,(\nabla_2.13-\nabla_3.12),$$

$$=\nabla_2^2\nabla_3\,\overline{12}^2\,\overline{13}^3\,23-\nabla_2\nabla_3^2\,\overline{13}^2\,\overline{12}^2\,\overline{23},$$

where each term, as containing an index 3, is sharp. To complete the reduction, observe that calling the expression $\mathfrak{A}-\mathfrak{B}$, then in the term $\mathfrak{A}$ interchanging the numbers 2 and 3 we obtain $\mathfrak{A}=-\mathfrak{B}$, and thence $\mathfrak{A}-\mathfrak{B}=2\mathfrak{A}$; so that the whole is $2\,\nabla_2^2\nabla_3\,\overline{12}^2\,\overline{13}^3\,\overline{23}$, viz. it is a multiple of $\overline{12}^2\,\overline{13}^3\,\overline{23}$.

I prove the general case, substantially in the manner used by Dr Clebsch, as follows. We assume that the theorem is proved up to a particular degree m: that is, we assume that every symbol belonging to a degree not exceeding m can be

expressed as a sum of terms each of which is sharp or inferior: and we have to prove this for the next following degree $m+1$, or writing for convenience p in place of $m+1$, (say for the degree p); that is, for a symbol

$$[12\ldots mp], = \overline{p1}^{\lambda_1}\,\overline{p2}^{\lambda_2}\ldots\overline{pm}^{\lambda_m}\,[12\ldots m]$$
$$= P\,[12\ldots m] \text{ suppose.}$$

I write as before $\sigma_1, \sigma_2, \ldots, \sigma_m$ for the index-sums of $[12\ldots m]$: those of $[12\ldots mp]$ are therefore $\sigma_1+\lambda_1, \sigma_2+\lambda_2, \ldots, \sigma_m+\lambda_m$, and (for the duads involving p) $\sigma_p = \lambda_1+\lambda_2\ldots+\lambda_m$.

If $[12\ldots m]$ is sharp, then $[12\ldots mp]$ is sharp, and the theorem is true.

If $\sigma_p < \frac{1}{2}n$, then $[12\ldots mp]$ is inferior in regard to p; and the theorem is true.

The only case requiring a proof is when $[12\ldots m]$ is not sharp (being therefore inferior) and when σ_p is $\overline{\overline{>}}\,\frac{1}{2}n$. And in this case if any one of the indices $\lambda_1, \ldots \lambda_m$ is $\overline{\overline{>}}\,\frac{1}{2}n$ (or say if P is sharp) then the theorem is true.

Consider the expression

$$\overline{p1}^{\lambda_1}\,\overline{p2}^{\lambda_2}\ldots\overline{pm}^{\lambda_m}\,[12\ldots m],$$

where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are as before the index-sums for $[12\ldots m]$ and therefore the numbers

$$n-\sigma_1-\lambda_1, \ldots, n-\sigma_m-\lambda_m$$

are none of them negative.

Assume that when $[12\ldots m]$ is inferior, and when $\lambda_1\ldots\lambda_m$ have any values such that their sum is not greater than a given value σ_p-1, the expression is a sum of terms each of which is inferior or sharp: we wish to show that when $\lambda_1+\lambda_2\ldots+\lambda_m$ has the next succeeding value, $=\sigma_p$, the case is still the same.

For this purpose, introducing the ∇'s I write

$$Q = \nabla_1^{n-\sigma_1-\lambda_1}\ldots\nabla_m^{n-\sigma_m-\lambda_m}\,\nabla_p^{n-\sigma_p}\,\overline{p1}^{\lambda_1}\,\overline{p2}^{\lambda_2}\ldots\overline{pm}^{\lambda_m}\,[12\ldots m];$$

then supposing for a moment that λ_1 is not $=n-\sigma_1$ and λ_2 not $=0$, the expression contains the factor $\nabla_1.p2$, which is equal to and may be replaced by $-\nabla_2.p1+\nabla_p.12$: we have thus

$$Q = Q' + \Omega,$$

where omitting the ∇'s

$$Q' = j\,\overline{p1}^{\lambda_1+1}\,\overline{p2}^{\lambda_2-1}\,\overline{p3}^{\lambda_3}\ldots\overline{pm}^{\lambda_m}\,[12\ldots m],$$
$$\Omega = k\,\overline{p1}^{\lambda_1}\,\overline{p2}^{\lambda_2-1}\,\overline{p3}^{\lambda_3}\ldots\overline{pm}^{\lambda_m}\,\overline{12}\,[12\ldots m].$$

Now for Ω the sum of the indices $\lambda_1, \lambda_2-1, \lambda_3\ldots\lambda_m$ is σ_p-1, so that by hypothesis Ω is inferior or sharp: that is, the difference $Q-Q'$ is inferior or sharp: so that to prove that Q is inferior or sharp, we have only to prove this of Q', where Q' is derived from Q by increasing by unity the index of $p1$, at the expense of that of $p2$

which is diminished by unity. Such change is possible so long as the index λ_1 has not attained its maximum value, $n-\sigma_1$ or σ_p as the case may be, and there is any other index $\lambda_2, \ldots, \lambda_m$ which is not $=0$: that is, we may pass from Q to Q', from Q' to Q'' and so on; and it will be sufficient to show that the last term of the series is inferior or sharp. We thus pass from Q to R, where

$$R = \overline{p1}^{n-\sigma_1}\, \overline{p2}^{\lambda_2-\alpha_2} \ldots \overline{pm}^{\lambda_m-\alpha_m}\, [12 \ldots m]$$

and $\alpha_2+\alpha_3 \ldots + \alpha_m = n-\sigma_1-\lambda_1$; or else to

$$R = \overline{p1}^{\sigma_p}\, [12 \ldots m],$$

according as $n-\sigma_1$ is not greater or is greater than σ_p.

Now let $[12 \ldots m]$ be inferior; suppose it to be so in regard to 1, that is, let σ_1 be less than $\frac{1}{2}n$ or $n-\sigma_1$ greater than $\frac{1}{2}n$. Then if σ_p be less than $\frac{1}{2}n$ it is less than $n-\sigma_1$, that is, we have for R the last-mentioned form which is inferior in regard to p, viz. R is inferior; if σ_p is equal to or greater than $\frac{1}{2}n$, then R, whichever its form may be, is sharp as to $p1$, viz. R is sharp. Hence in either case Q is a sum of terms which are inferior or sharp; that is, assuming the theorem for a form for which $\lambda_1+\lambda_2 \ldots + \lambda_m$ does not exceed a given value $\sigma_p - 1$, the theorem is true for the next succeeding value σ_p; or being true for the case $\sigma_p - 1 = 0$, it is true generally.

Cambridge, 24 *April*, 1872.

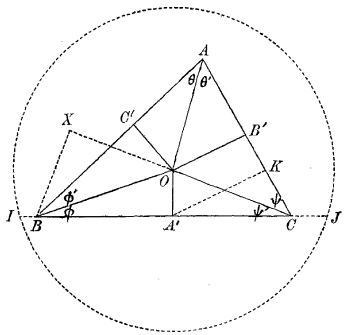
528.

ON THE NON-EUCLIDIAN GEOMETRY.

[From the *Mathematische Annalen*, vol. v. (1872), pp. 630—634.]

THE theory of the Non-Euclidian Geometry as developed in Dr Klein's paper "Ueber die Nicht-Euklidische Geometrie" may be illustrated by showing how in such a system we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the "hyperbolic" case of plane geometry; viz. the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to the points *within* the circle.

I use the simple letters a, A,.. to denote (linear or angular) distances measured in the ordinary manner; and the same letters, with a superscript stroke, $\bar{a}$, $\bar{A}$,.. to



denote the same distances measured according to the theory. The radius of the absolute is for convenience taken to be $=1$; the distance of any point from the centre can therefore be represented as the sine of an angle.

The distance $\overline{BC}$, or say $\bar{a}$, of any two points B, C is by definition as follows:

$$\begin{aligned}
&\text{Radius of circle} = 1:\\
\text{In } \triangle\ ABC,\ &\text{sides are } a,\ b,\ c:\\
&\text{angles } \text{,,}\ A,\ B,\ C:\\
&OA\ ,\ OB\ ,\ OC \quad \text{are} = \sin p,\ \sin q,\ \sin r:\\
&OA',\ OB',\ OC' \quad \text{,,}\ \text{,,}\ \sin a,\ \sin b,\ \sin c:\\
&\measuredangle\, BOC,\ COA,\ AOB \quad \text{,,}\ \text{,,}\ \alpha,\ \beta,\ \gamma.
\end{aligned}$$

$$\bar{a} = \tfrac{1}{2}\log\frac{BI\,.\,CJ}{BJ\,.\,CI},$$

(where I, J are the intersections of the line BC with the circle); that is,

$$e^{\bar{a}} + e^{-\bar{a}},\ \text{or}\ 2\cosh\bar{a} = \sqrt{\frac{\overline{BI\,.\,CJ}}{\overline{BJ\,.\,CI}}} + \sqrt{\frac{\overline{BJ\,.\,CI}}{\overline{BI\,.\,CJ}}},\quad = \frac{BI\,.\,CJ + BJ\,.\,CI}{\sqrt{\overline{BI\,.\,BJ}}\,\sqrt{\overline{CI\,.\,CJ}}},$$

where the numerator is

$$\begin{aligned}
BI\,(BJ - BC) + CI\,(BC + CJ),\quad &= BI\,.\,BJ + CI\,.\,CJ + BC\,(CI - BI),\\
&= BI\,.\,BJ + CI\,.\,CJ + BC^2.
\end{aligned}$$

Hence taking a for the distance BC, and $\sin q$, $\sin r$, for the distances OB, OC respectively, we have $BI\,.\,BJ = \cos^2 q$, $CI\,.\,CJ = \cos^2 r$; and the formula is

$$\cosh\bar{a} = \frac{\cos^2 q + \cos^2 r + a^2}{2\cos q\cos r},$$

or, what is the same thing, taking α for the angle BOC, and therefore

$$a^2 = \sin^2 q + \sin^2 r - 2\sin q\sin r\cos\alpha,$$

we have

$$\cosh\bar{a} = \frac{1 - \sin q\sin r\cos\alpha}{\cos q\cos r}.$$

In a similar manner, if $\sin\mathfrak{a}$ is the perpendicular distance from O on the line BC (that is, $a\sin\mathfrak{a} = \sin q\sin r\sin\alpha$) it can be shown that

$$\sinh\bar{a} = \frac{a\cos\mathfrak{a}}{\cos q\cos r},$$

the equivalence of the two formulæ appearing from the identity

$$\cos^2 q\cos^2 r = (1 - \sin q\sin r\cos\alpha)^2 - a^2 + a^2\sin^2\mathfrak{a},$$

which is at once verified.

Next for an angle; we have by definition

$$\bar{A} = \frac{1}{2i}\log\frac{\sin BAI\,.\,\sin CAJ}{\sin CAI\,.\,\sin BAJ},$$

where AI, AJ are the (imaginary) tangents from A to the circle; or writing for shortness BI &c. instead of BAI, &c. (the angular point being always at A),

$$\bar{A} = \frac{1}{2i} \log \frac{\sin BI \,.\, \sin CJ}{\sin CI \,.\, \sin BJ},$$

consequently

$$e^{i\bar{A}} - e^{-i\bar{A}} = 2i \sin \bar{A}$$

$$= \sqrt{\frac{\sin BI \,.\, \sin CJ}{\sin CI \,.\, \sin BJ}} - \sqrt{\frac{\sin CI \,.\, \sin BJ}{\sin BI \,.\, \sin CJ}}, \quad = \frac{\sin BI \sin CJ - \sin BJ \,.\, \sin CI}{\sqrt{\sin BI \,.\, \sin BJ}\sqrt{\sin CI \,.\, \sin CJ}},$$

where the numerator is

$$\sin BI \sin (BJ - BC) - \sin BJ \sin (BI + BC) = \sin BC \sin IJ,$$

or say $= \sin A \sin IJ$. Moreover taking the distance OA to be $= \sin p$, and the perpendicular distances from O on the lines AB, AC to be $\sin \mathfrak{c}$ and $\sin \mathfrak{b}$ respectively, then if for a moment the angle IJ is put $= 2\omega$, we have $\sin p \sin \omega = 1$: moreover

$$\sin BI \sin BJ = \sin (\omega - BO) \sin (\omega + BO) = \sin^2 \omega - \sin^2 BO;$$

and $\sin p \sin BO = \sin \mathfrak{c}$; that is, $\sin BI \sin BJ = \dfrac{1 - \sin^2 \mathfrak{c}}{\sin^2 p}$, $= \dfrac{\cos^2 \mathfrak{c}}{\sin^2 p}$: and similarly $\sin CI \sin CJ = \dfrac{\cos^2 \mathfrak{b}}{\sin^2 p}$; also

$$\sin IJ = -\sin 2\omega = 2 \sin \omega \cos \omega = \frac{2}{\sin p} \frac{i \cos p}{\sin p};$$

whence the required formula

$$\sin \bar{A} = \frac{\cos p \sin A}{\cos \mathfrak{b} \cos \mathfrak{c}}.$$

In the same way, or analytically from this value, we have

$$\cos \bar{A} = \frac{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}{\cos \mathfrak{b} \cos \mathfrak{c}},$$

and thence also

$$\tan \bar{A} = \frac{\cos p \sin A}{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}.$$

In particular, taking the line AC to pass through O, or writing in the formula $\mathfrak{b} = 0$, we have $\tan B\bar{O} = \cos p \tan BO = \cos p \tan \theta$; that is, $\bar{BO} = \tan^{-1} . \cos p \tan \theta$; and similarly $\bar{CO} = \tan^{-1} \cos p \tan \theta'$; we ought to have $\bar{A} = \bar{BO} + \bar{CO}$, that is,

$$A = \tan^{-1} \cos p \tan \theta + \tan^{-1} \cos p \tan \theta'$$

which, observing that $\sin p \sin \theta = \sin \mathfrak{c}$ and $\sin p \sin \theta' = \sin \mathfrak{b}$, also $A = \theta + \theta'$, is in fact equivalent to the above formula for $\tan \bar{A}$.

Observe in particular that when A is at the centre, p is $= 0$, and the formula becomes $\bar{A} = \theta + \theta'$, $= A$, or say for an angle at the centre, $\bar{O} = O$.

I return to the expression for $\cosh \bar{a}$; in explanation of its meaning, let the distances $\overline{OB}$, $\overline{OC}$ be $\bar{q}$, $\bar{r}$ respectively and let the angle $\overline{BOC}$ be $\bar{\alpha}$; to find $\bar{q}$ we have only to take C at O, that is, in the formula for $\cosh \bar{\alpha}$ to write $r=0$, we thus find $\cosh \bar{q} = \dfrac{1}{\cos q}$: and similarly $\cosh \bar{r} = \dfrac{1}{\cos r}$, whence also

$$\cos q = \operatorname{sech} \bar{q}, \quad \sin q = i \tanh \bar{q},$$
$$\cos r = \operatorname{sech} \bar{r}, \quad \sin r = i \tanh \bar{r},$$

also, as seen above, $\bar{\alpha} = \alpha$; the formula thus is

$$\cosh \bar{a} = \frac{1 + \tanh \bar{q} \tanh \bar{r} \cos \bar{\alpha}}{\operatorname{sech} \bar{q} \operatorname{sech} \bar{r}}$$
$$= \cosh \bar{q} \cosh \bar{r} + \sinh \bar{q} \sinh \bar{r} \cos \alpha,$$

or, what is the same thing, it is

$$\cos \bar{\alpha} = \frac{\cosh \bar{a} - \cosh \bar{q} \cosh \bar{r}}{\sinh \bar{q} \sinh \bar{r}},$$

viz. as will presently appear, this is the formula for $\cos \overline{BOC}$ in the triangle BOC.

From the above formulæ

$$\cosh \bar{a} = \frac{1 - \sin q \sin r \cos \alpha}{\cos q \cos r},$$

and

$$\sin \bar{A} = \frac{\cos p \sin A}{\cos \mathfrak{b} \cos \mathfrak{c}}, \quad \cos \bar{A} = \frac{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}{\cos \mathfrak{b} \cos \mathfrak{c}},$$

and the like formulæ for $\bar{b}$, $\bar{c}$, $\bar{B}$, $\bar{C}$, it may be shown that in the triangle ABC we have

$$\cosh \bar{a} = \frac{\cos \bar{A} + \cos \bar{B} \cos \bar{C}}{\sin \bar{B} \sin \bar{C}}.$$

In fact, substituting the foregoing values, this equation becomes

$$\frac{(1 - \sin^2 \mathfrak{a})(\cos A + \sin \mathfrak{b} \sin \mathfrak{c}) + (\cos B + \sin \mathfrak{c} \sin \mathfrak{a})(\cos C + \sin \mathfrak{a} \sin \mathfrak{b})}{\sin B \sin C \cos \mathfrak{b} \cos \mathfrak{c}} = \frac{1 - \sin q \sin r \cos \alpha}{\cos q \cos r},$$

that is,

$$\cos A + \cos B \cos C - \sin^2 \mathfrak{a} \cos A + \sin \mathfrak{a} \sin \mathfrak{b} \cos B + \sin \mathfrak{a} \sin \mathfrak{c} \cos C + \sin \mathfrak{b} \sin \mathfrak{c}$$
$$= \sin B \sin C (1 - \sin q \sin r \cos \alpha),$$

or, what is the same thing,

$$\sin^2 \mathfrak{a} (\cos B \cos C - \sin B \sin C) + \sin \mathfrak{a} \sin \mathfrak{b} \cos B + \sin \mathfrak{a} \sin \mathfrak{c} \cos C + \sin \mathfrak{b} \sin \mathfrak{c}$$
$$= - \sin B \sin C \sin q \sin r \cos \alpha,$$

that is,

$$(\sin \mathfrak{a} \cos B + \sin \mathfrak{c})(\sin \mathfrak{a} \cos C + \sin \mathfrak{b}) = \sin B \sin C (\sin^2 \mathfrak{a} - \sin q \sin r \cos \alpha),$$

a relation which I proceed to verify.

We may, from the formulæ

$$a^2 = \sin^2 q + \sin^2 r - 2 \sin q \sin r \cos \alpha, \quad a \sin \mathfrak{a} = \sin q \sin r \sin \alpha, \text{ \&c.},$$

but, more simply, geometrically as presently shown, deduce

$$\sin \mathfrak{a} \cos B + \sin \mathfrak{c} = \frac{1}{a} \sin B \sin q (\sin q - \sin r \cos \alpha),$$

$$\sin \mathfrak{a} \cos C + \sin \mathfrak{b} = \frac{1}{a} \sin C \sin r (\sin r - \sin q \cos \alpha),$$

and thence

$$\begin{aligned}(\sin \mathfrak{a} \cos B + \sin \mathfrak{c})(\sin \mathfrak{a} \cos C + \sin \mathfrak{b}) &= \frac{1}{a^2} \sin B \sin C \sin q \sin r \left\{\begin{matrix}\sin q \sin r (1 + \cos^2 \alpha) \\ - \cos \alpha (\sin^2 q + \sin^2 r)\end{matrix}\right\} \\ &= \frac{1}{a^2} \sin B \sin C \sin q \sin r (\sin q \sin r \sin^2 \alpha - a^2 \cos \alpha) \\ &= \sin B \sin C (\sin^2 \mathfrak{a} - \sin q \sin r \cos \alpha),\end{aligned}$$

which is the equation in question. For the subsidiary equations used in the demonstration, observe that the four points O, X, A', B lie in a circle, and consequently that $CO \,.\, CX = CA' \,.\, CB$; or multiplying each side by $\sin C$, then $CO \,.\, CX \,.\, \sin C = A'K \,.\, CB$, that is,

$$\sin r (\sin r - \sin q \cos \alpha) \sin C = a (\sin \mathfrak{a} \cos C + \sin \mathfrak{b}),$$

and the other of the equations in question is proved in the same manner.

From the formula for $\cosh \bar{a}$ we find

$$\sinh \bar{a} = \frac{1}{\sin \bar{B} \sin \bar{C}} \Delta,$$

where

$$\Delta^2 = -(1 - \cos^2 \bar{A} - \cos^2 \bar{B} - \cos^2 \bar{C} - 2 \cos \bar{A} \cos \bar{B} \cos \bar{C}),$$

whence also

$$\sinh \bar{a} : \sinh \bar{b} : \sinh \bar{c} = \sin \bar{A} : \sin \bar{B} : \sin \bar{C};$$

and we can also obtain

$$\cos \bar{A} = \frac{\cosh \bar{a} - \cosh \bar{b} \cosh \bar{c}}{\sinh \bar{b} \sinh \bar{c}} \text{ \&c.}$$

So that the formulæ are in fact similar to those of spherical trigonometry with only $\cosh \bar{a}$, $\sinh \bar{a}$ &c. instead of $\cos a$, $\sin a$ &c. The before-mentioned formula for $\cos \bar{\mathfrak{a}}$ in terms of $\bar{a}$, $\bar{q}$, $\bar{r}$ is obviously a particular case of the last-mentioned formula for $\cos \bar{A}$.

Cambridge, 11 *May*, 1872.

529.

A "SMITH'S PRIZE" PAPER[1]; SOLUTIONS.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. IV. (1868), pp. 201—226.]

1. *Find the form of a function of a given number of letters, which has two and only two values.*

It is required to find the general form of a function $\phi(a, b, c, \ldots, k)$, rational and integral, which for all permutations whatever of the letters has two and only two values.

Suppose that any particular permutation of the letters changes $\phi(a, b, c, \ldots, k)$ into $\phi_1(a, b, c, \ldots, k)$; then any permutation of the letters will either leave the functions ϕ, ϕ_1 each of them unaltered, or it will change ϕ into ϕ_1, and ϕ_1 into ϕ. Hence $\phi + \phi_1$ is a symmetrical function of all the letters, say

$$\phi + \phi_1 = 2L;$$

$\phi - \phi_1$ is a function which by any permutation of the letters is either unaltered, or simply changes its sign; and it is to be shown that, writing for shortness V to denote the product $(a-b)(a-c)\ldots(b-c)\ldots$ of the differences of the letters, and denoting by $2M$ a symmetrical function of the letters, we have

$$\phi - \phi_1 = 2VM.$$

These equations give

$$\phi = L + VM,$$

which is the general form required; viz. the function ϕ has then only the two values $L + VM$, $L - VM$.

To prove the subsidiary theorem, observe that there is at least one interchange of two letters which changes $\phi - \phi_1$ (for otherwise $\phi - \phi_1$ would be a symmetrical function);

[1] Set by me, for the Master of Trinity, Thursday, January 30, 1868.

let this be (a, b). Then $\phi - \phi_1$, changing its sign for the interchange in question, must vanish for $a = b$, that is, $\phi - \phi_1$ must contain the factor $(a-b)$; let it contain it in the power $(a-b)^s$; then the quotient $\phi - \phi_1 \div (a-b)^s$, not vanishing for $a = b$, cannot change its sign by the interchange in question, and as by supposition $\phi - \phi_1$ does change its sign, it appears that the exponent s must be odd. But $\phi - \phi_1$, containing the factor $(a-b)^s$, must contain every other like factor $(a-c)^s$ or $(c-d)^s$; in fact, writing $\phi - \phi_1 = K(a-b)^s$, then if the interchange (a, c) alters K into K_1, we have $K(a-b)^s = \pm K_1(a-c)^s$, and K consequently contains the factor $(a-c)^s$; it does not contain any higher power $(a-c)^{s+s'}$, for if it did, by reversing the process it would appear that $\phi - \phi_1$ contained (contrary to supposition) the factor $(a-b)^{s+s'}$. Similarly $\phi - \phi_1$ contains the factor $(c-d)^s$, but no higher power $(c-d)^{s+s'}$. Hence $\phi - \phi_1$ contains the product of all the factors $(a-b)^s$, that is, it contains V^s, and writing $\phi - \phi_1 = 2V^sM$, the quotient M does not contain any such factor as $(a-b)$; it therefore does not change its sign for any interchange whatever (a, b); and in consequence it remains unaltered for any such interchange, that is, M is a symmetrical function. Observing that any even power of V is a symmetrical function, we may without loss of generality include V^{s-1} in the symmetrical function M, and write therefore

$$\phi - \phi_1 = 2VM,$$

which is the subsidiary theorem in question.

2. *Express* $\dfrac{x^7-1}{x-1}$ *as the product of two cubic factors; and show generally that, for any prime exponent* p, *the function* $\dfrac{x^p-1}{x-1}$ *may be broken up into two factors each of the order* $\frac{1}{2}(p-1)$, *by means of a quadratic equation.*

Let r be any root of the given equation, then

$$r^6 + r^5 + r^4 + r^3 + r^2 + r + 1 = 0;$$

the roots of this equation are $r^1, r^2, r^3, r^4, r^5, r^6$.

Hence

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = (x - r^1)(x - r^2)(x - r^4) \,.\, (x - r^3)(x - r^5)(x - r^6),$$

and denoting the two cubic factors by y_1, y_2, or writing

$$y_1 = (x - r^1)(x - r^2)(x - r^4),$$
$$y_2 = (x - r^3)(x - r^5)(x - r^6),$$

we have

$$y_1 = x^3 - x^2(r^1 + r^2 + r^4) + x(r^3 + r^5 + r^6) - 1,$$
$$y_2 = x^3 - x^2(r^3 + r^5 + r^6) + x(r^1 + r^2 + r^4) - 1,$$

and thence

$$y_1 + y_2 = 2x^3 + x^2 - x - 2.$$

Hence y_1, y_2 are the roots of the quadratic equation

$$y^2 - y(2x^3 + x^2 - x - 2) + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

Solving the equation, observing that the quantity under the radical sign must of necessity be a square, and that its value is

$$(2x^3+x^2-x-2)^2-4(x^6+x^5+x^4+x^3+x^2+x+1),$$

which is in fact

$$=-7(x^2+x)^2,$$

we find that the roots y_1, y_2, that is, the required cubic factors, are

$$\tfrac{1}{2}\{2x^3+x^2-x-2\pm\sqrt{(-7)}(x^2+x)\}.$$

Generally for any prime exponent p, denoting any root by r, and writing

$$y_1=(x-r^{a_1})(x-r^{a_2})\ldots,$$
$$y_2=(x-r^{b_1})(x-r^{b_2})\ldots,$$

where a_1, $a_2\ldots$ are the quadratic residues, and b_1, $b_2\ldots$ the quadratic non-residues of p, we find

$$y^2-yP+Q=0,$$

where P is a function of x of the order $\frac{1}{2}(p-1)$, and Q is $=x^{p-1}+x^{p-2}\ldots+x+1$. Hence

$$y=\tfrac{1}{2}\{P\pm\sqrt{(P^2-4Q)}\},$$

where P^2-4Q is a perfect square, $=\epsilon pZ^2$, Z a rational function of a degree less than $\frac{1}{2}(p-1)$, and $\epsilon=+$ or $-$ according as p is $\equiv 1$ or $\equiv 3$ (mod. 4). Hence the required factors are

$$\tfrac{1}{2}\{P\pm\sqrt{(\epsilon p)}\,Z\}.$$

3. *In a Map of the World, wherein the meridians are projected into right lines meeting in a point, the inclination of any two of the lines being equal to that of the two meridians, and the parallels into circles about the point as centre, the radius of the circle being a given function of the colatitude: compare on the sphere and in the map* (1) *the corresponding elements of area,* (2) *the azimuths of corresponding linear elements; and explain what conveniences may be obtained by proper determinations of the above-mentioned function of the colatitude.*

Let the longitude and colatitude be

on the sphere l, c,

in the map l', c',

then the projection is such, that $l'=l$, $c'=f(c)$, a given function of c.

The lengths of corresponding linear elements in the direction of a meridian, and perpendicular to it, are

$$dc,\ \sin c\,dl,$$
$$dc',\quad c'dl:$$

whence, elements of area are

$$dc \,.\, \sin c\, dl,$$
$$dc'. \quad c' dl;$$

tangents of azimuth are

$$dc \div \sin c\, dl,$$
$$dc' \div \quad c' dl;$$

and substituting the values $l' = l$, $c' = f(c)$, we have $dl' = dl$, $dc' = f'(c)\, dc$; and thence

$$\text{elements of area are as } \sin c : f(c) f'(c),$$
$$\text{tangents of azimuth as } \frac{1}{\sin c} : \frac{f'(c)}{f(c)}.$$

By proper determinations of the function $f(c)$, we can make

(1) The ratio of the elements of area to be constant; this will be the case if

$$f(c) f'(c) = k \sin c, \text{ that is, } f^2(c) = \text{const.} - 2k \cos c$$
$$= 2k(1 - \cos c),$$

since $f(c)$, $= c'$, must vanish for $c = 0$; that is,

$$f(c) = 2\sqrt{(k)} \sin \tfrac{1}{2} c:$$

(2) corresponding azimuths to be equal; this will be the case if

$$\frac{f'(c)}{f(c)} = \frac{1}{\sin c}, \text{ that is, } \log f(c) = \text{const.} + \log \tan \tfrac{1}{2} c,$$

or say

$$f(c) = k \tan \tfrac{1}{2} c.$$

This is in fact the stereographic projection, in which (as is known) any indefinitely small figure is in the map represented without distortion.

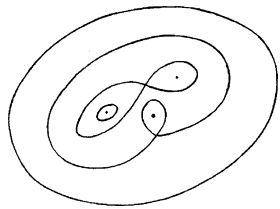
4. *Explain the general configuration of the contour and slope lines in a tract of Lake and Mountain country.*

A contour line is the locus of points having a given altitude.

To fix the ideas, consider an island forming a two-headed mountain. The contour line at the sea level is a closed curve; at a sufficiently great altitude the contour line consists of two closed curves surrounding the two summits respectively; the transition from one form to the other takes place at the altitude of the pass between the two summits, the contour line then having a node at the top of the pass, and being in form a figure of eight. At the altitude of the lower summit one of the closed curves is reduced to a point, and for greater altitudes it disappears, the contour line being then a single closed curve surrounding the higher summit; and at the altitude of the higher summit this reduces itself to a point. (See fig. 1.) If there is on the breast of the mountain a lake, the contour line at the lake level is (as in

the case of the pass) a curve with a node, being however here a closed curve with an interior loop as shown in the figure. The contour line for an altitude below the lake level includes as part of itself a closed curve lying within the contour of the lake,

Fig. 1.



and which for the altitude of the lowest point (or "imit") of the lake reduces itself to a point, and for smaller altitudes disappears.

The contour lines in the immediate neighbourhood of a summit are in general ellipses (geometrically, the indicatrix is an ellipse); they may however be circles (viz. if the summit be an umbilicus).

A slope line is a line of greatest inclination, and it is consequently an orthogonal trajectory to the series of contour lines. A slope line may be considered as always terminated in a summit or an imit; there is through each summit (or imit) an infinity of slope lines, and in general these all *touch* there; if, however, the contour lines in the immediate neighbourhood are circles, then the slope lines, instead of touching, pass from the point in all directions. Through the node at the top of a pass, or outlet of a lake, there are two intersecting slope lines, one ascending each way from the node, and being in general a "ridge-line"; the other descending each way, and being in general a "course-line," viz. in the case of a pass, it is in each direction the course of the principal stream of the valley; and in the case of a lake-outlet, it is in the direction away from the lake, the course of the out-flowing stream. The slope lines which thus pass through the several nodes mark out distinct regions, and so facilitate the tracing of the intermediate slope lines.

5. *Find the differential equations corresponding to the three integral equations respectively* (i) $(y+c)^2 = x(x-1)(x-2)$; (ii) $(y+c)^2 = x^2(x-1)$; *and* (iii) $(y+c)^2 = x^3$: *and discuss geometrically the singular solutions.*

Generally, for the equation $(y+c)^2 - X = 0$, the derived equation is $4X\left(\frac{dy}{dx}\right)^2 - X'^2 = 0$. If from the integral equation, differentiating in regard to c, we attempt to find the singular solution, we obtain $(y+c) = 0$; and thence $X = 0$ for the singular solution. It is however to be observed that, if X contain single and multiple factors,

$$X = (x+\alpha)(x+\beta)\dots(x+\gamma)^m\dots,$$

then it is only the single factors $x+\alpha=0$, $x+\beta=0$, ..., which are solutions of the differential equation when expressed in its proper form free from extraneous factors.

To explain how this is, write the differential equation in the form

$$X'^2\left(\frac{dx}{dy}\right)^2-4X=0;$$

for a single factor $x+\alpha$, X' does not contain the factor $x+\alpha$, and there is no division by $x+\alpha$. The equation $x+\alpha=0$ gives $\frac{dx}{dy}=0$, $X=0$, and the equation is thus satisfied; and by what precedes $x+\alpha=0$ is a singular solution. Contrariwise, for the multiple factor $(x+\gamma)^m$, X' will contain $(x+\gamma)^{m-1}$, and the equation $X'^2\left(\frac{dx}{dy}\right)^2-4X=0$, will divide by $(x+\gamma)^m$, and divested of this factor it will be of the form

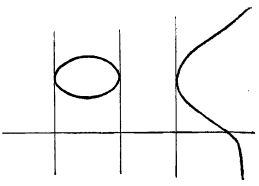
$$\frac{X'^2}{(x+\gamma)^m}\left(\frac{dx}{dy}\right)^2-\frac{4X}{(x+\gamma)^m}=0,$$

where $\frac{X'}{(x+\gamma)^m}$ contains the factor $(x+\gamma)^{m-2}$ (index is 0 or positive) but $\frac{X}{(x+\gamma)^m}$ does not contain $x+\gamma$; hence the equation $x+\gamma=0$, gives $\frac{dx}{dy}=0$, and therefore $\frac{X'}{(x+\gamma)^m}\left(\frac{dx}{dy}\right)^2=0$, but it does not give $\frac{X}{(x+\gamma)^m}=0$, and consequently fails to satisfy the differential equation. Reverting to the integral equation $(y+c)^2=(x+\alpha)(x+\beta)\ldots(x+\gamma)^m\ldots$, we see that $x+\alpha=0$ touches each of the series of curves, and is thus an envelope thereof: that $x+\gamma=0$ is not an envelope, but is the locus of a singular point on the series of curves.

Applying the foregoing considerations to the proposed question,

(i) The differential equation is (fig. 2),

Fig. 2.



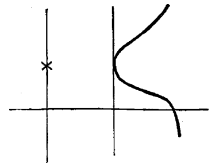
$$(3x^2-6x+2)^2\left(\frac{dx}{dy}\right)^2-4x(x-1)(x-2)=0,$$

having the singular solutions $x=0$, $x-1=0$, $x-2=0$.

(ii) The differential equation is (fig. 3),

$$(3x-2)^2\left(\frac{dx}{dy}\right)^2-4(x-1)=0,$$

Fig. 3.



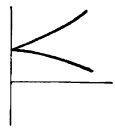
having the singular solution $x-1=0$. But $x=0$ is not a solution; it is the locus of the series of conjugate points of the curves $(y+c)^2=x^2(x-1)$.

(iii) The differential equation is (fig. 4)

$$9x\left(\frac{dx}{dy}\right)^2-4=0,$$

which has no singular solution. But $x=0$ is the locus of the cusp of the curves $(y+c)^2=x^3$.

Fig. 4.



6. *Show that the curve parallel to the parabola is of the order 6 and class 4; explain the reduction of class; and trace the system of parallel curves.*

The curve parallel to the parabola is the envelope of a circle of constant radius, having its centre on a parabola; taking the equation of the parabola to be $y^2=4x$, the coordinates of any point thereon may be taken to be α^2, 2α, and the equation of the variable circle is

$$(x-\alpha^2)^2+(y-2\alpha)^2-r^2=0,$$

that is,

$$\alpha^4+\alpha^2(-2x+4)+\alpha(-4y)+x^2+y^2-r^2=0;$$

or multiplying by 6, this is

$$(a,\ 0,\ c,\ d,\ e\between\alpha,\ 1)^4=0,$$

where

$$a=6,$$
$$c=-2x+4,$$
$$d=-6y,$$
$$e=6(x^2+y^2-r^2).$$

The equation of the envelope is obtained by eliminating α between the equation in α and the derived equation; or, what is the same thing, by equating to zero the discriminant of the equation in α; we thus obtain

$$(ae+3c^2)^3-27\,(ace-ad^2-c^3)^2=0,$$

which, substituting for a, c, d, e their values, is an equation in (x, y) of the order 6: the order of the curve is thus $=6$.

The class is most easily obtained by geometrical considerations. Seeking the tangents which can be drawn from a given point, it at once appears that, describing about this point a circle radius r, then to each tangent through the point to the parallel curve, there corresponds a common tangent of the circle and the parabola, and reciprocally; whence the class of the curve is equal to the number of common tangents of the circle and the parabola, that is, it is $=4$.

To the order 6 corresponds in general a class $=30$, and the reduction from 30 to 4, $=26$, is caused by the cusps and nodes of the curve.

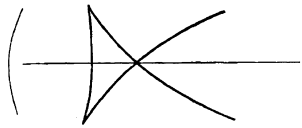
The cusps are given as the points of intersection of the curves $ae+3c^2=0$, $ace-ad^2-c^3=0$, which being respectively of the orders 2 and 3, give 6 cusps; it may be added that these cusps (2 real and 4 imaginary, or else all 6 imaginary) are, in regard to the parabola, the centres of curvature for those points at which the radius of curvature is $=r$.

There is on the axis a single point (always real) whose normal distance from the parabola is $=r$. Such point is a node of the parallel curve, viz., according to the value of r, either an acnode (conjugate point) or a crunode: we have thus 1 node. The parallel curve at infinity coincides with the two parabolas obtained by the displacement of the given parabola parallel to itself through the distances $+r$, $-r$ along the axis of y. Two such parabolas have at infinity on the axis of x a contact of the second order, equivalent to three coincident intersections; and the parallel curve has thus at infinity on the axis of x, a singular point equivalent to 3 nodes; the number of nodes is thus $=4$; and the 4 nodes and 6 cusps give the required reduction $2.4+3.6$, $=26$.

The parallel curve is most easily traced geometrically; there are two branches, one outside, the other inside the parabola, equidistant from it. The outside branch is a curve of continuous curvature, the form of which requires no explanation. As regards the inside branch, when r is small, this is also of continuous curvature, but there is on the axis of x inside the branch a real acnodal point (the node above referred to): when r becomes = radius of curvature at vertex (or twice the focal distance), the acnode coincides with the branch; and the point on the axis, although presenting no visible singularity, is really in the nature of a triple point composed of two cusps and a node; when r is greater than this limiting value, instead of the acnode we have a crunode;

and two real cusps present themselves; viz. the form of the inside branch is as shown in fig. 5.

Fig. 5.



7. *Show that a curve of the order n has at most $\frac{1}{2}(n-1)(n-2)$ double points; and that in any curve having this maximum number of double points, the coordinates (x, y, z) may be taken to be proportional to rational and integral functions of a variable parameter.*

A curve of the order n cannot have more than $\frac{1}{2}(n-1)(n-2)$ double points; for suppose it had one more, say

$$\tfrac{1}{2}(n-1)(n-2)+1, \; = \tfrac{1}{2}(n^2-3n)+2, \text{ double points};$$

then since a curve of the order $n-2$ can be drawn through

$$\tfrac{1}{2}(n-2)(n+1), \qquad = \tfrac{1}{2}(n^2-n)-1 \text{ points},$$

suppose it drawn through the

$$\tfrac{1}{2}(n^2-3n)+2 \text{ double points};$$

and besides through $\quad n-3$ points on the curve,

together $\quad \frac{1}{2}(n^2-n)-1$ points,

it will cut the curve of the order n in the double points considered as

$$n^2-3n+4 \text{ points},$$

and besides in $\quad n-3$ points,

together $\quad n^2-2n+1$ points,

that is, we should have a curve of order $n-2$ meeting the curve of the order n in $n(n-2)+1$ points, which is of course impossible.

Consider now a curve of the order n with $\frac{1}{2}(n-1)(n-2)$ double points; draw a curve of the order $(n-2)$ through the double points

$$\tfrac{1}{2}(n^2-3n)+1 \text{ points},$$

and besides through $\quad n-3$ points on the curve,

together $\quad \frac{1}{2}(n^2-n)-2$ points;

this imposes on the curve $\frac{1}{2}(n^2-n)-2$ conditions; but as the curve might have been determined to satisfy $\frac{1}{2}(n^2-n)-1$ conditions, its equation will contain an indeterminate parameter λ: it meets the curve n in the double points considered as

	n^2-3n+2	points,
in the	$n-3$	points,
and besides in	1	point,
altogether	n^2-2n	points.

Hence since for any value whatever of λ, there is only one remaining intersection, the coordinates of this point must be rationally determinable in terms of the parameter λ; that is, the coordinates of *any* point on the given curve will be rational functions of λ; or using trilinear coordinates, the coordinates x, y, z of any point on the given curve will be proportional to rational and integral functions of λ.

8. *Show that three bodies attracting each other according to the law of gravitation may move in a line in such wise that the mutual distances are in a constant ratio the value of which depends on the masses.*

Taking the masses m_1, m_2, m_3, to be arranged in this order at distances x_1, x_2, x_3 from a fixed origin, the equations of motion are

$$\frac{d^2x_1}{dt^2} = \frac{m_2}{(x_2-x_1)^2} + \frac{m_3}{(x_3-x_1)^2},$$

$$\frac{d^2x_2}{dt^2} = -\frac{m_1}{(x_1-x_2)^2} + \frac{m_3}{(x_3-x_2)^2},$$

$$\frac{d^2x_3}{dt^2} = -\frac{m_1}{(x_1-x_3)^2} - \frac{m_2}{(x_2-x_3)^2},$$

whence

$$\frac{d^2(x_2-x_1)}{dt^2} = -\frac{m_1+m_2}{(x_2-x_1)^2} + \frac{m_3}{(x_3-x_2)^2} - \frac{m_3}{(x_3-x_1)^2},$$

$$\frac{d^2(x_3-x_1)}{dt^2} = -\frac{m_1+m_3}{(x_3-x_1)^2} - \frac{m_2}{(x_3-x_2)^2} - \frac{m_2}{(x_2-x_1)^2}.$$

Assume

$$x_3-x_1 = \alpha(x_2-x_1),$$

and therefore

$$x_3-x_2 = x_3-x_1-(x_2-x_1) = (\alpha-1)(x_2-x_1);$$

then the equations become

$$\frac{d^2(x_2-x_1)}{dt^2} = \left\{-m_1-m_2+\frac{m_3}{(\alpha-1)^2}-\frac{m_3}{\alpha^2}\right\}\frac{1}{(x_2-x_1)^2},$$

and

$$\frac{d^2(x_3-x_1)}{dt^2} = \left\{-\frac{m_1+m_3}{\alpha^2}-\frac{m_2}{(\alpha-1)^2}-m_2\right\}\frac{1}{(x_2-x_1)^2},$$

which equations will be one and the same equation if only

$$\alpha\left\{-m_1-m_2+\frac{m_3}{(\alpha-1)^2}-\frac{m_3}{\alpha^2}\right\}=-\frac{m_1+m_3}{\alpha^2}-\frac{m_2}{(\alpha-1)^2}-m_2,$$

an equation which (multiplying out) is of the fifth order, and gives at least one real value of α: and α being thus determined, we may from either equation obtain an equation of the form $\frac{d^2(x_2-x_1)}{dt^2}=\frac{C}{(x_2-x_1)^2}$, which determines x_2-x_1 in terms of t, and then $x_3-x_1, =\alpha(x_2-x_1)$, is also known; that is, the relative motions of the three bodies are determined.

9. *Write down the integral equations for the elliptic motion of a planet; and if r, s, v_1 denote the radius vector, tangent of the latitude, and reduced longitude respectively, show that*

$$\frac{\sqrt{(1+s^2)}}{r}=\frac{1}{a_1(1-e_1^2)}\{\sqrt{(1+s^2)}+e_1\cos(v_1-\varpi_1)\},$$

explaining the significations of the constants a_1, e_1, ϖ_1.

Write

a, the mean distance,

n, the mean motion $=\frac{\sqrt{(\mu)}}{a^{\frac{3}{2}}}$,

e, the eccentricity,

θ, longitude of node,

ϖ, longitude of pericentre in orbit,

ϕ, the inclination,

v, the longitude in orbit.

The position in the orbit is determined in terms of the time by means of an auxiliary quantity u, viz. writing

$$nt+c=u-e\sin u.$$

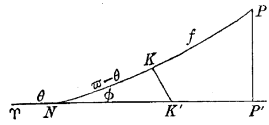
We then have the radius vector r, and the true anomaly f, given as functions of the time by the equations

$$\left.\begin{aligned}\cos f&=\frac{\cos u-e}{1-e\cos u},\\ \sin f&=\frac{\sqrt{(1-e^2)}\sin u}{1-e\cos u},\end{aligned}\right\}$$

$$r=\frac{a(1-e^2)}{1+e\cos f},\quad =a(1-e\cos u).$$

Take z, x, y the hypothenuse, base, and perpendicular of the right-angled triangle NPP', base angle $=\phi$ (see fig. 6), in which NP is the orbit, N the node, K the pericentre, P the planet; then

Fig. 6.



$z=\varpi-\theta+f$ gives z; and x, y are given in terms of z, ϕ, by the formulæ relating to the spherical triangle, so that we have

$$\begin{aligned}\text{longitude in orbit} &= z+\theta\,(=\varpi+f),\\ \text{reduced longitude} &= x+\theta,\\ \text{latitude} &= y.\end{aligned}$$

The expression for r, writing for f its value, $=v-\varpi$, gives

$$\frac{1}{r}=\frac{1}{a\,(1-e^2)}\{1+e\cos(v-\varpi)\},$$

and we have thence

$$\frac{\sqrt{(1+s^2)}}{r}=\frac{1}{a\,(1-e^2)}\{\sqrt{(1+s^2)}+\sqrt{(1+s^2)}\,e\cos(v-\varpi)\}.$$

But we have

$$v-\varpi=v-\theta-(\varpi-\theta),$$

and thence

$$\cos(v-\varpi)=\cos(v-\theta)\cos(\varpi-\theta)+\sin(v-\theta)\sin(\varpi-\theta).$$

Consequently

$$\begin{aligned}\sqrt{(1+s^2)}\cos(v-\varpi)=&\sqrt{(1+s^2)}\cos(\varpi-\theta)\cos(v-\theta)\\ &+\sqrt{(1+s^2)}\sin(\varpi-\theta)\sin(v-\theta);\end{aligned}$$

but by the right-angled triangle, we have

$$\sin(v-\theta)\sin\phi=\sin y \qquad =\frac{s}{\sqrt{(1+s^2)}},$$

$$\sin(v_1-\theta) \qquad =\cot\phi\tan y=s\cot\phi,$$

and thence

$$\sqrt{(1+s^2)}\sin(v-\theta)=\frac{s}{\sin\phi}=\frac{1}{\cos\phi}s\cot\phi=\frac{\sin(v_1-\theta)}{\cos\phi},$$

$$\cos(v-\theta)=\cos(v_1-\theta)\cos y=\frac{\cos(v_1-\theta)}{\sqrt{(1+s^2)}},$$

that is,

$$\sqrt{(1+s^2)}\cos(v-\theta)=\cos(v_1-\theta).$$

The formula thus becomes

$$\frac{\sqrt{(1+s^2)}}{r} = \frac{1}{a(1-e^2)}\left[\sqrt{(1+s^2)} + e\left\{\cos(v_1-\theta)\cos(\varpi-\theta) + \sin(v_1-\theta)\frac{\sin(\varpi-\theta)}{\cos\phi}\right\}\right]$$

$$= \frac{1}{a\sqrt{(1-e^2)}}\left[\sqrt{(1+s^2)} + e\frac{\cos(\varpi-\theta)}{\cos(\varpi_1-\theta)}\{\cos(v_1-\theta)\cos(\varpi-\theta) + \sin(v_1-\theta)\sin(\varpi-\theta)\right]$$

$$= \frac{1}{a_1(1-e_1^2)}\{\sqrt{(1+s^2)} + e_1\cos(v_1-\theta)\},$$

where

$$\frac{1}{\cos\phi}\tan(\varpi-\theta) = \tan(\varpi_1-\theta), \text{ gives } \varpi_1,$$

$$e_1 = e\frac{\cos(\varpi-\theta)}{\cos(\varpi_1-\theta)}, \text{ gives } e_1,$$

$$a_1(1-e_1^2) = a(1-e^2), \text{ gives } a_1.$$

The first equation shows that, drawing the arc KK' at right angles to NP, then

$$\varpi_1 - \theta = NK',$$

and therefore

$$\varpi_1 = \theta + (\varpi_1 - \theta) = \Upsilon K'.$$

The other two equations then give e_1 and a_1: it may be added, that from the right-angled triangle NKK' we have $\cos(\varpi_1-\theta) = \cos(\varpi-\theta)\cos KK'$, and consequently that e_1 is $= e \div \cos KK'$.

10. *Find the differential equations for the motion of a material line acted upon by any forces and moving in a given ruled surface.*

If a, b are the coordinates of the point of intersection of any line of the ruled surface with the plane of xy; α, β, γ the cosines of the inclinations of the line to the three axes respectively; then writing

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z}{\gamma}(=r),$$

and considering a, b, α, β, γ as given functions of a variable parameter θ, where α, β, γ satisfy the relation $\alpha^2+\beta^2+\gamma^2=1$, the equations in question, exclusive of the equation $(=r)$, determine the particular line on the surface; and taking account of the equation $(=r)$, they determine in terms of the parameters r, θ, the coordinates of a particular point in this line.

The motion of the material line is such that it comes successively to coincide with the several lines on the ruled surface; consider on the material line a fixed point, say its centre of gravity G, and imagine that in the course of the motion the material line comes to coincide with the line determined as above by the parameter θ, and the point G with the point determined as above by the parameters r, θ; consider on the

material line any other point P whose distance from G is $=s$; the parameters of P will be $r+s$, θ; and the coordinates will consequently be given in terms of the variable parameters r, θ by the equations

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z}{\gamma}=r+s,$$

that is, we have

$$x=a+(r+s)\,\alpha,\quad y=b+(r+s)\,\beta,\quad z=(r+s)\,\gamma,$$

where a, b, α, β, γ are given functions of θ; r, θ are parameters varying with the time, and s is a constant in regard to the time, but varies with the point under consideration.

Hence, writing as usual T to denote the *Vis Viva* function

$$=\tfrac{1}{2}\Sigma dm\,(x'^2+y'^2+z'^2),$$

where dm is the element of the material line, and x', y', z' are the velocities of the element, it only remains to express T as a function of r, θ; and the equations of motion will be

$$\frac{d}{dt}\frac{\partial T}{\partial r'}-\frac{\partial T}{\partial r}=0,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \theta'}-\frac{\partial T}{\partial \theta}=0.$$

Using a', b', α', β', γ' to denote the differential coefficients $\dfrac{\partial a}{\partial \theta}$, &c., but, as above, x', y', z', r', θ' to denote differential equations in regard to t, we have

$$\begin{aligned} x' &= [a'+(r+s)\,\alpha']\,\theta'+\alpha r',\\ y' &= [b'+(r+s)\,\beta']\,\theta'+\beta r',\\ z' &= [\qquad (r+s)\,\gamma']\,\theta'+\gamma r', \end{aligned}$$

and thence

$$x'^2+y'^2+z'^2=A\theta'^2+2Br'\theta'+r'^2,$$

if for shortness

$$\begin{aligned} A &= a'^2+b'^2+2(r+s)\,(\alpha'a'+\beta'b')+(r+s)^2(\alpha'^2+\beta'^2+\gamma'^2),\\ B &= a'\alpha+b'\beta+(r+s)\,(\alpha\alpha'+\beta\beta'+\gamma\gamma'),\\ &= a'\alpha+b'\beta, \end{aligned}$$

since

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'=0.$$

Hence we have

$$T=\tfrac{1}{2}\theta'^2\Sigma A dm+r'\theta'\Sigma B dm+\tfrac{1}{2}r'^2\Sigma dm,$$

and, observing that in the sum $\Sigma A dm$, the terms involving $\Sigma s dm$ vanish in consequence of G being the centre of gravity, we have

$$\Sigma A dm=[a'^2+b'^2+2r\,(\alpha'a'+\beta'b')+r^2(\alpha'^2+\beta'^2+\gamma'^2)]\,\Sigma dm+(\alpha'^2+\beta'^2+\gamma'^2)\,\Sigma s^2 dm.$$

Or writing $\Sigma dm = M$, $\Sigma s^2 dm = Mk^2$, (M being therefore the mass of the line, and Mk^2 its moment about the centre of gravity), we have

$$\Sigma A dm = \tfrac{1}{2} M\,[a'^2 + b'^2 + 2r\,(\alpha' a' + \beta' b') + (r^2 + k^2)\,(\alpha'^2 + \beta'^2 + \gamma'^2)].$$

Moreover

$$\Sigma B dm = M\,(a'\alpha + b'\beta),$$

and hence we have

$$T = \tfrac{1}{2} M \left\{\theta'^2 [a'^2 + b'^2 + 2r\,(\alpha' a' + \beta' b') + (r^2 + k^2)\,(\alpha'^2 + \beta'^2 + \gamma'^2)] + 2\theta' r'\,(a'\alpha + b'\beta) + r'^2\right\},$$

a given function of r, θ, r', θ'; and the differential equations are therefore given as above.

11. *Explain the mutual connexion of the three theorems in conics:* (1) *the theorem* ad quatuor lineas; (2) *Pascal's theorem;* (3) *the theorem of the anharmonic relation of four points.*

The theorem *ad quatuor lineas* is that the locus of a point, such that the product of its distances from two given lines is always in a given ratio to the product of its distances from two other given lines, is a conic.

Consider the lines as forming a quadrilateral $ABCD$. Then A, B, C, D are points on the conic, and writing PAB to denote the perpendicular distance of a point P from the line AB, and so in other cases, the theorem is that the expression

$$\frac{PAB\,.\,PCD}{PAC\,.\,PBD}$$

has a constant value for any point P whatever on the conic. Now the perpendicular distance PAB is $= 2\Delta PAB \div AB$, or, what is the same thing, it is $= PA\,.\,PB \sin PAB \div AB$; transforming the other perpendicular distances in the same manner, the distances PA, PB, PC, PD divide out and the foregoing expression becomes

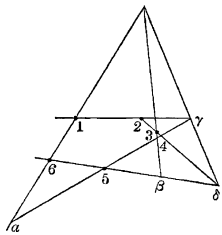
$$= \frac{AC\,.\,BD \sin PAB\,.\,\sin PCD}{AB\,.\,CD \sin PAC\,.\,\sin PBD},$$

viz. omitting the constant factor, it appears that the expression $\dfrac{\sin PAB\,.\,\sin PCD}{\sin PAC\,.\,\sin PBD}$ is constant for all points P on the conic; or, what is the same thing, that the anharmonic ratio of the pencil $P(A, B, C, D)$ is constant for all points P on the conic. This is the anharmonic property of the points of a conic.

Pascal's theorem is that for any six points 1, 2, 3, 4, 5, 6 (fig. 7) on a conic, the intersections of the lines 12 and 45, the lines 23 and 56, the lines 34 and 61 lie in a line. Marking the points α, β, γ, δ as shown in the figure, it appears by the theorem that we have the two lines $(65\beta\delta)$, $(\alpha 54\gamma)$ meeting in 5, and such that the lines through the corresponding points 6 and α, β and 4, δ and γ meet in a point. Hence the ranges $(65\beta\delta)$ and $(\alpha 54\gamma)$ have the same anharmonic ratio; or, what is the same thing, the pencils $3\,(65\beta\delta)$ and $1\,(\alpha 54\gamma)$ have the same anharmonic ratio; that is, the pencils $3\,(6542)$ and $1\,(6542)$ have the same anharmonic ratio; or, considering 1 as a variable

point, but 2, 3, 4, 5, 6 as fixed points on the conic, the pencil 1(6542) of lines from the point 1 to the four points 6, 5, 4, 2 has the same anharmonic ratio for all

Fig. 7.



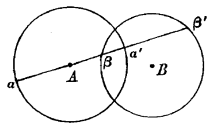
positions whatever of the variable point 1; which is the above-mentioned anharmonic property of the points of a conic.

12. *Show that any line through the centre of either of two orthotomic circles cuts the two circles harmonically; and connect this result with the theorem that the Jacobian of three circles is made up of the line infinity and the orthotomic circle.*

Drawing through the centre of the circle A, (fig. 8) the line $\alpha\beta\alpha'\beta'$, it appears by the figure that we have

$$A\beta \,.\, A\beta' = \text{square of tangential distance of } A \text{ from the circle } B,$$
$$= A\alpha'^2 \text{ since the circles cut at right angles,}$$
$$= A\alpha \,.\, A\alpha';$$

Fig. 8.



that is, the points β, β' are inverse points in regard to the circle on the diameter $\alpha\alpha'$; and the points α, α'; β, β' are thus harmonically related to each other.

Hence the polar of α, in regard to the circle B, passes through the point α', which is the opposite of α in regard to the circle A; and is consequently a fixed point for all the circles B orthotomic to the circle A. That is, considering any three circles orthotomic to the circle A, the circle A is a locus of points α such that the polars of α in regard to the three circles meet in a point.

Moreover, all circles meet the line at infinity in two fixed points I, J, the circular points at infinity: hence for any point α on the line infinity, the polar of α in regard to any circle whatever meets the line infinity in a fixed point, the harmonic of α in regard to the two points I, J; whence the line infinity is also a locus of points α such that the polars of α in regard to the three given circles meet in a point.

The Jacobian of any three conics is the locus of the points α, such that the polars of α in regard to the three conics meet in a point; and it is in general a cubic curve. Hence by what precedes it appears that the Jacobian of three circles is the cubic curve made up of the orthotomic circle and the line infinity.

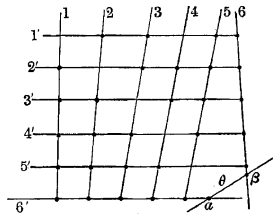
13. *If five given lines have a common transversal, then taking the remaining transversal of each four of the given lines, show by statical considerations or otherwise that the five transversals have a common transversal.*

Consider the line 6′ (fig. 9) meeting each of the lines 1, 2, 3, 4, 5, and take

1′	the remaining	transversal of	(2, 3, 4, 5),
2′	„	„	(3, 4, 5, 1),
3′	„	„	(4, 5, 1, 2),
4′	„	„	(5, 1, 2, 3),
5′	„	„	(1, 2, 3, 4);

it is to be shown that the lines 1′, 2′, 3′, 4′, 5′ have a common transversal 6.

Fig. 9.



Consider the line 6 as a line determined so as to meet the lines 2′, 3′, 4′ and 5′; and take any line θ meeting each of the lines 6, 6′; since the six lines 1, 2, 3, 4, 5, θ have a common transversal 6′, then considering the lines as fixed lines in a solid body, it is possible to find along the lines 1, 2, 3, 4, 5, θ forces which will be in equilibrium. Suppose for a moment that the line 6, determined as above, does not meet the line 1′; then we have the lines 1′, 2′, 3′, 4′, 5′, θ and 6 such that the line 6 meeting the lines 2′, 3′, 4′, 5′, and θ, does not meet the line 1. The six lines 1′, 2′, 3′, 4′, 5′, and θ would be independent lines, such that there do not exist along them forces in equilibrium, and a force acting in any line whatever may be resolved

into forces acting along the lines 1′, 2′, 3′, 4′, 5′, and θ. Hence the original forces along the lines 1, 2, 3, 4, 5 can be each of them so resolved; and combining with the resolved forces the original force along θ, we have a system of forces along the lines 1′, 2′, 3′, 4′, 5′, θ in equilibrium with each other; which is impossible if the lines in question are related as above; that is, the line 6 meeting the lines 2′, 3′, 4′, 5′, θ must also meet the line 1; which is the required theorem.

The statical principles assumed in the above demonstration are as follows:

(1) Six lines may be such that there exist along them forces in equilibrium; or say, for shortness, the six lines may be in involution.

(2) For any seven lines, no six or any less number of which are in involution, there exist along the lines forces in equilibrium; or, what is the same thing, given any six lines not in involution, and a seventh line; then a force along the seventh line may be resolved into forces along the six lines.

(3) Six lines which have a common transversal are in involution (remark in passing that it is not conversely true that if the six lines are in involution they have a common transversal), but if five of the six lines have a common transversal not meeting the sixth line, then the six lines are *not* in involution.

14. *In the theory of the variation of the arbitrary constants of a mechanical problem, state and explain the results obtained by Lagrange and Poisson respectively; and point out the peculiar advantage of Poisson's theory, in regard to the consequences which follow from the coefficients of his formulæ being independent of the time.*

In the theory of the variation of the arbitrary constants of a mechanical problem, it is assumed that the forces consist of principal and disturbing forces, each of them depending on a force function, say there is a principal force function and a disturbing function; (the assumption of a force function however in regard to the disturbing forces, though usual and convenient, is not essential); and that, when the disturbing forces are neglected, or say in the undisturbed problem, the equations of motion can be completely integrated; the theory consists herein that the same integral equations, taking the arbitrary constants to be variable, may be made to satisfy the disturbed equations of motion.

Suppose that the number of the coordinates $x, y, \ldots$ is $=p$, then since the differential equations are of the second order, the number of arbitrary constants $(a, b, c, \ldots)$ will be $=2p$. In Lagrange's solution it is assumed that the coordinates $x, y, \ldots$ (and consequently also the derived functions $x', y', \ldots$) are each of them given in terms of t and the $2p$ constants; the disturbing function Ω is given in the same form; and the expressions for the variations $\frac{da}{dt}$, &c. are obtained by the solution of a system of linear equations

$$\frac{dR}{da} = (a, b)\frac{db}{dt} + (a, c)\frac{dc}{dt} + \&c.,$$

where the coefficients (a, b) are functions involving $\frac{dx}{da}, \frac{dx'}{da}$, &c.; these coefficients are

thus *primâ facie* functions of the $2p$ constants *and of the time;* but it is a result of the theory that they are in fact functions of the constants only, the time disappearing of itself.

In Poisson's theory it is assumed that the integrals of the undisturbed problem are obtained in the form

$$a = \phi(x, y, \ldots x', y', \ldots, t), \text{ \&c.},$$

viz. that each constant is given in terms of the coordinates, their derived functions, and the time (so that, if all the integrals are known, this is equivalent to Lagrange's assumption). The expressions for the variations of the constants are given in the form

$$\frac{da}{dt} = [a, b]\frac{dR}{db} + [a, c]\frac{dR}{dc} + \text{\&c.},$$

where each coefficient $[a, b]$ is given as a function of $\frac{da}{dx}, \frac{da}{dx'}$, &c., $\frac{db}{dx}, \frac{db}{dx'}$, &c.: these coefficients are thus in the first instance expressed as functions of the coordinates $x, y, \ldots$, the derived functions $x', y', \ldots$, and the time; but it is a result of the theory that the coefficients $[a, b]$ are really constant; viz. that, if $x, y, \ldots, x', y', \ldots$ were expressed in terms of the $2p$ constants and the time, then that the time would disappear of itself and the coefficients $[a, b]$, &c. would be found to be functions of the constants only.

The formation of the value of any coefficient $[a, b]$ requires only the knowledge of the expressions of a, b in terms of $x, y, \ldots, x', y', \ldots, t$, or say the knowledge of the two integrals a, b. We thence obtain the expression of $[a, b]$ as a function of $x, y, \ldots, x', y', \ldots, t$, say $[a, b] = f(x, y, \ldots, x', y', \ldots, t)$. But, as already mentioned, $[a, b]$ is in fact a constant; calling it c, we have $c = f(x, y, \ldots, x', y', \ldots, t)$; that is, we have an integral of the equations of motion of the undisturbed problem. It may happen that the value of $[a, b]$ is found to be $= 0$; or to be a function of $x, y, \ldots, x', y', \ldots, t$, which in virtue of the given values of a, b in terms of these same quantities reduces itself to a function of (a, b); in either of these cases we obtain no new integral; but if (as may be) neither of the foregoing cases happen, then the equation $c = f(x, y, \ldots, x', y', \ldots, t)$ *is actually a new integral of the equations of motion* (in the undisturbed problem) *obtained by mere differentiations from the two given integrals* a, b. There is nothing analogous to this in Lagrange's theory.

15. *Write a short dissertation on the transformation of coordinates (rectangular in space of three dimensions); and in particular explain under what restriction it is true that two sets of rectangular axes about the same origin may be made to coincide by means of a rotation of either set about a certain axis; and from the formulae of transformation obtain expressions for the position of this axis and the amount of the rotation.*

Two sets of rectangular axes about the same origin (each axis considered, not as a line extending in two opposite senses, but as drawn from the origin in one sense only) are or are not displacements the one of the other; viz., making the axis of x_1 to coincide with that of x, and the axis of y_1 to coincide with that of y, then

either the axis of z_1 will coincide with that of z, or it will be in the opposite direction; in the former case the two sets are, in the latter case they are not, displacements one of the other. The restriction referred to in the question, is that the two sets shall be displacements one of the other

In the problem of transformation, the two sets of axes, if not displacements, can always be made so by simply reversing the direction of one of the axes (writing for example $-z$ for z); and there is thus no real loss of generality in considering the two sets as displacements the one of the other; and it is in general convenient to make this assumption.

The transformation between two sets of rectangular axes is at once given by the diagram

	x	y	z	
x_1	α	β	γ	
y_1	α'	β'	γ'	
z_1	α''	β''	γ''	

where α is the cosine of the inclination of the axes x, x_1; and so for the rest of the nine quantities. The relation between the two sets of axes is obtained at pleasure by reading the diagram horizontally $x_1 = \alpha x + \beta y + \gamma z$, &c.; or by reading it vertically $x = \alpha x_1 + \alpha' y_1 + \alpha'' z_1$, &c.

We must have identically $x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2$; and we thus obtain the two equivalent sets of equations

$$\begin{array}{l|l}
\alpha^2 + \beta^2 + \gamma^2 = 1, & \alpha^2 + \alpha'^2 + \alpha''^2 = 1, \\
\alpha'^2 + \beta'^2 + \gamma'^2 = 1, & \beta^2 + \beta'^2 + \beta''^2 = 1, \\
\alpha''^2 + \beta''^2 + \gamma''^2 = 1, & \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \\
\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0, \\
\alpha''\alpha + \beta''\beta + \gamma''\gamma = 0, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0, \\
\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0.
\end{array}$$

Either set leads to the relation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}^2 = 1, \text{ consequently } \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \pm 1,$$

the distinction between the two cases $+1$, -1 being that above explained; viz. if the axes are not displacements the one of the other, the sign is $-$; if they are, (and in all that follows this is assumed to be the case) then the sign is $+$. The equations give further $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c. (nine equations).

It is easy to see geometrically that, as stated in the question, two sets of axes (being as above mentioned displacements the one of the other) can be made to coincide by means of a rotation of either set about a certain axis; inasmuch as the position of the axis itself is not altered by the rotation, it is clear that for any point of the axis, the coordinates x, y, z and x_1, y_1, z_1 must be respectively equal; we thus have

$$\begin{aligned} (\alpha-1)\,x + \quad \beta y + \quad \gamma z &= 0, \\ \alpha' x + (\beta'-1)\,y + \quad \gamma' z &= 0, \\ \alpha'' x + \quad \beta'' y + (\gamma''-1)\,z &= 0, \end{aligned}$$

equations which must be equivalent to two equations; we in fact have

$$\begin{vmatrix} \alpha-1, & \beta & , & \gamma \\ \alpha' & , & \beta'-1, & \gamma' \\ \alpha'' & , & \beta'' & , & \gamma''-1 \end{vmatrix} = 0,$$

as is easily verified by means of the foregoing relations between the coefficients. Any two of the three equations will then determine the ratios $x:y:z$; taking the second and third, we have

$$x : y : z = (\beta'-1)(\gamma''-1) - \beta''\gamma' : \gamma'\alpha'' - \alpha'(\gamma''-1) : \alpha'\beta'' - \alpha''(\beta'-1),$$

reducible to

$$x : y : z = 1 + \alpha - \beta' - \gamma'' : \beta + \alpha' : \gamma + \alpha'',$$

and treating in the same way the third and first, and the first and second equations the system of formulæ is

$$\begin{aligned} x : y : z &= 1 + \alpha - \beta' - \gamma'' : \quad \beta + \alpha' \quad : \quad \gamma + \alpha'' \\ &= \quad \alpha' + \beta \quad : 1 - \alpha + \beta' - \gamma'' : \quad \gamma' + \beta'' \\ &= \quad \alpha'' + \gamma \quad : \quad \beta'' + \gamma' \quad : 1 - \alpha - \beta' - \gamma'', \end{aligned}$$

equations equivalent to each other; they determine the position of the axis in question, or resultant axis. The foregoing equations may also be written

$$x^2 : y^2 : z^2 : x^2 + y^2 + z^2 = 1 + \alpha - \beta' - \gamma'' : 1 - \alpha + \beta' - \gamma'' : 1 - \alpha - \beta' + \gamma'' : 3 - \alpha - \beta' - \gamma'',$$

and hence, if A, B, C be the inclinations of the resultant axis to the axes of x and x_1, y and y_1, z and z_1 respectively, we have

$$\cos^2 A = \frac{1 + \alpha - \beta' - \gamma''}{3 - \alpha - \beta' - \gamma''},$$

and thence also

$$\sin^2 A = \frac{2(1-\alpha)}{3-\alpha-\beta'-\gamma''},$$

$$\cos 2A = \frac{-1+3\alpha-\beta'-\gamma''}{3-\alpha-\beta'-\gamma''},$$

$$\alpha - \cos 2A = \frac{(1-\alpha)(1+\alpha+\beta'+\gamma'')}{3-\alpha-\beta'-\gamma''}.$$

Let θ be the amount of the rotation about the resultant axis; we have a spherical triangle the two sides whereof are A, A, the included angle θ, and the opposite side $\cos^{-1}\alpha$; that is, we have

$$\cos\theta = \frac{\alpha - \cos^2 A}{\sin^2 A},$$

and thence

$$4\cos^2\tfrac{1}{2}\theta = 2(1+\cos\theta) = \frac{2(\alpha - \cos 2A)}{\sin^2 A}$$

$$= 1 + \alpha + \beta' + \gamma'',$$

that is, the amount of the rotation is given by the formula

$$4\cos^2\tfrac{1}{2}\theta = 1 + \alpha + \beta' + \gamma''.$$

The nine cosines α, β, γ, &c. may be expressed in terms of the inclinations A, B, C, and the rotation θ, or putting λ, μ, ν equal to $\tan\frac{1}{2}\theta\cos A$, $\tan\frac{1}{2}\theta\cos B$, $\tan\frac{1}{2}\theta\cos C$, in terms of the three quantities λ, μ, ν; but the resulting formulæ for the transformation of coordinates can be more readily obtained by other methods.

530.

SOLUTION OF A SENATE-HOUSE PROBLEM.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. v. (1869), pp. 24—27.]

THE Problem, proposed January 7, 1869, is "If θ_1 and θ_2 are two values of θ which satisfy the equation

$$1+\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}=0,$$

show that θ_1 and θ_2, if substituted for θ and ϕ in this equation, will satisfy it."

That is, writing

$$\frac{\cos\theta_1\cos\phi}{a^2}+\frac{\sin\theta_1\sin\phi}{b^2}+1=0,$$

$$\frac{\cos\theta_2\cos\phi}{a^2}+\frac{\sin\theta_2\sin\phi}{b^2}+1=0,$$

where $a^2+b^2=1$, it is to be shown that

$$\frac{\cos\theta_1\cos\theta_2}{a^2}+\frac{\sin\theta_1\sin\theta_2}{b^2}+1=0.$$

From the given equations, we have

$$\frac{\cos\phi}{a^2}:\frac{\sin\phi}{b^2}:1=\sin\theta_1-\sin\theta_2:\cos\theta_2-\cos\theta_1:\sin(\theta_2-\theta_1),$$

which are

$$=\cos\tfrac{1}{2}(\theta_1+\theta_2):\sin\tfrac{1}{2}(\theta_1+\theta_2):-\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

Whence eliminating ϕ, we have

$$a^4 \cos^2 \tfrac{1}{2}(\theta_1 + \theta_2) + b^4 \sin^2 \tfrac{1}{2}(\theta_1 + \theta_2) - \cos^2 \tfrac{1}{2}(\theta_1 - \theta_2) = 0,$$

that is,

$$a^4 \{1 + \cos(\theta_1 + \theta_2)\} + b^4 \{1 - \cos(\theta_1 + \theta_2)\} - \{1 + \cos(\theta_1 - \theta_2)\} = 0,$$

or, what is the same thing,

$$a^4 + b^4 - 1 + (a^4 - b^4 - 1)\cos\theta_1 \cos\theta_2 + (-a^4 + b^4 - 1)\sin\theta_1 \sin\theta_2 = 0.$$

But from the equation $a^2 + b^2 = 1$, we have

$$a^4 + b^4 - 1 = -2a^2b^2,$$
$$a^4 - b^4 - 1 = -2b^2,$$
$$-a^4 + b^4 - 1 = -2a^2,$$

and the equation is thus

$$\frac{\cos\theta_1 \cos\theta_2}{a^2} + \frac{\sin\theta_1 \sin\theta_2}{b^2} + 1 = 0,$$

which is the required equation.

Stopping at the result obtained previous to the use of the relation $a^2 + b^2 = 1$, but making some obvious substitutions in the formulæ, the theorem may be presented in a more general form as follows; viz.:

If we have

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0,$$

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1 = 0,$$

where

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = 0, \quad \frac{x_1^2}{\alpha^2} + \frac{y_1^2}{\beta^2} - 1 = 0, \quad \frac{x_2^2}{\alpha^2} + \frac{y_2^2}{\beta^2} - 1 = 0,$$

then

$$\left(\frac{a^4}{\alpha^4} + \frac{b^4}{\beta^4} - 1\right) + \left(\frac{a^4}{\alpha^4} - \frac{b^4}{\beta^4} - 1\right)\frac{x_1x_2}{\alpha^2} + \left(-\frac{a^4}{\alpha^4} + \frac{b^4}{\beta^4} - 1\right)\frac{yy_2}{\beta^2} = 0,$$

a relation, the geometrical signification of which is: If (x, y) be a point on the conic $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = 0$, and if the polar hereof in regard to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ meet the first-mentioned conic in the points (x_1, y_1) and (x_2, y_2), then these points are harmonics in regard to the conic

$$\left(\frac{a^4}{\alpha^4} + \frac{b^4}{\beta^4} - 1\right) + \left(\frac{a^4}{\alpha^4} - \frac{b^4}{\beta^4} - 1\right)\frac{x^2}{\alpha^2} + \left(-\frac{a^4}{\alpha^4} + \frac{b^4}{\beta^4} - 1\right)\frac{y^2}{\beta^2} = 0;$$

and since the theorem is projective, it is seen that the first two conics may be any conics whatever, the third conic being a conic having with the other two a common system of conjugate points.

If to fix the ideas we write $\alpha=\beta=c$; then the theorem is, if the polar of a point on the circle $x^2+y^2=c^2$ in regard to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, meet the circle in two points, these are harmonics in regard to the conic

$$(a^4+b^4-c^4)+(a^4-b^4-c^4)\frac{x^2}{c^2}+(-a^4+b^4-c^4)\frac{y^2}{c^2}=0.$$

This last conic will be similar to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, if $c^4=(a^2+b^2)^2$; viz. if $c^2=a^2+b^2$, then the conic is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$; but if $c^2=-a^2-b^2$, the conic is not only similar to, but is the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Considering the given conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ to be an ellipse, the first case $(c^2=a^2+b^2)$ gives the two points (x_1, y_1), (x_2, y_2) harmonics in regard to the imaginary conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1=0$, but this is at once transformed into a real theorem, for we have $(-x_1, -y_1)$ and (x_2, y_2), or, what is the same thing, (x_1, y_1), $(-x_2, -y_2)$ harmonics in regard to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; and the theorem is: "Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and the circle $x^2+y^2=a^2+b^2$ (which is the locus of the intersection of a pair of orthotomic tangents of the ellipse), if the polar in regard to the ellipse of a point on the circle meet the circle in the points Q, R, and if the *opposite* points to these be Q_1, R_1, then (Q, R_1), or what is the same thing (Q_1, R) are harmonics in regard to the ellipse."

The second case $(c^2=-a^2-b^2)$ gives a real theorem if b^2 be negative; viz. writing $-b^2$ for b^2 we have the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $b^2=a^2+c^2$, an obtuse-angled hyperbola; the circle $x^2+y^2=a^2-b^2$, which is the locus of the intersection of a pair of orthotomic tangents of the hyperbola, is consequently imaginary; but the concentric orthotomic circle hereof, viz. the circle of the theorem, $x^2+y^2=b^2-a^2$, is a real circle; and the theorem is: "Given the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0\,(b^2>a^2)$ and the circle $x^2+y^2=b^2-a^2$ (the concentric orthotomic circle of the imaginary circle which is the locus of the intersection of a pair of orthotomic tangents of the hyperbola), if the polar in regard to the hyperbola of a point on the circle meet the circle in two points Q, R, then these are harmonics in regard to the hyperbola."

Of course, if reality be disregarded, the two theorems may each of them be stated of a conic generally; and observe, that in the first theorem the circle is the locus of the intersection of orthotomic tangents, and we have the opposites of the points Q, R; in the second theorem the circle is the concentric orthotomic circle of the circle, which is the locus of the intersection of orthotomic tangents, but we have the points Q, R themselves.

531.

A "SMITH'S PRIZE" PAPER(1); SOLUTIONS.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. v. (1869), pp. 40—64.]

1. *If* $\sqrt{\{(x-a)^2+(y-b)^2\}}$, $\sqrt{\{(x-a')^2+(y-b')^2\}}$, c, *are respectively real and positive, show that in the equation* $\pm\sqrt{\{(x-a)^2+(y-b)^2\}}\pm\sqrt{\{(x-a')^2+(y-b')^2\}}=c$, *considered as representing a curve, the signs cannot either of them be assumed at pleasure to be* $+$ *or to be* $-$: *and distinguish the cases of the ellipse and the hyperbola.*

Writing the equation in the form $\pm\sqrt{(S)}\pm\sqrt{(S')}=c$, we find

$$\pm 2c\sqrt{(S)}=c^2+S-S',$$
$$\pm 2c\sqrt{(S')}=c^2-S+S',$$

and thence

$$4c^2S=(c^2+S-S')^2,$$
$$4c^2S'=(c^2-S+S')^2,$$

either of which equations is the rational equation of the curve (the equation being of the second order, inasmuch as $S-S'$ is a linear function of the coordinates). But writing the rational equation under these two forms respectively and passing back to the last preceding forms, it is clear that for any given point of the curve, c^2+S-S' and c^2-S+S', *qua* rational functions of the coordinates, have each of them a completely determinate value; the ambiguous sign therefore cannot be assumed at pleasure, but in the equation $\pm 2c\sqrt{(S)}=c^2+S-S'$ it must be taken to be $+$ or to be $-$ according as the value of c^2+S-S' is positive or is negative; and the like for the other equation. It is with the signs so determined that the two irrational equations hold good, and inasmuch as from them we deduce the original equation $\pm\sqrt{(S)}\pm\sqrt{(S')}=c$, the signs in this equation are not arbitrary, but each of them has, at a given point of the curve, a determinate value, fixed as above.

[1] Set by me for the Master of Trinity, Feb. 3, 1869.

The condition for an ellipse is that the given length c shall be greater than the distance $\sqrt{\{(a-a')^2+(b-b')^2\}}$ between the two foci; for a hyperbola that it shall be less. In the former case, for any real point of the curve, it is obvious that the relation *must be* $\sqrt{(S)}+\sqrt{(S')}=c$; in the latter case, for any real point of the curve it *must be* either $\sqrt{(S)}-\sqrt{(S')}=c$ or $-\sqrt{(S)}+\sqrt{(S')}=c$, viz. one of these equations holds for one branch, the other for the other branch of the hyperbola. To see this *à posteriori*, observe that in the ellipse, starting from the equation

$$\pm\sqrt{\{(x-ae)^2+y^2\}}\pm\sqrt{\{(x+ae)^2+y^2\}}=2a,$$

we find

$$\pm\sqrt{\{(x-ae)^2+y^2\}}=a-ex,$$

$$\pm\sqrt{\{(x+ae)^2+y^2\}}=a+ex,$$

but here, e being less than 1, and for every real point of the curve x being less in absolute magnitude than a, we have $a-ex$, $a+ex$ each positive for any real point whatever of the curve; the two signs are therefore each of them $+$, or we have $\sqrt{\{(x-ae)^2+y^2\}}+\sqrt{\{(x+ae)^2+y^2\}}=2a$; and a like verification applies to the hyperbola.

2. *In a system of curves defined by an equation containing a variable parameter, investigate at any point the normal distance between two consecutive curves; and determine the form of the equation for a system of parallel curves.*

Consider the system of curves $f(x, y, c)=0$; then if the point x, y belongs to the curve $f(x, y, c)=0$, and the point $x+\delta x$, $y+\delta y$ to the curve $f(x, y, c+\delta c)=0$, we have

$$\frac{df}{dx}\delta x+\frac{df}{dy}\delta y+\frac{df}{dc}\delta c=0,$$

and if the point $x+\delta x$, $y+\delta y$ be on the normal at (x, y) to the curve $f(x, y, c)=0$, we have

$$\delta x\div\frac{df}{dx}=\delta y\div\frac{df}{dy},$$

or writing

$$\delta x=k\frac{df}{dx},\quad \delta y=k\frac{df}{dy},$$

we have

$$k\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}+\frac{df}{dc}\delta c=0,$$

wherefore

$$k=-\frac{df}{dc}\delta c\div\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\},$$

and the normal distance at (x, y) of the curves c and $c+\delta c$ is $=\sqrt{(\delta x^2+\delta y^2)}$, viz., it is $=\pm k\sqrt{\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}}$, or finally it is

$$=\pm\frac{df}{dc}\delta c\div\sqrt{\left\{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2\right\}},$$

where, if the distance in question be regarded as positive (that is, if we attend only to its absolute value), the sign is to be taken so that $\pm\frac{df}{dc}\delta c$ shall be positive.

If the system be given in the form $V-c=0$ (V a function of (x, y)), we have the normal distance

$$= \pm \delta c \div \sqrt{\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2\right\}}.$$

For parallel curves, the normal distance is everywhere the same, that is, $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2$ must have a constant value for all values of (x, y) satisfying the relation $V=c$, viz. we must have identically $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = \phi(V)$, ϕ arbitrary, a partial differential equation to be satisfied by V in order that $V=c$ may be the equation of a system of parallel curves. Assuming the equation to be satisfied for any particular form of ϕ, we may, it is clear, find U a function of V, such that $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 = 1$, and inasmuch as the equation $f(V) = Const.$ is the same thing as $V = Const.$, it follows that the equation of the system of parallel curves may be taken to be $V=c$, where

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1.$$

3. *Two cannons (each free to recoil) differ only in weight and in the weight of the ball; and it is assumed that at any instant during the explosion the explosive force depends only on the space occupied by the vapour of the gunpowder: compare the emerging velocities of the balls; and also the emerging velocities of balls fired from the same cannon when it is free to recoil, and when it is absolutely fixed.*

Consider a single cannon.

Take M for its mass, m for that of the ball, S, s for the spaces described, backwards by the cannon and forwards by the ball, at the time t during the explosion.

Then by hypothesis the explosive force, forwards on the ball and backwards on the cannon, is a function of $s+S$, $=\phi(s+S)$ suppose, or we have

$$m\frac{d^2s}{dt^2} = \phi(s+S),$$

$$M\frac{d^2S}{dt^2} = \phi(s+S),$$

and thence

$$\frac{d^2(s+S)}{dt^2} = \left(\frac{1}{m} + \frac{1}{M}\right)\phi(s+S).$$

Multiplying each side by $2\frac{d(s+S)}{dt}$, integrating from $s+S=0$, to $s+S=a$, where a is the length of the tube, and observing that the initial velocities are each $=0$, we find

$$\left\{\frac{d(s+S)}{dt}\right\}^2 = 2\left(\frac{1}{m} + \frac{1}{M}\right)\int_0^a \phi(s+S)(ds+dS),$$

where $\frac{ds}{dt}, \frac{dS}{dt}$ are the velocities of the ball and cannon respectively at the instant of emergence; $\int_0^a \phi(s+S)(ds+dS)$, is a constant (that is a quantity independent of the weights of the ball and cannon), and putting it $=\frac{1}{2}C$, the equation may be written

$$(v+V)^2=\left(\frac{1}{m}+\frac{1}{M}\right)C,$$

where v, V are the velocities at the instant of emergence.

But from the equation

$$m\frac{d^2s}{dt^2}=M\frac{d^2S}{dt^2},$$

(the initial values of the velocities being each $=0$), we have

$$mv=MV;$$

and hence from the foregoing equation we obtain

$$v=\frac{M}{M+m}(V+v), =\frac{M}{M+m}\frac{\sqrt{(M+m)}}{\sqrt{(Mm)}}\sqrt{(C)}=\frac{\sqrt{(M)}}{\sqrt{(m)}\sqrt{(M+m)}}\sqrt{(C)}=\frac{\sqrt{(C)}}{\sqrt{(m)}\sqrt{\left(1+\frac{m}{M}\right)}},$$

or say

$$v\sqrt{(m)}=\frac{\sqrt{(C)}}{\sqrt{\left(1+\frac{m}{M}\right)}}.$$

And similarly for the other cannon, if m_1, M_1 be the mass of the ball and cannon, v_1 the velocity of the ball, we have

$$v_1\sqrt{(m_1)}=\frac{\sqrt{(C)}}{\sqrt{\left(1+\frac{m_1}{M_1}\right)}},$$

whence

$$v\sqrt{(m)} : v_1\sqrt{(m_1)}=\sqrt{\left(1+\frac{m_1}{M_1}\right)} : \sqrt{\left(1+\frac{m}{M}\right)}.$$

If $m_1=m$, $M_1=\infty$, then v, v_1 may be taken to be the velocities of the same ball fired from the cannon, mass M, when it is free to recoil and when it is absolutely fixed; and we have

$$v : v_1=1 : \sqrt{\left(1+\frac{m}{M}\right)},$$

viz. the velocity in the former case is equal to that in the latter case divided by

$$\sqrt{\left(1+\frac{m}{M}\right)}.$$

4. *If* $X : Y : Z = \eta\zeta' - \eta'\zeta : \zeta\xi' - \zeta'\xi : \xi\eta' - \xi'\eta$, *where* $\xi^2 + \eta^2 + \zeta^2 = 0$, $\xi'^2 + \eta'^2 + \zeta'^2 = 0$, *show that* $\xi\xi'$, $\eta\eta'$, $\zeta\zeta'$, $\eta\zeta' + \eta'\zeta$, $\zeta\xi' + \zeta'\xi$, $\xi\eta' + \xi'\eta$ *are proportional to quadric functions of* X, Y, Z.

We have

$$Y^2 + Z^2 = (\zeta\xi' - \zeta'\xi)^2 + (\xi\eta' - \xi'\eta)^2$$
$$= \xi'^2(\eta^2 + \zeta^2) + \xi^2(\eta'^2 + \zeta'^2) - 2\xi\xi'(\eta\eta' + \zeta\zeta'),$$

which, in virtue of the equations

$$\xi^2 + \eta^2 + \zeta^2 = 0, \quad \xi'^2 + \eta'^2 + \zeta'^2 = 0,$$

is

$$= -2\xi^2\xi'^2 - 2\xi\xi'(\eta\eta' + \zeta\zeta')$$
$$= -2\xi\xi'(\xi\xi' + \eta\eta' + \zeta\zeta').$$

And again

$$YZ = (\zeta\xi' - \zeta'\xi)(\xi\eta' - \xi'\eta)$$
$$= \xi\xi'(\eta\zeta' + \eta'\zeta) - \xi'^2\eta\zeta - \xi^2\eta'\zeta',$$

which, in virtue of the same equations, is

$$= \xi\xi'(\eta\zeta' + \eta'\zeta) + \eta\zeta(\eta'^2 + \zeta'^2) + \eta'\zeta'(\eta^2 + \zeta^2)$$
$$= (\eta\zeta' + \eta'\zeta)(\xi\xi' + \eta\eta' + \zeta\zeta'),$$

and, forming the analogous equations by symmetry, the factor $(\xi\xi' + \eta\eta' + \zeta\zeta')$ divides out, and we have

$$Y^2 + Z^2 : Z^2 + X^2 : X^2 + Y^2 : -2YZ : -2ZX : -2XY$$
$$= \xi\xi' : \eta\eta' : \zeta\zeta' : \eta\zeta' + \eta'\zeta : \zeta\xi' + \zeta'\xi : \xi\eta' + \xi'\eta,$$

which is the required theorem.

5. *Two tangents of a conic are harmonically related to a second conic: find the locus of the intersection of the two tangents.*

In plane geometry the angle in a semicircle is, in spherical geometry it is not, a right angle: show how these conclusions follow from the solution of the above problem.

Let the equation of the first conic be $x^2 + y^2 + z^2 = 0$; its equation in line coordinates is therefore $u^2 + v^2 + w^2 = 0$; or what is the same thing, the line $ux + vy + wz = 0$ will be a tangent of the conic if only $u^2 + v^2 + w^2 = 0$. Hence the equations of the two tangents being

$$\xi x + \eta y + \zeta z = 0,$$
$$\xi' x + \eta' y + \zeta' z = 0,$$

we have $\xi^2 + \eta^2 + \zeta^2 = 0$, $\xi'^2 + \eta'^2 + \zeta'^2 = 0$; and using X, Y, Z for the coordinates of the point of intersection of these tangents, we have

$$X : Y : Z = \eta\zeta' - \eta'\zeta : \zeta\xi' - \zeta'\xi : \xi\eta' - \xi'\eta,$$

and thence, by the last question,

$$\begin{aligned}\xi\xi' : \eta\eta' : \zeta\zeta' : \eta\zeta'+\eta'\zeta : \zeta\xi'+\zeta'\xi : \xi\eta'+\xi'\eta \\ = Y^2+Z^2 : Z^2+X^2 : X^2+Y^2 : -2YZ : -2ZX : -2XY.\end{aligned}$$

Suppose that the equation of the second conic in line coordinates is

$$(a,\ b,\ c,\ f,\ g,\ h)(u,\ v,\ w)^2=0;$$

the condition in order that the two lines $\xi x+\eta y+\zeta z=0$, $\xi' x+\eta' y+\zeta' z=0$, may be harmonically related to this conic is

$$(a,\ b,\ c,\ f,\ g,\ h)(\xi,\ \eta,\ \zeta)(\xi',\ \eta',\ \zeta')=0;$$

or, what is the same thing, it is

$$a\xi\xi'+b\eta\eta'+c\zeta\zeta'+f(\eta\zeta'+\eta'\zeta)+g(\zeta\xi'+\zeta'\xi)+h(\xi\eta'+\xi'\eta)=0,$$

and by what precedes we have

$$a(Y^2+Z^2)+b(Z^2+X^2)+c(X^2+Y^2)-2fYZ-2gZX-2hXY=0,$$

or, what is the same thing,

$$(a+b+c)(X^2+Y^2+Z^2)-(a,\ b,\ c,\ f,\ g,\ h)(X,\ Y,\ Z)^2=0,$$

as the locus of the point of intersection of the two conics; the required locus is therefore a conic.

It is to be observed that the locus is that of a point which is such that the pairs of tangents from it to the two given conics respectively form a harmonic pencil; viz. if the equations of the given conics (in line coordinates) are

$$u^2+v^2+w^2=0,$$
$$(a,\ b,\ c,\ f,\ g,\ h)(u,\ v,\ w)^2=0;$$

then the equation of the required locus, or say the equation of the harmonic conic, is (in point coordinates)

$$(a+b+c)(x^2+y^2+z^2)-(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2=0.$$

In particular, if the second conic be a point-pair or, say, if its equation be

$$(\alpha u+\beta v+\gamma w)(\alpha' u+\beta' v+\gamma' w)=0,$$

then the equation of the harmonic conic is

$$(\alpha\alpha'+\beta\beta'+\gamma\gamma')(x^2+y^2+z^2)-(\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z)=0,$$

which is satisfied by writing $(x,\ y,\ z)=(\alpha,\ \beta,\ \gamma)$, or $=(\alpha',\ \beta',\ \gamma')$; viz. the harmonic conic passes through the two points of the point-pair. And so, if the first conic is also a point-pair, the harmonic conic passes through the four points of the two point-pairs.

The equation of the first conic in point coordinates is $x^2+y^2+z^2=0$; hence, supposing as above, that the second conic is a point-pair, the intersections of the first conic with the harmonic conic are given by

$$x^2+y^2+z^2=0, \quad (\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z)=0;$$

whence the harmonic conic has double contact with the first conic, only if each of the lines $\alpha x+\beta y+\gamma z$, $\alpha' x+\beta' y+\gamma' z=0$, touches the first conic; or, what is the same thing, if each of the points (α, β, γ), $(\alpha', \beta', \gamma')$ lies on the first conic.

In plane geometry, we have (in the plane) a point-pair, the two circular points at infinity; any conic through these points is a circle: the two lines harmonic in regard hereto are at right angles. Hence, by what precedes, taking one of the given conics to be the circular points at infinity, and the other conic to be any two points P, Q; the locus of the intersection of lines through P, Q, cutting each other at right angles, is a circle; this is evidently the circle standing on PQ as diameter, or the angle in a semicircle is a right angle.

In spherical geometry we have (on the sphere) an imaginary conic $x^2+y^2+z^2=0$, called the *absolute;* any conic having double contact herewith is a circle (small circle of the sphere); two lines (arcs of great circles), harmonic in regard hereto, are at right angles. Hence, taking one of the conics to be the conic $x^2+y^2+z^2=0$ and the other to be the two points P, Q; the locus of the intersections of the lines (arcs of great circles) through P, Q, which cut at right angles, is a spherical conic; but it is *not a circle* unless the points P, Q, are each of them on the conic $x^2+y^2+z^2=0$, viz. it is not a circle for any two real points whatever: that is, in spherical geometry, the angle in a semicircle is not a right angle.

6. *A mass M attached to the end A of a chain AC is placed (with the chain) on a horizontal plane, in such wise that a portion AB of the chain forms a straight line, the remaining portion BC being heaped up at B: the mass M is then set in motion in the direction B to A with a given velocity, and so moves in a straight line, dragging the chain: determine the motion; and explain the peculiarity of the dynamical problem.*

The mass attached to the end of the chain is taken to be M, and the mass of a unit of length of the chain to be $=m$; suppose also that at the commencement of the motion the distance CA is $=a$, and that at the end of the time t this distance is $=a+x$, so that the length of chain then in motion is $=a+x$, $(x<l-a$, if l be the whole length of the chain). Suppose also that the velocity at the time t is $=v$; then we have a mass $=M+m(a+x)$ moving with a velocity v; and, in the element of time dt, this sets in motion with the velocity v a length of chain $dx=vdt$, or mass of chain $=mvdt$; if then the impulse backwards on the mass $M+m(a+x)$, and forwards on the element $mvdt$ be $=R$, we have

$$\{M+m(a+x)\}\,dv=-R,$$
$$mvdt\,.\,v=\quad R,$$

that is,

$$\{M + m(a + x)\}\, dv + mv^2 dt = 0\,;$$

or, observing that $vdt = dx$, this may be written

$$(M + ma)\, dv + md\,.\,xv = 0,$$

that is,

$$\{M + m(a + x)\}\, v = \text{constant},$$
$$= (M + ma)\, V,$$

if V be the velocity at the commencement of the motion. The equation gives, in terms of the space x, the velocity v at any time t before the whole chain is set in motion: writing it in the form

$$\{M + m(a + x)\} \frac{dx}{dt} = (M + ma)\, V,$$

we have

$$(M + ma)\, x + \tfrac{1}{2} mx^2 = (M + ma)\, Vt,$$

and putting herein $x = l - a$, the equation gives the value of t at the instant when the whole chain is set in motion: after this epoch, the mass and chain will (it is clear) move on with a uniform velocity

$$= \frac{(M + ma)\, V}{M + ml}\,.$$

The foregoing equation

$$\{M + m(a + x)\}\, v = (M + ma)\, V$$

might have been obtained at once by the consideration that the momentum is constant throughout the motion; but the method employed puts more clearly in evidence the peculiarity of the dynamical problem; viz. it is, so to speak, a problem of continuous impulse: in each element of time dt an infinitesimal element of mass has its velocity abruptly altered (in the present problem from 0 to v), but, for the very reason that it is an infinitesimal element of mass which undergoes this abrupt change of velocity, the effect is a continuous, not an abrupt, change of velocity of the whole finite mass which is then in motion.

7. *Show how an ellipse may be constructed as the envelope of a variable circle having its centre upon either of the axes; and examine the geometrical peculiarities which occur according as the major or the minor axis is made use of.*

Considering the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation

$$(x - \alpha)^2 + y^2 = \gamma^2,$$

where α and γ are in the first instance arbitrary parameters, represents a circle having its centre on the major axis: in order that this may touch the ellipse, a relation must be established between α and γ. When this is done we obtain a variable circle

(the equation of which contains a single arbitrary parameter), and this circle has the ellipse for its envelope. (On account of the symmetry in regard to the axis of x, the circle will, it is clear, touch the ellipse in *two* points, situate symmetrically in regard to this axis.) Now to make the circle touch the ellipse, eliminating y, we have

$$(x-\alpha)^2+b^2\left(1-\frac{x^2}{a^2}\right)-\gamma^2=0,$$

that is,

$$x^2\left(1-\frac{b^2}{a^2}\right)-2\alpha x+\alpha^2-\gamma^2+b^2=0,$$

which equation, considered as an equation in x, must have equal roots; that is, we must have

$$\left(1-\frac{b^2}{a^2}\right)(\alpha^2-\gamma^2+b^2)-\alpha^2=0;$$

or, what is the same thing,

$$-b^2\alpha^2+(a^2-b^2)(b^2-\gamma^2)=0;$$

consequently

$$\gamma^2=b^2-\frac{b^2}{a^2-b^2}\alpha^2;$$

or the required equation of the circle is

$$(x-\alpha)^2+y^2=b^2-\frac{b^2}{a^2-b^2}\alpha^2;$$

or, as this may also be written

$$(x-\alpha)^2+y^2=(1-e^2)\left(a^2-\frac{\alpha^2}{e^2}\right).$$

It is to be observed that at the points of contact of the ellipse and circle, we have

$$x=\frac{a^2\alpha}{a^2-b^2}=\frac{\alpha}{e^2}.$$

By simply interchanging a and b, it appears that when the centre is on the minor axis, the equation of the variable circle is

$$x^2+(y-\beta)^2=a^2+\frac{a^2}{a^2-b^2}\beta^2,$$

and that at the points of contact of the ellipse and circle we have

$$y=-\frac{b^2\beta}{a^2-b^2}.$$

In the case where the centre is on the major axis, then attending to positive values of α, the circle remains real so long as α is $\not> ae$, but if α is $=ae$, that is, if the centre be at the focus, the radius of the circle is $=0$. But from the formula $x=\frac{\alpha}{e^2}$, we have $x=a$ for $\alpha=ae^2$, and when α is greater than ae^2, $x>a$, and the points of contact are imaginary. That is, as α passes from 0 to ae^2, the circle is real, and

has real contact with the ellipse; for $\alpha = ae^2$ the circle becomes the circle of maximum curvature, touching the ellipse at the extremity of the major axis; as α increases from ae^2 to ae, the circle is still real but its contact with the ellipse is imaginary; for $\alpha = ae$, the circle reduces itself to the focus considered as an evanescent circle; and for greater values of α, the circle is imaginary.

When the centre is on the minor axis it appears in like manner that the circle is always real; but, attending to negative values of β, the circle has real contact with the ellipse only so long as $\beta \not> b - \frac{a^2}{b}$; for this value of β, the circle touches the ellipse at the extremity of the minor axis, being in fact the circle of minimum curvature; and for greater negative values of β, the contact is imaginary.

8. *Show that the number of ways in which n things can be arranged so that no one of them occupies its original position is of the form $(n-1)A_n$; and that we have $A_1 = 0$, $A_2 = 1$, $A_n = (n-2)A_{n-1} + (n-3)A_{n-2}$: show also that*

$$A_n = (n-1)A_{n-1} - \frac{1}{n-1}\{A_{n-1} + (-)^{n-1}1\}.$$

Supposing the n things arranged as required, we may by a single transposition of two things bring a given thing, say n, into its original place (the last place): and conversely every arrangement of the required form can be obtained from an arrangement in which n occupies the last place, by a transposition of n with another of the things. Now in the arrangement which thus gives rise to an arrangement of the required form, either all the things $1, 2, 3, \ldots, (n-1)$ are out of their original places; and we can then transpose n with any one of the things $1, 2, 3, \ldots, n-1$: or else only one thing is in its original place; say 1 is in its original place, and we can then transpose 1 and n; or 2 is in its original place, and we can transpose 2 and n; ... or $n-1$ is in its original place, and we can transpose $n-1$ and n. And these are the only ways in which an arrangement of the required form is obtained. Hence if for n things we denote by U_n the number of arrangements in which no one thing occupies its original place, we have, by what precedes,

$$U_n = (n-1)(U_{n-1} + U_{n-2}),$$

and it thus appears that U_n is of the form $(n-1)A_n$; and writing accordingly $U_n = (n-1)A_n$, and therefore

$$U_{n-1} = (n-2)A_{n-1}, \quad U_{n-2} = (n-3)A_{n-2},$$

we have

$$A_n = (n-2)A_{n-1} + (n-3)A_{n-2};$$

which is the required equation for A_n; we have, it is clear, $A_1 = 0$, $A_2 = 1$, and the successive values $A_3 = 1$, $A_4 = 3$, &c. can then be calculated. But the equation of differences of the second order may be integrated into one of the first order, viz writing the equation in the form

$$\{(n-1)A_n - n(n-2)A_{n-1}\} + \{(n-2)A_{n-1} - (n-1)(n-3)A_{n-2}\} = 0,$$

the integral is

$$(n-1)A_n - n(n-2)A_{n-1} = (-)^{n-1}C,$$

and writing $n=2$, we have $C=-1$, wherefore

$$(n-1)A_n = n(n-2)A_{n-1} - (-)^{n-1}1,$$

or, what is the same thing,

$$A_n = (n-1)A_{n-1} - \frac{1}{n-1}\{A_{n-1} + (-)^{n-1}1\},$$

which is the other equation for A_n.

The foregoing very elegant proof of the equation

$$U_n = (n-1)(U_{n-1} + U_{n-2}) = (n-1)A_n$$

was unknown to me; but was given in the Examination; my own proof of the equation $U_n = (n-1)A_n$ was derived from the well-known formula

$$U_n = 1.2\ldots n\left\{1 - \frac{1}{1} + \frac{1}{1.2}\cdots + (-)^n\frac{1}{1.2.3\ldots n)}\right\}.$$

The theorem itself, and the two equations of differences for A_n are due to Euler, see his memoir, "Sur une espèce particulière de Carrés Magiques," *Comm. Arith. Coll.*, t. I., p. 359.

9. *If in any covariant of a binary form* $(a, b, c, \ldots, k)(x, y)^n$ *the coefficients* $a, b, \ldots, k$ *are replaced by* $ax+by, bx+cy, \ldots, kx+ly$ *respectively, show that the result is a covariant of the next superior form* $(a, b, c, \ldots, l)(x, y)^{n+1}$: *and determine the covariant obtained by thus operating on the discriminant of the cubic form* $(a, b, c, d)(x, y)^3$.

A covariant of the binary form $U, = (a, b, \ldots, k)(x, y)^n$, is either given by a single symbolical expression of the form $\overline{12}^\alpha\,\overline{13}^\beta\ldots\overline{23}^\gamma\ldots U_1U_2U_3\ldots$, or it is the sum of a number of such expressions, each multiplied by a constant (numerical factor): in the latter case each of the expressions in question is a covariant of U. Any such expression is at once seen to be a function of U and of its differential coefficients (i.e. it may contain the differential coefficients of each or any of the orders $0, 1, 2, \ldots, n$), homogeneous as regards the differential coefficients of the same order. But writing $U' = (a, b, c, \ldots, l)(x, y)^{n+1}$, the differential coefficients of any order of U are by the change of $a, b, \ldots, k$ into $ax+by, bx+cy, \ldots, kx+ly$, converted into the differential coefficients of the same order of U', each multiplied into the same merely numerical coefficient; the result (disregarding numerical factors) is thus the same derivative $\overline{12}^\alpha\,\overline{13}^\beta\ldots\overline{23}^\gamma\ldots U_1'U_2'U_3'\ldots$ of U'; viz. it is a covariant of U'; and making the like change in any sum of such expressions (each into a numerical factor), the result is a like sum of expressions referring to U'; that is, making the change in any covariant of U, the result is a covariant of U'.

Or the same thing may be otherwise proved thus; if to fix the ideas we write $U=(a,\ b,\ c,\ d)(x,\ y)^3$; any covariant Θ of U is reduced to zero by each of the operations $a\delta_b+2b\delta_c+3c\delta_d-y\delta_x$; $3b\delta_a+2c\delta_b+d\delta_c-x\delta_y$; and, conversely, any function Θ which is thus reduced to zero is a covariant of U. Attending to the first operator, we have for any covariant Θ of U,

$$(a\delta_b+2b\delta_c+3c\delta_d-y\delta_x)\,\Theta=0;$$

write $ax+by=a'$, $bx+cy=b'$, $cx+dy=c'$, $dx+ey=d'$, and let Θ' be the function obtained from Θ by changing $a,\ b,\ c,\ d$ into $a',\ b',\ c',\ d'$. We ought to have

$$(a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e-y\delta_x)\,\Theta'=0,$$

or considering Θ' as a function of $a',\ b',\ c',\ d',\ x,\ y$ this is

$$\{a\,(y\delta_{a'}+x\delta_{b'})+2b\,(y\delta_{b'}+x\delta_{c'})+3c\,(y\delta_{c'}+x\delta_{d'})+4dy\delta_{d'}-y\,(a\delta_{a'}+b\delta_{b'}+c\delta_{c'}+d\delta_{d'})-y\delta_x\}\,\Theta'=0,$$

or, what is the same thing,

$$\{(ax+by)\,\delta_{b'}+2\,(bx+cy)\,\delta_{c'}+3\,(cx+dy)\,\delta_{d'}-y\delta_x\}\,\Theta'=0,$$

that is,

$$\{a'\delta_{b'}+2b'\delta_{c'}+3c'\delta_{d'}-y\delta_x\}\,\Theta'=0,$$

an equation which, Θ' being the same function of $a',\ b',\ c',\ d',\ x,\ y$ that Θ is of $a,\ b,\ c,\ d,\ x,\ y$, is satisfied identically; and thus Θ' is reduced to zero by the operation $a\delta_b+2b\delta_c+3c\delta_d+4d\delta_e-y\delta_x$; and similarly it is reduced to zero by the operation $4b\delta_a+3c\delta_b+2d\delta_c+e\delta_d-x\delta_y$; and it is thus a covariant of $(a,\ b,\ c,\ d,\ e\between x,\ y)^4$.

The discriminant of $(a,\ b,\ c,\ d\between x,\ y)^3$ is

$$=a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2;$$

making the substitution in question, it is

$$=(ax+by)^2\,(dx+ey)^2+\&c.,$$

viz. it is

$$=(a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2)\,x^4+\&c.$$

But there is no such irreducible covariant of the quartic function $(a,\ b,\ c,\ d,\ e\between x,\ y)^4$; and observing that we have identically

$$(ae-4bd+3c^2)\,(ac-b^2)-(ace-ad^2-b^2e-c^3+2bcd)\,a=a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2,$$

the covariant in question must be

$$=(ae-4bd+3c^2)\,[(ac-b^2)\,x^4+\&c.]-(ace-ad^2-b^2e-c^3+2bcd)\,(a,\ b,\ c,\ d,\ e\between x,\ y)^4,$$

where $(ac-b^2)\,x^4+\&c.$ is the Hessian of the quartic function $(a,\ b,\ c,\ d,\ e\between x,\ y)^4$.

10. *From the equation of the curves of curvature of an ellipsoid, or otherwise, determine the curves of curvature of a paraboloid: show also that for the paraboloid $xy=cz$ (the parallel sections $z=Const.$ being rectangular hyperbolas) the curves of curvature are the intersections of the paraboloid by the system of surfaces*

$$h=\sqrt{(x^2+z^2)}\pm\sqrt{(y^2+z^2)}.$$

The curves of curvature of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are given as the intersection of the surface with the confocal surface

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1;$$

transforming to the vertex, or writing $z-c$ in place of z, these become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2-2cz}{c^2}=0,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2-2cz-\lambda}{c^2+\lambda}=0,$$

or multiplying by c and writing $\frac{a^2}{c}=l$, $\frac{b^2}{c}=m$, $\frac{\lambda}{c}=\theta$, the equations are

$$\frac{x^2}{l}+\frac{y^2}{m}+\frac{z^2}{c}-2z=0,$$

$$\frac{x^2}{l+\theta}+\frac{y^2}{m+\theta}+\frac{z^2}{c+\theta}-\frac{2z+\theta}{1+\frac{\theta}{c}}=0,$$

or, putting herein $c=\infty$, the equations are

$$\frac{x^2}{l}+\frac{y^2}{m}-2z=0,$$

$$\frac{x^2}{l+\theta}+\frac{y^2}{m+\theta}-2z-\theta=0,$$

viz. these equations, where θ is a variable parameter, determine the curves of curvature of the paraboloid $\frac{x^2}{l}+\frac{y^2}{m}-2z=0$. The second equation may be replaced by

$$\frac{x^2}{l(l+\theta)}+\frac{y^2}{m(m+\theta)}+1=0.$$

Write in the equations $l=-m=c$; they become

$$x^2-y^2-2cz=0,$$

$$\frac{x^2}{c+\theta}+\frac{y^2}{c-\theta}+c=0;$$

or, substituting herein $\frac{x+y}{\sqrt{2}}$ for x, and $\frac{x-y}{\sqrt{2}}$ for y, the equations are

$$xy-cz=0,$$

$$\frac{(x+y)^2}{c+\theta}+\frac{(x-y)^2}{c-\theta}+2c=0$$

the second of which is

$$c(x^2+y^2)-2\theta xy+c(c^2-\theta^2)=0,$$

which is the equation for determining the curves of curvature of the surface $xy-cz=0$.

Writing in this equation $\theta=\sqrt{(h^2+c^2)}$, it becomes

$$c(x^2+y^2)-2xy\sqrt{(h^2+c^2)}-ch^2=0,$$

or, as this may be written,

$$(c^2+x^2)(c^2+y^2)-x^2y^2-2cxy\sqrt{(h^2+c^2)}-c^2h^2-c^4=0,$$

that is,

$$\pm\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)}=xy+c\sqrt{(h^2+c^2)},$$

and thence

$$\begin{aligned}c^2h^2&=(c^2+x^2)(c^2+y^2)+x^2y^2-c^4\mp 2xy\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)}\\&=x^2(c^2+y^2)+y^2(c^2+x^2)\mp 2xy\sqrt{(c^2+x^2)}\sqrt{(c^2+y^2)},\end{aligned}$$

that is,

$$ch=x\sqrt{(c^2+y^2)}\pm y\sqrt{(c^2+x^2)};$$

or combining with the equation $xy-cz=0$ of the paraboloid, this is

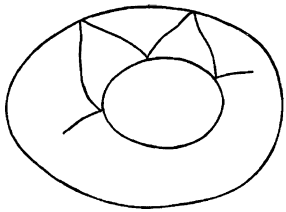
$$h=\sqrt{(x^2+z^2)}\mp\sqrt{(y^2+z^2)},$$

or the curves of curvature are given as the intersections of the paraboloid by the series of surfaces represented by this equation.

11. *Explain for a surface such as the ellipsoid the form of the curve of given constant slope; and in an ellipsoid having one of its principal planes horizontal determine the limits within which the curve is situate.*

At any point of a surface, the direction of the line of greatest slope is evidently at right angles to the level curve, or, what is the same thing, to the trace of the tangent plane on the horizontal plane; and the inclination of the element of the line of greatest slope is equal to that of the tangent plane to the horizontal plane. The line of given constant slope must therefore lie entirely on that portion of the surface for which the inclination of the tangent plane has a value not less than the given constant slope of the curve; viz. on a closed surface such as the ellipsoid, it will lie upon a certain zone of the surface, included between two boundaries, which boundaries are the curves such that at any point thereof the inclination of the tangent plane is *equal* to the given constant slope; and by what precedes, the direction of the line of given constant slope, at any point where it meets the boundary, is coincident with the direction of greatest slope; viz. it will in general meet the boundary at a finite angle; this implies that the point is a cusp on the curve of given constant slope, and the curve in question will be a curve as shown in the figure, passing continually from one boundary to the other, and when it reaches either boundary, turning back cusp-wise to the other boundary; the curve may in particular cases, after making a circuit, or any number of circuits of the zone, re-enter upon itself, forming a closed

curve; but in general this is not the case, and the curve will go on as above between the two boundaries, *ad infinitum.*



In the case of the ellipsoid, the plane of xy being as usual horizontal, and the equation being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then if λ be the given constant inclination, the boundaries are the series of points at which the inclination of the normal to the axes of z is $=\lambda$; that is, we have

$$\frac{z^2}{c^4}=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\cos^2\lambda,$$

or, what is the same thing,

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\cos^2\lambda-\frac{z^2}{c^4}\sin^2\lambda=0,$$

the boundaries are thus two detached ovals, meeting the principal sections $x=0$, $y=0$, in the points given by

$$\frac{z}{y}=\pm\frac{c^2}{b^2}\cot\lambda;\ \frac{z}{x}=\pm\frac{c^2}{a^2}\cot\lambda,$$

and the general form is thus at once perceived.

12. *To every point of space there corresponds a plane, viz. considering the several points as belonging to a solid body which is infinitesimally displaced in any manner, the plane which corresponds to a point is the plane drawn through the point at right angles to the direction of its motion; determine the plane which corresponds to a given point: and connect the result with any general geometrical theory.*

The displacements, in the directions of the axes, of a point whose coordinates are (x, y, z) are given by the ordinary formulæ

$$\delta x=a+qz-ry,$$
$$\delta y=b+rx-pz,$$
$$\delta z=c+py-qx,$$

where a, b, c, p, q, r are constants which determine the particular displacement; hence if X, Y, Z are current coordinates, the plane corresponding to the point (x, y, z) is the plane through this point at right angles to the line

$$\frac{X-x}{\delta x}=\frac{Y-y}{\delta y}=\frac{Z-z}{\delta z},$$

viz. it is the plane

$$(X-x)\,\delta x+(Y-y)\,\delta y+(Z-z)\,\delta z=0,$$

or, substituting for $\delta x, \delta y, \delta z$ their values, reducing and arranging, this is

$$\begin{aligned} & X(\ \ .\ \ -ry+qz+a) \\ +\ & Y(\ rx\ \ .\ \ -pz+b) \\ +\ & Z(-qx+py\ \ .\ \ +c) \\ +\ & \ \ (-ax-by-cz\ \ .\ \ )=0; \end{aligned}$$

that is, the equation of the plane corresponding to the point (x, y, z) is an equation linear in regard to the coordinates (x, y, z) of this point; and such that arranging as above in the form of a square, the coefficients which are symmetrically situate in regard to the dexter diagonal are equal and of opposite signs; or say that the coefficients form a skew symmetrical matrix.

More generally, corresponding to the point (x, y, z) we may have the plane

$$AX+BY+CZ+D=0,$$

where A, B, C, D are any linear functions whatever $(=\alpha x+\beta y+\gamma z+\delta$, &c.) of the coordinates (x, y, z); this is the general relation which is the analytical basis of the theory of duality; it in fact appears that if the point be a variable point situate in a line or a plane, then the corresponding plane is a variable plane passing through a line or a point, &c., &c. In the general case, to a given point there corresponds a plane not in general passing through the point; the case above considered is distinguished by the circumstance that for any point whatever the corresponding plane does pass through the point.

13. *Write down the Lagrangian equations of motion, explaining the notation, and mode of applying them; and by way of illustration deduce the equations of motion of three particles, connected so as to form an equilateral triangle (of variable magnitude), moving in a plane under the action of any forces.*

The Lagrangian equations of motion are

$$\frac{d}{dt}\cdot\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dU}{d\xi},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\eta'}-\frac{dT}{d\eta}=\frac{dU}{d\eta},$$

&c.,

$\xi, \eta, \ldots$ are here any independent coordinates (in the most general sense of the word) which serve to determine the position of the system at the time t; T is the *vis-viva* function, or half-sum of the mass of each particle of the system into the square of its velocity, expressed in terms of the coordinates $\xi, \eta, \ldots$ and of their differential coefficients $\xi', =\frac{d\xi}{dt}$; $\eta', =\frac{d\eta}{dt}$, &c.; T is thus a *given* function of $\xi, \eta, \ldots \xi', \eta', \ldots$, homogeneous of the second order as regards the quantities $\xi', \eta', \ldots$; $\frac{dT}{d\xi}, \frac{dT}{d\eta}$, &c., $\frac{dT}{d\xi'}, \frac{dT}{d\eta'}$, &c. are the partial differential coefficients of T in regard to $\xi, \eta, \ldots \xi', \eta', \ldots$ respectively, and they are thus given functions of $\xi, \eta, \ldots \xi', \eta', \ldots$; U is the force-function or sum

$$\Sigma dm \int (Xdx + Ydy + Zdz)$$

(it being assumed that $Xdx + Ydy + Zdz$ is a complete differential, that is, that there exists a force-function U) expressed in terms of the coordinates $\xi, \eta, \ldots$ and being thus a *given* function of these coordinates; $\frac{dU}{d\xi}, \frac{dU}{d\eta}$, &c. are the partial differential coefficients of U in regard to $\xi, \eta, \ldots$ respectively; and are thus given functions of these coordinates. The terms $\frac{d}{dt}\frac{dT}{d\xi'}$, contain, it is clear, (and that linearly) the second differential coefficients $\xi'', =\frac{d^2\xi}{dt^2}$, &c. ..., the equations thus establish between the coordinates $\xi, \eta, \ldots$, and their first and second differential coefficients in regard to the time a system of relations, the number of which is equal to that of the coordinates, and which therefore would by integration lead to the expression of the coordinates $\xi, \eta, \ldots$ in terms of the time.

It has been tacitly assumed that T, U were functions of $\xi, \eta, \ldots \xi', \eta', \ldots$, and of $\xi, \eta, \ldots$, not containing the time t; this is the ordinary case, but there are cases in which T and U or either of them may also contain the time t.

In the proposed case of the three particles, if m_1, m_2, m_3 be their masses, r the side of the equilateral triangle, x_1, y_1 the coordinates of m_1, θ, $\theta + 60°$ (or write for convenience $\theta + \alpha$) the inclinations of m_1m_2, m_1m_3 to the axis of x, then the coordinates of m_2, m_3 will be

$$\begin{array}{ll} x + r\cos\theta, & x + r\cos(\theta+\alpha), \\ y + r\sin\theta, & y + r\sin(\theta+\alpha). \end{array}$$

We have

$$\begin{aligned} T = {} & \tfrac{1}{2}m_1(x'^2 + y'^2) \\ & + \tfrac{1}{2}m_2[(x' + r'\cos\theta - r\sin\theta\,.\,\theta')^2 + (y' + r'\sin\theta + r\cos\theta\,.\,\theta')^2] \\ & + \tfrac{1}{2}m_3[\{x' + r'\cos(\theta+\alpha) - r\sin(\theta+\alpha)\,.\,\theta'\}^2 + \{y' + r'\sin(\theta+\alpha) + r\cos(\theta+\alpha)\,.\,\theta'\}^2] \\ = {} & \tfrac{1}{2}(m_1 + m_2 + m_3)(x'^2 + y'^2) \\ & + m_2[x'(r'\cos\theta - r\sin\theta\,.\,\theta') + y'(r'\sin\theta + r\cos\theta\,.\,\theta')] \\ & + m_3[x'\{r'\cos(\theta+\alpha) - r\sin(\theta+\alpha)\,.\,\theta'\} + y'\{r'\sin(\theta+\alpha) + r\cos(\theta+\alpha)\,.\,\theta'\}] \\ & + \tfrac{1}{2}(m_2 + m_3)(r'^2 + r^2\theta'^2), \end{aligned}$$

a given function of $x, y, r, \theta, x', y', r', \theta'$; also

$$U = m_1 \int (X dx + Y dy) + \&c.$$

will be a given function of x, y, r, θ; and the equations of motion will be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} = \frac{dU}{dx},$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} = \frac{dU}{dy},$$

$$\frac{d}{dt}\frac{dT}{dr'} - \frac{dT}{dr} = \frac{dU}{dr},$$

$$\frac{d}{dt}\frac{dT}{d\theta'} - \frac{dT}{d\theta} = \frac{dU}{d\theta}.$$

14. *From the integrals* $x = a\cos t + a'\sin t$, $y = b\cos t + b'\sin t$, *of the dynamical equations* $\frac{d^2x}{dt^2} = -x$, $\frac{d^2y}{dt^2} = -y$, *deduce a simultaneous solution of the two partial differential equations*

$$\frac{dS}{dt} + \tfrac{1}{2}\left\{\left(\frac{dS}{dx}\right)^2 + \left(\frac{dS}{dy}\right)^2\right\} = -\tfrac{1}{2}(x^2 + y^2),$$

$$\frac{dS}{dt} + \tfrac{1}{2}\left\{\left(\frac{dS}{da}\right)^2 + \left(\frac{dS}{db}\right)^2\right\} = -\tfrac{1}{2}(a^2 + b^2).$$

Writing $U = -\frac{1}{2}(x^2 + y^2)$, we have

$$x = a\cos t + a'\sin t,$$

$$y = b\cos t + b'\sin t,$$

as the integrals of the equations of motion

$$\frac{d^2x}{dt^2} = \frac{dU}{dx}, \quad \frac{d^2y}{dt^2} = \frac{dU}{dy};$$

moreover

$$x' = -a\sin t + a'\cos t,$$

$$y' = -b\sin t + b'\cos t,$$

where a, b, a', b' are the initial values of x, y, x', y' respectively $\left(x' = \frac{dx}{dt}, y' = \frac{dy}{dt}, \text{ as usual}\right)$.

Hence the Principal Function

$$S = \int_0^t (T + U)\,dt$$

$$= \tfrac{1}{2}\int_0^t (x'^2 + y'^2 - x^2 - y^2)\,dt,$$

expressed as a function of x, y, a, b, t, will satisfy simultaneously the proposed partial differential equations.

We have

$$x'^2 + y'^2 - x^2 - y^2 = (a'^2 + b'^2 - a^2 - b^2)\cos 2t - 2(aa' + bb')\sin 2t.$$

Hence

$$\begin{aligned}4S &= (a'^2+b'^2-a^2-b^2)\sin 2t + 2(aa'+bb')(\cos 2t - 1)\\ &= (a'^2+b'^2)\sin 2t - (a^2+b^2)\sin 2t - 4(aa'+bb')\sin^2 t.\end{aligned}$$

But

$$a'\sin t = x - a\cos t,$$
$$b'\sin t = y - b\cos t.$$

Hence

$$(aa'+bb')\sin t = ax+by-(a^2+b^2)\cos t,$$
$$(a'^2+b'^2)\sin^2 t = x^2+y^2-2(ax+by)\cos t + (a^2+b^2)\cos^2 t,$$

and substituting these values

$$\begin{aligned}4S = \frac{2\sin t\cos t}{\sin^2 t}\{x^2+y^2-2(ax+by)\cos t + &(a^2+b^2)\cos^2 t\}\\ &- (a^2+b^2)\,2\sin t\cos t\\ -4\sin t\quad\{\qquad ax+by &- (a^2+b^2)\cos t\}\\ = 2(x^2+y^2)\frac{\cos t}{\sin t} - \frac{4}{\sin t}(ax+by) &+ 2(a^2+b^2)\frac{\cos t}{\sin t};\end{aligned}$$

or, what is the same thing,

$$S = \tfrac{1}{2}(x^2+y^2+a^2+b^2)\cot t - 2(ax+by)\operatorname{cosec} t,$$

which value of S satisfies the two partial differential equations.

More generally, the two equations are satisfied by

$$S = c + \text{foregoing value},$$

(c an arbitrary constant) which new value, considered as a solution of the first equation, contains the three arbitrary constants c, a, b, and is thus a *complete* solution; and similarly considered as a solution of the second equation, it contains the three arbitrary constants c, x, y, and is thus a *complete* solution.

I venture to add a few remarks in illustration of what is required in the papers sent up in an Examination.

In the latter part of question (2) (form of the equation for a system of parallel curves) it is worse than useless to say $\frac{df}{dc} \div \left\{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2\right\}^{\frac{1}{2}}$ [p. 441] must be constant: a good and sufficient answer would be that it must be constant *in virtue of the given equation* $f(x, y, c) = 0$. So in question (13) (the Lagrangian equations of motion), it is *quite essential* to explain [p. 455] that T, U are *given* functions of $\xi, \eta, \ldots, \xi', \eta', \ldots$ and of $\xi, \eta, \ldots$ respectively; but for this the equations might be partial differential equations for the determination of T, U, or nobody knows what: it is natural and proper to explain further that T is homogeneous of the second order in regard to the derived functions $\xi', \eta', \ldots$. In question (14) the answer [p. 456] that $S = \frac{1}{2}\int_0^t (x'^2+y'^2-x^2-y^2)\,dt$ expressed as a function of x, y, a, b, t will satisfy simultaneously the proposed equations—would be, not of course a complete answer, but a good and creditable one; without the words "expressed as a function of x, y, a, b, t" it would be altogether worthless. A clear and precise indication of a process of solution is very much better than a detailed solution incorrectly worked out.

532.

NOTE ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS BY SERIES.

[From the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. v. (1869), pp. 77—82.]

THERE is a speciality in the integration of certain differential equations by series, which (though evidently quite familiar to those who have written on the subject—Ellis, Boole, Hargreave) has not, it appears to me, been exhibited in the clearest form. To fix the ideas, consider a linear differential equation of the second order integrable in the form $y = Ax^{\lambda} + Bx^{\lambda+1} + \ldots$; λ is determined by a quadratic equation, and for each value of λ the coefficients $B, C, \ldots$ are given multiples of A; we have thus the general solution

$$y = A\left(x^{\alpha} + \frac{B}{A}x^{\alpha+1}\ldots\right) + K\left(x^{\beta} + \frac{L}{K}x^{\beta+1} + \&c.\ \ldots\right).$$

The speciality referred to is, when the two roots differ by an integer number; suppose α is the smaller root, and $\beta = \alpha + k$ (k a positive integer) the larger. Then, inasmuch as the series

$$y = Ax^{\alpha} + Bx^{\alpha+1} + \&c.$$

is identical in form with the general solution as above written down, it is clear that, starting with the root $\lambda = \alpha$, the coefficients beginning with that of x^{β}, $= x^{\alpha+k}$, ought not to be any longer determinate multiples of A, but should contain a new arbitrary constant K, and thus that the series derived from the root $\lambda = \alpha$ should be the general solution containing two arbitrary constants. The most simple case is when the substitution of the series in the differential equation leads to a relation between *two* consecutive coefficients of the series. Here the values $\frac{B}{A}$, $\frac{C}{A}$, &c. are fractions

the numerators and denominators of which are factorial functions of α such that, for some coefficient $\frac{F}{A}$ preceding $\frac{K}{A}$ (if K is the coefficient of $x^{\alpha+k}$), and for all the succeeding coefficients $\frac{G}{A}$, &c. there is in the *numerators* one and the same evanescent factor; this being so, it is allowable to write $F=0$, $G=0$, &c. giving for the differential equation the finite solution

$$y = A\left(x^{\alpha} + \frac{B}{A}x^{\alpha+1} \ldots + \frac{E}{A}x^{\alpha+e}\right):$$

but if, notwithstanding the evanescent factor, we carry on the series, then in the coefficient of $x^{\alpha+k}$ there occurs in the *denominator* the same evanescent factor, so that the coefficient of this term presents itself in the form $A\frac{P}{Q}\cdot\frac{0}{0}$, = an arbitrary constant K (since the $\frac{0}{0}$ is essentially indeterminate), and the solution is thus obtained in the form

$$y = A\left(x^{\alpha} + \frac{B}{A}x^{\alpha+1} \ldots + \frac{E}{A}x^{\alpha+e}\right) + K\left(x^{\alpha+k} + \frac{L}{K}x^{\alpha+k+1} + \&c.\right),$$

viz. there is one particular solution which is finite.

Take for example the equation

$$\frac{d^2y}{dx^2} + q\frac{dy}{dx} - \frac{2}{x^2}y = 0 \qquad \text{(I)},$$

mentioned *Cambridge Math. Journal*, t. II. p. 176 (1840). If the integral is assumed to be

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + \&c.,$$

then we find

$$\begin{aligned}
(\alpha+1)(\alpha-2)A &= 0,\\
(\alpha-1)(\alpha+2)B + q\alpha A &= 0,\\
\alpha(\alpha+3)C + q(\alpha+1)B &= 0,\\
(\alpha+1)(\alpha+4)D + q(\alpha+2)C &= 0,\\
\&c.
\end{aligned}$$

Hence $\alpha = -1$, or else $\alpha = 2$;

$$B = \frac{-q\alpha}{(\alpha-1)(\alpha+2)}A,$$

$$C = \frac{q^2.\alpha(\alpha+1)}{(\alpha-1)\alpha(\alpha+2)(\alpha+3)}A,$$

$$D = \frac{-q^3\alpha(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}A,$$

&c.

Here taking $\alpha = -1$, we are at liberty to make C and all the following coefficients $= 0$: in fact, if we commence by assuming

$$y = Ax^{\alpha} + Bx^{\alpha+1},$$

the equations

$$\begin{aligned} (\alpha+1)(\alpha-2)\,A &= 0, \\ (\alpha-1)(\alpha+2)\,B + \qquad q\alpha A &= 0, \\ q(\alpha+1)\,B &= 0, \end{aligned}$$

are all satisfied if only $\alpha = -1$, $B = \dfrac{-q\alpha A}{(\alpha-1)(\alpha+2)}$; and we have thus the finite solution $y = A\left(\dfrac{1}{x} - \dfrac{q}{2}\right)$; but if we continue the series, retaining D to represent the indeterminate quantity $\dfrac{-q^3\alpha(\alpha+1)(\alpha+2)}{(\alpha-1)\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}A$, we have the solution

$$y = A\left(\frac{1}{x} - \tfrac{1}{2}q\right) + D\left(x^2 - \tfrac{1}{2}qx^3 + \&c.\right),$$

the second member of which is in fact the series derived from the root $\alpha = 2$. This series is expressible by means of an exponential, viz. we have

$$x^2 - \tfrac{1}{2}qx^3 + \&c. = \frac{12}{q^3}\left\{\left(\frac{1}{x} + \tfrac{1}{2}q\right)e^{-qx} - \left(\frac{1}{x} - \tfrac{1}{2}q\right)\right\},$$

and the complete integral is thus

$$y = A\left(\frac{1}{x} - \tfrac{1}{2}q\right) + B\left(\frac{1}{x} + \tfrac{1}{2}q\right)e^{-qx},$$

but this result is not immediately connected with the investigation. It may however be noticed that, writing $z = ye^{qx}$, the equation in z is

$$\frac{d^2z}{dx^2} - q\frac{dz}{dx} - \frac{2}{x^2}z = 0,$$

which only differs from the original equation in that it has $-q$ in place of q: there is consequently the particular solution $z = \dfrac{1}{x} + \tfrac{1}{2}q$, giving for y the particular solution $y = \left(\dfrac{1}{x} + \tfrac{1}{2}q\right)e^{-qx}$, and we have thence the complete solution as above.

Consider, secondly, the differential equation

$$\frac{d^2y}{dx^2} - \left(\tfrac{1}{4}q^2 + \frac{2}{x^2}\right)y = 0 \qquad \text{(II)},$$

(derived from the equation (I) by writing therein ye^{-qx} in place of y); this equation is satisfied by the series

$$y = Ax^{\alpha} + Bx^{\alpha+2} + Cx^{\alpha+4} + \&c.$$

Then $\alpha = -1$ or $\alpha = +2$, but here the series belonging to $\alpha = -1$ contains only odd powers, the other contains only even powers of x; hence the two series do not coalesce as in the former case, and the first series is obtained without the indeterminate symbol $\frac{0}{0}$ in any of the coefficients. We have in fact

$$\begin{aligned}(\alpha+1)(\alpha-2)\,A &= 0,\\ (\alpha+3)(\alpha\quad)\,B - \tfrac{1}{4}q^2A &= 0,\\ (\alpha+5)(\alpha+2)\,C - \tfrac{1}{4}q^2B &= 0,\\ &\vdots\end{aligned}$$

and there is not in the series in question (or in the other series) any evanescent factor, either in the numerators or in the denominators.

But consider, thirdly, the differential equation

$$\frac{d^2y}{dx^2} + (q-2\theta)\frac{dy}{dx} + \left(\theta^2 - q\theta - \frac{2}{x^2}\right)y = 0 \qquad \text{(III)},$$

(derived from (I) by writing therein $ye^{-\theta x}$ in place of y). This is satisfied by the series

$$y = Ax^\alpha + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + \&c.,$$

where $\alpha = -1$ or $\alpha = +2$ as before; in the series belonging to $\alpha = -1$, the coefficient D should become indeterminate. The relation between the coefficients is here a relation between three consecutive coefficients, viz. we have

$$\begin{aligned}&(\alpha+1)(\alpha-2)\,A = 0,\\ &(\alpha+2)(\alpha-1)\,B + (q-2\theta)(\quad\alpha\quad)\,A = 0,\\ &(\alpha+3)(\quad\alpha\quad)\,C + (q-2\theta)(\alpha+1)\,B + (\theta^2-q\theta)\,A = 0,\\ &(\alpha+4)(\alpha+1)\,D + (q-2\theta)(\alpha+2)\,C + (\theta^2-q\theta)\,B = 0,\\ &(\alpha+5)(\alpha+2)\,E + (q-2\theta)(\alpha+3)\,D + (\theta^2-q\theta)\,C = 0,\\ &\&c.\end{aligned}$$

It is to be shown that in the series for $\alpha = -1$, the expression $(q-2\theta)(\alpha+2)\,C + (\theta^2-q\theta)\,B$ contains the evanescent factor $(\alpha+1)$, and consequently that D is indeterminate; we have in fact

$$B = -\frac{(q-2\theta)\,\alpha}{(\alpha+2)(\alpha-1)}A,$$

$$C = \frac{1}{\alpha(\alpha+3)}\left\{\frac{(q-2\theta)^2\alpha(\alpha+1)}{(\alpha+2)(\alpha-1)} - (\theta^2-q\theta)\right\},$$

and thence

$$(q-2\theta)(\alpha+2)\,C + (\theta^2-q\theta)\,B$$

$$= \frac{1}{\alpha(\alpha+3)}\left\{-\frac{(q-2\theta)^3\alpha(\alpha+1)(\alpha+2)}{(\alpha+2)(\alpha+1)} - (q-2\theta)(\theta^2-q\theta)(\alpha+2)\right\}A - \frac{(\theta^2-q\theta)(q-2\theta)\,\alpha}{(\alpha+2)(\alpha-1)},$$

and then

$$\frac{\alpha+2}{\alpha(\alpha+3)}+\frac{\alpha}{(\alpha+2)(\alpha-1)}=\frac{2\alpha^3+6\alpha^2-4}{(\alpha-1)\alpha(\alpha+2)(\alpha+3)}=\frac{2(\alpha+1)(\alpha^2+2\alpha-2)}{(\alpha-1)\alpha(\alpha+2)(\alpha+3)},$$

so that the whole expression contains the factor $\alpha+1$. But observe that in the present case, if (as is allowable) we write $D=0$, the next coefficient E (depending not on D only, but on D and C) will not vanish; so that the solution obtained on the assumption $D=0$ will go on to infinity: and if instead of assuming $D=0$, we assume $D=$ an arbitrary quantity D', then E and the subsequent coefficients will contain terms depending on D'; and the complete form of the series belonging to $\alpha=-1$ will be

$$y=A\left(x^{-1}+\frac{B}{A}+\frac{C}{A}x+\frac{D}{A}x^2+\frac{E}{A}x^3+\&c.\right)+D'\left(x^2+\frac{E}{D'}x^3+\&c.\right),$$

where the second member is in fact the series belonging to $\alpha=2$. It is hardly necessary to remark that the solution thus obtained can be expressed by means of exponentials, viz. that the solution is

$$y=A\left(\frac{1}{x}-\tfrac{1}{2}q\right)e^{\theta x}+D'\frac{12}{q^3}\left\{\left(\frac{1}{x}+\tfrac{1}{2}q\right)e^{-qx}-\left(\frac{1}{x}-\tfrac{1}{2}q\right)\right\}e^{\theta x}.$$

533.

ON THE BINOMIAL THEOREM, FACTORIALS, AND DERIVATIONS.

[From the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. v. (1869), pp. 102—114.]

THE following was part of my course of lectures in the year 1867.

The proof commonly called "Euler's" of the binomial theorem is as follows: the theorem is assumed to be true for positive integer indices; that is, it is assumed that for any positive integer m we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.$$

This being so, since $(1+x)^m.(1+x)^n = (1+x)^{m+n}$, the equation

$$\left\{1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.\right\}\left\{1 + nx + \frac{n(n-1)}{1.2}x^2 + \&c.\right\}$$
$$= 1 + \frac{m+n}{1}x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \&c.$$

is true for any positive integer values whatever of the indices m, n; the equation is therefore true identically; and it is consequently true for all values whatever of the indices m, n. But any function ϕm of m, satisfying the functional equation $\phi m.\phi n = \phi(m+n)$, is an mth power, $= C^m$ suppose; that is, we have

$$C^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.,$$

where C is a constant, viz. it is independent of m; but the value of C will of course depend upon x; and if, in order to determine it, we write $m=1$, the equation gives $C = 1 + x$; that is, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.,$$

which is the binomial theorem in its general form.

It is to be observed that there is not in the demonstration any employment of the so called "principle of equivalent forms," we do not from the truth of an equation for positive integer values of m, n, infer the truth of it for any values whatever of m, n; there is the intermediate step, that being true for integers, it is true identically; and this identical truth of the equation depends on the circumstance that comparing on the two sides of the equation the coefficients of the successive powers x^0, x^1, x^2, &c., these coefficients are in every case finite, rational, and integral functions of m, n.

For instance, comparing the coefficients of x^2, we have

$$(m+n)(m+n-1) = m(m-1) + 2mn + n(n-1),$$

and any such equation, being true for all positive integer values of m, n, will be true identically; developing the two sides, the equation is in fact

$$m^2 + 2mn + n^2 - m - n = m^2 - m + 2mn + n^2 - n.$$

The reasoning is thus perfectly good; but I remark that it is quite as easy to prove the general equation of which the last mentioned equation is an example, as it is to prove the binomial theorem for positive integer indices; and consequently that we can *without the aid of the binomial theorem for positive integer indices* prove the fundamental equation

$$\left\{1 + mx + \frac{m(m-1)}{1.2}x^2 + \&c.\right\}\left\{1 + nx + \frac{n(n-1)}{1.2}x^2 + \&c.\right\}$$
$$= 1 + \frac{m+n}{1}x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \&c.$$

To show this I introduce the factorial notation and write

$$m(m-1)\ldots(m-r+1) = [m]^r;$$

this being so, the equation obtained by comparing the coefficients of x^r is readily found to be

$$[m+n]^r = [m]^r + \frac{r}{1}[m]^{r-1}[n]^1 + \frac{r(r-1)}{1.2}[m]^{r-2}[n]^2 + \&c.,$$

and I say that this, *the factorial binomial theorem for a positive integer index* r, is proved as easily and in the same manner as the binomial theorem for a like value of the index; or say as the equation

$$(m+n)^r = m^r + \frac{r}{1}m^{r-1}n^1 + \frac{r(r-1)}{1.2}m^{r-2}n^2 + \&c.$$

To show how this is, I form the values of $[m+n]^1$, $[m+n]^2$, $[m+n]^3$, &c. successively, by what may be termed the process of *varied multiplication:* we have

$$[m+n]^1 = m + n;$$

to obtain $[m+n]^2$ we have to multiply this by $m+n-1$; in regard to the first term m, I write the multiplier under the form $(m-1)+n$, and in regard to the second term n, under the form $m+(n-1)$; the process then stands

$$\begin{array}{lll}
[m+n]^1 & =m & +n \\
m+n-1 & =(m-1)+n & m+(n-1) \\
\hline
 & m(m-1) & +mn \\
 & & +mn \qquad +n(n-1) \\
\hline
[m+n]^2 & =m(m-1) & +2mn \qquad +n(n-1) \\
 & =[m]^2 & +2[m]^1[n]^1+[n]^2.
\end{array}$$

To form $[m+n]^3$ we have to multiply by $m+n-2$; in regard to the first term, this is written under the form $(m-2)+n$; in regard to the second term under the form $(m-1)+(n-1)$; and in regard to the third term under the form $m+(n-2)$; the product is thus obtained in the form

$$\begin{array}{l}
[m]^3+\ [m]^2[n]^1 \\
\quad +2[m]^2[n]^1+2[m]^1[n]^2 \\
\qquad\qquad +\ [m]^1[n]^2+[n]^3 \\
\hline
=[m]^3+3[m]^2[n]^1+3[m]^1[n]^2+[n]^3,
\end{array}$$

and so on; the law of the terms is obvious; and the numerical coefficients are in effect obtained as follows:

$$\begin{array}{l}
\quad 1,\ 1 \\
\times \text{ by } 1,\ 1 \\
\hline
\quad 1,\ 1 \\
\quad\ \ 1,\ 1 \\
\hline
1,\ 2,\ 1 \\
\times \text{ by } 1,\ 1 \\
\hline
1,\ 2,\ 1 \\
\quad 1,\ 2,\ 1 \\
\hline
1,\ 3,\ 3,\ 1 \\
\times \text{ by } 1,\ 1 \\
\&\text{c.,}
\end{array}$$

viz. by precisely the same process as is used in finding the numerical coefficients in the powers $(m+n)^1$, $(m+n)^2$, $(m+n)^3$, &c.; and we thus see that for any integer value r of the index, we have a factorial binomial theorem, wherein the numerical coefficients are the same as in the binomial theorem for the same index.

But the method of varied multiplication may be applied to the demonstration of a much more general theorem; viz. we may use it to develope a product such as

$$(h-a)(h-b)(h-c)(h-d)(h-e),$$

according to a series of products

$$\begin{aligned}
&(h-\alpha)(h-\beta)(h-\gamma)(h-\delta)(h-\epsilon),\\
&(h-\alpha)(h-\beta)(h-\gamma)(h-\delta),\\
&(h-\alpha)(h-\beta)(h-\gamma),\\
&(h-\alpha)(h-\beta),\\
&(h-\alpha),\\
&1.
\end{aligned}$$

For this purpose, starting with

$$h-a=h-\alpha+\alpha-a,$$

we multiply by $h-b$, written first under the form $h-\beta+\beta-b$, and then under the form $h-\alpha+(\alpha-b)$; we have thus

$$\begin{aligned}
(h-a)(h-b)=&(h-\alpha)(h-\beta)\\
&+(h-\alpha) \qquad \{(\alpha-a)+(\beta-b)\}\\
&+1 \qquad (\alpha-a)(\alpha-b),
\end{aligned}$$

and so on. It is easy to see that we may for instance write

$$\begin{aligned}
&(h-a)(h-b)(h-c)(h-d)(h-e)\\
&\quad=(h-\alpha)(h-\beta)(h-\gamma)(h-\delta)(h-\epsilon)\\
&\quad+(h-\alpha)(h-\beta)(h-\gamma)(h-\delta) \qquad \begin{pmatrix}\alpha, & \beta, & \gamma, & \delta, & \epsilon\\ a, & b, & c, & d, & e\end{pmatrix}_1\\
&\quad+(h-\alpha)(h-\beta)(h-\gamma) \qquad \begin{pmatrix}\alpha, & \beta, & \gamma, & \delta & \\ a, & b, & c, & d, & e\end{pmatrix}_2\\
&\quad+(h-\alpha)(h-\beta) \qquad \begin{pmatrix}\alpha, & \beta, & \gamma & & \\ a, & b, & c, & d, & e\end{pmatrix}_3\\
&\quad+h-\alpha \qquad \begin{pmatrix}\alpha, & \beta & & & \\ a, & b, & c, & d, & e\end{pmatrix}_4\\
&\quad+1 \qquad \begin{pmatrix}\alpha & & & & \\ a, & b, & c, & d, & e\end{pmatrix}_5,
\end{aligned}$$

where the symbols $(\ )_\theta$ denote sums of products of the differences of an upper letter and a lower letter, θ factors in each product; and the several sums being formed as follows:

For $\begin{pmatrix}\alpha, & \beta, & \gamma, & \delta, & \epsilon \\ a, & b, & c, & d, & e\end{pmatrix}_1$ write

α	a
β	b
γ	c
δ	d
ϵ	e;

the expression is

$$(\alpha - a) + (\beta - b) + (\gamma - c) + (\delta - d) + (\epsilon - e).$$

For $\begin{pmatrix}\alpha, & \beta, & \gamma, & \delta & \\ a, & b, & c, & d, & e\end{pmatrix}_2$ write

α, α	a, b
α, β	a, c
α, γ	a, d
β, β	b, c
α, δ	a, e
β, γ	b, d
β, δ	b, e
γ, γ	c, d
γ, δ	c, e
δ, δ	d, e;

viz. the expression is

$$(\alpha - a)(\alpha - b) + (\alpha - a)(\beta - c) + \ldots + (\delta - d)(\delta - e).$$

For $\begin{pmatrix}\alpha, & \beta, & \gamma & & \\ a, & b, & c, & d, & e\end{pmatrix}_3$ write

α, α, α	a, b, c
α, α, β	a, b, d
α, α, γ	a, b, e
α, β, β	a, c, d
α, β, γ	a, c, e
β, β, β	b, c, d
α, γ, γ	a, d, e
β, β, γ	b, c, e
β, γ, γ	b, d, e
γ, γ, γ	c, d, e;

viz. the expression is

$$(\alpha - a)(\alpha - b)(\alpha - c) + \ldots + (\gamma - c)(\gamma - d)(\gamma - e).$$

For $\begin{pmatrix} \alpha, \beta & \\ a, b, c, d, e \end{pmatrix}_4$ write

$$\begin{array}{r|l} \alpha, \alpha, \alpha, \alpha & a, b, c, d \\ \alpha, \alpha, \alpha, \beta & a, b, c, e \\ \alpha, \alpha, \beta, \beta & a, b, d, e \\ \alpha, \beta, \beta, \beta & a, c, d, e \\ \beta, \beta, \beta, \beta & b, c, d, e; \end{array}$$

viz. the expression is

$$(\alpha - a)(\alpha - b)(\alpha - c)(\alpha - d) \ldots + (\beta - b)(\beta - c)(\beta - d)(\beta - e).$$

Finally for

$$\begin{pmatrix} \alpha & \\ a, b, c, d, e \end{pmatrix}_5 \text{ write } \alpha, \alpha, \alpha, \alpha, \alpha \mid a, b, c, d, e;$$

viz. the expression is

$$(\alpha - a)(\alpha - b)(\alpha - c)(\alpha - d)(\alpha - e),$$

which explains the law of the formation of the several coefficients. It is to be observed that in forming the development of any symbol, for instance $\begin{pmatrix} \alpha, \beta, \gamma & \\ a, b, c, d, e \end{pmatrix}_3$, the first column contains the homogeneous products, 3 together, of α, β, γ; the second column the combinations (that is, combinations without repetitions) 3 together of a, b, c, d, e: the top line is $\alpha, \alpha, \alpha \mid a, b, c$ and to form the subsequent lines we must for any advance α into β, &c. of a greek letter make the like advance a into b, b into c, or c into d, of the corresponding latin letter.

Two particular cases of the theorem may be noticed: if the latin letters all vanish, we have, for example,

$$\begin{array}{ll} h^5 = (h-\alpha)(h-\beta)(h-\gamma)(h-\delta)(h-\epsilon). & \\ \quad + (h-\alpha)(h-\beta)(h-\gamma)(h-\delta) & .H_1(\alpha, \beta, \gamma, \delta, \epsilon) \\ \quad + (h-\alpha)(h-\beta)(h-\gamma) & .H_2(\alpha, \beta, \gamma, \delta) \\ \quad + (h-\alpha)(h-\beta) & .H_3(\alpha, \beta, \gamma) \\ \quad + (h-\alpha) & .H_4(\alpha, \beta) \\ \quad + 1 & .H_5(\alpha), \end{array}$$

where the symbols H denote the sum of the homogeneous products of the annexed letters, taken together according to the suffix number: the last coefficient $H_5(\alpha)$ is of course $= \alpha^5$. And if the greek letters all vanish, then we have in like manner

$$\begin{aligned} (h-a)(h-b)(h-c)(h-d)(h-e) = h^5 & \\ & - h^4 C_1(a, b, c, d, e) \\ & + h^3 C_2(a, b, c, d, e) \\ & - h^2 C_3(a, b, c, d, e) \\ & + h\, C_4(a, b, c, d, e) \\ & - \quad C_5(a, b, c, d, e), \end{aligned}$$

where the symbols C denote the combinations of the annexed letters taken together according to the suffix number; the last coefficient $C_5(a, b, c, d, e)$ is of course $= abcde$. This is the ordinary theorem giving the expression of an equation in terms of its roots.

Combining the two theorems, if in the first theorem we express the products $(h-\alpha)(h-\beta)(h-\gamma)(h-\delta)(h-\epsilon)$, &c. in powers of h by means of the second theorem; or if in the second theorem we express the powers h^5, h^4, &c. in terms of the products $(h-a)(h-b)(h-c)(h-d)(h-e)$, &c. by means of the first theorem; then in either case we obtain certain identical relations connecting the C, H of $(\alpha, \beta, \ldots)$ or of $(a, b, \ldots)$.

I have mentioned the factorial notation

$$[m]^r = m(m-1)\ldots(m-r+1),$$

where r is a positive integer; a consequence of this is

$$[m]^{r+s} = [m]^r [m-r]^s,$$

where r and s are positive integers; or as this may also be written

$$[m+r]^{r+s} = [m+r]^r [m]^s.$$

Assuming this to subsist for $s=0$, or a negative integer; first for $s=0$, we have $[m+r]^r = [m+r]^r [m]^0$; that is, $[m]^0$ is $=1$; and then for $s=-r$, we have $1 = [m+r]^r [m]^{-r}$; that is,

$$[m]^{-r} = \frac{1}{[m+r]^r},$$

and in particular, $r = 1, 2$, &c., we have

$$[m]^{-1} = \frac{1}{m+1},$$

$$[m]^{-2} = \frac{1}{(m+1)(m+2)},$$

&c.,

which explains the extension of the factorial notation to negative integer values of the index.

But the equation

$$[m]^{r+s} = [m]^r [m-r]^s$$

does not *in any determinate manner* lead to an extension of the factorial notation to fractional or other values of the index. In fact, assuming $[m]^r = \dfrac{\Pi m}{\Pi(m-r)}$, where Π is an arbitrary functional symbol, the equation in question becomes

$$\frac{\Pi m}{\Pi(m-r-s)} = \frac{\Pi m}{\Pi(m-r)} \frac{\Pi(m-r)}{\Pi(m-r-s)},$$

viz. the original equation is identically satisfied, without any condition whatever being imposed upon the function Π, and on this account we have not, in the notation of the factorial with an integer index, *any sufficient basis for a theory of general differentiation.*

A product

$$m(m-\alpha)\ldots\{m-(r-1)\alpha\}$$

can of course be expressed in the factorial notation, viz. it is

$$=\alpha^r\left[\frac{m}{\alpha}\right]^r,$$

and on this account it is not in general necessary to employ a notation such as $[m, \alpha]^r$ to denote such a factorial wherein the difference of the successive factors instead of being $=-1$ is $=-\alpha$; in particular cases where factorials of the kind in question are used, it may be convenient to employ such a notation. In particular it is sometimes convenient to use the notation $[m, -1]^r$ or better $\{m\}^r$ to denote the product

$$m(m+1)\ldots(m+r-1),$$

where the successive factors instead of being diminished, are increased by unity. It may be noticed, that reversing in this last product the order of the factors, we find

$$\{m\}^r=[m+r-1]^r;$$

a somewhat similar formula, but employing only the ordinary factorial notation, is obtained from the equation

$$[m]^r=m(m-1)\ldots(m-r+1),$$

by first changing the sign of m and then reversing the order of the factors; viz. we have

$$[-m]^r=(-)^r[m+r-1]^r.$$

Reverting to the process used for the development of the expressions $(\ )_\theta$, where there are two columns, the one of greek, the other of latin letters; it is to be remarked that although the order in which the successive lines are evolved is not material for the purpose of the theorem, yet that a certain definite order of evolution has been made use of; thus in regard to $\begin{pmatrix}\alpha, & \beta, & \gamma, & \delta \\ a, & b, & c, & d, & e\end{pmatrix}_2$, the column of greek letters, giving the homogeneous products of the second order in $(\alpha, \beta, \gamma, \delta)$, was

$$\begin{array}{cc}
\alpha & \alpha \\ \hline
\alpha & \beta \\ \hline
\alpha & \gamma \\
\beta & \beta \\ \hline
\alpha & \delta \\
\beta & \gamma \\ \hline
\beta & \delta \\
\gamma & \gamma \\ \hline
\gamma & \delta \\ \hline
\delta & \delta
\end{array}$$

this is evolved from the top term (α, α) by a process given implicitly in Arbogast's Calculus of Derivations, and which may be termed the rule *of the last and the last but one.* Let the direction "operate on any letter," be understood to mean that the letter in question is to be changed into that which immediately follows it, but in such wise that when the letter occurs more than once, e.g. as in α, α the operation affects only the letter in the right-hand place. Then operate on the α, α in regard to α, we obtain α, β; operate on this in regard to β, we obtain α, γ; and in regard to α, we obtain β, β. Again we operate on α, γ in regard to γ, and obtain α, δ; we do not operate on it in regard to α for the reason that α is not the letter immediately preceding γ. Operate on β, β in regard to β, we obtain β, γ. The next step, if the series extended to ϵ would be to operate on α, δ in regard to δ, giving α, ϵ; do not operate on it in regard to α, for the reason that α is not the letter immediately preceding δ. But in the example, since the series does not extend to ϵ, there is no operation on α, δ. Passing then to the next term β, γ, we operate in regard to γ, obtaining β, δ, and since β is the letter immediately preceding γ, we also operate in regard to β, obtaining γ, γ. Similarly if ϵ were admissible, β, δ would give β, ϵ, but it in fact gives nothing; γ, γ gives γ, δ; thus if ϵ were admissible would give γ, ϵ and δ, δ, but it in fact gives only δ, δ, and, ϵ being inadmissible, the process is here concluded. The rule is, operate on the last letter, and when the last but one letter is that which, in alphabetical order, immediately precedes the last letter (but in this case only) operate on the last but one letter.

Taking another example, but with numbers instead of letters, and supposing the highest admissible number to be 5, then from 111 we derive as follows:

111	112	113	114	115	125	135	145	155	255	355	455	555
		122	123	124	134	144	235	245	345	445		
			222	133	224	225	244	335	444			
				223	233	234	334	344				
						333,						

the original single column being here for greater convenience broken up into distinct columns; but the order of the terms, when the columns are taken one after the other in order, each being read from the top to the bottom, being the same as before; it will be noticed that the successive divisions are the partitions into 3 parts (no part exceeding 5) of the numbers 3, 4, ..., 15 respectively; the partitions being in each case obtained without repetition, and those of the same number being given, say in their numerical order (corresponding with the alphabetical order when letters are employed). It is necessary to show that the partitions will be obtained without repetitions; and that all the partitions will be obtained; for this purpose consider, for example, the partitions of 9; any one of these is either a partition 135 where the last number 5 is not a repeated number; and in this case there is a partition of 8, viz. 134, from which operating on the last we obtain 135; but there is no other partition of 8 which would give 135, the only such partition would be 125, but here, as 2 is not the number which immediately precedes 5, there is no operation on the last but one, and we do not from it obtain 135. Or else a partition of 9 is of the form 144

where the last letter is repeated; there exists in this case no partition of 8, such that operating on the last we obtain from it 144, but there does exist a partition of 8, viz. 134, such that operating on the last but one we obtain from it 144. That is, for any partition whatever of 9 there exists one (and only one) partition of 8, such that operating on the last or the last but one, we obtain from it the partition of 9; that is, taking the entire system of the partitions of 8, and operating on the last and the last but one, we obtain, and that without repetitions, the entire series of the partitions of 9; and so in general.

Translating the example into letters, but using for greater convenience a^2, &c. instead of a, a, &c. the process will be precisely the same; taking the letters to be a, b, c, d, e, we have

$$\begin{array}{ccccccccccccc}
a^2 & a^2b & a^2c & a^2d & a^2e & abe & ace & ade & ae^2 & be^2 & ce^2 & de^2 & e^3 \\
 & & ab^2 & abc & abd & acd & ad^2 & bce & bde & cde & d^2e & & \\
 & & & b^3 & ac^2 & b^2d & b^2e & bd^2 & c^2e & d^3 & & & \\
 & & & & b^2c & bc^2 & bcd & c^2d & cd^2 & & & & \\
 & & & & & & c^3. & & & & & &
\end{array}$$

Attributing *weights* to the several letters, viz. to a, b, c, d, e the weights 1, 2, 3, 4, 5 respectively, the several columns show the terms of the weights 3, 4, ... 15 respectively.

I have said that the foregoing rule is given implicitly in Arbogast's Calculus of Derivations; this calculus includes in fact a process for the expansion of a function

$$\phi\,(a + bx + cx^2 + dx^3 + \&c.)$$

in powers of x; the expansion in question may be obtained by means of Taylor's theorem, viz.

$$\begin{aligned}
&\phi\,(a + bx + cx^2 + dx^3 + \&c.) \\
&= \phi a \\
&+ \frac{\phi' a}{1}\quad (bx + cx^2 + dx^3 + \&c.) \\
&+ \frac{\phi'' a}{1.2}\quad (bx + cx^2 + dx^3 + \&c.)^2 \\
&+ \frac{\phi''' a}{1.2.3}(bx + cx^2 + dx^3 + \&c.)^3,
\end{aligned}$$

viz. expanding the several powers of the polynomial increment, and arranging in powers of x, this is

$$\begin{aligned}
&= \quad \phi a \\
&+ x\ (\phi' a\,.\,b) \\
&+ x^2\left(\phi' a\,.\,c + \phi'' a\,.\,\frac{b^2}{2}\right) \\
&+ x^3\left(\phi' a\,.\,d + \phi'' a\,.\,bc + \qquad \phi''' a\,.\,\frac{b^3}{6}\right) \\
&+ x^4\left\{\phi' a\,.\,e + \phi'' a\,.\,(bd + \tfrac{1}{2}c^2) + \phi''' a\,.\,\frac{b^2c}{2} + \phi'''' a\,.\,\frac{b^4}{24}\right\} \\
&+ \&c.
\end{aligned}$$

but the object of the rule is to obtain this last-mentioned result directly, and understanding "operate in regard to a letter" to mean differentiate with regard to this letter and integrate with respect to the next succeeding letter, then the coefficients of the successive powers of x are all obtained from the first coefficient ϕa, by operating thereon according to the rule of the last and the last but one; thus ϕa, operating on a gives $\phi' a \,.\, b$; this operating in regard to b gives $\phi' a \,.\, c$, and in regard to a gives $\phi'' a \,.\, \frac{b^2}{2}$; the term $\phi' a \,.\, c$ is to be operated upon in regard to c only, and it gives $\phi' a \,.\, d$; the other term $\phi'' a \,.\, \frac{b^2}{2}$ operated on in regard to b gives $\phi'' a \,.\, bc$, and in regard to a it gives $\phi''' a \,.\, \frac{b^3}{6}$; and so on. But attending only to the literal parts, the terms, for instance b^2, bc, bd, &c., which present themselves in the formula, are the homogeneous terms derived from $\overset{c^2}{b^2}$, by the rule, as originally stated, with a view to the derivation of such terms.

534.

A "SMITH'S PRIZE" PAPER[1]; SOLUTIONS.

[From the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol v. (1870), pp. 182—203.]

1. *Mention what form of given relation* $\phi(a, b, c, \ldots) = 0$ *between the roots of a given equation will in general serve for the rational determination of the roots; explain the case of failure; and state what information as to the roots is furnished by a given relation not of the form in question.*

In the given relation, $\phi(a, b, c, \ldots)$ must be a wholly unsymmetrical function of the roots; that is, a function altered by any permutation whatever of the roots; or, what is the same thing, by any interchange whatever of two roots.

For this being so, if $\alpha, \beta, \gamma, \ldots$ be the values of the roots, then for some one order, say $\alpha, \beta, \gamma, \ldots$, of these values the given relation $\phi(a, b, c, \ldots) = 0$ will be satisfied by writing therein $a = \alpha$, $b = \beta$, $c = \gamma$, &c.; but it will in general be satisfied for this order only, and not for any other order whatever (viz. it will not be satisfied by writing $a = \beta$, $b = \alpha$, $c = \gamma$, &c., or by any other such system). The given equation determines that the roots are equal to $\alpha, \beta, \gamma, \ldots$ in some order or other, but the given equation combined with the given relation $\phi(a, b, c, \ldots) = 0$, determines that a is $= \alpha$ and not equal to any other value, $b = \beta$ and not equal to any other value, &c.; and it thus appears *a priori*, that the two together must rationally determine each of the roots $a, b, c, \ldots$; the *a posteriori* verification, and actual rational determination of the values of $a, b, c, \ldots$ respectively, is a separate question which is not here considered.

The function $\phi(a, b, c, \ldots)$ may be of the proper form, and yet the particular values $\alpha, \beta, \gamma, \ldots$ be such that the given relation $\phi(a, b, c, \ldots) = 0$ is satisfied, not only for the single arrangement $a = \alpha$, $b = \beta$, $c = \gamma$, &c., but for some other arrangement,

[1] Set by me for the Master of Trinity, Feb. 3, 1870.

$a=\delta$, $b=\gamma$, $c=\beta, \ldots$ or for more than one such other arrangement. (For instance, if the given relation be $a+2b+3c-32=0$, and the roots are 3, 5, 7; the relation is satisfied by $a=5$, $b=3$, $c=7$, and also by $a=3$, $b=7$, $c=5$.) Here the given equation and relation do not completely determine each root, they only determine that a is $=\alpha$ or $=\delta$ (or as the case may be = some other one value); and similarly that b is $=\beta$ or $=\gamma$ (or as the case may be = some other one value), and so for the other roots $c, d, \ldots$; and it thus appears *a priori*, that in such a case each root is determined, not rationally, but by means of an equation, the order of which is equal to the number of the values of such root; we have here the case of failure of the general theorem.

When the given relation $\phi(a, b, c, \ldots)=0$ is not of the required form; that is, when $\phi(a, b, c, \ldots)$ is a partially symmetrical function, there will be in general several arrangements of $\alpha, \beta, \gamma, \ldots$, such that equating $a, b, c, \ldots$ to $\alpha, \beta, \gamma, \ldots$ according to each of these arrangements, the given relation $\phi(a, b, c, \ldots)=0$ will be satisfied; and it follows that each of the roots $a, b, c, \ldots$ is determined not rationally, but by means of an equation of a certain order (not necessarily the same order for each of the roots). Thus, if the relation be symmetrical as regards a pair of roots a and b; then if it be satisfied, suppose by $a=\alpha$, $b=\beta$, $c=\gamma, \ldots$, it will also be satisfied by $a=\beta$, $b=\alpha$, $c=\gamma, \ldots$, but not in general in any other manner; each of the roots a, b has here either of the values α, β, and the two roots a, b in question will be given, not rationally, but by means of the same quadratic equation. And observe, moreover, that any other function $\psi(a, b, c, \ldots)$ of the same form as ϕ, that is, symmetrical in regard to the two roots a, b, will for the two arrangements $a=\alpha$, $b=\beta$, $c=\gamma \ldots$, and $a=\beta$, $b=\alpha$, $c=\gamma, \ldots$ acquire not two distinct values, but one and the same value, that is, the value of $\psi(a, b, c, \ldots)$ will be determined *rationally;* and so in general.

There is for the partially symmetrical function $\phi(a, b, c, \ldots)$ a case of failure similar to that which arises for the completely unsymmetrical function, viz. the particular values $\alpha, \beta, \gamma \ldots$ may be such as to give more ways of satisfying the given relation $\phi(a, b, c, \ldots)=0$, than there would be but for such particular values of $\alpha, \beta, \gamma, \ldots$; and there is then a corresponding elevation of the order of the equation for the determination of the roots $a, b, c, \ldots$ or some of them.

2. *If the roots* $(\alpha, \beta, \gamma, \delta)$ *of the equation*

$$(a, b, c, d, e)(u, 1)^4=0$$

are no two of them equal; and if there exist unequal magnitudes θ, ϕ *such that*

$$(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4=(\phi+\alpha)^4 : (\phi+\beta)^4 : (\phi+\gamma)^4 : (\phi+\delta)^4;$$

show that the cubinvariant $ace-ad^2-b^2e-c^3+2bcd$ *is* $=0$; *and find the values of* θ, ϕ.

We have

$$\left(\frac{\theta+\alpha}{\phi+\alpha}\right)^4=\left(\frac{\theta+\beta}{\phi+\beta}\right)^4=\left(\frac{\theta+\gamma}{\phi+\gamma}\right)^4=\left(\frac{\theta+\delta}{\phi+\delta}\right)^4;$$

and we cannot have any two of the fourth roots, say $\dfrac{\theta+\alpha}{\phi+\alpha}$ and $\dfrac{\theta+\beta}{\phi+\beta}$ equal to each other; for this would imply $(\theta-\phi)(\alpha-\beta)=0$, that is, $\theta=\phi$, or else $\alpha=\beta$.

Hence assuming $\frac{\theta+\alpha}{\phi+\alpha}=\lambda$, we may write

$$\frac{\theta+\alpha}{\phi+\alpha}=\lambda,\quad \frac{\theta+\beta}{\phi+\beta}=-\lambda,\quad \frac{\theta+\gamma}{\phi+\gamma}=i\lambda,\quad \frac{\theta+\delta}{\phi+\delta}=-i\lambda,$$

$$\{i=\sqrt{(-1)} \text{ as usual}\},$$

viz. this is one of three systems of equations; the other two may be obtained therefrom by writing γ, δ, β and δ, β, γ successively in place of β, γ, δ. Hence assuming

$$v=\frac{\theta+u}{\phi+u},$$

the four values of u are α, β, γ, δ, and the corresponding four values of v are λ, $-\lambda$, $i\lambda$, $-i\lambda$; and v, u are linearly related to each other; the anharmonic ratio of $(\alpha, \beta, \gamma, \delta)$ is therefore equal to that of $(1, -1, i, -i)$, viz. we have

$$\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\delta)(\beta-\gamma)}=\frac{(1-i)(-1+i)}{(1+i)(-1-i)},\quad =\frac{(1-i)^2}{(1+i)^2},\quad =-1,$$

that is,

$$(\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)=0,$$

or, what is the same thing,

$$2(\alpha\beta+\gamma\delta)-(\alpha+\beta)(\gamma+\delta)\ =0,$$

viz. we have this relation, or else one of the like relations

$$2(\alpha\gamma+\delta\beta)-(\alpha+\gamma)(\delta+\beta)=0,$$
$$2(\alpha\delta+\beta\gamma)-(\alpha+\delta)(\beta+\gamma)=0,$$

that is, the product of the three functions $2(\alpha\beta+\gamma\delta)-(\alpha+\beta)(\gamma+\delta)$

$$\text{is } =0.$$

But the product in question is (save as to a numerical factor) the cubinvariant J of the quartic function; or the equation in question is the required equation $J=0$. More simply, the linear transformation $v=\frac{\theta+u}{\phi+u}$, gives for v the equation $v^4-\lambda^4=0$; which is $(1, 0, 0, 0, -\lambda^4 \between v, 1)^4=0$; the cubinvariant hereof is $=0$, and therefore also the cubinvariant of the original function $(a, b, c, d, e \between u, 1)^4$.

Reverting to the equations

$$\frac{\theta+\alpha}{\phi+\alpha}=\lambda,\quad \frac{\theta+\beta}{\phi+\beta}=-\lambda,\quad \frac{\theta+\gamma}{\phi+\gamma}=i\lambda,\quad \frac{\theta+\delta}{\phi+\delta}=-i\lambda,$$

(which, as we have seen, give $2(\alpha\beta+\gamma\delta)=(\alpha+\beta)(\gamma+\delta)$), the same equations give

$$\frac{\theta+\alpha}{\phi+\alpha}+\frac{\theta+\beta}{\phi+\beta}=0;\quad \frac{\theta+\gamma}{\phi+\gamma}+\frac{\theta+\delta}{\phi+\delta}=0,$$

that is,

$$2\theta\phi + 2\alpha\beta - (\theta+\phi)(\alpha+\beta) = 0,$$
$$2\theta\phi + 2\gamma\delta - (\theta+\phi)(\gamma+\delta) = 0,$$

or, what is the same thing,

$$\begin{aligned} 2\theta\phi : 2 : \theta+\phi &= -\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) \\ : \quad & \gamma+\delta \quad - \quad \alpha-\beta \\ : \quad & \gamma\delta \quad - \quad \alpha\beta, \end{aligned}$$

viz. we have thus the values of $\theta\phi$, $\theta+\phi$ (and thence of θ, ϕ) corresponding to the relation $2(\alpha\beta+\gamma\delta) = (\alpha+\beta)(\gamma+\delta)$ of the roots. And by cyclically permuting β, γ, δ as before, we have the values of $\theta\phi$, $\theta+\phi$ corresponding to the other two forms respectively of the relation between the roots.

3. *If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation*

$$\alpha . PAB + \beta . PBC + \gamma . PCD + \delta . PDA = 0$$

which connects the areas of the triangles PAB, &c.

Taking $(x, y, 1)$, $(x_1, y_1, 1)$, &c. for the coordinates of P, A, B, C, D respectively, we have

$$PAB = \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & x_2, & 1 \end{vmatrix}, = 012, \text{ suppose,}$$

$$PBC = 023, \text{ \&c.}$$

(where the values of the several determinants fix the signs of the several triangles). The identical equation then is

$$\alpha . 012 + \beta . 023 + \gamma . 034 + \delta . 041 = 0;$$

(that such an equation exists appears at once by the consideration that α, β, γ, δ can be determined so that the coefficients of x, y, and the constant term shall severally vanish); and in order actually to find the values we may make P coincide with the points A, B, C, D successively. We thus have

$$\beta . 123 + \gamma . 134 = 0,$$
$$\gamma . 234 + \delta . 241 = 0,$$
$$\delta . 341 + \alpha . 312 = 0,$$
$$\alpha . 412 + \beta . 423 = 0,$$

or, what is the same thing,

$$\beta . 123 + \gamma . 341 = 0,$$
$$\gamma . 234 + \delta . 412 = 0,$$
$$\delta . 341 + \alpha . 123 = 0,$$
$$\alpha . 412 + \beta . 234 = 0,$$

and these are at once seen to give

$$\alpha : \beta : \gamma : \delta = 234.341 : -341.412 : 412.123 : -123.341,$$

so that the required identical relation is

$$012.234.341 - 023.341.412 + 034.412.123 - 041.123.341 = 0,$$

in which 012, 023, 034, 041 stand for the triangles *PAB*, *PBC*, *PCD*, *PDA*, and 234, 341, 412, 123 for the triangles *BCD*, *CDA*, *DAB*, *ABC* respectively.

4. *Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point.*

Examine whether a like question applies to a point on a surface and the indicatrix section at such point.

Taking the origin at the point on the curve, the axis of x coinciding with the tangent and that of y with the normal; the equation of the curve taken to terms of the third order in x will by

$$y = bx^2 + cx^3,$$

and if, considering x as a small quantity of the first order, and therefore y as a small quantity of the second order, we regard y as given, and find the two values x_1, x_2, each of the order $\sqrt{(y)}$, which satisfy the equation, then, as will appear, $x_1 + x_2$ is a small quantity of the order x^2, and consequently $\dfrac{x_1 + x_2}{y}$ will have a finite value. And if ϕ be the required angle, then obviously $\tan\phi = \dfrac{\frac{1}{2}(x_1 + x_2)}{y}$.

We have as a first approximation $bx^2 = y$, or say $x = \dfrac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}}$, whence to a second approximation $bx^2 = y - \dfrac{cy^{\frac{3}{2}}}{b^{\frac{3}{2}}}$, $x^2 = \dfrac{y}{b}\left(1 - \dfrac{cy^{\frac{1}{2}}}{b^{\frac{3}{2}}}\right)$, whence $x = \dfrac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}}\left(1 - \dfrac{cy^{\frac{1}{2}}}{2b^{\frac{3}{2}}}\right)$, $= \dfrac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \dfrac{cy}{2b^2}$; say we have

$$x_1 = \frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \frac{cy}{2b^2},$$

$$x_2 = -\frac{y^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \frac{cy}{2b^2},$$

and thence

$$\tfrac{1}{2}(x_1 + x_2) = -\frac{cy}{2b^2};$$

whence

$$\tan\phi = -\frac{c}{2b^2},$$

which gives the value of the angle ϕ; it would be easy to express b, c in terms of the differential coefficients

$$d_x y,\ d_x^2 y,\ d_x^3 y.$$

It would at first sight appear that a like question might be asked as to a surface; viz. that it might be proposed to determine the angle between the normal and a line drawn from the point to the centre of the indicatrix conic. But this is not so; in fact, taking the origin at a point on the surface, the axes of x, y being in the tangent plane, and the axis of z coinciding with the normal: then to the third order we have

$$z = (A,\ B,\ C \between x,\ y)^2 + (a,\ b,\ c,\ d \between x,\ y)^3;$$

but here, regarding z as a given constant, if we take account of the terms of the third order, the section is not a conic but a cubic; and it has not in general any centre; and if (as in the ordinary theory) we neglect the terms of the third order, thus obtaining an indicatrix conic, the centre of this conic lies *on* the normal, and there is no angle corresponding to the angle ϕ of the plane problem.

The only case where there is such an angle is when the cubic terms $(a,\ b,\ c,\ d \between x,\ y)^3$ contain as a factor the quadric terms $(A,\ B,\ C \between x,\ y)^2$ (one relation between the coefficients A, B, C, a, b, c, d). For then we have

$$z = (A,\ B,\ C \between x,\ y)^2 (1 + 2lx + 2my), \text{ viz.}$$

$$z = (A,\ B,\ C \between x,\ y)^2 + 2\,(lxz + myz),$$

approximately to the third order; and then regarding z as a given constant, this last equation represents a conic having for the coordinates of its centre, say $x = \alpha z$, $y = \beta z$, and there is an angle $\phi = \tan^{-1} \sqrt{(\alpha^2 + \beta^2)}$; this is, in fact, what happens in the case of a quadric surface, for the section by a plane parallel and indefinitely near to the tangent plane is then a conic, the centre of which is not on the normal; and the angle ϕ (in the case of a central surface) is in fact the inclination of the normal to the radius from the centre.

I take the opportunity of adding a remark that the indicatrix is never a parabola, but in the separating case between the ellipse and the hyperbola it is a pair of parallel lines. The indicatrix, a parabola, is commonly obtained as follows: viz. taking the axes as before, but starting from an equation $U = 0$, the equation presents itself in the form

$$z = (A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2,$$

which, considering z as a given constant, represents a conic which, it is said, *may be a parabola*. But observe that z is of the order $(x,\ y)^2$, the terms $2Fyz + Gzx$, are consequently of the order $(x,\ y)^3$, but they are not all the terms of this order which would be obtained by the expansion of z as a function of $(x,\ y)$; there is consequently no meaning in retaining them, and they ought to be rejected; similarly the term in z^2 which is of the order $(x,\ y)^4$ ought to be rejected; the equation is thus reduced to

$$z = Ax^2 + 2Hxy + By^2,$$

which, when $AB - H^2 = 0$, represents not a parabola but a pair of parallel lines. On referring to Dupin's *Développements de Géométrie, &c.* (see p. 49) I find that he is quite accurate; his expression is, "elle peut cependant être une parabole; alors elle se présente sous la forme de deux droites parallèles *équidistantes de leur centre*": and he afterwards examines in particular "ce cas remarquable."

5. *Shew that a cubic surface has at most four conical points; and a quartic surface at most sixteen conical points.*

If a cubic surface has two conical points, then the line joining these has with the surface two intersections at each of the conical points, and therefore lies wholly in the surface. Hence, for a cubic surface with three conical points A, B, C, the lines AB, BC, CA lie wholly in the surface, and these three lines form the complete section of the surface by the plane ABC: it is clear that there cannot be in this plane a fourth conical point: but there may be, not in this plane, a fourth conical point D. Suppose that this is so, there cannot be a fifth conical point E; for if there were, the line DE would lie wholly in the surface, and would therefore meet the plane ABC at some point in the section of the surface by this plane; that is, at some point in one of the lines AB, AC, BC; say at a point in AB: but then the lines AB, DE would intersect, or the four conical points A, B, D, E would lie in a plane. Hence there cannot be any fifth conical point E.

For a quartic surface; suppose this has k conical points, and let any one of these be made the vertex of a cone circumscribing the surface; each generating line is a tangent of the surface; and considering any section by a plane through the vertex, and observing that from a double point of a quartic curve we may draw six tangents to the curve, it appears that the order of the cone is $=6$. It is easy to see that the lines from the vertex to the remaining $(k-1)$ conical points are each of them a double line of the cone, and that the cone has not any other double lines; the cone is therefore a cone of the order 6, with $(k-1)$ double lines. A proper cone of the order 6 has at most 10 double lines, but the cone need not be a proper one; it may, in fact, break up into 6 planes, and in this case the double lines are the 15 lines of intersections of the several pairs of planes. Hence $k-1$ is $=15$ at most: or k is $=16$ at most.

6. *Find the differential equation of the parallel surfaces of an ellipsoid.*

Let (x, y, z) be the coordinates of a point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; (X, Y, Z) the coordinates of a point on the normal at a distance $=k$ from the first-mentioned point. We have

$$\frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}}=\frac{Z-z}{\frac{z}{c^2}}, =\rho \text{ suppose};$$

that is,

$$X=x\left(1+\frac{\rho}{a^2}\right),\quad Y=y\left(1+\frac{\rho}{b^2}\right),\quad Z=z\left(1+\frac{\rho}{c^2}\right),$$

and thence

$$k^2=\rho^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right).$$

Moreover

$$x=\frac{a^2X}{a^2+\rho},\quad y=\frac{b^2Y}{b^2+\rho},\quad z=\frac{c^2Z}{c^2+\rho},$$

substituting these values in the equation of the ellipsoid, we have

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2},$$

which determines ρ as a function of X, Y, Z. The tangent plane of the ellipsoid at the point (x, y, z) and of the parallel surface at the point (X, Y, Z), are parallel to each other (or what is the same thing, the parallel surface cuts at right angles the normal of the ellipsoid), we have therefore

$$\frac{x}{a^2}dX + \frac{y}{b^2}dY + \frac{z}{c^2}dZ = 0,$$

or substituting for x, y, z their values, this is

$$\frac{XdX}{a^2+\rho} + \frac{YdY}{b^2+\rho} + \frac{ZdZ}{c^2+\rho} = 0,$$

where ρ denotes as above a function of (X, Y, Z) given by the equation

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2}.$$

We have thus the differential equation of the parallel surfaces. It may be remarked, that the integral equation (involving k as the constant of integration), is found by the elimination of x, y, z, ρ from the foregoing equations

$$x = \frac{a^2X}{a^2+\rho}, \quad y = \frac{b^2Y}{b^2+\rho}, \quad z = \frac{c^2Z}{c^2+\rho},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad k^2 = \rho^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right),$$

or, what is the same thing, by the elimination of ρ from the equations

$$\frac{k^2}{\rho^2} = \frac{X^2}{(a^2+\rho)^2} + \frac{Y^2}{(b^2+\rho)^2} + \frac{Z^2}{(c^2+\rho)^2},$$

$$1 = \frac{a^2X^2}{(a^2+\rho)^2} + \frac{b^2Y^2}{(b^2+\rho)^2} + \frac{c^2Z^2}{(c^2+\rho)^2};$$

these may be replaced by

$$\frac{X^2}{a^2+\rho} + \frac{Y^2}{b^2+\rho} + \frac{Z^2}{c^2+\rho} - \frac{k^2}{\rho} - 1 = 0,$$

$$\frac{X^2}{(a^2+\rho)^2} + \frac{Y^2}{(b^2+\rho)^2} + \frac{Z^2}{(c^2+\rho)^2} - \frac{k^2}{\rho^2} = 0,$$

or, since here the second equation is the derived equation of the first in regard to the parameter ρ, the parallel surface is the envelope of the quadric surface

$$\frac{X^2}{a^2+\rho} + \frac{Y^2}{b^2+\rho} + \frac{Z^2}{c^2+\rho} - \frac{k^2}{\rho} = 0,$$

where ρ is the variable parameter. Or analytically, we find the equation by equating to zero the discriminant in regard to ρ, of the quartic function

$$\rho(a^2+\rho)(b^2+\rho)(c^2+\rho)\left(1 + \frac{k^2}{\rho} - \frac{X^2}{a^2+\rho} - \frac{Y^2}{b^2+\rho} - \frac{Z^2}{c^2+\rho}\right).$$

7. *Explain wherein consists the peculiarity of the following problem, and solve it by geometrical considerations:—*

Determine the least circle inclosing three given points.

The peculiarity of the problem is that the variable parameters upon which the circle depends, (say α, β the coordinates of the centre and k the radius), are not subject to any equations, but only to the inequalities

$$k^2 > (\alpha - \alpha_1)^2 + (\beta - \beta_1)^2,$$
$$k^2 > (\alpha - \alpha_2)^2 + (\beta - \beta_2)^2,$$
$$k^2 > (\alpha - \alpha_3)^2 + (\beta - \beta_3)^2,$$

(α_1, β_1; α_2, β_2; α_3, β_3, the coordinates of the given points, and the sign $>$ including $=$). The problem therefore cannot be solved by the ordinary analytical method, but it is easily solved geometrically as follows: Let A, B, C be the three points; consider all the circles inclosing the three points, viz. O a circle not passing through any of them; A a circle through the point A, B a circle through the point B, AB a circle through the points A and B, &c. Then for any circle O, if the centre be fixed and the radius gradually diminish, the circle will at last pass through one of the points ABC; that is, every circle O is greater than some circle A, B, or C; and the circle O is therefore not a minimum. Taking next a circle A, we may imagine the centre to move from its original position in a straight line towards the point A, the circle thus gradually diminishing until it passes through one of the points B or C; that is, every circle A is greater than some circle AB or AC, and therefore no circle A is a minimum; and in like manner no circle B or C is a minimum. There remain the circles AB, AC, BC; if the triangle ABC is acute-angled, then in each series, the least circle is the circle ABC circumscribed about the triangle; and this is then the minimum circle inclosing the three points. But if the triangle is obtuse-angled, say at C, then the least circle CA or CB is the circle ABC circumscribed about the triangle; but this is not the least circle AB, viz. the circle AB, being diminished to ABC, may be further diminished until it becomes the circle on the diameter AB; but below this it cannot be diminished; and consequently the minimum circle inclosing the three points is in this case the circle on the diameter AB.

8. *A particle describes an ellipse under the simultaneous action of given central forces, each varying as (distance)$^{-2}$, at the two foci respectively: find the differential relation between the time and the eccentric anomaly.*

Taking the equation of the ellipse to be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the absolute forces at the two foci $(ae, 0)$, $(-ae, 0)$ to be μ, μ' respectively, the differential equations of motion will be

$$\frac{d^2x}{dt^2} = -\mu\frac{x-ae}{(a-ex)^3} - \mu'\frac{(x+ae)}{(a+ex)^3},$$

$$\frac{d^2y}{dt^2} = -\mu\frac{y}{(a-ex)^3} - \mu'\frac{y}{(a+ex)^3}.$$

But if u be the eccentric anomaly, then

$$x = a\cos u, \quad y = b\sin u, \quad = a\sqrt{(1-e^2)}\sin u,$$

and the equations become

$$-\sin u\frac{d^2u}{dt^2} - \cos u\left(\frac{du}{dt}\right)^2 = -\frac{\mu}{a^3}\frac{\cos u - e}{(1-e\cos u)^3} - \frac{\mu'}{a^3}\frac{\cos u + e}{(1+e\cos u)^3}$$

$$+\cos u\frac{d^2u}{dt^2} - \sin u\left(\frac{du}{dt}\right)^2 = -\frac{\mu}{a^3}\frac{\sin u}{(1-e\cos u)^3} - \frac{\mu'}{a^3}\frac{\sin u}{(1+e\cos u)^3},$$

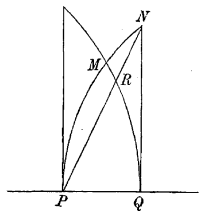
and multiplying by $-\cos u$, $-\sin u$ respectively, and adding, we have

$$\left(\frac{du}{dt}\right)^2 = \frac{\mu}{a^3}\frac{1}{(1-e\cos u)^2} + \frac{\mu'}{a^3}\frac{1}{(1+e\cos u)^2},$$

which is the required differential relation.

9. *Show that the attraction of an indefinitely thin double-convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.*

The figure represents the upper half only of the lens, but in speaking of any portion thereof, such as PRQ, we include the symmetrically situate portion of the under-half of the lens.



Let α, $=PQ$, be the thickness of the lens, $\angle NPQ = \lambda$, which angle is ultimately $=\frac{\pi}{2}$. Then it is at once seen that the attraction of the cone NPQ is $=2\pi\alpha(1-\cos\lambda)$: and from this it follows that the attraction of the infinite plate is $=2\pi\alpha$. The attraction of the whole infinite plate except the cone NPQ is $=2\pi\alpha\cos\lambda$, which is indefinitely small in regard to $2\pi\alpha$; and, *a fortiori*, the attraction of the portion MPR of the lens is indefinitely small in regard to $2\pi\alpha$. We have then only to show that the attraction of the solid NRQ is indefinitely small in regard to $2\pi\alpha$; for, this being so, the attraction of the lens may be taken to be equal to that of the cone NPQ, and will therefore ultimately be $=2\pi\alpha$, the attraction of the infinite plate.

Let the position of an element of the solid in question be determined by r its distance from P, θ the inclination of r to the axis PQ, and ϕ the azimuth in regard to any fixed plane through the axis; then $dm = r^2 \sin\theta\, dr\, d\theta\, d\phi$, and the attraction in the direction PQ is $= \int \sin\theta \cos\theta\, dr\, d\theta\, d\phi$, $= 2\pi \int \left(\frac{\alpha}{\cos\theta} - r\right) \sin\theta \cos\theta\, d\theta$, the integral in regard to ϕ having been taken from $\phi = 0$ to $\phi = 2\pi$, and that in regard to r from $r = r$ (value at the face MQ of the lens) to $r = \frac{\alpha}{\cos\theta}$ (value at the tangent plane QN). Taking the radius of the surface QM of the lens to be $= 1$, we have

$$(1 - \alpha + r\cos\theta)^2 + r^2 \sin^2\theta = 1,$$

that is,

$$r^2 + 2r\cos\theta\,(1-\alpha) = 2\alpha - \alpha^2,$$

$$\{r + (1-\alpha)\cos\theta\}^2 = (1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2,$$

or

$$r = -(1-\alpha)\cos\theta + \sqrt{\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}},$$

which is the value of r to be substituted in the formula

$$\frac{1}{2\pi} A = \int (\alpha \sin\theta - r\sin\theta\cos\theta)\, d\theta,$$

and the integral is to be taken from $\theta = 0$ to $\theta = \lambda$; viz. this is

$$\int [\alpha\sin\theta + (1-\alpha)\sin\theta\cos^2\theta - \sin\theta\cos\theta\sqrt{\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}}]\, d\theta,$$

$$= -\alpha\cos\theta - \tfrac{1}{3}(1-\alpha)\cos^3\theta + \frac{1}{3(1-\alpha)^2}\{(1-\alpha)^2\cos^2\theta + 2\alpha - \alpha^2\}^{\frac{3}{2}};$$

so that taking this between the limits in question, we have

$$\frac{1}{2\pi} A = \alpha(1-\cos\lambda) + \tfrac{1}{3}(1-\alpha)(1-\cos^3\lambda) + \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\cos^2\lambda + 2\alpha - \alpha^2\}^{\frac{3}{2}} - 1]$$

or writing for greater convenience $\lambda = \frac{\pi}{2} - \mu$, $(\mu = \angle PNQ)$, this is

$$\frac{1}{2\pi} A = \alpha(1-\sin\mu) + \tfrac{1}{3}(1-\alpha)(1-\sin^3\mu) + \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - 1]$$

$$= \alpha + \tfrac{1}{3}\left\{1 - \alpha - \frac{1}{(1-\alpha)^2}\right\} - \alpha\sin\mu$$

$$+ \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - (1-\alpha)^3\sin^3\mu]$$

$$= \frac{1}{3(1-\alpha)^2}(-3\alpha^2 + 2\alpha^3) - \alpha\sin\mu$$

$$+ \frac{1}{3(1-\alpha)^2}[\{(1-\alpha)^2\sin^2\mu + 2\alpha - \alpha^2\}^{\frac{3}{2}} - (1-\alpha)^3\sin^3\mu];$$

$\sin\mu$ is here an indefinitely small quantity of the order $\alpha^{\frac{1}{2}}$, all the terms are therefore at least of the order $\alpha^{\frac{3}{2}}$, and are to be neglected in comparison with α; or neglecting such terms we have $A=0$ (that is, the attraction of the solid NRQ is indefinitely small in regard to α); and the theorem is thus proved.

10. *Indicate in what manner the Lagrangian equations of motion*

$$\frac{d}{dt}\frac{dT}{d\xi'}-\frac{dT}{d\xi}=\frac{dV}{d\xi}, \text{ \&c.}$$

lead to the equations

$$A\frac{dp}{dt}+(C-B)qr=0, \text{ \&c.}$$

for the motion of a solid body about a fixed point.

The expression of the *vis viva* function T is

$$T=\tfrac{1}{2}(Ap^2+Bq^2+Cr^2),$$

but this expression will not by itself lead to the equations of motion; we require to know also the expressions of p, q, r in terms of certain coordinates λ, μ, ν, which determine the position of the body in regard to axes fixed in space, and of the differential coefficients λ', μ', ν' of these coordinates in regard to the time; each of the quantities p, q, r will be a linear function of λ', μ', ν' ($p=a\lambda'+b\mu'+c\nu'$, &c.), containing in any manner whatever the coordinates λ, μ, ν. This being so, the equations of motion will be

$$\frac{d}{dt}\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=0, \text{ \&c.}; \quad \frac{dT}{d\lambda'}=Ap\frac{dp}{d\lambda'}+Bq\frac{dq}{d\lambda'}+Cr\frac{dr}{d\lambda'},$$

where $\frac{dp}{d\lambda'}$, $\frac{dq}{d\lambda'}$, $\frac{dr}{d\lambda'}$ are each independent of λ', μ', ν'; hence, in the equation, the only terms containing the differential coefficients of p, q, r, are the terms

$$\frac{dp}{d\lambda'}.A\frac{dp}{dt}+\frac{dq}{d\lambda'}.B\frac{dq}{dt}+\frac{dr}{d\lambda'}.C\frac{dr}{dt}$$

of $\frac{d}{dt}.\frac{dT}{d\lambda'}$; and hence, assuming that the equations of motion are the known equations $A\frac{dp}{dt}+(C-B)qr=0$, it appears that the equation $\frac{d}{dt}.\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=0$ will assume the form

$$\frac{dp}{d\lambda'}\left\{A\frac{dp}{dt}+(C-B)qr\right\}+\frac{dq}{d\lambda'}\left\{B\frac{dq}{dt}+(A-C)rp\right\}+\frac{dr}{d\lambda'}\left\{C\frac{dr}{dt}+(B-A)pq\right\}=0;$$

there are of course two other equations only differing from this in that in place of λ', they contain μ' and ν' respectively; and since p, q, r regarded as functions of λ', μ', ν' are independent functions, the determinant formed with the differential coefficients

$\frac{dp}{d\lambda'}$, $\frac{dq}{d\lambda'}$, $\frac{dr}{d\lambda'}$, &c. is not $=0$; and the three equations are therefore equivalent (as they should be) to the equations

$$A\frac{dp}{dt}+(C-B)\,qr=0,\ \&c.$$

What precedes is a complete answer to the question, but in regard to the actual expressions of p, q, r, it may be remarked, that these quantities may be expressed very symmetrically in terms of the quantities

$$\lambda,\ \mu,\ \nu=\tan\tfrac{1}{2}\theta\cos f,\quad \tan\tfrac{1}{2}\theta\cos g,\quad \tan\tfrac{1}{2}\theta\cos h,$$

which determine the positions of the principal axes in regard to the axes fixed in space, by means of the angles of position ($\cos f$, $\cos g$, $\cos h$) of the resultant axis, and the rotation θ about this axis; viz. writing $\kappa=1+\lambda^2+\mu^2+\nu^2$, we then have

$$\begin{aligned}\kappa p&=2(\quad\lambda'+\nu\mu'-\mu\nu'),\\ \kappa q&=2(-\nu\lambda'+\ \mu'+\lambda\nu'),\\ \kappa r&=2(\ \mu\lambda'-\lambda\mu'+\ \nu'),\end{aligned}$$

and the above result may be verified *a posteriori* without any difficulty. See *Camb. Math. Jour.*, vol. III. (1843), [6], p. 224, [*Coll. Math. Papers*, vol. I. p. 33].

11. *Find in the Hamiltonian form,*

$$\frac{d\eta}{dt}=\frac{dH}{d\varpi},\quad \frac{d\varpi}{dt}=-\frac{dH}{d\eta},\ \&c.$$

the equations for the motion of a particle acted on by a central force.

Taking as coordinates r the radius vector, v the longitude, y the latitude, the equation of the *vis viva* function is

$$T=\tfrac{1}{2}\{r'^2+r^2(\cos^2 y\,.\,v'^2+y'^2)\},$$

hence

$$\begin{aligned}\frac{dT}{dr'}&=\quad r'\quad &&=\mathrm{r}\text{ suppose},\\ \frac{dT}{dv'}&=r^2\cos^2 y\,.\,v'=\mathrm{v} &&\ \ ,,\quad ,\\ \frac{dT}{dy'}&=\quad r^2y'\quad &&=\mathrm{y}\quad ,,\quad ,\end{aligned}$$

and the expression of T in terms of r, v, y, and of the new coordinates r, v, y is

$$T=\tfrac{1}{2}\left(\mathrm{r}^2+\frac{\mathrm{v}^2}{r^2\cos^2 y}+\frac{\mathrm{y}^2}{r^2}\right);$$

whence writing

$$H=\tfrac{1}{2}\left(\mathrm{r}^2+\frac{\mathrm{v}^2}{r^2\cos^2 y}+\frac{\mathrm{y}^2}{r^2}\right)-V,$$

the equations are

$$\frac{dH}{d\mathrm{r}} = \frac{dr}{dt}, \quad \frac{dH}{d\mathrm{v}} = \frac{dv}{dt}, \quad \frac{dH}{d\mathrm{y}} = \frac{dy}{dt},$$

$$\frac{dH}{dr} = -\frac{d\mathrm{r}}{dt}, \quad \frac{dH}{dv} = -\frac{d\mathrm{v}}{dt}, \quad \frac{dH}{dy} = -\frac{d\mathrm{y}}{dt}.$$

We have

$$\frac{dH}{d\mathrm{r}} = \mathrm{r}, \quad \frac{dH}{d\mathrm{v}} = \frac{\mathrm{v}}{r^2\cos^2 y}, \quad \frac{dH}{d\mathrm{y}} = \frac{\mathrm{y}}{r^2},$$

$$\frac{dH}{dr} = -\frac{1}{r^3}\left(\frac{\mathrm{v}^2}{\cos^2 y} + \mathrm{y}^2\right) - \frac{dV}{dr}, \quad \frac{dH}{dv} = 0, \quad \frac{dH}{dy} = -\frac{\mathrm{v}^2 \sin y}{r^2\cos^3 y};$$

and, substituting these values, the equations of motion present themselves as six equations of the first order between r, v, y, r, v, y, and t in the form

$$dt = \frac{dr}{\mathrm{r}} = \frac{dv}{\dfrac{\mathrm{v}}{r^2\cos^2 y}} = \frac{dy}{\dfrac{\mathrm{y}}{r^2}} = \frac{dr}{-\dfrac{1}{r^3}\left(\dfrac{\mathrm{v}^2}{\cos^2 y} + \mathrm{y}^2\right) - \dfrac{dV}{dr}} = \frac{d\mathrm{v}}{0} = \frac{d\mathrm{y}}{-\dfrac{\mathrm{v}^2\sin^2 y}{r^2\cos^3 y}}.$$

12. *An unclosed polygon of* $(m+1)$ *vertices is constructed as follows: viz. the abscissæ of the several vertices are* 0, 1, 2 ... m, *and, corresponding to the abscissa* k, *the ordinate is equal to the chance of* $(m+k)$ *heads in* $2m$ *tosses of a coin; and* m *then continually increases up to any very large value: what information in regard to the successive polygons, and to the areas of any portions thereof, is afforded by the general results of the Theory of Probabilities?*

It is somewhat more convenient to take account also of the abscissæ $-1, -2, \ldots, -m$, thereby obtaining a polygon of $2m+1$ vertices, symmetrical in regard to the axis of y. In such a polygon, the sum of the $2m+1$ ordinates is $=1$; the central ordinate is the largest, and the ordinates continually diminish as k increases: moreover for any large value of m the area of the whole polygon is very nearly, and may be regarded as being, $=1$; and the area between the ordinates corresponding to the abscissæ $+k$, $-k$ as being equal to the probability of a number of heads between $m+k$, $m-k$, in the $2m$ tosses of the coin. A general result of the Theory of Probabilities is that in a great number of trials the several events tend to happen in the proportion of their respective probabilities; viz. in the case of the $2m$ tosses there is a tendency to an equal number of heads and tails. But observe that this does not mean that the probability of m heads and m tails increases with the number $2m$ of the trials; or even that, α being any given number, the probability of a number of heads between $m+\alpha$ and $m-\alpha$ increases with the number $2m$ of trials; on the contrary, it diminishes; what it does mean is that taking the limit of deviation to vary with m, say a number of heads between $m+\alpha m$, $m-\alpha m$, the probability of such a number increases with m; viz. that taking α a fraction however small, m can be taken so large that the probability of a number of heads between $m+\alpha m$, $m-\alpha m$ in the $2m$ trials, shall be as nearly as we please $=1$.

The conclusion in regard to the areas of the polygons is that, taking k any given value whatever, however large, the ratio (m being of course $>k$) which the area between the ordinates to the abscissæ $m+k$, $m-k$ bears to the area of the whole polygon (or to unity) continually decreases as $2m$ increases, and ultimately vanishes; but contrarywise, taking α any given fraction whatever, however small, the ratio which the area between the ordinates to the abscissæ $m+\alpha m$, $m-\alpha m$ bears to the area of the whole polygon (or to unity) continually increases as $2m$ increases, and ultimately becomes $=1$.

13. *Show that for the quadric cones which pass through six given points the locus of the vertices is a quartic surface having upon it twenty-five right lines; and, thence or otherwise, that for the quadric cones passing through seven given points the locus of the vertices is a sextic curve.*

Suppose $U=0$, $V=0$, $W=0$, $S=0$ are any particular four quadric surfaces passing through the six points, say

$$\left(U=(a,\ldots)(x,\ y,\ z,\ w)^2,\quad V=(b,\ldots)(x,\ y,\ z,\ w)^2,\ \&c.\right);$$

then the equation of the general quadric surface through the six points will be

$$\alpha U+\beta V+\gamma W+\delta S=0,$$

and this surface will be a cone, having $(x,\ y,\ z,\ w)$ for the coordinates of its vertex, if only we have simultaneously

$$\alpha\frac{dU}{dx}+\beta\frac{dV}{dx}+\gamma\frac{dW}{dx}+\delta\frac{dS}{dx}=0,$$

$$\alpha\frac{dU}{dy}+\&c.\qquad\qquad=0,$$

$$\alpha\frac{dU}{dz}+\&c.\qquad\qquad=0,$$

$$\alpha\frac{dU}{dw}+\&c.\qquad\qquad=0.$$

Eliminating $(\alpha,\ \beta,\ \gamma,\ \delta)$ we have an equation $\nabla=0$, where ∇ is the Jacobian or functional determinant $\dfrac{d(U,\ V,\ W,\ S)}{d(x,\ y,\ z,\ w)}$ formed with the differential coefficients of the four functions $(U,\ V,\ W,\ S)$: the locus of the vertex is thus a quartic surface.

Calling the six points 1, 2, 3, 4, 5, 6, then taking as vertex any point in the line 12, the lines from such point to the points 1 and 2 coincide with the line 12, and we can through this line and the lines to the remaining points 3, 4, 5, 6 describe a quadric cone; the quartic surface therefore passes through the line 12; and similarly it passes through each of the fifteen lines 12, 13, ..., 56.

Again, taking the vertex anywhere in the line of intersection of the planes 123 and 456, we have an improper quadric cone, viz. the plane-pair formed by these two

planes; the line in question is therefore a line of the quartic surface; and similarly the quartic surface contains each of the ten lines 123.456, 124.356, ..., 156.234. We have thus in all 25 lines on the quartic surface.

In the case of seven points 1, 2, 3, 4, 5, 6, 7, the locus is the curve of intersection of the quartic surfaces which correspond to the points 1, 2, 3, 4, 5, 6 and the points 1, 2, 3, 4, 5, 7 respectively: these have in common the ten lines 12, 13, 14, 15, 23, 24, 25, 34, 35, 45 (which it is easy to see do not form part of the required locus), and they have therefore, as a residual intersection, a curve of the order $16-10, =6$, or sextic curve, which is the locus of the vertices of the cones which pass through the seven given points.

14. *Show that the envelope of a variable circle having its centre on a given conic and cutting at right angles a given circle is a bicircular quartic; which, when the given conic and circle have double contact, becomes a pair of circles; and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems:*

(1) *given two conics, to find a polygon of n sides inscribed in the one and circumscribed about the other;*

(2) *given two circles, to find a closed series of n circles each touching the two given circles and the two adjacent circles of the series.*

The equation of the given circle is taken to be

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2,$$

and that of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. This being so, we have $a\cos\theta$, $b\sin\theta$ as the coordinates of a point on the conic, which point may be taken to be the centre of the variable circle, and introducing the condition that the two circles cut at right angles, the equation of the variable circle is

$$(x-a\cos\theta)^2+(y-b\cos\theta)^2=(\alpha-a\cos\theta)^2+(\beta-b\sin\theta)^2-\gamma^2,$$

that is,

$$x^2+y^2-\alpha^2-\beta^2+\gamma^2-2ax\cos\theta-2by\sin\theta=0,$$

where θ is the variable parameter; and the equation of the envelope therefore is

$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2-4a^2x^2-4b^2y^2=0,$$

which is a quartic curve; and writing herein $\frac{x}{z}$, $\frac{y}{z}$ in place of x, y the equation would be of the second order in regard to x^2+y^2, z, and it thus appears that the curve has double points at each of the points $x^2+y^2=0$, $z=0$, viz. that the envelope is a bicircular quartic.

If the fixed circle touches the conic, then by a consideration of the figure it at once appears that the point of contact is a double point on the curve; and so if there is a double contact, then each of the points of contact is a double point on the curve. But in this case the curve is a bicircular quartic with *four* double points; viz. it is a pair of circles.

The porism in regard to the two conics is, that in general it is not possible to find any polygon of n sides satisfying the conditions; but that the conics may be such that there exists an infinity of polygons; viz. any point whatever of the one conic may then be taken as a vertex of the polygon, and then constructing the figure, the $(n+1)^{\text{th}}$ vertex will coincide with the first vertex, and there will be a polygon of n sides.

Now imagine that the conic touched by the sides is a circle having double contact with the other conic. Describe any one of the polygons, and with each vertex as centre describe the orthotomic circle, which will, it is clear, be a circle passing through the points of contact with the fixed circle of the sides through the vertex. We have thus a closed series of n circles, each touching the two adjacent circles of the series. And by considering any other polygon, we have a like series of n circles: and by what precedes the envelope of all the circles of the several series is a pair of circles; that is, the circles of every series touch these two circles. We have consequently two circles, such that there exists an infinity of closed series of n circles, each circle touching the two fixed circles, and also the two adjacent circles of the series; which is the porism arising out of the second problem.

535.

NOTE ON THE PROBLEM OF ENVELOPES.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 3, 4.]

THERE is a mode of looking at the problem of Envelopes, which, so far as I am aware, has not been explicitly noticed. Let $U=(x, y, z)^m$ be a function of the coordinates (x, y, z), $\Theta=\Theta'=(x, y, z)^a(x', y', z')^{a'}$ a function of the two sets of coordinates (x, y, z) and (x', y', z'); it being understood that when we write Θ we regard (x, y, z) as the current coordinates, when Θ' we regard (x', y', z') as the current coordinates. Suppose that we have $U=0$; the curve $\Theta'=0$ is then a curve the equation whereof contains as parameters the coordinates (x, y, z) of a point P on the curve $U=0$; and we may seek for the envelope of the curve $\Theta'=0$ as P describes the curve $U=0$; the required envelope is of course obtained as an equation in (x', y', z') given by the elimination of x, y, z, λ from the equations (equivalent to four equations only)

$$U=0, \quad \Theta'=0,$$
$$d_x\Theta'+\lambda d_x U=0,$$
$$d_y\Theta'+\lambda d_y U=0,$$
$$d_z\Theta'+\lambda d_z U=0.$$

But, observe that the required envelope is the locus of the points of intersection of the curve $\Theta'=0$ belonging to a particular point (x, y, z) of the curve $U=0$, by the curve $\Theta'=0$ which belongs to a consecutive point of U. The curve $\Theta=0$, considering therein (x', y', z') as the coordinates of a given point of the plane, determines by its intersection with $U=0$ those points (x, y, z) on the curve $U=0$, to each of which belongs a curve $\Theta'=0$ passing through the point in question (x', y', z'). Hence, if the curve $\Theta=0$ touch the curve $U=0$, the point of contact, coordinates (x, y, z), is a point such that to it and to the consecutive point there belong curves, each of them passing through the given point (x', y', z'). Hence expressing that the curves

$\Theta = 0$, $U = 0$ touch each other, we have a relation in (x', y', z') which is the locus of the point of intersection of the curves $\Theta' = 0$ belonging to two consecutive points of the curve $U = 0$; that is, the equation of the required envelope is obtained as the condition that the curves $U = 0$, $\Theta = 0$ shall touch each other. But when the curves touch each other, they have at the point of contact their derived functions proportional, or we have simultaneously

$$U = 0, \quad \Theta = 0,$$
$$d_x\Theta + \lambda d_x U = 0,$$
$$d_y\Theta + \lambda d_y U = 0,$$
$$d_z\Theta + \lambda d_z U = 0,$$

the same equations as before, since Θ and Θ' denote the same function.

It is to be added that, when $a = m$, the equations

$$d_x\Theta + \lambda d_x U = 0,$$
$$d_y\Theta + \lambda d_y U = 0,$$
$$d_z\Theta + \lambda d_z U = 0,$$

are homogeneous in (x, y, z), and we may by the elimination of (x, y, z) from these equations obtain an equation Disct. $(\Theta + \lambda U) = 0$, say for shortness $\Lambda = 0$, involving λ and also the coordinates (x', y', z'). Now it is a known theorem that the condition for the contact of the two curves $U = 0$, $\Theta = 0$ can be obtained by expressing that the equation $\Lambda = 0$ shall have a pair of equal roots, or, what is the same thing, by equating to zero the discriminant of the function Λ; this last-mentioned process leads therefore to the equation of the envelope of the curve $\Theta' = 0$, viz. (a being $= m$ as above) the equation of the envelope of the curve $\Theta' = 0$, is in fact

$$\text{Disct.}_{\lambda}\ \text{Disct.}_{(x, y, z)}\,(\Theta + \lambda U) = 0,$$

viz. we first take the discriminant of the function $\Theta + \lambda U$ in regard to the coordinates (x, y, z), and then taking the discriminant in regard to λ of this discriminant we equate it to zero. This is in many cases a more simple process than that of the direct elimination of x, y, z, λ from the five equations.

536.

NOTE ON LAGRANGE'S DEMONSTRATION OF TAYLOR'S THEOREM.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 22—24.]

I TAKE the occasion of the publication of the last edition of Mr Todhunter's *Treatise on the Differential Calculus* to make some remarks on the demonstration in question. Mr Todhunter proposes to himself to exhibit a comprehensive view of the Differential Calculus *on the method of Limits;* but he very properly introduces in some cases demonstrations founded upon other views of the subject, pointing out that this is the case, and explaining or indicating his objections. Thus (Chapter VI.) upon Taylor's Theorem, he remarks "Before we offer a strict demonstration of the theorem in question, we shall notice the method which it was usual to adopt in treatises on the Differential Calculus not based on the doctrine of limits," and then, after giving a demonstration depending on the relation $\frac{d}{dx}f(x+h)=\frac{d}{dh}f(x+h)$,[1] he goes on "There are numerous objections to the method of the preceding articles, and especially the use of an infinite series, without ascertaining that it is convergent, is inadmissible; we proceed then to a rigorous investigation," which investigation (after Mr Homersham Cox) is a demonstration of the equation

$$f(x+h)=f(x)+hf'(x)\ldots+\frac{h^n}{\lfloor n}f^n(x)+\frac{h^{n+1}}{\lfloor n+1}f^{n+1}(x+\theta h),$$

(θ between 0 and 1) whence "if the function $f^{n+1}(x+\theta h)$ is such that by making n sufficiently great the term $\frac{h^{n+1}}{\lfloor n+1}f^{n+1}(x+\theta h)$ can be made as small as we please, then by carrying on the series

$$f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\frac{h^3}{\lfloor 3}f'''(x)+\ldots$$

[1] This demonstration is similar in principle to Lagrange's but I think his is preferable; viz. the principle made use of by Lagrange is that the series has the same value whether x is changed into $x+k$, or h into $h+k$.

to as many terms as we please we obtain a result differing as little as we please from $f(x+h)$. Under these circumstances then we may assert the truth of Taylor's theorem."

I share Abel's horror of divergent series[1], and I maintain the validity of Lagrange's demonstration. When by an algebraic process we expand a function in a series, for instance the function $\frac{1}{1-x}$, by division

$$\begin{array}{l} 1-x)\,1 \qquad (1+x+x^2+\&c. \\ \quad\;\; \underline{1-x} \\ \qquad\quad x \\ \qquad\quad \underline{x-x^2} \\ \qquad\qquad x^2\ \&c. \end{array}$$

in the series $1+x+x^2+\&c.$, and write accordingly

$$\frac{1}{1-x}=1+x+x^2+\&c.$$

all that is (or ought to be) meant is that the algebraical operations continued as far as we please will give the series of terms $1, x, x^2, \ldots$ or say the series of coefficients $1, 1, 1, \ldots$ And of course with this meaning of the equation, the objection "*non constat* that the series is convergent" would be wholly irrelevant, we do not say that it is, we do not care whether it is so or not. In further illustration, remark that we frequently use such an equation merely as the means of expressing the law of a series of numbers $a_0, a_1, a_2, \ldots$, say $a_n=$ coeff. x^n in $f(x)$, where the function is assumed to be by a definite process expansible in the form $a_0+a_1x+a_2x^2+\&c.$ in question. Any objection that the series is not convergent would be simply irrelevant. Now any rational or irrational algebraic function $f(x+h)$ can by ordinary algebraical processes be expanded in the form $f(x)+$ terms in h, h^2 &c.... And if in regard to a function $f(x)$ we make the *single assumption* that $f(x+h)$ is *expansible in a form containing powers of h and reducing itself to $f(x)$ when h is put $=0$*, then Lagrange's demonstration shows that the powers of h are h, h^2, h^3, &c.... and that the expansion in fact is

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{1\,.\,2}f''(x)+\&c.;$$

viz. $f(x+h)$ acquires the same value $f(x+h+k)$ whether we change therein x into $x+k$ or h into $h+k$; and the expression on the right-hand side is the only series in h possessed of the same property. It is to be remarked that the equation contains in itself the definition of the operation of derivation, viz. the equation being true, $f'(x)$ can only denote the coefficient of h in the expansion of $f(x+h)$; and what

[1] Peut-on imaginer rien de plus horrible que de débiter

$$0=1^n-2^n+3^n-4^n+\text{etc.},$$

n étant un nombre entier positif?—*Œuvres*, t. II., p. 266; [Nouv. Éd., 1881, t. II., p. 257].

really is shown is that admitting such an operation to be possible in regard not only to $f(x)$, but to $f'(x)$, &c., then the coefficients $f'(x)$, $\frac{f''(x)}{1.2}$, &c., are obtained from $f(x)$ by the successive repetitions of this operation and by dividing by the proper numerical denominator.

By what precedes, any objection in regard to convergency, I regard as irrelevant; and if it is said that the above-mentioned single assumption is not granted, I would either ask "What is a function"—or I would content myself with the hypothetical statement—*if* $f(x)$ be such that $f(x+h)$ is expansible *ut suprà*, then Taylor's theorem.

In regard to the demonstration given by Mr Todhunter, it implicitly assumes that x and h are both real, and (although doubtless possible) it would be considerably more difficult to find an analogous demonstration of the formula involving $f^{n+1}(x+\theta h)$ in the case of x and h imaginary. But the formula *with the term in question* is not (nor does Mr Todhunter consider it as being) Taylor's theorem; to obtain from it Taylor's theorem, we require (in the foregoing point of view) the property that $h^{n+1}f(x+\theta h)$ is expansible in a series involving h^{n+1} and the higher powers of h, that is, the very property that $f(x+h)$ is expansible in positive powers of h.

Moreover admitting that the formula with the term $f^{n+1}(x+\theta h)$ is demonstrable for imaginary values of x, h, the formula is *meaningless* in the case where x, h are one or both a symbol or symbols of operation: θ would certainly have no definable numerical magnitude, and if it is considered as meaning anything, then the equation in question is a mere definition of what it does mean, and ceases to be a theorem in regard to $f(x+h)$. It is impossible, in a quantitative algebra such as is presupposed in the method of limits, to put any meaning on the equation

$$f\left(\frac{d}{dx}+h\right)=f\left(\frac{d}{dx}\right)+hf'\left(\frac{d}{dx}\right)+\&c.,$$

which however I regard as a legitimate particular form of Taylor's theorem.

537.

SOLUTIONS OF A SMITH'S PRIZE PAPER FOR 1871.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 37—47, 71—77, 89—95.]

1. *A point moves in a plane with a given velocity, and also with a given velocity about a fixed point in the plane: show that the locus is either a circle passing through the fixed point, or else a circle having the fixed point for its centre; and explain the relation between the two solutions.*

We have in general

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2,$$

and in the present question, taking the fixed point as the origin, and measuring θ from any fixed line through this point,

$$\frac{d\theta}{dt} = \omega, \quad V^2 = \left(\frac{dr}{dt}\right)^2 + r^2\omega^2,$$

where V, ω are given constants. Hence

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{dr}{dt}\right)^2 \div \left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{\omega^2} - r^2,$$

or, writing $V = a\omega$,

$$\left(\frac{dr}{d\theta}\right)^2 = a^2 - r^2,$$

therefore

$$d\theta = \frac{dr}{\sqrt{(a^2 - r^2)}},$$

or

$$\theta + \beta = \sin^{-1}\frac{r}{a}, \text{ (β the constant of integration)},$$

that is,

$$r = a \sin(\theta + \beta),$$

which is the equation of a circle (radius $= \frac{1}{2}a$) passing through the fixed point. In fact, the point moving in such a circle with a constant velocity, moves about the centre with a constant angular velocity, and about any fixed point in the circumference with an angular velocity which is one-half of that about the centre, and is therefore also constant.

Treating β as a variable parameter, to obtain the envelope we have

$$0 = a \cos(\theta + \beta),$$

that is, $\theta + \beta = \frac{\pi}{2}$ and therefore $r = a$, which is the equation of a circle (radius $= a$) having the fixed point for its centre. This is consequently the singular solution.

2. *Determine the system of curves which satisfy the differential equation*

$$dx\,\{\sqrt{(1+x^2)} + ny\} + dy\,\{\sqrt{(1+y^2)} + nx\} = 0\,;$$

and show that the curve which passes through the point $x = 0$, $y = n$ *contains as part of itself the conic*

$$x^2 + y^2 + 2xy\sqrt{(1+n^2)} - n^2 = 0.$$

The equation is integrable *per se*, viz. we have

$$x\sqrt{(1+x^2)} + \log\{x + \sqrt{(1+x^2)}\} + y\sqrt{(1+y^2)} + \log\{y + \sqrt{(1+y^2)}\} + 2nxy = C,$$

or, determining the constant so that for $x = 0$, y may be $= n$,

$$C = n\sqrt{(1+n^2)} + \log\{n + \sqrt{(1+n^2)}\},$$

and the equation may be written

$$x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)} + 2nxy - n\sqrt{(1+n^2)} + \log\frac{\{x + \sqrt{(1+x^2)}\}\{y + \sqrt{(1+y^2)}\}}{n + \sqrt{(1+n^2)}} = 0,$$

which is evidently a transcendental curve; it may however be shown that, if

$$x^2 + y^2 + 2xy\sqrt{(1+n^2)} - n^2 = 0,$$

then we have

$$x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)} + 2nxy - n\sqrt{(1+n^2)} = 0,$$

and

$$\{x + \sqrt{(1+x^2)}\}\{y + \sqrt{(1+y^2)}\} = n + \sqrt{(1+n^2)},$$

so that the equation of the curve is thus satisfied; wherefore the transcendental curve contains as part of itself the conic $x^2 + y^2 + 2xy\sqrt{(1+n^2)} - n^2 = 0$.

[As a simple illustration as to how this may happen, take the transcendental curve $y - \sin xy = 0$, which it is clear contains as part of itself the line $y = 0$.]

We have, from the equation of the conic

$$\{x + y\sqrt{(1+n^2)}\}^2 = n^2(1+y^2),$$

that is,

$$x + y\sqrt{(1+n^2)} = \pm n\sqrt{(1+y^2)},$$

but considering the radicals as positive, the sign must be taken so that we have simultaneously $x=0$, $y=n$. We have therefore

$$x + y\sqrt{(1+n^2)} = n\sqrt{(1+y^2)},$$

and similarly

$$y + x\sqrt{(1+n^2)} = n\sqrt{(1+x^2)}.$$

Then

$$\begin{aligned} n\{x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\} &= 2xy + (x^2+y^2)\sqrt{(1+n^2)} \\ &= 2xy + \sqrt{(1+n^2)}\{n^2 - 2xy\sqrt{(1+n^2)}\} \\ &= n^2\{\sqrt{(1+n^2)} - 2xy\}, \end{aligned}$$

which is the *first* of the relations in question; and

$$\begin{aligned} n^2\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\} \\ &= n^2xy + nx\{x + y\sqrt{(1+n^2)}\} + ny\{y + x\sqrt{(1+n^2)}\} \\ &\qquad + xy + (x^2+y^2)\sqrt{(1+n^2)} + xy(1+n^2) \\ &= \{n+\sqrt{(1+n^2)}\}\{x^2+y^2+2xy\sqrt{(1+n^2)}\} \\ &= \{n+\sqrt{(1+n^2)}\}n^2, \end{aligned}$$

which is the *second* of the two relations. And the theorem is thus proved.

[The foregoing is the easiest and most obvious solution, but it is interesting to consider the question differently, as follows:

Write

$$Q = \frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}},$$

we have

$$Q\ \{\sqrt{(n^2+1)}+n\} = \{\sqrt{(1+x^2)}+x\}\{\sqrt{(1+y^2)}+y\} = A+B,$$

$$Q^{-1}\{\sqrt{(n^2+1)}-n\} = \{\sqrt{(1+x^2)}-x\}\{\sqrt{(1+y^2)}-y\} = A-B,$$

if

$$A = \sqrt{(1+x^2)}\sqrt{(1+y^2)} + xy,$$

$$B = x\sqrt{(1+y^2)} + y\sqrt{(1+x^2)};$$

and then

$$\begin{aligned} AB &= x^2y\sqrt{(1+y^2)} + xy^2\sqrt{(1+x^2)} \\ &\qquad + y(1+x^2)\sqrt{(1+y^2)} + x(1+y^2)\sqrt{(1+x^2)} \\ &= x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)} + 2xyB, \end{aligned}$$

that is,

$$Q^2\{2n^2+1+2n\sqrt{(1+n^2)}\}-\frac{1}{Q^2}\{2n^2+1-2n\sqrt{(1+n^2)}\}$$
$$=4\{x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}\}$$
$$+4xy\left[Q\{\sqrt{(1+n^2)}+n\}-\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\right],$$

whence

$$4\{x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2nxy-n\sqrt{(1+n^2)}\}$$
$$=Q^2\{2n^2+1+2n\sqrt{(1+n^2)}\}-\frac{1}{Q^2}\{2n^2+1-2n\sqrt{(1+n^2)}\}$$
$$+8nxy-4n\sqrt{(1+n^2)}$$
$$-4xy\left[Q\{\sqrt{(1+n^2)}+n\}-\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\right]$$
$$=\left(Q^2-\frac{1}{Q^2}\right)(2n^2+1)+\left(Q^2+\frac{1}{Q^2}-2\right)2n\sqrt{(1+n^2)}$$
$$-\left(Q-\frac{1}{Q}\right)4xy\sqrt{(1+n^2)}-4\left(Q+\frac{1}{Q}-2\right)nxy$$
$$=(Q-1)\left\{\frac{(Q+1)(Q^2+1)}{Q^2}(2n^2+1)\right.$$
$$+\frac{(Q-1)(Q+1)^2}{Q^2}2n\sqrt{(1+n^2)}$$
$$-\frac{4(Q+1)}{Q}xy\sqrt{(1+n^2)}$$
$$\left.-4\frac{(Q-1)}{Q}nxy\right\}$$
$$=(Q-1)\,\Omega \text{ suppose,}$$

that is,

$$x\sqrt{(1+x^2)}+y\sqrt{(1+y^2)}+2nxy-n\sqrt{(1+n^2)}=\tfrac{1}{4}(Q-1)\,\Omega.$$

And the integral equation is

$$\tfrac{1}{4}(Q-1)\,\Omega+\log Q=C,$$

which, for $C=0$, is satisfied by $Q=1$.

Now starting from

$$Q=\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}},$$

we have

$$\sqrt{(1+x^2)}+x=Q\{\sqrt{(1+n^2)}+n\}\{\sqrt{(1+y^2)}-y\},$$
$$\sqrt{(1+x^2)}-x=\frac{1}{Q}\{\sqrt{(1+n^2)}-n\}\{\sqrt{(1+y^2)}+y\},$$

and thence

$$2x = K\sqrt{(1+y^2)} - Ly,$$

if

$$K = Q\{\sqrt{(1+n^2)}+n\} - \frac{1}{Q}\{\sqrt{(1+n^2)}-n\},$$

$$L = Q\{\sqrt{(1+n^2)}+n\} + \frac{1}{Q}\{\sqrt{(1+n^2)}-n\},$$

wherefore

$$L^2 - K^2 = 4.$$

Moreover

$$(2x+Ly)^2 = K^2(1+y^2),$$

that is,

$$4x^2 + (L^2-K^2)y^2 + 4Lxy = K^2,$$

or, what is the same thing,

$$x^2+y^2+Lxy = \tfrac{1}{4}(L^2-4),$$

which is the rationalised form of

$$Q = \frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}}.$$

And if $Q=1$ then $L = 2\sqrt{(1+n^2)}$, $\frac{1}{4}(L^2-4) = n^2$, so that this equation is

$$x^2+y^2+2xy\sqrt{(1+n^2)} - n^2 = 0;$$

or, when $C=0$, the complete integral is satisfied by

$$\frac{\{x+\sqrt{(1+x^2)}\}\{y+\sqrt{(1+y^2)}\}}{n+\sqrt{(1+n^2)}} = 1,$$

that is, by

$$x^2+y^2+2xy\sqrt{(1+n^2)} - n^2 = 0.$$

We may without difficulty rationalise, and present the result as follows: the equation

$$\left\{2\left(x+\frac{1}{x}\right)+\left(n-\frac{1}{n}\right)\left(y-\frac{1}{y}\right)\right\}\left(1+\frac{1}{x^2}\right)dx$$
$$+\left\{2\left(y+\frac{1}{y}\right)+\left(n-\frac{1}{n}\right)\left(x-\frac{1}{x}\right)\right\}\left(1+\frac{1}{y^2}\right)dy = 0,$$

has the complete integral

$$x^2-\frac{1}{x^2}+y^2-\frac{1}{y^2}+\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\left(n-\frac{1}{n}\right)-\left(n^2-\frac{1}{n^2}\right) = C+4\log\frac{xy}{n},$$

and a particular integral $xy-n=0$: the complete integral is in fact

$$(n-xy)\{-n^3x^2y^2+n^2xy(-x^2-y^2+1)+n(x^2y^2-x^2-y^2)-xy\} = x^2y^2n^2\left(C+4\log\frac{xy}{n}\right),$$

satisfied, for $C=0$, by $xy-n=0$.]

3. *Write* $\alpha = b - c$, $\beta = c - a$, $\gamma = a - b$; *then considering the three circles and the three conics*

$$(x-a)^2 + y^2 = -\beta\gamma, \quad \frac{x^2}{bc} + \frac{y^2}{K+bc} = 1,$$

$$(x-b)^2 + y^2 = -\gamma\alpha, \quad \frac{x^2}{ca} + \frac{y^2}{K+ca} = 1,$$

$$(x-c)^2 + y^2 = -\alpha\beta, \quad \frac{x^2}{ab} + \frac{y^2}{K+ab} = 1,$$

where K *is arbitrary; it is required to show that if a variable circle having its centre on one of the conics cuts at right angles the corresponding circle, the envelope is in each of the three cases one and the same bicircular quartic.*

Consider the circle $(x-a)^2 + y^2 = -\beta\gamma$ and the conic $\frac{x^2}{bc} + \frac{y^2}{K+bc} = 1$, the coordinates of a point on the conic are $\cos\theta\sqrt{(bc)}$, $\sin\theta\sqrt{(K+bc)}$, where θ is a variable parameter; say for a moment these values are p and q. The equation of the variable circle is

$$(x-p)^2 + (y-q)^2 = r^2,$$

and in order that this may cut at right angles the circle

$$(x-a)^2 + y^2 = -\beta\gamma,$$

we must have

$$(p-a)^2 + q^2 = r^2 - \beta\gamma,$$

or, substituting for r^2 its value from this equation, the equation of the variable circle is

$$(x-p)^2 + (y-q)^2 = (a-p)^2 + q^2 + \beta\gamma,$$

that is,

$$x^2 + y^2 - a^2 - \beta\gamma - 2p(x-a) - 2qy = 0,$$

viz. this is

$$(x^2 + y^2 - a^2 - \beta\gamma) - 2(x-a)\sqrt{(bc)}\cos\theta - 2y\sqrt{(K+bc)}\sin\theta = 0.$$

Hence taking the envelope in regard to θ, the equation is

$$(x^2 + y^2 - a^2 - \beta\gamma)^2 - 4(x-a)^2 bc - 4y^2(K+bc) = 0,$$

that is,

$$(x^2 + y^2 - ab - ac + bc)^2 - 4(x-a)^2 bc - 4y^2(K+bc) = 0,$$

or, what is the same thing,

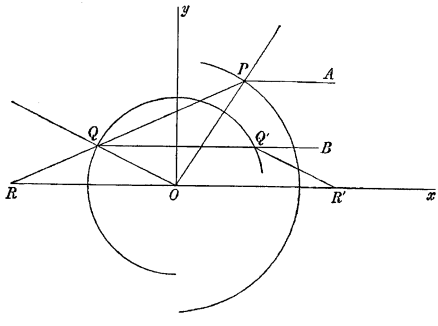
$$(x^2+y^2)^2 - 2(bc+ca+ab)(x^2+y^2) - 4Ky^2 + 8abcx$$
$$+ b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2b^2ca - 2c^2ab = 0,$$

viz. this equation, being symmetrical in regard to a, b, c, is the same equation as would have been obtained from either of the other conics and the corresponding circle; and from the form of the equation it is clear that the curve is a bicircular quartic.

4. *Show that the caustic by refraction for parallel rays of a circle, radius c, index of refraction μ, is the same curve as the caustic by refraction for parallel rays of the concentric circle, radius $\frac{c}{\mu}$, index of refraction $\frac{1}{\mu}$.*

Take as usual $\mu > 1$. Imagine the ray AP (fig. 1) parallel to the axis of x, incident at P on the circle radius c, and let the refracted ray after cutting the circle radius $\frac{c}{\mu}$, cut it again in Q, and then cut the axis in R. Take ϕ, ϕ' for the angles of incidence and refraction; $\sin\phi = \mu \sin\phi'$.

Fig. 1.



Moreover in the triangle PQO, we have $\sin Q : \sin P = c : \frac{c}{\mu}$; that is, $\sin Q = \mu \sin P$, $= \mu \sin\phi', = \sin\phi$; or $\angle Q = \phi$. And then in the triangle RQO, $\angle R = \phi - \phi'$, $\angle Q = 180° - \phi$, whence $\angle O = \phi'$, that is, $\angle QOR = \phi'$.

Consider now a ray BQ incident at Q and refracted in the direction QR; the index of refraction being $\frac{1}{\mu}$, that is, the denser medium being on the outside of the small circle. Taking θ, θ' for the angles of incidence and refraction, we have $\sin\theta = \frac{1}{\mu}\sin\theta'$; but, the refracted ray being by hypothesis QR, we have by what precedes $\theta' = \phi$, hence $\sin\theta = \frac{1}{\mu}\sin\phi = \sin\phi'$, that is, $\theta = \phi'$, or $\angle BQO = \angle QOR$, that is, the incident ray BQ is parallel to the axis of x. We have thus two pencils of rays each parallel to the axis, such that for any ray AP of the first pencil there is a corresponding ray BQ of the second pencil, the rays AP and BQ each giving rise to the same refracted ray PQR; hence the two pencils have the same caustic.

[It is proper to remark that for the ray BQ it has been assumed that the refraction takes place not at Q' where it first meets the small circle, but at Q; if

we consider the refraction at Q', then the index of refraction is still to be $=\frac{1}{\mu}$, that is, the denser medium must now be *inside* the small circle; the refracted ray is in the direction $R'Q'$ situate symmetrically with RQ on the opposite side of the axis of y; and it would at first sight appear that the caustic was a curve equal and similar to the original caustic, but situate on the opposite side of the axis of y. But geometrically the complete caustic consists of two equal and similar portions situate on opposite sides of the axis of y; so that we really obtain, not an equal and opposite caustic, but in each case one and the same caustic.

I originally obtained the theorem in a different manner; viz. the equation for the caustic for the first pencil of rays was found to be

$$(1-\mu^2)\frac{x}{c}=\left\{1-\mu^{\frac{4}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}}+\mu\left\{1-\mu^{-\frac{4}{3}}\left(\frac{y}{c}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}},$$

which equation (as is easily seen) remains unaltered when c, μ are changed into $\frac{c}{\mu}$, $\frac{1}{\mu}$ respectively.—See my "Memoir on Caustics," *Phil. Trans.*, t. CXLVII. (1857), [145], p. 285.]

5. *Given at each point of space the direction-cosines* (α, β, γ) *of a line through that point: it is required to find the conditions in order that the lines may be not a triple but a double system.*

For any given point P the values of the quantities α, β, γ which determine the direction of the line through that point are given as functions of the coordinates (x, y, z) of the point P. Hence passing from a point P to a consecutive point P' *on the line*, the coordinates of P' will be $x+\rho\alpha$, $y+\rho\beta$, $z+\rho\gamma$; and the values of α, β, γ for the point P' will be

$$\alpha+\rho\left(\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}\right),$$

$$\beta+\rho\left(\alpha\frac{d\beta}{dx}+\beta\frac{d\beta}{dy}+\gamma\frac{d\beta}{dz}\right),$$

$$\gamma+\rho\left(\alpha\frac{d\gamma}{dx}+\beta\frac{d\gamma}{dy}+\gamma\frac{d\gamma}{dz}\right).$$

But if the lines form a double system, we must have the same line for the point P, and for any other point P' on the line; and in particular the same line for the point P, and for the consecutive point P'. Hence as conditions for the double system we obtain

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\alpha}{dy}+\gamma\frac{d\alpha}{dz}=0,$$

$$\alpha\frac{d\beta}{dx}+\beta\frac{d\beta}{dy}+\gamma\frac{d\beta}{dz}=0,$$

$$\alpha\frac{d\gamma}{dx}+\beta\frac{d\gamma}{dy}+\gamma\frac{d\gamma}{dz}=0.$$

But in virtue of the relation $\alpha^2+\beta^2+\gamma^2=1$, we have

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\beta}{dx}+\gamma\frac{d\gamma}{dx}=0,$$

$$\alpha\frac{d\alpha}{dy}+\beta\frac{d\beta}{dy}+\gamma\frac{d\gamma}{dy}=0,$$

$$\alpha\frac{d\alpha}{dz}+\beta\frac{d\beta}{dz}+\gamma\frac{d\gamma}{dz}=0.$$

Hence subtracting the corresponding equations we have three equations, which are at once seen to be equivalent to the two equations

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy} : \frac{d\gamma}{dx}-\frac{d\alpha}{dz} : \frac{d\alpha}{dy}-\frac{d\beta}{dx}=\alpha : \beta : \gamma,$$

equations which must be satisfied identically, whatever are the values of (x, y, z). The equations have been obtained as necessary conditions; they are, in fact, the sufficient conditions for a double system; for the line being unaltered in passing from P to P', it remains unaltered when we pass to the following point P'', and so on; that is, for the passage to any point Q whatever on the line.

COR. If the equation $\alpha dx+\beta dy+\gamma dz=0$ be integrable by a factor, it must be integrable *per se:* in fact, the condition that it may be integrable by a factor is

$$\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)=0.$$

But we have

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=kz, \text{ \&c.},$$

and the equation thus becomes

$$k(\alpha^2+\beta^2+\gamma^2)=0,$$

that is, $k=0$, and therefore

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=0,\quad \frac{d\gamma}{dx}-\frac{d\alpha}{dz}=0,\quad \frac{d\alpha}{dy}-\frac{d\beta}{dx}=0.$$

Hence, also, if the lines cut at right angles a surface, we must have $\alpha dx+\beta dy+\gamma dz$ a complete differential.

The foregoing theory is given in Sir W. R. Hamilton's "Memoir on Ray-Systems."

6. *If* $X=0$, $Y=0$, $Z=0$, $W=0$ *are four given conics in the same plane and having a common point: show that, in the system of conics* $aX+bY+cZ+dW=0$, *there are in general four (improper) conics the equations of which may be taken to be* $x^2=0$, $y^2=0$, $xz=0$, $yz=0$.

Taking the conics to pass through the point $x=0$, $y=0$; their equations will be of the form

$$\begin{aligned} X &= a_1x^2 + 2h_1xy + b_1y^2 + 2f_1yz + 2g_1zx = 0, \\ Y &= a_2x^2 + 2h_2xy + b_2y^2 + 2f_2yz + 2g_2zx = 0, \\ Z &= a_3x^2 + 2h_3xy + b_3y^2 + 2f_3yz + 2g_3zx = 0, \\ W &= a_4x^2 + 2h_4xy + b_4y^2 + 2f_4yz + 2g_4zx = 0. \end{aligned}$$

Now multiplying by the indeterminate quantities α, β, γ, δ, the three ratios $\alpha:\beta:\gamma:\delta$ may be determined so that the terms in yz, zx shall vanish, and the terms in x^2, xy, y^2 be a perfect square: we thus arrive at a quadric equation for any one of the ratios, say $\alpha:\beta$, the remaining ratios being then linearly determined; viz. there are two sets of values of α, β, γ, δ: and changing the coordinates (x, y), the two resulting forms may be represented by $x^2=0$, $y^2=0$.

And it is clear that we thus have in the system of conics $\alpha X+\beta Y+\gamma Z+\delta W=0$, four conics the equations of which may be represented by

$$\begin{aligned} X' &= x^2 &&= 0, \\ Y' &= y^2 &&= 0, \\ Z' &= h_3xy + f_3yz + g_3zx &&= 0, \\ W' &= h_4xy + f_4yz + g_4zx &&= 0, \end{aligned}$$

where of course the coefficients f, g, h have new values.

We may then form the equations

$$\begin{aligned} \alpha X' + f_4Z' - f_3W' &= x\{\alpha x + (f_4h_3 - f_3h_4)\,y + (f_4g_3 - f_3g_4)\,z\}, \\ \beta Y' - g_4Z' + g_3W' &= y\{(g_3h_4 - g_4h_3)\,x + \beta y + (f_4g_3 - f_3g_4)\,z\}, \end{aligned}$$

so that, by writing $\alpha = g_3h_4 - g_4h_3$ and $\beta = f_4h_3 - f_3h_4$, the terms in $\{\ \}$ will be one and the same linear function of (x, y, z); that is, changing the z so as to denote the linear function in question by z, we have as conics of the series $xz=0$, and $yz=0$, that is, we have in the series the four conics $x^2=0$, $y^2=0$, $xz=0$, $yz=0$; whence also any other conic of the series, and consequently each of the original four conics, may be represented by an equation of the form

$$ax^2 + by^2 + 2fyz + 2gzx = 0.$$

7. *The coordinates (x, y, z, w) of a point P in space are connected with the coordinates (x', y', z') of a point P' in a plane by the equations*

$$x : y : z : w = X' : Y' : Z' : W',$$

where X', Y', Z', W' are quadric functions of (x', y', z') such that $X'=0$, $Y'=0$, $Z'=0$, $W'=0$ represent conics having a common point: show that the locus of P is a cubic scroll (skew surface of the third order): and find the curves in the plane which correspond to the generating lines of the scroll.

The equations $x : y : z : w = X' : Y' : Z' : W'$ are three equations containing the indeterminate parameters $x' : z'$ and $y' : z'$, so that eliminating these we have between (x, y, z, w) a single (homogeneous) equation representing a surface. To each point (x', y', z') of the plane, there corresponds a single point of the surface, and to each point (x, y, z, w) of the surface a single point (x', y', z') of the plane. The only exception is that for the common point of the four conics, the ratios $x : y : z : w$ are essentially indeterminate, and there is not corresponding hereto any determinate point of the surface.

To find the order of the surface, consider its intersection with any arbitrary line

$$ax + by + cz + dw = 0,$$
$$a_1x + b_1y + c_1z + d_1w = 0.$$

We have corresponding hereto in the plane the points of intersection of the conics

$$aX' + bY' + cZ' + dW' = 0,$$
$$a_1X' + b_1Y' + c_1Z' + d_1W' = 0,$$

viz. these are conics each of them passing through the common point of the four conics, and therefore they intersect besides in three points: that is, the order of the surface is $= 3$.

To show that the common point ought to be (as above) excluded, some further explanation is desirable. To the section of the surface by the plane $ax + by + cz + dw = 0$, corresponds the conic $aX' + bY' + cZ' + dW' = 0$; and similarly to the section by the plane $a_1x + b_1y + c_1z + d_1w = 0$, corresponds the conic $a_1X' + b_1Y' + c_1Z' + d_1W' = 0$. Now to the common point *considered as belonging to the first conic* there corresponds a determinate point of the surface; and to the common point *considered as belonging to the second conic* there corresponds a determinate point of the surface; but these are two distinct points on the surface: so that corresponding to the common point of the four conics, there is not on the surface any point of intersection of the two plane sections; but these intersect in only three points of the surface; viz. the line of intersection of the two planes meets the surface in three points: or the surface is a cubic surface.

The same result may be obtained, and it may be further shown that the surface is a scroll, by means of the property in the foregoing question 6; viz. it thereby appears that each of the functions X', Y', Z', W' may be taken to be of the form $ax'^2 + by'^2 + fy'z' + gz'x'$; hence replacing the original coordinates x, y, z, w, by properly selected linear functions of these coordinates, the given relations may be presented in the form

$$x : y : z : w = x'^2 : y'^2 : x'z' : y'z',$$

whence eliminating, we have

$$xw^2 - yz^2 = 0$$

the equation of a cubic scroll, having the line $z = 0$, $w = 0$ for a double line, and the line $x = 0$, $y = 0$ for a directrix line. The equations of a generating line of the scroll

are, it is clear, $z-\theta w=0$, $x-\theta^2 y=0$, where θ is a variable parameter; and corresponding hereto in the plane we have the line $x'-\theta y'=0$, viz. this is any line through the common intersection of the four conics.

8. *If U, V are binary functions of the form $(a, b, \ldots)(x, y)^m$ with arbitrary coefficients, and if the equations $U=0$, $V=0$ have a common root, show how this can be determined in terms of the derived functions of the Resultant in regard to the coefficients of either function.*

Show what results in regard to the common root can be obtained when the coefficients are not all of them arbitrary but (1) *each or either of the functions depends in any manner whatever on a set of arbitrary coefficients not entering into the other function,* (2) *the two functions depend in any manner whatever on one and the same set of arbitrary coefficients.*

How is the theory modified when, instead of the two equations, there is a single equation $U=0$ having a double root?

Suppose

$$U=(a, b, \ldots)(x, y)^m \left(=a x^m+\frac{m}{1} b x^{m-1} y+\&c.\right),$$

$$V=(a', b', \ldots)(x, y)^{m'} \left(=a' x^{m'}+\frac{m'}{1} b x^{m'-1} y+\&c.\right).$$

Then if R is the resultant, the equation $R=0$ is the relation which must exist between the coefficients $(a, b, \ldots)$ and $(a', b', \ldots)$ in order that the equations $U=0$ and $V=0$ may have a common root (that is, in order that the functions U, V may have a common factor $x-\alpha y$). Imagine the relation subsisting, and that x, y are the values belonging to the common root, or (what is the same thing) that we have $x-\alpha y=0$; we have then simultaneously $U=0$, $V=0$, $R=0$. Now suppose the coefficients $a, b, \ldots$ to be infinitesimally varied in such manner that U, V have still a common root; say the new values are $a+\delta a$, $b+\delta b, \ldots$: this implies between δa, $\delta b, \ldots$ the relation

$$\frac{dR}{da}\delta a+\frac{dR}{db}\delta b+\ldots=0.$$

But the common factor $x-\alpha y$ is a factor of the *unaltered* equation $V=0$; and the values of (x, y) are thus unaltered, viz. the equation $U=0$ is satisfied with the original values of (x, y); so that we have

$$\frac{dU}{da}\delta a+\frac{dU}{db}\delta b+\ldots=0,$$

or, what is the same thing,

$$x^m\delta a+m x^{m-1} y\delta b+\ldots=0,$$

an equation which must agree with the former one, that is, we have

$$x^m : m x^{m-1} y : \&c. = \frac{dR}{da} : \frac{dR}{db} : \&c.,$$

a series of equations giving the value of the common root $\frac{x}{y}(=\alpha)$ in the several forms

$$\frac{1}{m}\frac{x}{y}=\frac{dR}{da}\div\frac{dR}{db},\quad \frac{2}{m-1}\frac{x}{y}=\frac{dR}{db}\div\frac{dR}{dc},\ \&c.$$

And it is clear that we have in like manner

$$x^{m'} : m'x^{m'-1}y : \&c.=\frac{dR}{da'} : \frac{dR}{db'} : \&c.$$

It is clear that if U involves, *in any manner whatever*, the coefficients $a, b, \ldots$ which do not enter into the function V, then we have in precisely the same manner

$$\frac{dU}{da} : \frac{dU}{db} : \&c.=\frac{dR}{da} : \frac{dR}{db} : \&c.,$$

a system of equations satisfied by the values x, y which belong to the common root.

But if the coefficients $a, b, \ldots$ are contained in any manner whatever in both of the functions U, V; then by altering $a, b, \ldots$ we alter the common root; say that $x+\delta x, y+\delta y$ belong to its new value; then we have

$$\frac{dU}{dx}\delta x+\frac{dU}{dy}\delta y+\frac{dU}{da}\delta a+\frac{dU}{db}\delta b+\ldots=0,$$

$$\frac{dV}{dx}\delta x+\frac{dV}{dy}\delta y+\frac{dV}{da}\delta a+\frac{dV}{db}\delta b+\ldots=0.$$

Now the values of x, y which satisfy $U=0$, $V=0$ also satisfy

$$\frac{dU}{dx}\frac{dV}{dy}-\frac{dU}{dy}\frac{dV}{dx}=0;$$

hence from the foregoing two equations eliminating δx or δy, the other of these two quantities will disappear of itself, and we thus obtain an equation

$$A\delta a+B\delta b+\ldots=0,$$

which must agree with the above equation

$$\frac{dR}{da}\delta a+\frac{dR}{db}\delta b+\ldots=0,$$

or we have

$$\frac{dR}{da} : \frac{dR}{db} : \&c.=A : B : \&c.,$$

a system of equations satisfied by the values x, y which belong to the common root.

In the case of a single equation $U=0$ having a double root, the condition for this is $\Delta=0$, where Δ is the discriminant of the function U; and the like reasoning shows that for the values x, y which belong to the double root we have

$$\frac{dU}{da} : \frac{dU}{db} : \&c.=\frac{d\Delta}{da} : \frac{d\Delta}{db} : \ldots;$$

viz. if U is of the form $(a, b, \ldots)(x, y)^m$ with arbitrary coefficients, then we have thus a series of equations giving the required value of $\frac{x}{y}$; but if $(a, b, \ldots)$ are arbitrary coefficients contained in any manner whatever in the function U, then we have a series of equations satisfied by the values x, y which belong to the double root.

9. *The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section: show that the locus of the point of intersection is an ellipse having four (real or imaginary) contacts with the evolute of the principal section.*

The principal section is for convenience taken to be that in the plane of zx; the coordinates of any point thereof are therefore X, 0, Z where

$$\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1.$$

Consider the normal at a point X, Y, Z of the ellipsoid; taking x, y, z as current coordinates, the equations of the normal are

$$\frac{x-X}{\frac{X}{a^2}}=\frac{y-Y}{\frac{Y}{b^2}}=\frac{z-Z}{\frac{Z}{c^2}}.$$

Writing herein $y=0$, we have

$$x=X\left(1-\frac{b^2}{a^2}\right),\quad z=Z\left(1-\frac{b^2}{c^2}\right);$$

viz. x, z are here the coordinates of the point where the normal meets the plane of xz; and observing that the point in question lies on the normal at the point X, 0, Z, it is clear that x, y, z will be the coordinates of the intersection of the last-mentioned normal by the normal at the consecutive point not on the principal section.

Writing for shortness

$$\alpha=b^2-c^2,\quad \beta=c^2-a^2,\quad \gamma=a^2-b^2,$$

($\alpha+\beta+\gamma=0$, α and γ positive, β negative) the values are

$$x=\frac{\gamma X}{a^2},\quad z=-\frac{\alpha Z}{c^2},$$

wherefore

$$\frac{X}{a}=\frac{\alpha x}{\gamma},\quad \frac{Z}{c}=-\frac{cz}{\alpha};$$

or, substituting in

$$\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1,$$

we have

$$\frac{a^2x^2}{\gamma^2}+\frac{c^2z^2}{\alpha^2}=1,$$

the required locus, which is thus an ellipse.

If the point $(X, 0, Z)$ is an umbilicus, it is clear that the corresponding point of the locus will be a point of the evolute of the principal section; and to prove that the locus touches the evolute, it is only necessary to show that the tangent of the locus is also the tangent of the evolute; or what is the same thing, that the tangent of the locus passes through the umbilicus.

Now for the umbilicus we have

$$X^2=-a^2\frac{\gamma}{\beta},\quad Z^2=-c^2\frac{\alpha}{\beta};$$

the corresponding values of x, z being

$$x=\frac{\gamma X}{a^2},\quad z=-\frac{\alpha Z}{c^2}.$$

Take ξ, ζ as the current coordinates of a point on the tangent of the locus, we have

$$\frac{a^2x\xi}{\gamma^2}+\frac{c^2z\zeta}{\alpha^2}=1,$$

or, substituting for x, z the foregoing values,

$$\frac{X\xi}{\gamma}-\frac{Z\zeta}{\alpha}=1,$$

and these should be satisfied by ξ, $\zeta=X$, Z; viz. we ought to have

$$\frac{X^2}{\gamma}-\frac{Z^2}{\alpha}=1,$$

and this equation is in fact true for the values of X, Z at the umbilicus; viz. for these values we have

$$-\frac{a^2}{\beta}+\frac{c^2}{\beta}=1,$$

that is, $\beta=c^2-a^2$, which is in fact the value of β.

There is obviously a contact in each quadrant, that is, there are four contacts (in the present case all real) of the locus with the evolute.

The same theorem holds good in regard to the other principal sections; only for these, the umbilici being imaginary, the points of contact of the locus with the evolute are also imaginary.

Remark. There is a great convenience in questions relating to the ellipsoid, in the use of the foregoing notations α, β, γ.

10. *An endless heavy chain of given length is suspended from two fixed points in the same horizontal plane: show that (subject to a condition as to the length) the figure of equilibrium may consist of portions of two distinct catenaries.*

The two parts of the chain will each of them be a portion of a catenary, viz. they will either coincide with each other, forming a twice repeated portion of a catenary (which is always a possible position of equilibrium), or they will form portions of two distinct catenaries. That the latter form is in some cases possible, appears from the case of a very long chain. It is then clear that there is a position of equilibrium in which the upper catenary is nearly a straight line. It may be added, that, as the length of the chain diminishes, the two distinct catenaries approach more and more, and for a certain value of the length become coincident; for any smaller value of the length, the only position is that consisting of a twice repeated portion of a catenary. But to obtain the solution in a regular manner, observe that, in order to the existence of such a form of equilibrium, the necessary condition is, that the tension at A (or B) must be equal in the two catenaries. Now the tension at any point of a catenary is proportional to the height above the directrix of the catenary; hence the condition is, that there shall be through the points A, B two catenaries having the same directrix, and such that the sum of the lengths is equal to the given length of the chain.

Take $AB = 2a$, the length of the chain $= 2l$. Take β for the distance of the directrix below the points A, B; c for the parameter of the catenary (or distance of its lowest point above the directrix), β, c being of course unknown. Then taking the origin at the mid-point of the directrix, and the axis of y vertically upwards, the equation of the catenary is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

whence for the point A or B,

$$\beta = \frac{c}{2}\left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right),$$

and the arc measured from the lowest point is

$$s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Hence, assuming that there are two distinct catenaries, if the parameters are c, c', we have

$$\tfrac{1}{2}c\left(e^{\frac{a}{c}} + e^{-\frac{a}{c}}\right) = \tfrac{1}{2}c'\left(e^{\frac{a}{c'}} + e^{-\frac{a}{c'}}\right),$$

$$\tfrac{1}{2}c\left(e^{\frac{a}{c}} - e^{-\frac{a}{c}}\right) + \tfrac{1}{2}c'\left(e^{\frac{a}{c'}} - e^{-\frac{a}{c'}}\right) = l,$$

which are the conditions for the determination of c, c'; and it is to be shown that these can be satisfied otherwise than by taking $c = c'$.

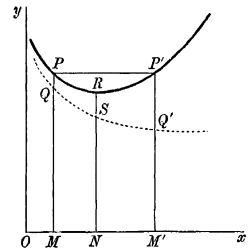
Trace the two curves

$$y = \frac{x}{2}\left(e^{\frac{a}{x}} + e^{-\frac{a}{x}}\right),$$

$$y' = \frac{x}{2}\left(e^{\frac{a}{x}} - e^{-\frac{a}{x}}\right),$$

shown respectively by the black line and the dotted line in fig. 2. Draw any line parallel to the axis of x, meeting the first curve in the points P, P' respectively, and let the ordinates MP, $M'P'$ meet the second curve in the points Q, Q' respectively;

Fig. 2.



then it is clear, that if for a given value of l the line PP' can be drawn in such-wise that $MQ + M'Q' = l$, there will be in fact the required two values $c = OM$ and $c' = OM'$.

And since for MP very large we have MQ, and therefore also $MQ + M'Q'$, very large, and as MP diminishes, $MQ + M'Q'$ also diminishes until it attains a certain minimum value, say $= \lambda$, it is clear that if l has any value greater than this minimum value, PP' can be so drawn that $QM + Q'M' = l$.

[The above remarkably elegant investigation in regard to the two values c, c' was given in the Examination; it seems to be the case that as PP' moves downwards, $MQ + M'Q'$ continually decreases (viz. MQ decreases more rapidly than $M'Q'$ increases), its value being least, and $= 2NS$ when PP' becomes a tangent to the first curve at its lowest point R; but it is not by any means easy to prove that this is so. The question depends on the form of the curve defined by the equations

$$X = \tfrac{1}{2}x_1\left(e^{\frac{a}{x_1}} - e^{-\frac{a}{x_1}}\right) + \tfrac{1}{2}x_2\left(e^{\frac{a}{x_2}} - e^{-\frac{a}{x_2}}\right),$$

$$Y = \tfrac{1}{2}x_1\left(e^{\frac{a}{x_2}} + e^{-\frac{a}{x_2}}\right) = \tfrac{1}{2}x_2\left(e^{\frac{a}{x_2}} + e^{-\frac{a}{x_2}}\right)$$

where X and Y are the current coordinates.]

11. *A particle is attracted to two centres of force, one of them at the origin, the other revolving about the origin in a circle in the plane of xy with a uniform angular velocity n′: find the equations of motion; and writing v for the velocity of the particle and A for the resolved area (about the fixed centre) in the plane of xy, show that there is a first integral giving the value of* $v^2 - 4n'\dfrac{dA}{dt}$ *in terms of the coordinates of the particle and of the revolving centre.*

Take ξ, η, 0 for the coordinates of the moving centre, so that

$$\xi = a\cos n't, \quad \eta = a\sin n't;$$

the equations of motion are

$$\frac{d^2x}{dt^2} = -\phi r\frac{x}{r} - \psi\rho\frac{x-\xi}{\rho},$$

$$\frac{d^2y}{dt^2} = -\phi r\frac{y}{r} - \psi\rho\frac{y-\eta}{\rho},$$

$$\frac{d^2z}{dt^2} = -\phi r\frac{z}{r} - \psi\rho\frac{z}{\rho},$$

where

$$r^2 = x^2 + y^2 + z^2,$$

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2 + z^2.$$

We have

$$r\,dr = x\,dx + y\,dy + z\,dz,$$

$$\rho\,d\rho = (x-\xi)(dx - d\xi) + (y-\eta)(dy - d\eta) + z\,dz.$$

But

$$d\xi = -n'a\sin n't\,dt = -n'\eta\,dt,$$

$$d\eta = \quad n'a\cos n't\,dt = \quad n'\xi\,dt,$$

whence

$$\begin{aligned}\rho\,d\rho &= (x-\xi)dx + (y-\eta)\,dy + z\,dz \\ &\qquad + n'[\eta(x-\xi) - \xi(y-\eta)]\,dt \\ &= (x-\xi)dx + (y-\eta)\,dy + z\,dz \\ &\qquad - n'[x(y-\eta) - y(x-\xi)]\,dt.\end{aligned}$$

Hence from the equations of motion

$$\begin{aligned}&2\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right) \\ &\quad - 2n'\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) \\ &= -\frac{\phi r}{r}2\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) \\ &\quad - \frac{\psi\rho}{\rho}\left[2\left\{(x-\xi)\frac{dx}{dt} + (y-\eta)\frac{dy}{dt} + z\frac{dz}{dt}\right\}\right. \\ &\quad \left. - 2n'\{x(y-\eta) - y(x-\xi)\}\right].\end{aligned}$$

But we have

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2,$$

$$x\frac{d^2y}{dt} - y\frac{d^2x}{dt^2} = \frac{d}{dt}r^2\frac{d\theta}{dt} = 2\frac{dA}{dt};$$

the foregoing equation may be written

$$\frac{d.v^2}{dt} - 4n'\frac{d^2A}{dt^2} = -2\phi r\frac{dr}{dt} - 2\psi\rho\frac{d\rho}{dt},$$

whence

$$v^2 - 4n'\frac{dA}{dt} = C - 2\int\phi r dr - 2\int\psi\rho d\rho,$$

the required result.

12. *If x, y are the coordinates of a particle moving* in plano *under the action of a central force varying as (distance)$^{-2}$: write down the expressions of the coordinates x, y in terms of the time t and of four arbitrary constants: and (in case of disturbed motion) starting from the equations*

$$\delta x = 0,\ \delta y = 0,\ \delta x' = \frac{d\Omega}{dx}dt,\ \delta y' = \frac{d\Omega}{dy}dt,$$

(the notation to be explained), indicate the process of finding the variations of the constants in terms of (1) $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$, (2) *the derived functions of Ω in regard to the constants.*

We have

$$x = a\left\{\frac{\cos u - e}{1 - e\cos u}\cos\varpi + \frac{\sqrt{(1-e^2)}\sin u}{1 - e\cos u}\sin\varpi\right\},$$

$$y = a\left\{\frac{\sqrt{(1-e^2)}\sin u}{1 - e\cos u}\cos\varpi - \frac{\cos u - e}{1 - e\cos u}\sin\varpi\right\},$$

where

$$n - e\sin u = t\sqrt{\left(\frac{\mu}{a^3}\right)} + c,$$

an equation serving to express u in terms of t and the constants a, e, c; the foregoing equations, therefore, in effect give x, y in terms of t and the four constants a, e, c, ϖ.

In the second part of the question, Ω is a given function of x, y, t, the differential coefficients $\frac{d\Omega}{dx}$, $\frac{d\Omega}{dy}$ being the partial ones in regard to x, y respectively. The equation $\delta x = 0$ signifies that the variation of x, in so far as it arises from the variation of the constants, is $= 0$; it in fact *means*

$$\frac{dx}{da}\frac{da}{dt} + \frac{dx}{de}\frac{de}{dt} + \frac{dx}{dc}\frac{dc}{dt} + \frac{dx}{d\varpi}\frac{d\varpi}{dt} = 0.$$

The value of $x'\left(=\frac{dx}{dt}\right)$ is therefore obtained from that of x by differentiating in regard to t alone, as if a, e, c, ϖ were constants: viz. x' will be a given function of t, a, e, c, ϖ; $\delta x'$ then denotes the variation of x' in so far as it arises from the variation of the constants, viz. the equation $\delta x' = \frac{d\Omega}{da}\,dt$ *means*

$$\frac{dx'}{da}\frac{da}{dt}+\frac{dx'}{de}\frac{de}{dt}+\frac{dx'}{dc}\frac{dc}{dt}+\frac{dx'}{d\varpi}\frac{d\varpi}{dt}=\frac{d\Omega}{dx}.$$

There are the like equations in regard to y, y', viz. in all, four equations which are linear in regard to $\frac{da}{dt}, \frac{de}{dt}, \frac{dc}{dt}, \frac{d\varpi}{dt}$; and which serve to determine these quantities in terms of $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}$.

Now considering the x, y as expressed in terms of a, e, c, ϖ, t, then Ω becomes a function of these quantities; the differential coefficients $\frac{d\Omega}{da}$, &c., being connected with the original differential coefficients $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}$ by the equations

$$\frac{d\Omega}{da}=\frac{d\Omega}{dx}\frac{dx}{da}+\frac{d\Omega}{dy}\frac{dy}{da},$$

$$\frac{d\Omega}{de}=\frac{d\Omega}{dx}\frac{dx}{de}+\frac{d\Omega}{dy}\frac{dy}{de},$$

&c.

As there are four equations, $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}$ can be expressed in an infinity of ways in terms of $\frac{d\Omega}{da}, \frac{d\Omega}{de}, \frac{d\Omega}{dc}, \frac{d\Omega}{d\varpi}$, and considering $\frac{da}{dt}$, &c., as given in terms of $\frac{d\Omega}{dx}, \frac{d\Omega}{dy}$, we can in an infinity of ways express $\frac{da}{dt}$, &c., as linear functions of $\frac{d\Omega}{da}, \frac{d\Omega}{de}, \frac{d\Omega}{dc}, \frac{d\Omega}{d\varpi}$. But there is one form (obtained by combining the equations in a particular manner) wherein the coefficients of the last-mentioned quantities are functions of a, e, c, ϖ *without* t; and this is the form actually employed for the expression of $\frac{da}{dt}, \frac{de}{dt}, \frac{dc}{dt}, \frac{d\varpi}{dt}$ in terms of $\frac{d\Omega}{da}, \frac{d\Omega}{de}, \frac{d\Omega}{dc}, \frac{d\Omega}{d\varpi}$, in the method wherein these quantities are made use of.

I remark upon the present question, that the answer *ought* to be *in substance perfectly familiar to every student in Physical Astronomy;* and that a student *ought* to be able to present it *in a clear and logical form:* the question being in fact intended as a test of ability *in this respect.*

13. *Explain the course of the geodesic lines on a spheroid of revolution: and in particular show that the condition is satisfied in virtue of which any geodesic line, considered as starting from a given point, ceases at some point of its course to be a shortest line.*

From each point on the surface a geodesic line may be drawn in any direction whatever along the surface, that is, through each point of the surface there is a singly infinite series of geodesic lines. A geodesic line undulates (in the manner of a sinusoid) between two parallels equidistant from the equator on opposite sides thereof; viz. considering it as starting from a point A on the equator, it arrives at a point V on the upper parallel (there touching the parallel), and passes downwards to cut the equator at A', and thence arrives at a point V' on the lower parallel (there touching the parallel), and again passes upwards to meet the equator at A'', and so on; the arcs AV, VA', $A'V'$, $V'A''$, &c., being similar and equal to each other (differing only in position). The equatoreal arc $AA'(=A'A''=\&c.)$ or difference of the longitudes A, A', is always less than 180°, its value increasing with the inclination at which the geodesic line cuts the equator, viz. when this angle is indefinitely small, the arc is $=\frac{c}{a}180°$ (c, a the polar and equatoreal axes respectively), and as the inclination becomes indefinitely near to 90°, the value of the arc becomes indefinitely near 180°. If the arc in question is commensurable with 180°, the geodesic line will be, it is clear, a closed curve; but if not, then it is not a closed curve, but proceeds undulating for ever between the two parallels. In the limiting case where the inclination is $=90°$, the geodesic line is obviously a meridian.

Considering a geodesic line starting in a given direction from a point A, and the geodesic line from the same point A in the consecutive direction, it appears from the foregoing account of the configuration of the lines, that the two lines will intersect each other in general an indefinite number of times: supposing that they first intersect in a point K, then by a general theorem of Jacobi's, the geodesic line AK is a shortest line from A to any point nearer than K, but it is not a shortest line from A to any point beyond K.

538.

EXTRACT FROM A LETTER TO MR. C. W. MERRIFIELD.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 87, 88.]

THE general integral of the equations

$$\frac{\alpha}{\beta}=\frac{\beta}{\gamma}=\frac{\gamma}{\delta},$$

$\left[\text{where } \alpha, \beta, \gamma, \delta=\frac{d^3z}{dx^3}, \frac{d^3z}{dx^2dy}, \frac{d^3z}{dx\,dy^2}, \frac{d^3z}{dy^3}\right]$, can, I think, be found, viz. $\frac{\alpha}{\beta}=\frac{\beta}{\gamma}$ gives $r=$ function s, and $\frac{\beta}{\gamma}=\frac{\gamma}{\delta}$ gives $s=$ function t. But $r=$ function s, is integrated as the equation of a developable surface (p instead of z), viz. we have

$$\left.\begin{aligned} p&=ax+hy+g\\ 0&=a'x+\ \ y+g' \end{aligned}\right\},$$

a and g functions of h, and

$$\left(a'=\frac{da}{dh},\quad g'=\frac{dg}{dh}\right);$$

similarly, $s=$ function t, gives

$$q=hx+by+f,$$

$$0=\ \ x+b'y+f',\quad \left(b'=\frac{db}{dh},\quad f'=\frac{df}{dh}\right).$$

Observe that the constants have been so taken, that $\frac{dp}{dy}=h$, $\frac{dq}{dx}=h$; but in order that h may, in the two pairs of equations, mean the same function of (x, y), we must have

$$a'=\frac{1}{b'}=\frac{g'}{f'},$$

that is,

$$b = \int \frac{dh}{a'}, \quad f = \int \frac{g'dh}{a'},$$

or, writing $a = \phi h$, $g = \chi h$, we have

$$p = x\phi h + yh + \chi h,$$

$$q = hx + y\int \frac{dh}{\phi' h} + \int \frac{\chi' h \,.\, dh}{\phi' h},$$

where

$$x\phi' h + y + \chi' h = 0.$$

The last equation gives h as a function of (x, y), and the values of p, q are then such that $dz = pdx + qdy$ is a complete differential, so that we obtain z by the integration of that equation.

A simple example is

$$p = \tfrac{1}{2}h^2 x - hq, \quad q = -hx + y\log h, \quad hx - y = 0,$$

that is,

$$p = -\tfrac{1}{2}\frac{y^2}{x}, \quad q = -y + y\log\frac{y}{x},$$

whence

$$z = \tfrac{1}{2}y^2\log\frac{y}{x} - \tfrac{3}{4}y^2,$$

we have

$$r = \tfrac{1}{2}\frac{y^2}{x^2}, \quad s = -\frac{y}{x}, \quad t = \log\frac{y}{x},$$

$$\alpha = -\frac{y^2}{x^3}, \quad \beta = \frac{y}{x^2}, \quad \gamma = -\frac{1}{x}, \quad \delta = \frac{1}{y},$$

or

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta}\left(= -\frac{y}{x}\right),$$

as it should be.

Cambridge, 28 *July*, 1871.

539.

FURTHER NOTE ON LAGRANGE'S DEMONSTRATION OF TAYLOR'S THEOREM.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 105, 106.]

THIS refers to a paper, Wilkinson, "Note on Taylor's Theorem," *Messenger of Mathematics*, same volume, pp. 36, 37, discussing the "Note on Lagrange's Demonstration of Taylor's Theorem," [536, ante, pp. 493—495].

540.

ON A PROPERTY OF THE TORSE CIRCUMSCRIBED ABOUT TWO QUADRIC SURFACES.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 111, 112.]

THE property mentioned by Mr Townsend in his paper(1) in the August No., "On a Property in the Theory of Confocal Quadrics," may be demonstrated in a form which, it appears to me, better exhibits the foundation and significance of the theorem.

Starting with two given quadric surfaces, the torse circumscribed about these touches each of a singly infinite series of quadric surfaces, any two of which may be used (instead of the two given surfaces) to determine the torse; in the series are included four conics, one of them in each of the planes of the self-conjugate tetrahedron of the two given surfaces; and if we attend to only two of these conics, the two conics are in fact any two conics whatever, and the torse is the circumscribed torse of the two conics; or, what is the same thing, it is the envelope of the common tangent-planes of the two conics.

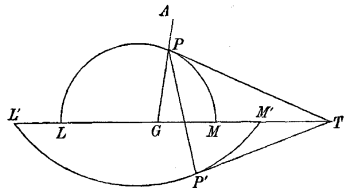
Consider now two conics U, U', the planes of which intersect in a line I; and let I meet U in the points L, M, and meet U' in the points L', M': take A the pole of I in regard to the conic U, and A' the pole of I' in regard to the conic U'.

Take T any point on I, and draw TP touching U in P, and TP' touching U' in P': the points P, P' may be considered as corresponding points on the two conics respectively.

Join AP and produce it to meet the line I in G; the line APG is in fact the polar of T in regard to the conic U (for T being a point on I, the polar of T passes through A; and this polar also passes through P); that is, the points T, G and L, M are harmonics on the line I; whence also, in the plane of the conic U',

[1 *Messenger of Mathematics*, same volume, pp. 49, 50.]

the lines $P'G$, $P'T$ and $P'L$, $P'M$ are harmonic lines through the point P'. It thus appears that in the particular case where the points L, M are the foci of the conic U', the line $P'G$ is the normal at the point P'; and we may say in general that $P'G$ is the quasi-normal at the point P' of the conic U'.



Consider now the torse circumscribed about the conics U, U'; the plane PTP' will represent any plane, and the line PP' any line of this torse: projecting on the plane of U' with the point A as centre of projection, the projection of PP' is the line $P'G$; which, as just seen, is the quasi-normal of the conic U' at the point P'.

The projection of the cuspidal curve is the envelope of line $P'G$, which is the projection of the generating line PP' of the torse—viz. this envelope is the quasi-evolute of the conic U'; which is the theorem in question.

541.

ON THE RECIPROCAL OF A CERTAIN EQUATION OF A CONIC.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 120, 121.]

THE following formula is useful in various problems relating to conics: the reciprocal equation of the conic

$$\lambda(ax+by+cz)(a'x+b'y+c'z)-\mu(a''x+b''y+c''z)(a'''x+b'''y+c'''z)=0$$

may be written indifferent in either of the forms

$$\left\{\lambda\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a, & b, & c\end{vmatrix}+\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a'', & b'', & c''\\ a''', & b''', & c'''\end{vmatrix}\right\}^2+4\lambda\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a, & b, & c\\ a''', & b''', & c'''\end{vmatrix}\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a'', & b'', & c''\end{vmatrix}=0,$$

and

$$\left\{\lambda\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a, & b, & c\end{vmatrix}-\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a'', & b'', & c''\\ a''', & b''', & c'''\end{vmatrix}\right\}^2+4\lambda\mu\begin{vmatrix}\xi, & \eta, & \zeta\\ a, & b, & c\\ a'', & b'', & c'\end{vmatrix}\begin{vmatrix}\xi, & \eta, & \zeta\\ a', & b', & c'\\ a''', & b''', & c'''\end{vmatrix}=0.$$

In fact, in the reciprocal equation, seeking for the coefficient of ξ^2, it is

$$\{\lambda(bc'+b'c)-\mu(b''c'''+b'''c'')\}^2-(2\lambda bb'-2\mu b''b''')(2\lambda cc'-2\mu c''c'''),$$

viz. this is

$$\lambda^2(bc'-b'c)^2+\mu^2(b''c'''-b'''c'')^2+2\lambda\mu\left\{\begin{matrix}2bb'c''c'''+2b''b'''cc'\\ -(bc'+b'c)(b''c'''+b'''c'')\end{matrix}\right\},$$

or, as it may be written,

$$\{\lambda(bc'-b'c)\pm\mu(b''c'''-b'''c'')\}^2+2\lambda\mu\left\{\begin{matrix}2bb'c''c'''+2b''b'''cc'\\-(bc'+b'c)(b''c'''+b'''c'')\\\mp(bc'-b'c)(b''c'''-b'''c'')\end{matrix}\right\}.$$

Taking the upper signs, this is

$$\{\lambda(bc'-b'c)+\mu(b''c'''-b'''c'')\}^2+4\lambda\mu\left(\begin{matrix}bb'c''c'''+b''b'''cc'\\-bc'b''c'''-b'cb'''c''\end{matrix}\right),$$

viz. the term in $\lambda\mu$ is

$$+4\lambda\mu(bc'''-b'''c)(b'c''-b''c').$$

Taking the lower signs, it is

$$\{\lambda(bc'-b'c)-\mu(b''c'''-b'''c'')\}^2+4\lambda\mu\left(\begin{matrix}bb'c''c'''+b''b'''cc'\\-bc'b'''c''-b'cb''c'''\end{matrix}\right),$$

viz. the term in $\lambda\mu$ is

$$+4\lambda\mu(bc''-b''c)(b'c'''-b'''c').$$

And it is thence easy to infer the forms of the other coefficients, and to arrive at the foregoing result.

542.

FURTHER NOTE ON TAYLOR'S THEOREM.

[From the *Messenger of Mathematics*, vol. I. (1872), p. 137.]

THIS paper refers to a "Further Note on Taylor's Theorem," *Messenger of Mathematics*, same volume, pp. 135—137.

543.

ON AN IDENTITY IN SPHERICAL TRIGONOMETRY.

[From the *Messenger of Mathematics*, vol. I. (1872), p. 145.]

In a spherical triangle, writing for shortness α, β, γ for the cosines and α', β', γ' for the sines, of the sides: also

$$\Delta^2 = 1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma\,;$$

we have

$$\cos A = \frac{\alpha - \beta\gamma}{\beta'\gamma'}, \quad \sin A = \frac{\Delta}{\beta'\gamma'},$$

with the like expressions in regard to the other two angles B, C respectively.

Hence

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \&c.$$
$$= \frac{(\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta) - \Delta^2(\alpha+\beta+\gamma-\beta\gamma-\gamma\alpha-\alpha\beta)}{(1-\alpha^2)(1-\beta^2)(1-\gamma^2)}.$$

The numerator is identically

$$= (1-\alpha)(1-\beta)(1-\gamma)[\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)],$$

viz. comparing the two expressions, we have

$$(1-\alpha)(1-\beta)(1-\gamma)\Delta^2 - (1-\alpha^2)(1-\beta^2)(1-\gamma^2)$$
$$= (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta) + \Delta^2(-\alpha-\beta-\gamma+\beta\gamma+\gamma\alpha+\alpha\beta)\,;$$

or, what is the same thing,

$$(1-\alpha\beta\gamma)\Delta^2 = (1-\alpha^2)(1-\beta^2)(1-\gamma^2) + (\alpha-\beta\gamma)(\beta-\gamma\alpha)(\gamma-\alpha\beta),$$

which is the identity in question and can be immediately verified. We have thus

$$\cos(A+B+C) = \frac{\Delta^2 - (1+\alpha)(1+\beta)(1+\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)},$$

and thence

$$1 + \cos(A+B+C) = \frac{\Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)},$$

$$1 - \cos(A+B+C) = \frac{2(1+\alpha)(1+\beta)(1+\gamma) - \Delta^2}{(1+\alpha)(1+\beta)(1+\gamma)},$$

giving at once the values of $\cos^2\frac{1}{2}(A+B+C)$, $\sin^2\frac{1}{2}(A+B+C)$, $\sin(A+B+C)$, and $\tan^2\frac{1}{2}(A+B+C)$: these are known expressions in regard to the spherical excess.

544.

ON A PENULTIMATE QUARTIC CURVE.

[From the *Messenger of Mathematics*, vol. I. (1872), pp. 178—180.]

I HAVE had occasion to consider with some particularity the form of a curve about to degenerate into a system of multiple curves; a simple instance is a trinodal quartic curve about to degenerate into the form $x^2y^2=0$, or say a "penultimate" of $x^2y^2=0$. To fix the ideas, take x, y, z to denote the perpendiculars on the sides of an equilateral triangle, altitude $=1$ (so that $x+y+z=1$), and let the curve be symmetrical in regard to the coordinates x, y, its equation being thus

$$(a, a, 1, f, f, h)(x, y, z)^2=0,$$

where a, f, h are ultimately all indefinitely small in regard to unity: to diminish the number of cases I further assume

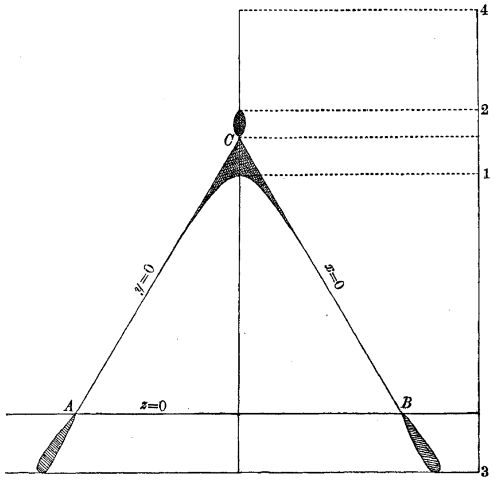
$$\begin{aligned} &a=+, \quad f \text{ and } h=-, \\ &h^2>a^2, \quad \text{that is, } a+h=-, \\ &f>a, \quad \text{,, ,, ,} \quad \sqrt{(a)}+f=-, \end{aligned}$$

but I do not in the first instance take a, f, h to be indefinitely small. Then if $-f$ is not too large, the curve is as shown in the figure[1], viz. it is a triloop curve, with two horizontal double tangents, 3 touching the curve in two real points, 4 touching it in two imaginary points. Imagine $-f$ increased: the new curve will have the same general form, intersecting the first curve at A and B but touching it at C, viz. it will pass inside the loop C but outside the loops A, B; and outside the remainder of the curve; and the 4 will also move downwards as shown. The new position of 4 will be below the first position.

Supposing that a, h have given values, and that $-f$ continually is increased in regard to $\sqrt{(a)}$; two things may happen. First, the double tangent 3 may move down

[1] The figure is drawn with very small values of a, f, h, in order to exhibit as nearly as may be one of the penultimate forms of the curve; but this is not in anywise assumed in the reasoning of the text. Observe in the figure that the points A, B are ordinary double points, and that there are at each of them two distinct tangents inclined at a small angle to each other.

to $z=-\infty$, the lower loops lengthening out and finally becoming each of them a pair of parabolic branches parallel at infinity; and then reappearing at $z=+\infty$, again move downwards, each loop becoming in this case a pair of hyperbolic branches touching two asymptotes at $z=-\infty$, and then again on the opposite sides thereof at $z=+\infty$,



and coming down as a single branch to touch the double tangent 3 which is now above 4. Secondly, the double tangent 4 may come to coincide with the horizontal tangent 2, at the instant of coincidence being a tangent of four-pointic contact; and becoming afterwards (being as before above 2) an ordinary double tangent with two real points of contact; viz. instead of a simple loop at C we have a heart-shaped loop.

But to investigate whether the two cases actually happen, and in what order of succession, we require the expressions of z for the several lines in question; we find, without difficulty,

$$\text{for line 1,}\quad z_1=\frac{1}{1+2\lambda_1},\quad \text{where}\quad \lambda_1=-2f+\sqrt{\{4f^2-2(a+h)\}},$$

$$\text{,, 2,}\quad z_2=\frac{1}{1-2\lambda_2},\quad \text{,,}\quad \lambda_2=2f+\sqrt{\{4f^2-2(a+h)\}},$$

$$\text{,, 3,}\quad z_3=\frac{1}{1-2\lambda_3},\quad \text{,,}\quad \lambda_3=\frac{a-h}{2\{-\sqrt{(a)}-f\}},$$

$$\text{,, 4,}\quad z_4=\frac{1}{1-2\lambda_4},\quad \text{,,}\quad \lambda_4=\frac{a-h}{2\{\sqrt{(a)}-f\}},$$

where λ_1, λ_2, λ_3, λ_4 are all positive. Observe that in the limiting case $-f=\sqrt{(a)}$, where, instead of the loops at A, B, we have cusps; z_1, z_2, and z_4 are (in general)

positive; $\lambda_3 = \infty$, and therefore $z_4 = 0$; that is, the line 3 coincides with AB, ceasing to be a double tangent; there is in this case the one double tangent 4.

First. z_3 becomes infinite for $1-2\lambda_3=0$; that is, $a-h=-\sqrt{(a)}-f$, or $-f=\sqrt{(a)}+(a-h)$; viz. for $-f=\sqrt{(a)}+(a-h)-\epsilon$, we have $z_3=-\infty$, and for $-f=\sqrt{(a)}+(a-h)+\epsilon$, we have $z_3=+\infty$.

Secondly. The lines 4 and 2 will coincide if

$$\lambda_4\left[=\frac{a-h}{2\{\sqrt{(a)}-f\}}\right]=2f+\sqrt{\{4f^2-2(a+h)\}},$$

that is, if

$$\lambda_4(\lambda_4-4f)=-2(a+h),$$

or, substituting for λ_4 its value,

$$(a-h)[(a-h)-8f\{\sqrt{(a)}-f\}]+8(a+h)\{\sqrt{(a)}-f\}^2=0,$$

the condition is

$$\{3a+h-4f\sqrt{(a)}\}^2=0, \text{ or } 4f\sqrt{(a)}=3a+h,$$

(observe that, f having been assumed negative, this implies $-h>3a$). That is, $3a+h$ being $=-$ but not otherwise, the double tangent 4 will, for the value $-f=\frac{-3a-h}{4\sqrt{(a)}}$, come to coincide with the line 2; and for any greater value of $-f$ will be as before above line 2, (being in this case an ordinary double tangent with real points of contact) as appears from the form, $U^2=0$, of the foregoing equation for the determination of f.

The passage of the line 3 to infinity, and the coincidence of the lines 4 and 2 may take place for the same value of f, viz. this will be the case if

$$\sqrt{(a)}+(a-h)=\frac{-3a-h}{4\sqrt{(a)}},$$

that is, if

$$7a+4\sqrt{(a)}+h\{1-4\sqrt{(a)}\}=0 \text{ or } -h=\frac{a\{7+4\sqrt{(a)}\}}{1-4\sqrt{(a)}},$$

or, a being small, for the value $-h=7a$ approximately. If $-h$ is less than the above value, then $\frac{-3a-h}{4\sqrt{(a)}}$ is less than $\sqrt{(a)}+a-h$, or $-f$ increasing from $\sqrt{(a)}$, the coincidence of the lines 2, 4 takes place before the line 3 goes off to infinity: contrarywise, if $-h$ is greater than the above value.

In any form of the curve (i.e. whatever be the value of f in regard to a, h), if we imagine a, h indefinitely diminished, the lines 1, 2 and 4 will continually approach C, and the curve will gather itself up into certain definite portions of the lines $x=0$, $y=0$. Thus any secant through A (not being indefinitely near to the line AC), which meets the curve in real points, will meet it in two points tending to coincide at the intersection of the secant with the line $x=0$; analytically there are always two intersections real or imaginary which (the secant not being indefinitely near the line AC) tend to coincide at the intersection of the secant with the line $x=0$; and we thus see how we ultimately arrive at the line $x=0$ twice repeated; and similarly for the line $y=0$.

545.

ON THE THEORY OF THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 6—12.]

I CONSIDER a differential equation under the form

$$\phi(x, y, p) = 0,$$

where

1°. ϕ is as to p, rational and integral of the degree n;

2°. it is, or is taken to be, one-valued in regard to (x, y);

3°. it has no mere (x, y) factor;

4°. it is indecomposable as regards p.

Considering (x, y) as the coordinates of a point *in plano*, the differential equation determines a system of curves, in general indecomposable, the system depending on a single variable parameter, and such that through each point of the plane there pass n curves.

Such a system is represented by an integral equation

$$f(x, y, c_1, c_2 \ldots c_m) = 0,$$

where

5°. f is rational and integral in regard to the m constants, which constants are connected by an algebraic $(m-1)$ fold relation;

6°. it is, or is taken to be, one-valued in regard to (x, y);

7°. it has no mere (x, y) factor;

8°. it is indecomposable as regards (x, y);

9°. Considering (x, y) as given, the equation $f=0$, together with the $(m-1)$ fold relation between the constants, must constitute a m-fold relation of the order n, that is, must give for the constants n sets of values. We may, if we please, take f to be linear in regard to the constants $c_1, c_2, \ldots, c_m$, and then the condition simply is, that the $(m-1)$ fold relation shall be of the order n.

I give in regard to these definitions such explanations as seem necessary.

2°. A one-valued function of (x, y) is either a rational function, or a function such as $e^x \sin y$, &c., which for any given values whatever, real or imaginary, of the variables, has only one value. A function is taken to be one-valued when, either for any values whatever of the variables, or for any class of values (e.g. all real values), we select for any given values of the variables one value, and attend exclusively to such one value, of the function.

Thus, U a rational function of (x, y), $\phi = p^2 - U = 0$, ϕ is one-valued in regard to (x, y). But if, U not being the square of a rational function, we take $\sqrt{(U)}$ to be a one-valued function (consider it as denoting, say for all real values of x, y for which U is positive, the positive square-root of U), then, $\phi = p + \sqrt{(U)} = 0$, ϕ is taken to be one-valued in regard to (x, y).

3°. The meaning is, that the equation $\phi = 0$ is not satisfied irrespectively of the value of p, by any relation between the variables x, y.

4°. The meaning is that ϕ is not the product of two factors, each rational and integral in regard to p, and being or being taken to be one-valued in regard to (x, y). Thus, if as before U is a rational function of (x, y) but is not the square of a rational function, and if we do not take any one-valued function, then the equation $\phi = p^2 - U = 0$ is indecomposable; but if we take $\sqrt{(U)}$ as one-valued, then we have $\phi = \{p - \sqrt{(U)}\}\{p + \sqrt{(U)}\} = 0$, and the equation breaks up into the two equations $p - \sqrt{(U)} = 0$ and $p + \sqrt{(U)} = 0$. I assume as an axiom, that the curves represented by the indecomposable differential equation are in general indecomposable; for supposing the differential equation satisfied in regard to a system of curves, the general curve breaking into two curves, each depending on the arbitrary parameter, then we have two distinct systems of curves; either the differential equation is satisfied in regard to each system separately, and in this case they are the same system twice repeated; or the differential equation is satisfied in regard to one system only, and in this case the other system is not part of the solution, and it is to be rejected. As an instance, take the equation $\phi = px + y = 0$, $(x\,dy + y\,dx = 0)$; if we choose to integrate this in the form $x^2y^2 - c = 0$, this equation represents the two hyperbolas $xy + \sqrt{(c)} = 0$, $xy - \sqrt{(c)} = 0$, but considering each of these separately, and giving to the constant theory any value whatever, we have simply the system of hyperbolas $xy - c = 0$ twice repeated. But if by any (faulty) process of integration the solution had been obtained in a form such as $(c+x)(c-xy)=0$, then the differential equation is not satisfied in regard to the system $c + x = 0$; and the factor $c + x$ is to be rejected. Observe that it is said, that the curves are *in general* indecomposable; particular curves of the system may very well be decomposable; thus in the foregoing example, where the system of curves is $xy - c = 0$, in the particular case $c = 0$, the hyperbola breaks up into the two lines $x = 0$, $y = 0$.

5°. It is *necessary* to consider a form $f=0$ involving the m constants connected by the $(m-1)$ fold relation. For taking such a system of constants, imagine an equation $f(x, y, c_1, c_2, \ldots, c_m)=0$ rational and integral in regard to (x, y), and representing an indecomposable curve; such an integral equation leads to a differential equation of the form $\phi(x, y, p)=0$, rational in regard to (x, y); whence, conversely, a differential equation of the form last referred to *may have* for its integral the equation $f(x, y, c_1, c_2, \ldots, c_m)=0$. And we cannot *in a proper form* exhibit this integral in terms of a single constant. For first consider for a moment $c_1, c_2, \ldots, c_m$ as the coordinates of a point in m-dimensional space; the curve is not in general unicursal, and unless it be so, we cannot express the quantities $c_1, c_2, \ldots, c_m$ rationally in terms of a parameter; that is, we cannot in general express $c_1, c_2, \ldots, c_m$ rationally in terms of a parameter. Secondly, if by means of the $(m-1)$ fold relation we sought to eliminate from the equation $f=0$ all but one of the m constants, we should indeed arrive at an equation $F(x, y, c)=0$ rational and integral as regards x and y, and also as regards c; but this equation *would not represent an indecomposable curve.*

6°. It is important to remark that, even in the case where $\phi(x, y, p)$ is one-valued in regard to (x, y) (2°), there is not in every case a form $f=0$ one-valued in regard to (x, y). A simple example shows this; let α, β be incommensurable (e.g. $\alpha=e$, $\beta=\pi$), then the equation $\phi=\beta xp+\alpha y=0\,(\alpha y\,dx+\beta x\,dy=0)$ has for its integral $c=x^\alpha y^\beta$, where $x^\alpha y^\beta$ is not a one-valued function of x, y, and we cannot in any way whatever transform the integral so as to express it in terms of one-valued functions of x and y. But taking $x^\alpha y^\beta$ to be a one-valued function—if e.g. for all real values of x, y we consider $x^\alpha y^\beta$ as representing the real value of $(\pm x)^\alpha(\pm y)^\beta$—, we shall have, without any loss of generality, the integral of the differential equation; the *whole* system of curves $c=x^\alpha y^\beta$, c any value whatever, is the same whether we attribute to $x^\alpha y^\beta$ its infinite series of values, or only one of these values.

7°. The meaning is, that the equation $f=0$ is not satisfied irrespectively of the values of $c_1, c_2, \ldots, c_m$ by any relation between x, y only.

8°. The meaning is, that the function f is not the product of two factors, rational or irrational in regard to $c_1, c_2, \ldots, c_m$, but each of them one-valued or taken to be one-valued in regard to x, y. Thus the function

$$f=x^2y^2-c, \quad =\{xy+\sqrt{(c)}\}\{xy-\sqrt{(c)}\},$$

is decomposable; but if we do not take any one-valued function, then $f=xy-c$ is indecomposable; if we take $\sqrt{(xy)}$ to be one-valued, then it is decomposable.

The case $f=x^\alpha y^\beta-c$ (α and β incommensurable) is to be noticed; starting from the differential equation $\alpha y\,dx+\beta x\,dy=0$, there is no reason for writing the integral $f=x^\alpha y^\beta-c=0$ rather than in either of the forms $f=x^{m\alpha}y^{m\beta}-c=0$, $f=x^{\frac{\alpha}{m}}y^{\frac{\beta}{m}}-c=0$, ($m$ an integer), (α, β), $(m\alpha, m\beta)$, $\left(\frac{\alpha}{m}, \frac{\beta}{m}\right)$ are, each pair as well as the others, two incommensurable magnitudes. If we choose to take $x^\alpha y^\beta$ but not $x^{\frac{\alpha}{m}}y^{\frac{\beta}{m}}$ as one-valued,

then $f = x^{\alpha}y^{\beta} - c$ and $f = x^{m\alpha}y^{m\beta} - c$ are each one-valued, and the former is, the latter is not, indecomposable; they would be each decomposable if we chose to take $x^{\frac{\alpha}{m}}y^{\frac{\beta}{m}}$ as one-valued; and if only $x^{m\alpha}y^{m\beta}$ were taken to be one-valued, then $f = x^{m\alpha}y^{m\beta} - c$ would be indecomposable.

9°. This is a mere statement of the condition in order that the system of curves represented by the integral equation may be such that, through a given point of the plane, there may pass n of these curves. Since the number of constants is unlimited, there is clearly no loss of generality in assuming that the equation is linear in regard to the several constants.

I consider now the differential equation

$$\phi(x, y, p) = 0,$$

(as already stated of the degree n as regards p), and its integral equation

$$f(x, y, c_1, c_2, \ldots, c_m) = 0.$$

I take (x, y) to be the coordinates of a point, say in the horizontal plane, and I use C to refer to the constants $c_1, c_2, \ldots, c_m$ collectively, thus for given values of x, y I speak of the n values of C, meaning thereby the n values of the set $(c_1, c_2, \ldots, c_m)$; of C having a two-fold value, meaning thereby that two of the sets $c_1, c_2, \ldots, c_m$ become identical; and so on.

The case $C = c$, where there is only a single constant c, is interesting as affording an easier geometrical conception; we may take $c = z$ to be a third coordinate; the equation $f(x, y, z) = 0$ thus represents a surface, such that its plane sections $z = c$, or say these sections projected by vertical ordinates on the horizontal plane $z = 0$ are the series of curves $f(x, y, c) = 0$. But the case is not really distinct from the general one.

The theory of singular solutions depends on the following considerations:

To a given point P on the horizontal plane belong n values of C, each determining a curve $f(x, y, C) = 0$ through P; and also n values of p, viz. these give the directions at P of the n curves respectively.

The curve $f(x, y, C) = 0$ may be such as to have in general a certain number of nodes and of cusps (either or each of these numbers being $= 0$): we may imagine C determined, say $C = C_0$, so that the curve shall have one additional node: this node I call a "level point." Take P at the level point, there are n values of C, viz. C_0 and $\overline{n-1}$ other values; that is, there are through P the nodal curve, and $n-1$ other curves, and therefore $2 + (n-1), = n+1$ directions of the tangent; but the directions are determined by the equation $\phi = 0$ of the order n: and the only way in which we can have more than n values is when this equation becomes an identity $0 = 0$; that is, P at the level point, the function $\phi(x, y, p)$ will vanish identically, irrespectively of the value of p.

The ordinary nodes (if any) on the curves $f(x, y, C)=0$ form a locus called the "nodal locus," and the cusps (if any) a locus called the "cuspidal locus." Take the point P a given point on the nodal locus, C has a two-fold value answering to the curve in regard to which P is a node, and $n-2$ other values; that is, the n curves through P are the nodal curve reckoned twice and $n-2$ other curves; the directions are the directions at the node, and the $n-2$ other directions, in all n directions, which are the directions given by the equation $\phi=0$; there is no peculiarity in regard to this equation.

Similarly, take P anywhere on the cuspidal locus: C has a two-fold value, answering to the curve in regard to which P is a cusp, and $n-2$ other values; that is, the n curves through P are the cuspidal curve reckoned twice and $n-2$ other curves; the directions are the direction at the cusp reckoned twice and $n-2$ other directions: in all n directions, which are the directions given by the equation $\phi=0$; this equation thus gives a two-fold value of p.

There is a locus (distinct from the nodal and cuspidal loci) which may be called the "envelope locus," such that taking P anywhere on this locus C has a two-fold value; for such position of P the n values of C are the value in question reckoned twice and $n-2$ other values; the n curves through P are that belonging to the two-fold value of C, or say the two-fold curve, and $n-2$ other values; and the n directions are the direction along the two-fold curve counted twice, and $n-2$ other directions; these are the n directions given by the equation $\phi=0$, viz. this equation gives a two-fold value of p.

The envelope locus may be an indecomposable curve, or it may break up into two or more curves; and it may happen that either the whole curve or one or more of the component curves may coincide with a particular curve or curves of the system $f(x, y, C)=0$.

There is a locus (distinct from the cuspidal and envelope loci) which may be called the tac-locus, such that taking P anywhere on this locus p has a two-fold value; for such position of P, there is no peculiarity as regards C, viz. C has n distinct values giving rise to n curves through P; but as the directions are given by the equation $\phi=0$, two of the curves touch each other, viz. the tac-locus is the locus of points, such that at any one of them two of the curves $f(x, y, C)=0$ through the point touch each other.

We may by an extension of the received notation write

$$\operatorname{disct}_C f(x, y, C)=0$$

to denote the equation between (x, y), such that, for any values of (x, y), which satisfy the condition, or say for any position of P on the C-discriminant locus, there is a two-fold value of C. By what precedes it appears that the C-discriminant locus is made up of the nodal, cuspidal, and envelope loci, and without going into the proof I infer that it is in fact made up of the nodal locus twice, the cuspidal locus three times, and the envelope locus once.

Writing moreover

$$\text{disct}_p\, \phi\,(x,\ y,\ p) = 0$$

to denote the equation between (x, y), such that for any values of (x, y) which satisfy the condition, or say for any position of P on the p-discriminant locus, there is a two-fold value of p. By what precedes, it appears that the p-discriminant locus is made up of the envelope locus, cuspidal locus, and the tac-locus; as I infer, each of them once.

The foregoing are the abstract principles: I consider the singular solution to be that given by the equation which belongs to the envelope-locus (viz. *I do not recognise any singular solution which is not of the envelope species*); and the result of the investigation is, when we seek in the ordinary way to obtain the singular solution, whether from the integral equation or from the differential equation, that we account for the extraneous factors which present themselves in the two processes respectively. I reserve for another communication the discussion of particular examples.

546.

THEOREMS IN RELATION TO CERTAIN SIGN-SYMBOLS.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 17—20.]

I FIND the following among my papers:

Let the latin letters $a, b, \ldots$ denote *lines* of n signs $\pm$, and the greek letters $\alpha, \beta, \ldots$ *columns* of the same number n of signs $\pm$; two symbols of the same kind are multiplied together by multiplying their corresponding terms, the product being thus a symbol of the same kind; in particular, the product of a symbol by itself, or square of a symbol, is a line (or column as the case may be) of +'s: and the symbol itself is thus a square root of a line (or column) of +'s. Thus n being $=5$, we say that the latin letters denote roots of $+ + + + +$ and the greek letters roots of

$$\begin{matrix} + \\ + \\ + \\ + \\ +. \end{matrix}$$

The roots a, b, c, d, e will be *independent* if no one of them is equal to the product of all or any of the others; and, this being so, the 32 roots are the terms of

$$(1+a)(1+b)(1+c)(1+d)(1+e);$$

it follows that, for any other system of independent roots a', b', c', d', e', we have

$$(1+a')(1+b')(1+c')(1+d')(1+e') = (1+a)(1+b)(1+c)(1+d)(1+e):$$

and conversely if either system be independent and this equation is satisfied, then the other system is also independent.

In particular a, b, c, d, e being independent, then a, b, c, bd, e (viz. any term d is replaced by its product by some other term b) is also independent; and by a similar transformation on the new series a, b, c, bd, e, and so on in succession we can pass from a given independent system a, b, c, d, e to *any other independent system whatever.*

A similar but more general theorem is the following: let a, b, c, d, e be independent, and l be equal to the product of all or any of these roots, but so that as regards, suppose (b, c, e), the number of these factors contained in l is even (or may be $=0$), e.g. l is $=a$, or abc, &c., but it is not $=ab$, or bce, &c.

Then a, lb, lc, d, le is an independent system: to show this we must show that

$$\overline{1+a}\,\overline{1+b}\,\overline{1+c}\,\overline{1+d}\,\overline{1+e}=\overline{1+a}\,\overline{1+lb}\,\overline{1+lc}\,\overline{1+d}\,\overline{1+le},$$

that is,

$$\overline{1+a}\,\overline{1+d}\,(\overline{1+lb}\,\overline{1+lc}\,\overline{1+le}-\overline{1+b}\,\overline{1+c}\,\overline{1+e})=0,$$

that is,

$$\overline{1+a}\,\overline{1+d}\,(\overline{l-1}\,\overline{b+c+e}+\overline{l^2-1}\,\overline{bc+be+ce}+\overline{l^3-1}\,bce)=0,$$

or, since $l^2=1$, $l^3=l$, this is

$$(1+a)(1+d)(l-1)(b+c+e+bce)=0,$$

which is easily verified under the assumed conditions as to l, e.g. $l=abc$,

$$(1+a)\,l=abc+a^2bc=abc+bc=(1+a)\,bc,$$

$$(1+a)(l-1)=(1+a)(bc-1),$$

and the equation is

$$(1+a)(1+d)(bc-1)(b+c+e+bce)=0;$$

and we in fact have

$$bc\,(b+c+e+bce)=c+b+ebc+e;$$

that is,

$$(bc-1)(b+c+e+bce)=0.$$

The proof is obviously quite general.

All that precedes applies also to the columns.

Now consider a square of 5×5 signs $\pm$; I say that, if this is *independent as to its lines,* it will be also *independent as to its columns.*

To prove this consider any particular square, say

	α	β	γ	δ	ϵ
a					
b	+	−	−	+	−
c					
d					
e					

independent as to its lines, and also independent as to its columns: I derive from this the square

	α	$\theta\beta$	$\theta\gamma$	δ	$\theta\epsilon$
a					
b					
c					
bd					
e					

viz. in the new square the line d is replaced by bd, the designation of the columns will be presently explained. This new square is, by what precedes, independent as to its lines; we have to show that it is also independent as to its columns.

As regards the columns, any column is either unchanged or it is changed in its fourth place only, according as the sign in b is for that column $+$ or $-$; that is, if we write $\theta = +$, the columns of the new square are (as above written down) α, $\theta\beta$, $\theta\gamma$, δ, $\theta\epsilon$;

$+$

$+$

$-$

$+$

and θ is a product of all or some of the original columns α, β, γ, δ, ϵ: but as regards β, γ, ϵ it contains an even number (or it may be 0) of these factors; for otherwise the sign in the second line of θ instead of being $+$ would be $-$. But these are the very conditions that show that the columns α, $\theta\beta$, $\theta\gamma$, δ, $\theta\epsilon$ are independent.

Hence starting from the square

$-$	$+$	$+$	$+$	$+$
$+$	$-$	$+$	$+$	$+$
$+$	$+$	$-$	$+$	$+$
$+$	$+$	$+$	$-$	$+$
$+$	$+$	$+$	$+$	$-$

which obviously is independent as to its lines and also as to its columns; and transforming as above any number of times in succession, we obtain ultimately a square which has for its lines *any system* whatever of independent roots, and by what precedes each of the new squares is also independent as to its columns; that is, every square independent as to its lines is also independent as to its columns. Q.E.D.

547.

ON THE REPRESENTATION OF A SPHERICAL OR OTHER SURFACE ON A PLANE: A SMITH'S PRIZE DISSERTATION.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 36, 37.]

IN the Smith's Prize Examination for 1871 I set as the subject for a dissertation:

The representation of a spherical or other surface on a plane.

I give the following as a specimen of the sort of answer required: an answer which, without so much as noticing that projection (in its restricted sense) is only one kind of representation, goes into the details of the constructions for the different projections of the sphere, and even into the demonstrations of these constructions, errs quite as much by excess as by defect, and is worth very little indeed.

The question is understood to refer to Chartography, viz. the kind of representation is taken to be such as that of a hemisphere or other portion of the earth's surface in a map.

An implied condition is that each point of the surface (viz. of the portion thereof comprised in the map) shall be represented by a single point on the map; and conversely, that each point on the map shall represent a single point on the surface. And further, any closed curve on the surface must be represented by a closed curve on the map, and the points within the one by the points within the other. If for shortness the term element is used to denote an infinitesimal area included within a closed curve, we may say that each element of the surface must be represented by an element of the map; and conversely, each element of the map must represent an element of the surface.

A map would be perfect if each element of the surface and the corresponding element of the map were of the same form, and were in a constant ratio as to magnitude; say if it were free from the defects of "distortion" and "inequality" (of scale); the condition as to form, or freedom from distortion, may be otherwise expressed by saying that any two contiguous elements of length on the surface and the corresponding two contiguous elements of length on the map must meet at the same angle (this at once appears by taking the two elements of area to be each of them a triangle). But for a spherical or other non-developable surface, it is not possible to construct a map free from the two defects.

An obvious and usual kind of representation is that by projection: viz. taking any fixed point and plane, the line joining any point P of the surface with the fixed point meets the fixed plane in a point P' which is taken to be the representation of the point P on the surface.

When the surface is a sphere the projection is called orthographic, gnomonic or stereographic according to the positions of the fixed point and plane: the last kind is here alone considered; viz. in the stereographic projection the fixed point is on the surface of the sphere, and the fixed plane is parallel to the tangent plane at that point, and is usually and conveniently taken to pass through the centre of the sphere.

The stereographic projection is one of those which is free from the defect of distortion; it is consequently, and that in a considerable degree, subject to the defect of inequality. It possesses in a high degree the important quality of facility of construction, viz. any great or small circle on the sphere is represented by a circle in the map; and from the general property of the equality of corresponding angles, or otherwise, there arise easy rules for the construction of such circles.

The so-called Mercator's projection is an instance of a representation which is not in the above restricted sense a projection; and which is free from the defect of distortion: viz. the (equidistant) meridians are here represented by a system of (equidistant) parallel lines; and the parallels of latitude by a set of lines at right angles thereto: the distance between consecutive parallels in the map being taken in such wise as is required to obtain freedom from distortion; for this purpose the increments of latitude and longitude must have in the map the same ratio that they have on the sphere, and since in the map the length of a degree of longitude (instead of decreasing with the latitude) remains constant, the lengths of the successive degrees of latitude in the map must increase with the latitude: the scale of the representation thus increases with the latitude, and would for the latitude $\pm 90°$ become infinite.

There is a simple representation of a hemisphere, due to M. Babinet, in which the defect of inequality is avoided, viz. the meridians are represented by ellipses having their major axes coincident with the diameter through the poles and dividing the equator into equal distances, and the parallels by straight lines parallel to the equator.

548.

ON LISTING'S THEOREM.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 81—89.]

LISTING'S theorem, (established in his Memoir*, *Die Census räumlicher Gestalten*), is a generalisation of Euler's theorem $S+F=E+2$, which connects the number of summits, faces, and edges in a polyhedron; viz. in Listing's theorem we have for a figure of any sort whatever

$$A-B+C-D-(p-1)=0,$$

or, what is the same thing,

$$A+C=B+D+(p-1),$$

where

$$\begin{aligned} A &= a, \\ B &= b-\kappa', \\ C &= c-\kappa''+\pi, \\ D &= d-\kappa''', \end{aligned}$$

in which theorem a relates to the points; b, κ' relate to the lines; c, κ'', π to the surfaces; d, κ''' to the spaces; and p relates to the detached parts of the figure, as will be explained.

a is the number of points; there is no question of multiplicity, but a point is always a single point. A point is either detached or situate on a line or surface.

b is the number of lines (straight or curved). A line is always finite, and if not reentrant there must be at each extremity a point: no attention is paid to cusps, inflexions, &c., and if the line cut itself there must be at each intersection a point;

* *Gött. Abh.* t. x. (1862).

and in general a point placed on a line constitutes a termination or boundary of the line. Thus a line is either an oval (that is, a non-intersecting closed curve of any form whatever), a punctate oval (oval with a single point upon it), or a biterminal (line terminated by two distinct points). For instance, a figure of eight is taken to be two punctate ovals; an oval, placing upon it two points, is thereby changed into two biterminals.

κ'. The definition, analogous to the subsequent definitions of κ'' and κ''', would be that κ' is the sum, for all the lines, of the number of circuits for each line; but inasmuch as for an oval the number of circuits is $=1$, and for any other line (punctate oval, or biterminal) it is $=0$, κ' is in fact the number of ovals.

c is the number of surfaces. A surface is always finite, and if not reentrant there must be at every termination thereof a line: no attention is paid to cuspidal lines, &c., and if the surface cut itself there must be at each intersection a point or a line; and in general a point or a line placed on a surface constitutes a termination or boundary thereof. It may be added that if a line intersects a surface there must be at the intersection a point, constituting a termination or boundary as well of the line as of the surface. Thus a surface is either an ovoid (simple closed surface, such as the sphere or the ellipsoid), a ring (surface such as the torus or anchor-ring), or other more complicated form of reentrant surface; or else it is a surface in part bounded by a point or points, line or lines. We may in particular consider a blocked surface having upon it one or more blocks: where by a block is meant a point, line, or connected superficial figure composed of points and lines in any manner whatever, the superficial area (if any) included within the block being disregarded as not belonging to the surface, or being, if we please, cut out from the surface. Thus an ovoid having upon it a point, and a segment or incomplete ovoid bounded by an oval, are each of them to be regarded as a one-blocked ovoid; the boundary being in the first case the point, and in the second case the oval; and so in general the blocked surface is bounded by the boundary or boundaries of the block or blocks. It will be understood from what precedes, and it is almost needless to mention, that for any surface we can pass along the surface from each point to each point thereof; any line which would prevent this would divide the surface into two or more distinct surfaces.

κ'' is the sum, for the several surfaces, of the number of circuits on each surface. The word circuit here signifies a path on the surface from any point to itself: all circuits which can by continuous variation be made to coincide being regarded as identical; and the evanescible circuit reducible to the point itself being throughout disregarded. Moreover, we count only the simple circuits, disregarding circuits which can be obtained by any repetition or combination of these. Thus for an ovoid, or for a one-blocked ovoid, there is only the evanescible circuit, that is, no circuit to be counted; but for a two-blocked ovoid there is besides one circuit, or we count this as one; and so for a n-blocked ovoid we count $n-1$ circuits. For a ring it is easy to see that (besides the evanescible circuit) there are, and we accordingly count, two circuits; and so in other cases.

π. It might be possible to find an analogous definition, but the most simple one is that π denotes the number of ovoids (unblocked ovoids) or other surfaces not bounded by any point or line.

d is the number of spaces, reckoning as one of them infinite space.

κ''' is the sum, for the several spaces, of the number of circuits in each space: the word circuit here signifying a path in the space from a point to itself; all circuits which can by continuous variation be made to coincide being considered as identical, and the evanescible circuit reducible to the point itself being throughout disregarded. Moreover, we count only the simple circuits, disregarding circuits which can be obtained by a repetition or combination of these. Thus for infinite space, or for the space within an ovoid, there is only the evanescible circuit, or there is no circuit to be counted; and the same is the case if within such space we have any number of ovoidal blocks (the term will, I think, be understood without explanation); but if within the space we have an oval, ring, or other ring-block of any kind whatever, then there is (besides the evanescible circuit) a circuit interlacing the ring-block, and we count one circuit; and so if there are n ring-blocks, either separate or interlacing each other in any manner, then there are, and we accordingly count, n circuits. So for the space inside a ring there is (besides the evanescible circuit), and we count, one circuit; and the case is the same if we have within the ring any number of ovoidal blocks whatever; but if there is within the ring an oval ring or other ring-block, then there is one new circuit, and we count in all (for the space in question) two circuits.

p is the number of detached parts of the figure; or, say the number of detached aggregations of points, lines, and surfaces. Observe, that rings interlacing each other in any manner (but not intersecting) are considered as detached; so also two closed surfaces, one within the other, are considered as detached. The figure may be infinite space alone; we have then $p=0$.

The examples which follow will further illustrate the meaning of the terms and nature of the theorem; and will also indicate in what manner a general demonstration of the theorem might be arrived at.

1. Infinite space.

$$\begin{array}{llll}
a=0, & & & A=0, \\
b=0, & \kappa'=0, & & B=\ \ 0 \\
c=0, & \kappa''=0, & \pi=0, & C=0, \\
d=1, & \kappa'''=0, & & D=\ \ 1 \\
p=0, & & & p-1=-1 \\
\hline
 & & & 0\ \ =\ \ 0.
\end{array}$$

2. Spherical surface.

$$\begin{array}{llll}
a=0, & & A=0, & \\
b=0, & \kappa'=0, & & B=0 \\
c=1, & \kappa''=0, \quad \pi=1, & C=2, & \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & =2:
\end{array}$$

viz. the effect is to increase C by 2 and D and $p-1$ each by 1.

3. Spherical surface, with point upon it.

$$\begin{array}{llll}
a=1, & & A=1, & \\
b=0, & \kappa'=0, & & B=0 \\
c=1, & \kappa''=0, \quad \pi=0, & C=1, & \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & =2:
\end{array}$$

viz. the effect is to increase a and diminish π each by 1; that is, A is increased and C diminished each by 1.

4. Spherical surface with two points.

$$\begin{array}{llll}
a=2, & & A=2, & \\
b=0, & \kappa=0, & & B=0 \\
c=1, & \kappa''=1, \quad \pi=0, & C=0, & \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & =2:
\end{array}$$

viz. the second point increases a and κ'' each by 1, that is, it increases A and diminishes C each by 1.

And for each new point on the spherical surface there is this same effect; so that we have, for the next case:

5. Spherical surface with n points ($n \geqq 2$).

$$\begin{array}{llll}
a=n, & & A=\quad n, & \\
b=0, & \kappa=0, & & B=0 \\
c=1, & \kappa''=n-1, \quad \pi=0, & C=2-n, & \\
d=2, & \kappa'''=0, & & D=2 \\
p=1, & & & p-1=0 \\
\hline
 & & 2 & =2.
\end{array}$$

Imagine that besides the n points there is an aperture (bounded by a closed curve); the case is:

6. Spherical surface with n points ($n \overset{=}{>} 2$) and aperture.

$$\begin{array}{llll}
a = n, & & A = \quad n, & \\
b = 1, & \kappa \ = 1 \quad , & & B = 0 \\
c = 1, & \kappa'' = n \quad , \quad \pi = 0, & C = 1 - n, & \\
d = 1, & \kappa''' = 0 \quad , & & D = 1 \\
p = 1, & & & p - 1 = 0 \\
\hline
 & & 1 & = 1:
\end{array}$$

viz. b and κ are each increased by 1, and therefore B is unaltered; κ'' is increased, and therefore C is diminished, by 1; but d is diminished, and therefore D also diminished, by 1.

7. Spherical surface with n points ($n \overset{=}{>} 2$) and two apertures.

$$\begin{array}{llll}
a = n, & & A = \quad n, & \\
b = 2, & \kappa \ = 2 \quad , & & B = 0 \\
c = 1, & \kappa'' = n + 1, \quad \pi = 0, & C = \ - n, & \\
d = 1, & \kappa''' = 1 \quad , & & D = 0 \\
p = 1, & & & p - 1 = 0 \\
\hline
 & & 0 & = 0:
\end{array}$$

viz. b and κ are each increased by 1, and thus B is still unaltered; κ'' is increased, and therefore C diminished, by 1; κ''' is increased, and therefore D diminished, by 1, and each new aperture produces the like effect. Thus we have:

8. Spherical surface with m apertures ($m \overset{=}{>} 2$).

$$\begin{array}{llll}
a = 0\,, & & A = 0, & \\
b = m, & \kappa' \ = m \quad , & & B = 0 \\
c = 1\,, & \kappa'' = m - 1, \quad \pi = 0, & C = 2 - m, & \\
d = 1\,, & \kappa''' = m - 1, & & D = 2 - m \\
p = 1\,, & & & p - 1 = 0 \\
\hline
 & & 2 - m & = 2 - m\,;
\end{array}$$

where, comparing with case 5, we see the different effects of a point and an aperture.

9. Spherical surface with m apertures ($m \overset{=}{>} 2$) and a point or points on each or any of the bounding curves of the aperture.

If on the bounding curve of any aperture we place a point, this increases a, and therefore A, by 1; the bounding curve is no longer a simple closed curve, and we

thus also have κ diminished, and therefore B increased by 1; and the balance still holds.

Placing on the same bounding curve a second point, a, and therefore A, is increased by 1; but the bounding curve is converted into two distinct curves; that is, b, and therefore B, is increased by 1; and the balance still holds. And the like for each new point on the same bounding curve.

10. Spherical surface with n points connected in any manner by lines.

Reverting to the cases 4 and 5, by joining any two points by a line, we increase b, and therefore B, by 1; but as regards κ'' the two united points take effect as a single point; that is, κ'' is diminished, and therefore C increased, by 1; the balance is therefore undisturbed.

The case is the same for each new line, if only we do not thereby produce on the surface a closed polygon, or partition an existing closed polygon; in each of these cases we still increase b, and therefore B, by 1; and instead of diminishing κ'', we increase c, by 1, and therefore still increase C by 1; and the balance continues to subsist.

By continuing to join the several points we at last arrive at a spherical surface partitioned into polygons in any manner whatever; or, what is the same thing, we have:

11. Closed polyhedral surface. Here, if S is the number of summits, F the number of faces, E the number of edges; then

$$\begin{array}{llll}
a = S, & & A = S, & \\
b = E, & \kappa = 0, & & B = E \\
c = F, & \kappa'' = 0, \quad \pi = 0, & C = & F, \\
d = 2, & \kappa''' = 0, & & D = \quad 2 \\
p = 1, & & & p - 1 = \quad 0 \\
\hline
 & & S + F & = E + 2,
\end{array}$$

so that we have Euler's theorem. Observe that this theorem (Euler's) does not apply to annular polyhedral surfaces, or to polyhedral shells. For instance, consider a shell, the exterior and interior surfaces of which are each of them a closed polyhedral surface; $S = S' + S''$, $F = F' + F''$, $E = E' + E''$, where $S' + F' = E' + 2$, $S'' + F'' = E'' + 2$, and therefore $S + F = E + 4$. Listing's theorem, of course, applies, viz. we have

12.
$$\begin{array}{lll}
a = S' + S'', & A = S' + S'', & \\
b = E' + E'', & B = & E' + E'' \\
c = F' + F'', & C = F' + F'', & \\
d = 3, & D = & 3 \\
p = 2, & p - 1 = & 1 \\
\hline
 & S + F & = E' + E'' + 4.
\end{array}$$

As another group of examples, consider a plane rectangle, for instance, a sheet of paper bounded by its four edges; here

13.
$$\begin{array}{llll} a=4, & & A=4, & \\ b=4, & \kappa'=0, & & B=4 \\ c=1, & \kappa''=0, \quad \pi=0, & C=1, & \\ d=1, & \kappa'''=0, & & D=1 \\ p=1, & & & p-1=0 \\ \hline & & 5 & =5. \end{array}$$

Let the paper be formed into a tube by uniting two opposite sides, the suture not being obliterated, but continuing as a line drawn lengthwise from one extremity of the tube to the other: here

14.
$$\begin{array}{llll} a=2, & & A=2, & \\ b=3, & \kappa'=0, & & B=3 \\ c=1, & \kappa''=0, \quad \pi=0, & C=1, & \\ d=1, & \kappa'''=1, & & D=0 \\ p=1, & & & p-1=0 \\ \hline & & 3 & =3. \end{array}$$

Let the suture be obliterated, so that we have simply a tube open at each end; here

15.
$$\begin{array}{llll} a=0, & & A=0, & \\ b=2, & \kappa'=2, & & B=0 \\ c=1, & \kappa''=1, \quad \pi=0, & C=0, & \\ d=1, & \kappa'''=1, & & D=0 \\ p=1, & & & p-1=0 \\ \hline & & 0 & =0. \end{array}$$

Let the tube be formed into an annulus by bending it round and joining the two extremities, the suture not being obliterated, but continuing as a closed curve round the tube; here

16.
$$\begin{array}{llll} a=0, & & A=0, & \\ b=1, & \kappa'=1, & & B=0 \\ c=1, & \kappa''=1, \quad \pi=0, & C=0, & \\ d=2, & \kappa'''=2, & & D=0 \\ p=1, & & & p-1=0 \\ \hline & & 0 & =0. \end{array}$$

Let the suture be obliterated, so that we have simply a tubular annulus; here

17.
$$\begin{array}{llll} a=0, & & & A=0, \\ b=0, & \kappa'=0, & & B=0 \\ c=1, & \kappa''=2, & \pi=1, & C=0, \\ d=2, & \kappa'''=2, & & D=0 \\ p=1, & & & p-1=0 \\ \hline & & & 0 \quad =0. \end{array}$$

We may compare herewith the case of a simple annulus or closed curve.

18.
$$\begin{array}{llll} a=0, & & & A=0, \\ b=1, & \kappa'=1, & & B=0 \\ c=0, & \kappa''=0, & \pi=0, & C=0, \\ d=1, & \kappa'''=1, & & D=0 \\ p=1, & & & p-1=0 \\ \hline & & & 0 \quad =0. \end{array}$$

Add to such an annulus, for instance, three radii meeting in the centre; then

19.
$$\begin{array}{llll} a=4, & & & A=4, \\ b=6, & \kappa'=0, & & B=6 \\ c=0, & \kappa''=0, & \pi=0, & C=0, \\ d=1, & \kappa'''=3, & & D=-2 \\ p=1, & & & p-1=0 \\ \hline & & & 4 \quad =4. \end{array}$$

Let the last-mentioned figure become tubular, all sutures being obliterated; then

20.
$$\begin{array}{llll} a=0, & & & A=0, \\ b=0, & \kappa'=0, & & B=0 \\ c=1, & \kappa''=6, & \pi=1, & C=-4, \\ d=2, & \kappa'''=6, & & D=-4 \\ p=1, & & & p-1=0 \\ \hline & & & -4 \quad =-4. \end{array}$$

And so if instead of the tubular figure, annulus with three radii, we had a tubular figure, annulus with diameter, then

21.
$$\begin{array}{llll} a=0, & & & A=0, \\ b=0, & \kappa'=0, & & B=0 \\ c=1, & \kappa''=4, & \pi=1, & C=-2, \\ d=2, & \kappa'''=4, & & D=-2 \\ p=1, & & & p-1=0 \\ \hline & & & -2 \quad =-2; \end{array}$$

and the like in other cases.

549.

NOTE ON THE MAXIMA OF CERTAIN FACTORIAL FUNCTIONS.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 129, 130.]

I CONSIDER the functions

$$\begin{aligned}
y_1 &= x(x-1),\\
y_2 &= x(x-\tfrac{1}{2})(x-1),\\
y_3 &= x(x-\tfrac{1}{3})(x-\tfrac{2}{3})(x-1),\\
&\vdots\\
y_n &= x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\dots\left(x-\frac{n-1}{n}\right)(x-1).
\end{aligned}$$

Attending only to the absolute values, disregarding the signs, y_n has n maxima, viz. if n be odd, $=2p+1$ suppose, these are

$$Y_1,\ Y_2, \dots,\ Y_p,\ Y_{p+1},\ Y_p, \dots Y_1,$$

where Y_{p+1} corresponds to the value $x=\frac{1}{2}$, and $Y_1,\ Y_2, \dots,\ Y_p$ to values of x between

$$0 \text{ and } \frac{1}{2p+1},\ \frac{1}{2p+1} \text{ and } \frac{2}{2p+1}, \dots,\ \frac{p-1}{2p+1} \text{ and } \frac{p}{2p+1}.$$

But if n be even, $=2p$ suppose, then the maxima are

$$Y_1,\ Y_2, \dots,\ Y_p,\ Y_p, \dots,\ Y_1,$$

where $Y_1,\ Y_2, \dots,\ Y_p$ correspond to values of x between

$$0 \text{ and } \frac{1}{2p},\ \frac{1}{2p} \text{ and } \frac{2}{2p}, \dots,\ \frac{p-1}{2p} \text{ and } \tfrac{1}{2}.$$

In every case the maxima decrease from Y_1 which is the greatest, to Y_p or Y_{p+1} which is the least; in particular, $n = 2p+1$, then

$$Y_{p+1} = \tfrac{1}{2}\left(\tfrac{1}{2} - \frac{1}{2p+1}\right) \dots (\tfrac{1}{2} - 1)$$

$$= \left(\tfrac{1}{2}\frac{2p-1}{2\,.\,2p+1} \cdots \frac{1}{2\,.\,2p+1}\right)^2$$

$$= \frac{[1\,.\,3 \dots (2p-1)]^2}{2^{2p+2}.\,(2p+1)^{2p}} = \tfrac{1}{4}\frac{(\tfrac{1}{2}\,.\,\tfrac{3}{2} \dots p - \tfrac{1}{2})^2}{(2p+1)^{2p}},$$

which is

$$= \tfrac{1}{4}\frac{\{\Gamma(p+\tfrac{1}{2}) \div \Gamma\tfrac{1}{2}\}^2}{(2p+1)^{2p}} = \frac{\Gamma^2(p+\tfrac{1}{2})}{4\pi(2p+1)^{2p}}.$$

Suppose p is large; then, as for large values of x,

$$\Gamma x = \sqrt{(2\pi)}\, x^{x-\frac{1}{2}} e^{-x},$$

we have

$$\Gamma(p+\tfrac{1}{2}) = \sqrt{(2\pi)}\,(p+\tfrac{1}{2})^p e^{-p-\frac{1}{2}}$$

$$= \sqrt{(2\pi)}\, p^p e^{p\log\left(1+\frac{1}{2p}\right)} e^{-p-\frac{1}{2}} = \sqrt{(2\pi)}\, p^p e^{-p},$$

$$(2p+1)^{2p} = (2p)^{2p}.\, e^{2p\log\left(1+\frac{1}{2p}\right)} = 2^{2p} p^{2p} e,$$

and so

$$Y_{p+1} = \frac{2\pi p^{2p} e^{-2p}}{4\pi 2^{2p} p^{2p} e} = \frac{p^2 e^{-2p-1}}{2^{2p-1}} = p^2\left(\frac{1}{2e}\right)^{2p+1}.$$

Also Y_1 corresponds *approximately* to

$$x = \tfrac{1}{2}\frac{1}{2p+1} = \frac{1}{2n},$$

$$Y_1 = \frac{1}{2n}\cdot\frac{1}{2n}\cdot\frac{3}{2n}\cdots\frac{2n-1}{2n} = \frac{1}{n^{n+1}}\tfrac{1}{2}\,.\,\tfrac{1}{2}\,.\,\tfrac{3}{2}\dots(n-\tfrac{1}{2})$$

$$= \frac{1}{n^{n+1}}\tfrac{1}{2}\frac{\Gamma(n+\tfrac{1}{2})}{\Gamma\tfrac{1}{2}} = \frac{1}{2\,(2p+1)^{2p+1}}\,\frac{\Gamma(2p+\tfrac{3}{2})}{\sqrt{(\pi)}}.$$

Now

$$\Gamma(2p+\tfrac{3}{2}) = \sqrt{(2\pi)}\,(2p+\tfrac{3}{2})\, e^{-2p-\frac{3}{2}} = \sqrt{(2\pi)}\,(2p)^{2p+2} e^{(2p+\frac{3}{2})\log\left(1+\frac{3}{4p}\right)} e^{-2p-\frac{3}{2}}$$

$$= \sqrt{(2\pi)}\, 2^{2p+2} p^{2p+2} e^{-2p},$$

and

$$(2p+1)^{2p+1} = (2p)^{2p+1} e^{(2p+1)\log\left(1+\frac{1}{2p}\right)} = (2p)^{2p+1} e;$$

so that

$$Y_1 = \frac{1}{2^{2p+2} p^{2p+1} e}\cdot\frac{\sqrt{(2\pi)}\,.\,2^{2p+2}.\,p^{2p+2} e^{-2p}}{\sqrt{(\pi)}}$$

$$= \frac{p\sqrt{(2)}}{e^{2p+1}},$$

so that, p being large, Y_1 is far larger than Y_{p+1}.

550.

PROBLEM AND HYPOTHETICAL THEOREMS IN REGARD TO TWO QUADRIC SURFACES.

[From the *Messenger of Mathematics*, vol. II. (1873), p. 137.]

Two conics may be circum-and-inscribable to an n-gon; viz. the conics may be such that there exists a singly infinite series of n-gons each inscribed in the first and circumscribed about the second of the conics. In particular they may be circum-and-inscribable to a triangle.

The following problem arises:

Consider two given quadric surfaces and a given line S; to find the planes through S, which cut the surfaces in two conics circum-and-inscribable to a triangle (it is presumed there are two or more such planes).

Let the surfaces be Θ, Θ', and let the line S a tangent to Θ' meet Θ in the points A, B; if through S we draw two planes as above, then in the first plane the tangents from A, B to the section of Θ' will meet in a point C of Θ; and in the second plane the tangents from A, B to the section of Θ' will meet in a point D of Θ. The points C, D being thus determined the lines AB, AC, BC, AD, BD all touch the surface Θ', and it is presumed that the surfaces Θ, Θ' may be such that CD also touches the surface Θ'; viz. in this case we have a tetrahedron $ABCD$, the summits of which lie in the surface Θ, and the edges touch the surface Θ'; and not only so, but it is further presumed that the surfaces may be such that starting from *any* point A of Θ and using either *any* tangent or a properly selected tangent AB of Θ', it shall be possible to complete the figure as above; or, what is the same thing, the surfaces may be such that there exists a *doubly* or a *triply* infinite series of tetrahedra, the summits of each lying in Θ and its edges touching Θ'. It is also presumed that the faces of the tetrahedra all touch one and the same quadric surface Θ''.

551.

TWO SMITH'S PRIZE DISSERTATIONS.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 145—149.]

WRITE dissertations on the following subjects:

1. The theory of interpolation, with a determination of the limits of error in the value of a function obtained by interpolation.

2. Determinants.

1. The general problem is to find y a function of x having given values for given values of x. The problem thus stated is of course indeterminate; in practice, we assume a certain form for the function y, the coefficients of which form are determined by the given conditions, viz. either y is known to be of the form in question, the actual value being then determined as above, or it is assumed that y is approximately equal to a function of the form in question, and the value is then approximately determined in such wise that, for the given values of x, the function y shall have its given values.

The ordinary case is when we have the values of y corresponding to n given values of x, and y is taken to be a function of the form $A + Bx + \ldots + Kx^{n-1}$.

Suppose to fix the ideas $n=4$, and that y_1, y_2, y_3, y_4 are the values of y corresponding to the values a, b, c, d of x. We may at once write down the expression

$$\begin{aligned} y = &\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}y_1 \\ &+\frac{(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}y_2 \\ &+\frac{(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}y_3 \\ &+\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}y_4, \end{aligned}$$

for clearly this is of the form in question $A+Bx+Cx^2+Dx^3$, and y becomes $=y_1$ for $x=a$; $=y_2$ for $x=b$, &c. And the like for any value of n. This is known as Lagrange's interpolation formula.

The given values of x may be equidistant, say they are 0, 1, 2, ..., $n-1$, and the corresponding values of y are $y_0, y_1, \ldots, y_{n-1}$; then writing down the expression

$$y_x = y_0 + \frac{x}{1}\Delta y_0 + \frac{x.\overline{x-1}}{1.2}\Delta^2 y_0 + \ldots + \frac{x.\overline{x-1}\ldots\overline{x-n+2}}{1.2\ldots\overline{n-1}}\Delta^{n-1}y_0,$$

where, as usual,

$$\Delta y_0 = y_1 - y_0, \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0, \text{ \&c.};$$

then for $x=0$, 1, 2, &c. the values of y are

$$\begin{aligned} &y_0, \\ &y_0 + \Delta y_0, \quad = y_1, \\ &y_0 + 2\Delta y_0 + \Delta^2 y_0, \quad = y_2, \\ &\text{\&c.}, \end{aligned}$$

or the required conditions are satisfied.

As regards the determination of the limits of error, taking a particular case $n=4$, suppose that we have the values y_0, y_1, y_2, y_3 of y corresponding to the values 0, 1, 2, 3 of x, and that the true value of y is known to be

$$= A + Bx + Cx^2 + Dx^3 + Kx^4,$$

where K is a function of x, which for any value of x within the given values (i.e. from $x=0$ to $x=3$) is known to be at least $=P$ and at most $=Q$, i.e., $K>P<Q$, where to fix the ideas P and Q are each positive, Q being the greater. Here calculating the interpolation value of $y-Kx^4$ (the last term Kx^4 by Lagrange's formula), we have

$$\begin{aligned} y = y_0 &+ \frac{x}{1}\Delta y_0 + \frac{x.\overline{x-1}}{1.2}\Delta^2 y_0 + \frac{x.\overline{x-1}\,\overline{x-2}}{1.2.3}\Delta^3 y_0 \\ &+ Kx^4 \\ &\qquad - \tfrac{1}{2}\, K_1 x(x-2)(x-3) \\ &\qquad + 8\, K_2 x(x-1)(x-3) \\ &\qquad - \tfrac{27}{2} K_3 x(x-1)(x-2), \end{aligned}$$

viz. this is the true value of y. Hence using the approximate formula as given by the first line, the last four lines give the error, viz. this is

$$= Kx^4 + K_2 \tfrac{81}{2} x^2 + K_2(8x^3 + 24x) + K_1 \tfrac{5}{2} x^2 - K_3(\tfrac{27}{2}x^3 + 27x) + K_2(32x^2) - K_1(\tfrac{1}{2}x^3 + 3x).$$

But K_1, K_2, K_3 being each $>P$ and $<Q$, this is

$$\begin{aligned} &> P(x^4 + \ \ 8x^3 + 43x^2 + 24x) \\ &- Q(\qquad 14x^3 + 32x^2 + 30x), \end{aligned}$$

and it is

$$\begin{aligned} &< Q(x^4 + \ \ 8x^3 + 43x^2 + 24x) \\ &- P(\qquad 14x^3 + 33x^2 + 30x), \end{aligned}$$

the difference of these limits being

$$= (Q - P)(x^4 + 22x^3 + 75x^2 + 54x).$$

2. A determinant is a function of n^2 letters; viz. arranging these in the form of a square, the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

is a function linear in regard to each of the n^2 letters, and such that interchanging any two entire lines, or any two entire columns, the sign of the determinant is reversed, its absolute value being unaltered.

The above definition leads to a rule for calculating the actual value of the determinant, which rule may be taken as a definition, viz. the determinant is the sum of $1.2.3\ldots n$ terms obtained as follows: starting from the term

$$+a_1b_2c_3\ldots,$$

we permute in every possible way the suffixes 1, 2, 3, ..., and give to the term a sign, $\pm$, which is that compounded of as many $-$ signs as there are cases in which an inferior number succeeds a superior number. Or, what is the same thing, any arrangement may be obtained by a succession of interchanges of two letters; and then taking for each interchange the sign $-$, we obtain the sign $\pm$ of the term in question. The positive and the negative terms are each of them $\frac{1}{2}(1.2.3\ldots n)$ in number.

To show the connexion of the two definitions, it is sufficient to observe that in the second definition, attending for instance only to the first and second columns, to any terms Ma_1b_2, Na_1b_3, &c., there always correspond other terms $-Ma_2b_1$, $-Na_3b_1$, &c., so that taking the pairs together, these are $M(a_1b_2 - a_2b_1)$, $N(a_1b_3 - a_3b_1)$, &c., terms which change their sign, but remain unaltered as to their absolute values by the interchange of the first and second columns.

Among the fundamental properties of determinants are as follows:

The properties are the same as regards lines and columns.

A determinant vanishes if any line vanishes (that is, if each term of the line is $=0$).

A determinant vanishes if two lines are identical.

A determinant is a linear function of its lines.

Whence—

Determinant having a line sA is $=s$ times the determinant having the line A (sA is here used to denote the line each term of which is s times the corresponding term of the line A).

Determinant having a line $A+A'$ = determinant with line A + determinant with line A'.

It follows that, if any line of a determinant is the sum of the other lines, each multiplied by an arbitrary coefficient, or, what is the same thing, if we can with any of the lines, each multiplied by an arbitrary coefficient, compose a line 0, then the determinant is $=0$.

The same principle leads to a theorem for the product of two determinants of the same order n, viz. it is found that the product is a determinant of the same order n, each term thereof being a sum of the products of the terms of a line of one of the factors into the corresponding terms of a line of the other factor. Starting with this expression of the product, we decompose it into a series of determinants each of which is either $=0$, or it is a product of a single term of the one factor into the other factor, and the sum of all these products is equal to the product of the two factors.

If we have n quantities $x, y, \ldots$ connected by as many linear equations

$$a_1x+b_1y+c_1z+\ldots=0,$$

then the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, \ldots \\ a_2, & b_2, & c_2, \ldots \\ a_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix} \text{ is } =0;$$

and so, if we have n linear equations

$$a_1x+b_1y+c_1z+\ldots=u,$$

then each of the quantities $x, y, z, \ldots$ is given as the quotient of two determinants, the denominator being in each case

$$\begin{vmatrix} a_1, & b_1, & c_1, \ldots \\ a_2, & b_2, & c_2, \ldots \\ a_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix},$$

and the numerators being (save as to their signs) that for x

$$\begin{vmatrix} u_1, & b_1, & c_1, \ldots \\ u_2, & b_2, & c_2, \ldots \\ u_3, & b_3, & c_3, \ldots \\ \vdots & & \end{vmatrix},$$

and the like for $y, z, \ldots$.

A determinant remains unaltered when the lines and columns are interchanged, the dexter diagonal ($\diagdown$) remaining unaltered.

A determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1, \dots \\ a_2, & b_2, & c_2, & d_2, \dots \\ a_3, & b_3, & c_3, & d_3, \dots \\ a_4, & b_4, & c_4, & d_4, \dots \\ \vdots & & & \end{vmatrix},$$

is a sum of products of complementary determinants

$$\Sigma \pm \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \begin{vmatrix} c_3, & d_3, \dots \\ c_4, & d_4, \dots \\ \vdots & \end{vmatrix},$$

or, say of products of complementary minors.

In particular, it is a sum

$$\Sigma \pm a_1 \begin{vmatrix} b_2, & c_2, & d_2, \dots \\ b_3, & c_3, & d_3, \dots \\ b_4, & c_4, & d_4, \dots \\ \vdots & & \end{vmatrix},$$

of products of first minors into single terms or $(n-1)^{\text{th}}$ minors.

This last theorem affords a convenient rule for the development of a determinant.

552.

ON A DIFFERENTIAL FORMULA CONNECTED WITH THE THEORY OF CONFOCAL CONICS.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 157, 158.]

THE following transformations present themselves in connexion with the theory of confocal conics.

The coordinates x, y of a point are considered as functions of the parameters h, k where

$$\frac{x^2}{a+h}+\frac{y^2}{b+h}=1,$$

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}=1;$$

and then assuming $\xi=x+iy$, $\eta=x-iy$ $(i=\sqrt{(-1)}$ as usual), and writing $c=a-b$, we find

$$h=\tfrac{1}{2}(-a-b+\xi\eta)+\tfrac{1}{2}\sqrt{\{(\xi^2-c)(\eta^2-c)\}},$$
$$k=\tfrac{1}{2}(-a-b+\xi\eta)-\tfrac{1}{2}\sqrt{\{(\xi^2-c)(\eta^2-c)\}},$$

whence if

$$H=(a+h)(b+h),\quad K=(a+k)(b+k),$$

we have

$$H=\tfrac{1}{4}\{\xi\sqrt{(\eta^2-c)}+\eta\sqrt{(\xi^2-c)}\}^2,$$
$$K=\tfrac{1}{4}\{\xi\sqrt{(\eta^2-c)}-\eta\sqrt{(\xi^2-c)}\}^2,$$

or, say

$$\sqrt{(H)}=\tfrac{1}{2}\{\xi\sqrt{(\eta^2-c)}+\eta\sqrt{(\xi^2-c)}\},$$
$$\sqrt{(K)}=\tfrac{1}{2}\{\xi\sqrt{(\eta^2-c)}-\eta\sqrt{(\xi^2-c)}\},$$

and thence

$$h + \tfrac{1}{2}(a+b) + \sqrt{(H)} = \tfrac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\}\{\eta + \sqrt{(\eta^2 - c)}\},$$

$$k + \tfrac{1}{2}(a+b) + \sqrt{(K)} = \tfrac{1}{2}\{\xi + \sqrt{(\xi^2 - c)}\}\{\eta - \sqrt{(\eta^2 - c)}\}$$

$$= \tfrac{1}{2}c\,\frac{\xi + \sqrt{(\xi^2 - c)}}{\eta + \sqrt{(\eta^2 - c)}}.$$

These also follow from the known differential formula

$$4(dx^2 + dy^2) = (h-k)\left(\frac{dh^2}{H} - \frac{dk^2}{K}\right),$$

that is,

$$\frac{4d\xi d\eta}{\sqrt{(\xi^2 - c)}\sqrt{(\eta^2 - c)}} = \frac{dh^2}{H} - \frac{dk^2}{K},$$

implying

$$\frac{2\alpha d\xi}{\sqrt{(\xi^2 - c)}} = \frac{dh}{\sqrt{(H)}} + \frac{dk}{\sqrt{(K)}},$$

$$\frac{2d\eta}{\alpha\sqrt{(\eta^2 - c)}} = \frac{dh}{\sqrt{(H)}} - \frac{dk}{\sqrt{(K)}},$$

where α is a constant. The foregoing integral formulæ give at once

$$\frac{dh}{\sqrt{(H)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} + \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

$$\frac{dk}{\sqrt{(K)}} = \frac{d\xi}{\sqrt{(\xi^2 - c)}} - \frac{d\eta}{\sqrt{(\eta^2 - c)}},$$

and substituting these values we find $\alpha = 1$, and the differential formulæ are then satisfied.

We thence have

$$\text{const.} = \sqrt{\{(a+h)(b+h)\}} \pm \sqrt{\{(a+k)(b+k)\}},$$

as the integral of the differential equation

$$\frac{dh}{\sqrt{(H)}} \pm \frac{dk}{\sqrt{(K)}} = 0.$$

553.

TWO SMITH'S PRIZE DISSERTATIONS.

[From the *Messenger of Mathematics*, vol. II. (1873), pp. 161—166.]

WRITE dissertations:

1. On the condition of the similarity of two dynamical systems.
2. On orthogonal surfaces.

1. We may consider two particles m, m', describing similarly two similar paths (which for convenience may be taken to be similarly situate in regard to two sets of rectangular axes respectively), viz. this means that the times t, t' of passage through corresponding arcs s, s' are proportional. The ratios $\frac{s'}{s}$, $\frac{t'}{t}$, are thus each of them constant; and this must also be the case with the ratio $\frac{v'}{v}$, of the velocities v, v' at corresponding points; since it is clear that we must have $\frac{s'}{s} = \frac{v'}{v} \cdot \frac{t'}{t}$.

Now in order that the two particles may move as above under the action of any forces upon the two particles respectively, it is clearly necessary that the forces F, F' at corresponding points shall act in the same direction, and be in a constant ratio of magnitude. To obtain this ratio, imagine the two particles, masses m, m', moving as above, in the corresponding infinitesimal elements of time τ, τ', with the velocities v, v' through the infinitesimal arcs σ, σ' respectively, $\left(\frac{\tau}{\tau'} = \frac{t}{t'},\ \frac{v}{v'} = \frac{v}{v'},\ \frac{\sigma}{\sigma'} = \frac{s}{s'}\right)$; the deflections from the tangent will be $\frac{1}{2}\frac{F}{m}\tau^2$, $\frac{1}{2}\frac{F'}{m'}\tau'^2$ respectively, and these must be in the ratio of the corresponding arcs σ, σ', viz. we must have

$$\frac{F\tau^2}{m} : \frac{F'\tau'^2}{m'} = \sigma : \sigma',$$

or, what is the same thing,

$$\frac{Ft^2}{m} : \frac{F't'^2}{m'} = s : s',$$

that is,

$$\frac{F'}{F} = \frac{m'}{m}\,\frac{s'}{s} \div \left(\frac{t'}{t}\right)^2,$$

and this relation subsisting, and the velocities at the beginnings of the elements of time τ, τ' being in the assumed ratio, it is clear that the velocities at the ends of these elements of time will be in the same ratio; and thus the two particles will go on moving in the manner in question.

All that has been said as to two particles, applies without alteration to any two systems of particles moving under the like geometrical conditions, and we thus arrive at the conclusion; given two similarly constituted systems, which at any instant are in a given magnitude-ratio $\frac{s'}{s}$, their component particles being in a given ratio $\frac{m'}{m}$ (the same for each pair of component particles), then if the particles of the two systems respectively are to move in similar paths of the same magnitude-ratio $\frac{s'}{s}$, the times of describing corresponding arcs being in a given constant ratio $\frac{t'}{t}$ $\left(\text{this implying as above that the ratio of the velocities at corresponding points is } \frac{v'}{v}, = \frac{s'}{s} \div \frac{t'}{t}\right)$, it is necessary that the forces on corresponding particles in corresponding positions shall act in the same directions, and shall be in the constant magnitude-ratio

$$\frac{F'}{F} = \frac{m'}{m} \cdot \frac{s'}{s} \div \left(\frac{t'}{t}\right)^2,$$

and this being so, the motion of the two systems will in fact be similar as above explained.

Taking $\frac{m'}{m}, = \mu$ for the mass-ratio, $\frac{s'}{s}, = \sigma$ for the length-ratio, and $\frac{t'}{t}, = \tau$ for the time-ratio; also $\frac{F'}{F}, = \phi$ for the force-ratio, the condition determining the force-ratio ϕ is thus

$$\phi = \frac{\mu\sigma}{\tau^2}.$$

It is to be observed, that if the forces are entirely internal, and proportional to homogeneous functions of the same order, say $-n$, of the coordinates of all or any of the particles; e.g. if they are central forces varying as the inverse nth power of the distances; then the condition as to the action of the forces in the two systems respectively can always be satisfied by giving a proper constant value to the ratio of the absolute forces (or forces at unity of distance); thus, if in the first system we

have two particles m_1, m_2 attracting or repelling each other with a force $\frac{km_1m_2}{r^n}$, and if in the second system the force is $\frac{k'm_1'm_2'}{r'^n}$; then the condition as to the direction of the forces at corresponding positions is satisfied *ipso facto;* and the condition as to magnitude is

$$\frac{k'm_1'm_2'}{r'^n} \times \frac{r^n}{km_1m_2} = \frac{m'}{m}\,\frac{s'}{s}\,\frac{t^2}{t'^2},$$

that is,

$$\frac{k'}{k} = \frac{m'}{m_1'm_2'}\,\frac{m_1m_2}{m}\,\frac{s'r'^n}{sr^n}\,\frac{t^2}{t'^2}$$

$$= \frac{m}{m'}\left(\frac{s'}{s}\right)^{n+1}\frac{t^2}{t'^2},$$

or, say

$$\frac{t'}{t} = \left(\frac{s'}{s}\right)^{\frac{n+1}{2}}\left(\frac{km}{k'm'}\right)^{\frac{1}{2}}.$$

In the case $n=2$, the present theorem (applying however only to the case of two elliptic orbits of the same eccentricity) agrees with Kepler's third law, or say with the theorem

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{(\mu)}},$$

that is,

$$T \propto \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}},$$

where observe that the μ, or mass in the sense of the formula, is the km, or attractive force on a unit of mass, of the theorem as above written down.

2. In a family of surfaces $F(x, y, z, p) = 0$, containing a single variable parameter p, there is through any given point of space, a surface or surfaces of the family; or (if more than one, confining the attention to one of these surfaces) we may say that there is, through any given point of space, a surface of the family.

Considering now two other families of surfaces, there will be through any given point of space, three surfaces, one of each family; and if (for every given point of space whatever) these intersect each other at right angles, we have a system of orthogonal surfaces.

Supposing the equations of the three families to be

$$F(x, y, z, p) = 0,$$

$$\Phi(x, y, z, q) = 0,$$

$$\Psi(x, y, z, r) = 0,$$

then the requisite conditions are

$$\frac{dF}{dx}\frac{d\Phi}{dx}+\frac{dF}{dy}\frac{d\Phi}{dy}+\frac{dF}{dz}\frac{d\Phi}{dz}=0,$$

$$\frac{dF}{dx}\frac{d\Psi}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

$$\frac{d\Phi}{dx}\frac{d\Psi}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

viz. these equations must be satisfied, not in general identically, but in virtue of the given equations $F=0$, $\Phi=0$, $\Psi=0$.

Or, what is more convenient, we may take the equations of the three families to be

$$p-f(x, y, z)=0, \quad q-\phi(x, y, z)=0, \quad r-\psi(x, y, z)=0;$$

and write the conditions in the form

$$\frac{dp}{dx}\frac{dq}{dx}+\frac{dp}{dy}\frac{dq}{dy}+\frac{dp}{dz}\frac{dq}{dz}=0,$$

$$\frac{dp}{dx}\frac{dr}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

$$\frac{dq}{dx}\frac{dr}{dx}+\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

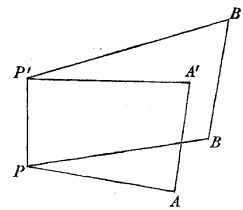
where of course p, q, r stand for their given functional values, $p=f(x, y, z)$, &c.; the equations in this form contain only (x, y, z), and not the parameters p, q, r; so that, if satisfied at all, they must be satisfied identically; and the required conditions therefore are that the last-mentioned system of equations shall be satisfied identically by the values p, q, r considered as given functions of (x, y, z).

The last-mentioned conditions lead to the theorem known as Dupin's; viz. it follows from them that the surfaces intersect along their curves of curvature; or more definitely, each surface of one family is intersected by the surfaces of the other two families in its two sets of curves of curvature respectively.

To indicate the geometrical ground of the theorem, consider on a surface of one family a point P, and at this point the normal meeting the consecutive surface in P'; the surfaces through P of the other two families respectively will pass through P', and meet the given surface in two curves PA, PB (viz. PA, PB represent infinitesimal arcs on these two curves respectively), the angle at P being a right angle.

Drawing at A, B normals to the given surface to meet the consecutive surface in the points A', B' respectively, the same two surfaces will meet the consecutive surface in the arcs $P'A'$, $P'B'$ respectively; and (the system being orthogonal), we must have the angle at P' a right angle. This imposes a condition upon the direction

(in the tangent plane at P) of the orthogonal directions PA, PB; viz. it is found that these must be such that the normals PP', AA' intersect, or, what is the same



thing, the normals PP', BB' (one of these conditions implying the other); that is, that the lines PA, PB shall be the directions of the two curves of curvature through P on the given surface.

Observe that PP', AA' intersecting each other, the four points P, P', A, A' are in the same plane, that is, PA, $P'A'$ intersect, these lines being the normals at P, P' respectively of the surface through P of one of the other two families; and similarly PP', BB' intersecting each other, the lines PB, $P'B'$ intersect; these being the normals at P, P' respectively of the surface through P of the other two families. We have through PP' two planes at right angles to each other; and these are met by a plane $A'P'B'$, in two lines $A'P'$, $B'P'$, the inclinations of which to the line PP' differ only infinitesimally from a right angle, say they are $90° - a$ and $90° - b$ respectively; hence if the angle $A'P'B'$ is $= 90° - c$, this is the hypotenuse of a right-angled spherical triangle, the sides whereof are $90° - a$, $90° - b$; wherefore $\sin c = \sin a \sin b$, viz. $\sin c$ is an infinitesimal of a higher order which may be neglected, or the angle P' will be $= 90°$; that is, the surfaces through P of the other two families, intersecting the given surface in the directions PA, PB of the two curves of curvature, will intersect the consecutive surface at P' in the two directions $P'A'$, $P'B'$ *at right angles to each other;* which is an *a posteriori* verification of Dupin's theorem.

In what precedes the given surface through P may be regarded as a surface assumed at pleasure; and it in effect appears that taking the consecutive surface also at pleasure (but varying only infinitesimally from the given surface), the condition in order that the two surfaces, which at P intersect each other and the given surface at right angles, shall at P' intersect the consecutive surface in two directions at right angles to each other, is that they shall intersect the given surface in the directions PA, PB of the two curves of curvature. But if we thus take the consecutive surface at pleasure,—or say if we construct it by measuring off along the normal at *each* point P of the given surface an infinitesimal distance PP', $= \rho$, where ρ an arbitrary function of the coordinates of the point P,—then although on the consecutive surface the lines $P'B'$, $P'A'$ are at right angles to each other, there is nothing to show, and it is not in fact the case, that these lines $P'A'$, $P'B'$ are the directions of the curves

of curvature on the consecutive surface. In the orthogonal system they must be so; and this imposes upon the infinitesimal normal distance ρ, a condition; viz. it is found that ρ considered as a function of (x, y, z) must satisfy a certain partial differential equation of the second order.

It hence appears that no one of the three families of surfaces can be assumed at pleasure; for taking the equation of a family to be $p-f(x, y, z)=0$, then p being the value of the parameter for the given surface of the foregoing investigation, and $p+\delta p$ the value of the parameter for the consecutive surface, the normal distance at the point (x, y, z) between the two surfaces is

$$=\delta p \div \sqrt{\left\{\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2\right\}};$$

viz. δp is here a constant; and we have

$$1 \div \sqrt{\left\{\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2\right\}},$$

satisfying the foregoing partial differential equation; or, what is the same thing, p considered as a function of x, y, z must satisfy a certain partial differential equation of the third order; viz. this is the condition to be satisfied in order that a family of surfaces $p-f(x, y, z)=0$ may belong to an orthogonal system.

554.

AN ELLIPTIC-TRANSCENDENT IDENTITY.

[From the *Messenger of Mathematics*, vol. II. (1873), p. 179.]

THE following is a singular identity:

$$\begin{aligned}&(1+q)(1+q^3)(1+q^5)(1+q^7)^2(1+q^9)\dots\\ &-(1-q)(1-q^3)(1-q^5)(1-q^7)^2(1-q^9)\dots\\ &=2q(1+q^2)(1+q^4)(1+q^6)(1+q^8)\ (1+q^{10})(1+q^{12})(1+q^{14})^2(1+q^{16})\dots,\end{aligned}$$

where in each of the three terms every factor has the exponent 1 or 2 according as the exponent of q is not, or is, divisible by 7.

555.

NOTICES OF COMMUNICATIONS TO THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

[From the *Reports of the British Association for the Advancement of Science*, 1865 to 1873, *Notices and Abstracts of Miscellaneous Communications to the Sections.*]

1. *On the Problem of the in-and-circumscribed Triangle.* Report, 1870, pp. 9, 10.

I HAVE recently accomplished the solution of this problem, which I spoke of at the Meeting in 1864. The problem is as follows: required the number of the triangles the angles of which are situate in a given curve or curves, and the sides of which touch a given curve or curves. There are in all 52 cases [see 514] of the problem, according as the curves which contain the angles and are touched by the sides are distinct curves, or are any or all of them the same curve. The first and easiest case is when the curves are all of them distinct; the number of triangles is here $=2aceBDF$, where a, c, e are the *orders* of the curves containing the angles (or, say, of the angle-curves) respectively; and B, D, F are the *classes* of the curves touched by the sides (or, say, of the side-curves) respectively. An interesting case is when the angle-curves are one and the same curve; or, say, $a=c=e$ (where the sign $=$ is used to denote the identity of the curves); the number of triangles is here $=\{2a(a-1)(a-2)+A\}\,BDF$, where a, A are the order and class of the curve $a=c=e$. In the reciprocal case, where the side-curves are one and the same curve, say $B=D=F$, we have of course a like formula, viz. the number of triangles is here $=\{2B(B-1)(B-2)+b\}\,ace$, where B, b are the class and order of the curve $B=D=F$. The last and most difficult case is when the six curves are all of them one and the same curve, say $a=c=e=B=D=F$; the number of triangles is here = one-sixth of

$$\begin{aligned}
& A^4(\ .\quad .\quad .\quad .\quad .\quad +\ 1),\\
+\ & A^3(\ .\quad 2a^3-18a^2+52a-46)\\
+\ & A^2(\ .\ -18a^3+162a^2-420a+221)\\
+\ & A\ (\ .\quad 52a^3-420a^2+704a+172)\\
+\ & \quad a^4-46a^3+221a^2+172a\\
+\ & a\left\{\begin{array}{l} A^2(\ .\quad .\quad .\ -\ 9)\\ +A\ (\ .\quad .-\ 12a+135)\\ \qquad -9a^2+135a-600\end{array}\right\},
\end{aligned}$$

where a is the order, A the class of the curve; α is the number, three times the class + the number of cusps, or (what is the same thing) three times the order + the number of inflexions.

---

2. *On a Correspondence of Points and Lines in Space.* Report, 1870, p. 10.

NINE points in a plane may be the intersection of two (and therefore of an infinite series of) cubic curves; say, that the nine points are an "ennead": and similarly nine lines through a point may be the intersection of two (and therefore of an infinite series of) cubic cones; say, the nine lines are an ennead. Now, imagine (in space) any 8 given points; taking a variable point P, and joining this with the 8 points, we have through P 8 lines, and there is through P a ninth line completing the ennead; this is said to be the corresponding line of P. We have thus to any point P a single corresponding line through the point P; this is the correspondence referred to in the heading, and which I would suggest as an interesting subject of investigation to geometers. Observe that, considering the whole system of points in space, the corresponding lines are a triple system of lines, *not* the whole system of lines in space. It is thus, not any line whatever, but only a line of the triple system, which has on it a corresponding point. But as to this some explanation is necessary; for starting with an arbitrary line, and taking upon it a point P, it would seem that P might be so determined that the given line and the lines from P to the eight points should form an ennead,—that is, that the arbitrary line would have upon it a corresponding point or points.

The question of the foregoing species of correspondence was suggested to me by the consideration of a system of 10 points, such that joining any one whatever of them with the remaining nine points, the nine lines thus obtained form an ennead; or, say, that each of the 10 points is the "enneadic centre" of the remaining nine. I have been led to such a system of 10 points by my researches upon Quartic Surfaces; but I do not as yet understand the theory.

---

3. *On the Number of Covariants of a Binary Quantic.* Report, 1871, pp. 9, 10.

THE author remarked [see 462] that it had been shown by Prof. Gordan that the number of the covariants of a binary quantic of any order was finite, and, in particular, that the numbers for the quintic and the sextic were 23 and 26 respectively. But the demonstration is a very complicated one, and it can scarcely be doubted that a more simple demonstration will be found. The question in its most simple form is as follows: viz. instead of the covariants we substitute their leading coefficients, each of which is a "seminvariant" satisfying a certain partial differential equation; say, the quantic is $(a, b, c, \ldots, k \between x, y)^n$, then the differential equation is $(a\partial_b + 2b\partial_c \ldots + nj\partial_k)\, u = 0$, which *quà* equation with $n+1$ variables admits of n independent solutions: for instance, if $n=3$, the equation is $(a\partial_b + 2b\partial_c + 3c\partial_d)\, u = 0$, and the solutions are a, $ac - b^2$,

$a^2d - 3abc + 2b^3$; the general value of u is $u =$ any function whatever of the last-mentioned three functions. We have to find the rational non-integral functions of these functions which are rational and integral functions of the coefficients; such a function is

$$\frac{1}{a^2}\{(a^2d - 3abc + 2b^3)^2 + 4(ac - b^2)^3\},$$

$$= a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,$$

and the original three solutions, together with the last-mentioned function $a^2d^2 +$ &c., constitute the complete system of the seminvariants of the cubic function; viz. every other seminvariant is a rational *and integral* function of these. And so, in the general case, the problem is to complete the series of the n solutions a, $ac - b^2$, $a^2d - 3abc + 2b^3$, $a^3e - 4a^2bd + 6ab^2c - 3b^4$, &c. by adding thereto the solutions which, being rational but non-integral functions of these, are rational and integral functions of the coefficients; and thus to arrive at a series of solutions such that every other solution is a rational and integral function of these.

---

4. *Note on certain Families of Surfaces.* Report, 1871, pp. 19, 20.

SEE the paper numbered 538, of which this Note is a duplicate.

---

5. *On the Mercator's Projection of a Surface of Revolution.* Report, 1873, p. 9.

THE theory of Mercator's projection is obviously applicable to any surface of revolution; the meridians and parallels are represented by two systems of parallel lines at right angles to each other, in such wise that for the infinitesimal rectangles included between two consecutive arcs of meridian and arcs of parallel the rectangle in the projection is similar to that on the surface. Or, what is the same thing, drawing on the surface the meridians at equal infinitesimal intervals of angular distance, we may draw the parallels at such intervals as to divide the surface into infinitesimal squares; the meridians and parallels are then in the projection represented by two systems of equidistant parallel lines dividing the plane into squares. And if the angular distance between two consecutive meridians instead of being infinitesimal is taken moderately small (5° or even 10°), then it is easy on the surface or *in plano*, using only the curve which is the meridian of the surface, to lay down graphically the series of parallels which are in the projection represented by equidistant parallel lines. The method is, of course, an approximate one, by reason that the angular distance between the two consecutive meridians is finite instead of infinitesimal.

I have in this way constructed the projection of a skew hyperboloid of revolution: viz. in one figure I show the hyperbola, which is the meridian section, and by means of it (taking the interval of the meridians to be $= 10°$) construct the positions of the

successive parallels; I complete the figure by drawing the hyperbolas which are the orthographic projections of the meridians, and the right lines which are the orthographic projections of the parallels; the figure thus exhibits the orthographic projection (on the plane of a meridian) of the hyperboloid divided into squares as above. The other figure, which is the Mercator's projection, is simply two systems of equidistant parallel lines dividing the paper into squares. I remark that in the first figure the projections of the right lines on the surface are the tangents to the bounding hyperbola; in particular, the projection of one of these lines is an asymptote of the hyperbola. This I exhibit in the figure, and by means of it trace the Mercator's projection of the right line on the surface; viz. this is a serpentine curve included between the right lines which represent two opposite meridians and having these lines for asymptotes. It is sufficient to show one of these curves, since obviously for any other line of the surface belonging to the same system the Mercator's projection is at once obtained by merely displacing the curve parallel to itself, and for any line of the other system the projection is a like curve in a reversed position.

A Mercator's projection might be made of a skew hyperboloid not of revolution; viz. the curves of curvature might be drawn so as to divide the surface into squares, and the curves of curvature be then represented by equidistant parallel lines as above; and the construction would be only a little more difficult. The projection presented itself to me as a convenient one for the representation of the geodesic lines on the surface, and for exhibiting them in relation to the right lines of the surface; but I have not yet worked this out. In conclusion, it may be remarked that a surface in general cannot be divided into squares by its curves of curvature, but that it may be in an infinity of ways divided into squares by two systems of curves on the surface, and any such system of curves gives rise to a Mercator's projection of the surface.

NOTES AND REFERENCES.

518. RIBAUCOUR, *C. R.*, t. LXXV. (1872), pp. 533—536, referring to my Note remarks that the condition can be (by means of the imaginary coordinates of M. Ossian Bonnet) expressed in a simple form communicated by him to the Philomathic Society, May, 1870. I reproduce this investigation, although it is not easy to present it in a quite intelligible form. We take $p = f(x, y, z)$ to represent a family of surfaces belonging to a triply orthotomic system, and consider two neighbouring surfaces (A) and (A') corresponding to the values z and $z + dz$; A and A' the two points where they meet the trajectories of the surfaces; AT, $A'T'$ the tangents to the curves of curvature of the same system at A, A' respectively. Then according to the remark of M. Lévy, it is to be expressed that these lines meet, and this is done by expressing that *along the trajectory AA', the variation of the angle of AT with the osculating plane at A is equal to the angle of the osculating planes at A, A' respectively.*

Let B' be the spherical image of A', the plane OBB' is parallel to the osculating plane at A of the trajectory, and the angle of the two osculating planes measures the geodesic curvature of BB': denote this by $d\gamma$.

Let β be the angle of BB' with BX, θ the angle of AT with BX, $\beta - \theta$ is the angle of AT with the osculating plane at A of the trajectory: $d\beta - d\theta = d\gamma$. Introducing the symmetric imaginary coordinates x and y, we write

$$a = \frac{dp}{\lambda^2 dx}, \quad b = \frac{dp}{\lambda^2 dy}, \quad c = \frac{1}{\lambda^2}\frac{d^2p}{dx\,dy}, \quad ds^2 = 4\lambda^2 \frac{da}{dx}\frac{db}{dy}\,dx\,dy.$$

But dx and dy being the increments of x, y corresponding to dz in the passage from A to A', then by a theorem of M. Liouville

$$d\gamma = d\beta - i\left(\frac{d\lambda}{\lambda\,dx}\,dx - \frac{d\lambda}{\lambda\,dy}\,dy\right);$$

the condition thus is

$$d\theta = i\left(\frac{d\lambda}{\lambda\,dx}\,dx - \frac{d\lambda}{\lambda\,dy}\,dy\right),$$

and the formula

$$e^{-2i\theta} = \pm\sqrt{\frac{da}{dx}} \div \sqrt{\frac{db}{dy}},$$

enables this to be written in the definitive form

$$dx\,\frac{d}{dx}\,l\left(\lambda^4\,\frac{db}{dy} \div \frac{da}{dx}\right) + dy\,\frac{d}{dy}\,l\left(\frac{db}{dy} \div \lambda^4\,\frac{da}{dx}\right) + dz\left\{\frac{d}{dz}\left(l\,\frac{db}{dy}\right) - \frac{d}{dz}\left(l\,\frac{da}{dx}\right)\right\} = 0.$$

We have

$$dx\,(\tfrac{1}{2}p + c) + dy\,\frac{db}{dy} \qquad + dz\,\frac{db}{dz} = 0,$$

$$dx\,\frac{da}{dx} \qquad + dy\,(\tfrac{1}{2}p + c) + dz\,\frac{da}{dz} = 0,$$

and thence eliminating dx, dy, dz, we have

$$\begin{vmatrix} \frac{d}{dx}\,l\left(\lambda^4\frac{db}{dy} \div \frac{da}{dx}\right), & \frac{d}{dy}\,l\left(\frac{db}{dy} \div \lambda^4\frac{da}{dx}\right), & \frac{d}{dz}\,l\left(\frac{db}{dy} \div \frac{da}{dx}\right) \\ \frac{1}{2}p + c \quad , & \frac{db}{dy} \quad , & \frac{db}{dz} \\ \frac{da}{dx} \quad , & \frac{1}{2}p + c \quad , & \frac{da}{dz} \end{vmatrix} = 0,$$

which defines the triply orthotomic system.

END OF VOL. VIII.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

CAMBRIDGE UNIVERSITY PRESS.

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. New and Enlarged Edition. Crown 8vo. 7*s.* 6*d.*

Solutions to the Examples in a Treatise on Elementary Dynamics. By the same Author. Crown 8vo. 7*s.* 6*d.*

Plane Trigonometry. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. 7*s.* 6*d.* Or in separate Parts. Part I. up to and including the Solution of Triangles. 5*s.* Part II. Analytical Trigonometry. 3*s.* 6*d.*

Geometrical Conics. By F. S. MACAULAY, M.A., Assistant Master at St Paul's School. Crown 8vo. 4*s.* 6*d.*

A Treatise on Elementary Hydrostatics. By J. GREAVES, M.A., Fellow and Lecturer of Christ's College. Crown 8vo. 5*s.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

A Treatise on the General Principles of Chemistry. By M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A., and D. J. CARNEGIE, M.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

Lectures on the Physiology of Plants. By S. H. VINES, Sc.D., Professor of Botany in the University of Oxford. Demy 8vo. With Illustrations. 21*s.*

CAMBRIDGE NATURAL SCIENCE MANUALS.

Elementary Palaeontology for Geological Students. By HENRY WOODS, M.A., F.G.S. Crown 8vo. 6*s.*

Heat and Light. By R. T. GLAZEBROOK, M.A., F.R.S., Assistant Director of the Cavendish Laboratory. Crown 8vo. 5*s.*

The Two Parts are also published separately.

Heat. 3*s.* **Light.** 3*s.*

Mechanics and Hydrostatics. By R. T. GLAZEBROOK, M.A. Crown 8vo.
Part I. **Dynamics.** 4*s.* Part II. **Statics.** 3*s.* Part III. **Hydrostatics.** *Preparing.*

Solution and Electrolysis. By W. C. D. WHETHAM, M.A., Fellow of Trinity College. Crown 8vo. 7*s.* 6*d.* [*Nearly ready.*

Other Volumes preparing.

PITT PRESS MATHEMATICAL SERIES.

Arithmetic for Schools. By C. SMITH, M.A., Master of Sidney Sussex College, Cambridge. Extra Fcap. 8vo. With or without Answers. Second Edition, 3*s.* 6*d.* Or in two Parts 2*s.* each.

Key to Smith's Arithmetic. By G. HALE, M.A. Crown 8vo. 7*s.* 6*d.*

Elementary Algebra. By W. W. ROUSE BALL, M.A., Fellow and Mathematical Lecturer of Trinity College, Cambridge. Extra Fcap. 8vo. 4*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Extra Fcap. 8vo. Books I.—IV. 3*s.* Books I.—VI. 4*s.* Books I. and II. 1*s.* 6*d.* Books III. and IV. 1*s.* 6*d.* Books V. and VI. 1*s.* 6*d.* Books XI. and XII. [*In the Press.*

Solutions to the Exercises in Taylor's Euclid. Books I.—IV. By W. W. TAYLOR, M.A. 6*s.*

The Elements of Statics and Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. Third Edition. Extra Fcap. 8vo. 7*s.* 6*d.*
Also published separately STATICS, 4*s.* 6*d.* DYNAMICS, 3*s.* 6*d.*

Solutions to the Examples in the Elements of Statics and Dynamics. By S. L. LONEY, M.A. 7*s.* 6*d.*

Mechanics and Hydrostatics for Beginners. By S. L. LONEY, M.A. 4*s.* 6*d.*

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow of Christ's College, Cambridge, and University Lecturer in Mathematics, and C. M. JESSOP, M.A., Fellow of Clare College. Extra Fcap. 8vo. 4*s.* 6*d.*

London: C. J. CLAY AND SONS,

CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Collected Mathematical Papers of the late Arthur Cayley, Sc.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 4to. 10 vols. Vols. I. II. III. IV. V. VI. VII. and VIII. 25*s.* each. [Vol. IX. *In the Press.*

The Scientific Papers of the late Prof. J. Clerk Maxwell. Edited by W. D. NIVEN, M.A., formerly Fellow of Trinity College. In 2 vols. Royal 4to. £3. 3*s.* (net).

Mathematical and Physical Papers, by Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., Pres.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Demy 8vo. Vol. I. 18*s.* Vol. II. 15*s.* Vol. III. 18*s.*

The Scientific Papers of the late John Couch Adams, M.A., Sc.D., D.C.L., LL.D., F.R.S., F.R.A.S., F.C.P.S., &c., Fellow of Pembroke College, Honorary Fellow of St John's College, and Lowndean Professor of Astronomy and Geometry in the University of Cambridge. Vol. I. Edited by WILLIAM GRYLLS ADAMS, M.A., Sc.D., F.R.S., &c., Professor of Natural Philosophy in King's College, London, late Fellow of St John's College, Cambridge. With a Memoir by J. W. L. GLAISHER, Sc.D., F.R.S., &c., Fellow of Trinity College, Cambridge. Demy 4to. [*In the Press.*

A Treatise on the Mathematical Theory of Elasticity, by A. E. H. LOVE, M.A., Fellow of St John's College, Cambridge. In Two Volumes. Demy 8vo. Vol. I. 12*s.* Vol. II. 12*s.*

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.* Vol. II. Saint-Venant to Lord KELVIN (Sir WILLIAM THOMSON). In 2 parts. By the same Editor. 30*s.*

Geometrical Optics, Treatise on, by R. A. HERMAN, M.A., Fellow of Trinity College, Cambridge. Demy 8vo. [*Preparing.*

A Treatise on the Lunar Theory. By E. W. BROWN, M.A., Fellow of Christ's College, Cambridge, Professor of Applied Mathematics in Haverford College, U.S.A. Royal 8vo. [*In the Press.*

An Introduction to Abel's Theorem and the allied theory. By H. F. BAKER, M.A., Fellow of St John's College, Cambridge. Royal 8vo. [*In the Press.*

The Steam Engine and other Heat Engines. By J. A. EWING, M.A., F.R.S., M.Inst. C.E., Professor of Mechanism and Applied Mathematics in the University of Cambridge. Demy 8vo. 15*s.*

A Treatise on Natural Philosophy. By Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Lord KELVIN (Sir W. THOMSON) and Professor P. G. TAIT. Demy 8vo. 9*s.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of Peterhouse, Cambridge. Demy 8vo. Vol. I. 14*s.* Vol. II. 10*s.*

Theory of Differential Equations. Part I. Exact Equations and Pfaff's Problem. By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, and Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 8vo. 12*s.*

A Treatise on the Theory of Functions of a Complex Variable, by A. R. FORSYTH, Sc.D., F.R.S. Royal 8vo. 21*s.* Net.

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A., F.R.S., late Fellow of Trinity College, Cambridge, Professor of Mathematics in the Owens College, Manchester. Demy 8vo. New Edition. 20*s. Net.*

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., Fellow and Tutor of Christ's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on Spherical Astronomy. By Sir ROBERT S. BALL, M.A., F.R.S., Lowndean Professor of Astronomy and Geometry. Demy 8vo. [*Preparing.*

Elementary Thermodynamics. By J. PARKER, M.A., Fellow of St John's College, Cambridge. Crown 8vo. 9*s.*

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

Electricity and Magnetism. By J. J. THOMSON, M.A., Cavendish Professor of Experimental Physics. Crown 8vo. [*In the Press.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Demy 4to. cloth (Vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half morocco 28*s.* (net). New Series for the years 1874—1883. Vols. 9—10, cloth 25*s.* (net) each, half morocco 32*s.* (net) each. [Vol. XI. *In the Press.*

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. IX.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1896

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 74 papers, numbered 556 to 629, published for the most part in the years 1874 to 1877.

The Table for the nine volumes is

Vol.	I.	Numbers	1	to	100,
,,	II.	,,	101	,,	158,
,,	III.	,,	159	,,	222,
,,	IV.	,,	223	,,	299,
,,	V.	,,	300	,,	383,
,,	VI.	,,	384	,,	416,
,,	VII.	,,	417	,,	485,
,,	VIII.	,,	486	,,	555,
,,	IX.	,,	556	,,	629.

A. R. FORSYTH.

17 *December* 1895.

CONTENTS.

PAGE

556. *On Steiner's surface* 1

Proc. Lond. Math. Society, t. v. (1873—1874), pp. 14—25

557. *On certain constructions for bicircular quartics* 13

Proc. Lond. Math. Society, t. v. (1873—1874), pp. 29—31

558. *A geometrical interpretation of the equations obtained by equating to zero the resultant and the discriminants of two binary quantics* 16

Proc. Lond. Math. Society, t. v. (1873—1874), pp. 31—33

559. [*Note on inversion*] 18

Proc. Lond. Math. Society, t. v. (1873—1874), p. 112

560. [*Addition to Lord Rayleigh's paper "On the numerical calculation of the roots of fluctuating functions"*] 19

Proc. Lond. Math. Society, t. v. (1873—1874), pp. 123, 124

561. *On the geometrical representation of Cauchy's theorems of root-limitation* 21

Camb. Phil. Trans., t. XII. Part II. (1877), pp. 395—413

562. *On a theorem in maxima and minima: addition* [*to Mr Walton's paper*] *by Professor Cayley* 40

Quart. Math. Jour., t. X. (1870), pp. 262, 263

563. *Note on the transformation of two simultaneous equations* . . 42

Quart. Math. Jour., t. XI. (1871), pp. 266, 267

564. *On a theorem in elimination* 43

Quart. Math. Jour., t. XII. (1873), pp. 5, 6

PAGE

565. *Note on the Cartesian* 45
Quart. Math. Jour., t. XII. (1873), pp. 16—19

566. *On the transformation of the equation of a surface to a set of chief axes* 48
Quart. Math. Jour., t. XII. (1873), pp. 34—38

567. *On an identical equation connected with the theory of invariants* 52
Quart. Math. Jour., t. XII. (1873), pp. 115—118

568. *Note on the integrals* $\int_0^\infty \cos x^2\,dx$ *and* $\int_0^\infty \sin x^2\,dx$ 56
Quart. Math. Jour., t. XII. (1873), pp. 118—126

569. *On the cyclide* 64
Quart. Math. Jour., t. XII. (1873), pp. 148—165

570. *On the superlines of a quadric surface in five-dimensional space* 79
Quart. Math. Jour., t. XII. (1873), pp. 176—180

571. *A demonstration of Dupin's theorem* 84
Quart. Math. Jour., t. XII. (1873), pp. 185—191

572. *Theorem in regard to the Hessian of a quaternary function* . 90
Quart. Math. Jour., t. XII. (1873), pp. 193—197

573. *Note on the* (2, 2) *correspondence of two variables* 94
Quart. Math. Jour., t. XII. (1873), pp. 197, 198

574. *On Wronski's theorem* 96
Quart. Math. Jour., t. XII. (1873), pp. 221—228

575. *On a special quartic transformation of an elliptic function* . 103
Quart. Math. Jour., t. XII. (1873), pp. 266—269

576. *Addition to Mr Walton's paper "On the ray-planes in biaxal crystals"*. 107
Quart. Math. Jour., t. XII. (1873), pp. 273—275

577. *Note in illustration of certain general theorems obtained by Dr Lipschitz* 110
Quart. Math. Jour., t. XII. (1873), pp. 346—349

PAGE

578. *A memoir on the transformation of elliptic functions* . . 113

Phil. Trans., t. CLXIV. (for 1874), pp. 397—456

579. *Address delivered by the President, Professor Cayley, on presenting the Gold Medal of the [Royal Astronomical] Society to Professor Simon Newcomb* 176

Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 224—233

580. *On the number of distinct terms in a symmetrical or partially symmetrical determinant; with an addition* 185

Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 303—307; p. 335

581. *On a theorem in elliptic motion*. 191

Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 337—339

582. *Note on the Theory of Precession and Nutation* . . . 194

Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 340—343

583. *On spheroidal trigonometry* 197

Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), p. 92

584. *Addition to Prof. R. S. Ball's paper "Note on a transformation of Lagrange's equations of motion in generalised coordinates, which is convenient in Physical Astronomy"* . 198

Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), pp. 269—271

585. *A new theorem on the equilibrium of four forces acting on a solid body* 201

Phil. Mag., t. XXXI. (1866), pp. 78, 79; Camb. Phil. Soc. Proc., t. I. (1866), p. 235

586. *On the mathematical theory of isomers* 202

Phil. Mag., t. XLVII. (1874), pp. 444—467

587. *A Smith's Prize dissertation* [1873] 205

Messenger of Mathematics, t. III. (1874), pp. 1—4

588. *Problem [on tetrahedra]* 209

Messenger of Mathematics, t. III. (1874), pp. 50—52

PAGE

589. *On residuation in regard to a cubic curve* 211

Messenger of Mathematics, t. III. (1874), pp. 62—65

590. *Addition to Prof. Hall's paper "On the motion of a particle toward an attracting centre at which the force is infinite"* . 215

Messenger of Mathematics, t. III. (1874), pp. 149—152

591. *A Smith's Prize paper and dissertation* [1874]; *solutions and remarks* 218

Messenger of Mathematics, t. III. (1874), pp. 165—183; t. IV. (1875), pp. 6—8

592. *On the Mercator's projection of a skew hyperboloid of revolution* 237

Messenger of Mathematics, t. IV. (1875), pp. 17—20

593. *A Sheepshanks' problem* (1866) 241

Messenger of Mathematics, t. IV. (1875), pp. 34—36

594. *On a differential equation in the theory of elliptic functions* . 244

Messenger of Mathematics, t. IV. (1875), pp. 69, 70

595. *On a Senate-House problem* 246

Messenger of Mathematics, t. IV. (1875), pp. 75—78

596. *Note on a theorem of Jacobi's for the transformation of a double integral* 250

Messenger of Mathematics, t. IV. (1875), pp. 92—94

597. *On a differential equation in the theory of elliptic functions* . 253

Messenger of Mathematics, t. IV. (1875), pp. 110—113

598. *Note on a process of integration* 257

Messenger of Mathematics, t. IV. (1875), pp. 149, 150

599. *A Smith's Prize dissertation* 259

Messenger of Mathematics, t. IV. (1875), pp. 157—160

600. *Theorem on the n-th Roots of Unity* 263

Messenger of Mathematics, t. IV. (1875), p. 171

601. *Note on the Cassinian* 264

Messenger of Mathematics, t. IV. (1875), pp. 187, 188

PAGE

602. *On the potentials of polygons and polyhedra* 266
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 20—34

603. *On the potential of the ellipse and the circle* 281
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 38—58

604. *Determination of the attraction of an ellipsoidal shell on an exterior point* 302
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 58—67

605. *Note on a point in the theory of attraction* 312
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 79—81

606. *On the expression of the coordinates of a point of a quartic curve as functions of a parameter* 315
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 81—83

607. *A memoir on prepotentials.* 318
Phil. Trans., t. CLXV. (for 1875), pp. 675—774

608. [*Extract from a*] *Report on Mathematical Tables* . . . 424
Brit. Assoc. Report, 1873, pp. 3, 4

609. *On the analytical forms called factions* 426
Brit. Assoc. Report, 1875, Notices of Communications to the Sections, p. 10

610. *On the analytical forms called Trees, with application to the theory of chemical combinations* 427
Brit. Assoc. Report, 1875, pp. 257—305

611. *Report on mathematical tables* 461
Brit. Assoc. Report, 1875, pp. 305—336

612. *Note sur une formule d'intégration indéfinie* 500
Comptes Rendus, t. LXXVIII. (1874), pp. 1624—1629

613. *On the group of points* G_4^1 *on a sextic curve with five double points* 504
Math. Ann., t. VIII. (1875), pp. 359—362

614. *On a problem of projection* 508
Quart. Math. Jour., t. XIII. (1875), pp. 19—29

PAGE

615. *On the conic torus* 519

Quart. Math. Jour., t. XIII. (1875), pp. 127—129

616. *A geometrical illustration of the cubic transformation in elliptic functions* 522

Quart. Math. Jour., t. XIII. (1875), pp. 211—216

617. *On the scalene transformation of a plane curve* 527

Quart. Math. Jour., t. XIII. (1875), pp. 321—328

618. *On the mechanical description of a Cartesian* 535

Quart. Math. Jour., t. XIII. (1875), pp. 328—330

619. *On an algebraical operation* 537

Quart. Math. Jour., t. XIII. (1875), pp. 369—375

620. *Correction of two numerical errors in Sohnke's paper respecting modular equations* 543

Crelle, t. LXXXI. (1876), p. 229

621. *On the number of the univalent radicals* C_nH_{2n+1} 544

Phil. Mag., Ser. 5, t. III. (1877), pp. 34, 35

622. *On a system of equations connected with Malfatti's problem* . 546

Proc. Lond. Math. Society, t. VII. (1876), pp. 38—42

623. *On three-bar motion* 551

Proc. Lond. Math. Society, t. VII. (1876), pp. 136—166

624. *On the bicursal sextic* 581

Proc. Lond. Math. Society, t. VII. (1876), pp. 166—172

625. *On the condition for the existence of a surface cutting at right angles a given set of lines* 587

Proc. Lond. Math. Society, t. VIII. (1877), pp. 53—57

626. *On the general differential equation* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, *where* X, Y *are the same quartic functions of* x, y *respectively* . . 592

Proc. Lond. Math. Society, t. VIII. (1877), pp. 184—199

PAGE

627. *Geometrical illustration of a theorem relating to an irrational function of an imaginary variable* 609

Proc. Lond. Math. Society, t. VIII. (1877), pp. 212—214

628. *On the circular relation of Möbius* 612

Proc. Lond. Math. Society, t. VIII. (1877), pp. 220—225

629. *On the linear transformation of the integral* $\int \frac{du}{\sqrt{U}}$ . . . 618

Proc. Lond. Math. Society, t. VIII. (1877), pp. 226—229

Plate . . . . . *to face p.* 460

CLASSIFICATION.

GEOMETRY:

Möbius' circular relation, 628
Malfatti's problem, 622
Inversion, 559
Projections, 592, 614
Mechanical construction of curves, 557, 618
Quartic curve and functions of a single parameter, 606
Cartesian, 565
Cassinian, 601
Sextic curves, 613, 624
Correspondence, residuation and transformation, 566, 573, 589, 617
Conformal representation, 627
Illustrations of algebraical theorems, 558, 561
Three-bar motion, 623
Spheroidal Trigonometry, 583
Dupin's theorem, 571
Cyclide, 569
Steiner's surface, 556
Surface orthogonal to set of lines, 625
Conic torus, 615
Quadrics in hyperdimensional space, 570

ASTRONOMY and DYNAMICS:

Presidential address, 579
Elliptic motion, 581
Precession and nutation, 582
General equations of dynamics, 577, 584, 590
Potentials, attractions and prepotentials, 602, 603, 604, 605, 607

ANALYSIS:

Determinant, number of terms in, 580
Elimination, 564
Invariants and covariants, 567, 572, 619
Mathematical tables, 608, 611
Trees, and their applications to chemistry, 586, 610, 621
Roots of Unity, 600
Transformation of equations, 563
Maxima and minima, 562
Factions, 609
Fluctuating functions, 560
Wronski's theorem of expansion, 574
Integration and definite integrals, 568, 596, 598, 612
Elliptic functions, 626, 629
Transformation of elliptic functions, 575, 578, 594, 597, 616, 620
Smith's Prize Dissertations and Solutions, 587, 591, 599
Miscellaneous, 576, 585, 588, 593, 595

556.

ON STEINER'S SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), pp. 14—25. Read December 11, 1873.]

I HAVE constructed a model and drawings of the symmetrical form of Steiner's Surface, viz. that wherein the four singular tangent planes form a regular tetrahedron, and consequently the three nodal lines (being the lines joining the mid-points of opposite edges) a system of rectangular axes at the centre of the tetrahedron. Before going into the analytical theory, I describe as follows the general form of the surface: take the tetrahedron, and inscribe in each face a circle (there will be, of course, two circles touching at the mid-point of each edge of the tetrahedron; each circle will contain, on its circumference at angular distances of 120°, three mid-points, and the lines joining these with the centre of the tetrahedron, produced beyond the centre, meet the opposite edges, and are in fact the before-mentioned lines joining the mid-points of opposite edges). Now truncate the tetrahedron by planes parallel to the faces so as to reduce the altitudes each to three-fourths of the original value, and from the centre of each new face round off symmetrically up to the adjacent three circles; and within each circle scoop down to the centre of the tetrahedron, the bounding surface of the excavation passing through the three right lines, and the sections (by planes parallel to the face) being in the neighbourhood of the face nearly circular, but as they approach the centre, assuming a trigonoidal form, and being close to the centre an indefinitely small equilateral triangle. We have thus the surface, consisting of four lobes united only by the lines through the mid-points of opposite edges, these lines being consequently nodal lines; the mid-points being pinch-points of the surface, and the faces singular planes, each touching the surface along the inscribed circle. The joining lines, produced indefinitely both ways, belong as nodal lines to the surface; but they are, outside the tetrahedron, mere acnodal lines not traversed by any real sheet of the surface.

We may imagine the tetrahedron placed in two different positions, (1) resting with one of its faces on the horizontal plane, (2) with two opposite edges horizontal, or say with the horizontal plane passing through the centre of the tetrahedron and being parallel to two opposite edges; or, what is the same thing, the nodal lines form a system of rectangular axes, one of them, say that of z, being vertical. And I proceed to consider, in the two cases respectively, the horizontal sections of the surface.

In the first case, the coordinates x, y, z, w may be taken to be the perpendicular distances of a point from the faces of the tetrahedron, w being the distance from the base. We have*, if the altitude be h,

$$x+y+z+w=h;$$

an equation which may be used to homogenize any equation not originally homogeneous; thus, for the plane $w=\lambda$, of altitude λ, we have

$$w=\frac{\lambda}{h}(x+y+z+w),$$

or, what is the same thing,

$$w=\frac{\lambda}{h-\lambda}(x+y+z).$$

The equation of the surface is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{w}=0,$$

and if we herein consider w as having the last-mentioned value, the equation will belong to the section by the plane $w=\lambda$. I remark that the section of the tetrahedron, by this plane, is an equilateral triangle, the side of which is to an edge of the tetrahedron as $h-\lambda : h$. For a point in the plane of the triangle, if X, Y, Z are the perpendiculars on the sides, then

$$X+Y+Z=P,$$

(if for a moment P is the perpendicular from a vertex on the opposite side of the triangle, viz. we have $P=\frac{h-\lambda}{h}p$, if p be the perpendicular for a face of the tetrahedron). And it is clear that x, y, z are proportional to X, Y, Z; we consequently have, for the equation of the section,

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{\frac{\lambda}{h-\lambda}(X+Y+Z)}=0,$$

* I take the opportunity of remarking that in a regular tetrahedron, if s be the length of an edge, p the perpendicular from a summit on an edge (or altitude of a face), h the perpendicular from a summit on a face (or altitude of the tetrahedron), and q the distance between the mid-points of opposite edges, then

$$s=\frac{\sqrt{3}}{\sqrt{2}}h,\quad p=\frac{3}{2\sqrt{2}}h,\quad q=\frac{\sqrt{3}}{2}h.$$

The tetrahedron can, by means of planes through the mid-points of the edges at right angles thereto, be divided into four hexahedral figures (8 summits, 6 faces, 12 edges, each face a quadrilateral); viz. in each such figure there are, meeting in a summit of the tetrahedron, three edges, each $=\frac{1}{2}s$; meeting in the centre three edges, each $=\frac{1}{4}h$; and six other edges, each $=\frac{1}{3}p$.

where the coordinates X, Y, Z are the perpendicular distances from the sides of the triangle which is the section of the tetrahedron. To simplify, I write

$$\frac{\lambda}{h-\lambda} = 2q+1,$$

that is,

$$q = \frac{2\lambda - h}{2h - 2\lambda};$$

the equation then is

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} + \sqrt{(2q+1)(X+Y+Z)} = 0;$$

or, proceeding to rationalize, we have first

$$q(X+Y+Z) = \sqrt{YZ} + \sqrt{ZX} + \sqrt{XY},$$

and thence

$$q^2(X+Y+Z)^2 - (YZ+ZX+XY) = 2\sqrt{XYZ}(\sqrt{X}+\sqrt{Y}+\sqrt{Z});$$

and finally

$$\{q^2(X+Y+Z)^2 - YZ - ZX - XY\}^2 = 4(2q+1)XYZ(X+Y+Z).$$

This is a quartic curve, having for double tangents the four lines $X=0$, $Y=0$, $Z=0$, $X+Y+Z=0$, the last of these being the line infinity touching the curve in two imaginary points, since obviously the whole real curve lies within the triangle. This is as it should be: the double tangents are the intersections of the plane $w=\lambda$ by the singular planes of the surface.

To find the points of contact, writing for instance $Z=0$, the equation becomes

$$q^2(X+Y)^2 - XY = 0,$$

that is,

$$X^2 + \left(2 - \frac{1}{q^2}\right)XY + Y^2 = 0;$$

whence

$$X = \left(-1 + \frac{1}{2q^2} \pm \sqrt{\frac{1}{4q^4} - \frac{1}{q^2}}\right)Z,$$

giving the two points of contact equi-distant from the centre; these are imaginary if $q > \frac{1}{2}$, but otherwise real, which agrees with what follows. (See the Table afterwards referred to.)

The nodal lines of the surface are $(x-y=0,\ z-w=0)$, $(y-z=0,\ x-w=0)$, $(z-x=0,\ y-w=0)$. Considering the first of these, we have for its intersection with the plane $w=\lambda$,

$$X = Y,\quad Z = \frac{\lambda}{h-\lambda}(X+Y+Z),\ = (2q+1)(X+Y+Z),$$

and the last equation gives

$$Z = (2q+1)(2X+Z),$$

that is,

$$0 = (2q+1)X + qZ,$$

so that for the point in question we have $X : Y : Z = -q : -q : 2q+1$; and taking the perpendicular from the vertex on a side as unity, the values $-q$, $-q$, $2q+1$ will be absolute magnitudes. We thus see that the curve must have the three nodes $(2q+1, -q, -q)$, $(-q, 2q+1, -q)$, $(-q, -q, 2q+1)$; and it is easy to verify that this is so.

The curve will pass through the centre $X = Y = Z$, if

$$(9q^2-3)^2 = 12(2q+1),$$

that is, if

$$3(3q^2-1)^2 - 4(2q+1) = 0,$$

or if

$$(3q+1)^2(q-1) = 0.$$

If $q = 1$, that is, $\lambda = 3(h-\lambda)$, or $\lambda = \frac{3}{4}h$, the equation is

$$(X^2+Y^2+Z^2+YZ+ZX+XY)^2 - 12XYZ(X+Y+Z) = 0,$$

where the curve is, in fact, a pair of imaginary conics meeting in the four real points $(3, -1, -1)$, $(-1, 3, -1)$, $(-1, -1, 3)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To verify this, observe that, writing

$$\begin{aligned} A &= (Y-Z)(2X+Y+Z), \\ B &= (Z-X)(X+2Y+Z), \\ C &= (X-Y)(X+Y+2Z), \end{aligned}$$

and therefore

$$A+B+C = 0,$$

the function in (X, Y, Z) is $= \frac{1}{2}(A^2+B^2+C^2)$, and thus the equation may be written in the equivalent forms

$$B^2+BC+C^2 = 0, \quad C^2+CA+A^2 = 0, \quad A^2+AB+B^2 = 0,$$

each of which shows that the curve breaks up into two imaginary conics. The foregoing value $q = 1$, or $\lambda = \frac{3}{4}h$, belongs to the summit or highest real point of the surface.

In the case $3q+1 = 0$, that is,

$$q = -\tfrac{1}{3} = \frac{2\lambda - h}{2h - 2\lambda}, \quad \text{or } \lambda = \tfrac{1}{4}h,$$

the equation is

$$\{(X+Y+Z)^2 - 9(YZ+ZX+XY)\}^2 = 108XYZ(X+Y+Z),$$

which is, in fact, the equation of a curve having the centre, or point $X = Y = Z$, for a triple point.

To verify this, write

$$\begin{aligned} X &= \beta - \gamma + u, \\ Y &= \gamma - \alpha + u, \\ Z &= \alpha - \beta + u; \end{aligned}$$

also

$$2\Delta = (\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2,$$
$$\Omega = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Then we have

$$X + Y + Z = 3u,$$
$$YZ + ZX + XY = 3u^2 - \Delta,$$
$$XYZ = u^3 - u\Delta + \Omega,$$

and the equation is

$$\{9u^2 - 9(3u^2 - \Delta)\}^2 - 324u(u^3 - u\Delta + \Omega) = 0,$$

that is,

$$(-2u^2 + \Delta)^2 - 4u(u^3 - u\Delta + \Omega) = 0,$$

or finally

$$\Delta^2 - 4u\Omega = 0,$$

where the lowest terms in $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ are of the order 3, and the theorem is thus proved. The case in question, $q = -\frac{1}{3}$ or $\lambda = \frac{1}{4}h$, is where the plane passes through the centre of the tetrahedron.

When $q = \frac{1}{2} = \dfrac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{2}{3}h$, the equation is

$$(X^2 + Y^2 + Z^2 - 2YZ - 2ZX - 2XY)^2 = 128XYZ(X + Y + Z).$$

Here each of the lines $X = 0$, $Y = 0$, $Z = 0$ is an osculating tangent having with the curve a 4-pointic intersection.

When $q = 0 = \dfrac{2\lambda - h}{2h - 2\lambda}$, or $\lambda = \frac{1}{2}h$, the equation is

$$(YZ + ZX + XY)^2 - 4XYZ(X + Y + Z) = 0,$$

that is,

$$Y^2Z^2 + Z^2X^2 + X^2Y^2 - 2XYZ(X + Y + Z) = 0;$$

viz. each angle of the triangle is here a cusp.

When $q = -\frac{1}{2}$, or $\lambda = 0$, the curve is

$$\{X^2 + Y^2 + Z^2 - 2(YZ + ZX + XY)\}^2 = 0,$$

viz. the plane is here the base of the tetrahedron, and the section is the inscribed circle taken twice.

For tracing the curves, it is convenient to find the intersections with the lines $Y - Z = 0$, $Z - X = 0$, $X - Y = 0$ drawn from the centre of the triangle to the vertices; each of these lines passes through a node, and therefore besides meets the curve in two points. Writing, for instance, $Y = X$, the equation becomes

$$\{q^2(2X + Z)^2 - 2XZ - X^2\}^2 - 4(2q + 1)X^2Z(2X + Z) = 0;$$

viz. this is

$$\{qZ + (2q + 1)X\}^2\{q^2Z^2 + (4q^2 - 2q - 4)XZ + (4q^2 - 4q + 1)X^2\} = 0,$$

where the first factor gives the node. Equating to zero the second factor, we have

$$\left\{qZ+\left(2q-1-\frac{2}{q}\right)X\right\}^2 = X^2\left\{\left(2q-1-\frac{2}{q}\right)^2-4q^2+4q-1\right\}$$

$$= X^2\frac{4}{q^2}(1-q)(1+2q);$$

or, finally,

$$qZ=\left\{-2q+1+\frac{2}{q}\pm\frac{2}{q}\sqrt{(1-q)(1+2q)}\right\}X,$$

giving two real values for all values of q from $q=1$ to $q=-\frac{1}{2}$. (See the Table afterwards referred to.)

We may recapitulate as follows:

$q>1$, or $\lambda>\frac{3}{4}h$; the curve is imaginary, but with three real acnodes, answering to the acnodal parts of the nodal lines:

$q=1$, or $\lambda=\frac{3}{4}h$; the summit appears as a fourth acnode:

$q<1>\frac{1}{2}$, or $\lambda<\frac{3}{4}h>\frac{2}{3}h$; the curve consists of three acnodes and a trigonoid lying within the triangle and having the sides of the triangle for bitangents of imaginary contact:

$q=\frac{1}{2}$, or $\lambda=\frac{2}{3}h$; the curve consists of three acnodes and a trigonoid having the sides of the triangle for osculating tangents:

$q<\frac{1}{2}>0$, or $\lambda<\frac{2}{3}h>\frac{1}{2}h$; the curve consists of three conjugate points and an indented trigonoid having the sides of the triangle for bitangents of real contact:

$q=0$, or $\lambda=\frac{1}{2}h$; curve has the summits of the triangle for cusps:

$q<0>-\frac{1}{2}$, or $\lambda<\frac{1}{2}h>\frac{1}{4}h$; curve has three crunodes, or say it is a cis-centric trifolium:

$q=-\frac{1}{3}$, or $\lambda=\frac{1}{4}h$; curve has a triple point, or say it is a centric trifolium:

$q<-\frac{1}{3}>-\frac{1}{2}$, or $\lambda<\frac{1}{4}h>0$; curve has three crunodes, or say it is a trans-centric trifolium:

$q=-\frac{1}{2}$, or $\lambda=0$; curve is a two-fold circle:

$q<-\frac{1}{2}$, or $\lambda<0$; the curve becomes again imaginary, consisting of three acnodes answering to the acnodal parts of the nodal lines.

For the better delineation of the series of curves, I calculated the following Table, wherein the first column gives a series of values of $\lambda : h$; the second the corresponding values of q, $=\dfrac{2\lambda-h}{2h-2\lambda}$; the third the positions of the point of contact, say with the side $Z=0$, the value of $X : Y$ being calculated from the foregoing formula,

$$X\div Y=-1+\frac{1}{2q^2}\pm\sqrt{\frac{1}{4q^4}-\frac{1}{q^2}};$$

and the fourth the apsidal distances, say for the radius vector $X = Y$, the value of $Z : X$ being calculated from the foregoing formula

$$Z \div X = -2 + \frac{1}{q} + \frac{2}{q^2} \pm \frac{2}{q}\sqrt{\left(\frac{1}{q} - 1\right)\left(2 + \frac{1}{q}\right)}.$$

The Table is:

	$\lambda : h$	q	Contact, $Z=0$, $X : Y=$			Apses, $X=Y$; $X : Z=$		
*	·75	1·00	imposs.			1·		
	·70	·6666	imposs.			·032	or	7·968
*	·666	·5	1·			0·		16·
	·65	·4285	·320	or	3·124	·006		22·44
	·6	·25	·0721		13·9279	·059		67·941
	·55	·1111	·011		78·988	·15		337·85
*	·5	0·	0	or	∞	·25		∞
	·45	−·0909	·005		118·99	·39		457·61
	·4	−·1666	·029		33·971	·496		127·504
	·35	−·2308	·060		16·72	·648		61·79
*	·3	−·2857	·099		10·151	·812		37·187
*	·25	−·3333	·141		6·854	1·		25
	·2	−·375	·207		4·904	1·218		17·89
	·15	−·4118	·276		3·622	1·48		13·25
	·10	−·4444	·372		2·690	1·813		9·937
	·05	−·4737	·515	or	1·941	3·15	or	6·46
*	·0	−·5	1·			4·		

where the asterisks show the critical values of $\lambda : h$.

It is worth while to transform the equation to new coordinates X', Y', Z' such that $X' = 0$, $Y' = 0$, $Z' = 0$ represent the sides of the triangle formed by the three nodes. Writing for shortness $X + Y + Z = P$, $YZ + ZX + XY = Q$, $XYZ = R$, the equation is

$$(q^2P - Q)^2 = 4(2q + 1)PR.$$

The expressions of X, Y, Z in terms of the new coordinates are of the form $X' + \theta P'$, $Y' + \theta P'$, $Z' + \theta P'$, where $P' = X' + Y' + Z'$; writing also $Q' = Y'Z' + Z'X' + X'Y'$, $R' = X'Y'Z'$, then the values of P, Q, R are

$$(1 + 3\theta)P', \quad Q' + (2\theta + 3\theta^2)P'^2, \quad R' + \theta P'Q' + (\theta^2 + \theta^3)P',$$

and the transformed equation is

$$[\{q^2(1+3\theta)^2-2\theta-3\theta^2\}P'^2-Q']^2=4(2q+1)(1+3\theta)P\{(\theta^2+\theta^3)P'^3+\theta P'Q'+R'\},$$

which is satisfied by $Q'=0$, $R'=0$, if only

$$\{q^2(1+3\theta)^2-2\theta-3\theta^2\}^2=4(2q+1)(1+3\theta)(\theta^2+\theta^3),$$

or, if for a moment $q(1+3\theta)=\Omega$, the equation is

$$(\Omega^2-2\theta-3\theta^2)^2=4(\theta^2+\theta^3)(2\Omega+1+3\theta),$$

that is,

$$\Omega^4+\Omega^2(-6\theta^2-4\theta)+\Omega(-8\theta^3-8\theta^2)-3\theta^4-4\theta^3=0,$$

that is,

$$(\Omega+\theta)^2(\Omega^2-2\theta\Omega-3\theta^2-4\theta)=0.$$

If the new axes pass through the nodes, then $\Omega+\theta=0$; that is, $q(1+3\theta)+\theta=0$, which equation gives the value of θ for which the new axes have the position in question; substituting in the first instance for q the value $\dfrac{-\theta}{3\theta+1}$, the equation becomes

$$\{2\theta(1+\theta)P'^2+Q'\}^2=4(1+\theta)P'\{\theta^2(1+\theta)P'^3+\theta P'Q'+R'\},$$

that is,

$$Q'^2=4(1+\theta)P'R';$$

or, finally, substituting for θ its value in terms of q, the required equation is,

$$Q'^2=4\frac{2q+1}{3q+1}P'R',$$

that is,

$$(Y'Z'+Z'X'+X'Y')^2=4\frac{2q+1}{3q+1}X'Y'Z'(X'+Y'+Z').$$

In particular, for $q=0$ the equation is

$$(Y'Z'+Z'X'+X'Y')^2-4X'Y'Z'(X'+Y'+Z')=0,$$

which is right, since, in the case in question (the tricuspidal curve), we have

$$X,\ Y,\ Z=X',\ Y',\ Z'.$$

I remark, in passing, that, taking the equation to be

$$(Y'Z'+Z'X'+X'Y')^2=mX'Y'Z'(X'+Y'+Z'),$$

we may write herein

$$Z'=\tfrac{1}{3}-x,$$

$$X'=\tfrac{1}{3}+\tfrac{1}{2}x-\frac{\sqrt{3}}{2}y,$$

$$Y'=\tfrac{1}{3}+\tfrac{1}{2}x+\frac{\sqrt{3}}{2}y;$$

where

$$x = \frac{2\sqrt{m(m-3)}}{9}\cos\theta - \frac{2(m-3)}{9}\cos 2\theta,$$

$$y = \frac{2\sqrt{m(m-3)}}{9}\sin\theta + \frac{2(m-3)}{9}\sin 2\theta,$$

which are the formulæ for the description of the trinodal quartic as a unicursal curve.

I consider now the second position; viz. the horizontal plane now passes through the centre of the tetrahedron, and is parallel to two opposite edges. The equations of the nodal lines are here $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$; and if for convenience we assume the distance of the mid-points of opposite edges to be $=2$, or the half of this $=1$, then the equations of the faces are

$$\begin{aligned} X &= x+y+z-1=0,\\ Y &= -x-y+z-1=0,\\ Z &= x-y-z-1=0,\\ W &= -x+y-z-1=0, \end{aligned}$$

and the equation of the surface is

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{W}=0.$$

Proceeding to rationalise, this is

$$X+Y+2\sqrt{XY}=Z+W+2\sqrt{ZW},$$

viz.

$$2z+\sqrt{XY}=\sqrt{ZW};$$

we thence have

$$4z^2+4z\sqrt{XY}+XY=ZW;$$

or, since

$$ZW-XY=4z+4xy,$$

this is

$$z+xy-z^2=z\sqrt{XY};$$

whence

$$(z+xy-z^2)^2=z^2\{(z-1)^2-(x+y)^2\};$$

or reducing,

$$2xyz+y^2z^2+z^2x^2+x^2y^2=0,$$

a form which puts in evidence the nodal lines. Considering z as constant, we have the equation of the section; this is a quartic having the node $(x=0,\ y=0)$, and two other nodes at infinity on the two axes respectively; moreover, the curve has for bitangents the intersections of its plane with the faces of the tetrahedron; or what is the same thing, attributing to z its constant value, the equations of the bitangents are

$$\begin{aligned} x+y+z-1&=0,\\ -x-y+z-1&=0,\\ x-y-z-1&=0,\\ -x+y-z-1&=0. \end{aligned}$$

These lines form a rectangle which is the section of the tetrahedron; observe that this is inscribed in the square the corners of which are $x=\pm 1$, $y=\pm 1$; viz. $z=+1$ (highest section), this is the dexter diagonal (considered as an indefinitely thin rectangle), and as z diminishes, the longer side decreases and the shorter increases until for $z=0$ (central section) the rectangle becomes a square; after which, for z negative it again becomes a rectangle in the conjugate direction, and finally, for $z=-1$ (lowest section) it becomes the sinister diagonal (considered as an indefinitely thin rectangle). But on account of the symmetry it is sufficient to consider the upper sections for which z is positive. The sides $\pm(x+y)+z-1=0$ parallel to the dexter diagonal of the square may for convenience be termed the dexter sides, and the others the sinister sides. In what follows I write c to denote the constant value of z.

We require to know whether the bitangents have real or imaginary contacts; and for this purpose to find the coordinates of the points of contact.

Take first a dexter bitangent $x+y+c-1=0$; the coordinates of any point hereof are

$$x=\tfrac{1}{2}(1-c+\theta),\quad y=\tfrac{1}{2}(1-c-\theta),$$

where θ is arbitrary; and substituting in the equation of the curve, we should have for θ a twofold quadric equation, giving the values of θ for the two points of contact respectively. We have

$$x^2+y^2=\tfrac{1}{2}\{(1-c)^2+\theta^2\},\quad xy=\tfrac{1}{4}\{(1-c)^2-\theta^2\},$$

and thence

$$8c^2\{(1-c)^2+\theta^2\}+8c\{(1-c)^2-\theta^2\}+\{(1-c)^2-\theta^2\}^2=0,$$

viz. this equation is

$$\{\theta^2-(1-c)(1+3c)\}^2=0,$$

a twofold quadric equation, as it should be; and the values of θ being $=\pm\sqrt{(1-c)(1+3c)}$, we see that these, and therefore the contacts, are real from $c=1$ to $c=-\frac{1}{3}$.

In exactly the same way for a sinister bitangent $\pm(x-y)-c-1=0$, we have

$$x=\tfrac{1}{2}(1+c+\phi),\quad -y=\tfrac{1}{2}(1+c-\phi),\quad \text{and}\quad \phi=\pm\sqrt{(1-3c)(1+c)},$$

viz. the values of ϕ, and therefore the contacts, are real from $c=\frac{1}{3}$ to $c=-1$.

That is,

	Contacts of Dexter Bitangents.	Contacts of Sinister Bitangents.
$c=1$ to $\frac{1}{3}$	real,	imaginary,
$c=\frac{1}{3}$ to $-\frac{1}{3}$	real,	real,
$c=-\frac{1}{3}$ to -1	imaginary,	real;

or say $c=1$ to $\frac{1}{3}$, the contacts are real, imaginary; but $c=\frac{1}{3}$ to 0, they are real, real. In the transition case, $c=\frac{1}{3}$, the sinister bitangents become osculating (4-pointic) tangents touching at points on the dexter diagonal. This can be at once verified.

Observe that when $c=1$, we have

$$(x+y)^2+x^2y^2=0,$$

so that the only real point is $x=0$, $y=0$; viz. this is a tacnode, having the real tangent $x+y=0$. For $c=0$ (central section) the equation becomes $x^2y^2=0$; viz. the curve is here the two nodal lines each twice.

It is now easy to trace the changes of form.

$c=1$; curve is a tacnode, as just mentioned, tangent the dexter diagonal.

$c<1>\frac{1}{3}$; curve is a figure of 8 inside the rectangle, having real contacts with the dexter sides, but imaginary contacts with the sinister sides.

$c=\frac{1}{3}$; curve is a figure of 8 having real contacts with the dexter sides, and osculating (4-pointic) contacts with the sinister sides.

$c<\frac{1}{3}>0$; curve is an indented figure of 8 having real contacts as well with the sinister as the dexter sides.

$c=0$; curve is squeezed up into a finite cross, being the crunodal parts of the nodal lines; and joined on to these we have the acnodal parts, so that the whole curve consists of the lines $x=0$, $y=0$ each as a twofold line.

For tracing the curve, it is convenient to turn the axes through an angle of 45°; viz. writing $\frac{y+x}{\sqrt{2}}$, $\frac{y-x}{\sqrt{2}}$ in place of x, y respectively, the equation becomes

$$c(y^2-x^2)+c^2(y^2+x^2)+\tfrac{1}{4}(y^2-x^2)^2=0;$$

$$x=0 \text{ gives } y^2=0 \text{ or } y^2=-4c(1+c),^*$$

$$y=0 \text{ gives } x^2=0 \text{ or } x^2=4c(1-c).$$

Moreover, we have

$$4(c-c^2)(y^2-x^2)+8c^2y^2+(y^2-x^2)^2=0,$$

viz.

$$(x^2-y^2+2c^2-2c)^2=4c^2\{(c-1)^2-2y^2\},$$

and similarly

$$(y^2-x^2+2c^2+2c)^2=4c^2\{(c+1)^2-2x^2\},$$

putting in evidence the bitangents, now represented by the equations $c-1=\pm y\sqrt{2}$ and $c+1=\pm x\sqrt{2}$ respectively. And for the first of these, or $c-1=\pm y\sqrt{2}$, we have for the points of contact $x^2=\frac{1}{2}(1-c)(1+3c)$; and for the second of them, or $c+1=\pm x\sqrt{2}$, the points of contact are $y^2=\frac{1}{2}(1+c)(1-3c)$.

I consider the circumscribed cone having its vertex at a point $(0, 0, \gamma)$ on the nodal line $(x=0, y=0)$. Writing in the equation of the surface $x=\lambda(z-\gamma)$, $y=\mu(z-\gamma)$, the equation, throwing out the factor $(z-\gamma)^2$, becomes

$$2\lambda\mu z+(\lambda^2+\mu^2)z^2+\lambda^2\mu^2(z-\gamma)^2=0,$$

that is,

$$\begin{aligned}&(\lambda^2\mu^2+\lambda^2+\mu^2)z^2\\&+2(-\gamma\lambda\mu+1)z\lambda\mu\\&+\qquad\gamma^2\ .\ \lambda^2\mu^2=0;\end{aligned}$$

* y always imaginary when c is positive.

and equating to zero the discriminant in regard to z, we have

$$\gamma^2(\lambda^2\mu^2+\lambda^2+\mu^2)-(-\gamma\lambda\mu+1)^2=0,$$

that is,

$$\gamma^2(\lambda^2+\mu^2)+2\gamma\lambda\mu-1=0;$$

and substituting herein the values $\lambda=\dfrac{x}{z-\gamma}$ and $\mu=\dfrac{y}{z-\gamma}$, we have the equation of the cone, viz. this is

$$\gamma^2(x^2+y^2)+2\gamma xy-(z-\gamma)^2=0,$$

or, what is the same thing,

$$\gamma^2(x^2+y^2-1)+2\gamma(xy+z)-z^2=0;$$

viz. this is a quadric cone having for its principal planes $z-\gamma=0$, $x+y=0$, $x-y=0$, these last being the planes through the nodal line and the two edges of the tetrahedron. In the particular case $\gamma=\infty$, the cone becomes the circular cylinder $x^2+y^2-1=0$.

The cone intersects the plane $z=0$ in the conic

$$\gamma^2(x^2+y^2-1)+2\gamma xy=0,$$

which is a conic passing through the corners of the square $(x=0,\ y=\pm 1)$, $(x=\pm 1,\ y=0)$. For $\gamma>1$, that is, for an exterior point, the conic is an ellipse having for the squares of the reciprocals of the semi-axes $1+\dfrac{1}{\gamma}$, $1-\dfrac{1}{\gamma}$ (this at once appears by writing in the equation $\dfrac{x+y}{\sqrt{2}}$, $\dfrac{x-y}{\sqrt{2}}$ in place of x, y respectively). In particular, for $\gamma=\infty$, the curve becomes the circle $x^2+y^2-1=0$. We have thus the apparent contour of the surface as seen from the point $z=\gamma$ on the nodal line, projected on the plane $z=0$ of the other two nodal lines.

To find the curve of contact of the cone and surface, or say the surface-contour from the same point, write for a moment

$$V=\gamma(x^2+y^2-1)+2\gamma(xy+z)-z^2,$$
$$U=(xy+z)^2+z^2(x^2+y^2-1);$$

then, substituting for x^2+y^2-1 its value in terms of V from the first equation, we find

$$U=\left(xy+z-\frac{z^2}{\gamma}\right)^2+\frac{z^2}{\gamma^2}V,$$

and the equations $U=0$, $V=0$ give therefore $xy+z-\dfrac{z^2}{\gamma}=0$, or say $\gamma(xy+z)-z^2=0$. The cone and surface therefore touch along the quadriquadric curve

$$\gamma^2(x^2+y^2-1)+2\gamma(xy+z)-z^2=0,$$
$$\gamma(xy+z)-z^2=0,$$

equations which may be replaced by

$$\gamma\ (x^2+y^2-1)+xy+z=0,$$
$$\gamma^2(x^2+y^2-1)+z^2\qquad=0.$$

In the case $\gamma=\infty$, the equations are $x^2+y^2-1=0$, $xy+z=0$, viz. the curve is the intersection of the hyperbolic paraboloid $xy+z=0$ by the cylinder $x^2+y^2-1=0$.

557.

ON CERTAIN CONSTRUCTIONS FOR BICIRCULAR QUARTICS.

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), pp. 29—31. Read March 12, 1874.]

I CALL to mind that if F, G are any two points and F', G' their antipoints; then the circle on the diameter FG and that on the diameter $F'G'$ are concentric orthotomics, viz. they have the same centre, and the sum of the squared radii is $=0$. Moreover, if the circles B, B' are concentric orthotomics, and the circle A is orthotomic to B, then it is a bisector of B', viz. it cuts B' at the extremities of a diameter of B'; and B' is then said to be a bifid circle in regard to A.

Given two real circles, these have an axial orthotomic, the circle, centre on the line of centres at its intersection with the radical axis, which cuts at right angles the given circles; viz. this axial orthotomic is real if the circles have no real intersection; but if the intersections are real, then the axial orthotomic is a pure imaginary, and instead thereof we may consider its concentric orthotomic, viz. this is the axial bifid of the two circles, or circle having its centre on the line of centres at the intersection thereof with the radical axis or common chord of the two circles, and having this common chord for its diameter.

If one of the circles is a pure imaginary, then we have still an axial orthotomic; viz. the pure imaginary circle is replaced by the concentric orthotomic; and the axial orthotomic is a bisector of the substituted circle; and so if each of the circles is a pure imaginary, then we have still an axial orthotomic, viz. each circle is replaced by the concentric orthotomic, and the axial orthotomic is a bisector of the substituted circles. And in either case the axial orthotomic of the original circles (one or each of them pure imaginary) *is real;* viz. this is given either as the axial bisector of one real circle and orthotomic of another real circle; or as the axial bisector of two circles, from which the reality thereof easily appears. Or we may verify it thus: Suppose

that the two circles are $(x-\alpha)^2+y^2=\beta^2$, $(x-\alpha')^2+y^2=\beta'^2$, and their axial orthotomic $(x-m)^2+y^2=k^2$, then we have $(m-\alpha)^2=\beta^2+k^2$, $(m-\alpha')^2=\beta'^2+k^2$; subtracting, it appears that m is real; and then if either β^2 or β'^2 is negative, the equation containing this quantity shows that k^2 is positive; viz. the circle $(x-m)^2+y^2=k^2$ is real.

The above remarks have an obvious application to the theory of bicircular quartics; viz. a bicircular quartic is the envelope of a variable circle, having its centre on a conic, and orthotomic to a circle: it may be that this circle is a pure imaginary. We then replace it by the concentric orthotomic, and say that the curve is the envelope of a variable circle having its centre on a conic and bisecting a circle. We have thus a real form for cases which originally present themselves under an imaginary form.

The Bicircular Quartic with given vertices.

First, if the vertices are real; let the vertices taken in order be F, G, H, K.

First construction: On FG as diameter describe a circle, and on HK as diameter a circle; on the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ_1, and describe the axial orthotomic circle Σ_1 of the two circles (viz. the centre of Σ_1 is on the axis of symmetry at its intersection with the radical axis of the two circles); then the curve is the envelope of a variable circle having its centre on Θ_1 and orthotomic to Σ_1.

Second construction: On FH as diameter describe a circle, and on GK as diameter a circle. On the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ_2, and describe the axial bifid circle Σ_2' of the two circles (viz. the centre of Σ_2' is on the axis of symmetry at its intersection with the radical axis or common chord of the two circles, and its diameter is this common chord); then the curve is the envelope of a variable circle having its centre on Θ_2 and bisecting Σ_2'.

Third construction: On FK as diameter describe a circle, and on GH as diameter a circle; and then, as in the first construction, a conic Θ_3 and a circle Σ_3; the curve is the envelope of a variable circle having its centre on Θ_3 and orthotomic to Σ_3.

Observe that in the three constructions the conics have always the same centre; and if the three conics are taken with the same foci, then the three constructions give one and the same bicircular quartic. The first and third constructions form a pair, and there is no reason for selecting one of them in preference to the other; but the second construction is unique; it is on this account natural to make use of it in discussing the series of curves with the given vertices.

In the particular case where the points F, G and H, K are situate symmetrically on opposite sides of a centre O ($OF=OK$, $OG=OH$), then in the third construction the centres each coincide with O, or the axis of the conic vanishes; hence the construction fails: the first and second constructions hold good, and in each of them the

conic and circle are concentric. The curve is in this case quadrantal: having, besides the original axis of symmetry, another axis of symmetry through O, at right angles thereto.

Secondly, if the vertices are two real, two imaginary, say f, $g=\alpha\pm\beta\iota$; h, k, we modify the first or third construction; viz. if F', G' are the antipoints of F, G; then on $F'G'$ as diameter describe a circle, and on HK as diameter a circle. On the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ_1, and describe the axial bisector-orthotomic circle Σ_1 of the two circles; viz. this is the circle (centre on the axis of symmetry) which bisects the circle $F'G'$, and cuts at right angles the circle HK; then the curve is the envelope of the variable circle having its centre on Θ_1 and orthotomic to Σ_1.

Thirdly, if the vertices are all imaginary, say f, $g=\alpha\pm\beta\iota$; h, $k=\gamma\pm\delta\iota$, we modify the first or third construction. Take F', G' the antipoints of F, G, and H', K' the antipoints of H, K; then on $F'G'$ as diameter describe a circle, and on $H'K'$ as diameter a circle; on the line terminated by the two centres (as transverse or conjugate axis) describe a conic Θ, and describe the axial bisector-circle Σ of the two circles (viz. this is a circle, centre on the axis of symmetry, bisecting each of the circles): the curve is the envelope of a variable circle, centre on the conic Θ and cutting at right angles the circle Σ.

558.

A GEOMETRICAL INTERPRETATION OF THE EQUATIONS OBTAINED BY EQUATING TO ZERO THE RESULTANT AND THE DISCRIMINANTS OF TWO BINARY QUANTICS.

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), pp. 31—33. Read March 12, 1874.]

CONSIDER the equations

$$U = (a, b, \ldots \between t, 1)^{\lambda} = 0,$$
$$U' = (a', b', \ldots \between t, 1)^{\lambda'} = 0;$$

and equating to zero the discriminants of the two functions respectively, and also the resultant of the two functions, let the equations thus obtained be

$$\Delta = (a, b, \ldots)^{2\lambda-2} = 0,$$
$$\Delta' = (a', b', \ldots)^{2\lambda'-2} = 0,$$
$$R = (a, b, \ldots)^{\lambda'} (a, b, \ldots)^{\lambda} = 0.$$

I take $(a, b, \ldots)$, $(a', b', \ldots)$ to be linear functions of the coordinates (x, y, z); and t to be an indeterminate parameter. Hence $U=0$ represents a line the envelope whereof is the curve $\Delta=0$, or, what is the same thing, the equation $U=0$ represents any tangent of the curve $\Delta=0$; this is a unicursal curve of the order $2\lambda-2$ and class λ, with $3(\lambda-2)$ cusps and $\frac{1}{2}(\lambda-2)(\lambda-3)$ nodes. Similarly $U'=0$ represents a line the envelope of which is the curve $\Delta'=0$: this is a unicursal curve of the order $2\lambda'-2$ and class λ', with $3(\lambda'-2)$ cusps and $\frac{1}{2}(\lambda'-2)(\lambda'-3)$ nodes; the equation $U'=0$ represents any tangent of this curve.

The equations $U=0$, $U'=0$ considered as existing simultaneously with the same value of t, establish a (1, 1) correspondence between the tangents (or if we please, between the points) of the two curves. The locus of the intersection of the corre-

sponding tangents is the curve $R=0$, a unicursal curve of the order $\lambda+\lambda'$, with $\frac{1}{2}(\lambda+\lambda'-1)(\lambda+\lambda'-2)$ nodes and no cusps; consequently of the class $2(\lambda+\lambda'-1)$.

It is to be shown that the curve $R=0$ touches the curve $\Delta=0$ in $\lambda'+2\lambda-2$ points, and similarly the curve $\Delta'=0$ in $2\lambda'+\lambda-2$ points.

In fact, consider any tangent T' of the curve Δ'; let this meet the curve Δ in a point A, and let Q be the tangent at A to the curve Δ; suppose, moreover, that T is the tangent of Δ corresponding to the tangent T' of Δ'. Then if Q and T coincide, the corresponding tangent of T' will be Q, and the curve R will pass through A. It is easy to see that in this case the curves R, Δ will touch at A. Again, if P be a tangent from A to the curve Δ, then, if P and T coincide, the corresponding tangent of T' will be P, and the curve R will pass through A; but in this case the point A will be a mere intersection, not a point of contact, of the two curves.

The tangents T, Q each correspond to T', and they consequently correspond to each other. For a given position of T we have a single position of T', and therefore $2\lambda-2$ positions of A, or, what is the same thing, of Q; that is, for a given position of T we have $2\lambda-2$ positions of Q. Again, to a given position of Q corresponds a single position of A, therefore λ' positions of T', therefore also λ' positions of T; that is, for a given position of Q we have λ' positions of T. The correspondence between T, Q is thus a $(\lambda', 2\lambda-2)$ correspondence, and the number of united tangents is therefore $\lambda'+2\lambda-2$, or the curves R, Δ touch in $\lambda'+2\lambda-2$ points.

The tangents T, P each correspond to T', and they therefore correspond to each other. For a given position of T we have a single position of T', and therefore $2\lambda-2$ positions of A, and thence $(2\lambda-2)(\lambda-2)$ positions of P; that is, for a given position of T we have $(2\lambda-2)(\lambda-2)$ positions of P. Again, to a given position of P correspond $2\lambda-4$ positions of A, therefore $(2\lambda-4)\lambda'$ positions of T' or of T; that is, for a given position of P we have $(2\lambda-4)\lambda'$ positions of T. The correspondence between T, P is thus a $[2\lambda'(\lambda-2), 2(\lambda-1)(\lambda-2)]$ correspondence, and the number of united tangents is $2(\lambda+\lambda'-1)(\lambda-2)$; or the curves R, Δ meet in $2(\lambda+\lambda'-1)(\lambda-2)$ points.

Reckoning the contacts twice, the total number of intersections of R, Δ is

$$2\lambda'+4\lambda-4+2(\lambda+\lambda'-1)(\lambda-2),\ =(\lambda+\lambda')(2\lambda-2),$$

as it should be.

In the particular case $\lambda=\lambda'=2$, the curves Δ, Δ' are conics, and the curve R is a quartic curve touching each of the conics 4 times; this is at once verified, since the equations here are

$$ac-b^2=0,\quad a'c'-b'^2=0,\quad 4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0.$$

559.

[NOTE ON INVERSION.]

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), p. 112.]

THE inverse of the anchor ring (in the foregoing paper* called the cyclide) is in fact the general binodal cyclide or binodal bicircular quartic; viz. assuming it to be a cyclide (bicircular quartic), to see that it is binodal, it need only be observed that the anchor ring is binodal (has two real or imaginary conic points, viz. these are the intersections of the circles in the several axial planes); and to see that it is the general binodal cyclide, we have only to count the constants; viz. the general cyclide or surface

$$(x^2+y^2+z^2)^2+(x^2+y^2+z^2)(\alpha x+\beta y+\gamma z)+(a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)(x,\ y,\ z,\ 1)^2=0$$

contains 13 constants, and therefore the binodal cyclide $13-2, =11$ constants. But the anchor ring, irrespective of position, contains 2 constants; centre of inversion, taken in given axial plane, has 2 constants; radius of inversion, 1 constant; in all $2+2+1, =5$ constants; or taking the inverse surface in an arbitrary position, the number of constants is $5+6, =11$.

* By Mr H. M. Taylor: Inversion, with special reference to the Inversion of an Anchor Ring or Torus, (*Lond. Math. Soc. Proc.*, same volume, pp. 105—112).

560.

[ADDITION TO LORD RAYLEIGH'S PAPER "ON THE NUMERICAL CALCULATION OF THE ROOTS OF FLUCTUATING FUNCTIONS."]

[From the *Proceedings of the London Mathematical Society*, vol. v. (1873—1874), pp. 123, 124. November 22.]

PROF. CAYLEY, to whom Lord Rayleigh's paper was referred, pointed out that a similar result may be attained by a method given in a paper by Encke, "Allgemeine Auflösung der numerischen Gleichungen," *Crelle*, t. XXII. (1841), pp. 193—248, as follows:

Taking the equation

$$0 = 1 - ax + bx^2 - cx^3 + dx^4 - ex^5 + fx^6 - gx^7 + hx^8 - \dots;$$

if the equation whose roots are the squares of these is

$$0 = 1 - a_1x + b_1x^2 - c_1x^3 + \dots,$$

then

$$\begin{aligned} a_1 &= a^2 - 2b, \\ b_1 &= b^2 - 2ac + 2d, \\ c_1^2 &= c^2 - 2bd + 2ae - 2f, \\ d_1^2 &= d^2 - 2ce + 2bf - 2ag + 2h, \text{ \&c.}; \end{aligned}$$

and we may in the same way derive a_2, b_2, c_2, &c. from a_1, b_1, c_1, &c., and so on.

As regards the function

$$J_n(z) = \frac{z^n}{2^n . \Gamma(n+1)} \left\{1 - \frac{z^2}{2 . 2n+2} + \frac{z^4}{2 . 4 . 2n+2 . 2n+4} - \dots\right\},$$

we have as follows:

$$
\begin{aligned}
a^{-1} &= 2^2 \,.\, n+1, \\
b^{-1} &= 2^5 \,.\, n+1 \,.\, n+2, \\
c^{-1} &= 2^7 \,.\, 3 \,.\, n+1 \ldots n+3, \\
d^{-1} &= 2^{11} \,.\, 3 \,.\, n+1 \ldots n+4, \\
e^{-1} &= 2^{13} \,.\, 3 \,.\, 5 \,.\, n+1 \ldots n+5, \\
f^{-1} &= 2^{16} \,.\, 3^2 \,.\, 5 \,.\, n+1 \ldots n+6, \\
g^{-1} &= 2^{18} \,.\, 3^2 \,.\, 5 \,.\, 7 \,.\, n+1 \ldots n+7, \\
h^{-1} &= 2^{23} \,.\, 3^2 \,.\, 5 \,.\, 7 \,.\, n+1 \ldots n+8, \\
a_1^{-1} &= 2^4 \,.\, (n+1)^2 \,.\, n+2, \\
b_1^{-1} &= 2^9 \,.\, (n+1 \,.\, n+2)^2 \,.\, n+3 \,.\, n+4, \\
c_1^{-1} &= 2^{13} \,.\, 3 \,.\, (n+1 \ldots n+3)^2 \,.\, n+4 \ldots n+6, \\
d_1^{-1} &= 2^{19} \,.\, 3 \,.\, (n+1 \ldots n+4)^2 \,.\, n+5 \ldots n+8,
\end{aligned}
$$

$$a_2 = \frac{5n+11}{2^8 \,.\, (n+1)^4 (n+2)^2 \, n+3 \,.\, n+4},$$

$$b_2 = \frac{25n^2+231n+542}{2^{17} \,.\, (n+1 \,.\, n+2)^4 (n+3 \,.\, n+4)^2 \, n+5 \ldots n+8},$$

$$a_3 = \frac{429n^5+7640n^4+53752n^3+185430n^2+311387n+202738}{2^{16} (n+1)^8 (n+2)^4 (n+3 \,.\, n+4)^2 \, n+5 \,.\, n+6 \,.\, n+7 \,.\, n+8}.$$

If $n=0$,

$$\Sigma p^{-16} = a_3 = \frac{101369}{2^{27} \,.\, 3^3 \,.\, 5 \,.\, 7} = p_1^{-16}, \text{ suppose};$$

whence

$$p_1 = 2{\cdot}404825.$$

[The quantities $p_1, p_2, \ldots$ are the roots of the function $J_n(x)$ in increasing order of magnitude, so that, as these roots are all real, it follows that for $J_0(x)$,

$$a = \Sigma p_1^{-2}, \quad a_1 = \Sigma p_1^{-4}, \quad a_2 = \Sigma p_1^{-8}, \quad a_3 = \Sigma p_1^{-16}, \ldots]$$

561.

ON THE GEOMETRICAL REPRESENTATION OF CAUCHY'S THEOREMS OF ROOT-LIMITATION.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XII. Part II. (1877), pp. 395—413. Read February 16, 1874.]

THERE is contained in Cauchy's Memoir "Calcul des Indices des Fonctions," *Journ. de l'École Polytech.* t. XV. (1837) a general theorem, which, though including a well-known theorem in regard to the imaginary roots of a numerical equation, seems itself to have been almost lost sight of. In the general theorem (say Cauchy's two-curve theorem) we have in a plane two curves $P=0$, $Q=0$, and the real intersections of these two curves, or say the "roots," are divided into two sets according as the Jacobian

$$d_xP\,.\,d_yQ - d_xQ\,.\,d_yP$$

is positive or negative, say these are the Jacobian-positive and the Jacobian-negative roots: and the question is to determine for the roots within a given contour or circuit, the difference of the numbers of the roots belonging to the two sets respectively.

In the particular theorem (say Cauchy's rhizic theorem) P and Q are the real part and the coefficient of i in the imaginary part of a function of $x+iy$ with, in general, imaginary coefficients (or, what is the same thing, we have $P+iQ=f(x+iy)+i\phi(x+iy)$, where f, ϕ are real functions of $x+iy$): the roots of necessity are of the same set: and the question is to determine the number of roots within a given circuit.

In each case the required number is theoretically given by the same rule, viz. considering the fraction $\frac{P}{Q}$, it is the excess of the number of times that the fraction changes from $+$ to $-$ over the number of times that it changes from $-$ to $+$, as the point (x, y) travels round the circuit, attending only to the changes which take place on a passage through a point for which P is $=0$.

In the case where the circuit is a polygon, and most easily when it is a rectangle the sides of which are parallel to the two axes respectively, the excess in question can be actually determined by means of an application of Sturm's theorem successively to each side of the polygon, or rectangle.

In the present memoir I reproduce the whole theory, presenting it under a completely geometrical form, viz. I establish between the two sets of roots the distinction of *right-* and *left-handed:* and (availing myself of a notion due to Prof. Sylvester *) I give a geometrical form to the theoretic rule, making it depend on the "intercalation" of the intersections of the two curves with the circuit: I also complete the Sturmian process in regard to the sides of the rectangle: the memoir contains further researches in regard to the curves in the case of the particular theorem, or say as to the rhizic curves $P=0$, $Q=0$.

The General Theory. Articles Nos. 1 to 19.

1. Consider in a plane two curves $P=0$, $Q=0$ (P and Q each a rational and integral function of x, y), which to fix the ideas I call the *red* curve and the *blue* curve respectively†: the curve $P=0$ divides the plane into two sets of regions, say a positive set for each of which P is positive, and a negative set for each of which P is negative: it is of course immaterial which set is positive and which negative, since writing $-P$ for P the two sets would be interchanged: but taking P to be given, the two sets are distinguished as above. And we may imagine the negative regions to be coloured red, the positive ones being left uncoloured, or say they are white. Similarly the curve $Q=0$ divides the plane into two sets of regions, the negative regions being coloured blue, and the positive ones being left uncoloured, or say they are white. Taking account of the twofold division, and considering the coincidence of red and blue as producing black, there will be four sets of regions, which for convenience may be spoken of as *sable, gules, argent, azure:* viz. in the figures we have

P	Q	
$-$	$-$	sable, shown by cross lines,
$-$	$+$	gules, „ „ vertical lines,
$+$	$+$	argent, left white,
$+$	$-$	azure, shown by horizontal lines,

sable and argent ($-\ -$ and $+\ +$) being thus positive colours, and gules and azure ($-\ +$ and $+\ -$) negative colours. See figures [pp. 32, 38] towards end of Memoir.

* See his memoir, "A theory of the Syzygetic relations, &c." *Phil. Trans.*, 1853. The Sturmian process is by Sturm and Cauchy applied to two independent functions ϕx, fx of a variable x; but the notion of an intercalation as applied to the order of succession of the roots of the equations $\phi(x)=0$, $f(x)=0$ is due to Sylvester, and it was he who showed that what the Sturmian process determined was in fact the intercalation of these roots: but, not being concerned with circuits, he was not led to consider the intercalation of a circuit.

† It is assumed throughout that the two curves have no points (or at least no real points) of multiple intersection; i.e. they nowhere touch each other, and neither curve passes through a multiple point of the other curve.

2. Consider any point of intersection of the two curves. There will be about this point four regions, sable and argent being opposite to each other, as also gules and azure; whence selecting an order

sable, gules, argent, azure;

if to have the colours in this order we have to go about the point, or root, right-handedly, the root is right-handed: but if left-handedly, then the root is left-handed: or, what is more convenient, going always right-handedly, then, if the order of the colours is

sable, gules, argent, azure,

the root is right-handed: but if the order is

sable, azure argent, gules,

the root is left-handed.

3. The distinction of right- and left-handed corresponds to the sign of the Jacobian

$$\frac{d(P, Q)}{d(x, y)} \; (= d_x P \,.\, d_y Q - d_x Q \,.\, d_y P);$$

we may (reversing if necessary the original sign of one of the functions) assume that for a right-handed root the Jacobian is positive, for a left-handed one, negative.

4. I consider a trajectory which may be either an unclosed curve not cutting itself, or else a circuit, viz. this is a closed curve not cutting itself. A circuit is considered as described right-handedly: an unclosed trajectory is considered as described according to a currency always determinate *pro hâc vice:* viz. one extremity is selected as the beginning and the other as the end of the trajectory: but the currency may if necessary or convenient be reversed: thus if an unclosed trajectory forms part of a circuit the currency is thereby determined: but the same unclosed trajectory may form part of two opposite circuits, and as such may have to be taken with opposite currencies. It is assumed that a trajectory does not pass through any intersection of the P and Q curves.

5. A trajectory has its P- and Q-sequence, viz. considering in order its intersections with the two curves, we write down a P for each intersection with the red curve and a Q for each intersection with the blue curve, thus obtaining an intermingled series of P's and Q's, which is the sequence in question. In the case of a circuit, the sequence is considered as a circuit, viz. the first and last terms are considered as contiguous, and it is immaterial at what point the sequence commences. The sequence will of course vanish if the trajectory does not meet either of the curves.

6. A P- and Q-sequence gives rise to an "intercalation," viz. if in the sequence there occur together any even number of the same letter, these are omitted (whence also any odd number of the same letter is reduced to the letter taken once): and if by reason of an omission there again occur an even number of the same letter, these

are omitted: and so on. The intercalation contains therefore only the letters P and Q alternately: viz. in the case of an unclosed trajectory the intercalation may contain an even number of letters, beginning with the one and ending with the other letter, and so containing the same number of each letter—or it may contain an odd number of letters, beginning and ending with the same letter, and so containing one more of this than of the other letter; say the intercalation is PQ or QP, or else PQP or QPQ. The intercalation may vanish altogether; thus if the sequence were $QPPQ$, this would be the case.

7. In the case of a circuit the intercalation cannot begin and end with the same letter, for these, as contiguous letters, would be omitted; and since any letter thereof may be regarded as the commencement it is PQ or QP indifferently. A little consideration will show that the whole number of letters must be evenly even, or, what is the same thing, the number of each letter must be even. Thus imagine the circuit beginning in sable, and let the intercalation begin with PQ; viz. P we pass from sable to azure, and Q we pass from azure to argent: in order to get back into sable we must either return the same way (Q argent to azure, P azure to sable), but then the sequence is $PQQP$, and the intercalation vanishes: here the number of letters is 0, an evenly even number: or else we must complete the cycle of colours P argent to gules, Q gules to sable: and the sequence and therefore also the intercalation then is $PQPQ$, where the number of letters is 4, an evenly even number.

8. In the case of any trajectory whatever, the half number of letters in the intercalation is termed the "index," viz. this is either an integer or an integer $+\frac{1}{2}$. But in the case of a circuit the index is an even integer, and the half-index is therefore an integer. The index may of course be $=0$.

9. But we require a further distinction: instead of a P- and Q-sequence we have to consider a $\pm P$- and Q-sequence. To explain this, observe that a passage over the red curve may be from a negative to a positive colour (azure to sable or gules to argent), this is $+P$, or from a positive to a negative colour (sable to azure or argent to gules), this is $-P$. And so the passage over the blue curve may be from a negative to a positive colour (gules to sable or azure to argent), this is $+Q$, or else from a positive to a negative colour (sable to gules or argent to azure), this is $-Q$. The sequence will contain the P and Q intermingled in any manner, but the signs will always be $+\ -$ *alternately;* for $+(P$ or $Q)$, denoting the passage into a positive colour, must always be immediately succeeded by $-(P$ or $Q)$, denoting the passage into a negative colour. Whence, knowing the sequence independently of the signs, we have only to prefix to the first letter the sign $+$ or $-$ as the case may be, and the sequence is then completely determined.

10. Passing to a $\pm$ intercalation, observe that in omitting any even number of P's or Q's, the omitted signs are always $+\ -\ +\ -$ &c. or else $-\ +\ -\ +$ &c., viz. the omitted signs begin with one sign and end with the opposite sign. Hence the signs being in the first instance alternate, they will after any omission remain alternate: and the letters being also alternate, the intercalation can contain only $+P$ and $-Q$

or else $-P$ and $+Q$. Hence in the case of a circuit the intercalation is either $(+P-Q)$, say this is a *positive* circuit, or else $(-P+Q)$, say this is a *negative* circuit. There is of course the *neutral* circuit $(PQ)_0$ for which the intercalation vanishes.

11. Consider a circuit not containing within it any root; as a simple example let the circuit lie wholly in one colour, or wholly in two adjacent colours, say sable and gules: in the former case the sequence, and therefore also the intercalation, vanishes: in the latter case the sequence is $+Q-Q$, and therefore the intercalation vanishes: viz. in either case the intercalation is $(PQ)_0$.

12. Consider next a circuit containing within it one right-handed root; for instance let the circuit lie wholly in the four regions adjacent to this root, cutting the two curves each twice; the sequence and therefore also the intercalation is $+P-Q+P-Q$; viz. this is a positive circuit $(+P-Q)_1$, where the subscript number is the half-index, or half of the number of P's or of Q's. Similarly if a circuit contains within it one left-handed root, for instance if the circuit lies wholly in the four regions adjacent to this root, cutting the two curves each twice, the sequence and therefore also the intercalation is $-P+Q-P+Q$, viz. this is a negative circuit $(-P+Q)_1$: and the consideration of a few more particular cases leads easily to the general and fundamental theorem:

13. *A circuit is positive* $(+P-Q)_\delta$ *or negative* $(-P+Q)_\delta$ *according as it contains within it more right-handed or more left-handed roots; and in either case the half-index* δ *is equal to the excess of the number of one over that of the other set of roots. If the circuit is neutral* $(PQ)_0$, *then there are within it as many left-handed as right-handed roots.*

14. The proof depends on a composition of circuits, but for this some preliminary considerations are necessary.

Imagine two unclosed trajectories forming a circuit, and write down in order the intercalation of each. The whole number of letters must be even: viz. the numbers for the two intercalations respectively must be both even or both odd. I say that if the terminal letter of the first intercalation and the initial letter of the second intercalation are different, then also the initial letter of the first intercalation and the terminal letter of the second intercalation will be different: if the same, then the same. In fact, the intercalations may be each PQ or each QP, or one PQ and the other QP: or each PQP, or each QPQ, or one PQP and the other QPQ. Supposing the letters in question are different, then the intercalations may be termed similar; but if the same, then the intercalations may be termed contrary.

15. In the first case, that is when the intercalations are similar, the two together form the intercalation of the circuit; the sum of their numbers of letters (that is, twice the sum of their indices) will be evenly even, and the half of this, or the sum of the indices, will be the index of the circuit; each intercalation will be $(+P-Q)$ or else each will be $(-P+Q)$; and the circuit will be $(+P-Q)$ or $(-P+Q)$ accordingly.

In the second case, that is, when the intercalations are contrary, they counteract each other in forming the intercalation of the circuit: it is the *difference* of the numbers of letters, or twice the difference of the indices, which is evenly even, and the half of this, or the difference of the indices, which is the index of the circuit: one intercalation is $(+P-Q)$, and the other is $(-P+Q)$: and the circuit will agree with that which has the larger index.

In particular if the circuit consist of a single unclosed trajectory, taken forwards and backwards; then the trajectory taken one way is $(+P-Q)$, taken the other way it is $(-P+Q)$; the number of terms is of course equal, and the circuit is $(PQ)_0$.

16. Consider now two circuits $ABCA$ and $ACDA$, having a common portion CA, or, more accurately, the common portions AC and CA: write down in order the intercalations of

$$ABC,\quad CA,\quad AC,\quad CDA:$$

the two mean terms destroy each other, and we can hence deduce the intercalation of the entire circuit $ABCDA$.

Suppose *first*, that ABC and CDA are similar; then if CA is similar to ABC it is also similar to CDA, that is, AC is contrary to CDA: and so if CA is contrary to ABC, then AC is similar to CDA.

To fix the ideas suppose CA similar to ABC, but AC contrary to CDA, then $ABCA$ is similar to CA; but $ACDA$ will be similar or contrary to AC, that is, contrary or similar to CA, that is, to $ABCA$, according as index of $AC >$ or $<$ index of CDA.

Suppose Ind. $AC <$ Ind. CDA, then $ACDA$ is similar to $ABCA$.

Now
$$\begin{aligned}\text{Ind. } ABCDA &= \text{Ind. } ABC + \text{Ind. } CDA,\\ \text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } AC,\\ \text{Ind. } ACDA &= \text{Ind. } CDA - \text{Ind. } AC,\end{aligned}$$
and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA + \text{Ind. } ACDA,$$
the whole circuit being in this case similar to each of the component ones.

But if Ind. $AC >$ Ind. CDA, then $ACDA$ is contrary to $ABCA$.

And
$$\begin{aligned}\text{Ind. } ABCDA &= \text{Ind. } ABC + \text{Ind. } CDA,\\ \text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } CA,\\ \text{Ind. } ACDA &= -\text{Ind. } CDA + \text{Ind. } AC,\end{aligned}$$
and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA - \text{Ind. } ACDA;$$
and the investigation is like hereto if CA is contrary to ABC but AC similar to CDA.

17. *Secondly*, if ABC and CDA are contrary, then if CA is similar to ABC it is contrary to CDA, that is, AC is similar to CDA; and so if CA is contrary to ABC it is similar to CDA, that is, AC is contrary to CDA.

Suppose CA similar to ABC, and AC similar to CDA; then $ABCA$ is also similar to ABC, and $ACDA$ similar to CDA; viz. ABC, CA and $ABCA$ are similar to each other, and contrary to AC, CDA, $ACDA$ which are also similar to each other.

Also
$$\begin{aligned}\text{Ind. } ABCDA &= \text{Ind. } ABC \sim \text{Ind. } CDA,\\ \text{Ind. } ABCA &= \text{Ind. } ABC + \text{Ind. } CA,\\ \text{Ind. } ACDA &= \text{Ind. } CDA + \text{Ind. } AC,\end{aligned}$$
and thence
$$\text{Ind. } ABCDA = \text{Ind. } ABCA \sim \text{Ind. } ACDA,$$
and the investigation is like hereto if CA is contrary to ABC and AC contrary to CDA.

18. It thus appears that in every case
$$\begin{aligned}\text{Ind. } ABCDA &= \text{Ind. } ABCA + \text{Ind. } ACDA,\\ \text{or } &= \text{Ind. } ABCA \sim \text{Ind. } ACDA,\end{aligned}$$
according as the component circuits are similar or contrary, and in the latter case the entire circuit is similar to that which has the largest index.

Moreover, any circuit whatever can be broken up into two smaller circuits, and these again continually into smaller circuits until we arrive at the before-mentioned elementary circuits, and the theorem as to the number of roots within a circuit is true as regards these elementary circuits; wherefore the theorem is true as regards any circuit whatever.

19. In the case where a trajectory is a finite right line, y is a given linear function of x, or the coordinates x, y can if we please be expressed as linear functions of a parameter u, so that as the describing point passes along the line, u varies between given limits, say from $u=0$ to $u=1$. The functions P, Q thus become given rational and integral functions of a single variable u (or it may be x or y), and the question of the P- and Q-sequence and intercalation relates merely to the order of succession of the roots of the equations $P=0$, $Q=0$, where P and Q denote functions of a single variable as above. To fix the ideas, let the trajectory be a line parallel to the axis of x; and in this case taking x as the parameter, and supposing that y_0 is the given value of y, P and Q are the functions of x obtained by writing y_0 for y in the original expressions of these functions. Of course the theory will be precisely the same for a line parallel to the axis of y: and by combining two lines parallel to each axis we have the case of a rectangular circuit. We require, for each side of the rectangle considered according to its proper currency, the intercalation PQ, QP, PQP or QPQ as the case may be, and also the sign $+$ or $-$ of the initial letter of the first intercalation; for then writing down the intercalations in order, with the signs for the several letters, $+$ and $-$ alternately (the first sign being $+$ or $-$ as the case may be), we have or deduce the intercalation of the circuit, and thus obtain the value of the difference of the numbers of the included right- and left-handed roots. We thus see how the whole theory depends on the case where the trajectory is a right line.

Intercalation-theory for a right line. Articles Nos. 20 to 31.

20. Considering then the case where the trajectory is a line parallel to the axis of x, P and Q will denote given rational functions of x; the curves $P=0$, $Q=0$ being of course each of them a set of right lines parallel to the axis of y: the regions will be bands each of them included between two such lines; and colouring them as explained in the general case, the colours will be as before, sable, gules, argent, azure, each region having in the neighbourhood of the trajectory (what we are alone concerned with) the same colour that it had in the original case where P and Q were functions of (x, y). We may regard the trajectory as described according to the currency $x=-\infty$ to $x=+\infty$: we have in regard to the trajectory a P- and Q-sequence and intercalation, a $\pm P$- and Q-sequence, &c., as in the original case. The intercalation may be as before PQ, QP, PQP or QPQ, and in each of these cases it may be positive, that is, $(+P-Q)$, or else negative, that is, $(-P+Q)$.

21. The question of sign may in the present case be disposed of without difficulty. For the initial point of the trajectory, we know the signs of P, Q, that is, the colour of the region: suppose for example that we have $P=-$, $Q=+$, or that the region is gules: then if the intercalation begin with P, this means that we either first pass a red line, or before doing so we pass an even number of blue lines: but in the last case the colours are sable, gules, sable, gules,... always ending in gules; and the passage over the red line is gules to argent, viz. this is $+P$; and so in general the initial P or Q of the intercalation has the sign opposite to that of the P or Q belonging to the commencement of the trajectory.

22. For the solution of the problem we connect with P, Q a set of functions R, S, T, &c.: the intercalation is in fact given by means of the gain or loss of changes of sign in these functions on substituting therein the initial and final values of the variable x. It is convenient to consider the functions as arranged in a column

$$\begin{matrix} P \\ Q \\ R \\ S \\ \vdots \end{matrix}$$

say this is the column $PQRS\ldots$, and to connect therewith a signaletic bicolumn: viz. the left-hand column is here the series of signs of these functions for the initial value of x, and the right-hand column is the series of signs for the terminal value of x: the bicolumn thus consisting of as many rows each of two signs, as there are functions. But such a bicolumn may be considered apart from any series of functions, as a set of rows each of two signs taken at pleasure.

We say that the "gain" of a bicolumn is

$$=-(\text{No. of changes of sign in left-hand column})+(\text{No. in right-hand ditto}),$$

the gain being of course positive or negative; and a negative gain being regarded as a loss. Also if a positive gain be converted into an equal negative gain or *vice versâ*, we may speak of the gain as *reversed*.

23. A bicolumn may be divided in any manner into parts, taking always the last row of any part as being also the first row of the next succeeding part. This being so, the gain of the whole bicolumn is equal to the sum of the gains of its parts.

In a bicolumn of two rows, if we reverse either row (that is, write therein $-$ for $+$ and $+$ for $-$), we reverse the gain: and hence dividing a bicolumn into bicolumns each of two rows, viz. first and second rows, second and third rows, and so on, it at once appears that if we reverse alternate rows (viz. either the first, third, fifth, &c., rows, or the second, fourth, sixth, &c., rows) we reverse the gain. It of course follows that reversing all the rows, we leave the gain unaltered.

24. If to any bicolumn we prefix at the top thereof the second row reversed, we either leave the gain unaltered or we alter it by ± 1. In fact, as regards either column, if this originally begin with a change, the process introduces no change therein; but if it begins with a continuation, then the process introduces a change. Hence if the columns begin each with a change or each with a continuation, the gain is unaltered: but if one begins with a change, and the other with a continuation, then the gain is altered by ± 1; viz. the left-hand column beginning with a continuation, the gain is altered by -1: and the right-hand column beginning with a continuation, the gain is altered by $+1$.

The column $PQRST\ldots$ is taken to satisfy the following conditions: two consecutive terms never vanish together (that is, for the same value of the variable): if for a given value of the variable, any term vanishes, the preceding and succeeding terms have then opposite signs; the last term, say V, is of constant sign.

25. Considering P, Q as given functions without a common measure, such a column of functions is obtained by the well-known process of seeking for the greatest common measure, reversing at each step the sign of the remainder: viz. we thus derive a set of functions R, S, $T, \ldots$ where

$$\begin{aligned} P &= \lambda Q - R, \\ Q &= \mu R - S, \\ R &= \nu S - T, \\ S &= \rho T - U, \\ &\vdots \end{aligned}$$

the degrees of the successive functions R, S, T, ..., being successively less and less, so that the last of them, say V, is an absolute constant: or we may stop the process as soon as we arrive at a function V, the sign of which remains unaltered for all values between the initial and final values of the variable. It may be observed that the process may be regarded as applicable in the case where the degree of Q exceeds that of P: viz. we then have $\lambda = 0$, $R = -P$, and the column begins $(P, Q, -P, S, \ldots)$, the subsequent terms being, except as to sign, the same as if P, Q had been interchanged.

Reversing the sign of P or Q, we reverse in the bicolumn a set of alternate rows, and thus reverse the gain: and reversing both signs we reverse all the rows,

and leave the gain unaltered—of course the intercalation (considered irrespectively of sign) is in each case unaltered. It is convenient to take the signs in such manner that for the initial value of x, the signs of P, Q shall be each positive: or, what is the same thing, taking P, Q with their proper signs, we may in the bicolumn, by reversing if necessary each or either set of alternate rows, make the left-hand column to begin with $+\,+$: then if there is a gain the intercalation begins with P, if a loss with Q, the gain or loss showing the number of P's. To find the number of Q's prefix at the top of the bicolumn the second row reversed—then the gain or loss (equal to or differing by unity from the original value) shows the number of Q's. It may happen that for P the gain is $=0$; then for Q the gain is 0 or ± 1, and the intercalation vanishes or is Q.

27. I give some simple examples.

	0	2	4
$P = x - 1$	$-$	$+$	$+$
$Q = x - 3$	$-$	$-$	$+$
$R = \ \ -1$	$-$	$-$	$-$

0 P 2 Q 4

	0	2	4
$P = x - 3$	$-$	$-$	$+$
$Q = x - 1$	$-$	$+$	$+$
$R = \ \ +1$	$+$	$+$	$+$

0 Q 2 P 4

In the left-hand example taking the intervals to be successively $0-2$, $0-4$, $2-4$, the bicolumns modified as above are

$0-2$		$0-4$		$2-4$	
$-$	$-$	$-$	$+$	$-$	$+$
$+$	$-$	$+$	$-$	$+$	$+$
$+$	$+$	$+$	$-$	$+$	$-$
$+$	$+$	$+$	$+$	$-$	$-$

viz.

Interval $0-2$; for P gain $=1$, P first; for Q gain $=0$; Intercalation is P;

" $0-4$ " " $=1$, " " " $=1$; " " PQ;

" $2-4$ " " $=0$ ..... " loss $=1$; " " Q.

And similarly in the right-hand example we have

$0-2$		$0-4$		$2-4$	
$-$	$+$	$-$	$+$	$-$	$-$
$+$	$+$	$+$	$-$	$+$	$-$
$+$	$-$	$+$	$-$	$+$	$+$
$-$	$-$	$-$	$-$	$-$	$-$

Interval $0-2$ for P gain $= 0$, ... for Q gain $=-1$; Intercalation is Q;
" $0-4$ " " $=-1$, Q first, " " $=-1$; " " QP;
" $2-4$ " " $=+1$, P first, " " $= 0$; " " P.

28. Or to take a slightly more complicated example,

	1	3	$5 \pm \epsilon$	7	9
$P = x^2 - 8x + 12$	+	−	−	+	+
$Q = x^2 - 12x + 32$	+	+	−	−	+
$R = - x + 5$	+	+	$\mp$	−	−
$S = + 1$	+	+	+	+	+

		P		Q		P		Q	
0	1	2	3	4	5	6	7	8	9

And hence for the several intervals,

$1-3$	$1-5$	$1-7$	$1-9$	$3-5$	$3-7$	$3-9$	$5-7$	$5-9$	$7-9$
− −	− +	− +	− −	− +	− +	− −	− −	− +	− +
+ −	+ −	+ +	+ +	+ +	+ −	+ −	+ −	+ −	+ +
+ +	+ −	+ −	+ +	+ −	+ −	+ +	+ +	+ −	+ −
+ +	+ $\mp$	+ −	+ −	− $\pm$	− +	− +	$\pm$ +	$\pm$ +	− −
+ +	+ +	+ +	+ +	+ +	+ +	+ +	− −	− −	− −
Showing P	PQ	PQP	$PQPQ$	Q	QP	QPQ	P	PQ	Q

For instance:—

Interval $1-9$ for P gain $=2$, P first, for Q gain $=2$: Intercalation is $PQPQ$.

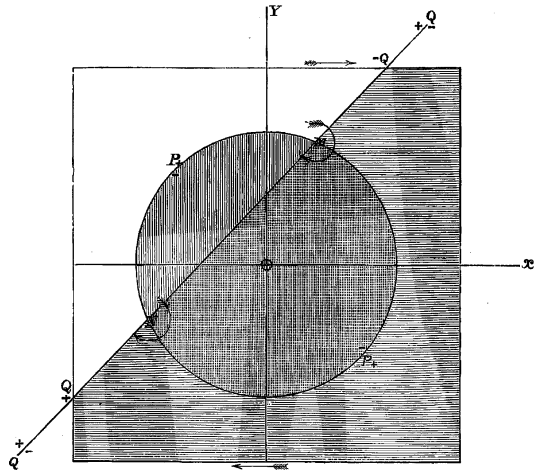
It may be added that P being $+$ for $x=1$, the $\pm$ intercalation is $+PQPQ$.

29. As an example of circuits take the following: curves are $P=0$, $Q=0$, where

$$P = x^2 + y^2 - 4,$$
$$Q = y - x - 1;$$

viz. $P=0$ (see figure) is a circle, centre the origin, radius $=2$: the inside hereof ($P=-$) being coloured red: and $Q=0$ is a right line cutting the axes of x, y at the points $(-1, 0)$ and $(0, 1)$ respectively, or say running N.E. and S.W., the lower region ($Q=-$) being coloured blue: the square is an arbitrary circuit ($x=\pm 3$, $y=\pm 3$) surrounding the circle, and the regions within the square are coloured by what precedes sable, gules, azure, argent, as shown in the figure: the line and circle intersect in two points M, N. Going right-handedly round these respectively, for M the order is sable, gules, argent, azure, viz. M is a right-handed root; while for N the order is

sable, azure, argent, gules, viz. N is a left-handed root: the two points are accordingly in the figure denoted by $+M$ and $-N$ respectively.



30. Now considering successively the four smaller squares of the figure, say these are the squares N.E., S.E., S.W., N.W.: and going right-handedly round each of these:

In the square N.E., the sequence and therefore also the intercalation is $+P-Q+P-Q$, viz. this is an intercalation $(+P-Q)$, showing an excess 1 of right-handed roots, and of course consisting with the single right-handed root M.

In the square S.E., the sequence is $-P+P$, viz. this is an intercalation $(PQ)_0$, showing an equality of right- and left-handed roots, and consisting with no root.

In the square S.W., the sequence and therefore also the intercalation is $-P+Q-P+Q$: viz. this is an intercalation $(-P+Q)_1$, showing an excess 1 of left-handed roots, and consisting with the single left-handed root N.

And in the square N.W., the sequence is $-Q+P-P+Q$, viz. this is an intercalation $(PQ)_0$, showing an equality of right- and left-handed roots, and consisting with no root.

Again take the whole large square: the sequence is $-Q+Q$: viz. the intercalation is $(PQ)_0$, showing an equality of right- and left-handed roots, and consisting with there being one of each.

So taking the squares N.E. and N.W. conjointly, the sequence and therefore also the intercalation is $-Q+P-Q+P$, viz. this is an intercalation $(+P-Q)_1$, as for the single square N.E.

31. As regards the analytical determination it will be sufficient to consider a single square, say N.E.: going round right-handedly, the trajectories will be

(1) $x=0,\ y=0$ to 3;

(2) $y=3,\ x=0$ to 3;

(3) $x=3,\ y=3$ to 0; or if $y'=-y$, then $y'=-3$ to 0;

(4) $y=0,\ x=3$ to 0; or if $x'=-x$, then $x'=-3$ to 0.

And we thus have

(1)
$$\begin{array}{rr|c|c|}
 & & 0 & 3 \\
P= & y^2-4 & - & + \\
Q= & y-1 & - & + \\
R= & -1 & + & -
\end{array}\text{, that is, }
\begin{array}{|c|c|}
- & + \\
\hline
+ & - \\
+ & - \\
- & -
\end{array}$$
, for P gain $=-1$, Q begins, $1P$;
" Q " $=-1$, " $1Q$.
Intercalation is QP, or since at origin $P=-$, $R=-$, or region is sable, it is $-Q+P$.

(2)
$$\begin{array}{rr|c|c|}
 & & 0 & 3 \\
P= & x^2+5 & + & + \\
Q= & -x+2 & + & - \\
R= & -1 & - & -
\end{array}\text{, that is, }
\begin{array}{|c|c|}
- & + \\
\hline
+ & + \\
+ & - \\
- & -
\end{array}$$
, for P gain $=0$,
" Q " $=-1$.
Intercalation is $-Q$.

(3)
$$\begin{array}{rr|c|c|}
 & & -3 & 0 \\
P= & y'^2+5 & + & + \\
Q= & -y'+4 & - & - \\
R= & -1 & - & -
\end{array}\text{, that is, }
\begin{array}{|c|c|}
- & - \\
\hline
+ & + \\
+ & + \\
- & -
\end{array}$$
, for P gain $=0$,
" Q " $=0$.
Intercalation vanishes.

(4)
$$\begin{array}{rr|c|c|}
 & & -3 & 0 \\
P= & x'^2-4 & + & - \\
Q= & x'-1 & - & - \\
R= & +1 & + & +
\end{array}\text{, that is, }
\begin{array}{|c|c|}
- & - \\
\hline
+ & - \\
+ & + \\
+ & +
\end{array}$$
, for P gain $=+1$, P first,
" Q " $=0$.
Intercalation is $+P$.

Hence for the four sides, combining the intercalations, we have $-Q+P-Q+P$, and since there are no terms to be omitted, this is the intercalation of the N.E. square: which is right.

The Rhizic Theory. Articles Nos. 32 to 38.

32. Consider now $F(z)=(*)(z, 1)^n$ a rational and integral function of z, of the order n with in general imaginary (complex) coefficients, or, what is the same thing, let $F(z)=f(z)+i\phi(z)$, where the functions f, ϕ are real*. Writing herein $z=x+iy$, let P, Q be the real part and the coefficient of the imaginary part in the function $F(x+iy)$: or, what is the same thing, assume

$$P+iQ=f(x+iy)+i\phi(x+iy),$$

then it is clear that to any root $\alpha+i\beta$ (real or imaginary) of the equation $F(z)=0$, there corresponds a real intersection, or root, $x=\alpha$, $y=\beta$, of the curves $P=0$, $Q=0$. The functions, P, Q, as thus serving for the determination of the roots of the equation $F(z)=0$, are termed "rhizic functions," and similarly the curves $P=0$, $Q=0$ are "rhizic curves." The assumed equation shows at once that we have

$$d_y(P+iQ)=i\,d_x(P+iQ),$$

or, what is the same thing,

$$d_yP=-d_xQ,\ d_xP=d_yQ.$$

And we hence see that

$$\frac{d(P,\ Q)}{d(x,\ y)},\ =(d_xP)^2+(d_yP)^2,\ \text{or}\ (d_xQ)^2+(d_yQ)^2,$$

is positive: viz. that the roots $P=0$, $Q=0$ are all of them right-handed (the essential thing is that they are same-handed; for by reversing the signs of P and Q they might be made left-handed: but it is convenient to take them as right-handed): hence the theorem—which in the general case, where P and Q are arbitrary functions, serves to determine the difference of the numbers of the right- and left-handed roots—in the particular case, where P and Q are rhizic functions, serves to determine the number of intersections of the curves $P=0$, $Q=0$: or, what is the same thing, the number of the (real or imaginary) roots of the equation $F(z)=0$: viz. we thus determine the number of roots within a given circuit.

33. The rhizic curves $P=0$, $Q=0$ have various properties. 1°. Each curve has n real points at infinity, or, what is the same thing, n real asymptotes: and the P and Q points at infinity succeed each other, a P-point and then a Q-point, and so on, alternately.

In fact, from the equation

$$P+iQ=(a'+ia'')(x+iy)^n+\ldots+(k'+k''i),$$

writing herein $a'+ia''=a(\cos\alpha+i\sin\alpha)$, and $x+iy=\rho(\cos\theta+i\sin\theta)$, we have

$$P+iQ=a\rho^n[\cos(n\theta+\alpha)+i\sin(n\theta+\alpha)]+\ldots+k'+k''i.$$

* It is assumed that the equation $F(z)=0$ has no equal roots: this being so, the curves $P=0$, $Q=0$, will have no point of multiple intersection; which accords with the assumption made in the general case of two arbitrary curves.

It thus appears that for the curve $P=0$, the points at infinity are given by the equation $\cos(n\theta+\alpha)=0$; while for the curve $Q=0$, they are given by the equation $\sin(n\theta+\alpha)=0$: which proves the theorem.

Representing infinity as a closed curve or circuit, each point at infinity must be represented by two opposite points on the circuit; so that writing down P for each P-point and Q for each Q-point, we have $2n$ P's and $2n$ Q's succeeding each other, a P-point and then a Q-point, and so on, alternately.

It may be assumed that taking the circuit right-handedly, the P's are $+$ and the Q's $-$, (this depends only on the colouring, but it corresponds with the foregoing assumption that the roots $P=0$, $Q=0$ are right-handed): the theorem just obtained then really is that for the circuit infinity, the intercalation is $(+P-Q)_n$: and we have herein a proof of the theorem that a numerical equation of the order n with real or imaginary coefficients has precisely n real or imaginary roots. But the force of this will more distinctly appear presently.

34. 2°. Neither of the curves $P=0$, $Q=0$ can include as part of itself a closed curve or circuit.

The foregoing relations between the differential coefficients give

$$d_x^2P+d_y^2P=0,\quad d_x^2Q+d_y^2Q=0,$$

which equations for the two curves respectively lead to the theorem in question. For as regards the curve $P=0$, take z a coordinate perpendicular to the plane of xy, and consider the surface $z=P$: if the curve $P=0$ included as part of itself a closed curve, then corresponding to some point (x, y) within the curve we should have z a proper maximum or minimum, viz. there would be a summit or an imit; at the point in question we should have $d_xP=0$, $d_xQ=0$; and also (as the condition of a summit or imit) $d_x^2P.d_y^2P-(d_xd_yP)^2=+$, implying that d_x^2P and d_y^2P have at this point the same sign: but this is inconsistent with the foregoing relation $d_x^2P+d_y^2P=0$.

35. 3°. The curves $P=0$, $Q=0$ have not in general any double (or higher multiple) points. A point which is a double (or higher multiple) point on one of these curves is not of necessity a point on the other curve: but being a point on the other curve it is on that curve a point of the same multiplicity. For changing if necessary the coordinates, the point in question may be taken to be at the origin: forming the equation

$$P+iQ=(a'+a''i)(x+iy)^n+\ldots+(k'+k''i)(x+iy)^2+(l'+l''i)(x+iy)+m'+m''i=0,$$

the point $x=0$, $y=0$ will not be a double point on the curve $P=0$, unless we have $m'=0$, $l'=0$, $l''=0$; these conditions being satisfied, it will not be a point on the curve $Q=0$ unless also $m''=0$; but this being so, it will be a double point on the curve $Q=0$: and the like for points of higher multiplicity. But a point which is a multiple point on each curve, represents four or more coincident intersections of the curves $P=0$, $Q=0$, that is, four or more equal roots of the equation $F(z)=0$; so that assuming that the equation has no equal roots, the case does not arise: and we in fact exclude it from consideration.

To fix the ideas assume that the curves $P=0$, $Q=0$ are each of them without double points. As already seen, neither of them includes as part of itself a closed curve. Hence in the figure the curve $P=0$ must consist of n branches each drawn from a point P in the circuit (viz. the circuit infinity) to another point P in the circuit; and in such manner that no two branches intersect each other: this implies that the two points P of the same branch must include between them an even number (which may of course be $=0$) of points P. And the like as regards the curve $Q=0$.

36. 4°. No branch of the P-curve can meet a branch of the Q-curve more than once. In fact, drawing the two branches to meet twice, the colouring would at once show that of the two intersections or roots, one must be right, the other left-handed: whence, the roots being all right-handed, the branches do not meet twice. And in exactly the same way it appears that no P-branch can meet two Q-branches, or any Q-branch meet two P-branches. And under these restrictions it requires only a consideration of a few successive cases to show that the n P-branches, and the n Q-branches can only be drawn on the condition that each P-branch shall intersect once and only once a single Q-branch; which of course implies that each Q-branch intersects once and once only a single P-branch: and further, that there shall be precisely n intersections: viz. the n P-branches and the n Q-branches must satisfy the conditions just stated. And the theorem of the n roots is thus obtained as a consequence of the impossibility (except under the same conditions) of drawing the n P-branches and the n Q-branches, so as to give rise to right-handed roots only. But the case of double or higher multiple points would need to be specially considered.

37. It is interesting for a given value of n to consider $\phi(n)$, the number of different ways in which the P-branches and the Q-branches can be drawn. We have $2n$ points P and $2n$ points Q, in all $4n$ points: starting from any point P, these may be numbered in order $1, 2, 3, \ldots, 4n$, the points P bearing odd numbers and the points Q even numbers. We may consider the P-branch which joins 1 with some P-point β, and (intersecting this) the Q-branch which joins some two Q-points α and γ: the numbers $1\alpha\beta\gamma$ are then in order of increasing magnitude: and excluding these four points there remain the points corresponding to numbers between 1 and α, between α and β, between β and γ, and between γ and 1. Now since the P-branch 1β meets the Q-branch $\alpha\gamma$, no branch from a point between 1 and α can meet either of these curves; hence these points form a system by themselves, capable of being connected together by P-branches and Q-branches: the number of them must therefore be a multiple of 4: and the like as to the points between α and β, between β and γ, and between γ and 1. Taking the number of the points in the four systems to be $4x$, $4y$, $4z$, and $4w$ respectively, we have $x+y+z+w=n-1$, and the first-mentioned four points bear the numbers

$$\begin{aligned} &1, \\ &\alpha = 4x+2, \\ &\beta = 4x+4y+3, \\ &\gamma = 4x+4y+4z+4. \end{aligned}$$

For the four systems the number of ways of drawing the P- and Q-branches are ϕx, ϕy, ϕz, ϕw respectively: that is, x, y, z, w being any partition whatever of $n-1$ (order attended to), and $\phi(0)$ being $=1$, we have

$$\phi(n) = \Sigma \phi(x)\,\phi(y)\,\phi(z)\,\phi(w),$$

which is the condition for the determination of ϕn.

Taking then θ for the value of the generating function

$$1 + t\phi(1) + t^2\phi(2) \ldots + t^n\phi(n) + \ldots,$$

it hereby appears that we have

$$\theta = 1 + t\theta^4;$$

or writing this for a moment $\theta = u + t\theta^4$, and expanding by Lagrange's theorem, but putting finally $u=1$, we have the value of θ, that is of the generating function,

$$= 1 + [4]^0 \frac{t}{1} + [8]^1 \frac{t^2}{1.2} + [12]^2 \frac{t^3}{1.2.3} \ldots\ldots + [4n]^{n-1} \frac{t^n}{1.2\ldots n} + \ldots$$

$$= 1 + t + 4t^2 + 22t^3 + 140t^4 + \ldots,$$

that is,

$$\phi(1) = 1, \quad \phi(2) = 4, \quad \phi(3) = 22, \quad \phi(4) = 140, \ldots$$

and generally

$$\phi(n) = \frac{[4n]^{n-1}}{[n]^n}, \quad = \frac{4n.4n-1\ldots 3n+2}{2.3\ldots n}.$$

The results are easily verified for the successive particular cases; thus $n=1$, the points are 1, 2, 3, 4, and the P- and Q-branches respectively are 13, 24: $\phi(1)=1$. Again $n=2$, the points are 1, 2, 3, 4, 5, 6, 7, 8: we may join 13, 24 or 13, 28 or 17, 28 or 17, 68, leaving in each case four contiguous numbers which may be joined in a single manner: that is, $\phi(2)=4$. Or, what is the same thing, the partitions of 1 are 0001, 0010, 0100, 1000, whence $\phi(2) = 4\{\phi(0)\}^3\phi(1) = 4$. Again $n=3$, the partitions of 2 are 0002, &c. (4 of this form) and 1100 (6 of this form): that is, $\phi(3) = 4\{\phi(0)\}^3\phi(2) + 6\{\phi(0)\}^2\{\phi(1)\}^2, = 4.4 + 6.1 = 22$, and so on.

38. Starting from the $4n$ points P and Q, and joining them in any manner subject to the foregoing conditions, we have a diagram representing two rhizic curves; and colouring the regions we verify that the n roots are all of them right-handed. We have for instance the annexed figure ($n=3$).

Having drawn such a figure we may, by a continuous variation of the several lines, in a variety of ways introduce a double point in the P-curve, or in the Q-curve: and by a continued repetition of the process introduce double points in each or either curve: thus for instance we may from the last figure derive a new figure in which the P-curve has a node at N. It will be observed that here it is no longer the case that each P-branch intersects one and only one Q-branch: the P-branch $1-9$ does not meet any Q-branch, but the P-branch $7-11$ meets two Q-branches. But looking at the figure in a different manner, and considering the P-branches through N as

being either $11-N-1$ and $7-N-9$, or $1-N-7$ and $9-N-11$, then in either case each P-branch intersects one and only one Q-branch: and in this way, in a

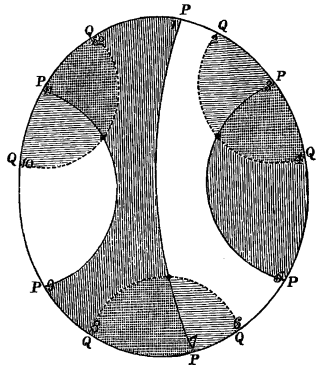
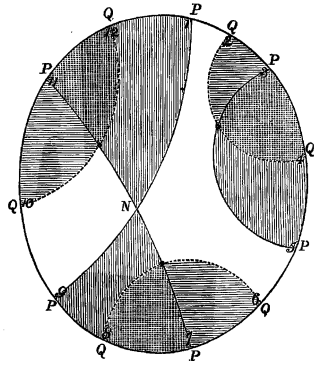


diagram in which the two curves have each or either of them double points, but neither curve passes through a double point of the other curve, the theorem may be regarded as remaining true—we in fact consider the diagram as the limit of a diagram wherein the curves have no double points. It will be recollected that, the equation



$F(z)$ being without equal roots, we cannot have either curve passing through a multiple point of the other curve. And we thus see that the various figures drawn as above without double points are, so to speak, the types of all the different forms of a system of rhizic curves $P=0$, $Q=0$.

In connexion with the present paper I give the following list of Memoirs:—

Cauchy. Calcul des Indices des fonctions. *Jour. de l'École Polyt.* t. XV. (1837), pp. 176—229. First part seems to have been written in 1833: second part is dated 20th June, 1837. Refers to a memoir presented to the Academy of Turin the 17th Nov. 1831, wherein the principles of the "Calcul des Indices des fonctions" are deduced from the theory of definite integrals: I have not seen this.

Sturm and Liouville. Démonstration d'un théorème de M. Cauchy relative aux racines imaginaires des équations. *Liouv.* t. I. (1836), pp. 278—289.

Sturm. Autres démonstrations du même théorème. *Liouv.* t. I. (1836), pp. 290—308.

These two papers contain proofs of the particular theorem relating to the roots of an equation $F(z)=0$, but do not refer to the general theorem relating to the intersection of the two curves $P=0$, $Q=0$: the special theorem of the existence of the n roots of the equation $F(z)=0$ is considered.

Sylvester. A theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions and that of the greatest algebraical common measure. *Phil. Trans.* t. CXLIII. (1853), pp. 407—548.

De Morgan. A proof of the existence of a root in every algebraic equation, with an examination and extension of Cauchy's theorem on imaginary roots, and remarks on the proofs of the existence of roots given by Argand and Mourey. *Camb. Phil. Trans.* t. X. (1858), pp. 261—270.

Contains the important remark that the two curves $P=0$, $Q=0$ are such that two branches, one of each curve, cannot inclose a space; also that the two curves always [i.e. at a simple intersection] intersect orthogonally.

Airy, G. B. Suggestion of a proof of the theorem that every algebraic equation has a root. *Camb. Phil. Trans.* t. X. (1859), pp. 283—289.

Cayley, A. Sketch of a proof of the theorem that every algebraic equation has a root. *Phil. Mag.* t. XVIII. (1859), [248], pp. 436—439.

Walton, W. On a theorem in maxima and minima. *Quart. Math. Jour.* t. X. (1870), pp. 253—262. **Cayley, A.** Addition thereto, [562], pp. 262, 263. (Relates to the curves $P=0$, $Q=0$.)

Walton, W. Note on rhizic curves. *Quart. Math. Jour.* t. XI. (1871), pp. 91—98. First use of the term "rhizic curves:" relates chiefly to the configuration of each curve at a multiple point, and of the two at a common multiple point.

Walton, W. On the spoke-asymptotes of rhizic curves. *Quart. Math. Jour.* t. XI. (1871), pp. 200—202.

Walton, W. On a property of the curvature of rhizic curves at multiple points. *Quart. Math. Jour.* t. XI. (1871), pp. 274—281.

Björling. Sur la séparation des racines d'équations algébriques. *Upsala, Nova Acta Soc. Sci.* (1870), pp. 1—35. (Contains delineations of some rhizic curves.)

562.

[ADDITION TO MR. WALTON'S PAPER "ON A THEOREM IN MAXIMA AND MINIMA."]

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 262, 263.]

In what follows I write x, y, z in place of Mr Walton's u, v, w: (so that if $i=\sqrt{(-1)}$, as usual, we have

$$f(x+iy)=P+iQ):$$

and I attend exclusively to the case where the second differential coefficients of P, Q do not vanish.

There are not on the surface $z=P$ any proper maxima or minima; but only level points, such as at the top of a pass: say there are not any summits or imits, but only cruxes; and moreover at any crux, the two crucial (or level) directions intersect at right angles. Every node of the curve $Q=0$ is subjacent to a crux of the surface $z=P$; and moreover the two directions of the curve $Q=0$ at the node are at right angles to each other; hence, considering the intersection of the surface $z=P$ by the cylinder $Q=0$, the path $Q=0$ on the surface has a node at the crux; or say there are at the crux two directions of the path; these cross at right angles, and are consequently separated the one from the other by the crucial directions; that is to say, there is one path ascending, and another path descending, each way from the crux. And the complete statement is; that the elevation of the path is then only a maximum or minimum when the path passes through a crux; and that at any crux there are two paths, one ascending, the other descending, each way from the crux.

The analytical demonstration is exceeding simple; we have

$$\left(\frac{dP}{dy}+i\frac{dQ}{dy}\right)=i\left(\frac{dP}{dx}+i\frac{dQ}{dx}\right);$$

that is,

$$\frac{dP}{dy} = -\frac{dQ}{dx}, \quad \frac{dQ}{dy} = \frac{dP}{dx},$$

and passing thence to the second differential coefficients, we may write

$$\frac{dP}{dx} = \frac{dQ}{dy} = L, \quad \frac{dP}{dy} = -\frac{dQ}{dx} = M,$$

$$\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dx^2} = \frac{d^2Q}{dy^2} = a,$$

$$\frac{d^2Q}{dx\,dy} = \frac{d^2P}{dx^2} = -\frac{d^2P}{dy^2} = b,$$

so that we have

$$\delta P = L\delta x + M\delta y, \qquad \delta Q = -M\delta x + L\delta y,$$

$$\delta^2 P = (b,\ a,\ -b \between \delta x,\ \delta y)^2, \quad \delta^2 Q = (-a,\ b,\ a \between \delta x,\ \delta y)^2.$$

Hence, for the maximum or minimum elevation of the path, we have $0 = \delta P$, where $\delta Q = 0$; that is, $0 = \frac{L^2 + M^2}{L}\delta x$, and therefore $L^2 + M^2 = 0$; that is, $L = 0$, $M = 0$; and at any such point $\delta z = 0$, that is, there is a crux of the surface $z = P$; and $\delta Q = 0$, that is, there is a node of the curve $Q = 0$. Moreover the crucial directions for the surface $z = P$ are given by the equation $(b,\ a,\ -b \between \delta x,\ \delta y)^2 = 0$, or these are at right angles to each other; and the nodal directions for the curve $Q = 0$ are given by $(-a,\ b,\ a \between \delta x,\ \delta y)^2 = 0$; or these are likewise at right angles to each other.

563.

NOTE ON THE TRANSFORMATION OF TWO SIMULTANEOUS EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 266, 267.]

WRITING in Mr Walton's equations (1) and (2)

$$\frac{a}{d},\ \frac{b}{d},\ \frac{c}{d},\ \frac{\alpha}{\delta},\ \frac{\beta}{\delta},\ \frac{\gamma}{\delta}$$

instead of a, b, c, α, β, γ respectively; and putting for shortness

$$\begin{aligned} A &= b\gamma - c\beta, & F &= a\delta - d\alpha, \\ B &= c\alpha - a\gamma, & G &= b\delta - d\beta, \\ C &= a\beta - b\alpha, & H &= c\delta - d\gamma, \end{aligned}$$

the equations become

$$\frac{a(b-c)}{F} + \frac{b(c-a)}{G} + \frac{c(a-b)}{H} = 0,$$

$$\frac{\alpha(\beta-\gamma)}{F} + \frac{\beta(\gamma-\alpha)}{G} + \frac{\gamma(\alpha-\beta)}{H} = 0.$$

Multiplying by FGH and effecting some obvious transformations, the equations become

$$\left.\begin{aligned} aAF + bBG + cCH &= 0 \\ \alpha AF + \beta BG + \gamma CH &= 0 \end{aligned}\right\} \quad \text{.....(18)};$$

whence also

$$AF^2 + BG^2 + CH^2 = 0 \quad \text{.....(19)}.$$

Now regarding $(\alpha, \beta, \gamma, \delta)$ as the coordinates of a point in space, the equations (18) and (19) represent each of them a cone having for vertex the point $\alpha : \beta : \gamma : \delta = a : b : c : d$, viz. (18) is a quadric cone, (19) a cubic cone; they intersect therefore in six lines; and it may be shown that these are

the line	$\alpha : \beta : \gamma = a : b : c$	(twice)	2
"	$\beta : \gamma : \delta = b : c : d$		1
"	$\gamma : \alpha : \delta = c : a : d$		1
"	$\alpha : \beta : \delta = a : b : d$		1
"	$\beta - \gamma : \gamma - \alpha : \alpha - \beta : \delta = b - c : c - a : a - b : d$		1
			6,

agreeing with Mr Walton's result.

564.

ON A THEOREM IN ELIMINATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 5, 6.]

I FIND among my papers the following example of a theorem in elimination communicated to me by Prof. Sylvester. Writing

$$\begin{aligned}
\phi &= ax^3 + 3bx^2y + 3cxy^2 + dy^3,\\
\phi_1 &= \qquad bx^2 \ + 2cxy \ + dy^2,\\
\phi_2 &= \qquad\qquad cx \ + dy\,,\\
\phi_3 &= \qquad\qquad\qquad d \ ;
\end{aligned}$$

$$\begin{aligned}
f &= bx^3 + 3cx^2y + 3dxy^2 + ey^3,\\
f_1 &= \qquad cx^2 \ + 2dxy \ + ey^2,\\
f_2 &= \qquad\qquad dx \ + ey\,,\\
f_3 &= \qquad\qquad\qquad e\ ,
\end{aligned}$$

then we have

$$\Delta_a \,.\, R(f,\ \phi) = \Delta f \,.\, R(\phi_1,\ f_1)^3\, R(\phi_1,\ f_2)^2,$$

viz. $R(f,\ \phi)$ is the resultant of the functions $(f,\ \phi)$, and similarly $R(\phi_1,\ f_1)$, $R(\phi_1,\ f_2)$. Moreover, Δf is the discriminant of f; and $\Delta_a R(f,\ \phi)$ is the discriminant of $R(f,\ \phi)$ in regard to a. The equation thus is

$$\begin{aligned}
&\Delta_a\,[(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e - c^3 + 2bcd)^2]\\
&\qquad\qquad = (b^2e^2 + 4bd^3 + 4c^3e - 3c^2d^2 - 6bcde)^3\,(d^3 - 2cde + be^2)^2\,;
\end{aligned}$$

or, what is the same thing, reversing the order of the letters $(a,\ b,\ c,\ d,\ e)$, it is

$$\begin{aligned}
&\Delta_e\,[(ae - 4bd + 3c^2)^3 - 27\,(ace - ad^2 - b^2e - c^3 + 2bcd)]\\
&\qquad\qquad = (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd)^3\,(b^3 - 2abc + a^2d)^2,
\end{aligned}$$

viz. arranging in powers of e, the function is

$$\begin{aligned} & e^3 \,.\quad a^3 \\ & + 3e^2 \,. - a^2 (4bd - 3c^2) \;-\; 9 (ac \;- b^2)^2 \\ & + 3e \;.\quad a \;(4bd - 3c^2)^2 + 18 (ac \;- b^2) (ad^2 - 2bcd + c^3) \\ & + 1 \;\;.- \quad (4bd - 3c^2)^3 - 27 (ad^2 - 2bcd + c^3)^2, \end{aligned}$$

which last coefficient is

$$= - d^2 (27a^2d^2 + 54ac^3 + 64b^3d - 36b^2c^2 - 108abcd),$$

and the discriminant of this cubic function of e is

$$= (a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd)^2 (b^3 - 2abc + a^2d)^2.$$

The occurrence of the factor

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd$$

is accounted for as the resultant in regard to e of the invariants I, J; we, in fact, have

$$\begin{aligned} (ac - b^2) I - aJ &= (ac - b^2) (- 4bd + 3c^2) - a (- ad^2 - c^3 + 2bcd) \\ &= a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd, \end{aligned}$$

and the identity itself may be proved without any particular difficulty.

565.

NOTE ON THE CARTESIAN.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 16—19.]

THE following are doubtless known theorems, but the form of statement, and the demonstration of one of them, may be interesting.

A point P on a Cartesian has three "opposite" points on the curve, viz. if the axial foci are A, B, C, then the opposite points are P_a, P_b, P_c where

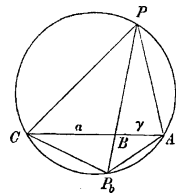
P_a is intersection of line PA with circle PBC,
P_b " " PB " PCA,
P_c " " PC " PAB.

And, moreover, supposing in the three circles respectively, the diameters at right angles to PA, PB, PC are $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ respectively, then the points α, α', β, β', γ, γ' lie by threes in two lines passing through P, viz. one of these, say $P\alpha\beta\gamma$, is the tangent, and the other $P\alpha'\beta'\gamma'$ the normal, at P; and then the tangents and normals at the opposite points are $P_a\alpha$ and $P_a\alpha'$, $P_b\beta$ and $P_b\beta'$, $P_c\gamma$, and $P_c\gamma'$ respectively.

There exists a second Cartesian with the same axial foci A, B, C, and passing through the points P, P_a, P_b, P_c (which are obviously opposite points in regard thereto); the tangent at P is $P\alpha'\beta'\gamma'$ and the normal is $P\alpha\beta\gamma$; and the tangent and the normal at the other points are $P_a\alpha'$ and $P_a\alpha$, $P_b\beta'$ and $P_b\beta$, $P_c\gamma'$ and $P_c\gamma$ respectively: viz. the two curves cut at right angles at each of the four points.

Starting with the foci A, B, C and the point P, the points P_a, P_b, P_c are constructed as above, without the employment of the Cartesian; there are through P with the foci A, B, C two and only two Cartesians; and if it is shown that these pass through one of the opposite points, say P_b, they must, it is clear, pass through

the other two points P_a, P_c. I propose to find the two Cartesians in question. To fix the ideas, let the points C, B, A be situate in order as shown in the figure, their



distances from a fixed point O being a, b, c, so that writing α, β, $\gamma = b-c$, $c-a$, $a-b$ respectively, we have $\alpha+\beta+\gamma=0$, and α, γ will represent the positive distances CB and BA respectively, and $-\beta$ the positive distance AC. Suppose, moreover, that the distances PA, PB, PC regarded as positive are R, S, T respectively; and that the distances P_bA, P_bB, P_bC regarded as positive are R', S', T' respectively.

Suppose that for a current point Q the distances QA, QB, QC regarded as indifferently positive, or negative, are r, s, t respectively; then the equation of a bicircular quartic having the points A, B, C for axial foci is

$$lr+ms+nt=0,$$

where l, m, n are constants; and this will be a Cartesian if only

$$\frac{l^2}{\alpha}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}=0.$$

We have the same curve whatever be the signs of l, m, n, and hence making the curve pass through P, we may, without loss of generality, write

$$lR+mS+nT=0,$$

R, S, T denoting the positive distances PA, PB, PC as above. We have thus for the ratios $l:m:n$, two equations, one simple, the other quadric; and there are thus two systems of values, that is, two Cartesians with the foci A, B, C, and passing through P.

I proceed to show that for one of these we have $-lR'+mS'+nT'=0$, and for the other $lR'+mS'-nT'=0$, or, what is the same thing, that the values of $l:m:n$ are

$$l:m:n=-(ST'+S'T):TR'+T'R:RS'-R'S,$$

and

$$l:m:n=\quad(ST'-S'T):-(TR'+T'R):RS'+R'S;$$

viz. that the equations of the two Cartesians are

$$\begin{vmatrix} r, & s, & t \\ R, & S, & T \\ -R', & S', & T' \end{vmatrix}=0, \text{ and } \begin{vmatrix} r, & s, & t \\ R, & S, & T \\ R', & S', & -T' \end{vmatrix}=0,$$

respectively; this being so each of the Cartesians will, it is clear, pass through the point P_b, and therefore also through P_a and P_c.

The geometrical relations of the figure give

$$\alpha R^2 + \beta S^2 + \gamma T^2 = -\alpha\beta\gamma,$$
$$\alpha R'^2 + \beta S'^2 + \gamma T'^2 = -\alpha\beta\gamma,$$
$$RT' + R'T = -\beta(S + S'),$$
$$\gamma\alpha = SS',$$
$$\gamma TT' = \alpha RR',$$

to which might be joined

$$R'^2S + \gamma^2(S + S') + R^2S' = SS'(S + S'),$$
$$T'^2S + \alpha^2(S + S') + T^2S' = SS'(S + S'),$$
$$SR'T' = S'RT,$$
$$SP'R' = S'PR,$$

but these are not required for the present purpose.

As regards the first Cartesian, we have to verify that

$$\frac{(ST' + S'T)^2}{\alpha} + \frac{(TR' + T'R)^2}{\beta} + \frac{(RS' - R'S)^2}{\gamma} = 0.$$

The left-hand side is

$$\frac{S^2T'^2 + S'^2T^2 + 2\gamma\alpha TT'}{\alpha} + \frac{\beta^2(S^2 + S'^2 + 2\gamma\alpha)}{\beta} + \frac{S^2R'^2 + S'^2R^2 - 2\gamma\alpha RR'}{\gamma},$$

viz. this is

$$= S^2\left(\frac{T'^2}{\alpha} + \beta + \frac{R'^2}{\gamma}\right) + S'^2\left(\frac{T^2}{\alpha} + \beta + \frac{R^2}{\gamma}\right) + 2\alpha\beta\gamma + 2(\gamma TT' - \alpha RR'),$$

which is

$$= S^2\left(\frac{-\beta S'^2}{\gamma\alpha}\right) + S'^2\left(\frac{-\beta S^2}{\gamma\alpha}\right) + 2\alpha\beta\gamma + 2(\gamma TT' - \alpha RR'),$$

and since the first and second terms are together $= -2\dfrac{\beta}{\gamma\alpha}S^2S'^2$, that is, $= -2\alpha\beta\gamma$, the whole is as it should be $= 0$.

In precisely the same manner we have

$$\frac{(ST' - S'T)^2}{\alpha} + \frac{(TR' + T'R)^2}{\beta} + \frac{(RS' + R'S)^2}{\gamma} = 0,$$

which is the condition for the second Cartesian: and the theorem in question is thus proved.

566.

ON THE TRANSFORMATION OF THE EQUATION OF A SURFACE TO A SET OF CHIEF AXES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 34—38.]

WE have at any point P of a surface a set of chief axes (PX, PY, PZ), viz. these are, say the axis of Z in the direction of the normal, and those of X, Y in the directions of the tangents to the two curves of curvature respectively. It may be required to transform the equation of the surface to the axes in question; to show how to effect this, take (x, y, z) for the original (rectangular) coordinates of the point P, $x+\delta x$, $y+\delta y$, $z+\delta z$ for the like coordinates of any other point on the surface, so that $(\delta x, \delta y, \delta z)$ are the coordinates of the point referred to the origin P; the equation of the surface, writing down only the terms of the first and second orders in the coordinates δx, δy, δz, is

$$A\delta x + B\delta y + C\delta z + \tfrac{1}{2}(a, b, c, f, g, h)(\delta x, \delta y, \delta z)^2 + \&c. = 0,$$

where (A, B, C) are the first derived functions and (a, b, c, f, g, h) the second derived functions of U for the values (x, y, z) which belong to the given point P, if $U=0$ is the equation of the surface in terms of the original coordinates (x, y, z); we have X, Y, Z linear functions of $(\delta x, \delta y, \delta z)$; say

$$\begin{array}{c|c|c|c|}
 & \delta x & \delta y & \delta z \\
\hline
X & \alpha_1 & \beta_1 & \gamma_1 \\
\hline
Y & \alpha_2 & \beta_2 & \gamma_2 \\
\hline
Z & \alpha & \beta & \gamma \\
\hline
\end{array}$$

that is, $X = \alpha_1\delta x + \beta_1\delta y + \gamma_1\delta z$, &c. and $\delta x = \alpha_1 X + \alpha_2 Y + \alpha Z$, &c. where the coefficients satisfy the ordinary relations in the case of transformation between two sets of rectangular axes; and the transformed equation is therefore

$$\begin{aligned} A(\alpha_1 X + \alpha_2 Y + \alpha Z) + B(\beta_1 X + \beta_2 Y + \beta Z) + C(\gamma_1 X + \gamma_2 Y + \gamma Z) \\ + (a, b, c, f, g, h)(\alpha_1 X + \alpha_2 Y + \alpha Z, \beta_1 X + \beta_2 Y + \beta Z, \gamma_1 X + \gamma_2 Y + \gamma Z)^2 = 0, \end{aligned}$$

or, as this may be written,

$$\begin{aligned} X(A\alpha_1+B\beta_1+C\gamma_1)+Y(A\alpha_2+B\beta_2+C\gamma_2)+Z(A\alpha+B\beta+C\gamma) \\ +\tfrac{1}{2}X^2\ (a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)^2 \\ +\tfrac{1}{2}Y^2\ (a,\ldots)(\alpha_2,\ \beta_2,\ \gamma_2)^2 \\ +\ XY(a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)(\alpha_2,\ \beta_2,\ \gamma_2) \\ +\ XZ(a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)(\alpha,\ \beta,\ \gamma) \\ +\ YZ(a,\ldots)(\alpha_2,\ \beta_2,\ \gamma_2)(\alpha,\ \beta,\ \gamma) \\ +\tfrac{1}{2}Z^2\ (a,\ldots)(\alpha,\ \beta,\ \gamma)^2 \qquad +\&c.=0, \end{aligned}$$

where the &c. refers to terms of the form $(X,\ Y,\ Z)^3$ and higher powers.

But in order that the new axes may be chief axes, we must have

$$\begin{aligned} A\alpha_1+B\beta_1+C\gamma_1=0, \\ A\alpha_2+B\beta_2+C\gamma_2=0, \\ (a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)(\alpha_2,\ \beta_2,\ \gamma_2)=0, \end{aligned}$$

so that putting for shortness

$$A\alpha+B\beta+C\gamma=\nabla,$$

the equation becomes

$$\begin{aligned} \nabla Z+\tfrac{1}{2}X^2\ (a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)^2+\tfrac{1}{2}Y^2(a,\ldots)(\alpha_2,\ \beta_2,\ \gamma_2)^2 \\ +\ XZ(a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)(\alpha,\ \beta,\ \gamma) \\ +\ YZ(a,\ldots)(\alpha_2,\ \beta_2,\ \gamma_2)(\alpha,\ \beta,\ \gamma) \\ +\tfrac{1}{2}Z^2\ (a,\ldots)(\alpha,\ \beta,\ \gamma)^2 \qquad +\&c.=0. \end{aligned}$$

We have

$$A\ :\ B\ :\ C=\beta_1\gamma_2-\beta_2\gamma_1\ :\ \gamma_1\alpha_2-\gamma_2\alpha_1\ :\ \alpha_1\beta_2-\alpha_2\beta_1,$$

that is,

$$=\quad \alpha\quad :\quad \beta\quad :\quad \gamma\quad ,$$

and thence

$$\alpha,\ \beta,\ \gamma=\frac{A}{\nabla},\ \frac{B}{\nabla},\ \frac{C}{\nabla};\ \nabla=\sqrt{(A^2+B^2+C^2)}.$$

I write

$$\frac{1}{\rho_1}=(a,\ldots)(\alpha_1,\ \beta_1,\ \gamma_1)^2,$$

and also for a moment

$$P=\left(a-\frac{1}{\rho_1},\quad h\quad ,\quad g\quad\right)(\alpha_1,\ \beta_1,\ \gamma_1),$$

$$Q=\left(\quad h\quad ,\quad b-\frac{1}{\rho_1},\quad f\quad\right)(\alpha_1,\ \beta_1,\ \gamma_1),$$

$$R=\left(\quad g\quad ,\quad f\quad ,\quad c-\frac{1}{\rho_1}\right)(\alpha_1,\ \beta_1,\ \gamma_1).$$

We find

$$P\alpha_1 + Q\beta_1 + R\gamma_1 = (a, \ldots)(\alpha_1, \beta_1, \gamma_1)^2 - \frac{1}{\rho_1}, = 0,$$

$$P\alpha_2 + Q\beta_2 + R\gamma_2 = (a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha_2, \beta_2, \gamma_2) - \frac{1}{\rho_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2), = 0,$$

and thence

$$\begin{aligned} P : Q : R &= \beta_1\gamma_2 - \beta_2\gamma_1 : \gamma_1\alpha_2 - \gamma_2\alpha_1 : \alpha_1\beta_2 - \alpha_2\beta_1 \\ &= \alpha \quad : \quad \beta \quad : \quad \gamma \quad , \end{aligned}$$

or say

$$P, Q, R = \theta_1 A, \theta_1 B, \theta_1 C;$$

we have thus the equations

$$\left(a - \frac{1}{\rho_1}, \quad h \quad , \quad g \quad\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 A,$$

$$\left(\quad h \quad , \quad b - \frac{1}{\rho_1}, \quad f \quad\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 B,$$

$$\left(\quad g \quad , \quad f \quad , \quad c - \frac{1}{\rho_1}\right)(\alpha_1, \beta_1, \gamma_1) = \theta_1 C,$$

and joining hereto

$$(A, B, C)(\alpha_1, \beta_1, \gamma_1) = 0,$$

we eliminate α_1, β_1, γ_1 and obtain the equation

$$\begin{vmatrix} a - \frac{1}{\rho_1}, & h \quad , & g \quad , & A \\ h \quad , & b - \frac{1}{\rho_1}, & f \quad , & B \\ g \quad , & f \quad , & c - \frac{1}{\rho_1}, & C \\ A \quad , & B \quad , & C \quad , & 0 \end{vmatrix} = 0,$$

and in like manner writing

$$\frac{1}{\rho_2} = (a, \ldots)(\alpha_2, \beta_2, \gamma_2)^2,$$

we have the same equation for ρ_2; wherefore ρ_1, ρ_2 are the roots of the quadric equation

$$\begin{vmatrix} a - \frac{1}{\rho}, & h \quad , & g \quad , & A \\ h \quad , & b - \frac{1}{\rho}, & f \quad , & B \\ g \quad , & f \quad , & c - \frac{1}{\rho}, & C \\ A \quad , & B \quad , & C \quad , & 0 \end{vmatrix} = 0.$$

Moreover, ρ_1, ρ_2 being thus determined, we have, α_1, β_1, γ_1, θ_1 proportional to the determinants formed with the matrix

$$\left|\begin{array}{cccc} a-\dfrac{1}{\rho_1}, & h\ , & g\ , & A \\ h\ , & b-\dfrac{1}{\rho_1}, & f\ , & B \\ g\ , & f\ , & c-\dfrac{1}{\rho_1}, & C \end{array}\right|,$$

say, α_1, β_1, γ_1, $\theta_1 = k\mathfrak{A}_1$, $k\mathfrak{B}_1$, $k\mathfrak{C}_1$, $k\Omega$, where $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$, Ω_1 are the determinants in question; and then $1 = k^2(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)$, or we have

$$\theta_1 = \frac{\Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}}.$$

But we find at once

$$(a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha, \beta, \gamma) = \theta_1(A\alpha + B\beta + C\gamma) = \theta_1 \nabla,$$

that is,

$$(a, \ldots)(\alpha_1, \beta_1, \gamma_1)(\alpha, \beta, \gamma) = \frac{\nabla \Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}},$$

and in the same manner

$$(a, \ldots)(\alpha_2, \beta_2, \gamma_2)(\alpha, \beta, \gamma) = \frac{\nabla \Omega_2}{\sqrt{(\mathfrak{A}_2^2 + \mathfrak{B}_2^2 + \mathfrak{C}_2^2)}}.$$

Hence the transformed equation is

$$\begin{aligned} &\nabla Z + \tfrac{1}{2}\frac{X^2}{\rho_1} + \tfrac{1}{2}\frac{Y^2}{\rho_2} \\ &\quad + XZ\frac{\nabla \Omega_1}{\sqrt{(\mathfrak{A}_1^2 + \mathfrak{B}_1^2 + \mathfrak{C}_1^2)}} + YZ\frac{\nabla \Omega_2}{\sqrt{(\mathfrak{A}_2^2 + \mathfrak{B}_2^2 + \mathfrak{C}_2^2)}} \\ &\quad + \tfrac{1}{2}Z^2\frac{(a, \ldots)(A, B, C)^2}{\nabla^2} + \&c. = 0, \end{aligned}$$

where it will be recollected that $\nabla = \sqrt{(A^2 + B^2 + C^2)}$. The &c. refers as before to the terms $(X, Y, Z)^3$ and higher powers, which are obtained from the corresponding terms in δx, δy, δz, by substituting for these their values $\delta x = \alpha_1 X + \alpha_2 Y + \alpha Z$, &c., where the coefficients have the values above obtained for them. It will be observed, that the radii of curvature are $\nabla\rho_1$, $\nabla\rho_2$, and that the process includes an investigation of the values of these radii of curvature similar to the ordinary one; the novelty is in the terms in XZ, YZ, and Z^2. But regarding X, Y as small quantities of the first order, Z is of the second order, and the terms in XZ, YZ are of the third order, and that in Z^2 of the fourth order.

567.

ON AN IDENTICAL EQUATION CONNECTED WITH THE THEORY OF INVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 115—118.]

WRITE

$$a = g - h,$$
$$b = h - f,$$
$$c = f - g,$$

equations implying a fourth equation forming with them the system

$$\begin{aligned} \cdot \quad -h + g - a &= 0, \\ h \quad \cdot \quad -f - b &= 0, \\ -g + f \quad \cdot \quad -c &= 0, \\ a + b + c \quad \cdot \quad &= 0, \end{aligned}$$

and also

$$af + bg + ch = 0.$$

Then, putting for shortness

$$P = (bg - ch)(ch - af)(af - bg),$$
$$Q = a^2g^2h^2 + b^2h^2f^2 + c^2f^2g^2 + a^2b^2c^2,$$
$$R = a^2f^2(a^2 + f^2) + b^2g^2(b^2 + g^2) + c^2h^2(c^2 + h^2),$$

we have

$$2P + Q - R = 0,$$

viz. substituting for a, b, c their values $g - h$, $h - f$, $f - g$, this is an identical equation.

The direct verification is however somewhat tedious, and the equation may be proved more easily as follows:

In the terms a^2+f^2, b^2+g^2, c^2+h^2 of R, substituting for a, b, c their values, we find

$$\begin{aligned} R = (f^2+g^2+h^2)(a^2f^2+b^2g^2+c^2h^2) \\ -2fgh(a^2f+b^2g+c^2h), \end{aligned}$$

which may be written

$$\begin{aligned} R = -2(f^2+g^2+h^2)(bcgh+cahf+abfg) \\ -2fgh(a^2f+b^2g+c^2h). \end{aligned}$$

We have then

$$2P = -2bcgh(bg-ch) - 2cahf(ch-af) - 2abfg(af-bg),$$

and thence

$$\begin{aligned} 2P - R = 2bcgh(f^2+g^2+h^2-bg+ch) \\ +2cahf(f^2+g^2+h^2-ch+af) \\ +2abfg(f^2+g^2+h^2-af+bg) \\ +2fgh(a^2f+b^2g+c^2h), \end{aligned}$$

which is at once converted into

$$\begin{aligned} 2P - R = 2bcgh\{a^2+f(f+g+h)\} \\ +2cahf\{b^2+g(f+g+h)\} \\ +2abfg\{c^2+h(f+g+h)\} \\ +2fgh(a^2f+b^2g+c^2h); \end{aligned}$$

or, what is the same thing,

$$2P - R = 2fgh\{(bc+ca+ab)(f+g+h)+a^2f+b^2g+c^2h\} + 2abc(agh+bhf+cfg),$$

where, since

$$agh+bhf+cfg = -abc,$$

the last term is

$$= -2a^2b^2c^2.$$

But from the equation last written down we deduce at once

$$Q = 2a^2b^2c^2 - 2fgh(bcf+cag+abh),$$

and we thence have

$$2P+Q-R = 2fgh\{(bc+ca+ab)(f+g+h)+(a^2f+b^2g+c^2h)-bcf-cag-abh\},$$

which is

$$= 2fgh(a+b+c)(af+bg+ch),$$

and consequently $=0$, the theorem in question.

Instead of a, b, c, f, g, h, I write $aW \div YZ$, $bW \div ZX$, $cW \div XY$, $f \div X$, $g \div Y$, $h \div Z$: we have therefore

$$\begin{aligned} . \quad -hY + gZ - aW &= 0, \\ hX \quad . \quad -fZ - bW &= 0, \\ -gX + fY \quad . \quad -cW &= 0, \\ aX + bY + cZ \quad . \quad &= 0, \end{aligned}$$

and as before

$$af + bg + ch = 0.$$

Moreover, omitting a common factor, the new values of P, Q, R are

$$\begin{aligned} P &= XYZW(bg - ch)(ch - af)(af - bg), \\ Q &= a^2g^2h^2X^4 + b^2h^2f^2Y^4 + c^2f^2g^2Z^4 + a^2b^2c^2W^4, \\ R &= a^2f^2(a^2X^2W^2 + f^2Y^2Z^2) + b^2g^2(b^2Y^2W^2 + g^2Z^2X^2) + c^2h^2(c^2Z^2W^2 + h^2X^2Y^2), \end{aligned}$$

and the identical equation is, as before,

$$2P + Q - R = 0.$$

Consider the operative symbols

$$\begin{aligned} &d_{x_1}, \ d_{x_2}, \ d_{x_3}, \ d_{x_4}, \\ &d_{y_1}, \ d_{y_2}, \ d_{y_3}, \ d_{y_4}, \end{aligned}$$

and write $a = d_{x_1}d_{y_2} - d_{y_1}d_{x_2} = 12$, &c., that is

$$\begin{aligned} a &= 23, \quad f = 14, \\ b &= 31, \quad g = 24, \\ c &= 12, \quad h = 34, \end{aligned}$$

and also $X = xd_{x_1} + yd_{y_1}$, &c. say

$$X = \nabla_1, \quad Y = \nabla_2, \quad Z = \nabla_3, \quad W = \nabla_4.$$

These values of $a, b, c, f, g, h, X, Y, Z, W$ satisfy the above written equations of connexion, and therefore the identical equation $2P + Q - R = 0$. Hence taking U to denote the quartic function $U = (a, b, c, d, e)(x, y)^4$, and therefore $U_1 = (a, \ldots)(x_1, y_1)^4$, &c., we have

$$(2P + Q - R)\, U_1U_2U_3U_4 = 0,$$

where, after the differentiations, $(x_1, y_1), \ldots, (x_4, y_4)$ are to be each of them replaced by (x, y).

Observe that P is the sum of three positive and three negative terms, but that after the omission of the suffixes each term taken with its proper sign becomes equal to the same quantity, and the value of P is $= 6$ times any one term thereof. Thus omitting for the moment the factor $\nabla_1\nabla_2\nabla_3\nabla_4$, two of the terms are $-(af)^2\, bg + af(bg)^2$, that is,

$$-(14 \,.\, 23)^2(24 \,.\, 31) + (14 \,.\, 23)(24 \,.\, 31)^2,$$

and, if in the first term we interchange 3 and 4, it becomes $-(13.24)^2(23.41)$, that is, $+(14.23)(24.31)^2$, viz. it becomes equal to the second term. As regards Q the terms are all positive and become equal to each other; and the like as regards R: hence we have

$$\{12\nabla_1\nabla_2\nabla_3\nabla_4(14.23)(24.31)^2+4\nabla_1^4(23)^2(34)^2(42)^2-6\nabla_1^2\nabla_4^2(43)^4(14)^2\}\,U_1U_2U_3U_4=0,$$

which, omitting a numerical factor $6.2.12^2.2.24^2.4, =3^5.2^{15}$, is in fact the well-known equation

$$\Omega+JU-IH=0,$$

where

$$\begin{aligned} U &= (a,\ b,\ c,\ d,\ e)(x,\ y)^4,\\ \Omega &= \text{disct.}\,(ax+by,\ bx+cy,\ cx+dy,\ dx+ey)(\xi,\ \eta)^3\\ &= (ax+by)^2(dx+ey)^2+\&\text{c.},\\ I &= ae-4bd+3c^2,\\ J &= ace-ad^2-b^2c-c^3+2bcd, \end{aligned}$$

viz. attending only to the coefficient of x^4, this equation is

$$a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd+a(ace-ad^2-b^2e-c^3+2bcd)+(ac-b^2)(ae-4bd+3c^2)=0.$$

568.

NOTE ON THE INTEGRALS $\int_0^\infty \cos x^2\,dx$ AND $\int_0^\infty \sin x^2\,dx$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 118—126.]

MR WALTON has raised, in relation to these integrals, a question which it is very interesting to discuss. Taking for greater convenience the limits to be $-\infty$, $+\infty$, and writing

$$2u = \int_{-\infty}^{\infty} \cos x^2 dx,\ 2v = \int_{-\infty}^{\infty} \sin x^2 dx,$$

then we have

$$4(u^2 - v^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(x^2 + y^2)\,dx\,dy,$$

$$8uv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin(x^2 + y^2)\,dx\,dy,$$

and writing herein $x = r\cos\theta$, $y = r\sin\theta$, and therefore $dxdy = rdr\,d\theta$, it would thence appear that we have

$$4(u^2 - v^2) = \int_0^\infty \int_0^{2\pi} \cos r^2 . rdrd\theta = 2\pi \int_0^\infty \cos r^2 . rdr,$$

$$8uv = \int_0^\infty \int_0^{2\pi} \sin r^2 . rdrd\theta = 2\pi \int_0^\infty \sin r^2 . rdr,$$

or, finally

$$4(u^2 - v^2) = \pi \sin\infty,$$

$$8uv = \pi(1 - \cos\infty);$$

that is, either the integrals have their received values $\left\{\text{each } = \frac{\sqrt{(\pi)}}{2\sqrt{(2)}}\right\}$, and then $\sin\infty = 0$, $\cos\infty = 0$; or else the integrals, instead of having their received values, are indeterminate.

The error is in the assumption as to the limits of r, θ; viz. in the original expressions for $4(u^2-v^2)$, $8uv$, we integrate over the area of an indefinitely large square (or rectangle); and the assumption is that we are at liberty, instead of this, to integrate over the area of an indefinitely large circle.

Consider in general in the plane of xy, a closed curve, surrounding the origin, depending on a parameter k, and such that each radius vector continually increases and becomes indefinitely large as k increases and becomes indefinitely large: the curve in question may be referred to as the bounding curve; and the area inside or outside this curve as the inside or outside area. And consider further an integral $\iint z\,dx\,dy$, where z is a given function of x, y, and the integration extends over the inside area. The function z may be such that, for a given form of the bounding curve, the integral, as k becomes indefinitely large, continually approaches to a determinate limiting value (this of course implies that z is indefinitely small for points at an indefinitely large distance from the origin); and we may then say that the integral taken over the infinite inside area has this determinate value; but it is by no means true that the value is independent of the form of the bounding curve; or even that, being determinate for one form of this curve, it is determinate for another form of the curve.

I remark, however, that if z is always of the same sign (say always positive) then the value, assumed to be determinate for a certain form of the bounding curve, is independent of the form of this curve and remains therefore unaltered when we pass to a different form of bounding curve. To fix the ideas, let the first form of bounding curve be a square ($x=\pm k$, $y=\pm k$), and the second form a circle ($x^2+y^2=k^2$). Imagine a square inside a circle which is itself inside another square; then z being always positive, the integral taken over the area of the circle is less than the integral over the area of the larger square, greater than the integral over the area of the smaller square. Let the sides of the two squares continually increase, then for each square the integral has ultimately its limiting value; that is, for the area included between the two squares the value is ultimately $=0$, and consequently for the circle the integral has ultimately the same value that it has for the square. When z is not always of the same sign the proof is inapplicable; and although, for certain forms of z, it may happen that the value of the integral is independent of the form of the bounding curve, this is not in general the case.

We have thus a justification of the well known process for obtaining the value of the integral $\int_0^\infty e^{-x^2}\,dx$, viz. calling this u, or writing

$$2u=\int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

then

$$4u^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy=\int_0^\infty\int_0^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$=2\pi\cdot\tfrac{1}{2},\quad\text{or}\quad u=\tfrac{1}{2}\sqrt{(\pi)},$$

but in consequence of the alternately positive and negative values of $\cos x^2$ and $\sin x^2$, we cannot infer that the like process is applicable to the integrals of these functions.

To show that it is in fact inapplicable, it will be sufficient to prove that the integrals in question have determinate values; for this being so, the double integrals $\iint \cos(x^2+y^2)\,dx\,dy$ and $\iint \sin(x^2+y^2)\,dx\,dy$, taken over an infinite square (or, if we please, over a rectangle the sides of which are both infinite, the ratio having any value whatever), will have determinate values; whereas, by what precedes, the values taken over an infinite circle are indeterminate. The thing may be seen in a very general sort of way thus: consider the surface $z=\sin(x^2+y^2)$, and let the plane of xy be divided into zones by the concentric circles, radii $\sqrt{(\pi)}$, $\sqrt{(2\pi)}$, $\sqrt{(3\pi)}$, &c. ..., then in the several zones z is alternately positive and negative, the maximum (positive or negative) value being ± 1; and though the breadths of the successive zones decrease, the areas and values of the integral remain constant for the successive zones; the integral over the circle radius $\sqrt{(n\pi)}$ is thus given as a neutral series having no determinate sum. But if the plane xy is divided in like manner into squares by the lines $x=\pm\sqrt{(n\pi)}$, $y=\pm\sqrt{(n\pi)}$, then in each of the bands included between successive squares, z has a succession of positive and negative values; the breadths continually diminish, and although the areas remain constant, yet, on account of the succession of the positive and negative values of z, there is a continual diminution in the values of the integral for the successive bands respectively, and the value of the integral for the whole square is given as a series which may very well be, and which I assume is in fact, convergent. Observe that I have not above employed this mode of integration (but by considering the single integral have in effect divided the square into indefinitely thin slices, and considered each slice separately); it would be interesting to carry out the analytical division of the square into bands, and show that we actually obtain a convergent series; but I do not pursue this inquiry.

Consider the integral

$$v=\int_0^\infty \sin x^2\,dx,$$

and taking for a moment the superior limit to be $(n+1)\pi$, then the quantity under the integral sign is positive from $x^2=0$ to $x^2=\pi$, negative from $x^2=\pi$ to $x^2=2\pi$, and so on; we may therefore write

$$\int_0^{(n+1)\pi} \sin x^2\,dx = A_0 - A_1 + A_2 \ldots + (-)^n A_n,$$

where

$$A_r, = (-)^r \int_{r\pi}^{(r+1)\pi} \sin x^2\,dx,$$

is positive. Writing herein $x^2=r\pi+u$, we have

$$A_r = \tfrac{1}{2}\int_0^\pi \frac{\sin u\,du}{\sqrt{(r\pi+u)}},$$

which, for r large, may be taken to be

$$=\tfrac{1}{2}\int_0^\pi \frac{\sin u du}{\sqrt{(r\pi)}},\ =\frac{1}{\sqrt{(r\pi)}},$$

viz. r being large, we have A_r differing from the above value $\frac{1}{\sqrt{(r\pi)}}$ by a quantity of the order $\frac{1}{r^{\frac{3}{2}}}$.

It is obviously immaterial whether we integrate from $x^2=0$ to $(n+1)\pi$ or to $(n+1)\pi+\epsilon$, where ϵ has any value less than π; for by so doing, we alter the value of the integral by a quantity less than A_{n+1}, and which consequently vanishes when n is indefinitely large. And similarly, it is immaterial whether we stop at an odd or an even value of n.

We have therefore

$$v=\int_0^{(n+1)\pi} \sin x^2\,dx = A_0 - A_1 + A_2 \ldots + (-)^n A_n,$$

or, taking n to be odd, this is

$$= A_0 - A_1 + A_2 \ldots - A_n,$$

or, say it is

$$= (A_0 - A_1) + (A_2 - A_3) \ldots + (A_{n-1} - A_n),$$

viz. n here denotes an indefinitely large odd integer.

If instead of $A_0 - A_1 + A_2 - A_3 +$ &c., the signs had been all positive, then the term A being ultimately as $\frac{1}{\sqrt{(r)}}$, the series would have been divergent, and would have had no definite sum: but with the actual alternate signs, the series is convergent, and the sum has a determinate value. To show this more distinctly, observe that we have

$$A_{r-1} - A_r = (-)^{r-1}\cdot\tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\sin(r\pi+u)\,du}{\sqrt{(r\pi+u)}},\ = -\tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\sin u\,dy}{\sqrt{(r\pi+u)}},$$

or, taking the integral from $-\pi$ to 0 and from 0 to π, and in the first integral writing $-u$ in place of u, then

$$A_{r-1} - A_r = \tfrac{1}{2}\int_0^\pi \sin u du \left\{\frac{1}{\sqrt{(r\pi-u)}} - \frac{1}{\sqrt{(r\pi+u)}}\right\},$$

where, r being large, expanding the term in { } in ascending powers of u, then $A_{r-1} - A_r$ is of the order $\frac{1}{r^{\frac{3}{2}}}$: and the series $(A_0 - A_1) + (A_2 - A_3) \ldots + (A_{n-1} - A_n)$ is therefore convergent, and the sum as n is increased approaches a definite limit. Hence the integral v has a definite value: and similarly, the integral u has a definite value.

The values of u, v being shown to be determinate, I see no ground for doubting that these are the values of the more general integrals

$$\int_0^\infty e^{-ax^2}\cos x^2\,dx, \quad \int_0^\infty e^{-ax^2}\sin x^2\,dx,$$

(a real and positive) when a is supposed to continually diminish and ultimately become $=0$. We have, in fact, (a as above)

$$\int_0^\infty e^{(-a+bi)y}\,y^{n-1}\,dy = \frac{\Gamma(n)\,e^{in\theta}}{(a^2+b^2)^{\frac{1}{2}n}},\,*$$

where $\theta = \tan^{-1}\dfrac{b}{a}$, an angle included between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. Writing herein $n=\frac{1}{2}$, $b=1$, $y=x^2$, then

$$\int_0^\infty e^{(-a+i)x^2}\,dx = \frac{\sqrt{(\pi)}\,e^{\frac{1}{2}i\theta}}{2\,(a^2+1)^{\frac{1}{4}}},$$

where $\theta = \tan^{-1}\dfrac{1}{a}$, an angle included between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$; or, putting herein $a=0$, we have $\theta=\frac{1}{2}\pi$, and therefore

$$\int_0^\infty e^{ix^2}\,dx = \tfrac{1}{2}\sqrt{(\pi)}\,e^{\frac{1}{4}i\pi};$$

that is, equating the real and imaginary parts,

$$u = v = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

which are the received values of the integrals

$$u = \int_0^\infty \cos x^2\,dx, \quad v = \int_0^\infty \sin x^2\,dx.$$

An important instance of the general theory presents itself in the theory of elliptic functions, viz. the integral

$$\iint \frac{dx\,dy}{(\Omega x + \Upsilon y)^2},$$

the ratio $\Omega : \Upsilon$ being imaginary, will, if the bounding curve be symmetrical in regard to the two axes respectively, have a determinate value *dependent on the form of the bounding curve;* if for instance this is a rectangle $x = \pm ak$, $y = \pm bk$, then the value of the integral will depend on the ratio $a : b$ of the infinite sides; and so if the bounding curve be an infinite ellipse, the value of the integral will depend on the ratio and position of the axes. See as to this my papers "On the inverse elliptic

* For brevity I take the integral under this form, but the real and imaginary parts might have been considered separately; and there would have been some advantage in following that course. The like remark applies to a subsequent investigation.

functions," *Camb. Math. Jour.*, t. IV. (1845), pp. 257—277, [24]; and "Mémoire sur les fonctions doublement périodiques," *Liouv.* t. X. (1845), pp. 385—420, [25].

A like theory applies to series, viz. as remarked by Cauchy, although the series $A_0+A_1+A_2+\ldots$ and $B_0+B_1+B_2+$ &c. are respectively convergent, then arranging the product in the form

$$\begin{aligned} & A_0B_0+A_0B_1+A_0B_2+\ldots \\ +\; & A_1B_0+A_1B_1+A_1B_2+\ldots \\ +\; & A_2B_0+A_2B_1+A_2B_2+\ldots \\ +\; & \ldots, \end{aligned}$$

say the general term is $C_{m,n}$, then if we sum this double series according to an assumed relation between the suffixes m, n (if, for instance, we include all those terms for which $m^2+n^2 \overline{\overline{<}} k^2$, making k to increase continually) it by no means follows that we approach a limit which is equal to the product of the sums of the original two series, or even that we approach a determinate limit.

Mr Walton, agreeing with the rest of the foregoing Note, wrote that he was unable to satisfy himself that the value of $\int_0^\infty e^{ix^2}\,dx$ is correctly deduced from that of $\int_0^\infty e^{(-a+bi)y}\,y^{n-1}\,dy$. Writing $n=\frac{1}{2}$, the question in fact is whether the formula

$$\int_0^\infty e^{(-a+bi)y}\,y^{-\frac{1}{2}}\,dy = \frac{\sqrt{(\pi)}\,e^{\frac{1}{2}i\theta}}{(a^2+b^2)^{\frac{1}{4}}}\left(\theta=\tan^{-1}\frac{b}{a},\ \text{angle between } \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi\right),$$

which is true when a is an indefinitely small positive quantity, is true when $a=0$; that is, taking b positive, whether we have

$$\int_0^\infty e^{iby}\,y^{-\frac{1}{2}}\,dy = \frac{\sqrt{(\pi)}\,e^{\frac{1}{4}i\pi}}{\sqrt{(b)}}.$$

Write in general

$$u=\int_0^\infty e^{(-a+bi)y}\,y^{-\frac{1}{2}}\,dy,$$

then, differentiating with respect to b, we have

$$\frac{du}{db}=\int_0^\infty iy^{\frac{1}{2}}\,e^{(-a+bi)y}\,dy,$$

or, integrating by parts,

$$\frac{du}{db}=\frac{i}{-a+bi}\,y^{\frac{1}{2}}\,e^{(-a+bi)y}-\frac{i}{2\,(-a+bi)}\int_0^\infty y^{-\frac{1}{2}}\,e^{(-a+bi)y}\,dy,$$

where the first term is to be taken between the limits ∞, 0; viz. this is

$$\frac{du}{db} = \left[\frac{i}{-a+bi}\, y^{\frac{1}{2}} e^{(-a+bi)y}\right]_0^\infty - \frac{i}{2(-a+bi)}\, u.$$

When a is not $=0$, the first term vanishes at each limit, and we have

$$\frac{du}{db} = \frac{-i}{2(-a+bi)}\, u.$$

The doubt was in effect whether this last equation holds good for the limiting value $a=0$. When a is $=0$, then in the original equation for $\frac{du}{db}$ the first term is indeterminate, and if the equation were true, it would follow that $\frac{du}{db}$ was indeterminate; the original equation for $\frac{du}{db}$ *is not true*, but we *truly* have

$$\frac{du}{db} = -\frac{1}{2b}\, u,$$

the same result as would be obtained from the general equation, rejecting the first term and writing $a=0$.

To explain this observe that for $a=0$, we have

$$u = \int_0^\infty y^{-\frac{1}{2}} e^{iby}\, dy,$$

which for a moment I write

$$u = \int_0^k y^{-\frac{1}{2}} e^{iby}\, dy,$$

where, as before, b is taken to be positive. Writing herein $by = x$, we have

$$u = \frac{1}{\sqrt{(b)}} \int_0^{bk} x^{-\frac{1}{2}} e^{ix}\, dx,$$

and assuming only that the integral $\int_0^M x^{-\frac{1}{2}} e^{ix}\, dx$ has a *determinate* limit as M becomes indefinitely large*, then supposing that k is indefinitely large, the integral in the last-mentioned expression for u has the value in question

$$\left(= \int_0^\infty x^{-\frac{1}{2}} e^{ix}\, dx\right),$$

* This is in fact the theorem $\int_0^\infty e^{ix^2}\, dx =$ a determinate value $\{= \frac{1}{2}\sqrt{(\pi)}\, e^{\frac{1}{4}i\pi}\}$, proved in the former part of the present Note.

which is independent of b, say this is

$$u = \frac{C}{\sqrt{(b)}},$$

and thence differentiating in regard to b, we find

$$\frac{du}{db} = -\frac{1}{2b}\,u,$$

the theorem in question.

But the value of $\frac{du}{db}$ *cannot be obtained by differentiating under the integral sign,* for this would give

$$\frac{du}{db} = \int_0^\infty iy^{\frac{1}{2}}\,e^{iby}\,dy,$$

and this integral is certainly indeterminate.

569.

ON THE CYCLIDE*.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 148—165.]

THE Cyclide, according to the original definition, is the envelope of a variable sphere which touches three given spheres, or, more accurately, the envelope of a variable sphere belonging to one of the four series of spheres which touch three given spheres. In fact, the spheres which touch three given spheres form four series, the spheres of each series having their centres on a conic; viz. if we consider the plane through the centres of the given spheres, and in this plane the eight circles which touch the sections of the given spheres, the centres of these circles form four pairs of points, or joining the points of the same pair, we have four chords which are the transverse axes of the four conics in question.

It thus appears, that one condition imposed on the variable sphere is, that its centre shall be in a plane; and a second condition, that the centre shall be on a conic in this plane; so that the original definition may be replaced first by the following one, viz.:

The cyclide is the envelope of a variable sphere having its centre on a given plane, and touching two given spheres.

Starting herefrom, it follows that the locus of the centre will be a conic in the given plane: the transverse axis of the conic being the projection on the given plane of the line joining the centres of the given spheres; and it, moreover, follows, that if in the perpendicular plane through the transverse axis we construct a conic having for vertices the foci, and for foci the vertices, of the locus-conic, then the conic so constructed will pass through the centres of the given spheres.

* I use the term in its original sense, and not in the extended sense given to it by Darboux, and employed by Casey in his recent memoir "On Cyclides and Spheroquartics," *Phil. Trans.* 1871, pp. 582—721. With these authors the Cyclide here spoken of is a Dupin's or tetranodal Cyclide.

Two conics related in the manner just mentioned are the flat-surfaces of a system of confocal quadric surfaces; they may for convenience be termed anti-conics (fig. 1); one of them is always an ellipse and the other a hyperbola; and the property of them is that, taking any two fixed points on the two branches, or on the same branch of the

Fig. 1.

hyperbola, and considering their distances from a variable point of the ellipse: in the first case the sum, in the second case the difference, of these two distances is constant. And similarly taking any two fixed points on the ellipse, and considering their distances from a variable point of the hyperbola, then the difference, first distance *less* second distance is a constant, $+\alpha$ for one branch, $-\alpha$ for the other branch of the hyperbola.

And we thus arrive at a third, and simplified definition of the cyclide, viz. considering any two anti-conics, the cyclide is the envelope of a variable sphere having its centre on the first anti-conic, and touching a given sphere whose centre is on the second anti-conic.

And it is to be added, that the same cyclide will be the envelope of a variable sphere having its centre on the second anti-conic and touching a given sphere whose centre is on the first anti-conic, such given sphere being in fact any particular sphere of the first series of variable spheres. And, moreover, the section of the surface by the plane of either of the anti-conics is a pair of circles, the surface being thus (as will further appear) of the fourth order.

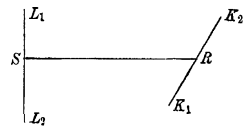
In the series of variable spheres the intersection of any two consecutive spheres is a circle, the centre of which is in the plane of the locus-anti-conic, and its plane perpendicular to that of the locus-anti-conic, this variable circle having for its diameter in the plane of the locus-anti-conic a line terminated by the two fixed circles in that plane. The cyclide is thus in two different ways the locus of a variable circle; and investigating this mode of generation, we arrive at a fourth definition as follows:—

Consider in a plane any two circles, and through either of the centres of symmetry draw a secant cutting the two circles; in the perpendicular plane through the secant, draw circles having for their diameters the chords formed by the two pairs of anti-parallel points on the secant (viz. each pair consists of two points, one on each circle, such that the tangents at the two points are not parallel to each other): the locus of the two variable circles is the cyclide.

Before going further it will be convenient to establish the definition of "skew anti-points": viz. if we have the points K_1, K_2 (fig. 2), mid-point R, and L_1, L_2, mid-point S, such that K_1K_2, RS and L_1L_2 are respectively at right angles to each other, and

$\overline{K_1R}^2+\overline{RS}^2+\overline{SL_1}^2=0$, &c.; or, what is the same thing, the distances $L_1K_1=L_1K_2=L_2K_1=L_2K_2$ are each $=0$, so that the points K_1, K_2 and L_1, L_2 are skew anti-points. Observe that the lines of the figure and the points R, S are taken to be real; but the distances $RK_1=RK_2$ and $SL_1=SL_2$ cannot be both real: it is assumed that one is real and

Fig. 2.



the other a pure imaginary, or else that they are both of them pure imaginaries. To fix the ideas we may in the figure consider the plane through K_1K_2, RS as horizontal, and that through RS, L_1L_2 as vertical.

Reverting now to the cyclide, suppose that we have (in the same plane) the two circles C, C' intersecting in K_1, K_2, and having S for a centre of symmetry, and let R be the mid-point of K_1, K_2.

The construction is:—through S draw a secant meeting the two circles in A, B and A', B' respectively, where A, A' and B, B' are parallel points, (therefore A, B' and A', B anti-parallel points), then the cyclide is the locus of the circles in the perpendicular plane on the diameters AB' and $A'B$ respectively.

The two circles have their radical axis passing through S, and not only so, but the points of intersection L_1, L_2 of the two circles are situate at a distance $SL_1=SL_2$, which is independent of the position of the secant: the points L_1, L_2 and K_1, K_2 being in fact a system of skew anti-points. And, moreover, the two circles have a centre of symmetry at the point where the plane of the two circles meets the line K_1K_2.

Consider in particular the two circles D, D' which are situate in the perpendicular plane through SR; these have the radical axis L_1L_2, and a centre of symmetry R; and if with these circles D, D' as given circles, and with R as the centre of symmetry, we obtain in a plane through K_1K_2 two circles having K_1K_2 for their radical axis, and having for a centre of symmetry the intersection of their plane with L_1L_2, the locus of these circles is the same cyclide as before; and, in particular, if their plane passes through RS, then the two circles are the before-mentioned circles C, C', having S for a centre of symmetry.

It will be noticed that, starting with the same two circles C, C' or D, D', we obtain two different cyclides according as we use in the construction one or other of the two centres of symmetry.

The cyclide is a quartic surface having the circle at infinity for a nodal line: viz. it is an anallagmatic or bicircular quartic surface; and it has besides the points

K_1, K_2, L_1, L_2, that is, a system of skew anti-points, for nodal points; these determine the cyclide save as to a single parameter. In fact, starting with the four points L_1, L_2, K_1, K_2, which give S, and therefore the plane of the circles C, C'; the circle C is then any one of the circles through K_1, K_2; and then drawing from S the two tangents to C, there is one other circle C' passing through K_1, K_2 and touching these tangents; C' is thus uniquely determined, and the construction is effected as above. Hence, with a given system of skew anti-points we have a single series of cyclides, say a series of conodal cyclides.

If in general we consider a quartic surface having a nodal conic and four nodes A, B, C, D, then it is to be observed that, taking the nodes in a proper order, we have a skew quadrilateral $ABCD$, the sides whereof AB, BC, CD, DA, lie wholly on the surface. In fact, considering the section by the plane ABC, this will be a quartic curve having the nodes A, B, C and two other nodes, the intersections of the plane with the nodal conic; the section is thus made up of a pair of lines and a conic; it follows that two of the sides of the triangle ABC, say the sides AB, BC, each meet the nodal conic, and that the section in question is made up of the lines AB, BC, and of a conic through the points A, C and the intersections of AB, BC with the nodal conic. Considering next the section by the plane through ACD, here (since AC is not a line on the surface) the lines CD, DA each meet the nodal conic, and the section is made up of the lines CD, DA and of a conic passing through the points A, C and the intersections of the lines CD, DA with the nodal conic. Thus the lines AB, BC, CD, DA each meet the nodal conic, and lie wholly on the surface; the lines AC, BD do not meet the conic or lie wholly on the surface.

A quartic surface depends upon 34 constants; it is easy to see that, if the surface has a given nodal conic, this implies 21 conditions, or say the postulation of a given nodal conic is $=21$, whence also the postulation of a nodal conic (not a given conic) is $=13$. Suppose that the surface has the given nodes A, B, C, D; the postulation hereof is $=16$; the nodal conic is then a conic meeting each of the lines AB, BC, CD, DA, viz. if the plane of the conic is assumed at pleasure, then the conic passes through 4 given points, and thus it still contains 1 arbitrary parameter; that is, in order that the nodal conic may be a given conic (satisfying the prescribed conditions) the postulation is $=4$. The whole postulation is thus $16+13+4$, $=33$, or the quartic surface which satisfies the condition in question (viz. which has for nodes the given points A, B, C, D, and for nodal conic a given conic meeting each of the lines AB, BC, CD, DA) contains still 1 arbitrary parameter: which agrees with the foregoing result in regard to the existence of a series of conodal cyclides.

It is to be added that, if a quartic surface has for a nodal line the circle at infinity and has four nodes, then the nodes form a system of skew anti-points and the surface is a cyclide. In fact, taking the nodes to be A, B, C, D, then each of the lines AB, BC, CD, DA meets the circle at infinity; but if the line AB meets the circle at infinity, then the distance AB is $=0$, and similarly the distances BC, CD, DA are each $=0$; that is, the nodes (A, C) and (B, D) are a system of skew anti-points.

Reverting to the cyclide, and taking (as before) the nodes to be K_1, K_2 and L_1, L_2, the line RS which joins the mid-points of K_1K_2 and L_1L_2 may be termed the axis of the cyclide, and the points where it meets the cyclide, or, what is the same thing, the circles C, C' or D, D', the vertices of the cyclide, say these are the points F, G, H, K. Supposing that the distances of these from a point on the axis are f, g, h, k, the origin may be taken so that $f+g+h+k=0$; the origin is in this case the "centre" of the cyclide. It is to be remarked, that given the vertices there are three series of cyclides; viz. we may in an arbitrary plane through the axis take for C, C' the circles standing on the diameters FG and HK respectively; and then, according as we take one or the other centre of symmetry, we have in the plane at right angles hereto for D, D' the circles on the diameters FH and GK, or else the circles on the diameters FK and GH respectively; there are thus three cases according as the two pairs of circles are the circles on the diameters

$$\begin{aligned} &FH,\ KG \text{ and } FK,\ GH,\\ &FK,\ GH \quad ,, \quad FG,\ HK,\\ &FG,\ HK \quad ,, \quad FH,\ KG. \end{aligned}$$

The equation of the cyclide expressed in terms of the parameters f, g, h, k assumes a peculiarly simple form; in fact, taking the origin at the centre, so that $f+g+h+k=0$, the axis of x coinciding with the axis of the cyclide, and those of y, z parallel to the lines K_1K_2 and L_1L_2, or L_1L_2 and K_1K_2 respectively: writing also

$$\begin{aligned} fg+hk&=G,\\ fh+kg&=H,\\ fk+gh&=K, \end{aligned}$$

then the equation of one of the cyclides is

$$(y^2+z^2)^2+2x^2(y^2+z^2)+Gy^2+Hz^2+(x-f)(x-g)(x-h)(x-k)=0,$$

which we may at once partially verify by observing that for $z=0$ this equation becomes

$$[y^2+(x-f)(x-g)][y^2+(x-h)(x-k)]=0,$$

and for $y=0$ it becomes

$$[z^2+(x-f)(x-h)][z^2+(x-k)(x-g)]=0,$$

viz. the equations of the circles C, C' are

$$y^2+(x-f)(x-g)=0,\quad y^2+(x-h)(x-k)=0,$$

and those of D, D'

$$z^2+(x-f)(x-h)=0,\quad z^2+(x-k)(x-g)=0.$$

Starting from these equations of the four circles, the points K_1, K_2 are given by

$$Y^2=-(P-f)(P-g)=-(P-h)(P-k),$$

and the points L_1, L_2 by

$$Z^2 = -(Q-f)(Q-h) = -(Q-k)(Q-g).$$

Now writing for a moment

$$\beta = f+g = -h-k,$$
$$\gamma = f+h = -k-g,$$
$$\delta = f+k = -g-h,$$

we have $P = -\frac{1}{2}\frac{\gamma\delta}{\beta}$, $Q = -\frac{1}{2}\frac{\beta\delta}{\gamma}$, and thence $PQ = \frac{1}{4}\delta^2$. Moreover

$$\begin{aligned}
&2Y^2 + 2Z^2 + 2(P-Q)^2 \\
&\quad = -(P-f)(P-g) - (P-h)(P-k) \\
&\qquad - (Q-f)(Q-h) - (Q-k)(Q-g) + 2(P-Q)^2 \\
&\quad = -(fg+hk+fh+gk) - 4PQ \\
&\quad = \delta^2 - 4PQ \\
&\quad = 0,
\end{aligned}$$

that is,

$$Y^2 + Z^2 + (P-Q)^2 = 0,$$

which equation expresses that the four points are a system of skew anti-points.

The point $x = Q$ should be a centre of symmetry of the circles C, C'; to verify that this is so, transforming to the point in question as origin, the equations are

$$y^2 + \{x + Q - \tfrac{1}{2}(f+g)\}^2 - \tfrac{1}{4}(f-g)^2 = 0,$$
$$y^2 + \{x + Q - \tfrac{1}{2}(h+k)\}^2 - \tfrac{1}{4}(k-h)^2 = 0,$$

that is,

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(\delta+\gamma)\right\}^2 - \tfrac{1}{4}(f-g)^2 = 0,$$

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(\delta-\gamma)\right\}^2 - \tfrac{1}{4}(k-h)^2 = 0.$$

But $\delta+\gamma = f-g$, $\delta-\gamma = k-h$, so that these equations are

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(f-g)\right\}^2 = \tfrac{1}{4}(f-g)^2,$$

$$y^2 + \left\{x - \tfrac{1}{2}\frac{\beta}{\gamma}(k-h)\right\}^2 = \tfrac{1}{4}(k-h)^2,$$

which are of the form

$$y^2 + (x - \alpha)^2 = c^2,$$
$$y^2 + (x - m\alpha)^2 = m^2c^2,$$

and consequently $x = Q$ is a centre of symmetry of the circles C, C'; and in like manner it would appear that $x = P$ is a centre of symmetry of the circles D, D'.

If in the last-mentioned equations of the circles C, C' we write $x = \Omega \cos\theta$, $y = \Omega \sin\theta$, and put for shortness

$$\rho = \alpha\cos\theta - \nabla, \quad \sigma = m(\alpha\cos\theta - \nabla),$$
$$\rho' = \alpha\cos\theta + \nabla, \quad \sigma' = m(\alpha\cos\theta + \nabla),$$

where $\nabla = \sqrt{(c^2 - \alpha^2\sin^2\theta)}$, then the values of Ω for the first circle are ρ, σ, and those for the second circle are ρ', σ'. Hence the equations of the generating circles are

$$z^2 + (r-\rho)(r-\sigma') = 0,$$
$$z^2 + (r+\rho')(r-\sigma) = 0,$$

where r is the abscissa in the plane of the circles, measured from the point $x = Q$. Attending say to the first of these equations, to find the equation of the cyclide, we must eliminate θ from the equations

$$z^2 + (r-\rho)(r-\sigma') = 0, \quad x = r\cos\theta, \quad y = r\sin\theta;$$

the first equation is

$$z^2 + r^2 + m(\alpha^2 - c^2) - r(\rho + \sigma') = 0,$$

and we have

$$\rho + \sigma' = (m+1)\alpha\cos\theta - (m-1)\sqrt{(c^2 - \alpha^2\sin^2\theta)},$$

and thence

$$(\rho + \sigma')r = (m+1)\alpha x - (m-1)\sqrt{\{c^2(x^2+y^2) - \alpha^2 y^2\}},$$

so that we have

$$z^2 + x^2 + y^2 + m(\alpha^2 - c^2) - (m+1)\alpha x + (m-1)\sqrt{\{c^2(x^2+y^2) - \alpha^2 y^2\}} = 0,$$

viz. this is the equation of the cyclide in terms of the parameters α, c, m, the origin being at the point $x = Q$, the centre of symmetry of the circles C, C'.

Reverting to the former origin at the centre of the cyclide, we must write $x - Q$ for x; the equation thus is

$$\{y^2 + z^2 + (x-Q)^2 - (m+1)\alpha(x-Q) + m(\alpha^2 - c^2)\}^2 - (m-1)^2[\{c^2(x-Q)^2 + (c^2 - \alpha^2)y^2\}] = 0,$$

where

$$Q = -\tfrac{1}{2}\frac{\beta\delta}{\gamma}, \quad \alpha = \tfrac{1}{2}\frac{\beta}{\gamma}(f-g), \quad c^2 = \tfrac{1}{4}(f-g)^2, \quad m = \frac{k-h}{f-g},$$

whence also

$$m+1 = \frac{2\delta}{f-g}, \quad m-1 = \frac{-2\gamma}{f-g}, \quad \alpha^2 - c^2 = \tfrac{1}{4}(f-g)^2\frac{(f-k)(g-h)}{\gamma^2}.$$

After all reductions, the equation assumes the before-mentioned form

$$(y^2+z^2)^2 + 2x^2(y^2+z^2) + Gy^2 + Hz^2 + (x-f)(x-g)(x-h)(x-k) = 0.$$

The equation may be written

$$(x^2+y^2+z^2)^2 + (G+H+K)x^2 + Gy^2 + Hz^2 - \beta\gamma\delta x + fghk = 0,$$

and if we express everything in terms of β, γ, δ by the formulæ

$$\begin{aligned} 2f &= \beta+\gamma+\delta, & 2G &= \beta^2-\gamma^2-\delta^2,\\ 2g &= \beta-\gamma-\delta, & 2H &= -\beta^2+\gamma^2-\delta^2,\\ 2h &= -\beta+\gamma-\delta, & 2K &= -\beta^2-\gamma^2+\delta^2,\\ 2k &= -\beta-\gamma+\delta, & 2(G+H+K) &= -\beta^2-\gamma^2-\delta^2; \end{aligned}$$

then we have

$$(x^2+y^2+z^2)^2+\tfrac{1}{2}(-\beta^2-\gamma^2-\delta^2)\,x^2+\tfrac{1}{2}(\beta^2-\gamma^2-\delta^2)\,y^2+\tfrac{1}{2}(-\beta^2+\gamma^2-\delta^2)\,z^2$$
$$-\beta\gamma\delta x+\tfrac{1}{16}(\beta^4+\gamma^4+\delta^4-2\beta^2\gamma^2-2\beta^2\delta^2-2\gamma^2\delta^2)=0;$$

or, what is the same thing,

$$(x^2+y^2+z^2+\tfrac{1}{4}\beta^2+\tfrac{1}{4}\gamma^2-\tfrac{1}{4}\delta^2)^2-(\beta^2+\gamma^2)\,x^2-\gamma^2y^2-\beta^2z^2-\beta\gamma\delta x-\tfrac{1}{4}\beta^2\gamma^2=0.$$

An equivalent form of equation may be obtained very simply as follows: the surface

$$(x^2+y^2+z^2)^2+2Ax^2+2By^2+2Cz^2+2Kx+L=0$$

will be a cyclide if only the section by each of the planes $y=0$, $z=0$ breaks up into a pair of circles. Now for $y=0$ the equation is

$$(x^2+z^2)^2+2Ax^2+2Cz^2+2Kx+L=0,$$

that is,

$$z^4+2z^2(x^2+C)+x^4+2Ax^2+2Kx+L=0,$$

or

$$(z^2+x^2+C)^2=2(C-A)\,x^2-2Kx+C^2-L,$$

which will be a pair of circles if only

$$2(C-A)(C^2-L)=K^2;$$

and similarly writing $z=0$, we obtain

$$2(B-A)(B^2-L)=K^2.$$

These equations give

$$\begin{aligned} L &= (B+C)^2-(BC+CA+AB),\\ K^2 &= -2(B-A)(C-A)(B+C), \end{aligned}$$

so that L, K having these values the surface is a cyclide; there are two cyclides corresponding to the two different values of K, which agrees with a former result.

Reverting to the equation in terms of β, γ, δ this may be written

$$\beta^2-\gamma^2+\sqrt{\{(2\gamma x+\beta\delta)^2-4(\beta^2-\gamma^2)\,y^2\}}+\sqrt{\{(2\beta x+\gamma\delta)^2+4(\beta^2-\gamma^2)\,z^2\}}=0.$$

[Compare herewith Kummer's form

$$b^2=\sqrt{\{(ax-ek)^2+b^2y^2\}}+\sqrt{\{(ex-ak)^2-b^2z^2\}},\text{ where } b^2=a^2-e^2.]$$

In fact, representing this for a moment by

$$\beta^2 - \gamma^2 + \sqrt{(\Theta)} + \sqrt{(\Phi)} = 0,$$

we have

$$(\beta^2 - \gamma^2)^2 + \Theta - \Phi = -2(\beta^2 - \gamma^2)\sqrt{(\Theta)},$$

or, substituting and dividing by $\beta^2 - \gamma^2$, we have

$$\beta^2 - \gamma^2 + \delta^2 - 4(x^2 + y^2 + z^2) + 2\sqrt{\{(2\gamma x + \beta\delta)^2 - 4(\beta^2 - \gamma^2)y^2\}} = 0,$$

or, similarly

$$\beta^2 - \gamma^2 - \delta^2 + 4(x^2 + y^2 + z^2) + 2\sqrt{\{(2\beta x + \gamma\delta)^2 + 4(\beta^2 - \gamma^2)z^2\}} = 0,$$

either of which leads at once to the rational form.

The irrational equation

$$\beta^2 - \gamma^2 + \sqrt{\{(2\gamma x + \beta\delta)^2 - 4(\beta^2 - \gamma^2)y^2\}} + \sqrt{\{(2\beta x + \gamma\delta)^2 + 4(\beta^2 - \gamma^2)z^2\}} = 0$$

is of the form

$$p + \sqrt{(qr)} + \sqrt{(st)} = 0,$$

which belongs to a quartic surface having the nodal conic $p = 0$, $qr - st = 0$ (in the present case the circle at infinity), and also the four nodes ($q = 0$, $r = 0$, $p^2 - st = 0$) and ($s = 0$, $t = 0$, $p^2 - qr = 0$), viz. these are

$$x = -\tfrac{1}{2}\frac{\beta\delta}{\gamma}, \quad y = 0, \quad z = \pm\tfrac{1}{2}\frac{1}{\gamma}\sqrt{\{(\beta^2 - \gamma^2)(\gamma^2 - \delta^2)\}},$$

and

$$x = -\tfrac{1}{2}\frac{\gamma\delta}{\beta}, \quad y = \pm\tfrac{1}{2}\frac{1}{\beta}\sqrt{\{(\gamma^2 - \beta^2)(\beta^2 - \delta^2)\}}, \quad z = 0,$$

and we hence again verify that the nodes form a system of skew anti-points, viz. the condition for this is

$$\delta^2\left(\frac{\beta}{\gamma} - \frac{\gamma}{\beta}\right)^2 + (\beta^2 - \gamma^2)\left(1 - \frac{\delta^2}{\gamma^2}\right) - (\beta^2 - \gamma^2)\left(1 - \frac{\delta^2}{\beta^2}\right) = 0,$$

that is,

$$\delta^2(\beta^2 - \gamma^2) + \beta^2(\gamma^2 - \delta^2) - \gamma^2(\beta^2 - \delta^2) = 0,$$

which is satisfied identically.

The cyclide has on the nodal conic or circle at infinity four pinch-points, viz. these are the intersections of the circle at infinity with the planes $\beta^2 y^2 + \gamma^2 z^2 = 0$.

If $\beta = 0$, the equation becomes

$$\tfrac{1}{2}\gamma + \sqrt{(x^2 + y^2)} + \sqrt{(\tfrac{1}{4}\delta^2 - z^2)} = 0,$$

viz. the cyclide has in this case become a torus; there are here two nodes on the axis ($x = 0$, $y = 0$), and two other nodes on the circle at infinity, viz. these are the circular points at infinity of the sections perpendicular to the axes, and the pinch-points coincide in pairs with the last-mentioned two nodes; viz. each of the circular points at infinity = node + two pinch-points.

The Parabolic Cyclide.

One of the circles C, C' and one of the circles D, D' may become each of them a line; the cyclide is in this case a cubic surface. The easier way would be to treat the case independently, but it is interesting to deduce it from the general case. For this purpose, starting from the equation

$$(y^2+z^2)^2+2x^2(y^2+z^2)+Gy^2+Hz^2+(x-f)(x-g)(x-h)(x-k)=0,$$

where $f+g+h+k=0$, $G=fg+hk$, $H=fh+gk$, I write $x-\alpha$ for x, and assume $\alpha+f$, $\alpha+g$, $\alpha+h$, $\alpha+k$, equal to f', g', h', k' respectively; whence $4\alpha=f'+g'+h'+k'$; and the equation is

$$(y^2+z^2)^2+2(x-\alpha)^2(y^2+z^2)+(f'g'+h'k'-2\alpha^2)y^2+(f'h'+g'k'-2\alpha^2)z^2$$
$$+(x-f')(x-g')(x-h')(x-k')=0,$$

or, what is the same thing,

$$(y^2+z^2)^2+(2x^2-4\alpha x)(y^2+z^2)+(f'g'+h'k')y^2+(f'h'+g'k')z^2$$
$$+(x-f')(x-g')(x-h')(x-k')=0.$$

Now assuming $k'=\infty$, we have $4\alpha=k'=\infty$, or writing 4α instead of k', and attending only to the terms which contain α, we have

$$x(y^2+z^2)-h'y^2-g'z^2+(x-f')(x-g')(x-h')=0,$$

or, what is the same thing,

$$(x-f')(x-g')(x-h')+(x-h')y^2+(x-g')z^2=0,$$

where by altering the origin we may make $f'=0$.

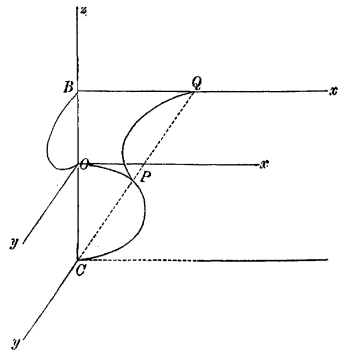
It is somewhat more convenient to take the axis of z (instead of that of x) as the axis of the cyclide; making this change, and writing also 0, β, γ in place of the original constants, I take the equation to be

$$z(z-\beta)(z-\gamma)+(z-\gamma)y^2+(z-\beta)x^2=0,$$

viz. this is a cubic surface having upon it the right lines $(z=\gamma,\ x=0)$, $(z=\beta,\ y=0)$; the section by a plane through either of these lines is the line itself and a circle; and in particular the circle in the plane $x=0$ is $z(z-\beta)+y^2=0$, and that in the plane $y=0$ is $z(z-\gamma)+x^2=0$. And it is easy to see how the surface is generated: if, to fix the ideas, we take β positive, γ negative, the lines and circles are as shown in fig. 3; and if we draw through Cy a plane cutting the circle CO and the line Bx in P, Q respectively, then the section is a circle on the diameter PQ; and similarly for the sections by the planes through Bx. It is easy to see that the whole surface is included between the planes $z=\beta$, $z=\gamma$; considering the sections parallel to these planes (that is, to the plane of xy) $z=\beta$, the section is the two-fold line $y=0$; $z=$ any smaller positive value, it is a hyperbola having the axis of y for its transverse axis; $z=0$, it is the pair of real lines $\gamma y^2+\beta x^2=0$; z negative and less in absolute

magnitude than $-\gamma$, it is a hyperbola having the axis of x for its transverse axis; and finally $z=\gamma$, it is the two-fold line $x=0$. It is easy to see the forms of the cubic curves which are the sections by any planes $x=$ const. or $y=$ const.

Fig. 3.



The before-mentioned circles are curves of curvature of the surface; to verify this *à posteriori*, write

$$U=z(z-\beta)(z-\gamma)+(z-\gamma)y^2+(z-\beta)x^2=0$$

for the equation of the surface; and put for shortness $P=3z^2-2z(\beta+\gamma)+\beta\gamma$, $P+x^2+y^2=L$, so that $d_zU=P+x^2+y^2$, $=L$. The differential equation for the curves of curvature is

$$\begin{vmatrix} 2x(z-\beta) & , & 2y(z-\gamma) & , & P+x^2+y^2 \\ xdz+(z-\beta)dx, & & ydz+(z-\gamma)dy, & & \frac{1}{2}P'dz+xdx+ydy \\ dx & , & dy & , & dz \end{vmatrix}=0,$$

or, say this is

$$\begin{aligned}\Omega = {} & dx^2 \,.\, 2xy(z-\gamma)-dy^2 \,.\, 2xy(z-\beta)+dz^2 \,.\, 2xy(\gamma-\beta) \\ & + dzdy \,.\, x[-2(z-\beta)(2z-\beta)+L] \\ & + dxdz \,.\, y[\quad 2(z-\gamma)(2z-\gamma)-L] \\ & + dxdy \,.\, [(\gamma-\beta)P+(2z-\beta-\gamma)(y^2-x^2)]=0.\end{aligned}$$

But in virtue of the equation $U=0$, we have identically

$$\begin{aligned}&\{2(z-\beta)x\,dx+2(z-\gamma)y\,dy+Ldz\}\times\left\{-\frac{z-\gamma}{z-\beta}y\,dx+\frac{z-\beta}{z-\gamma}x\,dy+\frac{xy(\gamma-\beta)}{(z-\beta)(z-\gamma)}dz\right\} \\ &+\Omega \\ &=(\gamma-\beta)\left\{2-\frac{L}{(z-\beta)(z-\gamma)}\right\}\times\{xydz^2-y(z-\gamma)dzdx-x(z-\beta)dzdy+(z-\beta)(z-\gamma)dxdy\}.\end{aligned}$$

Hence in virtue of the equations $U=0$, $dU=0$ the equation $\Omega=0$ becomes

$$xy\,dz^2 - y(z-\gamma)\,dz\,dx - x(z-\beta)\,dz\,dy + (z-\beta)(z-\gamma)\,dx\,dy = 0,$$

that is,

$$\{x\,dz - (z-\gamma)\,dx\}\{y\,dz - (z-\beta)\,dy\} = 0,$$

whence either $x - C(z-\gamma) = 0$ or $y - C'(z-\beta) = 0$; viz. the section of the surface by a plane of either series (which section is a circle) is a curve of curvature of the surface.

The equation of the cyclide can be elegantly expressed in terms of the ellipsoidal coordinates (λ, μ, ν) of a point (x, y, z); viz. writing for shortness $\alpha = b^2 - c^2$, $\beta = c^2 - a^2$, $\gamma = a^2 - b^2$, the coordinates (λ, μ, ν) are such that

$$\begin{aligned} -\beta\gamma x^2 &= (a^2+\lambda)(a^2+\mu)(a^2+\nu), \\ -\gamma\alpha y^2 &= (b^2+\lambda)(b^2+\mu)(b^2+\nu), \\ -\alpha\beta z^2 &= (c^2+\lambda)(c^2+\mu)(c^2+\nu), \end{aligned}$$

(see Roberts, *Comptes Rendus*, t. LIII. (Dec., 1861), p. 1119), whence

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + \lambda + \mu + \nu,$$

$$(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = b^2c^2 + c^2a^2 + a^2b^2 - \mu\nu - \nu\lambda - \lambda\mu.$$

The equation of the cyclide then is

$$\sqrt{(a^2+\lambda)} + \sqrt{(a^2+\mu)} + \sqrt{(a^2+\nu)} = \sqrt{(\delta)}.$$

In fact, starting from this equation and rationalising, we have

$$\begin{aligned} (3a^2+\lambda+\mu+\nu-\delta)^2 &= 4\left[\sqrt{\{(a^2+\mu)(a^2+\nu)\}} + \sqrt{\{(a^2+\nu)(a^2+\lambda)\}} + \sqrt{\{(a^2+\lambda)(a^2+\mu)\}}\right]^2 \\ &= 4\left[3a^4 + 2a^2(\lambda+\mu+\nu) + \mu\nu + \nu\lambda + \lambda\mu + 2\sqrt{\{(a^2+\lambda)(a^2+\mu)(a^2+\nu)\}}\sqrt{(\delta)}\right], \end{aligned}$$

which, substituting for

$$\lambda+\mu+\nu,\ \mu\nu+\nu\lambda+\lambda\mu \text{ and } \sqrt{\{(a^2+\lambda)(a^2+\mu)(a^2+\nu)\}}$$

their values, is

$$(x^2+y^2+z^2+\gamma-\beta-\delta)^2 = 4\{(\gamma-\beta)x^2 - \beta y^2 + \gamma z^2 - \beta\gamma - 2x\sqrt{(-\beta\gamma\delta)}\},$$

or, writing $-\frac{1}{4}\gamma^2$, $\frac{1}{4}\beta^2$, $\frac{1}{4}\delta^2$ in place of β, γ, δ respectively, this is

$$(x^2+y^2+z^2+\tfrac{1}{4}\gamma^2+\tfrac{1}{4}\beta^2-\tfrac{1}{4}\delta^2)^2 = (\gamma^2+\beta^2)x^2 + \beta^2z^2 + \gamma^2y^2 + \tfrac{1}{4}\beta^2\gamma^2 + \beta\gamma\delta x,$$

which agrees with a foregoing form of the equation.

The generating spheres of the cyclide cut at right angles each of a series of spheres; viz. each of these spheres passes through one and the same circle in the plane of, and having double contact with, the conic which contains the centres of the generating spheres; the centres of the orthotomic spheres being consequently in a line meeting an axis, and at right angles to the plane of the conic in question. Or, what is the same thing, starting with a conic, and a sphere having double contact therewith, the cyclide is the envelope of a variable sphere having its centre on the conic and cutting at right angles the fixed sphere.*

* I am indebted for this mode of generation of a Cyclide to the researches of Mr Casey.

It may be remarked, that if we endeavour to generalize a former generation of the cyclide, and consider the envelope of a variable sphere having its centre on a conic, and *touching* a fixed sphere, this is in general a surface of an order exceeding 4; it becomes a surface of the fourth order, viz. a cyclide, *only* in the case where the fixed sphere has its centre on the anti-conic. But if we consider the envelope of a variable sphere having its centre on a conic and cutting at right angles a fixed sphere, this is always a quartic surface having the circle at infinity for a double line; the surface has moreover two nodes, viz. these are the anti-points of the circle which is the intersection of the sphere by the plane of the conic. If the sphere touches the conic, then there is at the point of contact a third node; and similarly, if it has double contact with the conic, then there is at each point of contact a node; viz. in this case the surface has four nodes, and it is in fact a cyclide.

There is no difficulty in the analytical proof: consider the envelope of a variable sphere having its centre on the conic $Z=0$, $-\frac{X^2}{\beta}+\frac{Y^2}{\alpha}=1$, and which cuts at right angles the sphere $(x-l)^2+(y-m)^2+(z-n)^2=k^2$.

Take the equation of the variable sphere to be

$$(x-X)^2+(y-Y)^2+z^2=c^2,$$

then the orthotomic condition is

$$(X-l)^2+(Y-m)^2+n^2=c^2+k^2,$$

or, substituting this value of c^2, the equation of the variable sphere is

$$(x-X)^2+(y-Y)^2+z^2=-k^2+(X-l)^2+(Y-m)^2+n^2,$$

all which spheres pass through the points

$$x=l,\quad y=m,\quad z=\pm\sqrt{(n^2-k^2)};$$

that is,

$$x^2+y^2+z^2+k^2-l^2-m^2-n^2-2(x-l)X-2(y-m)Y=0,$$

and considering X, Y as variable parameters connected by the equation $-\frac{X^2}{\beta}+\frac{Y^2}{\alpha}=1$, the equation of the envelope is

$$(x^2+y^2+z^2+k^2-l^2-m^2-n^2)^2+4\beta(x-l)^2-4\alpha(y-m)^2=0,$$

viz. this is a bicircular quartic, having the two nodes $x=l$, $y=m$, $z=\pm\sqrt{(n^2-k^2)}$; these are the anti-points of the circle $(x-l)^2+(y-m)^2=k^2-n^2$, which is the intersection of the sphere $(x-l)^2+(y-m)^2+(z-n)^2=k^2$ by the plane of the conic.

The constants might be particularised so that the equation should represent a cyclide; but I treat the question in a somewhat different manner, by showing that the generating spheres of a cyclide cut at right angles each of a series of fixed spheres. Write α, β, $\gamma=b^2-c^2$, c^2-a^2, a^2-b^2; then if

$$\frac{X^2}{-\beta}+\frac{Y^2}{\alpha}=1;\quad \frac{X_1^2}{\gamma}-\frac{Z_1^2}{\alpha}=1,$$

the points $(X, Y, 0)$ and $(X_1, 0, Z_1)$ will be situate on a pair of anti-conics.

Consider the fixed sphere

$$(x-X_1)^2+y^2+(z-Z_1)^2=c_1^2,$$

then if this is touched by the variable sphere

$$(x-X)^2+(y-Y)^2+z^2=c^2,$$

the last-mentioned sphere will be a generating sphere of the cyclide. The condition of contact is

$$(X-X_1)^2+Y^2+Z_1^2=(c+c_1)^2,$$

that is,

$$\begin{aligned}(c+c_1)^2&=X^2-2XX_1+X_1^2+\alpha\left(1+\frac{X^2}{\beta}\right)+\alpha\left(\frac{X_1^2}{\gamma}-1\right)\\&=-\frac{\gamma}{\beta}X^2-2XX_1-\frac{\beta}{\gamma}X_1^2\\&=\Omega^2,\end{aligned}$$

if for a moment

$$\Omega=X\sqrt{\left(\frac{-\gamma}{\beta}\right)}+X_1\sqrt{\left(\frac{-\beta}{\gamma}\right)},$$

that is, $c=-c_1+\Omega$, and the equation of the variable sphere is

$$(x-X)^2+(y-Y)^2+z^2=(c_1-\Omega)^2,$$

where X, Y are variable parameters connected by

$$\frac{X^2}{-\beta}+\frac{Y^2}{\alpha}=1.$$

Suppose that the variable sphere is orthotomic to

$$(x-X_2)^2+y^2+(z-Z_2)^2=c_2^2,$$

the condition for this is

$$(X-X_2)^2+Y^2+Z_2^2=c^2+c_2^2,$$

or combining with the identical equation

$$(X-X_1)^2+Y^2+Z_1^2=(c+c_1)^2,$$

we have

$$\begin{aligned}-2X(X_2-X_1)+X_2^2-X_1^2+Z_2^2-Z_1^2&=c_2^2-c_1^2-2c_1(-c_1+\Omega)\\&=c_2^2+c_1^2-2c_1\Omega,\end{aligned}$$

or, substituting for Z_1, Ω their values, this is

$$-2X(X_2-X_1)+X_2^2-X_1^2+Z_2^2-\alpha\left(\frac{X_1^2}{\gamma}-1\right)=c_2^2+c_1^2-2c_1\left\{X\sqrt{\left(-\frac{\gamma}{\beta}\right)}+X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)}\right\},$$

viz. this will be identically true if

$$X_2 = X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)},$$

$$X_2^2 + Z_2^2 - c_2^2 = -\frac{\beta}{\gamma}X_1^2 + 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} - \alpha + c_1^2,$$

or, as this last equation may be written

$$Z_2^2 - c_2^2 = Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}.$$

The equation of the orthotomic sphere is thus found to be

$$\left\{x - X_1 - c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}^2 + y^2 + z^2 - 2zZ_2 + Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} = 0;$$

or, what is the same thing,

$$x^2 + y^2 + z^2 - 2zZ_2 - 2x\left\{X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} - X_1^2\frac{\beta}{\gamma} - 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} + c_1^2 - \alpha = 0,$$

or, as this may be written

$$\alpha\left(-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1\right) + z^2 - 2zZ_2 - \frac{\gamma}{\beta}x^2 - 2xX_1 - \frac{\beta}{\gamma}X_1^2 - 2c_1x\sqrt{\left(-\frac{\beta}{\gamma}\right)} - 2c_1X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} + c_1^2 = 0,$$

viz. this is

$$\alpha\left(-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1\right) + z^2 - 2zZ_2 + \left\{x\sqrt{\left(-\frac{\gamma}{\beta}\right)} + X_1\sqrt{\left(-\frac{\beta}{\gamma}\right)} - c_1\right\}^2 = 0,$$

where Z_2 is arbitrary. We have thus a series of orthotomic spheres; viz. taking any one of these, the envelope of a variable sphere having its centre on the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} - 1 = 0$, and cutting at right angles the orthotomic sphere, is a cyclide. The centre of the orthotomic sphere is a point at pleasure on the line

$$x = X_1 + c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)},\ y = 0;$$

and the sphere passes through the circle $z = 0$,

$$\left\{x - X_1 - c_1\sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\}^2 + y^2 + Z_1^2 - c_1^2\frac{\alpha}{\beta} + 2c_1X_1\left\{\sqrt{\left(-\frac{\beta}{\gamma}\right)} + \sqrt{\left(-\frac{\gamma}{\beta}\right)}\right\} = 0,$$

viz. this is a circle having double contact with the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} = 1$; or, what is the same thing, the orthotomic sphere is a sphere having its centre on the line in question, and having double contact with the conic $-\frac{x^2}{\beta} + \frac{y^2}{\alpha} = 1$.

570.

ON THE SUPERLINES OF A QUADRIC SURFACE IN FIVE-DIMENSIONAL SPACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 176—180.]

IN ordinary or three-dimensional space a quadric surface has upon it two singly infinite systems of lines, such that each line of the one system intersects each line of the other system, but that two lines of the same system do not intersect.

In five-dimensional space* a quadric surface has upon it two triply infinite systems of superlines, such that each superline of either system intersects each superline of the same system; a superline of the one system does not in general intersect a superline of the opposite system, but it may do so, and then it intersects it not in a mere point, but in a line.

The theory will be established by an independent analysis, but it is, in fact, a consequence of the correspondence which exists between the lines of ordinary space and the points of a quadric surface in five-dimensional space. Thus the correspondence is

In ordinary space.	In five-dimensional space.
Line.	Point on quadric surface.
Lines meeting a given line.	Points which lie in tangent plane at given point.
Pair of intersecting lines.	Two points such that each lies in the tangent plane at the other, or say, pair of harmonic points.
Lines meeting each of two given lines.	Points lying in the sub-plane common to the tangent planes at two given points.

* In explanation of the nomenclature, observe that in 5 dimensional geometry we have: space, surface, subsurface, supercurve, curve, and point-system, according as we have between the six coordinates 0, 1, 2, 3, 4, or 5 equations: and so when the equations are linear, we have: space, plane, subplane, superline, line, and point. Thus in the text a quadric surface is the locus determined by a single quadric equation between the coordinates; and the superline and line are the loci determined by three linear equations and four linear equations respectively.

But in ordinary space if the two given lines intersect, then the system of lines meeting these, breaks up into two systems, viz. that of the lines which pass through the point of intersection, and that of the lines which lie in the common plane of the two given lines. It follows that in the five-dimensional space the intersection of the quadric surface by the subplane common to the tangent planes at two harmonic points must break up into a pair of superlines, viz. that we have on the quadric two systems of superlines; a superline of the one kind answering in ordinary space to the lines which pass through a given point, and a superline of the other kind answering to the lines which lie in a given plane. (Observe that, as regards the five-dimensional geometry, this is no distinction of nature between the two kinds of superlines, they are simply correlative to each other, like the two systems of generating lines of a quadric in ordinary space.)

Moreover, considering two superlines of the first kind, then answering thereto in ordinary space we have the lines through one given point, and the lines through another given point; and these systems have a common line, that joining the two given points; whence the two superlines have a common point. And, similarly, two superlines of the second kind have a common point. But taking two superlines of opposite kinds, then in ordinary space we have the lines through a given point, and the lines in a given plane: and the two systems have not in general any common line; that is, the two superlines have no common point. If, however, the given point lies in the given plane, then there is not one common line, but a singly infinite series of common lines, viz. all the lines in the given plane and through the given point; and corresponding hereto we have as the intersection of the two superlines, not a mere point, but a line.

Passing now to the independent theory, I consider, for comparison, first the case of the lines on a quadric surface in ordinary space; the equation of the surface may be taken to be

$$u^2+v^2-x^2-y^2=0,$$

(u, v, x, y ordinary quadriplanar coordinates) and the equations of a line on the surface are

$$u=\alpha x+\beta y,$$
$$v=\alpha' x+\beta' y,$$

where α, β, α', β' are coefficients of a rectangular transformation, viz. we have $\alpha^2+\beta^2=1$, $\alpha'^2+\beta'^2=1$, $\alpha\alpha'+\beta\beta'=0$; and therefore $(\alpha\beta'-\alpha'\beta)^2=1$, consequently $\alpha\beta'-\alpha'\beta=\pm 1$; and the lines will be of one or the other kind, according as the sign is $+$ or $-$. It is rather more convenient to assume always $\alpha\beta'-\alpha'\beta=+1$, and write the equations

$$u=\alpha x+\beta y,$$
$$v=k(\alpha' x+\beta' y),$$

k denoting ± 1, and the lines being of the one kind or of the other kind, according as the sign is $+$ or $-$.

Thus considering any two lines, the equations may be written

$$u = \alpha x + \beta y, \quad u = - \ (ax + by),$$
$$v = \alpha' x + \beta' y, \quad v = -k(a'x + b'y),$$

where the lines will be of the same kind or of different kinds, according as k is $=+1$ or $=-1$. Observe that k is introduced into one equation only; if it had been introduced into both, there would be no change of kind. If the lines intersect we have

$$(\alpha + a)x + (\beta + b)y = 0,$$
$$(\alpha' + ka')x + (\beta' + kb')y = 0,$$

viz. the condition of intersection is

$$\begin{vmatrix} \alpha + a, & \beta + b \\ \alpha' + ka', & \beta' + kb' \end{vmatrix} = 0,$$

that is,

$$\alpha\beta' - \alpha'\beta + k(ab' - a'b) + a\beta' - a'\beta + k(ab' - a'b) = 0,$$

or, what is the same thing,

$$1 + \alpha\beta' - \alpha'\beta + k(1 + ab' - a'b) = 0.$$

But we have, say

$$\alpha = \cos\theta, \quad \beta = \sin\theta, \quad a = \cos\phi, \quad b = \sin\phi,$$
$$\alpha' = -\sin\theta, \quad \beta' = \cos\theta, \quad a' = -\sin\phi, \quad b' = \cos\phi,$$

and thence

$$\alpha\beta' - \alpha'\beta = \cos(\theta - \phi) = ab' - a'b,$$

and the equation is

$$(1 + k)\{1 + \cos(\theta - \phi)\} = 0,$$

viz. this is satisfied if $k = -1$, i.e. if the lines are of opposite kinds, but not if $k = +1$. And it is important to remark that there is no exception corresponding to the other factor, viz. if $k = +1$, and $1 + \cos(\theta - \phi) = 0$, for we then have $\theta - \phi = \pi$, $\cos\phi = -\cos\theta$, $\sin\phi = -\sin\theta$, and consequently the two sets of equations for u, v become identical; that is, for lines of the same kind a line meets itself only.

Passing to the five-dimensional space, the equation of the quadric surface may be taken to be

$$u^2 + v^2 + w^2 - x^2 - y^2 - z^2 = 0,$$

and for a superline on the surface we have

$$u = \alpha x + \beta y + \gamma z,$$
$$v = \alpha' x + \beta' y + \gamma' z,$$
$$w = \alpha'' x + \beta'' y + \gamma'' z,$$

where (α, β, γ), &c., are the coefficients of a rectangular transformation; the determinant formed with these coefficients is $= \pm 1$, and the superline is of the one kind or the

other, according as the sign is + or −. It is more convenient to take the determinant to be always +, and to write the equations in the form

$$\begin{aligned} u &= k(\alpha x + \beta y + \gamma z), \\ v &= k(\alpha' x + \beta' y + \gamma' z), \\ w &= k(\alpha'' x + \beta'' y + \gamma'' z), \end{aligned}$$

where $k = \pm 1$, and the superline is of the one or the other kind, according as the sign is + or −.

Now considering two superlines, we may write

$$\begin{aligned} u &= \alpha x + \beta y + \gamma z, & u &= -k(ax + by + cz), \\ v &= \alpha' x + \beta' y + \gamma' z, & v &= -k(a'x + b'y + c'z), \\ w &= \alpha'' x + \beta'' y + \gamma'' z, & w &= -k(a''x + b''y + c''z). \end{aligned}$$

If the superlines intersect, then

$$\begin{aligned} (\alpha + ka)x + (\beta + kb)y + (\gamma + kc)z &= 0, \\ (\alpha' + ka')x + (\beta' + kb')y + (\gamma' + kc')z &= 0, \\ (\alpha'' + ka'')x + (\beta'' + kb'')y + (\gamma'' + kc'')z &= 0, \end{aligned}$$

viz. the determinant formed with these coefficients must be $= 0$. The condition is at once reduced to

$$1 + k^3 + (k + k^2)(a\alpha + b\beta + c\gamma + a'\alpha' + b'\beta' + c'\gamma' + a''\alpha'' + b''\beta'' + c''\gamma'') = 0,$$

viz. it is satisfied when $k = -1$, that is, when the superlines are of the *same* kind; but not in general when $k = +1$.

If $k = +1$ the condition will be satisfied if

$$1 + a\alpha + b\beta + c\gamma + a'\alpha' + b'\beta' + c'\gamma' + a''\alpha'' + b''\beta'' + c''\gamma'' = 0,$$

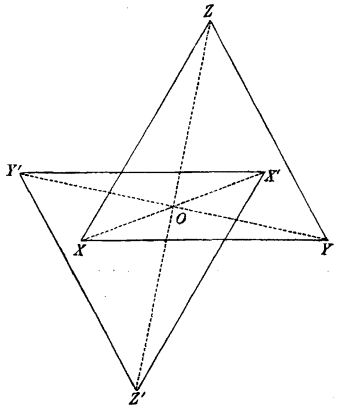
and it is to be shown that then the three equations reduce themselves not to two equations, but to a single equation.

It is allowable to take the second set of equations to be simply $u = -kx$, $v = -ky$, $w = -kz$; for this comes to replacing the analytically rectangular system $ax + by + cz$, $a'x + b'y + c'z$, $a''x + b''y + c''z$ by x, y, z. Writing also $k = +1$, the theorem to be proved is that the equations

$$\begin{aligned} (\alpha + 1)x + \beta y + \gamma z &= 0, \\ \alpha' x + (\beta' + 1)y + \gamma' z &= 0, \\ \alpha'' x + \beta'' y + \gamma'' z &= 0, \end{aligned}$$

reduce themselves to a single equation, provided only $1 + \alpha + \beta' + \gamma'' = 0$; or, what is the same thing, we have to prove that the expressions $\beta'' - \gamma'$, $\gamma - \alpha''$, $\alpha' - \beta$ each vanish, provided only $1 + \alpha + \beta' + \gamma'' = 0$. This is a known theorem depending on the

theory of the resultant axis, viz. the rotation round the resultant axis is then 180°, and we have $OX = OX'$, $OY = OY'$, $OZ = OZ'$, and thence we have evidently $YZ' = Y'Z$, $ZX' = Z'X$, $XY' = X'Y$.



But to prove it analytically, writing P, Q, R for $\beta'' - \gamma'$, $\gamma - \alpha''$, $\alpha' - \beta$ respectively, and Ω for $1 + \alpha + \beta' + \gamma''$, observe that we have identically

$$(\beta'' + \gamma)\,\Omega = QR,$$
$$(\gamma + \alpha')\,\Omega = RP,$$
$$(\alpha' + \beta'')\,\Omega = PQ,$$
$$(\beta'' + \gamma')\,P = (\gamma + \alpha'')\,Q = (\alpha' + \beta)\,R,$$

$$\begin{aligned}
(\alpha - 1)\,\Omega &= - \gamma Q + \beta R, \\
\alpha'\Omega &= - \gamma' Q + (1 + \beta')\,R, \\
\alpha''\Omega &= -(1 + \gamma'')\,Q + \beta'' R, \\
\beta\Omega &= -(1 + \alpha)\,R + \gamma P, \\
(\beta' - 1)\,\Omega &= - \alpha' R + \gamma' P, \\
\beta''\Omega &= - \alpha'' R + (1 + \gamma'')\,P, \\
\gamma\Omega &= - \beta P + (1 + \alpha)\,Q, \\
\gamma'\Omega &= -(1 + \beta')\,P + \alpha' Q, \\
(\gamma'' - 1)\,\Omega &= - \beta'' P + \alpha'' Q,
\end{aligned}$$

whence Ω being $= 0$, we have also $P = 0$, $Q = 0$, $R = 0$. The final conclusion is that the two superlines of opposite kinds, when they intersect, intersect in a line.

571.

A DEMONSTRATION OF DUPIN'S THEOREM.

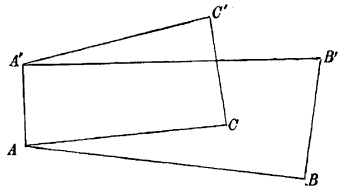
[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 185—191.]

THE theorem is that three families of surfaces intersecting everywhere at right angles intersect along their curves of curvature. The following demonstration puts in evidence the geometrical ground of the theorem.

I remark that it was suggested to me by the perusal of a most interesting paper by M. Lévy, "Mémoire sur les coordonnées curvilignes orthogonales et en particulier sur celles qui comprennent une famille quelconque de surfaces de second degré," (*Jour. de l'École Polyt.*, Cah. 43 (1870), pp. 157—200). It was known that a family of surfaces $\rho = f(x, y, z)$ where the function is arbitrary, does not in general form part of an orthogonal system, but that ρ considered as a function of (x, y, z) must satisfy a partial differential equation of the third order. M. Lévy obtains a theorem which, in fact, enables the determination of this partial differential equation; he does not himself obtain it, although he finds what the equation becomes on writing therein $\frac{d\rho}{dx} = 0$, $\frac{d\rho}{dy} = 0$; but I have, in a recent communication to the French Academy, found this equation.

Proceeding to the consideration of Dupin's theorem, on a surface of the first family take a point A and through it two elements of length on the surface, AB, AC, at right angles to each other; draw at A, B, C the normals meeting the consecutive surface in A', B', C' and join $A'B'$, $A'C'$. It is to be shown that the condition in order that $B'A'C'$ may be a right angle is the same as the condition for the intersection of the normals AA' and BB' (or of the normals AA' and CC'); for this being so, since by hypothesis $B'A'C'$ is a right angle, it follows that AA', BB' intersect;

that is, that AB is an element of one of the curves of curvature through the point A of the surface. And, similarly, that AA', CC' intersect; that is, that AC is an element of the other of the curves of curvature through the point A on the surface.



Take x, y, z for the coordinates of the point A; α, β, γ for the cosine inclinations of AA'; α_1, β_1, γ_1 for those of AB; and α_2, β_2, γ_2 for those of AC. Write also

$$\delta = \alpha\, d_x + \beta\, d_y + \gamma\, d_z,$$
$$\delta_1 = \alpha_1 d_x + \beta_1 d_y + \gamma_1 d_z,$$
$$\delta_2 = \alpha_2 d_x + \beta_2 d_y + \gamma_2 d_z;$$

then it will be shown that the condition for the intersection of the normals AA', BB' is

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$

and that these are equivalent to each other, and to the condition for the angle $B'A'C'$ being a right angle.

Taking l, l_1, l_2 for the lengths AA', AB, AC, the coordinates of A', B, C measured from the point A are

$$(l\alpha,\ l\beta,\ l\gamma),\quad (l_1\alpha_1,\ l_1\beta_1,\ l_1\gamma_1),\quad (l_2\alpha_2,\ l_2\beta_2,\ l_2\gamma_2) \text{ respectively.}$$

The equations of the normal at A may be written

$$X = x + \theta\alpha,$$
$$Y = y + \theta\beta,$$
$$Z = z + \theta\gamma,$$

where X, Y, Z are current coordinates, and θ is a variable parameter. Hence for the normal at B, passing from the coordinates x, y, z to $x + l_1\alpha_1$, $y + l_1\beta_1$, $z + l_1\gamma_1$, the equations are

$$X = x + l_1\alpha_1 + l_1\delta_1\,(\theta\alpha),$$
$$Y = y + l_1\beta_1 + l_1\delta_1\,(\theta\beta),$$
$$Z = z + l_1\gamma_1 + l_1\delta_1\,(\theta\gamma),$$

and if the two normals intersect in the point (X, Y, Z), then

$$\alpha_1 + \alpha\delta_1\theta + \theta\delta_1\alpha = 0,$$
$$\beta_1 + \beta\delta_1\theta + \theta\delta_1\beta = 0,$$
$$\gamma_1 + \gamma\delta_1\theta + \theta\delta_1\gamma = 0,$$

viz. eliminating θ and $\delta_1\theta$ the condition is

$$\begin{vmatrix} \alpha_1, & \alpha, & \delta_2\alpha \\ \beta_1, & \beta, & \delta_2\beta \\ \gamma_1, & \gamma, & \delta_1\gamma \end{vmatrix} = 0;$$

or, since

$$\alpha_2,\ \beta_2,\ \gamma_2 = \beta\gamma_1 - \beta_1\gamma,\ \gamma\alpha_1 - \gamma_1\alpha,\ \alpha\beta_1 - \alpha_1\beta,$$

this is

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0.$$

Similarly the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0.$$

We have

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = \alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma;$$

in fact, this equation is

$$(\alpha_2\delta_1 - \alpha_1\delta_2)\,\alpha + (\beta_2\delta_1 - \beta_1\delta_2)\,\beta + (\gamma_2\delta_1 - \gamma_1\delta_2)\,\gamma = 0,$$

which I proceed to verify.

In the first term the symbol $\alpha_2\delta_1 - \alpha_1\delta_2$ is

$$\alpha_2(\alpha_1 d_x + \beta_1 d_y + \gamma_1 d_z) - \alpha_1(\alpha_2 d_x + \beta_2 d_y + \gamma_2 d_z),$$

viz. this is

$$(\alpha_2\beta_1 - \alpha_1\beta_2)\,d_y + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\,d_z;$$

or, what is the same thing, it is

$$\beta d_z - \gamma d_y,$$

and the equation to be verified is

$$(\beta d_z - \gamma d_y)\,\alpha + (\gamma d_x - \alpha d_z)\,\beta + (\alpha d_y - \beta d_x)\,\gamma = 0,$$

viz. writing

$$\alpha,\ \beta,\ \gamma = \frac{X}{R},\ \frac{Y}{R},\ \frac{Z}{R},$$

where if $\rho = f(x, y, z)$ is the equation of the surface X, Y, Z are the derived functions $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{df}{dz}$, and $R = \sqrt{(X^2 + Y^2 + R^2)}$, the function on the left-hand consists of two parts; the first is

$$\frac{1}{R}\{(\beta d_z - \gamma d_y)\,X + (\gamma d_x - \alpha d_z)\,Y + (\alpha d_y - \beta d_x)\,Z\},$$

that is,

$$\frac{1}{R}\{\alpha(d_yZ-d_zY)+\beta(d_zX-d_xZ)+\gamma(d_xY-d_yX)\},$$

which vanishes; and the second is

$$-\frac{1}{R}\{\alpha(\beta d_z-\gamma d_y)+\beta(\gamma d_x-\alpha d_z)+\gamma(\alpha d_y-\beta d_x)\}R,$$

which also vanishes; that is, we have identically

$$\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma=\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma,$$

and the vanishing of the one function implies the vanishing of the other.

Proceeding now to the condition that the angle $B'A'C'$ shall be a right angle, the coordinates of B' are what those of A' become on substituting therein $x+l_1\alpha_1$, $y+l_1\beta_1$, $z+l_1\gamma_1$ in place of x, y, z; that is, these coordinates are

$$x+l\alpha+l_1\alpha_1+l_1\delta_1(l\alpha),\ \&c.,$$

or, what is the same thing, measuring them from A' as origin, the coordinates of B' are

$$\begin{aligned}&l_1(\alpha_1+l\delta_1\alpha+\alpha\delta_1 l),\\&l_1(\beta_1+l\delta_1\beta+\beta\delta_1 l),\\&l_1(\gamma_1+l\delta_1\gamma+\gamma\delta_1 l),\end{aligned}$$

and similarly those of C' measured from the same origin A' are

$$\begin{aligned}&l_2(\alpha_2+l\delta_2\alpha+\alpha\delta_2 l),\\&l_2(\beta_2+l\delta_2\beta+\beta\delta_2 l),\\&l_2(\gamma_2+l\delta_2\gamma+\gamma\delta_2 l).\end{aligned}$$

Hence the condition for the right angle is

$$\begin{aligned}&(\alpha_1+l\delta_1\alpha+\alpha\delta_1 l)(\alpha_2+l\delta_2\alpha+\alpha\delta_2 l)\\&+(\beta_1+l\delta_1\beta+\beta\delta_1 l)(\beta_2+l\delta_2\beta+\beta\delta_2 l)\\&+(\gamma_1+l\delta_1\gamma+\gamma\delta_1 l)(\gamma_2+l\delta_2\gamma+\gamma\delta_2 l)=0.\end{aligned}$$

Here the terms independent of l, $\delta_1 l$, $\delta_2 l$ vanish; and writing down only the terms which are of the first order in these quantities, the condition is

$$\begin{aligned}&\alpha_1(l\delta_2\alpha+\alpha\delta_2 l)+\alpha_2(l\delta_1\alpha+\alpha\delta_1 l)\\&+\beta_1(l\delta_2\beta+\beta\delta_2 l)+\beta_2(l\delta_1\beta+\beta\delta_1 l)\\&+\gamma_1(l\delta_2\gamma+\gamma\delta_2 l)+\gamma_2(l\delta_1\gamma+\gamma\delta_1 l)=0,\end{aligned}$$

where the terms in $\delta_1 l$, $\delta_2 l$ vanish; the remaining terms divide by l, and throwing out this factor, the condition is

$$(\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma)+(\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma)=0,$$

viz. by what precedes, this may be written under either of the forms

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

and the theorem is thus proved.

It may be remarked that if we had simply the first surface, and two other surfaces, or say a second and a third surface, cutting the first surface and each other at right angles, that is, cutting each other in AA' the element of the normal at A, and cutting the first surface in the elements AB, AC at right angles to each other, then the tangent plane of the second surface will be the plane $A'AB$, not in general passing through B'; and the tangent plane of the third surface will be the plane $A'AC$, not in general passing through C'. The condition, that the elements $A'B'$ and $A'C'$ on the surface consecutive to the first surface are at right angles, makes CC' and BB' each intersect AA'; and we then have, the tangent plane of the second surface is the plane through the elements AA', BB', the tangent plane of the second surface is the plane through the elements AA', CC'.

As already remarked, a family of surfaces $\rho = f(x, y, z)$ where the function is arbitrary cannot form part of an orthogonal system. In fact, if the surfaces do belong to an orthogonal system, we have AA', BB' in the same plane, and consequently AB and $A'B'$ intersect; and, similarly, AC and $A'C'$ intersect; that is, if from a point A on a given surface of the family we pass along the normal to the point A' on the consecutive surface; and if the lines AB, AC are the tangents to the curves of curvature at A, and $A'B'$, $A'C'$ the tangents to the curves of curvature at A', then AB intersects $A'B'$, or, what is the same thing, AC intersects $A'C'$; and, conversely, when this condition is satisfied in general (that is, for every surface of the family and the surface consecutive thereto), then the family forms part of an orthogonal system; this is, in fact, the fundamental theorem of M. Lévy's memoir. The analytical form of the condition, viewed in this manner, is

$$\alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 = 0, \quad \text{or} \quad \alpha_1\delta\alpha_2 + \beta_1\delta\beta_2 + \gamma_1\delta\gamma_2 = 0\,;$$

or, as it is convenient to write it,

$$\alpha_2\delta\alpha_1 + \beta_2\delta\beta_1 + \gamma_2\delta\gamma_1 - (\alpha_1\delta\alpha_2 + \beta_1\delta\beta_2 + \gamma_1\delta\gamma_2) = 0\,;$$

and it was by means of it that I obtained the partial differential equation of the third order above referred to. The condition written in the form

$$X_2\delta X_1 + Y_2\delta Y_1 + Z_2\delta Z_1 = 0, \quad \text{or} \quad X_1\delta X_2 + Y_1\delta Y_2 + Z_1\delta Z_2 = 0,$$

presents itself in the proof of Dupin's Theorem by R. L. Ellis, (given in Gregory's *Examples*, Cambridge, 1841), but the geometrical signification of it is not explained.

Closely connected with Dupin's, we have the following theorem: if two surfaces intersect at right angles along a curve which is a curve of curvature of one of them, it is a curve of curvature of the other of them. I remark hereon as follows:

Let the intersection be a curve of curvature on the first surface; the successive normals intersect, giving rise to a developable, and the intersection of the two surfaces, say I, is an involute of the edge of regression of this developable, say of the curve C. The successive normals of the second surface are the lines at the different points of I at right angles to the planes of the developable, that is, to the osculating planes of C; or, what is the same thing, they are lines parallel to the binormals of C (the line at any point of a curve, at right angles to the osculating plane, is termed the "binormal"). But if the intersection I is a curve of curvature on the second surface, then the successive lines intersect; that is, starting from the curve C, the theorem in effect is that at each point of the involute drawing a line parallel to the binormal of the corresponding point of the curve, the successive lines intersect, giving rise to a developable. To prove this, let the arc s be measured from any fixed point of the curve, and the coordinates x, y, z be considered as functions of s; and let x', x'', x''' denote $\frac{dx}{ds}, \frac{d^2x}{ds^2}, \frac{d^3x}{ds^3}$, and the like as regards y and z. Measuring off on the tangent at the point (x, y, z) a length $l-s$, the locus of the extremity is the involute; that is, for the point (x, y, z) on the curve, the coordinates of the corresponding point on the involute are $x+(l-s)x'$, $y+(l-s)y'$, $z+(l-s)z'$. Moreover, the cosine inclinations of the binormal are as $y'z''-y''z'$, $z'x''-z''x'$, $x'y''-x''y'$. Hence taking X, Y, Z as current coordinates, the equations of the line parallel to the binormal may be written

$$\begin{aligned} X &= x+(l-s)x'+\theta(y'z''-y''z'), \\ Y &= y+(l-s)y'+\theta(z'x''-z''x'), \\ Z &= z+(l-s)z'+\theta(x'y''-x''y'), \end{aligned}$$

and the condition of intersection is therefore

$$\begin{vmatrix} x'', & y'z''-y''z', & (y'z''-y''z')' \\ y'', & z'x''-z''x', & (z'x''-z''x')' \\ z'', & x'y''-x''y', & (x'y''-x''y')' \end{vmatrix} = 0.$$

Form a minor out of the first and second columns, e.g.

$$y''(x'y''-x''y')-z''(z'x''-z''x'),$$

this is,

$$x'(x''^2+y''^2+z''^2)-x''(x'x''+y'y''+z'z''),$$

or the last term being $=0$, and the factor $x''^2+y''^2+z''^2$ being common, the minors are as $x' : y' : z'$. Moreover $(y'z''-y''z')' = y'z'''-y'''z'$, &c., hence the determinant is

$$x'(y'z'''-y'''z')+y'(z'x'''-z'''x')+z'(x'y'''-x'''y'),$$

viz. this is $=0$, or the theorem is proved.

572.

THEOREM IN REGARD TO THE HESSIAN OF A QUATERNARY FUNCTION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 193—197.]

I WISH to put on record the following expression for the Hessian of $P^k + \lambda P'^{k'}$, where P, P' are quaternary functions of (x, y, z, w) of the degrees l, l' respectively, and λ is a constant; the demonstration is tedious enough, but presents no particular difficulty.

I write (A, B, C, D) for the first derived functions of P; and $(a, b, c, d, f, g, h, l, m, n)$ for the second derived functions; and similarly for P'. The Hessian of P is thus

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix},$$

which is denoted by $(abcd)$; moreover, if in this determinant we substitute the accented letters for the letters of each line successively, the result is denoted by $(abcd')$; and so if we substitute the accented letters for the letters of each pair of lines successively, the result is denoted by $(abc'd')$. Observe that

$$abcd' = (a'\delta_a + b'\delta_b + \ldots)\, abcd \text{ and } abc'd' = \tfrac{1}{2}(a'\delta_a + b'\delta_b + \ldots)^2\, abcd.$$

The notation $(abcD'^2)$ is used to denote the determinant

$$-\begin{vmatrix} & A', & B', & C', & D' \\ A', & a, & h, & g, & l \\ B', & h, & b, & f, & m \\ C', & g, & f, & c, & n \\ D', & l, & m, & n, & d \end{vmatrix},$$

and from it we derive the expression $(abc'D'^2)$, viz.

$$abc'D'^2 = (a'\delta_a + b'\delta_b + \ldots)\, abcD'^2.$$

The final result is expressed in terms of the several functions $abcd$, $abcd'$, $abc'd'$, $ab'c'd'$, $a'b'c'd'$, $abcD'^2$, $a'b'c'D^2$, $abc'D'^2$, $a'b'cD^2$, viz. we have

$$\begin{aligned}
\mathfrak{H}(P^k + \lambda P'^{k'}) = \quad & k^4(k-1)\left(\frac{1}{k-1} + \frac{l}{l-1}\right) P^{4k-4}.\, abcd \\
& + \lambda \left\{ \begin{array}{l} k^3(k-1)\left(\dfrac{1}{k-1} + \dfrac{l}{l-1}\right) k' P^{3k-3} \left\{ \begin{array}{r} P'^{k'-1}.\, abcd' \\ + (k'-1) P'^{k'-2}.\, abcD'^2 \end{array} \right\} \\ - \dfrac{k^3(k-1)k'}{(l-1)^2} [l'(l'-1) + l^2(k-1)] P^{3k-4} P'^{k'}.\, abcd \end{array} \right\} \\
& + \lambda^2 \left\{ \begin{array}{l} k^2 k'^2 \qquad P^{2k-2} P'^{2k'-2}\, abc'd' \\ + k^2 k'^2 (k'-1) P^{2k-2} P'^{2k'-3}\, abc'D'^2 \\ + k^2 k'^2 (k-1) P^{2k-3} P'^{2k'-2}\, a'b'cD^2 \\ + k^2 k'^2 (k-1)(k'-1) P^{2k-3} P'^{2k'-3} \left\{ \begin{array}{l} \dfrac{l(l-1)}{(l'-1)^2} a'b'c'D^2.\, P \\ + \dfrac{l'(l'-1)}{(l-1)^2} abcD'^2.\, P' \\ - \dfrac{l^2}{(l'-1)^2} ab'c'd'.\, P^2 \\ + \dfrac{ll'}{(l-1)(l'-1)} abc'd'.\, PP' \\ - \dfrac{l'^2}{(l-1)^2} abcd'.\, P'^2 \end{array} \right\} \end{array} \right\} \\
& + \lambda^3 \left\{ \begin{array}{l} k'^3(k'-1)\left(\dfrac{1}{k'-1} + \dfrac{l'}{l'-1}\right) k P'^{3k'-3} \left\{ \begin{array}{r} P^{k-1}.\, a'b'c'd \\ + (k-1) P^{k-2}.\, a'b'c'D^2 \end{array} \right\} \\ - \dfrac{k'^3(k'-1)k}{(l'-1)^2} [l(l-1) + l'^2(k'-1)] P^k P'^{3k'-4}.\, a'b'c'd' \end{array} \right\} \\
& + \lambda^4 .\, k'^4(k'-1)\left(\frac{1}{k'-1} + \frac{l'}{l'-1}\right) P'^{4k'-4}.\, a'b'c'd'.
\end{aligned}$$

In verification, I remark that, $\lambda = 0$, the formula becomes

$$\mathfrak{H}(P^k) = k^4(k-1)\left(\frac{1}{k-1} + \frac{l}{l-1}\right) P^{4k-4}.\, abcd,$$

that is

$$= \frac{k^4(kl-1)}{l-1} P^{4k-4}.\, abcd.$$

Hence, writing $P' = P$, which implies $k' = k$ and $l' = l$, we ought to have

$$\mathfrak{H}\{(1+\lambda)P^k\} = (1+\lambda)^4 . \frac{k^4(kl-1)}{l-1} P^{4k-4} . abcd.$$

But writing in the formula $P' = P$, it is to be observed that $abcd' = 4abcd$, $abc'd' = 6abcd$, $ab'c'd' = 4abcd$, $a'b'c'd' = abcd$: moreover that $abcD'^2$ and $a'b'c'D^2$ are each $= abcD^2$, but that $abc'D'^2$ and $a'b'cD^2$ are each $= 3abcD^2$, and (as is easily shown to be the case)

$$abcD^2 = \frac{l}{l-1} P . abcd.$$

Thus the whole coefficient of λ becomes

$$\left\{\begin{array}{l} k^4(k-1)\left(\frac{1}{k-1}+\frac{l}{l-1}\right)\left\{4+\frac{(k-1)l}{l-1}\right\} \\ -k^4\frac{(k-1)}{(l-1)^2}\{l(l-1)+l^2(k-1)\} \end{array}\right\} P^{4k-4} . abcd,$$

where the numerical factor is

$$= k^4(k-1)^2\left(\frac{1}{k-1}+\frac{l}{l-1}\right)\left(\frac{4}{k-1}+\frac{l}{l-1}-\frac{l}{l-1}\right)$$

$$= 4k^4(k-1)\left(\frac{1}{k-1}+\frac{l}{l-1}\right);$$

or, finally, it is

$$= \frac{4k^4(kl-1)}{l-1}.$$

The coefficient of λ^2 is

$$= k^4\left\{6 - \frac{2l^2(k-1)^2}{(l-1)^2}\right\} P^{4k-4} . abcd$$

$$+ k^4\left\{6(k-1) + \frac{2(k-1)^2 l}{l-1}\right\} P^{4k-5} . abcD^2;$$

or, substituting for $abcD^2$ its value $= \frac{l}{l-1} P . abcd$, the expression is equal to $P^{4k-4}abcd$ into a numerical coefficient, which is

$$k^4\left\{6 - \frac{2l^2(k-1)^2}{(l-1)^2} + \left(\frac{6(k-1)l}{l-1} + \frac{2(k-1)^2 l^2}{(l-1)^2}\right)\right\},$$

viz. this is

$$6k^4\left\{1 + \frac{(k-1)l}{l-1}\right\}$$

$$= 6\frac{k^4(kl-1)}{l-1},$$

and the coefficients of λ^3, and λ^4 are equal to those of λ and λ^0 respectively. Hence the formula gives, as it should do,

$$\mathfrak{H}\{(1+\lambda)P^k\} = (1+\lambda)^4 \frac{k^4(kl-1)}{l-1} P^{4k-4} . abcd.$$

Attending only to the form of the result, and representing the numerical factors by A, B, &c., we may write

$$\begin{aligned}
\mathfrak{H}(P^k+\lambda P'^{k'}) = \quad & A \quad P^{4k-4}\,abcd \\
+\lambda\,.\quad & B \left\{\begin{array}{l} P^{3k-3}\,P'^{k'-1}\,abcd' \\ +(k'-1)\,P^{2k-3}\,P'^{k'-2}\,abcD'^2 \end{array}\right\} \\
& +C \quad P^{3k-4}\,P'^{k'}\,abcd \\
+\lambda^2.\quad & D \quad P^{2k-2}\,P'^{2k'-2}\,abc'd' \\
& +E \quad P^{2k-2}\,P'^{2k'-3}\,abc'D'^2 \\
& +E' \quad P^{2k-3}\,P'^{2k'-2}\,a'b'cD^2 \\
& +F \quad P^{2k-3}\,P'^{2k'-3}\,(\Lambda P+\Lambda' P') \\
+\lambda^3.\quad & C' \quad P^k\,P'^{3k'-4}\,a'b'c'd' \\
& +B' \left\{\begin{array}{l} P^{k-1}\,P'^{3k-3}\,a'b'c'd \\ +(k-1)\,P^{k-2}\,P'^{3k'-3}\,a'b'c'D^2 \end{array}\right\} \\
+\lambda^4.\quad & A' \quad P'^{4k'-4}\,a'b'c'd',
\end{aligned}$$

where, for shortness, certain terms in λ^2 have been represented by $\Lambda P+\Lambda' P'$.

Suppose $k=k'=2$; then attending only to the terms of the lowest order in P, P' conjointly, we have

$$\begin{aligned}
\mathfrak{H}(P^2+\lambda P'^2) = \quad & \lambda B\,.\,P^2\,.\,abcD'^2 \\
& +\lambda^2\,.\,PP'(\Lambda P+\Lambda' P') \\
& +\lambda^3 B'\,.\,P'^3\,.\,a'b'c'D^2.
\end{aligned}$$

If the function operated upon with $\mathfrak{H}$ had been $UP^2+U'P'^2$, the lowest terms in P, P' would have been of the like form; and it thus appears that for a surface of the form $UP^2+U'P'^2=0$, the nodal curve $P=0$, $P'=0$ is a triple curve on the Hessian surface.

If $k=2$, $k'=3$, then attending only to the terms of the lowest order in P, P' conjointly, we have

$$\begin{aligned}
\mathfrak{H}(P^2+\lambda P'^3) = \quad & A\,.\,P^4\,.\,abcd \\
& +\lambda\,.\,2B\,.\,P^2P'\,.\,abcD'^2;
\end{aligned}$$

and the like result would be obtained if the function operated upon with $\mathfrak{H}$ had been $UP^2+U'P'^3$. It thus appears that for a surface of the form $UP^2+U'P'^3=0$, the cuspidal curve $P=0$, $P'=0$ is a 4-tuple curve on the Hessian surface, the form in the vicinity of this line, or direction of the tangent plane, being given by

$$P^3(A\,.\,P\,.\,abcd+2B\lambda\,.\,P'\,.\,abcD'^2)=0,$$

viz. there is a triple sheet $P^3=0$, coinciding with the direction of the surface in the vicinity of the cuspidal line; and a single sheet

$$A\,.\,P\,.\,abcd+2B\lambda\,.\,P'\,.\,abcD'^2=0.$$

At the points for which the osculating plane of the curve $P=0$, $P'=0$ coincides with the tangent plane of $P=0$ (or, what is the same thing, with that of the surface), we have $abcD'^2=0$, and the triple and single sheets then coincide in direction.

573.

NOTE ON THE (2, 2) CORRESPONDENCE OF TWO VARIABLES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 197, 198.]

In connection with my paper "On the porism of the in-and-circumscribed polygon and the (2, 2) correspondence of points on a conic," *Quar. Math. Jour.*, t. XI. (1871), pp. 83—91, [489], I remark that if (θ, ϕ) have a symmetrical (2, 2) correspondence, and also (ϕ, χ) the same symmetrical (2, 2) correspondence, then (θ, χ) will have a (not in general the same) symmetrical (2, 2) correspondence. In fact, to a given value θ there correspond, say the values ϕ_1, ϕ_2 of ϕ; then to ϕ_1 correspond the values θ, χ_1 of χ (viz. one of the two values is $=\theta$), and to ϕ_2 the values θ, χ_2 of χ (viz. one of the values is here again $=\theta$); that is, to the given value θ there correspond the two values χ_1, χ_2 of χ; and similarly to any value of χ there correspond two values of θ; viz. to χ_1 the value θ and say θ_1; to χ_2 the value θ and say θ_2; that is, the correspondence of θ, χ is a (2, 2) correspondence and is symmetrical.

Analytically, if we have

$$(a, b, c, f, g, h \between \theta\phi, \theta + \phi, 1)^2 = 0,$$

and

$$(a, b, c, f, g, h \between \phi\chi, \phi + \chi, 1)^2 = 0,$$

then writing

$$(a, \ldots \between \phi u, \phi + u, 1)^2 = 0,$$

the roots hereof are $u = \theta$, $u = \chi$; i.e. we have

$$(a, \ldots \between \phi u, \phi + u, 1)^2 = (a, \ldots \between \phi, 1, 0)^2 (u - \theta)(u - \chi);$$

or, what is the same thing, we have

$$1 : -(\theta+\chi) : \theta\chi = (a, \ldots \between \phi,\ 1,\ 0)^2 : 2\,(a, \ldots \between \phi,\ 1,\ 0 \between 0,\ \phi,\ 1) : (a, \ldots \between 0,\ \phi,\ 1)^2$$
$$= \quad a\phi^2 + 2h\phi + b \quad : \quad 2\,(h\phi^2 + \overline{b+g}\,\phi + f) \quad : \quad b\phi^2 + 2f\phi + c,$$

giving $\phi^2 : \phi : 1$ proportional to linear functions of $1,\ \theta+\chi,\ \theta\chi$, and therefore a quadric relation $(*\between \theta\chi,\ \theta+\chi,\ 1)^2 = 0$, with coefficients which are not in general $(a,\ b,\ c,\ f,\ g,\ h)$.

Suppose, however, that the coefficients have these values, or that the correspondence is

$$(a,\ b,\ c,\ f,\ g,\ h \between \theta\chi,\ \theta+\chi,\ 1)^2 = 0,$$

we must have

$$(a,\ b,\ c,\ f,\ g,\ h \between a\phi^2 + 2h\phi + b,\quad -2\,(h\phi^2 + \overline{b+g}\,\phi + f),\quad b\phi^2 + 2f\phi + c)^2 = 0,$$

that is,

$$(ac + b^2 + 2bg - 4fh)\,(a,\ b,\ c,\ f,\ g,\ h \between \phi^2,\ -2\phi,\ 1)^2 = 0,$$

or, we have

$$ac + b^2 + 2bg - 4fh = 0,$$

as the condition in order that the symmetrical (2, 2) correspondence between θ and χ may be the same correspondence as that between θ and ϕ, or between ϕ and χ.

574.

ON WRONSKI'S THEOREM.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 221—228.]

THE theorem, considered by the author as an answer to the question "En quoi consistent les Mathématiques? N'y aurait-il pas moyen d'embrasser par un seul problème, tous les problèmes de ces sciences et de resoudre généralement ce problème universel?" is given without demonstration in his *Réfutation de la Théorie de Fonctions Analytiques de Lagrange*, Paris, 1812, p. 30, and reproduced (with, I think, a demonstration) in the *Philosophie de la Technie*, Paris, 1815; and it is also stated and demonstrated in the *Supplément à la Réforme de la Philosophie*, Paris, 1847, p. CIX *et seq.*; the theorem, but without a demonstration, is given in *Montferrier's Encyclopédie Mathématique* (Paris, no date), t. III. p. 398.

The theorem gives the development of a function Fx of the root of an equation

$$0 = fx + x_1 f_1 x + x_2 f_2 x + \&c.,$$

but it is not really more general than that for the particular case $0 = fx + x_1 f_1 x$; or say when the equation is $0 = \phi x + \lambda fx$.* Considering then this equation

$$\phi x + \lambda fx = 0,$$

let a be a root of the equation $\phi x = 0$; the theorem is

$$\begin{aligned} Fx = F & \\ & - \frac{\lambda}{1} \frac{1}{\phi'} \left| \, (\int fF')' \, \right| \\ & + \frac{\lambda^2}{1.2} \frac{1}{\phi'^3} \left| \begin{matrix} \phi', & (\int f^2 F')' \\ \phi'', & (\int f^2 F')'' \end{matrix} \right| \frac{1}{1} \end{aligned}$$

* For in the result, as given in the text, instead of λfx write $x_1 f_1 x + x_2 fx + \&c.$, then expanding the several powers of this quantity, each determinant is replaced by a sum of determinants of the same order, and we have the expansion of Fx in powers of $x_1, x_2, \ldots$.

$$-\frac{\lambda^3}{1.2.3}\frac{1}{\phi'^6}\begin{vmatrix} \phi', & (\phi^2)', & (\int f^3F')' \\ \phi'', & (\phi^2)'', & (\int f^3F')'' \\ \phi''', & (\phi^2)''', & (\int f^3F')''' \end{vmatrix}\frac{1}{1.1.2}$$

$$+ \&c.,$$

where F, f, F', &c. denote $Fa, fa, F'a$, &c. and the accents denote differentiation in regard to a; the integral sign $\int$ is written instead of $\int_a$; this is introduced for symmetry only, and obviously disappears; in fact, we may equally well write

$$Fx = F$$

$$-\frac{\lambda}{1}\frac{1}{\phi'}fF'$$

$$+\frac{\lambda^2}{1.2}\frac{1}{\phi'^3}\begin{vmatrix} \phi', & f^2F' \\ \phi'', & (f^2F')' \end{vmatrix}\frac{1}{1}$$

$$-\frac{\lambda^3}{1.2.3}\frac{1}{\phi'^6}\begin{vmatrix} \phi', & (\phi^2)', & f^3F' \\ \phi'', & (\phi^2)'', & (f^3F')' \\ \phi''', & (\phi^2)''', & (f^3F')'' \end{vmatrix}\frac{1}{1.1.2}$$

$$+ \&c.$$

I stop for a moment to remark that Laplace's theorem is really equivalent to Lagrange's; viz. in the first mentioned theorem we have $x = \phi(a + \lambda fx)$, that is $\phi^{-1}x = a + \lambda f\phi \,.\, \phi^{-1}x$, and then $Fx = F\phi \,.\, \phi^{-1}x$, viz. by Lagrange's theorem

$$Fx = F\phi + \frac{\lambda}{1}F\phi' \,.\, f\phi + \frac{\lambda^2}{1.2}\{F\phi' \,.\, (f\phi)^2\}' + \&c.,$$

where on the right hand $F\phi$ and $f\phi$ are each regarded as one symbol, the argument being always a and the accents denoting differentiation in regard to a, thus $F\phi'$ is

$$d_a \,.\, F\phi a = F'\phi a \,.\, \phi' a, \ \&c.,$$

viz. this is Laplace's theorem.

Suppose in Wronski's theorem $\phi x = x - a$; that is, let the equation be

$$x - a + \lambda\phi x = 0,$$

then each determinant reduces itself to a single term: thus the determinant of the third order is

$$\begin{vmatrix} (x-a)', & \{(x-a)^2\}', & f^3F' \\ (x-a)'', & \{(x-a)^2\}'', & (f^3F')' \\ (x-a)''', & \{(x-a)^2\}''', & (f^3F')'' \end{vmatrix},$$

where in the first and second columns the accents denote differentiation in regard to x, which variable is afterwards put $= a$; the determinant is thus

$$= \begin{vmatrix} 1, & *, & * \\ 0, & 1.2, & * \\ 0, & 0, & (f^3F')'' \end{vmatrix},$$

viz. it is

$$= 1 \,.\, 1 \,.\, 2\,(f^3F')'',$$

and so in other cases; the formula is thus

$$Fx = F - \frac{\lambda}{1} fF' + \frac{\lambda^2}{1\,.\,2}(f^2F')' - \frac{\lambda^3}{1\,.\,2\,.\,3}(f^3F')'' + \&c.,$$

agreeing with Lagrange's theorem.

Suppose in general $\phi x = (x-a)\,\psi x$, or let the equation be

$$(x-a)\,\psi x + \lambda f x = 0,$$

that is,

$$x - a + \lambda \frac{fx}{\psi x} = 0:$$

we have then by Lagrange's theorem

$$Fx = F - \frac{\lambda}{1} F' \frac{f}{\psi} + \frac{\lambda^2}{1\,.\,2}\left\{F'\left(\frac{f}{\psi}\right)^2\right\}' - \frac{\lambda^3}{1\,.\,2\,.\,3}\left\{F'\left(\frac{f}{\psi}\right)^3\right\}'' + \&c.$$

Consider for example the term $\left\{F'\left(\frac{f}{\psi}\right)^3\right\}''$; this is

$$= \left\{F'x \,.\, \frac{(x-a)^3\,(fx)^3}{(\phi x)^3}\right\}'',$$

the accents denoting differentiation in regard to x, and x being ultimately put $= a$; or, what is the same thing, it is

$$= \left(\frac{d}{d\theta}\right)^2 \left[F'(a+\theta)\,\frac{\theta^3\,\{f(a+\theta)\}^3}{\{\phi(a+\theta)\}^3}\right],$$

the accents now denoting differentiation in regard to θ, and this being ultimately put $= 0$. This is

$$\left(\frac{d}{d\theta}\right)^2 \left[F'(a+\theta)\,\frac{\{f(a+\theta)\}^3}{\left(\phi' a + \frac{\theta}{1\,.\,2}\phi'' a + \ldots\right)^3}\right].$$

This may be written $\left(F'f^3\,\frac{1}{A^3}\right)''$, where

$$A = \phi' + \tfrac{1}{2}\theta\phi'' + \tfrac{1}{6}\theta^2\phi'' + \ldots,$$

it being understood that as regards $F'f^3$, which is expressed as a function of a only (θ having been therein put $= 0$), the exterior accents denote differentiations in respect to a, whereas in regard to A, $= \phi' + \tfrac{1}{2}\theta\phi'' + \&c.$, they denote differentiation in regard to θ, which is afterwards put $= 0$. And the theorem thus is

$$Fx = F - \frac{\lambda}{1}\left(F'f \,.\, \frac{1}{A}\right) + \frac{\lambda^2}{1\,.\,2}\left(F'f^2 \,.\, \frac{1}{A^2}\right)' - \frac{\lambda^3}{1\,.\,2\,.\,3}\left(F'f^3 \,.\, \frac{1}{A^3}\right)'' + \&c.$$

This must be equivalent to Wronski's theorem; it is in a very different, and, I think, a preferable form; but the results obtained from the comparison are very interesting, and I proceed to make this comparison.

Taking the foregoing coefficient $\left(F'f^3\frac{1}{A^3}\right)''$ this should be equal to Wronski's term

$$\frac{1}{1.1.2}\frac{1}{\phi'^6}\begin{vmatrix} \phi', & (\phi^2)', & f^3F' \\ \phi'', & (\phi^2)'', & (f^3F')' \\ \phi''', & (\phi^2)''', & (f^3F')'' \end{vmatrix};$$

or, what is the same thing, the determinant should be

$$=1.1.2\phi'^6\left(\frac{1}{A^3}f^3F'\right)''$$

$$=1.1.2\phi'^6\left\{f^3F'\left(\frac{1}{A^3}\right)''+2(f^3F')'\left(\frac{1}{A^3}\right)'+(f^3F')''\frac{1}{A^3}\right\},$$

that is, the values of

$$1.1.2\phi'^6\frac{1}{A^3},\quad 1.1.2\phi'^6 2\left(\frac{1}{A^3}\right)',\quad 1.1.2\phi'^6\left(\frac{1}{A^3}\right)''$$

should be

$$=\phi'(\phi^2)''-\phi''(\phi^2)',\quad \phi'''(\phi^2)'-\phi'(\phi^2)''',\quad \phi''(\phi^2)'''-\phi'''(\phi^2)''$$

respectively. Or, what is the same thing, if

$$\frac{1}{\left(\phi'+\frac{\theta}{2}\phi''+\frac{\theta^2}{2.3}\phi'''+\dots\right)^3}=A_0+\frac{1}{1}A_1\theta+\frac{1}{1.2}A_2\theta^2+\dots,$$

then the last mentioned functions should be

$$1.1.2\phi'^6A_0,\quad 1.1.2\phi'^6 2A_1,\quad 1.1.2\phi'^6A_2.$$

We have

$$A_0=\frac{1}{\phi'^3},\quad A_1=-\frac{3}{2}\frac{\phi''}{\phi'^4},\quad A_2=-\frac{\phi'''}{\phi'^4}+\frac{3\phi''^2}{\phi'^5},$$

or the identities are

$$\begin{aligned} 2\phi'^3 &= \phi'(\phi^2)''-\phi''(\phi^2)', &&= \phi'(2\phi\phi''+2\phi'^2)-\phi''.2\phi\phi', \\ -6\phi''\phi'^2 &= \phi'''(\phi^2)'-\phi'(\phi^2)''', &&= \phi'''.2\phi\phi'-\phi'(2\phi\phi''+6\phi'\phi''), \\ +6\phi''^2\phi'-2\phi'''\phi'^2 &= \phi''(\phi^2)'''-\phi'''(\phi^2)'', &&= \phi''(2\phi\phi'''+6\phi'\phi'')-\phi'''(2\phi\phi''+2\phi'^2), \end{aligned}$$

which is right. And in like manner to verify the coefficient of λ^4, we should have to compare the first four terms of the expansion of

$$\frac{1}{\left(\phi'+\frac{\theta}{2}\phi''+\frac{\theta^2}{2.3}\phi'''+\dots\right)^4}.$$

with the determinants formed out of the matrix

$$\begin{vmatrix} \phi', & \phi'', & \phi''', & \phi'''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''', & (\phi^2)'''' \\ (\phi^3)', & (\phi^3)'', & (\phi^3)''', & (\phi^3)'''' \end{vmatrix}.$$

The series of equalities may be presented as follows, writing as above A to denote the function

$$\phi' + \frac{\theta}{2}\phi'' + \frac{\theta^2}{2.3}\phi''' + \ldots,$$

$$\frac{1}{A} = \frac{1}{\phi'}.1,$$

$$\frac{1}{A^2} = \frac{-1}{\phi'^3}\begin{vmatrix} \theta, & 1 \\ \phi', & \phi'' \end{vmatrix}.\frac{1}{1},$$

$$\frac{1}{A^3} = \frac{+1}{\phi'^6}\begin{vmatrix} \frac{1}{2}\theta^2, & \frac{1}{2}\theta, & 1 \\ \phi', & \phi'', & \phi''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''' \end{vmatrix}.\frac{1}{1.1.2},$$

$$\frac{1}{A^4} = \frac{-1}{\phi'^{10}}\begin{vmatrix} \frac{1}{6}\theta^3, & \frac{1}{6}\theta^2, & \frac{1}{3}\theta, & 1 \\ \phi', & \phi'', & \phi''', & \phi'''' \\ (\phi^2)', & (\phi^2)'', & (\phi^2)''', & (\phi^2)'''' \\ (\phi^3)', & (\phi^3)'', & (\phi^3)''', & (\phi^3)'''' \end{vmatrix}.\frac{1}{1.1.2.1.2.3},$$

&c.,

where in each case the function on the left hand is to be expanded only as far as the power of θ which is contained in the determinant: the numerical coefficients in the top-lines of the several determinants are the reciprocals of

$$n(n-1)\ldots 2.1, \quad n(n-1)\ldots 2, \quad n(n-1), \quad n, \quad 1,$$

where n is the index of the highest power of θ. The demonstration of Wronski's theorem therefore ultimately depends on the establishment of the foregoing equalities As a verification, in the fourth formula, write $\phi = e^a\,(a=0)$, we have

$$\left(\frac{\theta}{e^\theta - 1}\right)^4 \text{ or } \frac{1}{(1+\frac{1}{2}\theta+\frac{1}{6}\theta^2+\frac{1}{24}\theta^3+\ldots)^4} = -\frac{1}{12}\begin{vmatrix} \frac{1}{6}\theta^3, & \frac{1}{6}\theta^2, & \frac{1}{3}\theta, & 1 \\ 1, & 1, & 1, & 1 \\ 2, & 4, & 8, & 16 \\ 3, & 9, & 27, & 81 \end{vmatrix},$$

where the right hand is

$$\begin{aligned} &= -\tfrac{1}{12}(-1.12 + \tfrac{1}{3}\theta.72 - \tfrac{1}{6}\theta^2.132 + \tfrac{1}{6}\theta^3.72) \\ &= \qquad 1 - 2\theta + \tfrac{11}{6}\theta^2 - \theta^3, \end{aligned}$$

and expanding the left hand as far as θ^3, this is

$$\begin{array}{rl} = 1 & = 1 \\ -\ 4\,(\tfrac{1}{2}\theta + \tfrac{1}{6}\theta^2 + \tfrac{1}{24}\theta^3) & -2\theta - \tfrac{2}{3}\theta^2 - \tfrac{1}{6}\theta^3 \\ +\ 10\,(\quad \tfrac{1}{4}\theta^2 + \tfrac{1}{6}\theta^3) & +\ \tfrac{5}{2}\theta^2 + \tfrac{5}{3}\theta^3 \\ -\ 20\,(\quad \tfrac{1}{8}\theta^3) & -\ \tfrac{5}{2}\theta^3 \\ \hline & 1 - 2\theta + \tfrac{11}{6}\theta^2 - \theta^3, \end{array}$$

which agrees.

Reverting to the above equations, and expanding the several terms $(\phi^2)' = 2\phi\phi'$, $(\phi^2)'' = 2\phi\phi'' + 2\phi'^2$, &c., then, since in each case the left-hand side contains ϕ', ϕ'', ϕ''', &c. but not ϕ, it is clear that on the right-hand side the terms involving ϕ must disappear of themselves; and assuming that this is so, the equality takes the more simple form obtained by writing in the foregoing expressions $\phi = 0$, viz. we thus have $(\phi^2)' = 0$, $(\phi^2)'' = 2\phi'^2$, &c. In order to simplify the formulæ, I replace the series ϕ', $\frac{1}{2}\phi''$, $\frac{1}{6}\phi'''$, $\frac{1}{24}\phi''''$, &c. by b, c, d, e, &c., and I thus find that they assume the following simple form, viz. writing

$$\Theta = b + c\theta + d\theta^2 + e\theta^3 + \&c.,$$

then we have

$$\frac{1}{\Theta} = \frac{1}{b}\cdot 1,$$

$$\frac{1}{\Theta^2} = -\frac{2}{b^3}\begin{vmatrix} \theta, & \frac{1}{2} \\ b, & c \end{vmatrix},$$

$$\frac{1}{\Theta^3} = +\frac{3}{b^6}\begin{vmatrix} \theta^2, & \frac{1}{2}\theta, & \frac{1}{3} \\ b\,, & c\,, & d \\ & b^2, & 2bc \end{vmatrix},$$

$$\frac{1}{\Theta^4} = -\frac{4}{b^{10}}\begin{vmatrix} \theta^3, & \frac{1}{2}\theta^2, & \frac{1}{3}\theta\,, & \frac{1}{4} \\ b\,, & c\,, & d\,, & e \\ & b^2, & 2bc, & 2bd + c^2 \\ & & b^3, & 3b^2c \end{vmatrix},$$

viz. for Θ^{-n} the right-hand gives the development as far as θ^{n-1}. It will be observed, that in the determinants the several lines are the coefficients in the expansions of Θ, Θ^2, Θ^3, &c. respectively.

The demonstration is very easy; it will be sufficient to take the equation for $\frac{1}{\Theta^4}$. Assume

$$\frac{1}{\Theta^4} = \ldots r\theta^6 + q\theta^5 + p\theta^4 + \beta\theta^3 + \tfrac{1}{2}\gamma\theta^2 + \tfrac{1}{3}\delta\theta + \tfrac{1}{4}\epsilon,$$

where clearly $\epsilon = \frac{4}{b^4}$, and write also

$$\begin{array}{rl} \Theta = & B_1 + C_1\theta + D_1\theta^2 + E_1\theta^3 + \ldots, \\ \Theta^2 = & \quad\ B_2 + C_2\theta + D_2\theta^2 + \ldots, \\ \Theta^3 = & \qquad\ B_3 + C_3\theta + \ldots, \end{array}$$

where $B_1 = b$, $B_2 = b^2$, $B_3 = b^3$; we wish to show that

$$\begin{aligned} \beta B_1 + \gamma C_1 + \delta D_1 + \epsilon E_1 &= 0, \\ \gamma B_2 + \delta C_2 + \epsilon D_2 &= 0, \\ \delta B_3 + \epsilon C_3 &= 0, \end{aligned}$$

for this being the case, neglecting the terms in θ^4, θ^5, &c., and writing

$$\beta\theta^3 + \tfrac{1}{2}\gamma\theta^2 + \tfrac{1}{3}\delta\theta + \epsilon\left(\tfrac{1}{4} - \frac{1}{\epsilon\Theta^4}\right) = 0,$$

then eliminating β, γ, δ, ϵ, we have

$$\begin{vmatrix} \theta^3, & \tfrac{1}{2}\theta^2, & \tfrac{1}{3}\theta, & \tfrac{1}{4} - \dfrac{1}{\epsilon\Theta^4} \\ B_1, & C_1, & D_1, & E_1 \\ & B_2, & C_2, & D_2 \\ & & B_3, & C_3 \end{vmatrix} = 0,$$

in which equation the term which contains

$$\frac{1}{\Theta^4} \text{ is } + \frac{1}{\epsilon} B_1 B_2 B_3 \frac{1}{\Theta^4}, \quad = \tfrac{1}{4} b^{10} \frac{1}{\Theta^4};$$

and the equation thus is $\dfrac{1}{\Theta^4} = -\dfrac{4}{b^{10}}$ multiplied by the determinant without the term in question (that is, with $\frac{1}{4}$ for its corner term).

To prove the subsidiary theorems, multiply the expression of $\dfrac{1}{\Theta^4}$ by $\dfrac{1}{\theta^4}$, and differentiate in regard to θ, we have

$$\frac{4(\theta\Theta)'}{(\theta\Theta)^5} = \ldots - 2r\theta - q + \frac{\beta}{\theta^2} + \frac{\gamma}{\theta^3} + \frac{\delta}{\theta^4} + \frac{\epsilon}{\theta^5}.$$

Multiplying by

$$\theta\Theta = B_1\theta + C_1\theta^2 + D_1\theta^3 + E_1\theta^4,$$

we see that $B_1\beta + C_1\gamma + D_1\delta + E_1\epsilon$ is the coefficient of $\dfrac{1}{\theta}$ in $\dfrac{4(\theta\Theta)'}{(\theta\Theta)^4}$; and similarly $B_2\gamma + C_2\delta + E_2\epsilon$ is the coefficient of $\dfrac{1}{\theta}$ in $\dfrac{4(\theta\Theta)'}{(\theta\Theta)^3}$, and $B_3\delta + C_3\epsilon$ that of $\dfrac{1}{\theta}$ in $\dfrac{4(\theta\Theta)'}{(\theta\Theta)^2}$. Now, m being any positive integer, $\dfrac{1}{(\Theta\theta)^m}$ expanded in ascending powers of θ contains negative and positive powers of θ, but of course no logarithmic term; hence differentiating in regard to θ, $\dfrac{(\Theta\theta)'}{(\Theta\theta)^{m+1}}$ contains no term in $\dfrac{1}{\theta}$;* and the expressions in question are thus each $= 0$; which completes the demonstration.

The foregoing formulæ giving the expansion of $\dfrac{1}{\Theta^n}$ up to θ^{n-1} in terms of the coefficients in the expansions of Θ, Θ^2, ... Θ^{n-1} are I think interesting.

* This is a well-known method made use of by Jacobi and Murphy.

575.

ON A SPECIAL QUARTIC TRANSFORMATION OF AN ELLIPTIC FUNCTION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 266—269.]

IT is remarked by Jacobi that a transformation of the order $n'n''$ may lead to a modular equation

$$\frac{\Delta'}{\Delta} = \frac{n'}{n''}\frac{K'}{K},$$

and in particular when $n' = n''$, or the order is square, then the equation may be $\frac{\Delta'}{\Delta} = \frac{K'}{K}$; viz. that instead of a transformation we may have a multiplication. A quartic transformation of the kind in question may be obtained as follows: writing

$$X = (a, b, c, d, e \between x, 1)^4 = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta),$$

H the Hessian, Φ the cubi-covariant, I and J the two invariants, then there is a well known quartic transformation

$$z = \frac{2H}{X},$$

leading to

$$\frac{dz}{\sqrt{(Z)}} = \frac{2\sqrt{(-2)}\,dx}{\sqrt{(X)}},$$

where $Z = z^3 - Iz + 2J$. In fact we have

$$Z = \frac{2}{X^3}(4H^3 - IH^2X + JX^3), \quad = \frac{-2}{X^3}\Phi^2,$$

that is,

$$\sqrt{(Z)} = \frac{\sqrt{(-2)}\,\Phi}{X^2}\sqrt{(X)},$$

so that, by Jacobi's general principle, it at once appears that we have a transformation of the form in question.

Now we may establish a linear transformation

$$z = \frac{py+q}{y-\delta},$$

such that to the roots z_1, z_2, z_3 of the equation $z^3 - Iz + 2J = 0$ correspond the values α, β, γ of y; and this being so, we have between y, z the relation

$$\frac{dz}{\sqrt{(Z)}} = \frac{\sqrt{(-2)}\,dy}{\sqrt{(Y)}},$$

where $Y = a(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)$, $=(a, b, c, d, e \between y, 1)^4$; that is, we have

$$\frac{py+q}{y-\delta} = \frac{2H}{X},$$

such that

$$\frac{dy}{\sqrt{(Y)}} = \frac{2dx}{\sqrt{(X)}},$$

which is a quartic transformation giving a duplication of the integral. The foundation of the theorem is that we can determine p, q in such wise that the functions

$$\frac{p\alpha+q}{\alpha-\delta}, \quad \frac{p\beta+q}{\beta-\delta}, \quad \frac{p\gamma+q}{\gamma-\delta}$$

shall be the roots z_1, z_2, z_3 of the equation $z^3 - Iz + 2J = 0$. For writing

$$A = (\beta-\gamma)(\alpha-\delta),$$
$$B = (\gamma-\alpha)(\beta-\delta),$$
$$C = (\alpha-\beta)(\gamma-\delta),$$

and observing the equations

$$I = \frac{a^2}{24}(A^2+B^2+C^2), \quad = -\frac{a^2}{12}(BC+CA+AB),$$

(since $A+B+C=0$) and

$$2J = -\frac{a^3}{216}(B-C)(C-A)(A-B),$$

the equation in z is

$$\{z - \tfrac{1}{6}a(B-C)\}\{z - \tfrac{1}{6}a(C-A)\}\{z - \tfrac{1}{6}a(A-B)\},$$

and the equations for the determination of p, q thus are

$$p\alpha + q = \tfrac{1}{6}a(\alpha-\delta)(B-C), = \tfrac{1}{6}a(\alpha-\delta)\{2(\alpha\delta+\beta\gamma)-(\alpha+\delta)(\beta+\gamma)\},$$
$$p\beta + q = \tfrac{1}{6}a(\beta-\delta)(C-A), = \tfrac{1}{6}a(\beta-\delta)\{2(\beta\delta+\gamma\alpha)-(\beta+\delta)(\gamma+\alpha)\},$$
$$p\gamma + q = \tfrac{1}{6}a(\gamma-\delta)(A-B), = \tfrac{1}{6}a(\gamma-\delta)\{2(\gamma\delta+\alpha\beta)-(\gamma+\delta)(\alpha+\beta)\},$$

giving

$$p = \tfrac{1}{6} a \{-3\delta^2 + 2\delta(\alpha+\beta+\gamma) - \beta\gamma - \gamma\alpha - \alpha\beta\},$$
$$q = \tfrac{1}{6} a \{\delta^2(\alpha+\beta+\gamma) - 2\delta(\beta\gamma+\gamma\alpha+\alpha\beta) + 3\alpha\beta\gamma\},$$

or, as these may also be written

$$p = \tfrac{1}{6} a \{(\beta-\delta)(\gamma-\delta) + (\gamma-\delta)(\alpha-\delta) + (\alpha-\delta)(\beta-\delta)\},$$
$$q = \tfrac{1}{6} a \{\alpha(\beta-\delta)(\gamma-\delta) + \beta(\gamma-\delta)(\alpha-\delta) + \gamma(\alpha-\delta)(\beta-\delta)\};$$

and observe also

$$p\delta + q = \tfrac{1}{2} a (\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

Taking X in the standard form $=(1-x^2)(1-k^2x^2)$, and writing

$$\gamma = -1, \quad \delta = 1, \quad \alpha = +\frac{1}{k}, \quad \beta = -\frac{1}{k},$$

we have

$$z = \frac{py+q}{y-1} = \frac{-\frac{1}{6}\{2k^2(1+k^2)(1+k^2x^4) + (1-10k^2+k^4)x^2\}}{(-x^2)(1-k^2x^2)},$$

$$A = -1 + \frac{2}{k} - \frac{1}{k^2},$$
$$B = \quad 1 + \frac{2}{k} + \frac{1}{k^2},$$
$$C = \quad -\frac{4}{k};$$

$$z_1 = \tfrac{1}{6}(1+6k+k^2),$$
$$z_2 = \tfrac{1}{6}(1-6k+k^2),$$
$$z_3 = -\tfrac{1}{3}(1+k^2);$$

$$Z = z^3 - \tfrac{1}{12}(1+14k^2+k^4)z + \tfrac{1}{108}(1+k^2)(1-34k^2+k^4)$$
$$= (z-z_1)(z-z_2)(z-z_3),$$
$$p = \tfrac{1}{6}(1-5k^2), \quad q = \tfrac{1}{6}(5-k^2), \quad p+q = 1-k^2;$$

giving as they should do

$$z_1 = \frac{\frac{p}{k}+q}{\frac{1}{k}-1}, \quad z_2 = \frac{-\frac{p}{k}+q}{-\frac{1}{k}-1}, \quad z_3 = \frac{-p+q}{-2}.$$

Write for shortness

$$-\tfrac{1}{6}\{2k^2(1+k^2)(1+x^4) + (1-10k^2+k^4)x^2\} = Q,$$

so that

$$\frac{py+q}{y-1} = \frac{Q}{X},$$

then

$$\frac{Q}{X} - z_1 = \frac{p+q}{k-1} \cdot \frac{ky-1}{y-1},$$

$$\frac{Q}{X} - z_2 = \frac{p+q}{k+1} \cdot \frac{ky+1}{y+1},$$

$$\frac{Q}{X} - z_3 = \frac{p+q}{2} \cdot \frac{y+1}{y-1}.$$

The last of these is

$$\frac{1-k^2}{2} \frac{y+1}{y-1} = \frac{\frac{1}{2}(1-k^2)^2 x^2}{(1-x^2)(1-k^2x^2)},$$

that is,

$$\frac{y+1}{y-1} = \frac{(1-k^2)x^2}{(1+x^2)(1-k^2x^2)},$$

from which the foregoing equation

$$\frac{dy}{\sqrt{(Y)}} = \frac{2dx}{\sqrt{(X)}}$$

may be at once verified.

576.

ADDITION TO MR WALTON'S PAPER "ON THE RAY-PLANES IN BIAXAL CRYSTALS."

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 273—275.]

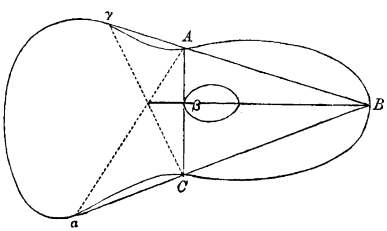
INSTEAD of Mr Walton's a^2, b^2, c^2 write a, b, c, and assume α, β, $\gamma = b-c$, $c-a$, $a-b$; δ, ϵ, $\zeta = b+c$, $c+a$, $a+b$. Also instead of his x^2, y^2, z^2 write x, y, z. Then instead of the octic cone we have the quartic cone, or say the quartic curve

$$\begin{aligned}
&\alpha^2\beta^2\gamma^2 \sec^2\theta \,.\, xyz\,(x+y+z) \\
&= \alpha^2 yz\,\{(bc-a^2)\,x + a\,(\beta y - \gamma z)\}^2 \\
&+ \beta^2 zx\,\{(ca-b^2)\,y + b\,(\gamma z - \alpha x)\}^2 \\
&+ \gamma^2 xy\,\{(ab-c^2)\,z + c\,(\alpha x - \beta y)\}^2,
\end{aligned}$$

viz. we may herein consider x, y, z as trilinear coordinates, the ratios $x : y : z$ being positive for a point within the fundamental triangle.

The curve passes through the angles of the triangle, and it touches the sides in the points $(x=0,\ \beta y - \gamma z = 0)$, $(y=0,\ \gamma z - \alpha x = 0)$, $(z=0,\ \alpha x - \beta y = 0)$ respectively. Moreover, the tangents at the angles of the triangle lie each of them outside the triangle. Hence, supposing a, b, c each positive, and $a > b > c$, we have α and γ each positive, β negative, and the form of the curve is as shown in the figure, or else the like form with the oval lying outside the triangle. And it is hence clear that, if the side $AC\ (y=0)$ instead of touching the curve meets it in a node, this is a conjugate point arising from the evanescence of the oval; and in this case no part of the curve lies within the triangle. Now considering any point $x : y : z = l : m : n$, we obtain a tetrad of points $x : y : z = \pm\sqrt{(l)} : \pm\sqrt{(m)} : \pm\sqrt{(n)}$ on the octic cone or curve; and in order that the point on the octic curve may be real, we must have l, m, n all of the

same sign; that is, the point on the quartic curve must lie within the triangle. Hence, when in the quartic curve the oval becomes a conjugate point, the octic curve has no real branch, but it consists wholly of conjugate points; viz. it consists of the points A, B, C as conjugate points; two imaginary conjugate points answering to the



point α of the figure, two other imaginary conjugate points answering to the point γ; and two conjugate points answering to the point β, these last being not ordinary conjugate points, but conjugate tacnodal points, or points of contact of two imaginary branches of the curve.

The case in question, β a conjugate point on the quartic curve, answers to Mr Walton's critical value of $\sec^2\theta$, viz. in the present notation $\sec^2\theta = \dfrac{4b^2+\alpha\gamma}{\alpha\gamma}$. To show this I consider the intersection of the curve by the line $\gamma z - \alpha x = 0$; and I write for convenience $\gamma z = \alpha x = \gamma\alpha u$, that is, $x = \gamma u$, $z = \alpha u$. Substituting these values, the equation divides by yu, or omitting this factor it is

$$\begin{aligned}
&\alpha^2\gamma^3\beta^2\sec^2\theta\,.\,u\,\{y+(\alpha+\gamma)\,u\}\\
&= \alpha^3\,\{\gamma u\,(bc-a^2-a\alpha)+a\beta y\}^2\\
&+ \beta^2\alpha\gamma\,.\,uy\,(ca-b^2)^2\\
&+ \gamma^3\,\{\alpha u\,(ab-c^2+c\gamma)-c\beta y\}^2,
\end{aligned}$$

or observing that we have $\alpha+\gamma=-\beta$, $bc-a^2-a\alpha=\zeta\beta$, $ab-c^2+c\gamma=-\delta\beta$, this becomes

$$\begin{aligned}
&\alpha^2\gamma^3\sec^2\theta u\,(y-\beta u)\\
&= \alpha^3\ (\gamma\zeta u+ay)^2\\
&+ \alpha\gamma\,(ca\ \ -b^2)^2\,uy\\
&+ \gamma^3\ (\alpha\delta u+cy)^2,
\end{aligned}$$

viz. this is

$$\begin{aligned}
&u^2\,\{\alpha^2\gamma^3\zeta^2+\alpha^2\gamma^3\delta^2+\alpha^2\gamma^3\sec^2\theta\,.\,\beta\}\\
&+ uy\,\{2\alpha^2a\gamma\zeta+2\gamma^2c\alpha\delta+\alpha\gamma\,(ca-b^2)^2-\alpha^2\gamma^3\sec^2\theta\}\\
&+ y^2\,(a^2\alpha^3+c^2\gamma^3)=0.
\end{aligned}$$

The required condition is that the coefficient of u^2 shall vanish; viz. we then have

$$\begin{aligned} -\alpha\beta\gamma\sec^2\theta &= \alpha\zeta^2 + \gamma\delta^2 \\ &= (b-c)(a+b)^2 + (a-b)(b+c)^2 \\ &= (a-c)\{3b^2 + b(a+c) - ac\} \\ &= -\beta(4b^2 + \alpha\gamma), \end{aligned}$$

that is,

$$\alpha\gamma\sec^2\theta = 4b^2 + \alpha\gamma,$$

agreeing with Mr Walton's value. Giving $\sec^2\theta$ this value, and throwing out the factor u, the equation becomes

$$\begin{aligned} &u\{2\alpha^3\alpha\gamma\zeta + 2\gamma^3 c\alpha\delta + \alpha\gamma(ca - b^2)^2 - \alpha^2\gamma^2(4b^2 + \alpha\gamma)\} \\ &+ y(\alpha^2\alpha^3 + c^2\gamma^3) = 0; \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} &\alpha\gamma u\{2a(a+b)(b-c)^2 + 2c(c+b)(b-a)^2 + (ca-b^2)^2 - (b-c)(a-b)4b^2 - (b-c)^2(b-a)^2\} \\ &+ y(\alpha^2\alpha^3 + c^2\gamma^3) = 0, \end{aligned}$$

say

$$\alpha\gamma Ku + (\alpha^2\alpha^3 + c^2\gamma^3)y = 0,$$

viz. this equation determines the remaining intersection of the curve by the line $\gamma z - \alpha x = 0$; the point in question lies outside the triangle, that is, $u : y$ should be negative; or α, γ, $\alpha^2\alpha^3 + c^2\gamma^3$ being each positive, we should have K positive; we in fact find

$$\begin{aligned} K &= 4b^4 + b^2(a^2 + c^2 - 6ac) + 4a^2c^2 \\ &= 4(b^2 - ac)^2 + b^2(a+c)^2, \end{aligned}$$

which is as it should be.

577.

NOTE IN ILLUSTRATION OF CERTAIN GENERAL THEOREMS OBTAINED BY DR LIPSCHITZ.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XII. (1873), pp. 346—349.]

THE paper by Dr Lipschitz, which follows the present Note [in the *Quarterly Journal, l.c.*], is supplemental to Memoirs by him in *Crelle*, vols. LXX., LXXII., and LXXIV.; and he makes use of certain theorems obtained by him in these memoirs; these theorems may be illustrated by the consideration of a particular example.

Imagine a particle not acted on by any forces, moving in a given surface; and let its position on the surface at the time t be determined by means of the general coordinates x, y. We have then the vis-viva function T, a given function of x, y, x', y'; and the equations of motion are

$$\frac{d}{dt}\frac{dT}{dx'}-\frac{dT}{dx}=0,\quad \frac{d}{dt}\frac{dT}{dy'}-\frac{dT}{dy}=0,$$

which equations serve to determine x, y in terms of t, and of four arbitrary constants; these are taken to be the initial values (or values corresponding to the time $t=t_0$) of x, y, x', y'; say the values are α, β, α', β'.

We have the theorem that x, y are functions of α, β, $\alpha'(t-t_0)$, $\beta'(t-t_0)$.

Suppose for example that x, y, z denote ordinary rectangular coordinates, and that the particle moves on the sphere $x^2+y^2+z^2=c^2$; to fix the ideas, suppose that the coordinates z are measured vertically upwards, and that the particle is on the upper hemisphere; that is, take $z=+\sqrt{(c^2-x^2-y^2)}$, we have

$$T=\tfrac{1}{2}(x'^2+y'^2+z'^2),$$

where z' denotes its value in terms of x, y, x', y'; viz. we have $xx'+yy'+zz'=0$, or

$$z'=-\frac{xx'+yy'}{z},\quad =-\frac{xx'+yy'}{\sqrt{(c^2-x^2-y^2)}};$$

the proper value of T is thus

$$= \tfrac{1}{2}\left\{x'^2 + y'^2 + \frac{(xx' + yy')^2}{c^2 - x^2 - y^2}\right\},$$

but it is convenient to retain z, z', taking these to signify throughout their foregoing values in terms of x, y, x', y'.

The constants of integration are, as before, α, β, α', β'; but we use also γ, γ' considered as signifying given functions of these constants, viz. we have

$$\gamma = \sqrt{(c^2 - \alpha^2 - \beta^2)} \text{ and } \gamma' = -\frac{\alpha\alpha' + \beta\beta'}{\sqrt{(c^2 - \alpha^2 - \beta^2)}},$$

(in fact, $\alpha^2 + \beta^2 + \gamma^2 = c^2$ and $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$; γ, γ' being thus the initial values of z, z').

Now, writing

$$\sigma = \frac{(t - t_0)\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}}{c},$$

the required values of x, y and the corresponding value of z are

$$x = \alpha \cos\sigma + \frac{c\alpha'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}} \sin\sigma,$$

$$y = \beta \cos\sigma + \frac{c\beta'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}} \sin\sigma,$$

$$z = \gamma \cos\sigma + \frac{c\gamma'}{\sqrt{(\alpha'^2 + \beta'^2 + \gamma'^2)}} \sin\sigma.$$

To verify that these are functions of α, β, $\alpha'(t - t_0)$, $\beta'(t - t_0)$, write $\alpha'(t - t_0) = u$, $\beta'(t - t_0) = v$; and take also $\gamma'(t - t_0) = w$; we have $\alpha u + \beta v + \gamma w = 0$, viz. $w, = -\frac{1}{\gamma}(\alpha u + \beta v)$, is a function of α, β, u, v; and then

$$\sigma = \frac{\sqrt{(u^2 + v^2 + w^2)}}{c},$$

and

$$x = \alpha \cos\sigma + \frac{u}{\sigma} \sin\sigma,$$

$$y = \beta \cos\sigma + \frac{v}{\sigma} \sin\sigma,$$

$$z = \gamma \cos\sigma + \frac{w}{\sigma} \sin\sigma;$$

so that x, y (and also z) are each of them a function of α, β, u, v, that is α, β, $\alpha'(t - t_0)$, $\beta'(t - t_0)$, which is the theorem in question.

The original variables are x, y; the quantities $\alpha'(t - t_0)$, $\beta'(t - t_0)$, or u, v are Dr Lipschitz' "Normal-Variables," and the theorem is that the original variables are functions of their initial values, and of the normal-variables.

The vis-viva function T may be expressed in terms of the normal-variables and their derived functions; viz. it is easy to verify that we have

$$T = \tfrac{1}{2}\left(\frac{1}{c^2\sigma^2} - \frac{\sin^2\sigma}{c^2\sigma^4}\right)(uu' + vv' + ww')^2$$

$$- \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma}(u'^2 + v'^2 + w'^2),$$

where w denotes $-\frac{1}{\gamma}(\alpha u + \beta v)$ and consequently w' denotes $-\frac{1}{\gamma}(\alpha u' + \beta v')$; introducing herein differentials instead of derived functions, or writing

$$\phi(du) = \tfrac{1}{2}\left(\frac{1}{c^2\sigma^2} - \frac{\sin^2\sigma}{c^2\sigma^4}\right)(udu + vdv + wdw)^2$$

$$+ \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma^2}(du^2 + dv^2 + dw^2),$$

where w, dw denote $-\frac{1}{\gamma}(\alpha u + \beta v)$, $-\frac{1}{\gamma}(\alpha du + \beta dv)$ respectively; then $\phi(du)$ is the function thus denoted by Dr Lipschitz: and writing herein $t - t_0 = 0$, and thence $u = 0$, $v = 0$, $w = 0$, $\sigma = 0$, the resulting value of $\phi(du)$ is

$$f_0(du), \quad = \tfrac{1}{2}(du^2 + dv^2 + dw^2),$$

where $f_0(du)$ is the function thus denoted by him; the corresponding value of $f_0(u)$ is $= \frac{1}{2}(u^2 + v^2 + w^2)$. We have thus an illustration of his theorem that the function $\phi(du)$ is such that we have identically

$$\phi(du) - \{d\sqrt{\{f_0(u)\}}\}^2 = \frac{m^2}{2f_0(u)}[f_0(du) - \{d\sqrt{(f_0 u)}\}^2],$$

where m is a function of u, v independent of the differentials du, dv; the value in the present example is in fact $m^2 = c^2\sin^2\sigma$; or the identity is

$$\phi(du) - \{d\sqrt{(f_0 u)}\}^2 = \frac{c^2\sin^2\sigma}{2(f_0 u)}[f_0(du) - \{d\sqrt{(f_0 u)}\}^2],$$

in verification whereof observe that we have

$$d\sqrt{(f_0 u)} = \frac{df_0(u)}{2\sqrt{(f_0 u)}} = \frac{udu + vdv + wdw}{\sqrt{(u^2 + v^2 + w^2)}}$$

$$= \frac{1}{c\sigma}(udu + vdv + wdw)^2.$$

The value of the left-hand side is thus

$$= -\frac{\sin^2\sigma}{c^2\sigma^4}(udu + vdv + wdw)^2 + \tfrac{1}{2}\frac{\sin^2\sigma}{\sigma^2}(du^2 + dv^2 + dw^2),$$

viz. this is

$$= \frac{c^2\sin^2\sigma}{c^2\sigma^2}\left\{\tfrac{1}{2}(du^2 + dv^2 + dw^2) - \frac{1}{c^2\sigma^2}(udu + vdv + wdw)^2\right\};$$

or, finally, it is

$$= \frac{c^2\sigma^2}{2f_0(u)}\left\{f_0(du) - [d\sqrt{\{f_0(u)\}}]^2\right\},$$

which is right.

578.

A MEMOIR ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXIV. (for the year 1874), pp. 397—456. Received November 14, 1873,—Read January 8, 1874.]

THE theory of Transformation in Elliptic Functions was established by Jacobi in the *Fundamenta Nova* (1829); and he has there developed, transcendentally, with an approach to completeness, the general case, n an odd number, but algebraically only the cases $n=3$ and $n=5$; viz. in the general case the formulæ are expressed in terms of the elliptic functions of the nth part of the complete integrals, but in the cases $n=3$ and $n=5$ they are expressed rationally in terms of u and v (the fourth roots of the original and the transformed moduli respectively), these quantities being connected by an equation of the order 4 or 6, the modular equation. The extension of this algebraical theory to any value whatever of n is a problem of great interest and difficulty: such theory should admit of being treated in a purely algebraical manner; but the difficulties are so great that it was found necessary to discuss it by means of the formulæ of the transcendental theory, in particular by means of the expressions involving Jacobi's q $\left(\text{the exponential of } -\frac{\pi K'}{K}\right)$, or say by means of the q-transcendents. Several important contributions to the theory have since been made:—Sohnke, "Equationes modulares pro transformatione functionum ellipticarum," *Crelle*, t. XVI. (1836), pp. 97—130, (where the modular equations are found for the cases $n=3$, 5, 7, 11, 13, 17, and 19); Joubert, "Sur divers équations analogues aux équations modulaires dans la théorie des fonctions elliptiques," *Comptes Rendus*, t. XLVII. (1858), pp. 337—345, (relating among other things to the multiplier equation for the determination of Jacobi's M); and Königsberger, "Algebraische Untersuchungen aus der Theorie der elliptischen Functionen," *Crelle*, t. LXXII. (1870), pp. 176—275; together with other papers by Joubert and by Hermite in later volumes of the *Comptes Rendus*, which need not be more particularly referred to. In the present Memoir I carry on the theory, algebraically, as far as I am able; and I have, it appears to me, put the purely

algebraical question in a clearer light than has hitherto been done; but I still find it necessary to resort to the transcendental theory. I remark that the case $n=7$ (next succeeding those of the *Fundamenta Nova*), on account of the peculiarly simple form of the modular equation $(1-u^3)(1-v^3)=(1-uv)^3$, presents but little difficulty; and I give the complete formulæ for this case, obtaining them as well algebraically as transcendentally; I also to a considerable extent discuss algebraically the case of the next succeeding prime value $n=11$. For the sake of completeness I reproduce Sohnke's modular equations, exhibiting them for greater clearness in a square form, and adding to them those for the non-prime cases $n=9$ and $n=15$; also a valuable table given by him for the powers of $f(q)$; and I give other tabular results which are of assistance in the theory.

The General Problem. Article Nos. 1 to 6.

1. Taking n a given odd number, I write

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{P-Qx}{P+Qx}\right)^2,$$

where P, Q are rational and integral functions of x^2, $P\pm Qx$ being each of them of the order $\frac{1}{2}(n-1)$, or, what is the same thing, $(1\pm x)(P\pm Qx)^2$ being each of them of the order n; that is,

		$n=4p+1$,	$n=4p+3$,
Order of P in x^2 is		p	p,
" Q	"	$p-1$;	p;

whence in the first case the number of coefficients in P and Q is $(p+1)+p$, $=\frac{1}{2}(n+1)$, and in the second case the number is $(p+1)+(p+1)$, $=\frac{1}{2}(n+1)$, as before. Taking

$$P=\alpha+\gamma x^2+\epsilon x^4+\ldots,$$

$$Q=\beta+\delta x^2+\zeta x^4+\ldots,$$

the formula is

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x+\gamma x^2-\ldots}{\alpha+\beta x+\gamma x^2+\ldots}\right)^2,$$

the number of coefficients being as just explained. Starting herefrom I reproduce in a somewhat altered form the investigation in the *Fundamenta Nova*, as follows.

2. If the coefficients are such that the equation remains true when we therein change simultaneously x into $\frac{1}{kx}$ and y into $\frac{1}{\lambda y}$, then the variables x, y will satisfy the differential equation

$$\frac{Mdy}{\sqrt{1-y^2.1-\lambda^2y^2}}=\frac{dx}{\sqrt{1-x^2.1-k^2x^2}},$$

$\left(M \text{ a constant, the value of which, as will appear, is given by } \frac{1}{M} = 1 + \frac{2\beta}{\alpha}\right)$; and the problem of transformation is thus to find the coefficients so that the equation may remain true on the above simultaneous change of the values of x, y.

In fact, observing that the original equation and therefore the new equation are each satisfied on changing therein simultaneously x, y into $-x$, $-y$, it follows that the equation may be written in the four forms

$$1 - y = (1 - x) A^2 (\div), \quad 1 + y = (1 + x) B^2 (\div),$$
$$1 - \lambda y = (1 - kx) C^2 (\div), \quad 1 + \lambda y = (1 + kx) D^2 (\div),$$

the common denominator being, say E, where A, B, C, D, E are all of them rational and integral functions of x; and this being so, the differential equation will be satisfied.

3. To develop the condition, observe that the assumed equation gives

$$y = \frac{x(P^2 + 2PQ + Q^2x^2)}{P^2 + 2PQx^2 + Q^2x^2}, \quad = \frac{x\mathfrak{A}}{\mathfrak{B}} \text{ suppose,}$$

where $\mathfrak{A}$, $\mathfrak{B}$ are functions each of them of the degree $\frac{1}{2}(n-1)$ in x^2. (Hence, if with Jacobi $\frac{1}{M}$ denotes the value $(y \div x)_{x=0}$, we have $\frac{1}{M} = \left(1 + \frac{2Q}{P}\right)_{x=0}$, $= 1 + \frac{2\beta}{\alpha}$, as mentioned.)

Suppose in general that U being any integral function $(1, x^2)^p$, we have

$$U^* = (k^2x^2)^p \left(1, \frac{1}{k^2x^2}\right)^p;$$

viz. let U^* be what U becomes when x is changed into $\frac{1}{kx}$ and the whole multiplied by $(k^2x^2)^p$.

Let y^* be the value of y obtained by writing $\frac{1}{kx}$ for x; then, observing that in the expression for y the degree of the numerator exceeds by unity that of the denominator, we have

$$y^* = \frac{1}{kx} \frac{\mathfrak{A}^*}{\mathfrak{B}^*},$$

whence

$$yy^* = \frac{1}{k} \frac{\mathfrak{A}\mathfrak{A}^*}{\mathfrak{B}\mathfrak{B}^*};$$

and the functions $\mathfrak{A}$, $\mathfrak{B}$ may be such that this shall be a constant value, $= \frac{1}{\lambda}$; viz. this will be the case if

$$\frac{\lambda}{k} = \frac{\mathfrak{B}\mathfrak{B}^*}{\mathfrak{A}\mathfrak{A}^*},$$

which being so, the required condition is satisfied.

4. I shall ultimately, instead of k, λ introduce Jacobi's u, v ($u=\sqrt[4]{k}$, $v=\sqrt[4]{\lambda}$); but it is for the present convenient to retain k, and instead of λ to introduce the quantity Ω connected with it by the equation $\lambda=k\Omega^2$; or say the value of Ω is $=v^2\div u^2$. The modular equation in its standard form is an equation between u, v, which, as will appear, gives rise to an equation of the same order between u^2, v^2; and writing herein $v^2=\Omega u^2$, the resulting equation contains only integer powers of u^4, that is, of k, and we have an Ωk-form of the modular equation, or say an Ωk-modular equation, of the same order in Ω as the standard form is in v; these Ωk-forms for $n=3$, 5, 7, 11 will be given presently.

5. Suppose then, Ω being a constant, that we have identically

$$\mathfrak{A}=\frac{1}{\Omega k^{\frac{1}{2}(n-1)}}\mathfrak{B}^*;$$

this implies

$$\mathfrak{B}=\frac{\Omega}{k^{\frac{1}{2}(n-1)}}\mathfrak{A}^*.$$

(In fact, if

$$\begin{aligned}\mathfrak{A}&=a+cx^2+\ldots+qx^{n-3}+sx^{n-1},\\ \mathfrak{B}&=b+dx^2+\ldots+rx^{n-3}+tx^{n-1},\end{aligned}$$

then

$$\begin{aligned}\mathfrak{A}^*&=s+qk^2x^2+\ldots+ck^{n-3}x^{n-3}+ak^{n-1}x^{n-1},\\ \mathfrak{B}^*&=t+rk^2x^2+\ldots+dk^{n-3}x^{n-3}+bk^{n-1}x^{n-1},\end{aligned}$$

and the assumed equation gives

$$a=\frac{1}{\Omega k^{\frac{1}{2}(n-1)}}t,\quad c=\frac{k^2}{\Omega k^{\frac{1}{2}(n-1)}}r,\ldots,q=\frac{k^{n-3}}{\Omega k^{\frac{1}{2}(n-1)}}d,\quad s=\frac{k^{n-1}}{\Omega k^{\frac{1}{2}(n-1)}}b,$$

that is,

$$b=\frac{\Omega}{k^{\frac{1}{2}(n-1)}}s,\qquad d=\frac{\Omega k^2}{k^{\frac{1}{2}(n-1)}}q,\ \ldots,\ r=\frac{\Omega k^{n-3}}{k^{\frac{1}{2}(n-1)}}c,\qquad t=\frac{\Omega k^{n-2}}{k^{\frac{1}{2}(n-1)}}a;$$

and therefore $\mathfrak{B}=\frac{\Omega}{k^{\frac{1}{2}(n-1)}}\mathfrak{A}^*$.)

From these equations $\frac{\mathfrak{B}\mathfrak{B}^*}{\mathfrak{A}\mathfrak{A}^*}=\Omega^2$, that is, $=\frac{\lambda}{k}$, as it should be; so that Ω signifying as above, the required condition will be satisfied if only $\mathfrak{A}=\frac{1}{\Omega k^{\frac{1}{2}(n-1)}}\mathfrak{B}^*$; or substituting for $\mathfrak{A}$, $\mathfrak{B}$ their values, if only

$$(P^2+2PQx^2+Q^2x^2)^*=\Omega k^{\frac{1}{2}(n-1)}(P^2+2PQ+Q^2x^2),$$

where each side is a function of x^2 of the order $\frac{1}{2}(n-1)$, or the number of terms is $\frac{1}{2}(n+1)$, the several coefficients being obviously homogeneous quadric functions of the $\frac{1}{2}(n+1)$ coefficients of P, Q. We have thus $\frac{1}{2}(n+1)$ equations, each of the form $U=\Omega V$, where U, V are given quadric functions of the coefficients of P, Q, say of the $\frac{1}{2}(n+1)$ coefficients α, β, γ, δ, &c., and where Ω is indeterminate.

6. We may from the $\frac{1}{2}(n+1)$ equations eliminate the $\frac{1}{2}(n-1)$ ratios $\alpha:\beta:\gamma:\ldots$, thus obtaining an equation in Ω (involving of course the parameter k) which is the Ωk-modular equation above referred to; and then Ω denoting any root of this equation, the $\frac{1}{2}(n+1)$ equations give a single value for the set of ratios $\alpha:\beta:\gamma:\delta:\ldots$, so that the ratio of the functions P, Q is determined, and consequently the value of y as given by the equation

$$\frac{1-y}{1+y}=\frac{(1-x)(P-Qx)^2}{(1+x)(P+Qx)^2}, \quad \text{or} \quad y=\frac{x(P^2+2PQ+Q^2x^2)}{P^2+2PQx^2+Q^2x^2}.$$

The entire problem thus depends on the solution of the system of $\frac{1}{2}(n+1)$ equations,

$$(P^2+2PQx^2+Q^2x^2)^*=\Omega k^{\frac{1}{2}(n-1)}(P^2+2PQ+Q^2x^2).$$

The Ωk-Modular Equations, $n=3, 5, 7, 11$. Article No. 7.

7. For convenience of reference, and to fix the ideas, I give these results, calculated, as above explained, from the standard or uv-forms.

$n=3$:

	k^2	k	1	
Ω^4		$+1$		$=0$
Ω^3	-4			
Ω^2		$+6$		
Ω			-4	
Ω^0		$+1$		
	-4	$+8$	-4	

$\Omega=1$, we have $-4(k-1)^2=0$.

$n=5$:

	k^4	k^3	k^2	k	1	
Ω^6			$+1$			$=0$
Ω^5	-16		$+10$			
Ω^4			$+15$			
Ω^3			-20			
Ω^2			$+15$			
Ω			$+10$		-16	
Ω^0			$+1$			
	-16		$+32$		-16	

$\Omega=1$, we have $-16(k^2-1)^2=0$.

$n = 7$:

	k^6	k^5	k^4	k^3	k^2	k	1	
Ω^8				$+1$				$=0$
Ω^7	-64		$+56$					
Ω^6		-112		$+140$				
Ω^5			-112		$+56$			
Ω^4				$+70$				
Ω^3			$+56$		-112			
Ω^2				$+140$		-112		
Ω					$+56$		-64	
Ω^0				$+1$				
	-64	-112	0	$+352$	0	-112	-64	

$\Omega = 1$, we have

$$-16(k-1)^2(4k^2+3k+1)(k^2+3k+4) = 0.$$

$n = 11$:

	k^{10}	k^9	k^8	k^7	k^6	k^5	k^4	k^3	k^2	k^1	k^0	
Ω^{12}						$+1$						$+1$
Ω^{11}	-1024		$+1408$		-396							-12
Ω^{10}		-5632		$+4400$		$+1298$						$+66$
Ω^9			-16192		$+16368$		-396					-220
Ω^8				-18656		$+19151$						$+495$
Ω^7			-16016		-1144		$+16368$					-792
Ω^6				$+4400$		-7876		$+4400$				$+924$
Ω^5					$+16368$		-1144		-16016			-792
Ω^4						$+19151$		-18656				$+495$
Ω^3					-396		$+16368$		-16192			-220
Ω^2						$+1298$		$+4400$		-5632		$+66$
Ω^1							-396		$+1408$		-1024	-12
Ω^0						$+1$						$+1$
			-32208	-18656	-1936	-7876	-1936	-18656	-32208			
			$+1408$	$+8800$	$+32736$	$+40900$	$+32736$	$+8800$	$+1408$			
	-1024	-5632	-30800	-9856	$+30800$	$+33024$	$+30800$	-9856	-30800	-5632	-1024	± 94624

Equation-systems for the cases $n = 3, 5, 7, 9, 11$. Art. Nos. 8 to 10.

8. $u = 3$, cubic transformation. $k = u^4$, $\Omega = \dfrac{v^2}{u^2}$ (here and in the other cases).

$P = \alpha$, $Q = \beta$. The condition here is

$$k^2x^2\alpha^2 + (2\alpha\beta + \beta^2) = \Omega k\,\{(\alpha^2 + 2\alpha\beta) + \beta^2x^2\},$$

and the system of equations thus is

$$k\alpha^2 = \Omega\beta^2,$$
$$2\alpha\beta + \beta^2 = \Omega k\,(\alpha^2 + 2\alpha\beta),$$

and similarly in the other cases; for these it will be enough to write down the equation-systems.

$n = 5$, quintic transformation.

$P = \alpha + \gamma x^2$, $Q = \beta$.

$$k^2\alpha^2 = \Omega\gamma^2,$$
$$2\alpha\gamma + 2\alpha\beta + \beta^2 = \Omega\,(2\alpha\gamma + 2\beta\gamma + \beta^2),$$
$$\gamma^2 + 2\beta\gamma = \Omega k^2\,(\alpha^2 + 2\alpha\beta).$$

$n = 7$, septic transformation.

$P = \alpha + \gamma x^2$, $Q = \beta + \delta x^2$.

$$k^3\alpha^2 = \Omega\delta^2,$$
$$k\,(2\alpha\gamma + 2\alpha\beta + \beta^2) = \Omega\,(\gamma^2 + 2\gamma\delta + 2\beta\delta),$$
$$\gamma^2 + 2\beta\gamma + 2\alpha\delta + 2\beta\delta = \Omega k\,(2\alpha\gamma + 2\beta\gamma + 2\alpha\delta + \beta^2),$$
$$\delta^2 + 2\gamma\delta = \Omega k^3\,(\alpha^2 + 2\alpha\beta).$$

$n = 9$, enneadic transformation.

$P = \alpha + \gamma x^2 + \epsilon x^4$, $Q = \beta + \delta x^2$.

$$k^4\alpha^2 = \Omega\epsilon^2,$$
$$k^2\,(2\alpha\gamma + 2\alpha\beta + \beta^2) = \Omega\,(2\gamma\epsilon + 2\epsilon\delta + \delta^2),$$
$$2\alpha\epsilon + \gamma^2 + 2\alpha\delta + 2\gamma\beta + 2\beta\delta = \Omega\,(2\alpha\epsilon + \gamma^2 + 2\gamma\delta + 2\epsilon\beta + 2\beta\delta),$$
$$2\gamma\epsilon + 2\gamma\delta + 2\epsilon\beta + \delta^2 = \Omega k^2\,(2\alpha\gamma + 2\alpha\delta + 2\gamma\beta + \beta^2),$$
$$\epsilon^2 + 2\delta\epsilon = \Omega k^4\,(\alpha^2 + 2\alpha\beta).$$

$n = 11$, endecadic transformation.

$P = \alpha + \gamma x^2 + \epsilon x^4$, $Q = \beta + \delta x^2 + \zeta x^4$.

$$k^5\alpha^2 = \Omega\zeta^2,$$
$$k^3\,(2\alpha\gamma + 2\alpha\beta + \beta^2) = \Omega\,(\epsilon^2 + 2\epsilon\zeta + 2\delta\zeta),$$
$$k\,(2\alpha\epsilon + \gamma^2 + 2\alpha\delta + 2\gamma\beta + 2\beta\delta) = \Omega\,(2\gamma\epsilon + 2\gamma\zeta + 2\epsilon\delta + 2\beta\delta + \delta^2),$$
$$2\gamma\epsilon + 2\alpha\zeta + 2\gamma\delta + 2\epsilon\beta + 2\beta\zeta + \delta^2 = \Omega k\,(2\alpha\epsilon + \gamma^2 + 2\alpha\zeta + 2\gamma\delta + 2\epsilon\beta + 2\beta\delta),$$
$$\epsilon^2 + 2\gamma\zeta + 2\epsilon\delta + 2\delta\zeta = \Omega k^3\,(2\alpha\gamma + 2\alpha\delta + 2\gamma\beta + \beta^2),$$
$$2\epsilon\zeta + \zeta^2 = \Omega k^5\,(\alpha^2 + 2\alpha\beta).$$

And so on.

9. It will be noticed that if the coefficients of $P+Qx$ taken in order are

$$\alpha,\ \beta,\ \ldots,\ \rho,\ \sigma,$$

then in every case the first and last equations are

$$k^{\frac{1}{2}(n-1)}\alpha^2 = \Omega\sigma^2,$$
$$2\rho\sigma + \sigma^2 = \Omega k^{\frac{1}{2}(n-1)}(\alpha^2 + 2\alpha\beta).$$

Putting in the first of these $k = u^4$, $\Omega = \frac{v^2}{u^2}$, the equation becomes

$$u^{2n}\alpha^2 = v^2\sigma^2,$$

where each side is a perfect square; and in extracting the square root we may without loss of generality take the roots positive, and write $u^n\alpha = v\sigma$.

This speciality, although it renders it proper to employ ultimately u, v in place of k, Ω, produces really no depression of order (viz. the Ωk-form of the modular equation is found to be of the same order in Ω that the standard or uv-form is in v), and is in another point of view a disadvantage, as destroying the uniformity of the several equations: in the discussion of order I consequently retain Ω, k. Ultimately these are to be replaced by u, v; the change in the equation-systems is so easily made that it is not necessary here to write them down in the new form in u, v.

10. The case $\alpha = 0$ has to be considered in the discussion of order, but we have thus only solutions which are to be rejected; in the proper solutions α is not $=0$, and it may therefore for convenience be taken to be $=1$. We have then $\sigma = u^n \div v$. The last equation becomes therefore

$$\frac{u^n}{v}\left(2\rho + \frac{u^n}{v}\right) = \frac{v^2}{u^2}u^{2n-2}(1+2\beta);$$

or recollecting that β is connected with the multiplier M by the relation $\frac{1}{M} = 1 + 2\beta$, that is,

$$2\beta = \frac{1}{M} - 1,$$

and substituting for $1+2\beta$ its value, the equation becomes

$$2\rho = v^3u^{n-4}\left(\frac{1}{M} - \frac{u^4}{v^4}\right);$$

that is, the first and the last coefficients are 1, $\frac{u^n}{v}$, and the second and the penultimate coefficients are each expressed in terms of v, M. The cases $n=3$, $n=5$ are so far peculiar, that the only coefficients are α, β, or α, β, γ; in the next case $n=7$, the only coefficients are α, β, γ, δ, and we have in this case all the coefficients expressed as above.

The Ωk-form—Order of the Systems. Art. Nos. 11 to 22.

11. In the general case, n an odd number, we have Ω and $\frac{1}{2}(n+1)$ coefficients connected by a system of $\frac{1}{2}(n+1)$ equations of the form

$$\Omega = \frac{U}{U'} = \frac{V}{V'} = \ldots,$$

where $U, V, \ldots, U', V', \ldots$ are given quadric functions of the coefficients. Omitting the $(\Omega =)$, there remains a system of $\frac{1}{2}(n-1)$ equations of the form $\frac{U}{U'} = \frac{V}{V'} = \ldots$, or say

$$\begin{vmatrix} U, & V, & W, \ldots \\ U', & V', & W', \ldots \end{vmatrix} = 0,$$

which determine the ratios $\alpha : \beta : \gamma : \ldots$ of the coefficients; and to each set of ratios there corresponds a single value of Ω. The order of the system, or number of sets of ratios, is $= \frac{1}{2}(n+1) . 2^{\frac{1}{2}(n-1)}$, $= (n+1) . 2^{\frac{1}{2}(n-3)}$; and this is consequently the number of values of Ω, or the order of the equation for the determination of Ω; viz. but for reduction, the order in Ω of the Ωk-modular equation would be $= (n+1) . 2^{\frac{1}{2}(n-3)}$. In the case $n = 3$, this is $= 4$, which is right, but for any larger value of n the order is far too high; in fact, assuming (as the case is) that the order is equal to the order in v of the uv-form, the order should for a prime value of n be $= n+1$, and for a composite value not containing any square factor be = the sum of the divisors of n. I do not attempt a general investigation, but confine myself to showing in what manner the reductions arise.

12. I will first consider the cubic transformation; here, writing for convenience $\frac{\alpha}{\beta} = \theta$, the equations give

$$\frac{k\theta^2}{2\theta+1} = \frac{1}{k(\theta^2+2\theta)}, \text{ that is, } k^2\theta^3(\theta+2) - (2\theta+1) = 0,$$

and

$$k\theta^2 = \Omega.$$

The equation in θ gives $(k^2\theta^4 - 1)^2 - 4\theta^2(k^2\theta^2 - 1)^2 = 0$, and we have thence

$$k(\Omega^2 - 1)^2 - 4\Omega(k\Omega - 1)^2 = 0,$$

that is,

$$k\Omega^4 - 4k^2\Omega^3 + 6k\Omega^2 - 4\Omega + k = 0,$$

the modular equation; and then $k^2\theta^4 - 1 + 2\theta(k^2\theta^2 - 1) = 0$, that is, $\Omega^2 - 1 + 2\theta(k\Omega - 1) = 0$, or $2\theta = -\frac{\Omega^2 - 1}{k\Omega - 1}$, which is $= \frac{2\alpha}{\beta}$, say we have $\alpha = \Omega^2 - 1$, $\beta = 2(1 - k\Omega)$; consequently

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left\{\frac{\Omega^2 - 1 + 2(k\Omega - 1)x}{\Omega^2 - 1 - 2(k\Omega - 1)x}\right\}^2,$$

$$\lambda = \Omega^2 k, \text{ and } \frac{1}{M} = \frac{\alpha^2 + 2\alpha\beta}{\alpha^2} = \frac{\theta + 2}{\theta} = \frac{\Omega^2 - 4k\Omega + 3}{\Omega^2 - 1},$$

which completes the theory.

13. Reproducing for this case the general theory, it appears *à priori* that Ω is determined by a quartic equation; in fact, from the original equations eliminating Ω, we have an equation

$$\begin{vmatrix} U, & V \\ U', & V' \end{vmatrix} = 0,$$

where U, U', V, V' are quartic functions of α, β; that is, the ratio $\alpha : \beta$ has four values, and to each of these there corresponds a single value of Ω; viz. Ω is determined by a quartic equation.

14. Considering next the case $n=5$, the quintic transformation; the elimination of Ω gives the equations

$$\frac{U}{U'} = \frac{V}{V'} = \frac{W}{W'},$$

where U, U', &c. are all quadric functions of α, β, γ. We have thence $4.4-2.2$, $=12$ sets of values of $\alpha : \beta : \gamma$; viz. considering α, β, γ as coordinates *in plano*, the curves $UV'-U'V=0$, $UW'-U'W=0$ are quartic curves intersecting in 16 points; but among these are included the four points $U=0$, $U'=0$ (in fact, the point $\alpha=0$, $\gamma=0$ four times), which are not points of the curve $VW'-V'W=0$; there remain therefore $16-4$, $=12$ intersections, agreeing with the general value $(n+1).2^{\frac{1}{2}(n-3)}$. Hence Ω is in the first instance determined by an equation of the order 12; but the proper order being $=6$, there must be a factor of the order 6 to be rejected. To explain this and to determine the factor, observe that the equations in question are

$$\begin{aligned} &k^2\alpha^2(2\alpha\gamma+2\beta\gamma+\beta^2)-\gamma^2(2\alpha\gamma+2\alpha\beta+\beta^2)=0, \\ &k^4\alpha^2(\alpha+2\beta) \qquad\qquad -\gamma^3(\gamma+2\beta) \qquad\qquad =0; \end{aligned}$$

at the point $\alpha=0$, $\gamma=0$, the first of these has a double point, the second a triple point; or there are at the point in question 6 intersections; but 4 of these are the points which give the foregoing reduction $16-4=12$; we have thus the point $\alpha=0$, $\gamma=0$, counting twice among the twelve points. Writing in the two equations $\beta=0$, the equations become $k^2\alpha^3\gamma-\alpha\gamma^3=0$, $k^4\alpha^4-\gamma^4=0$, viz. these will be satisfied if $k^2\alpha^2-\gamma^2=0$, that is, the curves pass through each of the two points ($\beta=0$, $\gamma=\pm k\alpha$), and these values satisfy (as in fact they should) the third equation

$$k^2(2\alpha\gamma+2\alpha\beta+\beta^2)\,\alpha(\alpha+2\beta)-\gamma(\gamma+2\beta)(2\alpha\gamma+2\beta+\beta^2)=0.$$

It is moreover easily shown that the three curves have at each of the points in question a common tangent; viz. taking A, B, C as current coordinates, the tangent at the point (α, β, γ) of the second curve has for its equation

$$A(2\alpha^3+3\alpha^2\beta)k^4+B(k^4\alpha^3-\gamma^3)-C(2\gamma^3+3\gamma^2\beta)=0;$$

and for $\beta=0$, $\gamma=\pm k\alpha$, this becomes $2kA+B(k\mp1)\mp2C=0$, viz. this is the line from the point ($\beta=0$, $\gamma=\pm k\alpha$) to the point $(1, -2, 1)$. And similarly for the other two curves we find the same equation for the tangent.

Hence among the 12 points are included the point ($\gamma=0$, $\alpha=0$) twice, and the points ($\beta=0$, $\gamma=\pm k\alpha$) each twice: we have thus a reduction $=6$.

15. Writing in the equations $\gamma=0$, $\alpha=0$, the first and third are satisfied identically, and the second becomes $\beta^2=\Omega\beta^2$, that is, the equations give $\Omega=1$; writing $\beta=0$, they become

$$k^2\alpha^2=\Omega\gamma^2,\quad \alpha\gamma=\Omega\alpha\gamma,\quad \gamma^2=\Omega k^2\alpha^2,$$

viz. putting herein $\gamma^2=k^2\alpha^2$, the equations again give $\Omega=1$; hence the factor of the order 6 is $(\Omega-1)^6$, and the equation of the twelfth order for the determination of Ω is

$$(\Omega-1)^6\{(\Omega,\ 1)^6\}=0,$$

where $(\Omega,\ 1)^6=0$ is the Ωk-modular equation above written down.

16. Reverting to the equation

$$\frac{1-y}{1+y}=\frac{(1-x)(P-Qx)^2}{(1+x)(P+Qx)^2},$$

it is to be observed that for $\alpha=0$, $\gamma=0$, that is, $P=0$, this becomes simply $y=x$, which is the transformation of the order 1; the corresponding value of the modulus λ is $\lambda=k$, and the equation $\lambda=\Omega^2 k$ then gives $\Omega^2=1$, which is replaced by $\Omega-1=0$.

If in the same equation we write $\beta=0$, that is, $Q=0$, then (without any use of the equation $\gamma^2=k^2\alpha^2$) we have $y=x$, the transformation of the order 1; but although this is so, the fundamental equation

$$(P^2+2PQx^2+Q^2x^2)^*=\Omega k^2(P^2+2PQ+Q^2x^2),$$

which, putting therein $Q=0$, becomes $(P^2)^*=\Omega k^2P^2$, that is, $(k^2x^2\alpha+\gamma)^2=\Omega k^2(\alpha+\gamma x^2)^2$, is not satisfied by the single relation $\Omega-1=0$, but necessitates the further relation $\gamma^2=k^2\alpha^2$.

The thing to be observed is that the extraneous factor $(\Omega-1)^6$, equated to zero, gives for Ω the value $\Omega=1$ corresponding to the transformation $y=x$ of the order 1.

17. Considering next $n=7$, the septic transformation; we have here between α, β, γ, δ a fourfold relation of the form

$$\begin{pmatrix} U, & V, & W, & Z \\ U', & V', & W', & Z' \end{pmatrix}=0,$$

where, as before, U, U', &c. are quadric functions, and the number of solutions is here 8.2^2, $=32$; to each of these corresponds a single value of Ω, or Ω is in the first instance determined by an equation of the order 32. But the order of the modular equation is $=8$; or representing this by $\{(\Omega,\ 1)^8\}=0$, the equation must be

$$(\Omega,\ 1)^{24}\{(\Omega,\ 1)^8\}=0,$$

viz. there must be a special factor of the order 24.

18. One way of satisfying the equations is to write therein $\alpha=0$, $\delta=0$; the equations thus become

$$k\beta^2=\Omega\gamma^2,$$
$$\gamma^2+2\beta\gamma=\Omega k(2\beta\gamma+\beta^2);$$

or putting β, $\gamma=\alpha'$, β',

$$k\alpha'^2=\Omega\beta'^2,$$
$$\beta'^2+2\alpha'\beta'=\Omega k(2\alpha'\beta'+\alpha'^2),$$

which (with α', β' instead of α, β) are the very equations which belong to the cubic transformation; hence a factor is $\{(\Omega,\ 1)^4\}$.

Observe that for the values in question $\alpha=0$, $\delta=0$, $P=\beta'x^2$, $Q=\alpha'$,

$$(P\pm Qx)^2=x^2(\alpha'\pm\beta'x)^2,\ =x^2(P'\pm Q'x)^2,\ \text{if}\ P'=\alpha',\ Q'=\beta',$$

and therefore

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{P'-Q'x}{P'+Q'x}\right)^2,$$

which is the formula for a cubic transformation.

19. The equations may also be satisfied by writing therein $\gamma=k\alpha$, $\delta=k\beta$; in fact, substituting these values, they become

$$k^2\alpha^2=\Omega k^2\beta^2,$$
$$2k^2\alpha^2+k(2\alpha\beta+\beta^2)=\Omega k^2(\alpha^2+2\alpha\beta)+2\Omega k\beta^2,$$
$$k^2\alpha^2+2k(\beta^2+2\alpha\beta)=2\Omega k^2(\alpha^2+2\alpha\beta)+\Omega k\beta^2,$$
$$k^2(\beta^2+2\alpha\beta)=\Omega k^2(\alpha^2+2\alpha\beta);$$

the first and last of these are

$$k\alpha^2=\Omega\beta^2,$$
$$\beta^2+2\alpha\beta=\Omega k(\alpha^2+2\alpha\beta),$$

which being satisfied the second and third equations are satisfied identically; and these are the formulæ for a cubic transformation; that is, we again have the factor $\{(\Omega,\ 1)^4\}$.

Observe that for the values in question $\gamma=k\alpha$, $\delta=k\beta$, we have $P=\alpha(1+kx^2)$, $Q=\beta(1+kx^2)$; so that, writing $P'=\alpha$, $Q'=\beta$, we have for y the value

$$\frac{1-y}{1+y}=\frac{(1-x)(P'-Q'x)^2}{(1+x)(P'+Q'x)^2},$$

which is the formula for a cubic transformation.

20. It is important to notice that we cannot by writing $\alpha=0$ or $\delta=0$ reduce the transformation to a quintic one; in fact, the equation $k^2\alpha^2=\Omega\delta^2$ shows that if either of these equations is satisfied the other is also satisfied; and we have then the foregoing case $\alpha=0$, $\delta=0$, giving not a quintic but a cubic transformation.

And for the same reason we cannot by writing $\alpha=0$, $\beta=0$, $\gamma=0$ or $\beta=0$, $\gamma=0$, $\delta=0$ reduce the transformation to the order 1. There is thus no factor $\Omega-1$.

21. As regards the non-existence of the factor $\Omega - 1$, I further verify this by writing in the equations $\Omega = 1$; they thus become

$$k^3\alpha^2 = \delta^2,$$
$$k(2\alpha\gamma + 2\alpha\beta + \beta^2) = \gamma^2 + 2\gamma\delta + 2\beta\delta,$$
$$\gamma^2 + 2\beta\gamma + 2\alpha\delta + 2\beta\delta = k(2\alpha\gamma + 2\beta\gamma + 2\alpha\delta + \beta^2),$$
$$\delta^2 + 2\gamma\delta = k^3(\alpha^2 + 2\alpha\beta),$$

which it is to be shown cannot be satisfied in general, but only for certain values of k.

Reducing the last equation, this is $\gamma\delta = k^3\alpha\beta$, which, combined with the first, gives $\alpha\gamma = \beta\delta$; and if for convenience we assume $\alpha = 1$, and write also $\theta = \pm\sqrt{k}$ (that is, $k = \theta^2$), then the values of α, β, γ, δ are $\alpha = 1$, $\beta = \gamma\theta^{-3}$, $\gamma = \gamma$, $\delta = \theta^3$; which values, substituted in the second and the third equations, give two equations in γ, θ; and from these, eliminating γ, we obtain an equation for the determination of θ, that is, of k. In fact, the second equation gives

$$\theta^2(2\gamma + 2\gamma\theta^{-3} + \gamma^2\theta^{-6}) = \gamma^2 + 2\gamma\theta^3 + 2\gamma;$$

or, dividing by γ and reducing,

$$\gamma(1 - \theta^4) = 2\theta^3(\theta^2 - 1)(\theta^2 - \theta + 1),$$

that is,

$$\gamma(1 + \theta^2) = -2\theta^3(\theta^2 - \theta + 1),$$

or, as this may also be written,

$$(\gamma + \theta^3)(1 + \theta^2) = -\theta^3(\theta - 1)^2,$$

that is,

$$\gamma + \theta^3 = \frac{-\theta^3(\theta - 1)^2}{\theta^2 + 1}.$$

Moreover the third equation gives

$$\gamma^2 + 2\gamma^2\theta^{-3} + 2\theta^3 + 2\gamma = \theta^2(2\gamma + 2\gamma^2\theta^{-3} + 2\theta^3 + \gamma^2\theta^{-6}),$$

that is,

$$\gamma^2(\theta^4 - 2\theta^3 + 2\theta - 1) - 2(\gamma + \theta^3)\theta^4(\theta^2 - 1) = 0;$$

or dividing by $\theta^2 - 1$, it is

$$\gamma^2(\theta - 1)^2 = 2\theta^4(\gamma + \theta^3);$$

whence also

$$\gamma^2 = \frac{-2\theta^7}{\theta^2 + 1}.$$

Also

$$4\theta^6(\theta^2 - \theta + 1)^2 = \gamma^2(\theta^2 + 1)^2,$$

wherefore

$$2(\theta^2 - \theta + 1)^2 = -\theta(\theta^2 + 1) \quad \text{or} \quad 2(\theta^2 - \theta + 1)^2 + \theta(\theta^2 + 1) = 0,$$

or

$$\theta(\theta^2 + 1)^2 + 2(\theta^2 - \theta + 1)^2 = 0,$$

that is,

$$2\theta^4 - 3\theta^3 + 6\theta^2 - 3\theta + 2 = 0,$$

or finally

$$(2\theta^2 - \theta + 1)(\theta^2 - \theta + 2) = 0.$$

We have thus $(2\theta^2+1)^2=\theta^2$, that is, $4\theta^4+3\theta^2+1=0$ or $4k^2+3k+1=0$, or else $(\theta^2+2)^2=\theta^3$, that is, $\theta^4+3\theta^2+4=0$ or $k^2+3k+4=0$; viz. the equation in k is

$$(4k^2+3k+1)(k^2+3k+4)=0,$$

these being in fact the values of k given by the modular equation on putting therein $\Omega=1$.

The equation of the order 32 thus contains the factor $\{(\Omega, 1)^4\}$ at least twice, and it does not contain either the factor $\Omega-1$, or the factor $\{(\Omega, 1)^3\}$ belonging to the quintic transformation; it may be conjectured that the factor $\{(\Omega, 1)^4\}$ presents itself six times, and that the form is

$$\{(\Omega, 1)^4\}^6(\Omega, 1)^3=0;$$

but I am not able to verify this, and I do not pursue the discussion further.

22. The foregoing considerations show the grounds of the difficulty of the purely algebraical solution of the problem; the required results, for instance the modular equation, are obtained not in the simple form, but accompanied with special factors of high order. The transcendental theory affords the means of obtaining the results in the proper form without special factors; and I proceed to develop the theory, reproducing the known results as to the modular and multiplier equations, and extending it to the determination of the transformation-coefficients α, β, ...

The Modular Equation. Art. Nos. 23 to 28.

23. Writing, as usual, $q=e^{-\frac{\pi K'}{K}}$, we have u, a given function of q, viz.

$$\begin{aligned} u &= \sqrt{2}q^{\frac{1}{8}}\frac{1+q^2.\,1+q^4.\,1+q^6..}{1+q\,.\,1+q^3.\,1+q^5..} \\ &= \sqrt{2}q^{\frac{1}{8}}(1-q+2q^2-3q^3+4q^4-6q^5+9q^6-12q^7+\ldots) \\ &= \sqrt{2}q^{\frac{1}{8}}f(q) \text{ suppose;} \end{aligned}$$

and this being so, the several values of v and of the other quantities in question are all given in terms of q.

The case chiefly considered is that of n an odd prime; and unless the contrary is stated it is assumed that this is so. We have then $n+1$ transformations corresponding to the same number $n+1$ of values of v; these may be distinguished as $v_0, v_1, v_2, \ldots, v_n$; viz. writing α to denote an imaginary nth root of unity, we have

$$v_0=(-)^{\frac{n^2-1}{8}}\sqrt{2}q^{\frac{n}{8}}f(q^n),\quad v_1=\sqrt{2}\,(\alpha q^{\frac{1}{n}})^{\frac{1}{8}}f(\alpha q^{\frac{1}{n}}),\quad v_2=\sqrt{2}\,(\alpha^2 q^{\frac{1}{n}})^{\frac{1}{8}}f(\alpha^2 q^{\frac{1}{n}}),\ \&c.,$$

$$v_n=\qquad\sqrt{2}q^{\frac{1}{8n}}f(q^{\frac{1}{n}}).$$

(Observe $(-)^{\frac{n^2-1}{8}}=+$ for $n=8p\pm1$, $-$ for $n=8p\pm3$.)

The occurrence of the fractional exponent $\frac{1}{8}$ is, as will appear, a circumstance of great importance; and it will be convenient to introduce the term "octicity," viz. an expression of the form $q^{\frac{f}{8}}F(q)$ ($f=0$, or a positive integer not exceeding 7, $F(q)$ a rational function of q) may be said to be of the octicity f.

24. The modular equation is of course

$$(v-v_0)(v-v_1)\ldots.(v-v_n)=0;$$

say this is

$$v^{n+1}-Av^n+Bv^{n-1}-\ldots=0,$$

so that $A=\Sigma v_0$, $B=\Sigma v_0 v_1$, &c. In the development of these expressions, the terms having a fractional exponent, with denominator n, would disappear of themselves, as involving symmetrically the several nth roots of unity; and each coefficient would be of the form $q^{\frac{g}{8}}F(q)$, F a rational and integral function of q. It is moreover easy to see that, for the several coefficients A, B, C,....., g will denote the positive residue (mod. 8) of n, $2n$, $3n$,... respectively.

Hence assuming, as the fact is, that these coefficients are severally rational and integral functions of q, it follows that the form is

$$au^g+bu^{g+8}+cu^{g+16}+\ldots.,$$

g having the foregoing values for the several coefficients respectively. And it being known that the modular equation is as regards u of the order $=n+1$, there is a known limit to the number of terms in the several coefficients respectively. We have thus for each coefficient an identity of the form

$$A=au^g+bu^{g+8}+\ldots.,$$

where A and u being each of them given in terms of q, the values of the numerical coefficients a, b,.. can be determined; and we thus arrive at the modular equation.

25. It is in effect in this manner that the modular equations are calculated in Sohnke's Memoir. Various relations of symmetry in regard to (u, v) and other known properties of the modular equation are made use of in order to reduce the number of the unknown coefficients to a minimum; and (what in practice is obviously an important simplification) instead of the coefficients Σv_0, $\Sigma v_0 v_1$, &c., it is the sums of powers Σv_0, Σv_0^3, &c., which are compared with their expressions in terms of u, in order to the determination of the unknown numerical coefficients a, b,... The process is a laborious one (although less so than perhaps might beforehand have been imagined), involving very high numbers; it requires the development up to high powers of q, of the high powers of the before-mentioned function $f(q)$; and Sohnke gives a valuable Table, which I reproduce here; adding to it the three columns which relate to ϕq.

ind. of q.	ϕq =	$\phi^5 q$ =	$\phi^{-3} q$ =	fq =	$f^2 q$ =	$f^3 q$ =	$f^4 q$ =	$f^5 q$ =	$f^6 q$ =	$f^7 q$ =	$f^8 q$ =	$f^9 q$ =	$f^{10} q$ =	$f^{11} q$ =
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	+2	+4	−4	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11
2	0	+4	+12	+2	+5	+9	+14	+20	+27	+35	+44	+54	+65	+77
3	0	0	−32	−3	−10	−22	−40	−65	−98	−140	−192	−255	−330	−418
4	+2	+4	+76	+4	+18	+48	+101	+185	+309	+483	+718	+1026	+1420	+1914
5	0	+8	−168	−6	−32	−99	−236	−481	−882	−1498	−2400	−3672	−5412	−7733
6	0	0	+352	+9	+55	+194	+518	+1165	+2330	+4277	+7352	+11997	+18765	+28336
7	0	0	−704	−12	−90	−363	−1080	−2665	−5784	−11425	−20992	−36414	−60270	−95931
8	0	+4		+16	+144	+657	+2162	+5820	+13644	+28889	+56549	+103977	+181645	+304062
9	+2	+4		−22	−226	−1155	−4180	−12220	−30826	−69734	−145008	−281911	−518660	−911240
10	0	+8		+29	+346	+1977	+7840	+24802	+67107	+161735	+356388	+730953	+1413465	+2601786
11	0	0		−38	−522	−3312	−14328	−48880	−141444	−362271	−844032	−1822689	−3697960	−7120136
12	0	0		+50	+777	+5443	+25591	+93865	+289746	+786877	+1934534	+4390824	+9331565	+18766759
13	0	+8		−64	−1138	−8787	−44776	−176125	−578646	−1662927	−4306368	−10256508	−22800050	−47830486
14	0	0		+82	+1648	+13968	+76918	+323685	+1129527	+3428770	+9337704	+23303025	+54112825	+118270746
15	0	0		−105	−2362	−21894	−129952	−583798	−2159774	−6913760	−19771392	−51631227	−125090220	−284527793
16	+2	+4		+132	+3348	+33873	+216240	+1035060	+4052721	+13660346	+40965362	+111804966	+282298020	+667553898
17	0	+8		−166	−4704	−51795	−354864	−1806600	−7474806	−26492361	−83207976	−237074742	−623185010	−1530587256
18	0	+4		+208	+6554	+78345	+574958	+3108085	+15063859*	+50504755	+165944732	+493063403	+1348033540	+3435726536
19	0	0		−258	−9056	−117412								
20	0	+8		+320	+12425	+174033								
21	0	0		−395	−16932	−255945								
22	0	0		+484										
23	0	0		−592										
24	0	0		+722										
25	+2	+12		−876										
26	0	+8		+1060										

* [Wrongly given by Sohnke as +3108085.]

ind. of q.	$f^{12}q$ =	$f^{13}q$ =	$f^{14}q$ =	$f^{15}q$ =	$f^{16}q$ =	$f^{17}q$ =	$f^{18}q$ =	$f^{19}q$ =	$f^{20}q$ =
0	1	1	1	1	1	1	1	1	1
1	− 12	− 13	− 14	− 15	− 16	− 17	− 18	− 19	− 20
2	+ 90	+ 104	+ 119	+ 135	+ 152	+ 170	+ 189	+ 209	+ 230
3	− 520	− 637	− 770	− 920	− 1088	− 1275	− 1482	− 1710	− 1960
4	+ 2523	+ 3263	+ 4151	+ 5205	+ 6444	+ 7888	+ 9558	+ 11476	+ 13665
5	− 10764	− 14651	− 19558	− 25668	− 33184	− 42330	− 53352	− 66519	− 82124
6	+ 41534	+ 59345	+ 82936	+ 113675	+ 153152	+ 203201	+ 265923	+ 343710	+ 439270
7	− 147720	− 221091	− 322828	− 461265	− 646528	− 890800	− 1208610	− 1617147	− 2136600
8	+ 490869	+ 768131	+ 1169847	+ 1739710	+ 2533070	+ 3619334	+ 5084478	+ 7034047	+ 9596460
9	− 1539472	− 2514551	− 3988292	− 6164345	− 9311664	− 13780540	− 20021534	− 28607673	− 40260300
10	+ 4592430	+ 7818200	+ 12896562	+ 20690964	+ 32387616	+ 49590581	+ 74438388	+ 109745767	+ 159174524
11	− 13111632	− 23233535	− 39809574	− 66222405	− 107299904	− 169812320	− 263104686		
12	+ 36006362	+ 66328961	+ 117921321	+ 203173760	+ 340436664	+ 556366922	+ 889020813		
13	− 95497116	− 182681916	− 336630840	− 600165795	− 1039026144	− 1752038020	− 2884990266		
14	+ 245457000	+ 487098378	+ 929461993	+ 1713196575	+ 3061896704	+ 5323089708	+ 9026077050		
15	− 613183064	− 1261118313	− 2489690882	− 4740491107	− 8739810688	− 15653783345	− 27314626158		
16	+ 1492474572	+ 3178449222	+ 6486711301	+ 12748926285	+ 24229115109	+ 44679433473	+ 80177033781*		
17	− 3546915228	− 7815313766	− 16475721276	− 33400680615	− 65390485328	− 124069449335	− 228831885054		
18	+ 8245677110	+ 18783535199	+ 40874694490	+ 85415669230	+ 172155210320	+ 335888162944	+ 636376573943		

* [In Sohnke, the figure 1 has] dropped out.

26. I give from Sohnke the series of modular equations, adding those for the composite cases $n=9$ and $n=15$, as to which see the remarks which follow the Table.

$n=3$.

	v^4	v^3	v^2	v	1
u^4					-1
u^3		$+2$			
u^2					
u				-2	
1	$+1$				
	1	$+2$	0	-2	-1

$= (v+1)^3(v-1)$.

$n=5$.

	v^6	v^5	v^4	v^3	v^2	v	1
u^6							-1
u^5		$+4$					
u^4					-5		
u^3							
u^2			$+5$				
u						-4	
1	$+1$						
	1	$+4$	$+5$	0	-5	-4	-1

$= (v+1)^5(v-1)$.

$n=7$.

	v^8	v^7	v^6	v^5	v^4	v^3	v^2	v	1
u^8	0								$+1$
u^7		-8							
u^6			$+28$						
u^5				-56					
u^4					$+70$				
u^3						-56			
u^2							$+28$		
u								-8	
1	$+1$								0
	1	-8	$+28$	-56	$+70$	-56	$+28$	-8	$+1$

$= (v-1)^8$.

$n = 9.$

	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v	1
u^{12}					0								+ 1
u^{11}				− 16								+ 8	
u^{10}			+ 16								+ 10		
u^{9}		− 16								− 24			
u^{8}	0								+ 15				
u^{7}								+ 48					
u^{6}							− 84						
u^{5}						+ 48							
u^{4}					+ 15								0
u^{3}				− 24								− 16	
u^{2}			+ 10								+ 16		
u		+ 8								− 16			
1	+ 1								0				
	1	− 8	+ 26	− 40	+ 15	+ 48	− 84	+ 48	+ 15	− 40	+ 26	− 8	+ 1

$= (v-1)^{10}(v+1)^{9}.$

$n = 11.$

	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v	1
u^{12}					0								− 1
u^{11}		+ 32								− 22			
u^{10}							− 44						
u^{9}				+ 88								+ 22	
u^{8}	0								− 165				
u^{7}						+ 132							
u^{6}			+ 44								− 44		
u^{5}								− 132					
u^{4}					+ 165								0
u^{3}		− 22								− 88			
u^{2}							+ 44						
u				+ 22								− 32	
1	+ 1								0				
	1	+ 10	+ 44	+ 110	+ 165	+ 132	0	− 132	− 165	− 110	− 44	− 10	− 1

$= (v+1)^{11}(v-1).$

	v^{14}	v^{13}	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v	1
u^{14}							0								-1
u^{13}		$+64$								-52					
u^{12}					0								-65		
u^{11}								$+208$							
u^{10}			0								-429				
u^{9}						$+520$								$+52$	
u^{8}	0								-429						
u^{7}				$+208$								-208			
u^{6}							$+429$								0
u^{5}		-52								-520					
u^{4}					$+429$								0		
u^{3}								-208							
u^{2}			$+65$								0				
u						$+52$								-64	
1	$+1$								0						
	1	$+12$	$+65$	$+208$	$+429$	$+572$	$+429$	0	-429	-572	-429	-208	-65	-12	-1

$n=13.$

$=(v+1)^{13}(v-1).$

	v^{24}	v^{23}	v^{22}	v^{21}	v^{20}	v^{19}	v^{18}	v^{17}	v^{16}	v^{15}	v^{14}	v^{13}	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v^{1}	v^{0}
u^{24}	0								0								0								+1
u^{23}		0								−128								+120							
u^{22}			0								−960								+980						
u^{21}				0								−2880								+2888					
u^{20}					−256								−3920								+4050				
u^{19}						−1920								−800								+2888			
u^{18}							−5824								+5040								+980		
u^{17}								−8960								+8160								+120	
u^{16}	0								−26970								+27209								0
u^{15}		−128								−6960								+8160							
u^{14}			−960								−5320								+5040						
u^{13}				−2880								+3120								−800					
u^{12}					−3920								+9660								−3920				
u^{11}						−800								+3120								−2880			
u^{10}							+5040								−5320								−960		
u^{9}								+8160								−6960								−128	
u^{8}	0								+27209								−26970								0
u^{7}		+120								+8160								−8960							
u^{6}			+980								+5040								−5824						
u^{5}				+2888								−800								−1920					
u^{4}					+4050								−3920								−256				
u^{3}						+2888								−2880								0			
u^{2}							+980								−960								0		
u								+120								−128								0	
1	+1								0								0								0
	+1	−8	+20	+8	−126	+168	+196	−680	+239	+1072	−1240	−560	+1820	−560	−1240	+1072	+239	−680	+196	+168	−126	+8	+20	−8	+1

$n=15.$

$$=(v-1)^8(v^2-1)^8.$$

	v^{18}	v^{17}	v^{16}	v^{15}	v^{14}	v^{13}	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v	1
u^{18}			0								0								+1
u^{17}		−256								+272								−34	
u^{16}	0								−272								+425		
u^{15}								+1632								−2448			
u^{14}							−4080								+7140				
u^{13}						+4896								−13464					
u^{12}					−4080								+22644						
u^{11}				+1632								−33456							
u^{10}			−272								+44030								0
u^{9}		+272								−49164								−272	
u^{8}	0								+44030								−272		
u^{7}								−33456								+1632			
u^{6}							+22644								−4080				
u^{5}						−13464								+4896					
u^{4}					+7140								−4080						
u^{3}				−2448								+1632							
u^{2}			+425								−272								0
u		−34								+272								−256	
1	+1								0								0		
	1	−18	+153	−816	+3060	−8568	+18564	−31824	+43758	−48520	+43758	−31824	+18564	−8568	+3060	−816	+153	−18	+1

$n=17.$

$=(v-1)^{18}.$

	v^{20}	v^{19}	v^{18}	v^{17}	v^{16}	v^{15}	v^{14}	v^{13}	v^{12}	v^{11}	v^{10}	v^{9}	v^{8}	v^{7}	v^{6}	v^{5}	v^{4}	v^{3}	v^{2}	v	1
u^{20}					0								0								-1
u^{19}		$+512$								-608								$+114$			
u^{18}							$+2432$								-2584						
u^{17}				-2432								$+3344$								-114	
u^{16}	0								$+3952$								-6859				
u^{15}						$+5472$								$+2280$							
u^{14}			-2432								-10488								-2584		
u^{13}								$+25536$								-2280					
u^{12}					-3952								-21242								0
u^{11}		-608								$+20748$								-3344			
u^{10}							$+10488$								-10488						
u^{9}				$+3344$								-20748								$+608$	
u^{8}	0								$+21242$								$+3952$				
u^{7}						$+2280$								-25536							
u^{6}			$+2584$								$+10488$								$+2432$		
u^{5}								-2280								-5472					
u^{4}					$+6859$								-3952								0
u^{3}		$+114$								-3344								$+2432$			
u^{2}							$+2584$								-2432						
u				-114								$+608$								-512	
1	1								0								0				
	1	$+18$	$+152$	$+798$	$+2907$	$+7752$	$+15504$	$+23256$	$+25194$	$+16796$	0	-16796	-25194	-23256	-15504	-7752	-2907	-798	-152	-18	-1

$n=19.$

$=(v+1)^{19}(v-1).$

Various remarks arise on the Tables. Attending first to the cases n a prime number; the only terms of the order $n+1$ in v or u are $v^{n+1} \pm u^{n+1}$, viz. $n \equiv 3$ or 5 (mod. 8) the sign is $-$, but $n \equiv 1$ or 7 (mod. 8) the sign is $+$. And there is in every case a pair of terms $v^n u^n$ and vu, having coefficients equal in absolute magnitude, but of opposite signs, or of the same sign, in the two cases respectively.

Each Table is symmetrical in regard to its two diagonals respectively, so that every non-diagonal coefficient occurs (with or without reversal of sign) 4 times; viz. in the case $n \equiv 1$ or 7 (mod. 8) this is a perfect symmetry, without reversal of sign; but in the case $n \equiv 3$ or 5 (mod. 8) it is, as regards the lines parallel to either diagonal, and in regard to the other diagonal, alternately a perfect symmetry without reversal of sign and a skew symmetry with reversal. Thus in the case $n = 19$, the lines parallel to the dexter diagonal are -1 (symmetrical), $+114$, -114 (skew), 0, -2584, -6859, -2584, 0 (symmetrical), and so on. The same relation of symmetry is seen in the composite cases $n = 9$ and $n = 15$, both belonging to $n \equiv 1$ or 7, mod. 8.

If, as before, n is prime, then putting in the modular equation $u = 1$, the equation in the case $n \equiv 1$ or 7 (mod. 8) becomes $(v-1)^{n+1} = 0$, but in the case $n \equiv 3$ or 5 (mod. 8) it becomes $(v+1)^n (v-1) = 0$.

27. In the case n a composite number not containing any square factor, then dividing n in every possible way into two factors $n = ab$ (including the divisions $n \,.\, 1$ and $1 \,.\, n$), and denoting by β an imaginary bth root of unity, a value of v is

$$\pm \sqrt{2}\,(\beta q^{\frac{a}{b}})^{\frac{1}{8}} f(\beta q^{\frac{a}{b}});$$

so that the whole number of roots (or order of the modular equation) is $= \nu$, if ν be the sum of the divisors of n. Thus $n = 15$, the values are

$$\begin{array}{cccc} \sqrt{2}q^{\frac{1}{8}} f(q^{15}), & -\sqrt{2}q^{\frac{1}{8}\cdot\frac{5}{3}} f(q^{\frac{5}{3}}), & -\sqrt{2}q^{\frac{1}{8}\cdot\frac{3}{5}} f(q^{\frac{3}{5}}), & \sqrt{2}q^{\frac{1}{8}\cdot\frac{1}{15}} f(q^{\frac{1}{15}}) \\ 1 \quad , & 3 \quad , & 5 \quad , & 15 \text{ roots}; \end{array}$$

and the order of the modular equation is $= 24$. The modular equation might thus be obtained as for a prime number; but it is easier to decompose n into its prime factors, and consider the transformation as compounded of transformations of these prime orders. Thus $n = 15$, the transformation is compounded of a cubic and a quintic one. If the v of the cubic transformation be denoted by θ, then we have

$$\theta^4 + 2\theta^3 u^3 - 2\theta u - u^4 = 0;$$

and to each of the four values of θ corresponds the six values of v belonging to the quintic transformation given by

$$v^6 + 4v^5\theta^6 + 5v^4\theta^2 - 5v^2\theta^4 - 4v\theta - \theta^6 = 0.$$

The equation in v is thus

$$(v^6 + 4v^5\theta_1^5 + \ldots - \theta_1^6)(v^6 + \ldots - \theta_2^6)(v^6 + \ldots - \theta_3^6)(v^6 + \ldots - \theta_4^6) = 0,$$

where θ_1, θ_2, θ_3, θ_4 are the roots of the equation in θ, viz. we have

$$\theta^4 + 2\theta^3 u^3 - 2\theta u - u^4 = (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4);$$

and it was in this way that the equation for the case $n = 15$ was calculated. Observe that writing $u = 1$, we have $(\theta + 1)^3(\theta - 1) = 0$, or say $\theta_1 = \theta_2 = \theta_3 = -1$, $\theta_4 = +1$. The equation in v thus becomes $\{(v-1)^5(v+1)\}^3(v+1)^5(v-1) = 0$, that is, $(v-1)^{16}(v+1)^8 = 0$.

28. The case where n has a square factor is a little different; thus $n = 9$, the values are

$$\begin{array}{ccc} \sqrt{2}q^{\frac{2}{3}}f(q^9), & -\sqrt{2}q^{\frac{1}{3}\cdot\frac{3}{3}}f(q^{\frac{3}{3}}), & \sqrt{2}q^{\frac{1}{3}\cdot\frac{1}{3}}f(q^{\frac{1}{3}}), \\ 1\ , & 3\ , & 9\ , \text{ roots;} \end{array}$$

but here ω being an imaginary cube root of unity, the second term denotes the three values,

$$\sqrt{2}q^{\frac{1}{3}}f(q), \quad \sqrt{2}(q\omega)^{\frac{1}{3}}f(\omega q), \quad \sqrt{2}(\omega^2 q)^{\frac{1}{3}}f(\omega^2 q),$$

the first of which is $= u$, and is to be rejected; there remain $1 + 2 + 9$, $= 12$ roots, or the equation is of the order 12.

Considering the equation as compounded of two cubic transformations, if the value of v for the first of these be θ, then we have

$$\theta^4 + 2\theta^3 u^3 - 2\theta u - u^4 = 0;$$

and to the four values of θ correspond severally the four values of v given by the equation

$$v^4 + 2v^3\theta^3 - 2v\theta - \theta^4 = 0.$$

One of these values is however $v = -u$, since the $v\theta$-equation is satisfied on writing therein $v = -u$; hence, writing

$$\theta^4 + 2\theta^3 u^3 - 2\theta u - u^4 = (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4),$$

we have an equation

$$(v^4 + 2v^3\theta_1^3 - 2v\theta_1 - \theta_1^4)(v^4 + \ldots - \theta_2^4)(v^4 + \ldots - \theta_3^4)(v^4 + \ldots - \theta_4^4) = 0,$$

which contains the factor $(v + u)^4$ and, divested hereof, gives the required modular equation of the order 12; it was in fact obtained in this manner.

Observe that writing $u = 1$, we have $(\theta + 1)^3(\theta - 1) = 0$, or say $\theta_1 = \theta_2 = \theta_3 = -1$, $\theta_4 = 1$; the modular equation then becomes

$$\{(v-1)^3(v+1)\}^3(v+1)^3(v-1) \div (v+1)^4 = 0,$$

that is,

$$(v-1)^{10}(v+1)^2 = 0.$$

The Multiplier Equation. Art. No. 29.

29. The theory is in many respects analogous to that of the modular equation. To each value of v there corresponds a single value of M; hence M, or what is the same thing $\frac{1}{M}$, is determined by an equation of the same order as v, viz. n being prime, the order is $=n+1$. The last term of the equation is constant, and the other coefficients are rational and integral functions of u^8, of a degree not exceeding $\frac{1}{2}(n-1)$; and not only so, but they are, $n \equiv 1$ (mod. 4), rational and integral functions of $u^8(1-u^8)$, and $n \equiv 3$ (mod. 4), alternately of this form and of the same form multiplied by the factor $(1-2u^8)$.

The values are in fact given as transcendental functions of q; viz. denoting by $M_0, M_1, M_2, \ldots, M_n$ the values corresponding to $v_0, v_1, v_2, \ldots, v_n$ respectively, and writing

$$\phi(q) = \frac{(1+q)(1+q^3)(1+q^5)\ldots(1-q^2)(1-q^4)(1-q^6)\ldots}{(1-q)(1-q^3)(1-q^5)\ldots(1+q^2)(1+q^4)(1+q^6)\ldots}$$
$$= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \ldots,$$

then we have

$$M_0 = \frac{(-)^{\frac{n-1}{2}}}{n}\frac{\phi^2(q)}{\phi^2(q^n)},$$

$$M_1 = \frac{\phi^2(q)}{\phi^2(\alpha q^{\frac{1}{n}})}, \ldots (\alpha \text{ an imaginary } n\text{th root of unity})$$

$$\vdots$$

$$M_n = \frac{\phi^2 q}{\phi^2(q^{\frac{1}{n}})}.$$

Hence, the form of the equation being known, the values of the numerical coefficients may be calculated; and it was in this way that Joubert obtained the following results. I have in some cases changed the sign of Joubert's multiplier, so that in every case the value corresponding to $u=0$ shall be $M=1$.

The equations are:

$$\begin{aligned} n=3,\quad & \frac{1}{M^4} && u=0, \text{ this is} \\ & + \frac{1}{M^3}\cdot\ 0 && \left(\frac{1}{M}-1\right)^3\left(\frac{1}{M}+3\right)=0; \\ & + \frac{1}{M^2}\cdot -6 && u=1, \text{ it is} \\ & + \frac{1}{M}\cdot\ 8(1-2u^8) && \left(\frac{1}{M}+1\right)^3\left(\frac{1}{M}-3\right)=0. \\ & -3=0. \end{aligned}$$

$n=5,\quad \dfrac{1}{M^6}$ $\qquad u=0$ or 1, this is

$+\dfrac{1}{M^5}\cdot -10$ $\qquad \left(\dfrac{1}{M}-1\right)^5\left(\dfrac{1}{M}-5\right)=0.$

$+\dfrac{1}{M^4}\cdot +35$

$+\dfrac{1}{M^3}\cdot -60$

$+\dfrac{1}{M^2}\cdot +55$

$\dfrac{1}{M}\cdot -26+256u^8(1-u^8)$

$+5=0.$

$n=7,\quad \dfrac{1}{M^8}$ $\qquad u=0$, this is

$+\dfrac{1}{M^7}\cdot 0$ $\qquad \left(\dfrac{1}{M}-1\right)^7\left(\dfrac{1}{M}-7\right)=0;$

$+\dfrac{1}{M^6}\cdot -28$ $\qquad u=1$, it is

$+\dfrac{1}{M^5}\cdot +112(1-2u^8)$ $\qquad \left(\dfrac{1}{M}+1\right)^7\left(\dfrac{1}{M}+7\right)=0.$

$+\dfrac{1}{M^4}\cdot -210$

$+\dfrac{1}{M^3}\cdot +224(1-2u^8)$

$+\dfrac{1}{M^2}\cdot -140-21\,.\,256u^8(1-u^8)$

$+\dfrac{1}{M}\cdot\{48+2048u^8(1-u^8)\}(1-2u^8)$

$+7=0.$

$n=11,\quad \dfrac{1}{M^{12}}$ $\qquad u=0$, this is

$+\dfrac{1}{M^{11}}\cdot 0$ $\qquad \left(\dfrac{1}{M}-1\right)^{11}\left(\dfrac{1}{M}+11\right)=0;$

$+\dfrac{1}{M^{10}}\cdot -66$ $\qquad u=1$, it is

$+\dfrac{1}{M^9}\cdot +440(1-2u^8)$ $\qquad \left(\dfrac{1}{M}+1\right)^{11}\left(\dfrac{1}{M}-11\right)=0.$

$$
\begin{aligned}
&+\frac{1}{M^8}\,.\,-1485\\
&+\frac{1}{M^7}\,.\,+3168\,(1-2u^8)\\
&+\frac{1}{M^6}\,.\,-4620-3\,.\,11^2\,.\,256u^8\,(1-u^8)\\
&+\frac{1}{M^5}\,.\,\{+4752+11\,.\,4096u^8\,(1-u^8)\}\,(1-2u^8)\\
&+\frac{1}{M^4}\,.\,-3465-3\,.\,7\,.\,11\,.\,512u^8\,(1-u^8)\\
&+\frac{1}{M^3}\,.\,\{+1760+11\,.\,83\,.\,2048u^8\,(1-u^8)\}\,(1-2u^8)\\
&+\frac{1}{M^2}\,.\,-594-9\,.\,11\,.\,37\,.\,256u^8\,(1-u^8)-3\,.\,11\,.\,131072\,\{u^8\,(1-u^8)\}^2\\
&+\frac{1}{M}\{120+15\,.\,4096u^8\,(1-u^8)-524288\,\{u^8\,(1-u^8)\}^2\}\,(1-2u^8)\\
&\qquad -11=0.
\end{aligned}
$$

The Multiplier as a rational function of u, v. Art. Nos. 30 to 36.

30. The multiplier M, as having a single value corresponding to each value of v, is necessarily a rational function of u, v; and such an expression of M can, as remarked by Königsberger, be deduced from the multiplier equation by means of Jacobi's theorem,

$$M^2=\frac{1}{n}\,\frac{\lambda\,(1-\lambda^2)}{k\,(1-k^2)}\,\frac{dk}{d\lambda};$$

viz. substituting for k, λ their values u^8, v^8, and observing that if the modular equation be $F(u,\,v)=0$ so that the value of $\frac{du}{dv}$ is $=-F'(v)\div F'(u)$, this is

$$M^2=-\frac{1}{n}\,\frac{(1-v^8)\,vF'v}{(1-u^8)\,uF'u};$$

and then in the multiplier equation separating the terms which contain the odd and even powers, and writing it in the form $\Phi(M^2)+M\Psi(M^2)=0$, this equation, substituting therein for M^2 its value, gives the value of M rationally.

The rational expression of M in terms of u, v is of course indeterminate, since its form may be modified in any manner by means of the equation $F(u,\,v)=0$; and in the expression obtained as above, the orders of the numerator and the denominator are far too high. A different form may be obtained as follows: for greater convenience I seek for the value not of M but of $\frac{1}{M}$.

31. Denoting, as above, by $M_0, M_1, \ldots, M_n$ the values which correspond to $v_0, v_1, \ldots, v_n$ respectively, and writing $S\frac{1}{M} = \frac{1}{M_0} + \frac{1}{M_1} + \ldots + \frac{1}{M_n}$, &c., we have $S\frac{1}{M}$, $S\frac{v}{M}$, &c., all of them expressible as determinate functions of u; and we have moreover the theorem that each of these is a rational and integral function of u: we have thus the series of equations

$$S\frac{1}{M} = A, \quad S\frac{v}{M} = B, \ \ldots, \ S\frac{v^n}{M} = H,$$

where $A, B, \ldots, H$ are rational and integral functions of u. These give linearly the different values of $\frac{1}{M}$; in fact, we have

$$(v_0 - v_1) \ldots (v_0 - v_n) \frac{1}{M_0} = H - GSv_1 + FSv_1v_2 - \ldots \pm Av_1v_2 \ldots v_n,$$

where Sv_1, Sv_1v_2, &c. denote the combinations formed with the roots $v_1, v_2, \ldots, v_n$ (these can be expressed in terms of the single root v_0); and we have also $(v_0 - v_1) \ldots (v_0 - v_n) = F'(v_0)$: the resulting equation is consequently $F'v_0 \frac{1}{M_0} = R(u, v_0)$, R a determinate rational and integral function of (u, v_0); but as the same formula exists for each root of the modular equation, we may herein write M, v in place of M_0, v_0; and the formula thus is

$$F'v \,.\, \frac{1}{M} = R(u, v),$$

viz. we thus obtain the required value of $\frac{1}{M}$ as a rational fraction, the denominator being the determinate function $F'v$, and the numerator being, as is easy to see, a determinate function of the order n as regards v.

32. The method is applicable when M is only known by its expression in terms of q; but if we know for M an expression in terms of v, u, then the method transforms this into a standard form as above. By way of illustration I will consider the case $n = 3$, where the modular equation is

$$v^4 + 2v^3u^3 - 2vu - u^4 = 0,$$

and where a known expression of M is $\frac{1}{M} = 1 + \frac{2u^3}{v}$. Here writing S_{-1}, $S_0 (= 4)$, S, &c. to denote the sum of the powers -1, 0, 1, &c. of the roots of the equation, we have

$$S\frac{1}{M} = S_0 + 2u^3S_{-1}, \ = 0 \quad \text{, as appears from the values presently given,}$$

$$S\frac{v}{M} = S_1 + 2u^3S_0 \quad , \ = 6u^3,$$

$$S\frac{v^2}{M} = S_2 + 2u^3S_1 \quad , \ = 0 \quad ,$$

$$S\frac{v^3}{M} = S_3 + 2u^3S_2 \quad , \ = 6u;$$

and observing that v_0 being ultimately replaced by v, we have

$$Sv_1 = Sv_0 - v, \qquad Sv_0v_1 = Sv_0v_1 - vSv_0 + v^2, \quad v_1v_2v_3 = Sv_0v_1v_2 - vSv_0v_1 + v_2Sv_0 - v^3,$$

that is,

$$Sv_1 = -2u^3 - v, \quad Sv_1v_2 = 2u^3v + v^2, \quad v_1v_2v_3 = 2u - 2u^3v^2 - v^3,$$

we have

$$\begin{aligned} F'v \,.\, \frac{1}{M} = \quad & (S_3 + 2u^3S_2) \\ & + (2u^3 + v)(S_2 + 2u^3S_1) \\ & + (2u^3v + v^2)(S_1 + 2u^3S_0) \\ & + (-2u + 2u^3v^2 + v^3)(S_0 + 2u^3S_{-1}), \end{aligned}$$

viz. this is

$$\begin{aligned} 2(2v^3 + 3v^2u^3 - u)\frac{1}{M} = \quad & v^3(S_0 + 2u^3S_{-1}) \\ & + v^2(S_1 + 4u^3S_0 + 4u^6S_{-1}) \\ & + v\,(S_2 + 4u^3S_1 + 4u^6S_0) \\ & + \quad (S_3 + 4u^3S_2 + 4u^6S_1 - 2uS_0 - 4u^4S_{-1}). \end{aligned}$$

But we have

$$S_{-1} = -\frac{2}{u^3}, \quad S_0 = 4, \quad S_1 = -2u^3, \quad S_2 = 4u^6, \quad S_3 = 6u - 8u^3;$$

and the equation thus is

$$(2v^3 + 3v^2u^3 - u)\frac{1}{M} = 3(v^2u^2 + 2u^5v + 1)u;$$

to verify which observe that, substituting herein for $\frac{1}{M}$ its value $1 + \frac{2u^3}{v}$, the equation becomes

$$(2v^3 + 3v^2u^3 - u)(v + 2u^3) - 3vu(v^2u^2 + 2u^5v + 1) = 0;$$

that is,

$$2v^4 + 4v^2u^3 - 4vu - 2u^4 = 0,$$

as it should do.

33. Any expression whatever of M in terms of u, v is in fact one of a system of four expressions; viz. we may simultaneously change

	u	v	$\frac{1}{M}$	that is, signs are $n \equiv 1$	$n \equiv 3$	$n \equiv 5$	$n \equiv 7$ (mod. 8)
into	v,	$(-)^{\frac{n^2-1}{8}} u$,	$(-)^{\frac{n-1}{2}} nM$;	+ + +	+ − −	+ − −	+ + −
or	$\frac{1}{u}$,	$\frac{1}{v}$,	$\frac{v^4}{u^4M}$;	+ + +	+ + +	+ + +	+ + +
or	$\frac{1}{v}$,	$(-)^{\frac{n^2-1}{8}} \frac{1}{u}$,	$(-)^{\frac{n-1}{2}} \frac{u^4}{v^4} nM$;	+ + +	+ − −	+ − +	+ + −

Thus $n=3$, starting from $\frac{1}{M}=1+\frac{2u^3}{v}$, we have

$$\frac{1}{M}=1+\frac{2u^3}{v},\quad -3M=1-\frac{2v^3}{u},\quad \frac{v^4}{u^4M}=1+\frac{2v}{u^3},\quad -\frac{u^4}{v^4}3M=1-\frac{2u}{v^3};$$

and of course if from any two of these we eliminate M, we have either an identity or the modular equation; thus we have the modular equation under the six different forms:

$$\begin{array}{lll}
(1,\ 2) & (v+2u^3)(u-2v^3)+3uv & =0,\\
(1,\ 3) & v^3(v+2u^2)-u(u^3+2v) & =0,\\
(1,\ 4) & (v+2u^3)(v^3-2u)+3u^4 & =0,\\
(2,\ 3) & (u-2v^3)(u^3+2v)+3v^4 & =0,\\
(2,\ 4) & v(v^3-2u)-u^3(u-2v^3) & =0,\\
(3,\ 4) & (u^3+2v)(v^3-2u)+3u^3v^3 & =0.
\end{array}$$

34. Next $n=5$. Here, starting from $\frac{1}{M}=\frac{v-u^5}{v(1-uv^3)}$, the changes give

$$\frac{1}{M}=\frac{v-u^5}{v(1-uv^3)},\quad 5M=\frac{u+v^5}{u(1+u^3v)},\quad \frac{v^4}{u^4M}=\frac{v^3(v-u^5)}{u^4(1-uv^3)},\quad \frac{u^4}{v^4}5M=\frac{u^3(u+v^5)}{v^4(1+u^3v)},$$

viz. the third and the fourth forms agree with the first and the second forms respectively; that is, there are only two independent forms, and the elimination of M from these gives

$$5uv(1-uv^3)(1+u^3v)-(v-u^5)(u+v^5)=0,$$

which is a form of the modular equation.

35. In the case $n=7$, starting from $\frac{1}{M}=\frac{-7u(1-uv)(1-uv+u^2v^2)}{u-v^7}$ (as to this see *post*, No. 43), the forms are

$$\frac{1}{M}=\frac{-7u(1-uv)(1-uv+u^2v^2)}{u-v^7}\ \text{.....(1)},$$

$$-7M=\frac{-7v(1-uv)(1-uv+u^2v^2)}{v-u^7}\ \text{.....(2)},$$

$$\frac{v^4}{u^4M}=\frac{-7v^4(1-uv)(1-uv+u^2v^2)}{u^3(u-v^7)}\ \text{.....(3)},$$

$$-\frac{u^4}{v^4}7M=\frac{-7u^4(1-uv)(1-uv+u^2v^2)}{v^3(v-u^7)}\ \text{.....(4)};$$

so that here again the third and the fourth forms are identical with the second and the third forms respectively; there are thus only two forms, and the elimination of M gives

$$(u-v^7)(v-u^7)+7uv(1-uv)^2(1-uv+u^2v^2)^2=0,$$

which is a form of the modular equation.

36. If in the foregoing equation

$$F'v \cdot \frac{1}{M} = R(u, v),$$

we make the change u, v, $\frac{1}{M}$ into v, $\pm u$, $\pm nM$, it becomes

$$\pm F'u \cdot nM = R(v, \pm u);$$

combining these equations, we have

$$\pm nM^2 \cdot \frac{F'u}{F'v} = \frac{R(v, \pm u)}{R(u, v)};$$

or, substituting herein the foregoing value

$$M^2 = -\frac{1}{n} \frac{(1-v^8)\, vF'v}{(1-u^8)\, uF'u},$$

this becomes

$$\mp \frac{v(1-v^8)}{u(1-u^8)} = \frac{R(v, \pm u)}{R(u, v)} \qquad \begin{array}{l} + \text{ for } n \equiv 3 \text{ or } 5 \text{ (mod. 8)}, \\ - \text{ for } n \equiv 1 \text{ or } 7 \text{ (mod. 8)}, \end{array}$$

which must agree with the modular equation. Thus in the last-mentioned case $n = 3$, we have

$$\tfrac{1}{2} F'v \cdot \frac{1}{M} = 3(v^2u^2 + 2u^5v + 1)\, u,$$

or, say

$$R(u, v) = (v^2u^2 + 2u^5v + 1)\, u,$$

and therefore

$$R(v, -u) = (v^2u^2 - 2uv^5 + 1)\, v;$$

the equation is

$$+\frac{v(1-v^8)}{u(1-u^8)} = \frac{(v^2u^2 - 2uv^5 + 1)\, v}{(v^2u^2 + 2u^5v + 1)\, u},$$

which is right; because Jacobi, p. 82, [*Ges. Werke*, t. I., p. 137], for the modular equation, gives

$$1 - u^8 = (1 - u^2v^2)(v^2u^2 + 2u^5v + 1), \quad 1 - v^8 = (1 - u^2v^2)(v^2u^2 - 2uv^5 + 1).$$

Observe that the general equation

$$\mp \frac{v(1-v^8)}{u(1-u^8)} = \frac{R(v, \pm u)}{R(u, v)}$$

no longer contains the functions $F'v$, $F'u$, which enter into Jacobi's expression of M^2.

Theorem in connexion with the multiplication of Elliptic Functions. Art. Nos. 37 to 40.

37. The theory of multiplication gives an important theorem in regard to transformation. Starting with the $n$thic transformation

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left(\frac{\alpha - \beta x + \gamma x^2 - \ldots}{\alpha + \beta x + \gamma x^2 + \ldots}\right)^2, \quad = \frac{1-x}{1+x}\left(\frac{P - Qx}{P + Qx}\right)^2,$$

we may form a like transformation,

$$\frac{1-z}{1+z}=\frac{1-y}{1+y}\left(\frac{\alpha'-\beta'y+\gamma'y^2-\ldots}{\alpha'+\beta'y+\gamma'y^2+\ldots}\right)^2, \quad =\frac{1-y}{1+y}\left(\frac{P'-Q'y}{P'+Q'y}\right)^2,$$

such that the combination of the two gives a multiplication, viz. for the relation between y, z, deriving w from v as v from u, we have $w=u$; and instead of M we have M', $=\pm\dfrac{1}{nM}$; that is, we have

$$\frac{dx}{\sqrt{1-x^2.1-u^8x^2}}=\frac{Mdy}{\sqrt{1-y^2.1-v^8y^2}},$$

$$\frac{dy}{\sqrt{1-y^2.1-v^8y^2}}=\frac{M'dz}{\sqrt{1-z^2.1-w^8z^2}},$$

and thence

$$\frac{dx}{\sqrt{1-x^2.1-u^8x^2}}=\frac{\pm\frac{1}{n}dz}{\sqrt{1-z^2.1-w^8z^2}};$$

or, writing $x=\operatorname{sn}\theta$, we have $z=\pm\operatorname{sn}n\theta$; $\pm$ is here $(-)^{\frac{n-1}{2}}$, viz. it is $-$ for $n\equiv 3$ or 7 (mod. 8), and $+$ for $n\equiv 1$ or 5 (mod. 8).

Now in part effecting the substitution, we have

$$\frac{1-z}{1+z}=\frac{1-x}{1+x}\left(\frac{P-Qx}{P+Qx}\right)^2\cdot\left(\frac{P'-Q'y}{P'+Q'y}\right)^2,$$

where y denotes its value in terms of x.

And from the theory of elliptic functions, replacing $\operatorname{sn}n\theta$, $\operatorname{sn}\theta$ by their values $\pm z$, x, we have an equation

$$\frac{1-z}{1+z}=\frac{1-x}{1+x}\left(\frac{A-Bx+Cx^2-\ldots}{A+Bx+Cx^2+\ldots}\right)^2,$$

where $A-Bx+Cx^2-\ldots$, $A+Bx+Cx^2+\ldots$ are given functions each of the order $\frac{1}{2}(n^2-1)$; viz. the coefficients are given functions of k, or, what is the same thing, of u^4. Comparing the two results, we see that in the $n$thic transformation the sought-for function, $\alpha+\beta x+\gamma x^2+\ldots$ of the order $\frac{1}{2}(n-1)$, is a factor of a given function $A+Bx+Cx^2+\ldots$ of the order $\frac{1}{2}(n^2-1)$.

38. Considering the modular equation as known, then by what precedes we have

$$\alpha+\beta x+\gamma x^2+\ldots=\alpha\left\{1+\frac{\beta}{\alpha}x+\ldots+\frac{u^n}{v}x^{\frac{1}{2}(n-1)}\right\};$$

that is, the given function $A+Bx+Cx^2+\ldots$ has a factor $1+\dfrac{\beta}{\alpha}x+\ldots+\dfrac{u^n}{v}x^{\frac{1}{2}(n-1)}$, of which one (the last) coefficient $\dfrac{u^n}{v}$ is known, and we are hence able theoretically to

determine all the other coefficients rationally in terms of u, v; that is, the modular equation being known, we can theoretically complete the solution of the transformation problem. I do not, however, see the way to obtaining a convenient solution in this manner.

39. The formula in question for $n=3$ is

$$\frac{1+\operatorname{sn}3\theta}{1-\operatorname{sn}3\theta}=\frac{1-\operatorname{sn}\theta}{1+\operatorname{sn}\theta}\left(\frac{1+2\operatorname{sn}\theta-2k^2\operatorname{sn}^3\theta-k^2\operatorname{sn}^4\theta}{1-2\operatorname{sn}\theta+2k^2\operatorname{sn}^3\theta-k^2\operatorname{sn}^4\theta}\right)^2,$$

which, putting therein $x=\operatorname{sn}\theta$, $z=-\operatorname{sn}3\theta$, and replacing k by u^4, may be written

$$1+z\,(\div)=(1+x)\,(1-2x+2u^8x^3-u^8x^4)^2\,(\div),$$

where the signs $(\div)$ indicate denominators which are obtained from the numerators by changing the signs of z, x respectively.

The theorem in regard to $n=3$ thus is, $1+\frac{u^2}{v}x$ is a factor of $1-2x+2u^8x^3-u^8x^4$; viz. writing in the last-mentioned function $x=-\frac{v}{u^3}$, we ought to have

$$0=1+2\frac{v}{u^3}-2\frac{v^3}{u}-\frac{v^4}{u^4},$$

that is,

$$u^4+2uv-2u^3v^3-v^4=0,$$

which is in fact the modular equation.

40. And so for $n=5$, if $x=\operatorname{sn}\theta$, $z=\operatorname{sn}5\theta$; and for $n=7$, if $x=\operatorname{sn}\theta$, $z=-\operatorname{sn}7\theta$; the formulæ are:—

$n=5$,

$$\begin{array}{lll}
1+z=(1+x)\,\{ & 1 & \\
(\div) & +\ 2 & x \\
 & -\ 4 & x^2 \\
 & -10u^8 & x^3 \\
 & +\ 5u^8 & x^4 \\
 & +4u^8\,(3+2u^8) & x^5 \\
 & +4u^8\,(1-\ u^8) & x^6 \\
 & -4u^8\,(2+3u^8) & x^7 \\
 & -\ 5u^{16} & x^8 \\
 & +10u^{16} & x^9 \\
 & +\ 4u^{24} & x^{10} \\
 & -\ 2u^{24} & x^{11} \\
 & -\ \ u^{24} & x^{12}\}^2\div
\end{array}$$

$n=7$,

$$\begin{array}{lll}
1+z=(1+x)\,\{ & 1 & \\
(\div) & -\ 4 & x \\
 & -\ 4 & x^2 \\
 & +\ 4\,(2+7u^8) & x^3 \\
 & -14u^8 & x^4 \\
 & -28u^8\,(3+2u^8) & x^5 \\
 & +28u^8\,(4+\ u^8) & x^6 \\
 & +\ 4u^8\,(\ 16+\ 51u^8+\ \ 8u^{16}) & x^7 \\
 & -\ \ u^8\,(144+305u^8+\ 16u^{16}) & x^8 \\
 & -\ 8u^8\,(\ \ 4+\ 25u^8+\ 16u^{16}) & x^9 \\
 & +\ 8u^8\,(\ \ 8+\ 57u^8+\ 46u^{16}) & x^{10} \\
 & +56u^{16}\,(\ 2+\ \ u^8) & x^{11} \\
 & -\ 4u^{16}\,(\ 56+161u^8+\ 56u^{16}) & x^{12}
\end{array}$$

Term in { } has factor

$$1+\frac{\beta}{\alpha}x+\frac{u^5}{v}x^2;$$

$u=1$, term in { } is

$$=(1+x)^7(1-x)^5.$$

$$\begin{aligned}
&+56u^{24}(1+2u^8\qquad\qquad)\,x^{13}\\
&+\ 8u^{24}(\ 46+\ 57u^8+\ \ 8u^{16})\,x^{14}\\
&-\ 8u^{24}(\ 16+\ 25u^8+\ \ 4u^{16})\,x^{15}\\
&-\ \ u^{24}(\ 16+305u^8+144u^{16})\,x^{16}\\
&+\ 4u^{24}(\ \ 8+\ 51u^8+\ 16u^{16})\,x^{17}\\
&+28u^{32}(\ \ 1+\ \ 4u^8)\qquad\qquad x^{18}\\
&-28u^{32}(\ \ 2+\ \ 3u^8)\qquad\qquad x^{19}\\
&-14u^{40}\qquad\qquad\qquad\qquad x^{20}\\
&+\ 4u^{40}(\ \ 7+\ \ 2u^8)\qquad\qquad x^{21}\\
&-\ 4u^{48}\qquad\qquad\qquad\qquad x^{22}\\
&-\ 4u^{48}\qquad\qquad\qquad\qquad x^{23}\\
&+\ \ u^{48}\qquad\qquad\qquad\qquad x^{24}\}^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\div)
\end{aligned}$$

Term in { } has factor

$$1+\frac{\beta}{\alpha}x+\frac{\gamma}{\alpha}x^2+\frac{u^7}{v}x^3;$$

$u=1$, term in { } is $(1+x)^{10}(1-x)^{14}$.

The transformations $n=3, 5, 7, 11$. Art. Nos. 41 to 51.

41. The cubic transformation, $n=3$.

I reproduce the results already obtained; since there are only two coefficients α, β, these are also the last but one and last coefficients ρ, σ. Hence, from the values of α, β, ρ, σ, we have

$$\alpha=1,\qquad 2\alpha=\frac{v^3}{u}\left(\frac{1}{M}-\frac{u^4}{v^4}\right),$$

$$2\beta=\frac{1}{M}-1,\qquad \beta=\frac{u^3}{v};$$

the two values of $\frac{1}{M}$ are thus $\frac{1}{M}=2\frac{u}{v^3}+\frac{u^4}{v^4}$, $=1+\frac{2u^3}{v}$, giving the modular equation

$$v^4+2v^3u^3-2vu-u^4=0;$$

and we then have

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{v-u^3x}{v+u^3x}\right)^2.$$

42. The quintic transformation, $n=5$.

Here there are the three coefficients α, β, γ, or β, γ are the last but one and last coefficients ρ, σ; we have

$$\alpha = 1, \qquad 2\beta = v^2 u\left(\frac{1}{M} - \frac{u^4}{v^4}\right),$$

$$2\beta = \frac{1}{M} - 1, \qquad \gamma = \frac{u^5}{v}.$$

Comparing the two values of β, we have $\frac{1}{M} = \frac{v - u^5}{v(1 - v^2 u)}$, and then

$$\alpha = 1, \quad 2\beta = \frac{u(v^4 - u^4)}{v(1 - v^2 u)}, \quad \gamma = \frac{u^5}{v},$$

so that only the modular equation remains to be determined.

The unused equation is

$$2\alpha\gamma + 2\alpha\beta + \beta^2 = \frac{v^2}{u^2}(2\alpha\gamma + 2\beta\gamma + \beta^2),$$

which, putting therein $\alpha = 1$, may be written

$$(2\gamma + \beta^2)(u^2 - v^2) = 2\beta(\gamma v^2 - u^2);$$

attending to the value of β, this divides by $u^2 - v^2$; in fact the equation may be written

$$2\gamma + \beta^2 = -\frac{u(v^2 + u^2)}{v(1 - v^2 u)}(\gamma v^2 - u^2);$$

and then completing the substitution, and integralizing, this becomes

$$\{8vu^3(1 - v^3 u)^2 + (v^4 - u^4)^2\} = 4uv(u^2 + v^2)(1 - u^3 v)(1 - uv^3),$$

viz. this is

$$4(1 - v^3 u)uv\{2u^2(1 - v^3 u) - (u^2 + v^2)(1 - vu^3)\} + (v^4 - u^4)^2 = 0;$$

and the term in { } being $= -(v^2 - u^2)(1 + vu^3)$, the whole again divides by $v^2 - u^2$, and the equation thus becomes

$$(v^2 + u^2)(v^4 - u^4) - 4uv(1 - v^3 u)(1 + vu^3) = 0,$$

which is the modular equation.

43. The septic transformation, $n = 7$.

I do not propose to complete the solution directly from the fundamental equations for α, β, γ, δ, but resort to the known modular equation, and to an expression of M which I obtain by means thereof.

The modular equation is

$$(1 - u^8)(1 - v^8) - (1 - uv)^8 = 0,$$

which may also be written

$$(v-u^7)(u-v^7)+7uv(1-uv)^2(1-uv+u^2v^2)^2=0,$$

as can be at once verified; but it also follows from Cauchy's identity

$$(x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)^2.$$

We then have

$$M^2=-\frac{1}{n}\frac{(1-v^8)\,vF'v}{(1-u^8)\,uF'u}.$$

Moreover

$$\begin{aligned} uF'u &= -u^8(1-v^8)+uv(1-uv)^7 \\ &= \frac{-u^8}{1-u^8}(1-uv)^8+uv(1-uv)^7 \\ &= \frac{(1-uv)^7}{1-u^8}u(v-u^7); \end{aligned}$$

and similarly

$$F'v=\frac{(1-uv)^7}{1-v^8}v(u-v^7),$$

whence

$$\frac{1}{M^2}=\frac{-7u}{v}\frac{(v-u^7)}{u-v^7}.$$

Writing this under the form

$$\frac{1}{M^2}=\frac{-7uv}{v^2}\frac{(v-u^7)(u-v^7)}{(u-v^7)^2},\quad =\frac{49u^2(1-uv)^2(1-uv+u^2v^2)^2}{(u-v^7)^2},$$

I find, as will appear, that the root must be taken with the sign $-$, and that we thus have $\frac{1}{M}=-\frac{7u(1-uv)(1-uv+u^2v^2)}{u-v^7}$, whence also $M=\frac{v(1-uv)(1-uv+u^2v^2)}{v-u^7}$.

44. Recurring now to the fundamental equations for the septic transformation, the coefficients are α, β, γ, δ, and we have

$$\alpha=1,\qquad 2\gamma=u^3v^3\left(\frac{1}{M}-\frac{u^4}{v^4}\right),$$

$$2\beta=\frac{1}{M}-1,\quad \delta=\frac{u^7}{v},$$

so that the coefficients are all given in terms of v, M. The unused equations are

$$\begin{aligned} u^6(2\alpha\gamma+2\alpha\beta+\beta^2) &= v^2(\gamma^2+2\gamma\delta+2\beta\delta), \\ u^{-2}(\gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta) &= v^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2), \end{aligned}$$

which, substituting therein for α, β, γ, δ the foregoing values, give two equations; from these, eliminating M, we should obtain the modular equation, and then M in terms of u, v.

Substituting in the first instance for α, δ their values, the equations are

$$u^6(2\beta+2\gamma+\beta^2)=v^2\left\{\gamma^2+2\frac{u^7}{v}(\beta+\gamma)\right\}$$

$$\gamma^2+2\beta\gamma+(2+2\beta)\frac{u^7}{v}=u^2v^2\left\{2\gamma+2\beta\gamma+2\frac{u^7}{v}+\beta^2\right\}.$$

The first of these is

$$4(1-uv)(2\beta+2\gamma)+4\beta^2-4\frac{v^2}{u^6}\gamma^2=0,$$

viz. this is

$$4(1-uv)\left(\frac{1}{M}-1+\frac{u^3v^3}{M}-\frac{u^7}{v}\right)+\left(\frac{1}{M}-1\right)^2-v^8\left(\frac{1}{M}-\frac{u^4}{v^4}\right)^2=0;$$

or observing that in this equation the coefficient of $\frac{2}{M}$ is

$$(1-u^2v^2)\{2-2uv+2u^2v^2-1-u^2v^2\},$$
$$=(1-u^2v^2)(1-uv)^2,\quad =(1-uv)^3(1+uv),$$

the equation becomes

$$(1-v^8)\frac{1}{M^2}+\frac{2}{M}(1-uv)^3(1+uv)+1-u^8-4(1-uv)\left(1+\frac{u^7}{v}\right)=0.$$

45. This should be satisfied identically by the foregoing value of $\frac{1}{M}$; viz. it should be satisfied on writing therein

$$\frac{1}{M^2}=-\frac{7u}{v}\,\frac{v-u^7}{u-v^7},$$

$$\frac{1}{M}=-\frac{7u(1-uv)(1-uv+u^2v^2)}{u-v^7};$$

that is, we should have

$$-7\frac{u}{v}(v-u^7)(1-v^8)-14u(1-uv)^4(1+u^3v^3)$$
$$+(u-v^7)\left\{1-u^8-4(1-uv)\left(1+\frac{u^7}{v}\right)\right\}=0,$$

where observe that the $-$ sign of the second term is the sign of the foregoing value of $\frac{1}{M}$; so that the identity being verified, it follows that the correct sign has been attributed to the value of $\frac{1}{M}$.

46. Multiplying by v, the equation is

$$-7(1-u^8-\overline{1-uv})(1-v^8)-14uv(1-uv)^4(1+u^3v^3)$$
$$+\{1-v^8-\overline{1-uv}\}\{-8(1-uv)+1-u^8\}+4(1-uv)(v-u^7)(u-v^7)=0,$$

viz. this is

$$\begin{aligned}&-7(1-u^8)(1-v^8)+7(1-uv)(1-v^8)-14uv(1-uv)^4(1+u^3v^3)\\&+\ \ (1-u^8)(1-v^8)-8(1-uv)(1-v^8)+\ \ 8\ \ \ (1-uv)^2\\&\qquad\qquad -1(1-uv)(1-u^8)+\ \ 4\ \ \ (1-uv)(v-u^7)(u-v^7)=0.\end{aligned}$$

In the second column the coefficient of $1-uv$ is $2-u^8-v^8$, viz. this is

$$=(1-u^8)(1-v^8)+1-(uv)^8, \text{ or it is } =(1-uv)^8+1-(uv)^8.$$

Reducing also the other two columns by means of the modular equation, the equation thus becomes

$$\begin{aligned}-6(1-uv)^8-(1-uv)\{(1-uv)^8+1-(uv)^8\}&-14uv(1-uv)^4(1+u^3v^3)\\&+\ \ 8\ \ \ (1-uv)^2\\&-28uv(1-uv)^3(1-uv+u^2v^2)^2=0.\end{aligned}$$

This is in fact an identity; to show it, writing for convenience θ in place of uv, and observing that the terms

$$\begin{aligned}&-(1-\theta)(1-\theta^8)+8(1-\theta)^2,\\=\ &(1-\theta)^2\{8-(1+\theta+\theta^2+\theta^3+\theta^4+\theta^5+\theta^6+\theta^7)\}\end{aligned}$$

are

$$=\ (1-\theta)^3(7+6\theta+5\theta^2+4\theta^3+3\theta^4+2\theta^5+\theta^6),$$

the whole equation divides by $(1-\theta)^3$; or throwing out this factor, it is

$$\begin{aligned}-6(1-\theta)^5-(1-\theta)^6+7+6\theta+5\theta^2+4\theta^3+3\theta^4+2\theta^5+\theta^6&\\-14\theta(1-\theta)(1+\theta^3)-28\theta(1-\theta+\theta^2)^2=0.&\end{aligned}$$

The first line is $=14\theta(3-5\theta+6\theta^2-3\theta^3+\theta^4)$; whence, throwing out the factor 14θ, the equation is

$$3-5\theta+6\theta^2-3\theta^3+\theta^4-(1-\theta)(1+\theta^3)-2(1-\theta+\theta^2)^2,$$

that is,

$$(1-\theta+\theta^2)(3-2\theta+\theta^2)-(1-\theta^2)(1-\theta+\theta^2)-2(1-\theta+\theta^2)^2=0;$$

or throwing out the factor $1-\theta+\theta^2$, the equation is

$$(3-2\theta+\theta^2)-(1-\theta^2)-2(1-\theta+\theta^2)=0,$$

which is an identity.

The other equation is

$$\gamma^2+2\beta\gamma+(2+2\beta)\frac{u^7}{v}=u^2v^2\left(2\gamma+2\beta\gamma+2\frac{u^7}{v}+\beta^2\right);$$

that is,

$$\gamma^2+2\beta\gamma-u^2v^2\beta^2+2(1+\beta)\left(\frac{u^7}{v}-\gamma u^2v^2\right)-2u^9v=0,$$

which might also be verified, but I have not done this.

47. The conclusion is

$$\alpha = 1, \quad \beta = \tfrac{1}{2}\left(\frac{1}{M} - 1\right), \quad \gamma = \tfrac{1}{2}u^3v^3\left(\frac{1}{M} - \frac{u^4}{v^4}\right), \quad \delta = \frac{u^7}{v},$$

where

$$\frac{1}{M} = \frac{-7u(1-uv)(1-uv+u^2v^2)}{u-v^7},$$

and of course

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left(\frac{1-\beta x+\gamma x^2-\delta x^3}{1+\beta x+\gamma x^2+\delta x^3}\right)^2;$$

but the resulting form may admit of simplification.

48. The endecadic transformation, $n = 11$.

I have not completed the solution, but the results, so far as I have obtained them, are interesting. The coefficients are α, β, γ, δ, ϵ, ζ; and we have, as in general,

$$\alpha = 1 \quad , \quad 2\epsilon = u^7v^3\left(\frac{1}{M} - \frac{u^4}{v^4}\right),$$

$$2\beta = \frac{1}{M} - 1, \quad \zeta = \frac{u^{11}}{v}.$$

The unused equations then are

$$u^{14}(2\alpha\gamma + 2\alpha\beta + \beta^2) = v^2(\epsilon^2 + 2\epsilon\zeta + 2\delta\zeta),$$

$$u^6(\gamma^2 + 2\alpha\epsilon + 2\alpha\delta + 2\beta\gamma + 2\beta\delta) = v^2(2\gamma\epsilon + 2\gamma\zeta + 2\delta\epsilon + 2\beta\zeta + \delta^2),$$

$$u^{-2}(2\gamma\epsilon + 2\alpha\zeta + 2\gamma\delta + 2\beta\epsilon + 2\beta\zeta + \delta^2) = v^2(\gamma^2 + 2\alpha\epsilon + 2\alpha\zeta + 2\gamma\delta + 2\beta\epsilon + 2\beta\delta),$$

$$u^{-10}(\epsilon^2 + 2\gamma\zeta + 2\delta\epsilon + 2\delta\zeta) = v^2(2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + \beta^2);$$

but I attend only to the first and the last, which, it will be observed, contain γ, δ linearly. If in the first instance we substitute only for α, ζ their values, the equations become

$$u^{12}\beta(2+\beta) - \frac{v^2}{u^2}\epsilon\left(\epsilon + 2\frac{u^{11}}{v}\right) \qquad + u^{12}.2\gamma \qquad - vu^9.2\delta = 0,$$

$$u^{-12}\epsilon^2 \qquad - \frac{v^2}{u^2}\beta^2 \qquad + \left\{\frac{1}{uv} - \frac{v^2}{u^2}(1+\beta)\right\}.2\gamma + \left\{\frac{1}{uv} - \frac{v^2}{u^2} + u^{-12}\epsilon\right\}.2\delta = 0;$$

say, for a moment, these are

$$A + P.2\gamma + Q.2\delta = 0,$$

$$B + R.2\gamma + S.2\delta = 0,$$

giving

$$1 : 2\gamma : 2\delta = PS - QR : QB - SA : RA - PB.$$

Here

$$PS - QR = \frac{u^{11}}{v} + \epsilon - u^{10}v^2 + u^8 - u^7v^3(1+\beta)$$

$$= \tfrac{1}{2}\left\{\frac{2u^{11}}{v} + \left(u^7v^3\frac{1}{M} - \frac{u^{11}}{v}\right) - 2u^{10}v^2 + 2u^8 - 2u^7v^3 - \left(u^7v^3\frac{1}{M} - u^7v^3\right)\right\},$$

where the terms containing $\frac{1}{M}$ disappear of themselves, viz. this is

$$= \tfrac{1}{2}\left(\frac{u^{11}}{v} - 2u^{10}v^2 + 2u^8 - u^7v^3\right)$$

$$= -\tfrac{1}{2}\frac{u^7}{v}(v^4 + 2v^3u^3 - 2vu - u^4);$$

observe that the term in ( ), equated to zero, gives the modular equation for the case $n=3$. It thus appears that γ and δ are given as fractions, having in their denominator this function $u^4 + 2uv - 2u^3v^3 - v^4$.

49. To complete the calculation, we have

$$QB - AS = -vu^9\left(\frac{\epsilon^2}{u^{12}} - \frac{v^2}{u^2}\beta^2\right)$$
$$-\left\{u^{12}\beta(2+\beta) - \frac{v^2}{u^2}\epsilon\left(\epsilon + 2\frac{u^{11}}{v}\right)\right\}\left\{\frac{1}{uv} - \frac{v^2}{u^2} + \frac{\epsilon}{u^{12}}\right\};$$

viz. multiplying by 8, and substituting for 2β, 2ϵ their values, this is

$$8(QB - AS) = -2vu^9\left\{u^2v^6\left(\frac{1}{M} - \frac{u^4}{v^4}\right)^2 - \frac{v^2}{u^2}\left(\frac{1}{M} - 1\right)^2\right\}$$
$$-\left\{u^{12}\left(\frac{1}{M} - 1\right)\left(\frac{1}{M} + 3\right) - u^{12}v^8\left(\frac{1}{M} - \frac{u^4}{v^4}\right)\left(\frac{1}{M} + \frac{3u^4}{v^4}\right)\right\}\left(\frac{1}{uv} - \frac{2v^2}{u^2} + \frac{v^3}{u^5}\frac{1}{M}\right),$$

or, what is the same thing,

$$-\frac{8v}{u^7}(QB - AS) = 2\left\{u^4v^8\left(\frac{1}{M} - \frac{u^4}{v^4}\right)^2 - v^4\left(\frac{1}{M} - 1\right)^2\right\}$$
$$+\left\{\left(\frac{1}{M} - 1\right)\left(\frac{1}{M} + 3\right) - v^8\left(\frac{1}{M} - \frac{u^4}{v^4}\right)\left(\frac{1}{M} + 3\frac{u^4}{v^4}\right)\right\}\left(u^4 - 2v^3u^3 + \frac{v^4}{M}\right),$$

viz. the left-hand side is

$$= 2\left\{-v^4(1 - u^4v^4)\frac{1}{M^2} + 2v^4(1 - u^8)\frac{1}{M} + u^{12} - v^4\right\}$$
$$+\left\{\frac{1}{M^2}(1 - v^8) + \frac{2}{M}(1 - u^4v^4) - 3(1 - u^8)\right\}\left\{\frac{1}{M} + u^3(u - 2v^3)\right\};$$

or, say we have $-\frac{8v}{u^7}(QB - SA) = \Pi$, where

$$\Pi = \frac{1}{M^2}\cdot v^4(1 - v^8)$$
$$+\frac{1}{M^2}\cdot u^3(u - 2v^3)(1 - v^8)$$
$$+\frac{1}{M}\cdot 4u^7v^2 + v^4(1 - 3u^8) - 4u^3v^3 + 2u^4$$
$$+ \quad \cdot -2v^4 + 6v^3u^3(1 - u^8) + u^4(-3 + 5u^8);$$

wherefore the value of 2γ is $=\frac{1}{2}\Pi\div(v^4+2v^3u^3-2vu-u^4)$. Similarly, writing

$$\begin{aligned}\Pi' = & \frac{1}{M^3}\cdot v^4(1-v^8)\\ & +\frac{1}{M^2}\cdot v(v^3-2u)(1-v^8)\\ & +\frac{1}{M}\cdot 4u^5v^5+v^4(3-u^8)+4uv-2u^4v^8\\ & + \quad v^4(-5+3u^8)+6vu(1-u^8)+2u^{12},\end{aligned}$$

we find

$$2\delta = \frac{1}{4}\frac{u^3}{v}\Pi'\div(v^4+2v^3u^3-2vu-u^4);$$

in verification whereof observe that this being so, the first equation gives the identity

$$\left\{\left(\frac{1}{M}-1\right)\left(\frac{1}{M}+3\right)-v^8\left(\frac{1}{M}-\frac{u^4}{v^4}\right)\left(\frac{1}{M}+\frac{3u^4}{v^4}\right)\right\}(v^4+2v^3u^3-2vu-u^4)+\Pi-\Pi'=0.$$

50. The result is that, writing for the moment $v^4+2v^3u^3-2vu-u^4=\Delta$, the values of the coefficients are

$$\begin{array}{cccccc}\alpha, & \beta\ , & \gamma\ , & \delta\ , & \epsilon\ , & \zeta\ ,\\ =1, & \frac{1}{2}\left(\frac{1}{M}-1\right), & \frac{1}{8}\frac{\Pi}{\Delta}, & \frac{1}{8}\frac{u^3}{v}\frac{\Pi'}{\Delta}, & \frac{1}{2}u^7v^3\left(\frac{1}{M}-\frac{u^4}{v^4}\right), & \frac{u^{11}}{v},\end{array}$$

and

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\beta x+\gamma x^2-\delta x^3+\epsilon x^4-\zeta x^5}{1+\beta x+\gamma x^2+\delta x^3+\epsilon x^4+\zeta x^5}\right)^2;$$

the modular equation is known, and to complete the solution we require only an expression for M in terms of u, v.

51. We may herein illustrate the following theorem, viz. we may simultaneously change u, v, $\frac{1}{M}$, $\alpha:\beta:\gamma:\delta:\epsilon:\zeta$; into $\frac{1}{u}$, $\frac{1}{v}$, $\frac{v^4}{u^4}\frac{1}{M}$, $\zeta:\epsilon:\delta:\gamma:\beta:\alpha$.

Thus making the change in the equation

$$\frac{\beta}{\alpha}=\frac{1}{2}\left(\frac{1}{M}-1\right),$$

we have

$$\frac{\epsilon}{\zeta}=\frac{1}{2}\left(\frac{v^4}{u^4}\frac{1}{M}-1\right);\ \text{that is, } \frac{1}{2}\frac{v^4}{u^4}\left(\frac{1}{M}-\frac{u^4}{v^4}\right)=\frac{1}{2}\left(\frac{v^4}{u^4}\frac{1}{M}-1\right),$$

which is right.

So in the equation $\frac{\gamma}{\alpha}=\frac{1}{8}\frac{\Pi}{\Delta}$, if for a moment (Π), (Δ) are what Π, Δ become, the

equation is $\frac{\delta}{\zeta}=\frac{1}{8}\frac{(\Pi)}{(\Delta)}$, that is, $\frac{1}{u^8}\frac{\Pi'}{\Delta}=\frac{(\Pi)}{(\Delta)}$, or $(\Pi)=\frac{1}{u^8}\frac{(\Delta)}{\Delta}\Pi'$; but obviously $\frac{(\Delta)}{\Delta}=-\frac{1}{u^4v^4}$; and the equation thus is $(\Pi)=-\frac{1}{u^{12}v^4}\Pi'$, or say $u^{12}v^4(\Pi)=-\Pi'$; that is,

$$\begin{aligned}-\Pi'=u^{12}v^4\Big\{&\frac{v^{12}}{u^{12}}\cdot\frac{1}{M^2}\cdot\frac{1}{v^4}\left(1-\frac{1}{v^3}\right)\\&+\frac{v^8}{u^8}\cdot\frac{1}{M^2}\cdot\frac{1}{u^3}\left(\frac{1}{u}-\frac{2}{v^3}\right)\left(1-\frac{1}{v^3}\right)\\&+\frac{v^4}{u^4}\cdot\frac{1}{M}\cdot\frac{4}{u^7v^7}+\frac{1}{v^4}\left(1-\frac{3}{u^3}\right)-\frac{4}{u^5v^3}+\frac{2}{u^4}\\&\qquad-\frac{2}{v^4}+\frac{6}{v^3u^3}\left(1-\frac{1}{u^3}\right)+\frac{1}{u^4}\left(-3+\frac{5}{u^3}\right)\Big\},\end{aligned}$$

which is right.

The general theory by q-transcendents. Art. Nos. 52 to 71.

52. I recur to the formula

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x+\gamma x^2+\ldots\pm\sigma n^{\frac{1}{2}(n-1)}}{\alpha+\beta x+\gamma x^2+\ldots+\sigma n^{\frac{1}{2}(n-1)}}\right)^2,$$

and seek to express the ratios $\alpha:\beta:\ldots:\sigma$ in terms of q. Writing with Jacobi

$$\omega=\frac{mK+m'iK'}{n},$$

we have in general

$$\alpha+\beta x+\gamma x^2+\ldots+\sigma x^{\frac{1}{2}(n-1)}=\alpha\left(1+\frac{x}{\operatorname{snc}2\omega}\right)\left(1+\frac{x}{\operatorname{snc}4\omega}\right)\ldots\left(1+\frac{x}{\operatorname{snc}(n-1)\omega}\right),$$

$$(\operatorname{snc}=\sin\operatorname{co}\operatorname{am};\ \text{viz. } \operatorname{snc}2\omega=\operatorname{sn}(K-2\omega),\ \&c.);$$

and the values of $\alpha, \beta, \ldots, \theta$ which correspond to the moduli $v_0, v_1, \ldots v_n$, or say the values $(\alpha_0, \beta_0, \ldots, \theta_0)$, $(\alpha_1, \beta_1, \ldots, \theta_1), \ldots, (\alpha_n, \beta_n, \ldots, \theta_n)$, are obtained by giving to ω the values

$$\begin{aligned}&\omega_0\ ,\quad \omega_1\ ,\quad \omega_2\ ,\ \ldots,\ \omega_n\ ,\\&=\frac{2K}{n},\ \frac{2K+iK'}{n},\ \frac{4K+iK'}{n},\ \ldots,\ \frac{iK'}{n},\end{aligned}$$

viz. the cases ω_0, ω_n correspond to Jacobi's first and second real transformations, and the others to the imaginary transformations.

I remark that $\omega=\omega_0$ gives for $\operatorname{snc}2g\omega$ an expression which is rational as regards q, but $\omega=\omega_n$ gives an expression involving $q^{\frac{1}{n}}$, the real nth root of q; the other values ω_1, $\omega_2, \ldots$ give the like expressions, involving $\alpha q^{\frac{1}{n}}$, $\alpha^2 q^{\frac{1}{n}}, \ldots$ (α an imaginary nth root of unity), the imaginary nth roots of q.

53. I consider first the expression

$$\frac{1}{\operatorname{snc} 2g\omega_0}, \quad = \frac{1}{\operatorname{sn}(K - 2g\omega_0)}, \quad = \frac{\operatorname{dn} 2g\omega_0}{\operatorname{cn} 2g\omega_0}.$$

Here, writing $2g\omega_0 = \frac{2K\xi}{\pi}$ (ξ for Jacobi's x, as x is being used in a different sense), that is,

$$\xi = \frac{\pi}{2K} \cdot 2g \cdot \frac{2K}{n}, \quad = \frac{2g\pi}{n},$$

(and thence $e^{i\xi} = e^{g\frac{2\pi i}{n}} = \alpha^g$, $e^{2i\xi} = \alpha^{2g}$, if $\alpha = e^{\frac{2\pi i}{n}}$, an imaginary nth root of unity), we have (Jacobi, p. 86, [*Ges. Werke*, t. I., p. 143])

$$\frac{1}{\operatorname{snc} 2g\omega_0} = \operatorname{dn} \frac{2K\xi}{\pi} \div \operatorname{cn} \frac{2K\xi}{\pi}$$

$$= \frac{C}{B} \cdot \frac{2e^{i\xi}}{1 + e^{2i\xi}} \cdot \frac{(1 + qe^{2i\xi}) \ldots (1 + qe^{-2i\xi}) \ldots}{(1 + q^2 e^{2i\xi}) \ldots (1 + q^2 e^{-2i\xi}) \ldots},$$

where

$$\frac{C}{B} = \left\{\frac{(1 + q^2) \ldots}{(1 + q) \ldots}\right\}^2 = f^2(q);$$

that is,

$$\frac{1}{\operatorname{snc} 2g\omega_0} = \frac{2\alpha^g}{1 + \alpha^{2g}} \cdot f^2(q) \cdot \frac{(1 + \alpha^{2g} q) \ldots (1 + \alpha^{n-2g} q) \ldots}{(1 + \alpha^{2g} q^2) \ldots (1 + \alpha^{n-2g} q) \ldots},$$

where, for shortness, I write $(1 + qe^{2i\xi}) \ldots$ to denote the infinite product

$$(1 + qe^{2i\xi})(1 + q^3 e^{2i\xi})(1 + q^5 e^{2i\xi}) \ldots,$$

and similarly $(1 + q^2 e^{2i\xi}) \ldots$ to denote the infinite product $(1 + q^2 e^{2i\xi})(1 + q^4 e^{2i\xi})(1 + q^6 e^{2i\xi}) \ldots$, and the like for the terms in $e^{-2i\xi}$: the notation, accompanied by its explanation, is quite intelligible, and it would be difficult to make one which would be at the same time complete and not cumbrous. Then attributing to g the values $1, 2, \ldots, \frac{1}{2}(n-1)$, and forming the symmetric functions of these expressions, we have the values of $\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}$, &c., or α being put $= 1$, say the values of $\beta, \gamma, \ldots, \sigma$.

54. I stop to notice a verification afforded by the value of β_0. Putting $u = 0$, that is, $q = 0$, we have

$$\frac{1}{\operatorname{snc} 2g\omega_0} = \frac{2\alpha^g}{1 + \alpha^{2g}},$$

and thence

$$\beta_0 = 2\left\{\frac{\alpha}{1 + \alpha^2} + \frac{\alpha^2}{1 + \alpha^4} + \frac{\alpha^3}{1 + \alpha^6} + \ldots + \frac{\alpha^{\frac{1}{2}(n-1)}}{1 + \alpha^{n-1}}\right\},$$

we have $2\beta_0 = \frac{1}{M_0} - 1$; and putting as above $u = 0$, the value of $\frac{1}{M_0}$ is $= (-)^{\frac{n-1}{2}} n$; whence

$$(-)^{\frac{1}{2}(n-1)} n - 1 = 4\left\{\frac{\alpha}{1 + \alpha^2} + \frac{\alpha^2}{1 + \alpha^4} + \frac{\alpha^3}{1 + \alpha^6} + \ldots + \frac{\alpha^{\frac{1}{2}(n-1)}}{1 + \alpha^{n-1}}\right\},$$

a theorem relating to the imaginary nth roots of unity, n an odd prime. In particular,

$$n=3,\quad -4=4\left\{\frac{\alpha}{1+\alpha^2}\right\},\text{ at once verified by } \alpha^2+\alpha+1=0;$$

$$n=5,\quad 4=4\left\{\frac{\alpha}{1+\alpha^2}+\frac{\alpha^2}{1+\alpha^4}\right\},\text{ verified by } \alpha^5-1=0,$$

viz. the theorem is also true for the real root $\alpha=1$; in fact, the term in { } is

$$\{\alpha(1+\alpha^4)+\alpha^2(1+\alpha^2)\}\div(1+\alpha^2)(1+\alpha^4),\text{ that is, } (\alpha+1+\alpha^2+\alpha^4)\div(1+\alpha^2+\alpha^4+\alpha),=1;$$

$$n=7,\quad -8=4\left\{\frac{\alpha}{1+\alpha^2}+\frac{\alpha^2}{1+\alpha^4}+\frac{\alpha^3}{1+\alpha^6}\right\},$$

which may be verified by means of $\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$; and so on.

55. I further remark that we have

$$\frac{1}{M_0}=(-)^{\frac{1}{2}(n-1)}\left\{\frac{\operatorname{sn}2\omega_0\,.\,\operatorname{sn}4\omega_0\,..\,\operatorname{sn}(n-1)\omega_0}{\operatorname{snc}2\omega_0\,.\,\operatorname{snc}4\omega_0\,..\,\operatorname{snc}(n-1)\omega_0}\right\}^2.$$

But Jacobi (p. 86, [*l.c.*]),

$$\operatorname{sn}2g\omega_0=\operatorname{sn}\frac{2K\xi}{\pi},$$

$$=\frac{AK}{\pi i}\,\frac{e^{2i\xi}-1}{e^{i\xi}}\,\frac{(1-q^2e^{2i\xi})..(1-q^2e^{-2i\xi})..}{(1-qe^{2i\xi})..(1-qe^{-2i\xi})..},$$

where (p. 89, [*l.c.*, p. 146])

$$\frac{AK}{\pi}=\frac{\sqrt[4]{q}}{\sqrt{k}},\quad =\tfrac{1}{2}\left\{\frac{(1+q)(1+q^3)\ldots}{(1+q^2)(1+q^4)\ldots}\right\}^2=\tfrac{1}{2}f^{-2}(q),$$

that is,

$$\operatorname{sn}2g\omega=f^{-2}q\,.\,\frac{\alpha^{2g}-1}{2i\alpha^g}\,\frac{(1-\alpha^{2g}q^2)..(1-\alpha^{n-2g}q^2)..}{(1-\alpha^{2g}q)..(1-\alpha^{n-2g}q)..}.$$

Hence

$$\frac{\operatorname{sn}2g\omega_0}{\operatorname{snc}2g\omega_0}=\frac{\alpha^{2g}-1}{i(\alpha^{2g}+1)}\,\frac{1-\alpha^{2g}q^2..}{1+\alpha^{2g}q^2..}\,\frac{1+\alpha^{2g}q..}{1-\alpha^{2g}q..}\,\frac{1-\alpha^{n-2g}q^2..}{1+\alpha^{n-2g}q^2..}\,\frac{1+\alpha^{n-2g}q..}{1-\alpha^{n-2g}q..};$$

and giving to g the values $1, 2, \ldots, \frac{1}{2}(n-1)$, and multiplying the several expressions, we have the value of $\frac{1}{M_0}$, viz. this is

$$\frac{1}{M_0}=(-)^{\frac{1}{2}(n-1)}\,\Pi\left\{\frac{(\alpha^{2g}+1)^2}{i^2(\alpha^{2g}+1)^2}\right\}R(q),$$

where $R(q)$ denotes the product of the several factors which contain q.

56. The (i^2) of the denominator gives a factor i^{n-1}, $=(-)^{\frac{n-1}{2}}$, which destroys the factor $(-)^{\frac{n-1}{2}}$. We have then a factor

$$\Pi\left(\frac{\alpha^{2g}-1}{\alpha^{2g}+1}\right)^2,\text{ which is }=(-)^{\frac{1}{2}(n-1)}\,n.$$

In fact, $n=3$, this is

$$\left(\frac{\alpha^2-1}{\alpha^2+1}\right)^2=-3,$$

viz. the numerator is $\alpha-2\alpha^2+1, =-3\alpha^2$, and the denominator is $(-\alpha)^2, =\alpha^2$.

So $n=5$, the formula is

$$\left(\frac{\alpha^2-1}{\alpha^2+1}\cdot\frac{\alpha^4-1}{\alpha^4+1}\right)^2=5, \text{ that is, } \frac{(\alpha^2-1)^4}{(\alpha^4+1)^2}=5;$$

or

$$\frac{\alpha^8-4\alpha^2+6\alpha^4-4\alpha+1}{\alpha^8+2\alpha^4+1}=5,$$

viz. this is $5(1+\alpha^8+2\alpha^4)-(1-4\alpha-4\alpha^2+\alpha^8+6\alpha^4)=0$, which is right; and so in other cases.

We thus have

$$\frac{1}{M_0}=(-)^{\frac{1}{2}(n-1)}\,n\,.\,R(q),$$

which, on putting therein $u=0$, that is, $q=0$, gives, as it should do, $\frac{1}{M_0}=(-)^{\frac{1}{2}(n-1)}\,n$.

57. As regards the expression of $R(q)$, observe that, giving to g its different values, the factors $1-\alpha^{2g}q^2$ and $1-\alpha^{n-2g}q^2$ are all the factors other than $1-q^2$ of $1-q^{2n}$, and so as to the other pairs of factors; viz. we have

$$R(q)=\left(\frac{1-q^{2n}..}{1-q^2\ ..}\,\frac{1+q^n..}{1+q\ ..}\,\frac{1+q^2\ ..}{1+q^{2n}..}\,\frac{1-q\ ..}{1-q^n..}\right)^2,$$

viz. this is

$$=\left(\frac{1-q^{2n}..}{1+q^{2n}..}\,\frac{1+q^n..}{1-q^n..}\right)^2\div\left(\frac{1-q^2..}{1+q^2..}\,\frac{1+q..}{1-q..}\right)^2,$$

that is,

$$\frac{1}{M_0}=(-)^{\frac{1}{2}(n-1)}\,n\,\frac{\phi^2(q^n)}{\phi^2(q)},$$

agreeing with a former result.

58. We have of course the identity $2\beta_0=\frac{1}{M_0}-1$; that is,

$$4S\frac{\alpha^g}{1+\alpha^{2g}}f^2(q)\cdot\frac{(1+\alpha^{2g}q\ )..(1+\alpha^{n-2g}q\ )..}{(1+\alpha^{2g}q^2)..(1+\alpha^{n-2g}q^2)..}=(-)^{\frac{1}{2}(n-1)}\,n\,\frac{\phi^2(q^n)}{\phi^2(q)}-1,$$

$\left(g=1,\ 2,..\frac{1}{2}(n-1)\right)$, which, putting therein $q=0$, is an identity before referred to; a form perhaps more convenient is obtained by dividing each side by $f^2(q)$.

59. I notice further that we have

$$v_0=u^n\{\operatorname{snc}2\omega_0\operatorname{snc}4\omega_0\ldots\operatorname{snc}(n-1)\omega_0\};$$

the term in { } is

$$\Pi\frac{1+\alpha^{2g}}{2\alpha^g}f^{-g}(q)\frac{(1+\alpha^{2g}q^2)..(1+\alpha^{n-2g}q)^2..}{(1+\alpha^g\,q\ )..(1+\alpha^{n-2g}q\ )\ ..},$$

where we have $\Pi\frac{1+\alpha^{2g}}{\alpha}=(-)^{\frac{1}{2}(n^2-1)}$. For example, $n=3$, the term is $\frac{1+\alpha^2}{\alpha}=-1$;

$n=5$, it is

$$\frac{(1+\alpha^2)(1+\alpha^4)}{\alpha\,.\,\alpha^2}, \quad =\frac{1+\alpha^2+\alpha^4+\alpha}{\alpha^3}, \quad =-1;$$

$n=7$, it is

$$\frac{(1+\alpha^2)(1+\alpha^4)(1+\alpha^6)}{\alpha\,.\,\alpha^2\,.\,\alpha^4}, \quad =\frac{1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+2\alpha^6}{\alpha^6}=1;$$

and so on. The term in question thus is

$$(-)^{\frac{1}{2}(n^2-1)}\,.\,\frac{1}{(\sqrt{2})^{n-1}}f^{-n+1}(q)\frac{1+q^{2n}\,..\,1+q\,..}{1+q^n\,..\,1+q^2..},$$

that is,

$$(-)^{\frac{1}{2}(n^2-1)}\frac{1}{(\sqrt{2})^{n-1}}f^{-n}(q)f(q^n).$$

This has to be multiplied by w^n, $=(\sqrt{2})^n q^{\frac{n}{8}} f^n(q)$, and we thus obtain

$$v_0=(-)^{\frac{1}{2}(n^2-1)}\sqrt{2}q^{\frac{n}{8}}f(q^n),$$

agreeing with a former result.

We have in what precedes a complete q-transcendental solution for the *transformatio prima;* viz. the original modulus $k^2(=w^8)$ being given as a function of q, then, as well the new modulus $\lambda_0^2(=v_0^8)$ and the multiplier M_0, as also the several functions which enter into the expression

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left\{\frac{\left(1-\frac{x}{\operatorname{snc}2\omega_0}\right)\cdots\left(1-\frac{x}{\operatorname{snc}(n-1)\omega_0}\right)}{\left(1+\frac{x}{\operatorname{snc}2\omega_0}\right)\cdots\left(1+\frac{x}{\operatorname{snc}(n-1)\omega_0}\right)}\right\}^2,$$

are all of them expressed as functions of q.

60. I consider in like manner the expression

$$\frac{1}{\operatorname{snc}2g\omega_n}\,\frac{1}{\operatorname{sn}(K-2g\omega_n)}, \quad =\frac{\operatorname{dn}2g\omega_n}{\operatorname{cn}2g\omega_n}.$$

Here, writing $2g\omega_n=\frac{2K\xi}{\pi}$ (ξ instead of Jacobi's x as before), that is,

$$\xi=\frac{\pi}{2K}\,.\,2g\,.\,\frac{iK'}{n}=\frac{g\pi iK'}{nK},$$

and thence

$$e^{i\xi}=e^{-\frac{g}{n}\frac{\pi K'}{K}}, \quad =q^{\frac{g}{n}},$$

we have

$$\frac{1}{\operatorname{snc}2g\omega_n}=\operatorname{dn}\frac{2K\xi}{\pi}\div\operatorname{cn}\frac{2K\xi}{\pi}$$

$$=f^2(q)\,.\,\frac{2q^{\frac{g}{n}}}{1+q^{\frac{2g}{n}}}\,.\,\frac{(1+q^{1+\frac{2g}{n}})..}{(1+q^{2+\frac{2g}{n}})..}\,\frac{(1+q^{1-\frac{2g}{n}})..}{(1+q^{2-\frac{2g}{n}})..},$$

where the notations are as follows:

$$(1+q^{1+\frac{2g}{n}})\,..\text{ is the infinite product }(1+q^{1+\frac{2g}{n}})(1+q^{3+\frac{2g}{n}})(1+q^{5+\frac{2g}{n}})\,..,$$

and

$$(1+q^{2+\frac{2g}{n}})\,..\text{ is the infinite product }(1+q^{2+\frac{2g}{n}})(1+q^{4+\frac{2g}{n}})(1+q^{6+\frac{2g}{n}})\,..;$$

and the like as to the expressions with exponents containing $-\frac{2g}{n}$.

And then attributing to g the values $1, 2, \ldots, \frac{1}{2}(n-1)$, and forming the symmetric functions of these expressions, we have the values of $\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \ldots, \frac{\sigma}{\alpha}$; or α being put $=1$, say the values of $\beta, \gamma, \ldots, \sigma$.

It is easy to see, and I do not stop to prove that, if instead of $\omega=\omega_n$ we have $\omega=\omega_1, \omega_2, \ldots$, or ω_{n-1}, we simply multiply $q^{\frac{1}{n}}$ by an imaginary nth root of unity; that is, we replace the real nth root $q^{\frac{1}{n}}$ by an imaginary nth root of q.

In the case $u=0$, that is, $q=0$, we have $\frac{1}{\operatorname{snc} 2g\omega_n}=0$, and thence $\beta=0$; and the like for the values $\omega_1, \omega_2, \ldots, \omega_{n-1}$: the equation $2\beta=\frac{1}{M}-1$ gives consequently for $\frac{1}{M}$, n values each $=1$, agreeing with the multiplier equation.

61. We have for M_n the formula

$$\frac{1}{M_n}=(-)^{\frac{1}{2}(n-1)}\left\{\frac{\operatorname{sn} 2\omega_n \operatorname{sn} 4\omega_n \ldots \operatorname{sn}(n-1)\omega_n}{\operatorname{snc} 2\omega_n \operatorname{snc} 4\omega_n \ldots \operatorname{snc}(n-1)\omega_n}\right\}^2,$$

and, as before,

$$\operatorname{sn} 2g\omega_n=f^{-2}(q)\,.\,\frac{q^{\frac{2g}{n}}-1}{2iq^{\frac{g}{n}}}\,\frac{(1-q^{2+\frac{2g}{n}})\,..}{(1-q^{1+\frac{2g}{n}})\,..}\,\frac{(1-q^{2-\frac{2g}{n}})\,..}{(1-q^{1-\frac{2g}{n}})\,..};$$

hence

$$\frac{\operatorname{sn} 2g\omega_n}{\operatorname{snc} 2g\omega_n}=\frac{q^{\frac{2g}{n}}-1}{i(q^{\frac{2g}{n}}+1)}\,.\,\frac{(1-q^{2+\frac{2g}{n}})\,..}{(1+q^{2+\frac{2g}{n}})\,..}\,\frac{(1+q^{1+\frac{2g}{n}})\,..}{(1-q^{1+\frac{2g}{n}})\,..}\,\frac{(1-q^{2-\frac{2g}{n}})\,..}{(1+q^{2-\frac{2g}{n}})\,..}\,\frac{(1+q^{1-\frac{2g}{n}})\,..}{(1-q^{1-\frac{2g}{n}})\,..},$$

and we thence derive the value of $\frac{1}{M_n}$; viz. observing that we have in the denominator $(i^2)^{\frac{1}{2}(n-1)}, =(-)^{\frac{1}{2}(n-1)}$ which destroys this factor in the expression of $\frac{1}{M_n}$, this is

$$\frac{1}{M_n}=\Pi\left\{\frac{1-q^{\frac{2g}{n}}}{1+q^{\frac{2g}{n}}}\,\frac{(1-q^{2+\frac{2g}{n}})\,..(1+q^{1+\frac{2g}{n}})\,..(1-q^{2-\frac{2g}{n}})\,..(1+q^{1-\frac{2g}{n}})\,..}{(1+q^{2+\frac{2g}{n}})\,..(1-q^{1+\frac{2g}{n}})\,..(1+q^{2-\frac{2g}{n}})\,..(1-q^{1-\frac{2g}{n}})\,..}\right\}^2.$$

Now, giving to g its values, it is easy to see that we have

$$\Pi(1-q^{\frac{2g}{n}})(1-q^{2+\frac{2g}{n}})\ldots(1-q^{2-\frac{2g}{n}})\ldots=\frac{(1-q^{\frac{2}{n}})\ldots}{(1-q^2)\ldots},$$

where $(1-q^{\frac{2}{n}})\ldots$ denotes $(1-q^{\frac{2}{n}})(1-q^{\frac{4}{n}})(1-q^{\frac{6}{n}})\ldots$, viz. it is the same function of $q^{\frac{1}{n}}$ that $(1-q^2)\ldots$ is of q; also

$$\Pi(1+q^{1+\frac{2g}{n}})\ldots(1+q^{1-\frac{2g}{n}})\ldots=\frac{(1+q^{\frac{1}{n}})\ldots}{(1+q)\ldots},$$

where $(1+q^{\frac{1}{n}})\ldots$ denotes $(1+q^{\frac{1}{n}})(1+q^{\frac{3}{n}})(1+q^{\frac{5}{n}})\ldots$, viz. it is the same function of $q^{\frac{1}{n}}$ that $(1+q)\ldots$ is of q; and the like as to the denominator factors: we thus have

$$\frac{1}{M_n}=\left\{\frac{(1-q^{\frac{2}{n}})\ldots}{(1-q^2)\ldots}\,\frac{(1+q^{\frac{1}{n}})\ldots}{(1+q\ )\ldots}\,\frac{(1+q^2)\ldots}{(1+q^{\frac{2}{n}})\ldots}\,\frac{(1-q\ )\ldots}{(1-q^{\frac{1}{n}})\ldots}\right\}^2,$$

viz. this is

$$=\left\{\frac{(1-q^{\frac{2}{n}})\ldots}{(1+q^{\frac{2}{n}})\ldots}\,\frac{(1+q^{\frac{1}{n}})\ldots}{(1-q^{\frac{1}{n}})\ldots}\right\}^2\div\left\{\frac{(1-q^2)\ldots(1+q)\ldots}{(1+q^2)\ldots(1-q)\ldots}\right\}^2;$$

or, we have

$$\frac{1}{M_n}=\phi^2(q^{\frac{1}{n}})\div\phi^2(q),$$

agreeing with a former result.

We have

$$2\mathcal{S}_n=\frac{1}{M_n}-1,$$

that is,

$$2\left\{\frac{1}{\operatorname{snc}2\omega_n}+\frac{1}{\operatorname{snc}4\omega_n}\ldots+\frac{1}{\operatorname{snc}(n-1)\omega_n}\right\}=\frac{\phi^2(q^{\frac{1}{n}})}{\phi^2(q)}-1,$$

a result which, substituting on the left-hand side the foregoing values of the several functions, must be identically true.

62. We have also

$$v_n=w^n\{\operatorname{snc}2\omega_n\operatorname{snc}4\omega_n\ldots\operatorname{snc}(n-1)\omega_n\},$$

where the term in { } is

$$=\Pi f^{-2}(q)\,\frac{(1+q^{\frac{2g}{n}})}{2q^{\frac{g}{n}}}\,\frac{(1+q^{2+\frac{2g}{n}})\ldots(1+q^{2-\frac{2g}{n}})\ldots}{(1+q^{1+\frac{2g}{n}})\ldots(1+q^{1-\frac{2g}{n}})\ldots};$$

or, observing that the sum of the exponents $\frac{g}{n}$ is $\frac{1}{n}\{1+2\ldots+\frac{1}{2}(n-1)\}=\frac{n^2-1}{8n}$, this is

$$=f^{-n+1}(q)\,.\,\frac{1}{(\sqrt{2})^{n-1}q^{\frac{n^2-1}{8n}}}\,\frac{(1+q^{\frac{2}{n}})\ldots(1+q\ )\ldots}{(1+q^2)\ldots(1+q^{\frac{1}{n}})\ldots};$$

or, the last factor being $f(q^{\frac{1}{n}}) \div f(q)$, the expression is

$$f^{-n}(q)\frac{1}{(\sqrt{2})^{n-1}}q^{-\frac{n}{8}+\frac{1}{8n}}f(q^{\frac{1}{n}});$$

or, multiplying by u^n, $=(\sqrt{2})^n q^{\frac{n}{8}} f^n(q)$, we have

$$v_n = \sqrt{2}q^{\frac{1}{8n}}f(q^{\frac{1}{n}}),$$

agreeing with a former result.

We have in what precedes the complete q-transcendental solution for the *transformatio secunda;* viz. the original modulus $k(=u^4)$ being given as a function of q, then, as well the new modulus $\lambda_n(=v_n^4)$ and the multiplier M_n, as also the several functions which enter into the formula

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left\{\frac{\left(1-\frac{x}{\operatorname{snc} 2\omega_n}\right)\cdots\left(1-\frac{x}{\operatorname{snc}(n-1)\omega_n}\right)}{\left(1+\frac{x}{\operatorname{snc} 2\omega_n}\right)\cdots\left(1+\frac{x}{\operatorname{snc}(n-1)\omega_n}\right)}\right\}^2,$$

are all expressed in terms of q. The expressions all contain $q^{\frac{1}{n}}$, and by substituting for this an imaginary nth root of q, we have the formulæ belonging to the several $(n-1)$ imaginary transformations.

63. As an illustration of the formulæ for the *transformatio secunda* I write $n=7$; and putting for greater convenience $q=r^7$, that is, $r=q^{\frac{1}{7}}$, then we have

$$v_7=\sqrt{2}r^{\frac{1}{8}}f(r),\quad \frac{1}{M_7}=\frac{\phi^2(r)}{\phi^2(r^7)},$$

$$\frac{1}{\operatorname{snc} 2\omega_7}=2f^2(r^7)A,\quad \frac{1}{\operatorname{snc} 4\omega_7}=2f^2(r^7)B,\quad \frac{1}{\operatorname{snc} 6\omega_7}=2f^2(r^7)C,$$

where

$$A=r\cdot\frac{5.19\ldots\ 9.23..}{2.16\ldots 12.26..},$$

$$B=r^2\cdot\frac{3.17..\ 11.25..}{4.18..\ 10.24..},$$

$$C=r^3\cdot\frac{1.15..\ 13.27..}{6.20..\ 8.22..},$$

where the numerator of A denotes $(1+r^5)(1+r^{19})..(1+r^9)(1+r^{23})..$, and so in other cases, the difference of the exponents being always $=14$. And we have, as mentioned, the identical equation

$$f^2(r^7)(A+B+C)=\tfrac{1}{4}\left\{\frac{\phi^2 r}{\phi^2 r^7}-1\right\}.$$

The values of the several expressions up to r^{50} are as follows: Mr J. W. L. Glaisher kindly performed for me the greater part of the calculation.

Ind. of r	A	B	C	Sum	Multiplied by $f^2(r^7)$	$\phi^2(r) \div \phi^2(r^7)$
0				0	0	+ 1
1	+ 1			+ 1	+ 1	+ 4
2		+ 1		+ 1	+ 1	+ 4
3	− 1		+ 1	0	0	0
4			+ 1	+ 1	+ 1	+ 4
5	+ 1	+ 1		+ 2	+ 2	+ 8
6	+ 1	− 1		0	0	0
7	− 1			− 1	− 1	− 4
8	− 1			− 1	− 3	− 12
9	+ 1	− 1	− 1	− 1	− 3	− 12
10	+ 2	+ 1	− 1	+ 2	+ 2	+ 8
11	− 1		− 1	− 2	− 4	− 16
12	− 2	− 1	− 1	− 4	− 8	− 32
13		+ 2		+ 2	+ 2	+ 8
14	+ 2	− 1		+ 1	+ 3	+ 12
15	+ 1	− 1	+ 1	+ 1	+ 8	+ 32
16	− 2	+ 2	+ 2	+ 2	+ 9	+ 36
17	− 2	− 2	+ 2	− 2	− 6	− 24
18	+ 1	+ 1	+ 2	+ 4	+ 13	+ 52
19	+ 2	+ 2	+ 2	+ 6	+ 24	+ 96
20		− 3	+ 1	− 2	− 6	− 24
21	− 2	+ 2	− 1	− 1	− 8	− 32
22	− 2	+ 1	− 2	− 3	− 20	− 80
23	+ 2	− 4	− 3	− 5	− 24	− 96
24	+ 3	+ 3	− 4	+ 2	+ 16	+ 64
25		− 1	− 4	− 5	− 33	− 132
26	− 4	− 3	− 3	− 10	− 62	− 248
27	− 2	+ 5	− 1	+ 2	+ 16	+ 64
28	+ 4	− 3	+ 1	+ 2	+ 19	+ 76
29	+ 5	− 1	+ 3	+ 7	+ 46	+ 184
30	− 3	+ 6	+ 5	+ 8	+ 56	+ 224
31	− 7	− 6	+ 7	− 6	− 40	− 160
32	+ 1	+ 1	+ 7	+ 9	+ 77	+ 308
33	+ 9	+ 5	+ 4	+ 18	+ 144	+ 576
34	+ 3	− 8	+ 1	− 4	− 38	− 152
35	− 9	+ 5	− 1	− 5	− 42	− 168
36	− 7	+ 2	− 5	− 10	− 99	− 396
37	+ 7	− 9	− 9	− 11	− 122	− 488
38	+ 11	+ 10	− 11	+ 10	+ 88	+ 352
39	− 4	− 3	− 10	− 17	− 168	− 672
40	− 13	− 8	− 7	− 28	− 310	− 1240
41	− 2	+ 13	− 3	+ 8	+ 82	+ 328
42	+ 13	− 8	+ 3	+ 8	+ 88	+ 352
43	+ 8	− 3	+ 9	+ 14	+ 204	+ 816
44	− 11	+ 14	+ 14	+ 17	+ 252	+ 1008
45	− 14	− 14	+ 16	− 12	− 182	− 728
46	+ 5	+ 4	+ 15	+ 24	+ 344	+ 1376
47	+ 17	+ 11	+ 12	+ 40	+ 632	+ 2528
48	+ 3	− 20	+ 5	− 12	− 168	− 672
49	− 17	+ 13	− 5	− 9	− 175	− 700
50	− 13	+ 5	− 14	− 22	− 401	− 1604

64. As already mentioned, the foregoing expressions of the coefficients in terms of q may be applied to the determination of the coefficients as rational functions of u, v.

Representing by θ any one of the coefficients $\alpha, \beta, \gamma, \ldots, \sigma$, consider the sum

$$S\frac{v^f\theta}{\alpha},$$

f a positive integer, and the summation extending as before to the $n+1$ values of v, and corresponding values of $\frac{\theta}{\alpha}$. This is a rational function of u, and it is also integral. As to this observe that the function, if not integral, must become infinite either for $u=0$ (this would mean that the expression contained a term or terms Au^{-a}) or for some finite value of u. But the function can only become infinite by reason of some term or terms of $Sv^f\frac{\theta}{\alpha}$ becoming infinite; viz. some term $\frac{1}{\operatorname{snc} 2g\omega}$ must become infinite; or attending to the equation

$$v = u^n\{\operatorname{snc} 2\omega \operatorname{snc} 4\omega \ldots \operatorname{snc}(n-1)\omega\},$$

it can only happen if $u=0$, or if $v=\infty$; and from the modular equation it appears that if $v=\infty$, then also $u=\infty$: the expression in question can therefore only become infinite if $u=0$, or if $u=\infty$. Now $u=0$ gives the ratios $\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \ldots$, each of them a determinate function of n, that is finite; and gives also $v=0$, so that the expression does not become infinite for $u=0$; hence it does not become infinite either for $u=0$ or for any finite value of u; wherefore it is integral. The like reasoning applies to the sum $Sv^{-f}\frac{\theta}{\alpha}$; viz. this is a rational function of u; and it is quasi-integral, viz. there are no terms having a denominator other than a power of u, the highest denominator being u^{nf}; viz. the expression contains negative and positive integer powers of u, the lowest power (highest negative power) being $\frac{1}{u^{nf}}$.

65. It is to be observed, further, that writing the expression in the form

$$v_0^f\frac{\theta_0}{\alpha_0} + S'v^f\frac{\theta}{\alpha},$$

(where S' refers to the values $v_1, v_2, \ldots, v_n$ of the modulus), and considering the several quantities as expressed in terms of q, then in the sum S' every term involving a fractional power $q^{\frac{h}{n}}$ acquires by the summation the coefficient $(1+\alpha+\alpha^2+\ldots+\alpha^{n-1})$, and therefore disappears; there remains only the radicality $q^{\frac{1}{8}}$ occurring in the expressions of the v's; and if $nf \equiv \mu \pmod{8}$, $\mu=0$, or a positive integer less than 8, then the form of the expression is $q^{\frac{\mu}{8}}$ into a rational function of q. Hence this, being a rational and integral function of u, must be of the form

$$Au^\mu + Bu^{\mu+8} + Cu^{\mu+16} + \&c.$$

66. We have thus in general

$$Sv^f \frac{\theta}{\alpha} = Au^\mu \quad + Bu^{\mu+8} \quad + \&c.;$$

and in like manner

$$Sv^{-f} \frac{\theta}{\alpha} = A'u^{-nf} \quad + B'u^{-nf+8} + \&c.$$

We may in these expressions find a limit to the number of terms, by means of the before-mentioned theorem that we may simultaneously interchange u, v; α, $\beta, \ldots, \rho$, σ into $\frac{1}{u}$, $\frac{1}{v}$; σ, $\rho, \ldots, \beta$, α. Starting from the expression of $Sv^f \frac{\theta}{\alpha}$, let ϕ be the corresponding coefficient to θ; viz. in the series α, $\beta, \ldots, \theta, \ldots, \phi, \ldots, \rho$, σ, let ϕ be as removed from σ as θ is from α; then the equation becomes

$$Sv^{-f} \frac{\phi}{\sigma} = Au^{-\mu} \quad + Bu^{-\mu-8} \quad + \&c.,$$

where $\frac{\phi}{\sigma} = \frac{\phi}{\alpha} \frac{\alpha}{\sigma} = \frac{v}{u^n} \frac{\phi}{\alpha}$; the equation thus is

$$Sv^{1-f} \frac{\phi}{\alpha} = Au^{n-\mu} \quad + Bu^{n-\mu-8} \quad + \&c.;$$

and by what precedes the series on the right-hand side can contain no negative power higher than $\frac{1}{u^{n(f-1)}}$; that is, the series of coefficients A, B, $C, \ldots$ goes on to a certain point only, the subsequent coefficients all of them vanishing.

In like manner from the equation for $Sv^{-f} \frac{\theta}{\alpha}$ we have

$$Sv^{f+1} \frac{\phi}{\alpha} = A'u^{(n+1)f} + B'u^{(n+1)f-8} + \&c.,$$

where the indices must be positive; viz. the series of coefficients A', $B', \ldots$ goes on to a certain point only, the subsequent coefficients all of them vanishing.

67. The like theory applies to the expression $\frac{1}{M}$. We have, putting as before $nf \equiv \mu$ (mod. 8),

$$Sv^f \frac{1}{M} = Au^\mu + Bu^{\mu+8} + \ldots,$$

$$Sv^{-f} \frac{1}{M} = A'u^{-nf} + B'u^{-nf+8} + \ldots$$

and we find a limit to the number of terms by the consideration that we may simultaneously change u, v, $\frac{1}{M}$ into $\frac{1}{u}$, $\frac{1}{v}$, $\frac{v^4}{u^4 M}$; the equations thus become

$$Sv^{4-f} \frac{1}{M} = Au^{4-\mu} + Bu^{-4-\mu} + \ldots$$

$\left(\text{where, if } f = \text{ or } < 4, \text{ there must be on the right-hand side no negative power of } u;\right.$ but if $f > 4$, then the highest negative power must be $\left.\frac{1}{u^{(f-4)n}}\right)$, and

$$Sv^{4+f}\frac{1}{M} = A'u^{nf+4} + B'u^{nf-4} + \ldots,$$

where on the right-hand side there must be no negative power of u.

68. It is to be remarked that β, ρ being always given linearly in terms of $\frac{1}{M}$ it is the same thing whether we seek in this manner for the values of β, ρ or for that of $\frac{1}{M}$; but the latter course is practically more convenient. Thus in the cases $n=5$, $n=7$ we require only the value of $\frac{1}{M}$.

In the case $n=11$, where the coefficients are α, β, γ, δ, ϵ, ζ, it has been seen that γ, δ are given as cubic functions of $\frac{1}{M}$: seeking for them directly, their values would (if the process be practicable) be obtained in a better form, viz. instead of the denominator $(F'v)^2$ there would be only the denominator $F'(v)$.

69. I consider for $\frac{1}{M}$ the cases $n=3$ and 5:

$$n=3,\ f=0,\ 1,\ 2,\ 3, \text{ then } \mu=0,\ 3,\ 6,\ 1;$$

and we write down the equations

$$S\frac{1}{M} = A, \quad \text{giving } S\frac{v^4}{M} = Au^4,$$

$$S\frac{v}{M} = A'u^3, \quad \text{,,} \quad S\frac{v^3}{M} = A'u,$$

$$S\frac{v^2}{M} = 0, \quad \text{,,} \quad S\frac{v^2}{M} = 0;$$

viz. if we had in the first instance assumed $S\frac{1}{M} = A + Bu^3 + \ldots$, this would have given $S\frac{v^4}{M} = Au^4 + Bu^{-4} + \ldots$, whence B and the succeeding coefficients all vanish; and so in other cases. We have here only the coefficients A, A'; and these can be obtained without the aid of the q-formulæ by the consideration that for $u=1$ the corresponding values of v, $\frac{1}{M}$ are

$$v = 1,\ -1,\ -1,\ -1,$$

$$\frac{1}{M} = 3,\ -1,\ -1,\ -1,$$

whence $A=0$, $A'=6$; or we have the equations

$$S\frac{1}{M}=0,\quad S\frac{v}{M}=6u^3,\quad S\frac{v^2}{M}=0,\quad S\frac{v^3}{M}=6u,$$

giving as before

$$(2v^3+3v^2u-u)\frac{1}{M}=3\,(v^2u^3+2u^5v+1)\,u,$$

reducible by means of the modular equation to $\frac{1}{M}=1+\frac{2u^3}{v}$.

70. $n=5$. Corresponding to $f=0, 1, 2, 3, 4, 5$, we have $\mu=0, 5, 2, 7, 4, 1$, and we find

$$\begin{array}{lll}
S\frac{1}{M}=A, & \text{giving } S\frac{v^4}{M} & =Au^4,\\
S\frac{v}{M}=0, & \quad\text{,,} \quad S\frac{v^3}{M} & =0,\\
S\frac{v^2}{M}=A'u^2, & \quad\text{,,} \quad S\frac{v^2}{M} & =A'u^2,\\
S\frac{v^5}{M}=A''u+B''u^9, & \quad\text{,,} \quad Sv^{-1}\frac{1}{M} & =A''u^3+B''u^{-5}.
\end{array}$$

But for $u=1$ the corresponding values of v, $\frac{1}{M}$ are

$$\begin{array}{l}
v\;=1,\;-1,\;-1,\;-1,\;-1,\;-1,\\
\frac{1}{M}=5,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1;
\end{array}$$

whence $A=A'=10$, $A''+B''=0$, or say the value of $S\frac{v^5}{M}$ is $=A''u\,(1-u^8)$.

The value of A'' is found very easily by the q-formulæ, viz. neglecting higher powers of q, we have

$$u=q^{\frac{1}{8}}\sqrt{2},\ v_0=q^{\frac{5}{8}}\sqrt{2},\ \frac{1}{M_0}=5;\ v_5=q^{\frac{1}{40}}\sqrt{2},\ \frac{1}{M_5}=1;$$

hence

$$S\frac{v^5}{M}=\frac{v_0^5}{M}+S'\frac{v^5}{M},\ =5q^{\frac{1}{8}}(\sqrt{2})^5=A''q^{\frac{1}{8}}\sqrt{2};$$

that is, $A''=20$, and the equations are

$$S\frac{1}{M}=10,\quad S\frac{v}{M}=0,\quad S\frac{v^2}{M}=10u^2,\quad S\frac{v^3}{M}=0,\quad S\frac{v^4}{M}=10u^4,\quad S\frac{v^5}{M}=20u\,(1-u^8);$$

whence

$$\begin{aligned}
F'v.\frac{1}{M}=\;& 20u\,(1-u^8)\\
&-10u^4\,(Sv_0-v)\\
&-10u^2\,(Sv_0v_1v_2-vSv_0v_1+v^2Sv_0-v^3)\\
&-10\;\;(Sv_0v_1v_2v_3v_4-vSv_0v_1v_2v_3+v^2Sv_0v_1v_2-v^3Sv_0v_1+v^4Sv_0-v^5),
\end{aligned}$$

where Sv_0, &c. are the coefficients of the equation

$$v^6 + 4v^5u^5 + 5v^4u^2 - 5v^2u^4 - 4vu - u^6 = 0,$$

viz.

$$Sv_0, \quad v_0v_1, \quad v_0v_1v_2, \quad v_0v_1v_2v_3, \quad v_0v_1v_2v_3v_4$$

are

$$-4u^5, \quad +5u^2, \quad 0\ , \quad -5u^4\ , \quad 4u\ ;$$

or the equation is

$$\begin{aligned} F'v\,.\,\frac{1}{M} = \ & 20u\,(1-u^8) \\ & -10u^4\,(-4u^5 - \quad v) \\ & -10u^2\,(\qquad -5u^2v - 4u^5v^2 - \ v^3) \\ & -10\ \ (\ \ 4u\ + 5u^4v \qquad\quad -5v^3u^2 - 4v^4u^5 - v^5), \end{aligned}$$

or, say

$$\tfrac{1}{2}F'v\,\frac{1}{M} = 5\,\{v^5 + \ 4v^4u^5 + \ 6v^3u^2 + 4v^2u^7 + \ vu^4 - 2u\,(1-u^8)\},$$

where

$$\tfrac{1}{2}F'v \quad = 3\ \ v^5 + 10v^4u^5 + 10v^3u^2 \qquad\quad -5vu^4 - 2u.$$

Hence also, reducing by the modular equation,

$$\tfrac{1}{2}vF'v\,\frac{1}{M} = 5u\,\{v^4u + 4v^3u^4 + 6v^2u^3 + 2v\,(1+u^8) + u^5\},$$

the one of which forms is as convenient as the other.

71. Making the change u, v, $\dfrac{1}{M}$ into v, $-u$, $-5M$, we have

$$-\tfrac{1}{2}F'u\,.\,5M = 5\,\{-u^5 + 4v^5u^4 - 6v^2u^3 + 4v^7u^2 - v^4u - 2v\,(1-v^8)\}\,;$$

and comparing with the equation

$$5M^2 = -\frac{(1-v^8)\,vF'v}{(1-u^8)\,uF'u},$$

we obtain

$$\frac{v\,(1-v^8)}{u\,(1-u^8)} = \frac{-2v\,(1-v^8) - v^4u + 4v^7u^2 - 6v^2u^3 + 4v^5u^4 - u^5}{-2u\,(1-u^8) + u^4v + 4u^7v^2 + 6u^2v^3 + 4u^5v^4 + v^5}.$$

Writing for a moment $M = u^4 + 6u^2v^2 + v^4$, $N = u^2 + v^2$, this is

$$-\frac{v\,(1-v^8)}{u\,(1-u^8)} = \frac{-2v\,(1-v^8) - uM + 4v^5u^2N}{-2u\,(1-u^8) + vM + 4v^2u^5N},$$

that is,

$$-4uv\,(1-u^8)\,(1-v^8) - \{u^2\,(1-u^8) - v^2\,(1-v^8)\}\,M + 4v^3u^3\,\{u^2\,(1-v^8) + v^2\,(1-u^8)\}\,N = 0.$$

But we have

$$u^2\,(1-u^8) - v^2\,(1-v^8) = (u^2 - v^2)\,\{1 - u^8 - u^6v^2 - u^4v^4 - u^2v^6 - v^8\},$$

$$u^2\,(1-v^8) + v^2\,(1-u^8) = (u^2 + v^2)\,\{1 - u^2v^2\,(u^4 - u^2v^2 + v^4)\}.$$

Hence, replacing M, N by their values, this is

$$\begin{aligned}&-4uv(1-u^8)(1-v^8)\\&-(u^2-v^2)(1-u^8-u^6v^2-u^4v^4-u^2v^6-v^8)(u^4+6u^2v^2+v^4)\\&+4u^3v^3(u^2+v^2)^2\{1-u^2v^2(u^4-u^2v^2+v^4)\}=0;\end{aligned}$$

viz. writing $u^2-v^2=A$, $uv=B$, this is

$$\begin{aligned}&-4B\{1-A^4-4A^2B^2-2B^4+B^8\}\\&-\ \ A\{1-A^4-5A^2B^2-3B^4\}(A^2+8B^2)\\&+4B^2(A^2+4B^2)\{1-A^2B^2-B^4\}=0,\end{aligned}$$

that is,

$$\begin{aligned}&-4B\{(1-A^4-4A^2B^2-2B^4+B^8)-B^2(A^2+4B^2)(1-A^2B^2-B^4)\}\\&-\ \ A\ (1-A^4-5A^2B^2-3B^4)(A^2+8B^2)=0;\end{aligned}$$

viz.

$$\begin{aligned}&-4B\ (1-A^4-5A^2B^2-3B^4)(1-B^4)\\&-\ \ A\ (1-A^4-5A^2B^2-3B^4)(A^2+8B^2)=0;\end{aligned}$$

or throwing out the factor $-(1-A^4-5A^2B^2-3B^4)$, this is

$$A(A^2+8B^2)+4B(1-B^4)=0,$$

the modular equation, which is right.

The four forms of the modular equation, and the curves represented thereby.
Art. Nos. 72 to 79.

72. The modular equation for any value of n has the property that it may be represented as an equation of the same order ($=n+1$, when n is prime) between u, v: or between u^2, v^2: or between u^4, v^4: or between u^8, v^8. As to this, remark that in general an equation $(u, v, 1)^m=0$ of the order m gives rise to an equation $(u^2, v^2, 1)^{2m}=0$ of the order $2m$ between u^2, v^2; viz. the required equation is

$$(u, v, 1)^m(u, -v, 1)^m(-u, v, 1)^m(-u, -v, 1)^m=0,$$

where the left-hand side is a rational function of u^2, v^2 of the form $(u^2, v^2, 1)^{2m}$; or again starting from a given equation $(u, v, w)^m=0$, and transforming by the equations $x:y:z=u^2:v^2:w^2$, the curve in (x, y, z) is of the order $2m$; in fact, the intersections of the curve by the arbitrary line $ax+by+cz=0$ are given by the equations $(u, v, w)^m=0$, $au^2+bv^2+cw^2=0$, and the number of them is thus $=2m$. Moreover, by the general theory of rational transformation, the new curve of the order $2m$ has the same deficiency as the original curve of the order m. The transformed curve in $x, y, z, =u^2, v^2, w^2$ may in particular cases reduce itself to a curve of the order m twice repeated; but it is important to observe that here, taking the single curve of the order m as the transformed curve, this has no longer the same deficiency as the original curve; and in particular the curves represented by the modular equation in its four several forms, writing therein successively u, v; u^2, v^2; u^4, v^4; u^8, v^8, $=x, y$, are not curves of the same deficiency.

73. The question may be looked at as follows: the quantities which enter rationally into the elliptic-function formulæ are k^2, $\lambda^2=u^8$, v^8; if a modular equation $(u, v)^\nu=0$ led to the transformed equation $(u^8, v^8)^{8\nu}=0$, then to a given value of u^8

would correspond 8 values of u, therefore 8ν values of v, giving the same number, 8ν, values of v^8; that is, the values of v^8 corresponding to a given value of v^8 would group themselves in eights corresponding to the 8 values of u. There is, in fact, no such grouping; the equations are $(u, v)^\nu = 0$, $(u^8, v^8)^\nu = 0$; to a given value of u^8 correspond 8 values of u, and therefore 8ν values of v, but these give in eights the same value of v^8, so that the number of values of v^8 is $= \nu$.

74. I consider the case $n = 3$: here, writing x, y for u, v, we have here the sextic curve

I. $y^4 - x^4 + 2xy(x^2y^2 - 1) = 0$;

and it is easy to see that the remaining forms wherein x, y denote u^2, v^2; u^4, v^4; and u^8, v^8 respectively, are derived herefrom as follows; viz.

II. $(y^2 - x^2)^2 - 4xy(xy - 1)^2 = 0$, that is,

$$y^4 + 6x^2y^2 + x^4 - 4xy(x^2y^2 + 1) = 0;$$

III. $(y^2 + 6xy + x^2)^2 - 16xy(xy + 1)^2 = 0$, that is,

$$y^4 + 6x^2y^2 + x^4 - 4xy(4x^2y^2 - 3x^2 - 3y^2 + 4) = 0;$$

IV. $(y^2 + 6xy + x^2)^2 - 16xy(4xy - 3x - 3y + 4)^2 = 0$, that is,

$$y^4 - 762x^2y^2 + x^4 - 4xy\{64x^2y^2 - 96x^2y - 96xy^2 + 33x^2 + 33y^2 - 96x - 96y + 64\} = 0,$$

where it may be noticed that the process is not again repeatable so as to obtain a sextic equation between x, y standing for u^{16}, v^{16} respectively.

The curve I. has a dp (fleflecnode) at the origin, viz. the branches are given by $y^3 - 2x = 0$, $-x^3 - 2y = 0$; and it has 2 cusps at infinity, on the axes $x = 0$, $y = 0$ respectively; viz. the infinite branches are given by $y + 2x^3 = 0$, $-x + 2y^3 = 0$ respectively. These same singularities present themselves in the other curves.

The curve II. has the four dps $(x^2 - y^2 = 0,\ xy - 1 = 0)$, that is,

$$(x = y = 1),\ (x = y = -1),\ (x = i,\ y = -i),\ (x = -i,\ y = i).$$

Corresponding hereto we have in the curve III. the 2 dps $(x = y = 1,\ x = y = -1)$, and in the curve IV. the dp $(x = y = 1)$.

The curve III. has besides the 4 dps $y^2 + 6xy + x^2 = 0$, $xy + 1 = 0$, that is,

$$(1 + \sqrt{2},\ 1 - \sqrt{2}),\ (1 - \sqrt{2},\ 1 + \sqrt{2}),\ (-1 - \sqrt{2},\ -1 + \sqrt{2}),\ (-1 + \sqrt{2},\ -1 - \sqrt{2});$$

and corresponding hereto in the curve IV. we have the 2 dps

$$(3 + 2\sqrt{2},\ 3 - 2\sqrt{2}),\ (3 - 2\sqrt{2},\ 3 + 2\sqrt{2}).$$

The curve IV. has besides the 4 dps $(y^2 + 6xy + x^2 = 0,\ 4xy - 3x - 3y + 4 = 0)$, or say $(2x - \frac{3}{2})(2y - \frac{3}{2}) + \frac{7}{4} = 0$, $2(x + \frac{9}{4})^2 + 2(y + \frac{9}{4})^2 - \frac{177}{8} = 0$. Hence the 4 curves have respectively the dps and deficiency following:—

dps.		dps.	Def.
2, 1	=	3,	7,
2, 1, 4	=	7,	3,
2, 1, 2, 4	=	9,	1,
2, 1, 1, 2, 4	=	10,	0;

viz. the curve IV. representing the equation between u^8 and v^8 is a unicursal sextic.

It may be noticed that, except the fleflecnode at the origin and the cusps at infinity, the dps in question are all acnodes (conjugate points).

75. The foregoing equations may be exhibited in the square diagrams:—

I.

	y^4	y^3	y^2	y	1
x^4					-1
x^3		$+2$			
x^2					
x				-2	
1	$+1$				

$1 \quad +2 \quad 0 \quad -2 \quad -1 = (y+1)^3(y-1);$

II.

y^4	y^3	y^2	y	1
				$+1$
	-4			
		$+6$		
			-4	
$+1$				

$1 \quad -4 \quad +6 \quad -4 \quad +1 = (y-1)^4;$

III.

	y^4	y^3	y^2	y	1
x^4					$+1$
x^3		-16		$+12$	
x^2			$+6$		
x		$+12$		-16	
1	$+1$				

$1 \quad -4 \quad +6 \quad -4 \quad +1 = (y-1)^4;$

IV.

y^4	y^3	y^2	y	1
				$+1$
	-256	$+384$	-132	
	$+384$	-762	$+384$	
	-132	$+384$	-256	
$+1$				

$1 \quad -4 \quad +6 \quad -4 \quad +1 = (y-1)^4;$

where the subscript line, showing in each case what the equation becomes on writing therein $x=1$, serves as a verification of the numerical values.

The curve IV. being unicursal, the coordinates may be expressed rationally in terms of a parameter; in fact, we have

$$x = \frac{\alpha^3(2+\alpha)}{1+2\alpha}, \quad y = \frac{\alpha(2+\alpha)^3}{(1+2\alpha)^3}.$$

These values give

$$\begin{aligned}
16xy &= 16\alpha^4(2+\alpha)^4 && \div (1+2\alpha)^4,\\
4+4xy-3x-3y &= (4,\ 8,\ 12,\ 32,\ 50,\ 32,\ 12,\ 8,\ 4\between 1,\ \alpha)^8 && \div (1+2\alpha)^4,\\
x^2+6xy+y^2 &= 4\alpha^2(2+\alpha)^3(4,\ 8,\ 12,\ 32,\ 50,\ 32,\ 12,\ 8,\ 4\between 1,\ \alpha)^8 && \div (1+2\alpha)^6,
\end{aligned}$$

and the equation of the curve is thus verified.

76. Considering in like manner the modular equation for the quintic transformation, we derive the four forms as follows:—

I. $x^6y^6 + 5x^2y^2(x^2 - y^2) + 4xy(1 - x^4y^4) = 0$;

II. $\{x^3 - y^3 + 5xy(x - y)\}^2 - 16xy(1 - x^2y^2)^2 = 0$, that is,

$$x^6 + 15x^4y^2 + 15x^2y^4 + y^6 - 2xy(8 - 5x^4 + 10x^2y^2 - 5y^4 + 8x^4y^4) = 0;$$

III. $(x^3 + 15x^2y + 15xy^2 + y^3)^2 - 4xy(8 - 5x^2 + 10xy - 5y^2 + 8x^2y^2)^2 = 0$, that is,

$$x^6 + 655x^4y^2 + 655x^2y^4 + y^6 - 640x^2y^2 - 640x^4y^4$$
$$+ xy(-256 + 320x^2 + 320y^2 - 70x^4 - 660x^2y^2 - 70y^4 + 320x^4y^2 + 320x^2y^4 - 256x^4y^4) = 0;$$

IV. $(x^3 + 655x^2y + 655xy^2 + y^3 - 640xy - 640x^2y^2)^2$

$$- xy(-256 + 320x + 320y - 70x^2 - 660xy - 70y^2 + 320x^2y + 320xy^2 - 256x^2y^2)^2 = 0:$$

or, expanding the two terms in the last equation separately, this is

xy		$-$ 65536	$= 0.$
x^2y		$+$ 163840	
xy^2		$+$ 163840	
x^3y		$-$ 138240	
x^2y^2	$+$ 409600	$-$ 542720	
xy^3		$-$ 138240	
x^4y	$-$ 1280	$+$ 44800	
x^3y^2	$-$ 838400	$+$ 631040	
x^2y^3	$-$ 838400	$+$ 631040	
xy^4	$-$ 1280	$+$ 44800	
x^6	$+$ 1		
x^5y	$+$ 1310	$-$ 4900	
x^4y^2	$+$ 430335	$-$ 297200	
x^3y^3	$+$ 1677252	$-$ 986072	
x^2y^4	$+$ 430335	$-$ 297200	
xy^5	$+$ 1310	$-$ 4900	
y^6	1		
x^5y^2	$-$ 1280	$+$ 44800	
x^4y^3	$-$ 838400	$+$ 631040	
x^3y^4	$-$ 838400	$+$ 631040	
x^2y^5	$-$ 1280	$+$ 44800	
x^5y^3		$-$ 138240	
x^4y^4	$+$ 409600	$-$ 542720	
x^3y^5		$-$ 138240	
x^5y^4		$+$ 163840	
x^4y^5		$+$ 163840	
x^5y^5		$-$ 65536	

77. The square diagrams are:—

I.

	y^6	y^5	y^4	y^3	y^2	y	1
x^6							-1
x^5		$+4$					
x^4					-5		
x^3							
x^2			$+5$				
x^1						-4	
1	$+1$						
	1	$+4$	$+5$	0	-5	-4	-1

$= (y+1)^5 (y-1)$;

II.

	y^6	y^5	y^4	y^3	y^2	y	1
							$+1$
		-16				$+10$	
					$+15$		
				-20			
			$+15$				
		$+10$				-16	
	$+1$						
	1	-6	$+15$	-20	$+15$	-6	$+1$

$= (y-1)^6$;

III.

	y^6	y^5	y^4	y^3	y^2	y	1
x^6							$+1$
x^5		-256		$+320$		-70	
x^4			-640		$+655$		
x^3		$+320$		-660		$+320$	
x^2			$+655$		-640		
x		-70		$+320$		-256	
1	$+1$						
	$+1$	-6	$+15$	-20	$+15$	-6	$+1$

$= (y-1)^6$;

IV.

	y^6	y^5	y^4	y^3	y^2	y	1
x^6							$+1$
x^5		-65536	$+163840$	-138240	$+43520$	-3590	
x^4		$+163840$	-133120	-207360	$+133135$	$+43520$	
x^3		-138240	-207360	$+691180$	-207360	-138240	
x^2		$+43520$	$+133135$	-207360	-133120	$+163840$	
x		-3590	$+43520$	-138240	$+163840$	-65536	
1	$+1$						
	$+1$	-6	$+15$	-20	$+15$	-6	$+1$

$= (y-1)^6$:

where the subscript line, showing in each case what the equation becomes on writing therein $x=1$, serves as a verification of the numerical values.

78. The curve I. has at the origin a dp in the nature of a flecnode, viz. the two branches are given by $x^5+4y=0$, $-y^5+4x=0$ respectively; and there are two singular points at infinity on the two axes respectively, viz. the infinite branches are given by $-y-4x^5=0$, $x-4y^5=0$ respectively. Writing the first of these in the form $-yz^4-4x^5=0$, we see that the point at infinity on the axis $x=0$ (i.e. the point $z=0$, $x=0$) is $=6$ dps; and similarly writing for the other branch $xz^4-4y^5=0$, the point at infinity on the axis $y=0$ (i.e. the point $z=0$, $y=0$) is $=6$ dps*.

Moreover, as remarked to me by Professor H. J. S. Smith, the curve has 8 other dps; viz. writing ω to denote an eighth root of -1, $(\omega^8+1=0)$, then a dp is $x=\omega$, $y=\omega^5$. To verify this, observe that these values give

$$\left.\begin{array}{rr} 6x^5 & \overbrace{=+\ 6}^{\omega^5} \\ +20x^3y^2 & -20 \\ -10xy^4 & -10 \\ +\ 4y & +\ 4 \\ -20x^4y^5 & +20 \end{array}\right| \left.\begin{array}{rr} -\ 6y^5 & \overbrace{=+\ 6}^{\omega} \\ +10x^4y & -10 \\ -20x^2y^3 & -20 \\ +\ 4x & +\ 4 \\ -20x^5y^4 & +20 \end{array}\right\| ;$$

or the derived functions each vanish. Thus I. has in all $1+12+8$, $=21$ dps.

In II. we have in like manner $1+12+4$, $=17$ dps; viz. instead of the 8 dps, we have the 4 dps $x=\omega^2$, $y=\omega^2$, $(\omega^8+1=0)$, or, what is the same thing, $x=\omega$, $y=-\omega$, where $\omega^4+1=0$. But we have besides the 12 dps given by

$$x^3-y^3+5xy\,(x-y)=0,\quad 1-x^2y^2=0,$$

viz. we have in all $1+12+4+12$, $=29$ dps.

In III. we thence have $1+12+2+6$, $=21$ dps; and, besides, the 12 dps given by

$$x^3+15x^2y+15xy^2+y^3=0,\quad 8-5x^2+10xy-5y^2+8x^2y^2=0,$$

in all $1+12+2+6+12$, $=33$ dps.

And in IV. we thence have $1+12+1+3+6$, $=23$ dps; and, besides, the 12 dps given by

$$x^3+655x^2y+655xy^2+y^3-640xy-640x^2y^2=0,$$
$$-256+320x+320y-70x^2-660xy-70y^2+320x^2y+320xy^2-256x^2y^2=0,$$

(these curves intersect in 16 points, 4 of them at infinity, in pairs on the lines $x=0$, $y=0$ respectively; and the intersections at infinity being excluded, there remain $16-4$, $=12$ intersections); there are thus in all $1+12+1+3+6+12$, $=35$ dps.

* These results follow from the general formulæ in the paper "On the Higher Singularities of Plane Curves," *Camb. and Dubl. Math. Journ.* t. VII. (1866), pp. 212—223, [374]; but they are at once seen to be true from the consideration that the curve $yz^4-x^5=0$, which has only the singularity in question, is unicursal; the singularity is thus $=6$ dps.

Arranging the results in a tabular form and adding the values of the deficiency, we have

	dps.	dps.	Def.
I.	$1+12+8$	$=21,$	$=15,$
II.	$1+12+4+12$	$29,$	$7,$
III.	$1+12+2+\ 6+12$	$33,$	$3,$
IV.	$1+12+1+\ 3+\ 6+12$	$35,$	$1,$

so that the curve IV. is a curve of deficiency 1, or bicursal curve. It appears by Jacobi's investigation for the quintic transformation (*Fund. Nov.* pp. 26—28, [*Ges. Werke*, t. I., pp. 77—79]) that we can in fact express x, y, that is, u^5, v^5, rationally in terms of the parameters α, β connected by the equation

$$\alpha^3 = 2\beta(1+\alpha+\beta),$$

which is that of a general cubic (deficiency = 1); in fact, we have

$$\frac{2-\alpha}{\alpha-2\beta} = \frac{v^4}{u^4}, \quad \beta = \frac{u^5}{v},$$

that is,

$$u^5(=x) = \beta^2\left(\frac{2-\alpha}{\alpha-2\beta}\right), \quad v^5(=y) = \beta^2\left(\frac{2-\alpha}{\alpha-2\beta}\right)^5,$$

where α, β satisfy the relation just referred to. The actual verification of the equation IV. by means of these values would be a work of some labour.

79. In the general case p an odd prime, then in I. we have at the origin one dp (in the nature of a flecnode) and at infinity two singular points each $=\frac{1}{2}(p-1)(p-2)$ dps. I infer, from a result obtained by Professor Smith, that there are besides $(p-1)(p-3)$ dps; but I have not investigated the nature of these. And the Table of dps and deficiency then is

		dps.	Def.
I.	$1+(p-1)(p-2)+\ (p-1)(p-3)$	$=\ 2p^2-7p+6,$	$4p-5,$
II.	$1+(p-1)(p-2)+\frac{1}{2}(p-1)(p-3)+\frac{1}{2}(p^2-1)$	$2p^2-5p+4,$	$2p-3,$
III.	$1+(p-1)(p-2)+\frac{1}{4}(p-1)(p-3)+\frac{1}{4}(p^2-1)+\frac{1}{2}(p^2-1)$	$2p^2-4p+3,$	$p-2,$
IV.	$1+(p-1)(p-2)+\frac{1}{8}(p-1)(p-3)+\frac{1}{8}(p^2-1)+\frac{1}{4}(p^2-1)+\frac{1}{2}(p^2-1)$	$2p^2-\frac{7}{2}p+\frac{5}{2},$	$\frac{1}{2}p-\frac{3}{2};$

viz. his values of the deficiencies being as in the last column, the total number of dps must be as in the last but one column.

579.

ADDRESS DELIVERED BY [PROFESSOR CAYLEY AS] THE PRESIDENT [OF THE ROYAL ASTRONOMICAL SOCIETY] ON PRESENTING THE GOLD MEDAL OF THE SOCIETY TO PROFESSOR SIMON NEWCOMB.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXIV. (1873—1874), pp. 224—233.]

THE Council have awarded the medal to Professor Simon Newcomb for his Researches on the Orbits of *Neptune* and *Uranus*, and for his other contributions to mathematical astronomy. And upon me, as President, the duty has devolved of explaining to you the grounds of their decision.

I think it right to remark that it appears to me that, in the award of their highest honour, the Council of a Society are not bound to institute a comparison between heterogeneous branches of a science, or classes of research—to weigh, for instance, mathematical against observational astronomy or astronomical physics; or, in the several branches respectively, the happy idea which originates a theory against the patience and the skilled labour which develope and carry it out; and still less to decide between the merits of different workers in the science. It is enough that the different branches of a science coming before them in different years, the medal should in every case be bestowed as a recognition of high merit in some important branch of the science.

Before speaking of the Tables, I will notice some of Professor Newcomb's other works.

Memoir "On the secular Variations and mutual Relations of the Orbits of the Asteroids," *Mem. American Academy*, vol. V. (1860), pp. 124—152. The object is to examine those circumstances of the forms, positions, variations, and general relations of the asteroid orbits which may serve as a test, complete or imperfect, of any hypothesis respecting the cause from which they originated, or the reason why they are in a

group by themselves. Every *a posteriori* test is founded on the supposition, that the hypothesis necessarily or probably implies that certain conditions must be satisfied by the asteroids or their orbits, viz. in the one case the conditions are those which follow necessarily and immediately from the hypothesis itself, in the other case those which are deducible from it by the principle of random distribution. The two principal hypotheses are that of Olbers, where the asteroids are supposed to be the fragments of a shattered single planet: and the hypothesis that they were formed by the breaking up of a ring of nebulous matter. On the first hypothesis the orbits of all the asteroids once intersected in a common point; the second affords no conclusion equally susceptible of an *a posteriori* test.

But for a rigorous or probable test of either hypothesis, what is needed is rigorous expressions in terms of the time for the eccentricity, inclination, and longitudes of perihelion and node of each of the asteroids considered, or, what is the same thing, the computation of the secular variations of the quantities h, l, p, q, which replace these elements. The investigation is applied to those asteroids the elements of which were determined with sufficient accuracy, and the eccentricities and inclinations of which were sufficiently small (limit taken is 11°). And the backbone of the memoir is the investigation of the h, l, p, q, for twenty-five asteroids included between the numbers (1) and (40). In this calculation, as was clearly necessary, the action of the asteroids on the larger planets and on each other was neglected; the expressions for the h, l, p, q, of the larger planets are regarded as given—they are, in fact, taken from Le Verrier (as calculated by him before the discovery of *Neptune*, but afterwards partially extended to that planet). The effect is that the differential coefficients $\frac{dh}{dt}$, &c. are given each of them as a sum of sines or cosines of arguments varying with the time; and thus, although the calculation is sufficiently laborious, the process is not one of the extreme labour and difficulty which it is in the case of the larger planets. The resulting table of the h, l, p, q, of the twenty-five asteroids has, of course, a value quite independent of the theoretical part of the memoir. Of this it is sufficient to say here that the conclusion is on the whole against Olbers's hypothesis. The subject is resumed, and more fully examined in a paper in the *Astronomische Nachrichten*, t. LVIII.

"Investigation of the Distance of the Sun and of the Elements which depend upon it, from the Observations of *Mars* made during the Opposition of 1862, and from other Sources," *Washington Observations for* 1865, Appendix II., pp. 1—29. The chief part of this valuable Memoir is occupied with a determination of the solar parallax by the discussion of the observations of *Mars* made in 1862 on the plan of Winnecke: three partial discussions had previously appeared, but these having been by comparisons of pairs of observations, one in each hemisphere, many observations in one hemisphere were lost by want of a corresponding observation in the other hemisphere; and out of a total of nearly 300 observations, only 125 were utilised. The idea is, the perturbations of the Earth and *Mars* being perfectly known for the period under consideration, every observation of the planet would lead rigorously to an equation of condition between its parallax, the six elements of its orbit, and the six elements of

the Earth's orbit—thus 13 or more observations, when compared with any theory, should suffice to correct the errors of that theory. But the observations extending only over a short interval, say one month, the coefficients would be so minute as to give no trustworthy value of the corrections; the equations only suffice to determine a few functions of the elements which, being determined, the equations will be satisfied by widely differing values of the elements, if only these values are such as to give to the functions their right values. And by fixing *a priori* the entire number of functions in question, and using them in place of the elements of the Earth and *Mars*, the equations will be practically as rigorous as if all the 13 unknown quantities had been introduced. By such considerations as these, each observation is made to give a relation between only 3 unknown quantities, the correction of the Sun's parallax being one of them.

The principle appears to be one of extended application, in regard to the proper mode of dealing with the constantly recurring problem of the determination of a set of corrections from a large number of linear equations; and it is used by the author in regard to the equations which present themselves in his theories of *Neptune* and *Uranus*.

Returning to the *Mars* observations, these were made at six Northern and three Southern Observatories, the total number being 154 Northern, and 143 Southern, together 297 observations. There was the difficulty of reducing to a concordant system the observations at the different Observatories, since (the whole number of comparison stars not being observed on each night) the adopted mean position of each of them was not unimportant. But this being carefully discussed and allowed for, the observations, extending from August 21 to November 3, 1862, are divided into five groups, and from these is deduced a correction to the provisional value 8″·9 of the parallax. The author then reproduces or discusses other determinations, from micrometric observations of *Mars*, the parallactic inequality of the Moon, the lunar equation of the Earth, the transit of 1769, and Foucault's experiment on Light—the last result, as not a strictly astronomical one, and with no means of assigning its probable error, is left out of consideration—and the combination of the remaining ones gives the author's concluded value of the parallax; from which other astronomical constants are deduced.

"On the Right Ascensions of the Equatoreal Fundamental Stars and the Corrections necessary to reduce the Right Ascensions of different Catalogues to a mean homogeneous System," *Washington Observations for* 1870, Appendix III., pp. 1—73.

This important Memoir is referred to in the Council Report for 1873. The object is to do for the right ascensions of the equatoreal and zodiacal Stars what had been done by Auwers for the declinations, namely, to furnish the data necessary to reduce the principal original catalogues of stars to a homogeneous system by freeing them of their systematic differences. The results are contained in two tables of corrections (as depending on the R.A. and N.P.D. respectively) to the several catalogues; and in a table of concluded mean right ascensions for the beginning of each fifth Besselian year,

1750 to 1900, of 32 fundamental Stars, and of periodic terms in the right ascensions of *Sirius* and *Procyon*.

The evil of systematic differences between the observations of different Observatories of course presents itself in every case where such observations have to be combined: for instance, in the just-mentioned determination of the solar parallax by the observations of *Mars*; and in the making of a set of planetary tables: and all that tends to remove or diminish it is most important to the progress of Astronomy. I cannot help thinking that there should be some confederation of Observatories, or Central calculating Board, for publishing the lunar and planetary observations, &c., reduced to a concordant system. It seems hard upon the maker of a set of planetary tables that he should not at least have, ready to hand for comparison with his theory, a single and entire series of the observations of the planet.

"Théorie des Perturbations de la Lune, qui sont dues à l'action des Planètes," *Liouville*, t. XVI. (1871), pp. 1—45. This is a very important theoretical Memoir on the disturbed motion of three bodies: a problem which, so far as I am aware, has not hitherto been considered at all. I have elsewhere remarked that the so-called "Problem of Three Bodies," as usually treated is not really this problem at all, but a different and more simple one—that of disturbed elliptic motion. Thus, in the planetary theory, each planet is considered as moving in an ellipse, and as disturbed by the action of forces represented by means of a disturbing function peculiar to the planet in question. An approach is made to the problem of three bodies when, as in memoirs by Hamilton and Jacobi, the (say) two planets are replaced by two fictitious bodies, and instead of a disturbing function peculiar to each planet, the motion of the system is made to depend on a single disturbing function. And there are memoirs by Jacobi, Bertrand, and Bour, which do relate to the proper problem of three bodies, viz. to their undisturbed motion. But in the present Memoir, Professor Newcomb starts from this problem as if it were actually solved, viz. he takes the coordinates of the three bodies (Sun, Earth, and Moon) as given in terms of the time and of 18 constants of integration *. And then considering the system as acted upon by the attraction of a planet, represented by means of a disturbing function, he applies to the system of the three bodies the method of the variation of the elements. The six elements which determine the motion of the centre of gravity of the system are left out of consideration; there remain to be considered 12 elements only; six of these are ϵ_0, π_0, θ_0, ϵ_0', π_0', θ_0' (initial mean longitudes and longitudes of pericentre and node): but the other six k_ϵ, k_π, &c., are functions the invention of which is a leading step in the theory, and it is in fact by means of them that the investigation is brought to a successful conclusion: the expressions of the last-mentioned six functions can, it is stated, be formed with facility by means of the developments (obtainable from the lunar theory) of the rectangular

* Of course the expressions actually used must be approximations: the centre of gravity of the Earth and Moon is regarded as moving round the Sun in an ellipse affected by a secular motion of perihelion (ultimately neglected); and the coordinates of the Moon in regard to the Earth are considered to be given by Delaunay's Lunar Theory. The centre of gravity of the whole system (in the undisturbed motion) moves uniformly in a right line, viz. the coordinates are $a+a't$, $b+b't$, $c+c't$; and we have thus the whole number $6+6+6$, $=18$, of arbitrary constants.

coordinates x, y, z, as periodic functions of the time. With these twelve elements, the expressions for the variations assume the canonical form

$$\frac{dk_\epsilon}{dt} = \frac{dR}{d\epsilon_0}, \quad \frac{d\epsilon_0}{dt} = -\frac{dR}{dk_\epsilon}, \text{ \&c.}$$

The concluding part of the Memoir contains approximate calculations which seems to show that the whole process is a very practicable one: but the author remarks that it is only doing justice to Delaunay to say that, starting from his (Delaunay's) final differential equations, and regarding the planet as adding new terms to the disturbing function, there would be obtained equations of the same degree of rigour as those of his own Memoir.

Everything in the Lunar Theory is laborious, and it is impossible to form an opinion as to the comparative facility of methods; but irrespectively of the possible applications of the method, the Memoir is, from the boldness of the conception and beauty of the results, a very remarkable one, and constitutes an important addition to Theoretical Dynamics *.

I come now to the planets *Neptune* and *Uranus*: it is well-known how, historically, the two are connected. The increasing and systematic inaccuracies of Bouvard's Tables of *Uranus* were found to be such as could be accounted for by the existence of an exterior disturbing planet; and it was thus that the planet *Neptune* was discovered by Adams and Le Verrier before it was seen in the telescope, in September 1846. It was afterwards ascertained that the planet had been seen twice by Lalande, in May 1795. The theory of *Neptune* was investigated by Peirce and Walker: viz. Walker, by means of the observations of 1795, and those of 1846—47, and using Peirce's formulæ for the perturbations produced by *Jupiter*, *Saturn*, and *Uranus*, determined successfully two sets of elliptic elements of the planet. The values first obtained showed that it was necessary to revise the perturbation-theory, which Peirce accordingly did, and with the new perturbations and revised normal places, the second set of elements (*Walker's Elliptic Elements II.*) was computed. With these elements and perturbations there was obtained for the planet from the time of its discovery a continuous ephemeris, published in the *Smithsonian Contributions*, *Gould's Astronomical Journal*, and since 1852 in the *American Ephemeris* and the *Nautical Almanac*. The theory was next considered by Kowalski in a work published at Kasan in the year 1855. The long period inequalities are dealt with by him in a manner different from that adopted by Peirce, so that the two theories are not directly comparable, but Professor Newcomb, by a comparison of the ephemerides with observation, arrives at the conclusion that the theory of Kowalski (although derived from observations up to 1853, when the planet had moved through an arc of 16°) was on the whole no nearer the truth than that of Walker;

* Since the above was written, Professor Newcomb has communicated to me some very interesting details as to the extent to which he has carried his computations, and in particular he mentions that, considering the action of each planet from *Mercury* to *Saturn*, he has (in regard to the terms the coefficients of which might become large by integration) estimated the probable limiting value of more than fifty such terms of period from a few years to several thousands without finding any which could become sensible, except the term leading to Hansen's first inequality produced by *Venus*.

he observed, however, that this failure is accounted for by an accidental mistake in the computation of the perturbations of the radius vector by *Jupiter*.

Professor Newcomb's theory of *Neptune* is published in the *Smithsonian Contributions* under the title "An Investigation of the Orbit of *Neptune*, with General Tables of its Motion," (accepted for publication, May 1865). The errors of the published ephemerides were increasing rapidly; in 1863 Walker's was in error by 33″, and Kowalski's by 22″; both might be in error by 5′ before the end of the century. The time was come when (the planet having moved through nearly 40°) the orbit could be determined with some degree of accuracy. The general objects of the work are stated to be:

(1) To determine the elements of the orbit of *Neptune* with as much exactness as a series of observations extending through an arc of 40° would admit of.

(2) To inquire whether the mass of *Uranus* can be concluded from the motion of *Neptune*.

(3) To inquire whether these motions indicate the action of an extra-Neptunian planet, or throw any light on the question of the existence of such planet.

(4) To construct general tables and formulæ, by which the theoretical place of *Neptune* may be found at any time, and more particularly between the years 1600 and 2000.

The formation of the tables of a planet may, I think, be considered as the culminating achievement of Astronomy: the need and possibility of the improvement and approximate perfection of the tables advance simultaneously with the progress of practical astronomy, and the accumulation of accurate observations; and the difficulty and labour increase with the degree of perfection aimed at. The leading steps of the process are in each case the same, and it is well-known what these are; but it will be convenient to speak of them in order, with reference to the present tables: they are *first* to decide on the form of the formulæ, whether the perturbations shall be applied to the elements or the coordinates—or partly to the elements and partly to the coordinates; and as to other collateral matters. These are questions to be decided in each case, in part by reference to the numerical values (in particular, the ratios and approach to commensurability of the mean motions), in part by the degree of accuracy aimed at, or which is attainable—the tables may be intended to hold good for a few centuries, or for a much longer period. The general theory as regards these several forms ought, I think, to be developed to such an extent, that it should be possible to select, according to the circumstances, between two or three ready-made theories; and that the substitution therein of the adopted numerical values should be a mere mechanical operation; but in the planetary theory in its present state, this is very far from being the case, and there is always a large amount of delicate theoretical investigation to be gone through in the selection of the form and development of the algebraical formulæ which serve as the basis of the tables. In Prof. Newcomb's theory the perturbations are applied to the elements; in particular, it was determined that the long inequality arising from the near approach of the mean motion of *Uranus* to twice that of *Neptune* (period about 4,300 years), should be developed as a perturbation, not of the coordinates, but of the elements. And it was best, (as for a theory designed

to remain of the highest degree of exactness for only a few centuries) to take not the mean values of the elements, but their values at a particular epoch during the period for which the theory is intended to be used. The adopted provisional elements of *Neptune*, and the elements of the disturbing planets, are accordingly not mean values, but values affected by secular and long inequalities, representing the actual values at the present time. *Secondly*, the form being decided on and the formulæ obtained, the numerical values of the adopted provisional elements of the planet, and of the elements of the disturbing planets and their masses, have to be substituted, so as to obtain the actual formulæ serving for the calculation of a provisional ephemeris; and such ephemeris, first of heliocentric, and then of geocentric positions, has to be computed for the period over which the observations extend. *Thirdly*, the ephemeris, computed as above, has to be compared with the observed positions; viz. in the present case these are, Lalande's two observations of 1795, and the modern observations at the Observatories of Greenwich, Cambridge, Paris, Washington, Hamburg, and Albany, extending over different periods from 1846 to 1864: these are discussed in reference to their systematic differences, and they are then corrected accordingly, so as to reduce the several series of observations to a concordant system. In this way is formed a series of 71 observed longitudes and latitudes (1795, and 1846 to 1864); the comparison of these with the computed values shows the errors of the provisional ephemeris. *Fourthly*, the errors of the provisional elements have to be corrected by means of the last-mentioned series of errors: as regards the longitudes, the comparison gives a series of equations between $\delta\epsilon$, δn, δh, δk, and μ (correction to the assumed mass of *Uranus*). The discussion of the equations shows that no reliable value of μ can be obtained from them; it indeed appears that, if *Uranus* had been unknown, its existence could scarcely have been detected from all the observations hitherto made of *Neptune* (far less is there any indication to be as yet obtained as to the existence of a trans-Neptunian planet): hence, finally, μ is taken $=0$, and the equations used for the determination of the remaining corrections. As regards the latitudes, the comparison gives a series of equations serving for the determination of the values of δp and δq. And applying the corrections to the provisional elements, the author obtains his concluded elements; viz. as already mentioned, these are the values, as affected by the long inequality, belonging to the epoch 1850. *Fifthly*, the tables are computed from the concluded elements, and the perturbations of the provisional theory.

After the elements of *Neptune* were ascertained, the question of its action on *Uranus* was considered by Peirce in a paper in the *Proc. American Acad.*, vol. I. (1848), pp. 334—337. This contains the results of a complete computation of the general perturbations of *Uranus* by *Neptune* in longitude and radius vector, but without any details of the investigation, or statement of the methods employed: it is accompanied by a comparison of the calculated and observed longitudes of *Uranus* (with three different masses of *Neptune*) for years at intervals from 1690 to 1845, and for one of these masses the residuals are so small that it appears that, using these perturbations by *Neptune* and Le Verrier's perturbations by *Jupiter* and *Saturn*, there existed a theory of *Uranus* from which quite accurate tables might have been constructed. But this was never done. The ephemeris of *Uranus* in the *American Ephemeris* was intended

to be founded on the theory, but the proper definitive elements do not seem to have been adopted: and in the *Nautical Almanac* for the years up to 1876, Bouvard's Tables of *Uranus* were still employed; for the year 1877 the ephemeris is derived from heliocentric places communicated by Prof. Newcomb.

An extended investigation of the subject was made by Safford, but only a brief general description of his results is published, *Monthly Notices, R.A.S.*, vol. XXII. (1862). The effect of *Neptune* was here computed by mechanical quadratures; and corrections were obtained for the mass of *Neptune* and elements of *Uranus*.

Professor Newcomb's Tables of *Uranus* have only recently appeared. They are published in the *Smithsonian Contributions* under the title "An Investigation of the Orbit of *Uranus*, with General Tables of its Motion," (accepted for publication February, 1873), forming a volume of about 300 pages. The work was undertaken as far back as 1859, but the labour devoted to it at first amounted to little more than tentative efforts to obtain numerical data of sufficient accuracy to serve as a basis of the theory, and to decide on a satisfactory way of computing the general perturbations. First, the elements of *Neptune* had to be corrected, and this led to the foregoing investigation of that planet: it then appeared that the received elements of *Uranus* also differed too widely from the truth to serve as the basis of the work, and they were provisionally corrected by a series of heliocentric longitudes, derived from observations extending from 1781 to 1861. Finally, it was found that the adopted method of computing the perturbations, that of the "variation of the elements," was practically inapplicable to the computation of the more difficult terms, viz. those of the second order in regard to the disturbing force. While entertaining a high opinion of Hansen's method as at once general, practicable, and fully developed, the author conceived that it was on the whole preferable to express the perturbations directly in terms of the time, owing to the ease with which the results of different investigations could be compared, and corrections to the theory introduced; and under these circumstances he worked out the method described in the first chapter of his treatise, not closely examining how much it contained that was essentially new. With these improved elements and methods the work was recommenced in 1868; the investigation has occupied him during the subsequent five years: and, though aided by computers, every part of the work has been done under his immediate direction, and as nearly as possible in the same way as if he had done it himself: a result in some cases obtained only by an amount of labour approximating to that saved by the employment of the computer.

The leading steps of the investigation correspond to those for *Neptune*: there is, *first*, the theoretical investigation already referred to; *secondly*, the formation of the provisional theory with assumed elements; *thirdly*, the comparison with observation; and here the observations are the accidental ones previous to the discovery of *Uranus* as a planet by Herschel in 1781, and the subsequent systematic ones of twelve Observatories, extending over intervals during periods from 1781 to 1872; all which have to be freed from systematic differences, and reduced to a concordant system as before: the operation is facilitated by the existence, since 1830, of ephemerides computed from Bouvard's Tables serving as an intermediate term for the comparison of

the observations with the provisional theory. *Fourthly*, the correction of the elements of the provisional theory, viz. the equations for the comparison of the longitudes give $\delta\epsilon$, δn, δh, δk, and a correction to the assumed mass of *Neptune*, which mass is thus brought out $= \frac{1}{19640}$. And the equations for the comparison of latitudes give δp, δq; there is thus obtained a corrected set of elements (Newcomb's Elements IV.), being for the year 1850, the elements as affected with the long inequality; these are the elements upon which the Tables are founded. But it is theoretically interesting to have the absolute mean values of the elements, and the author accordingly obtains these (his Elements V.) together with the corrections corresponding to a varied mass of *Neptune*, $\left(\text{that is, the terms in } \mu \text{ corresponding to a mass } \frac{1+\mu}{19700}\right)$; he remarks that, admitting the mass of *Neptune* to be uncertain by about one-fiftieth of its value, the mean longitude of the perihelion of *Uranus* is from this cause uncertain by more than two minutes, the mean longitude of the planet by nearly a minute, and the mean motion by nearly two seconds in a century. *Fifthly*, the formation of the tables, based on the Elements IV.; the tables calculated with these elements are intended to hold good for the period between the years 1000 and 2200; but by aid of the Elements V. they may be made applicable for a more extended period.

In what precedes I have endeavoured to give you an account of Professor Newcomb's writings: they exhibit all of them a combination, on the one hand, of mathematical skill and power, and on the other hand of good hard work—devoted to the furtherance of Astronomical Science. The Memoir on the Lunar Theory contains the successful development of a highly original idea, and cannot but be regarded as a great step in advance in the method of the variation of the elements and in theoretical dynamics generally; the two sets of planetary tables are works of immense labour, embodying results only attainable by the exercise of such labour under the guidance of profound mathematical skill—and which are needs in the present state of Astronomy. I trust that imperfectly as my task is accomplished, I shall have satisfied you that we have done well in the award of our medal.

The President then, delivering the medal to the Foreign Secretary, addressed him in the following terms:

Mr Huggins—I request that you will have the goodness to transmit to Professor Newcomb this medal, as an expression of the opinion of the Society of the excellence and importance of what he has accomplished; and to assure him at the same time of our best wishes for his health and happiness, and for the long and successful continuation of his career as a worker in our science.

580.

ON THE NUMBER OF DISTINCT TERMS IN A SYMMETRICAL OR PARTIALLY SYMMETRICAL DETERMINANT.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXIV. (1873—1874), pp. 303—307, and p. 335.]

THE determination of a set of unknown quantities by the method of least squares is effected by means of formulæ depending on symmetrical or partially symmetrical determinants; and it is interesting to have an expression for the number of distinct terms in such a determinant.

The terms of a determinant are represented as duads, and the determinant itself as a bicolumn; viz. we write, for instance,

$$\left\{\begin{matrix} aa \\ bb \\ pp' \\ qq' \end{matrix}\right\} \text{ to represent the determinants } \begin{vmatrix} aa, & ab, & ap', & aq' \\ ba, & bb, & bp', & bq' \\ pa, & pb, & pp', & pq' \\ qa, & qb, & qp', & qq' \end{vmatrix}.$$

This being so if the duads are such that in general $rs = sr$, then the determinant is wholly or partially symmetrical; viz. the determinant just written down, for which the bicolumn contains such symbols as pp' and qq', (each letter $p, q, \ldots$ being distinct from every letter $p', q', \ldots$) is partially symmetrical, but a determinant such as $\left\{\begin{matrix} aa \\ bb \\ cc \end{matrix}\right\}$ is wholly symmetrical. A determinant for which the bicolumn has m rows aa, bb, &c., and n rows pp', qq', &c. is called a determinant (m, n); and the number of distinct terms in the developed expression of the determinant is taken to be $\phi(m, n)$; the problem is to find the number of distinct terms $\phi(m, n)$.

Consider a determinant (m, n) where n is not $=0$; for instance, the determinant above written down, which is (2, 2); this contains terms multiplied by qa, qb, qp', qq' respectively: where, disregarding signs, the whole factor multiplied by qa is $\left\{\begin{matrix} bb \\ ap' \\ pq' \end{matrix}\right\}$, which is a determinant (1, 2), and similarly the whole factor multiplied by qb is a determinant (1, 2). But the whole factor multiplied by qp' is the determinant $\left\{\begin{matrix} aa \\ bb \\ pq' \end{matrix}\right\}$, which is a determinant (2, 1), and the whole factor multiplied by qq' is also a determinant (2, 1).

Hence, observing that qa, qb, qp', qq' are distinct terms occurring *only* in the last line of the determinant, the number of distinct terms is equal to the sum of the numbers of distinct terms in the several component parts, or we have

$$\phi(2, 2) = 2\phi(1, 2) + 2\phi(2, 1);$$

and so in general:

$$\phi(m, n) = m\phi(m-1, n) + n\phi(m, n-1).$$

Consider next a completely symmetrical determinant $(m, 0)$; for instance (4, 0), the determinant

$$\left\{\begin{matrix} aa \\ bb \\ cc \\ dd \end{matrix}\right\}, = \begin{vmatrix} aa, & ab, & ac, & ad \\ ba, & bb, & bc, & bd \\ ca, & cb, & cc, & cd \\ da, & db, & dc, & dd \end{vmatrix}.$$

We have *first* the terms containing dd; the whole factor is $\left\{\begin{matrix} aa \\ bb \\ cc \end{matrix}\right\}$, which is a determinant (3, 0); *secondly*, the terms containing $ad.da$, or the like combinations, $bd.db$ or $cd.dc$; the whole factor multiplied by $ad.da$ is $\left\{\begin{matrix} aa \\ bb \end{matrix}\right\}$, which is a determinant (2, 0); *thirdly*, the terms containing $ad.db + bd.da$, $= 2ad.bd$; or the like combinations $2ad.cd$ or $2bd.cd$: the whole factor multiplying the term $2ad.bd$ is $\left\{\begin{matrix} cc \\ ba \end{matrix}\right\}$, which is a determinant (1, 1). Hence observing that ad, bd, cd, $= da$, db, dc, and dd are terms occurring *only* in the last line and column of the original determinant, it is clear that the number of distinct terms in the original determinant is equal to the sum of the numbers of distinct terms in the component parts, or that we have $\phi(4, 0) = \phi(3, 0) + 3\phi(2, 0) + 3\phi(1, 1)$; and so in general:

$$\phi(m, 0) = \phi(m-1, 0) + m\phi(m-2, 0) + \frac{m.m-1}{2}\phi(m-3, 1).$$

The two equations of differences, together with the initial values $\phi(0, 0) = 1$, $\phi(1, 0) = \phi(0, 1) = 1$, $\phi(2, 0) = \phi(1, 1) = \phi(1, 2) = 2$, enable the calculation of the successive values of $\phi(m, n)$: viz. arranging these in the order

$$\begin{array}{c} \phi(0, 0), \\ \phi(1, 0), \quad \phi(0, 1), \\ \phi(2, 0), \quad \phi(1, 1), \quad \phi(0, 2), \\ \phi(3, 0), \text{ \&c., \&c.,} \end{array}$$

we calculate simultaneously the lines $\phi(m, 0)$, $\phi(m, 1)$; and thence successively the remaining lines $\phi(m, 2)$, $\phi(m, 3)$, &c.: the values up to $m + n = 6$ being in fact

$$\begin{array}{ccccccc} & & & 1, & & & \\ & & 1, & & 1, & & \\ & 2, & & 2, & & 2, & \\ 5, & & 6, & & 6, & & 6, \\ 17, & 23, & & 24, & & 24, & 24, \\ 73, & 109, & 118, & & 120, & 120, & 120, \\ 388, & 618, & 690, & 714, & 720, & 720, & 720: \end{array}$$

where the process for the first two lines is

$$\begin{array}{rl} 5 = & 2 + {}_2 \,.\; 1 + \;.\; 1, \\ 17 = & 5 + {}_3 \,.\; 2 + {}_3 \,.\; 2, \\ 73 = & 17 + {}_4 \,.\; 5 + {}_6 \,.\; 6, \\ 388 = & 73 + {}_5 \,.\, 17 + \;.\, 23, \\ & \vdots \end{array} \qquad \begin{array}{rl} 6 = & {}_2 \,.\; 2 + 2, \\ 23 = & {}_3 \,.\; 6 + 5, \\ 109 = & {}_4 \,.\; 23 + 17, \\ 618 = & {}_5 \,.\, 109 + 23, \\ & \vdots \end{array}$$

the larger figures being those of the two lines, and the smaller ones numerical multipliers. And then for the third line, fourth line, &c., we have

$$\begin{array}{rl} 6 = & {}_1 \,.\; 2 + {}_2 \,.\; 2, \\ 24 = & {}_2 \,.\; 6 + {}_2 \,.\; 6, \\ 118 = & {}_3 \,.\; 24 + {}_2 \,.\; 23, \\ 690 = & {}_4 \,.\, 118 + {}_2 \,.\, 109, \end{array} \qquad \begin{array}{rl} 120 = & {}_2 \,.\; 24 + {}_3 \,.\; 24, \\ 714 = & {}_3 \,.\, 120 + {}_4 \,.\, 118, \\ & \vdots \end{array}$$

and so on.

This is, in fact, the easiest way of obtaining the actual numerical values; but we may obtain an analytical formula. Considering the two equations

$$\phi(m, 1) = m\phi(m-1, 1) + \phi(m, 0),$$

$$\phi(m, 0) = \phi(m-1, 0) + m\phi(m-2, 0) + \frac{m \,.\, m-1}{2}\,\phi(m-3, 1);$$

and using the first of these to eliminate the term $\phi(m-3, 1)$ and resulting terms $\phi(m-4, 1)$, &c. which present themselves in the second equation, this, after a succession of reductions, becomes

$$\begin{aligned}\phi(m, 0) = \quad & \phi(m-1, 0) \\ & + (m-1)\phi(m-2, 0) \\ & + \frac{m.m-1}{2}\{\phi(m-3, 0) \\ & \qquad + (m-3)\phi(m-4, 0) \\ & \qquad \vdots \\ & \qquad + (m-3)\ldots 3.2\phi(1, 0) \\ & \qquad + (m-3)\ldots 3.2.1 \quad \};\end{aligned}$$

or, observing that the last term $(m-3)\ldots 3.2.1$ is, in fact, $=(m-3)\ldots 3.2.1\phi(0, 0)$, this may be written:

$$\begin{aligned}2\phi(m, 0) - \phi(m-1, 0) - (m-1)\phi(m-2, 0) = \quad & \phi(m-1, 0) \\ & + (m-1)\phi(m-2, 0) \\ & + (m-1)(m-2)\phi(m-3, 0) \\ & \vdots \\ & + (m-1)..3.2.1\phi(\quad 0, 0).\end{aligned}$$

And hence assuming

$$u = \phi(0, 0) + \frac{x}{1}\phi(1, 0) + \frac{x^2}{1.2}\phi(2, 0) + \ldots + \frac{x^m}{1.2\ldots m}\phi(m, 0) + \ldots,$$

we find at once

$$2\frac{du}{dx} - u - xu = \frac{u}{1-x},$$

that is,

$$2\frac{du}{u} = dx\left(1 + x + \frac{1}{1-x}\right),$$

or integrating and determining the constant so that u shall become $=1$ for $x=0$, we have

$$u = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{1-x}};$$

wherefore we have

$$\phi(m, 0) = 1.2\ldots m \text{ coefft. } x^m \text{ in } \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{1-x}}.$$

Developing as far as x^6, the numerical process is

	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{48}$	$\frac{1}{384}$	$\frac{1}{3840}$	$\frac{1}{46080}$
	1		$\frac{1}{4}$		$\frac{1}{32}$		$\frac{1}{384}$
	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{48}$	$\frac{1}{384}$	$\frac{1}{3840}$	$\frac{1}{46080}$
			$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{192}$	$\frac{1}{1536}$
					$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{256}$
							$\frac{1}{384}$
	1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{48}$	$\frac{25}{384}$	$\frac{27}{1280}$	$\frac{331}{46080}$
	1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{35}{128}$	$\frac{63}{256}$	$\frac{231}{1024}$
	1	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{48}$	$\frac{25}{384}$	$\frac{27}{1280}$	$\frac{331}{46080}$
		$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{7}{96}$	$\frac{25}{768}$	$\frac{27}{2560}$
			$\frac{3}{8}$	$\frac{3}{16}$	$\frac{9}{64}$	$\frac{7}{128}$	$\frac{25}{1024}$
				$\frac{5}{16}$	$\frac{5}{32}$	$\frac{15}{128}$	$\frac{35}{768}$
					$\frac{35}{128}$	$\frac{35}{256}$	$\frac{105}{1024}$
						$\frac{63}{256}$	$\frac{63}{512}$
							$\frac{231}{1024}$
	1	1	1	$\frac{5}{6}$	$\frac{17}{24}$	$\frac{73}{120}$	$\frac{97}{180}$
× by	1	1	2	6	24	120	720
	1	1	2	5	17	73	388,

agreeing with the former values.

The expression of $\phi(m, 0)$ once found, it is easy thence to obtain

$$\phi(m, 1) = 1 . 2 \ldots . m \text{ coefft. } x^m \text{ in } \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{3}{2}}}$$

$$\phi(m, 2) = 1 . 2 \ldots . m \text{ coefft. } x^m \text{ in } \frac{2e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{5}{2}}}$$

$$\phi(m, 3) = 1 . 2 \ldots . m \text{ coefft. } x^m \text{ in } \frac{2 . 3e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{7}{2}}}$$

and so on, the law being obvious.

[*Addition*, p. 335.] The generating function

$$u, = 1 + u_1 x + \ldots + u_n \frac{x^n}{1 . 2 \ldots n} + \ldots, = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{1-x}},$$

was obtained as the solution of the differential equation

$$2\frac{du}{dx} = u\left(1 + x + \frac{1}{1-x}\right).$$

Writing this in the form

$$2(1-x)\frac{du}{dx} = u(2-x^2),$$

we at once obtain for u_n the equation of differences,

$$u_n = nu_{n-1} - \tfrac{1}{2}(n-1)(n-2)u_{n-3};$$

and it thus appears that the values of u_n (number of distinct terms in a symmetrical determination of the order n) can be calculated the one from the other by the process

$$\begin{aligned}
n &= 1, & 1 &= {}_1 .\ 1,\\
&= 2, & 2 &= {}_2 .\ 1,\\
&= 3, & 5 &= {}_3 .\ 2 - {}_1 . 1,\\
&= 4, & 17 &= {}_4 .\ 5 - {}_3 . 1,\\
&= 5, & 73 &= {}_5 . 17 - {}_6 . 2,\\
&= 6, & 388 &= {}_6 . 73 - {}_{10} . 5,\\
& & &\text{\&c.}
\end{aligned}$$

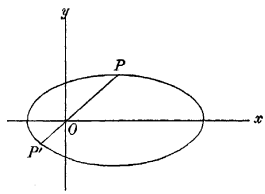
which is one of extreme facility.

581.

ON A THEOREM IN ELLIPTIC MOTION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXV. (1874—1875), pp. 337—339.]

LET a body move through apocentre between two opposite points of its orbit, say from the point P, eccentric anomaly u, to the point P', eccentric anomaly u', where



u, u' are each positive, $u < \pi$, $u' > \pi$. Taking the origin at the focus, and the axis of x in the direction through apocentre, then—

$$\text{Coordinates of } P \text{ are } x = a(-\cos u + e), \quad y = a\sqrt{1-e^2}\sin u,$$

$$\text{,, } \quad P' \text{ ,, } x = a(-\cos u' + e), \quad y = a\sqrt{1-e^2}\sin u';$$

whence, expressing that the points P, P' are in a line with the focus,

$$\sin u'(-\cos u + e) - \sin u(-\cos u' + e) = 0,$$

that is,

$$\sin(u' - u) = e(\sin u' - \sin u),$$

which is negative, viz. $u' - u$ is $> \pi$.

The time of passage from P to P' is

$$\begin{aligned} nt &= (u' - e\sin u') - (u - e\sin u), \\ &= u' - u - e(\sin u' - \sin u), \\ &= u' - u - \sin(u' - u), \end{aligned}$$

which, $u' - u$ being greater than π and $-\sin(u' - u)$ positive, is greater than π; viz. the time of passage is greater than one-half the periodic time. Of course, if P and P' are at pericentre and apocentre, the time of passage is equal one-half the periodic time.

The time of passage from P' to P through the pericentre is

$$nt = 2\pi - (u' - u) + \sin(u' - u),$$

which is

$$= 2\pi - (u' - u) - \sin\{2\pi - (u' - u)\},$$

where $2\pi - (u' - u)$, $= \alpha$ suppose, is an angle $< \pi$. Writing, then

$$nt = \alpha - \sin\alpha,$$

and comparing with the known expression for the time in the case of a body falling directly towards the centre of force, we see that the time of passage from P' to P through the pericentre, is equal to the time of falling directly towards the same centre of force from rest at the distance $2a$ to the distance $a(1 + \cos\alpha)$, where, as above $\alpha = 2\pi - (u' - u)$, $u' - u$ being the difference of the eccentric anomalies at the two opposite points P, P'. If $\alpha = \pi$, the times of passage are each $= \frac{\pi}{n}$, that is, one-half the periodic time.

The foregoing equation $\sin(u' - u) = e(\sin u' - \sin u)$ gives obviously

$$\cos\tfrac{1}{2}(u' - u) = e\cos\tfrac{1}{2}(u' + u);$$

that is,

$$1 + \tan\tfrac{1}{2}u\tan\tfrac{1}{2}u' = e(1 - \tan\tfrac{1}{2}u\tan\tfrac{1}{2}u'),$$

or,

$$-\tan\tfrac{1}{2}u\tan\tfrac{1}{2}u' = \frac{1-e}{1+e};$$

(in the figure $\tan\frac{1}{2}u$ is positive, $\tan\frac{1}{2}u'$ negative); and we thence obtain further

$$\begin{aligned} \sin\tfrac{1}{2}(u' - u) &= \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u(\tan\tfrac{1}{2}u' - \tan\tfrac{1}{2}u), \\ \sin\tfrac{1}{2}(u' + u) &= \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u(\tan\tfrac{1}{2}u' + \tan\tfrac{1}{2}u), \\ \cos\tfrac{1}{2}(u' - u) &= \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u \cdot \frac{2e}{1+e}, \\ \cos\tfrac{1}{2}(u' + u) &= \cos\tfrac{1}{2}u'\cos\tfrac{1}{2}u \cdot \frac{2}{1+e}; \end{aligned}$$

and thence also

$$\cos u + \cos u' = 2 \cos \tfrac{1}{2}(u' + u) \cos \tfrac{1}{2}(u' - u),$$

$$= \cos^2 \tfrac{1}{2} u' \cos^2 \tfrac{1}{2} u \cdot \frac{8e}{(1+e)^2}.$$

But we have

$$1 + \cos(u' - u) = 2 \cos^2 \tfrac{1}{2}(u' - u) = \cos^2 \tfrac{1}{2} u' \cos^2 \tfrac{1}{2} u \cdot \frac{8e^2}{(1+e)^2},$$

or, comparing with the last equation,

$$1 + \cos(u' - u) = e(\cos u + \cos u'),$$

or, what is the same thing,

$$1 - \cos(u' - u) = (1 - e \cos u') + (1 - e \cos u);$$

and in like manner,

$$1 + \cos(u' + u) = 2 \cos^2 \tfrac{1}{2}(u' + u) = \cos^2 \tfrac{1}{2} u' \,.\, \cos^2 \tfrac{1}{2} u \frac{8}{(1+e)^2};$$

or, comparing with the same equation,

$$1 + \cos(u' + u) = \frac{1}{e}(\cos u + \cos u'):$$

which are formulæ corresponding with the original equation

$$\sin(u' - u) = e(\sin u' - \sin u).$$

582.

NOTE ON THE THEORY OF PRECESSION AND NUTATION.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXV. (1874—1875), pp. 340—343.]

WE have in the dynamical theory of Precession and Nutation (see Bessel's *Fundamenta* (1818), p. 126),

$$C\frac{dp}{dt} + (B - A)\,qr = LS\,(x'y - xy')\,dm'\left(\frac{1}{\Delta^3} - \frac{1}{r^3}\right),$$

$$A\frac{dq}{dt} + (C - B)\,rp = LS\,(y'z - yz')\,dm'\left(\frac{1}{\Delta^3} - \frac{1}{r^3}\right),$$

$$B\frac{dr}{dt} + (A - C)\,pq = LS\,(z'x - zx')\,dm'\left(\frac{1}{\Delta^3} - \frac{1}{r^3}\right),$$

where L is the mass of the Sun or Moon, x, y, z the coordinates of its centre referred to the centre of the Earth as origin,

$$r = \sqrt{x^2 + y^2 + z^2},$$

the distance of its centre, and

$$\Delta = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

the distance of its centre from an element dm', coordinates (x', y', z') of the Earth's mass, the sum or integral S being extended to the whole mass of the Earth—I have written dm', r for Bessel's dm, r_1—, we have

$$\Delta^2 = r^2 - 2\,(xx' + yy' + zz') + x'^2 + y'^2 + z'^2;$$

and thence

$$\frac{1}{\Delta^3} - \frac{1}{r^3} = \frac{3}{r^5}\,(xx' + yy' + zz') - \tfrac{3}{2}\,\frac{1}{r^7}\,\{(x^2 + y^2 + z^2)\,(x'^2 + y'^2 + z'^2) - 5\,(xx' + yy' + zz')^2\} + \text{etc.}$$

The principal term is the first one,

$$\frac{3}{r^5}(xx' + yy' + zz');$$

but Bessel takes account also of the second term,

$$-\tfrac{3}{2}\frac{1}{r^7}\{(x^2+y^2+z^2)(x'^2+y'^2+z'^2) - 5(xx'+yy'+zz')^2\},$$

viz. considering the Earth as a solid of revolution (as to density as well as exterior form), he obtains in regard to it the following terms of $\sin\omega\dfrac{d\psi}{dt}$ and $\dfrac{d\omega}{dt}$ respectively;

$$\frac{3L}{4r^4}\frac{1}{Cn}.\,2(C-A)K(5\sin^2\delta - 1)\cos\delta\sin\alpha,$$

$$-\frac{3L}{4r^4}\frac{1}{Cn}.\,2(C-A)K(5\sin^2\delta - 1)\cos\delta\cos\alpha,$$

where

$$2(C-A)K = S(3\mu - 5\mu^3)\,2\pi\rho\, R^5 dR\, d\mu,$$

K being in fact a numerical quantity, relating to the Earth only, and the value of which is by pendulum observations ultimately found to be $= 0{\cdot}13603$.

Writing, for shortness,

$$(x^2+y^2+z^2)(x'^2+y'^2+z'^2) - 5(xx'+yy'+zz')^2 = \Omega,$$

then the foregoing terms of $\sin\omega\dfrac{d\psi}{dt}$ and $\dfrac{d\omega}{dt}$ depend, as regards their form, on the theorem that for any solid of revolution (about the axis of z) we have

$$\begin{aligned} &S(x'y - xy')\,\Omega dm', \quad S(y'z - yz')\,\Omega dm', \quad S(z'x - zx')\,\Omega dm' \\ &= 0, \\ &\tfrac{1}{2}y(x^2+y^2+z^2-5z^2)\,S[3(x'^2+y'^2+z'^2) - 5z'^2]\,z'dm', \\ &-\tfrac{1}{2}x(x^2+y^2+z^2-5z^2)\,S[3(x'^2+y'^2+z'^2) - 5z'^2]\,z'dm', \end{aligned}$$

respectively: viz. writing $x'^2+y'^2+z'^2 = R^2$, and $z' = R\mu$, also $x^2+y^2+z^2 = r^2$ and $x = r\cos\delta\cos\alpha$, $y = r\cos\delta\sin\alpha$, $z = r\sin\delta$, the values would be

$$\begin{aligned} &0, \\ &\tfrac{1}{2}r^3\cos\delta\sin\alpha\,(1 - 5\sin^2\delta)\,S(3 - 5\mu^2)\,\mu R^3 dm', \\ &-\tfrac{1}{2}r^3\cos\delta\sin\alpha\,(1 - 5\sin^2\delta)\,S(3 - 5\mu^2)\,\mu R^3 dm', \end{aligned}$$

which are of the form in question.

The verification is easy: the solid being one of revolution about the axis of z, any integral such as $Sx'z'^2dm'$ or $Sx'y'z'dm'$ which contains an odd power of x' or of y' is $=0$; while such integrals as $Sx'^2z'dm'$, $Sy'^2z'dm'$ are equal to each other, or, what is the same thing, each $=\frac{1}{2}S(x'^2+y'^2)\,z'dm'$. That we have $S(x'y - xy')\,\Omega dm' = 0$ is

at once seen to be true; considering the next integral $S(y'z - yz')\,\Omega dm'$, the terms of $(y'z - zy')\,\Omega$ which lead to non-evanescent integrals are

$$\begin{aligned}
&-yz'.(x^2+y^2+z^2)(x'^2+y'^2+z'^2),\\
&-5y'z.2yzy'z',\\
&+5yz'.(x^2x'^2+y^2y'^2+z^2z'^2);
\end{aligned}$$

giving in the integral the several terms

$$\begin{aligned}
&-y(x^2+y^2+z^2)\,S(x'^2+y'^2+z'^2)\,z'dm',\\
&-10yz^2.\tfrac{1}{2}\,S(x'^2+y'^2+z'^2-z'^2)\,z'dm',\\
&+5y(x^2+y^2+z^2-z^2).\tfrac{1}{2}\,S(x'^2+y'^2+z'^2-z'^2)\,z'dm',\\
&+yz^2Sz'^3\,dm',
\end{aligned}$$

viz. collecting, the value is

$$\begin{aligned}
(-1+\tfrac{5}{2}=)\quad &\tfrac{3}{2}\,(x^2+y^3+z^2)\,yS(x'^2+y'^2+z'^2)\,z'dm',\\
(-\tfrac{5}{2}=)\quad &-\tfrac{5}{2}\,(x^2+y^2+z^2)\,y\Sigma z'^3dm',\\
(-\tfrac{5}{2}-5=)\quad &-\tfrac{15}{2}\,yz^2S(x'^2+y'^2+z'^2)\,z'dm',\\
(+\tfrac{5}{2}+5+5=)\quad &+\tfrac{25}{2}\,yz^2Sz'^3dm';
\end{aligned}$$

which is

$$= \tfrac{1}{2}\,y(x^2+y^2+z^2-5z^2)\,S\,[3(x'^2+y'^2+z'^2)-5z'^2]\,z'dm';$$

and similarly the last term is

$$= -\tfrac{1}{2}\,x(x^2+y^2+z^2-5z^2)\,S\,[3(x'^2+y'^2+z'^2)-5z'^2]\,z'dm',$$

which completes the proof.

583.

ON SPHEROIDAL TRIGONOMETRY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXVII. (1876—1877), p. 92.]

THE fundamental formulæ of Spheroidal Trigonometry are those which belong to a right-angled triangle PSS_0, where P is the pole, PS, PS_0 arcs of meridian, and SS_0 a geodesic line cutting the meridian PS at a given angle, and the meridian PS_0 at right angles. We consider a spherical triangle PSS_0,

$$\begin{array}{llllll} \text{Sides} & PS, & PS_0, & SS_0 = \gamma, & \gamma_0, & s, \\ \text{Angles} & S_0, & S, & P = 90^\circ, & \theta, & l, \end{array}$$

where γ is the reduced colatitude of the point S on the spheroid (and thence also γ_0 the reduced colatitude of S_0) and θ the azimuth of the geodesic SS_0, or angle at which this cuts the meridian SP; and then if S be the length of the geodesic SS_0 measured as a circular arc, radius = Earth's equatoreal radius, and L be the angle SPS_0, S, L differ from the corresponding spherical quantities s, l by terms involving the excentricity of the spheroid, viz. calling this e and writing

$$k = \frac{e \cos \gamma_0}{\sqrt{1 - e^2 \sin^2 \gamma_0}},$$

then (see Hansen's "Geodätische Untersuchungen," *Abh. der K. Sächs. Gesell.*, t. VIII. (1865) pp. 15 and 23, but using the foregoing notation) we have, to terms of the sixth order in e,

$$\begin{aligned} \frac{S}{\sqrt{1-e^2}} = \quad & (1 + \tfrac{1}{4}k^2 + \tfrac{13}{64}k^4 + \tfrac{45}{256}k^6)\, s \\ & + (\tfrac{1}{8}k^2 + \tfrac{3}{32}k^4 + \tfrac{79}{1024}k^6) \sin 2s \\ & + (\tfrac{1}{256}k^4 + \tfrac{5}{1024}k^6) \sin 4s \\ & + \tfrac{1}{3072}k^6 \sin 6s; \end{aligned}$$

and

$$\begin{aligned} L \quad = l - \tfrac{1}{2}e^2 \sin \gamma_0 \{ & (1 - \tfrac{1}{8}k^2 + \tfrac{1}{4}e^2 - \tfrac{5}{64}k^4 + \tfrac{1}{8}e^4)\, s \\ & - (\tfrac{1}{16}k^2 + \tfrac{1}{32}k^4) \sin 2s \\ & + \tfrac{1}{256}k^4 \sin 4s \}, \end{aligned}$$

which are the formulæ in question.

584.

ADDITION TO PROF. R. S. BALL'S PAPER, "NOTE ON A TRANSFORMATION OF LAGRANGE'S EQUATIONS OF MOTION IN GENERALISED COORDINATES, WHICH IS CONVENIENT IN PHYSICAL ASTRONOMY."

[From the *Monthly Notices of the Royal Astronomical Society*, vol. XXXVII. (1876—1877), pp. 269—271.]

THE formulæ may be established in a somewhat different way, as follows:—

Consider the masses M_1, M_2, ...

Let X_1, Y_1, Z_1 be the coordinates (in reference to a fixed origin and axes) of the C.G. of M_1;

x_1, y_1, z_1 the coordinates (in reference to a parallel set of axes through the C.G. of M_1) of an element m_1 of the mass M_1, and similarly for the masses M_2, ...; the coordinates (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), ... all belonging to the same origin and axes;

And let $\dot{X}_1$, &c. denote the derived functions $\frac{dX_1}{dt}$, &c.

We have

$$\begin{aligned} T = \quad & S\tfrac{1}{2}m_1[(\dot{X}_1+\dot{x}_1)^2+(\dot{Y}_1+\dot{y}_1)^2+(\dot{Z}_1+\dot{z}_1)^2] \\ & +S\tfrac{1}{2}m_2[(\dot{X}_2+\dot{x}_2)^2+(\dot{Y}_2+\dot{y}_2)^2+(\dot{Z}_2+\dot{z}_2)^2] \\ & \quad \vdots \end{aligned}$$

or since $Sm_1x_1=0$, &c., and therefore also $Sm_1\dot{x}_1=0$, &c., this is

$$\begin{aligned} T = \quad & \tfrac{1}{2}M_1(\dot{X}_1^2+\dot{Y}_1^2+\dot{Z}_1^2) \qquad +S\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2+\dot{z}_1^2) \\ & +\tfrac{1}{2}M_2(\dot{X}_2^2+\dot{Y}_2^2+\dot{Z}_2^2) \qquad +S\tfrac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2+\dot{z}_2^2) \\ & \quad \vdots \qquad\qquad\qquad\qquad \vdots \end{aligned}$$

Write u, v, w for the coordinates of the C.G. of the whole system: then

$$M_1X_1 + M_2X_2 + \ldots = (M_1 + M_2 \ldots)\, u,$$
$$M_1Y_1 + M_2Y_2 + \ldots = (M_1 + M_2 \ldots)\, v,$$
$$M_1Z_1 + M_2Z_2 + \ldots = (M_1 + M_2 \ldots)\, w;$$

and thence

$$M_1\dot{X}_1 + M_2\dot{X}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{u},$$
$$M_1\dot{Y}_1 + M_2\dot{Y}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{v},$$
$$M_1\dot{Z}_1 + M_2\dot{Z}_2 + \ldots = (M_1 + M_2 \ldots)\, \dot{w};$$

and thence

$$\begin{aligned} T - \tfrac{1}{2}(M_1 + M_2 + \ldots)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) \\ = \frac{1}{M_1 + M_2 \ldots}\{M_1M_2[(\dot{X}_1 - \dot{X}_2)^2 + (\dot{Y}_1 - \dot{Y}_2)^2 + (\dot{Z}_1 - \dot{Z}_2)^2]\} \\ \vdots \\ + S\tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) \\ + S\tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \\ \vdots \end{aligned}$$

or, representing the function on the right-hand side by T', this is

$$T = \tfrac{1}{2}(M_1 + M_2 + \ldots)(\dot{u}_2 + \dot{v}_2 + \dot{w}_2) + T' \ldots, = T_0 + T'.$$

Suppose the positions are determined by means of the $6n$ coordinates $((q))$; the equations of motion are each of them of the form

$$\frac{d}{dt}\cdot\frac{dT_0}{d\dot{q}} - \frac{dT_0}{dq} + \frac{d}{dt}\cdot\frac{dT'}{d\dot{q}} - \frac{dT'}{dq} = -\frac{dV}{dq}.$$

But these admit of further reduction; the part in T_0 depends upon three terms, such as

$$\frac{d}{dt}\left(\dot{u}\frac{d\dot{u}}{d\dot{q}}\right) - \dot{u}\frac{d\dot{u}}{dq}, \quad = \frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}} + \dot{u}\left(\frac{d}{dt}\frac{d\dot{u}}{d\dot{q}} - \frac{d\dot{u}}{dq}\right).$$

But we have u a function of $((q))$, and thence

$$\frac{d\dot{u}}{d\dot{q}} = \frac{du}{dq}, \text{ or } \frac{d}{dt}\frac{d\dot{u}}{d\dot{q}} - \frac{d\dot{u}}{dq}, \quad = \frac{d}{dt}\frac{du}{dq} - \frac{d\dot{u}}{dq}, \quad = 0,$$

or the term is simply

$$= \frac{d\dot{u}}{dt}\frac{d\dot{u}}{dq}.$$

The equation thus becomes

$$(M_1 + M_2 \ldots)\left(\frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}} + \frac{d\dot{v}}{dt}\frac{d\dot{v}}{d\dot{q}} + \frac{d\dot{w}}{dt}\frac{d\dot{w}}{d\dot{q}}\right) + \frac{d}{dt}\frac{dT'}{d\dot{q}} - \frac{dT'}{dq} = -\frac{dV}{dq}.$$

Suppose now that T', V are functions of $6n-3$ out of the $6n$ coordinates $((q))$, and of the differential coefficients $\dot{q}$ of the same $6n-3$ coordinates, but are independent of the remaining three coordinates and of their differential coefficients; then, first, if q denotes any one of the three coordinates, the equation becomes

$$\frac{d\dot{u}}{dt}\frac{d\dot{u}}{d\dot{q}}+\frac{d\dot{v}}{dt}\frac{d\dot{v}}{d\dot{q}}+\frac{d\dot{w}}{dt}\frac{d\dot{w}}{d\dot{q}}=0;$$

or, better,

$$\frac{d\dot{u}}{dt}\frac{du}{dq}+\frac{d\dot{v}}{dt}\frac{dv}{dq}+\frac{d\dot{w}}{dt}\frac{dw}{dq}=0;$$

and the three equations of this form give

$$\frac{d\dot{u}}{dt}=0,\quad \frac{d\dot{v}}{dt}=0,\quad \frac{d\dot{w}}{dt}=0,$$

viz. these are the equations for the conservation of the motion of the centre of gravity.

And this being so, then, if q now denotes any one of the $6n-3$ coordinates, each of the remaining equations assumes the form

$$\frac{d}{dt}\cdot\frac{dT'}{d\dot{q}}-\frac{dT'}{dq}=-\frac{dV}{dq},$$

viz. we have thus $6n-3$ equations for the relative motion of the bodies of the system.

585.

A NEW THEOREM ON THE EQUILIBRIUM OF FOUR FORCES ACTING ON A SOLID BODY.

[From the *Philosophical Magazine*, vol. XXXI. (1866), pp. 78, 79; *Camb. Phil. Soc. Proc.* vol. I. (1866), p. 235.]

DEFINING the "moment of two lines" as the product of the shortest distance of the two lines into the sine of their inclination, then, if four forces acting along the lines 1, 2, 3, 4 respectively are in equilibrium, the lines must, as is known (Möbius), be four generating lines of an hyperboloid; and if 12 denote the moment of the lines 1 and 2, and similarly 13 the moment of the lines 1 and 3, &c., the forces are as

$$\sqrt{23.34.42} : \sqrt{34.41.13} : \sqrt{41.12.24} : \sqrt{12.23.31}.$$

Calling the four forces P_1, P_2, P_3, P_4, it follows as a corollary that we have

$$P_1P_2.12 = 12.34\sqrt{13.42}.\sqrt{14.23} = P_3P_4.34;$$

viz. the product of any two of the forces into the moment of the lines along which they act is equal to the product of the other two forces into the moment of the lines along which they act,—which is equivalent to Chasles's theorem, that, representing a force by a finite line of proportional magnitude, then in whatever way a system of forces is resolved into two forces, the volume of the tetrahedron formed by joining the extremities of the two representative lines is constant.

586.

ON THE MATHEMATICAL THEORY OF ISOMERS.

[From the *Philosophical Magazine*, vol. XLVII. (1874), pp. 444—446.]

I CONSIDER a "diagram," viz. a set of points H, O, N, C, &c. (any number of each), connected by links into a single assemblage under the condition that through each H there passes not more than one link, through each O not more than two links, through each N not more than three links, through each C not more than four links. Of course through every point there passes at least one link, or the points would not be connected into a single assemblage.

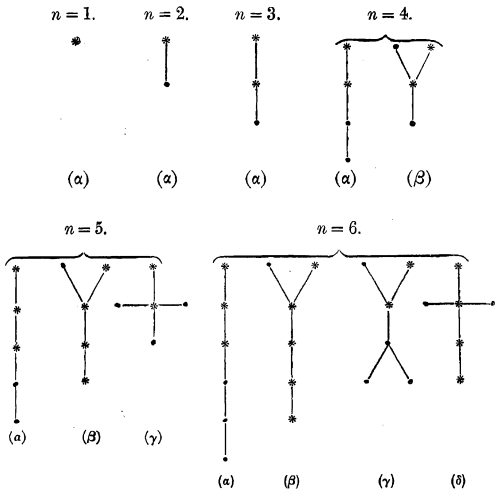
In such a diagram each point having its full number of links is saturate, or nilvalent: in particular, each point H is saturate. A point not having its full number of links is univalent, bivalent, or trivalent, according as it wants one, two, or three of its full number of links. If every point is saturate the diagram is saturate, or nilvalent; or, say, it is a "plerogram"; but if the diagram is susceptible of n more links, then it is n-valent; viz. the valency of the diagram is the sum of the valencies of the component points.

Since each H is connected by a single link (and therefore to a point O, C, &c. as the case may be, but not to another point H), we may without breaking up the diagram remove all the points H with the links belonging to them, and thus obtain a diagram without any points H: such a diagram may be termed a "kenogram": the valency is obviously that of the original diagram *plus* the number of removed H's.

If from a kenogram, we remove every point O, C, &c. connected with the rest of the diagram by a single link only (each with the link belonging to it), and so on indefinitely as long as the process is practicable, we arrive at last at a diagram in which every point O, C, &c. is connected with the rest of the diagram by two links at least: this may be called a "mere kenogram."

Each or any point of a mere kenogram may be made the origin of a "ramification"; viz. we have here links branching out from the original point, and then again from the derived points, and so on any number of times, and never again uniting. We can thus from the mere kenogram obtain (in an infinite variety of ways) a diagram. The diagram completely determines the mere kenogram; and consequently two diagrams cannot be identical unless they have the same mere kenogram. Observe that the mere kenogram may evanesce altogether; viz. this will be the case if the diagram or kenogram is a simple ramification.

A ramification of n points C is $(2n+2)$-valent: in fact, this is so in the most simple case $n=1$; and admitting it to be true for any value of n, it is at once seen to be true for the next succeeding value. But no kenogram of points C is so much as $(2n+2)$-valent; for instance, 3 points C linked into a triangle, instead of being 8-valent are only 6-valent. We have therefore plerograms of n points C and $2n+2$ points H, say plerograms CH^{2n+2}; and in any such plerogram the kenogram is of necessity a ramification of n points C; viz. the different cases of such ramifications are *



where the mathematical question of the determination of such forms belongs to the class of questions considered in my paper "On the Theory of the Analytical Forms called Trees," *Phil. Mag.* vol. XIII. (1857), [203], and vol. XVIII. (1859), [247], and in some papers on Partitions in the same Journal.

* The distinction in the diagrams of asterisks and dots is to be in the first instance disregarded; it is made in reference to what follows, the explanation as to the allotrious points.

The different forms of univalent diagrams C^nH^{2n+1} are obtained from the same ramifications by adding to each of them all but one of the $2n+2$ points H; that is, by adding to each point C except one its full number of points H, and to the excepted point one less than the full number of points H. The excepted point C must therefore be univalent at least; viz. it cannot be a saturate point, which presents itself for example in the diagrams $n=5\,(\gamma)$ and $n=6\,(\delta)$. And in order to count the number of distinct forms (for the diagrams C^nH^{2n+1}), we must in each of the above ramifications consider what is the number of distinct classes into which the points group themselves, or, say, the number of "allotrious" points. For instance, in the ramification $n=3$ there are two classes only; viz. a point is either terminal or medial; or, say, the number of allotrious points is $=2$: this is shown in the diagrams by means of the asterisks; so that in each case the points which may be considered allotrious are represented by asterisks, and the number of asterisks is equal to the number of allotrious points.

Thus, number of univalent diagrams C^nH^{2n+1}:

$n=1$,	.....		1
$n=2$,	.....		1
$n=3$,	.....		2
$n=4$,	$(\alpha)\,2$; $(\beta)\,2$;	together	4
$n=5$,	$(\alpha)\,3$; $(\beta)\,4$; $(\gamma)\,1$;	„	8
$n=6$,	$(\alpha)\,3$; $(\beta)\,5$; $(\gamma)\,2$; $(\delta)\,3$;	„	13

where it will be observed that, $n=5\,(\gamma)$, and $n=6\,(\delta)$, the numbers of allotrious points are 2 and 4 respectively; but since in each of these cases one point is saturate, they give only the numbers 1 and 3 respectively. It might be mathematically possible to obtain a general solution; but there would be little use in this; and for even the next succeeding case, No. of bivalent diagrams C^nH^{2n}; the extreme complexity of the question would, it is probable, prevent the attainment of a general solution.

Passing to the chemical signification of the formulæ, and instead of the radicals C^nH^{2n+1} considering the corresponding alcohols $C^nH^{2n+1}.OH$, then, $n=1, 2, 3, 4$, the numbers of known alcohols are 1, 1, 2, 4, agreeing with the foregoing theoretic number (see Schorlemmer's *Carbon Compounds*, 1874); but $n=4$, the number of known alcohols is $=2$, instead of the foregoing theoretic number 8. It is, of course, no objection to the theory that the number of theoretic forms should exceed the number of known compounds; the missing ones may be simply unknown; or they may be only capable of existing under conceivable, but unattained, physical conditions (for instance, of temperature); and if defect from the theoretic number of compounds can be thus accounted for, the theory holds good without modification. But it is also possible that the diagrams, in order that they may represent chemical compounds, may be subject to some as yet undetermined conditions; viz. in this case the theory would stand good as far as it goes, but would require modification.

587.

A SMITH'S PRIZE DISSERTATION.

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 1—4.]

WRITE a dissertation:

On the general equation of virtual velocities.

Discuss the principles of Lagrange's proof of it and employ it [the general equation] to demonstrate the Parallelogram of Forces.

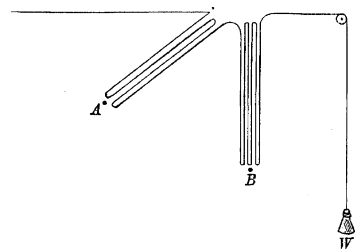
Imagine a system of particles connected with each other in any manner and subject to any geometrical conditions, for instance, two particles may be such that their distance is invariable, a particle may be restricted to move on a given surface, &c. And let each particle be acted upon by a force [this includes the case of several forces acting on the same particle, since we have only to imagine coincident particles each acted upon by a single force]. Imagine that the system has given to it any indefinitely small displacement consistent with the mutual connexions and geometrical conditions; and suppose that for any particular particle the force acting on it is P, and the displacement in the direction of the force (that is, the actual displacement multiplied into the cosine of the angle included between its direction and that of the force P) is $=\delta p$. Then δp is called the virtual velocity of the particle, and the principle of virtual velocities asserts that the sum of the products $P\delta p$, taken for all the particles of the system, and for any displacement consistent as above, is $=0$; say that we have

$$\Sigma P\delta p = 0.$$

This is also the general equation of virtual velocities: as to the mode of using it, observe that the displacements δp are not arbitrary quantities, but are in virtue of the mutual connexions and other geometrical conditions connected together by certain linear relations; or, what is the same thing, they are linear functions of certain independent arbitrary quantities δu. Substituting for δp their expressions in terms of δu

we have $\Sigma P\delta p = \Sigma U\delta u$, where the several expressions U are each of them a linear function of the forces P, and where on the right hand Σ refers to the several quantities δu; and the resulting equation is $\Sigma U\delta u = 0$; viz. since the quantities δu are independent, the equation divides itself into a set of equations $U_1 = 0$, $U_2 = 0, \ldots$ which are the equations of equilibrium of the system.

Lagrange imagines the forces produced by means of a weight W at the extremity of a string passing over a set of pulleys, as shown in the figure, viz. assuming the forces commensurable and equal to mW, nW, &c., we must have m strings at A,



n strings at B, and so on. Suppose any indefinitely small displacement given to the system; each string at A is shortened by δp, or the m strings at A by $m\delta p$; and the like for the other particles at B, &c.; hence, if $m\delta p + n\delta q + \ldots, = \frac{1}{W}(P\delta p + Q\delta q + \ldots)$, be positive, the weight W will descend through the space

$$\frac{1}{W}(P\delta p + Q\delta q + \ldots).$$

Now, in order that the system may be in equilibrium, W must be in its lowest position; or, what is the same thing, if there is any displacement allowing W to descend, W will descend, causing such displacement, and the original position is not a position of equilibrium. That is, if the system be in equilibrium, the sum $\Sigma P\delta p$ cannot be positive.

But it cannot be negative; since, if for any particular values of δp the sum $\Sigma P\delta p$ is negative, then reversing the directions of the several displacements, that is, giving to the several displacements δp the same values with opposite signs, then the sum $\Sigma P\delta p$ will be positive; and we *assume* that it is possible thus to reverse the directions of the several displacements. Hence, if the system be in a position of equilibrium, we cannot have $\Sigma P\delta p$ either positive or negative; that is, we obtain as the condition of equilibrium $\Sigma P\delta p = 0$.

The above is Lagrange's reasoning, and it seems completely unobjectionable. As regards the reversal of the directions of the displacements, observe that we consider

such conditions as a condition that the particle shall be always *in* a given plane, but exclude the condition that the particle shall lie *on* a given plane, i.e. that it shall be at liberty to move in one direction (but not in the opposite direction) off from the plane. But the pulley-proof is equally applicable to a case of this kind. Thus, imagine a particle resting on a horizontal plane, and let z be measured vertically downwards, x and y horizontally. Suppose the particle acted on by the forces X, Y, Z, and replacing these by a weight W as above, the condition of equilibrium is, that

$$X\delta x + Y\delta y + Z\delta z$$

shall not be positive. We may have δx and δy, each positive or negative; whence the conditions $X=0$ and $Y=0$. But δz is negative; hence the required condition is satisfied if only Z is positive; that is, if the vertical force acts downwards. Clearly this is right, for if it acted upwards it would lift the particle from the plane. The case considered by Lagrange is where the particle is always *in* the plane; here $\delta z = 0$, and there is no condition as to the force Z.

The only omission in Lagrange's proof is, that he does not expressly consider the case of unstable equilibrium, where the weight W is at a position, not of minimum, but of maximum altitude. In such a case, however, the sum $\Sigma P\delta p$ is still $=0$, taking account (as the proof does) of the displacements considered as infinitesimals of the first order; although taking account of higher powers, the sum $\Sigma P\delta p$ would have a positive value. An explanation as to this point might properly have been added to make the proof "refutation-tight," but the proof is not really in defect.

P.S. Lagrange excludes tacitly, not expressly, the case where the direction of a displacement is not reversible; he observes that the various displacements δp, when not arbitrary, are connected only by linear equations; and "par conséquent les valeurs de toutes ces quantités seront toujours telles qu'elles pourront changer de signe à la fois." The point was brought out more fully by Ostrogradsky, but I think there is no ground for the view that it was not brought out with sufficient clearness by Lagrange himself.

Parallelogram of forces.

Let P, Q, R be the forces, α, β, γ their inclinations to any line; then taking δs the displacement in the direction of this line, the displacements in the directions of the forces are $\delta s \cos\alpha$, $\delta s \cos\beta$, $\delta s \cos\gamma$, and the equation $\Sigma P\delta p = 0$ assumes the form

$$(P\cos\alpha + Q\cos\beta + R\cos\gamma)\,\delta s = 0,$$

that is, we have

$$P\cos\alpha + Q\cos\beta + R\cos\gamma = 0,$$

viz. this equation holds whatever be the fixed line to which the forces are referred. It is easy to see that, supposing it to hold in regard to any two lines, it will hold generally, and that the relation in question is thus equivalent to two independent conditions; and forming these we may obtain from them the theorem of the parallelogram of forces.

But to obtain this more directly, take A, B, C for the angles between the forces Q and R, R and P, P and Q respectively, then $A+B+C=2\pi$, and thence

$$\begin{aligned}\alpha &= \alpha,\\ \beta &= \alpha + C,\\ \gamma &= \alpha + C + A = \alpha + 2\pi - B,\end{aligned}$$

whence writing $\alpha = \frac{1}{2}\pi$, or taking the line of displacement at right angles to the force P, we have

$$\alpha = \tfrac{1}{2}\pi, \quad \beta = \tfrac{1}{2}\pi + C, \quad \gamma = 2\pi + \tfrac{1}{2}\pi - B,$$

and the equation becomes $0P - Q\sin C + R\sin B = 0$, that is, $Q : R = \sin B : \sin C$; and similarly $R : P = \sin C : \sin A$, that is,

$$P : Q : R = \sin A : \sin B : \sin C,$$

equations which in fact express that each force is equal and opposite to the diagonal of the parallelogram formed by the other two forces.

588.

PROBLEM.

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 50—52.]

It is required to place two given tetrahedra in perspective; or, what is the same thing, the tetrahedra being $ABCD$, $A'B'C'D'$ respectively, to place these so that the lines AA', BB', CC', DD' may meet in a point O.

The following considerations present themselves in regard to the solution of this problem. Take the tetrahedron $ABCD$ to be given in position, and the point O at pleasure; then drawing the lines OA, OB, OC, OD, we may in a determinate number of ways (viz. in 16 different ways) place the tetrahedron $A'B'C'D'$ in such manner that the summits A', B', C' shall be in the lines OA, OB, OC respectively. But the summit D' will then not be in general in the line OD; and in order that it may be so, a two-fold condition must be satisfied by the point O; viz. the locus of this point must be a certain curve in space.

Or again, we may look at the question thus: we have to place a point O in relation to the tetrahedron $ABCD$, and a point O' in relation to the tetrahedron $A'B'C'D'$, in such manner that the edges of the first tetrahedron subtend at O the same angles that the edges of the second tetrahedron subtend at O'; for this being done, then considering O' as rigidly connected with $A'B'C'D'$, we may move the figure $O'A'B'C'D'$ so that O' shall coincide with O, and the lines $O'A'$, $O'B'$, $O'C'$, $O'D'$ with OA, OB, OC, OD respectively. Take a, b, c, f, g, h, for the sides of the tetrahedron $ABCD$ (BC, CA, AB, AD, BD, $CD = a, b, c, f, g, h$ respectively), and take also x, y, z, w for the distances OA, OB, OC, OD respectively; and let $a', b', c', f', g', h', x', y', z', w'$ have the like significations in regard to the tetrahedron $A'B'C'D'$ and the point O', and write

$$\frac{y^2+z^2-a^2}{2yz},\quad \frac{z^2+x^2-b^2}{2zx},\quad \frac{x^2+y^2-c^2}{2xy},\quad \frac{x^2+w^2-f^2}{2xw},\quad \frac{y^2+w^2-g^2}{2yw},\quad \frac{z^2+w^2-h^2}{2zw},$$
$$=A,\qquad B,\qquad C,\qquad F,\qquad G,\qquad H,$$

respectively; and the like as regards the accented letters. Then A, B, C, F, G, H are the cosines of the angles which the edges of the tetrahedron $ABCD$ subtend at O; they are consequently the cosines of the six sides of the spherical quadrangle obtained by the projection of $ABCD$ on a sphere centre O; and they are therefore not independent, but are connected by a single equation; substituting for A, B, C, F, G, H their values, we have a relation between a, b, c, f, g, h, x, y, z, w; viz. this is the relation which connects the ten distances of the five points in space O, A, B, C, D (and which relation was originally obtained by Carnot in this very manner). There is of course the like relation between the accented letters.

The conditions as to the two tetrahedra are

$$A = A',\ B = B',\ C = C',\ F = F',\ G = G',\ H = H',$$

which, attending to the relations just referred to and therefore regarding w as a given function of x, y, z, and w' as a given function of x', y', z', are equivalent to five equations (or rather to a five-fold relation); the elimination of x', y', z' from the five-fold relation gives therefore a two-fold relation between x, y, z, that is, between the distances OA, OB, OC; or the locus of O is as before a curve in space.

The conditions may be written:

$$\begin{aligned} y'^2 + z'^2 - 2Ay'z' &= a'^2, & x'^2 + w'^2 - 2Fx'w' &= f'^2, \\ z'^2 + x'^2 - 2Bz'x' &= b'^2, & y'^2 + w'^2 - 2Gy'w' &= g'^2, \\ x'^2 + y'^2 - 2Cx'y' &= c'^2, & z'^2 + w'^2 - 2Hz'w' &= h'^2; \end{aligned}$$

whence eliminating x', y', z', w', and in the result regarding A, B, C, F, G, H as given functions of x, y, z, w, we have between x, y, z, and w a three-fold relation determining w as a function of x, y, z, and establishing besides a two-fold relation between x, y, z.

As a particular case: One of the tetrahedra may degenerate into a plane quadrangle, and we have then the problem: a given plane quadrangle $ABCD$ being assumed to be the perspective representation of a given tetrahedron $A'B'C'D'$, it is required to determine the positions in space of this tetrahedron and of the point of sight O.

A generalisation of the original problem is as follows: determine the two-fold relation which must subsist between the 4×6, $= 24$ coordinates of four lines, in order that it may be possible to place in the tetrad of lines a given tetrahedron; that is, to place in the four lines respectively the four summits of the given tetrahedron. It may be remarked that considering three of the four lines as given, say these lines are the loci of the summits A, B, C respectively, we can in 16 different ways place in these lines respectively the three summits, and for each of these there are two positions of the summit D; there are consequently 32 positions of D; and the two-fold relation, considered as a relation between the six coordinates of the remaining line, must in effect express that this line passes through some one of the 32 points.

589.

ON RESIDUATION IN REGARD TO A CUBIC CURVE.

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 62—65.]

THE following investigation of Prof. Sylvester's theory of Residuation may be compared with that given in Salmon's *Higher Plane Curves*, 2nd Edition (1873), pp. 133—137:

If the intersections of a cubic curve U_3 with any other curve V_n are divided in any manner into two systems of points, then each of these systems is said to be the residue of the other; and, in like manner, if starting with a given system of points on a cubic curve we draw through them a curve of any order V_n, then the remaining intersections of this curve with the cubic constitute a residue of the original system of points.

If the number of points in the original system is $=3p$, then the number of points in the residual system is $=3q$; and if we again take the residue, and so on indefinitely, the number of points in each residue will be $\equiv 0$ (Mod. 3); viz. we can never in this way arrive at a single point. But if the number of points in the original system be $3p \pm 1$, then that in the residual system will be $3q \mp 1$; and we may in an infinity of different ways arrive at a residue consisting of a single point; or say at a "residual point," viz. after an odd number of steps if the original number of points is $=3p-1$, but after an even number of steps if the original number of points is $=3p+1$. But starting from a given system of points on a given cubic curve, the residual point, however it is arrived at, will be one and the same point; this is Prof. Sylvester's theorem of the residuation of a cubic curve. For instance, starting with two given points on the cubic curve, the line joining these meets the curve in a third point, which is the residual point; any other process leading to a residual point must lead to the same point. Thus if through the 2 points we draw a conic, meeting the cubic besides in 4 points; through these a conic meeting the cubic besides

in 2 points; and through this a line meeting the cubic besides in 1 point; this will be the before-mentioned residual point.

The general proof is such as in the following example:

Take on the cubic U_3 a system of $3\kappa-2$ points, say the points α: through these a curve V_k, besides meeting the cubic in $3k-3\kappa+2$ points β: and through these a curve $P_{k-\kappa+1}$, besides meeting the cubic in a point C. And again through the $3\kappa-2$ points α a curve $W_{k'}$, besides meeting the cubic in $3k'-3\kappa+2$ points β': and through these a curve $Q_{k'-\kappa+1}$, besides meeting the cubic in a single point; this will be the point C.

The proof consists in showing that we have a curve $A_{k+k'-\kappa-2}$ such that

$$A_{k+k'-\kappa-2}\, U_3 = Q_{k'-\kappa+1}\, V_k + P_{k-\kappa+1}\, W_{k'}.$$

For this observe that

$Q_{k'-\kappa+1}$ meets $W_{k'}$ in $3k'-3\kappa+2$ points β' and besides in $k'^2-k'(\kappa+2)+3\kappa-2$ points ϵ';
$P_{k-\kappa+1}$ meets V_k in $3k-3\kappa+2$ points β and besides in $k'^2-k(\kappa+2)+3\kappa-2$ points ϵ;
$P_{k-\kappa+1}$, $Q_{k'-\kappa+1}$ meet in $(k-\kappa+1)(k'-\kappa+1)$ points C;
V_k, $W_{k'}$ meet in $3\kappa-2$ points α and $kk'-3\kappa+2$ points a;
$Q_{k'-\kappa+1}\, V_k$ and $P_{k-\kappa+1}\, W_{k'}$ meet in

$$\begin{array}{llll}
kk'-k(\kappa-1)-k'(\kappa-1)+(\kappa-1)^2 & \text{points} & C \\
\qquad\qquad 3k' \qquad -3\kappa+2 & \text{,,} & \beta' \\
k'^2 \qquad\quad -k'(\kappa+2)+3\kappa-2 & \text{,,} & \epsilon' \\
\qquad 3k \qquad\qquad -3\kappa+2 & \text{,,} & \beta \\
k^2-k(\kappa+2) \qquad +3\kappa-2 & \text{,,} & \epsilon \\
kk' \qquad\qquad -3\kappa+2 & \text{,,} & a \\
\qquad\qquad\qquad 3\kappa-2 & \text{,,} & \alpha \\
\hline
(k+k')^2-(2\kappa-2)(k+k')+(\kappa-1)^2 & & \\
= (k+k'-\kappa+1)^2 \text{ points.} & &
\end{array}$$

Every $(k+k'-\kappa+1)$thic through

$$\tfrac{1}{2}(k+k'-\kappa+1)(k+k'-\kappa+4)-1$$

of these points passes through all.

Now $A_{k+k'-\kappa-2}$ may be drawn to pass through

$$\tfrac{1}{2}(k+k'-\kappa-2)(k+k'-\kappa+1)$$

of the points a.

Hence $A_{k+k'-\kappa-2}\,U_3$ is a $(k+k'-\kappa+1)$thic through

$$\begin{array}{rrrl}
 & & \tfrac{1}{2}(k+k'-\kappa-2)(k+k'-\kappa+1) & \\
= \tfrac{1}{2}(k+k')^2 - \tfrac{1}{2}(2\kappa+1)(k+k') + \tfrac{1}{2}(\kappa^2+\kappa-2) & & & \text{points } \alpha \\
 & & 3\kappa-2 & \text{,,} \quad \alpha \\
 & 3k & -3\kappa+2 & \text{,,} \quad \beta \\
 & 3k' & -3\kappa+2 & \text{,,} \quad \beta' \\
\hline
\tfrac{1}{2}(k+k')^2 + (-\kappa+\tfrac{5}{2})(k+k') + \tfrac{1}{2}\kappa^2 - \tfrac{5}{2}\kappa + 1 & & & \\
= \tfrac{1}{2}\{(k+k')^2 + (k+k')(-2\kappa+5) + (\kappa-1)(\kappa+4) - 2\} & & & \\
= \tfrac{1}{2}(k+k'-\kappa+1)(k+k'-\kappa+4) - 1 & & &
\end{array}$$

of the points in question; and therefore through all. Whence

$$A_{k+k'-\kappa-2}\,U_3 = Q_{k'-\kappa+1}\,V_k + P_{k-\kappa+1}\,W_{k'}.$$

Also U_3 meets $Q_{k'-\kappa+1}\,V_k$ in $3(k+k'-\kappa+1)$ of the $(k+k'-\kappa+1)^2$ points, viz. these are

$$\begin{array}{rll}
3\kappa-2 & \text{points} & \alpha, \\
3k-3\kappa+2 & \text{,,} & \beta, \\
3k'-3\kappa+2 & \text{,,} & \beta', \\
1 & \text{,,} & C,
\end{array}$$

and $A_{k+k'-\kappa-2}$ meets $Q_{k'-\kappa+1}\,V_k$ in $(k+k'-\kappa-2)(k+k'-\kappa+1)$, that is, in

$$(k+k')^2 + (k+k')(-2\kappa-1) + \kappa^2 + \kappa - 2$$

of the

$$(k+k'-\kappa+1)^2 \text{ points,}$$

viz. these are

$$\begin{array}{lll}
kk' & +(k+k')(-\ \kappa+1) + \kappa^2 - 2\kappa & \text{points } C \\
k'^2 & -k'(\kappa+2) \qquad\qquad + 3\kappa - 2 & \text{,,} \quad \epsilon' \\
k^2 & -k(\kappa+2) \qquad\qquad + 3\kappa - 2 & \text{,,} \quad \epsilon \\
kk' & \qquad\qquad\qquad\qquad - 3\kappa + 2 & \text{,,} \quad \alpha \\
\hline
 & (k+k')^2 + (k+k')(-2\kappa-1) + \kappa^2 + \ \kappa - 2 & \text{points.}
\end{array}$$

Hence U_3 passes through 1 of the points C, that is, through an intersection of $Q_{k'-\kappa+1}$ and $P_{k-\kappa+1}$, that is, $Q_{k'-\kappa+1}$ and $P_{k-\kappa+1}$ intersect U_3 in a common point C; which was the theorem to be proved.

In the particular case $3\kappa-2=10$, $k=k'=4$, the theorem is, given on a cubic 10 points, if through these we draw a quartic meeting the cubic besides in 2 points;

and through these a line meeting the cubic besides in a point C; then this is a fixed point, independent of the particular quartic. And the proof is as follows: we have

U a cubic through 10 points α;

V a quartic through the 10 points, and besides meeting the cubic in 2 points β;

W a quartic through the 10 points, and besides meeting the cubic in 2 points β';

P the line joining the two points β, and besides meeting V in two points ϵ;

Q the line joining the two points β', and besides meeting W in two points ϵ'; P, Q meet in the point C;

U, V meet in the 10 points α, and besides in 6 points a;

A a conic through 5 of the points a.

Then quintics QV, PW meet in the 10 points α, 2 points β, 2 points ϵ, 2 points β', 2 points ϵ', 6 points a and 1 point C. Every quintic through 19 of these passes through the 25. But we have AU, a quintic through 5 points a, and the 10 points α, 2 points β and 2 points β'; hence AU passes through all the remaining points, or we have

$$AU = QV + PW,$$

P	passes through	$\beta, \beta, \epsilon, \epsilon, C,$	
Q	„	$\beta', \beta', \epsilon', \epsilon', C,$	
V	„	$\epsilon, \epsilon, \beta, \beta,$	6 points a, 10 points α,
W	„	$\epsilon', \epsilon', \beta', \beta',$	6 points a, 10 points α,
A	„	$\epsilon, \epsilon, \epsilon', \epsilon',$	6 points a,
U	„	$\beta, \beta, \beta', \beta', C,$	

or, what is the same thing,

A, P	intersect in	$\epsilon, \epsilon,$	
A, Q	„	$\epsilon', \epsilon',$	
A, V	„	$\epsilon, \epsilon,$	6 points a,
A, W	„	$\epsilon', \epsilon',$	6 points a,
U, P	„	$\beta, \beta, C,$	
U, Q	„	$\beta', \beta', C,$	
U, V	„	$\beta, \beta,$	10 points α,
U, W	„	$\beta', \beta',$	10 points α.

In particular U, P, Q intersect in the point C; that is, C as given by the intersection of U by the line P; and as given by the intersection of U by the line Q; is one and the same point.

590.

ADDITION TO PROF. HALL'S PAPER "ON THE MOTION OF A PARTICLE TOWARD AN ATTRACTING CENTRE AT WHICH THE FORCE IS INFINITE."

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 149—152.]

I DO not in the passage referred to* expressly profess to interpret Newton's idea. After referring to his investigation I say, "The method has the advantage of explaining the paradoxical result which presents itself in the case force $\propto$ (dist.)$^{-2}$, and in some other cases where the force becomes infinite. According to theory the velocity becomes infinite at the centre, but the direction of the motion is there abruptly reversed, so that the body in its motion does not pass through the centre, but on arriving there forthwith returns towards its original position; of course such a motion cannot occur in nature, where neither a force nor a velocity is actually infinite;" viz. while assuming that the analysis gives a motion as just described, or in Prof. Hall's figure, a reciprocating motion between A and C, I expressly state that the motion is not one that can occur in nature; in fact, my view is that the question (which, to render it precise, I state as follows: "What happens in nature when the moving point arrives at C") presupposes what is inconceivable. But I consider that the analysis gives a motion as above, viz. that it gives x, t each as a one-valued function of a parameter ϕ, such that this parameter ϕ increasing continuously, we have for the moving point a continuous series of positions corresponding to the motion in question, gives in fact the equations $x = a(1 - \cos\phi)$ and $\dfrac{t\sqrt{(\mu)}}{a^{\frac{3}{2}}} = \phi - \sin\phi$.

In explanation and justification of the assumption, it is interesting to show how the solution just referred to can be obtained from the equation of motion $\dfrac{d^2x}{dt^2} = -\dfrac{\mu}{x^2}$, without (in the process) the extraction of the square root of the two sides of an

[* By Professor Hall in his paper (p. 144, *l.c.*) quoted in the title. The passage is an extract from the British Association Report (1862) *On the progress of the solution of certain special problems of dynamics*, p. 186; [298], *Coll. Math. Papers*, vol. IV. p. 515.]

equation. Taking x as the independent variable and writing for a moment $\frac{dt}{dx} = t'$, $\frac{d^2t}{dx^2} = t''$, the equation is

$$\frac{1}{t'^3} t'' = \frac{\mu}{x^2},$$

and if we herein assume $x = a(1 - \cos\phi)$ and transform to ϕ as the independent variable, it becomes

$$\frac{a^2 \sin^2\phi}{\left(\frac{dt}{d\phi}\right)^3} \left\{\frac{1}{a \sin\phi} \frac{d^2t}{d\phi^2} - \frac{\cos\phi}{a \sin^2\phi} \frac{dt}{d\phi}\right\} = \frac{\mu}{a^2 (1 - \cos\phi)^2};$$

or, what is the same thing,

$$\sin\phi \frac{d}{d\phi}\left(\frac{dt}{d\phi}\right) - \cos\phi \left(\frac{dt}{d\phi}\right) = \frac{\mu}{a^3} \frac{1}{(1 - \cos\phi)^2} \left(\frac{dt}{d\phi}\right)^3,$$

a differential equation of the first order for the determination of $\frac{dt}{d\phi}$ as a function of ϕ. Since a is a constant of integration of the original equation, a particular integral only is required, but it is as well to obtain the general integral. For this purpose assume

$$\frac{dt}{d\phi} = \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}} z (1 - \cos\phi);$$

then, omitting from each side of the equation the factor $\frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}}$, the equation becomes

$$\sin\phi \left\{z \sin\phi + \frac{dz}{d\phi}(1 - \cos\phi)\right\} - \cos\phi \,.\, z (1 - \cos\phi) = (1 - \cos\phi) z^3,$$

viz. the left-hand side being $(1 - \cos\phi)\left(z + \frac{dz}{d\phi} \sin\phi\right)$, the whole equation contains the factor $(1 - \cos\phi)$, and omitting this, the equation becomes

$$z + \frac{dz}{d\phi} \sin\phi = z^3;$$

or, what is the same thing,

$$\frac{dz}{z^3 - z} = \frac{d\phi}{\sin\phi}.$$

The integral of this is

$$\log \frac{z^2 - 1}{z^2} = 2 \log k + 2 \log \tan \tfrac{1}{2}\phi;$$

or, what is the same thing,

$$\frac{z^2 - 1}{z^2} = k^2 \tan^2 \tfrac{1}{2}\phi,$$

where k is the constant of integration.

[In explanation of this constant k, observe that the equation gives

$$z = \frac{1}{\sqrt{(1 - k^2 \tan^2 \frac{1}{2}\phi)}},$$

and that we thence have

$$\frac{dt}{d\phi} = \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}} \frac{1 - \cos\phi}{\sqrt{(1 - k^2 \tan^2 \frac{1}{2}\phi)}};$$

that is,

$$\frac{dt}{dx} = \frac{\sqrt{(\mu)}}{a^{\frac{1}{2}}} \frac{\sin\phi}{1 - \cos\phi} \frac{1}{\sqrt{(1 - k^2 \tan^2 \frac{1}{2}\phi)}}, \quad = \frac{a^{\frac{1}{2}}}{\sqrt{(\mu)}} \frac{\tan \frac{1}{2}\phi}{\sqrt{(1 - k^2 \tan^2 \frac{1}{2}\phi)}},$$

or, since

$$\tan^2 \tfrac{1}{2}\phi = \frac{x}{2a - x},$$

this is

$$\frac{dt}{dx} = \frac{a^{\frac{1}{2}}}{\sqrt{(\mu)}} \frac{\sqrt{(x)}}{\sqrt{(2a - x - k^2 x)}};$$

or, what is the same thing,

$$\frac{dt}{dx} = \frac{\sqrt{\left(\frac{a}{1+k^2}\right)}}{\sqrt{(\mu)}} \frac{\sqrt{(x)}}{\sqrt{\left(2\frac{a}{1+k^2} - x\right)}},$$

viz. we in effect have $\frac{a}{1+k^2}$ as a constant of integration in place of the original constant a.]

Recurring to the general solution

$$\frac{z^2 - 1}{z^2} = k^2 \tan^2 \tfrac{1}{2}\phi,$$

we may take $z = 1$, as a particular solution answering to the value $k = 0$ of the constant; and we then have

$$\frac{dt}{d\phi} = \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}} (1 - \cos\phi),$$

viz. reckoning t from the epoch for which ϕ is $= 0$, we thus have

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{(\mu)}} (\phi - \sin\phi),$$

which, combined with the assumed equation

$$x = a(1 - \cos\phi),$$

gives the foregoing solution.

I quite admit that, considering (with Prof. Hall) the attracted particle as split into two equal particles placed at equal distances above and below the centre C, the motion when the distances become infinitesimal is a motion not as above, but backwards and forwards along the entire line AB; but it remains to be seen whether *at* the limit this can be brought out as an analytical solution of the differential equation $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$. Possibly this may be done, and I remark as an objection, not to the foregoing as an admissible solution of the problem but to its generality as the *only* solution, that, in writing $x = a(1 - \cos\phi)$ and assuming that ϕ is real, I in effect assume that x is always positive. But the burthen of the proof is with Prof. Hall, to show that there is an analytical solution in which x acquires negative values.

591.

A SMITH'S PRIZE PAPER AND DISSERTATION; SOLUTIONS AND REMARKS.

[From the *Messenger of Mathematics*, vol. III. (1874), pp. 165—183, vol. IV. (1875), pp. 6—8.]

1. *Find the triangular numbers which are also square.*

The "*mise en équation*" is immediate; we have to find n, m such that

$$\tfrac{1}{2}n(n+1) = m^2;$$

or, what is the same thing,

$$(2n+1)^2 - 8m^2 = 1.$$

Observing that this is satisfied by $n = m = 1$, that is, $2n+1 = 3$, $2m = 2$, we have the general solution given by

$$2n + 1 + 2m\sqrt{(2)} = \{3 + 2\sqrt{(2)}\}^p,$$

where p is any positive integer; viz. $2n+1$, $2m$ being rational, this implies

$$2n + 1 - 2m\sqrt{(2)} = \{3 - 2\sqrt{(2)}\}^p,$$

and thence the equation in question. The successive powers

$$3 + 2\sqrt{(2)},\quad 17 + 12\sqrt{(2)},\quad 99 + 70\sqrt{(2)},\ \&c.,$$

give the solutions

$$n,\ m = \ 1,\ 1\ ,\qquad 8,\ 6\ ,\qquad 48,\ 35\ ,\ \&c.;$$

viz. the square triangular numbers are

$$1^2,\ = \tfrac{1}{2}\,1{\cdot}2;\quad 6^2,\ = \tfrac{1}{2}\,8{\cdot}9;\quad 35^2,\ = \tfrac{1}{2}\,49{\cdot}50,\ \&c.$$

2. *Show how to express any symmetrical function of the roots of an equation in terms of the coefficients. What objection is there to the method which employs the sums of the powers of the roots?*

The ordinary method is that referred to, employing the sums of the powers of the roots; *but it is a very bad one.* In fact, writing

$$x^n - bx^{n-1} + cx^{n-2} - \&c., = (x-\alpha)(x-\beta)(x-\gamma)\ldots = 0,$$

leading to

$$\begin{aligned} S_1 &= b, \\ S_2 &= b^2 - 2c, \\ S_3 &= b^3 - 3bc + 3d, \\ &\vdots \end{aligned}$$

then if the method were employed throughout, we should have for instance to find $S\alpha\beta\gamma$, that is, d, from the formula

$$\begin{aligned} 6S\alpha\beta\gamma = \quad & S_1^3 = b^3 \\ - 3S_1S_2 \quad & - 3b(b^2 - 2c) \\ + 2 \quad S_3 \quad & + 2(b^3 - 3bc + 3d) \\ = \quad & 6d, \text{ which is right,} \end{aligned}$$

but the process introduces terms b^3 and bc each of a higher order than d (reckoning the order of each coefficient as unity), with numerical coefficients which destroy each other. And, so again, $S\alpha^2\beta$ would be calculated from the formula

$$\begin{aligned} S\alpha^2\beta = \quad & S_1S_2 = b(b^2 - 2c) \\ - \quad & S_3 \quad - (b^3 - 3bc + 3d) \\ = \quad & bc - 3d, \text{ which is right,} \end{aligned}$$

but there is here also a term b^3 of a higher order, with numerical coefficients which destroy each other. And the order in which the several expressions are derived the one from the other is a non-natural one; S_3 is required for the determination of $S\alpha^2\beta$, whereas (as will be seen) it is properly $S\alpha^2\beta$ which leads to the value of S_3.

The true method is as follows: we have

$$S\alpha = b, \quad S\alpha\beta = c, \quad S\alpha\beta\gamma = d, \ \&c.,$$

and we thence derive the sets of equations

$$\begin{aligned} b &= S\alpha; \\ c &= S\alpha\beta, \\ b^2 &= S\alpha^2 + 2S\alpha\beta; \\ d &= S\alpha\beta\gamma, \\ bc &= S\alpha^2\beta + 3S\alpha\beta\gamma, \\ b^3 &= S\alpha^3 + 3S\alpha^2\beta + 6S\alpha\beta\gamma; \\ &\vdots \end{aligned}$$

viz. we thus have 1 equation to give $S\alpha$; 2 equations to give $S\alpha\beta$ and $S\alpha^2$; 3 equations to give $S\alpha\beta\gamma$, $S\alpha^2\beta$, $S\alpha^3$; and so on. And taking for instance the third set of equations, the first equation gives $S\alpha\beta\gamma$, the second then gives $S\alpha^2\beta$, and the third then gives $S\alpha^3$, viz. we have

$$\begin{aligned} S\alpha\beta\gamma &= d, \\ S\alpha^2\beta &= bc - 3d, \\ S\alpha^3 &= b^3 - 3(bc - 3d) - 6d, \\ &= b^3 - 3bc + 3d. \end{aligned}$$

Of course the process for the formation of the successive sets of equations would require further explanation and development.

3. *Given a point P in the interior of an ellipsoid, show that it is possible to determine an exterior point Q such that for every chord RS through P, the relation $QR : QS = PR : PS$ may hold good.*

There is no difficulty in the analytical solution and in showing thereby that the point Q is determined as the intersection of the polar plane of P by the perpendicular let fall from P on this plane. But a simple and elegant geometrical solution was given in the Examination. Constructing Q as above, let the chord RS meet the polar plane of P in Z; then the polar plane of Z passes through P, that is, the line ZP is harmonically divided in R, S, or we have

$$ZR : ZS = PR : PS.$$

Again ZQP being a right angle, the sphere on ZP as diameter will pass through Q; and R, S being points on the diameter, and Z, Q points on the surface, $ZR : ZS = QR : QS$; whence the required relation $QR : QS = PR : PS$.

4. *Find the number of regions into which infinite space is divided by n planes.*

The number $\frac{1}{6}(n^3 + 5n + 6)$ is a known result, but not a generally known one, and I intended the question as a problem; I do not think it is a difficult one.

Consider the analogous problem for lines in a plane: the first line divides the plane into 2 regions.

The second line is by the first divided into 2 parts, and therefore adds 2 regions.

The third line is by the other two divided into 3 parts, and therefore adds 3 regions; and so on.

That is, the number of regions for

1 line is	$= 2$	$= 2$	regions,
2 lines	$= 2 + 2$	$= 4$	„
3 lines	$= 2 + 2 + 3$	$= 7$	„
$\vdots$			
n lines	$= 2 + 2 + 3 + \ldots + n$	$= \frac{1}{2}(n^2 + n + 2)$	„

In exactly the same way for the problem in space:

The first plane divides space into 2 regions.

The second plane is by the first plane divided into 2 regions, and therefore adds 2 regions.

The third plane is by the other two planes divided into 4 regions, and therefore adds 4 regions.

The fourth plane is by the other three planes divided into 7 regions, and therefore adds 7 regions: and so on.

That is the number of regions for

1 plane is	$=2$	$=$ 2 regions
2 planes	$=2+2$	$=$ 4 „
3 planes	$=2+2+4$	$=$ 8 „
4 planes	$=2+2+4+7$	$=15$ „
$\vdots$		
n planes	$=2+2+4+7+\ldots+\frac{1}{2}(n^2-n+2)=\frac{1}{6}(n^2+5n+6)$,	

where, for effecting the summation, observe that the series is

$$\begin{aligned}&=2+\{1+1+1\ldots(n-1)\text{ terms}\}\\&\quad+\{1+3+6\ldots+\tfrac{1}{2}n(n-1)\},\\&=2+(n-1)+\tfrac{1}{6}(n+1)n(n-1),\ =\text{as above.}\end{aligned}$$

5. *In the theory of Elliptic Functions, explain and connect together the notations* $F(\theta)$, $\operatorname{am} u$ ($\operatorname{sinam} u$, $\operatorname{cosam} u$, $\Delta\operatorname{am} u$), *illustrating them by reference to the circular functions**.

What is asked for is an explanation of the fundamental notations of Elliptic Functions. To a student acquainted with the subject, the only difficulty is to say enough to bring the meaning fully out, and not to say more than enough.

Defining $F(x)$ by the equation

$$F(x)=\int_0\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

(viz. the integral is taken from 0 up to the indefinite value x), then the fundamental property of elliptic functions (derived from consideration of the differential equation

$$\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}+\frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}}=0)$$

consists herein, that the functional relation

$$F(x)+F(y)=F(z)$$

* It would have been better in the question to have written $F(x)$ instead of $F(\theta)$.

is equivalent to an algebraic equation between the arguments x, y, z. $F(x)$ as defined by the foregoing equation is properly an inverse function; this at once appears from a particular case, viz. writing $k=0$, $F(x)=\sin^{-1}x$, and the theory of the function $F(x)$ in the general case corresponds to what the theory of circular functions would be, if writing $F(x)$ to denote $\sin^{-1}x$, we were to work with the equation

$$F(x)+F(y)=F(z)$$

as equivalent to the algebraical equations (one a transformation of the other)

$$z=x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)},$$
$$\sqrt{(1-z^2)}=\sqrt{(1-x^2)}\sqrt{(1-y^2)}-xy.$$

But in the actual theory of circular functions, we introduce the direct symbols sin, cos; writing $F(x)=\theta$, that is, $x=\sin\theta$, $\sqrt{(1-x^2)}=\cos\theta$, and similarly $F(y)=\phi$, that is, $y=\sin\phi$ and $\sqrt{(1-y^2)}=\cos\phi$, then the equation

$$F(x)+F(y)=F(z)$$

becomes $F(z)=\theta+\phi$, that is, $z=\sin(\theta+\phi)$, $\sqrt{(1-z^2)}=\cos(\theta+\phi)$, and the other two equations become

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta,$$
$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi,$$

viz. these are the addition-equations for the functions sin and cos.

In passing from the original notation $F(x)$ to the notation $\operatorname{am}u$, we make the like step of passing from an inverse to a set of direct functions; first modifying the meaning of F, so as to denote by $F(\theta)$ what was originally $F(\sin\theta)$, we have as the new definition

$$F(\theta)=\int_0\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}=\int_0\frac{d\theta}{\Delta(\theta)},$$

(if as usual $\Delta\theta$ denotes $\sqrt{(1-k^2\sin^2\theta)}$), and this being so, the relation $F(\theta)+F(\phi)=F(\mu)$ is equivalent to a relation between the sine, cosine, and Δ of θ, ϕ, μ. Writing then $F(\theta)=u$, and considering this equation as determining θ as a function of u, $\theta=\operatorname{am}u$, we have $\sin\theta=\sin.\operatorname{am}u$, $\cos\theta=\cos.\operatorname{am}u$, and $\Delta\theta=\Delta.\operatorname{am}u$, and similarly $F(\phi)=v$, $\phi=\operatorname{am}v$, &c., then the equation $F(\theta)+F(\phi)=F(\mu)$ becomes $F(\mu)=u+v$, that is, $\mu=\operatorname{am}(u+v)$; and the algebraic relation in its various forms gives the values of $\sin.\operatorname{am}(u+v)$, $\cos.\operatorname{am}(u+v)$, $\Delta.\operatorname{am}(u+v)$ in terms of the like functions of u, v respectively, viz. it is the addition-theorem for the function am.

Observe that $\operatorname{am}u$ is considered as a certain function of u, $\sin.\operatorname{am}u$ is its sine, $\cos.\operatorname{am}u$ its cosine, and

$$\Delta.\operatorname{am}u=\sqrt{(1-k^2\sin^2.\operatorname{am}u)},$$

a function analogous to a cosine. But making only a slight change in the point of view, we have $\operatorname{sinam}u$, a certain function of u, and

$$\operatorname{cosam}u\,\{=\sqrt{(1-\operatorname{sinam}^2u)}\},\ \Delta\operatorname{am}u\,\{=\sqrt{(1-k^2\operatorname{sinam}^2u)}\},$$

two allied functions, viz. $\operatorname{sinam} u$ is analogous to a sine, and the other two functions to cosines; the algebraical equations give the sinam, cosam, and $\Delta\mathrm{am}$ of $u+v$ in terms of the like functions of u and v respectively, viz. they constitute the addition-theorem for these functions.

6. *Find the differential equation satisfied by a hypergeometric series, and express by means of such series the coefficients of the expansion of* $(1-2a\cos\theta+a^2)^{-n}$ *according to multiple cosines of* θ.

I understand the expression "hypergeometric series" in the restricted sense in which it signifies the series

$$F(\alpha,\ \beta,\ \gamma,\ x)=1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\gamma(\gamma+1)}x^2+\&c.$$

I find it was understood in the more general sense of a series

$$u=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots,$$

where the coefficient a_{n+1} is given in terms of the preceding one a_n by an equation of the form $a_{n+1}=\phi(n)\,.\,a_n$. In this latter sense, but supposing for greater simplicity, that $\phi(n)$ is a rational and integral function of n, the solution is as follows: we operate on the series with the symbol $\phi\left(x\dfrac{d}{dx}\right)$; viz. $x\dfrac{d}{dx}$ is regarded as a single symbol of operation; $x\dfrac{d}{dx}\,.\,x^n=nx^n$, $\left(x\dfrac{d}{dx}\right)^2x^n=n^2x^n$, &c.; thus $x\dfrac{d}{dx}$ is, as regards x^n, $=n$, and therefore $\phi\left(x\dfrac{d}{dx}\right)=\phi(n)$. We thence have

$$\begin{aligned}\phi\left(x\frac{d}{dx}\right)u&=\phi(0)\,a_0+\phi(1)\,a_1x+\phi(2)\,a_2x^2\ldots+\phi(n)\,a_nx^n+\ldots\\&=\ a_1\ +\ a_2x\ +\ a_3x^2\ \ \ldots+\ a_{n+1}x^n\ +\ldots,\end{aligned}$$

and consequently

$$x\phi\left(x\frac{d}{dx}\right)u=u-a_0,$$

which is the required differential equation. This is equivalent to the process given in Boole, only he writes $x=e^\theta$, in order to reduce $x\dfrac{d}{dx}$ to a mere differentiation $\dfrac{d}{d\theta}$. I regard this introduction of a new variable θ *as most unfortunate;* the effect is entirely to conceal the real nature of the operation; the notion of $x\dfrac{d}{dx}$ as a single symbol of operation is quite as simple as that of $\dfrac{d}{d\theta}$; and by means of it we retain the original variable.

The process is substantially the same when $\phi(n)$ is a rational fraction, but I give the investigation directly for the hypergeometric series in the restricted sense, viz. writing u for the series $F(\alpha,\ \beta,\ \gamma,\ x)$, we find

$$x\left(x\frac{d}{dx}+\alpha\right)\left(x\frac{d}{dx}+\beta\right)u=x\frac{d}{dx}\left(x\frac{d}{dx}+\gamma-1\right)u;$$

or, what is the same thing,

$$\left\{\left(x\frac{d}{dx}+\alpha\right)\left(x\frac{d}{dx}+\beta\right)-\frac{d}{dx}\left(x\frac{d}{dx}+\gamma-1\right)\right\}u=0,$$

as at once appears by writing the general term successively under the two forms

$$\frac{\alpha\,.\,\alpha+1\ldots\alpha+n-1\,.\,\beta\,.\,\beta+1\ldots\beta+n-1}{1\,.\,2\ \ldots\ \ n\qquad\gamma\,.\,\gamma+1\ldots\gamma+n-1}x^n,$$

and

$$\frac{\alpha\,.\,\alpha+1\ldots\alpha+n\,.\,\beta\,.\,\beta+1\ldots\beta+n}{1\,.\,2\ \ldots n+1\,.\,\gamma\,.\,\gamma+1\ldots\gamma+n}x^{n+1}.$$

The differential equation may also be written

$$\left[(x^2-x)\frac{d^2}{dx^2}+\{(1+\alpha+\beta)\,x-\gamma\}\frac{d}{dx}+\alpha\beta\right]u=0.$$

Take next the function

$$\begin{aligned}
&(1-2a\cos\theta+a^2)^{-n},\\
&=\left\{1-a\left(x+\frac{1}{x}\right)+a^2\right\}^{-n}\\
&=\left\{(1-ax)\left(1-a\frac{1}{x}\right)\right\}^{-n},\ \text{if } x+\frac{1}{x}=2\cos\theta,\\
&=\left(1+\frac{n}{1}ax+\frac{n\,.\,n+1}{1\,.\,2}a^2x^2+\ldots\right)\left(1+\frac{n}{1}\frac{a}{x}+\frac{n\,.\,n+1}{1\,.\,2}\frac{a^2}{x^2}+\ldots\right)\\
&=\left\{1^2+\left(\frac{n}{1}\right)^2a^2+\left(\frac{n\,.\,n+1}{1\,.\,2}\right)^2a^4\ldots\right\}\\
&+\left\{1\,.\,\frac{n}{1}+\frac{n}{1}\frac{n\,.\,n+1}{1\,.\,2}a^2+\frac{n\,.\,n+1}{1\,.\,2}\frac{n\,.\,n+1\,.\,n+2}{1\,.\,2\,.\,3}a^4+\ldots\right\}a\left(x+\frac{1}{x}\right)(=2a\cos\theta)\\
&+\left\{1\,.\,\frac{n\,.\,n+1}{1\,.\,2}+\&c.\qquad\qquad\qquad\qquad\qquad\qquad\right\}a^2\left(x^2+\frac{1}{x^2}\right)(=2a^2\cos 2\theta),\\
&\quad\&c.\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\&c.
\end{aligned}$$

where the second term contains the factor $\frac{n}{1}a$, the third the factor $\frac{n\,.\,n+1}{1\,.\,2}a^2$, and so on. Throwing these out, the remaining factors are each of them a hypergeometric series, viz. representing the whole expression by

$$A_0+2A_1\cos\theta+2A_2\cos 2\theta+\&c.,$$

we have

$$A_0=F(n,\ n,\ 1,\ a^2),$$

$$A_1=\frac{n}{1}aF(n,\ n+1,\ 2,\ a^2),$$

$$\vdots$$

and generally

$$A_r=\frac{n\,.\,n+1\ldots n+r-1}{1\,.\,2\ldots r}a^rF(n,\ n+r,\ r+1,\ a^2).$$

7. *The function* $e^{-\frac{1}{(x-a)^2}}$ *has been suggested as an exception to the theorem that if a function and all its differential coefficients vanish for a given value of the variable, then the function is identically* $=0$; *discuss the question as regards the precise meaning of the theorem, and validity of the exception.*

The suggestion was made by Sir W. R. Hamilton; the following remarks arise in regard to it:

The function $e^{-\frac{1}{(x-a)^2}}$ is a function which *in a certain sense* satisfies the condition that for a given value $(=a)$ of the variable, the function and all its differential coefficients vanish; viz. each differential coefficient is of the form $Xe^{-\frac{1}{(x-a)^2}}$, where X is a finite series of negative powers of $x-a$; if then $x=a\pm r$, where r is real and positive, and if r continually diminishes to zero, then $(x-a)^2$, remaining always real and positive, continually diminishes to zero, that is, $-\frac{1}{(x-a)^2}$ remaining always real and negative continually increases to $-\infty$, and $e^{-\frac{1}{(x-a)^2}}$ remaining always real and positive continually diminishes to zero. And, moreover, (X containing only a finite series of negative powers of $x-a$) the expression $Xe^{-\frac{1}{(x-a)^2}}$ will in like manner, remaining always real, continually approximate to zero. But assume $x=a+r(\cos\theta+i\sin\theta)$, r real and positive, θ real; then $(x-a)^2=r^2(\cos 2\theta+i\sin 2\theta)$, and if $\cos 2\theta$ be positive, then the real part of $(x-a)^2$, being always positive, continually diminishes to zero, and the like conclusions follow. If however $\cos 2\theta$ be negative, then the real part of $(x-a)^2$ is negative, and the real part of $-\frac{1}{(x-a)^2}$ is positive, and as r diminishes continually approximates to $+\infty$; so far from $e^{-\frac{1}{(x-a)^2}}$ continually approximating to zero, it is in general an imaginary quantity continually approximating to infinity; and the like is the case with its successive differential coefficients; the conclusion is, it is not true *simpliciter* that the function $e^{-\frac{1}{(x-a)^2}}$, or any one of its successive differential coefficients, vanishes for the value a of the variable.

Generally, if a real or imaginary quantity $\alpha+\beta i$ is represented by the point whose rectangular coordinates are α, β; say if the value a of the variable x is represented by the point P, and any other value $a+h+ki$, by the point Q (h, k being therefore the coordinates of Q measured from the origin P), then a function $F(x)$ which as Q (no matter in what direction) approaches and ultimately coincides with P, tends to become and becomes ultimately $=0$, may be said to vanish *simpliciter* for the value a of the variable; but if this is only the case when Q approaches P in a certain direction or within certain limits of direction, the function not becoming zero when Q approaches in a different direction, then the function may be said to vanish *sub modo* for the value a of the variable.

Taking the theorem to mean "If for a given value a of the variable, a function and its differential coefficients vanish *sub modo*, the function is identically $=0$," the

instance of the function $e^{-\frac{1}{(x-a)^2}}$ shows that the theorem is certainly not true; but taking the theorem to mean "If for a given value a of the variable, the function and its differential coefficients vanish *simpliciter*, then the function is identically $=0$"; the instance does not apply to it, and the truth of the theorem remains an open question.

The above view is consistent with a theorem obtained by Cauchy and others, defining within what limits of h the expansion by Taylor's theorem of the function $F(a+h)$ is applicable, viz. a and h being in general imaginary as above, if the function (or ? the function and its successive differential coefficients) is (or are) finite and continuous so long as the distance PQ does not exceed a certain real and positive value ρ, then the expansion is applicable for any point Q, whose distance PQ does not exceed this value ρ: but it ceases to be applicable for a point Q, the distance of which is equal to or exceeds ρ. In the case of a function such as $e^{-\frac{1}{(x-a)^2}}$, discontinuity arises *at* the point P, that is, for the value $\rho=0$, and according to the theorem in question, the expansion is not applicable for any value of ρ however small.

I wish to remark on a view which appears to me to be founded on a radical misconception of the notion of convergence. Writing $F(x)=e^{-\frac{1}{(x-a)^2}}$, consider the series

$$F(a)+F'(a)\frac{h}{1}+F''(a)\frac{h^2}{1.2}+\&c.\ldots$$

Then admitting that the exponential $e^{-\frac{1}{(x-a)^2}}$ becomes $=0$ for $x=a$, the successive functions $F(a)$, $F'(a)$, $F''(a)$, ... are each $=0$ as containing this exponential: but inasmuch as the successive differentiations introduce negative powers of $x-a$, each successive function is regarded as an infinitesimal of a lower order than those which precede it; say $F(a)$ being $=0^{\mu}$, the successive terms are multiples 0^{μ}, $0^{\mu-3}$, $0^{\mu-6}$, $0^{\mu-9}$, &c. respectively; where however μ is infinite, so that the several exponents μ, $\mu-3$, $\mu-6$, &c., however far the series is continued, remain all of them positive. This being so, it is said that the series $F(a)+F'(a)\frac{h}{1}+\&c.$, as being really of the form $0^{\mu}+0^{\mu-3}+0^{\mu-6}+\ldots$ is divergent, and for this reason fails to give a correct value of $F(a+h)$. I apprehend that the notion of divergence is a strictly numerical one; a series of numbers $a+b+c+d+\ldots$ is divergent when the successive sums a, $a+b$, $a+b+c$, $a+b+c+d$, &c., are numbers not continually tending to a determinate limit. In the actual case the series is $0+0+0+0+\ldots$, viz. each term is by hypothesis an absolute zero; the successive sums 0, $0+0$, $0+0+0$, ... are each $=0$, and we cannot, by the process of numerical summation, make the sum of the series to be anything else than 0. If it could, there would be an end of all numerical equality between infinite series; for taking any convergent series $a+b+c+d+\ldots$, if 0 means 0, this is the same thing as the series, also a convergent one,

$$(a+0)+(b+0)+(c+0)+\&c.,$$

and their difference $0+0+0+\ldots$ must be $=0$. I regard the view as a mere failure to reconcile the equation

$$F(a+h)=Fa+\frac{h}{1}F'(a)+\&c.,$$

with the *supposed* fact in regard to the function $e^{-\frac{1}{(x-a)^2}}$.

8. *Find the value of the definite integrals*

$$\int e^{-x^2}\,dx,\ \int \sin x^2\,dx,\ \int \cos x^2\,dx,$$

the limits being in each case ∞, $-\infty$. *Examine whether the last two integrals can be found by a process such as Laplace's (depending on a double integral) for the first integral.*

Laplace's process for the integral $\int e^{-x^2}\,dx$ is as follows: write $u=\int e^{-x^2}\,dx$, then also $u=\int e^{-y^2}\,dy$, and thence

$$u^2=\iint e^{-(x^2+y^2)}\,dx\,dy,$$

which, considering x, y as rectangular coordinates and substituting for them the polar coordinates r, θ, becomes

$$u^2=\iint e^{-r^2}\,r\,dr\,d\theta;$$

and then considering the double integral as extending over the infinite plane, and taking the limits to be $r=0$ to $r=\infty$, $\theta=0$ to $\theta=2\pi$, we obtain

$$u^2=\left(-\tfrac{1}{2}e^{-r^2}\right)_0^\infty 2\pi,\ =\tfrac{1}{2}\,.\,2\pi,\ =\pi,$$

that is,

$$u=\int e^{-x^2}\,dx=\sqrt{(\pi)}.$$

There is an assumption the validity of which requires examination. We have u the limit of the integral $\int_{-\alpha}^{\alpha} e^{-x^2}\,dx$, as α approaches to ∞; and this being so, we have u^2 the limit of

$$\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} e^{-(x^2+y^2)}\,dx\,dy,$$

viz. u^2 is the integral of $e^{-(x^2+y^2)}$ taken over a square, the side of which is 2α, α being ultimately infinite. But making the transformation to polar coordinates, and integrating as above, we in fact take the integral over a circle radius $=\beta$, β being ultimately infinite. And we assume that the two values are equal; or, generally, that taking the integral over an area bounded by a curve which is such that the distance of every point from the origin is ultimately infinite, the value of the integral is independent of the form of the curve.

This is really the case under the following conditions: 1°. For a curve of a given form, the integral tends to a fixed limit, as the size is continually increased. 2°. The quantity under the integral sign is always of the same sign (say always positive); (the last condition is sufficient, but not necessary). For, to fix the ideas, let the curves be as before the square and the circle: take a square; surrounding this, a circle; and surrounding the circle, a square. Imagine the two squares and the circle continually to increase in magnitude; the integral over the smaller square and that over the larger square, each tend to the same fixed limit; consequently the integral over the area enclosed between the two squares tends to the limit zero; and *à fortiori* the integral over the area enclosed between the circle and either of the two squares tends to the limit zero; that is, the integral over the square, and that over the circle, tend to the same limit. In the case under consideration, the function $e^{-(x^2+y^2)}$ is always positive; and the integral $\iint e^{-(x^2+y^2)}\,dx\,dy$, taken over the circle, tends (as in effect shown above) to the limit π: hence the process is a legitimate one.

But endeavour to apply it to the other two integrals; write

$$\begin{aligned} u &= \int \sin x^2 dx \\ &= \int \sin y^2 dy, \end{aligned} \qquad \begin{aligned} v &= \int \cos x^2 dx \\ &= \int \cos y^2 dy, \end{aligned}$$

then

$$\iint \sin(x^2+y^2)\,dx\,dy = 2uv, \quad \iint \cos(x^2+y^2)\,dx\,dy = v^2 - u^2,$$

where the double integrals on the left-hand side really denote integrals taken over a square and are not equal to the like integrals taken over a circle. This appears *à posteriori* if we only assume that the integrals u, v have determinate values; for taking the integrals over a circle they would be

$$\iint \begin{matrix}\sin\\ \cos\end{matrix}\, r^2 . r\,dr\,d\theta,$$

and would involve the indeterminate functions $\begin{matrix}\sin\\ \cos\end{matrix}\,\infty$; that is, if it were allowable to take the integrals over a circle, we should have $2uv$ and $v^2 - u^2$ indeterminate instead of determinate.

A process of finding them is as follows: in the equation $\int e^{-x^2}\,dx = \sqrt{(\pi)}$, substituting in the first instance $x\sqrt{(a)}$ for x, a real and positive, we have

$$\int e^{-ax^2}\,dx = \frac{\sqrt{(\pi)}}{\sqrt{(a)}},$$

and if it be assumed that this equation extends to the case where $a = \alpha + \beta i$, *the*

real part a real and positive;* or, what is the same thing, $a = \rho(\cos\theta + i\sin\theta)$, ρ real and positive, θ between the limits 0 and $\frac{1}{2}\pi$, then we have

$$\int e^{-\rho(\cos\theta + i\sin\theta)x^2}\,dx = \frac{\sqrt{(\pi)}}{\sqrt{(\rho)}}(\cos\tfrac{1}{2}\theta - i\sin\tfrac{1}{2}\theta),$$

or, separating the real and imaginary parts and taking $\rho = 1$, we have

$$\int e^{-x^2\cos\theta}\cos(x^2\sin\theta)\,dx = \sqrt{(\pi)}\cos\tfrac{1}{2}\theta,$$

$$\int e^{-x^2\cos\theta}\sin(x^2\sin\theta)\,dx = \sqrt{(\pi)}\sin\tfrac{1}{2}\theta.$$

Admitting these formulæ to be true in general, there is still considerable difficulty in seeing that they hold good in the limiting case $\theta = \frac{1}{2}\pi$. But assuming that they do, the formulæ then become

$$\int \cos x^2\,dx = \frac{\sqrt{(\pi)}}{\sqrt{(2)}},\quad \int \sin x^2\,dx = \frac{\sqrt{(\pi)}}{\sqrt{(2)}},$$

which are the values of the integrals in question.

9. *Considering in a solid body a system of two, three, four, five, or six lines, determine in each case the relations between the lines in order that it may be possible to find along them forces to hold the body in equilibrium.*

If there are two lines, the condition obviously is that these must be one and the same line.

If three lines, then these must lie in a plane, and meet in a point.

The conditions in the other cases ought to be in the text-books; they in fact are not, and I assumed that they would not be known, and considered the question as a problem; it is, in regard to the cases of four and five lines, *a very easy problem when the solution is seen.*

In the case of four lines; imagine in the solid body an axis meeting any three of the lines, and let this axis be fixed; the condition of equilibrium about this axis is that the fourth line shall meet the axis. The required condition therefore is that every line meeting three of the four lines shall meet the fourth line; or, what is the same thing, the four lines must be generators (of the same kind) of a skew hyperboloid.

In the case of five lines, taking any four of them, we have two lines (tractors) each meeting the four lines; and taking either of the two lines as an axis, then for equilibrium the fifth line must also meet this axis; the required relations therefore are that the fifth line shall meet each of the two lines which meet the other four lines; or, what is the same thing, that there shall be two lines each meeting the five given lines.

* The equation is clearly *not true* unless this is so: for a being negative, then in virtue of the factor e^{-ax^2}, the exponential, instead of decreasing will increase, and ultimately become infinite as x increases to $\pm\infty$

The case of six lines is one the answer to which could not have been discovered in an examination; the relations in fact are that the six lines shall form an involution; viz. this is a system such that taking five of the lines as given, then if the sixth line is taken to pass through a given point it may be any line whatever in a determinate plane through this point; or, what is the same thing, if the sixth line is taken to be in a given plane, it may be any line whatever through a determinate point in this plane. But in a particular case, the answer is easy; suppose five of the six given lines to be met by a single line, then the sixth line may be any line whatever meeting this single line.

10. *If $X, Y, Z, \ldots$ are the roots of the equation*

$$(1, P, Q, \ldots)(c, 1)^n = 0,$$

show that the differential equation obtained by the elimination of c is $\zeta X'Y'Z' = 0$, where ζ denotes the product of the squared differences of the roots $X, Y, Z, \ldots$, and $X', Y', Z', \ldots$ are the derived functions of these roots; and connect this result with the theory of singular solutions.

We have identically

$$(1, P, Q \ldots)(c, 1)^n = (c - X)(c - Y)(c - Z)\ldots;$$

the original equation and its derived equation

$$(0, P', Q', \ldots)(c, 1)^n = 0$$

(the latter of them of degree $n-1$) may therefore be written

$$(c - X)(c - Y)(c - Z)\ldots = 0,$$
$$X'(c - Y)(c - Z)\ldots + Y'(c - X)(c - Z)\ldots + \&c. = 0.$$

To eliminate c, we have in the *nilfactum* of the second equation to substitute successively the values $c = X$, $c = Y$, &c., multiply the several functions together and equate the result to zero; the factors are evidently

$$X'(X - Y)(X - Z)\ldots, \quad Y'(Y - X)(Y - Z)\ldots, \ \&c.,$$

where each difference occurs twice, e.g. $X - Y$ under the two forms $X - Y$ and $Y - X$ respectively; the result thus is

$$X'Y'Z'\ldots(X - Y)^2(X - Z)^2(Y - Z)^2\ldots = 0;$$

that is,

$$\zeta . X'Y'Z'\ldots = 0.$$

Thus in particular in the case of a quadric equation

$$(1, P, Q)(c, 1)^2, = (c - X)(c - Y), = 0,$$

the differential equation is

$$(X - Y)^2 X'Y' = 0;$$

viz. since $X+Y=-P$, and $XY=Q$, this is

$$(P^2-4Q)\,X'Y'=0,$$

and writing also

$$X=-\tfrac{1}{2}\{P+\sqrt{(P^2-4Q)}\},\quad Y=-\tfrac{1}{2}\{P-\sqrt{(P^2-4Q)}\},$$

we find

$$X'Y'=\tfrac{1}{4}\left\{P'^2-\frac{(PP'-2Q')^2}{P^2-4Q}\right\};$$

the differential equation thus is

$$(P^2-4Q)\left\{P'^2-\frac{(PP'-2Q')^2}{P^2-4Q}\right\}=0.$$

The application to the theory of singular solutions is that, in the case where the function $(1,\ P,\ Q\ldots)(c,\ 1)^n$ breaks up into rational factors $c-X$, $c-Y,\ldots$, the factor $\zeta=(X-Y)^2(X-Z)^2\ldots$ divides out and should be rejected from the differential equation, which in its true form is $X'Y'Z'\ldots=0$; viz. this is what we obtain immediately, considering the given integral equation as meaning the system of curves $c-X=0$, $c-Y=0,\ldots$, and there is not really any singular solution; whereas in the case where the factors are not rational, the factor in question, when the product $X'Y'Z'\ldots$ is expressed in terms of the coefficients $P,\ Q,\ldots$, and their derived coefficients does not divide out from the equation; and in this case, equated to zero, it gives a proper singular solution of the equation.

11. *In the theory of elliptic motion, v denoting the mean anomaly and e the eccentricity, if m' be an angle such that* $\tan\frac{1}{2}v=\dfrac{1+e}{1-e}\tan\frac{1}{2}m'$, *find in terms of e, m' the mean anomaly m.*

Taking as usual u for the eccentric anomaly, to commence the solution write down

$$\tan\tfrac{1}{2}v=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}u$$

$$=\frac{1+e}{1-e}\tan\tfrac{1}{2}m',$$

that is,

$$\tan\tfrac{1}{2}u=\sqrt{\left(\frac{1+e}{1-e}\right)}\tan\tfrac{1}{2}m',$$

and u being given hereby as a function of m', we have by substitution in the equation $m=u-e\sin u$, to find m as a function of m'.

A creditable approximate solution would be $m=m'+0\,.\,e$, viz. this would be to show that neglecting terms in e^2, &c., we have $m=m'$. In fact, taking e small, we have

$$\tan\tfrac{1}{2}u=(1+e)\tan\tfrac{1}{2}m',$$

and thence if $u=m'+x$, we have

$$\tan\tfrac{1}{2}m'+\tfrac{1}{2}x\sec^2\tfrac{1}{2}m'=(1+e)\tan\tfrac{1}{2}m',$$

that is,

$$x = 2e\cos^2 \tfrac{1}{2}m' \tan \tfrac{1}{2}m' = e\sin m' ;\ \ u = m' + e\sin m',$$

and

$$\begin{aligned} m &= m' + e\sin m' \\ &\quad - e\sin(m' + \ldots) \\ &= m' + 0\,.\,e. \end{aligned}$$

The complete solution would be obtained by expanding u in terms of e, m' from the equation $\tan \frac{1}{2}u = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan \frac{1}{2}m'$ (which is of the form $\tan \frac{1}{2}u = n \tan \frac{1}{2}m'$, giving for u a known series $= m' +$ multiple sines of m'), and then observing that the same equation leads to

$$\sin u = \frac{\sqrt{(1-e^2)}\sin m'}{1 - e\cos m'},$$

we have

$$m = \text{series} - \frac{e\sqrt{(1-e^2)}\sin m'}{1 - e\cos m'},$$

where the second term has also to be expanded in a series of multiple sines of m'; which can be done without difficulty.

12. *If (u, v) are given functions of the coordinates (x, y), neither of them a maximum or a minimum at a given point O; and if through O we draw Ox' in the direction in which v is constant and u increases, and Oy' in the direction in which u is constant and v increases; then the rotation (through an angle not greater than π), from Ox' to Oy' is in the same direction with that from Ox to Oy, or in the contrary direction, according as $\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$ is positive or negative.*

The theorem has not, so far as I am aware, been noticed, and it seems to be one of some importance; there is no difficulty in it, but the answer requires some care in writing out; of course where the whole question is one of sign and direction, the omission to state that a subsidiary quantity is positive may render an answer worthless.

It depends on the following lemma: Consider the triangle $OX'Y'$, such that Ox, Oy being any rectangular axes through the origin O, the coordinates of X' are h, k, and those of Y' are h_1, k_1: then considering the area as positive, the double area is $= \pm(hk_1 - h_1k)$, viz. the sign is $+$ or $-$ according as the rotation from OX' to OY' (through an angle less than π) is in the same direction with that from Ox to Oy, or in the contrary direction; or, what is the same thing, $hk_1 - h_1k$ is in the first case positive and in the second case negative.

To show this, suppose for a moment that the lines OX', OY' are each of them in the quadrant xOy, say in the first quadrant, the inclination of OY' to Ox exceeding that of OX' to Ox; then h, k, h_1, k_1 are all positive, and $\frac{k_1}{h_1} > \frac{k}{h}$, that is, $hk_1 - h_1k$ is $+$,

and the rotation from OX' to OY' is in the same direction as that from Ox to Oy; or the lemma holds good. Now OX' remaining fixed, let OY' revolve in the direction Ox to Oy; so long as OY' remains in the first quadrant, $\frac{k_1}{h_1}$ continues to increase, and we have always $\frac{k_1}{h_1} > \frac{k}{h}$, and $hk_1 - h_1k = +$; when OY' comes into the second quadrant (h, k being always positive), h_1 is negative and k_1 positive, consequently $hk_1 - h_1k$ is the sum of two positive terms, and therefore $= +$; as OY' continues to revolve and passes into the third quadrant, we have h_1, k_1 each negative, but $\frac{k_1}{h_1} < \frac{k}{h}$, and therefore $hk_1 - h_1k$ still $= +$; when, however, OY' comes into the position opposite to OX', then $\frac{k_1}{h_1} = \frac{k}{h}$, and $hk_1 - h_1k$ is $= 0$; and when OY', continuing in the third quadrant, has passed the position in question, we have $\frac{k_1}{h_1} > \frac{k}{h}$, and therefore $hk_1 - h_1k = -$, but now the angle $X'OY'$ measured in the original direction has become $> \pi$, and the rotation OX' to OY' through an angle less than π will be in the opposite direction, that is, in the direction opposite to that from Ox to Oy; and, similarly, when OY' passes into the fourth quadrant, and until, passing into the first quadrant, it approaches the position OX', the sign of $hk_1 - h_1k$ will be $-$, and the rotation will be in the direction contrary to that from Ox to Oy. The lemma is thus true for any position of OX' in the first quadrant; and the like reasoning would show that it is true for any position of OX' in the second, the third, or the fourth quadrant; hence the lemma is true generally.

This being so, taking a new origin, let the coordinates of O be x, y; and drawing through O the axes Ox', Oy' as directed, let X' be the point belonging to the values $u + \delta u$, v of (u, v), and Y' the point belonging to the values u, $v + \delta v$ of (u, v); taking δu positive, X' will be on Ox' in the direction O to x', and similarly taking δv positive, Y' will be on Oy' in the direction O to y'. Taking as before (h, k) for the coordinates of X', and (h_1, k_1) for the coordinates of Y', these coordinates being measured from the point O as origin, we have

$$\delta u = \frac{du}{dx} h + \frac{du}{dy} k,$$

$$0 = \frac{dv}{dx} h + \frac{dv}{dy} k,$$

whence, writing for a moment $J = \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$, we have $Jh = +\frac{dv}{dy}\delta u$, $Jk = -\frac{dv}{dx}\delta u$. And in like manner

$$0 = \frac{du}{dx} h_1 + \frac{du}{dy} k_1,$$

$$\delta v = \frac{dv}{dx} h_1 + \frac{dv}{dy} k_1,$$

whence

$$Jh_1 = -\frac{du}{dy}\delta v, \quad Jk_1 = \frac{du}{dx}\delta v;$$

and hence

$$J^2(hk_1 - h_1k) = \left(\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right)\delta u\,\delta v, \; = J\delta u\delta v,$$

that is,

$$hk_1 - h_1k = \frac{1}{J}\,\delta u\delta v,$$

and δu, δv being as above each of them positive, J has the same sign as $hk_1 - h_1k$. But the rotation from OX' to OY' is in the same direction as that from Ox to Oy, or in the contrary direction, according as $hk_1 - h_1k$ is $+$ or $-$, that is, according as $J, = \frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$, is $+$ or $-$; which is the theorem in question.

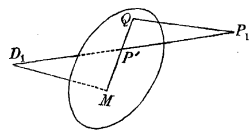
13. Write a dissertation on:

The theory and constructions of Perspective.

In Perspective we represent an object in space by means of its central projection upon a plane: viz. any point P_1* of the object is represented by P', the intersection with the plane of projection of the line D_1P_1 from the centre of projection (or say the eye) D_1 to the point P_1; and considering any line or curve in the object, this is represented by the line or curve which is the locus of the points P', the projections of the corresponding points P_1 of the line or the curve in the object.

The fundamental construction in perspective is derived from the following considerations: viz. considering through P_1 (fig. 1) a line meeting the plane of projection in Q, and drawing parallel thereto through D_1 a line to meet the plane of projection in M and joining the points M, Q, then the lines D_1M, MQ, QP_1 are in a plane; that is, the plane through D_1 and the line P_1Q meets the plane of projection in MQ;

Fig. 1.



and consequently the projection P' of any point P_1 in the line P_1Q lies in the line QM; and not only so, but considering only the points P_1 of this line which lie behind the plane of projection (D_1 being considered as in front of it), the projections of all these points lie on the terminated line MQ; viz. Q is the projection of the point Q, and M the projection of the point at infinity on the line QP_1; or, if we please, the finite line QM is the projection of the line $QP_1\infty$.

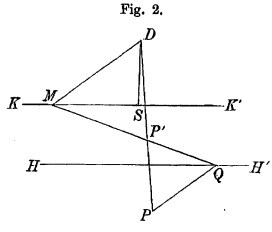
* The subscript unity is used to denote a point not in the plane of projection, considered as a point out of this plane; a point in the plane of projection, used in the constructions of perspective as a conventional representation of a point P_1, will be denoted by the same letter P without the subscript unity. And the like as regards D_1 and D.

If we consider a set of lines parallel to P_1Q, these all give rise to the same point M, and thus their projections MQ all pass through this point M, which is said to be the "vanishing point" of the system of parallel lines. Again, if we consider any two or more lines through P_1, to each of these there correspond different points M and Q, and, therefore, a different line MQ, but these all intersect in a common point P' which is the projection of P_1. If the lines are all in one and the same plane through P_1, then the locus of the points Q is a line, the intersection of this plane with the plane of projection, say the "trace" line; and the locus of the points M is a parallel line, the intersection of the parallel plane through D_1 with the plane of projection; say this is the "vanishing line" for the plane in question.

A construction in perspective presupposes a conventional representation on the plane of projection (or say on the paper) as well of the position of the eye as of the object to be projected. If for simplicity we suppose the object to be a figure in one plane, then this plane intersects the paper in a trace line, and we may imagine the plane made to rotate about the trace line until it comes to coincide with the paper, and we have thus the plane object conventionally represented on the paper. Similarly considering the parallel plane through the eye D_1, and regarding D_1 as a point of this plane, the plane meets the paper in the vanishing line, and we may imagine the plane made to rotate (in the direction opposite to that of the first rotation) until it comes to coincide with the paper, bringing the point D_1 to coincide with a point D of the paper. We have thus the "point of distance" D, being a conventional representation on the paper of the position of the eye D_1; but which point D has, observe, a different position for different directions of the plane of the object.

To fix the ideas, suppose the plane of projection to be vertical, and the plane of the object to be a horizontal plane situate below the eye. The trace line will be represented by a horizontal line HH' (fig. 2), and the object by a figure in the plane

Fig. 2.



of the paper below the line HH' such that, bending this portion of the paper backwards through a right angle round HH', the figure would be brought to coincide with the object*. The vanishing line will be a horizontal line KK' above HH', and the

* It is assumed in the text, that the figure on the paper is equal in magnitude to the object; but practically the figure is drawn on a reduced scale, the distance between the lines KK', HH', and the distance DS (representing respectively the distance between the parallel planes, and the distance of the eye from the plane of projection) being drawn on the same reduced scale.

eye will be represented by a point D above KK', in suchwise that, bending the upper part of the paper round KK' forwards through a right angle, the point D would come to coincide with the position D_1 of the eye. This being so, taking any line PQ in the representation of the object, we draw through D the parallel line DM, and then joining the points M and Q, we have MQ as the perspective representation of the line $QP\infty$, which represents a line $QP_1\infty$ of the object. And drawing through P any number of lines, each of these gives a point Q and a point M, but the lines MQ all meet in a common point P', which is the perspective representation of the point P; which point P' may, it is clear, be obtained as the intersection with any one line MQ of the line DP drawn to join P with the point of distance D. The plane of the object has for convenience been taken to be horizontal; but its position may be any whatever, and in particular the construction is equally applicable in the case where the plane is vertical.

In the case of an object not in one plane, any point Q_1 of the object may be determined by means of its projection by a vertical line upon a given horizontal plane, say this is P_1, and of its altitude Q_1P_1 above this plane. We in fact determine the object by means of its groundplan, and of the altitudes of the several points thereof. It is easy, from the foregoing principles, to see that, drawing through P the vertical line PQ equal to the altitude, and joining the points Q, D, then the vertical line through P' meets this line QD in a point Q', which will be the perspective representation of Q_1. We have thus a construction applicable to any solid figure whatever.

592.

ON THE MERCATOR'S-PROJECTION OF A SKEW HYPERBOLOID OF REVOLUTION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 17—20.]

IN a note "On the Mercator's-projection of a surface of revolution" read before the British Association, [555, (5)], I remarked that the surface might be, by its meridians and parallels, divided into infinitesimal squares; and that these would be on the map represented by two systems of parallel lines at right angles to each other, dividing the map into infinitesimal squares; and that, by taking the squares not infinitesimal but small, for instance, by considering the meridians at intervals of 10° or 5°, we might approximately construct a Mercator's-projection of the surface. But it is worth while, for the skew hyperboloid of revolution, to develop analytically the ordinary accurate solution.

Taking the equation of the surface to be

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1,$$

{or, if as usual $a^2+c^2=a^2e^2$, then $x^2+y^2-(e^2-1)z^2=a^2$}, and writing $x=r\cos\theta$, $y=r\sin\theta$, the meridians corresponding to the several longitudes θ are in the map represented by the parallel lines $X=a\theta$, and the parallels corresponding to the several values of z are in the map represented by a set of parallel lines $Z=f(z)$, the form of the function being so determined that the infinitesimal rectangles on the map are similar to those on the surface. The required relation is readily found to be

$$Z=\int\frac{\sqrt{\{(a^2+c^2)z^2+c^4\}}}{z^2+c^2}\,dz,$$

where the integral is taken from the value $z=0$.

The substitution which first presents itself is to write herein $z = \frac{c^2}{\sqrt{(a^2+c^2)}} \tan\phi$; or, what is the same thing,

$$z = a\left(e - \frac{1}{e}\right)\tan\phi,$$

where observe that $a\left(e-\frac{1}{e}\right)$ is the distance between a focus and its corresponding directrix. The equation of the surface is satisfied by writing therein $\sqrt{(x^2+y^2)} = a\sec\psi$, $z = c\tan\psi$, and ψ as thus defined is the "parametric latitude"; hence the foregoing angle ϕ is a deduced latitude connected with the parametric latitude ψ by the equation

$$\tan\phi = \frac{c}{a\left(e-\frac{1}{e}\right)}\tan\psi, \quad = \frac{e}{\sqrt{(e^2-1)}}\tan\psi.$$

The resulting formula in terms of ϕ is

$$Z = \int_0 \frac{c^2\sqrt{(a^2+c^2)}\,d\phi}{\cos\phi\,(c^2+a^2\cos^2\phi)},$$

or, if we write herein $\zeta = \tan\frac{1}{2}\phi$, the formula becomes

$$Z = 2c^2\sqrt{(a^2+c^2)}\int_0 \frac{(1+\zeta^2)^2}{1-\zeta^2}\,\frac{d\zeta}{c^2(1+\zeta^2)^2+a^2(1-\zeta^2)^2},$$

viz. the function under the integral sign is rational. The expression is, however, complicated, and a more simple formula is obtained by using instead of ϕ the parametric latitude ψ; viz. we have $z = c\tan\psi$, and thence

$$Z = \int \frac{\sqrt{(a^2\sin^2\psi + c^2)}}{\cos\psi}\,d\psi,$$

or, putting herein

$$\sin\psi = \frac{c}{a}\,\frac{u}{\sqrt{(1-u^2)}},$$

and therefore

$$\cos^2\psi = \frac{a^2-(a^2+c^2)u^2}{a^2(1-u^2)},$$

and

$$a^2\sin^2\psi + c^2 = \frac{c^2}{1-u^2}, \quad \cos\psi\,d\psi = \frac{c}{a}\,\frac{du}{(1-u^2)^{\frac{3}{2}}},$$

the formula becomes

$$Z = c^2 a\int_0 \frac{du}{(1-u^2)\{a^2-(a^2+c^2)u^2\}},$$

or, what is the same thing,

$$= a(e^2-1)\int_0 \frac{du}{(1-u^2)(1-e^2u^2)};$$

viz. we thus have

$$Z = \tfrac{1}{2}a\left\{\log\frac{1-u}{1+u} - e\log\frac{1-eu}{1+eu}\right\},$$

the logarithms being hyperbolic.

As already mentioned, u is connected with the parametric latitude ψ by the equation

$$\sin\psi = \frac{c}{a}\,\frac{u}{\sqrt{(1-u^2)}}, \quad = \frac{u\sqrt{(e^2-1)}}{\sqrt{(1-u^2)}},$$

that is,

$$\sin\psi = \sqrt{(e^2-1)}\tan p, \text{ if } u = \sin p,$$

or conversely

$$u = \frac{\sin\psi}{\sqrt{(e^2-1+\sin^2\psi)}},$$

so that the point passing to infinity along the branch of the hyperbola, or ψ passing from 0 to 90°, u passes from 0 to $\frac{1}{e}$; and for $u = \frac{1}{e}$ the value of Z becomes, as it should do, infinite. The value of z in terms of u is

$$z = \frac{(e^2-1)\,u}{\sqrt{(1-e^2u^2)}}, \text{ or conversely } u = \frac{z}{\sqrt{(e^2z^2+e^2-1)}},$$

and we have, moreover,

$$u = \frac{1}{e}\,\frac{2\zeta}{1+\zeta^2}, \quad = \frac{1}{e}\sin\phi, \quad = \text{(as before)}\ \frac{\sin\psi}{\sqrt{(e^2-1+\sin^2\psi)}}.$$

It will be recollected that, in the Mercator's-projection of the sphere, the longitude and latitude being θ, ϕ, the values of X, Z are

$$X = a\theta, \quad Z = \log\tan\left(\frac{\pi}{4} + \tfrac{1}{2}\phi\right),$$

the logarithm being hyperbolic.

In the case of the rectangular hyperbola $a = c$, $= 1$ suppose,

$$e = \sqrt{(2)}, \quad z = \tan\psi, \quad u = \frac{\sin\psi}{\sqrt{(1+\sin^2\psi)}}, \quad = \sin p, \text{ if } \sin\psi = \tan p;$$

whence

$$Z = \tfrac{1}{2}h\,.\,l\,\frac{\tan(45^\circ - \frac{1}{2}p)}{\tan(45^\circ + \frac{1}{2}p)} - \tfrac{1}{2}\sqrt{(2)}\,h\,.\,l\,\frac{\tan(22^\circ 30' - \frac{1}{2}p)}{\tan(22^\circ 30' + \frac{1}{2}p)},$$

the first term being of course

$$= h\,.\,l\tan(45^\circ - \tfrac{1}{2}p), \text{ or } -h\,.\,l\tan(45^\circ + \tfrac{1}{2}p).$$

Transforming to ordinary logarithms, this is

$$Z = \frac{1}{\sqrt{(2)}\log e}\left[-\sqrt{(2)}\log\tan(45^\circ + \tfrac{1}{2}p) + \{\log\tan(22^\circ 30' + \tfrac{1}{2}p) - \log\tan(22^\circ 30' - \tfrac{1}{2}p)\}\right],$$

say this is

$$Z = \frac{1}{\sqrt{(2)}\log e}(-A + B),$$

where

$$A = \sqrt{(2)} \log \tan (45^\circ + \tfrac{1}{2}p),$$

$$B = \log \tan (22^\circ 30' + \tfrac{1}{2}p) - \log \tan (22^\circ 30' - \tfrac{1}{2}p).$$

Taking ψ as the argument, I tabulate z, $= \tan \psi$, and $Z \,.\, \sqrt{(2)} \log e$, $= -A + B$, as shown in the annexed table: the last column of which gives, therefore, the positions of the several parallels of 5°, 10°, ..., 85°; the interval of 5° between two meridians is, on the same scale,

$$\sqrt{(2)} \log e \,.\, \frac{\pi}{36} = (1{\cdot}4142136)\,({\cdot}4342945)\,({\cdot}0872665), = {\cdot}05360;$$

viz. this is nearly equal to the arc of meridian 0° to 5°, and the table shows that the arcs 0°—5°, 5°—10°, &c. continually increase as in a Mercator's-projection of the sphere, but more rapidly; there is, however, nothing in this comparison, since the determination of latitude on the hyperboloid by the equation $z = \tan \psi$ is altogether arbitrary.

ψ	$z = \tan \psi$	p	A	B	$-A+B$
0°	0·	0°	0·	0·	0·
5	·08749	4°58′ 51″	·05348	·10719	·05370
10	·17632	9 51 0	·10611	·21439	·10827
15	·26795	14 30 40	·15724	·32174	·16450
20	·36397	18 53 0	·20619	·42943	·22324
25	·46631	22 54 30	·25342	·53780	·28539
30	·57735	26 34 0	·29557	·64758	·35201
35	·70020	29 50 20	·33538	·75959	·42421
40	·83910	32 44 0	·37170	·87506	·50337
45	1·00000	35 15 50	·40446	·99554	·59108
50	1·19175	37 27 20	·43360	1·12355	·68995
55	1·42815	39 19 20	·45913	1·26151	·80238
60	1·73205	40 53 40	·48117	1·41445	·93328
65	2·14450	42 11 10	·49971	1·58840	1·08869
70	2·74747	43 13 10	·51478	1·79504	1·28026
75	3·73205	44 0 30	·52646	2·05524	1·52877
80	5·67128	44 33 40	·53469	2·41347	1·87877
85	11·43005	44 53 30	·53973	3·02355	2·48381
90	∞	45° 0′ 0″	·54133	∞	∞

593.

A SHEEPSHANKS' PROBLEM (1866).

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 34—36.]

Apply the formulæ of elliptic motion to determine the motion of a body let fall from the top of a tower at the equator.

The earth is regarded as rotating with the angular velocity ω round a fixed axis, so that the body is in fact projected from the apocentre with an angular velocity $=\omega$; and we write α for the equatorial radius, β for the height of the tower; then g denoting the force of gravity, and μ, h, n, a, e, θ, as in the theory of elliptic motion, we have

$$\begin{aligned} \mu = \quad n^2a^3 &= g\alpha^2, \\ h = (\alpha+\beta)^2\,\omega &= na^2\sqrt{(1-e^2)}, \\ \alpha+\beta &= a\,(1+e); \end{aligned}$$

whence

$$\begin{aligned} (\alpha+\beta)^4\,\omega^2 &= g\alpha^2 a\,(1-e^2), \\ (\alpha+\beta) &= a\,(1+e), \\ \frac{(\alpha+\beta)^3\,\omega^2}{g\alpha^2} &= \frac{a\omega^2}{g}\left(1+\frac{\beta}{\alpha}\right)^3 = 1-e, \end{aligned}$$

where $\dfrac{a\omega^2}{g}=$ ratio of centrifugal force to gravity,

$$=\tfrac{1}{289},$$

so that $1-e$ is small;

$$r = \frac{a\,(1-e^2)}{1-e\cos\theta}, \quad = \frac{(\alpha+\beta)\,(1-e)}{1-e\cos\theta},$$

whence

$$1-e\cos\theta \quad = \frac{(\alpha+\beta)\,(1-e)}{r}.$$

Suppose

$$r = \alpha,$$

$$1 - e\cos\theta = (1-e)\left(1 + \frac{\beta}{\alpha}\right) = 1 - e + \frac{\beta}{\alpha}(1-e),$$

which is nearly

$$= 1 - e,$$

that is, θ is small, and therefore approximately

$$1 - e + \tfrac{1}{2}e\theta^2 = 1 - e + \frac{\beta}{\alpha}(1-e),$$

or

$$\theta^2 = \frac{2\beta}{\alpha}\,\frac{1-e}{e};$$

we then have

$$r^2 d\theta = h dt, \text{ or } dt = \frac{r^2 d\theta}{h} = \frac{(\alpha+\beta)^2(1-e)^2}{(\alpha+\beta)^2\,\omega\,(1-e\cos\theta)^2}\,d\theta$$

$$= \frac{(1-e)^2}{\omega^2(1-e\cos\theta)^2}\,d\theta$$

$$= \frac{(1-e)^2}{\omega\,(1-e+\frac{1}{2}e\theta^2)^2}\,d\theta$$

$$= \frac{1}{\omega\left(1 + \frac{\frac{1}{2}e}{1-e}\,\theta^2\right)^2}\,d\theta$$

$$= \frac{1}{\omega}\left(1 - \frac{e}{1-e}\,\theta^2\right)d\theta;$$

that is,

$$\omega\, dt = \left(1 - \frac{e}{1-e}\,\theta^2\right)d\theta.$$

Integrating, we have

$$\theta\left(1 - \frac{\frac{1}{3}e\theta^2}{1-e}\right) = \omega t,$$

where $\omega t =$ earth's rotation in time t, $= \phi$ suppose; therefore

$$\theta\left(1 - \frac{\frac{1}{3}e\theta^2}{1-e}\right) = \phi,$$

hence, if θ be as above, the angle described in falling to the surface,

$$\frac{e\theta^2}{1-e} = \frac{2\beta}{\alpha},$$

$$\theta\left(1 - \tfrac{2}{3}\frac{\beta}{\alpha}\right) = \phi,$$

$$\theta - \phi = \tfrac{2}{3}\frac{\beta}{\alpha}\theta = \tfrac{2}{3}\frac{\beta}{\alpha}\sqrt{\left(\frac{2\beta}{\alpha}\,\frac{1-e}{e}\right)}.$$

Writing herein

$$\frac{1-e}{e}, \ =1-e, \ =\frac{\alpha\omega^2}{g},$$

this is

$$=\tfrac{2}{3}\frac{\beta}{\alpha}\sqrt{\left(\frac{2\beta}{\alpha}\,\frac{\alpha\omega^2}{g}\right)}=\frac{(2\beta)^{\frac{3}{2}}}{3\alpha^{\frac{3}{2}}}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

viz.

$$\theta-\phi=\frac{2^{\frac{3}{2}}.\,\beta^{\frac{3}{2}}}{3\alpha^{\frac{3}{2}}}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

whence

$$\alpha(\theta-\phi)=\frac{2\sqrt{(2)}}{3}\sqrt{\left(\frac{\beta}{\alpha}\right)}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)}\beta,$$

or say

$$\frac{\alpha(\theta-\phi)}{\beta}=\frac{2\sqrt{(2)}}{3}\sqrt{\left(\frac{\beta}{\alpha}\right)}\sqrt{\left(\frac{\alpha\omega^2}{g}\right)},$$

where $\alpha(\theta-\phi)$ is the distance at which the body falls from the foot of the tower. Substituting for $\sqrt{\left(\frac{\alpha\omega^2}{g}\right)}$ its value, $=\frac{1}{17}$, we have

$$\frac{\alpha(\theta-\phi)}{\beta}=\frac{2\sqrt{(2)}}{51}\sqrt{\left(\frac{\beta}{\alpha}\right)}, \ =\cdot056\sqrt{\left(\frac{\beta}{\alpha}\right)}.$$

594.

ON A DIFFERENTIAL EQUATION IN THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 69, 70.]

THE following equation presented itself to me in connexion with the cubic transformation:

$$Q^2 - Q\left(k + \frac{1}{k}\right) - 3 = 3(1-k^2)\frac{dQ}{dk}.$$

Writing as usual $k = u^4$, I was aware that a solution was

$$Q = \frac{v^2}{u^2} + 2uv,$$

where u, v are connected by the modular equation

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0;$$

but it was no easy matter to verify that the differential equation was satisfied. After a different solution, it occurred to me to obtain the relation between (Q, u); or, what is the same thing, (Q, k), viz. eliminating v, we find

$$Q^4 - 6Q^2 - 4\left(u^4 + \frac{1}{u^4}\right)Q - 3 = 0,$$

or say

$$\frac{1}{Q}(Q^4 - 6Q^2 - 3) = 4\left(k + \frac{1}{k}\right),$$

whence also

$$\frac{1}{Q}(Q^4 - 6Q^2 \pm 8Q - 3) = 4\left(k \pm 2 + \frac{1}{k}\right),$$

that is,

$$\frac{1}{Q}(Q-1)^3(Q+3)=4\left\{\sqrt{k}+\frac{1}{\sqrt{k}}\right\}^2,$$

and

$$\frac{1}{Q}(Q+1)^3(Q-3)=4\left\{\sqrt{k}-\frac{1}{\sqrt{k}}\right\}^2,$$

and thence

$$\frac{(Q+1)^3(Q-3)}{(Q-1)^3(Q+3)}=\left(\frac{k-1}{k+1}\right)^2;$$

viz. the value of Q thus determined must satisfy the differential equation. This is easily verified, for, in virtue of the assumed integral, we have

$$Q^2-3-\tfrac{1}{4}(Q^4-6Q^2-3)=3(1-k^2)\frac{dQ}{dk};$$

that is,

$$Q^4-10Q^2+9=-12(1-k^2)\frac{dQ}{dk},$$

or finally

$$(Q^2-1)(Q^2-9)=-12(1-k^2)\frac{dQ}{dk},$$

an equation which is at once obtained by differentiating logarithmically the former result, and we have thus the verification of the solution. This is, however, a particular integral only; and it appears doubtful whether there exists a general integral of an algebraical form.

595.

ON A SENATE-HOUSE PROBLEM.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 75—78.]

THE following was given [5 Jan., 1874,] as a problem of elementary algebra: "Solve the equations

$$u(2a-x)=x(2a-y)=y(2a-z)=z(2a-u)=b^2,$$

and prove that unless $b^2=2a^2$, $x=y=z=u$, but that if $b^2=2a^2$, the equations are not independent."

This is really a very remarkable theorem in regard to the intersections of a certain set of four quadric surfaces in four-dimensional space; viz. slightly altering the notation, we may write the equations in the form

$$x(2\theta-y)=m\theta^2 \ldots (12),$$
$$y(2\theta-z)=m\theta^2 \ldots (23),$$
$$z(2\theta-w)=m\theta^2 \ldots (34),$$
$$w(2\theta-x)=m\theta^2 \ldots (41),$$

where, regarding (x, y, z, w, θ) as coordinates in four-dimensional space, each equation represents a quadric surface. I remark that in such a space we have the notions, point-system, curve, subsurface, surface, according as the number of equations is 4, 3, 2, or 1.

Four quadric surfaces intersect in general in 16 points. But for the system in question (m being arbitrary), the common intersection consists of two lines and the two points

$$x=y=z=w=\theta\{1\pm\sqrt{(1-m)}\};$$

and in the case where $m=2$, then the intersection consists of two lines and a certain unicursal quartic curve.

To obtain these results, I consider the four points

$$\theta=0,\quad x=0,\quad y=0,\quad z=0,\ \ldots\ 123,$$

$$\theta=0,\quad y=0,\quad z=0,\quad w=0,\ \ldots\ 234,$$

$$\theta=0,\quad z=0,\quad w=0,\quad x=0,\ \ldots\ 341,$$

$$\theta=0,\quad w=0,\quad x=0,\quad y=0,\ \ldots\ 412:$$

the two points

$$x=y=z=w=\theta\{1\pm\sqrt{(1-m)}\},\ \ldots\ PQ:$$

and the six lines

$$\theta=0,\quad x=0,\quad y=0,\ \ldots\ 12,$$

$$\theta=0,\quad y=0,\quad z=0,\ \ldots\ 23,$$

$$\theta=0,\quad z=0,\quad w=0,\ \ldots\ 34,$$

$$\theta=0,\quad w=0,\quad x=0,\ \ldots\ 41,$$

$$\theta=0,\quad x=0,\quad z=0,\ \ldots\ 13,$$

$$\theta=0,\quad y=0,\quad w=0,\ \ldots\ 24,$$

being the edges of a tetrahedron, the vertices of which are the four points, viz. the point 123 is the intersection of the lines 12, 13, 23, and so for the other points.

The surfaces contain the several lines, viz.

the surface	12	contains	$(12)^2$, 13, 14, 23, 24,
„	23	„	$(23)^2$, 12, 24, 13, 34,
„	34	„	$(34)^2$, 13, 23, 14, 24,
„	41	„	$(41)^2$, 24, 34, 12, 13,

where $(12)^2$ denotes that 12 is a double line on the surface, and so in other cases. And it thus appears that the surfaces pass all four of them through the lines 13, 24, so that these lines are a part of the common intersection. To obtain the residual intersection, observe that the equations give

$$x=2\theta-m\frac{\theta^2}{w}=\frac{m\theta^2}{2\theta-y},$$

$$z=2\theta-m\frac{\theta^2}{y}=\frac{m\theta^2}{2\theta-w},$$

whence

$$(2\theta-y)\left(2\theta-\frac{m\theta^2}{w}\right)=m\theta^2,$$

$$(2\theta-w)\left(2\theta-\frac{m\theta^2}{y}\right)=m\theta^2,$$

or omitting from each equation the factor θ, the equations become

$$(2\theta - y)(2w - m\theta) = m\theta w,$$

$$(2\theta - w)(2y - m\theta) = m\theta y,$$

that is,

$$(4 - 2m)\,\theta w - 2m\theta^2 - 2yw + m\theta\,(y + w) = 0,$$

$$(4 - 2m)\,\theta y - 2m\theta^2 - 2yw + m\theta\,(y + w) = 0.$$

Whence, m not being $= 2$, we have $y = w$, and then

$$w^2 - 2\theta w + m\theta^2 = 0,$$

or, what is the same thing,

$$2\theta - w = \frac{m\theta^2}{w},$$

giving $x = y = z = w = \theta\,\{1 \pm \sqrt{(1 - m)}\}$, viz. the surfaces each pass through the points P, Q. As regards the omitted factor θ, it is to be observed that, writing in the equations of the four surfaces $\theta = 0$, the equations become $xy = 0$, $yz = 0$, $zw = 0$, $wx = 0$, satisfied by $x = 0$, $z = 0$, or by $y = 0$, $w = 0$, we have thus ($\theta = 0$, $x = 0$, $z = 0$) and ($\theta = 0$, $y = 0$, $w = 0$), viz. the before-mentioned lines 13 and 24.

In the case $m = 2$, we have between y, w the single equation

$$yw - \theta\,(y + w) + 2\theta^2 = 0,$$

giving

$$y = \frac{\theta\,(w - 2\theta)}{w - \theta},$$

and thence

$$x = \frac{2\theta\,(w - \theta)}{w},$$

$$z = \frac{-2\theta^2}{w - \theta};$$

or, writing for convenience $\alpha = \dfrac{w}{\theta}$, then the equations are

$$\frac{w}{\theta} = \alpha,$$

$$\frac{y}{\theta} = \frac{\alpha - 2}{\alpha - 1},$$

$$\frac{z}{\theta} = \frac{-2}{\alpha - 2},$$

$$\frac{x}{\theta} = \frac{2\,(\alpha - 1)}{\alpha};$$

or, what is the same thing,

$$\begin{array}{lllll} x = & & 2(\alpha-1)^2(\alpha-2) & & \left(1-\dfrac{\alpha}{\infty}\right) \\ : y & : & \alpha \quad \ldots & (\alpha-2)^2 & \left(1-\dfrac{\alpha}{\infty}\right) \\ : z & : & -2\alpha(\alpha-1) \quad \ldots & & \left(1-\dfrac{\alpha}{\infty}\right)^2 \\ : w & : & \alpha^2(\alpha-1)(\alpha-2) & \ldots & \\ : \theta & : & \alpha(\alpha-1)(\alpha-2) & & \left(1-\dfrac{\alpha}{\infty}\right), \end{array}$$

where, for the sake of homogeneity, I have introduced the factors $\left(1-\dfrac{\alpha}{\infty}\right)$ and $\left(1-\dfrac{\alpha}{\infty}\right)^2$; viz. we have x, y, z, w, θ proportional to quartic functions of the arbitrary parameter α, or the curve is a unicursal quartic. Writing in the equations $\alpha = 0, 1, 2, \infty$ successively, we see that this quartic curve passes through the four points 123, 234, 341, 412 (intersecting at these points the lines 13 and 24 respectively); and writing also $\alpha = 1 \pm i$ we see that the curve passes through the points P, Q, the coordinates of which now are

$$x = y = z = w = (1 \pm i)\,\theta.$$

It should admit of being proved by general considerations that, in 4-dimensional geometry when 4 quadric surfaces partially intersect in two lines, the residual intersection consists of 2 points; and that, when they intersect in the two lines and in a unicursal quartic met twice by each of the lines, there is no residual intersection—but this theory has not yet been developed.

596.

NOTE ON A THEOREM OF JACOBI'S FOR THE TRANSFORMATION OF A DOUBLE INTEGRAL.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 92—94.]

JACOBI, in the Memoir "De Transformatione Integralis Duplicis..." &c., *Crelle*, t. VIII. (1832) pp. 253—279 and 321—357, [*Ges. Werke*, t. III., pp. 91—158], after establishing a theorem which includes the addition-theorem of elliptic functions, viz. this last is "the differential equation

$$\frac{d\eta}{\sqrt{(G'^2\cos^2\eta + G''^2\sin^2\eta - G^2)}} + \frac{d\theta}{\sqrt{(G'^2\cos^2\theta + G''^2\sin^2\theta - G^2)}},$$

has for its *complete* integral

$$G + G'\cos\eta\cos\theta + G''\sin\eta\sin\theta = 0,"$$

{observe, as to the integral being complete, that the differential equation contains only the constant $G^2 - G'^2 \div (G^2 - G''^2)$, whereas the integral equation contains the two constants $G' \div G$ and $G'' \div G$}, obtains a corresponding theorem for double integrals; viz. this, in the corresponding special case, is as follows: If the variables (ϕ, ψ) and (η, θ) are connected by the two equations

$$\begin{array}{l|l} \alpha \qquad\qquad\qquad\qquad = 0, & \beta \qquad\qquad\qquad\qquad = 0, \\ + \alpha' \cos\phi \qquad . \cos\eta & + \beta' \cos\phi \qquad . \cos\eta \\ + \alpha'' \sin\phi\cos\psi \,.\, \sin\eta\cos\theta & + \beta'' \sin\phi\cos\psi \,.\, \sin\eta\cos\theta \\ + \alpha''' \sin\phi\sin\psi \,.\, \sin\eta\sin\theta & + \beta''' \sin\phi\sin\psi \,.\, \sin\eta\sin\theta \end{array}$$

and if putting for shortness

$$\begin{array}{ll} \alpha''\beta''' - \alpha'''\beta'' = f, & \alpha\beta' - \alpha'\beta = a, \\ \alpha'''\beta' - \alpha'\beta''' = g, & \alpha\beta'' - \alpha''\beta = b, \\ \alpha'\beta'' - \alpha''\beta' = h, & \alpha\beta''' - \alpha'''\beta = c, \end{array}$$

$$(\text{whence } af + bg + ch = 0);$$

$$\begin{aligned} R^2 = \; & f^2 (\sin\phi \cos\psi)^2 (\sin\phi \sin\psi)^2 \\ & + g^2 (\sin\phi \cos\psi)^2 (\cos\phi)^2 \\ & + h^2 (\cos\phi)^2 (\sin\phi \cos\psi)^2 \\ & - a^2 (\cos\phi)^2 \\ & - b^2 (\sin\phi \cos\psi)^2 \\ & - c^2 (\sin\phi \sin\psi)^2, \end{aligned}$$

$$\begin{aligned} S^2 = \; & f^2 (\sin\eta \cos\theta)^2 (\sin\eta \sin\theta)^2 \\ & + g^2 (\sin\eta \sin\theta)^2 (\cos\eta)^2 \\ & + h^2 (\cos\eta)^2 (\sin\eta \cos\theta)^2 \\ & - a^2 (\cos\eta)^2 \\ & - b^2 (\sin\eta \cos\theta)^2 \\ & - c^2 (\sin\eta \sin\theta)^2, \end{aligned}$$

then we have

$$\frac{\sin\phi \, d\phi \, d\psi}{R} = \frac{\sin\eta \, d\eta \, d\theta}{S}.$$

And it may be added that the integral equations are, so to speak, a complete integral of the differential relation; viz. in virtue of the identity $af + bg + ch = 0$, the differential relation contains really only four constants; the integral relations contain the six constants $\alpha : \alpha' : \alpha'' : \alpha'''$ and $\beta : \beta' : \beta'' : \beta'''$, or we have *two* constants introduced by the integration.

The best form of statement is, in the first theorem, to write x, y for $\cos\eta, \sin\eta$, $(x^2+y^2=1)$, ξ, η for $\cos\theta$, $\sin\theta$, $(\xi^2+\eta^2=1)$, and similarly in the second theorem to introduce the variables x, y, z connected by $x^2+y^2+z^2=1$, and ξ, η, ζ connected by $\xi^2+\eta^2+\zeta^2=1$; then in the first theorem $d\eta$, $d\theta$ represent elements of circular arc, and in the second theorem $\sin\phi \, d\phi \, d\psi$ and $\sin\eta \, d\eta \, d\theta$ represent elements of spherical surface, and the theorems are:

I. If (x, y) are coordinates of a point on the circle $x^2+y^2=1$, and (ξ, η) coordinates of a point on the circle $\xi^2+\eta^2=1$, and if ds, $d\sigma$ are the corresponding circular elements, then

$$\frac{ds}{\sqrt{(ax^2+by^2-c)}} = \frac{d\sigma}{\sqrt{(a\xi^2+b\eta^2-c)}},$$

has for its complete integral

$$ax\xi + by\eta - c = 0.$$

II. If (x, y, z) are coordinates of a point on the sphere $x^2+y^2+z^2=1$, and (ξ, η, ζ) coordinates of a point on the sphere $\xi^2+\eta^2+\zeta^2=1$; and if ds, $d\sigma$ are the corresponding spherical elements, and

$$\beta\gamma'-\beta'\gamma=f, \quad \alpha\delta'-\alpha'\delta=a,$$
$$\gamma\alpha'-\gamma'\alpha=g, \quad \beta\delta'-\beta'\delta=b,$$
$$\alpha\beta'-\alpha'\beta=h, \quad \gamma\delta'-\gamma'\delta=c,$$
$$(\text{whence } af+bg+ch=0);$$

and for shortness

$$S^2=f^2y^2z^2+g^2z^2x^2+h^2x^2y^2-a^2x^2-b^2y^2-c^2z^2,$$
$$\Sigma^2=f^2\eta^2\zeta^2+g^2\zeta^2\xi^2+h^2\xi^2\eta^2-a^2\xi^2-b^2\eta^2-c^2\zeta^2,$$

then the differential relation

$$\frac{ds}{\sqrt{(S)}}=\frac{d\sigma}{\sqrt{(\Sigma)}},$$

has for its complete integral the system

$$\alpha x\xi+\beta y\eta+\gamma z\zeta+\delta=0,$$
$$\alpha' x\xi+\beta' y\eta+\gamma' z\zeta+\delta'=0,$$

where by complete integral is meant a system of two equations containing two arbitrary constants.

597.

ON A DIFFERENTIAL EQUATION IN THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 110—113.]

THE differential equation

$$Q^2 - Q\left(k + \frac{1}{k}\right) - 3 = 3(1-k^2)\frac{dQ}{dk},$$

considered *ante*, p. 69, [594, this volume, p. 244], belongs to a class of equations transformable into linear equations of the second order, and consequently is such that, knowing a particular solution, we can obtain the general solution.

In fact, assuming

$$Q = -3(1-k^2)\frac{1}{z}\frac{dz}{dk},$$

the equation becomes

$$9(1-k^2)^2\frac{1}{z^2}\left(\frac{dz}{dk}\right)^2 + 3(1-k^2)\left(k+\frac{1}{k}\right)\frac{1}{z}\frac{dz}{dk} - 3$$
$$= 3(1-k^2)\left\{3(1-k^2)\frac{1}{z^2}\frac{dz}{dk^2} + 6k\frac{1}{z}\frac{dz}{dk} - 3(1-k^2)\frac{1}{z}\frac{d^2z}{dk^2}\right\},$$

viz. omitting the terms in $\frac{1}{z^2}\left(\frac{dz}{dk}\right)^2$ which destroy each other, and dividing by $3(1-k^2)$, this is

$$\left(k+\frac{1}{k}\right)\frac{1}{z}\frac{dz}{dk} - \frac{1}{1-k^2} = 6k\frac{1}{z}\frac{dz}{dk} - 3(1-k^2)\frac{1}{z}\frac{d^2z}{dk^2},$$

or finally

$$3(1-k^2)\frac{d^2z}{dk^2} + \frac{1-5k^2}{k}\frac{dz}{dk} - \frac{1}{1-k^2}z = 0.$$

But knowing a particular value of Q we have

$$z = \exp.\left\{-\tfrac{1}{3}\int\frac{Qdz}{1-k^2}\right\},$$

a particular value of z, and thence in the ordinary manner the general value of z, giving the general value of Q.

The solution given in my former paper may be exhibited in a more simple form by introducing, instead of k, the variable α connected with it by the equation $k^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha}$. We have in fact, *Fundamenta Nova*, p. 25, [Jacobi's *Ges. Werke*, t. I., p. 76],

$$u^8 = \alpha^3\frac{2+\alpha}{1+2\alpha}, \quad = k^2,$$

$$v^8 = \alpha\left(\frac{2+\alpha}{1+2\alpha}\right)^3, \quad = \lambda^2,$$

viz. these expressions of u, v in terms of the parameter α, are equivalent to, and replace, the modular equation $u^4 - v^4 + 2uv(1-u^2v^2) = 0$. We thence obtain

$$u^8v^8 = \frac{\alpha^4(2+\alpha)^4}{(1+2\alpha)^4}, \quad \frac{v^8}{u^8} = \frac{(2+\alpha)^2}{\alpha^2(1+2\alpha)^2},$$

that is,

$$uv = \sqrt{(\alpha)}\sqrt{\left(\frac{2+\alpha}{1+2\alpha}\right)}, \quad \frac{v^2}{u^2} = \frac{1}{\sqrt{(\alpha)}}\sqrt{\left(\frac{2+\alpha}{1+2\alpha}\right)},$$

and the particular solution, $Q = \frac{v^2}{u^2} + 2uv$, becomes

$$Q = \frac{1}{\sqrt{(\alpha)}}\sqrt{(1+2\alpha\,.\,2+\alpha)}, \quad = \sqrt{\left\{5+2\left(\alpha+\frac{1}{\alpha}\right)\right\}}.$$

Introducing into the differential equation α in place of k, this is found to be

$$Q^2 - Q\frac{\frac{1}{\alpha^2}+\alpha^2+2\left(\frac{1}{\alpha}+\alpha\right)}{\sqrt{\left\{5+2\left(\alpha+\frac{1}{\alpha}\right)\right\}}} - 3 = (1-\alpha^2)\sqrt{\left\{5+2\left(\alpha+\frac{1}{\alpha}\right)\right\}}\frac{dQ}{d\alpha}.$$

But from this form it at once appears that it is convenient in place of α to introduce the new variable β, $=\alpha+\frac{1}{\alpha}$; the equation thus becomes

$$Q^2 + Q\frac{2-2\beta-\beta^2}{\sqrt{(5+2\beta)}} - 3 = (4-\beta^2)\sqrt{(5+2\beta)}\frac{dQ}{d\beta},$$

satisfied by $Q = \sqrt{(5+2\beta)}$; or, what is the same thing, writing $5+2\beta = \gamma^2$, that is, $\beta = -\frac{5}{2}+\gamma^2$, the equation becomes

$$4Q^2 + \frac{Q}{\gamma}(3+6\gamma^2-\gamma^4) - 12 = -(\gamma^2-1)(\gamma^2-9)\frac{dQ}{d\gamma},$$

satisfied by $Q = \gamma$.

Writing here

$$Q = \tfrac{1}{4}(\gamma^2-1)(\gamma^2-9)\frac{1}{z}\frac{dz}{d\gamma},$$

we have for z the equation

$$(\gamma^2-1)(\gamma^2-9)\frac{d^2z}{d\gamma^2} + (3\gamma^4-14\gamma^2+3)\frac{dz}{d\gamma} - \frac{48}{(\gamma^2-1)(\gamma^2-9)}z = 0,$$

satisfied by

$$z = \left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}}.$$

[In fact, this value gives

$$z = (\gamma^2-9)^{\frac{1}{4}}(\gamma^2-1)^{-\frac{1}{4}},$$

$$\frac{dz}{d\gamma} = 4\gamma(\gamma^2-9)^{-\frac{3}{4}}(\gamma^2-1)^{-\frac{5}{4}},$$

$$\frac{d^2z}{d\gamma^2} = (-12\gamma^4+57\gamma^2+36)(\gamma^2-9)^{-\frac{7}{4}}(\gamma^2-1)^{-\frac{9}{4}},$$

which verify the equation as they should do.]

Representing for a moment the differential equation by $A\dfrac{d^2z}{d\gamma^2} + B\dfrac{dz}{d\gamma} + Cz = 0$, and putting $z_1 = \left(\dfrac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}}$, then assuming $z = z_1\int y\,d\gamma$, we find

$$A\left(z_1\frac{dy}{d\gamma} + 2y\frac{dz_1}{d\gamma}\right) + Byz_1 = 0,$$

that is,

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + \frac{B}{A} = 0,$$

viz.

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + \frac{3\gamma^4-14\gamma^2+3}{(\gamma^2-1)(\gamma^2-9)} = 0,$$

or

$$\frac{1}{y}\frac{dy}{d\gamma} + \frac{2}{z_1}\frac{dz_1}{d\gamma} + 3 + \frac{1}{\gamma^2-1} + \frac{15}{\gamma^2-9} = 0;$$

whence, integrating

$$\log yz_1^2 + 3\gamma - \tfrac{1}{2}\log\frac{\gamma+1}{\gamma-1} - \tfrac{5}{2}\log\frac{\gamma+3}{\gamma-3} = 0,$$

that is,

$$y = e^{-3\gamma}\frac{1}{z_1^2}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\left(\frac{\gamma+3}{\gamma-3}\right)^{\frac{5}{2}}$$

$$= e^{-3\gamma}\left(\frac{\gamma-1\,.\,\gamma+1}{\gamma-3\,.\,\gamma+3}\right)^{\frac{1}{2}}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}}\left(\frac{\gamma+3}{\gamma-3}\right)^{\frac{5}{2}}$$

$$= \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3}e^{-3\gamma}.$$

Hence, the general value of z is

$$z = K\left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}} \int_{\gamma_0} \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3} e^{-3\gamma} d\gamma,$$

the constants of integration being K and γ_0, or, what is the same thing,

$$z = \left(\frac{\gamma^2-9}{\gamma^2-1}\right)^{\frac{1}{4}} \left\{C + D\int_{\infty} \frac{(\gamma+1)(\gamma+3)^2}{(\gamma-3)^3} e^{-3\gamma} d\gamma\right\},$$

the corresponding value of Q being

$$Q = \tfrac{1}{4}(\gamma^2-1)(\gamma^2-9)\frac{1}{z}\frac{dz}{d\gamma},$$

which contains the single arbitrary constant $\dfrac{D}{C}$; when this vanishes, we have the foregoing particular solution $Q = \gamma$.

I recall that the expression of γ is

$$\gamma = \sqrt{(5+2\beta)}, \quad = \sqrt{\left\{5 + 2\left(\alpha + \frac{1}{\alpha}\right)\right\}}, \quad = \frac{1}{\sqrt{(\alpha)}}\sqrt{\{(2+\alpha)(1+2\alpha)\}},$$

where α is connected with k by the relation

$$k^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha}.$$

598.

NOTE ON A PROCESS OF INTEGRATION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 149, 150.]

I HAD occasion to consider the integral

$$\int_0^R \frac{r^{s-1}\,dr}{\{r^2+e^2\}^{\frac{1}{2}s+q}},$$

where e is small in regard to R and q is negative. The integral is finite when $e=0$, and it might be imagined that it could be expanded in positive powers of e; and, assuming it to be thus expansible, that the process would simply be to expand under the integral sign in ascending powers of e, and integrate each term separately, so that the series would be in integer powers of e^2.

Take two particular cases. First, let

$$s=2,\quad q=-\tfrac{3}{2};$$

the integral is

$$\int_0^R r\sqrt{(r^2+e^2)}\,dr=\int_0^R dr\,(r^2+\tfrac{1}{2}e^2r^0-\tfrac{1}{8}e^4r^{-2}+\ldots)$$

$$=\tfrac{1}{3}R^3+\tfrac{1}{2}e^2R+\infty\,e^4+\ldots,$$

viz. the integral is not thus obtainable: the series is right as far as it goes, but the true expansion contains a term in e^3; and the failure of the series to give the true expansion is indicated by the appearance of infinite coefficients. In fact, the indefinite integral is $\frac{1}{3}(r^2+e^2)^{\frac{3}{2}}$; taking this between the limits, it is

$$\tfrac{1}{3}(R^2+e^2)^{\frac{3}{2}}-\tfrac{1}{3}e^3,\quad =\tfrac{1}{3}R^3+\tfrac{1}{2}e^2R+\ldots-\tfrac{1}{3}e^3.$$

Again, let $s=1$, $q=-2$; the integral is

$$\int_0^R (r^2+e^2)^{\frac{3}{2}}\,dr=\int_0^R (r^2+\tfrac{3}{2}e^2r+\tfrac{3}{8}e^4r^{-1}+\ldots)$$

$$=\tfrac{1}{4}R^4+\tfrac{3}{4}e^2R^2+\infty\,e^4+\ldots,$$

viz. the integral is not thus obtainable: the series is right as far as it goes, but the true expansion contains a term as $e^4 \log e$, and the failure is indicated by the infinite coefficients. In fact, the indefinite integral is

$$(\tfrac{1}{4}r^3 + \tfrac{5}{8}e^2 r)\sqrt{(r^2+e^2)} + \tfrac{3}{8}e^4 \log\{r + \sqrt{(r^2+e^2)}\},$$

which between the limits is

$$(\tfrac{1}{4}R^3 + \tfrac{5}{8}e^2 R)\sqrt{(R^2+e^2)} + \tfrac{3}{8}e^4 \log \frac{R+\sqrt{(R^2+e^2)}}{e},$$

$$= \tfrac{1}{4}R^4 + \tfrac{3}{4}e^2 R^2 + \ldots - \tfrac{3}{8}e^4 \log e.$$

In the general case, the term causing the failure is Ke^{-2q} when q is fractional, and $Ke^{-2q}\log e$ when q is integral. As a step towards determining the entire expansion, I notice that, writing $x = \dfrac{e^2}{e^2+r^2}$ or $r = ex^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$, the value of the integral is

$$= \tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

where

$$X = \frac{e^2}{e^2+R^2}.$$

599.

A SMITH'S PRIZE DISSERTATION.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 157—160.]

WRITE a dissertation on Bernoulli's Numbers and their use in Analysis.

The function $\frac{t}{e^t-1}+\frac{1}{2}t$ is an *even* function of t, as appears by expressing it in the form

$$\tfrac{1}{2}t\frac{e^t+1}{e^t-1}, \quad =\tfrac{1}{2}t\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}},$$

and its value for $t=0$ being obviously $=1$, we may write

$$\frac{t}{e^t-1}+\tfrac{1}{2}t=1+B_1\frac{t^2}{1.2}-B_2\frac{t^4}{1.2.3.4}+\text{\&c.};$$

or, what is the same thing,

$$\frac{t}{e^t-1}=1-\tfrac{1}{2}t+B_1\frac{t^2}{1.2}-B_2\frac{t^4}{1.2.3.4}+\ldots+(-)^{n-1}B_n\frac{t^{2n}}{1.2\ldots 2n}+\ldots,$$

where the several coefficients B_1, B_2, B_3, &c., are, as is at once seen, rational fractions, and, as it may be shown, are all of them positive. These numerical coefficients B_1, B_2, B_3, &c., are called Bernoulli's numbers.

There is no difficulty in calculating directly the first few terms; viz. we have

$$\begin{aligned}\frac{t}{e^t-1}=\frac{1}{1+(\frac{1}{2}t+\frac{1}{6}t^2+\frac{1}{24}t^3+\ldots)}&=1-t\ (\tfrac{1}{2}+\tfrac{1}{6}t+\tfrac{1}{24}t^2+\tfrac{1}{120}t^3+\ldots)\\&\quad+t^2(\tfrac{1}{4}+\tfrac{1}{6}t+\tfrac{5}{72}t^2+\ldots)\\&\quad-t^3(\tfrac{1}{8}+\tfrac{1}{8}t+\ldots)\\&\quad+t^4(\tfrac{1}{16}+\ldots)\\&=1+t(-\tfrac{1}{2})+t^2(-\tfrac{1}{6}+\tfrac{1}{4},\ =+\tfrac{1}{12})+t^3(-\tfrac{1}{24}+\tfrac{1}{6}-\tfrac{1}{8}=0)\\&\qquad+t^4(-\tfrac{1}{120}+\tfrac{5}{72}-\tfrac{1}{8}+\tfrac{1}{16}=-\tfrac{1}{720})+\ldots,\end{aligned}$$

viz.

$$=1-\tfrac{1}{2}t+\tfrac{1}{12}t^2-\tfrac{1}{720}t^4+\ldots,$$

which is therefore

$$= 1 - \tfrac{1}{2}t + \tfrac{1}{2}B_1 t^2 - \tfrac{1}{24}B_2 t^4 + \dots,$$

and consequently

$$B_1 = \tfrac{1}{6}, \quad B_2 = \tfrac{1}{30},$$

and so a few more terms might have been found.

But a more convenient method is to express the numbers in terms of the differences of 0^m by means of a general formula for the expansion of a function of e^t, viz. this is

$$\phi(e^t) = \phi(1+\Delta)\, e^{t \cdot 0},$$

where

$$e^{t \cdot 0} = 0^0 + \frac{t}{1} 0^1 + \frac{t^2}{1.2} 0^2 + \frac{t^3}{1.2.3} 0^3 + \&c.,$$

and the $\phi(1+\Delta)$ is to be applied to the terms 0^0, 0^1, 0^2, 0^3, &c. We have thus

$$\begin{aligned}\frac{t}{e^t - 1} &= \frac{\log(e^t)}{e^t - 1} \\ &= \frac{\log(1+\Delta)}{\Delta} e^{t \cdot 0} \\ &= \frac{\log(1+\Delta)}{\Delta}\left\{0^0 + \frac{t}{1} 0^1 + \frac{t^2}{1.2} 0^2 + \&c. \dots + \frac{t^{2n-1}}{1.2\dots 2n-1} 0^{2n-1} + \frac{t^{2n}}{1.2\dots 2n} 0^{2n} + \dots\right\}.\end{aligned}$$

We have, as may be at once verified,

$$\frac{\log(1+\Delta)}{\Delta} 0^0 = 1, \quad \frac{\log(1+\Delta)}{\Delta} 0^1 = -\tfrac{1}{2},$$

and by what precedes, since the coefficient of every higher odd power of t vanishes,

$$\frac{\log(1+\Delta)}{\Delta} 0^{2n-1} = 0;$$

and then, by comparing the even powers of t,

$$(-)^{n-1} B_n = \frac{\log(1+\Delta)}{\Delta} 0^{2n},$$

that is,

$$(-)^{n-1} B_n = \left(1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2 \dots + \frac{1}{2n+1}\Delta^{2n}\right) 0^{2n},$$

the series for $\dfrac{\log(1-\Delta)}{\Delta}$ being stopped at this point since $\Delta^{2n+1} 0^{2n} = 0$, &c. For instance, in the case $n = 1$, we have

$$\begin{aligned} B_1 = (1 - \tfrac{1}{2}\Delta + \tfrac{1}{3}\Delta^2)\, 0^2 = \;& 0^2 \\ & - \tfrac{1}{2}(1^2 - 0^2) \\ & + \tfrac{1}{3}(2^2 - 2.1^2 + 0^2) \\ = -\tfrac{1}{2} + \tfrac{2}{3}, \; &= \tfrac{1}{6} \text{ as above.}\end{aligned}$$

The formula shows, not only that B_n is a rational fraction but that its denominator is at most = least common multiple of the numbers $2, 3, \ldots, 2n+1$; the actual denominator of the fraction in its least terms is, however, much less than this, there being as to its value a theorem known as Staudt's theorem. It does not obviously show that the Numbers are positive, or afford any indication of the rate of increase of the successive terms of the series.

These last requirements are satisfied by an expression for B_n as the sum of an infinite numerical series, which expression is obtained by means of the function $\cot\theta$, as follows:

We have

$$\frac{t}{e^t-1}+\tfrac{1}{2}t, \quad =\tfrac{1}{2}t\,\frac{e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}}{e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}}=1+B_1\frac{t^2}{1\,.\,2}-B_2\frac{t^4}{1\,.\,2\,.\,3\,.\,4}+\&c.,$$

or, writing herein $t=2i\theta$ $\{i=\sqrt{(-1)}$ as usual$\}$, this is

$$\theta\cot\theta=1-B_1\frac{2^2\theta^2}{1\,.\,2}-B_2\frac{2^4\theta^4}{1\,.\,2\,.\,3\,.\,4}-\&c.$$

But we have

$$\log\sin\theta=\log\theta+\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\ldots,$$

and thence, by differentiation,

$$\theta\cot\theta=1-\frac{2\theta^2}{\pi^2}\left\{\frac{1}{1^2\left(1-\frac{\theta^2}{1^2\pi^2}\right)}+\frac{1}{2^2\left(1-\frac{\theta^2}{2^2\pi^2}\right)}+\ldots\right\}$$

$$\begin{aligned}&=1-\frac{2\theta^2}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\&c.\right\}\\&\quad-\frac{2\theta^4}{\pi^4}\left\{\frac{1}{1^4}+\frac{1}{2^4}+\&c.\right\}\\&\quad-\&c.\end{aligned}$$

Hence

$$B_n\frac{2^{2n}}{1\,.\,2\ldots2n}=\frac{2}{\pi^{2n}}\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\right\},$$

that is,

$$B_n=\frac{2\,(1\,.\,2\,..\,2n)}{(2\pi)^{2n}}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots\right),$$

showing first, that B_n is positive, and next, that it rapidly increases with n, viz. n being large, we have

$$B_n=\frac{2\,(1\,.\,2\ldots2n)}{(2\pi)^{2n}},$$

or, instead of $1\,.\,2\ldots2n$ writing its approximate value $\sqrt{(2\pi)}\,.\,(2n)^{2n+\frac{1}{2}}e^{-2n}$, this is

$$B_n=4\sqrt{(n\pi)}\left(\frac{n}{\pi e}\right)^{2n}.$$

The result may of course be considered from the opposite point of view, as giving a determination of the sum $\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+$ &c. ... in terms of Bernoulli's Numbers, assumed to be known, viz. we thus have

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\ldots=\frac{(2\pi)^{2n}}{2\,(1\,.\,2\ldots 2n)}\,B_n.$$

For instance, $n=1$,

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots=\frac{(2\pi)^2}{2\,.\,1\,.\,2}\cdot\tfrac{1}{6},\quad=\frac{\pi^2}{6},$$

and this is one and a good instance of the use of Bernoulli's Numbers in Analysis.

Another and very important one is in the summation of a series, or say in the determination of $\Sigma u_x,\ =u_0+u_1+\ldots+u_{x-1}$; viz. starting from

$$\frac{1}{e^t-1}=\frac{1}{t}-\tfrac{1}{2}+\frac{B_1}{1\,.\,2}\,t-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\,t^3+\text{\&c.},$$

and writing herein $t=d_x$, and therefore

$$\frac{1}{e^t-1}=\frac{1}{e^{d_x}-1},\quad=\frac{1}{\Delta}\text{ or }\Sigma,$$

and applying each side to a function u_x of x, we have

$$\Sigma u_x=C+\int dx\,u_x-\tfrac{1}{2}u_x+\frac{B_1}{1\,.\,2}\,d_xu_x-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\,d_x^3u_x+\ldots,$$

or taking the two sides each between the integer limits a, x,

$$u_a+u_{a+1}\ldots+u_{x-1}=\int_a^x dx\,u_x-\tfrac{1}{2}(u_x-u_a)+\frac{B_1}{1\,.\,2}\,(d_xu_x)_a^x-\frac{B_2}{1\,.\,2\,.\,3\,.\,4}\,(d_x^3u_x)_a^x+\ldots,$$

where if u_x is a rational and integral function the series on the right-hand side is finite. If for instance $u_x=x$, the equation is

$$a+(a+1)\ldots+(x-1)=\tfrac{1}{2}(x^2-a^2)-\tfrac{1}{2}(x-a),$$

viz.

$$\{1+2\ldots+(x-1)\}-\{1+2\ldots+(a-1)\}=\tfrac{1}{2}(x^2-n)-\tfrac{1}{2}(a^2-a),$$

which is right.

Applying the formula to the function $\log x$, we deduce theorems as to the Γ-function; and it is also interesting to apply it to $\frac{1}{x}$.

The above is given as a specimen of what might be expected in an examination: I remark as faults the omission to make it clear that B_n is a rational fraction; and the giving the series-formula as a formula for the convenient calculation of B_n. The omission to give the first-mentioned straightforward process of development strikes me as curious.

600.

THEOREM ON THE nth ROOTS OF UNITY.

[From the *Messenger of Mathematics*, vol. IV. (1875), p. 171.]

IF n be an odd prime, and α an imaginary nth root of unity, then

$$(-)^{\frac{1}{2}(n-1)}\, n - 1 = 4\left\{\frac{\alpha}{1+\alpha^2} + \frac{\alpha^2}{1+\alpha^4} + \frac{\alpha^3}{1+\alpha^6} \cdots + \frac{\alpha^{\frac{1}{2}(n-1)}}{1+\alpha^{n-1}}\right\};$$

for instance,

$$n = 3, \quad -4 = 4\,\frac{\alpha}{1+\alpha^2},$$

verified at once by means of the equation $1 + \alpha + \alpha^2 = 0$:

$$n = 5, \quad 4 = 4\left(\frac{\alpha}{1+\alpha^2} + \frac{\alpha^2}{1+\alpha^4}\right),$$

where the term in ( ) is

$$\frac{\alpha(1+\alpha^4) + \alpha^2(1+\alpha^2)}{(1+\alpha^2)(1+\alpha^4)},$$

that is,

$$= \frac{\alpha + 1 + \alpha^2 + \alpha^4}{1 + \alpha^2 + \alpha^4 + \alpha}, \quad = 1:$$

and so in other cases.

601.

NOTE ON THE CASSINIAN.

[From the *Messenger of Mathematics*, vol. IV. (1875), pp. 187, 188.]

A SYMMETRICAL bicircular quartic has in general on the axis two nodofoci and four ordinary foci; viz. joining a nodofocus with either of the circular points at infinity, the joining line is a tangent to the curve at the circular point (and, this being a node of the curve, the tangent has there a three-pointic intersection): and joining an ordinary focus with either of the circular points at infinity, the joining line is at some other point a tangent to the curve, viz. an ordinary tangent of two-pointic intersection. In the case of the Cassinian, each circular point at infinity is a fleflecnode (node with an inflexion on each branch); of the four ordinary foci on the axis, one coincides with one nodofocus, another with the other nodofocus, and there remain only two ordinary foci on the axis; the so-called foci of the Cassinian are in fact the nodofoci, viz. each of these points is by what precedes a nodofocus *plus* an ordinary focus, and the line from either of these points to a circular point at infinity, *quà* tangent at a fleflecnode, has there a four-pointic intersection with the curve.

The analytical proof is very easy; writing the equation under the homogeneous form

$$\{(x-az)^2+y^2\}\{(x+az)^2+y^2\}-c^4z^4=0,$$

then the so-called foci are the points $(x=az,\ y=0)$, $(x=-az,\ y=0)$; at either of these, say the first of them, the line drawn to one of the circular points at infinity is $x=az+iy$, and substituting this value in the equation of the curve we obtain $z^4=0$, viz. the line is a tangent of four-pointic intersection; this implies that there is an inflexion at the point of contact on the branch touched by the line $x=az+iy$; and there is similarly an inflexion at the point of contact on the branch touched by the line $x=-az+iy$; viz. the circular point $x=iy$, $z=0$ is a fleflecnode; and similarly the circular point $x=-iy$, $z=0$, is also a fleflecnode.

To verify that there are on the axis only two ordinary foci, we write in the equation $x = \alpha z + iy$, and determine α by the condition that the resulting equation for y (which equation, by reason that the circular point $z = 0$, $x = iy$, is a node, will be a quadric equation only) shall have two equal roots; the equation is in fact

$$\{(\alpha - a)^2 z^2 + 2(\alpha - a) iyz\} \{(\alpha + a)^2 z^2 - 2(\alpha + a) iyz\} - c^2 z^4 = 0,$$

viz. throwing out the factor z^2, this is

$$(\alpha^2 - a^2) \{(\alpha - a) z + 2iy\} \{(\alpha + a) z + 2iy\} - c^4 z^2 = 0,$$

or, what is the same thing, it is

$$(\alpha^2 - a^2) \{(\alpha z + 2iy)^2 - a^2 z^2\} - c^4 z^2 = 0,$$

viz. it is

$$(2iy + \alpha z)^2 - \left(a^2 + \frac{c^4}{\alpha^2 - a^2}\right) z^2 = 0.$$

The condition in order that this may have equal roots is

$$a^2 + \frac{c^4}{\alpha^2 - a^2} = 0, \text{ that is, } \alpha^2 = a^2 - \frac{c^4}{a^2};$$

hence α has only the two values $\pm \sqrt{\left(a^2 - \frac{c^4}{a^2}\right)}$, viz. there are only two ordinary foci.

602.

ON THE POTENTIALS OF POLYGONS AND POLYHEDRA.

[From the *Proceedings of the London Mathematical Society*, vol. VI. (1874—1875), pp. 20—34. Read December 10, 1874.]

THE problem of the attraction of polyhedra is treated of by Mehler, *Crelle*, t. LXVI. pp. 375—381 (1866); but the results here obtained are exhibited under forms, which are very different from his and which give rise to further developments of the theory.

General Formulæ for the Potentials of a Cone and a Shell.

1. The law of attraction is taken to be according to the inverse square of the distance; and I commence with the general case of a cone standing upon any portion of a surface Σ as its base, and attracting a point at its vertex, the cone being considered as a mass of density unity.

2. Considering, in the first instance, an element of mass, the position of which is determined by its distance r from the vertex (or origin) and by two angular coordinates defining the position of the radius vector r, then the element is $= r^2\,dr\,d\omega$ (where $d\omega$ is the element of solid angle, or surface of the unit-sphere), and the corresponding element of potential is $\frac{1}{r}\,r^2\,dr\,d\omega,\ = r\,dr\,d\omega$; whence

$$V = \int r\,dr\,d\omega,$$

which, integrating from $r = 0$ to $r =$ its value at the surface, is

$$= \tfrac{1}{2}\int r^2\,d\omega,$$

where r now denotes the radius vector at a point of the surface, being, therefore, a given function of the two angular coordinates: and the remaining (double) integration

is to be extended to all values of the angular coordinates belonging to a position of r within the conical surface which is the other boundary of the attracting mass, or say over the spherical aperture of the cone.

3. If the value of the radius vector at the surface is taken to be mr (m a constant), then we have obviously

$$V = \tfrac{1}{2} m^2 \int r^2 \, d\omega \, ;$$

and hence also, writing $m + dm$ instead of m, we obtain, for the potential of the portion of the shell lying between the similar and similarly situated surfaces Σ and Σ', belonging to the parameters m and $m + dm$ respectively, the value

$$V = m \, dm \int r^2 \, d\omega \, ;$$

this is $= 2 \dfrac{dm}{m}$ into the potential of the cone; and we thus see that it is the same problem to determine the potential of the cone, and that of the subtended portion of the indefinitely thin shell included between the two surfaces.

4. The same result may be arrived at as follows: the element of solid angle $d\omega$ determines on the surface an element of surface $d\Sigma$, and if $d\nu$ be the corresponding normal thickness of the shell, then the element of mass is $= d\nu \, d\Sigma$, and the element of potential is $= \dfrac{1}{mr} d\nu \, d\Sigma$ (mr being, as before, the radius vector at the surface). Take α the complement of the inclination of the radius vector to the tangent plane—that is, α the inclination of the radius vector to the normal, or, what is the same thing, to the perpendicular from the origin on the tangent plane (whence, also, if mp be the length of this perpendicular, then $p = r \cos \alpha$). The shell-thickness in the direction of the radius vector is $= r \, dm$, or we have $d\nu = r \, dm \cos \alpha$; the element of potential is therefore $= \dfrac{dm}{m} \cos \alpha \, d\Sigma$. But $d\omega$ being the spherical aperture of the cone subtending the element $d\Sigma$, the perpendicular section at the distance mr is $= m^2 r^2 d\omega$; we have therefore $d\Sigma = \dfrac{1}{\cos \alpha} m^2 r^2 \, d\omega$; and hence the element of potential is $= m \, dm \, . \, r^2 \, d\omega$, or the potential of the subtended portion of the shell is as before, $= m \, dm \int r^2 \, d\omega$.

5. It may be added that, integrating between the values m, n ($m > n$), we obtain $\tfrac{1}{2}(m^2 - n^2) \int r^2 \, d\omega$ for the potential of the shell-portion included between the surfaces mr, nr; and if $n = 0$, then, as before, the potential of the cone is $= \tfrac{1}{2} m^2 \int r^2 \, d\omega$.

Cone on a plane base, and plane figure.

6. Suppose that the surface Σ is a plane; the surface Σ' is, of course, a parallel plane. Taking here mp for the perpendicular distance of the plane Σ from the origin, then, if δ be the infinitesimal distance of the two planes from each other, we have $\delta = p\,dm$, that is, $dm = \frac{\delta}{p}$; the potential of the cone is, as before, $= \frac{1}{2}m^2 \int r^2\,d\omega$, and that of the plane figure, thickness δ, is $= \frac{m\delta}{p}\int r^2 d\omega$.

7. Taking, for greater convenience, $m = 1$, we have

$$\text{Potential of cone} \qquad = \tfrac{1}{2}\int r^2\,d\omega,$$

$$\text{Do. of plane figure} = \frac{\delta}{p}\int r^2\,d\omega,$$

where p is now the perpendicular distance of the plane from the vertex; or if, as regards the plane figure, the infinitesimal thickness δ is taken as unity, then

$$\text{Potential of plane figure} = \frac{1}{p}\int r^2\,d\omega.$$

In each case r is the value of the radius vector corresponding to a point of the plane figure which is the base of the cone, and the integration extends over the spherical aperture of the cone.

8. If the position of the radius vector is determined by the usual angular coordinates, θ its inclination to the axis of z, and ϕ its azimuth from the plane of zx—viz. if we have

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta;$$

then, as is well-known, $d\omega = \sin\theta\,d\theta\,d\phi$, and the integral $\int r^2\,d\omega$ is $= \int r^2 \sin\theta\,d\theta\,d\phi$.

Taking the inclination of p to the axes to be α, β, γ respectively, the equation of the plane which is the base of the cone is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p;$$

viz. we have

$$r\left[(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta\right] = p;$$

that is,

$$r = \frac{p}{(\cos\alpha\cos\phi + \cos\beta\sin\phi)\sin\theta + \cos\gamma\cos\theta},$$

and the integral $\int r^2\,d\omega$ is therefore

$$=p^2\int\frac{\sin\theta\,d\theta\,d\phi}{[(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta]^2};$$

and, in particular, if p coincide with the axis of z, so that the equation of the plane is $z=p$, then the integral is

$$=p^2\int\frac{\sin\theta\,d\theta\,d\phi}{\cos^2\theta}.$$

9. The integration in regard to θ can be at once performed; viz. in the latter case we have $\int\frac{\sin\theta\,d\theta}{\cos^2\theta}=\sec\theta$; and in the former case, writing, as we may do,

$$(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta=M\cos(\theta-N),$$

then

$$\int\frac{\sin\theta\,d\theta}{[(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta]^2}=\frac{1}{M^2}\int\frac{\sin(\theta-N+N)\,d\theta}{\cos^2(\theta-N)}$$

$$=\frac{1}{M^2}\left[\cos N\int\frac{\sin(\theta-N)\,d\theta}{\cos^2(\theta-N)}+\sin N\int\frac{d\theta}{\cos(\theta-N)}\right]$$

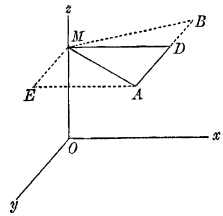
$$=\frac{1}{M^2}[\cos N\sec(\theta-N)+\sin N\log\tan\{\tfrac{1}{4}\pi+\tfrac{1}{2}(\theta-N)\}].$$

Case of a Polyhedron or a Polygon.

10. Consider now the pyramid, vertex the origin O, standing on a polygonal base. Letting fall from the vertex a perpendicular OM on the base of the pyramid, and drawing planes through OM and the several vertices of the polygon, we thus divide the pyramid into triangular pyramids; viz. AB being any side of the polygon, a component pyramid (or tetrahedron) will be $OMAB$, vertex O and base MAB, where MO is a perpendicular at M to the triangular base MAB. And drawing through MO a plane at right angles to AB, meeting it in D (viz. MD is the perpendicular from M on the base AB of the triangle), we divide the triangular pyramid into two pyramids $OMAD$, $OMBD$, each having for its base a right-angled triangle; viz. the vertex is O, the base is the triangle ADM (or, as the case may be, BDM) right-angled at D, and OM is a perpendicular at the vertex M to the plane of the triangle. It is to be observed that, in speaking of the original pyramid as thus divided, we mean that the pyramid is the sum of the component pyramids taken each with the proper sign, + or −, as the case may be.

11. In the case of a polyhedron, this is in the like sense divisible into pyramids having for the common vertex the origin or point O, and standing on the several faces respectively; hence the polyhedron is ultimately divisible into triangular pyramids such as $OADM$, where ADM is a triangle right-angled at D, and where OM is a perpendicular at M to the plane of the triangle. Hence the potential of the polyhedron

in regard to the point O depends upon that of the pyramid $OADM$; and (what is the same thing) the potential of any plane polygon in regard to the point O depends upon that of the right-angled triangle ADM, situate as above in regard to the point O. I



take $OM = h$, $MD = f$, $DA = g$; viz. supposing, as we may do, that the plane of the triangle is parallel to that of xy, the point M on the axis of z, and the side MD parallel to the axis of x, then f, g, h will be the coordinates of the point A.

Formulæ for component triangular Pyramid, and Triangle.

12. Writing, as above, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, and observing that h is the perpendicular distance originally called p, we have, for the potential of the pyramid,

$$V = \tfrac{1}{2}\int r^2\, d\omega = \tfrac{1}{2}h^2 \int \frac{\sin\theta\, d\theta\, d\phi}{\cos^2\theta}$$

$$= \tfrac{1}{2}h^2 \int d\phi\, (\sec\theta),$$

where, ϕ being regarded as a given angle, the integral expression $\sec\theta$ must be taken from $\theta = 0$ to the value of θ corresponding to a point in the side AD. For any such point we have $f = r\sin\theta\cos\phi$, $h = r\cos\theta$, that is, $\frac{f}{h} = \tan\theta\cos\phi$, or the required value of θ is $= \tan^{-1}\frac{f}{h\cos\phi}$, and consequently that of $\sec\theta$ is

$$\sqrt{1 + \frac{f^2}{h^2\cos^2\phi}}, \quad = \frac{1}{h\cos\phi}\sqrt{f^2 + h^2\cos^2\phi},$$

or, as this may also be written,

$$= \frac{1}{h}\sqrt{f^2 + h^2 + f^2\tan^2\phi};$$

hence

$$V = \tfrac{1}{2}h \int (\sqrt{f^2 + h^2 + f^2\tan^2\phi} - h)\, d\phi.$$

13. The first term of the integral, writing therein for a moment $\tan\phi = x$, is

$$\int \frac{(f^2+h^2+f^2x^2)\,dx}{(1+x^2)\sqrt{f^2+h^2+f^2x^2}},$$

$$= f^2\int \frac{dx}{\sqrt{f^2+h^2+f^2x^2}} + h^2\int \frac{dx}{(1+x^2)\sqrt{f^2+h^2+f^2x^2}}$$

$$= f\log\left(fx+\sqrt{f^2+h^2+f^2x^2}\right) + h\tan^{-1}\frac{hx}{\sqrt{f^2+h^2+f^2x^2}}.$$

Hence, replacing x by its value, we have

$$V = \tfrac{1}{2}h\left\{h\tan^{-1}\frac{h\tan\phi}{\sqrt{f^2+h^2+f^2\tan^2\phi}} + f\log\left(f\tan\phi+\sqrt{f^2+h^2+f^2\tan^2\phi}\right) - h\phi\right\},$$

to be taken from $\phi = 0$ to the value of ϕ corresponding to the point A; viz. we have here $f = r\sin\theta\cos\phi$, $g = r\sin\theta\sin\phi$, $h = r\cos\theta$, and thence $\tan\phi = \frac{g}{f}$ or $f\tan\phi = g$; whence, writing for shortness, $s = \sqrt{f^2+g^2+h^2}$ (viz. s denotes the distance OA), we have

$$V = \tfrac{1}{2}h\left\{h\tan^{-1}\frac{gh}{fs} + f\log\frac{s+g}{\sqrt{f^2+h^2}} - h\tan^{-1}\frac{g}{f}\right\};$$

or, observing that

$$\frac{s+g}{s-g} = \frac{(s+g)^2}{\sqrt{f^2+h^2}},$$

this is

$$V = \tfrac{1}{2}h\left\{h\tan^{-1}\frac{gh}{fs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} - h\tan^{-1}\frac{g}{f}\right\},$$

for the potential of the pyramid $OMDA$ in regard to the point O; by omitting the factor $\frac{1}{2}h$, we have

$$V = h\tan^{-1}\frac{gh}{fs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} - h\tan^{-1}\frac{g}{f}$$

for the potential of the triangle MDA. The expression $\tan^{-1}$ denotes, here and elsewhere, an arc included between the limits $-\frac{\pi}{2}$, $+\frac{\pi}{2}$; it is therefore $+$ or $-$ according as the tangent is $+$ or $-$.

Formulæ for rectangular Pyramid, and Rectangle.

14. Completing the rectangle $MDAE$, the potential of the triangle AME is obtained by interchanging the letters g and f; viz. we have

$$V = h\tan^{-1}\frac{fh}{gs} + \tfrac{1}{2}g\log\frac{s+f}{s-f} - h\tan^{-1}\frac{f}{g}$$

for the potential of the triangle MEA.

The sum of the two gives the potential of the rectangle $MDAE$; viz. for this rectangle, we have

$$V = h\left(\tan^{-1}\frac{gh}{fs} + \tan^{-1}\frac{fh}{gs} - \frac{\pi}{2}\right) + \tfrac{1}{2}f\log\frac{s+g}{s-g} + \tfrac{1}{2}g\log\frac{s+f}{s-f}.$$

But we have

$$\tan^{-1}\frac{gh}{fs} + \tan^{-1}\frac{hf}{gs} + \tan^{-1}\frac{fg}{hs} = \frac{\pi}{2};$$

for the function on the left hand is

$$= \tan^{-1}\frac{\dfrac{gh}{fs} + \dfrac{hf}{gs} + \dfrac{fg}{hs} - \dfrac{fgh}{s^3}}{1 - \dfrac{f^2}{s^2} - \dfrac{g^2}{s^2} - \dfrac{h^2}{s^2}};$$

viz. the denominator being $1 - \dfrac{f^2+g^2+h^2}{s^2}$, $=0$, the tangent of the arc is ∞, and the component arcs being each positive and less than $\dfrac{\pi}{2}$, the arc in question can only be $=\dfrac{\pi}{2}$. We have consequently

$$V = -h\tan^{-1}\frac{fg}{hs} + \tfrac{1}{2}f\log\frac{s+g}{s-g} + \tfrac{1}{2}g\log\frac{s+f}{s-f}$$

for the potential of the rectangle $MDAE$. And, multiplying this by $\frac{1}{2}h$, we have

$$V = -\tfrac{1}{2}h^2\tan^{-1}\frac{fg}{hs} + \tfrac{1}{4}hf\log\frac{s+g}{s-g} + \tfrac{1}{4}gh\log\frac{s+f}{s-f}$$

for the potential of the rectangular pyramid, vertex O and base $MDAE$.

Formula for the Cuboid.

15. Completing the rectangular parallelopiped, or, say for shortness, the "cuboid," the sides whereof are (f, g, h); this breaks up into three pyramids, standing on the rectangles fg, gh, and hf respectively; and the potentials for the last two pyramids are at once obtained from the last-mentioned expression of V by mere cyclical interchanges of the letters. Adding the three expressions, we obtain

$$V = \tfrac{1}{2}gh\log\frac{s+f}{s-f} + \tfrac{1}{2}hf\log\frac{s+g}{s-g} + \tfrac{1}{2}fg\log\frac{s+h}{s-h} - \tfrac{1}{2}f^2\tan^{-1}\frac{gh}{fs} - \tfrac{1}{2}g^2\tan^{-1}\frac{hf}{gs} - \tfrac{1}{2}h^2\tan^{-1}\frac{fg}{hs}$$

for the potential of the cuboid.

Group of Results, for Point, Line, Rectangle, and Cuboid.

16. It is convenient to prefix two results, that for the potential of the point A (mass taken to be unity), and that for the potential of the line AE (density taken to

be unity, or mass of an element of length dx, taken to be $=dx$). We have, the attracted point being always at O,

$$\text{Potential of point } A \qquad = \frac{1}{s}, \qquad (s = \sqrt{f^2+g^2+h^2}, \text{ as before}),$$

$$\text{Potential of line } AE \qquad = \tfrac{1}{2}\log\frac{s+f}{s-f},$$

$$\text{Potential of rectangle } MDAE = \tfrac{1}{2}g\log\frac{s+f}{s-f} + \tfrac{1}{2}f\log\frac{s+g}{s-g} - h\tan^{-1}\frac{fg}{hs},$$

$$\text{Potential of cuboid} \qquad = \tfrac{1}{2}gh\log\frac{s+f}{s-f} + \tfrac{1}{2}hf\log\frac{s+g}{s-g} + \tfrac{1}{2}fg\log\frac{s+h}{s-h}$$

$$- \tfrac{1}{2}f^2\tan^{-1}\frac{gh}{fs} - \tfrac{1}{2}g^2\tan^{-1}\frac{hf}{gs} - \tfrac{1}{2}h^2\tan^{-1}\frac{fg}{hs},$$

which functions may be called $A(f, g, h)$, $B(f, g, h)$, $C(f, g, h)$, and $D(f, g, h)$ respectively. It is to be observed that f, g, h are taken to be each of them positive, and that s denotes in every case the positive value of $\sqrt{f^2+g^2+h^2}$; for a symmetrically situated body, corresponding to negative values of each or any of these quantities, the potential has in each case its original value, without change of sign. But B is an odd function as regards f, C an odd function as regards f or g, D an odd function as regards f, g, or h; for example, $C(-f, g, \pm h)$ and $C(f, -g, \pm h)$ are each $=-C(f, g, h)$, and therefore of course $C(-f, -g, \pm h) = C(f, g, h)$.

Extension to case where the attracted point has an arbitrary position.

17. The attracted point has thus far been considered as in a definite position in regard to the attracting mass; but it is easy to pass to the general case of any relative position whatever. Thus, for a line AB, if M be the foot of the perpendicular let fall from the point O, and if, to fix the ideas, the order of succession of the three points is A, B, M, then, with respect to the point O,

$$\text{line } AB = \text{line } AM - \text{line } BM.$$

$$\underline{A \qquad B \quad M}$$

Taking the y- and z-coordinates to be b, c, the x-coordinates for the points A, B, M to be x_0, x_1, a respectively, and in the figure $a > x_1$, $x_1 > x_0$, then $a - x_0$, $a - x_1$ are each of them positive, $a - x_0$ being the greater, the potential of the line AM is $= B(a - x_0, b, c)$, that of BM is $= B(a - x_1, b, c)$, and the potential of the whole line is

$$= B(a - x_0, b, c) - B(a - x_1, b, c);$$

viz. this formula is proved for the case where M is situate as in the figure. But supposing that A and B retain their relative position (viz. $x_1 > x_0$), then the formula holds good for any other position of M; thus, if M be between the points A, B—viz. if the order is A, M, B—then

$$\text{line } AB = \text{line } AM + \text{line } BM,$$

and potential is

$$= B(a-x_0,\ b,\ c) + B(x_1-a,\ b,\ c),$$

where the second term is $= -B(a-x_1,\ b,\ c)$; and so, if the order is M, A, B, then

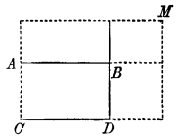
$$\text{line } AB = \text{line } BM - \text{line } AM,$$

and the potential is $B(x_1-a,\ b,\ c) - B(x_0-a,\ b,\ c)$, which is

$$= -B(a-x_1,\ b,\ c) + B(a-x_0,\ b,\ c).$$

18. Similarly for a rectangle $ABCD$, if M, the foot of the perpendicular from the point O, has the position shown in the figure, then

$$\begin{aligned}\text{rectangle } AD = &\ \text{rectangle } MC\\ &- \text{rectangle } MA\\ &- \text{rectangle } MD\\ &+ \text{rectangle } MB,\end{aligned}$$



where O is a point on the perpendicular at the common vertex M of the four rectangles; and the resulting expression for the rectangle AD will apply to any position of the point M.

19. And in like manner for a cuboid; taking the point O in any determinate position, the cuboid may be decomposed into eight cuboids (each with the sign + or − as the case may be) having the point O for a common vertex; and the resulting expression for the potential will apply to any position whatever of the point M.

20. The results may be collected and exhibited as follows:—the coordinates of the attracted point are a, b, c; and it is assumed that $x_1 > x_0$, $y_1 > y_0$, $z_1 > z_0$, (viz. for x the order is $+\infty$, x_1, x_0, $-\infty$, and so for y and z respectively).

Potential of point $(x,\ y,\ z)$ is $= A(a-x,\ b-y,\ c-z)$;

Potential of line $(x_1,\ y,\ z)$, $(x_0,\ y,\ z)$ is $= B(a-x_0,\ b-y,\ c-z) - B(a-x_1,\ b-y,\ c-z)$;

Potential of rectangle $(x_1,\ y_1,\ z)$, $(x_0,\ y_1,\ z)$, $(x_1,\ y_0,\ z)$, $(x_0,\ y_0,\ z)$ is

$$\begin{aligned}= &\ C(a-x_0,\ b-y_0,\ c-z)\\ &- C(a-x_0,\ b-y_1,\ c-z)\\ &- C(a-x_1,\ b-y_0,\ c-z)\\ &+ C(a-x_1,\ b-y_1,\ c-z);\end{aligned}$$

$$\begin{array}{lll}
\text{Potential of cuboid} & (x_1, y_1, z_1), (x_0, y_1, z_1) \text{ is } = & D(a-x_0,\ b-y_0,\ c-z_0) \\
 & (x_1, y_0, z_1), (x_0, y_0, z_1) & -D(a-x_1,\ b-y_0,\ c-z_0) \\
 & (x_1, y_1, z_0), (x_0, y_1, z_0) & -D(a-x_0,\ b-y_1,\ c-z_0) \\
 & (x_1, y_0, z_0), (x_0, y_0, z_0) & +D(a-x_1,\ b-y_1,\ c-z_0) \\
 & & -D(a-x_0,\ b-y_0,\ c-z_1) \\
 & & +D(a-x_1,\ b-y_0,\ c-z_1) \\
 & & +D(a-x_0,\ b-y_1,\ c-z_1) \\
 & & -D(a-x_1,\ b-y_1,\ c-z_1).
\end{array}$$

21. These are connected together as follows, viz.:—

$$\text{Potential of line} = \int_{x_0}^{x_1} dx \text{ Potential of point,}$$

$$\text{Potential of rectangle} = \int_{y_0}^{y_1} dy \text{ Potential of line,}$$

$$\text{Potential of cuboid} = \int_{z_0}^{z_1} dz \text{ Potential of rectangle,}$$

equations which are in fact of the form

$$B(x, y, z) = \int dx\, A(x, y, z),$$

$$C(x, y, z) = \int dy\, B(x, y, z),$$

$$D(x, y, z) = \int dz\, C(x, y, z).$$

Differential properties of the functions A, B, C, D.

22. These relations, with other allied ones, may be verified as follows. Writing $r = \sqrt{x^2+y^2+z^2}$, the fundamental forms are

$$\log\frac{r+x}{r-x}, \quad \text{and} \quad \tan^{-1}\frac{yz}{rx}.$$

We have $d_x r = \frac{x}{r}$, &c., and thence

$$d_x \log\frac{r+x}{r-x} = \frac{1+\frac{x}{r}}{r+x} - \frac{-1+\frac{x}{r}}{r-x}, \ = \frac{1}{r}+\frac{1}{r}, \ = \frac{2}{r};$$

$$d_y \log\frac{r+x}{r-x} = \frac{\frac{y}{r}}{r+x} - \frac{\frac{y}{r}}{r-x}, \ = \frac{y}{r}\left(\frac{1}{r+x} - \frac{1}{r-x}\right), \ = \frac{-2xy}{r(r^2-x^2)};$$

&c. &c. &c.

$$d_x \tan^{-1}\frac{yz}{rx} = \frac{-yz\left(r+\frac{x^2}{r}\right)}{r^2x^2+y^2z^2} = \frac{-yz}{r}\,\frac{r^2+x^2}{r^2x^2+y^2z^2},$$

or, since

$$r^2+x^2=(r^2-y^2)+(r^2-z^2) \text{ and } r^2x^2+y^2z^2=(r^2-y^2)(r^2-z^2),$$

this is

$$=-\frac{yz}{r}\left(\frac{1}{r^2-y^2}+\frac{1}{r^2-z^2}\right);$$

$$d_y \tan^{-1}\frac{yz}{rx}=\frac{rx\,.\,z-yz\,.\,\dfrac{xy}{r}}{r^2x^2+y^2z^2}=\frac{xz}{r}\,\frac{r^2-y^2}{r^2x^2+y^2z^2},$$

which, the denominator being, as before, $(r^2-y^2)(r^2-z^2)$, is

$$=\frac{xz}{r}\,\frac{1}{r^2-z^2}.$$

It is now easy to form the following results:—

23. First,

$$u=A(x,\ y,\ z)=\frac{1}{r} \text{ (symmetrical)},$$

$$d_xu=-\frac{x}{r^3}, \text{ \&c.},$$

$$d_x^2u=\frac{3x^2}{r^5}-\frac{1}{r^3}, \text{ \&c.}, \quad d_xd_yu=\frac{3xy}{r^5}, \text{ \&c.},$$

and thence

$$(d_x^2+d_y^2+d_z^2)\,u=0.$$

24. Secondly,

$$u=B(x,\ y,\ z)=\tfrac{1}{2}\log\frac{r+x}{r-x} \text{ (symmetrical as to } y,\ z);$$

then

$$d_xu=\frac{1}{r}\Big(=A(x,\ y,\ z)\Big), \quad d_yu=\frac{-xy}{r(r^2-x^2)}, \text{ \&c.},$$

$$d_x^2u=-\frac{x}{r^3}, \quad d_xd_yu=-\frac{y}{r^3}, \quad d_yd_zu=\frac{xyz}{r^3(r^2-x^2)}+\frac{2xyz}{r(r^2-x^2)^2}, \text{ \&c.},$$

$$d_y^2u=\frac{-x}{r(r^2-x^2)}+\frac{2xy^2}{r(r^2-x^2)^2}+\frac{xy^2}{r^3(r^2-x^2)}, \text{ \&c.};$$

and thence

$$(d_x^2+d_y^2+d_z^2)\,u=-\frac{x}{r^3}-\frac{2x}{r(r^2-x^2)}+\frac{2x}{r(r^2-x^2)}+\frac{x}{r^3}, \ =0.$$

25. Thirdly,

$$u=C(x,\ y,\ z)=\tfrac{1}{2}y\log\frac{r+x}{r-x}+\tfrac{1}{2}x\log\frac{r+y}{r-y}-z\tan^{-1}\frac{xy}{zr} \text{ (symmetrical as to } x,\ y)$$

$$d_yu=\tfrac{1}{2}\log\frac{r+x}{r-x}\Big(=B(x,\ y,\ z)\Big):$$

in verification whereof, observe that the remaining terms are

$$= -\frac{xy^2}{r\,.\,r^2-x^2} + \frac{x}{r} - \frac{xz^2}{r}\,\frac{r^2-y^2}{(r^2-x^2)(r^2-y^2)},$$

$$= \frac{x}{r}\left(-\frac{y^2}{r^2-x^2} + 1 - \frac{z^2}{r^2-x^2}\right),$$

$$= \frac{x}{r(r^2-x^2)}(-y^2+r^2-x^2-z^2),$$

which is $=0$;

$$d_z u = -\frac{xyz}{r(r^2-x^2)} - \frac{xyz}{r(r^2-y^2)} + \frac{xyz}{r}\,\frac{r^2-x^2+r^2-y^2}{(r^2-x^2)(r^2-y^2)} - \tan^{-1}\frac{xy}{zr},$$

$$= -\tan^{-1}\frac{xy}{zr},$$

$$d_x^2 u = -\frac{xy}{r(r^2-y^2)},$$

$$d_z^2 u = \frac{xy}{r}\,\frac{r^2-x^2+r^2-y^2}{(r^2-x^2)(r^2-y^2)}, \quad = \frac{xy}{r}\left(\frac{1}{r^2-x^2} + \frac{1}{r^2-y^2}\right),$$

and thence

$$(d_x^2 + d_y^2 + d_z^2)\,u = 0.$$

26. Fourthly,

$$u = D(x,\ y,\ z) = \tfrac{1}{2}\,yz\log\frac{r+x}{r-x} + \tfrac{1}{2}\,zx\log\frac{r+y}{r-y} + \tfrac{1}{2}\,xy\log\frac{r+z}{r-z}$$

$$-\tfrac{1}{2}\,x^2\tan^{-1}\frac{yz}{xr} - \tfrac{1}{2}\,y^2\tan^{-1}\frac{zx}{yr} - \tfrac{1}{2}\,z^2\tan^{-1}\frac{xy}{zr} \text{ (symmetrical)},$$

$$d_z u = \tfrac{1}{2}\,y\log\frac{r+x}{r-x} + \tfrac{1}{2}\,x\log\frac{r+y}{r-y} - z\tan^{-1}\frac{xy}{zr} = C(x,\ y,\ z),$$

$$d_z^2 u = -\tan^{-1}\frac{xy}{zr};$$

and thence

$$(d_x^2 + d_y^2 + d_z^2)\,u = -\tan^{-1}\frac{yz}{xr} - \tan^{-1}\frac{zx}{yr} - \tan^{-1}\frac{xy}{zr}$$

$$= -\tan^{-1}\frac{\dfrac{yz}{xr} + \dfrac{zx}{yr} + \dfrac{xy}{zr} - \dfrac{xyz}{r^3}}{1 - \dfrac{x^2}{r^2} - \dfrac{y^2}{r^2} - \dfrac{z^2}{r^2}};$$

viz. the denominator being $=0$, the arc is $\pm\dfrac{\pi}{2}$, or we have

$$(d_x^2 + d_y^2 + d_z^2)\,u = \mp\frac{\pi}{2},$$

the value being $-\dfrac{\pi}{2}$ if x, y, z are all three of them, or only one, positive; but $+\dfrac{\pi}{2}$ if they are all three of them, or only one, negative.

Application to the Potentials of the Point, the Line, the Rectangle, and the Cuboid.

27. Take now V to denote in succession the foregoing expressions of the potential of a point, a line, a rectangle, or a cuboid, at the point (a, b, c). In the first three cases respectively, each of the component terms is reduced to zero by the operator $d_a^2 + d_b^2 + d_c^2$; and we have, therefore,

$$(d_a^2 + d_b^2 + d_c^2)\, V = 0,$$

which is as it should be. But in the case of the cuboid, each of the eight component terms is by the operator reduced to $\mp \frac{\pi}{2}$, and we have therefore

$$(d_a^2 + d_b^2 + d_c^2)\, V = \Sigma \left\{ \pm \left(\mp \frac{\pi}{2} \right) \right\};$$

Σ denoting the sum of eight terms, the $\pm$ denoting $+$ or $-$, according to the sign of the term in the formula (viz. in four cases this is $+$, and in four cases it is $-$), and the $\mp \frac{\pi}{2}$ denoting the value $\frac{\pi}{2}$ with its proper sign depending on the signs of the quantities $(a - x_0,\ b - y_0,\ c - z_0)$, &c., as explained in the preceding Number.

Suppose for a moment $a > x_1$, $b > y_1$, $c > z_1$, or the attracted point in one of the regions exterior to the cuboid; then $\mp \frac{\pi}{2}$ will in each case be $= -\frac{\pi}{2}$, and the sign $\mp$, being $+$ for four of the terms and $-$ for the four remaining terms, the sum is $= 0$. And similarly, in all cases where the attracted point is exterior to the cuboid, the sum of the eight terms is $= 0$. But when the attracted point is interior, that is, when $a > x_0 < x_1$, $b > y_0 < y_1$, $c > z_0 < z_1$, then it is found that, for the four terms which have the sign $+$, the value of $\mp \frac{\pi}{2}$ is $= -\frac{\pi}{2}$; and for the four terms which have the sign $-$, its value is $= +\frac{\pi}{2}$; whence, in the sum, each term is $= -\frac{\pi}{2}$, or the value is $= -4\pi$. Hence, in the case of the cuboid, we have

$$(d_a^2 + d_b^2 + d_c^2)\, V = 0 \text{ or } -4\pi,$$

according as the attracted point is external or internal.

Verification in regard to the Rectangle.

28. I start from the formula

$$\begin{aligned} V = \quad & C(a - x_0,\ b - y_0,\ c) \\ & - C(a - x_1,\ b - y_0,\ c) \\ & - C(a - x_0,\ b - y_1,\ c) \\ & + C(a - x_1,\ b - y_1,\ c), \end{aligned}$$

where, as before, $x_1 > x_0$, $y_1 > y_0$. V is here a function of (a, b, c), satisfying the partial differential equation

$$(d_a^2 + d_b^2 + d_c^2)\, V = 0,$$

and (as is easily verified) vanishing when any one of the variables a, b, c becomes infinite; it does not become infinite for any finite values of a or b, or any positive value of c. Hence, by a theorem of Green's *, there exists on the plane $z = 0$ a distribution of matter giving rise to the potential V; and not only so, but the density at any point (x, y) of the plane is given by the formula

$$\rho = -\frac{1}{2\pi}\left(\frac{dW}{dc}\right)_{c=0},$$

where W is what V becomes on writing therein x, y in place of a, b, and $c = 0$ is regarded as an indefinitely small positive quantity.

We have

$$d_c C(x, y, c) = -\tan^{-1}\frac{xy}{cr}, \text{ where } r = \sqrt{x^2 + y^2 + c^2}.$$

And hence

$$\begin{aligned} d_c W = &-\tan^{-1}\frac{(x - x_0)(y - y_0)}{c\sqrt{(x - x_0)^2 + (y - y_0)^2 + c^2}} \\ &+\tan^{-1}\frac{(x - x_1)(y - y_0)}{c\sqrt{(x - x_1)^2 + (y - y_0)^2 + c^2}} \\ &+\tan^{-1}\frac{(x - x_0)(y - y_1)}{c\sqrt{(x - x_0)^2 + (y - y_1)^2 + c^2}} \\ &-\tan^{-1}\frac{(x - x_1)(y - y_1)}{c\sqrt{(x - x_1)^2 + (y - y_1)^2 + c^2}}. \end{aligned}$$

Putting $c = 0$, as above, each arc is $= \frac{\pi}{2}$ or $-\frac{\pi}{2}$, according as the fraction under the $\tan^{-1}$ is positive or negative—that is, according as the numerator is positive or negative. Suppose for a moment $x > x_1$, $y > y_1$, viz. the point (x, y) is here in a region exterior to the rectangle (x_1, y_1), (x_1, y_0), (x_0, y_1), (x_0, y_0): the value of $d_c W$ is $= -\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}, = 0$; and similarly, for every other position of the point (x, y)

* The theorem in question is a particular case of Green's, $4\pi\rho = -\frac{dV}{dw} - \frac{dV'}{dw}$ ("Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" (1828), see p. 31 of the *Collected Works*); viz. the surface is here a plane, and $V = V'$. And it is also a particular case of the formula $\rho' = \frac{-\Gamma\frac{1}{2}(n-1)}{2\pi^{\frac{1}{2}s}\,\Gamma\frac{1}{2}(n-s+1)}\, P'$ ("Memoir on the Determination of the Exterior and Interior Attraction of Ellipsoids of Variable Densities" (1835), see p. 199 of the *Collected Works*); viz. s is taken $= 2$; and Green's extra-spatial coordinate u then becomes the coordinate z of ordinary tri-dimensional space.

exterior to the rectangle, the value is $=0$. But for a point interior to the rectangle, we have $x<x_1>x_0$, $y<y_1>y_0$, and in this case the value is

$$-\frac{\pi}{2}+\left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}, \ =-2\pi.$$

Hence

$$\rho, \ =-\frac{1}{2\pi}(d_c W)_{c=0}, \text{ is } =0 \text{ or } 1,$$

according as the point is exterior or interior to the rectangle, viz. the distribution producing the potential in question is a uniform distribution (density unity) over the rectangle, which is as it should be.

Potential of a Cuboidal Surface.

29. The preceding formulæ lead to the expression of the potential of a cuboidal surface (viz. the surface composed of the six faces of a cuboid, each of them being considered as a plate of the same uniform density) upon a point a, b, c. Writing, for convenience,

$$E(f, g, h)=\tfrac{1}{2}(g+h)\log\frac{s+f}{s-f}+\tfrac{1}{2}(h+f)\log\frac{s+g}{s-g}+\tfrac{1}{2}(f+g)\log\left(\frac{s+h}{s-h}\right)$$
$$-f\tan^{-1}\frac{gh}{fs}-g\tan^{-1}\frac{hf}{gs}-h\tan^{-1}\frac{fg}{hs},$$

where each term is supposed to have (compounded with its expressed sign) a sign $\pm$, as follows: viz. in any fg term $\left(\tfrac{1}{2}f\log\frac{s+g}{s-g}, \ \tfrac{1}{2}g\log\frac{s+f}{s-f}, \text{ or } h\tan^{-1}\frac{fg}{hs}\right)$, this sign $\pm$ is $+$ if f and g are both positive or both negative, but is $-$ if f and g are the one of them positive and the other negative; and the like as to the gh terms and the hf terms respectively. And this being so, the expression for the potential (applying as well to an interior as to an exterior point) is

$$\begin{aligned} V = \ & E(a-x_0, \ b-y_0, \ c-z_0) \\ & +E(a-x_1, \ b-y_0, \ c-z_0) \\ & +E(a-x_0, \ b-y_1, \ c-z_0) \\ & +E(a-x_1, \ b-y_1, \ c-z_0) \\ & +E(a-x_0, \ b-y_0, \ c-z_1) \\ & +E(a-x_1, \ b-y_0, \ c-z_1) \\ & +E(a-x_0, \ b-y_1, \ c-z_1) \\ & +E(a-x_1, \ b-y_1, \ c-z_1). \end{aligned}$$

It is, in fact, easy to verify that the final result, interpreted as above, represents the sum of the six positive values, which are the values of the potential for the six faces of the cuboid respectively.

603.

ON THE POTENTIAL OF THE ELLIPSE AND THE CIRCLE.

[From the *Proceedings of the London Mathematical Society*, vol. VI. (1874—1875), pp. 38—58. Read January 14, 1875.]

The Potential of the Ellipse.

1. I CONSIDER the potential of an ellipse (or say an elliptic plate of uniform density); viz. this is

$$V = \int \frac{dx\,dy}{\sqrt{(a-x)^2 + (b-y)^2 + c^2}},$$

the limits being given by the equation $\frac{x^2}{f^2} + \frac{y^2}{g^2} = 1$.

Writing herein $x = mf\cos u$, $y = mg\sin u$, we have $dxdy = fg\, m dm du$; and consequently

$$V = fg \int \frac{m\,dm\,du}{\sqrt{(a - mf\cos u)^2 + (b - mg\sin u)^2 + c^2}},$$

where the integrations are to be taken from $m = 0$ to $m = 1$, and from $u = 0$ to $u = 2\pi$.

2. It is to be remarked that, by first performing the integration in regard to m, we may reduce the potential to the form $\int du\,.\,F$, where F is an *algebraic* function of $\cos u$, $\sin u$; and that the result so obtained, although in the general case too complex to be manageable, is a useful one in the case $f = g$, where the ellipse becomes a circle. The case of the circle will be treated of separately, but in the general case it will be sufficient to show that the integral is of the form in question.

3. To accomplish this, writing

$$A = a^2 + b^2 + c^2,$$
$$B = af\cos u + bg\sin u,$$
$$C = f^2\cos^2 u + g^2\sin^2 u,$$

then the integral in regard to m is

$$\int \frac{m\,dm}{\sqrt{A - 2Bm + Cm^2}},$$

which is

$$= \frac{1}{C}\sqrt{A - 2Bm + Cm^2} + \frac{B}{C\sqrt{C}}\log\left\{Cm - B + \sqrt{C}\sqrt{A - 2Bm + Cm^2}\right\}.$$

Taken between the limits 0 and 1, this is

$$= \frac{1}{C}(\sqrt{A - 2B + C} - \sqrt{A}) + \frac{B}{C\sqrt{C}}\log\left\{\frac{C - B + \sqrt{C}\sqrt{A - 2B + C}}{-B + \sqrt{C}\sqrt{A}}\right\};$$

and we have therefore

$$V = fg\int du\,\frac{1}{C}(\sqrt{A - 2B + C} - \sqrt{A}) + fg\int du\,\frac{(af\cos u + bg\sin u)}{(f^2\cos^2 u + g^2\sin^2 u)^{\frac{3}{2}}}\log\Upsilon,$$

where, for greater clearness, the value of the coefficient $\frac{B}{C\sqrt{C}}$ of the logarithmic term has been written at full length.

4. But this coefficient admits of algebraic integration, viz. we have

$$fg\int du\,\frac{af\cos u + bg\sin u}{(f^2\cos^2 u + g^2\sin^2 u)^{\frac{3}{2}}} = \frac{ag\sin u - bf\cos u}{(f^2\cos^2 u + g^2\sin^2 u)^{\frac{1}{2}}};$$

hence, integrating the second term by parts, we have

$$V = fg\int du\,\frac{1}{C}\{\sqrt{A - 2B + C} - \sqrt{A}\}$$
$$+ \frac{ag\sin u - bf\cos u}{(f^2\cos^2 u + g^2\sin^2 u)^{\frac{1}{2}}}\log\Upsilon$$
$$- \int du\,\frac{ag\sin u - bf\cos u}{(f^2\cos^2 u + g^2\sin^2 u)^{\frac{1}{2}}}\cdot\frac{\Upsilon'}{\Upsilon},$$

where the second term, taken between the limits $u = 0$, $u = 2\pi$, is $= 0$; and $\frac{\Upsilon'}{\Upsilon}$ being an algebraic function of $\sin u$, $\cos u$, the potential is expressed in the form in question.

5. But we may, by means of a transformation upon u (that made use of in Gauss' Memoir * on the attraction of an elliptic ring), transform the expression so as

* [*Ges. Werke*, t. III., pp. 333—355; in particular, *l.c.*, p. 338].

to obtain the integral in regard to m under a much more simple form. We, in fact, assume

$$\cos u = \frac{\alpha + \alpha' \cos T + \alpha'' \sin T}{\gamma + \gamma' \cos T + \gamma'' \sin T},$$

$$\sin u = \frac{\beta + \beta' \cos T + \beta'' \sin T}{\gamma + \gamma' \cos T + \gamma'' \sin T},$$

where the nine coefficients are such that identically

$$(\alpha + \alpha' \cos T + \alpha'' \sin T)^2 + (\beta + \beta' \cos T + \beta'' \sin T)^2 - (\gamma + \gamma' \cos T + \gamma'' \sin T)^2 = \cos^2 T + \sin^2 T - 1,$$

(this of course renders the two equations consistent); and also that

$$(a - mf \cos u)^2 + (b - mg \sin u)^2 + c^2 = \frac{1}{(\gamma + \gamma' \cos T + \gamma'' \sin T)^2} (G + G' \cos^2 T + G'' \sin^2 T).$$

This last condition gives, for the determination of the coefficients G, G', G'', the identity

$$(\theta - G)(\theta + G')(\theta + G'') = -(\theta + m^2 f^2)(\theta + m^2 g^2)\, \theta \left\{ \frac{a^2}{\theta + m^2 f^2} + \frac{b^2}{\theta + m^2 g^2} + \frac{c^2}{\theta} - 1 \right\};$$

or, what is the same thing, G, $-G'$, $-G''$ are the roots of the equation

$$\frac{a^2}{\theta + m^2 f^2} + \frac{b^2}{\theta + m^2 g^2} + \frac{c^2}{\theta} - 1 = 0.$$

This equation has one positive root, which may be taken to be G, and two negative roots, which will then be $-G'$, $-G''$; viz. G, G', G'' are thus all positive; and G denotes the positive root of the last-mentioned equation.

6. We have

$$du = \frac{dT}{(G + G' \cos T + G'' \sin T)^2},$$

and thence

$$V = fg \int m\, dm \int \frac{dT}{(G + G' \cos^2 T + G'' \sin^2 T)^{\frac{1}{2}}},$$

the integral in regard to T being taken from 0 to 2π; or, what is the same thing, we may multiply by 4 and take the integral only from 0 to $\frac{\pi}{2}$; viz. we thus have

$$V = 4fg \int m\, dm \int_0^{\frac{1}{2}\pi} \frac{dT}{(G + G' \cos^2 T + G'' \sin^2 T)^{\frac{1}{2}}},$$

where the integral in regard to T can be at once reduced to the standard form of an elliptic function, or it might be calculated by Gauss' method of the arithmetico-geometrical mean.

7. But, for the present purpose, a further reduction is required. Writing

$$t = G + (G + G') \cot^2 T,$$

we have

$$t - G = (G + G')\frac{\cos^2 T}{\sin^2 T},$$

$$t + G' = (G + G')\frac{1}{\sin^2 T},$$

$$t + G'' = (G + G'\cos^2 T + G''\sin^2 T)\frac{1}{\sin^2 T}:$$

whence

$$\sqrt{t - G\,.\,t + G'\,.\,t + G''} = (G + G')(G + G'\cos^2 T + G''\sin^2 T)^{\frac{1}{2}}\frac{\cos T}{\sin^2 T};$$

moreover

$$dt = -2(G + G')\frac{\cos T}{\sin^2 T}dT.$$

Hence

$$\frac{dt}{\sqrt{t - G\,.\,t + G'\,.\,t + G''}} = \frac{-2dT}{\{G + G'\cos^2 T + G''\sin^2 T\}^{\frac{1}{2}}};$$

and, observing that to the limits 0, $\frac{\pi}{2}$ of T correspond the limits ∞, G of t, we thence obtain

$$V = 2fg\int m\,dm\int_G^\infty \frac{dt}{\sqrt{t - G\,.\,t + G'\,.\,t + G''}};$$

or, what is the same thing,

$$V = 2fg\int m\,dm\int_G^\infty \frac{dt}{\sqrt{t(t + m^2f^2)(t + m^2g^2)\left(1 - \frac{a^2}{t + m^2f^2} - \frac{b^2}{t + m^2g^2} - \frac{c^2}{t}\right)}},$$

where G denotes, as before, the positive root of the equation

$$\frac{a^2}{\theta + m^2f^2} + \frac{b^2}{\theta + m^2g^2} + \frac{c^2}{\theta} - 1 = 0.$$

8. Writing for t, m^2t, and for G, m^2G, the formula becomes

$$V = 2fg\int m\,dm\int_G^\infty \frac{dt}{\sqrt{t\,.\,t + f^2\,.\,t + g^2\left(m^2 - \frac{a^2}{t + f^2} - \frac{b^2}{t + g^2} - \frac{c^2}{t}\right)}},$$

where G now denotes the positive root of the equation

$$\frac{a^2}{\theta + f^2} + \frac{b^2}{\theta + g^2} + \frac{c^2}{\theta} - m^2 = 0.$$

Thus G is a function of m; but it is to be remarked that the integration in respect to m can be performed through the integral sign $\int_G^\infty dt$ in precisely the same way as if G were constant, and that we, in fact, have

$$V = 2fg\left[\int_G^\infty dt \sqrt{m^2 - \frac{a^2}{t+f^2} - \frac{b^2}{t+g^2} - \frac{c^2}{t}}\,\frac{1}{\sqrt{t\,.\,t+f^2\,.\,t+g^2}}\right],$$

where the function of m is to be taken between the limits 0 and 1. The reason is that, differentiating this last integral in respect to m, the term depending on the variation of the limit G is

$$-\sqrt{m^2 - \frac{a^2}{G+f^2} - \frac{b^2}{G+g^2} - \frac{c^2}{G}}\,\frac{1}{\sqrt{G\,.\,G+f^2\,.\,G+g^2}}\,\frac{dG}{dm},$$

which is $=0$ in virtue of the equation which defines G; hence the whole result is the term arising from the variation of m in so far as it appears explicitly.

9. Proceeding next to take the function of m between the two limits: for $m=0$ we have $G=\infty$, and the integral vanishes; for $m=1$ we have G the positive root of the equation

$$\frac{a^2}{\theta+f^2} + \frac{b^2}{\theta+g^2} + \frac{c^2}{\theta} - 1 = 0,$$

or, using θ to denote the positive root of this equation, the value is $G=\theta$; we thus finally obtain

$$V = 2fg\int_\theta^\infty dt \sqrt{1 - \frac{a^2}{t+f^2} - \frac{b^2}{t+g^2} - \frac{c^2}{t}}\,\frac{1}{\sqrt{t\,.\,t+f^2\,.\,t+g^2}}$$

as the expression for the potential of the ellipse semiaxes (f, g) on the point (a, b, c).

Case where the Attracted Point is on the Focal Hyperbola.

10. The result becomes very simple when the attracted point is in the focal hyperbola of the ellipse, viz. when we have $b=0$ and $\frac{a^2}{f^2-g^2} - \frac{c^2}{g^2} = 1$. The function $1 - \frac{a^2}{t+f^2} - \frac{b^2}{t+g^2} - \frac{c^2}{t}$ is here

$$\begin{aligned}
&= \frac{a^2}{f^2-g^2} - \frac{c^2}{g^2} - \frac{a^2}{f^2+t} - \frac{c^2}{t} \\
&= (t+g^2)\left\{\frac{a^2}{(t+f^2)(f^2-g^2)} - \frac{c^2}{g^2 t}\right\} \\
&= (t+g^2)\left\{\left(1+\frac{c^2}{g^2}\right)\frac{1}{t+f^2} - \frac{c^2}{g^2 t}\right\} \\
&= \frac{t+g^2}{t\,(t+f^2)}\left(t - \frac{c^2 f^2}{g^2}\right).
\end{aligned}$$

Hence also $\theta = \frac{c^2 f^2}{g^2}$; introducing this value, the function in question becomes

$$= \frac{(t+g^2)(t-\theta)}{t(t+f^2)},$$

and we have

$$V = 2fg \int_\theta^\infty dt \frac{\sqrt{t+g^2 . t-\theta}}{\sqrt{t . t+f^2}} \frac{1}{\sqrt{t . t+f^2 . t+g^2}}$$

$$= 2fg \int_\theta^\infty \frac{dt \sqrt{t-\theta}}{t . t+f^2},$$

which, writing $t = x^2 + \theta$, becomes

$$V = 4fg \int_0^\infty \frac{x^2 dx}{x^2+\theta . x^2+\theta+f^2}$$

$$= \frac{4fg}{f^2} \int_0^\infty \left(\frac{\theta+f^2}{x^2+\theta+f^2} - \frac{\theta}{x^2+\theta} \right) dx$$

$$= \frac{4g}{f} \left(\sqrt{\theta+f^2} \tan^{-1} \frac{x}{\sqrt{\theta+f^2}} - \sqrt{\theta} \tan^{-1} \frac{x}{\sqrt{\theta}} \right)_0^\infty$$

$$= 2\pi \frac{g}{f} (\sqrt{\theta+f^2} - \sqrt{\theta});$$

or, substituting for θ its value $\frac{c^2 f^2}{g^2}$, this is

$$V = 2\pi (\sqrt{c^2+g^2} - c),$$

which is, in fact, the potential of the circle $x^2+y^2=g^2$ on the axial point $(0, 0, c)$; and, observing that the value is independent of f, we have at once the theorem that, considering f as variable, and taking the attracted point at the constant altitude c in the focal hyperbola $\frac{x^2}{f^2-g^2} - \frac{z^2}{g^2} = 1$, the potential is the same, whatever is the value of the semi-axis major f of the ellipse.

11. A point in the focal hyperbola determines, with the ellipse, a right circular cone having for its axis the tangent to the hyperbola; viz. the tangent in question is equally inclined to the two lines joining the point with the foci of the hyperbola, or with the extremities of the major axis of the ellipse. Taking θ for the inclination of the tangent to either of these lines, viz. θ is the semi-aperture of the cone, and γ for the inclination of the tangent to the axis of z, then it is easy to show that

$$\sqrt{c^2+g^2} = c \frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}};$$

and we thence have

$$V = 2\pi c \left(\frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}} - 1 \right),$$

viz. the ellipse is here considered as the section of a right cone of semi-aperture θ, the perpendicular distance from the vertex being $=c$, and the inclination of this distance to the axis of the cone being $=\gamma$; and this being so, the potential is then expressed by the last preceding equation. It will be observed that, when $\gamma=\frac{\pi}{2}-\theta$, the section becomes a parabola, and the potential is infinite; for any larger value of γ, the section is a hyperbola, and the formula ceases to be applicable.

12. I originally obtained the result by thus considering the ellipse as the section of a right cone. Consider for a moment, in the case of any cone whatever, the plate included between the plane, perpendicular distance from the vertex $=c$, and the consecutive parallel plane, distance $=c+dc$. Let $d\Sigma$ denote an element of the first plane, r its distance from the vertex, and $r+dr$ the distance produced to meet the second plane; also let $d\omega$ denote the subtended solid angle. We have $d\Sigma\, dc=r^2\, dr\, d\omega$, or, since $\frac{dc}{c}=\frac{dr}{r}$, we obtain $d\Sigma=\frac{1}{c}r^3\, d\omega$, or $\frac{1}{r}\, d\Sigma=\frac{1}{c}r^3\, d\omega$; wherefore the potential of the plane section is $V=\frac{1}{c}\int r^2\, d\omega$, where r denotes the value at a point of the plane section, and the integration extends over the spherical aperture of the cone.

13. Let the position of r be determined by means of its inclination θ to the axis of the cone, and the azimuth ϕ of the plane through r and the axis of the cone; viz. taking the axis of the cone for the axis of z, suppose, as usual, $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$. We have then, as usual, $d\omega=\sin\theta\, d\theta\, d\phi$; and if the equation of the plane be $x\cos\alpha+y\cos\beta+z\cos\gamma=c$, then the value of r is obtained from the equation

$$r\,\{(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta\}=c\,;$$

so that we have for the potential

$$V=c\int\frac{\sin\theta\, d\theta\, d\phi}{\{(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta\}^2},$$

where the integration is extended over the whole spherical aperture of the cone; viz. in the case of a right cone of semi-aperture θ, the limits are from $\theta=0$ to $\theta=\theta$ and from $\phi=0$ to $\phi=2\pi$.

14. Write

$$(\cos\alpha\cos\phi+\cos\beta\sin\phi)\sin\theta+\cos\gamma\cos\theta=M\cos(\theta-N),$$

where M, N are given functions of ϕ; then we have

$$V=c\int\frac{d\phi}{M^2}\int\frac{\sin\theta\, d\theta}{\cos^2(\theta-N)}$$

and the θ-integral is

$$\int\frac{[\sin(\theta-N)\cos N+\cos(\theta-N)\sin N]d\theta}{\cos^2(\theta-N)},$$

$$=\cos N\sec(\theta-N)+\sin N\log\tan\{\tfrac{1}{4}\pi+\tfrac{1}{2}(\theta-N)\},$$

which between the limits is

$$= \cos N \{\sec(\theta - N) - \sec N\} + \sin N \{\log\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta - N)] - \log\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}N)\},$$

θ now denoting the semi-aperture of the right cone. And we have

$$V = c\int \frac{d\phi}{M^2}\left\{\cos N\left(\frac{1}{\cos(N-\theta)} - \frac{1}{\cos N}\right) + \sin N\,[\log\tan\{\tfrac{1}{4}\pi + \tfrac{1}{2}(\theta - N)\} - \log\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}N)]\right\}.$$

We may without loss of generality write $\cos\beta = 0$, and therefore $\cos\alpha = \sin\gamma$, where γ now is the inclination of the perpendicular on the plane to the axis of the cone. We thus have

$$\cos\gamma\cos\theta + \sin\gamma\cos\phi\sin\theta = M\cos(\theta - N),$$

that is,

$$\cos\gamma = M\cos N,$$

$$\sin\gamma\cos\phi = M\sin N;$$

whence

$$\tan N = \tan\gamma\cos\phi \quad\text{or}\quad N = \tan^{-1}(\tan\gamma\cos\phi),$$

$$M^2 = \cos^2\gamma + \sin^2\gamma\cos^2\phi = 1 - \sin^2\gamma\sin^2\phi,$$

and

$$\frac{\cos N}{\cos(N-\theta)} = \frac{1}{\cos\theta + \sin\theta\tan\gamma\cos\phi}.$$

15. We have, therefore,

$$V = c\int \frac{d\phi}{1-\sin^2\gamma\sin^2\phi}\left(\frac{1}{\cos\theta + \sin\theta\tan\gamma\cos\phi} - 1\right)$$

$$+ c\int \frac{d\phi\sin\gamma\cos\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\{\log\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}\theta - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$- \log\tan[\tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}.$$

But

$$\int \frac{d\phi\cos\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{3}{2}}} = \frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}};$$

hence the second line is

$$c\sin\gamma\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\{\log\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}\theta - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$- \log\tan[\tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}$$

$$- c\sin\gamma\int d\phi\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\frac{d}{d\phi}\{\log\tan[\tfrac{1}{4}\pi + \tfrac{1}{2}\theta - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$- \log\tan[\tfrac{1}{4}\pi - \tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}.$$

But, restoring for a moment N in place of $\tan^{-1}(\tan\gamma\cos\phi)$, we have

$$\frac{d}{d\phi}\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta - N) = -\frac{dN}{d\phi}\frac{1}{\cos(N-\theta)} = \frac{\sin\gamma\cos\gamma\sin\phi}{1-\sin^2\gamma\sin^2\phi}\frac{1}{\cos(N-\theta)},$$

$$\frac{d}{d\phi}\log\tan(\tfrac{1}{4}\pi - N) = -\frac{dN}{d\phi}\frac{1}{\cos N} = \frac{\sin\gamma\cos\gamma\sin\phi}{1-\sin^2\gamma\sin^2\phi}\frac{1}{\cos N}.$$

And then, in place of $\dfrac{1}{\cos(N-\theta)}-\dfrac{1}{\cos N}$, writing

$$\frac{1}{\cos\gamma\sqrt{1-\sin^2\gamma\sin^2\phi}}\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right),$$

the expression in question becomes

$$c\sin\gamma\frac{\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\{\log\tan[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$-\log\tan[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}$$
$$-c\int d\phi\,\frac{\sin^2\gamma\sin^2\phi}{1-\sin^2\gamma\sin^2\phi}\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right).$$

And we have

$$V=\frac{c\sin\gamma\sin\phi}{(1-\sin^2\gamma\sin^2\phi)^{\frac{1}{2}}}\{\log\tan[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$-\log\tan[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}+c\int d\phi\left(\frac{1}{\cos\theta+\sin\theta\tan\gamma\cos\phi}-1\right).$$

16. The integral is here

$$=\int d\phi\left\{\frac{\cos\gamma(\cos\theta\cos\gamma-\sin\theta\sin\gamma\cos\phi)}{\cos^2\theta\cos^2\gamma-\sin^2\theta\sin^2\gamma\cos^2\phi}-1\right\}$$
$$=\cos^2\gamma\cos\theta\int\frac{d\phi}{\cos^2\theta\cos^2\gamma-\sin^2\theta\sin^2\gamma\cos^2\phi}$$
$$-\cos\gamma\sin\gamma\sin\theta\int\frac{\cos\phi\,d\phi}{\cos^2\theta\cos^2\gamma-\sin^2\theta\sin^2\gamma\cos^2\phi}-\int d\phi$$
$$=\frac{\cos\gamma}{\sqrt{\cos^2\gamma-\sin^2\theta}}\tan^{-1}\frac{\cos\theta\cos\gamma\tan\phi}{\sqrt{\cos^2\gamma-\sin^2\theta}}$$
$$-\frac{\cos\gamma}{\sqrt{\cos^2\gamma-\sin^2\theta}}\tan^{-1}\frac{\sin\theta\sin\gamma\sin\phi}{\sqrt{\cos^2\gamma-\sin^2\theta}}-\phi,$$

as may be immediately verified.

Hence

$$V=\frac{c\sin\gamma\sin\phi}{\sqrt{1-\sin^2\gamma\sin^2\phi}}\{\log\tan[\tfrac{1}{4}\pi+\tfrac{1}{2}\theta-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]$$
$$-\log\tan[\tfrac{1}{4}\pi-\tfrac{1}{2}\tan^{-1}(\tan\gamma\cos\phi)]\}$$
$$+\frac{c\cos\gamma}{\sqrt{\cos^2\theta-\sin^2\gamma}}\tan^{-1}\frac{\cos\theta\cos\gamma\tan\phi}{\sqrt{\cos^2\theta-\sin^2\gamma}}$$
$$-\frac{c\cos\gamma}{\sqrt{\cos^2\theta-\sin^2\gamma}}\tan^{-1}\frac{\sin\theta\sin\gamma\sin\phi}{\sqrt{\cos^2\theta-\sin^2\gamma}}$$
$$-c\phi,$$

which is to be taken between the limits 0 and 2π; or, what is the same thing, the integral may be taken between the limits 0, π, and multiplied by 2. But as ϕ passes

from 0 to π, the arc of the form $\tan^{-1}(A\tan\phi)$ passes through the values $0, \frac{\pi}{2}, -\frac{\pi}{2}, 0$, but the other arc of the form $\tan^{-1}(B\sin\phi)$ through the values $0, \frac{\pi}{2}, \frac{\pi}{2}, 0$; the first arc gives therefore a term π, the second arc a term 0, and the final result is

$$V = 2c\pi\left(\frac{\cos\gamma}{\sqrt{\cos^2\gamma - \sin^2\theta}} - 1\right),$$

which is right.

The Potential of the Circle.

17. In the case of the circle we have $g=f$; the terms containing a^2, b^2 unite throughout into a single term containing a^2+b^2, and there is obviously no loss of generality in assuming $b=0$, and so reducing this to a^2; viz. we take the axis of x to pass through the projection of the attracted point, the coordinates of this point being therefore $(a, 0, c)$. We in fact consider the potential

$$V = \int \frac{dx\,dy}{\sqrt{(a-x)^2+y^2+c^2}}$$

over the circle $x^2+y^2=f^2$; or, writing $x = mf\cos\phi$, $y = mf\sin\phi$, we have $dxdy = f^2 m dm d\phi$, and therefore

$$V = f^2\int \frac{m\,dm\,d\phi}{\sqrt{a^2+c^2+m^2f^2-2maf\cos\phi}},$$

the integral being taken from $m=0$ to $m=1$, and $\phi=0$ to $\phi=2\pi$.

Writing in the general formula $g=f$ and $b=0$, we have

$$V = 2f^2\int_\theta^\infty \frac{dt\sqrt{1-\frac{a^2}{t+f^2}-\frac{c^2}{t}}}{(t+f^2)\sqrt{t}},$$

where θ denotes the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}-\frac{c^2}{\theta}=0;$$

or, observing that

$$\begin{aligned}
1-\frac{a^2}{t+f^2}-\frac{c^2}{t} &= a^2\left(\frac{1}{\theta+f^2}-\frac{1}{t+f^2}\right)+c^2\left(\frac{1}{\theta}-\frac{1}{t}\right)\\
&= (t-\theta)\left\{\frac{a^2}{(\theta+f^2)(t+f^2)}+\frac{c^2}{\theta t}\right\}\\
&= \frac{t-\theta}{t.t+f^2}\left\{\left(1-\frac{c^2}{\theta}\right)t+\frac{c^2}{\theta}(t+f^2)\right\}\\
&= \frac{(t-\theta)\left(t+\frac{c^2f^2}{\theta}\right)}{t.t+f^2},
\end{aligned}$$

we have also

$$V = 2f^2 \int_\theta^\infty \frac{\sqrt{\left(t - \theta \,.\, t + \frac{c^2 f^2}{\theta}\right)} dt}{t(t+f^2)\sqrt{t+f^2}}.$$

18. The present particular case gives rise to some interesting investigations. We may, in the first place, complete the process of first integrating directly in regard to m.

Writing

$$V = f \int \frac{[(mf - a\cos\phi) + a\cos\phi]\, dm\, d\phi}{\{(mf - a\cos\phi)^2 + a^2\sin^2\phi + c^2\}^{\frac{1}{2}}},$$

the integral in regard to m is

$$= \frac{1}{f}\{\sqrt{(mf - a\cos\phi)^2 + a^2\sin^2\phi + c^2} + a\cos\phi \log\{mf - a\cos\phi + \sqrt{(mf - a\cos\phi)^2 + a^2\sin^2\phi + c^2}\}\}$$

to be taken from $m = 0$ to $m = 1$; and we thus obtain

$$V = \int d\phi\, \{\sqrt{a^2 + c^2 + f^2 - 2af\cos\phi} - \sqrt{a^2 + c^2}$$
$$+ a\cos\phi\, [\log(f - a\cos\phi + \sqrt{a^2 + c^2 + f^2 - 2af\cos\phi}) - \log(-a\cos\phi + \sqrt{a^2 + c^2})]\}.$$

Writing for shortness $\sqrt{a^2 + c^2 + f^2 - 2af\cos\phi} = \Delta$, the second line of this is

$$a\sin\phi\, [\log(f - a\cos\phi + \Delta) - \log(-a\cos\phi + \sqrt{a^2 + c^2})]$$
$$- \int d\phi\, a^2\sin^2\phi \left\{\frac{f + \Delta}{\Delta(f - a\cos\phi + \Delta)} - \frac{1}{-a\cos\phi + \sqrt{a^2 + c^2}}\right\},$$

and we thus have

$$V = a\sin\phi\, \{\log(f - a\cos\phi + \Delta) - \log(-a\cos\phi + \sqrt{a^2 + c^2})\}$$
$$+ \int d\phi \left\{\Delta - \sqrt{a^2 + c^2} - \frac{a^2\sin^2\phi\,(f + \Delta)}{\Delta(f - a\cos\phi + \Delta)} + \frac{a^2\sin^2\phi}{-a\cos\phi + \sqrt{a^2 + c^2}}\right\}.$$

19. We have

$$\frac{f + \Delta}{\Delta(f - a\cos\phi + \Delta)} = \frac{(f + \Delta)(f - a\cos\phi - \Delta)}{\Delta\{(f - a\cos\phi)^2 - \Delta^2\}},$$

the numerator of which is $f^2 - \Delta^2 - a\cos\phi\,(f + \Delta)$,

$$= f^2 + \Delta^2 + a\cos\phi\,(f - a\cos\phi - \Delta) - 2af\cos\phi + a^2\cos^2\phi,$$
$$= -c^2 - a^2\sin^2\phi + a\cos\phi\,(f - a\cos\phi - \Delta),$$

and the denominator is $= -\Delta(c^2 + a^2\sin^2\phi)$. The second line of V is thus

$$= \int d\phi \left\{\Delta - \sqrt{a^2 + c^2} - \frac{a^2\sin^2\phi}{\Delta} + \frac{a^3\sin^2\phi\cos\phi}{\Delta}\,\frac{f - a\cos\phi - \Delta}{c^2 + a^2\sin^2\phi} + \frac{a^2\sin^2\phi\,(\sqrt{a^2 + c^2} + a\cos\phi)}{c^2 + a^2\sin^2\phi}\right\},$$

which is easily reduced to

$$\int d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} - \frac{c^2\sqrt{a^2+c^2}}{c^2+a^2\sin^2\phi} \right\},$$

the last term of which is $= -c\tan^{-1}\dfrac{\sqrt{a^2+c^2}\tan\phi}{c}$; and we thus have

$$V = a\sin\phi\,\{\log(f-a\cos\phi+\Delta) - \log(-a\cos\phi+\sqrt{a^2+c^2})\} - c\tan^{-1}\frac{\sqrt{a^2+c^2}\tan\phi}{c}$$
$$+\int d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} \right\}$$

between the limits 0, 2π; or, finally,

$$V = -2c\pi + 2\int_0^\pi d\phi \left\{ \frac{c^2+f^2-af\cos\phi}{\Delta} - \frac{c^2a\cos\phi\,(f-a\cos\phi)}{(c^2+a^2\sin^2\phi)\,\Delta} \right\};$$

in partial verification whereof observe that for $a=0$ we have $\Delta=\sqrt{c^2+f^2}$, and the value becomes

$$V = 2\pi(\sqrt{c^2+f^2}-c),$$

which, writing therein g in place of f, agrees with a foregoing result.

20. The process applied to finding the Potential of the Ellipse is really applicable step by step to the Circle; but if we begin by assuming $g=f$, it presents itself under a different and simplified form. Starting from

$$V = f^2\int m\,dm\int\frac{d\phi}{\sqrt{a^2+c^2+m^2f^2-2maf\cos\phi}},$$

for convenience we assume

$$P^2+Q^2 = a^2+c^2+m^2f^2,$$
$$PQ = maf,$$

thereby converting the radical into $\sqrt{P^2+Q^2-2PQ\cos\phi}$. Writing also

$$\Omega = a^4+c^4+m^4f^4+2a^2c^2+2m^2c^2f^2-2m^2a^2f^2, \quad = (P^2-Q^2)^2,$$

and hence assuming $P^2-Q^2=\sqrt{\Omega}$, and combining with the foregoing equation

$$P^2+Q^2 = a^2+c^2+m^2f^2,$$

we have

$$P^2 = \tfrac{1}{2}(a^2+c^2+m^2f^2+\sqrt{\Omega}),$$
$$Q^2 = \tfrac{1}{2}(a^2+c^2+m^2f^2-\sqrt{\Omega}).$$

21. This being so, the transformation-equations to the new variable T are

$$\cos\phi = \frac{P\cos T+Q}{P+Q\cos T}, \text{ whence } \cos T = \frac{P\cos\phi-Q}{P-Q\cos\phi},$$

$$\sin\phi = \frac{\sqrt[4]{\Omega}\sin T}{P+Q\cos T}, \qquad \sin T = \frac{\sqrt[4]{\Omega}\sin\phi}{P-Q\cos\phi};$$

and also

$$\sqrt{\Omega} = (P + Q\cos T)(P - Q\cos\phi), \quad = P^2 - Q^2.$$

We find moreover

$$d\phi = \frac{\sqrt[4]{\Omega}\, dT}{P + Q\cos T}, \qquad dT = \frac{\sqrt[4]{\Omega}\, d\phi}{P - Q\cos\phi},$$

and

$$P^2 + Q^2 - 2PQ\cos\phi = \frac{\sqrt{\Omega}\,(P - Q\cos T)}{P + Q\cos T},$$

whence

$$\frac{d\phi}{\sqrt{P^2 + Q^2 - 2PQ\cos\phi}} = \frac{dT}{\sqrt{P^2 - Q^2\cos^2 T}};$$

and hence

$$V = f^2 \int m\, dm \int \frac{dT}{\sqrt{P^2 - Q^2\cos^2 T}},$$

where the limits of T are from 0 to 2π, or, what is the same thing, we may multiply by 4, and take them to be 0, $\frac{1}{2}\pi$.

22. Assuming next

$$t = P^2 - m^2 f^2 + (P^2 - Q^2)\cot^2 T,$$

we have

$$t - P^2 + m^2 f^2 = (P^2 - Q^2)\frac{\cos^2 T}{\sin^2 T},$$

$$t - Q^2 + m^2 f^2 = (P^2 - Q^2)\frac{1}{\sin^2 T},$$

$$t \qquad + m^2 f^2 = (P^2 - Q^2\cos^2 T)\frac{1}{\sin^2 T},$$

and thence

$$\sqrt{t - P^2 + m^2 f^2 . t - Q^2 + m^2 f^2 . t + m^2 f^2} = (P^2 - Q^2)\frac{\cos T}{\sin^3 T}\sqrt{P^2 - Q^2\cos^2 T};$$

also

$$dt = -2(P^2 - Q^2)\frac{\cos T}{\sin^3 T}\, dT;$$

and consequently

$$\frac{dt}{\sqrt{t - P^2 + m^2 f^2 . t - Q^2 + m^2 f^2 . t + m^2 f^2}} = \frac{-2dT}{\sqrt{P^2 - Q^2\cos^2 T}}.$$

$T = 0$ gives $t = \infty$, and $T = \frac{1}{2}\pi$ gives $t = P^2 - m^2 f^2$, $= G$ suppose; and we thus have

$$V = 2f^2 \int m\, dm \int_G^\infty \frac{dt}{\sqrt{t - P^2 + m^2 f^2 . t - Q^2 + m^2 f^2 . t + m^2 f^2}}.$$

23. We have

$$(t - P^2 + m^2 f^2)(t - Q^2 + m^2 f^2) = t^2 + (m^2 f^2 - a^2 - c^2)t - m^2 c^2 f^2,$$

or, putting $m^2 t$ in the place of t, this is

$$= m^2\{m^2 t^2 + (m^2 f^2 - a^2 - c^2)t - c^2 f^2\},$$

or, what is the same thing,

$$= m^2 t(t+f^2)\left\{m^2 - \frac{a^2}{t+f^2} - \frac{c^2}{t}\right\};$$

whence, completing the substitution, we have

$$V = 2f^2 \int m\,dm \int_\theta^\infty \frac{1}{\sqrt{m^2 - \frac{a^2}{t+f^2} - \frac{c^2}{t}}} \frac{dt}{\sqrt{t}(t+f^2)},$$

where the inferior limit θ, $= \frac{1}{m^2} G$, $= \frac{1}{m^2} P^2 - f^2$ is, in fact, the positive root of the equation

$$m^2 - \frac{a^2}{\theta+f^2} - \frac{c^2}{\theta} = 0.$$

24. We may hence integrate in regard to m, through the sign $\int dt$, in the same way as if θ were constant; viz. we have

$$V = 2f^2 \left[\int_\theta^\infty \sqrt{m^2 - \frac{a^2}{t+f^2} - \frac{c^2}{t}} \frac{dt}{\sqrt{t}(t+f^2)}\right],$$

where the function of m is to be taken between the limits 0, 1: for $m=0$, we have $\theta = \infty$, and the function vanishes; hence, writing $m=1$, we obtain

$$V = 2f^2 \int_\theta^\infty \sqrt{1 - \frac{a^2}{t+f^2} - \frac{c^2}{t}} \frac{dt}{\sqrt{t}(t+f^2)},$$

where θ now denotes the positive root of

$$1 - \frac{a^2}{\theta+f^2} - \frac{c^2}{\theta} = 0.$$

25. But it is interesting to reverse the transformation, so as to bring the radical back into its original form. For this purpose, taking now

$$P^2 + Q^2 = a^2 + c^2 + f^2,$$
$$PQ = af,$$

and consequently

$$P^2 = \tfrac{1}{2}(a^2 + c^2 + f^2 + \sqrt{\Omega}),$$
$$Q^2 = \tfrac{1}{2}(a^2 + c^2 + f^2 - \sqrt{\Omega}),$$

where

$$\Omega = a^4 + c^4 + f^4 + 2a^2c^2 + 2c^2f^2 - 2a^2f^2,$$

and writing

$$t = P^2 - f^2 + (P^2 - Q^2)\cot^2 T,$$

we first obtain

$$V = f^2 \int_0^{2\pi} \frac{\Omega \cos^2 T\,dT}{(P^2 - Q^2\cos^2 T - f^2 \sin^2 T)(P^2 - Q^2 \cos^2 T)^{\frac{3}{2}}};$$

and then, writing

$$\cos T = \frac{P\cos\phi - Q}{P - Q\cos\phi},$$

$$\sin T = \frac{\sqrt[4]{\Omega}\sin\phi}{P - Q\cos\phi},$$

we bring in the variable ϕ. But it is important to remark that this is not the quantity which was, at the beginning of the investigation, represented by this letter, and that it is not easy to see the connexion between the two quantities ϕ. We find

$$V = f^2\int_0^{2\pi}\frac{(P - Q\cos\phi)^2(P\cos\phi - Q)^2\,d\phi}{(a^2 + c^2 + f^2\cos^2\phi - 2af\cos\phi)(a^2 + c^2 + f^2 - 2af\cos\phi)^{\frac{3}{2}}}.$$

26. To reduce this, write as before

$$\Delta = \sqrt{a^2 + c^2 + f^2 - 2af\cos\phi},$$

and also

$$\Phi = a^2 + c^2 - 2af\cos\phi + f^2\cos^2\phi,$$

so that the denominator in the integral is $= \Phi\Delta^3$.

We have

$$\begin{aligned}(P - Q\cos\phi)^2(P\cos\phi - Q)^2 &= (\Delta^2 - Q^2\sin^2\phi)(\Delta^2 - P^2\sin^2\phi),\\ &= \Delta^4 - (a^2 + c^2 + f^2)\Delta^2\sin^2\phi + a^2f^2\sin^4\phi,\\ &= \Delta^2\{\Delta^2 - (c^2 + f^2)\sin^2\phi\} - a^2\sin^2\phi\,(\Delta^2 - f^2\sin^2\phi),\\ &= \Delta^2\{\Delta^2 - (c^2 + f^2)\sin^2\phi\} - a^2\sin^2\phi\,.\,\Phi,\end{aligned}$$

and hence

$$V = \int\frac{f^2[\Delta^2 - (c^2 + f^2)\sin^2\phi]\,d\phi}{\Phi\Delta} - a^2f^2\int\frac{\sin^2\phi\,d\phi}{\Delta^3},$$

the limits being always 0, 2π. But we have identically

$$\frac{d}{d\phi}\frac{\sin\phi}{\Delta} = \frac{\cos\phi}{\Delta} - \frac{af\sin^2\phi}{\Delta^3},$$

and thence

$$\int\frac{\sin^2\phi\,d\phi}{\Delta^3} = -\frac{1}{af}\left(\frac{\sin\phi}{\Delta}\right) - \frac{1}{af}\int\frac{\cos\phi\,d\phi}{\Delta},$$

where the term $\left(\frac{\sin\phi}{\Delta}\right)$ is to be taken between the limits, but for the present I retain it as it stands. Moreover, $\Delta^2 = \Phi + f^2\sin^2\phi$, and consequently

$$\Delta^2 - (c^2 + f^2)\sin^2\phi = \Phi - c^2\sin^2\phi,$$

and we thus obtain the result

$$V = af\left(\frac{\sin\phi}{\Delta}\right) - af\int\frac{\cos\phi\,d\phi}{\Delta} + f^2\int\frac{d\phi}{\Delta} - c^2f^2\int\frac{\sin^2\phi\,d\phi}{\Phi\Delta},$$

where the denominators under the integral signs are

$$\Delta, = \sqrt{a^2 + c^2 + f^2 - 2af\cos\phi}, \text{ and } \Phi\Delta, = (a^2 + c^2 - 2af\cos\phi + f^2\cos^2\phi)\,\Delta.$$

27. We may, by a transformation such as that for the change of parameter in an elliptic integral of the third kind, make the denominators to be Δ and $(c^2+a^2\sin^2\phi)\,\Delta$; viz. for this purpose we assume $\Lambda=\tan^{-1}\dfrac{B\Delta}{A}$, where B and A are functions of ϕ such that we have identically $A^2+B^2\Delta^2=(c^2+a^2\sin^2\phi)\,(a^2+c^2-2af\cos\phi+f^2\cos^2\phi)$; the values of B, A are found to be $c\cos\phi$ and $\sin\phi\,(a^2+c^2-af\cos\phi)$, whence, dividing each of these for greater convenience by $\sin\phi$, we have

$$\Lambda=\tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right),$$

so that, writing now B, $A=c\cot\phi$ and $a^2+c^2-af\cos\phi$ respectively, the value is

$$\Lambda=\tan^{-1}\left(\frac{B\Delta}{A}\right),$$

where

$$A^2+B^2\Delta^2=\frac{1}{\sin^2\phi}\,\Pi\Phi\,;$$

and, as before, $\Phi=a^2+c^2-2af\cos\phi+f^2\cos^2\phi$, and also $\Pi=c^2+a^2\sin^2\phi$. We have

$$\frac{d\Lambda}{d\phi}=\frac{(AB'-A'B)\,\Delta^2+\frac{1}{2}AB\,(\Delta^2)'}{(A^2+B^2\Delta^2)\,\Delta},\quad\left(A'=\frac{dA}{d\phi},\ \&c.\right);$$

and then

$$AB'-A'B=\frac{a}{\sin^2\phi}(-a^2-c^2+af\cos^3\phi),$$

$$\tfrac{1}{2}AB\,(\Delta^2)'=\frac{c\sin\phi\cos\phi}{\sin^2\phi}\,(a^2+c^2-af\cos\phi)\,af\sin\phi,$$

and the numerator thus is

$$\frac{c}{\sin^2\phi}\{(-a^2-c^2+af\cos^3\phi)\,(a^2+c^2+f^2-2af\cos\phi)$$
$$+af\cos\phi\,(1-\cos^2\phi)\,(a^2+c^2-af\cos\phi)\},$$

which is in fact

$$=\frac{c}{\sin^2\phi}\{-(c^2+a^2\sin^2\phi)\,(a^2+c^2+f^2-2af\cos\phi)$$
$$+(af\cos\phi-a^2\cos^2\phi)\,(a^2+c^2-2af\cos\phi+f^2\cos^2\phi)\},$$
$$=\frac{c}{\sin^2\phi}\{-\Pi\Delta^2+(af\cos\phi-a^2\cos^2\phi)\,\Phi\}\,;$$

or, what is the same thing,

$$=\frac{c}{\sin^2\phi}\{-\Pi\phi-\Pi f^2\sin^2\phi+(af\cos\phi-a^2\cos^2\phi)\,\Phi\},$$

and the denominator, by what precedes, is

$$=\frac{1}{\sin^2\phi}\,.\,\Pi\Phi\Delta.$$

We thus have

$$\frac{1}{c}\frac{d\Lambda}{d\phi}=-\frac{1}{\Delta}-\frac{f^2\sin^2\phi}{\Phi\Delta}+\frac{af\cos\phi-a^2\cos^2\phi}{\Pi\Delta},$$

whence, by integration,

$$\frac{1}{c}\tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right)=-\int\frac{d\phi}{\Delta}+\int\frac{(af\cos\phi-a^2\cos^2\phi)\,d\phi}{\Pi\Delta}-f^2\int\frac{\sin^2\phi\,d\phi}{\Phi\Delta},$$

which is the required formula of transformation.

28. Multiplying by c^2, and subtracting from the value of V, we find

$$V=c\tan^{-1}\left(\frac{c\cot\phi\,\Delta}{a^2+c^2-af\cos\phi}\right)+af\left(\frac{\sin\phi}{\Delta}\right)$$
$$+\int\frac{(c^2+f^2-af\cos\phi)d\phi}{\Delta}-c^2a\int\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta},$$

which is to be taken between the limits 0 and 2π; viz. we thus have

$$V=-2c\pi+2\int_0^\pi\frac{(c^2+f^2-af\cos\phi)\,d\phi}{\Delta}-2c^2a\int_0^\pi\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta},$$

agreeing with a former result.

29. But this former result, previous to the final step of taking the integrals between the limits, was

$$V=2a\sin\phi\log\left(\frac{f-a\cos\phi+\Delta}{-a\cos\phi+\sqrt{a^2+c^2}}\right)-c\tan^{-1}\left(\frac{\sqrt{a^2+c^2}\tan\phi}{c}\right)$$
$$+\int\frac{(c^2+f^2-af\cos\phi)\,d\phi}{\Delta}-c^2a\int\frac{\cos\phi\,(f-a\cos\phi)\,d\phi}{(c^2+a^2\sin^2\phi)\,\Delta};$$

viz. the integrals are the same, but the integrated terms are altogether different; the explanation of course is that the ϕ's are different in the two formulæ, which therefore do not correspond element by element but only in their ultimate value between the limits.

30. In order to discuss numerically the Potential of the Circle,

$$V=2f^2\int_\theta^\infty\frac{\sqrt{\left(t-\theta\,.\,t+\frac{c^2f^2}{\theta}\right)}\,dt}{t\,(t+f^2)\sqrt{t+f^2}},$$

this must be reduced to elliptic functions. Writing $t=\theta+x^2$, we have

$$V=4f^2\int_0^\infty\frac{x^2\sqrt{x^2+\beta^2}\,dx}{(x^2+\theta)\,(x^2+\alpha^2)^{\frac{3}{2}}};$$

if for shortness

$$\theta + f^2 = \alpha^2,$$

$$\theta + \frac{c^2 f^2}{\theta} = \beta^2.$$

The constants α, β, θ may be considered as replacing the original constants a, c, f; viz. from the last two equations and the equation

$$\frac{a^2}{\theta + f^2} + \frac{c^2}{\theta} = 1,$$

we deduce

$$a^2 = \frac{\alpha^2(\alpha^2 - \beta^2)}{\alpha^2 - \theta}, \quad c^2 = \frac{\theta(\beta^2 - \theta)}{\alpha^2 - \theta}, \quad f^2 = \alpha^2 - \theta;$$

showing that α^2, β^2, θ are in order of decreasing magnitude; viz. $\alpha^2 - \beta^2$, $\beta^2 - \theta$, $\alpha^2 - \theta$ are all positive. The formula may be written

$$\tfrac{1}{4} V = (\alpha^2 - \theta) \int_0^\infty \frac{x^2(x^2 + \beta^2)\, dx}{(x^2 + \theta)(x^2 + \alpha^2)\sqrt{x^2 + \alpha^2 \,.\, x^2 + \beta^2}};$$

which, in virtue of the identity

$$(\alpha^2 - \theta)\, x^2 (x^2 + \beta^2) = (\alpha^2 - \theta)(x^2 + \theta)(x^2 + \alpha^2) - \alpha^2(\alpha^2 - \beta^2)(x^2 + \theta) - \theta(\beta^2 - \theta)(x^2 + \alpha^2),$$

becomes

$$\begin{aligned} \tfrac{1}{4} V = \quad & (\alpha^2 - \theta) \int_0^\infty \frac{dx}{\sqrt{x^2 + \alpha^2 \,.\, x^2 + \beta^2}} \\ & - \alpha^2(\alpha^2 - \beta^2) \int_0^\infty \frac{dx}{(x^2 + \alpha^2)\sqrt{x^2 + \alpha^2 \,.\, x^2 + \beta^2}} \\ & - \theta(\beta^2 - \theta) \int_0^\infty \frac{dx}{(x^2 + \theta)\sqrt{x^2 + \alpha^2 \,.\, x^2 + \beta^2}}. \end{aligned}$$

31. Writing here $x = \alpha \cot u$, and therefore $dx = -\alpha \operatorname{cosec}^2 u\, du$, to the values $x = \infty$, 0 correspond $u = 0$, $\frac{1}{2}\pi$, and we have

$$\tfrac{1}{4} V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{\alpha^2 \cos^2 u + \beta^2 \sin^2 u}} \left\{ \alpha^2 - \theta - (\alpha^2 - \beta^2) \sin^2 u - \frac{\theta(\beta^2 - \theta) \sin^2 u}{\alpha^2 \cos^2 u + \theta \sin^2 u} \right\}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{\alpha^2 \cos^2 u + \beta^2 \sin^2 u}} \left\{ \alpha^2 - \theta + \frac{\theta(\beta^2 - \theta)}{\alpha^2 - \theta} - (\alpha^2 - \beta^2) \sin^2 u - \frac{\alpha^2 \theta(\beta^2 - \theta)}{\alpha^2 - \theta} \frac{1}{\alpha^2 \cos^2 u + \theta \sin^2 u} \right\}.$$

Writing $k^2 = 1 - \dfrac{\beta^2}{\alpha^2}$, we have

$$\sqrt{\alpha^2 \cos^2 u + \beta^2 \sin^2 u} = \alpha \sqrt{1 - k^2 \sin^2 u},$$

and thence

$$\tfrac{1}{4} V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} \times \left\{ \alpha(1 - k^2 \sin^2 u) - \frac{k^2 \theta}{1 - \frac{\theta}{\alpha^2}} - \frac{\frac{\theta}{\alpha} \frac{\beta^2 - \theta}{\alpha^2}}{1 - \frac{\theta}{\alpha^2}} \frac{1}{1 - \left(1 - \frac{\theta}{\alpha^2}\right) \sin^2 u} \right\};$$

viz. writing $n = -1 + \frac{\theta}{\alpha^2}$ (so that n is negative and in absolute magnitude < 1), and moreover $\beta^2 = \alpha^2 k'^2$ and $\theta = (n+1)\alpha^2$, this is

$$\tfrac{1}{4} V = \int_0^{\frac{1}{2}\pi} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} \times \left\{(1 - k^2 \sin^2 u)\,\alpha + \frac{k^2}{n}(n+1)\,\alpha - \frac{n+1\,.\,n+k^2}{n}\,\alpha\,\frac{1}{1 + n \sin^2 u}\right\};$$

viz. this is

$$= \alpha \left\{E_1 k + k^2 \frac{n+1}{n} F_1 k - \frac{n+1\,.\,n+k^2}{n} \Pi_1(n, k)\right\}.$$

32. This may be further reduced by substituting for the complete function $\Pi_1(n, k)$, its value; viz. writing

$$n = \left(-1 + \frac{\theta}{\alpha^2}\right) = -1 + k'^2 \sin^2 \lambda,$$

that is, $\sin^2 \lambda = \frac{\theta}{\beta^2}$; then, writing the value first in the form

$$\alpha \left\{E_1 k - (n+1) F_1 k - \frac{n+1\,.\,n+k^2}{n} [\Pi_1(n, k) - F_1 k]\right\},$$

and observing that

$$\frac{n+1\,.\,n+k^2}{n} [\Pi_1(n, k) - F_1 k] = \frac{k'^4 \sin^2 \lambda \cos^2 \lambda}{1 - k'^2 \sin^2 \lambda} [\Pi_1(n, k) - F_1 k]$$

$$= \frac{k'^2 \sin \lambda \cos \lambda}{\sqrt{1 - k'^2 \sin^2 \lambda}} \left\{\tfrac{1}{2}\pi + (F_1 k - E_1 k)\, F(k', \lambda) - F_1 k\,.\,E(k', \lambda)\right\},$$

we have

$$\tfrac{1}{4} V = \alpha \left\{E_1 k - k'^2 \sin^2 \lambda\, F_1 k - \frac{k'^2 \sin \lambda \cos \lambda}{\sqrt{1 - k'^2 \sin^2 \lambda}} [\tfrac{1}{2}\pi + (F_1 k - E_1 k)\, F(k', \lambda) - F_1 k\,.\,E(k', \lambda)]\right\},$$

where

$$\alpha^2 = \theta + f^2, \quad k^2 = 1 - \frac{\beta^2}{\alpha^2}, \quad = 1 - \frac{\theta + \frac{c^2 f^2}{\theta}}{\theta + f^2}, \quad = \frac{f^2\left(1 - \frac{c^2}{\theta}\right)}{\theta + f^2},$$

or, what is the same thing,

$$k = \frac{af}{f^2 + \theta}, \quad \sin^2 \lambda = \frac{\theta}{\beta^2}, \quad = \frac{1}{1 + \frac{c^2 f^2}{\theta^2}},$$

θ being, it will be recollected, the positive root of

$$\frac{a^2}{f^2 + \theta} + \frac{c^2}{\theta} = 1.$$

33. Thus when in particular $a = 0$, we have $\theta = c^2$, and thence

$$\alpha = \sqrt{c^2 + f^2}, \quad k = 0, \quad k' = 1, \quad \sin \lambda = \frac{c}{\sqrt{c^2 + f^2}};$$

whence

$$\tfrac{1}{4}V = \tfrac{1}{2}\pi\sqrt{c^2+f^2}\,\{1-\sin^2\lambda-\sin\lambda\,(1-\sin\lambda)\},$$
$$= \tfrac{1}{2}\pi\sqrt{c^2+f^2}\,(1-\sin\lambda), \quad = \tfrac{1}{2}\pi\,(\sqrt{c^2+f^2}-c),$$

or

$$V = 2\pi\,(\sqrt{c^2+f^2}-c),$$

which is right.

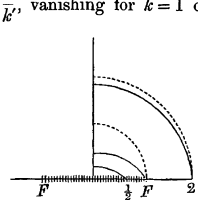
34. If $c=0$, a being $>f$, then $\theta = a^2-f^2$, $k=\dfrac{f}{a}$, $\lambda=\tfrac{1}{2}\pi$, $\alpha=a$; so that, retaining k as standing for its value $\dfrac{f}{a}$, we have

$$\tfrac{1}{4}V = a\,(E_1k - k'^2F_1k), \quad \text{or} \quad V = 4a\,(E_1k - k'^2\,F_1k),$$

which may easily be verified.

If $c=0$, a being $<f$, then, recurring to the original equation for the determination of θ, viz. $(\theta+f^2)^2\,\theta\left(\dfrac{a^2}{\theta+f^2}+\dfrac{c^2}{\theta}-1\right)=0$, which for $c=0$ becomes $\theta\,(\theta+f^2)\,(\theta-a^2+f^2)=0$, we have (as the positive root of this equation) $\theta=0$; whence $\alpha=f$; also, observing that $1-\dfrac{c^2}{\theta}=\dfrac{a^2}{f^2}$, $k=\dfrac{a}{f}$, and $\sin^2\lambda=\dfrac{\theta}{\theta+\dfrac{c^2}{\theta}f^2}$ $\left(\text{where } \dfrac{c^2}{\theta} \text{ is finite}\right)$, $=0$, and retaining k to denote its value $=\dfrac{a}{f}$, we obtain $\tfrac{1}{4}V=fE_1k$, or $V=4fE_1k$.

If $a=f$, then in each of the formulæ $k=1$; and since in the first formula k'^2F_1k, k nearly $=1$, is $=k'^2\log\dfrac{4}{k'}$, vanishing for $k=1$ or $k'=0$, we have $V=4fE_11, =4f$.



Section of Equipotential surfaces of a Circle.

It would be interesting to consider the value of the potential at different points of the ellipse $\dfrac{a^2}{f^2+\theta}+\dfrac{c^2}{\theta}=1$ (θ constant, a, c current coordinates). For this purpose writing $a=\sqrt{f^2+\theta}\cos q$, $c=\sqrt{\theta}\sin q$, we should have $\alpha=\sqrt{f^2+\theta}$ (a constant), and

$$k=\frac{f\cos q}{\sqrt{f^2+\theta}}, \quad k'=\frac{\sqrt{\theta+f^2\sin^2 q}}{\sqrt{f^2+\theta}},$$

$$\sin\lambda=\frac{\sqrt{\theta}}{\sqrt{\theta+f^2\sin^2 q}}, \quad \cos\lambda=\frac{f\sin q}{\sqrt{\theta+f^2\sin^2 q}};$$

and then V through k, k', λ, is a given function of q.

35. Suppose, to fix the ideas, $f=1$, and consider the points $(0, c)$ and $(a, 0)$, which have equal potentials. First, if $a>f$ (that is, $a>1$), then writing $k=\frac{1}{a}$, the relation is

$$2\pi(\sqrt{1+c^2}-c)=\frac{4}{k}(E_1k-k'^2F_1k);$$

and we have

$$F_1 30^\circ=1{\cdot}68575,\quad E_1 30^\circ=1{\cdot}46746,\quad \frac{4}{\pi}=1{\cdot}27324.$$

Secondly, if $a<f$ (that is, $a<1$), then writing $k=a$, the relation is

$$2\pi(\sqrt{1+c^2}-c)=4E_1k.$$

(1) In particular $a=\frac{1}{2}$, $=\sin 30^\circ$, this is

$$\sqrt{1+c^2}-c=\frac{2}{\pi}E_1 30^\circ \qquad = \cdot 93421.$$

(2) $a=1$, then

$$\sqrt{1+c^2}-c=\frac{2}{\pi} \qquad = \cdot 63662.$$

(3) $a=2$, $k=\frac{1}{2}$, $=\sin 30^\circ$,

$$\sqrt{1+c^2}-c=\frac{4}{\pi}\{E_1(30^\circ)-\tfrac{3}{4}F_1(30^\circ)\} = \cdot 25866.$$

But if $\sqrt{1+c^2}-c=m$, then $c=\frac{1}{2}\left(\frac{1}{m}-m\right)$; whence

a	c
0	·0
$\frac{1}{2}$	·06810
1	·46709
2	1·80376

for the values of c, corresponding to the foregoing values of a.

604.

DETERMINATION OF THE ATTRACTION OF AN ELLIPSOIDAL SHELL ON AN EXTERIOR POINT.

[From the *Proceedings of the London Mathematical Society*, vol. VI. (1874—1875), pp. 58—67. Read January 14, 1875.]

THE shell in question is the indefinitely thin shell included between two concentric, similar, and similarly situated ellipsoidal surfaces, the density being uniform and the attraction varying as the inverse square of the distance.

It was shown by Poisson that the attraction was in the direction of the axis of the circumscribed cone, and expressible in finite terms; the theorem as to the direction of the attraction was afterwards demonstrated geometrically by Steiner, *Crelle*, t. XII. (1834), his method being to divide the shell into elements by means of conical surfaces having their vertices at an interior point Q; and the investigation was about two years ago completed by Prof. Adams, so as to obtain from it the finite expression for the attraction of the shell. The process was explained in a lecture at which I was present: I did not particularly attend to the details of it; and I now reproduce the solution in my own form, stating, in the first place, the geometrical theorems on which it depends.

Statement of the Geometrical Theorems.

1. We consider (see figure, p. 305) an ellipsoid, and two corresponding points, an external point P, and an internal point Q; as will appear, the correspondence is not a reciprocal one. The points are such that each of them is, in regard to the ellipsoid, in the polar plane of the other; moreover PQ is the perpendicular at P to the polar plane of Q; that is, Q being regarded as given, then P is determined as the foot of the perpendicular let fall from Q upon its polar plane; to a given position of Q there corresponds thus a single position of P. It follows that PQ is the normal at

P to the confocal ellipsoid through this point; that is, given the position of P, then Q is the intersection of the polar plane of P by the normal at P to the confocal ellipsoid. Analytically, to a given position of P, there correspond three positions of Q, viz. these are the intersections of the polar plane of P by the normals at P to the three confocal surfaces through this point, and the correspondence of the points P, Q is a (1, 3) correspondence; but the other two positions of Q are external to the ellipsoid, and we are not concerned with them; we determine Q as above by means of the normal to the confocal ellipsoid.

2. If through the point Q we draw at pleasure a chord $R'QR''$, and join the extremities R', R'' with P, then the line PQ bisects the angle $R'PR''$; whence also $PR' : QR' = PR'' : QR''$, or writing QR', $QR'' = r'$, r'' and PR', $PR'' = \rho'$, ρ'', then $\frac{\rho'}{r'} = \frac{\rho''}{r''}$. Putting each of these equal ratios $= \frac{\Omega}{R}$, where Ω is a length depending on the position of Q but independent of the direction of the chord $R'QR''$, then R will be a length depending on the direction of the chord, and if along the chord (say in the sense Q to R') we measure off from Q a length $QT, = R$, thence the locus of the extremity T of this line will be an ellipsoid, centre Q, similarly situate to the given ellipsoid, say this is the "auxiliary ellipsoid."

Consider now the given ellipsoid and a concentric and similarly situated similar ellipsoid, exterior to and indefinitely near it. To fix the ideas, let the semi-axes of the given ellipsoid be mf, mg, mh, and those of the consecutive ellipsoid be $(m+dm)f$, $(m+dm)g$, $(m+dm)h$. Producing the chord $R'R''$ to meet the consecutive ellipsoid in S', S'', then the radial thicknesses $R'S'$, $R''S''$ of the included shell will be equal to each other, or say each $= \Lambda dm$, where Λ is a quantity dependent as well on the position of the point Q as on the direction of the chord $R'R''$ through this point.

3. Let 2ϕ denote the angle $R'PR''$, or, what is the same thing, let ϕ denote either of the equal angles $R'PQ$, $R''PQ$; then, R, Λ being as above, it is found that $\cos\phi = \frac{mR}{\Lambda}$.

Determination of the Attraction of the Shell.

4. We may now solve the attraction-problem. We consider the indefinitely thin shell (density unity) included between the given ellipsoid and the consecutive ellipsoid, and attracting the exterior point P. We determine the corresponding interior point Q, and then dividing the shell into elements by means of indefinitely thin cones having their vertices at Q, we consider in conjunction the elements determined by any two opposite cones, say the two opposite cones, having for their axis the chord $R'QR''$ and a spherical aperture $= d\omega$. The shell-element at R' is

$$r'^2 d\omega \,.\, R'S' = r'^2 \Lambda d\omega\, dm;$$

its attraction on P is therefore

$$\frac{r'^2}{\rho'^2}\Lambda\, d\omega\, dm, = \frac{R^2}{\Omega^2}\Lambda\, dm\, d\omega,$$

and the attractions in the directions QR' and PQ are this quantity multiplied by $\sin\phi$ and $\cos\phi$ respectively.

5. But the shell-element at R'' exerts upon P the same attraction $\frac{R^2}{\Omega^2}\Lambda\, dm\, d\omega$, and the attractions in the directions QR'' and PQ are this quantity multiplied by $\sin\phi$ and $\cos\phi$ respectively: hence the attractions in the directions QR', QR'' exactly counterbalance each other, and there remain only the two equal attractions in the direction PQ; viz. this, for either of the elements in question, say for the element at R', is

$$= \frac{R^2}{\Omega^2}\Lambda\cos\phi\, dm\, d\omega,$$

or, substituting for $\cos\phi$ its value, $= \frac{mR}{\Lambda}$, this is

$$= \frac{m\, dm}{\Omega^2} R^3\, d\omega.$$

Hence the whole attraction of the shell is in the direction PQ, its value being

$$\frac{m\, dm}{\Omega^2}\iint R^3\, d\omega,$$

over the whole solid angle at Q; and recollecting that R denotes the radius vector in the auxiliary ellipsoid, we have the volume of this ellipsoid

$$= \iiint r^2\, dr\, d\omega = \tfrac{1}{3}\iint R^3\, d\omega,$$

that is, $\iint R^3\, d\omega =$ thrice the volume of the auxiliary ellipsoid, $= 4\pi FGH$, if F, G, H are the semiaxes of the auxiliary ellipsoid. That is,

$$\text{Attraction of shell} = \frac{m\, dm}{\Omega^2} 4\pi FGH.$$

The problem is now solved; but it remains to prove the geometrical theorems, and to determine the values of the quantities Ω, F, G, H, which enter into the expression for the attraction; and we may also deduce the formula for the attractions of a solid ellipsoid.

Proof of the Geometrical Theorems.

6. I take

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = m^2$$

for the equation of the ellipsoid; a, b, c for the coordinates of P; ξ, η, ζ for those of Q; α, β, γ for the cosine-inclinations of the radius QR' to the axes. Hence, in

the equation of the ellipsoid, substituting for x, y, z the values $\xi + r\alpha$, $\eta + r\beta$, $\zeta + r\gamma$, and writing for shortness

$$A = \frac{\alpha^2}{f^2} + \frac{\beta^2}{g^2} + \frac{\gamma^2}{h^2},$$

$$B = \frac{\alpha\xi}{f^2} + \frac{\beta\eta}{g^2} + \frac{\gamma\zeta}{h^2},$$

$$C = \frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2, \text{ (C being therefore negative)},$$

we have r', $-r''$ as the roots of the equation

$$Ar^2 + 2Br + C = 0;$$

viz.

$$\frac{2B}{A} = -r' + r'', \quad \frac{C}{A} = -r'r'',$$

and thence

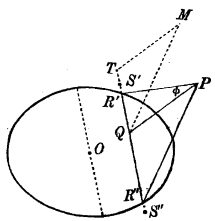
$$r' = \frac{-B + \sqrt{B^2 - AC}}{A}, \quad r'' = \frac{B + \sqrt{B^2 - AC}}{A}, \quad r' + r'' = \frac{2\sqrt{B^2 - AC}}{A}.$$

7. Suppose for a moment that the semidiameter parallel to $R'R''$ is $= mv$; we have evidently $v^2 = \frac{1}{A}$. And then, if in the central section through $R'R''$ the conjugate semidiameter is mu, the equation of the section referred to these conjugate axes will be $\frac{x^2}{m^2u^2} + \frac{y^2}{m^2v^2} = 1$, or say, $y^2 = m^2v^2 - \frac{v^2}{u^2}x^2$, where y is the coordinate parallel to $R'R''$, so that, taking the coordinate to belong to the point R', we have

$$y = \tfrac{1}{2}(r' + r'') = \frac{\sqrt{B^2 - AC}}{A}.$$

For the exterior surface of the shell, m is to be changed into $m + dm$; hence, y and m alone varying, we have

$$y\,dy = mv^2\,dm, \; = m\,dm\,\frac{1}{A},$$



that is,

$$dy = m\,dm\,\frac{1}{\sqrt{B^2 - AC}};$$

viz. this is the value of the radial thickness $R'S'$ of the shell; or, since the same process applies to the point R'', we have

$$R'S' = R''S'' = m\,dm\,\frac{1}{\sqrt{B^2 - AC}},$$

or, calling this, as above, $\Lambda\,dm$, the value of Λ is $= \dfrac{m}{\sqrt{B^2 - AC}}$.

8. The points P and Q are connected by the condition that, for every direction whatever of the chord $R'R''$, we have

$$PR' : PR'' = QR' : QR'',$$

or, what is the same thing, that the line QP bisects the angle $R'PR''$. Taking $PR' = \rho'$, $PR'' = \rho''$, the condition is $\rho' : r' = \rho'' : r''$; and taking (a, b, c) as the coordinates of the point P, we have

$$\begin{aligned}\rho'^2 &= (\xi + r'\alpha - a)^2 + (\eta + r'\beta - b)^2 + (\zeta + r'\gamma - c)^2\\ &= \sigma^2 + 2r'U + r'^2,\end{aligned}$$

if, for shortness,

$$\begin{aligned}\sigma^2 &= (\xi - a)^2 + (\eta - b)^2 + (\zeta - c)^2,\ (= \overline{QP}^2),\\ U &= \alpha(\xi - a) + \beta(\eta - b) + \gamma(\zeta - c);\end{aligned}$$

and similarly

$$\rho''^2 = \sigma^2 - 2r''U + r''^2.$$

The required condition therefore is

$$\frac{\sigma^2}{r'^2} + \frac{2}{r'}U = \frac{\sigma^2}{r''^2} - \frac{2}{r''}U,$$

viz. this is

$$\sigma^2\left(\frac{1}{r'^2} - \frac{1}{r''^2}\right) + 2U\left(\frac{1}{r'} + \frac{1}{r''}\right) = 0,$$

so that, omitting a factor, it becomes

$$\sigma^2\left(\frac{1}{r'} - \frac{1}{r''}\right) + 2U = 0,$$

that is,

$$\sigma^2 . \frac{-2B}{C} + 2U = 0, \text{ or } U = \frac{\sigma^2 B}{C},$$

which must be satisfied independently of the values of α, β, γ.

9. Writing, for greater convenience, $\dfrac{\sigma^2}{C} = -\theta$, the equation is $U = -\theta B$, viz. substituting for U, B their values, this gives $\xi - a + \dfrac{\theta\xi}{f^2} = 0$, &c., or say,

$$a = \xi\left(1 + \frac{\theta}{f^2}\right),$$

$$b = \eta\left(1 + \frac{\theta}{g^2}\right),$$

$$c = \zeta\left(1 + \frac{\theta}{h^2}\right);$$

and the assumed relation $\frac{\sigma^2}{C} = -\theta$ is

$$(\xi - a)^2 + (\eta - b)^2 + (\zeta - c)^2 = -\theta\left(\frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2\right);$$

viz. substituting for a, b, c the foregoing values, and omitting a factor θ, this is

$$\theta\left(\frac{\xi^2}{f^4} + \frac{\eta^2}{g^4} + \frac{\zeta^2}{h^4}\right) = -\left(\frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2} - m^2\right);$$

or, writing for shortness

$$\frac{1}{\Omega^2} = \left(\frac{\xi^2}{f^4} + \frac{\eta^2}{g^4} + \frac{\zeta^2}{h^4} - m^2\right),$$

the equation is

$$\theta = -\Omega^2 C.$$

We thus see that, (ξ, η, ζ) being given, θ, and therefore also (a, b, c), are uniquely determined. It may be added that, writing $C = -\frac{\sigma^2}{\theta}$, we have $\theta^2 = \Omega^2\sigma^2$, or say $\Omega\sigma = \theta$.

10. We have, moreover,

$$\frac{a^2}{\theta + f^2} = \frac{\xi^2}{f^4}(\theta + f^2), \text{ \&c.},$$

and

$$\frac{\xi^2}{f^4}(\theta + f^2) + \frac{\eta^2}{g^4}(\theta + g^2) + \frac{\zeta^2}{h^4}(\theta + h^2) = \frac{1}{\Omega^2}\theta + \frac{\xi^2}{f^2} + \frac{\eta^2}{g^2} + \frac{\zeta^2}{h^2}$$
$$= -C + m^2 + C, = m^2;$$

whence

$$\frac{a^2}{\theta + f^2} + \frac{b^2}{\theta + g^2} + \frac{c^2}{\theta + h^2} = m^2,$$

or, regarding (a, b, c) as given, θ is determined as a function of (a, b, c) by this cubic equation; and θ being (in accordance with the foregoing equation $\theta = -\Omega^2 C$) assumed to be positive, we have θ the positive root of this equation, and $m^2(\theta + f^2)$, $m^2(\theta + g^2)$, $m^2(\theta + h^2)$ as the squared semiaxes of the confocal ellipsoid through the point P. And θ being known, ξ, η, ζ are, by the foregoing equations $a = \xi\left(1 + \frac{\theta}{f^2}\right)$, &c., determined in terms of ξ, η, ζ; that is, starting from the given external point P, we have the internal point Q. And it appears that PQ is the normal at P to the confocal ellipsoid, or, what is the same thing, the axis of the circumscribed cone, vertex P.

11. The foregoing equation

$$\frac{\xi^2}{f^4}(\theta + f^2) + \frac{\eta^2}{g^4}(\theta + g^2) + \frac{\zeta^2}{h^4}(\theta + h^2) = m^2,$$

considering a, b, c, and therefore θ, as given, shows further that the point Q is situate on an ellipsoid which is the inverse of the confocal ellipsoid $\frac{x^2}{\theta + f^2} + \frac{y^2}{\theta + g^2} + \frac{z^2}{\theta + h^2} = m^2$ in regard to the given ellipsoid $\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = m^2$.

12. Expressing Ω in terms of a, b, c, we have

$$\frac{1}{\Omega^2}=\frac{a^2}{(\theta+f^2)^2}+\frac{b^2}{(\theta+g^2)^2}+\frac{c^2}{(\theta+h^2)^2}.$$

We have $\sigma^2=\dfrac{\theta^2}{\Omega^2}$, $=C^2\Omega^2$, and

$$U=\alpha(\xi-a)+\beta(\eta-b)+\gamma(\zeta-c),$$

$$=-\theta\left(\frac{\alpha\xi}{f^2}+\frac{\beta\eta}{g^2}+\frac{\gamma\zeta}{h^2}\right),\quad =-\theta B,\quad =BC\Omega^2;$$

whence

$$\frac{\rho'^2}{r'^2}=\frac{\sigma^2}{r'^2}+2U\frac{1}{r'}+1,\quad =\frac{C^2\Omega^2}{r'^2}+\frac{2BC\Omega^2}{r'}+1,$$

$$=C\Omega^2\left(\frac{C}{r'^2}+\frac{2B}{r'}\right)+1;$$

or, since

$$A+2B\frac{1}{r'}+C\frac{1}{r'^2}=0,$$

this is

$$\frac{\rho'^2}{r'^2}=-AC\Omega^2+1=\Omega^2\left(\frac{1}{\Omega^2}-AC\right)$$

$$=\frac{\Omega^2}{R^2},\ \text{if}\ \frac{1}{R^2}=\frac{1}{\Omega^2}-AC.$$

This last equation may also be written

$$\frac{1}{R^2}=\frac{1}{\Omega^2}(\alpha^2+\beta^2+\gamma^2)-C\left(\frac{\alpha^2}{f^2}+\frac{\beta^2}{g^2}+\frac{\gamma^2}{h^2}\right);$$

or, what is the same thing,

$$\frac{1}{R^2}=\frac{\alpha^2}{F^2}+\frac{\beta^2}{G^2}+\frac{\gamma^2}{H^2};$$

if for shortness

$$\frac{1}{F^2}=\frac{1}{\Omega^2}-\frac{C}{f^2},$$

$$\frac{1}{G^2}=\frac{1}{\Omega^2}-\frac{C}{g^2},$$

$$\frac{1}{H^2}=\frac{1}{\Omega^2}-\frac{C}{h^2};$$

viz. substituting herein for C its value $-\dfrac{\theta}{\Omega^2}$, these equations give

$$F=\frac{\Omega f}{\sqrt{\theta+f^2}},\quad G=\frac{\Omega g}{\sqrt{\theta+g^2}},\quad H=\frac{\Omega h}{\sqrt{\theta+h^2}};$$

where Ω stands for its expression in terms of a, b, c.

13. The expression for $\frac{1}{R^2}$ shows that R is the radius vector, cosine-inclinations α, β, γ, in an ellipsoid semi-axes F, G, H, which may be regarded as having its centre at Q; viz. this is the "auxiliary ellipsoid." And this being so, we have

$$\frac{\rho'}{r'} = \frac{\rho''}{r''} = \frac{\Omega}{R}.$$

It appears from these equations that, drawing from Q parallel to PR'' a line QM, $=\Omega$, and from its extremity M parallel to PQ a line to meet QR' in T, the locus of T is the auxiliary ellipsoid.

14. By what precedes, the angles $R'PQ$, $R''PQ$ are equal to each other, say each is $=\phi$; the triangle $R'PR''$ gives

$$\cos 2\phi = \frac{\rho'^2 + \rho''^2 - (r' + r'')^2}{2\rho'\rho''},$$

that is,

$$\cos^2\phi = \frac{(\rho' + \rho'')^2 - (r' + r'')^2}{4\rho'\rho''};$$

viz. this is

$$= \left(\frac{\Omega^2}{R^2} - 1\right)(r' + r'')^2 \div 4\frac{\Omega^2}{R^2} r'r'',$$

$$= R^2\left(\frac{1}{R^2} - \frac{1}{\Omega^2}\right)\frac{(r' + r'')^2}{4r'r''}$$

$$= -ACR^2 . \frac{4(B^2 - AC)}{A^2} . \frac{-A}{4C}$$

$$= R^2(B^2 - AC);$$

or say

$$\cos\phi = R\sqrt{B^2 - AC};$$

a remarkable equation which may also be written

$$\cos\phi = \frac{R}{v^2} . \tfrac{1}{2}(r' + r''),$$

if, as before, v is the semi-diameter parallel to $R'R''$.

In virtue of the equation $\Lambda = \frac{m}{\sqrt{B^2 - AC}}$ which defines Λ, the equation becomes

$$\cos\phi = \frac{mR}{\Lambda};$$

and we thus complete the demonstration of the several geometrical theorems upon which the investigation was founded.

Analytical Expressions for the Attraction of the Shell, and for the Resolved Attractions.

15. The attraction of the shell was shown to be

$$= \frac{m\,dm}{\Omega^2} 4\pi FGH;$$

or, since the mass of the shell, the density being unity, is

$$\frac{4\pi}{3} fgh \,.\, 3m^2 dm = 4m^2\, dm\, \pi fgh,$$

we have

$$\text{Attraction} \div \text{Mass} = \frac{1}{m\Omega^2} \frac{FGH}{fgh};$$

which, by what precedes, is

$$= \frac{\Omega}{m\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

where

$$\frac{1}{\Omega^2} = \frac{a^2}{(f^2+\theta)^2} + \frac{b^2}{(g^2+\theta)^2} + \frac{c^2}{(h^2+\theta)^2},$$

θ being the positive root of

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \frac{c^2}{h^2+\theta} = m^2.$$

16. It is to be observed that the cosine-inclinations of the line PQ to the axes are

$$\frac{a\Omega}{f^2+\theta},\ \frac{b\Omega}{g^2+\theta},\ \frac{c\Omega}{h^2+\theta},$$

respectively; so that, considering, for instance, the attraction parallel to the axis of x, we have

$$\text{Resolved Attraction} \div \text{Mass} = \frac{a\Omega^2}{m(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}}.$$

Resolved Attractions of the Ellipsoid $\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = 1.$

17. We may find the attraction of the solid ellipsoid

$$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \frac{z^2}{h^2} = 1.$$

For this purpose, dividing it into shells, semi-axes mf, mg, mh, and $(m+dm)f$, $(m+dm)g$, $(m+dm)h$ respectively, we have for the shell in question

$$\text{Resolved Attraction} \div \text{Mass} = \frac{a\Omega^2}{m(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

and the mass of the shell is $\frac{4\pi}{3} fgh \,.\, 3m^2 dm$, where the first factor is the mass of the ellipsoid; whence

$$\text{Resolved Attraction} \div \text{Mass of Ellipsoid} = \frac{a \,.\, 3m\Omega^2\, dm}{(f^2+\theta)\sqrt{(f^2+\theta)(g^2+\theta)(h^2+\theta)}},$$

θ being here a function of m, and m extending from 0 to 1. But taking θ as the variable in place of m, the equation

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \frac{c^2}{h^2+\theta} = m^2$$

gives

$$-\frac{1}{\Omega^2} d\theta = 2m\, dm\,; \text{ that is, } 3m\Omega^2\, dm = -\tfrac{3}{2} d\theta.$$

Moreover $m=0$ gives $\theta=\infty$, and $m=1$ gives $\theta=$ its value as defined by the equation

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \frac{c^2}{h^2+\theta} = 1,$$

so that, reversing the sign, the limits are ∞, θ; or, finally, writing under the integral sign ϕ in place of θ, the formula is

$$\text{Resolved Attraction} \div \text{Mass of Ellipsoid} = \tfrac{3}{2} a \int_\theta^\infty \frac{d\phi}{(f^2+\phi)\sqrt{(f^2+\phi)(g^2+\phi)(h^2+\phi)}},$$

which is a known formula.

605.

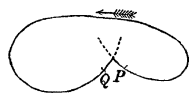
NOTE ON A POINT IN THE THEORY OF ATTRACTION.

[From the *Proceedings of the London Mathematical Society*, vol. VI. (1874—1875), pp. 79—81. Read February 11, 1875.]

CONSIDER a mass of matter distributed in any manner on a surface, and attracting points P, Q not on the surface. Consider a point Q accessible from P, viz. such that we can pass continuously from P to Q without passing through the surface. (It is hardly necessary to remark that, if for example the matter is distributed over a hemisphere or segment of a closed surface, then by the surface we mean the hemisphere or segment, not the whole closed surface.) The potential and its differential coefficients *ad infinitum*, in regard to the coordinates of the attracted point, all vary continuously as we pass from P to Q; and it follows that the potential is one and the same analytical function of (a, b, c), the coordinates of the attracted point, for the whole series of points accessible from the original point P; in particular, if the surface be an unclosed surface, for instance a hemisphere or segment of a sphere, then every point Q whatever not on the surface is accessible from P; and the theorem is that the potential is one and the same analytical function of (a, b, c), the coordinates of the attracted point, for any position whatever of this point (not being a point on the surface). But this seems to give rise to a difficulty. Consider the matter as uniformly distributed over a closed surface, and divide the closed surface into two segments: the potential of the whole shell is the sum of the potentials of the two segments; and the potential of the first segment being always one and the same function of (a, b, c), whatever may be the position of the attracted point, and similarly the potential of the second segment being always one and the same function of (a, b, c), whatever may be the position of the attracted point; then the potential of the whole shell is one and the same function of (a, b, c), whatever may be the position of the attracted point. This we know is not the case for a uniform spherical

shell; for the potential is a different function for external and interior points, viz. for internal points it is a constant, $= M \div$ radius; for external points it is $= \dfrac{M}{\sqrt{a^2+b^2+c^2}}$, if a, b, c are the coordinates measured from the centre of the sphere.

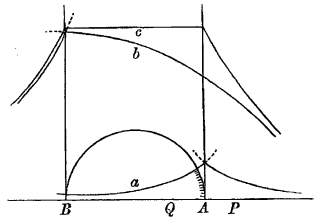
The difficulty is rather apparent than real. Reverting to the case of an unclosed surface or segment, and considering the continuous curve from P to Q, let this be completed by a curve from Q to P through the segment; viz. we thus have P, Q points on a closed curve or circuit meeting the segment in a single point. To fix the ideas, the circuit may be taken to be a plane curve, and the position of a point on the circuit may be determined by means of its distance s from a fixed point on the circuit. Considering this circuit as drawn on a cylinder, we may at each point of the circuit measure off, say upwards, along the generating line of the cylinder, a length or ordinate z, proportional to the potential of the point on the circuit, the extremities of these distances forming a curve on the cylinder, say the potential curve. We may draw a figure representing this curve only; the points P, Q being marked



as if they were points on the curve (viz. at the upper instead of the lower extremities of the corresponding ordinates z): the generating lines of the cylinder, and the plane section which is the circuit, not being shown in the figure. The potential curve is then, as shown in the figure, a continuous curve, viz. we pass from P to Q in the direction of the arrow, or along that part of the circuit which does not meet the segment, a curve without any abrupt change in the value of the ordinate z or of any of its differential coefficients, $\dfrac{dz}{ds}$, $\dfrac{d^2z}{ds^2}$, &c.; but there is, corresponding to the point where the circuit meets the surface, an abrupt change in the direction of the potential curve or value of the differential coefficient $\dfrac{dz}{ds}$, viz. the point on the curve is really a node, the two branches crossing at an angle, as shown by the dotted lines, but without any potentials corresponding to these dotted lines.

In the case of two segments forming a closed surface, or say two segments forming a complete spherical shell; then, if the points P, Q are one of them internal, the other external, the circuit, assuming it to meet the first segment in one point only, will meet the second segment in at least one point; the potential curves corresponding to the two segments respectively will have each of them, at the point corresponding to the intersection of the circuit with the segment, a node; and it hence appears how, in the potential curve corresponding to the whole shell (for which curve the ordinate z is the sum of the ordinates belonging to the two segments respectively), there will be a discontinuity of form corresponding to the passage from an exterior to an interior point.

This is best shown by the annexed figure, which represents a uniform spherical shell made up of two segments, one of which is taken to be a small segment or disc having the point A for its centre, the other the large segment B, which is the remainder of the shell; the circuit is taken to be the right line ..$PAQB$.. through



the centre of the sphere (viz. we may imagine the two extremities meeting at infinity, or we may, outside the sphere, bend the line so as to unite the two extremities, thus forming a closed curve). The curve (a) represents the potential curve for the segment A, the curve (b) that for the segment B, these two curves having, as shown by the dotted lines, nodes corresponding to the points A, B respectively (but these dotted portions not indicating any potentials); and then, drawing at each point the ordinate which is the sum of those for the curves (a), (b) respectively, we have the discontinuous curve (c), composed of a horizontal portion and two hyperbolic branches, which is the potential curve for the whole spherical shell.

Practically the figure is constructed by drawing the curves (c), (a), and from them deducing the curve (b). As regards the curve (a) it may be noticed that, treating the segment (a) as a plane disc, the curve (a) is made up of portions of two hyperbolas; viz. it breaks up into two curves, instead of being, as assumed in the discussion, a single curve; this is a mere accident, not affecting the theory; and, in fact, taking the segment to be what it really is, the segment of a sphere the potential curve does not thus break up.

606.

ON THE EXPRESSION OF THE COORDINATES OF A POINT OF A QUARTIC CURVE AS FUNCTIONS OF A PARAMETER.

[From the *Proceedings of the London Mathematical Society*, vol. VI. (1874—1875), pp. 81—83. Read February 11, 1875.]

THE present short Note is merely the development of a process of Prof. Sylvester's. It will be recollected that the general quartic curve has the deficiency 3 (or it is 4-cursal); the question is therefore that of the determination of the subrational* functions of a parameter which have to be considered in the theory of curves of the deficiency 3.

Taking the origin at a point of the curve, the equation is

$$(x,\ y)^4 + (x,\ y)^3 + (x,\ y)^2 + (x,\ y) = 0\,;$$

and writing herein $y = \lambda x$, the equation, after throwing out the factor x, becomes

$$(1,\ \lambda)^4 x^3 + (1,\ \lambda)^3 x^2 + (1,\ \lambda)^2 x + (1,\ \lambda) = 0\,;$$

or, say

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

where we write for shortness

$$a,\ b,\ c,\ d = (1,\ \lambda)^4,\ \tfrac{1}{3}(1,\ \lambda)^3,\ \tfrac{1}{3}(1,\ \lambda)^2,\ (1,\ \lambda)\,;$$

viz. a, b, c, d stand for functions of λ of the degrees 4, 3, 2, and 1 respectively.

The equation may be written

$$(ax+b)^3 - 3(b^2 - ac)(ax+b) + a^2 d - 3abc + 2b^3 = 0\,;$$

* The expression "subrational" includes irrational, but it is more extensive; if Y, X are rational functions, the same or different, of y, x respectively and Y is determined as a function of x by an equation of the form $Y = X$, then y is a subrational function of x. The notion is due to Prof. Sylvester.

viz. writing for a moment $ax+b=2\sqrt{b^2-ac}\,.\,u$, this is

$$4u^3-3u+\frac{a^2d-3abc+2b^3}{2\,(b^2-ac)\sqrt{b^2-ac}}=0.$$

Hence, assuming

$$-\cos\phi=\frac{a^2d-3abc+2b^3}{2\,(b^2-ac)\sqrt{b^2-ac}},$$

then we have $4u^3-3u-\cos\phi=0$; consequently u has the three values $\cos\frac{1}{3}\phi$, $\cos\frac{1}{3}(\phi+2\pi)$, $\cos\frac{1}{3}(\phi-2\pi)$, and we may regard $\cos\frac{1}{3}\phi$ as representing any one of these values.

We have thus $ax+b=2\sqrt{b^2-ac}\cos\frac{1}{3}\phi$, and $y=\lambda x$, giving x and y as functions of λ and ϕ, that is, of λ. But for their expression in this manner we introduce the irrationality $\sqrt{b^2-ac}$, which is of the form $\sqrt{(1,\ \lambda)^6}$, and the trisection or derivation of $\cos\frac{1}{3}\phi$ from a given value of $\cos\phi$; viz. we have, as above, $-\cos\phi$, a function of λ of the form

$$(1,\ \lambda)^9\div(1,\ \lambda)^6\sqrt{(1,\ \lambda)^6}.$$

The equation for ϕ may be expressed in the equivalent forms

$$\sin\phi=\frac{a\sqrt{-(a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2)}}{(b^2-ac)\sqrt{b^2-ac}},$$

$$-\tan\phi=\frac{a\sqrt{-(a^2d^2+4ac^3+4b^3d-6abcd-3b^2c^2)}}{a^2d-3abc+2b^3};$$

and inasmuch as we have

$$2\sqrt{b^2-ac}=-\frac{a^2d-3abc+2b^3}{(b^2-ac)\cos\phi},$$

we may, instead of

$$ax+b=2\sqrt{b^2-ac}\cos\tfrac{1}{3}\phi,$$

write

$$ax+b=-\frac{(a^2d-3abc+2b^3)\cos\frac{1}{3}\phi}{(b^2-ac)\cos\phi};$$

or, what is the same thing,

$$=\frac{-(a^2d-3abc+2b^3)}{(b^2-ac)\,(4\cos^2\frac{1}{3}\phi-3)}.$$

The formulæ may be simplified by introducing μ, a function of λ, determined by the equation

$$c\mu^2-2b\mu+a=0;$$

viz. this equation is

$$\tfrac{1}{3}(1,\ \lambda)^2\mu^2-\tfrac{2}{3}(1,\ \lambda)^3\mu+(1,\ \lambda)^4=0,$$

so that $(\lambda,\ \mu)$ may be regarded as coordinates of a point on a nodal quartic curve, or a quartic curve of the next inferior deficiency 2. And we then have

$$(c\mu-b)=\sqrt{b^2-ac},$$

and consequently

$$-\cos\phi = \frac{a^2d - 3abc + 2b^3}{2(c\mu - b)^3};$$

viz. $\cos\phi$ is given as a rational function of the coordinates (λ, μ); there is, as before, the trisection; and we then have

$$ax + b = 2(c\mu - b)\cos\tfrac{1}{3}\phi, \quad y = \lambda x,$$

giving x and y as functions of λ, μ, ϕ; that is, ultimately, as functions of λ. I have not succeeded in obtaining in a good geometrical form the relation between the point (x, y) on the given quartic and the point (λ, μ) on the nodal quartic.

Reverting to the expression of $\tan\phi$, it may be remarked that $a = 0$ gives the values of λ which correspond to the four points at infinity on the given quartic curve; $a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2 = 0$, the values corresponding to the ten tangents from the origin; and $a^3d - 3abc + 2b^3 = 0$, the values corresponding to the nine lines through the origin, which are each such that the origin is the centre of gravity of the other three points on the line.

I take the opportunity of mentioning a mechanical construction of the Cartesian. The equation $r' = -A\cos\theta - N$ represents a limaçon (which is derivable mechanically from the circle $r' = -A\cos\theta$), and if we effect the transformation $r' = r + \frac{B}{r}$, the new curve is $r + \frac{B}{r} + A\cos\theta + N = 0$; that is, $r^2 + r(A\cos\theta + N) + B = 0$, which is, in fact, the equation of a Cartesian. The assumed transformation $r' = r + \frac{B}{r}$ can be effected immediately by a Peaucellier cell.

607.

A MEMOIR ON PREPOTENTIALS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXV. Part II. (1875), pp. 675—774. Received April 8,—Read June 10, 1875.]

THE present Memoir relates to multiple integrals expressed in terms of the $(s+1)$ ultimately disappearing variables $(x, \ldots, z, w)$, and the same number of parameters $(a, \ldots, c, e)$; they are of the form

$$\int \frac{\rho\, d\varpi}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

where ρ and $d\varpi$ depend only on the variables $(x, \ldots, z, w)$. Such an integral, in regard to the index $\frac{1}{2}s+q$, is said to be "prepotential," and in the particular case $q=-\frac{1}{2}$ to be "potential."

I use throughout the language of hyper-tridimensional geometry: $(x, \ldots, z, w)$ and $(a, \ldots, c, e)$ are regarded as coordinates of points in $(s+1)$-dimensional space, the former of them determining the position of an element $\rho\, d\varpi$ of attracting matter, the latter being the attracted point; viz. we have a mass of matter $=\int \rho\, d\varpi$ distributed in such manner that, $d\varpi$ being the element of $(s+1)$- or lower-dimensional volume at the point $(x, \ldots, z, w)$, the corresponding density is ρ, a given function of $(x, \ldots, z, w)$, and that the element of mass $\rho\, d\varpi$ exerts on the attracted point $(a, \ldots, c, e)$ a force inversely proportional to the $(s+2q+1)$th power of the distance $\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}}$. The integration is extended so as to include the whole attracting mass $\int \rho\, d\varpi$; and the integral is then said to represent the Prepotential of the mass in regard to the point $(a, \ldots, c, e)$. In the particular case $s=2$, $q=-\frac{1}{2}$, the force is as the inverse square of the distance, and the integral represents the Potential in the ordinary sense of the word.

The element of volume $d\varpi$ is usually either the element of solid (spatial or $(s+1)$-dimensional) volume $dx \ldots dz\, dw$, or else the element of superficial (s-dimensional) volume dS. In particular, when the surface (s-dimensional locus) is the (s-dimensional)

plane $w=0$, the superficial element dS is $=dx\ldots dz$. The cases of a less-than-s-dimensional volume are in the present memoir considered only incidentally. It is scarcely necessary to remark that the notion of density is dependent on the dimensionality of the element of volume $d\varpi$: in passing from a spatial distribution, $\rho\, dx\ldots dz\, dw$, to a superficial distribution, $\rho\, dS$, we alter the signification of ρ. In fact, if, in order to connect the two, we imagine the spatial distribution as made over an indefinitely thin layer or stratum bounded by the surface, so that at any element dS of the surface the normal thickness is $d\nu$, where $d\nu$ is a function of the coordinates $(x, \ldots, z, w)$ of the element dS, the spatial element is $=d\nu\, dS$, and the element of mass $\rho\, dx\ldots dz\, dw$ is $=\rho\, d\nu\, dS$; and then changing the signification of ρ, so as to denote by it the product $\rho\, d\nu$, the expression for the element of mass becomes $\rho\, dS$, which is the formula in the case of the superficial distribution.

The space or surface over which the distribution extends may be spoken of as the material space or surface; so that the density ρ is not $=0$ for any finite portion of the material space or surface; and if the distribution be such that the density becomes $=0$ for any point or locus of the material space or surface, then such point or locus, considered as an infinitesimal portion of space or surface, may be excluded from and regarded as not belonging to the material space or surface. It is allowable, and frequently convenient, to regard ρ as a discontinuous function, having its proper value within the material space or surface, and having its value $=0$ beyond these limits; and this being so, the integrations may be regarded as extending as far as we please beyond the material space or surface (but so always as to include the whole of the material space or surface)—for instance, in the case of a spatial distribution, over the whole $(s+1)$-dimensional space; and in the case of a superficial distribution, over the whole of the s-dimensional surface of which the material surface is a part.

In all cases of surface-integrals it is, unless the contrary is expressly stated, assumed that the attracted point does not lie on the material surface; to make it do so is, in fact, a particular supposition. As to solid integrals, the cases where the attracted point is not, and is, in the material space may be regarded as cases of coordinate generality; or we may regard the latter one as the general case, deducing the former one from it by supposing the density at the attracted point to become $=0$.

The present memoir has chiefly reference to three principal cases, which I call A, C, D, and a special case, B, included both under A and C: viz. these are:—

A. The prepotential-plane case; q general, but the surface is here the plane $w=0$, so that the integral is

$$\int \frac{\rho\, dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}.$$

B. The potential-plane case; $q=-\frac{1}{2}$, and the surface the plane $w=0$, so that the integral is

$$\int \frac{\rho\, dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

C. The potential-surface case; $q=-\frac{1}{2}$, the surface arbitrary, so that the integral is

$$\int \frac{\rho\, dS}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

D. The potential-solid case; $q=-\frac{1}{2}$, and the integral is

$$\int \frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

It is, in fact, only the prepotential-plane case which is connected with the partial differential equation

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)V=0,$$

considered in Green's memoir* "On the Attractions of Ellipsoids" (1835), and called here "the prepotential equation." For this equation is satisfied by the function

$$\frac{1}{\{a^2+\ldots+c^2+e^2\}^{\frac{1}{2}s+q}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

and consequently by the integral

$$\int \frac{\rho\, dx \ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

that is, by the prepotential-plane integral; but the equation is *not* satisfied by the value

$$\frac{1}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

nor, therefore, by the prepotential-solid, or general superficial, integral.

But if $q=-\frac{1}{2}$, then, instead of the prepotential equation, we have "the potential equation"

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=0;$$

and this is satisfied by

$$\frac{1}{\{a^2+\ldots+c^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

and therefore also by

$$\frac{1}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

Hence it is satisfied by

$$\int \frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(D)},$$

* [*Green's Mathematical Papers*, pp. 185—222.]

the potential-solid integral, *provided that the point* $(a, \ldots, c, e)$ *does not lie within the material space:* I would rather say that the integral does *not* satisfy the equation, but of this more hereafter; and it is satisfied by

$$\int \frac{\rho\, dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{C}),$$

the potential-surface integral. The potential-plane integral (B), as a particular case of (C), of course also satisfies the equation.

Each of the four cases give rise to what may be called a distribution-theorem; viz. given V a function of $(a, \ldots, c, e)$ satisfying certain prescribed conditions, but otherwise arbitrary, then the form of the theorem is that there exists and that we can find an expression for ρ, the density or distribution of matter over the space or surface to which the theorem relates, such that the corresponding integral V has its given value: viz. in A and B there exists such a distribution over the plane $w=0$, in C such a distribution over a given surface, and in D such a distribution in space. The establishment, and exhibition in connexion with each other, of these four distribution-theorems is the principal object of the present memoir; but the memoir contains other investigations which have presented themselves to me in treating the question. It is to be noticed that the theorem A belongs to Green, being in fact the fundamental theorem of his memoir of 1835, already referred to. Theorem C, in the particular case of tridimensional space, belongs also to him, being given in his "Essay on the Application of Mathematical Analysis to the theories of Electricity and Magnetism" (Nottingham, 1828*), being partially rediscovered by Gauss** in the year 1840; and theorem D, in the same case of tridimensional space, to Lejeune-Dirichlet: see his memoir "Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque homogène ou hétérogène," *Crelle*, t. XXXII. pp. 80—84 (1840). I refer more particularly to these and other researches by Gauss, Jacobi, and others in an Annex to the present memoir.

On the Prepotential Surface-integral. Art. Nos. 1 to 18.

1. In what immediately follows we require

$$V = \int \frac{dx \ldots dz}{(x^2 + \ldots + z^2 + e^2)^{\frac{1}{2}s+q}},$$

limiting condition $x^2 + \ldots + z^2 = R^2$, the prepotential of a uniform (s-coordinal) circular disk†, radius R, in regard to a point $(0, \ldots, 0, e)$ on the axis; and in particular the

* [Also *Crelle*, t. XXXIX., pp. 73—89, t. XLIV., pp. 356—374, t. XLVII., pp. 161—221; *Green's Mathematical Papers*, pp. 1—115.]

** ["Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungskräfte," *Ges. Werke*, t. V., pp. 195—242.]

† It is to be throughout borne in mind that $x, \ldots, z$ denotes a set of s coordinates, $x, \ldots, z, w$ a set of $s+1$ coordinates; the adjective coordinal refers to the number of coordinates which enter into the equation; thus, $x^2 + \ldots + z^2 + w^2 = f^2$ is an $(s+1)$-coordinal sphere (observe that the surface of such a sphere is s-dimensional); $x^2 + \ldots + z^2 = f^2$, according as we tacitly associate with it the condition $w=0$, or w arbitrary, is an s-coordinal circle, or cylinder, the surface of such circle or cylinder being s-dimensional, but the circumference of the circle $(s-1)$-dimensional; or if we attend only to the s-dimensional space constituted by the plane $w=0$, the locus may be considered as an s-coordinal sphere, its surface being $(s-1)$-dimensional.

value is required in the case where the distance e (taken to be always positive) is indefinitely small in regard to the radius R.

Writing $x = r\xi, \ldots, z = r\zeta$, where the s new variables $\xi, \ldots, \zeta$ are such that $\xi^2 + \ldots + \zeta^2 = 1$, the integral becomes

$$\int \frac{r^{s-1}\,dr\,dS}{(r^2+e^2)^{\frac{1}{2}s+q}}, \quad = \int dS \int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

where dS is the element of surface of the s-dimensional unit-sphere $\xi^2 + \ldots + \zeta^2 = 1$; the integral $\int dS$ denotes the entire surface of this sphere, which (see Annex I.) is $= \dfrac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$. The other factor,

$$\int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

is the r-integral of Annex II.

2. We now consider the prepotential-surface integral

$$V = \int \frac{\rho\,dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}}.$$

As already mentioned, it is only a particular case of this, the prepotential-plane integral, which is specially discussed; but at present I consider the general case, for the purpose of establishing a theorem in relation thereto. The surface (s-dimensional surface) S is any given surface whatever.

Let the attracted point P be situate indefinitely near to the surface, on the normal thereto at a point N, say the normal distance NP is $= \varepsilon$*; and let this point N be taken at the centre of an indefinitely small circular (s-dimensional) disk or segment (of the surface), the radius of which R, although indefinitely small, is indefinitely large in comparison with the normal distance ε. I proceed to determine the prepotential of the disk; for this purpose, transforming to new axes, the origin being at N and the axes of $x, \ldots, z$ in the tangent-plane at N, then the coordinates of the attracted point P will be $(0, \ldots, 0, \varepsilon)$, and the expression for the prepotential of the disk will be

$$V = \int \frac{\rho\,dx \ldots dz}{\{x^2 + \ldots + z^2 + \varepsilon^2\}^{\frac{1}{2}s+q}},$$

where the limits are given by $x^2 + \ldots + z^2 < R^2$.

Suppose for a moment that the density at the point N is $= \rho'$, then the density throughout the disk may be taken $= \rho'$, and the integral becomes

$$V = \rho' \int \frac{dx \ldots dz}{\{x^2 + \ldots + z^2 + \varepsilon^2\}^{\frac{1}{2}s+q}},$$

where instead of ρ' I write ρ; viz. ρ now denotes the density at the point N. Making this change, then (by what precedes) the value is

$$= \rho\,\frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)} \int_0^R \frac{r^{s-1}\,dr}{\{r^2+\varepsilon^2\}^{\frac{1}{2}s+q}}.$$

* ε is positive; in afterwards writing $\varepsilon = 0$, we mean by 0 the limit of an indefinitely small positive quantity.

$q = \textit{Positive}$. Art. Nos. 3 to 7.

3. I consider first the case where q is positive. The value is here

$$= \rho \frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)} \frac{1}{2s^{2q}} \left\{ \frac{\Gamma\frac{1}{2}s\Gamma q}{\Gamma(\frac{1}{2}s+q)} - \int_0^{\frac{s^2}{R^2}} \frac{x^{q-1}\,dx}{(1+x)^{\frac{1}{2}s+q}} \right\};$$

or, since $\frac{s}{R}$ is indefinitely small, the x-integral may be neglected, and the value is

$$= \frac{1}{s^{2q}} \rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s+q)}.$$

Observe that this value is independent of R, and that the expression is thus the same as if (instead of the disk) we had taken the whole of the infinite tangent-plane, the density at every point thereof being $= \rho$. It is proper to remark that the neglected terms are of the orders

$$\frac{1}{s^{2q}} \left\{ \left(\frac{s}{R}\right)^{2q}, \left(\frac{s}{R}\right)^{2q+2}, \text{\&c.} \right\};$$

so that the complete value multiplied by s^{2q} is equal to the constant $\rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s+q)}$ + terms of the orders $\left(\frac{s}{R}\right)^{2q}$, $\left(\frac{s}{R}\right)^{2q+2}$, &c.

4. Let us now consider the prepotential of the remaining portion of the surface; every part thereof is at a distance from P exceeding, in fact far exceeding, R; so that imagining the whole mass $\int \rho\, dS$ to be collected at the distance R, the prepotential of the remaining portion of the surface is less than

$$\frac{\int \rho\, dS}{R^{s+2q}};$$

viz. we have thus, in the case where the mass $\int \rho\, dS$ is finite, a superior limit to the prepotential of the remaining portion of the surface. This will be indefinitely small in comparison with the prepotential of the disk, provided only s^{2q} is indefinitely small compared with R^{s+2q}, that is, s indefinitely small in comparison with $R^{1+\frac{s}{2q}}$. The proof assumes that the mass $\int \rho\, dS$ is finite; but considering the very rough manner in which the limit $\frac{\int \rho\, dS}{R^{s+2q}}$ was obtained, it can scarcely be doubted that, if not universally, at least for very general laws of distribution, even when $\int \rho\, dS$ is infinite, the same thing is true; viz. that by taking s sufficiently small in regard to R, we can make the

prepotential of the remaining portion of the surface vanish in comparison with that of the disk. But without entering into the question I assume that the prepotential of the remaining portion does thus vanish; the prepotential of the whole surface in regard to the indefinitely near point P is thus equal to the prepotential of the disk; viz. its value is

$$= \frac{1}{\mathscr{s}^{2q}} \rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

which, observe, is infinite for a point P on the surface.

5. Considering the prepotential V at an arbitrary point $(a, .., c, e)$ as a given function of $(a, .., c, e)$ the coordinates of this point, and taking $(x, .., z, w)$ for the coordinates of the point N, which is, in fact, an arbitrary point on the surface, then the value of V at the point P indefinitely near to N will be $= W$, if W denote the same function of $(x, .., z, w)$ that V is of $(a, .., c, e)$. The result just obtained is therefore

$$W = \frac{1}{\mathscr{s}^{2q}} \rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s+q)}, \ (\mathscr{s}=0),$$

or, what is the same thing,

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q} (\mathscr{s}^{2q} W)_{\mathscr{s}=0}.$$

As to this, remark that V is not an arbitrary function of $(a, .., c, e)$: *non constat* that there is any distribution of matter, and still less that there is any distribution of matter on the surface, which will produce at the point $(a, .., c, e)$, that is, at every point whatever, a prepotential the value of which shall be a function assumed at pleasure of the coordinates $(a, .., c, e)$. But suppose that V, the given function of $(a, .., c, e)$, is such that there does exist a corresponding distribution of matter on the surface, (viz. that V satisfies the conditions, whatever they are, required in order that this may be the case), then the foregoing formula determines the distribution, viz. it gives the expression of ρ, that is, the density at any point of the surface.

6. The theorem may be presented in a somewhat different form; regarding the prepotential as a function of the normal distance $\mathscr{s}$, its derived function in regard to $\mathscr{s}$ is

$$= -\frac{2q}{\mathscr{s}^{2q+1}} \rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

that is,

$$= -\frac{1}{\mathscr{s}^{2q+1}} \rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)};$$

and we thus have

$$\frac{dW}{d\mathscr{s}} = -\frac{1}{\mathscr{s}^{2q+1}} \rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}, \ (\mathscr{s}=0),$$

or, what is the same thing,

$$\rho = -\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s \Gamma(q+1)} \left(\mathscr{s}^{2q+1} \frac{dW}{d\mathscr{s}}\right)_{\mathscr{s}=0},$$

where, however, W being given as a function of $(x, \ldots, z, w)$, the notation $\frac{dW}{ds}$ requires explanation. Taking $\cos\alpha, \ldots, \cos\gamma$ to be the inclinations of the normal at N, in the direction NP in which the distance s is measured, to the positive parts of the axes of $(x, \ldots, z)$, viz. these cosines denote the values of

$$\frac{dS}{dx}, \ldots, \frac{dS}{dz},$$

each taken with the same sign + or −, and divided by the square root of the sum of the squares of the last-mentioned quantities, then the meaning is

$$\frac{dW}{ds} = \frac{dW}{dx}\cos\alpha + \ldots + \frac{dW}{dz}\cos\gamma.$$

7. The surface S may be the plane $w = 0$, viz. we have then the prepotential-plane integral

$$V = \int \frac{\rho\, dx \ldots dz}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s-q}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)},$$

where e (like s) is positive. In afterwards writing $e = 0$, we mean by 0 the limit of an indefinitely small positive quantity.

The foregoing distribution-formulæ then become

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\,\Gamma q}\left(e^{2q}\,W\right)_{e=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)},$$

and

$$\rho = -\frac{\Gamma(\frac{1}{2}s+q)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_{e=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A*)},$$

which will be used in the sequel.

It will be remembered that in the preceding investigation it has been assumed that q is positive, the limiting case $q = 0$ being excluded†.

$q = -\frac{1}{2}$. Art. Nos. 8 to 13.

8. I pass to the case $q = -\frac{1}{2}$, viz. we here have the potential-surface integral

$$V = \int \frac{\rho\, dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}:$$

it will be seen that the results present themselves under a remarkably different form.

The potential of the disk is, as before,

$$\rho\,\frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int \frac{r^{s-1}\,dr}{(r^2+s^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

† This is, as regards q, the case throughout; a limiting value, if not expressly stated to be included, is always excluded.

where ρ here denotes the density at the point N; and the value of the r-integral

$$= R\left(1 + \text{terms in } \frac{\varepsilon^2}{R^2}, \frac{\varepsilon^4}{R^4}, \ldots\right) - \varepsilon \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}.$$

Observe that this is indefinitely small, and remains so for a point P on the surface; the potential of the remaining portion of the surface (for a point P near to or on the surface) is finite, that is, neither indefinitely large nor indefinitely small, and it varies continuously as the attracted point passes through the disk (or aperture in the material surface now under consideration); hence the potential of the whole surface is finite for an attracted point P on the surface, and it varies continuously as P passes through the surface.

It will be noticed that there is in this case a term in V independent of ε; and it is on this account necessary, instead of the potential, to consider its derived function in regard to ε; viz. neglecting the indefinitely small terms which contain powers of $\frac{\varepsilon}{R}$, I write

$$\frac{dV}{d\varepsilon} = -\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho.$$

The corresponding term arising from the potential of the other portion of the surface, viz. the derived function of the potential in regard to ε, is not indefinitely small; and calling it Q, the formula for the whole surface becomes

$$\frac{dV}{d\varepsilon} = Q - \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho.$$

9. I consider positions of the point P on the two opposite sides of the point N, say at the normal distances ε', ε'', these being positive distances measured in opposite directions from the point N. The function V, which represents the potential of the surface in regard to the point P, is or may be a different function of the coordinates $(a, \ldots, c, e)$ of the point P, according as the point is situate on the one side or the other of the surface (as to this more presently). I represent it in the one case by V', and in the other case by V''; and in further explanation state that ε' is measured *into* the space to which V' refers, ε'' *into* that to which V'' refers; and I say that the formulæ belonging to the two positions of the point P are

$$\frac{dW'}{d\varepsilon'} = Q' - \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho,$$

$$\frac{dW''}{d\varepsilon''} = Q'' - \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho,$$

where, instead of V', V'', I have written W', W'', to denote that the coordinates, as well of P' as of P'', are taken to be the values $(x, \ldots, z, w)$ which belong to the point N. The symbols denote

$$\frac{dW'}{d\varepsilon'} = \frac{dW'}{dx}\cos\alpha' + \ldots + \frac{dW'}{dz}\cos\gamma',$$

$$\frac{dW''}{d\varepsilon''} = \frac{dW''}{dx}\cos\alpha'' + \ldots + \frac{dW''}{dz}\cos\gamma'',$$

where $(\cos\alpha', \ldots, \cos\gamma')$ and $(\cos\alpha'', \ldots, \cos\gamma'')$ are the cosine-inclinations of the normal distances s', s'' to the positive parts of the axes of $(x, \ldots, z)$; since these distances are measured in opposite directions, we have $\cos\alpha'' = -\cos\alpha', \ldots, \cos\gamma'' = -\cos\gamma'$. If we imagine a curve through N cutting the surface at right angles, or, what is the same thing, an element of the curve coinciding in direction with the normal element $P'NP''$, and if s denote the distance of N from a fixed point of the curve, and for the point P' if s become $s + \delta's$, while for the point P'' it becomes $s - \delta''s$, or, what is the same thing, if s increase in the direction of NP' and decrease in that of NP'', then if any function Θ of the coordinates $(x, \ldots, z, w)$ of N be regarded as a function of s, we have

$$\frac{d\Theta}{ds} = \frac{d\Theta}{ds'}, \quad \frac{d\Theta}{ds} = -\frac{d\Theta}{ds''}.$$

10. In particular, let Θ denote the potential of the remaining portion of the surface, that is, of the whole surface exclusive of the disk; the curve last spoken of is a curve which does not pass through the material surface, viz. the portion to which Θ has reference: and there is no discontinuity in the value of Θ as we pass along this curve through the point N. We have $Q' =$ value of $\frac{d\Theta}{ds'}$ at the point P', and $Q'' =$ value of $\frac{d\Theta}{ds''}$ at the point P''; and the two points P', P'' coming to coincide together at the point N, we have then

$$Q' = \frac{d\Theta}{ds'}, \quad = \frac{d\Theta}{ds},$$

$$Q'' = \frac{d\Theta}{ds''}, \quad = -\frac{d\Theta}{ds}.$$

We have in like manner $\frac{dW'}{ds'} = \frac{dW'}{ds}$, $\frac{dW''}{ds''} = -\frac{dW''}{ds}$; and the equation obtained above may be written

$$\frac{dW'}{ds} = \frac{d\Theta}{ds} - \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho,$$

$$\frac{dW''}{ds} = \frac{d\Theta}{ds} + \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\rho;$$

in which form they show that as the attracted point passes through the surface from the position P' on the one side to P'' on the other, there is an abrupt change in the value of $\frac{dW}{ds}$, or say of $\frac{dV}{ds}$, the first derived function of the potential in regard to the orthotomic arc s, that is, in the rate of increase of V in the passage of the attracted point normally to the surface. It is obvious that, if the attracted point traverses the surface obliquely instead of normally, viz. if the arc s cuts the surface obliquely, there is the like abrupt change in the value of $\frac{dV}{ds}$.

Reverting to the original form of the two equations, and attending to the relation $Q'+Q''=0$, we obtain

$$\frac{dW'}{ds'}+\frac{dW''}{ds''}=\frac{-4\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\,\rho,$$

or, what is the same thing,

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4\,(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW'}{ds'}+\frac{dW''}{ds''}\right)\text{ .....(C).}$$

11. I recall the signification of the symbols:—V', V'' are the potentials, it may be different functions of the coordinates $(a, \ldots, c, e)$ of the attracted point, for positions of this point on the two sides of the surface (as to this more presently): and W', W'' are what V', V'' respectively become when the coordinates $(a, \ldots, c, e)$ are replaced by $(x, \ldots, z, w)$, the coordinates of a point N on the surface. The explanation of the symbols $\frac{dW'}{ds'}$, $\frac{dW''}{ds''}$ is given a little above; ρ denotes the density at the point $(x, \ldots, z, w)$.

12. The like remarks arise as with regard to the former distribution theorem (A); the functions V', V'' cannot be assumed at pleasure; *non constat* that there is any distribution in space, and still less any distribution on the surface, which would give such values to the potential of a point $(a, \ldots, c, e)$ on the two sides of the surface respectively; but assuming that the functions V', V'' are such that they do arise from a distribution on the surface, or say that they satisfy all the conditions, whatever they are, required in order that this may be so, then the formula determines the distribution, viz. it gives the value of ρ, the density at a point $(x, \ldots, z, w)$ of the surface.

13. In the case where the surface is the plane $w=0$, viz. in the case of the potential-plane integral,

$$V=\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s-\frac{1}{2}}}\text{ .....(B),}$$

(e assumed to be positive); then, since the conformation is symmetrical on the two sides of the plane, V' and V'' are the same functions of $(a, \ldots, c, e)$, say they are each $=V$; W', W'' are each of them the same function, say they are each $=W$, of $(x, \ldots, z, e)$ that V is of $(a, \ldots, c, e)$; the distribution-formula becomes

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{2\,(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW}{de}\right)_{e=0}\text{ ..... (B),}$$

viz. this is also what one of the prepotential-plane formulæ becomes on writing therein $q=-\frac{1}{2}$.

$q=0$, *or Negative.* Art. Nos. 14 to 18.

14. Consider the case $q=0$. The prepotential of the disk is

$$\rho\,\frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\,(\log R+N-\log s\ldots);$$

to get rid of the constant term we must consider the derived function in regard to $\varkappa$, viz. this is

$$= -\rho \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} \cdot \frac{1}{\varkappa},$$

and we have thus for the whole surface

$$\frac{dV}{d\varkappa} = Q - \rho \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} \frac{1}{\varkappa},$$

where Q, which relates to the remaining portion of the surface, is finite; we have thence, writing, as before, W in place of V,

$$\varkappa \frac{dW}{d\varkappa} = -\rho \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s},$$

or say

$$\rho = -\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s} \left(\varkappa \frac{dW}{d\varkappa}\right)_{\varkappa=0}.$$

15. Consider the case q negative, but $-q < \frac{1}{2}$. The prepotential of the disk is here

$$= \rho \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} \left\{\frac{R^{-2q}}{-2q} + \frac{1}{2}\varkappa^{-2q} \frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)} + \ldots\right\};$$

to get rid of the first term we must consider the derived function in regard to $\varkappa$, viz. this is

$$-\varkappa^{-2q-1} \rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)};$$

whence, for the potential of the whole surface,

$$\frac{dV}{d\varkappa} = Q - \varkappa^{-2q-1} \rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)},$$

where Q, the part relating to the remaining portion of the surface, is finite. Multiplying by $\varkappa^{2q+1}$ (where the index $2q+1$ is positive), the term in Q disappears; and writing, as before, W in place of V, this is

$$\varkappa^{2q+1} \frac{dW}{d\varkappa} = -\rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma\frac{1}{2}s+q},$$

or, say

$$\rho = -\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s \Gamma(q+1)} \left(\varkappa^{2q+1} \frac{dW}{d\varkappa}\right)_{\varkappa=0};$$

viz. we thus see that the formula (A*) originally obtained for the case q positive extends to the case $q=0$, and $q=-$ but $-q<\frac{1}{2}$; but, as already seen, it does not extend to the limiting case $q=-\frac{1}{2}$.

16. If q be negative and between $-\frac{1}{2}$ and -1, we have in like manner a formula

$$\frac{dV}{d\varkappa} = Q - \rho \frac{2(\Gamma\frac{1}{2})^s \Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)} \varkappa^{-2q-1};$$

but here, $2q+1$ being negative, the term $s^{2q+1}Q$ does not disappear: the formula has to be treated in the same way as for $q=-\frac{1}{2}$, and we arrive at

$$\left\{s'^{2q+1}\frac{dW'}{ds'}+s''^{2q+1}\frac{dW''}{ds''}\right\}=-\frac{4(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\rho;$$

viz. the formula is of the same form as for the potential case $q=-\frac{1}{2}$. Observe that the formula does not hold good in the limiting case $q=-1$.

17. We have, in fact, for $q=-1$, the potential of the disk

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\rho\left\{\frac{R^2}{2}-s^2\log s\frac{\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s-1)}\right\};$$

whence

$$\frac{dV}{ds}=Q-\frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s-1)}\rho(2s\log s),$$

since, in the complete differential coefficient $s+2s\log s$, the term s vanishes in comparison with $2s\log s$. Then, proceeding as before, we find

$$\frac{1}{s'\log s'}\frac{dW'}{ds'}+\frac{1}{s''\log s''}\frac{dW''}{ds''}=\frac{-8(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s-1)}\rho;$$

but I have not particularly examined this formula.

18. If q be negative and >-1 (that is, $-q>1$), then the prepotential for the disk is

$$=\rho\frac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\left\{\frac{R^{-2q}}{-2q}+\frac{\frac{1}{2}s+q}{1}\cdot\frac{R^{-2q-2}}{-2q-2}\cdot s^2+\ldots+Ks^{-2q}\right\};$$

and it would seem that, in order to obtain a result, it would be necessary to proceed to a derived function higher than the first; but I have not examined the case.

Continuity of the Prepotential-surface Integral. Art. Nos. 19 to 25.

19. I again consider the prepotential-surface integral

$$\int\frac{\rho\,dS}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-q}}$$

in regard to a point $(a,\ldots,c,e)$ not on the surface; q is either positive or negative, as afterwards mentioned.

The integral or prepotential and all its derived functions, first, second, &c. *ad infinitum*, in regard to each or all or any of the coordinates $(a,\ldots,c,e)$, are all finite. This is certainly the case when the mass $\int\rho\,dS$ is finite, and possibly in other cases also; but to fix the ideas we may assume that the mass is finite. And the prepotential and its derived functions vary continuously with the position of the attracted

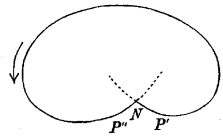
point $(a, \ldots, c, e)$, so long as this point in its course does not traverse the material surface. For greater clearness we may consider the point as moving along a continuous curve (one-dimensional locus), which curve, or the part of it under consideration, does not meet the surface; and the meaning is that the prepotential and each of its derived functions vary continuously as the point $(a, \ldots, c, e)$ passes continuously along the curve.

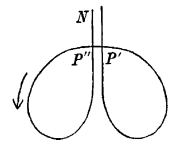
20. Consider a "region," that is, a portion of space any point of which can be, by a continuous curve not meeting the material surface, connected with any other point of the region. It is a legitimate inference, from what just precedes, that the prepotential is, for any point $(a, \ldots, c, e)$ whatever within the region, one and the same function of the coordinates $(a, \ldots, c, e)$, viz. the theorem, rightly understood, is true; but the theorem gives rise to a difficulty, and needs explanation.

Consider, for instance, a closed surface made up of two segments, the attracting matter being distributed in any manner over the whole surface (as a particular case $s+1=3$, a uniform spherical shell made up of two hemispheres); then, as regards the first segment (now taken as the material surface), there is no division into regions, but the whole of the $(s+1)$-dimensional space is one region; wherefore the prepotential of the first segment is one and the same function of the coordinates $(a, \ldots, c, e)$ of the attracted point for any position whatever of this point. But in like manner the prepotential of the second segment is one and the same function of the coordinates $(a, \ldots, c, e)$ for any position whatever of the attracted point. And the prepotential of the whole surface, being the sum of the prepotentials of the two segments, is consequently one and the same function of the coordinates $(a, \ldots, c, e)$ of the attracted point for any position whatever of this point; viz. it is the same function for a point in the region inside the closed surface and for a point in the outside region. That this is not in general the case we know from the particular case, $s+1=3$, of a uniform spherical shell referred to above.

21. Consider in general an unclosed surface or segment, with matter distributed over it in any manner; and imagine a closed curve or circuit cutting the segment once; and let the attracted point $(a, \ldots, c, e)$ move continuously along the circuit. We may consider the circuit as corresponding to (in ordinary tridimensional space) a plane curve of equal periphery, the corresponding points on the circuit and the plane curve being points at equal distances s along the curves from fixed points on the two curves respectively; and then treating the plane curve as the base of a cylinder, we may represent the potential as a length or ordinate, $V=y$, measured upwards from the point on the plane curve along the generating line of the cylinder, in such wise that the upper extremity of the length or ordinate y traces out on the cylinder a curve, say the prepotential curve, which represents the march of the prepotential. The attracted point may, for greater convenience, be represented as a point *on* the prepotential curve, viz. by the upper instead of the lower extremity of the length or ordinate y; and the ordinate, or height of this point above the base of the cylinder, then represents the value of the prepotential. The before-mentioned continuity-theorem is that the prepotential curve, corresponding to any portion (of the circuit) which

does not meet the material surface, is a continuous curve: viz. that there is no abrupt change of value either in the ordinate y $(= V)$ of the prepotential curve, or in the first or any other of the derived functions $\frac{dy}{ds}$, $\frac{d^2y}{ds^2}$, &c. We have thus (in each of the two figures) a continuous curve as we pass (in the direction of the arrow) from





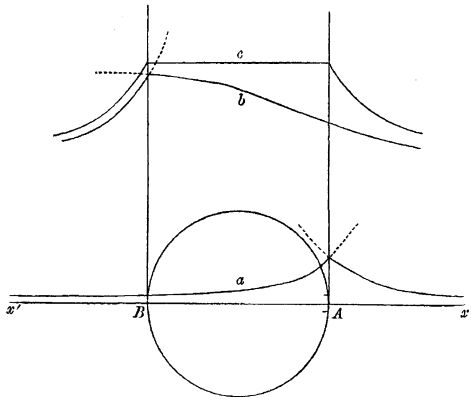
a point P' on one side of the segment to a point P'' on the other side of the segment; but this continuity does not exist in regard to the remaining part, from P'' to P', of the prepotential curve corresponding to the portion (of the circuit) which traverses the material surface.

22. I consider first the case $q = -\frac{1}{2}$ (see the left-hand figure): the prepotential is here a potential. At the point N, which corresponds to the passage through the material surface, then, as was seen, the ordinate y (= the Potential V) remains finite and continuous; but there is an abrupt change in the value of $\frac{dy}{ds}$, that is, in the direction of the curve: the point N is really a node with two branches crossing at this point, as shown in the figure; but the dotted continuations have only an analytical existence, and do not represent values of the potential. And by means of this branch-to-branch discontinuity at the point N, we escape from the foregoing conclusion as to the continuity of the potential on the passage of the attracted point through a closed surface.

23. To show how this is, I will for greater clearness examine the case $(s+1) = 3$, in ordinary tridimensional space, of the uniform spherical shell attracting according to the inverse square of the distance; instead of dividing the shell into hemispheres, I divide it by a plane into any two segments (see the figure, wherein A, B represent the centres of the two segments respectively, and where for graphical convenience the segment A is taken to be small).

We may consider the attracted point as moving along the axis xx', viz. the two extremities may be regarded as meeting at infinity, or we may outside the sphere bend the line round, so as to produce a closed circuit. We are only concerned with what happens at the intersections with the spherical surface. The ordinates represent the potentials, viz. the curves are a, b, c for the segments A, B, and the whole spherical surface respectively. Practically, we construct the curves c, a, and deduce the curve b by taking for its ordinate the difference of the other two ordinates. The curve c is, as we know, a discontinuous curve, composed of a horizontal line and two hyperbolic branches; the curve a can be laid down approximately by treating the segment A as a plane circular disk; it is of the form shown in the figure, having a node at the point corresponding to A. (In the case where the segment A is

actually a plane disk, the curve is made up of portions of branches of two hyperbolas; but taking the segment A as being what it is, the segment of a spherical surface, the curve is a single curve, having a node as mentioned above.) And from the



curves c and a, deducing the curve b, we see that this is a curve without any discontinuity corresponding to the passage of the attracted point through A (but with an abrupt change of direction or node corresponding to the passage through B). And conversely, using the curves a, b to determine the curve c, we see how, on the passage of the attracted point at A into the interior of the sphere, in consequence of the branch-to-branch discontinuity of the curve a, the curve c, obtained by combination of the two curves, undergoes a change of law, passing abruptly from a hyperbolic to a rectilinear form, and how similarly on the passage of the attracted point at B from the interior to the exterior of the sphere, in consequence of the branch-to-branch discontinuity of the curve b, the curve c again undergoes a change of law, abruptly reverting to the hyperbolic form.

24. In the case q positive, the prepotential curve is as shown by the right-hand figure on p. 332, viz. the ordinate is here infinite at the point N corresponding to the passage through the surface; the value of the derived function changes between + infinity and − infinity; and there is thus a discontinuity of value in the derived function. It would seem that, when q is fractional, this occasions a change of law on passage through the surface: but that there is no change of law when q is integral.

In illustration, consider the closed surface as made up of an infinitesimal circular disk, as before, and of a residual portion; the potential of the disk at an indefinitely near point is found as before, and the prepotential of the whole surface is

$$= \frac{1}{s^{2q}} \rho \frac{(\Gamma\frac{1}{2})^s \Gamma q}{\Gamma(\frac{1}{2}s + q)} + V_1,$$

where V_1, the prepotential of the remaining portion of the surface, is a function which varies (and its derived functions vary) continuously as the attracted point traverses the disk. To fix the ideas, we may take the origin at the centre of the disk, and the axis of e as coinciding with the normal, so that s, which is always positive, is $= \pm e$; the expression for the prepotential at a point $(a, \ldots, c, e)$ on the normal through the centre of the disk is

$$= \frac{1}{(\pm e)^{2q}} \cdot \rho \frac{(\Gamma\frac{1}{2})^s \cdot \Gamma q}{\Gamma(\frac{1}{2}s + q)} + V_1;$$

viz. when q is fractional there is the discontinuity of law, inasmuch as the term changes from $\frac{1}{(+e)^{2q}}$ to $\frac{1}{(-e)^{2q}}$: but when q is integral this discontinuity disappears. The like considerations, using of course the proper formula for the attraction of the disk, would apply to the case $q = 0$ or negative.

25. Or again, we might use the formulæ which belong to the case of a uniform $(s+1)$-coordinal spherical shell (see Annex No. III.), viz. we decompose the surface as follows,

$$\text{surface} = \text{disk} + \text{residue of surface};$$

and then, considering a spherical shell touching the surface at the point in question (so that the disk is, in fact, an element common to the surface and the spherical shell), and being of a uniform density equal to that of the disk, we have

$$\text{disk} = \text{spherical shell} - \text{residue of spherical shell};$$

and consequently

$$\text{surface} = \text{spherical shell} - \text{residue of spherical shell} + \text{residue of surface};$$

and then, considering the attracted point as passing through the disk, it does not pass through either of the two residues, and there is not any discontinuity, as regards the prepotentials of these residues respectively; there is consequently, as regards the prepotential of the surface, the same discontinuity that there is as regards the prepotential of the spherical shell. But I do not further consider the question from this point of view.

The Potential Solid Integral. Art. No. 26.

26. We have further to consider the prepotential (and in particular the potential) of a material space; to fix the ideas, consider for the moment the case of a distribution over the space included within a closed surface, the exterior density being zero, and the interior density being, supposed for the moment, constant; we consider the discontinuity which takes place as the attracting point passes from the exterior space through the bounding surface into the interior material space. We may imagine the interior space divided into indefinitely thin shells by a series of closed surfaces similar, if we please, to the bounding surface; and we may conceive the matter included between any two consecutive surfaces as concentrated on the exterior of the

two surfaces, so as to give rise to a series of consecutive material surfaces; the quantity of such matter is infinitesimal, and the density of each of the material surfaces is therefore also infinitesimal. As the attracted point comes from the external space to pass through the first of the material surfaces—suppose, to fix the ideas, it moves continuously along a curve the arc of which measured from a fixed point is $=s$—there is in the value of V (or, as the case may be, in the values of its derived functions $\frac{dV}{ds}$, &c.) the discontinuity due to the passage through the material surface; and the like as the attracted point passes through the different material surfaces respectively. Take the case of a potential, $q=-\frac{1}{2}$; then, if the surface-density were finite, there would be no finite change in the value of V, but there would be a finite change in the value of $\frac{dV}{ds}$; as it is, the changes are to be multiplied by the infinitesimal density, say ρ, of the material surface; there is consequently no finite change in the value of the first derived function; but there is, or may be, a finite change in the value of $\frac{d^2V}{ds^2}$ and the higher derived functions. But there is in V an infinitesimal change corresponding to the passage through the successive material surfaces respectively; that is, as the attracted point enters into the material space, there is a change in the law of V considered as a function of the coordinates $(a, \ldots, c, e)$ of the attracted point; but by what precedes this change of law takes place without any abrupt change of value either of V or of its first derived function; which derived function may be considered as representing the derived function in regard to any one of the coordinates $a, \ldots, c, e$. The suppositions, that the density outside the bounding surface was zero and inside it constant, were made for simplicity only, and were not essential; it is enough if the density, changing abruptly at the bounding surface, varies continuously in the material space within the bounding surface*. The conclusion is that V', V'' being the values at points within and without the bounding surface, V' and V'' are in general different functions of the coordinates $(a, \ldots, c, e)$ of the attracted point; but that *at* the surface we have not only $V'=V''$, but that the first derived functions are also equal, viz. that we have

$$\frac{dV'}{da}=\frac{dV''}{da}, \ldots, \quad \frac{dV'}{dc}=\frac{dV''}{dc}, \quad \frac{dV'}{de}=\frac{dV''}{de}.$$

27. In the general case of a Potential, we have

$$V=\int\frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

If ρ does not vanish at the attracted point $(a, \ldots, c, e)$, but has there a value ρ' different from zero, we may consider the attracting $(s+1)$-dimensional mass as made

* It is, indeed, enough if the density varies continuously within the bounding surface in the neighbourhood of the point of passage through the surface; but the condition may without loss of generality be stated as in the text, it being understood that for each abrupt change of density within the bounding surface we must consider the attracted point as passing through a new bounding surface, and have regard to the resulting discontinuity.

up of an indefinitely small sphere, radius ϵ and density ρ', which includes within it the attracted point, and of a remaining portion external to the attracted point. Writing ∇ to denote $\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}$, then, as regards the potential of the sphere, we have $\nabla V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho'$ (see Annex III. No. 67), and as regards the remaining portion $\nabla V=0$; hence, as regards the whole attracting mass, ∇V has the first-mentioned value, that is, we have

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}\rho',$$

where ρ' is the same function of the coordinates $(a,\ldots,c,e)$ that ρ is of $(x,\ldots,z,w)$; viz. the potential of an attracting mass distributed not on a surface, but over a portion of space, *does not satisfy the potential equation*

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V=0,$$

but it satisfies the foregoing equation, which only agrees with the potential equation in regard to a point $(a,\ldots,c,e)$ outside the material space, and for which, therefore, ρ' is $=0$.

The equation may be written

$$\rho'=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V;$$

or, considering V as a given function of $(a,\ldots,c,e)$, in general a discontinuous function but subject to certain conditions as afterwards mentioned, and taking W the same function of $(x,\ldots,z,w)$ that V is of $(a,\ldots,c,e)$, then we have

$$\rho=-\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{d^2}{dx^2}+\ldots+\frac{d^2}{dz^2}+\frac{d^2}{dw^2}\right)W \quad\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{D}),$$

viz. this equation determines ρ as a function, in general a discontinuous function, of $(x,\ldots,z,w)$ such that the corresponding integral

$$V=\int\frac{\rho\,dx\ldots dz\,dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+\theta}}$$

may be the given function of the coordinates $(a,\ldots,c,e)$. The equation is, in fact, the distribution-theorem D.

28. It is to be observed that the given function of $(a,\ldots,c,e)$ must satisfy certain conditions as to value at infinity and continuity, but it is not (as in the distribution-theorems A, B, and C it is) required to satisfy a partial differential equation; the function, except as regards the conditions as to value at infinity and continuity, is absolutely arbitrary.

The potential (assuming that the matter which gives rise to it lies wholly within a finite closed surface) must vanish for points at an infinite distance: or, more accurately, it must for indefinitely large values of $a^2 + \ldots + c^2 + e^2$ be of the form, Constant $\div (a^2 + \ldots + c^2 + e^2)^{\frac{1}{2}s-\frac{1}{2}}$. It may be a discontinuous function; for instance, outside a given closed surface it may be one function, and inside the same surface a different function of the coordinates $(a, \ldots, c, e)$; viz. this may happen in consequence of an abrupt change of the density of the attracting matter on the one and the other side of the given closed surface, but not in any other manner; and, happening in this manner, then V' and V'' being the values for points within and without the surface respectively, it has been seen to be necessary that, at the surface, not only $V' = V''$, but also $\frac{dV'}{da} = \frac{dV''}{da}, \ldots, \frac{dV'}{dc} = \frac{dV''}{dc}, \frac{dV'}{de} = \frac{dV''}{de}$. Subject to these conditions as to value at infinity and continuity, V may be any function whatever of the coordinates $(a, \ldots, c, e)$; and then taking W, the same function of $(x, \ldots, z, w)$, the foregoing equation determines ρ, viz. determines it to be $= 0$ for those parts of space which do not belong to the material space, and to have its proper value as a function of $(x, \ldots, z, w)$ for the remaining or material space.

The Prepotential-Plane Theorem A. Art. Nos. 29 to 36.

29. We have seen that, if there exists on the plane $w = 0$ a distribution of matter producing at the point $(a, \ldots, c, e)$ a given prepotential V—viz. V is to be regarded as a given function of $(a, \ldots, c, e)$—, then the distribution or density ρ is given by a determinate formula; but it was remarked that the prepotential V cannot be a function assumed at pleasure: it must be a function satisfying certain conditions. One of these is the condition of continuity; the function V and all its derived functions must vary continuously as we pass, without traversing the material plane, from any given point to any other given point. But it is sufficient to attend to points on one side of the plane, say the upperside, or that for which e is positive; and since any such point is accessible from any other such point by a path which does not meet the plane, it is sufficient to say that the function V must vary continuously for a passage by such path from any such point to any such point; the function V must therefore be one and the same function (and that a continuous one in value) for all values of the coordinates $(a, \ldots, c)$ and positive values of the coordinate e.

If, moreover, we assume that the distribution which corresponds to the given potential V is a distribution of a finite mass $\int \rho\, dx \ldots dz$ over a finite portion of the plane $w = 0$, viz. over a portion or area such that the distance of a point within the area from a fixed point, or say from the origin $(a, \ldots, c) = (0, \ldots, 0)$, is always finite; this being so, we have the further condition that the prepotential V must, for indefinitely large values of all or any of the coordinates $(a, \ldots, c, e)$, reduce itself to the form

$$\left(\int \rho\, dx \ldots dz\right) \div (a^2 + \ldots + c^2 + e^2)^{\frac{1}{2}s+q}.$$

The assumptions upon which this last condition is obtained are perhaps unnecessary; instead of the condition in the foregoing form we, in fact, use only the condition that the prepotential vanishes for a point at infinity, that is, when all or any one or more of the coordinates $(a, \ldots, c, e)$ are or is infinite.

Again, as we have seen, the prepotential V must satisfy the prepotential equation

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)V=0.$$

These conditions satisfied, to the given prepotential V there corresponds, on the plane $w=0$, a distribution given by the foregoing formula; it will be a distribution over a finite portion of the plane, as already mentioned.

30. The proof depends upon properties of the prepotential equation

$$\left(\frac{d^2}{dx^2}+\ldots+\frac{d^2}{dz^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)W=0,$$

or, what is the same thing,

$$\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right)+\ldots+\frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right)+\frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right)=0,$$

say, for shortness, $\square W=0$.

Consider, in general, the integral

$$\int dx\ldots dz\,de\,.\,e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2+\ldots+\left(\frac{dW}{dz}\right)^2+\left(\frac{dW}{de}\right)^2\right\},$$

taken over a closed surface S lying altogether on the positive side of the plane $e=0$, the function W being in the first instance arbitrary.

Writing the integral under the form

$$\int dx\ldots dz\,de\left(e^{2q+1}\frac{dW}{dx}\cdot\frac{dW}{dx}+\ldots+e^{2q+1}\frac{dW}{dz}\cdot\frac{dW}{dz}+e^{2q+1}\frac{dW}{de}\cdot\frac{dW}{de}\right),$$

we reduce the several terms by an integration by parts as follows:—

The term in $\frac{dW}{dx}$ is $=\int dy\ldots dz\,de\,We^{2q+1}\frac{dW}{dx}-\int dx\ldots dz\,de\,W\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right)$,

$\vdots$

$\frac{dW}{dz}$ is $=\int dx\ldots\ldots de\,We^{2q+1}\frac{dW}{dz}-\int dx\ldots dz\,de\,W\frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right)$,

$\frac{dW}{de}$ is $=\int dx\ldots\ldots dz\,We^{2q+1}\frac{dW}{de}-\int dx\ldots dz\,de\,W\frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right)$.

Write dS to denote an element of surface at the point $(x, \ldots, z, e)$. Then taking $\alpha, \ldots, \gamma, \delta$ to denote the inclinations of the interior normal at that point to the positive axes of coordinates, we have

$$\begin{aligned} dy \ldots dz\, de &= -dS \cos\alpha, \\ &\vdots \\ dx \ldots\ldots de &= -dS \cos\gamma, \\ dx \ldots\ldots dz &= -dS \cos\delta; \end{aligned}$$

and the first terms are together

$$= -\int e^{2q+1} W \left(\frac{dW}{dx}\cos\alpha + \ldots + \frac{dW}{dz}\cos\gamma + \frac{dW}{de}\cos\delta\right) dS,$$

W here denoting the value at the surface, and the integration being extended over the whole of the closed surface: this may also be written

$$= -\int e^{2q+1} W \frac{dW}{d\nu} dS,$$

where ν denotes an element of the internal normal.

The second terms are together

$$= -\int dx \ldots dz\, de\, W \left\{\frac{d}{dx}\left(e^{2q+1}\frac{dW}{dx}\right) + \ldots + \frac{d}{dz}\left(e^{2q+1}\frac{dW}{dz}\right) + \frac{d}{de}\left(e^{2q+1}\frac{dW}{de}\right)\right\} = -\int dx \ldots dz\, de\, W \square W.$$

We have consequently

$$\int dx \ldots dz\, de \,.\, e^{2q+1} \left\{\left(\frac{dW}{dx}\right)^2 + \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\}$$

$$= -\int e^{2q+1} W \frac{dW}{d\nu} dS - \int dx \ldots dz\, de\, e^{2q+1} W \square W.$$

31. The second term vanishes if W satisfies the prepotential equation $\square W = 0$; and this being so, if also $W = 0$ for all points of the closed surface S, then the first term also vanishes, and we therefore have

$$\int dx \ldots dz\, de \,.\, e^{2q+1} \left\{\left(\frac{dW}{dx}\right)^2 + \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\} = 0,$$

where the integration extends over the whole space included within the closed surface; whence, W being a real function,

$$\frac{dW}{dx} = 0, \ldots, \frac{dW}{dz} = 0, \quad \frac{dW}{de} = 0,$$

for all points within the closed surface; consequently, since W vanishes at the surface, $W = 0$ for all points within the closed surface.

32. Considering W as satisfying the equation $\square W = 0$, we may imagine the closed surface to become larger and larger, and ultimately infinite, at the same time flattening

itself out into coincidence with the plane $e=0$, so that it comes to include the whole space above the plane $e=0$: say the surface breaks up into the surface positive infinity and the infinite plane $e=0$.

The integral $\int e^{2q+1} W \frac{dW}{dv} dS$ separates itself into two parts, the first relating to the surface positive infinity, and vanishing if $W=0$ at infinity (that is, if all or any of the coordinates $x, \ldots, z, e$ are infinite); the second, relating to the plane $e=0$, is $\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz$, W here denoting its value at the plane, that is, when $e=0$, and the integral being extended over the whole plane. The theorem thus becomes

$$\int dx \ldots dz\, de \,.\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 + \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\} = -\int W\left(e^{2q+1}\frac{dW}{de}\right) dx \ldots dz.$$

Hence also, if $W=0$ at all points of the plane $e=0$, the right-hand side vanishes, and we have

$$\int dx \ldots dz\, de \,.\, e^{2q+1}\left\{\left(\frac{dW}{dx}\right)^2 + \ldots + \left(\frac{dW}{dz}\right)^2 + \left(\frac{dW}{de}\right)^2\right\} = 0.$$

Consequently $\frac{dW}{dx}=0, \ldots, \frac{dW}{dz}=0, \frac{dW}{de}=0$, for all points whatever of positive space; and therefore also $W=0$ for all points whatever of positive space.

33. Take next U, W, each of them a function of $(x, \ldots, z, e)$, and consider the integral

$$\int dx \ldots dz\, de \,.\, e^{2q+1}\left(\frac{dU}{dx}\frac{dW}{dx} + \ldots + \frac{dU}{dz}\frac{dW}{dz} + \frac{dU}{de}\frac{dW}{de}\right),$$

taken over the space within a closed surface S; treating this in a similar manner, we find it to be

$$= -\int e^{2q+1} W \frac{dU}{dv} dS - \int dx \ldots dz\, de \,.\, e^{2q+1} W \square U,$$

where the integration extends over the whole of the closed surface S; and by parity of reasoning it is also

$$= -\int e^{2q+1} U \frac{dW}{dv} dS - \int dx \ldots dz\, de \,.\, e^{2q+1} U \square W,$$

with the same limits of integration; that is, we have

$$\int e^{2q+1} W \frac{dU}{dv} dS + \int dx \ldots dz\, de \,.\, e^{2q+1} W \square U = \int e^{2q+1} U \frac{dW}{dv} dS + \int dx \ldots dz\, de \,.\, e^{2q+1} U \square W,$$

which, if U and W each satisfy the prepotential equation, becomes

$$\int e^{2q+1} W \frac{dU}{dv} dS = \int e^{2q+1} U \frac{dW}{dv} dS.$$

And if we now take the closed surface S to be the surface positive infinity, together with the plane $e=0$, then, provided only U and V vanish at infinity, for each integral

the portion belonging to the surface positive infinity vanishes, and there remains only the portion belonging to the plane $e=0$; we have therefore

$$\int e^{2q+1} W \frac{dU}{de} dx \ldots dz = \int e^{2q+1} U \frac{dW}{de} dx \ldots dz,$$

where the functions U, W have each of them the value belonging to the plane $e=0$: viz. in U, W considered as given functions of $(x, \ldots, z, e)$ we regard e as a positive quantity ultimately put $=0$; and where the integrations extend each of them over the whole infinite plane.

34. Assume

$$U = \frac{1}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}},$$

an expression which, regarded as a function of $(x, \ldots, z, e)$, satisfies the prepotential equation in regard to these variables, and which vanishes at infinity when all or any of these coordinates $(x, \ldots, z, e)$ are infinite.

We have

$$\frac{dU}{de} = \frac{-2(\frac{1}{2}s+q)\, e}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q+1}};$$

and we have consequently

$$\int W \frac{-2(\frac{1}{2}s+q)\, e^{2q+2}}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q+1}} dx \ldots dz = \int \left(e^{2q+1} \frac{dW}{de}\right) \frac{dx \ldots dz}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q}},$$

where it will be recollected that e is ultimately $=0$; to mark this, we may for W write W_0.

Attend to the left-hand side; take V_0 the same function of $a, \ldots, c$, $e=0$, that W_0 is of $x, \ldots, z$, $e=0$; then, first writing the expression in the form

$$V_0 \int \frac{-2(\frac{1}{2}s+q)\, e^{2q+2}\, dx \ldots dz}{\{(a-x)^2 + \ldots + (c-z)^2 + e^2\}^{\frac{1}{2}s+q+1}},$$

write $x = a + e\xi, \ldots, z = c + e\zeta$, the expression becomes

$$= V_0 \int \frac{-2(\frac{1}{2}s+q)\, e^{2q+2} \cdot e^s d\xi \ldots d\zeta}{\{e^2(1+\xi^2 + \ldots + \zeta^2)\}^{\frac{1}{2}s+q+1}}, \quad = -2(\tfrac{1}{2}s+q)\, V_0 \int \frac{d\xi \ldots d\zeta}{\{1+\xi^2 + \ldots + \zeta^2\}^{\frac{1}{2}s+q+1}},$$

where the integral is to be taken from $-\infty$ to $+\infty$ for each of the new variables $\xi, \ldots, \zeta$.

Writing $\xi = r\alpha, \ldots, \zeta = r\gamma$, where $\alpha^2 + \ldots + \gamma^2 = 1$, we have $d\xi \ldots d\zeta = r^{s-1} dr\, dS$: also $\xi^2 + \ldots + \zeta^2 = r^2$, and the integral is

$$= \int \frac{r^{s-1} dr dS}{(1+r^2)^{\frac{1}{2}s+q+1}}, \quad = \int dS \int_0^\infty \frac{r^{s-1} dr}{(1+r^2)^{\frac{1}{2}s+q+1}},$$

where $\int dS$ denotes the surface of the s-coordinal unit sphere $\alpha^2 + \ldots + \gamma^2 = 1$, and the r-integral is to be taken from $r = 0$ to $r = \infty$; the values of the two factors thus are

$$\int dS = \frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\,(\frac{1}{2}s)}, \text{ and } \int \frac{r^{s-1}\,dr}{(1+r^2)^{\frac{1}{2}s+q+1}} = \frac{\frac{1}{2}\,\Gamma\frac{1}{2}s\,\Gamma\,(q+1)}{\Gamma\,(\frac{1}{2}s+q+1)}.$$

Hence the expression in question is

$$-2\,(\tfrac{1}{2}s+q)\,V_0\,\frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\,\frac{\frac{1}{2}\,\Gamma\frac{1}{2}s\,\Gamma\,(q+1)}{\Gamma\,(\frac{1}{2}s+q+1)}, \quad = \frac{-2\,(\Gamma\frac{1}{2})^s\,\Gamma\,(q+1)}{\Gamma\,(\frac{1}{2}s+q)}\,V_0,$$

and we have

$$\int \left(e^{2q+1}\frac{dW}{de}\right)_0 \frac{dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}} = \frac{-2\,(\Gamma\frac{1}{2})^s\,\Gamma\,(q+1)}{\Gamma\,(\frac{1}{2}s+q)}\,V_0;$$

or, what is the same thing,

$$V_0 = \int \frac{-\,\Gamma\,(\frac{1}{2}s+q)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma\,(q+1)}\,\frac{\left(e^{2q+1}\frac{dW}{de}\right)_0 dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}.$$

35. Take now V a function of $(a, \ldots, c, e)$ satisfying the prepotential equation in regard to these variables, always finite, and vanishing at infinity; and let W be the same function of $(x, \ldots, z, e)$, W therefore satisfying the prepotential equation in regard to the last-mentioned variables. Consider the function

$$V - \int \frac{-\,\frac{\Gamma\,(\frac{1}{2}s+q)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma\,(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0 dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where the integral is taken over the infinite plane $e = 0$; then this function ($V -$ the integral) satisfies the prepotential equation (for each term separately satisfies it), is always finite, and it vanishes at infinity. It also, as has just been seen, vanishes for any point whatever of the plane $e = 0$. Consequently it vanishes for all points whatever of positive space. Or, what is the same thing, if we write

$$V = \int \frac{\rho\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

where ρ is a function of $(x, \ldots, z)$, and the integral is taken over the whole infinite plane, then if V is a function of $(a, \ldots, c, e)$ satisfying the above conditions, there exists a corresponding value of ρ; viz. taking W the same function of $(x, \ldots, z, e)$ which V is of $(a, \ldots, c, e)$, the value of ρ is

$$\rho = -\frac{\Gamma\,(\frac{1}{2}s+q)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma\,(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

where e is to be put $= 0$ in the function $e^{2q+1}\dfrac{dW}{de}$. This is the prepotential-plane theorem; viz. taking for the prepotential in regard to a given point $(a, \ldots, c, e)$ a function of $(a, \ldots, c, e)$ satisfying the prescribed conditions, but otherwise arbitrary, there exists on the plane $e = 0$ a distribution ρ given by the last-mentioned formula.

36. It is assumed in the proof that $2q+1$ is positive or zero; viz. q is positive, or if negative then $-q \not> \frac{1}{2}$; the limiting case $q=-\frac{1}{2}$ is included.

It is to be remarked that, by what precedes, if q be positive (but excluding the case $q=0$), the density ρ is given by the equivalent more simple formula

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s \Gamma q}(e^{2q}W)_0.$$

The foregoing proof is substantially that given in Green's memoir on the Attraction of Ellipsoids; it will be observed that the proof only imposes upon V the condition of vanishing at infinity, without obliging it to assume for large values of $(a, \ldots, c, e)$ the form

$$\frac{M}{\{a^2+\ldots+c^2+e^2\}^{\frac{1}{2}s+q}}.$$

The Potential-surface Theorem C. Art. Nos. 37 to 42.

37. In the case $q=-\frac{1}{2}$, writing here $\nabla = \frac{d^2}{dx^2}+\ldots+\frac{d^2}{dz^2}+\frac{d^2}{de^2}$, we have, precisely as in the general case,

$$\int W\frac{dU}{ds}\,dS + \int dx\ldots dz\,de\,W\nabla U = \int U\frac{dW}{ds}\,dS + \int dx\ldots dz\,de\,U\nabla W;$$

and if the functions U, W satisfy the equations $\nabla U=0$, $\nabla W=0$, then (subject to the exception presently referred to) the second terms on the two sides respectively each of them vanish.

But, instead of taking the surface to be the surface positive infinity together with the plane $e=0$, we now leave it an arbitrary closed surface, and for greater symmetry of notation write w in place of e; and we suppose that the functions U and W, or one of them, may become infinite at points within the closed surface; then, on this last account, the second terms do not in every case vanish.

38. Suppose, for instance, that U at a point indefinitely near the point $(a, \ldots, c, e)$ within the surface becomes

$$= \frac{1}{\{(x-a)^2+\ldots+(z-c)^2+(w-e)^2\}^{\frac{1}{2}s-\frac{1}{2}}};$$

then if V be the value of W at the point $(a, \ldots, c, e)$, we have

$$\int dx\ldots dz\,dw\,W\nabla U = V\int dx\ldots dz\,dw\,\nabla U;$$

and since $\nabla U=0$, except at the point in question, the integral may be taken over any portion of space surrounding this point, for instance, over the space included within the

sphere, radius R, having the point $(a, \ldots, c, e)$ for its centre; or taking the origin at this point, we have to find $\int dx \ldots dz\, dw\, \nabla U$, where

$$U = \frac{1}{\{x^2 + \ldots + z^2 + w^2\}^{\frac{1}{2}s - \frac{1}{2}}},$$

and the integration extends over the space within the sphere $x^2 + \ldots + z^2 + w^2 = R^2$.

39. This may be accomplished most easily by means of a particular case of the last-mentioned theorem; viz. writing $W = 1$, we have

$$\int \frac{dU}{ds}\, dS + \int dx \ldots dz\, dw\, \nabla U = 0,$$

or the required value is $= -\int \frac{dU}{ds}\, dS$ over the surface of the last-mentioned sphere. We have, if for a moment $r^2 = x^2 + \ldots + z^2 + w^2$,

$$\frac{dU}{ds} = -\left(\frac{x}{r}\frac{d}{dx} + \ldots + \frac{z}{r}\frac{d}{dz} + \frac{w}{r}\frac{d}{dw}\right) U, \quad = -\frac{dU}{dr}\left\{\left(\frac{x}{r}\frac{d}{dx} + \ldots + \frac{z}{r}\frac{d}{dz} + \frac{w}{r}\frac{d}{dw}\right) r\right\}, \quad = -\frac{dU}{dr},$$

that is, $\frac{dU}{ds} = \frac{s-1}{r^s}$, $= \frac{s-1}{R^s}$ at the surface; and hence

$$\int \frac{dU}{ds}\, dS = \frac{s-1}{R^s} \int dS,$$

where $\int dS$ is the whole surface of the sphere $x^2 + \ldots + z^2 + w^2 = R^2$, viz. it is $= R^3$, multiplied by the surface of the unit-sphere $x^2 + \ldots + z^2 + w^2 = 1$. This spherical surface, say $\int d\Sigma$, is

$$= \frac{2\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}(s+1)}, \quad = \frac{4\,(\Gamma\frac{1}{2})^{s+1}}{(s-1)\,\Gamma\frac{1}{2}(s-1)},$$

and we have thus $\int \frac{dU}{ds}\, dS = \frac{4\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}(s-1)}$, and consequently

$$\int dx \ldots dz\, dw\, \nabla U = -\frac{4\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}.$$

40. Treating in like manner the case, where W at a point indefinitely near the point $(\mathrm{a}, \ldots, \mathrm{c}, \mathrm{e})$ within the surface becomes

$$= \frac{1}{\{(x-\mathrm{a})^2 + \ldots + (z-\mathrm{c})^2 + (w-\mathrm{e})^2\}^{\frac{1}{2}s - \frac{1}{2}}},$$

and writing T to denote the same function of $(\mathrm{a}, \ldots, \mathrm{c}, \mathrm{e})$ that U is of $(x, \ldots, z, w)$, we have, instead of the foregoing, the more general theorem

$$\int W \frac{dU}{ds}\, dS + \int dx \ldots dz\, dw\, W \nabla U - \frac{4\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\, V$$
$$= \int U \frac{dW}{ds}\, dS + \int dx \ldots dz\, dw\, U \nabla W - \frac{4\,(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})}\, T,$$

where, in the two solid integrals, we exclude from consideration the space in the immediate neighbourhood of the two critical points $(a, \ldots, c, e)$ and $(\mathrm{a}, \ldots, \mathrm{c}, \mathrm{e})$ respectively.

Suppose that W is always finite within the surface, and that U is finite except at the point $(a, \ldots, c, e)$: and moreover that U, W are such that $\nabla U = 0$, $\nabla W = 0$; then the equation becomes

$$\int W \frac{dU}{ds} dS - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma\frac{1}{2}s - \frac{1}{2}} V = \int U \frac{dW}{ds} dS.$$

In particular, this equation holds good if U is

$$= \frac{1}{\{(a-x)^2 + \ldots + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

41. Imagine now on the surface S a distribution $\rho\, dS$ producing at a point $(a', \ldots, c', e')$ within the surface a potential V', and at a point $(a'', \ldots, c'', e'')$ without the surface a potential V''; where, by what precedes, V'' is in general not the same function of $(a'', \ldots, c'', e'')$ that V' is of $(a', \ldots, c', e')$.

It is further assumed that at a point $(a, \ldots, c, e)$ on the surface we have $V' = V''$:

that V', or any of its derived functions, are not infinite for any point $(a', \ldots, c', e')$ within the surface:

that V'', or any of its derived functions, are not infinite for any point $(a'', \ldots, c'', e'')$ without the surface:

and that $V'' = 0$ for any point at infinity.

Consider V' as a given function of $(a, \ldots, c, e)$; and take W' the same function of $(x, \ldots, z, w)$. Then if, as before,

$$U = \frac{1}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we have

$$\left(\frac{d^2}{dx^2} + \ldots + \frac{d^2}{dz^2} + \frac{d^2}{dw^2}\right) U = 0,$$

$$\int U \frac{dW'}{ds'} dS = \int W' \frac{dU}{ds'} dS - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V'.$$

Similarly, considering V'' as a given function of $(a, \ldots, c, e)$, take W'' the same function of $(x, \ldots, z, e)$. Then, by considering the space outside the surface S, or say between this surface and infinity, and observing that U does not become infinite for any point in this space, we have

$$\int U \frac{dW''}{ds''} dS = \int W'' \frac{dU}{ds''} dS;$$

adding these two equations, we have

$$\int U \left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) dS = \int \left(W' \frac{dU}{ds'} + W'' \frac{dU}{ds''}\right) dS - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V'.$$

But in this equation the functions W' and W'' each of them belong to a point $(x, \ldots, z, w)$ on the surface, and we have at the surface $W' = W'', = W$ suppose; the term on the right-hand side thus is $\int W\left(\frac{dU}{ds'} + \frac{dU}{ds''}\right) dS$, which vanishes in virtue of $\frac{dU}{ds'} + \frac{dU}{ds''} = 0$; and the equation thus becomes

$$\int U\left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) dS = -\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V':$$

that is, the point $(a, \ldots, c, e)$ being interior, we have

$$V' = \int \frac{-\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}} \left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) \frac{dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

In exactly the same way, if $(a, \ldots, c, e)$ be an exterior point, then we have

$$\int U \frac{dW'}{ds'} dS = \int W' \frac{dU}{ds'} dS,$$

$$\int U \frac{dW''}{ds''} dS = \int W'' \frac{dU}{ds''} dS - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V'';$$

adding, and omitting the terms which vanish,

$$\int U\left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) dS = -\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V'',$$

that is,

$$V'' = \int \frac{-\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}} \left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) \frac{dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}.$$

42. Comparing the two results with

$$V = \int \frac{\rho\, dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we see that, V' and V'' satisfying the foregoing conditions, there exists a distribution ρ on the surface, producing the potentials V' and V'' at an interior point and an exterior point respectively; the value of ρ in fact being

$$\rho = -\frac{\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}} \left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)},$$

where W', W'' are respectively the same functions of $(x, \ldots, z, w)$ that V', V'' are of $(a, \ldots, c, e)$.

The Potential-solid Theorem D. Art. No. 43.

43. We have as before (No. 40),

$$\int W \frac{dU}{ds} dS + \int dx \ldots dz\, dw\, W \nabla U - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V$$

$$= \int U \frac{dW}{ds} dS + \int dx \ldots dz\, dw\, U \nabla W - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} T,$$

where, assuming first that W is not infinite for any point $(x, \ldots, z, w)$ whatever, we have no term in T; and taking next $U = \dfrac{1}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$ as before, we have $\nabla U = 0$; the equation thus becomes

$$\int W \frac{dU}{ds} dS - \int U \frac{dW}{ds} dS - \frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V = \int dx \ldots dz\, dw\, U \nabla W,$$

where W may be a discontinuous function of the coordinates $(x, \ldots, z, w)$, provided only there is no abrupt change in the value either of W or of any of its first derived functions $\dfrac{dW}{dx}, \ldots, \dfrac{dW}{dz}, \dfrac{dW}{dw}$, viz. it may be any function which can represent the potential of a solid mass on an attracted point $(x, \ldots, z, w)$; the resulting value of ∇W is of course discontinuous. Taking, then, for the closed surface S the boundary of infinite space, U and W each vanish at this boundary, and the equation becomes

$$-\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} V = \int dx \ldots dz\, dw\, U \nabla W;$$

viz. substituting for U its value, and comparing with

$$V = \int \frac{\rho\, dx \ldots dz\, dw}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

where the integral in the first instance extends to the whole of infinite space, but the limits may be ultimately restricted by ρ being $= 0$, we see that the value of ρ is

$$\rho = -\frac{\Gamma(\frac{1}{2}s - \frac{1}{2})}{(\Gamma\frac{1}{2})^{s+1}} \left(\frac{d^2}{dx^2} + \ldots + \frac{d^2}{dz^2} + \frac{d^2}{dw^2}\right) W,$$

W being the same function of $(x, \ldots, z, w)$ that V is of $(a, \ldots, c, e)$: which is the theorem D.

Examples of the foregoing Theorems. Art. Nos. 44 to 50.

44. It will be remarked, as regards all the theorems, that we do not start with known limits; we start with V a function of $(a, \ldots, c, e)$, the coordinates of the attracted point, satisfying certain prescribed conditions, and we thence find ρ, a function of the coordinates $(x, \ldots, z)$ or $(x, \ldots, z, w)$, as the case may be, which function is found to be $= 0$ for values of $(x, \ldots, z)$ or $(x, \ldots, z, w)$ lying beyond certain limits, and to have a determinate non-evanescent value for values of $(x, \ldots, z)$ or $(x, \ldots, z, w)$ lying within these limits; and we thus, as a result, obtain these limits for the limits of the multiple integral V.

45. Thus in theorem A, in the example where the limiting equation is ultimately found to be $x^2 + \ldots + z^2 = f^2$, we start with V a certain function of $a^2 + \ldots + c^2$ $(= \kappa^2$ suppose) and e^2, viz. V is a function of these quantities through θ, which denotes the positive root of the equation

$$\frac{\kappa^2}{f^2 + \theta} + \frac{e^2}{\theta} = 1,$$

the value in fact being $V=\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\,dt$, and the resulting value of ρ is found to be $=0$ for values of $(x,\ldots,z)$ for which $x^2+\ldots+z^2>f^2$. Hence V denotes an integral

$$\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $x^2+\ldots+z^2=f^2$: say this is the s-coordinal sphere.

And similarly, in the examples where the limiting equation is ultimately found to be $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$, we start with V a certain function of $a,\ldots,c,e$ through θ (or directly and through θ), where θ denotes the positive root of the equation

$$\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

and the resulting value of ρ is found to be $=0$ for values of $(x,\ldots,z)$ for which

$$\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}>1.$$

Hence V denotes an integral

$$\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

the limiting equation being $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$: say this is the s-coordinal ellipsoid. It is clear that this includes the before-mentioned case of the s-coordinal sphere; but, on account of the more simple form of the θ-equation, it is worth while to work out directly an example for the sphere.

46. Three examples are worked out in Annex IV.; the results are as follows:—

First, θ defined for the sphere as above; $q+1$ positive;

$$V=\int\frac{\left(1-\frac{x^2+\ldots+z^2}{f^2}\right)^q dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the sphere $x^2+\ldots+y^2=f^2$,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}f^s\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\,dt.$$

This is included in the next-mentioned example for the ellipsoid.

Secondly, θ defined for the ellipsoid as above; $q+1$ positive;

$$V=\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\cdot f\ldots h\int_\theta^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt.$$

This result is included in the next-mentioned example; but the proof for the general value of m is not directly applicable to the value $m=0$ for the case in question.

Thirdly, θ defined for the ellipsoid as above; $q+1$ positive; $m=0$ or positive, and apparently in other cases,

$$V=\int\frac{\left(1+\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

over the ellipsoid as above,

$$=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\Gamma(1+m)}\cdot f\ldots h\int_0^\infty\left(1-\frac{a^2}{f^2+\theta}-\ldots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta}\right)^m t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt.$$

And we have in Annex V. a fourth example; here θ and the ellipsoid are as above: the result involves the Greenian functions.

47. We may in the foregoing results write $e=0$; the results,—writing therein $s+1$ for s, and in the new forms taking $(a,\ldots,c,e)$ and $(x,\ldots,z,w)$ for the two sets of coordinates respectively, also writing $q-\frac{1}{2}$ for q—, would give integrals of the form

$$\int\frac{\rho\,dx\ldots dz\,dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}}$$

for the $(s+1)$-coordinal sphere and ellipsoid $x^2+\ldots+z^2+w^2=f^2$ and $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}+\frac{w^2}{k^2}=1$: say these are prepotential-solid integrals; and then, writing $q=-\frac{1}{2}$, we should obtain potential-solid integrals, such as are also given by the theorem D. The change can be made if necessary; but it is more convenient to retain the results in their original forms, as relating to the s-coordinal sphere and ellipsoid.

There are two cases, according as the attracted point $(a,\ldots,c)$ is external or internal.

For the sphere:—For an external point $\kappa^2>f^2$; writing $e=0$, the equation $\frac{\kappa^2}{f^2+\theta}=1$ has a positive root, viz. this is $\theta=\kappa^2-f^2$; and θ will have, or it may be replaced by, this value κ^2-f^2: for an internal point $\kappa^2<f^2$; as e approaches zero, the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

For the ellipsoid:—For an external point $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1$; writing $e=0$, the equation $\frac{a^2}{\theta+f^2}+\ldots+\frac{c^2}{\theta+h^2}=1$ has a positive root, and θ will denote this positive root: for an

internal point $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}<1$; as e approaches zero the positive root of the original equation gradually diminishes and becomes ultimately $=0$, viz. in the formulæ θ is to be replaced by this value 0.

The resulting formulæ for the sphere $x^2+\ldots+z^2=f^2$ may be compared with formulæ for the spherical shell, Annex VI., and each set with formulæ obtained by direct integration in Annex III.

We may in any of the formulæ write $q=-\frac{1}{2}$, and so obtain examples of theorem B.

48. As regards theorem C, we might in like manner obtain examples of potentials relating to the surfaces of the $(s+1)$-coordinal sphere $x^2+\ldots+z^2+w^2=f^2$, and ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}+\frac{w^2}{k^2}=1$, or say to spherical and ellipsoidal shells; but I have confined myself to the sphere. We have to assume values V' and V'' belonging to the cases of an internal and an external point respectively, and thence to obtain a value ρ, or distribution over the spherical surface, which shall produce these potentials respectively. The result (see Annex VI.) is

$$\int\frac{dS}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over the surface of the $(s+1)$-coordinal sphere $x^2+\ldots+z^2+w^2=f^2$,

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{\kappa^{s-1}} \text{ for exterior point } \kappa>f,$$

and

$$=\frac{2(\Gamma\frac{1}{2})^{s+1}f^s}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{1}{f^{s-1}} \text{ for interior point } \kappa<f,$$

where $\kappa^2=a^2+\ldots+c^2+e^2$. Observe that for the interior point the potential is a mere constant multiple of f.

The same Annex VI. contains the case of the s-coordinal cylinder $x^2+\ldots+z^2=f^2$, which is peculiar in that the cylinder is not a finite closed surface; but the theorem C is found to extend to it.

49. As regards theorem D, we might in like manner obtain potentials relating to the $(s+1)$-coordinal sphere $x^2+\ldots+z^2+w^2=f^2$ and ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}+\frac{w^2}{k^2}=1$; but I confine myself to the case of the sphere (see Annex VII.). We here assume values V' and V'' belonging to an internal and an external point respectively, and thence obtain a value ρ, or distribution over the whole $(s+1)$-dimensional space, which density is found to be $=0$ for points outside the sphere. The result obtained is

$$V=\int\frac{dx\ldots dz\,dw}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

over the $(s+1)$-coordinal sphere $x^2+\ldots+z^2+w^2=f^2$,

$$=\frac{(\Gamma\frac{1}{2}s)^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\cdot\frac{f^{s+1}}{\kappa^{s-1}} \text{ for an exterior point } \kappa>f,$$

$$=\frac{(\Gamma\frac{1}{2}s)^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\{(\tfrac{1}{2}s+\tfrac{1}{2})f^2-(\tfrac{1}{2}s-\tfrac{1}{2})\kappa^2\} \text{ for an interior point } \kappa<f,$$

where $\kappa^2=a^2+\ldots+c^2+e^2$.

50. The remaining Annexes VIII. and IX. have no immediate reference to the theorems A, B, C, D, which are the principal objects of the memoir. The subjects to which they relate will be seen from the headings and introductory paragraphs.

ANNEX I. *Surface and Volume of Sphere* $x^2+\ldots+z^2+w^2=f^2$. Art. Nos. 51 and 52.

51. We require in $(s+1)$-dimensional space, $\int dx\ldots dz\,dw$, the volume of the sphere $x^2+\ldots+z^2+w^2=f^2$, and $\int dS$, the surface of the same sphere.

Writing $x=f\sqrt{\xi},\ldots,z=f\sqrt{\zeta}$, $w=f\sqrt{\omega}$, we have

$$dx\ldots dz\,dw=\frac{1}{2^{s+1}}f^{s+1}\xi^{-\frac{1}{2}}\ldots\zeta^{-\frac{1}{2}}\omega^{-\frac{1}{2}}d\xi\ldots d\zeta\,d\omega,$$

with the limiting condition $\xi+\ldots+\zeta+\omega=1$; but in order to take account as well of the negative as the positive values of $x,\ldots,z$, w, we must multiply by 2^{s+1}. The value is therefore

$$=f^{s+1}\int\xi^{-\frac{1}{2}}\ldots\zeta^{-\frac{1}{2}}\omega^{-\frac{1}{2}}d\xi\ldots d\zeta\,d\omega,$$

extended to all positive values of $\xi,\ldots,\zeta$, ω, such that $\xi+\ldots+\zeta+\omega<1$; and we obtain this by a known theorem, viz.

$$\text{Volume of } (s+1)\text{-dimensional sphere}=f^{s+1}\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}.$$

Writing $x=f\xi,\ldots,z=f\zeta$, $w=f\omega$, we obtain $dS=f^s d\Sigma$, where $d\Sigma$ is the element of surface of the unit-sphere $\xi^2+\ldots+\zeta^2+\omega^2=1$; we have element of volume $d\xi\ldots d\zeta\,d\omega=r^s\,dr\,d\Sigma$, where r is to be taken from 0 to 1, and thence

$$\int d\xi\ldots d\zeta\,d\omega=\int_0^1 r^s\,dr\int d\Sigma=\frac{1}{s+1}\int d\Sigma,$$

that is,

$$\int d\Sigma=(s+1)\int d\xi\ldots d\zeta\,d\omega,\ =2(\tfrac{1}{2}s+\tfrac{1}{2})\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}=\frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

consequently $\int dS=\text{surface of } (s+1)\text{-dimensional sphere}=f^s\dfrac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$.

52. Writing $s-1$ for s, we have

$$\text{Volume of } (s-1)\text{-dimensional sphere} = f^s \frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+1)},$$

$$\text{Surface of} \qquad \text{do.} \qquad = f^{s-1} \frac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)},$$

which forms are sometimes convenient.

Writing in the first forms $s+1=3$, or in the second forms $s=3$, we find in ordinary space

$$\text{Volume of sphere} = f^3 \frac{(\Gamma\frac{1}{2})^3}{\Gamma(\frac{3}{2})} = f^3 \frac{\pi^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}, = \frac{4\pi f^3}{3},$$

and

$$\text{Surface of sphere} = f^2 \frac{2(\Gamma\frac{1}{2})^3}{\Gamma\frac{3}{2}} = f^2 \frac{2\pi^{\frac{3}{2}}}{\frac{1}{2}\sqrt{\pi}}, \qquad = 4\pi f^2,$$

as they should be.

Annex II. *The Integral* $\int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$. Art. Nos. 53 to 63.

53. The integral in question (which occurs *antè*, No. 2) may also be considered as arising from a prepotential integral in tridimensional space; the prepotential of an element of mass dm is taken to be $=\frac{dm}{d^{s+2q}}$, where d is the distance of the element from the attracted point P. Hence if the element of mass be an element of the plane $z=0$, coordinates (x, y), ρ being the density, and if the attracted point be situate in the axis of z at a distance e from the origin, the prepotential is

$$V = \int \frac{\rho\,dx\,dy}{(x^2+y^2+e^2)^{\frac{1}{2}s+q}}.$$

For convenience, it is assumed throughout that e is positive.

Suppose that the attracting body is a circular disk, radius R, having the origin for its centre (viz. that bounded the curve $x^2+y^2=R^2$); then writing $x=r\cos\theta$, $y=r\sin\theta$, we have

$$V = \int \frac{\rho r\,dr\,d\theta}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which, if ρ is a function of r only, is

$$= 2\pi \int \frac{\rho r\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$$

and in particular, if $\rho = r^{s-2}$, then the value is

$$= 2\pi \int \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

the integral in regard to r being taken from $r=0$ to $r=R$. It is assumed that $s-1$ is not negative, viz. it is positive or (it may be) zero. I consider the integral

$$\int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}},$$

which I call the r-integral, more particularly in the case where e is small in comparison with R. It is to be observed that e not being $=0$, and R being finite, the integral contains no infinite element, and is therefore finite, whether q is positive, negative, or zero.

54. Writing $r=e\sqrt{v}$, the integral is

$$=\tfrac{1}{2}e^{-2q}\int\frac{v^{\frac{1}{2}s-1}\,dv}{(1+v)^{\frac{1}{2}s+q}},$$

the limits being $\dfrac{R^2}{e^2}$ and 0.

In the case where q is positive, this is

$$=\tfrac{1}{2}e^{-2q}\left(\int_0^\infty-\int_{\frac{R^2}{e^2}}^\infty\right)\frac{v^{\frac{1}{2}s-1}\,dv}{(1+v)^{\frac{1}{2}s+q}};$$

viz. the first term of this is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\frac{1}{2}s\,.\,\Gamma q}{\Gamma(\frac{1}{2}s+q)},$$

and the second term is a term expansible in a series containing the powers $2q$, $2q+2$, &c. of the small quantity $\dfrac{e^2}{R^2}$, as appears by effecting therein the substitution $v=\dfrac{1}{x}$; viz. the value of the entire integral is by this means found to be

$$=\tfrac{1}{2}e^{-2q}\left\{\frac{\Gamma\frac{1}{2}s\,.\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}-\int_0^{\frac{e^2}{R^2}}\frac{x^{q-1}\,dx}{(1+x)^{\frac{1}{2}s+q}}\right\}.$$

55. In the case where q is $=0$, or negative, the formula fails by reason that the element $\dfrac{v^{\frac{1}{2}s-1}\,dv}{(1+v)^{\frac{1}{2}s+q}}$ of the integrals $\int_0^\infty$, $\int_{\frac{R^2}{e^2}}^\infty$ becomes infinite for indefinitely large values of v. Recurring to the original form $\int_0^R\dfrac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$, it is to be observed that the integral has a finite value when $e=0$; and it might therefore at first sight be imagined that the factor $(r^2+e^2)^{-\frac{1}{2}s-q}$ might be expanded in ascending powers of e^2, and the value of the integral consequently obtained as a series of positive powers of e^2. But the series thus obtained is of the form $e^{2k}\int_0^R r^{-2q-2k-1}\,dr$, where $2q$ being positive, the exponent $-2q-2k-1$ is for a sufficiently small value of k at first positive, or if negative less than -1, and the value of the integral is finite; but as k increases the exponent becomes negative, and equal or greater than -1, and the value of the

integral is then infinite. The inference is that the series *commences* in the form $A + Be^2 + Ce^4 \ldots$: but that we come at last when q is fractional to a term of the form Ke^{-2q}, and when q is $=0$ or is integral, to a term of the form $Ke^{-2q}\log e$; the process giving the coefficients $A, B, C, \ldots$, so long as the exponent of the corresponding term $e^0, e^2, e^4, \ldots$ is less than $-2q$ (in particular $q=0$, there is a term $k\log e$, *and the expansion-process does not give any term of the result*), and the failure of the series after this point being indicated by the values of the subsequent coefficients coming out $=\infty$.

56. In illustration, we may consider any of the cases in which the integral can be obtained in finite terms. For instance,

$s=2,\ q=-\frac{3}{2}$,

$$\text{Integral is } \int r(r^2+e^2)^{\frac{1}{2}}\,dr,\ =\tfrac{1}{3}(r^2+e^2)^{\frac{3}{2}}, \text{ from } 0 \text{ to } R,$$
$$=\tfrac{1}{3}(R^2+e^2)^{\frac{3}{2}}-\tfrac{1}{3}e^3;$$

viz. expanding in ascending powers of e, this is

$$=\tfrac{1}{3}R^3+\tfrac{1}{2}Re^2-\ldots-\tfrac{1}{3}e^3,$$

or we have here a term in e^3. And so,

$s=1,\ q=-2$,

$$\text{Integral is } \int (r^2+e^2)^{\frac{3}{2}}\,dr,\ =(\tfrac{1}{4}r^2+\tfrac{5}{8}e^2)\,r\sqrt{r^2+e^2}+\tfrac{3}{8}e^4\log(r+\sqrt{r^2+e^2}), \text{ from } 0 \text{ to } R,$$
$$=(\tfrac{1}{4}R^2+\tfrac{5}{8}e^2)\,R\sqrt{R^2+e^2}+\tfrac{3}{8}e^4\log\frac{R+\sqrt{R^2+e^2}}{e};$$

viz. expanding in ascending powers of e, this is

$$=\tfrac{1}{4}R^4+\tfrac{3}{4}R^2e^2+\ldots+\tfrac{3}{8}e^4\log\frac{R}{e}\text{*},$$

or we have here a term in $e^4\log e$.

57. Returning to the form

$$\tfrac{1}{2}e^{-2q}\int_0^{\frac{R^2}{e^2}}\frac{v^{\frac{1}{2}s-1}\,dv}{(1+v)^{\frac{1}{2}s+q}},$$

and writing herein $v=\dfrac{1-x}{x}$, or, what is the same thing, $x=\dfrac{1}{1+v}$, and for shortness $X=\dfrac{e^2}{e^2+R^2}$, $=\dfrac{1}{1+\dfrac{R^2}{e^2}}$, the value is

$$=\tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

where observe that $q-1$ is 0 or negative, but X being a positive quantity less than 1, the function $x^{q-1}(1-x)^{\frac{1}{2}s-1}$ is finite for the whole extent of the integration.

* Term is $\frac{3}{8}e^4\log\frac{2R}{e}$, $=\frac{3}{8}e^4\left(\log\frac{R}{e}+\log 2\right)$, which, $\frac{R}{e}$ being large, is reduced to $\frac{3}{8}e^4\log\frac{R}{e}$.

58. If $q=0$, this is

$$=\tfrac{1}{2}\int_X^1 \frac{1-\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx}{x}$$

$$=\tfrac{1}{2}\log X-\tfrac{1}{2}\int_X^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx}{x}$$

$$=\tfrac{1}{2}\log\sqrt{1+\frac{R^2}{e^2}}-\tfrac{1}{2}\int_0^1 \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx}{x}+\tfrac{1}{2}\int_0^X \frac{\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx}{x}:$$

where observe that, in virtue of the change made from $\frac{1}{x}(1-x)^{\frac{1}{2}s-1}$ to $\frac{1}{x}\{1-(1-x)^{\frac{1}{2}s-1}\}$ (a function which becomes infinite, to one which does not become infinite, for $x=0$), it has become allowable in place of $\int_X^1$ to write $\int_0^1-\int_0^X$.

When e is small, the integral which is the third term of the foregoing expression is obviously a quantity of the order e^2; the first term is $\tfrac{1}{2}\left(\log\frac{R}{e}+\log\sqrt{1+\frac{e^2}{R^2}}\right)$, which, neglecting terms in e^2, is $=\tfrac{1}{2}\log\frac{R}{e}$, and hence the approximate value of the r-integral $\int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s}}$ is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dx\,\frac{1-(1-x)^{\frac{1}{2}s-1}}{x},$$

or, what is the same thing, it is

$$=\log\frac{R}{e}-\tfrac{1}{2}\int_0^1 dy\,\frac{1-y^{\frac{1}{2}s-1}}{1-y},$$

where the integral in this expression is a mere numerical constant, which, when $\frac{1}{2}s-1$ is a positive integer, has the value

$$\tfrac{1}{1}+\tfrac{1}{2}+\ldots+\frac{1}{\frac{1}{2}s-1};$$

neglecting this in comparison with the logarithmic term, the approximate value is

$$=\log\frac{R}{e}.$$

59. I consider also the case $q=-\frac{1}{2}$; the integral is here

$$\tfrac{1}{2}e\int_X^1 x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}\,dx$$

$$=\tfrac{1}{2}e\int_X^1 x^{-\frac{3}{2}}(1-\{1-(1-x)^{\frac{1}{2}s-1}\})\,dx$$

$$=e\,(X^{-\frac{1}{2}}-1)+\tfrac{1}{2}e\int_X^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx;$$

and the first term of this being $=\sqrt{e^2+R^2}-e$, this is consequently

$$=\sqrt{R^2+e^2}+\tfrac{1}{2}e\int_0^X x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx-e\left(1+\tfrac{1}{2}\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx\right).$$

As regards the second term of this, we have

$$-2x^{-\frac{1}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}+2(\tfrac{1}{2}s-1)\int x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}s-2}\,dx=\int x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx;$$

or, taking each term between the limits 1, 0,

$$-2+2(\tfrac{1}{2}s-1)\frac{\Gamma\frac{1}{2}\Gamma(\frac{1}{2}s-1)}{\Gamma(\frac{1}{2}s-\frac{1}{2})}=\int_0^1 x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx;$$

viz. this integral has the value

$$-2+\frac{2\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})};$$

and the value of the r-integral $\int_0^R \frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}}$ is consequently

$$=\sqrt{R^2+e^2}+\tfrac{1}{2}e\int_0^X x^{-\frac{3}{2}}\{1-(1-x)^{\frac{1}{2}s-1}\}\,dx-e\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

which is of the form

$$R\left\{1+\text{terms in }\frac{e^2}{R^2},\ \frac{e^4}{R^4},\ldots\right\}-e\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})};$$

say the approximate value is

$$R-e\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-\frac{1}{2})},$$

where the first term R is the term $\int_0^R dr$, given by the expansion in ascending powers of e^2; the second term is the term in e^{-2q}. And observe that the term is the value of

$$\tfrac{1}{2}e\int_0^1 x^{-\frac{3}{2}}(1-x)^{\frac{1}{2}s-1}\,dx,$$

calculated by means of the ordinary formula for a Eulerian integral (which formula, on account of the negative exponent $-\frac{3}{2}$, is not really applicable, the value of the integral being $=\infty$) on the assumption that the Γ of a negative q is interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. the value thus calculated is

$$=\tfrac{1}{2}e\frac{\Gamma(-\frac{1}{2})\,\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s-\frac{1}{2})},\ =-e\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

on the assumption $\Gamma\frac{1}{2}=-\frac{1}{2}\Gamma(-\frac{1}{2})$; and this agrees with the foregoing value.

60. It is now easy to see in general how the foregoing transformed value $\frac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx$, where q is negative and fractional, gives at once the value of

the term in e^{-2q}. Observe that in the integral x is always between 1 and X $\left(=\dfrac{e^2}{e^2+R^2}\right.$, a positive quantity less than $\left.1\right)$; the function to be integrated never becomes infinite. Imagine for a moment an integral $\int_X^1 x^\alpha\,dx$, where α is positive or negative. We may conventionally write this $=\int_0^1 x^\alpha\,dx-\int_0^X x^\alpha\,dx$, understanding the first symbol to mean $\dfrac{1^{1+\alpha}}{1+\alpha}$, and the second to mean $\dfrac{X^{1+\alpha}}{1+\alpha}$; they of course properly mean $\dfrac{1^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$ and $\dfrac{X^{1+\alpha}-0^{1+\alpha}}{1+\alpha}$; but the terms in $0^{1+\alpha}$, whether zero or infinite, destroy each other, the original form $\int_X^1 x^\alpha\,dx$, in fact, showing that no such terms can appear in the result.

In accordance with the convention, we write

$$\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx=\int_0^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx-\int_0^X x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx;$$

and it follows that the term in e^{-2q} is

$$\tfrac{1}{2}e^{-2q}\int_0^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

this last expression (wherein q, it will be remembered, is a negative fraction) being understood according to the convention; and so understanding it, the value of the term is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma\left(\frac{1}{2}s+q\right)},$$

where the Γ of the negative q is to be interpreted in accordance with the equation $\Gamma(q+1)=q\Gamma q$; viz. we have $\Gamma q=\dfrac{1}{q}\Gamma(q+1),=\dfrac{1}{q(q+1)}\Gamma(q+2)$, &c., so as to make the argument of the Γ positive. Observe that under this convention we have

$$\Gamma q\Gamma(1-q)=\frac{\Gamma^2\left(\frac{1}{2}\right)}{\sin q\pi}:\text{ or the term is }\tfrac{1}{2}e^{-2q}.\frac{\Gamma^2\left(\frac{1}{2}\right)}{\sin q\pi}\frac{\Gamma\frac{1}{2}s}{\Gamma\left(\frac{1}{2}s+q\right)\Gamma(1-q)}.$$

61. An example in which $\frac{1}{2}s-1$ is integral will make the process clearer, and will serve instead of a general proof. Suppose $q=-\frac{1}{7}$, $\frac{1}{2}s-1=4$, the expression

$$\int_0^1 x^{-\frac{8}{7}}(1-x)^4\,dx=\int_0^1\left(x^{-\frac{8}{7}}-4x^{-\frac{1}{7}}+6x^{\frac{6}{7}}-4x^{\frac{13}{7}}+x^{\frac{22}{7}}\right)dx$$

is used, in accordance with the convention, to denote the value

$$\begin{aligned}&-7-\tfrac{14}{3}+\tfrac{42}{13}-\tfrac{7}{5}+\tfrac{7}{27}\\&=7\left(-1-\tfrac{2}{3}+\tfrac{6}{13}-\tfrac{1}{5}+\tfrac{1}{27}\right),=7\left(-\tfrac{44}{27}-\tfrac{1}{3}+\tfrac{6}{13}\right),=\frac{-7\,.\,2401}{5\,.\,13\,.\,27},=\frac{-7^5}{5\,.\,13\,.\,27}.\end{aligned}$$

But we have

$$\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)}=\frac{\Gamma 5\,\Gamma(-\frac{1}{7})}{\Gamma(5-\frac{1}{7})}=\frac{24\,\Gamma(-\frac{1}{7})}{\frac{27}{7}\,.\,\frac{20}{7}\,.\,\frac{13}{7}\,.\,\frac{6}{7}\,.\,\frac{-1}{7}\,\Gamma(-\frac{1}{7})}=\frac{-7^5}{5\,.\,13\,.\,27},$$

agreeing with the former value.

62. The case of a negative integer is more simple. To find the logarithmic term of

$$\tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(1-x)^{\frac{1}{2}s-1}\,dx,$$

we have only to expand the factor $(1-x)^{\frac{1}{2}s-1}$ so as to obtain the term involving x^{-q}. We have thus the term

$$\tfrac{1}{2}e^{-2q}\int_X^1 x^{q-1}(-)^q\frac{\Gamma\frac{1}{2}s}{\Gamma(1-q)\,\Gamma(\frac{1}{2}s+q)}\,x^{-q}\,dx$$

$$=\tfrac{1}{2}(-)^q e^{-2q}\frac{\Gamma\frac{1}{2}s}{\Gamma(1-q)\,\Gamma(\frac{1}{2}s+q)}\log\frac{1}{X},$$

where $\log\frac{1}{X}=\log\left(1+\frac{R^2}{e^2}\right)$, $=2\log\frac{R}{e}+2\log\sqrt{1+\frac{e^2}{R^2}}$; so that, neglecting the terms in $\frac{e^2}{R^2}$, &c., this is $=2\log\frac{R}{e}$, and the term in question is

$$=(-)^q e^{-2q}\frac{\Gamma\frac{1}{2}s}{\Gamma(1-q)\,\Gamma(\frac{1}{2}s+q)}\log\frac{R}{e}.$$

The general conclusion is that q being negative, the r-integral

$$\int_0^R\frac{r^{s-1}\,dr}{(r^2+e^2)^{\frac{1}{2}s+q}}$$

has for its value a series proceeding in powers of e^2, which series up to a certain point is equal to the series obtained by expanding in ascending powers of e^2 and integrating each term separately; viz. the series to the point in question is

$$\frac{R^{-2q}}{-2q}-\frac{\frac{1}{2}s+q}{1}\,\frac{R^{-2q-2}}{-2q-2}\,e^2+\frac{\frac{1}{2}s+q\,.\,\frac{1}{2}s+q+1}{1\,.\,2}\,\frac{R^{-2q-4}}{-2q-4}\,e^4+\ldots,$$

continued so long as the exponent of e is less than $-2q$; together with a term Ke^{-2q} when q is fractional, and $Ke^{-2q}\log\frac{R}{e}$ when q is integral; viz. q fractional, this term is

$$=\tfrac{1}{2}e^{-2q}\frac{\Gamma\frac{1}{2}s\,\Gamma q}{\Gamma(\frac{1}{2}s+q)},\quad=\tfrac{1}{2}e^{-2q}\frac{\Gamma^2\frac{1}{2}}{\sin q\pi}\,\frac{\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)\,\Gamma(1-q)},$$

and q integral, it is

$$=(-)^q e^{-2q}\frac{\Gamma\frac{1}{2}s}{\Gamma(1-q)\,\Gamma(\frac{1}{2}s+q)}\log\frac{R}{e}.$$

63. It has been tacitly assumed that $\frac{1}{2}s+q$ is positive; but the formulæ hold good if $\frac{1}{2}s+q$ is $=0$ or negative. Suppose $\frac{1}{2}s+q$ is 0 or a negative integer, then $\Gamma(\frac{1}{2}s+q)=\infty$, and the special term involving e^{-2q} or $e^{-2q}\log e$ vanishes; in fact, in this case the r-integral is

$$=\int_0^R r^{\frac{1}{2}s-1}(r^2+e^2)^{-(\frac{1}{2}s+q)}\,dr,$$

where $(r^2+e^2)^{-(\frac{1}{2}s+q)}$ has for its value a finite series, and the integral is therefore equal to a finite series $A+Be^2+Ce^4+$ &c. If $\frac{1}{2}s+q$ be fractional, then the Γ of the negative quantity $\frac{1}{2}s+q$ must be understood as above, or, what is the same thing, we may, instead of $\Gamma(\frac{1}{2}s+q)$, write

$$\frac{(\Gamma\frac{1}{2})^2}{\sin(\frac{1}{2}s+q)\,\pi\,\Gamma(1-q-\frac{1}{2}s)};$$

thus, q being integral, the exceptional term is

$$=(-)^q\,e^{-2q}\,\frac{\Gamma\frac{1}{2}s\sin(\frac{1}{2}s+q)\,\pi\,.\,\Gamma(1-q-\frac{1}{2}s)}{(\Gamma\frac{1}{2})^2\,\Gamma(1-q)}\log\frac{R}{e}.$$

For instance, $s=1$, $q=-2$, the term is

$$\tfrac{1}{2}e^4\,\frac{\Gamma\frac{1}{2}\sin(-\frac{3}{2}\pi)\,\Gamma\frac{5}{2}}{(\Gamma\frac{1}{2})^2\,.\,\Gamma 3}\log\frac{R}{e};$$

or, since $\Gamma\frac{5}{2}=\frac{3}{2}\,.\,\frac{1}{2}\Gamma\frac{1}{2}$, and $\Gamma 3=2$, the term is $+\frac{3}{8}e^4\log\frac{R}{e}$, agreeing with a preceding result.

ANNEX III. *Prepotentials of Uniform Spherical Shell and Solid Sphere.* Art. Nos. 64 to 92.

64. The prepotentials in question depend ultimately upon two integrals, which also arise, as will presently appear, from prepotential problems in two-dimensional space, and which are for convenience termed the ring-integral and the disk-integral respectively. The analytical investigation in regard to these, depending as it does on a transformation of a function allied with the hypergeometric series, is I think interesting.

65. Consider first the prepotential of a uniform $(s+1)$-dimensional spherical shell. This is

$$V=\int\frac{dS}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s+q}},$$

the equation of the surface being $x^2+\ldots+z^2+w^2=f^2$; and there are the two cases of an internal point, $a^2+\ldots+c^2+e^2<f^2$, and an external point, $a^2+\ldots+c^2+e^2>f^2$.

The value is a function of $a^2+\ldots+c^2+e^2$, say this is $=\kappa^2$. Taking the axes so that the coordinates of the attracted point are $(0,\ldots,0,\kappa)$, the integral is

$$=\int\frac{dS}{\{x^2+\ldots+z^2+(\kappa-w)^2\}^{\frac{1}{2}s+q}},$$

where the equation of the surface is still $x^2 + \ldots + z^2 + w^2 = f^2$. Writing $x = f\xi, \ldots, z = f\zeta$, $w = f\omega$, where $\xi^2 + \ldots + \zeta^2 + \omega^2 = 1$, we have $dS = \dfrac{f^s\, d\xi \ldots d\zeta}{\omega}$, or the integral is

$$= f^s \int \frac{d\xi \ldots d\zeta}{\omega\,(f^2 - 2\kappa f\omega + \kappa^2)^{\frac{1}{2}s+q}}.$$

Assume $\xi = px, \ldots, \zeta = pz$, where $x^2 + \ldots + z^2 = 1$; then $p^2 + \omega^2 = 1$. Moreover, $d\xi \ldots d\zeta$. $= p^{s-1}\, dp\, d\Sigma$, where $d\Sigma$ is the element of surface of the s-dimensional unit-sphere $x^2 + \ldots + z^2 = 1$; or for p, substituting its value $\sqrt{1-\omega^2}$, we have $dp = \dfrac{-\omega d\omega}{\sqrt{1-\omega^2}}$; and thence $d\xi \ldots d\zeta = -(1-\omega^2)^{\frac{1}{2}s-1}\,\omega\, d\omega\, d\Sigma$. The integral as regards p is from $p = -1$ to $+1$, or as regards ω from 1 to -1; whence reversing the sign, the integral will be from $\omega = -1$ to $+1$; and the required integral is thus

$$= f^s \int_{-1}^{1} \frac{(1-\omega^2)^{\frac{1}{2}s-1}\, d\omega\, d\Sigma}{(f^2 - 2\kappa f\omega + \kappa^2)^{\frac{1}{2}s+q}}, \quad = f^s \int d\Sigma \int_{-1}^{1} \frac{(1-\omega^2)^{\frac{1}{2}s-1}\, d\omega}{(f^2 - 2\kappa f\omega + \kappa^2)^{\frac{1}{2}s+q}},$$

where $\int d\Sigma$ is the surface of the s-dimensional unit-sphere (see Annex I.), $= \dfrac{2\,(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$; and for greater convenience transforming the second factor by writing therein $\omega = \cos\theta$, the required integral is $= \dfrac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}$ multiplied by

$$2f^s \int_0^\pi \frac{\sin^{s-1}\theta\, d\theta}{(f^2 - 2\kappa f\cos\theta + \kappa^2)^{\frac{1}{2}s+q}},$$

which last expression—including the factor $2f^s$, but without the factor $\dfrac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$—is the ring-integral discussed in the present Annex. It may be remarked that the value can be at once obtained in the particular case $s = 2$, which belongs to tridimensional space: viz. we then have

$$\begin{aligned} V &= 2\pi f^2 \int_0^\pi \frac{\sin\theta\, d\theta}{(f^2 - 2\kappa f\cos\theta + \kappa^2)^{q+1}} \\ &= \frac{2\pi f^2}{2\kappa f q}\,(f^2 - 2\kappa f\cos\theta + \kappa^2)^{-q} \\ &= \frac{\pi f}{\kappa q}\,\{(f-\kappa)^{-2q} - (f+\kappa)^{-2q}\}, \end{aligned}$$

which agrees with a result given, *Mécanique Céleste*, Book XII. Chap. II.

66. Consider next the prepotential of the uniform solid $(s+1)$-dimensional sphere,

$$V = \int \frac{dx \ldots dz\, dw}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+q}},$$

the equation of the surface being $x^2 + \ldots + z^2 + w^2 = f^2$; there are the two cases of an internal point $\kappa < f$, and an external point $\kappa > f$ ($a^2 + \ldots + c^2 + e^2 = \kappa^2$ as before).

Transforming so that the coordinates of the attracted point are $0, \ldots, 0, \kappa$, the integral is

$$= \int \frac{dx \ldots dz\, dw}{\{x^2 + \ldots + z^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}},$$

where the equation is still $x^2 + \ldots + z^2 + w^2 = f^2$. Writing here $x = r\xi, \ldots, z = r\zeta$, where $\xi^2 + \ldots + \zeta^2 = 1$, we have $dx \ldots dz = r^{s-1}\, dr\, d\Sigma$, where $d\Sigma$ is an element of surface of the s-dimensional unit-sphere $\xi^2 + \ldots + \zeta^2 = 1$; the integral is therefore

$$= \int \frac{r^{s-1}\, dr\, d\Sigma\, dw}{\{r^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}}$$

$$= \int d\Sigma \int \frac{r^{s-1}\, dr\, dw}{\{r^2 + (\kappa - w)^2\}^{\frac{1}{2}s+q}},$$

where, as regards r and w, the integration extends over the circle $r^2 + w^2 = f^2$. The value of the first factor (see Annex I.) is $= \dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$; writing y and x in place of r and w respectively, the integral is $= \dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}$ multiplied by

$$\int \frac{y^{s-1}\, dx\, dy}{\{(x-\kappa)^2 + y^2\}^{\frac{1}{2}s+q}}$$

over the circle $x^2 + y^2 = f^2$; viz. this last expression $\left(\text{without the factor } \dfrac{2(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)}\right)$ is the disk-integral discussed in the present Annex.

67. We find, for the value in regard to an internal point $\kappa < f$,

$$V = \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} f^{s+1} \int_0^\infty (t + f^2 - \kappa^2)^{\frac{1}{2}-q}\, t^{-\frac{1}{2}-q}\, (t+f^2)^{-\frac{1}{2}s+q-1}\, dt,$$

which, in the particular case $q = -\frac{1}{2}$, is

$$= \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} f^{s+1} \int_0^\infty (t + f^2 - \kappa^2)\,(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\, dt;$$

viz. the integral in t is here

$$= \int_0^\infty \{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}} - \kappa^2 (t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\}\, dt, \quad = \frac{1}{f^{s+1}}\left(\frac{f^2}{\frac{1}{2}s - \frac{1}{2}} - \frac{\kappa^2}{\frac{1}{2}s + \frac{1}{2}}\right),$$

or we have

$$V = \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s - \frac{1}{2})} \left(\frac{f^2}{\frac{1}{2}s - \frac{1}{2}} - \frac{\kappa^2}{\frac{1}{2}s + \frac{1}{2}}\right).$$

It may be added that, in regard to an external point $\kappa > f$, the value is

$$V = \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} f^{s+1} \text{ and } \int_{\kappa^2 - f^2}^\infty (t + f^2 - \kappa^2)^{\frac{1}{2}-q}\, t^{-\frac{1}{2}-q}\, (t+f^2)^{-\frac{1}{2}s+q-1}\, dt,$$

which, in the same case $q=-\frac{1}{2}$, is

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}f^{s+1} \text{ and } \int_{\kappa^2-f^2}^{\infty}(t+f^2-\kappa^2)(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\,dt,$$

where the t-integral is

$$=\int_{\kappa^2-f^2}^{\infty}\{(t+f^2)^{-\frac{1}{2}s-\frac{1}{2}}-\kappa^2(t+f^2)^{-\frac{1}{2}s-\frac{3}{2}}\}\,dt,\quad =\frac{\kappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}}-\frac{\kappa^2.\,\kappa^{-s-1}}{\frac{1}{2}s+\frac{1}{2}},\quad =\frac{\kappa^{-s+1}}{\frac{1}{2}s-\frac{1}{2}\cdot\frac{1}{2}s+\frac{1}{2}};$$

and the value of V is therefore

$$=\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\frac{f^{s+1}}{\kappa^{s-1}}.$$

Recurring to the case of the internal point; then, writing $\nabla=\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}$, and observing that $\nabla(\kappa^2)=4(\frac{1}{2}s+\frac{1}{2})$, we have

$$\nabla V=-\frac{4(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s-\frac{1}{2})}:$$

(in particular, for ordinary space $s+1=3$, or the value is $\frac{-4\pi^{\frac{3}{2}}}{\sqrt{\pi}}$, $=-4\pi$, which is right).

68. The integrals referred to as the ring-integral and the disk-integral arise also from the following integrals in two-dimensional space, viz. these are

$$\int\frac{y^{s-1}\,dS}{\{(x-\kappa)^2+y^2\}^{\frac{1}{2}s+q}},\quad \int\frac{y^{s-1}\,dx\,dy}{\{(x-\kappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

in the first of which dS denotes an element of arc of the circle $x^2+y^2=f^2$, the integration being extended over the whole circumference, and in the second the integration extends over the circle $x^2+y^2=f^2$; y^{s-1} is written for shortness instead of $(y^2)^{\frac{1}{2}(s-1)}$, viz. this is considered as always positive, whether y is positive or negative; it is moreover assumed that $s-1$ is zero or positive.

Writing in the first integral $x=f\cos\theta$, $y=f\sin\theta$, the value is

$$=f^s\int\frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{\frac{1}{2}s+q}};$$

viz. this represents the prepotential of the circumference of the circle, density varying as $(\sin\theta)^{s-1}$, in regard to a point $x=\kappa$, $y=0$ in the plane of the circle; and similarly the second integral represents the prepotential of the circular disk, density of the element at the point $(x, y)=y^{s-1}$, in regard to the same point $x=\kappa$, $y=0$; it being in each case assumed that the prepotential of an element of mass $\rho\,d\varpi$ at a point at distance d is $=\frac{\rho\,d\varpi}{d^{s+2q}}$.

69. In the case of the circumference, it is assumed that the attracted point is not on the circumference, κ not $=f$; and the function under the integral sign, and therefore the integral itself, is in every case finite. In the case of the circle, if κ be an interior point, then if $2q-1$ be $=0$ or positive, the element at the attracted point becomes infinite; but to avoid this we consider, not the potential of the whole circle, but the potential of the circle *less* an indefinitely small circle radius ϵ having the attracted point for its centre; which being so, the element under the integral sign, and consequently the integral itself, remains finite.

It is to be remarked that the two integrals are connected with each other; viz. the circle of the second integral being divided into rings by means of a system of circles concentric with the bounding circle $x^2+y^2=f^2$, then the prepotential of each ring or annulus is determined by an integral such as the first integral; or, analytically, writing in the second integral $x=r\cos\theta$, $y=r\sin\theta$, and therefore $dx\,dy=r\,dr\,d\theta$, the second integral is

$$=\int r^s\,dr\int\frac{(\sin\theta)^{s-1}\,d\theta}{(r^2+\kappa^2-2\kappa r\cos\theta)^{\frac{1}{2}s+q}},$$

viz. the integral in regard to θ is here the same function of r, κ that the first integral is of f, κ; and the integration in regard to r is of course to be taken from $r=0$ to $r=f$. But the θ-integral is not, in its original form, such a function of r as to render possible the integration in regard to r; and I, in fact, obtain the second integral by a different and in some respects a better process.

70. Consider first the ring-integral which, writing therein as above $x=f\cos\theta$, $y=f\sin\theta$, and multiplying by 2 in order that the integral, instead of being taken from 0 to 2π, may be taken from 0 to π, becomes

$$=2f^s\int\frac{(\sin\theta)^{s-1}\,d\theta}{(f^2-2\kappa f\cos\theta+\kappa^2)^{\frac{1}{2}s+q}}.$$

Write $\cos\frac{1}{2}\theta=\sqrt{x}$; then $\sin\frac{1}{2}\theta=\sqrt{1-x}$, $\sin\theta=2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$; $d\theta=-x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$; $\cos\theta=-1+2x$; $\theta=0$ gives $x=1$, $\theta=\pi$ gives $x=0$, and the integral is

$$=2^{s-1}f^s\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{\{(f+\kappa)^2-4\kappa fx\}^{\frac{1}{2}s+q}},$$

$$=\frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}}\int_0^1\frac{x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx}{(1-ux)^{\frac{1}{2}s+q}},$$

if for shortness $u=\dfrac{4\kappa f}{(\kappa+f)^2}$, so that obviously $u<1$.

The integral in x is here an integral belonging to the general form

$$\Pi(\alpha,\ \beta,\ \gamma,\ u)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}\,dx,$$

viz. we have

$$\text{Ring-integral}=\frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}}\,\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u\right).$$

71. The general function $\Pi(\alpha, \beta, \gamma, u)$ is

$$\Pi(\alpha, \beta, \gamma, u) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)} F(\alpha, \gamma, \alpha+\beta, u),$$

or, what is the same thing,

$$F(\alpha, \beta, \gamma, u) = \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)} \Pi(\alpha, \gamma-\alpha, \beta, u),$$

and consequently transformable by means of various theorems for the transformation of the hypergeometric series, in particular, by the theorems

$$F(\alpha, \beta, \gamma, u) = F(\beta, \alpha, \gamma, u),$$
$$F(\alpha, \beta, \gamma, u) = (1-u)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, u);$$

and if $v = \left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)^2$, or, what is the same thing, $u = \frac{4\sqrt{v}}{(1+\sqrt{v})^2}$, then

$$F(\alpha, \beta, 2\beta, u) = (1+\sqrt{v})^{2\alpha} F(\alpha, \alpha-\beta+\tfrac{1}{2}, v).$$

In verification, observe that if $u=1$ then also $v=1$, and that with these values, calculating each side by means of the formulæ

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma\gamma\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}, \qquad \Pi(\alpha, \beta, \gamma, 1) = \frac{\Gamma\alpha\,\Gamma(\beta-\gamma)}{\Gamma(\alpha+\beta-\gamma)},$$

the resulting equation, $F(\alpha, \beta, 2\beta, 1) = 2^{2\alpha} F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, 1)$, becomes

$$\frac{\Gamma 2\beta\,\Gamma(\beta-\alpha)}{\Gamma(2\beta-\alpha)\,\Gamma\beta} = 2^{2\alpha} \frac{\Gamma(\beta+\frac{1}{2})\,\Gamma(2\beta-2\alpha)}{\Gamma(2\beta-\alpha)\,\Gamma(\beta-\alpha+\frac{1}{2})},$$

that is,

$$\frac{\Gamma 2\beta}{\Gamma\beta\,\Gamma(\beta+\frac{1}{2})} = 2^{2\alpha} \frac{\Gamma(2\beta-2\alpha)}{\Gamma(\beta-\alpha)\,\Gamma(\beta-\alpha+\frac{1}{2})},$$

which is true, in virtue of the relation $\frac{\Gamma 2x\,\Gamma\frac{1}{2}}{\Gamma x\,\Gamma(x+\frac{1}{2})} = 2^{2x-1}$.

72. The foregoing formulæ, and in particular the formula which I have written $F(\alpha, \beta, 2\beta, u) = (1+\sqrt{v})^{2\alpha} F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, v)$, are taken from Kummer's Memoir, "Ueber die hypergeometrische Reihe," *Crelle*, t. XV. (1836), viz. the formula in question is, under a slightly different form, his formula (41), p. 76; the formula (43), p. 77, is intended to be equivalent thereto; but there is an error of transcription, $2\alpha-2\beta+1$, in place of $\beta+\frac{1}{2}$, which makes the formula (43) erroneous.

It may be remarked as to the formulæ generally that, although very probably $\Pi(\alpha, \beta, \gamma, u)$ may denote a proper function of u, whatever be the values of the indices (α, β, γ), and the various transformation-theorems hold good accordingly (the Γ-function of a negative argument being interpreted in the usual manner by means of the equation $\Gamma x = \frac{1}{x}\Gamma(1+x)$, $= \frac{1}{x(x+1)}\Gamma(2+x)$ &c.), yet that the function $\Pi(\alpha, \beta, \gamma, u)$,

used as denoting the definite integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\gamma}\,dx$, has no meaning except in the case where α and β are each of them positive.

In what follows we obtain for the ring-integral and the disk-integral various expressions in terms of Π-functions, which are afterwards transformed into t-integrals with a superior limit ∞ and inferior limit 0, or κ^2-f^2; but for values of the variable index, q lying beyond certain limits, the indices α and β, or one of them, of the Π-function will become negative, viz. the integral represented by the Π-function, or, what is the same thing, the t-integral, will cease to have a determinate value, and at the same time, or usually so, the argument or arguments of one or more of the Γ-functions will become negative. It is quite possible that in such cases the results are not without meaning, and that an interpretation for them might be found; but they have not any obvious interpretation, and we must in the first instance consider them as inapplicable.

73. We require further properties of the Π-functions. Starting with the foregoing equation

$$F(\alpha,\ \beta,\ 2\beta,\ u)=(1+\sqrt{v})^{2\alpha}F(\alpha,\ \alpha-\beta+\tfrac{1}{2},\ \beta+\tfrac{1}{2},\ v),$$

each side may be expressed in a fourfold form:—

$$\begin{array}{l|l}
 & = \\
F(\alpha,\ \beta,\ 2\beta,\ u) & (1+\sqrt{v})^{2\alpha}F(\alpha,\ \alpha-\beta+\tfrac{1}{2},\ \beta+\tfrac{1}{2},\ v) \\
=F(\beta,\ \alpha,\ 2\beta,\ u) & =(1+\sqrt{v})^{2\alpha}F(\alpha-\beta+\tfrac{1}{2},\ \alpha,\ \beta+\tfrac{1}{2},\ v) \\
=(1-u)^{\beta-\alpha}F(2\beta-\alpha,\ \beta,\ 2\beta,\ u) & =(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(\beta-\alpha+\tfrac{1}{2},\ 2\beta-\alpha,\ \beta+\tfrac{1}{2},\ v) \\
=(1-u)^{\beta-\alpha}F(\alpha,\ 2\beta-\alpha,\ 2\beta,\ u) & =(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}F(2\beta-\alpha,\ \beta-\alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2},\ v),
\end{array}$$

where, instead of $(1+\sqrt{v})^{2\alpha}(1-v)^{2\beta-2\alpha}$, it is proper to write $(1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}$; and then to each form applying the transformation

$$F(\alpha,\ \beta,\ \gamma,\ u)=\frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}\Pi(\alpha,\ \gamma-\alpha,\ \beta,\ u),$$

we have

$$\begin{aligned}
&\frac{\Gamma 2\beta}{\Gamma\alpha\,\Gamma(2\beta-\alpha)}\Pi(\alpha,\ 2\beta-\alpha,\ \beta,\ u)\\
=&\frac{\Gamma 2\beta}{\Gamma\beta\,\Gamma\beta}\Pi(\beta,\ \beta,\ \alpha,\ u)\\
=&(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma(2\beta-\alpha)\Gamma\alpha}\Pi(2\beta-\alpha,\ \alpha,\ \beta,\ u)\\
=&(1-u)^{\beta-\alpha}\frac{\Gamma 2\beta}{\Gamma\alpha\,\Gamma(2\beta-\alpha)}\Pi(\alpha,\ 2\beta-\alpha,\ 2\beta-\alpha,\ u);
\end{aligned}$$

$$= (1+\sqrt{v})^{2\alpha} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma\alpha\,\Gamma(\beta-\alpha+\frac{1}{2})} \Pi(\alpha,\ \beta-\alpha+\tfrac{1}{2},\ \alpha-\beta+\tfrac{1}{2},\ v)$$

$$= (1+\sqrt{v})^{2\alpha} \frac{\Gamma(\beta+\frac{1}{2})\,1}{\Gamma(\alpha-\beta+\frac{1}{2})\,\Gamma(2\beta-\alpha)} \Pi(\alpha-\beta+\tfrac{1}{2},\ 2\beta-\alpha,\ \alpha,\ v)$$

$$= (1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta-\alpha+\frac{1}{2})\,\Gamma\alpha} \Pi(\beta-\alpha+\tfrac{1}{2},\ \alpha,\ 2\beta-\alpha,\ v)$$

$$= (1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(2\beta-\alpha)\,\Gamma(\alpha-\beta+\frac{1}{2})} \Pi(2\beta-\alpha,\ \alpha-\beta+\tfrac{1}{2},\ \beta-\alpha+\tfrac{1}{2},\ v).$$

I select the second of the first four forms; equating it successively to each of the second four forms, and attending to the relation $\frac{\Gamma\beta\,\Gamma(\beta+\frac{1}{2})}{\Gamma 2\beta} = 2^{1-2\beta}\,\Gamma\tfrac{1}{2}$, we find

$$\Pi(\beta,\ \beta,\ \alpha,\ u) = (1+\sqrt{v})^{2\alpha}\,2^{1-2\beta} \frac{\Gamma\beta\,\Gamma\frac{1}{2}}{\Gamma\alpha\,\Gamma(\beta-\alpha+\frac{1}{2})} \Pi(\alpha,\ \beta-\alpha+\tfrac{1}{2},\ \alpha-\beta+\tfrac{1}{2},\ v)$$

$$= (1+\sqrt{v})^{2\alpha}\,2^{1-2\beta} \frac{\Gamma\beta\,\Gamma\frac{1}{2}}{\Gamma(\alpha-\beta+\frac{1}{2})\,\Gamma(2\beta-\alpha)} \Pi(\alpha-\beta+\tfrac{1}{2},\ 2\beta-\alpha,\ \alpha,\ v)$$

$$= (1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\,2^{1-2\beta} \frac{\Gamma\beta\,\Gamma\frac{1}{2}}{\Gamma(\beta-\alpha+\frac{1}{2})\,\Gamma\alpha} \Pi(\beta-\alpha+\tfrac{1}{2},\ \alpha,\ 2\beta-\alpha,\ v)$$

$$= (1+\sqrt{v})^{2\beta}(1-\sqrt{v})^{2\beta-2\alpha}\,2^{1-2\beta} \frac{\Gamma\beta\,\Gamma\frac{1}{2}}{\Gamma(2\beta-\alpha)\,\Gamma(\alpha-\beta+\frac{1}{2})} \Pi(2\beta-\alpha,\ \alpha-\beta+\tfrac{1}{2},\ \beta-\alpha+\tfrac{1}{2},\ v).$$

Putting herein $\beta = \frac{1}{2}s$, $\alpha = \frac{1}{2}s+q$, the formulæ become

$$\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u) = (1+\sqrt{v})^{s+2q}\,2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} \Pi(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ v) \quad \text{.....(I.)}$$

$$= (1+\sqrt{v})^{s+2q}\,2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)} \Pi(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q,\ \tfrac{1}{2}s+q,\ v) \quad \text{.....(II.)}$$

$$= (1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}\,2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)} \Pi(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ v) \quad \text{.....(III.)}$$

$$= (1+\sqrt{v})^{s}(1-\sqrt{v})^{-2q}\,2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)} \Pi(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ v) \quad \text{.....(IV.)},$$

where observe that on the right-hand side the Π-functions in I. and IV. only differ by the sign of q, and so also the Π-functions in II. and III. only differ by the sign of q. We hence have

$$\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s-q,\ u) = (1+\sqrt{v})^{s-2q}\,2^{1-s}\cdot \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)} \Pi(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ v);$$

and comparing with (IV.),

$$\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u\right)=\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^{2q}\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s-q,\ u\right).$$

74. The foregoing formula,

$$\text{Ring-integral}=\frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}}\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u\right),$$

where $u=\dfrac{4\kappa f}{(f+\kappa)^2}$, gives, as well in the case of an exterior as an interior point, a convergent series for the integral; but this series proceeds according to the powers of $\dfrac{4\kappa f}{(f+\kappa)^2}$. We may obtain more convenient formulæ applying to the cases of an internal and an external point respectively.

75. For an internal point $\kappa<f$, $\sqrt{1-u}=\dfrac{f-\kappa}{f+\kappa}$, and therefore $v=\dfrac{\kappa^2}{f^2}$.

$$\begin{aligned}
\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u\right)&=\left(\frac{f+\kappa}{f}\right)^{s+2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{2}s+q\right)\Gamma\left(\tfrac{1}{2}-q\right)}\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s+2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{2}+q\right)\Gamma\left(\tfrac{1}{2}s-q\right)}\Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q,\ \tfrac{1}{2}s+q,\ \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{2}-q\right)\Gamma\left(\tfrac{1}{2}s+q\right)}\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{\kappa^2}{f^2}\right)\\
&=\left(\frac{f+\kappa}{f}\right)^{s}\left(\frac{f-\kappa}{f}\right)^{-2q}2^{1-s}\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{2}s-q\right)\Gamma\left(\tfrac{1}{2}+q\right)}\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right):
\end{aligned}$$

where the Π-functions on the right-hand side are respectively

$$\begin{aligned}
&=f^{2q+1}\int_0^1\frac{x^{\frac{1}{2}s+q-1}(1-x)^{-q-\frac{1}{2}}\,dx}{(f^2-\kappa^2x)^{q+\frac{1}{2}}} &&=\frac{f^{2q+1}}{(f^2-\kappa^2)^{2q}}\int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}\,dt\\
&=f^{s+2q}\int_0^1\frac{x^{q-\frac{1}{2}}(1-x)^{\frac{1}{2}s-q-1}\,dx}{(f^2-\kappa^2x)^{\frac{1}{2}s+q}} &&=\frac{f^{s+2q}}{(f^2-\kappa^2)^{2q}}\int_0^\infty t^{q-\frac{1}{2}}(t+f^2-\kappa^2)^{q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s-q}\,dt\\
&=f^{s-2q}\int_0^1\frac{x^{-q-\frac{1}{2}}(1-x)^{\frac{1}{2}s+q-1}\,dx}{(f^2-\kappa^2x)^{\frac{1}{2}s-q}} &&=\frac{f^{s-2q}}{(f^2-\kappa^2)^{-2q}}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt\\
&=f^{-2q+1}\int_0^1\frac{x^{\frac{1}{2}s-q+1}(1-x)^{q-\frac{1}{2}}\,dx}{(f^2-\kappa^2x)^{-q+\frac{1}{2}}} &&=\frac{f^{-2q+1}}{(f^2-\kappa^2)^{-2q}}\int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}}\,dt,
\end{aligned}$$

the t-forms being obtained by means of the transformation $x=\dfrac{t}{t+f^2-\kappa^2}$; viz. this gives

$$1-x=\frac{f^2-\kappa^2}{t+f^2-\kappa^2},\quad f^2-\kappa^2x=\frac{(f^2-\kappa^2)(t+f^2)}{t+f^2-\kappa^2},\quad dx=\frac{(f^2-\kappa^2)\,dt}{(t+f^2-\kappa^2)^2},$$

whence the results just written down.

We hence have

$$\text{Ring-integral} = \frac{f}{(f^2-\kappa^2)^{2q}} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} \int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s+q}(t+f^2)^{-q-\frac{1}{2}}\,dt$$

$$= \frac{f^s}{(f^2-\kappa^2)^{2q}} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)} \int_0^\infty t^{q-\frac{1}{2}}\quad(t+f^2-\kappa^2)^{q-\frac{1}{2}}\quad(t+f^2)^{-\frac{1}{2}s-q}\,dt$$

$$= \quad f^s \quad \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)} \int_0^\infty t^{-q-\frac{1}{2}}\quad(t+f^2-\kappa^2)^{-q-\frac{1}{2}}\,(t+f^2)^{-\frac{1}{2}s+q}\,dt$$

$$= \quad f \quad \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)} \int_0^\infty t^{\frac{1}{2}s-q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s-q}(t+f^2)^{q-\frac{1}{2}}\,dt.$$

As a verification write $\kappa = 0$, the four integrals are

$$\int_0^\infty \frac{t^{\frac{1}{2}s+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{2q-1}\,\frac{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}+q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{2q-s}\,\frac{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{-2q-s}\,\frac{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^\infty \frac{t^{\frac{1}{2}s-q-1}\,dt}{(t+f^2)^{\frac{1}{2}s+\frac{1}{2}}}, \quad = f^{-2q-1}\,\frac{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

hence from each of them

$$\text{Ring-integral} = \frac{1}{f^{2q}}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

which is, in fact, the value obtained from

$$\text{Ring-integral} = \frac{2^{s-1}f^s}{(f+\kappa)^{s+2q}}\,\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ \frac{4\kappa f}{(\kappa+f)^2}\right)$$

on putting therein $\kappa = 0$; viz. the value is

$$= \frac{2^{s-1}}{f^{2q}}\int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\,dx, \quad = \frac{1}{f^{2q}}\,\frac{2^{s-1}\,\Gamma\frac{1}{2}s\,.\,\Gamma\frac{1}{2}s}{\Gamma s}.$$

76. For an external point $\kappa > f$, $\sqrt{1-u} = \dfrac{\kappa-f}{\kappa+f}$, and therefore $v = \dfrac{f^2}{\kappa^2}$.

$$\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u) = \left(\frac{\kappa+f}{\kappa}\right)^{s+2q} \quad 2^{1-s}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}\,\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}\ -q,\ \tfrac{1}{2}\ +q,\ \frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s+2q} \quad 2^{1-s}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)}\,\Pi\left(\tfrac{1}{2}\ +q,\ \tfrac{1}{2}s-q,\ \tfrac{1}{2}s+q,\ \frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s}\left(\frac{\kappa-f}{\kappa}\right)^{-2q} 2^{1-s}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)}\,\Pi\left(\tfrac{1}{2}\ -q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{f^2}{\kappa^2}\right)$$

$$= \left(\frac{\kappa+f}{\kappa}\right)^{s}\left(\frac{\kappa-f}{\kappa}\right)^{-2q} 2^{1-s}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)}\,\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}\ +q,\ \tfrac{1}{2}\ -q,\ \frac{f^2}{\kappa^2}\right):$$

where the Π-functions on the right hand are respectively

$$= \kappa^{2q+1} \int_0^1 \frac{x^{\frac{1}{2}s+q-1}(1-x)^{-q-\frac{1}{2}}\,dx}{(\kappa^2-f^2x)^{q+\frac{1}{2}}} \qquad = \frac{\kappa^{2q+1}}{(\kappa^2-f^2)^{2q}} \int_{\kappa^2-f^2}^{\infty} t^{-\frac{1}{2}s+q}(t+f^2-\kappa^2)^{\frac{1}{2}s+q-1}(t+f^2)^{-q-\frac{1}{2}}\,dt,$$

$$= \kappa^{s+2q} \int_0^1 \frac{x^{q-\frac{1}{2}}(1-x)^{\frac{1}{2}s-q-1}\,dx}{(\kappa^2-f^2x)^{\frac{1}{2}s+q}} \qquad = \frac{\kappa^{s+2q}}{(\kappa^2-f^2)^{2q}} \int_{\kappa^2-f^2}^{\infty} t^{q-\frac{1}{2}} \quad (t+f^2-\kappa^2)^{q-\frac{1}{2}} \quad (t+f^2)^{-\frac{1}{2}s-q}\,dt,$$

$$= \kappa^{s-2q} \int_0^1 \frac{x^{-q-\frac{1}{2}}(1-x)^{\frac{1}{2}s+q-1}\,dx}{(\kappa^2-f^2x)^{\frac{1}{2}s-q}} \qquad = \frac{\kappa^{s-2q}}{(\kappa^2-f^2)^{-2q}} \int_{\kappa^2-f^2}^{\infty} t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}} \quad (t+f^2)^{-\frac{1}{2}s+q}\,dt,$$

$$= \kappa^{-2q+1} \int_0^1 \frac{x^{\frac{1}{2}s-q+1}(1-x)^{q-\frac{1}{2}}\,dx}{(\kappa^2-f^2x)^{-q+\frac{1}{2}}} \qquad = \frac{\kappa^{-2q+1}}{(\kappa^2-f^2)^{-2q}} \int_{\kappa^2-f^2}^{\infty} t^{-\frac{1}{2}s-q}(t+f^2-\kappa^2)^{\frac{1}{2}s-q-1}(t+f^2)^{q-\frac{1}{2}} \quad dt.$$

We have then

$$\text{Ring-integral} = \frac{f^s\,\kappa^{1-s}}{(\kappa^2-f^2)^{2q}} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} \int_{\kappa^2-f^2}^{\infty} t^{-\frac{1}{2}s+q}(t+f^2-\kappa^2)^{\frac{1}{2}s+q-1}(t+f^2)^{-q-\frac{1}{2}}\,dt$$

$$= \frac{f^s}{(\kappa^2-f^2)^{2q}} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)} \int_{\kappa^2-f^2}^{\infty} t^{q-\frac{1}{2}} \quad (t+f^2-\kappa^2)^{q-\frac{1}{2}} \quad (t+f^2)^{-\frac{1}{2}s-q}\,dt$$

$$= \quad f^s \quad \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)} \int_{\kappa^2-f^2}^{\infty} t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}} \quad (t+f^2)^{-\frac{1}{2}s+q}\,dt$$

$$= \frac{f^s\,\kappa^{1-s}}{(\kappa^2-f^2)^{2q}} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)} \int_{\kappa^2-f^2}^{\infty} t^{-\frac{1}{2}s-q}(t+f^2-\kappa^2)^{\frac{1}{2}s-q-1}(t+f^2)^{q-\frac{1}{2}} \quad dt.$$

Observe that in II. and III. the integrals, except as to the limits, are the same as in the corresponding formulæ for the interior point.

If in the t-integrals we put $t+\kappa^2-f^2$ in place of t, and ultimately suppose κ indefinitely large in comparison with f, they severally become

$$\int_0^{\infty} (t+\kappa^2-f^2)^{-\frac{1}{2}s+q}\, t^{\frac{1}{2}s+q-1}(t+\kappa^2)^{-q-\frac{1}{2}}\,dt = \int_0^{\infty} \frac{t^{\frac{1}{2}s+q-1}\,dt}{(t+\kappa^2)^{\frac{1}{2}s+\frac{1}{2}}} = \kappa^{2q-1} \frac{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^{\infty} (t+\kappa^2-f^2)^{q-\frac{1}{2}} \quad t^{q+\frac{1}{2}} \quad (t+\kappa^2)^{-\frac{1}{2}s-q}\,dt = \int_0^{\infty} \frac{t^{\frac{1}{2}+q-1}\,dt}{(t+\kappa^2)^{\frac{1}{2}s+\frac{1}{2}}} = \kappa^{2q-s} \frac{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^{\infty} (t+\kappa^2-f^2)^{-q-\frac{1}{2}}\, t^{-q-\frac{1}{2}} \quad (t+\kappa^2)^{-\frac{1}{2}s+q}\,dt = \int_0^{\infty} \frac{t^{\frac{1}{2}-q-1}\,dt}{(t+\kappa^2)^{\frac{1}{2}s+\frac{1}{2}}} = \kappa^{-2q-s} \frac{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

$$\int_0^{\infty} (t+\kappa^2-f^2)^{-\frac{1}{2}s-q}\, t^{\frac{1}{2}s-q-1}(t+\kappa^2)^{q-\frac{1}{2}} \quad dt = \int_0^{\infty} \frac{t^{\frac{1}{2}s-q-1}\,dt}{(t+\kappa^2)^{\frac{1}{2}s+\frac{1}{2}}} = \kappa^{-2q-1} \frac{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

and they all four give

$$\text{Ring-integral} = \frac{f^s}{\kappa^{s+2q}} \cdot \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

which agrees with the value

$$\frac{2^{s-1}f^s}{(\kappa+f)^{s+2q}}\,\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ \frac{4\kappa f}{(\kappa+f)^2}\right), \quad = \frac{2^{s-1}f^s}{\kappa^{s+2q}}\,\Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ 0\right),$$

when $\frac{\kappa}{f}$ is indefinitely large.

77. We come now to the disk-integral,

$$\int \frac{y^{s-1}\,dx\,dy}{\{(x-\kappa)^2+y^2\}^{\frac{1}{2}s+q}},$$

over the circle $x^2+y^2=f^2$. Writing $x=\kappa+\rho\cos\phi$, $y=\rho\sin\phi$, we have $dx\,dy=\rho\,d\rho\,d\phi$, and the integral therefore is

$$\iint \frac{\sin^{s-1}\phi\,d\rho\,d\phi}{\rho^{2q}},$$

where the integration in regard to ρ is performed at once; viz. the integral is

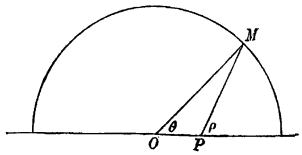
$$=\frac{1}{1-2q}\int(\rho^{1-2q})\sin^{s-1}\phi\,d\phi;$$

or multiplying by 2, in order that the integration may be taken only over the semicircle, $y=$ positive, this is

$$=\frac{1}{\frac{1}{2}-q}\int(\rho^{1-2q})\sin^{s-1}\phi\,d\phi,$$

the term (ρ^{1-2q}) being taken between the proper limits.

78. Consider first an interior point $\kappa<f$. As already mentioned, we exclude an indefinitely small circle radius ϵ, and the limits for ρ are from $\rho=\epsilon$ to $\rho=$ its value at



the circumference; viz. if here $x=f\cos\theta$, $y=f\sin\theta$, then we have $f\cos\theta=\kappa+\rho\cos\phi$, $f\sin\theta=\rho\sin\phi$, and consequently

$$\rho^2=\kappa^2+f^2-2\kappa f\cos\theta,$$

$$\sin\phi=\frac{f}{\rho}\sin\theta,\quad =\frac{f\sin\theta}{\sqrt{\kappa^2+f^2-2\kappa f\cos\theta}},$$

and the integral therefore is

$$=\frac{1}{\frac{1}{2}-q}\left(\frac{f^{s-1}\sin^{s-1}\theta}{\{\kappa^2+f^2-2\kappa f\cos\theta\}^{\frac{1}{2}s+q-1}}-\epsilon^{1-2q}\sin^{s-1}\phi\right)d\phi.$$

As regards the second term, this is $=-\dfrac{\epsilon^{1-2q}}{\frac{1}{2}-q}\displaystyle\int\sin^{s-1}\phi\,d\phi$, from $\phi=0$ to $\phi=\pi$; or, what is the same thing, we may multiply by 2 and take the integral from $\phi=0$ to $\phi=\dfrac{\pi}{2}$.

Writing then $\sin\phi = \sqrt{x}$, and consequently $\sin^{s-1}\phi\, d\phi = \frac{1}{2}x^{\frac{1}{2}s-1}(1-x)^{-\frac{1}{2}}\, dx$, the term is $= -\dfrac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\dfrac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$: the value of the disk-integral is

$$= \frac{f^{s-1}}{\frac{1}{2}-q}\int \frac{\sin^{s-1}\theta\, d\phi}{(\kappa^2+f^2-2\kappa f\cos\theta)^{\frac{1}{2}s+q-1}} - \frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

But we have

$$\sin\phi = \frac{f\sin\theta}{\rho},\quad \cos\phi = \frac{f\cos\theta-\kappa}{\rho},$$

and thence

$$\tan\phi = \frac{f\sin\theta}{f\cos\theta-\kappa},\quad \sec^2\phi\, d\phi = \frac{f(f-\kappa\cos\theta)\, d\theta}{(f\cos\theta-\kappa)^2};$$

that is,

$$d\phi = \frac{f(f-\kappa\cos\theta)\, d\theta}{\rho^2},\quad = \frac{f(f-\kappa\cos\theta)\, d\theta}{f^2+\kappa^2-2\kappa f\cos\theta},$$

or, what is the same thing,

$$= \frac{\frac{1}{2}\{(f^2-\kappa^2)+(f^2+\kappa^2-2\kappa f\cos\theta)\}}{f^2+\kappa^2-2\kappa f\cos\theta};$$

the expression for the disk-integral is therefore

$$= \frac{\frac{1}{2}f^{s-1}}{\frac{1}{2}-q}\int_0^\pi \frac{\sin^{s-1}\theta\,\{(f^2-\kappa^2)+(f^2+\kappa^2-2\kappa f\cos\theta)\}\, d\theta}{\{f^2+\kappa^2-2\kappa f\cos\theta\}^{\frac{1}{2}s+q}} - \frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

79. Writing as before $\cos\frac{1}{2}\theta = \sqrt{x}$, $\sin\frac{1}{2}\theta = \sqrt{1+x}$, &c., and $u = \dfrac{4\kappa f}{(\kappa+f)^2}$, this is

$$= \frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\kappa+f)^{s+2q-2}}\left\{\frac{(f^2-\kappa^2)}{(\kappa+f)^2}\,\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u) + \Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q-1, u)\right\} - \frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}.$$

As a verification, observe that, if $\kappa = 0$, each of the Π-functions becomes

$$= \int_0^1 x^{\frac{1}{2}s-1}(1-x)^{\frac{1}{2}s-1}\, dx,\quad = \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}s}{\Gamma s};$$

hence the whole first term is $= \dfrac{2\,.\,2^{s-2}\,.\,f^{1-2q}}{\frac{1}{2}-q}\,.\,\dfrac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}s}{\Gamma s}$, viz. this is $= \dfrac{f^{1-2q}}{\frac{1}{2}-q}\,\dfrac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}$, and the complete value is

$$= \frac{1}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\{f^{1-2q}-\epsilon^{1-2q}\},$$

vanishing, as it should do, if $f = \epsilon$.

80. In the case of an exterior point $\kappa > f$, the process is somewhat different; but the result is of a like form. We have

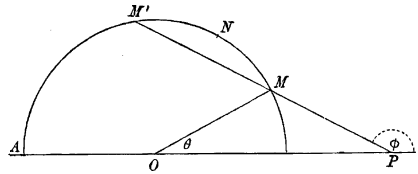
$$\text{Disk-integral} = \frac{1}{\frac{1}{2}-q}\int(\rho_1^{1-2q}-\rho^{1-2q})\sin^{s-1}\phi\, d\phi,$$

where ρ_1 refers to the point M' and ρ to the point M. Attending first to the integral $\int \rho^{1-2q} \sin^{s-1}\phi\, d\phi$, and writing as before $f\cos\theta = \kappa + \rho\cos\phi$, $f\sin\theta = \rho\sin\phi$, this is

$$= f^{s-1} \int \frac{\sin^{s-1}\theta\, d\phi}{\{\kappa^2 + f^2 - 2\kappa f\cos\theta\}^{\frac{1}{2}s+q}}$$

$$= \tfrac{1}{2} f^{s-1} \int \frac{\sin^{s-1}\theta\, \{(f^2-\kappa^2) + (f^2+\kappa^2-2\kappa f\cos\theta)\}\, d\theta}{(f^2+\kappa^2-2f\kappa\cos\theta)^{\frac{1}{2}s+q}},$$

the inferior and the superior limits being here the values of θ which correspond to the points N, A respectively, say $\theta + \alpha$, and $\theta = 0$; hence, reversing the sign and inter-



changing the two limits, the value of $-\int \rho^{1-2q} \sin^{s-1}\theta\, d\phi$ is the above integral taken from 0 to α. But similarly the value of $+\int \rho_1^{1-2q} \sin^{s-1}\theta\, d\phi$ is the same integral taken from α to π. For the two terms together, the value is the same integral from 0 to π; viz. we thus find

$$\text{Disk-integral} = \frac{\frac{1}{2} f^{s-1}}{\frac{1}{2} - q} \int_0^\pi \frac{\sin^{s-1}\theta\, \{-(\kappa^2 - f^2) + (f^2+\kappa^2-2\kappa f\cos\theta)\}\, d\theta}{(f^2+\kappa^2-2f\kappa\cos\theta)^{\frac{1}{2}s+q}};$$

or, writing as before $\cos\frac{1}{2}\theta = \sqrt{x}$, &c., and $u = \dfrac{4\kappa f}{(\kappa+f)^2}$, this is

$$= \frac{2^{s-2} f^{s-1}}{(\frac{1}{2}-q)(\kappa+f)^{s+2q-2}} \left\{ -\frac{\kappa^2 - f^2}{(\kappa+f)^2} . \Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u\right) + \Pi\left(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q-1\right) \right\}.$$

81. As a verification, suppose that κ is indefinitely large: we must recur to the last preceding formula; the value is thus

$$= \frac{f^s}{(\frac{1}{2}-q)\,\kappa^{s+2q-1}} \int_0^\pi \frac{\sin^{s-1}\theta \left(-\cos\theta + \dfrac{f}{\kappa}\right)}{\left(1 - \dfrac{2f}{\kappa}\cos\theta\right)^{\frac{1}{2}s+q}}\, d\theta;$$

viz. this is

$$= \frac{f^s}{(\frac{1}{2}-q)\,\kappa^{s+2q-1}} \int_0^\pi \sin^{s-1}\theta \left\{ -\cos\theta + [1-(s+2q)\cos^2\theta]\,\frac{f}{\kappa} \right\} d\theta,$$

where the integral of the first term vanishes; the value is thus

$$= \frac{f^{s+1}}{(\frac{1}{2}-q)\,\kappa^{s+2q}} \int_0^\pi \sin^{s-1}\theta\, [1-(s+2q)\cos^2\theta]\, d\theta,$$

where we may multiply by 2 and take the integral from 0 to $\frac{\pi}{2}$. Writing then $\sin\theta = \sqrt{x}$, the value is

$$= \frac{f^{s+1}}{(\frac{1}{2}-q)\,\kappa^{s+2q}} \int_0^1 x^{\frac{1}{2}s-\frac{1}{2}} \{1-(s+2q)(1-x)\}(1-x)^{-\frac{1}{2}}\,dx,$$

where the integral is

$$= \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})} \left\{1 - \frac{\frac{1}{2}(s+2q)}{\frac{1}{2}s+\frac{1}{2}}\right\}, \quad = \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})} \cdot \frac{\frac{1}{2}-q}{\frac{1}{2}s+\frac{1}{2}},$$

and hence the value is

$$= \frac{f^{s+1}}{\kappa^{s+2q}} \cdot \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{3}{2})};$$

viz. this is $= \frac{1}{\kappa^{s+2q}} \int y^{s-1}\,dx\,dy$, over the circle $x^2+y^2=f^2$, as is easily verified.

82. Reverting to the interior point $\kappa < f$,

Disk-integral

$$= \frac{2^{s-2} f^{s-1}}{(\frac{1}{2}-q)(\kappa+f)^{s+2q-2}} \left\{\frac{f-\kappa}{f+\kappa}\,\Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q, u) + \Pi(\tfrac{1}{2}s, \tfrac{1}{2}s, \tfrac{1}{2}s+q-1, u)\right\} - \frac{\epsilon^{1-2q}}{\frac{1}{2}-q} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{1}{2})};$$

then reducing the expression in { } by the transformations for $\Pi(\frac{1}{2}s, \frac{1}{2}s, \frac{1}{2}s+q, u)$ and the like transformations for $\Pi(\frac{1}{2}s, \frac{1}{2}s, \frac{1}{2}s+q-1, u)$, the term in { } may be expressed in the four forms:—

$$2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)} \frac{(f+\kappa)^{s+2q-2}}{f^{s+2q-2}} \text{ multiplied by}$$

$$\left[\left(1-\frac{\kappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q, \tfrac{1}{2}-q, \tfrac{1}{2}+q, \frac{\kappa^2}{f^2}\right) + \frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\,\Pi\left(\tfrac{1}{2}s+q-1, \tfrac{3}{2}-q, -\tfrac{1}{2}+q, \frac{\kappa^2}{f^2}\right)\right],$$

$$2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)} \frac{(f+\kappa)^{s+2q-2}}{f^{s+2q-2}} \text{ multiplied by}$$

$$\left[\left(1-\frac{\kappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}+q, \tfrac{1}{2}s-q, \tfrac{1}{2}s+q, \frac{\kappa^2}{f^2}\right) + \frac{-\frac{1}{2}+q}{\frac{1}{2}s+q}\,\Pi\left(-\tfrac{1}{2}+q, \tfrac{1}{2}s-q+1, \tfrac{1}{2}s+q-1, \frac{\kappa^2}{f^2}\right)\right],$$

$$2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)} \frac{(f+\kappa)^{s-1}(f-\kappa)^{1-2q}}{f^{s-2q}} \text{ multiplied by}$$

$$\left[\Pi\left(\tfrac{1}{2}-q, \tfrac{1}{2}s+q, \tfrac{1}{2}s-q, \frac{\kappa^2}{f^2}\right) + \left(1-\frac{\kappa^2}{f^2}\right)\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\,\Pi\left(\tfrac{3}{2}-q, \tfrac{1}{2}s+q-1, \tfrac{1}{2}s-q+1, \frac{\kappa^2}{f^2}\right)\right],$$

$$2^{1-s} \frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)} \frac{(f+\kappa)^{s-1}(f-\kappa)^{1-2q}}{f^{s-2q}} \text{ multiplied by}$$

$$\left[\Pi\left(\tfrac{1}{2}s-q, \tfrac{1}{2}+q, \tfrac{1}{2}-q, \frac{\kappa^2}{f^2}\right) + \left(1-\frac{\kappa^2}{f^2}\right)\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\,\Pi\left(\tfrac{1}{2}s-q+1, -\tfrac{1}{2}+q, \tfrac{3}{2}-q, \frac{\kappa^2}{f^2}\right)\right].$$

83. The first and the fourth of these are susceptible of a reduction which does not appear to be applicable to the second and the third. Consider in general the function

$$(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\,\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v);$$

the second Π-function is here

$$\int_0^1 x^{\alpha-2}(1-x\,.\,1-vx)^\beta\,dx;$$

viz. this is

$$=\frac{x^{\alpha-1}}{\alpha-1}(1-x\,.\,1-vx)^\beta-\frac{1}{\alpha-1}\int_0^1 x^{\alpha-1}\frac{d}{dx}(1-x\,.\,1-vx)^\beta\,dx,$$

or, since the first term vanishes between the limits, this is

$$=\frac{\beta}{\alpha-1}\int_0^1 x^{\alpha-1}\,.\,(1-x\,.\,1-vx)^{\beta-1}(1+v-2vx)\,dx,$$

$$=\frac{\beta}{\alpha-1}\left\{(1+v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)-2v\int_0^1 x^\alpha(1-x\,.\,1-vx)^{\beta-1}\,dx\right\}.$$

Hence the two Π-functions together are

$$=(1-v+1+v)\int_0^1 x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}\,dx-2\int_0^1 vx\,.\,x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}\,dx,$$

$$=2\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-vx)^\beta\,dx,$$

that is,

$$(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\,\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)=2\Pi(\alpha,\ \beta,\ -\beta,\ v).$$

We have therefore

$$\left(1-\frac{\kappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right)+\frac{\tfrac{1}{2}s+q-1}{\tfrac{1}{2}-q}\,\Pi\left(\tfrac{1}{2}s+q-1,\ \tfrac{3}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right)$$
$$=2\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right):$$

and from the same equation, written in the form

$$\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)+\frac{\beta}{\alpha-1}(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)=2\,\frac{\beta}{\alpha-1}\,\Pi(\alpha,\ \beta,\ -\beta,\ v),$$

we obtain

$$\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right)+\frac{-\tfrac{1}{2}+q}{\tfrac{1}{2}s-q}\left(1-\frac{\kappa^2}{f^2}\right)\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\ \frac{\kappa^2}{f^2}\right)$$
$$=\frac{2(-\tfrac{1}{2}+q)}{\tfrac{1}{2}s-q}\,\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right).$$

84. Hence the terms in [ ] in the first and the fourth expressions in No. 82 are

$$=\frac{2^{2-s}\,\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+q)\,\Gamma(\tfrac{1}{2}-q)}\,.\,\frac{(f+\kappa)^{s+2q-2}}{f^{s+2q-2}}\,.\,\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right),$$

$$=\frac{2^{2-s}(-\tfrac{1}{2}+q)\,.\,\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q+1)\,\Gamma(\tfrac{1}{2}+q)}\,\frac{(f+\kappa)^{s-1}(f-\kappa)^{1-2q}}{f^{s-2q}}\,\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right),$$

respectively; the corresponding values of the disk-integral are

$$\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{3}{2}-q)\,\Gamma(\tfrac{1}{2}s+q)}\,f^{1-2q}\,.\,\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right) \qquad -\frac{\epsilon^{1-2q}}{\tfrac{1}{2}-q}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

$$\frac{-\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s-q+1)\,\Gamma(\tfrac{1}{2}+q)}\left(\frac{f^2-\kappa^2}{f}\right)^{1-2q}.\,\Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right)-\frac{\epsilon^{1-2q}}{\tfrac{1}{2}-q}\,\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

which we may again verify by writing therein $\kappa=0$, viz. the Π-functions thus become

$$\frac{\Gamma(\tfrac{1}{2}s+q)\,.\,\Gamma(\tfrac{1}{2}-q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})} \text{ and } \frac{\Gamma(\tfrac{1}{2}s-q+1)\,\Gamma(-\tfrac{1}{2}+q)}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})},$$

and consequently the integral is

$$=\frac{1}{\tfrac{1}{2}-q}\cdot\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}s+\tfrac{1}{2})}\,(f^{1-2q}-\epsilon^{1-2q}).$$

85. But the forms nevertheless belong to a system of four. In the formulæ

$$\begin{aligned}
&\Pi(\alpha,\ \beta,\ \gamma,\ v)\\
=\ &\frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\gamma\,\Gamma(\alpha+\beta-\gamma)}\,\Pi(\gamma,\ \alpha+\beta-\gamma,\ \alpha,\ v)\\
=(1-v)^{\beta-\gamma}\ &\Pi(\beta,\ \alpha,\ \alpha+\beta-\gamma,\ v)\\
=(1-v)^{\beta-\gamma}\ &\frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta-\gamma)\,\Gamma\gamma}\,\Pi(\alpha+\beta-\gamma,\ \gamma,\ \beta,\ v),
\end{aligned}$$

writing $\alpha=\tfrac{1}{2}s+q$, $\beta=\tfrac{1}{2}-q$, $\gamma=-\tfrac{1}{2}+q$, we deduce

$$\begin{aligned}
&\Pi(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ v)\\
=\ &\frac{\Gamma(\tfrac{1}{2}s+q)\,\Gamma(\tfrac{1}{2}-q)}{\Gamma(-\tfrac{1}{2}+q)\,\Gamma(\tfrac{1}{2}s-q+1)}\,\Pi(-\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\ v)\\
=(1-v)^{1-2q}\ &\Pi(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\ v)\\
=(1-v)^{1-2q}\ &\frac{\Gamma(\tfrac{1}{2}s+q)\,\Gamma(\tfrac{1}{2}-q)}{\Gamma(\tfrac{1}{2}s-q+1)\,\Gamma(-\tfrac{1}{2}+q)}\,\Pi(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ v);
\end{aligned}$$

and the last-mentioned values of the disk-integral may thus be written in the four forms:

$$\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{3}{2}-q)\,\Gamma(\tfrac{1}{2}s+q)}\quad f^{1-2q}\quad \Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{\kappa^2}{f^2}\right)\quad -\text{term in }\epsilon,$$

$$\frac{-\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\,\Gamma(\tfrac{1}{2}s-q+1)}\quad f^{1-2q}\quad \Pi\left(-\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\ \frac{\kappa^2}{f^2}\right)-\quad ,,\quad ,$$

$$\frac{\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{3}{2}-q)\,\Gamma(\tfrac{1}{2}s+q)}\quad \left(f-\frac{\kappa^2}{f}\right)^{1-2q}\quad \Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\ \frac{\kappa^2}{f^2}\right)\quad -\quad ,,\quad ,$$

$$\frac{-\Gamma\tfrac{1}{2}s\,\Gamma\tfrac{1}{2}}{\Gamma(\tfrac{1}{2}+q)\,\Gamma(\tfrac{1}{2}s-q+1)}\left(f-\frac{\kappa^2}{f}\right)^{1-2q}\quad \Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{\kappa^2}{f^2}\right)-\quad ,,\quad ;$$

and since the last of these is in fact the second of the original forms, it is clear that, if instead of the first we had taken the second of the original forms, we should have obtained again the same system of four forms.

86. Writing as before $x = \dfrac{t}{t+f^2-\kappa^2}$, &c., the forms are

$$\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\,\Gamma(\frac{1}{2}s+q)}\quad (f^2-\kappa^2)^{1-2q}\quad \int_0^\infty t^{\frac{1}{2}s+q-1}(t+f^2-\kappa^2)^{-\frac{1}{2}s+q-1}(t+f^2)^{-\frac{1}{2}s-q}\ dt - \text{term in } \epsilon,$$

$$\frac{-\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q+1)}f^{s+1}(f^2-\kappa^2)^{1-2q}\int_0^\infty t^{-\frac{3}{2}+q}\ (t+f^2-\kappa^2)^{\ q-\frac{1}{2}}\ (t+f^2)^{-\frac{1}{2}s-q}\ dt - \quad ,,\quad ,$$

$$\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\,\Gamma(\frac{1}{2}s+q)}\quad f^{s+1}\quad \int_0^\infty t^{-q-\frac{1}{2}}\ (t+f^2-\kappa^2)^{-q+\frac{1}{2}}\ (t+f^2)^{-\frac{1}{2}s+q-1}\,dt - \quad ,,\quad ,$$

$$\frac{-\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\,\Gamma(\frac{1}{2}+q)}\quad \int_0^\infty t^{\frac{1}{2}s-q}\ (t+f^2-\kappa^2)^{-\frac{1}{2}s-q}\ (t+f^2)^{-\frac{1}{2}+q}\ dt - \quad ,,\quad .$$

87. The third of these possesses a remarkable property. Write mf instead of f, and at the same time change t into m^2t: the integral becomes

$$\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\,\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\{m^2(t+f^2)-\kappa^2\}^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt - \text{term in } \epsilon;$$

and hence, writing $mf = f+\delta f$ or $m = 1+\dfrac{\delta f}{f}$, and therefore $m^2 = 1+2\dfrac{\delta f}{f}$, the value is

$$\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\,\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}\left\{t+f^2-\kappa^2+\frac{2\delta f}{f}(t+f^2)\right\}^{-q+\frac{1}{2}}.(t+f^2)^{-\frac{1}{2}s+q-1}\,dt - \text{term in } \epsilon.$$

Hence the term in δf is

$$= 2(-q+\tfrac{1}{2})\frac{\delta f}{f}\cdot\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}-q)\,\Gamma(\frac{1}{2}s+q)}f^{s+1}\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt,$$

$$= \delta f \text{ into expression } \frac{2\Gamma\frac{1}{2}s\,.\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)}f^s\int_0^\infty t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt,$$

where the factor which multiplies δf is, as it should be, the ring-integral; it in fact agrees with one of the expressions previously obtained for this integral.

88. Similarly for an exterior point $\kappa > f$; starting in like manner from, Disk-integral

$$= \frac{2^{s-2}f^{s-1}}{(\frac{1}{2}-q)(\kappa+f)^{s+2q-2}}\left\{-\frac{\kappa-f}{\kappa+f}\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q,\ u)+\Pi(\tfrac{1}{2}s,\ \tfrac{1}{2}s,\ \tfrac{1}{2}s+q-1,\ u)\right\},$$

and reducing in like manner, the term in { } may be expressed in the four forms

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}\ \frac{(\kappa+f)^{s+2q-2}}{\kappa^{s+2q-2}}\ \text{multiplied by}$$

$$\left[-\left(1-\frac{f^2}{\kappa^2}\right)\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{1}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\kappa^2}\right)+\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\Pi\left(\tfrac{1}{2}s+q-1,\ \tfrac{3}{2}-q,\ -\tfrac{1}{2}+q,\ \frac{f^2}{\kappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q)}\frac{(\kappa+f)^{s+2q-2}}{\kappa^{s+2q-2}} \text{ multiplied by}$$

$$\left[-\left(1-\frac{f^2}{\kappa^2}\right)\Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q,\ \tfrac{1}{2}s+q,\ \frac{f^2}{\kappa^2}\right)+\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\ \Pi\left(-\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q-1,\ \frac{f^2}{\kappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}s+q)}\left(\frac{\kappa+f}{\kappa}\right)^{s-1}\left(\frac{\kappa-f}{\kappa}\right)^{-2q+1} \text{ multiplied by}$$

$$\left[-\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{f^2}{\kappa^2}\right)+\left(1-\frac{f^2}{\kappa^2}\right)\frac{\frac{1}{2}s+q-1}{\frac{1}{2}-q}\,\Pi\left(\tfrac{1}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q,\ \frac{f^2}{\kappa^2}\right)\right],$$

$$2^{1-s}\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q)\,\Gamma(\frac{1}{2}+q)}\left(\frac{\kappa+f}{\kappa}\right)^{s-1}\left(\frac{\kappa-f}{\kappa}\right)^{-2q+1} \text{ multiplied by}$$

$$\left[-\Pi\left(\tfrac{1}{2}s-q,\ \tfrac{1}{2}+q,\ \tfrac{1}{2}-q,\ \frac{f^2}{\kappa^2}\right)+\left(1-\frac{f^2}{\kappa^2}\right)\frac{-\frac{1}{2}+q}{\frac{1}{2}s-q}\ \Pi\left(\tfrac{1}{2}s-q+1,\ -\tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\ \frac{f^2}{\kappa^2}\right)\right].$$

89. For the reduction of the first and the fourth of these, we have to consider

$$-(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\,\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v);$$

viz. this is

$$\begin{aligned}&(-1+v+1+v)\int_0^1 x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}\,dx-2\int_0^1 vx\,.\,x^{\alpha-1}(1-x\,.\,1-vx)^{\beta-1}\,dx,\\ &=2v\int_0^1 x^{\alpha-1}(1-x)(1-x\,.\,1-vx)^{\beta-1}\,dx,\\ &=2v\,\Pi(\alpha,\ \beta+1,\ -\beta+1,\ v);\end{aligned}$$

that is,

$$-(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\,\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)=2v\Pi(\alpha,\ \beta+1,\ -\beta+1,\ v).$$

I repeat, for comparison, the foregoing equation

$$+(1-v)\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)+\frac{\alpha-1}{\beta}\,\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)=2\Pi(\alpha,\ \beta,\ -\beta,\ v);$$

by adding and subtracting these we obtain two new formulæ; for reduction of the fourth formula, the equation may be written

$$-\Pi(\alpha-1,\ \beta+1,\ -\beta,\ v)+(1-v)\frac{\beta}{\alpha-1}\,\Pi(\alpha,\ \beta,\ 1-\beta,\ v)=-2\frac{\beta}{\alpha-1}v\Pi(\alpha,\ \beta+1-\beta+1,\ v).$$

90. But it is sufficient to consider the first formula; the term in [ ] is

$$=\frac{2^{2-s}\,\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}\quad\left(\frac{\kappa+f}{\kappa}\right)^{s+2q-2}\frac{f^2}{\kappa^2}\ \Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\kappa^2}\right),$$

and the corresponding value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\kappa^{s+2q}}\qquad\Pi\left(\tfrac{1}{2}s+q,\ \tfrac{3}{2}-q,\ \tfrac{1}{2}+q,\ \frac{f^2}{\kappa^2}\right),$$

which we may again verify by taking therein κ indefinitely large; viz. the value is then $=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\frac{f^{s+1}}{\kappa^{s+2q}}$, as above. It is the first of a system of four forms, the others of which are

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q+1)}\frac{f^{s+1}}{\kappa^{s+2q}}\qquad \Pi\left(\tfrac{1}{2}+q,\ \tfrac{1}{2}s-q+1,\ \tfrac{1}{2}s+q,\ \frac{f^2}{\kappa^2}\right),$$

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}\frac{f^{s+1}}{\kappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{3}{2}-q,\ \tfrac{1}{2}s+q,\ \tfrac{1}{2}s-q+1,\ \frac{f^2}{\kappa^2}\right),$$

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\,\Gamma(\frac{1}{2}+q)}\frac{f^{s+1}}{\kappa^{s+2q}}\left(1-\frac{f^2}{\kappa^2}\right)^{1-2q}\Pi\left(\tfrac{1}{2}s-q+1,\ \tfrac{1}{2}+q,\ \tfrac{3}{2}-q,\ \frac{f^2}{\kappa^2}\right).$$

And hence, writing as before $x=\frac{t+f^2-\kappa^2}{t}$, &c., the four values are

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}\quad\frac{f^{s+1}}{\kappa^{s-1}}(\kappa^2-f^2)^{1-2q}\int_{\kappa^2-f^2}^{\infty}t^{-\frac{1}{2}s+q-1}(t+f^2-\kappa^2)^{\frac{1}{2}s+q-1}(t+f^2)^{-\frac{1}{2}-q}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+q)\,\Gamma(\frac{1}{2}s-q+1)}f^{s+1}(\kappa^2-f^2)^{1-2q}\int_{\kappa^2-f^2}^{\infty}t^{q-\frac{3}{2}}\quad(t+f^2-\kappa^2)^{q-\frac{1}{2}}\quad(t+f^2)^{-\frac{1}{2}s-q}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}\quad f^{s+1}\qquad\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}\quad(t+f^2-\kappa^2)^{-q+\frac{1}{2}}\quad(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s-q+1)\,\Gamma(\frac{1}{2}+q)}\frac{f^{s+1}}{\kappa^{s-1}}\qquad\int_{\kappa^2-f^2}^{\infty}t^{-\frac{1}{2}s-q}\quad(t+f^2-\kappa^2)^{\frac{1}{2}s-q}\quad(t+f^2)^{q-\frac{3}{2}}\qquad dt,$$

where we may in the integrals write $t+\kappa^2-f^2$ in place of t, making the limits ∞, 0; but the actual form is preferable.

91. In the third form, for f write mf, at the same time changing t into mt; the new value of the disk-integral is

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}f^{s+1}\int_{\frac{\kappa^2}{m^2}-f^2}^{\infty}t^{-q-\frac{1}{2}}\{m^2(t+f^2)-\kappa^2\}^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt.$$

Writing here $mf=f+\delta f$, that is, $m=1+\frac{\delta f}{f}$, $m^2=1+\frac{2\delta f}{f}$, and observing that, if $-q+\frac{1}{2}$ be positive, the factor $\{m^2(t+f^2)-\kappa^2\}^{-q+\frac{1}{2}}$ vanishes for the value $t=\frac{\kappa^2}{m^2}-f^2$ at the lower limit, we see that on this supposition, $-q+\frac{1}{2}$ positive, the value is

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}f^{s+1}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}\left\{t+f^2-\kappa^2+\frac{2\delta f}{f}(t+f^2)\right\}^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt;$$

viz. the term in δf is $=\delta f$ multiplied by the expression

$$2\left(\tfrac{1}{2}-q\right)\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}f^{s}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

that is, multiplied by

$$2\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{1}{2}-q)}f^s\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q-\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q}\,dt,$$

which is in fact $=\delta f$ multiplied by the value of the ring-integral.

92. Comparing for the cases of an interior point $\kappa<f$ and an exterior point $\kappa>f$, the four expressions for the disk-integral, it will be noticed that only the third expressions correspond precisely to each other; viz. these are: interior point, $\kappa<f$; the value is

$$\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}f^{s+1}\int_0^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt-\frac{\epsilon^{1-2q}}{\frac{1}{2}-q}\,\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}s}{\Gamma(\frac{1}{2}s+q)},$$

where, if $\frac{1}{2}-q$ be positive (which is, in fact, a necessary condition in order to the applicability of the formula), the term in ϵ vanishes, and may therefore be omitted: and exterior point, $\kappa>f$; the value is

$$=\frac{\Gamma\frac{1}{2}s\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}s+q)\,\Gamma(\frac{3}{2}-q)}f^{s+1}\int_{\kappa^2-f^2}^{\infty}t^{-q-\frac{1}{2}}(t+f^2-\kappa^2)^{-q+\frac{1}{2}}(t+f^2)^{-\frac{1}{2}s+q-1}\,dt,$$

differing only from the preceding one in the inferior limit κ^2-f^2 in place of 0 of the integral. We have $\frac{1}{2}-q$ positive, and also $\frac{1}{2}s+q$ positive; viz. q may have any value diminishing from $\frac{1}{2}$ to $-\frac{1}{2}s$, the extreme values *not* admissible.

ANNEX IV. *Examples of Theorem* A. Art. Nos. 93 to 112.

93. It is remarked in the text that, in the examples which relate to the s-coordinal sphere and ellipsoid respectively, we have a quantity θ, a function of the coordinates $(a, \ldots, c, e)$ of the attracted point; viz. in the case of the sphere, writing $a^2+\ldots+c^2=\kappa^2$, we have

$$\frac{\kappa^2}{f^2+\theta}+\frac{e^2}{\theta}=1:$$

in the case of the ellipsoid, we have

$$\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1,$$

the equations having in each case a positive root which is called θ. The properties of the equation are the same in each case; but for the sphere, the equation being a quadric one, can be solved. The equation in fact is

$$\theta^2-\theta(e^2+\kappa^2-f^2)-e^2f^2=0,$$

and the positive root is therefore

$$\theta=\tfrac{1}{2}\{e^2+\kappa^2-f^2+\sqrt{(e^2+\kappa^2-f^2)^2+4e^2f^2}\}.$$

Suppose e to diminish gradually and become $=0$; for an exterior point, $\kappa>f$, the value of the radical is $=\kappa^2-f^2$, and we have $\theta=\kappa^2-f^2$; for an interior point, $\kappa<f$,

the value of the radical, supposing e only indefinitely small, is $=f^2-\kappa^2+\frac{f^2+\kappa^2}{f^2-\kappa^2}e^2$, and we have $\theta=\frac{1}{2}e^2\left(1+\frac{f^2+\kappa^2}{f^2-\kappa^2}\right)$, $=\frac{e^2f^2}{f^2-\kappa^2}$, or, what is the same thing, $\frac{e^2}{\theta}=\left(1-\frac{\kappa^2}{f^2}\right)$; viz. the positive root of the equation continually diminishes with e, and becomes ultimately $=0$.

If κ or e be indefinitely large, then the radical may be taken $=e^2+\kappa^2$, and we have θ indefinitely large, $=e^2+\kappa^2$.

94. The result is similar for the general equation

$$\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1;$$

the left-hand side is $=0$ for $\theta=\infty$, and (as θ decreases) continually increases, becoming infinite for $\theta=0$; there is consequently a single positive value of θ for which the value is $=1$; viz. the equation has a single positive root, and θ is taken to denote this root.

In the last-mentioned equation, let e gradually diminish and become $=0$; then for an exterior point, viz. if

$$\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1, \text{ the equation } \frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}=1$$

has (as is at once seen) a single positive root, and θ becomes equal to the positive root of this equation; but for an interior point, or $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}<1$, the equation just written down has no positive root, and θ becomes $=0$, that is, the positive root of the original equation continually diminishes with e, and for $e=0$ becomes ultimately $=0$; its value for e small is, in fact, given by $\frac{e^2}{\theta}=\left(1-\frac{a^2}{f^2}-\ldots-\frac{c^2}{h^2}\right)$. Also $a,\ldots,c,e$ (or any of them) indefinitely large, θ is indefinitely large, $=a^2+\ldots+c^2+e^2$.

95. We have an interesting geometrical illustration in the case $s+1=2$; θ is here determined by the equation

$$\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}+\frac{e^2}{\theta}=1;$$

viz. θ is the squared z-semiaxis of the ellipsoid, confocal with the conic $\frac{x^2}{f^2}+\frac{y^2}{g^2}=1$, which passes through the point (a, b, e). Taking $e=0$, the point in question, if $\frac{a^2}{f^2}+\frac{b^2}{g^2}>1$, is a point in the plane of xy, outside the ellipse, and we have through the point a proper confocal ellipsoid, whose squared z-semiaxis does not vanish; but if $\frac{a^2}{f^2}+\frac{b^2}{g^2}<1$, then the point is within the ellipse, and the only confocal ellipsoid through the point is the indefinitely thin ellipsoid, squared semiaxes $(f^2, g^2, 0)$, which in fact coincides with the ellipse.

96. The positive root θ of the equation

$$J, = 1 - \frac{a^2}{f^2+\theta} - \dots - \frac{c^2}{h^2+\theta} - \frac{e^2}{\theta}, = 0$$

has certain properties which connect themselves with the function

$$\Theta, = \theta^{-q-1}\{(\theta+f^2)\dots(\theta+h^2)\}^{-\frac{1}{2}}.$$

We have, the accents denoting differentiations in regard to θ,

$$J'\frac{d\theta}{da} - \frac{2a}{\theta+f^2} = 0, \quad \text{or} \quad \frac{d\theta}{da} = \frac{1}{J'}\frac{2a}{\theta+f^2},$$

where

$$J' = \frac{a^2}{(f^2+\theta)^2} + \dots + \frac{c^2}{(h^2+\theta)^2} + \frac{e^2}{\theta^2}:$$

and we have the like formulæ for .., $\frac{d\theta}{dc}$, $\frac{d\theta}{de}$.

We deduce

$$\frac{a}{\theta+f^2}\frac{d\theta}{da} + \dots + \frac{c}{\theta+h^2}\frac{d\theta}{dc} + \frac{e}{\theta}\frac{d\theta}{de} = \frac{2}{J'}\left\{\frac{a^2}{(\theta+f^2)^2} + \dots + \frac{c^2}{(\theta+h^2)^2} + \frac{e^2}{\theta}\right\}, = 2;$$

and to this we may join, η being arbitrary,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da} + \dots + \frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc} + \frac{e}{\theta+\eta}\frac{d\theta}{de} = \frac{2}{J'}\left\{\frac{a^2}{\theta+f^2 . \theta+\eta+f^2} + \dots \right.$$
$$\left. + \frac{c^2}{\theta+h^2 . \theta+\eta+h^2} + \frac{e^2}{\theta . \theta+\eta}\right\}.$$

Again, defining $\nabla_1\theta$ and $\Box\theta$ as immediately appears, we have

$$\nabla_1\theta, = \left(\frac{d\theta}{da}\right)^2 + \dots + \left(\frac{d\theta}{dc}\right)^2, \quad = \frac{1}{J'^2} . 4J', \quad = \frac{4}{J'};$$

and passing to the second differential coefficients, we have

$$\frac{d^2\theta}{da^2} = \frac{2}{J'(\theta+f^2)} - \frac{8a^2}{J'^2(\theta+f^2)^3} - \frac{4a^2J''}{J'^3(\theta+f^2)^2},$$

where

$$J'' = -2\left\{\frac{a^2}{(\theta+f^2)^3} + \dots + \frac{c^2}{(\theta+h^2)^3} + \frac{e^2}{\theta^3}\right\},$$

and the like formulæ for .., $\frac{d^2\theta}{dc^2}$, $\frac{d^2\theta}{de^2}$. Joining to these $\frac{2q+1}{e}\frac{d\theta}{de} = \frac{4q+2}{J'\theta}$, we obtain

$$\Box\theta, = \left(\frac{d^2\theta}{da^2} + \dots + \frac{d^2\theta}{dc^2} + \frac{d^2\theta}{de^2} + \frac{2q+1}{e}\frac{d\theta}{de}\right),$$
$$= \frac{2}{J'}\left\{\frac{1}{\theta+f^2} + \dots + \frac{1}{\theta+h^2} + \frac{1+(2q+1)}{\theta}\right\}$$
$$- \frac{8}{J'^2}(-\tfrac{1}{2}J'') - \frac{4J''}{J'^3}(J'),$$

where the last two terms destroy each other; observing that we have

$$\frac{\Theta'}{\Theta} = -\tfrac{1}{2}\left(\frac{1}{\theta+f^2}+\ldots+\frac{1}{\theta+h^2}+\frac{2q+2}{\theta}\right),$$

the result is

$$\square\theta = \frac{2}{J'}\left(-\frac{2\Theta'}{\Theta}\right), \quad = -\frac{4\Theta'}{J'\Theta}.$$

97. First example. $\kappa^2 = a^2+\ldots+c^2$, and θ the positive root of $\dfrac{\kappa^2}{f^2+\theta}+\dfrac{e^2}{\theta}=1$.

V is assumed $=\displaystyle\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\,dt$, where $q+1$ is positive.

I do not work the example out; it corresponds step by step with, and is hardly more simple than, the next example, which relates to the ellipsoid. The result is

$$\rho = 0, \text{ if } x^2+\ldots+z^2 > f^2,$$

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\,f^{-s}\left(1-\frac{x^2+\ldots+z^2}{f^2}\right)^q, \text{ if } x^2+\ldots+z^2 < f^2;$$

hence the integral

$$\int\frac{\left(1-\dfrac{x^2+\ldots+z^2}{f^2}\right)^q dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

taken over the sphere $x^2+\ldots+z^2=f^2$,

$$=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\int_\theta^\infty t^{-q-1}(t+f^2)^{-\frac{1}{2}s}\,dt.$$

98. Second example. θ the positive root of $\dfrac{a^2}{f^2+\theta}+\ldots+\dfrac{c^2}{h^2+\theta}+\dfrac{e^2}{\theta}=1$; $q+1$ positive.

Consider here the function

$$V=\int_\theta^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt;$$

this satisfies the prepotential equation. We have in fact

$$\frac{dV}{da} = -\Theta\frac{d\theta}{da};\quad \frac{d^2V}{da^2} = -\Theta\frac{d^2\theta}{da^2}-\Theta'\left(\frac{d\theta}{da}\right)^2,$$

with the like expressions for ..., $\dfrac{d^2V}{dc^2}$, $\dfrac{d^2V}{de^2}$; also

$$\frac{2q+1}{e}\frac{dV}{de} = -\Theta\frac{2q+1}{e}\frac{d\theta}{de}.$$

Hence

$$\square V = -\Theta\square\theta-\Theta'\nabla_1\theta,$$

or, substituting for $\square\theta$ and $\nabla_1\theta$ their values, this is

$$= -\Theta\left(-\frac{4\Theta'}{J'\Theta}\right) - \Theta'.4J, = 0.$$

Moreover V does not become infinite for any values of $(a, \ldots, c, e)$, e not $=0$; and it vanishes for points at ∞. And not only so, but for indefinitely large values of any of the coordinates $(a, \ldots, c, e)$ it reduces itself to a numerical multiple of $(a^2 + \ldots + c^2 + e^2)^{-\frac{1}{2}s+q}$; in fact, in this case θ is indefinitely large, $= a^2 + \ldots + c^2 + e^2$. Consequently throughout the integral, t is indefinitely large, and we may therefore write

$$V = \int_\theta^\infty t^{-q-1}.\, t^{-\frac{1}{2}s}\, dt, = -\frac{1}{\frac{1}{2}s+q}\left(t^{-\frac{1}{2}s-q}\right)_\theta^\infty, = \frac{1}{\frac{1}{2}s+q}\,\theta^{-\frac{1}{2}s-q},$$

that is,

$$V = \frac{1}{\frac{1}{2}s+q}\,(a^2 + \ldots + c^2 + e^2)^{-\frac{1}{2}s-q}.$$

The conditions of the theorem are thus satisfied, and we have for ρ either of the formulæ

$$\rho = \frac{\Gamma\left(\frac{1}{2}s+q\right)}{(\Gamma\frac{1}{2})^s\,\Gamma q}\,(e^{2q}W)_0, \quad \rho = \frac{-\Gamma\left(\frac{1}{2}s+q\right)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0:$$

in the former of them q must be positive; in the latter it is sufficient if $q+1$ be positive.

99. We have W the same function of $(x, \ldots, z, e)$ that V is of $(a, \ldots, c, e)$; viz. writing λ for the positive root of

$$\frac{x^2}{f^2+\lambda} + \ldots + \frac{z^2}{h^2+\lambda} + \frac{e^2}{\lambda} = 1,$$

the value of W is

$$= \int_\lambda^\infty t^{-q-1}\,\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt.$$

Considering the formula which involves $e^{2q}W$,—first, if $\frac{x^2}{f^2} + \ldots + \frac{z^2}{h^2} > 1$, then, when e is $=0$ the value of λ is not $=0$; the integral W is therefore finite (not indefinitely large), and we have $e^{2q}W = 0$, consequently $\rho = 0$.

But if $\frac{x^2}{f^2} + \ldots + \frac{z^2}{h^2} < 1$, then, when e is indefinitely small λ is also indefinitely small; viz. we then have $\frac{e^2}{\lambda} = 1 - \frac{x^2}{f^2} - \ldots - \frac{z^2}{h^2}$; the value of W is

$$W = (f\ldots h)^{-1}\int_\lambda^\infty t^{-q-1}\,dt, = (f\ldots h)^{-1}\,\frac{1}{q}\,\lambda^{-q},$$

and hence

$$\rho = \frac{\Gamma\left(\frac{1}{2}s+q\right)}{(\Gamma\frac{1}{2})^s\,\Gamma q}\,\frac{1}{q}\left(\frac{e^2}{\lambda}\right)^q(f\ldots h)^{-1}, \quad = \frac{\Gamma\left(\frac{1}{2}s+q\right)}{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}\,(f\ldots h)^{-1}\left(1 - \frac{x^2}{f^2} - \ldots - \frac{z^2}{h^2}\right)^q.$$

100. Again, using the formula which involves $\left(e^{2q+1}\dfrac{dW}{de}\right)$; we have here $\dfrac{dV}{de}=-\Theta\dfrac{d\theta}{de}$, or substituting for Θ and $\dfrac{d\theta}{de}$ their values and multiplying by e^{2q+1}, we find

$$e^{2q+1}\frac{dV}{de}=2e^{2q+2}\,\theta^{-1}\,J'^{-1}\,\Theta,$$

$$=2e^{2q+2}\,\theta^{-q-2}\left[\frac{a^2}{(f^2+\theta)^2}+\ldots+\frac{c^2}{(h^2+\theta)^2}+\frac{e^2}{\theta^2}\right]^{-1}\{(\theta+f^2)\ldots(\theta+h^2)\}^{-\frac{1}{2}},$$

and therefore

$$e^{2q+1}\frac{dW}{de}=2e^{2q+2}\,\lambda^{-q-2}\left[\frac{x^2}{(f^2+\lambda)^2}+\ldots+\frac{z^2}{(h^2+\lambda)^2}+\frac{e^2}{\lambda^2}\right]^{-1}\{(\lambda+f^2)\ldots(\lambda+h^2)\}^{-\frac{1}{2}}.$$

Hence, writing $e=0$: first, for an exterior point or $\dfrac{x^2}{f^2}+\ldots+\dfrac{z^2}{h^2}>1$, λ is not $=0$, and the expression vanishes in virtue of the factor e^{2q+2}, whence also $\rho=0$; next, for an interior point or $\dfrac{x^2}{f^2}+\ldots+\dfrac{z^2}{h^2}<1$, λ is $=0$, hence also $\dfrac{e^2}{\lambda^2}=\dfrac{1}{\lambda}\left(1-\dfrac{x^2}{f^2}-\ldots-\dfrac{z^2}{h^2}\right)$ is infinite; neglecting in comparison with it the other terms $\dfrac{x^2}{(f^2+\lambda)^2}+\ldots$, the value is

$$2\left(\frac{e^2}{\lambda}\right)^q(f\ldots h)^{-1},=2\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q(f\ldots h)^{-1},$$

and we have, as before,

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}(f\ldots h)^{-1}\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q.$$

101. Hence in the formula

$$V=\int\frac{\rho\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

$$=\int_\theta^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt,$$

ρ has the value just found, or, what is the same thing, we have

$$\int\frac{\left(1-\dfrac{x^2}{f^2}-\ldots-\dfrac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

taken over ellipsoid $\dfrac{x^2}{f^2}+\ldots+\dfrac{z^2}{h^2}=1$,

$$=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt.$$

102. We may in this result write $e=0$. There are two cases, according as the attracted point is exterior or interior: if it is exterior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1$, θ will denote the positive root of the equation $\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}=1$; if it be interior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}<1$, θ will be $=0$; and we thus have

$$\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}$$

$$=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt, \text{ for exterior point } \frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1:$$

$$=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_0^\infty t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\,dt, \text{ for interior point } \frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}<1;$$

but as regards the value for an interior point it is to be observed that, unless q be negative (between 0 and -1, since $1+q$ is positive by hypothesis), the two sides of the equation will be each of them infinite.

103. Third example. We assume here

$$V=\int_\theta^\infty dt\, I^m\, T,$$

where

$$I=1-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t},$$

$$T=t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}};$$

as before, θ is the positive root of the equation

$$J=1-\frac{a^2}{f^2+\theta}-\ldots-\frac{c^2}{h^2+\theta}-\frac{e^2}{\theta},\ =0,$$

and $\frac{1}{2}s+q$ is positive in order that the integral may be finite; also m is positive.

104. In order to show that V satisfies the prepotential equation $\square V=0$, I shall, in the first place, consider the more general expression,

$$V=\int_{\theta+\eta}^\infty dt\, I^m T,$$

where η is a constant positive quantity which will be ultimately put $=0$. The functions previously called J and Θ will be written J_θ and Θ_θ, and J, Θ will now denote

$$J,\ =1-\frac{a^2}{\theta+\eta+f^2}-\ldots-\frac{c^2}{\theta+\eta+h^2}-\frac{e^2}{\theta+\eta},$$

$$\Theta,\ =(\theta+\eta)^{-q-1}\{(\theta+\eta+f^2)\ldots(\theta+\eta+h^2)\}^{-\frac{1}{2}};$$

whence also, subtracting from J the evanescent function J_0, we have

$$J=\eta\left(\frac{a^2}{\theta+f^2.\,\theta+\eta+f^2}+\ldots+\frac{c^2}{\theta+h^2.\,\theta+\eta+h^2}+\frac{e^2}{\theta.\,\theta+\eta}\right),$$

say this is

$$J=\eta P;$$

and we have thence, by former equations and in the present notation,

$$\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}+\ldots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}=\frac{2}{J_0}P,$$

$$\nabla_1\theta=\frac{4}{J_0},$$

$$\square\theta=\frac{-4\Theta_0'}{J_0'\Theta_0}.$$

In virtue of the equation which determines θ, we have

$$\frac{dV}{da}=\int_{\theta+\eta}^{\infty}dt\,mI^{m-1}\frac{-2a}{t+f^2}T-J^m\Theta\frac{d\theta}{da};$$

and thence

$$\begin{aligned}\frac{d^2V}{da^2}=\int_{\theta+\eta}^{\infty}dt\,&\left\{mI^{m-1}\frac{-2}{t+f^2}+m(m-1)I^{m-2}\frac{4a^2}{(t+f^2)^2}\right\}T\\&-mJ^{m-1}\left(-\frac{2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\\&-mJ^{m-1}\left(\frac{-2a}{\theta+\eta+f^2}\right)\Theta\frac{d\theta}{da}\\&-\frac{d}{d\theta}(J^m\Theta)\left(\frac{d\theta}{da}\right)^2\\&-J^m\Theta\frac{d^2\theta}{da^2};\end{aligned}$$

with like expressions for $\ldots,\ \frac{d^2V}{dc^2},\ \frac{d^2V}{de^2}$. Also

$$\frac{2q+1}{e}\frac{dV}{de}=\int_{\theta+\eta}^{\infty}dt\,mI^{m-1}\frac{-4q-2}{t}T-\frac{2q+1}{e}J^m\Theta\frac{d\theta}{de};$$

and hence

$$\begin{aligned}\square V=\int_{\theta+\eta}^{\infty}dt\,&\Bigg[-2m\,I^{m-1}\left\{\frac{1}{t+f^2}-\ldots-\frac{1}{t+h^2}+\frac{1+(2q+1)}{t}\right\}T\\&\qquad+m(m-1)I^{m-2}.\,4\left\{\frac{a^2}{(t+f^2)^2}+\ldots+\frac{c^2}{(t+h^2)^2}+\frac{e^2}{t^2}\right\}T\Bigg]\\&+4m\,J^{m-1}\Theta\left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da}+\ldots+\frac{c}{\theta+\eta+h^2}\frac{d\theta}{dc}+\frac{e}{\theta+\eta}\frac{d\theta}{de}\right)\\&-\frac{d}{d\theta}(J^m\Theta)\left\{\left(\frac{d\theta}{da}\right)^2+\ldots+\left(\frac{d\theta}{dc}\right)^2+\left(\frac{d\theta}{de}\right)^2\right\}\\&-J^m\Theta\left(\frac{d^2\theta}{da^2}+\ldots+\frac{d^2\theta}{dc^2}+\frac{d^2\theta}{de^2}+\frac{2q+1}{e}\frac{d\theta}{de}\right).\end{aligned}$$

105. Writing I', T' for the first derived coefficients of I, T in regard to t, we have

$$I' = \frac{a^2}{(t+f^2)^2} + \ldots + \frac{c^2}{(t+h^2)^2} + \frac{e^2}{t^2}, \qquad \frac{T'}{T} = -\tfrac{1}{2}\left(\frac{1}{t+f^2} + \ldots + \frac{1}{t+h^2} + \frac{2q+2}{t}\right).$$

The integral is therefore

$$\int_{\theta+\eta}^{\infty} dt \left\{2m\, I^{m-1} \frac{2T'}{T}\, T + m(m-1)\, I^{m-2} \,.\, 4I'T\right\},$$

$$= \int_{\theta+\eta}^{\infty} dt \left\{4m\, I^{m-1}\, T' + 4m(m-1)\, I^{m-2}\, I'T\right\},$$

$$= \int_{\theta+\eta}^{\infty} dt\, 4m \frac{d}{dt}(I^{m-1}\, T);$$

viz. $I^{m-1}\, T$ vanishing for $t = \infty$, this is

$$= -4m\, J^{m-1}\, \Theta.$$

Hence, writing $(J^m\, \Theta)'$ instead of $\frac{d}{d\theta}(J^m\, \Theta)$, we have

$$\begin{aligned}\square V = &-4m\, J^{m-1}\, \Theta \\ &+ 4m\, J^{m-1}\, \Theta \left(\frac{a}{\theta+\eta+f^2}\frac{d\theta}{da} + \ldots + \frac{c}{\theta+\eta+f^2}\frac{d\theta}{dc} + \frac{e}{\theta+\eta}\frac{d\theta}{de}\right) \\ &- (J^m\, \Theta)'\, \nabla_1 \theta \\ &- J^m\, \Theta\, \square \theta;\end{aligned}$$

viz. this is

$$\begin{aligned}\square V = &-4m\, J^{m-1}\, \Theta \\ &+ 8m\, J^{m-1}\, \Theta \,.\, \frac{P}{J_0'} \\ &- 4\,(J^m\, \Theta)' \frac{1}{J_0'} \\ &+ 4 J^m\, \Theta\, \frac{\Theta_0'}{J_0' \Theta_0};\end{aligned}$$

or, writing $m\, J^{m-1}\, J'\Theta + J^m \Theta'$ instead of $(J^m \Theta)'$, this is

$$\square V = -\frac{4m\, J^{m-1}\Theta}{J_0'}(J' - 2P + J) - \frac{4J^m}{J_0'\Theta_0}(\Theta'\Theta_0 - \Theta\Theta_0').$$

We have here

$$\begin{aligned}J' - 2P + J &= a^2\left\{\frac{1}{(\theta+\eta+f^2)^2} - \frac{2}{(\theta+\eta+f^2)(\theta+f^2)} + \frac{1}{(\theta+f^2)^2}\right\} + \ldots + e^2\left\{\frac{1}{(\theta+\eta)^2} - \frac{2}{(\theta+\eta)\theta} + \frac{1}{\theta^2}\right\} \\ &= \eta^2\left\{\frac{a^2}{(\theta+f^2)^2(\theta+\eta+f^2)^2} + \ldots + \frac{c^2}{(\theta+h^2)^2(\theta+\eta+h^2)^2} + \frac{e^2}{\theta^2(\theta+\eta)^2}\right\} \\ &= \eta^2\, Q, \text{ suppose.}\end{aligned}$$

Also $\Theta'\Theta_0 - \Theta\Theta_0'$ contains the factor η, is $= \eta M$ suppose.

106. Substituting for J, $J'-2P+J$, and $\Theta'\Theta_0-\Theta\Theta_0'$ their values ηP, ηQ, and ηM, the whole result contains the factor η^{m+1}, viz. we have

$$\square V=-\frac{4\eta^{m+1}P^{m-1}}{J_0'}\left(Q\Theta+\frac{PM}{\Theta_0}\right).$$

If here, except in the term η^{m+1}, we write $\eta=0$, we have

$$P=\frac{a^2}{(\theta+f^2)^2}+\ldots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2},\ =J_0,$$

$$Q=\frac{a^2}{(\theta+f^2)^4}+\ldots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4},\ =\tfrac{1}{6}J_0''',$$

$$M=\Theta_0\Theta_0''-\Theta_0'^2;$$

the formula becomes

$$\square V=-4\eta^{m+1}J_0'^{m-2}\left\{\tfrac{1}{6}J_0'''\Theta_0+J_0'\left(\Theta_0''-\frac{\Theta_0'^2}{\Theta_0}\right)\right\};$$

or (instead of J_0, Θ_0) using now J, Θ in their original significations

$$J=1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta},\ \text{and}\ \Theta=\theta^{-q-1}\{(\theta+f^2)\ldots(\theta+h^2)\}^{-\frac{1}{2}},$$

this is

$$\square V=-4\eta^{m+1}J'^{m-2}\left\{\tfrac{1}{6}J'''\Theta+J'\left(\Theta''-\frac{\Theta'^2}{\Theta}\right)\right\},$$

or, what is the same thing,

$$=-4\eta^{m+1}J'^{m-2}\Theta\left\{\tfrac{1}{6}J'''+J'\left(\frac{\Theta'}{\Theta}\right)'\right\};$$

viz. the expression in { } is

$$=\left[\frac{a^2}{(\theta+f^2)^4}+\ldots+\frac{c^2}{(\theta+h^2)^4}+\frac{e^2}{\theta^4}\right]$$

$$-\left[\frac{a^2}{(\theta+f^2)^2}+\ldots+\frac{c^2}{(\theta+h^2)^2}+\frac{e^2}{\theta^2}\right]\left[\frac{1}{(\theta+f^2)^2}+\ldots+\frac{1}{(\theta+h^2)^2}+\frac{2q+2}{\theta^2}\right].$$

We thus see that, η being infinitesimal, $\square V$ is infinitesimal of the order η^{m+1}; and hence, η being $=0$, we have

$$\square V=0;$$

viz. the prepotential equation is satisfied by the value

$$V=\int_\theta^\infty dt\, I^m\, T,$$

where $m+1$ is positive.

107. We have consequently a value of ρ corresponding to the foregoing value of V; and this value is

$$\rho=-\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_{e=0},$$

where, writing λ for the positive root of

$$1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}=0,$$

we have

$$W=\int_\lambda^\infty dt\left(1-\frac{x^2}{t+f^2}-\dots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m t^{-q-1}\{(t+f^2)\dots(t+h^2)\}^{-\frac{1}{2}};$$

we thence obtain

$$\frac{dW}{de}=\int_\lambda^\infty dt\,.-\frac{2me}{t}\left(1-\frac{x^2}{t+f^2}-\dots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m-1}t^{-q-1}\{(t+f^2)\dots(t+h^2)\}^{-\frac{1}{2}}$$
$$-\left(1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m\lambda^{-q-1}\{(\lambda+f^2)\dots(\lambda+h^2)\}^{-\frac{1}{2}}\frac{d\lambda}{de};$$

or, multiplying by e^{2q+1} and substituting for $\frac{d\lambda}{de}$ its value

$$=\frac{\frac{2e}{\lambda}}{\left\{\frac{x^2}{(\lambda+f^2)^2}+\dots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}},$$

we have

$$e^{2q+1}\frac{dW}{de}=\int_\lambda^\infty dt\,.-\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}-\dots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^{m+1}\{(t+f^2)\dots(t+h^2)\}^{-\frac{1}{2}}$$
$$-\frac{\frac{2e^{2q+2}}{\lambda^{q+2}}}{\left\{\frac{x^2}{(\lambda+f^2)^2}+\dots+\frac{z^2}{(\lambda+h^2)^2}+\frac{e^2}{\lambda^2}\right\}}\left(1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m\{(\lambda+f^2)\dots(\lambda+h^2)\}^{-\frac{1}{2}},$$

where the second term, although containing the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m,$$

is for the present retained.

108. I attend to the second term.

1°. Suppose $\frac{x^2}{f^2}+\dots+\frac{z^2}{h^2}>1$; then, as e diminishes and becomes $=0$, λ does not become zero, but it becomes the positive root of the equation

$$1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}=0;$$

hence the term, containing as well the evanescent factor e^{2q+2} as the other evanescent factor $\left(1-\frac{x^2}{\lambda+f^2}-\dots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m$, is $=0$.

2°. Suppose $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}<1$; then, as e diminishes to zero, λ tends to become $=0$, but $\frac{e^2}{\lambda}$ is finite and $=1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}$, whence $\frac{e^2}{\lambda^2}$ is indefinitely large; and since $\frac{x^2}{(\lambda+f^2)^2}+\ldots+\frac{z^2}{(\lambda+h^2)^2}$ becomes $=\frac{x^2}{f^4}+\ldots+\frac{z^2}{h^4}$, which is finite, the denominator may be reduced to $\frac{e^2}{\lambda^2}$, and the term therefore is

$$=-2\left(\frac{e^2}{\lambda}\right)^q\left(1-\frac{x^2}{\lambda+f^2}-\ldots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m\{(\lambda+f^2)\ldots(\lambda+h^2)\}^{-\frac{1}{2}},$$

$$=-2\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q\left(1-\frac{x^2}{\lambda+f^2}-\ldots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m(f\ldots h)^{-1},$$

which, the other factor being finite, vanishes in virtue of the evanescent factor

$$\left(1-\frac{x^2}{\lambda+f^2}-\ldots-\frac{z^2}{\lambda+h^2}-\frac{e^2}{\lambda}\right)^m.$$

Hence the second term always vanishes, and we have (e being $=0$)

$$e^{2q+1}\frac{dW}{de}=\int_\lambda^\infty dt.-\frac{2me^{2q+2}}{t^{q+2}}\left(1-\frac{x^2}{t+f^2}-\ldots-\frac{z^2}{t+h^2}-\frac{e^2}{t}\right)^m\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}.$$

109. Considering first the case $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}>1$: then, as e diminishes to zero, λ does not become $=0$; the integral contains no infinite element, and it consequently vanishes in virtue of the factor e^{2q+2}.

But if $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}<1$, then, introducing instead of t the new variable ξ, $=\frac{e^2}{t}$, that is, $t=\frac{e^2}{\xi}$, $dt=\frac{-e^2\,d\xi}{\xi^2}$, and writing for shortness

$$R=1-\frac{x^2}{f^2+\frac{e^2}{\xi}}-\ldots-\frac{z^2}{h^2+\frac{e^2}{\xi}},$$

the term becomes

$$=\int d\xi\,.\,2m\,(R-\xi)^{m-1}\xi^q\left\{\left(f^2+\frac{e^2}{\xi}\right)\ldots\left(h^2+\frac{e^2}{\xi}\right)\right\}^{-\frac{1}{2}},$$

where, as regards the limits, corresponding to $t=\infty$ we have $\xi=0$, and corresponding to $t=\lambda$ we have ξ the positive root of $R-\xi=0$. But e is indefinitely small; except for indefinitely small values of ξ, we have

$$R=1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2},\ \text{and}\ \left\{\left(f^2+\frac{e^2}{\xi}\right)\ldots\left(h^2+\frac{e^2}{\xi}\right)\right\}^{-\frac{1}{2}}=(f\ldots h)^{-1};$$

and if ξ be indefinitely small, then, whether we take the accurate or the reduced expressions, the elements are finite, and the corresponding portion of the integral is indefinitely small. We may consequently reduce as above; viz. writing now

$$R = 1 - \frac{x^2}{f^2} - \ldots - \frac{z^2}{h^2},$$

the formula is

$$e^{2q+1}\frac{dW}{de} = \int_R^0 d\xi \,.\, 2m\,(R-\xi)^{m-1}\xi^q\,(f\ldots h)^{-1},$$

$$= -2m\,(f\ldots h)^{-1}\int_0^R d\xi \,.\, \xi^q\,(R-\xi)^{m-1};$$

or writing $\xi = Ru$, the integral becomes $= R^{q+m}\int_0^1 du \,.\, u^q\,(1-u)^{m-1}$, which is

$$= \frac{\Gamma(1+q)\,\Gamma(m)}{\Gamma(1+q+m)}\,R^{q+m};$$

that is, we have

$$e^{2q+1}\frac{dW}{de} = -2\,(f\ldots h)^{-1}\frac{\Gamma(1+q)\,\Gamma(1+m)}{\Gamma(1+q+m)},$$

and consequently

$$\rho = \frac{\Gamma(\frac{1}{2}s+q)}{2\,(\Gamma\frac{1}{2})^s\,\Gamma(1+q)}\,2\,(f\ldots h)^{-1}\frac{\Gamma(1+q)\,\Gamma(1+m)}{\Gamma(1+q+m)}\,R^{q+m},$$

that is,

$$\rho = (f\ldots h)^{-1}\frac{\Gamma(\frac{1}{2}s+q)\,\Gamma(1+m)}{(\Gamma\frac{1}{2})^s\,\Gamma(1+q+m)}\,R^{q+m},$$

viz. ρ has this value for values of $(x,\ldots,z)$ such that $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2} < 1$, but is $=0$ if $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2} > 1$.

110. Multiplying by a constant factor so as to reduce ρ to the value R^{q+m}, the final result is that the integral

$$V = \int \frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+m} dx\ldots dz}{[(a-x)^2+\ldots+(c-z)^2+\ldots+e^2]^{\frac{1}{2}s+q}},$$

the limits being given by the equation

$$\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2} = 1,$$

is equal to

$$\frac{\Gamma(\frac{1}{2})^s\,\Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt \,.\, t^{-q-1}\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}\right)^m\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}},$$

where θ is the positive root of

$$1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta} = 0.$$

In particular, if $e=0$, or

$$V=\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

there are two cases:

exterior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1$, θ is positive root of $1-\frac{a^2}{f^2}-\ldots-\frac{c^2}{h^2}=0$,

interior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}<1$, θ vanishes, viz. the limits in the integral are ∞, 0;

q must be *negative*, $1+q$ positive as before, in order that the t-integral may not be infinite in regard to the element $t=0$.

It is assumed in the proof that m and $1+q$ are each of them positive; but, as appears by the second example, the theorem is true for the extreme value $m=0$; it does not, however, appear that the proof can be extended to include the extreme value $q=-1$. The formula seems, however, to hold good for values of m, q beyond the foregoing limits; and it would seem that the only necessary conditions are $\frac{1}{2}s+q$, $1+m$, and $1+q+m$, each of them positive. The theorem is, in fact, a particular case of the following one, proved Annex X. No. 162, viz.

$$V=\int\frac{\phi\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

taken over the ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$, is equal to

$$\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\,\Gamma(\frac{1}{2}s+q)}\int_\theta^\infty dt\,t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}(1-\sigma)^{-q}\int_0^1 x^{-q-1}\,\phi\{\sigma+(1-\sigma)x\}\,dx,$$

where σ denotes $\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$: assuming $\phi u=(1-u)^{q+m}$, we have

$$\phi\{\sigma+(1-\sigma)x\}=(1-\sigma)^{q+m}(1-x)^{q+m},$$

and the theorem is thus proved.

111. Particular cases: $m=0$;

$$\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q dx\ldots dz}{[(a-x)^2+\ldots+(c-z)^2+e^2]^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\,t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}$$

Cor. In a somewhat similar manner it may be shown that

$$\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q x\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\,\frac{af^2}{t+f^2}t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}.$$

Multiplying the first by a and subtracting it from the second, we have

$$\int \frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q (a-x)\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}} = \frac{(\Gamma\frac{1}{2})^s\,\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\,\frac{a}{t+f^2}t^{-q}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}};$$

or, writing $q+1$ for q, this is

$$\int \frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+1} (a-x)\,dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q+1}} = \frac{(\Gamma\frac{1}{2})^s\,\Gamma(2+q)}{\Gamma(\frac{1}{2}s+q+1)}(f\ldots h)\int_\theta^\infty dt\,\frac{a}{t+f^2}t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}};$$

and we have similar formulæ with $\ldots, (c-z), e$, instead of $(a-x)$, in the numerator.

112. If $m=1$, we have

$$\int \frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+1} dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

$$= \frac{(\Gamma\frac{1}{2})^s\,\Gamma(2+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty dt\left\{1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}},$$

which, differentiated in respect to a, gives the $(a-x)$-formula; hence conversely, assuming the $a-x, \ldots, c-z$, e-formulæ, we obtain by integration the last preceding formula to a constant *près*, viz. we thereby obtain the multiple integral $= C +$ right-hand function, where C is independent of $(a, \ldots, c, e)$; by taking these all infinite, and observing that then $\theta=\infty$, the two integrals each vanish, and we obtain $C=0$.

In particular, when $s=3$, $q=-1$, then

$$\int \frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2+e^2\}^{\frac{1}{2}}} = \pi fgh\int_\theta^\infty dt\left\{1-\frac{a^2}{t+f^2}-\frac{b^2}{t+g^2}-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right\}(t+f^2.t+g^2.t+h^2)^{-\frac{1}{2}},$$

which, putting therein $e=0$, gives the potential of an ellipsoid for the cases of an exterior point and an interior point respectively.

Annex V. Green's *Integration of the Prepotential Equation* $\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}+\frac{2q+1}{e}\frac{d}{de}\right)V=0$. Art. Nos. 113 to 128.

113. In the present Annex, I in part reproduce Green's process for the integration of this equation by means of a series of functions, which are analogous to Laplace's Functions and may be termed "Greenians" (see his Memoir on the Attraction of Ellipsoids, referred to above, p. 320); each such function gives rise to a Prepotential Integral.

Green shows, by a complicated and difficult piece of general reasoning, that there exist solutions of the form $V=\Theta\phi$ (see *post*, No. 116), where ϕ is a function of the

s new variables $\alpha, \beta, \ldots, \gamma$ without θ, such that $\nabla\phi = \kappa\phi$, κ being a function of θ only; these functions ϕ of the variables $\alpha, \beta, \ldots, \gamma$ are in fact the Greenian Functions in question. The function of the order 0 is $\phi = 1$; those of the order 1 are $\phi = \alpha$, $\phi = \beta, \ldots, \phi = \gamma$; those of the order 2 are $\phi = \alpha\beta$, &c., and s functions each of the form

$$\tfrac{1}{2}\{A\alpha^2 + B\beta^2 + \ldots + C\gamma^2\} + D.$$

The existence of the functions just referred to other than the s functions involving the squares of the variables is obvious enough; the difficulty first arises in regard to these s functions; and the actual development of them appears to me important by reason of the light which is thereby thrown upon the general theory. This I accomplish in the present Annex; and I determine by Green's process the corresponding prepotential integrals. I do not go into the question of the Greenian Functions of orders superior to the second.

114. I write for greater clearness $(a, b, \ldots, c, e)$ instead of $(a, \ldots, c, e)$ to denote the series of $(s+1)$ variables; viz. $(a, b, \ldots, c)$ will denote a series of s variables; corresponding to these we have the semiaxes $(f, g, \ldots, h)$, and the new variables $(\alpha, \beta, \ldots, \gamma)$; these last, with the before-mentioned function θ, are the $s+1$ new variables of the problem; and, for convenience, there is introduced also a quantity ϵ; viz. we have

$$\begin{aligned} a &= \sqrt{f^2+\theta}\ \alpha, \\ b &= \sqrt{g^2+\theta}\ \beta, \\ &\vdots \\ c &= \sqrt{h^2+\theta}\ \gamma, \\ e &= \sqrt{\theta}\qquad \epsilon, \end{aligned}$$

where $1 = \alpha^2 + \beta^2 + \ldots + \gamma^2 + \epsilon^2$.

That is, we have θ a function of $a, b, \ldots, c, e$, determined by

$$\frac{a^2}{f^2+\theta} + \frac{b^2}{g^2+\theta} + \ldots + \frac{c^2}{h^2+\theta} + \frac{e^2}{\theta} = 1;$$

and then $\alpha, \beta, \ldots, \gamma$ are given as functions of the same quantities $a, b, \ldots, c, e$ by the equations

$$\alpha^2 = \frac{a^2}{f^2+\theta},\quad \beta^2 = \frac{b^2}{g^2+\theta}, \ldots, \gamma^2 = \frac{c^2}{h^2+\theta};$$

also ϵ, considered as a function of the same quantities, is

$$= \sqrt{1 - \frac{a^2}{f^2+\theta} - \frac{b^2}{g^2+\theta} - \ldots - \frac{c^2}{h^2+\theta}}.$$

115. Introducing instead of $a, b, \ldots, c, e$ the new variables $\alpha, \beta, \ldots, \gamma, \theta$, the transformed differential equation is

$$4\theta\frac{d^2V}{d\theta^2} + 2\frac{dV}{d\theta}\left(s + 2q + 2 - \frac{f^2}{f^2+\theta} - \ldots - \frac{h^2}{h^2+\theta}\right) + \nabla V = 0$$

where for shortness

$$\begin{aligned}\nabla V = &\frac{1}{f^2+\theta}\left\{-\alpha^2-\frac{g^2}{g^2+\theta}\beta^2-\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2V}{d\alpha^2}\\ &+\frac{1}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\beta^2-\ldots-\frac{h^2}{h^2+\theta}\gamma^2+1\right\}\frac{d^2V}{d\beta^2}\\ &\quad\vdots\\ &+\frac{1}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2-\frac{g^2}{g^2+\theta}\beta^2-\ldots-\gamma^2+1\right\}\frac{d^2V}{d\gamma^2}\\ &-\frac{2\theta}{f^2+\theta\,.\,g^2+\theta}\frac{d^2V}{d\alpha\,d\beta}-\&c.\\ &+\frac{1}{f^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{g^2+\theta}+\ldots+\frac{1}{h^2+\theta}\right)\right\}\alpha\frac{dV}{d\alpha}\\ &+\frac{1}{g^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}+\ldots+\frac{1}{h^2+\theta}\right)\right\}\beta\frac{dV}{d\beta}\\ &\quad\vdots\\ &+\frac{1}{h^2+\theta}\left\{-2q-2-\theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}+\ldots\right)\right\}\gamma\frac{dV}{d\gamma}.\end{aligned}$$

Also

$$e^{2q+1}\frac{dV}{de}=-\theta^{q+1}e^{2q+2}\left\{1-\frac{f^2\alpha^2}{f^2+\theta}-\ldots-\frac{h^2\gamma^2}{\gamma^2+\theta}\right\}^{-1}\left(\frac{1}{f^2+\theta}\alpha\frac{dV}{d\alpha}+\ldots+\frac{1}{h^2+\theta}\gamma\frac{dV}{d\gamma}-2\frac{dV}{d\theta}\right).$$

116. To integrate the equation for V, we assume

$$V=\Theta\phi,$$

where Θ is a function of θ only, and ϕ a function of $\alpha, \beta, \ldots, \gamma$ (without θ), such that

$$\nabla\phi=\kappa\phi,$$

κ being a function of θ only. Assuming that this is possible, the remaining equation to be satisfied is obviously

$$4\theta\frac{d^2\Theta}{d\theta^2}+2\frac{d\Theta}{d\theta}\left\{2q+2+\theta\left(\frac{1}{f^2+\theta}+\ldots+\frac{1}{h^2+\theta}\right)\right\}+\kappa\Theta=0.$$

Solutions of the form in question are

$$\begin{aligned}&\phi=1\ ,\quad \kappa=0,\\ &\phi=\alpha\ ,\quad \kappa=\frac{1}{f^2+\theta}\left\{-2q-2-\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right\},\\ &\phi=\beta\ ,\quad \kappa=\quad ,, \qquad\qquad ,,\\ &\quad\vdots\\ &\phi=\alpha\beta,\quad \kappa=\frac{-2\theta}{f^2+\theta\,.\,g^2+\theta}+\frac{1}{f^2+\theta}\left\{-2q-2-\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right\}\\ &\qquad\qquad\qquad\qquad+\frac{1}{g^2+\theta}\left\{-2q-2-\frac{\theta}{f^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right\};\end{aligned}$$

and it can be shown next that there is a solution of the form

$$\phi = \tfrac{1}{2}(A\alpha^2 + B\beta^2 + \ldots + C\gamma^2) + D.$$

117. In fact, assuming that this satisfies $\nabla\phi - \kappa\phi = 0$, we must have identically

$$\begin{aligned}
&\frac{A}{f^2+\theta}\left\{-\alpha^2 - \frac{g^2}{g^2+\theta}\beta^2 - \ldots - \frac{h^2}{h^2+\theta}\gamma^2 + 1\right\}\\
&+\frac{B}{g^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2 - \beta^2 - \ldots - \frac{h^2}{h^2+\theta}\gamma^2 + 1\right\}\\
&\quad\vdots\\
&+\frac{C}{h^2+\theta}\left\{-\frac{f^2}{f^2+\theta}\alpha^2 - \frac{g^2}{g^2+\theta}\beta^2 - \ldots - \gamma^2 + 1\right\}\\
&+\frac{A}{f^2+\theta}\left\{-s-2q-1+\frac{g^2}{g^2+\theta} + \ldots + \frac{h^2}{h^2+\theta}\right\}\\
&+\frac{B}{g^2+\theta}\left\{-s-2q-1+\frac{f^2}{f^2+\theta} + \ldots + \frac{h^2}{h^2+\theta}\right\}\\
&\quad\vdots\\
&+\frac{C}{h^2+\theta}\left\{-s-2q-1+\frac{f^2}{f^2+\theta} + \frac{g^2}{g^2+\theta} + \ldots\right\}\\
&+\kappa\left\{\tfrac{1}{2}(A\alpha^2 + B\beta^2 + \ldots + C\gamma^2) \quad + \quad D \quad\right\};
\end{aligned}$$

so that, from the term in α^2, we have

$$\frac{A}{f^2+\theta}\left\{-s-2q-2+\frac{g^2}{g^2+\theta}+\ldots+\frac{h^2}{h^2+\theta}\right\} - \tfrac{1}{2}\kappa A - \frac{Bf^2}{f^2+\theta\,.\,g^2+\theta} - \ldots - \frac{Cf^2}{f^2+\theta\,.\,h^2+\theta} = 0;$$

or, what is the same thing,

$$A\left\{-2q-3-\frac{\theta}{g^2+\theta}+\ldots+\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\kappa(f^2+\theta)\right\} - B\frac{f^2}{g^2+\theta} - \ldots - \frac{Cf^2}{h^2+\theta} = 0,$$

with the like equations from $\beta^2, \ldots, \gamma^2$; and from the constant term we have

$$A\frac{1}{f^2+\theta} + B\frac{1}{g^2+\theta} + \ldots + C\frac{1}{h^2+\theta} - \kappa D = 0.$$

118. Multiplying this last by f^2, and adding it to the first, we obtain

$$A\left\{-2q-2-\frac{\theta}{f^2+\theta}-\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}-\tfrac{1}{2}\kappa(f^2+\theta)\right\} - \kappa f^2 D = 0;$$

viz. putting for shortness $\Omega = \theta\left(\frac{1}{f^2+\theta}+\frac{1}{g^2+\theta}+\ldots+\frac{1}{h^2+\theta}\right)$, this is

$$A\{2q+2+\Omega+\tfrac{1}{2}\kappa(f^2+\theta)\} + \kappa f^2 D = 0;$$

and similarly

$$\begin{aligned}
&B\{2q+2+\Omega+\tfrac{1}{2}\kappa(g^2+\theta)\} + \kappa g^2 D = 0,\\
&\quad\vdots\\
&C\{2q+2+\Omega+\tfrac{1}{2}\kappa(h^2+\theta)\} + \kappa h^2 D = 0.
\end{aligned}$$

To these we join the foregoing equation

$$\frac{A}{f^2+\theta}+\frac{B}{g^2+\theta}+\ldots+\frac{C}{h^2+\theta}-\kappa D=0.$$

Eliminating $A, B, \ldots, C, D$, we have an equation which determines κ as a function of θ; and the equations then determine the ratios of $A, B, \ldots, C, D$, so that these quantities will be given as determinate multiples of an arbitrary quantity M. The equation for κ is in fact

$$\frac{f^2}{(f^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(f^2+\theta)\}}+\frac{g^2}{(g^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(g^2+\theta)\}}+\ldots$$
$$+\frac{h^2}{(h^2+\theta)\{2q+2+\Omega+\frac{1}{2}\kappa(h^2+\theta)\}}+1=0;$$

and the values of $A, B, \ldots, C, D$ are then

$$\frac{Mf^2}{2q+2+\Omega+\frac{1}{2}\kappa(f^2+\theta)},\ \frac{Mg^2}{2q+2+\Omega+\frac{1}{2}\kappa(g^2+\theta)},\ldots,\frac{Mh^2}{2q+2+\Omega+\frac{1}{2}\kappa(h^2+\theta)},\ -\frac{M}{\kappa},$$

values which seem to be dependent on θ: if they were so, it would be fatal to the success of the process; but they are really independent of θ.

119. That they are independent of θ depends on the theorems; that we have

$$\kappa=\frac{(2q+2+\Omega)\kappa_0}{2q+2-\frac{1}{2}\kappa_0\theta},$$

where κ_0 is a quantity independent of θ determined by the equation

$$\frac{1}{2q+2+\frac{1}{2}\kappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\kappa_0 g^2}+\ldots+\frac{1}{2q+2+\frac{1}{2}\kappa_0 h^2}+1=0,$$

(κ_0 is in fact the value of κ on writing $\theta=0$): and that, omitting the arbitrary multiplier, the values of $A, B, \ldots, C, D$ then are

$$\frac{f^2}{2q+2+\frac{1}{2}\kappa_0 f^2},\ \frac{g^2}{2q+2+\frac{1}{2}\kappa_0 g^2},\ldots,\frac{h^2}{2q+2+\frac{1}{2}\kappa_0 h^2},\ -\frac{1}{\kappa_0};$$

or, what is the same thing, the value of ϕ is

$$=\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\kappa_0 f^2}+\frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\kappa_0 g^2}+\ldots+\frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\kappa_0 h^2}-\frac{1}{\kappa_0}.$$

120. To explain the ground of the assumption

$$\kappa=\frac{(2q+2+\Omega)\kappa_0}{2q+2-\frac{1}{2}\kappa_0\theta},$$

observe that, assuming

$$\frac{2q+2+\Omega+\frac{1}{2}\kappa(f^2+\theta)}{2q+2+\frac{1}{2}\kappa_0 f^2}=\frac{2q+2+\Omega+\frac{1}{2}\kappa(g^2+\theta)}{2q+2+\frac{1}{2}\kappa_0 g^2},$$

then multiplying out and reducing, we obtain

$$\tfrac{1}{2}\kappa_0(2q+2+\Omega)(g^2-f^2)+(2q+2)\tfrac{1}{2}\kappa(f^2-g^2)+\tfrac{1}{4}\kappa_0\kappa(g^2-f^2)\theta=0;$$

viz. the equation divides out by the factor g^2-f^2, thereby becoming

$$\kappa_0(2q+2+\Omega)-(2q+2)\alpha+\tfrac{1}{2}\kappa\kappa_0\theta=0,$$

that is, it gives for κ the foregoing value: hence clearly, κ having this value, we obtain by symmetry

$$2q+2+\Omega+\tfrac{1}{2}\kappa(f^2+\theta),\ 2q+2+\Omega+\tfrac{1}{2}\kappa(g^2+\theta),\ldots,2q+2+\Omega+\tfrac{1}{2}\kappa(h^2+\theta),$$

proportional to

$$2q+2+\tfrac{1}{2}\kappa_0 f^2,\qquad 2q+2+\tfrac{1}{2}\kappa_0 g^2,\ldots,\qquad 2q+2+\tfrac{1}{2}\kappa_0 h^2;$$

viz. the ratios, not only of $A:B$, but of $A:B:\ldots:C$ will be independent of θ.

121. To complete the transformation, starting with the foregoing value of κ, we have

$$2q+2+\Omega+\tfrac{1}{2}\kappa(f^2+\theta)=(2q+2+\Omega)\frac{2q+2+\frac{1}{2}\kappa_0 f^2}{2q+2-\frac{1}{2}\kappa_0\theta},\ \&c.;$$

so that we have

$$\begin{aligned}&A\{2q+2+\tfrac{1}{2}\kappa_0 f^2\}+\kappa_0 f^2 D=0,\\&B\{2q+2+\tfrac{1}{2}\kappa_0 g^2\}+\kappa_0 g^2 D=0,\\&\ \vdots\\&C\{2q+2+\tfrac{1}{2}\kappa_0 h^2\}+\kappa_0 h^2 D=0,\end{aligned}$$

and

$$\frac{A}{f^2+\theta}+\frac{B}{g^2+\theta}+\ldots+\frac{C}{h^2+\theta}-\frac{(2q+2+\Omega)\kappa_0 D}{2q+2-\frac{1}{2}\kappa_0\theta}=0.$$

Substituting for $A, B,\ldots, C$ their values, this last becomes

$$-\frac{\kappa_0 D}{2q+2+\frac{1}{2}\kappa_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\kappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}-\ldots-\frac{\kappa_0 D}{2q+2-\frac{1}{2}\kappa_0\theta}\left\{\frac{2q+2}{2q+2+\frac{1}{2}\kappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}$$
$$-\frac{\kappa_0 D}{2q+2-\frac{1}{2}\kappa_0\theta}\{2q+2+\Omega\}=0;$$

viz. this is

$$\left\{\frac{2q+2}{2q+2+\frac{1}{2}\kappa_0 f^2}-\frac{\theta}{f^2+\theta}\right\}+\ldots+\left\{\frac{2q+2}{2q+2+\frac{1}{2}\kappa_0 h^2}-\frac{\theta}{h^2+\theta}\right\}+2q+2+\Omega=0;$$

or, substituting for Ω its value, and dividing out by $2q+2$, we have

$$\frac{1}{2q+2+\frac{1}{2}\kappa_0 f^2}+\frac{1}{2q+2+\frac{1}{2}\kappa_0 g^2}+\ldots+\frac{1}{2q+2+\frac{1}{2}\kappa_0 h^2}+1=0,$$

the equation for the determination of κ_0.

122. The equation for κ_0 is of the order s; there are consequently s functions of the form in question, and each of the terms $\alpha^2, \beta^2, \ldots, \gamma^2$ can be expressed as a linear function of these. It thus appears that any quadric function of $\alpha, \beta, \ldots, \gamma$ can be expressed as a sum of Greenian functions; viz. the form is

$$\begin{aligned}
&A \\
&+ B\alpha + \&c. \\
&+ C\alpha\beta + \&c. \\
&+ D'\left(\frac{\frac{1}{2}f^2\alpha^2}{2q+2+\frac{1}{2}\kappa_0' f^2} + \frac{\frac{1}{2}g^2\beta^2}{2q+2+\frac{1}{2}\kappa_0' g^2} + \ldots + \frac{\frac{1}{2}h^2\gamma^2}{2q+2+\frac{1}{2}\kappa_0' h^2} - \frac{1}{\kappa_0'}\right) \\
&+ D''(\qquad \text{,,} \qquad\qquad \text{,,} \qquad\qquad \text{,,} \qquad\qquad) \\
&\quad (s \text{ lines}),
\end{aligned}$$

viz. the terms multiplied by D', D'', &c. respectively are those answering to the roots κ_0', κ_0'', .. of the equation in κ_0.

The general conclusion is that any rational and integral function of $\alpha, \beta, \ldots, \gamma$ can be expressed as a sum of Greenian functions.

123. We have next to integrate the equation

$$4\theta\frac{d^2\Theta}{d\theta^2} + 2\frac{d\Theta}{d\theta}\left(2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}+\ldots+\frac{\theta}{h^2+\theta}\right) - \kappa\Theta = 0.$$

Suppose $\kappa = 0$, a particular solution is $\Theta = 1$. Next, suppose

$$\kappa = \frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right); \text{ a particular solution is } \frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2+\ldots+h^2}}:$$

in fact, omitting the constant denominator, or writing $\Theta = \sqrt{f^2+\theta}$, and therefore

$$\frac{d\Theta}{d\theta} = \frac{1}{2\sqrt{f^2+\theta}}, \quad \frac{d^2\Theta}{d\theta^2} = -\frac{1}{4(f^2+\theta)^{\frac{3}{2}}},$$

the equation to be verified is

$$\begin{aligned}
&-\frac{\theta}{(f^2+\theta)^{\frac{3}{2}}} + \frac{1}{\sqrt{f^2+\theta}}\left\{2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}+\ldots+\frac{\theta}{h^2+\theta}\right\} \\
&\qquad + \frac{1}{\sqrt{f^2+\theta}}\left\{-2q-2 \qquad -\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right\} = 0, \text{ which is right.}
\end{aligned}$$

Again, suppose $\kappa = \dfrac{-2\theta}{f^2+\theta \,.\, g^2+\theta} + \&c.$ (value belonging to $\phi = \alpha\beta$, see No. 116); a particular solution is $\dfrac{\sqrt{f^2+\theta}\sqrt{g^2+\theta}}{f^2+g^2+\ldots+h^2}$: in fact, omitting the constant factor, or writing

$$\Theta = \sqrt{f^2+\theta}\sqrt{g^2+\theta},$$

and therefore

$$\frac{d\Theta}{d\theta} = \tfrac{1}{2}\left\{\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}} + \frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right\},$$

$$\frac{d^2\Theta}{d\theta^2} = \tfrac{1}{4}\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}} + \frac{2}{\sqrt{f^2+\theta}\sqrt{g^2+\theta}} - \frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\},$$

the equation to be verified is

$$\begin{aligned}
&\theta\left\{-\frac{\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}} + \frac{2}{\sqrt{f^2+\theta}\sqrt{g^2+\theta}} - \frac{\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}}\right\} \\
&+\left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}} + \frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)\left\{2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}+\ldots+\frac{\theta}{h^2+\theta}\right\} \\
&+\sqrt{f^2+\theta}\sqrt{g^2+\theta}\quad\left\{\frac{-2\theta}{f^2+\theta\,.\,g^2+\theta}+\frac{1}{f^2+\theta}\left(-2q-2-\frac{\theta}{g^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right)\right. \\
&\qquad\qquad\left.+\frac{1}{g^2+\theta}\left(-2q-2-\frac{\theta}{f^2+\theta}-\ldots-\frac{\theta}{h^2+\theta}\right)\right\} = 0;
\end{aligned}$$

or putting for shortness $\Omega = \dfrac{\theta}{f^2+\theta} + \dfrac{\theta}{g^2+\theta} + \ldots + \dfrac{\theta}{h^2+\theta}$, this is

$$\begin{aligned}
&-\frac{\theta\sqrt{g^2+\theta}}{(f^2+\theta)^{\frac{3}{2}}} + \frac{2\theta}{\sqrt{f^2+\theta}\sqrt{g^2+\theta}} - \frac{\theta\sqrt{f^2+\theta}}{(g^2+\theta)^{\frac{3}{2}}} \qquad + \left(\frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}} + \frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\right)(2q+2+\Omega) \\
&-\frac{2\theta}{\sqrt{f^2+\theta}\sqrt{g^2+\theta}} + \frac{\sqrt{g^2+\theta}}{\sqrt{f^2+\theta}}\left(-2q-2+\frac{\theta}{f^2+\theta}-\Omega\right) + \frac{\sqrt{f^2+\theta}}{\sqrt{g^2+\theta}}\left(-2q-2+\frac{\theta}{g^2+\theta}-\Omega\right) = 0,
\end{aligned}$$

which is true. And, generally, the particular solution is deduced from the value of ϕ by writing therein

$$\frac{\sqrt{f^2+\theta}}{\sqrt{f^2+g^2+\ldots+h^2}},\ \frac{\sqrt{g^2+\theta}}{\sqrt{f^2+g^2+\ldots+h^2}},\ldots,\ \frac{\sqrt{h^2+\theta}}{\sqrt{f^2+g^2+\ldots+h^2}}$$

in place of $\alpha, \beta, \ldots, \gamma$ respectively: say the value thus obtained is $\Theta = H$, where H is what ϕ becomes by the above substitution.

124. Represent for a moment the equation in Θ by

$$4\theta\frac{d^2\Theta}{d\theta^2} + 2\frac{d\Theta}{d\theta}P + \kappa\Theta = 0,$$

and assume that this is satisfied by $\Theta = H\int z\,d\theta$. Then we have

$$\begin{aligned}
&4\theta\left(\frac{d^2H}{d\theta^2}\int z\,d\theta + 2\frac{dH}{d\theta}z + H\frac{dz}{d\theta}\right) \\
&+2P\left(\frac{dH}{d\theta}\int z\,d\theta + \quad H z\right) \\
&+\quad \kappa H\int z\,d\theta \qquad\qquad = 0,
\end{aligned}$$

and therefore

$$\left(8\theta\frac{dH}{d\theta}+2PH\right)z+4\theta H\frac{dz}{d\theta}=0;$$

viz. multiplying by $\frac{H}{4\theta}$, this is

$$\frac{d}{d\theta}(H^2z)+\frac{1}{2\theta}PH^2z=0,$$

or

$$\frac{1}{H^2z}\frac{d}{d\theta}(H^2z)+\frac{1}{2\theta}P=0;$$

viz. substituting for P its value, this is

$$\frac{1}{H^2z}\frac{d}{d\theta}(H^2z)+\frac{1}{2\theta}\left(2q+2+\frac{\theta}{f^2+\theta}+\frac{\theta}{g^2+\theta}+\ldots+\frac{\theta}{h^2+\theta}\right)=0.$$

Hence, integrating,

$$H^2z=\frac{C\theta^{-q-1}}{\sqrt{f^2+\theta\,.\,g^2+\theta\ldots h^2+\theta}},\qquad C \text{ an arbitrary constant,}$$

and

$$\Theta=CH\int_\chi\frac{\theta^{-q-1}\,d\theta}{H^2\sqrt{f^2+\theta\,.\,g^2+\theta\ldots h^2+\theta}},\quad \chi \text{ arbitrary,}$$

where the constants of integration are C, λ; or, what is the same thing, taking T the same function of t that H is of θ (viz. T is what ϕ becomes on writing therein

$$\frac{\sqrt{f^2+t}}{\sqrt{f^2+g^2+\ldots+h^2}},\ \frac{\sqrt{g^2+t}}{\sqrt{f^2+g^2+\ldots+h^2}},\ldots\ \frac{\sqrt{h^2+t}}{\sqrt{f^2+g^2+\ldots+h^2}},$$

in place of $\alpha, \beta, \ldots, \gamma$ respectively), then

$$\Theta=-CH\int_\theta^\chi\frac{t^{-q-1}\,dt}{T^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}},$$

where χ may be taken $=\infty$: we thus have

$$V=\Theta\phi=-CH\phi\int_\theta^\infty\frac{t^{-q-1}\,dt}{T^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}}.$$

Recollecting that

$$1=\frac{a^2}{f^2+\theta}+\frac{b^2}{g^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta},$$

so that for $\theta=\infty$ we have $a^2+b^2+\ldots+c^2+e^2=\theta$, the assumption $\chi=\infty$ comes to making V vanish for infinite values of $(a, b, \ldots, c, e)$.

125. We have to find the value of ρ corresponding to the foregoing value of V; viz. W being the value of V, on writing therein $(x, y, \ldots, z)$ in place of $(a, b, \ldots, e)$, then (theorem A)

$$\rho=-\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\left(e^{2q+1}\frac{dW}{de}\right)_0.$$

Take λ the same function of $(x, y, \ldots, z, e)$ that θ is of $(a, b, \ldots, c, e)$: viz. take λ the positive root of

$$\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}+\ldots+\frac{z^2}{h^2+\lambda}-\frac{e^2}{\lambda}=1;$$

and let $(\xi, \eta, \ldots, \zeta, \tau)$ correspond to $(\alpha, \beta, \ldots, \gamma, \epsilon)$, viz.

$$\xi=\frac{x}{\sqrt{f^2+\lambda}},\ \eta=\frac{y}{\sqrt{g^2+\lambda}},\ldots,\zeta=\frac{z}{\sqrt{h^2+\lambda}},\ \tau=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}-\ldots-\frac{z^2}{h^2+\lambda}},$$

so that W is the same function of $(\xi, \eta, \ldots, \lambda)$ that V is of $(\alpha, \beta, \ldots, \theta)$: say this is

$$W=-C\Lambda\psi\int_\lambda^\infty\frac{t^{-q-1}\,dt}{T^2\sqrt{f^2+t\,.\,g^2+t\ldots h^2+t}};$$

then we have for ρ the value

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{2(\Gamma\frac{1}{2})^s\Gamma(q+1)}\lambda^{q+1}\tau^{2q+2}\left(1-\frac{f^2\xi_0^2}{f^2+\lambda}-\ldots-\frac{h^2\zeta_0^2}{h^2+\lambda}\right)^{-1}$$
$$\times\left(\frac{1}{f^2+\lambda}\xi\frac{dW}{d\xi}+\ldots+\frac{1}{h^2+\lambda}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right),$$

where e is to be put $=0$.

126. Suppose e is $=0$; then, if $\frac{x^2}{f^2}+\frac{y^2}{g^2}+\ldots+\frac{z^2}{h^2}>1$, λ is not $=0$ but is the positive root of $\frac{x^2}{f^2+\lambda}+\frac{y^2}{g^2+\lambda}+\ldots+\frac{z^2}{h^2+\lambda}=1$: $\tau,=\sqrt{1-\frac{x^2}{f^2+\lambda}-\frac{y^2}{g^2+\lambda}-\ldots-\frac{z^2}{h^2+\lambda}}$, is $=0$: and we have $\rho=0$, viz. ρ is $=0$ for all points outside the ellipsoid $\frac{x^2}{f^2}+\frac{y^2}{g^2}+\ldots+\frac{z^2}{h^2}=1$.

But if $\frac{x^2}{f^2}+\frac{y^2}{g^2}+\ldots+\frac{z^2}{h^2}<1$, then, on writing $e=0$, we have $\lambda=0$, $\tau^2=\frac{e^2}{\lambda}$,

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\lambda^{q+1}\frac{e^{2q+2}}{\lambda^{q+1}}\cdot\frac{\lambda}{e^2}\left(\frac{1}{f^2}\xi\frac{dW}{d\xi}+\frac{1}{g^2}\eta\frac{dW}{d\eta}+\ldots+\frac{1}{h^2}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right)_{\lambda=0}$$
$$=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\quad e^{2q}\lambda\ .\ \left(\frac{1}{f^2}\xi\frac{dW}{d\xi}+\frac{1}{g^2}\eta\frac{dW}{d\eta}+\ldots+\frac{1}{h^2}\zeta\frac{dW}{d\zeta}-2\frac{dW}{d\lambda}\right)_{\lambda=0},$$

where the term in ( ) is

$$=-C\Lambda_0\psi_0\left(+\frac{2\lambda^{-q-1}}{\Lambda_0^2fg\ldots h}\right)$$
$$=-2C\frac{\psi_0}{\Lambda_0fg\ldots h}\cdot\frac{1}{\lambda^{q+1}}.$$

Hence

$$\rho=\frac{\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\frac{2C\psi_0}{\Lambda_0fg\ldots h}\left(\frac{e^2}{\lambda}\right)^q$$
$$=\frac{-\Gamma(\frac{1}{2}s+q)}{2\pi^{\frac{1}{2}s}\Gamma(q+1)}\cdot\frac{2C\psi_0}{\Lambda_0fg\ldots h}\left(1-\frac{x^2}{f^2}-\frac{y^2}{g^2}-\ldots-\frac{z^2}{h^2}\right)^q,$$

where ψ_0, Λ_0 are what ψ, Λ become on writing therein $\lambda=0$. It will be remembered that Λ is what H becomes on changing therein θ into λ; hence Λ_0 is what H becomes on writing therein $\theta=0$.

Moreover ψ is what ϕ becomes on changing therein $\alpha, \beta, \ldots, \gamma$ into $\xi, \eta, \ldots, \zeta$: writing $\lambda=0$, we have $\xi=\frac{x}{f}$, $\eta=\frac{y}{g}, \ldots, \zeta=\frac{z}{h}$; hence ψ_0 is what ϕ becomes on changing therein $\alpha, \beta, \ldots, \gamma$ into $\frac{x}{f}, \frac{y}{g}, \ldots, \frac{z}{h}$. And it is proper in ϕ to restore the original variables by writing $\frac{a}{\sqrt{f^2+\theta}}, \frac{b}{\sqrt{g^2+\theta}}, \ldots, \frac{c}{\sqrt{h^2+\theta}}$ in place of $\alpha, \beta, \ldots, \gamma$.

127. Recapitulating,

$$V=\int \frac{\rho\, dx \ldots dz}{[(a-x)^2+\ldots+(c-z)^2+e^2]^{\frac{1}{2}s+q}},$$

where, since for the value of V about to be mentioned ρ vanishes for points outside the ellipsoid, the integral is to be taken over the ellipsoid

$$\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1:$$

and then, transferring a constant factor, if

$$V=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(q+1)}{\Gamma(\frac{1}{2}s+q)}\Lambda_0(f\ldots h).H\phi\int_\theta^\infty \frac{t^{-q-1}\,dt}{T^2\sqrt{(t+f^2)\ldots(t+h^2)}},$$

the corresponding value of ρ is

$$\rho=\psi_0\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q,$$

where Λ_0 is what H becomes on writing therein $\theta=0$, and ψ_0 is what ψ becomes on writing

$$\frac{x}{f}, \ldots, \frac{z}{h} \text{ in place of } \alpha, \ldots, \gamma.$$

128. Thus, putting for shortness $\Omega=t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}$, we have in the three several cases $\phi=1$, $\phi=\frac{a}{\sqrt{f^2+\theta}}$, $\phi=\frac{ab}{\sqrt{f^2+\theta}.\sqrt{g^2+\theta}}$ respectively,

$$H=1, \qquad \rho=\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^q, \quad V=\frac{(\Gamma\frac{1}{2})^s\Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)}(f\ldots h)\int_\theta^\infty \qquad \Omega dt,$$

$$H=\frac{\sqrt{f^2+\theta}}{\sqrt{f^2+\ldots+h^2}}, \quad \rho=x\,(\;\text{"}\qquad\text{"}\;)^q, \quad V=\quad\text{"}\quad\text{"}\quad a\int_\theta^\infty \frac{f^2}{f^2+t} \qquad \Omega dt,$$

$$H=\frac{\sqrt{f^2+\theta}.\sqrt{g^2+\theta}}{f^2+\ldots+h^2}, \quad \rho=xy\,(\;\text{"}\qquad\text{"}\;)^q, \quad V=\quad\text{"}\quad\text{"}\quad ab\int_\theta^\infty \frac{f^2g^2}{f^2+t\,.\,g^2+t}\,\Omega dt.$$

For the case last considered

$$\phi = \frac{\frac{1}{2}\frac{f^2a^2}{f^2+\theta}}{2q+2+\frac{1}{2}\kappa_0 f^2} + \ldots + \frac{\frac{1}{2}\frac{h^2c^2}{h^2+\theta}}{2q+2+\frac{1}{2}\kappa_0 h^2} - \frac{1}{\kappa_0},$$

$$H = \frac{\frac{1}{2}f^2(f^2+\theta)}{2q+2+\frac{1}{2}\kappa_0 f^2} + \ldots + \frac{\frac{1}{2}h^2(h^2+\theta)}{2q+2+\frac{1}{2}\kappa_0 h^2} - \frac{1}{\kappa_0}, \quad T \text{ same function with } t \text{ for } \theta,$$

$$\psi_0 = \frac{\frac{1}{2}x^2}{2q+2+\frac{1}{2}\kappa_0 f^2} + \ldots + \frac{\frac{1}{2}z^2}{2q+2+\frac{1}{2}\kappa_0 h^2} - \frac{1}{\kappa_0},$$

$$\Lambda_0 = \frac{\frac{1}{2}f^4}{2q+2+\frac{1}{2}\kappa_0 f^2} + \ldots + \frac{\frac{1}{2}h^4}{2q+2+\frac{1}{2}\kappa_0 h^2} - \frac{1}{\kappa_0},$$

where κ_0 is the root of the equation

$$\frac{1}{2q+2+\frac{1}{2}\kappa_0 f^2} + \ldots + \frac{1}{2q+2+\frac{1}{2}\kappa_0 h^2} + 1 = 0,$$

$$\rho = \left(1 - \frac{x^2}{f^2} - \ldots - \frac{z^2}{h^2}\right)^2 \psi_0, \quad V = \frac{(\Gamma\frac{1}{2})^s \Gamma(1+q)}{\Gamma(\frac{1}{2}s+q)} (f \ldots h)\, \Lambda_0 H \phi \int_\theta^\infty T^{-2} t^{-q-1} \{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}} dt.$$

ANNEX VI. *Examples of Theorem* C. Art. Nos. 129 to 132.

129. First example: relating to the $(s+1)$-coordinal sphere $x^2 + \ldots + z^2 + w^2 = f^2$. Assume

$$V' = \frac{M}{(a^2 + \ldots + c^2 + e^2)^{\frac{1}{2}(s-1)}}, \quad V'' = \frac{M}{f^{s-1}}, \text{ (a constant)};$$

these values each satisfy the potential equation.

V' is not infinite for any point outside the surface, and for indefinitely large distances it is of the proper form.

V'' is not infinite for any point inside the surface; and at the surface $V' = V''$.

The conditions of the theorem are therefore satisfied. Writing

$$V = \int \frac{\rho\, dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}},$$

we have

$$\rho = -\frac{\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}} \left(\frac{dW'}{ds'} + \frac{dW''}{ds''}\right),$$

where

$$W' = \frac{M}{(x^2 + \ldots + z^2 + w^2)^{\frac{1}{2}s-\frac{1}{2}}}, \quad W'' = \frac{M}{f^{s-1}}; \text{ hence } \frac{dW''}{ds''} = 0,$$

$$\frac{dW'}{ds'} = \left(\frac{x}{f}\frac{d}{dx} + \ldots + \frac{z}{f}\frac{d}{dz} + \frac{w}{f}\frac{d}{dw}\right) \frac{M}{(x^2 + \ldots + z^2 + w^2)^{\frac{1}{2}s-\frac{1}{2}}}$$

$$= -\frac{(s-1)\frac{1}{f}(x^2 + \ldots + z^2 + w^2)\, M}{(x^2 + \ldots + z^2 + w^2)^{\frac{1}{2}s+\frac{1}{2}}},$$

which at the surface is

$$= \frac{-(s-1)M}{f^s}.$$

Hence

$$\rho = \frac{(s-1)\,\Gamma(\frac{1}{2}s-\frac{1}{2})\,M}{4\,(\Gamma\frac{1}{2})^{s+1} f^s}, \quad = \frac{\Gamma(\frac{1}{2}s+\frac{1}{2})\,M}{2\,(\Gamma\frac{1}{2})^{s+1} f^s}: \text{ (viz. } \rho \text{ is constant).}$$

130. Writing for convenience $M = \dfrac{2\,(\Gamma\frac{1}{2})^{s+1} f^s}{\Gamma(\frac{1}{2}s+1)}\,\delta f$ (δf a constant which may be put $=1$), also $a^2+\ldots+c^2+e^2=\kappa^2$, we have $\rho=\delta f$, and consequently

$$\int \frac{\delta f\, dS}{\{(a-x)^2+\ldots+(c-z)^2+(e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$= \frac{2\,(\Gamma\frac{1}{2})^{s+1} f^s\,\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\,\frac{1}{\kappa^{s-1}} \text{ for exterior point } \kappa > f,$$

$$= \frac{2\,(\Gamma\frac{1}{2})^{s+1} f^s\,\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\,\frac{1}{f^{s-1}} \text{ for interior point } \kappa < f.$$

By making $a, \ldots, c, e$ all indefinitely large, we find

$$\int \delta f\, dS = \frac{2\,(\Gamma\frac{1}{2})^{s+1} f^s\,\delta f}{\Gamma(\frac{1}{2}s+\frac{1}{2})},$$

viz. the expression on the right-hand side is here the mass of the shell thickness δf.

Taking $s=3$, we have the ordinary formulæ for the Potential of a uniform spherical shell.

131. Suppose $s=3$, but let the surface be the infinite cylinder $x^2+y^2=f^2$. Take here

$$V' = M \log \sqrt{a^2+b^2}, \quad V'' = M \log f,$$

each satisfying the potential equation $\dfrac{d^2V}{da^2}+\dfrac{d^2V}{db^2}=0$; but V', instead of vanishing, is infinite at infinity, and the conditions of the theorem are not satisfied; the Potential of the cylinder is in fact infinite. But the failure is a mere consequence of the special value of s, viz. this is such that $s-2$, instead of being positive, is $=0$. Reverting to the general case of $(s+1)$-dimensional space, let the surface be the infinite cylinder $x^2+\ldots+z^2=f^2$; and assume

$$V' = \frac{M}{(a^2+\ldots+c^2)^{\frac{1}{2}(s-2)}}, \quad V'' = \frac{M}{f^{s-2}} \text{ (a constant).}$$

These satisfy the potential equation; viz. as regards V', we have

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{de^2}\right)V'=0, \text{ that is, } \left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}\right)V'=0.$$

V' is not infinite at any point outside the cylinder; and it vanishes at infinity, except indeed when only the coordinate e is infinite, and its form at infinity is not

$$= M \div (a^2 + \ldots + c^2 + e^2)^{\frac{1}{2}(s-1)}.$$

V'' is not infinite for any point within the cylinder; and at the surface we have $V' = V''$.

We have

$$\rho = -\frac{\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}}\left(\frac{dW'}{d\theta'} + \frac{dW''}{d\theta''}\right),$$

where

$$\frac{dW'}{d\theta'} = \frac{-(s-2)\frac{1}{f}(x^2 + \ldots + z^2)M}{(x^2 + \ldots + z^2)^{\frac{1}{2}s}}, \quad = \frac{-(s-2)M}{f^{s-1}} \text{ at the surface}; \quad \frac{dW''}{d\theta''} = 0,$$

and therefore

$$\rho = \frac{(s-2)\Gamma(\frac{1}{2}s - \frac{1}{2})M}{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}}: \text{ (viz. } \rho \text{ is constant)};$$

or, what is the same thing, writing $M = \dfrac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma\frac{1}{2}(s-1)}$, whence $\rho = \delta f$, and writing also $a^2 + \ldots + c^2 = \kappa^2$, we have

$$\int \frac{\delta f\, dS}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s-\frac{1}{2}}}$$

$$= \frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s - \frac{1}{2})}\frac{1}{\kappa^{s-2}} \text{ for an exterior point } \kappa > f,$$

$$= \frac{4(\Gamma\frac{1}{2})^{s+1}f^{s-1}\delta f}{(s-2)\Gamma(\frac{1}{2}s - \frac{1}{2})}\frac{1}{f^{s-2}} \text{ for an interior point } \kappa < f.$$

132. This is right; but we can without difficulty bring it to coincide with the result obtained for the $(s+1)$-dimensional sphere with only $s-1$ in place of s; we may in fact, by a single integration, pass from the cylinder $x^2 + \ldots + z^2 = f^2$ to the s-dimensional sphere or circle $x^2 + \ldots + z^2 = f^2$, which is the base of this cylinder. Writing first $dS = d\Sigma\, dw$, where $d\Sigma$ refers to the s variables $(x, \ldots, z)$ and the sphere $x^2 + \ldots + z^2 = f^2$; or using now dS in this sense, then in place of the original dS we have $dS\, dw$: and the limits of w being $\infty, -\infty$, then in place of $e - w$ we may write simply w. This being so, and putting for shortness $(a-x)^2 + \ldots + (c-z)^2 = A^2$, the integral is

$$\int_{-\infty}^{\infty} dw \int \frac{\delta f\, dS\, dw}{(A^2 + w^2)^{\frac{1}{2}(s-1)}};$$

and we have without difficulty

$$\int_{-\infty}^{\infty} \frac{dw}{(A^2 + w^2)^{\frac{1}{2}(s-1)}} = \frac{1}{A^{s-2}}\,\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}(s-2)}{\Gamma\frac{1}{2}(s-1)}.$$

To prove it, write $w = A \tan\theta$, then the integral is in the first place converted into $\frac{2}{A^{s-2}}\int_0^{\frac{\pi}{2}} \cos^{s-2}\theta\, d\theta$, which, putting $\cos\theta = \sqrt{x}$ and therefore $\sin\theta = \sqrt{1-x}$, becomes

$$= \frac{1}{A^{s-2}}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}(s-2)-1}\, dx,$$

which has the value in question.

Hence, replacing A by its value, we have

$$\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}(s-2)}{\Gamma\frac{1}{2}(s-1)}\int \frac{\delta f\, dS}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}} = \frac{4\pi^{\frac{1}{2}s}\,\Gamma(\frac{1}{2})f^{s-1}\,\delta f}{(s-2)\,\Gamma\frac{1}{2}(s-1)}\left\{\frac{1}{(a^2+\ldots+c^2)^{\frac{1}{2}(s-2)}} \text{ or } \frac{1}{f^{s-2}}\right\};$$

that is,

$$\int \frac{\delta f\, dS}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}(s-2)}} = \frac{4\pi^{\frac{1}{2}s} f^{s-1}\,\delta f}{(s-2)\,\Gamma\frac{1}{2}(s-2)}\left\{\frac{1}{(a^2+\ldots+c^2)^{\frac{1}{2}(s-2)}} \text{ or } \frac{1}{f^{s-2}}\right\}$$

$$= \frac{2\pi^{\frac{1}{2}s} f^{s-1}\,\delta f}{\Gamma\frac{1}{2}s}\left\{\frac{1}{(a^2+\ldots+c^2)^{\frac{1}{2}(s-2)}} \text{ or } \frac{1}{f^{s-2}}\right\};$$

viz. this is the formula for the sphere with $s-1$ instead of s.

Annex VII. *Example of Theorem* D. Art. Nos. 133 and 134.

133. The example relates to the $(s+1)$-dimensional sphere $x^2+\ldots+z^2+w^2=f^2$. Instead of at once assuming for V a form satisfying the proper conditions as to continuity, we assume a form with indeterminate coefficients, and make it satisfy the conditions in question. Write

$$V = \frac{M}{(a^2+\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}} \quad \text{for } a^2+\ldots+c^2+e^2 > f^2;$$

$$= A\,(a^2+\ldots+c^2+e^2)+B \quad \text{for } a^2+\ldots+c^2+e^2 < f^2.$$

In order that the two values may be equal at the surface, we must have

$$\frac{M}{f^{s-1}} = Af^2+B:$$

in order that the derived functions $\frac{dV}{da}$, &c. may be equal, we must have

$$\frac{-(s-1)\,aM}{f^{s+1}} = 2Aa, \text{ \&c.},$$

viz. these are all satisfied if only $\frac{-(s-1)\,M}{f^{s+1}} = 2A$.

We have thus the values of A and B; or the exterior potential being as above

$$= \frac{M}{(a^2+\ldots+c^2+e^2)^{\frac{1}{2}s-\frac{1}{2}}},$$

the value of the interior potential must be

$$= \frac{M}{f^{s-1}} \left\{ (\tfrac{1}{2}s + \tfrac{1}{2}) - (\tfrac{1}{2}s - \tfrac{1}{2}) \frac{a^2 + \ldots + c^2 + e^2}{f^2} \right\}.$$

The corresponding values of W are of course

$$\frac{M}{(x^2 + \ldots + z^2 + w^2)^{\frac{1}{2}s-\frac{1}{2}}} \text{ and } \frac{M}{f^{s-1}} \left\{ (\tfrac{1}{2}s + \tfrac{1}{2}) - (\tfrac{1}{2}s - \tfrac{1}{2}) \frac{x^2 + \ldots + z^2 + w^2}{f^2} \right\};$$

and we thence find

$$\rho = 0, \qquad \text{if } x^2 + \ldots + z^2 + w^2 > f^2,$$

$$\rho = -\frac{\Gamma(\frac{1}{2}s - \frac{1}{2})}{4(\Gamma\frac{1}{2})^{s+1}} \{-4(\tfrac{1}{2}s - \tfrac{1}{2})(\tfrac{1}{2}s + \tfrac{1}{2})\} \frac{M}{f^{s+1}}, \quad = \frac{\Gamma(\frac{1}{2}s + \frac{3}{2})}{(\Gamma\frac{1}{2})^{s+1}} \frac{M}{f^{s+1}},$$

$$\text{if } x^2 + \ldots + z^2 + w^2 < f^2.$$

Assuming for M the value $\frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s + \frac{3}{2})} f^{s+1}$, the last value becomes $\rho = 1$; writing for shortness $a^2 + \ldots + c^2 + e^2 = \kappa^2$, we have

$$V = \int \frac{dx \ldots dz\, dw}{\{(a-x)^2 + \ldots + (c-z)^2 + (e-w)^2\}^{\frac{1}{2}s+\frac{1}{2}}} \text{ over } (s+1)\text{-dimensional sphere } x^2 + \ldots + z^2 + w^2 = f^2,$$

$$= \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s + \frac{3}{2})} \frac{f^{s+1}}{\kappa^{s-1}} \qquad \text{, for an exterior point } \kappa > f,$$

$$= \frac{(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s + \frac{3}{2})} \{(\tfrac{1}{2}s + \tfrac{1}{2}) f^2 - (\tfrac{1}{2}s - \tfrac{1}{2}) \kappa^2\}, \text{ for an interior point } \kappa < f.$$

134. The case of the ellipsoid $\frac{x^2}{f^2} + \ldots + \frac{z^2}{h^2} = 1$ for $s+1$-dimensional space may be worked out by the theorem; this is, in fact, what is done in tridimensional space by Lejeune-Dirichlet in his Memoir of 1846 above referred to (p. 321).

ANNEX VIII. *Prepotentials of the Homaloids.* Art. Nos. 135 to 137.

135. We have in tridimensional space the series of figures—the plane, the line, the point; and there is in like manner in $(s+1)$-dimensional space a corresponding series of $(s+1)$ terms; the $(s+1)$-coordinal plane—the line, the point: say these are the homaloids or homaloidal figures. And, taking the density as uniform, or, what is the same thing, $= 1$, we may consider the prepotentials of these several figures in regard to an attracted point, which, for greater simplicity, is taken not to be on the figure.

136. The integral may be written

$$V = \int \frac{dw \ldots dt}{\{(a-x)^2 + \ldots + (c-z)^2 + (d-w)^2 + \ldots + (e-t)^2 + u^2\}^{\frac{1}{2}s+q}},$$

which still relates to a $(s+1)$-dimensional space: the $(s+1)$ coordinates of the attracted point are $(a, \ldots, c, d, \ldots, e, u)$, instead of being $(a, \ldots, c, e)$; viz. we have the

s' coordinates $(a, \ldots, c)$, the $s-s'$ coordinates $(d, \ldots, e)$, and the $(s+1)$th coordinate u: and the integration is extended over the $(s-s')$-dimensional figure $w=-\infty$ to $+\infty, \ldots, t=-\infty$ to $+\infty$. And it is also assumed that q is positive.

It is at once clear that we may reduce the integral to

$$V=\int \frac{dw \ldots dt}{\{(a-x)^2+\ldots+(c-z)^2+u^2+w^2+\ldots+t^2\}^{\frac{1}{2}s+q}},$$

say for shortness

$$=\int \frac{dw \ldots dt}{(A^2+w^2+\ldots+t^2)^{\frac{1}{2}s+q}},$$

where $A^2, =(a-x)^2+\ldots+(c-z)^2+u^2$, is a constant as regards the integration, and where the limits in regard to each of the $s-s'$ variables are $-\infty, +\infty$.

We may for these variables write $r\xi, \ldots, r\zeta$, where $\xi^2+\ldots+\zeta^2=1$; and we then have $w^2+\ldots+t^2=r^2$, $dw \ldots dt=r^{s-s'-1}\,dr\,dS$, where dS is the element of surface of the $(s-s')$-coordinal unit-sphere $\xi^2+\ldots+\zeta^2=1$. We thus obtain

$$V=\int \frac{r^{s-s'-1}\,dr}{\{A^2+r^2\}^{\frac{1}{2}s+q}}\int dS,$$

where the integral in regard to r is taken from 0 to ∞, and the integral $\int dS$ over the surface of the unit-sphere; hence by Annex I. the value of this last factor is $=\dfrac{2\,(\Gamma\frac{1}{2})^{s-s'}}{\Gamma\frac{1}{2}(s-s')}$. The integral represented by the first factor will be finite, provided only $\frac{1}{2}s'+q$ be positive; which is the case for any value whatever of s', if only q be positive.

The first factor is an integral such as is considered in Annex II.; to find its value we have only to write $r=A\sqrt{x}$, and we thus find it to be

$$=\frac{1}{(A^2)^{\frac{1}{2}s'+q}}\,\frac{1}{2}\int_0^\infty \frac{x^{\frac{1}{2}s-\frac{1}{2}s'-1}\,dx}{(1+x)^{\frac{1}{2}s+q}}, \text{ viz.} = \frac{1}{A^{s'+2q}}\,\frac{\frac{1}{2}\Gamma\frac{1}{2}(s-s')\,\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

and we thus have

$$V=\frac{1}{A^{s'+2q}}\,\frac{(\Gamma\frac{1}{2})^{s-s'}\,\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)},$$

$$=\frac{(\Gamma\frac{1}{2})^{s-s'}\,\Gamma(\frac{1}{2}s'+q)}{\Gamma(\frac{1}{2}s+q)}\,\frac{1}{\{(a-x)^2+\ldots+(c-z)^2+u^2\}^{\frac{1}{2}s'+q}}.$$

137. As a verification, observe that the prepotential equation $\square V=0$, that is,

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{dd^2}+\ldots+\frac{d^2}{de^2}+\frac{d^2}{du^2}+\frac{2q+1}{u}\,\frac{d}{du}\right)V=0,$$

for a function V, which contains only the $s'+1$ variables $(a, \ldots, c, u)$, becomes

$$\left(\frac{d^2}{da^2}+\ldots+\frac{d^2}{dc^2}+\frac{d^2}{du^2}+\frac{2q+1}{u}\,\frac{d}{du}\right)V=0,$$

which is satisfied by V, a constant multiple of $\{(a-x)^2+\ldots+(c-z)^2+u^2\}^{\frac{1}{2}-s'-q}$.

ANNEX IX. *The* GAUSS-JACOBI *Theory of Epispheric Integrals.* Art. No. 138.

138. The formula obtained (Annex IV. No. 110) is proved only for positive values of m; but writing therein $q=0$, $m=-\frac{1}{2}$, it becomes

$$\int \frac{dx \ldots dz}{\sqrt{1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}}\,\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s}}$$

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} f \ldots h \int^{\infty} dt \,.\, t^{-1}\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right)^{-\frac{1}{2}} \{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}},$$

a formula which is obtainable as a particular case of the more general formula

$$\int \frac{dS}{\{(*\between x,\ldots,z,\ w)^2\}^{\frac{1}{2}s}} = \frac{2\,(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s)} \int_{-\Delta}^{\infty} dt \frac{1}{\sqrt{-\text{Disct.}\,\{(*\between X,\ldots,Z,W,T)^2+t\,(X^2+\ldots+Z^2+W^2+T^2)\}}},$$

(notation to be presently explained), being a result obtained by Jacobi by a process which is in fact the extension to any number of variables of that used by Gauss* in his Memoir "Determinatio attractionis quam ..... exerceret planeta, &c." (1818). I proceed to develop this theory.

139. Jacobi's process has reference to a class of s-tuple integrals (including some of those here previously considered) which may be termed "epispheric": viz. considering the $(s+1)$ variables $(x,\ldots,z,\ w)$ connected by the equation $x^2+\ldots+z^2+w^2=1$, or say they are the coordinates of a point on a $(s+1)$-tuple unit-sphere, then the form is $\int U dS$, where dS is the element of the surface of the unit-sphere, and U is any function of the $s+1$ coordinates; the integral is taken to be of the form $\int \frac{dS}{\{(*\between x,\ldots,z,\ w,\ 1)^2\}^{\frac{1}{2}s}}$, and we then obtain the general result above referred to.

Before going further it is convenient to remark that, taking as independent variables the s coordinates $x,\ldots,z$, we have $dS=\frac{dx\ldots dz}{dw}$, where w stands for $\pm\sqrt{1-x^2-\ldots-z^2}$; we must in obtaining the integral take account of the two values of w, and finally extend the integral to the values of $x,\ldots,z$ which satisfy $x^2+\ldots+z^2<1$.

If, as is ultimately done, in place of $x,\ldots,z$ we write $\frac{x}{f},\ldots,\frac{z}{h}$ respectively, then the value of dS is $=\frac{1}{f\ldots h}\frac{dx\ldots dz}{w}$, where w now stands for $\pm\sqrt{1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}}$; we must, in finding the value of the integral, take account of the two values of w, and finally extend the integral to the values of $x,\ldots,z$ which satisfy $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}<1$.

* [*Ges. Werke*, t. iii, pp. 331—355.]

140. The determination of the integral depends upon formulæ for the transformation of the spherical element dS, and of the quadric function $(x, y, .., z, w, 1)^2$.

First, as regards the spherical element dS; let the $s+1$ variables $x, y, .., z, w$ which satisfy $x^2+y^2+\ldots+z^2+w^2=1$ be regarded as functions of the s independent variables $\theta, \phi, .., \psi$; then we have

$$dS = \begin{vmatrix} x, & y, .., & z, & w \\ \frac{dx}{d\theta}, & \frac{dy}{d\theta}, .., & \frac{dz}{d\theta}, & \frac{dw}{d\theta} \\ \frac{dx}{d\phi}, & \frac{dy}{d\phi}, .., & \frac{dz}{d\phi}, & \frac{dw}{d\phi} \\ \vdots & & & \\ \frac{dx}{d\psi}, & \frac{dy}{d\psi}, .., & \frac{dz}{d\psi}, & \frac{dw}{d\psi} \end{vmatrix} d\theta\, d\phi \ldots d\psi, = \frac{\partial(x, y, .., z, w)}{\partial(\theta, \phi, .., \psi, *)} d\theta\, d\phi \ldots d\psi, \text{ for shortness.}$$

Suppose we effect on the $s+1$ variables $(x, y, .., z, w)$ a transformation

$$x, y, .., z, w = \frac{X}{T}, \frac{Y}{T}, .., \frac{Z}{T}, \frac{W}{T},$$

thus introducing for the moment $s+2$ variables $X, Y, .., Z, W, T$, which satisfy identically $X^2+Y^2+\ldots+Z^2+W^2-T^2=0$; then, considering these as functions of the foregoing s independent variables $\theta, \phi, .., \psi$, we have

$$dS = \frac{1}{T^{s+1}} \begin{vmatrix} X, & Y, .., & Z, & W \\ \frac{dX}{d\theta}, & \frac{dY}{d\theta}, .., & \frac{dZ}{d\theta}, & \frac{dW}{d\theta} \\ \frac{dX}{d\phi}, & \frac{dY}{d\phi}, .., & \frac{dZ}{d\phi}, & \frac{dW}{d\phi} \\ \vdots & & & \\ \frac{dX}{d\psi}, & \frac{dY}{d\psi}, .., & \frac{dZ}{d\psi}, & \frac{dW}{d\psi} \end{vmatrix} d\theta\, d\phi \ldots d\psi = \frac{1}{T^{s+1}} \frac{\partial(X, Y, .., Z, W)}{\partial(\theta, \phi, .., \psi, *)} d\theta\, d\phi \ldots d\psi.$$

141. Considering next the $s+2$ variables $X, Y, .., Z, W, T$ as linear functions (with constant terms) of the $s+1$ new variables $\xi, \eta, .., \zeta, \omega$, or say as linear functions of the $s+2$ quantities $\xi, \eta, .., \zeta, \omega, 1$: which implies between them a linear relation

$$aX+bY+\ldots+cZ+dW+eT=1:$$

and assuming that we have *identically*

$$X^2+Y^2+\ldots+Z^2+W^2-T^2=\xi^2+\eta^2+\ldots+\zeta^2+\omega^2-1,$$

so that, in consequence of the left-hand side being $=0$, the right-hand side is also $=0$; viz. $\xi, \eta, .., \zeta, \omega$ are connected by

$$\xi^2+\eta^2+\ldots+\zeta^2+\omega^2=1:$$

let $d\Sigma$ represent the spherical element belonging to the coordinates $\xi, \eta, \ldots, \zeta, \omega$. Considering these as functions of the foregoing s independent variables $\theta, \phi, \ldots, \psi$, we have

$$d\Sigma = \begin{vmatrix} \xi, & \eta, \ldots, & \zeta, & \omega \\ \frac{d\xi}{d\theta}, & \frac{d\eta}{d\theta}, \ldots, & \frac{d\zeta}{d\theta}, & \frac{d\omega}{d\theta} \\ \frac{d\xi}{d\phi}, & \frac{d\eta}{d\phi}, \ldots, & \frac{d\zeta}{d\phi}, & \frac{d\omega}{d\phi} \\ \vdots & & & \\ \frac{d\xi}{d\psi}, & \frac{d\eta}{d\psi}, \ldots, & \frac{d\zeta}{d\psi}, & \frac{d\omega}{d\psi} \end{vmatrix} d\theta\, d\phi \ldots d\psi = \frac{\partial(\xi, \eta, \ldots, \zeta, \omega)}{\partial(\theta, \phi, \ldots, \psi, *)} d\theta\, d\phi \ldots d\psi.$$

142. In this expression we have $\xi, \eta, \ldots, \zeta, \omega$, each of them a linear function of the $s+2$ quantities $X, Y, \ldots, Z, W, T$; the determinant is consequently a linear function of $s+2$ like determinants obtained by substituting for the variables any $s+1$ out of the $s+2$ variables $X, Y, \ldots, Z, W, T$; but in virtue of the equation

$$X^2 + Y^2 + \ldots + Z^2 + W^2 - T^2 = 0,$$

these $s+2$ determinants are proportional to the quantities $X, Y, \ldots, Z, W, T$ respectively, and the determinant thus assumes the form

$$\frac{aX + bY + \ldots + cZ + dW + eT}{T} \Delta,$$

where Δ is the like determinant with $(X, Y, \ldots, Z, W)$, and where the coefficients $a, b, \ldots, c, d, e$ are precisely those of the linear relation $aX + bY + \ldots + cZ + dW + eT = 1$; the last-mentioned expression is thus $= \frac{1}{T}\Delta$, or, substituting for Δ its value, we have

$$d\Sigma = \frac{1}{T} \frac{\partial(X, Y, \ldots, Z, W)}{\partial(\theta, \phi, \ldots, \psi, *)} d\theta\, d\phi \ldots d\psi;$$

viz. comparing with the foregoing expression for dS we have

$$dS = \frac{1}{T^s} d\Sigma,$$

which is the requisite formula for the transformation of dS.

143. Consider the integral

$$\int \frac{dS}{\{(*\,\text{Q}\,x, y, \ldots, z, w, 1)^2\}^{\frac{1}{2}s}},$$

which, from its containing a single quadric function, may be called "one-quadric." Then effecting the foregoing transformation,

$$x, y, \ldots, z, w = \frac{X}{T}, \frac{Y}{T}, \ldots, \frac{Z}{T}, \frac{W}{T},$$

and observing that

$$(*\between x,\ y, \ldots, z,\ w,\ 1)^2 = \frac{1}{T^2}(*\between X,\ Y, \ldots, Z,\ W,\ T)^2,$$

the integral becomes

$$= \int \frac{d\Sigma}{\{(*\between X,\ Y, \ldots, Z,\ W,\ T)^2\}^{\frac{1}{2}s}},$$

where $X,\ Y, \ldots, Z,\ W,\ T$ denote given linear functions (with constant coefficients) of the $s+1$ variables $\xi,\ \eta, \ldots, \zeta,\ \omega$, or, what is the same thing, given linear functions of the $s+2$ quantities $\xi,\ \eta, \ldots, \zeta,\ \omega,\ 1$, such that identically

$$X^2 + Y^2 + \ldots + Z^2 + W^2 - T^2 = \xi^2 + \eta^2 + \ldots + \zeta^2 + \omega^2 - 1.$$

We have then $\xi^2 + \eta^2 + \ldots + \zeta^2 + \omega^2 - 1 = 0$, and $d\Sigma$ as the corresponding spherical element.

144. We may have $X,\ Y, \ldots, Z,\ W,\ T$ such linear functions of $\xi,\ \eta, \ldots, \zeta,\ \omega,\ 1$ that not only

$$X^2 + Y^2 + \ldots + Z^2 + W^2 - T^2 = \xi^2 + \eta^2 + \ldots + \zeta^2 + \omega^2 - 1$$

as above, but also

$$(*\between X,\ Y, \ldots, Z,\ W,\ T)^2 = A\xi^2 + B\eta^2 + \ldots + C\zeta^2 + E\omega^2 - L;$$

this being so, the integral becomes

$$\int \frac{d\Sigma}{\{A\xi^2 + B\eta^2 + \ldots + C\zeta^2 + E\omega^2 - L\}^{\frac{1}{2}s}},$$

where the $s+2$ coefficients $A,\ B, \ldots, C,\ E,\ L$ are given by means of the identity

$$-(\theta + A)(\theta + B) \ldots (\theta + C)(\theta + E)(\theta + L)$$
$$= \text{Disct.}\ \{(*\between X,\ Y, \ldots, Z,\ W,\ T)^2 + \theta(X^2 + Y^2 + \ldots + Z^2 + W^2 - T^2)\};$$

viz. equating the discriminant to zero, we have an equation in θ, the roots whereof are $-A,\ -B, \ldots, -C,\ -E,\ -L$.

The integral is

$$\int \frac{d\Sigma}{\{(A-L)\xi^2 + (B-L)\eta^2 + \ldots + (C-L)\zeta^2 + (E-L)\omega^2\}^{\frac{1}{2}s}},$$

which is of the form

$$\int \frac{d\Sigma}{\{a\xi^2 + b\eta^2 + \ldots + c\zeta^2 + e\omega^2\}^{\frac{1}{2}s}},$$

where I provisionally assume that $a,\ b, \ldots, c,\ e$ are all positive.

145. To transform this, in place of the $s+1$ variables $\xi,\ \eta, \ldots, \zeta,\ \omega$ connected by $\xi^2 + \eta^2 + \ldots + \zeta^2 + \omega^2 = 1$, we introduce the $s+1$ variables $x,\ y, \ldots, z,\ w$, such that

$$x = \frac{\xi\sqrt{a}}{\rho},\quad y = \frac{\eta\sqrt{b}}{\rho}, \ldots,\quad z = \frac{\zeta\sqrt{c}}{\rho},\quad w = \frac{\omega\sqrt{d}}{\rho},$$

where

$$\rho^2 = a\xi^2 + b\eta^2 + \ldots + c\zeta^2 + e\omega^2,$$

and consequently

$$x^2 + y^2 + \ldots + z^2 + w^2 = 1.$$

Hence, writing dS to denote the spherical element corresponding to the point $(x, y, \ldots, z, w)$, we have, by a former formula,

$$dS = \frac{1}{\rho^{s+1}} \frac{\partial(\xi\sqrt{a},\ \eta\sqrt{b}, \ldots,\ \zeta\sqrt{c},\ \omega\sqrt{e})}{\partial(\theta,\ \phi, \ldots,\ \psi,\ *)} d\theta\, d\phi \ldots d\psi$$

$$= \frac{(ab \ldots ce)^{\frac{1}{2}}}{\rho^{s+1}} d\Sigma;$$

or, what is the same thing,

$$\frac{d\Sigma}{\{a\xi^2 + b\eta^2 + \ldots + c\zeta^2 + e\omega^2\}^{\frac{1}{2}(s+1)}} = \frac{1}{(ab \ldots ce)^{\frac{1}{2}}} dS.$$

Hence, integrating each side, and observing that $\int dS$, taken over the whole spherical surface $x^2 + y^2 + \ldots + z^2 + w^2 = 1$, is $= 2(\Gamma\frac{1}{2})^{s+1} \div \Gamma(\frac{1}{2}s + \frac{1}{2})$, we have

$$\int \frac{d\Sigma}{\{a\xi^2 + b\eta^2 + \ldots + c\zeta^2 + e\omega^2\}^{\frac{1}{2}(s+1)}} = \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma(\frac{1}{2}s + \frac{1}{2})} \frac{1}{(ab \ldots ce)^{\frac{1}{2}}}.$$

146. For $a, b, \ldots, c, e$ write herein $a + \theta, b + \theta, \ldots, c + \theta, e + \theta$ respectively, and multiply each side by θ^{q-1}, where q is any positive integer or fractional number less than $\frac{1}{2}s$: integrate from $\theta = 0$ to $\theta = \infty$. On the left-hand side, attending to the relation $\xi^2 + \eta^2 + \ldots + \zeta^2 + \omega^2 = 1$, the integral in regard to θ is

$$\int_0^\infty \frac{\theta^{q-1}\, d\theta}{\{\rho^2 + \theta\}^{\frac{1}{2}(s+1)}},$$

where $\rho^2, = a\xi^2 + b\eta^2 + \ldots + c\zeta^2 + e\omega^2$, is independent of θ as before; the value of the definite integral is

$$= \frac{\Gamma\{\frac{1}{2}(s+1) - q\}\,\Gamma(q)}{\Gamma\frac{1}{2}(s+1)} \frac{1}{\rho^{s+1-2q}},$$

which, replacing ρ by its value and multiplying by $d\Sigma$, and prefixing the integral sign, gives the left-hand side; hence, forming the equation and dividing by a numerical factor, we have

$$\int \frac{d\Sigma}{(a\xi^2 + \ldots + c\zeta^2 + e\omega^2)^{\frac{1}{2}(s+1)-q}} = \frac{2(\Gamma\frac{1}{2})^{s+1}}{\Gamma q\, \Gamma\{\frac{1}{2}(s+1) - q\}} \int_0^\infty dt\,.\, t^{-q-1}\{(t+a) \ldots (t+c)(t+e)\}^{-\frac{1}{2}}.$$

In particular, if $q = -\frac{1}{2}$, then

$$\int \frac{d\Sigma}{(a\xi^2 + \ldots + c\zeta^2 + e\omega^2)^{\frac{1}{2}s}} = \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s} \int_0^\infty dt\,.\, t^{-\frac{1}{2}}\{(t+a) \ldots (t+c)(t+e)\}^{-\frac{1}{2}};$$

or, if for $a, \ldots, c, e$ we restore the values $A-L, \ldots, C-L, E-L$, then

$$\int \frac{d\Sigma}{(A\xi^2+\ldots+C\zeta^2+E\omega^2-L)^{\frac{1}{2}s}} = \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_0^\infty dt\,.\,t^{-\frac{1}{2}}\{(t+A-L)\ldots(t+C-L)(t+E-L)\}^{-\frac{1}{2}},$$

$$= \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-L}^\infty dt\,\{(t+A)\ldots(t+C)(t+E)(t+L)\}^{-\frac{1}{2}};$$

viz. we thus have

$$\int \frac{dS}{\{(*\between w, \ldots, z, w, 1)^2\}^{\frac{1}{2}s}} = \frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_{-L}^\infty dt\,\{(t+A)\ldots(t+C)(t+E)(t+L)\}^{-\frac{1}{2}},$$

where $(t+A)\ldots(t+C)(t+E)(t+L)$ is in fact a given rational and integral function of t; viz. it is

$$= -\text{Disct.}\,\{(*\between X, \ldots, Z, W, T)^2 + t(X^2+\ldots+Z^2+W^2-T^2)\}.$$

147. Consider, in particular, the integral

$$\int \frac{dS}{\{(a-fx)^2+\ldots+(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}};$$

here

$$\begin{aligned}(*\between X, \ldots, Z, W, T)^2 &+ t(X^2+\ldots+Z^2+W^2-T^2)\\ &= (aT-fX)^2+\ldots+(cT-hZ)^2+(eT-kW)^2+l^2T^2+t(X^2+\ldots+Z^2+W^2-T^2)\\ &= (f^2+t)X^2+\ldots+(h^2+t)Z^2+(k^2+t)W^2+(a^2+\ldots+c^2+e^2+l^2-t)T^2\\ &\qquad -2afXT-\ldots-2chZT-2ekWT;\end{aligned}$$

viz. the discriminant taken negatively is

$$\begin{vmatrix} t+f^2, \ldots & , & -af \\ \vdots & & \\ & \ldots, t+h^2, & -ch \\ -af, \ldots & -ch, & -(a^2+\ldots+c^2+e^2+l^2)+t \end{vmatrix},$$

which is

$$= (t+f^2)\ldots(t+h^2)(t+k^2)\left(t-a^2-\ldots-c^2-e^2-l^2+\frac{a^2f^2}{t+f^2}+\ldots+\frac{c^2h^2}{t+h^2}+\frac{e^2k^2}{t+k^2}\right),$$

$$= \{t(t+f^2)\ldots(t+h^2)(t+k^2)\}\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)$$

$$= (t+A)\ldots(t+C)(t+E)(t+L);$$

and consequently $-A, \ldots, -C, -E, -L$ are the roots of the equation

$$1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}=0.$$

148. The roots are all real; moreover there is one and only one positive root. Hence, taking $-L$ to be the positive root, we have $A, \ldots, C, E, -L$ all positive, and therefore *à fortiori* $A-L, \ldots, C-L, E-L$ all positive: which agrees with a foregoing

provisional assumption. Or, writing for greater convenience θ to denote the positive quantity $-L$, that is, taking θ to be the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}-\frac{l^2}{\theta}=0,$$

we have

$$\int\frac{dS}{\{(a-fx)^2+\ldots+(c-hz)^2+(e-kw)^2+l^2\}^{\frac{1}{2}s}}$$
$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\,\frac{1}{\sqrt{t(t+f^2)\ldots(t+h^2)(t+k^2)\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)}};$$

or, what is the same thing, we have

$$\frac{1}{f\ldots h}\int\frac{dx\ldots dz}{\pm w\{(a-x)^2+\ldots+(c-z)^2+(e\mp kw)^2+l^2\}^{\frac{1}{2}s}}$$
$$=\frac{\Gamma\frac{1}{2}s}{2(\Gamma\frac{1}{2})^s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}-\frac{l^2}{t}\right)^{-\frac{1}{2}}\{t(t+f^2)\ldots(t+h^2)(t+k^2)\}^{-\frac{1}{2}},$$

where on the left-hand side w now denotes $\sqrt{1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}}$, and the limiting equation is $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$.

149. Suppose $l=0$: then, if

$$\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1,$$

the equation

$$1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0$$

has a positive root differing from zero, which may be represented by the same letter θ; but if

$$\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}<1,$$

then the positive root of the original equation becomes $=0$; viz. as l gradually diminishes to zero, the positive root θ also diminishes and becomes ultimately zero.

Hence, writing $l=0$, we have

$$\int\frac{dS}{\{(a-fx)^2+\ldots+(c-hz)^2+(e-kw)^2\}^{\frac{1}{2}s}},$$

or, what is the same thing,

$$\frac{1}{f\ldots h}\int\frac{dx\ldots dz}{\pm w\{(a-x)^2+\ldots+(c-z)^2+(e\mp kw)^2\}^{\frac{1}{2}s}},$$
$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t+k^2}\right)^{-\frac{1}{2}}\{t(t+f^2)\ldots(t+h^2)(t+k^2)\}^{-\frac{1}{2}},$$

θ now denoting either the positive root of the equation

$$1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta+k^2}=0,$$

or else 0, according as

$$\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1 \text{ or } <1.$$

In the case $\frac{a^2}{f^2}+\ldots+\frac{e^2}{k^2}<1$, the inferior limit being then 0, this is, in fact, Jacobi's theorem (*Crelle*, t. XII. p. 69, 1834); but Jacobi does not consider the general case where l is not $=0$, nor does he give explicitly the formula in the other case

$$l=0,\ \frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}+\frac{e^2}{k^2}>1.$$

150. Suppose $k=0$, e being in the first instance not $=0$: then the former alternative holds good; and observing, in regard to the form which contains $\pm w$ in the denominator, that we can now take account of the two values by simply multiplying by 2, we have

$$\int\frac{dS}{\{(a-fx)^2+\ldots+(c-hz)^2+e^2\}^{\frac{1}{2}s}},\ =\frac{2}{f\ldots h}\int\frac{dx\ldots dz}{w\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s}},$$

(w on the right-hand side denoting $\sqrt{1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}}$, and the limiting equation being $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$), each

$$=\frac{2(\Gamma\frac{1}{2})^s}{\Gamma\frac{1}{2}s}\int_\theta^\infty dt\left(1-\frac{a^2}{t+f^2}-\ldots-\frac{c^2}{t+h^2}-\frac{e^2}{t}\right)^{-\frac{1}{2}}t^{-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}},$$

where θ is here the positive root of the equation $1-\frac{a^2}{\theta+f^2}-\ldots-\frac{c^2}{\theta+h^2}-\frac{e^2}{\theta}=0$, which is the formula referred to at the beginning of the present Annex. We may in the formula write $e=0$, thus obtaining the theorem under two different forms for the cases $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1$ and <1 respectively.

ANNEX X. *Methods of* LEJEUNE-DIRICHLET *and* BOOLE. Art. Nos. 151 to 162.

151. The notion, that the density ρ is a discontinuous function vanishing for points outside the attracting mass, has been made use of in a different manner by Lejeune-Dirichlet (1839) and Boole (1857): viz. supposing that ρ has a given value $f(x,\ldots,z)$ within a given closed surface S and is $=0$ outside the surface, these geometers in the expression of a potential or prepotential integral replace ρ by a definite integral which possesses the discontinuity in question, viz. it is $=f(x,\ldots,z)$ for points inside

the surface and $=0$ for points outside the surface; and then in the potential or prepotential integral they extend the integration over the whole of infinite space, thus getting rid of the equation of the surface as a limiting equation for the multiple integral.

152. Lejeune-Dirichlet's paper "Sur une nouvelle méthode pour la détermination des intégrales multiples" is published in *Comptes Rendus*, t. VIII. pp. 155—160 (1839), and *Liouville*, t. IV. pp. 164—168 (same year). The process is applied to the form

$$-\frac{1}{p-1}\frac{d}{da}\int\frac{dx\,dy\,dz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}(p-1)}}$$

taken over the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$; but it would be equally applicable to the triple integral itself, or say to the s-tuple integral

$$\int\frac{dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

or, indeed, to

$$\int\frac{dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

taken over the ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$; but it may be as well to attend to the first form, as more resembling that considered by the author.

153. Since $\frac{2}{\pi}\int_0^\infty\frac{\sin\phi}{\phi}\cos\lambda\phi\,d\phi$ is $=1$ or 0, according as λ is <1 or >1, it follows that the integral is equal to the real part of the following expression,

$$\frac{2}{\pi}\int_0^\infty d\phi\,\frac{\sin\phi}{\phi}\int e^{i\left(\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}\right)}\frac{dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}},$$

where the integrations in regard to $x,\ldots,z$ are now to be extended from $-\infty$ to $+\infty$ for each variable. A further transformation is necessary: since

$$\frac{1}{\sigma^r}=\frac{1}{\Gamma r}e^{-r\pi i}\int_0^\infty d\psi\,.\,\psi^{r-1}e^{i\sigma\psi},\qquad \sigma \text{ positive, and } r \text{ positive and } <1,$$

writing herein $(a-x)^2+\ldots+(c-z)^2$ for σ, and $\frac{1}{2}s+q$ for r, we have

$$\frac{1}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}=\frac{1}{\Gamma\left(\frac{1}{2}s+q\right)}e^{-\left(\frac{1}{2}s+q\right)\pi i}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}e^{i\psi\{(a-x)^2+\ldots+(c-z)^2\}},$$

and the value is thus

$$=\frac{2}{\pi\Gamma\left(\frac{1}{2}s+q\right)}e^{-\left(\frac{1}{2}s+q\right)\frac{\pi i}{2}}\int_0^\infty d\phi\,\frac{\sin\phi}{\phi}\int_0^\infty d\psi\,.\,\psi^{\frac{1}{2}s+q-1}\int e^{i\left(\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}\right)\phi}e^{-i\psi\{(a-x)^2+\ldots+(c-z)^2\}}\,dx\ldots dz,$$

where the integral in regard to the variables $(x, \ldots, z)$ is

$$= e^{i\psi(a^2+\ldots+c^2)} \int dx^{i\left\{\left(\psi+\frac{\phi}{f^2}\right)x^2+2a\psi x\right\}} \ldots \int dz\, e^{i\left\{\left(\psi+\frac{\phi}{h^2}\right)z^2-2c\psi z\right\}};$$

and here the x-integral is

$$= e^{\frac{1}{4}i\pi} \sqrt{\frac{f^2\pi}{f^2\psi+\phi}}\, e^{-\frac{a^2f^2\psi^2 i}{f^2\psi+\phi}},$$

and the like for the other integrals up to the z-integral. The resulting value is thus

$$= \frac{2}{\pi\Gamma(\frac{1}{2}s+q)} e^{-\frac{1}{2}q\pi i} \int_0^\infty \frac{\sin\phi}{\phi} d\phi \int_0^\infty d\psi \,.\, \psi^{\frac{1}{2}s+q-1} e^{\psi\phi i\left(\frac{a^2}{\phi+f^2\psi}+\ldots+\frac{c^2}{\phi+h^2\psi}\right)} \frac{\pi^{\frac{1}{2}s} f \ldots h}{\sqrt{(\phi+f^2\psi)\ldots(\phi+h^2\psi)}},$$

which, putting therein $\psi = \frac{\phi}{t}$, $d\psi = -\frac{\phi}{t^2}dt$, is

$$= \frac{2\pi^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s+q)} (f \ldots h)\, e^{-\frac{1}{2}q\pi i} \int_0^\infty dt \frac{t^{-q-1}}{\sqrt{(f^2+t)\ldots(h^2+t)}} \int_0^\infty e^{i\phi\left(\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}\right)} \sin\phi \,.\, \phi^{q-1} d\phi.$$

154. But we have to consider only the real part of this expression; viz. writing for shortness $\sigma = \frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}$, we require the real part of

$$e^{-\frac{1}{2}q\pi i} \int_0^\infty e^{i\sigma\phi} \phi^{q-1} \sin\phi\, d\phi.$$

Writing here for $\sin\phi$ its exponential value $\frac{1}{2i}(e^{i\phi}-e^{-i\phi})$, and using the formula

$$\frac{1}{\sigma^q} = \frac{1}{\Gamma q} e^{-q\pi i} \int_0^\infty d\phi \,.\, \phi^{q-1} e^{i\sigma\phi} \qquad (\sigma \text{ positive}),$$

and the like one

$$\frac{1}{(-\sigma)^q} = \frac{1}{\Gamma q} e^{q\pi i} \int_0^\infty d\phi \,.\, \phi^{q-1} e^{i\sigma\phi} \qquad (\sigma \text{ negative}),$$

(in which formulæ q must be positive and less than 1), we see that the real part in question is $= 0$, or is

$$-\frac{\Gamma q \sin(q+1)\pi}{2(1-\sigma)^q}, \quad = \frac{\pi}{2\Gamma(1-q)} \frac{1}{(1-\sigma)^q},$$

according as $\sigma > 1$ or $\sigma < 1$.

155. If the point is interior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2} < 1$, and consequently also $\sigma < 1$, and the value, writing $(\Gamma\frac{1}{2})^2$ instead of π, is

$$= \frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\Gamma(1-q)} (f \ldots h) \int_0^\infty dt \,.\, t^{-q-1} \{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}} \left(1-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}\right)^{-q}.$$

But if the point be exterior, $\frac{a^2}{f^2}+\ldots+\frac{c^2}{h^2}>1$, and hence, writing θ for the positive root of the equation, $\sigma=1$; viz. θ is the positive root of the equation $\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}=1$; then $t=0$, σ is greater than 1, and continues so as t increases, until, for $t=\theta$, σ becomes $=1$, and for larger values of t we have $\sigma<1$; and the expression thus is

$$=\frac{(\Gamma\frac{1}{2})^s}{\Gamma(\frac{1}{2}s+q)\,\Gamma(1-q)}(f\ldots h)\int_\theta^\infty dt\,.\,t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}\right)^{-q};$$

viz. the two expressions, in the cases of an interior point and an exterior point respectively, give the value of the integral

$$\int\frac{dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2\}^{\frac{1}{2}s+q}}.$$

This is, in fact, the formula of Annex IV. No. 110, writing therein $e=0$ and $m=-q$.

156. Boole's researches are contained in two memoirs dated 1846, "On the Analysis of Discontinuous Functions," *Trans. Royal Irish Academy*, vol. XXI. (1848), pp. 124—139, and "On a certain Multiple Definite Integral," do. pp. 140—150 (the particular theorem about to be referred to is stated in the postscript of this memoir), and in the memoir "On the Comparison of Transcendents, with certain applications to the theory of Definite Integrals," *Phil. Trans.* vol. CXLVII. (1857), pp. 745—803, the theorem being the third example, p. 794. The method is similar to, and was in fact suggested by, that of Lejeune-Dirichlet; the auxiliary theorem made use of in the memoir of 1857 for the representation of the discontinuity being

$$\frac{f(x)}{t^i}=\frac{1}{\pi\Gamma i}\int_{-\infty}^\infty\int_0^\infty\int_0^\infty da\,dv\,ds\cos\{(a-x-ts)\,v+\tfrac{1}{2}i\pi\}\,v^i s^{i-1}f(a),$$

which is a deduction from Fourier's theorem.

Changing the notation (and in particular writing s and $\frac{1}{2}s+q$ for his n and i), the method is here applied to the determination of the s-tuple integral

$$V=\int dx\ldots dz\,\frac{\phi\left(\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}\right)}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}},$$

where ϕ is an arbitrary function, taken over the ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$.

157. The process is as follows: we have

$$\frac{\phi\left(\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}\right)}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}=\frac{1}{\pi\,\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\,v^{\frac{1}{2}s+q}\,t^{\frac{1}{2}s+q-1}$$

$$\cos\left\{\left(u-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}-\tau\{(a-x)^2+\ldots+(c-z)^2+e^2\}\right)v+\tfrac{1}{2}(\tfrac{1}{2}s+q)\,\pi\right\}\phi u;$$

viz. the right-hand side is here equal to the left-hand side or is $=0$, according as $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}<1$ or >1. V is consequently obtained by multiplying the right-hand side by $dx\ldots dz$ and integrating from $-\infty$ to $+\infty$ for each variable.

Hence, changing the order of the integration,

$$V=\frac{1}{\pi\,\Gamma\left(\frac{1}{2}s+q\right)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,d\tau\,v^{\frac{1}{2}s+q}\,\tau^{\frac{1}{2}s+q-1}\,\phi u\,.\,\Omega,$$

where

$$\Omega=\int dx\ldots dz\cos\left\{\left(u-e^2\tau-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}+\tau\left\{(a-x)^2+\ldots+(c-z)^2\right\}\right)v+\tfrac{1}{2}\left(\tfrac{1}{2}s+q\right)\pi\right\}.$$

Now

$$\frac{x^2}{f^2}+\tau(a-x)^2=\frac{1+f^2\tau}{f^2}\,\xi^2+\frac{\tau a^2}{1+f^2\tau},\quad\ldots,\quad\frac{z^2}{h^2}+\tau(c-z)^2=\frac{1+h^2\tau}{h^2}\,\zeta^2+\frac{\tau c^2}{1+h^2\tau},$$

if

$$\xi=x-\frac{f^2\tau a}{1+f^2\tau},\qquad\qquad\ldots,\ \zeta=z-\frac{h^2\tau c}{1+h^2\tau}.$$

158. Substituting, and integrating with respect to $\xi,\ldots,\zeta$ between the limits $-\infty$, $+\infty$, we have

$$\Omega=\frac{(f\ldots h)\,\pi^{\frac{1}{2}s}}{\left\{(1+f^2\tau)\ldots(1+h^2\tau)\right\}^{\frac{1}{2}}\,v^{\frac{1}{2}s}}\cos\left\{\left(u-e^2\tau-\frac{a^2\tau}{1+f^2\tau}-\ldots-\frac{c^2\tau}{1+h^2\tau}\right)v+\tfrac{1}{2}q\pi\right\};$$

or, what is the same thing, writing $\frac{1}{t}$ in place of τ, this is

$$\Omega=\frac{(f\ldots h)\,\pi^{\frac{1}{2}s}\,t^{\frac{1}{2}}}{\left\{(f^2+t)\ldots(h^2+t)\right\}^{\frac{1}{2}}\,v^{\frac{1}{2}s}}\cos\left\{\left(u-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)v+\tfrac{1}{2}q\pi\right\};$$

that is, writing

$$\sigma=\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t},$$

we have

$$V=\frac{\pi^{\frac{1}{2}s-1}\,(f\ldots h)}{\Gamma\left(\frac{1}{2}s+q\right)}\int_0^1\int_0^\infty\int_0^\infty du\,dv\,dt\,\frac{t^{-q-1}\,v^q\cos\left\{(u-\sigma)\,v+\frac{1}{2}q\pi\right\}\,\phi u}{\left\{(t+f^2)\ldots(t+h^2)\right\}^{\frac{1}{2}}};$$

or, writing $\pi^{\frac{1}{2}s-1}=\frac{1}{\pi}\left(\Gamma\frac{1}{2}\right)^s$, this is

$$=\frac{\left(\Gamma\frac{1}{2}\right)^s(f\ldots h)}{\Gamma\left(\frac{1}{2}s+q\right)}\int_0^\infty dt\,.\,t^{-q-1}\left\{(t+f^2)\ldots(t+h^2)\right\}^{-\frac{1}{2}}\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,.\,v^q\cos\left\{(u-\sigma)\,v+\tfrac{1}{2}q\pi\right\}\phi u.$$

159. Boole writes

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\,v^q\cos\left\{(u-\sigma)\,v+\tfrac{1}{2}q\pi\right\}\phi u=\left(-\frac{d}{d\sigma}\right)^q\phi(\sigma);$$

viz. starting from Fourier's theorem,

$$\frac{1}{\pi}\int_0^1\int_0^\infty du\,dv\cos(u-\sigma)\,v\,.\,\phi u=\phi(\sigma),$$

where $\phi(\sigma)$ is regarded as vanishing except when σ is between the limits 0, 1, and the limits of u are taken to be 1, 0 accordingly, then, according to an admissible theory of general differentiation, we have the result in question. He has in the formula $\frac{1}{s}$ instead of my t; and he proceeds, "Here σ increases continually with s. As s varies from 0 to ∞, σ also varies from 0 to ∞. To any positive limits of σ will correspond positive limits of s; and these, as will hereafter appear—this refers to his note B—, will in certain cases replace the limits 0 and ∞ in the expression for V."

160. It seems better to deal with the result in the following manner, as in part shown p. 803 of Boole's memoir. Writing the integral in the form

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\pi\,\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\,.\,t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\phi(u)\int_0^\infty dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\},$$

effect the integration in regard to v; viz. according as u is greater or less than σ, then

$$\int_0^\infty dv\,.\,v^q\cos\{(u-\sigma)v+\tfrac{1}{2}q\pi\}=\frac{\Gamma(q+1)\sin(q+1)\pi}{(u-\sigma)^{q+1}},\text{ or }0,$$

$$=\frac{\pi}{\Gamma(-q)(u-\sigma)^{q+1}}\quad,\text{ or }0;$$

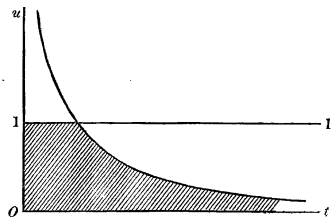
and consequently, writing for σ its value,

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\,\Gamma(\frac{1}{2}s+q)}\int_0^1\int_0^\infty du\,dt\left\{t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\right.$$
$$\left.\left(u-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u\right\},\text{ or 0, as above.}$$

161. To further explain this, consider t as an x-coordinate and u as a y-coordinate; then, tracing the curve

$$y=\frac{a^2}{f^2+x}+\ldots+\frac{c^2}{h^2+x}+\frac{e^2}{x},$$

for positive values of x this is a mere hyperbolic branch, as shown in the figure, viz. $x=0$, $y=\infty$; and as x continually increases to ∞, y continually decreases to zero.



The limits are originally taken to be from $u=0$ to $u=1$ and $t=0$ to $t=\infty$, viz. over the infinite strip bounded by the lines tO, $O1$, 11; but within these limits the

function under the integral sign is to be replaced by zero whenever the values u, t are such that u is less than $\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$, viz. when the values belong to a point in the shaded portion of the strip; the integral is therefore to be extended only over the unshaded portion of the strip; viz. the value is

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\,\Gamma(\frac{1}{2}s+q)}\iint du\,dt\,.\,t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\left(u-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^{-q-1}\phi u,$$

the double integral being taken over the unshaded portion of the strip; or, what is the same thing, the integral in regard to u is to be taken from $u=\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$ (say from $u=\sigma$) to $u=1$, and then the integral in regard to t is to be taken from $t=\theta$ to $t=\infty$, where, as before, θ is the positive root of the equation $\sigma=1$, that is, of $\frac{a^2}{f^2+\theta}+\ldots+\frac{c^2}{h^2+\theta}+\frac{e^2}{\theta}=1$.

162. Write $u=\sigma+(1-\sigma)x$, and therefore $u-\sigma=(1-\sigma)x$, $1-u=(1-\sigma)(1-x)$ and $du=(1-\sigma)dx$; then the limits (1, 0) of x correspond to the limits (1, σ) of u, and the formula becomes

$$V=\frac{(\Gamma\frac{1}{2})^s(f\ldots h)}{\Gamma(-q)\,\Gamma(\frac{1}{2}s+q)}\int_\theta^\infty dt.t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}(1-\sigma)^{-q-1}\int_0^1 dx\,.\,x^{-q-1}\phi\,\{\sigma+(1-\sigma)x\},$$

where σ is retained in place of its value $\frac{a^2}{f^2+t}+\ldots+\frac{c^2}{h^2+t}+\frac{e^2}{t}$. This is, in fact, a form (deduced from Boole's result in the memoir of 1846) given by me, *Cambridge and Dublin Mathematical Journal*, vol. II. (1847), p. 219, [44].

If in particular $\phi u=(1-u)^{q+m}$, then $\phi\{\sigma+(1-\sigma)x\}=(1-\sigma)^{q+m}(1-x)^{q+m}$, and thence

$$\int_0^1 x^{-q-1}\{\phi\sigma+(1-\sigma)x\}\,dx=(1-\sigma)^m\int_0^1 x^{-q-1}(1-x)^{q+m}\,dx,$$

$$=\frac{\Gamma(-q)\,\Gamma(1+q+m)}{\Gamma(1+m)}(1-\sigma)^m;$$

and then, restoring for σ its value, we have

$$V=\frac{(\Gamma\frac{1}{2})^s\,\Gamma(1+q+m)}{\Gamma(\frac{1}{2}s+q)\,\Gamma(1+m)}(f\ldots h)\int_\theta^\infty dt.t^{-q-1}\{(t+f^2)\ldots(t+h^2)\}^{-\frac{1}{2}}\left(1-\frac{a^2}{f^2+t}-\ldots-\frac{c^2}{h^2+t}-\frac{e^2}{t}\right)^m$$

as the value of the integral

$$\int\frac{\left(1-\frac{x^2}{f^2}-\ldots-\frac{z^2}{h^2}\right)^{q+m}dx\ldots dz}{\{(a-x)^2+\ldots+(c-z)^2+e^2\}^{\frac{1}{2}s+q}}$$

taken over the ellipsoid $\frac{x^2}{f^2}+\ldots+\frac{z^2}{h^2}=1$. This is, in fact, the theorem of Annex IV. No. 110 in its general form; but the proof assumes that q is positive.

608.

[EXTRACT FROM A] REPORT ON MATHEMATICAL TABLES.

[From the *Report of the British Association for the Advancement of Science*, (1873), pp. 3, 4.]

It was necessary as a preliminary to form a classification of mathematical (numerical) tables; and the following classification was drawn up by Prof. Cayley and adopted by the Committee.

A. Auxiliary for non-logarithmic computations.

1. Multiplication.

2. Quarter-squares.

3. Squares, cubes, and higher powers, and reciprocals.

B. Logarithmic and circular.

4. Logarithms (Briggian) and antilogarithms (do.); addition and subtraction logarithms, &c.

5. Circular functions (sines, cosines, &c.), natural, and lengths of circular arcs.

6. Circular functions (sines, cosines, &c.), logarithmic.

C. Exponential.

7. Hyperbolic logarithms.

8. Do. antilogarithms (e^x) and $\mathrm{h.l}\tan(45^\circ + \frac{1}{2}\phi)$, and hyperbolic sines, cosines, &c., natural and logarithmic.

D. Algebraic constants.

9. Accurate integer or fractional values. Bernoulli's Numbers, $\Delta^n 0^m$, &c. Binomial coefficients.

10. Decimal values auxiliary to the calculation of series.

E. 11. Transcendental constants, e, π, γ, &c., and their powers and functions.

F. Arithmological.

12. Divisors and prime numbers. Prime roots. The Canon arithmeticus, &c.

13. The Pellian equation.

14. Partitions.

15. Quadratic forms a^2+b^2, &c., and partition of numbers into squares, cubes, and biquadrates.

16. Binary, ternary, &c. quadratic, and higher forms.

17. Complex theories.

G. Transcendental functions.

18. Elliptic.

19. Gamma.

20. Sine-integral, cosine-integral, and exponential-integral.

21. Bessel's and allied functions.

22. Planetary coefficients for given $\frac{a}{a'}$.

23. Logarithmic transcendental.

24. Miscellaneous.

Several of these classes need some little explanation. Thus D 9 and 10 are intended to include the same class of constants, the only difference being that in 9 accurate values are given, while in 10 they are only approximate; thus, for example, the accurate Bernoulli's numbers as vulgar fractions, and the decimal values of the same to (say) ten places are placed in different classes, as the former are of theoretical interest, while the latter are only of use in calculation. It is not necessary to enter into further detail with respect to the classification, as in point of fact it is only very partially followed in the Report.

609.

ON THE ANALYTICAL FORMS CALLED FACTIONS.

[From the *Report of the British Association for the Advancement of Science*, (1875), p. 10.]

A FACTION is a product of differences such that each letter occurs the same number of times; thus we have a quadrifaction where each letter occurs twice, a cubifaction where each letter occurs three times, and so on. A broken faction is one which is a product of factions having no common letter; thus

$$(a-b)^2(c-d)(d-e)(e-c)$$

is a broken quadrifaction, the product of the quadrifactions

$$(a-b)^2 \text{ and } (c-d)(d-e)(e-c).$$

We have, in regard to quadrifactions, the theorem that every quadrifaction is a sum of broken quadrifactions such that each component quadrifaction contains two or else three letters. Thus we have the identity

$$2(a-b)(b-c)(c-d)(d-a) = (b-c)^2.(a-d)^2 - (c-a)^2.(b-d)^2 + (a-b)^2.(c-d)^2,$$

which verifies the theorem in the case of a quadrifaction of four letters; but the verification even in the next following case of a quadrifaction of five letters is a matter of some difficulty.

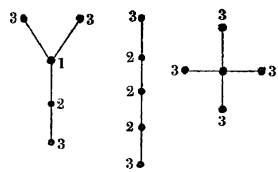
The theory is connected with that of the invariants of a system of binary quantics.

610.

ON THE ANALYTICAL FORMS CALLED TREES, WITH APPLICATION TO THE THEORY OF CHEMICAL COMBINATIONS.

[From the *Report of the British Association for the Advancement of Science*, (1875), pp. 257—305.]

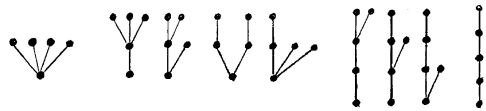
I HAVE in two papers "On the Analytical forms called Trees," *Phil. Mag.* vol. XIII. (1857), pp. 172—176, [203], and ditto, vol. XX. (1859), pp. 374—378, [247], considered this theory; and in a paper "On the Mathematical Theory of Isomers," ditto, vol. XLVII. (1874), p. 444, [586], pointed out its connexion with modern chemical theory. In particular, as regards the paraffins C_nH_{2n+2}, we have n atoms of carbon connected by $n-1$ bands, under the restriction that from each carbon-atom there proceed at most 4 bands (or, in the language of the papers first referred to, we have n knots connected by $n-1$ branches), in the form of a tree; for instance, $n=5$, such forms (and the only such forms) are



And if, under the foregoing restriction of only 4 bands from a carbon-atom, we connect with each carbon-atom the greatest possible number of hydrogen-atoms, as shown in the diagrams by the affixed numerals, we see that the number of hydrogen-atoms is 12 $(=2.5+2)$; and we have thus the representations of three different paraffins, C_5H_{12}. It should be observed that the tree-symbol of the paraffin is

completely determined by means of the tree formed with the carbon-atoms, or say of the carbon-tree, and that the question of the determination of the theoretic number of the paraffins C_nH_{2n+2} is consequently that of the determination of the number of the carbon-trees of n knots, viz. the number of trees with n knots, subject to the condition that the number of branches from each knot is at most $=4$.

In the paper of 1857, which contains no application to chemical theory, the number of branches from a knot was unlimited; and, moreover, the trees were considered as issuing each from one knot taken as a root, so that, $n=5$, the trees regarded as distinct (instead of being as above only 3) were in all 9, viz. these were



which, regarded as issuing from the bottom knots, are in fact distinct; while, taking them as issuing each from a properly selected knot, they resolve themselves into the above-mentioned 3 forms. The problem considered was in fact that of the "general root-trees with n knots"—*general*, inasmuch as the number of branches from a knot was without limit; *root-trees*, inasmuch as the enumeration was made on the principle last referred to. It was found that for

knots.....	1,	2,	3,	4,	5,	6,	7,	8,.....
No. of trees was ...	1,	1,	2,	4,	9,	20,	48,	115,.....
=	1,	A_1,	A_2,	A_3,	A_4,	A_5,	A_6,	A_7,.....;

the law being given by the equation

$$(1-x)^{-1}(1-x^2)^{-A_1}(1-x^3)^{-A_2}(1-x^4)^{-A_3}\ldots = 1 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots;$$

but the next following numbers A_8, A_9, A_{10}, the correct values of which are 286, 719, 1842, were given erroneously as 306, 775, 2009. I have since calculated two more terms, A_{11}, $A_{12}=4766$, 12486.

The other questions considered in the paper of 1857 and in that of 1859 have less immediate connexion with the present paper, but for completeness I reproduce the results in a Note*.

* In the paper of 1857 I also considered the problem of finding B_r the number with r free branches, with bifurcations at least: this was given by a like formula

$$(1-x)^{-1}(1-x^2)^{-B_2}(1-x^3)^{-B_3}(1-x^4)^{-B_4}\ldots = 1 + x + 2B_2 x^2 + 2B_3 x^3 + 2B_4 x^4 \ldots,$$

leading to

$B_r=$	1,	2,	5,	12,	33,	90, ...

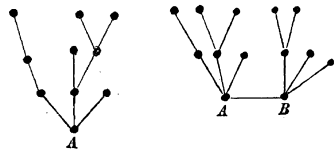
for

$r=$	2,	3,	4,	5,	6,	7, ...

In the paper of 1859, the question is to find the number of trees with a given number m of terminal knots: we have here

$$\phi m = 1.2.3\ldots(m-1) \text{ coefficient of } x^{m-1} \text{ in } \frac{1}{2-e^x},$$

To count the trees on the principle first referred to, we require the notions of "centre" and "bicentre," due, I believe, to Sylvester; and to establish these we require the notions of "main branch" and "altitude": viz. in a tree, selecting any knot at pleasure as a root, the branches which issue from the root, each with all the branches that belong to it, are the main branches, and the distance of the furthest knot, measured by the number of intermediate branches, is the altitude of the main



branch. Thus in the left-hand figure, taking A as the root, there are 3 main branches of the altitudes 3, 3, 1 respectively: in the right-hand figure, taking A as the root, there are 4 main branches of the altitudes 2, 2, 1, 3 respectively; and we have then the theorem that in every tree there is either one and only one centre, or else one and only one bicentre; viz. we have (as in the left-hand figure) a centre A which is such that there issue from it two or more main branches of altitudes equal to each other and superior to those of the other main branches (if any); or else (as in the right-hand figure) a bicentre AB, viz. two contiguous knots, such that issuing from A (but not counting AB), and issuing from B (but not counting BA), we have two or more main branches, one at least from A and one at least from B, of altitudes equal to each other and superior to those of the other main branches in question (if any). The theorem, once understood, is proved without difficulty: we consider two terminal knots, the distance of which, measured by the number of intermediate branches, is greater than or equal to that of any other two terminal knots; if, as in the left-hand figure, the distance is even, then the central knot A is the centre of the tree; if, as in the right-hand figure, the distance is odd, then the two central knots AB form the bicentre of the tree.

In the former case, observe that if G, H are the two terminal knots, the distance of which is $=2\lambda$, then the distance of each from A is $=\lambda$, and there cannot be

giving the values

$\phi m=$	1,	1,	3,	13,	75,	541,	4683,	47293, ...
for $m=$	1,	2,	3,	4,	5,	6,	7,	8, ...

But if from each non-terminal knot there ascend two and only two branches, then in this case $\phi m=$coefficient of x^{m-1} in $\dfrac{1-\sqrt{1-4x}}{2x}$, viz. we have the very simple form

$$\phi m=\frac{1\,.\,3\,.\,5\ldots 2m-3}{1\,.\,2\,.\,3\ldots m}\,2^{m-1},$$

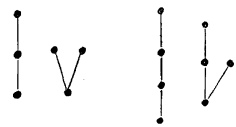
giving

$\phi m=$	1,	1,	2,	5,	14,	42, ...	
for $m=$	1,	2,	3,	4,	5,	7, ...	

any other terminal knot I, the distance of which from A is greater than λ (for, if there were, then the distance of I from G or else from H would be greater than 2λ); there cannot be any two terminal knots I, J, the distance of which is greater than 2λ; and if there are any two knots I, J, the distance of which is $=2\lambda$, then these belong to different main branches, the distance of each of them from A being $=\lambda$; whence, starting with I, J (instead of G, H), we obtain the same point A as centre. Similarly, in the latter case, there is a single bicentre AB.

Hence, since in any tree there is a unique centre or bicentre, the question of finding the number of distinct trees with n knots is in fact that of finding the number of centre- and bicentre-trees with n knots; or say it is the problem of the "general centre- and bicentre-trees with n knots:" *general*, inasmuch as the number of branches from a knot is as yet taken to be without limit; or since (as will appear) the number of the bicentre-trees can be obtained without difficulty when the problem of the root-trees is solved, the problem is that of the "general centre-trees with n knots." It will appear that the solution depends upon and is very readily derived from that of the foregoing problem of general root-trees, so that this last has to be considered, not only for its own sake, but with a view to that of the centre-trees. And in each of the two problems we doubly divide the whole system of trees according to the number of the main branches, issuing from the root or centre as the case may be, and according to the altitude of the longest main branch or branches, or say the altitude of the tree; so that the problem really is, for a given number of knots, a given number of main branches, and a given altitude, to find the number of root-trees, or (as the case may be) centre-trees.

We next introduce the restriction that the number of branches from any knot is equal to a given number at most; viz. according as this number is $=2$, 3 or 4, we have, say oxygen-trees, boron-trees*, and carbon-trees respectively; and these are, as before, root-trees or centre- or bicentre-trees, as the case may be. The case where the number is 2 presents no difficulty: in fact, if the number of knots be $=n$, then the number of root-trees is either $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$; viz. $n=3$ and $n=4$, the root-trees are



and the number of centre- or bicentre-trees is always $=1$: viz. n odd, there is one centre-tree; and n even, one bicentre-tree; it is only considered as a particular case of the general theorem. The case where the number is $=3$ is analytically interesting: although there may not exist, for any 3-valent element, a series of hydrogen compounds

* I should have said nitrogen-trees; but it appears to me that nitrogen is of necessity 5-valent, as shown by the compound, Ammonium-Chloride, $=NH_4Cl$. Of course, the word boron is used simply to stand for a 3-valent element.

B_nH_{n+2} corresponding to the paraffins. The case, where the number is $=4$ or say the carbon-trees, is that which presents the chief chemical interest, as giving the paraffins C_nH_{2n+2}; and I call to mind here that the theory of the carbon-root trees is established as an analytical result for its own sake and as the foundation for the other case, but that it is the number of the carbon centre- and bicentre-trees which is the number of the paraffins.

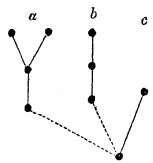
The theory extends to the case where the number of branches from a knot is at most $=5$, or $=$ any larger number; but I have not developed the formula.

I pass now to the analytical theory: considering first the case of general root-trees, we endeavour to find for a given altitude N the number of trees of a given number of knots n and main branches α, or say the generating function

$$\Sigma\Omega t^\alpha x^n,$$

where the coefficient Ω gives the number of the trees in question. And we assume that the problem is solved for the cases of the several inferior altitudes $0, 1, 2, 3, \ldots, N-1$.

This being so, observe that a tree of altitude N can be built up as shown in the figure, which I call the edification diagram, by combining one or more trees of altitude $N-1$ with a single tree of altitude not exceeding $N-1$; viz. in the figure, $N=3$, we have the two trees a, b, each of altitude 2, combined, as shown by the



dotted lines, with the tree c of altitude 1: the whole number of knots in the resulting tree is the sum of the number of knots on the three trees a, b, c: the number of main branches is equal to the number of the trees a, b, plus the number of main branches of the tree c. It is to be observed that the tree c may reduce itself to the tree $(\cdot)$ of one knot and of altitude zero; but each of the trees a, b, as being of the altitude $N-1$, must contain at least N knots.

Taking $N=2$ or any larger number, it is hence easy to see that the required generating function $\Sigma\Omega t^\alpha x^n$ is

$$= (1-tx^N)^{-1}(1-tx^{N+1})^{-l_1}(1-tx^{N+2})^{-l_2}\ldots[t^{1\ldots\infty}] \qquad \text{(first factor)},$$

$$x+(t)\,x^2+(t,\ t^2)\,x^3+(t,\ t^2,\ t^3)\,x^4+\ldots \qquad \text{(second factor)}.$$

As regards the first factor, the exponents taken with reversed sign, that is, as positive, are $1=$ no. of trees, altitude $N-1$, of N knots; $l_1=$ ditto, same altitude, of $(N+1)$ knots; $l_2=$ ditto, same altitude, of $N+2$ knots, and so on; and where the

symbol $[t^{1\cdots\infty}]$ denotes that, in the function or product of factors which precedes it, the terms to be taken account of are those in $t^1, t^2, t^3, \ldots$; viz. it denotes that the term in t^0, or constant term ($=1$ in fact), is to be rejected.

In the second factor, the expressions x, $(t)\,x^2$, $(t,\ t^2)\,x^3, \ldots$ represent, for given exponents of t, x, denoting the number of main branches and the number of knots respectively, the number of trees of altitude not exceeding $N-1$: thus x, $=1\,t^0x^1$ represents the number of such trees, 1 knot, 0 main branch, $=1$; and so, if the value of $(t,\ t^2,\ t^3,\ t^4)\,x^5$ be $(\alpha t+\beta t^2+\gamma t^3+\delta t^4)\,x^5$, then for trees of an altitude not exceeding $N-1$, and of 5 knots, α represents the number of trees of 1 main branch, β that of trees of 2 main branches, γ that of trees of 3 main branches, δ that of trees of 4 main branches. It is clear that the number of trees satisfying the given conditions and of an altitude not exceeding $N-1$ is at once obtained by addition of the numbers of the trees satisfying the given conditions, and of the altitudes 0, 1, 2, . . , $N-1$; all which numbers are taken to be known.

It is to be remarked that the first factor,

$$(1-tx^N)^{-1}\,(1-tx^{N+1})^{-l_1}\,(1-tx^{N+2})^{-l_2}\ldots[t^{1\cdots\infty}],$$

shows by its development the number of combinations of trees a, b,. . of the altitude $N-1$; one such tree at least *must* be taken, and the symbol $[t^{1\cdots\infty}]$ gives effect to this condition: the second factor $x+(t)\,x^2+(t,\ t^2)\,x^3+\ldots$ shows the number of the trees c of altitude not exceeding $N-1$. And this being so, there is no difficulty in seeing how the product of the two factors is the generating function for the trees of altitude N.

In the case $N=0$, the generating function, or GF, is $=x$; viz. altitude 0, there is only the tree $(\cdot)$, 1 knot, 0 main branch.

When $N=1$, the GF is $=(1-tx)^{-1}\,[t^{1\cdots\infty}]\,x$, $=tx^2+t^2x^3+t^3x^4\ldots$,

viz. altitude 1, there is 1 tree tx^2, 2 knots, 1 main branch; 1 tree t^2x^3, 3 knots, 2 main branches; and so on.

Hence $N=2$, we obtain

$$GF=(1-tx^2)^{-1}\,(1-tx^3)^{-1}\,(1-tx^4)^{-1}\ldots[t^{1\cdots\infty}]\,.\,(x+tx^2+t^2x^3+t^3x^4+\ldots);$$

viz. as regards the second factor, altitude not exceeding 1, that is, $=0$ or 1, there is altitude 0, 1 tree x, and altitude 1, 1 tree tx^2, 1 tree t^2x^3, and so on. And we hence derive the GF's for the higher values $N=3$, 4, &c.: the details of the process will be afterwards more fully explained.

So far, we have considered root-trees; but referring to the last diagram, it is at once seen that the assumed root will be a centre, provided only that (instead of, it may be, only a single tree a of the altitude $N-1$), we take always two or more trees of the altitude $N-1$ to form the new tree of the altitude N. And we give effect

to this condition by simply writing in place of $[t^{1\cdots\infty}]$ the new symbol $[t^{2\cdots\infty}]$, which denotes that only the terms $t^2, t^3, t^4, \ldots$ are to be taken account of; viz. that the terms in t^0 and t^1 are to be rejected. The component trees of the altitude $N-1$ are, it is to be observed, as before, root-trees; hence the second factor of the generating function is unaltered: the theorem is that for the centre-trees of altitude N we have the same generating function as for the root-trees, writing only $[t^{2\cdots\infty}]$ in place of $[t^{1\cdots\infty}]$. Or, what is the same thing, supposing that the first factor, unaffected by either symbol, is

$$= 1 + x^N(\alpha t + \beta t^2 + \ldots) + x^{N+1}(\alpha' t + \beta' t^2 + \ldots) + \ldots,$$

then, affecting it with $[t^{1\cdots\infty}]$, the value for the root-trees is

$$= x^N(\alpha t + \beta t^2 + \ldots) + x^{N+1}(\alpha' t + \beta' t^2 + \ldots) + \ldots,$$

and, affecting it with $[t^{2\cdots\infty}]$, the value for the centre-trees is

$$= x^N(\beta t^2 + \ldots) + x^{N+1}(\beta' t^2 + \ldots) + \ldots$$

It thus appears how the fundamental problem is that of the root-trees, its solution giving at once that of the centre-trees; whereas we cannot conversely solve the problem of the root-trees by means of that of the centre-trees.

As regards the bicentre-trees, it is to be remarked that, starting from a centre-tree of altitude $N+1$ with two main branches, then by simply striking out the centre, so as to convert into a single branch the two branches which issue from it, we obtain a bicentre-tree of altitude N. Observe that the altitude of a bicentre-tree is measured by that of the longest main branch from A or B, not reckoning AB or BA as a main branch. Hence the number of bicentre-trees, altitude N, is = number of centre-trees of two main branches, altitude $N+1$.

This is, in fact, the convenient formula, provided only the number of centre-trees of two main branches has been calculated up to the altitude $N+1$. But we can find independently the number of bicentre-trees of a given altitude N: the bicentre-tree is, in fact, formed by taking the two connected points A, B each as the root of a root-tree altitude N (the number of knots of the bicentre-tree being thus, it is clear, equal to the sum of the numbers of knots of the two root-trees respectively); and it is thus an easy problem of combinations to find the number of bicentre-trees of a given altitude N. Write

$$x^{N+1}(1 + \beta x + \gamma x^2 + \delta x^3 + \ldots)$$

as the generating function of the root-trees of altitude N; viz. for such trees, 1 = no. of trees with $N+1$ knots, β = no. with $N+2$ knots, and so on; then the generating function of the bicentre-trees of the same altitude N is

$$= x^{2N+2}(1 + \beta_{,} x + \gamma_{,} x^2 + \delta_{,} x^3 + \ldots),$$

where

$$\beta_{,} = \beta,$$
$$\gamma_{,} = \gamma + \tfrac{1}{2}\beta(\beta+1),$$
$$\delta_{,} = \delta + \beta\gamma,$$
$$\epsilon_{,} = \epsilon + \beta\delta + \tfrac{1}{2}\gamma(\gamma+1),$$
$$\zeta_{,} = \zeta + \beta\epsilon + \gamma\delta,$$

and so on; or, what is the same thing, calling the first generating function ϕx, then the second generating function is $= \frac{1}{2}\{(\phi x)^2 + \phi(x^2)\}$.

It will be noticed that the bicentre-trees are not, as were the centre-trees, divided according to the number of their main branches; they might be thus divided according to the sum of the number of the main branches issuing from the two points of the bicentre respectively; a more complete division would be according to the number of main branches issuing from the two points respectively; thus we might consider the bicentre-trees (2, 3), with 2 main branches from one point, and 3 main branches from the other point of the bicentre; but the whole theory of the bicentre-trees is comparatively easy, and I do not go into it further.

We have yet to consider the case of the limited trees where the number of branches from a knot is equal to a given number at most: to fix the ideas, say the carbon-trees, where this number is $=4$. The distinction as to root-trees and centre- and bicentre-trees is as before; and the like theory applies to the two cases respectively. Considering first the case of the root-trees, and referring to the former figure for obtaining the trees of altitude N from those of inferior altitudes, then the trees $a, b, \ldots$ of altitude $N-1$ must be each of them a carbon-tree of not more than $(4-1=)\,3$ main branches: this restriction is necessary, inasmuch as, if for any such tree the number of main branches was $=4$, then there would be from the root of such tree 4 branches *plus* the new branch shown by the dotted line, in all 5 branches; and similarly, inasmuch as there is at least one component tree a contributing one main branch, the number of main branches of the tree c must be $(4-1=)\,3$ at most: the mode of introducing these conditions will appear in the explanation of the actual formation of the generating functions (see explanation preceding Tables III., IV., &c.). The number of main branches is $=4$ at most, and the generating functions have only to be taken up to the terms in t^4; the first factor is consequently in each case affected with a symbol $[t^{1\cdots4}]$, denoting that the only terms to be taken account of are those in t, t^2, t^3, t^4; hence as there is a factor t at least, and the whole is required only up to t^4, the second factor is in each case required only up to t^3.

As regards the centre-trees, the generating functions have here the same expressions as for the root-trees, except that, instead of the symbol $[t^{1\cdots4}]$, we have the symbol $[t^{2\cdots4}]$, denoting that in the first factor the only terms to be taken account of are those in t^2, t^3, t^4; hence as there is a factor t^2 at least, and the whole is required only up to t^4, the second factor is in each case required up to t^2; and we then complete the theory by obtaining the bicentre-trees. The like remarks apply of course to

the boron-trees, number of branches $=3$ at most, and to the oxygen-trees, number $=2$ at most; but, as already remarked, this last case is so simple, that the general method is applied to it only for the sake of seeing what the general method becomes in such an extreme case.

We thus form the Tables, which I proceed to explain.

Table I. of general root-trees is in fact a Table of triple entry, viz. it gives for any given number of knots from 1 to 13 the number of root-trees corresponding to any given number of main branches and to any given altitude. In each compartment, that is, for any given number of knots, the totals of the columns give the number of the trees for each given altitude, and the totals of the lines give the number of the trees for each given number of main branches: the corner grand totals of these totals respectively show for each given number of knots the whole number of root-trees:—

viz. knots ...	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,
numbers are ...	1,	1,	2,	4,	9,	20,	48,	115,	286,	719,	1842,	4766,	12486,

as already mentioned: these numbers were calculated by an independent method.

Table II. of general centre- and bicentre-trees consists of a centre part and a bicentre part: the centre part is arranged precisely in the same manner as the root-table. As to the bicentre part, where it will be observed there is no division for number of main branches, the calculation of the several columns is effected by the before-mentioned formula,

$$\phi_{,}x = \tfrac{1}{2}\{(\phi x)^2 + \phi(x^2)\};$$

thus column 2, we have by Table I. (totals of column 2)

$$\phi x = x^3 + 2x^4 + 4x^5 + 6x^6 + 10x^7 + 14x^8 + 21x^9 + 29x^{10} + \ldots,$$

and thence

$$\phi_{,}x = x^6 + 2x^7 + 7x^8 + 14x^9 + 32x^{10} + 58x^{11} + 110x^{12} + 187x^{13} + \ldots$$

As already mentioned, each column of Table I. is calculated by means of a generating function given as a product of two factors, each of which is obtained from the columns which precede the column in question; and Table II., the centre part of it, is calculated by means of the same generating functions slightly modified: these generating functions serving for the calculation of the two Tables are given in the table entitled "Subsidiary Table for the calculation of the GF's of Tables I. and II.," which immediately follows these two Tables, and will be further explained.

TABLE I.—General Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	1	1												
	Total	1	1												
2	1		1	1											
	Total		1	1											
3	1			1	1										
	2		1		1										
	Total		1	1	2										
4	1			1	1	2									
	2			1		1									
	3		1			1									
	Total		1	2	1	4									
5	1			1	2	1	4								
	2			2	1		3								
	3			1			1								
	4		1				1								
	Total		1	4	3	1	9								
6	1			1	4	3	1	9							
	2			2	3	1		6							
	3			2	1			3							
	4			1				1							
	5		1					1							
	Total		1	6	8	4	1	20							
7	1			1	6	8	4	1	20						
	2			3	8	4	1		16						
	3			3	3	1			7						
	4			2	1				3						
	5			1					1						
	6		1						1						
	Total		1	10	18	13	5	1	48						
8	1			1	10	18	13	5	1	48					
	2			3	15	13	5	1		37					
	3			4	9	4	1			18					
	4			3	3	1				7					
	5			2	1					3					
	6			1						1					
	7		1							1					
	Total		1	14	38	36	19	6	1	115					
9	1			1	14	38	36	19	6	1	115				
	2			4	30	36	19	6	1		96				
	3			5	19	14	5	1			44				
	4			5	9	4	1				19				
	5			3	3	1					7				
	6			2	1						3				
	7			1							1				
	8		1								1				
	Total		1	21	76	93	61	26	7	1	286				

TABLE I. (*continued*).

Index z, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	13
10	1			1	21	76	93	61	26	7	1	286			
	2			4	51	89	61	26	7	1		239			
	3			7	42	41	20	6	1			117			
	4			6	20	14	5	1				46			
	5			5	9	4	1					19			
	6			3	3	1						7			
	7			2	1							3			
	8			1								1			
	9		1									1			
	Total		1	29	147	225	180	94	34	8	1	719			
11	1			1	29	147	225	180	94	34	8	1	719		
	2			5	90	210	180	94	34	8	1		622		
	3			8	79	110	67	27	7	1			299		
	4			9	46	42	20	6	1				124		
	5			7	20	14	5	1					47		
	6			5	9	4	1						19		
	7			3	3	1							7		
	8			2	1								3		
	9			1									1		
	10		1										1		
	Total		1	41	277	528	498	308	136	43	9	1	1842		
12	1			1	41	277	528	498	308	136	43	9	1	1842	
	2			5	145	467	493	308	136	43	9	1		1607	
	3			10	152	278	208	101	35	8	1			793	
	4			11	91	115	68	27	7	1				320	
	5			10	47	42	20	6	1					126	
	6			7	20	14	5	1						47	
	7			5	9	4	1							19	
	8			3	3	1								7	
	9			2	1									3	
	10			1										1	
	11		1											1	
	Total		1	55	509	1198	1323	941	487	188	53	10	1	4766	
13	1			1	55	509	1198	1323	941	487	188	53	10	1	4766
	2			6	238	1012	1524	941	487	188	53	10	1		4460
	3			12	272	669	376	344	144	44	9	1			1871
	4			15	184	299	213	102	35	8	1				857
	5			13	95	116	68	27	7	1					327
	6			11	47	42	20	6	1						127
	7			7	20	14	5	1							47
	8			5	9	4	1								19
	9			3	3	1									7
	10			2	1										3
	11			1											1
	12		1												1
	Total		1	76	924	2666	3405	2744	1615	728	251	64	11	1	12486

TABLE II.—General Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1	1							1	1	0						
	Total	1	1															
2										0	1	1	1					
3	2		1	1														
	Total		1	1						1	1	0						
4	2																	
	3		1	1														
	Total		1	1						1	2	1		1				
5	2			1	1													
	3																	
	4		1		1													
	Total		1	1	2					2	3	1		1				
6	2			1	1													
	3			1	1													
	4																	
	5		1		1													
	Total		1	2	3					3	6	3		2	1			
7	2			2	1	3												
	3			2		2												
	4			1		1												
	5																	
	6		1			1												
	Total		1	5	1	7				7	11	4		2	2			
8	2			2	2	4												
	3			3	1	4												
	4			2		2												
	5			1		1												
	6																	
	7		1			1												
	Total		1	8	3	12				12	23	11		3	7	1		
9	2			3	7	1	11											
	3			4	3		7											
	4			4	1		5											
	5			2			2											
	6			1			1											
	7																	
	8		1				1											
	Total		1	14	11	1	27			27	47	20		3	14	3		

TABLE II. (*continued*).

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
10	2			3	14	3	20											
	3			6	11	1	18											
	4			5	3		8											
	5			4	1		5											
	6			2			2											
	7			1			1											
	8																	
	9		1				1											
	Total		1	21	29	4	55			55	106	51		4	32	14	1	
11	2			4	32	14	1	51										
	3			7	26	4		37										
	4			8	12	1		21										
	5			6	3			9										
	6			4	1			5										
	7			2				2										
	8			1				1										
	9																	
	10		1					1										
	Total		1	32	74	19	1	127		127	235	108		4	58	42	4	
12	2			4	58	42	4	108										
	3			9	63	19	1	92										
	4			10	30	4		44										
	5			9	12	1		22										
	6			6	3			9										
	7			4	1			5										
	8			2				2										
	9			1				1										
	10																	
	11		1					1										
	Total		1	45	167	66	5	284		284	551	267		5	110	128	23	1
13	2			5	110	128	23	1	267									
	3			11	132	66	5		214									
	4			14	78	20	1		113									
	5			12	31	4			47									
	6			10	12	1			23									
	7			6	3				9									
	8			4	1				5									
	9			2					2									
	10			1					1									
	11																	
	12		1						1									
	Total		1	65	367	219	29	1	682	682	1301	619		5	187	334	88	5

Subsidiary Table for *GF*'s of Tables I. and II.

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													*GF*, column 0.
*	.	−1													*GF*, column 1.
0	(1)														First factor.
1		1													
2			1												
3				1											
4					1										
5						1									
6							1								
7								1							
8									1						
9										1					
10											1				
11												1			
12													1		
0		1													Second factor.
*			−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1		*GF*, column 2.
0	(1)														First factor.
1			1	1	1	1	1	1	1	1	1	1	1		
2					1	1	2	2	3	3	4	4	5		
3							1	1	2	3	4	5	7		
4									1	1	2	3	5		
5											1	1	2		
6													1		
0		1													Second factor.
1			1												
2				1											
3					1										
4						1									
5							1								
6								1							
7									1						
8										1					
9											1				
10												1			
11													1		
12														1	
*				−1	−2	−4	−6	−10	−14	−21	−29	−41	−55	−76	*GF*, column 3.
0	(1)														First factor.
1		.	.	1	2	4	6	10	14	21	29	41	55		
2							1	2	7	14	32	58	110		
3										1	2	7	18		
4													1		
0		1													Second factor.
1			1	1	1	1	1	1	1	1	1	1	1	1	
2				1	1	2	2	3	3	4	4	5	5	6	
3					1	1	2	3	4	5	7	8	10	12	
4						1	1	2	3	5	6	9	11	15	
5							1	1	2	3	5	7	10	13	
6								1	1	2	3	5	7	11	
7									1	1	2	3	5	7	
8										1	1	2	3	5	
9											1	1	2	3	
10												1	1	2	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*					−1	−3	−8	−18	−38	−76	−147	−277	−509	−924	*GF*, column 4.
0	(1)														First factor.
1				.	1	3	8	18	38	76	147	277	509		
2									1	3	14	42	128		
3													1		
0		1													Second factor.
1			1	1	2	3	5	7	11	15	22	30	42	56	
2				1	1	3	5	11	18	34	55	95	150	244	
3					1	1	3	6	13	24	49	87	162	284	
4						1	1	3	6	14	26	55	102	199	
5							1	1	3	6	14	27	57	108	
6								1	1	3	6	14	27	58	
7									1	1	3	6	14	27	
8										1	1	3	6	14	
9											1	1	3	6	
10												1	1	3	
11													1	1	
12														1	
*	.				.	−1	−4	−13	−36	−93	−225	−528	−1198	−2666	*GF*, column 5.
0	(1)														First factor.
1						1	4	13	36	93	225	528	1198		
2											1	4	23		
0		1													Second factor.
1			1	1	2	4	8	15	29	53	98	177	319	565	
2				1	1	3	6	15	31	70	144	305	617	1256	
3					1	1	3	7	17	38	90	197	440	953	
4						1	1	3	7	18	40	97	217	498	
5							1	1	3	7	18	41	99	224	
6								1	1	3	7	18	41	100	
7									1	1	3	7	18	41	
8										1	1	3	7	18	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*	.	.	.	.	.	.	−1	−5	−19	−61	−180	−498	−1323	−3405	*GF*, column 6.
0	(1)														First factor.
1							1	5	19	61	180	498	1323		
2													1		
0		1													Second factor.
1			1	1	2	4	9	19	42	89	191	402	847	1763	
2				1	1	3	6	16	36	89	205	485	1110	2780	
3					1	1	3	7	18	43	110	264	648	1329	
4						1	1	3	7	19	45	117	285	711	
5							1	1	3	7	19	46	119	292	
6								1	1	3	7	19	46	120	
7									1	1	3	7	19	46	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								−1	−6	−26	−94	−308	−941	−2744	*GF*, column 7.
0	(1)														First factor.
1								1	6	26	94	308	941		
0		1													Second factor.
1			1	1	2	4	9	20	47	108	252	582	1345	3086	
2				1	1	3	6	16	37	95	231	579	1418	3721	
3					1	1	3	7	18	44	116	291	749	1673	
4						1	1	3	7	19	46	123	312	813	
5							1	1	3	7	19	47	125	319	
6								1	1	3	7	19	47	126	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*									−1	−7	−34	−136	−487	−1615	*GF*, column 8.
0	(1)														First factor.
1									1	7	34	136	487	1615	
0		1													Second factor.
1			1	1	2	4	9	20	48	114	278	676	1653	4027	
2				1	1	3	6	16	37	96	238	613	1554	4208	
3					1	1	3	7	18	44	117	298	784	1817	
4						1	1	3	7	19	46	124	319	848	
5							1	1	3	7	19	47	126	326	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*										−1	−8	−43	−188	−728	*GF*, column 9.
0	(1)														First factor.
1										1	8	43	188	728	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	285	710	1789	4514	
2				1	1	3	6	16	37	96	239	621	1597	4396	
3					1	1	3	7	18	44	117	299	792	1861	
4						1	1	3	7	19	46	124	320	856	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

Subsidiary Table for *GF*'s of Tables I. and II. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*											−1	−9	−53	−251	*GF*, column 10.
0	(1)														First factor.
1											1	9	53	251	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	718	1832	4702	
2				1	1	3	6	16	37	96	239	622	1606	4449	
3					1	1	3	7	18	44	117	299	793	1870	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*												−1	−10	−64	*GF*, column 11.
0	(1)														First factor.
1												1	10	64	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1841	4755	
2				1	1	3	6	16	37	96	239	622	1607	4459	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*													−1	−11	*GF*, column 12.
0	(1)														First factor.
1													1	11	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1842	4765	
2				1	1	3	6	16	37	96	239	622	1607	4460	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	
*														−1	*GF*, column 13.
0	(1)														First factor.
1														1	
0		1													Second factor.
1			1	1	2	4	9	20	48	115	286	719	1842	4766	
2				1	1	3	6	16	37	96	239	622	1607	4460	
3					1	1	3	7	18	44	117	299	793	1871	
4						1	1	3	7	19	46	124	320	857	
5							1	1	3	7	19	47	126	327	
6								1	1	3	7	19	47	127	
7									1	1	3	7	19	47	
8										1	1	3	7	19	
9											1	1	3	7	
10												1	1	3	
11													1	1	
12														1	

I proceed to explain the Subsidiary Table, first in its application to Table I.

The Subsidiary Table is divided into sections, giving the GF's of the successive columns of Table I., each section being given by means of the preceding columns of Table I.; for instance, that for column 3 by means of columns 0, 1, 2 of Table I.

As regards column 0, the Table shows that the GF is $=x$.

As regards column 1, it shows that the GF has a first factor,

$$(1-tx)^{-1}, =(1)+tx+t^2x^2+t^3x^3+\ldots,$$

which is operated on by the symbol $[t^{1\ldots\infty}]$, viz. the constant term (1) is to be rejected; and that it has a second factor, $=x$: the product of these, viz. $(tx+t^2x^2+t^3x^3+\ldots)\times x$, is the required GF, the coefficients of which are accordingly given in column 1 of Table I.

As regards column 2, it shows that the GF has a first factor,

$$(1-tx^2)^{-1}(1-tx^3)^{-1}(1-tx^4)^{-1}\ldots,$$

where the indices $-1, -1, -1, \ldots$ are the sums of the numbers in column 1, Table I., (with their signs changed): which first factor is

$$1+tx^2+tx^3+\begin{pmatrix} t \\ +t^2 \end{pmatrix}x^4+\ldots,$$

and it is as before to be operated on with $[t^{1\ldots\infty}]$, viz. the constant term is to be rejected; and further, that there is a second factor $=x+tx^2+t^2x^3+\ldots$, the coefficients of which are obtained by summation of the numbers in the several lines of columns 0, 1 of Table I. We have thence column 2 of Table I.

As regards column 3, it shows that the GF has a first factor,

$$(1-tx^3)^{-1}(1-tx^4)^{-2}(1-tx^5)^{-4}\ldots,$$

where the indices $-1, -2, -4, \ldots$ are the sums of the numbers in column 2 of Table I., (with their signs changed): which first factor is

$$=1+tx^3+2tx^4+4tx^5+\begin{pmatrix} 6t \\ +t^2 \end{pmatrix}x^6+\ldots,$$

and it is as before to be operated on with $[t^{1\ldots\infty}]$, viz. the constant term is to be rejected; and that there is a second factor

$$=x+tx^2+\begin{pmatrix} t \\ +t^2 \end{pmatrix}x^3+\begin{pmatrix} t \\ +t^2 \\ +t^3 \end{pmatrix}x^4+\ldots,$$

the coefficients of which are obtained by summation of the numbers in the several lines of columns 0, 1, 2 of Table I.: we have thence column 3 of Table I.

And similarly, by means of columns 0, 1, 2, 3 of Table I., we form the GF of column 4; that is, we obtain column 4 of Table I., and so on indefinitely.

To apply the Subsidiary Table to the calculation of the GF's of Table II., the only difference is that the first factors are to be taken without the terms in t^1: thus for Table II. column 3, the first factor of the GF

$$= t^2x^6 + 2t^2x^7 + 7t^2x^8 + \begin{pmatrix} 14t^2 \\ + t^3 \end{pmatrix} x^9 + \&c.,$$

the second factor being as for Table I.

$$= x + tx^2 + \begin{pmatrix} t \\ + t^2 \end{pmatrix} x^3 + \&c.$$

The remaining Tables are Tables III. and IV., oxygen root-trees and centre- and bicentre-trees, followed by a Subsidiary Table for the calculation of the GF's: Tables V. and VI., boron root-trees and centre- and bicentre-trees, followed by a Subsidiary Table; and Tables VII. and VIII., carbon root-trees and centre- and bicentre-trees, followed by a Subsidiary Table. The explanations given as to Tables I., II. and the Subsidiary Table apply *mutatis mutandis* to these; and but little further explanation is required: that given in regard to the Subsidiary Table of Tables III. and IV. shows how this limiting case comes under the general method. As to the Subsidiary of Tables V. and VI., it is to be observed that each * line of the Table is calculated from a column of Table V., rejecting the numbers which belong to t^3; thus Table V., column 4, the numbers are

t^1	1	3	5	7	8	9 ...
t^2		1	4	10	21	36 ...
t^3			1	4	11	26 ...;

and taking the sums for the first and second lines only, these are

$$1,\ 4,\ 9,\ 17,\ 29,\ 45, \ldots,$$

which, taken with a negative sign, are the numbers of the line $*GF$, column 5.

And so as to the Subsidiary of Tables VII. and VIII., each * line of the Table is calculated from a column of Table VII., rejecting the numbers which belong to t^4; thus Table VII., column 4, the numbers are

t^1	1	3	8	15	27	43 ...
t^2		1	4	13	33	74 ...
t^3			1	4	14	38 ...
t^4				1	4	14 ...;

and taking the sums for the first, second, and third lines only, these are

$$1,\ 4,\ 13,\ 32,\ 74,\ 155, \ldots,$$

which, taken with a negative sign, are the numbers of the line $*GF$, column 5.

Referring to the foregoing "Edification Diagram," the effect is that we thus introduce the conditions that in a boron-tree the number of component trees $a, b, \ldots$ is at most $(3 - 1 =) 2$ and that in a carbon-tree the number of component trees $a, b, \ldots$ is at most $(4 - 1 =) 3$.

TABLE III.—Oxygen Root-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.												
		0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1												
2	1		1											
3	1 2		1	1										
4	1 2			1	1									
5	1 2			1	1	1								
6	1 2				1	1	1							
7	1 2				1	1	1	1						
8	1 2					1	1	1	1					
9	1 2					1	1	1	1	1				
10	1 2						1	1	1	1	1			
11	1 2						1	1	1	1	1	1		
12	1 2							1	1	1	1	1	1	
13	1 2							1	1	1	1	1	1	1

TABLE IV.—Oxygen Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1								1	1	0						
2										0	1	1	1					
3	2		1							1	1	0						
4										0	1	1		1				
5	2			1						1	1	0						
6										0	1	1			1			
7	2				1					1	1	0						
8										0	1	1				1		
9	2					1				1	1	0						
10										0	1	1					1	
11	2						1			1	1	0						
12										0	1	1						1
13	2							1		1	1	0						

Subsidiary Table for *GF*'s of Tables III. and IV.

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0															*GF*, column 0.
*		−1													*GF*, column 1.
0	1														First factor.
1		1													
2			1												
0		1													Second factor.
*			−1												*GF*, column 2.
0	1														First factor.
1			1												
2					1										
0		1													Second factor.
1			1												
*				−1											*GF*, column 3.
0	1														First factor.
1				1	.	.	.								
2							1								
0		1													Second factor.
1			1	1											
*					−1										*GF*, column 4.
0	1														First factor.
1					1										
2						.	.	.	1						
0		1													Second factor.
1			1	1	1										
*						−1									*GF*, column 5.
0	1														First factor.
1						1									
2											1				
0		1													Second factor.
1			1	1	1	1									

and so on indefinitely; viz. observing that the first factors, as shown by the Table, are $(1-tx)^{-1}[t^{1\cdot 2}]$, $(1-tx^2)^{-1}[t^{1\cdot 2}]$, &c., the Table in fact shows that as regards Table III. the GF's are for

$$\begin{array}{lll}
\text{column} & 0: & x, \\
\text{,,} & 1: & tx + t^2x^2 \,.\, x, \\
\text{,,} & 2: & tx^2 + t^2x^4 \,.\, x + tx^2, \\
\text{,,} & 3: & tx^3 + t^2x^5 \,.\, x + t\,(x^2 + x^3), \\
\text{,,} & 4: & tx^4 + t^2x^6 \,.\, x + t\,(x^2 + x^3 + x^4), \\
\text{,,} & 5: & tx^5 + t^2x^{10} \,.\, x + t\,(x^2 + x^3 + x^4 + x^5);
\end{array}$$

viz. developing as far as t^2, that the successive GF's are

$$\begin{array}{lll}
\text{column} & 0: & x, \\
\text{,,} & 1: & tx^2 + t^2x^3, \\
\text{,,} & 2: & tx^3 + t^2\,(x^4 + x^5), \\
\text{,,} & 3: & tx^4 + t^2\,(x^5 + x^6 + x^7), \\
\text{,,} & 4: & tx^5 + t^2\,(x^6 + x^7 + x^8 + x^9), \\
\text{,,} & 5: & tx^6 + t^2\,(x^7 + x^8 + x^9 + x^{10} + x^{11});
\end{array}$$

&c., agreeing with Table III.

And so also it shows that, as regards Table IV. (centre part), the GF's of the successive columns are for

$$\begin{array}{lll}
\text{column} & 0: & x, \\
\text{,,} & 1: & t^2x^2 \,.\, x, \\
\text{,,} & 2: & t^2x^4 \,.\, x, \\
\text{,,} & 3: & t^2x^6 \,.\, x, \\
\text{,,} & 4: & t^2x^8 \,.\, x, \\
\text{,,} & 5: & t^2x^{10} \,.\, x; \\
 & \vdots &
\end{array}$$

viz. that the successive GF's are x, t^2x^3, t^2x^5, t^2x^7, t^2x^9, $t^2x^{11}, \ldots$, agreeing in fact with Table IV.

TABLE V.—Boron Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	
1	0	1	1												
	Total	1	1												
2	1		1	1											
	Total		1	1											
3	1			1	1										
	2		1		1										
	Total		1	1	2										
4	1			1	1	2									
	2			1		1									
	3		1			1									
	Total		1	2	1	4									
5	1				2	1	3								
	2			2	1		3								
	3			1			1								
	Total			3	3	1	7								
6	1				2	3	1	6							
	2			1	3	1		5							
	3			2	1			3							
	Total			3	6	4	1	14							
7	1				1	5	4	1	11						
	2			1	6	4	1		12						
	3			2	3	1			6						
	Total			3	10	10	5	1	29						
8	1				1	7	9	5	1	23					
	2				7	10	5	1		23					
	3			2	7	4	1			14					
	Total			2	15	21	15	6	1	60					
9	1					8	17	14	6	1	46				
	2				9	21	15	6	1		52				
	3			1	11	11	5	1			29				
	Total			1	20	40	37	21	7	1	127				
10	1					9	29	32	20	7	1	98			
	2				7	36	37	21	7	1		109			
	3			1	18	26	16	6	1			68			
	Total			1	25	71	82	59	28	8	1	275			
11	1					7	45	66	53	27	8	1	207		
	2				7	59	82	59	28	8	1		244		
	3				21	53	43	22	7	1			147		
	Total				28	119	170	147	88	36	9	1	598		
12	1					7	66	127	125	81	35	9	1	451	
	2				4	82	165	147	88	36	9	1		532	
	3				26	102	105	66	29	8	1			337	
	Total				30	191	336	340	242	125	45	10	1	1320	
13	1					4	89	231	274	213	117	44	10	1	983
	2				3	114	316	340	242	125	45	10	1		1196
	3				26	175	236	177	96	37	9	1			757
	Total				29	293	641	748	612	375	171	55	11	1	2936

TABLE VI.—Boron Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1	1															
	Total	1	1							1	1	0						
2																		
										0	1	1	1					
3	2		1	1														
	Total		1	1						1	1	0						
4	2		1	1														
	Total		1	1						1	2	1		1				
5	2			1	1													
	Total			1	1					1	2	1		1				
6	2			1	1													
	3			1	1													
	Total			2	2					2	4	2		1	1			
7	2			1	1	2												
	3			2		2												
	Total			3	1	4				4	6	2			2			
8	2				2	2												
	3			2	1	3												
	Total			2	3	5				5	11	6			5	1		
9	2				5	1	6											
	3			1	3		4											
	Total			1	8	1	10			10	18	8			5	3		
10	2				5	3	8											
	3			1	9	1	11											
	Total			1	14	4	19			19	37	18			6	11	1	
11	2				6	11	1	18										
	3				14	4		18										
	Total				20	15	1	36		36	66	30			4	22	4	
12	2				4	22	4	30										
	3				21	16	1	38										
	Total				25	38	5	68		68	135	67			3	44	19	1
13	2				3	44	19	1	67									
	3				24	42	5		71									
	Total				27	86	24	1	138	138	265	127			1	68	53	5

Subsidiary Table for GF's of Tables V. and VI.

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													GF, column 0.
*		−1													GF, column 1.
0	1														First factor.
1		1													
2			1												
3				1											
0		1													Second factor.
*			−1	−1											GF, column 2.
0	1														First factor.
1			1	1											
2					1	1	1								
3							1	1	1	1					
0		1													Second factor.
1			1												
2				1											
*				−1	−2	−2	−1	−1							GF, column 3.
0	1														First factor.
1				1	2	2	1	1							
2							1	2	5	5	6	4	3		
3										1	2	5	9		
0		1													Second factor.
1			1	1	1										
2				1	1	2	1	1							
*					−1	−3	−5	−7	−8	−9	−7	−7	−4	−3	GF, column 4.
0	1														First factor.
1					1	3	5	7	8	9	7	7	4		
2									1	3	11	22	44		
3													1		
0		1													Second factor.
1			1	1	2	2	2	1	1						
2				1	1	3	4	7	7	7	7	7	4	3	
*						−1	−4	−9	−17	−29	−45	−66	−89	−118	GF, column 5.
0	1														First factor.
1						1	4	9	17	29	45	66	89		
2											1	4	19		
0		1													Second factor.
1			1	1	2	3	5	6	8	8	9	7	7	4	
2				1	1	3	5	11	17	30	43	66	86	117	
*							−1	−5	−14	−32	−66	−127	−231	−405	GF, column 6.
0	1														First factor.
1							1	5	14	32	66	127	231		
2													1		
0		1													Second factor.
1			1	1	2	3	6	10	17	25	38	52	73	93	
2				1	1	3	5	12	22	45	80	148	251	433	

Subsidiary Table for *GF*'s of Tables V. and VI. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								−1	−6	−20	−53	−125	−274	−571	*GF*, column 7.
0 1	1							1	6	20	53	125	274		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	22 23	39 51	70 101	118 207	200 398	324 773	Second factor.
*									−1	−7	−27	−81	−213	−516	*GF*, column 8.
0 1	1								1	7	27	81	213		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	45 52	90 108	171 235	325 486	598 1015	Second factor.
*										−1	−8	−35	−117	−338	*GF*, column 9.
0 1	1									1	8	35	117		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	46 52	97 109	198 243	406 522	811 1140	Second factor.
*											−1	−9	−44	−162	*GF*, column 10.
0 1	1										1	9	44		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	46 52	98 109	206 244	441 531	928 1185	Second factor.
*												−1	−10	−54	*GF*, column 11.
0 1	1											1	10		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	46 52	98 109	207 244	450 532	972 1195	Second factor.
*													−1	−11	*GF*, column 12.
0 1	1												1		First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	46 52	98 109	207 244	451 532	982 1196	Second factor.
*														−12	*GF*, column 13.
0	1														First factor.
0 1 2		1	1	1 1	2 1	3 3	6 5	11 12	23 23	46 52	98 109	207 244	451 532	983 1196	Second factor.

TABLE VII.—Carbon Root-trees.

Index x, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	
1	0	1	1												
	Total	1	1												
2	1		1	1											
	Total		1	1											
3	1			1	1										
	2		1		1										
	Total		1	1	2										
4	1			1	1	2									
	2			1		1									
	3		1			1									
	Total		1	2	1	4									
5	1			1	2	1	4								
	2			2	1		3								
	3			1			1								
	4		1				1								
	Total		1	4	3	1	9								
6	1				4	3	1	8							
	2			2	3	1		6							
	3			2	1			3							
	4			1				1							
	Total			5	8	4	1	18							
7	1				4	8	4	1	17						
	2			2	8	4	1		15						
	3			3	3	1			7						
	4			2	1				3						
	Total			7	16	13	5	1	42						

TABLE VII. (*continued*).

Index z, or number of knots.	Index t, or number of main branches.	Altitude or number of column.													
		0	1	2	3	4	5	6	7	8	9	10	11	12	
8	1				5	15	13	5	1	39					
	2			1	13	13	5	1		33					
	3			3	9	4	1			17					
	4			3	3	1				7					
	Total			7	30	33	19	6	1	96					
9	1				4	27	32	19	6	1	89				
	2			1	22	33	19	6	1		82				
	3			3	17	14	5	1			40				
	4			4	9	4	1				18				
	Total			8	52	78	57	26	7	1	229				
10	1				4	43	74	56	26	7	1	211			
	2				29	74	57	26	7	1		194			
	3			3	34	38	20	6	1			102			
	4			4	18	14	5	1				42			
	Total			7	85	169	156	89	34	8	1	549			
11	1				3	67	155	151	88	34	8	1	507		
	2				40	154	156	89	34	8	1		482		
	3			2	54	95	63	27	7	1			249		
	4			5	38	39	20	5	1				108		
	Total			7	135	355	394	272	130	43	9	1	1346		
12	1				2	97	316	374	267	129	43	9	1	1238	
	2				46	297	389	273	130	43	9	1		1188	
	3			1	88	218	184	96	35	8	1			631	
	4			4	66	100	64	27	7	1				269	
	Total			5	202	712	953	770	439	181	53	10	1	3326	
13	1				1	136	612	889	743	432	180	52	10	1	3056
	2				55	550	929	770	439	181	53	10	1		2988
	3			1	127	474	491	309	138	44	9	1			1594
	4			4	117	239	190	97	35	8	1				691
	Total			5	300	1399	2222	2065	1355	665	243	63	11	1	8329

TABLE VIII.—Carbon Centre- and Bicentre-Trees.

Index x, or number of knots.	Index t, or number of main branches.	Centre-Trees. Altitude or number of column.								Centre.	Grand Total.	Bicentre.	Bicentre-Trees. Altitude.					
		0	1	2	3	4	5	6					0	1	2	3	4	5
1	0	1	1															
	Total	1	1							1	1	0						
2																		
										0	1	1	1					
3	2		1	1														
	Total		1	1						1	1	0						
4	2		1	1														
	Total		1	1						1	2	1		1				
5	2		1	1	2													
	Total		1	1	2					2	3	1		1				
6	2			1	1													
	3			1	1													
	Total			2	2					2	5	3		2	1			
7	2			2	1	3												
	3			2		2												
	4			1		1												
	Total			5	1	6				6	9	3		1	2			
8	2			1	2	3												
	3			3	1	4												
	4			2		2												
	Total			6	3	9				9	18	9		1	7	1		
9	2			1	7	1	9											
	3			3	3		6											
	4			4	1		5											
	Total			8	11	1	20			20	35	15			12	3		
10	2				12	3	15											
	3			3	11	1	15											
	4			4	3		7											
	Total			7	26	4	37			37	75	38			23	14	1	
11	2				23	14	1	38										
	3			2	24	4		30										
	4			5	12	1		18										
	Total			7	59	19	1	86		86	159	73			30	39	4	
12	2				30	39	4	73										
	3			1	54	19	1	75										
	4			4	27	4		35										
	Total			5	111	62	5	183		183	357	174			42	108	23	1
13	2				42	108	23	1	174									
	3			1	88	63	5		157									
	4			4	63	20	1		88									
	Total			5	193	191	29	1	419	419	799	380			47	244	84	5

Subsidiary Table for *GF*'s of Tables VII. and VIII.

Index t.	Index of x. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	
0		1													*GF*, column 0.
*		−1													*GF*, column 1.
0	1														First factor.
1		1													
2			1												
3				1											
4					1										
0		1													Second factor.
*			−1	−1	−1										*GF*, column 2.
0	(1)														First factor.
1			1	1	1										
2					1	1	2	1	1						
3							1	1	2	2	2	1	1		
4									1	1	2	2	3		
0		1													Second factor.
1			1												
2				1											
3					1										
*				−1	−2	−4	−4	−5	−4	−4	−3	−2	−1	−1	*GF*, column 3.
0	(1)														First factor.
1				1	2	4	4	5	4	4	3	2	1		
2							1	2	4	12	23	30	42		
3										1	2	7	16		
4													1		
0		1													Second factor.
1			1	1	1	1									
2				1	1	2	2	2	1	1					
3					1	1	2	3	3	3	3	2	1	1	
*					−1	−3	−8	−15	−27	−43	−67	−97	−136	−183	*GF*, column 4.
0	(1)														First factor.
1					1	3	8	15	27	43	67	97	136		
2									1	3	14	39	108		
3													1		
0		1													Second factor.
1			1	1	2	3	4	4	5	4	4	3	2	1	
2				1	1	3	5	10	14	23	29	40	46	55	
3					1	1	3	6	12	20	37	56	89	128	
*						−1	−4	−13	−32	−74	−155	−316	−612	−1160	*GF*, column 5.
0	(1)														First factor.
1						1	4	13	32	74	155	316	612		
2											1	4	23		
0		1													Second factor.
1			1	1	2	4	7	12	20	31	47	70	99	137	
2				1	1	3	6	14	27	56	103	194	343	605	
3					1	1	3	7	16	34	75	151	307	602	
*							−1	−5	−19	−56	−151	−374	−889	−2032	*GF*, column 6.
0	1														First factor.
1							1	5	19	56	151	374	889		
2													1		
0		1													Second factor.
1			1	1	2	4	8	16	33	63	121	225	415	749	
2				1	1	3	6	15	32	75	160	350	732	1534	
3					1	1	3	7	17	39	95	214	491	1093	

Subsidiary Table for *GF*'s of Tables VII. and VIII. (*continued*).

Index t.	Index of x.														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
*								−1	−6	−26	−88	−267	−743	−1968	*GF*, column 7.
0	(1)														First factor.
1								1	6	26	88	267	743		
0		1													Second factor.
1			1	1	2	4	8	17	38	82	177	376	789	1638	
2				1	1	3	6	15	33	81	186	439	1005	2304	
3					1	1	3	7	17	40	101	241	587	1402	
*									−1	−7	−34	−129	−432	−1320	*GF*, column 8.
0	(1)														First factor.
1									1	7	34	129	432		
0		1													Second factor.
1			1	1	2	4	8	17	39	88	203	464	1056	2381	
2				1	1	3	6	15	33	82	193	473	1135	2743	
3					1	1	3	7	17	40	102	248	622	1540	
*										−1	−8	−43	−180	−657	*GF*, column 9.
0	(1)														First factor.
1										1	8	43	180		
0		1													Second factor.
1			1	1	2	4	8	17	39	89	210	498	1185	2813	
2				1	1	3	6	15	33	82	194	481	1178	2924	
3					1	1	3	7	17	40	102	249	630	1584	
*											−1	−9	−53	−242	*GF*, column 10.
0	(1)														First factor.
1											1	9	53		
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	506	1228	2993	
2				1	1	3	6	15	33	82	194	482	1187	2977	
3					1	1	3	7	17	40	102	249	631	1593	
*												−1	−10	−63	*GF*, column 11.
0	(1)														First factor.
1												1	10	63	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1237	3048	
2				1	1	3	6	15	33	82	194	482	1188	2987	
3					1	1	3	7	17	40	102	249	631	1594	
*													−1	−11	*GF*, column 12.
0	(1)														First factor.
1													1	11	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1238	3055	
2				1	1	3	6	15	33	82	194	482	1188	2988	
3					1	1	3	7	17	40	102	249	631	1594	
*														−1	*GF*, column 13.
0	(1)														First factor.
1														1	
0		1													Second factor.
1			1	1	2	4	8	17	39	89	211	507	1238	3056	
2				1	1	3	6	15	33	82	194	482	1188	2988	
3					1	1	3	7	17	40	102	249	634	1594	

I annex the following two Tables of (centre- and bicentre-) trees as far as I have completed them.

TABLE A.

Knots.	Valency not greater than									Gen.
	0	1	2 Oxygen.	3 Boron.	4 Carbon.	5	6	7	8	
1	1	1	1	1	1	1	1	1	1	1
2		1	1	1	1	1	1	1	1	1
3			1	1	1	1	1	1	1	1
4			1	2	2	2	2	2	2	2
5			1	2	3	3	3	3	3	3
6			1	4	5	6	6	6	6	6
7			1	6	9	10	11	11	11	11
8			1	11	18	21	22	23	23	23
9			1	18	35	42	45	46	47	47
10			1	37	75					106
11			1	66	159					235
12			1	135	357					551
13			1	265	799					1301

TABLE B.

Knots.	Actual Valency.								
	0	1	2	3	4	5	6	7	8
1	1								
2		1							
3			1						
4			1	1					
5			1	1	1				
6			1	3	1	1			
7			1	5	3	1	1		
8			1	10	7	3	1	1	
9			1	17	17	7	3	1	1
10			1	36	38				
11			1	65	93				
12			1	134	222				
13			1	264	534				

In A, the columns 2, 3, 4, and the last column are the totals given by the Tables IV., VI., VIII., and II., and the remaining numbers of columns 5, 6, 7, 8 have been found by trial; and, in B, the several columns are the differences of the

columns of A. The signification is obvious; for instance, if the number of knots is $=9$, then Table A, if the valency, or the maximum number of branches from a knot,

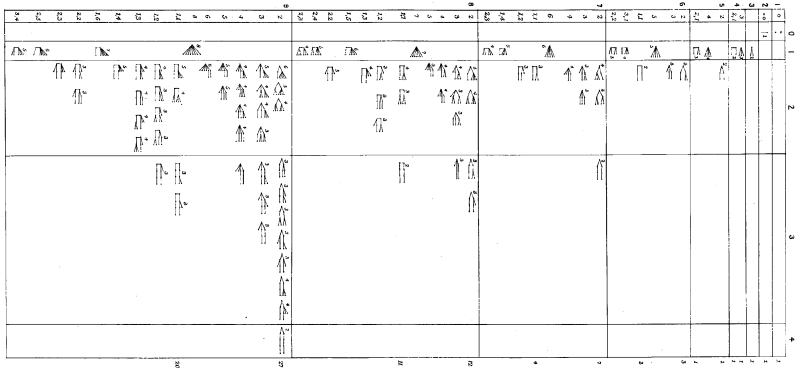
is =	2,	3,	4,	5,	6,	7,	8 or any greater number,
No. of trees =	1,	18,	35,	42,	45,	46,	47:

viz. with 9 knots the tree can have at most 8 branches from a knot, so that the number of trees having at most 8 branches from a knot is $=47$, the whole number of trees with 9 knots; and so the number of knots being as before $=9$, Table B shows that the number of 47 is made up of the numbers

1, 17, 17, 7, 3, 1, 1;

viz. 1	is the	No. of trees,	at most	2	branches	from a knot,		
17	"	"	3	"	"	at least	one 3-branch	knot.
17	"	"	4	"	"	"	4	"
7	"	"	5	"	"	"	5	"
3	"	"	6	"	"	"	6	"
1	"	"	7	"	"	"	7	"
1	"	"	8	"	"	"	8	"

I annex also a plate showing the figures of the $1+1+2+3+6+11+23+47$ trees of 1, 2, 3, . . , 9 knots, classified according to their altitudes and number of main branches; and as to the bicentre-trees, according to the number of main branches from *each* point of the bicentre. The affixed numbers show in each case the greatest number of branches from a knot; so that when this is (2), the knots may be oxygen-, boron-, carbon-, &c., atoms; when (3), boron-, carbon-, &c., atoms; when (4), carbon-, &c., atoms; and so on.


Altitude
2
3
4

611.

REPORT OF THE COMMITTEE ON MATHEMATICAL TABLES:

CONSISTING OF PROFESSOR CAYLEY, F.R.S., PROFESSOR STOKES, F.R.S., PROFESSOR SIR W. THOMSON, F.R.S., PROFESSOR H. J. S. SMITH, F.R.S., AND J. W. L. GLAISHER, F.R.S.

[From the *Report of the British Association for the Advancement of Science* (1875), pp. 305—336.]

THE present Report (say Report 1875) is in continuation of that by Mr Glaisher, published in the volume for 1873, and here cited as Report 1873.

Report 1873 extends to all those tables which are at p. 3 (*l.c.*) included under the headings:—

A, auxiliary for non-logarithmic calculation, 1, 2, 3;

B, logarithmic and circular, 4, 5, 6;

C, exponential, 7, 8 (but only partially to C. 8), other than those tables of C referred to as "$\mathrm{h}\,.\,\mathrm{l}\tan(45^\circ + \frac{1}{2}\phi)$"; and also partially (see Art. 24, pp. 81—83) to the tables included under the heading "E. 11, transcendental constants ϵ, π, γ, &c., and their powers and functions."

A future Report will comprise the tables, or further tables, included under the headings:—

C. 8. Hyperbolic antilogarithms (e^x) and $\mathrm{h}\,.\,\mathrm{l}\tan(45^\circ + \frac{1}{2}\phi)$, and hyperbolic sines, cosines, &c.

D. Algebraic constants.

9. Accurate integer or fractional values. Bernoulli's Numbers, $\Delta^n 0^m$, &c. Binomial coefficients.

10. Decimal values auxiliary to the calculation of series.

E. 11. Transcendental constants ϵ, π, γ, &c., and their powers and functions.

The present Report (1875) comprises the tables included under the headings:—

F. Arithmological.

12. Divisors and prime numbers. Prime roots. The Canon arithmeticus, &c.

13. The Pellian equation.

14. Partitions.

15. Quadratic forms $a^2 + b^2$, &c., and partitions of numbers into squares, cubes, and biquadrates.

16. Binary, ternary, &c., quadratic and higher forms.

17. Complex theories:

which divisions are herein referred to, for instance, as [F. 12. Divisors, &c.].

Report 1873 consists of six sections (§) divided into articles, which are separately numbered (see contents, p. 174); the present Report 1875 forms a single section (§ 7), divided in like manner into articles, which are separately numbered; but besides this the paragraphs are numbered, and that continuously, through the present Report 1875, so that any paragraph may be cited as Report 1875, No. —, as the case may be.

[F. 12. *Divisors, &c.*] *Divisors and Prime Numbers.* Art. I.

1. As to divisors and prime numbers see Report 1873, Art. 8 (Tables of Divisors—factor tables—and Tables of Primes), pp. 34—40. The tables there referred to, such as Chernac, Burckhardt, Dase, Dase and Rosenberg, are chiefly tables running up to very high numbers (the last of them the ninth million): wherein, to save space, multiples of 2, 3, 5 are frequently omitted, and in some of them only the least divisor is given. It would be for many purposes convenient to have a small table, going up say to 10,000, showing in every case all the prime factors of the number. Such a table might be arranged, 500 numbers in a page, in some such form as the following:—

Factor Table | 1 to 500 |

	0	1	2	3	4	5	6	7	8	9
39	2.3.5.13	17.23	$2^3.7^2$	3.131	2.197	5.79	$2^2.3^2.11$	397*	2.199	3.7.19

where the top line shows the units, and the left-hand column the remaining figures, viz. the specimen exhibits the composition of the several numbers from 390 to 399: a prime number, e.g. 397, would be sufficiently indicated by the absence of any decomposition, or it may be further indicated by an asterisk.

It may be noticed that, in the theory of numbers, the decomposition is specially required when the next following number is a prime, viz. that of $p-1$, p being a

prime: also, that this is given incidentally, for prime numbers p up to 1000, in Jacobi's *Canon Arithmeticus*, *post*, No. 20, and up to 15,000 in Reuschle's Tables, V. (a, b, c) *post*, No. 22.

2. It may be proper to remark here that any table of a binary form is really a factor-table in the complex theory connected with such binary form. Thus in a table of the form a^2+b^2, a number of this form has a factor $a+bi$ ($i=\sqrt{-1}$ as usual); and the table, in fact, shows the complex factor $a+bi$ of the number in question: a well arranged table would give all the prime complex factors $a+bi$ of the number. But as to this more hereafter; at present, we are concerned with the real theory only, not with any complex theory.

3. Connected with a factor-table, we have (i) a Table of the number of less relative primes; viz. such a table would show, for every number, the number of inferior integers having no common factor with the number itself. The formula is a well-known one: for a number $N=a^\alpha b^\beta c^\gamma \ldots$, ($a$, b, ... the distinct prime factors of N), the number of less relative primes is

$$\varpi(N),\ =a^{\alpha-1}b^{\beta-1}\ldots(a-1)(b-1)\ldots,$$

or, what is the same thing, $=N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\ldots$ A small table ($N=1$ to 100), occupying half a page, is given by

Euler, *Op. Arith. Coll.* t. II. p. 128; viz. this is $\pi 1=0$, $\pi 2=1, \ldots$, $\pi 100=40$.

4. But it would be interesting to have such a table of the same extent with the proposed factor-table. The table might be of like form; for instance,

Number of less relative Primes Table | 1 to 500 |

	0	1	2	3	4	5	6	7	8	9
29	112	192	144	292	84	232	144	198	148	264

It would be of still greater interest to have an inverse table showing the values of N which belong to a given value of $\varpi(N)$; for instance,

$\varpi=$	$N=$
40	41, 55, 75, 82, 88, 100, 110,
42	43, 49, 86, 98,
44	69, 92,
46	47, 94,
48	65, 104, 105, 112,
⋮	

where, observe, that ϖ is of necessity even.

5. Again, connected with a factor-table, we have (ii) a Table of the Sum of the divisors of a Number. The formula is also a well-known one; for a number $N = a^\alpha b^\beta \ldots$, ($a, b, \ldots$ the distinct prime factors of N), the required sum

$$\int N \text{ is } = (1 + a + \ldots + a^\alpha)(1 + b + \ldots + b^\beta) \ldots,$$

or, what is the same thing,

$$= \frac{a^{\alpha+1} - 1}{a - 1} \cdot \frac{b^{\beta+1} - 1}{b - 1} \ldots,$$

where, observe, that the number itself is reckoned as a divisor.

6. Such a table was required by Euler in his researches on Amicable Numbers (see *post*, No. 10), and he accordingly gives one of a considerable extent, viz.

Euler, *Op. Arith. Coll.* t. I. pp. 104—109.

It is to be remarked that, inasmuch as $\int N$ is obviously $= \int a^\alpha \int b^\beta \ldots$, the function need only be tabulated for the different integer powers a^α of each prime number a. The range of Euler's table is as follows:—

$a =$	$\alpha =$
2	1 to 36,
3	1 „ 15,
5	1 „ 9,
7	1 „ 10,
11	1 „ 9,
13	1 „ 7,
17	1 „ 5,
19	1 „ 5,
23	1 „ 4,
29 to 997	1 „ 3,

viz. for the several prime numbers from 29 to 997 the table gives $\int a$, $\int a^2$, and $\int a^3$. And it is to be noticed that the values of the sum are exhibited, decomposed into their prime factors: thus a specimen of the table is

Num.	Summa Divisorum.
139	$2^2 . 5 . 7$
139^2	$3 . 13 . 499$
139^3	$2^3 . 5 . 7 . 9661$

7. The form of the above table is adapted to its particular purpose (the theory of amicable numbers); but Euler gives also,

Euler, *Op. Arith. Coll.* t. I. p. 147—in the paper "Observatio de Summis Divisorum," (1752), pp. 146—154,—a short table of about half a page, $N=1$ to 100, of the form $\int 1=1$, $\int 2=3, \ldots, \int 100=217$. The paper contains interesting analytical researches on the subject of $\int N$ which connect themselves with the theory of the Partition of Numbers.

8. It would be interesting to carry the last-mentioned table to the same extent as the proposed factor-table; and to add to it an inverse table, as suggested in regard to the number of less relative primes table.

9. *Perfect Numbers.*—A perfect number is a number which is equal to the sum of its divisors, the number itself not being reckoned as a divisor; e.g.

$$6=1+2+3, \text{ and } 28=1+2+4+7+14.$$

Such numbers are indicated by a table of the sums of divisors $\int 6=12$, $\int 28=56$, these two being, as appears by the table, Art. 7, the only perfect numbers less than 100.

10. *Amicable Numbers.*—These are pairs of numbers such that each is equal to the sum of the divisors of the other, not reckoning the other number as a divisor; that is, each has the same sum of divisors, the number being here reckoned as a divisor; say $\int' A=B$, $\int' B=A$; or, what is the same thing, $\int A=\int B\,(=A+B)$. Thus for the numbers 220, 284,

$$\int' 220=(1+2+4)(1+5)(1+11)-220, \quad =284,$$

$$\int' 284=(1+2+4)(1+71)-284, \quad =220;$$

or, what is the same thing,

$$\int 220=(1+2+4)(1+5)(1+11)=504=(1+2+4)(1+71)=\int 284.$$

11. A catalogue of 61 pairs of numbers is given by

Euler, *Op. Arith. Coll.* t. I. pp. 144—145; it occupies about one page. The paper, "De Numeris Amicabilibus," pp. 102—145, contains an elaborate investigation of the theory, by means whereof all but two of the pairs of numbers are obtained. The first pair is the above-mentioned one, $2^2.5.11$ and $2^2.71$ (=220 and 284); and the fifty-ninth pair is the high numbers

$$3^5.7^2.13.19.53.6959 \text{ and } 3^5.7^2.13.19.179.2087.$$

The last two pairs are referred to as "formæ diversæ a precedentibus;" viz. these are

$$\begin{cases} 2^3 . 19 . 41 \\ 2^5 . 199 \end{cases} \text{and} \begin{cases} 2^3 . 41 . 467 \\ 2^5 . 19 . 233. \end{cases}$$

12. A Table of the Frequency of Primes is given by

Gauss, Tafel der Frequenz der Primzahlen, *Werke*, t. II. pp. 436—443; viz. this extends to 3,000,000.

The first part, extending to 1,000,000, = 1000 thousand, shows how many primes there are in each thousand: a specimen is

1, 168:
2, 135:
3, 127:
4, 120:
5, 119:
&c.;

viz. in the first thousand there are 168 primes, in the second thousand 135 primes, and so on.

For the second and third millions the frequency is given for each ten thousand: a specimen is

1,000,000 to 1,100,000.

	0	1	2	3	4	5	6	7	8	9	
1	1										1
2		1				1		1	1		4
3		4	2	2	3	1	2	3	3	1	21
4	2	8	5	4	3	6	9	4	5	8	54
5	11	10	8	18	12	10	10	12	15	8	114
6	14	14	18	21	16	22	19	15	17	15	171
7	26	17	23	23	24	24	17	22	20	21	217
8	19	19	21	7	14	15	20	17	15	17	164
9	11	13	9	13	14	14	12	13	11	16	126
10	8	6	8	5	9	5	5	9	7	9	71
11	6	6	4	6	3	1	3	1	4	5	39
12	1	1	2	1	1	1	2	2	1		12
13	1	1			1		1	1	1		6
14											
15											
16											
	752	719	732	700	731	698	713	722	706	737	7210,

$$\int \frac{dx}{\log x} = 7212{\cdot}99;$$

viz. in the interval 1,000,000 to 1,010,000, 100 hundreds, there is 1 hundred containing 1 prime, there are 2 hundreds each containing 4 primes, 11 hundreds each containing 5 primes, .., 1 hundred containing 13 primes, so that, as

$$\begin{array}{rcl} 1 \times 1 & = & 1, \\ 4 \times 2 & = & 8, \\ 5 \times 11 & = & 55, \\ \vdots & & \\ 13 \times 1 & = & 13, \\ \hline 100 & & 752, \end{array}$$

the whole 10,000 contains 752 primes; the next 10,000 contains 719 primes, and so on; the whole 100,000 thus containing $752+719+\&c.\ldots=7210$ primes, which number is at the foot compared with the theoretic approximate value

$$\int \frac{dx}{\log x} \text{ (limits 1,000,000 to 1,010,000)} = 7212{\cdot}99.$$

The integral in question is represented by the notation Li. x or li. x.

p. 443. We have the like tables 1,000,000 to 2,000,000 and 2,000,000 to 3,000,000, showing for each 100,000 how many hundreds there are containing 0 prime, 1 prime, 2 primes, up to (the largest number) 17 primes.

13. It is noticed that

the 26,379th hundred contains no prime,

the 27,050th hundred contains 17 primes.

It may be observed that, if $N=2.3.5\ldots p$, the product of all the primes up to p, then each of the numbers $N+1$ and $N+q$ (if q be the prime next succeeding p) is or is not a prime; but the intermediate numbers $N+2$, $N+3, \ldots$, $N+q-1$ are certainly composite; viz. we thus have at least $q-2$ consecutive composites. To obtain in this manner 99 consecutive composites, the value of N would be $=2.3.5\ldots97$, viz. this is a number far exceeding 2,637,900; but, in fact, the hundred numbers 2,637,901 to 2,638,000 are all composite.

Legendre, in his *Essai sur la Théorie des Nombres* (1st edit., 1798; 2nd edit., 1808, supplement, 1816: references to this edition), gives for the number of primes inferior to a given limit x the approximate formula

$$\frac{x}{\log x - 1{\cdot}08366};$$

and p. 394, and supplement, p. 62, he compares for each 10,000 up to 100,000, and for each 100,000 up to 1,000,000, the values as computed by this formula with the actual numbers of primes exhibited by the tables of Wega and Chernac. Thus for $x=1{,}000{,}000$, the computed value is 78,543, the actual value 78,493.

He shows, p. 414, that the number of integers, which are less than n and are not divisible by any of the numbers θ, λ, $\mu, \ldots$, is approximately

$$=n\left(1-\frac{1}{\theta}\right)\left(1-\frac{1}{\lambda}\right)\left(1-\frac{1}{\mu}\right)\ldots;$$

and taking θ, λ, $\mu, \ldots$ the successive primes 3, 5, 7, ... he gives the values of the function in question, or, say, the function

$$\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdots\frac{\omega-1}{\omega},$$

ω a prime, for the several prime values $\omega=3$ to 1229 in the Table IX. (one page) at the end of the work.

14. A table of frequency is given by

Glaisher, J. W. L., *British Association Report* for 1872, p. 20. This gives for the second and the ninth millions, respectively divided into intervals of 50,000, the actual number of primes in each interval, as compared with the theoretic value $\mathrm{li}\,x'-\mathrm{li}\,x$; and also deduced therefrom, by the formula $\log\frac{1}{2}(x'+x)$, a table of the average interval between two consecutive primes; this average interval increases very slowly: at the beginning and the end of the second million the values are 13·76 and 14·58 (theoretic values 13·84 and 14·50); at the beginning and the end of the ninth million 16·02 and 15·95 (theoretic values 15·90 and 16·01).

15. Coming under the head of Divisor Tables, some tables by Reuschle and Gauss may be here referred to. These are:—

Reuschle, *Mathematische Abhandlung, zahlentheoretische Tabellen sammt einer dieselben treffenden Correspondenz mit der verewigten C. G. J. Jacobi*, 4°, pp. 1—61* (1856). The tables belonging to the present subject are

A. Tafeln zur Zerlegung von a^n-1 (pp. 18—22).

I. Table of the prime factors of 10^n-1, viz.

(a. pp. 18—19.) Complete decomposition of 10^n-1, $n=1$ to 42: and 10^n+1, $n=1$ to 21. Some values of n are omitted.

A specimen is

$$10^{13}-1=3^2\,.\,53\,.\,79\,.\,265371653,$$
$$10^{13}+1=11\,.\,189\,.\,1058313049.$$

(b. p. 19.) List of the specific prime factors f of 10^n-1, or the prime factors of the residue after separation of the analytical factors, for those values of n for which the complete decomposition is unknown, and omitting those values for which no factor is known, $n=25$ to 243.

* Titlepage missing in my copy; but I find from Prof. Kummer's notice of the work, *Crelle*, t. LIII. (1857), p. 379, that it appeared as a Programm of the Stuttgart Gymnasium, Michaelmas, 1856, and was separately printed by Liesching and Co., Stuttgart.

A specimen is

n	f
25	21401.

The meaning seems to be, residue of $10^{25}-1$ is $1+10^5+10^{10}+10^{15}+10^{20}$, and this contains the prime factor 21401; but it is not clear why this is the "specific prime factor."

II. Prime factors of a^n-1 for different values of a and n.

(a. p. 20) gives for 41 values of a (2, 3, &c. at intervals to 100) and for the following values of n the decompositions of the residues or specific factors of a^n-1; viz. these are

$n=$ 1, $a-1$:
„ 2, $a+1$:
„ 3, a^2+a+1:
„ 6, a^2-a+1:
„ 4, a^2+1:
„ 5, $a^4+a^3+a^2+a+1$:
„ 10, $a^4-a^3+a^2-a+1$:
„ 8, a^4+1:
„ 12, a^4-a^2+1.

A specimen is

a	$a-1$	a^2-1	a^3-1	a^6-1	a^4-1	a^5-1	$a^{10}-1$	a^8-1	$a^{12}-1$
10	3^2	11	$3^3.37$	7.13	101	41.271	9091	73.137	9901

(b. p. 21.) Specific prime factors for the numbers 2, 3, 5, 6, 7, 10, (the powers 4, 8, 9 being omitted as coming under 2 and 3), for the exponents 1 to 42.

A specimen is

n	2^n-1	3^n-1	5^n-1	6^n-1	7^n-1	10^n-1
19	524287	1597.363889	$191.x$	$191.x$	$419.x$	

where the x denotes that the other factor is not known to be prime. And so, where no number is given, as in $10^{19}-1$, it is not known whether the number $(=1+10^1+10^2+\ldots+10^{18})$ is or is not prime.

Addition, p. 22. For $a=2$, the complete decomposition of the prime factor of 2^n-1 is given for values of n, $=44, 45, \ldots$ at intervals to 156.

A specimen is

n	f
44	397.2113,

viz.

$$2^{20}-2^{18}+2^{16}-\ldots-2^2+1, \; =838861=397.2113.$$

$n=31$, Fermat's prime. $n=37$, the first case for which the decomposition is not given completely. $n=41$, the first case for which no factor is known.

16. **Gauss,** Tafel zur Cyclotechnie, *Werke*, t. II. pp. 478—495, shows, for 2452 numbers of the several forms a^2+1, a^2+4, $a^2+9, \ldots$, a^2+81, the values of a such that the number in question is a product of prime factors no one of which exceeds 200, and exhibits all the odd prime factors of each such number. The table is in nine parts, zerlegbare a^2+1, zerlegbare a^2+4, &c., with to each part a subsidiary table, as presently mentioned. Thus a specimen is

zerlegbare a^2+9.

1	5
2	13
4	5 . 5
5	17
7	29
8	73
⋮	⋮
1411168679	5 . 5 . 13 . 17 . 17 . 89 . 113 . 157 . 173 . 197 . 197 :

viz.

1^2+9, odd prime factor is 5,

2^2+9, „ „ 13,

4^2+9, „ factors are 5, 5,

and so on.

And the subsidiary table is

5	1, 4, 79
13	2, 11, 41
17	5, 29, 46, 379, 1042
⋮	⋮

showing that the numbers a for which the largest factor is 5 are 1, 4, 79; those for which it is 13 are 2, 11, 41; and so on.

The object of the table is explained in the *Bemerkungen*, (*l. c.*, p. 523), by Schering, the editor of the volume, viz. it is to facilitate the calculation of the circular arcs the cotangents of which are rational numbers. To take a simple example, it appears to be by means of it that Gauss obtained, among other formulæ, the following:

$$\frac{\pi}{4} = 12 \text{ arctan } \tfrac{1}{18} + 8 \text{ arctan } \tfrac{1}{57} - 5 \text{ arctan } \tfrac{1}{239},$$

and

$$= 12 \text{ arctan } \tfrac{1}{38} + 20 \text{ arctan } \tfrac{1}{57} + 7 \text{ arctan } \tfrac{1}{239} + 24 \text{ arctan } \tfrac{1}{268}.$$

[F. 12. *Divisors, &c.*] *continued. Prime Roots. The Canon Arithmeticus, Quadratic residues.* Art. II.

17. *Prime Roots.*—Let p be a prime number; then there exist $\varpi(p-1)$ inferior integers g, such that all the numbers $1, 2, \ldots, p-1$ are, to the modulus p,

$$\equiv 1,\ g,\ g^2, \ldots,\ g^{p-2}\ (g^{p-1} \text{ is of course } \equiv 1).$$

This being so, g is said to be a prime root of p; and moreover the several numbers g^α, where α is any number whatever less than and prime to $p-1$, constitute the series of the $\varpi(p-1)$ prime roots of p. It may be added that, if β be an integer number less than $p-1$, and having with it a greatest common measure $=k$, so that

$$(g^\beta)^{\frac{p-1}{k}} \equiv g^{\frac{\beta}{k}(p-1)}, \equiv 1, \left(\text{since } \frac{\beta}{k} \text{ is an integer, and } g^{p-1} \equiv 1\right),$$

then g^β has the indicatrix $\frac{p-1}{k}$: the prime roots are those numbers which have the indicatrix $p-1$.

The like theory exists as to any number N of the form p^m or $2p^m$. There are here $\varpi(N), = N\left(1-\frac{1}{p}\right)$ or $\frac{1}{2}N\left(1-\frac{1}{p}\right)$, in the two cases respectively, numbers less than N and prime to it; and we have then $\varpi(\varpi(N))$ numbers g such that, to the modulus N, all these numbers are $\equiv 1,\ g,\ g^2 \ldots g^{\varpi(N)-1}$ ($g^{\varpi(N)}$ is of course $\equiv 1$). This being so, g may be regarded as a prime root of N ($=p^m$ or $2p^m$, as the case may be); and moreover the several numbers g^α, where α is any number whatever less than and prime to $\varpi(N)$, constitute the series of the $\varpi(\varpi(N))$ prime roots of N. Thus $N=3^2=9$, $\varpi(N)=6$; we have

$$\begin{array}{l} 1,\quad 2^1,\quad 2^2,\quad 2^3,\quad 2^4,\quad 2^5, \\ \equiv 1,\quad 2,\quad 4,\quad 8,\quad 7,\quad 5,\ \text{mod. } 9; \end{array}$$

or the prime roots of 9 are 2^1 and 2^5, $=2$ and 5.

So also $N=2\,.\,3^2=18$, $\varpi(N)=6$; we have

$$\begin{array}{l} 1,\quad 5^1,\quad 5^2,\quad 5^3,\quad 5^4,\quad 5^5, \\ \equiv 1,\quad 5,\quad 7,\quad 17,\quad 13,\quad 11,\ \text{mod. } 18; \end{array}$$

and 5^1 and 5^5, $=5$ and 11 are the prime roots of 18.

18. A small table of prime roots, $p=3$ to 37, is given by

Euler, *Op. Arith. Coll.* t. I. pp. 525—526. The Memoir is entitled "Demonstrationes circa residua e divisione potestatum per numeros primos resultantia," pp. 516—537 (1772).

19. A table, p and p^m, 3 to 97, is given by

Gauss, "Disquisitiones Arithmeticæ," 1801, (*Werke*, t. I. p. 468). This gives in each case a prime root, and it shows the exponents in regard thereto of the several prime numbers less than p or p^m. Thus a specimen is

		2	3	5	7	11	13	17	19	23	29 &c.
27	2	1	*	5	16	13	8	15	12	11	
29	10	11	27	18	20	23	2	7	15	24	

viz. for 27 we have 2 a prime root, and $2 \equiv 2^1$, $5 \equiv 2^5$, $7 \equiv 2^{16}$, $11 \equiv 2^{13}$, &c.; and so also for 29 we have 10 a prime root, and $2 \equiv 10^{11}$, $3 \equiv 10^{27}$, $5 \equiv 10^{18}$, &c.

20. Small tables are probably to be found in many other places; but the most extensive and convenient table is **Jacobi's** *Canon Arithmeticus*, the complete title of which is

Canon Arithmeticus sive tabula quibus exhibentur pro singulis numeris primis vel primorum potestatibus infra 1000 *numeri ad datos indices et indices ad datos numeros pertinentes.* Edidit C. G. J. Jacobi. Berolini, 1839. 4°.

The contents are as follows:—

	Pages
Introductio	i to xl
Tabulæ numerorum ad indices datos pertinentium et indicum numero dato correspondentium pro modulis primis minoribus quam 1000	1—221
Tabulæ residuorum et indicum sibi mutuo respondentium pro modulis minoribus quam 1000 qui sunt numerorum primorum potestates	222—238
Hujus tabula ea pars quæ pertinet ad modulos formæ 2^n, invenitur	239—240.

The following is a specimen of the principal tables:—

$$p = 19,\ p - 1 = 2 \,.\, 3^2.$$

Numeri.

I	0	1	2	3	4	5	6	7	8	9
		10	5	12	6	3	11	15	17	18
	9	14	7	13	16	8	4	2	1	

Indices.

N	0	1	2	3	4	5	6	7	8	9
		18	17	5	16	2	4	12	15	10
1	1	6	3	13	11	7	14	8	9	

where the first table gives the values of the powers of the prime root 10 (that 10 is the root appears by its index being given as $=1$) to the modulus 19, viz. $10^1 \equiv 10$, $10^2 \equiv 5$, $10^3 \equiv 12$, &c.; and the second table gives the index of the power to which the same prime root must be raised in order that it may be, to the modulus 19, congruent with a given number: thus $10^{18} \equiv 1$, $10^{17} \equiv 2$, $10^5 \equiv 3$, &c. The units of the index or number, as the case may be, are contained in the top line of the table, and the tens or hundreds and tens in the left-hand column.

21. There is given by

Jacobi, *Crelle,* t. XXX. (1846), pp. 181, 182, a table of m' for the argument m, such that

$$1 + g^m \equiv g^{m'} \text{ (mod. } p), \quad p = 7 \text{ to } 103, \text{ and } m = 0 \text{ to } 102.$$

A specimen is

p	7	11	13	17	19	23	29	31	37 ... to 103
g	3	2	6	10	10	10	10	17	5
m									
11	.	.	6	4	7	*	27	21	34

for instance, $p = 19$, $1 + 10^{11} \equiv 10^7$ (mod. 19).

Jacobi remarks that this table was calculated for him by his class during the winter course of 1836—37; and that, by means of the *Canon Arithmeticus* since published (in 1839), the same might easily be extended to all primes under 1000. In fact, for any such number p, putting any number of the table "Indices" $= m$, the next following number of the table gives the value of m'.

22. We have next, in **Reuschle's** Memoir (*ante,* No. 15), the following relating to prime roots:—

C. Tafeln für primitive Wurzeln und Hauptexponenten, oder V. erweiterte und bereicherte Burkhardtsche Tafel, pp. 41—61, being divided into three parts; viz. these are

a. Table of the Hauptexponenten of the six roots 10, 5, 2, 6, 3, 7 for all prime numbers of the first 1000, together with the least primitive root of each of these numbers (pp. 42—46).

A specimen is as follows:—

		10		5		2		6		3		7		w
p	$p-1$	e	n	e	n	e	n	e	n	e	n	e	n	
53	$2^2.13$	13	4	52	1	52	1	26	2	52	1	26	2	2

,

where e is the Hauptexponent or indicatrix of the root (10, 5, 2, 6, 3, 7, as the case may be), $n = \frac{p-1}{e}$, w the least primitive root; thus

$$p = 53, \quad 10^{13} \equiv 1, \quad 5^{52} \equiv 1, \quad 2^{52} \equiv 1,$$

(2 being accordingly the least prime root),

$$6^{36} \equiv 1, \quad 3^{32} \equiv 1, \quad 7^{36} \equiv 1.$$

The number w of the last column is the least primitive root. It is, of course, not always (as in the present case) one of the numbers 10, 5, 2, 6, 3, 7 to which the table relates: the first exception is $p = 191$, $w = 19$: the highest value of w is $w = 21$, corresponding to $p = 409$.

b. The like table for the roots 10 and 2 for all prime numbers from 1000 to 5000, together with as convenient as possible a prime root (and in some cases two prime roots) for each such number (pp. 47—53).

A specimen is:—

		10		2		
p	$p-1$	e	n	e	n	w
1289	$2^3 . 7 . 23$	92	14	161	8	6, 11

viz. here, mod. 1289, $10^{92} \equiv 1$, $2^{161} \equiv 1$; and two prime roots are 6, 11. We have thus by the present tables a prime root for every prime number not exceeding 5000.

c. The like table for the root 10 for all prime numbers between 5000 and 15000, (no column for w, nor any prime root given), pp. 53—61.

A specimen is

p	$p-1$	e	n
9859	$2 . 3 . 31 . 53$	3286	3:

viz., mod. 9859, we have $10^{3286} \equiv 1$. But in a large number of cases we have $n = 1$, and therefore 10 a prime root. For example,

9887	$2 . 4983$	9886	1.

23. For a composite number n, if $N = \varpi(n)$ be the number of integers less than n and prime to it, then if x be any number less than n and prime to it, we have $x^N \equiv 1$ (mod. n). But we have in this case no analogue of a prime root—there is no number x, such that its several powers $x^1, x^2, \ldots, x^{N-1}$ (mod. n) are all different from unity; or, what is the same thing, there is for each value of x some submultiple of N, say N', such that $x^{N'} \equiv 1$ (mod. n). And these several numbers N' have a least common multiple I, which is not $= N$, but is a submultiple of N; and this being so, then for all the several values of x, I is said to be the maximum indicator. For instance, $n = 12$, $N = \varpi(n)$; the numbers less than 12 and prime to it are 1, 5, 7, 11. We have, (mod. 12), $1^1 \equiv 1$, $5^2 \equiv 1$, $7^2 \equiv 1$, $11^2 = 1$, or the values of N' are 1, 2, 2, 2; their least common multiple is 2, and we have accordingly $I = 2$: viz. $x^2 \equiv 1$ (mod. 12) has the $\varpi(12)$ roots 1, 5, 7, 11. So $n = 24$, $\varpi(n) = 8$; the maximum indicator I is in this case also $= 2$.

A table of the maximum indicator $n=1$ to 1000 is given by

Cauchy, *Exer. d'Analyse &c.*, t. II. (1841), pp. 36—40, contained in the "Mémoire sur la résolution des équations indéterminées du premier degré en nombres entiers," pp. 1—40.

24. It thus appears that for a composite number n, the $\varpi(n)$ numbers less than n and prime to it cannot be expressed as $\equiv$ (mod. n) to the power of a single root; but for the expression of them it is necessary to employ two or more roots. A small table, $n=1$ to 50, is given by

Cayley, Specimen Table $M \equiv a^\alpha b^\beta$ (mod. N) for any prime or composite modulus; *Quart. Math. Journ.* vol. IX. (1868), pp. 95, 96, and folding sheet, [397].

A specimen is

Nos.	12
roots	5, 7
Ind.	2, 2
M.I.	2
ϕ	4
1	0, 0
2	
3	
4	
5	1, 0
6	
7	0, 1
8	
9	
10	
11	1, 1

viz. for the modulus 12 the roots are 5, 7, having respectively the indicators 2, 2, viz. $5^2 \equiv 1$ (mod. 12), $7^2 \equiv 1$ (mod. 12). Hence also the maximum indicator is $=2$. $\phi\,(=\varpi(n))=4$ is the number of integers less than 12 and prime to it, viz. these are 1, 5, 7, 11, which in terms of the roots 5, 7 and to mod. 12 are respectively $\equiv 5^0 . 7^0$, $5^1 . 7^0$, $5^0 . 7^1$, and $5^1 . 7^1$.

25. *Quadratic Residues.*—In regard to a given prime number p, a number N is or is not a quadratic residue according as the index of N is even or odd, viz. g being a prime root and $N \equiv g^\alpha$, then N is or is not a quadratic residue according as α is even or odd. But the quadratic residues can, of course, be obtained directly without the consideration of prime roots.

A small table, $p=3$ to 97 and $N=-1$ and (prime values) 3 to 97, is given by

Gauss, "Disquisitiones Arithmeticæ," 1801; Table II. (*Werke*, t. I. p. 469): I notice here a misprint in the top line of the original; it should be $-1, +2, +3$, &c., instead of

1, + 2, + 3, &c.; the − 1 is printed correctly on p. 499 of the French translation *Recherches Arithmétiques,* Paris, 1807 and on p. 469 of vol. I. of *Werke,* (Göttingen, 1870).

A specimen is

	− 1	+ 2	+ 3	+ 5	+ 7	+ 11	+ 13	+ 17	+ 19	+ 23	&c.
19				−	−	−		−	−	−	

viz. − 1, 2, 3, 13 are not, 5, 7, 11, 17 &c. are, quadratic residues of 19. The residues taken positively and less than 19 are, in fact, 1, 4, 5, 6, 7, 11, 16, 17.

The same table carried from $p = 3$ to 503, and prime values $N = 3$ to 997, is given by **Gauss,** *Werke,* t. II. pp. 400—409. A specimen is

	2	3	5	7	11	13	17	19	23	&c.;
19			−	−	−		−	−	−	

viz. the arrangement is the same, except only that the − 1 column is omitted.

26. We have also by **Gauss**

"Disquisitiones Arithmeticæ" Table III. (*Werke,* t. I. p. 470), for the conversion into decimals of a vulgar fraction, denominator p or p^μ, not exceeding 100. The explanation is given in Art. 314 *et seq.* of the same work.

But this table, carried to a greater extent, is given by **Gauss,** *Werke,* t. II. pp. 412—434, "Tafel zur Verwandlung gemeiner Brüche mit Nennern aus dem ersten Tausend in Decimalbrüche;" viz. the denominators are here primes or powers of primes, p^μ up to 997.

To explain the table, consider a modulus p^μ (where μ may be $=1$); if 10 is not a prime root of p^μ, consider a prime root r, which is such that $r^e \equiv 10$ (mod. p^μ), e being a submultiple of $p^{\mu-1}(p-1)$; say we have $ef = p^{\mu-1}(p-1)$: then $10^f \equiv 1$ (mod. p^μ). Consider any fraction $\frac{N}{p^\mu}$; then we may write $N \equiv r^{kl+l}$ (mod. p^μ), k from 0 to $f-1$ and l from 0 to $e-1$, $\equiv 10^k r^l$, and consequently $\frac{N}{p^\mu}$ and $\frac{10^k r^l}{p^\mu}$ have the same mantissa (decimal part regarded as an integer); hence, in order to know the mantissa of every fraction whatever of $\frac{N}{p^\mu}$, it is sufficient to know the mantissa of $\frac{rl}{p^\mu}$, that is, the mantissæ of $\frac{1}{p^\mu}, \frac{r}{p^\mu}, \frac{r^2}{p^\mu}, \ldots, \frac{r^{e-1}}{p^\mu}$, or, what is the same thing, the mantissæ of $\frac{10}{p^\mu}, \frac{10r}{p^\mu}, \ldots, \frac{10r^{e-1}}{p^\mu}$.

For instance, $p^\mu = 11$, $10^2 \equiv 1$ (mod. 11), whence $f = 2$, $e = 5$; and taking $r = 2$, we have $10 \equiv r^3$ (mod. 11).

The required mantissæ, denoted in the table by

$$(0),\quad (1),\quad (2),\quad (3),\quad (4),$$

are those of

$$\frac{10}{11},\ \frac{10.2}{11},\ \frac{10.2^2}{11},\ \frac{10.2^3}{11},\ \frac{10.2^4}{11},$$

viz. these fractions are respectively =

$$\begin{array}{ccccc} (0), & (1), & (2), & (3), & (4), \\ \cdot 9090\ldots, & 1\cdot 8181\ldots, & 3\cdot 6363\ldots, & 7\cdot 2727\ldots, & 14\cdot 5454\ldots; \end{array}$$

or their mantissæ are 90, 81, 63, 27, 54.

And we accordingly have as a specimen

$$11 \,\Big|\, (1)\ldots 81,\quad (2)\ldots 63,\quad (3)\ldots 27,\quad (4)\ldots 54,\quad (0)\ldots 90.$$

Or again, as another specimen, $r=2$:—

$$27 \,\Big|\, (1)\ldots 740,\quad (2)\ldots 481,\quad (3)\ldots 962,\quad (4)\ldots 925,\quad (5)\ldots 851,\quad (0)\ldots 370.$$

The table in this form extends to $p^\mu = 463$; the values of r (not given in the body of the table) are annexed, p. 420.

In the latter part of the table $p^\mu = 467$ to 997, we have only the mantissæ of $\frac{100}{p^\mu}$. A specimen is

$$547 \,\Big|\, \begin{array}{llll} 1828153564 & 8994515539 & 3053016453 & 3820840950 \\ 6398537477 & 1480804387 & 5685557586 & 8372943327 \\ 2394881170 & \mathbf{0182815356}, & & \end{array}$$

viz. the fraction $\frac{100}{547} = \cdot 182815\ldots$ has a period of 91, $= \frac{1}{6}546$, figures.

[F. 13. *The Pellian Equation.*] Art. III.

27. The Pellian equation is $y^2 = ax^2 + 1$, a being a given integer number, which is not a square (or rather, if it be, the only solution is $y=1$, $x=0$), and x, y being numbers to be determined: what is required is the least values of x, y, since these, being known, all other values can be found. A small table $a=2$ to 68 is given by

Euler, *Op. Arith. Coll.* t. I. p. 8. The Memoir is "Solutio problematum Diophanteorum per numeros integros," pp. 4—10, 1732—33. The form of the table is

a	$x\,(=p)$	$y\,(=q)$
2	2	3
3	1	2
5	4	9
⋮	⋮	⋮
68	4	33.

Even here, for some of the values of a, the values of x, y are extremely large; thus $a=61$, $x=226{,}153{,}980$, $y=1{,}766{,}399{,}049$.

And probably tables of a like extent may be found elsewhere; in particular, a table of the solution of $y^2=ax^2\pm 1$ (− when the value of a is such that there is a solution of $y^2=ax^2-1$, and + for other values of a), $a=2$ to 135, is given by **Legendre,** *Théorie des Nombres*, 2nd ed. 1808, in the Table X. (one page) at the end of the work. For the before-mentioned number 61, the equation is $y^2=61\,x^2-1$, and the values are $x=3805$, $y=29718$; much smaller than Euler's values for the equation $y^2=61\,x^2+1$.

28. The most extensive table, however, is given by

Degen, *Canon Pellianus, sive Tabula simplicissimam equationis celebratissimæ: $y^2=ax^2+1$, solutionem, pro singulis numeri dati valoribus ab* 1 *usque ad* 1000 *in numeris rationalibus, iisdemque integris exhibens.* Auctore Carolo Ferdinando Degen. Hafn (Copenhagen) apud Gerhardum Bonnarum, 1817. 8vo. pp. iv to xxiv and 1 to 112.

The first table (pp. 3—106) is entitled as "Tabula I. Solutionem Equationis $y^2-ax^2-1=0$ exhibens." It, in fact, also gives the expression of $\sqrt{a}$ as a continued fraction; thus a specimen is

209	14	2	5	3	(2)
	1	13	5	8	11
	3220				
	46551				

Here the first line gives the continued fraction, viz.

$$\sqrt{209}=14+\frac{1}{2}+\frac{1}{5}+\frac{1}{3}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{2}+\frac{1}{28}+\frac{1}{2}+\&c.,$$

the period being (2, 5, 3, 2, 3, 5, 2) indicated by 2, 5, 3 (2). [The number of terms in the period is here odd, but it may be even; for instance, the period (1, 1, 5, 5, 1, 1) is indicated by 1, 1 (5, 5).]

The second line contains auxiliary numbers presenting themselves in the process; thus, if $R^2=239$, we have $R=14+\frac{1}{\alpha}$,

$$\alpha=\frac{1}{R-14}=\frac{1\,(R+14)}{209-14^2}=\frac{R+14}{13}=2+\frac{1}{\beta},$$

$$\beta=\frac{13}{R-12}=\frac{13\,(R+12)}{209-12^2}=\frac{R+12}{5}=5+\frac{1}{\gamma},$$

$$\gamma=\frac{5}{R-13}=\frac{5\,(R+13)}{209-13^2}=\frac{R+13}{8}=3+\frac{1}{\delta},$$

&c.,

where the second line 1, 13, 5,... shows the numerical factors of the third column. The value of this second line as a result is not very obvious.

The third line gives x, and the fourth line y.

29. The second table, pp. 109—112, is entitled "Tabula II. Solutionem æquationis $y^2 - ax^2 + 1 = 0$, quotiescunque valor ipsius a talem admiserat, exhibens"; viz. it is remarked that this is only possible (but see *infrà*) for those values of a which in Table I. correspond to a period of an even number of terms, as shown by two equal numbers in brackets; thus $a = 13$, the period of $\sqrt{13}$ given in Table I. is (1, 1, 1, 1) as shown by the top line 3, 1 (1, 1), and accordingly 13 is one of the numbers in Table II.; and we have there 13 $\left|\begin{array}{l} 5 \\ 18. \end{array}\right.$

Or take another specimen, 241 $\left|\begin{array}{l} 4574225 \\ 71011068; \end{array}\right.$ viz. the first line gives the value of x, and the second line the value of y (least values), for which $y^2 - ax^2 = -1$.

It is to be noticed that $a = 2$ and $a = 5$, for which we have obviously the solutions ($x = 1$, $y = 1$) and ($x = 1$, $y = 2$) respectively, are exceptional numbers not satisfying the test above referred to; and (apparently for this reason) the values in question, 2 and 5, are omitted from the table.

30. **Cayley,** "Table des plus petites solutions impaires de l'équation $x^2 - Dy^2 = \pm 4$, $D \equiv 5$ (mod. 8)." *Crelle*, t. LIII. (1857), page 371 (one page), [231].

As regards the theory of quadratic forms, it is important to know whether for a given value of $D (\equiv 5$, mod. 8) there does or does not exist a solution, in odd numbers, of the equation $x^2 - Dy^2 = 4$. As remarked in the paper, "Note sur l'équation $x^2 - Dy^2 = \pm 4$, $D \equiv 5$ (mod. 8)," pp. 369—371, [231], this can be determined for values of D of the form in question up to $D = 997$ by means of Degen's Table; and the solutions, when they exist, of the equation $x^2 - Dy^2 = 4$, as also of the equation $x^2 - Dy^2 = -4$, can be obtained up to the same value of D. Observe that when the equation $x^2 - Dy^2 = -4$ is possible, the equation $x^2 - Dy^2 = 4$ is also possible, and that its least solution is obtained very readily from that of the other equation; it is therefore sufficient to tabulate the solution of $x^2 - Dy^2 = \pm 4$, the sign being $-$ when the corresponding equation is possible, and being in other cases $+$. Hence the form of the Table: viz. as a specimen we have

D	$\pm$	x	y
757		imposs.	
765	+	83	3
773	−	139	5
781		imposs.	
⋮			

that is, if $D=757$ or 781, there is no solution of either $x^2-Dy^2=+4$ or $=-4$; if $D=765$, there is a solution $x=83$, $y=3$ of $x^2-Dy^2=+4$, but none of $x^2-Dy^2=-4$; if $D=773$, there is a solution $x=139$, $y=5$ of $x^2-Dy^2=-4$, and therefore also a solution of $x^2-Dy^2=+4$; and so in other cases.

[F. 14. *Partitions.*] Art. IV.

31. The problem of Partitions is closely connected with that of Derivations. Thus if it be asked in how many ways can the number n be expressed as a sum of three parts, the parts being 0, 1, 2, 3, and each part being repeatable an indefinite number of times, it is clear that n is at most $=9$, and that for the values of $m, =0, 1, .., 9$ shown by the top line of the annexed table, the number of partitions has the values shown by the bottom line thereof:—

0	1	2	3	4	5	6	7	8	9
a^3	a^2b	a^2c ab^2	a^3d abc b^3	abd ac^2 b^2c	acd b^2d bc^2	ad^2 bcd c^3	bd^2 c^2d	cd^2	d^3
1	1	2	3	3	3	3	2	1	1

But taking a, b, c, d to stand for 0, 1, 2, 3 respectively, the actual partitions of the required form are exhibited by the literal terms of the table (these being obtained, each column from the preceding one, by the method of derivations, or say by the rule of the last and last but one), and the numbers of the bottom line are simply the number of terms in the several columns respectively.

32. A set of such literal tables, say of tables $\begin{pmatrix} a, b, c, .., k \\ =0, 1, 2, .., m \end{pmatrix}^n$, for different values of n and m (where the number of letters is $=m+1$), would be extremely interesting and valuable. The tables for a given value of m and for different values of n are, it is clear, the proper foundation of the theory of the binary quantic $(a, b, c, .., k \between x, 1)^m$, which corresponds to such value of m. Prof. Cayley regrets that he has not in his covariant tables given in every case the complete series of literal terms; viz. the literal terms which have zero coefficients are, for the most part, though not always, omitted in the expressions of the several covariants.

33. But the question at present is as to the *number* of terms in a column, that is, as to the number of the partitions of a given form: the analytical theory has been investigated by Euler and others. The expression for the number of partitions is usually obtained as equal to the coefficient of x^n in the development, in ascending powers of x, of a given rational function of x: for instance, if there is no limitation as to the number of the parts, but if the parts are 1, 2, 3, m (viz. a part may have any value not exceeding m), each part being repeatable an indefinite number of times, then

Number of partitions of n = coefficient of x^n in $\dfrac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^m)}$,

and we can, by actual development, obtain for any given values of m, n the number of partitions.

These have been tabulated $m=1, 2, \ldots, 20$, and $m=\infty$ (viz. there is in this case no limit as to the largest part), and $n=1$ to 59, by

Euler, *Op. Arith. Coll.* t. I. pp. 97—101, given in the paper "De Partitione Numerorum," pp. 73—101, (1750); the heading is "Tabula indicans quot variis modis numerus n e numeris 1, 2, 3, 4,.., m, per additionem exhibi potest, seu exhibens valores formulæ $n^{(m)}$." The successive lines are, in fact, the coefficients of the several powers $x^0, x^1, \ldots, x^9$ in the expansions of the functions

$$\frac{1}{1-x}, \quad \frac{1}{1-x\,.\,1-x^2}, \ldots, \quad \frac{1}{1-x\,.\,1-x^2\ldots 1-x^{20}}.$$

34. The generating function for any given value of m is, it is clear, $=\dfrac{1}{1-x^m}$ multiplied by that for the next preceding value of m, and it thus appears how each line of the table is calculated from that which precedes it. The auxiliary numbers are *printed*; thus a specimen is

Valores numeri n.

m	0	1	2	3	4	5	6	7	8	9	10	
					1	1	2	3	5	6	9	
4	1	1	2	3	5	6	9	11	15	18	23	
						1	1	2	3	5	7	
5	1	1	2	3	5	7	10	13	18	23	30	

viz. suppose the numbers in the second 4-line known: then simply moving these each five steps onward we have the (auxiliary) numbers of the first 5-line; and thence by a mere addition the required series of numbers shown by the second 5-line. And similarly from this is obtained the second 6-line, and so on.

35. More extensive tables are contained in the memoir by

Marsano, *Sulle leggi delle derivate generali delle funzioni di funzioni et sulla teoria delle forme di partizione dei numeri intieri,* (4°. Genova, 1870), pp. 1—281; and three tables paged separately, described merely as "Tavole dei numeri $C_{q,r}$, $S_{q,e}$, $S'_{q,e}$ citate nel testo colle indicazioni di Tavole I., II., III., ai n^i 77, 79, 81"; viz. the reader is referred to these articles for the explanations of what the tabulated functions are; and there is not even then any explicit statement, but the investigation itself has to be studied to make out what the tables are. It is, in fact, easier to make this out from the tables themselves; the explanation is as follows:—

Table I. (16 pages) is, in fact, Euler's table, showing in how many ways the number n can be made up with the parts 1, 2, 3, .., m; but the extent is greater, viz. n is from 1 to 103, and m from 1 to 102. The auxiliary numbers given in Euler's table are omitted, as also certain numbers which occur in each successive line; thus a specimen is

$n =$	1	2	3	4	5	6	7	8	9	10	&c.
$C_{0,n}$	1	0	0	0	0	0	0	0	0	0	
$C_{1,n}$		1	1	1	1	1	1	1	1	1	
$C_{2,n}$			2	2	3	3	4	4	5	5	
$C_{3,n}$				3	4	5	7	8	10	12	
$C_{4,n}$					5	6	9	11	15	18	
&c.											

where the line $C_{4,n}$ (ways of making up $n-1$ with the parts 1, 2, 3, 4) is 1, 1, 2, 3, 5, 6, 9, 11, 15, 18, &c., viz. we read from the corner diagonally downwards as far as the 5, and then horizontally along the line: this saves a large number of figures. The table is printed in ordinary quarto pages, which are taken to come in in tiers of seven, five, and three pages one under the other, as shown by a prefixed diagram; and the necessity of a large folding plate is thus avoided.

The successive lines give, in fact, the coefficients in the expansions of

$$\frac{1}{1-x},\ \frac{1}{1-x\,.\,1-x^2},\ \frac{1}{1-x\,.\,1-x^2\,.\,1-x^3},\ldots,\ \frac{1}{1-x\,.\,1-x^2\ldots 1-x^{102}},$$

each expanded as far as x^{103}.

Table II. (6 pages). The successive lines give the coefficients in the expansions of

$$S,\ \frac{S}{1-x},\ \frac{S}{1-x\,.\,1-x^2},\ldots,\ \frac{S}{1-x\,.\,1-x^2\ldots 1-x^{35}},$$

where

$$S=\frac{1}{(1-x)(1-x^2)(1-x^3)}\ldots \text{ad inf.},$$

each expanded as far as x^{53}, and further continued as regards the first ten lines, that is, the expansions of

$$S,\ \frac{S}{1-x},\ \frac{S}{1-x\,.\,1-x^2},\ldots,\ \frac{S}{1-x\,.\,1-x^2\ldots 1-x^9},$$

each as far as x^{107}.

Table III. (2 pages). The successive lines give the coefficients in the expansions of

$$S^2,\ \frac{S^2}{1-x},\ \frac{S^2}{1-x\,.\,1-x^2},\ldots,\ \frac{S^2}{1-x\,.\,1-x^2\ldots 1-x^5},$$

each expanded as far as x^{55}.

36. As regards Tables II. and III., the analytical explanations have been given in the first instance; but it is easy to see that the tables give numbers of partitions. Thus, in Table II., the second line gives the coefficients in the development of

$$\frac{1}{(1-x)^2(1-x^2)(1-x^3)\ldots};$$

viz. these are 1, 2, 4, 7 12, 19, 30, ..., being the number of ways in which the numbers 0, 1, 2, 3, 4, &c. respectively can be made up with the parts 1, 1′, 2, 3, 4, &c.; thus

	Partitions.	No. =
1	1	2
	1′	
2	2	4
	1, 1	
	1, 1′	
	1′, 1′	
3	3	7
	2, 1	
	2, 1′	
	1, 1, 1	
	1, 1, 1′	
	1, 1′, 1′	
	1′, 1′, 1′	
&c.		&c.

Similarly, the third line shows the number of ways in which these numbers respectively can be made up with the parts 1, 1′, 2, 2′, 3, 4, 5, &c.; the fourth line with the parts 1, 1′, 2, 2′, 3, 3′, 4, 5, &c.; and so on.

And in like manner in Table III., the first line shows the number of ways when the parts are 1, 1′, 2, 2′, 3, 3′, ...; the second line when they are 1, 1′, 1″, 2, 2′, 3, 3′, &c.; the third when they are 1, 1′, 1″, 2, 2′, 2″, 3, 3′, &c.; and so on.

It is clear that the series of tables might be continued indefinitely, viz. there might be a Table IV. giving the developments of

$$S^3, \frac{S^3}{1-x}, \frac{S^3}{1-x\,.\,1-x^2}; \text{ and so on.}$$

An interesting table would be one composed of the first lines of the above series, viz. a table giving in its successive lines the developments of S, S^2, S^3, S^4, &c.

There are throughout the work a large number of numerical results given in a quasi-tabular form; but the collection of these, with independent explanations of the significations of the tabulated numbers, would be a task of considerable labour.

[F. 15. *Quadratic forms* a^2+b^2, *&c., and Partitions of Numbers into squares, cubes, and biquadrates.*] Art. V.

37. The forms here referred to present themselves in the various complex theories. Thus $N=a^2+b^2$, $=(a+bi)(a-bi)$; this means that, in the theory of the complex numbers $a+bi$ (a and b integers), N is not a prime but a composite number. It is well known that an ordinary prime number $\equiv 3$, mod. 4, is not expressible as a sum a^2+b^2, being, in fact, a prime in the complex theory as well as in the ordinary one: but that an ordinary prime number $\equiv 1$, mod. 4, is (in one way only) $=a^2+b^2$; so that it is in the complex theory a composite number. A number whose prime factors are each of them $\equiv 1$, mod. 4, or which contains, if at all, an even number of times any prime factor $\equiv 3$, mod. 4, can be expressed in a variety of ways in the form a^2+b^2; but these are all easily deducible from the expressions in the form in question of its several factors $\equiv 1$, mod. 4, so that the required table is a table of the form $p=a^2+b^2$, p an ordinary prime number $\equiv 1$, mod. 4: a and b are one of them odd, the other even; and to render the decomposition definite a is taken to be odd.

$p=a^2+b^2$; viz. decomposition of the primes of the form $4n+1$ into the sum of two squares: a table extending from $p=5$ to 11981 (calculated by *Zornow*) is given by

Jacobi, *Crelle*, t. XXX. (1846), pp. 174—176.

This is carried by Reuschle, as presently mentioned, up to $p=24917$. Reuschle notices that $2713=3^2+52^2$ is omitted, also $6997=39^2+74^2$, and that 8609 should be $=47^2+80^2$.

38. Similarly, primes of the form $6n+1$ are expressible in the form $p=a^2+3b^2$. Observe that, ω being an imaginary cube root of unity, this is connected with $p'=(a+b\omega)(a+b\omega^2)$, $=a^2-ab+b^2$, viz. we have $4p'=(2a-b)^2+3b^2$; or the form a^2+3b^2 is connected with the theory of the complex numbers composed of the cube roots of unity.

$p=a^2+3b^2$; viz. decomposition of the primes of the form $6n+1$ into the form a^2+3b^2: a table extending from $p=7$ to 12007 (calculated also by *Zornow*) is given by

Jacobi, *Crelle*, t. XXX. (1846), *ut suprà*, pp. 177—179.

This is carried by Reuschle up to $p=13369$, and for certain higher numbers up to 49999, as presently mentioned. Reuschle observes that $6427=80^2+3.3^2$ is by accident omitted, and that 6481 should be $=41^2+3.40^2$.

39. Again, primes of the form $8n+1$ are expressible in the form $p=a^2+2b^2$ (or say $=c^2+2d^2$), the theory being connected with that of the complex numbers composed with the 8th roots of unity $\left(\text{fourth root of } -1, =\frac{1+i}{\sqrt{2}}\right)$.

$p=c^2+2d^2$; viz. decomposition of primes of the form $8n+1$ into the form c^2+2d^2:

a table extending from $p=16$ to 5943 (extracted from a MS. table calculated by *Struve*) is given by

Jacobi, *Crelle*, t. XXX. (1846), *ut suprà*, p. 180.

This is carried by Reuschle up to $p=12377$, and for certain higher numbers up to 24889, as presently mentioned.

40. Reuschle's tables of the forms in question are contained in the work:—

Reuschle, *Mathematische Abhandlung*, &c. (see *ante* No. 15), under the heading "B. Tafeln zur Zerlegung der Primzahlen in Quadrate" (pp. 22—41). They are as follows:—

Table III. for the primes $6n+1$.

The first part gives $p=A^2+3B^2$ and $4p=L^2+27M^2$, from $p=7$ to 5743. The table gives A, B, L, M; those numbers which have 10 for a cubic residue are distinguished by an asterisk. A specimen is

p	A	B	L	M
37*	5	2	11	1

viz. $37=5^2+3\,.\,2^2$, $148=11^2+27\,.\,1^2$; the asterisk shows that $x^3\equiv+10$ (mod. 37) is possible: in fact $34^3\equiv 10$ (mod. 37).

The second part gives $p=A^2+3B^2$ only, from $p=5749$ to 13669. The table gives A, B; and the asterisk implies the same property as before.

The third part gives $p=A^2+3B^2$, but only for those values of p which have 10 for a cubic residue, viz. for which $x^3\equiv 10$ (mod. p) is possible, from $p=13689$ to 49999. The table gives A, B; the asterisk, as being unnecessary, is not inserted.

Table IV. for the primes $4n+1$ in the form A^2+B^2, and for those which are also $8n+1$ in the form C^2+2D^2.

The first part gives $p=A^2+B^2$, $=C^2+2D^2$, from $p=5$ to 12377. The table gives A, B, C, D; those numbers which have 10 for a biquadratic residue, viz. for which $x^4\equiv 10$ (mod. p) is possible, are distinguished by an asterisk; those which have also 10 for an octic residue, viz. for which $x^8\equiv 10$ (mod. p) is possible, by a double asterisk. A specimen is

p	A	B	C	D
229	15	2	—	—
233	13	8	15	2
241**	15	4	13	6

The second part gives $p=A^2+B^2$, from $p=12401$ to 24917 for all those values of p which have 10 for a biquadratic residue ($x^4\equiv 10$ (mod. p) possible). The table gives A, B; those values of p which have 10 for an octic residue, viz. for which $x^8\equiv 10$ (mod. p) is possible, are distinguished by an asterisk.

The third part gives $p = C^2 + 2D^2$, from $p = 12641$ to 24889 for all those values of p which have 10 for a biquadratic residue. The table gives C, D; those values of p which have 10 as an octic residue are distinguished by an asterisk.

41. A table by **Zornow,** *Crelle,* t. XIV. (1835), pp. 279, 280 (belonging to the Memoir "De Compositione numerorum e Cubis integris positivis," pp. 276—280), shows for the numbers 1 to 3000 the least number of cubes into which each of these numbers can be decomposed. Waring gave, without demonstration, the theorem that every number can be expressed as the sum of at most 9 cubes. The present table seems to show that 23 is the only number for which the number of cubes is $= 9\,(= 2.2^3 + 7.1^3)$; that there are only fourteen numbers for which the number of cubes is $= 8$, the largest of these being 454; and hence that every number greater than 454 can be expressed as a sum of at most 7 cubes; and further, that every number greater than 2183 can be expressed as a sum of at most 6 cubes. A small subsidiary table (p. 276) shows that the number of numbers requiring 6 cubes gradually diminishes—e.g. between 12^3 and 13^3 there are seventy-five such numbers, but between 13^3 and 14^3 only sixty-four such numbers; and the author conjectures "that for numbers beyond a certain limit every number can be expressed as a sum of at most 5 cubes."

42. For the decomposition of a number into biquadrates we have

Bretschneider, "Tafeln für die Zerlegung der Zahlen bis 4100 in Biquadrate," *Crelle,* t. XLVI. (1853), pp. 3—23.

Table I. gives the decompositions, thus:—

N	1^4,	2^4,	3^4,	4^4, 5^4,
696	6	1	2	2
	3	2	5	1
	0	3	8	

viz. $696 = 6 . 1^4 + 1 . 2^4 + 2 . 3^4 + 2 . 4^4$, &c.

And Table II. enumerates the numbers which are sums of at least 2, 3, 4,.., 19 biquadrates. There is at the end a summary showing for the first 4100 numbers how many numbers there are of these several forms respectively: 28 numbers are each of them a sum of 2 biquadrates, 75 a sum of 3, ..., 7 a sum of 19 biquadrates. The seven numbers, each of them a sum of 19 biquadrates, are 79, 159, 239, 319, 399, 479, 559.

[F. 16. *Binary, Ternary, &c. quadratic and higher forms.*] Art. VI.

43. Euler worked with the quadratic forms $ax^2 \pm cy^2$ (p and q integers), particularly in regard to the forms of the divisors of such numbers. It will be sufficient to refer to his memoir:—

Euler, "Theoremata circa divisores numerorum in hac forma $pa^2 \pm qb^2$ contentorum," (*Op. Arith. Coll.* pp. 35—61, 1744), containing fifty-nine theorems, exhibiting in a quasi-tabular form the linear forms of the divisors of such numbers. As a specimen:—

"Theorema 13. Numerorum in hac forma $a^2 + 76b^2$ contentorum divisores primi omnes sunt vel 2, vel 7, vel in una sex formularum

$$\begin{array}{ll} 28m+1, & 28m+11, \\ 28m+9, & 28m+15, \\ 28m+25, & 28m+23, \end{array}$$

seu in una harum trium

$$\begin{array}{l} 14m+1, \\ 14m+9, \\ 14m+11, \end{array}$$

sunt contenti"; viz. the forms are the three $14m+1$, $14m+9$, $14m+11$.

But Euler did not consider, or if at all very slightly, the trinomial forms $ax^2 + bxy + cy^2$, nor attempt the theory of the reduction of such forms. This was first done by Lagrange in the memoir

Lagrange, *Mém. de Berlin*, 1773. And the theory is reproduced by

Legendre, *Théorie des Nombres*, Paris, 1st ed. 1798; 2nd ed. 1808, § 8, "Réduction de la formule $Ly^2 + Myz + Nz^2$ à l'expression la plus simple," (2nd ed. pp. 61—67).

44. But the classification of quadratic forms, as established by Legendre, is defective as not taking account of the distinction between proper and improper equivalence; and the ulterior theory as to orders and genera, and the composition of forms (although in the meantime established by Gauss), are not therein taken into account; for this reason the **Legendre's** Tables I. to VIII. relating to quadratic forms, given after p. 480 (thirty-two pages not numbered), are of comparatively little value, and it is not necessary to refer to them in detail.

The complete theory was established by

Gauss, *Disquisitiones Arithmeticæ*, 1801.

It is convenient to refer also to the following memoir:

Lejeune Dirichlet, "Recherches sur diverses applications de l'Analyse à la théorie des Nombres," *Crelle*, t. XIX. (1839), p. 338, [*Ges. Werke*, t. I. p. 427], as giving a succinct statement of the principle of classification, and in particular a table of the characters of the genera of the properly primitive order, according to the four forms $D = PS^2$, $P \equiv 1$ or 3 (mod. 4), and $D \equiv 2PS^2$, $P \equiv 1$ or 3 (mod. 4), of the determinant.

45. Tables of quadratic forms arranged on the Gaussian principle are given by

Cayley, *Crelle*, t. LX. (1862), pp. 357—372, [335]; viz. the tables are—

Table I. des formes quadratiques binaires ayant pour déterminants les nombres négatifs depuis $D = -1$ jusqu'à $D = -100$. (Pp. 360—363: [*Coll. Math. Papers*, t. V. pp. 144—147].)

A specimen is

D	Classes	α	β	δ	ϵ	$\delta\epsilon$	Cp
$=26$	1, 0, 26	+				+	1
	3, −1, 9	+				+	g^2
	3, 1, 9	+				+	g^4
	5, 2, 6	−				−	g
	2, 0, 13	−				−	g^3
	5, −2, 6	−				−	g^5

where α, β denote, as there explained, the characters in regard to the odd prime factors of D; δ, ϵ, $\delta\epsilon$ those in regard to the numbers 4 and 8. The last column shows that the forms in the two genera respectively are 1, g^2, g^4 and g, g^3, g^5, where $g^6=1$, viz. the form g, six times compounded, gives the principal form (1, 0, 26).

Table II. des formes quadratiques binaires ayant pour déterminants les nombres positifs non-carrés depuis $D=2$ jusqu'à $D=99$. (Pp. 364—369: [*l.c.*, pp. 148—153].)

The arrangement is the same, except that there is a column "Périodes" showing, in an easily understood abbreviated form, the period of each form. Thus $D=7$, the period of the principal form (1, 0, −7), is given as 1, 2, −3, 1, 2, 1, −3, 2, 1, which represents the series of forms (1, 2, −3), (−3, 1, 2) (2, 1, −3), (−3, 2, 1).

Table III. des formes quadratiques binaires pour les treize déterminants négatifs irréguliers du premier millier. (Pp. 370—372: [*l.c.*, pp. 154—156].)

The arrangement is the same as in Table I. It may be mentioned that the thirteen numbers, and the forms for the principal genus for these numbers, respectively are:—

$-D=$	Principal genus
576, 580, 820, 900	$(1, e^2)(1, e_1^2)$
884	$(1, e^2)(1, i^2, i^4, i^6)$
243, 307, 339, 459, 675, 891	$(1, d, d^2)(1, d_1, d_1^2)$
755, 974	$(1, d, d^2)(1, d_1, d_1^2)(1, e^2)$,

where $d^3=d_1^3=1$, $e^4=e_1^4=1$, $i^8=1$, viz. $(1, e^2)(1, e_1^2)$ denotes four forms, $1, e^2, e_1^2, e^2e_1^2$; and so in the other cases.

46. **Gauss** must have computed quadratic forms to an enormous extent; but, for the reasons (rather amusing ones) mentioned in a letter of May 17, 1841, to Schumacher (quoted in Prof. Smith's Report on "The Theory of Numbers," *Brit. Assoc. Report* for 1862, p. 526, [and Smith's *Coll. Math. Papers*, t. I. p. 261]), he did not preserve his results in detail, but only in the form appearing in the

"Tafel der Anzahl der Classen binärer quadratischer Formen," *Werke*, t. II. pp. 449—476; see editor's remarks, pp. 521—523.

This relates almost entirely to negative determinants, only three quarters of p. 475 and p. 476 to positive ones; for negative determinants, it gives the number of genera and classes, as also the index of irregularity for the determinants of the hundreds 1 to 30, 43, 51, 61, 62, 63, 91 to 100, 117 to 120; then, in a different arrangement, for the thousands 1, 3 and 10, for the first 800 numbers of the forms $-(15n+7)$ and $-(15n+13)$; also for some very large numbers, and for positive determinants of the hundreds 1, 2, 3, 9, 10, and for some others.

A specimen is

		Centas I.	
G II.	(58) ...		(280)
1.	5,	6,	8,
	9,	10,	12,
	13,	15,	16,
	18,	22,	25,
	28,	37,	58,
2.	14,	17,	20,
	.....		
Summa	233		477
Irreg.	0	Impr.	74;

viz. this shows, as regards the negative determinants 1 to 100, that the determinants belonging to G II. 1, viz. those which have two genera each of one class, are 5, 6, 8, 9, &c., in all fifteen determinants; those belonging to G II. 2, viz. those which have two genera each of two classes, are 14, 17, 20, &c.; and so on. The head numbers (58) ... (280) show the number of determinants, each having two genera, and the number of classes; thus,

G II.	$1 \times 15 =$		15
	$2 \times 17 =$		34
	$3 \times 17 =$		51
	$4 \times 6 =$		24
	$5 \times 2 =$		10
	$6 \times 1 =$		6
		58	140
			$\times 2$
			$= 280$;

and the bottom numbers show the total number of genera and of classes, thus

G I.	$17 \times 1 =$		17	61
II.	$58 \times 2 =$		116	280
IV.	$25 \times 4 =$		100	136
	100		233	477;

viz. seventeen determinants, each of one genus, and together of sixty-one classes; fifty-eight determinants, each of two genera, and together of 280 classes: and twenty-five determinants, each of four genera, and together 136 classes, give in all 233 genera and 477 classes. These are exclusive of 74 classes belonging to the improperly primitive order; and the number of irregular determinants (in the first hundred) is = 0.

The irregular determinants are indicated thus:

243(*3*),
307(*3*), 339(*3*),
459(*),
576(*2*), 580(*2*),
675(*3*),
755(*3*),
891(*3*), 820(*2*), 900(*2*), 884(*2*), 974(*3*),
3 243, 307, 339, 459?, 675, 755, 891,
2 576, 589, 820, 884, 900, 974,

which is a notation not easily understood.

As regards the positive determinants, a specimen is

Centas I.
Excedunt determinantis
quadrati 10.

G I. ... (12),
1. 2, 5, 13,
17, 29, 41,
53, 61, 73,
89, 97,
3. 37;

viz. in the first hundred, the positive determinants having one genus of one class are 2, 5, 13, &c.... (eleven in number); that having one genus of three classes is 37, (one in number); $11+1=12$. The irregular determinants, if any, are not distinguished.

47. *Binary cubic forms.*—The earliest table is given by

Arndt, "Tabelle der reducirten binären kubischen Formen und Klassen für alle negativen Determinanten $-D$ von $D=3$ bis $D=2000$," *Grunert's Archiv*, t. XXXI. 1858, pp. 369—448.

The memoir is a sequel to one in t. XVII. (1851). The binary cubic form (a, b, c, d), of determinant $-D\left(=(bc-ad)^2-4(b^2-ac)(c^2-bd)\right)$, is said to be reduced when its characteristic ϕ, $=(A, B, C)$, $=\left(2(b^2-ac), bc-ad, 2(c^2-bd)\right)$, is a reduced quadratic form, that is, when in regard to absolute values B is not $>\frac{1}{2}A$, C not $<A$.

A specimen is

D	Reduced forms, with characters	Classes
44	(0, 1, 0, −11) (1, −1, −2, 0) (2, 0, 22 6, 2, 8)	(0, −1, 0, 11) (0, −2, ±1, 1)

Two subsidiary tables are given, pp. 351, 352, and 353—368.

48. It appeared suitable to remodel a part of this table in the manner made use of for quadratic forms in my tables above referred to; and it is accordingly divided into the three tables given by

Cayley, *Quart. Math. Journ.* t. XI. (1871), where the notation &c. is explained, pp. 251—261, [496]; viz. these are:—

Table I. of the binary cubic forms, the determinants of which are the negative numbers $\equiv 0$ (mod. 4) from -4 to -400 (pp. 251—258; [*Coll. Math. Papers*, t. VIII., pp. 55—61]).

A specimen is

Det. 4 ×	Classes.	Order.	Charact.	Comp.
11.	0, −1, 0, 11	on	1, 0, 11	1
	0, −2, −1, 1	*pp pp*	3, 1, 4	d
	0, −2, 1, 1		3, −1, 4	d^2.

Table II. of the binary cubic forms the determinants of which (taken positively) are $\equiv 1$ (mod. 4) from -3 to -99, the original heading is here corrected, [*l.c.*, pp. 61, 62]; and

Table III. of the binary cubic forms the determinants of which are the negative numbers -972, -1228, -1336, -1836, and -2700, [*l.c.*, pp. 63, 64]; viz. $-972 = 4 \times -243, \ldots, -2700 = 4 \times -675$, where $-243, \ldots, -675$ are the first six irregular numbers for quadric forms.

4×-675, $= -2700$ is beyond the limits of Arndt's tables, and for this number the calculation had to be made anew; the table gives nine classes $(1, d, d^2)$ $(1, d_1, d_1^2)$ of the order ip on pp, but it is remarked that there may possibly be other cubic classes based on a non-primitive characteristic; the point was left unascertained.

49. The theory of ternary quadratic forms was discussed and partially established by Gauss in the *Disquisitiones Arithmeticæ*. It is proper to recall that a ternary quadratic form is either determinate, viz. always positive, such as $x^2 + y^2 + z^2$, or always negative, such as $-x^2 - y^2 - z^2$; or else it is indeterminate, such as $x^2 + y^2 - z^2$. But as regards determinate forms, the negative ones are derived from the positive ones by simply reversing the signs of all the coefficients, so that it is sufficient to attend to the positive forms; and practically the two cases are positive forms (meaning thereby positive determinate forms) and indeterminate forms; but the theory for positive forms was first established completely, and so as to enable the formation of tables, in the work

Seeber, *Ueber die Eigenschaften der positiven ternären quadratischen Formen*, (4to. Freiburg, 1831),

which is reviewed by Gauss in the *Gött. Gelehrte Anzeigen*, 1831, July 9 (see Gauss, *Werke*, t. II. pp. 188—193). The author gives (pp. 220—243) tables "of the classes of positive ternary forms represented by means of the corresponding reduced forms" for the determinants 1 to 100. A specimen is

$$\text{Det. } 6 \qquad \begin{pmatrix} 1, & 1, & 2 \\ 0, & 0, & -1 \end{pmatrix}, \begin{pmatrix} 1, & 1, & 2 \\ -1, & -1, & 0 \end{pmatrix},$$

$$\begin{matrix}\text{Zugeordnete} \\ \text{Formen}\end{matrix} \begin{pmatrix} 8, & 8, & 3 \\ 0, & 0, & 8 \end{pmatrix}, \begin{pmatrix} 7, & 7, & 4 \\ 4, & 4, & 2 \end{pmatrix},$$

where it is to be observed that Seeber admits odd coefficients for the terms in yz, zx, xy; viz. his symbol $\begin{pmatrix} a, & b, & c \\ f, & g, & h \end{pmatrix}$ denotes

$$ax^2 + by^2 + cz^2 + fyz + gzx + hxy,$$

and his determinant is

$$4abc - af^2 - bg^2 - ch^2 + fgh.$$

Also his adjoint form is

$$\begin{pmatrix} 4bc - f^2, & 4ca - g^2, & 4ab - h^2 \\ 2gh - 4af, & 2hf - 4bg, & 2fg - 4ch \end{pmatrix}, = (4bc - f^2)\,x^2 + \ldots + (2gh - 4af)\,yz + \ldots$$

In the notation of the *Disquisitiones Arithmeticæ*, followed by Eisenstein and others, the symbol $\begin{pmatrix} a, & b, & c \\ f, & g, & h \end{pmatrix}$ denotes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy;$$

the determinant is

$$= -(abc - af^2 - bg^2 - ch^2 + 2fgh),$$

a positive form having thus always a negative determinant. And the adjoint form is

$$-\begin{pmatrix} bc - f^2, & ca - g^2, & ab - h^2 \\ gh - af, & hf - bg, & fg - ch \end{pmatrix}, = -(bc - f^2)\,x^2 - \ldots - 2\,(gh - af)\,yz - \ldots$$

Hence Seeber's determinant is $= -4$ multiplied by that of Gauss, and his tables really extend between the values -1 and -25 of the Gaussian determinant.

50. Tables of greater extent, and in the better form just referred to, are given by

Eisenstein, *Crelle*, t. XLI. (1851), pp. 169—190; viz. these are

I. "Tabelle der eigentlich primitiven positiven ternären Formen für alle negativen Determinanten von -1 bis -100," (pp. 169—185).

A specimen is

D	Anzahl	Reducirte Formen für $-D$
10	3	$\begin{pmatrix} 1, & 1, & 10 \\ 0, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} 1, & 2, & 5 \\ 0, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} 2, & 2, & 3 \\ 0, & -1, & 0 \end{pmatrix}$.
		$\delta = 8$ $\quad$ $\delta = 4$ $\quad$ $\delta = 4$

II. "Tabelle der *uneigentlich* primitiven positiven ternären Formen für alle negativen Determinanten von -2 bis -100," (pp. 186—189).

A specimen is

D	Anzahl	Reducirte Formen für $-D$
10	1	$\begin{pmatrix} 2, & 2, & 4 \\ 1, & 1, & 1 \end{pmatrix}$. $\delta = 6$.

And there is given (p. 190) a table of the reduced forms for the determinant $-385 (= -5 . 7 . 11)$, selected merely as a largish number with three factors; viz. there are in all fifty-nine forms, corresponding to values 1, 2, 4, 6, 8 of δ.

It may be remarked that δ denotes, for any given form, the number of ways in which this is linearly transformable into itself, this number being always 1, 2, 4, 6, 8, 12, or 24. The theory as to this and other points is explained in the memoir (pp. 141—168), and various subsidiary tables are contained therein and in the *Anhang* (pp. 227—242); and there is given a small table relating to *indeterminate forms*, viz. this is

"C. Versuch einer Tabelle der nicht äquivalenten unbestimmten (indifferenten) ternären quadratischen Formen für die Determinanten ohne quadratischen Theiler unter 20," (pp. 239, 240).

A specimen is

D	Indifferente ternäre quadratische Formen
10	$\begin{pmatrix} 0, & 1, & 10 \\ 0, & 0, & 1 \end{pmatrix}$, $\begin{pmatrix} 1, & 2, & -5 \\ 0, & 0, & 0 \end{pmatrix}$, $\begin{pmatrix} 0, & 0, & 10 \\ 0, & 0, & 1 \end{pmatrix}$,

where, when the determinant is even, the forms in the second line are always improperly primitive forms.

[F. 17. *Complex Theories.*] Art. VII.

51. The theory of binary quadratic forms (a, b, c), with complex coefficients of the form $\alpha + \beta i$, ($i = \sqrt{-1}$ as usual, α and β integers), has been studied by Lejeune Dirichlet, Prof. H. J. S. Smith, and possibly others; but no tables have, it is believed, been calculated. The calculations would be laborious; but tables of a small extent only would be a sufficient illustration of the theory, and would, it is thought, be of great interest.

The theory of complex numbers of the last-mentioned form $\alpha + \beta i$, or say of the numbers formed with the fourth root of unity, had previously been studied by Gauss; and the theory of the numbers formed with the cube roots of unity ($\alpha + \beta\omega$, $\omega^2 + \omega + 1 = 0$, α and β integers) was studied by Eisenstein; but the general theory of the numbers involving the nth roots of unity (n an odd prime) was first studied by Kummer. It will be sufficient to refer to his memoir,

Kummer, "Zur Theorie der complexen Zahlen," *Berl. Monatsb.*, March, 1845; and *Crelle*, t. xxxv. (1847), pp. 319—326; also "Ueber die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren," same volume, pp. 327—367, where the astonishing theory of "Ideal Complex Numbers" is established.

52. It may be recalled that, p being an odd prime, and ρ denoting a root of the equation $\rho^{p-1} + \rho^{p-2} + \ldots + \rho + 1 = 0$, then the numbers in question are those of the form $a + b\rho + \ldots + k\rho^{n-2}$, where $(a, b, \ldots, k)$ are integers; or (what is in one point of view more, and in another less, general) if $\eta, \eta_1, \ldots, \eta_{e-1}$ are "periods" composed with the powers of ρ (e any factor of $p-1$), then the form considered is $a\eta + b\eta_1 + \ldots + h\eta_{e-1}$. For any value of p or e there is a corresponding complex theory. A number (real or complex) is in the complex theory prime or composite, according as it does not, or does, break up into factors of the form under consideration. For p a prime number under 23, if in the complex theory N is a prime, then any power of N (to fix the ideas say N^3) has no other factors than N or N^2; but if $p = 23$ (and similarly for higher values of p), then N may be such that, for instance, N^3 has complex factors other than N or N^2 (for $p = 23$, $N = 47$ is the first value of N, viz. 47^3 has factors other than 47 and 47^2); say N^3 has a complex prime factor A, or we have $\sqrt[3]{A}$ as an ideal complex factor of N. Observe that by hypothesis N is not a perfect cube, viz. there is no complex number whose cube is $= A$. In the foregoing general statement, made by way of illustration only, all reference to the complex factors of unity is purposely omitted, and the statement must be understood as being subject to correction on this account.

What precedes is by way of introduction to the account of **Reuschle's** Tables (*Berliner Monatsberichte*, 1859—60), which give in the different complex theories $p = 5$, 7, 11, 13, 17, 19, 23, 29 the complex factors of the decomposable real primes up to in some cases 1000.

It should be remarked that the form of a prime factor is to a certain extent indeterminate, as the factor can without injury be modified by affecting it with a complex factor of unity; but in the tables the choice of the representative form is made according to definite rules, which are fully explained, and which need not be here referred to.

53. The following synopsis is convenient:—

	Theory of the pth roots	Form of real prime to mod. $p \equiv$	No. of factors in complex theory	Extent of table; all primes under	Equation of periods
1859. pp. 488–491.	5	1 4 2, 3	4 2 (not tabulated) prime.	2500	$\alpha^4+\ldots+\alpha+1=0.$ $y^2+y-1=0.$
694–697.	7	1 6 2, 4 3, 5	6 3 2 prime.	1000	$\alpha^6+\ldots+\alpha+1=0.$ $y^3+y^2-2y-1=0.$ $y^2+y+2=0.$
1860. pp. 190–194.	11	1 10 3, 4, 5, 9 2, 6, 7, 8	10 5 2 1	1000	$\alpha^{10}+\ldots+\alpha+1=0.$ $y^5+y^4-4y^3-3y^2+3y+1=0.$ $y^2+y+3=0.$
194–199.	13	1 12 3, 9 5, 8 4, 10 2, 6, 7, 11	12 6 4 3 2 prime.	1000	$\alpha^{12}+\ldots+\alpha+1=0.$ $y^6+y^5-5y^4-4y^3+6y^2+3y-1=0.$ $y^4+y^3+2y^2-4y+3=0.$ $y^3+y^2-4y+1=0.$ $y^2+y-3=0.$
714–719.	17	1 16 4, 13 2, 8, 9, 15 3, 5, 6, 7, 11, 12, 14	16 8 4 2 prime.	1000	$\alpha^{16}+\ldots+\alpha+1=0.$ $y^8+y^7-7y^6-6y^5+15y^4+10y^3-10y^2-4y+1=0.$ $y^4+y^3-6y^2+1=0.$ $y^2+y-4=0.$
719–725.	19	1 18 7, 11 8, 12 4, 5, 6, 9, 13, 17 2, 3, 10, 13, 14, 15	18 9 6 3 2 prime.	1000	$\alpha^{18}+\ldots+\alpha+1=0.$ $y^9+y^8-8y^7-7y^6+21y^5+15y^4-20y^3-10y^2+5y+1=0.$ $y^6+y^5+2y^4-8y^3-y^2+5y+7=0.$ $y^3+y^2-6y-7=0.$ $y^2+y+5=0.$
725–729.	23	1 22 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 ?5, 7, 10, 11, 14, 15, 17, 19, 20, 21	22 11 2 prime.	1000	$\alpha^{22}+\ldots+\alpha+1=0.$ $y^{11}+y^{10}-10y^9-9y^8+36y^7+28y^6-56y^5-35y^4+35y^3+15y^2-6y-1=0.$ $y^2+y+6=0.$
729–734.	29	1 28 12, 17 7, 16, 20, 23, 24, 25 4, 5, 6, 9, 13, 22 2, 3, 8, 10, 11, 14, 15, 18, 19, 21, 26, 27	28 14 7 4 2 prime.	1000	$\alpha^{28}+\ldots+\alpha+1=0.$ $y^{14}+y^{13}-13y^{12}-12y^{11}+66y^{10}+55y^9-165y^8-120y^7+210y^6+126y^5-126y^4-56y^3+28y^2+7y-1=0.$ $y^7+y^6-12y^5-7y^4+28y^3+14y^2-9y+1=0.$ $y^4+y^3+4y^2+20y+23=0.$ $y^2+y-7=0.$

The foregoing synopsis of Reuschle's tables in the *Berliner Monatsberichte* was written previous to the publication of Reuschle's far more extensive work. It is

allowed to remain, but some explanations which were given have been struck out, and were instead given in reference to the larger work, which is

Reuschle, *Tafeln complexer Primzahlen, welche aus Wurzeln der Einheit gebildet sind.* Berlin, 4° (1875), pp. iii—vi and 1—671.

This work (the mass of calculation is perfectly wonderful) relates to the roots of unity, the degree being any prime or composite number, as presently mentioned, having all the values up to and a few exceeding 100; viz. the work is in five divisions, relating to the cases:

I. (pp. 1—171), degree any odd prime of the first 100, viz. 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97;

II. (pp. 173—192), degree the power of an odd prime 9, 25, 27, 49, 81;

III. (pp. 193—440), degree a product of two or more odd primes or their powers, viz. 15, 21, 33, 35, 39, 45, 51, 55, 57, 63, 65, 69, 75, 77, 85, 87, 91, 93, 95, 99, 105;

IV. (pp. 441—466), degree an even power of 2, viz. 4, 8, 16, 32, 64, 128;

V. (pp. 467—671), degree divisible by 4, viz. 12, 20, 24, 28, 36, 40, 44, 48, 52, 56, 60, 68, 72, 76, 80, 84, 88, 92, 96, 100, 120;

the only excluded degrees being those which are the double of an odd prime, these, in fact, coming under the case where the degree is the odd prime itself.

It would be somewhat long to explain the specialities which belong to degrees of the forms II., III., IV., V.; and what follows refers only to Division I., degree an odd prime.

For instance, if $\lambda = 7$, $\lambda - 1 = 2.3$; the factors of 6 being 6, 3, 2, 1, there are accordingly four divisions, viz.

I. α a prime seventh root, that is, a root of $\alpha^6 + \alpha^5 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$;

II. $\eta_0 = \alpha + \alpha^{-1}$, $\eta_1 = \alpha^2 + \alpha^{-2}$, $\eta_2 = \alpha^3 + \alpha^{-3}$, or η a root of $\begin{cases} \eta_0^2 = 2 + \eta_1, \ \eta_1^2 = 2 + \eta_2, \ \&c. \\ \eta_0\eta_1 = \eta_0 + \eta_2, \ \&c. \\ \eta_0\eta_2 = \eta_1 + \eta_2, \ \&c.; \end{cases}$
$\eta^3 + \eta^2 - 2\eta - 1 = 0$,

III. $\eta_0 = \alpha + \alpha^2 + \alpha^4$, $\eta_1 = \alpha^3 + \alpha^5 + \alpha^6$, or η a root of $\eta^2 + \eta + 2 = 0$;

IV. Real numbers.

I. $p = 7m + 1$. First, it gives for the several prime numbers of this form 29, 43, .., 967 the congruence roots, mod. p; for instance,

p	α	α^2	α^3	α^4	α^5	α^6
29	− 5	− 4	− 9	− 13	+ 7	− 6
43	+ 11	− 8	− 2	+ 21	+ 16	+ 4.
⋮						

This means that, if $\alpha \equiv -5$ (mod. 29), then $\alpha^2 \equiv 25$, $\equiv -4$, $\alpha^3 \equiv 20$, $\equiv -9$, &c., values which satisfy the congruence $\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 \equiv 0$ (mod. 29).

Secondly, it gives, under the simple and the primary forms, the prime factors $f(\alpha)$ of these same numbers 29, 43,.., 967; for instance,

p	$f(\alpha)$ simple.	$f(\alpha)$ primary.
29	$\alpha + \alpha^2 - \alpha^3$	$2 + 3\alpha - \alpha^2 + 5\alpha^3 - 2\alpha^4 + 4\alpha^5$
43	$\alpha^2 + 2\alpha^6$	$2\alpha - 2\alpha^2 + 4\alpha^4 - \alpha^5 - 5\alpha^6$.

The definition of a primary form is a form for which $f(\alpha)f(\alpha^{-1}) \equiv f(1)^2$ mod. λ, and $f(\alpha) \equiv f(1)$ mod. $(1-\alpha)^2$. The simple forms are also chosen so as to satisfy this *last* condition; thus $f(\alpha) = \alpha + \alpha^2 - \alpha^3$, then $f(1) - f(\alpha) = 1 - \alpha - \alpha^2 + \alpha^3 = (1-\alpha)^2(1+\alpha)$, $\equiv 0$ mod. $(1-\alpha)^2$.

II. $p = 7m - 1$. First, it gives for the several prime numbers of this form 13, 41,.., 937 the congruence roots, mod. p; for instance,

p	η_0	η_1	η_2
13	-3	-6	-5
41	-4	$+14$	-11;
⋮			

and secondly, it gives, under the simple and the primary forms, the prime factors $f(\eta)$ of these same numbers 13, 41,.., 937; for instance,

p	$f(\eta)$ simple.	$f(\eta)$ primary.
13	$\eta_0 + 2\eta_2$	$3 + 7\eta_1$
41	$4 + \eta_0$	$-11 + 7\eta_1 - 7\eta_2$.
⋮		

Thus $13 = (\eta_0 + 2\eta_2)(\eta_1 + 2\eta_0)(\eta_2 + 2\eta_1)$, as is easily verified; the product of first and second factors is $= 4 + 3\eta_0 + 8\eta_1 + 5\eta_2$, and then multiplying by the third factor, the result is $42 + 29(\eta_0 + \eta_2)$, $= 13$.

III. $p = 7m + 2$ or $7m + 4$. First, it gives for the several prime numbers of this form 2, 11,.., 991 the congruence roots, mod. p; for instance,

p	η_0	η_1
2	0	-1
11	4	-5;
⋮		

and secondly, it gives the primary prime factors $f(\eta)$ of these same numbers; for instance,

p	$f(\eta)$
2	η_0
11	$1 - 2\eta_1$.

IV. $p = 7m + 3$ or $7m + 5$. The prime numbers of these forms, viz. 3, 5, 17, 19,.., 997, are primes in the complex theory, and are therefore simply enumerated.

The arrangement is the same for the higher prime numbers $\lambda = 23$, &c., for which ideal factors make their appearance; but it presents itself under a more complicated form. Thus $\lambda = 23$, $\lambda - 1 = 2 . 11$, and the factors of 22 are 22, 11, 2, 1. There are thus four sections.

I. α a prime root, or $\alpha^{22} + \alpha^{21} + \ldots + \alpha^2 + \alpha + 1 = 0$;

II. $\eta_0 = \alpha + \alpha^{-1}, \ldots, \eta_{10} = \alpha^{11} + \alpha^{-11}$, or η a root of $\eta^{11} + \eta^{10} - 10\eta^9 + \ldots + 15\eta^2 - 6\eta - 1 = 0$;

III. $\eta_0 = \alpha + \alpha^2$, $\eta_1 = \alpha^{-1} + \alpha^{-2}$, or η a root of $\eta^2 + \eta + 6 = 0$;

IV. Real numbers.

I. $p = 23m + 1$. First, it gives for the prime numbers of this form 47, 139,.., 967 congruence roots, mod. p, and *also* congruence roots, mod. p^3 *; these last in the form $a + bp + cp^2$, where a is given in the former table; thus first table:—

p	α	α^2	$\alpha^3 \ldots$	α^{22}
47	6	-11	$-19\ldots$	$+8$;

and second table—

p	α	α^2	α^3	$\ldots$	α^{22}
47	$+p - 2p^2$	$+13p - 23p^2$	$+19p - 8p^2 \ldots$		$+22p + 22p^2$.

The meaning is that, $p = 47$, the roots of the congruence

$$\alpha^{22} + \alpha^{21} + \ldots + \alpha^2 + \alpha + 1 \equiv 0 \pmod{47^3}$$

are

$$\alpha = 6 + p - 2p^2, \ \alpha^2 = -11 + 13p - 23p^2, \ \&c.$$

Secondly, it then gives $f(\alpha)$, the actual ideal prime factor of these same primes 47, 139,.., 967; viz. the whole of this portion of the table $\lambda = 23$, I. (2) is,

having actual prime factors,

p	$f(\alpha)$
599	$\alpha + \alpha^{16} - \alpha^{17}$
691	$\alpha^3 + \alpha^{21} + \alpha^{22}$
829	$\alpha^2 + \alpha^5 + \alpha^{46}$;

having ideal factors, their third powers actual,

p	$f^3(\alpha)$
47	$\alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10} + \alpha^{16} - \alpha^{20} + \alpha^{22}$
139	$1 - \alpha^3 - \alpha^7 + \alpha^9 + \alpha^{11} + \alpha^{14} + \alpha^{15} + \alpha^{17} + \alpha^{18} + \alpha^{20} + \alpha^{21}$
277	$\alpha^2 - \alpha^4 - \alpha^6 + \alpha^7 - \alpha^{10} - \alpha^{15} - \alpha^{17} + \alpha^{21} + \alpha^{22}$
461	$\alpha - \alpha^2 + \alpha^3 - \alpha^9 + \alpha^{14} - 2\alpha^{15}$
967	$\alpha^2 - \alpha^3 - \alpha^5 + \alpha^{10} + \alpha^{16} - 2\alpha^{16} + \alpha^{17} + \alpha^{19}$.

I repeat the explanation that, for the number 47, this means $f(\alpha) f(\alpha^3) \ldots f(\alpha^{22}) = 47^3$.

* Where, as presently appearing, 3 is the index of ideality or power to which the ideal factors have to be raised in order to become actual.

And the like further complication presents itself in the part III. of the same table, $\lambda = 23$ (not, as it happens, in part II., nor of course in the concluding part IV., which is a mere enumeration of real primes). Thus III. (1), we have congruences, (mod. p^3),

$$p = 2, \quad \eta \equiv -2, \quad p = 3, \quad \eta_0 = +12, \text{ \&c.};$$

and having actual prime factors,

p	$f(\eta)$
59	$5 - 2\eta_1$
101	$1 - 4\eta_1$;
⋮	

and having ideal prime factors, their third powers actual,

p	$f^3(\eta)$
2	$1 - \eta_1$
3	$1 - 2\eta_1$;
⋮	

as regards these last the signification being

$$2^3 = (1 - \eta_0)(1 - \eta_1), \ \eta_0 + \eta_1 = -1, \ \eta_0\eta_1 = 6 \text{ (as is at once verified)},$$
$$3^3 = (1 - 2\eta_0)(1 - 2\eta_1);$$

but the simple numbers 2, 3 are neither of them of the form $(a + b\eta_0)(a + b\eta_1)$.

Contents of Report 1875 on Mathematical Tables.

§ 7. *Tables* F. *Arithmological.*

		Page
Art. I.	Divisors and Prime Numbers	462
II.	Prime Roots. The Canon Arithmeticus, Quadratic residues	471
III.	The Pellian Equation	477
IV.	Partitions	480
V.	Quadratic forms $a^2 + b^2$ &c., and Partitions of Numbers into squares, cubes, and biquadrates	484
VI.	Binary, Ternary, &c. quadratic and higher forms . . .	486
VII.	Complex Theories	493

612.

NOTE SUR UNE FORMULE D'INTÉGRATION INDÉFINIE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, tom. LXXVIII. (*Janvier—Juin*, 1874), pp. 1624—1629.]

EN étudiant les Mémoires de M. Serret (*Journal de Liouville*, t. X., 1845) par rapport à la représentation géométrique des fonctions elliptiques, avec les remarques de M. Liouville sur ce sujet, je suis parvenu à une formule d'intégration indéfinie qui me paraît assez remarquable, savoir: en prenant θ entier positif quelconque, je dis que l'intégrale

$$\int \frac{(x+p)^{m+n-\theta}(x+q)^{\theta}\,dx}{x^{m+1}(x+p+q)^{n+1}}$$

a une valeur algébrique

$$(x+p)^{m+n-\theta+1}(x+p+q)^{-n}x^{-m}(A+Bx+Cx^2+\ldots+Kx^{\theta-1}),$$

pourvu qu'une seule condition soit satisfaite par les quantités m, n, p, q. Cette condition s'écrit sous la forme symbolique

$$([m]p^2+[n]q^2)^{\theta}=0,$$

en dénotant ainsi l'équation

$$[m]^{\theta}p^{2\theta}+\frac{\theta}{1}[m]^{\theta-1}[n]^1p^{2\theta-2}q^2+\ldots+[n]^{\theta}q^{2\theta}=0,$$

où, comme à l'ordinaire, $[m]^{\theta}$ signifie $m(m-1)\ldots(m-\theta+1)$.

Je rappelle que les formules de M. Serret ne contiennent que des exposants entiers, et celles de M. Liouville qu'un seul exposant quelconque: la nouvelle formule contient deux exposants quelconques, m, n. Je remarque aussi l'analogie de la condition $([m]p^2+[n]q^2)^{\theta}=0$ avec celle-ci

$$\frac{1}{\zeta^{n-m}}\left(\frac{d}{d\zeta}\right)^m\zeta^n(\zeta-1)^m=0$$

(m étant un entier positif), qui figure dans les Mémoires cités.

Pour démontrer la formule, j'écris

$$u = x^{-m}(A + Bx + Cx^2 + \ldots + Kx^{\theta-1}),$$

et aussi pour abréger

$$X = (x+p)^{m+n-\theta+1}(x+p+q)^{-n},$$

ce qui donne

$$\frac{X}{(x+p)(x+p+q)} = (x+p)^{m+n-\theta}(x+p+q)^{-n-1}.$$

L'équation à vérifier est donc

$$Xu = \int \frac{X(x+q)^\theta\,dx}{x^{m+1}(x+p)(x+p+q)},$$

ou, en différentiant et divisant par X,

$$\frac{X'}{X}u + u' = \frac{(x+q)^\theta}{x^{m+1}(x+p)(x+p+q)},$$

ou enfin

$$[(m+n-\theta+1)(x+p+q) - n(x+p)]\,u + (x+p)(x+p+q)\,u' = \frac{(x+q)^\theta}{x^{m+1}},$$

où u' dénote $\frac{du}{dx}$. Il ne s'agit donc que d'exprimer que cette équation ait une intégrale

$$u = x^{-m}(A + Bx + Cx^2 + \ldots + Kx^{\theta-1}).$$

En supposant que cela soit ainsi, et en effectuant la substitution, les termes en $x^{-m+\theta}$ se détruisent, et l'on obtient une équation qui contient des termes en x^{-m-1}, $x^{-m}, \ldots, x^{-m+\theta-1}$, savoir $(\theta+1)$ termes. On a ainsi, entre les θ coefficients $A, B, C, \ldots, K$ un système de $(\theta+1)$ équations linéaires, ce qui implique une condition entre les constantes m, n, p, q; mais, cette condition satisfaite, les équations se réduisent à θ équations indépendantes, et les coefficients seront ainsi déterminés.

Par exemple, soit $\theta = 2$; l'équation différentielle est

$$[\overline{m-1}\,p + \overline{m+n-1}\,q + \overline{m-1}\,x]\,u + [p^2 + pq + x(2p+q) + x^2]\,u' = x^{-m-1}(q+x)^2,$$

laquelle doit être satisfaite par $u = Ax^{-m} + Bx^{-m+1}$. Cela donne

$$\begin{array}{cccc|c}
\overbrace{\phantom{x^{-m-1}}} & \overbrace{x^{-m}}^{(\overline{m-1}\,p+\overline{m+n-1}\,q)A,} & \overbrace{x^{-m+1}}^{(\overline{m-1}\,p+\overline{m+n-1}\,q)B,} & \overbrace{x^{-m+2}} & \\
\text{"}, & \text{"}, & (m-1)A, & (m-1)B & \\
-m(p^2+pq)A, & -(m-1)(p^2+pq)B, & \text{"}, & \text{"} & \\
\text{"}, & -m(2p+q)A, & -(m-1)(2p+q)B, & \text{"} & = 0, \\
\text{"}, & \text{"}, & -mA, & -(m-1)B & \\
-q^2, & -2q, & -1, & \text{"} &
\end{array}$$

c'est-à-dire

$$\begin{aligned}
&1 + [(m-1)p - nq]\,B + 1A = 0,\\
&2q + (m-1)(p^2+pq)\,B + [(m+1)p-(n-1)q]\,A = 0,\\
&q^2 + m(p^2+pq)\,A = 0,
\end{aligned}$$

ce qui donne une condition, déterminant $=0$, et l'on a une condition de cette même forme pour une valeur quelconque de θ.

En formant, puis en réduisant les expressions de ces déterminants, on obtient pour toutes les valeurs $\theta = 1, 2, 3, 4, \ldots$ respectivement la suite d'équations que voici :

$$0 = \begin{vmatrix} 1, & mp-nq \\ q, & m(p^2+pq) \end{vmatrix} = [m]^1 p(p+q) - 1.\, q([m]p-[n]q)^2, \quad = ([m]p^2+[n]q^2)^1,$$

$$0 = \begin{vmatrix} 1, & \overline{m-1}\,p-nq, & 1 \\ 2q, & \overline{m-1}\,(p^2+pq), & \overline{m+1}\,p-\overline{n-1}\,q \\ q^2, & \cdot, & m(p^2+pq) \end{vmatrix} = [m]^2 p^2 (p+q)^2 - 2.\,[m]^1 p(p+q)\,q([m-1]p-[n]q)^1 + 1.\, q^2([m]\quad p-[n]q)^2, \quad = ([m]p^2+[n]q^2)^2,$$

$$0 = \begin{vmatrix} 1, & \overline{m-2}\,p-nq, & 1, & \cdot \\ 3q, & \overline{m-2}\,(p^2+pq), & mp-\overline{n-1}\,q, & 2 \\ 3q^2, & \cdot, & \overline{m-1}\,(p^2+pq), & \overline{m+2}\,p-\overline{n-2}\,q \\ q^3, & \cdot, & \cdot, & m(p^2+pq) \end{vmatrix} = [m]^3 p^3 (p+q)^3 - 3.\,[m]^2 p^2 (p+q)^2 q\,([m-2]p-[n]q)^1 + 3.\,[m]^1 p\,(p+q)\,q^2([m-1]p-[n]q)^2 - 1.\, q^3([m]\quad p-[n]q)^3, \quad = ([m]p^2+[n]q^2)^3,$$

$$0 = \begin{vmatrix} 1, & \overline{m-3}\,p-nq, & 1, & \cdot, & \cdot \\ 4q, & \overline{m-3}\,(p^2+pq), & \overline{m-1}\,p-\overline{n-1}\,q, & 2, & \cdot \\ 6q^2, & \cdot, & \overline{m-2}\,(p^2+pq), & \overline{m+1}\,p-\overline{n-2}\,q, & 3 \\ 4q^3, & \cdot, & \cdot, & \overline{m-1}\,(p^2+pq), & \overline{m+3}\,p-\overline{n-3}\,q \\ q^4, & \cdot, & \cdot, & \cdot, & m(p^2+pq) \end{vmatrix} = [m]^4 p^4 (p+q)^4 - 4.\,[m]^3 p^3 (p+q)^3 q\,([m-3]p-[n]q)^1 + 6.\,[m]^2 p^2 (p+q)^2 q^2([m-2]p-[n]q)^2 - 4.\,[m]^1 p\,(p+q)\,q^3([m-1]p-[n]q)^3 + 1.\, q^4([m]\quad p-[n]q)^4, \quad = ([m]p^2+[n]q^2)^4;$$

et ainsi de suite. Les notations $([m]p-[n]q)^1$, $([m]p-[n]q)^2, \ldots$ ont des significations semblables à celles de $([m]p^2+[n]q^2)^1$, $([m]p^2+[n]q^2)^2, \ldots$, auparavant expliquées. On a, par exemple,

$$([m]p-[n]q)^2=[m]^2p^2-2[m]^1[n]^1pq+[n]^2q^2.$$

Considérons, par exemple, le deuxième déterminant: ceci contient trois termes en 1, $2q$, q^2 respectivement; le premier terme est

$$1\,.\,(m-1)(p^2+pq)\,.\,m(p^2+pq),$$

c'est-à-dire

$$[m]^2p^2(p+q)^2;$$

le deuxième terme est

$$2q\,.\,-m(p^2+pq)[(m-1)p-nq],$$

c'est-à-dire

$$-2[m]^1p(p+q)q([m-1]p-[n]q)^1;$$

le troisième terme est

$$q^2[(\overline{m-1}\,p-nq)(\overline{m+1}\,p-\overline{n-1}q)-(m-1)(p^2+pq)],$$

c'est-à-dire

$$q^2[(m^2-m)p^2-2mnpq+(n^2-n)q^2]=q^2([m]p-[n]q)^2.$$

Et de même le troisième déterminant est composé de quatre termes en 1, $3q$, $3q^2$, q^3 respectivement, lesquels sont les quatre termes de la première expression transformée; et ainsi pour le quatrième déterminant, etc. Au moyen de ces premières transformées, on obtient sans peine les expressions finales $([m]p^2+[n]q^2)^1$, $([m]p^2+[n]q^2)^2, \ldots$.

En écrivant $z-\frac{1}{2}(p+q)$ au lieu de x, et puis $\frac{1}{2}(p+q)=\alpha$, $\frac{1}{2}(p-q)=a$, la formule devient

$$\int\frac{(z-a)^\theta(z+a)^{m+n-\theta}\,d\theta}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}},$$

et la valeur algébrique

$$=(z+a)^{m+n-\theta-1}(z-\alpha)^{-m}(z+\alpha)^{-n}(A'+B'z+\ldots+K'z^{\theta-1}),$$

pourvu qu'on ait entre les quantités m, n, a, α la relation

$$\{[m](\alpha+a)^2+[n](\alpha-a)^2\}^\theta=0.$$

En écrivant $\theta=m$, on a la formule de MM. Serret et Liouville, laquelle, en y écrivant $\frac{(\alpha+a)^2}{4a\alpha}=\zeta$ et $\frac{(\alpha-a)^2}{4a\alpha}=\zeta-1$, peut s'écrire sous la forme $\{[m]\zeta+[n](\zeta-1)\}^\theta=0$. Je remarque que l'équation en ζ ne donne pas *toujours* pour ζ des valeurs réelles, positives et plus grandes que l'unité: par exemple, pour $\theta=1$, on a $\zeta=\frac{n}{m+n}$, valeur qui ne peut pas satisfaire à ces conditions. Je n'ai pas cherché dans quel cas ces conditions (qui ont rapport à l'application des formules à la représentation des fonctions elliptiques) subsistent.

613.

ON THE GROUP OF POINTS G_4^1 ON A SEXTIC CURVE WITH FIVE DOUBLE POINTS.

[From the *Mathematische Annalen*, vol. VIII. (1875), pp. 359—362.]

THE present note relates to a special group of points considered incidentally by MM. Brill and Nöther in their paper "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," *Math. Annalen*, t. VII. pp. 268—310 (1874).

I recall some of the fundamental notions. We have a basis-curve which to fix the ideas may be taken to be of the order n, $=p+1$, with $\frac{1}{2}p(p-3)$ dps, and therefore of the "Geschlecht" or deficiency p; any curve of the order $n-3$, $=p-2$ passing through the $\frac{1}{2}p(p-3)$ dps is said to be an adjoint curve. We may have, on the basis-curve, a special group G_Q^q of Q points ($Q \not> 2p-2$); viz. this is the case when the Q points are such that every adjoint curve through $Q-q$ of them—that is, every curve of the order $p-2$ through $\frac{1}{2}p(p-3)$ dps and the $Q-q$ points—passes through the remaining q points of the group: the number q may be termed the "speciality" of the group: if $q=0$, the group is an ordinary one.

It may be observed that a special group G_Q^q is chiefly noteworthy in the case where $Q-q$ is so small that the adjoint curve is not completely determined: thus if $p=5$, viz. if the basis-curve be a sextic with 5 dps, then we may have a special group G_6^2, but there is nothing remarkable in this; the 6 points are intersections with the sextic of an arbitrary cubic through the 5 dps—the cubic of course intersects the sextic in the 5 dps counting as 10 points, and in 8 other points—and such cubic is completely determined by means of the 5 dps and any 4 of the 6 points. But contrariwise, there is something remarkable in the group G_4^1 about to be considered: viz. we have here on the sextic 4 points, such that every cubic through the 5 dps and through 3 of the 4 points (through 8 points in all) passes through the remaining one of the 4 points.

The whole number of intersections of the basis-curve with an adjoint, exclusive of the dps counting as $p(p-3)$ points, is of course $=2p-2$: hence an adjoint through the Q points of a group G_Q^q meets the basis-curve besides in R, $=2p-2-Q$,

points; we have then the "Riemann-Roch" theorem that these R points form a special group G_R^r, where

$$Q + R = 2p - 2,$$

as just mentioned, and

$$Q - R = 2q - 2r;$$

viz. dividing in any manner the $2p-2$ intersections of the basis-curve by an adjoint into groups of Q and R points respectively, these will be special groups, or at least one of them will be a special group, G_Q^q, G_R^r, such that their specialities q, r are connected by the foregoing relation $Q - R = 2q - 2r$.

The Authors give (*l.c.*, p. 293) a Table showing for a given basis-curve, or given value of p, and for a given value of r, the least value of R and the corresponding values of q, Q: this table is conveniently expressed in the following form.

The least value of

$$R = p - \frac{p}{r+1} + r;$$

and then

$$q = \frac{p}{r+1} - 1,$$

$$Q = p + \frac{p}{r+1} - r - 2,$$

where $\frac{p}{r+1}$ denotes the integer equal to or next less than the fraction.

It is, I think, worth while to present the table in the more developed form:

n	p	Dps	$r=$ 1	2	3	4	5	6
4	3	0	G_3^1	G_4^2	.	.	.	.
			G_1^0	G_0^0	.	.	.	.
5	4	2	G_3^1	G_5^2	G_6^3	.	.	.
			G_3^1	G_1^0	G_0^0	.	.	.
6	5	5	G_4^1	G_6^2	G_7^3	G_8^4	.	.
			G_4^1	G_2^0	G_1^0	G_0^0	.	.
7	6	9	G_4^1	G_6^2	G_8^3	G_9^4	G_{10}^5	.
			G_6^2	G_4^1	G_2^0	G_1^0	G_0^0	.
8	7	14	G_5^1	G_7^2	G_9^3	G_{10}^4	G_{11}^5	G_{12}^6
			G_7^2	G_5^1	G_3^0	G_2^0	G_1^0	G_0^0
⋮								

where the table shows the values of $\begin{matrix} G_R^r \\ G_Q^q \end{matrix}$ for any given values of p, r.

I recur to the case $p=5$ and the group G_4^1, which is the subject of the present note: viz. we have here a sextic curve with 5 dps, and on it a group of 4 points G_4^1, such that every cubic through the 5 dps and through 3 points of the group, 8 points in all, passes through the remaining 1 point.

MM. Brill and Nöther show (by consideration of a rational transformation of the whole figure) that, given 2 points of the group, it is possible, and possible in 5 different ways, to determine the remaining 2 points of the group.

I remark that the 5 dps and the 4 points of the group form "an ennead" or system of the nine intersections of two cubic curves: and that the question is, given the 5 dps and 2 points on the sextic, to show how to determine on the sextic a pair of points forming with the 7 points an ennead: and to show that the number of solutions is $=5$.

We have the following "Geiser-Cotterill" theorem:

If seven of the points of an ennead are fixed, and the eighth point describes a curve of the order n passing $a_1, a_2, \ldots, a_7$ times through the seven points respectively, then will the ninth point describe a curve of the order ν passing $\alpha_1, \alpha_2, \ldots, \alpha_7$ times through the seven points respectively: where

$$\begin{aligned} \nu &= 8n - 3\Sigma a, \\ \alpha_1 &= 3n - a_1 - \Sigma a, \\ &\vdots \\ \alpha_7 &= 3n - a_7 - \Sigma a, \end{aligned}$$

and conversely

$$\begin{aligned} n &= 8\nu - 3\Sigma\alpha, \\ a_1 &= 3\nu - \alpha_1 - \Sigma\alpha, \\ &\vdots \\ a_7 &= 3\nu - \alpha_7 - \Sigma\alpha, \end{aligned}$$

(Geiser, *Crelle-Borchardt*, t. LXVII. (1867), pp. 78—90; the complete form, as just stated, and which was obtained by Mr Cotterill, has not I believe been published): and also Geiser's theorem "the locus of the coincident eighth and ninth points is a sextic passing twice through each of the seven points."

The sextic and the curve n intersect in $6n$ points, among which are included the seven points counting as $2\Sigma a$ points: the number of the remaining points is $=6n-2\Sigma a$. Similarly, the sextic and the curve ν intersect in 6ν points, among which are included the seven points counting as $2\Sigma\alpha$ points: the number of the remaining points is $6\nu-2\Sigma\alpha$ $(=6n-2\Sigma a)$. The points in question are, it is clear, common intersections of the sextic, and the curves n, ν: viz. of the intersections of the curves n, ν, a number $6n-2\Sigma a$, $=6\nu-2\Sigma\alpha$, $=3n+3\nu-\Sigma a-\Sigma\alpha$ lie on the sextic.

The curves n, ν intersect in $n\nu$ points, among which are included the seven points counting $\Sigma a\alpha$ times: the number of the remaining intersections is therefore

$n\nu - \Sigma a\alpha$, but among these are included the $3n + 3\nu - \Sigma a - \Sigma\alpha$ points on the sextic; omitting these, there remain $n\nu - 3(n+\nu) - \Sigma a\alpha + \Sigma a + \Sigma\alpha$ points, or, what is the same thing, $(n-3)(\nu-3) - \Sigma(a-1)(\alpha-1) - 2$ points: it is clear that these must form pairs such that, the eighth point being either point of a pair, the ninth point will be the remaining point of the pair: the number of pairs is of course

$$\tfrac{1}{2}[(n-3)(\nu-3) - \Sigma(a-1)(\alpha-1) - 2],$$

and we have thus the solution of the question, given the seven points to determine the number of pairs of points on the curve n (or on the curve ν) such that each pair may form with the seven points an ennead.

In particular, if $n = 6$; $a_1, a_2, a_3, a_4, a_5, a_6, a_7 = 2, 2, 2, 2, 2, 1, 1$ respectively, viz. if the curve be a sextic having 5 of the points for dps, and the remaining two for simple points, then we find $\nu = 12$; $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 = 4, 4, 4, 4, 4, 5, 5$ respectively, and the number of pairs is

$$= \tfrac{1}{2}[3 \,.\, 9 - 5(2-1)(4-1) - 2], = \tfrac{1}{2}(27 - 15 - 2), = 5,$$

viz. starting with the 5 dps and any 2 points of the group G_4^1 we can, in 5 different ways, determine the remaining 2 points of the group.

In reference to the number $3p-3$ of parameters in the curves belonging to a given value of p, it may be remarked as follows. Such a curve is rationally transformable into a curve of the order $p+1$ with $\tfrac{1}{2}p(p-3)$ dps, and therefore containing $\tfrac{1}{2}(p+1)(p+4)$, $-\tfrac{1}{2}p(p-3)$, $= 4p+2$ parameters. Employing an arbitrary homographic transformation to establish any assumed relations between the parameters, the number is diminished to $4p+2-8$, $= 4p-6$; and again employing a rational transformation by means of adjoint curves of the order $p-2$ drawn through the dps and $p-3$ points of the curve—thereby transforming the curve into one of the same order $p+1$ and deficiency p—then, assuming that the $p-3$ parameters (or constants on which depend the positions of the $p-3$ points) can be disposed of so as to establish $p-3$ relations between the parameters and so further diminish the number by $p-3$, the required number of parameters will finally be $4p-6-(p-3) = 3p-3$.

Cambridge, 26th October, 1874.

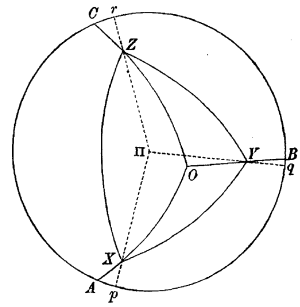
614.

ON A PROBLEM OF PROJECTION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 19—29.]

I MEASURE off on three rectangular axes the distances $\Omega X = \Omega Y = \Omega Z, = \theta$; and then, in a plane through Ω drawing in arbitrary directions the three lines ΩA, ΩB, ΩC, $= a, b, c$ respectively, I assume that A, B, C (fig. 1) are the parallel projections of X, Y, Z respectively; viz. taking ΩO as the direction of the projecting lines, then ΩA, ΩB, ΩC being given in position and magnitude, we have to find θ, and the position of the line ΩO.

Fig. 1.



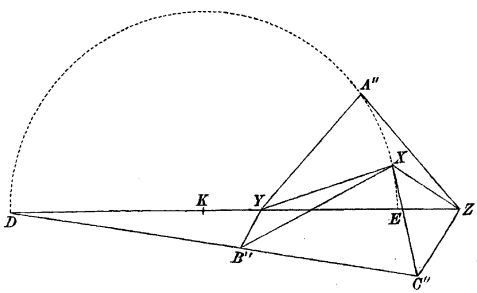
This is in fact a case of a more general problem solved by Prof. Pohlke in 1853, (see the paper by Schwarz, "Elementarer Beweis des Pohlke'schen Fundamentalsatzes der Axonometrie," *Crelle*, t. LXIII. (1864), pp. 309—314), viz. the three lines ΩX, ΩY, ΩZ may be any three axes given in magnitude and direction, and their parallel projection

is to be similar to the three lines ΩA, ΩB, ΩC. Schwarz obtains a very elegant construction, which I will first reproduce. We may imagine through Ω a plane cutting at right angles the projecting lines, say in the points X', Y', Z'; we have then in plano a triad of lines $\Omega X'$, $\Omega Y'$, $\Omega Z'$ which are an orthogonal projection of ΩX, ΩY, ΩZ; and are also an orthogonal projection of a plane triad similar to ΩA, ΩB, ΩC; quà such last-mentioned projection, the triangles $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, must be proportional to the triangles ΩBC, ΩCA, ΩAB; that is, we have to find an orthogonal projection of ΩX, ΩY, ΩZ, such that the triangles $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, which are the projections of ΩYZ, ΩZX, ΩXY respectively, shall be in given ratios. There is no difficulty in the solution of this problem; referring everything to a sphere centre Ω, let the normals to the planes ΩYZ, ΩZX, ΩXY, meet the sphere in the points X'', Y'', Z'' respectively, and the projecting line through Ω meet the sphere in the point O, then the projection of ΩYZ is to ΩYZ as $\cos OX'' : 1$; and the like as to the projections of ΩZX and ΩXY; that is, in the given spherical triangle $X''Y''Z''$, we have to find a point O, such that the cosines of the distances OX'', OY'', OZ'' are in given ratios: we have at once, through X'', Y'', Z'' respectively, three arcs meeting in the required point O.

The projecting lines being thus obtained, say these are the three parallel lines X', Y', Z', we have next to draw through Ω a plane meeting these in the points A', B', C' such that the triangle $A'B'C'$ is similar to the given triangle ABC; for this being so, the triangles $\Omega B'C'$, $\Omega C'A'$, $\Omega A'B'$ being the projections of, and therefore proportional to $\Omega Y'Z'$, $\Omega Z'X'$, $\Omega X'Y'$, that is, proportional to ΩBC, ΩCA, ΩAB, will, it is clear, be similar to these triangles respectively; that is, we have the triad $\Omega A'$, $\Omega B'$, $\Omega C'$, a projection of ΩX, ΩY, ΩZ, and similar to the triad ΩA, ΩB, ΩC, which is what was required.

It remains only to show how the given three parallel lines X', Y', Z', not in the same plane, can be cut by a plane in a triangle similar to a given triangle ABC.

Fig. 2.



Imagine the three lines at right angles to the plane of the paper, meeting the plane of the paper in the given points X, Y, Z (fig. 2) respectively. On the base

YZ describe a triangle $A''YZ$ similar to the given triangle ABC; and through A'', X with centre on the line YZ, describe a circle meeting this line in the points D and E. Then in the plane, through YZ at right angles to the plane of the paper, we may draw a line meeting the lines Y, Z in the points B'', C'' respectively, such that joining XB'', XC'' we obtain a triangle $XB''C''$ similar to $A''YZ$, that is, to the given triangle ABC.

Taking K the centre of the circle, suppose that its radius is $=l$, and that we have $KY=\beta$, $KZ=\gamma$; also $YX=\sigma$, $ZX=\tau$; $YA''=\sigma''$, $ZA''=\tau''$. If for a moment x, y denote the coordinates of X, then

$$\sigma^2=(x-\beta)^2+y^2,\ =l^2+\beta^2-2\beta x,$$
$$\tau^2=(x-\gamma)^2+y^2,\ =l^2+\gamma^2-2\gamma x,$$

and thence

$$\gamma\sigma^2-\beta\tau^2=\gamma(l^2+\beta^2)-\beta(l^2+\gamma^2),$$

that is,

$$\gamma\sigma^2-\beta\tau^2=(\gamma-\beta)(l^2-\beta\gamma);$$

viz. this is the equation of the circle in terms of the vectors σ, τ; we have therefore in like manner

$$\gamma\sigma''^2-\beta\tau''^2=(\gamma-\beta)(l^2-\beta\gamma).$$

We may determine θ so as to satisfy the two equations

$$\sigma''^2=\sigma^2\cos^2\theta+(l+\beta)^2\sin^2\theta,$$
$$\tau''^2=\tau^2\cos^2\theta+(l+\gamma)^2\sin^2\theta;$$

in fact, these equations give

$$\gamma\sigma''^2-\beta\tau''^2=(\gamma\sigma^2-\beta\tau^2)\cos^2\theta+\{\gamma(l^2+\beta^2)-\beta(l^2+\gamma^2)\}\sin^2\theta,$$

which, the left-hand side and the coefficients of $\cos^2\theta$, and $\sin^2\theta$ on the right-hand side being each $=(\gamma-\beta)(l^2-\beta\gamma)$, is, in fact, an identity.

But in the figure, if θ, determined as above, denote the angle at D, then

$$(XB'')^2=XY^2+YB''^2=\sigma^2+(l+\beta)^2\tan^2\theta,$$
$$(ZC'')^2=XZ^2+ZC''^2=\tau^2+(l+\gamma)^2\tan^2\theta,$$

that is,

$$XB''=\sigma''\sec\theta,\ ZC''=\tau''\sec\theta,$$

or, since $B''C''=YZ\sec\theta\,\{=(\gamma-\beta)\sec\theta\}$, the triangle $XB''C''$ is, as mentioned, similar to the triangle $A''YZ$.

I was not acquainted with the foregoing construction when my paper was written; but the analytical investigation of the particular case is nevertheless interesting, and I proceed to consider it.

Taking (fig. 1) Ω as the centre of a sphere and projecting on this sphere, we have A, B, C given points on a great circle; and we have to find the point O, such

that there may be a trirectangular triangle XYZ, the vertices of which lie in OA, OB, OC respectively, and for which

$$\frac{\sin OX}{\sin OA}=\frac{a}{\theta},\quad \frac{\sin OY}{\sin OB}=\frac{b}{\theta},\quad \frac{\sin OZ}{\sin OC}=\frac{c}{\theta}.$$

I take the arcs BC, CA, $AB=\alpha$, β, γ respectively, $\alpha+\beta+\gamma=2\pi$; and the required arcs OA, OB, OC are taken to be ξ, η, ζ respectively; these are connected by the relation

$$\sin\alpha\cos\xi+\sin\beta\cos\eta+\sin\gamma\cos\zeta=0,$$

to obtain which, observe that from the triangles OAB, OAC, we have

$$\cos A=\frac{\cos\eta-\cos\xi\cos\gamma}{\sin\xi\sin\gamma}=-\frac{\cos\zeta-\cos\xi\cos\beta}{\sin\xi\sin\beta},$$

that is,

$$\sin\beta(\cos\eta-\cos\xi\cos\gamma)+\sin\gamma(\cos\zeta-\cos\xi\cos\beta)=0,$$

which, with $\sin\alpha=-\sin(\beta+\gamma)$, gives the required relation. We have

$$\sin OX=\frac{a}{\theta}\sin\xi,\ \sin OY=\frac{b}{\theta}\sin\eta,\ \sin OZ=\frac{c}{\theta}\sin\zeta;$$

and then from the triangles OBC, OCA, OAB, and the quadrantal triangles OYZ, OZX, OXY, we have

$$\cos BOC=\frac{\cos\alpha-\cos\eta\cos\zeta}{\sin\eta\sin\zeta}=-\frac{\sqrt{\left(1-\frac{b^2}{\theta^2}\sin^2\eta\right)}\sqrt{\left(1-\frac{c^2}{\theta^2}\sin^2\zeta\right)}}{\frac{bc}{\theta^2}\sin\eta\sin\zeta},\ \&c.;$$

that is,

$$bc(\cos\alpha-\cos\eta\cos\zeta)=-\sqrt{(\theta^2-b^2\sin^2\eta)}\sqrt{(\theta^2-c^2\sin^2\zeta)},$$
$$ca(\cos\beta-\cos\zeta\cos\xi)=-\sqrt{(\theta^2-c^2\sin^2\zeta)}\sqrt{(\theta^2-a^2\sin^2\xi)},$$
$$ab(\cos\gamma-\cos\xi\cos\eta)=-\sqrt{(\theta^2-a^2\sin^2\xi)}\sqrt{(\theta^2-b^2\sin^2\eta)},$$

which, when rationalized, are quadric equations in $\cos\xi$, $\cos\eta$, $\cos\zeta$. The first equation, in fact, gives

$$b^2c^2(\cos\alpha-\cos\eta\cos\zeta)^2=(\theta^2-b^2+b^2\cos^2\eta)(\theta^2-c^2+c^2\cos^2\zeta),$$

that is,

$$(\theta^2-b^2)(\theta^2-c^2)-b^2c^2\cos^2\alpha+(\theta^2-b^2)c^2\cos^2\zeta+(\theta^2-c^2)b^2\cos^2\eta+2b^2c^2\cos\alpha\cos\eta\cos\zeta=0,$$

or, what is the same thing,

$$-\left(1-\frac{b^2c^2\cos^2\alpha}{b^2-\theta^2\,.\,c^2-\theta^2}\right)+\frac{c^2}{c^2-\theta^2}\cos^2\zeta+\frac{b^2}{b^2-\theta^2}\cos^2\eta-\frac{2b^2c^2}{(b^2-\theta^2)(c^2-\theta^2)}\cos\alpha\cos\eta\cos\zeta=0.$$

Completing the system, we have

$$-\left(1-\frac{c^2a^2\cos^2\beta}{c^2-\theta^2\,.\,a^2-\theta^2}\right)+\frac{a^2}{a^2-\theta^2}\cos^2\xi+\frac{c^2}{c^2-\theta^2}\cos^2\zeta-\frac{2c^2a^2}{(c^2-\theta^2)(a^2-\theta^2)}\cos\beta\cos\zeta\cos\xi=0,$$

$$-\left(1-\frac{a^2b^2\cos^2\gamma}{a^2-\theta^2\,.\,b^2-\theta^2}\right)+\frac{b^2}{b^2-\theta^2}\cos^2\eta+\frac{a^2}{a^2-\theta^2}\cos^2\xi-\frac{2a^2b^2}{(a^2-\theta^2)(b^2-\theta^2)}\cos\gamma\cos\xi\cos\eta=0,$$

and, as above,

$$\sin\alpha\cos\xi + \sin\beta\cos\eta + \sin\gamma\cos\zeta = 0.$$

It seems difficult from these equations to eliminate ξ, η, ζ, so as to obtain an equation in θ; but I employ some geometrical considerations.

Taking Π as the pole of the circle ABC, and drawing ΠX, ΠY, ΠZ to meet the circle in p, q, r respectively, then, if α'', β'', γ'' are the cosine-inclinations of O to X, Y, Z respectively, we have

$$\sin Xp,\quad \sin Yq,\quad \sin Zr = \alpha'',\ \beta'',\ \gamma''.$$

From the right parallel triangles BYq and CZr, we have

$$\sin Yq = \sin BY \sin B,$$
$$\sin Zr = \sin CZ \sin C,$$

and, thence,

$$\frac{\sin Yq}{\sin Zr} = \frac{\sin BY}{\sin CZ}\cdot\frac{\sin OC}{\sin OB};$$

or, since

$$BY = OB - OY,\quad CZ = OC - OZ,$$

and thence

$$\sin BY = \frac{\sin\eta}{\theta}\{\sqrt{(\theta^2 - b^2\sin^2\eta)} - b\cos\eta\},$$

$$\sin CZ = \frac{\sin\zeta}{\theta}\{\sqrt{(\theta^2 - c^2\sin^2\zeta)} - c\cos\zeta\},$$

we obtain

$$\frac{\beta''}{\gamma''} = \frac{\sqrt{(\theta^2 - b^2\sin^2\eta)} - b\cos\eta}{\sqrt{(\theta^2 - c^2\sin^2\zeta)} - c\cos\zeta}.$$

We have thence

$$\beta''\sqrt{(\theta^2 - c^2\sin^2\zeta)} - \gamma''\sqrt{(\theta^2 - b^2\sin^2\eta)} = \beta'' c\cos\zeta - \gamma'' b\cos\eta,$$

or, squaring and reducing

$$\beta''^2(\theta^2 - c^2) + \gamma''^2(\theta^2 - b^2) + 2\beta''\gamma''\{-\sqrt{(\theta^2 - c^2\sin^2\zeta)}\sqrt{(\theta^2 - b^2\sin^2\eta)} + bc\cos\eta\cos\zeta\} = 0.$$

that is,

$$\beta''^2(\theta^2 - c^2) + \gamma''^2(\theta^2 - b^2) + 2\beta''\gamma''\,.\,bc\cos\alpha = 0;$$

and, similarly,

$$\gamma''^2(\theta^2 - a^2) + \alpha''^2(\theta^2 - c^2) + 2\gamma''\alpha''\,.\,ca\cos\beta = 0,$$
$$\alpha''^2(\theta^2 - b^2) + \beta''^2(\theta^2 - a^2) + 2\alpha''\beta''\,.\,ab\cos\gamma = 0,$$

or, what is the same thing,

$$\frac{\beta''^2}{b^2 - \theta^2} + \frac{\gamma''^2}{c^2 - \theta^2} - \frac{2bc\cos\alpha}{b^2 - \theta^2\,.\,c^2 - \theta^2}\beta''\gamma'' = 0,$$

$$\frac{\gamma''^2}{c^2 - \theta^2} + \frac{\alpha''^2}{a^2 - \theta^2} - \frac{2ca\cos\beta}{c^2 - \theta^2\,.\,a^2 - \theta^2}\gamma''\alpha'' = 0,$$

$$\frac{\alpha''^2}{a^2 - \theta^2} + \frac{\beta''^2}{b^2 - \theta^2} - \frac{2ab\cos\gamma}{a^2 - \theta^2\,.\,b^2 - \theta^2}\alpha''\beta'' = 0;$$

writing

$$\alpha'',\ \beta'',\ \gamma'' = X\sqrt{(a^2-\theta^2)},\quad Y\sqrt{(b^2-\theta^2)},\quad Z\sqrt{(c^2-\theta^2)},$$

and

$$\frac{bc\cos\alpha}{\sqrt{(b^2-\theta^2 . c^2-\theta^2)}},\ \frac{ca\cos\beta}{\sqrt{(c^2-\theta^2 . a^2-\theta^2)}},\ \frac{ab\cos\gamma}{\sqrt{(a^2-\theta^2 . b^2-\theta^2)}} = f,\ g,\ h,$$

the equations are

$$Y'^2 + Z'^2 - 2fY'Z' = 0,$$
$$Z'^2 + X'^2 - 2gZ'X' = 0,$$
$$X'^2 + Y'^2 - 2hX'Y' = 0.$$

Writing the last two under the form

$$X'^2 - 2gZ'X' + Z'^2 = 0,$$
$$X'^2 - 2hY'X' + Y'^2 = 0,$$

and eliminating X', we have

$$-4(1-g^2)(1-h^2)Y'^2Z'^2 + (Y'^2 + Z'^2 - 2ghY'Z')^2 = 0,$$

which, in virtue of the first equation, is

$$-4(1-g^2)(1-h^2)Y'^2Z'^2 + 4(gh-f)^2 Y'^2Z'^2 = 0,$$

that is,

$$(1-g^2)(1-h^2) - (gh-f)^2 = 0;$$

or, what is the same thing,

$$1 - f^2 - g^2 - h^2 + 2fgh = 0.$$

I remark that we may write

$$gh - f = \sqrt{(1-g^2)}\sqrt{(1-h^2)},$$
$$hf - g = \sqrt{(1-h^2)}\sqrt{(1-f^2)},$$
$$fg - h = \sqrt{(1-f^2)}\sqrt{(1-g^2)},$$

the signs on the right-hand side being either all +, or else one + and two −, so that the product is +. In fact, multiplying the assumed equations, we have

$$f^2g^2h^2 - fgh(f^2+g^2+h^2) + g^2h^2 + h^2f^2 + f^2g^2 - fgh = 1 - f^2 - g^2 - h^2 + g^2h^2 + h^2f^2 + f^2g^2 - f^2g^2h^2,$$

that is,

$$1 - f^2 - g^2 - h^2 + fgh(1 + f^2 + g^2 + h^2) - 2f^2g^2h^2 = 0,$$

or,

$$(1 - f^2 - g^2 - h^2 + 2fgh)(1 - fgh) = 0,$$

which is right; but with a different combination of signs the result would not have been obtained.

Substituting for f, g, h their values, we have

$$(a^2-\theta^2)(b^2-\theta^2)(c^2-\theta^2) - b^2c^2(a^2-\theta^2)\cos^2\alpha - c^2a^2(b^2-\theta^2)\cos^2\beta$$
$$- a^2b^2(c^2-\theta^2)\cos^2\gamma + 2a^2b^2c^2\cos\alpha\cos\beta\cos\gamma = 0,$$

where the term independent of θ is

$$a^2b^2c^2(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma),$$

which is $=0$ in virtue of $\alpha+\beta+\gamma=2\pi$. We have, therefore, for θ^2 the quadric equation

$$b^2c^2\sin^2\alpha+c^2a^2\sin^2\beta+a^2b^2\sin^2\gamma-(a^2+b^2+c^2)\,\theta^2+\theta^4=0,$$

giving for θ^2 the two real positive values

$$\theta^2=\tfrac{1}{2}\{a^2+b^2+c^2\pm\sqrt{(\Omega)}\},$$

where

$$\begin{aligned}\Omega^2&=(a^2+b^2+c^2)^2-4\,(b^2c^2\sin^2\alpha+c^2a^2\sin^2\beta+a^2b^2\sin^2\gamma)\\&=a^4+b^4+c^4+2b^2c^2\cos 2\alpha+2c^2a^2\cos 2\beta+2a^2b^2\cos 2\gamma\\&=(a^2+b^2\cos 2\gamma+c^2\cos 2\beta)^2+(b^2\sin 2\gamma-c^2\sin 2\beta)^2.\end{aligned}$$

I write now

$$\frac{a\cos\xi}{\sqrt{(a^2-\theta^2)}},\quad \frac{b\cos\eta}{\sqrt{(b^2-\theta^2)}},\quad \frac{c\cos\zeta}{\sqrt{(c^2-\theta^2)}}=X,\ Y,\ Z,$$

and also

$$\frac{\sqrt{a^2-\theta^2}}{a}\sin\alpha,\quad \frac{\sqrt{b^2-\theta^2}}{b}\sin\beta,\quad \frac{\sqrt{c^2-\theta^2}}{c}\sin\gamma=A,\ B,\ C.$$

The equations for $\cos\xi$, $\cos\eta$, $\cos\zeta$ become

$$\begin{aligned}Y^2+Z^2-2fYZ-(1-f^2)&=0,\\Z^2+X^2-2gZX-(1-g^2)&=0,\\X^2+Y^2-2hXY-(1-h^2)&=0,\end{aligned}$$

and

$$AX+BY+CZ=0,$$

in virtue of the relation between f, g, h. The first three equations are satisfied by a two-fold relation between X, Y, Z; viz. treating these as coordinates, the equations represent three quadric cylinders having a common conic.

To prove this, I write

$$1-f^2,\ 1-g^2,\ 1-h^2,\ gh-f,\ hf-g,\ fg-h=\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}.$$

We have, as usual,

$$\mathrm{bc}-\mathrm{f}^2,\ \mathrm{ca}-\mathrm{g}^2,\ \mathrm{ab}-\mathrm{h}^2,\ \mathrm{gh}-\mathrm{af},\ \mathrm{hf}-\mathrm{bg},\ \mathrm{fg}-\mathrm{ch},\ \text{each }=0:$$

the equations

$$\mathrm{a}X+\mathrm{h}Y+\mathrm{g}Z=0,\quad \mathrm{h}X+\mathrm{b}Y+\mathrm{f}Z=0,\quad \mathrm{g}X+\mathrm{f}Y+\mathrm{c}Z=0,$$

represent each of them one and the same plane, which I say is that of the conic in question.

The three given equations are

$$Y^2 + Z^2 - 2fYZ \quad - \mathrm{a} = 0,$$
$$Z^2 + X^2 - 2gZX \quad - \mathrm{b} = 0,$$
$$X^2 + Y^2 - 2hXY - \mathrm{c} = 0,$$

say these are $U=0$, $V=0$, $W=0$; it is to be shown that $\mathrm{c}V - \mathrm{b}W$, $\mathrm{a}W - \mathrm{c}U$, $\mathrm{b}U - \mathrm{a}V$, each contain the linear factor in question. We have

$$\mathrm{c}V - \mathrm{b}W = (\mathrm{c} - \mathrm{b})X^2 - \mathrm{b}Y^2 + \mathrm{c}Z^2 - 2\mathrm{c}gZX + 2\mathrm{b}hXY;$$

or, what is the same thing,

$$\mathrm{a}(\mathrm{c}V - \mathrm{b}W) = \mathrm{a}(\mathrm{c} - \mathrm{b})X^2 - \mathrm{h}^2Y^2 + \mathrm{g}^2Z^2 - 2g\mathrm{g}^2ZX + 2h\mathrm{h}^2XY.$$

Assuming this

$$= (\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z)(\lambda X - \mathrm{h}Y + \mathrm{g}Z),$$

we have

$$\mathrm{a}\lambda = \quad \mathrm{a}(\mathrm{c} - \mathrm{b}),$$
$$\mathrm{g}(\quad \mathrm{a} + \lambda) = -2g\mathrm{g}^2,$$
$$\mathrm{h}(-\mathrm{a} + \lambda) = \quad 2h\mathrm{h}^2,$$

that is,

$$\lambda = \mathrm{c} - \mathrm{b}, \quad \mathrm{a} + \lambda = -2g\mathrm{g}, \quad -\mathrm{a} + \lambda = 2h\mathrm{h}:$$

but $\lambda = \mathrm{c} - \mathrm{b}$, $= -\mathrm{h}^2 + \mathrm{g}^2$, and the other two equations are $\mathrm{a} + \mathrm{c} - \mathrm{b} + 2g\mathrm{g} = 0$, $\mathrm{a} + \mathrm{b} - \mathrm{c} + 2h\mathrm{h} = 0$, which are identically true.

The values of X, Y, Z are thus determined as the coordinates of the intersection of the conic with the plane $AX + BY + CZ = 0$; or, what is the same thing, of the line

$$AX + BY + CZ = 0,$$
$$\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z = 0,$$

with any one of the three cylinders.

We may, however, complete the analytical solution in a different manner as follows:

Assuming as above $\sqrt{(\mathrm{bc})} = \mathrm{f}$, $\sqrt{(\mathrm{ca})} = \mathrm{g}$, $\sqrt{(\mathrm{ab})} = \mathrm{h}$, and thence $\mathrm{h}\sqrt{(\mathrm{c})} - \mathrm{g}\sqrt{(\mathrm{b})} = 0$, we obtain from the second and the third equations

$$Y = hX + \sqrt{(\mathrm{c})}\sqrt{(1 - X^2)}, \quad Z = gX - \sqrt{(\mathrm{b})}\sqrt{(1 - X^2)},$$

(the signs are one $+$ the other $-$, in order that this may consist with the equation $\mathrm{a}X + \mathrm{h}Y + \mathrm{g}Z = 0$). Substituting in $AX + BY + CZ = 0$, we have

$$(A + Bh + Cg)X + \{B\sqrt{(\mathrm{c})} - C\sqrt{(\mathrm{b})}\}\sqrt{(1 - X^2)} = 0,$$

that is,

$$(A + Bh + Cg)^2X^2 - (B^2\mathrm{c} + C^2\mathrm{b} - 2BC\mathrm{f})(1 - X^2) = 0,$$

or say

$$(A+Bh+Cg)^2X^2+\{B^2(1-h^2)+C^2(1-g^2)-2BC(gh-f)\}(X^2-1)=0,$$

that is,

$$(A^2+B^2+C^2+2BCf+2CAg+2ABh)X^2=\{B^2+C^2+2BCf-(Bh+Cg)^2\},$$

or writing

$$A^2+B^2+C^2+2BCf+2CAg+2ABh=\Delta,$$

say we have

$$\Delta X^2=B^2+C^2+2BCf-(Bh+Cg)^2,$$
$$\Delta Y^2=C^2+A^2+2CAg-(Cf+Ah)^2,$$
$$\Delta Z^2=A^2+B^2+2ABh-(Ag+Bf)^2.$$

Now attending to the values of A, B, C, f, g, h, we have

$$BCf,\ CAg,\ ABh=\sin\beta\sin\gamma\cos\alpha,\quad \sin\gamma\sin\alpha\cos\beta,\quad \sin\alpha\sin\beta\cos\gamma,$$

and thence

$$\Delta=\sin^2\alpha\left(1-\frac{\theta^2}{a^2}\right)+\sin^2\beta\left(1-\frac{\theta^2}{b^2}\right)+\sin^2\gamma\left(1-\frac{\theta^2}{c^2}\right)$$
$$+2(\sin\beta\sin\gamma\cos\alpha+\sin\gamma\sin\alpha\cos\beta+\sin\alpha\sin\beta\cos\gamma);$$

in virtue of $\alpha+\beta+\gamma=2\pi$, the last term is

$$=2(\cos\alpha\cos\beta\cos\gamma-1),$$

whence

$$\Delta=-\theta^2\left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),\text{ say this is }=-\theta^2\Lambda.$$

Moreover $Bh+Cg=\dfrac{a\sin\alpha}{\sqrt{(a^2-\theta^2)}}$, whence the value of ΔX^2 is

$$=\sin^2\beta\left(1-\frac{\theta^2}{b^2}\right)+\sin^2\gamma\left(1-\frac{\theta^2}{c^2}\right)+2\sin\beta\sin\gamma\cos\alpha-\left(1-\frac{\theta^2}{a^2}\right)\sin^2\alpha.$$

Here the constant term is

$$=\sin^2\beta+\sin^2\gamma+2\sin\beta\sin\gamma\cos\alpha,$$

that is,

$$=1-(1-\sin^2\beta)(1-\sin^2\gamma)+\sin^2\beta\sin^2\gamma+2\sin\beta\sin\gamma\cos\alpha$$
$$=1-\cos^2\beta\cos^2\gamma-\cos^2\alpha+(\cos\alpha+\sin\beta\sin\gamma)^2$$
$$=1-\cos^2\alpha,\ =\sin^2\alpha,$$

or the whole is

$$\sin^2\alpha\left(1-\frac{a^2}{a^2-\theta^2}\right)-\theta^2\left(\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),$$

which is

$$= -\theta^2\left(\frac{\sin^2\alpha}{a^2-\theta^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right),$$

so that we have

$$\Lambda X^2 = \left(\frac{\sin^2\alpha}{a^2-\theta^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right).$$

Similarly,

$$\Lambda Y^2 = \left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2-\theta^2}+\frac{\sin^2\gamma}{c^2}\right),$$

$$\Lambda Z^2 = \left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2-\theta^2}\right);$$

and hence also

$$\Lambda(1-X^2) = \frac{-\theta^2\sin^2\alpha}{a^2(a^2-\theta^2)},\quad \Lambda(1-Y^2) = \frac{-\theta^2\sin^2\beta}{b^2(b^2-\theta^2)},\quad \Lambda(1-Z^2) = \frac{-\theta^2\sin^2\gamma}{c^2(c^2-\theta^2)},$$

where

$$\Lambda = \frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}.$$

The equation in X is

$$\Lambda\left(1-\frac{a^2\cos^2\xi}{a^2-\theta^2}\right) = \frac{-\theta^2\sin^2\alpha}{a^2(a^2-\theta^2)},$$

that is,

$$\Lambda(a^2\sin^2\xi-\theta^2) = -\theta^2\frac{\sin^2\alpha}{a^2},$$

or

$$a^2\sin^2\xi = \theta^2\left(1-\frac{\sin^2\alpha}{a^2\Lambda}\right),$$

and the like for η, ζ. Writing for greater convenience $\frac{\sin^2\alpha}{a^2}$, $\frac{\sin^2\beta}{b^2}$, $\frac{\sin^2\gamma}{c^2} = p, q, r$, then $\Lambda = p+q+r$, and we have

$$\sin^2\xi = \frac{\theta^2}{a^2}\frac{q+r}{p+q+r},\quad \sin^2\eta = \frac{\theta^2}{b^2}\frac{r+p}{p+q+r},\quad \sin^2\zeta = \frac{\theta^2}{c^2}\frac{p+q}{p+q+r},$$

(whence also $a^2\sin^2\xi+b^2\sin^2\eta+c^2\sin^2\zeta = 2\theta^2$: as a simple verification, observe that, if the projection is rectangular, the axes being all equally inclined to the plane of projection, then $\xi=\eta=\zeta=90°$, $a=b=c=\theta\sin s$, and the equation is $3\sin^2 s = 2$; s, s are here the sides of an isosceles quadrantal triangle, the included angle being 120°, that is, we have $\cos 120° (= -\frac{1}{2}) = -\cot^2 s$, that is, $\cot^2 s = \frac{1}{2}$, or $\sin^2 s = \frac{2}{3}$, which is right).

I remark, that a geometrical solution may be obtained upon very different principles. We have on a sphere the trirectangular triangle XYZ, which by parallel lines is projected into ABC. Every great circle of the sphere is projected into an ellipse having double

contact at the extremities of a diameter with the ellipse which is the apparent contour of the sphere. Moreover, if the arc of great circle XY is a quadrant, then the radius through X and the tangent at Y are parallel to each other, whence, if Ω be the projection of the centre, and AB the projection of the arc XY, then in the projection the line ΩA and the tangent at B are parallel to each other. It is now easy to derive a construction: with centre Ω, and conjugate semi-axes $(\Omega B, \Omega C)$, $(\Omega C, \Omega A)$, $(\Omega A, \Omega B)$ respectively, describe three ellipses; and find a concentric ellipse having double contact with each of these (there are in fact two such ellipses, one touching the three ellipses internally, and giving an imaginary solution; the other touching them externally, which is the ellipse intended). Drawing then through the ellipse a right cylinder (there are two such cylinders, but only one of them is real), and inscribing in it a sphere, and projecting on to the surface of the sphere by lines parallel to the axis of the cylinder, the three ellipses are projected into three great circles cutting at right angles, or, say, the elliptic arcs BC, CA, AB are projected into the trirectangular triangle XYZ.

615.

ON THE CONIC TORUS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 127—129.]

THE equation $p+\sqrt{(qr)}+\sqrt{(st)}=0$, where p, q, r, s, t are linear functions of the coordinates (x, y, z, w), and as such are connected by a linear relation, belongs to a quartic surface having a nodal conic $(p=0, qr-st=0)$; and four nodes (conical points), viz. these are the intersections of the line $q=0$, $r=0$ with the quadric surface $p^2-qr-st=0$, and of the line $r=0$, $s=0$ with the same surface. The quartic surface has also four tropes (planes which touch the surface along a conic); viz. these are the planes $q=0$, $r=0$, $s=0$, $t=0$, the conic of contact or tropal conic in each plane being the intersection of the plane with the before-mentioned quadric surface $p^2-qr-st=0$. The planes $q=0$, $r=0$, and also the conics in these planes pass through two of the nodes, say A, C; and the planes $s=0$, $t=0$, and also the conics in these planes pass through the remaining two nodes, say B, D; so that the relations of the surface are as is shown in fig. 1. It is to be added that AB, BC, CD, DA (but not AC or BD) are lines on the surface.

The planes $q=0$, $r=0$, which contain the tropal conics through A, C, are in general distinct from the planes ABC, ADC which contain the line-pairs BA, BC and DA, DC respectively: and so also the planes $s=0$, $t=0$, which contain the tropal conics through B, D, are in general distinct from the planes ABD, CBD which contain the line-pairs AB, AD and CB, CD respectively.

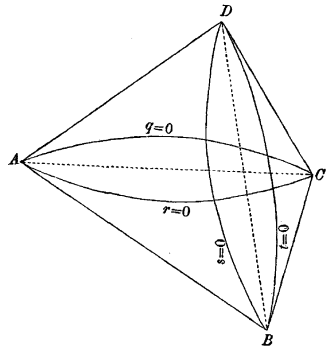
If, however, the identical linear relation contain only p, s, t, then the planes $q=0$, $r=0$ will be the planes ABC, ADC respectively: and the tropal conics in these planes will consequently be the line-pairs BA, BC, and DA, DC respectively. But the planes $s=0$, $t=0$ will continue to be distinct from the planes ABD, CBD: and the tropal conics in the planes $s=0$, $t=0$ will remain proper conics.

A surface of the last-mentioned form is

$$mz + \sqrt{(xy)} + \sqrt{(w^2 - z^2)} = 0,$$

viz. this has the nodal conic $z=0$, $xy - w^2 = 0$, the nodes $\{x=0, y=0, (m^2+1)z^2 - w^2 = 0\}$, and $(z=0, w=0, x=0)$, $(z=0, w=0, y=0)$, and the tropes $x=0$, $y=0$, $z+w=0$, $z-w=0$; but the planes $z+w=0$ and $z-w=0$ are ordinary tropal planes each touching the surface in a proper conic; the planes $x=0$, $y=0$ special planes each touching along a line-pair.

Fig. 1.



The equation in question, writing therein $w=1$ and $x+iy$, $x-iy$ in place of (x, y) respectively, is

$$\{\sqrt{(x^2+y^2)} + mz\}^2 = 1 - z^2,$$

which is derived from

$$(x+mz)^2 = 1 - z^2,$$

by the change of x into $\sqrt{(x^2+y^2)}$; and the surface is consequently the torus generated by the rotation of the conic $(x+mz)^2 = 1 - z^2$ about its diameter. Or, what is the same thing, the surface

$$mz + \sqrt{(xy)} + \sqrt{(w^2 - z^2)} = 0,$$

regarding therein (x, y) as circular coordinates and w as being $=1$, is a torus. The rational equation is $U=0$, where we have

$$\begin{aligned} U &= \{(m^2+1)z^2 - w^2 + xy\}^2 - 4m^2z^2xy \\ &= \{xy + (1-m^2)z^2 - w^2\}^2 + 4m^2z^2(z^2 - w^2) \\ &= x^2y^2 + (m^2+1)^2z^4 + w^4 + (2-2m^2)z^2xy - (2+2m^2)z^2w^2 - 2xyw^2. \end{aligned}$$

I find that the Hessian H of this function U contains the factor $xy + (1-m^2)z^2 - w^2$, viz. that we have

$$H = \{xy + (1-m^2)z^2 - w^2\}H',$$

where

$$\begin{aligned} H' = \; & x^2y^2\,(1-m^2) \\ & + x^2y^2\{(3+8m^2+m^4)\,z^2+(-3+m^2)\,w^2\} \\ & + xy\;\{(3+11m^2+9m^4+m^6)\,z^4+(-6-12m^2+6m^4)\,z^2w^2+(3+m^2)\,w^4\} \\ & + \quad (1+m^2)\{(1-m^2)\,z^2-w^2\}\{(1+m^2)\,z^2-w^2\}^2, \end{aligned}$$

giving without much difficulty

$$\begin{aligned} H' = \; & z^6\,(1+m^2)^2(1-m^2) \\ & + 2z^4[(1+4m^2+m^4)\,xy-(1-m^4)\,w^2](1+m^2) \\ & + \; z^2\,(xy-w^2)\,[(1+12m^2-m^4)\,xy-(1-m^4)\,w^2] \\ & + \quad [(1-m^2)\,xy-(1+m^2)\,w^2]\,U; \end{aligned}$$

say this is

$$= z^2H'' + [(1-m^2)\,xy-(1+m^2)\,w^2]\,U,$$

where

$$\begin{aligned} H'' = \; & z^4(1+m^2)^3\;(1-m^2) \\ & + 2z^2(1+m^2)\;[(1+\;4m^2+m^4)\,xy-(1-m^4)\,w^2] \\ & + \quad (xy-w^2)\,[(1+12m^2-m^4)\,xy-(1-m^4)\,w^2], \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} H'' = \; & x^2y^2(1+12m^2-m^4) \\ & + 2xy\,[(1+4m^2+m^4)(1+m^2)\,z^2+(-1+6m^2)\,w^2] \\ & + \quad (1-m^4)\{(1+m^2)\,z^2-w^2\}^2. \end{aligned}$$

It consequently appears that the complete spinode curve or intersection of the quartic surface and its Hessian, being of order 4×8, $=32$, breaks up into

$$U=0,\; xy+(1-m^2)\,z^2-w^2=0,$$

that is,

conic $z=0,\; xy-w^2=0$ twice,	order	4
conic $z+w=0,\; xy-m^2w^2=0,$	„	2
conic $z-w=0,\; xy-m^2w^2=0,$	„	2

and

$$U=0,\; z^2H''=0,$$

that is,

$U=0,\; z^2=0,$ conic $z=0,\; xy-w^2=0$ four times,	„	8
proper spinode curve $U=0,\; H''=0,$	„	16
		32;

viz. the intersection is made up of the conic $z=0,\; xy-w^2=0$ six times, the conics $z\pm w=0,\; xy-m^2w^2=0$ each twice, and the proper spinode curve of the order 16.

616.

A GEOMETRICAL ILLUSTRATION OF THE CUBIC TRANSFORMATION IN ELLIPTIC FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 211—216.]

CONSIDER the cubic curve

$$x^3 + y^3 + z^3 + 6lxyz = 0.$$

If through one of the inflexions $z=0$, $x+y=0$, we draw an arbitrary line $z = u(x+y)$, we have at the other intersections of this line with the curve

$$u\{u^2(x+y)^2 + 6lxy\} + x^2 - xy + y^2 = 0;$$

that is,

$$(u^3+1)(x^2+y^2) + 2xy(u^3 + 3lu - \tfrac{1}{2}) = 0;$$

and from this equation it appears that the ratio $x : y$ is given as a function involving the square root of

$$(u^3 + 3lu - \tfrac{1}{2})^2 - (u^3+1)^2,$$

which, rejecting a factor 3, is

$$= (2u^3 + 3lu + \tfrac{1}{2})(lu - \tfrac{1}{2}).$$

It may be noticed that $lu - \frac{1}{2} = 0$ gives the value of u, which in the equation $z = u(x+y)$ belongs to the tangent at the inflexion; and $2u^3 + 3lu + \frac{1}{2} = 0$ gives the values which belong to the three tangents from the inflexion.

It thus appears that the coordinates x, y, z of any point of the curve can be expressed as proportional to functions of u involving the radical

$$\sqrt{\{(lu - \tfrac{1}{2})(2u^3 + 3lu + \tfrac{1}{2})\}},$$

and the theory of the curve is connected with that of a quasi-elliptic integral depending on this radical.

Taking ω an imaginary cube root of unity, write

$$\begin{aligned} \omega x + \omega^2 y - 2lz &= x', \\ \omega^2 x + \omega y - 2lz &= y', \\ x + y - 2lz &= z'; \end{aligned}$$

then we have

$$x'y'z' = x^3 + y^3 - 8l^3z^3 + 6lxyz = x^3 + y^3 + z^3 + 6lxyz - (1 + 8l^3)\, z^3.$$

Also

$$-6lz = x' + y' + z', \quad z^3 = \frac{-1}{216l^3}(x' + y' + z')^3,$$

whence

$$(x' + y' + z')^3 - \frac{216l^3}{1 + 8l^3}\, x'y'z' = \frac{216l^3}{1 + 8l^3}(x^3 + y^3 + z^3 + 6lxyz);$$

so that, putting

$$m^3 = \frac{-l^3}{1 + 8l^3},$$

or, what is the same thing,

$$8l^3m^3 + l^3 + m^3 = 0,$$

the equation of the curve is

$$(x' + y' + z')^3 + 216m^3x'y'z' = 0;$$

and if we write

$$x' : y' : z' = X^3 : Y^3 : Z^3,$$

then the original curve is transformed into

$$(X^3 + Y^3 + Z^3)^3 + 216m^3X^3Y^3Z^3 = 0,$$

a curve of the ninth order breaking up into three cubic curves, one of which is

$$X^3 + Y^3 + Z^3 + 6mXYZ = 0,$$

and for the other two we write herein $m\omega$ and $m\omega^2$ respectively in place of m. Attending only to the first curve, we have

$$\begin{aligned} x^3 + y^3 + z^3 + 6lxyz &= 0, \\ X^3 + Y^3 + Z^3 + 6mXYZ &= 0, \end{aligned}$$

as corresponding curves, the corresponding points being connected by the relation

$$\omega x + \omega^2 y - 2lz : \omega^2 x + \omega y - 2lz : x + y - 2lz = X^3 : Y^3 : Z^3,$$

or, for convenience, we may write

$$\begin{aligned} \omega x + \omega^2 y - 2lz &= X^3, \quad \text{giving} \quad & 3x &= \omega^2 X^3 + \omega Y^3 + Z^3, \\ \omega^2 x + \omega y - 2lz &= Y^3, & 3y &= \omega X^3 + \omega^2 Y^3 + Z^3, \\ x + y - 2lz &= Z^3, & -6lz &= X^3 + Y^3 + Z^3. \end{aligned}$$

This is a (1, 3) correspondence; viz. to a given point on the curve (m), there corresponds one point on (l); but to a given point on (l), three points on (m). As to the first case, this is obvious. As to the second case, if the point (x, y, z) is given, then the corresponding point (X, Y, Z) on the other curve will lie on one of the three lines

$$Y^3(\omega x + \omega^2 y - 2lz) - X^3(\omega^2 x + \omega y - 2lz) = 0;$$

each of these intersects the curve (m) in three points: but of the points in the same line it is only one which is a corresponding point of (x, y, z), and the number of the corresponding points is consequently the same as the number of lines, viz. it is $=3$.

We infer that the above equations lead to a cubic transformation of the quasi-elliptic integral

$$\int du \div \sqrt{\{(lu - \tfrac{1}{2})(2u^3 + 3lu + \tfrac{1}{2})\}},$$

into one of the like form

$$\int dv \div \sqrt{\{(mv - \tfrac{1}{2})(2v^3 + 3mv + \tfrac{1}{2})\}};$$

and this is now to be verified.

We have, as before, the line $z = u(x+y)$ meeting the curve (l) in the points

$$(u^3+1)(x^2+y^2) + 2xy(u^3 + 3lu - \tfrac{1}{2}) = 0;$$

and if similarly through an inflexion of the curve (m) we take the line $Z = v(X+Y)$, this meets the curve in the points

$$(v^3+1)(X^2+Y^2) + 2XY(v^3 + 3mv - \tfrac{1}{2}) = 0.$$

Then if (x, y, z), (X, Y, Z) are taken to be the corresponding points as above, we can obtain v as a function of u. We, in fact, have

$$-2lu = \frac{-2lz}{x+y} = \frac{X^3+Y^3+Z^3}{-X^3-Y^3+2Z^3} = \frac{X^3+Y^3+v^3(X+Y)^3}{-(X^3+Y^3)+2v^3(X+Y)^3}$$

$$= \frac{X^2-XY+Y^2+v^3(X+Y)^2}{-X^2+XY-Y^2+2v^3(X+Y)^2},$$

$$= \frac{(v^3+1)(X^2+Y^2)+(2v^3-1)XY}{(2v^3-1)(X^2+Y^2)+(4v^3+1)XY};$$

or, since we have

$$(v^3+1)(X^2+Y^2) + 2XY(v^3+3mu-\tfrac{1}{2}) = 0,$$

that is,

$$X^2+Y^2 : XY = -2v^3 - 6mv + 1 : v^3+1,$$

the equation becomes

$$-2lu = \frac{-6mv(v^3+1)}{(2v^3-1)(-2v^3-6mv+1)+(4v^3+1)(v^3+1)}$$

$$= \frac{-6mv(v^3+1)}{-3v(4mv^3-3v^2-2m)};$$

or say,

$$-lu = m(v^3+1)(\div), \text{ where the denominator } = 4mv^3 - 3v^2 - 2m.$$

This may also be written

$$-(lu - \tfrac{1}{2}) = 3v^2(mv - \tfrac{1}{2}) \div .$$

Proceeding to calculate $2u^3 + 3lu + \frac{1}{2}$, omitting the denominator $(4mv^3 - 3v^2 - 2m)^3$, this is

$$-\frac{2m^3}{l^3}(v^3+1)^3 - 3m(v^3+1)(4mv^3 - 3v^2 - 2m)^2 + \tfrac{1}{2}(4mv^3 - 3v^2 - 2m)^3;$$

or, observing that

$$m^3 = \frac{-l^3}{1+8l^3},$$

that is,

$$l^3 = \frac{-m^3}{1+8m^3} \text{ or } -\frac{m^3}{l^3} = 1 + 8m^3,$$

the numerator is

$$= 2(1+8m^3)(v^3+1)^3 - 3m(v^3+1)(4mv^3 - 3v^2 - 2m)^2 + \tfrac{1}{2}(4mv^3 - 3v^2 - 2m)^3,$$

which is found to be identically

$$= (2v^3 + 3mv + \tfrac{1}{2})(v^3 + 6mv - 2)^2;$$

viz. we have

$$2u^3 + 3lu + \tfrac{1}{2} = (2v^3 + 3mv + \tfrac{1}{2})(v^3 + 6mv - 2)^2 \div (4mv^3 - 3v^2 - 2m)^3,$$

and hence

$$(lu - \tfrac{1}{2})(2u^3 + 3lu + \tfrac{1}{2}) = -3(mv - \tfrac{1}{2})(2v^3 + 3mv + \tfrac{1}{2})(v^3 + 6mv - 2)^2 v^2 \div (4mv^3 - 3v^2 - 2m)^4.$$

Moreover, we find

$$ldu = 3mdv \,.\, v(v^3 + 6mv - 2) \div (4mv^3 - 3v^2 - 2m)^2,$$

and we thence have

$$\frac{ldu}{\sqrt{\{(lu - \frac{1}{2})(2u^3 + 3lu + \frac{1}{2})\}}} = \sqrt{(-3)}\,\frac{mdv}{\sqrt{\{(mv - \frac{1}{2})(2v^3 + 3mv + \frac{1}{2})\}}};$$

viz. this differential equation corresponds to the integral equation

$$-lu = m(v^3+1) \div (4mv^3 - 3v^2 - 2m),$$

where $8l^3m^3 + l^3 + m^3 = 0$, which corresponds to the modular equation.

It may be remarked that, if v is the same function of u', l, m that u is of v, m, l; viz. if

$$-mv = l(u'^3 + 1) \div (4lu'^3 - 3u'^2 - 2m'),$$

then

$$\frac{mdv}{\sqrt{\{(mv - \frac{1}{2})(2v^3 + 3mv + \frac{1}{2})\}}} = \sqrt{(-3)}\,\frac{-ldu'}{\sqrt{\{(lu' - \frac{1}{2})(2u'^3 + 3lu' + \frac{1}{2})\}}},$$

and consequently

$$\frac{du}{\sqrt{\{(lu - \frac{1}{2})(2u^3 + 3lu + \frac{1}{2})\}}} = \frac{-3du'}{\sqrt{\{(lu' - \frac{1}{2})(2u'^3 + 3lu' + \frac{1}{2})\}}},$$

which accords with the general theory of the cubic transformation.

We may inquire into the relation between the absolute invariants of the two curves. Taking the absolute invariant to be

$$\Omega = \frac{64S^3 - T^2}{64S^3},$$

where S and T bear the usual significations, we have for the one curve

$$\Omega = \frac{(1+8l^3)^3}{64l^3(1-l^3)^3},$$

and for the other curve

$$\Omega' = \frac{(1+8m^3)^3}{64m^3(1-m^3)^3},$$

and, as above, $8l^3m^3 + l^3 + m^3 = 0$: writing herein

$$l^3 = -\frac{1}{8\alpha'},\ m^3 = -\frac{1}{8\beta'},$$

the relation between α', β' is simply $\alpha' + \beta' = 1$; and the values of Ω, Ω' are found to be

$$\Omega = \frac{64\alpha'(1-\alpha')^3}{(1+8\alpha')^3},\ \Omega' = \frac{64\beta'(1-\beta')^3}{(1+8\beta')^3};$$

viz. the required relation is given by the elimination of α', β' from these three equations. Or, what is the same thing, writing $\alpha' = \frac{1}{2} + \theta$, and therefore $\beta' = \frac{1}{2} - \theta$, we have

$$(5+8\theta)^3\,\Omega = 4(1+2\theta)\,(1-2\theta)^3,$$
$$(5-8\theta)^3\,\Omega' = 4(1+2\theta)^3(1-2\theta),$$

and the elimination of θ from these equations gives the required relation between Ω and Ω'.

It of course follows that, if we have a *cubic* transformation

$$\frac{dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)\}^4}} = \frac{C\,dx'}{\sqrt{\{(a',\ b',\ c',\ d',\ e' \between x',\ 1)\}^4}},$$

then the absolute invariants Ω, Ω' of the two quartic functions are connected by the above relation. I have obtained this result, by reducing the radicals to the standard forms

$$\sqrt{(1-x^2 \,.\, 1-k^2x^2)},\ \sqrt{(1-x'^2 \,.\, 1-\lambda^2x'^2)},$$

from the known modular equation as represented by the equations

$$\lambda^2 = \frac{\alpha^3(2+\alpha)}{1+2\alpha},\ k^2 = \frac{\alpha(2+\alpha)^3}{(1+2\alpha)^3};$$

viz. the values of the absolute invariants

$$\left(=1 - \frac{27J^2}{I^3},\ 1 - \frac{27J'^2}{I'^3}\right),$$

are

$$\Omega = \frac{108k^2(1-k^2)^4}{(k^4+14k^2+1)^3},\ \Omega' = \frac{108\lambda^2(1-\lambda^2)^4}{(\lambda^4+14\lambda^2+1)^3},$$

but the method of effecting this is by no means obvious.

617.

ON THE SCALENE TRANSFORMATION OF A PLANE CURVE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 321—328.]

THE transformation by reciprocal radius vectors can be effected mechanically by Sylvester's Peaucellier-cell. But, employing a more general cell (considered incidentally by him) which may be called the scalene-cell, we have the scalene transformation in question*; viz. if, in two curves, r, r' are radius vectors belonging to the same angle (or say opposite angles) θ, then the relation between r, r' is

$$rr'(r+r')+(m^2-l^2)r+(m^2-n^2)r'=0;$$

or, as this may also be written,

$$r^2+\left(r'-\frac{l^2-m^2}{r'}\right)r+m^2-n^2=0.$$

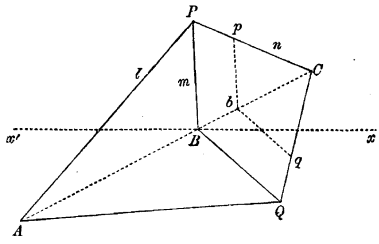
The transformation is, it will be seen, an interesting one for its own sake, independently of the remarkably simple mechanical construction, viz. the scalene cell is simply a system of 3 pairs of equal rods PA, QA; PB, QB; PC, QC (fig. 1, p. 528), jointed together at and capable of rotating about the points P, Q, A, B, C; the three lengths PA, PB, PC (say these are $=l$, m, n) are all of them unequal: in the case of any two of them equal, we have Peaucellier or isosceles cell. The effect of the arrangement is that the points A, B, C are retained in a right line, the distances BA, $=r'$, and BC, $=r$, being connected by the above-mentioned equation; so that taking B as a fixed point, if the point A describe any given curve, the point C will describe the corresponding or transformed curve.

In the case where the given curve is a right line or a circle, we may through B draw at right angles to the curve the axis $x'Bx$: viz. in the case of the circle,

* The transformation itself, and doubtless many of the results obtained by means of it, are familiar to Prof. Sylvester; and I abandon all claim to priority.

the axis $x'Bx$ passes through its centre; and we measure the angle θ from this line, viz. we write $\angle xBC = \angle x'BA = \theta$.

Fig. 1.



Suppose, first, that the locus of A is a right line, or a circle passing through B. Its equation is $r' = \dfrac{c}{\cos\theta}$ or $= c\cos\theta$; and we accordingly have for the transformed curve

$$r^2 + \left(\frac{c}{\cos\theta} - \frac{l^2 - m^2}{c\cos\theta}\right) r + m^2 - n^2 = 0,$$

or else

$$r^2 + \left(c\cos\theta - \frac{l^2 - m^2}{c\cos\theta}\right) r + m^2 - n^2 = 0;$$

viz. multiplying in each case by $r\cos\theta$, and then writing $r\cos\theta = x$, $r^2 = x^2 + y^2$, the equations become

$$x(x^2+y^2) + c(x^2+y^2) - \frac{l^2-m^2}{c}x^2 \qquad + (m^2-n^2)x = 0,$$

and

$$x(x^2+y^2) + cx^2 \qquad - \frac{l^2-m^2}{c}(x^2+y^2) + (m^2-n^2)x = 0;$$

viz. in each case the curve is a circular cubic passing through the origin B and having an asymptote parallel to the axis of y. The curve is nodal, if $m = n$, viz. in this case the origin is a node: or if $c = \sqrt{(l^2 - n^2)} + \sqrt{(m^2 - n^2)}$.

Suppose next that the locus of A is a circle, centre at a distance $= \gamma$ along Bx' and radius $= h$: we have

$$r'^2 - 2\gamma r'\cos\theta + \gamma^2 - h^2 = 0,$$

viz. if

$$\gamma^2 - h^2 = -(l^2 - m^2),$$

or, what is the same thing,

$$h^2 + m^2 = \gamma^2 + l^2,$$

then we have

$$r' - \frac{l^2 - m^2}{r'} = 2\gamma\cos\theta,$$

and the transformed curve is

$$r^2 + 2\gamma r \cos\theta + m^2 - n^2 = 0,$$

or, as this may be written,

$$r^2 + 2\gamma r \cos\theta + \gamma^2 - f^2 = 0,$$

where $\gamma^2 - f^2 = m^2 - n^2$, that is, $f^2 + m^2 = \gamma^2 + n^2$; viz. this is a concentric circle radius f.

The theorem may be presented as follows. Consider two concentric circles, centre O and radii h, f respectively; take an arbitrary point B, distance $OB = \gamma$; and taking m arbitrary, determine l, n by the equations

$$l^2 = m^2 + h^2 - \gamma^2,\ n^2 = m^2 + f^2 - \gamma^2;$$

then drawing through B an arbitrary line to meet the circles in A, C respectively; also describing a circle, centre B and radius $= m$; and through O drawing a line perpendicular to ABC to meet the last-mentioned circle in two points P, Q: for these points, the distances from the points A, B, C are $= l, m, n$ respectively.

To verify this, take O as the origin, OB for the axis of x, θ the inclination of ABC to this axis, $BA = r'$, $BC = r$; the coordinates of C, B, A are

$$\begin{array}{lll} \gamma + r\cos\theta, & r\sin\theta, \\ \gamma \quad , & 0 \quad , \\ \gamma + r'\cos\theta, & -r'\sin\theta, \end{array}$$

whence, taking (x, y) for the coordinates of P (or Q), the equations to be verified are

$$\begin{aligned} (x - \gamma - r\cos\theta)^2 + (y - r\sin\theta)^2 &= n^2, \\ (x - \gamma)^2 \qquad\quad + y^2 \qquad\quad &= m^2, \\ (x - \gamma + r'\cos\theta)^2 + (y + r'\sin\theta)^2 &= l^2. \end{aligned}$$

By means of the second equation, the other two become

$$\begin{aligned} -2(x-\gamma)r\cos\theta - 2yr\sin\theta + r^2 &= n^2 - m^2, \\ 2(x-\gamma)r'\cos\theta + 2yr'\sin\theta + r'^2 &= l^2 - m^2; \end{aligned}$$

or, substituting for $n^2 - m^2$, $l^2 - m^2$ the values $f^2 - \gamma^2$ and $h^2 - \gamma^2$, the equations are

$$\begin{aligned} -2xr\cos\theta - 2yr\sin\theta + r^2 + 2\gamma r\cos\theta + \gamma^2 - f^2 &= 0, \\ 2xr'\cos\theta + 2yr'\sin\theta + r'^2 - 2\gamma r'\cos\theta + \gamma^2 - h^2 &= 0, \end{aligned}$$

viz. in virtue of the equations of the two circles, these reduce themselves each of them to

$$x\cos\theta + y\sin\theta = 0,$$

which equation, together with the second equation

$$(x - \gamma)^2 + y^2 = m^2,$$

determine (x, y) as above.

Reverting to the case where the locus of A is the circle

$$r'^2 - 2\gamma r' \cos\theta + \gamma^2 - h^2 = 0,$$

this gives

$$r' = \gamma\cos\theta + \sqrt{(h^2 - \gamma^2\sin^2\theta)},$$

$$\frac{1}{r'} = \frac{\gamma\cos\theta - \sqrt{(h^2 - \gamma^2\sin^2\theta)}}{\gamma^2 - h^2};$$

so that for the transformed curve we have

$$r^2 + r\left(1 - \frac{l^2 - m^2}{\gamma^2 - h^2}\right)\gamma\cos\theta + r\left(1 + \frac{l^2 - m^2}{\gamma^2 - h^2}\right)\sqrt{(h^2 - \gamma^2\sin^2\theta)} + m^2 - n^2 = 0.$$

Putting for shortness $\frac{l^2 - m^2}{\gamma^2 - h^2} = \lambda$, and for r, $r\cos\theta$, $r\sin\theta$, writing $\sqrt{(x^2+y^2)}$, x, y respectively, this is

$$x^2 + y^2 + (1-\lambda)\gamma x + (1+\lambda)\sqrt{\{h^2(x^2+y^2) - \gamma^2 y^2\}} + m^2 - n^2 = 0,$$

or, what is the same thing,

$$\{x^2 + y^2 + (1-\lambda)\gamma x + m^2 - n^2\}^2 = (1+\lambda)^2\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

a bicircular quartic. In the case $\lambda = -1$, it reduces itself to the circle

$$x^2 + y^2 + 2\gamma x + m^2 - n^2 = 0$$

twice, which is the case considered above; and in the case $\lambda = 1$, or $l^2 + h^2 = m^2 + \gamma^2$, the equation is

$$(x^2 + y^2 + m^2 - n^2)^2 = 4\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

so that the curve is symmetrical in regard to each axis. In the case $\gamma = 0$, the locus is a pair of concentric circles, centre B.

The equation

$$\{x^2 + y^2 + (1-\lambda)\gamma x + m^2 - n^2\}^2 = (1+\lambda)^2\{h^2(x^2+y^2) - \gamma^2 y^2\},$$

which contains the four constants λ, γ, h and $m^2 - n^2$, may be written in the form

$$(x^2 + y^2 + Ax + B)^2 = ax^2 + ey^2,$$

(where the constants A, B, a, e are also arbitrary). This is, in fact, the equation of the general symmetrical bicircular quartic, referred to a properly-selected point on the axis as origin, viz. the origin is the centre of any one of the three involutions formed by the vertices (or points on the axis); say it is any one of the three involution-centres of the curve.

To show this, assume

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4 - px^3 + qx^2 - rx + s:$$

then, taking B arbitrary, the equation of the symmetrical bicircular quartic having for vertices the points $x=\alpha$, $x=\beta$, $x=\gamma$, $x=\delta$, is

$$(x^2+y^2-\tfrac{1}{2}px+B)^2=(2B+\tfrac{1}{4}p^2-q)\,x^2+(r-pB)\,x+(-s+B^2);$$

in fact, this is the form of the general equation, and writing therein $y=0$, it becomes $x^4-px^3+qx^2-rx+s=0$, that is, $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$. Hence, writing for convenience

$$\begin{aligned} A &= -\tfrac{1}{2}p, \\ a &= 2B+\tfrac{1}{4}p^2-q, \\ b &= r-pB, \\ c &= -s+B^2, \end{aligned}$$

the equation is

$$(x^2+y^2+Ax+B)^2=ax^2+bx+c.$$

This may be written

$$(x^2+y^2+Ax+B+\theta)^2=(a+2\theta)\,x^2+2\theta y^2+(b+2\theta A)\,x+c+2B\theta+\theta^2,$$

viz. assuming $\theta=-\dfrac{b}{2A}$ in order on the right-hand side to destroy the term in x, the equation is

$$\left(x^2+y^2+Ax+B-\frac{b}{2A}\right)^2=\left(a-\frac{b}{A}\right)x^2-\frac{b}{A}\,y^2+\frac{1}{4A^2}(b^2-4ABb+4A^2c),$$

which is of the form

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2+f;$$

and if $f=0$, that is, if $b^2-4ABb+4A^2c=0$, then it is of the required form

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2.$$

We have

$$\begin{aligned} b^2-4ABb+4A^2c &= (r-pB)^2+2pB\,(r-pB)+p^2\,(-s+B^2) \\ &= r^2-p^2s, \end{aligned}$$

or the required condition is $r^2-p^2s=0$. But we have

$$p^2s-r^2=(\alpha\delta-\beta\gamma)\,(\beta\delta-\gamma\alpha)\,(\gamma\delta-\alpha\beta),$$

as is easily verified by writing

$$p=\delta+p_0,\quad q=\delta p_0+q_0,\quad r=\delta q_0+r_0,\quad s=\delta r_0,$$

where p_0, q_0, r_0 stand for

$$\alpha+\beta+\gamma,\quad \beta\gamma+\gamma\alpha+\alpha\beta,\quad \alpha\beta\gamma,$$

respectively. The required condition thus is

$$(\alpha\delta-\beta\gamma)(\beta\delta-\gamma\alpha)(\gamma\delta-\alpha\beta)=0,$$

viz. the origin (that is, the fixed point B of the cell) must be at one of the three involution-centres.

Comparing the equation

$$\{x^2+y^2+(1-\lambda)\gamma x+m^2-n^2\}^2=(1+\lambda)^2\{h^2(x^2+y^2)-\gamma^2y^2\}$$

with the equation

$$\{x^2+y^2+Ax+B\}^2=ax^2+ey^2,$$

we have

$$A=(1-\lambda)\gamma,$$

$$B=m^2-n^2,$$

$$a=(1+\lambda)^2h^2,$$

$$e=(1+\lambda)^2(h^2-\gamma^2),$$

and thence $a-e=(1+\lambda)^2\gamma^2$. Consequently $\dfrac{a-e}{A^2}=\left(\dfrac{1+\lambda}{1-\lambda}\right)^2$, which gives λ: and then $h^2=\dfrac{a}{(1+\lambda)^2}$, $\gamma^2=\dfrac{a-e}{(1+\lambda)^2}$, $m^2-n^2=B$; viz. we thus have the values of λ, h, γ and m^2-n^2 for the description of a given curve $(x^2+y^2+Ax+B)^2=ax^2+ey^2$. In order that the description may be possible, a and $a-e$ must be each of them positive.

For the Cartesian a is $=e$, whence $1+\lambda=0$, and the equation becomes

$$(x^2+y^2+2\gamma x+m^2-n^2)^2=0,$$

which is a twice repeated circle; hence the Cartesian cannot be constructed by means of a cell as above.

To obtain a construction of the Cartesian, it may be remarked that, if a symmetrical bicircular quartic be inverted in regard to an axial focus, viz. if the focus be taken as the centre of inversion, we obtain a Cartesian. The axial foci of the curve

$$(x^2+y^2+Ax+B)^2=ax^2+ey^2$$

are points on the axis, the abscissa $x=\theta$ being determined by the equation

$$e(\theta^2+A\theta+B)^2-a(\theta^2-B)^2-ae\theta^2=0.$$

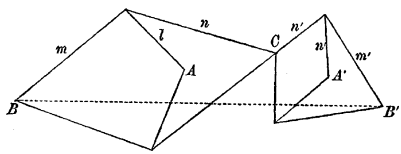
The equation referred to a focus as origin is therefore

$$\{x^2+y^2+(A+2\theta)x+B+\theta^2\}^2=ax^2+ey^2+2a\theta x+\theta^2;$$

then inverting, viz. for x, y writing $\dfrac{k^2x}{r^2}$, $\dfrac{k^2y}{r^2}$ (k arbitrary), we have, as may be verified the equation of a Cartesian.

The inversion can be performed mechanically by an ordinary Peaucellier-cell; the complete apparatus for the construction of a Cartesian is therefore as in fig. 2, viz. we have a cell BAC as before, B a fixed point, locus of A a circle (for convenience of drawing, the arrangement has been made BAC instead of ABC), and we connect with C a Peaucellier-cell $CA'B'$, arms n', n', m', the fixed point B' being on the axis, which is the line joining B with the centre of the circle described by A. This being so, then A describing a circle, C will describe a symmetrical bicircular quartic, and A' will describe the inverse of this, being in general a like curve; but if the position of B' be properly determined, viz. if B' be at a focus of the first-mentioned quartic,

Fig. 2.



then A' will describe a Cartesian. A further investigation would be necessary in order to determine how to adapt the apparatus to the description of a given Cartesian.

A more convenient mechanical description of a Cartesian is, however, that given in the paper which follows the present one [618].

The equation

$$\{x^2+y^2+(1-\lambda)\gamma x+m^2-n^2\}^2=(1+\lambda)^2\{h^2(x^2+y^2)-\gamma^2y^2\}$$

may also be written

$$\{x^2+y^2+(1-\lambda)\gamma x-\tfrac{1}{2}(1+\lambda)^2(h^2-\gamma^2)+m^2-n^2\}^2$$
$$=(1+\lambda)^2\{\gamma^2x^2-(1-\lambda)(h^2-\gamma^2)\gamma x+\tfrac{1}{4}(1+\lambda)^2(h^2-\gamma^2)^2-(m^2-n^2)(h^2-\gamma^2)\},$$

viz. the equation is now brought into the form

$$(x^2+y^2+Ax+B)^2=ax^2+bx+c.$$

Expressing the coefficients A, B, a, b, c in terms of λ, γ, h, m^2-n^2, it appears by what precedes, that we should have identically $b^2-4ABb+4A^2c=0$, viz. this is the equation which expresses that the origin is an involution-centre.

If, instead of the original cell, we consider a new cell obtained by substituting for the arms PB, BQ, the arms pb, bq, jointed on to the points p, q on the arms CP, CQ respectively, and instead of B, making b the fixed point; then writing $C\mathrm{p}=kn$, $\mathrm{pb}=km$, so that the parameters of the cell are l, m, n, k, and taking $C\mathrm{b}=s$, $\mathrm{b}A=s'$,

we have $s = kr$, $s + s' = r + r'$, that is, $r = \frac{s}{k}$, $r' = \frac{k-1}{k} s + s'$. Substituting in the equation between r, r', written for greater convenience in the form

$$(r + r')(rr' + m^2 - l^2) + (l^2 - n^2) r' = 0,$$

the relation between s, s' is found to be

$$(s + s')\left(\frac{k-1}{k^2} s^2 + \frac{ss'}{k} + m^2 - l^2\right) + (l^2 - n^2)\left(\frac{k-1}{k} s + s'\right) = 0.$$

On account of the term in s^3, this equation in its general form does not, it would appear, give rise to transformations of much elegance. If, however, $l = n$, then the relation becomes

$$(k-1) s^2 + kss' + k^2 (m^2 - l^2) = 0;$$

and in particular, if $k = 2$, then

$$s^2 + 2ss' = 4(l^2 - m^2), \text{ or say } (s + s')^2 - s'^2 = 4(l^2 - m^2),$$

viz. taking A instead of b as the fixed point, the relation between the radii AC, Ab is $\rho^2 - \rho'^2 = 4(l^2 - m^2)$; the cell is in this case Sylvester's "quadratic-binomial extractor."

618.

ON THE MECHANICAL DESCRIPTION OF A CARTESIAN.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 328—330.]

SUPPOSE that in two different curves the radius vectors r, r', which belong to the same angle θ, are connected by the equation

$$r^2 + \left(Mr' + N + \frac{P}{r'}\right) r + B = 0;$$

then, taking one of the curves to be the circle

$$Mr' + \frac{P}{r'} = A \cos\theta,$$

the other curve is

$$r^2 + (A \cos\theta + N) r + B = 0,$$

viz. this is a Cartesian. It perhaps would not be difficult to contrive a mechanical arrangement to connect the radius vectors in accordance with the foregoing equation; but the required result may be obtained equally well by means of a particular case of the relation in question; viz. taking this to be

$$r^2 + (-r' + N) r + B = 0,$$

then, taking the one curve to be the circle $r' = -A \cos\theta$, the other curve is the Cartesian,

$$r^2 + (2l \cos\theta + B) r + D = 0, \text{ that is, } r^2 + (A \cos\theta + N) r + B = 0.$$

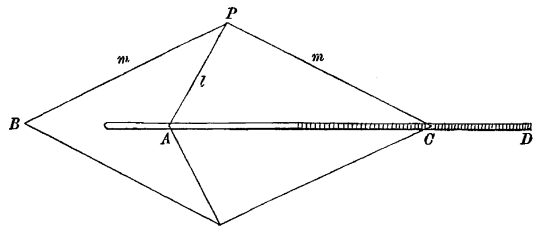
The relation between the radius vectors may in this case be written

$$r' = N + r + \frac{B}{r},$$

which can be constructed mechanically by a simple addition to the Peaucellier-cell, viz. if we joint on to C (fig. 1, p. 536) a rod CDA, having a slot, working on a pin at A, so that the rod is thereby kept always in the line BAC, then, making B the

fixed point and taking $BA = r$, we have $AC = \frac{m^2 - l^2}{r}$, whence $BC = r + \frac{m^2 - l^2}{r}$, or D being a point at the distance $CD, = \alpha$, from the point C, and denoting BD by r', we have $r' = r + \frac{m^2 - l^2}{r} + \alpha$, which is an equation of the required form; whence, if the point D describe a circle passing through B, then the point A will describe a Cartesian.

Fig. 1.



The equation of the Cartesian is $r^2 + (A \cos\theta + N) r + B = 0$, viz. this is $x^2 + y^2 + Ax + B = -N\sqrt{(x^2 + y^2)}$, or writing $N^2 = a$, it is

$$(x^2 + y^2 + Ax + B)^2 = ax^2 + ay^2,$$

which is the form considered in the preceding paper. It may be further observed in regard to it that, starting from the focal equation $r = ls + m$, where r, s are the distances of a point (x, y) of the Cartesian from any two of its three foci, this equation gives $r^2 - l^2s^2 + m^2 = 2mr$, or writing $r^2 = x^2 + y^2$, $s^2 = (x - \alpha)^2 + y^2$, the function on the left-hand is of the form $(1 - l^2)(x^2 + y^2 + Ax + B)$, whence, assuming $\frac{2m}{1 - l^2} = \sqrt{(a)}$, the equation becomes as above

$$(x^2 + y^2 + Ax + B)^2 = a(x^2 + y^2).$$

Taking the distance $r = \sqrt{(x^2 + y^2)}$ to be measured from a given focus, it is easy to see that, no matter which of the other two foci we associate with it, we obtain the *same* equation $(x^2 + y^2 + Ax + B)^2 = a(x^2 + y^2)$; viz. starting with any one focus, we connect with it a determinate circle $x^2 + y^2 + Ax + B = 0$, and a determinate coefficient a, such that taking this focus as the origin, the equation of the curve is

$$(x^2 + y^2 + Ax + B)^2 = a(x^2 + y^2);$$

but there are for the given curve three such forms of equation, according as the origin is taken at one or other of the three foci.

(Addition, Feb. 1875.) It is obviously the same thing, but I find that it is mechanically more convenient to derive the Cartesian from the Limaçon $r' = -N - A\cos\theta$, by the transformation $r' = r + \frac{B}{r}$: I have on this principle constructed an apparatus whereby the Cartesian is described on a rotating board by a pencil moving in a fixed line.

619.

ON AN ALGEBRAICAL OPERATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIII. (1875), pp. 369—375.]

I CONSIDER

$$\Omega F(a, x),$$

an operation Ω performed upon $F(a, x)$ a rational function of (a, x); viz. F being first expanded or regarded as expanded in ascending powers of a, the coefficients of the several powers are then to be expanded or regarded as expanded in ascending powers of x, and the operation consists in the rejection of all negative powers of x.

In the cases intended to be considered, F contains only positive powers of a: but this restriction is not necessary to the theory.

The investigation has reference to the functions $A(x)$ of my "Ninth Memoir on Quantics," *Phil. Trans.*, t. CLXI. (1871), pp. 17—50, [462]; for instance, we there have as regards the covariants of a quadric

$$A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right) = \frac{1 - x^{-2}}{1 - ax^2 \,.\, 1 - a \,.\, 1 - ax^{-2}},$$

and consequently, in the present notation,

$$A(x) \qquad = \Omega \frac{1 - x^{-2}}{1 - ax^2 \,.\, 1 - a \,.\, 1 - ax^{-2}};$$

by a process of development and summation, the value of this expression was found to be

$$= \frac{1}{1 - ax^2 \,.\, 1 - a^2},$$

and in the other more complicated cases the value of $A(x)$ was found only by trial and verification. What I purpose now to show is that the operation Ω can be performed without any development in an infinite series; or say that it depends on finite algebraical operations only.

It is clear that if $F(a, x)$, considered to be developed as above contains only positive powers of x, then

$$\Omega F(a, x) = F(a, x);$$

and if it contains only negative powers of x, then $\Omega F(a, x) = 0$.

Consider now $\Omega \dfrac{\phi(x)}{x-a}$, where $\phi(x)$ is a function containing only positive powers of x; we have

$$\frac{\phi(x)}{x-a} = \frac{\phi(x)-\phi(a)}{x-a} + \frac{\phi(a)}{x-a},$$

and thence

$$\Omega \frac{\phi(x)}{x-a} = \Omega \frac{\phi(x)-\phi(a)}{x-a} + \Omega \frac{\phi(a)}{x-a}$$

$$= \frac{\phi(x)-\phi(a)}{x-a},$$

since $\dfrac{\phi(x)-\phi(a)}{x-a}$ is a rational and integral function of a, which when developed contains only positive powers of x, and $\dfrac{\phi(a)}{x-a}$ when developed contains only negative powers of x.

Consider next $\Omega \dfrac{\phi(x)}{x^2-a}$, where $\phi(x)$ is a rational and integral function of x; writing this $= f(x^2) + xg(x^2)$, we have

$$\Omega \frac{\phi(x)}{x^2-a} = \Omega \frac{f(x^2)}{x^2-a} + \Omega \frac{xg(x^2)}{x^2-a}$$

$$= \frac{f(x^2)-f(a)}{x^2-a} + \frac{x\{g(x^2)-g(a)\}}{x^2-a}.$$

As regards the last term, notice that

$$\Omega \frac{xg(x^2)}{x^2-a} = \Omega \frac{x\{g(x^2)-g(a)\}}{x^2-a} + \Omega \frac{xg(a)}{x^2-a},$$

in which $\dfrac{x\{g(x^2)-g(a)\}}{x^2-a}$ is a rational and integral function of (a, x), and therefore when developed contains only positive powers of x, while $\dfrac{xg(a)}{x^2-a}$ when developed contains only negative powers of x.

We thus have

$$\Omega \frac{\phi(x)}{x^2-a} = \frac{f(x^2)+xg(x^2)-f(a)-xg(a)}{x^2-a}$$

$$= \frac{\phi(x)-f(a)-xg(a)}{x^2-a}.$$

Similarly, if $\phi(x) = f(x^2) + xg(x^2) + x^2h(x^2)$, then

$$\Omega \frac{\phi(x)}{x^2 - a} = \frac{\phi(x) - f(a) - xg(a) - x^2h(a)}{x^2 - a};$$

and so on.

Consider now the above-mentioned function

$$A(x), = \Omega \frac{1 - x^{-2}}{1 - ax^2 . 1 - a . 1 - ax^{-2}}.$$

Writing

$$\frac{1 - x^{-2}}{1 - ax^2 . 1 - a . 1 - ax^{-2}} = \frac{P}{1 - ax^2} + \frac{Q}{1 - a} + \frac{R}{1 - ax^{-2}},$$

we have

$$P = \left(\frac{1 - x^{-2}}{1 - a . 1 - ax^{-2}}\right)_{a = x^{-2}}, \quad = \frac{1}{1 - x^{-4}}, \qquad = \frac{-x^4}{1 - x^4},$$

$$Q = \left(\frac{1 - x^{-2}}{1 - ax^2 . 1 - ax^{-2}}\right)_{a = 1}, \quad = \frac{1}{1 - x^2},$$

$$R = \left(\frac{1 - x^{-2}}{1 - ax^2 . 1 - a}\right)_{a = x^2}, \quad = \frac{1 - x^{-2}}{1 - x^4 . 1 - x^2}, \quad = \frac{-1}{x^2(1 - x^4)};$$

that is,

$$\frac{1 - x^{-2}}{1 - ax^2 . 1 - a . 1 - ax^{-2}} = \frac{-x^4}{1 - x^4} \frac{1}{1 - ax^2} + \frac{1}{1 - x^2} \frac{1}{1 - a} - \frac{1}{1 - x^4} \frac{1}{x^2 - a},$$

and thence

$$\Omega \frac{1 - x^{-2}}{1 - ax^2 . 1 - a . 1 - ax^{-2}} = \frac{-x^4}{1 - x^4} \frac{1}{1 - ax^2} + \frac{1}{1 - x^2 . 1 - a} + \Omega \frac{-1}{1 - x^4} \frac{1}{x^2 - a}.$$

Here, as regards the last term,

$$\phi(x^2) = \frac{-1}{1 - x^4}, \quad = f(x^2), \quad g(x^2) = 0,$$

$$f(a) = -\frac{1}{1 - a^2},$$

$$\frac{\phi(x^2) - f(a)}{x^2 - a} = \frac{1}{x^2 - a}\left(\frac{-1}{1 - x^4} + \frac{1}{1 - a^2}\right) = -\frac{x^4 - a^2}{x^2 - a . 1 - x^4 . 1 - a^2} = -\frac{x^2 + a}{1 - x^4 . 1 - a^2},$$

and we have

$$\Omega \frac{1 - x^{-2}}{1 - ax^2 . 1 - a . 1 - ax^{-2}} = \frac{-x^4}{1 - x^4 . 1 - ax^2} + \frac{1}{1 - x^2 . 1 - a} - \frac{x^2 + a}{1 - x^4 . 1 - a^2}.$$

The second term is $= \dfrac{1 + x^2 . 1 + a}{1 - x^4 . 1 - a^2}$: combining this with the third term, the two together are $= \dfrac{1 + ax^2}{1 - x^4 . 1 - a^2}$.

Hence the value is

$$=\frac{1}{1-x^4}\left(\frac{-x^4}{1-ax^2}+\frac{1+ax^2}{1-a^2}\right),$$

which is

$$=\frac{1}{1-ax^2\,.\,1-a^2},$$

being, in fact, the expression for this function when decomposed into partial fraction of the denominators $1-ax^2$ and $1-a^2$ respectively. Hence finally

$$A(x)=\Omega\frac{1-x^{-2}}{1-ax^2\,.\,1-a\,.\,1-ax^{-2}}=\frac{1}{1-ax^2\,.\,1-a^2},$$

as it should be.

For the cubic function, we have

$$A(x)=\Omega\frac{1-x^{-2}}{1-ax^3\,.\,1-ax\,.\,1-ax^{-1}\,.\,1-ax^{-3}};$$

the function operated upon, when decomposed into partial fractions, is

$$=\frac{x^{10}}{1-x^4\,.\,1-x^6}\frac{1}{1-ax^3}-\frac{x^4}{1-x^2\,.\,1-x^4}\frac{1}{1-ax}$$

$$+\frac{x}{1-x^2\,.\,1-x^4}\frac{1}{x-a}+\frac{-x}{1-x^4\,.\,1-x^6}\frac{1}{x^3-a}.$$

Hence we require

$$\Omega\frac{x}{1-x^2\,.\,1-x^4}\frac{1}{x-a}+\Omega\frac{-x}{1-x^4\,.\,1-x^6}\frac{1}{x^3-a}.$$

The first of these is

$$=\frac{1}{x-a}\left\{\frac{x}{1-x^2\,.\,1-x^4}-\frac{a}{1-a^2\,.\,1-a^4}\right\},$$

which is

$$=\frac{1}{1-x^2\,.\,1-x^4\,.\,1-a^2\,.\,1-a^4}\left\{\begin{array}{l}1\\+x\,(a+a^3-a^5)\\+x^2\,(a^2-a^4)\\+x^3\,(a-a^3)\\-x^4a^2\\-x^5a\end{array}\right\}.$$

As regards the second, the function operated on may be expressed in the form

$$\frac{-x^9-x-x^5}{1-x^{12}\,.\,1-x^6}\frac{1}{x^3-a},$$

whence $f(a)$, $g(a)$, $h(a)$, and therefore $f(a)+xg(a)+x^2h(a)$, respectively, are $=-a^3$, -1, $-a$, $-a^2-x-ax^2$, each divided by $1-a^2.1-a^4$; or the term is

$$=\frac{1}{x^3-a}\left\{\frac{-x}{1-x^4.1-x^6}+\frac{a^3+x+ax^2}{1-a^2.1-a^4}\right\},$$

which is

$$=\frac{1}{1-x^4.1-x^6.1-a^2.1-a^4}\left\{\begin{array}{l}-a^2\\ +x(-a-a^3+a^5)\\ +x^2.-1\\ +x^3.-a\\ +x^4.(a^4-1)\\ +x^5.0\\ +x^6.a^2\\ +x^7.a^3\\ +x^8.1\\ +x^9.a\end{array}\right\}.$$

To combine the two terms, we multiply the numerator and denominator of the first by $1+x^2+x^4$, thereby reducing its denominator to $1-x^4.1-x^6.1-a^2.1-a^4$, the denominator of the second term; then the sum of the numerators is found to be

$$\begin{array}{ll}=1-a^2 & \text{viz. this is } =(1-a^2)\left\{\begin{array}{l}1\\ +a^2x^2\\ +(a+a^3)x^3\\ +(a+a^3)x^5\\ +a^2x^6\\ +x^8.\end{array}\right.\\ +x(a^2-a^4) & \\ +x^3(a-a^5) & \\ +x^5(a-a^5) & \\ +x^6(a^2-a^4) & \\ +x^8(1-a^2), & \end{array}$$

Hence we have

$$A(x)=\Omega\frac{1-x^{-2}}{1-ax^3.1-ax.1-ax^{-1}.1-ax^{-3}}$$

$$=\frac{x^{10}}{1-x^4.1-x^6}\frac{1}{1-ax^3}$$

$$+\frac{-x^4}{1-x^2.1-x^4}\frac{1}{1-ax}$$

$$+\frac{1+x^6+(x^3+x^5)a+(x^2+x^6)a^2+(x^3+x^5)a^3}{1-x^4.1-x^6}\frac{1}{1-a^4},$$

which is, in fact, the expression for $\dfrac{1-ax+a^2x^2}{1-ax^3.1-ax.1-a^4}$ decomposed into partial

fractions with the denominators $1-ax^2$, $1-ax$, $1-a^4$ respectively. This is most easily seen by completing the decomposition, viz. we have

$$4\{1+x^6+(x^3+x^5)a+(x^2+x^6)a^2+(x^3+x^5)a^3\}$$

$$=(1+x^3)^2(1+x^2)(1+a)(1+a^2)+(1-x^3)^2(1+x^2)(1-a)(1+a^2)+2(1-x^2)(1-x^6)(1-a^2),$$

and thence the expression is

$$=\frac{x^{10}}{1-x^4\,.\,1-x^6}\frac{1}{1-ax^2}$$

$$+\frac{-x^4}{1-x^2\,.\,1-x^4}\frac{1}{1-ax}$$

$$+\tfrac{1}{4}\frac{1+x^3}{1-x^2\,.\,1-x^3}\frac{1}{1-a}+\tfrac{1}{4}\frac{1-x^3}{1-x^2\,.\,1+x^3}\frac{1}{1+a}+\tfrac{1}{2}\frac{1}{1+x^2}\frac{1}{1+a^2}$$

$$=\frac{1-ax+a^2x^2}{1-ax^3\,.\,1-ax\,.\,1-a^4},$$

as above. Hence finally

$$A(x)=\Omega\frac{1-x^{-2}}{1-ax^3\,.\,1-ax\,.\,1-ax^{-1}\,.\,1-ax^{-3}}$$

$$=\frac{1-ax+a^2x^2}{1-ax^3\,.\,1-ax\,.\,1-a^4}$$

$$=\frac{1+a^3x^3}{1-ax^3\,.\,1-a^2x^2\,.\,1-a^4}$$

$$=\frac{1-a^6x^6}{1-ax^3\,.\,1-a^2x^2\,.\,1-a^3x^3\,.\,1-a^4}.$$

620.

CORRECTION OF TWO NUMERICAL ERRORS IN SOHNKE'S PAPER RESPECTING MODULAR EQUATIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXI. (1876), p. 229.]

IN Sohnke's paper "Aequationes modulares pro transformatione functionum ellipticarum," *Crelle*, t. XVI. (1837), there is, on p. 113, an obvious error in the expression of u^6, viz. the term q^{18} is given with the same numerical coefficient as it had in u^5: this remark was made to me by Mr W. Barrett Davis, who finds that the term of u^6 should be

$$+13569463\, q^{18}.$$

In the expression of u^{18} (*l. c.*, p. 115), I had remarked that, in the coefficient of q^{16}, a figure must have dropped out. Mr Davis has verified this, and finds that the figure omitted is a 1 in the unit's place, and thus that the correct value* is

$$+80177033781\, q^{16}.$$

Cambridge, 26 *October*, 1875.

* [The former of these numbers should replace the number 15063859 in the Table, p. 128 of this volume; the latter has been introduced in the Table, p. 129.]

621.

ON THE NUMBER OF THE UNIVALENT RADICALS C_nH_{2n+1}.

[From the *Philosophical Magazine*, series 5, vol. III. (1877), pp. 34, 35.]

I HAVE just remarked that the determination is contained in my paper "On the Analytical Forms called Trees, &c.," *British Association Report*, 1875, [610]; in fact, in the form C_nH_{2n+1}, there is one carbon atom distinguished from the others by its being combined with (instead of 4, only) 3 other atoms; viz. these are 3 carbon atoms, 2 carbon atoms and 1 hydrogen atom, or else 1 carbon atom and 2 hydrogen atoms (CH_3, methyl, is an exception; but here the number is $=1$). The number of carbon atoms thus combined with the first-mentioned atom is the number of main branches, which is thus $=3$, 2, or 1; hence we have, number of radicals C_nH_{2n+1} is =

No. of carbon root-trees C_n with one main branch,
+ No. of " " with two main branches,
+ No. of " " with three main branches;

and the three terms for the values $n=1$ to 13 are given in Table VII. (pp. 454, 455 of this volume) of the paper referred to.

Thus, if $n=5$, an extract from the Table (p. 454 of this volume), is

Index x, or number of knots	Index t, or number of main branches	Altitude					
		0	1	2	3	4	
5	1			1	2	1	4
	2			2	1		3
	3			1			1
	4		1				1
	Total ...		1	4	3	1	9

and the number of the radicals C_5H_{11} (isomeric amyls) is $4+3+1=8$: or, what is the same thing, it is $9-1$, the corner-total less the number immediately above it. The tree-forms corresponding to the numbers 1, 2, 1; 2, 1; 1 in the body of the Table are the trees 2 to 9 in the figure, p. 428 of this volume.

The numbers of the radicals C_nH_{2n+1}, as obtained from the Table in the manner just explained, are:—

$n=$	Number of radicals C_nH_{2n+1}.			
1	1		= 1	Methyl.
2	1		1	Ethyl.
3	1		1	Propyl.
4	4		4	Butyls.
5	9	− 1	8	Amyls.
6	18	− 1	17	Hexyls.
7	42	− 3	39	Heptyls.
8	96	− 7	89	Octyls.
9	229	− 18	211	Nonyls.
10	549	− 42	507	Decyls.
11	1346	− 108	1238	Undecyls.
12	3326	− 269	3057	Dodecyls.
13	8329	− 691	7638	Tridecyls.

The question next in order, that of the determination of the number of the bivalent radicals C_nH_{2n}, might be solved without much difficulty.

Cambridge, November 20, 1876.

622.

ON A SYSTEM OF EQUATIONS CONNECTED WITH MALFATTI'S PROBLEM.

[From the *Proceedings of the London Mathematical Society*, vol. VII. (1875—1876), pp. 38—42. Read December 9, 1875.]

I CONSIDER the equations

$$X, = by^2 + cz^2 - 2fyz - a(bc - f^2), = 0,$$
$$Y, = cz^2 + ax^2 - 2gzx - b(ca - g^2), = 0,$$
$$Z, = ax^2 + by^2 - 2hxy - c(ab - h^2), = 0,$$

where the constants (a, b, c, f, g, h) are such that

$$K, = abc - af^2 - bg^2 - ch^2 + 2fgh, = 0.$$

Hence, writing as usual (A, B, C, F, G, H) to denote the inverse coefficients

$$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

we have $(A, B, C, F, G, H \between x, y, z)^2 =$ the square of a linear function, $= (\alpha x + \beta y + \gamma z)^2$ suppose; that is,

$$(A, B, C, F, G, H) = (\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta).$$

It is to be shown that the three quadric surfaces $X = 0$, $Y = 0$, $Z = 0$ intersect in a conic Θ lying in the plane $a\alpha x + b\beta y + c\gamma z = 0$, and in two points I, J; or more completely, that

the surfaces Y, Z meet in the conic Θ and a conic P,
" Z, X " " " Q,
" X, Y " " " R,

where the conics P, Q, R each pass through the two points I, J, and meet the conic Θ in two points, viz.,

the conics P, Θ meet in two points P_1, P_2,
" Q, Θ " " Q_1, Q_2,
" R, Θ " " R_1, R_2.

For this purpose, writing

$$\nabla = Aa - Ff, = Bb - Gg, = Cc - Hh, = abc - fgh,$$
$$\Omega = \tfrac{1}{2}(X + Y + Z), = ax^2 + by^2 + cz^2 - fyz - gzx - hxy - \nabla,$$
$$\theta = a\alpha x + b\beta y + c\gamma z,$$
$$\xi = \frac{Aa}{\alpha}x + \frac{Ff}{\beta}y + \frac{Ff}{\gamma}z,$$
$$\eta = \frac{Gg}{\alpha}x + \frac{Bb}{\beta}y + \frac{Gg}{\gamma}z,$$
$$\zeta = \frac{Hh}{\alpha}x + \frac{Hh}{\beta}y + \frac{Cc}{\gamma}z,$$

then we have identically

$$aA\Omega - \nabla X = \theta\xi,$$
$$bB\Omega - \nabla Y = \theta\eta,$$
$$cC\Omega - \nabla Z = \theta\zeta.$$

In fact, the first of these equations, written at full length, is

$$aA\,(ax^2 + by^2 + cz^2 - fyz - gzx - hxy - \nabla) - \nabla\,(by^2 + cz^2 - 2fyz - aA)$$
$$= (a\alpha x + b\beta y + c\gamma z)\left(\frac{Aa}{\alpha}x + \frac{Ff}{\beta}y + \frac{Ff}{\gamma}z\right),$$

where on the left-hand side the constant term is $= 0$. Comparing, first, the coefficients of x^2, y^2, z^2, on the two sides respectively, these are Aa^2, $(Aa - \nabla)b$, $(Aa - \nabla)c$, and Aa^2, Ffb, Ffc, which are equal. Comparing the coefficients of yz, zx, xy, the equations which remain to be verified are

$$-(aA - 2\nabla)f = Ff\left(c\frac{\gamma}{\beta} + b\frac{\beta}{\gamma}\right),$$
$$-aAg \qquad = Faf\frac{\alpha}{\gamma} + Aac\frac{\gamma}{\alpha},$$
$$-aAh \qquad = Faf\frac{\alpha}{\beta} + Aab\frac{\beta}{\alpha};$$

or, as these may be written,

$$-(aA - 2\nabla)\beta\gamma = F(c\gamma^2 + b\beta^2),$$
$$-Ag\gamma\alpha \qquad = Ff\alpha^2 + Ac\gamma^2,$$
$$-Ah\alpha\beta \qquad = Ff\alpha^2 + Ab\beta^2;$$

and, substituting for α^2, β^2, γ^2, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ their values, these may be verified without difficulty.

It thus appears that the equations of the three quadric surfaces may be written in the form

$$aA\Omega - \theta\xi = 0,\ bB\Omega - \theta\eta = 0,\ cC\Omega - \theta\zeta = 0;$$

and we thus obtain the conics Θ, P, Q, R as the intersections of the surface $\Omega = 0$ by the four planes

$$\theta = 0,\quad \frac{\eta}{Bb} - \frac{\zeta}{Cc} = 0,\quad \frac{\zeta}{Cc} - \frac{\xi}{Aa} = 0,\quad \frac{\xi}{Aa} - \frac{\eta}{Bb} = 0,$$

respectively. There is no difficulty in verifying that the conics intersect as mentioned above, and that the coordinates of their points of intersection are

$$P,\ P_1 : \left(\sqrt{bc},\ \frac{ch}{\sqrt{bc}},\ \frac{bg}{\sqrt{bc}}\right),\ \left(-\sqrt{bc},\ -\frac{ch}{\sqrt{bc}},\ -\frac{bg}{\sqrt{bc}}\right);$$

$$Q,\ Q_1 : \left(\frac{ch}{\sqrt{ca}},\ \sqrt{ca},\ \frac{af}{\sqrt{ca}}\right),\ \left(-\frac{ch}{\sqrt{ca}},\ -\sqrt{ca},\ -\frac{af}{\sqrt{ca}}\right);$$

$$R,\ R_1 : \left(\frac{bg}{\sqrt{ab}},\ \frac{af}{\sqrt{ab}},\ \sqrt{ab}\right),\ \left(-\frac{bg}{\sqrt{ab}},\ -\frac{af}{\sqrt{ab}},\ -\sqrt{ab}\right);$$

$$I,\ J\ :\ (f,\ g,\ h),\qquad (-f,\ -g,\ -h).$$

In a paper "On a system of Equations connected with Malfatti's Equation and on another Algebraical System," *Camb. and Dublin Math. Journal*, vol. IV. (1849), pp. 270—275, [79], I considered a system of equations which, writing therein $\theta = 1$, and changing the signs of (f, g, h), are the equations here considered, $X = 0$, $Y = 0$, $Z = 0$: only the constants (a, b, c, f, g, h) are not connected by the equation $K = 0$, but are perfectly arbitrary. The three quadric surfaces intersect therefore in 8 points, the coordinates of which are obtained in the paper just referred to, viz. making the above changes of notation, the values are

$$x^2 = \frac{1}{2a}(abc + fgh - f\sqrt{BC} + g\sqrt{CA} + h\sqrt{AB}),$$

$$y^2 = \frac{1}{2b}(abc + fgh + f\sqrt{BC} - g\sqrt{CA} + h\sqrt{AB}),$$

$$z^2 = \frac{1}{2c}(abc + fgh + f\sqrt{BC} + g\sqrt{CA} - h\sqrt{AB}),$$

$$\begin{aligned} yz &= \tfrac{1}{2}(gh + af + \sqrt{BC}),\\ zx &= \tfrac{1}{2}(hf + bg + \sqrt{CA}),\\ xy &= \tfrac{1}{2}(fg + ch + \sqrt{AB}); \end{aligned}$$

where the radicals are such that $\sqrt{BC}.\sqrt{CA}.\sqrt{AB} = ABC$, so that the system $(x^2, y^2, z^2, yz, zx, xy)$ has four values only, and consequently (x, y, z) has eight values.

It is very remarkable that, introducing the foregoing relation $K=0$, there is not in the solution any indication that the intersection has become a conic and two points, but the solution gives eight determinate points, viz. the before-mentioned points P, P_1, Q, Q_1, R, R_1, and I, J.

To develope the solution, remark that, in virtue of the relation in question, we have

$$\sqrt{BC}=\pm F,\quad \sqrt{CA}=\pm G,\quad \sqrt{AB}=\pm H,$$

where the signs must be such that the product is $=FGH$ (viz. they must be all positive, or else one positive and the other two negative); for, taking the product to be $+FGH$, the equations give

$$0=ABC-FGH,$$

that is,

$$0=A(BC-F^2)-F(GH-AF),\ =K(Aa-Ff),\ =K\nabla,$$

which is true in virtue of the relation $K=0$. Taking the signs all positive, we have for x^2, y^2, z^2, yz, zx, xy, the values f^2, g^2, h^2, gh, hf, fg, viz. we have thus the points

$$(f,\ g,\ h),\quad (-f,\ -g,\ -h),$$

which are the points I, J. Taking the signs one positive and the other two negative, say $\sqrt{BC}=F$, $\sqrt{CA}=-G$, $\sqrt{AB}=-H$, we find for x^2, y^2, z^2, yz, zx, xy the values bc, $\frac{ch^2}{b}$, $\frac{bg^2}{c}$, gh, bg, ch, viz. we have thus the points

$$\left(\sqrt{bc},\ \frac{ch}{\sqrt{bc}},\ \frac{bg}{\sqrt{bc}}\right),\quad \left(-\sqrt{bc},\ -\frac{ch}{\sqrt{bc}},\ -\frac{bg}{\sqrt{bc}}\right),$$

which are the points P, P_1; and the other two combinations of sign give of course the points Q, Q_1 and R, R_1 respectively.

If the coefficients $(a,\ b,\ c,\ f,\ g,\ h)$, instead of the foregoing relation $K=0$, satisfy the relation

$$abc-af^2-bg^2-ch^2-2fgh=0,\ \text{say}\ K'=0,$$

the quadric surfaces intersect in 8 points, the coordinates of which are given by the general formulæ: but the expressions assume a very simple form. Writing for shortness

$$F'=gh+af,\quad G'=hf+bg,\quad H'=fg+ch,$$

then, in virtue of the assumed relation,

$$\sqrt{BC}=\pm F',\quad \sqrt{CA}=\pm G',\quad \sqrt{AB}=\pm H',$$

where the signs are such that the product of the three terms is positive, viz. they must be all positive, or else one positive and the other two negative. For, assuming it to be so, we have

$$0=ABC-F'G'H',$$

that is,

$$\begin{aligned} 0 &= A(BC - F'^2) - F'(G'H' - AF'), \\ &= K'(Aa + F'f), \ = K'(abc + fgh); \end{aligned}$$

which is right, in virtue of the relation $K' = 0$. Taking the signs all positive, we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values (A, B, C, F', G', H'), giving two points of intersection

$$\left(\sqrt{A}, \frac{H'}{\sqrt{A}}, \frac{G'}{\sqrt{A}}\right) \text{ and } \left(-\sqrt{A}, -\frac{H'}{\sqrt{A}}, -\frac{G'}{\sqrt{A}}\right).$$

Taking the signs one positive and the other two negative, say

$$\sqrt{BC} = F', \quad \sqrt{CA} = -G', \quad \sqrt{AB} = -H',$$

we find for $(x^2, y^2, z^2, yz, zx, xy)$ the values

$$\left(0, \frac{Cc}{b}, \frac{Bb}{c}, F', 0, 0\right),$$

viz. we have thus two intersections

$$\left(0, \sqrt{\frac{Cc}{b}}, F'\sqrt{\frac{b}{Cc}}\right), \quad \left(0, -G'\sqrt{\frac{Cc}{b}}, -F'\sqrt{\frac{b}{Cc}}\right);$$

and the other combinations of signs give the remaining two pairs of intersections

$$\left(G'\sqrt{\frac{c}{Aa}}, 0, \sqrt{\frac{Aa}{c}}\right), \quad \left(-G'\sqrt{\frac{c}{Aa}}, 0, -\sqrt{\frac{Aa}{c}}\right),$$

and

$$\left(\sqrt{\frac{Bb}{a}}, H'\sqrt{\frac{a}{Bb}}, 0\right), \quad \left(-\sqrt{\frac{Bb}{a}}, -H'\sqrt{\frac{a}{Bb}}, 0\right).$$

But the most convenient statement of the result is that the values of $(ax^2, by^2, cz^2, yz, zx, xy)$, for the four pairs of points respectively, are

$$\begin{array}{llllll} (aA, & bB, & cC, & F', & G', & H'), \\ (\ 0\ , & cC, & bB, & F', & 0\ , & 0\ ), \\ (cC\ , & 0\ , & aA, & 0\ , & G', & 0\ ), \\ (bB\ , & aA, & 0\ , & 0\ , & 0\ , & H'); \end{array}$$

there is no difficulty in substituting these values in the original equations, and in verifying that the equations are in each case satisfied.

623.

ON THREE-BAR MOTION.

[From the *Proceedings of the London Mathematical Society*, vol. VII. (1875—1876), pp. 136—166. Read March 10, 1876.]

THE discovery by Mr Roberts of the triple generation of a Three-Bar Curve, throws a new light on the whole theory, and is a copious source of further developments *. The present paper gives in its most simple form the theorem of the triple generation; it also establishes the relation between the nodes and foci; and it contains other researches. I have made on the subject a further investigation, which I give in a separate paper, "On the Bicursal Sextic," [624]; but the two papers are intimately related and should be read in connection.

The Three-Bar Curve is derived from the motion of a system of three bars of given lengths pivoted to each other, and to two fixed points, so as to form the three sides of a quadrilateral, the fourth side of which is the line joining the two fixed points; the curve is described by a point rigidly connected with the middle bar; or, what is more convenient, we take the middle bar to be a triangle pivoted at the extremities of the base to the other two bars (say, the radial bars), and having its vertex for the describing point.

Including the constants of position and magnitude, the Three-Bar Curve thus depends on nine parameters; viz. these are the coordinates of the two fixed points, the lengths of the connecting bars, and the three sides of the triangle. It is known that the curve is a tricircular trinodal sextic, and the equation of such a curve contains $27-6-6-3, =12$ constants. Imposing on the curve the condition that the three nodes lie upon a given curve, the number of constants is reduced to $12-3, =9$: and it is in this way that the Three-Bar Curve is distinguished from the general tricircular

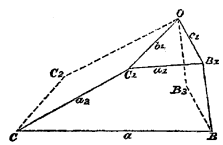
* See his paper "On Three-Bar Motion in Plane Space," *l.c.*, vol. VII., pp. 15—23, which contains more than I had supposed of the results here arrived at. There is no question as to Mr Roberts' priority in all his results.

trinodal sextic; viz. in the Three-Bar Curve the two fixed points are foci, and they determine a third focus*; and the condition is that the nodes are situate on the circle through the 3 foci.

The nodes are two of them arbitrary points on the circle; and the third of them is a point such that, measuring the distances along the circle from any fixed point of the circumference, the sum of the distances of the nodes is equal to the sum of the distances of the foci. Considering the two fixed points as given, the curve depends upon five parameters, viz. the lengths of the connecting bars and the sides of the triangle. Taking the *form* of the triangle as given, there are then only three parameters, say the lengths of the connecting bars and the base of the triangle; in this case the third focus is determined, and therefore the circle through the three foci; we may then take two of the nodes as given points on this circle, and thereby establish two relations between the three parameters, in fact, we thereby determine the differences of the squares of the lengths in question: but the third node is then an absolutely determined point on the circle, and we cannot make use of it for completing the determination of the parameters; viz. one parameter remains arbitrary. Or, what is the same thing, given the three foci and also the three nodes, consistently with the foregoing conditions, viz. the nodes lie in the centre through the three foci, the sum of the distances of the nodes being equal to the sum of the distances of the foci: we have a singly infinite series of three-bar curves.

In reference to the notation proper for the theorem of the triple generation, I shall, when only a single node of generation is attended to, take the curve to be generated as shown in the annexed Figure 1; viz. O is the generating point, OC_1B_1 the triangle, C, B the fixed points, CC_1 and BB_1 the radial bars. The sides of the

Fig. 1.



triangle are a_1, b_1, c_1; its angles are $O, = A$, $B_1, = B$, $C_1, = C$: the bars CC_1 and BB_1 are $= a_2$ and a_3 respectively, and the distance CB is $= a$. The sides a_1, b_1, c_1 may be put $= k_1(\sin A, \sin B, \sin C)$, and the lines a_1, a_2, $a_3 = (k_1, k_2, k_3) \sin A$, viz. the original data a_1, b_1, c_1, a_1, a_2, a_3, may be replaced by the angles A, B, C $(A + B + C = \pi)$ and the lines k_1, k_2, k_3. And it is convenient to mention at once that the third focus A is then a point such that ABC is a triangle similar and congruent to OB_1C_1.

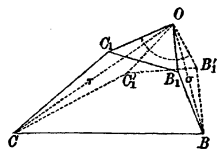
* A focus is a point, given as the intersection of a tangent to the curve from one circular point at infinity with a tangent from the other circular point at infinity; if the circular points are simple or multiple points on the curve, then the tangent or tangents *at* a circular point should be excluded from the tangents *from* the point; and the intersection of two such tangents at the two circular points respectively is not an ordinary focus; but, as the points in question are the only kind of foci occurring in the present paper, I have in the text called them foci.

It may be remarked that, producing CC_1 and BB_1 to meet in a point α, this is the centre of instantaneous rotation of the triangle, and therefore αO is the normal to the curve at O.

I proceed to show that the three nodes F, G, H are in the circle circumscribed about ABC, and that their positions are such that (the distances being measured along the circle as before) we have the property, Sum of the distances of F, G, H is equal to the Sum of the distances of A, B, C.

Supposing O to be at a node F, we have then the two equal triangles FB_1C_1, $FB_1'C_1'$, such that C_1, C_1' are equidistant from C, and B_1, B_1' equidistant from B. Hence the angles B_1FB_1', C_1FC_1' are equal; consequently the halves of these angles CFC_1' and

Fig. 2.



BFB_1' are equal; whence the angle CFB is equal to the angle $C_1'FB_1'$, that is, to the angle A; or F lies on a circle through B, C such that the segment upon BC contains the angle A, that is, upon the circle through A, B, C. To complete the investigation of the nodes, suppose $CF = \tau$, $BF = \sigma$: then the condition $\angle CFC_1' = \angle BFB_1'$ gives

$$\frac{b_1^2 + \tau^2 - a_2^2}{2b_1\tau} = \frac{c_1^2 + \sigma^2 - a_3^2}{2c_1\sigma},$$

that is,

$$c_1\sigma(b_1^2 + \tau^2 - a_2^2) - b_1\tau(c_1^2 + \sigma^2 - a_3^2) = 0;$$

and the condition that F is on the circle gives

$$\sigma^2 + \tau^2 - 2\sigma\tau\cos A = a^2.$$

These equations give six values of (σ, τ) corresponding in pairs to each other; viz. if (σ_1, τ_1) is a solution, then $(-\sigma_1, -\tau_1)$ is also a solution; and to each pair of solutions corresponds a single point on the circle, viz. we have thus the three nodes F, G, H.

Writing the foregoing equation in the form

$$\{c_1(b_1^2 - a_2^2)\sigma - b_1(c_1^2 - a_3^2)\tau\}(\sigma^2 + \tau^2 - 2\sigma\tau\cos A) + a^2(c_1\sigma\tau^2 - b_1\sigma^2\tau) = 0,$$

and putting the left-hand side $= M(\sigma - p_1\tau)(\sigma - p_2\tau)(\sigma - p_3\tau)$; then, if α, β, γ denote $\cos A + i\sin A$, $\cos B + i\sin B$, $\cos C + i\sin C$ respectively, putting first $\sigma = \alpha\tau$ and next $\sigma = \frac{\tau}{\alpha}$, and dividing one of these results by the other, we find

$$\frac{c_1 - b_1\alpha}{c_1\alpha - b_1} = \frac{\alpha - p_1 . \alpha - p_2 . \alpha - p_3}{1 - p_1\alpha . 1 - p_2\alpha . 1 - p_3\alpha}.$$

The left-hand side is here

$$=\frac{\sin C-\alpha\sin B}{\alpha\sin C-\sin B}=\frac{\sin C-\sin B(\cos A+i\sin A)}{\sin C(\cos A+i\sin A)-\sin B}$$

$$=\frac{\sin A(\cos B-i\sin B)}{-\sin A(\cos C-i\sin C)}=-\frac{\gamma}{\beta},$$

or the equation is

$$\frac{\alpha-p_1\,.\,\alpha-p_2\,.\,\alpha-p_3}{1-p_1\alpha\,.\,1-p_2\alpha\,.\,1-p_3\alpha}=-\frac{\gamma}{\beta}.$$

Also, writing f for the angle FCB, we have $\sigma=\frac{\sin f}{\sin(A+f)}\tau$, viz. the values of p_1, p_2, p_3 are $\frac{\sin f}{\sin(A+f)}$, $\frac{\sin g}{\sin(A+g)}$, $\frac{\sin h}{\sin(A+h)}$. We thence find

$$\frac{\alpha-p_1}{1-\alpha p_1}=\frac{\sin(A+f)(\cos A+i\sin A)-\sin f}{\sin(A+f)-(\cos A+i\sin A)\sin f}=\frac{\cos(A+f)+i\sin(A+f)}{\cos f-i\sin f}$$

$$=\cos(A+2f)+i\sin(A+2f);$$

with the like values for the other two values. Hence, writing also

$$-\frac{\gamma}{\beta}=-\cos(C-B)-i\sin(C-B)=\cos(\pi+C-B)+i\sin(\pi+C-B),$$

the equation becomes

$$\cos(3A+2f+2g+2h)+i\sin(3A+2f+2g+2h)=\cos(\pi+C-B)+i\sin(\pi+C-B),$$

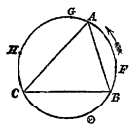
that is,

$$3A+2f+2g+2h=\pi+C-B,$$

or, what is the same thing,

$$2f+2g+2h=\pi+C-B-3A.$$

Fig. 3.



Reckoning the angles round the centre from a point Θ on the circumference, if A', B', C', F', G', H' are the angles belonging to the points A, B, C, F, G, H respectively, then

$$A'=\lambda+2C,\qquad F'=\lambda+2f,$$
$$B'=\lambda,\qquad G'=\lambda+2g,$$
$$C'=\lambda+2C+2B,\quad H'=\lambda+2h;$$

and therefore

$$A'+B'+C'=3\lambda+4C+2B,\quad F'+G'+H'=3\lambda+2f+2g+2h,\quad =3\lambda+\pi+C-B-3A;$$

that is, $A'+B'+C'-F'-G'-H'=-\pi+3(A+B+C)$; or, omitting an angle 2π, this is $A'+B'+C'=F'+G'+H'$, the equation which determines the relation between the three nodes on the circle ABC.

Reverting to the equation $c_1\sigma(b_1^2+\tau^2-a_2^2)-b_1\tau(c_1^2+\sigma^2-a_3^2)=0$, which belongs to a node: if we consider the form of the triangle as given, and write b_1, $c_1=k_1\sin B$, $k_1\sin C$, this becomes

$$\sigma\sin C(b_1^2-a_2^2)-\tau\sin B(c_1^2-a_3^2)+\sigma\tau(\tau\sin C-\sigma\sin B)=0;$$

viz. considering the node as given, then the values of σ, τ are given, and the equation establishes a relation between the values of $b_1^2-a_2^2$ and $c_1^2-a_3^2$*. If a second node be given, we have a second relation between these same quantities, and the two equations give the values of the two quantities, viz. the values of $k_1^2\sin^2 B-k_2^2\sin^2 A$, $k_1^2\sin^2 C-k_3^2\sin^2 A$, or, what is the same thing, the value of $\dfrac{k_1^2}{\sin^2 A}-\dfrac{k_2^2}{\sin^2 B}$, $\dfrac{k_1^2}{\sin^2 A}-\dfrac{k_3^2}{\sin^2 B}$. It thus appears that, if l_1, l_2, l_3 are any values of k_1, k_2, k_3 belonging to a given system of three nodes, the general values of k_1, k_2, k_3 belonging to the same system of three nodes are

$$k_1^2=l_1^2+u\sin^2 A,\quad k_2^2=l_2^2+u\sin^2 B,\quad k_3^2=l_3^2+u\sin^2 C,$$

where u is an arbitrary constant.

It may be added that there will be a node at B, if the equation is satisfied by $\tau=0$, $\sigma=a$, for the condition is $b_1^2-a_2^2=0$: that is, if $\dfrac{k_1}{\sin A}=\dfrac{k_2}{\sin B}$; similarly, there will be a node at C, if $c_1^2-a_3^2=0$, that is, if $\dfrac{k_1}{\sin A}=\dfrac{k_3}{\sin B}$; and a node at A, if $\dfrac{k_2}{\sin C}=\dfrac{k_3}{\sin B}$. If two of these equations are satisfied, the third equation is also satisfied, viz. we then have $\dfrac{k_1}{\sin A}=\dfrac{k_2}{\sin B}=\dfrac{k_3}{\sin C}$; and the three nodes coincide with the three foci respectively.

If, in Figure 2 (p. 553), the points C_1, C_1' coincide on the line CF, and therefore also the points B_1, B_1' coincide on the line BF, then, instead of a node at F, we have a cusp. We have in this case a triangle the sides of which are a_2+b_1, a_3+c_1, a, and the included angle between the first two sides is $=A$: we have, therefore, the relation

$$a^2=(a_2+b_1)^2+(a_3+c_1)^2-2(a_2+b_1)(a_3+c_1)\cos A.$$

Substituting herein for a, a_2, b_1, &c., the values $k\sin A$, $k_2\sin A$, $k_1\sin B$, &c., the equation is

$$k^2\sin^2 A=(k_1\sin B+k_2\sin A)^2+(k_1\sin C+k_3\sin A)^2$$
$$-2(k_1\sin B+k_2\sin A)(k_1\sin C+k_3\sin A)\cos A.$$

* Considering, in the equation, a_2 and a_3 as the distances of a variable point P from the points C and B respectively, the equation represents a circle having its centre on the line CB. Similarly, when a second node is given, the corresponding equation represents another circle, having its centre on the line CB, and the intersections of the two circles determine a_2 and a_3, the lengths of the radial bars, in order that the curve may have the given nodes.

Expanding the right-hand side and reducing by means of $A+B+C=\pi$, the whole becomes divisible by $\sin^2 A$, and we have

$$k^2 = k_1^2 + k_2^2 + k_3^2 - 2k_2k_3 \cos A + 2k_3k_1 \cos B + 2k_1k_2 \cos C;$$

viz. considering A, B, C, k_1, k_2, k_3 as given, this equation determines k so that the curve may have a cusp. The equation is one of the system of four equations

$$k^2 = k_1^2 + k_2^2 + k_3^2 - 2k_2k_3 \cos A + 2k_3k_1 \cos B + 2k_1k_2 \cos C,$$
$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos A - 2k_3k_1 \cos B + 2k_1k_2 \cos C,$$
$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos A + 2k_3k_1 \cos B - 2k_1k_2 \cos C,$$
$$k^2 = k_1^2 + k_2^2 + k_3^2 - 2k_2k_3 \cos A - 2k_3k_1 \cos B - 2k_1k_2 \cos C,$$

which belong to the different arrangements CC_1F or CFC_1, BB_1F or BFB_1, of the three points on the lines BF and CF; if k has any of these four values, the curve will have a cusp. If two of the equations subsist together, we have a curve with two cusps. Taking k_1, k_2, k_3, and also $\cos A$, $\cos B$, $\cos C$, as positive, viz. assuming that the triangle is acute-angled, the fourth equation cannot subsist with any one of the others: but two of the others may subsist together, for instance, the first and second will do so, if $k_2k_3 \cos A = k_3k_1 \cos B$, that is, if $\dfrac{k_1}{\cos A} = \dfrac{k_2}{\cos B}$, and then $k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 \cos C$: the curve has then two cusps. Similarly, the three equations may subsist together, viz. we must then have

$$\frac{k_1}{\cos A} = \frac{k_2}{\cos B} = \frac{k_3}{\cos C}, \quad k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos A:$$

writing herein k_1, k_2, $k_3 = \lambda \cos A$, $\lambda \cos B$, $\lambda \cos C$, we find

$$k^2 = \lambda^2 (\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C) = \lambda^2;$$

viz. if k_1, k_2, k_3 are respectively $= k \cos A$, $k \cos B$, $k \cos C$, the curve has then three cusps. It will be recollected that, if

$$k_1 : k_2 : k_3 = \sin A : \sin B : \sin C,$$

the nodes coincide with the foci; the two sets of conditions subsist together, if $A = B = C = 60°$; $k_1 = k_2 = k_3 = \frac{1}{2}k$, viz. we have then a curve with three cusps coinciding with the three foci respectively.

Before going further, I will establish the theorem for the triple generation of the curve.

The theorem which gives the triple generation may be stated as follows. See Figures 4, 5, 6*.

Imagine a triangle ABC and a point O, through which point are drawn lines parallel to the sides dividing the triangle into three triangles OB_1C_1, OC_2A_2, OA_3B_3,

* Figure 6 (substantially the same as Fig. 5) belongs to the same curve as Figures 1 and 2, and it exhibits the triple generation of this curve: the generating point O being taken at a node (the same node as in Figure 2), and the two positions OB_1C_1 and $OB_1'C_1'$ of one of the triangles being shown in the figure.

similar *inter se* and to the original triangle, and into three parallelograms OA_2AA_3, OB_3BB_1, OC_1CC_2. Then, considering the three triangles as pivoted together at the point O, and replacing the exterior sides of the parallelograms by pairs of bars A_2AA_3, B_3BB_1, C_1CC_2 pivoted together at A, B, C, and to the triangles at A_2, A_3, B_3, B_1, C_1, C_2, the figure thus consisting of the three triangles and the six bars; let the

Fig. 4.

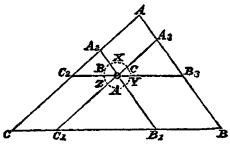


Fig. 5.

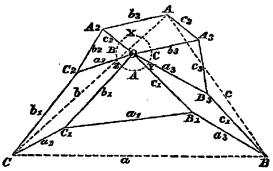
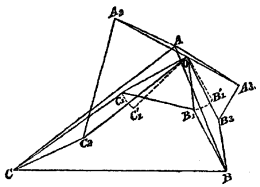


Fig. 6.



three triangles be turned at pleasure about the point O, so as to displace in any manner the points A, B, C: we have the theorem that the triangle ABC will remain always similar to the original triangle ABC, that is, to each of the three triangles OB_1C_1, OC_2A_2, OA_3B_3: and further, that, starting from any given positions of the three triangles, we may so move them as not to alter the triangle ABC in magnitude: whence, conversely, fixing the three points A, B, C, the point O will be moveable in a curve.

Assuming this, it is clear that the locus of the point O is simultaneously the locus given by

The triangle	OB_1C_1,	connected by bars	B_1B	and	C_1C	to fixed points	B, C,
„	OC_2A_2,	„	C_2C	„	A_2A	„	C, A,
„	OA_3B_3,	„	A_3A	„	B_3B	„	A, B;

or, that we have a triple generation of the same three-bar curve. It may be remarked that the intersection of the lines BB_1 and CC_1 is the axis of the instantaneous rotation of the triangle OB_1C_1, so that, joining this intersection with the point O, we have the normal at O to the locus; and similarly for the other two triangles. It of course follows that the intersections of BB_1 and CC_1, of CC_2 and AA_2, and of AA_3 and BB_3, lie on a line through O, viz. this line is the normal at O.

The result depends on the following theorem: viz. starting with the similar triangles OB_1C_1, A_2OC_2, A_3B_3O, say, the angles of these are A, B, C, so that the sides are

$$k_1(\sin A,\ \sin B,\ \sin C),\ k_2(\sin A,\ \sin B,\ \sin C),\ k_3(\sin A,\ \sin B,\ \sin C);$$

then it follows that the sides of the triangle ABC are

$$k(\sin A,\ \sin B,\ \sin C),$$

the value of k being given by the equation

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3\cos(X-A) + 2k_3k_1\cos(Y-B) + 2k_1k_2\cos(Z-C),$$

where X, Y, Z denote the angles A_2OA_3, B_3OB_1, C_1OC_2 respectively: whence, since $A+B+C=\pi$, we have also $X+Y+Z=\pi$. If therefore the angles X, Y, Z vary in any manner subject to this last relation and to the equation $k^2=$ const., the triangle ABC will be constant in magnitude.

There is no difficulty in proving the theorem. Writing $OC=\tau$, and $OB=\sigma$, also $\angle COC_1=\psi$, and $\angle BOB_1=\phi$, we have

$$\tau^2=b_1^2+a_2^2+2b_1a_2\cos Z,\quad \frac{\sin\psi}{a_2}=\frac{\sin Z}{\tau},\quad \cos\psi=\frac{b_1+a_2\cos Z}{\tau},$$

$$\sigma^2=c_1^2+a_3^2+2c_1a_3\cos Y,\quad \frac{\sin\phi}{a_3}=\frac{\sin Y}{\sigma},\quad \cos\phi=\frac{c_1+a_3\cos Y}{\sigma};$$

and then

$$\begin{aligned}
a^2&=\tau^2+\sigma^2-2\tau\sigma\cos(A+\psi+\phi)\\
&=\tau^2+\sigma^2-2\tau\sigma\cos A\cos\phi\cos\psi+2\tau\sigma\cos A\sin\phi\sin\psi\\
&\qquad\qquad+2\tau\sigma\sin A\sin\psi\cos\phi+2\tau\sigma\sin A\cos\psi\sin\phi\\
&=b_1^2+c_1^2+a_2^2+a_3^2+2b_1a_2\cos Z+2c_1a_3\cos Y\\
&\quad-2\cos A\,(b_1+a_2\cos Z)(c_1+a_3\cos Y)\\
&\quad+2\cos A\,.\,a_2a_3\sin Y\sin Z\\
&\quad+2\sin A\sin Z\,.\,a_2(c_1+a_3\cos Y)\\
&\quad+2\sin A\sin Y\,.\,a_3(b_1+a_2\cos Z)\\
&=b_1^2+c_1^2-2b_1c_1\cos A+a_2^2+a_3^2\\
&\quad+2a_2a_3\,[-\cos A\,(-\sin Y\sin Z+\cos Y\cos Z)\\
&\qquad\qquad+\sin A\,(\sin Y\cos Z+\cos Y\sin Z)]\\
&\quad+2a_3\,[(c_1-b_1\cos A)\cos Y+b_1\sin A\sin Y]\\
&\quad+2a_2\,[(b_1-c_1\cos A)\cos Z+c_1\sin A\sin Z].
\end{aligned}$$

We have here $b_1^2+c_1^2-2b_1c_1\cos A=a_1^2$: the second line is $=-2a_2a_3\cos(A+Y+Z)$ which, by virtue of $Y+Z=\pi-X$, is $=2a_2a_3\cos(X-A)$: and in the third and fourth lines

$$c_1-b_1\cos A=a_1\cos B,\quad b_1\sin A=a_1\sin B,$$
$$b_1-c_1\cos A=a_1\cos C,\quad c_1\sin A=a_1\sin C;$$

whence these lines are $2a_3a_1\cos(Y-B)$, $2a_1a_2\cos(Z-C)$: the equation therefore is

$$a^2=a_1^2+a_2^2+a_3^2+2a_2a_3\cos(X-A)+2a_3a_1\cos(Y-B)+2a_1a_2\cos(Z-C),$$

which, putting therein for a_1, a_2, a_3 the values $k_1\sin A$, $k_2\sin A$, $k_3\sin A$, and assuming as above

$$k^2=k_1^2+k_2^2+k_3^2+2k_2k_3\cos(X-A)+2k_3k_1\cos(Y-B)+2k_1k_2\cos(Z-C),$$

becomes $a^2=k^2\sin^2 A$, or say $a=k\sin A$; and similarly $b=k\sin B$, $c=k\sin C$, that is, $(a,\ b,\ c)=k\,(\sin A,\ \sin B,\ \sin C)$, the required theorem.

Before proceeding to find the equation of the curve, I insert, by way of lemma, the following investigation:—

Three triads (A, B, C), (F, G, H), (I, J, K) of points in a line, or of lines through a point, may be in cubic involution; viz. representing A, B, &c. by the equations $x-ay=0$, $x-by=0$, &c., then this is the case when the cubic functions

$$(x-ay)(x-by)(x-cy),\ (x-fy)(x-gy)(x-hy),\ (x-iy)(x-jy)(x-ky),$$

are connected by a linear equation. Regarding I, J, K as given, the condition establishes between (A, B, C) and (F, G, H) two relations: viz. these are

$$\begin{aligned}&(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c) : (k-a)(k-b)(k-c)\\ =&(i-f)(i-g)(i-h) : (j-f)(j-g)(j-h) : (k-f)(k-g)(k-h).\end{aligned}$$

But, if K be regarded as indeterminate, then the condition establishes only the single relation

$$\begin{aligned}&(i-a)(i-b)(i-c) : (j-a)(j-b)(j-c)\\ =&(i-f)(i-g)(i-h) : (j-f)(j-g)(j-h),\end{aligned}$$

which relation, if $i=0$, $j=\infty$, takes the form $abc=fgh$. When K is thus indeterminate, we may say that the triads (A, B, C), (F, G, H) are in cubic involution with the duad I, J.

If A, B, &c. are points on a conic, then, considering the pencils obtained by joining these points with a point Θ on the conic, if the cubic involution exists for any particular position of Θ, it will exist for every position whatever of Θ; hence, considering triads of points on a conic, we may have a cubic involution between three triads, or between two triads and a duad, as above.

Taking $x=0$, $y=0$ for the equations of the tangents at the points I, J respectively, and $z=0$ for the equation of the line joining these two points, the equation of the conic may be taken to be $xy-z^2=0$, and consequently the coordinates of any point A on the conic may be taken to be $x : y : z = \alpha : \frac{1}{\alpha} : 1$. It is then readily shown that α, β, γ, f, g, h referring to the points A, B, C, F, G, H respectively, the condition for the cubic involution of (A, B, C), (F, G, H) with the duad (I, J) is $\alpha\beta\gamma=fgh$.

And we thence at once prove the theorem, that there exists a cubic curve $J_AI_BI_CFGH$, viz. a cubic curve passing through J, and having there the tangent JA, having at I a node with the tangents IB, IC to the two branches respectively, and passing through the points F, G, H; viz. that the triads (A, B, C), (F, G, H) being in cubic involution with (I, J) as above, there exists a cubic curve satisfying these $2+5+3, =10$ conditions. In fact, the equation of the cubic curve is

$$\begin{aligned}&J_AI_BI_CFGH;\ \left(y-\frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)\\ &\quad+\frac{z}{\alpha x}\{(x-\alpha z)(x-\beta z)(x-\gamma z)-(x-fz)(x-gz)(x-hz)\}=0,\end{aligned}$$

where observe that second term is an integral function $\frac{1}{\alpha}z^2(-Mx+Nz)$, if, for shortness,

$$M=\alpha+\beta+\gamma-f-g-h,$$
$$N=\beta\gamma+\gamma\alpha+\alpha\beta-gh-hf-fg.$$

In fact, the equations of the lines JA, IB, IC are $y-\frac{z}{\alpha}=0$, $x-\beta z=0$, $x-\gamma z=0$, respectively, and we at once see that these lines are tangents at the points I, J respectively; moreover, at the point F, we have $x, y, z=f, \frac{1}{f}, 1$. Substituting these values, the equation becomes

$$\left(\frac{1}{f}-\frac{1}{\alpha}\right)(f-\beta)(f-\gamma)+\frac{1}{\alpha f}(f-\alpha)(f-\beta)(f-\gamma)=0,$$

viz. the equation is satisfied identically, or the curve passes through F; and similarly the curve passes through G and H.

In precisely the same manner there exists a cubic curve $I_AJ_BJ_CFGH$; viz. this is

$$I_AJ_BJ_CFGH;\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)$$
$$+\frac{\alpha z}{y}\left\{\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)-\left(y-\frac{z}{f}\right)\left(y-\frac{z}{g}\right)\left(y-\frac{z}{h}\right)\right\}=0,$$

where the second term is an integral function, $\alpha z^2(-M'y+N'z)$; if, for shortness,

$$M'=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h}=\frac{1}{\alpha\beta\gamma}N,$$
$$N'=\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}-\frac{1}{gh}-\frac{1}{hf}-\frac{1}{fg}=\frac{1}{\alpha\beta\gamma}M,$$

in virtue of the relation $\alpha\beta\gamma=fgh$; so that the second term is in fact $=\frac{z^2}{\beta\gamma}(-Cx+Bz)$.

Writing for shortness J_A, I_A to denote these two cubics respectively, we have four other like cubics, $J_B(=J_BI_CI_AFGH)$, $I_B(=I_BJ_CJ_AFGH)$, $J_C(=J_CI_AI_BFGH)$, and $I_C(=I_CJ_AJ_BFGH)$; the equations being

$$J_A;\ \left(y-\frac{z}{\alpha}\right)(x-\beta z)(x-\gamma z)+\frac{z^2}{\alpha}(-Mx+Nz)=0,$$

$$J_B;\ \left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)+\frac{z^2}{\beta}(-Mx+Nz)=0,$$

$$J_C;\ \left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)+\frac{z^2}{\gamma}(-Mx+Nz)=0,$$

$$I_A;\ (x-\alpha z)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)+\frac{z^2}{\beta\gamma}(-Ny+Mz)=0,$$

$$I_B;\ (x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right)+\frac{z^2}{\gamma\alpha}(-Ny+Mz)=0,$$

$$I_C;\ (x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)+\frac{z^2}{\alpha\beta}(-Ny+Mz)=0.$$

We require the differences of the products $I_A J_A$, $I_B J_B$, $I_C J_C$. We find

$$I_B J_B = (x-\alpha z)(x-\beta z)(x-\gamma z)\left(y-\frac{z}{\alpha}\right)\left(y-\frac{z}{\beta}\right)\left(y-\frac{z}{\gamma}\right)+\frac{z^4}{\alpha\beta\gamma}(-Mx+Nz)(-Ny+Mz)$$

$$+\frac{z^2}{\gamma\alpha}(-Ny+Mz)\left(y-\frac{z}{\beta}\right)(x-\gamma z)(x-\alpha z)$$

$$+\frac{z^2}{\beta}(-Mx+Nz)(x-\beta z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\alpha}\right);$$

let Ω denote the sum of the two expressions in the first line. Similarly, we have

$$I_C J_C = \Omega + \frac{z^2}{\alpha\beta}(-My+Nz)\left(y-\frac{z}{\gamma}\right)(x-\alpha z)(x-\beta z)$$

$$+\frac{z^2}{\gamma}(-Mx+Nz)(x-\gamma z)\left(y-\frac{z}{\gamma}\right)\left(y-\frac{z}{\beta}\right).$$

We have thence

$$I_B J_B - I_C J_C = z^2\left\{\frac{1}{\alpha}(x-\alpha z)(-Ny+Mz)-\left(y-\frac{z}{\alpha}\right)(-Mx+Nz)\right\}$$

$$\times\left\{\frac{1}{\gamma}\left(y-\frac{z}{\beta}\right)(x-\gamma z)-\frac{1}{\beta}\left(y-\frac{z}{\gamma}\right)(x-\beta z)\right\}:$$

the factors in { } are respectively

$$=\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(xy-z^2) \text{ and } \left(M-\frac{N}{\alpha}\right)(xy-z^2),$$

so that we have

$$I_B J_B - I_C J_C = \left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)z^2(xy-z^2)^2.$$

The constant factor

$$\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{N}{\alpha}-M\right)$$

is

$$=\left(\frac{N}{\alpha\beta}+\frac{M}{\gamma}\right)-\left(\frac{N}{\gamma\alpha}+\frac{M}{\beta}\right), =P_3-P_2,$$

if P_1, P_2, P_3 denote respectively the functions

$$\frac{N}{\beta\gamma}+\frac{M}{\alpha},\ \frac{N}{\gamma\alpha}+\frac{M}{\beta},\ \frac{N}{\alpha\beta}+\frac{M}{\gamma}.$$

Attending to the equation $\alpha\beta\gamma = fgh$, it appears that we have

$$P_1=\frac{1}{\alpha}(\alpha+\beta+\gamma-f-g-h)+\alpha\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{f}-\frac{1}{g}-\frac{1}{h}\right),$$

with like values for P_2 and P_3.

We have thus

$$I_B J_B - I_C J_C = -(P_2 - P_3)\, z^2 (xy - z^2)^2,$$

and similarly

$$I_C J_C - I_A J_A = -(P_3 - P_1)\, z^2 (xy - z^2)^2,$$
$$I_A J_A - I_B J_B = -(P_1 - P_2)\, z^2 (xy - z^2)^2.$$

Any function $I_A J_A + \lambda z^2 (xy - z^2)^2$, where λ is arbitrary, can of course be expressed in the form $I_A J_A + (\theta + P_1)\, z^2 (xy - z^2)^2$, where θ is arbitrary, and therefore in the three equivalent forms

$$I_A J_A + (\theta + P_1)\, z^2 (xy - z^2)^2,$$
$$I_B J_B + (\theta + P_2)\, z^2 (xy - z^2)^2,$$
$$I_C J_C + (\theta + P_3)\, z^2 (xy - z^2)^2.$$

We have $z = 0$, the line IJ: and $xy - z^2 = 0$, the conic $IJABCFGH$. The equation $I_A J_A + \lambda z^2 (xy - z^2)^2 = 0$ may thus be written in the more complete form

$$I_A J_B J_C FGH \,.\, J_A I_B I_C FGH + \lambda (IJ)^2 (IJABCFGH)^2 = 0,$$

and we hence see that it is the equation of a sextic curve, having a triple point at I, the tangents there being IA, IB, IC; having a triple point at J, the tangents there being JA, JB, JC; and having a node (double point) at each of the points F, G, H. There are thus in all $(6+3)+(6+3)+3+3+3$, $=27$ conditions, and these would in general be sufficient to determine the sextic. The data are, however, related in a special manner; viz. regarding the points I, J, F, G, H as arbitrary, the lines IA, IB, IC, JA, JB, JC are not arbitrary, but satisfy the conditions that A, B are arbitrary points, and C a determinate point, on the conic $IJABC$. And the foregoing result shows that, this being so, there exists a sextic satisfying the foregoing conditions, but containing in its equation an arbitrary constant λ or θ, and that the equation may be presented under the three forms

$$I_A J_B J_C FGH \,.\, J_A I_B I_C FGH + (\theta + P_1)(IJ)^2 (IJABCFGH)^2 = 0, \text{ \&c.},$$

corresponding to the partitions A, BC; B, CA; C, AB of the three points A, B, C.

In the case where I, J are the circular points at infinity, the conic $IJABCFGH$ is a circle passing through the six points A, B, C, F, G, H; and the condition of the cubic involution of the triads (A, B, C) and (F, G, H) with the points (I, J) is easily seen to be equivalent to the following relation, viz. the sum of the distances (measured along the circle from any fixed point of the circumference) of the three points A, B, C is equal to the sum of the distances of the three points F, G, H.

The sextic is a tricircular sextic having the three points A, B, C for foci, and having three nodes F, G, H, on the circle ABC, two of them being arbitrary points, and the third of them a determinate point on this circle. And it appears that there exists a sextic satisfying the foregoing conditions, and containing in its equation an arbitrary parameter.

I proceed to find the equation of the curve.

Consider the curve (see Fig. 1, p. 552) as generated by the point O, the vertex of the triangle OC_1B_1, connected by the bars C_1C and B_1B with the fixed points C and B respectively; and suppose, as before, $CB = a$, $C_1C = a_2$, $B_1B = a_3$, $B_1C_1 = a_1$, $OC_1 = b_1$, $OB_1 = c_1$; and draw as in the figure the parallelograms C_1CC_2O and B_1BB_3O; then O may be considered as the intersection of a circle, centre C_2 and radius C_2O, with a circle, centre B_3 and radius B_3O. Take $\angle C_2CB = \theta$, $\angle B_3BC = \phi$: the lines CC_2, BB_3 are parallel to OC_1, OB_1 respectively, and consequently $\theta + \phi = \pi - A$, a relation between the two variable angles θ, ϕ.

Taking the origin at C and the axis of x along the line CB, that of y being at right angles to it: the coordinates of C_2 are $(b_1 \cos\theta,\ b_1 \sin\theta)$, and those of B_3 are $(a - c_1 \cos\phi,\ c_1 \sin\phi)$; the equations of the circles thus are

$$(x - b_1 \cos\theta)^2 \qquad + (y - b_1 \sin\theta)^2 = a_2^2,$$

$$(x - a + c_1 \cos\phi)^2 + (y - c_1 \sin\phi)^2 = a_3^2;$$

whence

$$+ 2b_1 x \cos\theta + 2b_1 y \sin\theta = \qquad x^2 + y^2 + b_1^2 - a_2^2,$$

$$- 2c_1 (x - a) \cos\phi + 2c_1 y \sin\phi = (x - a)^2 + y^2 + c_1^2 - a_3^2,$$

which equations, writing therein for θ its value $= \pi - A - \phi$ and eliminating the single parameter ϕ, give the equation of the curve.

We in fact have

$$- 2b_1 x \cos(A + \phi) + 2b_1 y \sin(A + \phi) = \qquad x^2 + y^2 + b_1^2 - a_2^2,$$

$$- 2c_1 (x - a) \cos\phi \qquad + 2c_1 y \sin\phi \qquad = (x - a)^2 + y^2 + c_1^2 - a_3^2;$$

or say these are

$$P \cos\phi + Q \sin\phi = R,$$

$$P' \cos\phi + Q' \sin\phi = R',$$

where

$$P = - 2b_1 x \cos A + 2b_1 y \sin A,\ P' = - 2c_1 (x - a),$$

$$Q = \quad 2b_1 x \sin A + 2b_1 y \cos A,\ Q' = \quad 2c_1 y,$$

$$R = \quad x^2 + y^2 + b_1^2 - a_2^2, \qquad R' = (x - a)^2 + y^2 + c_1^2 - a_3^2.$$

The equations give therefore

$$\cos\phi : \sin\phi : -1 = QR' - Q'R : RP' - R'P : PQ' - P'Q,$$

whence

$$(QR' - Q'R)^2 + (RP' - R'P)^2 = (PQ' - P'Q)^2;$$

and it hence follows that the nodes are the common intersections of the three curves

$$QR' - Q'R = 0,\ RP' - R'P = 0,\ PQ' - P'Q = 0.$$

We have, retaining R and R' to denote their values,

$$QR' - Q'R = -2\,[(Rc_1 - R'b_1 \cos A)\,y - R'b_1 \sin A\,.\,x],$$
$$RP' - R'P = -2\,[(Rc_1 - R'b_1 \cos A)\,(x-a) + R'b_1 \sin A\,(y - a\cot A)],$$
$$PQ' - P'Q = -4\,b_1c_1\,[x\,(x-a) + y\,(y - a\cot A)].$$

Observing that $R=0$, $R'=0$ are circles; the equation $QR'-Q'R=0$ is a circular cubic through the point $x=0$, $y=0$; the equation $RP'-R'P=0$, a circular cubic through the point $x=a$, $y=a\cot A$; and the equation $PQ'-P'Q=0$, a circle through these two points (and also the points $x=0$, $y=a\cot A$; $x=a$, $y=0$). Hence the first and third curves intersect in the point ($x=0$, $y=0$), in the circular points at infinity, and in three other points which are the nodes; viz. the curve has three nodes, say these are F, G, H. The second and third curves intersect in the point ($x=0$, $y=a\cot A$), in the circular points at infinity, and in the three nodes. As regards the first and second curves, it is readily shown that these touch at the circular points at infinity; viz. they intersect in these points each twice, in the two finite intersections of the circles $R=0$, $R'=0$, and in the three nodes.

The three nodes F, G, H thus lie in the circle

$$x\,(x-a) + y\,(y - a\cot A) = 0,$$

which passes through the points ($x=0$, $y=0$) and ($x=a$, $y=0$), that is, the points C and B. Assuming $b = \dfrac{a\sin A}{\sin B}$, the circle also passes through the point $x = b\cos C$, $y = b\sin C$, that is, the point A of the figure. Thus the three nodes F, G, H lie in the circle circumscribed about the triangle ABC.

Writing, for greater convenience,

$$R = x^2 + y^2 - e^2,\ R' = x^2 + y^2 - 2ax - f^2,$$

the nodes F, G, H lie on the two curves

$$c_1y\,(x^2 + y^2 - e^2) - b_1 \sin A\,(x + y\cot A)\,(x^2 + y^2 - 2ax - f^2) = 0,$$
$$x^2 + y^2 = a\,(x + y\cot A).$$

The first of these is

$$\begin{aligned}&[c_1y - b_1 \sin A\,(x + y\cot A)]\,(x^2 + y^2)\\ &+ [b_1 \sin A\,(x + y\cot A)\,f^2 - c_1e^2y]\\ &+ 2ab_1 \sin A\,(x + y\cot A)\,x = 0.\end{aligned}$$

We may combine these equations so as to obtain the equation of the triad of lines CF, CG, CH; viz. multiplying the second and the third terms of the first equation by $\dfrac{(x^2+y^2)^2}{a^2\,(x + y\cot A)^2}$ and $\dfrac{x^2+y^2}{a\,(x + y\cot A)}$ (each $=1$ in virtue of the second equation), the equation becomes divisible by x^2+y^2: and, throwing this out, the equation is

$$\begin{aligned}&c_1y - b_1 \sin A\,(x + y\cot A)\\ &+ [b_1 \sin A\,(x + y\cot A)\,f^2 - c_1e^2y]\,\frac{x^2+y^2}{a^2\,(x + y\cot A)^2}\\ &+ 2b_1x \sin A = 0,\end{aligned}$$

where the first and the third terms together are $=(c_1 - b_1 \cos A)\, y + b_1 x \sin A$, viz. this is $= a_1 \sin B\,(x + y \cot B)$. Hence, writing also in the second term $a_1 \sin B$ for $b_1 \sin A$, the equation is

$$(x + y \cot A)^2 (x + y \cot B) + \frac{1}{a^2}\left\{(x + y \cot A) f^2 - \frac{c_1 e^2}{a_1 \sin B}\, y\right\}(x^2 + y^2) = 0\,;$$

or say this is

$$(x \sin A + y \cos A)^2 (x \sin B + y \cos B)$$
$$+ \frac{\sin A \sin B}{a^2}\left\{(x \sin A + y \cos A) f^2 - \frac{c_1 e^2}{b_1}\, y\right\}(x^2 + y^2) = 0\,;$$

viz. there is a term in $x^2 + y^2$, and another term

$$(x \sin A + y \cos A)^2 (x \sin B + y \cos B).$$

Suppose for a moment that the angles FBC, GBC, HBC are called F, G, H; then the function on the left hand must be

$$= M\,(x \sin F - y \cos F)\,(x \sin G - y \cos G)\,(x \sin H - y \cos H).$$

Writing in the identity $x = iy$, we have

$$(\cos A + i \sin A)^2 (\cos B + i \sin B) = -M\,(\cos F - i \sin F)\,(\cos G - i \sin G)\,(\cos H - i \sin H)\,;$$

and similarly, writing $x = -iy$, we have the like equation with $-i$ instead of $+i$; whence, dividing the two equations and taking the logarithms,

$$4A + 2B = 2m\pi - F - G - H,$$

which leads as before to the relation $A' + B' + C' = F' + G' + H'$.

In completion of the investigation, observe that M is determinately $+1$ or -1: and that

$$\frac{\sin A \sin B}{a^2}\left\{(x \sin A + y \cos A) f^2 - \frac{c_1}{b_1} e^2 y\right\}$$

is the linear factor of

$$M\,(x \sin F - y \cos F)\,(x \sin G - y \cos G)\,(x \sin H - y \cos H)$$
$$- (x \sin A + y \cos A)^2 (x \sin B + y \cos B),$$

which remains after throwing out the factor $x^2 + y^2$. Calling this linear factor $px + qy$, we have

$$\frac{a^2 p}{\sin A \sin B} = f^2 \sin A, \quad \frac{a^2 q}{\sin A \sin B} = f^2 \cos A - \frac{c_1}{b_1} e^2,$$

or, as this last equation may be written,

$$\frac{a^2 q}{\sin A \sin B} = f^2 \cos A - \frac{\sin C}{\sin B}\, e^2.$$

Hence, writing $a = k \sin A$, we have

$$f^2 = \frac{k^2 p}{\sin B}, \quad e^2 = \frac{k^2}{\sin C}\,(p \cos A - q \sin A)\,;$$

substituting for f^2 and e^2 their values, we have

$$-k_1^2 \sin^2 C + k_3^2 \sin^2 A = \frac{k^2 p}{\sin B} + k^2 \sin^2 A,$$

$$-k_1^2 \sin^2 B + k_2^2 \sin^2 A = \frac{k^2}{\sin C}(p \cos A - q \sin A),$$

or, what is the same thing,

$$-\frac{k_1^2}{\sin^2 A} + \frac{k_3^2}{\sin^2 C} = \frac{k^2}{\sin^2 A \sin B \sin^2 C}(p + \sin^2 A \sin B)$$

$$= \frac{k^2 \sin F \sin G \sin H}{\sin A \sin C . \sin A \sin B \sin C},$$

$$-\frac{k_1^2}{\sin^2 A} + \frac{k_2^2}{\sin^2 B} = \frac{k^2}{\sin^2 A \sin^2 B \sin C}(p \cos A - q \sin A)$$

$$= \frac{k^2 \sin (A - F) \sin (A - G) \sin (A - H)}{\sin A \sin B . \sin A \sin B \sin C},$$

which are the relations connecting k_1, k_2, k_3, when the foci and nodes are given.

It is to be remarked that if, for instance, $F = 0$ and $G = A$, then $k_1 : k_2 : k_3 = \sin A : \sin B : \sin C$; the nodes in this case coincide with the foci. A simple example is when $A = B = C$; the three triangles are here equal equilateral triangles. The general equations show that, if l_1, l_2, l_3 are values of k_1, k_2, k_3 belonging to a given set of nodes and foci, then the values $k_1^2 = l_1^2 + u \sin^2 A$, $k_2^2 = l_2^2 + u \sin^2 B$, $k_3^2 = l_3^2 + u \sin^2 C$ (where u is arbitrary) will belong to the same set of nodes and foci.

I write the equation of the curve in the form

$$\{(QR' - Q'R) + i(RP' - R'P)\}\{QR' - Q'R - i(RP' - R'P)\} - (PQ' - P'Q)^2 = 0,$$

where

$$(QR' - Q'R) + i(RP' - R'P) = (Rc_1 - R'b_1 \cos A)\, i(x - a - iy) - R'b_1 \sin A\, \{x - i(y - a \cot A)\}.$$

Calling I, J the circular points $(\infty, x + iy = 0)$ and $(\infty, x - iy = 0)$, this is a nodal circular cubic having I for an ordinary point, but J for a node. Moreover, one of the tangents at J is the line $x - iy = 0$, that is, the line JB; in fact, writing as before

$$R = x^2 + y^2 - e^2, \quad R' = x^2 + y^2 - 2ax - f^2,$$

then, when $x - iy = 0$, we have $R = -e^2$, $R' = -2ax - f^2$, and the equation becomes

$$\{-ce^2 + b_1 \cos A\,(2ax + f^2)\}(-ia) + b_1 \sin A\,(2ax + f^2)(ia \cot A) = 0;$$

viz. the term in x here disappears, or the three intersections are at infinity. The other tangent at J is the line $x - a - iy = 0$, that is, the line JC; in fact, when $x - a - iy = 0$, that is, $y = -i(x - a)$, we have $R = 2ax - a^2 - e^2$, $R' = -a^2 - f^2$, and the equation becomes

$$\{c_1(2ax - a^2 - e^2) + b_1 \cos A\,(a^2 + f^2)\} . 0 + b_1 \sin A\,(a^2 + f^2) . a(1 + i \cot A) = 0,$$

viz. the three intersections are here at infinity. The tangent at I is the line $x - b\cos C + i(y - b\sin C) = 0$, that is, the line IA; in fact, writing this in the form

$$y = ix - ib(\cos C + i\sin C) = ix - ib\gamma,$$

(if for a moment $\cos C + i\sin C = \gamma$, and similarly $\cos A + i\sin A = \alpha$, $\cos B + i\sin B = \beta$); then, y having this value, we find

$$R = 2bxy - b^2\gamma^2 - e^2,\ R' = 2(b\gamma - a)x - b^2\gamma^2 - f^2,$$
$$= -\frac{2c}{\beta}x - b^2\gamma^2 - f^2;$$

and the equation becomes

$$\left\{\begin{array}{c} C_1(2b\gamma x - b^2\gamma^2 - e^2) \\ -b_1\cos A\left(-\dfrac{2c}{\beta}x - b^2\gamma^2 - e^2\right)\end{array}\right\} i(2x - a - b\gamma)$$
$$-b_1\sin A\left(-\frac{2c}{\beta}x - b^2\gamma^2 - e^2\right)(2x + ia\cot A - b\gamma) = 0.$$

The coefficient of x^2 is here

$$2i\left(bc_1\gamma + \frac{b_1c}{\beta}\cdot\frac{1}{\alpha}\right);$$

or, since $b_1c = bc_1$, this is

$$= 2ibc_1\left(\gamma + \frac{1}{\alpha\beta}\right),\ = 0,$$

in virtue of the relation $A + B + C = \pi$, giving $\alpha\beta\gamma = -1$: hence there is only one finite intersection, or the line IA is a tangent.

The cubic in question

$$QR' - Q'R + i(RP' - R'P) = 0$$

is thus a nodal circular cubic which it is convenient to represent in the form

$$(I_AJ_BJ_CFGH) = 0;$$

viz. this is a cubic, through I with the tangent IA, having J as a node with the tangents JB, JC, and through the points F, G, H. Observe that, if F, G, H were arbitrary, this would be $2 + 5 + 3, = 10$ conditions. The before-mentioned relation is, in fact, the condition in order to the existence of the cubic.

Similarly the cubic

$$QR' - Q'R - i(RP' - R'P) = 0$$

is the cubic

$$(J_AI_BI_CFGH) = 0.$$

The circle $PQ' - P'Q = 0$ is the conic through I, J, A, B, C, F, G, H; or it may in like manner be written $(IJABCFGH) = 0$; and we may write $(IJ) = 0$, as the equation of the line infinity. The functions denoted as above contain implicitly

constant multipliers which give, in the equation of the three-bar curve, one arbitrary parameter—and the equation thus is

$$(I_A J_B J_C FGH)(J_A I_B I_C FGH) - \theta (IJ)^2 (IJABCFGH)^2 = 0,$$

a form which puts in evidence that I, J are triple points having the tangents IA, IB, IC, and JA, JB, JC respectively (whence also A, B, C are foci), and that F, G, H are nodes; viz. the result is as follows:—

Taking A, B, C, F, G, H points in a circle, such that, Sum of the distances (being the angular distances from a fixed point in the circumference) of A, B, C is equal to the sum of the distances of F, G, H: then there exist the cubics $(I_A J_B J_C FGH) = 0$, $(J_A I_B I_C FGH) = 0$, and the sextic is as above.

Writing for shortness

$$(I_A J_B J_C FGH) = I_A, \quad (J_A I_B I_C FGH) = J_A,$$

then the above form is clearly one of three equivalent forms

$$\begin{aligned} U &= I_A J_A - \theta_1 \Omega^2, \\ &= I_B J_B - \theta_2 \Omega^2, \\ &= I_C J_C - \theta_3 \Omega^2. \end{aligned}$$

This implies an identical linear relation between the functions $I_A J_A$, $I_B J_B$, $I_C J_C$; whence also U and Ω^2 are each of them a linear function of any two of these quantities.

I originally obtained the equation of the curve in a form which, though far less valuable than the preceding one, is nevertheless worth preserving; viz. the equation

$$(QR' - Q'R)^2 + (RP' - R'P)^2 = (PQ' - P'Q)^2$$

may be written

$$(R^2 - P^2 - Q^2)(R'^2 - P'^2 - Q'^2) - (RR' - PP' - QQ')^2 = 0,$$

which equation, substituting therein for P, Q, R, P', Q', R' their values, gives the form in question.

Proceeding to the reduction, we have

$$\begin{aligned} R^2 - P^2 - Q^2 &= (x^2 + y^2 + b_1^2 - a_2^2)^2 - 4b_1^2 (x^2 + y^2) \\ &= (x^2 + y^2)^2 - 2(b_1^2 + a_2^2)(x^2 + y^2) + (b_1^2 - a_2^2)^2 \\ &= (x^2 + y^2 - \overline{b_1 + a_2}^2)(x^2 + y^2 - \overline{b_1 - a_2}^2); \end{aligned}$$

$$\begin{aligned} R'^2 - P'^2 - Q'^2 &= (\overline{x - a}^2 + y^2 + c_1^2 - a_3^2)^2 - 4c_1^2 (\overline{x - a}^2 + y^2) \\ &= (\overline{x - a}^2 + y^2)^2 - 2(c_1^2 + a_3^2)(\overline{x - a}^2 + y^2) + (c_1^2 - a_3^2)^2 \\ &= (\overline{x - a}^2 + y^2 - \overline{c_1 + a_3}^2)(\overline{x - a}^2 + y^2 - \overline{c_1 - a_3}^2). \end{aligned}$$

But the reduction of $RR' - PP' - QQ'$ is somewhat longer. We have

$$RR' - PP' - QQ' = (x^2 + y^2 + b_1^2 - a_2^2)(\overline{x-a}^2 + y^2 + c_1^2 - a_3^2)$$
$$- 4b_1c_1(x\overline{x-a} + y^2)\cos A - 4b_1c_1 ay \sin A :$$

and here

$$2b_1c_1\cos A = b_1^2 + c_1^2 - a_1^2,\quad 2x(x-a) + 2y^2 = x^2 + y^2 + (x-a)^2 + y^2 - a^2,$$

also $b_1c_1 \sin A = a_1p_1$, if p_1 be the perpendicular distance of O from the base B_1C_1. Hence the second line is

$$-(b_1^2 + c_1^2 - a_1^2)(x^2 + y^2 + \overline{x-a}^2 + y^2 - a^2) - 4aa_1p_1y,$$

and the whole is

$$\begin{aligned} = \quad & (x^2+y^2)(\overline{x-a}^2 + y^2) \\ & + (x^2+y^2)(a_1^2 - b_1^2 - a_3^2) \\ & + (\overline{x-a}^2 + y^2)(a_1^2 - c_1^2 - a_2^2) \\ & + (b_1^2 - a_2^2)(c_1^2 - a_3^2) + a^2(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y; \end{aligned}$$

whence, finally, we have

$$RR' - PP' - QQ' = (x^2 + y^2 + a_1^2 - a_2^2 - c_1^2)(\overline{x-a}^2 + y^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y.$$

Hence the equation of the curve is

$$(x^2 + y^2 - \overline{b_1 + a_2}^2)(x^2 + y^2 - \overline{b_1 - a_2}^2)(\overline{x-a}^2 + y^2 - \overline{c_1 + a_3}^2)(\overline{x-a}^2 + y^2 - \overline{c_1 - a_3}^2)$$
$$- \{(x^2 + y^2 + a_1^2 - a_2^2 - c_1^2)(\overline{x-a}^2 + y^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y\}^2 = 0,$$

where p_1 is given in terms of the constants a_1, b_1, c_1 by the equation

$$2a_1p_1 = \sqrt{2b_1^2c_1^2 + 2c_1^2a_1^2 + 2a_1^2b_1^2 - a_1^4 - b_1^4 - c_1^4}.$$

There are in the equation two terms, $(x^2+y^2)^2$, $(\overline{x-a}^2 + y^2)^2$, which destroy each other, and the remaining terms are of the order 6 at most. Hence the curve is a sextic; and it is, moreover, readily seen that the curve is tricircular. Assuming this, it appears at once that the lines $x + iy = 0$, $x - iy = 0$ are tangents to the curve at the two circular points at infinity. In fact, assuming either of these equations, we have $x^2 + y^2 = 0$, and the equation becomes

$$(b_1^2 - a_2^2)(-2ax + a^2 - \overline{c_1 + a_3}^2)(-2ax + a^2 - \overline{c_1 - a_3}^2)$$
$$- \{(a_1^2 - a_2^2 - c_1^2)(-2ax + a^2 + a_1^2 - a_3^2 - b_1^2)$$
$$+ (a^2 + a_1^2 - a_2^2 - a_3^2)(b_1^2 + c_1^2 - a_1^2) - 4aa_1p_1y\}^2 = 0,$$

a quadric equation. Hence there are on each of the two lines only two finite intersections, or the number of intersections at infinity is $= 4$; viz. the line is a tangent

to one of the branches at the triple point. Similarly, the lines $x-a+iy=0$, $x-a-iy=0$ are tangents. Thus the points C and B are foci. It might with somewhat more difficulty be shown from the equation that the point $x=b\cos C$, $y=b\sin C$ $\left(\text{where, as before, } b=\dfrac{a\sin A}{\sin B}\right)$, viz. the point A of the figure, is a focus; but I have not verified this directly. It clearly follows, if we generate the curve by means of the triangle OA_2C_2 and the fixed points C, A. Hence A, B, C are a triad of foci, and the theorem as to the nodes is that these lie on the circle drawn through the three foci A, B, C.

I prove in a somewhat different manner, for the sake of the further theory which arises, the theorem of the triple generation; for this purpose, constructing the foregoing Figure 2 (p. 553) by means of the three triangles OB_1C_1, OC_2A_2, OA_3B_3, but without assuming anything as to the form or position of the triangle ABC, I draw through O a line Ox, the position of which is in the first instance arbitrary, say its inclination to OC_2 is $=v$; and drawing Oy at right angles to Ox, I proceed, in regard to these axes, to find the coordinates of the points C, B. We have, for C,

$$x=a_2\cos v+b_1\cos(v+Z),\quad y=a_2\sin v+b_1\sin(v+Z);$$

for B,

$$x=c_1\cos(v+A+Z)+a_3\cos(v+A+Z+Y),$$
$$y=c_1\sin(v+A+Z)+a_3\sin(v+A+Z+Y);$$

or, writing for $Y+Z$ the value $\pi-X$, so that

$$v+A+Z+Y=\pi+v+A-X,$$

the coordinates of B are

$$x=c_1\cos(v+A+Z)-a_3\cos(v+A-X),$$
$$y=c_1\sin(v+A+Z)-a_3\sin(v+A-X).$$

Taking the two values of y equal to each other, the equation to determine v is

$$a_2\sin v+b_1\sin(v+Z)-c_1\sin(v+A+Z)+a_3\sin(v+A-X)=0.$$

We make the line Ox parallel to BC, so that, writing

$$x=a_2\cos v\qquad\qquad+b_1\cos(v+Z),$$
$$x-a=c_1\cos(v+A+Z)-a_3\cos(v+A-X),$$

we have

$$a=a_2\cos v+b_1\cos(v+Z)-c_1\cos(v+A+Z)+a_3\cos(v+A-X),$$

which determines the distance BC, $=a$. And moreover, writing

$$y=a_2\sin v\qquad\qquad+b_1\sin(v+Z),$$
$$=c_1\sin(v+A+Z)-a_3\sin(v+A-X),$$

we have y as the perpendicular distance of O from BC, and x and $(a-x)$ as the two parts into which BC is divided by the foot of this perpendicular. In the reduction of the formulæ we assume that the three triangles are similar; viz. we write

$$(a_1,\ b_1,\ c_1),\ (a_2,\ b_2,\ c_2),\ (a_3,\ b_3,\ c_3)$$
$$= k_1(\sin A,\ \sin B,\ \sin C),\ k_2(\sin A,\ \sin B,\ \sin C),\ k_3(\sin A,\ \sin B,\ \sin C);$$

and we use when required the relation $A+B+C=\pi$.

The equation for v becomes

$$k_1 \sin(v-C+Z)+k_2 \sin v+k_3 \sin(v+A-X)=0,$$

which may be written

$$L \sin v - M \cos v = 0,$$

where

$$L = k_2 + k_1 \cos(Z-C) + k_3 \cos(X-A),$$
$$M = \quad -k_1 \sin(Z-C) + k_3 \sin(X-A);$$

hence, putting

$$k^2 = k_1^2 + k_2^2 + k_3^2 + 2k_2k_3 \cos(X-A) + 2k_3k_1 \cos(Y-B) + 2k_1k_2 \cos(Z-C),$$

we have $L^2+M^2=k^2$, that is, $\sqrt{L^2+M^2}=k$, and therefore

$$k \sin v = M, \quad k \cos v = L,$$

which gives the value of v; and then, after all reductions,

$$\begin{aligned} kx = {} & k_1^2 \sin B \cos C + k_2^2 \sin A + k_3^2 . 0 + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[-\sin B \cos(Y+A)] \\ & + k_1k_2[\sin(B-A)\cos(Z+A) + 2 \sin A \sin B \sin(Z+A)], \end{aligned}$$

$$\begin{aligned} k(a-x) = {} & k_1^2 \sin C \cos B + k_2^2 . 0 + k_3^2 \sin A + k_2k_3 \sin A \cos(X-A) \\ & + k_3k_1[\sin(C-A)\cos(Y+A) + 2 \sin A \sin C \sin(Y+A)] \\ & + k_1k_2[-\sin C \cos(Z+A)], \end{aligned}$$

and

$$ky = k_1^2 \sin B \sin C + k_2k_3 \sin A \sin(X-A) + k_3k_1 \sin B \sin(Y+A) + k_1k_2 \sin C \sin(Z+A).$$

The first and second equations give $ka = k^2 \sin A$, that is, $a = k \sin A$; and, similarly, $b = k \sin B$, $c = k \sin C$; viz. we have

$$(a,\ b,\ c) = k(\sin A,\ \sin B,\ \sin C),$$

or the triangle ABC is similar to the other three triangles, its magnitude being given by the foregoing equation for k^2. These are the properties which give the triple generation.

Changing the notation of the coordinates, and writing (x, y, z) for the perpendicular distances from O on the sides of the triangle ABC, we have, as above,

$$kx = k_1^2 \sin B \sin C + k_2 k_3 \sin A \sin (X - A) + k_3 k_1 \sin B \sin (Y + A) + k_1 k_2 \sin C \sin (Z + A),$$

and therefore

$$ky = k_2^2 \sin B \sin C + k_2 k_3 \sin A \sin (X + B) + k_3 k_1 \sin B \sin (Y - B) + k_1 k_2 \sin C \sin (Z + B),$$

$$kz = k_3^2 \sin C \sin A + k_2 k_3 \sin A \sin (X + C) + k_3 k_1 \sin B \sin (Y + C) + k_1 k_2 \sin C \sin (Z - C),$$

values which give, as they should do,

$$x \sin A + y \sin B + z \sin C = k^2 \sin A \sin B \sin C.$$

Taking (x, y, z) as simply proportional (instead of equal) to the perpendicular distances, then (x, y, z) will be a system of trilinear coordinates in which the equation of the line infinity is

$$x \sin A + y \sin B + z \sin C = 0;$$

and considering (x, y, z) as proportional to the foregoing values, and in these X, Y, Z as connected by the equation $X + Y + Z = \pi$ and by the equation which determines k^2, the coordinates (x, y, z) are given as proportional to functions of a single parameter, so that the equations in effect determine the curve which is the locus of O.

But to determine the order, &c., the trigonometrical functions must be expressed algebraically; and this is done most readily by introducing instead of X, Y, Z the functions

$$\cos X + i \sin X, \quad \cos Y + i \sin Y, \quad \cos Z + i \sin Z, \quad = \xi, \eta, \zeta;$$

and we may at the same time, in place of A, B, C, introduce the functions

$$\cos A + i \sin A, \quad \cos B + i \sin B, \quad \cos C + i \sin C, \quad = \alpha, \beta, \gamma.$$

The relation $X + Y + Z = \pi$ gives $\xi\eta\zeta = -1$; and similarly $A + B + C = \pi$ gives $\alpha\beta\gamma = -1$.

We have

$$\cos (X - A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right), \quad i \sin (X - A) = \tfrac{1}{2}\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right), \text{ \&c.};$$

the equation $k^2 = k_1^2 + \&c.$ becomes

$$k^2 = k_1^2 + k_2^2 + k_3^2 + k_2 k_3 \left(\frac{\xi}{\alpha} + \frac{\alpha}{\xi}\right) + k_3 k_1 \left(\frac{\eta}{\beta} + \frac{\beta}{\eta}\right) + k_1 k_2 \left(\frac{\zeta}{\gamma} + \frac{\gamma}{\zeta}\right),$$

or, as this may be written,

$$(-k^2 + k_1^2 + k_2^2 + k_3^2) + k_2 k_3 \left(\frac{\xi}{\alpha} - \alpha\eta\zeta\right) + k_3 k_1 \left(\frac{\eta}{\beta} - \beta\zeta\xi\right) + k_1 k_2 \left(\frac{\zeta}{\gamma} - \gamma\xi\eta\right) = 0.$$

Also the value of x is proportional to

$$k_1^2 \left(\beta - \frac{1}{\beta}\right)\left(\gamma - \frac{1}{\gamma}\right) + k_2 k_3 \left(\alpha - \frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha} - \frac{\alpha}{\xi}\right) + k_3 k_1 \left(\beta - \frac{1}{\beta}\right)\left(\alpha\eta - \frac{1}{\alpha\eta}\right) + k_1 k_2 \left(\gamma - \frac{1}{\gamma}\right)\left(\alpha\zeta - \frac{1}{\alpha\zeta}\right),$$

or, what is the same thing, to

$$k_1^2\left(\beta-\frac{1}{\beta}\right)\left(\gamma-\frac{1}{\gamma}\right)+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\frac{\xi}{\alpha}+\alpha\eta\zeta\right)+k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\alpha\eta+\frac{\zeta\xi}{\alpha}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha\zeta+\frac{\xi\eta}{\alpha}\right);$$

with the like expressions as to the values of y and z. Introducing for homogeneity a quantity ω, viz. writing $\frac{\xi}{\omega}$, $\frac{\eta}{\omega}$, $\frac{\zeta}{\omega}$ in place of ξ, η, ζ, we have the parameters $(\xi, \eta, \zeta, \omega)$ connected by the homogeneous equations

$$\xi\eta\zeta+\omega^3=0,$$

$$(-k^2+k_1^2+k_2^2+k_3^2)\,\omega^2+k_2k_3\left(\frac{\xi\omega}{\alpha}-\alpha\eta\zeta\right)+k_3k_1\left(\frac{\eta\omega}{\beta}-\beta\zeta\xi\right)+k_1k_2\left(\frac{\zeta\omega}{\gamma}-\gamma\xi\eta\right)=0,$$

and the ratios of the coordinates are

$$\begin{aligned}
x : y : z = {} & k_1^2\left(\beta-\frac{1}{\beta}\right)\left(\gamma-\frac{1}{\gamma}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\frac{\xi\omega}{\alpha}+\alpha\eta\zeta\right)\\
& \qquad +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\alpha\eta\omega+\frac{\zeta\xi}{\alpha}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha\zeta\omega+\frac{\xi\eta}{\alpha}\right)\\
: {} & k_2^2\left(\gamma-\frac{1}{\gamma}\right)\left(\alpha-\frac{1}{\alpha}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\beta\xi\omega+\frac{\eta\zeta}{\beta}\right)\\
& \qquad +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\frac{\eta\omega}{\beta}+\beta\zeta\xi\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\beta\zeta\omega+\frac{\xi\eta}{\beta}\right)\\
: {} & k_3^2\left(\alpha-\frac{1}{\alpha}\right)\left(\beta-\frac{1}{\beta}\right)\omega^2+k_2k_3\left(\alpha-\frac{1}{\alpha}\right)\left(\gamma\xi\omega+\frac{\eta\zeta}{\gamma}\right)\\
& \qquad +k_3k_1\left(\beta-\frac{1}{\beta}\right)\left(\gamma\eta\omega+\frac{\zeta\xi}{\gamma}\right)+k_1k_2\left(\gamma-\frac{1}{\gamma}\right)\left(\frac{\zeta\omega}{\gamma}+\gamma\xi\eta\right).
\end{aligned}$$

Suppose, for shortness, these are $x : y : z = P : Q : R$. Observe that the form of the equations is $\xi\eta\zeta+\omega^3=0$, $\Omega=0$, and $x : y : z = P : Q : R$, where Ω and P, Q, R are each of them a quadric function of the form $(\omega^2, \omega\xi, \omega\eta, \omega\zeta, \eta\zeta, \zeta\xi, \xi\eta)$, the terms in ξ^2, η^2, ζ^2 being wanting.

Treating $(\xi, \eta, \zeta, \omega)$ as the coordinates of a point in space, the equation $\xi\eta\zeta+\omega^3=0$ is a cubic surface having a binode at each of the points $(\xi=0, \omega=0)$, $(\eta=0, \omega=0)$, $(\zeta=0, \omega=0)$, and the second equation is that of a quadric surface passing through these three points; hence the two equations together represent a sextic in space, or say a skew sextic, having a node at each of these three points. The equations $x : y : z = P : Q : R$ establish a (1, 1) correspondence between the locus of O and this skew sextic. To find the degree of the locus we intersect it by the arbitrary line $ax+by+cz=0$; viz. we intersect the skew sextic by the quadric surface $aP+bQ+cR=0$. This is a surface passing through the three nodes of the skew sextic, and it therefore besides intersects the skew sextic in $12-2.3$, $=6$ points. Hence the locus is (as it should be) a sextic.

I consider the point $\eta=0$, $\zeta=0$, $\omega=0$, or say the point (1, 0, 0, 0), of the skew sextic. This is a node, and for the consecutive point on one branch we have $\eta : \zeta : \omega = m\epsilon : l\epsilon^2 : n\epsilon$, where ϵ is infinitesimal. The equation of the cubic surface gives $lm+n^2=0$, and the equation of the quadric surface gives $k_2k_3 . \frac{\omega}{\alpha} - k_1k_2\gamma\eta = 0$, that is, $k_3\omega = \alpha\gamma k_1\eta$, which, in fact, determines the ratio $l : m$; but it will be convenient to retain the equation in this form. For the corresponding values of (x, y, z) we have

$$\begin{aligned} x : y : z = {} & k_3\left(\alpha - \frac{1}{\alpha}\right)\frac{\omega}{\alpha} + k_1\left(\gamma - \frac{1}{\gamma}\right)\frac{\eta}{\alpha} \\ : {} & k_3\left(\alpha - \frac{1}{\alpha}\right)\beta\omega + k_1\left(\gamma - \frac{1}{\gamma}\right)\frac{\eta}{\beta} \\ : {} & k_3\left(\alpha - \frac{1}{\alpha}\right)\gamma\omega + k_1\left(\gamma - \frac{1}{\gamma}\right)\gamma\eta, \end{aligned}$$

which, writing for $k_3\omega$ its value $=\alpha\gamma k_1\eta$, become

$$\begin{aligned} x : y : z = {} & \left(\alpha - \frac{1}{\alpha}\right)\gamma + \left(\gamma - \frac{1}{\gamma}\right)\frac{1}{\alpha} = & & \alpha\gamma - \frac{\gamma}{\alpha} + \frac{\gamma}{\alpha} - \frac{1}{\gamma\alpha} \\ : {} & \left(\alpha - \frac{1}{\alpha}\right)\alpha\beta\gamma + \left(\gamma - \frac{1}{\gamma}\right)\frac{1}{\beta} & : & -\alpha + \frac{1}{\alpha} - \alpha\gamma^2 + \alpha \\ : {} & \left(\alpha - \frac{1}{\alpha}\right)\alpha\gamma^2 + \left(\gamma - \frac{1}{\gamma}\right)\gamma & : & \alpha^2\gamma^2 - \gamma^2 + \gamma^2 - 1, \end{aligned}$$

the last set of values being obtained by aid of the relation $\alpha\beta\gamma = -1$; viz. we thus have

$$x : y : z = \frac{1}{\alpha\gamma}(-1+\alpha^2\gamma^2) : -\frac{1}{\alpha}(-1+\alpha^2\gamma^2) : (-1+\alpha^2\gamma^2),$$

that is,

$$x : y : z = \frac{1}{\gamma} : -1 : \alpha,$$

which are, in fact, the values belonging to one of the circular points at infinity. For the consecutive point on the other branch we should obtain in like manner $x : y : z = \gamma : -1 : \frac{1}{\alpha}$, which are the values belonging to the other circular point at infinity; viz. the node (1, 0, 0, 0) of the skew sextic corresponds to the circular points at infinity. But, in like manner, the other two nodes (0, 1, 0, 0) and (0, 0, 1, 0) each correspond to the circular points at infinity, or say we have in the skew sextic the three nodes each corresponding to one circular point at infinity, and the same three nodes each corresponding to the other circular point at infinity; viz. we thus prove that each of the circular points at infinity is a triple point on the locus of O.

In order not to interrupt the demonstration, I have assumed the formulæ which, in the system of coordinates defined by taking x, y, z proportional to the perpendiculars on the sides of a triangle ABC, or where the equation of the line infinity is

$$x \sin A + y \sin B + z \sin C = 0,$$

give the circular points at infinity; viz. writing

$$\cos A + i \sin A,\ \cos B + i \sin B,\ \cos C + i \sin C = \alpha,\ \beta,\ \gamma,$$

the coordinates for the two points respectively are

$$\begin{aligned} x : y : z &= -1 : \quad \gamma : \frac{1}{\beta} \\ &= \frac{1}{\gamma} : -1 : \alpha \\ &= \beta : \frac{1}{\alpha} : -1, \end{aligned} \quad \text{and} \quad \begin{aligned} x : y : z &= -1 : \quad \frac{1}{\gamma} : \beta \\ &= \gamma : -1 : \frac{1}{\alpha} \\ &= \frac{1}{\beta} : \alpha : -1, \end{aligned}$$

the three values for each point being equivalent in virtue of the relation $\alpha\beta\gamma = -1$. This is, in fact, under a different form, the theorem given in my Smith's Prize paper for 1875; viz. the theorem was: If λ, μ, ν are the inclinations to a fixed line of the perpendiculars let fall from an interior point on the sides of the fundamental triangle ABC, then, in the system of trilinear coordinates in which the coordinates of a point P are proportional to the triangles PBC, PCA, PAB (or where the equation of the line infinity is $x+y+z=0$), the coordinates of the circular points at infinity are proportional, those of the one point to $e^{i\lambda}\sin(\mu-\nu)$, $e^{i\mu}\sin(\nu-\lambda)$, $e^{i\nu}\sin(\lambda-\mu)$, and those of the other point to $e^{-i\lambda}\sin(\mu-\nu)$, $e^{-i\mu}\sin(\nu-\lambda)$, $e^{-i\nu}\sin(\lambda-\mu)$.

In the plane curve, the lines drawn from A, B, C to the circular points at infinity are:

	To the one point.	To the other point.
From A,	$\alpha y + z = 0$,	$y + \alpha z = 0$;
„ B,	$\beta z + x = 0$,	$z + \beta x = 0$;
„ C,	$\gamma x + y = 0$,	$x + \gamma y = 0$.

Each of these lines, *quà* tangent at a triple point, meets the curve in the circular point at infinity counted four times, and in two other points. The corresponding points on the skew sextic should be a node counted twice, the two other nodes counted each once, and two other points. The proof that this is so would show that the points A, B, C are a triad of foci. There is also the question of the determination of the values of $(\xi, \eta, \zeta, \omega)$ which correspond to the nodes of the plane curve. But I have not further pursued the theory.

ADDITION.—Since writing the foregoing paper, I have found that the relation between the nodes and foci (sum of angular distances of the foci = sum of angular distances of the nodes) may be expressed in a different form; viz. the triangle of the foci and the triangle of the nodes are circumscribed to a parabola (having its focus on the circle); and I have made in relation to the question the following further investigations:—

Considering a circle: and a parabola having its focus at K, a point of the circle; then if, as usual, I, J are the circular points at infinity, we have IJK a triangle inscribed in the circle and circumscribed to the parabola; hence there exists a

singly-infinite series of in- and circumscribed triangles, so that, drawing from a point A of the circle tangents to the parabola again meeting the circle in the points B and C respectively, BC will be a tangent to the parabola; or, what is the same thing, starting with the triangle ABC inscribed in the circle, we can, with the arbitrary point K on the circle as focus, describe a parabola touching the three sides of the triangle ABC; viz. the parabola described to touch two of the sides of the triangle will touch the third side.

Taking, then, a circle radius $\frac{1}{2}k$, and upon it the three points A, B, C determined by the angles 2α, 2β, 2γ respectively (viz. the coordinates of A are x, $y = \frac{1}{2}k\cos 2\alpha$, $\frac{1}{2}k\sin 2\alpha$, &c.), and a point K determined by the angle 2κ (suppose for a moment the origin is at K), the equation of a parabola having K for its focus will be

$$x^2 + y^2 = (x\cos 2\theta + y\sin 2\theta - p)^2,$$

or, what is the same thing,

$$(x\sin 2\theta - y\cos 2\theta)^2 + 2p\,(x\cos 2\theta + y\sin 2\theta) - p^2 = 0,$$

where θ, p are in the first instance arbitrary; and the condition in order that $\xi x + \eta y + \zeta = 0$ may be a tangent is easily found to be

$$p\,(\xi^2 + \eta^2) + 2\xi\cos 2\theta + 2\eta\sin 2\theta = 0.$$

It is to be shown that p, θ can be determined so that the parabola shall touch each of the lines BC, CA, AB.

Taking the origin at the centre, the equation of BC is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k\cos(\beta-\gamma) = 0,$$

as is at once verified by showing that this equation is satisfied by the values

$$x,\ y = \tfrac{1}{2}k\cos 2\beta,\ \tfrac{1}{2}k\sin 2\beta,\ \text{and}\ = \tfrac{1}{2}k\cos 2\gamma,\ \tfrac{1}{2}k\sin 2\gamma.$$

Hence, transforming to the point K as origin, the equation is

$$[x + \tfrac{1}{2}k\cos 2\kappa]\cos(\beta+\gamma) + [y + \tfrac{1}{2}k\sin 2\kappa]\sin(\beta+\gamma) - \tfrac{1}{2}k\cos(\beta-\gamma) = 0;$$

viz. this is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - \tfrac{1}{2}k[\cos(\beta-\gamma) - \cos(\beta+\gamma-2\kappa)] = 0;$$

or, finally, it is

$$x\cos(\beta+\gamma) + y\sin(\beta+\gamma) - k\sin(\kappa-\beta)\sin(\kappa-\gamma) = 0.$$

Hence the condition of contact with the line BC is

$$p = 2k\sin(\kappa-\beta)\sin(\kappa-\gamma)\cos(2\theta-\beta-\gamma);$$

and, similarly, the condition of contact with the line CA is

$$p = 2k\sin(\kappa-\gamma)\sin(\kappa-\alpha)\cos(2\theta-\gamma-\alpha);$$

viz. these conditions determine the unknown quantities p, θ. It is at once seen that we have

$$2\theta - \beta - \gamma = \tfrac{1}{2}\pi - (\kappa - \alpha), \text{ that is, } 2\theta = \tfrac{1}{2}\pi - \kappa + \alpha + \beta + \gamma;$$

and then

$$p = 2k \sin(\kappa - \alpha) \sin(\kappa - \beta) \sin(\kappa - \gamma);$$

from symmetry, we see that the parabola touches also the side AB.

Suppose, next, F, G are points on the circle determined by the angles $2f$, $2g$; retaining p and θ to denote their values,

$$p = 2k \sin(\kappa - \alpha) \sin(\kappa - \beta) \sin(\kappa - \gamma), \text{ and } 2\theta = \tfrac{1}{2}\pi - \kappa + \alpha + \beta + \gamma,$$

the condition, in order that FG may be a tangent, is

$$p = 2k \sin(\kappa - f) \sin(\kappa - g) \cos(2\theta - f - g);$$

viz. determining h by the equation

$$\alpha + \beta + \gamma = f + g + h,$$

this is

$$p = 2k \sin(\kappa - f) \sin(\kappa - g) \sin(\kappa - h),$$

or, what is the same thing,

$$\sin(\kappa - \alpha) \sin(\kappa - \beta) \sin(\kappa - \gamma) = \sin(\kappa - f) \sin(\kappa - g) \sin(\kappa - h);$$

viz. this equation, considering therein h as standing for $\alpha + \beta + \gamma - f - g$, is the relation which must subsist between f and g, in order that the line FG may be a tangent to the parabola. And then, h being determined as above, and the point H on the circle being determined by the angle $2h$, it is clear that the lines GH, HF will also be tangents to the parabola; viz. FGH will be an in- and circumscribed triangle, provided only f, g, h satisfy the above-mentioned two equations. The latter of these, if f, g, h satisfy only the relation $\alpha + \beta + \gamma = f + g + h$, serves to determine κ; and then, θ and κ denoting as above, the equation of the parabola is

$$x^2 + y^2 = (x \cos 2\theta + y \sin 2\theta - p)^2;$$

and it thus appears that the condition in question, $\alpha + \beta + \gamma = f + g + h$, is equivalent to the condition that the triangles ABC, FGH shall be circumscribed to the same parabola.

It is to be remarked that the distances KA, KB, &c. are equal to $k \sin(\kappa - \alpha)$, $k \sin(\kappa - \beta)$, &c.; hence the condition

$$\sin(\kappa - \alpha) \sin(\kappa - \beta) \sin(\kappa - \gamma) = \sin(\kappa - f) \sin(\kappa - g) \sin(\kappa - h)$$

becomes

$$KA \,.\, KB \,.\, KC = KF \,.\, KG \,.\, KH;$$

viz. the focus K is a point on the circle such that the product of its (linear) distances from the foci A, B, C is equal to the product of its (linear) distances from the nodes F, G, H.

It is to be remarked that the foregoing equation in κ determines a single position of the point K; viz. it determines $\tan\kappa$, and therefore $\sin 2\kappa$ and $\cos 2\kappa$, linearly. The equation is, in fact, a cubic equation in $\tan\kappa$, satisfied identically by $\tan\kappa = i$ and $\tan\kappa = -i$, and therefore reducible to a linear equation.

Write for a moment $\tan\kappa = \omega$, and

$$(\tan\kappa - \tan\alpha)(\tan\kappa - \tan\beta)(\tan\kappa - \tan\gamma) = \omega^3 - p\omega^2 + q\omega - r,$$

$$(\tan\kappa - \tan f)(\tan\kappa - \tan g)(\tan\kappa - \tan h) = \omega^3 - p'\omega^2 + q'\omega - r';$$

also

$$M = \cos f \cos g \cos h \div \cos\alpha \cos\beta \cos\gamma.$$

Then we have

$$\omega^3 - p\omega^2 + q\omega - r = M(\omega^3 - p'\omega^2 + q'\omega - r'),$$

where

$$p = r + M(r' - p),\quad q = 1 + M(q' - 1).$$

Substituting these values, the equation becomes

$$\omega^3 - r\omega^2 + \omega - r = M(\omega^3 - r'\omega^2 + \omega - r'),$$

viz. dividing by $\omega^2 + 1$, this is $\omega - r = M(\omega - r')$; or substituting for r, r', M their values,

$$(\cos\alpha\cos\beta\cos\gamma - \cos f\cos g\cos h)\tan\kappa = (\sin\alpha\sin\beta\sin\gamma - \sin f\sin g\sin h),$$

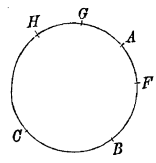
which is the value of $\tan\kappa$, and then

$$\sin 2\kappa = \frac{2\tan\kappa}{1 + \tan^2\kappa},\quad \cos 2\kappa = \frac{1 - \tan^2\kappa}{1 + \tan^2\kappa}.$$

It may be further noticed that, if the parabola intersect the circle in a point L, and the tangent at L to the parabola again meet the circle in M, then, if $2l$, $2m$ are the angles for the points L, M, we have l, m, m for values of f, g, h, whence l, m are determined by the equations

$$l + 2m = \alpha + \beta + \gamma,\ \sin(\kappa - l)\sin^2(\kappa - m) = \sin(\kappa - \alpha)\sin(\kappa - \beta)\sin(\kappa - \gamma);$$

but as the circle intersects the parabola not only in two real points, but in two other imaginary points, there is no simple formula for the determination of l and m.



To determine the linkage when the nodes are given, suppose that, in the generation by O, the vertex of the triangle OB_1C_1, we have O at the node F: then, if τ, σ are the distances of C, B from the node in question, we have, as in the memoir,

$$(b_1^2 + \tau^2 - a_2^2)\, c_1\sigma = (c_1^2 + \sigma^2 - a_3^2)\, b_1\tau,$$

that is,

$$(b_1^2 - a_2^2)\, c_1\sigma - (c_1^2 - a_3^2)\, b_1\tau + \sigma\tau\,(c_1\tau - b_1\sigma) = 0,$$

or, what is the same thing,

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + \sigma\tau\,(c\tau - b\sigma) = 0.$$

Suppose, as in the figure, that F is between B and A; then, if $AF = \rho$, we have $c\tau = b\sigma + a\rho$, and the equation becomes

$$(b_1^2 - a_2^2)\, c\sigma - (c_1^2 - a_3^2)\, b\tau + a\rho\sigma\tau = 0.$$

Similarly, if, as in the figure, G is on the other side of A, that is, between A and C, and if ρ', σ', τ' be the distances AG, BG, CG, then $b\sigma' = c\tau' + a\rho'$, that is, $c\tau' - b\sigma' = -a\rho'$, and the corresponding equation is

$$(b_1^2 - a_2^2)\, c\sigma' - (c_1^2 - a_3^2)\, b\tau' - a\rho'\sigma'\tau' = 0.$$

We hence find

$$(b_1^2 - a_2^2)\, c\,(\sigma\tau' - \sigma'\tau) + a\tau\tau'\,(\rho\sigma + \rho'\sigma') = 0,$$

$$(c_1^2 - a_3^2)\, b\,(\sigma\tau' - \sigma'\tau) + a\sigma\sigma'\,(\rho\tau + \rho'\tau') = 0.$$

But we have

$$BG\,.\,CF = BF\,.\,CG + BC\,.\,FG,$$

that is,

$$\sigma'\tau = \sigma\tau' + a\,.\,FG, \text{ or } \sigma\tau' - \sigma'\tau = -a\,.\,FG;$$

also,

$$\rho\sigma + \rho'\sigma' = AF\,.\,BF + AG\,.\,BG, = FG\,.\,CH, = FG\,.\,\tau'',$$

and

$$\rho\tau + \rho'\tau' = AF\,.\,CF + AG\,.\,CG, = FG\,.\,BH, = FG\,.\,\sigma'',$$

as may be shown without difficulty, ρ'', σ'', τ'' being the distances AH, BH, CH. Hence the equations become

$$c\,(b_1^2 - a_2^2) - \tau\tau'\tau'' = 0,$$

$$b\,(c_1^2 - a_3^2) - \sigma\sigma'\sigma'' = 0,$$

showing that, the foci being as in the figure, $b_1^2 - a_2^2$ and $c_1^2 - a_3^2$ are each of them positive; viz. that, in the generation by the triangle OC_1B_1, the radial bars a_2, a_3 are shorter than the sides b_1, c_1 respectively. Substituting for b_1, &c. the values $k_1 \sin B$, &c.; also, instead of k_1, k_2, k_3, introducing the quantities λ_1, λ_2, λ_3, where

$$k_1,\ k_2,\ k_3 = \lambda_1 \sin A,\ \lambda_2 \sin B,\ \lambda_3 \sin C,$$

these equations become

$$c\,(\lambda_1^2 - \lambda_2^2) \sin^2 A \sin^2 B = \tau\tau'\tau'',$$

$$b\,(\lambda_1^2 - \lambda_3^2) \sin^2 A \sin^2 C = \sigma\sigma'\sigma'';$$

or, as these may be written, putting for shortness $M = \sin A \sin B \sin C$,

$$M^2k^2\,(\lambda_1^2 - \lambda_2^2) = c\,\tau\tau'\tau'',$$

$$M^2k^2\,(\lambda_1^2 - \lambda_3^2) = b\,\sigma\sigma'\sigma''.$$

All the quantities have so far been regarded as positive, and the formulæ are applicable to the particular figure; but, to present them in a form applicable to any order of the nodes and foci, we have only to write the equations in the forms

$$M^2(\lambda_1^2-\lambda_2^2)=k^2\sin(\alpha-\beta)\sin(f-\gamma)\sin(g-\gamma)\sin(h-\gamma),$$
$$M^2(\lambda_1^2-\lambda_3^2)=k^2\sin(\alpha-\gamma)\sin(f-\beta)\sin(g-\beta)\sin(h-\beta);$$

and these may be replaced by the system

$$M^2(\lambda_3^2-\lambda_2^2)=k^2\sin(\beta-\gamma)\sin(f-\alpha)\sin(g-\alpha)\sin(h-\alpha),$$
$$M^2(\lambda_3^2-\lambda_1^2)=k^2\sin(\gamma-\alpha)\sin(f-\beta)\sin(g-\beta)\sin(h-\beta),$$
$$M^2(\lambda_1^2-\lambda_2^2)=k^2\sin(\alpha-\beta)\sin(f-\gamma)\sin(g-\gamma)\sin(h-\gamma),$$

since the first of these equations is implied in the other two; and then, reverting to the original form, we may write

$$M^2k^2(\lambda_2^2-\lambda_3^2)=BC\,.\,FA\,.\,GA\,.\,HA,$$
$$M^2k^2(\lambda_3^2-\lambda_1^2)=CA\,.\,FB\,.\,GB\,.\,HB,$$
$$M^2k^2(\lambda_1^2-\lambda_2^2)=AB\,.\,FC\,.\,GC\,.\,HC,$$

it being understood that the distances BC, FA, &c., which enter into these equations, are not all positive, but that they stand for $k\sin(\beta-\gamma)$, $k\sin(f-\alpha)$, &c., and that their signs are to be taken accordingly. Or, again, these may be written

$$BC\,(c_2^2-b_3^2)=FA\,.\,GA\,.\,HA,$$
$$CA\,(a_3^2-c_1^2)=FB\,.\,GB\,.\,HB,$$
$$AB\,(b_1^2-a_2^2)=FC\,.\,GC\,.\,HC,$$

where the signs are as just mentioned. We may say that $\pm(c_2^2-b_3^2)$ is the modulus for the focus A; and the formula then shows that this modulus, taken positively, is equal to the product of the distances FA, GA, HA of A from the three nodes respectively, divided by BC, the distance of the other two foci from each other.

624.

ON THE BICURSAL SEXTIC.

[From the *Proceedings of the London Mathematical Society*, vol. VII. (1875—1876), pp. 166—172. Read March 10, 1876.]

IN the paper "On the mechanical description of certain sextic curves," *Proceedings of the London Mathematical Society*, vol. IV. (1872), pp. 105—111, [504], I obtained the bicursal sextic as a rational transformation of a binodal quartic. The theory was in effect as follows: taking Ω, P, Q, R, each of them a function of λ, μ of the form $(*\between 1, \lambda)^2(1, \mu)^2$, and considering (λ, μ) as connected by the equation $\Omega = 0$, (viz. λ, μ being coordinates, this represents a binodal quartic), then, if we assume $x : y : z = P : Q : R$, the locus of the point (x, y, z) is a curve rationally connected with the binodal quartic, viz. the points of the two curves have with each other a (1, 1) correspondence; whence the locus in question, say the curve $U = 0$, is bicursal. The degree is obtained as the number of the intersections of the curve by an arbitrary line, or, what is the same thing, the number of the variable intersections of the corresponding $\lambda\mu$-curves

$$\Omega = 0, \quad \alpha P + \beta Q + \gamma R = 0,$$

viz. each of these being a quartic curve having the same two nodes, the nodes each count as 4 intersections, and the number of the remaining intersections is $4 . 4 - 2 . 4, = 8$, and thus the curve $U = 0$ is in general of the order 8. But if the curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ have (besides the nodes) k common intersections, then these are also fixed intersections of the two curves $\Omega = 0$, $\alpha P + \beta Q + \gamma R = 0$, and the number of variable intersections is reduced to $8 - k$; we have thus $8 - k$ as the order of the curve $U = 0$. In particular, if $k = 2$, then the curve is a bicursal sextic.

The theory assumes a different and more simple form if, in the several functions Ω, P, Q, R, we suppose that the terms in λ^2, μ^2 are wanting. The curves $\Omega = 0$, $P = 0$, $Q = 0$, $R = 0$ are here cubics having two common points; the curve $U = 0$, *quà*

rational transformation of the cubic $\Omega=0$, is still a bicursal curve; but its order is given as the number of the variable intersections of the cubics

$$\Omega=0,\quad \alpha P+\beta Q+\gamma R=0,$$

viz. this is $=3.3-2$, $=7$. But if the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ have (besides the before-mentioned two common points) k other common points, then the number of the variable intersections is $=7-k$: and this is therefore the order of the curve $U=0$. In particular, if $k=1$, then the curve is a bicursal sextic. And, in the present paper, I consider the binodal sextic as thus obtained, viz. as given by the equations $\Omega=0$, $x:y:z=P:Q:R$, where $\Omega=0$, $P=0$, $Q=0$, $R=0$ are cubics, having (in all) three common points.

The bicursal sextic has in general 9 nodes; but 3 of these may unite together into a triple point: this will be the case if, in the series of curves $\alpha P+\beta Q+\gamma R=0$, there are any two curves which have 3 common intersections with the curve $\Omega=0$. (Observe that we throughout disregard the 3 common points of the curves $\Omega=0$, $P=0$, $Q=0$, $R=0$, and attend only to the 6 variable points of intersection of the curves $\Omega=0$ and $\alpha P+\beta Q+\gamma R=0$,—the meaning is, that there are two curves of the series such that, attending only to the 6 variable intersections of each of them with the curve $\Omega=0$, there are three common intersections.) For, supposing the two curves to be $\alpha P+\beta Q+\gamma R=0$ and $\alpha' P+\beta' Q+\gamma' R=0$, then any curve whatever

$$\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$$

has the same three intersections with the curve $\Omega=0$, say these are the points A_1, A_2, A_3, the coordinates of which are independent of θ. Hence the line

$$(\alpha x+\beta y+\gamma z)+\theta(\alpha' x+\beta' y+\gamma' z)=0$$

intersects the curve $U=0$ in six points, three of which, as corresponding to the points A_1, A_2, A_3, are independent of θ, viz. they are the same three points for any line whatever of the series; and this means that the curve $U=0$ has at the point

$$(\alpha x+\beta y+\gamma z=0,\quad \alpha' x+\beta' y+\gamma' z=0)$$

a triple point; and that to this triple point correspond the three points A_1, A_2, A_3.

We may, in the series of lines $\alpha x+\beta y+\gamma z+\theta(\alpha' x+\beta' y+\gamma' z)=0$, rationally determine θ so that one of the three variable points of intersection shall correspond to A_1, A_2, or A_3; viz. θ must be such that the curve $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$ shall touch the curve $\Omega=0$ at one of the points A_1, A_2, A_3. The three lines thus determined are the three tangents to the curve at the triple point: and the three branches may be considered as corresponding to the three points A_1, A_2, A_3, respectively.

There is no loss of generality in assuming that the triple point is the point $(x=0,\ z=0)$; the condition then simply is that the curves $P=0$, $R=0$ shall have three common intersections with the curve $\Omega=0$; and the tangents at the triple point are $x+\theta z=0$, θ being so determined that one of the three variable points of intersection shall correspond to one of the three points A_1, A_2, A_3: in particular, if this is the case for the line $x=0$, then this line will be one of the tangents at the triple point.

The bicursal sextic may have a second triple point, viz. three other nodes may unite together into a triple point. The theory is precisely the same: we must have two other curves $\alpha P+\beta Q+\gamma R=0$, $\alpha' P+\beta' Q+\gamma' R=0$, having with the curve $\Omega=0$ three common intersections B_1, B_2, B_3: there is then a second triple point

$$(\alpha x+\beta y+\gamma z=0,\quad \alpha' x+\beta' y+\gamma' z=0);$$

and, to find the tangents at this point, we must determine θ so that one of the variable points of intersection of the line

$$\alpha x+\beta y+\gamma z+\theta(\alpha' x+\beta' y+\gamma' z)=0$$

with the sextic shall correspond with B_1, B_2, or B_3; viz. θ must be such that the curve $\alpha P+\beta Q+\gamma R+\theta(\alpha' P+\beta' Q+\gamma' R)=0$ shall touch the curve $\Omega=0$ at one of the points B_1, B_2, B_3. In particular, if, as before, the curves $P=0$, $R=0$ have three common intersections with the curve $\Omega=0$, and if, moreover, the curves $Q=0$, $R=0$ have three common intersections with the curve $\Omega=0$, then the bicursal sextic will have the two triple points $(x=0,\ z=0)$ and $(y=0,\ z=0)$; and it may further happen that the line $x=0$ is a tangent at the first triple point, and the line $y=0$ a tangent at the second triple point. The sextic may in like manner have a third triple point, but this is a special case which I do not at present consider.

I write for greater convenience $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ in place of λ, μ, so as to make Ω, P, Q, R each of them a homogeneous cubic function of (λ, μ, ν); and I give to these functions, not the most general values belonging to a bicursal sextic with two triple points, but the values in the form obtained for them, as appearing further on, in the problem of three-bar motion; viz. the equations $\Omega=0$, $x:y:z=P:Q:R$ are respectively taken to be

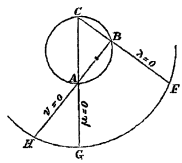
$$\nu(h\nu\lambda+f\lambda^2)+\mu(g\nu^2+e\nu\lambda+g\lambda^2)+\mu^2(f\nu+h\lambda)=0,$$

$$x:y:z=\lambda\mu(a\lambda+b\mu):\nu^2(c\lambda+d\mu):\lambda\mu\nu.$$

The four curves $\Omega=0$, $P=0$, $Q=0$, $R=0$ have thus the three common intersections

$$(\mu=0,\ \nu=0),\ (\nu=0,\ \lambda=0),\ (\lambda=0,\ \mu=0),$$

represented in the figure by the points A, B, C; the curve drawn in the figure is the curve $\Omega=0$, and the points F, G, H are the third points of intersection of the cubic with the lines BC, CA, AB respectively.



The equation $P+\theta R=0$ is here $\lambda\mu(a\lambda+b\mu+\theta\nu)=0$, which intersects $\Omega=0$ in the points C, A, G, C, B, F, and the three intersections by the line $a\lambda+b\mu+\theta\nu=0$;

viz. excluding the fixed points A, B, C, the six intersections are C, F, G, and the three intersections by the line. Hence, of the six intersections, we have C, F, G independent of θ, or we have $(x=0,\ z=0)$ a triple point, say I, corresponding to the three points C, F, G, viz. these are the points

$$(\lambda,\ \mu,\ \nu)=(0,\ 0,\ 1),\quad (0,\ g,\ -f),\quad (h,\ 0,\ -f),\qquad (C,\ F,\ G).$$

The equation $Q+\theta R=0$ is $\nu(c\lambda\nu+d\mu\nu+\theta\lambda\mu)=0$; viz. the line $\nu=0$ meets $\Omega=0$ in the three points A, B, H, and the conic $c\lambda\nu+d\mu\nu+\theta\lambda\mu=0$ meets $\Omega=0$ in the points A, B, C, and three other points: hence, rejecting the points A, B, C, the six points of intersection are the points A, B, H, and the three variable points of intersection by the conic; or we have $(y=0,\ z=0)$ a triple point, say J, corresponding to the three points A, B, H, viz. these are the points

$$(\lambda,\ \mu,\ \nu)=(1,\ 0,\ 0),\quad (0,\ 1,\ 0),\quad (h,\ -g,\ 0),\qquad (A,\ B,\ H).$$

To find the tangents at the triple point I, these are $x+\theta z=0$, where θ is to be successively determined by the conditions that the line $a\lambda+b\mu+\theta\nu=0$ shall pass through the points C, F, G*; viz. we thus have

$\theta=0$, $\quad x=0$, the tangent corresponding to the point C, $(0,\ 0,\ 1)$,

$\theta=+\dfrac{bg}{f}$, $\quad fx+bgz=0$, $\quad$ " " " " F, $(0,\ g,\ -f)$,

$\theta=+\dfrac{ah}{f}$, $\quad fx+ahz=0$, $\quad$ " " " " G, $(h,\ 0,\ -f)$.

And similarly, at the triple point J, the tangents are $y+\theta z=0$, where θ is to be successively determined by the conditions that the conic $c\nu\lambda+d\nu\mu+\theta\lambda\mu=0$ shall pass through the point H, and shall touch the cubic at the points A, B; viz. we thus have

$\theta=0$, $\quad y=0$, the tangent corresponding to the point H, $(h,\ -g,\ 0)$,

$\theta=\dfrac{cg}{f}$, $\quad fy+cgz=0$, $\quad$ " " " " A, $(1,\ 0,\ 0)$,

$\theta=\dfrac{dh}{f}$, $\quad fy+dhz=0$, $\quad$ " " " " B, $(0,\ 1,\ 0)$.

The two last values of θ are obtained by the consideration that the equations of the tangents to $\Omega=0$ at the points A, B respectively, are $g\mu+f\nu=0$, $h\lambda+f\nu=0$, where λ, μ, ν are current coordinates of a point on the tangent: it may be added that the equation of the tangent at the point C is $h\lambda+g\mu=0$.

* Observe the somewhat altered form of the condition: θ is to be determined so that the cubic $\lambda\mu(a\lambda+b\mu+\theta\nu)=0$ shall touch the cubic $\Omega=0$ at one of the points C, F, G: but, as the first-mentioned cubic breaks up, and the component curve $a\lambda+b\mu+\theta\nu=0$ does not pass through any one of these points, this can only mean that θ shall be so determined as that the line shall pass through one of these points, viz. that there shall be at the point, not a proper contact, but a double intersection, arising from a node of the cubic $\lambda\mu(a\lambda+b\mu+\theta\nu)=0$. And the like case happens for the other triple point; viz. there the cubic $\nu(c\nu\lambda+d\nu\mu+\theta\lambda\mu)=0$ is to touch the cubic $\Omega=0$ at one of the points A, B, H; the component conic $c\nu\lambda+d\nu\mu+\theta\lambda\mu=0$ passes through the points A and B but not through H; hence the conditions for θ are, that the conic shall touch the cubic at A or B, or that it shall pass through H.

The three-bar curve may be represented by means of a system of equations of the last-mentioned form, viz. $x : y : z = \lambda\mu(a\lambda + b\mu) : \nu^2(c\lambda + d\mu) : \lambda\mu\nu$, where λ, μ, ν are connected as above; or, taking X, Y as ordinary rectangular coordinates, x, y, and z are here the circular coordinates $\frac{x}{z} = X + iY$, $\frac{y}{z} = X - iY$, and $z = 1$; and the parameters $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ denote like functions $\cos\theta + i\sin\theta$, $\cos\phi + i\sin\phi$ of angles which are the inclinations of two bars to a fixed line. Using, for convenience, Figure 2 of my paper on Three-bar Motion, (p. 553 of this volume), the curve is considered as the locus of the vertex O of the triangle OC_1B_1, connected by the bars C_1C and B_1B with the fixed points B and C respectively; and we have $CC_1 = a_2$, $OC_1 = b_1$, $OB_1 = c_1$, $C_1B_1 = a_1$, $B_1B = a_3$. Also, to avoid confusion with the foregoing notation of the present paper, instead of calling it a, I take $BC = a_0$: the angle OC_1B_1 is $= C_1$, and $\cos C_1 + i\sin C_1$ is taken $= \gamma$.

Hence, taking the origin at C, the axis of X coinciding with CB and that of Y being at right angles to it: taking also θ, ϕ, ψ for the inclinations of CC_1, C_1B_1, and B_1B to CB, we have

$$a_2\cos\theta + a_1\cos\phi - a = -a_3\cos\psi,$$
$$a_2\sin\theta + a_1\sin\phi \quad = \quad a_3\sin\psi;$$

viz. writing $\cos\theta + i\sin\theta = \lambda$, $\cos\phi + i\sin\phi = \mu$, these give

$$a_2\lambda + a_1\mu - a_0 = -a_3(\cos\psi - i\sin\psi),$$
$$a_2\frac{1}{\lambda} + a_1\frac{1}{\mu} - a_0 = -a_3(\cos\psi + i\sin\psi),$$

that is,

$$(a_2\lambda + a_1\mu - a)\left(a_2\frac{1}{\lambda} + a_1\frac{1}{\mu} - a_0\right) - a_3^2 = 0;$$

viz.

$$(a_0^2 + a_1^2 + a_2^2 - a_3^2) + a_1a_2\left(\frac{\mu}{\lambda} + \frac{\lambda}{\mu}\right) - a_0a_2\left(\lambda + \frac{1}{\lambda}\right) - a_0a_1\left(\mu + \frac{1}{\mu}\right) = 0,$$

for the relation between the parameters λ, μ. And then

$$X = a_2\cos\theta + b_1\cos(\phi + C_1),$$
$$Y = a_2\sin\theta + b_1\sin(\phi + C_1);$$

viz. if x, $y = X + iY$, $X - iY$, then

$$\begin{cases} x = a_2\lambda + b_2\gamma_1\mu, \\ y = a_2\frac{1}{\lambda} + b_2\frac{1}{\gamma_1\mu}, \end{cases}$$

which equations determine the coordinates (x, y) in terms of the parameters λ, μ connected by the foregoing relation.

Writing for homogeneity $\frac{\lambda}{\nu}$, $\frac{\mu}{\nu}$ in place of λ, μ, and $\frac{x}{z}$, $\frac{y}{z}$ in place of x, y, the equations become

$$(a_0^2+a_1^2+a_2^2-a_3^2)\lambda\mu\nu+a_1a_2(\lambda^2+\mu^2)\nu-a_0a_2\mu(\nu^2+\lambda^2)-a_0a_1\lambda(\mu^2+\nu^2)=0,$$

and

$$x : y : z=(a_2\lambda+b_2\gamma_1\mu)\lambda\mu : \left(\frac{b_2}{\gamma_1}\lambda+a_2\mu\right)\nu^2 : \lambda\mu\nu.$$

Comparing with the foregoing equations

$$e\lambda\mu\nu+f(\lambda^2+\mu^2)\nu+g(\nu^2+\lambda^2)\mu+h(\mu^2+\nu^2)\lambda=0,$$

and

$$x : y : z=(a\lambda+b\mu)\lambda\mu : (c\lambda+d\mu)\nu^2 : \lambda\mu\nu,$$

the equations agree together, and we have

$$\begin{cases} e=a_0^2+a_1^2+a_2^2-a_3^2, \\ f=+a_1a_2, \\ g=-a_0a_2, \\ h=-a_0a_1, \\ a=\quad a_2, \\ b=\quad b_1\gamma_1, \\ c=\quad \dfrac{b_1}{\gamma_1}, \\ d=\quad a_2. \end{cases}$$

The tangents at the triple points thus are

$$\begin{aligned} x&=0, & y&=0, \\ a_1x-a_0b_1\gamma_1z&=0, & a_1y-\frac{a_0b_1}{\gamma_1}z&=0, \\ x-a_0z&=0, & y-a_0z&=0; \end{aligned}$$

viz. restoring the rectangular coordinates, and for γ substituting the value $\cos C+i\sin C$, for a_0 writing a, and taking $b=\frac{ab_1}{a_1}$, we have

$$\begin{aligned} X+iY&=0, & X-iY&=0, \\ X+iY&=b(\cos C+i\sin C), & X-iY&=b(\cos C-i\sin C), \\ X+iY&=a_0, & X-iY&=a_0; \end{aligned}$$

viz. the first two intersect in the point (0, 0), the second two in the point ($b\cos C$, $b\sin C$), the third two in the point (a, 0): the first and third of these are the points B and C, the second of them is the point A of the figure; viz. the formulæ give the point A, forming, with B and C, a triad of foci.

625.

ON THE CONDITION FOR THE EXISTENCE OF A SURFACE CUTTING AT RIGHT ANGLES A GIVEN SET OF LINES.

[From the *Proceedings of the London Mathematical Society*, vol. VIII. (1876—1877), pp. 53—57. Read December 14, 1876.]

In a congruency or doubly infinite system of right lines, the direction-cosines α, β, γ of the line through any given point (x, y, z), are expressible as functions of x, y, z; and it was shown by Sir W. R. Hamilton in a very elegant manner that, in order to the existence of a surface (or, what is the same thing, a set of parallel surfaces) cutting the lines at right angles, $\alpha dx + \beta dy + \gamma dz$ must be an exact differential: when this is so, writing $V = \int (\alpha dx + \beta dy + \gamma dz)$, we have $V = c$, the equation of the system of parallel surfaces each cutting the given lines at right angles.

The proof is as follows:—If the surface exists, its differential equation is $\alpha dx + \beta dy + \gamma dz = 0$, and this equation must therefore be integrable by a factor. Now the functions α, β, γ are such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, and they besides satisfy a system of partial differential equations which Hamilton deduces from the geometrical notion of a congruency; viz. passing from the point (x, y, z) to the consecutive point on the line, that is, to the point whose coordinates are $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$ (ρ infinitesimal), the line belonging to this point is the original line; and consequently α, β, γ, considered as functions of x, y, z, must remain unaltered when these variables are changed into $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$, respectively. We thus obtain the equations

$$\alpha \frac{d\alpha}{dx} + \beta \frac{d\alpha}{dy} + \gamma \frac{d\alpha}{dz} = 0,$$

$$\alpha \frac{d\beta}{dx} + \beta \frac{d\beta}{dy} + \gamma \frac{d\beta}{dz} = 0,$$

$$\alpha \frac{d\gamma}{dx} + \beta \frac{d\gamma}{dy} + \gamma \frac{d\gamma}{dz} = 0.$$

Combining herewith the equations obtained by differentiation of $\alpha^2+\beta^2+\gamma^2=1$, viz.

$$\alpha\frac{d\alpha}{dx}+\beta\frac{d\beta}{dx}+\gamma\frac{d\gamma}{dx}=0,$$

$$\alpha\frac{d\alpha}{dy}+\beta\frac{d\beta}{dy}+\gamma\frac{d\gamma}{dy}=0,$$

$$\alpha\frac{d\alpha}{dz}+\beta\frac{d\beta}{dz}+\gamma\frac{d\gamma}{dz}=0,$$

and subtracting the corresponding equations, we obtain three equations which may be written

$$\alpha:\beta:\gamma=\frac{d\beta}{dz}-\frac{d\gamma}{dy}:\frac{d\gamma}{dx}-\frac{d\alpha}{dz}:\frac{d\alpha}{dy}-\frac{d\beta}{dx},$$

or, what is the same thing,

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy},\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz},\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}=k\alpha,\ k\beta,\ k\gamma,$$

and, multiplying by α, β, γ, and adding,

$$k=\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right).$$

We thus see that, if the function on the right-hand vanish, then $k=0$, and consequently also

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy},\ \frac{d\gamma}{dx}-\frac{d\alpha}{dz},\ \frac{d\alpha}{dy}-\frac{d\beta}{dx}\ \text{each}=0;$$

viz. if the equation $\alpha dx+\beta dy+\gamma dz=0$ be integrable, then $\alpha dx+\beta dy+\gamma dz$ is an exact differential; which is the theorem in question.

But it is interesting to obtain the first mentioned set of differential equations from the analytical equations of a congruency, viz. these are $x=mz+p$, $y=nz+q$, where m, n, p, q are functions of two arbitrary parameters, or, what is the same thing, p, q are given functions of m, n; and therefore, from the three equations, m, n are given functions of x, y, z. And it is also interesting to express in terms of these quantities m, n, considered as functions of x, y, z, the condition for the existence of the set of surfaces.

We have

$$\alpha,\ \beta,\ \gamma=\frac{m}{R},\ \frac{n}{R},\ \frac{1}{R},\ \text{where}\ R=\sqrt{1+m^2+n^2};$$

and thence without difficulty

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\alpha=\frac{1}{R^4}\left[(1+n^2)\left(m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}\right)\quad-mn\quad\left(m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\beta=\frac{1}{R^4}\left[-mn\ \left(\qquad\text{"}\qquad\right)+(1+m^2)\left(\qquad\text{"}\qquad\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\gamma=\frac{1}{R^4}\left[-\ \left(\qquad\text{"}\qquad\right)-\left(\qquad\text{"}\qquad\right)\right],$$

so that the required equations in α, β, γ will be satisfied if only

$$m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}=0,$$

$$m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}=0,$$

and it is to be shown that these equations hold good.

Writing for shortness $dp=A\,dm+B\,dn$, $dq=C\,dm+D\,dn$, the equations of the line give

$$\begin{array}{l|l} 1=z\dfrac{dm}{dx}+A\dfrac{dm}{dx}+B\dfrac{dn}{dx}, & 0=z\dfrac{dn}{dx}+C\dfrac{dm}{dx}+D\dfrac{dn}{dx}, \\ 0=z\dfrac{dm}{dy}+A\dfrac{dm}{dy}+B\dfrac{dn}{dy}, & 1=z\dfrac{dn}{dy}+C\dfrac{dm}{dy}+D\dfrac{dn}{dy}, \\ -m=z\dfrac{dm}{dz}+A\dfrac{dm}{dz}+B\dfrac{dn}{dz}, & -n=z\dfrac{dn}{dz}+C\dfrac{dm}{dz}+D\dfrac{dn}{dz}; \end{array}$$

or, writing

$$\lambda,\ \mu,\ \nu=\frac{dm}{dy}\frac{dn}{dz}-\frac{dm}{dz}\frac{dn}{dy},\quad \frac{dm}{dz}\frac{dn}{dx}-\frac{dm}{dx}\frac{dn}{dz},\quad \frac{dm}{dx}\frac{dn}{dy}-\frac{dm}{dy}\frac{dn}{dx},$$

so that identically

$$\lambda\frac{dm}{dx}+\mu\frac{dm}{dy}+\nu\frac{dm}{dz}=0,$$

$$\lambda\frac{dn}{dx}+\mu\frac{dn}{dy}+\nu\frac{dn}{dz}=0,$$

then in each set, multiplying by λ, μ, ν and adding, so as to eliminate A, B, C, D, we find

$$\lambda-m\nu=0,\quad \mu-n\nu=0.$$

Substituting these values of λ, μ in the last preceding equations, ν divides out, and we have the two equations in question.

The foregoing equations give further

$$A,\ B,\ C,\ D=-z+\frac{1}{\nu}\frac{dn}{dy},\quad -\frac{1}{\nu}\frac{dm}{dy},\quad -\frac{1}{\nu}\frac{dn}{dx},\quad -z+\frac{1}{\nu}\frac{dm}{dx}.$$

Taking for α, β, γ the before-mentioned values, we find

$$\frac{d\alpha}{dy}-\frac{d\beta}{dx}=\frac{1}{R}\left(\frac{dm}{dy}-\frac{dn}{dx}\right)-\frac{m}{R^3}\left(m\frac{dm}{dy}+n\frac{dn}{dy}\right)-\frac{n}{R^3}\left(m\frac{dm}{dx}+n\frac{dn}{dx}\right)$$

$$=\frac{1}{R^3}\left\{(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)\right\};$$

and similarly, but using the equations

$$m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}=0,\quad m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}=0,$$

to eliminate the coefficients $\frac{dm}{dz}$, $\frac{dn}{dz}$ which in the first instance present themselves, we find

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy}=\frac{m}{R^2}\left\{(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)\right\},$$

$$\frac{d\gamma}{dx}-\frac{d\alpha}{dz}=\frac{n}{R^2}\left\{\quad\text{"}\quad\text{"}\quad\text{"}\quad\right\}:$$

whence, multiplying by γ, α, β, and adding,

$$\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)$$

$$=\frac{1}{1+m^2+n^2}\left\{(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)\right\};$$

or we have

$$(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)=0$$

as the condition for the existence of the set of surfaces.

It is clear that the condition is satisfied when the lines are the normals of a given surface: seeking the surfaces which cut the lines at right angles, we obtain the parallel surfaces; and we are led to the theorem that any parallel surface is the locus of the extremity of a line of constant length measured off from each point of the surface along the normal—or, what is equivalent thereto, the parallel surface is the envelope of a sphere of constant radius having its centre on the surface. I will verify the theorem for the case of the ellipsoid. Taking X, Y, Z as the coordinates of a point on the ellipsoid $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$, and x, y, z as current coordinates, the equations of the normal are

$$\frac{a^2}{X}(x-X)=\frac{b^2}{Y}(y-Y)=\frac{c^2}{Z}(z-Z),\ (=\lambda \text{ suppose}).$$

We have therefore

$$X,\ Y,\ Z=\frac{a^2x}{a^2+\lambda},\ \frac{b^2y}{b^2+\lambda},\ \frac{c^2z}{c^2+\lambda},$$

and thence

$$\frac{a^2x^2}{(a^2+\lambda)^2}+\frac{b^2y^2}{(b^2+\lambda)^2}+\frac{c^2z^2}{(c^2+\lambda)^2}=1,$$

an equation which determines λ as a function of x, y, z.

The direction-cosines α, β, γ of the normal are proportional to $\frac{X}{a^2}$, $\frac{Y}{b^2}$, $\frac{Z}{c^2}$, that

is, to $\frac{x}{a^2+\lambda}$, $\frac{y}{b^2+\lambda}$, $\frac{z}{c^2+\lambda}$, and the equation $\alpha^2+\beta^2+\gamma^2=1$ then determines their absolute magnitudes: the equation $\alpha dx+\beta dy+\gamma dz=dV$ thus is

$$\frac{\frac{x\,dx}{a^2+\lambda}+\frac{y\,dy}{b^2+\lambda}+\frac{z\,dz}{c^2+\lambda}}{\sqrt{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}}}=dV,$$

viz. the left-hand side, considering therein λ as a given function of V, is an exact differential. We verify this by finding the value of V, viz. writing down the two equations

$$\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}-\frac{V^2}{\lambda^2}=0,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-\frac{V^2}{\lambda}=1,$$

these are equivalent in virtue of the equation that determines λ; and it is to be shown that, regarding V as given by either of them, say by the second equation, we have for dV its foregoing value. In fact, differentiating the second equation, the term in $d\lambda$ disappears by virtue of the first equation, and the result is

$$\frac{x\,dx}{a^2+\lambda}+\frac{y\,dy}{b^2+\lambda}+\frac{z\,dz}{c^2+\lambda}-\frac{V\,dV}{\lambda}=0,$$

in which substituting for $\frac{V}{\lambda}$ its value from the first equation, we have for dV the value in question. Regarding V as a given constant, the two equations give, by elimination of λ, an equation $\phi(x, y, z, V)=0$, which is, in fact, the surface parallel to the ellipsoid and at a constant normal distance $=V$ from it.

626.

ON THE GENERAL DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, WHERE X, Y ARE THE SAME QUARTIC FUNCTIONS OF x, y RESPECTIVELY.

[From the *Proceedings of the London Mathematical Society*, vol. VIII. (1876—1877), pp. 184—199. Read February 8, 1877.]

WRITE $\Theta=a+b\theta+c\theta^2+d\theta^3+e\theta^4$, the general quartic function of θ; and let it be required to integrate by Abel's theorem the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

We have

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix}=0,$$

a particular integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0;$$

and consequently the above equation, taking therein z, w as constants, is the general integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

viz. the two constants z, w must enter in such wise that the equation contains only a single constant; whence also, attributing to w any special value, we have the general integral with z as the arbitrary constant.

Take $w=\infty$; the equation becomes

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ 1, & 0, & 0, & \sqrt{e} \end{vmatrix}=0,$$

a relation between x, y, z which may be otherwise expressed by means of the identity

$$e(\theta^2+\beta\theta+\gamma)^2-(e\theta^4+d\theta^3+c\theta^2+b\theta+a)=(2\beta e-d)(\theta-x)(\theta-y)(\theta-z),$$

or, what is the same thing,

$$\begin{aligned} e(2\gamma+\beta^2)-c &= -(2\beta e-d)(x+y+z), \\ e\ 2\beta\gamma \quad -b &= \quad (2\beta e-d)(yz+zx+xy), \\ e\ \gamma^2 \quad -a &= -(2\beta e-d)xyz, \end{aligned}$$

where β, γ are indeterminate coefficients which are to be eliminated.

Write

$$x^2-\frac{\sqrt{X}}{\sqrt{e}}=P, \quad y^2-\frac{\sqrt{Y}}{\sqrt{e}}=Q;$$

then we have

$$\beta x+\gamma+P=0, \quad \beta y+\gamma+Q=0;$$

giving

$$\beta : \gamma : 1 = Q-P : Py-Qx : x-y.$$

Substituting these values in the first of the preceding three equations, we have

$$e\frac{2(Py-Qx)(x-y)+(Q-P)^2}{(x-y)^2}-c=-\left\{\frac{2(Q-P)e}{x-y}-d\right\}(x+y+z),$$

that is,

$$e\left\{\frac{2(Qy-Px)}{x-y}+\frac{(Q-P)^2}{(x-y)^2}+\frac{2(Q-P)}{x-y}z\right\}=c+d(x+y+z);$$

or, reducing by

$$Qy-Px=y^3-x^3+\frac{x\sqrt{X}-y\sqrt{Y}}{\sqrt{e}},$$

$$Q-P=y^2-x^2+\frac{\sqrt{X}-\sqrt{Y}}{\sqrt{e}}, \quad =y^2-x^2+(y-x)\frac{M}{\sqrt{e}}, \text{ if } M=\frac{\sqrt{X}-\sqrt{Y}}{x-y},$$

this is

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}+2xy+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}-2(x+y)z+2z\frac{M}{\sqrt{e}}\right\}$$
$$=c+d(x+y+z)+e(x+y)^2.$$

We have Euler's solution in the far more simple form

$$M^2=C+d(x+y)+e(x+y)^2,$$

where C is the arbitrary constant. It is to be observed that, in the particular case where $e=0$, the first equation becomes

$$M^2=c+d(x+y+z);$$

and the two results for this case agree on putting $C=c+dz$.

But it is required to identify the two solutions in the general case where e is not $=0$. I remark that I have, in my *Treatise on Elliptic Functions*, Chap. XIV., further developed the theory of Euler's solution, and have shown that, regarding C as variable, and writing

$$\mathfrak{C}=ad^2+b^2e-2bcd+C[-4ae+bd+(C-c)^2],$$

then the given equation between the variables x, y, C corresponds to the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0,$$

a result which will be useful for effecting the identification. The Abelian solution may be written

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}-x^2-y^2+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}\right\}-c-d(x+y)=z\{d+2e(x+y)-2M\sqrt{e}\};$$

and substituting for M its value, and multiplying by $(x-y)^2$, the equation becomes

$$\begin{aligned}2\sqrt{e}(x-y)(x\sqrt{X}-y\sqrt{Y})-e(x^2+y^2)(x-y)^2+(\sqrt{X}-\sqrt{Y})^2\\ -2(x^2-y^2)(\sqrt{X}-\sqrt{Y})\sqrt{e}-c(x-y)^2-d(x+y)(x-y)^2\\ =z(x-y)\{d(x-y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}.\end{aligned}$$

On the left-hand side, the rational part is

$$X+Y+c(-x^2+2xy-y^2)+d(-x^3+x^2y+xy^2-y^3)+e(-x^4+2x^3y-2x^2y^2+2xy^3-y^4),$$

which, substituting therein for X, Y their values, becomes

$$=2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2);$$

and the irrational part is at once found to be

$$=2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}.$$

The equation thus is

$$z=\frac{\left\{\begin{array}{c}2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\\ +2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\end{array}\right\}}{(x-y)\{d(x-y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}},$$

which equation is thus a form of the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, and also a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$.

Multiplying the numerator and the denominator by

$$d(x-y)+2e(x^2-y^2)+2(\sqrt{X}-\sqrt{Y})\sqrt{e},$$

the denominator becomes

$$=(x-y)^3\left[\{d+2e(x+y)\}^2-4e\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2\right],$$

which, introducing herein the C of Euler's equation, is

$$=(x-y)^3(d^2-4eC).$$

We have therefore

$$z(x-y)^3(d^2-4eC)=\{2a+b(x+y)+c\,.\,2xy+d\,xy(x+y)+e\,.\,2xy(x^2-xy+y^2)$$
$$+2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\times\{d(x-y)+2e(x^2-y^2)+2\sqrt{e}(\sqrt{X}-\sqrt{Y})\}.$$

Using $\mathfrak{C}$ to denote the same value as before, the function on the right-hand is, in fact,

$$=(x-y)^3\{2be-cd+dC+2\sqrt{e}\sqrt{\mathfrak{C}}\};$$

and, this being so, the required relation between z, C is

$$z(d^2-4eC)=\{2be-cd+dC+2\sqrt{e}\sqrt{\mathfrak{C}}\}.$$

To prove this, we have first, from the equation

$$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=C+d(x+y)+e(x+y)^2,$$

to express $\mathfrak{C}$ as a function of x, y. This equation, regarding therein C as a variable, gives

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0;$$

and we have therefore

$$-\sqrt{\mathfrak{C}}=\sqrt{X}\frac{dC}{dx}=\sqrt{Y}\frac{dC}{dy},$$

viz. $\sqrt{X}\frac{dC}{dx}$ will be a symmetrical function of x, y. Putting, as before

$$M=\frac{\sqrt{X}-\sqrt{Y}}{x-y},$$

we have

$$C=M^2-d(x+y)-e(x+y)^2,$$

and thence

$$\frac{dC}{dx}=2M\frac{dM}{dx}-d-2e(x+y).$$

We have

$$\frac{dM}{dx}=\frac{1}{x-y}\,\frac{X'}{2\sqrt{X}}-\frac{\sqrt{X}-\sqrt{Y}}{(x-y)^2},$$

and hence

$$\sqrt{\mathfrak{C}}\,(x-y)^2 = -\sqrt{X}\,(x-y)^2\left\{2M\frac{dM}{dx}-d-2e\,(x+y)\right\}$$

$$= -(x-y)\,X'\,(\sqrt{X}-\sqrt{Y})+2\,(X+Y-2\sqrt{XY})\sqrt{X}$$
$$+(d+2e\,\overline{x+y})\,(x-y)^2\sqrt{X}$$

$$= \quad [(x-y)\,X'+2X+2Y+(d+2e\,\overline{x+y})\,(x-y)^2]\sqrt{X}$$
$$+[(x-y)\,X'-4X]\sqrt{Y}.$$

We obtain at once the coefficient of $\sqrt{Y}$, and with little more difficulty that of $\sqrt{X}$; and the result is

$$\sqrt{\mathfrak{C}}\,(x-y)^2 = -[4a+3bx+2cx^2+dx^3+y\,(b+2cx+3dx^2+4ex^3)]\sqrt{Y}$$
$$+[4a+3by+2cy^2+dy^2+x\,(b+2cy+3dy^2+4ey^3)]\sqrt{X}.$$

We have also

$$C\,(x-y)^2 = (\sqrt{X}-\sqrt{Y})^2-d\,(x+y)\,(x-y)^2-e\,(x+y)^2\,(x-y)^2$$
$$= X+Y-d\,(x^3-x^2y-xy^2+y^3)-e\,(x^4-2x^2y^2+y^4)-2\sqrt{XY}$$
$$= 2a+b\,(x+y)+c\,(x^2+y^2)+d\,xy\,(x+y)+2e\,x^2y^2-2\sqrt{XY},$$

or, say

$$C\,(x-y)^2 = 2a\,(x-y)+b\,(x^2-y^2)+c\,(x^3-x^2y+xy^2-y^3)+d\,xy\,(x^2-y^2)$$
$$+2e\,x^2y^2\,(x-y)-2\,(x-y)\sqrt{XY}.$$

We can hence form the expression of

$$(x-y)^3\,\{2be-cd+dC+2\sqrt{e}\sqrt{\mathfrak{C}}\},$$

viz. this is

$$= (2be-cd)\,(x-y)^2+2ad\,(x-y)+bd\,(x^2-y^2)+cd\,(x^3-x^2y+xy^2-y^3)+d^2\,xy\,(x^2-y^2)$$
$$+2de\,x^2y^2\,(x-y)-2d\,(x-y)\sqrt{XY}$$
$$+2\sqrt{e}\,\{[-(4a+3bx+2cx^2+dx^3)-y\,(b+2cx+3dx^2+4ex^3)]\sqrt{Y}$$
$$+[(4a+3by+2cy^2+dy^2)+x\,(b+2cy+3dy^2+4ey^3)]\sqrt{X}\},$$

and this should be

$$= \{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)$$
$$+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\times\{d\,(x-y)+2e\,(x^2-y^2)+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\}.$$

The function on the right-hand is, in fact,

$$= \{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$\times\{d\,(x-y)+2e\,(x^2-y^2)\}+4e\,(x-y)\,(\sqrt{X}-\sqrt{Y})\,(x\sqrt{Y}-y\sqrt{X})$$
$$+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\,\{2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})\,\{d\,(x-y)+2e\,(x^2-y^2)\},$$

viz. this is

$$
\begin{aligned}
&= \{2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\} \\
&\qquad\qquad \times\{d(x-y)+2e(x^2-y^2)\}+4e(x-y)(-xY-yX) \\
&\quad -2\sqrt{XY}\{d(x-y)+2e(x^2-y^2)\}+4e(x-y)(x+y)\sqrt{XY} \\
&\quad +2\sqrt{e}\left\{\begin{array}{l}\sqrt{X}\,\{2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2) \\ \qquad +2Y-(x-y)\,y\,[d(x-y)+2e(x^2-y^2)]\} \\ -\sqrt{Y}\,\{2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2) \\ \qquad +2X-(x-y)\,x\,[d(x-y)+2e(x^2-y^2)]\}\end{array}\right\},
\end{aligned}
$$

which is, in fact, equal to the expression on the left-hand side.

To complete the theory, we require to express $\sqrt{Z}$ as a function of x, y. It would be impracticable to effect this by direct substitution of the foregoing value of z; but, observing that the value in question is a solution of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$, or, what is the same thing, that $\frac{1}{\sqrt{X}}+\frac{1}{\sqrt{Z}}\frac{dz}{dx}=0$, $\frac{1}{\sqrt{Y}}+\frac{1}{\sqrt{Z}}\frac{dz}{dy}=0$, we can from either of these equations, considering therein z as a given function of x, y, calculate $\sqrt{Z}$.

Writing for shortness

$$z=\frac{J-2\sqrt{e}\,y\,(x-y)\sqrt{X}+2\sqrt{e}\,x\,(x-y)\sqrt{Y}-2\sqrt{XY}}{R-2\sqrt{e}\,(x-y)\sqrt{X}+2\sqrt{e}\,(x-y)\sqrt{Y}},$$

where

$$R=(x-y)^2\{d+2e(x+y)\},$$

$$J=2a+b(x+y)+2c\,xy+d\,xy\,(x+y)+2e\,xy\,(x^2-xy+y^2);$$

or, if for a moment $z=\frac{N}{D}$, then

$$\frac{dz}{dx}=\frac{1}{D^2}\left(D\frac{dN}{dx}-N\frac{dD}{dx}\right)=-\frac{\sqrt{Z}}{\sqrt{X}},$$

that is,

$$\sqrt{Z}=\frac{\sqrt{X}}{D^2}\left(N\frac{dD}{dx}-D\frac{dN}{dx}\right),\ =\frac{\Omega}{D^2}\text{ suppose};$$

or, writing for shortness X', R', J to denote the derived functions $\frac{dX}{dx}$, $\frac{dR}{dx}$, $\frac{dJ}{dx}$, (Y' is afterwards written to denote $\frac{dY}{dy}$, but as the final formulæ contain only X', $=\frac{dX}{dx}$, and Y', $=\frac{dY}{dy}$, this does not occasion any defect of symmetry), we find

$$
\begin{aligned}
\Omega = \ & N\{R'\sqrt{X}-2\sqrt{e}\,X-\sqrt{e}\,(x-y)\,X'+2\sqrt{e}\sqrt{XY}\} \\
& -D\{J'\sqrt{X}-2\sqrt{e}\,yX-\sqrt{e}\,(x-y)\,yX'+2\sqrt{e}\,(2x-y)\sqrt{XY}-X'\sqrt{Y}\};
\end{aligned}
$$

and substituting herein for N, D their values, and arranging the terms, we find

$$\Omega=\sqrt{e}\,\mathfrak{A}+\mathfrak{B}\sqrt{X}+\mathfrak{C}\sqrt{Y}+\sqrt{e}\,\mathfrak{D}\sqrt{XY},$$

where

$$\begin{aligned}\mathfrak{A}=&-J\{2X+(x-y)X'\}\\&-2(x-y)yR'X\\&-4XY\\&+Ry\{2X+(x-y)X'\}\\&+2(x-y)XJ'\\&+2(x-y)X'Y,\end{aligned}\qquad\begin{aligned}\mathfrak{B}=&\;JR'\\&+2e(x-y)y\{2X+(x-y)X'\}\\&+4ex(x-y)Y\\&-RJ'\\&-2e(x-y)y\{2X+(x-y)X'\}\\&-4e(x-y)(2x-y)Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{C}=&-4ey(x-y)X\\&-2e(x-y)x\{2X+(x-y)X'\}\\&-2R'X\\&+RX'\\&+2e(x-y)y\{2X+(x-y)X'\}\\&+4e(x-y)(2x-y)X,\end{aligned}\qquad\begin{aligned}\mathfrak{D}=&\;2J\\&+2(x-y)xR'\\&+2\{2X+(x-y)X'\}\\&-2(2x-y)R\\&-2(x-y)X'\\&-2(x-y)J',\end{aligned}$$

where the terms have been written down as they immediately present themselves; but, collecting and arranging, we have

$$\begin{aligned}\mathfrak{A}&=\;2X(-J+Ry-2Y)+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\},\\\mathfrak{B}&=\;JR'-J'R-4e(x-y)^2Y,\\\mathfrak{C}&=-2XR'+X'R+4e(x-y)^2X-2e(x-y)^3X',\\\mathfrak{D}&=\;2J+4X-2Rx+2(x-y)(xR'-R-J').\end{aligned}$$

To reduce these expressions, writing

$$\begin{aligned}M&=d+2e(x+y),\\\Lambda&=c+\;d(x+y)+e(x^2+y^2),\end{aligned}$$

we have $R=(x-y)^2M$, and therefore $R'=2(x-y)M+2e(x-y)^2$; also

$$J=X+Y-(x-y)^2\Lambda;$$

also, from the original form,

$$J'=b+2cy+d(2xy+y^2)+e(6x^2y-4xy^2+2y^3).$$

The final values are

$$\begin{aligned}\mathfrak{A}&=-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\},\\\mathfrak{B}&=(x-y)M\{4Y+(x-y)Y'\}+2e(x-y)^3Y',\\\mathfrak{C}&=-(x-y)M\{4X-(x-y)X'\}-2e(x-y)^3X',\\\mathfrak{D}&=4(X+Y)+4e(x-y)^4,\end{aligned}$$

which, once obtained, may be verified without difficulty.

Verification of 21.—The equation is

$$-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)M+xyM^2\}$$
$$=2X(-J+Ry-2Y)+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};$$

or, putting for shortness

$$\Lambda^2+(-b+dxy)M+xyM^2=\nabla,$$

this is

$$\begin{aligned}(x-y)^4\nabla = \;& X^2+6XY+Y^2\\ &+2X\{-X-3Y+(x-y)^2\Lambda+(x-y)^2yM\}\\ &+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\},\\ =\;& -X^2+Y^2+2(x-y)^2X\Lambda+2(x-y)^2yXM\\ &+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};\end{aligned}$$

we have $-X^2+Y^2=-(X-Y)(X+Y)$, where $X-Y$ divides by $x-y$, $=(x-y)\Omega$ suppose; hence, throwing out the factor $x-y$, the equation becomes

$$\begin{aligned}(x-y)^2\nabla = \;& -\Omega(X+Y)+2(x-y)X\Lambda+2(x-y)yXM\\ &+2XJ'+2X'Y-X'\{X+Y-(x-y)^2\Lambda\}\\ &-2yX\{2(x-y)M+2(x-y)^2e\}+(x-y)^2yMX',\\ =\;& -\Omega(X+Y)+2XJ'-X'(X-Y)\\ &+2(x-y)X\Lambda-2(x-y)yXM\\ &+(x-y)^2X'\Lambda-4(x-y)^2eyX+(x-y)^2yMX'.\end{aligned}$$

We have $2XJ'=J'(X+Y)+J'(X-Y)$, and hence the first line is

$$=(-\Omega+J')(X+Y)+J'(X-Y);$$

$-\Omega+J'$, as will be shown, divides by $x-y$, or say it is $=(x-y)\Phi$, and, as before, $X-Y$ is $=(x-y)\Omega$; hence, throwing out the factor $x-y$, the equation becomes

$$(x-y)^2\nabla=\Phi(X+Y)+\Omega(J'-X')+2X\Lambda-2yXM+(x-y)\{X'\Lambda-4eyX+yMX'\}.$$

We have

$$\Omega=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3),$$

and thence

$$-\Omega+J'=c(-x+y)+d(-x^2+xy)+e(-x^3+5x^2y-5xy^2+y^3);$$

or, dividing this by $(x-y)$, we find

$$\Phi=-c-dx-e(x^2-4xy+y^2),$$

or, as this may be written,

$$\Phi=-\Lambda+dy+4exy.$$

We find, moreover,

$$J'-X'=2c(-x+y)+d(-3x^2+2xy+y^2)+e(-4x^3+6x^2y-4xy^2+2y^3),$$

which divides by $(x-y)$, the quotient being

$$-2c-d(3x+y)-e(4x^2-2xy+2y^2),$$

viz. this is

$$=-2\Lambda-(x-y)(d+2ex).$$

Hence the equation now is

$$\begin{aligned}(x-y)^2\nabla=&(X+Y)\{-\Lambda+dy+4exy\}+2X\Lambda-2yXM\\&+(x-y)\Omega\{-2\Lambda-(x-y)(d+2ex)\}\\&+(x-y)\quad\{\quad X'\Lambda-4eyX+yMX'\quad\}.\end{aligned}$$

The first line is

$$(X+Y)\{-\Lambda+yM+2(x-y)ye\}+2X\Lambda-2yXM,$$

which is

$$=(\Lambda-yM)(X-Y)+2(x-y)ey(X+Y);$$

hence, throwing out the factor $x-y$, the equation becomes

$$\begin{aligned}(x-y)\nabla&=(\Lambda-yM)\Omega+2ey(X+Y)-2\Lambda\Omega+X'\Lambda-4eyX+yMX'-(x-y)\Omega(d+2ex)\\&=(\Lambda+yM)(-\Omega+X')-2ey(X-Y)-(x-y)\Omega(d+2ex).\end{aligned}$$

We have

$$-\Omega+X'=c(x-y)+d(2x^2-xy-y^2)+e(3x^3-x^2y-xy^2-y^3),$$

which is $=(x-y)(\Lambda+xM)$: also $(X-Y)=(x-y)\Omega$, as before; whence, throwing out the factor $x-y$, the equation is

$$\nabla=(\Lambda+xM)(\Lambda+yM)-2ey\Omega-(d+2ex)\Omega,$$

that is,

$$\nabla=(\Lambda+xM)(\Lambda+yM)-M\Omega;$$

viz. substituting for ∇ its value, reducing, and throwing out the factor M, the equation becomes

$$-b+dxy=(x+y)\Lambda-\Omega,$$

which is right.

Verification of $\mathfrak{B}$.—The equation is

$$\begin{aligned}J\{2(x-y)M+2e(x-y)^2\}-J'(x-y)^2M-4e(x-y)^2Y\\=4(x-y)MY+(x-y)^2MY'+2e(x-y)^3Y',\end{aligned}$$

which, throwing out the factor $x-y$, is

$$0=2M(-J+2Y)+(x-y)M(J'+Y')+2e(x-y)(-J+2Y)+2e(x-y)^2Y'.$$

Here $-J+2Y, =-(X-Y)+(x-y)^2\Lambda$, is divisible by $(x-y)$: hence, throwing out the factor $x-y$, the equation is

$$\begin{aligned}0 = M\{&-2b-2c(x+y)-2d(x^2+xy+y^2)-2e(x^3+x^2y+xy^2+y^3)\}\\&+M(J'+Y')+2M(x-y)\Lambda+2e(-J+2Y)+2e(x-y)Y'.\end{aligned}$$

In the first and second terms, the factor which multiplies M is

$$c(-2x+2y)+d(-2x^2+2y^2)+e(-2x^3+4x^2y-6xy^2+4y^3),$$

which is divisible by $x-y$; also $-J+2Y, =-(X-Y)+(x-y)^2\Lambda$, is divisible by $(x-y)$: hence, throwing this factor out, the equation is

$$\begin{aligned}0 = M\{&-2c+d(-2x-2y)+e(-2x^2+2xy-4y^2)\}+2M\Lambda\\&+2e\{-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)\}\\&+2e(x-y)\Lambda+2eY'.\end{aligned}$$

Here in the first line the coefficient of M is $=e(2xy-2y^2)$: hence, throwing out the constant factor $2e$, the equation is

$$0=-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)+Y'+(x-y)yM+(x-y)\Lambda.$$

The first five terms are

$$=c(-x+y)+d(-x^2-xy+2y^2)+e(-x^3-x^2y-xy^2+3y^3),$$

which is divisible by $x-y$; throwing out this factor, the equation is

$$0=-c-d(x+2y)-e(x^2+2xy+3y^2)+\Lambda+yM,$$

which is right.

Verification of $\mathfrak{C}$.—We have

$$\begin{aligned}-2X\{2(x-y)M+2e(x-y)^2\}+(x-y)^2X'M+4e(x-y)^2X-2e(x-y)^3X'\\=-(x-y)M\{4X-(x-y)X'\}-2e(x-y)^3X',\end{aligned}$$

which is, in fact, an identity.

Verification of $\mathfrak{D}$.—The equation may be written

$$\begin{aligned}4X+4Y+4e(x-y)^4\\=\ &2X+2Y-2(x-y)^2\Lambda\\&+4X-2x(x-y)^2M\\&+2(x-y)\{2(x-y)xM+2ex(x-y)^2-M(x-y)^2-J'\},\end{aligned}$$

viz. this is

$$\begin{aligned}0=2X-2Y-4e(x-y)^4-2(x-y)^2\Lambda+2x(x-y)^2M\\+4ex(x-y)^3-2M(x-y)^3-2(x-y)J'.\end{aligned}$$

The first term $2(X-Y)$ is divisible by $2(x-y)$; throwing this factor out, the equation becomes

$$\begin{aligned}0 = b + c(x+y) + d(x^2+xy+y^2) + e(x^3+x^2y+xy^2+y^3) - J' \\ - 2e(x-y)^3 - (x-y)\Lambda + x(x-y)M + 2ex(x-y)^2 - M(x-y)^2.\end{aligned}$$

Substituting for J' its value, the first line becomes

$$c(x-y) + d(x^2-xy) + e(x^3-5x^2y+5xy^2-y^3),$$

which is divisible by $(x-y)$; hence, throwing out this factor, the equation is

$$0 = c + dx + e(x^2-4xy+y^2) - \Lambda + xM - 2e(x-y)^2 + 2ex(x-y) - M(x-y),$$

where the sum of all the terms but the last is $= d(x-y) + e(2x^2-2xy)$: hence, again throwing out the factor $x-y$, the equation becomes

$$0 = d + 2ex - 2e(x-y) + 2ex - M,$$

which is right.

Recapitulating, we have for the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, or for a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$,

$$z = \frac{J - 2\sqrt{e}(x-y)\,y\sqrt{X} + 2\sqrt{e}(x-y)\,x\sqrt{Y} - 2\sqrt{XY}}{(x-y)^2M - 2\sqrt{e}(x-y)\sqrt{X} + 2\sqrt{e}(x-y)\sqrt{Y}},$$

the corresponding value of $\sqrt{Z}$ being

$$\sqrt{Z} = \frac{\begin{aligned}&\sqrt{e}\,[-X^2 - 6XY - Y^2 + (x-y)^4\{\Lambda^2 + (-b+dxy)M + xyM^2\}] \\ &+ [\{4Y + (x-y)Y'\}M + 2e(x-y)^2Y'](x-y)\sqrt{X} \\ &- [\{4X - (x-y)X'\}M + 2e(x-y)^2X'](x-y)\sqrt{Y} \\ &+ [\quad 4(X+Y) + 4e(x-y)^4] \quad \sqrt{XY}\end{aligned}}{\{(x-y)^2M - 2\sqrt{e}(x-y)\sqrt{X} + 2\sqrt{e}(x-y)\sqrt{Y}\}^2},$$

where, as before,

$$\begin{aligned}M &= d + 2e(x+y), \\ \Lambda &= c + d(x+y) + e(x^2+y^2), \\ J &= 2a + b(x+y) + 2cxy + dxy(x+y) + exy(x^2-xy+y^2):\end{aligned}$$

also X is the general quartic function $a + bx + cx^2 + dx^3 + ex^4$, and Y, Z are the same functions of y, z respectively.

In connexion with what precedes, I give some investigations relating to the more simple form $\Theta = a + c\theta^2 + e\theta^4$, or, as it will be convenient to write it, $\Theta = 1 - l\theta^2 + \theta^4$.

We have

$$\left.\begin{array}{l}\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix}=0 \text{ a particular integral} \\ \begin{vmatrix} x^2, & x, & \sqrt{X} \\ y^2, & y, & \sqrt{Y} \\ z^2, & z, & \sqrt{Z} \end{vmatrix}=0 \text{ the general integral} \end{array}\right\} \text{of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

$$\left.\begin{array}{l}\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W} \\ \cdots & & \cdots & \end{vmatrix}=0 \text{ the general integral} \end{array}\right\} \text{of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0,$$

$$\text{a particular integral} \quad \text{of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

and so on; viz. in taking

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0 \text{ as the general integral of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

we consider z as the constant of integration: and so in other cases.

It is to be remarked that it is an essentially different problem to verify a particular integral and to verify a general integral, and that the former is the more difficult one. In fact, if $U=0$ is a particular integral of the differential equation $Mdx+Ndy=0$, then we must have $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, not identically but in virtue of the relation $U=0$, or we have to consider whether two given relations between x and y are in fact one and the same relation. In the case of a general solution, this is theoretically reducible to the form $c=U$, c being the constant of integration, and we have then the equation $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, satisfied identically, or, what is the same thing, U a solution of this partial differential equation.

Hence it is theoretically easier to verify that

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0$$

is a general solution, than to verify that

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix}=0$$

is a particular solution of the differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$. Moreover, taking the first equation in the before mentioned form

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}},$$

and writing therein $z=\infty$, we see that the second equation

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix}=0$$

is, in fact, a particular case of the first equation, so that we only require to verify the first equation; or, what is the same thing, to verify that

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}}$$

is the general integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}.$$

To verify this, we have to show that $dz=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}\right)$, viz. that $\sqrt{X}\frac{dz}{dx}=\Omega$, a symmetrical function of (x, y); for then $\sqrt{Y}\frac{dz}{dy}=\Omega$, and we have the relation in question.

We have

$$\begin{aligned}(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}&=\sqrt{X}\left\{(x^2-y^2)\left(\sqrt{Y}-\frac{yX'}{2\sqrt{X}}\right)-2x(x\sqrt{Y}-y\sqrt{X})\right\}\\&=\sqrt{X}\left\{(x^2-y^2-2x^2)\sqrt{Y}-\frac{(x^2-y^2)yX'}{2\sqrt{X}}+2xy\sqrt{X}\right\}\\&=-(x^2+y^2)\sqrt{XY}+2xyX-\tfrac{1}{2}(x^2-y^2)yX'.\end{aligned}$$

Writing here $X=1-lx^2+x^4$, then $X'=-2lx+4x^3$, and we have the last two terms

$$\begin{aligned}&=2xy(1-lx^2+x^4)+(x^2-y^2)xy(l-2x^2)\\&=xy\{2-2lx^2+2x^4+(x^2-y^2)(l-2x^2)\}\\&=xy\{2-l(x^2+y^2)+2x^2y^2\}.\end{aligned}$$

Hence the equation is

$$(x\sqrt{Y}-y\sqrt{X})^2\sqrt{X}\frac{dz}{dx}=-(x^2+y^2)\sqrt{XY}+xy\{2-l(x^2+y^2)+2x^2y^2\},$$

or we have

$$\Omega=\frac{1}{(x\sqrt{Y}-y\sqrt{X})^2}\{-(x^2+y^2)\sqrt{XY}+xy(2-l(x^2+y^2)+2x^2y^2)\},$$

which is symmetrical in (x, y), as it should be. And observe, further, that since the equation is a particular solution of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$, we must have $\Omega=-\sqrt{Z}$; viz. we have

$$\sqrt{Z}(x\sqrt{Y}-y\sqrt{X})^2=-(x^2+y^2)\sqrt{XY}+xy\{2-l(x^2+y^2)+2x^2y^2\}.$$

Proceeding to the next case, where we have between x, y, z, w a relation which may be written

$$(x^3, x, x^2\sqrt{X}, \sqrt{X})=0,$$

then here a, b, c, d can be determined so that

$$(c\theta^2+d)^2(1+\beta\theta^2+\gamma\theta^4)-(a\theta^3+b\theta)^2=c^2\gamma(\theta^2-x^2)(\theta^2-y^2)(\theta^2-z^2)(\theta^2-w^2),$$

viz. we have $d^2=c^2\gamma x^2y^2z^2w^2$, or say $d=c\sqrt{\gamma}\,xyzw$. And, supposing the ratios of a, b, c, d determined by the three equations which contain (x, y, z) respectively, we have

$$a:b:c:d=(x, x^2\sqrt{X}, \sqrt{X}):-(x^3, x^2\sqrt{X}, \sqrt{X}):(x^3, x, \sqrt{X}):-(x^3, x, x^2\sqrt{X}),$$

or in particular

$$\frac{d}{c}=\frac{-(x^3, x, x^2\sqrt{X})}{(x^3, x, \sqrt{X})}, \quad =\frac{-xyz(x^2, 1, x\sqrt{X})}{(x^3, x, \sqrt{X})};$$

whence we have

$$w=-\frac{(x^2, 1, x\sqrt{X})}{(x^3, x, \sqrt{X})}$$

as a new form of the integral equation; viz. written at full length, this is

$$-w=\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix};$$

and taking $w=0$ and $=\infty$ respectively, we thus see how

$$\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix}=0, \qquad \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0,$$

are each of them a particular integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0.$$

Reverting to the general form

$$w=-\frac{(x^2, 1, x\sqrt{X})}{(x^3, x, \sqrt{X})},$$

this will be a general integral if only

$$dw=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}\right),$$

viz. if we have

$$-\sqrt{X}\,\frac{d}{dx}\,\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})}=\Omega,\ \text{a symmetrical function of } (x,\ y,\ z).$$

The expression is

$$\Omega=\frac{1}{(x^3,\ x,\ \sqrt{X})^2}\left\{(x^2,\ 1,\ x\sqrt{X})\sqrt{X}\,\frac{d}{dx}(x^3,\ x,\ \sqrt{X})-(x^3,\ x,\ \sqrt{X})\sqrt{X}\,\frac{d}{dx}(x^2,\ 1,\ x\sqrt{X})\right\},$$

or, writing for shortness

$$\begin{aligned}\alpha&=x(y^2-z^2), & a&=yz(y^2-z^2),\\ \beta&=y(z^2-x^2), & b&=zx(z^2-x^2),\\ \gamma&=z(x^2-y^2), & c&=xy(x^2-y^2),\end{aligned}$$

we have

$$\begin{aligned}(x^2,\ 1,\ x\sqrt{X})&=\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z},\\ (x^3,\ x,\ \ \sqrt{X})&=a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z};\end{aligned}$$

and the formula is

$$\begin{aligned}&(x^3,\ x,\ \sqrt{X})^2\,\Omega\\ &=(\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})\{(y^3z-yz^3)\tfrac{1}{2}X'+(-3x^2z+z^3)\sqrt{X}Y+(3x^2y-y^3)\sqrt{XZ}\}\\ &\quad-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})\{(y^2-z^2)(X+\tfrac{1}{2}X'x)-2xy\sqrt{X}Y-2xz\sqrt{XZ}\}\\ &=(\alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})(L+M\sqrt{XY}+N\sqrt{XZ})\\ &\quad-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})(P+Q\sqrt{XY}+R\sqrt{XZ}),\ \text{suppose,}\end{aligned}$$

$$=\begin{array}{llll}\sqrt{X} & +\sqrt{Y} & +\sqrt{Z} & +\sqrt{XYZ}\\ \hline \alpha L & +\alpha MX & +\alpha NX & \\ +\beta MY & +\beta L & & +\beta N\\ +\gamma NZ & & +\gamma L & +\gamma M\\ -aP & -aQX & -aRX & \\ -bQY & -bP & & -bR\\ -cRZ & & -cP & -cQ\end{array};$$

viz. this is

$$\begin{aligned}=\ &\{\alpha L-aP+Y(\beta M-bQ)+Z(\gamma N-cR)\}\sqrt{X}\\ &+\{X(\alpha M-aQ)+\beta L-bP\ \}\sqrt{Y}\\ &+\{X(\alpha N-aR)+\gamma L-cP\ \}\sqrt{Z}\\ &+(\beta N+\gamma M-bR-cQ\ )\sqrt{XYZ}.\end{aligned}$$

The coefficient of $\sqrt{XYZ}$ is here

$$\begin{aligned} &= y(z^2-x^2)(3x^2y-y^3) && = y^2(z^2-x^2)(3x^2-y^2)\\ &\quad + z(x^2-y^2)(-3x^2z+z^3) && \quad + z^2(x^2-y^2)(-3x^2+z^2)\\ &\quad - zx(z^2-x^2)(2xz) && \quad - 2x^2z^2(z^2-x^2)\\ &\quad - xy(x^2-y^2)(-2xy) && \quad + 2x^2y^2(x^2-y^2), \end{aligned}$$

which is

$$= 6x^2y^2z^2 - y^2z^4 - y^4z^2 - z^2x^4 - z^4x^2 - x^4y^2 - x^2y^4.$$

The coefficient of $\sqrt{Y}$ is

$$\begin{aligned} &= [x(y^2-z^2)(-3x^2z+z^3)+yz(y^2-z^2)2xy]X\\ &\quad + y(z^2-x^2)\tfrac{1}{2}X'(y^2z-yz^3) - zx(z^2-x^2)(y^2-z^2)(X+\tfrac{1}{2}X'x)\\ &= -2xz(x^2-y^2)(y^2-z^2)X - z(x^2-y^2)(y^2-z^2)(z^2-x^2)\tfrac{1}{2}X'\\ &= -(x^2-y^2)(y^2-z^2)z\{2xX+\tfrac{1}{2}(z^2-x^2)X'\}, \end{aligned}$$

where the term in { } is

$$\begin{aligned} &= 2x(1-lx^2+x^4)+(z^2-x^2)(-lx+2x^3),\\ &= x\{2-l(z^2+x^2)+2z^2x^2\}, \end{aligned}$$

or the whole coefficient is

$$= -(x^2-y^2)(y^2-z^2)zx\{2-l(z^2+x^2)+2z^2x^2\}.$$

We obtain in like manner the coefficient of $\sqrt{Z}$, and with a little more trouble that of $\sqrt{X}$; and the final result is

$$\begin{aligned} \Omega(x^3, x, \sqrt{X})^2 = &-(z^2-x^2)(x^2-y^2)yz\{2-l(y^2+z^2)+2y^2z^2\}\sqrt{X}\\ &-(x^2-y^2)(y^2-z^2)zx\{2-l(z^2+x^2)+2z^2x^2\}\sqrt{Y}\\ &-(y^2-z^2)(z^2-x^2)xy\{2-l(x^2+y^2)+2x^2y^2\}\sqrt{Z}\\ &+(6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^2y^4-x^4y^2)\sqrt{XYZ}. \end{aligned}$$

And inasmuch as the equation is a solution of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

it follows that $\Omega=-\sqrt{W}$, viz. that $\sqrt{W}$ is by the foregoing equation expressed as a function of x, y, z.

The equation $(x^3, x, x^2\sqrt{X}, \sqrt{X})=0$, that is,

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X}\\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y}\\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z}\\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W} \end{vmatrix}=0,$$

gives

$$w=\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},$$

where the numerator and the denominator are determinants formed with the variables x, y, z.

Writing $\frac{1}{w}$ for w, it follows that the equation

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w, & w^3, & \sqrt{W}, & w^2\sqrt{W} \end{vmatrix}=0$$

gives

$$w=\frac{(x^3,\ x,\ \sqrt{X})}{(x^2,\ 1,\ x\sqrt{X})},$$

which last equation is a transformation of

$$\begin{vmatrix} x^4, & x^2, & 1, & x\sqrt{X} \\ y^4, & y^2, & 1, & y\sqrt{Y} \\ z^4, & z^2, & 1, & z\sqrt{Z} \\ w^4, & w^2, & 1, & w\sqrt{W} \end{vmatrix}=0.$$

The two equations, involving these determinants of the order 4, are consequently equivalent equations.

627.

GEOMETRICAL ILLUSTRATION OF A THEOREM RELATING TO AN IRRATIONAL FUNCTION OF AN IMAGINARY VARIABLE.

[From the *Proceedings of the London Mathematical Society*, vol. VIII. (1876—1877), pp. 212—214. Read May 11, 1876.]

If we have v, a function of u, determined by an equation $f(u, v)=0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, $x'+iy'$ of v: and for the complete understanding of the relation between the two imaginary variables, we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$, of v, u respectively. We must for this purpose take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P' in another plane Π'. The series of values $x+iy$ of u is then represented by means of a curve in the first plane, and the series of values $x'+iy'$ of v by means of a corresponding curve in the second plane. The correspondence between the two points P and P' is of course established by the two equations into which the given equation $f(x+iy, x'+iy')=0$ breaks up, on the assumption that x, y, x', y' are all of them real. If we assume that the coefficients in the equation are real, then the two equations are

$$f(x+iy, x'+iy')+f(x-iy, x'-iy')=0,$$

$$f(x+iy, x'+iy')-f(x-iy, x'-iy')=0;$$

viz. if in these equations we regard either set of coordinates, say (x, y), as constants, then the other set (x', y') are the coordinates of any real point of intersection of the curves represented by these equations respectively.

I consider the particular case where the equation between u, v is $u^2+v^2=a^2$: we have here $(x+iy)^2+(x'+iy')^2=a^2$: so that, to a given point P in the first plane, there

correspond in general two points P_1', P_2' in the second plane: but to each of the points A and B, coordinates $(a, 0)$ and $(-a, 0)$, there corresponds only a single point in the second plane.

We have here a particular case of a well-known theorem: viz. if from a given point P we pass by a closed curve, not containing within it either of the points A or B, back to the initial point P, we pass in the other plane from P_1' by a closed curve back to P_1'; and similarly from P_2' by a closed curve back to P_2': but if the closed curve described by P contain within it A or B, then, in the other plane, we pass continuously from P_1' to P_2'; and also continuously from P_2' to P_1'.

The relations between (x, y), (x', y') are

$$x'^2 - y'^2 = a^2 - (x^2 - y^2),$$
$$x'y' = -xy,$$

whence also

$$(x'^2 + y'^2)^2 = a^4 - 2a^2(x^2 - y^2) + (x^2 + y^2)^2.$$

And if the point (x, y) describe a curve $x^2 + y^2 = \phi(x^2 - y^2)$, then will the point (x', y') describe a curve $x'^2 + y'^2 = \psi(x'^2 - y'^2)$, obtained by the elimination of $x^2 - y^2$ from the two equations

$$x'^2 - y'^2 = a^2 - (x^2 - y^2),$$
$$(x'^2 + y'^2)^2 = a^4 - 2a^2(x^2 - y^2) + \phi(x^2 - y^2);$$

viz. this is

$$(x'^2 + y'^2)^2 = -a^4 + 2a^2(x'^2 - y'^2) + \phi\{a^2 - (x'^2 - y'^2)\}.$$

In particular, if the one curve be $(x^2 + y^2)^2 = \alpha + \beta(x^2 - y^2)$; then the other curve is

$$(x'^2 + y'^2)^2 = -a^4 + 2a^2(x'^2 - y'^2) + \alpha + \beta\{a^2 - (x'^2 - y'^2)\},$$

that is,

$$(x'^2 + y'^2)^2 = \alpha' + \beta'(x'^2 - y'^2),$$

where

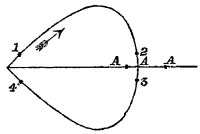
$$\alpha' = -a^4 + \beta a^2 + \alpha, \quad \beta' = 2a^2 - \beta.$$

Writing for greater simplicity $a = 1$, then $\alpha' = -1 + \alpha + \beta$, $\beta' = 2 - \beta$; in particular, if $\alpha = 0$, then $\alpha' = -1 + \beta$, $\beta' = 2 - \beta$.

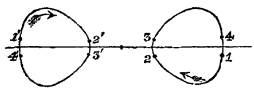
Supposing successively $\beta < 1$, $\beta = 1$, and $\beta > 1$, then in each case P describes a closed curve or half figure-of-eight, as shown in the annexed P-figure; but in the first case the point A is inside the curve, in the second case on it, and in the third case outside it, as shown by the letters A, A, A of the figure; and, corresponding to the three cases respectively, we have the three P'-figures, the curve in the first of them consisting of two ovals, in the second of them being a figure of eight, and in the third a twice-indented or pinched oval: the small figures 1, 2, 3, 4 in the P-figure, and 1, 2, 3, 4 and 1′, 2′, 3′, 4′ in the P'-figures serve to show the corresponding

positions of the points P and P_1', P_2' respectively; and the courses are further indicated by the arrows. And we thus see how the two separate closed curves described

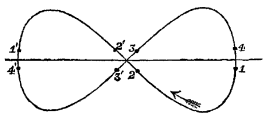
P-Figure.



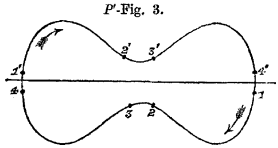
P'-Fig. 1.



P'-Fig. 2.



P'-Fig. 3.



by P_1' and P_2', as in figure 1, change into the single closed curve described one half of it by P_1' and the other half of it by P_2' as in figure 3.

628.

ON THE CIRCULAR RELATION OF MÖBIUS.

[From the *Proceedings of the London Mathematical Society*, vol. VIII. (1876—1877), pp. 220—225. Read April 12, 1877.]

IN representing a given imaginary or complex quantity $u, =x+iy$, by means of the point whose coordinates are x, y, we assume in the first instance that x, y are real,—but in the results this restriction may be abandoned—for instance, if the imaginary quantities u, u', c are connected by the equation $u^2+u'^2=c^2$; then, writing $u=x+iy$, $u'=x'+iy'$, $c=a+bi$, we have $x^2-y^2+x'^2-y'^2=a^2-b^2$, $xy+x'y'=ab$, equations connecting the points U, U', C which serve to represent the quantities u, u', c, and which (regarding C as a fixed point) establish a correspondence between the two variable points U, U': any given value $u, =x+iy$, is represented by the point U, and corresponding hereto we have (in the present case) two points U', viz. these are the real intersections of the curves $x'^2-y'^2=a^2-b^2-(x^2-y^2)$, $x'y'=ab-xy$, and then the coordinates x', y' of either of these give the value $x'+iy'$ of u'.

But, the two curves once arrived at, we may for other purposes be concerned with their intersections as well imaginary as real; or, still more generally, all the quantities entering into the two equations may be regarded as imaginary.

Theoretically we seem to require two imaginary roots of unity, incommensurable and convertible, viz. taking these to be i, I, then $i^2=-1$, $I^2=-1$, $iI=Ii$, but without any relation between i, I: thus, in what precedes, writing I instead of i, viz. $u, u', c=x+Iy, x'+Iy', a+bI$, here each of the quantities x, y, x', y', a, b can be *ab initio* an imaginary quantity of the form $\lambda+\mu i$. But, conforming to the ordinary practice, I use i only, writing for instance $u=x+iy$, without any express statement that x, y are real; on the understanding that any equation containing such quantities, and therefore ultimately of the form $P+iQ=0$, denotes the two equations $P=0$, $Q=0$ (or, what is the same thing, that we have not only the original equation, but, in addition to it, the like equation with each such quantity $x+iy$ replaced by the con-

jugate quantity $x-iy$): and the further understanding that, in the pair of equations, each of the quantities x, y, &c. entering therein may itself be considered as an imaginary quantity of the form $\lambda+\mu i$.

The foregoing explanation is required, for otherwise it would appear as if the circular relation of Möbius* about to be explained was of necessity a relation between real points: I hold that this is *not* the case. But in all that follows I do, in fact, consider primarily the case of real points; and indeed the occasion does not arise for any explicit consideration of the case of imaginary points.

The circular relation is as follows. If in the first instance we have four points U, A, B, C on a line, and u, a, b, c their distances from any fixed point on that line; and again, U', A', B', C' four other points on a line (the same or a different line), and u', a', b', c' their distances from any fixed point on that line; then the same equation between u, a, b, c, u', a', b', c' which expresses the homographic relation between the two ranges of points U, A, B, C and U', A', B', C', expresses, when differently interpreted, the circular relation between the four points U, A, B, C in a plane, and the four other points U', A', B', C' in the same or a different plane—viz. for the new interpretation, u is used as denoting $x+iy$, the linear function of the coordinates x, y of the point U, and the like as regards the remaining quantities a, b, c, u', a', b', c'.

As in the homographic theory (but of course without the condition of being in a line), we have A, A'; B, B'; C, C' given pairs of corresponding points: the equation now represents two equations; and these, when either of the points U, U' is given, determine the corresponding point U' or U.

The homographic relation may be written in the forms

$$\begin{vmatrix} 1, & u, & u', & uu' \\ 1, & a, & a', & aa' \\ 1, & b, & b', & bb' \\ 1, & c, & c', & cc' \end{vmatrix} = 0,$$

$$u-a\,.\,b-c : u-b\,.\,c-a : u-c\,.\,a-b = u'-a'\,.\,b'-c' : u'-b'\,.\,c'-a' : u'-c'\,.\,a'-b',$$

viz. these are forms of one and the same equation: and it may be added that, if ω in the first system corresponds to ∞ in the second system, and ω' in the second system to ∞ in the first system (of course ω, ω' are not corresponding values in the two systems respectively); so that

$$\begin{vmatrix} & 1, & \omega \\ 1, & a, & a', & aa' \\ 1, & b, & b', & bb' \\ 1, & c, & c', & cc' \end{vmatrix} = 0, \qquad \begin{vmatrix} 1, & \omega' \\ 1, & a, & a', & aa' \\ 1, & b, & b', & bb' \\ 1, & c, & c', & cc' \end{vmatrix} = 0,$$

$$\omega-a\,.\,b-c : \omega-b\,.\,c-a : \omega-c\,.\,a-b = b'-c' : c'-a' : a'-b';$$

$$b-c : \qquad c-a : \qquad a-b = \omega'-a'\,.\,b'-c' : \omega'-b'\,.\,c'-a' : \omega'-c'\,.\,a'-b';$$

* [Möbius, *Ges. Werke;* t. II., pp. 243—314, and elsewhere.]

whence also, if

$$\Lambda = \begin{vmatrix} 1, & 1, & 1 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}, \quad = bc' - b'c + ca' - c'a + ab' - a'b,$$

then

$$-\Lambda . \omega - a = b - c . c' - a' . a' - b', \quad \Lambda . \omega' - a' = b' - c' . c - a . a - b,$$
$$-\Lambda . \omega - b = c - a . a' - b' . b' - c', \quad \Lambda . \omega' - b' = c' - a' . a - b . b - c,$$
$$-\Lambda . \omega - c = a - b . b' - c' . c' - a', \quad \Lambda . \omega' - c' = a' - b' . b - c . c - a.$$

Then, ω, ω' being thus determined, we have

$$\omega - a . \omega' - a' = \omega - b . \omega' - b' = \omega - c . \omega' - c' = \omega - u . \omega' - u'$$
$$= -\frac{b - c . c - a . a - b . b' - c' . c' - a' . a' - b'}{\Lambda^2} \; (= \Delta \text{ suppose}),$$

viz we have $\omega - u . \omega' - u' =$ a given value; which is the most simple form of the relation between u, u'.

Interpreting everything in the first instance in regard to the homographic ranges, the equations show that there is in the first range a point O, and in the second range a point O', such that OA, &c. denoting distances, we have

$$OA . O'A' = OB . O'B' = OC . O'C' = OU . O'U' \; (= \Delta);$$

or, what is the same thing, if in the line of the first range we construct A_1, B_1, C_1, U_1 by the formulæ

$$OA . OA_1 = OB . OB_1 = OC . OC_1 = OU . OU_1 = \Delta,$$

that is, invert the first range in regard to the centre O and squared radius Δ, then we have a range O, A_1, B_1, C_1, U_1 *equal* to the range O', A', B', C', U'; viz. the distances of corresponding points are equal in the two cases: or say a range O, A_1, B_1, C_1, U_1 imposable upon O', A', B', C', U'.

The like result holds for the circular relation, but the interpretation must be explained more in detail. And, first, it is to be remarked that O in the first figure is the point corresponding to any point whatever at infinity in the second figure; viz. writing $u' = \xi' + i\eta'$, $= \infty$, then, whatever value we give to the ratio of the two infinite quantities ξ', η', we obtain the same complex value of ω, that is, the same coordinates for the point O. And, similarly, O' in the second figure is the point corresponding to any point whatever at infinity in the first figure.

To determine O, we have the equation

$$\frac{\omega - a}{\omega - b} = \frac{b' - c'}{c' - a'} \cdot \frac{c - a}{a - b}.$$

Any such equation gives at once the geometrical construction, viz. $\omega - a = OAe^{iOAx}$, where OA is the distance of the points O, A regarded as positive, and OAx is the

inclination of the line OA regarded as drawn from A to O to the line Ax, such angle being measured in the sense Ax to Ay; where Ax, Ay are the lines drawn from A in the senses x positive and y positive respectively: and so in other cases.

The equation is therefore equivalent to the two equations

$$\frac{OA}{OB} = \frac{B'C'}{C'A'} \cdot \frac{CA}{AB},$$

and

$$\angle OAx - \angle OBx = \angle B'C'x - \angle C'A'x + \angle CAx - \angle ABx.$$

The former of these expresses that O is in a certain circle which, having its centre on the line AB, cuts AB and AB produced in the one or the other sense; the latter that it is in the segment described on a determinate side of AB and containing a given angle: hence O, as the intersection of the segment with the first-mentioned circle, is a uniquely determined point. Similarly O' is a uniquely determined point.

It is not obvious how to construct Λ, from its original value as given above (but, ω being known, we can without difficulty construct it from the value

$$-\Lambda \,.\, \omega - a = b - c \,.\, c' - a' \,.\, a' - b'),$$

nor consequently Δ from its expression in terms of Λ: but, ω and ω' being known, we can construct Δ from the expression $\omega - a \,.\, \omega' - a' = \Delta$; supposing it thus constructed, $= ke^{i\theta}$ suppose, then if, with centre O and squared radius k, we invert the first figure, thereby obtaining the points A_1, B_1, C_1, U_1 such that

$$OA \,.\, OA_1 = OB \,.\, OB_1 = OC \,.\, OC_1 = OU \,.\, OU_1 = k,$$

(the points A_1, B_1, C_1, U_1 being on the lines OA, OB, OC, OU respectively,) then the equations

$$\omega - a \,.\, \omega' - a' = \omega - b \,.\, \omega' - b' = \omega - c \,.\, \omega' - c' = \omega - u \,.\, \omega' - u' = ke^{i\theta}$$

give

$$\omega - a \,.\, \omega' - a' = OA \,.\, OA_1 e^{i\theta},$$

that is,

$$OA \,.\, O'A' = OA \,.\, OA_1, \text{ or, simply, } O'A' = OA_1,$$

and

$$\angle AOx + \angle AO'x' = \theta,$$

or, what is the same thing,

$$\angle A_1Ox + \angle A'Ox' = \theta,$$

and so for the other letters, viz. we have

$$O'A',\ O'B',\ O'C',\ O'U' = OA_1,\ OB_1,\ OC_1,\ OU_1,$$

respectively; and further

$$\angle\text{'s } A_1Ox,\ B_1Ox,\ C_1Ox,\ U_1Ox = \theta - A'O'x',\ \theta - B'O'x',\ \theta - C'O'x',\ \theta - U'O'x',$$

respectively; viz. the system of points O, A_1, B_1, C_1, U_1 is *equal* to the system O', A', B', C', U', that is, the distances of corresponding points and magnitudes of corresponding angles are severally equal—but the angles A_1Ox and $A'O'x'$, &c. are in opposite senses, as appears by the just mentioned equations $A_1Ox = \theta - A'O'x'$, &c.; that is, the two figures are symmetrically equal: but the one of them is not, except by a turning over of its plane, imposable upon the other.

The conclusion is, the two figures A, B, C, U and A', B', C', U' are each of them equal by symmetry, but not superimposably, to a figure which is the inverse of the other of them; viz. there exists in the first figure a point O, and in the second figure a point O', such that, inverting say the first figure, with centre O and a squared radius of determinate magnitude, we obtain the points A_1, B_1, C_1, U_1, forming with O a figure equal by symmetry, but not superimposably, to the second figure A', B', C', U', O'. Hence also to any line in the first figure corresponds in the second figure a circle through O', and to any line in the second figure there corresponds in the first figure a circle through O; or, more generally, to any circle in either figure there corresponds a circle in the other figure.

There is a particular case of peculiar interest, viz. writing for greater convenience d, d' as corresponding values in place of u, u', the system a, b, c, d corresponds homographically to itself in three different ways; that is, we may have

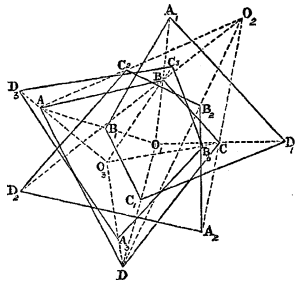
$$(a', b', c', d') = (b, a, d, c),\ (c, d, a, b) \text{ or } (d, c, b, a).$$

To fix the ideas, attending to the first case, we have thus the range of points (A, B, C, D) corresponding homographically to (B, A, D, C), viz. here $\omega' = \omega$, and $\omega - a \,.\, \omega - b = \omega - c \,.\, \omega - d$, that is, the corresponding points U, U' belong to the involution where A and B and also C and D are corresponding points. The like theory applies to the circular transformation: viz. the points (A, B, C, D) may correspond to (B, A, D, C), viz. there exists a point O (or say O_1) and squared radius k_1, such that, inverting the figure and marking the inverse points of A, B, C, D as B_1, A_1, D_1, C_1 respectively, the new figure O_1, A_1, B_1, C_1, D_1 is equal by symmetry, but not superimposably, to the original figure $OABCD$. The equation $\omega_1 - a \,.\, \omega_1 - b = \omega_1 - c \,.\, \omega_1 - d$ gives the geometrical definition of the point O_1, viz. this is a point such that $O_1A \,.\, O_1B = O_1C \,.\, O_1D$ and further that AB and CD subtend at O_1 equal angles: we have $\omega_1 = \dfrac{ab - cd}{a + b - c - d}$, giving for $\omega_1 - a$, $\omega_1 - b$, $\omega_1 - c$, $\omega_1 - d$ convenient expressions the first of which is $\omega_1 - a = \dfrac{c - a \,.\, d - a}{c + d - a - b}$. We hence obtain a convenient construction for O, viz. taking M for the middle point of AB and N for the middle point of CD, and drawing from A in the sense M to N a line AP, $= 2MN$, then this equation may be written $\omega_1 - a = \dfrac{c - a \,.\, d - a}{p - a}$ (p the function $x + iy$ which belongs to the point P); thence $O_1A = \dfrac{CA \,.\, DA}{PA}$ and

$$\angle O_1Ax = \angle CAx + \angle DAx - \angle PAx,$$

conditions which determine uniquely the position of O_1.

We may have (A, B, C, D) corresponding to (C, D, A, B) and (D, C, B, A), the inversions for these depending on the points O_2 and O_3 respectively: I annex a figure showing the three inversions of the same four points A, B, C, D.



629.

ON THE LINEAR TRANSFORMATION OF THE INTEGRAL $\int \frac{du}{\sqrt{U}}$.

[From the *Proceedings of the London Mathematical Society*, vol. VIII. (1876—1877), pp. 226—229. Read April 12, 1877.]

THE quartic function U is taken to be $=\epsilon . u-a . u-b . u-c . u-d$, where a, b, c, d are imaginary values represented in the usual manner by means of the points A, B, C, D; viz. if $a=\alpha_0+\alpha_1 i$, then A is the point whose rectangular coordinates are α_0, α_1; and the like as regards B, C, D. And I consider chiefly the definite integrals such as $\int_a^b \frac{du}{\sqrt{U}}$ where the path is taken to be the right line from A to B. There is here nothing to fix the sign of the radical; but if at any particular point of the path we assign to it at pleasure one of its two values, then (the radical varying continuously) this determines the value at every other point of the path; and the integral defined as above is completely determinate except as to its sign, which might be fixed as above, but which is better left indeterminate. The integral, thus determinate except as to its sign, is denoted by (AB).

I wish to establish the theorem that, if the points A, B, C, D taken in this order form a convex quadrilateral, then

$$(AB)=\pm(CD),\quad (AD)=\pm(BC),\ \text{but not}\ (AC)=\pm(BD);$$

whereas, if the four points form a triangle and interior point, then the three equations all hold good. I regard the theorem as the precise statement of Bouquet and Briot's theorem, $A-B+C-D=0$, or say $(OA)-(OB)+(OC)-(OD)=0$, where the four terms are the rectilinear integrals taken from a point O to the four points A, B, C, D respectively. The two cases may be called, for shortness, the convex and the reentrant cases respectively.

To prove in the case of a convex quadrilateral that (AC) is not $= \pm (BD)$, it is sufficient to consider the integral $\int \frac{du}{\sqrt{u^4-1}}$, where A, B, C, D are the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ respectively, and where, writing $v = iu$, it at once appears that we have

$$\int_{-1}^{1} \frac{du}{\sqrt{u^4-1}} = \pm i \int_{-i}^{i} \frac{du}{\sqrt{u^4-1}},$$

that is,

$$(AC) = \pm i\,(BD), \text{ not } (AC) = \pm (BD).$$

But I consider the general question of the linear transformation. If a', b', c', d' correspond homographically to a, b, c, d, then to represent these values a', b', c', d' we have the points A', B', C', D', connected with A, B, C, D according to the circular relation of Möbius; and then, making u', a', b', c', d' correspond homographically to u, a, b, c, d, and representing in like manner the variables u, u' by the points U, U' respectively, we have the circular relation between the two systems U, A, B, C, D and U', A', B', C', D'.

Before going further I remark that the distinction of the convex and reentrant cases is not an invariable one; the figures are transformable the one into the other. Thus, taking C on the line BD (that is, between B and D, not on the line produced), there is not this relation between B', C', D', and the figure $A'B'C'D'$ is convex or reentrant as the case may be. Giving to C an infinitesimal displacement to the one side or the other of the line BD, we have in the one case a convex figure, in the other case a reentrant figure $ABCD$; but the corresponding displacement of C' being infinitesimal, the figure $A'B'C'D'$ remains for either displacement, convex or reentrant, as it originally was; that is, we have a convex figure $ABCD$ and a reentrant figure $ABCD$, each corresponding to the figure $A'B'C'D'$ (which is convex, or else reentrant, as the case may be).

Writing for convenience

$$\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} = b-c,\ c-a,\ a-b,\ a-d,\ b-d,\ c-d,$$

$$\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' = b'-c',\ c'-a',\ a'-b',\ a'-d',\ b'-d',\ c'-d',$$

so that identically

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0, \quad \mathrm{a'f'} + \mathrm{b'g'} + \mathrm{c'h'} = 0,$$

then the homographic relation between (a, b, c, d), (a', b', c', d') may be written in the forms

$$\mathrm{af} : \mathrm{bg} : \mathrm{ch} = \mathrm{a'f'} : \mathrm{b'g'} : \mathrm{c'h'},$$

or, what is the same thing, there exists a quantity N such that

$$\frac{\mathrm{a'f'}}{\mathrm{af}} = \frac{\mathrm{b'g'}}{\mathrm{bg}} = \frac{\mathrm{c'h'}}{\mathrm{ch}} = N^2.$$

The relation between u, u' may be written in the forms

$$\frac{u'-a'}{u'-d'} = P\frac{u-a}{u-d}, \quad \frac{u'-b'}{u'-d'} = Q\frac{u-b}{u-d}, \quad \frac{u'-c'}{u'-d'} = R\frac{u-c}{u-d};$$

and then, writing for u, u' their corresponding values, we find

$$P = \frac{\mathrm{b'h}}{\mathrm{bh'}} = \frac{\mathrm{c'g}}{\mathrm{cg'}}, \quad Q = \frac{\mathrm{c'f}}{\mathrm{cf'}} = \frac{\mathrm{a'h}}{\mathrm{ah'}}, \quad R = \frac{\mathrm{a'g}}{\mathrm{ag'}} = \frac{\mathrm{b'f}}{\mathrm{bf'}},$$

giving

$$\mathrm{f}^2PN^2 = \mathrm{f'}^2QR, \quad \mathrm{g}^2QN^2 = \mathrm{g'}^2RP, \quad \mathrm{h}^2RN^2 = \mathrm{h'}^2PQ, \quad \sqrt{PQR} = \frac{\mathrm{fgh}}{\mathrm{f'g'h'}}N^3.$$

Differentiating any one of the equations in (u, u'), for instance the first of them, we find

$$\frac{\mathrm{f'}du'}{(u'-d')^2} = \frac{\mathrm{f}Pdu}{(u-d)^2};$$

then, forming the equation

$$\frac{\sqrt{\epsilon\,.\,u'-a'\,.\,u'-b'\,.\,u'-c'\,.\,u'-d'}}{(u'-d')^2} = \pm\frac{\sqrt{PQR}\sqrt{\epsilon\,.\,u-a\,.\,u-b\,.\,u-c\,.\,u-d}}{(u-d)^2},$$

and attending to the relation $\mathrm{f}^2PN^2 = \mathrm{f'}^2QR$, we obtain

$$\pm\frac{Ndu'}{\sqrt{U'}} = \frac{du}{\sqrt{U}},$$

which is the differential relation between u, u'.

We have in connection with A, B, C, D the point O, and in connection with A', B', C', D' the point O'. As U describes the right line AB, U' describes the arc not containing O' of the circle $A'B'O'$; for observe that O' corresponds in the second figure to the point at infinity on the line AB, viz. as U passes from A to B, not passing through the point at infinity, U' must pass from A' to B', not passing through the point O', that is, it must describe, not the arc $A'O'B'$, but the remaining arc $2\pi - A'O'B'$, say this is the arc $\overset{\smile}{A'B'}$. The integral in regard to u' is thus not the rectilinear integral $(A'B')$, but the integral along the just-mentioned circular arc, say this is denoted by $(\overset{\smile}{A'B'})$; and we thus have

$$(AB) = \pm N(\overset{\smile}{A'B'}).$$

But we have $(\overset{\smile}{A'B'})$ = or not $= (A'B')$, according as the chord $A'B'$ and the arc $\overset{\smile}{A'B'}$ do not include between them either of the points C', D', or include between them one or both of these points; and in the same cases respectively

$$(AB) = \text{or not} = \pm N(A'B').$$

Of course we may in any way interchange the letters, and write under the like circumstances

$$(AC) = \text{or not} = \pm N(A'C'), \text{ \&c.}$$

Suppose now that $ABCD$ is a convex quadrilateral, and consider first in regard to (AB), and next in regard to (AC), the three transformations $A'B'C'D' = BADC$, $= CDAB$, and $= DCBA$, respectively. We have here a figure as in the paper "On the circular relation of Möbius," [628], p. 617 of this volume, the points O_1, O_2, O_3 belonging to the three cases respectively. It will be observed in the figure, and it is easy to see generally, that the points O_1 and O_3 are interior, the point O_2 exterior. We have $N = 1$, and therefore

$$(AB) = \text{or not} = \pm (AB), = \text{or not} = \pm (CD), = \text{or not} = \pm (CD),$$

according as

(1) the chord AB and the arc $\overset{\smile}{AB}$ of ABO_1 do not or do inclose C and D or either of them;

(2) the chord CD and the arc $\overset{\smile}{CD}$ of CDO_2 do not or do inclose A and B or either of them;

(3) the chord CD and the arc $\overset{\smile}{CD}$ of CDO_3 do not or do inclose A and B or either of them.

The first test gives merely the identity $(AB) = \pm (AB)$; the other two each of them give $(AB) = \pm (CD)$, as is seen from the positions of the points O_1, O_2, O_3.

Next, apply the test to AC; we have

$$(AC) = \text{or not} = \pm (BD), = \text{or not} = \pm (AC), = \text{or not} = \pm (BD),$$

according as

(1) the chord AC and the arc $\overset{\smile}{AC}$ of ACO_1 do not or do inclose B and D or either of them;

(2) the chord BD and the arc $\overset{\smile}{BD}$ of BDO_2 do not or do inclose A and C or either of them;

(3) the chord BD and the arc $\overset{\smile}{BD}$ of BDO_3 do not or do inclose A and C or either of them.

In the second case, neither A nor C is inclosed, but we have merely the identity $(AC) = \pm (AC)$; in the first case, B is inclosed and, in the third case, C is inclosed; and the tests each give (AC) not $= \pm (BD)$.

I have not taken the trouble of drawing the figure for a reentrant quadrilateral $ABCD$; the mere symmetry is here enough to show that, having one, we have all three, of the relations in question

$$(AD) = \pm (BC), \quad (BD) = \pm (CA), \quad (CD) = \pm (AB).$$

END OF VOL. IX.

Cambridge:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

SELECT SCIENTIFIC PUBLICATIONS OF

THE CAMBRIDGE UNIVERSITY PRESS.

The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S., late Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 4to. Thirteen Volumes. Vols. I. II. III. IV. V. VI. VII. VIII., edited by Professor CAYLEY; vol. IX., edited by A. R. FORSYTH, Sc.D., F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge; 25*s.* each. [Vol. X. *In the Press.*

The Scientific Papers of James Clerk Maxwell. Edited by W. D. NIVEN, M.A., F.R.S., formerly Fellow of Trinity College. In Two Volumes. Royal 4to. £3. 3*s.* (net).

Mathematical and Physical Papers, by Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., Professor of Natural Philosophy in the University of Glasgow. Collected from different Scientific Periodicals from May 1841, to the present time. Demy 8vo. Vol. I. 18*s.* Vol. II. 15*s.* Vol. III. 18*s.*

The Scientific Papers of John Couch Adams, M.A., Sc.D., D.C.L., LL.D., F.R.S., F.R.A.S., F.C.P.S., Fellow of Pembroke College, Honorary Fellow of St John's College, and late Lowndean Professor of Astronomy and Geometry in the University of Cambridge. Vol. I. Edited by WILLIAM GRYLLS ADAMS, Sc.D., F.R.S., Professor of Natural Philosophy in King's College, London, late Fellow of St John's College, Cambridge. With a Memoir by J. W. L. GLAISHER, Sc.D., F.R.S., Fellow of Trinity College, Cambridge. Demy 4to. [*In the Press.*

A Treatise on the Mathematical Theory of Elasticity, by A. E. H. LOVE, M.A., F.R.S., Fellow of St John's College, Cambridge. In Two Volumes. Demy 8vo. Vol. I. 12*s.* Vol. II. 12*s.*

A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639—1850. By the late I. TODHUNTER, Sc.D., F.R.S., edited and completed by Professor KARL PEARSON, M.A. Demy 8vo. 25*s.*
Vol. II. Saint-Venant to Lord KELVIN (Sir WILLIAM THOMSON). In 2 parts. By the same Editor. 30*s.*

Hydrodynamics, a Treatise on the Mathematical Theory of the Motion of Fluids, by H. LAMB, M.A., F.R.S., late Fellow of Trinity College, Cambridge, Professor of Mathematics in the Owens College, Manchester. Demy 8vo. New Edition. 20*s. Net.*

A Treatise on the Lunar Theory. By E. W. BROWN, M.A., late Fellow of Christ's College, Cambridge, Professor of Applied Mathematics in Haverford College, U.S.A. Royal 8vo. [*In the Press.*

An Introduction to Abel's Theorem and the allied theory. By H. F. BAKER, M.A., Fellow of St John's College, Cambridge. Royal 8vo. [*In the Press.*

The Steam Engine and other Heat Engines. By J. A. EWING, M.A., F.R.S., M.Inst. C.E., Professor of Mechanism and Applied Mathematics in the University of Cambridge. Demy 8vo. 15*s.*

A Treatise on Natural Philosophy. By Lord KELVIN (Sir W. THOMSON), LL.D., D.C.L., F.R.S., and P. G. TAIT, M.A. **Part I.** Demy 8vo. 16*s.* **Part II.** Demy 8vo. 18*s.*

Elements of Natural Philosophy. By Lord KELVIN (Sir W. THOMSON) and Professor P. G. TAIT. Demy 8vo. 9*s.*

A Treatise on Analytical Statics. By E. J. ROUTH, Sc.D., F.R.S., Fellow of the University of London, Honorary Fellow of Peterhouse, Cambridge. Demy 8vo. Vol. I. 14*s.* Vol. II. 10*s.*

Theory of Differential Equations. Part I. Exact Equations and Pfaff's Problem. By A. R. FORSYTH, Sc.D., F.R.S., Fellow of Trinity College, and Sadlerian Professor of Pure Mathematics in the University of Cambridge. Demy 8vo. 12*s.*

A Treatise on the Theory of Functions of a Complex Variable, by A. R. FORSYTH, Sc.D., F.R.S. Royal 8vo. 21*s.* Net.

A Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S., Fellow and Tutor of Christ's College, Cambridge. Demy 8vo. 12*s.*

A Treatise on Spherical Astronomy. By Sir ROBERT S. BALL, M.A., F.R.S., Lowndean Professor of Astronomy and Geometry. Demy 8vo. [*Preparing.*

Elementary Thermodynamics. By J. PARKER, M.A., late Fellow of St John's College, Cambridge. Crown 8vo. 9*s.*

Geometrical Optics, Treatise on, by R. A. HERMAN, M.A., Fellow of Trinity College, Cambridge. Demy 8vo. [*Preparing.*

A Treatise on Geometrical Optics. By R. S. HEATH, M.A., late Fellow of Trinity College, Cambridge, Professor of Mathematics in Mason Science College, Birmingham. Demy 8vo. 12*s.* 6*d.*

Elements of the Mathematical Theory of Electricity and Magnetism. By J. J. THOMSON, M.A., F.R.S., Fellow of Trinity College, Cambridge, and Cavendish Professor of Experimental Physics. Crown 8vo. 10*s.*

Catalogue of Scientific Papers compiled by the Royal Society of London: Vols. 1—6 for the years 1800—1863, Demy 4to. cloth (Vol. 1 in half morocco) £4 (net); half morocco £5. 5*s.* (net). Vols. 7—8 for the years 1864—1873, cloth £1. 11*s.* 6*d.* (net); half morocco £2. 5*s.* (net). Single volumes cloth 20*s.* or half morocco 28*s.* (net). New Series for the years 1874—1883. Vols. 9—10, cloth 25*s.* (net) each, half morocco 32*s.* (net) each. [Vol. XI. *Just ready.*

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

CAMBRIDGE UNIVERSITY PRESS.

A Treatise on Elementary Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. New and Enlarged Edition. Crown 8vo. 7*s.* 6*d.*

Solutions to the Examples in a Treatise on Elementary Dynamics. By the same Author. Crown 8vo. 7*s.* 6*d.*

Plane Trigonometry. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. 7*s.* 6*d.* Or in separate Parts. Part I. up to and including the Solution of Triangles. 5*s.* Part II. Analytical Trigonometry. 3*s.* 6*d.*

Geometrical Conics. By F. S. MACAULAY, M.A., Assistant Master at St Paul's School. Crown 8vo. 4*s.* 6*d.*

A Treatise on Elementary Hydrostatics. By J. GREAVES, M.A., Fellow and Lecturer of Christ's College. Crown 8vo. 5*s.*

An Elementary Treatise on Geometrical Optics. By R. S. HEATH, M.A. Cr. 8vo. 5*s.*

A Treatise on the General Principles of Chemistry. By M. M. PATTISON MUIR, M.A., Fellow of Gonville and Caius College. Demy 8vo. *New Edition.* 15*s.*

Elementary Chemistry. By M. M. PATTISON MUIR, M.A., and CHARLES SLATER, M.A., M.B. Crown 8vo. 4*s.* 6*d.*

Practical Chemistry. A Course of Laboratory Work. By M. M. PATTISON MUIR, M.A., and D. J. CARNEGIE, M.A. Crown 8vo. 3*s.*

Notes on Qualitative Analysis. Concise and Explanatory. By H. J. H. FENTON, M.A., F.I.C., Demonstrator of Chemistry in the University of Cambridge. Cr. 4to. *New Edition.* 6*s.*

CAMBRIDGE NATURAL SCIENCE MANUALS.

PHYSICAL SERIES. General Editor R. T. GLAZEBROOK, M.A., F.R.S., Assistant Director of the Cavendish Laboratory.

Heat and Light. By R. T. GLAZEBROOK, M.A., F.R.S. Crown 8vo. 5*s.*

The Two Parts are also published separately.

Heat. 3*s.* **Light.** 3*s.*

Mechanics and Hydrostatics. By R. T. GLAZEBROOK, M.A., F.R.S. Crown 8vo. 8*s.* 6*d.*

Also separately.

Part I. **Dynamics.** 4*s.* Part II. **Statics.** 3*s.* Part III. **Hydrostatics.** 3*s.*

Solution and Electrolysis. By W. C. D. WHETHAM, M.A., Fellow of Trinity College. Crown 8vo. 7*s.* 6*d.*

Other Volumes preparing.

PITT PRESS MATHEMATICAL SERIES.

Arithmetic for Schools. By C. SMITH, M.A., Master of Sidney Sussex College, Cambridge. Extra Fcap. 8vo. With or without Answers. Second Edition, 3*s.* 6*d.* Or in two Parts 2*s.* each.

Key to Smith's Arithmetic. By G. HALE, M.A. Crown 8vo. 7*s.* 6*d.*

Elementary Algebra. By W. W. ROUSE BALL, M.A., Fellow and Tutor of Trinity College, Cambridge. Extra Fcap. 8vo. 4*s.* 6*d.*

Euclid's Elements of Geometry. Edited by H. M. TAYLOR, M.A., Fellow and formerly Tutor of Trinity College, Cambridge. Extra Fcap. 8vo. Books I—VI. XI. XII. 5*s.* Books I.—VI. 4*s.* Books I.—IV. 3*s.* Books I. and II. 1*s.* 6*d.* Books III. and IV. 1*s.* 6*d.* Books V. and VI. 1*s.* 6*d.* Books XI. and XII. 1*s.* 6*d.*

Solutions to the Exercises in Taylor's Euclid. Books I.—IV. By W. W. TAYLOR, M.A. 6*s.*

The Elements of Statics and Dynamics. By S. L. LONEY, M.A., late Fellow of Sidney Sussex College. Third Edition. Extra Fcap. 8vo. 7*s.* 6*d.*

Also published separately. STATICS, 4*s.* 6*d.* DYNAMICS, 3*s.* 6*d.*

Solutions to the Examples in the Elements of Statics and Dynamics. By S. L. LONEY, M.A. 7*s.* 6*d.*

Mechanics and Hydrostatics for Beginners. By S. L. LONEY, M.A. 4*s.* 6*d.*

An Elementary Treatise on Plane Trigonometry. By E. W. HOBSON, Sc.D., F.R.S., Fellow of Christ's College, and University Lecturer in Mathematics, and C. M. JESSOP, M.A., late Fellow of Clare College. Extra Fcap. 8vo. 4*s.* 6*d.*

London: C. J. CLAY AND SONS,

CAMBRIDGE UNIVERSITY PRESS WAREHOUSE, AVE MARIA LANE.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. X.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1896

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 76 papers, numbered 630 to 705, published for the most part in the years 1876 to 1880.

The Table for the ten volumes is

Vol.	I.	Numbers	1	to	100,
,,	II.	,,	101	,,	158,
,,	III.	,,	159	,,	222,
,,	IV.	,,	223	,,	299,
,,	V.	,,	300	,,	383,
,,	VI.	,,	384	,,	416,
,,	VII.	,,	417	,,	485,
,,	VIII.	,,	486	,,	555,
,,	IX.	,,	556	,,	629,
,,	X.	,,	630	,,	705.

A. R. FORSYTH.

11 *June*, 1896.

CONTENTS.

[An Asterisk means that the paper is not printed in full.]

PAGE

630. *On an expression for* $1 \pm \sin(2p+1)u$ *in terms of* $\sin u$ . . 1
Messenger of Mathematics, t. v. (1876), pp. 7, 8

631. *Synopsis of the theory of equations* 3
Messenger of Mathematics, t. v. (1876), pp. 39—49

632. *On Aronhold's integration-formula* 12
Messenger of Mathematics, t. v. (1876), pp. 88—90

*633. *Note on Mr. Martin's paper "On the integrals of some differentials"* 15
Messenger of Mathematics, t. v. (1876), p. 163

634. *Theorems in trigonometry and on partitions* 16
Messenger of Mathematics, t. v. (1876), p. 164, p. 188

635. *Note on the demonstration of Clairaut's theorem* . . . 17
Messenger of Mathematics, t. v. (1876), pp. 166, 167

636. *On the theory of the singular solutions of differential equations of the first order* 19
Messenger of Mathematics, t. vi. (1877), pp. 23—27

637. *On a differential equation in the theory of elliptic functions* . 24
Messenger of Mathematics, t. vi. (1877), p. 29

638. *On a q-formula leading to an expression for* E_1 . . . 25
Messenger of Mathematics, t. vi. (1877), pp. 63—66

PAGE

639. *An elementary construction in optics* 28
Messenger of Mathematics, t. VI. (1877), pp. 81, 82

*640. *Further note on Mr. Martin's paper* 29
Messenger of Mathematics, t. VI. (1877), pp. 82, 83

641. *On the flexure of a spherical surface* 30
Messenger of Mathematics, t. VI. (1877), pp. 88—90

642. *On a differential relation between the sides of a quadrangle* . 33
Messenger of Mathematics, t. VI. (1877), pp. 99—101

643. *On a quartic curve with two odd branches* 36
Messenger of Mathematics, t. VI. (1877), pp. 107, 108

644. *Note on magic squares* 38
Messenger of Mathematics, t. VI. (1877), p. 168

645. *A Smith's Prize Paper*, 1877 39
Messenger of Mathematics, t. VI. (1877), pp. 173—182

646. *On the general equation of differences of the second order* . 47
Quart. Math. Journ., t. XIV. (1877), pp. 23—25

647. *On the quartic surfaces represented by the equation, symmetrical determinant* $=0$ 50
Quart. Math. Journ., t. XIV. (1877), pp. 46—52

648. *Algebraical theorem* 57
Quart. Math. Journ., t. XIV. (1877), p. 53

649. *Addition to Mr. Glaisher's Note on Sylvester's paper "Development of an idea of Eisenstein"* 58
Quart. Math. Journ., t. XIV. (1877), pp. 83, 84

650. *On a quartic surface with twelve nodes* 60
Quart. Math. Journ., t. XIV. (1877), pp. 103—106

651. *On a special surface of minimum area* 63
Quart. Math. Journ., t. XIV. (1877), pp. 190—196

652. *On a sextic torse* 68
Quart. Math. Journ., t. XIV. (1877), pp. 229—235

PAGE

653. *On a torse depending on the elliptic functions* 73
Quart. Math. Journ., t. XIV. (1877), pp. 235—241

654. *On certain octic surfaces* 79
Quart. Math. Journ., t. XIV. (1877), pp. 249—264

655. *A memoir on differential equations* 93
Quart. Math. Journ., t. XIV. (1877), pp. 292—339

656. *On the theory of partial differential equations* 134
Mathematische Annalen, t. XI. (1877), pp. 194—198

657. *Note on the theory of elliptic integrals* 139
Mathematische Annalen, t. XII. (1877), pp. 143—146

658. *On some formulæ in elliptic integrals* 143
Mathematische Annalen, t. XII. (1877), pp. 369—374

659. *A theorem on groups* 149
Mathematische Annalen, t. XIII. (1878), pp. 561—565

660. *On the correspondence of homographies and rotations* . . 153
Mathematische Annalen, t. XV. (1879), pp. 238—240

661. *On the double* ϑ*-functions* 155
Proc. Lond. Math. Soc., t. IX. (1878), pp. 29, 30

662. *On the double* Θ*-functions in connexion with a* 16*-nodal quartic surface* 157
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 210—219

663. *Further investigations on the double* ϑ*-functions* . . . 166
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 220—233

664. *On the* 16*-nodal quartic surface* 180
Crelle's Journal der Mathem., t. LXXXIV. (1878), pp. 238—241

665. *A memoir on the double* ϑ*-functions* 184
Crelle's Journal der Mathem., t. LXXXV. (1878), pp. 214—245

666. *Sur un exemple de réduction d'intégrales abéliennes aux fonctions elliptiques* 214
Comptes Rendus, t. LXXXV. (1877), pp. 265—268, 373—376, 426—429, 472—475

PAGE

667. *On the bicircular quartic—Addition to Professor Casey's memoir: "On a new form of tangential equation"* . . 223
Phil. Trans., t. 167 (for 1877), pp. 441—460

668. *On compound combinations* 243
Proceedings of the Lit. Phil. Soc. Manchester, t. XVI. (1877), pp. 113, 114; Memoirs, ib., Ser. III., t. VI. (1879), pp. 99, 100

669. *On a problem of arrangements* 245
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 338—342

670. [*Note on Mr. Muir's solution of a "problem of arrangement"*] 249
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 388—391

671. *On a sibi-reciprocal surface* 252
Berlin, Akad. Monatsber., (1878), pp. 309—313

672. *On the game of mousetrap* 256
Quart. Math. Journ., t. XV. (1878), pp. 8—10

673. *Note on the theory of correspondence* 259
Quart. Math. Journ., t. XV. (1878), pp. 32, 33

674. *Note on the construction of Cartesians* 261
Quart. Math. Journ., t. XV. (1878), p. 34

675. *On the fleflecnodal planes of a surface* 262
Quart. Math. Journ., t. XV. (1878), pp. 49—51

676. *Note on a theorem in determinants* 265
Quart. Math. Journ., t. XV. (1878), pp. 55—57

677. [*Addition to Mr. Glaisher's paper "Proof of Stirling's theorem"*] 267
Quart. Math. Journ., t. XV. (1878), pp. 63, 64

678. *On a system of quadric surfaces* 269
Quart. Math. Journ., t. XV. (1878), pp. 124, 125

679. *On the regular solids* 270
Quart. Math. Journ., t. XV. (1878), pp. 127—131

680. *On the Hessian of a quartic surface* 274
Quart. Math. Journ., t. XV. (1878), pp. 141—144

PAGE

681. *On the derivatives of three binary quantics* 278
Quart. Math. Journ., t. xv. (1878), pp. 157—168

682. *Formulæ relating to the right line* 287
Quart. Math. Journ., t. xv. (1878), pp. 169—171

683. *On the function* arc sin $(x+iy)$. 290
Quart. Math. Journ., t. xv. (1878), pp. 171—174

684. *On a relation between certain products of differences* . . 293
Quart. Math. Journ., t. xv. (1878), pp. 174, 175

685. *On Mr. Cotterill's goniometrical problem* 295
Quart. Math. Journ., t. xv. (1878), pp. 196—198

686. *On a functional equation* 298
Quart. Math. Journ., t. xv. (1878), pp. 315—325; Proc. Lond. Math. Soc., t. IX. (1878), p. 29

687. *Note on the function* $\vartheta(x) = a^2(c-x) \div \{c(c-x) - b^2\}$ . . 307
Quart. Math. Journ., t. xv. (1878), pp. 338—340

688. *Geometrical considerations on a solar eclipse* 310
Quart. Math. Journ., t. xv. (1878), pp. 340—347

689. *On the geometrical representation of imaginary variables by a real correspondence of two planes* 316
Proc. Lond. Math. Soc., t. IX. (1878), pp. 31—39

690. *On the theory of groups* 324
Proc. Lond. Math. Soc., t. IX. (1878), pp. 126—133

691. *Note on Mr. Monro's paper "On flexure of spaces"* . . 331
Proc. Lond. Math. Soc., t. IX. (1878), pp. 171, 172

692. *Addition to* [578] *memoir on the transformation of elliptic functions* 333
Phil. Trans., vol. 169, Part II. (for 1878), pp. 419—424

693. *A tenth memoir on quantics* 339
Phil. Trans., vol. 169, Part II. (for 1878), pp. 603—661

PAGE

694. *Desiderata and Suggestions* 401

No. 1. *The theory of groups;*
American Journal of Mathematics, t. I. (1878), pp. 50—52

No. 2. *The theory of groups; graphical representation;*
American Journal of Mathematics, t. I. (1878), pp. 174—176

No. 3. *The Newton-Fourier imaginary problem;*
American Journal of Mathematics, t. II. (1879), p. 97

No. 4. *The mechanical construction of conformable figures;*
American Journal of Mathematics, t. II. (1879), p. 186

695. *A link-work for x^2: extract from a letter to Mr. Sylvester* . 407
American Journal of Mathematics, t. I. (1878), p. 386

696. *Calculation of the minimum N.G.F. of the binary seventhic* . 408
American Journal of Mathematics, t. II. (1879), pp. 71—84

697. *On the double ϑ-functions* 422
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 74—81

698. *On a theorem relating to covariants* 430
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 82, 83

699. *On the triple ϑ-functions* 432
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 134—138

700. *On the tetrahedroid as a particular case of the* 16*-nodal quartic surface* 437
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 161—164

701. *Algorithm for the characteristics of the triple ϑ-functions* . 441
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 165—169

702. *On the triple ϑ-functions* 446
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 190—198

703. *On the addition of the double ϑ-functions* 455
Crelle's Journal der Mathem., t. LXXXVIII. (1880), pp. 74—81

704. *A memoir on the single and double theta-functions* 463
Phil. Trans., vol. 171, Part III. (for 1880), pp. 897—1002

705. *Problems and Solutions* 566
Mathematical Questions with their Solutions from the Educational Times, vols. XIV. to LXI. (1871—1894); for contents, see p. 615

CLASSIFICATION.

GEOMETRY :

Geometrical constructions : (i) in optics, 639.
(ii) for solar eclipses, 688.

Quartic curves, 643, 667.

Construction of Cartesians, 674.

Correspondence, 673, 683, 689, 694.

Quadric surfaces, 678.

Quartic surfaces, 647, 650, 662, 664, 680, 700.

Sextic and other torses, 652, 653.

Octic surfaces, 654, 671.

Line-geometry, 682.

Fleflecnodal planes, 675.

Flexure of surfaces, 641, 691.

Minimal surface, 651.

Regular solids, 679.

ANALYSIS :

Trigonometry, 630, 634.

Theory of equations, 631, 694.

Groups, 659, 660, 690, 694.

Combinatory analysis, 634, 644, 668, 669, 670, 672.

Invariants and covariants, 681, 696, 698.

Quantics, 693.

Integration, 632, 633, 640.

Differential equations, 636, 637, 642, 655, 656.

Finite differences, 646, 677, 686, 687.

Elliptic functions and elliptic integrals, 637, 638, 653, 657, 658, 692.

Double theta functions, 661, 662, 663, 665, 697, 703, 704.

Triple theta functions, 699, 701, 702.

Reduction of transcendental integrals, 666.

Problems and solutions, 645, 705.

Miscellaneous, 635, 648, 649, 676, 684, 685, 695.

630.

ON AN EXPRESSION FOR $1 \pm \sin(2p+1)u$ IN TERMS OF $\sin u$.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 7, 8.]

WRITE $\sin u = x$, then we have

$$\begin{aligned} \sin u &= x, & \cos u &= \sqrt{(1-x^2)}, \\ \sin 3u &= 3x - 4x^3, & \cos 3u &= (1-4x^2)\sqrt{(1-x^2)}, \\ \sin 5u &= 5x - 20x^3 + 16x^5, & \cos 5u &= (1-12x^2+16x^4)\sqrt{(1-x^2)}, \\ &\&c. & &\&c. \end{aligned}$$

It is hence clear, that in general

$$1 - \sin(2p+1)u = (1 \pm x)\{(1, \quad x)^p\}^2,$$
$$1 + \sin(2p+1)u = (1 \mp x)\{(1, -x)^p\}^2,$$

where $(1, x)^p$ denotes a rational and integral function of x of the order p, and $(1, -x)^p$ the same function of $-x$; for it is only in this manner that we can have

$$\cos^2(2p+1)u = (1-x^2)\{[1, x^2]^p\}^2.$$

We, in fact, find

$$\begin{aligned} 1 + \sin u &= 1 + x, \\ 1 - \sin 3u &= (1+x)(1-2x)^2, \\ 1 + \sin 5u &= (1+x)(1+2x-4x^2)^2, \\ 1 - \sin 7u &= (1+x)(1-4x-4x^2+8x^3)^2, \\ &\&c. \end{aligned}$$

and it thus appears that the form is

$$1 + (-)^p \sin(2p+1)u = (1+x)\{(1, x)^p\}^2.$$

To find herein the expression of the factor $(1, x)^p$, write $u = \frac{1}{2}\pi - \theta$ and consequently $x = \cos\theta$; we have therefore

$$1 + \cos(2p+1)\theta = (1 + \cos\theta)\{(1, x)^p\}^2,$$

where in the second factor on the right-hand side x is retained to stand for its value $\cos\theta$. This gives

$$2\cos^2(p + \tfrac{1}{2})\theta = 2\cos^2\tfrac{1}{2}\theta\{(1, x)^p\}^2,$$

or, what is the same thing,

$$(1, x)^p = \frac{\cos(p + \frac{1}{2})\theta}{\cos\frac{1}{2}\theta},$$

viz. this is

$$= \cos p\theta - \sin p\theta \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta},$$

which is

$$= \cos p\theta - \sin p\theta \frac{1 - \cos\theta}{\sin\theta}.$$

We have

$$\cos p\theta + i\sin p\theta = \{x + i\sqrt{(1 - x^2)}\}^p$$
$$= X + i\sqrt{(1 - x^2)}\, Y, \text{ suppose,}$$

where X, Y are rational and integral functions of x of the orders p and $p - 1$ respectively; that is,

$$\cos p\theta = X, \quad \sin p\theta = \sin\theta \,.\, Y,$$

and we have therefore

$$(1, x)^p = X - Y(1 - x),$$

which is the required expression for $(1, x)^p$. For instance

$$p = 3, \quad X + i\sqrt{(1 - x^2)}\, Y = \{x + i\sqrt{(1 - x^2)}\}^3;$$

that is,

$$\begin{array}{rl} X = & -3x \qquad + 4x^3 \\ Y = -1 + 4x^2, \text{ and } \therefore & -(1 - x)\,Y = 1 - \ x - 4x^2 + 4x^3 \\ \hline & X - (1 - x)\,Y = 1 - 4x - 4x^2 + 8x^3, = (1, x)^3, \end{array}$$

so that

and hence

$$1 - \sin 7u = (1 + x)(1 - 4x - 4x^2 + 8x^3)^2,$$

which agrees with a result already obtained.

The foregoing value of $(1, x)^p$ may also be written

$$(1, x)^p = \frac{1}{\sin\theta}\{\sin(p+1)\theta - \sin p\theta\},$$

which however is not practically so convenient.

The formula corresponds to a like formula in elliptic functions, viz. writing $\operatorname{sinam} u = x$, the numerator of $1 + (-)^p \operatorname{sinam}(2p+1)u$ is

$$= (1 + x)\{(1, x)^{2p(p+1)}\}^2,$$

which is $(1 + x)$ multiplied by the square of a rational and integral function of x.

631.

SYNOPSIS OF THE THEORY OF EQUATIONS.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 39—49.]

THE following was proposed as one of the subjects of a Dissertation for the Trinity Fellowships:

Synopsis of the theory of equations; i.e. a statement in a logical order, of the divisions of the subject and the leading questions and theorems, but without demonstrations.

In the subject "Theory of Equations," the term equation is used to denote an equation of the form $x^n - p_1 x^{n-1} + \ldots \pm p_n = 0$, where $p_1, p_2, \ldots, p_n$ are regarded as known, and x as a quantity to be determined; for shortness, the equation is written $f(x) = 0$.

The equation may be *numerical;* that is, the coefficients $p_1, p_2, \ldots, p_n$ are then numbers; understanding by number, a quantity of the form $\alpha + \beta i$, where α and β have any positive or negative real values whatever; or say, each of these is regarded as susceptible of continuous variation from an indefinitely large negative to an indefinitely large positive value: and i denotes $\sqrt{(-1)}$.

Or the equation may be *algebraic;* viz. the coefficients are then not restricted to denote, or are not explicitly considered as denoting, numbers.

I. We consider first numerical equations.

A number a (real or imaginary), such that substituted for x it makes the function $x^n - p_1 x^{n-1} + \ldots \pm p_n$ to be $= 0$, or say, such that it satisfies the equation, is said to be a root of the equation; viz. a being a root, we have

$$a^n - p_1 a^{n-1} + \ldots \pm p_n = 0, \text{ or say } f(a) = 0;$$

and it is then shown that $x - a$ is a factor of the function $f(x)$, viz. that we have $f(x) = (x-a) f_1(x)$, where $f_1(x)$ is a function $x^{n-1} - q_1 x^{n-2} + \ldots \pm q_{n-1}$, of the order $n-1$, with numerical coefficients $q_1, q_2, \ldots, q_{n-1}$.

In general, a is not a root of the equation $f_1(x)=0$; but it may be so, viz. $f_1(x)$ may contain the factor $x-a$; when this is so, $f(x)$ will contain the factor $(x-a)^2$; writing then $f(x)=(x-a)^2 f_2(x)$, and assuming that a is not a root of the equation $f_2(x)=0$, $x=a$ is then said to be a double root of the equation. Similarly, $f(x)$ may contain the factor $(x-a)^3$ and no higher power, and then $x=a$ is said to be a triple root; and so on.

Supposing, in general, that $f(x)=(x-a)^\alpha F(x)$, where α is a positive integer which may be $=1$, and Fx is of the order $n-\alpha$, then if b is a root different from a, we shall have $x-b$ a factor (in general a simple one, but it may be a multiple one) of $F(x)$, and $f(x)$ will in this case become $=(x-a)^\alpha(x-b)^\beta\,\Phi(x)$, where β is a positive integer which may be $=1$, and Φx is of the order $n-\alpha-\beta$. The original equation $fx=0$ is in this case said to have α roots each $=a$, β roots each $=b$, and so on.

We have the *theorem*, a numerical equation of the order n has in every case n roots, viz. there exist n numbers $a, b, \ldots$ (in general, all of them distinct, but they may arrange themselves in groups of equal values) such that

$$f(x)=(x-a)(x-b)(x-c)\ldots \text{ identically.}$$

If an equation has equal roots, these can in general be determined; the case is at any rate a special one, which may be here omitted from consideration. It is therefore, in general, assumed that the equation $f(x)=0$ under consideration has all its roots unequal. If the coefficients $p_1, p_2, \ldots$ are all or any one or more of them imaginary, then the equation $f(x)=0$, separating the real and imaginary parts, may be written $F(x)+i\Phi(x)=0$, where $F(x)$, $\Phi(x)$ are each of them a function with real coefficients; and it thus appears that the equation $f(x)=0$ with imaginary coefficients has not in general any real root; supposing it to have a real root a, this must be at once a root of each of the equations $F(x)=0$ and $\Phi(x)=0$.

But an equation with real coefficients may have as well imaginary as real roots; and we have further the *theorem* that for such an equation the imaginary roots enter in pairs, viz. $\alpha+\beta i$ being a root, then will also $\alpha-\beta i$ be a root.

Considering an equation with real coefficients, the question arises as to the number and situation of its real roots; this is completely resolved by means of *Sturm's theorem*, viz. we form a series of functions $f(x)$, $f'(x)$, $f_2(x), \ldots, f_n(x)$ (a constant) of the degrees n, $n-1, \ldots$, 2, 1, 0 respectively; and substituting therein for x any two real values a and b, we find by means of the resulting signs of these functions how many real roots of $f(x)$ lie between the limits a, b.

The same thing can frequently be effected with greater facility by other means, but the only general method is the one just referred to.

In the general case of an equation with imaginary (it may be real) coefficients, the like question arises as to the situation of the (real or imaginary) roots, viz. if for facility of conception we regard the constituents α, β of a root $\alpha+\beta i$ as the coordinates of a point *in plano*, and accordingly represent the root by such point; then drawing in the plane any closed curve or "contour," the question is how many roots lie within such contour.

This is solved *theoretically* by means of a theorem of Cauchy's, viz. writing in the original equation $x+iy$ in place of x, the function $f(x+iy)$ becomes $=P+iQ$, where P and Q are each of them a rational and integral function (with real coefficients) of (x, y). Imagining the point (x, y) to travel along the contour, and considering the number of changes of sign from $-$ to $+$ and from $+$ to $-$ of the fraction $\frac{P}{Q}$ corresponding to passages of the fraction through zero (that is, to values for which P becomes $=0$, disregarding those for which Q becomes $=0$), the difference of these numbers determines the number of roots within the contour. The investigation leads to a proof of the before-mentioned theorem, that a numerical equation of the order n has precisely n roots.

But, for the actual determination, it is necessary to consider a rectangular contour, and to apply to each of its sides separately a method such as that of Sturm's theorem; and thus the actual determination ultimately depends on a method such as that of Sturm's theorem.

Recurring to the case of an equation with real coefficients, it is important to *separate* the real roots, viz. to determine limits, such that each real root lies alone by itself between two limits l and m. This can be done (with more or less difficulty according to the nearness of the real roots) by repeated applications of Sturm's theorem, or otherwise.

The same thing would be useful, and can theoretically be effected, in regard to the roots of an equation generally, viz. we may, by lines parallel to the axes of x and y respectively, divide the plane into rectangles such that each (real or imaginary) root lies alone by itself in a given rectangle; but the ulterior theory, even as regards the imaginary roots of an equation with real coefficients, has not been developed, and the remarks which immediately follow have reference only to equations with real coefficients, and to the real roots of such equations.

Supposing the roots separated as above, so that a certain root is known to lie alone by itself between two given limits, then it is possible by various processes (Horner's, or Lagrange's method of continued fractions) to obtain to any degree of approximation the numerical value of the real root in question, and thus to obtain (approximately as above) the values of the several real roots.

The real roots can also frequently be obtained, without the necessity of a previous separation of the roots, by other processes of approximation—Newton's, as completed by Fourier, or by a method given by Encke—and the problem of their determination to any degree of approximation may be regarded as completely solved. But this is far from being practically the case even as regards the imaginary roots of such equations, or as regards the roots of an equation with imaginary coefficients.

A class of numerical equations which need to be considered, are the binomial equations $x^n-a=0$, where $a, =\alpha+\beta i$, is a complex number. The foregoing conclusions apply, viz. there are always n roots, which it may be shown are all unequal. Supposing

one of these is θ, so that $\theta^n = a$, then, assuming $x = \theta y$, we have $y^n - 1 = 0$, which equation (like the more general one $x^n - a = 0$) has precisely n roots; it is shown that these are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where ω is a complex number $\alpha + \beta i$ such that $\alpha^2 + \beta^2 = 1$, or, what is the same thing, a complex number of the form $\cos\theta + i\sin\theta$; and it then at once appears that θ may be taken $= \frac{2\pi}{n}$. We have thus the trigonometrical solution of the equation $x^n - 1 = 0$. We may also obtain a like trigonometrical solution of the first-mentioned equation $x^n - a = 0$. We are thus led to the notion (a numerical) of the radical $a^{\frac{1}{n}}$, regarded as an n-valued function, viz. any one of these being denoted by $\sqrt[n]{(a)}$, then the series of values is

$$\sqrt[n]{(a)},\ \omega\sqrt[n]{(a)}, \ldots,\ \omega^{n-1}\sqrt[n]{(a)}.$$

Or we may, if we please, use $\sqrt[n]{(a)}$, instead of $a^{\frac{1}{n}}$, as a symbol to denote the n-valued function.

It is not necessary, as regards the equation $x^n - 1 = 0$, to refer here to the distinctions between the cases n a prime, and a composite, number.

As the coefficients of an algebraical equation may be numerical, all which follows in regard to algebraical equations, is (with, it may be, some few modifications) applicable to numerical equations; and hence, concluding for the present this subject, it will be convenient to pass on to algebraical equations.

II. We consider, secondly, an algebraical equation

$$x^n - p_1 x^{n-1} + \ldots = 0,$$

and we here *assume* the existence of roots, viz. we assume that there are n quantities $a, b, c, \ldots$ (in general, all of them different, but in particular cases they may become equal in sets in any manner), such that

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b)\ldots.$$

Or, looking at the question in a different point of view, and starting with the roots $a, b, c, \ldots$ as given, we express the product of the n factors $x-a, x-b, \ldots$ in the foregoing form, and thus arrive at an equation of the order n having the n roots $a, b, c, \ldots$. In either case, we have

$$p_1 = \Sigma a,\ p_2 = \Sigma ab, \ldots,\ p_n = abc\ldots,$$

viz. regarding the coefficients $p_1, p_2, \ldots, p_n$ as given, then we assume the existence of roots $a, b, c, \ldots$ such that $p_1 = \Sigma a$, &c., or regarding the roots as given, then we write p_1, p_2, &c., to denote the functions Σa, Σab, &c.

It is to be noticed that, in virtue of

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b),\ \&c.,$$

or of the equivalent equations $p_1 = \Sigma a$, &c., then

$$\begin{aligned} a^n - p_1 a^{n-1} + \ldots &= 0, \\ b^n - p_1 b^{n-1} + \ldots &= 0, \\ \&c., \end{aligned}$$

(viz. it is for this reason that $a, b, \ldots$ are said to be roots of $x^n - p_1 x^{n-1} + \ldots = 0$); and, moreover, that conversely from the last-mentioned equations, assuming that $a, b, \ldots$ are all different, we deduce

$$p_1 = \Sigma a,\ p_2 = \Sigma ab,\ \&c.,$$

and

$$x^n - p_1 x^{n-1} + \ldots = (x-a)(x-b)\ldots.$$

Observe that, if for instance $a = b$, then the two equations $a^n - p_1 a^{n-1} + \ldots = 0$, $b^n - p_1 b^{n-1} + \ldots = 0$ would reduce themselves to a single equation, which would not of itself express that a was a double root, that is, that $(x-a)^2$ was a factor of $x^n - p_1 x^{n-1} + \&c.$; but by considering b as the limit of $a + h$, h indefinitely small, we obtain a second equation

$$n a^{n-1} - (n-1) p_1 a^{n-2} + \ldots = 0,$$

which, with the first, expresses that a is a double root; and then the whole system of equations leads, as before, to the equations $p_1 = \Sigma a$, &c. But this in passing: the general case is when the roots are all unequal.

We have then the *theorem* that every rational symmetrical function of the roots is a rational function of the coefficients; this is an easy consequence from the less general theorem, every rational and integral symmetrical function of the roots is a rational and integral function of the coefficients.

In particular, the sums of powers Σa^2, Σa^3, &c., are rational and integral functions of the coefficients.

An ordinary process, as regards the expression of other functions $\Sigma a^\alpha b^\beta$, &c., in terms of the coefficients, is to make them depend on the functions Σa^α, &c., but this is *very objectionable;* the true theory consists in showing that we have systems of equations

$$p_1 = \Sigma a,$$

$$\left\{\begin{aligned} p_2 &= \Sigma ab, \\ p_1^2 &= \Sigma a^2 + 2\Sigma ab, \end{aligned}\right.$$

$$\left\{\begin{aligned} p_3 &= \Sigma abc, \\ p_1 p_2 &= \Sigma a^2 b + 3\Sigma abc, \\ p_1^3 &= \Sigma a^3 + 3\Sigma a^2 b + 6\Sigma abc, \end{aligned}\right.$$

&c., &c.

where, in each system, there are precisely as many equations as there are root-functions on the right-hand side, e.g. 3 equations and 3 functions Σabc, $\Sigma a^2 b$, Σa^3. Hence, in each system, the root-functions can be determined linearly in terms of the powers and products of the coefficients.

It follows that it is possible to determine an equation (of an assignable order) having for roots any given (unsymmetrical) functions of the roots of a given equation. For example, in the case of a quartic equation, roots (a, b, c, d), it is possible to find an equation having the roots ab, ac, ad, bc, bd, cd, being therefore a sextic equation; viz. in the product $(y-ab)(y-ac)(y-ad)(y-bc)(y-bd)(y-cd)$, the coefficients of the several powers of y will be symmetrical functions of a, b, c, d, and therefore rational and integral functions of the coefficients of the original quartic equation.

In connexion herewith, the question arises as to the number of values (obtained by permutations of the roots) of given unsymmetrical functions of the roots; for instance, with roots (a, b, c, d) as before, how many values are there of the function $ab+cd$; or, better, how many functions are there of this form; the answer is 3, viz. $ab+cd$, $ac+bd$, $ad+bc$; or, again, we may ask whether it is possible to obtain functions of a given number of values, 3-valued, 4-valued functions, &c.

We have, moreover, the very important *theorem* that, given the value of any unsymmetrical function, e.g. $ab+cd$, it is in general possible to determine rationally the value of any similar function, e.g. $(a+b)^3+(c+d)^3$.

The *à priori* ground of this theorem may be illustrated by means of a numerical equation. Suppose, e.g. that the roots of a quartic equation are 1, 2, 3, 4; then if it is given that $ab+cd=14$, this in effect determines a, b to be 1, 2 (viz. $a=1$, $b=2$, or else $a=2$, $b=1$) and c, d to be 3, 4 (viz. $c=3$, $d=4$, or else $c=4$, $d=3$); and it therefore in effect determines $(a+b)^3+(c+d)^3$ to be $=370$, and not any other value. And we can in the same way account for cases of failure as regards particular equations; thus, the roots being 1, 2, 3, 4, as above, $a^2b=2$ determines a to be $=1$ and b to be $=2$; but if the roots had been 1, 2, 4, 16, then $a^2b=16$ does not uniquely determine a and b, but only makes them to be 1 and 16, or else 2 and 4, respectively.

As to the *à posteriori* proof, assume, for instance, $t_1=ab+cd$, $y_1=(a+b)^3+(c+d)^3$, and so $t_2=ac+db$, $y_2=(a+c)^3+(d+b)^3$, &c.—in the present case there are only the functions t_1, t_2, t_3 and y_1, y_2, y_3—then $y_1+y_2+y_3$, $t_1y_1+t_2y_2+t_3y_3$, $t_1^2y_1+t_2^2y_2+t_3^2y_3$ will be respectively symmetrical functions of the roots of the quartic, and therefore rational and integral functions of its coefficients, that is, they will be known.

Imagine, in the first instance, that t_1, t_2, t_3 are all known; then the equations being linear in y_1, y_2, y_3, these can be expressed rationally in terms of known functions of the coefficients and of t_1, t_2, t_3, that is, y_1, y_2, y_3 will be known. But observe further, that y_1 is obtained as a function of t_1, t_2, t_3 symmetrical as regards t_2, t_3; it can consequently be expressed as a rational function of t_1 and of t_2+t_3, t_2t_3, or, what is the same thing, of t_1 and $t_1+t_2+t_3$, $t_1t_2+t_1t_3+t_2t_3$, $t_1t_2t_3$; but these last will be symmetrical functions of the roots, and as such expressible rationally in terms of the coefficients: that is, y_1 will be expressed as a rational function of t_1 and of the coefficients, or, t_1 being known, y_1 will be rationally determined.

We may consider now the question of the algebraical solution of equations, or, more accurately, that of the *solution of equations by radicals.*

In the case of a quadric equation $x^2+px+q=0$, we can find for x, by the assistance of the sign $\sqrt{(\ )}$ or $(\ )^{\frac{1}{2}}$, an expression for x as a two-valued function of the coefficients p, q, such that, substituting this value in the equation, the equation is thereby identically satisfied, viz. we have

$$x=-\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-q)},$$

giving

$$\begin{array}{rl} x^2= & \tfrac{1}{2}p^2-q\mp p\sqrt{(\tfrac{1}{4}p^2-q)} \\ +px= & -\tfrac{1}{2}p^2 \qquad \pm p\sqrt{(\tfrac{1}{4}p^2-q)} \\ +q\ = & \qquad\quad +q \\ \hline x^2+px+q\ = & 0, \end{array}$$

and the equation is on this account said to be algebraically solvable, or, more accurately, to be *solvable by radicals*. Or we may, by writing $x=-\frac{1}{2}p+z$, reduce the equation to $z^2=\frac{1}{4}p^2-q$, viz. to an equation of the form $z^2=a$, and, in virtue of its being thus reducible, we may say that the equation is solvable by radicals. And the question for an equation of any higher order is, say of the order n, can we by means of radicals, that is, by aid of the sign $\sqrt[m]{(\ )}$ or $(\ )^{\frac{1}{m}}$, using as many as we please of such signs and with any values of m, find an n-valued function (or any function) of the coefficients, which substituted for x in the equation shall satisfy it identically.

It will be observed that the coefficients $p, q, \ldots$ are not explicitly considered as numbers, but that even if they do denote numbers, the question whether a numerical equation admits of solution by radicals is wholly unconnected with the before-mentioned theorem of the existence of the n roots of such an equation. It does not even follow that, in the case of a numerical equation solvable by radicals, the algebraical expression of x gives the numerical solution; but this requires explanation. Consider, first, a numerical quadric equation with imaginary coefficients; in the formula $x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2-q)}$, substituting for p, q their given numerical values we obtain for x an expression of the form $x=\alpha+\beta i\pm\sqrt{(\gamma+\delta i)}$, where α, β, γ, δ are real numbers; this value substituted in the numerical equation would satisfy it identically and it is thus an algebraical solution; but there is no obvious *à priori* reason why the expression $\sqrt{(\gamma+\delta i)}$ should have a value $=c+di$, where c and d are real numbers calculable by the extraction of a root or roots of real numbers; it appears upon investigation that $\sqrt{(\gamma+\delta i)}$ has such a value calculable by means of the radical expression $\sqrt{[\sqrt{(\gamma^2+\delta^2)}\pm\gamma]}$; and hence that the algebraical solution of a quadric equation does in every case give the numerical solution of a numerical quadric. The case of a numerical cubic will be considered presently.

A cubic equation can be solved by radicals, viz. taking for greater simplicity the cubic in the reduced form $x^3-qx-r=0$, and writing $x=a+b$, this will be a solution if only $3ab=q$, and $a^3+b^3=r$, or say $\frac{1}{2}(a^3+b^3)=\frac{1}{2}r$; whence

$$\tfrac{1}{2}(a^3-b^3)=\pm\sqrt{(\tfrac{1}{4}r^2-\tfrac{1}{27}q^3)},$$

and therefore

$$a = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}},$$

a six-valued function of q, r. But then writing $b = \dfrac{q}{3a}$, we have, as may be shown, $a+b$ a three-valued function of the coefficients; it would have been wrong to complete the solution by writing $b = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}}$, since here $(a+b)$ would be given as a 9-valued function, having only 3 of its values roots, and the other 6 values being irrelevant. An interesting variation of the solution is to write $x = ab\,(a+b)$, giving $a^3b^3(a^3+b^3) = r$ and $3a^3b^3 = q$, or say $\tfrac{1}{2}(a^3+b^3) = \tfrac{3}{2}\dfrac{r}{q}$, $a^3b^3 = \tfrac{1}{3}q$; whence

$$\{\tfrac{1}{2}(a^3 - b^3)\}^2 = \frac{9}{q^2}(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3),$$

and therefore

$$a = \sqrt[3]{\left\{\tfrac{3}{2}\frac{r}{q} \pm \frac{3}{q}\sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\right\}}, \quad b = \sqrt[3]{\left\{\tfrac{3}{2}\frac{r}{q} \mp \frac{3}{q}\sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\right\}},$$

and here although a, b are each of them a 6-valued function, yet, as may be shown, $ab\,(a+b)$ is only a 3-valued function.

In the case of a numerical cubic, even when the coefficients are real, substituting their values in the expression

$$x = \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}} + [\tfrac{1}{3}q \div \sqrt[3]{\{\tfrac{1}{2}r \pm \sqrt{(\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3)}\}}],$$

this *may* depend on an expression of the form $\sqrt[3]{(\gamma + \delta i)}$, where γ and δ are real numbers (viz. it will do so if $\tfrac{1}{4}r^2 - \tfrac{1}{27}q^3$ is a negative number), and here we *cannot* by the extraction of any root or roots of real numbers reduce $\sqrt[3]{(\gamma + \delta i)}$ to the form $c + di$, c and d real numbers; hence, here the algebraical solution does not give the numerical solution. It is to be added that the case in question, called the "irreducible case," is that wherein the three roots of the cubic equation are all real; if the roots are one real and two imaginary, then, contrariwise, the quantity under the cube root is real, and the algebraical solution gives the numerical one.

The irreducible case is solvable by a trigonometrical formula, but this is not a solution by radicals; it consists, in effect, in reducing the given numerical cubic (not to a cubic of the form $z^3 = a$, solvable by the extraction of a cube root, but) to a cubic of the form $4x^3 - 3x = a$, corresponding to the equation $4\cos^3\theta - 3\cos\theta = \cos 3\theta$ which serves to determine $\cos\theta$ when $\cos 3\theta$ is known.

A quartic equation is solvable by radicals; and it may be remarked, that the existence of such a solution depends on the existence of 3-valued functions such as $ab + cd$, of the four roots (a, b, c, d); by what precedes, $ab + cd$ is the root of a cubic equation, which equation is solvable by radicals; hence $ab + cd$ can be found by radicals; and since $abcd$ is a given value, ab and cd can each be found by radicals. But by what precedes, if ab be known, then any similar function, say $a + b$, is obtainable rationally; and, consequently, from the values of $a + b$ and ab we may by radicals

obtain the value of a or b, that is, an expression for a root of the given quartic expression; the expression finally obtained is 4-valued, corresponding to the different values of the several radicals which enter therein, and we have therefore the expression by radicals of each of the four roots of the quartic equation. But when the quartic is numerical, the same thing arises as in the cubic: the algebraical expression does not in every case give the numerical one.

It will be understood from the foregoing explanation as to the quartic, how in the next following case, that of a quintic equation, the question of the solvability by radicals depends on the existence or non-existence of k-valued functions of the five roots (a, b, c, d, e); a fundamental theorem on the subject is that a rational function of 5 letters, if it has less than 5, cannot have more than 2 values; viz. that there are no 3-valued, or 4-valued, functions of 5 letters; and by reasoning, depending in part upon this theorem, Abel showed that a general quintic equation is not solvable by radicals: and *à fortiori* the general equation of any order higher than 5 is not solvable by radicals.

The general theory of the solvability of an equation by radicals depends very much on Vandermonde's remark, that supposing an equation is solvable (by radicals) and that we have therefore an algebraical expression of x in terms of the coefficients, then substituting for the coefficients their values in terms of the roots, the resulting value of the expression must reduce itself to any one at pleasure of the roots $a, b, c, \ldots$; thus in the case of the quadric equation where the solution is $x = +\frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 - q)}$, writing for p, q their values $a+b$, ab, this is $x = \frac{1}{2}[(a+b) \pm \sqrt{[(a-b)^2]}]$, $= a$ or b according to the value of the radical. But it is not considered necessary in the present sketch to go further into the theory of the solvability of an equation by radicals. It may be proper to remark that, for quintic equations, there are solutions analogous to the trigonometrical solution of a cubic equation, viz. the quintic equation is here in effect reduced to some special form of quintic equation; for instance, to Jerrard's form $x^5 + ax + b = 0$ or to some form presenting itself in the theory of elliptic functions; but the solutions in question are not solutions by radicals. And there are various other interesting parts of the theory which have been excluded from consideration.

632.

ON ARONHOLD'S INTEGRATION-FORMULA.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 88—90.]

THE fundamental theorem in Aronhold's Memoir, "Ueber eine neue algebraische Behandlungsweise der Integrale...$\Pi(x, y)\,dx$, &c.," *Crelle*, t. LXI. (1863), pp. 95—145, is a theorem of *indefinite* integration. The form is

$$\Lambda\int\frac{dx}{(\alpha x+\beta y+\gamma)(hx+by+f)}=\log\frac{(a\xi+h\eta+g)\,x+(h\xi+b\eta+f)\,y+g\xi+f\eta+c}{\alpha x+\beta y+\gamma},$$

where y is a certain irrational function of x, determined by a quadric equation, and the other symbols denote constants connected by certain relations; viz. writing, for shortness,

$$U=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2,\quad =(a,\ldots\between x,\ y,\ 1)^2 \text{ for shortness,}$$

that is,

$$=ax^2+2hxy+by^2+2fy+2gx+c\,;$$

$$W=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1\between\xi,\ \eta,\ 1),\quad =(a,\ldots\between x,\ y,\ 1\between\xi,\ \eta,\ 1),$$

that is,

$$=(ax+hy+g)\,\xi+(hx+by+f)\,\eta+gx+fy+c,$$

or

$$(a\xi+h\eta+g)\,x+(h\xi+b\eta+f)\,y+g\xi+f\eta+c\,;$$

$$(P\ ,\ Q\ ,\ R\ )=(ax+hy+g,\quad hx+by+f,\quad gx+fy+c),$$

$$(P_0,\ Q_0,\ R_0)=(a\xi+h\eta+g,\quad h\xi+b\eta+f,\quad g\xi+f\eta+c),$$

$$\Omega\ =\alpha x+\beta y+\gamma,$$

$$\Omega_0=\alpha\xi+\beta\eta+\gamma,$$

$$(A,\ B,\ C,\ F,\ G,\ H)=(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch),$$

then y is determined as a function of x by the equation $U=0$, that is,

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ 1)^2 = 0;$$

or, what is the same thing,

$$by = -\{hx + f + \sqrt{(-Cx^2 + 2Gx - A)}\};$$

the constants α, β, ξ, η are such that

$$(a,\ b,\ c,\ f,\ g,\ h \between \xi,\ \eta,\ 1)^2 = 0,$$
$$\alpha\xi + \beta\eta + \gamma = 0,$$

that is,

$$\Omega_0 = 0;$$

and the value of Λ is given by

$$\Lambda^2 = -(A,\ B,\ C,\ F,\ G,\ H \between \alpha,\ \beta,\ \gamma)^2.$$

The theorem may therefore be written

$$\Lambda \int \frac{dx}{\Omega Q} = \log \frac{W}{\Omega},$$

where the several symbols have the significations explained above.

The verification is as follows. We ought to have

$$\frac{\Lambda\, dx}{\Omega Q} = \frac{P_0 dx + Q_0 dy}{W} - \frac{\alpha\, dx + \beta\, dy}{\Omega},$$

when dx, dy satisfy the relation $P\,dx + Q\,dy = 0$, viz. substituting for dy the value $-\dfrac{P\,dx}{Q}$, the equation becomes

$$\frac{\Lambda}{\Omega} = \frac{P_0 Q - P Q_0}{W} - \frac{\alpha Q - \beta P}{\Omega},$$

that is, substituting for Ω its value,

$$\Lambda W = (P_0 Q - P Q_0)(\alpha x + \beta y + \gamma) - (\alpha Q - \beta P)\, W.$$

On the right-hand side, substituting for W its value,

$$\text{coeff. } \alpha = x(P_0 Q - P Q_0) - Q(P_0 x + Q_0 y + R_0), \quad = Q_0 R - Q R_0,$$
$$\text{coeff. } \beta = y(P_0 Q - P Q_0) + P(P_0 x + Q_0 y + R_0), \quad = R_0 P - R P_0,$$

(as at once appears by aid of the relation $U = Px + Qy + R = 0$),

$$\text{coeff. } \gamma \qquad = P_0 Q - P Q_0.$$

The equation to be verified thus is

$$\Lambda W = \begin{vmatrix} \alpha, & \beta, & \gamma \\ P_0, & Q_0, & R_0 \\ P, & Q, & R \end{vmatrix},$$

which, substituting therein for P, Q, R, P_0, Q_0, R_0, their values, and writing

$$(\lambda,\ \mu,\ \nu) = (\eta - y,\ x - \xi,\ \xi y - \eta x),$$

is in fact

$$\Lambda W = (A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma).$$

We have identically

$$(a, \ldots \between x,\ y,\ 1)^2 . (a, \ldots \between \xi,\ \eta,\ 1)^2 - W^2 = (A, \ldots \between \lambda,\ \mu,\ \nu)^2,$$

which, in virtue of $(a, \ldots \between \xi,\ \eta,\ 1)^2 = 0$, gives

$$W^2 = -(A, \ldots \between \lambda,\ \mu,\ \nu)^2;$$

and since $\Lambda^2 = -(A, \ldots \between \alpha,\ \beta,\ \gamma)^2$, the equation is thus

$$\sqrt{\{-(A, \ldots \between \alpha,\ \beta,\ \gamma)^2\}} . \sqrt{\{-(A, \ldots \between \lambda,\ \mu,\ \nu)^2\}} = (A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma),$$

that is,

$$(A, \ldots \between \alpha,\ \beta,\ \gamma)^2 . (A, \ldots \between \lambda,\ \mu,\ \nu)^2 - [(A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma)]^2 = 0.$$

The left-hand side is here identically

$$= K\,(a, \ldots \between \gamma\mu - \beta\nu,\ \alpha\nu - \gamma\lambda,\ \beta\lambda - \alpha\mu)^2:$$

substituting for λ, μ, ν their values, we find

$$(\gamma\mu - \beta\nu,\ \alpha\nu - \gamma\lambda,\ \beta\lambda - \alpha\mu) = (x\Omega_0 - \xi\Omega,\ y\Omega_0 - \eta\Omega,\ z\Omega_0 - \zeta\Omega);$$

viz. in virtue of $\Omega_0 = 0$, these are $= -\xi\Omega,\ -\eta\Omega,\ -\zeta\Omega$, and the quadric function is $= K\Omega^2\,(a, \ldots \between \xi,\ \eta,\ 1)^2$, vanishing in virtue of the relation $(a, \ldots \between \xi,\ \eta,\ 1)^2 = 0$.

The equation in question

$$\sqrt{\{-(A \ldots \between \alpha,\ \beta,\ \gamma)^2\}} . \sqrt{\{-(A \ldots \between \lambda,\ \mu,\ \nu)^2\}} = (A \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma)$$

is thus verified, and the theorem is proved.

633.

NOTE ON MR MARTIN'S PAPER, "ON THE INTEGRALS OF SOME DIFFERENTIALS."

[From the *Messenger of Mathematics*, vol. v. (1876), p. 163.]

The Note refers to a detail in a process of integration.

634.

THEOREMS IN TRIGONOMETRY AND ON PARTITIONS.

[From the *Messenger of Mathematics*, vol. v. (1876), p. 164, and p. 188.]

IF

$$A+B+C+F+G+H=0,$$

then

$$\begin{vmatrix} \sin\overline{A+F}\sin\overline{B+F}\sin\overline{C+F}, & \cos F, & \sin F \\ \sin\overline{A+G}\sin\overline{B+G}\sin\overline{C+G}, & \cos G, & \sin G \\ \sin\overline{A+H}\sin\overline{B+H}\sin\overline{C+H}, & \cos H, & \sin H \end{vmatrix} = 0.$$

Let u_n = number of partitions of n, no part less than 2, the order attended to; e.g. if $n=7$, the partitions are 7, 52, 25, 43, 34, 322, 232, 223, $u_7=8$; the series is

$$\begin{aligned} u_2 &= 1, \\ u_3 &= 1, \\ u_4 &= 2, \\ u_5 &= 3, \\ u_6 &= 5, \\ u_7 &= 8, \\ u_8 &= 13, \\ u_9 &= 21, \end{aligned}$$

where each term is the sum of the next preceding two terms.

635.

NOTE ON THE DEMONSTRATION OF CLAIRAUT'S THEOREM.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 166, 167.]

It seems worth while to indicate what the leading steps of the demonstration are.

The potential of the Earth's mass upon an external or superficial point is taken to be

$$V, \quad = \frac{V_0}{r} + \frac{V_1}{r^2} + \frac{V_2}{r^3} + \&c.,$$

where V_1, V_2, $V_3, \ldots$ are Laplace's functions of the angular coordinates.

The surface is assumed to be a nearly spherical surface $r = a(1+u)$, where $u = u_1 + u_2 + \&c.$, and u_1, $u_2, \ldots$ are Laplace's functions of the angular coordinates. To be a surface of equilibrium, with an equation $V + \frac{1}{2}\omega^2 r^2 \sin^2\theta = C$, the latter must be equivalent to the equation $r = a(1+u)$, and *it follows* that we have

$$\begin{aligned} V_1 &= V_0 a u_1, \\ V_2 &= V_0 a^2 u_2 - \tfrac{1}{2}\omega^2 a^5 (\tfrac{1}{3} - \cos^2\theta), \\ V_3 &= V_0 a^3 u_3, \\ &\&c., \end{aligned}$$

which values are to be substituted in the expression for V.

The whole force of gravity (due to the attraction and the centrifugal force) is taken to be $g, = -\frac{d}{dr}(V + \frac{1}{2}\omega^2 r^2 \sin^2\theta)$, and *it follows* that

$$g = \frac{V_0}{a^2}(1 + u_2 + 2u_3 + \ldots) - \tfrac{2}{3}\omega^2 a - \tfrac{5}{2}\omega^2 a (\tfrac{1}{3} - \cos^2\theta),$$

which is of the form

$$g = G\left\{1 + u_2 - \tfrac{5}{2}\frac{\omega^2 a}{G}(\tfrac{1}{3} - \cos^2\theta) + 2u_3 + \ldots\right\}.$$

Taking the Earth to be the spheroid of revolution

$$r = a\{1 + \epsilon(\tfrac{1}{3} - \cos^2\theta)\},$$

then

$$u_2 = \epsilon(\tfrac{1}{3} - \cos^2\theta), \quad u_3 = 0, \text{ \&c.},$$

and the equation is

$$g = G\left\{1 - \left(\tfrac{5}{2}\frac{\omega^2 a}{G} - \epsilon\right)(\tfrac{1}{3} - \cos^2\theta)\right\},$$

or say

$$g = G\{1 - (\tfrac{5}{2}m - \epsilon)(\tfrac{1}{3} - \cos^2\theta)\},$$

where $m, = \dfrac{\omega^2 a}{G}$, is the ratio of the centrifugal force at the equator to the force of gravity, which is the theorem in question. The expression "it follows" has been twice used as meaning it follows as a mere analytical consequence, in the proper degree of approximation, the steps of the deduction being purposely omitted.

636.

ON THE THEORY OF THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 23—27.]

IN continuation of the former paper with this title (*Messenger*, vol. II., 1873, pp. 6—12, [545]), I propose to discuss various particular examples, chiefly of cases in which the differential equation is of the form $(L, M, N \between p, 1)^2 = 0$, where L, M, N are rational and integral functions of (x, y), and whether it admits or does not admit of an integral equation $(P, Q, R \between c, 1)^2 = 0$, where P, Q, R are rational and integral functions of (x, y).

The singular solution of the differential equation

$$(L, M, N \between p, 1)^2 = 0,$$

if there be a singular solution, is $S = 0$, where S is either $= LN - M^2$, or a factor of $LN - M^2$. But in general $LN - M^2$ is an indecomposable function, such that $LN - M^2 = 0$ is not a solution of the differential equation, and this being so, there is no singular solution; viz. a differential equation $(L, M, N \between p, 1)^2 = 0$, where L, M, N are rational and integral functions of (x, y), has not in general any singular solution.

Consider now a system of algebraical curves $U = 0$, where U is as regards (x, y) a rational and integral function of the order m, and depends in any manner on an arbitrary parameter C*. *I say that there is always a proper envelope,* which envelope is the singular solution of the differential equation obtained by the elimination of C from the equation $U = 0$, and the derived equation in regard to (x, y). *It follows that the differential equation* $(L, M, N \between p, 1)^2 = 0$, *which has no singular solution, does not admit of an integral of the form in question* $U = 0$, *viz. an integral representing a system of algebraic curves.*

* The expressions in the text may be understood as extending to the case where U is a function of any number (α) of constants $c_1, c_2, \ldots, c_\alpha$, connected by an $(\alpha - 1)$fold relation, U thus virtually depending on a single arbitrary parameter.

The theorem just referred to, that the system of algebraic curves $U=0$ has always an envelope, is an interesting theorem, which I proceed to prove. Assume that in general, that is, for an arbitrary value of the parameter, the equation $U=0$ represents a curve of the order m, with δ nodes and κ cusps (and therefore of the class n, with i inflexions and τ double tangents, the numbers $m, \delta, \kappa, n, \tau, i$ being connected by Plücker's equations); for particular values of the parameter, the values of δ and κ may be increased, or the curve may break up, but this is immaterial.

The consecutive curve $U+\delta c d_c U=0$ is a curve of the same order m, with δ nodes and κ cusps, consecutive to the nodes and cusps of the original curve U, and the two curves intersect in m^2 points; but of these, there are 2 coinciding with each node, and 3 coinciding with each cusp of the curve $U=0$, as at once appears by drawing a curve with a node or a cusp, and the consecutive curve with a consecutive node or cusp; the number of the remaining intersections is $=m^2-2\delta-3\kappa$, and the envelope is the locus of these $m^2-2\delta-3\kappa$ points. Observe that the two curves have in common n^2 tangents; but of these, 2 coincide with each double tangent and 3 coincide with each stationary tangent of the curve $U=0$, viz. the number of the remaining common tangents is $=n^2-2\tau-3i$ (which is $=m^2-2\delta-3\kappa$): and that these $n^2-2\tau-3i$ common tangents are indefinitely near to the $m^2-2\delta-3\kappa$ common points respectively, and are in fact the tangents of the envelope at the $m^2-2\delta-3\kappa$ points respectively. Now in an algebraic curve we have $m+n=m^2-2\delta-3\kappa$, viz. the number $m^2-2\delta-3\kappa$ cannot be $=0$, and we have therefore always an envelope the locus of the system of the $m^2-2\delta-3\kappa$ points. It might be thought that the conclusion extends to transcendental curves; if this were so, the result would prove too much, viz. it would follow that a differential equation $(L, M, N \between p, 1)^2=0$ without a singular solution had no general integral; but it will appear by an example that the theorem as to the envelope does not extend to transcendental curves.

Ex. 1.

$$p^2-(1-y^2)=0, \text{ that is, } dy^2-(1-y^2)\,dx^2=0.$$

Here there is no algebraical integral, but there is a quasi-algebraical integral of the form $(P, Q, R \between c, 1)^2=0$; viz. starting with the form $y=\sin(x+C)$ and expressing $\sin C$ and $\cos C$ rationally in terms of a new parameter, this is

$$c^2(y+\cos x)-2c\sin x+(y-\cos x)=0,$$

where the coefficients are one-valued functions of (x, y). The discriminant of the differential equation in regard to p and that of the integral equation in regard to c are each $=y^2-1$, and we have a true singular solution $y^2-1=0$.

Ex. 2.

$$(1-x^2)p^2-(1-y^2)=0,$$

that is,

$$(1-x^2)\,dy^2-(1-y^2)\,dx^2=0.$$

We have here an algebraic integral of the proper form, which is at once derived from the circular form

$$C=\cos^{-1}x+\cos^{-1}y$$

by changing the constant, viz. this is

$$c^2 - 2cxy - (1 - x^2 - y^2) = 0.$$

The two discriminants are here each $=(x^2-1)(y^2-1)$, and we have

$$(x^2-1)(y^2-1)=0$$

as a true singular solution. The curves are in fact the system of conics (ellipses and hyperbolas) each touching the four lines $x=1$, $x=-1$, $y=1$, $y=-1$.

Ex. 3.

$$(1-y^2)p^2-1=0, \text{ that is, } (1-y^2)\,dy^2-dx^2=0.$$

This is an extremely interesting example: the curve is the orthogonal trajectory of the system of sinusoids $y=\sin(x+c)$, which is the integral of Example 1; and we thus at once see that the real portion of the curve is wholly included between the lines $y=-1$, $y=+1$, being an infinite continuous curve, having a series of equidistant cusps alternately at the one and the other line, and obtained by the continued repetition of the finite portion included between two consecutive cusps on the same line. The discriminant of the differential equation equated to zero gives $y^2-1=0$, the equation of the two lines in question; but this does not satisfy the differential equation, and it is consequently not a singular solution; by what precedes, it appears that it is, in fact, a cusp-locus.

We thus see that the curves which represent the integral equation have no *real* envelope; but it is to be further shown that there is no imaginary envelope, and that the curve obtained by the elimination of the parameter is, in fact, made up of a (imaginary) node-locus and of the foregoing cusp-locus.

The curve is properly represented by taking x, y each of them a one-valued function of the parameter θ, viz. we may write

$$y=\cos\theta,$$
$$x=c+\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta.$$

In fact, these values give

$$\frac{dy}{d\theta}=-\sin\theta,\ \frac{dx}{d\theta}=\tfrac{1}{2}(1-\cos 2\theta)=\sin^2\theta,$$

and therefore

$$p=-\frac{1}{\sin\theta}=\frac{-1}{\sqrt{(1-y^2)}},$$

that is, $(1-y^2)p^2-1=0$, the differential equation.

It is obvious that to a given value of the parameter there corresponds a single point of the curve; and it is to be shown that, conversely, to a given point of the curve corresponds in general a single value of the parameter.

Suppose the coordinates of the given point are $y = \cos\alpha$, $x = c + \frac{1}{2}\alpha - \frac{1}{2}\sin 2\alpha$, where α is a determinate quantity; then, to find θ, we have

$$\cos\theta = \cos\alpha, \quad 2\theta - \sin 2\theta = 2\alpha - \sin 2\alpha.$$

The first equation gives $\theta = 2m\pi \pm \alpha$, and the second equation then is

$$4m\pi \pm 2\alpha \mp \sin 2\alpha = 2\alpha - \sin 2\alpha;$$

viz. taking the upper signs, this is $4m\pi = 0$, giving $m = 0$ and $\theta = \alpha$; and, taking the lower signs, it is $m\pi = \alpha - \sin\alpha$, which, α being given, is not in general satisfied; hence to the given point there corresponds *only* the value α of the parameter θ. If, however, α is such that $\alpha - \sin\alpha$ is equal to a multiple of π, say $r\pi$, then the last-mentioned equation is satisfied by the value $m = r$, so that to the given point of the curve correspond the two values α and $2r\pi - \alpha$ of the parameter; these values are in general unequal, and the point is then a node; but they may be equal, viz. this is so if $\alpha = r\pi$ (the point on the curve being then $y = \cos r\pi$, $= \pm 1$, $x = c + \frac{1}{2}r\pi$), and the point is then a cusp; showing what was known, that there are on each of the lines $y = -1$, $y = +1$, an infinite series of equidistant cusps.

More definitely, suppose $\alpha = r\pi \pm \beta$, where β is a root of the equation $2\beta - \sin 2\beta = 0$, then

$$\sin 2\alpha = \pm \sin 2\beta, \quad 2\alpha - \sin 2\alpha = 2r\pi \pm (2\beta - \sin 2\beta) = 2r\pi,$$

and to the given point on the curve correspond the two values α and $2r\pi - \alpha$ of the parameter. If $\beta = 0$, we have, as above, the cusps on the two lines $y = +1$, $y = -1$ respectively; but if β be an imaginary root of the equation $2\beta - \sin 2\beta = 0$, then we have an infinite series of nodes on the imaginary line $y = \cos r\pi \cos\beta$; and there are an infinite number of such lines corresponding to the different imaginary roots of the equation $2\beta - \sin 2\beta = 0$.

From the form in which the equation of the curve is given, we cannot directly form the equation of the envelope by equating to zero the discriminant in regard to the constant c; but we may determine the intersections of the curve by the consecutive curve (corresponding to a value $c + \delta c$ of the constant), and thus determine the locus of these intersections.

Consider for a moment the curves belonging to the constants c, c_1, and let θ, θ_1 be the values of the parameter θ belonging to the points of intersection; we have $\cos\theta = \cos\theta_1$, $4c + 2\theta - \sin 2\theta = 4c_1 + 2\theta_1 - \sin 2\theta_1$; we have $\theta_1 = 2r\pi + \theta$, but we cannot thereby satisfy the second equation; or else $\theta_1 = 2r\pi - \theta$, giving

$$4c + 2\theta - \sin 2\theta = 4c_1 + 4r\pi - 2\theta + \sin 2\theta,$$

that is, $2\theta - \sin 2\theta = 2c_1 - 2c + 2r\pi$; and we have thus corresponding to any given value of r a series of values of θ, viz. these are $\theta = r\pi + \beta$, where β is any root of the equation

$$2\beta - \sin 2\beta = 2c_1 - 2c.$$

In particular, taking $c_1 = c$, the intersections are given by $\theta = r\pi + \beta$, where β is any root of the equation $2\beta - \sin 2\beta = 0$; viz. we have thus an infinite number of intersections lying on each of the lines $y = \cos r\pi \cos \beta$. If $\beta = 0$, the intersections lie on the two lines $y = 1$, $y = -1$ respectively; if β be an imaginary root of the equation $2\beta - \sin 2\beta = 0$, then they lie on the imaginary lines $y = \cos r\pi \cos \beta$. But by what precedes, it is clear that in the former case the intersections are nothing else than the cusps on the lines $y = 1$, $y = -1$; and in the latter case nothing else than the nodes on the lines $y = \cos r\pi \cos \beta$; viz. there is no proper envelope, but instead thereof we have lines of cusps and of nodes.

Ex. 4.

$$(1 - y^2) p^2 - (1 - x^2) = 0,$$

that is,

$$(1 - y^2) dy^2 - (1 - x^2) dx^2 = 0.$$

I have not examined this; the curve is the series of orthogonal trajectories of the conics of Example 2, and the integral equation may be represented by $y = \cos \theta$, $x = \cos \phi$, where $c = (2\theta - \sin 2\theta) - (2\phi - \sin 2\phi)$.

Equating to zero the discriminant of the differential equation, we have $(1-y^2)(1-x^2) = 0$, viz. the four lines $x = 1$, $x = -1$, $y = 1$, $y = -1$; this is not an envelope, but a locus of cusps.

637.

ON A DIFFERENTIAL EQUATION IN THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. VI. (1877), p. 29.]

IN the differential equation

$$Q^2 - Q\left(k + \frac{1}{k}\right) - 3 = 3\,(1 - k^2)\,\frac{dQ}{dk},$$

considered *Messenger*, t. IV., pp. 69 and 110, [594] and [597], writing $Q = x$ and $k + \frac{1}{k} = y$, the equation becomes

$$dy = \frac{3\,(y^2 - 4)\,dx}{3 + xy - x^2},$$

and we have, as a particular solution,

$$y = \tfrac{1}{4}\left(x^3 - 6x - \frac{3}{x}\right).$$

To verify this, observe that from the value of y

$$dy = \frac{3}{4x^2}\,(x^2 - 1)^2\,dx, \quad 3 + xy - x^2 = \tfrac{1}{4}\,(x^2 - 1)\,(x^2 - 9),$$

and the equation becomes

$$\frac{3}{4x^2}\,(x^2 - 1)^2 = \frac{\dfrac{3}{16x^2}\,\{(x^4 - 6x^2 - 3)^2 - 64x^2\}}{\tfrac{1}{4}\,(x^2 - 1)\,(x^2 - 9)},$$

viz. this is

$$(x^2 - 1)^3\,(x^2 - 9) = (x^4 - 6x^2 - 3)^2 - 64x^2,$$

which is right.

638.

ON A q-FORMULA LEADING TO AN EXPRESSION FOR E_1.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 63—66.]

IT is to be shown that we have identically

$$(1+2q+2q^4+2q^9+\ldots)^4-16\left(\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\frac{3q^3}{1-q^6}+\ldots\right)$$

$$=\frac{1-9q-25q^3+49q^6+81q^{10}-\ldots}{1-q-q^3+q^6+q^{10}-\ldots}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (A);$$

or, what is the same thing,

$$(1-2q+2q^4-2q^9+\ldots)^4-16\left(\frac{-q}{1-q^2}+\frac{2q^2}{1-q^4}-\frac{3q^3}{1-q^6}+\ldots\right)$$

$$=\frac{1+9q+25q^3+49q^6+81q^{10}+\ldots}{1+q+q^3+q^6+q^{10}+\ldots}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (B),$$

where the form (A) is that intended to be made use of, but the form (B) is rather more convenient for the demonstration.

We have

$$(1-2q+2q^4-2q^9+\ldots)^4=1+8\left\{\frac{-q}{1+q}+\frac{2q^2}{1+q^2}-\frac{3q^3}{1+q^3}+\ldots\right\},$$

(Jacobi, *Fund. Nova*, p. 188, *Ges. Werke*, t. I., p. 239), taking the formula as there written down, and changing q into $-q$.

Also, if for a moment

$$X=1+q+q^3+q^6+q^{10}+\&c.,$$

and

$$X'=\frac{dX}{dq},$$

so that

$$qX' = q + 3q^3 + 6q^6 + 10q^{10} + \&c.,$$

then

$$X + 8qX' = 1 + 9q + 25q^3 + 49q^6 + 81q^{10} + \&c.,$$

so that the right-hand side of (B) is

$$\frac{X + 8qX'}{X}, \quad = 1 + 8q\frac{X'}{X}.$$

But (*Fund. Nova*, p. 185, *Ges. Werke*, t. I., p. 237),

$$X = \frac{1 - q^2 \,.\, 1 - q^4 \,.\, 1 - q^6 \ldots}{1 - q \,.\, 1 - q^3 \,.\, 1 - q^5 \ldots},$$

so that

$$\frac{X'}{X} = \frac{-2q}{1-q^2} - \frac{4q^3}{1-q^4} - \frac{6q^5}{1-q^6} - \ldots$$
$$+ \frac{1}{1-q} + \frac{3q^2}{1-q^3} + \frac{5q^4}{1-q^5} + \ldots.$$

And the equation (B) intended to be proved thus becomes

$$1 + 8\left\{\frac{-q}{1+q} + \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \ldots\right\}$$
$$- 16\left\{\frac{-q}{1-q^2} + \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \ldots\right\}$$
$$= 1 + 8q\left\{\frac{-2q}{1-q^2} - \frac{4q^3}{1-q^4} - \frac{6q^5}{1-q^6} - \ldots\right.$$
$$\left. + \frac{1}{1-q} + \frac{3q^2}{1-q^3} + \frac{5q^4}{1-q^5} + \ldots\right\};$$

viz. omitting the terms unity, dividing by $8q$, and transposing, this is

$$-\frac{1}{1+q} + \frac{2q}{1+q^2} - \frac{3q^2}{1+q^3} + \ldots$$
$$+ \frac{2}{1-q^2} - \frac{4q}{1-q^4} + \frac{6q^2}{1-q^6} - \ldots$$
$$+ \frac{2q}{1-q^2} + \frac{4q^3}{1-q^4} + \frac{6q^5}{1-q^6} + \ldots$$
$$- \frac{1}{1-q} - \frac{3q^2}{1-q^3} - \frac{5q^4}{1-q^5} - \ldots = 0.$$

The second and third lines unite together, and the equation becomes

$$-\frac{1}{1+q} + \frac{2q}{1+q^2} - \frac{3q^2}{1+q^3} + \frac{4q^3}{1+q^4} - \ldots$$
$$+ \frac{2}{1-q} - \frac{4q}{1+q^2} + \frac{6q^2}{1-q^3} - \frac{8q^3}{1+q^4} + \ldots$$
$$- \frac{1}{1-q} - \frac{3q^2}{1-q^3} - \frac{5q^4}{1-q^5} - \frac{7q^6}{1-q^7} - \ldots = 0;$$

or, collecting and arranging,

$$\begin{aligned} &-\frac{1}{1+q}-\frac{2q}{1+q^2}-\frac{3q^3}{1+q^3}-\frac{4q^3}{1+q^4}-\frac{5q^4}{1+q^5}-\dots \\ &+\frac{1}{1-q} \qquad\quad +\frac{3q^2}{1-q^3} \qquad\quad +\frac{5q^4}{1-q^5}+\dots=0, \end{aligned}$$

an identity which it is easy to verify to any number of terms. But to prove it directly, we have only to add the pairs of terms in the alternate columns; calling the left-hand side Fq, we thus obtain

$$\begin{aligned} Fq=2q\Big\{&-\frac{1}{1+q^2}-\frac{2q^2}{1+q^4}-\frac{3q^4}{1+q^6}-\dots \\ &+\frac{1}{1-q^2} \qquad\quad +\frac{3q^4}{1-q^6}+\dots\Big\}; \end{aligned}$$

viz. this equation is $Fq=2qF(q^2)$; and thence

$$Fq=2^2q^{1+2}F(q^4)=2^3q^{1+2+4}F(q^8)=\&c.;$$

we thus have $Fq=0$.

The equation (B), or, what is the same thing, the equation (A) is thus proved.

Reverting to the equation (A), we have

$$(1+2q+2q^4+\dots)^4=\frac{4K^2}{\pi^2},$$

(Jacobi, *Fund. Nova*, p. 188, *Ges. Werke*, t. I., p. 239),

$$\left(\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\dots\right)=\frac{K^2}{2\pi^2}\left(1-\frac{E_1}{K}\right),$$

(*ib.*, p. 135; *ib.*, p. 189),

if $q=e^{-\frac{\pi K'}{K}}$, and K, E_1 are the complete functions F_1k, E_1k.

The left-hand side of the equation is thus

$$\frac{4K^2}{\pi^2}-\frac{8K^2}{\pi^2}\left(1-\frac{E_1}{K}\right), \quad =\frac{4K^2}{\pi^2}\left(-1+\frac{2E_1}{K}\right),$$

and we have

$$\left(-1+\frac{2E_1}{K}\right)=\frac{\pi^2}{4K^2}\cdot\frac{1-9q^1-25q^3+49q^6+81q^{10}-\dots}{1-q^1-q^3+q^6+q^{10}-\dots},$$

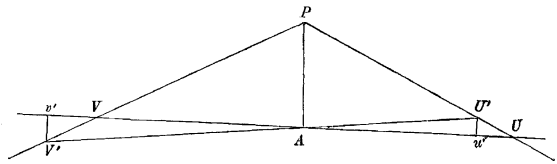
which is a new expression for E_1 as a q-function. The expression on the right-hand side presents itself, Clebsch, *Theorie der Elasticität* (Leipzig, 1862), p. 162, and must have been obtained by him as a value for $\left(-1+\frac{2E_1}{K}\right)$; but there is no statement that this is so, nor anything to show how this form of q-function was arrived at. Mr Todhunter called my attention to the passage in Clebsch.

639.

AN ELEMENTARY CONSTRUCTION IN OPTICS.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 81, 82.]

CONSIDER two lines meeting at a point P, and a point A; through A, draw at right angles to AP, a line meeting the two lines in the points U, V respectively; and through the same point A draw any other line meeting the two lines in the



points U', V' respectively; also let the points u', v' be the feet of the perpendiculars let fall from U', V' respectively on the line UV; then we have

$$\frac{1}{Au'}+\frac{1}{Av'}=\frac{1}{AU}+\frac{1}{AV}.$$

The theorem can be proved at once without any difficulty. It answers to the optical construction, according to which, if UPV represents the path of a ray through a convex lens AP, then the thin pencil, axis $U'P$ and centre U', converges after refraction to the point V', where $U'V'$ are *in lineâ* with A the centre of the lens; considering as usual the inclinations to the axis as small, we have approximately $AV'=Av'$, $AU'=Au'$, and the theorem is

$$\frac{1}{AU'}+\frac{1}{AV'}=\frac{1}{AU}+\frac{1}{AV}, \quad =\frac{1}{AF},$$

if AF is the focal length of the lens.

In the original theorem, the line UV need not be at right angles to AP, but may be any line whatever; the projecting lines $U'u'$ and $V'v'$ must then be parallel to AP, and the theorem remains true.

640.

FURTHER NOTE ON MR MARTIN'S PAPER "ON THE INTEGRALS OF SOME DIFFERENTIALS."

[From the *Messenger of Mathematics*, vol. VI. (1877), p. 82.]

See paper, Number 633; this further note relates also to a detail.

641.

ON THE FLEXURE OF A SPHERICAL SURFACE.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 88—90.]

IT is known that an inextensible spherical surface, or to fix the ideas the spherical quadrilateral included between two arcs of meridian and two arcs of parallel, may be bent in suchwise as to be part of a surface of revolution, the meridians and parallels of the spherical surface being meridians and parallels of the new surface, and, moreover, the radius of each parallel of the spherical surface being in the new surface altered in the constant ratio k to 1. We have, in fact, on the spherical surface, writing p for the latitude and q for the longitude, and the radius being unity,

$$x = \cos p \cos q,$$
$$y = \cos p \sin q,$$
$$z = \sin p,$$

values which give

$$dx^2 + dy^2 + dz^2 = dp^2 + \cos^2 p \, dq^2.$$

This last equation is satisfied by the values

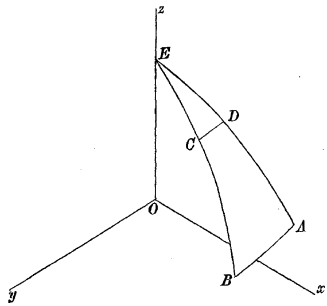
$$x = \cos p \cos \frac{q}{k},$$
$$y = \cos p \sin \frac{q}{k},$$
$$z = E(k, p),$$

where $E(k, p), = \int_0 \sqrt{(1 - k^2 \sin^2 p)}\, dp$, is the elliptic function of the second kind; or rather, this is so when $k < 1$, but the same notation may be used when $k > 1$. These values give the deformation in question.

The two cases to be considered are $k < 1$, and $k > 1$; we take in each case a spherical quadrilateral $ABCD$ (fig. 1), bounded by AB (an arc of the equator), the arc of parallel CD, and the two arcs of meridian AD and BC. In the first case, there is no limit to the latitude $AD, = BC$, or taking these $= 90°$, we may in place

of the quadrilateral $ABCD$ consider the birectangular triangle ABE; the new form of this is $A'B'E'$, where the radius OA', $= k \,.\, OA$, $= k$, is less than the original radius

Fig. 1.



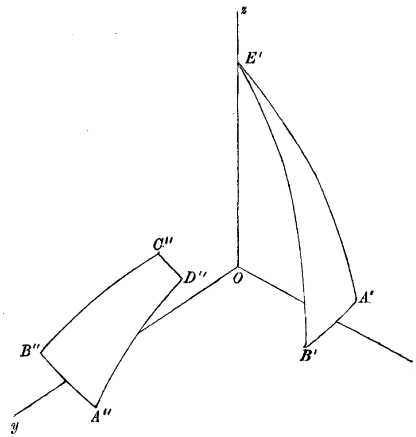
unity, but OE', $= E_1k$, is greater than the same radius unity. The surface has at E' a conical point, the semi-aperture of the cone being $= \tan^{-1}\frac{1}{k'}$, if as usual $k' = \sqrt{(1-k^2)}$; to verify this, writing for convenience $q = 0$, we have for the meridian section $x = \cos p$, $z = E(k, p)$, and thence $\frac{dx}{dz} = -\frac{\sqrt{(1-k^2\sin^2 p)}}{\sin p}$, which for $p = 90°$ gives $\frac{dz}{dx} = -k'$. Observe also that for $p = 0$, $\frac{dz}{dx} = \infty$, viz. the surface of revolution cuts the plane of the equator at right angles.

There is no limit to the arc AB, it may be $= 360°$, viz. we must in this case cut the hemisphere along a meridian to allow of the deformation; or it may exceed 360°, the hemisphere spherical surface being in this case conceived of as wrapping indefinitely over itself, and we may instead of the half lune $E'A'B'$, consider the lune included between two meridians extending from pole to pole, and therefore the whole spherical surface, conceived of as wrapping indefinitely over itself; the result is, that this may be deformed into a surface of revolution, which, in its general form, resembles that obtained by the revolution of an arc less than a semi-circle round its chord; the half-chord being greater, and the versed-sine less than the radius of the original sphere.

If $k > 1$, there is obviously a limit to the latitude AD, $= BC$, of the spherical quadrilateral; viz. this is equal to $\sin^{-1}\frac{1}{k}$. Supposing that in the quadrilateral $ABCD$ (fig. 1) the latitude has this limiting value, then (see fig. 2) the new form is $A''B''C''D''$, where along the bounding arc $C''D''$ the tangent plane is horizontal; viz. as before $\frac{dz}{dx} = -\frac{\sqrt{(1-k^2\sin^2 p)}}{\sin p}$, $= 0$ for $p = \sin^{-1}\frac{1}{k}$. It is to be observed, that the radii for the parallels $A''B''$ and $C''D''$ are k and $k\cos p$ respectively; the difference of

these is $k(1-\cos p)$, which, however great k is, must be less than the arc of meridian $A''D''$, $=p$; substituting for k the value $\frac{1}{\sin p}$, the condition is $\frac{1-\cos p}{\sin p}<p$, viz. this

Fig. 2.



is $\tan\frac{1}{2}p<p$, which is true for every value up to $p=90°$. But, more than this, we should have

$$k^2(1-\cos p)^2+E^2(k,\ p)<p^2,$$

viz. writing as before $k=\frac{1}{\sin p}$, this is

$$E^2\left(\frac{1}{\sin p},\ p\right)<p^2-\tan^2\tfrac{1}{2}p\,;$$

this must be true, although (relating as it does to a form of E for which k is greater than 1) there might be some difficulty in verifying it.

There is, as in the first case, no limit to the value of AB, viz. this may be $=360°$, the spherical zone being then cut along a meridian, or it may be greater than $360°$; and, moreover, the spherical quadrilateral may extend south of the equator, but of course so that the limiting south latitude does not extend beyond the foregoing value $\sin^{-1}\frac{1}{k}$: viz. we may have a zone between the latitudes $\pm\sin^{-1}\frac{1}{k}$, which may be a complete zone from longitude $0°$ to $360°$ or to any greater value than $360°$. The result is, that the zone is deformed into a surface of revolution, which in its general form resembles that obtained by the revolution of a half-circle or half-ellipse about a line parallel to and beyond its bounding diameter, the bounding half-diameter being less, and the greatest radius of rotation greater, than the radius of the original sphere.

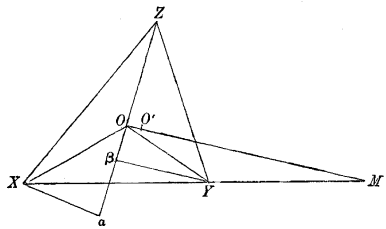
642.

ON A DIFFERENTIAL RELATION BETWEEN THE SIDES OF A QUADRANGLE.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 99—101.]

LET the sides and diagonals YZ, ZX, XY, OX, OY, OZ of a quadrangle be f, g, h, a, b, c, and let the component triangles be denoted as follows:

$$\begin{aligned} A &= \Delta YZO, &&= (b, c, f), \\ B &= \Delta ZXO, &&= (c, a, g), \\ C &= \Delta XYO, &&= (a, b, h), \\ \Omega &= \Delta XYZ, &&= (f, g, h), \end{aligned}$$



viz. A, B, C, Ω are the triangles whose sides are (b, c, f), (c, a, g), (a, b, h), (f, g, h) respectively, so that $\Omega = A + B + C$. Then we have between (a, b, c, f, g, h) an equation giving rise to a differential relation, which may be written

$$\Omega\,(Aada + Bbdb + Ccdc) - (BCfdf + CAgdg + ABhdh) = 0.$$

This may be proved geometrically and analytically. First, for the geometrical proof, it is enough to prove that, when a and b alone vary, the relation between the increments is $Aada + Bbdb = 0$; for then a and g alone varying, the relation between

the increments will be $\Omega ada - Cgdg = 0$ (as to the negative sign it is clear from the figure that a, g will increase or diminish together): and we thence at once infer the general relation.

We have consequently to prove that, considering a and b as alone variable,

$$Aada + Bbdb = 0;$$

or, what is the same thing,

$$ada : -bdb = XOZ : YOZ.$$

The points XYZ remain fixed; but O moves through the infinitesimal arc OO', centre Z, which may be considered as situate in the right line OM drawn from O at right angles to ZO, and meeting XY produced in the point M. And then, writing for a moment $\angle OXY = X$, $\angle OYX = Y$, $\angle OMY = M$, we find at once

$$da = OO' \cos(X + M),$$
$$-db = OO' \cos(Y - M);$$

that is,

$$-\frac{da}{db} = \frac{\cos(X+M)}{\cos(Y-M)}, \text{ or } -\frac{ada}{bdb} = \frac{a\cos(X+M)}{b\cos(Y-M)}.$$

But drawing $X\alpha$, $Y\beta$ each of them at right angles to ZO, we have $a\cos(X+M) = X\alpha$, $b\cos(Y-M) = Y\beta$, and evidently $XOZ : YOZ = X\alpha : Y\beta$; whence the equation is $-\frac{ada}{bdb} = \frac{XOZ}{YOZ}$, which is the required relation.

For the analytical proof, it is to be observed that the relation between a, b, c, f, g, h is a quadric relation in the quantities a^2, b^2, c^2, f^2, g^2, h^2 respectively; this may be written

$$u = \begin{array}{c|c|c|c|} & 1 & a^2 & a^4 \\ \hline 1 & b^2g^4 + b^4g^2 + c^2h^4 + c^4h^2 - (b^2+c^2)g^2h^2 - (g^2+h^2)b^2c^2 & -(b^2-c^2)(g^2-h^2) & \\ \hline f^2 & +(b^2-h^2)(c^2-g^2) & -b^2-c^2-g^2-h^2 & +1 \\ \hline f^4 & & +1 & \\ \hline \end{array} = 0:$$

say for a moment this is $A + Ba^2 + Ca^4 = 0$, where

$$\begin{aligned} A &= b^2g^4 + b^4g^2 + c^2h^4 + c^4h^2 \quad + f^2(b^2-h^2)(c^2-g^2) \\ &\quad -(b^2+c^2)g^2h^2 - (g^2+h^2)b^2c^2 \\ B &= -(b^2-c^2)(g^2-h^2) \quad + f^2(-b^2-c^2-g^2-h^2) + f^4, \\ C &= \quad f^2; \end{aligned}$$

then we have as usual

$$\frac{du}{d.a^2}ada + \frac{du}{d.b^2}bdb + \&c. = 0,$$

where

$$\tfrac{1}{2}\frac{du}{d\,.\,a^2} = Ca^2 + \tfrac{1}{2}B.$$

But in virtue of $u=0$, we have

$$(Ca^2 + \tfrac{1}{2}B)^2 = C(Ca^4 + Ba^2 + A) + \tfrac{1}{4}(B^2 - AC),$$

that is, $\dfrac{du}{d\,.\,a^2} = \sqrt{(B^2 - 4AC)}$; and here $B^2 - 4AC$ is a quartic function of f^2, which is easily seen to reduce itself to the form

$$f^2 - (g+h)^2 f^2 - (g-h)^2 f^2 - (b+c)^2 f^2 - (b-c)^2.$$

The coefficients of bdb, cdc, &c., are given as expressions of the like form; substituting their values, the differential relation is

$$\sqrt{\{f^2 - (g+h)^2 f^2 - (g-h)^2 f^2 - (b+c)^2 f^2 - (b-c)^2\}}\, ada + \&c. = 0,$$

which is, in fact, the foregoing result.

It is right to notice that there are in all 16 linear factors,

$$\begin{array}{rrrr|l}
f+g+h, & b+c+f, & c+a+g, & a+b+h & \text{say } d\ ,\ f\ ,\ g\ ,\ h\ , \\
-f+g+h, & -b+c+f, & -c+a+g, & -a+b+h & \quad d'\ ,\ f'\ ,\ g'\ ,\ h'\ , \\
f-g+h, & b-c+f, & c-a+g, & a-b+h & \quad d''\ ,\ f''\ ,\ g''\ ,\ h''\ , \\
f+g-h, & b+c-f, & c+a-g, & a+b-h & \quad d'''\ ,\ f'''\ ,\ g'''\ ,\ h'''\ ;
\end{array}$$

and this being so, the coefficients of ada, bdb, cdc, fdf, gdg, hdh, are

$$\begin{array}{ll}
\sqrt{(dd'd''d'''\,.\,ff'f''f''')}, & -\sqrt{(gg'g''g'''\,.\,hh'h''h''')}, \\
\sqrt{(dd'd''d'''\,.\,gg'g''g''')}, & -\sqrt{(hh'h''h'''\,.\,ff'f''f''')}, \\
\sqrt{(dd'd''d'''\,.\,hh'h''h''')}, & -\sqrt{(ff'f''f'''\,.\,gg'g''g''')},
\end{array}$$

respectively.

We may imagine the quadrilateral $ZOXY$ composed of the four rods ZO, OX, XY, YZ (lengths c, a, h, f as before) jointed together at the angles, and kept in equilibrium by forces B, G acting along the diagonals $OY(=b)$, $ZX(=a)$ respectively. We have c, a, h, f given constants, and the relation $\phi(a, b, c, f, g, h) = 0$, which connects the six quantities is the relation between the two variable diagonals (g, b); by what precedes, the differential relation $\phi'g\,.\,dg + \phi'b\,.\,db = 0$ is equivalent to $\Omega Bbdb - CAgdg = 0$. By virtual velocities we have as the condition of equilibrium $Bdb + Gdg = 0$; hence, eliminating db, dg we have

$$\frac{B}{\Omega Bb} = -\frac{G}{CAg};$$

or, say

$$\frac{B}{b} : -\frac{G}{g} = \frac{1}{\Delta XYO\,.\,\Delta ZYO} : \frac{1}{\Delta XYZ\,.\,\Delta ZXO},$$

viz. the forces, divided by the diagonals along which they act, are proportional to the reciprocals of the products of the two pairs of triangles which stand on these diagonals respectively. The negative sign shows, what is obvious, that the forces must be, one of them a pull, the other a push.

643.

ON A QUARTIC CURVE WITH TWO ODD BRANCHES.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 107, 108.]

It is a known theorem that the branches of a plane curve are even or odd; viz. two even branches, or an even and an odd branch (whether of the same curve or of different curves) intersect in an even number (it may be 0, and this is to be understood throughout) of real points; but two odd branches (of the same curve or of different curves) intersect in an odd number of real points *.

In particular, a right line is an odd branch, and hence it meets any even branch of a curve in an even number of real points, and an odd branch in an odd number of real points; or (what is the same thing) an even branch is one which is met by any right line whatever in an even number of real points; and an odd branch is one that is met by any right line whatever in an odd number of real points.

It is to be observed, that the simple term branch is used to denote what has been called a *complete* branch, viz. the partial branches which touch an asymptote at its opposite extremities are considered as parts of one and the same branch, and so in other cases. Thus a quadric curve, whether ellipse, parabola, or hyperbola, is one even branch; a cubic curve is either one odd branch, or else it is an odd branch and an even branch; and generally a curve of an odd order has always an odd number of odd branches, and a curve of an even order has always an even number of odd branches.

A curve without nodes has at most one odd branch; for if there were two, these would intersect in a real point, which would be a real node on the curve. In particular, a quartic curve having two odd branches must have a real node; this however may be, as in the instance about to be given, a node at infinity.

A simple instance of a quartic curve with two odd branches is that represented by the equation

$$(x^2-1)(y^2+1)-2mxy=0,$$

* The two branches must be distinct branches; a branch whether odd or even does not of necessity intersect itself (have upon it any real node), but it may intersect itself in an odd, or an even, number of real points.

or, what is the same thing,

$$y = \frac{1}{x^2-1}\left[mx \pm \sqrt{\left\{-(x^2-\alpha^2)\left(x^2-\frac{1}{\alpha^2}\right)\right\}}\right],$$

where

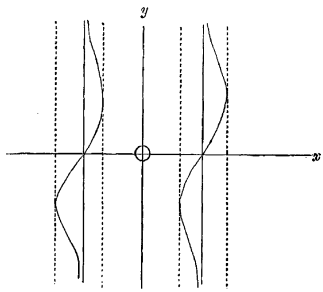
$$\alpha^2 + \frac{1}{\alpha^2} = 2 + m^2,$$

or say

$$\alpha^2 = \tfrac{1}{2}\{2 + m^2 + m\sqrt{(4+m^2)}\},$$

$$\frac{1}{\alpha^2} = \tfrac{1}{2}\{2 + m^2 - m\sqrt{(4+m^2)}\},$$

so that m being positive $\alpha > 1$, and the curve consists of two real branches included between the lines $x = \alpha$, $x = \frac{1}{\alpha}$, and the lines $x = -\alpha$, $x = -\frac{1}{\alpha}$ respectively; each of these lines touches the curve in a real point, viz. x having any one of the last-mentioned values, the value of y at the point of contact is $y = \frac{mx}{x^2-1}$; and between each pair of lines we have the asymptote $x = +1$ or $x = -1$. Hence the curve has the form shown in the figure, and it is thereby evident, that each branch of the curve is met by



any real right line whatever in one real point, or else in three real points. The numerical values in the figure are $\alpha = \frac{3}{2}$, $m = \frac{5}{6}$, whence also $x = \alpha$ or $-\frac{1}{\alpha}$, $y = 1$, and $x = -\alpha$ or $\frac{1}{\alpha}$, $y = -1$.

The curve has two nodes at infinity, viz. writing the equation in the form

$$(x^2 - z^2)(y^2 + z^2) - mxyz^2 = 0,$$

that is,

$$x^2y^2 + z^2(x^2 - y^2 - mxy) + z^4 = 0,$$

it appears that the points $(z = 0,\ x = 0)$, $(z = 0,\ y = 0)$ are each of them a node. The first of these $(z = 0,\ x = 0)$ is the real intersection of the two odd branches: the other of them is a conjugate point.

644.

NOTE ON MAGIC SQUARES.

[From the *Messenger of Mathematics*, vol. VI. (1877), p. 168.]

In a magic square of any odd order, formed according to the ordinary process, there is a tolerably simple analytical expression for the number which occupies any given compartment; thus taking the square of 21, let the dexter diagonals (N.W. to S.E.) commencing from the N.E. corner compartment, be numbered 1, 2, 3,.., 20, 21, 20′, 19′,.., 2′, 1′, the diagonals of course containing these numbers of compartments respectively; and in any diagonal let the compartments reckoning from the top line be numbered 1, 2, 3,.., respectively; then if $D_{\theta,\phi}$ (or $D'_{\theta,\phi}$ as the case may be) denotes the number in the compartment ϕ of the diagonal θ or θ', we have

$$\begin{aligned}
D_{2\theta+1,\phi} &= 20\theta + 10 + \phi,\\
D_{2\theta\ \ ,\phi} &= 20\theta + 231 + \phi\,(-21),\\
D'_{2\theta+1,\phi} &= -22\theta + 430 + \phi,\\
D'_{2\theta\ \ ,\phi} &= -22\theta + 231 + \phi\,(-21),
\end{aligned}$$

where in the second and fourth expressions the term -21 is to be retained only if $\phi > \theta$; if $\phi \not> \theta$, it is to be omitted. There would be a like formulæ for a square of any odd order, and it would be easy to write down the formulæ for the general value $2n+1$: but I have preferred to give them for a specific case.

645.

A SMITH'S PRIZE PAPER, 1877.

[From the *Messenger of Mathematics*, vol. VI. (1877), pp. 173—182.]

THE paper was as follows:

1. Show (independently of the theory of roots) how, if x satisfies an equation of the order n, a given rational function of x can in general be expressed as a rational and integral function of the order $n-1$. State the theorem in a more precise form, so as to make it true universally.

2. Investigate the form of the factors of $1 \pm \sin(2n+1)x$ considered as a function of $\sin x$; and give the formulæ in the two cases, $2n+1=3$ and 5 respectively.

3. Write down the substitutions which do not alter the function $ab+cd$; and explain the constitution of the group.

4. Find in a form adapted for calculation an approximate value for the sum of the middle $2\alpha+1$ terms of the expansion of $(1+1)^{2n}$, n being a large number, and α small in comparison therewith.

Obtain thence a complete and precise statement of the theorem that in a large number of tosses the numbers of heads and tails will probably be nearly equal.

5. A point in space is represented on a given plane by its projections from two fixed points. Show how a problem relating to points, lines, and planes, is thereby reduced to a problem *in plano;* and apply the method to construct the line of intersection of two planes each passing through three given points.

6. A weight is supported on a tripod of three unequal legs resting on a smooth horizontal plane, their feet connected in pairs by strings of given lengths. Show how to determine the tensions of the several strings.

7. Explain the ordinary configuration of a system of isoparametric lines on a spherical surface; for instance, what is the configuration when there are two points of minimum value, and one point of maximum value, of the parameter?

8. Find the attraction of an infinite circular cylinder, of uniform density, on a given exterior or interior point.

9. Determine the number of arbitrary constants contained in the equation of a surface of the order r which passes through the curve of intersection of two given surfaces of the orders m and n respectively.

10. Find, for the several values of p, the number of the conics passing through p given points and touching $5-p$ given lines; and, in each case, show how to obtain (in point-coordinates or line-coordinates, as may be most simple) the equations of the conics satisfying the conditions in question.

11. Investigate the theory of the linear transformation of a ternary quadric function into itself.

12. Explain the theory of the solution of a partial differential equation, given function of $x, y, z, p, q, r =$ arbitrary constant H; where p, q, r are the differential coefficients of the dependent variable u in regard to the independent variables x, y, z respectively.

I propose, not (as in former years) to give complete solutions, but only to notice in more or less detail the leading points in the several questions.

1. The expression is of course required, not only for a given integral function of x, but for a given fractional function. The case where the given function is integral presents no difficulty; when the given function is fractional, the most simple case is where it is $=\frac{1}{x-a}$; supposing the equation to be $f(x)=0$, here dividing $f(x)$ by $x-a$, we have a quotient $R(x)$ which is a rational and integral function of an order not exceeding $n-1$, and a remainder which is $=f(a)$; that is,

$$\frac{f(x)}{x-a}=R(x)+\frac{f(a)}{x-a};$$

or, in virtue of the given equation $\frac{f(a)}{x-a}=-R(x)$, viz. we have thus $\frac{1}{x-a}$ in the required form. But if $f(a)=0$, then we do not obtain such an expression of $\frac{1}{x-a}$. It has to be shown that the like considerations apply to any fractional function, and the precise form of the theorem is, that any rational function of x which does not become infinite for any value of x satisfying the given equation, can be expressed as a rational and integral function of an order not exceeding $n-1$.

2. The function $1-\sin(2n+1)x$ is a rational and integral function of $\sin x$, of the order $2n+1$; which if n is even (or $2n+1=4p+1$) contains, as is at once seen, the factor $1-\sin x$, but if n is odd (or $2n+1=4p-1$) the factor $(1+\sin x)$. Suppose that any other factor is $1-\frac{\sin x}{\sin \alpha}$, where $\sin\alpha$ not $=\pm 1$; then this will be a double factor if only $\sin x=\sin\alpha$ satisfies the condition

$$0=\frac{d}{d\,.\,\sin x}\{1-\sin(2n+1)x\},$$

that is, $0 = \frac{\cos(2n+1)x}{\cos x}$; the value in question gives $\sin(2n+1)x = 1$, and therefore $\cos(2n+1)x = 0$; and it does not give $\cos x = 0$; hence every such factor $1 - \frac{\sin x}{\sin \alpha}$ is a double factor, or we have

$$1 - \sin(2n+1)x = (1 \pm \sin x)\,\Pi\left(1 - \frac{\sin x}{\sin \alpha}\right)^2.$$

Or the like result might be obtained by considering instead of $1 - \sin(2n+1)x$, the more general function

$$\sin(2n+1)a \pm \sin(2n+1)x,$$

and finally assuming $a = \frac{1}{2}\pi$.

3. Relates to a theory which is not, but ought to be, treated of in the text books of the University. See Serret's *Algèbre Supérieure*, t. II., Sect. IV.

The substitutions which leave $ab + cd$ unaltered are

1	1, that is, the letters remain unchanged,
α	(ab), that is, a and b are interchanged,
β	(cd), that is, c and d are interchanged,
γ	$(ab)\,(cd)$, that is, a and b and also c and d are simultaneously interchanged,
δ	$(ac)\,(bd)$, same with a and c, b and d,
ϵ	$(ad)\,(bc)$, same with a and d, b and c,
ζ	$(acbd)$, that is, we cyclically change a into c, c into b, b into d, and d into a,
θ	$(adbc)$, that is, we cyclically change a into d, d into b, b into c, and c into a,

viz. we have eight substitutions $1, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta$ forming a group; that is, the product of any two of them, in either order, is a substitution of the group (or, what is the same thing, the effect of the successive performance of the two upon any arrangement $abcd$ is the same as that of the performance thereon of some other substitution of the group); thus we have $\alpha^2 = 1$, $\beta^2 = 1$, $\gamma^2 = 1$, $\alpha\beta = \beta\alpha = \gamma$, &c.; the system of these equations, which verify that the set of substitutions form a group, defines the constitution of the group—thus to take a more simple instance, a group of 4 may be $1, \alpha, \alpha^2, \alpha^3$ $(\alpha^4 = 1)$ or $1, \alpha, \beta, \alpha\beta$, $(\alpha^2 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha)$.

4. The expression of the general coefficient is

$$= \frac{1 \,.\, 2 \ldots 2n}{1 \,.\, 2 \ldots n - \alpha \,.\, 1 \,.\, 2 \ldots n + \alpha},$$

which can be transformed by the well-known formula

$$1 \,.\, 2 \ldots n = n^{n+\frac{1}{2}} \sqrt{(\pi)}\, e^{-n},$$

viz. the coefficient thus becomes

$$=\frac{2^{2n}}{\sqrt{(n\pi)}}\frac{1}{\left(1-\frac{\alpha}{n}\right)^{n-\alpha+\frac{1}{2}}\left(1+\frac{\alpha}{n}\right)^{n+\alpha+\frac{1}{2}}}.$$

Now α is supposed small in comparison with n, and the factors in the denominator have the logarithms

$$(n-\alpha+\tfrac{1}{2})\log\left(1-\frac{\alpha}{n}\right),\ =(n-\alpha+\tfrac{1}{2})\left(-\frac{\alpha}{n}+\tfrac{1}{2}\frac{\alpha^2}{n^2}\right),\ =-\alpha+\tfrac{1}{2}\frac{\alpha^2}{n},$$

and

$$(n+\alpha+\tfrac{1}{2})\log\left(1+\frac{\alpha}{n}\right),\qquad\qquad =\ \alpha+\tfrac{1}{2}\frac{\alpha^2}{n};$$

hence the denominator is $=e^{\frac{\alpha^2}{n}}$, and the final approximate value of the coefficient is

$$=\frac{2^{2n}}{\sqrt{(n\pi)}}e^{-\frac{\alpha^2}{n}}.$$

Hence, converting as usual the sum into a definite integral, we have the sum of the $2\alpha+1$ coefficients

$$=\frac{2^{2n}}{\sqrt{(n\pi)}}\int_{-\alpha}^{\alpha}e^{-\frac{\alpha^2}{n}}d\alpha,$$

or, what is the same thing,

$$=\frac{2^{2n}}{\sqrt{(\pi)}}\int_{-\frac{\alpha}{\sqrt{n}}}^{\frac{\alpha}{\sqrt{n}}}e^{-x^2}dx.$$

For the chance that the number of tosses lies between $n+\alpha$ and $n-\alpha$, this has merely to be divided by 2^{2n}; hence writing $\alpha=kn$, the chance that the number may be between $n(1+k)$ and $n(1-k)$ is

$$=\frac{1}{\sqrt{(\pi)}}\int_{-k\sqrt{n}}^{k\sqrt{n}}e^{-x^2}dx,$$

where observe that the integral, taken with the limits ∞, $-\infty$ has the value $\sqrt{(\pi)}$.

Considering k as a given fraction however small, by increasing n we make $k\sqrt{(n)}$ as large as we please, and therefore the integral, as nearly as we please $=\sqrt{(\pi)}$, or the chance as nearly as we please $=1$; and hence the complete and precise statement of the theorem, viz. by sufficiently increasing the number of tosses, the probability that the deviation from equality shall be any given percentage (as small as we please) of the whole number of tosses, can be made as nearly as we please equal to certainty.

Further, restoring α in place of kn, the chance of a number between $n+\alpha$ and $n-\alpha$ is

$$=\frac{1}{\sqrt{(\pi)}}\int_{-\frac{\alpha}{\sqrt{n}}}^{\frac{\alpha}{\sqrt{n}}}e^{-x^2}dx,$$

which when $\frac{\alpha}{\sqrt{(n)}}$ is small is $=\frac{2\alpha}{\sqrt{(n\pi)}}$, $\left(\text{more accurately } \frac{2\alpha+1}{\sqrt{(n\pi)}}, \text{ when } \alpha \text{ is small}\right)$; hence, however large α is, the chance of a deviation from equality not exceeding $\pm\alpha$, continually *diminishes* with n, and by making n sufficiently large becomes as small as we please.

5. The point is represented in the given plane by two points which lie *in lineâ* with a fixed point (say O) of that plane, viz. O is the intersection of the given plane by the line which joins the two projecting points.

A line is represented on the given plane by two lines, viz. these are the projections of the line from the two given points; each point of the line is represented by the points of intersection of the two lines by any line through O.

A plane may be represented on the given plane by means of its trace thereon, and of the two points (*in lineâ* with O) which represent any point of the plane.

Thus any problem relating to points, lines, and planes, in space is converted into a problem of plane geometry. For instance, to find the trace on the given plane of a plane through three given points A, B, C, the three points are represented by means of the pairs of points A_1, A_2; B_1, B_2; C_1, C_2, the points of each pair lying *in lineâ* with O; the required trace passes through the intersections with the given plane of the lines BC, CA, AB respectively, and we hence find it as the line through the three points which are the intersections of B_1C_1, B_2C_2, of C_1A_1, C_2A_2, and of A_1B_1, A_2B_2 respectively; that these points are in a line is a theorem of plane geometry, which, if not previously known, would have at once been given by the construction.

6. The solution ought obviously to be obtained from the principle of virtual velocities; taking a, b, c for the lengths of the legs, f, g, h for the lengths of the strings, and z for the height of the summit, z is a known function of a, b, c, f, g, h, $\left(z \text{ is in fact } =\frac{3V}{\Delta}, \text{ where } V, \text{ the volume of the tetrahedron, is a given function of } a, b, c, f, g, h; \text{ and } \Delta, \text{ the area of the base, is a given function of } f, g, h\right)$. Writing then F, G, H for the tensions, and W for the weight, and regarding z, f, g, h as variable, the principle gives

$$Wdz + Fdf + Gdg + Hdh = 0,$$

that is,

$$F,\ G,\ H, = -W\frac{dz}{df},\ -W\frac{dz}{dg},\ -W\frac{dz}{dh},$$

respectively.

7. The ordinary case is when an isoparametric line has on one side of it larger values, on the other side of it smaller values of the parameter; the case where the isoparametric line is a line of maximum, or of minimum, parameter is excluded.

The lines in the neighbourhood of a point of maximum, or of minimum, parameter are ovals surrounding the point in question, each oval being itself surrounded by the consecutive oval. Supposing that there are two points of minimum parameter, we have round each of them a series of ovals, until at length an oval belonging to the one of them comes to unite itself with an oval belonging to the other, the two ovals altering themselves into a figure of eight. Surrounding this we have a closed curve (in the first instance a deeply twice-indented oval) which (in the case supposed of there being, besides the two points of minimum parameter, a single point of maximum parameter) is in fact an oval surrounding the point of maximum parameter, and the remaining curves are the series of ovals surrounding that point. If we project stereographically from the point of maximum parameter (so that this point is represented by the points at infinity) we have a figure of eight, each loop containing within it a series of continually diminishing closed curves, and the figure of eight itself surrounded by a series of continually increasing closed curves.

8. The investigation by means of the Potential presents the difficulty that the Potential of the infinite cylinder has no determinate value, as at once appears from the limiting case where the cylinder is reduced to a right line; the difficulty is perhaps rather apparent than real, inasmuch as the partial differential equations contain only differential coefficients $\frac{dV}{dr}$, $\frac{d^2V}{dr^2}$, where $\frac{dV}{dr}$ as representing an attraction, and therefore also $\frac{d^2V}{dr^2}$, are determinate. But it is safer to work directly with the Attraction; the Attraction of an infinite line acts in the perpendicular plane through the attracted point, and is inversely proportional to the distance; the problem is thus reduced to the plane problem of a circle of uniform density, force varying as $(\text{distance})^{-1}$, attracting a point in its own plane. This is precisely similar to the case of a sphere with the ordinary law of attraction; dividing the circle into rings, each ring exerts an attraction $=0$ upon an interior point, and an attraction as if collected at the centre upon an exterior point. Hence, writing a for the radius of the cylinder, and r for the distance of the attracted point, the attraction is $=\pi r$ for an interior point, and $=\frac{\pi a^2}{r}$ for an exterior point.

9. The theory is precisely the same as for curves; taking the surfaces to be $U=0$ of the order m, and $V=0$ of the order n, the general form of the equation of a surface of the order r (r not less than m or n) is $LU+MV=0$, where L is the general function of the order $r-m$, and M the general function of the order $r-n$; and so long as r is less than $m+n$, we obtain the required number of arbitrary constants as the sum of the numbers of the coefficients of L and of M, less unity. But as soon as r is $=m+n$ a modification arises, viz. we obtain here an identity by assuming $L=V$, $M=-U$, and so for any larger value of r, we have an identity by assuming $L=V\phi$, $M=-U\phi$, where ϕ is the general function of the order $r-m-n$.

10. The numbers are known to be 1, 2, 4, 4, 2, 1, which values are obtained most easily (though not in the way which is theoretically most interesting) by finding for

the first three cases the equation of the required conic in point-coordinates; and then, by changing these into line-coordinates, we have the equations for the remaining three cases.

$p=5$: 5 points. The equation of the conic is

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2=0,$$

and we have 5 linear equations to determine the ratios of the coefficients; the number is therefore $=1$.

$p=4$: 4 points and 1 line. Taking $U=0$ and $V=0$, the equations of any two conics each passing through the four points, the equation of the required conic will be $U+\lambda V=0$, and the condition of touching a given line gives a quadric equation for λ; the number is therefore $=2$.

$p=3$: 3 points and 2 lines. In the same manner, by taking $U=0$, $V=0$, $W=0$, for the equations of any three conics through the three points; or if the equations of the lines containing the three points in pairs are $x=0$, $y=0$, $z=0$, then the equations of the three conics are $yz=0$, $zx=0$, $xy=0$, and the equation of any conic through these points is $fyz+gzx+hxy=0$; the conditions of touching two given lines $\xi x+\eta y+\zeta z=0$ and $\xi' x+\eta' y+\zeta' z=0$, are

$$\sqrt{f}\sqrt{\xi}+\sqrt{g}\sqrt{\eta}+\sqrt{h}\sqrt{\zeta}=0,\quad \sqrt{f}\sqrt{\xi'}+\sqrt{g}\sqrt{\eta'}+\sqrt{h}\sqrt{\zeta'}=0;$$

we have thus the ratios $\sqrt{f}:\sqrt{g}:\sqrt{h}$ linearly determined in terms of $\sqrt{\xi}$, $\sqrt{\eta}$, &c.; there is no loss of generality in taking $\sqrt{\xi}$, $\sqrt{\xi'}$ each with a determinate sign, the signs of $\sqrt{\eta}$, &c. being then arbitrary, we have $2^4, =16$ values of $\sqrt{f}:\sqrt{g}:\sqrt{h}$, and therefore one-fourth of this $=4$, for the number of values of $f:g:h$; that is, the number is $=4$.

11. This is a known theory; taking x_1, y_1, z_1 for the linear functions of x, y, z, which are such that

$$(a,\ b,\ c,\ f,\ g,\ h \between x_1,\ y_1,\ z_1)^2=(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2,$$

then assuming $x_1,\ y_1,\ z_1=2\xi-x,\ 2\eta-y,\ 2\zeta-z$ respectively, we have

$$(a,\ldots \between 2\xi-x,\ 2\eta-y,\ 2\zeta-z)^2=(a,\ldots \between x,\ y,\ z)^2,$$

or, omitting terms which destroy each other, and throwing out the factor 4, this is

$$(a,\ldots \between \xi,\ \eta,\ \zeta)^2=(a,\ldots \between \xi,\ \eta,\ \zeta \between x,\ y,\ z),$$

an equation which is satisfied identically by assuming

$$\begin{aligned} a\xi+h\eta+g\zeta&=ax+hy+gz\quad .\quad -\nu\eta+\mu\zeta,\\ h\xi+b\eta+f\zeta&=hx+by+fz+\nu\xi\quad .\quad -\lambda\zeta,\\ g\xi+f\eta+c\zeta&=gx+fy+cz-\mu\xi+\lambda\eta\quad .\ , \end{aligned}$$

where λ, μ, ν are arbitrary; viz. multiplying by ξ, η, ζ, and adding we have the equation in question. The three equations determine ξ, η, ζ as linear functions of x, y, z; and we have thence x_1, y_1, z_1 as linear functions of x, y, z; viz. this is a solution containing three arbitrary constants λ, μ, ν.

12. The partial differential equation might equally well have been proposed in the form, given function of $x, y, z, p, q, r = 0$, viz. the equation then is $\phi(x, y, z, p, q, r) = 0$, the general partial differential equation involving the three independent variables x, y, z, and the derived functions p, q, r of the dependent variable u, *but not involving the dependent variable* u. The question is therefore in effect as follows: to find p, q, r functions of x, y, z connected by the foregoing equation, and, moreover, such that $pdx + qdy + rdz$ is an exact differential; for then writing $u = \int(pdx + qdy + rdz)$, we have the solution of the given partial differential equation.

Whatever be the method adopted, it comes out that the solution depends on the integration of the system of ordinary differential equations

$$\frac{dp}{-\frac{d\phi}{dx}} = \frac{dq}{-\frac{d\phi}{dy}} = \frac{dr}{-\frac{d\phi}{dz}} = \frac{dx}{\frac{d\phi}{dp}} = \frac{dy}{\frac{d\phi}{dq}} = \frac{dz}{\frac{d\phi}{dr}},$$

and the answer consists first in showing this, and secondly, in showing how from an integral or integrals of the system we pass to the solution of the partial differential equation.

Considering the partial differential equation in the form actually proposed, we may instead of ϕ write H, where H will stand for that given function of x, y, z, p, q, r which is the value of the arbitrary constant H; making this change, and putting the foregoing equal quantities equal to the differential dt of a new variable, the system of ordinary differential equations is

$$\frac{dp}{dt} = -\frac{dH}{dx}, \quad \frac{dq}{dt} = -\frac{dH}{dy}, \quad \frac{dr}{dt} = -\frac{dH}{dz},$$

$$\frac{dx}{dt} = \frac{dH}{dp}, \quad \frac{dy}{dt} = \frac{dH}{dq}, \quad \frac{dz}{dt} = \frac{dH}{dr},$$

where H is a given function of x, y, z, p, q, r. This is, in fact, the Hamiltonian system of equations; and it was in view to the connexion that the partial differential equation was proposed in its actual form.

646.

ON THE GENERAL EQUATION OF DIFFERENCES OF THE SECOND ORDER.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 23—25.]

CONSIDER the equation of differences

$$u_x = a_{x-1}\, u_{x-1} + b_{x-2}\, u_{x-2},$$

viz. we have

$$\begin{aligned}
u_2 &= a_1 u_1 + b_0 u_0,\\
u_3 &= a_2 u_2 + b_1 u_1,\\
u_4 &= a_3 u_3 + b_2 u_2,\\
u_5 &= a_4 u_4 + b_3 u_3,\\
u_6 &= a_5 u_5 + b_4 u_4,\\
&\&c.,
\end{aligned}$$

and thence

$$u_3 = \left|\begin{array}{l} a_2 a_1 \\ \quad + b_1 \end{array}\right| u_1 + \quad a_2 \quad b_0 u_0,$$

$$u_4 = \left|\begin{array}{r} a_3 a_2 a_1 \\ + a_3 b_1 \\ + a_1 b_2 \end{array}\right| u_1 + \left|\begin{array}{r} a_3 a_2 \\ + b_2 \end{array}\right| b_0 u_0,$$

$$u_5 = \left|\begin{array}{r} a_4 a_3 a_2 a_1 \\ + a_4 a_3 b_1 \\ + a_4 a_1 b_2 \\ + a_3 a_1 b_3 \\ + b_1 b_3 \end{array}\right| u_1 + \left|\begin{array}{r} a_4 a_3 a_2 \\ + a_4 b_2 \\ + a_2 b_3 \end{array}\right| b_0 u_0,$$

$$
\begin{array}{r|l|l|l}
u_6 = & a_5a_4a_3a_2a_1 & u_1 + \quad a_5a_4a_3a_2 & b_0u_0, \\
 & + a_5a_4a_3b_1 & \quad\quad + a_5a_4b_2 & \\
 & + a_5a_4a_1b_2 & \quad\quad + a_5a_2b_3 & \\
 & + a_5a_3a_1b_3 & \quad\quad + a_3a_2b_4 & \\
 & + a_3a_2a_1b_4 & \quad\quad + b_4b_2 & \\
 & + a_3b_3b_1 & & \\
 & + a_3b_4b_1 & & \\
 & + a_1b_4b_2 & &
\end{array}
$$

&c.

It is now easy to see the law; viz. writing for instance

$$u_6 = 54321 \,.\, u_1 + 5432 \,.\, b_6u_0,$$

then 54321 has a leading term $a_5a_4a_3a_2a_1$: it has terms derived from this by changing any pair a_2a_1 into b_1, a_3a_2 into b_2, a_4a_3 into b_3, a_5a_4 into b_4: it has terms derived by changing any two pairs a_4a_3, a_2a_1 into b_3b_1; a_5a_4, a_2a_1 into b_4b_1; a_5a_4, a_3a_2 into b_4b_2, and so on; where observe that the expression a pair denotes the product of two consecutive a's.

And, similarly, 5432 has a leading term $a_5a_4a_3a_2$; the other terms being derived from this in the same manner precisely.

The solution of $u_x = lx\,(au_{x-1} - u_{x-2})$ is included in, and might be deduced from the foregoing, but it is convenient to obtain it separately. Supposing for greater simplicity that $u_{-1} = 0$, $u_0 = 1$ (or, what is the same thing, $u_0 = 1$, $u_1 = l_1a$), then we find

$$u_0 = 1,$$

$$u_1 = l_1a,$$

$$u_2 = l_2l_1a^2 - l_2,$$

$$u_3 = l_3l_2l_1a^3 - \left|\begin{array}{r} l_3l_2 \\ + l_3l_1 \end{array}\right| a.$$

$$u_4 = l_4l_3l_2l_1a^4 - \left|\begin{array}{r} l_4l_3l_2 \\ + l_4l_3l_1 \\ + l_4l_2l_1 \end{array}\right| a^2 + l_4l_2,$$

$$u_5 = l_5l_4l_3l_2l_1a^5 - \left|\begin{array}{r} l_5l_4l_3l_2 \\ + l_5l_4l_3l_1 \\ + l_5l_4l_2l_1 \\ + l_5l_3l_2l_1 \end{array}\right| a^3 + \left|\begin{array}{r} l_5l_4l_2 \\ + l_5l_3l_2 \\ + l_5l_3l_1 \end{array}\right| a,$$

&c.,

viz. we may for example write

$$u_5 = l_5\, 4321\,.\, a^5 - 4321\,(\cdot)\, a^3 + 4321\,(:)\, a\,;$$

where

$$4321 \text{ denotes } l_4 l_3 l_2 l_1:$$

in 4321 (·), we omit successively each number, viz. we thus obtain

$$432 + 431 + 421 + 321\,,$$
$$= l_4 l_3 l_2 + l_4 l_3 l_1 + l_4 l_2 l_1 + l_3 l_2 l_1:$$

in 4321 (:), we omit successively each two non-consecutive numbers, viz. the omitted numbers being 1, 3; 1, 4; 2, 4, we obtain

$$42 + 32 + 31,$$
$$= l_4 l_2 + l_3 l_2 + l_3 l_1:$$

and so on, the omissions being each three numbers, each four numbers, &c., no two of them being consecutive; thus in 654321 (∴), the omissions are 5, 3, 1, and 6, 4, 2; or the symbol is

$$642 + 531\,,$$
$$= l_6 l_4 l_2 + l_5 l_3 l_1.$$

As an application, a solution of the differential equation $\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x-a)\,y = 0$ is $y = u_0 + u_1 x + u_2 x^2 +$ &c., where $n^2 u_n = a u_{n-1} - u_{n-2}$, and in particular $1^2 u_1 = a u_0$; the equation of differences is thus of the form in question, and retaining l_n in place of its value, $= n^2$, the solution is $u_0 = 1$, $u_1 = l_1 a$, $u_2 = l_2 l_1 a^2 - l_2$, &c. *ut suprà*. The differential equation was considered by the Rev. H. J. Sharpe, who mentioned it to Prof. Stokes.

647.

ON THE QUARTIC SURFACES REPRESENTED BY THE EQUATION, SYMMETRICAL DETERMINANT = 0.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 46—52.]

CONSIDER the equation

$$\nabla = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} = 0,$$

where for the moment $(a, b, \ldots)$ denote linear functions of the coordinates (x, y, z, w). This is a quartic surface having 10 nodes; viz. if we write $(A, B, \ldots)$ for the first minors of the determinant, then the cubic surfaces $A = 0$, $B = 0, \ldots$ have in common 10 points which are nodes of the quartic surface.

Suppose that (a, b, c, f, g, h) are linear functions of the form (x, y); then, observing that every term of ∇ contains as a factor

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

or one of its first minors, it is clear that the line $x = 0$, $y = 0$ is a double line on the surface. But the number of nodes is now less than 10; in fact, writing $(x = 0, y = 0)$, we make each of the first minors of ∇ to vanish; that is, the cubic surfaces, which by their intersection determine the nodes, have in common the line $(x = 0, y = 0)$, and there is a diminution in the number of their common intersections. I do not pursue the enquiry, but pass to a different question.

I, in fact, take the terms $(a, \ldots)$ of the determinant to be homogeneous functions of (x, y, z, w) of the degrees

$$\begin{vmatrix} 0, & 1, & 1, & 0 \\ 1, & 2, & 2, & 1 \\ 1, & 2, & 2, & 1 \\ 0, & 1, & 1, & 0 \end{vmatrix},$$

respectively, viz. a, d, l are constants, g, h, m, n linear functions, and b, c, f quadric functions of the coordinates; $\nabla = 0$ still represents a quartic surface; and it appears by a general formula that the number of nodes is $= 8$. But we can easily show this directly; and further, that the 8 nodes are the intersections of three quadric surfaces; or say that the quartic surface is *octadic*. For denoting as before the first minors by $A, \ldots$, then B, C, F are each of them a quadric function of the coordinates, viz. we have

$$\begin{aligned} B &= d(ac - g^2) - cl^2 - an^2 + 2gln, \\ C &= d(ab - h^2) - am^2 - bl^2 + 2hlm, \\ F &= d(gh - af) + l^2 f + mna - nlh - lmg, \end{aligned}$$

and we have identically

$$BC - F^2 = (ad - l^2)\nabla,$$

so that throwing out the constant factor $ad - l^2$, the equation of the surface is

$$BC - F^2 = 0,$$

and it has 8 nodes, the intersections of the three quadric surfaces $B = 0$, $C = 0$, $F = 0$. By equating to zero any other minor of the determinant ∇, we have a surface passing through the 8 nodes; we have for instance the quartic surface

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

Suppose now (and in all that follows) that, the degrees being as already mentioned, we further assume that b, c, f are quadric functions of the form $(x, y)^2$, g, h linear functions of the form (x, y); then since each term of ∇ contains either

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

or one of its first minors, it is clear that the line $(x = 0, y = 0)$ is a double line on the surface. But in the present case there is not any diminution in the number of the nodes; in fact, writing $x = 0$, $y = 0$, and therefore b, c, f, g, h each $= 0$ (but not $a = 0$), the minors B, C, F none of them vanish; that is, the line $x = 0$, $y = 0$ is not a line on any one of the quadric surfaces, and the quadric surfaces intersect as before in an octad of points.

The equation $\nabla = 0$ thus represents a quartic surface having a double line, and also 8 nodes forming an octad.

We may without loss of generality write $d = 0$; in fact, the determinant is unaltered if we add to the fourth column θ times the first column, and then to the fourth line θ times the first line; the determinant is thus of the original form, but in place of d it has $d + 2\theta l + \theta^2 a$, which by properly determining θ can be made $= 0$. And then changing the original l, m, n, the equation is

$$\nabla = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0.$$

Or, writing for shortness,

$$K = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

and denoting the minors hereof by (a, b, c, f, g, h), then the equation is

$$\nabla = (\mathrm{a, b, c, f, g, h} \between l, m, n)^2 = 0,$$

where the degree of K is 4, and the degrees of a, b, c, f, g, h are 4, 2, 2, 2, 3, 3 respectively, those of l, m, n being 0, 1, 1 respectively.

The nodes are, as before, the intersections of the quadric surfaces $B = 0$, $C = 0$, $F = 0$, viz. (d being now $= 0$) the values are

$$\begin{aligned} -B &= cl^2 - 2gln + an^2, \\ -C &= bl^2 - 2hlm + am^2, \\ F &= fl^2 - glm - hln + amn. \end{aligned}$$

But, according to a previous remark, the nodes lie also on the quartic surface $K = 0$; viz. this is a set of four planes intersecting in the line $x = 0$, $y = 0$.

Now, in general, any plane through the line $x = 0$, $y = 0$ meets the surface in this line twice and in a conic; if the plane is $y = \theta x$, we have

$$a, b, c, f, g, h = a', b'x^2, c'x^2, f'x^2, g'x, h'x,$$

where a', b', c', f', g', h' are functions of θ of the degrees (0, 2, 2, 2, 1, 1) respectively; and thence also

$$\mathrm{a, b, c, f, g, h} = \mathrm{a}'x^4, \mathrm{b}'x^2, \mathrm{c}'x^2, \mathrm{f}'x^2, \mathrm{g}'x^3, \mathrm{h}'x^3,$$

where a′, b′, c′, f′, g′, h′ are functions of θ of the degrees 4, 2, 2, 2, 3, 3 respectively; the equation of the surface thus becomes $(\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' \between lx, m, n)^2 = 0$; viz. this is a quadric equation which, combined with the equation $y - \theta x = 0$, determines the

conic in question. But for each of the planes $K=0$, we have $(\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' \between lx, m, n)^2$ a perfect square, or the conic a two-fold line; we have thus the 8 nodes lying in pairs on four lines, say the four "rays," in the four planes $K=0$ respectively; each of these rays meets the double line $x=0$, $y=0$ in a point; and we have thus on the double line 4 points, which are in fact pinch-points of the surface (as to this presently). It has just been stated that for the plane passing through the nodal line and a ray, the conic is a two-fold line (the ray twice) containing upon it a pair of nodes; more properly, the conic is the point-pair composed of the two nodes.

We can find through the nodes four different plane-pairs; in fact, forming the equation

$$-B+2\lambda F-\lambda^2 C=0,$$

this is

$$l^2(c+2\lambda f+\lambda^2 b)-2l(g+\lambda h)(n+\lambda m)+a(n+\lambda m)^2=0;$$

or, as this may also be written,

$$[a(n+\lambda m)-l(g+\lambda h)]^2+l^2(\mathrm{b}-2\lambda\mathrm{f}+\lambda^2\mathrm{c})=0,$$

where b, c, f and therefore also $\mathrm{b}-2\lambda\mathrm{f}+\lambda^2\mathrm{c}$ are of the form $(x, y)^2$; say that we have $\mathrm{b}-2\lambda\mathrm{f}+\lambda^2\mathrm{c}=(p, q, r \between x, y)^2$, where p, q, r are of course quadric functions of λ; determining λ by the quartic equation $pr-q^2=0$, we have $\mathrm{b}-2\lambda\mathrm{f}+\lambda^2\mathrm{c}$ a perfect square, $=(\alpha x+\beta y)^2$ suppose; and we have thus the plane-pair

$$[a(n+\lambda m)-l(g+\lambda h)]^2-l^2(\alpha x+\beta y)^2=0$$

containing the eight nodes; viz. there are four such plane-pairs. The two planes of a plane-pair intersect in a line called an "axis"; that is, we have four axes each meeting the nodal line; and we have thus also through the nodal line and the four axes respectively four planes, which are "pinch-planes" of the quartic surface (as to this presently).

It has just been seen that the equation $B-2\lambda F+C\lambda^2=0$ (where λ is arbitrary) is expressible in the form

$$[a(n+\lambda m)-l(g+\lambda h)]^2+l^2(p, q, r \between x, y)^2=0,$$

viz. this is the equation of a quadric cone having its vertex on the nodal line at the point $x=0$, $y=0$, $an-lg+\lambda(am-lh)=0$; this is, in fact, a cone *touching* the surface, as at once appears by writing the equation of the cone in the form

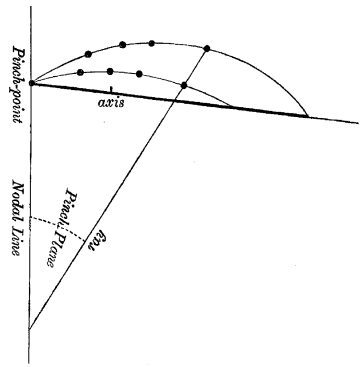
$$\frac{1}{C}\{BC-F^2+(\lambda C-F)^2\}=0,$$

that is,

$$\frac{1}{C}\{-l^2\nabla+(\lambda C-F)^2\}=0;$$

we thus see that, taking for vertex any point whatever on the nodal line, there is a circumscribed quadric cone.

For each of the above-mentioned four values of λ, the quadric cone breaks up into a plane-pair; each plane of the plane-pair is thus a "trope" or plane touching the surface along a conic; viz. this is the conic passing through the intersection of the plane (or say of an axis) with the nodal line and through four nodes of the surface. We have thus 8 tropes, intersecting in pairs in the four axes (and the intersection of each axis with the nodal line being a pinch-point). Moreover, joining the nodes in pairs, we have four rays, each meeting the nodal line, the plane through it and the nodal line being a pinch-plane; this is illustrated in the figure.



As to the pinch-planes and pinch-points, remark first that a plane through the nodal line is in general a bitangent plane, its two points of contact being the points where the conic in such plane meets the nodal line. When the two points of contact come to coincide, the plane is a pinch-plane; viz. this happens when the plane passes through a ray, the conic being then the ray twice repeated. And secondly, at a point on the nodal line there are in general two tangent planes, viz. these are the tangent planes to the quadric cone belonging to such point; when the two tangent-planes come to coincide the point is a pinch-point, and this happens when the point is the intersection of the nodal line with an axis, for then (the quadric cone breaking up into the two tropes through the axis) the two tangent planes become the plane through the axis taken twice.

Each section through the nodal line is a conic, and the polar of the nodal line in regard to this conic is a point; the locus of this point (for different sections through the nodal line) is a right line which may be called simply the "polar." To prove this, considering the section by the plane $y=\theta x$, we have to find the pole of the line $x=0$ in regard to the conic

$$(\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' \between lx, m, n)^2 = 0;$$

this is $lx : m : n = a' : h' : g'$, viz. if $g = g_0x + g_1y$, $h = h_0x + h_1y$, this is

$$lx : m : n = a : g_0 + g_1\theta : h_0 + h_1\theta,$$

or joining hereto the equation $y = \theta x$, we have

$$lx : ly : m : n = a : a\theta : g_0 + g_1\theta : h_0 + h_1\theta,$$

where l, a, g_0, g_1, h_0, h_1 are constants; m, n are linear functions of the coordinates (x, y, z, w). The equations represent, it is clear, a right line which is the polar in question; and they may be written

$$\frac{lx}{a} = \frac{h_1m - g_1n}{h_1g_0 - h_0g_1}, \quad \frac{ly}{a} = -\frac{h_0m - g_0n}{h_1g_0 - h_0g_1}.$$

When the plane passes through a ray, the conic becomes, as was stated, the point-pair composed of the two nodes in such ray; the harmonic in regard to these two points of the intersection of the ray with the nodal line is thus a point on the polar: that is, the polar meets the ray; and the two nodes are situate harmonically in regard to the intersections of the ray with the nodal line and the polar respectively.

The polar may be arrived at in a different manner, viz. if instead of a plane through the nodal line we consider a point on the nodal line, this is the vertex of a circumscribed quadric cone; and taking the polar plane of the nodal line in regard to this cone, then considering the point as variable, the different polar planes all pass through a line which is the polar in question. And hence, taking for the point the intersection of the nodal line with an axis, it appears that the axis meets the polar; and, moreover, that the two tropes through the axis are harmonics in regard to the planes through the axis, and the polar and nodal line respectively.

Collecting the foregoing results, we have a quartic surface as follows:

We have two lines, a nodal line and a polar; meeting each of these, four lines called "rays" and four other lines called "axes." On each ray, harmonically in regard to its intersections with the nodal line and the polar, two nodes of the surface (in all 8 nodes): through each axis, harmonically in regard to the planes through it and the nodal line and the axis respectively, two tropes of the surface (in all 8 tropes). In each trope (or, what is the same thing, in its conic of contact) are 4 nodes; through each node (or, what is the same thing, touching its tangential quadricone) are 4 tropes; the relation of the nodes and tropes may be thus represented, viz. taking the pairs of nodes to be 1, 2; 3, 4; 5, 6; 7, 8; and those of tropes to be I, II; III, IV; V, VI; VII, VIII; then we have

	I	II	III	IV	V	VI	VII	VIII
1	.		.		.		.	
2		.		.		.		.
3	.		.			.		.
4		.		.	.		.	
5	.			.	.			.
6		.	.			.	.	
7	.			.		.	.	
8		.	.		.			.

viz. reading horizontally or vertically, the dots show the tropes through each node, or the nodes in each trope.

The plane through any ray and the nodal line is a pinch-plane of the surface, its point of contact being the intersection of the ray with the nodal line; and the intersection of each axis with the nodal line is a pinch-point of the surface, the tangent plane being the plane through the axis and the nodal line; the surface has thus 4 pinch-planes and 4 pinch-points.

648.

ALGEBRAICAL THEOREM.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), p. 53.]

I WISH to put on record the following theorem, given by me as a Senate-House Problem, January, 1851.

If $\{\alpha+\beta+\gamma+\ldots\}^p$ denote the expansion of $(\alpha+\beta+\gamma+\ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\ldots$ only in which

$$b+c+d+\ldots \not> p-1, \quad c+d+\ldots \not> p-2, \text{ \&c., \&c.},$$

then

$$x^n = (x+\alpha)^n - n\{\alpha\}^1(x+\alpha+\beta)^{n-1} + \tfrac{1}{2}n(n-1)\{\alpha+\beta\}^2(x+\alpha+\beta+\gamma)^{n-2}$$
$$-\tfrac{1}{6}n(n-1)(n-2)\{\alpha+\beta+\gamma\}^3(x+\alpha+\beta+\gamma+\delta)^{n-3} + \text{ \&c.}$$

The theorem, in a somewhat different and imperfectly stated form, is given, Burg, *Crelle*, t. I. (1826), p. 368, as a generalisation of Abel's theorem,

$$(x+\alpha)^n = x^n + n\alpha(x+\beta)^{n-1} + \tfrac{1}{2}n(n-1)\alpha(\alpha-2\beta)(x+2\beta)^{n-2}$$
$$+\tfrac{1}{6}(n-1)(n-2)(n-3)\alpha(\alpha-3\beta)^2(x+3\beta)^2 + \text{ \&c.}$$

649.

ADDITION TO MR GLAISHER'S NOTE ON SYLVESTER'S PAPER, "DEVELOPMENT OF AN IDEA OF EISENSTEIN."

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 83, 84.]

THE formula (11) [in the Note], under a slightly different form, is demonstrated by me in an addition [263] to Sir J. F. W. Herschel's paper "On the formulæ investigated by Dr Brinkley, &c.," *Phil. Trans.* t. CL., 1860, pp. 321—323. The demonstration is in effect as follows: let u denote a series of the form $1+bx+cx^2+dx^3+\ldots$, and let u^i (where i is positive or negative, integer or fractional) denote the development of the i-th power of u, continued up to the term which involves x^n, the terms involving higher powers of x being rejected; $u^0, u^1, u^2, \ldots$, and generally u^s will denote in like manner the developments of these powers up to the terms involving x^n, or, what is the same thing, they will be the values of u^i corresponding to $i=0, 1, 2, \ldots, s$. By the formula $u^i=1+\frac{i}{1}(u-1)+\frac{i.i-1}{1.2}(u-1)^2+\ldots$ as far as the term involving $(u-1)^n$, u^i is a rational and integral function of i of the degree n, and can therefore be expressed in terms of the values $u^0, u^1, u^2, \ldots, u^n$ which correspond to $i=0, 1, 2, \ldots, n$. Let s have any one of the last-mentioned values, then the expression

$$\frac{i.i-1.i-2\ldots i-n}{i-s}\,\frac{1}{s.s-1\ldots 2.1.-1.-2\ldots-(n-s)},$$

which as regards i is a rational and integral function of the degree n (the factor $i-s$ which occurs in the numerator and denominator being of course omitted), vanishes for each of the values $i=0, 1, 2, \ldots, n$, except only for the value $i=s$, in which case it becomes equal to unity. The required formula is thus seen to be

$$u^i=\Sigma\left\{\frac{i.i-1.i-2\ldots i-n}{i-s}\,\frac{1}{s.s-1\ldots 2.1.-1.-2\ldots-(n-s)}\,u^s\right\},$$

where the summation extends to the several values $s=0, 1, 2, \ldots, n$; or, what is the same thing, it is

$$u^i = \Sigma \left\{ \frac{i \,.\, i-1 \,.\, i-2 \ldots i-n}{i-s} \, \frac{(-)^{n-s} 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots (n-s)} \, u^s \right\},$$

or, changing the sign of i, it is

$$u^{-i} = \Sigma \left\{ \frac{i \,.\, i+1 \,.\, i+2 \ldots i+n}{i+s} \, \frac{(-)^s 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots n-s} \, u^s \right\},$$

where, as before, s has the values $0, 1, 2, \ldots, n$ successively. Or, what is the same thing, we have

$$C_{-i,\,n} = \Sigma \left\{ \frac{i \,.\, i+1 \,.\, i+2 \ldots i+n}{i+s} \, \frac{(-)^s 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots n-s} \, C_{s,\,n} \right\},$$

where the term corresponding to $s=0$, as containing the factor $C_{0,\,n}$ vanishes except in the case $n=0$ (for which it is $=1$); and omitting this evanescent term, this is in fact the formula (11).

650.

ON A QUARTIC SURFACE WITH TWELVE NODES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 103—106.]

WRITE for shortness

$$a = \beta - \gamma, \quad f = \alpha - \delta, \quad af = p,$$
$$b = \gamma - \alpha, \quad g = \beta - \delta, \quad bg = q,$$
$$c = \alpha - \beta, \quad h = \gamma - \delta, \quad ch = r;$$

then, θ being a variable parameter, the surface in question is the envelope of the quadric surface

$$(\alpha + \theta)^2 agh X^2 + (\beta + \theta)^2 bhf Y^2 + (\gamma + \theta)^2 cfg Z^2 + (\delta + \theta)^2 abc W^2 = 0;$$

viz. this is

$$\Sigma \alpha^2 agh X^2 . \Sigma agh X^2 - \Sigma \alpha agh X^2 = 0.$$

There are no terms in X^4, &c.; the coefficient of Y^2Z^2 is

$$\gamma^2 cfg . bfh + \beta^2 bfh . cfg - 2\beta bfh . \gamma cfg,$$

which is

$$= bcf^2gh(\beta - \gamma)^2, \; = a^2bcf^2gh, \; = abcfgh . p.$$

Hence the whole equation divides by $abcfgh$, and throwing out this factor, the result is

$$p(Y^2Z^2 + X^2W^2) + q(Z^2X^2 + Y^2W^2) + r(X^2Y^2 + Z^2W^2) = 0,$$

or, observing that $p + q + r = 0$, this may also be written

$$p(YZ + XW)^2 + q(ZX + YW)^2 + r(XY + ZW)^2 = 0,$$

and also

$$p(YZ-XW)^2+q(ZX-YW)^2+r(XY-ZW)^2=0.$$

The more general equation

$$(p,\ q,\ r,\ l,\ m,\ n\between YZ+XW,\ ZX+YW,\ XY+ZW)^2=0$$

represents a quartic surface (octadic) having the 8 nodes

$$\begin{array}{ll} (1,\ 0,\ 0,\ 0), & (\bar{1},\ 1,\ 1,\ 1), \\ (0,\ 1,\ 0,\ 0), & (1,\ \bar{1},\ 1,\ 1), \\ (0,\ 0,\ 1,\ 0), & (1,\ 1,\ \bar{1},\ 1), \\ (0,\ 0,\ 0,\ 1), & (1,\ 1,\ 1,\ \bar{1}). \end{array}$$

We have

$d_XU=$		$d_YU=$	
$p.$	XW^2+YZW	$p.$	YZ^2+XZW
$q.$	YW^2+YZW	$q.$	YW^2+XZW
$r.$	ZW^2+YZW	$r.$	YX^2+XZW
$l.$	$2XYZ+W(Y^2+Z^2)$	$l.$	$2XYW+Z(W^2+X^2)$
$m.$	$2XYW+Z(W^2+Y^2)$	$m.$	$2YZX+W(Z^2+X^2)$
$n.$	$2XZW+Y(W^2+Z^2)$,	$n.$	$2YZW+X(W^2+Z^2)$,

$d_ZU=$		$d_WU=$	
$p.$	Y^2Z+XYW	$p.$	X^2W+XYZ
$q.$	X^2Z+XYW	$q.$	Y^2W+XYZ
$r.$	W^2Z+XYW	$r.$	Z^2W+XYZ
$l.$	$2ZXW+Y(W^2+X^2)$	$l.$	$2WYZ+X(Y^2+Z^2)$
$m.$	$2YZW+X(W^2+Y^2)$	$m.$	$2WZX+Y(Z^2+X^2)$
$n.$	$2ZXY+W(X^2+Y^2)$,	$n.$	$2WXY+Z(X^2+Y^2)$.

Hence there will be a node

$$\begin{array}{l} 1,\ \bar{1},\ \bar{1},\ 1,\ \text{if } p+q+r+2l-2m-2n=0, \\ \bar{1},\ 1,\ \bar{1},\ 1,\ \ldots\ p+q+r-2l+2m-2n=0, \\ \bar{1},\ \bar{1},\ 1,\ 1,\ \ldots\ p+q+r-2l-2m+2n=0, \\ 1,\ 1,\ 1,\ 1,\ \ldots\ p+q+r+2l+2m+2n=0; \end{array}$$

or say there will be

$$\begin{array}{ll} 1 \text{ of these nodes if} & p+q+r+2l+2m+2n=0, \\ 2\ \ldots\ldots\ldots\ldots\ldots\ldots & p+q+r+2l=0,\ m+n=0, \\ 3\ \ldots\ldots\ldots\ldots\ldots\ldots & p+q+r=2l=-2m=-2n, \\ 4\ \ldots\ldots\ldots\ldots\ldots\ldots & p+q+r=0,\ l=0,\ m=0,\ n=0; \end{array}$$

viz. the surface having the 12 nodes is the original surface

$$p(YZ+XW)^2+q(ZX+YW)^2+r(XY+ZW)^2,$$

where

$$p+q+r=0.$$

The Jacobian of the quadrics

$$YZ+XW=0,\ ZX+YW=0,\ XY+ZW=0,$$

is

$$\left\| \begin{matrix} W, & Z, & Y, & X \\ Z, & W, & X, & Y \\ Y, & X, & W, & Z \end{matrix} \right\| = 0;$$

viz. the equations are

$$\begin{aligned} X^3 - X(Y^2+Z^2+W^2)+2YZW &= 0, \\ Y^3 - Y(Z^2+X^2+W^2)+2ZXW &= 0, \\ Z^3 - Z(X^2+Y^2+W^2)+2XYW &= 0, \\ W^3 - W(X^2+Y^2+Z^2)+2XYW &= 0, \end{aligned}$$

each of which is satisfied in virtue of any one of the pairs of equations

$$\begin{array}{l|l} (Y-Z=0,\ X-W=0) & (Y+Z=0,\ X+W=0), \\ (Z-X=0,\ Y-W=0) & (Z+X=0,\ Y+W=0), \\ (X-Y=0,\ Z-W=0) & (X+Y=0,\ Z+W=0), \end{array}$$

so that the Jacobian curve is, in fact, the six lines represented by these equations.

Any two of the three tetrads form an octad, the 8 points of intersection of three quadric surfaces; a figure representing the relation of the 12 points to each other may be constructed without difficulty.

Each tetrad is a sibi-conjugate tetrad *quoad* the quadric $X^2+Y^2+Z^2+W^2=0$. The three tetrads are not on the same quadric surface.

651.

ON A SPECIAL SURFACE OF MINIMUM AREA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 190—196.]

A VERY remarkable form of the surface of minimum area was obtained by Prof. Schwarz in his memoir "Bestimmung einer speciellen Minimal-fläche," Berlin, 1871, [*Ges. Werke*, t. I., pp. 6—125], crowned by the Academy of Sciences at Berlin. The equation of the surface is

$$1 + \mu\nu + \nu\lambda + \lambda\mu = 0,$$

where λ, μ, ν are functions of x, y, z respectively, viz.

$$x = -\int_{\lambda}^{\infty} \frac{d\theta}{\sqrt{(\frac{3}{4}\theta^4 + \frac{5}{2}\theta^2 + \frac{3}{4})}},$$

and y, z are the same functions of μ, ν respectively. A direct verification of the theorem that this is a surface of minimum area, satisfying, that is, the differential equation

$$r(1 + q^2) - 2pqs + t(1 + p^2) = 0,$$

is given in the memoir; but the investigation may be conducted in quite a different manner, so as to be at once symmetrical and somewhat more general, viz. we may enquire whether there exists a surface of minimum area

$$1 + \mu\nu + \nu\lambda + \lambda\mu = 0,$$

where the determining equations are

$$\lambda'^2 = a\lambda^4 + b\lambda^2 + c,$$
$$\mu'^2 = a\mu^4 + b\mu^2 + c,$$
$$\nu'^2 = a\nu^4 + b\nu^2 + c,$$

$\left(\lambda' = \frac{d\lambda}{dx}, \text{ \&c.}\right)$. I find that the coefficients a, b, c must satisfy four homogeneous quadric equations, which, in fact, admit of simultaneous solution, and that in three distinct ways; viz. assuming $a = 1$, the solutions are

$$\begin{aligned} a &= 1, \quad b = \tfrac{10}{3}, \quad c = 1, \\ a &= 1, \quad b = -2, \quad c = 1, \\ a &= 1, \quad b = -\tfrac{3}{2}, \quad c = -\tfrac{1}{3}; \end{aligned}$$

that is,

$$\lambda'^2 = \lambda^4 + \tfrac{10}{3}\lambda^2 + 1 \left\{= \tfrac{4}{3}\left(\tfrac{3}{4}\lambda^4 + \tfrac{5}{2}\lambda^2 + \tfrac{3}{4}\right)\right\},$$

which gives Schwarz's surface:

$$\lambda'^2 = \lambda^4 - 2\lambda^2 + 1 \quad \text{or} \quad \lambda' = \pm(\lambda^2 - 1),$$

which, it is easy to see, gives only $x + y + z = \text{const.}$; and

$$\lambda'^2 = \lambda^4 - \tfrac{2}{3}\lambda^2 - \tfrac{1}{3}, \quad = (\lambda^2 - 1)(\lambda^2 + \tfrac{1}{3}),$$

which is a surface similar in its nature to Schwarz's surface.

The investigation is as follows: the condition to be satisfied by a surface of minimum area $U = 0$ is

$$(\mathrm{a} + \mathrm{b} + \mathrm{c})(X^2 + Y^2 + Z^2) - (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between X, Y, Z)^2 = 0,$$

where (X, Y, Z) are the first derived coefficients and (a, b, c, f, g, h) the second derived coefficients of U in regard to the coordinates. Considering U as a function of λ, μ, ν, which are functions of x, y, z respectively, and writing (L, M, N) and (a, b, c, f, g, h) for the first and second derived functions of U in regard to λ, μ, ν, also λ', λ'' for the first and second derived functions of λ in regard to x, and so for μ', μ'' and ν', ν'': we have

$$(X, Y, Z) = (L\lambda', M\mu', N\nu'),$$

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}) = (a\lambda'^2 + L\lambda'', b\mu'^2 + M\mu'', c\nu'^2 + N\nu'', f\mu'\nu', g\nu'\lambda', h\lambda'\mu'),$$

and for the particular surface $U = 1 + \mu\nu + \nu\lambda + \lambda\mu = 0$, the values are

$$(L, M, N, a, b, c, f, g, h) = (\mu + \nu, \nu + \lambda, \lambda + \mu, 0, 0, 0, 1, 1, 1).$$

Hence the condition is found to be

$$\begin{aligned} & 2\mu'^2\nu'^2(\lambda + \mu)(\lambda + \nu) \\ + & 2\nu'^2\lambda'^2(\mu + \nu)(\mu + \lambda) \\ + & 2\lambda'^2\mu'^2(\nu + \lambda)(\nu + \mu) \\ - & \lambda''(\mu + \nu)\{(\lambda + \nu)^2\mu'^2 + (\lambda + \mu)^2\nu'^2\} \\ - & \mu''(\nu + \lambda)\{(\mu + \lambda)^2\nu'^2 + (\mu + \nu)^2\lambda'^2\} \\ - & \nu''(\lambda + \mu)\{(\nu + \mu)^2\lambda'^2 + (\nu + \lambda)^2\mu'^2\} = 0, \end{aligned}$$

or say this is

$$2\Sigma\mu'^2\nu'^2(\lambda+\mu)(\lambda+\nu)$$
$$-\Sigma\lambda''(\mu+\nu)\{(\lambda+\nu)^2\mu'^2+(\lambda+\mu)^2\nu'^2\}=0.$$

We have to write in this equation $\lambda'^2=a\lambda^4+b\lambda^2+c$, and therefore $\lambda''=2a\lambda^3+b\lambda$, &c.; the left-hand side, call it Ω, is a symmetrical function of λ, μ, ν, and is consequently expressible as a rational function of

$$p, =\lambda+\mu+\nu,$$
$$q, =\mu\nu+\nu\lambda+\lambda\mu,$$
$$r, =\lambda\mu\nu.$$

We ought to have $\Omega=0$, not identically, but in virtue of the equation $1+q=0$, that is, Ω should divide by $1+q$; or, what is the same thing, Ω should vanish on writing therein $q=-1$.

To effect the reduction as easily as possible, observe that we have $(\lambda+\mu)(\lambda+\nu)=\lambda^2+q$; and therefore

$$\Sigma\mu'^2\nu'^2(\lambda+\mu)(\lambda+\nu)=\Sigma\lambda^2\mu'^2\nu'^2+q\Sigma\mu'^2\nu'^2.$$

Similarly, in the second term,

$$(\mu+\nu)(\lambda+\nu)^2=(\nu+\lambda)(\nu^2+q) \text{ and } (\mu+\nu)(\lambda+\mu)^2=(\mu+\lambda)(\mu^2+q).$$

The complete value of Ω thus is

$$\Omega=2(Aq+B)-[(C+D)q+E+F],$$

where

$$A=\Sigma\lambda^2\mu'^2\nu'^2, \qquad B=\Sigma\mu'^2\nu'^2,$$
$$C=\Sigma\lambda\lambda''(\nu^2\mu'^2+\mu^2\nu'^2), \quad D=\Sigma\lambda''(\nu^2\mu'^2+\mu^2\nu'^2),$$
$$E=\Sigma\lambda\lambda''(\mu'^2+\nu'^2), \qquad F=\Sigma\lambda''(\nu\mu'^2+\mu\nu'^2).$$

We find without difficulty

$$\begin{aligned}
A = \; & a^2(\;\; q^4-4q^2pr+4qr^2+2p^2r^2)\\
& +ab(-2q^3+\;q^2p^2+4qpr-3r^2-2p^3r)\\
& +ac(\;\;4q^2-8qp^2+8pr+2p^4)\\
& +b^2(\;\;q^2-2pr)\\
& +bc(-4q+2p^2)\\
& +c^2(\;\;3),\\
B = \; & a^2(\;\;q^2r^2+2pr^3)\\
& +ab(-4qr^2+2p^3r^2)\\
& +ac(-2q^3+q^2p^2+4qpr-3r^2-2p^3r)\\
& +b^2(\;\;3r^2)\\
& +bc(\;\;2q^2-4pr)\\
& +c^2(-2q+p^2),
\end{aligned}$$

$$
\begin{aligned}
C = \; & a^2 \; (\; 4q^4 - 16q^2pr + 16qr^2 + 8p^2r^2) \\
& + ab \, (- 6q^3 + 3q^2p^2 + 12qpr - 9r^2 - 6p^3r) \\
& + ac \, (\; 8q^2 - 16qp^2 + 16pr + 4p^4) \\
& + b^2 \; (\; 2q^2 - 4pr) \\
& + bc \, (- 4q + 2p^2), \\
D = \; & a^2 \; (\; 2q^2pr - 2qr^2 - 4p^2r^2) \\
& + ab \, (- 4qpr + 2p^3r) \\
& + ac \, (- 4q^2 + 2qp^2 - 2pr) \\
& + b^2 \; (\; 2pr) \\
& + bc \, (\; 2q), \\
E = \; & + a^2 \; (\; 4q^2r^2 - 8pr^3) \\
& + ab \, (- 12qr^2 + 6p^2r^2) \\
& + ac \, (- 4q^3 + 2q^2p^2 + 8qpr - 6r^2 - 4p^3r) \\
& + b^2 \; (\; 6r^2) \\
& + bc \, (\; 2q^2 - 4pr), \\
F = \; & a^2 \; (\; 4pr^3) \\
& + ab \, (\; q^2pr + 3qr^2 - 2p^2r^2) \\
& + ac \, (\; 4q^3 - 12qpr + 12r^2) \\
& + b^2 \; (\; qpr - 3r^2) \\
& + bc \, (- 2q^2 + qp^2 - pr),
\end{aligned}
$$

where in each line the terms are arranged according to their order in p, r.

Substituting, we find

$$
\begin{aligned}
\Omega = \; & a^2 \; (- 2q^3 + 6q^2pr - 8q^2r^2 \qquad) \\
& + ab \, (\; 2q^4 - q^3p^2 - q^2pr + 4qr^2) \\
& + ac \, (\qquad - 2q^2p^2 + 14qpr - 12r^2) \\
& + b^2 \; (\qquad\qquad - 3qpr + 3r^2) \\
& + bc \, (- 2q^2 + qp^2 - 3pr \qquad) \\
& + c^2 \; (\; 2q + 2p^2 \qquad\qquad) ;
\end{aligned}
$$

viz. writing $q = -1$, this is

$$
\begin{aligned}
\Omega = \; & a^2 \; (\; 2 - 6pr - 8r^2) \\
& + ab \, (\; 2 + p^2 - pr - 4r^2) \\
& + ac \, (\qquad - 2p^2 - 14pr - 12r^2) \\
& + b^2 \; (\qquad\qquad 3pr + 3r^2) \\
& + bc \, (- 2 - p^2 - 3pr \qquad) \\
& + c^2 \; (- 2 - 2p^2 \qquad\qquad) ;
\end{aligned}
$$

or, what is the same thing, it is

$$\begin{aligned} = \quad & (\ 2a^2 + 2ab \qquad\qquad - 2bc - 2c^2) \\ & + p^2\,(\qquad ab - 2ac \qquad - bc + 2c^2) \\ & + pr\,(-6a^2 - ab - 14ac + 3b^2 - 3bc \qquad) \\ & + r^2\,(-8a^2 - 4ab - 12ac + 3b^2 \qquad\qquad); \end{aligned}$$

so that, writing for convenience $a = 1$, the equations to be satisfied are

$$\begin{aligned} 2 - 2c^2 + 2(1 - c)\ b &= 0, \\ -2c + 2c^2 + (1 - c)\ b &= 0, \\ -6 - 14c + 3b^2 - (1 + 3c)\ b &= 0, \\ -8 - 12c + 3b^2 - \qquad 4b &= 0. \end{aligned}$$

The first and second are $(1-c)(2+2c+2b) = 0$ and $(1-c)(-2c+b) = 0$; viz. they give $c = 1$, or else $b = -\frac{2}{3}$, $c = \frac{1}{3}$. In the former case, the third and fourth equations each become $3b^2 - 4b - 20 = 0$, that is $(3b-10)(b-2) = 0$; in the latter case, they are satisfied identically; hence we have for a, b, c the three systems of values mentioned at the beginning.

This completes the investigation; but it is interesting to find the values assumed by the other factor of Ω on substituting therein for a, b, c the foregoing several systems of values. We have in general

$$\begin{aligned} \Omega = \quad & -2a^2q^3 + 2abq^4 - 2bcq^2 + 2c^2q \\ & + p^2\,(-abq^3 - 2acq^2 + bcq + 2c^2 \qquad) \\ & + pr\,(6a^2q^3 - abq^2 + 14acq - 3b^2q - 3bc) \\ & + r^2\,(-8a^2q^2 + 4abq - 12ac + 3b^2 \qquad) \\ = \ & -2a^2(q^5+1) \qquad + 2ab(q^4-1) - 2bc(q^2-1) + 2c^2(q+1) \\ & + p^2\ \{-ab\ (q^3+1) - 2ac(q^2-1) + bc(q+1)\} \\ & + pr\ \{\ 6a^2(q^3+1) - ab(q^2-1) + (14ac - 3b^2)(q+1)\} \\ & + r^2\ \{\qquad\qquad -8a^2\ (q^2-1) + 4ab(q+1)\} \\ = \ & (q+1)\left\{\begin{aligned} & -2a^2(q^4-q^3+q^2-q+1) + 2ab(q^3-q^2+q-1) - 2bc(q-1) + 2c^2 \\ & + p^2\ \{-ab(q^2-q+1) - 2ac(q-1) + bc\} \\ & + pr\ \{\ 6a^2(q^2+q+1) - ab(q-1) + (14ac - 3b^2)\} \\ & + r^2\ \{\qquad\qquad -8a^2(q-1) + 4ab\} \end{aligned}\right\}. \end{aligned}$$

Hence writing, first, $a = c = 1$, $b = \frac{10}{3}$, we obtain, after some reductions,

$$\Omega = (q+1)\{-2q(q-1)(q^2 - \tfrac{10}{3}q + 1) + p^2(q-1)(-\tfrac{10}{3}q - 2) + pr(6q^2 - \tfrac{28}{3}q - 10) + r^2 - 8q + \tfrac{64}{3}\};$$

secondly, writing $a = c = 1$, $b = -2$, we obtain

$$\Omega = (q+1)\{-2(q+1)^2(q^2+1) + p^2 . 2(q-1)^2 + 2pr(3q^2 - 2q + 6) - 8r^2q\};$$

and, thirdly, writing $a = 1$, $b = -\frac{1}{3}$, $c = -\frac{2}{3}$, we obtain

$$\Omega = (q+1)\{(-2q^4 + \tfrac{4}{3}q^3 - \tfrac{4}{3}q^2 + \tfrac{8}{9}q) + p^2(-\tfrac{1}{3}q^2 + \tfrac{5}{3}q - \tfrac{13}{9}) + pr(6q^2 - \tfrac{17}{3}q - \tfrac{10}{3}) + r^2(-8q + \tfrac{20}{3})\}.$$

652.

ON A SEXTIC TORSE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 229—235.]

THE torse having for its edge of regression or cuspidal edge the curve defined by the equations $x=\cos\phi$, $y=\sin\phi$, $z=\cos 2\phi$, is an interesting and convenient one for the construction of a model, and it is here considered partly from that point of view.

The edge is a quadriquadric curve, the intersection of the cylinder $x^2+y^2=1$ with the parabolic hyperboloid $z=x^2-y^2$; the cylinder regarded as a cone having its vertex at infinity on the line $x=0$, $y=0$, viz. the vertex is on the hyperboloid, or the curve is a nodal quadriquadric (the node being thus an isolated point at infinity on the line in question), and the torse is consequently of the order $8-2, =6$, viz. it is a sextic torse.

The edge is a bent oval situate on the cylinder $x^2+y^2=1$, such that, regarding ϕ as the azimuth (or angle measured along the circular base from its intersection with the axis of x), the altitude z is given by the equation $z=\cos 2\phi$; viz. there are in the plane xz, or, say in the planes xz, $x'z$, two maxima altitudes $z=1$, and in the plane yz, or, say in the planes yz and $y'z$, two minima altitudes $z=-1$. The sections by these principal planes are, as is seen at once, nodal curves on the surface; they are, in fact, the cubic curves $z=3-\dfrac{2}{x^2}$, viz. here as x increases from ± 1 to $\pm\infty$, z increases from the before-mentioned value 1 to 3, and $z=-3+\dfrac{2}{y^2}$, viz. as y increases from ± 1 to $\pm\infty$, z decreases from the before-mentioned value -1 to -3. The two half-sheets (which meet in the cuspidal edge) intersect each other along these nodal lines, in suchwise that the section of the surface by any axial plane (plane through the line $x=0$, $y=0$) is a curve having a cusp on the cuspidal edge, and such that when the axial plane coincides with either of the principal planes $x=0$, $y=0$, the

two half-branches of the curve coincide together with the portions which lie outside the cylinder $x^2+y^2=1$, in fact, the portions referred to above, of the nodal curve in the plane in question; the portions which lie inside the cylinder are acnodal or isolated curves without any real sheet through them. It may be added, in the way of general description, that the section of the surface by any cylinder $x^2+y^2=c^2$ $(c>1)$ is a curve of the form $z=C\cos(2\theta\pm B)$, θ the angle along the base of the cylinder from the intersection with the axis of x; C, B are functions of c; viz. we have for the two half sheets respectively

$$z=C\cos(2\theta+B) \text{ and } z=C\cos(2\theta-B),$$

each curve having thus the two maxima $+C$, and the two minima $-C$; and the two curves intersect each other at the four points in the two principal planes respectively; viz. the points for which $\theta=0, 90°, 180°, 270°$, and $z=C\cos B, -C\cos B, C\cos B, -C\cos B$ accordingly.

Proceeding to discuss the surface analytically, we have for the equations of a generating line

$$\frac{x-\cos\phi}{-\sin\phi}=\frac{y-\sin\phi}{\cos\phi}=\frac{z-\cos 2\phi}{-2\sin 2\phi}, \quad =\rho \text{ suppose,}$$

or say

$$\begin{aligned} x&=\cos\phi-\rho\sin\phi,\\ y&=\sin\phi+\rho\cos\phi,\\ z&=\cos 2\phi-2\rho\sin 2\phi, \end{aligned}$$

which equations, considering therein ρ, ϕ as arbitrary parameters, determine the surface.

Writing $x=0$, we find $y=\dfrac{1}{\sin\phi}$, and then $z=-3+2\sin^2\phi$, viz. we have

$$x=0,\quad z=-3+\frac{2}{y^2}, \text{ for section in plane } yz;$$

and, similarly, writing $y=0$, we find $x=\dfrac{1}{\cos\phi}$, and then $z=3-2\cos^2\phi$, viz.

$$y=0,\quad z=3-\frac{2}{x^2} \text{ for section by plane } xz.$$

By what precedes, these are nodal curves, crunodal for the portions

$$(y=\pm 1 \text{ to } \pm\infty,\ z=-1 \text{ to } -3) \text{ and } (x=\pm 1 \text{ to } \pm\infty,\ z=1 \text{ to } 3)$$

respectively, acnodal for the remaining portions $y<\pm 1$, $x<\pm 1$ respectively.

Writing $x=r\cos\theta$, $y=r\sin\theta$, so that the coordinates of a point on the surface are r, θ, z, where $r=\sqrt{(x^2+y^2)}$ is the projected distance, θ is the azimuth from the axis of x, and z is the altitude, we have

$$\begin{aligned} r\cos\theta&=\cos\phi-\rho\sin\phi,\\ r\sin\theta&=\sin\phi+\rho\cos\phi,\\ z&=\cos 2\phi-2\rho\sin 2\phi. \end{aligned}$$

We have $r^2 = 1 + \rho^2$; and thence also, if $\tan\alpha = 2\rho$, $= \pm 2\sqrt{(r^2-1)}$, that is,

$$\cos\alpha = \frac{1}{\sqrt{(4r^2-3)}}, \quad \sin\alpha = \pm\frac{2\sqrt{(r^2-1)}}{\sqrt{(4r^2-3)}},$$

then

$$z = \sqrt{(4r^2-3)}\cos(2\phi+\alpha),$$

showing that for a given value of r (or section by the cylinder $x^2+y^2=r^2$) the maximum and minimum values of z are $z = \pm\sqrt{(4r^2-3)}$.

But proceeding to eliminate ϕ, we find

$$r^2\cos 2\theta = (1-\rho^2)\cos 2\phi - 2\rho\sin 2\phi,$$
$$r^2\sin 2\theta = 2\rho\cos 2\phi + (1-\rho^2)\sin 2\phi;$$

or multiplying these by $1+3\rho^2$ and $2\rho^2$ and adding

$$r^2\{(1+3\rho^2)\cos 2\theta + 2\rho^3\sin 2\theta\} = (1+\rho^2)^2(\cos 2\phi - 2\rho\sin 2\phi),$$

that is,

$$r^2\{(3r^2-2)\cos 2\theta \pm 2(r^2-1)^{\frac{3}{2}}\sin 2\theta\} = r^4 z;$$

or, finally,

$$r^2 z = (3r^2-2)\cos 2\theta \pm 2(r^2-1)^{\frac{3}{2}}\sin 2\theta,$$

which is the equation of the surface in terms of the coordinates r, θ, z.

Observing that $(3r^2-2)^2 + 4(r^2-1)^3 = r^4(4r^2-3)$, we may write

$$r^2\sqrt{(4r^2-3)}\cos\beta = 3r^2-2,$$
$$r^2\sqrt{(4r^2-3)}\sin\beta = 2(r^2-1)^{\frac{3}{2}},$$

and therefore also

$$\tan\beta = \frac{2(r^2-1)^{\frac{3}{2}}}{3r^2-2},$$

and the equation thus becomes

$$z = \sqrt{(4r^2-3)}\cos(2\theta \mp \beta),$$

where z is the altitude belonging to the azimuth θ in the cylindrical section, radius r. The maxima and minima altitudes are $\pm\sqrt{(4r^2-3)}$, and these correspond to the values $\theta = \pm\frac{1}{2}\beta$, $\frac{1}{2}\pi \pm \frac{1}{2}\beta$, $\pi \pm \frac{1}{2}\beta$, $\frac{3}{2}\pi \pm \frac{1}{2}\beta$; it is to be further noticed that when $r=1$, we have $\beta = 0$, but as r increases and becomes ultimately infinite, β increases to $\frac{1}{2}\pi$, that is, $\frac{1}{2}\beta$ increases from 0 to $\frac{1}{4}\pi$.

It may be noticed that the surface is a peculiar kind of deformation, obtained by giving proper rotations to the several cylindrical sections of the surface $z = \sqrt{(4r^2-3)}\cos 2\theta$; viz. in rectangular coordinates this is $r^2 z = \sqrt{(4r^2-3)}(x^2-y^2)$, that is,

$$(x^2+y^2)^2 z^2 - \{4(x^2+y^2)-3\}(x^2-y^2)^2 = 0.$$

To obtain the equation in rectangular coordinates, we have

$$\left\{r^2 z - \frac{3r^2-2}{r^2}(x^2-y^2)\right\}^2 - 16(r^2-1)^3\frac{x^2y^2}{r^4} = 0,$$

viz. this is

$$r^4 z^2 - 2z(3r^2-2)(x^2-y^2) + (3r^2-2)^2\left(1 - \frac{4x^2y^2}{r^4}\right) - 16(r^2-1)^3\frac{x^2y^2}{r^4} = 0,$$

or, what is the same thing, it is

$$r^4z^2 - 2z(3y^2-2)(x^2-y^2) + (3r^2-2)^2 - \frac{4x^2y^2}{r^4}\{4(r^2-1)^2 + (3r^2-2)^2\} = 0,$$

viz. the term in { } being $r^4(4r^2-3)$, this is

$$r^4z^2 - 2z(3r^2-2)(x^2-y^2) + (3r^2-2)^2 - 4x^2y^2(4r^2-3) = 0,$$

or say

$$z^2(x^2+y^2)^2 - 2z(3x^2+3y^2-2)(x^2-y^2) + (3x^2+3y^2-2)^2 - 4x^2y^2(4x^2+4y^2-3) = 0.$$

This may also be written

$$\{z(x^2-y^2) - 3x^2 - 3y^2 + 2\}^2 + 4x^2y^2(z^2 - 4x^2 - 4y^2 + 3) = 0,$$

a form which puts in evidence the nodal curves

$$x = 0,\ xy^2 = -3y^2 + 2, \text{ and } y = 0,\ zx^2 = 3x^2 - 2.$$

It shows also that the quadric cone $z^2 - 4x^2 - 4y^2 + 3 = 0$ touches the surface along the curve of intersection with the surface $z(x^2-y^2) - 3(x^2+y^2) + 2 = 0$. This is, in fact, the curve of maxima and minima of the cylindrical sections, viz. reverting to the form $z = \sqrt{(4r^2-3)}\cos(2\theta \mp \beta)$, or, if for greater clearness, attending only to one sheet of the surface, we write it $z = \sqrt{(4r^2-3)}\cos(2\theta - \beta)$, we have a maximum, $z = \sqrt{(4r^2-3)}$, for $2\theta = \beta$ (or $2\pi + \beta$), giving

$$\cos 2\theta = \cos\beta,\ = \frac{3r^2-2}{r^2\sqrt{(4r^2-3)}},\ = \frac{3r^2-2}{r^2 z}:$$

and a minimum, $z = -\sqrt{(4r^2-3)}$, for $2\theta = \pi + \beta$ (or $3\pi + \beta$), giving

$$\cos 2\theta = -\cos\beta = -\frac{3r^2-2}{r^2\sqrt{(4r^2-3)}},\ = \frac{3r^2-2}{r^2 z};$$

viz. the locus is $z^2 = 4(r^2-3)$, $z(x^2-y^2) = 3r^2-2$; and for $z = \sqrt{(4r^2-3)}\cos(2\theta+\beta)$ we find the same locus, viz. the equations of the locus are

$$z^2 - 4x^2 - 4y^2 + 3 = 0,\quad z(x^2-y^2) - 3x^2 - 3y^2 + 2 = 0,$$

as above.

To put in evidence the cuspidal edge, write for a moment $\zeta = z - x^2 + y^2$, the equation becomes

$$\{\zeta(x^2-y^2) + (r^2-1)(r^2-2) - 4x^2y^2\}^2 + 4x^2y^2\{\zeta^2 + 2\zeta(x^2-y^2) + (r^2-1)(r^2-3) - 4x^2y^2\} = 0;$$

viz. this is

$$\zeta^2r^4 + 2\zeta(x^2-y^2)(r^2-1)(r^2-2) + (r^2-1)^2(r^2-2)^2 - 4x^2y^2(r^2-1)^2 = 0,$$

or writing the last term thereof in the form

$$-\{r^2 - (x^2-y^2)^2\}(r^2-1)^2,$$

and then putting $r^2 = 1 + U$, the equation is

$$\zeta^2(1 + 2U + U^2) + 2\zeta U(U-1)(x^2-y^2) + U^2(U-1)^2 - U^2\{(U+1)^2 - (x^2-y^2)^2\} = 0;$$

viz. this is

$$\{\zeta - U(x^2-y^2)\}^2 + 2U\{\zeta^2 + \zeta U(x^2-y^2) - 2U^2\} + \zeta^2U^2 = 0,$$

showing the cuspidal edge $\zeta=0$, $U=0$, viz. $z=x^2-y^2$, $x^2+y^2=1$. Moreover, along the cuspidal edge the surface is touched by $\zeta-U(x^2-y^2)=0$, that is, by $z-(x^4-y^4)=0$; and at the points where this tangent surface again meets the surface we have $(x^2-y^2)^2(x^2+y^2+3)-4=0$; viz. the surface contains upon itself the curve represented by this last equation, and $z-(x^4-y^4)=0$.

As a verification, in the form

$$\{z(x^2-y^2)-3x^2-3y^2+2\}^2+4x^2y^2(z^2-4x^2-4y^2+3)=0$$

of the equation of the surface, write $z=x^4-y^4$. If for a moment $x^2+y^2=\lambda$, $x^2-y^2=\mu$, then the value of z is $z=\lambda\mu$, and the equation becomes

$$(\lambda\mu^2-3\lambda+2)^2+(\lambda^2-\mu^2)(\lambda^2\mu^2-4\lambda+3)=0,$$

that is,

$$\mu^2(\lambda^4-6\lambda^2+8\lambda-3)-4\lambda^3+12\lambda^2-12\lambda+4=0;$$

or, what is the same thing,

$$(\lambda-1)^3\{\mu^2(\lambda+3)-4\}=0,$$

so that we have $(\lambda-1)^3=0$, or else $\mu^2(\lambda+3)-4=0$; viz. $(x^2+y^2-1)^3=0$, or else $(x^2-y^2)^2(x^2+y^2+3)-4=0$, agreeing with the former result.

In polar coordinates, the surface is touched along the cuspidal edge by the surface $z=r^4\cos 2\theta$, and where this again meets the surface we have $r^4(r^2+3)\cos^2 2\theta-4=0$.

For the model, taking the unit to be 1 inch, I suppose that for the edge of regression we have

$$x=2\cos\phi,\ y=2\sin\phi,\ z=5+(\cdot45)\cos 2\phi;$$

viz. the curve is situate on a cylinder radius 2 inches. And I construct in zinc-plate the cylindric sections, or say the templets, for one sheet of the surface, for the several radii 2, 3,.., 8 inches; taking the radius as k inches, the circumference of the cylinder, or entire base of the flattened templet, is $=2k\pi$; and the altitude, writing 2θ in place of $2\theta-\beta$ as above, is given by the formula $z=5+(\cdot45)\sqrt{(k^2-3)}\cos 2\theta$, so that the half altitude of the wave is $=(\cdot45)\sqrt{(k^2-3)}$; having this value, the curve is at once constructed geometrically. We have, moreover, $\cos\beta=\dfrac{3k^2-8}{k^2\sqrt{(k^2-3)}}$; the numerical values then are

k	$2k\pi$	$(\cdot45)\sqrt{(k^2-3)}$	$\dfrac{3k^2-8}{k^2\sqrt{(k^2-3)}}$	$\frac{1}{2}\beta$
2	12·57	0·45	1·00	0°
3	18·85	1·10	·86	15
4	25·13	1·62	·69	23
5	31·42	2·11	·57	27½
6	37·70	2·59	·48	30½
7	43·98	3·05	·42	32½
8	50·27	3·51	·36	34

the altitudes in the successive templets being thus included between the limits $5\pm0\cdot45$, $5\pm1\cdot10$,.., $5\pm3\cdot51$.

653.

ON A TORSE DEPENDING ON THE ELLIPTIC FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 235—241.]

ON attempting to cover with paper one half-sheet of the foregoing sextic torse, [652], I found that the paper assumed approximately the form of a circular annulus of an angle exceeding 360°, and this led me to consider the general theory of the construction of a torse in paper, and, in particular, to consider the torses such that when developed into a plane the edge of regression becomes a circular arc. It is scarcely necessary to remark that, to construct in paper a circular annulus of an angle exceeding 360°, we have only to take a complete annulus, cut it along a radius, and then insert (gumming it on to the two terminal radii) a portion of an equal circular annulus; drawing from each point of the inner circular boundary a half-tangent, and considering these half-tangents as rigid lines, the paper will bend round them so as to form the half-sheet of a torse having for its edge of regression this inner boundary, which will assume the form of a closed curve with two equal and opposite maxima and two equal and opposite minima, described on a cylinder, and being *approximately* such as the curve given by the equations

$$x = \cos\theta,\ y = \sin\theta,\ z = m\cos 2\theta.$$

Considering, in general, an arc PQ (without inflexions) of any curve, and drawing at the consecutive points P, P', P'', &c. the several half-tangents PT, $P'T'$, $P''T''$, ..., then, considering these as rigid lines and bending the paper round them, we have the half-sheet of a torse, having for its edge of regression the curve in question now bent into a curve of double curvature. It is, moreover, clear that the edge of regression has at each point thereof the same radius of absolute curvature as the original plane curve; in fact, if in the plane curve $PP' = ds$, and the angle $T'PT$ between the consecutive half-tangents PT and $P'T'$ be $= d\phi$, these quantities ds and

$d\phi$ remain unaltered in the curve of double curvature; and the radius of absolute curvature is given by the equation $\rho d\phi = ds$. In particular when, as above, the arc is a circular one, say of radius $=\alpha$, then, however the paper is bent, the edge of regression has at each point thereof the radius of absolute curvature $=\alpha$.

Consider on any given surface, at a given point P thereof, and in a given direction, an element of length PP', then (under the restrictions presently mentioned) we can determine the consecutive element $P'P''$, such that the curve $PP'P''\ldots$ shall have at P a radius of absolute curvature $=\alpha$; in fact, r being the radius of curvature of the normal section of the surface through the element PP', the radius of curvature of the section inclined at an angle θ to the normal section is $=r\cos\theta$; so that we have only to take the section at the inclination θ, $=\cos^{-1}\frac{\alpha}{r}$ to the normal section, and we have the consecutive element $P'P''$ such that the radius of absolute curvature of the curve $PP'P''$ is $=\alpha$. The necessary restriction, of course, is that $r>\alpha$; thus, if at the given point P the two principal radii of curvature are of the same sign (to fix the ideas, let the two principal radii and also α be each of them positive), then we may on the surface determine a direction PQ, for which the radius of curvature of the normal section is $=\alpha$; and then the direction of the element PP' may be any direction between PQ and the direction PR, corresponding to the greatest of the two principal radii.

Having obtained the element $P'P''$, we may, if the radius of absolute curvature at P' be given, construct the next element $P'P''$, and so on; that is to say, on a given surface starting from a given point P and given initial direction PP', we can (under a restriction, as above, as to the curvature at the different points of the surface) construct a curve having at the successive points thereof given values of the radius of absolute curvature; viz., the value may be given either as a function of the coordinates of the point on the surface, or as a function of the length of the curve measured say from the initial point P; it is in this last manner that in what follows the value of the radius of absolute curvature is assumed to be given.

We may thus, taking on paper an arc PQ with its half-tangents, apply it to a given surface, the point P to a given point, and the infinitesimal arc PP' to an element PP' in a given direction from the given point; and we thus obtain the half-sheet of a torse having for its edge of regression a determinate curve upon the surface. In particular, the arc PQ may be circular of the radius α, and the surface be a circular cylinder of radius a; and we thus obtain the torse having for edge of regression a curve on the cylinder radius α, and such that the radius of absolute curvature is at each point $=a$. There are three cases according as $a>\alpha$, $a=\alpha$, or $a<\alpha$; it is to be remarked that if $a>\alpha$, then the curve must at each point cut the generating line of the cylinder at an angle not exceeding $\cos^{-1}\frac{\alpha}{a}$, but that in the other two cases the angle may have any value whatever; and, further, that in every case when the angle is $=0$, viz. when the curve touches a generating line of the cylinder, then the osculating plane of the curve coincides with the tangent plane of the cylinder.

The analytical theory is very simple. Taking x, y, z as functions of the length s, we have

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1;$$

the condition, which expresses that the radius of absolute curvature is $= a$, then is

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{1}{a^2}.$$

By what precedes, the point (x, y, z) may be taken to be upon a given surface, say upon the cylinder $x^2 + y^2 = \alpha^2$; and we may then write $x = \alpha \cos\theta$, $y = \alpha \sin\theta$. Taking instead of s any independent variable u whatever, and using accents to denote the derived functions in regard to u, the equations become

$$x'^2 + y'^2 + z'^2 = s'^2,$$

$$x''^2 + y''^2 + z''^2 - s''^2 = \frac{1}{a^2} s'^4,$$

$$x = \alpha \cos\theta,\ y = \alpha \sin\theta.$$

From the last two equations we obtain

$$x'^2 + y'^2 = \alpha^2\theta'^2,\ x''^2 + y''^2 = \alpha^2(\theta''^2 + \theta'^4),$$

and the first two equations thus become

$$\alpha^2\theta'^2 + z'^2 = s'^2,$$

$$\alpha^2(\theta''^2 + \theta'^4) + z''^2 - s''^2 = \frac{1}{a^2} s'^4,$$

and from the first of these we find

$$s'' = \frac{\alpha^2\theta'\theta'' + z'z''}{(\alpha^2\theta'^2 + z'^2)^{\frac{1}{2}}},$$

whence the second equation is

$$\alpha^2(\theta''^2 + \theta'^4) + z''^2 - \frac{(\alpha^2\theta'\theta'' + z'z'')^2}{(\alpha^2\theta'^2 + z'^2)} = \frac{(\alpha^2\theta'^2 + z'^2)^2}{a^2},$$

or reducing, this is

$$(\alpha^2\theta'^2 + z'^2)(\theta''^2 + \theta'^4) + (\theta'^2 z''^2 - 2\theta'\theta'' z' z'' - \alpha^2\theta'^2\theta''^2) = \frac{1}{a^2\alpha^2}(\alpha^2\theta'^2 + z'^2)^3.$$

Taking here θ as the independent variable, we have $\theta' = 1$, $\theta'' = 0$, and the equation becomes

$$(\alpha^2 + z'^2) + z''^2 = \frac{1}{a^2\alpha^2}(\alpha^2 + z'^2)^3;$$

or, what is the same thing,

$$z''^2 = \frac{1}{a^2\alpha^2}(\alpha^2 + z'^2)^3 - (\alpha^2 + z'^2)^2.$$

Write here

$$\alpha^2 + z'^2 = \Omega^2,$$

then

$$z'' = \frac{\Omega\Omega'}{\sqrt{(\Omega^2 - \alpha^2)}},$$

and the equation becomes

$$\frac{\Omega'^2}{\Omega^2 - \alpha^2} = \frac{\Omega^4}{a^2\alpha^2} - 1,$$

or say

$$\frac{a\alpha d\Omega}{\sqrt{(\Omega^2 - \alpha^2 \,.\, \Omega^4 - a^2\alpha^2)}} = d\theta,$$

viz. this equation determines Ω as a function of θ, and we then have

$$\left\{\begin{aligned} ds &= \Omega d\theta, \\ dz &= \sqrt{(\Omega^2 - \alpha^2)}\, d\theta, \\ x &= \alpha \cos\theta, \\ y &= \alpha \sin\theta, \end{aligned}\right.$$

equations which determine x, y, z, s as functions of the parameter θ, and give thus the edge of regression of the torse in question.

It is clear that the formulæ are very much simplified in the case $a = \alpha$, where the radius of absolute curvature a is equal to the radius α of the cylinder; but it is worth while to develope the general case somewhat further.

Considering the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, to the modulus $k\,(= k') = \frac{1}{\sqrt{(2)}}$, assume

$$\Omega = -\frac{\sqrt{(a\alpha)}}{k} \frac{\operatorname{dn} u}{\operatorname{sn} u},$$

then

$$d\Omega = -\frac{\sqrt{(a\alpha)}}{k} \frac{\operatorname{cn} u\, du}{\operatorname{sn}^2 u},$$

$$\begin{aligned} \Omega^2 - \alpha^2 &= \frac{a\alpha}{k^2 \operatorname{sn}^2 u}\left(\operatorname{dn}^2 u - \frac{\alpha}{a} k^2 \operatorname{sn}^2 u\right), \\ &= \frac{a\alpha}{k^2 \operatorname{sn}^2 u}\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\}, \\ \Omega^4 - a^2\alpha^4 &= \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u}(\operatorname{dn}^4 u - k^4 \operatorname{sn}^4 u), \\ &= \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u}(1 - 2k^2 \operatorname{sn}^2 u), = \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u} \operatorname{cn}^2 u, \end{aligned}$$

and hence

$$d\theta = \frac{k^2 \operatorname{sn} u\, du}{\sqrt{\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\}}},$$

$$ds = \frac{k\sqrt{(a\alpha)} \operatorname{dn} u\, du}{\sqrt{\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\}}},$$

$$dz = k\sqrt{(a\alpha)}\, du.$$

We have thus $z = k\sqrt{(a\alpha)}\, u$, no constant of integration being required, viz. u is a mere constant multiple of z: and the first and second equations then give s and θ as functions of u, that is, of z; but it is obviously convenient to retain u instead of expressing it in terms of z. As regards the form of these integrals observe that, writing $\operatorname{sn} u = \lambda$, we have

$$du = \frac{d\lambda}{\sqrt{\{1-\lambda^2 . 1 - k^2\lambda^2\}}},$$

and thence

$$d\theta = \frac{k^2\lambda d\lambda}{\sqrt{\left\{1-\lambda^2 . 1 - k^2\lambda^2 . 1 - \left(1+\frac{\alpha}{a}\right)k^2\lambda^2\right\}}},$$

$$ds = \frac{k\sqrt{(a\alpha)}\, d\lambda}{\sqrt{\left\{1-\lambda^2 . 1 - \left(1+\frac{\alpha}{a}\right)k^2\lambda^2\right\}}},$$

each of which is in fact reducible to elliptic integrals, but I do not further pursue this general case.

In the particular case $a = \alpha$, we have

$$1 - \left(1+\frac{\alpha}{a}\right)k^2 \operatorname{sn}^2 u = \operatorname{cn}^2 u,$$

and the equations become

$$d\theta = \frac{k^2 \operatorname{sn} u\, du}{\operatorname{cn} u},\quad ds = \frac{k\alpha \operatorname{dn} u\, du}{\operatorname{cn} u},$$

which admit of immediate integration; viz. we have

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'},$$

or determining the constant so that θ may vanish for $u = 0$, say

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right);$$

and

$$s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right);$$

viz. to verify these results we have

$$\frac{d\theta}{du} = \tfrac{1}{2}\frac{k^2}{k'} . k^2 \operatorname{sn} u \operatorname{cn} u\left\{\frac{1}{\operatorname{dn} u + k'} - \frac{1}{\operatorname{dn} u - k'}\right\},$$

$$= \frac{k^4 \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u - k'^2},\ = k^2\frac{\operatorname{sn} u}{\operatorname{cn} u},$$

and

$$\frac{ds}{du} = \tfrac{1}{2}k\alpha . \operatorname{cn} u \operatorname{dn} u\left\{\frac{1}{1+\operatorname{sn} u} + \frac{1}{1-\operatorname{sn} u}\right\},$$

$$= \frac{k\alpha \operatorname{cn} u \operatorname{dn} u}{1-\operatorname{sn}^2 u},\ = \frac{k\alpha \operatorname{dn} u}{\operatorname{cn} u}.$$

Hence, recurring to the original equations, and writing for convenience $a = \alpha = 1$, we see that a solution of the simultaneous equations

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = 1,$$

is

$$x = \cos\theta,\ y = \sin\theta,\ z = ku,$$

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),\ s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right),$$

where, as before, $k = k' = \dfrac{1}{\sqrt{(2)}}$.

Restoring the radius α, and writing the system in the form

$$x = \alpha\cos\theta,\ y = \alpha\sin\theta,\ z = k\alpha u,$$

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),\ s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right),$$

we see that, as u passes from $u = 0$ to $u = K$, and therefore z from $z = 0$ to $z = k\alpha K$ $\left(K \text{ the complete function } F_1\left\{\frac{1}{\sqrt{(2)}}\right\}\right)$, then θ and s each pass from 0 to ∞; and, similarly, as u passes from $u = 0$ to $u = -K$, that is, as z passes from 0 to $-k\alpha K$, then θ passes from 0 to ∞, and s from $s = 0$ to $s = -\infty$; viz. the curve makes in each direction an infinity of revolutions about the cylinder. Developing the cylinder, $\alpha\theta$ becomes an x-coordinate; viz. we have thus the plane curve

$$z = k\alpha u,$$

$$x = \tfrac{1}{2}\frac{k^2\alpha}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),$$

which is a curve extending from the origin in the direction x positive, to touch at infinity the two parallel asymptotes $z = \pm k\alpha K$; and conversely, when such a plane curve is wound about the cylinder, there will be in each direction an infinity of revolutions round the cylinder.

654.

ON CERTAIN OCTIC SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 249—264.]

I. CONSIDER the torse generated by the tangents of the quadriquadric curve, the intersection of the two quadric surfaces

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}w^2 = 0,$$
$$\mathrm{a}'x^2 + \mathrm{b}'y^2 + \mathrm{c}'z^2 + \mathrm{d}'w^2 = 0;$$

then, writing

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = a', \quad \mathrm{ad}' - \mathrm{a}'\mathrm{d} = f',$$
$$\mathrm{ca}' - \mathrm{c}'\mathrm{a} = b', \quad \mathrm{bd}' - \mathrm{b}'\mathrm{d} = g',$$
$$\mathrm{ab}' - \mathrm{a}'\mathrm{b} = c', \quad \mathrm{cd}' - \mathrm{c}'\mathrm{d} = h',$$

and therefore

$$a'f' + b'g' + c'h' = 0,$$

the equation of the torse, writing for greater convenience (a, b, c, f, g, h) in place of (a', b', c', f', g', h'), but understanding these letters as signifying the accented letters (a', b', c', f', g', h'), is

$$\begin{aligned}
& a^4f^2y^4z^4 + b^4g^2z^4x^4 + c^4h^2x^4y^4 \\
& + a^2f^4x^4w^4 + b^2g^4y^4w^4 + c^2h^4z^4w^4 \\
& + 2b^2c^2ghx^4y^2z^2 - 2c^2f^2ahx^4y^2w^2 + 2b^2f^2agx^4z^2w^2 \\
& + 2c^2a^2hfy^4z^2x^2 - 2a^2g^2bfy^4z^2w^2 + 2c^2g^2bhy^4x^2w^2 \\
& + 2a^2b^2fgz^4x^2y^2 - 2b^2h^2cgz^4x^2w^2 + 2a^2h^2cfz^4y^2w^2 \\
& - 2bcg^2h^2w^4y^2z^2 - 2cah^2f^2w^4z^2x^2 - 2abf^2g^2w^4x^2y^2 \\
& + 2(bg - ch)(ch - af)(af - bg)x^2y^2z^2w^2 = 0.
\end{aligned}$$

If in this equation we write

$$a'^2f' = a, \quad a'f'^2 = f,$$
$$b'^2g' = b, \quad b'g'^2 = g,$$
$$c'^2h' = c, \quad c'h'^2 = h;$$

and therefore

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = \frac{a}{\sqrt[3]{(af)}}, \quad \mathrm{ad}' - \mathrm{a}'\mathrm{d} = \frac{f}{\sqrt[3]{(af)}},$$

$$\mathrm{ca}' - \mathrm{c}'\mathrm{a} = \frac{b}{\sqrt[3]{(bg)}}, \quad \mathrm{bd}' - \mathrm{b}'\mathrm{d} = \frac{g}{\sqrt[3]{(bg)}},$$

$$\mathrm{ab}' - \mathrm{a}'\mathrm{b} = \frac{c}{\sqrt[3]{(ch)}}, \quad \mathrm{cd}' - \mathrm{c}'\mathrm{d} = \frac{h}{\sqrt[3]{(ch)}},$$

and consequently

$$(af)^{\frac{1}{3}} + (bg)^{\frac{1}{3}} + (ch)^{\frac{1}{3}} = 0;$$

then the equation becomes

$$\begin{aligned}
& a^2y^4z^4 + b^2z^4x^4 + c^2x^4y^4 \\
& + f^2x^4w^4 + g^2y^4w^4 + h^2z^4w^4 \\
& + 2bcx^4y^2z^2 - 2cfx^4y^2w^2 + 2bfx^4z^2w^2 \\
& + 2cay^4z^2x^2 - 2agy^4z^2w^2 + 2cgy^4x^2w^2 \\
& + 2abz^4x^2y^2 - 2bhz^4x^2w^2 + 2ahz^4y^2w^2 \\
& - 2ghw^4y^2z^2 - 2hfw^4z^2x^2 - 2fgw^4x^2y^2 \\
& + 2\{(bg)^{\frac{1}{3}} - (ch)^{\frac{1}{3}}\}\{(ch)^{\frac{1}{3}} - (af)^{\frac{1}{3}}\}\{(af)^{\frac{1}{3}} - (ch)^{\frac{1}{3}}\}\, x^2y^2z^2w^2 = 0.
\end{aligned}$$

This same equation, without the relation

$$(af)^{\frac{1}{3}} + (bg)^{\frac{1}{3}} + (ch)^{\frac{1}{3}} = 0,$$

and with an arbitrary coefficient for $x^2y^2z^2w^2$; or say, the equation

$$\begin{aligned}
& a^2y^4z^4 + b^2z^4x^4 + c^2x^4y^4 \\
& + f^2x^4w^4 + g^2y^4w^4 + h^2z^4w^4 \\
& + 2bcx^4y^2z^2 - 2cfx^4y^2w^2 + 2bfx^4z^2w^2 \\
& + 2cay^4z^2x^2 - 2agy^4z^2w^2 + 2cgy^4x^2w^2 \\
& + 2abz^4x^2y^2 - 2bhz^4x^2w^2 + 2ahz^4y^2w^2 \\
& - 2ghw^4y^2z^2 - 2hfw^4z^2x^2 - 2fgw^4x^2y^2 \\
& + 2kx^2y^2z^2w^2 = 0,
\end{aligned}$$

where a, b, c, f, g, h, k are arbitrary coefficients, is the general equation of an octic surface having the four nodal curves

$$\begin{aligned}
x &= 0, \quad . \quad hz^2w^2 - gw^2y^2 + ay^2z^2 = 0, \\
y &= 0, \quad -hz^2w^2 \quad . \quad + fw^2x^2 + bz^2x^2 = 0, \\
z &= 0, \quad gy^2w^2 - fw^2x^2 \quad . \quad + cx^2y^2 = 0, \\
w &= 0, \quad -ay^2z^2 - bz^2x^2 - cx^2y^2 \quad . \quad = 0.
\end{aligned}$$

In fact, the equation of the surface may be written in the form

$$\begin{aligned} & w^4\{f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgx^2y^2\} \\ + 2w^2 & \left\{\begin{aligned} & -cfx^4y^2-agy^4z^2-bhz^4x^2+2kx^2y^2z^2 \\ & +bfx^4z^2+cgy^4x^2+ahx^4y^2 \end{aligned}\right\} \\ + & \{ay^2z^2+bz^2x^2+cx^2y^2\}^2=0, \end{aligned}$$

which puts in evidence the nodal curve

$$w=0,\ -ay^2z^2-bz^2x^2-cx^2y^2=0:$$

there are three similar forms which put in evidence the other three nodal curves.

The four curves are so related to each other that every line which meets three of them meets also the fourth curve; there is consequently a singly infinite series of lines meeting each of the four curves; these break up into four series of lines each forming an octic scroll, and each scroll has the four curves for nodal curves respectively; that is, each scroll is a surface included under the foregoing general equation, and derived from it by assigning a proper value to the constant k. To determine these values, write

$$\left\{\begin{aligned} & \lambda+\mu+\nu=0, \\ & \frac{af}{\lambda^2}+\frac{bg}{\mu^2}+\frac{ch}{\nu^2}=0, \end{aligned}\right.$$

equations which give four systems of values for the ratios $(\lambda:\mu:\nu)$. We have then

$$k=af\frac{\nu-\mu}{\lambda}+bg\frac{\lambda-\nu}{\mu}+ch\frac{\mu-\nu}{\lambda},$$

viz. k has four values corresponding to the several values of $(\lambda:\mu:\nu)$.

The scroll in question is M. De La Gournerie's scroll Σ_1; the equation of the scroll Σ_1 is consequently obtained from the octic equation by writing therein the last-mentioned value of k.

It is to be noticed that k is, in effect, determined by a quartic equation; and, that, for a certain relation between the coefficients, this equation will have a twofold root. Assuming that this relation is satisfied, and assigning to k its twofold value, the resulting scroll becomes a torse; that is, two of the four scrolls coincide together and degenerate into a torse; corresponding to the remaining two values of k we have two scrolls, *companions* of the torse. In order to a twofold value of k, we must have

$$\frac{af}{\lambda^3}=\frac{bg}{\mu^3}=\frac{ch}{\nu^3};$$

and thence

$$(af)^{\frac{1}{3}}+(bg)^{\frac{1}{3}}+(ch)^{\frac{1}{3}}=0;$$

or, what is the same thing,

$$(af+bg+ch)^3-27abcfgh=0.$$

If for a moment we write $af=\alpha^3$, $bg=\beta^3$, $ch=\gamma^3$, and, therefore, $\alpha+\beta+\gamma=0$; then for the twofold root, we have $\lambda : \mu : \nu = \alpha : \beta : \gamma$, and consequently

$$\begin{aligned} k &= \alpha^2(\gamma-\beta)+\beta^2(\alpha-\gamma)+\gamma^2(\beta-\alpha) \\ &= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha), \end{aligned}$$

that is,

$$k = \{(af)^{\frac{1}{3}}-(bg)^{\frac{1}{3}}\}\{(bg)^{\frac{1}{3}}-(ch)^{\frac{1}{3}}\}\{(ch)^{\frac{1}{3}}-(af)^{\frac{1}{3}}\},$$

which agrees with the result in regard to the octic torse.

If in the octic equation we write (x, y, z, w) in place of (x^2, y^2, z^2, w^2), then we have the quartic equation

$$\begin{aligned} & a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 \\ & + f^2x^2w^2 + g^2y^2w^2 + h^2z^2w^2 \\ & + 2bcx^2yz - 2cfx^2yw + 2bfx^2zw \\ & + 2cay^2zx - 2agy^2zw + 2cgy^2xw \\ & + 2abz^2xy - 2bhz^2xw + 2ahz^2yw \\ & - 2ghw^2yz - 2hfw^2zx - 2fgw^2xy \\ & + 2kxyzw = 0, \end{aligned}$$

which is the equation of a quartic surface touched by the planes $x=0$, $y=0$, $z=0$, $w=0$, in the four conics

$$\begin{aligned} x &= 0, \quad . \quad hzw - gwy + ayz = 0, \\ y &= 0, \; -hzw \quad . \quad + fwx + bzx = 0, \\ z &= 0, \quad gyw - fwx \quad . \quad + cxy = 0, \\ w &= 0, \; -ayz - bzx - cxy \quad . \quad = 0, \end{aligned}$$

respectively.

II. The octic surface

$$\begin{aligned} U = {} & b^2c^2f^2x^8 + c^2a^2g^2y^8 + a^2b^2h^2z^8 + f^2g^2h^2w^8 \\ & - 2a^2cg\,(bg-ch)\,y^6z^2 - 2b^2ah\,(ch-af)\,z^6x^2 - 2c^2bf\,(af-bg)\,x^6y^2 \\ & + 2a^2bh\,(\quad ,, \quad)\,y^2z^6 + 2b^2cf\,(\quad ,, \quad)\,z^2x^6 + 2c^2ag\,(\quad ,, \quad)\,x^2y^6 \\ & - 2f^2bc\,(\quad ,, \quad)\,x^6w^2 - 2g^2ca\,(\quad ,, \quad)\,y^6w^2 - 2h^2ab\,(\quad ,, \quad)\,z^6w^2 \\ & + 2f^2gh\,(\quad ,, \quad)\,x^2w^6 + 2g^2hf\,(\quad ,, \quad)\,y^2w^6 + 2h^2fg\,(\quad ,, \quad)\,z^2w^6 \\ & + f^2(b^2g^2+c^2h^2-4bgch)\,w^4x^4 + g^2(c^2h^2+a^2f^2-4chaf)\,w^4y^4 + h^2(a^2f^2+b^2g^2-4abfg)\,w^4z^4 \\ & + a^2(\quad ,, \quad)\,y^4z^4 + b^2(\quad ,, \quad)\,z^4x^4 + c^2(\quad ,, \quad)\,x^4y^4 \\ & - 2gh\,(bcgh-a^2f^2-2afbg-2afch)\,w^4y^2z^2 \\ & - 2bh\,(\quad ,, \quad)\,z^4x^2w^2 \\ & + 2cg\,(\quad ,, \quad)\,y^4x^2w^2 \\ & + 2bc\,(\quad ,, \quad)\,x^4y^2z^2 \end{aligned}$$

$$
\begin{aligned}
&- 2hf\,(cahf - b^2g^2 - 2bgaf - 2bgch)\,w^4z^2x^2\\
&- 2cf\,(\qquad\qquad ,, \qquad\qquad)\,x^4y^2w^2\\
&+ 2ah\,(\qquad\qquad ,, \qquad\qquad)\,z^4y^2w^2\\
&+ 2ca\,(\qquad\qquad ,, \qquad\qquad)\,y^4x^2z^2\\
&- 2fg\,(abfg - c^2h^2 - 2chaf - 2chbg)\,w^2x^2y^2\\
&- 2ag\,(\qquad\qquad ,, \qquad\qquad)\,y^4z^2w^2\\
&+ 2bf\,(\qquad\qquad ,, \qquad\qquad)\,x^4z^2w^2\\
&+ 2ab\,(\qquad\qquad ,, \qquad\qquad)\,z^4x^2y^2\\
&+ 2\Omega x^2y^2z^2w^2 = 0,
\end{aligned}
$$

where the values of the coefficients indicated by (,,) are at once obtained by the proper interchanges of the letters, and where Ω is an arbitrary coefficient, is a surface having the four nodal conics

$$
\begin{aligned}
x &= 0, \quad . \quad cy^2 - bz^2 + fw^2 = 0,\\
y &= 0, \; -cx^2 \quad . \quad + az^2 + gw^2 = 0,\\
z &= 0, \quad bx^2 - ay^2 \quad . \quad hw^2 = 0,\\
w &= 0, \; -fx^2 - gy^2 - hz^2 \quad . \quad = 0.
\end{aligned}
$$

In fact, writing the equation under the form

$$w^2\Theta + (fx^2 + gy^2 + hz^2)^2 \times (b^2c^2x^4 + c^2a^2y^4 + a^2b^2z^4 - 2a^2bcy^2z^2 - 2b^2caz^2x^2 - 2c^2abx^2y^2) = 0,$$

we put in evidence the nodal conic $w = 0$, $fx^2 + gy^2 + hz^2 = 0$: and similarly for the other nodal conics.

It is to be observed, that the complete section by the plane $w = 0$ is the conic $fx^2 + gy^2 + hz^2 = 0$, twice repeated, and the quartic

$$b^2c^2x^4 + c^2a^2y^4 + a^2b^2z^4 - 2a^2bcy^2z^2 - 2ab^2cz^2x^2 - 2abc^2x^2y^2 = 0:$$

the latter being the system of four lines

$$\frac{x}{\sqrt{(a)}} + \frac{y}{\sqrt{(b)}} + \frac{z}{\sqrt{(c)}} = 0, \quad \frac{x}{\sqrt{(a)}} + \frac{y}{\sqrt{(b)}} - \frac{z}{\sqrt{(c)}} = 0,$$

$$\frac{x}{\sqrt{(a)}} - \frac{y}{\sqrt{(b)}} + \frac{z}{\sqrt{(c)}} = 0, \quad \frac{x}{\sqrt{(a)}} - \frac{y}{\sqrt{(b)}} - \frac{z}{\sqrt{(c)}} = 0.$$

The plane in question, $w = 0$, meets the other nodal conics in the six points

$$(x = 0,\ by^2 - cz^2 = 0), \quad (y = 0,\ cz^2 - ax^2 = 0), \quad (z = 0,\ ax^2 - by^2 = 0),$$

which six points are the angles of the quadrilateral formed by the above-mentioned four lines.

The four conics are such, that every line meeting three of these conics meets also the fourth conic. The lines in question form a double system: each of these

systems has, in reference to any pair of nodal conics, a homographic property as follows; viz. considering for example the two conics in the planes $z=0$ and $w=0$ respectively, if a line meets these conics in the points P and Q respectively, and through these points respectively and the line $x=0$, $y=0$ we draw planes, then the system of the P planes and the system of the Q planes correspond homographically to each other, the coincident planes of the two systems being the planes $x=0$ and $y=0$ respectively.

Conversely, if through the line $(x=0,\ y=0)$ we draw the two homographically related planes meeting the two conics in the points P and Q respectively, then, for a proper value (determined by a quadratic equation) of the constant $k(=\Theta\div\theta)$ which determines the homographic relation, the line PQ will be a line meeting each of the four conics, and will belong to one or other of the above-mentioned two systems, as k is equal to one or the other of the two roots of the quadratic equation. The scroll generated by the lines meeting each of the four conics, or what is the same thing, any three of these conics, is *primâ facie* a scroll of the order 16; but by what precedes, it appears that this scroll breaks up into two scrolls, which will be each of the order 8. Moreover, each scroll has the four conics for nodal curves; and since the equation $U=0$ is the general equation of an octic surface having the four conics for nodal curves, it follows, that the equation of the scroll is derived from that of the octic surface $U=0$, by assigning a proper value to the indeterminate coefficient Ω; so that there are in fact two values of Ω, for each of which the surface $U=0$ becomes a scroll.

To sustain the foregoing conclusions, take $x=\theta' y$, $x=\theta y$ for the equations of the two planes through the line $(x=0,\ y=0)$, which meet the z-conic and w-conic in the points P and Q respectively; then the equations of the line PQ are

$$\sqrt{(f\theta^2+g)}\,(x-\theta' y)+\sqrt{(-h)}\,(\theta'-\theta)\,z=0,$$
$$-\sqrt{(b\theta'^2-a)}\,(x-\theta y)^2+\sqrt{(-h)}\,(\theta'-\theta)\,w=0,$$

or, writing therein $\theta'=k\theta$, the equations are

$$\sqrt{(f\theta^2+g)}\,(x-k\theta y)+\sqrt{(-h)}\,(k-1)\,\theta z=0,$$
$$-\sqrt{(bk^2\theta^2-a)}\,(x-\theta y)+\sqrt{(-h)}\,(k-1)\,\theta w=0.$$

To find where the line in question meets the plane $y=0$, we have

$$\sqrt{(f\theta^2+g)}\,x+\sqrt{(-h)}\,(k-1)\,\theta z=0,$$
$$-\sqrt{(bk^2\theta^2-a)}\,x+\sqrt{(-h)}\,(k-1)\,\theta w=0,$$

and thence

$$(f\theta^2+g)\,x^2+h\,(k-1)^2\,\theta^2 z^2=0,$$
$$(bk^2\theta^2-a)\,x^2+h\,(k-1)^2\,\theta^2 w^2=0,$$

or multiplying a, g and adding

$$(af+bgk^2)\,x^2+h\,(k-1)^2\,(az^2+gw^2)=0,$$

or assuming

$$af+bgk^2+ch\,(k-1)^2=0,$$

the equation is

$$-cx^2+az^2+gw^2=0.$$

That is, k being determined by the quadric equation $af+bgk^2+ch(k-1)^2=0$, the line PQ meets the y-conic $y=0$, $-cx^2+az^2+gw^2=0$; and, in a similar manner, it appears that the line PQ also meets the x-conic $x=0$, $cy^2-bz^2+fw^2=0$.

Writing for greater symmetry $1:-k:k-1=\lambda:\mu:\nu$, we have

$$\lambda+\mu+\nu=0,$$

$$af\lambda^2+bg\mu^2+ch\nu^2=0,$$

so that there are two systems of values of (λ, μ, ν) corresponding to, and which may be used in place of, the two values of k respectively.

Starting now from the equations

$$(f\theta^2+g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2=0,$$

$$(bk^2\theta^2-a)(\theta y-x)^2+h(k-1)^2\theta^2w^2=0,$$

the elimination of θ from these equations leads to an equation $U=0$, of the above mentioned form but with a determinate value of the coefficient.

The process, although a long one, is interesting and I give it in some detail.

Elimination of θ from the foregoing equations.

We have

$$U=M\Pi\,[(f\theta^2+g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2],$$

where Π denotes the product of the expressions corresponding to the four roots of the equation

$$(bk^2\theta^2-a)(\theta y-x)^2+h(k-1)^2\theta^2w^2=0.$$

Observing that this equation does not contain z, and that the expression under the sign Π does not contain w, it is at once seen that the product Π is in regard to (z, w) a rational and integral function of the form $(z^2, w^2)^4$; and since, in regard to (z, w), U is also a rational and integral function of the same form $(z^2, w^2)^4$, it is clear that the factor M does not contain z or w, but is a function of only (x, y). To determine it we may write $z=0$, $w=0$: this gives

$$c^2(bx^2-ay^2)^2(fx^2+gy^2)^2=M\Pi\,(f\theta^2+g)(k\theta y-x)^2,$$

where

$$(bk^2\theta^2-a)(\theta y-x)^2=0,$$

and the values of θ are therefore $+\dfrac{\sqrt{(a)}}{k\sqrt{(b)}}$, $-\dfrac{\sqrt{(a)}}{k\sqrt{(b)}}$, $\dfrac{x}{y}$, $\dfrac{x}{y}$. Hence substituting and observing that

$$c^2h^2(k-1)^4=(af+bgk^2)^2,$$

it is easy to find

$$M = \frac{y^4}{x^4} \frac{b^4k^4}{h^2(k-1)^8},$$

that is, we have

$$\frac{x^4}{y^4} \frac{h^2(k-1)^8}{b^4k^4} U = \Pi\,[(f\theta^2+g)(k\theta y - x)^2 + h(k-1)^2\theta^2 z^2],$$

where

$$(bk^2\theta^2 - a)(\theta y - x)^2 + h(k-1)^2\theta^2 w^2 = 0.$$

If for greater convenience we write $\theta = \frac{x}{y}\phi$, then this formula becomes

$$\frac{y^4}{x^4} \frac{h^2(k-1)^8}{b^4k^4} U = \Pi\,[(fx^2\phi^2+gy^2)(k\phi-1)^2 + h(k-1)^2 z^2\phi^2],$$

where

$$(bk^2x^2\phi^2 - ay^2)(\phi-1)^2 + h(k-1)^2 w^2\phi^2 = 0,$$

or, what is the same thing,

$$\left(\phi^2 - \frac{ay^2}{bk^2x^2}\right)(\phi-1)^2 + \frac{h(k-1)^2w^2}{bk^2x^2}\phi^2 = 0.$$

Suppose that the terms in U which contain z^2 are $= \Theta z^2$; then we have

$$\frac{y^4}{x^4} \frac{h(k-1)^6}{b^4k^4}\Theta = \Sigma\phi_1^2\Pi'(fx^2\phi^2+gy^2)(k\phi-1)^2,$$

or, what is the same thing,

$$\Theta = \frac{b^4k^4}{h(k-1)^6}\frac{x^4}{y^4}\Sigma\phi_1^2\Pi'(fx^2\phi^2+gy^2)(k\phi-1)^2,$$

where Π' refers to the remaining three roots ϕ_2, ϕ_3, ϕ_4; this may also be written

$$\Theta = \frac{b^4k^4}{h(k-1)^6}\frac{x^4}{y^4}\Pi(fx^2\phi^2+gy^2)(k\phi-1)^2 . \Sigma\frac{\phi^2}{(fx^2\phi^2+gy^2)(k\phi-1)^2}.$$

Hence, observing that we have identically

$$\left(\phi^2 - \frac{ay^2}{bk^2x^2}\right)(\phi-1)^2 + \frac{h(k-1)^2w^2}{bk^2x^2}\phi^2 = (\phi-\phi_1)(\phi-\phi_2)(\phi-\phi_3)(\phi-\phi_4),$$

and writing $\phi = \pm\frac{iy\sqrt{(g)}}{x\sqrt{(f)}}$, $\phi = \frac{1}{k}$, $\{i = \sqrt{(-1)}$ as usual$\}$, we find

$$\Pi\,\{\phi x\sqrt{(f)} \pm iy\sqrt{(g)}\} = \frac{h(k-1)^2}{bk^2}[c\{x\sqrt{(f)} \pm iy\sqrt{(g)}\}^2 fgw^2]\,y^2,$$

$$\Pi\,(k\phi-1) \qquad = \frac{(k-1)^2}{bx^2}(bx^2 - ay^2 + hw^2);$$

whence, writing for shortness

$$\begin{aligned}\Delta &= [c\{x\sqrt{(f)} + iy\sqrt{(g)}\}^2 - fgw^2][c\{x\sqrt{(f)} - iy\sqrt{(g)}\}^2 - fgw^2],\\ &= c^2f^2x^4 + c^2g^2y^4 + f^2g^2w^4 + 2cfg^2y^2w^2 - 2cf^2gx^2w^2 + 2c^2fgx^2y^2,\end{aligned}$$

we find

$$\Pi\,(fx^2\phi^2+gy^2)=\frac{h^2\,(k-1)^4}{b^2k^4}\,\Delta y^4,$$

$$\Pi\,(k\phi-1)^2=\frac{(k-1)^4}{b^2}\,(bx^2-ay^2+hw^2)\,\frac{1}{x^4},$$

and thence

$$\Pi\,(fx^2\phi^2+gy^2)\,(k\phi-1)^2=\frac{h^2\,(k-1)^8}{b^4k^4}\,\Delta\,(bx^2-ay^2+hw^2)^2\,\frac{y^4}{x^4},$$

and consequently

$$\Theta=h\,(k-1)^2\,\Delta\,(bx^2-ay^2+hw^2)^2.\;\Sigma\,\frac{\phi^2}{(fx^2\phi^2+gy^2)\,(k\phi-1)^2}.$$

Hence, writing

$$\frac{\phi^2}{(fx^2\phi^2+gy^2)\,(k\phi-1)^2}=\frac{A}{(k\phi-1)^2}+\frac{B}{k\phi-1}+\frac{C}{x\phi\,\sqrt{(f)}+iy\,\sqrt{(g)}}+\frac{D}{x\phi\,\sqrt{(f)}-iy\,\sqrt{(g)}},$$

we may calculate separately the terms

$$\Sigma\left\{\frac{A}{(k\phi-1)^2}+\frac{B}{k\phi-1}\right\},$$

and

$$\Sigma\left\{\frac{C}{x\phi\,\sqrt{(f)}+iy\,\sqrt{(g)}}+\frac{D}{x\phi\,\sqrt{(f)}-iy\,\sqrt{(g)}}\right\}.$$

The first of these is

$$=\frac{1}{(k-1)^2\,(fx^2+k^2gy)^2\,(bx^2-ay^2+hw^2)^2}\,\{x,\;y,\;w\}^6,$$

if for shortness

$$\begin{aligned}\{x,\;y,\;w\}^6=(fx^2+k^2gy^2)\,[&4\,\{(2-k)\,bx^2-ay^2+h\,(1-k)\;w^2\}^2\\&-2\,(bx^2-ay^2+hw^2)\,\{(6-6k+k^2)\,bx^2-ay^2+h\,(1-k)^2\,w^2\}]\\&+4k^2\,(k-1)\,gy^2\,(bx^2-ay^2+hw^2)\,\{(2-k)\,bx^2-ay^2+h\,(1-k)\;w^2\}:\end{aligned}$$

the second is

$$=\frac{2}{(k-1)^2\,(fx^2+k^2gy^2)^2\,\Delta}\,(x,\;y,\;w)^6,$$

if for shortness

$$\begin{aligned}(x,\;y,\;w)^6=\{&(fx^2-k^2gy^2)\,(cfx^2-cgy^2-fgw^2)-4ckfgx^2y^2\}\\&\times\{fgbk^2x^2+[2ch\,(k-1)^2+af\,]\,gy^2+fgh\,(k-1)^2\,w^2\}\\&+2\,\{k^2bg-ch\,(k-1)^2\}\,fgx^2y^2\,\{c\,(k+1)\,(fx^2-kgy^2)-kfgw^2\};\end{aligned}$$

and hence

$$\Theta=\frac{1}{(fx^2+k^2gy^2)^2}\,[h\Delta\,\{x,\;y,\;w\}^6+2\,(bx^2-ay^2+hw^2)^2\,(x,\;y,\;w)^6],$$

which must be a rational and integral function of $(x,\;y,\;w)$.

In partial verification of this, observe that, because U contains the terms

$$2b^2cf\,(ch-af\,)\,x^6z^2+2\Omega x^2y^2z^2w^2,$$

Θ should contain the terms

$$2b^2cf(ch-af)\,x^6+2\Omega x^2y^2w^2,$$

viz. in Θ the term in x^6 should be $=2b^2cf(ch-af)\,x^6$.

Now writing $y=0$, $w=0$, we have

$$\begin{aligned}\Delta &= c^2f^2x^4,\\ \{x,\ y,\ w\}^6 &= b^2f\{4(2-k)^2-2(6-6k+k^2)\}\,x^6,\\ &= b^2f(4-4k+2k^2)\,x^6,\\ (x,\ y,\ w)^6 &= bcf^2gk^2x^6;\end{aligned}$$

and hence the required term of Θ is x^6 multiplied by

$$b^2c^2fh(4-4k+2k^2)+2b^3cfgk^2:$$

viz. the coefficient is

$$\begin{aligned}&= 2b^2cf[ch(2-2k+k^2)+bgk^2],\\ &= 2b^2cf[ch+ch(1-k)^2+bgk^2],\end{aligned}$$

which in virtue of the relation $af+bgk^2+ch(1-k)^2$ becomes, as it should do,

$$=2b^2cf(ch-af).$$

The actual division by $(fx^2+k^2gy^2)^2$ would, however, be a very tedious process, and it is to be observed, that we only require to know the term $2\Omega x^2y^2w^2$ of Θ. We may therefore adopt a more simple course as follows: the terms of Θ which contain w^2 are $=(Ax^4+2\Omega x^2y^2+By^4)\,w^2$, hence writing for a moment

$$\{x,\ y,\ w\}^6=P+Qw^2,\quad (x,\ y,\ w)^6=R+Sw^2,$$

and observing that we have

$$\begin{aligned}\Delta &= c^2(fx^2+gy^2)^2-2c^2fg(fx^2-gy^2)\,w^2+\&c.,\\ (bx^2-ay^2+hw^2)^2 &= (bx^2-ay^2)^2+2h(bx^2-ay^2)\,w^2+\&c.,\end{aligned}$$

we have

$$\begin{aligned}(fx^2+k^2gy^2)^2(Ax^4+2\Omega x^2y^2+By^4) &= c^2h(fx^2+gy^2)^2\,Q-2c^2fgh(fx^2-gy^2)\,P\\ &\quad+(bx^2-ay^2)^2\,S+2h(bx^2-ay^2)\,R.\end{aligned}$$

But in this identical equation we may write $x^2=a$, $y^2=b$, which gives

$$(af+k^2bg)^2(Aa^2+2\Omega ab+Bb^2)=c^2h(af+bg)^2\,Q-2c^2fgh(af-bg)\,P;$$

and from the equation

$$\{x,\ y,\ w\}^6=P+Qw^2,$$

we have

$$\begin{aligned}P+Qw^2 &= (af+bgk^2)\begin{bmatrix}4\{(1-k)\,ab+(1-k)\,hw^2\}^2\\ -2hw^2(5-6k+k^2)\,ab\end{bmatrix}\\ &\quad+4k^2(k-1)\,bghw^2(1-k)\,ab,\\ &= -ch(k-1)^2\{4(k-1)^2(a^2b^2+2w^2abh)-2(5-6k+k^2)\,hw^2\}\\ &\quad-4k^2(k-1)^2\,ab^3ghw^2,\end{aligned}$$

that is,

$$P = -4ch(k-1)^4 a^2b^3,$$
$$Q = \quad (k-1)^2 abh\,\{ch(-6k^2+4k+2) - 4k^2b\},$$

whence

$$(k-1)^2(Aa^2+2\Omega ab+Bb^2) = ab(af+bg)^2[ch(-6k^2+4k+2)-4k^2bg]$$
$$+ 8fg(af-bg)(k-1)^2(ab)^2.$$

But we have

$$Aa^2+Bb^2 = -2ab(af-bg)(-afbg+c^2h^2+2chaf+2chbg),$$

and thence

$$2(k-1)^2\Omega = (af+bg)^2[ch(-6k^2+4k+2)-4k^2bg]$$
$$+(k-1)^2(af-bg)\begin{pmatrix}-2afbg+2c^2h^2+4chaf+4chbg\\+8afbg\end{pmatrix},$$

or

$$(k-1)^2\Omega = (af+bg)^2[ch(-3k^2+2k+1)-2k^2bg]$$
$$+(k-1)^2(af-bg)[3afbg+c^2h^2+2chaf+2chbg].$$

Writing $-3k^2+2k+1 = -3(k-1)^2-4(k-1)$, this is

$$\Omega = (af+bg)^2\left[ch\left(-3-\frac{4}{k-1}\right)-\frac{2k^2}{(k-1)^2}bg\right]+(af-bg)[3afbg+c^2h^2+2chaf+2chbg],$$

or since $1 : -k : k-1 = \lambda : \mu : \nu$; and writing for shortness $(af, bg, ch) = (\alpha, \beta, \gamma)$, this is

$$\Omega = (\alpha+\beta)^2\left\{\gamma\left(-3-\frac{4\lambda}{\nu}\right)-\frac{2\mu^2}{\nu^2}\beta\right\}+(\alpha-\beta)\{3\alpha\beta+\gamma^2+2\gamma\alpha+2\gamma\beta\},$$

which is the value of Ω: viz. the conclusion arrived at is that, eliminating θ from the equations

$$(f\theta^2 \quad +g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2 = 0,$$
$$(bk^2\theta^2-a)(\quad \theta y-x)^2+h(k-1)^2\theta^2w^2 = 0:$$

where k denotes a determinate function of af, bg, ch, viz. writing $af, bg, ch = \alpha, \beta, \gamma$ and $1 : -k : k-1 = \lambda : \mu : \nu$, we have

$$\lambda + \mu + \nu = 0,$$
$$\alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = 0,$$

equations which serve to determine k: the result of the elimination is the octic equation

$$b^2c^2f^2x^8+\ldots+2\Omega x^2y^2z^2w^2 = 0,$$

where Ω has the last-mentioned value.

The value of Ω is unsymmetrical in its form, and there are apparently six values; viz. writing

$$A = (\beta+\gamma)^2\left\{\alpha\left(-3-\frac{4\mu}{\lambda}\right)-\frac{2\nu^2}{\lambda^2}\gamma\right\}+(\beta-\gamma)(S+\beta\gamma+\alpha^2),$$

$$B = (\gamma+\alpha)^2\left\{\beta\left(-3-\frac{4\nu}{\mu}\right)-\frac{2\lambda^2}{\mu^2}\alpha\right\}+(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

$$C = (\alpha+\beta)^2\left\{\gamma\left(-3-\frac{4\lambda}{\nu}\right)-\frac{2\mu^2}{\nu^2}\beta\right\}+(\alpha-\beta)(S+\alpha\beta+\gamma^2),$$

$$A_1 = (\beta+\gamma)^2\left\{\alpha\left(-3-\frac{4\nu}{\lambda}\right)-\frac{2\mu^2}{\lambda^2}\beta\right\}-(\beta-\gamma)(S+\beta\gamma+\alpha^2),$$

$$B_1 = (\gamma+\alpha)^2\left\{\beta\left(-3-\frac{4\lambda}{\mu}\right)-\frac{2\nu^2}{\mu^2}\gamma\right\}-(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

$$C_1 = (\alpha+\beta)^2\left\{\gamma\left(-3-\frac{4\mu}{\nu}\right)-\frac{2\lambda^2}{\nu^2}\alpha\right\}-(\alpha-\beta)(S+\alpha\beta+\gamma^2),$$

where for shortness $S = 2(\beta\gamma+\gamma\alpha+\alpha\beta)$, the six values would be A, B, C, A_1, B_1, C_1. But we have really

$$A = B = C = -A_1 = -B_1 = -C_1;$$

so that Ω has really only two values, equal and of opposite signs, or, what is the same thing, Ω^2 has a unique value. In fact, writing for shortness

$$\lambda+\mu+\nu = P, \quad \alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = X,$$

we find at once the identity

$$\lambda^2(A+A_1) = (\beta+\gamma)^2(-2X-4\lambda\alpha P),$$

so that $A = -A_1$, in value of $P = 0$, $X = 0$. And similarly $B = -B_1$, $C = -C_1$.

But the demonstration of the equation $A = B$ is more complicated. We have

$$A-B = -3\alpha(\beta+\gamma)^2-4\alpha(\beta+\gamma)^2\frac{\mu}{\lambda}-2\gamma(\beta+\gamma)^2\frac{\nu^2}{\lambda^2}+(\beta-\gamma)(S+\beta\gamma+\alpha^2)$$
$$+3\beta(\gamma+\alpha)^2+4\beta(\gamma+\alpha)^2\frac{\nu}{\mu}+2\alpha(\gamma+\alpha)^2\frac{\lambda^2}{\mu^2}-(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

that is,

$$\begin{aligned}\lambda^2\mu^2(A-B) = \{&-3\alpha(\beta+\gamma)^2+3\beta(\gamma+\alpha)^2+(\beta-\gamma)(S+\beta\gamma+\alpha^2)-(\gamma-\alpha)(S+\gamma\alpha+\beta^2)\}\lambda^2\mu^2\\ &-4\alpha(\beta+\gamma)^2\lambda\mu^3\\ &-2\gamma(\beta+\gamma)^2\nu^2\mu^2\\ &+4\beta(\gamma+\alpha)^2\nu\lambda^2\mu\\ &+2\alpha(\gamma+\alpha)^2\lambda^4,\end{aligned}$$

or, denoting for a moment the coefficient of $\lambda^2\mu^2$ by K, and writing also $\gamma\nu^2 = X - \alpha\lambda^2 - \beta\mu^2$, $\nu = P - \lambda - \mu$, this is

$$\begin{aligned}
&= \quad K\lambda^2\mu^2 \\
&\quad - 4\alpha\,(\beta+\gamma)^2\,\lambda\mu^2 \\
&\quad - 2\ \ (\beta+\gamma)^2\,\mu^2\,(X - \alpha\lambda^2 - \beta\mu^2) \\
&\quad + 4\beta\,(\gamma+\alpha)^2\,\lambda^2\mu\,(P - \lambda - \mu) \\
&\quad + 2\alpha\,(\gamma+\alpha)^2\,\lambda^4, \\
&= -2\ \ (\beta+\gamma)^2\,\mu^2 X + 4\beta\,(\gamma+\alpha)^2\,\lambda^2\mu P \\
&\quad + 2\alpha\,(\gamma+\alpha)^2\,\lambda^4 \\
&\quad - 4\beta\,(\gamma+\alpha)^2\,\lambda^3\mu \\
&\quad + \{-4\beta\,(\gamma+\alpha)^2 + K + 2\alpha\,(\beta+\gamma)^2\}\,\lambda^2\mu^2 \\
&\qquad - 4\alpha\,(\beta+\gamma)^2\,\lambda\mu^3 \\
&\qquad + 2\beta\,(\beta+\gamma)^2\,\mu^4,
\end{aligned}$$

and here the coefficient of $\lambda^2\mu^2$ is found to be

$$= 2\,\{\alpha\beta\,(\alpha+\beta) + \gamma\,(\alpha-\beta)^2 - 3\gamma^2\,(\alpha+\beta)\}.$$

Hence, the terms without X or P are $= 2\nabla$, where

$$\begin{aligned}
\nabla = \quad & \alpha\,(\gamma+\alpha)^2\lambda^4 \\
& - 2\beta\,(\gamma+\alpha)^2\lambda^3\mu \\
& + \{\alpha\beta\,(\alpha+\beta) + \gamma\,(\alpha-\beta)^2 - 3\gamma^2\,(\alpha+\beta)\}\,\lambda^2\mu^2 \\
& - 2\alpha\,(\beta+\gamma)^2\,\lambda\mu^3 \\
& + \beta\,(\beta+\gamma)^2\,\mu^3,
\end{aligned}$$

and this is identically

$$= \left.\begin{aligned} &(\alpha+\gamma)\,\lambda^2 \\ + \quad &2\gamma\lambda\mu \\ + &(\beta+\gamma)\,\mu^2 \end{aligned}\right\} \times \left\{\begin{aligned} &\alpha\,(\gamma+\alpha)\,\lambda^2 \\ - &2\,(\beta\gamma+\gamma\alpha+\alpha\beta)\,\lambda\mu \\ + &\beta\,(\beta+\gamma)\,\mu^2, \end{aligned}\right.$$

where observing that

$$\alpha\lambda^2 + \beta\mu^2 + \gamma\,(P - \lambda - \mu)^2 = X,$$

we have the first factor

$$(\alpha+\gamma)\,\lambda^2 + (\beta+\gamma)\,\mu^2 + 2\gamma\lambda\mu = X - \gamma P^2 + 2\gamma P\,(\lambda+\mu),$$

and consequently

$$\begin{aligned}
\lambda^2\mu^2\,(A - B) = & -2\,(\beta+\gamma)^2\,\mu^2 X + 4\beta\,(\gamma+\alpha)^2\lambda^2\mu P \\
& + 2\,\{X - \gamma P^2 + 2\gamma P\,(\lambda+\mu)\}\,\{\alpha\,(\gamma+\alpha)\,\lambda^2 - 2\,(\beta\gamma+\gamma\alpha+\alpha\beta)\,\lambda\mu + \beta\,(\beta+\gamma)\,\mu^2\};
\end{aligned}$$

viz. in virtue of $P = 0$, $X = 0$, we have $A = B$. And thus

$$A = B = C = -A_1 = -B_1 = -C_1:$$

so that the only values of Ω are, say, A and $-A$.

Reverting to the original equations

$$(f\theta^2 + g)(k\theta y - x)^2 + h(k-1)^2\theta^2 z^2 = 0,$$
$$(bk^2\theta^2 - a)(\theta y - x)^2 + h(k-1)^2\theta^2 w^2 = 0,$$

say these are

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e} \between \theta, 1)^4 = 0,$$
$$(\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{d}', \mathrm{e}' \between \theta, 1)^4 = 0,$$

then the coefficients in the two equations have the values

$$\begin{array}{ll} fk^2y^2, & bk^2y^2, \\ -2kfxy, & -2bk^2xy, \\ fx^2 + gky^2 + h(k-1)^2 z^2, & bk^2x^2 - ay^2 + h(k-1)^2 w^2, \\ -2gkxy, & 2axy, \\ gx^2, & -ax^2, \end{array}$$

where observe that only c contains z^2, and only c' contains w^2. The result of the elimination is

$$\left|\begin{array}{cccccccc} & & & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e} \\ & & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & \\ & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & & \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & & & \\ & & & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}' \\ & & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & \\ & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & & \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & & & \end{array}\right| = 0;$$

viz. here the only terms which contain z^8 and w^8 are

$$\mathrm{c}^2\mathrm{a}'^2\mathrm{e}'^2 + \mathrm{c}'^2\mathrm{a}^2\mathrm{e}^2,$$

and hence the terms in z^8 and w^8 are

$$h^4(k-1)^8 z^8 . a^2b^2k^4x^4y^4 + h^4(k-1)^8 w^8 . f^2g^2k^4x^4y^4,$$

viz. these are

$$= h^2k^4(k-1)^8 x^4y^4 (a^2b^2h^2z^8 + f^2g^2h^2w^8),$$

or assuming that the determinant contains as a factor the function $b^2c^2f^2x^8 + \ldots + 2\Omega x^2y^2z^2w^2$, with a properly determined value of Ω, we see that the other factor is $= h^2k^4(k-1)^8 x^4y^4$, which agrees with a preceding result.

655.

A MEMOIR ON DIFFERENTIAL EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 292—339.]

WE have to do with a set of variables, which is either unipartite $(x, y, z, \ldots)$, or else bipartite $(x, y, z, \ldots; p, q, r, \ldots)$, the variables in the latter case corresponding in pairs x and p, y and q, &c.

A letter not otherwise explained denotes a function of the variables. Any such letter may be put = const., viz. we thereby establish a relation between the variables; and when this is so, we use the *same* letter to denote the constant value of the function. Thus the set being $(x, y, z; p, q, r)$, H may denote a given function $pqr - xyz$; and then, if $H = \text{const.}$, we have $pqr - xyz = H$ (a constant). This notation, when once clearly understood, is I think a very convenient one.

The present memoir relates chiefly to the following subjects:

A. Unipartite set $(x, y, z, \ldots)$. The differential system

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \ldots,$$

and connected therewith the linear partial differential equation

$$X\frac{d\theta}{dx} + Y\frac{d\theta}{dy} + Z\frac{d\theta}{dz} + \ldots = 0:$$

also the lineo-differential

$$Xdx + Ydy + Zdz + \ldots.$$

B. Bipartite set $(x, y, z, \ldots; p, q, r, \ldots)$. The Hamiltonian system

$$\frac{dx}{\frac{dH}{dp}} = \frac{dy}{\frac{dH}{dq}} = \frac{dz}{\frac{dH}{dr}} = \ldots = \frac{dp}{-\frac{dH}{dx}} = \frac{dq}{-\frac{dH}{dy}} = \frac{dr}{-\frac{dH}{dz}} = \ldots,$$

and connected therewith the linear partial differential equation

$$\frac{dH}{dp}\frac{d\theta}{dx}-\frac{dH}{dx}\frac{d\theta}{dp}+\frac{dH}{dq}\frac{d\theta}{dy}-\frac{dH}{dy}\frac{d\theta}{dq}+\ldots=0,$$

otherwise written

$$(H,\ \theta),\ =\frac{d\,(H,\ \theta)}{d\,(p,\ x)}+\frac{d\,(H,\ \theta)}{d\,(q,\ y)}+\ldots,\ =0,$$

where H denotes a given function of the variables: also the Hamiltonian system as augmented by an equality $=dt$, and as augmented by this and another equality

$$=dV\div\left(p\frac{dH}{dp}+q\frac{dH}{dq}+r\frac{dH}{dr}\ldots\right).$$

C. Bipartite set $(x,\ y,\ z,\ldots;\ p,\ q,\ r,\ldots)$. The partial differential equation $H=\text{const.}$, where, as before, H is a given function of the variables, but $p,\ q,\ r,\ldots$ are now the differential coefficients in regard to $x,\ y,\ z,\ldots$ respectively of a function V of these variables, or, what is the same thing, there exists a function

$$V=\int(pdx+qdy+rdz+\ldots),$$

of the variables $x,\ y,\ z,\ldots$.

In what precedes, I have written $(x,\ y,\ z,\ldots)$ to denote a set of any number n of variables, and $(x,\ y,\ z,\ldots;\ p,\ q,\ r,\ldots)$ to denote a set of any even number $2n$ of variables, and the investigations are for the most part applicable to these general cases. But for greater clearness and facility of expression, I usually consider the case of a set $(x,\ y,\ z,\ w)$, or $(x,\ y,\ z;\ p,\ q,\ r)$, &c., as the case may be, consisting of a definite number of variables.

The greater part of the theory is not new, but I think that I have presented it in a more compact and intelligible form than has hitherto been done, and I have added some new results.

Introductory Remarks. Art. Nos. 1 to 3.

1. As already noticed, a letter not otherwise explained is considered as denoting a function of the variables of the set; but when necessary we indicate the variables by a notation such as $z=z(x,\ y)$; z is here a function (known or unknown as the case may be) of the variables $x,\ y$, the z on the right-hand side being in fact a functional symbol. And thus also $z=z(x,\ y)$, $=\text{const.}$ denotes that the function $z(x,\ y)$ of the variables $x,\ y$ has a constant value, which constant value is $=z$, viz. we thus indicate a relation between the variables $x,\ y$.

2. The variables $x,\ y$, &c., may have infinitesimal increments $dx,\ dy$, &c.; and the equations of connexion between the variables then give rise to linear relations between these increments, the coefficients therein being differential coefficients and,

as such, represented in the usual notation; thus if $z = z(x, y)$, we have $dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy$, where $\frac{dz}{dx}$, $\frac{dz}{dy}$ are the so-called partial differential coefficients of z in regard to x, y respectively. If we have $y = y(x)$, then also $dy = \frac{dy}{dx}dx$, and the foregoing equation becomes

$$dz = \left(\frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx}\right)dx;$$

but considering the two equations $z = z(x, y)$ and $y = y(x)$ as determining z as a function of x, say $z = z(x)$, we have $dz = \frac{d(z)}{dx}dx$; whence comparing the two formulæ

$$\frac{d(z)}{dx} = \frac{dz}{dx} + \frac{dz}{dy}\frac{dy}{dx},$$

where $\frac{d(z)}{dx}$ is the so-called total differential coefficient of z in regard to x. The distinction is best made, not by any difference of notation $\frac{d(z)}{dx}$, $\frac{dz}{dx}$, but by appending in any case of doubt the equations or equation used in the differentiation. Thus we have $\frac{dz}{dx}$ where $z = z(x, y)$; or, as the case may be, $\frac{dz}{dx}$ where $z = z(x, y)$ and $y = y(x)$.

3. A relation between increments is always really a relation between differential coefficients: but we use the increments for symmetry and conciseness, as in the case of a differential system $\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$, or in a question relating to the lineo-differential $Xdx + Ydy + Zdz$, for instance in the question whether this can be put $= du$.

Notations. Art. Nos. 4 to 6.

4. Functional determinants. If $a, b, c, \ldots$ are functions of the variables $x, y, z, w, \ldots$, then the determinants

$$\begin{vmatrix} \frac{da}{dx}, & \frac{da}{dy} \\ \frac{db}{dx}, & \frac{db}{dy} \end{vmatrix}, \quad \begin{vmatrix} \frac{da}{dx}, & \frac{da}{dy}, & \frac{da}{dz} \\ \frac{db}{dx}, & \frac{db}{dy}, & \frac{db}{dz} \\ \frac{dc}{dx}, & \frac{dc}{dy}, & \frac{dc}{dz} \end{vmatrix}, \text{ \&c.,}$$

are for shortness represented by

$$\frac{d(a, b)}{d(x, y)}, \quad \frac{d(a, b, c)}{d(x, y, z)}, \text{ \&c.,}$$

the notation being especially used in the first-mentioned case where the symbol is $\frac{d(a, b)}{d(x, y)}$. It is sometimes convenient to extend this notation, and for instance use $\frac{d(a, b)}{d(x, y, z)}$ to denote the series of determinants

$$\begin{vmatrix} \frac{da}{dx}, & \frac{da}{dy}, & \frac{da}{dz} \\ \frac{db}{dx}, & \frac{db}{dy}, & \frac{db}{dz} \end{vmatrix},$$

which can be formed by selecting in every way two columns to form thereout a determinant; the equation

$$\frac{d(a, b)}{d(x, y, z)} = 0$$

will then denote that each of these determinants is $= 0$.

The analogous notation

$$\frac{d(a, b, c)}{d(x, y)}$$

would denote non-existent determinants, viz. there are here not columns enough to form with them a determinant: and the notation is not required.

5. In the case of a bipartite set $(x, y, z, \ldots; p, q, r, \ldots)$, if a, b are any functions of these variables, we consider the derivative

$$(a, b) = \frac{d(a, b)}{d(p, x)} + \frac{d(a, b)}{d(q, y)} + \frac{d(a, b)}{d(r, z)} + \ldots,$$

viz. (a, b) is used to denote the sum of the functional determinants on the right hand.

6. Taking again $(x, y, z, w, \ldots)$ as the variables, then in the theory of the lineo-differential $Xdx + Ydy + Zdz + Wdw + \ldots$, we use certain derivative functions analogous to Pfaffians. They may be thus defined; viz. considering the numbers 1, 2, 3, 4, ... as corresponding to the variables $x, y, z, w, \ldots$ respectively, we have

$$1 = X,\ 2 = Y,\ 3 = Z,\ 4 = W, \text{ \&c.},$$

$$12 = \frac{dX}{dy} - \frac{dY}{dx},\ 13 = \frac{dX}{dz} - \frac{dZ}{dx}, \text{ \&c.},$$

$$\begin{aligned} 123 &= 1.23 + 2.31 + 3.12 \\ &= X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right), \end{aligned}$$

$$\begin{aligned} 1234 &= 12.34 + 13.42 + 14.23 \\ &= \left(\frac{dX}{dy} - \frac{dY}{dx}\right)\left(\frac{dZ}{dw} - \frac{dW}{dz}\right) + \left(\frac{dX}{dz} - \frac{dZ}{dx}\right)\left(\frac{dW}{dy} - \frac{dY}{dw}\right) + \left(\frac{dX}{dw} - \frac{dW}{dx}\right)\left(\frac{dY}{dz} - \frac{dZ}{dy}\right), \end{aligned}$$

and, adding for greater distinctness the next following cases,

$$12345 = 1\,.\,2345 + 2\,.\,3451 + 3\,.\,4512 + 4\,.\,5123 + 5\,.\,1234,$$

$$123456 = 12\,.\,3456 + 13\,.\,4561 + 14\,.\,5612 + 15\,.\,6123 + 16\,.\,2345,$$

where of course 2345, &c., have the significations mentioned above.

Dependency of Functions. Art. Nos. 7 and 8.

7. Two or more functions of the same variables may be independent, or else dependent or connected; viz. in the latter case any one of the functions is a function of the others $a = a(x)$, $b = b(x)$, the functions a, b are dependent, but if

$$a = a(x, y), \quad b = b(x, y),$$

then the condition of dependency is

$$\frac{d(a, b)}{d(x, y)} = 0,$$

and, similarly, if $a = a(x, y, z)$, $b = b(x, y, z)$, then the conditions of dependency are

$$\frac{d(a, b)}{d(x, y, z)} = 0,$$

viz. if the equations thus represented are all of them satisfied, the functions are dependent, but if not, then they are independent.

Observe that, when $a = a(x, y, z)$, $b = b(x, y, z)$ as above, if we choose to attend only to the variables x, y, treating z as a mere constant, there is then a single condition of dependency $\frac{d(a, b)}{d(x, y)} = 0$, and so if we attend only to the variable x, treating y, z as mere constants, then a and b are dependent. Thus when $a = x$, $b = x^2 + y$, the functions a, b are independent if we attend to both the variables x, y; dependent if y be regarded as a constant.

8. Further when $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, the functions a, b, c are dependent; but when $a = a(x, y, z)$, $b = b(x, y, z)$, $c = c(x, y, z)$, the condition of dependency is

$$\frac{d(a, b, c)}{d(x, y, z)} = 0:$$

and so when $a = a(x, y, z, w)$, $b = b(x, y, z, w)$, $c = c(x, y, z, w)$, the conditions of dependency are

$$\frac{d(a, b, c)}{d(x, y, z, w)} = 0;$$

viz. if all the equations thus represented are satisfied, the functions are dependent; but if not, then they are independent. And so in other cases.

The General Differential System. Art. Nos. 9 to 22.

9. Taking the set of variables to be (x, y, z, w), the system is

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{dw}{W},$$

and we associate with this the linear partial differential equation

$$X\frac{d\theta}{dx} + Y\frac{d\theta}{dy} + Z\frac{d\theta}{dz} + W\frac{d\theta}{dw} = 0.$$

10. It is tolerably evident that the differential equations establish between x, y, z, w a threefold relation depending upon three arbitrary constants; in fact, regarding (x, y, z, w) as the coordinates of a point in four-dimensional space, and starting from any given point, the differential equations determine the ratios of the increments dx, dy, dz, dw, that is, the direction of passage to a consecutive point; and then again taking for x, y, z, w the coordinates of this point, the same equations give the direction of passage to the next consecutive point, and so on. The locus of the point is therefore a curve, or we have between the coordinates a threefold relation, and (the initial point being arbitrary) we have a curve of the system through each point of the four-dimensional space, viz. the relation must involve three arbitrary constants. But this being so, the constants will be expressible as functions of the coordinates, viz. the threefold relation involving the three constants will be expressible in the form $a = \text{const.}$, $b = \text{const.}$, $c = \text{const.}$, where a, b, c denote respectively functions of the coordinates (x, y, z, w).

11. Supposing that one of the relations is $a = \text{const.}$, it is clear that the increment

$$da, = \frac{da}{dx}dx + \frac{da}{dy}dy + \frac{da}{dz}dz + \frac{da}{dw}dw,$$

must become $= 0$, on substituting therein for dx, dy, dz, dw, the values X, Y, Z, W to which by virtue of the differential equations they are proportional, viz. that we must have identically

$$X\frac{da}{dx} + Y\frac{da}{dy} + Z\frac{da}{dz} + W\frac{da}{dw} = 0.$$

Conversely, when this is so, we have $da = 0$, by virtue of the differential equation.

We say that a is a solution of the partial differential equation, and an integral of the differential equations, viz. any solution of the partial differential equation is an integral of the differential equations, and any integral of the differential equations is a solution of the partial differential equation, or, this being so, we may in general without risk of ambiguity, say simply a is an integral*; similarly b and c are integrals, and, by what precedes, there are three integrals a, b, c.

* Viz. we use indifferently, in regard to the differential equations and to the partial differential equation, the term integral, which is appropriate to the differential equations; the appropriate term in regard to the partial differential equation would be solution.

Observe that, in speaking of an integral a, we mean a function of the variables; the differential equations give between the variables the relation $a = \text{const.}$, and when this is so, we use the same letter a to denote the constant value of this function.

12. In speaking of the three integrals a, b, c we mean independent integrals; any function whatever ϕa of an integral a, or any function whatever $\phi(a, b)$ of two integrals a, b, is an integral (viz. it is an integral of the differential equations, and also a solution of the partial differential equation), but such dependent integrals give nothing new, and we require a third independent integral c, viz. we need this to express the threefold relation between the variables, given by the differential equations, and also to express the general solution $\phi(a, b, c)$ of the partial differential equation.

13. By what precedes the analytical condition, in order that the integrals a, b, c may be independent, is that they are such as not to satisfy the relations

$$\frac{d(a, b, c)}{d(x, y, z, w)} = 0.$$

14. We moreover see *à posteriori*, that there cannot be more than three independent integrals; in fact, if a, b, c, d are integrals, then, considering them as solutions of the partial differential equation, we have four equations which by the elimination therefrom of X, Y, Z, W, give

$$\frac{d(a, b, c, d)}{d(x, y, z, w)} = 0,$$

and this is the very equation which expresses that a, b, c, d are not independent.

15. The notion of the integrals may be arrived at somewhat differently thus: take a, b, c, d any functions of the variables, and write

$$A = X\frac{da}{dx} + Y\frac{da}{dy} + Z\frac{da}{dz} + W\frac{da}{dw},$$

and the like for B, C, D; then replacing the original variables x, y, z, w by the new variables a, b, c, d, the differential equations become

$$\frac{da}{A} = \frac{db}{B} = \frac{dc}{C} = \frac{dd}{D},$$

where A, B, C, D are to be (by means of the given values of a, b, c, d as functions of x, y, z, w) expressed as functions of a, b, c, d. If then $A = 0$, $B = 0$, $C = 0$, the differential equations become

$$\frac{da}{0} = \frac{db}{0} = \frac{dc}{0} = \frac{dd}{D};$$

viz. we have $da = 0$, $db = 0$, $dc = 0$, and therefore $a = \text{const.}$, $b = \text{const.}$, $c = \text{const.}$, that is, we have the integrals a, b, c as before.

16. There is no general process for obtaining an integral a of the differential equations. Supposing such integral known, we can introduce it as a variable, in place of one of the original variables, say w, viz. we thus reduce the system to

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{da}{0},$$

where X, Y, Z now denote the values assumed by these functions upon expressing therein w as a function of x, y, z, a, viz. they are now functions of x, y, z, a. The system thus breaks up into $da=0$ and the system

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

in which last (by virtue of the first equation, or $a=\text{const.}$) a is to be regarded as a constant; the original system of three equations between four variables is thus reduced to a system of two equations between three variables. Supposing b to be an integral of this reduced system, b is given as a function of x, y, z, a, but upon substituting herein for a its value as a function of x, y, z, w, we have b a function of the original variables x, y, z, w, and b is then a second integral of the original system.

17. In like manner supposing a and b to be known, we reduce the system to the single equation

$$\frac{dx}{X}=\frac{dy}{Y},$$

where X, Y are now functions of x, y, a, b; supposing an integral hereof to be c, we have c a function of x, y, a, b; but upon substituting herein for a, b their values as functions of x, y, z, w, we have c a function of x, y, z, w, and as such it is the third integral of the original system.

18. It may be remarked that if, to the original system, we join on an equality $=dt$, viz. if we consider the system

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{dw}{W}\,(=dt),$$

where X, Y, Z, W are as before functions of the variables (x, y, z, w), then the integrals a, b, c of the original system being known, we can by means of them express for instance X as a function of x, a, b, c, and we have then, $\text{const.}=t-\int\frac{dx}{X}$, where the integration is to be performed regarding a, b, c as constants; writing $\int\frac{dx}{X}=\tau$, but after the integration replacing a, b, c by their values as functions of x, y, z, w, we have τ a function of x, y, z, w; and we say that $t-\tau$ is an integral; putting it $=\text{const.}$ we use also τ to denote the constant value of the integral $t-\int\frac{dx}{X}$ in question. Observe that here, the integrals a, b, c being known, the last integral $t-\tau$ is obtained by a quadrature.

19. The result would have been similar, if the adjoined equality had been $=\frac{dt}{T}$ (T a function of x, y, z, w), but in reference to subsequent matter, I retain the equality $=dt$, and adjoin a second equality $=\frac{dV}{\Omega}$ (Ω a function of x, y, z, w); we have then the integral $t-\tau$ as before, and another integral $V-\int\frac{\Omega dx}{X}$, where Ω, X are first expressed as functions of x, a, b, c, but after the integration a, b, c are replaced by their values as functions of (x, y, z, w), say this is the integral $V-\lambda$; this, when the integrals a, b, c are known, is (like $t-\tau$) obtained by a quadrature.

20. Attending only to the adjoined equality $=dt$, we can by means of the four integrals express each of the variables x, y, z, w as a function of $a, b, c, t-\tau$; viz. these four equations, regarding therein $t-\tau$ as a variable parameter, are in fact equivalent to the equations $a=\text{const.}$, $b=\text{const.}$, $c=\text{const.}$, which connect the variables x, y, z, w with the integrals a, b, c regarded as constants.

21. All that precedes is of course applicable to a system of $n-1$ equations between n variables, the number of independent integrals being $=n-1$.

22. I take an example with the three variables x, y, z; the differential equations being

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)},$$

and therefore the partial differential equation

$$x(y-z)\frac{d\theta}{dx}+y(z-x)\frac{d\theta}{dy}+z(x-y)\frac{d\theta}{dz}=0.$$

The integrals are $a=x+y+z$, $b=xyz$; and it will be shown how either of these integrals being known, the system is reduced to a single equation between two variables, say x, y.

First, a being known, $=x+y+z$ as before, we have

$$x(y-z)=x(x+2y-a),\quad y(z-x)=y(a-2x-y),$$

and the system is

$$\frac{dx}{x(x+2y-a)}=\frac{dy}{y(a-2x-y)},$$

which has the integral $b=xy(a-x-y)$; observe that this is a solution of the partial differential equation

$$x(x+2y-a)\frac{d\theta}{dx}+y(a-2x-y)\frac{d\theta}{dy}=0.$$

For a putting its value we find $b=xyz$.

Secondly, b being known, $=xyz$ as before, we have

$$x(y-z)=xy-\frac{y}{b},\quad y(z-x)=\frac{b}{x}-xy,$$

and the system is

$$\frac{dx}{xy-\frac{b}{y}}=\frac{dy}{\frac{b}{x}-xy},$$

which has the integral $a=x+y+\frac{b}{xy}$; observe that this is a solution of the partial differential equation

$$\left(xy-\frac{b}{y}\right)\frac{d\theta}{dx}+\left(\frac{b}{x}-xy\right)\frac{d\theta}{dy}=0.$$

For b putting its value, we find $a=x+y+z$.

The Multiplier. Art. Nos. 23 to 29.

23. First, if there are only two variables (x, y), the system consists of the single equation

$$\frac{dx}{X}=\frac{dy}{Y},$$

which may be written

$$Ydx-Xdy=0.$$

Hence, if a be an integral, we have

$$\frac{da}{dx}dx+\frac{da}{dy}dy=0;$$

the two will agree if there exists a function M such that

$$\frac{da}{dx}=MY,\quad \frac{da}{dy}=-MX,$$

and thence, in virtue of the identity

$$\frac{d}{dy}\frac{da}{dx}=\frac{d}{dx}\frac{da}{dy},$$

we find

$$\frac{dMX}{dx}+\frac{dMY}{dy}=0;$$

or, as this may also be written,

$$X\frac{dM}{dx}+Y\frac{dM}{dy}+M\left(\frac{dX}{dx}+\frac{dY}{dy}\right)=0,$$

as the condition to determine the multiplier M. Supposing M known, we have $M(Ydx-Xdy)=da$, or say $a=\int M(Ydx-Xdy)$, viz. the integral a is determined by a quadrature.

24. In the case of three variables (x, y, z), the system is

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

or, writing these in the form

$$Ydz-Zdy=0,\quad Zdx-Xdz=0,\quad Xdy-Ydx=0,$$

the course which immediately suggests itself is to seek for factors L, M, N, such that, a being an integral, we may have

$$L(Ydz-Zdy)+M(Zdx-Xdz)+N(Xdy-Ydx)=da,$$

but this does not lead to any result. The course taken by Jacobi is quite a different one: he, in fact, determines a multiplier M connected with *two* integrals a, b.

25. Supposing that a, b are independent integrals, we have

$$X\frac{da}{dx}+Y\frac{da}{dy}+Z\frac{da}{dz}=0,$$

$$X\frac{db}{dx}+Y\frac{db}{dy}+Z\frac{db}{dz}=0;$$

and determining from these equations the ratio of the quantities X, Y, Z, we may, it is clear, write

$$MX,\ MY,\ MZ=\frac{d(a,\ b)}{d(y,\ z)},\quad \frac{d(a,\ b)}{d(z,\ x)},\quad \frac{d(a,\ b)}{d(x,\ y)}.$$

It may be shown that we have identically

$$\frac{d}{dx}\frac{d(a,\ b)}{d(y,\ z)}+\frac{d}{dy}\frac{d(a,\ b)}{d(z,\ x)}+\frac{d}{dz}\frac{d(a,\ b)}{d(x,\ y)}=0,$$

and we thence deduce

$$\frac{d(MX)}{dx}+\frac{d(MY)}{dy}+\frac{d(MZ)}{dz}=0;$$

or, what is the same thing,

$$X\frac{dM}{dx}+Y\frac{dM}{dy}+Z\frac{dM}{dz}+M\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=0,$$

as the condition for determining the multiplier M.

26. The use is as follows: supposing that M is known, and supposing also that one integral a of the system is known, we can then by a quadrature determine the other integral b. Thus, supposing that we know the integral a, $=a(x, y, z)$, we can by means of this integral express z in terms of x, y, a; and hence we may regard the unknown integral b as expressed in the like form, $b=b(x, y, a)$. The original values of $\frac{db}{dx}$, $\frac{db}{dy}$, $\frac{db}{dz}$ become on this supposition

$$\frac{db}{dx}+\frac{db}{da}\frac{da}{dx},\quad \frac{db}{dy}+\frac{db}{da}\frac{da}{dy},\quad \frac{db}{da}\frac{da}{dz},$$

and we thence find

$$\frac{d(a,\ b)}{d(y,\ z)},\quad \frac{d(a,\ b)}{d(z,\ x)},\quad \frac{d(a,\ b)}{d(x,\ y)} = -\frac{da}{dz}\frac{db}{dy},\quad \frac{da}{dz}\frac{db}{dx},\quad \frac{d(a,\ b)}{d(x,\ y)}.$$

We have therefore

$$MX,\ MY = -\frac{da}{dz}\frac{db}{dy},\quad \frac{da}{dz}\frac{db}{dx},$$

and, consequently,

$$db = \frac{db}{dx}dx + \frac{db}{dy}dy,\quad = \frac{M}{\frac{da}{dz}}(Ydx - Xdy);$$

viz. M, $\frac{da}{dz}$, Y, X being all of them expressed as functions of x, y, a, the expression on the right-hand is a complete differential, and we have

$$b = \int \frac{M}{\frac{da}{dz}}(Ydx - Xdy);$$

that is, the integral b is determined by a quadrature.

27. Thus, in the example No. 22,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

and a value of the multiplier is $=1$. Supposing that the given integral is $a = x + y + z$, then $\frac{da}{dz} = 1$, and we have accordingly 1 as the multiplier of the equation

$$y(a - 2x - y)\,dx + x(a - x - 2y)\,dy = 0,$$

viz. this equation is integrable *per se*. Supposing the given integral to be $b = xyz$, then $\frac{db}{dz} = xy$, viz. we have $\frac{1}{xy}$ as the multiplier of the equation

$$\left(\frac{b}{x} - xy\right)dx + \left(\frac{b}{y} - xy\right)dy = 0,$$

and we thus in each case obtain the other integral as before.

28. The foregoing result may be presented in a more symmetrical form by taking in place of x, y any two variables $u = u(x,\ y,\ z)$, $v = v(x,\ y,\ z)$.

Supposing the integral a known as before, the system then is

$$\frac{du}{U} = \frac{dv}{V} = \frac{da}{0},$$

where U, $V = X\frac{du}{dx} + Y\frac{du}{dy} + Z\frac{du}{dz}$, $X\frac{dv}{dx} + Y\frac{dv}{dy} + Z\frac{dv}{dz}$, these being expressed as functions of u, v, a; or, what is the same thing, we have $Vdu - Udv = 0$, a being in this equation regarded as a constant.

From the foregoing values of MX, MY, MZ, we deduce

$$MU,\ MV = \frac{d(u,\ a,\ b)}{d(x,\ y,\ z)},\quad \frac{d(v,\ a,\ b)}{d(x,\ y,\ z)}.$$

But forming the values of du, dv, da, db, we have an equation, determinant $= 0$, which equation may be written

$$du\frac{d(v,\ a,\ b)}{d(x,\ y,\ z)} - dv\frac{d(a,\ b,\ u)}{d(x,\ y,\ z)} + da\frac{d(b,\ u,\ v)}{d(x,\ y,\ z)} - db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)} = 0;$$

or, writing herein $da = 0$, this is

$$du\frac{d(v,\ a,\ b)}{d(x,\ y,\ z)} - dv\frac{d(u,\ a,\ b)}{d(x,\ y,\ z)} - db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)} = 0,$$

viz. this is

$$M(Vdu - Udv) = db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)},$$

or say

$$b = \int\left\{M \div \frac{d(u,\ v,\ a)}{d(x,\ y,\ z)}\right\}(Vdu - Udv),$$

where, on the right-hand side, everything must be expressed in terms of u, v, a. It thus appears that on expressing the final equation as a relation $Vdu - Udv = 0$ between the variables u and v, the multiplier hereof is $M \div \frac{d(u,\ v,\ a)}{d(x,\ y,\ z)}$. If $u,\ v = x,\ y$, this agrees with a foregoing result.

29. The theory is precisely the same for any number of variables. Thus, if there are four variables x, y, z, w, we have

$$MX,\ MY,\ MZ,\ MW = \frac{d(a,\ b,\ c)}{d(y,\ z,\ w)},\quad -\frac{d(a,\ b,\ c)}{d(z,\ w,\ x)},\quad \frac{d(a,\ b,\ c)}{d(w,\ x,\ y)},\quad -\frac{d(a,\ b,\ c)}{d(x,\ y,\ z)};$$

and, we have between the functions on the right-hand an identical relation, in virtue of which

$$\frac{d(MX)}{dx} + \frac{d(MY)}{dy} + \frac{d(MZ)}{dz} + \frac{d(MW)}{dw} = 0;$$

then, supposing that a value of M is known, and also any two integrals a, b, and that by means of these the equation to be finally integrated is expressed as a relation $Vdu - Udv = 0$ between any two variables u and v, the multiplier of this is

$$= M \div \frac{d(u,\ v,\ a,\ b)}{d(x,\ y,\ z,\ w)},$$

where U, V and this multiplier are to be expressed in terms of u, v, a, b.

The general result is that, given a value of the multiplier, and also all but one of the integrals, the final integral is expressible by a quadrature.

Pfaffian Theorem. Art. No. 30.

30. According as

the variables are	we have	
x,	Xdx	$= du$,
x, y,	$Xdx + Ydy$	$= \lambda du$,
x, y, z,	$Xdx + Ydy + Zdz$	$= \lambda du + dv$,
x, y, z, w,	$Xdx + Ydy + Zdz + Wdw$	$= \lambda du + \mu dv$,

and so on; viz. the theorem is that, taking for instance two variables, a given lineo-differential $Xdx + Ydy$ is $= \lambda du$, that is, there exist λ, u functions of x, y, which verify this identity, or, what is the same thing, such that we have

$$X,\ Y = \lambda \frac{du}{dx},\ \lambda \frac{du}{dy};$$

and so, in the case of three variables, there exist λ, u, v functions of x, y, z, such that

$$X,\ Y,\ Z = \lambda \frac{du}{dx} + \frac{dv}{dx},\ \lambda \frac{du}{dy} + \frac{dv}{dy},\ \lambda \frac{du}{dz} + \frac{dv}{dz}.$$

The problem of determining the functions on the right-hand side is known as the Pfaffian Problem; this I do not at present consider, but only assume that there exist such functions.

The Hamiltonian System, its derivation from the general System. Art. Nos. 31 to 34.

31. Considering a bipartite set $(x, y, z : p, q, r)$, the general system of differential equations may be written

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{dp}{-X} = \frac{dq}{-Y} = \frac{dr}{-Z}.$$

But by the Pfaffian theorem we may write

$$Xdx + Ydy + Zdz + Pdp + Qdq + Rdr = \xi d\rho + \eta d\sigma + \zeta d\tau,$$

viz. there exist ξ, η, ζ, ρ, σ, τ functions of the variables x, y, z, p, q, r, such that we have

$$X = \xi \frac{d\rho}{dx} + \eta \frac{d\sigma}{dx} + \zeta \frac{d\tau}{dx}, \ldots,\quad P = \xi \frac{d\rho}{dp} + \eta \frac{d\sigma}{dp} + \zeta \frac{d\tau}{dp},\ \ldots,$$

and we have the foregoing general system expressed by means of these given functions ξ, η, ζ, ρ, σ, τ of the variables.

32. But the lineo-differential

$$Xdx + Ydy + Zdz + Pdp + Qdq + Rdr$$

may be of a more special form; for instance, it may be a sum of two terms $=\xi d\rho+\eta d\sigma$: or, finally, it may be a single term $=\xi d\rho$, and in this case we have the Hamiltonian system, viz. writing H in place of ρ, if we have

$$Xdx+Ydy+Zdz+Pdp+Qdq+Rdr=\xi dH,$$

where H is a given function of the variables, then the system is

$$\frac{dx}{\frac{dH}{dp}}=\frac{dy}{\frac{dH}{dq}}=\frac{dz}{\frac{dH}{dr}}=\frac{dp}{-\frac{dH}{dx}}=\frac{dq}{-\frac{dH}{dy}}=\frac{dr}{-\frac{dH}{dz}},$$

which is the system in question.

33. Any integral a of the system is a solution of

$$\frac{dH}{dp}\frac{d\theta}{dx}+\frac{dH}{dq}\frac{d\theta}{dy}+\frac{dH}{dr}\frac{d\theta}{dz}-\frac{dH}{dx}\frac{d\theta}{dp}-\frac{dH}{dy}\frac{d\theta}{dq}-\frac{dH}{dz}\frac{d\theta}{dr}=0;$$

viz. writing, as above,

$$(H,\ \theta)=\frac{d(H,\ \theta)}{d(p,\ x)}+\frac{d(H,\ \theta)}{d(q,\ y)}+\frac{d(H,\ \theta)}{d(r,\ z)}$$

$$=\text{last-mentioned expression},$$

the partial differential equation is $(H,\ \theta)=0$; and, conversely, any solution of this equation is an integral of the differential equations.

34. It is obvious that a solution of $(H,\ \theta)=0$ is H; hence the entire system of independent solutions may be taken to be $H,\ a,\ b,\ c,\ d$; or, if we choose to consider a set of five independent solutions $a,\ b,\ c,\ d,\ e$, then we have $H=H(a,\ b,\ c,\ d,\ e)$ a function of these solutions.

An Identity in regard to the Functions $(H,\ \theta)$. Art. Nos. 35 and 36.

35. Taking the variables to be $(x,\ y,\ z,\ p,\ q,\ r)$, and $H,\ a,\ b$ to be any functions of these variables, we have the identity

$$(H,\ (a,\ b))+(a,\ (b,\ H))+(b,\ (H,\ a))=0,$$

which is now to be proved. For this purpose we write it in the slightly different form

$$((a,\ b),\ H)=(a,\ (b,\ H))-(b,\ (a,\ H)).$$

The first term on the right-hand side is

$$\left(\frac{da}{dp}\frac{d}{dx}+\frac{da}{dq}\frac{d}{dy}+\frac{da}{dr}\frac{d}{dz}-\frac{da}{dx}\frac{d}{dp}-\frac{da}{dy}\frac{d}{dq}-\frac{da}{dz}\frac{d}{dr}\right)$$

operating upon

$$\left(\frac{db}{dp}\frac{dH}{dx}+\frac{db}{dq}\frac{dH}{dy}+\frac{db}{dr}\frac{dH}{dz}-\frac{db}{dx}\frac{dH}{dp}-\frac{db}{dy}\frac{dH}{dq}-\frac{db}{dz}\frac{dH}{dr}\right);$$

and if we herein attend to the terms which contain the second differential coefficients of H, these are symmetrical functions of a, b. For instance,

$$\begin{array}{lll}
\dfrac{d^2H}{dx^2}, & \text{coefficient is} & \dfrac{da}{dp}\dfrac{db}{dp}, \\[2ex]
\dfrac{d^2H}{dx\,dy} & \quad\text{"} \qquad\text{"} & \dfrac{da}{dp}\dfrac{db}{dq}+\dfrac{da}{dq}\dfrac{db}{dp}, \\[2ex]
\dfrac{d^2H}{dx\,dp} & \quad\text{"} \qquad\text{"} & -\dfrac{da}{dp}\dfrac{db}{dx}-\dfrac{da}{dx}\dfrac{db}{dp}, \\[2ex]
\dfrac{d^2H}{dx\,dq} & \quad\text{"} \qquad\text{"} & -\dfrac{da}{dp}\dfrac{db}{dy}-\dfrac{da}{dy}\dfrac{db}{dp}.
\end{array}$$

Hence, forming the like terms of the second terms $(b, (a, H))$ and subtracting, the terms in question all vanish: and we thus see that $(a, (b, H))-(b, (a, H))$ is a linear function of the differential coefficients

$$\frac{dH}{dx}, \quad \frac{dH}{dy}, \quad \frac{dH}{dz}, \quad \frac{dH}{dp}, \quad \frac{dH}{dq}, \quad \frac{dH}{dr}.$$

36. Attending to any one of these, suppose $\dfrac{dH}{dx}$, the coefficient of this

$$\begin{array}{ll}
\text{in } (a, (b, H)) \text{ is} & =\left(a, \dfrac{db}{dp}\right) \\[2ex]
\text{in } (b, (a, H)) \quad \text{"} & \left(b, \dfrac{da}{dp}\right), \quad =-\left(\dfrac{da}{dp}, b\right),
\end{array}$$

wherefore, in the difference of these, it is

$$\left(a, \frac{db}{dp}\right)+\left(\frac{da}{dp}, b\right), \quad =\frac{d}{dp}(a, b).$$

Hence, for the several terms

$$\frac{dH}{dx}, \quad \frac{dH}{dy}, \quad \frac{dH}{dz}, \quad \frac{dH}{dp}, \quad \frac{dH}{dq}, \quad \frac{dH}{dr},$$

the coefficients are

$$\left(\frac{d}{dp}, \frac{d}{dq}, \frac{d}{dr}, -\frac{d}{dx}, -\frac{d}{dy}, -\frac{d}{dz}\right)(a, b);$$

or, what is the same thing, we have

$$(a, (b, H))-(b, (a, H))=((a, b), H),$$

the identity in question.

The Poisson-Jacobi Theorem. Art. Nos. 37 to 39.

37. The foregoing identity shows that if $(H, a)=0$, and $(H, b)=0$, then also $(H, (a, b))=0$; or, what is the same thing, if a and b are solutions of the partial differential equation $(H, \theta)=0$, then also (a, b) is a solution; or, say, if a, b are integrals, then also (a, b) is an integral.

Supposing that the set is (x, y, z, p, q, r), so that there are in all five integrals a, b, c, d, e, then the theorem may be otherwise stated, we have (a, b) a function of the integrals a, b, c, d, e.

Observe that, knowing only the integrals a and b, we find (a, b) as a function of x, y, z, p, q, r, this may be $=0$, or a determinate constant, or it may be such a function that by virtue of the given values of a and b it reduces itself to a function of a and b; in any of these cases the theorem does *not* determine a new integral. But if contrariwise the value of (a, b), obtained as above as a function of the variables, is not a function of a, b, then it is a new integral which may be called c.

38. To obtain in this way a new integral, we require two integrals a, b other than H; for knowing only the integrals a, H, the theorem gives only (a, H) an integral, and we have of course $(a, H)=0$, viz. we do not obtain a new integral.

But starting from two integrals a, b other than H, we *may* obtain as above a new integral c; and then again (a, c) and (b, c) will be integrals, one or both of which may be new. And it may therefore happen that in this way we obtain all the independent integrals a, b, c, d, e; or the process may on the other hand terminate, without giving all the independent integrals.

The theory is obviously applicable throughout to the case of a bipartite set $(x, y, z, \ldots, p, q, r, \ldots)$ of $2n$ variables.

39. It may be remarked here that, in the Hamiltonian system, a value of the multiplier is $M=1$; and consequently, if in any way all but one of the integrals, that is, $2n-2$ integrals, be known, the remaining integral can be found by a quadrature.

It is further to be noticed that, if we adjoin a new variable t and a term $=dt$ to the system of equations; then the $2n-1$ integrals of the original system being known, all the original variables can be expressed in terms of the $2n-1$ integrals regarded as constants and of one of the variables say x: we then have

$$dt = dx \div \frac{dH}{dp},$$

or

$$t - \epsilon = \int dx \div \frac{dH}{dp},$$

or say

$$\epsilon = t - \int dx \div \frac{dH}{dp},$$

viz. if after the integration we suppose the $2n-1$ integrals replaced each of them by its value, we have

$$\epsilon = t - \phi(x, y, z, \ldots, p, q, r, \ldots),$$

which is the remaining or $2n$th integral of the original system as augmented by the term $=dt$.

The Poisson-Jacobi theorem peculiar to the Hamiltonian Form. Art. Nos. 40 to 45.

40. Taking for greater simplicity the set (x, y, p, q), and writing

$$Xdx + Ydy + Pdp + Qdq = \xi d\rho + \eta d\sigma,$$

then the general system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dp}{-X} = \frac{dq}{-Y},$$

becomes

$$\frac{dx}{\xi\frac{d\rho}{dp} + \eta\frac{d\sigma}{dp}} = \frac{dy}{\xi\frac{d\rho}{dq} + \eta\frac{d\sigma}{dq}} = \frac{dp}{-\left(\xi\frac{d\rho}{dx} + \eta\frac{d\sigma}{dx}\right)} = \frac{dq}{-\left(\xi\frac{d\rho}{dy} + \eta\frac{d\sigma}{dy}\right)};$$

and the corresponding partial differential equation (θ the independent variable) is

$$\xi(\rho, \theta) + \eta(\sigma, \theta) = 0,$$

where

$$(\rho, \theta) = \frac{d(\rho, \theta)}{d(p, x)} + \frac{d(\rho, \theta)}{d(q, y)}, \quad (\sigma, \theta) = \frac{d(\sigma, \theta)}{d(p, x)} + \frac{d(\sigma, \theta)}{d(q, y)}.$$

It is to be shown that if a, b are solutions, viz. if we have

$$\xi(\rho, a) + \eta(\sigma, a) = 0,$$
$$\xi(\rho, b) + \eta(\sigma, b) = 0,$$

implying of course

$$(\rho, a)(\sigma, b) - (\rho, b)(\sigma, a) = 0,$$

then it is *not* in general true that we have (a, b) a solution; that is, *not* in general true that

$$\xi(\rho, (a, b)) + \eta(\sigma, (a, b)) = 0;$$

the condition for the truth of this equation is in fact $\frac{\eta}{\xi}$ = a function of ρ, σ, but when this is so, $\xi d\rho + \eta d\sigma$ is λdH, viz. there exist λ, H functions of ρ, σ (and therefore ultimately of x, y, p, q) satisfying this equation, and the system is really Hamiltonian.

41. We consider whether it is true that

$$\xi(\rho, (a, b)) + \eta(\sigma, (a, b)) = 0.$$

We have identically

$$((a, b), \rho) + ((b, \rho), a) + ((\rho, a), b) = 0,$$
$$((a, b), \sigma) + ((b, \sigma), a) + ((\sigma, a), b) = 0,$$

so that multiplying by ξ, η, and adding, the equation in question is

$$\xi[((b, \rho), a) + ((\rho, a), b)] + [((b, \sigma), a) + ((\sigma, a), b)] = 0.$$

But in virtue of the equations satisfied by a, b, we may write

$$(\rho, a) = l\eta, \quad (b, \rho) = -(\rho, b) = -m\eta,$$
$$(\sigma, a) = -l\xi, \quad (b, \sigma) = -(\sigma, b) = m\xi,$$

where l, m are indeterminate functions of x, y, p, q; and the equation in question now becomes

$$\xi[-(m\eta,\ a)+(l\eta,\ b)]+\eta[(m\xi,\ a)-(l\xi,\ b)]=0;$$

that is,

$$\begin{aligned}&\xi[-m(\eta,\ a)-\eta(m,\ a)+l(\eta,\ b)+\eta(l,\ b)]\\&+\eta[\ \ m(\xi,\ a)+\xi(m,\ a)-l(\xi,\ b)-\xi(l,\ b)]=0;\end{aligned}$$

viz. omitting the terms which destroy each other, this is

$$-m\xi(\eta,\ a)+l\xi(\eta,\ b)+m\eta(\xi,\ a)-l\eta(\xi,\ b)=0.$$

Substituting for $m\xi$, &c., their values, we have

$$(\sigma,\ b)(\eta,\ a)-(\sigma,\ a)(\eta,\ b)+(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)=0;$$

and the question is whether this is implied in the equations

$$\begin{aligned}\xi(\rho,\ a)+\eta(\sigma,\ a)&=0,\\ \xi(\rho,\ b)+\eta(\sigma,\ b)&=0.\end{aligned}$$

42. Write $\eta=\kappa\xi$, the equation in question is

$$(\sigma,\ b)(\kappa\xi,\ a)-(\sigma,\ a)(\kappa\xi,\ b)+(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)=0;$$

that is,

$$\left\{\begin{aligned}&(\sigma,\ b)\,\kappa(\xi,\ a)+\xi(\sigma,\ b)(\kappa,\ a)\\&-(\sigma,\ a)\,\kappa(\xi,\ b)-\xi(\sigma,\ a)(\kappa,\ b)\end{aligned}\right\}+\{(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)\}=0;$$

viz.

$$(\xi,\ a)[(\rho,\ b)+\kappa(\sigma,\ b)]-(\xi,\ b)[(\rho,\ a)+\kappa(\sigma,\ a)]+\xi[(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)]=0;$$

and we wish to see whether this is implied in

$$\begin{aligned}(\rho,\ a)+\kappa(\sigma,\ a)&=0,\\ (\rho,\ b)+\kappa(\sigma,\ b)&=0,\end{aligned}$$

which give

$$(\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)=0;$$

or, what is the same thing, whether these last equations imply

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=0.$$

Suppose κ is a function of ρ, σ, then, as is at once seen,

$$\begin{aligned}(\kappa,\ a)&=\frac{d\kappa}{d\rho}(\rho,\ a)+\frac{d\kappa}{d\sigma}(\sigma,\ a),\\ (\kappa,\ b)&=\frac{d\kappa}{d\rho}(\rho,\ b)+\frac{d\kappa}{d\sigma}(\sigma,\ b),\end{aligned}$$

and thence

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=\frac{d\kappa}{d\rho}[(\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)];$$

viz. κ being a function of ρ and σ, the two equations imply the third.

43. But we wish to prove the converse, viz. that, if the two equations imply the third, then κ is a function of ρ, σ.

Now the equations

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=0,\quad (\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)=0,$$

are transformable into

$$\begin{array}{ll} \dfrac{d(\sigma,\ \kappa)}{d(p,\ x)}\dfrac{d(b,\ a)}{d(p,\ x)}=0, & \dfrac{d(\sigma,\ \rho)}{d(p,\ x)}\dfrac{d(b,\ a)}{d(p,\ x)}=0 \\ +q,\ y, & +q,\ y, \\ +p,\ q, & +p,\ q, \\ +p,\ y, & +p,\ y, \\ +x,\ q, & +x,\ q, \\ +x,\ y, & +x,\ y, \end{array}$$

the lines after the first being the corresponding terms with q, y, &c. instead of p, x. And if independently of the values of a, b, one of these equations implies the other, we must have

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)},\ \frac{d(\sigma,\ \kappa)}{d(q,\ y)},\ \frac{d(\sigma,\ \kappa)}{d(p,\ q)},\ \frac{d(\sigma,\ \kappa)}{d(p,\ y)},\ \frac{d(\sigma,\ \kappa)}{d(x,\ q)},\ \frac{d(\sigma,\ \kappa)}{d(x,\ y)},$$

proportional to the like expressions with σ, ρ instead of σ, κ; say these are

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ x)},\ \text{\&c.}$$

44. Assume κ a function of ρ, σ, x, y; we have

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)}=\frac{d\sigma}{dp}\left(\frac{d\kappa}{d\rho}\frac{d\rho}{dx}+\frac{d\kappa}{d\sigma}\frac{d\sigma}{dx}+\frac{d\kappa}{dx}\right)-\frac{d\sigma}{dx}\left(\frac{d\kappa}{d\rho}\frac{d\rho}{dp}+\frac{d\kappa}{d\sigma}\frac{d\sigma}{dp}\right)=\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dp},\ \text{\&c.};$$

the equations thus become

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dp}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ x)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(q,\ y)}+\frac{d\kappa}{dy}\frac{d\sigma}{dq}=\Lambda\frac{d(\sigma,\ \rho)}{d(q,\ y)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ q)}+\quad 0\quad =\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ q)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ y)}+\frac{d\kappa}{dy}\frac{d\sigma}{dp}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ y)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(q,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dq}=\Lambda\frac{d(\sigma,\ \rho)}{d(q,\ x)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(x,\ y)}-\frac{d(\sigma,\kappa)}{d(x,y)}=\Lambda\frac{d(\sigma,\ \rho)}{d(x,\ y)}.$$

Hence, unless $\frac{d(\sigma, \rho)}{d(p, q)} = 0$, we have $\Lambda = \frac{d\kappa}{dp}$. The remaining five equations then are

$$\frac{d\kappa}{dx}\frac{d\sigma}{dp} = 0, \quad \frac{d\kappa}{dy}\frac{d\sigma}{dq} = 0,$$

$$\frac{d\kappa}{dy}\frac{d\sigma}{dp} = 0, \quad \frac{d\kappa}{dx}\frac{d\sigma}{dq} = 0, \quad \frac{d\sigma}{dx}\frac{d\kappa}{dy} - \frac{d\sigma}{dy}\frac{d\kappa}{dx} = 0,$$

which give, and are satisfied by $\frac{d\kappa}{dx} = 0$, $\frac{d\kappa}{dy} = 0$, viz. we then have κ a function of ρ, σ without x, y which is the theorem in question.

45. The proof fails if $\frac{d(\sigma, \rho)}{d(p, q)} = 0$. But here, unless also $\frac{d(\sigma, \rho)}{d(x, y)} = 0$, we can, by assuming in the first instance κ a function of ρ, σ, p, q, prove in like manner that κ is a function of only ρ and σ; if however we have as well $\frac{d(\sigma, \rho)}{d(p, q)} = 0$ and $\frac{d(\sigma, \rho)}{d(x, y)} = 0$, the last-mentioned process would also fail, but it can be shown the conclusion holds good in this case also; hence the conclusion that the Poisson-Jacobi theorem holds good only for a Hamiltonian system.

Conjugate Integrals of the Hamiltonian System. Art. Nos. 46 to 51.

46. For greater clearness, let $n = 4$, or let the variables be x, y, z, w, p, q, r, s; the system of differential equations therefore is

$$\frac{dx}{\frac{dH}{dp}} = \frac{dy}{\frac{dH}{dq}} = \frac{dz}{\frac{dH}{dr}} = \frac{dw}{\frac{dH}{ds}} = \frac{dp}{-\frac{dH}{dx}} = \frac{dq}{-\frac{dH}{dy}} = \frac{dr}{-\frac{dH}{dz}} = \frac{ds}{-\frac{dH}{dw}},$$

and any integral hereof is as before a solution of $(H, \theta) = 0$. Assume that the integrals are H, a, b, c, d, e, f, so that

$$(H, a) = 0, \quad (H, b) = 0, \quad (H, c) = 0, \quad (H, d) = 0, \quad (H, e) = 0, \quad (H, f) = 0.$$

Considering here a as denoting any integral whatever, that is, any solution whatever of the partial differential equation $(H, \theta) = 0$, it is to be shown that it is possible to determine θ so as to satisfy as well this equation $(H, \theta) = 0$, as also the new equation $(a, \theta) = 0$.

47. We, in fact, satisfy the first equation by taking

$$\theta, = \theta(H, a, b, c, d, e, f),$$

any function whatever of the seven integrals. But, θ having this value, we find

$$(a, \theta) = (a, H)\frac{d\theta}{dH} + (a, a)\frac{d\theta}{da} + (a, b)\frac{d\theta}{db} + (a, c)\frac{d\theta}{dc} + (a, d)\frac{d\theta}{dd} + (a, e)\frac{d\theta}{de} + (a, f)\frac{d\theta}{df};$$

or, since the first two terms on the right-hand vanish, the equation $(a, \theta)=0$ thus becomes

$$(a, b)\frac{d\theta}{db}+(a, c)\frac{d\theta}{dc}+(a, d)\frac{d\theta}{dd}+(a, e)\frac{d\theta}{de}+(a, f)\frac{d\theta}{df}=0.$$

But by the Poisson-Jacobi theorem (a, b), &c., are each of them a solution of $(H, \theta)=0$, viz. they are each of them a function of H, a, b, c, d, e, f. This is then a linear partial differential equation wherein the variables are H, a, b, c, d, e, f; or, since there are no terms in $\frac{d\theta}{dH}, \frac{d\theta}{da}$, we may regard a, H as constants, and treat it as a linear partial differential equation in b, c, d, e, f, the solutions of the equation being in fact the integrals, or any functions of the integrals, of

$$\frac{db}{(a, b)}=\frac{dc}{(a, c)}=\frac{dd}{(a, d)}=\frac{de}{(a, e)}=\frac{df}{(a, f)}.$$

48. Suppose any four integrals are b', c', d', e', so that a general integral is $\phi(H, a, b', c', d', e')$, then b', c', d', e' quà functions of H, a, b, c, d, e, f are integrals of the original equation $(H, \theta)=0$; hence *changing the notation* and writing b, c, d, e in place of these accented letters we have (H, a, b, c, d, e) as solutions of the two equations $(H, \theta)=0$, $(a, \theta)=0$; viz. a being any integral of the first of these equations, we see how to find four other integrals (b, c, d, e) which are such that

$$(H, a)=0,\quad (H, b)=0,\quad (H, c)=0,\quad (H, d)=0,\quad (H, e)=0,$$
$$(a, b)=0,\quad (a, c)=0,\quad (a, d)=0,\quad (a, e)=0.$$

49. We proceed in the same course and endeavour to find θ, so that not only $(H, \theta)=0$, $(a, \theta)=0$, but also $(b, \theta)=0$. Assuming here $\theta=\theta(H, a, b, c, d, e)$ an arbitrary function of the integrals, the first and second equations are satisfied; for the third equation, we have

$$(b, \theta)=(b, H)\frac{d\theta}{dH}+(b, a)\frac{d\theta}{da}+(b, b)\frac{d\theta}{db}+(b, c)\frac{d\theta}{dc}+(b, d)\frac{d\theta}{dd}+(b, e)\frac{d\theta}{de};$$

viz. the first three terms here vanish, and the equation $(b, \theta)=0$ becomes

$$(b, c)\frac{d\theta}{dc}+(b, d)\frac{d\theta}{dd}+(b, e)\frac{d\theta}{de}=0,$$

where, b, c, d, e being solutions as well of $(H, \theta)=0$ as of $(a, \theta)=0$, we have (b, c) a solution of these two equations, and as such a function of H, a, b, c, d, e; and so (b, d) and (b, e) are each of them a function of the same variables. The above is therefore a linear partial differential equation wherein the variables are H, a, b, c, d, e, but as the equation does not contain $\frac{d\theta}{dH}, \frac{d\theta}{da}$, or $\frac{d\theta}{db}$, we may regard H, a, b as constants; and the solutions of the equation are, in fact, the integrals of

$$\frac{dc}{(b, c)}=\frac{dd}{(b, d)}=\frac{de}{(b, e)}.$$

50. Supposing that any two integrals are c', d', so that a general integral is $\phi(H, a, b, c', d')$, then c', d' quà functions of H, a, b, c, d, e are integrals of the former equations $(H, \theta)=0$, $(a, \theta)=0$, so that again *changing the notation*, and writing c, d instead of the accented letters, we have (H, a, b, c, d) as solutions of the three equations $(H, \theta)=0$, $(a, \theta)=0$, $(b, \theta)=0$, viz. a being any solution of the first equation, and b any solution of the first and second equations, we see how to find two others c, d, of the same two equations, which are such that

$$\begin{aligned} (H, a)=0,\quad (H, b)=0,\quad (H, c)=0,\quad (H, d)=0,\\ (a, b)=0,\quad (a, c)=0,\quad (a, d)=0,\\ (b, c)=0,\quad (b, d)=0; \end{aligned}$$

or, attending only to the integrals H, a, b, c, these are integrals of the equations $(H, \theta)=0$, $(a, \theta)=0$, $(b, \theta)=0$, such that

$$(H, a)=0,\quad (H, b)=0,\quad (H, c)=0,\quad (a, b)=0,\quad (a, c)=0,\quad (b, c)=0.$$

We here say that H, a, b, c are a system of conjugate solutions. Attempting to continue the process, it would appear that there is not any new independent integral d, such that $(H, d)=0$, $(a, d)=0$, $(b, d)=0$, $(c, d)=0$ (the first three of these are satisfied by the integral d found above, but the last of them is not); we may, however, taking d an arbitrary function of H, a, b, c, replace H by d; viz. we thus have the four integrals a, b, c, d, such that

$$(a, b)=0,\quad (a, c)=0,\quad (a, d)=0,\quad (b, c)=0,\quad (b, d)=0,\quad (c, d)=0,$$

and which are consequently said to form a conjugate system.

51. The process is of course general, and it shows how, in the case of a Hamiltonian system of $2n$ variables, it is possible to find a system $H, a, b, \ldots, f$ consisting of H and $n-1$ other integrals, or, if we please, a system of n integrals $a, b, \ldots, f, g$, such that the derivative of any two integrals whatever of the system is $=0$; any such system is termed a conjugate system.

Hamiltonian System—the function V. Art. Nos. 52 to 58.

52. Taking a Hamiltonian system with the original variables x, y, z, p, q, r, we adjoin the two new variables t, V, forming the extended system

$$\frac{dx}{\frac{dH}{dp}}=\frac{dy}{\frac{dH}{dq}}=\frac{dz}{\frac{dH}{dr}}=\frac{dp}{-\frac{dH}{dx}}=\frac{dq}{-\frac{dH}{dy}}=\frac{dr}{-\frac{dH}{dz}}=dt=\frac{dV}{p\frac{dH}{dp}+q\frac{dH}{dq}+r\frac{dH}{dr}}.$$

Supposing the integrals of the original system to be a, b, c, d, e, we have $H=H(a, b, c, d, e)$ a determinate function of these integrals; also an integral $\tau=t-\phi(x, y, z, p, q, r)$ and an integral $\lambda=V-\psi(x, y, z, p, q, r)$; these integrals, exclusive of the last of them, serve to express x, y, z, p, q, r as functions of $a, b, c, d, e, t-\tau$; and the last integral then gives $V=\lambda+$ a function of the last-mentioned quantities.

53. We consider the differential expression

$$dV - p\,dx - q\,dy - r\,dz,$$

which, treating the integrals as constants, that is, in the expressions of V, x, y, z, regarding t as the only variable, is at once seen to be $=0$; hence, if we regard all the integrals as variables, the value is

$$= d\lambda + Ada + Bdb + Cdc + Ddd + Ede,$$

without any term in $d\tau$, since this enters originally in the form $dt - d\tau$, and therefore disappears with dt.

The coefficients A, B, C, D, E are of course functions of a, b, c, d, e, $t-\tau$; it is to be shown that they contain $t-\tau$ linearly, viz. that in these coefficients respectively the coefficients of $t-\tau$ are

$$\frac{dH}{da},\ \frac{dH}{db},\ \frac{dH}{dc},\ \frac{dH}{dd},\ \frac{dH}{de},$$

where H is expressed as above in the form $H(a, b, c, d, e)$; this being so, the entire term in $t-\tau$ will be $(t-\tau)\,dH$; each coefficient, for instance A, has besides a part A', which is a function of a, b, c, d, e without $t-\tau$, or *changing the notation* and writing the unaccented letters to denote these parts of the original coefficients, the final result is

$$dV - p\,dx - q\,dy - r\,dz = (t-\tau)\,dH + d\lambda + Ada + Bdb + Cdc + Ddd + Ede,$$

where H stands for its value $H(a, b, c, d, e)$, and A, B, C, D, E are functions of a, b, c, d, e without $t-\tau$.

54. To prove the theorem, we have

$$A = \frac{dV}{da} - p\frac{dx}{da} - q\frac{dy}{da} - r\frac{dz}{da},$$

and thence

$$\begin{aligned}\frac{dA}{dt} &= \frac{d^2V}{da\,dt} - \frac{dp}{dt}\frac{dx}{da} - \frac{dq}{dt}\frac{dy}{da} - \frac{dr}{dt}\frac{dz}{da} - p\frac{d^2x}{da\,dt} - q\frac{d^2y}{da\,dt} - r\frac{d^2z}{da\,dt}\\ &= \frac{d}{da}\left\{\frac{dV}{dt} - p\frac{dx}{dt} - q\frac{dy}{dt} - r\frac{dz}{dt}\right\}\\ &\qquad + \frac{dp}{da}\frac{dx}{dt} + \frac{dq}{da}\frac{dy}{dt} + \frac{dr}{da}\frac{dz}{dt} - \frac{dp}{dt}\frac{dx}{da} - \frac{dq}{dt}\frac{dy}{da} - \frac{dr}{dt}\frac{dz}{da};\end{aligned}$$

and then substituting for $\frac{dV}{dt}$, &c., their values from the system of differential equations, the first line vanishes, and the second line becomes

$$\begin{aligned}&= \frac{dH}{dp}\frac{dp}{da} + \frac{dH}{dq}\frac{dq}{da} + \frac{dH}{dr}\frac{dr}{da} + \frac{dH}{dx}\frac{dx}{da} + \frac{dH}{dy}\frac{dy}{da} + \frac{dH}{dz}\frac{dz}{da},\\ &= \frac{dH}{da};\end{aligned}$$

and hence $A=(t-\tau)\dfrac{dH}{da}+A'$, and the like for the other coefficients B, C, D, E, which is the theorem in question.

55. We may have between two coefficients of the formula, for instance, D and E, a relation $\dfrac{dD}{de}=\dfrac{dE}{dd}$, and I will for the present assume, without proving it, the theorem that if a, b, c are conjugate integrals, then this relation $\dfrac{dD}{de}-\dfrac{dE}{dd}=0$, holds good, merely mentioning that the proof depends on the consideration of certain symbols $[a, b]$, which are the converses, so to speak, of the symbols (a, b), viz. considering the variables x, y, z, p, q, r as given functions of a, b, c, d, e, $t-\tau$, then we have

$$[a, b]=\frac{d(p, x)}{d(a, b)}+\frac{d(q, y)}{d(a, b)}+\frac{d(r, z)}{d(a, b)}.$$

The assumption is used only in the two following Nos. 56 and 57.

56. Supposing then that a, b, c are conjugate integrals, we have $\dfrac{dD}{de}-\dfrac{dE}{dd}=0$, and there exists therefore ϕ, a function of a, b, c, d, e, such that

$$d\phi=A'\,da+B'\,db+C'\,dc+D\,dd+E\,de,$$

(A', B', C' functions of the same quantities a, b, c, d, e), we have therefore

$$dV-p\,dx-q\,dy-r\,dz=d\lambda+(t-\tau)\,dH+d\phi+(A-A')\,da+(B-B')\,db+(C-C')\,dc.$$

Taking as above a, b conjugate integrals $(a, b)=0$, and c any function whatever of a, b, H, then a, b, c are conjugate integrals, and the formula holds good. Suppose further that a, b, H are absolute constants, then $dH=0$, $da=0$, $db=0$, $dc=0$, and the formula becomes

$$dV-p\,dx-q\,dy-r\,dz=d\lambda+d\phi;$$

or, writing this under the form,

$$p\,dx+q\,dy+r\,dz=dV-d\lambda-d\phi,$$

it follows that $p\,dx+q\,dy+r\,dz$ is an exact differential, a theorem which may be stated as follows: viz. if a, b are conjugate integrals of the Hamiltonian system, and if from the equations $H=\text{const.}$, $a=\text{const.}$, $b=\text{const.}$, we express p, q, r as functions of x, y, z, then $p\,dx+q\,dy+r\,dz$ is an exact differential; or, what is the same thing, p, q, r are the differential coefficients $\dfrac{dU}{dx}$, $\dfrac{dU}{dy}$, $\dfrac{dU}{dz}$ of U a function of x, y, z. This is, in fact, a fundamental theorem in regard to the partial differential equation $H=\text{const.}$, and it will presently be proved in a different manner.

57. If, as before, a and b are conjugate integrals, then, writing as we may do λ in place of $\lambda+\phi$, and finding V as a function of x, y, z, a, b, H from the equation

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

and again treating a, b, H as variable, we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + A\,da + B\,db,$$

where A, B are functions of the integrals a, b, c, d, e, that is, they are themselves integrals, which may be taken for the integrals d, e, or we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + d\,da + e\,db;$$

we have therefore

$$\frac{dV}{da} = d, \quad \frac{dV}{db} = e,$$

equations which, on substituting therein for a, b, H their values as functions of x, y, z, p, q, r, determine the integrals d, e, which with a, b, H or a, b, c, are the remaining integrals of the Hamiltonian system; and further

$$\frac{dV}{dH} = t - \tau,$$

which, when in like manner, we substitute therein for a, b, H, their values as functions of x, y, z, p, q, r, determines τ, the remaining integral of the system as increased by the equality $= dt$.

58. Reverting to the general theorem No. 52, let x_0, y_0, z_0, p_0, q_0, r_0, t_0 be corresponding values of the variables x, y, z, p, q, r, t; and let a_0, &c., ..., V_0 be the same functions of x_0, y_0, z_0, p_0, q_0, r_0, t_0 that a, &c., ..., V are of the variables; we have $a = a_0, \ldots, e = e_0$, and corresponding to the equation

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + A\,da + \ldots + E\,de,$$

the like equation

$$dV_0 - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0 = d\lambda + (t_0-\tau)\,dH + A\,da + \ldots + E\,de.$$

Hence, subtracting

$$dV - dV_0 = (t-t_0)\,dH + p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0,$$

or, considering only H as an absolute constant,

$$dV - dV_0 = \qquad p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0;$$

viz. if from the equations $H = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$, we express p, q, r, p_0, q_0, r_0 as functions of x, y, z, x_0, y_0, z_0, H, then

$$p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0$$

will be an exact differential. And in particular regarding x_0, y_0, z_0 as constants, then $p\,dx + q\,dy + r\,dz$ is an exact differential, viz. there exists a function

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz).$$

We have thus again arrived at a solution of the partial differential equation $H = \text{const.}$

The Partial Differential Equation $H = const.$ Art. Nos. 59 to 70.

59. In what just precedes we have, in fact, brought the theory of the Hamiltonian system into connexion with a partial differential equation, viz. we have determined the variables p, q, r as functions of x, y, z such that $p\,dx + q\,dy + r\,dz$ is an exact differential $= dV$; but we now consider the subject in a more regular manner.

The partial differential equation is $H = \text{const.}$ viz. here H denotes, in the first instance, a given function of p, q, r, x, y, z, where p, q, r are the differential coefficients of a function V of x, y, z, or, what is the same thing, there exists a function V of x, y, z such that $p\,dx + q\,dy + r\,dz = dV$; and then, this function H being constant, we use the same letter H to denote the constant value of the function. The equation $H = \text{const.}$ is the most general form of a partial differential equation of the first order which contains the independent variable only through its differential coefficients p, q, r, and it is for convenience put in a form containing the arbitrary constant H, which constant might without loss of generality be put $= 0$ or $=$ any other determinate value.

60. We seek to determine p, q, r as functions of x, y, z, satisfying the given equation $H = \text{const.}$, and such that we have $p\,dx + q\,dy + r\,dz$ an exact differential $= dV$; this would be done if we can find two other equations $K = \text{const.}$ and $L = \text{const.}$, such that the values of p, q, r obtained from the three equations give p, q, r functions having the property in question. Attending to only two of the equations, say $H = \text{const.}$ and $K = \text{const.}$, we have here p, q, r functions of x, y, z, such that $p\,dx + q\,dy + r\,dz$ is an exact differential, and two of the equations which serve to determine p, q, r as functions of x, y, z are $H = \text{const.}$, $K = \text{const.}$ We have to prove the following fundamental theorem, viz. that $(H, K) = 0$.

61. In fact, from the equations $H = \text{const.}$, $K = \text{const.}$, treating x, y, z as independent variables, we have

$$\frac{dH}{dx} + \frac{dH}{dp}\frac{dp}{dx} + \frac{dH}{dq}\frac{dq}{dx} + \frac{dH}{dr}\frac{dr}{dx} = 0,$$

$$\frac{dK}{dx} + \frac{dK}{dp}\frac{dp}{dx} + \frac{dK}{dq}\frac{dq}{dx} + \frac{dK}{dr}\frac{dr}{dx} = 0;$$

and if from these equations in order to eliminate $\frac{dp}{dx}$ we multiply by $\frac{dK}{dp}$, $-\frac{dH}{dp}$, and add, we find

$$\frac{d(K, H)}{d(p, x)} + \qquad\qquad + \frac{d(K, H)}{d(p, q)}\frac{dq}{dx} + \frac{d(K, H)}{d(p, r)}\frac{dr}{dx} = 0;$$

and, in precisely the same way,

$$\frac{d(K, H)}{d(q, y)} + \frac{d(K, H)}{d(q, p)}\frac{dp}{dy} + \qquad\qquad + \frac{d(K, H)}{d(q, r)}\frac{dr}{dy} = 0,$$

$$\frac{d(K, H)}{d(r, z)} + \frac{d(K, H)}{d(r, p)}\frac{dp}{dz} + \frac{d(K, H)}{d(r, q)}\frac{dq}{dz} + \qquad\qquad = 0.$$

Adding these together, we have

$$(K, H)+\frac{d(K, H)}{d(q, r)}\left(\frac{dr}{dy}-\frac{dq}{dz}\right)+\frac{d(K, H)}{d(r, p)}\left(\frac{dp}{dz}-\frac{dr}{dx}\right)+\frac{d(K, H)}{d(p, q)}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)=0;$$

viz. if $p\,dx+q\,dy+r\,dz$ be an exact differential, then $(H, K)=0$, which is the theorem in question.

62. In the case where the variables are (x, y, p, q), we have simply

$$(K, H)+\frac{d(K, H)}{d(p, q)}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)=0;$$

viz. $p\,dx+q\,dy$ being a complete differential, $(K, H)=0$. Conversely, if $(K, H)=0$, then $\frac{dq}{dx}-\frac{dp}{dy}=0$, and $p\,dx+q\,dy$ is an exact differential; viz. this is so unless $\frac{d(K, H)}{d(p, q)}=0$; this equation would imply that K, H considered as functions of p, q, are functions one of the other: and, supposing it to hold good, we could not from the equations $H=0$, $K=0$ determine p, q as functions of x, y, for, eliminating one of the variables p, q, the other would disappear of itself. We hence obtain the complete statement of the converse theorem, viz. the functions H, K being such that it is possible from the equations $H=0$, $K=0$ to express p, q as functions of x, y, then, if $(H, K)=0$, we have $p\,dx+q\,dy$ an exact differential.

63. Returning to the case of the variables (x, y, z, p, q, r), if p, q, r are determined as functions of x, y, z by the three equations $H=0$, $K=0$, $L=0$, then, by what precedes, in order that $p\,dx+q\,dy+r\,dz$ may be a complete differential, we must have $(H, K)=0$, $(H, L)=0$, $(K, L)=0$; and it further appears that if these equations are satisfied, then we have, conversely,

$$\frac{dr}{dy}-\frac{dq}{dz}=0,\quad \frac{dp}{dz}-\frac{dr}{dx}=0,\quad \frac{dq}{dx}-\frac{dp}{dy}=0,$$

that is, $p\,dx+q\,dy+r\,dz$ is an exact differential; viz. this is the case unless we have between H, K, L the relation

$$\begin{vmatrix} \frac{d(H, K)}{d(q, r)}, & \frac{d(H, K)}{d(r, p)}, & \frac{d(H, K)}{d(p, q)} \\ H, L\ , & H, L\ , & H, L \\ K, L\ , & K, L\ , & K, L \end{vmatrix}=0,$$

where in the determinant the second and third lines are the same functions of H, L and K, L respectively that the first line is of H, L.

The determinant is, in fact, equal to the square of

$$\frac{d(H, K, L)}{d(p, q, r)},$$

and, if it vanish, it is impossible, by means of the equations $H=0$, $K=0$, $L=0$, to determine p, q, r as functions of x, y, z. Hence, if the last-mentioned equations are such that by means of them it is possible to effect the determination, and if, moreover, $(H, K)=0$, $(H, L)=0$, $(K, L)=0$, then $p\,dx+q\,dy+r\,dz$ will be an exact differential.

64. Considering H as given, we have, by what precedes, K, L solutions of the linear partial differential equation $(H, \theta)=0$; and since also K, L must be such that $(K, L)=0$, they are conjugate solutions; or in conformity with what precedes, using the small letters a, b instead of K, L, we have the following theorem for the integration of the partial differential equation $H=\text{const.}$, where as before H is a given function of x, y, z, p, q, r.

Find a and b, such that H, a, b are a system of conjugate solutions of the linear partial differential equation $(H, \theta)=0$: then from the equations $H=\text{const.}$, $a=\text{const.}$, $b=\text{const.}$, determining p, q, r as functions of a, b, H, and in the result treating these quantities as constants, we have $p\,dx+q\,dy+r\,dz$ an exact differential $=dV$, and thence

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

an expression for V containing the three arbitrary constants λ, a, b, and therefore a complete solution of the given partial differential equation $H=\text{const.}$

The theorem applies to the case where n has any value whatever, viz. if there are n variables $x, y, z, \ldots$, then we have to find the $n-1$ integrals $a, b, c, \ldots$, constituting with H a system of conjugate integrals; and the theorem holds good.

In particular, if $n=2$, or the independent variables are x and y, then we find any solution a of the partial differential equation $(H, \theta)=0$; the values p, q derived from the equations $H=\text{const.}$, $a=\text{const.}$, give $V=\lambda+\int(p\,dx+q\,dy)$, a complete solution.

65. But there is a different solution depending on the consideration of corresponding values; viz. if the independent variables be as before x, y, z, p, q, r, and if x_0, y_0, z_0, p_0, q_0, r_0 are corresponding values of x, y, z, p, q, r, then, taking a, b, c, d, e to be integrals of $(H, \theta)=0$: so that H is here a given function of a, b, c, d, e, since the number of independent variables is $=5$: and representing by a_0, b_0, c_0, d_0, e_0 the like functions of x_0, y_0, z_0, p_0, q_0, r_0, we form the equations

$$H=\text{const.},\ a=a_0,\ b=b_0,\ c=c_0,\ d=d_0,\ e=e_0.$$

We have the theorem that, expressing by means of these equations p, q, r, as functions of x, y, z, x_0, y_0, z_0, H, and regarding therein x_0, y_0, z_0, H as constants, we have $p\,dx+q\,dy+r\,dz$ an exact differential, and therefore

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

a solution of the equation $H=\text{const.}$ involving the arbitrary constants λ, x_0, y_0, z_0 (one more than required for a complete solution).

The theorem is here stated in the form proper for the solution of the partial differential equation $H=\text{const.}$; a more general statement will be given further on.

66. I take first $n=2$, or the independent variables to be x, y; here p, q are determined by the equations $a=a_0$, $b=b_0$, $c=c_0$, $H=\text{const.}$, and it is to be shown that $p\,dx+q\,dy=dV$.

Considering p, q, p_0, q_0 as functions of the independent variables x, y, then differentiating in regard to x, and eliminating $\frac{dp}{dx}$, $\frac{dp_0}{dx}$, $\frac{dq_0}{dx}$, we find

$$\begin{vmatrix} \frac{da}{dx}+\frac{da}{dq}\frac{dq}{dx}, & \frac{da}{dp}, & \frac{da_0}{dp_0}, & \frac{da_0}{dq_0} \\ \frac{db}{dx}+\frac{db}{dq}\frac{dq}{dx}, & \frac{db}{dp}, & \frac{db_0}{dp_0}, & \frac{db_0}{dq_0} \\ \frac{dc}{dx}+\frac{dc}{dq}\frac{dq}{dx}, & \frac{dc}{dp}, & \frac{dc_0}{dp_0}, & \frac{dc_0}{dq_0} \\ \frac{dH}{dx}+\frac{dH}{dq}\frac{dq}{dx}, & \frac{dH}{dp}, & 0\,, & 0 \end{vmatrix}=0,$$

viz. this is

$$\frac{d(a_0,\,b_0)}{d(p_0,\,q_0)}\left\{\frac{d(H,\,c)}{d(p,\,x)}+\frac{d(H,\,c)}{d(p,\,q)}\frac{dq}{dx}\right\}+\&\text{c.}=0.$$

But in the same way

$$\frac{d(a_0,\,b_0)}{d(p_0,\,q_0)}\left\{\frac{d(H,\,c)}{d(q,\,y)}+\frac{d(H,\,c)}{d(q,\,p)}\frac{dp}{dy}\right\}+\&\text{c.}=0\,;$$

adding these two equations we have

$$\frac{d(a_0,\,b_0)}{d(p_0,\,q_0)}\left\{(H,\,c)+\frac{d(H,\,c)}{d(p,\,q)}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)\right\}+\&\text{c.}=0,$$

the terms denoted by the &c. being the like terms with b, c, a and c, a, b in place of a, b, c. We have $(H,\,a)=0$, $(H,\,b)=0$, $(H,\,c)=0$, and the equation, in fact, is

$$\left\{\Sigma\frac{d(a_0,\,b_0)}{d(p_0,\,q_0)}\frac{d(H,\,c)}{d(p,\,q)}\right\}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)=0\,;$$

viz. we have $\frac{dq}{dx}-\frac{dp}{dy}=0$, the condition for an exact differential.

67. Coming now to the case where the independent variables are x, y, z, we proceed in the same way with the equations $H=\text{const.}$, $a=a_0$, $b=b_0$, $c=c_0$, $d=d_0$, $e=e_0$. Differentiating in regard to x, and eliminating

$$\frac{dp}{dx},\ \frac{dq}{dx},\ \frac{dp_0}{dx},\ \frac{dq_0}{dx},\ \frac{dr_0}{dx},$$

we find for $\frac{dr}{dx}$ the equation

$$\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}\left\{\frac{dr}{dx}\frac{d(a, b, H)}{d(r, p, q)} + \frac{d(a, b, H)}{d(x, p, q)}\right\} + \&c. = 0.$$

We have in the same way for $\frac{dp}{dz}$ the equation

$$\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}\left\{\frac{dp}{dz}\frac{d(a, b, H)}{d(p, r, q)} + \frac{d(a, b, H)}{d(z, r, q)}\right\} + \&c. = 0,$$

whence, adding, we obtain

$$\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}\left\{\left(\frac{dr}{dx} - \frac{dp}{dz}\right)\frac{d(a, b, H)}{d(r, p, q)} + \frac{d(a, b, H)}{d(x, p, q)} + \frac{d(a, b, H)}{d(r, z, q)}\right\} + \&c. = 0,$$

where the terms denoted by the &c. are the like terms corresponding to the different permutations of the letters a, b, c, d, e.

The equation may be simplified; we have identically

$$-\frac{da}{dq}(b, H) - \frac{db}{dq}(H, a) - \frac{dH}{dq}(a, b) = \frac{d(a, b, H)}{d(x, p, q)} + \frac{d(a, b, H)}{d(z, r, q)};$$

or, since $(H, a) = 0$, $(H, b) = 0$, the left-hand side is simply $-\frac{dH}{dq}(a, b)$, and the equation becomes

$$\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}\left\{\left(\frac{dr}{dx} - \frac{dp}{dz}\right)\frac{d(a, b, H)}{d(r, p, q)} - \frac{dH}{dq}(a, b)\right\} + \&c. = 0.$$

68. This ought to give $\frac{dr}{dx} - \frac{dp}{dz} = 0$; it will, if only

$$\Sigma\left\{\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}(a, b)\right\} = 0,$$

which is thus the condition which has to be proved. By the Poisson-Jacobi theorem, (a, b) is a function of a, b, c, d, e: if we write

$$(a_0, b_0) = \frac{d(a_0, b_0)}{d(p_0, x_0)} + \frac{d(a_0, b_0)}{d(q_0, y_0)} + \frac{d(a_0, b_0)}{d(r_0, z_0)},$$

then (a_0, b_0) is the same function of a_0, b_0, c_0, d_0, e_0; but these are equal to a, b, c, d, e respectively, and we then have $(a, b) = (a_0, b_0)$, and the theorem to be proved is

$$\Sigma\left\{\frac{d(c_0, d_0, e_0)}{d(p_0, q_0, r_0)}(a_0, b_0)\right\} = 0.$$

But, substituting for (a_0, b_0) its value, the function on the left-hand side is, it is easy to see, the sum of the three functional determinants

$$\frac{d(a_0, b_0, c_0, d_0, e_0)}{d(p_0, q_0, r_0, p_0, x_0)}, \quad \frac{d(a_0, b_0, c_0, d_0, e_0)}{d(p_0, q_0, r_0, q_0, y_0)}, \quad \frac{d(a_0, b_0, c_0, d_0, e_0)}{d(p_0, q_0, r_0, r_0, z_0)},$$

and each of these, as containing the same letter twice in the denominator, that is, as having two identical columns, is $=0$; the theorem is thus proved. And in the same way $\frac{dp}{dy}-\frac{dq}{dx}$, $\frac{dq}{dz}-\frac{dr}{dy}$ are each $=0$; that is, $p\,dx+q\,dy+r\,dz=dV$.

69. The proof would fail if the factors multiplying $\frac{dr}{dx}-\frac{dp}{dy}$, &c., or any one of these factors, were $=0$. I have not particularly examined this, but the meaning must be that here the equations $a=a_0$, &c., $H=$ const., fail to give for p, q, r expressions as functions of x, y, z, x_0, y_0, z_0, H; whenever such expressions are obtainable, we have

$$p\,dx+q\,dy+r\,dz=dV.$$

The proof in the case of a greater number of variables, say in the next case where the independent variables are x, y, z, w, would probably present greater difficulty, but I have not examined this.

70. Taking the independent variables to be x and y, we may from the equations $a=a_0$, $b=b_0$, $c=c_0$, $H=$ const. (which last equation may also be written $H=H_0=$ const.) find p, q, p_0, q_0 as functions of x, y, x_0, y_0, H; and we have then the theorem that, considering only H as a constant,

$$p\,dx+q\,dy-p_0\,dx_0-q_0\,dy_0=dV.$$

To show this, we have to prove the further equations $\frac{dp}{dx_0}+\frac{dp_0}{dx}=0$, &c.; we find

$$\frac{dp}{dx_0}\,\Sigma\left\{\frac{d(b_0,\ c_0)}{d(p_0,\ q_0)}\,\frac{d(a,\ H)}{d(p,\ q)}\right\}-\frac{dH}{dq}\,\frac{d(a_0,\ b_0,\ c_0)}{d(x_0,\ p_0,\ q_0)}=0,$$

$$\frac{dp_0}{dx}\,\Sigma\left\{\frac{d(b,\ c)}{d(p,\ q)}\,\frac{d(a_0,\ H_0)}{d(p_0,\ q_0)}\right\}-\frac{dH_0}{dq_0}\,\frac{d(a,\ b,\ c)}{d(x,\ p,\ q)}=0,$$

and it is to be shown that the coefficients of $\frac{dp}{dx_0}$, $\frac{dp_0}{dx}$ are equal and of opposite signs, and that the other two terms are equal; viz. this being so, subtracting the two equations, we have the required relation $\frac{dp}{dx_0}+\frac{dp_0}{dx}=0$. Now H, H_0 are the same functions of a, b, c and of a_0, b_0, c_0; and there is no real loss of generality in assuming $c=H$, $c_0=H_0$; but this being so, the first coefficient is

$$\frac{d(b_0,\ H_0)}{d(p_0,\ q_0)}\,\frac{d(a,\ H)}{d(p,\ q)}+\frac{d(H_0,\ a_0)}{d(p_0,\ q_0)}\,\frac{d(b,\ H)}{d(p,\ q)},$$

and the second is

$$\frac{d(b,\ H)}{d(p,\ q)}\,\frac{d(a_0,\ H_0)}{d(p_0,\ q_0)}+\frac{d(H,\ a)}{d(p,\ q)}\,\frac{d(b_0,\ H_0)}{d(p_0,\ q_0)},$$

which only differ by their signs. As regards the other two terms, we have identically

$$\frac{da}{dq}(b,\ H)+\frac{db}{dq}(H,\ a)+\frac{dH}{dq}(a,\ b)=\frac{d(a,\ b,\ H)}{d(x,\ p,\ q)},$$

which, in virtue of $(a, H)=0$, $(b, H)=0$, becomes

$$\frac{dH}{dq}(a, b)=\frac{d(a, b, H)}{d(x, p, q)};$$

similarly,

$$\frac{dH_0}{dq_0}(a_0, b_0)=\frac{d(a_0, b_0, H_0)}{d(x_0, p_0, q_0)}.$$

Hence the terms in question are

$$-\frac{dH}{dq}\frac{dH_0}{dq_0}(a_0, b_0), \quad -\frac{dH}{dq}\frac{dH_0}{dq_0}(a, b),$$

which are equal in virtue of $(a, b)=(a_0, b_0)$; and, similarly, the other conditions might be proved. But the proof would be more difficult in the case of a greater number of variables.

Examples. Art. Nos. 71 to 79.

71. The variables are taken to be x, y, z, p, q, r. As a first example, which will serve as an illustration of most of the preceding theorems, suppose $pqr-1=H$; the Hamiltonian system, with the adjoined equalities, is here

$$\frac{dx}{qr}=\frac{dy}{rp}=\frac{dz}{pq}=\frac{dp}{0}=\frac{dq}{0}=\frac{dr}{0}=dt=\frac{dV}{3pqr}.$$

The integrals of the original system may be taken to be

$$\begin{aligned} a&=p,\\ b&=q,\\ c&=r,\\ d&=qy-px,\\ e&=rz-px, \end{aligned}$$

and there is of course the integral $H=pqr-1$, which is connected with the foregoing five integrals by the relation $H=abc-1$.

We form at once the equations

$$\begin{aligned} (a, b)&=0, & (a, c)&=0, & (a, d)&=-a, & (a, e)&=-a,\\ & & (b, c)&=0, & (b, d)&=b, & (b, e)&=0,\\ & & & & (c, d)&=0, & (c, e)&=c,\\ & & & & & & (d, e)&=0; \end{aligned}$$

hence it happens that no two of these integrals a, b, c, d, e give by the Poisson-Jacobi theorem a new integral. To show how the theorem might have given a new integral, suppose that the known integrals had been $\alpha=p+q$, and $e=rz-px$, then $(\alpha, e)=-p$: or the theorem gives the new integral $\alpha=p$.

We have as a conjugate system a, b, c; also the conjugate systems H, a, b; H, a, c; H, b, c; H, b, e; H, c, d; H, d, e; but the first three of these, considering therein H as standing for its value $abc-1$, are substantially equivalent to the first-mentioned system (a, b, c).

72. Postponing the consideration of the augmented system, we now consider the partial differential equation $pqr = 1 + H$, where H is a given constant and p, q, r denote the differential coefficients of a function V. The most simple solution is that given by the conjugate system H, a, b, viz. here p, q, r are determined by the equations $p = a$, $q = b$, $pqr = 1 + H$, that is, $r = \dfrac{1+H}{ab}$; or, introducing for symmetry the constant c, where $abc = 1 + H$ as before, then $r = c$, and we have

$$V = \lambda + \int (adx + bdy + cdz), \quad = \lambda + ax + by + cz,$$

where a, b, c are connected by the just-mentioned equation $abc = 1 + H$. This is therefore a solution containing say the arbitrary constants λ, a, b, and, as such, is a complete solution.

But any other conjugate system gives a complete solution, and a very elegant one is obtained from the system H, d, e. Writing for symmetry $\beta - \alpha$, $\gamma - \alpha$ in place of d, e, we have here to find p, q, r from the equations

$$H = pqr - 1, \quad qy - px = \beta - \alpha, \quad rz - px = \gamma - \alpha;$$

or, if we assume $\theta = px - \alpha$, then

$$H = pqr - 1; \; px, \; qy, \; rz = \theta + \alpha, \; \theta + \beta, \; \theta + \gamma$$

respectively, whence

$$(1 + H)xyz = (\theta + \alpha)(\theta + \beta)(\theta + \gamma),$$

which equation determines θ as a function of x, y, z (in fact, it is a function of the product xyz), and then

$$p, \; q, \; r = \frac{\theta + \alpha}{x}, \quad \frac{\theta + \beta}{y}, \quad \frac{\theta + \gamma}{z},$$

and we have

$$V = \lambda + \int \left(\frac{\theta + \alpha}{x} dx + \frac{\theta + \beta}{y} dy + \frac{\theta + \gamma}{z} dz \right).$$

There is no difficulty in effecting the integration directly by introducing θ as a new variable, and we find

$$V = \lambda + 3\theta - \alpha \log \frac{\theta + \alpha}{x} - \beta \log \frac{\theta + \beta}{y} - \gamma \log \frac{\theta + \gamma}{z}.$$

Or, starting from this form, we may verify it by differentiation; the value of dV is

$$d\theta \left(3 - \frac{\alpha}{\theta + \alpha} - \frac{\beta}{\theta + \beta} - \frac{\gamma}{\theta + \gamma} \right) + \frac{\alpha\, dx}{x} + \frac{\beta\, dy}{y} + \frac{\gamma\, dz}{z},$$

where the term in $d\theta$ is

$$= \theta\, d\theta \left(\frac{1}{\theta+\alpha} + \frac{1}{\theta+\beta} + \frac{1}{\theta+\gamma}\right),$$

which, from the equation which determines θ, is

$$= \theta \left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right),$$

and the value of dV is thus

$$= \frac{\theta+\alpha}{x}\, dx + \frac{\theta+\beta}{y}\, dy + \frac{\theta+\gamma}{z}\, dz.$$

The solution contains apparently the four constants λ, α, β, γ, but there is no loss of generality in writing, for instance, $\alpha = 0$, and the number of constants really contained in the solution may be regarded as 3.

73. To show how the equations $H = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$ give a solution; remarking that these equations are $pqr - 1 = H$, $p = p_0$, $q = q_0$, $r = r_0$, $qy - px = q_0 y_0 - p_0 x_0$, $rz - px = r_0 z_0 - p_0 x_0$, we find

$$p\,(x - x_0) = q\,(y - y_0) = r\,(z - z_0),$$

and consequently p, q, $r =$

$$\sqrt[3]{(1+H)}\,\frac{(y-y_0)^{\frac{1}{3}}(z-z_0)^{\frac{1}{3}}}{(x-x_0)^{\frac{2}{3}}},\quad \sqrt[3]{(1+H)}\,\frac{(z-z_0)^{\frac{1}{3}}(x-x_0)^{\frac{1}{3}}}{(y-y_0)^{\frac{2}{3}}},\quad \sqrt[3]{(1+H)}\,\frac{(x-x_0)^{\frac{1}{3}}(y-y_0)^{\frac{1}{3}}}{(z-z_0)^{\frac{2}{3}}},$$

respectively: whence

$$V = \lambda + \int (p\, dx + q\, dy + r\, dz),$$

$$= \lambda + 3\sqrt[3]{(1+H)}\,(x-x_0)^{\frac{1}{3}}(y-y_0)^{\frac{1}{3}}(z-z_0)^{\frac{1}{3}},$$

which is the solution involving the four constants λ, x_0, y_0, z_0.

If in the foregoing value of V we consider x_0, y_0, z_0 as variables, then p, q, r having the values just mentioned, and p_0, q_0, r_0 being equal to these respectively, we obviously have

$$dV = p\, dx + q\, dy + r\, dz - p_0\, dx_0 - q_0\, dy_0 - r_0\, dz_0.$$

74. Considering now the augmented Hamiltonian system, we join to the foregoing integrals a, b, c, d, e, the new integrals $t - \tau = \dfrac{x}{qr}$ and $V - \lambda = 3px$. And then expressing all the quantities in terms of $t - \tau$,

$$\begin{aligned}
x &= bc\,(t-\tau),\\
y &= ca\,(t-\tau) + \frac{d}{b},\\
z &= ab\,(t-\tau) + \frac{e}{c}.\\
p &= a,\ q = b,\ r = c,\ H = abc - 1,\\
V &= \lambda + 3abc\,(t-\tau).
\end{aligned}$$

Forming from these the expression for $dV - p\,dx - q\,dy - r\,dz$, the term in $dt - d\tau$ disappears; there is a term in $t - \tau$, the coefficient of which is

$$3d\,.\,abc - a\,d\,.\,bc - b\,d\,.\,ca - c\,d\,.\,ab,$$

which is $= d\,.\,abc$, or the term is $(t-\tau)\,dH$; and we have, finally,

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH - bd\frac{d}{b} - cd\frac{e}{c};$$

viz. t enters only in the combination $(t-\tau)\,dH$, which is the fundamental theorem. Considering H as a determinate constant, this term disappears.

We may show how this formula leads to the solution of the partial differential equation $pqr = 1 + H$; treating H as a definite constant, then in order that the formula may give $dV - p\,dx - q\,dy - r\,dz = d\lambda$, or $V = \lambda + \int (p\,dx + q\,dy + r\,dz)$, as before, the last two terms of the formula must disappear; this will be the case if $\frac{d}{b}$ and $\frac{e}{c}$ are constants, or, say, $d = b\beta$, $e = c\gamma$, β and γ being constants. But, this being so, we have $q\beta = qy - px$, $r\gamma = rz - px$, that is, $px = q(y-\beta) = r(z-\gamma)$, $pqr = 1 + H$, giving the values of p, q, r; and then

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz), \quad = \lambda + 3\sqrt[3]{(1+H)}\,x^{\frac{1}{3}}(y-\beta)^{\frac{1}{3}}(z-\gamma)^{\frac{1}{3}},$$

which is substantially the same solution as is obtained above by a different process. Or, again, observing that we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH - dd - de - \frac{d}{b}\,db - \frac{e}{c}\,dc,$$

then, taking H, b, c constants, we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda - dd - de,$$

which, changing the value of λ, gives the before-mentioned solution

$$V = \lambda + ax + by + cz, \quad (abc = 1 + H).$$

75. As a second example, suppose

$$H = \tfrac{1}{2}(p^2 + q^2 + r^2 - x^2 - y^2 - z^2);$$

the augmented system is

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r} = \frac{dp}{x} = \frac{dq}{y} = \frac{dr}{z} = dt = \frac{dV}{p^2 + q^2 + r^2},$$

corresponding to the dynamical problem of the motion of a particle acted upon by a repulsive central force equal to the distance.

The integrals of the original system may be expressed in various forms, viz. the quotient of any two of the expressions $x + p$, $y + q$, $z + r$, or of any two of the

expressions $x-p$, $y-q$, $z-r$ is an integral, or again the product of any expression of the first set into any expression of the second set is an integral: we may take as integrals

$$\alpha = x^2 - p^2, \quad \beta = y^2 - q^2, \quad \gamma = z^2 - r^2, \quad \delta = \frac{y+q}{x+p}, \quad \epsilon = \frac{z+r}{x+p}.$$

We have then

$$dt = \frac{dx}{\sqrt{(x^2-\alpha)}}, \text{ that is, } t - \tau = \log\{x + \sqrt{(x^2-\alpha)}\} = \log(x+p),$$

giving $x+p = e^{t-\tau}$, and thence the other quantities $x-p$, $y+q$, &c. For greater symmetry, I introduce a new set of constants a, b, c, a', b', c', and I write also $e^{t-\tau} = T$, $e^{-t+\tau} = T'$ (where $TT' = 1$). We then have

$$\begin{aligned} x &= aT + a'T', & p &= aT - a'T', \\ y &= bT + b'T', & q &= bT - b'T', \\ z &= cT + c'T', & r &= cT - c'T'; \end{aligned}$$

also, comparing with the values obtained as above,

$$a = \tfrac{1}{2}, \quad b = \tfrac{1}{2}\delta, \quad c = \tfrac{1}{2}\epsilon,$$

$$a' = \tfrac{1}{2}\alpha, \quad b' = \tfrac{1}{2}\frac{\beta}{\delta}, \quad c' = \tfrac{1}{2}\frac{\gamma}{\epsilon}.$$

We have, moreover,

$$H = -2(aa' + bb' + cc') = -\tfrac{1}{2}(\alpha + \beta + \gamma).$$

76. We find

$$p^2 + q^2 + r^2 = H + (a^2 + b^2 + c^2)\,T^2 + (a'^2 + b'^2 + c'^2)\,T'^2,$$

and thence

$$\begin{aligned} V &= \lambda + \int (p^2 + q^2 + r^2)\,dt \\ &= \lambda + H(t-\tau) + \tfrac{1}{2}(a^2 + b^2 + c^2)\,T^2 - \tfrac{1}{2}(a'^2 + b'^2 + c'^2)\,T'^2. \end{aligned}$$

We may from this obtain the expression for

$$dV - p\,dx - q\,dy - r\,dz,$$

when everything is variable. The terms in $(dt - d\tau)$, as is obvious, disappear; omitting these from the beginning, we have

$$dV = d\lambda + (t-\tau)\,dH + (a\,da + b\,db + c\,dc)\,T^2 - (a'da' + b'db' + c'dc')\,T'^2:$$

also

$$\begin{aligned} p\,dx &= (aT - a'T')(T da + T' da'), \\ &= da\,(aT^2 - a') + da'\,(-a'T'^2 + a): \end{aligned}$$

thence forming the analogous expressions for $q\,dy$ and $r\,dz$, we have

$$\begin{aligned} p\,dx + q\,dy + r\,dz = {} & (a\,da + b\,db + c\,dc)\,T^2 - (a'da' + b'db' + c'dc')\,T'^2 \\ & - (a'da + b'db + c'dc) \quad + (a\,da' + b\,db' + c\,dc'), \end{aligned}$$

whence

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + a'da + b'db + c'dc - a\,da' - b\,db' - c\,dc';$$

or, in place of a, b, c, a', b', c', introducing α, β, γ, δ, ϵ, and attending to the value of H,

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + \tfrac{1}{2}\,dH + \tfrac{1}{2}\frac{\beta}{\delta}\,d\delta + \tfrac{1}{2}\frac{\gamma}{\epsilon}\,d\epsilon.$$

77. Suppose H, δ, ϵ absolute constants, this becomes

$$d(V-\lambda) = p\,dx + q\,dy + r\,dz,$$

or

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz),$$

and we have thus a solution of the partial differential equation

$$p^2 + q^2 + r^2 = x^2 + y^2 + z^2 + 2H;$$

viz. p, q, r are here to be determined as functions of x, y, z by the equations

$$\begin{aligned} p^2 + q^2 + r^2 &= x^2 + y^2 + z^2 + 2H, \\ y + q &= \delta(x+p), \\ z + r &= \epsilon(x+p). \end{aligned}$$

We have

$$2H + x^2 + y^2 + z^2 = p^2 + \{y - \delta(x+p)\}^2 + \{z - \epsilon(x+p)\}^2;$$

or, on the right-hand side, writing $p^2 = (x+p)^2 - 2x(x+p) + x^2$,

" left " " $x^2 = (x-p)^2 - 2x(x+p) + p^2$,

the equation is

$$(1 + \delta^2 + \epsilon^2)(x+p)^2 - 2(x + \delta y + \epsilon z)(x+p) - 2H = 0,$$

which gives p as a function of x, y, z. But the result is a complicated one, except in the case $H=0$; we then have

$$x + p = \frac{2(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

$$y + q = \frac{2\delta(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

$$z + r = \frac{2\epsilon(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

and thence

$$V = \lambda - \tfrac{1}{2}(x^2 + y^2 + z^2) + \frac{(x + \delta y + \epsilon z)^2}{1 + \delta^2 + \epsilon^2},$$

a complete solution of the partial differential equation

$$p^2 + q^2 + r^2 = x^2 + y^2 + z^2.$$

More symmetrically, we have the solution

$$V = \lambda - \tfrac{1}{2}(x^2 + y^2 + z^2) + \frac{(ax + by + cz)^2}{a^2 + b^2 + c^2},$$

as can be at once verified.

78. In the same particular case $H = 0$, introducing the corresponding values p_0, q_0, r_0, x_0, y_0, z_0, we find a very simple expression for $V - V_0$, as a function of x, y, z, x_0, y_0, z_0. We have, writing $T_0 = e^{t_0 - \tau}$, $T_0' = e^{-t_0 + \tau}$, and therefore $T_0 T_0' = 1$,

$$\begin{aligned} x_0 &= aT_0 + a'T_0', & p_0 &= aT_0 - a'T_0', \\ y_0 &= bT_0 + b'T_0', & q_0 &= bT_0 - b'T_0', \\ z_0 &= cT_0 + c'T_0', & r_0 &= cT_0 - c'T_0', \end{aligned}$$

and thence

$$x - x_0 = a(T - T_0) + a'\left(\frac{1}{T} - \frac{1}{T_0}\right), = (T - T_0)\left(a - \frac{a'}{TT_0}\right),$$

$$x + x_0 = a(T + T_0) + a'\left(\frac{1}{T} + \frac{1}{T_0}\right), = (T - T_0)\left(a + \frac{a'}{TT_0}\right).$$

Forming the analogous quantities $y - y_0$, &c., we deduce

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (T - T_0)^2 \left\{a^2 + b^2 + c^2 + (a'^2 + b'^2 + c'^2)\frac{1}{T^2 T_0^2}\right\},$$

$$(x + x_0)^2 + (y + y_0)^2 + (z + z_0)^2 = (T + T_0)^2 \left\{a^2 + b^2 + c^2 + (a'^2 + b'^2 + c'^2)\frac{1}{T^2 T_0^2}\right\}.$$

But we have

$$\begin{aligned} V - V_0 &= \tfrac{1}{2}\left\{(a^2 + b^2 + c^2)(T^2 - T_0^2) - (a'^2 + b'^2 + c'^2)\left(\frac{1}{T^2} - \frac{1}{T_0^2}\right)\right\} \\ &= \tfrac{1}{2}(T^2 - T_0^2)\left\{a^2 + b^2 + c^2 + (a'^2 + b'^2 + c'^2)\frac{1}{T^2 T_0^2}\right\}, \end{aligned}$$

and hence the required formula

$$V - V_0 = \tfrac{1}{2}\sqrt{\{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\}}\sqrt{\{(x + x_0)^2 + (y + y_0)^2 + (z + z_0)^2\}},$$

or, say, for shortness,

$$= \tfrac{1}{2}\sqrt{(R)}\sqrt{(S)}.$$

79. We ought, therefore, to have

$$\tfrac{1}{2} d\sqrt{(R)}\sqrt{(S)} = p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0,$$

where p, q, r, p_0, q_0, r_0 denote as above, and consequently

$$p^2 + q^2 + r^2 = x^2 + y^2 + z^2, \quad p_0^2 + q_0^2 + r_0^2 = x_0^2 + y_0^2 + z_0^2.$$

We have in fact

$$p = \tfrac{1}{2}\left\{ \frac{\sqrt{(S)}}{\sqrt{(R)}}(x - x_0) + \frac{\sqrt{(R)}}{\sqrt{(S)}}(x + x_0)\right\}, \text{ \&c.,}$$

$$p_0 = \tfrac{1}{2}\left\{ -\frac{\sqrt{(S)}}{\sqrt{(R)}}(x - x_0) + \frac{\sqrt{(R)}}{\sqrt{(S)}}(x + x_0)\right\}, \text{ \&c.;}$$

and thence

$$p^2 + q^2 + r^2 = \tfrac{1}{4}\{R + S + 2(x^2 + y^2 + z^2 - x_0^2 - y_0^2 - z_0^2)\}, = x^2 + y^2 - z^2,$$

$$p_0^2 + q_0^2 + r_0^2 = \tfrac{1}{4}\{R + S - 2(x^2 + y^2 + z^2 - x_0^2 - y_0^2 - z_0^2)\}, = x_0^2 + y_0^2 + z_0^2,$$

or the last-mentioned results are thus verified.

Partial Differential Equation containing the Dependent Variable: Reduction to Standard Form. Art. Nos. 80, 81.

80. The equation $H = \text{const.}$ is the most general form of a partial differential equation not containing the dependent variable V; but if a partial differential equation does contain the independent variable, we can, by regarding this as one of the dependent variables, and in place of it introducing a new independent variable, exhibit the equation in the standard form $H = \text{const.}$, H being here a homogeneous function of the order zero in the differential coefficients. Thus, if the independent variables are x, y, the dependent variable z, and its differential coefficients p, q, then the given partial differential equation may be $H, = H(\mathrm{p}, \mathrm{q}, x, y, z), = \text{const.}$ But we may determine z as a function of x, y by an equation $V = \text{const.}$, V being a desired function of x, y, z; and then writing p, q, r for the differential coefficients $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$, we have $\mathrm{p} = -\frac{p}{r}$, $\mathrm{q} = -\frac{q}{r}$, and the proposed partial differential equation becomes

$$H\left(-\frac{p}{r}, -\frac{q}{r}, x, y, z\right) = \text{const.}$$

viz. this is an equation containing only the differential coefficients p, q, r of the dependent variable V, a function of x, y, z. And, moreover, H is homogeneous of the order zero in p, q, r; consequently

$$p\frac{dH}{dp} + q\frac{dH}{dq} + r\frac{dH}{dr} = 0,$$

or, in the augmented Hamiltonian system, the last equality is $= \frac{dV}{0}$, so that an integral is $V = \text{const.}$; as already stated, this is the equation by which z is determined as a function of x, y.

81. Thus, if the given partial differential equation be $\mathrm{pq} - z = H$, we here consider the equation $\frac{pq}{r^2} - z = H$. The Hamiltonian system is

$$\frac{r^2dx}{q} = \frac{r^2dy}{p} = \frac{-r^2dz}{2pq} = \frac{dp}{0} = \frac{dq}{0} = \frac{dr}{1}\left(= \frac{dV}{0}\right),$$

having the integrals

$$a = p,$$

$$b = q,$$

$$c = px - qy,$$

$$d = \frac{1}{r} + \frac{x}{q},$$

$$e = \frac{z}{pq} - \frac{1}{r^2},$$

(whence $H = -abe$). We have H, a, b, a system of conjugate integrals and, in terms of these,

$$p = a,\ q = b,\ r = \sqrt{\left(\frac{ab}{z+H}\right)};$$

hence, writing λ for the constant value of V, we have

$$\lambda = \int\left\{a\,dx + b\,dy + \sqrt{\left(\frac{ab}{z+H}\right)}\,dz\right\},$$

that is,

$$\lambda = ax + by + 2\sqrt{\{ab\,(z+H)\}},$$

or say,

$$4ab\,(z+H) = (ax + by - \lambda)^2,$$

a solution containing really the two constants λ and $\frac{a}{b}$, and thus a complete solution of the given equation $\mathrm{pq} - z = H$. We have, in fact,

$$2ab\,\mathrm{p} = a\,(ax + by - \lambda),$$

$$2ab\,\mathrm{q} = b\,(ax + by - \lambda);$$

that is,

$$4a^2b^2\mathrm{pq} = ab\,(ax + by - \lambda)^2,\ = 4a^2b^2\,(z + H),$$

or

$$\mathrm{pq} = z + H,$$

as it should be.

656.

ON THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS.

[From the *Mathematische Annalen*, t. XI. (1877), pp. 194—198.]

IN what follows, any letter not otherwise explained denotes a function of certain variables (x, y, p, q), or (x, y, z, p, q, r), &c., as will be stated in each particular case.

An equation $a = \text{const.}$ denotes that the function a of the variables is, in fact, a constant (viz. by such equation we establish a relation between the variables): and when this is so, we use the same letter a to denote the constant value of the function in question; I find this a very convenient notation.

Thus if the variables are x, y, z, p, q, r and if p, q, r are the differential coefficients in regard to x, y, z respectively of a function V of x, y, z, then H (as a letter not otherwise explained) denotes a function of x, y, z, p, q, r and considering it as a given function,

$$\mathrm{H} = \text{const.}$$

will be a partial differential equation containing the constant H. For instance, if H denote the function $pqr - xyz$, $\mathrm{H} = \text{const.}$ is the partial differential equation, $pqr - xyz = \mathrm{H}$ (a given constant).

The integration of the partial differential equation, $\mathrm{H} = \text{const.}$, depends upon that of the linear partial differential equation

$$(\mathrm{H}, \Theta) = 0,$$

where as usual (H, Θ) signifies

$$\frac{\partial(\mathrm{H}, \Theta)}{\partial(p, x)} + \frac{\partial(\mathrm{H}, \Theta)}{\partial(q, y)} + \frac{\partial(\mathrm{H}, \Theta)}{\partial(r, z)}.$$

It can be effected if we know two conjugate solutions a, b of the equation $(\mathrm{H}, \Theta) = 0$, viz. a, b as solutions are such that $(\mathrm{H}, a) = 0$, $(\mathrm{H}, b) = 0$, and (as conjugate solutions) are also such that $(a, b) = 0$; in this case if from the equations

$$\mathrm{H} = \text{const.}, \quad a = \text{const.}, \quad b = \text{const.}$$

we determine p, q, r as functions of x, y, z, the resulting value of $p\,dx+q\,dy+r\,dz$ is an exact differential, and we have

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

a solution containing three arbitrary constants, λ, a, b, and therefore a complete solution of the proposed partial differential equation $\mathrm{H}=\text{const.}$

But (as is known) there is a different process of integration, for which the conjugate solutions are not required, and which has reference to a system of initial values x_0, y_0, z_0, p_0, q_0, r_0: viz. if the independent solutions of $(\mathrm{H},\,\Theta)=0$, are a, b, c, d, e, and if a_0, b_0, c_0, d_0, e_0 denote respectively the same functions of the initial variables that a, b, c, d, e are of x, y, z, p, q, r, then if from the equations

$$a=a_0,\quad b=b_0,\quad c=c_0,\quad d=d_0,\quad e=e_0,\quad \mathrm{H}=\text{const.}$$

we express p, q, r as functions of x, y, z and of x_0, y_0, z_0, H, these last being regarded as constants, we have $p\,dx+q\,dy+r\,dz$ an exact differential, and

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

a solution containing the constants λ, x_0, y_0, z_0 (that is, one supernumerary constant), and as such a complete solution.

It is interesting to prove directly that $p\,dx+q\,dy+r\,dz$ is an exact differential.

I consider first the more simple case where the variables are p, q, x, y. Here p, q are to be found from the equations

$$a=a_0,\quad b=b_0,\quad c=c_0,\quad \mathrm{H}=\text{const.}$$

and it is to be shown that $p\,dx+q\,dy$ is an exact differential.

Considering p, q, p_0, q_0 as functions of the independent variables x, y, then differentiating in regard to x, and eliminating $\frac{dp}{dx}$, $\frac{dp_0}{dx}$, $\frac{dq_0}{dx}$, we have

$$\begin{vmatrix} \frac{da}{dx}+\frac{da}{dq}\frac{dq}{dx}, & \frac{da}{dp}, & \frac{da_0}{dp_0}, & \frac{da_0}{dq_0} \\ \frac{db}{dx}+\frac{db}{dq}\frac{dq}{dx}, & \frac{db}{dp}, & \frac{db_0}{dp_0}, & \frac{db_0}{dq_0} \\ \frac{dc}{dx}+\frac{dc}{dq}\frac{dq}{dx}, & \frac{dc}{dp}, & \frac{dc_0}{dp_0}, & \frac{dc_0}{dq_0} \\ \frac{d\mathrm{H}}{dx}+\frac{d\mathrm{H}}{dq}\frac{dq}{dx}, & \frac{d\mathrm{H}}{dp}, & 0\ , & 0 \end{vmatrix}=0,$$

or introducing a well-known notation for functional determinants, and expanding the determinant, this is

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ x)}+\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\frac{dq}{dx}\right\}+\&\text{c.}=0.$$

But in the same way

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ y)}+\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ p)}\frac{dp}{dy}\right\}+\&\text{c.}=0;$$

or adding these, attending to the value of (H, c), and observing that $\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ p)}=-\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}$ we have

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{(\mathrm{H},\ c)+\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)\right\}+\&\text{c.}=0,$$

the terms denoted by the &c. being the like terms with b, c, a and c, a, b in place of a, b, c. We have (H, a) = 0, (H, b) = 0, (H, c) = 0, and the equation in fact is

$$\left\{\Sigma\frac{\partial(a_0,\ b_0)}{\partial(p,\ q)}\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\right\}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)=0;$$

viz. we have $\frac{dq}{dx}-\frac{dp}{dy}=0$, the condition for the exact differential.

Coming now to the case where the variables are x, y, z, p, q, r, and in the six equations treating p, q, r, p_0, q_0, r_0 as functions of the independent variables x, y, z,—then differentiating with regard to x and proceeding as before, we find for $\frac{dr}{dx}$ the equation

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\frac{dr}{dx}\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(r,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(x,\ p,\ q)}\right\}+\&\text{c.}=0.$$

We have, in the same way, for $\frac{dp}{dz}$ the equation

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\frac{dp}{dz}\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(p,\ r,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)}\right\}+\&\text{c.}=0;$$

or, adding the two equations,

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(r,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(x,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)}\right\}+\&\text{c.}=0,$$

where the terms denoted by the &c. indicate the like terms corresponding to the different partitions of the letters a, b, c, d, e.

The equation may be simplified; we have identically

$$-\frac{da}{dq}(b,\ \mathrm{H})-\frac{db}{dq}(\mathrm{H},\ a)-\frac{d\mathrm{H}}{dq}(a,\ b)=\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(x,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)},$$

or since $(\mathrm{H}, a)=0$, $(b, \mathrm{H})=0$, the left-hand side is simply $-\frac{d\mathrm{H}}{dq}(a, b)$, and the equation becomes

$$\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{\partial(a, b, \mathrm{H})}{\partial(r, p, q)}-\frac{d\mathrm{H}}{dq}(a, b)\right\}+\&c.=0.$$

This ought to give $\frac{dr}{dx}-\frac{dp}{dz}=0$, and it will do so if only

$$\Sigma\left\{\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}(a, b)\right\}=0;$$

this is then the equation which has to be proved. By the Poisson-Jacobi theorem, (a, b) is a function of a, b, c, d, e: if we write

$$(a_0, b_0)=\frac{\partial(a_0, b_0)}{\partial(p_0, x_0)}+\frac{\partial(a_0, b_0)}{\partial(q_0, y_0)}+\frac{\partial(a_0, b_0)}{\partial(r_0, z_0)},$$

then (a_0, b_0) is the same function of a_0, b_0, c_0, d_0, e_0; but these are $=a, b, c, d, e$ respectively, and we thence have $(a, b)=(a_0, b_0)$, and the theorem to be proved is

$$\Sigma\left\{\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}(a_0, b_0)\right\}=0.$$

But substituting for (a_0, b_0) its value, the function on the left-hand is (it is easy to see) the sum of the three functional determinants

$$\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, p_0, x_0)}+\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, q_0, y_0)}+\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, r_0, z_0)},$$

each of which vanishes as containing the same letter twice in the denominator, that is, as having two identical columns; and the theorem in question is thus proved. And in the same way $\frac{dp}{dy}-\frac{dq}{dx}$, $\frac{dq}{dz}-\frac{dr}{dy}$ are each $=0$: or we have $p\,dx+q\,dy+r\,dz$ an exact differential.

The proof would fail if the factors multiplying $\frac{dq}{dx}-\frac{dp}{dy}$, &c., or if any one of these factors, were $=0$; I have not particularly examined this, but the meaning would be, that here the equations in question $a=a_0$, &c., $\mathrm{H}=\text{const.}$, are such as not to give rise to expressions for p, q, r as functions of x, y, z, x_0, y_0, z_0, H, as assumed in the theorem; whenever such expressions are obtainable, then we have $p\,dx+q\,dy+r\,dz$ an exact differential.

The proof in the case of a greater number of variables, say in the next case where the variables are x, y, z, w, p, q, r, s, would present more difficulty—but I have not proceeded further in the question.

It is worth while to put the two processes into connexion with each other: taking in each case the variables to be x, y, z, p, q, r, and the partial differential equation to be $\mathrm{H}=\text{const.}$;

In the one case, a, b being conjugate solutions of $(\mathrm{H}, \Theta) = 0$,

from the equations $\mathrm{H} = \text{const.}$, $a = \text{const.}$, $b = \text{const.}$,

we find p, q, r functions of x, y, z, H, a, b:

and then $p\,dx + q\,dy + r\,dz$ is an exact differential.

In the other case, a, b, c, d, e being the solutions of $(\mathrm{H}, \Theta) = 0$,

from the equations $\mathrm{H} = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$,

we find p, q, r functions of x_0, y_0, z_0, H:

and then $p\,dx + q\,dy + r\,dz$ is an exact differential.

It may be added that, if from the last mentioned equations we determine also p_0, q_0, r_0 as functions of x, y, z, x_0, y_0, z_0, then considering only H as a constant, we ought to have $p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0$ an exact differential; I have not examined the direct proof.

Cambridge, 28 *Nov.*, 1876.

657.

NOTE ON THE THEORY OF ELLIPTIC INTEGRALS.

[From the *Mathematische Annalen*, t. XII. (1877), pp. 143—146.]

THE equation

$$\frac{Mdy}{\sqrt{1-y^2 \,.\, 1-k^2y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}$$

is integrable algebraically when M is rational: and so long as the modulus is arbitrary, then conversely, in order that the equation may be integrable algebraically, M must be rational. For particular values however of the modulus, the equation is integrable algebraically for values of the form M, or (what is the same thing) $\frac{1}{M}$, = a rational quantity $\pm$ square root of a negative rational quantity, say $= \frac{1}{p}(l + m\sqrt{-n})$, where l, m, n, p are integral and n is positive; we may for shortness call this a half-rational numerical value. The theory is considered by Abel in two Memoirs in the *Astr. Nach.* Nos. 138 & 147 (1828), being the Memoirs * XIII & XIV in the *Œuvres Complètes* (Christiania 1839). I here reproduce the investigation in a somewhat altered (and, as it appears to me, improved) form.

Putting the two differentials each $= du$, we have $x = \operatorname{sn}(u+\alpha)$, $y = \operatorname{sn}\left(\frac{u}{M}+\beta\right)$; and the question is whether there exists an algebraical relation between these functions, or, what is the same thing, an algebraical relation between the functions $x = \operatorname{sn} u$ and $y = \operatorname{sn}\frac{u}{M}$.

Suppose that A and B are independent periods of $\operatorname{sn} u$; so that $\operatorname{sn}(u+A) = \operatorname{sn} u$, $\operatorname{sn}(u+B) = \operatorname{sn} u$, and that every other period is $= mA + nB$, where m and n are integers. Then if u has successively the values u, $u+A$, $u+2A$, etc., the value of x

[* They are the Memoirs XIX. and XX. in the *Œuvres Complètes*, t. I., Christiania, 1881.]

remains always the same, and if x and y are algebraically connected, y can have only a finite number of values: there are consequently integer values p', p'' for which $\operatorname{sn}\frac{1}{M}(u+p'A)=\operatorname{sn}\frac{1}{M}(u+p''A)$: or writing $u-p'A$ for u and putting $p''-p'=p$, there is an integer value p for which $\operatorname{sn}\frac{1}{M}(u+pA)=\operatorname{sn}\frac{1}{M}u$.

Similarly there is an integer value q for which $\operatorname{sn}\frac{1}{M}(u+qB)=\operatorname{sn}\frac{1}{M}u$; and we are at liberty to assume $q=p$; for if the original values are unequal, we have only in the place of each of them to substitute their least common multiple.

We have thus an integer p, for which

$$\operatorname{sn}\frac{1}{M}(u+pA)=\operatorname{sn}\frac{1}{M}u,$$

$$\operatorname{sn}\frac{1}{M}(u+pB)=\operatorname{sn}\frac{1}{M}u.$$

There are consequently integers m, n, r, s such that

$$\frac{pA}{M}=mA+nB,$$

$$\frac{pB}{M}=rA+sB,$$

equations which will constitute a single relation $\frac{p}{M}=m$, if $m=s$, $r=n=0$; but in every other case will be two independent relations. In the case first referred to, the modulus is arbitrary and M is rational.

But excluding this case, the equations give

$$B(mA+nB)=A(rA+sB),$$

or, what is the same thing,

$$rA^2-(m-s)AB-nB^2=0,$$

an equation which implies that the modulus has some one value out of a set of given values. The ratio $A:B$ of the two periods is of necessity imaginary, and hence the integers m, n, r, s must be such that $(m-s)^2+nr$ is negative.

The foregoing equations may be written

$$\left(m-\frac{p}{M}\right)A+\qquad nB=0,$$

$$rA+\left(s-\frac{p}{M}\right)B=0,$$

whence eliminating A and B we have

$$\left(m-\frac{p}{M}\right)\left(s-\frac{p}{M}\right)-nr=0,$$

that is,

$$\left(\frac{p}{M}\right)^2 - (m+s)\frac{p}{M} + ms - nr = 0,$$

and consequently

$$\frac{p}{M} = \tfrac{1}{2}(m+s) \pm \tfrac{1}{2}\sqrt{(m-s)^2 + nr},$$

where, by what precedes, the integer under the radical sign is negative: and we have thus the above mentioned theorem.

As a very general example, consider the two rational transformations

$$z = (x, u, v);\ \text{mod. eq. } Q(u, v) = 0;\ \frac{Ndz}{\sqrt{1-z^2 . 1-v^8z^2}} = \frac{dx}{\sqrt{1-x^2 . 1-u^8x^2}},$$

$$y = (z, v, w);\ \text{mod. eq. } P(v, w) = 0;\ \frac{Mdy}{\sqrt{1-y^2 . 1-w^8y^2}} = \frac{dz}{\sqrt{1-z^2 . 1-v^8z^2}}:$$

viz. z is taken to be a rational function of x, and of the modular fourth roots u, v; and y to be a rational function of z, and of the modular fourth roots v, w; the transformations being (to fix the ideas) of different orders. We have y a rational function of x, corresponding to the differential relation

$$\frac{MNdy}{\sqrt{1-y^2 . 1-w^8y^2}} = \frac{dx}{\sqrt{1-x^2 . 1-u^8x^2}}.$$

Suppose here $w^8 = u^8$, or say $w = \theta u$, θ being an eighth root of unity: we then have $Q(u, v) = 0$, $P(v, \theta u) = 0$, equations which determine u. The differential equation is then

$$\frac{MNdy}{\sqrt{1-y^2 . 1-u^8y^2}} = \frac{dx}{\sqrt{1-x^2 . 1-u^8x^2}},$$

an equation the algebraical integral of which is $y =$ a rational function of x as above: hence, by what precedes, we have

$$\frac{1}{MN} = \frac{1}{2p}\{m + s \pm \sqrt{(m-s)^2 + nr}\},$$

a half-rational numerical value, as above.

To explain what the algebraical theorem implied herein is, observe that the equations $Q(u, v) = 0$, $P(v, \theta u) = 0$, give for u an algebraical equation. Admitting θ as an adjoint radical, suppose that an irreducible factor is $\phi(u)$, and take u to be determined by the equation $\phi u = 0$; then v, and consequently also any rational function $\frac{1}{MN}$ of u, v, can be expressed as a rational integral function of u, of a degree which is at most equal to the degree of the function ϕu less unity. The theorem is that, in virtue of the equation $\phi u = 0$, this rational function of u becomes equal to a half-rational numerical value as above. Thus in a simple case, which actually presented itself, the equation $\phi u = 0$ was $u^2 - 4u + 1 = 0$; and $\frac{1}{MN}$ had the value $u - 2$, which in virtue of this equation becomes $= \pm\sqrt{-3}$.

Thus if the second transformation be the identity $z=y$, $w=v$, $M=1$: we have $v=\theta u$; and the equations are

$$y=(x,\ u,\ \theta u),\quad Q(u,\ \theta u)=0,\quad \frac{Ndy}{\sqrt{1-y^2\,.\,1-u^8y^2}}=\frac{dx}{\sqrt{1-x^2\,.\,1-u^8x^2}}.$$

In particular, if the relation between y, x be given by the cubic transformation

$$y=\frac{\dfrac{v+2u^3}{v}x+\dfrac{u^6}{v^2}x^3}{1+vu^2(v+2u^3)x^2},$$

so that the modular equation $Q(u,\ v)=0$ is $u^4-v^4+2uv(1-u^2v^2)=0$; then, writing herein $v=\theta u$, and taking θ a prime eighth root of unity, that is, a root of $\theta^4+1=0$, we have

$$Q(u,\ \theta u)=-2\theta^3u^2(\theta u^2+\theta^3+u^4);$$

viz. disregarding the factor u^2, the equation for u is $u^4+\theta u^2+\theta^3=0$; or, if ω be an imaginary cube root of unity $(\omega^2+\omega+1=0)$, this is $(u^2-\omega\theta)(u^2-\omega^2\theta)=0$; so that a value of u^2 is $u^2=-\omega\theta$.

Assuming then $\theta^4+1=0$, $v=\theta u$ and $u^2=-\omega\theta$, we have $(v+2u^3)v=\theta^3\omega(1+2\omega)$, $=\theta^3\omega(\omega-\omega^2)$; $\dfrac{v+2u^3}{v}=\omega-\omega^2$; $\dfrac{u^6}{v^2}=\omega^2$, $(v+2u^3)vu^2=-\omega^2(\omega-\omega^2)$, $u^8=\omega^4\theta^4=-\omega$; and the formula becomes

$$y=\frac{(\omega-\omega^2)x+\omega^2x^3}{1-\omega^2(\omega-\omega^2)x^2},$$

giving

$$\frac{dy}{\sqrt{1-y^2\,.\,1+\omega y^2}}=\frac{(\omega-\omega^2)\,dx}{\sqrt{1-x^2\,.\,1+\omega x^2}},$$

where as before $\omega^2+\omega+1=0$, a result which can be at once verified. We have $(\omega-\omega^2)^2=-3$; or the coefficient $\omega-\omega^2$ in the differential equation is $=\sqrt{-3}$, which is of the form mentioned in the general theorem.

We might, instead of $z=y$, have assumed between y and z the relation corresponding to any other of the six linear transformations of an elliptic integral, and thus have obtained in each case, for a properly determined value of the modulus, a cubic transformation to the same modulus.

Cambridge, 10 *April*, 1877.

658.

ON SOME FORMULÆ IN ELLIPTIC INTEGRALS.

[From the *Mathematische Annalen*, t. XII. (1877), pp. 369—374.]

I REPRODUCE in a modified form an investigation contained in the memoir, Zolotareff, "Sur la méthode d'intégration de M. Tchébychef," *Mathematische Annalen*, t. V. (1872), pp. 560—580.

Starting from the quartic

$$(a, b, c, d, e)(x, 1)^4, \quad = a\,.\,x - \alpha\,.\,x - \beta\,.\,x - \gamma\,.\,x - \delta,$$

we derive from it the quartic

$$(a_1, b_1, c_1, d_1, e_1)(x_1, 1)^4 = a_1\,.\,x_1 - \alpha_1\,.\,x_1 - \beta_1\,.\,x_1 - \gamma_1\,.\,x_1 - \delta_1,$$

where, writing for shortness

$$\begin{aligned} \lambda &= -\alpha + \beta + \gamma - \delta, \\ \mu &= \alpha - \beta + \gamma - \delta, \\ \nu &= \alpha + \beta - \gamma - \delta, \end{aligned}$$

the roots of the new quartic are

$$\begin{aligned} \alpha_1 &= \theta + \frac{\mu\nu}{2\lambda}, \\ \beta_1 &= \theta + \frac{\nu\lambda}{2\mu}, \\ \gamma_1 &= \theta + \frac{\lambda\mu}{2\nu}, \\ \delta_1 &= \theta, \end{aligned}$$

θ being arbitrary: the differences of the roots α_1, β_1, γ_1, δ_1 are, it will be observed, functions of the differences of the roots α, β, γ, δ.

We assume $a_1 = a = 1$, nevertheless retaining in the formulæ a_1 or a (each meaning 1), whenever, for the sake of homogeneity, it is convenient to do so. The relations between the remaining coefficients b_1, c_1, d_1, e_1, and b, c, d, e, are of course to be calculated from the formulæ $-4b = \Sigma\alpha$, $6c = \Sigma\alpha\beta$, &c., and the like formulæ $-4b_1 = \Sigma\alpha_1$, $6c_1 = \Sigma\alpha_1\beta_1$, &c. We thus have

$$-4b_1 = 4\theta + \tfrac{1}{2}\ \Sigma\frac{\mu\nu}{\lambda},$$

$$6c_1 = 6\theta^2 + \tfrac{3}{2}\theta\ \Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{4}\Sigma\lambda^2,$$

$$-4d_1 = 4\theta^3 + \tfrac{3}{2}\theta^2\Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{2}\theta\Sigma\lambda^2 + \tfrac{1}{8}\lambda\mu\nu,$$

$$e_1 = \theta^4 + \tfrac{1}{2}\theta^3\Sigma\frac{\mu\nu}{\lambda} + \tfrac{1}{4}\theta^2\Sigma\lambda^2 + \tfrac{1}{8}\theta\lambda\mu\nu,$$

where $\Sigma\dfrac{\mu\nu}{\lambda} = \dfrac{1}{\lambda\mu\nu}\Sigma\lambda^2\mu^2$.

Writing, for shortness,

$$\begin{aligned}
C &= ac - b^2,\\
D &= a^2d - 3abc + 2b^3,\\
E &= a^3e - 4a^2bd + 6ab^2c - 3b^4 = a^2I - 3C^2,\\
I &= ae - 4bd + 3c^2,\\
J &= ace - ad^2 - b^2e + 2bcd - c^3,\\
B &= \frac{-a^2I + 12C^2}{4D},
\end{aligned}$$

we have

$$\begin{aligned}
\Sigma\lambda &= -4(b+\delta),\\
\Sigma\lambda^2 &= -48C,\\
\Sigma\lambda\mu &= 24C + 8(b+\delta)^2,\\
\lambda\mu\nu &= 32D,\\
\Sigma\lambda^2\mu^2 &= 64(-a^2I + 12C^2),
\end{aligned}$$

where the last equation may be verified by means of the formula

$$(\Sigma\lambda\mu)^2 = \Sigma\lambda^2\mu^2 + 2\lambda\mu\nu\,\Sigma\lambda.$$

And we hence obtain

$$\begin{aligned}
a_1 &= 1,\\
b_1 &= -\theta - B,\\
c_1 &= \theta^2 + 2B\theta - 2C,\\
d_1 &= -\theta^3 - 3B\theta^2 + 6C\theta - D,\\
e_1 &= \theta^4 + 4B\theta^3 - 12C\theta^2 + 4D\theta.
\end{aligned}$$

And consequently

$$(a_1,\ b_1,\ c_1,\ d_1,\ e_1)(x_1,\ 1)^4 = (1,\ -B,\ -2C,\ -D,\ 0)(x_1-\theta,\ 1)^4.$$

Hence also

$$\begin{aligned} I_1 &= a_1e_1 - 4b_1d_1 + 3c_1^2 = -4BD + 12C^2 = a^2I\,; \\ J_1 &= a_1c_1e_1 - a_1d_1^2 - b_1^2e_1 + 2b_1c_1d_1 - c_1^3 = -D^2 + 8C^3 - 4BCD \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -D^2 + 8C^3 + C(a^2I - 12C^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = a^2CI - 4C^3 - D^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = a^3J\,; \end{aligned}$$

where, as regards this last equation $a^2CI - 4C^3 - D^2 = a^3J$, observe that C and D are the leading coefficients of the Hessian H and the cubicovariant Φ of the quartic function U, and hence that the identity $-\Phi^2 = JU^3 - IU^2\mathrm{H} + 4\mathrm{H}^3$, attending only to the term in x^6, becomes $-D^2 = a^3J - a^2CI + 4C^3$, which is the equation in question.

We thus have $I_1 = I$, $J_1 = J$; viz. the functions $(a, b, c, d, e)(x, 1)^4$, $(a_1, b_1, c_1, d_1, e_1)(x_1, 1)^4$, are linearly transformable the one into the other, and that by a unimodular substitution $x_1 = \rho x + \sigma$, $y_1 = \rho' x + \sigma'$, where $\rho\sigma' - \rho'\sigma = 1$. It may be remarked that we have $(a, b, c, d, e)(x, 1)^4 = (1, 0, C, D, E)(x + b, 1)^4$; and hence the theorem may be stated in the form: the quartic functions $(1, 0, C, D, E)(x, 1)^4$, and $(1, -B, -2C, -D, 0)(x_1, 1)^4$, are transformable the one into the other by a unimodular substitution: or again, substituting for E its value $a^3I - 3C^2$, $= -4BD + 9C^2$, the quartic functions

$$(1,\ 0,\ C,\ D,\ -4BD + 9C^2)(x,\ 1)^4, \text{ and } (1,\ -B,\ -2C,\ -D,\ 0)(x_1,\ 1)^4$$

are linearly transformable the one into the other by a unimodular substitution. In this last form B, C, D are arbitrary quantities; it is at once verified that the invariants I, J have the same values for the two functions respectively; and the theorem is thus self-evident.

Reverting to the expressions for α_1, β_1, γ_1, δ_1 we obtain

$$\begin{aligned} \alpha_1 - \delta_1 &= \frac{\mu\nu}{2\lambda}\,; \quad \beta_1 - \gamma_1 = \frac{\lambda}{2\mu\nu}(\nu^2 - \mu^2), \quad = \frac{\alpha - \delta\,.\,\beta - \gamma}{\alpha_1 - \delta_1}, \\ \beta_1 - \delta_1 &= \frac{\nu\lambda}{2\mu}\,; \quad \gamma_1 - \alpha_1 = \frac{\mu}{2\nu\lambda}(\lambda^2 - \nu^2), \quad = \frac{\beta - \delta\,.\,\gamma - \alpha}{\beta_1 - \delta_1}, \\ \gamma_1 - \delta_1 &= \frac{\lambda\mu}{2\nu}\,; \quad \alpha_1 - \beta_1 = \frac{\nu}{2\lambda\mu}(\mu^2 - \lambda^2), \quad = \frac{\gamma - \delta\,.\,\alpha - \beta}{\gamma_1 - \delta_1}. \end{aligned}$$

Hence also

$$\begin{aligned} &\alpha - \delta\,.\,\beta - \gamma\,, \quad \beta - \delta\,.\,\gamma - \alpha\,, \quad \gamma - \delta\,.\,\alpha - \beta \\ &= \alpha_1 - \delta_1\,.\,\beta_1 - \gamma_1, \quad \beta_1 - \delta_1\,.\,\gamma_1 - \alpha_1, \quad \gamma_1 - \delta_1\,.\,\alpha_1 - \beta_1, \end{aligned}$$

which agrees with the foregoing equations $I_1 = I$ and $J_1 = J$, since I, J are functions of the first set of quantities and I_1, J_1 the like functions of the second set; in fact, $I = \frac{1}{24}(P^2 + Q^2 + R^2)$, and $J = \frac{1}{432}(Q - R)(R - P)(P - Q)$, if for a moment the quantities are called P, Q, R.

We consider now the differential expression $\dfrac{dx}{\sqrt{x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta}}$; to transform this into the elliptic form, assume

$$k^2 = -\frac{\alpha-\beta\,.\,\gamma-\delta}{\gamma-\alpha\,.\,\beta-\delta},\quad \operatorname{sn}^2 a = \frac{\gamma-\alpha}{\gamma-\delta};$$

(where a is of course not the coefficient, $=1$, heretofore represented by that letter: as a will only occur under the functional signs sn, cn, dn, there is no risk of ambiguity). And then further

$$x = \frac{\alpha \operatorname{sn}^2 u - \delta \operatorname{sn}^2 a}{\operatorname{sn}^2 u - \operatorname{sn}^2 a}.$$

Forming the equations

$$k^2 \operatorname{sn}^2 a = -\frac{\alpha-\beta}{\beta-\delta},\quad k^2 \operatorname{sn}^4 a = -\frac{\gamma-\alpha\,.\,\alpha-\beta}{\gamma-\delta\,.\,\beta-\delta},$$

we deduce without difficulty

$$\operatorname{sn}^2 a = \frac{\gamma-\alpha}{\gamma-\delta},\quad \frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 a} = \frac{x-\delta}{x-\alpha},$$

$$\operatorname{cn}^2 a = \frac{\alpha-\delta}{\gamma-\delta},\quad \frac{\operatorname{cn}^2 u}{\operatorname{cn}^2 a} = \frac{x-\gamma}{x-\alpha},$$

$$\operatorname{dn}^2 a = \frac{\alpha-\delta}{\beta-\delta},\quad \frac{\operatorname{dn}^2 u}{\operatorname{dn}^2 a} = \frac{x-\beta}{x-\alpha},$$

$$1 - k^2 \operatorname{sn}^4 a = \frac{(\alpha-\delta)(-\alpha+\beta+\gamma-\delta)}{\beta-\delta\,.\,\gamma-\delta} = \frac{\lambda(\alpha-\delta)}{\beta-\delta\,.\,\gamma-\delta},$$

the use of which last equation will presently appear.

We hence obtain

$$2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\, du = -(\alpha-\delta) \operatorname{sn}^2 a \frac{dx}{(x-\alpha)^2},$$

$$\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \quad = \quad \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{\sqrt{x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta}}{(x-\alpha)^2},$$

and consequently

$$2du = -\frac{(\alpha-\delta)\operatorname{sn} a}{\operatorname{cn} a \operatorname{dn} a} \frac{dx}{\sqrt{x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta}},$$

or, reducing the coefficient,

$$\frac{dx}{\sqrt{x-\alpha\,.\,x-\beta\,.\,x-\gamma\,.\,x-\delta}} = \frac{-2}{\sqrt{\gamma-\alpha\,.\,\beta-\delta}}\, du,$$

which is the required formula.

We next have

$$\operatorname{sn}^2 2a = \frac{4 \operatorname{sn}^2 a \operatorname{cn}^2 a \operatorname{dn}^2 a}{(1-k^2 \operatorname{sn}^4 a)^2} = \frac{4\beta-\delta\,.\,\gamma-\alpha}{\lambda^2} = \frac{\gamma_1-\alpha_1}{\gamma_1-\delta_1},$$

in virtue of the foregoing values

$$\gamma_1 - \alpha_1 = \frac{2\mu}{\nu\lambda}(\beta - \delta)(\gamma - \alpha) \text{ and } \gamma_1 - \delta_1 = \frac{\lambda\mu}{2\nu}.$$

Moreover

$$k^2 = -\frac{\alpha - \beta \,.\, \gamma - \delta}{\gamma - \alpha \,.\, \beta - \delta} = -\frac{\alpha_1 - \beta_1 \,.\, \gamma_1 - \delta_1}{\gamma_1 - \alpha_1 \,.\, \beta_1 - \delta_1}.$$

Hence the like formulæ with the same value of k^2, and with $2a$ in place of a, will be applicable to the like differential expression in x_1: viz. assuming

$$x_1 = \frac{\alpha_1 \operatorname{sn}^2 u_1 - \delta_1 \operatorname{sn}^2 2a}{\operatorname{sn}^2 u_1 - \operatorname{sn}^2 2a}$$

we have

$$\frac{dx_1}{\sqrt{x_1 - \alpha_1 \,.\, x_1 - \beta_1 \,.\, x_1 - \gamma_1 \,.\, x_1 - \delta_1}} = \frac{-2}{\sqrt{\gamma_1 - \alpha_1 \,.\, \beta_1 - \delta_1}}\, du_1.$$

We have thus the integral of the differential equation

$$\frac{dx_1}{\sqrt{x_1 - \alpha_1 \,.\, x_1 - \beta_1 \,.\, x_1 - \gamma_1 \,.\, x_1 - \delta_1}} = \frac{dx}{\sqrt{x - \alpha \,.\, x - \beta \,.\, x - \gamma \,.\, x - \delta}}$$

(the two quartic functions being of course connected as before); viz. assuming x, x_1 functions of u, u_1 respectively as above and recollecting that $\gamma_1 - \alpha_1 \,.\, \beta_1 - \delta_1 = \gamma - \alpha \,.\, \beta - \delta$, we have $du_1 = du$; and therefore $u_1 = u + f$ (f an arbitrary constant); the required integral is thus given by the equations

$$\frac{\operatorname{sn}^2 u}{\operatorname{sn}^2 a} = \frac{x - \delta}{x - \alpha}; \quad \frac{\operatorname{sn}^2 (u + f)}{\operatorname{sn}^2 2a} = \frac{x_1 - \delta_1}{x_1 - \alpha_1}; \quad (f \text{ the constant of integration}).$$

Using the formula

$$\operatorname{sn}(u + f) = \frac{\operatorname{sn}^2 u - \operatorname{sn}^2 f}{\operatorname{sn} u \operatorname{cn} f \operatorname{dn} f - \operatorname{sn} f \operatorname{cn} u \operatorname{dn} u},$$

we obtain

$$\frac{x_1 - \delta_1}{x_1 - \alpha_1} \operatorname{sn}^2 2a = \frac{\{(x - \delta) \operatorname{sn}^2 a - (x - \alpha) \operatorname{sn}^2 f\}^2}{\{\sqrt{x - \alpha \,.\, x - \delta} \operatorname{sn} a \operatorname{cn} f \operatorname{dn} f - \sqrt{x - \beta \,.\, x - \gamma} \operatorname{sn} f \operatorname{cn} a \operatorname{dn} a\}^2},$$

which is the general integral.

We obtain a particular integral of a very simple form by assuming $f = a$, viz. this is

$$\frac{x_1 - \delta_1}{x_1 - \alpha_1} \operatorname{sn}^2 2a = \frac{\operatorname{sn}^2 a}{\operatorname{cn}^2 a \operatorname{dn}^2 a} \frac{(\alpha - \delta)^2}{\{\sqrt{x - \alpha \,.\, x - \delta} - \sqrt{x - \beta \,.\, x - \gamma}\}^2};$$

this is

$$\frac{x_1 - \delta_1}{x_1 - \alpha_1} \frac{\gamma_1 - \alpha_1}{\gamma_1 - \delta_1} = \frac{\gamma - \alpha \,.\, \beta - \delta}{\{\sqrt{x - \alpha \,.\, x - \delta} - \sqrt{x - \beta \,.\, x - \gamma}\}^2},$$

or writing $\gamma-\alpha.\beta-\delta=\gamma_1-\alpha_1.\beta_1-\delta_1$, reducing and inverting, we have

$$\frac{x_1-\alpha_1}{x_1-\delta_1}=\frac{1}{\beta_1-\delta_1.\gamma_1-\delta_1}\{\sqrt{x-\alpha.x-\delta}-\sqrt{x-\beta.x-\gamma}\}^2,$$

which may also be written in the equivalent forms

$$\frac{x_1-\beta_1}{x_1-\delta_1}=\frac{1}{\gamma_1-\delta_1.\alpha_1-\delta_1}\{\sqrt{x-\beta.x-\delta}-\sqrt{x-\gamma.x-\alpha}\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1}=\frac{1}{\alpha_1-\delta_1.\beta_1-\delta_1}\{\sqrt{x-\gamma.x-\delta}-\sqrt{x-\alpha.x-\beta}\}^2.$$

In fact, from the first equation we have

$$\frac{\alpha_1-\delta_1.\beta_1-\delta_1.\gamma_1-\delta_1}{x_1-\delta_1}=(\beta_1-\delta_1)(\gamma_1-\delta_1)-\{\sqrt{x-\alpha.x-\delta}-\sqrt{x-\beta.x-\gamma}\}^2,$$

where the expression on the right-hand side is

$$\delta_1^2-\delta_1(\alpha_1+\beta_1+\gamma_1)+\alpha_1\delta_1+\beta_1\gamma_1-2x^2+x(\alpha+\beta+\gamma+\delta)-\alpha\delta-\beta\gamma+2\sqrt{X},$$

X having here the value

$$X=x-\alpha.x-\beta.x-\gamma.x-\delta.$$

Writing for a moment

$$\begin{aligned} P&=\alpha\delta+\beta\gamma, & P_1&=\alpha_1\delta_1+\beta_1\gamma_1,\\ Q&=\beta\delta+\gamma\alpha, & Q_1&=\beta_1\delta_1+\gamma_1\alpha_1,\\ R&=\gamma\delta+\alpha\beta, & R_1&=\gamma_1\delta_1+\alpha_1\beta_1,\end{aligned}$$

then, by what precedes, Q_1-R_1, R_1-P_1, P_1-Q_1 are equal to $Q-R$, $R-P$, $P-Q$ respectively; that is, $P_1-P=Q_1-Q=R_1-R$, $=$(suppose) Ω, a function symmetrical in regard to α_1, β_1, γ_1; α, β, γ: the equation therefore is

$$\frac{\alpha_1-\delta_1.\beta_1-\delta_1.\gamma_1-\delta_1}{x_1-\delta_1}=\delta_1(\delta_1-\alpha_1-\beta_1-\gamma_1)-2x^2+x(\alpha+\beta+\gamma+\delta)+2\sqrt{X}+\Omega,$$

or the relation is symmetrical in regard to α_1, β_1, γ_1; α, β, γ: and the first form implies therefore each of the other two forms.

Cambridge, 8 *May*, 1877.

659.

A THEOREM ON GROUPS.

[From the *Mathematische Annalen*, t. XIII. (1878), pp. 561—565.]

THE following theorem is very simple; but it seems to belong to a class of theorems, the investigation of which is desirable.

I consider a substitution-group of a given order upon a given number of letters; and I seek to *double* the group, that is to derive from it a group of twice the order upon twice the number of letters. This can be effected for *any* group, in a manner which is self-evident and in nowise interesting: but in a different manner for a *commutative* group (or group such that any two of its substitutions satisfy the condition $AB = BA$): it is to be observed that the double group is not in general commutative.

Let the letters of the original group be $abcde\ldots$, we may for shortness write $U = abcde\ldots$; and take U as the primitive arrangement: and let the group then be $1, A, B, \ldots$ where $A, B, \ldots$ represent substitutions: the corresponding arrangements are $U, AU, BU, \ldots$ and these may for shortness be represented by $1, A, B, \ldots$; viz. $1, A, B, \ldots$ represent, properly and in the first instance, substitutions; but when it is explained that they represent arrangements, then they represent the arrangements $U, AU, BU, \ldots$.

For the double group the letters are taken to be $a_1b_1c_1d_1e_1\ldots$ and $a_2b_2c_2d_2e_2\ldots$, $= U_1$ and U_2 suppose, and U_1U_2 is regarded as the primitive arrangement; A_1 and A_2 denote the same substitutions in regard to U_1 and U_2 respectively, that A denotes in regard to U: and so for B_1, B_2, etc.; moreover 12 denotes the substitution $(a_1a_2)(b_1b_2)(c_1c_2)(d_1d_2)(e_1e_2)\ldots$, or interchange of the suffixes 1 and 2. The substitutions A_1, A_2, or any powers of these A_1^α, A_2^β, are obviously commutative; applying them to the primitive arrangement U_1U_2, we have $A_1^\alpha A_2^\beta U_1U_2$ and $A_2^\beta A_1^\alpha U_1U_2$ each $= A_1^\alpha U_1 A_2^\beta U_2$. But A_1^α, A_2^β are not commutative with 12: we have for instance $12A_1^\alpha . U_1U_2 = 12A_1^\alpha U_1 . U_2 = A_1^\alpha U_2 . U_1$, but $A_1^\alpha 12 U_1U_2 = A_1^\alpha . U_2U_1 = U_2 . A_1^\alpha U_1$. If instead of the substitutions we consider the arrangements obtained by operating upon U_1U_2, then we

may for shortness consider for instance A_1A_2 as denoting the arrangement $A_1U_1 . A_2U_2$. But observe that in this use of the symbols the A_1, A_2 are not commutative, A_2A_1 would denote the different arrangement $A_2U_2 . A_1U_1$: in this use of the symbols, 1 would denote U_1U_2, and 12 would denote U_2U_1, but it would be clearer to use 12, 21 as denoting U_1U_2 and U_2U_1 respectively.

These explanations having been given, I remark that in every case the substitution-group 1, A, B, ... gives the double group

$$\begin{array}{lll} 1, & A_1A_2, & B_1B_2, \ldots \\ 12, & 12A_1A_2, & 12B_1B_2, \ldots \end{array}$$

as is at once seen to be true: but further when the original group 1, A, B, ... is commutative, then if m be any integer number, such that $m^2 \equiv 1$ (mod. the order of the original group), we have also the double group

$$\begin{array}{lll} 1, & A_1A_2^m, & B_1B_2^m, \ldots \\ 12, & 12A_1A_2^m, & 12B_1B_2^m, \ldots \end{array}$$

where of course if the order of the original group ($=\mu$ suppose) be prime, we have $m \equiv 1$ or else $m \equiv -1$ (mod. μ), say $m=1$ or $\mu-1$; but if the order μ be composite, then the number of solutions may be greater.

The condition in order to the existence of the double group of course is that, in the system of substitutions just written down, the combination of any two substitutions may give a substitution of the system. And this is in fact the case in virtue of the formulæ

$$\begin{array}{ll} 1^\circ. & A_1A_2^m . \quad B_1B_2^m = A_1B_1(A_2B_2)^m, \\ 2^\circ. & A_1A_2^m . \; 12B_1B_2^m = 12A_1^mB_1(A_2^mB_2)^m, \\ 3^\circ. & 12A_1A_2^m . \quad B_1B_2^m = 12(A_1B_1)(A_2B_2)^m, \\ 4^\circ. & 12A_1A_2^m . \; 12B_1B_2^m = A_1^mB_1(A_2^mB_2)^m, \end{array}$$

inasmuch as 1, A, B, ... being a group, AB and A^mB are each of them a substitution of the group, $=C$ suppose; we have of course in like manner $A_1B_1=C_1$, $A_2B_2=C_2$, etc., and the right-hand sides of the four formulæ are thus of the forms $C_1C_2^m$, $12C_1C_2^m$, $12C_1C_2^m$, $C_1C_2^m$ respectively, viz. these are substitutions of the system.

To prove for instance the formula 2°, considering the arrangements obtained by operating upon U_1U_2, we have

$$B_1B_2^mU_1U_2 = B_1B_2^m, \; 12B_1B_2^mU_1U_2 = B_2B_1^m, \quad A_1A_2^m \, 12B_1B_2^mU_1U_2 = A_2^mB_2 \, A_1B_1^m,$$

where of course the expressions on the right-hand side denote arrangements. By reason that the original group is commutative $(A^mB)^m$ is $=A^{m^2}B^m$ or since $m^2 \equiv 1$ (mod. μ) this is $=AB^m$; hence also $(A_2^mB_2)^m = A_2B_2^m$: hence, considering as before the arrangements obtained by operating on U_1U_2, we have

$$(A_2{}^mB_2)^m\, U_1U_2 = 1\,.\, A_2B_2{}^m;\quad A_1{}^mB_1\,(A_2{}^mB_2)^m\, U_1U_2 = A_1{}^mB_1A_2B_2{}^m,$$

and

$$12A_1{}^mB_1\,(A_2{}^mB_2)^m\, U_1U_2 = A_2{}^mB_2A_1B_1{}^m,$$

where of course the right-hand sides denote arrangements. Hence in the formula 2°, the two substitutions operating on U_1U_2 give each of them the same arrangement $A_2{}^mB_2\,A_1B_1{}^m$, that is, the two substitutions are equal. And similarly the other formulæ 1°, 3°, 4° may be proved.

By interchanging A and B, in the formulæ I obtain

$$1^\circ.\quad A_1A_2{}^m\,.\quad B_1B_2{}^m = A_1B_1\,(A_2B_2)^m;$$

$$B_1B_2{}^m\,.\quad A_1A_2{}^m = B_1A_1\,(B_2A_2)^m = A_1B_1\,(A_2B_2)^m,$$

which is

$$= A_1A_2{}^m\,.\,B_1B_2{}^m;$$

$$2^\circ \text{ and } 3^\circ.\quad A_1A_2{}^m\,.\,12B_1B_2{}^m = 12A_1{}^mB_1\,(A_2{}^mB_2)^m;$$

$$12B_1B_2{}^m\,.\quad A_1A_2{}^m = 12B_1A_1\,(B_2A_2)^m = 12A_1B_1\,(A_2B_2)^m,$$

which is not

$$= A_1A_2{}^m\,.\,12B_1B_2{}^m;$$

$$3^\circ \text{ and } 2^\circ.\quad 12A_1A_2{}^m\,.\quad B_1B_2{}^m = 12A_1B_1\,(A_2B_2)^m;$$

$$B_1B_2{}^m\,.\,12A_1A_2{}^m = 12A_1B_1{}^m\,(A_2B_2{}^m)^m = 12A_1B_1{}^m\,A_2{}^mB_2,$$

which is not

$$= 12A_1A_2{}^m\,.\,B_1B_2{}^m;$$

$$4^\circ.\quad 12A_1A_2{}^m\,.\,12B_1B_2{}^m = A_1{}^mB_1\,(A_2{}^mB_2)^m;$$

$$12B_1B_2{}^m\,.\,12A_1A_2{}^m = A_1B_1{}^m\,(A_2B_2{}^m)^m = (A_1{}^mB_1)^m\,A_2{}^mB_2,$$

which is not

$$= 12A_1A_2{}^m\,.\,12B_1B_2{}^m.$$

That is, in the double group any two substitutions of the form $A_1A_2{}^m$ are commutative, but a substitution of this form is not in general commutative with a substitution of the form $12B_1B_2{}^m$, nor are two substitutions of the last-mentioned form $12A_1A_2{}^m$ in general commutative with each other; hence the double group is not in general commutative.

In the formula 4°, writing $B = A$, we have

$$(12A_1A_2{}^m)^2 = A_1{}^{m+1}\,A_2{}^{m^2+m} = A_1{}^{m+1}\,.\,A_2{}^{m+1};$$

hence, if λ is the least integer value such that

$$\lambda\,(m+1) \equiv 0 \pmod{\mu},$$

we have $(12A_1A_2{}^m)^{2\lambda} = 1$, viz. in the double group the substitutions of the second row are each of them of an order not exceeding 2λ, the substitution 12 being of course of the order 2. In particular, if $m = \mu - 1$, then $\lambda = 1$: and the substitutions of the second row are each of them of the order 2.

As the most simple instance of the theorem, suppose that the original group is the group 1, (abc), (acb), or say 1, Θ, Θ^2, of the cyclical substitutions upon the 3 letters abc. Here $m^2 \equiv 1$ (mod. 3) or except $m=1$ the only solution is $m=2$, and thence $\lambda=1$. The double group is a group of the order 6 on the letters $a_1b_1c_1a_2b_2c_2$: viz. writing $\Theta=(abc)$, and therefore $\Theta_1=(a_1b_1c_1)$, $\Theta_1^2=(a_1c_1b_1)$, $\Theta_2=(a_2b_2c_2)$, $\Theta_2^2=(a_2c_2b_2)$, also writing $12=\alpha$, the substitutions are

$$\begin{array}{lll} 1, & \Theta_1\Theta_2^2, & \Theta_1^2\Theta_2, \\ \alpha, & \alpha\Theta_1\Theta_2^2, & \alpha\Theta_1^2\Theta_2, \end{array}$$

the arrangements corresponding to the second row of substitutions are $a_2b_2c_2a_1b_1c_1$, $b_2c_2a_2c_1a_1b_1$, $c_2a_2b_2b_1c_1a_1$, viz. the substitutions are $(a_1a_2)(b_1b_2)(c_1c_2)$, $(a_1b_2)(b_1c_2)(c_1a_2)$, $(a_1c_2)(b_1a_2)(c_1b_2)$, each of them of the second order as they should be.

I take the opportunity of mentioning a further theorem. Let μ be the order of the group, and a the order of any term A thereof, a being of course a submultiple of μ: and let the term A be called quasi-positive when $\mu\left(1-\frac{1}{a}\right)$ is even, quasi-negative when $\mu\left(1-\frac{1}{a}\right)$ is odd. The theorem is that the product of two quasi-positive terms, or of two quasi-negative terms, is quasi-positive; but the product of a quasi-positive term and a quasi-negative term is quasi-negative. And it follows hence that, either the terms of a group are all quasi-positive, or else one half of them are quasi-positive and the other half of them are quasi-negative.

The proof is very simple: a term A of the group operating on the μ terms $(1, A, B, C, \ldots)$ of the group, gives these same terms in a different order, or it may be regarded as a substitution upon the μ symbols $1, A, B, C, \ldots$; so regarded it is a *regular* substitution (this is a fundamental theorem, which I do not stop to prove), and hence since it must be of the order a it is a substitution composed of $\frac{\mu}{a}$ cycles, each of a letters. But in general a substitution is positive or negative according as it is equivalent to an even or an odd number of inversions; a cyclical substitution upon a letters is positive or negative according as $a-1$ is even or odd; and the substitution composed of the $\frac{\mu}{a}$ cycles is positive or negative according as $\frac{\mu}{a}(a-1)$, that is, $\mu\left(1-\frac{1}{a}\right)$, is even or odd. Hence by the foregoing definition, the term A, according as it is quasi-positive or quasi-negative, corresponds to a positive substitution or to a negative substitution; and such terms combine together in like manner with positive and negative substitutions.

Cambridge, 3rd April, 1878.

660.

ON THE CORRESPONDENCE OF HOMOGRAPHIES AND ROTATIONS.

[From the *Mathematische Annalen*, t. XV. (1879), pp. 238—240.]

It is a fundamental notion in Prof. Klein's theory of the "Icosahedron" that homographies correspond to rotations (of a solid body about a fixed point): in such wise that, considering the homographies which correspond to two given rotations, the homography compounded of these corresponds to the rotation compounded of the two rotations.

Say the two homographies are $A + Bp + Cq + Dpq = 0$, $A_1 + B_1q + C_1r + D_1qr = 0$, then, eliminating q, the compound homography is $A_2 + B_2p + C_2r + D_2pr = 0$, where

$$A_2,\ B_2,\ C_2,\ D_2 = B_1A - A_1C,\ B_1B - A_1D,\ D_1A - C_1C,\ D_1B - C_1D;$$

and the theorem is that, corresponding to these, we have rotations depending on the parameters (λ, μ, ν), $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ respectively, such that the third rotation is that compounded of the first and second rotations. The question arises to find the expression for the parameters of the homography in terms of the parameters of the corresponding rotation.

The rotation (λ, μ, ν) is taken to denote a rotation through an angle ϑ about an axis the inclinations of which to the axes of coordinates are f, g, h, the values of λ, μ, ν then being $= \tan\frac{1}{2}\vartheta \cos f$, $\tan\frac{1}{2}\vartheta \cos g$, $\tan\frac{1}{2}\vartheta \cos h$ respectively: $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ have of course the like significations; and then, if (λ, μ, ν) refer to the first rotation, and $(\lambda_1, \mu_1, \nu_1)$ to the second rotation, the values of $(\lambda_2, \mu_2, \nu_2)$ for the rotation compounded of these are taken to be*:

$$\begin{aligned}\lambda_2 &= \lambda + \lambda_1 + \mu\nu_1 - \mu_1\nu,\\ \mu_2 &= \mu + \mu_1 + \nu\lambda_1 - \nu_1\lambda,\\ \nu_2 &= \nu + \nu_1 + \lambda\mu_1 - \lambda_1\mu,\end{aligned}$$

* The numerators might equally well have been $\lambda + \lambda_1 - (\mu\nu_1 - \mu_1\nu)$, etc., but there is no essential difference: we pass from one set of formulæ to the other by reversing the signs of all the symbols: and hence, by properly fixing the sense of the rotations, the signs may be made to be + as in the text. Assuming this to be so, if we then reverse the order of the component rotations, we have for the new compound rotation the like formulæ with the signs − instead of +; but this in passing. The formulæ, virtually due to Rodrigues, are given in my paper "On the motion of a solid body," *Camb. Math. Journal*, t. III. (1843), [6].

each divided by

$$1 - \lambda\lambda_1 - \mu\mu_1 - \nu\nu_1;$$

and if we then write for λ, μ, ν, the quotients x, y, z each divided by w, and in like manner for λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2, the quotients x_1, y_1, z_1 each divided by w_1, and x_2, y_2, z_2 each divided by w_2, the formulæ for the composition of the rotations are

$$\begin{aligned} x_2 &= xw_1 + x_1 w + yz_1 - y_1 z, \\ y_2 &= yw_1 + y_1 w + zx_1 - z_1 x, \\ z_2 &= zw_1 + z_1 w + xy_1 - x_1 y, \\ w_2 &= ww_1 - xx_1 - yy_1 - zz_1; \end{aligned}$$

and the question is to express A, B, C, D as functions of (x, y, z, w), such that A_1, B_1, C_1, D_1 denoting the like functions of (x_1, y_1, z_1, w_1), A_2, B_2, C_2, D_2 shall denote the like functions of (x_2, y_2, z_2, w_2).

It is found that the required conditions are satisfied by assuming

$$A,\ B,\ C,\ D = ix - y,\quad -iz + w,\quad -iz - w,\quad -ix - y,$$

(where $i = \sqrt{-1}$ as usual): in fact, we then have

$$\begin{aligned} A_2 &= B_1 A - A_1 C \\ &= (-iz_1 + w_1)(ix - y) - (ix_1 - y_1)(-iz - w) \\ &= i\,(xw_1 + x_1 w + yz_1 - y_1 z) - (yw_1 + y_1 w + zx_1 - z_1 x) \\ &= -y_2 + ix_2, \end{aligned}$$

as it should be: and we verify in like manner the values of B_2, C_2 and D_2 respectively.

The result consequently is that we have the homography

$$(ix - y) + (-iz + w)\,p + (-iz - w)\,q + (-ix - y)\,pq = 0$$

corresponding to the rotation $\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$: where $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$ are the parameters of rotation, $\tan\frac{1}{2}\vartheta\cos f$, $\tan\frac{1}{2}\vartheta\cos g$, $\tan\frac{1}{2}\vartheta\cos h$.

I remark as regards the geometrical theory that, if we consider two lines J and K fixed in space, and a third line L fixed in the solid body and moveable with it; then, for any given position of the solid body, the three lines J, K, L are directrices of a hyperboloid, the generatrices whereof meet each of the three lines: and these generatrices determine, say on the fixed lines J and K, two series of points corresponding homographically to each other: that is, corresponding to any given position of the solid body we have a homography. But it is not immediately obvious how we can thence obtain the foregoing analytical formulæ.

Cambridge, 3 *April*, 1879.

661.

ON THE DOUBLE ϑ-FUNCTIONS.

[From the *Proceedings of the London Mathematical Society*, vol. IX. (1878), pp. 29, 30.]

PROF. CAYLEY gave an account of researches* on which he is engaged upon the double ϑ-functions. In regard to these, he establishes in a strictly analogous manner the theory of the single ϑ-functions, the process for the single functions being in fact as follows:—Considering u, x as connected by the differential relation

$$\delta u = \frac{\delta x}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}},$$

then, if A, B, C, D, Ω denote functions of u, viz. for shortness, the single letters are used, instead of writing them as functional symbols, $A(u)$, $B(u)$, &c., then, by way of definition of these functions (called, the first four of them ϑ-functions, and the last an ω-function), we assume

$$A,\ B,\ C,\ D = \Omega\sqrt{a-x},\ \Omega\sqrt{b-x},\ \Omega\sqrt{c-x},\ \Omega\sqrt{d-x}$$

respectively, together with one other equation, as presently mentioned. Without in any wise defining the meaning of Ω, we then obtain a set of equations of the form

$$A\delta B - B\delta A = \Omega^2\sqrt{c-x\,.\,d-x}\,\delta u,$$

(mere constant coefficients are omitted), or, what is the same thing,

$$A\delta B - B\delta A = CD\,\delta u,$$

which are differential equations defining the nature of the ratio-functions $A : B : C : D$. If, proceeding to second differential coefficients, we attempt to form the expressions for $A\delta^2 A - (\delta A)^2$, &c., these involve multiples of $\Omega\delta^2\Omega - (\delta\Omega)^2$; in order to obtain a con-

[* See paper, number 665.]

venient form, we assume $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2$, where M is a function of x. We thus obtain an equation $A\delta^2 A - (\delta A)^2 = \Omega^2 \mathfrak{A}(\delta u)^2$, where the value of $\mathfrak{A}$ depends upon that of M. The value of M has to be taken so as to simplify as much as may be the expression of $\mathfrak{A}$, but so that M shall be a symmetrical function of the constants a, b, c, d: this last condition is assigned in order that the like simplification may present itself in the analogous relations $B\delta^2 B - (\delta B)^2 = \Omega^2 \mathfrak{B}(\delta u)^2$, &c. The proper expression of M is found to be

$$M = -2x^2 + x(a+b+c+d) + a^2 + b^2 + c^2 + d^2 - 2bc - 2ca - 2ab - 2ad - 2bd - 2cd,$$

viz. M having this value, the one other equation above referred to is

$$\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2;$$

and we then have a system of four equations such as

$$A\delta^2 A - (\delta A)^2 = \Omega^2 \mathfrak{A}(\delta u)^2,$$

where $\mathfrak{A}$ is a linear function of x, and where consequently $\Omega^2\mathfrak{A}$ can be expressed as a linear function of any two of the four squares A^2, B^2, C^2, D^2.

To establish the connexion with the Jacobian H and Θ functions, the differential relation between u, x may be taken to be

$$\delta u = \frac{\delta x}{\sqrt{x \,.\, 1 - x \,.\, 1 - k^2 x}};$$

viz. we have here $d = \infty$, and to adapt the formulæ to this value it is necessary to write $\frac{u}{\sqrt{d}}$ instead of u, and make other easy changes. The result is that Ω differs from D by a constant factor only, so that, instead of the five functions A, B, C, D, Ω, we have only the four functions A, B, C, D. The equation $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2$ is replaced by an equation $D\delta^2 D - (\delta D)^2 = D^2 \mathfrak{D}(\delta u)^2$, or say $\delta^2(\log D) = \mathfrak{D}(\delta u)^2$, which gives a result of the form

$$D = e^{u + \lambda^2 \int \delta u \int \delta u \frac{A^2}{D^2}},$$

showing that D differs from Jacobi's $\Theta(u)$ only by an exponential factor of the form $Ce^{\lambda u^2}$. And it then further appears that A, B, C differ only by factors of the like form from the three numerator functions $\mathrm{H}(u)$, $\mathrm{H}(u+K)$, $\Theta(u+K)$, so that, neglecting constant factors, the functions

$$\frac{A}{D},\ \frac{B}{D},\ \frac{C}{D} \text{ are equal to } \frac{\mathrm{H}(u)}{\Theta(u)},\ \frac{\mathrm{H}(u+K)}{\Theta(u)},\ \frac{\Theta(u+K)}{\Theta(u)};$$

that is, to the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$.

662.

ON THE DOUBLE Θ-FUNCTIONS IN CONNEXION WITH A 16-NODAL QUARTIC SURFACE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXIII. (1877), pp. 210—219.]

I HAVE before me Göpel's memoir, "Theoriae transcendentium Abelianarum primi ordinis adumbratio levis," Crelle's Journal, t. XXXV. (1847), pp. 277—312. Writing P_1, P_2, P_3, etc., in place of his P', P'', P''', etc., also α, β, γ, δ, X', Y', Z', W', in place of his t, u, v, w, T, U, V, W, the system of 16 equations (given p. 287) is

$$
\begin{array}{lll}
(1) & P^2 = (\alpha, & -\beta, \quad -\gamma, \quad \delta)(X', Y', Z', W'), \\
(4) & P_1^2 = (\alpha, & \beta, \quad -\gamma, \quad -\delta)(X', Y', Z', W'), \\
(9) & P_2^2 = (\alpha, & -\beta, \quad \gamma, \quad -\delta)(X', Y', Z', W'), \\
(12) & P_3^2 = (\alpha, & \beta, \quad \gamma, \quad \delta)(X', Y', Z', W'), \\
(3) & Q^2 = (\beta, & -\alpha, \quad -\delta, \quad \gamma)(X', Y', Z', W'), \\
(2) & Q_1^2 = (\beta, & \alpha, \quad -\delta, \quad -\gamma)(X', Y', Z', W'), \\
(11) & Q_2^2 = (\beta, & -\alpha, \quad \delta, \quad -\gamma)(X', Y', Z', W'), \\
(10) & Q_3^2 = (\beta, & \alpha, \quad \delta, \quad \gamma)(X', Y', Z', W'), \\
(13) & R^2 = (\gamma, & -\delta, \quad -\alpha, \quad \beta)(X', Y', Z', W'), \\
(16) & R_1^2 = (\gamma, & \delta, \quad -\alpha, \quad -\beta)(X', Y', Z', W'), \\
(5) & R_2^2 = (\gamma, & -\delta, \quad \alpha, \quad -\beta)(X', Y', Z', W'), \\
(8) & R_3^2 = (\gamma, & \delta, \quad \alpha, \quad \beta)(X', Y', Z', W'), \\
(15) & S^2 = (\delta, & -\gamma, \quad -\beta, \quad \alpha)(X', Y', Z', W'), \\
(14) & S_1^2 = (\delta, & \gamma, \quad -\beta, \quad -\alpha)(X', Y', Z', W'), \\
(7) & S_2^2 = (\delta, & -\gamma, \quad \beta, \quad -\alpha)(X', Y', Z', W'), \\
(6) & S_3^2 = (\delta, & \gamma, \quad \beta, \quad \alpha)(X', Y', Z', W');
\end{array}
$$

viz. we have $P^2 = \alpha X' - \beta Y' - \gamma Z' + \delta W'$, etc. The reason for the apparently arbitrary manner in which I have numbered these equations, will appear further on. I recall that the sixteen double Θ-functions, that is, Θ-functions of two arguments u, u', are*

$$\begin{array}{cccc} P, & P_1, & P_2, & P_3, \\ iQ, & Q_1, & iQ_2, & Q_3, \\ iR, & iR_1, & R_2, & R_3, \\ S, & iS_1, & iS_2, & S_3, \end{array}$$

the factor i, $=\sqrt{-1}$, being introduced in regard to the six functions which are odd functions of the arguments. But disregarding the sign, I speak of P^2, P_1^2, ..., Q^2, etc., as the squared functions, or simply as the squares; α, β, γ, δ are constants, depending of course on the parameters of the Θ-functions; X', Y', Z', W', which are however to be eliminated, are themselves Θ-functions to a different set of parameters: the 16 equations express that the squared functions P^2, P_1^2, etc., are linear functions of X', Y', Z', W', and they consequently serve to obtain linear relations between the squared functions: viz. by means of them, Göpel expresses the remaining 12 squares, each in terms of the selected four squares P_1^2, P_2^2, S_1^2, S_2^2, which are linearly independent: that is, he obtains linear relations between five squares, and he seems to have assumed that there were not any linear relations between fewer than five squares.

It appears however by Rosenhain's "Mémoire sur les fonctions de deux variables et à quatre périodes etc.", *Mém. Sav. Étrangers*, t. XI. (1851), pp. 364—468, that there are, in fact, linear relations between four squares, viz. that there exist sixes of squares such that, selecting at pleasure any three out of the six, each of the remaining three squares can be expressed as a linear function of these three squares. Knowing this result, it is easy to verify it by means of the sixteen equations, and moreover to show that there are in all 16 such sixes: these are shown by the following scheme which I copy from Kummer's memoir "Ueber die algebraischen Strahlensysteme u. s. w." *Berlin. Abh.* (1866), p. 66: viz. the scheme is

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11.

* The same functions in Rosenhain's notation are

$$\begin{array}{cccc} 00, & 02, & 20, & 22, \\ 01, & 03, & 21, & 23, \\ 10, & 12, & 30, & 32, \\ 11, & 13, & 31, & 33; \end{array}$$

viz. the figures here written down are the suffixes of his ϑ-functions, $00 = \vartheta_{0,0}(v, w)$, etc.

In fact, to show that any four of the squares, for instance 1, 9, 13, 8, that is, P^2, P_2^2, R^2, R_3^2, are linearly connected, it is only necessary to show that the determinant of coefficients

$$\begin{vmatrix} \alpha, & -\beta, & -\gamma, & \delta \\ \alpha, & -\beta, & \gamma, & -\delta \\ \gamma, & -\delta, & -\alpha, & \beta \\ \gamma, & \delta, & \alpha, & \beta \end{vmatrix}$$

is $=0$, or what is the same thing, that there exists a linear function of the new variables (X, Y, Z, W), which will become $=0$ on putting for these variables the values in any line of this determinant: we have such a function, viz. this is

$$\beta X + \alpha Y - \delta Z - \gamma W,$$

or say

$$[1] \quad (\beta, \alpha, -\delta, -\gamma)(X, Y, Z, W).$$

This function also vanishes if for (X, Y, Z, W) we substitute the values

$$\begin{matrix} \delta, & -\gamma, & \beta, & -\alpha, \\ \delta, & \gamma, & \beta, & \alpha, \end{matrix}$$

which belong to 7, 6, that is, S_2^2 and S_3^2 respectively. It thus appears that 1, 9, 13, 8, 7, 6, that is, P^2, P_2^2, R^2, R_3^2, S_2^2, S_3^2, are a set of six squares having the property in question. I remark that the process of forming the linear functions is a very simple one; we write down six lines, and thence directly obtain the result, thus

$$\begin{array}{rcccc} 1 & \alpha, & -\beta, & -\gamma, & \delta \\ 9 & \alpha, & -\beta, & \gamma, & -\delta \\ 13 & \gamma, & -\delta, & -\alpha, & \beta \\ 8 & \gamma, & \delta, & \alpha, & \beta \\ 7 & \delta, & -\gamma, & \beta, & -\alpha \\ 6 & \delta, & \gamma, & \beta, & \alpha \\ \hline & \beta, & \alpha, & -\delta, & -\gamma: \end{array}$$

viz. β, α, δ, γ are the letters not previously occurring in the four columns respectively: the first letter β is taken to have the sign +, and then the remaining signs are determined by the condition that, combining the last line with any line above it (e.g. with the line next above it $\beta\delta + \alpha\gamma - \delta\beta - \gamma\alpha$), the sum must be zero.

We find in this way, as the conditions for the existence of the 16 sixes respectively,

$$\begin{aligned} &[1] \quad (\beta, \alpha, -\delta, -\gamma)(X, Y, Z, W) = 0, \\ &[2] \quad (\alpha, -\beta, -\gamma, \delta)(X, Y, Z, W) = 0, \\ &[3] \quad (\alpha, \beta, -\gamma, -\delta)(X, Y, Z, W) = 0, \\ &[4] \quad (\beta, -\alpha, -\delta, \gamma)(X, Y, Z, W) = 0, \end{aligned}$$

$$[5]\quad (\delta,\ \gamma,\ \beta,\ \alpha)(X,\ Y,\ Z,\ W)=0,$$
$$[6]\quad (\gamma,\ -\delta,\ \alpha,\ -\beta)(X,\ Y,\ Z,\ W)=0,$$
$$[7]\quad (\gamma,\ \delta,\ \alpha,\ \beta)(X,\ Y,\ Z,\ W)=0,$$
$$[8]\quad (\delta,\ -\gamma,\ \beta,\ -\alpha)(X,\ Y,\ Z,\ W)=0,$$
$$[9]\quad (\beta,\ \alpha,\ \delta,\ \gamma)(X,\ Y,\ Z,\ W)=0,$$
$$[10]\quad (\alpha,\ -\beta,\ \gamma,\ -\delta)(X,\ Y,\ Z,\ W)=0,$$
$$[11]\quad (\alpha,\ \beta,\ \gamma,\ \delta)(X,\ Y,\ Z,\ W)=0,$$
$$[12]\quad (\beta,\ -\alpha,\ \delta,\ -\gamma)(X,\ Y,\ Z,\ W)=0,$$
$$[13]\quad (\delta,\ \gamma,\ -\beta,\ -\alpha)(X,\ Y,\ Z,\ W)=0,$$
$$[14]\quad (\gamma,\ -\delta,\ -\alpha,\ \beta)(X,\ Y,\ Z,\ W)=0,$$
$$[15]\quad (\gamma,\ \delta,\ -\alpha,\ -\beta)(X,\ Y,\ Z,\ W)=0,$$
$$[16]\quad (\delta,\ -\gamma,\ -\beta,\ \alpha)(X,\ Y,\ Z,\ W)=0.$$

I repeat in a new order the sets of coefficients which belong to the several squares, viz. these are

$$(1)\quad P^2\ (\alpha,\ -\beta,\ -\gamma,\ \delta),$$
$$(2)\quad Q_1^2\ (\beta,\ \alpha,\ -\delta,\ -\gamma),$$
$$(3)\quad Q^2\ (\beta,\ -\alpha,\ -\delta,\ \gamma),$$
$$(4)\quad P_1^2\ (\alpha,\ \beta,\ -\gamma,\ -\delta),$$
$$(5)\quad R_2^2\ (\gamma,\ -\delta,\ \alpha,\ -\beta),$$
$$(6)\quad S_3^2\ (\delta,\ \gamma,\ \beta,\ \alpha),$$
$$(7)\quad S_2^2\ (\delta,\ -\gamma,\ \beta,\ -\alpha),$$
$$(8)\quad R_3^2\ (\gamma,\ \delta,\ \alpha,\ \beta),$$
$$(9)\quad P_2^2\ (\alpha,\ -\beta,\ \gamma,\ -\delta),$$
$$(10)\quad Q_3^2\ (\beta,\ \alpha,\ \delta,\ \gamma),$$
$$(11)\quad Q_2^2\ (\beta,\ -\alpha,\ \delta,\ -\gamma),$$
$$(12)\quad P_3^2\ (\alpha,\ \beta,\ \gamma,\ \delta),$$
$$(13)\quad R^2\ (\gamma,\ -\delta,\ -\alpha,\ \beta),$$
$$(14)\quad S_1^2\ (\delta,\ \gamma,\ -\beta,\ -\alpha),$$
$$(15)\quad S^2\ (\delta,\ -\gamma,\ -\beta,\ \alpha),$$
$$(16)\quad R_1^2\ (\gamma,\ \delta,\ -\alpha,\ -\beta).$$

And I remark that, if we connect these with the multipliers $(Y,\ -X,\ W,\ -Z)$, we obtain, except that there is sometimes a reversal of all the signs, the *same* linear functions of $(X,\ Y,\ Z,\ W)$ as are written down under the same numbers in square brackets above: thus (1) gives

$(\alpha,\ -\beta,\ -\gamma,\ \delta)(Y,\ -X,\ W,\ -Z)$, which is $(\beta,\ \alpha,\ -\delta,\ -\gamma)(X,\ Y,\ Z,\ W)$, $=[1]$;

and so (2) gives

$$(\beta,\ \alpha,\ -\delta,\ -\gamma)(Y,\ -X,\ W,\ -Z),\ \text{which is}\ (-\alpha,\ \beta,\ \gamma,\ -\delta)(X,\ Y,\ Z,\ W),$$

or, reversing the signs,

$$(\alpha,\ -\beta,\ -\gamma,\ \delta)(X,\ Y,\ Z,\ W),\ =[2].$$

Comparing with the geometrical theory in Kummer's Memoir, it appears that the several systems of values (1), (2), ..., (16) are the coordinates of the nodes of a 16-nodal quartic surface, which nodes lie by sixes in the singular tangent planes, in the manner expressed by the foregoing scheme, wherein each top number may refer to a singular tangent plane, and then the numbers below it show the nodes in this plane: or else the top number may refer to a node, and then the numbers below it show the singular planes through this node.

And, from what precedes, we have the general result: the 16 squared double Θ-functions correspond (one to one) to the nodes of a 16-nodal quartic surface, in such wise that linearly connected squared functions correspond to nodes in the same singular tangent plane.

The question arises, to find the equation of the 16-nodal quartic surface, having the foregoing nodes and singular tangent planes. Starting from one of the irrational forms, say

$$\sqrt{A\,[1]\,[5]}+\sqrt{B\,[2]\,[6]}+\sqrt{C\,[3]\,[7]}=0,$$

the coefficients A, B, C are readily determined; and the result written at full length is

$$\begin{aligned}&\sqrt{2(\alpha\beta-\gamma\delta)(\alpha\delta+\beta\gamma)(\beta X+\alpha Y-\delta Z-\gamma W)(\delta X+\gamma Y+\beta Z+\alpha W)}\\ &+\sqrt{(\alpha^2-\beta^2-\gamma^2+\delta^2)(\alpha\gamma-\beta\delta)(\alpha X-\beta Y-\gamma Z+\delta W)(\gamma X-\delta Y+\alpha Z-\beta W)}\\ &+\sqrt{(\alpha^2+\beta^2-\gamma^2-\delta^2)(\alpha\gamma+\beta\delta)(\alpha X+\beta Y-\gamma Z-\delta W)(\gamma X+\delta Y+\alpha Z+\beta W)}=0.\end{aligned}$$

It is a somewhat long, but nevertheless interesting, piece of algebraical work to rationalise the foregoing equation: the result is

$$\begin{aligned}&(\beta^2\gamma^2-\alpha^2\delta^2)(\gamma^2\alpha^2-\beta^2\delta^2)(\alpha^2\beta^2-\gamma^2\delta^2)(X^4+Y^4+Z^4+W^4)\\ &+(\gamma^2\alpha^2-\beta^2\delta^2)(\alpha^2\beta^2-\gamma^2\delta^2)(\alpha^4+\delta^4-\beta^4-\gamma^4)(Y^2Z^2+X^2W^2)\\ &+(\alpha^2\beta^2-\gamma^2\delta^2)(\beta^2\gamma^2-\alpha^2\delta^2)(\beta^4+\delta^4-\gamma^4-\alpha^4)(Z^2X^2+Y^2W^2)\\ &+(\beta^2\gamma^2-\alpha^2\delta^2)(\gamma^2\alpha^2-\beta^2\delta^2)(\gamma^4+\delta^4-\alpha^4-\beta^4)(X^2Y^2+Z^2W^2)\\ &-2\alpha\beta\gamma\delta(\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha^2+\delta^2-\beta^2-\gamma^2)(\beta^2+\delta^2-\alpha^2-\gamma^2)(\gamma^2+\delta^2-\alpha^2-\beta^2)XYZW=0;\end{aligned}$$

or, if we write for shortness

$$\begin{aligned}L&=\beta^2\gamma^2-\alpha^2\delta^2, & F&=\alpha^2+\delta^2-\beta^2-\gamma^2,\\ M&=\gamma^2\alpha^2-\beta^2\delta^2, & G&=\beta^2+\delta^2-\gamma^2-\alpha^2,\\ N&=\alpha^2\beta^2-\gamma^2\delta^2, & H&=\gamma^2+\delta^2-\alpha^2-\beta^2,\\ & & \Delta&=\alpha^2+\beta^2+\gamma^2+\delta^2,\end{aligned}$$

then the result is

$$\begin{aligned}&LMN(X^4+Y^4+Z^4+W^4)\\&+MN(F\Delta+2L)(Y^2Z^2+X^2W^2)\\&+NL(G\Delta+2M)(Z^2X^2+Y^2W^2)\\&+LM(H\Delta+2N)(X^2Y^2+Z^2W^2)\\&-2\alpha\beta\gamma\delta FGH\Delta XYZW=0.\end{aligned}$$

It may be easily verified that any one of the sixteen points, for instance $(\alpha, \beta, \gamma, \delta)$, is a node of the surface. Thus to show that the derived function in respect to X, vanishes for $X, Y, Z, W=\alpha, \beta, \gamma, \delta$; the derived function here divides by 2α, and omitting this factor, the equation to be verified is

$$LMN\,.\,2\alpha^2+MN(F\Delta+2L)\delta^2+NL(G\Delta+2M)\gamma^2+LM(H\Delta+2N)\beta^2-\beta^2\gamma^2\delta^2FGH\Delta=0,$$

viz. the whole coefficient of LMN is $2(\alpha^2+\beta^2+\gamma^2+\delta^2), =2\Delta$; hence throwing out the factor Δ, the equation becomes

$$2LMN+MNF\delta^2+NLG\gamma^2+LMH\beta^2-\beta^2\gamma^2\delta^2FGH=0.$$

Writing this in the form

$$L(2MN+NG\gamma^2+MH\beta^2)=F\delta^2(GH\beta^2\gamma^2-MN),$$

we find without difficulty $GH\beta^2\gamma^2-MN=-(\beta^2-\gamma^2)^2L$; hence throwing out the factor L, the equation becomes

$$N(2M+G\gamma^2)+MH\beta^2+F\delta^2(\beta^2-\gamma^2)^2=0;$$

we find

$$\begin{aligned}MH\beta^2+F\delta^2(\beta^2-\gamma^2)^2&=(\alpha^2\beta^2-\gamma^2\delta^2)(2\beta^2\delta^2-\gamma^2(\alpha^2+\beta^2+\delta^2)+\gamma^4)\\&=N(2\beta^2\delta^2-\gamma^2(\alpha^2+\beta^2+\delta^2)+\gamma^4),\end{aligned}$$

or throwing out the factor N, the equation becomes

$$2M+G\gamma^2+2\beta^2\delta^2-\gamma^2(\alpha^2+\beta^2+\delta^2)+\gamma^4=0,$$

which is at once verified: and similarly it can be shown that the other derived functions vanish, and the point $(\alpha, \beta, \gamma, \delta)$ is thus a node.

The surface seems to be the general 16-nodal surface, viz. replacing X, Y, Z, W by any linear functions of four coordinates, we have thus $4.4-1, =15$ constants, and the equation contains besides the three ratios $\alpha:\beta:\gamma:\delta$, that is, in all 18 constants: the general quartic surface has 34 constants, and therefore the general 16-nodal surface $34-16, =18$ constants: but the conclusion requires further examination.

Göpel and Rosenhain each connect the theory with that of the ultra-elliptic functions involving the radical $\sqrt{X}, =\sqrt{x.1-x.1-lx.1-mx.1-nx}$; viz. it appears by their formulæ (more completely by those of Rosenhain) that the ratios of the 16 squares can be expressed rationally in terms of the two variables x, x', and the radicals

$\sqrt{X}$, $\sqrt{X'}$, X' being the same function of x' that X is of x. We may instead of the preceding form take X to be the general quintic function, or what is better take it to be the sextic function $a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x$; and we thus obtain a remarkable algebraical theorem: viz. I say that the 16 squares, each divided by a proper constant factor, are proportional to six functions of the form

$$a-x\,.\,a-x',$$

and ten functions of the form

$$\frac{1}{(x-x')^2}\{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x'\,.\,e-x'\,.\,f-x'}-\sqrt{a-x'\,.\,b-x'\,.\,c-x'\,.\,d-x\,.\,e-x\,.\,f-x}\}^2,$$

and consequently that these 16 algebraical functions of x, x' are linearly connected in the manner of the 16 squares; viz. there exist 16 sixes such that, in each six, the remaining three functions can be linearly expressed in terms of any three of them.

To further develop the theory, I remark that the six functions may be represented by A, B, C, D, E, F respectively: any one of the ten functions would be properly represented by $ABC\,.\,DEF$, but isolating one letter F, and writing DE to denote DEF, this function $ABC\,.\,DEF$ may be represented simply as DE; and the ten functions thus are AB, AC, AD, AE, BC, BD, BE, CD, CE, DE.

Writing for shortness a, b, c, d, e, f, to denote $a-x$, $b-x$, etc., and similarly a', b', c', d', e', f', to denote $a-x'$, $b-x'$, etc., we thus have

$$(13)\quad A=aa',$$

$$(9)\quad B=bb',$$

$$(7)\quad C=cc',$$

$$(8)\quad D=dd',$$

$$(6)\quad E=ee', \qquad (=\bar{E}),$$

$$(1)\quad F=ff', \qquad (=\bar{F}),$$

$$(3)\quad DE=\frac{1}{(x-x')^2}\{\sqrt{abcd'e'f'}-\sqrt{a'b'c'def}\}^2, \quad (=\bar{D}),$$

$$(4)\quad CE=\frac{1}{(x-x')^2}\{\sqrt{abdc'e'f'}-\sqrt{a'b'd'cef}\}^2, \quad (=\bar{E}),$$

$$(2)\quad CD=\frac{1}{(x-x')^2}\{\sqrt{abec'd'f'}-\sqrt{a'b'e'cdf}\}^2,$$

$$(14)\quad BE=\frac{1}{(x-x')^2}\{\sqrt{acdb'e'f'}-\sqrt{a'c'd'bef}\}^2, \quad (=\bar{B}),$$

$$(16)\quad BD=\frac{1}{(x-x')^2}\{\sqrt{aceb'd'f'}-\sqrt{a'c'e'bdf}\}^2,$$

$$(15)\quad BC=\frac{1}{(x-x')^2}\{\sqrt{adeb'c'f'}-\sqrt{a'd'e'bcf}\}^2,$$

$$(10)\quad AE=\frac{1}{(x-x')^2}\{\sqrt{bcda'e'f'}-\sqrt{b'c'd'aef}\}^2, \quad (=\bar{A}),$$

$$(12)\quad AD = \frac{1}{(x-x')^2}\{\sqrt{bcea'd'f'} - \sqrt{b'c'e'adf}\}^2,$$

$$(11)\quad AC = \frac{1}{(x-x')^2}\{\sqrt{bdea'c'f'} - \sqrt{b'd'e'acf}\}^2,$$

$$(5)\quad AB = \frac{1}{(x-x')^2}\{\sqrt{cdea'b'f'} - \sqrt{c'd'e'abf}\}^2,$$

where the numbers are in accordance with the foregoing scheme; viz. the scheme becomes

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
F	CD	DE	CE	AB	E	C	D	B	AE	AC	AD	A	BE	BC	BD
B	AE	AC	AD	A	BE	BC	BD	F	CD	DE	CE	AB	E	C	D
A	BE	BC	BD	B	AE	AC	AD	AB	E	C	D	F	CD	DE	CE
D	C	E	AB	CE	DE	CD	F	BD	BC	BE	A	AD	AC	AE	B
C	D	AB	E	DE	CE	F	CD	BC	BD	A	BE	AC	AD	B	AE
E	AB	D	C	CD	F	CE	DE	DE	A	BD	BC	AE	B	AD	AC.

There is of course the six A, B, C, D, E, F; for each of these is a linear function of $1, x+x', xx'$, and there is thus a linear relation between any four of them. It would at first sight appear that the remaining sixes were of two different forms, A, B, AB, CE, CD, DE, and F, A, AB, AC, AD, AE; but these are really identical, for taking *any* two letters E, F, the six is E, F, AE, BE, CE, DE, or, as this might be written, E, F, AEF, BEF, CEF, DEF, where AEF means $BCD\,.\,AEF$, etc.; and we thus obtain each of the remaining fifteen sixes. The six just referred to, viz. E, F, AE, BE, CE, DE, or changing the notation say $E, F, \bar{A}, \bar{B}, \bar{C}, \bar{D}$ as indicated in the table, thus represents any one of the sixes other than the rational six A, B, C, D, E, F; and there is no difficulty in actually finding each of the fifteen relations between four functions of the six in question, $E, F, \bar{A}, \bar{B}, \bar{C}, \bar{D}$. It is to be observed that every such function as $\bar{A}$ contains the same irrational part

$$\frac{2}{(x-x')^2}\sqrt{abcdefa'b'c'd'e'f'},$$

and that the linear relations involve therefore only the differences $\bar{A}-\bar{B}$, $\bar{A}-\bar{C}$, etc., which are rational. Proceeding to calculate these differences, we have for instance

$$\bar{C}-\bar{D} = \frac{1}{(x-x')^2}(cefa'b'd' + c'e'f'abd - defa'b'c' - d'e'f'abc) = \frac{1}{(x-x')^2}(cd'-c'd)(efa'b'-e'f'ab);$$

or, substituting for a, a', etc. their values $a-x$, $a-x'$, etc., we have

$$cd'-c'd = (x-x')(c-d),$$

$$efa'b'-e'f'ab = (x-x')\begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+b, & ab \\ 1, & e+f, & ef \end{vmatrix},$$

or say for shortness

$$= (x-x')\,[xx'abef].$$

We have therefore

$$\bar{C} - \bar{D} = (c - d)\,[xx'abef]\,;$$

and in like manner we obtain the equations

$$\bar{B} - \bar{C} = (b - c)\,[xx'adef], \quad \bar{A} - \bar{D} = (a - d)\,[xx'bcef],$$
$$\bar{C} - \bar{A} = (c - a)\,[xx'bdef], \quad \bar{B} - \bar{D} = (b - d)\,[xx'caef],$$
$$\bar{A} - \bar{B} = (a - b)\,[xx'cdef], \quad \bar{C} - \bar{D} = (c - d)\,[xx'abef].$$

It is now easy to form the system of formulæ

E	F	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{D}$	
		$ae\,.\,af\,.\,bcd$	$-be\,.\,bf\,.\,cda$	$+\,ce\,.\,cf\,.\,dab$	$-\,de\,.\,df\,.\,abc$	$=0$
$ad\,.\,bf\,.\,cf$	$-\,ad\,.\,be\,.\,ce$	$+\,ef$			$-\,ef$	$=0$
$bd\,.\,cf\,.\,af$	$-\,bd\,.\,ce\,.\,ae$		$+\,ef$	$-\,ef$		$=0$
$cd\,.\,af\,.\,bf$	$-\,cd\,.\,ae\,.\,be$			$+\,ef$	$-\,ef$	$=0$
$bc\,.\,af\,.\,df$	$-\,bc\,.\,ae\,.\,de$		$+\,ef$	$-\,ef$		$=0$
$ca\,.\,bf\,.\,df$	$-\,ca\,.\,be\,.\,de$	$-\,ef$		$+\,ef$		$=0$
$ab\,.\,cf\,.\,df$	$-\,ab\,.\,ce\,.\,de$	$+\,ef$	$-\,ef$			$=0$
$-\,af\,.\,bcd$			$+\,be\,.\,cd$	$+\,ce\,.\,db$	$+\,de\,.\,bc$	$=0$
$-\,bf\,.\,cda$		$+\,ae\,.\,cd$		$+\,ce\,.\,da$	$+\,de\,.\,ac$	$=0$
$-\,cf\,.\,dab$		$+\,ae\,.\,bd$	$+\,be\,.\,da$		$+\,de\,.\,ab$	$=0$
$-\,df\,.\,abc$		$+\,ae\,.\,bc$	$+\,be\,.\,ca$	$+\,ce\,.\,ab$		$=0$
	$-\,ae\,.\,bcd$		$+\,bf\,.\,cd$	$+\,cf\,.\,db$	$+\,df\,.\,bc$	$=0$
	$-\,be\,.\,cda$	$+\,af\,.\,cd$		$+\,cf\,.\,da$	$+\,df\,.\,ac$	$=0$
	$-\,ce\,.\,dab$	$+\,af\,.\,bd$	$+\,bf\,.\,da$		$+\,df\,.\,ab$	$=0$
	$-\,df\,.\,abc$	$+\,af\,.\,bc$	$+\,bf\,.\,ca$	$+\,cf\,.\,ab$		$=0,$

where for shortness ab, ac, etc., are written to denote $a-b$, $a-c$, etc.; also abc, etc., to denote $(b-c)(c-a)(a-b)$, etc.: the equations contain all of them only the differences of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$; thus the first equation is equivalent to

$$ae\,.\,af\,.\,bcd\,(\bar{A} - \bar{D}) - be\,.\,bf\,.\,cde\,(\bar{B} - \bar{D}) + ce\,.\,cf\,.\,dab\,(\bar{C} - \bar{D}) = 0,$$

and so in other cases.

Cambridge, 14 *March*, 1877.

663.

FURTHER INVESTIGATIONS ON THE DOUBLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXIII. (1877), pp. 220—233.]

I CONSIDER six letters

$$a,\quad b,\quad c,\quad d,\quad e,\quad f;$$

a duad ab not containing f may be completed into the triad abf, and then into the double triad $abf.cde$; there are in all ten double triads, represented by the duads

$$ab,\quad ac,\quad ad,\quad ae,\quad bc,\quad bd,\quad be,\quad cd,\quad ce,\quad de,$$

and the whole number of letters and of double triads is $=16$.

Taking x, x' as variables, I form sixteen functions; viz. these are

$$[a] = a-x\,.\,a-x',$$

$$[ab] = \frac{1}{(x-x')^2}\left\{\sqrt{\frac{a-x\,.\,b-x\,.\,f-x}{c-x'\,.\,d-x'\,.\,e-x'}} \pm \sqrt{\frac{a-x'\,.\,b-x'\,.\,f-x'}{c-x\,.\,d-x\,.\,e-x}}\right\}^2,$$

where the function under each radical sign is the product of six factors, the arrangement in two lines being for convenience only: the sign $\pm$ has the same value in all the functions, and it will be observed that the irrational part is

$$= \pm\frac{2}{(x-x')^2}\sqrt{\frac{a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x}{a-x'\,.\,b-x'\,.\,c-x'\,.\,d-x'\,.\,e-x'\,.\,f-x'}},$$

viz. this has the same value in all the functions.

The general property of the double ϑ-functions is that the squares of the sixteen functions are proportional to constant multiples of the sixteen functions $[a]$, $[ab]$; but this theorem may be presented in a much more definite form, viz. we can determine, and

that very simply, the actual expressions for the constant factors; and so we can enunciate the theorem as follows; the squares of the sixteen double ϑ-functions are proportional to sixteen functions $-\{a\}$, $+\{ab\}$; where, in a notation about to be explained,

$$\{a\} = \sqrt{a}\,[a], \quad \{ab\} = \sqrt{ab}\,[ab].$$

Here in the radical $\sqrt{a}$, a is to be considered as standing in the first place for the pentad $bcdef$, which is to be interpreted as a product of differences,

$$= bc\,.\,bd\,.\,be\,.\,bf\,.\,cd\,.\,ce\,.\,cf\,.\,de\,.\,df\,.\,ef,$$

(where bc, bd, etc., denote the differences $b-c$, $b-d$, etc.). Similarly, in the radical $\sqrt{ab}$, ab is to be considered as standing in the first instance for the double triad $abf\,.\,cde$, which is to be interpreted as a product of differences, $= ab\,.\,af\,.\,bf\,.\,cd\,.\,ce\,.\,de$, (where ab, af, etc., denote the differences $a-b$, $a-f$, etc.).

It is convenient to consider a, b, c, d, e, f as denoting real magnitudes taken in decreasing order: in all the products $bcdef$, etc., and in each term abf or cde of a product $abf\,.\,cde$, the letters are to be written in alphabetical order; the differences bc, bd, etc., ab, af, etc., which present themselves in the several products, are thus all of them positive; and the radicals, being all of them the roots of positive quantities, may themselves be taken to be positive.

We have to consider the values of the functions $[a]$, $[ab]$, or $\{a\}$, $\{ab\}$, in the case where the variables x, x' become equal to any two of the letters a, b, c, d, e, f; it is clearly the same thing whether we have for instance $x=b$, $x'=c$, or $x=c$, $x'=b$, etc.: we have therefore to consider for x, x' the fifteen values ab, ac, ..., af, ..., ef; there is besides a sixteenth set of values x, x' each infinite, without any relation between the infinite values.

Taking this case first, x, x' each infinite, and in $[ab]$, etc., the sign $\pm$ to be $+$, we have

$$[a] = xx', \quad [ab] = \frac{4x^3x'^3}{(x-x')^2},$$

or, attending only to the ratios of these values,

$$[a] = 1, \qquad [ab] = \frac{4x^2x'^2}{(x-x')^2},$$

where $\dfrac{4x^2x'^2}{(x-x')^2}$ is infinite, and the values may finally be written

$$[a] = 0, \qquad [ab] = 1;$$

whence also, for x, x' infinite,

$$\{a\} = 0, \qquad \{ab\} = \sqrt{ab},$$

the radical $\sqrt{ab}$ being understood as before.

Suppose next that x, x' denote any two of the letters, for instance a, b; then two of the functions $[a]$ vanish, viz. these are $[a]$, $[b]$, but the remaining four functions acquire determinate values; and moreover four of the functions $[ab]$ vanish, viz. these are $[ab]$, $[cd]$, $[ce]$, $[de]$, for each of which the xx' letters a, b occur in the same triad (the

double triads for the four functions are, in fact, $abf.cde$, $cdf.abe$, $cef.abd$, $def.abc$); but the other six functions $[ab]$, for which the letters a, b occur in separate triads, acquire determinate values.

It is important to attend to the signs: for example, if x, $x' = b$, e, we have

$$[c] = ce.cb, \quad = -bc.ce$$

$$[ce] = \frac{1}{(be)^2}\frac{cb.eb.fb}{ae.be.de}, \quad = -\frac{cb.fb}{ae.de}, \quad = -\frac{bc.bf}{ae.de}.$$

TABLE I. OF THE VALUES OF $[a]$, $[ab]$, ETC.,

x,	$x' = \infty$ ∞	ab	ac	ad	ae	af	bc	bd
$[a]$	0	0	0	0	0	0	$+ ab.ac$	$+ ab.ad$
$[b]$	0	0	$- ab.bc$	$- ab.bd$	$- ab.be$	$- ab.bf$	0	0
$[c]$	0	$+ ac.bc$	0	$- ac.cd$	$- ac.ce$	$- ac.cf$	0	$- bc.cd$
$[d]$	0	$+ ad.bd$	$+ ad.cd$	0	$- ad.de$	$- ad.df$	$+ bd.cd$	0
$[e]$	0	$+ ae.be$	$+ ae.ce$	$+ ae.de$	0	$- ae.ef$	$+ be.ce$	$+ be.de$
$[f]$	0	$+ af.bf$	$+ af.cf$	$+ af.df$	$+ af.ef$	0	$+ bf.cf$	$+ bf.df$
$[ab]$	$+ abf.cde$	0	$+ \frac{ad.ae}{bc.cf}$	$+ \frac{ac.ae}{bd.df}$	$+ \frac{ac.ad}{be.ef}$	0	$+ \frac{ac.bd}{be.cf}$	$+ \frac{ad.bc}{be.df}$
$[ac]$	$+ acf.bde$	$- \frac{ad.ae}{bc.bf}$	0	$+ \frac{ab.ae}{cd.cf}$	$+ \frac{ab.ad}{ce.ef}$	0	$+ \frac{ab.bf}{cd.ce}$	0
$[ad]$	$+ adf.bce$	$- \frac{ac.ae}{bd.bf}$	$- \frac{ab.ae}{cd.cf}$	0	$+ \frac{ab.ac}{de.ef}$	0	0	$- \frac{ab.bf}{cd.de}$
$[ae]$	$+ aef.bcd$	$- \frac{ac.ae}{be.bf}$	$- \frac{ab.ad}{ce.cf}$	$- \frac{ab.ac}{de.df}$	0	0	0	0
$[bc]$	$+ bcf.ade$	$- \frac{ac.af}{bd.be}$	$- \frac{ab.af}{cd.ce}$	0	0	$- \frac{ab.ac}{df.ef}$	0	$- \frac{ab.be}{cd.df}$
$[bd]$	$+ bdf.ace$	$- \frac{ad.af}{bc.be}$	0	$+ \frac{ab.af}{cd.de}$	0	$- \frac{ab.ad}{cf.ef}$	$+ \frac{ab.be}{cd.cf}$	0
$[be]$	$+ bef.acd$	$- \frac{ae.af}{bc.bd}$	0	0	$- \frac{ab.af}{ce.de}$	$- \frac{ab.ae}{cf.df}$	$+ \frac{ab.bd}{ce.cf}$	$+ \frac{ab.bc}{de.df}$
$[cd]$	$+ cdf.abe$	0	$+ \frac{ad.af}{bc.ce}$	$+ \frac{ac.af}{bd.de}$	0	$- \frac{ac.ad}{bf.ef}$	$+ \frac{ac.bd}{bf.ce}$	$+ \frac{ad.bc}{bf.de}$
$[ce]$	$+ cef.abd$	0	$+ \frac{ae.af}{bc.cd}$	0	$- \frac{ac.af}{be.de}$	$- \frac{ac.ae}{bf.df}$	$+ \frac{ac.be}{bf.cd}$	0
$[de]$	$+ def.abc$	0	0	$- \frac{ae.af}{bc.cd}$	$- \frac{ad.af}{be.ce}$	$- \frac{ad.ae}{bf.cf}$	0	$- \frac{ad.be}{bf.cd}$

Here the symbols *be, ce,* etc., denote differences; [*ce*] is the product of four differences: the arrangement in two lines is for convenience only.

We thus obtain the series of values of [*a*], [*ab*], etc., which although only required as subsidiary to the determination of the corresponding values of {*a*}, {*ab*}, I nevertheless give in a table.

The signs are given as they were actually obtained, but as we are concerned only with the ratios of the functions, it is allowable to change all the signs in any

FOR THE SIXTEEN SPECIAL VALUES OF x, x'.

be	bf	cd	ce	cf	de	df	ef
$+ ab\,.\,ae$	$+ ab\,.\,af$	$+ ac\,.\,ad$	$+ ac\,.\,ae$	$+ ac\,.\,af$	$+ ad\,.\,ae$	$+ ad\,.\,af$	$+ ae\,.\,af$
0	0	$+ bc\,.\,bd$	$+ bc\,.\,be$	$+ bc\,.\,bf$	$+ bd\,.\,be$	$+ bd\,.\,bf$	$+ be\,.\,bf$
$- bc\,.\,ce$	$- bc\,.\,cf$	0	0	0	$+ cd\,.\,ce$	$+ cd\,.\,df$	$+ ce\,.\,cf$
$- bd\,.\,de$	$- bd\,.\,bf$	0	$- cd\,.\,de$	$- cd\,.\,df$	0	0	$+ de\,.\,df$
0	$- be\,.\,ef$	$+ ce\,.\,de$	0	$- ce\,.\,ef$	0	$- de\,.\,ef$	0
$+ bf\,.\,ef$	0	$+ cf\,.\,df$	$+ cf\,.\,ef$	0	$+ df\,.\,ef$	0	0
$+ \dfrac{ac\,.\,bc}{bd\,.\,ef}$	0	0	0	$- \dfrac{ac\,.\,bc}{df\,.\,ef}$	0	$- \dfrac{ad\,.\,bd}{cf\,.\,ef}$	$- \dfrac{ae\,.\,be}{cf\,.\,df}$
0	$+ \dfrac{ab\,.\,bc}{df\,.\,ef}$	$- \dfrac{ad\,.\,bc}{ce\,.\,df}$	$- \dfrac{ae\,.\,bc}{cd\,.\,ef}$	0	0	$- \dfrac{ad\,.\,bf}{cd\,.\,ef}$	$- \dfrac{ae\,.\,bf}{ce\,.\,df}$
0	$+ \dfrac{ab\,.\,bd}{cf\,.\,ef}$	$- \dfrac{ac\,.\,bd}{cf\,.\,de}$	0	$+ \dfrac{ac\,.\,bf}{cd\,.\,ef}$	$+ \dfrac{ae\,.\,bd}{cd\,.\,ef}$	0	$- \dfrac{ae\,.\,bf}{cf\,.\,de}$
$+ \dfrac{ab\,.\,bf}{ce\,.\,de}$	$+ \dfrac{ab\,.\,be}{cf\,.\,df}$	0	$+ \dfrac{ac\,.\,be}{cf\,.\,de}$	$+ \dfrac{ac\,.\,bf}{ce\,.\,df}$	$+ \dfrac{ad\,.\,be}{ce\,.\,df}$	$+ \dfrac{ad\,.\,bf}{cf\,.\,de}$	0
$- \dfrac{ab\,.\,bd}{ce\,.\,ef}$	0	$- \dfrac{ac\,.\,bd}{ce\,.\,df}$	$- \dfrac{ac\,.\,be}{cd\,.\,ef}$	0	0	$- \dfrac{af\,.\,bd}{cd\,.\,ef}$	$- \dfrac{af\,.\,be}{ce\,.\,df}$
$- \dfrac{ad\,.\,bc}{de\,.\,ef}$	0	$- \dfrac{ad\,.\,bc}{cf\,.\,de}$	0	$+ \dfrac{af\,.\,bc}{cd\,.\,ef}$	$+ \dfrac{ad\,.\,be}{cd\,.\,ef}$	0	$- \dfrac{af\,.\,be}{cf\,.\,de}$
0	0	0	$+ \dfrac{ae\,.\,bc}{cf\,.\,de}$	$+ \dfrac{af\,.\,bc}{ce\,.\,df}$	$+ \dfrac{ae\,.\,bd}{ce\,.\,df}$	$+ \dfrac{af\,.\,bd}{cf\,.\,de}$	0
0	$- \dfrac{af\,.\,bc}{bd\,.\,ef}$	0	$+ \dfrac{ac\,.\,bc}{de\,.\,ef}$	0	$+ \dfrac{ad\,.\,bd}{ce\,.\,ef}$	0	$- \dfrac{af\,.\,bf}{ce\,.\,de}$
$- \dfrac{ae\,.\,bc}{bf\,.\,de}$	$- \dfrac{af\,.\,bc}{be\,.\,df}$	$- \dfrac{ac\,.\,bc}{de\,.\,df}$	0	0	$+ \dfrac{ae\,.\,be}{cd\,.\,df}$	$+ \dfrac{af\,.\,bf}{cd\,.\,de}$	0
$- \dfrac{ae\,.\,bd}{bf\,.\,ce}$	$- \dfrac{af\,.\,bd}{be\,.\,cf}$	$- \dfrac{ad\,.\,bd}{ce\,.\,cf}$	$- \dfrac{ae\,.\,be}{cd\,.\,cf}$	$- \dfrac{af\,.\,bf}{cd\,.\,ce}$	0	0	0

column: and it appears that there are four columns in each of which the signs are or can be made all $+$; whereas in each of the remaining twelve columns the signs are or can be made six of them $+$, the other four $-$.

Passing to the values of $\{a\}$, $\{ab\}$, etc., we have for example, from the ab column of the foregoing table,

$$\begin{aligned} \{c\} &= +\sqrt{c}\,.\,ac\,.\,bc, \\ \{d\} &= +\sqrt{d}\,.\,ad\,.\,bd, \\ &\vdots \\ \{ac\} &= -\sqrt{ac}\,.\,\begin{matrix} ac\,.\,ae \\ bc\,.\,bf \end{matrix}, \\ &\vdots \end{aligned}$$

where (since the radicals are all positive) the signs are correct: substituting for the quantities under the radical signs their full values, and squaring the rational parts in order to bring them also under the radical signs, this is

$$\begin{aligned} \{c\} &= +\sqrt{ab\,.\,ad\,.\,ae\,.\,af\,.\,bd\,.\,be\,.\,bf\,.\,de\,.\,df\,.\,ef\,.\,ac^2\,.\,bc^2}, \\ \{d\} &= +\sqrt{ab\,.\,ac\,.\,ae\,.\,af\,.\,bc\,.\,be\,.\,bf\,.\,ce\,.\,cf\,.\,ef\,.\,ad^2\,.\,bd^2}, \\ &\vdots \\ \{ac\} &= -\sqrt{ac\,.\,af\,.\,cf\,.\,bd\,.\,be\,.\,de\,.\,ac^2\,.\,ae^2\,.\,bc^2\,.\,bf^2}, \end{aligned}$$

where all the expressions of this (the ab-column) have a common factor,

$$ac\,.\,ad\,.\,ae\,.\,af\,.\,bc\,.\,bd\,.\,be\,.\,bf.$$

Omitting this factor, we find

$$\begin{aligned} \{c\} &= +\sqrt{ab\,.\,ac\,.\,bc\,.\,de\,.\,df\,.\,ef}, \\ \{d\} &= +\sqrt{ab\,.\,ad\,.\,bd\,.\,ce\,.\,cf\,.\,ef}, \\ &\vdots \\ \{ac\} &= -\sqrt{ad\,.\,ae\,.\,de\,.\,bc\,.\,bf\,.\,cf}; \end{aligned}$$

viz. recurring to the foregoing condensed notation, this is

$$\begin{aligned} \{c\} &= +\sqrt{de}, \\ \{d\} &= +\sqrt{ce}, \\ &\vdots \\ \{ac\} &= -\sqrt{bc}, \end{aligned}$$

and, in fact, the terms in the several columns have only the ten values $\sqrt{ab}$, $\sqrt{ac}$, etc. each with its proper sign. I repeat the meaning of the notation: ab stands in the first instance for the double triad $abf\,.\,cde$, and then this denotes a product of differences $ab\,.\,af\,.\,bf\,.\,cd\,.\,ce\,.\,de$. We have thus the following table in which I have in several cases changed the signs of entire columns.

TABLE II. OF THE FUNCTIONS $\{a\}$, $\{ab\}$, ETC., FOR THE SIXTEEN SPECIAL VALUES OF x, x'.

$x, x' =$	$\infty\ \infty$	ab	ac	ad	ae	af	bc	bd	be	bf	cd	ce	cf	de	df	ef
$\{a\}$	0	0	0	0	0	0	$+\sqrt{de}$	$+\sqrt{ce}$	$-\sqrt{cd}$	$-\sqrt{ab}$	$-\sqrt{be}$	$+\sqrt{bd}$	$+\sqrt{ac}$	$+\sqrt{bc}$	$+\sqrt{ad}$	$-\sqrt{ae}$
$\{b\}$	0	0	$-\sqrt{de}$	$-\sqrt{ce}$	$+\sqrt{cd}$	$+\sqrt{ab}$	0	0	0	0	$-\sqrt{ae}$	$+\sqrt{ad}$	$+\sqrt{bc}$	$+\sqrt{ac}$	$+\sqrt{bd}$	$-\sqrt{be}$
$\{c\}$	0	$-\sqrt{de}$	0	$-\sqrt{be}$	$+\sqrt{bd}$	$+\sqrt{ac}$	0	$-\sqrt{ae}$	$+\sqrt{ad}$	$+\sqrt{bc}$	0	0	0	$+\sqrt{ab}$	$+\sqrt{cd}$	$-\sqrt{ce}$
$\{d\}$	0	$-\sqrt{ce}$	$+\sqrt{be}$	0	$+\sqrt{bc}$	$+\sqrt{ad}$	$+\sqrt{ae}$	0	$+\sqrt{ac}$	$+\sqrt{bd}$	0	$-\sqrt{ab}$	$-\sqrt{cd}$	0	0	$-\sqrt{de}$
$\{e\}$	0	$-\sqrt{cd}$	$+\sqrt{bd}$	$+\sqrt{bc}$	0	$+\sqrt{ae}$	$+\sqrt{ad}$	$+\sqrt{ac}$	0	$+\sqrt{be}$	$-\sqrt{ab}$	0	$-\sqrt{ce}$	0	$-\sqrt{de}$	0
$\{f\}$	0	$-\sqrt{ab}$	$+\sqrt{ac}$	$+\sqrt{ad}$	$-\sqrt{ae}$	0	$+\sqrt{bc}$	$+\sqrt{bd}$	$-\sqrt{be}$	0	$-\sqrt{cd}$	$+\sqrt{ce}$	0	$+\sqrt{de}$	0	0
$\{ab\}$	$+\sqrt{ab}$	0	$+\sqrt{bc}$	$+\sqrt{bd}$	$-\sqrt{be}$	0	$+\sqrt{ac}$	$+\sqrt{ad}$	$-\sqrt{ae}$	0	0	0	$-\sqrt{de}$	0	$-\sqrt{ce}$	$+\sqrt{cd}$
$\{ac\}$	$+\sqrt{ac}$	$+\sqrt{bc}$	0	$+\sqrt{cd}$	$-\sqrt{ce}$	0	$+\sqrt{ab}$	0	0	$-\sqrt{de}$	$+\sqrt{ad}$	$-\sqrt{ae}$	0	0	$-\sqrt{be}$	$+\sqrt{bd}$
$\{ad\}$	$+\sqrt{ad}$	$+\sqrt{bd}$	$-\sqrt{cd}$	0	$-\sqrt{de}$	0	0	$-\sqrt{ab}$	0	$-\sqrt{ce}$	$+\sqrt{ac}$	0	$+\sqrt{be}$	$+\sqrt{ae}$	0	$+\sqrt{bc}$
$\{ae\}$	$+\sqrt{ae}$	$+\sqrt{be}$	$-\sqrt{ce}$	$-\sqrt{de}$	0	0	0	0	$-\sqrt{ab}$	$-\sqrt{cd}$	0	$+\sqrt{ac}$	$+\sqrt{bd}$	$+\sqrt{ad}$	$+\sqrt{bc}$	0
$\{bc\}$	$+\sqrt{bc}$	$+\sqrt{ac}$	$-\sqrt{ab}$	0	0	$+\sqrt{de}$	0	$-\sqrt{cd}$	$+\sqrt{ce}$	0	$+\sqrt{bd}$	$-\sqrt{be}$	0	0	$-\sqrt{ae}$	$+\sqrt{ad}$
$\{bd\}$	$+\sqrt{bd}$	$+\sqrt{ad}$	0	$+\sqrt{ab}$	0	$+\sqrt{ce}$	$+\sqrt{cd}$	0	$+\sqrt{de}$	0	$+\sqrt{bc}$	0	$+\sqrt{ae}$	$+\sqrt{be}$	0	$+\sqrt{ac}$
$\{be\}$	$+\sqrt{be}$	$+\sqrt{ae}$	0	0	$+\sqrt{ab}$	$+\sqrt{cd}$	$+\sqrt{ce}$	$+\sqrt{de}$	0	0	0	$+\sqrt{bc}$	$+\sqrt{ad}$	$+\sqrt{bd}$	$+\sqrt{ac}$	0
$\{cd\}$	$+\sqrt{cd}$	0	$+\sqrt{ad}$	$+\sqrt{ac}$	0	$+\sqrt{be}$	$+\sqrt{bd}$	$+\sqrt{bc}$	0	$+\sqrt{ae}$	0	$+\sqrt{de}$	0	$+\sqrt{ce}$	0	$+\sqrt{ab}$
$\{ce\}$	$+\sqrt{ce}$	0	$+\sqrt{ae}$	0	$+\sqrt{ac}$	$+\sqrt{bd}$	$+\sqrt{be}$	0	$+\sqrt{bc}$	$+\sqrt{ad}$	$+\sqrt{de}$	0	0	$+\sqrt{cd}$	$+\sqrt{ab}$	0
$\{de\}$	$+\sqrt{de}$	0	0	$-\sqrt{ae}$	$+\sqrt{ad}$	$+\sqrt{bc}$	0	$-\sqrt{be}$	$+\sqrt{bd}$	$+\sqrt{ac}$	$+\sqrt{ce}$	$-\sqrt{cd}$	$-\sqrt{ab}$	0	0	0

Referring now to Göpel's memoir, *Crelle*, t. XXXV. (1847), pp. 277—312, we have the sixteen double ϑ-functions

$$P,\ P_1,\ P_2,\ P_3;\quad iQ,\ Q_1,\ iQ_2,\ Q_3;\quad iR,\ iR_1,\ R_2,\ R_3;\quad S,\ iS_1,\ iS_2,\ S_3,$$

where the six functions affected with the $i\ (=\sqrt{-1})$ are odd functions, vanishing for the values $u=0$, $u'=0$ of the arguments. It is convenient to take ∞, ∞ as the values of x, x' corresponding to these values $u=0$, $u'=0$: the expressions $\{a\}$ will thus correspond to the six squares $-Q^2$, $-Q_2^2$, $-R^2$, $-R_1^2$, $-S_1^2$, $-S_2^2$, and the expressions $\{ab\}$ to the remaining ten squares P^2, P_1^2, ..., S_3^2; and after some *tâtonnement*, I succeed in establishing the correspondence as follows

$$\begin{array}{cccccccccccccccc} S_2^2, & S_1^2, & R_1^2, & R^2, & Q^2, & Q_2^2, & Q_1^2, & P_1^2, & P^2, & S^2, & P_2^2, & P_3^2, & S_3^2, & Q_3^2, & R_3^2, & R_2^2, \\ = \{a\}, & \{b\}, & \{c\}, & \{d\}, & \{e\}, & \{f\}, & \{ab\}, & \{ac\}, & \{ad\}, & \{ae\}, & \{bc\}, & \{bd\}, & \{be\}, & \{cd\}, & \{ce\}, & \{de\}, \end{array}$$

viz. the sixteen squared double ϑ-functions are proportional to the sixteen expressions $-\{a\}$, $+\{ab\}$, as hereby appearing.

TABLE III. OF THE SIXTEEN FORMS OF

	0	A	B	$A+B$	K	$K+A$	$K+B$	$K+A+B$
$-S_2^2$	$-S_2^2=a$	$-S_3^2=-be$	$-S^2=-ae$	$-S_1^2=b$	$+R_2^2=de$	$R_3^2=ce$	$R^2=-d$	$R_1^2=-c$
$-S_1^2$	$-S_1^2=b$	$-S^2=-ae$	$-S_3^2=-be$	$-S_2^2=a$	$-R_1^2=c$	$-R^2=d$	$-R_3^2=-ce$	$-R_2^2=-de$
$-R_1^2$	$-R_1^2=c$	$-R^2=+d$	$-R_3^2=-ce$	$-R_2^2=-de$	$-S_1^2=b$	$-S^2=-ae$	$-S_3^2=-be$	$-S_3^2=a$
$-R^2$	$-R^2=d$	$-R_1^2=+c$	$-R_2^2=-de$	$-R_3^2=-ce$	$S^2=ae$	$S_1^2=-b$	$S_2^2=-a$	$S_3^2=be$
$-Q^2$	$-Q^2=e$	$-Q_1^2=-ab$	$-Q_2^2=+f$	$-Q_3^2=-cd$	$P^2=ad$	$P_1^2=ac$	$P_2^2=bc$	$P_3^2=bd$
$-Q_2^2$	$-Q_2^2=f$	$-Q_3^2=-cd$	$-Q^2=+e$	$-Q_1^2=-ab$	$P_2^2=bc$	$P_3^2=bd$	$P^2=ad$	$P_1^2=ac$
Q_1^2	$Q_1^2=ab$	$Q^2=-e$	$Q_3^2=cd$	$Q_2^2=-f$	$P_1^2=ac$	$P^2=ad$	$P_3^2=bd$	$P_2^2=bc$
P_1^2	$P_1^2=ac$	$P^2=ad$	$P_3^2=bd$	$P_2^2=bc$	$Q_1^2=ab$	$Q^2=-e$	$Q_3^2=cd$	$Q_2^2=-f$
P^2	$P^2=ad$	$P_1^2=ac$	$P_2^2=bc$	$P_3^2=bd$	$-Q^2=e$	$-Q_1^2=-ab$	$-Q_2^2=f$	$-Q_3^2=-cd$
S^2	$S^2=ae$	$S_1^2=-b$	$S_2^2=-a$	$S_3^2=be$	$-R^2=d$	$-R_1^2=c$	$-R_2^2=-de$	$-R_3^2=-ce$
P_2^2	$P_2^2=bc$	$P_3^2=bd$	$P^2=ad$	$P_1^2=ac$	$-Q_2^2=f$	$-Q_3^2=-cd$	$-Q^2=e$	$-Q_1^2=-ab$
P_3^2	$P_3^2=bd$	$P_2^2=bc$	$P_1^2=ac$	$P^2=ad$	$Q_3^2=cd$	$Q_2^2=-f$	$Q_1^2=ab$	$Q^2=-e$
S_3^2	$S_3^2=be$	$S_2^2=-a$	$S_1^2=-b$	$S^2=ae$	$R_3^2=ce$	$R_2^2=de$	$R_1^2=c$	$R^2=-d$
Q_3^2	$Q_3^2=cd$	$Q_2^2=-f$	$Q_1^2=ab$	$Q^2=-e$	$P_3^2=bd$	$P_2^2=bc$	$P_1^2=ac$	$P^2=ad$
R_3^2	$R_3^2=ce$	$R_2^2=de$	$R_1=-c$	$R^2=-d$	$S_3^2=be$	$S_2^2=-a$	$S_1^2=-b$	$S^2=ae$
R_2^2	$R_2^2=de$	$R_3^2=ce$	$R^2=-d$	$R_1^2=-c$	$-S_2^2=a$	$-S_3^2=-be$	$-S^2=-ae$	$-S_1^2=-b$
	$\infty\infty$	cd	ef	ab	bc	bd	ad	ac

We have, after Göpel (*l.c.* p. 283), a table showing how the ratios of the double ϑ-functions are altered, when the arguments are increased by the quarter-periods

$$A,\ B,\ A+B,\ K,\ L,\ K+L,$$

that is, when u, u' are simultaneously changed into $u+A$, $u'+A'$ or into $u+B$, $u'+B'$ etc. If instead, we consider the squared functions, the table is very much simplified, inasmuch as in place of the coefficients ± 1, $\pm i$, it will contain only the coefficients ± 1: and we may complete the table by extending it to all the combinations 0, A, B, $A+B$, K, $K+A$, $K+B$, $K+A+B$, L, $L+A$, $L+B$, $L+A+B$, $K+L$, $K+L+A$, $K+L+B$, $K+L+A+B$ of the quarter-periods: we have thus a table included in the annexed Table III., viz. attending herein only to the capital letters P, Q, R, S, the sixteen columns of the table show how the ratios of the terms $-S_2^2$, $-S_1^2$, etc., of the first column are altered when the arguments are increased by the foregoing combinations of quarter-periods, as indicated by the headings 0, A, B, etc., of the several columns.

THE SQUARED DOUBLE ϑ-FUNCTIONS.

L	$L+A$	$L+B$	$L+A+B$	$K+L$	$K+L+A$	$K+L+B$	$K+L+A+B$
$-Q_2^2=f$	$-Q_3^2=-cd$	$-Q^2=e$	$-Q_1^2=-ab$	$P_3^2=bc$	$P_3^2=bd$	$P^2=ad$	$P_1^2=ac$
$Q_1^2=ab$	$Q^2=-e$	$Q_3^2=cd$	$Q_2^2=-f$	$P_1^2=ac$	$P^2=ad$	$P_3^2=bd$	$P_2^2=bc$
$P_1^2=ac$	$P^2=ad$	$P_3^2=bd$	$P_2^2=bc$	$Q_1^2=ab$	$Q^2=-e$	$Q_3^2=cd$	$Q_2^2=-f$
$P^2=ad$	$P_1^2=ac$	$P_2^2=bc$	$P_3^2=bd$	$-Q^2=e$	$-Q_1^2=-ab$	$-Q_2^2=f$	$-Q_3^2=-cd$
$S^2=ae$	$S_1^2=-b$	$S_2^2=-a$	$S_3^2=be$	$-R^2=d$	$-R_1^2=c$	$-R_2^2=-de$	$-R_3^2=-ce$
$-S_2^2=a$	$-S_3^2=-be$	$-S^2=-ae$	$-S_1^2=b$	$R_2^2=de$	$R_3^2=ce$	$R^2=-d$	$R_1^2=-c$
$-S_1^2=b$	$-S^2=-ae$	$-S_3^2=-be$	$-S_2^2=a$	$-R_1^2=c$	$-R^2=d$	$-R_2^2=-ce$	$-R_2^2=-de$
$-R_1^2=c$	$-R^2=d$	$-R_3^2=-ce$	$-R_2^2=-de$	$-S_1^2=b$	$-S^2=-ae$	$-S_2^2=-be$	$-S_2^2=a$
$-R^2=d$	$-R_1^2=c$	$-R_2^2=-de$	$-R_3^2=-ce$	$S^2=ae$	$S_1^2=b$	$S_2^2=-a$	$S_3^2=be$
$-Q^2=e$	$-Q_1^2=-ab$	$-Q_2^2=f$	$-Q_3^2=-cd$	$P^2=ad$	$P_1^2=ac$	$P_2^2=bc$	$P_3^2=bd$
$R_2^2=de$	$R_3^2=ce$	$R^2=-d$	$R_1^2=-c$	$-S_2^2=a$	$-S_3^2=-be$	$-S^2=-ae$	$-S_1^2=b$
$R_3^2=ce$	$R_2^2=de$	$R_1^2=-c$	$R^2=-d$	$S_3^2=be$	$S_2^2=-a$	$S_1^2=-b$	$S^2=ae$
$Q_3^2=cd$	$Q_2^2=-f$	$Q_1^2=ab$	$Q^2=-e$	$P_3^2=bd$	$P_3^2=bc$	$P_1^2=ac$	$P^2=ad$
$S_3^2=be$	$S_2^2=-a$	$S_1^2=-b$	$S^2=ae$	$R_3^2=ce$	$R_2^2=de$	$R_1^2=-c$	$R^2=-d$
$P_3^2=bd$	$P_2^2=bc$	$P_1^2=ac$	$P^2=ad$	$Q_3^2=cd$	$Q_2^2=-f$	$Q_1^2=ab$	$Q^2=-e$
$P_2^2=bc$	$P_3^2=bd$	$P^2=ad$	$P_1^2=ac$	$-Q_2^2=f$	$-Q_3^2=-cd$	$-Q^2=e$	$-Q_1^2=-ab$
af	be	ae	bf	de	ce	df	cf

But I have also in the table inserted the values to which $-S_2^2$, $-S_1^2$, etc., are respectively proportional, viz. the table runs $-S_2^2 = a$, $-S_1^2 = b$, etc., (read $-S_2^2 = \{a\}$, $-S_1^2 = \{b\}$, etc., the brackets $\{\ \}$ having been for greater brevity omitted throughout the table), and where it is of course to be understood that $-S_2^2$, $-S_1^2$, etc., are proportional only, not absolutely equal to $\{a\}$, $\{b\}$, etc. And I have also at the foot of the several columns inserted suffixes $\infty\infty$, ab, cd, etc., which refer to the columns of Table II.

Comparing the first with any other column of the table, for instance with the second column, the two columns respectively signify that

$$\begin{array}{l|l} -S_2^2(u) = \{a\}, & -S_2^2(u+A) = -\{be\}, \\ -S_1^2(u) = \{b\}, & -S_1^2(u+A) = -\{ae\}, \\ \vdots & \vdots \\ Q_1^2(u) = \{ab\}, & Q_1^2(u+A) = -\{e\}, \\ \vdots & \vdots \end{array}$$

where, as before, the sign $=$ means only that the terms are proportional; u is written for shortness instead of (u, u'), and so $u+A$ for $(u+A, u'+A')$, etc.: the variables in the functions $\{a\}$, $\{be\}$, etc. are in each case x, x'. But if in the second column we write $u-A$ for A, then the variables x, x' will be changed into new variables y, y', or the meaning will be

$$\begin{array}{c|c} x,\ x' & y,\ y' \\ -S_2^2(u) = \{a\}, & -S_2^2(u) = -\{be\}, \\ -S_1^2(u) = \{b\}, & -S_1^2(u) = -\{ae\}, \\ \vdots & \vdots \\ Q_1^2(u) = \{ab\}, & Q_1^2(u) = -\{e\}, \\ \vdots & \vdots \end{array}$$

so that, omitting from the table the terms which contain the capital letters P, Q, R, S, except only the outside left-hand column $-S_2^2$, $-S_1^2$, etc., the table indicates that these functions $-S_2^2$, $-S_1^2$, etc., are proportional to the functions $\{a\}$, $\{b\}$, etc., of x, x' given in the first column; also to the functions $-\{be\}$, $-\{ae\}$, etc., of y, y' given in the second column; also to the functions $-\{ae\}$, $-\{be\}$, etc., of z, z' given in the third column; and so on, with a different pair of variables in each of the 16 columns.

Thus comparing any two columns, for instance the first and second, it appears that we can have simultaneously

$$\begin{array}{c} x,\ x' \quad y,\ y' \\ \{a\} = -\{be\}, \\ \{b\} = -\{ae\}, \\ \vdots \\ \{ab\} = -\{e\}, \\ \vdots \end{array}$$

(fifteen equations, since the meaning is that the terms are only proportional, not absolutely equal), equivalent to two equations serving to determine x and x' in terms of y and y',

or conversely y and y' in terms of x and x'. The functions in each column form in fact 16 sixes, such that any four belonging to the same six are linearly connected; and in any such linear relation between four functions in the left-hand column, substituting for these their values as functions in the right-hand column, we have the corresponding relations between four functions out of a set of six belonging to the right-hand column, or we have an identity $0=0$. I will presently verify this in a particular case.

If in any column we give to the variables the values ∞, ∞ we obtain for the terms in the column the values which the terms of the first column assume on giving to x, x' the values shown at the foot of the column in question; thus, in the second column giving to the variables the values ∞, ∞, the column becomes

$$-\sqrt{be},\ -\sqrt{ae},\ 0,\ 0,\ -\sqrt{ab},\ -\sqrt{cd},\ 0,\ \sqrt{ad},\ \sqrt{ac},\ 0,\ \sqrt{bd},\ \sqrt{bc},\ 0,\ 0,\ \sqrt{de},\ \sqrt{ce}$$

which is, in fact, the cd-column of Table II.: this is of course as it should be, for the values in question are those of the functions $-S_2^2$, $-S_1^2$, etc., on writing therein

$$x,\ x'=c,\ d.$$

The formulæ show that

$$\sqrt{ab},\ \sqrt{ac},\ \sqrt{ad},\ \sqrt{ae},\ \sqrt{bc},\ \sqrt{bd},\ \sqrt{be},\ \sqrt{cd},\ \sqrt{ce},\ \sqrt{de},$$

are, in fact, proportional to

$$k_1^2,\quad \varpi_1^2,\quad \varpi^2,\quad \sigma^2,\quad \varpi_2^2,\quad \varpi_3^2,\quad \sigma_3^2,\quad k_3^2,\quad \rho_3^2,\quad \rho_2^2,$$

(k_1, k_2, ... are Göpel's k', k'', ...). This gives rise to a remarkable theorem, for the ten squares are functions of only four quantities α, β, γ, δ (Göpel's t, u, v, w). For greater clearness, I introduce single letters A, B, ..., J and write

$$\begin{aligned}
A &= abc\,.\,def = (\sqrt{de})^2 = \rho_2^4, &&= (\alpha^2-\beta^2+\gamma^2-\delta^2)^2,\\
B &= abd\,.\,cef = (\sqrt{ce})^2 = \rho_3^4, &&= 4(\alpha\gamma+\beta\delta)^2,\\
C &= abe\,.\,cdf = (\sqrt{cd})^2 = k_3^4, &&= 4(\alpha\delta+\beta\gamma)^2,\\
D &= abf\,.\,cde = (\sqrt{ab})^2 = k_1^4, &&= (\alpha^2-\beta^2-\gamma^2+\delta^2)^2,\\
E &= acd\,.\,bef = (\sqrt{be})^2 = \sigma_3^4, &&= 4(\alpha\beta+\gamma\delta)^2,\\
F &= ace\,.\,bdf = (\sqrt{bd})^2 = \varpi_3^4, &&= (\alpha^2+\beta^2+\gamma^2+\delta^2)^2,\\
G &= acf\,.\,bde = (\sqrt{ac})^2 = \varpi_1^4, &&= 4(\alpha\delta-\beta\gamma)^2,\\
H &= ade\,.\,bcf = (\sqrt{bc})^2 = \varpi_2^4, &&= 4(\alpha\gamma-\beta\delta)^2,\\
I &= adf\,.\,bce = (\sqrt{ad})^2 = \varpi^4, &&= 4(\alpha\beta-\gamma\delta)^2,\\
J &= aef\,.\,bdc = (\sqrt{ae})^2 = \sigma^4, &&= (\alpha^2+\beta^2-\gamma^2-\delta^2)^2;
\end{aligned}$$

viz. it has to be shown that A, B, ..., J, considered as given functions of the six letters a, b, c, d, e, f, are really functions of four quantities α, β, γ, δ; or, what is the same thing, that A, B, ..., J, considered as functions of a, b, c, d, e, f satisfy all those relations which they satisfy when considered as given functions of α, β, γ, δ.

Now considering them as given functions of α, β, γ, δ, they ought to satisfy six relations; and inasmuch as, so considered, they are, in fact, linear functions of

$$\alpha^4+\beta^4+\gamma^4+\delta^4,\quad \alpha^2\beta^2+\gamma^2\delta^2,\quad \alpha^2\gamma^2+\beta^2\delta^2,\quad \alpha^2\delta^2+\beta^2\gamma^2,\quad \alpha\beta\gamma\delta,$$

five of these relations will be linear: there is a sixth non-linear relation, expressible in a variety of different forms, one of them, as is easily verified, being

$$\sqrt{AJ}\pm\sqrt{CG}\pm\sqrt{DF}=0.$$

Now considering A, B, ..., J as given functions of a, b, c, d, e, f, there exist between them linear relations which may be obtained by the consideration of identities of the form

$$\left|\begin{matrix} abcd \\ abcdef \end{matrix}\right|=0,$$

where the left-hand side is used for shortness to denote the determinant

$$\left|\begin{matrix} 1, & 1, & 1, & 1 & & \\ a, & b, & c, & d & & \\ a^2, & b^2, & c^2, & d^2 & & \\ 1, & 1, & 1, & 1, & 1, & 1 \\ a, & b, & c, & d, & e, & f \\ a^2, & b^2, & c^2, & d^2, & e^2, & f^2 \end{matrix}\right|=0.$$

We thus obtain between them a system of fifteen linear relations, which present themselves in the form

$$\begin{aligned}
&(1) && A-J+E-B=0,\\
&(2) && -A-I+F-C=0,\\
&(3) && A-H+G-D=0,\\
&(4) && -B-G+H+C=0,\\
&(5) && B-F+I+D=0,\\
&(6) && C-E+J-D=0,\\
&(7) && -E-D-H+E=0,\\
&(8) && E-C-I+G=0,\\
&(9) && F-B-J-G=0,\\
&(10) && H-A+J-I=0,\\
&(11) && -J+D-G+I=0,\\
&(12) && J+C-F+H=0,\\
&(13) && I+B-E-H=0,\\
&(14) && G+A+E-F=0,\\
&(15) && D-A+B-C=0,
\end{aligned}$$

and these are all included in the equations (10), (4), (12), (15), (6), which serve to express G, B, E, F, I in terms of D, H, C, A, J, i.e. ac, ce, eb, bd, da in terms of ab, bc, cd, de, ea, if for the moment we write $G = ac$, etc. But the five linear relations in question are, it is at once seen, satisfied by $A, B, \ldots, J$ considered as given functions of $\alpha, \beta, \gamma, \delta$.

The equation $\sqrt{AJ} \pm \sqrt{DF} \pm \sqrt{CG} = 0$, substituting for $A, B, \ldots, J$ their values in terms of a, b, c, d, e, f, becomes

$$\sqrt{abc\,.\,def\,.\,aef\,.\,bcd} \pm \sqrt{abf\,.\,cde\,.\,ace\,.\,bdf} \pm \sqrt{abe\,.\,cdf\,.\,acf\,.\,bde} = 0,$$

which (omitting common factors) becomes $\sqrt{bc^2\,.\,ef^2} \pm \sqrt{bf^2\,.\,ce^2} \pm \sqrt{be^2\,.\,cf^2} = 0$; or, taking the proper signs, this is the identity $bc\,.\,ef + be\,.\,fc + bf\,.\,ce = 0$.

It is to be noticed that

$$\begin{array}{ccc} \delta^2 + \alpha^2 - \beta^2 - \gamma^2, & 2(\alpha\beta - \gamma\delta), & 2(\gamma\alpha + \beta\delta), \\ 2(\alpha\beta + \gamma\delta), & \delta^2 + \beta^2 - \gamma^2 - \alpha^2, & 2(\beta\gamma - \alpha\delta), \\ 2(\gamma\alpha - \beta\delta), & 2(\beta\gamma + \alpha\delta), & \delta^2 + \gamma^2 - \alpha^2 - \beta^2, \end{array}$$

each divided by $\delta^2 + \alpha^2 + \beta^2 + \gamma^2$, form a system of coefficients in the transformation between two sets of rectangular coordinates. We have therefore

$$\begin{array}{ccc} \sqrt{ab}, & \sqrt{ad}, & \sqrt{ce}, \\ \sqrt{be}, & \sqrt{de}, & \sqrt{ac}, \\ \sqrt{bc}, & \sqrt{cd}, & \sqrt{ae}, \end{array}$$

each divided by $\sqrt{bd}$ and the several terms taken with proper signs, as a system of coefficients in the transformation between two sets of rectangular axes: a result which seems to be the same as that obtained by Hesse in the Memoir, "Transformations-Formeln für rechtwinklige Raum-Coordinaten"; *Crelle*, t. LXIII. (1864), pp. 247—251.

The composition of the last mentioned system of functions is better seen by writing them under the fuller form $\sqrt{abf\,.\,cde}$, etc.; viz. omitting the radical signs, the terms are

$$\begin{array}{ccc} abf\,.\,cde, & adf\,.\,bce, & abd\,.\,cef, \\ bef\,.\,acd, & def\,.\,abc, & acf\,.\,bde, \\ bcf\,.\,ade, & cdf\,.\,abe, & aef\,.\,bcd, \end{array}$$

each divided by $bdf\,.\,ace$; or, in an easily understood algorithm, the terms are

$$\begin{array}{r|ccc} & bf\,.\,d & df\,.\,b & bd\,.\,f \\ \hline a\,.\,ce & bf\,.\,d & df\,.\,b & bd\,.\,f \\ e\,.\,ac & bf\,.\,d & df\,.\,b & bd\,.\,f \\ c\,.\,ae & bf\,.\,d & df\,.\,b & bd\,.\,f, \end{array}$$

each divided by $bdf\,.\,ace$.

Reverting to the before-mentioned comparison of the first and second columns of Table III., four of the equations are

$$\begin{array}{rlll} x,\ x' & y,\ y' & & x,\ x' \qquad y,\ y' \\ \{c\} = & \{d\}, & \text{that is,} & \sqrt{c}\,[c] = \sqrt{d}\,[d], \\ \{d\} = & \{c\}, & \text{that is,} & \sqrt{d}\,[d] = \sqrt{c}\,[c], \\ \{e\} = & -\{ab\}, & \text{that is,} & \sqrt{e}\,[e] = -\sqrt{ab}\,[ab], \\ \{f\} = & -\{cd\}, & \text{that is,} & \sqrt{f}\,[f] = -\sqrt{cd}\,[cd]; \end{array}$$

viz. the four terms on the left-hand side are not absolutely equal, but only proportional, to those on the right-hand side. Substituting for $\sqrt{c}$, $\sqrt{d}$, etc., their values, and introducing on the right-hand side the factor

$$\sqrt{ac\,.\,bc\,.\,ce\,.\,cf\,.\,ad\,.\,bd\,.\,de\,.\,df},$$

the equations become

$$\begin{array}{rl} xx' & yy' \\ [c] = & ac\,.\,bc\,.\,ce\,.\,ef\,[d], \\ [d] = & ad\,.\,bd\,.\,de\,.\,df\,[c], \\ [e] = - & ce\,.\,de\,[ab], \\ [f] = - & cf\,.\,df\,[cd]. \end{array}$$

The functions on the left-hand satisfy the identity

$$def\,[c] - efc\,[d] + fcd\,[e] - cde\,[f] = 0,$$

or, as this may also be written,

$$def\,[c] - cef\,[d] + cdf\,[e] - cde\,[f] = 0.$$

Hence substituting the right-hand values, the whole equation divides by $ce\,.\,de\,.\,cf\,.\,df$; omitting this factor, it becomes

$$ef\,.\,ac\,.\,bc\,[d] - ef\,.\,ad\,.\,bd\,[c] - cd\,\{[ab] - [cd]\} = 0,$$

where the variables are y, y': it is to be shown that this is in fact an identity, and (as it is thus immaterial what the variables are) I change them into x, x'.

We have

$$\begin{aligned} ac\,.\,bc\,[d] - ad\,.\,bd\,[c] &= (a-c)(b-c)(d-x)(d-x') \\ &\quad -(a-d)(b-d)(c-x)(c-x') \\ &= (c-d)\begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix} \\ &= cd\,[xx'abcd], \end{aligned}$$

suppose.

We have moreover

$$[ab]-[cd]=\frac{1}{(x-x')^2}\left\{\begin{matrix} abf.c'd'e'+a'b'f'.cde \\ -cdf.a'b'e'-c'd'f'.abe \end{matrix}\right\}$$

$$=\frac{1}{(x-x')^2}(abc'd'-a'b'cd)(e'f-ef''),$$

where for the moment a, b, a', etc., are written to denote $a-x$, $b-x$, $a-x'$, etc.; we have then

$$e'f-ef''=(e-x')(f-x)-(e-x)(f-x')$$
$$=-(e-f)(x-x')=-ef(x-x'),$$

and

$$\begin{aligned} abc'd'-a'b'cd &= (a-x)(b-x)(c-x')(d-x')=-(x-x') \\ &\quad -(a-x')(b-x')(c-x)(d-x) \end{aligned} \begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}$$

$$=-(x-x')[xx'abcd].$$

Hence $[ab]-[cd]=ef[xx'abcd]$, and the equation to be verified becomes

$$(ef.cd-cd.ef)[xx'abcd]=0,$$

viz. this is, in fact, an identity.

Cambridge, 14 *March*, 1877.

664.

ON THE 16-NODAL QUARTIC SURFACE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXIV. (1877), pp. 238—241.]

PROF. BORCHARDT in the Memoir "Ueber die Darstellung u. s. w." *Crelle*, t. LXXXIII. (1877), pp. 234—243, shows that the coordinates x, y, z, w may be taken as proportional to four of the double ϑ-functions, and that the equation of the surface is then Göpel's relation of the fourth order between these four functions: and he remarks at the end of the memoir that it thus appears that the coordinates x, y, z, w of a point on the surface can be expressed as proportional to algebraic functions, involving square roots only, of two arbitrary parameters ξ, ξ'.

It is interesting to develope the theory from this point of view. Writing, as in my paper, "Further investigations on the double ϑ-functions," pp. 220—233, [663],

$$\begin{aligned}
[a] &= aa', \\
[b] &= bb', \\
[c] &= cc', \\
[d] &= dd', \\
[e] &= ee', \\
[f] &= ff', \\
[ab] &= \frac{1}{(\xi-\xi')^2}(\sqrt{abfc'd'e'} - \sqrt{a'b'f'cde})^2, \\
&\text{etc.,}
\end{aligned}$$

where on the right-hand sides $a, b, \ldots, a', \ldots$ denote $a-\xi, b-\xi, \ldots, a-\xi', \ldots$ (ξ, ξ' being here written in place of the x, x' of my paper), then the sixteen functions

are proportional to constant multiples of the square-roots of these expressions; viz. the correspondence is

$$\begin{array}{lllll}
S_2 = \vartheta_{12}, & S_1 = \vartheta_{24}, & R_1 = \vartheta_3, & R = \vartheta_{04}, & Q = \vartheta_1, \quad Q_2 = \vartheta_{02}, \\
i\sqrt[4]{a}\sqrt{[a]}, & i\sqrt[4]{b}\sqrt{[b]}, & i\sqrt[4]{c}\sqrt{[c]}, & i\sqrt[4]{d}\sqrt{[d]}, & i\sqrt[4]{e}\sqrt{[e]}, \quad i\sqrt[4]{f}\sqrt{[f]}; \\
Q_1 = \vartheta_2, & P_1 = \vartheta_{34}, & P = \vartheta_{01}, & S = -\vartheta_{14}, & P_2 = \vartheta_{12}, \quad P_3 = \vartheta_5, \\
\sqrt[4]{ab}\sqrt{[ab]}, & \sqrt[4]{ac}\sqrt{[ac]}, & \sqrt[4]{ad}\sqrt{[ad]}, & \sqrt[4]{ae}\sqrt{[ae]}, & \sqrt[4]{bc}\sqrt{[bc]}, \quad \sqrt[4]{bd}\sqrt{[bd]}; \\
 & S_3 = \vartheta_{23}, & Q_3 = \vartheta_0, & R_3 = \vartheta_4, & R_2 = \vartheta_{03}, \\
 & \sqrt[4]{be}\sqrt{[be]}, & \sqrt[4]{cd}\sqrt{[cd]}, & \sqrt[4]{ce}\sqrt{[ce]}, & \sqrt[4]{de}\sqrt{[de]};
\end{array}$$

where, under the signs $\sqrt[4]{\ }$, a signifies $bcdef$, that is, $bc\,.\,bd\,.\,be\,.\,bf\,.\,cd\,.\,ce\,.\,cf\,.\,de\,.\,df\,.\,ef$, and ab signifies $abf\,.\,cde$, that is, $ab\,.\,af\,.\,bf\,.\,cd\,.\,ce\,.\,de$, in which expressions $bc, bd, \ldots, ab, af, \ldots$ signify the differences $b-c, b-d, \ldots, a-b, a-f, \ldots$ But in what follows, we are not concerned with the values of these constant multipliers.

Prof. Borchardt's coordinates x, y, z, w are

$$x = \vartheta_0 = P;\ y = \vartheta_{23} = S_3;\ z = \vartheta_{14} = -S;\ w = \vartheta_5 = P_3;$$

viz. P, S, P_3, S_3 are a set connected by Göpel's relation of the fourth order—and this relation can be found (according to Göpel's method) by showing that Q^2 and R^2 are each of them a linear function of the four squares P^2, P_3^2, S^2, S_3^2, and further that QR is a linear function of PS and P_3S_3; for then, squaring the expression of QR, and for Q^2 and R^2 substituting their values, we have the required relation of the fourth order between P, S, P_3, S_3.

Now we have $P, S, P_3, S_3, Q, R =$ constant multiples of $\sqrt{[ac]}, \sqrt{[ab]}, \sqrt{[cd]}, \sqrt{[bd]}, \sqrt{[b]}, \sqrt{[c]}$ respectively: and it of course follows that we must have the like relations between these six quantities; viz. we must have $[b], [c]$ each of them a linear function of $[ac], [ab], [cd], [bd]$; and moreover $\sqrt{[b]}\sqrt{[c]}$ a linear function of $\sqrt{[ac]}\sqrt{[ab]}$ and $\sqrt{[bd]}\sqrt{[cd]}$.

As regards this last relation, starting from the formulæ

$$\sqrt{[ac]} = \frac{1}{\xi - \xi'}\{\sqrt{acfb'd'e'} + \sqrt{a'c'f'bde}\},$$

$$\sqrt{[bd]} = \frac{1}{\xi - \xi'}\{\sqrt{bdfa'c'e'} + \sqrt{b'd'f'ace}\},$$

$$\sqrt{[ab]} = \frac{1}{\xi - \xi'}\{\sqrt{abfc'd'e'} + \sqrt{a'b'f'cde}\},$$

$$\sqrt{[cd]} = \frac{1}{\xi - \xi'}\{\sqrt{cdfa'b'e'} + \sqrt{c'd'f'abe}\},$$

we have at once

$$\sqrt{[ac]}\sqrt{[ab]}=\frac{1}{(\xi-\xi')^2}\{(afd'e'+a'f'de)\sqrt{bcb'c'}+(bc'+b'c)\sqrt{adea'd'e'}\},$$

$$\sqrt{[bd]}\sqrt{[cd]}=\frac{1}{(\xi-\xi')^2}\{(dfa'e'+d'f'ae)\sqrt{bcb'c'}+(bc'+b'c)\sqrt{adea'd'e'}\};$$

the difference of these two expressions is

$$=\frac{1}{(\xi-\xi')^2}(ad'-a'd)(fe'-f'e)\sqrt{bcb'c'},$$

where substituting for $a, d, e, f, a', \ldots$ their values $a-\xi, d-\xi, e-\xi, f-\xi, a-\xi', \ldots$ we have $ad'-a'd=(a-d)(\xi-\xi')$, $fe'-f'e=(f-e)(\xi-\xi')$; also $\sqrt{bcb'c'}=\sqrt{[b]}\sqrt{[c]}$; and we have thus the required relation

$$\sqrt{[ac]}\sqrt{[ab]}-\sqrt{[bd]}\sqrt{[cd]}=-(a-d)(e-f)\sqrt{[b]}\sqrt{[c]}.$$

As regards the first mentioned relation, if for greater generality, θ being arbitrary, we write $[\theta]=\theta\theta'$, that is, $=(\theta-\xi)(\theta-\xi')$, then it is easy to see that there exists a relation of the form

$$\nabla\,[\theta]=A\,[ab]+B\,[ac]+C\,[bd]+D\,[cd],$$

where $A+B+C+D=0$. The right-hand side is thus a linear function of the differences $[ab]-[ac]$, $[ab]-[bd]$, $[ab]-[cd]$; and each of these, as the irrational terms disappear and the rational terms divide by $(\xi-\xi')^2$, is a mere linear function of 1, $\xi+\xi'$, $\xi\xi'$; whence there is a relation of the form in question. I found without much difficulty the actual formula; viz. this is

$$(a-d)(b-c)(e-f)\begin{vmatrix}1, & e+f, & ef\\ 1, & b+c, & bc\\ 1, & a+d, & ad\end{vmatrix}[\theta]$$

$$=\begin{vmatrix}1, & e, & f, & ef\\ 1, & b, & c, & bc\\ 1, & d, & a, & ad\\ 1, & \theta, & \theta, & \theta^2\end{vmatrix}[ac]-\begin{vmatrix}1, & e, & f, & ef\\ 1, & c, & b, & bc\\ 1, & d, & a, & ad\\ 1, & \theta, & \theta, & \theta^2\end{vmatrix}[ab]-\begin{vmatrix}1, & e, & f, & ef\\ 1, & b, & c, & bc\\ 1, & a, & d, & ad\\ 1, & \theta, & \theta, & \theta^2\end{vmatrix}[cd]+\begin{vmatrix}1, & e, & f, & ef\\ 1, & c, & b, & bc\\ 1, & a, & d, & ad\\ 1, & \theta, & \theta, & \theta^2\end{vmatrix}[bd],$$

where observe that on the right-hand side the last three determinants are obtained from the first one by interchanging b, c: or a, d: or b, c and a, d simultaneously: a single interchange gives the sign $-$, but for two interchanges the sign remains $+$.

Writing successively $\theta = b$ and $\theta = c$, we obtain

$$(a-d)(e-f)\begin{vmatrix} 1, & e+f, & ef \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}[b]$$

$$= (a-f)(b-d)(b-e)[ac] - (a-b)(b-f)(d-e)[ab]$$
$$+ (a-b)(b-e)(d-f)[cd] - (a-e)(b-d)(b-f)[bd];$$

$$(a-d)(e-f)\begin{vmatrix} 1, & e+f, & ef \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}[c]$$

$$= -(a-c)(c-f)(d-e)[ac] + (a-f)(c-d)(c-e)[ab]$$
$$-(a-e)(c-d)(c-f)[cd] + (a-c)(c-e)(d-f)[bd];$$

which values of $[b]$ and $[c]$, combined with the foregoing equation

$$(a-d)(e-f)\sqrt{[b]}\sqrt{[c]} = -\sqrt{[ac]}\sqrt{[ab]} + \sqrt{[cd]}\sqrt{[bd]},$$

give the required quartic equation between $\sqrt{[ac]}$, $\sqrt{[ab]}$, $\sqrt{[cd]}$, $\sqrt{[bd]}$.

Cambridge, 2 *August*, 1877.

665.

A MEMOIR ON THE DOUBLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXV. (1878), pp. 214—245.]

I RESUME my investigations on these functions; see my two papers, *Crelle*, t. LXXXIII. (1877), pp. 210—233; [662] and [663]. But it is proper in the first instance to develope in a corresponding manner, the theories of the circular (or exponential) functions, and of the single ϑ-functions.

Part I. Preliminary investigations.

Starting from the differential relation

$$\partial u = \frac{\partial x}{\sqrt{a-x\,.\,b-x}}$$

between the variables u and x, I write for shortness the single letters A, B, Ω, instead of functional forms $A(u)$, $B(u)$, $\Omega(u)$, to denote functions of u; and I assume as definitions the equations

$$A = \Omega\sqrt{a-x},$$

$$B = \Omega\sqrt{b-x},$$

and another equation to be presently mentioned: these two equations imply between A, B, Ω the algebraical relation

$$A^2 - B^2 = \Omega^2(a-b).$$

Differentiating, we obtain

$$\partial A = \partial\Omega\,.\,\sqrt{a-x} - \frac{\Omega}{2\sqrt{a-x}}\sqrt{a-x\,.\,b-x}\,\partial u,$$

that is,

$$\partial A = \frac{A}{\Omega}\partial\Omega - \tfrac{1}{2}B\,\partial u,$$

and similarly

$$\partial B = \frac{B}{\Omega}\partial\Omega - \tfrac{1}{2}A\,\partial u,$$

whence

$$A\,\partial B - B\,\partial A = -\tfrac{1}{2}(A^2 - B^2)\,\partial u.$$

Proceeding to a second differentiation, we find

$$\partial^2 A = \frac{A}{\Omega}\partial^2\Omega - \frac{B}{\Omega}\partial\Omega\,\partial u + \tfrac{1}{4}A\,(\partial u)^2,$$

$$\partial^2 B = \frac{B}{\Omega}\partial^2\Omega - \frac{A}{\Omega}\partial\Omega\,\partial u + \tfrac{1}{4}B\,(\partial u)^2,$$

and thence

$$A\,\partial^2 A - (\partial A)^2 = \frac{A^2}{\Omega^2}\{\Omega\,\partial^2\Omega - (\partial\Omega)^2\} + \tfrac{1}{4}(A^2 - B^2)\,(\partial u)^2,$$

$$B\,\partial^2 B - (\partial B)^2 = \frac{B^2}{\Omega^2}\{\Omega\,\partial^2\Omega - (\partial\Omega)^2\} + \tfrac{1}{4}(B^2 - A^2)\,(\partial u)^2.$$

To simplify these we assume (as the third equation above referred to)

$$\Omega\,\partial^2\Omega - (\partial\Omega)^2 = 0.$$

The last-mentioned two equations then become

$$A\,\partial^2 A - (\partial A)^2 = \tfrac{1}{4}(A^2 - B^2)\,(\partial u)^2,$$
$$B\,\partial^2 B - (\partial B)^2 = \tfrac{1}{4}(B^2 - A^2)\,(\partial u)^2,$$

which several equations contain the theory of the functions A, B, Ω: we have as their general integrals

$$\begin{aligned} A &= \tfrac{1}{2}\Lambda e^{\lambda u}\sqrt{a-b}\,\{e^{\frac{1}{2}(u+v)} + e^{-\frac{1}{2}(u+v)}\},\\ B &= -\tfrac{1}{2}\Lambda e^{\lambda u}\sqrt{a-b}\,\{e^{\frac{1}{2}(u+v)} - e^{-\frac{1}{2}(u+v)}\},\\ \Omega &= \Lambda e^{\lambda u}, \end{aligned}$$

where Λ, λ, v are arbitrary constants. Forming the quotients $A : \Omega$, $B : \Omega$, and introducing the notations cosh, sinh, of the hyperbolic sine and cosine, also writing for simplicity $v = 0$, the equations give

$$\begin{aligned} \sqrt{a-x} &= \sqrt{a-b}\cosh\tfrac{1}{2}u,\\ \sqrt{b-x} &= -\sqrt{a-b}\sinh\tfrac{1}{2}u, \end{aligned}$$

which express the integral of the differential relation

$$\partial u = \frac{\partial x}{\sqrt{a-x\,.\,b-x}}.$$

Instead of considering in like manner the radical $\sqrt{a-x.b-x.c-x}$, I pass at once to the radical $\sqrt{a-x.b-x.c-x.d-x}$; and starting from the differential relation

$$\partial u = \frac{\partial x}{\sqrt{a-x.b-x.c-x.d-x}},$$

and using the single letters A, B, C, D, Ω to denote functions of u, I assume as definitions

$$A = \Omega\sqrt{a-x},$$
$$B = \Omega\sqrt{b-x},$$
$$C = \Omega\sqrt{c-x},$$
$$D = \Omega\sqrt{d-x},$$

and another equation to be presently mentioned; A, B, C, D are called ϑ-functions, and Ω is called the ω-function.

But before proceeding further I introduce some locutions which will be useful. In reference to a given set of squares or products, I use the expression *a sum of squares* to denote the sum of all or any of the squares each multiplied by an arbitrary coefficient; and in like manner *a sum of products* to denote the sum of all or any of the products each multiplied by an arbitrary coefficient: in particular, the set may consist of a single square or product only, and the sum of squares or products will then denote the single term multiplied by an arbitrary coefficient. In the present case, we have the quantities $\sqrt{a-x}$, $\sqrt{b-x}$, $\sqrt{c-x}$, $\sqrt{d-x}$, and the squares are $a-x$, $b-x$, etc., which belong all to the same set; but the products (meaning thereby products of *two* quantities) $\sqrt{a-x.b-x}$, etc., are considered as being each of them a set by itself. A sum of squares is thus a linear function $\lambda+\mu x$, and conversely any such function is a sum of squares; a sum of products means a single term $\nu\sqrt{a-x.b-x}$ (or $\nu\sqrt{a-x.c-x}$, etc., as the case may be), and conversely any such function is a sum of products: the coefficients λ, μ, ν may depend upon or contain Ω, and in differential expressions (∂u being therein considered constant) the coefficients λ, μ, ν may contain the factor ∂u or $(\partial u)^2$—and if convenient we may of course express such factor by writing the coefficients in the form $\lambda\,\partial u$, or $\lambda(\partial u)^2$ etc., as the case may be.

We may now explain very simply the form, as well of the algebraical relations, as of the differential relations of the first and second orders respectively, which connect the functions A, B, C, D.

The functions A^2, B^2, C^2, D^2 are each of them a sum of squares, and hence there exists a linear relation between any three of these squares. But the products AB, AC, etc., are each of them a sum of products (meaning thereby a single term, as already explained); and hence there is not any linear relation between these products.

Considering the first derived functions ∂A, ∂B, etc., these each contain a term in $\partial\Omega$, which however disappears (as is obvious) from the combinations $A\,\partial B - B\,\partial A$,

etc.; and, without in any wise fixing the value of Ω, we in fact find that each of these expressions is a sum of products; the form is, as will appear,

$$A\,\partial B - B\,\partial A = \alpha\Omega^2\sqrt{c-x\,.\,d-x} = \nu\,CD, \text{ etc.*}$$

Passing to the second derived functions and forming the combinations $A\,\partial^2 A - (\partial A)^2$, etc., each of these will contain a multiple of $\Omega\,\partial^2\Omega - (\partial\Omega)^2$, but if we assume this expression $\Omega\,\partial^2\Omega - (\partial\Omega)^2 = \Omega^2 M$, where M is $= (\partial u)^2$ multiplied by a properly determined function of x, then it is found that each of the expressions in question $A\,\partial^2 A - (\partial A)^2$, etc., becomes equal to a sum of squares, that is, to a linear function $\Omega^2(\lambda + \mu x)$: viz. it is equal to a sum of squares formed with the squares A^2, B^2, C^2, D^2.

The foregoing equation

$$\Omega\,\partial^2\Omega - (\partial\Omega)^2 = \Omega^2 M,$$

where M has its proper value, is the other equation above referred to, which, with the equations $A = \Omega\sqrt{a-x}$, etc., serves for the definition of the functions A, B, C, D, Ω; it may be mentioned at once that the proper value is

$$M = \tfrac{1}{4}(\partial u)^2\{-2x^2 + x(a+b+c+d) + \kappa\},$$

where κ is a constant, symmetrical as regards a, b, c, d, which may be taken $= 0$, but which is better put

$$= a^2 + b^2 + c^2 + d^2 - ab - ac - ad - bc - bd - cd.$$

For the proof of the formula, I introduce and shall in general employ the abbreviations (a, b, c, d) to denote the differences $a-x$, $b-x$, $c-x$, $d-x$: the differential relation between x, u thus becomes $\partial x = \partial u\sqrt{\mathrm{abcd}}$. I use also the abbreviations $\Omega\,\partial^2\Omega - (\partial\Omega)^2 = \Delta\Omega$, etc.

We have

$$A\,\partial B - B\,\partial A = \Omega^2(\sqrt{\mathrm{a}}\,\partial\sqrt{\mathrm{b}} - \sqrt{\mathrm{b}}\,\partial\sqrt{\mathrm{a}}),$$

the terms in $\partial\Omega$ disappearing: viz. observing that $\partial\mathrm{a} = \partial\mathrm{b} = -\partial x$, this is

$$= -\tfrac{1}{2}\Omega^2\left(\frac{\sqrt{\mathrm{a}}}{\sqrt{\mathrm{b}}} - \frac{\sqrt{\mathrm{b}}}{\sqrt{\mathrm{a}}}\right)\partial x,$$

$$= \tfrac{1}{2}\Omega^2\frac{\mathrm{a}-\mathrm{b}}{\sqrt{\mathrm{ab}}}\,\partial x;$$

or observing that $\mathrm{a} - \mathrm{b} = a - b$, and writing for ∂x its value $= \sqrt{\mathrm{abcd}}\,\partial u$, this is

$$A\,\partial B - B\,\partial A = -\tfrac{1}{2}(a-b)\,\Omega^2\sqrt{\mathrm{cd}}\,\partial u,$$

$$= -\tfrac{1}{2}(a-b)\,\Omega^2\sqrt{c-x\,.\,d-x}\,\partial u,$$

which is the equation expressing $A\,\partial B - B\,\partial A$ as a sum of products: it is further obvious that the value is

$$= -\tfrac{1}{2}(a-b)\,CD\,\partial u.$$

* It is hardly necessary to remark that α, ν contain each of them the factor ∂u; and the like in other cases.

Proceeding next to find the value of ΔA, $= A\,\partial^2 A - (\partial A)^2$, $= A^2\partial^2 \log A$, it is to be remarked that we have in general

$$\Delta PQ = P^2\Delta Q + Q^2\Delta P,$$

and therefore also $\Delta P^2 = 2P^2\Delta P$, and consequently $\Delta\sqrt{P} = \frac{\frac{1}{2}}{P}\Delta P$. Hence starting from $A = \Omega\sqrt{\mathrm{a}}$, we have

$$\Delta A = \mathrm{a}\Delta\Omega + \Omega^2\frac{\frac{1}{2}}{\mathrm{a}}\Delta\mathrm{a},$$

where $\Delta\mathrm{a} = -\mathrm{a}\,\partial^2 x - (\partial x)^2$. I assume that we have $\Delta\Omega = \Omega^2 M = \frac{1}{4}\Omega^2 S(\partial u)^2$, where S denotes a function of x which is to be determined: the equation thus becomes

$$\Delta A = \tfrac{1}{4}\Omega^2\{\mathrm{a}S(\partial u)^2 - 2\partial^2 x - 2(\partial x)^2\};$$

we have $(\partial x)^2 = \mathrm{abcd}\,(\partial u)^2$, and thence, differentiating and omitting on each side the factor ∂x, we obtain

$$2\partial^2 x = -(\mathrm{abc} + \mathrm{abd} + \mathrm{acd} + \mathrm{bcd}),$$

and the equation becomes

$$\Delta A = \tfrac{1}{4}\Omega^2\{\mathrm{a}(S + \mathrm{bc} + \mathrm{bd} + \mathrm{cd}) - \mathrm{bcd}\}(\partial u)^2,$$

which is to be simplified by assuming a proper value for S; in order that the same simplification may apply to the formulæ for ΔB, etc., it is necessary that S be symmetrical in regard to a, b, c, d.

Writing for the moment b', c', d' to denote $b-a$, $c-a$, $d-a$ respectively, we have $b', c', d' = \mathrm{b}-\mathrm{a}, \mathrm{c}-\mathrm{a}, \mathrm{d}-\mathrm{a}$, and thence

$$b'c'd' = \mathrm{bcd} - \mathrm{a}(\mathrm{bc} + \mathrm{bd} + \mathrm{cd}) + \mathrm{a}^2(\mathrm{b} + \mathrm{c} + \mathrm{d}) - \mathrm{a}^3,$$

and consequently

$$\mathrm{a}(\mathrm{bc} + \mathrm{bd} + \mathrm{cd}) - \mathrm{bcd} = -b'c'd' + \mathrm{a}^2(\mathrm{b} + \mathrm{c} + \mathrm{d} - \mathrm{a}):$$

hence, in the expression of ΔA, the factor which multiplies $\frac{1}{2}\Omega^2(\partial u)^2$ is

$$\mathrm{a}\{S + \mathrm{a}(\mathrm{b} + \mathrm{c} + \mathrm{d} - \mathrm{a})\} - b'c'd',$$

viz. the expression added to S is

$$(a - x)(b + c + d - a - 2x),$$

$$= a(b + c + d - a) - x(a + b + c + d) + 2x^2.$$

Hence assuming

$$S = -2x^2 + x(a + b + c + d) + \kappa,$$

κ being a constant symmetrical in regard to a, b, c, d, which may be at once taken to be $= a^2 + b^2 + c^2 + d^2 - ab - ac - ad - bc - bd - cd$; then writing also

$$\lambda = b^2 + c^2 + d^2 - bc - bd - cd, \quad \mu = -b'c'd' = a - b\,.\,a - c\,.\,a - d,$$

the expression $\mathrm{a}\{S + \mathrm{a}(\mathrm{b} + \mathrm{c} + \mathrm{d} - \mathrm{a})\} - b'c'd'$ becomes $= \mathrm{a}\lambda + \mu$; and the sought for equation thus is

$$\Delta A = A\,\partial^2 A - (\partial A)^2 = \tfrac{1}{4}\Omega^2(\mathrm{a}\lambda + \mu)(\partial u)^2,$$

the equation in Ω being of course

$$\Delta\Omega = \Omega\,\partial^2\Omega - (\partial\Omega)^2 = \tfrac{1}{4}\Omega^2\{-2x^2 + x(a+b+c+d) + \kappa\}(\partial u)^2.$$

The theory in regard to the second derivatives is thus completed.

To adapt the formulæ to elliptic integrals, and ordinary H and Θ functions, the radical must be brought to the form $\sqrt{x\,.\,1-x\,.\,1-k^2x}$. Writing for this purpose

$$a,\ b,\ c,\ d = -k^2I^2,\ 0,\ 1,\ \frac{1}{k^2},\quad (I=\infty),$$

substituting also $\dfrac{2u}{I}$ for u, and $ikI\,.\,A$, iB ($i=\sqrt{-1}$ as usual) for A, B respectively, we find $\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x} = I\sqrt{x\,.\,1-x\,.\,1-k^2x}$; and then

$$2\partial u = \frac{\partial x}{\sqrt{x\,.\,1-x\,.\,1-k^2x}},$$

and

$$A=\Omega,\ B=\Omega\sqrt{x},\ C=\Omega\sqrt{1-x},\ D=\frac{1}{k}\Omega\sqrt{1-k^2x}.$$

Ω is in this case $=A$, a ϑ-function: and in the equation for $\Delta\Omega$, writing A in place of Ω, the equation becomes

$$A\,\partial^2A - (\partial A)^2 = \tfrac{1}{4}A^2\{-2x^2 + x(-k^2I^2) + \kappa\}\frac{4(\partial u)^2}{I^2},$$

viz. replacing $\dfrac{\kappa}{I^2}$ by a new constant, $=\lambda$ suppose and finally putting $I=\infty$, this is

$$A\,\partial^2A - (\partial A)^2 = A^2(\lambda - k^2x)(\partial u)^2.$$

The differential equation is satisfied by $x=\operatorname{sn}^2u$, giving $1-x=\operatorname{cn}^2u$, $1-k^2x=\operatorname{dn}^2u$; and the equation for A then is

$$\partial^2\log A = (\lambda - k^2\operatorname{sn}^2u)(\partial u)^2,$$

or say

$$A = Le^{\frac{1}{2}\lambda u^2 - k^2\int_0 du\int_0 du\operatorname{sn}^2u},$$

viz. by properly assuming the constants L, λ, we shall have $A=$ Jacobi's function Θu: and then $\operatorname{sn}u=\dfrac{B}{A}$, $\operatorname{cn}u=\dfrac{C}{A}$, $\operatorname{dn}u=\dfrac{kD}{A}$, which will give the ordinary expressions of sn, cn, dn in terms of H, Θ.

Part II. The double ϑ-functions.

Passing now to the double ϑ-functions, and writing for a moment

$$\sqrt{X} = \sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x},$$
$$\sqrt{Y} = \sqrt{a-y\,.\,b-y\,.\,c-y\,.\,d-y\,.\,e-y\,.\,f-y},$$

the differential equations which connect u, v with x, y are

$$\partial u = \frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}},$$

$$\partial v = \frac{x\,\partial x}{\sqrt{X}} + \frac{y\,\partial y}{\sqrt{Y}}.$$

There are here sixteen ϑ-functions A, B, C, D, E, F, AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, and an associated ω-function Ω, where for shortness I use the single and double letters A, B, ..., AB, ..., Ω, instead of functional expressions $A(u, v)$, $B(u, v)$,.., $AB(u, v)$, .., $\Omega(u, v)$, to denote functions of the two letters u, v. Writing also (a, b, c, d, e, f) for the differences $a-x$, $b-x$, etc., and ($\mathrm{a_1}$, $\mathrm{b_1}$, $\mathrm{c_1}$, $\mathrm{d_1}$, $\mathrm{e_1}$, $\mathrm{f_1}$) for the differences $a-y$, $b-y$, etc., whence $\sqrt{X} = \sqrt{\mathrm{abcdef}}$ and $\sqrt{Y} = \sqrt{\mathrm{a_1b_1c_1d_1e_1f_1}}$, and θ for the difference $x-y$, we have sixteen xy-functions which are represented by

$$\sqrt{a},\ \sqrt{b},\ \sqrt{c},\ \sqrt{d},\ \sqrt{e},\ \sqrt{f},\ \sqrt{ab},\ \sqrt{ac},\ \sqrt{ad},\ \sqrt{ae},\ \sqrt{bc},\ \sqrt{bd},\ \sqrt{be},\ \sqrt{cd},\ \sqrt{ce},\ \sqrt{de},$$

the values of which are

$$\sqrt{a} = \sqrt{\mathrm{aa_1}},\ \text{(six equations)},$$
$$\vdots$$
$$\sqrt{ab} = \frac{1}{\theta}\{\sqrt{\mathrm{abfc_1d_1e_1}} - \sqrt{\mathrm{a_1b_1f_1cde}}\},\ \text{(ten equations)},$$
$$\vdots$$

and the definitions of the sixteen ϑ-functions and the ω-function are

$$A = \Omega\sqrt{a},\quad \text{(six equations)},$$
$$\vdots$$
$$AB = \Omega\sqrt{ab},\ \text{(ten equations)},$$
$$\vdots$$

and one other equation to be afterwards mentioned.

I call to mind that, in a binary symbol such as $\sqrt{ab}$, it is always f that accompanies the two expressed letters a, b: the duad ab is, in fact, an abbreviated expression for the double triad $abf.\,cde$: and I remark also that I have for greater simplicity omitted certain constant factors which, in my second paper above referred to, were introduced as multipliers of the foregoing functions $\sqrt{a}$, ..., $\sqrt{ab}$, ... I remark also that, to avoid confusion, the square of any one of these functions $\sqrt{a}$ or $\sqrt{ab}$ is always written (not a or ab, but) $(\sqrt{a})^2$ or $(\sqrt{ab})^2$.

I use ∂ as a symbol of total differentiation: thus

$$\partial A = \frac{dA}{du}\partial u + \frac{dA}{dv}\partial v,\quad \partial^2 A = \frac{d^2A}{du^2}(\partial u)^2 + 2\frac{d^2A}{du\,dv}(\partial u\,\partial v) + \frac{d^2A}{dv^2}(\partial v)^2,\ \text{etc.}$$

Moreover I consider ∂u and ∂v as constants, and use single letters λ, L, etc., to denote linear functions $\alpha\,\partial u + \beta\,\partial v$, or quadric functions $\alpha(\partial u)^2 + 2\beta\,\partial u\,\partial v + \gamma(\partial v)^2$ (as the

case may be) of these differentials; thus, in speaking of $A\,\partial B - B\,\partial A$ as a sum of products, it is implied that the coefficients of the several products are linear functions of ∂u, ∂v; and so in speaking of $A\,\partial^2 A - (\partial A)^2$ as a sum of squares, it is in like manner implied that the coefficients of the several squares are quadric functions of ∂u, ∂v.

An xy-function is simplex, such as $\sqrt{a}$, or complex, such as $\sqrt{ab}$; the square of the former is $\mathrm{aa_1} = a^2 - a(x+y) + xy$, which is of the form $\lambda + \mu(x+y) + \nu xy$; the square of the latter is

$$= \frac{1}{\theta^2}\{\mathrm{abfc_1d_1e_1 + a_1b_1f_1cde} - 2\sqrt{XY}\},$$

where observe that the irrational part $-\frac{2}{\theta^2}\sqrt{XY}$ is the same for all these squares: so that, taking any two such squares, their difference is $=\frac{1}{\theta^2}$ multiplied by a rational function of xy: this rational function in fact divides by θ^2, the quotient being a rational and integral function of the foregoing form $\lambda + \mu(x+y) + \nu xy$. Hence selecting any one of the complex functions, say $\sqrt{de}$, the square of any other of the complex functions is equal to the square of this *plus* a term $\lambda + \mu(x+y) + \nu xy$; and hence the square of any function simplex or complex is of the form $\lambda + \mu(x+y) + \nu xy + \rho(\sqrt{de})^2$; this being so, the squares of the xy-functions may be regarded as forming a single set; every sum of squares is a function of this form $\lambda + \mu(x+y) + \nu xy + \rho(\sqrt{de})^2$; and conversely every function of this form is a sum of squares. A sum of squares thus depends upon four arbitrary coefficients λ, μ, ν, ρ; and we may, in an infinity of ways, select four out of the 16 squares such that every sum of squares can be represented as a sum of these four squares each multiplied by the proper coefficient; say as a sum of the selected four squares: in particular, each of the remaining squares can be expressed as a sum of the selected four squares. It appears, by the first of my papers above referred to, that there are systems of four squares connected together by a linear equation: we are not here concerned with such systems; only of course the four selected squares must not belong to such a system.

We have the products of the xy-functions, where by product is meant a product of *two* functions. The number of products is of course $=120$, but distinguishing these according to the radicals which they respectively contain, they form 30 different sets. Thus we have

$$\sqrt{b}\sqrt{ab} = \frac{1}{\theta}\{\mathrm{b}\sqrt{\mathrm{afb_1c_1d_1e_1}} - \mathrm{b_1}\sqrt{\mathrm{a_1f_1bcde}}\},$$

$$\sqrt{c}\sqrt{ac} = \frac{1}{\theta}\{\mathrm{c} \quad \text{,,} \quad -\mathrm{c_1} \quad \text{,,} \quad \},$$

$$\sqrt{d}\sqrt{ad} = \frac{1}{\theta}\{\mathrm{d} \quad \text{,,} \quad -\mathrm{d_1} \quad \text{,,} \quad \},$$

$$\sqrt{e}\sqrt{ae} = \frac{1}{\theta}\{\mathrm{e} \quad \text{,,} \quad -\mathrm{e_1} \quad \text{,,} \quad \},$$

which four expressions form a set, and there are 15 such sets. The set written down may be called the set af: and the fifteen sets are of course ab, ac, etc.

Again, we have

$$\sqrt{a}\sqrt{b} = \sqrt{\mathrm{aba_1b_1}},$$

$$\sqrt{ac}\sqrt{bc} = \frac{1}{\theta^2}\{(\mathrm{cfd_1e_1 + c_1f_1de})\sqrt{\mathrm{aba_1b_1}} - (\mathrm{ab_1 + a_1b})\sqrt{\mathrm{cdefc_1d_1e_1f_1}}\},$$

$$\sqrt{ad}\sqrt{bd} = \frac{1}{\theta^2}\{(\mathrm{dfc_1e_1 + d_1f_1ce}) \quad \text{,,} \quad - \quad \text{,,} \quad \text{,,} \quad \},$$

$$\sqrt{ae}\sqrt{be} = \frac{1}{\theta^2}\{(\mathrm{efc_1d_1 + e_1f_1cd}) \quad \text{,,} \quad - \quad \text{,,} \quad \text{,,} \quad \},$$

which four expressions form a set, and there are 15 such sets. The set written down may be called the set $\mathrm{aba_1b_1}$: and the fifteen sets are of course $\mathrm{aba_1b_1}$, $\mathrm{aca_1c_1}$, etc. The 15 and 15 sets make in all 30 sets as mentioned above.

The expression, a sum of products, means as already explained a sum of products belonging to the same set; and there are thus 30 forms of a sum of products. The products of the same set are connected by two linear relations, so that, selecting at pleasure any two of the products, the other two products can be expressed each of them as a linear function of these; hence a sum of products contains only two arbitrary coefficients.

Reverting now to the equations $A = \Omega\sqrt{a}$, etc., we see at once the form of the algebraical equations which connect the 16 ϑ-functions. Every squared function $A^2, \ldots, (AB)^2, \ldots$ is a sum of squares, whence selecting (as may be done in a great number of ways) four of these squared functions, each of the remaining 12 squares is a sum of these four squares each multiplied by the proper coefficient; or say it is a sum of the four selected squares. And in like manner the 120 products of two of the 16 functions form 30 sets, such that selecting at pleasure two of the set, the remaining two of the set are each of them a linear function of these.

Considering the first derived functions $\partial A, \partial B, \ldots, \partial AB, \ldots$, each of these contains a term in $\partial\Omega$; but $\partial\Omega$ disappears (as is obvious) from the several combinations $I\partial J - J\partial I$ (I write I and similarly J to denote indifferently a single letter A or a double letter AB): and, without in any wise fixing the value of Ω, we in fact find that each of these expressions is a sum of products.

Passing to the second derived functions, and forming the combinations $A\,\partial^2 A - (\partial A)^2$, etc., or to include the two cases of the single and the double letter, say $I\,\partial^2 I - (\partial I)^2$, each of these will contain a multiple of $\Omega\,\partial^2\Omega - (\partial\Omega)^2$; but if we assume this expression $\Omega\,\partial^2\Omega - (\partial\Omega)^2 = \Omega^2 M$, where M is a quadric function of ∂u, ∂v, the coefficients of $(\partial u)^2$, $\partial u\,\partial v$, $(\partial v)^2$ being properly determined functions of xy, then it is found that each of the expressions in question $I\,\partial^2 I - (\partial I)^2$ becomes equal to a sum of squares.

It is to be observed that M is not altogether arbitrary: the equation as containing terms in $(\partial u)^2$, $\partial u\,\partial v$, and $(\partial v)^2$, in fact represents three partial differential

equations, which for an arbitrary value of M would be inconsistent with each other: it is therefore necessary to verify that the value assigned to M is such as to render the three equations consistent with each other, and this will accordingly be done.

The foregoing equation

$$\Omega\,\partial^2\Omega - (\partial\Omega)^2 = \Omega^2 M,$$

where M has its proper value, (or say the three partial differential equations into which this breaks up), constitutes the other equation above referred to, which with the original equations $A = \Omega\sqrt{a}$, etc., serve to define the sixteen ϑ-functions and Ω.

The remainder of the present memoir is occupied with the analytical investigation of the foregoing theorems. Although the mere algebraical work is very long, yet it appears to me interesting, and I have thought it best to give it in detail.

The analytical theory: various subheadings.

The equations

$$\partial u = \frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}},$$

$$\partial v = \frac{x\,\partial x}{\sqrt{X}} + \frac{y\,\partial y}{\sqrt{Y}},$$

give

$$\frac{\theta\,\partial x}{\sqrt{X}} = \partial v - y\,\partial u, \qquad -\frac{\theta\,\partial y}{\sqrt{Y}} = \partial v - x\,\partial u,$$

which determine ∂x, ∂y in terms of ∂u, ∂v. A different form is sometimes convenient; writing $\partial\varpi = \partial v - a\,\partial u$, and recollecting that a, a_1 denote $a-x$, $a-y$ respectively, the equations become

$$\frac{\theta\,\partial x}{\sqrt{X}} = \partial\varpi + \mathrm{a}_1\,\partial u, \qquad -\frac{\theta\,\partial y}{\sqrt{Y}} = \partial\varpi + \mathrm{a}\,\partial u.$$

Expression for $\partial\sqrt{a}$.

We have

$$\partial\sqrt{a} = \partial\sqrt{\mathrm{aa}_1} = \frac{1}{2\sqrt{\mathrm{aa}_1}}(\mathrm{a}\,\partial\mathrm{a}_1 + \mathrm{a}_1\,\partial\mathrm{a}) = -\frac{1}{2\sqrt{\mathrm{aa}_1}}(\mathrm{a}\,\partial y + \mathrm{a}_1\,\partial x)$$

$$= \frac{1}{2\sqrt{\mathrm{aa}_1}}\,\frac{1}{\theta}\{\mathrm{a}\sqrt{Y}(\partial v - x\,\partial u) - \mathrm{a}_1\sqrt{X}(\partial v - y\,\partial u)\};$$

substituting for $\sqrt{X}$, $\sqrt{Y}$ their values $\sqrt{\mathrm{abcdef}}$, $\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}$, this is

$$\partial\sqrt{a} = \frac{1}{\theta}\{\sqrt{\mathrm{ab}_1\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}(\partial v - x\,\partial u) - \sqrt{\mathrm{a}_1\mathrm{bcdef}}(\partial v - y\,\partial u)\},$$

and by the mere interchange of letters we can of course find $\partial\sqrt{b}$, etc.

Expression for $\partial\sqrt{ab}$.

We have next to find

$$\partial\sqrt{ab} = \partial\frac{1}{\theta}\{\sqrt{\mathrm{abfc_1d_1e_1}} - \sqrt{\mathrm{a_1b_1f_1cde}}\};$$

here

$$\partial\theta = \partial x - \partial y, \quad = \frac{1}{\theta}\{\sqrt{\mathrm{abcdef}}(\partial v - y\,\partial u) + \sqrt{\mathrm{a_1b_1c_1d_1e_1f_1}}\,(\partial v - x\,\partial u)\},$$

and consequently $\partial\sqrt{ab}$ contains a term

$$-\frac{\partial\theta}{\theta^2}\{\sqrt{\mathrm{abfc_1d_1e_1}} - \sqrt{\mathrm{a_1b_1f_1cde}}\},$$

which is

$$= \frac{1}{\theta^3}\{\ (\ -\mathrm{abf}\sqrt{\mathrm{cdec_1d_1e_1}} + \mathrm{cde}\ \sqrt{\mathrm{abfa_1b_1f_1}})\,(\partial v - y\,\partial u)$$
$$+ (-\mathrm{c_1d_1e_1}\sqrt{\mathrm{abfa_1b_1f_1}} + \mathrm{a_1b_1f_1}\sqrt{\mathrm{cdec_1d_1e_1}})\,(\partial v - x\,\partial u)\},$$

or, what is the same thing,

$$= \frac{1}{\theta^2}\left\{\ \sqrt{\mathrm{abfa_1b_1f_1}}\left(\frac{\mathrm{cde} - \mathrm{c_1d_1e_1}}{\theta}\,\partial v + \frac{-\mathrm{cde}\,y + \mathrm{c_1d_1e_1}\,x}{\theta}\,\partial u\right)\right.$$
$$\left. + \sqrt{\mathrm{cdec_1d_1e_1}}\left(\frac{-\mathrm{abf} + \mathrm{a_1b_1f_1}}{\theta}\,\partial v + \frac{\mathrm{abf}\,y - \mathrm{a_1b_1f_1}\,x}{\theta}\,\partial u\right)\right\}.$$

Now

$$\frac{\mathrm{cde} - \mathrm{c_1d_1e_1}}{\theta} = -(cd + ce + de) + (c + d + e)(x + y) - x^2 - xy - y^2,$$

$$\frac{-\mathrm{cde}\,y + \mathrm{c_1d_1e_1}\,x}{\theta} = cde - (c + d + e)\,xy + xy\,(x + y);$$

with the like formulæ with a, b, f instead of c, d, e. Hence the foregoing, or say the first, part of $\partial\sqrt{ab}$ is

$$= \frac{1}{\theta^2}[\sqrt{\mathrm{abfa_1b_1f_1}}\,(\{-(cd + ce + de) + (c + d + e)(x + y) - x^2 - xy - y^2\}\,\partial v$$
$$+ \{\ cde - (c + d + e)\,xy + xy\,(x + y)\}\,\partial u)$$
$$+ \sqrt{\mathrm{cdec_1d_1e_1}}\,(\{\ ab + af + bf\ - (a + b + f)(x + y) + x^2 + xy + y^2\}\,\partial v$$
$$+ \{-abf + (a + b + f)\,xy - xy\,(x + y)\}\,\partial u)].$$

The other or second part of $\partial\sqrt{ab}$, using for shortness an accent to denote differentiation in regard to x or to y, according as it is applied to a function of x or of y, is readily found to be

$$= \frac{1}{\theta^2}[\sqrt{\mathrm{abfa_1b_1f_1}}\,(\{-(\mathrm{cde})' - (\mathrm{c_1d_1e_1})'\}\,\partial v + \{\ y\,(\mathrm{cde})' + x\,(\mathrm{c_1d_1e_1})'\}\,\partial u)$$
$$+ \sqrt{\mathrm{cdec_1d_1e_1}}\,(\{\ (\mathrm{abf})' + (\mathrm{a_1b_1f_1})'\}\,\partial v + \{-y\,(\mathrm{abf})' - x\,(\mathrm{a_1b_1f_1})'\}\,\partial u)].$$

Hence uniting the two terms so as to form the complete value of $\partial\sqrt{ab}$, we have first, a term $\frac{1}{\theta^2}\sqrt{\mathrm{abfa_1b_1f_1}}\,\partial v$, the coefficient of which is

$$= -(cd+ce+de)+(c+d+e)(x+y)-x^2-xy-y^2$$
$$-\tfrac{1}{2}\{(\mathrm{cde})'+(\mathrm{c_1d_1e_1})'\}:$$

this second line is

$$= cd+ce+de-(c+d+e)(x+y)+\tfrac{3}{2}x^2+\tfrac{3}{2}y^2,$$

or the coefficient is $-\frac{1}{2}x^2+xy-\frac{1}{2}y^2$, $=-\frac{1}{2}\theta^2$; the term is thus

$$=\tfrac{1}{2}\sqrt{\mathrm{abfa_1b_1f_1}}\,\partial v.$$

Secondly, a term in $\frac{1}{\theta^2}\sqrt{\mathrm{cdec_1d_1e_1}}\,\partial v$ which is in like manner found to be

$$=-\tfrac{1}{2}\sqrt{\mathrm{cdec_1d_1e_1}}\,\partial v.$$

Thirdly, a term $\frac{1}{\theta^2}\sqrt{\mathrm{abfa_1b_1f_1}}\,\partial u$, the coefficient of which is

$$= cde-(c+d+e)\,xy+x^2y+xy^2$$
$$+\tfrac{1}{2}\{y\,(\mathrm{cde})'+x\,(\mathrm{c_1d_1e_1})'\}:$$

this second line is

$$=-(cd+ce+de)\,\tfrac{1}{2}(x+y)+(c+d+e)\,2xy-\tfrac{3}{2}x^2y-\tfrac{3}{2}xy^2,$$

and the coefficient is thus

$$=\tfrac{1}{2}\{2\,cde-(cd+ce+de)(x+y)+(c+d+e)(x^2+y^2)-x^3\qquad -y^3$$
$$-(c+d+e)(x-y)^2+x^3-x^2y-xy^2+y^3\},$$

which is

$$=\tfrac{1}{2}\{\mathrm{cde}+\mathrm{c_1d_1e_1}-(c+d+e)\,\theta^2+(x+y)\,\theta^2\},$$

or the term is

$$=\tfrac{1}{2}\sqrt{\mathrm{abfa_1b_1f_1}}\left\{\frac{\mathrm{cde}+\mathrm{c_1d_1e_1}}{\theta^2}-(c+d+e)+x+y\right\}\partial u.$$

And, fourthly, a term in $\frac{1}{\theta^2}\sqrt{\mathrm{cdec_1d_1e_1}}\,\partial u$, which is in like manner found to be

$$=-\tfrac{1}{2}\sqrt{\mathrm{cdec_1d_1e_1}}\left\{\frac{\mathrm{abf}+\mathrm{a_1b_1f_1}}{\theta^2}-(a+b+f)+x+y\right\}\partial u.$$

Hence combining these several terms, we have finally

$$\partial\sqrt{ab}=\tfrac{1}{2}\sqrt{\mathrm{abfa_1b_1f_1}}\left[\;\partial v+\left(\frac{\mathrm{cde}+\mathrm{c_1d_1e_1}}{\theta^2}-c-d-e+x+y\right)\partial u\right]$$
$$+\tfrac{1}{2}\sqrt{\mathrm{cdec_1d_1e_1}}\left[-\partial v-\left(\frac{\mathrm{abf}+\mathrm{a_1b_1f_1}}{\theta^2}-a-b-f+x+y\right)\partial u\right];$$

and by the mere interchange of letters we can of course find $\partial\sqrt{ac}$, etc.

Expression for $A\,\partial B - B\,\partial A$.

Starting now from the equations $A = \Omega\sqrt{a}$, $B = \Omega\sqrt{b}$, we obtain

$$A\partial B - B\partial A = \Omega^2\{\sqrt{a}\,\partial\sqrt{b} - \sqrt{b}\,\partial\sqrt{a}\}$$

$$= \frac{\frac{1}{2}\Omega^2}{\theta}(\sqrt{\mathrm{aa_1}}\,\{\sqrt{\mathrm{ba_1c_1d_1e_1f_1}}\,(\partial v - x\,\partial u) - \sqrt{\mathrm{b_1acdef}}\,(\partial v - y\,\partial u)\}$$

$$- \sqrt{\mathrm{bb_1}}\,\{\sqrt{\mathrm{ab_1c_1d_1e_1f_1}}\,(\quad ,, \quad) - \sqrt{\mathrm{a_1bcdef}}\,(\quad ,, \quad)\}),$$

$$= \frac{\frac{1}{2}\Omega^2}{\theta}\{(\mathrm{a_1 - b_1})\sqrt{\mathrm{abc_1d_1e_1f_1}}\,(\partial v - x\,\partial u) - (\mathrm{a - b})\sqrt{\mathrm{a_1b_1cdef}}\,(\partial v - y\,\partial u)\};$$

or since $\mathrm{a_1 - b_1 = a - b} = a - b$, this is

$$A\,\partial B - B\,\partial A = \frac{\frac{1}{2}\Omega^2}{\theta}(a-b)\,\{\sqrt{\mathrm{abc_1d_1e_1f_1}}\,(\partial v - x\,\partial u) - \sqrt{\mathrm{a_1b_1cdef}}\,(\partial v - y\,\partial u)\},$$

which is a sum of products of the set ab: in fact, the four products of this set are

$$\sqrt{f}\sqrt{ab} = \frac{1}{\theta}\{\quad \mathrm{f}\sqrt{\mathrm{abc_1d_1e_1f_1}} - \mathrm{f_1}\sqrt{\mathrm{a_1b_1cdef}}\},$$

$$\sqrt{c}\sqrt{de} = \frac{1}{\theta}\{-\mathrm{c} \quad ,, \quad +\mathrm{c_1} \quad ,, \quad \},$$

$$\sqrt{d}\sqrt{ce} = \frac{1}{\theta}\{-\mathrm{d} \quad ,, \quad +\mathrm{d_1} \quad ,, \quad \},$$

$$\sqrt{e}\sqrt{cd} = \frac{1}{\theta}\{-\mathrm{e} \quad ,, \quad +\mathrm{e_1} \quad ,, \quad \};$$

choosing any two of these at pleasure, for instance the first and second, multiplying by $\partial v - c\,\partial u$, $\partial v - f\,\partial u$ and adding, we have

$$\left.\begin{array}{r}(\partial v - c\,\partial u)\sqrt{f}\sqrt{ab}\\ +(\partial v - f\,\partial u)\sqrt{c}\sqrt{de}\end{array}\right\} = \left\{\begin{array}{l}\frac{1}{\theta}\{\quad (\partial v - c\,\partial u)\,\mathrm{f}\sqrt{\mathrm{abc_1d_1e_1f_1}} - (\partial v - c\,\partial u)\,\mathrm{f_1}\sqrt{\mathrm{a_1b_1cdef}}\}\\ \frac{1}{\theta}\{-(\partial v - f\,\partial u)\,\mathrm{c} \quad ,, \quad +(\partial v - f\,\partial u)\,\mathrm{c_1} \quad ,, \quad \},\end{array}\right.$$

where the coefficients $\mathrm{f}(\partial v - c\,\partial u) - \mathrm{c}(\partial v - f\,\partial u)$, and $\mathrm{f_1}(\partial v - c\,\partial u) - \mathrm{c_1}(\partial v - f\,\partial u)$, by substituting for f, c, $\mathrm{f_1}$, $\mathrm{c_1}$ their values, become $= (f - c)(\partial v - x\,\partial u)$ and $(f - c)(\partial v - y\,\partial u)$; and the expression is thus

$$= \frac{f-c}{\theta}\{\sqrt{\mathrm{abc_1d_1e_1f_1}}\,(\partial v - x\,\partial u) - \sqrt{\mathrm{a_1b_1cdef}}\,(\partial v - y\,\partial u)\}.$$

Reverting to the original expression for $A\,\partial B - B\,\partial A$, it may be remarked that, if we write $\partial v - a\,\partial u = \partial\varpi$, $\partial v - b\,\partial u = \partial\sigma$, then

$$(a-b)(\partial v - x\,\partial u) = \mathrm{a}\,\partial\sigma - \mathrm{b}\,\partial\varpi, \quad (a-b)(\partial v - y\,\partial u) = \mathrm{a_1}\,\partial\sigma - \mathrm{b_1}\,\partial\varpi,$$

and the formula thus becomes

$$A\,\partial B - B\,\partial A = \frac{\frac{1}{2}\Omega^2}{\theta}\,\{\sqrt{\mathrm{abc_1d_1e_1f_1}}\,(\mathrm{a}\,\partial\sigma - \mathrm{b}\,\partial\varpi) - \sqrt{\mathrm{a_1b_1cdef}}\,(\mathrm{a_1}\,\partial\sigma - \mathrm{b_1}\,\partial\varpi)\}:$$

but I shall not in the sequel use this formula, or the notation $\partial v - b\,\partial u = \partial\sigma$ introduced for obtaining it.

Expression for $A\,\partial AB - AB\,\partial A$.

Starting from the equations $A = \Omega\sqrt{a}$ and $AB = \Omega\sqrt{ab}$, we have

$$A\,\partial AB - AB\,\partial A = \Omega^2\,\{\sqrt{a}\,\partial\sqrt{ab} - \sqrt{ab}\,\partial\sqrt{a}\},$$

where the term in { } is

$$= \sqrt{\mathrm{aa_1}}\Bigg[\ \tfrac{1}{2}\sqrt{\mathrm{abfa_1b_1f_1}}\ \Bigg\{\ \partial v + \left(\frac{\mathrm{cde} + \mathrm{c_1d_1e_1}}{\theta^2} - c - d - e + x + y\right)\partial u\Bigg\}$$

$$+ \tfrac{1}{2}\sqrt{\mathrm{cdec_1d_1e_1}}\left\{-\partial v - \left(\frac{\mathrm{abf} + \mathrm{a_1b_1f_1}}{\theta^2} - a - b - f + x + y\right)\partial u\right\}\Bigg]$$

$$-\frac{\frac{1}{2}}{\theta^2}\,(\sqrt{\mathrm{abfc_1d_1e_1}} - \sqrt{\mathrm{a_1b_1f_1cde}})\,[\sqrt{\mathrm{ab_1c_1d_1e_1f_1}}\,(\partial v - x\,\partial u) - \sqrt{\mathrm{a_1bcdef}}\,(\partial v - y\,\partial u)].$$

To reduce this, I write $\partial v - a\,\partial u = \partial\varpi$, and therefore

$$\partial v - x\,\partial u = \partial\varpi + \mathrm{a}\,\partial u,\quad \partial v - y\,\partial u = \partial\varpi + \mathrm{a_1}\,\partial u;$$

then for convenience multiplying by $2\theta^2$, the term is

$$= \mathrm{aa_1}\sqrt{\mathrm{bfb_1f_1}}\,\{\ \theta^2\,\partial\varpi + [(a - c - d - e + x + y)\,\theta^2 + \mathrm{cde} + \mathrm{c_1d_1e_1}]\,\partial u\}$$

$$+ \sqrt{\mathrm{acdea_1c_1d_1e_1}}\,\{-\theta^2\,\partial\varpi + [\qquad (b + f - x - y)\,\theta^2 - \mathrm{abf} - \mathrm{a_1b_1f_1}]\,\partial u\}$$

$$-(\sqrt{\mathrm{abfc_1d_1e_1}} - \sqrt{\mathrm{a_1b_1f_1cde}})\,\{\sqrt{\mathrm{ab_1c_1d_1e_1f_1}}\,(\partial\varpi + \mathrm{a}\,\partial u) - \sqrt{\mathrm{a_1bcdef}}\,(\partial\varpi + \mathrm{a_1}\,\partial u)\}.$$

The last line hereof is

$$= \qquad \sqrt{\mathrm{bfb_1f_1}}\,\{-\mathrm{ac_1d_1e_1}\,(\partial\varpi + \mathrm{a}\,\partial u) - \mathrm{a_1cde}\,(\partial\varpi + \mathrm{a_1}\,\partial u)\}$$

$$+ \sqrt{\mathrm{acdea_1c_1d_1e_1}}\,\{\qquad \mathrm{b_1f_1}\,(\partial\varpi + \mathrm{a}\,\partial u) + \qquad \mathrm{bf}\,(\partial\varpi + \mathrm{a_1}\,\partial u)\}.$$

Hence we have first, a term in $\sqrt{\mathrm{acdea_1c_1d_1e_1}}$, the coefficient of which is

$$= -\theta^2\,\partial\varpi + [(b + f - x - y)\,\theta^2 - \mathrm{abf} - \mathrm{a_1b_1f_1}]\,\partial u + \mathrm{b_1f_1}\,(\partial\varpi + \mathrm{a}\,\partial u) + \mathrm{bf}\,(\partial\varpi + \mathrm{a_1}\,\partial u),$$

viz. this is

$$= \partial\varpi\,(-\theta^2 + \mathrm{b_1f_1} + \mathrm{bf}) + \partial u\,[-(\mathrm{a} - \mathrm{a_1})\,(\mathrm{bf} - \mathrm{b_1f_1}) + (b + f - x - y)\,\theta^2],$$

where $(\mathrm{b} - \mathrm{b_1})\,(\mathrm{f} - \mathrm{f_1}) = \theta^2$, that is, $\mathrm{bf} + \mathrm{b_1f_1} - \theta^2 = \mathrm{bf_1} + \mathrm{b_1f}$, also

$$(\mathrm{a} - \mathrm{a_1})\,(\mathrm{bf} - \mathrm{b_1f_1}) = (b + f - x - y)\,\theta^2,$$

or the coefficient is $= (\mathrm{bf_1} + \mathrm{b_1f})\,\partial\varpi$: viz. the term in question is

$$= \sqrt{\mathrm{acdea_1c_1d_1e_1}}\,(\mathrm{bf_1} + \mathrm{b_1f})\,\partial\varpi.$$

We have then, secondly, a term in $\sqrt{\mathrm{bfb_1f_1}}$, the coefficient of which is

$$= \mathrm{aa_1}\{\theta^2\partial\varpi + [(a-c-d-e+x+y)\,\theta^2 + \mathrm{cde} + \mathrm{c_1d_1e_1}]\,\partial u\}$$
$$- \mathrm{ac_1d_1e_1}(\partial\varpi + \mathrm{a}\,\partial u) - \mathrm{a_1cde}(\partial\varpi + \mathrm{a_1}\,\partial u),$$

viz. this is

$$= (\mathrm{aa_1}\theta^2 - \mathrm{ac_1d_1e_1} - \mathrm{a_1cde})\,\partial\varpi + [\mathrm{aa_1}(a-c-d-e+x+y)\,\theta^2 + (\mathrm{a_1cde} - \mathrm{ac_1d_1e_1})(\mathrm{a}-\mathrm{a_1})]\,\partial u.$$

We have $\mathrm{a}-\mathrm{a_1} = -\theta$; also $\mathrm{a_1cde} - \mathrm{ac_1d_1e_1}$

$$= \theta\,\{cde - a(cd+ce+de) + (c+d+e)[a(x+y)-xy] - a(x^2+xy+y^2) + xy(x+y)\},$$

where the coefficient of θ is

$$= -(a-c)(a-d)(a-e) - (c+d+e)[a^2 - a(x+y)+xy] + (a+x+y)[a^2 - a(x+y)+xy],$$

viz. it is

$$= -(a-c)(a-d)(a-e) + \mathrm{aa_1}(a-c-d-e+x+y).$$

Hence the coefficient in question is

$$= (\mathrm{aa_1}\theta^2 - \mathrm{ac_1d_1e_1} - \mathrm{a_1cde})\,\partial\varpi + (a-c)(a-d)(a-e)\,\theta^2\,\partial u,$$

and the second term is $= \sqrt{\mathrm{bfb_1f_1}}$, multiplied by this coefficient.

Hence, observing that the whole has to be multiplied by $\frac{\frac{1}{2}}{\theta^2}\Omega^2$, we find

$$A\,\partial AB - AB\,\partial A = \frac{\frac{1}{2}}{\theta^2}\Omega^2\,\{\sqrt{\mathrm{acdea_1c_1d_1e_1}}\,(\mathrm{bf_1}+\mathrm{b_1f})\,\partial\varpi$$
$$+ \sqrt{\mathrm{bfb_1f_1}}\,[(\mathrm{aa_1}\theta^2 - \mathrm{ac_1d_1e_1} - \mathrm{a_1cde})\,\partial\varpi + (a-c)(a-d)(a-e)\,\theta^2\,\partial u]\},$$

where I retain $\partial\varpi$ in place of its value, $= \partial v - a\,\partial u$.

This is a sum of products of the set $\mathrm{bfb_1f_1}$: we, in fact, have

$$\sqrt{ac}\,\sqrt{de} = \frac{1}{\theta^2}\{(\mathrm{bf_1}+\mathrm{b_1f})\sqrt{\mathrm{acdea_1c_1d_1e_1}} - (\mathrm{acd_1e_1} + \mathrm{a_1c_1de})\sqrt{\mathrm{bfb_1f_1}}\},$$
$$\sqrt{ad}\,\sqrt{ce} = \text{,,}\ \{\quad \text{,,} \qquad \text{,,} \qquad -(\mathrm{adc_1e_1} + \mathrm{a_1d_1ce}) \qquad \text{,,}\ \},$$
$$\sqrt{ae}\,\sqrt{cd} = \text{,,}\ \{\quad \text{,,} \qquad \text{,,} \qquad -(\mathrm{aec_1d_1} + \mathrm{a_1e_1cd}) \qquad \text{,,}\ \},$$
$$\sqrt{b}\,\sqrt{f} = \text{,,}\ \{\qquad\qquad\qquad\qquad +\theta^2 \qquad \text{,,}\ \},$$

and selecting any two of these, for instance the first and the fourth, the coefficient of $\frac{\frac{1}{2}}{\theta^2}\Omega^2$ is at once seen to be of the form $\partial\varpi\sqrt{ac}\,\sqrt{de} + K\sqrt{b}\,\sqrt{f}$; and for the determination of K, we have

$$(-\mathrm{acd_1e_1} - \mathrm{a_1c_1de})\,\partial\varpi + K\theta^2 = (\mathrm{aa_1}\theta^2 - \mathrm{ac_1d_1e_1} - \mathrm{a_1cde})\,\partial\varpi + (a-c)(a-d)(a-e)\,\theta^2\,\partial u,$$

viz. this gives

$$K\theta^2 = \{\mathrm{aa_1}\theta^2 + (\mathrm{c}-\mathrm{c_1})(\mathrm{ad_1e_1} - \mathrm{a_1de})\}\,\partial\varpi + (a-c)(a-d)(a-e)\,\theta^3\,\partial u.$$

We then have

$$(\mathrm{c}-\mathrm{c}_1)(\mathrm{ad}_1\mathrm{e}_1-\mathrm{a}_1\mathrm{de})=\theta^2\{-\mathrm{aa}_1+(a-d)(a-e)\},$$

and the whole equation divides by θ^2; substituting for $\partial\varpi$ its value, we find

$$K=(a-d)(a-e)(\partial v-c\,\partial u).$$

Expression for $AC\partial AB-AB\partial AC$.

Starting in like manner from the equations $AB=\Omega\sqrt{ab}$, $AC=\Omega\sqrt{ac}$, we have

$$AC\partial AB-AB\partial AC=\frac{1}{\theta}\frac{1}{2}\Omega^2,$$

multiplied by

$$\left(\sqrt{\mathrm{acfb}_1\mathrm{d}_1\mathrm{e}_1}-\sqrt{\mathrm{a}_1\mathrm{c}_1\mathrm{f}_1\mathrm{bde}}\right)\cdot\left\{\begin{array}{l}\sqrt{\mathrm{abfa}_1\mathrm{b}_1\mathrm{f}_1}\left[\partial\varpi+\left(a-c-d-e+x+y+\dfrac{\mathrm{cde}+\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}{\theta^2}\right)\partial u\right]\\+\sqrt{\mathrm{cdec}_1\mathrm{d}_1\mathrm{e}_1}\left[-\partial\varpi+\left(b+f-x-y-\dfrac{\mathrm{abf}+\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1}{\theta^2}\right)\partial u\right]\end{array}\right\}$$

$$+\left(-\sqrt{\mathrm{abfc}_1\mathrm{d}_1\mathrm{e}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1\mathrm{cde}}\right)\cdot\left\{\begin{array}{l}\sqrt{\mathrm{acfa}_1\mathrm{c}_1\mathrm{f}_1}\left[\partial\varpi+\left(a-b-d-e+x+y+\dfrac{\mathrm{bde}+\mathrm{b}_1\mathrm{d}_1\mathrm{e}_1}{\theta^2}\right)\partial u\right]\\+\sqrt{\mathrm{bdeb}_1\mathrm{d}_1\mathrm{e}_1}\left[-\partial\varpi+\left(c+f-x-y-\dfrac{\mathrm{acf}+\mathrm{a}_1\mathrm{c}_1\mathrm{f}_1}{\theta^2}\right)\partial u\right]\end{array}\right\},$$

which, omitting the factor $\frac{1}{\theta}\frac{1}{2}\Omega^2$, is

$$\begin{aligned}=\{&\ \mathrm{afb}_1\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}-\mathrm{a}_1\mathrm{f}_1\mathrm{b}\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}\}\left[\partial\varpi+\left(a-c-d-e+x+y+\frac{\mathrm{cde}+\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}{\theta^2}\right)\partial u\right]\\+\{&\ \mathrm{cd}_1\mathrm{e}_1\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}-\mathrm{c}_1\mathrm{de}\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}\}\left[-\partial\varpi+\left(b+f-x-y-\frac{\mathrm{abf}+\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1}{\theta^2}\right)\partial u\right]\\+\{&-\mathrm{afc}_1\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}+\mathrm{a}_1\mathrm{f}_1\mathrm{c}\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}\}\left[\partial\varpi+\left(a-b-d-e+x+y+\frac{\mathrm{bde}+\mathrm{b}_1\mathrm{d}_1\mathrm{e}_1}{\theta^2}\right)\partial u\right]\\+\{&-\mathrm{bde}_1\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}+\mathrm{b}_1\mathrm{de}\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}\}\left[-\partial\varpi+\left(c+f-x-y-\frac{\mathrm{acf}+\mathrm{a}_1\mathrm{c}_1\mathrm{f}_1}{\theta^2}\right)\partial u\right];\end{aligned}$$

and here the whole coefficient of $\partial\varpi$ is

$$=(\mathrm{b}_1-\mathrm{c}_1)(\mathrm{af}-\mathrm{de})\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}-(\mathrm{b}-\mathrm{c})(\mathrm{a}_1\mathrm{f}_1-\mathrm{d}_1\mathrm{e}_1)\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}},$$

viz. observing that $\mathrm{b}_1-\mathrm{c}_1=\mathrm{b}-\mathrm{c}=b-c$, this is

$$=(b-c)\{[af-de-(a+f-d-e)x]\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}-[af-de-(a+f-d-e)y]\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}\},$$

or, what is the same thing, it is

$$\begin{aligned}=(b-c)\{&[-(a-d)(a-e)+(a+f-d-e)\,\mathrm{a}]\sqrt{\mathrm{bca}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1}\\&-[-(a-d)(a-e)+(a+f-d-e)\,\mathrm{a}_1]\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{adef}}\}.\end{aligned}$$

The coefficient of ∂u contains the factor $\sqrt{\mathrm{bca_1d_1e_1f_1}}$, multiplied by

$$\begin{aligned}
&\mathrm{afb_1}\left(a-c-d-e+x+y+\frac{\mathrm{cde}+\mathrm{c_1d_1e_1}}{\theta^2}\right)\\
&-\mathrm{c_1de}\left(\qquad b+f-x-y-\frac{\mathrm{abf}+\mathrm{a_1b_1f_1}}{\theta^2}\right)\\
&-\mathrm{afc_1}\left(a-b-d-e+x+y+\frac{\mathrm{bde}+\mathrm{b_1d_1e_1}}{\theta^2}\right)\\
&+\mathrm{b_1de}\left(\qquad c+f-x-y-\frac{\mathrm{acf}+\mathrm{a_1c_1f_1}}{\theta^2}\right);
\end{aligned}$$

here the terms divided by θ^2 destroy each other, and the expression of the coefficient of $\sqrt{\mathrm{bca_1d_1e_1f_1}}$ becomes

$$=(\mathrm{b_1}-\mathrm{c_1})\,[\mathrm{af}\,(a-d-e+x+y)+\mathrm{de}\,(f-x-y)]+(\mathrm{af}-\mathrm{de})\,(b\mathrm{c_1}-c\mathrm{b_1}),$$

or since $\mathrm{b_1}-\mathrm{c_1}=b-c$, $b\mathrm{c_1}-c\mathrm{b_1}=-(b-c)\,y$, this is

$$=(b-c)\,[\mathrm{af}\,(a-d-e+x+y)+\mathrm{de}\,(f-x-y)-(\mathrm{af}-\mathrm{de})\,y],$$

which is

$$=(b-c)\,[\mathrm{af}\,(a-d-e)+\mathrm{de}f+(\mathrm{af}-\mathrm{de})\,x],$$

and is readily reduced to

$$(b-c)\,[(a-d)\,(a-e)f-(a-d)\,(a-e)\,x],\quad =(b-c)\,(a-d)\,(a-e)\,\mathrm{f}:$$

viz. the coefficient of ∂u contains the term $(b-c)\,(a-d)\,(a-e)\,\mathrm{f}\sqrt{\mathrm{bca_1d_1e_1f_1}}$. There is a like term $-(b-c)\,(a-d)\,(a-e)\,\mathrm{f_1}\sqrt{\mathrm{b_1c_1adef}}$, and the two terms together form the whole coefficient of ∂u.

Hence, restoring the outside factor $\frac{1}{\theta}\Omega^2$, we have

$AC\partial AB-AB\partial AC$

$$\begin{aligned}
=\frac{1}{\theta}\Omega^2(b-c)\Big[&\{[-(a-d)\,(a-e)+(a+f-d-e)\,\mathrm{a}]\sqrt{\mathrm{bca_1d_1e_1f_1}}\\
&-[-(a-d)\,(a-e)+(a+f-d-e)\,\mathrm{a_1}]\sqrt{\mathrm{b_1c_1adef}}\}\,\partial\varpi\\
&+(a-d)\,(a-e)\,\{\mathrm{f}\sqrt{\mathrm{bca_1d_1e_1f_1}}-\mathrm{f_1}\sqrt{\mathrm{b_1c_1adef}}\}\,\partial u\Big],
\end{aligned}$$

where, as before, I retain $\partial\varpi$ instead of its value $=\partial v-a\,\partial u$. This is a sum of products of the set bc: the products, in fact, are

$$\begin{aligned}
\sqrt{a}\sqrt{de}&=\frac{1}{\theta}\{-\mathrm{a}\sqrt{\mathrm{bca_1d_1e_1f_1}}+\mathrm{a_1}\sqrt{\mathrm{b_1c_1adef}}\},\\
\sqrt{d}\sqrt{ae}&=\text{,,}\ \{-\mathrm{d}\qquad\text{,,}\qquad+\mathrm{d_1}\qquad\text{,,}\qquad\},\\
\sqrt{e}\sqrt{ad}&=\text{,,}\ \{-\mathrm{e}\qquad\text{,,}\qquad+\mathrm{e_1}\qquad\text{,,}\qquad\},\\
\sqrt{f}\sqrt{bc}&=\text{,,}\ \{+\mathrm{f}\qquad\text{,,}\qquad-\mathrm{f_1}\qquad\text{,,}\qquad\},
\end{aligned}$$

whence, observing that $\mathrm{a}-\mathrm{f}=\mathrm{a}_1-\mathrm{f}_1=a-f$, we have

$$\sqrt{a}\sqrt{de}+\sqrt{f}\sqrt{bc}=-\frac{a-f}{\theta}\{\sqrt{\mathrm{bca_1d_1e_1f_1}}-\sqrt{\mathrm{b_1c_1adef}}\}:$$

it is clear that the term in question is at once expressible as a sum formed with the products $\sqrt{a}\sqrt{de}$ and $\sqrt{f}\sqrt{bc}$.

It is to be remarked that there are 15 expressions such as $A\,\partial B-B\,\partial A$, and 45 expressions such as $AC\,\partial AB-AB\,\partial AC$; and that each of these $(15+45=)\,60$ expressions is a sum of products of a set such as ab: and that there are also 60 expressions of the form $A\,\partial AB-AB\,\partial A$, and that each of these is a sum of products of a set such as $\mathrm{aba_1b_1}$.

Expression of $\Omega\,\partial^2\Omega-(\partial\Omega)^2,\quad =\frac{1}{4}M\Omega^2.$

We assume $\Omega\,\partial^2\Omega-(\partial\Omega)^2=\frac{1}{4}M\Omega^2$, where M is a quadric function of ∂u, ∂v; suppose

$$M=\mathfrak{A}\,(\partial u)^2+2\mathfrak{B}\,\partial u\,\partial v+\mathfrak{C}\,(\partial v)^2.$$

It is to be noticed that the $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are not all of them arbitrary functions of (x, y) or (u, v); we, in fact, have $\frac{1}{4}M=\frac{\partial^2\Omega}{\Omega}-\frac{(\partial\Omega)^2}{\Omega^2}=\partial^2\log\Omega$; and hence $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ satisfy the conditions

$$\frac{d\mathfrak{A}}{dv}=\frac{d\mathfrak{B}}{du},\quad \frac{d\mathfrak{B}}{dv}=\frac{d\mathfrak{C}}{du}.$$

Taking $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ as functions of x, y, these become

$$\left(\frac{d\mathfrak{A}}{dx}+y\frac{d\mathfrak{B}}{dx}\right)\sqrt{X}=\left(\frac{d\mathfrak{A}}{dy}+x\frac{d\mathfrak{B}}{dy}\right)\sqrt{Y},$$

$$\left(\frac{d\mathfrak{B}}{dx}+y\frac{d\mathfrak{C}}{dx}\right)\sqrt{X}=\left(\frac{d\mathfrak{B}}{dy}+x\frac{d\mathfrak{C}}{dy}\right)\sqrt{Y}.$$

Putting for the moment

$$\begin{aligned}
&\lambda=a+b+c, && \rho=d+e+f, && p=\lambda+\rho,\\
&\mu=ab+ac+bc, && \sigma=de+df+ef, && q=\mu+\sigma,\\
&\nu=abc, && \tau=def, && r=\nu+\tau,
\end{aligned}$$

I found it convenient to assume

$$\mathfrak{C}=-2\,(x^2+xy+y^2)+p\,(x+y),$$

where observe that $p,=a+b+c+d+e+f$, is symmetrical in regard to the constants a, b, c, d, e, f. And then, $\mathfrak{C}$ having this value, there exists (as is seen at once) a value of $\mathfrak{B},=2\,(x^2y+xy^2)-pxy$, for which

$$\frac{d\mathfrak{B}}{dx}+y\frac{d\mathfrak{C}}{dx}=0,\quad \frac{d\mathfrak{B}}{dy}+x\frac{d\mathfrak{C}}{dy}=0,$$

and which thus satisfies the second of the above-mentioned conditions.

Assuming now

$$\mathfrak{A} = -2x^2y^2 + qxy - r(x+y) - \mu\sigma + \Theta,$$

where Θ has to be determined so as that the first of the same conditions may be also satisfied, then substituting this value of $\mathfrak{A}$, we have

$$\left(2y^3 - py^2 + qy - r + \frac{d\Theta}{dx}\right)\sqrt{X} = \left(2x^3 - px^2 + qx - r + \frac{d\Theta}{dy}\right)\sqrt{Y};$$

that is,

$$\left(-\mathrm{a_1b_1c_1} - \mathrm{d_1e_1f_1} + \frac{d\Theta}{dx}\right)\sqrt{X} = \left(-\mathrm{abc} - \mathrm{def} + \frac{d\Theta}{dy}\right)\sqrt{Y},$$

viz. in the terms independent of Θ writing for $\sqrt{X}$, $\sqrt{Y}$ their values, this is

$$(\mathrm{abc} + \mathrm{def})\sqrt{\mathrm{a_1b_1c_1d_1e_1f_1}} - (\mathrm{a_1b_1c_1} + \mathrm{d_1e_1f_1})\sqrt{\mathrm{abcdef}} + \frac{d\Theta}{dx}\sqrt{X} - \frac{d\Theta}{dy}\sqrt{Y} = 0,$$

or, what is the same thing,

$$-(\sqrt{\mathrm{abca_1b_1c_1}} - \sqrt{\mathrm{defd_1e_1f_1}})(\sqrt{\mathrm{defa_1b_1c_1}} - \sqrt{\mathrm{d_1e_1f_1abc}}) + \frac{d\Theta}{dx}\sqrt{X} - \frac{d\Theta}{dy}\sqrt{Y} = 0.$$

But treating Θ as a function of u and v, we have

$$\frac{d\Theta}{dv} = \frac{d\Theta}{dx}\frac{dx}{dv} + \frac{d\Theta}{dy}\frac{dy}{dv} = \frac{1}{\theta}\left(\frac{d\Theta}{dx}\sqrt{X} - \frac{d\Theta}{dy}\sqrt{Y}\right):$$

also

$$\sqrt{de} = \frac{1}{\theta}(\sqrt{\mathrm{defa_1b_1c_1}} - \sqrt{\mathrm{d_1e_1f_1abc}});$$

and we thus reduce the equation to

$$-(\sqrt{\mathrm{abca_1b_1c_1}} - \sqrt{\mathrm{defd_1e_1f_1}})\sqrt{de} + \frac{d\Theta}{dv} = 0.$$

But, referring to the expression for $\partial\sqrt{ab}$, we have, by a mere interchange of letters,

$$\frac{d}{dv}\sqrt{de} = -\tfrac{1}{2}(\sqrt{\mathrm{abca_1b_1c_1}} - \sqrt{\mathrm{defd_1e_1f_1}}),$$

and the formula thus becomes

$$2\sqrt{de}\frac{d}{dv}\sqrt{de} + \frac{d\Theta}{dv} = 0;$$

consequently

$$\Theta = -(\sqrt{de})^2 = -\frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1} + \mathrm{a_1b_1c_1def} - 2\sqrt{XY}),$$

and the value of $\mathfrak{A}$ thus is

$$\mathfrak{A} = \frac{1}{\theta^2}\{-\mathrm{abcd_1e_1f_1} - \mathrm{a_1b_1c_1def} + \theta^2(-2x^2y^2 + qxy - r(x+y) - \mu\sigma)\} + \frac{2}{\theta^2}\sqrt{XY},$$

or, as this may be written,

$$\mathfrak{A} = \frac{1}{\theta^2}(\mathrm{abc} - \mathrm{a_1b_1c_1})(\mathrm{def} - \mathrm{d_1e_1f_1}) - 2x^2y^2 + qxy - r(x+y) - \mu\sigma - \frac{1}{\theta^2}(\sqrt{X} - \sqrt{Y})^2.$$

Here

$$\begin{aligned}\mathrm{abc} - \mathrm{a_1b_1c_1} &= (\nu - \mu x + \lambda x^2 - x^3) - (\nu - \mu y + \lambda y^2 - y^3)\\ &= \theta\,[-\mu + \lambda(x+y) - (x^2 + xy + y^2)],\end{aligned}$$

and similarly

$$\mathrm{def} - \mathrm{d_1e_1f_1} = \theta\,[-\sigma + \rho(x+y) - (x^2 + xy + y^2)];$$

the expression of $\mathfrak{A}$ contains therefore the terms

$$[\mu - \lambda(x+y) + x^2 + xy + y^2]\,[\sigma - \rho(x+y) + x^2 + xy + y^2] - \mu\sigma - r(x+y) + q\,xy - 2x^2y^2,$$

viz. for r, q substituting their values $\nu + \tau$, $\mu + \rho$, these terms are

$$\begin{aligned}&= -(\mu\rho + \sigma\lambda + \nu + \tau)(x+y) + (\mu + \sigma + \lambda\rho)(x+y)^2\\ &\quad - (\lambda + \rho)(x+y)(x^2 + xy + y^2) + (x^2 + xy + y^2)^2 - 2x^2y^2.\end{aligned}$$

The coefficients $\mu\rho + \sigma\lambda + \nu + \tau$, $\mu + \sigma + \lambda\rho$, $\lambda + \rho$ are, in fact, symmetrical functions of a, b, c, d, e, f, viz. writing

$$\begin{aligned}X &= a - x\,.\,b - x\,.\,c - x\,.\,d - x\,.\,e - x\,.\,f - x,\\ &= \mathrm{A} - \mathrm{B}x + \mathrm{C}x^2 - \mathrm{D}x^3 + \mathrm{E}x^4 - \mathrm{F}x^5 + x^6,\end{aligned}$$

that is,

$$\mathrm{A} = abcdef,\quad \mathrm{B} = \Sigma\, abcde,\quad \mathrm{C} = \Sigma\, abcd,\quad \mathrm{D} = \Sigma\, abc,\quad \mathrm{E} = \Sigma\, ab,\quad \mathrm{F} = \Sigma\, a,$$

($\mathrm{F} = a + b + c + d + e + f$, which has in fact previously been called p), we have

$$\mu\rho + \sigma\lambda + \nu + \tau = \mathrm{D},\quad \mu + \sigma + \lambda\rho = \mathrm{E},\quad \lambda + \rho = \mathrm{F},$$

and the terms are

$$= -(x+y)\,\{\mathrm{D} - \mathrm{E}(x+y) + \mathrm{F}(x^2 + xy + y^2)\} + x^4 + 2x^3y + x^2y^2 + 2xy^3 + y^4;$$

viz. we have

$$\begin{aligned}\mathfrak{A} &= -\frac{1}{\theta^2}(\sqrt{X} - \sqrt{Y})^2\\ &\quad - (x+y)\,\{\mathrm{D} - \mathrm{E}(x+y) + \mathrm{F}(x^2 + xy + y^2)\} + (x^4 + 2x^3y + x^2y^2 + 2xy^3 + y^4).\end{aligned}$$

To this I join the foregoing values of $\mathfrak{B}$, $\mathfrak{C}$; viz. writing F in place of p, these are

$$\begin{aligned}\mathfrak{B} &= -\mathrm{F}\,xy + 2(x^2y + xy^2),\\ \mathfrak{C} &= \mathrm{F}(x+y) - 2(x^3 + xy + y^2),\end{aligned}$$

where it will be noticed that the values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are all of them symmetrical in regard to the constants a, b, c, d, e, f.

I recall the original form of $\mathfrak{A}$, viz. this was

$$\begin{aligned}\mathfrak{A} &= -\mu\sigma - r(x+y) + q\,xy - 2x^2y^2 - (\sqrt{de})^2\\ &= -(ab + ac + bc)(de + df + ef) - (abc + def)(x+y)\\ &\quad + (ab + ac + bc + de + df + ef)\,xy - 2x^2y^2 - (\sqrt{de})^2\\ &= \mathfrak{A}_0 - (\sqrt{de})^2,\end{aligned}$$

suppose; and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ denoting as above, we have

$$M = \mathfrak{A}(\partial u)^2 + 2\mathfrak{B}\,\partial u\,\partial v + \mathfrak{C}(\partial v)^2, \quad \Omega\,\partial^2\Omega - (\partial\Omega)^2 = \tfrac{1}{4}M\Omega^2.$$

For the subsequent calculation of $A\,\partial^2 A - (\partial A)^2$, it is convenient to transform this expression by introducing therein $\partial\varpi$ in place of ∂v, and a, a_1 in place of x, y. We have

$$\begin{aligned} M &= \{\mathfrak{A}_0 - (\sqrt{de})^2\}(\partial u)^2 + 2\mathfrak{B}\,\partial u\,(\partial\varpi + a\,\partial u) + \mathfrak{C}(\partial\varpi + a\,\partial u)^2 \\ &= \{\mathfrak{A}_0' - (\sqrt{de})^2\}(\partial u)^2 + 2\mathfrak{B}'\,\partial u\,\partial\varpi \qquad\qquad + \mathfrak{C}'(\partial\varpi)^2, \end{aligned}$$

suppose, where

$$\begin{aligned} \mathfrak{C}' &= \mathfrak{C}, \\ \mathfrak{B}' &= \mathfrak{B} + a\mathfrak{C}, \\ \mathfrak{A}_0' &= \mathfrak{A}_0 + 2a\mathfrak{B} + a^2\mathfrak{C}. \end{aligned}$$

Writing

$$x = a - \mathrm{a}, \quad y = a - \mathrm{a}_1,$$

we find

$$\begin{aligned} \mathfrak{C} &= -6a^2 + 2a\mathrm{F} + (6a - \mathrm{F})(\mathrm{a} + \mathrm{a}_1) - 2(\mathrm{a}^2 + \mathrm{a}\mathrm{a}_1 + \mathrm{a}_1^2), \\ \mathfrak{B} &= 4a^3 - a^2\mathrm{F} + (-6a^2 + a\mathrm{F})(\mathrm{a} + \mathrm{a}_1) + 2a(\mathrm{a}^2 + \mathrm{a}_1^2) + (8a - \mathrm{F})\mathrm{a}\mathrm{a}_1 - 2(\mathrm{a}^2\mathrm{a}_1 + \mathrm{a}\mathrm{a}_1^2), \\ \mathfrak{A}_0 &= -(ab + ac + bc)(de + df + ef) \\ &\quad - (abc + def)(2a - \mathrm{a} - \mathrm{a}_1) \\ &\quad + (ab + ac + bc + de + df + ef)[a^2 - a(\mathrm{a} + \mathrm{a}_1) + \mathrm{a}\mathrm{a}_1] \\ &\quad - 2[a^2 - a(\mathrm{a} + \mathrm{a}_1) + \mathrm{a}\mathrm{a}_1]^2, \end{aligned}$$

the developed value of which is

$$\begin{aligned} &= -2a^4 + a^3(b + c) + a^2(-bc + de + df + ef) \\ &\quad + a\{-2def - (b + c)(de + df + ef)\} - bc(de + df + ef) \\ &\quad + \{4a^3 - a^2(b + c) - a(de + df + ef) + def\}(\mathrm{a} + \mathrm{a}_1) \\ &\quad - 2a^2(\mathrm{a}^2 + \mathrm{a}_1^2) + \{-8a^2 + a(b + c) + bc + de + df + ef\}\mathrm{a}\mathrm{a}_1 \\ &\quad + 4a(\mathrm{a}^2\mathrm{a}_1 + \mathrm{a}\mathrm{a}_1^2) \\ &\quad - 2\mathrm{a}^2\mathrm{a}_1^2, \end{aligned}$$

and thence without difficulty

$$\begin{aligned} \mathfrak{C}' &= -6a^2 + 2a\mathrm{F} + (6a - \mathrm{F})(\mathrm{a} + \mathrm{a}_1) - 2(\mathrm{a}^2 + \mathrm{a}\mathrm{a}_1 + \mathrm{a}_1^2), \\ \mathfrak{B}' &= -8a^3 + a^2\mathrm{F} + (6a - \mathrm{F})\mathrm{a}\mathrm{a}_1 - 2(\mathrm{a}^2\mathrm{a}_1 + \mathrm{a}\mathrm{a}_1^2), \\ \mathfrak{A}_0' &= a^3(b + c) + a^2(-bc + de + df + ef) + a\{-2def - (b + c)(de + df + ef)\} - bc(de + df + ef) \\ &\quad + \{-a^3 + a^2(d + e + f) - a(de + df + ef) + def\}(\mathrm{a} + \mathrm{a}_1) \\ &\quad + \{4a^2 + a(-b - c - 2d - 2e - 2f) + bc + de + df + ef\}\mathrm{a}\mathrm{a}_1 \\ &\quad - 2\mathrm{a}^2\mathrm{a}_1^2, \end{aligned}$$

which are the required values.

Expression for $A\,\partial^2 A - (\partial A)^2$: several subheadings.

Writing for shortness $A\,\partial^2 A - (\partial A)^2 = \Delta A$, as before, and so in other cases: then in general $\Delta PQ = P^2\Delta Q + Q^2\Delta P$, and thence $\Delta P^2 = 2P^2\Delta P$ or $\Delta\sqrt{P} = \frac{\frac{1}{2}}{P}\Delta P$. Hence starting from $A = \Omega\sqrt{a} = \Omega\sqrt{\mathrm{aa}_1}$, we have

$$\Delta A = \Delta\Omega\sqrt{\mathrm{aa}_1} = \mathrm{aa}_1\,\Delta\Omega + \frac{\frac{1}{2}\Omega^2}{\mathrm{aa}_1}(\mathrm{a}^2\Delta\mathrm{a}_1 + \mathrm{a}_1^2\Delta\mathrm{a}),$$

where

$$\Delta\mathrm{a} = \mathrm{a}\,\partial^2\mathrm{a} - (\partial\mathrm{a})^2 = -\mathrm{a}\,\partial^2 x - (\partial x)^2,\ \ \Delta\mathrm{a}_1 = -\mathrm{a}\,\partial^2 y - (\partial y)^2.$$

Hence writing

$$\Delta\Omega = \tfrac{1}{4}M\Omega^2,$$

we have

$$\frac{1}{\Omega^2}\Delta A = \tfrac{1}{4}\mathrm{aa}_1 M - \tfrac{1}{2}(\mathrm{a}_1\,\partial^2 x + \mathrm{a}\,\partial^2 x_1) - \tfrac{1}{2}\left\{\frac{\mathrm{a}_1}{\mathrm{a}}(\partial x)^2 + \frac{\mathrm{a}}{\mathrm{a}_1}(\partial y)^2\right\}.$$

But we have

$$\partial x = \frac{\sqrt{X}}{\theta}(\partial v - y\,\partial u),\quad \partial y = -\frac{\sqrt{Y}}{\theta}(\partial v - x\,\partial u);$$

squaring the first of these and differentiating, we find

$$2\partial x\,\partial^2 x = \left[\left(-\frac{2X}{\theta^3} + \frac{X'}{\theta^2}\right)\partial x + \frac{2X}{\theta^3}\partial y\right](\partial v - y\,\partial u)^2 - 2\partial y\,\partial u\,\frac{X}{\theta^2}(\partial v - y\,\partial u),$$

where as regards X the accent denotes differentiation as to x (and further on, as regards Y, it denotes differentiation to y), viz. this is

$$= \left[\left(-\frac{2X}{\theta^3} + \frac{X'}{\theta^2}\right)\partial x + \frac{2X}{\theta^3}\partial y\right](\partial v - y\,\partial u)^2 - 2\partial y\,\partial u\,\frac{X}{\theta^2}(\partial v - y\,\partial u),$$

$$= \left(-\frac{2X}{\theta^3} + \frac{X'}{\theta^2}\right)\partial x\,(\partial v - y\,\partial u)^2 + \frac{2X}{\theta^3}(\partial v - y\,\partial u - \theta\,\partial u)(\partial v - y\,\partial u)\,\partial y,$$

where the second term is

$$\frac{2X}{\theta^3}(\partial v - x\,\partial u)(\partial v - y\,\partial u)\,\partial y,$$

which is

$$= -\frac{2\sqrt{XY}}{\theta^3}(\partial v - x\,\partial u)^2\,\partial x:$$

hence dividing by $2\partial x$, the equation is

$$\partial^2 x = \left(-\frac{X}{\theta^3} + \frac{X'}{2\theta^2}\right)(\partial v - y\,\partial u)^2 - \frac{\sqrt{XY}}{\theta^3}(\partial v - x\,\partial u)^2;$$

and similarly

$$\partial^2 y = \frac{\sqrt{XY}}{\theta^3}(\partial v - y\,\partial u)^2 + \left(\frac{Y}{\theta^3} + \frac{Y'}{2\theta^2}\right)(\partial v - x\,\partial u)^2;$$

and we may in these values in place of $\partial v - y\,\partial u$ and $\partial v - x\,\partial u$ write $\partial\varpi + \mathrm{a}_1\,\partial u$ and $\partial\varpi + \mathrm{a}\,\partial u$ respectively.

Hence in $\frac{1}{\Omega^2}\Delta A$ the irrational part is

$$\tfrac{1}{2}\frac{\sqrt{XY}}{\theta^3}\{\mathrm{a}_1(\partial\varpi+\mathrm{a}\,\partial u)^2-\mathrm{a}(\partial\varpi+\mathrm{a}_1\,\partial u)^2\}$$

$$=\frac{\tfrac{1}{2}\sqrt{XY}}{\theta^3}(\mathrm{a}_1-\mathrm{a})\{(\partial\varpi)^2-\mathrm{aa}_1(\partial u)^2\}=\frac{\tfrac{1}{2}\sqrt{XY}}{\theta^2}\{(\partial\varpi)^2-\mathrm{aa}_1(\partial u)^2\}.$$

But we have

$$(\sqrt{de})^2=\frac{1}{\theta^2}\{\mathrm{abcd_1e_1f_1}+\mathrm{a_1b_1c_1def}-2\sqrt{XY}\},$$

whence

$$\frac{\sqrt{XY}}{\theta^2}=\frac{\tfrac{1}{2}}{\theta^2}(\mathrm{abcd_1e_1f_1}+\mathrm{a_1b_1c_1def})-\tfrac{1}{2}(\sqrt{de})^2;$$

and the term thus is

$$\left[\frac{\tfrac{1}{4}}{\theta^2}(\mathrm{abcd_1e_1f_1}+\mathrm{a_1b_1c_1def})-\tfrac{1}{4}(\sqrt{de})^2\right]\{(\partial\varpi)^2-\mathrm{aa}_1(\partial u)^2\}.$$

Joining hereto the rational part of $\frac{1}{\Omega^2}\Delta A$, and multiplying the whole by 4, we have

$$\begin{aligned}\frac{4}{\Omega^2}\Delta A=\mathrm{aa}_1M&+\left[\mathrm{a}_1\left(\frac{2X}{\theta^3}-\frac{X'}{\theta^2}\right)-\frac{2\mathrm{a}_1}{\mathrm{a}}\frac{X}{\theta^2}\right](\partial\varpi+\mathrm{a}_1\partial u)^2\\&+\left[\mathrm{a}\left(-\frac{2Y}{\theta^3}-\frac{Y'}{\theta^2}\right)-\frac{2\mathrm{a}}{\mathrm{a}_1}\frac{Y}{\theta^2}\right](\partial\varpi+\mathrm{a}\,\partial u)^2\\&+\left[\frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1}+\mathrm{a_1b_1c_1def})-(\sqrt{de})^2\right]\{(\partial\varpi)^2-\mathrm{aa}_1(\partial u)^2\},\end{aligned}$$

where M has its foregoing value $=\{2\mathfrak{A}_0'-(\sqrt{de})^2\}(\partial u)^2+2\mathfrak{B}'\,\partial u\,\partial\varpi+\mathfrak{C}'(\partial\varpi)^2$.

First step of the reduction.

Writing $\mathrm{bcdef}=U$, $\mathrm{b_1c_1d_1e_1f_1}=U_1$, then $X=\mathrm{a}U$, $Y=\mathrm{a}_1U_1$, and consequently

$$X'=-U+\mathrm{a}U',\quad Y'=-U_1+\mathrm{a}_1U_1',$$

the accents in regard to U, U_1 denoting differentiations as to x, y respectively: then

$$\mathrm{a}_1\left(\frac{2X}{\theta^3}-\frac{X'}{\theta^2}\right)-\frac{2\mathrm{a}_1}{\mathrm{a}}\frac{X}{\theta^2}=\mathrm{a}_1\left(\frac{2\mathrm{a}U}{\theta^3}+\frac{U-\mathrm{a}U'}{\theta^2}\right)-\frac{2\mathrm{a}_1}{\mathrm{a}}\frac{\mathrm{a}U}{\theta^2}=\mathrm{aa}_1\left(\frac{2U}{\theta^3}-\frac{U'}{\theta^3}\right)-\frac{\mathrm{a}_1U}{\theta^2};$$

and similarly

$$\mathrm{a}\left(-\frac{2Y}{\theta^3}-\frac{Y'}{\theta^2}\right)-\frac{2\mathrm{a}}{\mathrm{a}_1}\frac{Y}{\theta^2}=\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ =\mathrm{aa}_1\left(-\frac{2U_1}{\theta^3}-\frac{U_1'}{\theta^2}\right)-\frac{\mathrm{a}U_1}{\theta^2}.$$

The formula thus becomes

$$\frac{4}{\Omega^2}\Delta A = \mathrm{aa_1}\left[M + \left(\frac{2U}{\theta^3} - \frac{U'}{\theta^2}\right)(\partial\varpi + \mathrm{a_1}\,\partial u)^2 + \left(-\frac{2U_1}{\theta^3} - \frac{U_1'}{\theta^2}\right)(\partial\varpi + \mathrm{a}\,\partial u)^2\right.$$
$$\left. - \left\{\frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1 + a_1b_1c_1def}) - (\sqrt{de})^2\right\}(\partial u)^2\right]$$
$$- \frac{\mathrm{a_1}U}{\theta^2}(\partial\varpi + \mathrm{a_1}\,\partial u)^2 - \frac{\mathrm{a}U_1}{\theta^2}(\partial\varpi + \mathrm{a}\,\partial u)^2$$
$$+ \left\{\frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1 + a_1b_1c_1def}) - (\sqrt{de})^2\right\}(\partial\varpi)^2,$$

viz. substituting for M its value, the term in $(\sqrt{de})^2(\partial u)^2$ disappears, and the formula is

$$\frac{4}{\Omega^2}\Delta A = \mathrm{aa_1}\left[\mathfrak{A}_0'(\partial u)^2 + 2\mathfrak{B}'\partial u\,\partial\varpi + \mathfrak{C}'(\partial\varpi)^2 + \left(\frac{2U}{\theta^3} - \frac{U'}{\theta^2}\right)(\partial\varpi + \mathrm{a_1}\,\partial u)^2\right.$$
$$\left. + \left(-\frac{2U_1}{\theta^3} - \frac{U_1'}{\theta^2}\right)(\partial\varpi + \mathrm{a}\,\partial u)^2 - \frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1 + a_1b_1c_1def})(\partial u)^2\right]$$
$$- \frac{\mathrm{a_1}U}{\theta^2}(\partial\varpi + \mathrm{a_1}\,\partial u)^2 - \frac{\mathrm{a}U_1}{\theta^2}(\partial\varpi + \mathrm{a}\,\partial u)^2$$
$$+ \left\{\frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1 + a_1b_1c_1def}) - (\sqrt{de})^2\right\}(\partial\varpi)^2;$$

say for shortness this is

$$\frac{4}{\Omega^2}\Delta A$$
$$= \mathrm{aa_1}\Sigma - \frac{\mathrm{a_1}U}{\theta^2}(\partial\varpi + \mathrm{a_1}\,\partial u)^2 - \frac{\mathrm{a}U_1}{\theta^2}(\partial\varpi + \mathrm{a}\,\partial u)^2 + \frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1 + a_1b_1c_1def}) - (\sqrt{de})^2(\partial\varpi)^2.$$

Second step of the reduction.

In the reductions which follow, we make as many terms as may be to contain the factor $\mathrm{aa_1}$, so as to simplify as much as possible the portion not containing this factor.

We have $\partial\varpi + \mathrm{a_1}\,\partial u = (\partial\varpi + \theta\,\partial u) + \mathrm{a}\,\partial u$, and consequently

$$(\partial\varpi + \mathrm{a_1}\,\partial u)^2 = (\partial\varpi + \theta\,\partial u)^2 + \mathrm{a}P,$$

where $P = 2\,\partial u\,\partial\varpi + (\mathrm{a} + 2\theta)(\partial u)^2$: similarly $\partial\varpi + \mathrm{a}\,\partial u = (\partial\varpi - \theta\,\partial u) + \mathrm{a_1}\,\partial u$, and therefore

$$(\partial\varpi + \mathrm{a}\,\partial u)^2 = (\partial\varpi - \theta\,\partial u)^2 + \mathrm{a_1}P_1,$$

where $P_1 = 2\,\partial u\,\partial\varpi + (\mathrm{a_1} - 2\theta)(\partial u)^2$: the values may also be written

$$P = 2\,\partial u\,\partial\varpi + (2\mathrm{a_1} - \mathrm{a})(\partial u)^2, \quad P_1 = 2\,\partial u\,\partial\varpi + (2\mathrm{a} - \mathrm{a_1})(\partial u)^2.$$

The formula thus becomes

$$\frac{4}{\Omega^2}\Delta A = \mathrm{aa}_1\left\{\Sigma - \frac{U}{\theta^2}P - \frac{U_1}{\theta^2}P_1\right\}$$
$$-\frac{\mathrm{a}_1 U}{\theta^2}(\partial\varpi + \theta\,\partial u)^2 - \frac{\mathrm{a}U_1}{\theta^2}(\partial\varpi - \theta\,\partial u)^2 + \frac{1}{\theta^2}(\mathrm{abcd}_1\mathrm{e}_1\mathrm{f}_1 + \mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{def})(\partial\varpi)^2$$
$$-(\sqrt{de})^2(\partial\varpi)^2.$$

The second line here is

$$-(\mathrm{a}_1 U + \mathrm{a}U_1)(\partial u)^2 - \frac{2}{\theta}(\mathrm{a}_1 U - \mathrm{a}U_1)\,\partial u\,\partial\varpi$$
$$+\frac{1}{\theta^2}\{-\mathrm{a}_1 U - \mathrm{a}U_1 + \mathrm{abcd}_1\mathrm{e}_1\mathrm{f}_1 + \mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{def}\}(\partial\varpi)^2,$$

and the coefficient herein of $(\partial\varpi)^2$ is $= \frac{1}{\theta^2}(\mathrm{ad}_1\mathrm{e}_1\mathrm{f}_1 - \mathrm{a}_1\mathrm{def})(\mathrm{bc} - \mathrm{b}_1\mathrm{c}_1)$. Writing for the moment $d-a$, $e-a$, $f-a = d'$, e', f', we have

$$\frac{1}{\theta}(\mathrm{ad}_1\mathrm{e}_1\mathrm{f}_1 - \mathrm{a}_1\mathrm{def}) = \frac{1}{\mathrm{a}_1 - \mathrm{a}}\{\mathrm{a}(d' + \mathrm{a}_1 . e' + \mathrm{a}_1 . f' + \mathrm{a}_1) - \mathrm{a}_1(d' + \mathrm{a} . e' + \mathrm{a} . f' + \mathrm{a})\}$$
$$= -d'e'f' + \mathrm{aa}_1(d' + e' + f' + \mathrm{a} + \mathrm{a}_1),$$
$$\frac{1}{\theta}(\mathrm{bc} - \mathrm{b}_1\mathrm{c}_1) = -(b' + c' + \mathrm{a} + \mathrm{a}_1).$$

The whole term in $(\partial\varpi)^2$ is thus

$$= \{(b' + c' + \mathrm{a} + \mathrm{a}_1)d'e'f' + \mathrm{aa}_1(b' + c' + \mathrm{a} + \mathrm{a}_1)(d' + e' + f' + \mathrm{a} + \mathrm{a}_1)\}(\partial\varpi)^2.$$

The coefficient of $\partial u\,\partial\varpi$ is $-\frac{2}{\theta}(\mathrm{a}_1 U - \mathrm{a}U_1)$: viz. this is

$$= \frac{2}{\mathrm{a}_1 - \mathrm{a}}\{\mathrm{a}_1(b' + \mathrm{a} . c' + \mathrm{a} . d' + \mathrm{a} . e' + \mathrm{a} . f' + \mathrm{a}) - \mathrm{a}(b' + \mathrm{a}_1 . c' + \mathrm{a}_1 . d' + \mathrm{a}_1 . e' + \mathrm{a}_1 . f' + \mathrm{a}_1)\},$$

and if

$$b' + \mathrm{a} . c' + \mathrm{a} . d' + \mathrm{a} . e' + \mathrm{a} . f' + \mathrm{a} = \mathrm{B}' + \mathrm{C}'\mathrm{a} + \mathrm{D}'\mathrm{a}^2 + \mathrm{E}'\mathrm{a}^3 + \mathrm{F}'\mathrm{a}^4 + \mathrm{a}^5,$$

that is,

$$\mathrm{B}' = b'c'd'e'f', \quad \mathrm{C}' = \Sigma b'c'd'e', \quad \mathrm{D}' = \Sigma b'c'd', \quad \mathrm{E}' = \Sigma b'c',$$
$$\mathrm{F}' = \Sigma b' = b' + c' + d' + e' + f',$$

this is

$$= 2\{\mathrm{B}' - \mathrm{D}'\mathrm{aa}_1 - \mathrm{E}'\mathrm{aa}_1(\mathrm{a} + \mathrm{a}_1) - \mathrm{F}'\mathrm{aa}_1(\mathrm{a}^2 + \mathrm{aa}_1 + \mathrm{a}_1^2) - \mathrm{aa}_1(\mathrm{a}^3 + \mathrm{a}^2\mathrm{a}_1 + \mathrm{aa}_1^2 + \mathrm{a}_1^3)\}:$$

or say for shortness it is $= -2(\mathrm{B}' - \mathrm{aa}_1\Phi)$ where

$$\Phi = \mathrm{D}' + \mathrm{E}'(\mathrm{a} + \mathrm{a}_1) + \mathrm{F}'(\mathrm{a}^2 + \mathrm{aa}_1 + \mathrm{a}_1^2) + \mathrm{a}^3 + \mathrm{a}^2\mathrm{a}_1 + \mathrm{aa}_1^2 + \mathrm{a}_1^3;$$

the term in question thus is $-2(\mathrm{B}' - \mathrm{aa}_1\Phi)\,\partial u\,\partial\varpi$.

The coefficient of $(\partial u)^2$ is $-(\mathrm{a}_1 U + \mathrm{a}U_1)$, viz. this is

$$-\mathrm{a}_1(b' + \mathrm{a} . c' + \mathrm{a} . d' + \mathrm{a} . e' + \mathrm{a} . f' + \mathrm{a}) - \mathrm{a}(b' + \mathrm{a}_1 . c' + \mathrm{a}_1 . d' + \mathrm{a}_1 . e' + \mathrm{a}_1 . f' + \mathrm{a}_1),$$

which is $= -(\mathrm{a} + \mathrm{a}_1)\mathrm{B}' - \mathrm{aa}_1\Psi$, where

$$\Psi = 2\mathrm{C}' + \mathrm{D}'(\mathrm{a} + \mathrm{a}_1) + \mathrm{E}'(\mathrm{a}^2 + \mathrm{a}_1^2) + \mathrm{F}'(\mathrm{a}^3 + \mathrm{a}_1^3) + \mathrm{a}^4 + \mathrm{a}_1^4.$$

The formula thus is

$$\frac{4}{\Omega^2}\Delta A = \mathrm{aa_1}\left\{\Sigma - \frac{U}{\theta^2}P - \frac{U_1}{\theta^2}P_1 - (b'+c'+\mathrm{a}+\mathrm{a_1})(d'+e'+f'+\mathrm{a}+\mathrm{a_1})(\partial\varpi)^2 + 2\Phi\,\partial u\,\partial\varpi - \Psi(\partial u)^2\right\}$$
$$-(\mathrm{a}+\mathrm{a_1})\mathrm{B}'(\partial u)^2 - 2\mathrm{B}'\partial u\,\partial\varpi + (b'+c'+\mathrm{a}+\mathrm{a_1})d'e'f'(\partial\varpi)^2 - (\sqrt{de})^2(\partial\varpi)^2.$$

The whole coefficient of $\mathrm{aa_1}$, substituting for Σ, P, P_1, Φ, Ψ their values, and arranging according to $\partial\varpi$, ∂u, is

$$=(\partial\varpi)^2\left\{\mathfrak{C}' + \frac{2U}{\theta^3} - \frac{U'}{\theta^2} + \left(-\frac{2U_1}{\theta^3} - \frac{U_1'}{\theta^2}\right) - (b'+c'+\mathrm{a}+\mathrm{a_1})(d'+e'+f'+\mathrm{a}+\mathrm{a_1})\right\}$$
$$+2\,\partial\varpi\,\partial u\left\{\mathfrak{B}' + \mathrm{a_1}\left(\frac{2U}{\theta^3} - \frac{U'}{\theta^2}\right) + \mathrm{a}\left(-\frac{2U_1}{\theta^3} - \frac{U_1'}{\theta^2}\right) - \frac{U}{\theta^2} - \frac{U_1}{\theta^2}\right.$$
$$\left. + \mathrm{D}' + \mathrm{E}'(\mathrm{a}+\mathrm{a_1}) + \mathrm{F}'(\mathrm{a}^2+\mathrm{aa_1}+\mathrm{a_1}^2) + \mathrm{a}^3 + \mathrm{a}^2\mathrm{a_1} + \mathrm{aa_1}^2 + \mathrm{a_1}^3\right\}$$
$$+(\partial u)^2\left\{\mathfrak{A}_0' + \mathrm{a_1}^2\left(\frac{2U}{\theta^3} - \frac{U'}{\theta^2}\right) + \mathrm{a}^2\left(-\frac{2U_1}{\theta^3} - \frac{U_1'}{\theta^2}\right) + (\mathrm{a}-2\mathrm{a_1})\frac{U_1}{\theta^2}\right.$$
$$\left. - \frac{1}{\theta^2}(\mathrm{abcd_1e_1f_1} + \mathrm{a_1b_1c_1def}) - 2\mathrm{C}' - \mathrm{D}'(\mathrm{a}+\mathrm{a_1}) - \mathrm{E}'(\mathrm{a}^2+\mathrm{a_1}^2) - \mathrm{F}'(\mathrm{a}^3+\mathrm{a_1}^3) - \mathrm{a}^4 - \mathrm{a_1}^4\right\}:$$

and we have to reduce separately the three coefficients of this formula.

Third step of the reduction.

First, for the coefficient of $(\partial\varpi)^2$; recollecting that $\theta = \mathrm{a_1} - \mathrm{a}$, we have

$$2\,\frac{U-U_1}{\theta} = -2\mathrm{C}' - 2\mathrm{D}'(\mathrm{a}+\mathrm{a_1}) - 2\mathrm{E}'(\mathrm{a}^2+\mathrm{aa_1}+\mathrm{a_1}^2) - 2\mathrm{F}'(\mathrm{a}^3+\mathrm{a}^2\mathrm{a_1}+\mathrm{aa_1}^2+\mathrm{a_1}^3)$$
$$-2(\mathrm{a}^4+\mathrm{a}^3\mathrm{a_1}+\mathrm{a}^2\mathrm{a_1}^2+\mathrm{aa_1}^3+\mathrm{a_1}^4),$$
$$-(U'+U_1') = 2\mathrm{C}' + 2\mathrm{D}'(\mathrm{a}+\mathrm{a_1}) + 3\mathrm{E}'(\mathrm{a}^2+\mathrm{a_1}^2) + 4\mathrm{F}'(\mathrm{a}^3+\mathrm{a_1}^3) + 5(\mathrm{a}^4+\mathrm{a_1}^4).$$

Adding these, the right-hand side divides by $(\mathrm{a_1}-\mathrm{a})^2$, that is, by θ^2; and the resulting value is

$$=\mathrm{E}' + 2\mathrm{F}'(\mathrm{a}+\mathrm{a_1}) + 3\mathrm{a}^2 + 4\mathrm{aa_1} + 3\mathrm{a_1}^2.$$

The term $-(b'+c'+\mathrm{a}+\mathrm{a_1})(d'+e'+f'+\mathrm{a}+\mathrm{a_1})$, attending to the values of E' and F', is

$$=b'c'+d'e'+d'f'+e'f'-\mathrm{E}'-\mathrm{F}'(\mathrm{a}+\mathrm{a_1})-\mathrm{a}^2-2\mathrm{aa_1}-\mathrm{a_1}^2;$$

hence the whole coefficient of $(\partial\varpi)^2$ is

$$=\mathfrak{C}'+b'c'+d'e'+d'f'+e'f'+\mathrm{F}'(\mathrm{a}+\mathrm{a_1})-2(\mathrm{a}^2+\mathrm{aa_1}+\mathrm{a_1}^2),$$

or substituting for b', c', d', e', f' their values, this is

$$=\mathfrak{C}'+4a^2-a(b+c+2d+2e+2f)+bc+de+df+ef+(\mathrm{F}-6a)(\mathrm{a}+\mathrm{a_1})-2(\mathrm{a}^2+\mathrm{aa_1}+\mathrm{a_1}^2).$$

Proceeding next to reduce the coefficient of $2\,\partial\varpi\,\partial u$, observing as before that $\theta = \mathrm{a_1}-\mathrm{a}$, we have

$$\frac{2\mathrm{a_1}U - 2\mathrm{a}U_1}{\theta} = 2\mathrm{B}' - 2\mathrm{D}'\mathrm{aa_1} - 2\mathrm{E}'\mathrm{aa_1}(\mathrm{a}+\mathrm{a_1}) - 2\mathrm{F}'\mathrm{aa_1}(\mathrm{a}^2+\mathrm{aa_1}+\mathrm{a_1}^2) - 2\mathrm{aa_1}(\mathrm{a}^3+\mathrm{a}^2\mathrm{a_1}+\mathrm{aa_1}^2+\mathrm{a_1}^3),$$

also

$$-(a_1U' + U) - (aU_1' + U_1) =$$
$$-2B' + D'(-a^2 + 4aa_1 - a_1^2) + E'(-a^3 + 3a^2a_1 + 3aa_1^2 - a_1^3) + F'(-a^4 + 4a^3a_1 + 4aa_1^3 - a_1^4)$$
$$-a^5 + 5a^4a_1 + 5aa_1^4 - a_1^5;$$

adding these two expressions, the right-hand side divides by $(a_1 - a)^2$, that is, by θ^2, and the resulting value is

$$= -\ D' - E'(a + a_1) - F'(a^2 + a_1^2) \qquad -a^3 + a^2a_1 + aa_1^2 - a_1^3.$$

To this is to be added

$$+2D' + E'(a + a_1) + F'(a^2 + aa_1 + a_1^2) + a^3 + a^2a_1 + aa_1^2 + a_1^3;$$

we thus see that the whole coefficient of $2\partial u\,\partial\varpi$ is

$$= D' + \qquad F'aa_1 + 2(a^2a_1 + aa_1^2),$$

or say it is

$$= D' + (F - 6a)\,aa_1 + 2(a^2a_1 + aa_1^2).$$

Lastly, for the coefficient of $(\partial u)^2$; we have

$$\frac{2a_1^2U - 2a^2U_1}{\theta} = 2B'(a + a_1) + 2C'aa_1 - 2E'a^2a_1^2 - 2F'a^2a_1^2(a + a_1) - 2a^2a_1^2(a^2 + aa_1 + a_1^2),$$

and also

$$-a_1^2U' + (a - 2a_1)\,U - a^2U_1' + (a_1 - 2a)\,U =$$
$$-B'(a + a_1) + C'(2a^2 - 4aa_1 + 2a_1^2) + D'(a^3 + a_1^3) + E'(a^4 - 2a^3a_1 + 6a^2a_1^2 - 2aa_1^3 + a_1^4)$$
$$+F'(a^5 - 2a^4a_1 + 4a^3a_1^2 + 4a^2a_1^3 - 2aa_1^4 + a_1^5) + (a^6 - 2a^5a_1 + 5a^4a_1^2 + 5a^2a_1^4 - 2aa_1^5 + a_1^6),$$

whence the sum of these two expressions is

$$= B'(a + a_1) + C'(2a^2 - 2aa_1 + 2a_1^2) + D'(a^3 + a_1^3) + E'(a^4 - 2a^3a_1 + 4a^2a_1^2 - 2aa_1^3 + a_1^4)$$
$$+F'(a^5 - 2a^4a_1 + 2a^3a_1^2 + 2a^2a_1^3 - 2aa_1^4 + a_1^5) + a^6 - 2a^5a_1 + 3a^4a_1^2 - 2a^3a_1^3 + 3a^2a_1^4 - 2aa_1^5 + a_1^6.$$

We must to this add the term $-(abcd_1e_1f_1 + a_1b_1c_1def)$, that is,

$$-a(b' + a.c' + a.d' + a_1.e' + a_1.f' + a_1) - a_1(b' + a_1.c' + a_1.d' + a.e' + a.f' + a).$$

Putting for the moment

$$d' + a.e' + a.f' + a = \tau' + \sigma'a + \rho'a^2 + a^3,$$

that is,

$$\tau' = d'e'f', \quad \sigma' = d'e' + d'f' + e'f', \quad \rho' = d' + e' + f',$$

the term is

$$-b'c'\tau'(a + a_1) - (b' + c')\,\tau'(a^2 + a_1^2) - \tau'(a^3 + a_1^3) - (b'c' + \sigma')(a^3a_1 + aa_1^3)$$
$$-(b' + c' + \rho')(a^3a_1^2 + a^2a_1^3) - 2a^3a_1^3$$
$$-2b'c'\sigma'aa_1 - [(b' + c')\,\sigma' + b'c'\rho'](a^2a_1 + aa_1^2) - 2(b' + c')\,\rho'a^2a_1^2.$$

Adding it to the preceding expression, the sum is

$$
\begin{aligned}
= (\mathrm{B}' - b'c'\tau')(\mathrm{a}+\mathrm{a}_1) &+ \{2\mathrm{C}' - (b'+c')\tau'\}(\mathrm{a}^2+\mathrm{a}_1^2) + (\mathrm{D}'-\tau')(\mathrm{a}^3+\mathrm{a}_1^3) + \mathrm{E}'(\mathrm{a}^4+\mathrm{a}_1^4) \\
&+ \{-2\mathrm{C}' - 2b'c'\sigma'\}\,\mathrm{aa}_1 - \{(b'+c')\sigma' + b'c'\rho'\}(\mathrm{a}^2\mathrm{a}_1 + \mathrm{aa}_1^2) - (2\mathrm{E}' + b'c' + \sigma')(\mathrm{a}^3\mathrm{a}_1 + \mathrm{aa}_1^3) \\
&\qquad\qquad + \{4\mathrm{E}' - 2(b'+c')\rho'\}\,\mathrm{a}^2\mathrm{a}_1^2 \\
&+ \mathrm{F}'(\mathrm{a}^5+\mathrm{a}_1^5) \qquad\qquad + \mathrm{a}^6+\mathrm{a}_1^6 \\
&- 2\mathrm{F}'(\mathrm{a}^4\mathrm{a}_1 + \mathrm{aa}_1^4) \qquad\qquad - 2(\mathrm{a}^5\mathrm{a}_1 + \mathrm{aa}_1^5) \\
&+ (2\mathrm{F}' - b' - c' - \rho')(\mathrm{a}^3\mathrm{a}_1^2 + \mathrm{a}^2\mathrm{a}_1^3) + 3(\mathrm{a}^4\mathrm{a}_1^2 + \mathrm{a}^2\mathrm{a}_1^4) - 4\mathrm{a}^3\mathrm{a}_1^3.
\end{aligned}
$$

This is, in fact, divisible by $(\mathrm{a}_1 - \mathrm{a})^2$, that is, by θ^2: for we have between the symbols the relations

$$
\begin{aligned}
\mathrm{F}' &= b' + c' + \rho', \\
\mathrm{E}' &= b'c' + (b'+c')\rho' + \sigma', \\
\mathrm{D}' &= b'c'\rho' + (b'+c')\sigma' + \tau', \\
\mathrm{C}' &= b'c'\sigma' + (b'+c')\tau', \\
\mathrm{B}' &= b'c'\tau',
\end{aligned}
$$

and we thus reduce the expression to

$$
\begin{aligned}
&\{2\mathrm{C}' - (b'+c')\tau'\}(\mathrm{a}^2 - 2\mathrm{aa}_1 + \mathrm{a}_1^2) + (\mathrm{D}'-\tau')(\mathrm{a}^3 - \mathrm{a}^2\mathrm{a}_1 - \mathrm{aa}_1^2 + \mathrm{a}_1^3) \\
&\quad + \mathrm{E}'(\mathrm{a}^4 - 3\mathrm{a}^3\mathrm{a}_1 + 4\mathrm{a}^2\mathrm{a}_1^2 - 3\mathrm{aa}_1^3 + \mathrm{a}_1^4) + (b'+c')\rho'(\mathrm{a}^3\mathrm{a}_1 - 2\mathrm{a}^2\mathrm{a}_1^2 + \mathrm{aa}_1^3) \\
&\quad + \mathrm{F}'(\mathrm{a}^5 - 2\mathrm{a}^4\mathrm{a}_1 + \mathrm{a}^3\mathrm{a}_1^2 + \mathrm{a}^2\mathrm{a}_1^3 - 2\mathrm{aa}_1^4 + \mathrm{a}_1^5) \\
&\quad + (\mathrm{a}^6 - 2\mathrm{a}^5\mathrm{a}_1 + 3\mathrm{a}^4\mathrm{a}_1^2 - 4\mathrm{a}^3\mathrm{a}_1^3 + 3\mathrm{a}^2\mathrm{a}_1^4 - 2\mathrm{aa}_1^5 + \mathrm{a}_1^6),
\end{aligned}
$$

viz. effecting the division, the quotient is

$$= 2\mathrm{C}' - (b'+c')\tau' + (\mathrm{D}'-\tau')(\mathrm{a}+\mathrm{a}_1) + \mathrm{E}'(\mathrm{a}^2+\mathrm{a}_1^2) + \mathrm{F}'(\mathrm{a}^3+\mathrm{a}_1^3) + \mathrm{a}^4 + 2\mathrm{a}^2\mathrm{a}_1^2 + \mathrm{a}_1^4 - (b'c'+\sigma')\,\mathrm{aa}_1.$$

To this must be added

$$-2\mathrm{C}' \qquad - \mathrm{D}'(\mathrm{a}+\mathrm{a}_1) \qquad - \mathrm{E}'(\mathrm{a}^2+\mathrm{a}_1^2) - \mathrm{F}'(\mathrm{a}^3+\mathrm{a}_1^3) - (\mathrm{a}^4+\mathrm{a}_1^4);$$

and we thus obtain the coefficient of $(\partial u)^2$ in the form

$$\mathfrak{A}_0' - (b'+c')\tau' - \tau'(\mathrm{a}+\mathrm{a}_1) - (b'c'+\sigma')\,\mathrm{aa}_1 + 2\mathrm{a}^2\mathrm{a}_1^2,$$

viz. this is

$$
\begin{aligned}
&= \mathfrak{A}_0' + (b+c-2a)(a-d)(a-e)(a-f) + (a-d)(a-e)(a-f)(\mathrm{a}+\mathrm{a}_1) \\
&\quad + \{-(a-b)(a-c) - (a-d)(a-e) - (a-d)(a-f) - (a-e)(a-f)\}\,\mathrm{aa}_1 + 2\mathrm{a}^2\mathrm{a}_1^2,
\end{aligned}
$$

or finally it is

$$\begin{aligned}
= \mathfrak{A}_0' - 2a^4 &+ a^3(b+c+2d+2e+2f) + a^2\{-(b+c)(d+e+f) - 2(de+df+ef)\} \\
&+ a\{(b+c)(de+df+ef)+2def\} - (b+c)\,def \\
&+ \{a^3 - a^2(d+e+f) + a(de+df+ef) - def\}(\mathrm{a}+\mathrm{a}_1) \\
&+ \{-4a^2 + a(b+c+2d+2e+2f) - bc - de - df - ef\}\,\mathrm{aa}_1 \\
&+ 2\mathrm{a}^2\mathrm{a}_1^2.
\end{aligned}$$

It is to be observed that the investigation thus far has been entirely independent of the values of $\mathfrak{A}_0'$, $\mathfrak{B}'$, $\mathfrak{C}'$: these values are, in fact, such as to make the coefficients of $(\partial\varpi)^2$, $\partial\varpi\,\partial u$, $(\partial u)^2$ each equal to a constant, and it was really by such a condition that the value of $\mathfrak{C}(=\mathfrak{C}')$ was determined; but if we had thus also determined the values of $\mathfrak{A}_0'$ and $\mathfrak{B}'$, it would not have been apparent that the values of $\mathfrak{A}_0'$, $\mathfrak{B}'$ and $\mathfrak{C}'$ thus determined would be consistent with each other: the foregoing investigation of these values was therefore prefixed.

Completion of the reduction and final expression for ΔA.

But now substituting the values of $\mathfrak{A}_0'$, $\mathfrak{B}'$, $\mathfrak{C}'$, we find

$$\begin{aligned}
\text{coeff. of } (\partial\varpi)^2 &= ab+ac+bc+de+df+ef, \\
\text{,, ,, } 2\partial\varpi\,\partial u &= -a^2(a-b-c-d-e-f), \\
\text{,, ,, } (\partial u)^2 &= -2a^4 + 2a^3(b+c+d+e+f) \\
&\quad - a^2(bc+bd+be+bf+cd+ce+cf+de+df+ef) \\
&\quad - (bcde+bcdf+bcef+bdef+cdef),
\end{aligned}$$

viz. these coefficients belong to the portion which contains the factor aa_1 of the expression for $\frac{4}{\Omega^2}\Delta A$: the other portion was

$$(b'+c'+\mathrm{a}+\mathrm{a}_1)\,d'e'f'(\partial\varpi)^2 - (\sqrt{de})^2(\partial\varpi)^2 - 2\mathrm{B}'\,\partial u\,\partial\varpi - (\mathrm{a}+\mathrm{a}_1)\,\mathrm{B}'\,(\partial u)^2,$$

where

$$\mathrm{B}' = b'c'd'e'f', \quad b' = b - a, \text{ etc.}$$

We have thus the complete result, viz. this is

$$\begin{aligned}
\frac{4}{\Omega^2}\Delta A = \mathrm{aa}_1\,[&(ab+ac+bc+de+df+ef)(\partial\varpi)^2 \\
&- a^2(a-b-c-d-e-f)\,2\partial\varpi\,\partial u \\
&+ \left\{\begin{matrix} -2a^4 + 2a^3(b+c+d+e+f) \\ -a^2(bc+bd+be+bf+cd+ce+cf+de+df+ef) \\ -(bcde+bcdf+bcef+bdef+cdef) \end{matrix}\right\}(\partial u)^2] \\
&- (-2a+b+c+\mathrm{a}+\mathrm{a}_1)(a-d)(a-e)(a-f)(\partial\varpi)^2 - (\sqrt{de})^2(\partial\varpi)^2 \\
&+ (a-b)(a-c)(a-d)(a-e)(a-f)\,2\partial u\,\partial\varpi \\
&+ (\mathrm{a}+\mathrm{a}_1)(a-b)(a-c)(a-d)(a-e)(a-f)(\partial u)^2,
\end{aligned}$$

which is obviously a sum of squares.

As a partial verification, I remark that ΔA should be symmetrical in regard to the constants b, c, d, e, f; this is obviously the case as regards the terms in $\partial u \partial \varpi$ and $(\partial u)^2$, and it must also be so in regard to the term in $(\partial \varpi)^2$. The whole coefficient of $(\partial \varpi)^2$ is

$$= \mathrm{aa_1}(ab + ac + bc + de + df + ef)$$
$$-(-2a + b + c + \mathrm{a} + \mathrm{a_1})(a - d)(a - e)(a - f) - (\sqrt{de})^2,$$

and if we interchange for instance b and d, this coefficient becomes

$$= \mathrm{aa_1}(ad + ac + cd + be + bf + ef)$$
$$-(-2a + d + c + \mathrm{a} + \mathrm{a_1})(a - b)(a - e)(a - f) - (\sqrt{be})^2.$$

These two expressions must be equal; viz. we must have

$$(\sqrt{be})^2 - (\sqrt{de})^2 = -\mathrm{aa_1}(b - d)(a + c - e - f) + (a - e)(a - f)(b - d)(-a + c + \mathrm{a} + \mathrm{a_1}):$$

the left-hand side is

$$= \frac{1}{\theta^2}(\mathrm{bd_1} - \mathrm{b_1d})(\mathrm{efa_1c_1} - \mathrm{e_1f_1ac}),$$

and we have

$$\mathrm{bd_1} - \mathrm{b_1d} = (b - d)\theta;$$

hence, throwing out the factor $b - d$, the equation to be verified becomes

$$\frac{1}{\theta}(\mathrm{efa_1c_1} - \mathrm{e_1f_1ac}) = -\mathrm{aa_1}(a + c - e - f) + (a - e)(a - f)(-a + c + \mathrm{a} + \mathrm{a_1}).$$

Writing

$$\mathrm{e} = e' + \mathrm{a}, \text{ etc.}, \quad \theta = \mathrm{a_1} - \mathrm{a},$$

the left-hand side is

$$(\mathrm{a} + \mathrm{a_1})e'f' + \mathrm{aa_1}(e' + f') + c'e'f' - c'\mathrm{aa_1},$$

and the right-hand side is

$$-\mathrm{aa_1}(c' - e' - f') + e'f'(c' + \mathrm{a} + \mathrm{a_1}),$$

and these are equal.

There are of course, in all, six expressions such as ΔA, each of them being by what precedes a sum of squares. And there are besides ten expressions such as

$$\Delta AB, \quad = AB\partial^2 AB - (\partial AB)^2,$$

each of which should be a sum of squares: but I have not as yet effected the calculation of this expression ΔAB.

Cambridge, 7th December, 1877.

666.

SUR UN EXEMPLE DE RÉDUCTION D'INTÉGRALES ABÉLIENNES AUX FONCTIONS ELLIPTIQUES.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, t. LXXXV. (Juillet—Décembre, 1877), pp. 265—268; 373, 374; 426—429; 472—475.]

JE reprends l'investigation de M. Hermite par rapport aux intégrales réductibles

$$\int \frac{(1, x)\, dx}{\sqrt{x \,.\, 1 - x \,.\, 1 + ax \,.\, 1 + bx \,.\, 1 - abx}},$$

publiée sous ce même titre: "Sur un exemple, etc.", (*Annales de la Société scientifique de Bruxelles*, 1876).

Nous avons les constantes a, b et les variables x, y, u, v; et en posant

$$X = x \,.\, 1 - x \,.\, 1 + ax \,.\, 1 + bx \,.\, 1 - abx,$$
$$Y = y \,.\, 1 - y \,.\, 1 + ay \,.\, 1 + by \,.\, 1 - aby$$

(et $c = \sqrt{1 + a \,.\, 1 + b}$), M. Hermite a effectué l'intégration, par fonctions elliptiques, des équations différentielles

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = -\frac{2}{c}\,(du + dv),$$

$$\frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} = -\frac{2}{c\sqrt{ab}}\,(du - dv);$$

il a en effet trouvé les expressions, au moyen des fonctions elliptiques de u, v, des fonctions symétriques $x + y$, xy, et, de là, des cinq fonctions a, b, c, d, e dont je vais parler.

Au cas d'une fonction X du sixième ordre, on a dans la théorie seize fonctions, savoir six fonctions a, b, c, d, e, f, et dix fonctions abf . cde, ..., ou (avec une notation

plus simple) ab, ac, ad, ae, bc, bd, be, cd, ce, de: dans le cas d'une fonction du cinquième ordre, et ainsi dans le cas actuel, l'une des six fonctions, disons f, se réduit à l'unité, et l'on a les cinq fonctions a, b, c, d, e, et les dix fonctions ab, ..., de.

Présentement, ces fonctions sont

$$\mathrm{a} = xy,$$
$$\mathrm{b} = 1 - x \,.\, 1 - y,$$
$$\mathrm{c} = 1 + ax \,.\, 1 + ay,$$
$$\mathrm{d} = 1 + bx \,.\, 1 + by,$$
$$\mathrm{e} = 1 - abx \,.\, 1 - aby.$$

$$\mathrm{ab} = (\sqrt{x \,.\, 1 - x \,.\, 1 + ay \,.\, 1 + by \,.\, 1 - aby} - \sqrt{y \,.\, 1 - y \,.\, 1 + ax \,.\, 1 + bx \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{ac} = (\sqrt{x \,.\, 1 + ax \,.\, 1 - y \,.\, 1 + by \,.\, 1 - aby} - \sqrt{y \,.\, 1 + ay \,.\, 1 - x \,.\, 1 + bx \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{ad} = (\sqrt{x \,.\, 1 + bx \,.\, 1 - y \,.\, 1 + ay \,.\, 1 - aby} - \sqrt{y \,.\, 1 + by \,.\, 1 - x \,.\, 1 + ax \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{ae} = (\sqrt{x \,.\, 1 - abx \,.\, 1 - y \,.\, 1 + ay \,.\, 1 + by} - \sqrt{y \,.\, 1 - aby \,.\, 1 - x \,.\, 1 + ax \,.\, 1 + bx})^2 \div (x - y)^2,$$
$$\mathrm{bc} = (\sqrt{1 - x \,.\, 1 + ax \,.\, y \,.\, 1 + by \,.\, 1 - aby} - \sqrt{1 - y \,.\, 1 + ay \,.\, x \,.\, 1 + bx \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{bd} = (\sqrt{1 - x \,.\, 1 + bx \,.\, y \,.\, 1 + ay \,.\, 1 - aby} - \sqrt{1 - y \,.\, 1 + by \,.\, x \,.\, 1 + ax \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{be} = (\sqrt{1 - x \,.\, 1 - abx \,.\, y \,.\, 1 + ay \,.\, 1 + by} - \sqrt{1 - y \,.\, 1 - aby \,.\, x \,.\, 1 + ax \,.\, 1 - bx})^2 \div (x - y)^2,$$
$$\mathrm{cd} = (\sqrt{1 + ax \,.\, 1 + bx \,.\, y \,.\, 1 - y \,.\, 1 - aby} - \sqrt{1 + ay \,.\, 1 + by \,.\, x \,.\, 1 - x \,.\, 1 - abx})^2 \div (x - y)^2,$$
$$\mathrm{ce} = (\sqrt{1 + ax \,.\, 1 - abx \,.\, y \,.\, 1 - y \,.\, 1 + by} - \sqrt{1 + ay \,.\, 1 - aby \,.\, x \,.\, 1 - x \,.\, 1 + bx})^2 \div (x - y)^2,$$
$$\mathrm{de} = (\sqrt{1 + bx \,.\, 1 - abx \,.\, y \,.\, 1 - y \,.\, 1 + ay} - \sqrt{1 + by \,.\, 1 - aby \,.\, x \,.\, 1 - x \,.\, 1 + ax})^2 \div (x - y)^2,$$

et je remarque que la différence de deux quelconques des fonctions ab, ac, ... est une fonction rationnelle et entière de x, y. On a, par exemple:

$$\mathrm{ac} - \mathrm{ad} = a - b \,.\quad 1 - ab\,xy,$$
$$\mathrm{bc} - \mathrm{bd} = a - b \,.\, -1 + ab\,(x + y) - ab\,xy,$$
$$\mathrm{be} - \mathrm{cd} = 1 + a \,.\quad 1 + b - 1 + ab\,xy,$$
$$\mathrm{ce} - \mathrm{de} = a - b \,.\, -1 + (x + y) - ab\,xy.$$

En faisant, comme auparavant, $c = \sqrt{1 + a \,.\, 1 + b}$, et puis

$$ck = \sqrt{a} + \sqrt{b}, \qquad cl = \sqrt{a} - \sqrt{b},$$
$$ck' = 1 - \sqrt{ab}\,, \qquad cl' = 1 + \sqrt{ab}\,;$$
$$\sigma = \mathrm{sn}\,(u, k)\,, \qquad \sigma_1 = \mathrm{sn}\,(v, l),$$
$$\gamma = \mathrm{cn}\,(u, k)\,, \qquad \gamma_1 = \mathrm{cn}\,(v, l),$$
$$\delta = \mathrm{dn}\,(u, k)\,, \qquad \delta_1 = \mathrm{dn}\,(v, l),$$

(où j'écris sn, cn, dn pour sin am, cos am, Δ am), et, pour un moment,

$$\xi = \sqrt{ab}\,(\gamma\sigma_1\delta_1 + \gamma_1\sigma\delta), \quad \eta = c\,(-k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta), \quad \zeta = \gamma\sigma_1\delta_1 - \gamma_1\sigma\delta\,{}^*,$$

x, y sont donnés au moyen des fonctions elliptiques σ, γ, δ, σ_1, γ_1, δ_1 de u, v par les équations

$$x + y = \frac{\xi^2 + \zeta^2 - \eta^2}{\xi^2}, \quad xy = \frac{\zeta^2}{\xi^2},$$

ou, ce qui est la même chose, on a identiquement

$$\xi^2 z^2 - (\xi^2 + \zeta^2 - \eta^2)\,z + \zeta^2 = \xi^2 . z - x . z - y,$$

de manière que x, y sont les racines de l'équation quadrique

$$\xi^2 z^2 - (\xi^2 + \zeta^2 - \eta^2)\,z + \zeta^2 = 0.$$

On a l'identité (due à M. Hermite)

$$\begin{aligned}(Pz^3 + Qz^2 + Rz + S)^2 - c^6\delta^2\delta_1^2\,(\sigma^2 - \sigma_1^2)^2\,Z \\ = [\sigma^2(1+az)(1+bz) - c^2 z]\,[\sigma_1^2(1+az)(1+bz) - c^2 z] \times [\xi^2 z^2 - (\xi^2 + \zeta^2 - \eta^2)\,z + \zeta^2],\end{aligned}$$

ou

$$Z = z . 1 - z . 1 + az . 1 + bz . 1 - abz;$$

et alors les valeurs de P, Q, R, S sont

$$\begin{aligned}
P &= -ab\sqrt{ab}\,\sigma\sigma_1(\gamma\sigma_1\delta_1 + \gamma_1\sigma\delta),\\
Q &= \sqrt{ab}\,\sigma\sigma_1[-(a+b-\sqrt{ab})\,\gamma\sigma_1\delta_1 - (a+b+\sqrt{ab})\,\gamma_1\sigma\delta] + c^2\sqrt{ab}\,(\delta\sigma_1\gamma_1 + \delta_1\sigma\gamma),\\
R &= \sigma\sigma_1[(a+b-\sqrt{ab})\,\gamma\sigma_1\delta_1 - (a+b+\sqrt{ab})\,\gamma_1\sigma\delta] + c^2(\delta\sigma_1\gamma_1 - \delta_1\sigma\gamma),\\
S &= \sigma\sigma_1(\gamma\sigma_1\delta_1 - \gamma_1\sigma\delta),
\end{aligned}$$

lesquelles peuvent aussi s'écrire comme il suit:

$$\begin{aligned}
P &= -ab\sigma\sigma_1\xi,\\
Q &= -ab\sigma\sigma_1\zeta - c^2\sqrt{ab}\,\sigma\sigma_1(l^2\gamma\sigma_1\delta_1 + k^2\gamma_1\sigma\delta) + c^2\sqrt{ab}\,(\delta\sigma_1\gamma_1 + \delta_1\sigma\gamma),\\
R &= \sigma\sigma_1\xi + c^2\sigma\sigma_1(l^2\gamma\sigma_1\delta_1 - k^2\gamma_1\sigma\delta) + c^2(\delta\sigma_1\gamma_1 - \delta_1\sigma\gamma),\\
S &= \sigma\sigma_1\zeta,
\end{aligned}$$

et je remarque l'équation

$$\begin{aligned}P + Q + R + S &= c^3\gamma\gamma_1(-k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta)\\ &= c^3\gamma\gamma_1\eta.\end{aligned}$$

En écrivant successivement $z = x$, $z = y$, et en choisissant convenablement les signes des radicaux, on obtient

$$\begin{aligned}Px^3 + Qx^2 + Rx + S &= c\delta\delta_1(\sigma^2 - \sigma_1^2)\sqrt{X},\\ Py^3 + Qy^2 + Ry + S &= c\delta\delta_1(\sigma^2 - \sigma_1^2)\sqrt{Y};\end{aligned}$$

on conçoit sans peine que c'est à cause de ces expressions rationnelles des radicaux que l'intégration des équations différentielles réussit.

* En écrivant

$$\xi' = \gamma\sigma_1\delta_1 + \gamma_1\sigma\delta, \quad \eta' = -k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta, \quad \zeta' = \gamma\sigma_1\delta_1 - \gamma_1\sigma\delta,$$

on a

$$\xi = \sqrt{ab}\,\xi', \quad \eta = c\eta', \quad \zeta = \zeta';$$

je me sers, dans la suite, de ce symbole

$$\xi_1' = \gamma\sigma_1\delta_1 + \gamma_1\sigma\delta.$$

[Pp. 373—376*; 426—429.] Les valeurs de $x+y$, xy donnent sans beaucoup de peine celles de a, b, c, d, e; mais les réductions pour obtenir les valeurs des dix fonctions ab, ..., de sont très pénibles; je donne seulement les résultats. Ces valeurs sont

$$\sqrt{\mathrm{a}} = \frac{1}{ab\xi'} \,.\, \gamma\sigma_1\delta_1 \quad - \gamma_1\sigma\delta,$$

$$\sqrt{\mathrm{b}} = \frac{c}{\sqrt{ab}\,\xi'} \,.\, - k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta,$$

$$\sqrt{\mathrm{c}} = \frac{c}{\sqrt{b}\,\xi'} \,.\, l\delta\sigma_1\gamma_1 \quad - k\delta_1\sigma\gamma,$$

$$\sqrt{\mathrm{d}} = \frac{c}{\sqrt{a}\,\xi'} \,.\, l\delta\sigma_1\gamma_1 \quad + k\delta_1\sigma\gamma,$$

$$\sqrt{\mathrm{e}} = \frac{c}{\xi'} \,.\, k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta,$$

où

$$\xi' = \gamma\sigma_1\delta_1 + \gamma_1\sigma\delta\,;$$

et puis

$$\sqrt{\mathrm{ab}} = \frac{c}{\xi'} \,.\, \gamma\gamma_1\delta\delta_1 - k'l'\sigma\sigma_1,$$

$$\sqrt{\mathrm{ac}} = \frac{c}{\xi'(l\delta\sigma_1\gamma_1 - k\delta_1\sigma\gamma)} \,.\, k\,(l'^2 + l^2\gamma_1^4)\,\sigma\gamma\delta - l\,(k'^2 + k^2\gamma^4)\,\sigma_1\gamma_1\delta_1,$$

$$\sqrt{\mathrm{ad}} = \frac{c}{\xi'(l\delta\sigma_1\gamma_1 + k\delta_1\sigma\gamma)} \,.\, k\,(l'^2 + l^2\gamma_1^4)\,\sigma\gamma\delta + l\,(k'^2 + k^2\gamma^4)\,\sigma_1\gamma_1\delta_1,$$

$$\sqrt{\mathrm{ae}} = \frac{c}{\xi'(k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta)} \,.\, k'\,(l'^2 + l^2\gamma_1^4)\,\sigma\gamma\delta + l'\,(k'^2 + k^2\gamma^4)\,\sigma_1\gamma_1\delta_1,$$

$$\sqrt{\mathrm{bc}} = \frac{\frac{1}{2}c^2}{\xi'} \,.\, k'\delta_1^2 + l'\delta^2 - kl\,(k'\sigma^2\gamma_1^2 + l'\sigma_1^2\gamma^2),$$

$$\sqrt{\mathrm{bd}} = \frac{\frac{1}{2}c^2}{\xi'} \,.\, k'\delta_1^2 + l'\delta^2 + kl\,(k'\sigma^2\gamma_1^2 + l'\sigma_1^2\gamma^2),$$

$$\sqrt{\mathrm{be}} = \frac{c}{\xi'} \,.\, - \sigma\sigma_1\delta\delta_1 - \gamma\gamma_1,$$

$$\sqrt{\mathrm{cd}} = \frac{c}{\xi'} \,.\, - \sigma\sigma_1\delta\delta_1 + \gamma\gamma_1,$$

$$\sqrt{\mathrm{ce}} = \frac{c}{\xi'} \,.\, 1 - \frac{1+a}{c\sqrt{a}}\,k\sigma^2 - \frac{1+a}{c\sqrt{a}}\,l\sigma_1^2 + kl\sigma^2\sigma_1^2,$$

$$\sqrt{\mathrm{de}} = \frac{c}{\xi'} \,.\, 1 - \frac{1+b}{c\sqrt{b}}\,k\sigma^2 + \frac{1+b}{c\sqrt{b}}\,l\sigma_1^2 - kl\sigma^2\sigma_1^2.$$

* Voir la note, p. 426 du volume.—Dans la seconde Communication (p. 373), une erreur de composition a fait placer, à la suite de la treizième ligne de la page 374, deux pages et demie de texte qui ne devaient trouver place que dans la Communication suivante. Nous rétablissons intégralement cette seconde Communication: la troisième sera insérée dans le prochain numéro.

Les valeurs de a, b, c, d, e donnent

$$\sqrt{X}\sqrt{Y} = \sqrt{\mathrm{a}}\sqrt{\mathrm{b}}\sqrt{\mathrm{c}}\sqrt{\mathrm{d}}\sqrt{\mathrm{e}}$$

$$= \frac{c^4}{(ab)^2\,\xi'^5}(\gamma\sigma_1\delta_1 - \gamma_1\sigma\delta)(-k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta)$$

$$\times (l\delta\sigma_1\gamma_1 - k\delta_1\sigma\gamma)(l\delta\sigma_1\gamma_1 + k\delta_1\sigma\gamma)(k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta),$$

$$= \frac{c^4}{(ab)^2\,\xi'^6}(\gamma^2\sigma_1^2\delta_1^2 - \gamma_1^2\sigma^2\delta^2)(-k'^2\sigma^2\gamma_1^2\delta_1^2 + l'^2\sigma_1^2\gamma^2\delta^2)(l^2\delta^2\sigma_1^2\gamma_1^2 - k^2\delta_1^2\sigma^2\gamma^2);$$

j'ai vérifié que le signe s'accorde avec celui de la valeur obtenue au moyen des expressions rationnelles de $\sqrt{X}$, $\sqrt{Y}$.

On vérifie en partie les valeurs des fonctions $\sqrt{\mathrm{ab}}$, $\sqrt{\mathrm{ac}}, \ldots$, en considérant les différences des carrés de ces fonctions; mais ce calcul n'est pas toujours facile. Par exemple, nous avons

$$\mathrm{ac} - \mathrm{ad} = (a-b)(1-ab\,xy)$$

$$= \frac{c^3kl}{\xi'^2}(\xi'^2 - \zeta^2)$$

$$= \frac{4c^3kl}{\xi'^5}\sigma\sigma_1\gamma\gamma_1\delta\delta_1;$$

et cette valeur doit ainsi être égale à

$$\frac{c^2}{\xi'^2}\left\{\frac{1}{(l\delta\sigma_1\gamma_1 - k\delta_1\sigma\gamma)^2}[k(l'^2 + l^2\gamma_1^4)\sigma\gamma\delta - l(k'^2 + k^2\gamma^4)\sigma_1\gamma_1\delta_1]^2\right.$$

$$\left. - \frac{1}{(l\delta\sigma_1\gamma_1 + k\delta_1\sigma\gamma)^2}[k(l'^2 + l^2\gamma_1^4)\sigma\gamma\delta + l(k'^2 + k^2\gamma^4)\sigma_1\gamma_1\delta_1]^2\right\}.$$

Pour voir cela, j'écris pour le moment

$$A = k(l'^2 + l^2\gamma_1^4)\sigma\gamma\delta, \qquad B = l(k'^2 + k^2\gamma^4)\sigma_1\gamma_1\delta_1,$$

$$\alpha = l\delta\sigma_1\gamma_1, \qquad \beta = k\delta_1\sigma\gamma;$$

l'équation devient ainsi

$$4kl\sigma\sigma_1\gamma\gamma_1\delta\delta_1(\alpha^2 - \beta^2)^2 = (\alpha+\beta)^2(A-B)^2 - (\alpha-\beta)^2(A+B)^2,$$

$$= 4[\alpha\beta(A^2 + B^2) - AB(\alpha^2 + \beta^2)];$$

or, en remarquant que AB et $\alpha\beta$ contiennent chacun le facteur $kl\,\sigma\sigma_1\gamma\gamma_1\delta\delta_1$, cette équation devient

$$(\alpha^2 - \beta^2)^2 = k^2(l'^2 + l^2\gamma_1^4)^2\sigma^2\gamma^2\delta^2 + l^2(k'^2 + k^2\gamma^4)^2\sigma_1^2\gamma_1^2\delta_1^2$$

$$- (l'^2 + l^2\gamma_1^4)(k'^2 + k^2\gamma_1^4)(l^2\delta^2\sigma_1^2\gamma_1^2 + k^2\delta_1^2\sigma^2\gamma^2),$$

c'est-à-dire

$$(\alpha^2 - \beta^2)^2 = [k^2(l'^2 + l^2\gamma_1^4)\sigma^2\gamma^2 - l^2(k'^2 + k^2\gamma^4)\sigma_1^2\gamma_1^2][(l'^2 + l^2\gamma_1^4)\delta^2 - (k'^2 + k^2\gamma^4)\delta_1^2];$$

or les deux facteurs à droite se réduisant l'un et l'autre à

$$k^2\sigma^2\gamma^2\delta_1^2 - l^2\sigma_1^2\gamma_1^2\delta^2,$$

c'est-à-dire à $(\alpha^2-\beta^2)$, la vérification est ainsi complétée.

La différence $\mathrm{be}-\mathrm{cd}$ donne un exemple beaucoup plus simple; on a

$$\begin{aligned}\mathrm{be}-\mathrm{cd} &= 1 + a\,.\,1 + b\,(-1 + ab\,xy)\\ &= \frac{c^2}{\xi'^2}(-\xi'^2 + \zeta^2)\\ &= \frac{c^2}{\xi'^2}(-4\sigma\sigma_1\gamma\gamma_1\delta\delta_1);\end{aligned}$$

l'équation à vérifier est ainsi

$$-4\sigma\sigma_1\gamma\gamma_1\delta\delta_1 = (-\sigma\sigma_1\delta\delta_1 - \gamma\gamma_1)^2 - (-\sigma\sigma_1\delta\delta_1 + \gamma\gamma_1)^2,$$

ce qui est juste.

[Pp. 472—475.] Je donne quelques autres formules dont je me suis servi dans le cours de cette recherche. Partant des expressions de ξ, η, ζ, on a

$$\begin{aligned}d\xi = \lambda du + \lambda_1 dv = \sqrt{ab}\,\{ &\ [-\sigma\delta\sigma_1\delta_1 + \gamma\gamma_1(1-2k^2\sigma^2)]\,du\\ &+ [\gamma\gamma_1(1-2l^2\sigma_1^2) - \sigma\sigma_1\delta\delta_1\ ]\,dv\},\\ d\eta = \mu du + \mu_1 dv = c\,\{ &\ [-k'\gamma\delta\gamma_1\delta_1 + l'\sigma\sigma_1(-1-k^2+2k^2\sigma^2)]\,du\\ &+ [k'\sigma\sigma_1(1+l^2-2l^2\sigma_1^2) + l'\gamma\delta\gamma_1\delta_1\ ]\,dv\},\\ d\zeta = \nu du + \nu_1 dv = \ \{ &\ [-\sigma\delta\sigma_1\delta_1 - \gamma\gamma_1(1-2k^2\sigma^2)]\,du\\ &+ [\gamma\gamma_1(1-2l^2\sigma_1^2) + \sigma\delta\sigma_1\delta_1\ ]\,dv\};\end{aligned}$$

en prenant pour A, B, C des fonctions telles que

$$A\,d\xi + B\,d\eta + C\,d\zeta = du + dv,$$

on a

$$\begin{aligned}A\lambda + B\mu + C\nu &= 1,\\ A\lambda_1 + B\mu_1 + C\nu_1 &= 1.\end{aligned}$$

Je pose aussi

$$A\xi + B\eta + C\zeta = 0,$$

et au moyen de ces équations, j'obtiens pour A, B, C les valeurs

$$\begin{aligned}A\nabla &= \frac{1}{2\sqrt{ab}}(-U+W),\\ B\nabla &= \frac{1}{2c}V,\\ C\nabla &= \tfrac{1}{2}(-U-W),\end{aligned}$$

où

$$U = l'\delta^2(\delta\sigma_1\gamma_1 + \delta_1\sigma\gamma) + k'\sigma^2\gamma_1^2(l^2\delta\sigma_1\gamma_1 + k^2\delta_1\sigma\gamma),$$
$$W = k'\delta_1^2(\delta\sigma_1\gamma_1 + \delta_1\sigma\gamma) + l'\sigma_1^2\gamma^2(l^2\delta\sigma_1\gamma_1 + k^2\delta_1\sigma\gamma),$$
$$V = 2\,[(l'^2 + l^2\gamma_1^4)\,\sigma\gamma\delta + (k'^2 + k^2\gamma^4)\,\sigma_1\gamma_1\delta_1],$$
$$\nabla = (k'\sigma\gamma_1\delta_1 + l'\sigma_1\gamma\delta)\,(l\delta\sigma_1\gamma_1 - k\delta_1\sigma\gamma)\,(l\delta\sigma_1\gamma_1 + k\delta_1\sigma\gamma);$$

et de là aussi

$$-U + W = -\frac{2\sqrt{ab}}{c}(\delta\delta_1\gamma\gamma_1 - k'l'\sigma\sigma_1)(\gamma\sigma_1\delta_1 + \gamma_1\sigma\delta),$$

$$U + W = \frac{2}{c}(\{1 + l^2\sigma^2\sigma_1^2 - \sqrt{ab}\,[(1 + k'l')\,\sigma^2 - l^2\sigma_1^2]\}\,\delta\sigma_1\gamma_1$$
$$+ \{1 - k^2\sigma^2\sigma_1^2 + \sqrt{ab}\,[(1 + k'l')\,\sigma_1^2 - k^2\sigma^2]\}\,\delta_1\sigma\gamma).$$

En admettant l'équation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = -\frac{2}{c}(du + dv),$$

on obtient sans peine les relations

$$A\xi = \frac{c}{x-y}\left(\frac{x^2 - x}{\sqrt{X}} - \frac{y^2 - y}{\sqrt{Y}}\right),$$
$$B\frac{\xi^2}{\eta} = \frac{c}{x-y}\left(\frac{x}{\sqrt{X}} - \frac{y}{\sqrt{Y}}\right),$$
$$C\frac{\xi^2}{\zeta} = \frac{c}{x-y}\left(\frac{-x+1}{\sqrt{X}} - \frac{-y+1}{\sqrt{Y}}\right),$$

et, en multipliant par

$$c^2\delta\delta_1(\sigma^2 - \sigma_1^2)\sqrt{XY}, \quad = \frac{c^4ab\eta\zeta\delta\delta_1(\sigma^2 - \sigma_1^2)}{\xi^5}\nabla,$$

et dans les seconds membres, au lieu de

$$c^2\delta\delta_1(\sigma^2 - \sigma_1^2)\sqrt{X}, \quad c^2\delta\delta_1(\sigma^2 - \sigma_1^2)\sqrt{Y},$$

substituant les valeurs

$$Px^3 + Qx^2 + Rx + S, \quad Py^3 + Qy^2 + Ry + S,$$

on obtient, après quelques réductions simples, les équations

$$c^4ab\delta\delta_1(\sigma^2 - \sigma_1^2)\,\nabla A = ab\sigma\sigma_1\xi\eta\zeta - \sigma\sigma_1\xi^2\eta + c^2\gamma\gamma_1\xi^3\zeta,$$
$$\text{,,} \qquad \nabla B = ab\sigma\sigma_1\zeta(\xi^2 + \zeta^2 - \eta^2) + \sigma\sigma_1\xi^3 - Q\xi\zeta,$$
$$\text{,,} \qquad \nabla C = ab\sigma\sigma_1\eta(-2\xi^2 - \zeta^2 + \eta^2) + Q\xi\eta - c^2\gamma\gamma_1\xi^3,$$

lesquelles satisfont, comme cela doit être, à la condition

$$A\xi + B\eta + C\zeta = 0.$$

Réciproquement, en vérifiant ces identités, ce qui est assez pénible, on obtient une démonstration de l'équation différentielle

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = -\frac{2}{c}(du + dv).$$

En écrivant, pour plus de simplicité,

$$A\nabla = -\frac{1}{c}\mathfrak{A}'\xi', \quad B\nabla = \frac{1}{c}\mathfrak{B}, \quad C\nabla = -\frac{1}{c}\mathfrak{C},$$

les valeurs de $\mathfrak{A}'$, $\mathfrak{B}$, $\mathfrak{C}$ sont

$$\begin{aligned}
\mathfrak{A}' &= \gamma\gamma_1\delta\delta_1 - k'l'\sigma\sigma_1,\\
\mathfrak{B} &= (l'^2 + l^2\gamma^4)\,\sigma\gamma\delta + (k'^2 + k^2\gamma^4)\,\sigma_1\gamma_1\delta_1,\\
\mathfrak{C} &= \{1 - l^2\sigma^2\sigma_1^2 - \sqrt{ab}\,[(1 + k'l')\,\sigma^2 - l^3\sigma_1^3]\}\,\delta\sigma_1\gamma_1\\
&\quad + \{1 - k^2\sigma^2\sigma_1^2 + \sqrt{ab}\,[(1 + k'l')\,\sigma_1^2 - k^3\sigma^3]\}\,\delta_1\sigma\gamma\,;
\end{aligned}$$

et des trois équations pour $A\xi$, $B\frac{\xi^2}{\eta}$, $C\frac{\xi^2}{\zeta}$, on déduit

$$-\frac{\mathfrak{A}'}{\sqrt{ab}} + B\eta\zeta = \Omega\left(\frac{x^2}{\sqrt{X}} - \frac{y^2}{\sqrt{Y}}\right),$$

$$B\zeta = \Omega\left(\frac{x}{\sqrt{X}} - \frac{y}{\sqrt{Y}}\right),$$

$$-\mathfrak{C}\eta + B\zeta = \Omega\left(\frac{1}{\sqrt{X}} - \frac{1}{\sqrt{Y}}\right),$$

où

$$\Omega = \frac{c^2\nabla\eta\zeta}{ab\xi'(x-y)}\,;$$

et c'est au moyen de ces équations que j'ai trouvé les valeurs ci-dessus données pour $\sqrt{\mathrm{ab}}$, $\sqrt{\mathrm{ac}}$, ... ; on a, par exemple,

$$\sqrt{\mathrm{ab}} = \frac{\sqrt{X}\sqrt{Y}}{\sqrt{\mathrm{a}}\sqrt{\mathrm{b}}\,(x-y)}\left(\frac{x-x^2}{\sqrt{X}} - \frac{y-y^2}{\sqrt{Y}}\right) = \frac{\sqrt{X}\sqrt{Y}}{\sqrt{\mathrm{a}}\sqrt{\mathrm{b}}\,(x-y)\,\Omega}\,\frac{\mathfrak{A}'\eta\zeta}{\sqrt{ab}},$$

ce qui se réduit sans peine à $\sqrt{\mathrm{ab}} = \frac{c}{\xi'}\mathfrak{A}'$. Les dix fonctions contiennent de cette manière les facteurs suivants:

$$\sqrt{\mathrm{ab}}, \quad \mathfrak{A}',$$

$$\sqrt{\mathrm{ac}}, \quad (1+a)\,\mathfrak{B} - \sqrt{\frac{a}{b}}\,\mathfrak{A}'\eta,$$

$$\sqrt{\mathrm{ad}}, \quad (1+b)\,\mathfrak{B} - \sqrt{\frac{a}{b}}\,\mathfrak{A}'\eta,$$

$$\sqrt{\mathrm{ae}}, \quad (1-ab)\,\mathfrak{B} + \sqrt{ab}\,\mathfrak{A}'\eta,$$

$$\sqrt{\mathrm{bc}}, \quad -\mathfrak{C} + \sqrt{\frac{a}{b}}\,\mathfrak{A}'\zeta,$$

$$\sqrt{\mathrm{bd}},\quad -\mathfrak{C}+\sqrt{\frac{b}{a}}\,\mathfrak{A}'\zeta',$$

$$\sqrt{\mathrm{be}},\quad -\mathfrak{C}-\sqrt{ab}\,\mathfrak{A}'\zeta,$$

$$\sqrt{\mathrm{cd}},\quad \frac{1}{c}\left(-\sqrt{ab}\,\eta\zeta\mathfrak{A}'+c^2\mathfrak{B}\zeta-\mathfrak{C}\eta\right),$$

$$\sqrt{\mathrm{ce}},\quad \frac{1}{c}\left[a\sqrt{ab}\,\eta+\mathfrak{A}'\zeta(1+a)(1-ab)\,\mathfrak{B}\zeta-\mathfrak{C}\eta\right],$$

$$\sqrt{\mathrm{de}},\quad \frac{1}{c}\left[b\sqrt{ab}\,\eta\zeta\mathfrak{A}'+(1+b)(1-ab)\,\mathfrak{B}\zeta-\mathfrak{C}\eta\right]:$$

mais il y a des dénominateurs variables qui contiennent des facteurs dont quelques-uns divisent les numérateurs, et la réduction aux formes ci-dessus données m'a coûté assez de peine.

667.

ON THE BICIRCULAR QUARTIC: ADDITION TO PROFESSOR CASEY'S MEMOIR "ON A NEW FORM OF TANGENTIAL EQUATION."

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXVII. Part II. (1877), pp. 441—460. Received January 24,—Read February 22, 1877.]

PROFESSOR CASEY communicated to me the MS. of the Memoir referred to, and he has permitted me to make to it the present Addition, containing further developments on the theory of the bicircular quartic.

Starting from his theory of the fourfold generation of the curve, Prof. Casey shows that there exist series of inscribed quadrilaterals $ABCD$ whereof the sides AB, BC, CD, DA pass through the centres of the four circles of inversion respectively; or (as it is convenient to express it) the pairs of points (A, B), (B, C), (C, D), (D, A) belong to the four modes of generation respectively, and may be regarded as depending upon certain parameters (his θ, θ', θ'', θ''', or say) ω_1, ω_2, ω_3, ω respectively, any three of these being in fact functions of the fourth. Considering a given quadrilateral $ABCD$, and giving to it an infinitesimal variation, we have four infinitesimal arcs AA', BB', CC', DD'; these are differential expressions, AA' and BB' being of the form $M_1 d\omega_1$, BB' and CC' of the form $M_2 d\omega_2$, CC' and DD' of the form $M_3 d\omega_3$, DD' and AA' of the form $M d\omega$; or, what is the same thing, AA' is expressible in the two forms $M d\omega$ and $M_1 d\omega_1$, BB' in the two forms $M_1 d\omega_1$ and $M_2 d\omega_2$, &c., the identity of the two expressions for the same arc of course depending on the relation between the two parameters. But any such monomial expression $M d\omega$ of an arc AA' would be of a complicated form, not obviously reducible to elliptic functions; Casey does not obtain these monomial expressions at all, but he finds geometrically monomial expressions for the differences and sum $BB' - AA'$, $CC' - BB'$, $DD' + CC'$, $DD' - AA'$ (they cannot be all of them differences), and thence a quadrinomial expression $AA' = N_1 d\omega_1 + N_2 d\omega_2 + N_3 d\omega_3 + N d\omega$ (his $ds' = \rho\, d\theta + \rho'\, d\theta' + \rho''\, d\theta'' + \rho'''\, d\theta'''$); and that without any explicit consideration of the relations which connect the parameters.

I propose to complete the analytical theory by establishing the monomial equations $AA' = Md\omega = M_1 d\omega_1$, &c., and the relations between the parameters ω, ω_1, ω_2, ω_3 which belong to an inscribed quadrilateral $ABCD$, so as to show what the process really is by which we pass from the monomial form to a quadrinomial form

$$AA' \text{ (or } dS) = Nd\omega + N_1 d\omega_1 + N_2 d\omega_2 + N_3 d\omega_3,$$

wherein each term is separately expressible as the differential of an elliptic integral; and further to develop the theory of the transformation to elliptic integrals. We require to establish for these purposes the fundamental formulæ in the theory of the bicircular quartic.

I remark that in the various formulæ f, g, θ, θ_1, θ_2, θ_3 are constants which enter only in the combinations $f+\theta$, $f-g$, $\theta_1-\theta$, $\theta_2-\theta$, $\theta_3-\theta$: that X, Y are taken as current coordinates, and these letters, or the same letters with suffixes, are taken as coordinates of a point or points on the bicircular quartic: and that the letters (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) are used throughout as variable parameters, viz. we have

$$\begin{aligned}(f+\theta)\,x^2 + (g+\theta)\,y^2 &= 1,\\ (f+\theta_1)\,x_1^2 + (g+\theta_1)\,y_1^2 &= 1,\\ (f+\theta_2)\,x_2^2 + (g+\theta_2)\,y_2^2 &= 1,\\ (f+\theta_3)\,x_3^2 + (g+\theta_3)\,y_3^2 &= 1;\end{aligned}$$

so that $x, y = \dfrac{\cos\omega}{\sqrt{f+\theta}}, \dfrac{\sin\omega}{\sqrt{g+\theta}}$, are functions of a single parameter ω, and similarly (x_1, y_1), (x_2, y_2), (x_3, y_3) are functions of the parameters ω_1, ω_2, ω_3 respectively. We sometimes use these or similar expressions of (x, y), &c., as trigonometrical functions of a single parameter; but we more frequently retain the pair of quantities, considered as connected by an equation as above and so as equivalent to a single variable parameter.

Formulæ for the fourfold generation of the Bicircular Quartic. Art. Nos. 1 to 5.

1. We have four systems of a dirigent conic and circle of inversion, each giving rise to the same bicircular quartic: viz. the bicircular quartic is the envelope of a generating circle, having its centre on a dirigent conic, and cutting at right angles the corresponding circle of inversion; or, what is the same thing, it is the locus of the extremities of a chord of the generating circle, which chord passes through the centre of the circle of inversion, and cuts at right angles the tangent (at the centre of the generating circle) to the dirigent conic; the two extremities of the chord are thus inverse points in regard to the circle of inversion. The four systems are represented by letters without suffixes, or with the suffixes 1, 2, 3 respectively; and we say that the system, or mode of generation, is 0, 1, 2, or 3 accordingly.

2. The dirigent conics are confocal, and their squared semiaxes may therefore be represented by $f+\theta$, $g+\theta$: $f+\theta_1$, $g+\theta_1$: $f+\theta_2$, $g+\theta_2$: $f+\theta_3$, $g+\theta_3$, (which are, in

fact, functions of the five quantities $f+\theta$, $f-g$, $\theta_1-\theta$, $\theta_2-\theta$, $\theta_3-\theta$); and we can in terms of these data express the equations as well of the dirigent conics as of the circles of inversion; viz. taking X, Y as current coordinates, the equations are

$$\frac{X^2}{f+\theta}+\frac{Y^2}{g+\theta}=1,\ (X-\alpha)^2+(Y-\beta)^2-\gamma^2=0,\ \text{or}\ X^2+Y^2-2\alpha X-2\beta Y+k=0,$$

$$\frac{X^2}{f+\theta_1}+\frac{Y^2}{g+\theta_1}=1,\ (X-\alpha_1)^2+(Y-\beta_1)^2-\gamma_1^2=0,\ \text{or}\ X^2+Y^2-2\alpha_1 X-2\beta_1 Y+k_1=0,$$

$$\frac{X^2}{f+\theta_2}+\frac{Y^2}{g+\theta_2}=1,\ (X-\alpha_2)^2+(Y-\beta_2)^2-\gamma_2^2=0,\ \text{or}\ X^2+Y^2-2\alpha_2 X-2\beta_2 Y+k_2=0,$$

$$\frac{X^2}{f+\theta_3}+\frac{Y^2}{g+\theta_3}=1,\ (X-\alpha_3)^2+(Y-\beta_3)^2-\gamma_3^2=0,\ \text{or}\ X^2+Y^2-2\alpha_3 X-2\beta_3 Y+k_3=0,$$

where

$$\sqrt{\frac{f+\theta\,.\,f+\theta_1\,.\,f+\theta_2\,.\,f+\theta_3}{f-g}}=(f+\theta)\,\alpha=(f+\theta_1)\,\alpha_1=(f+\theta_2)\,\alpha_2=(f+\theta_3)\,\alpha_3,$$

$$\sqrt{\frac{g+\theta\,.\,g+\theta_1\,.\,g+\theta_2\,.\,g+\theta_3}{g-f}}=(g+\theta)\,\beta=(g+\theta_1)\,\beta_1=(g+\theta_2)\,\beta_2=(g+\theta_3)\,\beta_3,$$

$$f+\theta\ .\ g+\theta\ .\ \gamma^2=\theta-\theta_1\,.\,\theta\ -\theta_2\,.\,\theta\ -\theta_3,$$

$$f+\theta_1\,.\,g+\theta_1\,.\,\gamma_1^2=\theta_1-\theta\ .\ \theta_1-\theta_2\,.\,\theta_1-\theta_3,$$

$$f+\theta_2\,.\,g+\theta_2\,.\,\gamma_2^2=\theta_2-\theta\ .\ \theta_2-\theta_1\,.\,\theta_2-\theta_3,$$

$$f+\theta_3\,.\,g+\theta_3\,.\,\gamma_3^2=\theta_3-\theta\ .\ \theta_3-\theta_1\,.\,\theta_3-\theta_2,$$

$$f+g+\theta+\theta_1+\theta_2+\theta_3=k+2\theta=k_1+2\theta_1=k_2+2\theta_2=k_3+2\theta_3.$$

3. The geometrical relations between the dirigent conics and circles of inversion are all deducible from the foregoing formulæ; in particular, the conics are confocal, and as such intersect each two of them at right angles; the circles intersect each two of them at right angles. Considering a dirigent conic and the corresponding circle of inversion, the centres of the remaining three circles are conjugate points in regard as well to the first-mentioned conic, as to the first-mentioned circle; or, what is the same thing, they are the centres of the quadrangle formed by the intersections of the conic and circle.

4. The centre of the conics and the centres of the four circles lie on a rectangular hyperbola, having its asymptotes parallel to the axes of the conics. Given the centres of three of the circles (this determines the centre of the fourth circle) and also the centre of the conic, these four points determine a rectangular hyperbola (which passes also through the centre of the fourth circle); and the axes of the conics are then the lines through the centre, parallel to the asymptotes of the hyperbola.

5. The equation of the bicircular quartic may be expressed in the four forms

$$(X^2+Y^2-k)^2-4[(f+\theta)(X-\alpha)^2+(g+\theta)(Y-\beta)^2]=0,$$
$$(X^2+Y^2-k_1)^2-4[(f+\theta_1)(X-\alpha_1)^2+(g+\theta_1)(Y-\beta_1)^2]=0,$$
$$(X^2+Y^2-k_2)^2-4[(f+\theta_2)(X-\alpha_2)^2+(g+\theta_2)(Y-\beta_2)^2]=0,$$
$$(X^2+Y^2-k_3)^2-4[(f+\theta_3)(X-\alpha_3)^2+(g+\theta_3)(Y-\beta_3)^2]=0,$$

the equivalence of which is easily verified by means of the foregoing relations.

Determination as to Reality. Art. Nos. 6 and 7.

6. To fix the ideas, suppose that $f-g$ is positive; then in order that the centres of the four circles of inversion may be real, we must have $f+\theta.f+\theta_1.f+\theta_2.f+\theta_3$ positive, but $g+\theta.g+\theta_1.g+\theta_2.g+\theta_3$ negative; and this will be the case if $f+\theta$, $f+\theta_1$, $f+\theta_2$, $f+\theta_3$ are all positive, but $g+\theta$, $g+\theta_1$, $g+\theta_2$, $g+\theta_3$ one of them negative, and the other three positive. In reference to a figure which I constructed, I found it convenient to take θ_3, θ_1, θ_0, θ_2 to be in order of increasing magnitude: this being so, we have $f+\theta_3$ positive, $g+\theta_3$ negative; and the other like quantities $f+\theta_1$, $f+\theta_0$, $f+\theta_2$, $g+\theta_1$, $g+\theta_0$, $g+\theta_2$ all positive: we then have γ_3^2 and γ_1^2 each positive, γ_0^2 negative, γ_2^2 positive: viz. the conics and circles are

Hyperbola	H_3,	corresponding to	real circle C_3,
Ellipse	E_1,	"	real circle C_1,
"	E_0,	"	imaginary circle C_0,
			(viz. the radius is a pure imaginary),
"	E_2,	"	real circle C_2,

and the confocal ellipses E_1, E_0, E_2 are in order of increasing magnitude. The centre C_0 is here a point within the triangle formed by the remaining three centres C_1, C_2, C_3. It will be convenient to adopt throughout the foregoing determination as to reality.

7. It may be remarked that a circle of a pure imaginary radius $\gamma, =i\lambda$, where λ is real, may be indicated by means of the concentric circle radius λ, which is the concentric orthotomic circle; and that a circle which cuts at right angles the original circle cuts diametrally (that is, at the extremities of a diameter) the substituted circle radius λ; we have thus a real construction in relation to a circle of inversion of pure imaginary radius.

Investigation of dS. Art. Nos. 8 to 17.

8. The coordinates of a point on the dirigent conic $\dfrac{X^2}{f+\theta}+\dfrac{Y^2}{g+\theta}=1$ may be taken to be $(f+\theta)x$ $(g+\theta)y$: and we hence prove as follows the fundamental

theorem for the generation of the bicircular quartic. Consider the generating circle, centre $(f+\theta)x$, $(g+\theta)y$, which cuts at right angles the circle of inversion

$$(X-\alpha)^2+(Y-\beta)^2=\gamma^2.$$

If for a moment the radius is called δ, then the equation of the generating circle is

$$(X-\overline{f+\theta}x)^2+(Y-\overline{g+\theta}y)^2=\delta^2;$$

the condition for the intersection at right angles is

$$(\alpha-\overline{f+\theta}x)^2+(\beta-\overline{g+\theta}y)^2=\gamma^2+\delta^2,$$

and hence eliminating δ^2, the equation of the generating circle is

$$X^2+Y^2-k-2(X-\alpha)(f+\theta)x-2(Y-\beta)(g+\theta)y=0;$$

and considering herein x, y as variable parameters connected by the foregoing equation $(f+\theta)x^2+(g+\theta)y^2=1$, we have as the envelope of this circle the required bicircular quartic.

9. It is convenient to write $R=\frac{1}{2}(X^2+Y^2-k)$. The equation then is

$$R-(X-\alpha)(f+\theta)x-(Y-\beta)(g+\theta)y=0;$$

the derived equation is

$$(X-\alpha)(f+\theta)\,dx+(Y-\beta)(g+\theta)\,dy=0;$$

and from these two equations, together with the equation in (x, y) and its derivative, we find $X-\alpha=Rx$, $Y-\beta=Ry$; from these last equations, and the equations $R=\frac{1}{2}(X^2+Y^2-k)$, $(f+\theta)x^2+(g+\theta)y^2=1$, eliminating x, y, R, we have

$$(f+\theta)(X-\alpha)^2+(g+\theta)(Y-\beta)^2=R^2,$$

that is,

$$(X^2+Y^2-k)^2-4[(f+\theta)(X-\alpha)^2+(g+\theta)(Y-\beta)^2]=0,$$

the required equation of the bicircular quartic.

10. We have thus $X-\alpha=Rx$, $Y-\beta=Ry$, as the equations which serve to determine the bicircular quartic: if from these equations, together with $R=\frac{1}{2}(X^2+Y^2-k)$, we eliminate X and Y, we have R expressed as a function of x, y; and thence also X, Y expressed in terms of x, y; that is, in effect the coordinates X, Y of a point of the bicircular quartic expressed as functions of a single variable parameter. The process gives $2R+k=(\alpha+Rx)^2+(\beta+Ry)^2$, viz. this is

$$R^2(x^2+y^2)-2(1-\alpha x-\beta y)R+\gamma^2=0,$$

or putting for shortness

$$\Omega=(1-\alpha x-\beta y)^2-\gamma^2(x^2+y^2),$$

this is

$$R=\frac{1-\alpha x-\beta y+\sqrt{\Omega}}{x^2+y^2},$$

or say the two values are

$$R = \frac{1 - \alpha x - \beta y + \sqrt{\Omega}}{x^2 + y^2}, \quad R' = \frac{1 - \alpha x - \beta y - \sqrt{\Omega}}{x^2 + y^2};$$

to preserve the generality it is proper to consider $\sqrt{\Omega}$ as denoting a determinate value (the positive or the negative one, as the case may be) of the radical.

11. Considering the root R', we have $X = \alpha + R'x$, $Y = \beta + R'y$; from these equations we obtain

$$dX = R'dx + x\,dR',$$
$$dY = R'dy + y\,dR'.$$

But from the equation for R' we have

$$[R'(x^2 + y^2) - (1 - \alpha x - \beta y)]\,dR' + R'^2(x\,dx + y\,dy) + R'(\alpha\,dx + \beta\,dy) = 0,$$

that is,

$$-\sqrt{\Omega}\,dR' + R'(X dx + Y dy) = 0;$$

whence

$$dX = R'dx + \frac{R'x}{\sqrt{\Omega}}(X dx + Y dy),$$

$$dY = R'dy + \frac{R'y}{\sqrt{\Omega}}(X dx + Y dy).$$

12. The differentials dx, dy can be expressed in terms of a single differential $d\omega$, viz. writing

$$x = \frac{\cos\omega}{\sqrt{f+\theta}}, \quad y = \frac{\sin\omega}{\sqrt{g+\theta}},$$

and

$$\Theta = (f+\theta)(g+\theta),$$

then we have

$$dx = -\frac{g+\theta}{\sqrt{\Theta}}\,y\,d\omega, \quad dy = \frac{f+\theta}{\sqrt{\Theta}}\,x\,d\omega.$$

It is to be observed that, when the dirigent conic is an ellipse, ω is a real angle, and Θ is positive (whence also $\sqrt{\Theta}$ is real and positive); but when the dirigent conic is a hyperbola, ω is imaginary, and Θ is negative; we have, however, in either case

$$dx^2 + dy^2 = \frac{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}{\Theta}\,d\omega^2,$$

and we may therefore write

$$\frac{d\omega}{\sqrt{\Theta}} = \frac{ds}{\sqrt{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}},$$

where $\sqrt{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}$ is positive; ds is the increment of arc on the conic $(f+\theta)x^2 + (g+\theta)y^2 = 1$, this arc being measured in a determinate sense, and therefore ds being positive or negative as the case may be: $\frac{d\omega}{\sqrt{\Theta}}$ has thus a real positive or negative value, even when ω is imaginary, and it is convenient to retain it in the formulæ.

13. It may further be noticed that, if v denote the inclination to the axis of x of the tangent to the dirigent conic at the point $\sqrt{f+\theta}\cos\omega$, $\sqrt{g+\theta}\sin\omega$, where v is Casey's θ, then

$$x=\frac{\cos v}{\sqrt{U}},\quad y=\frac{\sin v}{\sqrt{U}},\quad \text{where } U=(f+\theta)\cos^2 v+(g+\theta)\sin^2 v,$$

viz. we have

$$\frac{\cos\omega}{\sqrt{f+\theta}}=\frac{\cos v}{U},\quad \frac{\sin\omega}{\sqrt{g+\theta}}=\frac{\sin v}{U},$$

giving, as is easily verified, $\dfrac{d\omega}{\sqrt{\Theta}}=\dfrac{dv}{U}$; we have therefore

$$\frac{d\omega}{(x^2+y^2)\sqrt{\Theta}}=\frac{dv}{v(x^2+y^2)}=dv,$$

or

$$\frac{d\omega}{\sqrt{\Theta}}=(x^2+y^2)\,dv,$$

which is another interpretation of $\dfrac{d\omega}{\sqrt{\Theta}}$.

14. Substituting for dx, dy their values, the formulæ become

$$dX=\frac{R'}{\sqrt{\Theta}}\left\{-(g+\theta)y+\frac{x}{\sqrt{\Omega}}\left(-(g+\theta)yX+(f+\theta)xY\right)\right\}d\omega,$$

$$dY=\frac{R'}{\sqrt{\Theta}}\left\{(f+\theta)x+\frac{y}{\sqrt{\Omega}}\left(-(g+\theta)yX+(f+\theta)xY\right)\right\}d\omega.$$

We have

$$\begin{aligned} xX+yY&=\alpha x+\beta y+(x^2+y^2)R'\\ &=1-\sqrt{\Omega}, \end{aligned}$$

that is,

$$1=\frac{1-xX-yY}{\sqrt{\Omega}};$$

and consequently the foregoing expressions of dX, dY become

$$\begin{aligned} dX&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(g+\theta)y(xX+yY-1)+x\left(-(g+\theta)yX+(f+\theta)xY\right)\right\}\\ &=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(\overline{g+\theta}\,y^2+\overline{f+\theta}\,x^2)Y-(g+\theta)y\right\}, \end{aligned}$$

$$\begin{aligned} dY&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(f+\theta)x(1-xX-yY)+y\left(-(g+\theta)yX+(f+\theta)xY\right)\right\}\\ &=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(f+\theta)x-\left((f+\theta)x^2+(g+\theta)y^2\right)X\right\}, \end{aligned}$$

or finally

$$dX = \frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{Y-(g+\theta)y\} = \frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{R'y+\beta-(g+\theta)y\},$$

$$dY = \frac{-R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{X-(f+\theta)x\} = \frac{-R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{R'x+\alpha-(f+\theta)x\}.$$

15. We have

$$\begin{aligned}(R'x+\alpha-\overline{f+\theta}\,x)^2+(R'y+\beta-\overline{g+\theta}\,y)^2 \\ = R'^2(x^2+y^2)-2R'(1-\alpha x-\beta y) \\ +(\alpha-\overline{f+\theta}\,x)^2+(\beta-\overline{g+\theta}\,y)^2;\end{aligned}$$

viz. this is

$$\begin{aligned}&=(\alpha-\overline{f+\theta}\,x)^2+(\beta-\overline{g+\theta}\,y)^2-\gamma^2 \\ &=\delta^2, \text{ the radius of the generating circle.}\end{aligned}$$

Hence if $dS, =\sqrt{dX^2+dY^2}$, be the element of arc of the bicircular quartic, this element being taken to be positive, we have

$$dS = \frac{\epsilon' R'\delta\, d\omega}{\sqrt{\Omega}\sqrt{\Theta}},$$

where ϵ' denotes a determinate sign, + or −, as the case may be.

16. I stop to consider the geometrical interpretation; introducing $d\upsilon$, the formula may be written

$$dS = \frac{\epsilon'.R'(x^2+y^2)\,\delta\, d\upsilon}{\sqrt{\Omega}},$$

and we have $(x^2+y^2)R' = 1-\alpha x-\beta y-\sqrt{\Omega}$, or

$$\frac{(x^2+y^2)R'}{\sqrt{\Omega}} = \frac{1-\alpha x-\beta y}{\sqrt{\Omega}}-1.$$

Here $\frac{1-\alpha x-\beta y}{\sqrt{x^2+y^2}}$ is the perpendicular from the centre of the circle of inversion upon the tangent to the dirigent conic, and $\frac{\sqrt{\Omega}}{\sqrt{x^2+y^2}}$ is the half-chord which this perpendicular forms with the generating circle. Hence $\frac{1-\alpha x-\beta y}{\sqrt{\Omega}}-1$ = (perpendicular − half-chord) ÷ half-chord, the numerator being in fact the distance of the element dS (or point X, Y) from the centre of inversion: the formula thus is

$$dS = \pm\frac{\rho.\delta}{\frac{1}{2}c}\,d\upsilon,$$

where δ is the radius of the generating circle, ρ the distance of the element from the centre of the circle of inversion, and c the chord which this distance forms with

the generating circle. If we consider the two points on the generating circle, and write dS' for the element at the other point, then we have

$$(dS \pm dS') = \pm \frac{(\rho - \rho')\,\delta\,dv}{\frac{1}{2}c} = 2\delta\,dv,$$

which is Casey's formula $ds' - ds = 2\rho\,d\phi$ (273).

17. The foregoing forms of dX, dY are those which give most directly the required value of dS: but I had previously obtained them in a different form. Writing

$$\Delta = \beta x - \alpha y + (f - g)\,xy,$$

then

$$x\Delta = \beta x^2 - \alpha xy + [(f + \theta)\,x^2 - (g + \theta)\,y^2];$$

or since

$$(f + \theta)\,x^2 = 1 - (g + \theta)\,y^2,$$

this is

$$x\Delta = \beta x^2 - \alpha xy + [1 - (g + \theta)(x^2 + y^2)] = y\,(1 - \alpha x - \beta y) + (x^2 + y^2)\,(\beta - (g + \theta)\,y)$$
$$= (x^2 + y^2)\,\{yR' + \beta - (g + \theta)\,y\} + y\sqrt{\Omega},$$

that is,

$$x\Delta - y\sqrt{\Omega} = (x^2 + y^2)\,\{yR' + \beta - (g + \theta)\,y\};$$

and similarly

$$-y\Delta - x\sqrt{\Omega} = (x^2 + y^2)\,\{xR' + \alpha - (f + \theta)\,x\}.$$

We have therefore

$$dX = \frac{R'd\omega}{(x^2 + y^2)\sqrt{\Theta}\sqrt{\Omega}}(x\Delta - y\sqrt{\Omega}),$$

$$dY = \frac{R'd\omega}{(x^2 + y^2)\sqrt{\Theta}\sqrt{\Omega}}(y\Delta + x\sqrt{\Omega}),$$

and thence a value of dS which, compared with the former value, gives

$$\Omega + \Delta^2 = (x^2 + y^2)\,\delta^2,$$

an equation which may be verified directly.

Formulæ for the Inscribed Quadrilateral. Art. Nos. 18 to 22.

18. We consider on the curve four points, A, B, C, D, forming a quadrilateral, $ABCD$. The coordinates are taken to be (X, Y), (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) respectively. It is assumed that (A, B), (B, C), (C, D), (D, A) belong to the generations 1, 2, 3, 0, and depend on the parameters (x_1, y_1), (x_2, y_2), (x_3, y_3), (x, y) respectively.

We write

$$\Omega = (1 - \alpha x - \beta y)^2 - \gamma^2 (x^2 + y^2),$$
$$\Omega_1 = (1 - \alpha_1 x_1 - \beta_1 y_1)^2 - \gamma_1^2 (x_1^2 + y_1^2),$$
$$\Omega_2 = (1 - \alpha_2 x_2 - \beta_2 y_2)^2 - \gamma_2^2 (x_2^2 + y_2^2),$$
$$\Omega_3 = (1 - \alpha_3 x_3 - \beta_3 y_3)^2 - \gamma_3^2 (x_3^2 + y_3^2);$$

and then, $\sqrt{\Omega}$ denoting as above a determinate value, positive or negative as the case may be, of the radical, and similarly $\sqrt{\Omega_1}$, $\sqrt{\Omega_2}$, $\sqrt{\Omega_3}$ denoting determinate values of these radicals respectively, each radical having its own sign at pleasure, we further write

$$(x^2+y^2)R' = 1-\alpha x-\beta y-\sqrt{\Omega}, \quad (x_1^2+y_1^2)R_1 = 1-\alpha_1 x_1-\beta_1 y_1+\sqrt{\Omega_1},$$
$$(x_1^2+y_1^2)R_1' = 1-\alpha_1 x_1-\beta_1 y_1-\sqrt{\Omega_1}, \quad (x_2^2+y_2^2)R_2 = 1-\alpha_2 x_2-\beta_2 y_2+\sqrt{\Omega_2},$$
$$(x_2^2+y_2^2)R_2' = 1-\alpha_2 x_2-\beta_2 y_2-\sqrt{\Omega_2}, \quad (x_3^2+y_3^2)R_3 = 1-\alpha_3 x_3-\beta_3 y_3+\sqrt{\Omega_3},$$
$$(x_3^2+y_3^2)R_3' = 1-\alpha_3 x_3-\beta_3 y_3-\sqrt{\Omega_3}, \quad (x^2+y^2)R = 1-\alpha x-\beta y+\sqrt{\Omega};$$

and this being so, we must have

$$X=\alpha+R'x=\alpha_1+R_1x_1, \quad Y=\beta+R'y=\beta_1+R_1y_1, \quad R'=\tfrac{1}{2}(X^2+Y^2-k), \quad R_1=\tfrac{1}{2}(X^2+Y^2-k_1),$$
$$X_1=\alpha_1+R_1'x_1=\alpha_2+R_2x_2, \quad Y_1=\beta_1+R_1'y_1=\beta_2+R_2y_2, \quad R_1'=\tfrac{1}{2}(X_1^2+Y_1^2-k_1), \quad R_2=\tfrac{1}{2}(X_1^2+Y_1^2-k_2),$$
$$X_2=\alpha_2+R_2'x_2=\alpha_3+R_3x_3, \quad Y_2=\beta_2+R_2'y_2=\beta_3+R_3y_3, \quad R_2'=\tfrac{1}{2}(X_2^2+Y_2^2-k_2), \quad R_3=\tfrac{1}{2}(X_2^2+Y_2^2-k_3),$$
$$X_3=\alpha_3+R_3'x_3=\alpha+Rx, \quad Y_3=\beta_3+R_3'y_3=\beta+Ry, \quad R_3'=\tfrac{1}{2}(X_3^2+Y_3^2-k_3), \quad R=\tfrac{1}{2}(X_3^2+Y_3^2-k);$$

and then from the values of X, Y, R', R, we have

$$\alpha-\alpha_1+R'x-R_1x_1=0,$$
$$\beta-\beta_1+R'y-R_1y_1=0,$$
$$(\theta-\theta_1)+R'-R_1=0,$$

giving

$$(\beta-\beta_1)(x-x_1)-(\alpha-\alpha_1)(y-y_1)+(\theta-\theta_1)(xy_1-x_1y)=0;$$

and similarly

$$(\beta_1-\beta_2)(x_1-x_2)-(\alpha_1-\alpha_2)(y_1-y_2)+(\theta_1-\theta_2)(x_1y_2-x_2y_1)=0,$$
$$(\beta_2-\beta_3)(x_2-x_3)-(\alpha_2-\alpha_3)(y_2-y_3)+(\theta_2-\theta_3)(x_2y_3-x_3y_2)=0,$$
$$(\beta_3-\beta)(x_3-x)-(\alpha_3-\alpha)(y_3-y)+(\theta_3-\theta)(x_3y-xy_3)=0,$$

which are the relations connecting the parameters (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) of the quadrilateral.

19. We have thus apparently four equations for the determination of four quantities, or the number of quadrilaterals would be finite; but if from the first and second equations we eliminate (x_1, y_1), and if from the third and fourth equations we eliminate (x_3, y_3), we find in each case the same relation between (x, y), (x_2, y_2), viz. this is found to be

$$\Omega\Omega_2=(1-\alpha x_2-\beta y_2)^2(1-\alpha_2 x-\beta_2 y)^2;$$

and we have thus the singly infinite series of quadrilaterals. We have, of course, between (x_1, y_1), (x_3, y_3) the like relation,

$$\Omega_1\Omega_3=(1-\alpha_1x_3-\beta_1y_3)^2(1-\alpha_3x_1-\beta_3y_1)^2.$$

20. The relation between (x, y), (x_1, y_1) may be expressed also in the two forms:

$$1-\alpha(x+x_1)-\beta(y+y_1)+(f+\theta_1)xx_1+(g+\theta_1)yy_1+\frac{x^2+y^2}{xy_1-x_1y}(\overline{\alpha-\alpha_1}y_1-\overline{\beta-\beta_1}x_1)=0,$$

$$1-\alpha_1(x+x_1)-\beta_1(y+y_1)+(f+\theta)xx_1+(g+\theta)yy_1+\frac{x_1^2+y_1^2}{x_1y-xy_1}(\overline{\alpha_1-\alpha}y-\overline{\beta_1-\beta}x)=0.$$

In fact, the first of these equations is

$$\{1+(f+\theta_1)xx_1+(g+\theta_1)yy_1\}(xy_1-x_1y)-\{\alpha(x+x_1)+\beta(y+y_1)\}(xy_1-x_1y)$$
$$+\{(\alpha-\alpha_1)y_1-(\beta-\beta_1)x_1\}(x^2+y^2)=0,$$

which, by virtue of the original form of relation, is

$$-\{1+(f+\theta_1)xx_1+(g+\theta_1)yy_1\}\frac{(\beta-\beta_1)(x-x_1)-(\alpha-\alpha_1)(y-y_1)}{\theta-\theta_1}$$
$$-\{\alpha(x+x_1)+\beta(y+y_1)\}(xy_1-x_1y)+\{(\alpha-\alpha_1)y_1-(\beta-\beta_1)x_1\}(x^2+y^2)=0;$$

or, in the first term, writing

$$-\frac{\beta-\beta_1}{\theta-\theta_1}=\frac{\beta}{g+\theta_1},\quad \frac{\alpha-\alpha_1}{\theta-\theta_1}=\frac{-\alpha}{f+\theta_1},$$

and in the third term

$$\alpha-\alpha_1=-\frac{(\theta-\theta_1)\alpha}{f+\theta_1},\quad -(\beta-\beta_1)=\frac{(\theta-\theta_1)\beta}{g+\theta_1},$$

this is

$$(1+(f+\theta_1)xx_1+(g+\theta_1)yy_1)\left(\frac{\beta(x-x_1)}{g+\theta_1}-\frac{\alpha(y-y_1)}{f+\theta_1}\right)$$
$$-\{\alpha(x+x_1)+\beta(y+y_1)\}(xy_1-x_1y)-\left\{\frac{\alpha(\theta-\theta_1)}{f+\theta_1}y_1-\frac{\beta(\theta-\theta_1)}{g+\theta_1}x_1\right\}(x^2+y^2)=0.$$

In this equation the coefficients of α and of β are separately $=0$: in fact, the coefficient of β is

$$\frac{x-x_1}{g+\theta_1}+\frac{f+\theta_1}{g+\theta_1}xx_1(x-x_1)+(x-x_1)yy_1-(y+y_1)(xy_1-x_1y)+\frac{\theta-\theta_1}{g+\theta_1}x_1(x^2+y^2)$$
$$=\frac{x}{g+\theta_1}\{1-(f+\theta_1)x_1^2-(g+\theta_1)y_1^2\}-\frac{x_1}{g+\theta_1}\{1-(f+\theta)x^2-(g+\theta)y^2\}=0;$$

and similarly the coefficient of α is $=0$.

And in like manner the second equation may be verified.

21. The two equations are:

$$1-\alpha x-\beta y-(x^2+y^2)R'=\alpha x_1+\beta y_1-(f+\theta_1)xx_1-(g+\theta_1)yy_1,$$
$$1-\alpha_1x_1-\beta_1y_1-(x_1^2+y_1^2)R_1=\alpha_1x+\beta_1y-(f+\theta)xx_1-(g+\theta)yy_1;$$

or, substituting for R' and R_1 their values, these are

$$\sqrt{\Omega}=\alpha x_1+\beta y_1-(f+\theta_1)xx_1-(g+\theta_1)yy_1,\quad \sqrt{\Omega_1}=-\alpha_1x-\beta_1y+(f+\theta)xx_1+(g+\theta)yy_1;$$

and similarly

$$\sqrt{\Omega_1} = \alpha_1 x_2 + \beta_1 y_2 - (f+\theta_2)\, x_1 x_2 - (g+\theta_2)\, y_1 y_2, \quad \sqrt{\Omega_2} = -\alpha_2 x_1 - \beta_2 y_1 + (f+\theta_1)\, x_1 x_2 + (g+\theta_1)\, y_1 y_2$$

$$\sqrt{\Omega_2} = \alpha_2 x_3 + \beta_2 y_3 - (f+\theta_3)\, x_2 x_3 - (g+\theta_3)\, y_2 y_3, \quad \sqrt{\Omega_3} = -\alpha_3 x_2 - \beta_3 y_2 + (f+\theta_2)\, x_2 x_3 + (g+\theta_2)\, y_2 y_3,$$

$$\sqrt{\Omega_3} = \alpha_3 x + \beta_3 y - (f+\theta)\, x_3 x - (g+\theta)\, y_3 y, \quad \sqrt{\Omega} = -\alpha x_3 - \beta y_3 + (f+\theta_3)\, x_3 x + (g+\theta_3)\, y_3 y.$$

Differentiating the equation

$$(\beta-\beta_1)(x-x_1) - (\alpha-\alpha_1)(y-y_1) + (\theta-\theta_1)(xy_1 - x_1 y) = 0,$$

we have

$$[(\beta-\beta_1) + (\theta-\theta_1)\, y_1]\, dx - [(\alpha-\alpha_1) + (\theta-\theta_1)\, x_1]\, dy$$
$$- [(\beta-\beta_1) + (\theta-\theta_1) y]\, dx_1 + [(\alpha-\alpha_1) + (\theta-\theta_1)\, x]\, dy_1 = 0;$$

and writing herein

$$dx = -\frac{(g+\theta)}{\sqrt{\Theta}}\, y\, d\omega, \quad dx_1 = \frac{-(g+\theta_1)}{\sqrt{\Theta_1}}\, y_1\, d\omega_1,$$

$$dy = \frac{(f+\theta)}{\sqrt{\Theta}}\, x\, d\omega, \quad dy_1 = \frac{(f+\theta_1)}{\sqrt{\Theta_1}}\, x_1\, d\omega_1,$$

we find

$$-\frac{d\omega}{\sqrt{\Theta}}\{(g+\theta)(\beta-\beta_1)\, y + (f+\theta)(\alpha-\alpha_1)\, x + (\theta-\theta_1)((f+\theta)\, xx_1 + (g+\theta)\, yy_1)\}$$

$$+\frac{d\omega_1}{\sqrt{\Theta_1}}\{(g+\theta_1)(\beta-\beta_1)\, y_1 + (f+\theta_1)(\alpha-\alpha_1)\, x_1 + (\theta-\theta_1)((f+\theta_1)\, xx_1 + (g+\theta_1)\, yy_1)\} = 0;$$

viz., dividing by $\theta-\theta_1$, this becomes

$$-\sqrt{\Omega_1}\frac{d\omega}{\sqrt{\Theta}} - \sqrt{\Omega}\frac{d\omega_1}{\sqrt{\Theta_1}} = 0, \text{ that is, } \frac{d\omega}{\sqrt{\Theta}\sqrt{\Omega}} + \frac{d\omega_1}{\sqrt{\Theta_1}\sqrt{\Omega_1}} = 0;$$

or, completing the system, we have

$$\frac{d\omega}{\sqrt{\Theta}\sqrt{\Omega}} = \frac{-d\omega_1}{\sqrt{\Theta_1}\sqrt{\Omega_1}} = \frac{d\omega_2}{\sqrt{\Theta_2}\sqrt{\Omega_2}} = \frac{-d\omega_3}{\sqrt{\Theta_3}\sqrt{\Omega_3}},$$

which are the differential relations between the parameters ω, ω_1, ω_2, ω_3, or (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3).

22. From the equations $X = \alpha + R'x$, $Y = \beta + R'y$, we found

$$dX = \frac{R' d\omega}{\sqrt{\Omega}\sqrt{\Theta}}\{Y - (g+\theta)\, y\},$$

$$dY = \frac{R' d\omega}{\sqrt{\Omega}\sqrt{\Theta}}\{X - (f+\theta)\, x\};$$

the new values, $X = \alpha_1 + R_1 x_1$ and $Y = \beta_1 + R_1 y_1$, give in like manner

$$dX = -\frac{R_1 d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}\{Y - (g+\theta_1)\, y_1\},$$

$$dY = -\frac{R_1 d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}\{X - (f+\theta_1)\, x_1\};$$

in virtue of the relation just found between $d\omega$ and $d\omega_1$, these two sets of values will agree together if only

$$R'\{Y-(g+\theta)y\}=R_1\{Y-(g+\theta_1)y_1\},$$
$$R'\{X-(f+\theta)x\}=R_1\{X-(f+\theta_1)x_1\}.$$

These are easily verified: the first is

$$R'Y-(g+\theta)(Y-\beta)=(R'-\theta+\theta_1)Y-(g+\theta_1)(Y-\beta_1),$$

viz. this is $(g+\theta)\beta-(g+\theta_1)\beta_1=0$, which is right; and similarly the second equation gives $(f+\theta)\alpha-(f+\theta_1)\alpha_1=0$, which is right.

From the first values of dX, dY, we have, as above,

$$dS=\frac{\epsilon' R'\delta\, d\omega}{\sqrt{\Omega}\sqrt{\Theta}};$$

and the second values give in like manner

$$dS=\frac{\epsilon_1 R_1\delta_1\, d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},$$

where ϵ_1 is $=\pm 1$. It will be observed that we have in effect, by means of the relation $(\beta-\beta_1)(x-x_1)-(\alpha-\alpha_1)(y-y_1)+(\theta-\theta_1)(xy_1-x_1y)=0$, proved the identity of the two values of dS.

Considering the quadrilateral $ABCD$, and giving it an infinitesimal variation, so as to change it into $A'B'C'D'$, then dS is the element of arc AA'; and writing in like manner dS_1, dS_2, dS_3 for the elements of arc BB', CC', DD', we have, of course, a like pair of values for each of the elements dS_1, dS_2, dS_3.

Formulæ for the elements of Arc dS, dS_1, dS_2, dS_3. Art. Nos. 23 to 27.

23. The formulæ are

$$dS=\epsilon' R'\delta\,\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}}=\epsilon_1 R_1\delta_1\frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},$$

$$dS_1=\epsilon_1' R_1'\delta_1\frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}=\epsilon_2 R_2\delta_2\frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}},$$

$$dS_2=\epsilon_2' R_2'\delta_2\frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}}=\epsilon_3 R_3\delta_3\frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}},$$

$$dS_3=\epsilon_3' R_3'\delta_3\frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}}=\epsilon R\delta\,\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}},$$

where the ϵ's each denote ± 1. Supposing as above that γ^2 is negative, but that γ_1^2, γ_2^2, γ_3^2 are positive; then R', R have opposite signs: but R_1', R_1 have the same sign,

as have also R_2' and R_2, and R_3' and R_3. We may take δ, δ_1, δ_2, and δ_3 as each of them positive: the signs of

$$\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}},\ \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},\ \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}},\ \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}} \text{ are } +,\ -,\ +,\ -, \text{ or } -,\ +,\ -,\ +:$$

hence to make dS, dS_1, dS_2, dS_3 all positive,

$$\epsilon',\quad \epsilon_1',\quad \epsilon_2',\quad \epsilon_3',\quad \epsilon_1,\quad \epsilon_2,\quad \epsilon_3,\quad \epsilon,$$

must have either the signs of

$$R',\ -R_1',\ R_2',\ -R_3',\ -R_1,\ R_2,\ -R_3,\ R,$$

or else the reverse signs: hence in either case $\epsilon' = -\epsilon$, $\epsilon_1' = \epsilon_1$, $\epsilon_2' = \epsilon_2$, $\epsilon_3' = \epsilon_3$; or the equations are

$$\begin{aligned}
dS &= -\epsilon R'\delta \frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}} = \epsilon_1 R_1 \delta_1 \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},\\
dS_1 &= \epsilon_1 R_1' \delta_1 \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}} = \epsilon_2 R_2 \delta_2 \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}},\\
dS_2 &= \epsilon_2 R_2' \delta_2 \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}} = \epsilon_3 R_3 \delta_3 \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}},\\
dS_3 &= \epsilon_3 R_3' \delta_3 \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}} = \epsilon R\delta \frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}}.
\end{aligned}$$

24. But we have $R' - R = \dfrac{-2\sqrt{\Omega}}{x^2+y^2}$, &c.; and hence, putting for shortness

$$\frac{\delta}{(x^2+y^2)\sqrt{\Theta}},\ \frac{\delta_1}{(x_1^2+y_1^2)\sqrt{\Theta_1}},\ \frac{\delta_2}{(x_2^2+y_2^2)\sqrt{\Theta_2}},\ \frac{\delta_3}{(x_3^2+y_3^2)\sqrt{\Theta_3}} = P,\ P_1,\ P_2,\ P_3,$$

$$\begin{aligned}
dS + dS_3 &= +2\epsilon P\, d\omega,\\
dS_1 - dS &= -2\epsilon_1 P_1 d\omega_1,\\
dS_2 - dS_1 &= -2\epsilon_2 P_2 d\omega_2,\\
dS_3 - dS &= -2\epsilon_3 P_3 d\omega_3,
\end{aligned}$$

and consequently

$$\begin{aligned}
dS &= \epsilon P d\omega + \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3,\\
dS_1 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3,\\
dS_2 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 - \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3,\\
dS_3 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 - \epsilon_2 P_2 d\omega_2 - \epsilon_3 P_3 d\omega_3,
\end{aligned}$$

which are the required formulæ for the elements of arc.

25. The determination of the signs has been made by means of the particular figure; but it is easy to see that the pairs of terms could not for instance be $dS - dS_3$, $dS_1 - dS$, $dS_2 - dS_1$, $dS_3 - dS$, or any other pairs such that it would be possible to eliminate dS, dS_1, dS_2, dS_3, and thus obtain an equation such as

$$\epsilon P d\omega + \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3 = 0;$$

this would, by virtue of the relations between $d\omega$, $d\omega_1$, $d\omega_2$, $d\omega_3$, become

$$\epsilon\frac{\delta\sqrt{\Omega}}{x^2+y^2}-\epsilon_1\frac{\delta_1\sqrt{\Omega_1}}{x_1^2+y_1^2}+\epsilon_2\frac{\delta_2\sqrt{\Omega_2}}{x_2^2+y_2^2}-\epsilon_3\frac{\delta_3\sqrt{\Omega_3}}{x_3^2+y_3^2}=0,$$

an equation not deducible from the relations which connect ω, ω_1, ω_2, ω_3, and which therefore cannot be satisfied by the variable quadrilateral.

26. The differentials of the formulæ are, it will be observed, of the form $Pd\omega$

$$=\frac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}},$$

where $\sqrt{\Theta}, =\sqrt{f+\theta\,.\,g+\theta}$, is a mere constant,

$$x,\ y=\frac{\cos\omega}{\sqrt{f+\theta}},\ \frac{\sin\omega}{\sqrt{g+\theta}},$$

and

$$\delta^2=\{(f+\theta)x-\alpha\}^2+\{(g+\theta)y-\beta\}^2-\gamma^2;$$

viz. the form is

$$\frac{\sqrt{(\cos\omega\sqrt{f+\theta}-\alpha)^2+(\sin\omega\sqrt{g+\theta}-\beta)^2-\gamma^2}}{\sqrt{\Theta}\,.\left(\frac{\cos^2\omega}{f+\theta}+\frac{\sin^2\omega}{g+\theta}\right)}d\omega,$$

which is, in fact, the same as Casey's form in ϕ, equation (300), his ϕ being $=90^\circ-\omega$.

Writing as before v in place of his θ, the differential expression becomes simply $=\delta\, dv$: but δ^2 expressed as a function of v is an irrational function $M+N\sqrt{U}$, and δ would be the root of such a function; so that, if the form originally obtained had been this form $\delta\, dv$, it would have been necessary to transform it into the first-mentioned form $\frac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}}$, in which δ is expressed as a function of (x, y), that is, of ω.

27. The system of course is

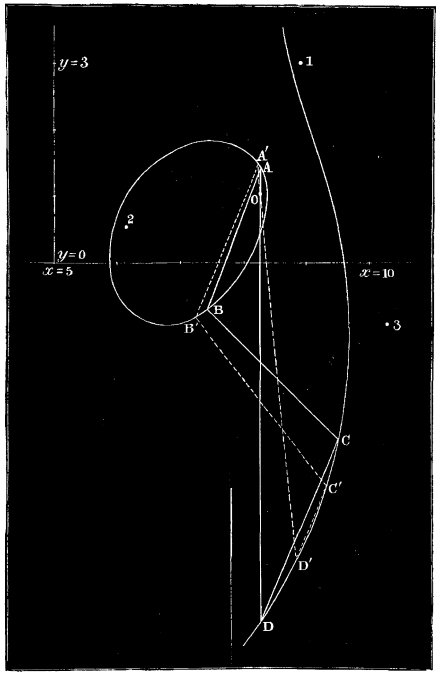
$$\begin{aligned}
dS &= \epsilon\delta dv+\epsilon_1\delta_1 dv_1+\epsilon_2\delta_2 dv_2+\epsilon_3\delta_3 dv_3,\\
dS_1 &= \epsilon\delta dv-\epsilon_1\delta_1 dv_1+\epsilon_2\delta_2 dv_2+\epsilon_3\delta_3 dv_3,\\
dS_2 &= \epsilon\delta dv-\epsilon_1\delta_1 dv_1-\epsilon_2\delta_2 dv_2+\epsilon_3\delta_3 dv_3,\\
dS_3 &= \epsilon\delta dv-\epsilon_1\delta_1 dv_1-\epsilon_2\delta_2 dv_2-\epsilon_3\delta_3 dv_3,
\end{aligned}$$

where $dv=\frac{d\omega}{(x^2+y^2)\sqrt{\Theta}}$, &c.; and this is the most convenient way of writing it.

Reference to Figure. Art. No. 28.

28. I constructed a bicircular quartic consisting of an exterior and interior oval with the following numerical data: ($f+\theta_3=48$, $f+\theta_1=56$, $f+\theta_0=60$, $f+\theta_2=80$; $g+\theta_3=-6$, $g+\theta_1=2$, $g+\theta_0=6$, $g+\theta_2=26$),—not very convenient ones, inasmuch as

the exterior oval came out too large. The annexed figure shows 0, 1, 2, 3, the centres of the circles of inversion, the interior oval, and a portion of the exterior



oval, also the origin and axes; it will be seen that the centres 0, 2 lie inside the interior oval, the centres 1, 3 outside the exterior oval: I add further the values

$$\sqrt{f+\theta_3}=6{\cdot}93,\quad \sqrt{-(g+\theta_3)}=2{\cdot}45,\quad \alpha_3=10{\cdot}18,\quad \beta_3=-\ {\cdot}98,$$

$$\sqrt{f+\theta_1}=7{\cdot}48,\quad \sqrt{g+\theta_1}\ =1{\cdot}41,\quad \alpha_1=\ 8{\cdot}73,\quad \beta_1=+2{\cdot}94,$$

$$\sqrt{f+\theta_0}=7{\cdot}75,\quad \sqrt{g+\theta_0}\ =2{\cdot}45,\quad \alpha_0=\ 8{\cdot}15,\quad \beta_0=+\ {\cdot}98,$$

$$\sqrt{f+\theta_2}=8{\cdot}94,\quad \sqrt{g+\theta_2}\ =5{\cdot}09,\quad \alpha_2=\ 6{\cdot}10,\quad \beta_2=+\ {\cdot}23.$$

We thus see how there exists a series of quadrilaterals $ABCD$, where A, B are situate on the interior oval, C, D on the exterior oval. Considering the sides as

drawn in the senses A to B, B to C, C to D, D to A: and representing the inclinations, measured from the positive infinity on the axis of x in the sense x to y, by v_1, v_2, v_3, v respectively: then, in passing to the consecutive quadrilateral $A'B'C'D'$, we have v_1 and v_2 decreasing, v_3 and v increasing, that is, dv_1 and dv_2 negative, dv_3 and dv positive; so that, reckoning the elements AA', BB', CC', DD', that is, dS_1, dS_2, dS_3, dS, as each of them positive, we have

$$dS_2 - dS_1 = -2\delta_1 dv_1,$$

$$dS_3 - dS_2 = -2\delta_2 dv_2,$$

$$dS - dS_3 = +2\delta_3 dv_3,$$

$$dS_1 + dS = +2\delta\, dv\,,$$

and thence

$$dS = \delta dv - \delta_1 dv_1 - \delta_2 dv_2 + \delta_3 dv_3,$$

$$dS_1 = \delta dv + \delta_1 dv_1 + \delta_2 dv_2 - \delta_3 dv_3,$$

$$dS_2 = \delta dv - \delta_1 dv_1 + \delta_2 dv_2 - \delta_3 dv_3,$$

$$dS_3 = \delta dv - \delta_1 dv_1 - \delta_2 dv_2 - \delta_3 dv_3,$$

which are the correct signs in regard to the particular figure.

Reduction of $\int \frac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}}$ *to Elliptic Integrals.* Art. No. 29.

29. The expression in question is

$$\int d\omega \,.\, \frac{\sqrt{(\cos\omega\sqrt{f+\theta}-\alpha)^2+(\sin\omega\sqrt{g+\theta}-\beta)^2-\gamma^2}}{\left\{\frac{\cos^2\omega}{f+\theta}+\frac{\sin^2\omega}{g+\theta}\right\}\sqrt{\Theta}},$$

where $\sqrt{\Theta}$ is a mere constant; and we may apply it to the Gaussian transformation,

$$\cos\omega = \frac{a+a'\cos T+a''\sin T}{c+c'\cos T+c''\sin T},$$

$$\sin\omega = \frac{b+b'\cos T+b''\sin T}{c+c'\cos T+c''\sin T},$$

where the coefficients a, b, c, a', b', c', a'', b'', c'' are such that identically

$$\cos^2\omega+\sin^2\omega-1 = \frac{1}{(c+c'\cos T+c''\sin T)^2}\{\cos^2 T+\sin^2 T-1\}:$$

and also

$$(\cos\omega\sqrt{f+\theta}-\alpha)^2+(\sin\omega\sqrt{g+\theta}-\beta)^2-\gamma^2,$$

that is,

$$\cos^2\omega\,(f+\theta)+\sin^2\omega\,(g+\theta)-2\alpha\sqrt{f+\theta}\cos\omega-2\beta\sqrt{g+\theta}\sin\omega+k,$$
$$=\frac{1}{(c+c'\cos T+c''\sin T)^2}(G_1-G_2\cos^2 T-G_3\sin^2 T).$$

30. It is found that G_1, G_2, G_3 are the roots of a cubic equation

$$(G+\theta-\theta_1)(G+\theta-\theta_2)(G+\theta-\theta_3),$$

which being so, we may assume $G_1=\theta_1-\theta$, $G_2=\theta_2-\theta$, $G_3=\theta_3-\theta$; the second condition, in fact, then is

$$(f+\theta)\cos^2\omega+(g+\theta)\sin^2\omega-2\alpha\sqrt{f+\theta}\cos\omega-2\beta\sqrt{g+\theta}\sin\omega+k$$
$$=\frac{1}{(c+c'\cos T+c''\sin T)^2}\{\theta_1-\theta-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T\};$$

and this being so, we find without difficulty the values

$$a^2=\frac{g+\theta_1\,.\,f+\theta_2\,.\,f+\theta_3}{f-g\,.\,\theta_1-\theta_2\,.\,\theta_1-\theta_3},\quad b^2=\frac{f+\theta_1\,.\,g+\theta_2\,.\,g+\theta_3}{g-f\,.\,\theta_1-\theta_2\,.\,\theta_1-\theta_3},\quad c^2=\frac{f+\theta_1\,.\,g+\theta_1}{\theta_1-\theta_2\,.\,\theta_1-\theta_3},$$

$$a'^2=-\frac{g+\theta_2\,.\,f+\theta_1\,.\,f+\theta_3}{f-g\,.\,\theta_2-\theta_1\,.\,\theta_2-\theta_3},\quad b'^2=-\frac{f+\theta_2\,.\,g+\theta_1\,.\,g+\theta_3}{g-f\,.\,\theta_2-\theta_1\,.\,\theta_2-\theta_3},\quad c'^2=-\frac{f+\theta_2\,.\,g+\theta_2}{\theta_2-\theta_1\,.\,\theta_2-\theta_3},$$

$$a''^2=-\frac{g+\theta_3\,.\,f+\theta_1\,.\,f+\theta_2}{f-g\,.\,\theta_3-\theta_1\,.\,\theta_3-\theta_2},\quad b''^2=-\frac{f+\theta_3\,.\,g+\theta_1\,.\,g+\theta_2}{g-f\,.\,\theta_3-\theta_1\,.\,\theta_3-\theta_2},\quad c''^2=-\frac{f+\theta_3\,.\,g+\theta_3}{\theta_3-\theta_1\,.\,\theta_3-\theta_2}.$$

To make these positive, the order of ascending magnitude must, however, be not as heretofore θ_3, θ_1, θ_2, but θ_3, θ_2, θ_1, viz. we must have $f+\theta_1$, $f+\theta_2$, $f+\theta_3$, $g+\theta_1$, $g+\theta_2$, $-(g+\theta_3)$, $\theta_1-\theta_3$, $\theta_1-\theta_2$, $\theta_2-\theta_3$ all positive.

31. The above are the values of the squares of the coefficients; we must have definite relations between the signs of the products aa', bb', ab, &c., viz. we may have

$$a'a''=\frac{f+\theta_1}{f-g\,.\,\theta_2-\theta_3}\sqrt{\frac{\Theta_2\Theta_3}{\theta_3-\theta_1\,.\,\theta_1-\theta_2}},\quad a''a=\frac{f+\theta_3}{f-g\,.\,\theta_3-\theta_1}\sqrt{\frac{-\Theta_3\Theta_1}{\theta_1-\theta_2\,.\,\theta_2-\theta_3}},$$

$$b'b''=\frac{g+\theta_1}{g-f\,.\,\theta_2-\theta_3}\sqrt{\quad\text{"}\quad},\quad b''b=\frac{g+\theta_3}{g-f\,.\,\theta_3-\theta_1}\sqrt{\quad\text{"}\quad},$$

$$c'c''=\frac{1}{\theta_2-\theta_3}\sqrt{\quad\text{"}\quad},\quad c''c=\frac{1}{\theta_3-\theta_1}\sqrt{\quad\text{"}\quad},$$

$$aa'=\frac{f+\theta_2}{f-g\,.\,\theta_1-\theta_2}\sqrt{\frac{-\Theta_1\Theta_2}{\theta_2-\theta_3\,.\,\theta_3-\theta_1}},$$

$$bb'=\frac{g+\theta_2}{g-f\,.\,\theta_1-\theta_2}\sqrt{\quad\text{"}\quad},$$

$$cc'=\frac{1}{\theta_1-\theta_2}\sqrt{\quad\text{"}\quad},$$

and further

$$ab = \frac{1}{f-g.\theta_3-\theta_1.\theta_1-\theta_2}\sqrt{-\Theta_1\Theta_2\Theta_3}, \quad bc = -\frac{f+\theta_1}{\theta_3-\theta_1.\theta_1-\theta_2}\sqrt{\frac{g+\theta_1.g+\theta_2.g+\theta_3}{g-f}},$$

$$a'b' = \frac{-1}{f-g.\theta_1-\theta_2.\theta_2-\theta_3}\sqrt{\quad ,, \quad}, \quad b'c' = \frac{f+\theta_2}{\theta_1-\theta_2.\theta_3-\theta_3}\sqrt{\quad ,, \quad},$$

$$a''b'' = \frac{-1}{f-g.\theta_2-\theta_3.\theta_3-\theta_1}\sqrt{\quad ,, \quad}, \quad b''c'' = \frac{f+\theta_3}{\theta_2-\theta_3.\theta_3-\theta_1}\sqrt{\quad ,, \quad},$$

$$ca = -\frac{g+\theta_1}{\theta_3-\theta_1.\theta_1-\theta_2}\sqrt{\frac{f+\theta_1.f+\theta_2.f+\theta_3}{f-g}},$$

$$c'a' = \frac{g+\theta_2}{\theta_1-\theta_2.\theta_2-\theta_3}\sqrt{\quad ,, \quad},$$

$$c''a'' = \frac{g+\theta_3}{\theta_2-\theta_3.\theta_3-\theta_1}\sqrt{\quad ,, \quad};$$

and also

$$b'c''+b''c' = \frac{2g+\theta_2+\theta_3}{\theta_2-\theta_3}\sqrt{\frac{g+\theta_1.f+\theta_2.f+\theta_3}{g-f.\theta_3-\theta_1.\theta_1-\theta_2}}, \quad c'a''+c''a' = \frac{2f+\theta_2+\theta_3}{\theta_2-\theta_3}\sqrt{\frac{f+\theta_1.g+\theta_2.g+\theta_3}{f-g.\theta_3-\theta_1.\theta_1-\theta_2}},$$

$$b''c+bc'' = \frac{2g+\theta_3+\theta_1}{\theta_3-\theta_1}\sqrt{-\frac{g+\theta_2.f+\theta_3.f+\theta_1}{g-f.\theta_1-\theta_3.\theta_2-\theta_3}}, \quad c''a+ca'' = \frac{2f+\theta_3+\theta_1}{\theta_3-\theta_1}\sqrt{-\frac{f+\theta_2.g+\theta_3.g+\theta_1}{f-g.\theta_1-\theta_2.\theta_2-\theta_3}},$$

$$bc'+b'c = \frac{2g+\theta_1+\theta_2}{\theta_1-\theta_2}\sqrt{-\frac{g+\theta_3.f+\theta_1.f+\theta_2}{g-f.\theta_2-\theta_3.\theta_3-\theta_1}}, \quad ca'+c'a = \frac{2f+\theta_1+\theta_2}{\theta_1-\theta_2}\sqrt{-\frac{f+\theta_3.g+\theta_1.g+\theta_2}{f-g.\theta_2-\theta_3.\theta_3-\theta_1}}.$$

32. These values, in fact, satisfy the several relations which exist between the nine coefficients; viz. the original expressions of $\cos\omega$, $\sin\omega$, in terms of $\cos T$, $\sin T$ give conversely expressions of $\cos T$, $\sin T$ in terms of $\cos\omega$, $\sin\omega$, the two sets being

$$\cos\omega = \frac{a+a'\cos T+a''\sin T}{c+c'\cos T+c''\sin T}, \quad \cos T = -\frac{a'\cos\omega+b'\sin\omega-c'}{a\cos\omega+b\sin\omega-c},$$

$$\sin\omega = \frac{b+b'\cos T+b''\sin T}{c+c'\cos T+c''\sin T}, \quad \sin T = -\frac{a''\cos\omega+b''\sin\omega-c''}{a\cos\omega+b\sin\omega-c}:$$

and we have then the relations

$$\cos^2\omega+\sin^2\omega-1 = \frac{1}{(c+c'\cos T+c''\sin T)^2}(\cos^2 T+\sin^2 T-1),$$

$$\cos^2 T+\sin^2 T-1 = \frac{1}{(a\cos\omega+b\sin\omega-c)^2}(\cos^2\omega+\sin^2\omega-1),$$

$$(\theta+f)\cos^2\omega+(\theta+g)\sin^2\omega-2\alpha\sqrt{\theta+f}\cos\omega-2\beta\sqrt{\theta+g}\sin\omega+k$$

$$= \frac{1}{(c+c'\cos T+c''\sin T)^2}\{(\theta_1-\theta)-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T\},$$

$$(\theta_1-\theta)-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T$$

$$= \frac{1}{(a\cos\omega+b\sin\omega-c)^2}\{(\theta+f)\cos^2\omega+(\theta+g)\sin^2\omega-2\alpha\sqrt{\theta+f}\cos\omega-2\beta\sqrt{\theta+g}\sin\omega+k\},$$

giving the four sets each of six equations

$$\begin{aligned}
a^2 + b^2 - c^2 &= -1, & a'a'' + b'b'' - c'c'' &= 0,\\
a'^2 + b'^2 - c'^2 &= +1, & a''a + b''b - c''c &= 0,\\
a''^2 + b''^2 - c''^2 &= +1, & aa' + bb' - cc' &= 0,\\
-a^2 + a'^2 + a''^2 &= +1, & -bc + b'c' + b''c'' &= 0,\\
-b^2 + b'^2 + b''^2 &= +1, & -ca + c'a' + c''a'' &= 0,\\
-c^2 + c'^2 + c''^2 &= -1, & -ab + a'b' + a''b'' &= 0,
\end{aligned}$$

$$\begin{aligned}
(\theta+f)a^2 + (\theta+g)b^2 - 2\alpha\sqrt{\theta+f}\,ac - 2\beta\sqrt{\theta+g}\,bc + kc^2 &= \theta_1+\theta,\\
(\theta+f)a'^2 + (\theta+g)b'^2 - 2\alpha\sqrt{\theta+f}\,a'c' - 2\beta\sqrt{\theta+g}\,b'c' + kc'^2 &= -\theta_2+\theta,\\
(\theta+f)a''^2 + (\theta+g)b''^2 - 2\alpha\sqrt{\theta+f}\,a''c'' - 2\beta\sqrt{\theta+g}\,b''c'' + kc''^2 &= -\theta_3+\theta,
\end{aligned}$$

$$\begin{aligned}
(\theta+f)a'a'' + (\theta+g)b'b'' - \alpha\sqrt{\theta+f}(a'c'' + a''c') - \beta\sqrt{\theta+g}(b'c'' + b''c') + kc'c'' &= 0,\\
(\theta+f)a''a + (\theta+g)b''b - \alpha\sqrt{\theta+f}(a''c + ac'') - \beta\sqrt{\theta+g}(b''c + bc'') + kc''c &= 0,\\
(\theta+f)aa' + (\theta+g)bb' - \alpha\sqrt{\theta+f}(ac' + a'c) - \beta\sqrt{\theta+g}(bc' + b'c) + kcc' &= 0,
\end{aligned}$$

$$\begin{aligned}
(\theta_1-\theta)a^2 - (\theta_2-\theta)a'^2 - (\theta_3-\theta)a''^2 &= \theta+f, \text{ or say } (\theta_1+f)a^2 - (\theta_2+f)a'^2 - (\theta_3+f)a''^2 = 0,\\
(\theta_1-\theta)b^2 - (\theta_2-\theta)b'^2 - (\theta_3-\theta)b''^2 &= \theta+g, \quad \text{,,} \quad (\theta_1+g)b^2 - (\theta_2+g)b'^2 - (\theta_3+g)b''^2 = 0,\\
(\theta_1-\theta)c^2 - (\theta_2-\theta)c'^2 - (\theta_3-\theta)c''^2 &= k, \quad \text{,,} \quad \theta_1c^2 - \theta_2c'^2 - \theta_3c''^2 = k+\theta,
\end{aligned}$$

$$\begin{aligned}
-(\theta_1-\theta)bc + (\theta_2-\theta)b'c' + (\theta_3-\theta)b''c'' &= -\beta\sqrt{\theta+g},\\
-(\theta_1-\theta)ca + (\theta_2-\theta)c'a' + (\theta_3-\theta)c''a'' &= -\alpha\sqrt{\theta+f},\\
-(\theta_1-\theta)ab + (\theta_2-\theta)a'b' + (\theta_3-\theta)a''b'' &= 0;
\end{aligned}$$

all which formulæ are in fact satisfied by the foregoing values of the expressions a^2, b^2, a'^2, &c.

33. We then have

$$d\omega = \frac{dT}{c + c'\cos T + c''\sin T};$$

the radical which multiplies $d\omega$ being

$$= \frac{1}{c + c'\cos T + c''\sin T}\sqrt{\theta_1 - \theta_2\cos^2 T - \theta_3\sin^2 T},$$

the differential becomes

$$= \frac{dT\sqrt{\theta_1 - \theta_2\cos^2 T - \theta_3\sin^2 T}}{\left(\dfrac{\cos^2\omega}{f+\theta} + \dfrac{\sin^2\omega}{g+\theta}\right)(c + c'\cos T + c''\sin T)^2\sqrt{\Theta}},$$

that is,

$$= \frac{dT\sqrt{\theta_1 - \theta_2\cos^2 T - \theta_3\sin^2 T}}{\left\{\dfrac{1}{f+\theta}(a + a'\cos T + a''\sin T)^2 + \dfrac{1}{g+\theta}(b + b'\cos T + b''\sin T)^2\right\}\sqrt{\Theta}}.$$

The denominator could, of course, be reduced to the form $(*\between 1, \cos T, \sin T)^2$; but the actual form seems preferable, inasmuch as it puts in evidence the linear factors

$$\frac{1}{\sqrt{f+\theta}}(a + a'\cos T + a''\sin T) \pm \frac{i}{\sqrt{g+\theta}}(b + b'\cos T + b''\sin T),$$

and there seems to be no advantage in further reducing the integral.

668.

ON COMPOUND COMBINATIONS.

[From the *Proceedings of the Lit. Phil. Soc. Manchester*, t. XVI. (1877), pp. 113, 114; *Memoirs, ib.*, Ser. III., t. VI. (1879), pp. 99, 100.]

PROF. CLIFFORD'S paper, "On the Types of Compound Statement involving Four Classes," [volume of *Proceedings* quoted, pp. 88—101; *Mathematical Papers*, pp. 1—13], relates mathematically to a question of compound combinations; and it is worth while to consider its connexion with another question of compound combinations, the application of which is a very different one.

Starting with four symbols, A, B, C, D, we have sixteen combinations of the *five types* 1, A, AB, ABC, $ABCD$, $(1+4+6+4+1=16$ as before). But in Prof. Clifford's question 1 means $A'B'C'D'$, A means $AB'C'D'$, &c.; viz. each of the symbols means an aggregate of four assertions; and the 16 symbols are thus *all of the same type*. Considering them in this point of view, the question is as to the number of types of the binary, ternary, &c., combinations of the sixteen combinations; for, according as these are combined,

$$\text{No. of types} = \frac{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14,\ 15}{1,\ 4,\ 6,\ 19,\ 27,\ 47,\ 55,\ 78,\ 55,\ 47,\ 27,\ 19,\ 6,\ 4,\ 1}$$

together.

In the first mentioned point of view the like question arises, in regard to the sets belonging to the five different types separately or in combination with each other; for instance, taking only the six symbols of the type AB, these may be taken 1, 2, 3, 4, or 5 together, and we have in these cases respectively

$$\text{No. of types} = \frac{1,\ 2,\ 3,\ 4,\ 5}{1,\ 2,\ 2,\ 2,\ 1},$$

as is very easily verified; but if the number of letters $A, B, \ldots$ be greater (say this $= 8$), or, instead of letters, writing the numbers 1, 2, 3, 4, 5, 6, 7, 8, then the question is that of the number of types of combination of the 28 duads 12, 13, ..., 78, taken 1, 2, 3, ..., 27 together, a question presenting itself in geometry in regard to the bitangents of a quartic curve (see Salmon's *Higher Plane Curves*, Ed. 2 (1873), pp. 222 *et seq.*): the numbers, so far as they have been obtained, are

$$\text{No. of types} = \frac{1,\ 2,\ 3,\ \ 4, \ldots,\ 24,\ 25,\ 26,\ 27}{1,\ 2,\ 5,\ 11, \ldots,\ 11,\ \ 5,\ \ 2,\ \ 1}.$$

It might be interesting to complete the series, and, more generally, to determine the number of the types of combination of the $\frac{1}{2}n(n-1)$ duads of n letters.

669.

ON A PROBLEM OF ARRANGEMENTS.

[From the *Proceedings of the Royal Society of Edinburgh*, t. IX. (1878), pp. 338—342.]

It is a well-known problem to find for n letters the number of the arrangements in which no letter occupies its original place; and the solution of it is given by the following general theorem:—viz., the number of the arrangements which satisfy any r conditions is

$$(1-1)(1-2)\ldots\ldots(1-r),$$
$$=1-\Sigma(1)+\Sigma(12)-\ldots\ldots\pm(12\ldots r),$$

where 1 denotes the whole number of arrangements; (1) the number of them which fail in regard to the first condition; (2) the number which fail in regard to the second condition; (12) the number which fail in regard to the first condition, and also in regard to the second condition; and so on: $\Sigma(1)$ means $(1)+(2)+\ldots+(r)$: $\Sigma(12)$ means $(12)+(13)+(2r)+\ldots+(r-1,\ r)$; and so on, up to $(12\ldots r)$, which denotes the number failing in regard to each of the r conditions.

Thus, in the special problem, the first condition is that the letter in the first place shall not be a; the second condition is that the letter in the second place shall not be b; and so on; taking $r=n$, we have the known result,

$$\text{No.}=\Pi n-\frac{n}{1}\Pi(n-1)+\frac{n.n-1}{1.2}\Pi(n-2).+\ldots\pm\frac{n.n-1\ldots2.1}{1.2\ldots n},$$
$$=1.2.3\ldots n\left\{1-\frac{1}{1}+\frac{1}{1.2}-\frac{1}{1.2.3}+\ldots\pm\frac{1}{1.2.3\ldots n}\right\},$$

giving for the several cases

$$n=2,\ 3,\ 4,\ \ 5,\ \ \ 6,\ \ \ \ 7,\ldots$$
$$\text{No.}=1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ldots$$

I proceed to consider the following problem, suggested to me by Professor Tait, in connexion with his theory of knots: to find the number of the arrangements of n letters $abc\ldots jk$, when the letter in the first place is not a or b, the letter in the second place not b or $c,\ldots$, the letter in the last place not k or a.

Numbering the conditions $1, 2, 3, \ldots, n$, according to the places to which they relate, a single condition is called [1]; two conditions are called [2] or [1, 1], according as the numbers are consecutive or non-consecutive: three conditions are called [3], [2, 1], or [1, 1, 1], according as the numbers are all three consecutive, two consecutive and one not consecutive, or all non-consecutive; and so on: the numbers which refer to the conditions being always written in their natural order, and it being understood that they follow each other cyclically, so that 1 is consecutive to n. Thus, $n=6$, the set 126 of conditions is [3], as consisting of 3 consecutive conditions; and similarly 1346 is [2, 2].

Consider a single condition [1], say this is 1; the arrangements which fail in regard to this condition are those which contain in the first place a or b; whichever it be, the other $n-1$ letters may be arranged in any form whatever; and there are thus $2\Pi(n-1)$ failing arrangements.

Next for two conditions; these may be [2], say the conditions are 1 and 2: or else [1, 1], say they are 1 and 3. In the former case, the arrangements which fail are those which contain in the first and second places ab, ac, or bc: and for each of these, the other $n-2$ letters may be arranged in any order whatever; there are thus $3\Pi(n-2)$ failing arrangements. In the latter case, the failing arrangements have in the first place a or b, and in the third place c or d,—viz. the letters in these two places are $a.c$, $a.d$, $b.c$, or $b.d$, and in each case the other $n-2$ letters may be arranged in any order whatever: the number of failing arrangements is thus $=2.2.\Pi(n-2)$. And so, in general, when the conditions are $[\alpha, \beta, \gamma, \ldots]$, the number of failing arrangements is

$$=(\alpha+1)(\beta+1)(\gamma+1)\ldots\Pi(n-\alpha-\beta-\gamma\ldots).$$

But for $[n]$, that is, for the entire system of the n conditions, the number of failing arrangements is (not as by the rule it should be $=n+1$, but) $=2$,—viz. the only arrangements which fail in regard to each of the n conditions are (as is at once seen), $abc\ldots jk$, and $bc\ldots jka$.

Changing now the notation so that [1], [2], [1, 1], &c., shall denote the *number* of the conditions [1], [2], [1, 1], &c., respectively, it is easy to see the form of the general result. If, for greater clearness, we write $n=6$, we have

$$\begin{array}{lllll}
1 & -\Sigma(1) & +\Sigma(12) & -\Sigma(123) & \\
\text{No.} = 720 - \{([1]=6)\,2\}\,120 + & \left\{\begin{array}{l}([2]\ \ =6)\,3\\ +([1,1]=9)\,2.2\end{array}\right\}24 - & \left\{\begin{array}{l}([3]\ \ =6)\,4\\ +([2,1]\ \ =12)\ \ 3.2\\ +([1,1,1]=\ 2)\,2.2.2\end{array}\right\}6 & & \\
 & +\Sigma(1234) & -\Sigma(12345) & +(123456) & \\
 & + \left\{\begin{array}{l}([4]\ \ =6)\,5\\ +([3,1]=6)\,4.2\\ +([2,2]=3)\,3.3\end{array}\right\}2 & -\{([5]=6)\,6\}\,1 & +\{([6]=1)\,2\}; &
\end{array}$$

or, reducing into numbers, this is

$$\text{No.} = 720 - 1440 + 1296 - 672 + 210 - 36 + 2, \qquad = 80.$$

The formula for the next succeeding case, $n = 7$, gives

$$\text{No.} = 5040 - 10080 + 9240 - 5040 + 1764 - 392 + 49 - 2, \quad = 579.$$

Those for the preceding cases, $n = 3, 4, 5$, respectively are

$$\text{No.} = 6 - 12 + 9 - 2 \qquad = 1,$$

$$\text{No.} = 24 - 48 + 40 - 16 + 2 \qquad = 2,$$

$$\text{No.} = 120 - 240 + 210 - 100 + 25 - 2 \qquad = 13.$$

We have in general $[1] = n$, $[2] = n$, $[1, 1] = \frac{1}{2}n(n-3)$; and in the several columns of the formulæ the sums of the numbers thus represented are equal to the coefficients of $(1+1)^2$: thus, when $n = 6$ as above, the sums are 6, 15, 20, 15, 6, 1. As regards the calculation of the numbers in question, any symbol $[\alpha, \beta, \gamma]$ is a sum of symbols $[\alpha - \alpha' + \beta - \beta' + \gamma - \gamma' + \ldots]$, where $\alpha' + \beta' + \gamma' + \ldots$ is any partition of $n - (\alpha + \beta + \gamma + \ldots)$; read, of the series of numbers $1, 2, 3, \ldots, n$, taken in cyclical order beginning with any number, retain α, omit α', retain β, omit β', retain γ, omit $\gamma', \ldots$. Thus in particular, $n = 6$, $[1, 1]$ is a sum of symbols $[1 - 3 + 1 - 1]$ and $[1 - 2 + 1 - 2]$; it is clear that any such symbol $[\alpha - \alpha' + \beta - \beta' + \ldots]$ is $= n$ or a submultiple of n (in particular, if n be prime, the symbol is always $= n$): and we thus in every case obtain the value of $[\alpha, \beta, \gamma, \ldots]$ by taking for the negative numbers the several partitions of

$$n - (\alpha + \beta + \gamma + \ldots),$$

and for each symbol

$$[\alpha - \alpha' + \beta - \beta' + \gamma - \gamma' + \ldots],$$

writing its value, $= n$ or a given submultiple of n, as just mentioned. There would, I think, be no use in pursuing the matter further, by seeking to obtain an analytical expression for the symbols $[\alpha, \beta, \gamma, \ldots]$.

For the actual formation of the required arrangements, it is of course easy, when all the arrangements are written down, to strike out those which do not satisfy the prescribed conditions, and so obtain the system in question. Or introducing the notion of substitutions*, and accordingly considering each arrangement as derived by a substitution from the primitive arrangement $abcd \ldots jk$, we can write down the substitutions which give the system of arrangements in which no letter occupies its original place: viz. we must for this purpose partition the n letters into parts, no part less than 2, and then in each set taking one letter (say the first in alphabetical order) as fixed, permute in every possible way the other letters of the set; we thus obtain

* In explanation of the notation of substitutions, observe that $(abcde)$ means that a is to be changed into b, b into c, c into d, d into e, e into a; and similarly $(ab)(cde)$ means that a is to be changed into b, b into a, c into d, d into e, e into c.

all the substitutions which move every letter. Thus when $n=5$, we obtain the 44 substitutions for the letters *abcde*, viz. these are

(*abcde*), &c., 24 symbols obtained by permuting in every way the four letters *b, c, d, e*;

(*ab*)(*cde*), &c., 20 symbols corresponding to the 10 partitions *ab, cde*, and for each of them 2 arrangements such as *cde, ced*.

And then if we reject those symbols which contain in any ( ) two consecutive letters, we have the substitutions which give the arrangements wherein the letter in the first place is not *a* or *b*, that in the second place not *b* or *c*, and so on. In particular, when $n=5$, rejecting the substitutions which contain in any ( ), *ab, bc, cd, de*, or *ea*, we have 13 substitutions, which may be thus arranged:—

(*acbed*), (*ac*)(*bed*), (*acebd*), (*adbec*), (*aedbc*),
(*aedbc*), (*bd*)(*aec*),
(*acedb*), (*ce*)(*adb*),
(*aecbd*), (*ad*)(*bec*),
(*adceb*), (*be*)(*adc*).

Here in the first column, performing on the symbol (*acbed*) the substitution (*abcde*), we obtain (*bdcae*), $=$(*aebdc*), the second symbol; and so again and again operating with (*abcde*), we obtain the remaining symbols of the column; these are for this reason said to be of the same type. In like manner, symbols of the second column are of the same type; but the symbols in the remaining three columns are each of them a type by itself; viz. operating with (*abcde*) upon (*acebd*), we obtain (*bdace*), $=$(*acebd*); and the like as regards (*adbec*) and (*aedbc*) respectively. The 13 substitutions are thus of 5 different types, or say the arrangements to which they belong, viz.

cebad, ceabd, cdeab, deabc, eabcd,
edacb, edabc,
caebd, daebc,
edbac, debac,
daecb, deacb,

are of 5 different types. The question to determine for any value of n, the number of the different types, is, it would appear, a difficult one, and I do not at present enter upon it.

670.

[NOTE ON MR MUIR'S SOLUTION OF A "PROBLEM OF ARRANGEMENT."]

[From the *Proceedings of the Royal Society of Edinburgh*, t. IX. (1878), pp. 388—391.]

THE investigation may be carried further: writing for shortness u_3, u_4, &c., in place of $\Psi(3)$, $\Psi(4)$, &c., the equations are

$$\begin{aligned}
u_3 &= 1,\\
u_4 &= 2u_3,\\
u_5 &= 3u_4 + 6u_3 + 1,\\
u_6 &= 4u_5 + 8u_4 + 12u_3,\\
u_7 &= 5u_6 + 10u_5 + 15u_4 + 18u_3 + 1.
\end{aligned}$$

Hence assuming

$$u = u_3 + u_4x + u_5x^2 + u_6x^3 + u_7x^4 + \ldots,$$

we have

$$\begin{aligned}
u = \frac{1}{1-x^2} &+ u_3(2x + 6x^2 + 12x^3 + 18x^4 + \ldots)\\
&+ u_4(3x^2 + 8x^3 + 15x^4 + 22x^5 + \ldots)\\
&+ u_5(4x^3 + 10x^4 + 18x^5 + 26x^6 + \ldots)\\
&+ u_6(5x^4 + 12x^5 + 21x^6 + 30x^7 + \ldots);
\end{aligned}$$

so that, forming the equation

$$\begin{aligned}
u'\frac{x^2}{(1-x)^2} = \quad & u_4(x^2 + 2x^3 + 3x^4 + 4x^5 + \ldots)\\
&+ u_5(2x^3 + 4x^4 + 6x^5 + 8x^6 + \ldots)\\
&+ u_6(3x^4 + 6x^5 + 9x^6 + 12x^7 + \ldots),
\end{aligned}$$

where u' denotes $\dfrac{du}{dx}$, we have

$$u - u'\frac{x^3}{(1-x)^2} = \frac{1}{1-x^2} + (u_3 + u_4 x + u_5 x^2 + \ldots)(2x + 6x^2 + 12x^3 + 18x^4 + \ldots)$$
$$= \frac{1}{1-x^2} + u(2x + 6x^2 + 12x^3 + 18x^4 + \ldots);$$

or, what is the same thing,

$$u - u'\frac{x^2}{(1-x)^2} = \frac{1}{1-x^2} + u\left\{\frac{2x}{(1-x)^2} - \frac{2x^4}{(1-x)^2(1+x)}\right\};$$

that is,

$$\left\{1 - \frac{2x}{(1-x)^2} + \frac{2x^4}{(1-x)^2(1+x)}\right\}u - \frac{x^2}{(1-x)^2}u' = \frac{1}{1-x^2}.$$

This equation may be simplified: write

$$u = -\frac{1-x^2}{x^4}Q, \quad = \left(-\frac{1}{x^4} + \frac{1}{x^2}\right)Q,$$

then

$$u' = \left(\frac{4}{x^5} - \frac{2}{x^3}\right)Q + \frac{1-x^2}{x^4}Q',$$

and the equation is

$$\left\{-\frac{1-x^2}{x^4} + \frac{2}{x^3}\frac{1+x}{(1+x)^2} - \frac{2}{(1-x)^2} - \frac{4}{x^3}\frac{1}{(1-x)^2} + \frac{2}{x(1-x)^2}\right\}Q + \frac{1+x}{(1+x)x^2}Q' = \frac{1}{1-x^2};$$

that is,

$$\left\{-\frac{1}{x^4} + \frac{1}{x^2} - \frac{2}{x^3(1-x)^2} + \frac{2}{x^2(1-x)^2} + \frac{2}{x(1-x)^2} - \frac{2}{(1-x)^2}\right\}Q + \frac{1+x}{(1-x)x^2}Q' = \frac{1}{1-x^2},$$

viz. this is

$$\left\{-\frac{(1-x)^2}{x^4} + \frac{(1-x)^2}{x^2} - \frac{2}{x^3} + \frac{2}{x^2} + \frac{2}{x} - 2\right\}Q + \frac{1-x^2}{x^2}Q' = \frac{1-x}{1+x},$$

that is,

$$\left\{-\frac{1}{x^4} + \frac{2}{x^2} - 1\right\}Q + \frac{1-x^2}{x^2}Q' = \frac{1-x}{1+x},$$

or

$$-\frac{(1-x^2)^2}{x^4}Q + \frac{1-x^2}{x^2}Q' = \frac{1-x}{1+x};$$

or finally,

$$Q\left(1 - \frac{1}{x^2}\right) + Q' = \frac{x^2}{(1+x)^2},$$

giving

$$Q = e^{-\left(x+\frac{1}{x}\right)}\int\frac{x^2}{(x+1)^2}e^{x+\frac{1}{x}}\,dx,$$

and thence

$$u = \frac{x^2-1}{x^4}e^{-\left(x+\frac{1}{x}\right)}\int\frac{x^2}{(x+1)^2}e^{\left(x+\frac{1}{x}\right)}\,dx,$$

which is the value of the generating function

$$u = u_3 + u_4 x + u_5 x^2 + \&c.$$

But for the purpose of calculation it is best to integrate by a series the differential equation for Q: assuming

$$Q = -q_3x^4 - q_4x^5 - q_5x^6 - \dots,$$

we find

$$\begin{aligned} q_4 &= 4q_3 & & -2, \\ q_5 &= 5q_4 & +q_3 & +3, \\ q_6 &= 6q_5 & +q_4 & -4, \\ q_7 &= 7q_6 & +q_5 & +5, \\ &\vdots \\ q_n &= nq_{n-1} + q_{n-2} + (-)^{n-1}(n-2). \end{aligned}$$

We have thus for $q_3, q_4, q_5, \dots$ the values 1, 2, 14, 82, 593, 4820, ..., and thence

$$u = (1 - x^2)(1 + 2x + 14x^2 + 82x^3 + 593x^4 + 4820x^5 + \dots),$$

viz. writing

1	2	14	82	593	4820...
		−1	−2	−14	−82

the values of $u_3, u_4, \dots$ are 1, 2, 13, 80, 579, 4738, ...,

agreeing with the results found above.

In the more simple problem, where the arrangements of the n things are such that no one of them occupies its original place, if u_n be the number of arrangements, we have

$$\begin{aligned} u_2 &= 1 & &= 1, \\ u_3 &= 2\,u_2 & &= 2, \\ u_4 &= 3\,(u_3 + u_2) & &= 9, \\ u_5 &= 4\,(u_4 + u_3) & &= 44, \\ &\vdots \\ u_{n+1} &= n\,(u_n + u_{n-1}), \end{aligned}$$

and writing

$$u = u_2 + u_3x + u_4x^2 + \dots,$$

we find

$$u = 1 + (2x + 3x^2)\,u + (x^2 + x^3)\,u';$$

that is,

$$(-1 + 2x + 3x^2)\,u + (x^2 + x^3)\,u' = -1,$$

or, what is the same thing,

$$u' + \left(\frac{3}{x} - \frac{1}{x^2}\right)u = -\frac{1}{x^2(1+x)},$$

whence

$$u = x^{-3}\,e^{-\frac{1}{x}}\int\frac{-x}{1+x}\,e^{\frac{1}{x}}\,dx.$$

But the calculation is most easily performed by means of the foregoing equation of differences, itself obtained from the differential equation written in the foregoing form,

$$(-1 + 2x + 3x^2)\,u + (x^2 + x^3)\,u' = -1.$$

671.

ON A SIBI-RECIPROCAL SURFACE.

[From the *Berlin. Akad. Monatsber.*, (1878), pp. 309—313.]

THE question of the generation of a sibi-reciprocal surface—that is, a surface the reciprocal of which is of the same order and has the same singularities as the original surface—was considered by me in the year 1868, see *Proc. London Math. Soc.* t. II. pp. 61—63, [part of 387], where it is remarked that if a surface be considered as the envelope of a quadric surface varying according to given conditions, then the reciprocal surface is given as the envelope of a quadric surface varying according to the reciprocal conditions; whence, if the conditions be sibi-reciprocal, it follows that the surface is a sibi-reciprocal surface. And I gave as instances the surface which is the envelope of a quadric surface touching each of 8 given lines; and also the surface called the "tetrahedroid," which is a homographic transformation of Fresnel's Wave Surface and a particular case of the quartic surface with 16 nodes.

The interesting surface of the order 8, recently considered by Herr Kummer, *Berl. Monatsber.*, Jan. 1878, pp. 25—36, is included under the theory. In fact, if we consider a line L, whereof the six coordinates

$$a,\ b,\ c,\ f,\ g,\ h,$$

satisfy each of the three linear relations

$$\begin{aligned} f_1a + g_1b + h_1c + a_1f + b_1g + c_1h &= 0, \\ f_2a + g_2b + h_2c + a_2f + b_2g + c_2h &= 0, \\ f_3a + g_3b + h_3c + a_3f + b_3g + c_3h &= 0, \end{aligned}$$

the locus of this line is a quadric surface the equation of which is

$$\begin{aligned} T = (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 \\ + [(abg) - (cah)]\,xw + [(bfg) + (chf)]\,yz \\ + [(bch) - (abf)]\,yw + [(cgh) + (afg)]\,zx \\ + [(caf) - (bcg)]\,zw + [(ahf) + (bgh)]\,xy = 0, \end{aligned}$$

where (agh) is used to denote the determinant $\begin{vmatrix} a_1, & g_1, & h_1 \\ a_2, & g_2, & h_2 \\ a_3, & g_3, & h_3 \end{vmatrix}$, and so for the other symbols. Considering the reciprocal of the line L in regard to the quadric surface $X^2+Y^2+Z^2+W^2=0$, the six coordinates of the reciprocal line are

$$f,\ g,\ h,\ a,\ b,\ c,$$

and it is hence at once seen that the locus of the reciprocal line is the quadric surface obtained from the equation $T=0$ by interchanging therein the symbolical quantities a, b, c and f, g, h: viz. writing also $(\xi, \eta, \zeta, \omega)$ in place of (x, y, z, w), the new equation is

$$\begin{aligned} T' = {} & (fbc)\,\xi^2 + (gca)\,\eta^2 + (hab)\,\zeta^2 + (fgh)\,\omega^2 \\ & + [(fgb) - (hfc)]\,\xi\omega + [(fab) + (hca)]\,\eta\zeta \\ & + [(ghc) - (fga)]\,\eta\omega + [(gbc) + (fab)]\,\zeta\xi \\ & + [(hfa) - (ghb)]\,\zeta\omega + [(hca) + (gbc)]\,\xi\eta = 0\,; \end{aligned}$$

or, what is the same thing, this equation $T'=0$ is the equation of the original quadric surface (the locus of L) expressed in terms of the plane-coordinates ξ, η, ζ, ω.

Now considering each of the quantities a_1, b_1, c_1, f_1, g_1, h_1, a_2, b_2, etc., a_3, b_3, etc., as a given linear function of a variable parameter λ, say $a_1 = a_1' + a_1''\lambda$, $b_1 = b_1' + b_1''\lambda$, etc., the equation $T=0$ takes the form

$$A\lambda^3 + 3B\lambda^2 + 3C\lambda + D = 0,$$

where A, B, C, D are given quadric functions of the coordinates x, y, z, w; and the envelope of the quadric surface $T=0$ is Herr Kummer's surface of the eighth order

$$(AD - BC)^2 - 4\,(AC - B^2)\,(BD - C^2) = 0.$$

In like manner the equation $T'=0$ takes the form

$$A'\lambda^3 + 3B'\lambda^2 + 3C'\lambda + D' = 0,$$

where A', B', C', D' are given functions of the coordinates ξ, η, ζ, ω; and we have

$$(A'D' - B'C')^2 - 4\,(A'C' - B'^2)\,(B'D' - C'^2) = 0,$$

as the equation of the reciprocal surface; or (what is the same thing) as that of the original surface, regarding ξ, η, ζ, ω as plane-coordinates.

In regard to the foregoing equation $T=0$, it is to be noticed that, if a_1, b_1, c_1, f_1, g_1, h_1; a_2, b_2, etc., a_3, b_3, etc., instead of being arbitrary coefficients, were the coordinates of three given lines L_1, L_2, L_3 respectively; that is, if we had

$$a_1 f_1 + b_1 g_1 + c_1 h_1 = 0,$$
$$a_2 f_2 + b_2 g_2 + c_2 h_2 = 0,$$
$$a_3 f_3 + b_3 g_3 + c_3 h_3 = 0,$$

then the three linear relations satisfied by (a, b, c, f, g, h) would express that the line L was a line meeting each of the three given lines L_1, L_2, L_3: the locus is therefore the quadric surface which passes through these three lines; and I have in my paper "On the six coordinates of a Line," *Camb. Phil. Trans.*, t. XI. (1869), pp. 290—323, [435], found the equation to be the foregoing equation $T=0$. But it is easy to see that the same equation subsists in the case where the three equations $a_1f_1+b_1g_1+c_1h_1=0$, etc., are not satisfied. For the several coefficients being perfectly general, any one of the three linear relations may be replaced by a linear combination of these equations; that is, in place of a_1, b_1, c_1, f_1, g_1, h_1, we may write a_1', b_1', c_1', f_1', g_1', h_1', where $a_1'=\theta_1a_1+\theta_2a_2+\theta_3a_3$, $b_1'=\theta_1b_1+\theta_2b_2+\theta_3b_3$, etc.; and these factors θ_1, θ_2, θ_3 may be conceived to be such that the condition in question $a_1'f_1'+b_1'g_1'+c_1'h_1'=0$ is satisfied. Similarly the second set of coefficients may be replaced by a_2', b_2', c_2', f_2', g_2', h_2', where $a_2'=\phi_1a_1+\phi_2a_2+\phi_3a_3$, etc., and the condition $a_2'f_2'+b_2'g_2'+c_2'h_2'=0$ is satisfied: and the third set by a_3', b_3', c_3', f_3', g_3', h_3', where $a_3'=\psi_1a_1+\psi_2a_2+\psi_3a_3$, etc., and the condition $a_3'f_3'+b_3'g_3'+c_3'h_3'=0$ is satisfied. We have therefore an equation $0=(a'g'h')\,x^2+$ etc., which only differs from the equation $T=0$ by having therein the accented letters in place of the unaccented ones: and, substituting for the accented letters their values, the whole divides by the determinant $(\theta\phi\psi)$, and throwing this out we obtain the required equation $T=0$.

But it is easier to obtain the equation $T=0$ directly. We have

$$\begin{aligned} \cdot\quad hy-gz+aw&=0,\\ -hx\quad\cdot\quad+fz+bw&=0,\\ gx-fy\quad\cdot\quad+cw&=0,\\ -ax-by-cz\quad\cdot\quad&=0; \end{aligned}$$

viz. in virtue of the equation $af+bg+ch=0$ which connects the six coordinates, these four equations are equivalent to two independent equations which are the equations of the line (a, b, c, f, g, h): or, what is the same thing, any three of these equations imply the fourth equation and also the relation $af+bg+ch=0$.

We might, from the three linear relations and any three of the last-mentioned four equations, eliminate a, b, c, f, g, h and so obtain the required equation $T=0$; but it is better, introducing the arbitrary coefficients α, β, γ, δ, to employ all the four equations. The result of the elimination is thus given in the form

$$\begin{vmatrix} \alpha, & w, & & & -z, & y \\ \beta, & & w, & z, & & -x \\ \gamma, & & & w, & -y, & x, \\ \delta, & x, & y, & z, & & \\ & f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ & f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ & f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix}=0,$$

viz. the left-hand side here contains the factor $-(\alpha x+\beta y+\gamma z+\delta w)$; throwing this out, we obtain the required quadric equation $T=0$. If for the calculation of T we compare the terms containing δ, we have

$$Tw=\begin{vmatrix} w, & & & -z, & y \\ w, & w, & z, & & -x \\ & & w, & -y, & x, \\ f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix},$$

where observe that, writing $w=0$, the right-hand side vanishes as containing the factor

$$\begin{vmatrix} & -z, & y \\ z, & & -x \\ -y, & x, & \end{vmatrix}.$$

Hence the right-hand side divides by w; and one of its terms being evidently $w^3(abc)$, T contains as it should do the term $(abc)w^2$: the remaining terms can be found without any difficulty, and the foregoing expression for T is thus verified.

672.

ON THE GAME OF MOUSETRAP.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 8—10.]

In the note "A Problem in Permutations," *Quarterly Mathematical Journal*, t. I. (1857), p. 79, [161], I have spoken of the problem of permutations presented by this game.

A set of cards—ace, two, three, &c., say up to thirteen—are arranged (in any order) in a circle with their faces upwards; you begin at any card, and count one, two, three, &c., and if upon counting, suppose the number five, you arrive at the card five, the card is thrown out; and beginning again with the next card, you count one, two, three, &c., throwing out (if the case happen) a new card as before, and so on until you have counted up to thirteen, without coming to a card which has to be thrown out. The original question proposed was: for any given number of cards to find the arrangement (if any) which would throw out all the cards in a given order; but (instead of this) we may consider *all* the different arrangements of the cards, and inquire how many of these there are in which all or any given smaller number of the cards will be thrown out; and (in the several cases) in what orders the cards are thrown out. Thus to take the simple case of four cards, the different arrangements, with the cards thrown out in each, are

1, 2, 3, 4	1,
1, 2, 4, 3	1, 3, 4, 2,
1, 3, 2, 4	1,
1, 3, 4, 2	1,
1, 4, 2, 3	1, 2, 3, 4,
1, 4, 3, 2	1,
2, 1, 3, 4	3, 4,
2, 1, 4, 3	—
2, 3, 4, 1	—
2, 3, 1, 4	4,
2, 4, 1, 3	—
2, 4, 3, 1	3, 2,
3, 1, 2, 4	4,
3, 1, 4, 2	—
3, 2, 1, 4	4, 2, 1, 3,
3, 2, 4, 1	2, 3,
3, 4, 1, 2	—
3, 4, 2, 1	—
4, 1, 2, 3	—
4, 1, 3, 2	3,
4, 2, 1, 3	2, 1, 3, 4,
4, 2, 3, 1	3, 1, 2, 4,
4, 3, 1, 2	—
4, 3, 2, 1	—.

Classifying these so as to show in how many arrangements a given card or permutation of cards is thrown out, we have the table

No.	Thrown out.
9	none
4	1
1	3
2	4
1	3, 2
1	2, 3
1	3, 4
1	1, 3, 4, 2
1	1, 2, 3, 4
1	4, 2, 1, 3
1	2, 1, 3, 4
1	3, 1, 2, 4,
24,	

viz. there are nine arrangements in which no card is thrown out, four arrangements in which only the card 1 is thrown out, one arrangement in which only the card 3 is thrown out, and so on.

It will be observed that there are five arrangements in which all the cards are thrown out, each throwing them out in a different order; there are thus only five orders in which all the cards are thrown out.

The general question is of course to form a like table for the numbers 5, 6, ..., or any greater number of cards.

673.

NOTE ON THE THEORY OF CORRESPONDENCE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 32, 33.]

If the point P on a given curve U of the order m, and the point Q on a given curve V of the order m', have a (1, 1) correspondence, this implying that the two curves have the same deficiency; then if PQ intersects the consecutive line $P'Q'$ in a point R, the locus of R is a curve W of the class $m+m'$, and the point R on this curve has, in general (but not universally), a (1, 1) correspondence with the point P on U or with the point Q on V. For, considering the correspondence of the points P and R, to a given position of P there corresponds, it is clear, a single position of R; on the other hand, starting from R, the tangent at this point to the curve W meets the curve U in m points and the curve V in m' points, but it is in general only one of the m points and only one of the m' points which are corresponding points on the curves U and V; that is, it is only one of the m points which is a point P; and the correspondence of (P, R) is thus a (1, 1) correspondence.

But the curves U, V may be such that the correspondence of (P, R) is not a (1, 1) but a $(k, 1)$ correspondence; viz., that to a given position of P there corresponds a single position of R, but to a given position of R, k positions of P. To show that this is so, imagine through P a line Π having therewith a $(k, 1)$ correspondence; P being, as above, a point on the curve U, the line in question envelopes a curve W; and the correspondence is such that, for any given position of P on the curve U, we have through it a single position of the line: but, for a given tangent of the curve W, we have upon it k positions of the point P, viz. k of the m intersections of the line with the curve U are points corresponding to the line; this, of course, implies that the curve U is not any curve whatever of the order m, but a curve of a peculiar nature.

Imagine now that we have on the line Π a point Q, having with P a (1, 1) correspondence of a given nature: to fix the ideas, suppose P, Q are harmonics in regard to a given conic: since on each of the lines Π there are k positions of P, there are also on the line k positions of Q, and the locus of these k points Q is a curve V, say of the order m'.

The point P on the curve U and the point Q on the curve V have a (1, 1) correspondence. For, consider P as given: there is a single position of the line Π intersecting V in m' points, but obviously only one of these is the point Q. And consider Q as given: then through Q we have say μ tangents of the curve W; each of these tangents intersects the curve U in m points, k of which are points P, but for a tangent taken at random no one of these is the correspondent of Q; it is, in general, only one of the μ tangents which has upon it k points P, one of them being the point corresponding to Q; that is, to a given position of Q there corresponds a single position of P; and the correspondence of the points (P, Q) is thus a (1, 1) correspondence.

We have thus the point P on the curve U and the point Q on the curve V, which points have with each other a (1, 1) correspondence; and the line Π is the line PQ joining these points; this intersects the consecutive line in a point R; and the locus of R is the curve W. To a given position of P there corresponds a single line Π, and therefore a single position of R; but to a given position of R there correspond k positions of P, viz. drawing at R the tangent to the curve W, this is a line Π having upon it k points P, or the correspondence of (P, Q) is, as stated, a $(k, 1)$ correspondence.

The foregoing considerations were suggested to me by the theory of parallel curves. Take a curve parallel to a given curve, for example, the ellipse; this is a curve of the order δ, such that every normal thereto is a normal at two distinct points; and the curve has as its evolute the evolute of the ellipse, *or, more accurately, the evolute of the ellipse taken twice;* but, attending only to the evolute taken once, each tangent of the evolute is a normal of the parallel curve at two distinct points thereof, and the points of the parallel curve have with those of the evolute not a (1, 1) but a (2, 1) correspondence.

674.

NOTE ON THE CONSTRUCTION OF CARTESIANS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), p. 34.]

If $\rho = a + b\cos\theta$, and $r = \frac{1}{2}\{\rho \pm \sqrt{(\rho^2 - c^2)}\}$, then obviously $r^2 - r\rho + \frac{1}{4}c^2 = 0$, that is,

$$r^2 - r(a + b\cos\theta) + \tfrac{1}{4}c^2 = 0,$$

which is the equation of a Cartesian. Here $\rho = a + b\cos\theta$ is the equation of a limaçon or nodal Cartesian, having the origin for the node; and for any given value of θ, deducing from the radius vector of the limaçon the new radius vector r by the above formula $r = \frac{1}{2}\{\rho \pm \sqrt{(\rho^2 - c^2)}\}$, we obtain a Cartesian, or by giving different values to c, a series of Cartesians having the origin for a common focus. The construction is a very convenient one.

675.

ON THE FLEFLECNODAL PLANES OF A SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 49—51.]

IF at a node (or double point) of a plane curve there is on one of the branches an inflexion, (that is, if the tangent has a 3-pointic intersection with the branch), the node is said to be a flecnode; and if there is on each of the branches an inflexion, then the node is said to be a fleflecnode. The tangent plane of a surface intersects the surface in a plane curve having at the point of contact a node; if this is a flecnode or a fleflecnode, the tangent plane is said to be a flecnodal or a fleflecnodal plane accordingly. For a quadric surface each tangent plane is fleflecnodal; this is obvious geometrically (since the section is a pair of lines), and it will presently appear that the analytical condition for such a plane is satisfied. In fact, if the origin be taken at a point of a surface, so that $z=0$ shall be the equation of the tangent plane, then in the neighbourhood of the point we have

$$z=(x,\ y)^2+(x,\ y)^3+\&c.;$$

and the condition for a fleflecnodal plane is that the term $(x,\ y)^2$ shall be a factor of the succeeding term $(x,\ y)^3$. Now for a quadric surface the equation is

$$z=\tfrac{1}{2}\{ax^2+2hxy+by^2+2(fy+gx)z+cz^2\};$$

that is,

$$z(1-fy-gx-\tfrac{1}{2}cz)=\tfrac{1}{2}(ax^2+2hxy+by^2),$$

or developing as far as the third order in $(x,\ y)$, we have

$$z=\tfrac{1}{2}(ax^2+2hxy+by^2)(1+fy+gx),$$

so that the condition in question is satisfied.

In what follows, I take for greater simplicity $h=0$, (viz. $x=0$, $y=0$ are here the tangents to the two curves of curvature at the point in question), and to avoid fractions write $2f$, $2g$ in place of f, g respectively; the developed equation of the quadric surface is thus

$$z=\tfrac{1}{2}(ax^2+by^2)+(ax^2+by^2)(gx+fy).$$

I consider the parallel surface, obtained by measuring off on the normal a constant length k. If, as usual, p, q denote $\frac{dz}{dx}$ and $\frac{dz}{dy}$ respectively, then, in general, (X, Y, Z) being the coordinates of the point on the parallel surface,

$$Z=z+\frac{k}{\sqrt{(1+p^2+q^2)}},$$

$$X=x-\frac{kp}{\sqrt{(1+p^2+q^2)}},$$

$$Y=y-\frac{kq}{\sqrt{(1+p^2+q^2)}}.$$

But in the present case

$$p=ax+3agx^2+2afxy+\ bgy^2,$$

$$q=by+\ afx^2+2bgxy+3bfy^2,$$

whence

$$X=x-k(ax+3agx^2+2afxy+\ bgy^2),$$

$$Y=y-k(by+\ afx^2+2bgxy+3bfy^2);$$

or, putting for convenience,

$$X=(1-ka)\xi,\quad Y=(1-kb)\eta,$$

then, for a first approximation $x=\xi$, $y=\eta$; whence, writing

$$P=3ag\xi^2+2af\xi\eta+\ bg\eta^2,$$

$$Q=\ af\xi^2+2bg\xi\eta+3bf\eta^2,$$

we find

$$x=\xi+\frac{kP}{1-ka},\quad y=\eta+\frac{kQ}{1-kb},$$

and thence

$$p=a\left(\xi+\frac{k}{1+ka}P\right)+P=a\xi+\frac{P}{1-ka},$$

$$q\qquad\qquad\qquad\qquad\qquad\qquad=b\eta+\frac{Q}{1-kb}.$$

Hence

$$Z=\tfrac{1}{2}(a\xi^2+b\eta^2)+\frac{ka}{1-ka}\xi P+\frac{kb}{1-kb}\eta Q+(a\xi^2+b\eta^2)(g\xi+f\eta)$$
$$+k\left\{1-\tfrac{1}{2}(a^2\xi^2+b^2\eta^2)-\frac{a\xi P}{1-ka}-\frac{b\eta Q}{1-kb}\right\};$$

or, finally,

$$Z-k=\tfrac{1}{2}\{a(1-ka)\xi^2+b(1-kb)\eta^2\}+(a\xi^2+b\eta^2)(g\xi+f\eta),$$

where, changing the origin to the point $x=0$, $y=0$, $z=k$ on the parallel surface, the coordinates of the consecutive point are $Z-k$, $X, =(1-ka)\,\xi$, and $Y, =(1-kb)\,\eta$.

We cannot, by any determination of the value of k, make the plane $Z-k=0$ a fleflecnodal plane of the parallel surface; but if

$$k=\frac{af^2+bg^2}{a^2f^2+b^2g^2},$$

then

$$1-ka=\frac{bg^2(b-a)}{a^2f^2+b^2g^2},\quad 1-kb=\frac{af^2(a-b)}{a^2f^2+b^2g^2},$$

and the equation becomes

$$Z-k=\tfrac{1}{2}\frac{ab(b-a)}{a^2f^2+b^2g^2}(g^2\xi^2-f^2\eta^2)+(a\xi^2+b\eta^2)(g\xi+f\eta);$$

viz. the term of the second has here a factor $g\xi+f\eta$ which divides the term of the third order, and the plane $Z-k=0$ is a flecnodal plane of the parallel surface.

676.

NOTE ON A THEOREM IN DETERMINANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 55—57.]

It is well known that if 12, &c., denote the determinants formed with the matrix

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix} \right\|,$$

then, identically,

$$12 \,.\, 34 + 13 \,.\, 42 + 14 \,.\, 23 = 0.$$

The proper proof of the theorem is obtained by remarking that we have

$$0 = \left| \begin{matrix} \alpha, & \beta, & \gamma, & . \\ \alpha', & \beta', & \gamma', & . \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix} \right|,$$

as at once appears by subtracting the first and second lines from the third and fourth lines respectively; and, this being so, the development of the determinant gives the theorem. The theorem might, it is clear, have been obtained in four different forms according as in the determinant the missing terms were taken to be as above (δ, δ'), or to be (α, α'), (β, β'), or (γ, γ'); but the four results are equivalent to each other.

There is obviously a like theorem for the sums of products of determinants formed with the matrix

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \zeta \\ \alpha', & \beta', & \gamma', & \delta', & \epsilon', & \zeta' \\ \alpha'', & \beta'', & \gamma'', & \delta'', & \epsilon'', & \zeta'' \end{matrix} \right\|,$$

viz. the theorem is obtained by development of the determinant in an identical equation, such as

$$0 = \begin{vmatrix} \alpha, & \beta, & \gamma, & \delta, & ., & . \\ \alpha', & \beta', & \gamma', & \delta', & ., & . \\ \alpha'', & \beta'', & \gamma'', & \delta'', & ., & . \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \zeta \\ \alpha', & \beta', & \gamma', & \delta', & \epsilon', & \zeta' \\ \alpha'', & \beta'', & \gamma'', & \delta'', & \epsilon'', & \zeta'' \end{vmatrix};$$

but we thus obtain 15 results which are not all equivalent.

If, for shortness, we write

$$\begin{aligned} A &= 123\,.\,456, \\ -B &= 124\,.\,356, \\ -C &= 125\,.\,346, \\ D &= 126\,.\,345, \\ -E &= 134\,.\,256, \\ -F &= 135\,.\,246, \\ G &= 136\,.\,245, \\ -H &= 145\,.\,236, \\ I &= 146\,.\,235, \\ J &= 156\,.\,234, \end{aligned}$$

then the fifteen results are

$$\begin{aligned} A + B - C - D &= 0, \\ A + B - E - J &= 0, \\ A - C + F - I &= 0, \\ A - D + G - H &= 0, \\ A - E + F + G &= 0, \\ A - H - I - J &= 0, \\ B - C - G + H &= 0, \\ B - D - F + I &= 0, \\ B - E + H + I &= 0, \\ B - F - G - J &= 0, \\ C + D - E - J &= 0, \\ C - E + G + I &= 0, \\ C - F - H - J &= 0, \\ D - E + F + H &= 0, \\ D - G - I - J &= 0, \end{aligned}$$

which are all satisfied if only

$$\begin{aligned} A &= . \quad . + H + I + J, \\ B &= F + G \quad . \quad . + J, \\ C &= F \quad . + H \quad . + J, \\ D &= . \quad G \quad . + I + J, \\ E &= F + G + H + I + J; \end{aligned}$$

and we thus have these five relations between the ten products of determinants $A, B, C, D, E, F, G, H, I, J$.

677.

[ADDITION TO MR GLAISHER'S PAPER "PROOF OF STIRLING'S THEOREM."]

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 63, 64.]

It is easy to extend Mr Glaisher's investigation so as to obtain from it the more approximate value

$$\Pi n = \sqrt{(2\pi)}\, n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}.$$

We, in fact, have

$$\psi x = e^{2nx+ax^2+bx^3+\dots},$$

where $a, b, \dots$ are given functions of n, viz.

$$a = \tfrac{2}{3}\left\{\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2n+1)^2}\right\},$$

$$b = \tfrac{2}{5}\left\{\frac{1}{3^4}+\frac{1}{5^4}+\dots+\frac{1}{(2n+1)^4}\right\},$$

&c.

And hence writing $x = 1$, we have

$$\psi(1) = \tfrac{1}{2}\frac{1}{2^{2n}\,\Pi^2(n)}(2n+2)^{2n+1} = e^{2n+a+b+\dots},$$

that is,

$$\Pi n = \left(\frac{2n+2}{2}\right)^{\frac{2n+1}{2}} e^{-n-\frac{1}{2}(a+b+\dots)}$$

$$= (n+1)^{n+\frac{1}{2}} e^{-n-\frac{1}{2}(a+b+\dots)}$$

$$= n^{n+\frac{1}{2}}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} e^{-n-\frac{1}{2}(a+b+\dots)}.$$

Hence for $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ writing $e^{-(n+\frac{1}{2})\log\left(1+\frac{1}{n}\right)}$, the whole exponent of e is

$$(n+\tfrac{1}{2})\log\left(1+\frac{1}{n}\right)-n-\tfrac{1}{2}(a+b+\ldots)$$

$$=(n+\tfrac{1}{2})\left(\frac{1}{n}-\tfrac{1}{2}\frac{1}{n^2}+\tfrac{1}{3}\frac{1}{n^3}-\ldots\right)-n-\tfrac{1}{2}(a+b+\ldots)$$

$$=-n+1+\frac{1}{3.4}\frac{1}{n^2}-\frac{2}{4.6}\frac{1}{n^3}+\frac{3}{5.8}\frac{1}{n^4}-\ldots$$

$$-\tfrac{1}{2}(a+b+\ldots).$$

We have

$$\frac{1}{1^2}+\frac{1}{3^2}+\ldots+\frac{1}{(2n+1)^2}=\text{const.}-\frac{1}{4n}+\text{terms in }\frac{1}{n^2},\ \frac{1}{n^3},\ \&c.$$

(the constant is in fact $=\frac{1}{8}\pi^2$, but the value is not required), hence $a=\text{const.}-\frac{1}{6n}$ + terms in $\frac{1}{n^2}$, $\frac{1}{n^3}$, &c.; as regards b, c, &c., there are no terms in $\frac{1}{n}$, but we have $b=\text{const.}+\text{terms in }\frac{1}{n^2}$, &c., $c=\text{const.}+\text{terms in }\frac{1}{n^3}$, &c. Hence the whole exponent of e is

$$=-n+C+\frac{1}{12n}+\text{terms in }\frac{1}{n^2},\ \&c.$$

As in Mr Glaisher's investigation, it is shown that $e^{-C}=\sqrt{(2\pi)}$, and hence neglecting the terms in $\frac{1}{n^2}$, &c., the final result is

$$\Pi n=\sqrt{(2\pi)}\,n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}}.$$

678.

ON A SYSTEM OF QUADRIC SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 124, 125.]

THE following theorem was communicated to me by Dr Klein; "given in regard to a quadric surface two sibi-reciprocal line-pairs, the two tractors (or lines meeting each of the four lines) form a sibi-reciprocal line-pair." This may be presented under a more general form as a theorem relating to the tractors of any two line-pairs. In fact, if a given line-pair is taken to be sibi-reciprocal in regard to a quadric surface, we thereby establish only a four-fold relation between the coefficients of the surface, and the surface will still depend on five arbitrary parameters. Whence if two given line-pairs are taken to be each of them sibi-reciprocal in regard to one and the same quadric surface, we thereby establish only an eight-fold relation and the surface will still depend upon one arbitrary parameter. The theorem thus is: given any two line-pairs, then each of these, and also the pair of tractors, are sibi-reciprocal in regard to a singly infinite system of quadric surfaces.

The question arises, what is this system of quadric surfaces? It is, in fact, the system of surfaces having in common a skew quadrilateral constructed as follows: starting from the two given line-pairs, construct the two tractors, each of them intersected by the given line-pairs in two point-pairs; and on each tractor construct the double or sibi-reciprocal points of the involution thus determined; these double points are the vertices (those on the same tractor being opposite vertices) of the skew quadrilateral; which is consequently at once obtained by joining the two double points on the one tractor with the two double points on the other tractor. The construction is an immediate consequence of the following theorem: consider a skew quadrilateral, and drawing its two diagonals, take a pair of lines cutting each diagonal harmonically; these will be sibi-reciprocal in regard to any quadric surface through the skew quadrilateral.

The condition of passing through a skew quadrilateral is that of passing through a certain system of eight points; in fact, the eight points may be taken to be the four vertices and any four points on the four sides respectively. But observe that the system of the quadric surfaces through *any* eight points has the characteristics (1, 2, 3); viz. there are in the system 1 surface passing through a given point, 2 touching a given line, 3 touching a given plane; the system of surfaces through the same skew quadrilateral has the characteristics (1, 2, 1).

679.

ON THE REGULAR SOLIDS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 127—131.]

In a regular solid, or say in the spherical figure obtained by projecting such solid, by lines from the centre, on the surface of a concentric sphere, we naturally consider 1° the summits, 2° the centres of the faces, 3° the mid-points of the sides. But, imagining the five regular figures drawn in proper relation to each other on the same spherical surface, the only points which have thus to be considered are 12 points A, 20 points B, 30 points Θ, and 60 points Φ. These may be, in the first instance, described by reference to the dodecahedron; viz. the points A are the centres of the faces, the points B are the summits, the points Θ are the mid-points of the sides, and the points Φ are the mid-points of the diagonals of the faces (viz. there are thus 5 points Φ in each face of the dodecahedron, or in all 60 points Φ). But reciprocally we may describe them in reference to the icosahedron; viz. the points A are the summits, the points B the centres of the faces, the points Θ the mid-points of the sides, (viz. each point Θ is the common mid-point of a side of the dodecahedron and a side of the icosahedron, which sides there intersect at right angles), and the points Φ are points lying by 3's on the faces of the icosahedron, each point Φ of the face being given as the intersection of a perpendicular $A\Theta$ of the face by a line BB, joining the centres of two adjacent faces and intersecting $A\Theta$ at right angles.

The points A lie opposite to each other in pairs in such wise that, taking any two opposite points as poles, the relative situation is as follows:

A	Longitudes.
1	— ,
5	0°, 72°, 144°, 216°, 288°,
5	36°, 108°, 180°, 252°, 324°,
1	— ,

where the points A in the same horizontal line form a zone of points equidistant from the point taken as the North Pole. And the points B lie also opposite to

each other in such wise that, taking two opposite points as poles, the relative situation is as follows:

B	Longitudes.
1	— ,
3	0°, 120°, 240° ,
6	(0°, 120°, 240°) ± 22° 14′,
6	(60°, 180°, 300°) ± 22° 14′,
3	60°, 180°, 300° ,
1	— ,

where the points B in the same horizontal line form a zone of points equidistant from the point taken as the North Pole. Neglecting the $3+3$ points B which lie adjacent to the poles, the remaining 14 points B may be arranged as follows ($\beta = 22° \; 14'$ as above):

B	Longitudes.	
1	—	
6	β, $120° + \beta$, $240° + \beta$	$-\beta$, $120° - \beta$, $240° - \beta$,
6	$60° + \beta$, $180° + \beta$, $300° + \beta$	$60° - \beta$, $180° - \beta$, $300° - \beta$.
1	—	

And taking the two poles separately with each system of the remaining poles, we have 2 systems each of 8 points B, which are, in fact, the summits of a cube (hexahedron); each point B taken as North Pole thus belongs to two cubes; but inasmuch as the cube has 8 summits, the number of the cubes thus obtained is $20 \times 2 \div 8, = 5$; viz. the 20 points B form the summits of 5 cubes, each point B of course belonging to 2 cubes.

It is to be added that, considering the 5 points B which form a face of the dodecahedron, any diagonal BB of this dodecahedron is a side of a cube. We have thus $12 \times 5, = 60$, the number of the sides of the 5 cubes.

It is at once seen that the centres of the faces of a cube are points Θ, and that the mid-points of the sides of the cube are points Φ.

To each cube there corresponds of course an octahedron, the summits being points Θ, the centres of the faces points B, and the mid-points of the sides points Φ; thus, for the five octahedra the summits are the $5 \times 6, = 30$, points Θ; the centres of the faces are $5 \times 8, = 40$, points B (each point B being thus a centre of face for two octahedra), and the mid-points of the sides being the $5 \times 12, = 60$, points Φ.

Finally, considering the 8 points B which belong to a cube, we can, in four different ways, select thereout 4 points B which are the summits of a tetrahedron;

the remaining 4 points B are then the centres of the faces, and the mid-points of the sides are points Θ: there are thus 5×4, $= 20$, tetrahedra having 20×4 summits which are the 20 points B each 4 times; 20×4 centres of faces which are the 20 points B each 4 times; and 20×6 mid-points of sides which are the 30 points Θ each 4 times.

It thus appears that, as mentioned above, the five regular figures depend only on the points A, B, Θ, and Φ.

We might take as poles two opposite points A, B, Θ, or Φ; and in each case determine in reference to these the positions of the other points; but for brevity I consider only the case in which we take as poles two opposite points A. We have the following table:

Poles two opposite points A.

	N. P. D.	Longitudes.
A_0	0°	—
$5A_1$	63° 26′	0°, 72°, 144°, 216°, 288°
$5A_2$	116° 34′	36°, 108°, 180°, 252°, 324°
A_3	180°	—
$5B_1$	37° 22′	36°, 108°, ..., 324°
$5B_2$	79° 12′	36°, 108°, ..., 324°
$5B_3$	100° 48′	0°, 72°, ..., 288°
$5B_4$	142° 38′	0°, 72°, ..., 288°
$5\Theta_1$	31° 43′	0°, 72°, ..., 288°
$5\Theta_2$	58° 77′	36°, 108°, ..., 324°
$10\Theta_3$	90°	(0°, 72°, ..., 288°) ± 18°
$5\Theta_4$	121° 43′	0°, 72°, ..., 288°
$5\Theta_5$	148° 17′	36°, 108°, ..., 324°
$5\Phi_1$	13° 16′	36°, 108°, ..., 324°
$10\Phi_2$	52° 52′	(0°, 72°, ..., 288°) ± 9° 44′
$10\Phi_3$	68° 10′	(0°, 72°, ..., 288°) ± 13° 35′
$5\Phi_4$	76° 42′	0°, 72°, ..., 288°
$5\Phi_5$	103° 18′	36°, 108°, ..., 324°
$10\Phi_6$	111° 50′	(36°, 108°, ..., 324°) ± 13° 35′
$10\Phi_7$	127° 8′	(36°, 108°, ..., 324°) ± 9° 44′
$5\Phi_8$	166° 44′	0°, 72°, ..., 108°.

I add for greater completeness the following results, some of which were used in the calculation of the foregoing table. Considering successively (1) the tetrahedral triangle, summits 3 points B, centre a point B; (2) the hexahedral square, summits 4 points B, centre a point Θ; (3) the octahedral triangle, summits 3 points Θ, centre a point B; (4) the icosahedral triangle, summits 3 points A, centre a point B; (5) the dodecahedral pentagon, summits 5 points, centre a point B; and (6), what may be called the small pentagon, summits 5 points Φ lying within a dodecahedral pentagon, and having therewith the common centre B; we may in each case write s the side, r the radius or distance of the centre from a summit, p the perpendicular or distance of the centre from a side. And the values then are

	s	r	p
Tet. Δ	109° 30′	70° 30′	54° 45′
Hex. square	70 30	54 45	45
Oct. Δ	90	54 45	35 15
Icos. Δ	63 26	37 22	20 55
Dod. pentagon	41 50	37 22	31 43
Small pentagon	15 30	13 16	10 48

680.

ON THE HESSIAN OF A QUARTIC SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 141—144.]

THE surface considered is

$$U = k^2w^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)-(x^2+y^2+z^2)^2 = 0,$$

or say

$$U = k^2w^2P - Q^2 = 0,$$

viz. this may be considered as the central inverse of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0.$$

The values of the second derived functions, or terms of the Hessian determinant

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix},$$

are

$$\begin{vmatrix} \frac{k^2}{a^2}w^2-2Q-4x^2, & -4xy, & -4xz, & \frac{2k^2}{a^2}wx \\ -4xy, & \frac{k^2}{b^2}w^2-2Q-4y^2, & -4yz, & \frac{2k^2}{b^2}wy \\ -4xz, & -4yz, & \frac{k^2}{c^2}w^2-2Q-4z^2, & \frac{2k^2}{c^2}wz \\ \frac{2k^2}{a^2}wx, & \frac{2k^2}{b^2}wy, & \frac{2k^2}{c^2}wz, & k^2P \end{vmatrix}$$

and we thence have

$$bc - f^2 = \frac{k^4}{b^2c^2} - \frac{k^2w^2}{b^2}(2Q + 4z^2) - \frac{k^2w^2}{c^2}(2Q + 4y^2) + 4Q^2 + 8Q(y^2 + z^2),$$

$$gh - af = 4yz\left(\frac{k^2w^2}{a^2} - 2Q\right),$$

whence, forming the analogous quantities $ca - g^2$, &c., it is easy to obtain

$$\begin{aligned} abc - af^2 - bg^2 - ch^2 + 2fgh \\ = \frac{k^8w^6}{a^2b^2c^2} \\ - k^4w^4\left\{2Q\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right) + 4\left(\frac{x^2}{b^2c^2} + \frac{y^2}{c^2a^2} + \frac{z^2}{a^2b^2}\right)\right\} \\ + k^2w^2\left\{12Q^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 8QP\right\} \\ - 24Q^3, \end{aligned}$$

which is to be multiplied by d, $= k^2P$. And

$$\begin{aligned} &-[l^2(bc - f^2) + m^2(ca - g^2) + n^2(ab - h^2) \\ &+ 2mn(gh - af) + 2nl(hf - bg) + 2lm(fg - ch)] \\ &= -\frac{4k^8w^6P}{a^2b^2c^2} \\ &- 4k^6w^4\left[2Q\left\{\frac{x^2}{a^4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{y^2}{b^4}\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + \frac{z^2}{c^4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\}\right. \\ &\qquad \left. + \frac{4y^2z^2}{a^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)^2 + \frac{4z^2x^2}{b^2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^2 + \frac{4x^2y^2}{c^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2\right] \\ &+ 4k^4w^2\left[4Q^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right. \\ &\qquad \left. + 8Q\left\{y^2z^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right)^2 + z^2x^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^2 + x^2y^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2\right\}\right], \end{aligned}$$

which is

$$\begin{aligned} &= -\frac{4k^8w^6P}{a^2b^2c^2} \\ &+ k^6w^4\left\{8\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right)PQ - \frac{24}{a^2b^2c^2}Q^2 + 16\left(\frac{x^2}{b^2c^2} + \frac{y^2}{c^2a^2} + \frac{z^2}{a^2b^2}\right)P\right\} \\ &+ k^4w^2\left\{-48\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)Q^2 + 32P^2Q\right\}. \end{aligned}$$

Hence, uniting the two parts, we have

$$
\begin{aligned}
H = \quad & k^8 w^6 \left(-\frac{3}{a^2b^2c^2}P\right) \\
& + k^8 w^4 \left\{ \begin{array}{l} \quad 6\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)PQ \\ - \quad \frac{24}{a^2b^2c^2}Q^2 \\ + 12\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right)P \end{array} \right\} \\
& + k^4 w^2 \left\{ \begin{array}{l} \quad 12\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)PQ^2 \\ - 48\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right)Q^3 \\ + 24P^2Q \end{array} \right\} \\
& + k^2 \quad \{-24PQ^3\}.
\end{aligned}
$$

Writing herein $Q^2 = k^2w^2P - U$, and transposing all the terms which contain U, we have

$$
\begin{aligned}
H + k^2U\left\{-\frac{24k^4w^4}{a^2b^2c^2} + 12k^2w^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P - 48k^2w^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) - 24PQ\right\} \\
= k^8w^4P \left\{ \begin{array}{l} - \quad \frac{27k^2}{a^2b^2c^2}w^2 \\ + 16\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q \\ + 12\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P \\ + 12\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right) \\ - 48\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) \end{array} \right\}
\end{aligned}
$$

where, in the term in { }, the last four lines are

$$
\begin{aligned}
= \quad & 18\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q \\
& - 36\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right).
\end{aligned}
$$

Hence, writing for shortness

$$
\Theta = -\frac{2k^4}{a^2b^2c^2}w^4 + k^2w^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P - 4k^2w^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) - 2PQ,
$$

we have

$$H + 12k^2\Theta U = 9k^6w^4P\left\{-\frac{3k^2}{a^2b^2c^2}w^2 + 2\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q - 4\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\right\}.$$

Hence, recollecting that $U = k^2w^2P - Q^2$, the Hessian curve of the order 32 breaks up into

$U = 0$, $w^4 = 0$, that is, $Q^2 = 0$, $w^4 = 0$, or the nodal conic,

$w = 0$, $Q = 0$, 8 times (order 16),

$U = 0$, $P = 0$, that is, $Q^2 = 0$, $P = 0$, or the quadriquadric,

$P = 0$, $Q = 0$, 2 times (order 8),

and into a curve (order 8) which is

$$k^2w^2P - Q^2 = 0,$$

$$-\frac{3k^2}{a^2b^2c^2}w^2 + 2\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q - 4\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) = 0,$$

viz. this, the intersection of the surface with a quadric surface, is the proper Hessian curve.

681.

ON THE DERIVATIVES OF THREE BINARY QUANTICS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 157—168.]

FOR a reason which will appear, instead of the ordinary factorial notation, I write $\{\alpha 012\}$ to denote the factorial $\alpha . \alpha+1 . \alpha+2$, and so in other cases; and I consider the series of equations

$$(1) = X,$$
$$(2) = (\{\alpha 0\},\ \{\beta 0\} \between Y,\ -Y'),$$
$$(3) = (\{\alpha 01\},\ 2\{\alpha 1\}\{\beta 1\},\ \{\beta 01\} \between Z,\ -Z',\ Z''),$$
$$(4) = (\{\alpha 012\},\ 3\{\alpha 12\}\{\beta 2\},\ 3\{\alpha 2\}\{\beta 12\},\ \{\beta 012\} \between W,\ -W',\ -W'',\ -W'''),$$
&c.

where

$$X = Y + Y',$$
$$Y = Z + Z',\ Y' = Z' + Z'',$$
$$Z = W + W',\ Z' = W' + W'',\ Z'' = W'' + W''',$$
&c.

We have thus a series of linear equations serving to determine X; Y, Y'; Z, Z', Z''; W, W', W'', W'''; &c. We require in particular the values of X; Y, Y'; Z, Z''; W, W'''; &c., and I write down the results as follow:

$$X = (1),$$

$$\begin{array}{lcl} & & \overline{(1)\quad(2)} \\ \{\alpha+\beta 0\}\, Y & = & \{\beta 0\},\ +1, \\ \{\quad ,, \quad\}\, Y' & = & \{\alpha 0\},\ -1; \end{array}$$

$$\begin{array}{lcllll}
 & & \{\alpha+\beta 2\}(1), & \{\alpha+\beta 1\}(2), & \{\alpha+\beta 0\}(3), & \\
\hline
\{\alpha+\beta 012\}Z & = & \{\beta 01\}\ , & +2\{\beta 1\}\ , & +1\ , & \\
\{\quad\text{,,}\quad\}Z'' & = & \{\alpha 01\}\ , & -2\{\alpha 1\}\ , & +1\ ; & \\
 & & \{\alpha+\beta 34\}(1), & \{\alpha+\beta 14\}(2), & \{\alpha+\beta 03\}(3), & \{\alpha+\beta 01\}(4); \\
\hline
\{\alpha+\beta 01\ldots 4\}W & = & \{\beta 012\}\ , & +3\{\beta 12\}\ , & +3\{\beta 2\}\ , & +1\ , \\
\{\quad\text{,,}\quad\}W'''' & = & \{\alpha 012\}\ , & -3\{\alpha 12\}\ , & +3\{\alpha 2\}\ , & -1\ ; \\
 & & \{\alpha+\beta 456\}(1), & \{\alpha+\beta 156\}(2), & \{\alpha+\beta 036\}(3), & \{\alpha+\beta 015\}(4), \quad \{\alpha+\beta 012\}(5); \\
\hline
\{\alpha+\beta 01\ldots 6\}U & = & \{\beta 0123\}\ , & +4\{\beta 123\}\ , & +6\{\beta 23\}\ , & +4\{\beta 3\}\ , \quad +1\ , \\
\{\quad\text{,,}\quad\}U'''' & = & \{\alpha 0123\}\ , & -4\{\alpha 123\}\ , & +6\{\alpha 23\}\ , & -4\{\alpha 3\}\ , \quad +1\ ;
\end{array}$$

&c.

read

$$\alpha+\beta\,.\,Y=\beta\,(1)+(2),$$
$$\text{,,}\quad .\,Y'=\alpha\,(1)-(2),$$
$$\alpha+\beta.\alpha+\beta+1.\alpha+\beta+2.Z=\beta.\beta+1.\alpha+\beta+2.(1)+2.\beta+1.\alpha+\beta+1.(2)+\alpha+\beta.(3),$$
$$\text{,,}\quad\text{,,}\quad\text{,,}\quad .Z''=\alpha.\alpha+1.\alpha+\beta+2.(1)+2.\alpha+1.\alpha+\beta+1.(2)+\alpha+\beta.(3),$$

&c.,

the law being obvious, except as regards the numbers which in the top lines occur in connexion with $\alpha+\beta$ in the { } symbols. As regards these, we form them by successive subtractions as shown by the diagrams

34	34	456	456	5678	5678	&c.;
2	14	3	156	4	1678	
11	03	12	036	13	0378	
2	01	21	015	22	0158	
		3	012	31	0127	
				4	0123	

and the statement of the result is now complete.

In part verification, starting from the Y-formulæ (which are obtained at once), assume

$$\begin{array}{lcllll}
 & & \{\alpha+\beta 2\}(1), & \{\alpha+\beta 1\}(2), & \{\alpha+\beta 0\}(3), \\
\hline
\{\alpha+\beta 012\}Z & = & \lambda\ , & \mu\ , & \nu \\
\{\quad\text{,,}\quad\}Z' & = & \lambda'\ , & \mu'\ , & \nu' \\
\{\quad\text{,,}\quad\}Z'' & = & \lambda''\ , & \mu''\ , & \nu''
\end{array}$$

we must have

$$\begin{array}{lcll}
 & & & (1)\quad(2) \\
\hline
\{\alpha+\beta 012\}\,.\,Z+Z' & = & \{\alpha+\beta 012\}\,Y\,, & =\{\alpha+\beta 12\}\,(\{\beta 0\},\ +1) \\
\{\quad\text{,,}\quad\}\,.\,Z'+Z'' & = & \{\quad\text{,,}\quad\}\,Y'\,, & =\{\quad\text{,,}\quad\}\,(\{\alpha 0\},\ -1)
\end{array}$$

that is,

$$\{\alpha+\beta 2\}\,.\,\lambda+\lambda'=\{\alpha+\beta 12\}\,\{\beta 0\},$$
$$\{\quad\text{,,}\quad\}\,.\,\lambda'+\lambda''=\{\quad\text{,,}\quad\}\,\{\alpha 0\},$$

and further

$$\{\alpha+\beta 2\}\,(\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between\lambda,\ \lambda',\ \lambda'')=0,$$

or, what is the same thing,

$$\begin{aligned}\lambda+\lambda' &= \{\alpha+\beta 1\}\,\{\beta 0\},\\ \lambda'+\lambda'' &= \{\quad ,, \quad\}\,\{\alpha 0\},\\ (\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between\lambda,\ \lambda',\ \lambda'') &= 0.\end{aligned}$$

And in like manner we have

$$\begin{aligned}\mu+\mu' &= \{\alpha+\beta 2\}\,.\ \ 1,\\ \mu'+\mu'' &= \{\quad ,, \quad\}\,.-1,\\ (\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between\mu,\ \mu',\ \mu'') &= 0;\end{aligned}$$

and

$$\begin{aligned}\nu+\nu' &= 0,\\ \nu'+\nu'' &= 0,\\ (\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between\nu,\ \nu',\ \nu'') &= 0.\end{aligned}$$

We hence find without difficulty

$$\begin{aligned}\lambda\,,\ \mu\,,\ \nu &= \beta\,.\,\beta+1,\quad 2\,.\,\beta+1,\ +1, = \{\beta 01\}\ ,\ 2\,\{\beta 1\},\ +1,\\ \lambda',\ \mu',\ \nu' &= \alpha\,.\,\beta\ ,\quad \alpha-\beta\ ,\ -1, = \{\alpha 0\}\,\{\beta 0\},\ \alpha-\beta\ ,\ -1,\\ \lambda'',\ \mu'',\ \nu'' &= \alpha\,.\,\alpha+1\,,\ -2\,.\,\alpha+1,\ +1, = \{\alpha 01\}\ ,\ 2\,\{\alpha 1\},\ +1;\end{aligned}$$

viz. for verification of the λ-equations we have

$$\beta\,.\,\beta+1\,.+\ \alpha\,.\,\beta\ ,\ \text{that is, } \alpha+\beta+1\,.\,\beta, = \{\alpha+\beta 1\}\,\{\beta 0\},$$

$$\alpha\,.\,\beta\,.\ +\alpha\,.\,\alpha+1,\quad ,,\quad \alpha+1+\beta\,.\,\alpha, = \{\quad ,, \quad\}\,\{\alpha 0\},$$

and

$$(\alpha\,.\,\alpha+1,\ -2\,.\,\alpha+1\,.\,\beta+1,\ \beta\,.\,\beta+1\between\beta\,.\,\beta+1,\ \alpha\,.\,\beta,\ \alpha\,.\,\alpha+1)=0,$$

that is,

$$\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1\,.-2\,.\,\alpha+1\,.\,\beta+1\,.\,\alpha\,.\,\beta\,.+\beta\,.\,\beta+1\,.\,\alpha\,.\,\alpha+1=0;$$

and similarly the μ- and ν-equations may be verified.

We have thus for the Z's the equations

		$\{\alpha+\beta 2\}$ (1),	$\{\alpha+\beta 1\}$ (2),	$\{\alpha+\beta 0\}$ (3),
$\{\alpha+\beta 012\}\,Z$	=	$\{\beta 01\}$,	$2\,\{\beta 1\}$,	$+1$,
$\{\quad ,, \quad\}\,Z'$	=	$\{\alpha 0\}\,\{\beta 0\}$,	$\alpha-\beta$,	-1 ,
$\{\quad ,, \quad\}\,Z''$	=	$\{\alpha 01\}$,	$-2\,\{\alpha 1\}$,	$+1$,

which include the foregoing expressions for Z and Z''.

We may then take the expressions for the W's to be

		$\{\alpha+\beta 34\}$ (1),	$\{\alpha+\beta 14\}$ (2),	$\{\alpha+\beta 03\}$ (3),	$\{\alpha+\beta 01\}$ (4),
$\{\alpha+\beta 0123\}\,W$	=	λ ,	μ ,	ν ,	ρ ,
$\{\quad ,, \quad\}\,W'$	=	λ' ,	μ' ,	ν' ,	ρ' ,
$\{\quad ,, \quad\}\,W''$	=	λ'' ,	μ'' ,	ν'' ,	ρ'' ,
$\{\quad ,, \quad\}\,W'''$	=	λ''' ,	μ''' ,	ν''' ,	ρ''' ;

and we obtain in like manner the equations

$$\begin{aligned}
\lambda + \lambda' &= \{\alpha + \beta 234\}\,\{\beta 01\},\\
\lambda' + \lambda'' &= \{\quad ,, \quad\}\,\{\alpha 0\}\,\{\beta 0\},\\
\lambda'' + \lambda''' &= \{\quad ,, \quad\}\,\{\alpha 01\},
\end{aligned}$$

$$(\{\alpha 012\},\ -3\,\{\alpha 12\}\,\{\beta 2\},\ +3\,\{\alpha 2\}\,\{\beta 12\},\ -\{\beta 012\} \between \lambda,\ \lambda',\ \lambda'',\ \lambda''') = 0;$$

$$\begin{aligned}
\mu + \mu' &= \{\alpha + \beta 134\}\,.\ \ 2\,\{\beta 1\},\\
\mu' + \mu'' &= \{\quad ,, \quad\}\,.\ \ \alpha - \beta,\\
\mu'' + \mu''' &= \{\quad ,, \quad\}\,.\ -2\,\{\alpha 1\},
\end{aligned}$$

$$(\{\alpha 012\},\ -3\,\{\alpha 12\}\,\{\beta 2\},\ +3\,\{\alpha 2\}\,\{\beta 12\},\ -\{\beta 012\} \between \mu,\ \mu',\ \mu'',\ \mu''') = 0;$$

$$\begin{aligned}
\nu + \nu' &= \{\alpha + \beta 034\}\,.\ \ 1,\\
\nu' + \nu'' &= \{\quad ,, \quad\}\,.\ -1,\\
\nu'' + \nu''' &= \{\quad ,, \quad\}\,.\ \ 1,
\end{aligned}$$

$$(\{\alpha 012\},\ -3\,\{\alpha 12\}\,\{\beta 2\},\ +3\,\{\alpha 2\}\,\{\beta 12\},\ -\{\beta 012\} \between \nu,\ \nu',\ \nu'',\ \nu''') = 0;$$

$$\begin{aligned}
\rho + \rho' &= 0,\\
\rho' + \rho'' &= 0,\\
\rho'' + \rho''' &= 0,
\end{aligned}$$

$$(\{\alpha 012\},\ -3\,\{\alpha 12\}\,\{\beta 2\},\ +3\,\{\alpha 2\}\,\{\beta 12\},\ -\{\beta 012\} \between \rho,\ \rho',\ \rho'',\ \rho''') = \{\alpha + \beta 01234\}.$$

These give for the $\lambda\rho'''$ square the values

$$\begin{array}{llll}
\{\beta 012\}\ , & 3\,\{\beta 12\}\ , & 3\,\{\beta 2\}\ , & +1,\\
\{\alpha 0\}\,\{\beta 01\}, & 2\alpha - \beta\,.\,\{\beta 1\}, & \alpha - 2\beta - 2, & -1,\\
\{\alpha 01\}\,\{\beta 0\}, & \alpha - 2\beta\,.\,\{\alpha 1\}, & -2\alpha + \beta - 2, & +1,\\
\{\alpha 012\}\ , & -3\,\{\alpha 12\}\ , & +3\,\{\alpha 2\}\ , & -1,
\end{array}$$

and so on; the law however of the terms in the intermediate lines is not by any means obvious.

Consider now the binary quantics P, Q, R, of the forms $(*\between x,\ y)^p$, $(*\between x,\ y)^q$, $(*\between x,\ y)^r$; we have for any, for instance for the fourth, order, the derivates

$$P(Q,\ R)^4,\quad (P,\ (Q,\ R)^3)^1,\quad (P,\ (Q,\ R)^2)^2,\quad (P,\ (Q,\ R)^1)^3,\quad (P,\ QR)^4;$$

and it is required to express

$$Q(P,\ R)^4 \text{ and } R(P,\ Q)^4,$$

each of them as a linear function of these.

I recall that we have $(P, Q)^0 = PQ$, so that the first and the last terms of the series might have been written $(P, (Q, R)^4)^0$ and $(P, (Q, R)^0)^4$ respectively; and, further, that $(P, Q)^1$ denotes $d_xP \,.\, d_yQ - d_yP \,.\, d_xQ$; $(P, Q)^2$ denotes

$$d_x^2P \,.\, d_y^2Q - 2d_xd_yP \,.\, d_xd_yQ + d_y^2P \,.\, d_x^2Q;$$

and so on.

I write (a, b, c, d, e) for the fourth derived functions of any quantic $U, = (*\between x, y)^m$; we have, in a notation which will be at once understood,

$$\begin{aligned}
U &= (a, b, c, d, e\between x, y)^4 \div [m]^4, \\
(d_x, d_y)\, U &= (a, b, c, d), (b, c, d, e)(x, y)^3 \div [m-1]^3, \\
(d_x, d_y)^2\, U &= (a, b, c), (b, c, d), (c, d, e)(x, y)^2 \div [m-2]^2, \\
(d_x, d_y)^3\, U &= (a, b), (b, c), (c, d), (d, e)(x, y)^1 \div [m-3]^1, \\
(d_x, d_y)^4\, U &= (a, b, c, d, e);
\end{aligned}$$

and then, taking

$$(a_1, b_1, c_1, d_1, e_1), \quad (a_2, b_2, c_2, d_2, e_2), \quad (a_3, b_3, c_3, d_3, e_3),$$

to belong to P, Q, R, respectively, we must, instead of m, write p, q, r for the three functions respectively.

If we attend only to the highest terms in x, we have

$$\begin{aligned}
U &= ax^4 \div [m]^4, \\
(d_x, d_y)\, U &= (a, b)\, x^3 \div [m-1]^3, \\
(d_x, d_y)^2\, U &= (a, b, c)\, x^2 \div [m-2]^2, \\
(d_x, d_y)^3\, U &= (a, b, c, d)\, x \div [m-3]^1, \\
(d_x, d_y)^4\, U &= (a, b, c, d, e).
\end{aligned}$$

Consider now $P(Q, R)^4$, $(P, (Q, R)^3)^1$, &c.; in each case attending only to the term in a_1, and in this term to the highest term in x, we have

$$\begin{aligned}
&(1) \quad [p]^4\, P(Q, R)^4 &&= a_2e_3 - 4b_2d_3 + 6c_2c_3 - 4d_2b_3 + e_2a_3 \quad (X), \\
&(2) \quad [p-1]^3\,[q-3]^1\,[r-3]^1\,(P, (Q, R)^3)^1 &&= [q-3]^1 \,.\, b_2d_3 - 3c_2c_3 + 3d_2b_3 - e_2a_3 \quad (-Y'), \\
& && + [r-3]^1 \,.\, a_2e_3 - 3b_2d_3 + 3c_2c_3 - d_2b_3 \quad (Y), \\
&(3) \quad [p-2]^2\,[q-2]^2\,[r-2]^2\,(P, (Q, R)^2)^2 &&= [q-2]^2 \,.\, c_2c_3 - 2d_2b_3 + e_2a_3 \quad (Z''), \\
& && + 2[q-2]^1\,[r-2]^1 \,.\, b_2d_3 - 2c_2c_3 + d_2b_3 \quad (-Z'), \\
& && + [r-2]^2 \,.\, a_2e_3 - 2b_2d_3 + c_2c_3 \quad (Z), \\
&(4) \quad [p-3]^1\,[q-1]^3\,[r-1]^3\,(P, (Q, R)^1)^3 &&= [q-1]^3 \,.\, d_2b_3 - e_2a_3 \quad (-W'''), \\
& && + 3[q-1]^3\,[r-1]^1 \,.\, c_2c_3 - d_2b_3 \quad (W''), \\
& && + 3[q-1]^1\,[r-1]^3 \,.\, b_2d_3 - c_2c_3 \quad (-W'), \\
& && + [r-1]^3 \,.\, a_2e_3 - b_2d_3 \quad (W),
\end{aligned}$$

$$
\begin{array}{llll}
(5) & [p-4]^0\,[q]^4\,[r]^4\,(P,\ QR)^4 = & [q]^4 \quad . e_2a_3 & (U'''') \\
 & & +4\,[q]^3\,[r]^1\,.\,d_2b_3 & (-U''') \\
 & & +6\,[q]^2\,[r]^2\,.\,c_2c_3 & (U'') \\
 & & +4\,[q]^1\,[r]^3\,.\,b_2d_3 & (-U') \\
 & & +\quad [r]^4\,.\,a_2e_3 & (U).
\end{array}
$$

Thus, for the second of these equations,

$$(P,\ (Q,\ R)^3)^1 = d_xP\,.\,d_y(Q,\ R)^3 - \&c.;$$

the term in a_1 is $d_y(Q,\ R)^3, = (d_xQ,\ R)^3 + (Q,\ d_yR)^3$, the whole being divided by $[p-1]^3$; where attending only to the highest terms in x, the two terms are respectively

$$(b_2d_3 - 3c_2c_3 + 3d_2b_3 - e_2a_3) \div [r-3]^1,$$

and

$$(a_2e_3 - 3b_2d_3 + 3c_2c_3 - d_2b_3) \div [q-3]^1,$$

which are each divided by $[p-1]^3$ as above; whence, multiplying by

$$[p-1]^3\,[q-1]^2\,[r-1]^1,$$

we have the formula in question; and so for the other cases.

Writing now (1), (2), (3), (4), (5) for the left-hand sides of the five equations respectively; and

$$
\begin{array}{r}
X: \\
-Y',\ \ Y: \\
Z'',\ \ Z',\ \ Z: \\
-W''',\ \ W'',\ -W',\ \ W: \\
U'''',\ -U''',\ \ U'',\ -U',\ \ U:
\end{array}
$$

for the literal parts on the right-hand sides of the same equations respectively; then we have

$$
\begin{array}{l}
X = Y + Y', \\
Y = Z + Z', \quad Y' = Z' + Z'', \\
\&c.,
\end{array}
$$

and the equations become

$$
\begin{array}{ll}
(1) = & X \qquad\qquad\qquad\

Also, attending as before only to the terms in a, and therein to the highest power of x, we have

$$Q(R,\ P)^4 = a_2 e_3 \div [q]^4,$$
$$R(P,\ Q)^4 = a_2 e_2 \div [r]^4;$$

that is,

$$[q]^4\, Q(R,\ P)^4 = U, \quad [r]^4\, R(P,\ Q)^4 = U'''';$$

and, observing that $\{q+r-6,\ 01\ldots 6\}$ is $=[q+r]$, and that $\{q+r-6,\ 456\}$, &c., may be written $\{q-r,\ \overline{21}\overline{1}0\}$, &c., where the superscript bars are the signs $-$, the formulæ become

$$\begin{array}{lllllll}
 & \{q+r,\ \overline{21}\overline{1}0\}(1), & \{q+r,\ \overline{5}\overline{1}0(2), & \{q+r,\ \overline{6}\overline{3}0\}(3), & \{q+r,\ \overline{651}\}(4), & \{q+r,\ \overline{654}\}(5), \\
\hline
[q+r]^7[q]^4Q(P,R)^4= & [q]^4 \quad , & +4[q]^3 \quad , & +6[q]^2 \quad , & +4[q]^1 \quad , & +1 \quad , \\
[q+r]^7[r]^4R(P,Q)^4= & [r]^4 \quad , & -4[r]^3 \quad , & +6[r]^2 \quad , & -4[r]^1 \quad , & +1 \quad .
\end{array}$$

Written at full length, the first of these equations (which, as being the fourth in a series, I mark 4th equation) is

$$\begin{array}{lll}
q+r]^7[q]^4Q(P,R)^4= & 1.q+r \quad .q+r-1.q+r-2. \quad [p]^4\ [q]^4 & .P,(Q,R)^4 \quad \text{(4th equat} \\
 & +4.q+r \quad .q+r-1.q+r-5.[p-1]^3[q]^3[q-3]^1\,[r-1]^1.(P,(Q,R)^3)^1 & \\
 & +6.q+r \quad .q+r-3.q+r-6.[p-2]^2[q]^2[q-2]^2\,[r-2]^2.(P,(Q,R)^2)^2 & \\
 & +4.q+r-1.q+r-5.q+r-6.[p-3]^1[q]^1[q-1]^3\,[r-1]^3.(P,(Q,R)^1)^3 & \\
 & +1.q+r-1.q+r-5.q+r-6. \qquad [q]^4 \quad [r]^4 & .P,(Q,R)^4\ ,
\end{array}$$

and the other is, in fact, the same equation with $q,\ Q,\ r,\ R$ interchanged with $r,\ R,\ q,\ Q$; the alternate $+$ and $-$ signs arise evidently from the terms

$$(R,\ Q)^4,\ =(Q,\ R)^4;\ (R,\ Q)^3,\ =-(Q,\ R)^3;\ \text{\&c.},$$

which present themselves on the right-hand side.

It will be observed that the identity has been derived from the comparison of the terms in a, which are the highest terms in x, the other terms not having been written down or considered; but it is easy to see that an identity of the form in question exists, and, this being admitted, the process is a legitimate one.

The preceding equations of the series are

$$\begin{array}{lll}
[q+r]^1\,[q]^1\,Q(P,R)^1= & 1.\ [p]^1 \quad [q]^1 & P\,(Q,\ R)^1 \quad \text{(1st equation)} \\
 & +1. \qquad [q]^1 \quad [r]^1 & (P,\ QR)^1; \\
[q+r]^3\,[q]^2Q(P,R)^2= & 1.q+r \quad .\ [p]^2 \quad [q]^2 & P,(Q,\ R)^2 \quad \text{(2nd equation)} \\
 & +2.q+r-1.[p-1]^1\,[q]^1\,[q-1]^1\,[r-1]^1 & (P,\ (Q,\ R)^1)^1 \\
 & +1.q+r-2. \qquad [q]^2 \quad [r]^2 & (P,\ QR)^2; \\
[q+r]^5\,[q]^3Q(P,R)^2= & 1.q+r \quad .q+r-1.\ [p]^3 \quad [q]^3 & P,(Q,\ R)^3 \quad \text{(3rd equation)} \\
 & +3.q+r \quad .q+r-3.[p-1]^2[q]^2[q-2]^1[r-2]^1\ (P,\ (Q,\ R)^2)^1 & \\
 & +3.q+r-1.q+r-4.[p-2]^1[q]^1[q-1]^2[r-1]^2(P,\ QR)^3 & \\
 & +1.q+r-3.q+r-4. \qquad [q]^3 \quad [r]^3 & (P,\ QR)^3.
\end{array}$$

From these four equations the law is evident, except as to the numbers subtracted from $q+r$. These are obtained, as explained above, in regard to the numbers added to $\alpha+\beta$ in the { } symbols; transforming the diagrams so as to be directly applicable to the case now in question, they become

0	0	01	01	012	012	0123	0123
1	1	2	03	3	015	4	0127
1	2	11	14	21	036	31	0158
		2	34	12	156	22	0378
				3	456	13	1678
						4	5678,

showing how the numbers are obtained for the equations 2, 3, 4, 5 respectively. The first equation is

$$(q^2+qr)\,Q(P,R)=pq\,P(Q,R)+qr\,[Q(P,R)+R(P,Q)],$$

viz. this is

$$\begin{aligned}0&=pq\,P(Q,R)-qr\,Q(RP)+qr\,R(P,Q)\\&\qquad+(q^2+qr)\,Q(R,P);\end{aligned}$$

or, dividing by q, this is

$$0=pP(Q,R)+qQ(R,P)+rR(P,Q),$$

which is a well-known identity.

We may verify any of the equations, though the process is rather laborious, for the particular values

$$P=x^{\frac{1}{2}(p+\alpha)}\,y^{\frac{1}{2}(p-\alpha)},\quad Q=x^{\frac{1}{2}(q+\beta)}\,y^{\frac{1}{2}(q-\beta)},\quad R=x^{\frac{1}{2}(r+\gamma)}\,y^{\frac{1}{2}(r-\gamma)};$$

thus, taking the second equation, we have, omitting common factors,

$$\begin{aligned}(Q,R)^2=&\quad q+\beta\,.\,q+\beta-2\,.\,r-\gamma\,.\,r-\gamma-2\\&-2\qquad.\,q+\beta\qquad.\,q-\beta\,.\,r+\gamma\,.\,r-\gamma\\&+\,.\,q-\beta\,.\,q-\beta-2\,.\,r+\gamma\,.\,r+\gamma-2\\=&\quad\beta^2(r^2-r)+\gamma^2(q^2-q)-2\beta\gamma(q-1)(r-1)-qr(q+r-2),\end{aligned}$$

$$\begin{aligned}(P,(Q,R)^2)^1&=(q+\beta\,.\,r-\gamma\,.-.\,q-\beta\,.\,r+\gamma)(p+\alpha\,.\,q+r-\beta-\gamma-2\,.-.\,p-\alpha\,.\,q+r+\beta+\gamma-2)\\&=(\beta r-q\gamma)(\alpha\,.\,q+r-2\,.-p\,.\,\beta+\gamma)\\&=\alpha\beta r(r+q-2)-\alpha\gamma q(q+r-2)-pr\beta^2+p(q-r)\beta\gamma+pq\gamma^2,\end{aligned}$$

and from the first of these the expressions of $Q(P,R)^2$ and $(P,QR)^2$ are at once obtained. The identity to be verified then becomes

$$\begin{aligned}[q+r]^3[q]^2\{&\alpha^2(r^2-r)+\gamma^2(p^2-p)-2\alpha\gamma(p-1)(r-1)-pr(p+r-2)\}\\=&\quad(q+r)[q]^2[p]^2\{\beta^2(r^2-r)+\gamma^2(q^2-q)-2\beta\gamma(q-1)(r-1)-qr(q+r-2)\}\\&+2(q+r-1)[q]^2(p-1)(r-1)\{\alpha\beta r(q+r-2)-\alpha\gamma q(q+r-2)\\&\qquad-pr\beta^2+p(q-r)\beta\gamma+pq\gamma^2\}\\&+(q+r-2)[q]^2[r]^2\{\alpha^2(q+r)(q+r-1)+(\beta+\gamma)^2(p^2-p)\\&\qquad-2\alpha(\beta+\gamma)(p-1)(q+r-1)-p(q+r)(p+q+r-2)\},\end{aligned}$$

which is easily verified, term by term; for instance, the terms with α, β, or γ, give

$$\begin{aligned}[q+r]^3[q]^2pr(p+r-2) = \quad & (q+r)[q]^2[p]^2qr(q+r-2) \\ & +(q+r-2)[q]^2[r]^2p(q+r)(p+q+r-2),\end{aligned}$$

which, omitting the factor $(q+r)(q+r-2)[q]^2pr$, is

$$(q+r-1)(p+r-2) = (p-1)q+(r-1)(p+q+r+2);$$

viz. the right-hand side is

$$(p-1)q+(r-1)q+(r-1)(p+r-2), \quad =(q+r-1)(p+r-2),$$

as it should be.

The equations are useful for the demonstration of a subsidiary theorem employed in Gordan's demonstration of the finite number of the covariants of any binary form U. Suppose that a system of covariants (including the quantic itself) is

$$P,\ Q,\ R,\ S,\ldots;$$

this may be the complete system of covariants; and if it is so, then, T and V being any functions of the form $P^\alpha Q^\beta R^\gamma\ldots$, every derivative $(T,\ V)^\theta$ must be a term or sum of terms of the like form $P^\alpha Q^\beta R^\gamma\ldots$; the subsidiary theorem is that in order to prove that the case is so, it is sufficient to prove that every derivative $(P,\ Q)^\theta$, where P and Q are any two terms of the proposed system, is a term or sum of terms of the form in question $P^\alpha Q^\beta R^\gamma\ldots$.

In fact, supposing it shown that every derivative $(T,\ V)^\theta$ up to a given value θ_0 of θ is of the form $P^\alpha Q^\beta R^\gamma\ldots$, we can by successive application of the equation for $Q(P,\ R)^{\theta+1}$, regarded as an equation for the reduction of the last term on the right-hand side $(P,\ QR)^{\theta+1}$, bring first $(P,\ QR)^{\theta+1}$, and then $(P,\ QRS)^{\theta+1},\ldots$, and so ultimately any function $(P,\ V)^{\theta+1}$, and then again any functions $(PQ,\ V)^{\theta+1}$, $(PQR,\ V)^{\theta+1},\ldots$, and so ultimately any function $(T,\ V)^{\theta+1}$, into the required form $P^\alpha Q^\beta R^\gamma\ldots$: or the theorem, being true for θ, will be true for $\theta+1$; whence it is true generally.

682.

FORMULÆ RELATING TO THE RIGHT LINE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 169—171.]

1. LET λ, μ, ν be the direction-angles of a line; α, β, γ the coordinates of a point on the line; and write

$$a = \cos\lambda, \quad f = \beta\cos\nu - \gamma\cos\mu,$$
$$b = \cos\mu, \quad g = \gamma\cos\lambda - \alpha\cos\nu,$$
$$c = \cos\nu, \quad h = \alpha\cos\mu - \beta\cos\lambda,$$

whence

$$a^2 + b^2 + c^2 = 1,$$
$$af + bg + ch = 0,$$

or the six quantities (a, b, c, f, g, h), termed the coordinates of the line, depend upon four arbitrary parameters.

2. It is at once shown that the condition for the intersection of any two lines (a, b, c, f, g, h), (a', b', c', f', g', h'), is $af' + bg' + ch' + a'f + b'g + c'h = 0$.

3. Given two lines (a, b, c, f, g, h), (a', b', c', f', g', h'), it is required to find their shortest distance, and the coordinates of their line of shortest distance.

Let

$$Ax + By + Cz + D = 0,$$
$$Ax + By + Cz + D' = 0,$$

be parallel planes containing the two lines respectively; then the first plane contains the point $\alpha + r\cos\lambda$, $\beta + r\cos\mu$, $\gamma + r\cos\nu$, and the second contains the point $\alpha' + r'\cos\lambda'$, $\beta' + r'\cos\mu'$, $\gamma' + r'\cos\nu'$; that is, we have

$$A\alpha + B\beta + C\gamma + D = 0,$$
$$A\alpha' + B\beta' + C\gamma' + D' = 0,$$
$$A\cos\lambda + B\cos\mu + C\cos\nu = 0,$$
$$A\cos\lambda' + B\cos\mu' + C\cos\nu' = 0,$$

which last equations may be written

$$Aa + Bb + Cc = 0,$$
$$Aa' + Bb' + Cc' = 0,$$

giving

$$A : B : C = bc' - b'c : ca' - c'a : ab' - a'b,$$

or, if we write

$$\theta = aa' + bb' + cc',$$

and assume, as is convenient,

$$A^2 + B^2 + C^2 = 1,$$

then

$$A,\ B,\ C = \frac{bc' - b'c}{\sqrt{(1-\theta^2)}},\ \frac{ca' - c'a}{\sqrt{(1-\theta^2)}},\ \frac{ab' - a'b}{\sqrt{(1-\theta^2)}},$$

where θ, = cosine-inclination, $= aa' + bb' + cc'$.

Hence, shortest distance $= D - D'$

$$= A(\alpha - \alpha') + B(\beta - \beta') + C(\gamma - \gamma')$$
$$= \frac{1}{\sqrt{(1-\theta^2)}}\{(bc' - b'c)(\alpha - \alpha') + (ca' - c'a)(\beta - \beta') + (ab' - a'b)\}$$
$$= \frac{1}{\sqrt{(1-\theta^2)}}\{a'(c\beta - b\gamma) + b'(a\gamma - c\alpha) + c'(b\alpha - a\beta)$$
$$+ a(c'\beta' - b'\gamma') + b(a'\gamma' - c'\alpha') + c(b'\alpha' - a'\beta')\}$$
$$= \frac{1}{\sqrt{(1-\theta^2)}}(af' + bg' + ch' + a'f + b'g + c'h),\ = \delta \text{ suppose.}$$

The six coordinates of the line of shortest distance are A, B, C, F, G, H, where A, B, C denote as before, and F, G, H are to be determined.

Since the line meets each of the given lines, we have

$$Af + Bg + Ch + Fa + Gb + Hc = 0,$$
$$Af' + Bg' + Ch' + Fa' + Gb' + Hc' = 0,$$

and we have also

$$FA + GB + HC = 0,$$

which equations give F, G, H. Multiplying the first equation by $b'C - c'B$, the second by $Bc - Cb$, and the third by $bc' - b'c$, we find

$$(b'C - c'B)(Af + Bg + Ch) + (Bc - Cb)(Af' + Bg' + Ch') + F \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ A, & B, & C \end{vmatrix} = 0.$$

Here

$$b'C - c'B = \frac{1}{\sqrt{(1-\theta^2)}}\{b'(ab' - a'b) - c'(ca' - c'a)\}$$
$$= \frac{1}{\sqrt{(1-\theta^2)}}\{a(a'^2 + b'^2 + c'^2) - a'(aa' + bb' + cc')\}$$
$$= \frac{1}{\sqrt{(1-\theta^2)}}(a - a'\theta),$$

and similarly

$$cB - bC = \frac{1}{\sqrt{(1-\theta^2)}}(a' - a\theta).$$

Also, putting for shortness

$$\Omega = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ f, & g, & h \end{vmatrix}, \quad \Omega' = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ f', & g', & h' \end{vmatrix},$$

we have

$$Af + Bg + Ch = \frac{1}{\sqrt{(1-\theta^2)}}\Omega, \quad Af' + Bg' + Ch' = \frac{1}{\sqrt{(1-\theta^2)}}\Omega',$$

and finally, the determinant which multiplies F is

$$\frac{1}{\sqrt{(1-\theta^2)}}\{(bc' - b'c)^2 + (ca' - c'a)^2 + (ab' - a'b)^2\} = \frac{1}{\sqrt{(1-\theta^2)}}(1-\theta^2), = \sqrt{(1-\theta^2)}.$$

We have thus the value of F; forming in the same way those of G and H, we find

$$F = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(a - a'\theta)\,\Omega + (a' - a\theta)\,\Omega'\},$$

$$G = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(b - b'\theta)\,\Omega + (b' - b\theta)\,\Omega'\},$$

$$H = \frac{-1}{(1-\theta^2)^{\frac{3}{2}}}\{(c - c'\theta)\,\Omega + (c' - c\theta)\,\Omega'\},$$

which, with the foregoing equations for A, B, C, give the six coordinates of the line of shortest distance.

683.

ON THE FUNCTION $\operatorname{arc\,sin}(x+iy)$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 171—174.]

THE determination of the function in question, the arc to a given imaginary sine, is considered in Cauchy's *Exercises d'Analyse, &c.*, t. III. (1844), p. 382; but it appears, by two hydrodynamical papers by Mr Ferrers and Mr Lamb, *Quarterly Mathematical Journal*, t. XIII. (1874), p. 115, and t. XIV. (1875), p. 40, that the question is connected with the theory of confocal conics.

Taking $c=\sqrt{(a^2-b^2)}$ a positive real quantity which may ultimately be put $=1$, the question is to find the real quantities ξ, η, such that

$$\xi+i\eta=\operatorname{arc\,sin}\frac{1}{c}(x+iy),$$

or say

$$x+iy=c\sin(\xi+i\eta),$$

so that

$$x=c\sin\xi\cos i\eta,\quad iy=c\cos\xi\sin i\eta.$$

It is convenient to remark that if a value of $\xi+i\eta$ be $\xi'+i\eta'$, then the general value is $2m\pi+\xi'+i\eta'$ or $(2m+1)\pi-(\xi'+i\eta')$; hence, η may be made positive or negative at pleasure; $\cos i\eta$ is in each case positive, but $\frac{1}{i}\sin i\eta$ has the same sign as η; hence $\cos\xi$ has the same sign as x, but $\sin\xi$ has the same sign as y or the reverse sign, according as η is positive or negative; for any given values of x and y, we obtain, as will appear, determinate positive values of $\sin^2\xi$ and $\cos^2\xi$; and the square roots of these must therefore be taken so as to give to $\sin\xi$, $\cos\xi$ their proper signs respectively.

Suppose that λ, μ are the elliptic coordinates of the point (x, y); viz. that we have

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1,$$

where $a^2+\lambda$, $b^2+\lambda$, and $a^2+\mu$ are positive, but $b^2+\mu$ is negative. Calling ρ, σ the distances of the point x, y from the points $(c, 0)$ and $(-c, 0)$, that is, assuming

$$\rho=\sqrt{\{(x-c)^2+y^2\}},$$

$$\sigma=\sqrt{\{(x+c)^2+y^2\}},$$

then we have

$$\sqrt{(a^2+\lambda)}=\tfrac{1}{2}(\sigma+\rho), \text{ whence also } \sqrt{(b^2+\lambda)}=\tfrac{1}{2}\sqrt{\{(\sigma+\rho)^2-4c^2\}},$$

$$\sqrt{(a^2+\mu)}=\tfrac{1}{2}(\sigma\sim\rho), \quad \text{,,} \quad \sqrt{(b^2+\mu)}=\tfrac{1}{2}\sqrt{\{(\sigma\sim\rho)^2-4c^2\}},$$

which equations determine λ, μ as functions of x, y.

Now we have

$$\rho\sigma=\sqrt{\{(x^2+y^2-c^2)^2-4c^2x^2\}}=\sqrt{\{(x^2-y^2-c^2)^2+4x^2y^2\}},$$

$$\rho^2+\sigma^2=2(x^2+y^2+c^2);$$

substituting herein for x, y their values

$$c\sin\xi\cos i\eta,\ -ci\cos\xi\sin i\eta,$$

we find

$$x^2-y^2-c^2=c^2\{\sin^2\xi\cos^2 i\eta+\cos^2\xi\sin^2 i\eta-(\sin^2\xi+\cos^2\xi)(\sin^2 i\eta+\cos^2 i\eta)\}$$

$$=-c^2(\sin^2\xi\sin^2 i\eta+\cos^2\xi\cos^2 i\eta),$$

whence

$$\begin{aligned}(x^2-y^2-c^2)^2= &\ c^4(\cos^2\xi\cos^2 i\eta+\sin^2\xi\sin^2 i\eta)^2\\ +4x^2y^2 &\ -4c^4\sin^2\xi\cos^2\xi\sin^2 i\eta\cos^2 i\eta\\ = &\ c^4(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta)^2.\end{aligned}$$

Hence

$$2\rho\sigma=2c^2(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta),$$

and

$$\rho^2+\sigma^2=2c^2(\sin^2\xi\cos^2 i\eta-\cos^2\xi\sin^2 i\eta+1);$$

hence

$$(\rho+\sigma)^2=2c^2(\cos^2 i\eta-\sin^2 i\eta+1),\ =4c^2\cos^2 i\eta,$$

$$(\rho-\sigma)^2=2c^2(\sin^2\xi-\cos^2\xi+1),\ =4c^2\sin^2\xi.$$

Consequently

$$a^2+\lambda=c^2\cos^2 i\eta, \text{ and thence } b^2+\lambda=-c^2\sin^2 i\eta,$$

$$a^2+\mu=c^2\sin^2\xi, \quad \text{,,} \quad b^2+\mu=-c^2\cos^2\xi,$$

values which verify as they should do the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1,$$

viz. these become

$$\frac{x^2}{c^2\cos^2 i\eta}+\frac{y^2}{-c^2\sin^2 i\eta}=\sin^2\xi\ +\cos^2\xi,\ =1,$$

$$\frac{x^2}{c^2\sin^2\xi}\ +\ \frac{y^2}{-c^2\cos^2\xi}=\cos^2 i\eta+\sin^2 i\eta,\ =1.$$

The same equations, or as we may also write them,

$$\lambda=-a^2\sin^2 i\eta-b^2\cos^2 i\eta,$$
$$\mu=-a^2\cos^2\xi\ -b^2\sin^2\xi,$$

determine η as a function of λ, and ξ as a function of μ; λ, μ being by what precedes, given functions of x, y.

Or more simply, starting from the last-mentioned values of λ, μ, and substituting these in the expressions

$$x^2=\frac{a^2+\lambda\,.\,a^2+\mu}{a^2-b^2},\quad y^2=\frac{b^2+\lambda\,.\,b^2+\mu}{b^2-a^2},$$

we find

$$x^2=c^2\sin^2\xi\cos^2 i\eta,\quad y^2=-c^2\cos^2\xi\sin^2 i\eta,$$

or say

$$x=c\sin\xi\cos i\eta,\quad iy=c\cos\xi\sin i\eta,$$

whence

$$x+iy=c\sin(\xi+i\eta),$$

the original relation between x, y and ξ, η.

684.

ON A RELATION BETWEEN CERTAIN PRODUCTS OF DIFFERENCES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 174, 175.]

CONSIDER the function

$$3\left\{\begin{matrix} abc\,.\,de \\ +\,bcd\,.\,ea \\ +\,cde\,.\,ab \\ +\,dea\,.\,bc \\ +\,eab\,.\,cd \end{matrix}\right\} - \left\{\begin{matrix} abd\,.\,ce \\ +\,bce\,.\,da \\ +\,cda\,.\,eb \\ +\,deb\,.\,ac \\ +\,eac\,.\,bd \end{matrix}\right\},$$

where

$$abc = (a-b)(b-c)(c-a),$$
$$ab = (a-b)(b-a), = -(a-b)^2,$$
$$\&c.;$$

therefore

$$abc = bca = cab = -bac, \ \&c.;$$
$$ab = ba.$$

It is to be shown that the function vanishes if $e = d$. Writing $e = d$, the value is

$$\begin{aligned} 3\,(bcd\,.\,da + dab\,.\,cd) &- abd\,.\,cd \\ &- bcd\,.\,da \\ &- cda\,.\,db \\ &- dac\,.\,bd, \end{aligned}$$

viz. this is

$$\begin{aligned} & 3\ bcd\,.\,ad - \ abd\,.\,cd \\ & + 3\ abd\,.\,cd - \ bcd\ .\,ad \\ & \qquad\qquad - 2acd\ .\,bd \\ & = 2\ bcd\,.\,ad - 2acd\ .\,bd + 2abd\,.\,cd \\ & = 2\,(bcd\,.\,ad + \ cad\ .\,bd + \ abd\,.\,cd), \end{aligned}$$

which is easily seen to vanish; the value is

$$\begin{aligned} (b-c)(c-d)(d-b)(a-d)^2 &= -(b-c)(a-d)^2(b-d)\ (c-d) \\ + (c-a)(a-d)(d-c)(b-d)^2 &\quad -(c-a)(a-d)\ (b-d)^2(c-d) \\ + (a-b)(b-d)(d-a)(c-d)^2 &\quad -(a-b)(a-d)\ (b-d)\ (c-d)^2: \end{aligned}$$

viz. omitting the factor $(a-d)(b-d)(c-d)$, this is

$$\begin{aligned} &= -(b-c)(a-d) \\ &\quad -(c-a)(b-d) \\ &\quad -(a-b)(c-d), \end{aligned}$$

which vanishes. Hence the function also vanishes if $e=a$, or $a=b$ or $b=c$, or $c=d$; and it is thus a mere numerical multiple of $(a-b)(b-c)(c-d)(d-e)(e-a)$, or say it is $=Mabcde$.

To find M write $e=c$, the equation becomes

$$\begin{aligned} & 3abc\,.\,dc - cda\,.\,cb = Mabcdc, = Mabc\,.\,dc, \\ & + 3bcd\,.\,ca - ac \\ & + 3dca\,.\,bc \\ & + 3cab\ .\,cd, \end{aligned}$$

viz. this is

$$6abc\,.\,dc + 4dbc\,.\,ac + 4adc\,.\,bc = M\,.\,abc\,.\,dc,$$

giving $M=10$. In fact, we then have

$$-4abc\,.\,dc + 4dbc\,.\,ac + 4adc\,.\,bc = 0,$$

that is,

$$-\ abc\,.\,dc - \ bdc\,.\,ac - \ dac\,.\,bc = 0,$$

which is right. And we have thus the identity

$$3\left\{\begin{array}{l} abc\,.\,de \\ + bcd\,.\,ea \\ + cde\,.\,ab \\ + dea\,.\,bc \\ + eab\,.\,cd \end{array}\right\} - \left\{\begin{array}{l} abd\,.\,ce \\ + bce\ .\,da \\ + cda\,.\,eb \\ + deb\,.\,ac \\ + eac\,.\,bd \end{array}\right\} = 10\,.\,abcde,$$

or say

$$3\,[abcde] - [acebd] = 10\,\{abcde\}.$$

685.

ON MR COTTERILL'S GONIOMETRICAL PROBLEM.

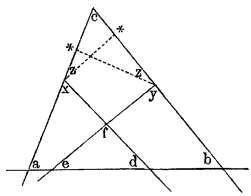
[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 196—198.]

THE very remarkable formulæ contained in Mr Cotterill's paper, "A goniometrical problem, to be solved analytically in one move, or more simply synthetically in two moves," *Quarterly Mathematical Journal*, t. VII. (1866), pp. 259—272, are presented in a form which, to say the least, is not as easily intelligible as might be; and they have not, I think, attracted the attention which they well deserve.

Using his notation, except that I write for angles small roman letters, in order to be able to have the corresponding italic small letters and capitals for the sines and cosines respectively of the same angles, we consider nine angles

$$\begin{matrix} a, & b, & c, \\ d, & e, & f, \\ x, & y, & z, \end{matrix}$$

which are such that the sum of three angles in the same line, or in the same column, is an odd multiple of π. Of course, any four angles such as a, b, d, e are



arbitrary, and each of the remaining angles is then determinate save as to an even multiple of π. And it may be remarked that these angles a, b, d, e may represent the inclinations of any four lines to a fifth line, and that the remaining angles are then at once obtained, as in the figure. The small roman letters are here used to denote as well angles as points, being so placed as to show what the angles are which they respectively denote; the points $*$, $*$ are constructed as the intersections of the lines ac, bc by the circle circumscribed about fxy, and the angle z is the angle which the points $*$, $*$ subtend at x or y. It will be observed that the sum of the three angles in a line or column is in each case $= \pi$.

But this in passing: the analytical theorem is, *first*, we can form with the sines and cosines of the angles in any two lines or columns a function S presenting itself under two distinct forms, which are in fact equal in value, or say S is a symmetrical function of the two lines or columns, viz. for the first and second lines this is

$$S\begin{pmatrix} \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{d}, & \mathrm{e}, & \mathrm{f} \end{pmatrix} = d^2 Abc + e^2 Bca + f^2 Cab$$
$$= a^2 Def + b^2 Efd + c^2 Fde,$$

where, as already mentioned, a, A denote sin a, cos a, and so for the other letters.

Secondly, if to the S of any two lines or columns we add twice the product of the six sines, we obtain a sum M which has the same value from whichever two lines or columns we obtain it; or, say M is a symmetrical function of the matrix of the nine angles. Thus

$$M = S\begin{pmatrix} \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{d}, & \mathrm{e}, & \mathrm{f} \end{pmatrix} + 2abcdef,$$

which is one of a system of six forms each of which (on account of the two forms of the S contained in it) may be regarded as a double form, and the twelve values are all of them equal. There are, moreover, 15 other forms, of M, viz. 3 line-forms, such as

$$bcdx + caey + abfz \text{ (belongs to line a, b, c)},$$

3 column-forms, such as

$$dxbc + xaef + adyz \text{ (belongs to column a, d, x)},$$

and 9 term-forms, such as

$$e^2z^2 + f^2y^2 + 2efyzA \text{ (belongs to term a)},$$

and the $12 + 15, = 27$ values are all equal.

The several identities can of course be verified by means of the relations between the nine angles, or rather the derived sine- and cosine-relations

$$C = ab - AB,$$
$$c = aB + bA, \text{ \&c.}$$

Thus, as regards the two forms of $S\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix}$, the identity to be verified may be written

$$c\,(d^2Ab + e^2Ba - cFde) = f\,(a^2De + b^2Ed - fCab).$$

Proceeding to reduce the factor $a^2De + b^2Ed - fCab$, if we first write herein $f = eD + dE$, it becomes

$$a^2De + b^2Ed - (eD + dE)\,Cab,$$

which is

$$= aDe\,(a - bC) + bEd\,(b - aC),$$

and then writing $C = ab - AB$, we have $a - bC = a\,(1 - b^2) + bAB$, $= B\,(aB + bA)$, $= Bc$; and, similarly, $b - aC = Ac$; whence the term is $= c\,(aeBD + bdAE)$; or, in the equation to be verified, the right-hand side is $= cf\,(aeBD + bdAE)$, and by a similar reduction, the left-hand side is found to have the same value.

The paper contains various other interesting results.

686.

ON A FUNCTIONAL EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 315—325; *Proceedings of the London Mathematical Society*, vol. IX. (1878), p. 29.]

I WAS led by a hydrodynamical problem to consider a certain functional equation; viz. writing for shortness $x_1 = \dfrac{ax+b}{cx+d}$, this is

$$\phi x - \phi x_1 = (x - x_1)\frac{Ax+B}{Cx+D}.$$

I find by a direct process, which I will afterwards explain, the solution

$$\phi x = \frac{A}{C}x + \frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C\,(dC-cD)}\int_0^\infty \frac{\sin \xi t \sin \eta t\, dt}{\sin \zeta t \sinh \pi t};$$

where ζ is a constant, but ξ, η are complicated logarithmic functions of x (ξ, η, ζ depend also on the quantities a, b, c, d, C, D); $\sinh \pi t$ denotes as usual the hyperbolic sine, $\frac{1}{2}(e^{\pi t} - e^{-\pi t})$.

The values of ξ, η, ζ are given by the formulæ

$$\lambda + \frac{1}{\lambda} = \frac{a^2+d^2+2bc}{ad-bc},$$

$$\mathrm{a} = ax+b, \quad \mathrm{b} = -dx+b,$$

$$\mathrm{c} = cx+d, \quad \mathrm{d} = cx-a,$$

$$W = C\mathrm{a} + D\mathrm{c},$$

$$Z = C\mathrm{b} + D\mathrm{d},$$

$$R = \lambda\mathrm{c} + \lambda\mathrm{d},$$

$$S = -\mathrm{c} - \mathrm{d},$$

$$R' = W + \frac{1}{\lambda}Z,$$

$$S' = -W - \lambda Z,$$

which determine λ, R, S, R', S' and then

$$\xi = \tfrac{1}{2}\log\frac{RS'}{R'S}, \quad \eta = \tfrac{1}{2}\left(\log\lambda + \log\frac{RR'}{SS'}\right), \quad \zeta = \tfrac{1}{2}\log\lambda.$$

There is some difficulty as to the definite integral, on account of the denominator factor $\sin\zeta t$, which becomes $=0$ for the series of values $t = \frac{m\pi}{\zeta}$, but this is a point which I do not enter into.

I will in the first instance verify the result. Writing x_1 in place of x, and taking ξ_1, η_1 to denote the corresponding values of ξ, η, it will be shown that

$$\xi_1 = \xi, \quad \eta_1 = \eta + 2\zeta, \qquad \text{see } \textit{post}, (1).$$

Hence in the difference $\phi x - \phi x_1$ we have the integral

$$\int \frac{\sin\xi t\,\{\sin\eta t - \sin(\eta + 2\zeta)\,t\}\,dt}{\sin\zeta t\sinh\pi t},$$

(where and in all that follows the limits are ∞, 0 as before); here, since

$$\sin\eta t - \sin(\eta + 2\zeta)\,t = -2\sin\zeta t\cos(\eta + \zeta)\,t,$$

the factor $\sin\zeta t$ divides out, and the numerator is

$$= -2\sin\xi t\cos(\eta + \zeta)\,t,$$

which is

$$= \sin(\eta + \zeta - \xi)\,t - \sin(\eta + \zeta + \xi)\,t.$$

Hence the integral in question is

$$= \int \frac{\sin(\eta + \zeta - \xi)\,t\,dt}{\sinh\pi t} - \int \frac{\sin(\eta + \zeta + \xi)\,t\,dt}{\sinh\pi t}.$$

Now we have in general

$$\frac{1}{1 + \exp.\,\alpha} = \tfrac{1}{2} - \int \frac{\sin\alpha t\,dt}{\sinh\pi t};$$

(this is, in fact, Poisson's formula

$$-\frac{1}{1 + k\beta^{2n}} = \tfrac{1}{2} - 2\int \frac{\sin(2n\log\beta + \log k)\,t\,.\,dt}{e^{\pi t} - e^{-\pi t}},$$

in the second Memoir on the distribution of Electricity, &c., *Mém. de l'Inst.*, 1811, p. 223); and hence the value is

$$-\frac{1}{1 + \exp.\,(\eta + \zeta - \xi)} + \frac{1}{1 + \exp.\,(\eta + \zeta + \xi)},$$

or since

$$\eta + \zeta = \log\lambda + \tfrac{1}{2}\log\frac{RR'}{SS'}, \quad \xi = \tfrac{1}{2}\log\frac{RS'}{R'S},$$

we have

$$\eta + \zeta - \xi = \log\lambda + \tfrac{1}{2}\log\frac{R'^2}{S'^2} = \log\lambda\frac{R'}{S'},$$

$$\eta + \zeta + \xi = \log\lambda + \tfrac{1}{2}\log\frac{R^2}{S^2} = \log\lambda\frac{R}{S},$$

and the value is thus

$$=-\frac{1}{1+\lambda\dfrac{R'}{S'}}+\frac{1}{1+\lambda\dfrac{R}{S}},\quad =-\frac{(RS'-R'S)\lambda}{(\lambda R'+S')(\lambda R+S)}.$$

Hence, from the assumed value of ϕx, we obtain

$$\phi x-\phi x_1=\frac{A}{C}(x-x_1)-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)\,(RS'-R'S)\,\lambda}{C(dC-cD)(\lambda R'+S')(\lambda R+S)}.$$

We have

$$RS'-R'S=\frac{(\lambda-1)(a+d)^2}{ad-bc}(dC-cD)\{cx^2+(d-a)x-b\},$$

$$R\lambda+\quad S=(\lambda^2-1)(cx+d),\qquad\qquad \text{see } \textit{post}, (2),$$

$$R'\lambda+\quad S'=(\lambda\ -1)(a+d)(Cx+D),$$

or since

$$\frac{cx^2+(d-a)x-b}{cx+d}=x-x_1,$$

this is

$$\phi x-\phi x_1=\frac{A}{C}(x-x_1)-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C}\,\frac{(a+d)\lambda}{(ad-bc)(\lambda^2-1)(Cx+D)}(x-x_1).$$

But from the value of λ,

$$\frac{\lambda}{\lambda^2-1}=\frac{ad-bc}{(a+d)\sqrt{\{(a-d)^2+4bc\}}},$$

and the equation thus is

$$\phi x-\phi x_1=(x-x_1)\left\{\frac{A}{C}-\frac{AD-BC}{C(Cx+D)}\right\},\quad =(x-x_1)\frac{Ax+B}{Cx+D},$$

as it should be.

(1) For the foregoing values of ξ_1, η_1, we require R_1, S_1, R_1', S_1', the values which R, S, R', S' assume on writing therein x_1 for x. We have

$$R_1=\quad\lambda(cx_1+d)+\quad(cx_1-a),$$

$$S_1=-\quad(cx_1+d)-\lambda(cx_1-a):$$

substituting for x_1 its value, we find

$$R_1(cx+d)=(a+d)\lambda(cx+d)-(ad-bc)(\lambda+1),$$

or writing herein

$$ad-bc=\frac{(a+d)^2\lambda}{(\lambda+1)^2},$$

this is

$$R_1(cx+d)=\frac{(a+d)\lambda}{\lambda+1}R;$$

and similarly

$$S_1(cx+d)=\frac{a+d}{\lambda+1}S.$$

We have in like manner

$$R_1' = \quad W_1 + \frac{1}{\lambda} Z_1, \text{ where } W_1 = C(\quad ax_1 + b) + D(cx_1 + d),$$

$$S_1' = - W_1 - \lambda Z_1, \text{ where } Z_1 = C(-dx_1 + b) + D(cx_1 - a).$$

Substituting for x_1 its value, we find

$$W_1(cx+d) = C[(a+d)(ax+b) - (ad-bc)x] + D[(a+d)(cx+d) - (ad-bc)],$$

$$Z_1(cx+d) = C[\qquad\qquad - (ad-bc)x] + D[\qquad\qquad - (ad-bc)]:$$

hence, substituting for $ad - bc$ as before,

$$W_1(cx+d) = \frac{a+d}{(\lambda+1)^2}\{(\lambda+1)^2 W - (a+d)\lambda(Cx+D)\},$$

$$Z_1(cx+d) = \frac{a+d}{(\lambda+1)^2}\{\qquad\qquad - (a+d)\lambda(Cx+D)\},$$

whence without difficulty

$$R_1'(cx+d) = \frac{(a+d)\lambda}{\lambda+1} R',$$

$$S_1'(cx+d) = \quad \frac{a+d}{\lambda+1} S':$$

consequently

$$\frac{R_1 S_1'}{R_1' S_1} = \quad \frac{RS'}{R'S}, \text{ that is, } \xi_1 = \xi,$$

$$\frac{R_1 R_1'}{S_1 S_1'} = \lambda^2 \frac{RR'}{SS'}, \quad \text{,,} \quad , \eta_1 = \log\lambda + \eta, = 2\zeta + \eta,$$

which are the formulæ in question.

(2) For the value of $RS' - R'S$, we have

$$RS' - R'S = (\lambda \mathrm{c} + \mathrm{d})(-W - \lambda Z) - (-\lambda \mathrm{d} - \mathrm{c})\left(W + \frac{Z}{\lambda}\right)$$

$$= \left(-\lambda^2 + \frac{1}{\lambda}\right)\mathrm{c}Z + (\lambda+1)\{(\mathrm{d}-\mathrm{c})W - \mathrm{d}R\}$$

$$= -(\lambda-1)\left\{\left(1 + \lambda + \frac{1}{\lambda}\right)\mathrm{c}Z + (\mathrm{c}-\mathrm{d})W + \mathrm{d}Z\right\};$$

or substituting for $\lambda + \frac{1}{\lambda}$, Z and W their values, this is

$$= \frac{-(\lambda-1)}{ad-bc}\{(a^2 + d^2 + ad + bc)\,\mathrm{c}\,(\mathrm{b}C + \mathrm{d}D)$$
$$+ (ad-bc)[(\mathrm{c}-\mathrm{d})(\mathrm{a}C + \mathrm{c}D) + \mathrm{d}(\mathrm{b}C + \mathrm{d}D)]\}.$$

In the term in { }, the coefficient of C is

$$[(a^2 + d^2 + ad + bc)\,\mathrm{b} + (ad-bc)\,\mathrm{a}]\,\mathrm{c} - \mathrm{d}\,(\mathrm{a}-\mathrm{b})(ad-bc)$$
$$= (a+d)(d\mathrm{b} - b\mathrm{d})\,\mathrm{c} - (a+d)\,\mathrm{d}x\,(ad-bc),$$

and similarly the coefficient of D is

$$[(a^2+d^2+ad+bc)\,\mathrm{d}+(ad-bc)\,\mathrm{c}]\,\mathrm{c}-\mathrm{d}\,(\mathrm{c}-\mathrm{d})\,(ad-bc)$$
$$=(a+d)\,(\mathrm{ad}-\mathrm{cb})\,\mathrm{c}-(a+d)\,\mathrm{d}\,(ad-bc).$$

Hence the whole term in { } is

$$=(a+d)\,\{[(d\mathrm{b}-b\mathrm{d})\,\mathrm{c}-\mathrm{d}\,(ad-bc)\,x]\,C+[(\mathrm{ad}-\mathrm{cb})\,\mathrm{c}-\mathrm{d}\,(ad-bc)]\,D\},$$

which is readily reduced to

$$(a+d)\,(\mathrm{ad}-\mathrm{bc})\,(-dC+cD);$$

also

$$\mathrm{ad}-\mathrm{bc}=(a+d)\,\{cx^2+(d-a)\,x-b\};$$

so that we have

$$RS'-R'S=\frac{(\lambda-1)\,(a+d)^2}{ad-bc}\,(dC-cD)\,[cx^2+(d-a)\,x-b],$$

which is the required value of $RS'-R'S$; and there is no difficulty in obtaining the other two formulæ,

$$R\lambda+S=(\lambda^2-1)\,(cx+d),$$
$$R'\lambda+S'=(\lambda-1)\,(a+d)\,(Cx+D);$$

the verification is thus completed.

To show how the formula was directly obtained, we have

$$\frac{Ax+B}{Cx+D}=\frac{A}{C}-\frac{AD-BC}{C}\,\frac{1}{Cx+D}$$
$$=\frac{A}{C}+\beta x \text{ suppose};$$

the equation then is

$$\phi x-\phi x_1=\frac{A}{C}\,(x-x_1)+(x-x_1)\,\beta x.$$

Hence, if x_1, x_2, x_3, ... denote the successive functions ϑx, $\vartheta^2 x$, $\vartheta^3 x$, &c., we have

$$\phi x_1-\phi x_2=\frac{A}{C}\,(x_1-x_2)+(x_1-x_2)\,\beta x_1,$$

$$\phi x_2-\phi x_3=\frac{A}{C}\,(x_2-x_3)+(x_2-x_3)\,\beta x_2,$$

whence adding, and neglecting ϕx_∞ and x_∞, we have

$$\phi x=\frac{A}{C}\,x+[(x-x_1)\,\beta x+(x_1-x_2)\,\beta x_1+(x_2-x_3)\,\beta x_2+\ldots],$$

where the term in [ ], regarding therein x_1, x_2, x_3, ... as given functions of x, is itself a given function of x; and it only remains to sum the series.

Starting from

$$x_1=\vartheta x=\frac{ax+b}{cx+d},$$

and writing

$$\lambda+\frac{1}{\lambda}=\frac{a^2+d^2+2bc}{ad-bc},$$

then the nth function is given by the formula

$$x_n = \vartheta_n x = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(cx-a)}$$

$$= \frac{(\lambda^{n+1}-1)\,\mathrm{a}+(\lambda^n-\lambda)\,\mathrm{b}}{(\lambda^{n+1}-1)\,\mathrm{c}+(\lambda^n-\lambda)\,\mathrm{d}}$$

$$= \frac{\lambda^n P+Q}{\lambda^n R+S},$$

if $P=\lambda\mathrm{a}+\mathrm{b}$, $Q=-\mathrm{a}-\lambda\mathrm{b}$, and as before $R=\lambda\mathrm{c}+\mathrm{d}$, $S=-\mathrm{c}-\lambda\mathrm{d}$.

I stop to remark that λ being real, then if $\lambda>1$ we have λ^n very large for n very large, and $x^n=\frac{P}{R}$ which is independent of n; the value in question is

$$x_n = \frac{\lambda(ax+b)+(-dx+b)}{\lambda(cx+d)+(\quad cx-a)},$$

which, observing that the equation in λ may be written

$$\frac{\lambda a-d}{c(\lambda+1)} = \frac{b(\lambda+1)}{\lambda d-a},$$

is, in fact, independent of x, and is $=\frac{\lambda a-d}{c(\lambda+1)}$ or $\frac{b(\lambda+1)}{\lambda d-a}$; we have $x_{n-1}=x_n$, or calling each of these two equal values x, we have

$$x = \frac{ax+b}{cx+d},$$

which is the same equation as is obtainable by the elimination of λ from the equations

$$x = \frac{\lambda a-d}{c(\lambda+1)} = \frac{b(\lambda+1)}{\lambda d-a}.$$

The same result is obtained by taking $\lambda<1$ and consequently $x_n=\frac{Q}{S}$.

We find

$$x_{n-1}-x_n = \frac{\lambda^{n-1}P+Q}{\lambda^{n-1}R+S} - \frac{\lambda^n P+Q}{\lambda^n R+S},$$

$$= \frac{-\lambda^{n-1}(\lambda-1)(PS-QR)}{(\lambda^{n-1}R+S)(\lambda^n R+S)},$$

where

$$PS-QR = -(\lambda^2-1)(\mathrm{a}d-\mathrm{b}c) = -(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\};$$

and therefore

$$x_{n-1}-x_n = \frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}\lambda^n}{\lambda(\lambda^{n-1}R+S)(\lambda^n R+S)}.$$

Also

$$\beta x_{n-1} = -\frac{AD-BC}{C}\,\frac{1}{Cx_{n-1}+D},$$

where

$$Cx_{n-1}+D=\frac{C(\lambda^{n-1}P+Q)+D(\lambda^{n-1}R+S)}{\lambda^{n-1}R+S}=\frac{R'\lambda^n+S'}{R\lambda^{n-1}+S},$$

where

$$R'=\frac{CP}{\lambda}+\frac{DR}{\lambda},\ =C\left(\mathrm{a}+\frac{\mathrm{b}}{\lambda}\right)+D\left(\mathrm{c}+\frac{\mathrm{d}}{\lambda}\right),$$

$$S'=CQ+DS,\ =C(-\mathrm{a}-\mathrm{b}\lambda)+D(-\mathrm{c}-\mathrm{d}\lambda);$$

viz.

$$R'=W+\frac{1}{\lambda}Z,\ \ S'=-W-\lambda Z,$$

where Z and W denote $\mathrm{a}C+\mathrm{c}D$ and $\mathrm{b}C+\mathrm{d}D$ as before.

We hence obtain

$$(x_{n-1}-x_n)\,\beta x_n=\frac{-(AD-BC)}{C}$$

$$\times\frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}}{\lambda}\,\frac{\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}$$

$$=\frac{-(AD-BC)}{C}$$

$$\times\frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}}{\lambda(RS'-R'S)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

or, substituting for $RS'-R'S$ its value in the denominator, this is

$$(x_{n-1}-x_n)\,\beta x_n=-\frac{AD-BC}{C}\,\frac{(ad-bc)(\lambda^2-1)}{(a+d)\lambda(cD-dC)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}$$

$$=-\frac{\sqrt{\{(a-d)^2+4bc\}}(AD-BC)}{C(cD-dC)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

and thence

$$\phi x=\frac{A}{C}x-\frac{\sqrt{\{(a-d)^2+4bc\}}(AD-BC)}{C(cD-dC)}\,\Sigma\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

the summation extending from 1 to ∞.

Now the before-mentioned integral formula gives

$$\frac{1}{1+k\lambda^n}=\tfrac{1}{2}-\int\frac{\sin(n\log\lambda+\log k)\,t\,dt}{\sinh\pi t},$$

$$\frac{1}{1+k'\lambda^n}=\tfrac{1}{2}-\int\frac{\sin(n\log\lambda+\log k')\,t\,dt}{\sinh\pi t}.$$

Taking the difference, and then writing $k=\frac{R}{S}$, $k'=\frac{R'}{S'}$, we have under the integral sign

$$\sin\left(n\log\lambda+\log\frac{R}{S}\right)t-\sin\left(n\log\lambda+\log\frac{R'}{S'}\right)t,$$

which is

$$= 2\sin\tfrac{1}{2}\left(\log\frac{RS'}{R'S}\right)t\cos\left(n\log\lambda+\tfrac{1}{2}\log\frac{RR'}{SS'}\right)t,$$

which attending to the before-mentioned values of ξ, η, ζ is

$$= 2\sin\xi t\cos(2n\zeta-\zeta+\eta)\,t,$$

and the formula thus is

$$\frac{S}{R\lambda^n+S}-\frac{S'}{R'\lambda^n+S'},=-\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}=-\int\frac{2\sin\xi t\cos(2n\zeta-\zeta+\eta)\,t\,dt}{\sinh\pi t}.$$

We have here

$$\cos(2n\zeta-\zeta+\eta)\,t=\cos 2n\zeta t\cos(\eta-\zeta)\,t-\sin 2n\zeta t\sin(\eta-\zeta)\,t,$$

whence summing from 1 to ∞ by means of the formulæ

$$\cos 2\zeta t+\cos 4\zeta t+\ldots=-\tfrac{1}{2},$$
$$\sin 2\zeta t+\sin 4\zeta t+\ldots=\tfrac{1}{2}\cot\zeta t,$$

(which series however are not convergent), the numerator under the integral sign becomes

$$\sin\xi t\{-\cos(\eta-\zeta)\,t-\cot\zeta t\sin(\eta-\zeta)\,t\},$$

which is

$$=-\frac{\sin\xi t\sin\eta t}{\sin\zeta t},$$

and the formula thus is

$$\Sigma\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}=-\int\frac{\sin\xi t\sin\eta t\,dt}{\sin\zeta t\sinh\pi t};$$

and we therefore find

$$\phi x=\frac{A}{C}x+\frac{\sqrt{\{(a-d)^2+4bc\}}(AD-BC)}{C(cD-dC)}\int\frac{\sin\xi t\sin\eta t\,dt}{\sin\zeta t\sinh\pi t},$$

which is the result in question.

The solution is a particular one; calling it for a moment (ϕx), then, if the general solution be $\phi x=\Phi x+(\phi x)$, it at once appears that we must have $\Phi x-\Phi x_1=0$; and as it has been shown that $\frac{RS'}{R'S}$ is a function of x which remains unaltered by the change of x into x_1, this is satisfied by assuming $\Phi x=f\left(\frac{RS'}{R'S}\right)$, an arbitrary function of $\frac{RS'}{R'S}$. Hence we may to the foregoing expression of ϕx add this term $f\left(\frac{RS'}{R'S}\right)$.

Postscript. The new formula

$$\vartheta^n x=\frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\quad cx-a)},$$

where

$$\lambda+\frac{1}{\lambda}=\frac{a^2+d^2+2bc}{ad-bc},$$

for the nth repetition of ϑx, $=\dfrac{ax+b}{cx+d}$, is a very interesting one. It is to be remembered that, when n is even the numerator and denominator each divide by $\lambda-1$, but when n is odd they each divide by λ^2-1; after such division, then further dividing by a power of λ, they each consist of terms of the form $\lambda^a+\dfrac{1}{\lambda^a}$, that is, they are each of them a rational function of $\lambda+\dfrac{1}{\lambda}$. Substituting and multiplying by the proper power of $ad-bc$, the numerator and denominator become each of them a rational and integral function of a, b, c, d of the order $n+1$ when n is even, but of the order n when n is odd; in the former case, however, the numerator and denominator each divide by $a+d$, so that ultimately, whether n be even or odd, the order is $=n$ as it should be.

For example, when $n=2$, the value is

$$\frac{(\lambda^3-1)\,\mathrm{a}+(\lambda^2-\lambda)\,\mathrm{b}}{(\lambda^3-1)\,\mathrm{c}+(\lambda^2-\lambda)\,\mathrm{d}},\quad =\frac{(\lambda^2+\lambda+1)\,\mathrm{a}+\lambda\mathrm{b}}{(\lambda^2+\lambda+1)\,\mathrm{c}+\lambda\mathrm{d}},\quad =\frac{\left(\lambda+\dfrac{1}{\lambda}+1\right)\mathrm{a}+\mathrm{b}}{\left(\lambda+\dfrac{1}{\lambda}+1\right)\mathrm{c}+\mathrm{d}},$$

or, as this may be written,

$$=\frac{\left(\lambda+\dfrac{1}{\lambda}+2\right)\mathrm{a}-\mathrm{a}+\mathrm{b}}{\left(\lambda+\dfrac{1}{\lambda}+2\right)\mathrm{c}-\mathrm{c}+\mathrm{d}},$$

where, observing that

$$\lambda+\frac{1}{\lambda}+2=\frac{(a+d)^2}{ad-bc},\quad -\mathrm{a}+\mathrm{b}=-(\mathrm{a}+\mathrm{d})\,x,\quad -\mathrm{c}+\mathrm{d}=-(a+d),$$

the numerator and denominator each divide by $a+d$, and the final value is

$$=\frac{(a+d)(ax+b)-(ad-bc)\,x}{(a+d)(cx+d)-(ad-bc)},\quad =\frac{(a^2+bc)\,x+b\,(a+d)}{c\,(a+d)\,x+bc+d^2},$$

which is the proper value of $\vartheta^2 x$. But, when $n=3$, the value is

$$\frac{(\lambda^4-1)\,\mathrm{a}+(\lambda^3-\lambda)\,\mathrm{b}}{(\lambda^4-1)\,\mathrm{c}+(\lambda^3-\lambda)\,\mathrm{d}},\quad =\frac{(\lambda^2+1)\,\mathrm{a}+\lambda\mathrm{b}}{(\lambda^2+1)\,\mathrm{c}+\lambda\mathrm{d}},\quad =\frac{\left(\lambda+\dfrac{1}{\lambda}\right)\mathrm{a}+\mathrm{b}}{\left(\lambda+\dfrac{1}{\lambda}\right)\mathrm{c}+\mathrm{d}};$$

and this is

$$=\frac{(a^2+d^2+2bc)(ax+b)+(ad-bc)(-dx+b)}{(a^2+d^2+2bc)(cx+d)+(ad-bc)(\quad cx-a)},$$

or finally

$$=\frac{(a^3+2abc+bcd)\,x+b\,(a^3+ad+bc+d^3)}{c\,(a^2+ad+bc+d^2)\,x+(abc+2bcd+d^3)},$$

which is the proper value of $\vartheta^3 x$.

687.

NOTE ON THE FUNCTION $\vartheta x = a^2(c-x) \div \{c(c-x)-b^2\}$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XV. (1878), pp. 338—340.]

STARTING from the general form

$$\vartheta x = \frac{\alpha x + \beta}{\gamma x + \delta},$$

we have

$$\vartheta^n x = \frac{(\lambda^{n+1}-1)(\alpha x+\beta)+(\lambda^n-\lambda)(-\delta x+\beta)}{(\lambda^{n+1}-1)(\gamma x+\delta)+(\lambda^n-\lambda)(\gamma x-\alpha)},$$

where

$$\lambda + \frac{1}{\lambda} = \frac{\alpha^2+\delta^2+2\beta\gamma}{\alpha\delta-\beta\gamma}.$$

For the function in question

$$\vartheta x = \frac{a^2(c-x)}{c(c-x)-b^2},$$

(a form which presents itself in the problem of the distribution of electricity upon two spheres), the values of α, β, γ, δ are

$$\alpha = -a^2, \quad \beta = a^2c, \quad \gamma = -c, \quad \delta = c^2-b^2;$$

the equation for λ therefore is

$$\lambda + \frac{1}{\lambda} = \frac{a^4+(c^2-b^2)^2-2a^2c^2}{a^2b^2};$$

or, what is the same thing,

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a^2+b^2-c^2)^2}{a^2b^2}.$$

Suppose that a, b, c are the sides of a triangle the angles whereof are A, B, C; then $c^2 = a^2 + b^2 - 2ab\cos C$, or we have

$$\frac{(\lambda+1)^2}{\lambda} = 4\cos^2 C;$$

or, writing this under the form

$$\sqrt{(\lambda)} + \frac{1}{\sqrt{(\lambda)}} = 2\cos C,$$

the value of λ is at once seen to be $= e^{xiC}$; and it is interesting to obtain the expression of the nth function in terms of the sides and angles of the triangle.

The numerator and the denominator are

$$\lambda^n P + Q,$$
$$\lambda^n R + S,$$

where

$$P = \lambda(\alpha x + \beta) + (-\delta x + \beta), \quad R = \lambda(\gamma x + \delta) + \gamma x - \alpha,$$
$$Q = -(\alpha x + \beta) - \lambda(-\delta x + \beta), \quad S = -(\gamma x + \delta) - \lambda(\gamma x - \alpha).$$

Hence, writing the numerator and the denominator in the forms

$$\lambda^{\frac{1}{2}n} P + \lambda^{-\frac{1}{2}n} Q,$$
$$\lambda^{\frac{1}{2}n} R + \lambda^{-\frac{1}{2}n} S,$$

these are

$$(P+Q)\cos nC + (P-Q)\, i \sin nC,$$
$$(R+S)\cos nC + (R-S)\, i \sin nC;$$

viz. they are

$$(\lambda-1)(\alpha+\delta)\, x \cos nC + (\lambda+1)\{(\alpha-\delta)x + 2\beta\}\, i \sin nC,$$
$$(\lambda-1)(\alpha+\delta)\,.\cos nC + (\lambda+1)\{2\gamma x - (\alpha-\delta)\}\, i \sin nC,$$

or, observing that $\frac{\lambda-1}{\lambda+1} = i\tan C$ and removing the common factor $i(\lambda+1)$, they may be written

$$\tan C\,(\alpha+\delta)\, x \cos nC + \{(\alpha-\delta)x + 2\beta\} \sin nC,$$
$$\tan C\,(\alpha+\delta)\,.\cos nC + \{2\gamma x - (\alpha-\delta)\} \sin nC.$$

Substituting for α, β, γ, δ their values, these are

$$\tan C\,\{(c^2 - a^2 - b^2)\, x \cos nC\} + \{(b^2 - a^2 - c^2)\, x + 2a^2c\} \sin nC,$$
$$\tan C\,\{(c^2 - a^2 - b^2)\,.\cos nC\} + \{-2cx - (b^2 - a^2 - c^2)\} \sin nC,$$
$$= \tan C\,\{-ab\cos Cx \cos nC\} + \{-ac\cos B\,.\,x + a^2c\} \sin nC,$$
$$\tan C\,\{-ab\cos Cx \cos nC\} + \{-cx + ac\cos B\} \sin nC,$$
$$= x\,\{-ab\sin C\cos nC - ac\cos B\sin nC\} + a^2c\sin nC,$$
$$-cx\sin nC \qquad + \{ac\cos B\sin nC - ab\sin C\cos nC\};$$

or, writing herein $b \sin C = c \sin B$, these are

$$\begin{aligned} &- acx\,\{\sin B \cos nC + \cos B \sin nC\} && + a^2c \sin nC, \\ &- cx \sin nC && + ac\,\{\cos B \sin nC - \sin B \cos nC\}, \end{aligned}$$

whence finally

$$\vartheta^n x = \frac{a^2 \sin nC - ax \sin(nC + B)}{a \sin(nC - B) - x \sin nC}.$$

As a verification, writing $n = 1$, we have

$$\vartheta x = \frac{a^2 \sin C - ax \sin A}{a \sin(C - B) - x \sin C}$$

$$= \frac{a^2c - acx \dfrac{\sin A}{\sin C}}{ac \dfrac{\sin(C-B)}{\sin C} - cx},$$

or observing that

$$ac \frac{\sin(C-B)}{\sin C} = c^2 - b^2,$$

(for this is $\sin A \sin(C - B) = \sin^2 C - \sin^2 B$), we have

$$\vartheta x = \frac{a^2(c-x)}{c^2 - b^2 - cx}$$

as it should be. If in the formula for $\vartheta^n x$ we write $x = 0$, we have a formula given in the Senate-House Problems, January 14, 1878: it was thus that I was led to investigate the general expression.

688.

GEOMETRICAL CONSIDERATIONS ON A SOLAR ECLIPSE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. xv. (1878), pp. 340—347.]

I CONSIDER, from a geometrical point of view, the phenomena of a solar eclipse over the earth generally; attending at present only to the penumbral cone, the vertex of which I denote by V. It is convenient to regard the earth as fixed, and the sun and moon as moving each of them with its proper motion, and also with the diurnal motion. The penumbral cone meets the earth's surface in a curve which may be called the penumbral curve; viz. when the cone is not completely traversed by the earth's surface, (that is, when only some of the generating lines of the cone meet the earth's surface), the penumbral curve is a single (convex or hour-glass-shaped) oval; separated, as afterwards mentioned, into two parts, one of them lying away from the sun, and having no astronomical significance; but when the cone is completely traversed by the earth's surface, then the penumbral curve consists of two separate (convex) ovals; one of them lying away from the sun and having no astronomical significance, the other lying towards the sun. The intermediate case is when the cone just traverses the earth's surface, or is touched internally by the earth's surface; the penumbral curve is then a figure of eight, one portion of which lies away from the sun, and has no astronomical significance: there is another limiting case when the cone is touched externally by the earth's surface, the penumbral curve being then a mere point.

It is necessary to consider on the earth's surface a curve which may for shortness be termed the horizon; viz. this is the curve of contact of the cone, vertex V, circumscribed about the earth; it is a small circle nearly coincident with the great circle, which is the intersection by a plane through the centre of the earth at right angles to the line from this point to the centre of the sun.

Regarding V as a point in the heavens, capable of being viewed notwithstanding the interposition of the moon; the horizon, as above defined, is the curve separating

the portions of the earth's surface for which V is visible and invisible respectively. The horizon does or does not meet the penumbral curve, according as this last consists of a single oval or of two distinct ovals; viz. in the latter case the horizon lies between the two ovals, in the former case the horizon traverses the area of the oval (separating this area into two parts), thus meeting the oval, or penumbral curve, in two points, or say these points separate the oval into two parts; from any point of the one part V is visible, from any point of the other part V is invisible; and from each of the two points themselves V is visible as a point on the horizon in the ordinary sense of the word; that is, there is an exterior contact of the sun and moon visible on the horizon. It is to be observed that, in the limiting cases where the penumbral curve is a mere point and a figure of eight respectively, the horizon passes through the mere point and through the node of the figure of eight respectively.

The two points of intersection of the penumbral curve with the horizon may for shortness be termed critic points. The lines which present themselves in a diagram of a solar eclipse, (see *Nautical Almanac*) are the "northern and south lines of simple contact," say for shortness the "limits"; viz. these are the envelope or, geometrically, a portion of the envelope of the penumbral curve; and the lines of "eclipse begins or ends at sunrise or sunset," say for shortness the critic lines; viz. these are the locus of the critic points.

The point V considered as a point in the heavens is a point occupying a position intermediate between those of the centres of the sun and moon; hence referring it to the surface of the earth by means of a line drawn from the centre, its position on the earth's surface is nearly coincident with that point to which the sun is then vertical; and its motion on the earth's surface is from east to west approximately along the parallel of latitude = sun's declination, and with a velocity of approximately 15° per hour. For any given position of V on the earth's surface, describing with a given angular radius nearly = 90° a small circle (nearly a great circle), this is the horizon; as V moves upon the surface of the earth, the horizon envelopes a curve which is very nearly a parallel, angular radius = sun's declination (there are two such curves in the northern and southern hemispheres respectively, but I attend only to one of them in the proper hemisphere, as will be explained), say this is the horizon-envelope; the horizon in each of its successive positions is thus a curvilinear tangent (nearly a great circle) to this horizon-envelope. If for a given position of V, and also for the consecutive position we consider the corresponding horizons, these intersect in a point K on the horizon-envelope, and the horizon for V is the circle centre V and angular radius VK; K is a point which is very nearly upon, and which may be taken to be upon, the meridian through V; the horizon may be regarded as a tangent which sweeps round the horizon-envelope; to each position thereof there corresponds a position of V, and consequently also a penumbral curve; and (when this is a single oval) the horizon meets it in two points, which are the critic points. It is to be added that, if for a given position of the horizon we consider as well K as the opposite point K_1, (viz., K_1 lies on the great circle KV), then the points K and K_1 divide the horizon into two portions; for any point on one of these portions

V (considered as a point in the heavens) is rising, for a point on the other of them it is setting; and for the points K and K_1 respectively it is moving horizontally; that is, first rising and then setting, or *vice versâ*.

A solar eclipse is of one of two classes; viz. either the penumbral cone completely traverses the earth, so that towards the middle of the eclipse the penumbral curve consists of two separate ovals: or the penumbral cone does not completely traverse the earth, so that throughout the eclipse the penumbral curve consists of a single oval only. In the former case, we have to consider the commencement, during which the penumbral curve passes from a mere point to a figure of eight: the middle, during which it passes from a figure of eight through two ovals to a figure of eight: and the termination, during which it passes from a figure of eight to a mere point. In the latter case, we consider the whole eclipse during which the penumbral curve passes from a mere point through a single oval to a mere point.

In an eclipse of the first class: for the commencement, the penumbral curve is at first a mere point (point of first contact); it then becomes a convex oval, each oval in the first instance inclosing the preceding ones, so that there is not any intersection of two consecutive ovals. We come at last to an oval which is touched north by its consecutive oval, and to an oval which is touched south by its consecutive oval (I presume that the contacts north and south do not take place on the same oval, but I am not sure); and after this, the ovals assume the hour-glass form, each oval intersecting the consecutive oval in two points north and two points south; the ovals thus beginning to form an envelope or limit. There are on each of the ovals two critic points, and we have thus a critic curve commencing at the mere point (point of first contact) and extending in each direction from this point. The point, where an oval is touched by the consecutive oval, is not so far as appears a critic point; that is, the critic curve does not at this point unite itself with the envelope or limit. But the critic curve comes subsequently to unite itself each way with the limit; and, since clearly it cannot intersect the limit, it will at each of these points touch the limit; that is, we have a critic curve extending each way from the point of first contact until it touches the northern limit and until it touches the southern limit. Observe that the penumbral curve, as being at first a mere point or an indefinitely small oval, does not at first contain within itself the point K or K_1: it can only come to do this by passing through a position where the curve passes through K or K_1; viz. K or K_1 would then be a critic point; and I assume for the present that this does not take place. The critic curve at the point of first contact is a curve "eclipse begins at sunrise," and as not coming to pass through a point or K_1, it cannot alter its character; that is, the critic curve, as extending each way from the point of first contact until it comes to touch the northern and southern limits respectively, is a curve "eclipse begins at sunrise"; at the terminal points in question, there is a mere contact of the sun and moon, so that they are points, where the eclipse begins and simultaneously ends at sunrise. Continuing the series of ovals until we arrive at the figure of eight, there are on each of them two critic points, which ultimately unite in the node of the figure of eight; these constitute a critic curve, extending each way from the node of the figure of eight to

the contacts with the northern and southern limits respectively. There is, as before, no passage through a point K or K_1, the curve in question thus retains throughout the same character; and by consideration of the two terminal points it at once appears that it is a curve "eclipse ends at sunrise." The above-mentioned critic curves form together an oval touching the northern and southern limits respectively; say this is the sunrise oval.

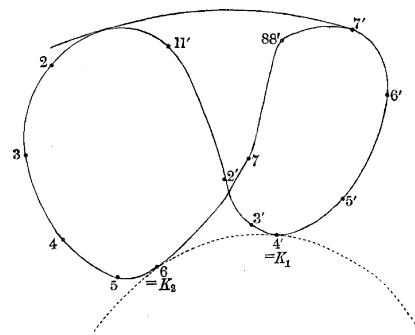
The termination of the eclipse is similar to this, only the events happen in the reverse order; we have a critic line starting from the node of the figure of eight and extending each way until it comes to touch the northern and southern limits respectively, viz. this is the line "eclipse begins at sunset"; and then, extending each way from the points of contact to reunite itself at the point of last contact, this being the line "eclipse ends at sunset," and the two portions together form an oval touching the northern and southern limits respectively; say this is the sunset oval. It is to be noticed that certain portions of the two limits are generated as the envelope of the penumbral curve during the commencement and during the termination of the eclipse.

For the middle of the eclipse; the penumbral curve, in the first instance a figure of eight, breaks up into two ovals, but only one of these is attended to; and ultimately the oval unites itself with another oval so as to give rise to a new figure of eight. There is thus throughout the middle of the eclipse a single oval; this has, north and south, an envelope which joins itself on to the portions enveloped during the latter part of the commencement and the former part of the termination of the eclipse, and constitutes therewith the northern and southern limits respectively, viz. each of these is considered as extending from a point of contact with the sunrise oval to a point of contact with the sunset oval.

The line K_1VK, or say the meridian line through V, travels westwardly, while the penumbral curve travels eastwardly; the two come to touch each other, and there are then two intersections which ultimately come to the northern and southern limits respectively: the locus of these is a line of "eclipse commences at midday"; as the motion continues, the points of intersection move away from the two limits respectively and ultimately unite at the point where the line KVK_1 again touches the penumbral curve; the locus is the line of "eclipse terminates at midday," the two lines together forming an oval which touches the northern and southern limits respectively and which may be termed the midday oval. In all that precedes, no distinction has been made between the two portions of the horizon-envelope, or the points K and K_1, and either curve and point indifferently may be alone attended to.

Considering now an eclipse of the second kind, the penumbral curve is at first a mere point (the point of first contact) and it then becomes an oval, the successive ovals not at first intersecting each other, but each oval inclosing within itself the preceding ones. Any oval is met by the corresponding horizon in two points P and P', at first coinciding with each other at the point of first contact, and then separating from each other, one of them, say P, moving down towards and ultimately arriving at one of the horizon-envelopes, say to fix the ideas the southern one (which curve

is henceforth selected as being, and is called, the horizon-envelope, and the points on this curve are taken to be the points K), viz. P is then a point K on the penumbral curve, I call it K_1. The successive ovals will in the meantime have begun to intersect each other so as to give rise to a northern limit; this will touch the critic line (locus of P, P'), and we have a portion of the critic line extending from the point of first contact, in one direction to the point of contact with the northern limit, and in the other direction to the point K_1 on the horizon-envelope; this is the line "eclipse begins at sunrise." As the horizon continues to sweep on, the other point P', which has not yet reached the horizon-envelope, will gradually approach and ultimately arrive at the horizon-envelope, say at the point K_2; we have thus a second portion of the critic line extending from the contact with the northern envelope to the point K_2; this is the line "eclipse ends at sunrise." The horizon continuing to sweep on, the point P beginning with the position K_1, which is now on the other side of the point of contact of the horizon with the horizon-envelope, will trace out a portion of the critic curve extending from K_1 to a second point of contact with the northern limit; this will be the line of "eclipse begins at sunset." And, finally, the point P from the last-mentioned point of contact, and the point P' from its position K_2, which is now on the other side of the point of contact of the horizon with the horizon-envelope, (that is, P, P' have now each passed through the point of contact of the horizon with the horizon-envelope, and are both of them on the same side thereof, viz. the side opposite to their original side), will come to unite at the point of the last contact; we have thus a fourth portion of the critic curve extending from K_2 to the second point of contact with the northern limit, viz. this is the line "eclipse ends at sunset." The description will be more intelligible by means of the figure, in which 1, 1′, 2, 2′, ..., 8, 8′ represent successive corresponding



positions of the points P, P', the successive positions of the horizon being given by the right lines 11′, 22′, &c., all of them tangents to the dotted circle or horizon-envelope.

The entire critic line is thus a figure of eight, twice touching the horizon-envelope

and also twice touching the limit. If we consider, as before, the intersections of KV with the corresponding penumbral curve, this will be a curve extending from K_1 so as to touch the limit, and thence onward to K_2, the portion from K_1 to the contact with the limit being the line "eclipse begins at transit," and the portion from the limit to K_2 the line "eclipse ends at transit." I say "transit" instead of midday, since for a circumpolar place the phenomenon may happen at one or the other transit of the sun over the meridian. It is to be remarked, that the node of the figure of eight is a point, such that the eclipse there begins at sunrise and ends at sunset; this point does not appear to be an important one in the geometrical theory.

The two loops of the critic line may be of very unequal magnitudes, and in particular one of them may actually vanish; viz. the points K_1 and K_2 then coincide together, and the critic curve is a closed cuspidal curve touching the horizon-envelope at the cusp; moreover, instead of two contacts with the limit there is one proper contact, and an improper contact at the cusp, that is, the limit simply passes through the cusp. And through this special separating case, we pass to the case where, instead of the figure of eight, we have a single oval, not touching the horizon-envelope (viz. the points K_1, K_2 have become imaginary), but still touching the limit twice; this is a distinct type for an eclipse of the second class.

And, similarly, in an eclipse of the first class, where the points K_1, K_2 do not in general exist (viz. geometrically they are imaginary), these points may present themselves in the first instance as two coincident points, viz. instead of the sunrise oval or the sunset oval (as the case may be), we have then a cuspidal curve; or they may be two real points, viz. instead of the same oval, we have then a figure of eight touching the horizon-envelope twice, and also touching each of the two limits. These are thus the several cases.

When the Earth traverses the penumbral cone, the critic curve is

1. A pair of ovals:
2. An oval and a cuspidate oval:
3. An oval and a figure of eight.

And when the Earth does not traverse the penumbral cone, the critic curve is

4. A figure of eight:
5. A cuspidate oval:
6. An oval.

To which may be added the transition case which separates 1 and 4, viz. here the Earth just has an internal contact with the penumbral cone, and the critic curve is

7. Two ovals touching each other.

But of course 2, 5, and 7 are so special that they may be disregarded altogether; and 3 and 6 are of rare occurrence. I have not sufficiently examined the conditions for the occurrence of these forms 3 and 6; my attention was called to them, and indeed to the whole theory, by a question proposed by Prof. Adams in the Cambridge Smith's Prize Examination for 1869.

689.

ON THE GEOMETRICAL REPRESENTATION OF IMAGINARY VARIABLES BY A REAL CORRESPONDENCE OF TWO PLANES.

[From the *Proceedings of the London Mathematical Society*, vol. IX. (1878), pp. 31—39. Read December 13, 1877.]

IN my recently published paper, "Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable," *Proceedings of the London Mathematical Society*, t. VIII. (1877), pp. 212—214, [627], I remark as follows:—"If we have v a function of u determined by an equation $f(u, v) = 0$, then to any given imaginary value $x + iy$ of u there belong two or more values, in general imaginary, of v; and for the complete understanding of the relation between the two imaginary variables we require to know the series of values $x' + iy'$ which correspond to a given series of values $x + iy$ of v, u respectively. We must, for this purpose, take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P' in another plane Π'";—and I then proceed to consider the particular case where the equation between u, v is $u^2 + v^2 = a^2$, that is, where

$$(x + iy)^2 + (x' + iy')^2 = a^2.$$

The general case is that of an equation $(*)\,(u, 1)^m\,(v, 1)^n = 0$, where to each given value, real or imaginary, of u, there correspond n real or imaginary values of v; and to each given value, real or imaginary, of v, there correspond m real or imaginary values of u. And then, writing $u = x + iy$ and $v = x' + iy'$, and regarding (x, y), (x', y') as the coordinates of the points P, P' in the two planes Π, Π' respectively, we have a *real* (m, n) correspondence between the two planes; viz. to each real point P in the first plane there correspond n real points P' in the second plane, and to each real point P' in the second plane there correspond m real points P in the first plane. But such real correspondence of two planes does not of necessity arise from an equation between the two imaginary variables u, v; and the

question of the real correspondence of two planes may be considered in itself, without any reference to such origin.

I was under the impression that the theory was a known one; but I have not found it anywhere set out in detail. It is to be noticed that, although intimately connected with, it is quite distinct from (and seems to me to go beyond) that of a Riemann's surface. Riemann represents the value $u, = x + iy$, by a point P whose coordinates are x, y; but he considers $u', = x' + iy'$, as a given imaginary value attached to the point P, without representing this value by a point P', coordinates x', y'.

I proceed to consider the general theory of the real (m, n) correspondence. Points in the first plane are denoted by the unaccented letters $P, Q, \ldots$; and the corresponding points in the second plane are in general denoted by the *same* letters accented; but there are, as will be explained, special points V, W where the letters are interchanged; viz. to the points V or W in the first plane correspond points W' or V' in the second plane.

1. To a point P there correspond in general n distinct points P'; and as P varies continuously, each of the points P' also varies continuously.

2. There are certain points V called branch-points (Verzweigungspunkte), such that to each point V there correspond two united points, represented by (W'), and $n-2$ other distinct points W'. The points (W') are called cross-points, and the number of them is of course equal to that of the branch-points V.

It is throughout assumed that a point denoted by a letter other than V is not a point V.

3. If the point P, moving continuously, describe a closed curve so as to return to its original position, then, if this curve includes within it no point V (or all the points V)*, each of the corresponding points P' will describe continuously a closed curve returning into its original position. Supposing that the curve described by P is an oval (non-autotomic closed curve), and taking this to be in the first instance an indefinitely small oval, then the curves described by the points P' will in the first instance be each of them an indefinitely small oval; but it is worth while to notice how, as the oval described by P increases, any one of the ovals described by a point P' may become autotomic; viz. if the oval described by P passes through two points Q, Q of the m points Q which correspond in the first plane to the same Q' in the second plane, then Q' will be a node in the closed curve described by that point P' which in the course of its motion comes to pass through Q'. This curve is in general an inloop curve composed of two loops, one wholly within the other (united at the point Q'), and such that they each include one and the same point V' (viz. V' is included within the inner loop): as to this, see *post*, Nos. 9 and 10. It will be observed that this node Q' is not a point (W') nor any other special point of the second plane.

* The two cases of the closed curve including no point V, and including all the points V, are really identical, as the discontinuity at infinity may be disregarded. It is to be observed that, this being so, it follows that the number of the points V must be even.

4. Consider, as before, P as describing a closed curve which does not include within it any point V, and the corresponding points P' as describing each of them a closed curve. As the curve described by P approaches a point V, the curves described by two of the points P' will approach the corresponding point (W'); and when the curve described by P passes through V, the curves described by the two points P' will unite together at this point (W') as a node; viz. they will form a figure of eight*, the crossing being at the cross-point (W'), which corresponds to the branch-point V. And, corresponding to the closed curve described by P, we have this figure of eight (replacing two of the original n closed curves), and $n-2$ closed curves described by the other points P'.

5. Supposing, next, that the closed curve described by P (instead of passing through the point V) includes within it the point V, then the figure of eight transforms itself into a twice-indented oval*. There are on this curve two of the points P' which correspond to the given point P; and as P, moving continuously in its closed curve, returns to its original position, the first of these points P', moving continuously along a portion of the curve, comes to coincide with the original position of the second point P'; while the second point P', moving continuously along the remaining portion of the curve, comes to coincide with the original position of the first point P'; viz. the two portions of the curve are described by the two points P' respectively. The curve may thus be regarded as a bifid curve, belonging to these two points P'. And, corresponding to the closed curve described by P, we have this bifid curve belonging to the two points P', and $n-2$ single closed curves belonging to the other $n-2$ points P' respectively.

6. If the closed curve described by P (including within it a point V) comes to pass through a second point V, the effect will be a new node at the corresponding point (W'); viz. at this point (W') either the bifid curve unites itself with one of the single curves, or two of the single curves unite together, or the bifid curve there cuts itself. And, if the curve described by P comes to include within it this second point V, then in the three cases respectively:—the bifid curve takes to itself the single curve, so that the system then is a trifid curve and $n-3$ single curves; or the two single curves give rise to a bifid curve, so that the system is two bifid curves and $n-4$ single curves; or, lastly, the bifid curve breaks up into two single curves, so that the system resumes its original form of n single curves.

7. We thus see how the closed curve described by P, including within it certain of the points V, may be such as to have corresponding to it an α-fid curve, a β-fid curve, &c., $(\alpha+\beta+\ldots=n)$; viz. an α-fid curve contains upon it α of the points P' which correspond to the original position of P; and then, as P describes

* The name *figure of eight* refers to the case where the two curves which come to unite at (W') are proper ovals (non-autotomic closed curves). They might have one or both of them a node or nodes, as explained in No. 3; and the term would then be inappropriate. And so, lower down, the name *twice-indented oval* is used to express the form into which a proper figure of eight is changed by the disappearance of the node.

continuously its closed curve, returning to its original position, each of these points P' describes a portion of the α-fid curve, passing from its original position to the original position of a point P' next to it upon the α-fid curve; and the like as to a β-fid curve, &c. The numbers α, β, ... are not of necessity unequal, and we may have sets of equal numbers in any manner. It is hardly necessary to remark that, if the curve described by P passes through any point or points V, then two of the curves described by the points P' will unite together, or it may be that one of these will cut itself at the corresponding point or points (W'); and further that, as in No. 3, if the curve described by P passes through two or more of the points Q which correspond to the same point Q', then any such point Q' will present itself as a node upon the curve belonging to some point, or set of points, P'. But the order of succession in which the original n single curves unite themselves together into multifid curves, or again break up into single curves, cannot, it would appear, be explained in any general manner, and would in each case depend on the nature of the particular correspondence.

8. We may consider the case where the closed curve described by P cuts itself. The curve may here be considered as made up of two or more ovals, or, to use a more appropriate term, say loops, each such loop being a curve not cutting itself; and the case is thus reducible to that before considered, where the curve does not cut itself. Thus, to fix the ideas, let the curve be a figure of eight, the initial position of P being at the crossing, and let neither of the loops contain within it a point V. Then, as P passes continuously along one of the loops, returning to its original position, each of the corresponding points P' describes a closed curve, which will be in the nature of a loop, viz. the initial and final directions of the motion of P not being continuous with each other, the initial and final directions of the motions of each point P' will not be continuous with each other, or there will be at the point P' an abrupt change in the direction of the curve. Similarly, as P describes the other loop of the figure of eight, each of the points P' will describe another loop; and the two loops belonging to the same point P' will unite together so as to form a figure of eight; viz. to the figure of eight described by P there will correspond figures of eight described by the n points P' respectively.

9. But consider next the case where the two loops of the curve described by P include each of them one and the same point V. This implies that one of the two loops lies inside the other, or that the curve is what has been called an inloop curve. As P, which is in the first instance taken to be at the node, passes continuously along one of the loops and returns to its original position, there are two of the points P' such that the first of these passes from its original position to the original position of the second, and the second of them passes from its original position to the original position of the first of them. We have thus two arcs between these two points P'; but inasmuch as the initial and the final directions of motion of the point P are not continuous with each other, these two arcs are not continuous in direction at the two points P', but at each of these points P' the two arcs meet at an angle. As P describes the other loop, we have in like manner two arcs between the same two points P', these arcs at each of the points P' meeting at an

angle; but they join on to the first-mentioned two arcs in such manner as to form two ovals intersecting each other in the two points P'. Corresponding to the inloop curve described by P, we have this pair of intersecting ovals described by two of the points P', and $n-2$ other curves described by the other points P', and being each of them (I assume) an inloop curve.

10. If we attend only to one of the two intersecting ovals, we have in the first plane an inloop curve, and corresponding thereto in the second plane an oval passing through two of the points P' which correspond to the node P of the inloop curve. Interchanging the two planes, and writing Q instead of P, we have in the first plane an oval passing through two of the points Q which correspond to a point Q'; and corresponding to this oval we have in the second plane an inloop curve having this point Q for its node, viz. these are the corresponding figures mentioned in No. 3.

11. Consider a given point Q; and let the corresponding points Q' be called (selecting the suffixes at pleasure) Q_1', Q_2', .., Q_n'. Taking then a point O indefinitely near to Q, the corresponding points O' will be indefinitely near to Q_1', Q_2', .., Q_n' respectively, and they will be called O_1', O_2', .., O_n' accordingly. It is to be observed that by the indefinitely near point O is meant a point such that the distance from O to Q is indefinitely small in comparison with the distance of either of these points from any point V; so that we cannot have from Q to O two indefinitely short paths including between them a point V; or say so that the indefinitely short path from Q to O is determinate.

Proceeding in this manner from Q to O, and so through a succession of indefinitely near points to a distant point S, we seem to determine the suffixes of the corresponding points S'; but, by what precedes, it appears that such determination for a point S is dependent on the path from Q to S; and consequently that we do not thus obtain a proper determination of the suffixes of the points S'. In fact, if we were to pass from Q by a path including one or more of the points V back to Q, we should obtain for the several points Q' respectively suffixes which are in general different from the suffixes originally given to these points respectively.

12. The difficulty is got over as follows:—Considering as before the given point Q, and calling the corresponding points Q_1', Q_2', .., Q_n' at pleasure, we pass from Q to the indefinitely near point O, and thence, by so many paths chosen at pleasure, to the several branch-points V; these paths from O to the several points V are called *barriers*. To fix the ideas, we may consider these as non-autotomic non-intersecting lines drawn from O to the several points V. Consider the barrier from O to one of these points V; as P passes along this barrier from O to V, two of the corresponding points P' will pass from two of the corresponding points O' to the corresponding cross-point (W'); the paths of these two points are called the *counter-barrier* corresponding to the barrier in question; and we have thus in the second plane a system of counter-barriers, each drawn from two points O' to meet in a point (W'). By what precedes, the points O' have each of them a determinate suffix; a counter-barrier is thus drawn from two points with given suffixes, suppose O_1' and O_2', to a

point (W'), and this may be distinguished accordingly as a counter-barrier 12; and in like manner the cross-point (W') through which it passes will be called a cross-point (W_{12}'); and the barrier corresponding hereto, and the branch-point V at which it terminates, will in like manner be called a barrier 12, and a branch-point V_{12}. Each barrier and branch-point will thus have a pair of suffixes; and the corresponding counter-barrier and cross-point will have the same pair of suffixes. It is to be observed that two or more of these corresponding figures may very well have the same pair of suffixes; but that such corresponding figures must be distinguished from each other; thus, if there are two branch-points V_{12}, these may be distinguished as the branch-points αV_{12} and βV_{12}, and the barriers, counter-barriers, and cross-points by means of these same letters α and β, (or otherwise), as may be convenient. It would seem that not only the number of the points V must be even, but the number of each set of points V_{12} must also be even (see *post*, No. 15).

13. It is also to be noticed that the determination of the suffixes of the several points V, &c., depends *first* upon the arbitrary choice of the suffixes of the points Q', and *next* on the choice of the system of barriers; but that, *these being assumed*, the suffixes of the several points V, &c., are completely determinate.

14. Taking now any point S whatever, and supposing that P moves from Q continuously to S by a path which does not meet a barrier, the points P' will move from the several points Q' to the several points S' by paths not meeting the counter-barriers; viz. to each point S' there will be a path from some point Q'; and giving to such point S' the suffix of the point Q', the suffixes of the several points S' which correspond to any point whatever, S, will be completely determined. The determination depends of course on the assumptions referred to No. 13, but not in anywise on the position of the point S.

It will be noticed that, as all the points V are connected together by the barriers, the only closed paths from a point to itself are paths not including any, or including all, of the points V; and that between such paths there is no real distinction.

15. Consider a point P moving continuously in any manner. The several corresponding points $P_1', P_2', \ldots, P_n'$ will each of them move continuously, but the suffixes interchange; viz. when P arrives at and then passes over a barrier $\alpha\beta$, the corresponding points P_α' and P_β' will each arrive at the corresponding counter-barrier $\alpha\beta$, and, on passing over this, P_α' will be changed into P_β' and P_β' into P_α', the other points P' remaining unchanged; and the like in other cases. This in fact includes the whole or the greater part of the foregoing theory. Thus, if P describe a closed curve not cutting any barrier, there will be no change of suffix; and when P returns to its original position each of the corresponding points $P_1', P_2', \ldots, P_n'$ will describe a closed curve, returning to its original position. But suppose that P describes a closed curve, cutting once only a barrier 12; suppose that the path is from P to Q, and then crossing the barrier to R, and thence again to P; P_1' passes to Q_1', and then crossing the counter-barrier it passes from R_2' to P_2'; while at the same time P_2' passes to Q_2', and then crossing the counter-barrier it passes from R_1' to P_1';

viz. we have P_1', P_2' describing the two portions of a bifid curve. If there were only a single branch-point V_{12}, and therefore only a single barrier OV_{12}, then we might have through P a closed curve cutting OV_{12} once only, and including within it the point O, but not including within it the point V_{12}; and here there ought not to be a bifid curve, but the points P_1', P_2' ought to describe each of them a single curve. But suppose there are two points V_{12}, and consequently two barriers OV_{12} (meeting in O); then the closed curve, meeting once only a barrier 12, (viz. it meets only one such barrier, and that once only), must include within one and only one of the two points V_{12}; and in this case there ought to be a bifid curve. It is by such reasoning as this that I infer the foregoing theorem (No. 12), that the number of each set of points V_{12} is even.

16. We may consider how the suffixes are affected when, instead of the original system of barriers, we have a new system of barriers. I suppose that we have in the two cases respectively the same point Q, and the same suffixes for the points Q_1', Q_2', .., Q_n' which correspond thereto. In the first case, passing from Q to an indefinitely near point O, say the red O, we draw from this point to the several points V a set of barriers, say the red barriers; while in the second case, passing from Q to an indefinitely near point O, say the blue O, we draw from this point to the several points V a set of barriers, say the blue barriers; and we then proceed as before, viz. in the first case, drawing from Q to the point S a curve which does not meet any of the red barriers, we determine accordingly the suffixes (say the red suffixes) of the several corresponding points S'; and in the second case, drawing in like manner from Q to S a curve which does not meet any of the blue barriers, we determine accordingly the suffixes (say the blue suffixes) of the same points S'. Now the curve drawn from Q to S so as not to cut any of the red barriers, and which is used for the determination of the red suffixes of the several points S', will in general cut certain of the blue barriers; and, by examining the suffixes of the blue barriers which are thus cut, we determine the blue suffixes of the same points S'; the result of course depending only on the situation of S in one or other of the regions formed by the red barriers and the blue barriers conjointly. In particular, the point S may be so situate that we can from Q to S draw a curve not meeting any red barrier or any blue barrier; and in this case the red suffixes and the blue suffixes are identical.

17. We may imagine the first plane as consisting of n superimposed planes or sheets, say the sheets 1, 2, .., n. Each barrier 12 is considered as a line drawn in the two sheets 1 and 2; and so on in other cases. The point P is considered as a set of superimposed points P_1, P_2, .., P_n moving in the several sheets respectively; under the convention that P_1 moving in the sheet 1, and coming to cross a barrier 12, passes into the sheet 2 and becomes P_2; and the like in other cases. And this being so, we say that to a point P, considered as a point P_α in the sheet α, there corresponds in the second plane one and only one point P_α'; and that P moving continuously in any manner (subject to the change of sheet as just explained), each of the n corresponding points P' will also move continuously, and so that each such point P_α' will return

to its original position, upon the corresponding point P_a returning to its original position and sheet. This is, in fact, Riemann's theory, only instead of the points P' we must speak of the values $x' + iy'$ of the irrational function of $x + iy$.

18. Everything is of course symmetrical as regards the two planes; we have therefore, in the second plane, a system of points V' and of barriers, and in the first plane a system of points (W) and of counter-barriers. To a given point P' in the second plane there correspond m points P in the first plane; and we can (the determination depending on the system of barriers in the second plane) assign to the m points suffixes, thereby distinguishing them as the corresponding points $P_1, P_2, \ldots, P_m$. And we may imagine the second plane as consisting of m superimposed planes or sheets, say the sheets $1, 2, 3, \ldots, m$; the general theorem then is that to a point P or P' in either plane, considered as a point P_a or P_a' in the sheet a or a', there corresponds in the other plane one and only one point P_a' or $P_{a'}$; and that the first-mentioned point in either plane moving continuously in any manner (subject to the proper change of sheet), the corresponding point in the other plane will also move continuously, and will return to its original position and sheet, upon the first-mentioned point returning to its original position and sheet.

19. In all that precedes it has been assumed that, to a branch-point V, there correspond two united points represented by (W') and $n-2$ distinct points W'; the cases of a point (W') composed of three or more united points, or of the points W' uniting themselves in sets in any other manner, would give rise to further specialities.

690.

ON THE THEORY OF GROUPS.

[From the *Proceedings of the London Mathematical Society*, t. IX. (1878), pp. 126—133. Read May 9, 1878.]

I RECAPITULATE the general theory so far as is necessary in order to render intelligible the quasi-geometrical representation of it which will be given.

Let $\alpha, \beta, ..$ be functional symbols each operating upon one and the same number of letters, and producing as its result the same number of functions of these letters. For instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and combination;

$$\alpha^2(x, y, z) = \alpha(X, Y, Z),$$

or

$$\beta\alpha(x, y, z) = \beta(X, Y, Z),$$

in each case equal to three given functions of (x, y, z); and similarly for α^3, $\alpha^2\beta$, etc.

The symbols are not in general commutative, $\alpha\beta$ not $=\beta\alpha$; but they are associative, $\alpha\beta.\gamma = \alpha.\beta\gamma$, each $=\alpha\beta\gamma$, which has thus a determinate meaning.

Unity as a functional symbol denotes that the letters are unaltered,

$$1(x, y, z) = (x, y, z);$$

whence

$$1\alpha = \alpha 1 = \alpha.$$

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order. Substitutions can be represented by the notation $\left(\begin{matrix} yzx \\ zxy \end{matrix}\right)$, the substitution which changes xyz into yzx, or, as products of cyclical substitutions, $\alpha = \left(\begin{matrix} yzx\ wvu \\ xyz\ uvw \end{matrix}\right)$, $= (xyz)(uw)$, the product of the cyclical substitutions x into y, y into z, z into x, and u into w, w into u, the letter v being unaltered.

A set of symbols α, β, γ,.., such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ and $\beta\alpha$) is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols 1, α, β, γ,.. satisfying the foregoing condition is a group. When the number of symbols (or terms) is $=n$, then the group is of the order n; and each symbol α is such that $\alpha^n=1$, so that a group of the order n is in fact a group of symbolical nth roots of unity.

A group is defined by means of the laws of combinations of its symbols. For the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of distinct functional symbols; or else, using distinct letters for the several terms of the group, employ a square diagram, as presently mentioned.

Thus, in the first mode, a group is 1, β, β^2, α, $\alpha\beta$, $\alpha\beta^2$, ($\alpha^2=1$, $\beta^3=1$, $\alpha\beta=\beta^2\alpha$), where observe that these conditions imply also $\alpha\beta^2=\beta\alpha$.

Or in the second mode, calling the symbols (1, α, β, $\alpha\beta$, β^2, $\alpha\beta^2$) of the same group (1, α, β, γ, δ, ϵ), or, if we please, (a, b, c, d, e, f), the laws of combination are given by one or other of the square diagrams:

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	γ	β	ϵ	δ
β	β	ϵ	δ	α	1	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	γ	1	ϵ	β	α
ϵ	ϵ	β	α	δ	γ	1

,

a	b	c	d	e	f
b	a	d	c	f	e
c	f	e	b	a	d
d	e	f	a	b	c
e	d	a	f	c	b
f	c	b	e	d	a

where, taking for greater symmetry the second form of the square, observe that the square is such that no letter occurs twice in the same line, or in the same column (or what is the same thing, each of the lines and of the columns contains all the letters). But this is not sufficient in order that the square may represent a group; the square must be such that the substitutions by means of which its several lines are derived from any line thereof are (in a different order) the same substitutions by which the lines are derived from a particular line, or say from the top line. These, in fact, are:

1
$ab\,.\,cd\,.\,ef$,
$ace\ .\ bfd$,
$ad\,.\,be\,.\,cf$,
$aec\ .\ bdf$,
$af\,.\,bc\,.\,de$,

where, for shortness, *ab*, *ace*, &c., are written instead of (*ab*), (*ace*), &c., to denote the cyclical substitutions *a* into *b*, *b* into *a*; and *a* into *c*, *c* into *e*, *e* into *a*, &c.; and it is at once seen that by the same substitutions the lines may be derived from any other line.

It will be noticed that in the foregoing substitution-group each substitution is *regular*, that is, composed of cyclical substitutions each of the same number of letters; and it is easy to see that this property is a general one; each substitution of the substitution-group must be regular.

By what precedes, the group of any order composed of the functional symbols is replaced by a substitution-group upon a set of letters the number of which is equal to the order of the group, and wherein all the substitutions are regular.

The general theory being thus explained, I endeavour to form a substitution-group with the twelve letters *abcdefghijkl*; and I assume that there is one substitution, such as *abc*.*def*.*ghi*.*jkl*, and another substitution, such as *agj*.*bfi*.*cek*.*dhl*. Observe that, if the twelve letters are to be thus arranged in two different ways as a set of four triads, without repetition of any duad, all the ways in which this can be done are essentially similar, and there is no loss of generality in taking the two sets of triads to be those just written down. But the substitution to be formed with either set of triads will be different according as any triad thereof, for instance *agj*, is written in this form or in the reversed form *ajg*. There are thus in all sixteen substitutions which can be formed with the first set of triads, and sixteen substitutions which can be formed with the second set of triads; and the relation of a triad of the first set to a triad of the second set is by no means independent of the selection of the triads out of the two sets respectively. To show this, take the two substitutions quite at random; suppose they are those written down above, say

$$\alpha = abc\,.\,def\,.\,ghi\,.\,jkl, \quad \beta = agj\,.\,bfi\,.\,cek\,.\,dhl;$$

and perform these in succession on the primitive arrangement $\Omega = abcdefghijkl$. The operation stands thus:

$$\beta\alpha\Omega = fegkihlbjcda,$$
$$\alpha\Omega = bcaefdhigklj,$$
$$\Omega = abcdefghijkl,$$

whence

$$\beta\alpha, = afhbeijcgl\,.\,dk,$$

is not a regular substitution; and, by what precedes, α, β cannot belong to a group.

But take the substitutions to be

$$\alpha \text{ (as before)} = abc\,.\,def\,.\,ghi\,.\,jkl, \quad \beta = ajg\,.\,bif\,.\,cek\,.\,dhl,$$

then we have

$$\beta\alpha\Omega = iejkbhlfacdg,$$
$$\alpha\Omega = bcaefdhigklj,$$
$$\Omega = abcdefghijkl,$$

whence

$$\beta\alpha = ai\,.\,be\,.\,cj\,.\,dk\,.\,fh\,.\,gl,$$

a regular substitution; and, for anything that appears to the contrary, α, β may belong to a group. It is convenient to mention at once that these two substitutions do, in fact, give rise to a group; viz. the square diagram is

a	b	c	d	e	f	g	h	i	j	k	l
b	c	a	e	f	d	h	i	g	k	l	j
c	a	b	f	d	e	i	g	h	l	j	k
d	l	h	a	g	j	e	c	k	f	i	b
e	j	i	b	h	k	f	a	l	d	g	c
f	k	g	c	i	l	d	b	j	e	h	a
g	f	k	l	c	i	j	d	b	a	e	h
h	d	l	j	a	g	k	e	c	b	f	i
i	e	j	k	b	h	l	f	a	c	d	g
j	i	e	h	k	b	a	l	f	g	c	d
k	g	f	i	l	c	b	j	d	h	a	e
l	h	d	g	j	a	c	k	e	i	b	f

and the substitutions, obtained therefrom by writing successively each line over the top line, are

$$\begin{array}{ll}
1 & = 1, \\
abc\,.\,def\,.\,ghi\,.\,jkl & \alpha, \\
acb\,.\,dfe\,.\,gih\,.\,jlk & \alpha^2, \\
ad\,.\,bl\,.\,ch\,.\,eg\,.\,fj\,.\,ik & \beta^2\alpha\beta^2, \\
aeh\,.\,bjd\,.\,cil\,.\,fkg & \beta\alpha^2, \\
afl\,.\,bkh\,.\,cgd\,.\,eij & \beta^2\alpha, \\
agj\,.\,bfi\,.\,cke\,.\,dlh & \beta^2, \\
ahe\,.\,bdj\,.\,cli\,.\,fgk & \beta\alpha^2\beta\alpha^2, \\
ai\,.\,be\,.\,cj\,.\,dk\,.\,fh\,.\,gl & \beta\alpha, \\
ajg\,.\,bif\,.\,cek\,.\,dhl & \beta, \\
ak\,.\,bg\,.\,cf\,.\,di\,.\,el\,.\,hj & \beta^2\alpha^2, \\
alf\,.\,bkh\,.\,cdg\,.\,eji & \beta^2\alpha\beta^2\alpha.
\end{array}$$

To explain the theory, I introduce the notion of a hemipolyhedron, or say a hemihedron, viz. this is a figure obtained from a polyhedron by the removal of certain faces. In a polyhedron each edge occurs twice (more properly it occurs in the two forms ab and ba), as belonging to two faces; but in a hemihedron one of these faces must always be removed, so that the edge may occur once only; and again (what is apparently, although not really, a different thing), we may remove two intersecting faces, leaving their edge of intersection; this edge is, in fact, then considered as a bilateral face $ab = ab.ba$, just as abc is a trilateral face $abc = ab.bc.ca$. Thus, if in a prism we remove the lateral faces, leaving the lateral edges, and leaving also the terminal faces, we have a hemihedron: thus, the prism being trilateral, say the faces of the hemihedron are abc, def, ad, be, cf, where ad, be, cf are the edges regarded as bilateral faces. And, for the present purpose, abc denotes the cyclical substitution a into b, b into c, c into a; and ad denotes in like manner the cyclical substitution (or interchange) a into d, d into a.

But the hemihedron about to be considered has no bilateral faces; it is, in fact, the figure composed of the 8 triangular faces of the octo-hexahedron or figure obtained by truncating the summits of a hexahedron (or of an octahedron) so as to obtain a polyhedron of 8 triangular faces and 6 square faces, representing the faces of the octahedron and the hexahedron respectively. The faces of the octo-hexahedron may be taken to be

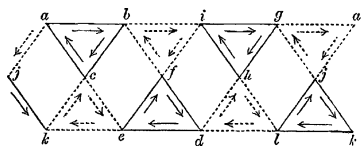
$$abc,\quad def,\quad ghi,\quad jkl,$$

$$ajg,\quad bif,\quad cek,\quad dhl,$$

$$cbfe,\quad fihd,\quad hgjl,\quad jack,\quad agib,\quad klde,$$

(where I observe in passing that the symbols are written in such manner that each edge ab occurs under the two opposite forms ab in abc and ba in $agib$). And then, omitting the square faces, represented by the third line, we have the hemihedron, wherein as before abc denotes the cyclical substitution a into b, b into c, c into a; and so for the other faces.

I represent this by a diagram, the lines of which were red and black, and they



will be thus spoken of, but the black lines are in the woodcut continuous lines, and the red lines broken lines: each face indicates a cyclical substitution, as shown by the arrows. The figure should be in the first instance drawn with the arrows, but without the letters, and these may then be affixed to the several points in a perfectly arbitrary manner; but I have in fact affixed them in such wise that the group given

by the diagram, as presently appearing, may (instead of being any other equivalent group) be that group which contains the before-mentioned substitution

$$\alpha = abc\,.\,def\,.\,ghi\,.\,jkl, \text{ and } \beta = ajg\,.\,bif\,.\,cek\,.\,dhl.$$

Observe that in the diagram, considering the lines to be drawn as shown by the arrows, there is *from* any given point whatever only one black line, and only one red line. Let B denote motion along a black line, R motion along a red line (always from a point to the next point); then B^2 will denote motion along two black lines successively, BR (any such symbol being read always from right to left) will denote motion first along a red line, and then along a black line, and so in other cases; a symbol or "route" $\ldots R^\beta B^\alpha$ has thus a perfectly definite signification, determining the path when the initial point is given.

The diagram has the property that every route, leading from any one letter to itself, leads also from every other letter to itself; or say a route leading from a to a, leads also from b to b, from c to $c, \ldots$, from l to l; and we can thus in the diagram speak absolutely (that is, without restriction as to the initial point) of a route as leading from a point to itself, or say as being equal to unity; *it is in virtue of this property* that the diagram gives a group.

For, assuming the property, it at once follows (1) that two routes, each leading say from the point a to the same point f, lead also from any other point b to one and the same point g. Such routes are said to be equivalent, or equal to each other; and the number of distinct routes (including the route unity) is thus equal to the numbers of the letters, viz. we have only the routes from a to a, to $b, \ldots$, to l, respectively; (2) a route, leading from a point a to a point f, leads from any other point b to a different point g; and (3) two routes, leading from the same point a to different points b and c, lead also from any other point f to different points k and l. Hence a given route leads from the several points $abc \ldots l$ successively to the same series of points taken in a different order, or we thus obtain a new arrangement of the points; and dealing in this manner successively with the routes from a to a, to $b, \ldots$, to l, we obtain so many distinct arrangements, beginning with the letters $a, b, c, \ldots, l$ respectively, such that in no two of them does the same letter occupy the same place; we thus obtain a square of 12 such as that already written down, and which is, in fact, the same square, the several routes of course corresponding to the substitutions of the square. The hemihedron thus gives the foregoing group of 12.

Observe that the diagram is composed of the four black triangles representing the substitution $abc\,.\,def\,.\,ghi\,.\,jkl$, and of the four red triangles representing the substitution $ajg\,.\,bif\,.\,cek\,.\,dhl$; viz. these are independent substitutions which by their powers and products serve to express all the substitutions of the group; that they are sufficient appears by the diagram itself, in that every point thereof is (by black and red lines) connected with every other point thereof. The group might have contained three or more independent substitutions, and the diagram would then have contained the like number of differently coloured sets of lines. The essential characters are that the lines of any given colour shall form polygons of the same number of sides (but for different

colours the polygons may have different numbers of sides; in particular, for any given colour or colours, the polygons may be bilaterals, represented each by a line with a double arrow pointing opposite ways); that there shall be *from* each point only one line of the same colour; that every point shall be connected with every other point; and finally, that every route leading from one point to itself shall lead also from every other point to itself. When these conditions are satisfied the foregoing investigation in fact shows that the diagram, or say the hemihedron, gives rise to a group.

It may be remarked that we can, if we please, introduce into the diagram a set of lines of a new colour to represent any dependent substitution of the group; thus, in the example considered, a substitution is $aeh.bjd.cil.fkg$, and if we draw these triangles in green (the arrows being from a to e, e to h, h to a, &c.), then there will be *from* each point one black line, one red line, and one green line; any route $\ldots G^{\gamma}R^{\beta}B^{\alpha}$ will thus be perfectly definite, and will have the same properties as a route composed of black and red lines only; and the theory thus subsists without alteration.

I remark, in conclusion, that the group of 12 considered above is, in fact, the group of 12 positive substitutions upon 4 letters $abcd$; viz. the substitutions are 1, abc, acb, abd, adb, acd, adc, bcd, bdc, $ab.cd$, $ac.bd$, $ad.bc$; the groups each contain unity, three substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3, and their identity can be easily verified.

691.

NOTE ON MR MONRO'S PAPER "ON FLEXURE OF SPACES."

[From the *Proceedings of the London Mathematical Society*, vol. IX. (1878), pp. 171, 172. Read June 13, 1878.]

CONSIDER an element of surface, surrounding a point P; the flexure of the element may be interfered with by the continuity round P, and it is on this account proper to regard the element as cut or slit along a radius drawn from P to the periphery of the element. This being understood, we have the well-known theorem that, considering in the neighbourhood of the origin elements of the surfaces

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2), \quad \text{and} \quad z' = \tfrac{1}{2}(a'x'^2 + 2h'x'y' + b'y'^2),$$

these will be applicable the one on the other, provided only $ab - h^2 = a'b' - h'^2$. But in connexion with Mr Monro's paper it is worth while to give the proof in detail.

It is to be shown that z, z' denoting the above-mentioned functions of (x, y) and (x', y') respectively, it is possible to find (for small values) x', y' functions of x, y such that identically

$$dx'^2 + dy'^2 + dz'^2 = dx^2 + dy^2 + dz^2.$$

The solution is taken to be $x' = x + \xi$, $y' = y + \eta$, where ξ, η denote cubic functions of x, y. We have then, attending only to the terms of an order not exceeding 3 in x, y,

$$\begin{aligned} dx^2 + dy^2 + 2(dx\,d\xi + dy\,d\eta) + \{(a'x + h'y)\,dx + (h'x + b'y)\,dy\}^2 \\ = dx^2 + dy^2 + \{(ax + hy)\,dx + (hx + by)\,dy\}^2, \end{aligned}$$

so that the terms $dx^2 + dy^2$ disappear; and then writing

$$d\xi = \frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy, \quad d\eta = \frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy,$$

the equation will be satisfied identically as regards dx, dy if only

$$2\frac{d\xi}{dx} = (ax+hy)^2 - (a'x+h'y)^2,$$

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} = (ax+hy)(hx+by) - (a'x+h'y)(h'x+b'y),$$

$$2\frac{d\eta}{dy} = (hx+by)^2 - (h'x+b'y)^2.$$

Calling the terms on the right-hand side $2\mathfrak{A}$, $\mathfrak{B}$, $2\mathfrak{C}$ respectively, we have

$$\frac{d^2\mathfrak{A}}{dy^2} - \frac{d^2\mathfrak{B}}{dx\,dy} + \frac{d^2\mathfrak{C}}{dx^2} = 0,$$

that is,

$$(h^2-h'^2) + (h^2-h'^2) - \{(ab+h^2) - (a'b'+h'^2)\} = 0,$$

or, what is the same thing,

$$a'b' - h'^2 = ab - h^2,$$

a relation which must exist between the constants (a, b, h) and (a', b', h').

It is easy to find the actual values of ξ, η; viz. these are

$$\xi = \tfrac{1}{6}(a^2 - a'^2)x^3 + \tfrac{1}{2}(ah - a'h')x^2y + \tfrac{1}{2}(h^2 - h'^2)x^2y + \tfrac{1}{6}(bh - b'h')y^3,$$

$$\eta = \tfrac{1}{6}(ah - a'h')x^3 + \tfrac{1}{2}(h^2 - h'^2)x^2y + \tfrac{1}{2}(bh - b'h')x^2y + \tfrac{1}{6}(b^2 - b'^2)y^3,$$

or, what is the same thing, we have

$$\xi = \tfrac{1}{24}\frac{d\Omega}{dx}, \quad \eta = \tfrac{1}{24}\frac{d\Omega}{dy},$$

where

$$\Omega = (a^2 - a'^2)x^4 + 4(ah - a'h')x^3y + 6(h^2 - h'^2)x^2y^2 + 4(bh - b'h')xy^3 + (b^2 - b'^2)y^4,$$

$$= (ax^2 + 2hxy + by^2)^2 - (a'x^2 + 2h'xy + b'y^2)^2 = 4(z^2 - z'^2),$$

in virtue of the relation $ab - h^2 = a'b' - h'^2$. The resulting values $x' = x + \xi$, $y' = y + \eta$ are obviously the first terms of two series which, if continued, would contain higher powers of (x, y).

692.

ADDITION TO THE MEMOIR ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXIX. Part II. (1878), pp. 419—424. Received February 6,—Read March 7, 1878.]

I HAVE recently succeeded in completing a theory considered in my "Memoir on the Transformation of Elliptic Functions," *Phil. Trans.*, vol. CLXIV. (1874), pp. 397—456, [578],—that of the septic transformation, $n=7$. We have here

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x+\gamma x^2-\delta x^3}{\alpha+\beta x+\gamma x^2+\delta x^3}\right)^2,$$

a solution of

$$\frac{Mdy}{\sqrt{1-y^2\,.\,1-v^2y^2}}=\frac{dx}{\sqrt{1-x^2\,.\,1-u^2x^2}},$$

where $\frac{1}{M}=1+\frac{2\beta}{\alpha}$; and the ratios $\alpha:\beta:\gamma:\delta$, and the uv-modular equation are determined by the equations

$$u^{14}\alpha^2=v^2\delta^2,$$

$$u^6(2\alpha\gamma+2\alpha\beta+\beta^2)=v^2(\gamma^2+2\gamma\delta+2\beta\delta),$$

$$\gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta=v^2u^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),$$

$$\delta^2+2\gamma\delta=v^2u^{10}(\alpha^2+2\alpha\beta);$$

or, what is the same thing, writing $\alpha=1$, the first equation may be replaced by $\delta=\frac{u^7}{v}$, and then, α, δ having these values, the last three equations determine β, γ and the modular equation. If instead of β we introduce M, by means of the relation

$\frac{1}{M}=1+2\beta$, that is, $2\beta=\frac{1}{M}-1$, then the last equation gives $2\gamma=u^3v^3\left(\frac{1}{M}-\frac{u^4}{v^4}\right)$; and α, β, γ, δ having these values, we have the residual two equations

$$u^6(2\alpha\gamma+2\alpha\beta+\beta^2)= \quad v^2(\gamma^2+2\gamma\delta+2\beta\delta),$$
$$\gamma^2+2\beta\gamma+2\alpha\delta+\beta\delta=v^2u^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),$$

viz. each of these is a quadric equation in $\frac{1}{M}$; hence eliminating $\frac{1}{M}$, we have the modular equation; and also (linearly) the value of $\frac{1}{M}$, and thence the values of α, β, γ, δ in terms of u, v.

Before going further it is proper to remark that, writing as above $\alpha=1$, then if $\delta=\beta\gamma$, we have

$$1-\beta x+\gamma x^2-\delta x^3=(1-\beta x)(1+\gamma x^2),$$
$$1+\beta x+\gamma x^2+\delta x^3=(1+\beta x)(1+\gamma x^2),$$

and the equation of transformation becomes

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\beta x}{1+\beta x}\right)^2,$$

viz. this belongs to the cubic transformation. The value of β in the cubic transformation was taken to be $\beta=\frac{u^3}{v}$, but for the present purpose it is necessary to pay attention to an omitted double sign, and write $\beta=\pm\frac{u^3}{v}$; this being so, $\delta=\beta\gamma$, and giving to γ the value $\mp u^4$, δ will have its foregoing value $=\frac{u^7}{v}$. And from the theory of the cubic equation, according as $\beta=\frac{u^3}{v}$ or $=-\frac{u^3}{v}$, the modular equation must be

$$u^4-v^4+2uv(1-u^2v^2)=0, \text{ or } u^4-v^4-2uv(1-u^2v^2)=0.$$

We thus see *à priori*, and it is easy to verify that the equations of the septic transformation are satisfied by the values

$$\alpha=1,\ \beta= \quad \frac{u^3}{v},\ \gamma= \quad u^4,\ \delta=\frac{u^7}{v}, \text{ and } u^4-v^4+2uv(1-u^2v^2)=0;$$

$$\alpha=1,\ \beta=-\frac{u^3}{v},\ \gamma=-u^4,\ \delta=\frac{u^7}{v}, \text{ and } u^4-v^4-2uv(1-u^2v^2)=0;$$

and it hence follows that in obtaining the modular equation for the septic transformation, we shall meet with the factors $u^4-v^4\pm 2uv(1-u^2v^2)$. Writing for shortness $uv=\theta$, these factors are $u^4-v^4\pm 2\theta(1-\theta^2)$; the factor for the proper modular equation is $u^8+v^8-\Theta$, where

$$\Theta=8\theta-28\theta^2+56\theta^3-70\theta^4+56\theta^5-28\theta^6+8\theta^7,$$

viz. the equation $(1-u^8)(1-v^8)-(1-uv)^8=0$ is $u^8+v^8-\Theta=0$; and the modular equation, as obtained by the elimination from the two quadric equations, presents itself in the form

$$(u^4-v^4+2\theta-2\theta^3)^2(u^4-v^4-2\theta+2\theta^3)^2(u^8+v^8-\Theta)=0.$$

Proceeding to the investigation, we substitute the values

$$\alpha=1,\ \beta=\tfrac{1}{2}\left(\frac{1}{M}-1\right),\ \gamma=\tfrac{1}{2}u^3v^2\left(\frac{1}{M}-\frac{u^4}{v^4}\right),\ \delta=\frac{u^7}{v},$$

in the residual two equations, which thus become

$$\begin{aligned}\frac{1}{M^2}(1-v^8)\qquad\qquad &+\frac{2}{M}(1-uv)^3(1+uv)\\ &+\left\{(1-u^8)-4(1-uv)\left(1+\frac{u^7}{v}\right)\right\}=0,\\ \frac{1}{M^2}\left\{-u^2v^2(1-uv)^3(1+uv)\right\}+\frac{2}{M}&\left\{u^2v^2(1-u^8)+\frac{u^3}{v}(1+u^2v^2)(u^4-v^4)\right\}\\ &+\left\{\frac{u^{14}}{v^2}+6\frac{u^7}{v}(1-u^3v^3)-u^2v^2\right\}\quad=0,\end{aligned}$$

the first of which is given p. 432 of the "Memoir," [*Coll. Math. Papers*, vol. IX., p. 150]. Calling them

$$\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\right)\left(\frac{1}{M},\ 1\right)^2=0,\ \left(\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}'\right)\left(\frac{1}{M},\ 1\right)^2=0,$$

we have

$$\frac{1}{M^2}:\frac{2}{M}:1=\mathrm{bc}'-\mathrm{b}'\mathrm{c}:\mathrm{ca}'-\mathrm{c}'\mathrm{a}:\mathrm{ab}'-\mathrm{a}'\mathrm{b},$$

and the result of the elimination therefore is

$$(\mathrm{ca}'-\mathrm{c}'\mathrm{a})^2-4(\mathrm{bc}'-\mathrm{b}'\mathrm{c})(\mathrm{ab}'-\mathrm{a}'\mathrm{b})=0.$$

Write as before $uv=\theta$. In forming the expressions $\mathrm{ca}'-\mathrm{c}'\mathrm{a}$, &c., to avoid fractions we must in the first instance introduce the factor v^2: thus

$$\begin{aligned}v^2(\mathrm{ca}'-\mathrm{c}'\mathrm{a})&=v\{v(1-u^8)-4(1-\theta)(v+u^7)\}\{-\theta^2(1+\theta)(1-\theta)^3\}\\ &\quad-\{u^{14}+6u^6\theta(1-\theta^2)-v^2\theta^2\}\{1-v^8\},\\ &=-\theta^2(1+\theta)(1-\theta)^3\{v^2(-3+4\theta)+u^6(-4\theta+3\theta^2)\}\\ &\quad-\{u^{14}+6u^5(\theta-\theta^3)-v^2\theta^2\}(1-v^8);\end{aligned}$$

but instead of θ^2v^2 writing u^2v^4, the expression on the right-hand side becomes divisible by u^2; and we find

$$\begin{aligned}\frac{v^2}{u^2}(\mathrm{ca}'-\mathrm{c}'\mathrm{a})&=-(1+\theta)(1-\theta)^3\{v^4(-3+4\theta)+u^4(-4\theta^3+3\theta^4)\}\\ &\quad-\{u^{12}+6u^4(\theta-\theta^3)-v^4\}(1-v^8),\end{aligned}$$

and thence

$$-\frac{v^2}{u^2}(\mathrm{ca'}-\mathrm{c'a}) = u^{12} + u^4(6\theta - 10\theta^3 + 11\theta^4 - 6\theta^5 - 8\theta^6 + 10\theta^7 - 4\theta^8)$$
$$+ v^4(-4 + 10\theta - 8\theta^2 - 6\theta^3 + 11\theta^4 - 10\theta^5 + 6\theta^7) + v^{12}.$$

Similarly we have

$$\frac{v^2}{u^2}(\mathrm{bc'}-\mathrm{b'c}) = u^{12}(5 - 5\theta + 4\theta^2 - 5\theta^3 + 2\theta^4) + u^4(9\theta - 16\theta^2 + \theta^3 + 10\theta^4 + \theta^5 - 16\theta^6 + 9\theta^7)$$
$$+ v^4(2 - 5\theta + 4\theta^2 - 5\theta^3 + 5\theta^4),$$
$$\frac{v^2}{u^2}(\mathrm{ab'}-\mathrm{a'b}) = u^4(\theta + \theta^3 - \theta^4) + v^4(2 - 5\theta + 4\theta^2 + 3\theta^3 - 10\theta^4 + 3\theta^5 + 4\theta^6 - 5\theta^7 + 2\theta^8)$$
$$+ v^{12}(-1 + \theta + \theta^3);$$

say these values are

$$u^{12} + pu^4 + qv^4 + v^{12}, \quad \lambda u^{12} + \mu u^4 + \nu v^4, \quad \rho u^4 + \sigma v^4 + \tau v^{12}.$$

The required equation is thus

$$0 = (u^{12} + pu^4 + qv^4 + v^{12})^2 - 4(\lambda u^{12} + \mu u^4 + \nu v^4)(\rho u^4 + \sigma v^4 + \tau v^{12}),$$

viz. the function is

$$\begin{aligned}
& u^{24} \\
& + u^{16}(2p - 4\lambda\rho) \\
& + u^8\ (2q\theta^4 + p^2 - 4\lambda\sigma\theta^4 - 4\mu\rho) \\
& + \quad (2\theta^{12} + 2pq\theta^4 - 4\lambda\tau\theta^{12} - 4\mu\sigma\theta^4 - 4\nu\rho\theta^4) \\
& + v^8\ (2p\theta^4 + q^2 - 4\mu\tau\theta^4 - 4\nu\sigma) \\
& + v^{16}(2q - 4\nu\tau) \\
& + v^{24},
\end{aligned}$$

or say it is

$$= (1,\ b,\ c,\ d,\ e,\ f,\ 1 \between u^{24},\ u^{16},\ u^8,\ 1,\ v^8,\ v^{16},\ v^{24}).$$

Supposing that this has a factor $u^8 - \Theta + v^8$, the form is

$$(u^{16} + Bu^8 + C + Dv^8 + v^{16})(u^8 - \Theta + v^8);$$

and comparing coefficients we have

$$\begin{aligned}
B - \Theta &= b, \\
C - \Theta B + \theta^8 &= c, \\
D\theta^8 - \Theta C + B\theta^8 &= d, \\
\theta^8 - \Theta D + C &= e, \\
-\Theta + D &= f,
\end{aligned}$$

where Θ has the before-mentioned value

$$= (8,\ -28,\ +56,\ -70,\ +56,\ -28,\ +8 \between \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5,\ \theta^6,\ \theta^7).$$

From the first, second, and fifth equations, $B = b + \Theta$, $C = c + \Theta B - \theta^8$, $D = f + \Theta$; and

the third and fourth equations should then be verified identically. Writing down the coefficients of the different powers of θ, we find

$$\begin{array}{rl}
2p = & 0 + 12 \quad 0 - 20 + 22 - 12 - 16 + 20 - 8 \; (\theta^0, \ldots, \theta^8) \\
-4\lambda\rho = & 0 - 20 + 20 - 36 + 60 - 44 + 36 - 28 + 8 \quad ,, \\
\hline
b = & 0 - \;\; 8 + 20 - 56 + 82 - 56 + 20 - \;\; 8 \quad 0 \quad ,, \\
\Theta = & 0 + \;\; 8 - 28 + 56 - 70 + 56 - 28 + \;\; 8 \quad 0 \quad ,, \\
\hline
\therefore \; B = & 0 \quad 0 - \;\; 8 \quad 0 + 12 \quad 0 - \;\; 8 \quad 0 \quad 0 \quad ,,
\end{array}$$

that is,

$$B = -8\theta^2 + 12\theta^4 - 8\theta^6;$$

and in precisely the same way the fifth equation gives

$$D = -8\theta^2 + 12\theta^4 - 8\theta^6.$$

We find similarly C from the second equation: writing down first the coefficients of p^2, $2q\theta^4$, $-4\lambda\sigma\theta^4$, and $-4\mu\rho$, the sum of these gives the coefficients of c; and then writing underneath these the coefficients of $B\Theta$ and of $-\theta^8$, the final sum gives the coefficients of C: the coefficients of each line belong to $(\theta^0, \theta^1, \ldots, \theta^{16})$.

$$\begin{array}{lllllllllllllllll}
0 & 0 & 36 & 0 - 120 + 132 + & 28 - 316 + 361 - & 20 - 340 + 396 - 144 - 112 + 164 - 80 + 16 \\
 & & & - \;\; 8 + \;\; 20 - \;\; 16 - \;\; 12 + \;\; 22 - \;\; 20 \quad 0 + \;\; 12 \\
 & & & - \;\; 40 + 140 - 212 + 140 + & 80 - 188 + 168 - \;\; 92 - & 64 + 176 - 164 + 80 - 16 \\
 & & - 36 + 64 - & 40 + \;\; 60 - \;\; 72 + \;\; 28 \quad 0 + \;\; 68 - 100 + \;\; 36 \\
\hline
0 & 0 & 0 + 64 - 208 + 352 - 272 - 160 + 463 - 160 - 272 + 352 - 208 + \;\; 64 & 0 & 0 & 0 \\
0 & 0 & 0 - 64 + 224 - 352 + 224 + 160 - 392 + 160 + 224 - 352 + 224 - \;\; 64 & 0 & 0 & 0 \\
 & & & - \;\; 1 \\
\hline
0 & 0 & 0 \quad 0 + \;\; 16 \quad 0 - \;\; 48 \quad 0 + \;\; 70 \quad 0 - \;\; 48 \quad 0 + \;\; 16 & 0 & 0 & 0 & 0,
\end{array}$$

that is,

$$C = 16\theta^4 - 48\theta^6 + 70\theta^8 - 48\theta^{10} + 16\theta^{12};$$

and in precisely the same way this value of C would be found from the fourth equation. There remains to be verified only the fourth equation $(D + B)\,\theta^8 - \Theta C = d$, that is,

$$2\theta^8(-8\theta^2 + 12\theta^4 - 8\theta^6) - \Theta C = (2 - 4\lambda\tau)\,\theta^{12} + (2pq - 4\mu\sigma - 4\nu\rho)\,\theta^4,$$

and this can be effected without difficulty.

The factor of the modular equation thus is

$$u^{16} + v^{16} + (-8\theta^2 + 12\theta^4 - 8\theta^6)(u^8 + v^8) + 16\theta^4 - 48\theta^6 + 70\theta^8 - 48\theta^{10} + 16\theta^{12},$$

viz. this is

$$\begin{aligned}&(u^8+v^8)^2+(-4\theta^2+6\theta^4-4\theta^6)\,2\,(u^8+v^8)+16\theta^4-48\theta^6+68\theta^8-48\theta^{10}+16\theta^{12}\\&=(u^8+v^8-4\theta^2+6\theta^4-4\theta^6)^2\\&=\{(u^4-v^4)^2-4\theta^2(1-\theta^2)^2\}^2,\end{aligned}$$

that is,

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\,\{u^4-v^4+2\theta(1-\theta^2)\}^2;$$

or the modular equation is

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\,\{u^4-v^4+2\theta(1-\theta^2)\}^2\,(u^8+v^8-\Theta)=0;$$

viz. the first and second factors belong to the cubic transformation; and we have for the proper modular equation in the septic transformation $u^8+v^8-\Theta=0$, or what is the same thing $(1-u^8)(1-v^8)-(1-\theta)^8=0$, that is, $(1-u^8)(1-v^8)-(1-uv)^8=0$, the known result; or, as it may also be written,

$$(\theta-u^8)(\theta-v^8)+7\theta^2(1-\theta)^2(1-\theta+\theta^2)^2=0.$$

The value of M is given by the foregoing relations

$$\frac{1}{M^2}:\frac{2}{M}:1=\lambda u^{12}+\mu u^4+\nu v^4:-(u^{12}+pu^4+qv^4+v^{12}):\rho u^4+\sigma v^4+\tau v^{12};$$

but these can be, by virtue of the proper modular equation $u^8+v^8-\Theta=0$, reduced into the form

$$\frac{1}{M^2}:\frac{2}{M}:1=7(\theta-u^8):14(\theta-2\theta^2+2\theta^3-\theta^4):-\theta+v^8,$$

viz. the equality of these two sets of ratios depends upon the following identities,

$$\begin{aligned}&(-\theta+v^8)(u^{12}+pu^4+qv^4+v^{12})+14(\theta-2\theta^2+2\theta^3-\theta^4)(\rho u^4+\sigma v^4+\tau v^{12})\\&\qquad=\{-\theta u^4+(1-\theta)(-4-\theta+5\theta^2-\theta^3-4\theta^4)v^4+v^{12}\}(u^8-\Theta+v^8),\\&-7(\theta-u^8)(\rho u^4+\sigma v^4+\tau v^{12})-(\theta-v^8)(\lambda u^{12}+\mu u^4+\nu v^4)\\&\qquad=\{(2\theta+5\theta^2+3\theta^3-2\theta^4-2\theta^5)u^4+(2+2\theta-3\theta^2-5\theta^3-2\theta^4)v^4\}(u^8-\Theta+v^8),\\&-2(\theta-2\theta^2+2\theta^3-\theta^4)(\lambda u^{12}+\mu u^4+\nu v^4)+(u^8-\theta)(u^{12}+pu^4+qv^4+v^{12})\\&\qquad=\{u^{12}+\theta(1-\theta)(3+5\theta+3\theta^2)u^4-\theta v^4\}(u^8-\Theta+v^8),\end{aligned}$$

which can be verified without difficulty: from the last-mentioned system of values, replacing θ by its value uv, we then have

$$\frac{1}{M^2}:\frac{2}{M}:1=7u(v-u^7):14uv(1-uv)(1-uv+u^2v^2):-v(u-v^7),$$

which agree with the values given p. 482 of the "Memoir"; and the analytical theory is thus completed.

693.

A TENTH MEMOIR ON QUANTICS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. CLXIX., Part II. (1878), pp. 603—661. Received June 12,—Read June 20, 1878.]

THE present Memoir, which relates to the binary quintic $(*\between x, y)^5$, has been in hand for a considerable time: the chief subject-matter was intended to be the theory of a canonical form which was discovered by myself and is briefly noticed in Salmon's *Higher Algebra*, 3rd Ed. (1876), pp. 217, 218; writing $a, b, c, d, e, f, g, \ldots, u, v, w$ to denote the 23 *covariants* of the quintic, then a, b, c, d, f are connected by the relation

$$f^2 = -a^3d + a^2bc - 4c^3;$$

and the form contains these covariants thus connected together, and also e; it, in fact, is

$$(1,\ 0,\ c,\ f,\ a^2b - 3c^2,\ a^3e - 2cf \between x, y)^5.$$

But the whole plan of the Memoir was changed by Sylvester's discovery of what I term the Numerical Generating Function (N.G.F.) of the covariants of the quintic, and my own subsequent establishment of the Real Generating Function (R.G.F.) of the same covariants. The effect of this was to enable me to establish for any given degree in the coefficients and order in the variables, or as it is convenient to express it, for any given deg-order whatever, a selected system of powers and products of the covariants, say a system of "segregates": these are asyzygetic, that is, not connected together by any linear equation with numerical coefficients; and they are also such that every other combination of covariants of the same deg-order, say every "congregate" of the same deg-order, can be expressed (and that, obviously, in one way only) as a linear function, with numerical coefficients, of the segregates of that deg-order. The number of congregates of a given deg-order is precisely equal to the number of the independent syzygies of the same deg-order, so that these syzygies give in effect the congregates in terms of the segregates: and the proper form in which to exhibit the

syzygies is thus to make each of them give a single congregate in terms of the segregates: viz. the left-hand side can always be taken to be a monomial congregate $a^\alpha b^\beta \ldots$ or, to avoid fractions, a numerical multiple of such form; and the right-hand side will then be a linear function, with numerical coefficients, of the segregates of the same deg-order. Supposing such a system of syzygies obtained for a given deg-order, any covariant function (rational and integral function of covariants) is at once expressible as a linear function of the segregates of that deg-order: it is, in fact, only necessary to substitute therein for every monomial congregate its value as a linear function of the segregates. Using the word covariant in its most general sense, the conclusion thus is that every covariant can be expressed, and that in one way only, as a linear function of segregates, or say in the segregate form.

Reverting to the theory of the canonical form, and attending to the relation

$$f^2 = -a^3d + a^2bc - 4c^3,$$

it thereby appears that every covariant multiplied by a power of the quintic itself a, can be expressed, and that in one way only, as a rational and integral function of the covariants a, b, c, d, e, f, linear as regards f: say every covariant multiplied by a power of a can be expressed, and that in one way only, in the "standard" form: as an illustration, take

$$a^2h = 6acd + 4bc^2 + ef.$$

Conversely, an expression of the standard form, that is, a rational and integral function of a, b, c, d, e, f, linear as regards f, not explicitly divisible by a, may very well be really divisible by a power of a (the expression of the quotient of course containing one or more of the higher covariants g, h, &c.), and we say that in this case the expression is divisible, and has for its divided form the quotient expressed as a rational and integral function of covariants. Observe that in general the divided form is not perfectly definite, only becoming so when expressed in the before-mentioned segregate form, and that this further reduction ought to be made. There is occasion, however, to consider these divided forms, whether or not thus further reduced; and moreover it sometimes happens that the non-segregate form presents itself, or can be expressed, with integer numerical coefficients, while the coefficients of the corresponding segregate form are fractional.

The canonical form is peculiarly convenient for obtaining the expressions of the several derivatives (Gordan's *Uebereinanderschiebungen*) $(a, b)^1$, $(a, b)^2$, &c., (or as I propose to write them $ab1$, $ab2$, &c.), which can be formed with two covariants, the same or different, as rational and integral functions of the several covariants. It will be recollected that in Gordan's theory these derivatives are used in order to establish the system of the 23 covariants: but it seems preferable to have the system of covariants, and by means of them to obtain the theory of the derivatives.

I mention at the end of the Memoir two expressions (one or both of them due to Sylvester) for the N.G.F. of a binary sextic.

The several points above adverted to are considered in the Memoir; the paragraphs are numbered consecutively with those of the former Memoirs upon Quantics.

The Numerical and Real Generating Functions. Art. Nos. 366 to 374, and Table No. 96.

366. I have, in my Ninth Memoir (1871) [462], given what may be called the Numerical Generating Function (N.G.F.) of the covariants of a quartic; this was

$$A(x) = \frac{1 - a^6x^{12}}{1 - ax^4 . 1 - a^2x^4 . 1 - a^2 . 1 - a^3 . 1 - a^3x^6},$$

the meaning being that the number of asyzygetic covariants $a^\theta x^\mu$, of the degree θ in the coefficients and order μ in the variables, or say of the deg-order $\theta . \mu$, is equal to the coefficient of $a^\theta x^\mu$ in the development of this function. And I remarked that the formula indicated that the covariants were made up of $(ax^4, a^2x^4, a^2, a^3, a^3x^6)$, the quartic itself, the Hessian, the quadrinvariant, the cubinvariant, and the cubicovariant, these being connected by a syzygy a^6x^{12} of the degree 6 and order 12. Calling these covariants a, b, c, d, e, so that these italic small letters stand for covariants,

Deg-order.	
1 . 4	a,
2 . 0	b,
2 . 4	c,
3 . 0	d,
3 . 6	e,

then it is natural to consider what may be called the Real Generating Function (R.G.F.): this is

$$\frac{1 - e^2}{1 - a . 1 - b . 1 - c . 1 - d . 1 - e};$$

the development of this contains, as it is easy to see, only terms of the form $a^\alpha b^\beta c^\gamma d^\delta$ and $a^\alpha b^\beta c^\gamma d^\delta e$, each with the coefficient $+1$, so that the number of terms of a given deg-order $\theta . \mu$ is equal to the coefficient of $a^\theta x^\mu$ in the first-mentioned function: and these terms of a given deg-order represent the asyzygetic covariants of that deg-order: any other covariant of the same deg-order is expressible as a linear function of them. For instance, deg-order 6 . 12, the terms of the R.G.F. are a^3d, a^2bc, c^3: there is one more term e^2 of the same deg-order; hence e^2 must be a linear function of these: and in fact

$$e^2 = -a^3d + a^2bc - 4c^3,$$

viz. this is the equation

$$\Phi^2 = -U^3J + U^2IH - 4H^3.$$

367. Sylvester obtained an expression for the N.G.F. of the quintic: this is

$$
\begin{array}{ll}
a^0 . & 1 \\
+a^3 . & x^3 + x^5 + x^9 \\
+a^4 . & x^4 + x^6 \\
+a^5 . & x + x^3 + x^7 - x^{11} \\
+a^6 . & x^2 + x^4 \\
+a^7 . & x + x^5 - x^9 \\
+a^8 . & x^2 + x^4 \\
+a^9 . & x^3 + x^5 - x^7 \\
+a^{10} . & x^2 + x^4 - x^{10} \\
+a^{11} . & x + x^3 - x^9 \\
+a^{12} . & x^2 - x^8 - x^{10} \\
+a^{13} . & x - x^7 - x^9 \\
+a^{14} . & x^4 - x^6 - x^8 \\
+a^{15} . & - x^7 - x^9 \\
+a^{16} . & x^2 - x^6 - x^{10} \\
+a^{17} . & - x^7 - x^9 \\
+a^{18} . & 1 - x^4 - x^8 - x^{10} \\
+a^{19} . & - x^5 - x^7 \\
+a^{20} . & - x^2 - x^6 - x^8 \\
+a^{23} . & - x^{11}
\end{array}
$$

$$1 - ax^5 . 1 - a^2x^2 . 1 - a^2x^6 . 1 - a^4 . 1 - a^8 . 1 - a^{12};$$

viz. expanding this function in ascending powers of a, x, then, if a term is $Na^\theta x^\mu$, this means that there are precisely N asyzygetic covariants of the deg-order $\theta . \mu$.

368. It is known that the number of the irreducible covariants of the binary quintic is $= 23$; representing these by the letters $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w$, ($a$ the quintic itself), the deg-orders of these, and the references* to the tables which give them are

[* See also the paper, 143, in the second volume of this collection.]

Deg-order.		Tab.	Mem.	
1.5	a	13	2	
2.2	b	14	„	
„.6	c	15	„	
3.3	d	16	„	
„.5	e	17	„	
„.9	f	18	„	
4.0	g	19	„	
„.4	h	20	„	
„.6	i	21	„	
5.1	j	22	„	
„.3	k	23	„	
„.7	l	24	„	
6.2	m	83	8	
„.4	n	84	„	
7.1	o	90*	9	
„.5	p	91	„	
8.0	q	25	2	[See also paper 143]
„.2	r	92	9	
9.3	s†			
11.1	t	94	9	
12.0	u	29	3	
13.1	v	95	9	
18.0	w	29A	5.	

Starting from the foregoing expression of the N.G.F. of the quintic, we can, instead of each term $a^\theta x^\mu$, introduce a covariant or product of covariants of the proper deg-order $\theta.\mu$: the mode of doing this depends of course on the different admissible partitions of θ, μ, and it is for some of the terms very indeterminate: for instance, a^5x^{11} is ai, bf, or ce. I found it possible to perform the whole process so as to satisfy a condition which will be presently referred to; and I found

[* See vol. VII. of this collection, p. 348.]

† See end of Memoir. The S of Table 93 has the value $-96(D, M)+16BO-7GK$, but it is better to use the simple value $-(D, M)$; and the S of the present Memoir has this value, say $S=-(d, m)$.

R.G.F. of quintic =	Deg-orders.
$1 \;.1-b^5$	$0.0-10.10$
$+d \;.1-ag^2$	$3.3-12.\;8$
$+e \;.1-b^2$	$3.5-\;7.\;9$
$+f \;.1-b$	$3.9-\;5.11$
$+h \;.1-ag^2$	$4.4-13.\;9$
$+i \;.1-b^2g$	$4.6-12.10$
$+j \;.1-ag^2$	$5.1-14.\;6$
$+k \;.1-b^2$	$5.3-\;9.\;7$
$+l \;.1-bg$	$5.7-11.\;9$
$+m \;.1-ag^2$	$6.2-15.\;7$
$+n \;.1-b^2g$	$6.4-14.\;8$
$+o \;.1-b^3$	$7.1-13.\;7$
$+p \;.1-b^2g$	$7.5-15.\;9$
$+r \;.1-b^2g$	$8.2-16.\;6$
$+dj \;.1-ag^2$	$8.4-17.\;9$
$+s \;.1-abg$	$9.3-16.10$
$+hj \;.1-ag^2$	$9.5-18.10$
$+j^2 \;.1-ag^2$	$10.2-19.\;7$
$+jk \;.1-b^2g$	$10.4-18.\;8$
$+t \;.1-b^3$	$11.1-17.\;7$
$+jm \;.1-ag^2$	$11.3-20.\;8$
$+jo \;.1-bg$	$12.2-18.\;4$
$+v \;.1-b^5$	$13.1-23.11$
$+js \;.1-bg$	$14.4-20.\;6$
$+jt \;.1-g$	$16.2-20.\;2$
$+w \;.1-a$	$18.0-19.\;5$

$$1-a.1-b.1-c.1-g.1-q.1-u,$$

where observe that each negative term of the numerator is equal to a positive term multiplied by a power or product of terms a, b, g, contained in the denominator: this is the condition above referred to. The expansion thus consists only of terms each with the coefficient $+1$; for instance, a part of the function is

$$\frac{s(1-abg)}{1-a.1-b.1-c.1-g.1-q.1-u}, \quad =\frac{s}{1-c.1-q.1-u}\cdot\frac{1-abg}{1-a.1-b.1-g},$$

where the first factor is the entire series of terms $sc^3q^\epsilon u^\zeta$, and the second factor is the series of terms $a^\alpha b^\beta g^\gamma$ omitting only those terms which are divisible by abg: and in the product of the two factors the terms are all distinct, so that the coefficients are still each $=1$. The same thing is true for every other pair of numerator terms: and since the terms arising from each such pair are distinct from each other, in the expansion of the entire function the coefficients are each $=+1$. Hence (as in the case of the quartic) for any given deg-order, the terms in the expansion of the R.G.F. may be taken for the asyzygetic covariants of that deg-order; and if there are any other terms of the same deg-order, each of these must be a linear function, with numerical coefficients, of these asyzygetic covariants: thus deg-order 6.14, the expansion contains only the terms a^2h, acd, bc^2; there is besides a term of the same deg-order, ef, which is not a term of the expansion, and hence ef must be a linear function of a^2h, acd, bc^2; we in fact have $ef = a^2h - 6acd - 4bc^2$.

The terms in the expansion of the R.G.F. may be called "segregates," and the terms not in the expansion "congregates"; the theorem thus is: every congregate is a linear function, with determinate numerical coefficients, of the segregates of the same deg-order.

369. I stop to remark that the numerator of the R.G.F. may be written in the more compendious form

$$\begin{aligned}
&(1-b^5)(1-v)+(1-b^3)(o+t)+(1-b^2)(e+k)+(1-b)f\\
&+(1-ag^2)(d+h+j+m+dj+hj+j^2+jm)\\
&+(1-bg)(l+jo+js)\\
&+(1-b^2g)(i+n+p+jk)\\
&+(1-abg)s\\
&+(1-g)jt\\
&+(1-a)w;
\end{aligned}$$

but the first-mentioned form is, I think, the more convenient one.

370. It is to be noticed that the positive terms of the numerator are unity, the seventeen covariants $d, e, f, h, i, j, k, l, m, n, o, p, r, s, t, v, w$, and the products of j by (d, h, j, k, m, o, s, t), where j^2 is reckoned as a product; in all, 26 terms. Disregarding the negative terms of the numerator the expansion would consist of these 26 terms, each multiplied by every combination whatever $a^\alpha b^\beta c^\gamma g^\delta q^\epsilon u^\zeta$ of the denominator terms a, b, c, g, q, u (which for this reason might be called "reiterative"): the effect of the negative terms of the numerator is to remove from the expansion certain of the terms in question, thereby diminishing the number of the segregates: thus as regards the terms belonging to unity, any one of these which contains the factor b^5 is not a segregate but a congregate: and so as regards the terms belonging to d, any one of these which contains the factor ag^2 is a congregate: and the like in other cases.

For a given deg-order we have a certain number of segregates and a certain number of congregates: and the number of independent syzygies of that deg-order is

precisely equal to the number of congregates: viz. each such syzygy may be regarded as giving a congregate in terms of the segregates: we have on the left-hand side a congregate, or, to avoid fractions, a numerical multiple of the congregate, and on the right-hand side a linear function, with numerical coefficients, of the segregates.

371. The syzygy is irreducible or reducible; and in the latter case it is, or is not, simply divisible: viz. if the congregate on the left-hand side contains any congregate factor (the other factor being literal), then the syzygy is reducible: it is, in fact, obtainable from the syzygy (of a lower deg-order) which gives the value of such congregate factor. But there are here two cases; multiplying the lower syzygy by the proper factor, the right-hand side may still contain segregates only, and then no further step is required: the original syzygy is nothing else than this lower syzygy, each side multiplied by the factor in question, and it is accordingly said to be simply divisible (S.D.). But contrariwise, the right-hand side, as multiplied, may contain congregates which have to be replaced by their values in terms of the segregates of the same deg-order: the resulting expression is then no longer explicitly divisible by the introduced factor: and the original syzygy, although arising as above from a lower syzygy, is not this lower syzygy each side multiplied by a factor: viz. it is in this case not simply divisible.

For example (see the subsequent Table No. 96, under the indicated deg-orders) (6.6), from the syzygy

$$9d^2 = aj - b^3 + 2bh - cg,$$

we deduce (7.11) the syzygy

$$9ad^2 = a^2j - ab^3 + 2abh - acg,$$

which (all the terms on the right-hand being segregates) requires no further reduction: it is a reducible and simply divisible syzygy. But we have (6.8) a syzygy giving de, and also (6.10) a syzygy giving e^2; multiplying the former of these by e or the latter of them by d, we obtain values of de^2, but in each case the right-hand sides contain terms which are not segregates, and have thus to be further reduced; the final formula (9.13) is

$$3de^2 = -4a^2bj + 3a^3dg + 4ab^4 - 8ab^2h + 4abcg - 12b^2cd,$$

which is not divisible by any factor: the syzygy is thus reducible, but not simply divisible.

A syzygy, which is not in the sense explained reducible, is said to be irreducible.

372. The number of irreducible syzygies is obviously finite: it has, however, the large value 179 as appears from the annexed diagram, showing the congregates determined by these several syzygies, and the deg-orders of the syzygies:—

		1	d	e	f	h	i	j	k	l	m	n	o	p	r	s	t	v	w	j^2	
		b^5	ag^2	b^2	b	ag^2	b^2g	ag^2	b^2	bg	ag^2	b^2g	b^3	b^2g	b^2g	abg	b^3	b^5	a		
0.0	1	10.10	12.8	7.9	5.11	13.9	12.10	14.6	9.7	11.9	15.7	14.8	13.7	15.9	16.6	16.10	17.7	23.11	19.5		1
								ag^2													
3.3	d		6.6	6.8	6.12	7.7	7.9	17.9	8.6	8.10	9.5	9.7	10.4	10.8	11.5	12.6	14.4	16.4	21.3	13.5	d
3.5	e			6.10	6.14	7.9	7.11	8.6	8.8	8.12	9.7	9.9	10.6	10.10	11.7	12.8	14.6	16.6	21.5		e
3.9	f				6.18	7.13	7.15	8.10	8.12	8.16	9.11	9.13	10.10	10.14	11.11	12.12	14.10	16.10	21.9		f
								ag^2													
4.4	h					8.8	8.10	18.10	9.7	9.11	10.6	10.8	11.5	11.9	12.6	13.7	15.5	17.5	22.4	14.6	h
4.6	i						8.12	9.7	9.9	9.13	10.8	10.10	11.7	11.11	12.8	13.9	15.7	17.7	22.6		i
								ag^2	b^2g		ag^2		bg			bg	g				
5.1	j							19.7	18.8	10.8	20.8	11.5	18.4	12.6	13.3	20.6	20.2	18.2	23.1	15.3	j
5.3	k								10.6	10.10	11.5	11.7	12.4	12.8	13.5	14.6	16.4	18.4	23.3	15.5	k
5.7	l									10.14	11.9	11.11	12.8	12.12	13.9	14.10	16.8	18.8	23.7		l
6.2	m										12.4	12.6	13.3	13.7	14.4	15.5	17.3	19.3	24.2	16.4	m
6.4	n											12.8	13.5	13.9	14.6	15.7	17.5	19.5	24.4		n
7.1	o												14.2	14.6	15.3	16.4	18.2	20.2	25.1	17.3	o
7.5	p													14.10	15.7	16.8	18.6	20.6	25.5		p
8.2	r														16.4	17.5	19.3	21.3	26.2		r
9.3	s															18.6	20.4	22.4	27.3	19.5	s
11.1	t																22.2	24.2	29.1	21.3	t
13.1	v																	26.2	31.1		v
18.0	w																		36.0		w

Each term inside this diagram is a deg-order indicating the congregate determined by an irreducible syzygy: viz. the congregate is the product of the outside covariants in the line and column containing the deg-order, and of the literal factor (if any) placed immediately above the deg-order. Thus, line d and column i, 7.9 indicates the congregate di, but, same line and column j, 17.9 indicates the congregate $dj \,.\, ag^2$, $= adg^2j$.

Observe as regards the foregoing diagram, that dj^2 is irreducible (since neither dj nor j^2 is segregate), and similarly j^2h, j^3, &c., are irreducible: we have thus the last or j^2 column of the diagram.

The simply divisible syzygies are infinite in number, as are also the reducible syzygies not simply divisible. There is obviously no use in writing down a simply divisible syzygy; but as regards the reducible syzygies not simply divisible, these require a calculation, and it is proper to give them as far as they have been obtained.

373. The following Table, No. 96, replaces Tables 88 and 89 of my Ninth Memoir. The arrangement is according to deg-orders, and the table is complete up to the deg-order 8.40: it shows for each deg-order the segregate covariants, and also the congregate covariants (if any), and the syzygies which are the expressions of these in terms of the segregates. When there are only segregates these are given in the same horizontal line with the deg-order; for instance, $|\ 5.9\ |\ ab^2,\ ah,\ cd$, shows that for the deg-order 5.9 the only covariants are the segregates ab^2, ah, cd; but when there are also congregates, the segregates are arranged in the same horizontal line with the deg-order, and the congregates, each in its own horizontal line together with its expression as a linear function of the segregates: thus $\left|\ 5.11\ \right|\ \begin{matrix} * \\ bf \end{matrix}\left|\ \begin{matrix} ai & ce \\ -1 & +1 \end{matrix}\right.$, the segregates are ai, ce, and there is a congregate bf which is a linear function of these, $= -ai + ce$. The table gives the irreducible syzygies and also the reducible syzygies which are not simply divisible, but the simply divisible syzygies are indicated each by a reference to the divided syzygy which occurs previously in the table.

374. Any syzygy might of course be directly verified by substituting for the several covariants contained therein their expressions in terms of the coefficients and facients of the quintic. But it is to be remarked that among the syzygies, or easily deducible from them, we have (6.18) the before-mentioned equation $f^2 = -a^3d + a^2bc - 4c^3$, and also a set of 17 syzygies, the left-hand sides of which are the covariants $g, h, \ldots, u, v, w$, each multiplied by a or a^2, and which lead ultimately to the standard expressions of these covariants respectively, viz. each covariant multiplied by a proper power of a can be expressed as a rational and integral function of a, b, c, d, e, f, linear as regards f. Supposing them thus expressed, a far more simple verification of any syzygy would consist in substituting therein for the several covariants their expressions in the standard form, reducing if necessary by the equation $f^2 = -a^3d + a^2bc - 4c^3$: but of course, as to the syzygies used for obtaining the standard forms, this is only a verification if the standard forms have been otherwise obtained, or are assumed to be correct.

The 17 syzygies above referred to are

Deg-ord.		
6 . 10	$a^2g =$	$12abd + 4b^3c + e^2,$
6 . 14	$a^3h =$	$6acd + 4bc^2 + ef,$
5 . 11	$ai =$	$-\ bf + ce,$
6 . 6	$aj =$	$b^3 - 2bh + cg + 9d^2,$
6 . 8	$ak =$	$-\ 2bi + 3de,$
6 . 12	$al =$	$2ci - 3df,$
7 . 7	$am =$	$-\ 2b^3d - cj + 3dh,$
7 . 9	$an =$	$b^3e - 6bl - 2ck - fg,$
8 . 6	$ao =$	$2bn + ej,$
8 . 10	$ap =$	$-\ 2cn - fj,$
9 . 5	$aq =$	$-\ 2b^2j + bdg - 12dm + hj,$
9 . 7	$ar =$	$b^2k + bp - co + hk,$
10 . 8	$as =$	$3bdk + 3dp + 2im,$
12 . 6	$at =$	$bjk + jp - 2mn,$
13 . 5	$18au =$	$2agq + b^2gj + 6bmj - 6dj^2 - ghj + no,$
14 . 6	$3av =$	$2b^3q - 8b^2j^2 - 2b^2gm + 6bdgj - 12bm^2 + 3et,$
19 . 5	$18aw =$	$3b^3gt + b^2go - 4bj^3o - bgmo + 18bmt + 3dgjo - 18djt - 3ght - 6m^2o,$

the last four of these being, however, beyond the limits of the table: the expressions of g, h, i are here in the standard form: the standard forms of the other covariants j, k,.., u, v, w, will be given further on.

TABLE No. 96 (Segregates, Congregates, and Syzygies).

Deg-ord.	Congs.	Segregates.
1. 1		1
3		
5		a
2. 0		
2		b
4		
6		c
8		
10		a^2
3. 1		
3		d
5		e
7		ab
9		f
11		ac
13		
15		a^3

TABLE No. 96 (*continued*).

Deg-ord.	Congs.	Segregates.
4. 0		g
2		
4		b^2, h
6		i
8		ad, bc
10		ae
12		a^2b, c^2
14		af
16		a^2c
18		
20		a^4
5. 1		j
3		k
5		ag, bd
7		be, l
9		ab^2, ah, cd
11	*	ai, ce
	bf	-1 $+1$
13		a^2d, abc
15		a^2e, cf
17		a^3b, ac^2
19		a^2f
21		a^3c
23		
25		a^5
6. 0		
2		bg, m
4		n
6	*	aj, b^3, bh, cg
	$d^2 . 9$	$+1$ -1 $+2$ -1
8	*	ak, bi
	$de . 3$	$+1$ $+2$
10	*	a^2g, abd, b^2c, ch
	e^2	$+1$ -12 -4 .
12	*	abe, al, ci
	$df . 3$	. -1 $+2$
14	*	a^2b^2, a^2h, acd, bc^2
	ef	. $+1$ -6 -4
16	*	a^2i, ace
	abf	. . . . . S.D. 5.11, bf

TABLE No. 96 (*continued*).

Deg-ord.	Congs.	Segregates.
6. 18	*	$a^3d,\ a^2bc,\ c^3$
	f^2	$-1\ \ +1\ \ -4$
20		$a^2e,\ acf$
22		$a^4b,\ a^2c^2$
24		a^3f
26		a^4c
28		
30		a^5
7. 1		o
3		$bj,\ dg$
5		$bk,\ eg,\ p$
7	*	$abg,\ am,\ b^2d,\ cj$
	dh. 3	$0\ \ +1\ \ +2\ \ +1$
9	*	$an,\ bl,\ ck,\ fg$
	b^2e	$+1\ \ +6\ \ +2\ \ +1$
	di. 3	$0\ \ +1\ \ +1$
	eh	$0\ \ +4\ \ +2\ \ +1$
11	*	$a^2j,\ ab^3,\ abh,\ acg,\ bcd$
	ad^2	. . . . . S.D. 6.6, d^2
	ei	$.\ \ .\ \ -1\ \ +6\ \ -6$
13	*	$a^2k,\ abi,\ bce,\ cl$
	ade	. . . . . S.D. 6.8 , de
	b^2f	. . . . . S.D. 5.11, bf
	fh. 3	$-1\ \ -2\ \ +3\ \ -6$
15	*	$a^3g,\ a^2bd,\ ab^2c,\ ach,\ c^2d$
	ae^2	. . . . . S.D. 6.10, e^2
	fi	$.\ \ +1\ \ -1\ \ +1\ \ -6$
17	*	$a^2be,\ a^3l,\ aci,\ c^2e$
	adf	. . . . . S.D. 6.12, df
	bcf	. . . . . S.D. 5.11, bf
19	*	$a^3b^2,\ a^3h,\ a^2cd,\ abc^2$
	aef	. . . . . S.D. 6.14, ef
21	*	$a^2i,\ a^2ce,\ c^2f$
	a^2bf	. . . . . S.D. 5.11, bf
23	*	$a^4d,\ a^3bc,\ ac^3$
	af^2	. . . . . S.D. 6.18, f^2

TABLE No. 96 (*continued*).

Deg-ord.	Congs.	Segregates.	
7. 25		a^4e, a^2cf	
27		a^5b, a^3c^2	
29		a^4f	
31		a^5c	
33			
35		a^7	
8. 0		g^2, q	
2		r	
4		b^2g, bm, dj, gh	
6	*	ao, bn, gi	
	dk . 3	+1 −3 −1	
	ej	+1 −2	
8	*	abj, adj, b^4, b^2h, bcg, cm	
	bd^2	. . . . .	S.D. 6.6, d^2
	ek	−4 +3 +4 −6 +2	
	h^2 . 3	+4 −3 −4 +8 −1 +12	
10	*	abk, aeg, ap, b^2i, cn	
	bde	. . . . .	S.D. 6.8, de
	dl . 9	+2 . +3 +1 +3	
	fj	. . −1 . −2	
	hi . 3	+1 . . +2 −3	
12	*	a^2bg, a^2m, ab^2d, acj, b^3c, bch, c^2g	
	adh	. . . . .	S.D. 7.7, dh
	be^2	. . . . .	S.D. 6.10, e^2
	cd^2	. . . . .	S.D. 6.6, d^2
	el	. −1 −2 +1 −2 +2	
	fk	. +1 . −1 +4 −6 +2	
	i^2	. . −1 . +1 −2 +1	
14	*	a^2n, abl, ack, afg, bci	
	ab^2e	. . . . .	S.D. 7.9, b^2e
	adi	. . . . .	S.D. 7.9, di
	aeh	. . . . .	S.D. 7.9, eh
	bdf	. . . . .	S.D. 6.12, df
	cde	. . . . .	S.D. 6.8, de
16	*	a^3j, a^2b^3, a^5bh, a^2cg, $abcd$, b^2c^2, c^2h	
	a^2d^2	. . . . .	S.D. 6.6, d^2
	aei	. . . . .	S.D. 7.11, ei
	bef	. . . . .	S.D. 6.14, ef
	ce^2	. . . . .	S.D. 6.10, e^2
	fl . 3	+1 −1 +2 −1 −3 −6 +6	

TABLE No. 96 (*continued*).

Deg-ord.	Congs.	Segregates.							
8. 18	*	a^3k,	a^2bi,	$abce$,	acl,	c^2i			
	a^2de	.	.	.	.	.	.	.	S.D. 6.8, de
	ab^2f	.	.	.	.	.	.	.	S.D. 5.11, bf
	afh	.	.	.	.	.	.	.	S.D. 7.13, fh
	cdf	.	.	.	.	.	.	.	S.D. 6.12, df
20	*	a^4g,	a^3bd,	a^2b^2c,	a^2ch,	ac^2d,	bc^3		
	a^2e^2	.	.	.	.	.	.	.	S.D. 6.10, e^2
	afi	.	.	.	.	.	.	.	S.D. 7.15, fi
	bf^2	.	.	.	.	.	.	.	S.D. 6.18, f^2
	cef	.	.	.	.	.	.	.	S.D. 6.14, ef
22	*	a^5be,	a^3l,	a^2ci,	$abcf$,	ac^2e			
	a^3df	.	.	.	.	.	.	.	S.D. 6.12, df
24	*	a^4b^2,	a^4h,	a^3cd,	$a^2b^2c^2$,	c^4			
	a^2ef	.	.	.	.	.	.	.	S.D. 6.14, ef
	cf^2	.	.	.	.	.	.	.	S.D. 6.18, f^2
26	*	a^4i,	a^3ce,	ac^2f					
	a^3bf	.	.	.	.	.	.	.	S.D. 5.11, bf
28	*	a^5d,	a^4bc,	a^3c^3					
	a^2f^2	.	.	.	.	.	.	.	S.D. 6.18, f^2
30		a^5e,	a^3cf						
32		a^6b,	a^4c^2						
34		a^5f							
36		a^6c							
38									
40		a^8							
9. 1		gj							
3		bo,	gk,	s					
5	*	ag^2,	aq,	b^2j,	bdg,	hj			
	dm . 12	.	-1	-2	$+1$	$+1$			
7	*	ar,	beg,	bp,	co,	gl			
	b^2k . 3	$+1$	.	-5	-1	$+1$			
	dn . 3	-1	.	-1	-1				
	em . 3	$+2$	.	$+2$	$+1$	-1			
	hk . 3	$+2$	.	$+2$	$+4$	-1			
	ij	.	.	$+1$	$+1$				
9	*	ab^3g,	abm,	adj,	agh,	b^3d,	bcj,	cdg	
	bdh . 3	.	$+1$	.	.	$+2$	$+1$		
	d^3 . 27	.	$+2$	$+3$	.	$+1$	$+2$	-3	
	en	$+1$	$+6$	.	-1	.	$+2$		
	ik	.	-1	.	.	.	-3	$+3$	

TABLE No. 96 (*continued*).

Deg-ord.	Congs.	Segregates.													
9. 11	*	a^2o,	abn,	agi,	b^2l,	bck,	ceg,	cp							
	adk	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.6, dk
	aej	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.6, ej
	b^3e	.	+1	−1	+6	+2	+1								
	bdi	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 7.9, bi
	beh	.	.	−1	+4	+2	+1								
	bfg	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 5.11, bf
	d^2e . 9	+1	−3	−1	+2	+2									
	fm . 3	+1	+3	+1	.	−3	.	−3							
	hl . 3	+1	−3	−1	+2	+5	.	−6							
13	*	a^2bj,	a^2dg,	ab^4,	ab^2h,	$abcg$,	acm,	b^2cd,	c^2j						
	abd^2	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 6.6, d^2
	aek	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.8, ek
	ah^2	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.8, h^2
	bei	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 7.11, ei
	cdh	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 7.7, dh
	de^2 . 3	−4	+3	+4	−8	+4	.	−12							
	fn	−2	+1	+2	−3	+1	−2	.	+2						
	il	−1	.	+1	−2	+1	−3	−3	+3						
10. 0															
2		bg^2,	bq,	gm,	j^2										
4	*	br,	gn,	jk											
	do . 3	+2	−1												
6	*	agj,	b^2g,	b^2m,	bdj,	bgh,	cg^2,	cq							
	d^2g	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 6.6, d^2
	eo	+1	+2	+12	−12	−2									
	hm . 3	−1	+1	+6	+12	−2	+1	−3							
	k^2	+1	−1	−4	−12	+2	−1								
8	*	abo,	agk,	as,	b^2n,	bgi,	cr								
	bdk	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.6, dk
	bej	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 8.6, ej
	deg	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 6.8, de
	dp . 9	−5	.	+3	+15	+5	−6								
	hn . 3	−6	+1	.	+18	+5	−12								
	im . 3	+1	.	.	−3	−1	+3								
	jl . 3	−5	.	+3	+15	+5	−12								
10	*	a^2g^2,	a^2q,	ab^2j,	$abdg$,	ahj,	b^3h,	b^2cg,	bcm,	cdj,	cgh				
	adm	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 9.5, dm
	b^5 . 8	.	+1	+10	−9	+3	+12	.	+32	−12	−4				
	b^3d^2 . 72	.	−1	−2	+9	−3	+4	−8	−32	+12	+4				
	bek . 2	.	+1	+2	−3	+3	.	+4	+32	−12	−4				
	bh^2 . 6	.	−1	−2	+3	−3	+4	−2	−8	+12	−4				
	d^2h . 27	.	−1	−2	+3	.	+1	−2	−8	+12	+1				
	e^2g	.	.	.	.	.	.	.	.	.	.	.	.	.	S.D. 6.10, e^2
	ep	.	.	.	.	−1	.	−2	−12	+6	+2				
	fo . 2	.	+1	+2	−3	+7	.	+4	+24	−24	−4				
	in . 4	.	−1	−2	+3	−5	.	.	.	+12					
	kl . 4	.	+1	+2	−1	−1	.	.	+16	+12					

TABLE No. 96 (*concluded*).

Deg-ord.	Congs.	Segregates.
11. 1		go, t
3		bgj, dg^2, dq, jm
5	*	b^2o, bgk, bs, eg^2, eq, gp
	dr . 18	− 2 + 5 − 6 . − 3 + 3
	ho . 3	− 2 + 11 − 24 . − 3 + 6
	jn	+ 1 − 3 + 6 . + 1 − 2
	km . 6	− 2 + 5 − 12 . − 3 + 3
12. 0		g^3, g^2q, u
2		gr, jo
4	*	b^2g^2, b^2q, bgm, bj^2, dgj, g^2h, hg
	ko	. − 2 − 2 − 4 + 3
	m^2 . 12	. + 2 + 1 + 4 − 3 . − 3
13. 1		g^2j, jq, v
3	*	bgo, bt, g^2k, gs, kg
	jr . 2	. − 2 . + 1 − 1
	mo . 2	. − 4 . + 1 − 1
14. 0		
2		bg^3, bgq, bu, g^2m, gj^2, mq, o^2
4	*	bgr, bjo, g^2n, gjk, js, nq
	dgo	. . . . . S.D. 10.4, do
	dt . 18	+ 1 + 2 . − 1 + 6 + 3
	mr . 12	+ 1 + 2 . − 1 . + 3

Theory of the Canonical Form. Art. Nos. 375 to 381, and Tables Nos. 97 and 98.

375. As the small italic letters have been used to represent the covariants, different letters are required for the coefficients of the quintic: using also new letters for the facients, I take the quintic to be $(\mathrm{a, b, c, d, e, f}\between\xi, \eta)^5$. Considering a linear transformation of $\frac{1}{\mathrm{a}}(\mathrm{a, b, c, d, e, f}\between\xi, \eta)^5$, viz.

$$\frac{1}{\mathrm{a}}(\mathrm{a, b, c, d, e, f}\between\xi - \mathrm{b}\eta, \mathrm{a}\eta)^5,$$

this is

	ξ^5	$5\xi^4\eta$	$10\xi^3\eta^2$	$10\xi^2\eta^3$	$5\xi\eta^4$	η^5
$=$	1	$-b$	$+b^2$	$-b^3$	$+b^4$	$-b^5$
$+b\,($		1	$-2b$	$+3b^2$	$-4b^3$	$+5b^4)$
$+ac\,($			1	$-3b$	$+6b^2$	$-10b^3)$
$+a^2d\,($				1	$-4b$	$+10b^2)$
$+a^3e\,($					1	$-5b\,)$
$+a^4f\,($						$1\,)$,

which is

$$=\left(\begin{array}{cccccc} 1 & 0 & ac+1 & a^2d+1 & a^3e+1 & a^4f+1 \\ & & b^2-1 & abc-3 & a^2bd-4 & a^3be-5 \\ & & & b^3+2 & ab^2c+6 & a^2b^2d+10 \\ & & & & b^4-3 & ab^3c-10 \\ & & & & & b^5+4 \end{array}\right|\,\between\xi,\ \eta)^5.$$

The values of a, b, c, d, e, f, considered for a moment as denoting the leading coefficients of the several covariants ultimately represented by these letters respectively, are

a	b	c	d	e	f
$a+1$	$ae+1$	$ac+1$	$ace+1$	a^2f+1	a^2d+1
	$bd-4$	b^2-1	ad^2-1	$abe+5$	$abc-3$
	c^2+3		b^2e-1	$acd+2$	b^3-2
			$bcd+2$	b^2d+8	
			c^3-1	bc^2-10	

satisfying, as they should do, the relation

$$f^2=-a^3d+a^2bc-4c^3.$$

Hence forming the values of a^2b-3c^2 and a^2e-2cf, it appears that the value of the last-mentioned quintic function is

$$(1,\ 0,\ c,\ f,\ a^2b-3c^2,\ a^2e-2cf\between\xi,\ \eta)^5.$$

Writing herein x, y in place of ξ, η, and now using a, b, c, d, e, f to denote, not the leading coefficients but the covariants themselves (a denoting the original quintic, with ξ, η as facients), we have the form

$$A=(1,\ 0,\ c,\ f,\ a^2b-3c^2,\ a^2e-2cf\between x,\ y)^5,$$

a new quintic, which is the canonical form in question: the covariants hereof (reckoning the quintic itself as a covariant) will be written A, B, $C, \ldots, V$, W, and will be spoken of as capital covariants.

376. The fundamental property is: Every capital covariant, say I, has for its leading coefficient the corresponding covariant i multiplied by a power of a: and this follows as an immediate consequence of the foregoing genesis of A. The covariant i of the form

$$\frac{1}{\mathrm{a}}(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f}\between\xi,\ \eta)^5$$

has a leading coefficient

$$=\frac{1}{\mathrm{a}^4}(\mathrm{a}^2\mathrm{cf}-\mathrm{a}^2\mathrm{de}+\&\mathrm{c.}),$$

which, when $a, b, c, d, e, f, \ldots, i$ denote leading coefficients, is $=i$ multiplied by a power of a: and upon substituting for the quintic the linear transformation thereof

$$(1,\ 0,\ c,\ f,\ a^2b-3c^2,\ a^2e-2cf\between\xi,\ \eta)^5,$$

(observing that, in the transformation ξ, η into $\xi-\mathrm{b}\eta$, $\mathrm{a}\eta$, the determinant of substitution is $=\mathrm{a}$), the value is still $=i$ multiplied by a power of a; or using the relation $a=\mathrm{a}$, say the value is $=i$ multiplied by a power of a. Now the covariant i is the same function of the covariants a, b, c, d, e, f that the leading coefficient i is of the leading coefficients a, b, c, d, e, f; hence, the italic letters now denoting covariants, the leading coefficient still is $=i$ multiplied by a power of a: which is the above-mentioned theorem.

377. To show how the transformation is carried out, consider, for example, the covariant B. This is obtained from the corresponding covariant of $(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f}\between\xi,\ \eta)^5$, that is,

$$\left(\begin{array}{c|c|c} \mathrm{ae}\quad 1 & \mathrm{af}\quad 1 & \mathrm{bf}\quad 1 \\ \mathrm{bd}-4 & \mathrm{be}-3 & \mathrm{ce}-4 \\ \mathrm{c}^2+1 & \mathrm{cd}+1 & \mathrm{d}^2+3 \end{array}\right.\between\xi,\ \eta)^2,$$

by changing the variables, and for the coefficients

$$\mathrm{a},\quad \mathrm{b},\quad \mathrm{c},\quad \mathrm{d},\quad \mathrm{e},\quad \mathrm{f}$$

writing

$$1,\quad 0,\quad c,\quad f,\quad a^2b-3c^2,\quad a^2e-2cf;$$

thus the coefficients are

First.	Second.	Third.
$1\,(a^2b-3c^2)$	$1\,(a^2e-2cf)$	$-4c\,(a^2b-3c^2)$
$+3c^2$	$+2cf$	$+3f^2$
$=\quad a^2b$	$=\quad a^2e$	$=-4a^2bc+12c^3$
		$+3\,(-a^3d+a^2bc-4c^3)$
		$=\quad a^3(-3ad-bc)$;

and we have thus the expression of B (see the Table No. 97); and similarly for the other capital covariants $C, D, \ldots, V, W$: in every case the coefficients are obtained in the standard form, that is, as rational and integral functions of a, b, c, d, e, f, linear as regards f.

378. It will be observed that there is in each case a certain power of a which explicitly divides all the coefficients and is consequently written as an exterior factor: disregarding these exterior factors, the leading coefficients for B, C, D, E, F are b, c, ad, e, f respectively; that for G is $12abd + 4b^2c + e^2$, which must be $= g$ multiplied by a power of a, and (in Table 97) is given as $= a^2g$; similarly, that for H is $6acd + 4bc^2 + ef$, which must be $= h$ multiplied by a power of a, and is given as $= a^3h$: and so in the other cases. The index of a is at once obtained by means of the deg-order, which is in each case inserted at the foot of the coefficient.

For A, B, C, E, F there is no power of a as an interior factor: and for the invariants G, Q, U we may imagine the interior factor thrown together with the exterior factor, ($G = a^6g$, &c.): whence disregarding the exterior factors, we may say that for A, B, C, E, F, G, Q, U the standard forms are also "divided" forms. But take any other covariant—for instance, D: the leading coefficient is ad, having the interior factor a; and this being so it is found that all the following coefficients will divide by a (the quotients being of course expressible only in terms of the covariants subsequent to f): thus the second coefficient of D is $-bf + ce$, and (5.11) we have $-bf + ce = ai$, or the coefficient divided by a is $= i$; and so for the other coefficients of D; or throwing out the factor a, we obtain for D an expression of the form $(d, i, \ldots \between x, y)^3$, see the Table 98: this is the "divided" form of D: and we have similarly a divided form for every other capital covariant. All that has been required is that each coefficient of the divided form shall be expressed as a rational and integral function of the covariants $a, b, c, \ldots, v, w$: and the form is not hereby made definite: to render it so, the coefficient must be expressed in the segregate form. But there is frequently the disadvantage that we thus introduce fractions; for instance, the last coefficient of D is $= -ci + df$, where to get rid of the congregate term df we have (6.12), $3df = -al + 2ci$, and the segregate form of the coefficient is $= -\frac{1}{3}al + \frac{2}{3}ci$.

379. We have in regard to the canonical form, a differential operator which is analogous to the two differential operators $xd_y - \{xd_y\}$, $yd_x - \{yd_x\}$ considered in the Introductory Memoir (1854), [139]. Let δ denote a differentiation in regard to the constants under the conditions

$$\begin{aligned}
\delta a &= 0, \\
\delta b &= e, \\
\delta c &= 3f, \\
\delta d &= \frac{1}{a}(-bf + ce), \ (= i), \\
\delta e &= -6ad - 10bc, \\
\delta f &= 2a^3b - 18c^2,
\end{aligned}$$

which (as is at once verified) are consistent with the fundamental relation

$$f^2 = -a^3d + a^2bc - 4c^3;$$

then it is easy to verify that

$$\left(x\frac{d}{dy} - 4cy\frac{d}{dx} - \delta\right)A = 0;$$

and this being so, any other covariant whatever, expressed in the like standard form, is reduced to zero by the operator

$$x\frac{d}{dy} - 4cy\frac{d}{dx} - \delta;$$

and we have thus the means of calculating the covariant when the leading coefficient is known.

Thus, considering the covariant B, the expression of which has just been obtained, $=(B_0, B_1, B_2 \between x, y)^2$, suppose: the equation to be satisfied is

$$\begin{aligned} & x\,(B_1x + 2B_2y \qquad\quad) \\ & -4cy\,(\qquad\quad 2B_0x + B_1y) \\ & -x^2\delta B_0 \;\; -xy\delta B_1 - y^2\delta B_2 \;\; = 0, \end{aligned}$$

viz. we have

$$\begin{aligned} B_1 \qquad\quad - \delta B_0 &= 0, \\ 2B_2 - 8cB_0 - \delta B_1 &= 0, \\ -4cB_1 - \delta B_2 &= 0; \end{aligned}$$

which (omitting, as we may do, the outside factor a^2) are satisfied by the foregoing values $B_0, B_1, B_2, = b, e, -3ad - bc$. And if we assume only $B_0 = b$, then the first equation gives at once the value $B_1 = e$, the second equation then gives $B_2 = -3ad - 3bc$; and the third equation is satisfied identically, viz. the equation is

$$-4ce + \delta\,(3ad + bc) = 0,$$

that is,

$$\begin{aligned} & -4ce \;\; = -4ce \qquad\qquad = 0, \\ & +c\delta b \qquad + c\,.\,e \\ & +b\delta e \qquad + b\,.\,3f \\ & +3a\delta d \quad + 3\,(-bf + ce) \end{aligned}$$

which is right.

Of course every invariant must be reduced to zero by the operation δ: thus we have, see the Table No. 97,

$$\begin{aligned} a^2g = \;\; & 12abd \\ & + \;\; 4b^3c \\ & + \;\; 1e^2, \end{aligned}$$

and thence

$$
\begin{array}{llll}
 & & & \quad\quad\quad\

It is obvious that for every covariant whatever written in the denumerate form $(I_0, I_1, \ldots \between x, y)^n$, the second coefficient is equal to the first coefficient operated upon by δ; so that the foregoing formulæ give, in fact, the second coefficients of the several covariants.

381. It is worth noticing how very much the formulæ of Table No. 97 simplify themselves, if one of the covariants b, c, d, e vanishes, in particular, if b vanishes. Suppose $b=0$; writing also (although this makes but little difference) $a=1$, we have

$$\begin{aligned}
a &= 1,\\
b &= 0,\\
c &= c,\\
d &= d,\\
e &= e,\\
f^2 &= -d-4c^3,\\
g &= e^2,\\
h &= 6cd+ef,\\
i &= ce,\\
j &= 9d^2+ce^2,\\
k &= 3de,\\
l &= -3df+2c^2e,\\
m &= 9cd^2+3def-c^2e^2,\\
n &= -6cde-e^2f,\\
o &= 9d^2e+ce^3,\\
p &= -9d^2f+12c^2de+ce^2f,\\
q &= -54cd^3-27d^2ef+18c^3de^2+ce^3f,\\
r &= 9cd^2e+3de^3f-c^2e^3,\\
s &= -27d^3f+54c^2d^2e+9cde^2f-2c^3e^3,\\
t &= -81d^4f-6d^2e^3+216c^2d^3e+54cd^2e^2f-24c^3de^3-c^2e^4f,\\
u &= -27d^5-18cd^3e^2-4d^2e^3f+c^2de^4,\\
v &= -81d^4ef-6d^2e^4+216c^2d^3e^2+54cd^2e^3f-24c^3de^4-1c^2e^5f,\\
w &\text{ (not calculated).}
\end{aligned}$$

These values are very convenient for the verification of syzygies, &c. Take, for instance, the before-mentioned relation $\delta v = -6dt-6mr+nq$, that is, if $V=(V_0, V_1 \between x, y)$, then $V_1 = -6dt-6mr+nq$: calculating the three products on the right-hand side, observing

that f^2 when it occurs is to be replaced by its value $-d-4c^3$, and taking their sum, the figures are as follows:

	$-6dt$	$-6mr$	$+nq$	Sum
d^5f	$+486$			$+486$
d^3e^3	$+36$	$+54$	-27	$+63$
c^3d^4e	-1296	-486	$+324$	-1458
cd^3e^2f	-324	-324	$+216$	-432
cde^5			$+1$	$+1$
$c^3d^2e^3$	$+144$	$+324$	-216	$+252$
c^2de^4f	$+6$	$+36$	-24	$+18$
c^4e^6		-6	$+4$	-2

where the last column is, in fact, what V_1 becomes on writing therein $a=1$, $b=0$. The verification would not of course apply to terms which contain b; thus, (13.3), a derived syzygy is $jr=bt+mo$; and the foregoing values give, as they should do, $jr=mo$: we might for the verification of most of the terms in b use values a, b, c, d, e, $f^2=1$, b, 0, d, e, $-d$: the only failure would be for terms containing bc.

TABLE No. 97 (Covariants of A, in the af- or standard forms: W is not given).

The several covariants are—

$A=($

0.0	1.3	2.6	3.9	4.12	5.15
1	0	$c+10$	$f+10$	a^2b+5 c^2-15	a^2e+1 $cf-2$

$)(x, y)^5$

$B=a^2($

2.2	3.5	4.8
$b+1$	$e+1$	$ad-3$ $bc-1$

$)(x, y)^2$

$C=($

2.6	3.9	4.12	5.15	6.18	7.21	8.24
$c+1$	$f+1$	a^2b+3 c^2-15	a^2e+1 $cf-10$	a^3d+6 a^2bc-3 c^3+15	a^3bf-3 $,,ce+3$ a^0cf+3	a^4b^2-1 a^3cd+2 a^2bc^2+4 $,,ef+1$ a^0c^4-1

$)(x, y)^6$

TABLE No. 97 (*continued*).

$D = a^3 ($

$ad + 1$	$bf - 1$ $ce + 1$	$a^2b^2 - 1$ $acd + 3$ $a^0bc^2 + 4$,, $ef + 1$	$adf + 1$ $a^0bcf + 1$,, $c^2e - 1$
4.8	5.11	6.14	7.17

$\mathfrak{X}(x, y)^3$

$E = a^2 ($

$e + 1$	$ad - 6$ $a^0bc - 10$	$bf - 12$ $ce + 2$	$a^2b^2 - 8$ $acd - 36$ $a^0bc^2 + 12$,, $ef - 2$	$a^2be - 5$ $adf - 24$ $a^0bcf - 4$,, $c^2e + 2$	$a^2bd - 6$ $a^2b^2c - 2$,, $e^2 - 1$ $ac^2d + 18$ $a^0bc^3 + 6$,, $cef + 2$
3.5	4.8	5.11	6.14	7.17	8.20

$\mathfrak{X}(x, y)^5$

$F = ($

$f+1$	$a^2b + 2$ $a^0c^2 - 18$	$a^2e + 1$ $a^0cf - 36$	$a^3d + 34$ $a^2bc - 42$ $a^0c^3 + 168$	$a^3bf - 40$,, $ce + 5$ $a^0c^2f - 126$	$a^4b^2 - 16$ $a^3cd + 6$ $a^2bc^2 + 134$,, $ef - 5$ $a^0c^4 - 252$	$a^4be - 7$ $a^3df + 8$ $a^2bcf + 8$,, $c^2e + 55$ $a^0c^3f - 84$	$a^5bd + 6$ $a^4b^2c - 22$,, $e^2 - 1$ $a^3c^2d + 54$ $a^2bc^2 + 66$,, $cef + 38$ $a^0c^5 + 72$	$a^5de - 12$ $a^4bce + 11$,, $b^2f - 9$ $a^3cdf + 24$ $a^2bc^2f + 32$,, $c^3e - 45$ $a^0c^4f + 9$	$a^6b^3 - 2$ $a^5bcd + 6$ $a^4b^2c^2 + 12$,, $bef + 3$,, $ce^2 - 1$ $a^3c^3d - 14$ $a^2bc^4 - 16$,, $c^2ef - 5$ $a^0c^6 - 2$
3.9	4.12	5.15	6.18	7.21	8.24	9.27	10.30	11.33	12.36

$\mathfrak{X}(x, y)^9$

$G = a^4$

$abd + 12$ $a^0b^2c + 4$,, $e^2 + 1$. $= a^2g$

6.10

TABLE No. 97 (*continued*).

$H = a^2($

$acd + 6$	$a^2be + 2$	$a^2b^3c + 4$	$a^3de + 2$	$a^4b^3 + 2$	$\between(x, y)^4$
$a^0bc^2 + 4$	$adf + 12$	,, $e^2 + 1$	$a^2b^2f + 4$	,, $d^2 + 6$	
,, $ef + 1$	$a^0bcf - 8$	$ac^2d - 36$	,, $bce - 6$	$a^3bcd - 2$	
	,, $c^2e - 8$	$a^0bc^3 - 24$	$acdf - 12$	$a^2b^2c^2 - 8$	
		,, $cef - 6$	$a^0bc^2f - 8$	,, $bef - 3$	
			,, $c^3e + 8$	,, $ce^2 + 1$	
				$ac^3d + 6$	
				$a^0bc^4 + 4$	
$= a^2h$				,, $c^2ef + 1$	
6.14	7.17	8.20	9.23	10.26	

$I = a^2($

$bf - 1$	$a^2b^2 - 2$	$adf - 15$	$a^3bd - 20$	$a^3de - 5$	$a^4b^3 + 2$	$a^4b^2e + 1$	$\between(x, y)^6$
$ce + 1$	$acd - 6$	$a^0bcf + 5$	$ac^2d + 60$	$a^2b^2f + 5$	,, $d^2 - 12$	$a^3bdf + 3$	
	$a^0bc^2 + 8$	,, $c^2e - 5$		,, $bce - 5$	$a^3bcd - 2$	,, $cde - 5$	
	,, $ef - 2$			$acdf + 30$	$a^2b^2c^2 - 6$	$a^2b^2cf + 1$	
				$a^0bc^2f + 5$	,, $ce^2 - 2$	,, $bc^2e - 5$	
				,, $c^3e - 5$	$ac^3d - 30$	,, $e^2f - 1$	
					$a^0bc^4 - 8$	$ac^2df - 3$	
					,, $c^2ef - 2$	$a^0bc^3f - 1$	
$= ai$						,, $c^4e + 1$	
5.11	6.14	7.17	8.20	9.23	10.26	11.29	

$J = a^4($

$a^2b^3 + 1$	$a^2b^2e - 1$	$\between(x, y)^1$
,, $d^2 + 9$	$a\,bdf - 6$	
$a^0b^2c^2 - 4$	,, $cde + 6$	
,, $bef - 2$	$a^0b^2cf - 4$	
,, $ce^2 + 1$	,, $bc^2e + 8$	
	,, $c^2f + 1$	
$= a^3j$		
8.16	9.19	

TABLE No. 97 (*continued*).

$K = a^4($	$ade + 3$	$a^2b^3 + 4$	$a^2b^2e + 1$	$a^3b^2d + 6$	$\mathfrak{X}(x, y)^3$
	$a^0b^2f + 2$	$,, d^2 - 18$	$abdf + 6$	$a^2b^3c + 2$	
	$,, bce - 2$	$abcd - 18$	$acde - 15$	$,, cd^2 - 18$	
		$a^0b^3c^2 - 16$	$a^0b^2cf - 2$	$abc^2d - 30$	
		$,, bef - 5$	$,, bc^2e - 2$	$adef - 9$	
		$,, ce^2 + 1$	$,, e^2f - 1$	$a^0b^2e^3 - 8$	
				$,, bcef - 5$	
	$= a^2k$			$,, c^2e^2 + 3$	
	7.13	8.16	9.19	10.22	

$L = a^2($	$adf - 3$	$a^3bd - 3$	$a^3de - 12$	$a^4b^3 - 6$	$a^4b^2e - 1$	$a^3b^2d + 15$	$a^3bde - 7$	$a^3b^4 - 2$	$\mathfrak{X}(x, y)^7$
	$a^0bcf - 2$	$a^2b^2c - 7$	$a^2b^2f - 9$	$,, d^2 - 39$	$a^3bdf + 39$	$a^4b^3c - 9$	$a^4b^2f - 7$	$,, bd^2 + 3$	
	$,, c^2e + 2$	$ac^2d + 42$	$,, bce + 9$	$a^2bcd + 40$	$,, cde - 14$	$,, cd^2 + 18$	$,, b^2ce + 14$	$a^3b^2cd + 10$	
		$a^0bc^3 + 28$	$acdf + 63$	$a^2b^2c^2 + 59$	$a^2b^2cf + 16$	$a^2bc^2d - 33$	$,, d^2f + 12$	$,, de^2 + 2$	
		$,, cef + 7$	$a^0bc^2f + 42$	$,, bef + 7$	$,, bc^2e - 12$	$,, def - 3$	$a^2bcdf + 23$	$a^4b^3c^2 + 13$	
			$,, c^3e - 42$	$,, ce^2 - 1$	$,, e^2f + 1$	$a^2b^2c^3 + 15$	$,, c^2de - 26$	$,, b^2ef + 4$	
				$ac^3d - 210$	$ac^2df - 105$	$,, bcef + 21$	$a^2b^2c^2f + 25$	$,, bce^2 - 2$	
				$a^0bc^4 - 140$	$a^0bc^3f - 70$	$,, c^2e^2 - 12$	$,, bc^3e - 53$	$,, c^3d^2 - 15$	
				$,, c^2ef - 35$	$,, c^4e + 70$	$ac^4d + 126$	$,, ce^2f - 7$	$a^2bc^3d - 28$	
						$a^0bc^5 + 84$	$ac^3df + 21$	$,, cdef - 7$	
						$,, c^3ef + 21$	$a^0bc^4f + 14$	$a^2b^2c^4 - 19$	
							$,, c^3e - 14$	$,, bc^2ef - 10$	
								$,, c^3e^2 + 5$	
								$ac^3d - 6$	
								$a^0bc^6 - 4$	
	$= a^2l$							$,, c^4ef - 1$	
	7.17	8.20	9.23	10.26	11.29	12.32	13.35	14.38	

$M = a^4($	$a^3b^2d - 2$	$a^3bde - 1$	$a^4b^4 - 1$	$\mathfrak{X}(x, y)^2$
	$a^2b^3c - 1$	$a^2b^2f - 1$	$,, bd^2 + 3$	
	$,, cd^2 + 9$	$,, b^2ce + 2$	$a^3b^2cd + 6$	
	$a\,bc^2d + 12$	$,, d^2f + 9$	$,, de^2 + 1$	
	$,, def + 3$	$a\,bcdf + 12$	$a^2b^3c^2 + 5$	
	$a^0b^2c^3 + 4$	$,, c^3de - 12$	$,, b^2ef + 2$	
	$,, bcef + 2$	$a^0b^2c^2f + 4$	$,, bce^2 - 1$	
	$,, c^2e^2 - 1$	$,, bc^3e - 8$	$,, c^2d^2 - 9$	
		$,, ce^2f - 1$	$a\,bc^3d - 12$	
			$,, cdef - 3$	
			$a^0b^2c^4 - 4$	
			$,, bc^2ef - 2$	
	$= a^4m$		$,, c^2e^2 + 1$	
	10.22	11.25	12.28	

TABLE No. 97 (*continued*).

$N = a^4($

$a^2b^2e + 1$	$a^2b^3c + 4$	$a^3bde + 12$	$a^4b^4 + 4$	$a^4b^3e + 2$
$a\,bdf + 6$	,, $cd^2 + 36$	$a^2b^3f + 6$	,, $bd^2 + 12$	,, $d^2e + 9$
,, $cde - 6$	$a^0b^2c^3 - 16$	,, $b^2ce - 6$	$a^3b^2cd + 8$	$a^3b^2df + 6$
$a^0b^2cf + 4$	,, $bcef - 8$	,, $d^2f + 54$	,, $de^2 + 4$	,, $bcde - 10$
,, $bc^3e - 8$	,, $c^2e^2 + 4$	$a\,bcdf + 36$	$a^2b^3c^2 - 12$	$a^2b^3cf - 2$
,, $c^2f - 1$		,, $c^2de - 36$	,, $b^2ef - 4$	,, $b^3c^2e - 11$
			,, $c^2d^2 - 108$	,, $bc^2f - 3$
			$a\,bc^2d - 96$	,, $cd^2f - 18$
			,, $cdef - 24$	,, $ce^3 + 1$
			$a^0b^2c^4 - 16$	$a\,bc^2df - 18$
			,, $bc^2ef - 8$	,, $c^3de + 18$
			,, $c^3e^2 + 4$	$a^0b^2c^2f - 4$
				,, $bc^4e + 8$
$= a^3n$				,, $ce^2f + 1$
9.19	10.22	11.25	12.28	13.31

$)(x, y)^2$

$O = a^3($

$a^2b^3e + 3$	$a^3b^3d - 6$
,, $d^2e + 9$	,, $d^3 - 54$
$a\,b^3df + 12$	$a^2b^4c - 2$
,, $bcde - 12$	,, $b^2e^2 + 1$
$a^0b^3cf + 8$	,, $bcd^2 - 18$
,, $b^2c^2e - 20$	$a\,b^2c^3d + 24$
,, $be^2f - 4$	,, $bdef + 18$
,, $ce^3 + 1$	,, $cde^2 - 12$
	$a^0b^3c^3 + 8$
	,, $b^2cef - 8$
	,, $bc^2e^2 - 10$
$= a^4o$	,, $e^3f - 1$
11.21	12.24

$)(x, y)^1$

$P = a^4($

$a^3bde - 2$	$a^4b^4 - 2$	$a^4b^3e - 5$	$a^5b^2d - 8$	$a^5b^2de + 6$	$a^6b^5 + 2$
$a^2b^3f - 1$	,, $bd^2 + 12$	,, $d^2e - 9$	,, $d^3 - 72$	$a^4b^4f + 5$	,, $b^2d^2 + 12$
,, $d^2f - 9$	$a^3b^2cd - 10$	$a^3b^2df - 24$	$a^4b^4c + 4$	,, $b^3ce - 13$	$a^3b^3cd - 2$
$a\,bcdf - 12$	,, $de^2 - 5$	,, $bcde + 44$	,, $b^2e^2 - 1$	,, $bd^2f + 21$	,, $bde^2 - 3$
,, $c^2de + 12$	$a^2b^3c^2 - 2$	$a^2b^3cf - 6$	,, $bcd^2 - 24$	,, $cd^2e - 21$	,, $cd^3 - 36$
$a^0b^2c^2f - 4$	,, $b^2e^2f - 1$	,, $b^2c^2e + 36$	$a^3b^2cd + 52$	$a^3b^2cdf - 4$	$a^4b^4c^2 - 12$
,, $bc^3e + 8$	,, $bce^2 + 3$	,, $be^2f + 6$	,, $bdef + 10$	,, $bc^3de + 10$	,, $b^3ef - 5$
,, $ce^2f + 1$	,, $c^3d^2 + 90$	,, $cd^2f + 90$	,, $cde^2 + 8$	,, $de^2f + 4$	,, $b^3ce^2 + 4$
	$a\,bc^3d + 120$	,, $ce^3 - 1$	$a^2b^3c^3 + 4$	$a^2b^3c^2f - 17$	,, $bc^3d^2 - 66$
	,, $cdef + 30$	$a\,bc^2df + 120$	,, $b^2cef - 2$	,, $b^2c^3e + 44$	,, $d^2ef - 18$
	$a^0b^3c^4 + 40$	,, $c^3de - 120$	,, $bc^2e^2 + 2$	,, $bce^2f + 13$	$a^3b^3c^3d - 10$
	,, $bc^3ef + 20$	$a^0b^3c^3f + 40$	,, $c^2d^2 - 180$	,, $c^2d^2f - 45$	,, $bcdef - 4$
	,, $c^5e^2 - 10$	,, $bc^4e - 80$	,, $e^3f + 1$	,, $c^2e^3 - 5$	,, $c^3de^2 + 1$
		,, $c^2e^3f - 10$	$a\,bc^4d - 240$	$a\,bc^3df - 60$	$a^2b^3c^4 + 14$
			,, $c^3def - 60$	,, $c^4de + 60$	,, $b^3c^2ef + 11$
			$a^0b^3c^5 - 80$	$a^0b^3c^4f - 20$	,, $bc^3e^2 - 11$
			,, $bc^3ef - 40$	,, $bc^5e + 40$	,, $c^4d^2 + 18$
			,, $c^4e^2 + 20$	,, $c^3e^2f - 5$	,, $ce^3f - 1$
					$a\,bc^5d + 26$
					,, $c^3def + 6$
					$a^0b^2c^6 + 8$
					,, $bc^4ef + 4$
$= a^4p$					,, $c^5e^2 - 2$
11.25	12.28	13.31	14.34	15.37	16.40

$)(x, y)^5$

TABLE No. 97 (*continued*).

$Q = a^6$

a^4b^5	$-\ 2$
,, b^3d^2	$+\ 18$
a^3b^3cd	$+\ 22$
,, bde^2	$+\ 3$
,, cd^3	$-\ 54$
$a^2b^4c^2$	$+\ 12$
,, b^3ef	$+\ 5$
,, b^2ce^2	$-\ 4$
,, bc^2d^2	$-\ 108$
,, d^3ef	$-\ 27$
$a\,b^2c^3d$	$-\ 72$
,, $bcdef$	$-\ 36$
,, c^3de^2	$+\ 18$
$a^0b^3c^4$	$-\ 16$
,, b^2c^2ef	$-\ 12$
,, bc^3e^2	$+\ 12$
,, ce^3f	$+\ 1$
$= a^6q$	

14 . 30

$R = a^6\,($

a^3bde	$-\ 1$	a^4b^5	$+\ 2$
a^2b^4f	$+\ 1$	,, b^3d^2	$+\ 6$
,, b^3ce	$-\ 3$	a^3b^3cd	$-\ 2$
,, bd^2f	$-\ 9$	,, bde^2	$-\ 2$
,, cd^2e	$+\ 9$	,, cd^3	$-\ 54$
$a\,b^3cdf$	$-\ 12$	$a^2b^4c^2$	$-\ 8$
,, bc^2de	$+\ 24$	,, b^3ef	$-\ 4$
,, de^2f	$+\ 3$	,, b^2ce^2	$+\ 2$
$a^0b^3c^2f$	$-\ 4$	,, bc^2d^2	$-\ 72$
,, b^3c^3e	$+\ 12$	,, d^3ef	$-\ 18$
,, bce^2f	$+\ 3$	$a\,b^3c^2d$	$-\ 24$
,, c^2e^3	$-\ 1$	,, $bcdef$	$-\ 12$
		,, c^2de^2	$+\ 6$
$= a^5r$			

13 . 27 14 . 30

$\mathfrak{Q}(x,\ y)^2$

a^4b^4e	$+\ 1$
a^3b^3df	$+\ 3$
,, b^2cde	$-\ 11$
,, d^3f	$-\ 27$
,, de^3	$-\ 1$
a^2b^4cf	$+\ 1$
,, b^3c^2e	$-\ 7$
,, b^2e^2f	$-\ 2$
,, bcd^2f	$-\ 45$
,, bce^3	$+\ 1$
,, c^2d^2e	$+\ 45$
$a\,b^2c^2df$	$-\ 24$
,, bc^3de	$+\ 48$
a^0b^3cf	$-\ 4$
,, b^2c^4e	$+\ 12$
,, bc^2e^2f	$+\ 3$
,, c^3e^3	$-\ 1$

15 . 33

$S = a^6\,($

a^4bd^2e	$+\ 9$
a^3b^3df	$+\ 7$
,, b^2cde	$-\ 12$
,, d^3f	$-\ 27$
a^2b^4cf	$+\ 2$
,, b^3c^2e	$-\ 6$
,, bcd^2f	$-\ 54$
,, c^2d^2e	$+\ 54$
$a\,b^2c^2df$	$-\ 36$
,, bc^2de	$+\ 72$
,, cde^2f	$+\ 9$
$a^0b^3c^3f$	$-\ 8$
,, b^2c^4e	$+\ 24$
,, bc^2e^2f	$+\ 6$
,, c^3e^3	$-\ 2$
$= a^6s$	

15 . 33

a^5b^4d	$+\ 15$
,, bd^3	$-\ 27$
a^4b^5c	$+\ 3$
,, b^2cd^2	$-\ 99$
,, d^3e^2	$-\ 18$
$a^3b^3c^2d$	$-\ 114$
,, b^2def	$-\ 33$
,, $bcde^2$	$+\ 12$
,, c^2d^3	$+\ 162$
$a^2b^4c^2$	$-\ 24$
,, b^3cef	$-\ 9$
,, $b^2c^2e^2$	$+\ 9$
,, bc^3d^2	$+\ 324$
,, cd^2ef	$+\ 69$
$a\,b^3c^4d$	$+\ 216$
,, bc^2def	$+\ 120$
,, c^3de^2	$-\ 54$
$a^0b^3c^5$	$+\ 48$
,, b^2c^3ef	$+\ 36$
,, bc^4e^2	$-\ 36$
,, c^2e^2f	$-\ 3$

16 . 36

a^5b^2de	$-\ 6$
,, d^3e	$-\ 27$
a^4b^3f	$-\ 3$
,, b^4ce	$+\ 9$
,, b^2d^2f	$-\ 9$
,, bcd^2e	$-\ 18$
a^3b^3cdf	$-\ 9$
,, b^2c^2de	$+\ 24$
,, bde^2f	$+\ 3$
,, cd^3f	$+\ 81$
,, cde^3	$-\ 3$
$a^2b^4c^2f$	$+\ 6$
,, b^3c^3e	$-\ 18$
,, b^2ce^2f	$-\ 9$
,, bc^3d^2f	$+\ 162$
,, bc^2e^3	$+\ 3$
,, c^2d^2e	$-\ 162$
$a\,b^2c^3df$	$+\ 108$
,, bc^4de	$-\ 216$
,, c^2de^2f	$-\ 27$
$a^0b^3c^4f$	$+\ 24$
,, b^2c^5e	$-\ 72$
,, bc^3e^2f	$-\ 18$
,, c^4e^3	$+\ 6$

17 . 39

$\mathfrak{Q}(x,\ y)^3$

a^6b^6	$-\ 2$
,, b^3d^2	$+\ 9$
,, d^4	$-\ 27$
a^5b^4cd	$+\ 18$
,, b^3de^2	$+\ 6$
,, bcd^3	$-\ 54$
$a^4b^5c^2$	$+\ 15$
,, b^4ef	$+\ 6$
,, $b^3c^2d^2$	$-\ 36$
,, b^5ce^2	$-\ 6$
,, $b^2c^3d^2$	$-\ 27$
,, bd^2ef	$-\ 9$
,, cd^2e^2	$-\ 9$
$a^5b^3c^3d$	$-\ 54$
,, b^2cdef	$-\ 27$
,, bc^2de^2	$+\ 3$
,, c^3d^3	$-\ 54$
,, de^3f	$-\ 2$
$a^2b^4c^4$	$-\ 24$
,, b^3ce^2f	$-\ 21$
,, $b^3c^3e^2$	$+\ 21$
,, bc^4d^2	$-\ 108$
,, bce^3f	$+\ 2$
,, c^3d^2ef	$-\ 27$
$a\,b^3c^3d$	$-\ 72$
,, bc^3def	$-\ 36$
,, c^4de^2	$+\ 18$
$a^0b^3c^6$	$-\ 16$
,, b^2c^4ef	$-\ 12$
,, bc^5e^2	$+\ 12$
,, c^3e^3f	$+\ 1$

18 . 42

TABLE No. 97 (*continued*).

$T = ($

a^5b^4de	$+\ 7$
,, bd^3e	$+\ 27$
a^4b^6f	$+\ 1$
,, b^5ce	$-\ 2$
,, b^3d^2f	$+\ 24$
,, b^3cd^2e	$-\ 54$
,, d^4f	$-\ 81$
,, d^2e^3	$-\ 6$
a^3b^4cdf	$+\ 16$
,, b^3c^2de	$-\ 76$
,, b^2de^2f	$-\ 12$
,, bcd^3f	-216
,, $bcde^3$	$+\ 5$
,, c^3d^3e	$+216$
$a^2b^4c^3e$	$-\ 8$
,, $b^3c^2d^2f$	-216
,, $b^2c^2e^3$	$+\ 2$
,, bc^3d^2e	$+432$
,, cd^3e^2f	$+\ 54$
$a\ b^3c^3df$	$-\ 96$
,, b^3c^4de	$+288$
,, bc^2de^2f	$+\ 72$
,, c^3de^3	$-\ 24$
$a^0b^4c^4f$	$-\ 16$
,, b^3c^5e	$+\ 64$
,, $b^3c^3e^2f$	$+\ 24$
,, bc^4e^3	$-\ 16$
,, c^2e^4f	$-\ 1$
$= a^8t$	

19 . 41

a^6b^7	$+\ 2$
,, b^4d^2	$+\ 6$
a^5b^5cd	$-\ 10$
,, b^3de^2	$-\ 8$
,, b^3cd^2	$-\ 54$
,, d^3e^2	$-\ 27$
$a^4b^6c^2$	$-\ 14$
,, b^5ef	$-\ 7$
,, b^4ce^2	$+\ 9$
,, $b^3c^2d^2$	$-\ 84$
,, b^3d^2ef	$-\ 27$
,, bcd^2e^2	$+\ 9$
,, c^2d^4	$+162$
$a^3b^4c^3d$	$-\ 8$
,, b^3cdef	$+\ 4$
,, $b^2c^2de^2$	$+\ 18$
,, bc^3d^3	$+432$
,, bde^3f	$+\ 3$
,, cd^3ef	$+108$
,, cde^4	$-\ 1$
$a^2b^5c^4$	$+\ 16$
,, b^4c^2ef	$+\ 20$
,, $b^3c^3e^2$	$-\ 24$
,, $b^2c^4d^2$	$+432$
,, b^2ce^3f	$-\ 5$
,, bc^2d^2ef	$+216$
,, bc^2e^4	$+\ 1$
,, $c^3d^2e^2$	-108
$a\ b^3c^5d$	$+192$
,, b^2c^3def	$+144$
,, bc^4de^2	-144
,, c^2de^3f	$-\ 12$
$a^0b^4c^6$	$+\ 32$
,, b^3c^4ef	$+\ 32$
,, $b^2c^5e^2$	$-\ 48$
,, bc^3e^3f	$-\ 8$
,, c^4e^4	$+\ 2$

20 . 44

$)(x, y)^1$

$U =$

a^5b^6d	$-\ 3$
,, b^3d^3	$+\ 14$
,, d^5	$-\ 27$
a^4b^7c	$-\ 1$
,, b^4cd^2	$+\ 34$
,, $b^2d^2e^2$	$+\ 11$
,, bcd^4	$-\ 81$
$a^3b^5c^2d$	$+\ 32$
,, b^4def	$+\ 10$
,, b^3cde^2	$-\ 6$
,, $b^2c^2d^3$	-144
,, bd^3ef	$-\ 18$
,, cd^3e^2	$-\ 18$
$a^2b^6c^2$	$+\ 8$
,, b^5cef	$+\ 4$
,, $b^4c^2e^2$	$-\ 6$
,, $b^3c^3d^2$	-152
,, b^3cd^2ef	$-\ 60$
,, $bc^2d^2e^2$	$+\ 6$
,, d^2e^3f	$-\ 4$
$a\ b^4c^4d$	$-\ 80$
,, b^3c^2def	$-\ 56$
,, $b^3c^3de^2$	$+\ 48$
,, $bcde^3f$	$+\ 2$
,, c^2de^4	$+\ 1$
$a^0b^5c^5$	$-\ 16$
,, b^4c^3ef	$+\ 16$
,, $b^3c^4e^2$	$+\ 24$
,, $b^2c^2e^3f$	$+\ 4$
,, bc^3e^4	$+\ 1$
$= a^9u$	

21 . 45

TABLE No. 97 (*concluded*).

$V = a^{10}($

a^6b^5	$-\ 4$	a^3b^7e	$-\ 2$
,, b^5d^2	$-\ 12$	,, b^4d^3e	$-\ 48$
a^5b^6cd	$+\ 20$	,, bd^4e	$-\ 162$
,, b^4de^2	$+\ 23$	a^3b^6df	$-\ 6$
,, b^3cd^3	$+\ 108$	,, b^5cde	$+\ 8$
,, bd^3e^2	$+\ 81$	,, b^3d^3f	$-\ 144$
$a^4b^7c^2$	$+\ 28$	,, b^3de^3	$+\ 8$
,, b^5ef	$+\ 15$	,, b^3cd^3e	$+\ 324$
,, b^5ce^2	$-\ 20$	,, d^5f	$+\ 486$
,, $b^4c^3d^2$	$+\ 168$	,, d^3e^3	$+\ 63$
,, b^3d^2ef	$+\ 78$	a^4b^7cf	$-\ 2$
,, $b^3cd^2e^2$	$-\ 72$	,, b^6c^2e	$+\ 18$
,, bc^2d^4	$-\ 324$	,, b^5e^2f	$+\ 7$
,, d^4ef	$-\ 81$	,, b^4cd^2f	$-\ 144$
,, d^2e^4	$-\ 6$	,, b^4ce^3	$-\ 9$
$a^3b^5c^3d$	$+\ 16$	,, $b^3c^2d^2e$	$+\ 648$
,, b^4cdef	$+\ 8$	,, $b^2d^3e^2f$	$+\ 99$
,, $b^3c^2de^2$	$-\ 112$	,, bcd^4f	$+\ 1458$
,, $b^2c^3d^3$	$-\ 864$	,, bcd^2e^3	$-\ 27$
,, b^3de^3f	$-\ 18$	,, c^2d^3e	$-\ 1458$
,, bcd^3ef	$-\ 432$	$a^3b^5c^2df$	$-\ 32$
,, $bcde^4$	$+\ 7$	,, b^4c^3de	$+\ 208$
,, $c^2d^3e^2$	$+\ 216$	,, b^3cde^2f	$+\ 20$
$a^2b^6c^4$	$-\ 32$	,, $b^3c^2d^2f$	$+\ 1728$
,, b^5c^2ef	$-\ 40$	,, $b^2c^2de^3$	$-\ 40$
,, $b^4c^3e^2$	$+\ 40$	,, bc^3d^3e	$-\ 3456$
,, $b^3c^4d^2$	$-\ 864$	,, bde^4f	$-\ 3$
,, b^3ce^3f	$+\ 10$	,, cd^3e^2f	$-\ 432$
,, $b^2c^2d^2ef$	$-\ 648$	,, cde^5	$+\ 1$
,, $bc^3d^2e^2$	$+\ 648$	$a^2b^4c^2e^2f$	$-\ 20$
,, cd^2e^3f	$+\ 54$	,, $b^3c^3d^2f$	$+\ 1008$
$a\ b^4c^2d$	$-\ 384$	,, $b^3c^3e^3$	$+\ 20$
,, b^3c^3def	$-\ 384$	,, $b^2c^4d^2e$	$-\ 3024$
,, $b^2c^4de^2$	$+\ 576$	,, b^3ce^4f	$+\ 5$
,, bc^2de^3f	$+\ 96$	,, $bc^3d^2e^2f$	$-\ 756$
,, c^3de^4	$-\ 24$	,, bc^2e^5	$-\ 1$
$a^2b^5c^6$	$-\ 64$	,, $c^3d^2e^3$	$+\ 252$
,, b^4c^4ef	$-\ 80$	$a\ b^4c^4df$	$+\ 288$
,, $b^3c^3e^2$	$+\ 160$	,, b^3c^3de	$-\ 1152$
,, $b^2c^3e^3f$	$+\ 40$	,, $b^2c^3de^2f$	$-\ 432$
,, bc^4e^4	$-\ 20$	,, bc^4de^3	$+\ 288$
,, c^2e^5f	$-\ 1$	,, c^2de^4f	$+\ 18$
		a^2b^3cf	$+\ 32$
		,, b^4c^2e	$-\ 160$
		,, $b^3c^4e^2f$	$-\ 80$
		,, $b^2c^2e^3$	$+\ 80$
		,, bc^3e^4f	$+\ 10$
$= a^{11}v$		,, c^4e^5	$-\ 2$
22 . 46		23 . 49	

$\mathfrak{X}(x,\ y)^2$

TABLE No. 98. Covariants of A, divided and (except as to a few coefficients) segregate.

A and B as given in Table 97 were divided and segregate.

C was divided but not segregate: the divided and segregate form is

$C = ($	$c+1$	$f+3$	a^2b+3 a^0c^2-15	a^2e+1 a^0cf-10	a^3d+6 a^0bc-3 $,, c^3+15$	a^3i+3 a^0c^2f+3	a^4b^2-1 $,, h+1$ a^3cd-4 a^0c^4-1	$\mathfrak{X}(x,y)^6.$
	2.6	3.9	4.12	5.15	6.18	7.21	8.24	

D divided and segregate is

$\div 3$

$D = a^3($	$d+1$	$i+1$	ab^2-1 $,, h+1$ a^0cd-3	$al-1$ a^0ci-1	$\mathfrak{X}(x,y)^3,$
	3.3	4.6	5.9	6.12	

an integer non-segregate form of the fractional coefficient is

$ci-1$ $df+1$

E was divided but not segregate: the divided and segregate form is

$E = ($	$e+1$	$ad-6$ a^0bc-10	$ai+12$ a^0ce-10	a^2b^2-8 $,, h-2$ $acd-24$ a^0bc^2+20	a^2be-5 $,, l+8$ $aci-12$ a^0c^2e+5	a^4g-1 a^3bd+6 a^2b^2c+2 $,, ch+2$ ac^2d+6 a^0bc^3-2	$\mathfrak{X}(x,y)^5.$
	3.5	4.8	5.11	6.14	7.17	8.20	

TABLE No. 98 (*continued*).

F was divided but not segregate: the divided and segregate form is

						$\div\ 3$				
$F=($	$f+1$	a^2b+2	a^2e+1	a^3d+34	a^3i+40	a^4b^2-16	a^4be-21	a^5g-1	a^5k-4	a^6b^3-2 $)(x,y)^9$,
		a^0c^2-18	a^0cf-36	a^2bc-42	a^2ce-35	,, $h-5$	a^3l-8	a^5bd+18	a^5bi+1	,, $bh+3$
				a^0c^3+168	a^0c^2f+126	a^3cd+46	,, $ci-16$	a^4b^2c-18	a^4bce+2	,, $cg-1$
						a^2bc^2+155	a^2c^2e+189	,, $ch+38$	,, $cl-8$	$a^4b^2c^2+4$
						a^0c^4+252	a^0c^3f-252	a^3c^2d-174	a^3c^2i-16	,, c^2h-5
								a^2bc^3-86	a^2c^3e-13	a^3c^3d+16
								a^0c^5+72	a^0c^4f+9	a^2bc^4+4
										a^0c^6-2
	3.9	4.12	5.15	6.18	7.21	8.24	9.27	10.30	11.33	12.36

where for an integer non-segregate value of the fractional coefficient, see the original form of F.

G as an invariant was divided and segregate, $G = a^6\ g$.
4.0

H divided and segregate is

				$\div\ 3$	$\div\ 3$	
$H=a^4($	$h+1$	$be+2$	a^2g+1	a^3k+2	a^3j+2	$)(x,y)^4$,
		$l-4$	$abd-12$	$abi-8$	a^2b^3+4	
			a^0ch-6	a^0bce-6	,, $bh-5$	
				,, $cl+12$	,, $cg+1$	
					$abcd+12$	
					a^0c^2h+3	
	4.4	5.7	6.10	7.13	8.16	

where the fractional coefficients are =

$ade+2$	a^2b^3+2
a^0b^2f+4	,, d^2+6
,, $bce-6$	$abcd-2$
,, $cl+4$	$a^0b^2c^2-8$
	,, $bef-3$
	,, c^2h+1
	,, ce^2+1

TABLE No. 98 (*continued*).

I divided and segregate is

					$\div 3$	$\div 3$	$\div 3$	
$I = a^3($	$i+1$	$ab^2 - 2$	$al + 5$	$a^2bd - 20$	$a^3k - 5$	$a^4j - 4$	$a^3b^2e + 3$	$\between x, y)^6,$
		$,, h + 2$	$a^0ci - 15$	$a^0c^2d + 60$	$a^2bi - 25$	$a^3b^3 + 10$	$,, bl + 9$	
		$a^0cd - 18$			$acl - 30$	$,, bh - 8$	$,, ck - 5$	
					$a^0c^2i + 45$	$,, cg + 4$	$,, fg - 3$	
						$a^2bcd - 6$	$a^2bci - 8$	
						$a\,bc^2 - 18$	$a\,b^2cf - 9$	
						$,, c^2h - 6$	$,, bc^2e - 15$	
						$a^0c^3d - 54$	$,, c^3l + 3$	
							$a^0c^3i - 3$	
	4.6	5.9	6.12	7.15	8.18	9.21	10.24	

where the fractional coefficients are =

$a^2de - 5$	$a^3b^3 + 2$	$a^3b^2e + 1$
$a\,b^2f + 5$	$,, d^2 - 12$	$a^2bdf + 3$
$,, bce - 5$	$a^2bcd - 2$	$,, cde - 5$
$a^0c^2i - 5$	$a\,b^3c^2 - 6$	$a\,b^2cf + 1$
$,, cdf + 30$	$,, c^2h - 2$	$,, bc^2e - 5$
	$,, ce^2 - 2$	$,, e^2f - 1$
	$a^0c^3d - 18$	$a^0c^3i + 1$
		$,, c^2df - 3$

J divided and segregate is

$$J = a^7\,(j, -n\between x, y)^1.$$
$$5.6 \quad 6.4$$

K divided and segregate is

$K = a^6($	$k+1$	$aj - 2$	$an + 1$	$a^2m - 3$	$\between x, y)^3.$
		$a^0b^3 + 6$	$a^0ck - 3$	$acj + 1$	
		$,, bh - 9$		$a^0b^3c - 2$	
		$,, cg + 3$		$,, bch + 3$	
				$,, c^2g - 1$	
	5.3	6.6	7.9	8.12	

TABLE No. 98 (*continued*).

L divided and (as to first six coefficients) segregate is

				÷ 3	÷ 3			
$L = a^4($	$l+1$	$abd-3$	a^2k-4	a^3j-13	a^3n-3	a^4m-1	a^3bde-7	a^4b^4-2
		a^0b^3c-7	$abi+1$	a^2b^3-5	a^2bl-45	a^3b^2d+13	a^2b^3f-7	,, bd^2+3
		,, $ch+7$	a^0cl-21	,, $bh-5$	,, $ck-20$	,, $cj+3$	,, b^2ce+14	a^3b^2cd+10
				,, $cg+10$	$abci-10$	a^2b^3c-13	,, d^2f+12	,, de^2+2
				$abcd+30$	a^0c^2l+105	,, $bch+29$	$abcdf+23$	$a^2b^3c^2+13$
				$a^0b^2c^2+105$		,, c^2g-16	,, c^3de-24	,, b^2ef+4
				,, $ch-105$		abc^2d-3	a^0b^2cf+25	,, bce^2-2
						$a^0b^2c^3-21$	,, bc^2e-55	,, c^2d^2-15
						,, c^3h+21	,, c^3l-3	abc^3d-28
							,, c^2fh+2	,, $cdef-7$
							,, ce^2f-7	$a^0b^2c^4-19$
								,, bc^2ef-10
								,, c^4h-1
								,, c^3e^2+5
	5 . 7	6 . 10	7 . 13	8 . 16	9 . 19	10 . 22	11 . 25	12 . 28

$(x, y)^7$,

where the fractional coefficients are =

a^2b^3-6	a^2b^2e-1
,, d^2+3	$abdf+39$
$abcd+26$	,, $cde-22$
$a^0b^2c^2+31$	a^0b^2cf+16
,, $bef+7$	,, bc^2e-4
,, c^2h-7	,, c^2l+19
,, ce^2-1	,, $cfh-8$
,, $fl-14$	,, e^2f+1

the last two coefficients have not been reduced to the segregate form.

M divided and segregate is

$M = a^3($	$m+1$	$bk-1$	$abj-1$	$(x, y)^2$.
		$p-1$	$adg+1$	
			,, $^0cm-1$	
	6 . 2	7 . 5	8 . 8	

TABLE No. 98 (*continued*).

N divided and segregate is

$N = a^7 ($

$n + 1$	$cj + 4$	$ap - 6$ $cn - 6$	$a^2bj - 4$,, $dg + 4$ $ab^4 + 8$,, $b^2h - 12$,, $bcg + 4$,, $cm - 8$ $a^0c^2j - 4$	$a^3o + 1$ $a^2bn - 1$ $acp + 2$ $a^0c^2n + 1$
6 . 4	7 . 7	8 . 10	9 . 13	10 . 16

$\mathfrak{X}(x, y)^4$.

O divided and segregate is

$O = a^{10} ($

$o + 1$	$b^2g + 1$ $bm + 6$ $df - 6$ $gh - 1$
7 . 1	8 . 4

$\mathfrak{X}(x, y)^1$.

P divided and (as to first three coefficients) segregate is

$P = a^8 ($

$p + 1$	$abj + 8$,, $dg - 5$ $a^0b^4 - 14$,, $b^2h + 15$,, $bcg - 5$,, $cm + 10$	$a^2o + 7$ $abn - 2$ $a^0cp - 14$	$a^2bm + 8$,, $dj + 3$ $ab^3d + 9$,, $bcj + 13$,, $bdh - 9$,, $cdg - 12$,, $d^3 - 81$ $a^0b^4c - 5$,, $b^2ch - 8$,, $b^2e^2 - 1$,, $bc^2g - 1$,, $bcd^2 - 9$,, $bel - 1$,, $bfk + 1$,, $c^2m - 12$,, $dei + 9$,, $e^2h + 1$,, $fp - 2$	$a^2b^2k - 3$,, $bp - 3$,, $em + 2$ $ab^2de + 3$,, $bdl + 3$,, $bfj + 8$,, $d^2i - 27$,, $deh - 3$,, $dfg - 8$ $a^0b^3ce - 7$,, $b^2cl + 1$,, $bc^3k + 9$,, $bcdi - 9$,, $bceh + 10$,, $c^2eg - 3$,, $c^2p + 9$,, $cd^2e + 18$,, $cfm + 4$,, $efk + 3$	$a^3b^2j + 2$,, $bdg - 2$ $a^2bcm + 2$,, $bek - 1$,, $ep - 1$ $ab^3cd + 24$,, $bc^2j - 6$,, $bcdh - 33$,, $c^2dg + 15$,, $cd^3 - 54$,, $dfk - 9$ $a^0b^4c^2 + 8$,, $b^2c^2h - 11$,, $bc^3g + 3$,, $bc^3d^2 - 18$,, $bcfk - 1$,, $c^3m - 6$,, $cfp + 2$
7 . 5	8 . 8	9 . 11	10 . 14	11 . 17	12 . 20

$\mathfrak{X}(x, y)^5$,

the last three coefficients have not been reduced to the segregate form.

TABLE No. 98 (*continued*).

Q as an invariant was divided and segregate, $Q = a^{12}\, q$.

8.0

R divided and segregate is

		$\div 2$	$\div 3$	
$R = a^{11}($	$r + 1$	$aq \quad + 1$	$abo \quad - 1$	$\between(x, y)^3,$
		$a^0b^2j \quad + 6$	$,, gk \quad - 1$	
		$,, bdg \quad - 5$	$,, s \quad + 3$	
		$,, hj \quad - 1$	$a^0cr \quad - 3$	
	8.2	9.5	10.8	

where the fractional coefficients are =

$b^2j \quad + 2$	$bdk + 3$
$bdg - 2$	$bej \quad + 1$
$dm - 6$	$cr \quad + 1$
	$deg - 1$
	$dp \quad + 3$

S divided and (as to the first three coefficients) segregate is

				$\div 2$	
$S = a^{12}($	$s + 1$	$agj \quad - 2$	$a\,br - 1$	$a^2bq \quad + 4$	$\between(x, y)^3,$
		$a^0b^3g \quad + 2$	$,, do - 1$	$a\,b^3j \quad + 4$	
		$,, b^2m \quad + 3$	$a^0cs - 3$	$,, b^3dg \quad - 4$	
		$,, bdj \quad + 21$		$,, bdm \quad - 31$	
		$,, bgh \quad - 4$		$,, d^2j \quad - 3$	
		$,, cg^2 \quad + 2$		$a^0b^6 \quad + 4$	
		$,, cq \quad - 3$		$,, b^3d^2 \quad + 16$	
				$,, bd^2h \quad - 24$	
				$,, den \quad + 4$	
				$,, fs \quad - 1$	
	9.3	10.6	11.9	12.12	

but the last coefficient is neither segregate nor integer.

TABLE No. 98 (*concluded*).

T divided and segregate is

$$T = a^{16}\left(\begin{array}{c|c} & \div 2 \\ \hline t + 1 & \begin{array}{l} bgm + 1 \\ bj^2 + 4 \\ dgj - 3 \\ hq - 1 \end{array} \\ \hline 11.1 & 12.4 \end{array}\right)(x, y)^1,$$

where the fractional coefficient is =

$$\begin{array}{|l|} \hline b^2q - 1 \\ hq + 1 \\ m^2 + 6 \\ \hline \end{array}$$

U as an invariant was divided and segregate, $U = a^{18}\ u.$
12.0

V divided and segregate is

$$V = a^{19}\left(\begin{array}{c|c} & \div 6 \\ \hline v + 1 & \begin{array}{l} bgr - 5 \\ bjo - 10 \\ gjk + 5 \\ js - 12 \\ nq - 9 \end{array} \\ \hline 13.1 & 14.4 \end{array}\right)(x, y)^1,$$

where the fractional coefficient is =

$$\begin{array}{|l|} \hline dt - 6 \\ mr - 6 \\ nq + 1 \\ \hline \end{array}$$

W as an invariant was divided and segregate, $W = a^{27}\ w.$
18.0

Derivatives. Art. Nos. 382 to 384, and Tables Nos. 99 and 100.

382. I call to mind that any two covariants a, b, the same or different, give rise to a set of derivatives $(a, b)^1$, $(a, b)^2$, $(a, b)^3$, &c., or, as I propose to write them, $ab1$, $ab2$, $ab3$, &c., viz.:

$$\begin{aligned}
ab1 &= d_x a \,.\, d_y b - \quad d_y a \,.\, d_x b,\\
ab2 &= d_x^2 a \,.\, d_y^2 b - 2d_x d_y a \,.\, d_x d_y b + \quad d_y^2 a \,.\, d_x^2 b,\\
ab3 &= d_x^3 a \,.\, d_y^3 b - 3d_x^2 d_y a \,.\, d_x d_y^2 b + 3d_x d_y^2 a \,.\, d_x^2 d_y b - d_y^3 a \,.\, d_x^3 b,\\
&\&c.;
\end{aligned}$$

or, as these are symbolically written,

$$ab1 = \overline{12}a_1 b_2, \quad ab2 = \overline{12}^2 a_1 b_2, \quad ab3 = \overline{12}^3 a_1 b_2, \ \&c.;$$

where

$$12 = \xi_1\eta_2 - \xi_2\eta_1, \quad = \frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1},$$

the differentiations $\frac{d}{dx_1}$, $\frac{d}{dy_1}$ applying to the a_1 and the $\frac{d}{dx_2}$, $\frac{d}{dy_2}$ applying to the b_2, but the suffixes being ultimately omitted: hence if θ be the index of derivation, the derivative is thus a linear function of the differential coefficients of the order θ of the two covariants a and b respectively: and we have the general property that any such derivative, if not identically vanishing, is a covariant. If the a and the b are one and the same covariant, then obviously every odd derivative is $=0$; so that in this case the only derivatives to be considered are the even derivatives $aa2$, $aa4$, &c.: moreover, if the index of derivation θ exceeds the order of either of the component covariants, then also the derivative is $=0$: in particular, neither of the covariants must be an invariant. The degree of the derivative is evidently equal to the sum of the degrees of the component covariants; the order is equal to the sum of the orders less twice the index of derivation.

383. It was by means of the theory of derivatives that Gordan proved (for a binary quantic of any order) that the number of covariants was finite, and, in the particular case of the quintic, established the system of the 23 covariants. Starting from the quantic itself a, then the system of derivatives $aa2$, $aa4$, &c., must include among itself all the covariants of the second degree, and if the entire system of these is, suppose, b, c, &c., then the derivatives $ab1$, $ab2$, &c., $ac1$, $ac2$, &c., must include among them all the covariants of the third degree, and so on for the higher degrees; and in this way, limiting by general reasoning the number of the independent covariants of each degree obtained by the successive steps, the foregoing conclusion is arrived at. But returning to the quintic, and supposing the system of the 23 covariants established, then knowing the deg-order of a derivative we know that it must be a linear function of the segregates of that deg-order; and we thus confirm, *à posteriori*, the results of the derivation theory. I annex the following Table No. 99, showing all the derivatives which present themselves, and for each of them the

covariants as well congregate as segregate of the same deg-order: the congregates are distinguished each by two prefixed dots, $..bf$, &c. No further explanation of the arrangement is, I think, required. We see from the table in what manner the different covariants present themselves in connexion with the derivation-theory. Thus starting with the quintic itself a, we have the two derivatives $aa4$, $aa2$, which are in fact the covariants of the second degree (deg-orders 2.2 and 2.6 respectively) b and c. For the third degree we have the derivatives $ab2$, $ab1$, $ac5$, $ac4$, $ac3$, $ac2$, $ac1$: the deg-order of $ac5$ is 3.1, and there being no covariants of this deg-order, $ac5$ must, it is clear, vanish identically: $ab2$ and $ac4$ are each of them of the deg-order 3.3, but for this deg-order we have only the covariant d, and hence $ab2$ and $ac4$ must be each of them a numerical multiple of d; similarly, deg-order 3.5, $ab1$ and $ac3$ must be each of them a numerical multiple of e; deg-order 3.7, $ac2$ must be a numerical multiple of ab; and deg-order 3.9, $ac1$ must be a numerical multiple of f: the 7 derivatives, which *primâ facie* might give, each of them, a covariant of the third degree, thus give in fact only the 3 covariants d, e, f; and in order to show according to the theory of derivations that this is so, it is necessary to prove—1°, that $ac5=0$; 2°, that $ac4$ and $ab2$ differ only by a numerical factor; 3°, that $ab1$ and $ac3$ differ only by a numerical factor; 4°, that $ac2$ is a numerical multiple of ab: which being so, we have the 3 new covariants. The table shows that

for degrees	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,	20,	21,	22,	23,	24
No. of derivatives =	2,	7,	19,	29,	41,	46,	52,	46,	44,	35,	26,	19,	17,	12,	13,	6,	6,	3,	3,	1,	1,	0,	1

so that the whole number of derivatives is 429, giving the 22 covariants $b, c, \ldots, w$. While it is very remarkable that (by general reasoning, as already mentioned, and with a very small amount of calculation) Gordan should have been able in effect to show this, the great excess of the number of derivatives over that of the covariants seems a reason why the derivations ought not to be made a basis of the theory.

It is to be remarked that we may consider derivatives $pq1$, $pq2$, &c., where p, q instead of being simple covariants are powers or products of covariants, but that these may be made to depend upon the derivatives formed with the simple covariants. (As to this see my paper "On the Derivatives of Three Binary Quantics," *Quart. Math. Journal*, t. xv. (1877), pp. 157—168, [681].)

TABLE No. 99 (Index Table of Derivatives).

Deg.	2				3				
Ord.	0	2	4	6	1	3	5	7	9
		b		c		d	e	ab	f
	aa	4		2	ab	2	1		
					ac	5	4	3	2	1
	2 derivs.				7 derivs.				

TABLE No. 99 (*continued*).

Deg.	4						
Ord.	0	2	4	6	8	10	12
	g		b^3	i	ad	ae	a^2b
			h		bc		c^2
ad		3	2	1			
ae	5	4	3	2	1		
af			5	4	3	2	1
bb	2						
bc			2	1			
cc	6		4		2		

19 derivs.

Deg.	5						
Ord.	1	3	5	7	9	11	13
	j	k	ag	be	ab^2	ai	a^2d
			bd	l	ah	.. bf	abc
					cd	ce	
ah	4	3	2	1			
ai	5	4	3	2	1		
bd	2	1					
be		2	1				
bf				2	1		
cd		3	2	1			
ce	5	4	3	2	1		
cf		6	5	4	3	2	1

29 derivs.

Deg.	6							
Ord.	0	2	4	6	8	10	12	14
		bg	n	aj	ak	a^2g	abe	a^2b^2
		m		b^3	bi	abd	al	a^2h
				bh	.. de	b^2c	ci	acd
				cg		ch	.. df	bc^2
				.. d^2		.. e^2		.. ef
aj			1					
ak		3	2	1				
al		5	4	3	2	1		
bh		2	1					
bi			2	1				
ch		4	3	2	1			
ci	6	5	4	3	2	1		
dd		2						
de		3	2	1				
df				3	2	1		
ee		4		2				
ef			5	4	3	2	1	
ff		8		6		4		2

41 derivs.

TABLE No. 99 (*continued*).

Deg.	7						
Ord.	1	3	5	7	9	11	13
	o	bj	bk	abg	an	a^2j	a^2k
		dg	eg	am	$..b^2e$	ab^3	abi
			p	b^2d	bl	abh	$..ade$
				cj	ck	acg	$..b^2f$
				$..dh$	$..di$	$..ad^2$	bce
					$..eh$	bcd	cl
					fg	$..ei$	$..fh$
am		2	1				
an	4	3	2	1			
bj	1						
bk	2	1					
bl			2	1			
cj			1				
ck		3	2	1			
cl	6	5	4	3	2	1	
dh	3	2	1				
di		3	2	1			
eh	4	3	2	1			
ei	5	4	3	2	1		
fh			4	3	2	1	
fi		6	5	4	3	2	1

46 derivs.

TABLE No. 99 (*continued*).

Deg.	8							
Ord.	0	2	4	6	8	10	12	14
	g^2	r	b^2g	as	abj	abk	a^2bg	a^2n
	q		bm	bn	adg	aeg	a^2m	. . ab^2e
			dj	. . dk	b^4	ap	ab^2d	abl
			gh	. . ej	b^2h	b^2i	acj	ack
				gi	bcg	. . bde	. . adh	. . adi
					. . bd^2	cn	b^3c	. . aeh
					cm	. . dl	bch	afg
					. . ek	. . fj	. . be^2	bci
					. . h^2	. . hi	c^2g	. . bdf
							. . cd^2	. . cde
							. . el	
							. . fk	
							. . i^2	
ao			1					
ap	5	4	3	2	1			
bm	2	1						
bn		2	1					
cm			2	1				
cn		4	3	2	1			
dj		1						
dk	3	2	1					
dl			3	2	1			
ej			1					
ek		3	2	1				
el		5	4	3	2	1		
fj					1			
fk				3	2	1		
fl		7	6	5	4	3	2	1
hh	4		2					
hi		4	3	2	1			
ii	6		4		2			

52 derivs.

TABLE No. 99 (*continued*).

Deg.	9					
Ord.	1	3	5	7	9	11
	gj	bo gk s	ag^2 aq b^2j bdg . . dm hj	ar . . b^2k beg bp co . . dn . . em gl . . hk . . ij	ab^2g abm adj agh b^3d bcj . . bdh cdg . . d^3 . . en . . ik	a^2o abn . . adk . . aej agi . . b^3e b^2l bck . . bdi . . beh . . bfg ceg cp . . d^2e . . fm . . hl
ar		2	1			
bo	1					
bp		2	1			
co			1			
cp	5	4	3	2	1	
dm	2	1				
dn	3	2	1			
em		2	1			
en	4	3	2	1		
fm				2	1	
fn			4	3	2	1
hj		1				
hk	3	2	1			
hl		4	3	2	1	
ij			1			
ik		3	2	1		
il	6	5	4	3	2	1

46 derivs.

TABLE No. 99 (*continued*).

Deg.	10						
Ord.	0	2	4	6	8	10	12
		bg^2	br	agj	abo	a^2g^2	a^2r
		bq	. . do	b^3g	agk	a^2q	. . ab^2k
		gm	gn	b^2m	as	ab^2j	$abeg$
		j^2	jk	bdj	b^2n	$abdg$	abp
				bgh	. . bdk	. . adm	aco
				cg^2	. . bej	ahj	. . adn
				cq	bgi	. . b^5	. . aem
				. . d^2g	cr	b^3h	agl
				. . eo	. . deg	b^2cg	. . ahk
				. . hm	. . dp	. . b^2d^2	. . aij
				. . k^2	. . hn	bcm	b^3i
					. . im	. . bek	. . b^2de
					. . jl	. . bh^2	bcn
						cdj	. . bdl
						cgh	. . bfj
						. . d^2h	. . bhi
						. . e^2g	. . cdk
						. . ep	. . cej
						. . fo	cgi
						. . in	. . d^2i
						. . kl	. . deh
							. . dfg
as		3	2	1			
br	2	1					
cr			2	1			
do		1					
dp		3	2	1			
eo			1				
ep	5	4	3	2	1		
fo					1		
fp			5	4	3	2	1
hm		2	1				
hn	4	3	2	1			
im			2	1			
in		4	3	2	1		
jk		1					
jl				1			
kk		2					
kl			3	2	1		
ll		6		4		2	

44 derivs.

TABLE No. 99 (*continued*).

Deg.	11				
Ord.	1	3	5	7	9
	go	bgj	b^2o	abg^2	abr
	t	dg^2	bgk	abq	. . ado
		dq	bs	agm	agn
		jm	. . dr	aj^2	ajk
			eg^2	b^2j	. . b^3k
			eq	b^2dg	. . b^3eg
			gp	. . bdm	b^3p
			. . ho	bhj	bco
			. . jn	cgj	. . bdn
			. . km	. . d^2j	. . bem
				. . dgh	bgl
				. . er	. . bhk
				. . io	. . bij
				. . kn	cjk
					cs
					. . d^2k
					. . dej
					. . dgi
					. . egh
					fg^2
					fq
					. . hp
					. . lm
bs	2	1			
cr		3	2	1	
dr	2	1			
er		2	1		
fr				2	1
ho		1			
hp	4	3	2	1	
io			1		
ip	5	4	3	2	1
jm	1				
jn		1			
km	2	1			
kn	3	2	1		
lm	2	1			
ln		4	3	2	1

35 derivs.

TABLE No. 99 (*continued*).

Deg.	12					
Ord.	0	2	4	6	8	10
	g^3	gr	b^2g^2	ago	$abgj$	ab^2o
	g^9	jo	b^2q	at	. . adg^2	$abgk$
	u		bgm	b^2r	adq	abs
			bj^2	. . bdo	ajm	. . adr
			dgj	bgn	b^4g	aeg^2
			g^2h	bjk	b^3m	aeq
			hq	. . dgk	b^3dj	agp
			. . ko	. . ds	b^2gh	. . aho
			. . m^2	. . egj	bcg^2	. . ajn
				g^2i	bcq	. . akm
				. . hr	. . bd^3g	b^3n
				iq	. . beo	. . b^3dk
				. . jp	. . bhm	. . b^2ej
				. . mn	. . bk^2	. . b^2gi
					cgm	bcr
					cj^2	. . $bdeg$
					. . d^3m	. . bdp
					. . dhj	. . bhn
					. . egk	. . bim
					. . es	. . bjl
					. . gh^2	. . cdo
					. . ir	cgn
					. . kp	cjk
					. . lo	. . d^2n
					. . n^2	. . dem
						. . dgl
						. . dhk
						. . dij
						. . ehj
						. . fgj
						. . ghi
at			1			
ds	3	2	1			
es		3	2	1		
fs				3	2	1
hr		2	1			
ir			2	1		
jo	1					
jp			1			
ko		1				
kp		3	2	1		
lo				1		
lp		5	4	3	2	1

26 derivs.

TABLE No. 99 (*continued*).

Deg.	13			
Ord.	1	3	5	7
	g^2j	bgo	ag^3	agr
	jq	bt	agq	ajo
	v	g^2k	au	. . b^3o
		gs	b^3gj	. . b^3gk
		. . jr	bdg^2	b^3s
		kq	bdq	. . bdr
		. . mo	bjm	beg^2
			. . dgm	beq
			. . dj^2	bgp
			ghj	. . bho
			. . kr	. . bjn
			. . no	. . bkm
				cgo
				ct
				. . d^2o
				. . dgn
				. . djk
				. . egm
				. . ej^2
				g^2l
				. . ghk
				. . gij
				. . ks
				. . lq
				. . mp
bt	1			
ct			1	
hs	3	2	1	
is		3	2	1
jr	1			
kr		1		
lr				1
mo	1			
mp		2	1	
no		1		
np	4	3	2	1

19 derivs.

TABLE No. 99 (*continued*).

Deg.	14					15		
Ord.	0	2	4	6	8	1	3	5
		bg^3	bgr	$..ag^2j$	$abgo$	g^2o	bg^2j	$..b^2go$
		bgq	bjo	ajq	abt	gt	bjq	b^2t
		bu	$..dgo$	av	ag^2k	oq	bv	bg^2k
		g^2m	$..dt$	b^3g^2	ags		dg^3	bgs
		gj^2	g^2n	b^2q	$..ajr$		dgq	$..bjr$
		mq	gjk	b^2gm	akq		$..du$	bkq
		$..o^2$	js	b^2j^2	$..amo$		gjm	$..bmo$
			$..mr$	$bdgj$	b^3r		$..j^3$	$..dgr$
			nq	bg^2h	$..b^2do$		$..or$	$..djo$
				bhq	$..b^2gm$			eg^3
				$..bko$	b^2jk			egq
				$..bm^2$	$..bdgk$			$..eu$
				cg^2	$..bds$			g^2p
				cgq	$..begj$			$..gho$
				cu	bg^2i			$..gjn$
				$..d^2g^2$	$..bhr$			$..gkm$
				$..d^2q$	biq			$..ht$
				$..djm$	$..bjp$			$..j^2k$
				$..ego$	$..bmn$			$..ms$
				$..et$	cgr			pq
				$..ghm$	cjo			
				$..gk^2$	$..d^2r$			
				$..hj^2$	$..deg^2$			
				$..ks$	$..deq$			
				$..nr$	$..dgp$			
				$..op$	$..dho$			
					$..djn$			
					$..dkm$			
					$..ejm$			
					$..ghn$			
					$..gim$			
					$..gjl$			
					$..hjk$			
					$..ij^2$			
dt			1			bv	1	
et				1		cv		1
ft					1	ht		1
js		1				it		1
ks	3	2	1			ms	2	1
ls			3	2	1	ns	3	2	1
mr	2	1				or	1	
nr		2	1			pr		2	1
op			1					
pp		4		2				

17 derivs. 12 derivs.

TABLE No. 99 (*continued*).

Deg.	16					
Ord.	0	2	4	6	8	Col. 8 concl.
	g^4	g^2r	b^2g^3	ag^2o	. . abg^2j	. . ekq
	g^2q	gjo	b^2gq	agt	$abjq$	. . emo
	gu	jt	b^2u	aoq	abv	. . g^2h^2
	q^2	qr	bg^2m	. . b^2gr	. . adg^3	. . gir
			bgj^2	b^2jo	$adgq$	. . gkp
			bmq	. . $bdgo$	. . adu	. . glo
			. . bo^2	. . bdt	$agjm$	. . gn^2
			dg^2j	bg^2n	. . aj^3	. . h^2q
			djq	$bgjk$	. . aor	. . hko
			. . dv	bjs	b^4g^2	. . hm^2
			g^3h	. . bmr	b^4q	. . ijo
			ghq	. . bnq	b^3gm	. . jkn
			. . gko	. . dg^2k	b^3j^2	. . k^2m
			. . gm^2	. . dgs	b^2dgj	. . lt
			. . hu	. . djr	b^2g^2h	. . ps
			. . j^2m	. . dkq	b^2hq	
			. . kt	. . dmo	. . b^2ko	
			. . os	. . eg^2j	. . b^2m^2	
			. . r^2	. . egj	bcg^3	
				. . ev	$bcgq$	
				g^3i	bcu	
				. . ghr	. . bd^2g^2	
				giq	. . bd^2q	
				. . gjp	. . $bdjm$	
				. . gmn	. . $bego$	
				. . hjo	. . bet	
				. . iu	. . $bghm$	
				. . j^2n	. . bgk^2	
				. . jkm	. . bhj^2	
					. . bks	
					. . bnr	
					. . bop	
					cg^2m	
					cgj^2	
					cmq	
					. . co^2	
					. . d^2gm	
					. . d^2j^2	
					. . $dghj$	
					. . dkr	
					. . dno	
					. . eg^2k	
					. . egs	
					. . ejr	
dv		1				
ev			1			
fv					1	
jt	1					
kt		1				
lt				1		
op			1			
os		1				
pp		4		2		
ps		3	2	1		

13 derivs.

TABLE No. 99 (*continued*).

Deg.	17			18				19	
Ord.	1	3	5	0	2	4	6	1	3
	g^3j	bg^2o	ag^4	w	bg^4	bg^2r	$\ldots ag^2j$	g^3o	bg^2j
	gjq	bgt	ag^2q		bg^2q	$\ldots bgjo$	$agjq$	g^2t	$bgjq$
	gv	boq	agu		bgu	bjt	agv	gqo	bgv
	ju	g^3k	aq^2		bq^2	bqr	aju	ou	bju
		g^2s	b^3g^2j		g^3m	$\ldots dg^2o$	b^3g^3	qt	dg^4
		$\ldots gjr$	b^2jq		g^2j^2	$\ldots dgt$	b^2gq		dg^2q
		gkq	b^2v		gmq	$\ldots doq$	b^2u		$\ldots djt$
		$\ldots gmo$	bdg^3		$\ldots go^2$	g^3n	b^2g^2m		dq^2
		$\ldots j^2o$	$bdgq$		j^2q	g^2jk	b^2gj^2		g^2jm
		ku	bdu		$\ldots jv$	gjs	b^2mq		$\ldots gj^3$
		$\ldots mt$	$bgjm$		mu	$\ldots gmr$	$\ldots b^2o^2$		$\ldots gor$
		qs	$\ldots bj^3$		$\ldots ot$	gnq	bdg^2j		jmq
			$\ldots bor$			$\ldots j^2r$	$bdjq$		$\ldots jo^2$
			$\ldots dg^2m$			jkq	$\ldots bdv$		$\ldots mv$
			$\ldots dgj^2$			$\ldots jmo$	bg^3h		$\ldots rt$
			$\ldots dmq$			$\ldots kv$	$bghq$		
			$\ldots do^2$			nu	$\ldots bgko$	mv 1	
			g^2hj				$\ldots bgm^2$	nv	1
			$\ldots gkr$				bhu	rt 1	
			$\ldots gno$				$\ldots bj^2m$		
			hjq				$\ldots bkt$		
			$\ldots hv$				$\ldots bos$		
			$\ldots jko$				$\ldots br^2$		
			$\ldots jm^2$				cg^4		
			$\ldots nt$				cg^2q		
			$\ldots rs$				cgu		
							cq^2		
	hv	1					$\ldots d^2g^3$		
	iv		1				$\ldots d^2gq$		
	mt 1						$\ldots d^2u$		
	nt	1					$\ldots dgjm$		
	rs 2	1					$\ldots dj^3$		
							$\ldots dor$		
							$\ldots eg^2o$		
							$\ldots egt$		
							$\ldots eoq$		
							$\ldots g^2hm$		
							$\ldots g^2k^2$		
							$\ldots ghj^2$		
							$\ldots gks$		
							$\ldots gnr$		
							$\ldots gop$		
							$\ldots hmq$		
							$\ldots ho^2$		
							$\ldots jkr$		
							$\ldots jno$		
							$\ldots k^2q$		
							$\ldots kmo$		
							$\ldots m^3$		
							$\ldots pt$		
							$\ldots s^2$		
				jv 1					
				kv	1				
				lv			1		
				ot 1					
				pt		1			
				ss	2				
	6 derivs.			6 derivs.				3 derivs.	

TABLE No. 99 (*concluded*).

Deg.	20			21		22		24	
Ord.	0	2	4		1	0	2		0
	g^5	bw	b^2g^4		g^4j	gw	bg^5		g^3
	g^3q	g^3r	b^2g^2q		g^2jq		bg^3q		g^4q
	g^2u	g^2jo	b^2gu		g^2v		bg^2u		g^3u
	gq^2	$\ldots gjt$	b^2q^2		gju		bgq^2		g^2q^2
	qu	gqr	bg^3m		jq^2		bqu		gqu
		$\ldots joq$	bg^2j^2		qv		g^4m		q^2
		$\ldots ov$	$bgmq$				g^2j^2		u^2
		ru	$\ldots bgo^2$	ro	1		g^2mq		
			bj^2q				$\ldots g^2o^2$	tv	1
			$\ldots bjv$				gj^2q		
			bmu				$\ldots gjv$		
			$\ldots bot$				gmu		
			dg^2q				$\ldots got$		
			$dgjq$				j^2u		
			$\ldots dgv$				mq^2		
			dju				$\ldots o^2q$		
			g^4h				$\ldots t^2$		
			g^2hq			sv	1		
			$\ldots g^2ko$						
			$\ldots g^2m^2$						
			ghu						
			$\ldots gj^2m$						
			$\ldots gkt$						
			$\ldots gos$						
			$\ldots gr^2$						
			hq^2						
			$\ldots j^4$						
			$\ldots jor$						
			$\ldots koq$						
			$\ldots m^2q$						
			$\ldots mo^2$						
			$\ldots st$						
ov	1								
pv			1						
sv		1							
	3 derivs.			1 deriv.		1 deriv.		1 deriv.	

384. The Canonical form (using the divided expressions, Table No. 98) is peculiarly convenient for the calculation of the derivatives. Some attention is required in regard to the numerical determination: it will be observed that A is given in the standard form $(A_0, A_1, A_2, A_3, A_4, A_5 \between x, y)^6$, while the other covariants are given in the denumerate forms $B=(B_0, B_1, B_2 \between x, y)^2$ &c.: these must be converted into the other form $B=(B_0, \frac{1}{2}B_1, B_2 \between x, y)^2$, $C=(C_0, \frac{1}{6}C_1, \frac{1}{15}C_2, \frac{1}{20}C_3, \frac{1}{15}C_4, \frac{1}{6}C_5, C_6 \between x, y)^6$, &c., the numerical coefficients being of course the reciprocals of the binomial coefficients. We thus have, for instance, the leading coefficients,

$$l.c. \text{ of } AC2 = A_0 . \tfrac{1}{15}C_2 - 2 . \; A_1 . \tfrac{1}{6}C_1 + A_2 . C_0,$$

but

$$\text{,, ,, } BC2 = B_0 . \tfrac{1}{15}C_2 - 2 . \tfrac{1}{2}B_1 . \tfrac{1}{6}C_1 + B_2 . C_0.$$

Moreover, as regards the covariants $AA2$, $AA4$, &c., we take what are properly the half-values,

$$l.c. \text{ of } AA2 = A_0A_2 - \; A_1^2 \qquad \text{(instead of } A_0A_2 - 2A_1A_1 + A_2A_0),$$

$$\text{,, ,, } AA4 = A_0A_4 - 4A_1A_3 + 3A_2^2 \text{ (instead of } A_0A_4 - 4A_1A_3 + 6A_2A_2 - 4A_3A_1 - A_4A_0),$$

&c.,

and similarly

$$l.c. \text{ of } BB2 = B_0B_2 - (\tfrac{1}{2}B_1)^2,$$

$$\text{,, ,, } CC2 = C_0 . \tfrac{1}{15}C_2 - (\tfrac{1}{6}C_1)^2,$$

&c.

Any one of these leading coefficients, for instance $l.c.$ of $AC2$, is equal to the corresponding covariant derivative, multiplied, it may be, by a power of a: the index of this power being at once found by comparing the deg-orders, these in fact differing by a multiple of 1.5 the deg-order of a. Thus

$aa2$, $A_0A_2 - \; A_1^2$, deg-orders are 2.6, 2.6: or $aa2 = A_0A_2 - A_1^2$,

$aa4$, $A_0A_4 - 4A_1A_3 + 3A_2^2$, deg-orders are 2.2, 4.12: or $aa4 = \dfrac{1}{a^2}(A_0A_4 - 4A_1A_3 + 3A_2^2)$;

we have in fact

$$A_0A_2 - \; A_1^2 = 1 . c - 0^2 = c \qquad \text{: and } aa2 = c,$$

$$A_0A_4 - 4A_1A_3 + 3A_2^2 = 1 . (a^2b - 3c^2) - 4 . 0 . f + 3 . c^2, \; = a^2b \text{: and } aa4 = b.$$

As another instance, and for the purpose of showing how the calculation is actually effected, consider the derivative $ch2$, which is to be calculated from the leading coefficient of $CH2$, $= C_0 . \frac{1}{6}H_2 - 2 . \frac{1}{6}C_1 . \frac{1}{4}H_1 + \frac{1}{15}C_2 . H_0$: this is

$$\begin{aligned} = \; & c\,(\tfrac{1}{6}a^2g - 2abd - ch) \\ & - 2 . \tfrac{1}{2}f\,(\tfrac{1}{2}be - l) \\ & + (\tfrac{1}{3}a^2b - c^2)\,h \end{aligned}$$

= column next written down; but this column contains congregate terms which have to be replaced by their segregate values (see Table No. 96, deg-order 8.16); and we thus obtain

	a^3j	a^2b^3	a^2bh	a^2cg	$abcd$	b^3c^2	c^2h
$\frac{1}{3}a^2bh$			$+\frac{1}{3}$				
$+\frac{1}{6}a^2cg$				$+\frac{1}{6}$			
$-2abcd$					-2		
$-\frac{1}{2}bef$			$-\frac{1}{2}$		$+3$	$+2$	
$-2c^2h$							-2
$+fl$	$\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	$+2$
$=$	$\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{11}{30}$	$-\frac{1}{6}$	0	0	0

viz. the terms other than those divisible by a^2 all disappear: we may either abbreviate the calculation by omitting them *ab initio*, or retain them for the sake of the verification afforded by their disappearance. The factor a^2 divides out, and the final result is

$$ch2 = \tfrac{1}{3}aj - \tfrac{1}{3}b^3 + \tfrac{11}{30}bh - \tfrac{1}{6}cg,$$

which is the proper segregate expression of the derivative $ch2$: of course, we have deg-order $CH2 = 8.16$, deg-order $ch2 = 6.6$, and the difference is 2.10, the double of 1.5, so that the factor a^2 is as it ought to be.

TABLE No. 100 (The Derivatives up to the Sixth Order).

Degree 2.

2.2	b
$aa4$	$+1$

2.6	c
$aa2$	$+1$

Degree 3.

3.1	
$ac5$	0

3.3	d
$ab2$	-3
$ac4$	$+\frac{12}{5}$

3.5	e
$ab1$	$+\frac{1}{2}$
$ac3$	$+\frac{1}{20}$

3.7	ab
$ac2$	$+\frac{1}{5}$

3.9	f
$ac1$	$+\frac{1}{2}$

Degree 4.

4.0	g
$ae5$	-2
$bb2$	$-\frac{1}{4}$
$cc6$	$-\frac{1}{40}$

4.2	
$ad3$	0
$ae4$	0

4.4	b^2	h
$ad2$	$-\frac{1}{3}$	$+\frac{1}{3}$
$ae3$	$-\frac{4}{5}$	$-\frac{6}{5}$
$af5$	$+\frac{62}{63}$	$-\frac{83}{63}$
$bc2$	$+\frac{1}{5}$	$-\frac{1}{2}$
$cc4$	$+\frac{3}{25}$	$-\frac{1}{10}$

4.6	i
$ad1$	$+\frac{1}{3}$
$ae2$	$+\frac{6}{5}$
$af4$	$+\frac{183}{62}$
$bc1$	$+\frac{1}{2}$

TABLE No. 100 (*continued*).

4.8	ad	bc
$ae\,1$	$-\frac{6}{5}$	-2
$af\,3$	$+\frac{59}{42}$	$-\frac{5}{6}$
$cc\,2$	$+\frac{1}{4}$	$-\frac{1}{20}$

4.10	ae
$af\,2$	$+1$

4.12	a^2b	c^2
$af\,1$	$+\frac{2}{9}$	-2

Degree 5.

5.1	j
$ah\,4$	$+2$
$ai\,5$	$-\frac{4}{3}$
$bd\,2$	$-\frac{1}{3}$
$ce\,5$	$-\frac{8}{5}$

5.3	k
$ah\,3$	$+\frac{1}{2}$
$ai\,4$	$+\frac{1}{3}$
$bd\,1$	$-\frac{1}{6}$
$be\,2$	$-\frac{3}{5}$
$cd\,3$	$+\frac{3}{20}$
$ce\,4$	$+\frac{2}{25}$
$cf\,6$	$-\frac{19}{42}$

5.5	ag	bd
$ah\,2$	$+\frac{1}{6}$	-2
$ai\,3$	$+0$	-2
$be\,1$	$-\frac{1}{2}$	$+\frac{24}{5}$
$cd\,2$	0	$-\frac{2}{15}$
$ce\,3$	$-\frac{1}{20}$	$-\frac{48}{5}$
$df\,5$	$-\frac{1}{72}$	$+\frac{8}{63}$

5.7	be	l
$ah\,1$	-2	-4
$ai\,2$	0	$+\frac{1}{3}$
$bf\,2$	$-\frac{7}{36}$	$+1$
$cd\,1$	0	$+\frac{1}{6}$
$ce\,2$	$\frac{1}{5}$	$-\frac{2}{3}$
$df\,4$	$-\frac{1}{90}$	$+\frac{43}{315}$

5.9	ab^2	ah	cd
$ai\,1$	$-\frac{1}{3}$	$+\frac{1}{3}$	$+3$
$bf\,1$	$+\frac{2}{9}$	$-\frac{1}{2}$	$+3$
$ce\,1$	0	$-\frac{1}{2}$	$+\frac{9}{5}$
$df\,3$	$+\frac{2}{15}$	$-\frac{11}{120}$	$-\frac{439}{420}$

5.11	ai	ce
$df\,2$	$+\frac{1}{45}$	$+\frac{1}{180}$

5.13	a^2d	abc
$df\,1$	$+\frac{1}{2}$	$-\frac{5}{8}$

Degree 6.

6.0	
$ci\,6$	0

6.2	bg	m
$ak\,3$	0	-4
$al\,5$	0	$+\frac{20}{7}$
$bh\,2$	$-\frac{1}{3}$	-2
$ch\,4$	$\frac{1}{10}$	$-\frac{2}{5}$
$ci\,5$	0	$+\frac{2}{3}$
$dd\,2$	0	$+\frac{1}{9}$
$de\,3$	0	$-\frac{4}{5}$
$ee\,4$	-1	$-\frac{48}{25}$
$ff\,8$	$-\frac{5}{324}$	$-\frac{68}{567}$

6.4	n
$aj\,1$	-1
$ak\,2$	$+\frac{1}{3}$
$al\,4$	$-\frac{1}{35}$
$bh\,1$	$+\frac{1}{2}$
$bi\,2$	$+\frac{1}{3}$
$ch\,3$	$+\frac{3}{10}$
$ci\,4$	$+\frac{1}{15}$
$de\,2$	$-\frac{1}{3}$
$ef\,5$	$-\frac{64}{63}$

6.6	aj	b^3	bh	cg
$a\,1$	$-\frac{2}{3}$	$+2$	-3	$+1$
$al\,3$	$-\frac{16}{35}$	$+\frac{2}{7}$	$-\frac{5}{7}$	$+\frac{3}{7}$
$bi\,1$	$-\frac{1}{3}$	$+\frac{5}{6}$	$-\frac{1}{2}$	0
$ch\,2$	$+\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{11}{36}$	$-\frac{1}{6}$
$ci\,3$	$-\frac{1}{6}$	$-\frac{1}{30}$	$+\frac{11}{3}$	$+\frac{7}{60}$
$de\,1$	$-\frac{2}{15}$	$+\frac{2}{15}$	$+\frac{1}{15}$	$-\frac{1}{5}$
$df\,3$	$+\frac{59}{378}$	$-\frac{143}{378}$	$+\frac{425}{756}$	$-\frac{139}{756}$
$ee\,2$	$-\frac{4}{25}$	$+\frac{4}{25}$	$-\frac{38}{25}$	$+\frac{9}{25}$
$ef\,4$	$+\frac{236}{315}$	$-\frac{4}{105}$	$+\frac{10}{63}$	$-\frac{71}{105}$
$ff\,6$	$-\frac{1529}{7938}$	$+\frac{2873}{7938}$	$+\frac{3533}{15876}$	$-\frac{5591}{31752}$

TABLE No. 100 (*concluded*).

6 . 8	ak	bi
$al\,2$	$-\frac{4}{21}$	$+\frac{1}{21}$
$ch\,1$	$+\frac{1}{6}$	$+\frac{1}{3}$
$ci\,2$	$+\frac{1}{9}$	$+\frac{4}{45}$
$df\,2$	$-\frac{11}{108}$	$-\frac{1}{54}$
$ef\,3$	$+\frac{32}{315}$	$+\frac{64}{315}$

6 . 10	a^2g	abd	b^2c	ch
$al\,1$	0	$-\frac{3}{7}$	-1	$+1$
$ci\,1$	0	$-\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{6}$
$df\,1$	0	$-\frac{1}{9}$	$+\frac{1}{3}$	$-\frac{1}{3}$
$ef\,2$	$+\frac{1}{36}$	$+\frac{7}{5}$	$-\frac{19}{45}$	$-\frac{4}{5}$
$ff\,4$	$+\frac{1}{432}$	$-\frac{53}{756}$	$+\frac{89}{756}$	$-\frac{8}{63}$

6 . 12	abe	al	ci
$ef\,1$	$\frac{2}{9}$	$-\frac{2}{5}$	$-\frac{0}{5}$

6 . 14	a^2b^2	a^2h	acd	bc^2
$ff\,2$	$-\frac{4}{81}$	$+\frac{1}{36}$	$+\frac{5}{6}$	$-\frac{2}{9}$

which is complete to the sixth degree. I had calculated the derivatives up to the tenth degree, but the results were not in the segregate form.

On the form of the Numerical Generating Functions: the N.G.F. of a Sextic.

Art. Nos. 385, 386.

385. It is to be remarked that the R.G.F. is derived not from the fraction in its least terms, which is algebraically the most simple form of the N.G.F., but from a form which contains common factors in the numerator and denominator: thus for the quadric, the cubic, and the quartic, writing down the two forms (identical in the case of the quadric) these are—

Quadric

$$\text{N.G.F.} = \frac{1}{1-ax^2\,.\,1-a^2}.$$

Cubic

$$\text{N.G.F.} = \frac{1-ax+a^2x^2}{1-a^4\,.\,1-ax^3\,.\,1-ax} = \frac{1-a^6x^6}{1-a^4\,.\,1-ax^3\,.\,1-a^2x^2\,.\,1-a^3x^3}.$$

Quartic

$$\text{N.G.F.} = \frac{1-ax^2+a^2x^4}{1-a^2\,.\,1-a^3\,.\,1-ax^4\,.\,1-ax^2} = \frac{1-a^8x^{12}}{1-a^2\,.\,1-a^3\,.\,1-ax^4\,.\,1-a^2x^4\,.\,1-a^3x^6}.$$

For the quintic the two forms are, N.G.F. =

$(\ 1$			$-\ a^6$			$+\ a^{12})\ x^0$
$+\ (-\ 1$		$+\ a^4$	$+\ 2a^6$			$-\ a^{12})\ ax^1$
$+\ ($	$+\ a^2$			$-\ a^8$		$+\ a^{10})\ x^2$
$+\ (-\ 1$		$+\ a^4$	$+\ a^6$	$+\ a^8$	$-\ a^{10}$	$-\ a^{12})\ ax^3$
$+\ (+\ 1$	$+\ a^2$	$-\ a^4$	$-\ a^6$	$-\ a^8$		$+\ a^{12})\ a^2x^4$
$+\ ($	$-\ a^2$	$+\ a^4$			$-\ a^{10}$	$)\ a^3x^5$
$+\ (+\ 1$			$-\ 2a^6$	$-\ a^8$		$+\ a^{12})\ a^2x^6$
$+\ (-\ 1$			$+\ a^6$			$-\ a^{12})\ a^3x^7$

divided by

$$1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - ax^5 . 1 - ax^3 . 1 - ax\,;$$

and

$(\ 1$									$+\ a^{18})\ x^0$
$($		a^4	$+\ a^6$		$+\ a^{10}$	$+\ a^{12}$			$)\ ax$
$($		a^4	$+\ a^6$	$+\ a^8$	$+\ a^{10}$		$+\ a^{14}$		$-\ a^{18})\ a^2x^2$
$(\ 1$	$+\ a^2$	$+\ a^4$		$+\ a^8$					$)\ a^3x^3$
$(\ 1$	$+\ a^2$	$+\ a^4$	$+\ a^6$		$+\ a^{10}$		$-\ a^{14}$		$)\ a^4x^4$
$(\ 1$		$+\ a^4$	$+\ a^6$					$-\ a^{16}$	$)\ a^3x^5$
$($	a^2					$-\ a^{12}$	$-\ a^{14}$		$)\ a^2x^6$
$($		a^4		$-\ a^8$		$-\ a^{12}$	$-\ a^{14}$	$-\ a^{16}$	$-\ a^{18})\ ax^7$
$($					$-\ a^{10}$	$-\ a^{12}$	$-\ a^{14}$	$-\ a^{16}$	$-\ a^{18})\ a^2x^8$
$($		$-\ a^4$		$-\ a^8$	$-\ a^{10}$	$-\ a^{12}$	$-\ a^{14}$		$)\ a^3x^9$
$($			$-\ a^6$	$-\ a^8$		$-\ a^{12}$	$-\ a^{14}$		$)\ a^4x^{10}$
$(-1$									$-\ a^{18})\ a^5x^{11}$

divided by

$$1 - a^4 . 1 - a^8 . 1 - a^{12} . 1 - ax^5 . 1 - a^2x^2 . 1 - a^2x^6 :$$

this last being in fact equivalent to that used for the determination of the R.G.F.

386. For the sextic the forms are, N.G.F. =

$(1$	$+a$		$-a^3$	$-a^4$	$-a^5$		$+a^7$	$+a^8)\,x^0$
$(-1$	$-a$	$+a^2$	$+2a^3$	$+2a^4$	$+a^5$		$-a^7$	$-a^8)\,ax^2$
$(-1$		$+a^2$	$+a^3$	$+a^4$	$+a^5$		$-a^7$	$-a^8)\,ax^4$
$(1$	$+a$		$-a^3$	$-a^4$	$-a^5$	$-a^6$		$+a^8)\,a^2x^6$
$(1$	$+a$		$-a^3$	$-2a^4$	$-2a^5$	$-a^6$	$+a^7$	$+a^8)\,a^2x^8$
$(-1$	$-a$		$+a^3$	$+a^4$	$+a^5$		$-a^7$	$-a^8)\,a^3x^{10}$

divided by

$$1+a\,.\,1-a^2\,.\,1-a^3\,.\,1-a^4\,.\,1-a^5\,.\,1-ax^6\,.\,1-ax^4\,.\,1-ax^2;$$

and

$(1$															$+a^{15})\,x^0$
$+(1$		$+a^2$		$+a^4$	$+a^5$		$+a^7$		$+a^9$						$)\,a^2x^2$
$+($		$+a^2$	$+a^3$	$+a^4$	$+a^5$	$+a^6$	$+a^7$	$+a^8$	$+a^9$		$+a^{11}$				$)\,a^2x^4$
$+(1$	$+a$		$+2a^3$		$+a^5$	$+a^6$		$+a^8$					$-a^{13}$		$)\,a^2x^6$
$+(+a^{.5}$		$+a^{2.5}$		$+a^{4.5}$						$-a^{10.5}$		$-a^{12.5}$		$-a^{14.5}$	$)\,a^{2.5}x^8$
$+($		$+a^2$					$-a^7$		$-a^9$	$-a^{10}$		$-2a^{12}$		$-a^{14}$	$-a^{15})\,a^2x^{10}$
$+($				$-a^4$		$-a^6$	$-a^7$	$-a^8$	$-a^9$	$-a^{10}$	$-a^{11}$	$-a^{12}$	$-a^{13}$		$)\,a^3x^{12}$
$+($						$-a^6$		$-a^8$		$-a^{10}$	$-a^{11}$		$-a^{13}$		$-a^{15})\,a^2x^{14}$
$+(-1$															$-a^{15})\,a^5x^{16}$

divided by

$$1-a^2\,.\,1-a^4\,.\,1-a^6\,.\,1-a^{10}\,.\,1-ax^6\,.\,1-a^2x^4\,.\,1-a^2x^8,$$

where observe that in the middle term, although for symmetry $a^{.5}\,(=\sqrt{a})$ has been introduced into the expression, the coefficient is really rational, viz. the term is

$$(a^3+a^5+a^7-a^{13}-a^{15}-a^{17})\,x^8.$$

The second form or one equivalent to it is due to Sylvester: I do not know whether he divided out the common factors so as to obtain the first form. I assume that it would be possible from this second form to obtain a R.G.F., and thence to establish for the 26 covariants of the sextic a theory such as has been given for the 23 covariants of the quintic: but I have not entered upon this question.

TABLE No. 93 *bis* (The covariant S, adopted form $= -(D, M)$).

In this Table, a, b, c, d, e, f denote, as in the tables of former memoirs, the coefficients of the quintic form $(a, b, c, d, e, f \between x, y)^5$.

$S = ($

$a^3b^6c^2f^3$ − 2	$a^2b^6c^3df^3$ − 3	$a^3b^6cd^2f^3$ + 3	$a^3b^6d^3f^3$ + 2
c^3def^2 + 15	$c^3e^2f^2$ + 3	cde^3f^2 − 6	$d^2e^2f^2$ − 6
c^3e^3f − 9	cd^3ef^2 + 24	ce^4f + 3	de^4f + 6
cd^3f^2 − 9	cde^3f − 42	d^3ef^2 − 3	e^6 − 2
cd^2e^2f − 6	ce^5 + 18	d^2e^2f + 6	$a^2b\ cd^2f^3$ − 15
cde^4 + 9	d^4f^2 − 18	de^5 − 3	cde^2f^2 + 30
d^4ef + 9	d^3e^2f + 33	$a^2b^2d^2f^3$ − 3	ce^4f − 15
d^3e^3 − 7	d^2e^4 − 15	de^2f^2 + 6	d^3ef^2 + 15
$a^2b^3c^2f^3$ + 6	$a^2b^2cdf^3$ + 6	e^4f − 3	d^2e^2f − 30
$cdef^2$ − 30	ce^2f^2 − 6	,, $b\ c^3df^3$ − 24	de^5 + 15
ce^2f + 18	d^2ef^2 − 24	$c^2e^2f^2$ + 24	,, $b^6c^3df^3$ + 9
d^2f^2 + 9	de^3f + 42	cd^2ef^2 + 78	$c^3e^2f^2$ − 9
d^2e^2f + 6	e^5 − 18	cde^3f − 108	$c^3d^2ef^2$ − 21
de^4 − 9	,, $b\ c^3f^3$ + 3	ce^5 + 30	c^3de^3f + 15
,, $b\ c^3ef^2$ − 15	c^3def^2 − 78	d^4f^2 − 24	c^3e^2 + 6
$c^3d^2f^2$ + 21	c^3e^3f + 69	d^3e^2f + 24	cd^4f^2 + 3
c^3de^2f − 6	cd^3f^2 + 93	,, $b^6c^4f^3$ + 18	cd^2e^2f + 21
c^3e^4 + 18	cd^2e^2f − 51	c^3def^2 − 93	cd^2e^4 − 24
cd^3ef + 30	cde^4 − 33	c^3e^3f + 21	d^5ef − 9
cd^3e^3 − 51	d^4ef − 57	$c^3d^3f^2$ + 36	d^4e^3 + 9
d^5f − 36	d^3e^3 + 54	$c^3d^2e^2f$ + 123	$a\ b^3d^3f^3$ + 9
d^4e^2 + 39	,, $b^6c^4ef^2$ + 24	c^3de^4 − 51	de^2f^2 − 18
,, $b^6c^4df^2$ − 3	$c^3d^2f^2$ − 36	cd^4ef − 111	e^4f + 9
c^4e^2f + 45	c^3de^2f − 9	cd^3e^3 + 39	,, $b^3c^3df^3$ + 6
c^3d^2ef − 84	c^3e^4 − 54	d^6f + 27	$c^2e^2f^2$ − 6
c^3de^3 − 63	c^2d^3ef + 24	d^5e^2 − 9	cd^2ef^2 + 6
c^2d^4f + 45	$c^2d^2e^3$ + 129	$a\ b^3cdf^3$ + 42	cde^2f − 24
$c^2d^3e^2$ + 150	cd^5f + 9	ce^2f^2 − 42	ce^5 + 18
cd^5e − 117	cd^4e^2 − 114	d^2ef^2 − 69	d^4f^2 − 45
d^7 + 27	d^5e + 27	de^2f + 96	d^3e^2f + 96
$a^1b^4ef^3$ − 6	$a^1b^4df^3$ − 3	e^5 − 27	d^2e^4 − 51
def^2 + 15	e^2f^2 + 3	,, $b^2c^3f^3$ − 33	,, $b\ c^4f^3$ − 9
e^3f − 9	,, $b^6c^2f^3$ − 6	c^3def^2 + 51	c^3def^2 − 30
,, $b^3c^2ef^2$ + 30	$cdef^2$ + 108	c^2e^3f + 48	c^3e^2f + 66
cd^2f^2 − 15	ce^3f − 96	cd^3f^2 + 9	$c^2d^2f^2$ + 84
cde^2f + 24	d^3f^2 − 21	cd^2e^2f − 147	$c^2d^2e^2f$ − 36
ce^4 − 45	d^2e^2f − 48	cde^4 + 39	c^2de^4 − 102
d^3ef − 66	de^4 + 63	d^4ef + 78	cd^4ef − 174
d^2e^3 + 72	,, $b^6c^3ef^2$ − 24	d^3e^3 − 45	cd^3e^3 + 210
,, $b^3c^2df^2$ − 21	c^2d^2f − 123	,, $b\ c^4ef^2$ + 57	d^6f + 63
c^3e^2f − 96	c^2de^2f + 147	$c^3d^2f^2$ − 24	d^5e^2 − 72
c^3d^2ef + 36	c^2e^4 + 66	c^3de^2f − 78	,, $b^3c^2ef^2$ + 36
c^2de^3 + 213	cd^3ef + 78	c^3e^4 − 60	$c^3d^2f^2$ − 45
cd^4f + 120	cd^2e^3 − 186	c^2d^3ef + 36	c^4de^2f − 120
cd^3e^2 − 303	d^5f + 51	$c^2d^2e^3$ + 108	c^4e^4 − 6
d^5e + 51	d^4e^2 − 9	cd^5f − 24	c^3d^3ef + 204

$\between x, y)^3$

(*continued on next page.*)

(*continued from last page.*)

$a^1b\,c^5f^2$	+ 9	$a^1b\,c^4df^2$	+ 111	$a\,b\,cd^4e^2$	− 6	$a\,b^0c^3d^2e^3$	+ 120
c^4def	+ 174	c^4e^2f	− 78	d^5e	− 9	c^2d^5f	− 66
c^4e^3	− 36	c^3d^2ef	− 36	,, $b^0c^5df^2$	− 9	$c^2d^4e^2$	− 240
c^2d^3f	− 204	c^3de^3	− 54	c^5e^2f	− 51	cd^5e	+ 144
$c^3d^2e^2$	− 174	c^3d^4f	− 96	c^4d^2ef	+ 96	d^6	− 27
c^3d^4e	+ 330	$c^3d^3e^2$	+ 150	c^4de^3	+ 111	$a^0b^4cdf^3$	− 9
cd^6	− 99	cd^5e	+ 30	c^3d^4f	− 27	ce^2f^2	+ 9
,, b^0c^6ef	− 63	d^7	− 27	$c^3d^3e^2$	− 234	d^2ef^2	− 18
c^5d^2f	+ 66	,, $b^0c^6f^2$	− 27	c^2d^5e	+ 141	de^2f	+ 45
c^5de^2	+ 99	c^5def	+ 24	cd^7	− 27	e^5	− 27
c^4d^2e	− 147	c^5e^3	+ 54	$a^0b^5df^3$	− 18	,, $b^0c^2f^3$	+ 7
c^3d^5	+ 45	c^4d^3f	+ 27	e^2f^2	+ 18	c^3def^2	+ 51
$a^0b^6f^3$	+ 2	$c^4d^2e^2$	− 93	,, $b^4c^2f^3$	+ 15	c^2e^2f	− 72
,, b^5cef^2	− 15	c^3d^4e	+ 6	$cdef^2$	+ 33	cd^2f^2	+ 63
d^2f^2	− 6	c^2d^5	+ 9	ce^3f	− 63	cd^2e^2f	− 213
de^2f	− 18	$a^0b^5cf^3$	+ 3	d^3f^2	+ 54	cde^4	+ 171
e^4	+ 27	def^2	− 30	d^2e^2f	− 66	d^4ef	+ 36
,, $b^4c^3df^2$	+ 24	e^3f	+ 27	de^4	+ 27	d^3e^3	− 43
c^2e^2f	+ 51	,, $b^4cd^2f^2$	+ 51	,, $b^3c^3ef^2$	− 54	,, $b^2c^4ef^2$	− 39
cd^2ef	+ 102	cde^2f	− 39	$c^2d^2f^2$	− 129	$c^3d^2f^2$	− 150
cde^3	− 171	ce^4	− 27	c^2de^2f	+ 186	c^2de^3f	+ 303
d^4f	+ 6	d^3ef	+ 60	c^2e^4	+ 45	c^2e^4	− 18
d^3e^2	+ 18	d^2e^3	− 45	cd^3ef	+ 54	c^2d^3ef	+ 174
,, $b^3c^4f^2$	− 9	,, $b^3c^3df^2$	− 39	cd^2e^3	− 96	$c^2d^2e^2$	− 345
c^3def	− 210	c^3e^2f	+ 45	d^3f	− 54	cd^3f	− 99
c^3e^3	+ 43	c^3d^2ef	− 108	d^2e^2	+ 48	cd^2e^2	+ 192
c^2d^3f	− 120	c^3de^3	+ 96	,, $b^2c^4df^2$	+ 114	d^3e	− 18
$c^2d^2e^2$	+ 345	cd^4f	− 111	c^4e^2f	+ 9	,, $b\,c^5df^2$	+ 117
cd^4e	− 87	cd^3e^2	+ 147	c^3d^2ef	− 150	c^5e^2f	− 51
d^5	− 2	d^5e	− 30	c^3de^3	− 147	c^4d^2ef	− 330
,, b^0c^5ef	+ 72	,, $b^2c^5f^2$	+ 9	c^3d^4f	+ 93	c^4de^3	+ 87
c^4d^2f	+ 240	c^4def	+ 6	$c^2d^3e^2$	+ 150	c^3d^4f	+ 147
c^4de^2	− 192	c^4e^3	− 48	cd^5e	− 87	$c^3d^3e^2$	+ 186
c^3d^3e	− 186	c^3d^3f	+ 234	d^7	+ 18	c^2d^5e	− 201
c^2d^5	+ 96	$c^3d^2e^2$	− 150	,, $b\,c^6f^2$	− 27	cd^7	+ 45
,, $b\,c^6df$	− 144	c^2d^4e	− 108	c^5def	− 30	,, $b^0c^7f^2$	− 27
c^6e^2	+ 18	cd^6	+ 57	c^5e^3	+ 30	c^6def	+ 99
c^5d^2e	+ 201	,, $b\,c^6ef$	+ 9	c^4d^3f	− 6	c^6e^3	+ 2
c^4d^4	− 87	c^5d^2f	− 141	$c^4d^2e^2$	+ 108	c^5d^3f	− 45
,, b^0c^8f	+ 27	c^5de^2	+ 87	c^3d^4e	− 96	$c^5d^2e^2$	− 96
c^7de	− 45	c^4d^3e	+ 96	c^3d^5	+ 21	c^4d^4e	+ 87
c^6d^3	+ 20	c^3d^5	− 51	,, b^0c^7ef	+ 27	c^3d^6	− 20
		,, b^0c^7df	+ 27	c^6d^2f	− 9		
		c^7e^2	− 18	c^5de^2	− 57		
		c^6d^2e	− 21	c^5d^3e	+ 51		
		c^5d^4	+ 12	c^4d^5	− 12		

I remark that I calculated the first two coefficients S_0, S_1, and deduced the other two S_2 from S_1, and S_3 from S_0, by reversing the order of the letters (or which is the same thing, interchanging a and f, b and e, c and d) and reversing also the signs of the numerical coefficients. This process for S_2, S_3 is to a very great extent a verification of the values of S_0, S_1. For, as presently mentioned, the

terms of S_0 form subdivisions such that in each subdivision the sum of the numerical coefficients is $=0$: in passing by the reversal process to the value of S_3, the terms are distributed into an entirely new set of subdivisions, and then in each of these subdivisions the sum of the numerical coefficients is found to be $=0$; and the like as regards S_1 and S_2.

If in the expressions for S_0, S_1, S_2, S_3 we first write $d=e=f=1$, thus in effect combining the numerical coefficients for the terms which contain the same powers in a, b, c, we find

$$\begin{aligned}
S_0 = \;& a^3(-2c^3+6c^2-6c+2)\\
&+a^2\{b^2(6c^2-12c-6)+b(-15c^3+33c^2-21c+3)\\
&\qquad\qquad+b^0(42c^4-147c^3+195c^2-117c+27)\}\\
&+a\;\{b^4.0+b^3(30c^2-36c+6)+b^2(-117c^3+249c^2-183c+51)\\
&\qquad+b(9c^5+138c^4-378c^3+330c^2-99c)+b^0(-63c^6+165c^5-147c^4+45c^3)\}\\
&+a^0\{b^6.2+b^5(-15c+3)+b^4(75c^2-69c+24)+b^3(-9c^4-167c^3+225c^2-87c-2)\\
&\qquad+b^2(72c^5+48c^4-186c^3+96c^2)+b(-126c^6+201c^5-87c^4)\\
&\qquad+b^0(27c^8-45c^7+20c^6)\};
\end{aligned}$$

which for $c=1$ becomes

$$=2b^6-12b^5+30b^4-40b^3+30b^2-12b+2, \text{ that is, } 2(b-1)^6,$$

and for $b=1$ becomes $=0$.

$$\begin{aligned}
S_2 = \;& a^3(0c^3+0c+0)\\
&+a^2\{b^2(0c+0)+b(3c^3-9c^2+9c-3)+b^0(24c^4-99c^3+153c^2-105c+27)\}\\
&+a\;\{b^4.0\;+b^3(-6c^2+12c-6)+b^2(-24c^3+90c^2-108c+42)\\
&\qquad+b(33c^4-90c^3+54c^2+30c-27)+b^0(-27c^6+78c^5-66c^4+6c^3+9c^2)\}\\
&+a^0\{b^5(3c-3)+b^4(-15c+15)+b^3(6c^3-12c^2+36c-30)\\
&\qquad+b^2(9c^5-42c^4+84c^3-108c^2+57c)+b(9c^6-54c^5+96c^4-51c^3)\\
&\qquad+b^0(9c^7-9c^6)\}:
\end{aligned}$$

which for $c=1$ becomes $=0$.

$$\begin{aligned}
S_3 = \;& a^3(0c+0)\\
&+a^2\{b^2.0+b(0c^2+0c+0)+b^0(18c^4-72c^3+108c^2-72c+18)\}\\
&+a\;\{b^3(0c+0)+b^2(-33c^3+99c^2-99c+33)+b(57c^4-162c^3+144c^2-30c-9)\\
&\qquad+b^0(-60c^5+207c^4-261c^3+141c^2-27c)\}\\
&+a^0\{b^5.0+b^4(15c^2-30c+15)+b^3(-54c^3+102c^2-42c-6)\\
&\qquad+b^2(123c^4-297c^3+243c^2-87c+18)+b(-27c^5+102c^4-96c^3+21c^2)\\
&\qquad+b^0(27c^7-66c^6+51c^5-12c^4)\}:
\end{aligned}$$

which for $c=1$ becomes $=0$.

$$
\begin{aligned}
S_4 = \quad & a^3 . 0 \\
& + a^2 \{b\ (0c + 0) + b^0 (0c^3 + 0c^2 + 0c + 0)\} \\
& + a\ \{b^3 . 0\ + b^2 (0c^2 + 0c + 0) + b (-9c^4 + 36c^3 - 54c^2 + 36c - 9) \\
& \qquad + b^0 (36c^5 - 171c^4 + 324c^3 - 306c^2 + 144c - 27)\} \\
& + a^0 \{b^4 (0c + 0) + b^3 (7c^3 - 21c^2 + 21c - 7) + b^2 (-39c^4 + 135c^3 - 171c^2 + 93c - 18) \\
& \qquad + b (66c^5 - 243c^4 + 333c^3 - 201c^2 + 45c) \\
& \qquad + b^0 (-27c^7 + 101c^6 - 141c^5 + 87c^4 - 20c^3)\} :
\end{aligned}
$$

which for $c = 1$ becomes $= 0$.

It follows that, for $c = d = e = f = 1$, the value of the covariant S is $= 2(b-1)^6 a^3$, which might be easily verified.

694.

DESIDERATA AND SUGGESTIONS.

No. 1. The theory of groups.

[From the *American Journal of Mathematics*, t. I. (1878), pp. 50—52.]

Substitutions, and (in connexion therewith) groups, have been a good deal studied; but only a little has been done towards the solution of the general problem of groups. I give the theory so far as is necessary for the purpose of pointing out what appears to me to be wanting.

Let $\alpha, \beta, \ldots$ be functional symbols, each operating upon one and the same number of letters and producing as its result the same number of functions of these letters; for instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and of combination;

$$\alpha^2(x, y, z) = \alpha(X, Y, Z), \text{ or } \beta\alpha(x, y, z) = \beta(X, Y, Z),$$

$=$ in each case three given functions of (x, y, z); and similarly for α^3, $\alpha^2\beta$, &c.

The symbols are not in general commutative, $\alpha\beta$ not $=\beta\alpha$; but they are associative, $\alpha\beta.\gamma = \alpha.\beta\gamma$, each $=\alpha\beta\gamma$, which has thus a determinate signification.

The associativeness of such symbols arises from the circumstance that the definitions of $\alpha, \beta, \gamma, \ldots$ determine the meanings of $\alpha\beta$, $\alpha\gamma$, &c.: if $\alpha, \beta, \gamma, \ldots$ were quasi-quantitative symbols such as the quaternion imaginaries i, j, k, then $\alpha\beta$ and $\beta\gamma$ might have by definition values δ and ϵ such that $\alpha\beta.\gamma$ and $\alpha.\beta\gamma$ ($=\delta\gamma$ and $\alpha\epsilon$ respectively) have unequal values.

Unity as a functional symbol denotes that the letters are unaltered, $1(x, y, z) = (x, y, z)$; whence $1\alpha = \alpha 1 = \alpha$.

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order: substitutions can be represented by the notation $\alpha = \frac{yzx}{xyz}$, the substitution which changes xyz into yzx, or as products of cyclical substitutions, $\alpha = \frac{yzx\ wu}{xyz\ uw}$, $= (xyz)(uw)$, the product of the cyclical interchanges x into y, y into z, and z into x; and u into w, w into u.

A set of symbols $\alpha, \beta, \gamma, \ldots$, such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ or $\beta\alpha$), is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols, $1, \alpha, \beta, \gamma, \ldots$ satisfying the foregoing condition is a group. When the number of the symbols (or terms) is $= n$, then the group is of the nth order; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is, in fact, a group of symbolical nth roots of unity.

A group is defined by means of the laws of combination of its symbols: for the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of independent functional symbols, or else, using distinct letters for the several terms of the group, employ a square diagram as presently mentioned.

Thus, in the first mode, a group is $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ $(\alpha^2 = 1,\ \beta^3 = 1,\ \alpha\beta = \beta^2\alpha)$; where observe that these conditions imply also $\alpha\beta^2 = \beta\alpha$.

Or, in the second mode, calling the same group $(1, \alpha, \beta, \gamma, \delta, \epsilon)$, the laws of combination are given by the square diagram

	1	α	β	γ	δ	ϵ
1	1	α	β	γ	δ	ϵ
α	α	1	γ	β	ϵ	δ
β	β	ϵ	δ	α	1	γ
γ	γ	δ	ϵ	1	α	β
δ	δ	γ	1	ϵ	β	α
ϵ	ϵ	β	α	δ	γ	1

,

for the symbols $(1, \alpha, \beta, \gamma, \delta, \epsilon)$ are in fact $= (1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2)$.

The general problem is to find all the groups of a given order n; thus if $n = 2$, the only group is $1, \alpha$ $(\alpha^2 = 1)$; if $n = 3$, the only group is $1, \alpha, \alpha^2$ $(\alpha^3 = 1)$; if $n = 4$, the groups are $1, \alpha, \alpha^2, \alpha^3$ $(\alpha^4 = 1)$, and $1, \alpha, \beta, \alpha\beta$ $(\alpha^2 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha)$; if $n = 6$, there are three groups, a group $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5$ $(\alpha^6 = 1)$, and two groups $1, \beta, \beta^2, \alpha, \alpha\beta,$

$\alpha\beta^2$ ($\alpha^2=1$, $\beta^3=1$); viz. in the first of these $\alpha\beta=\beta\alpha$, while in the other of them (that mentioned above) we have $\alpha\beta=\beta^2\alpha$, $\alpha\beta^2=\beta\alpha$.

But although the theory as above stated is a general one, including as a particular case the theory of substitutions, yet the general problem of finding all the groups of a given order n, is really identical with the apparently less general problem of finding all the groups of the same order n, which can be formed with the substitutions upon n letters; in fact, referring to the diagram, it appears that 1, α, β, γ, δ, ϵ may be regarded as substitutions performed upon the six letters 1, α, β, γ, δ, ϵ, viz. 1 is the substitution unity which leaves the order unaltered, α the substitution which changes $1\alpha\beta\gamma\delta\epsilon$ into $\alpha 1\gamma\beta\epsilon\delta$, and so for β, γ, δ, ϵ. This, however, does not in any wise show that the best or the easiest mode of treating the general problem is thus to regard it as a problem of substitutions: and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions.

Cambridge, 26th November, 1877.

No. 2. The theory of groups; graphical representation.

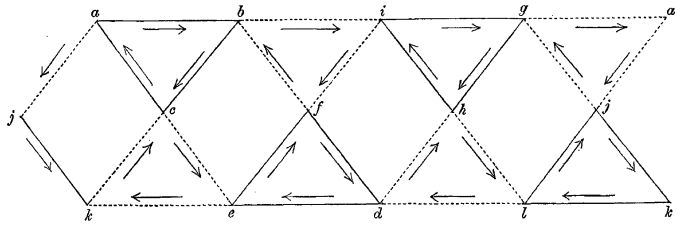
[From the *American Journal of Mathematics*, t. I. (1878), pp. 174—176.]

In regard to a substitution-group of the order n upon the same number of letters, I omitted to mention the important theorem that every substitution is *regular* (that is, either cyclical or composed of a number of cycles each of them of the same order). Thus, in the group of 6 given in No. 1, writing a, b, c, d, e, f in place of 1, α, β, γ, δ, ϵ, the substitutions of the group are 1, $ace.bfd$, $aec.bdf$, $ab.cd.ef$, $ad.be.cf$, $af.bc.de$.

Let the letters be represented by points; a change a into b will be represented by a directed line (line with an arrow) joining the two points; and therefore a cycle abc, that is, a into b, b into c, c into a, by the three sides of the trilateral abc, with the three arrows pointing accordingly, and similarly for the cycles $abcd$, &c.: the cycle ab means a into b, b into a, and we have here the line ab with a two-headed arrow pointing both ways; such a line may be regarded as a bilateral. A substitution is thus represented by a multilateral or system of multilaterals, each side with its arrow; and in the case of a regular substitution the multilaterals (if more than one) have each of them the same number of sides. To represent two or more substitutions we require different colours, the multilaterals belonging to any one substitution being of the same colour.

In order to represent a group we need to represent only independent substitutions thereof; that is, substitutions such that no one of them can be obtained from the others by compounding them together in any manner. I take as an example a group

of the order 12 upon 12 letters, where the number of independent substitutions is $=2$. See the diagram, wherein the continuous lines represent black lines, and the dotted lines, red lines.



The diagram is drawn, in the first instance, with the arrows but without the letters, which are then affixed *at pleasure;* viz. the *form of the group* is quite independent of the way in which this is done, though the group itself is of course dependent upon it. The diagram shows two substitutions, each of them of the third order: one is represented by the black triangles, and the other by the dotted triangles. It will be observed that there is *from* each point of the diagram (that is, in the direction of the arrow) one and only one black line, and one and only one dotted line; hence a symbol B, "move along a black line," B^2, "move successively along two black lines," BR (read always from right to left), "move first along a dotted line and then along a black line," has in every case a perfectly definite meaning and determines the path when the initial point is given; any such symbol may be spoken of as a "route."

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	1
b	*c*	*a*	*e*	*f*	*d*	*h*	*i*	*g*	*k*	*l*	*j*	*abc . def . ghi . jkl* (= *B*)
c	*a*	*b*	*f*	*d*	*e*	*i*	*g*	*h*	*l*	*j*	*k*	*acb . dfe . gih . jlk*
d	*l*	*h*	*a*	*g*	*j*	*e*	*c*	*k*	*f*	*i*	*b*	*ad . bl . ch . eg . fj . ik*
e	*j*	*i*	*b*	*h*	*k*	*f*	*a*	*l*	*d*	*g*	*c*	*aeh . bjd . cil . fkg*
f	*k*	*g*	*c*	*i*	*l*	*d*	*b*	*j*	*e*	*h*	*a*	*afl . bkh . cgd . eij*
g	*f*	*k*	*l*	*c*	*i*	*j*	*d*	*b*	*a*	*e*	*h*	*agj . bfi . cke . dlh*
h	*d*	*l*	*j*	*a*	*g*	*k*	*e*	*c*	*b*	*f*	*i*	*ahe . bdj . cli . fgk*
i	*e*	*j*	*k*	*b*	*h*	*l*	*f*	*a*	*c*	*d*	*g*	*ai . be . cj . dk . fh . gl*
j	*i*	*e*	*h*	*k*	*b*	*a*	*l*	*f*	*g*	*c*	*d*	*ajg . bif . cek . dhl* (= *R*)
k	*g*	*f*	*i*	*l*	*c*	*b*	*j*	*d*	*h*	*a*	*e*	*ak . bg . cf . di . el . hj*
l	*h*	*d*	*g*	*j*	*a*	*c*	*k*	*e*	*i*	*b*	*f*	*alf . bhk . cdg . eji*

The diagram has a remarkable property, *in virtue whereof it in fact represents a group.* It may be seen that any route leading from some one point a to itself, leads also from every other point to itself, or say from b to b, from c to c,..., and from l to l. We hence see that a route, applied in succession to the whole series of initial points or letters *abcdefghijkl*, gives a new arrangement of these letters, wherein no one of them occupies its original place; a route is thus, in effect, a substitution. Moreover, we may regard as distinct routes, those which lead from a to a, to b, to c,..., to l, respectively. We have thus 12 substitutions (the first of them, which leaves the arrangement unaltered, being the substitution unity), and these 12 substitutions form a group. I omit the details of the proof; it will be sufficient to give the square obtained by means of the several routes, or substitutions, performed upon the primitive arrangement *abcdefghijkl*, and the cyclical expressions of the substitutions themselves: it will be observed that the substitutions are unity, 3 substitutions of the order (or index) 2, and 8 substitutions of the order (or index) 3.

It may be remarked that the group of 12 is really the group of the 12 positive substitutions upon 4 letters *abcd*, viz. these are 1, *abc*, *acb*, *abd*, *adb*, *acd*, *adc*, *bcd*, *bdc*, *ab . cd*, *ac . bd*, *ad . bc*.

Cambridge, 16*th May*, 1878.

No. 3. The Newton-Fourier imaginary problem.

[From the *American Journal of Mathematics*, t. II. (1879), p. 97.]

The Newtonian method as completed by Fourier, or say the Newton-Fourier method, for the solution of a numerical equation by successive approximations, relates to an equation $f(x)=0$, with real coefficients, and to the determination of a certain real root thereof a by means of an assumed approximate real value ξ satisfying prescribed conditions: we then, from ξ, derive a nearer approximate value ξ_1 by the formula $\xi_1=\xi-\frac{f(\xi)}{f'(\xi)}$; and thence, in like manner, ξ_1, ξ_2, ξ_3,... approximating more and more nearly to the required root a.

In connexion herewith, throwing aside the restrictions as to reality, we have what I call the Newton-Fourier Imaginary Problem, as follows.

Take $f(u)$, a given rational and integral function of u, with real or imaginary coefficients; ξ, a given real or imaginary value, and from this derive ξ_1 by the formula $\xi_1=\xi-\frac{f(\xi)}{f'(\xi)}$, and thence ξ_1, ξ_2, ξ_3,... each from the preceding one by the like formula.

A given imaginary quantity $x+iy$ may be represented by a point the coordinates of which are (x, y): the roots of the equation are thus represented by given points

$A, B, C, \ldots$, and the values $\xi, \xi_1, \xi_2, \ldots$ by points $P, P_1, P_2, \ldots$ the first of which is assumed at pleasure, and the others each from the preceding one by the like given geometrical construction. The problem is to determine the regions of the plane such that, P being taken at pleasure anywhere within one region, we arrive ultimately at the point A; anywhere within another region at the point B; and so for the several points representing the roots of the equation.

The solution is easy and elegant in the case of a quadric equation: but the next succeeding case of the cubic equation appears to present considerable difficulty.

Cambridge, March 3rd, 1879.

No. 4. The mechanical construction of conformable figures.

[From the *American Journal of Mathematics*, t. II. (1879), p. 186.]

Is it possible to devise an apparatus for the mechanical construction of conformable figures; that is, figures which are similar as regards corresponding infinitesimal areas? The problem is to connect mechanically two points P_1 and P_2 in such wise that P_1 (1) shall have two degrees of freedom (or be capable of moving over a plane area) its position always determining that of P_2: (2) that if P_1, P_2 describe the infinitesimal lengths P_1Q_1, P_2Q_2, then the ratio of these lengths, and their mutual inclination, shall depend upon the position of P_1, but be independent of the direction of P_1Q_1: or what is the same thing, that if P_1 describe uniformly an indefinitely small circle, then P_2 shall also describe uniformly an indefinitely small circle, the ratio of the radii, and the relative position of the starting points in the two circles respectively, depending on the position of P_1.

Of course a pentagraph is a solution, but the two figures are in this case similar; and this is not what is wanted. Any unadjustable apparatus would give one solution only: the complete solution would be by an apparatus containing, suppose, a flexible lamina, so that P_1 moving in a given right line, the path of P_2 could be made to be any given curve whatever.

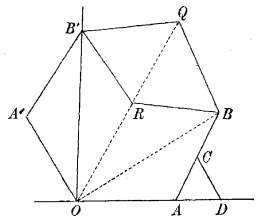
Cambridge, July 9th, 1879.

695.

A LINK-WORK FOR x^2: EXTRACT FROM A LETTER TO MR. SYLVESTER.

[From the *American Journal of Mathematics*, t. I. (1878), p. 386.]

I SUPPOSE the following is substantially your link-work for x^2. I use a slot to make D move in the line OA; but this could be replaced by proper link-work. Supposing O and A fixed; the line OB is movable, and I wanted to have the



distance OB measured in a fixed direction. This can be done by a hexagon $OABQB'A'$ with equal sides, and two other equal links $B'R$, BR: then of course, if O, R, Q are in lineâ, the hexagon will be symmetrical as to OQ, and OB' will be equal to OB, and B' may be made to move in the fixed line OB'. If

$$BOA = \tfrac{1}{2}\theta, \quad OA = AB = a, \quad AC = CD = \tfrac{1}{2}a,$$

then

$$OB = 2a\cos\tfrac{1}{2}\theta, \quad OD = a(1+\cos\theta) = 2a\cos^2\tfrac{1}{2}\theta,$$

or

$$2a\,.\,OD = (OB)^2.$$

November 30, 1877.

696.

CALCULATION OF THE MINIMUM N.G.F. OF THE BINARY SEVENTHIC.

[From the *American Journal of Mathematics*, t. II. (1879), pp. 71—84.]

FOR the binary seventhic $(a, \ldots \between x, y)^7$ the number of the asyzygetic covariants $(a, \ldots)^\theta (x, y)^\mu$, or say of the deg-order $(\theta . \mu)$, is given as the coefficient of $a^\theta x^\mu$ in the function

$$\frac{1 - x^{-2}}{1 - ax^7 . 1 - ax^5 . 1 - ax^3 . 1 - ax . 1 - ax^{-1} . 1 - ax^{-3} . 1 - ax^{-5} . 1 - ax^{-7}}$$

developed in ascending powers of a. See my "Ninth Memoir on Quantics," *Phil. Trans.*, t. CLXI. (1871), pp. 17—50, [462].

This function is in fact

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where, developing in ascending powers of a, the second term $-\frac{1}{x^2} A\left(\frac{1}{x}\right)$ contains only negative powers of x, and it may consequently be disregarded: the number of asyzygetic covariants of the deg-order $(\theta . \mu)$ is thus equal to the coefficient of $a^\theta x^\mu$ in the function $A(x)$, which function is for this reason called the Numerical Generating Function (N.G.F.) of the binary seventhic; and the function $A(x)$ expressed as a fraction in its least terms is said to be the minimum N.G.F.

According to a theorem of Professor Sylvester's (*Proc. Royal Soc.*, t. XXVIII. (1878), pp. 11—13), this minimum N.G.F. is of the form

$$\frac{Z_0 + aZ_1 + a^2 Z_2 + \ldots + a^{36} Z_{36}}{1 - ax . 1 - ax^3 . 1 - ax^5 . 1 - ax^7 . 1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - a^{10} . 1 - a^{12}}$$

where $Z_0, Z_1, \ldots, Z_{36}$ are rational and integral functions of x of degrees not exceeding 14; and where, as will presently be seen, there is a symmetry in regard to the terms Z_0, Z_{36}; Z_1, Z_{35}; &c., equidistant from the middle term Z_{18}, such that the terms $Z_0, \ldots, Z_{18}$ being known, the remaining terms $Z_{19}, \ldots, Z_{36}$ can be at once written down.

Using only the foregoing properties, I obtained for the N.G.F. an expression which I communicated to Professor Sylvester, and which is published, *Comptes Rendus*, t. LXXXVII. (1878), p. 505, but with an erroneous value for the coefficient of a^7 and for that of the corresponding term a^{29}.* The correct value is

Numerator of Minimum N.G.F. is =

$$
\begin{aligned}
& 1 \\
& + a\ (-x - x^3 - x^5) \\
& + a^2\ (x^2 + x^4 + 2x^6 + x^8 + x^{10}) \\
& + a^3\ (-x^7 - x^9 - x^{11} - x^{13}) \\
& + a^4\ (2x^4 + x^8 + x^{14}) \\
& + a^5\ (x + 2x^3 - x^9 - x^{11}) \\
& + a^6\ (-1 + 2x^2 - x^4 - x^8 - x^{10} + x^{12}) \\
& + a^7\ (4x + x^3 + 3x^5 - x^9 + x^{11}) \\
& + a^8\ (2 - x^2 - 3x^6 - 3x^8 - x^{10} - x^{12}) \\
& + a^9\ (x + 3x^3 + x^5 - x^7 + 2x^9 + 2x^{13}) \\
& + a^{10}\ (-1 + 4x^2 - x^6 - 2x^8 - 2x^{10} - x^{14}) \\
& + a^{11}\ (5x + 3x^3 + 2x^5 - x^7 - 2x^9 - x^{11} + x^{13}) \\
& + a^{12}\ (5 + x^2 - 4x^6 - 6x^8 - 4x^{10} - x^{12} + 2x^{14}) \\
& + a^{13}\ (x - 4x^5 - 4x^7 - x^9 + x^{11} + 4x^{13}) \\
& + a^{14}\ (2 + 5x^2 + x^4 + x^6 - 2x^8 + 3x^{12} - x^{14}) \\
& + a^{15}\ (3x - x^3 - x^5 - 7x^7 - 5x^9 - x^{11} - x^{13}) \\
& + a^{16}\ (6 + 3x^3 + 3x^4 - 4x^6 - 3x^8 - x^{12} + 5x^{14}) \\
& + a^{17}\ (-x - 2x^3 - 9x^5 - 8x^7 - 4x^9 - 3x^{11} + 4x^{13}) \\
& + a^{18}\ (2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14}) \\
& + a^{19}\ (4x - 3x^3 - 4x^5 - 8x^7 - 9x^9 - 2x^{11} - x^{13}) \\
& + a^{20}\ (5 - x^2 - 3x^6 - 4x^8 + 3x^{10} + 3x^{12} + 6x^{14}) \\
& + a^{21}\ (-x - x^3 - 5x^5 - 7x^7 - x^9 - x^{11} + 3x^{13}) \\
& + a^{22}\ (-1 + 3x^2 - 2x^4 + x^8 + x^{10} + 5x^{12} + 2x^{14}) \\
& + a^{23}\ (4x + x^3 - x^5 - 4x^7 - 4x^9 + x^{13})
\end{aligned}
$$

* The existence of these errors was pointed out to me by Professor Sylvester in a letter dated 13th November, 1878.

$$
\begin{aligned}
&+ a^{24}(2 - x^2 - 4x^4 - 6x^6 - 4x^8 + x^{10} + 5x^{14}) \\
&+ a^{25}(x - x^3 - 2x^5 - x^7 + 2x^9 + 3x^{11} + 5x^{13}) \\
&+ a^{26}(-1 - 2x^4 - 2x^6 - x^8 + 4x^{10} - x^{14}) \\
&+ a^{27}(2x + 2x^5 - x^7 + x^9 + 3x^{11} + x^{13}) \\
&+ a^{28}(-x^2 - x^4 - 3x^6 - 3x^8 - x^{12} + 2x^{14}) \\
&+ a^{29}(x^3 - x^5 + 3x^9 + x^{11} + 4x^{13}) \\
&+ a^{30}(x^2 - x^4 - x^6 - x^{10} + 2x^{12} - x^{14}) \\
&+ a^{31}(-x^3 - x^5 + 2x^{11} + x^{13}) \\
&+ a^{32}(1 + x^6 + 2x^{10}) \\
&+ a^{33}(-x - x^3 - x^5 - x^7) \\
&+ a^{34}(x^4 + x^6 + 2x^8 + x^{10} + x^{12}) \\
&+ a^{35}(-x^9 - x^{11} - x^{13}) \\
&+ a^{36} \,.\, x^{14}.
\end{aligned}
$$

Denominator (as mentioned before) is

$$= 1 - ax \,.\, 1 - ax^3 \,.\, 1 - ax^5 \,.\, 1 - ax^7 \,.\, 1 - a^4 \,.\, 1 - a^6 \,.\, 1 - a^8 \,.\, 1 - a^{10} \,.\, 1 - a^{12}.$$

The method of calculation is as follows: write for a moment

$$\phi(a, x) = \frac{1 - x^{-2}}{1 - ax^7 \,.\, 1 - ax^5 \,.\, 1 - ax^3 \,.\, 1 - ax \,.\, 1 - ax^{-1} \,.\, 1 - ax^{-3} \,.\, 1 - ax^{-5} \,.\, 1 - ax^{-7}};$$

then $\phi(a, x)$, developed in ascending powers of a, and rejecting from the result all negative powers of x, is

$$= \frac{Z_0 + aZ_1 + \ldots + a^{36}Z_{36}}{1 - ax \,.\, 1 - ax^3 \,.\, 1 - ax^5 \,.\, 1 - ax^7 \,.\, 1 - a^4 \,.\, 1 - a^6 \,.\, 1 - a^8 \,.\, 1 - a^{10} \,.\, 1 - a^{12}},$$

developed in like manner in ascending powers of a; for the determination of the Z's up to Z_{18} we require only the development of $\phi(a, x)$ up to a^{18}; and, assuming that each Z is at most of the degree 14 in x, we require the coefficients of the different powers of a in $\phi(a, x)$ only up to x^{14}. Assuming then that $\phi(a, x)$ developed in ascending powers of a, up to a^{18}, rejecting all negative powers of x, and all positive powers greater than x^{14}, is

$$= X_0 + aX_1 + \ldots + a^{18}X_{18},$$

we have

$$X_0 + aX_1 + \ldots + a^{18}X_{18} = \frac{Z_0 + aZ_1 + \ldots + a^{18}Z_{18}}{1 - ax \,.\, 1 - ax^3 \,.\, 1 - ax^5 \,.\, 1 - ax^7 \,.\, 1 - a^4 \,.\, 1 - a^6 \,.\, 1 - a^8 \,.\, 1 - a^{10} \,.\, 1 - a^{12}},$$

or say

$$Z_0 + aZ_1 + \ldots + a^{18}Z_{18} = 1 - a^4 \,.\, 1 - a^6 \,.\, 1 - a^8 \,.\, 1 - a^{10} \,.\, 1 - a^{12} \,.$$
$$1 - ax \,.\, 1 - ax^3 \,.\, 1 - ax^5 \,.\, 1 - ax^7 \,.\, (X_0 + aX_1 + \ldots + a^{18}X_{18});$$

viz. developing here the right-hand side as far as a^{18}, but in each term rejecting the powers of x above x^{14}, the coefficients of the several powers $a^0, a^1, \ldots, a^{18}$ give the

required values $Z_0, Z_1, \ldots, Z_{18}$. We require, therefore, only to know the values of these functions $X_0, X_1, \ldots, X_{18}$.

To make a break in the calculation, it is convenient to write

$$1-ax\,.\,1-ax^3\,.\,1-ax^5\,.\,1-ax^7\,(X_0+aX_1+\ldots+a^{18}X_{18})=Y_0+aY_1+\ldots+a^{18}Y_{18};$$

putting then

$$1-ax\,.\,1-ax^3\,.\,1-ax^5\,.\,1-ax^7=1-ap+a^2q-a^3r,$$

where (up to x^{14})

$$p=x+x^2+x^3+x^7,$$
$$q=x^4+x^6+2x^8+x^{10}+x^{12},$$
$$r=x^9+x^{11}+x^{13},$$

we have

$$Y_0+aY_1+a^2Y_2+\ldots+a^{18}Y_{18}=(1-ap+a^2q-a^3r)\,(X_0+aX_1+a^2X_2+\ldots+a^{18}X_{18}).$$

The values of $Y_0, Y_1, \ldots, Y_{18}$ then are

$$\begin{array}{rrrrrrr}
 & Y_0 & Y_1 & Y_2 & Y_3 \;\ldots\; & & Y_{18} \\
\hline
= & X_0 & X_1 & X_2 & X_3 & & X_{18} \\
 & & -pX_0 & -pX_1 & -pX_2 & & -pX_{17} \\
 & & & +qX_0 & +qX_1 & & +qX_{16} \\
 & & & & -rX_0 & & -rX_{15}
\end{array}$$

the values being taken to x^{14} only; and we then have

$$Z_0+aZ_1+a^2Z_2+\ldots+a^{18}Z_{18}=1-a^4\,.\,1-a^6\,.\,1-a^8\,.\,1-a^{10}\,.\,1-a^{12}\,(Y_0+aY_1+\ldots+a^{18}Y_{18});$$

viz. the values of $Z_0, Z_1, \ldots, Z_{18}$ are

$$\begin{array}{rrrrrrrrrrr}
 & Z_0 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 & Z_9 \\
\hline
= & Y_0 & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 \\
 & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 & -Y_4 & -Y_5 \\
 & & & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 \\
 & & & & & & & & & -Y_0 & -Y_1
\end{array}$$

$$\begin{array}{rrrrrrrrrr}
 & Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\
\hline
= & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} \\
 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} & -Y_{13} & -Y_{14} \\
 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} \\
 & -Y_2 & -Y_3 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} \\
 & & & & & +2Y_0 & +2Y_1 & +2Y_2 & +2Y_3 & +2Y_4 \\
 & & & & & & & +2Y_0 & +2Y_1 & +2Y_2 \\
 & & & & & & & & & +Y_0.
\end{array}$$

The rule of symmetry, before referred to, is that the coefficient Z_{36-p} of a^{36-p} is obtained from the coefficient Z_p of a^p by changing each power x^q into x^{14-q}, the coefficients being unaltered; in particular Z_{18}, the coefficient of a^{18}, must remain

unaltered when each power x^q is changed into x^{14-q}; and the verification thus obtained of the value

$$Z_{18} = 2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14}$$

is in fact almost a complete verification of the whole work. Some other verifications, which present themselves in the course of the work, will be referred to further on.

We have, therefore, to calculate the coefficients $X_0, X_1, \ldots, X_{18}$; the function $\phi(a, x)$ regarded as a function of a is at once decomposed into simple fractions; viz. we have

$$\begin{aligned}
\phi(a, x) = & \frac{1 - x^{-2}}{1 - ax^7 . 1 - ax^5 . 1 - ax^3 . 1 - ax . 1 - ax^{-1} . 1 - ax^{-3} . 1 - ax^{-5} . 1 - ax^{-7}} \\
= & \frac{x^{54}}{1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12} . 1 - x^{14}} \frac{1}{1 - ax^7} \\
& - \frac{x^{40}}{1 - x^2 . 1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12}} \frac{1}{1 - ax^5} \\
& + \frac{x^{28}}{1 - x^2 . (1 - x^4)^2 . 1 - x^6 . 1 - x^8 . 1 - x^{10}} \frac{1}{1 - ax^3} \\
& - \frac{x^{18}}{1 - x^2 . (1 - x^4)^2 . (1 - x^6)^2 . 1 - x^8} \frac{1}{1 - ax} \\
& + \frac{x^{10}}{1 - x^2 . (1 - x^4)^2 . (1 - x^6)^2 . 1 - x^8} \frac{1}{1 - ax^{-1}} \\
& - \frac{x^4}{1 - x^2 . (1 - x^4)^2 . 1 - x^6 . 1 - x^8 . 1 - x^{10}} \frac{1}{1 - ax^{-3}} \\
& + \frac{1}{1 - x^2 . 1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12}} \frac{1}{1 - ax^{-5}} \\
& - \frac{x^{-2}}{1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12} . 1 - x^{14}} \frac{1}{1 - ax^{-7}} .
\end{aligned}$$

In order to obtain the expansion of $\phi(a, x)$ in the assumed form of an expansion in ascending powers of a, we must of course expand the simple fractions $\frac{1}{1 - ax^7}$, &c., in ascending powers of a, but it requires a little consideration to see that we must also expand the x-coefficients of these simple fractions *in ascending powers of* x. For instance, as regards the term independent of a, here developing the several coefficients as far as x^{18}, the last five terms give (see *post*)

$$\begin{array}{rrrrrrrrrrr}
 & & & & & & & & & & -\ x^{18} \\
 & & & & & & +\ x^{10} + & x^{12} + & 3x^{14} + & 5x^{16} + & 9x^{18} \\
 & & & & & -\ x^4 - & x^6 - & 3x^8 - & 4x^{10} - & 8x^{12} - & 11x^{14} - 18x^{16} - 24x^{18} \\
 & & & 1 + & x^2 + & 2x^4 + & 3x^6 + & 5x^8 + & 7x^{10} + & 11x^{12} + & 14x^{14} + 20x^{16} + 26x^{18} \\
-\ x^{-2} & . & -\ x^2 - & x^4 - & 2x^6 - & 2x^8 - & 4x^{10} - & 4x^{12} - & 6x^{14} - & 7x^{16} - & 10x^{18} \\
\hline
= -x^{-2} + 1 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

viz. the sum is $= 1 - x^{-2}$ as it should be*.

* To give the last degree of perfection to the beautiful method of Professor Cayley it would seem desirable that a proof should be given of the principle illustrated by the example in the text, and the nature of the mischief resulting from its neglect clearly pointed out.—EDS. of the A. J. M.

The expansion is required only as far as x^{14}: the first four terms are therefore to be disregarded, and, writing for shortness

$$E = \frac{1}{1-x^2 . (1-x^4)^2 (1-x^6)^2 . 1-x^8},$$

$$F = \frac{1}{1-x^2 . (1-x^4)^2 . 1-x^6 . 1-x^8 . 1-x^{10}},$$

$$G = \frac{1}{1-x^2 . 1-x^4 . 1-x^6 . 1-x^8 . 1-x^{10} . 1-x^{12}},$$

$$H = \frac{1}{1-x^4 . 1-x^6 . 1-x^8 . 1-x^{10} . 1-x^{12} . 1-x^{14}},$$

we have

$$\phi(a, x) = \frac{x^{10}E}{1-ax^{-1}} - \frac{x^4F}{1-ax^{-3}} + \frac{G}{1-ax^{-5}} - \frac{x^{-2}H}{1-ax^{-7}},$$

which is

$$\begin{aligned} = \quad & x^{10}E\,(1+ax^{-1}+a^2x^{-2}+\ldots) \\ - \quad & x^4 F\,(1+ax^{-3}+a^2x^{-6}+\ldots) \\ + \quad & G\,(1+ax^{-5}+a^2x^{-10}+\ldots) \\ - \quad & x^{-2}H\,(1+ax^{-7}+a^2x^{-14}+\ldots), \end{aligned}$$

where the several series are to be continued up to a^{18}, and, after substituting for E, F, G, H their expansions in ascending powers of x, we are to reject negative powers of x, and also powers higher than x^{14}. The functions E, F, G, H contain each of them only even powers of x, and it is easy to see that we require the expansions up to x^{22}, x^{64}, x^{104} and x^{142} respectively. For the sake of a verification, I in fact calculated E, F up to x^{64} and G, H up to x^{142}: viz. we have

$$(1-x^6)\,E = (1-x^{10})\,F,$$

from the coefficients of E we have those of $(1-x^6)\,E$, and in the process of calculating F we have at the last step but one the coefficients of $(1-x^{10})\,F$, the agreement of the two sets being the verification; similarly,

$$(1-x^2)\,G = (1-x^{14})\,H$$

gives a verification. The process for the calculation of E,

$$= \frac{1}{1-x^2 . (1-x^4)^2 (1-x^6)^2 . 1-x^8},$$

is as follows:

	Ind. x 0	2	4	6	8	10	12	14	16	18	20	22
$(1-x^2)^{-1}$	1	1	1	1	1	1	1	1	1	1	1	1
			1	1	2	2	3	3	4	4	5	5
$(1-x^4)^{-1}$	1	1	2	2	3	3	4	4	5	5	6	6
			1	1	3	3	6	6	10	10	15	15
$(1-x^4)^{-1}$	1	1	3	3	6	6	10	10	15	15	21	21
				1	1	3	4	7	9	14	17	24
$(1-x^5)^{-1}$	1	1	3	4	7	9	14	17	24	29	38	45
				1	1	3	5	8	12	19	25	36
$(1-x^6)^{-1}$	1	1	3	5	8	12	19	25	36	48	63	81
					1	1	3	5	9	13	22	30
$E=(1-x^8)^{-1}$	1	1	3	5	9	13	22	30	45	61	85	111

the alternate lines giving the developments of the functions

$$(1-x^2)^{-1},\ (1-x^2)^{-1}(1-x^4)^{-1},\ (1-x^2)^{-1}(1-x^4)^{-2},\ \ldots,$$

which are the products of the x-functions down to any particular line. And in like manner we have the expansions of the other functions F, G, H respectively. I give first the expansions of E, F, G, H; next the calculation of the X's; then the calculation of the Y's: and from these the Z's up to Z_{18}, or coefficients of the powers a^0, a^1, ..., a^{18} in the numerator of the N.G.F. are at once found; and the coefficients of the remaining powers a^{19}, ..., a^{36} are then deduced, as already mentioned.

Writing in the formula $x=0$, we have, for the numerator of the N.G.F. of the *invariants*, the expression

$$1-a^6+2a^8-a^{10}+5a^{12}+2a^{14}+6a^{16}+2a^{18}+5a^{20}-a^{22}+2a^{24}-a^{26}+a^{32},$$

agreeing with a result in my "Second Memoir on Quantics," *Phil. Trans.*, t. CXLVI. (1856), [Number 141, vol. II. in this Collection, p. 266]; this, then, was a known result, and it affords a verification, not only of the terms in x^0, but also of those in x^{14}. Thus, in calculating the foregoing expression of the numerator, we obtain $Z_4=(2x^4+x^8+x^{14})$, viz. the term is

$$a^4(2x^4+x^8+x^{14}),$$

and we thence have the corresponding term $a^{22}(1+x^8+2x^{10})$, which, when $x=0$, becomes $=a^{22}$, a term of the numerator for the invariants: and the term $1x^{14}$ of Z_4

is thus verified, viz. so soon as, in the calculation, we arrive at this term, we know that it is right, and the calculation up to this point is, to a considerable extent, verified. And similarly, in continuing the calculation, we arrive at other terms which are verified in the like manner.

Expansions of the Functions E, F, G, H.

Ind. x	E	F	G	H	Ind. x	E	F	G	H
0	1	1	1	1	16	45	36	20	6
2	1	1	1	0	18	61	47	26	7
4	3	3	2	1	20	85	66	35	10
6	5	4	3	1	22	111	84	44	11
8	9	8	5	2	24		113	58	16
10	13	11	7	2	26		141	71	17
12	22	18	11	4	28		183	90	23
14	30	24	14	4	30		225	110	26

Ind. x	F	G	H	Ind. x	G	H	Ind. x	H
32	284	136	33	70	2172	419	108	2265
34	344	163	37	72	2432	472	110	2426
36	425	199	47	74	2702	515	112	2623
38	508	235	52	76	3009	576	114	2807
40	617	282	64	78	3331	629	116	3026
42	729	331	72	80	3692	699	118	3232
44	872	391	86	82	4070	760	120	3479
46	1020	454	96	84	4494	843	122	3708
48	1205	532	115	86	4935	913	124	3981
50	1397	612	127	88	5427	1007	126	4240
52	1632	709	149	90	5942	1091	128	4541
54	1877	811	166	92	6510	1197	130	4828
56	2172	931	192	94	7104	1293	132	5164
58	2480	1057	212	96	7760	1416	134	5481
60	2846	1206	245	98	8442	1525	136	5850
62	3228	1360	269	100	9192	1663	138	6204
64	3677	1540	307	102	9975	1790	140	6609
66		1729	338	104	10829	1945	142	6998
68		1945	382	106		2088		

Calculation of the X's.

Ind. x even or odd according as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
						1	1	3
			− 1	− 1	− 3	− 4	− 8	− 11
	1	1	2	3	5	7	11	14
		− 1	− 1	− 2	− 2	− 4	− 4	− 6
$X_0 =$	1	0	0	0	0	0	0	0
					1	1	3	
	− 1	− 1	− 3	− 4	− 8	− 11	− 18	
	3	5	7	11	14	20	26	
	− 2	− 4	− 4	− 6	− 7	− 10	− 11	
$X_1 =$	0	0	0	+ 1	0	0	0	
					1	1	3	5
	− 1	− 3	− 4	− 8	− 11	− 18	− 24	− 36
	7	11	14	20	26	35	44	58
	− 6	− 7	− 10	− 11	− 16	− 17	− 23	− 26
$X_2 =$	0	+ 1	0	+ 1	0	+ 1	0	+ 1
				1	1	3	5	
	− 4	− 8	− 11	− 18	− 24	− 36	− 47	
	20	26	35	44	58	71	90	
	− 16	− 17	− 23	− 26	− 33	− 37	− 47	
$X_3 =$	0	+ 1	+ 1	+ 1	+ 2	+ 1	+ 1	
				1	1	3	5	9
	− 8	− 11	− 18	− 24	− 36	− 47	− 66	− 84
	35	44	58	71	90	110	136	163
	− 26	− 33	− 37	− 47	− 52	− 64	− 72	− 86
$X_4 =$	1	0	+ 3	+ 1	+ 3	+ 2	+ 3	+ 2
			1	1	3	5	9	
	− 18	− 24	− 36	− 47	− 66	− 84	− 113	
	71	90	110	136	163	199	235	
	− 52	− 64	− 72	− 86	− 96	− 115	− 127	
$X_5 =$	1	+ 2	+ 3	+ 4	+ 4	+ 5	+ 4	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
			1	1	3	5	9	13
	− 24	− 36	− 47	− 66	− 84	− 113	− 141	− 183
	110	136	163	199	235	282	331	391
	− 86	− 96	− 115	− 127	− 149	− 166	− 191	− 212
$X_6 =$	0	+ 4	+ 2	+ 7	+ 5	+ 8	+ 8	+ 9
		1	1	3	5	9	13	
	− 47	− 66	− 84	− 113	− 141	− 183	− 225	
	199	235	282	331	391	454	532	
	− 149	− 166	− 192	− 212	− 245	− 269	− 307	
$X_7 =$	3	+ 4	+ 7	+ 9	+ 10	+ 11	+ 13	
		1	1	3	5	9	13	22
	− 66	− 84	− 113	− 141	− 183	− 225	− 284	− 344
	282	331	391	454	532	612	709	811
	− 212	− 245	− 269	− 307	− 338	− 382	− 419	− 472
$X_8 =$	4	+ 3	+ 10	+ 9	+ 16	+ 14	+ 19	+ 17
	1	1	3	5	9	13	22	
	− 113	− 141	− 183	− 225	− 284	− 344	− 425	
	454	532	612	709	811	931	1057	
	− 338	− 382	− 419	− 472	− 515	− 576	− 629	
$X_9 =$	4	+ 10	+ 13	+ 17	+ 21	+ 24	+ 25	
	1	1	3	5	9	13	22	30
	− 141	− 183	− 225	− 284	− 344	− 425	− 508	− 617
	612	709	811	931	1057	1206	1360	1540
	− 472	− 515	− 576	− 629	− 699	− 760	− 843	− 913
$X_{10} =$	0	+ 12	+ 13	+ 23	+ 23	+ 34	+ 31	+ 40
	1	3	5	9	13	22	30	
	− 225	− 284	− 344	− 425	− 508	− 617	− 729	
	931	1057	1206	1360	1540	1729	1945	
	− 699	− 760	− 843	− 913	− 1007	− 1091	− 1197	
$X_{11} =$	8	+ 16	+ 24	+ 31	+ 38	+ 43	+ 49	
	1	3	5	9	13	22	30	45
	− 284	− 344	− 425	− 508	− 617	− 729	− 872	− 1020
	1206	1360	1540	1729	1945	2172	2432	2702
	− 913	− 1007	− 1091	− 1197	− 1293	− 1416	− 1525	− 1663
$X_{12} =$	10	+ 12	+ 29	+ 33	+ 48	+ 49	+ 65	+ 64

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	3	5	9	13	22	30	45	
	− 425	− 508	− 617	− 729	− 872	− 1020	− 1205	
	1729	1945	2172	2432	2702	3009	3331	
	− 1293	− 1416	− 1525	− 1663	− 1790	− 1945	− 2088	
$X_{13}=$	14	+ 26	+ 39	+ 53	+ 62	+ 74	+ 83	
	3	5	9	13	22	30	45	61
	− 508	− 617	− 729	− 872	− 1020	− 1205	− 1397	− 1632
	2172	2432	2702	3009	3331	3692	4070	4494
	− 1663	− 1790	− 1945	− 2088	− 2265	− 2426	− 2623	− 2807
$X_{14}=$	4	+ 30	+ 37	+ 62	+ 68	+ 91	+ 95	+ 116
	5	9	13	22	30	45	61	
	− 729	− 872	− 1020	− 1205	− 1397	− 1632	− 1877	
	3009	3331	3692	4070	4494	4935	5427	
	− 2265	− 2426	− 2623	− 2807	− 3026	− 3232	− 3479	
$X_{15}=$	20	+ 42	+ 62	+ 80	+ 101	+ 116	+ 132	
	5	9	13	22	30	45	61	85
	− 872	− 1020	− 1205	− 1397	− 1632	− 1877	− 2172	− 2480
	3692	4070	4494	4935	5427	5942	6510	7104
	− 2807	− 3026	− 3232	− 3479	− 3708	− 3981	− 4240	− 4541
$X_{16}=$	18	+ 33	+ 70	+ 81	+ 117	+ 129	+ 159	+ 168
	9	13	22	30	45	61	85	
	− 1205	− 1397	− 1632	− 1877	− 2172	− 2480	− 2846	
	4935	5427	5942	6510	7104	7760	8442	
	− 3708	− 3981	− 4240	− 4541	− 4828	− 5164	− 5481	
$X_{17}=$	31	+ 62	+ 92	+ 122	+ 149	+ 177	+ 200	
	9	13	22	30	45	61	85	111
	− 1397	− 1632	− 1877	− 2172	− 2480	− 2846	− 3228	− 3677
	5942	6510	7104	7760	8442	9192	9975	10829
	− 4541	− 4828	− 5164	− 5481	− 5850	− 6204	− 6609	− 6998
$X_{18}=$	13	+ 63	+ 85	+ 137	+ 157	+ 203	+ 223	+ 265

Calculation of the Y's.

Ind. x even or odd as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	1							
$Y_0 =$	1							
				1				
	− 1	− 1	− 1	− 1				
$Y_1 =$	− 1	− 1	− 1	0	0	0	0	
	0	1	0	1	0	1	0	1
					− 1	− 1	− 1	− 1
			1	1	2	1	1	
$Y_2 =$	0	1	1	2	1	1	0	0
		1	1	1	2	1	1	
		− 1	− 1	− 2	− 2	− 2	− 2	
						1	1	
					− 1	− 1	− 1	
$Y_3 =$	0	0	0	− 1	− 1	− 1	− 1	
	1	0	3	1	3	2	3	2
			− 1	− 2	− 3	− 5	− 5	− 5
				1	1	3	2	4
$Y_4 =$	1	0	+ 2	0	+ 1	0	0	+ 1
	1	2	3	4	4	5	4	
	− 1	− 1	− 4	− 5	− 7	− 9	− 9	
				1	2	4	6	
						− 1	− 1	
$Y_5 =$	0	+ 1	− 1	0	− 1	− 1	0	
		4	2	7	5	8	7	9
		− 1	− 3	− 6	− 10	− 13	− 16	− 17
			1	1	5	5	11	10
							− 1	− 2
$Y_6 =$	0	+ 3	0	+ 2	0	0	+ 1	0

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	3	4	7	9	10	11	13	
		− 4	− 6	− 13	− 18	− 22	− 27	
			1	3	7	12	17	
					− 1	− 1	− 4	
$Y_7 =$	3	0	+ 2	− 1	− 2	0	− 1	
	4	3	10	9	16	14	19	17
		− 3	− 7	− 14	− 23	− 30	− 37	− 43
				4	6	17	20	33
						− 1	− 3	− 6
$Y_8 =$	4	0	+ 3	− 1	− 1	0	− 1	+ 1
	4	10	13	17	21	24	25	
	− 4	− 7	− 17	− 26	− 38	− 49	− 58	
			3	7	17	27	40	
						− 4	− 6	
$Y_9 =$	0	+ 3	− 1	− 2	0	− 2	+ 1	
		12	13	23	23	34	31	40
		− 4	− 14	− 27	− 44	− 61	− 75	− 87
			4	7	21	29	52	61
						− 3	− 7	− 14
$Y_{10} =$	0	+ 8	+ 3	+ 3	0	− 1	+ 1	0
	8	16	24	31	38	43	49	
		− 12	− 25	− 48	− 71	− 93	− 111	
			4	14	31	54	78	
					− 4	− 7	− 17	
$Y_{11} =$	8	+ 4	+ 3	− 3	− 6	− 3	− 1	
	10	12	29	33	48	49	65	64
		− 8	− 24	− 48	− 79	− 109	− 136	− 161
				12	25	60	84	128
						− 4	− 14	− 27
$Y_{12} =$	10	+ 4	+ 5	− 3	− 6	− 4	− 1	+ 4

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	14	26	39	53	62	74	83	
	− 10	− 22	− 51	− 84	− 122	− 159	− 195	
			8	24	56	95	141	
						− 12	− 25	
$Y_{13}=$	4	+ 4	− 4	− 7	− 4	− 2	+ 4	
	4	30	37	62	68	91	95	116
		− 14	− 40	− 79	− 132	− 180	− 228	− 272
			10	22	61	96	161	204
						− 8	− 24	− 48
$Y_{14}=$	4	+ 16	+ 7	+ 5	− 3	− 1	+ 4	0
	20	42	62	80	101	116	132	
	− 4	− 34	− 71	− 133	− 197	− 258	− 316	
			14	40	93	158	233	
					− 10	− 22	− 51	
$Y_{15}=$	16	+ 8	+ 5	− 13	− 13	− 6	− 2	
	18	33	70	81	117	129	159	168
		− 20	− 62	− 124	− 204	− 285	− 359	− 429
			4	34	75	163	238	350
						− 14	− 40	− 79
$Y_{16}=$	18	+ 13	+ 12	− 9	− 12	− 7	− 2	+ 10
	31	62	92	122	149	177	200	
	− 18	− 51	− 121	− 202	− 301	− 397	− 486	
			20	62	144	246	367	
					− 4	− 34	− 71	
$Y_{17}=$	13	+ 11	− 9	− 18	− 12	− 8	+ 10	
	13	63	85	137	157	203	223	265
		− 31	− 93	− 185	− 307	− 425	− 540	− 648
			18	51	139	235	389	511
						− 20	− 62	− 124
$Y_{18}=$	13	+ 32	+ 10	+ 3	− 11	− 7	+ 10	+ 4

Cambridge, December 7th, 1878.

697.

ON THE DOUBLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 74—81.]

I HAVE sought to obtain, in forms which may be useful in regard to the theory of the double ϑ-functions, the integral of the elliptic differential equation

$$\frac{dx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}}+\frac{dy}{\sqrt{a-y\,.\,b-y\,.\,c-y\,.\,d-y}}=0:$$

the present paper has immediate reference only to this differential equation; but, on account of the design of the investigation, I have entitled it as above.

We may for the general integral of the above equation take a particular integral of the equation

$$\frac{dx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}}+\frac{dy}{\sqrt{a-y\,.\,b-y\,.\,c-y\,.\,d-y}}\pm\frac{dz}{\sqrt{a-z\,.\,b-z\,.\,c-z\,.\,d-z}}=0;$$

viz. this particular integral, regarding therein z as an arbitrary constant, will be the general integral of the first mentioned equation. And we may further assume that z is the value of y corresponding to the value a of x.

I write for shortness

$$a-x,\ b-x,\ c-x,\ d-x=\mathrm{a}\,,\ \mathrm{b}\,,\ \mathrm{c}\,,\ \mathrm{d}\,,$$
$$a-y,\ b-y,\ c-y,\ d-y=\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1;$$

and I write also $(xy,\ bc,\ ad)$, or more shortly $(bc,\ ad)$ to denote the determinant

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix};$$

we have of course $(ad,\ bc)=-(bc,\ ad)$, and there are thus the three distinct determinants $(ad,\ bc)$, $(bd,\ ac)$ and $(cd,\ ab)$.

We have then for each of the functions

$$\sqrt{\frac{a-z}{d-z}},\quad \sqrt{\frac{b-z}{d-z}},\quad \sqrt{\frac{c-z}{d-z}}$$

a set of four equivalent expressions, the whole system being

$$\sqrt{\frac{a-z}{d-z}}=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{adb_1c_1}}+\sqrt{\mathrm{a_1d_1bc}}\}}{(bc,\ ad)}=\frac{\sqrt{a-b\,.\,a-c}\,(x-y)}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{abc_1d_1}}+\sqrt{\mathrm{a_1b_1cd}}\}}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{acb_1d_1}}+\sqrt{\mathrm{a_1c_1bd}}\}}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}};$$

$$\sqrt{\frac{b-z}{d-z}}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{(a-c)\sqrt{\mathrm{bdb_1d_1}}+(b-d)\sqrt{\mathrm{aca_1c_1}}\}}{(bc,\ ad)}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{\sqrt{\mathrm{abc_1d_1}}-\sqrt{\mathrm{a_1b_1cd}}\}}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{\frac{a-b}{a-d}}\,(cd,\ ab)}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{(a-d)\sqrt{\mathrm{bcb_1c_1}}+(b-c)\sqrt{\mathrm{ada_1d_1}}\}}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}};$$

$$\sqrt{\frac{c-z}{d-z}}=\frac{\sqrt{\frac{a-c}{a-d}}\,\{(a-b)\sqrt{\mathrm{cdc_1d_1}}+(c-d)\sqrt{\mathrm{aba_1b_1}}\}}{(bc,\ ad)}=\frac{\sqrt{\frac{a-c}{a-d}}\,\{\sqrt{\mathrm{acb_1d_1}}-\sqrt{\mathrm{a_1c_1bd}}\}}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{\frac{a-c}{a-d}}\,\{(a-d)\sqrt{\mathrm{bcb_1c_1}}-(b-c)\sqrt{\mathrm{ada_1d_1}}\}}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{\frac{a-c}{a-d}}\,(bd,\ ac)}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}}.$$

The expressions in the like fourfold form for the functions $\mathrm{sn}\,(u+v)$, $\mathrm{cn}\,(u+v)$, $\mathrm{dn}\,(u+v)$ are given p. 63 of my *Treatise on Elliptic Functions.*

It is easy to verify first that the four expressions for the same function of z are identical, and next that the expressions for the three several functions

$$\sqrt{\frac{a-z}{d-z}},\quad \sqrt{\frac{b-z}{d-z}},\quad \sqrt{\frac{c-z}{d-z}},$$

are consistent with each other. For instance, comparing the first and second expressions of $\sqrt{\frac{a-z}{d-z}}$, the equation to be verified is

$$\mathrm{adb_1c_1}-\mathrm{a_1d_1bc}=(x-y)\,(bc,\ ad),$$

which is at once shown to be true. Again comparing the first and second expressions for $\sqrt{\frac{b-z}{d-z}}$, we ought to have

$$\{(a-c)\sqrt{\mathrm{bdb_1d_1}}+(b-d)\sqrt{\mathrm{aca_1c_1}}\}\,\{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}\}=(bc,\ ad)\,\{\sqrt{\mathrm{abc_1d_1}}-\sqrt{\mathrm{a_1b_1cd}}\}.$$

Here the product on the left-hand side is

$$= (a-c)\{b_1 d\sqrt{abc_1d_1} - bd_1\sqrt{a_1b_1cd}\} + (b-d)\{-a_1c\sqrt{abc_1d_1} + ac_1\sqrt{a_1b_1cd}\},$$

viz. this is

$$= \sqrt{abc_1d_1}\,\{(a-c)\,b_1d - (b-d)\,a_1c\} - \sqrt{a_1b_1cd}\,\{(a-c)\,bd_1 - (b-d)\,ac_1\},$$

and in this last expression the two terms in { } are at once shown to be each $=(bc,\ ad)$; whence the identity in question.

Comparing in like manner the first expressions for $\sqrt{\dfrac{a-z}{d-z}}$ and $\sqrt{\dfrac{b-z}{d-z}}$ respectively, we have

$$(b-d)\,(bc,\ ad)^2\,\frac{a-z}{d-z} = (a-b)\,(a-c)\,(b-d)\,\{adb_1c_1 + a_1d_1bc + 2\sqrt{abcda_1b_1c_1d_1}\},$$

$$(d-a)\,(bc,\ ad)^2\,\frac{b-z}{d-z} =$$

$$-(a-b)\,\{(a-c)^2\,bdb_1d_1 + (b-d)^2\,aca_1c_1 + 2\,(a-c)\,(b-d)\sqrt{abcda_1b_1c_1d_1}\},$$

whence, adding, the radical on the right-hand side disappears; the whole equation divides by $-(a-b)$, and omitting this factor, the relation to be verified is

$$(bc,\ ad)^2 = (a-c)^2\,bdb_1d_1 + (b-d)^2\,aca_1c_1 - (a-c)\,(b-d)\,(adb_1c_1 + a_1d_1bc);$$

the right-hand side is here

$$= \{(a-c)\,b_1d - (b-d)\,a_1c\}\,\{(a-c)\,bd_1 - (b-d)\,ac_1\},$$

and each of the two factors being $=(bc,\ ad)$, the identity is verified. It thus appears that the twelve equations are in fact equivalent to a single equation in x, y, z.

Writing in the several formulæ $x=a$, b, c, d successively, they become

$x=a$,	$x=b$,	$x=c$,	$x=d$,
$\dfrac{a-z}{d-z} = \dfrac{a_1}{d_1}$,	$-\dfrac{c-a}{d-b}\cdot\dfrac{b_1}{c_1}$,	$-\dfrac{b-a}{d-c}\cdot\dfrac{c_1}{b_1}$,	$\dfrac{a-b\,.\,a-c}{d-b\,.\,d-c}\cdot\dfrac{d_1}{a_1}$,
$\dfrac{b-z}{d-z} = \dfrac{b_1}{d_1}$,	$-\dfrac{c-b}{d-a}\cdot\dfrac{a_1}{c_1}$,	$\dfrac{b-a\,.\,b-c}{d-a\,.\,d-c}\cdot\dfrac{d_1}{b_1}$,	$-\dfrac{a-b}{d-c}\cdot\dfrac{c_1}{a_1}$,
$\dfrac{c-z}{d-z} = \dfrac{c_1}{d_1}$,	$\dfrac{c-a\,.\,c-b}{d-a\,.\,d-b}\cdot\dfrac{d_1}{c_1}$,	$-\dfrac{b-c}{d-a}\cdot\dfrac{a_1}{b_1}$,	$-\dfrac{a-c}{d-b}\cdot\dfrac{b_1}{a_1}$,

viz. for $x=a$, the relation is $z=y$, but in the other three cases respectively the relation is a linear one, $z=\dfrac{\alpha y+\beta}{\gamma y+\delta}$.

Rationalising the first equation for $\sqrt{\dfrac{a-z}{d-z}}$, we have

$$(bc,\ ad)^2\,(a-z) = (a-b)\,(a-c)\,(d-z)\,\{adb_1c_1 + a_1d_1bc + 2\sqrt{abcda_1b_1c_1d_1}\},$$

and thence

$$\{(bc,\ ad)^2\,(a-z) - (a-b)\,(a-c)\,(d-z)\,(adb_1c_1 + a_1d_1bc)\}^2$$
$$= (a-b)^2\,(a-c)^2\,(d-z)^2\,.\,4abcda_1b_1c_1d_1.$$

Expanding, and observing that

$$(\mathrm{adb_1c_1 + a_1d_1bc})^2 = (\mathrm{adb_1c_1 - a_1d_1bc})^2 + 4\mathrm{abcda_1b_1c_1d_1} = (bc,\ ad)^2\,(x-y)^2 + 4\mathrm{abcda_1b_1c_1d_1},$$

the whole equation becomes divisible by $(bc,\ ad)^2$, and omitting this factor, the equation is

$$(bc,\ ad)^2\,(a-z)^2 - 2\,(a-b)\,(a-c)\,(a-z)\,(d-z)\,(\mathrm{adb_1c_1 + a_1d_1bc}) \\ + (a-b)^2\,(a-c)^2\,(d-z)^2\,(x-y)^2 = 0,$$

or, as this may also be written,

$$\begin{aligned} & z^2\,\{(bc,\ ad)^2 \ - 2\,(a-b)\,(a-c)\,(\mathrm{adb_1c_1 + a_1d_1bc}) \qquad + (a-b)^2\,(a-c)^2\,(x-y)^2 \ \} \\ - & 2z\ \{(bc,\ ad)\,a - \ (a-b)\,(a-c)\,(\mathrm{adb_1c_1 + a_1d_1bc})\,(a+d) + (a-b)^2\,(a-c)^2\,(x-y)^2\,d\ \} \\ + & \ \{(bc,\ ad)\,a^2 - 2\,(a-b)\,(a-c)\,(\mathrm{adb_1c_1 + a_1d_1bc})\,ad \qquad + (a-b)^2\,(a-c)^2\,(x-y)^2\,d^2\} = 0. \end{aligned}$$

This is really a symmetrical equation in x, y, z of the form

$$\begin{aligned} & A \\ & + 2B\,(x+y+z) \\ & + \ C\,(x^2+y^2+z^2) \\ & + 2D\,(yz+zx+xy) \\ & + 2E\,(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ & + 4Fxyz \\ & + 2G\,(x^2yz+xy^2z+xyz^2) \\ & + \ H\,(y^2z^2+z^2x^2+x^2y^2) \\ & + 2I\,(xy^2z^2+x^2yz^2+x^2y^2z) \\ & + \ Jx^2y^2z^2 = 0\,; \end{aligned}$$

the several coefficients being symmetrical as regards b, c, d, but the a entering unsymmetrically: the actual values are

$$\begin{aligned} A = & \quad a^4\,\{b^2c^2+b^2d^2+c^2d^2-2bcd\,(b+c+d)\} + 2a^3bcd\,(bc+bd+cd) - 3a^2b^2c^2d^2, \\ B = & \quad 2a^4bcd - a^3\,(b^2c^2+b^2d^2+c^2d^2) + ab^2c^2d^2, \\ C = & -4a^3bcd + a^2\,(bc+bd+cd)^2 - 2abcd\,(bc+bd+cd) + b^2c^2d^2, \\ D = & - \ a^4\,(bc+bd+cd) + a^3\,(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2-2bcd) \\ & \qquad + a^2\,\{b^2c^2+b^2d^2+c^2d^2-bcd\,(b+c+d)\} - b^2c^2d^2, \\ E = & \quad a^3\,(bc+bd+cd) - a^2\,(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2) + abcd\,(b+c+d), \\ F = & \quad a^4\,(b+c+d) - a^3\,(b^2+c^2+d^2+bc+bd+cd) + 6a^2bcd \\ & \qquad - a\,\{b^2c^2+b^2d^2+c^2d^2+bcd\,(b+c+d)\} + bcd\,(bc+bd+cd), \\ G = & - \ a^4 + a^2\,(b^2+c^2+d^2-bc-bd-cd) + a\,(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2-2bcd) \\ & \qquad - bcd\,(b+c+d), \\ H = & \quad a^4 - 2a^3\,(b+c+d) + a^2\,(b+c+d)^2 - 4abcd, \\ I = & \quad a^3 - a\,(b^2+c^2+d^2) + 2bcd, \\ J = & - 3a^2 + 2a\,(b+c+d) + b^2+c^2+d^2 - 2\,(bc+bd+cd). \end{aligned}$$

It may be remarked by way of verification that the equation remains unaltered on substituting for x, y, z, a, b, c, d their reciprocals and multiplying the whole by $a^4b^2c^2d^2x^2y^2z^2$.

I further remark that, writing $a=0$, we have

$$A=0,\quad B=0,\quad C=b^2c^2d^2,\quad D=-b^2c^2d^2,\quad E=0,\quad F=bcd\,(bc+bd+cd),$$

$$G=-bcd\,(b+c+d),\quad H=0,\quad I=2bcd,\quad J=b^2+c^2+d^2-2\,(bc+bd+cd);$$

and writing also

$$\epsilon=1,\quad -\delta=(b+c+d),\quad \gamma=bc+bd+cd,\quad -\beta=bcd,$$

(whence

$$a-x\,.\,b-x\,.\,c-x\,.\,d-x=\beta x+\gamma x^2+\delta x^3+\epsilon x^4),$$

we have the formula

$$\begin{aligned}&\beta^2\,(x^2+y^2+z^2-2yz-2zx-2xy)\\ &-4\beta\gamma\,xyz\\ &-2\beta\delta\,xyz\,(x+y+z)\\ &-4\beta\epsilon\,xyz\,(yz+zx+xy)\\ &+\ (\delta^2-4\gamma\epsilon)\,x^2y^2z^2=0,\end{aligned}$$

given p. 348 of my *Elliptic Functions* as a particular integral of the differential equation when the radical is $\sqrt{\beta x+\gamma x^2+\delta x^3+\epsilon x^4}$.

Let the equation in (x, y, z) be called $u=0$; u has been given in the form $u=\mathfrak{C}z^2-2\mathfrak{B}z+\mathfrak{A}$, and we thence have $\frac{1}{2}\frac{du}{dz}=\mathfrak{C}z-\mathfrak{B}$ which, in virtue of the equation $u=0$ itself, becomes $\frac{1}{2}\frac{du}{dz}=\sqrt{\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}}$; we find easily

$$\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=(a-b)^2\,(a-c)^2\,(a-d)^2\,\{(\mathrm{adb_1c_1}+\mathrm{a_1d_1bc})^2-(bc,\ ad)^2\,(x-y)^2\},$$

or, attending to the relation

$$\begin{aligned}(\mathrm{adb_1c_1}+\mathrm{a_1d_1bc})^2&=(\mathrm{adb_1c_1}-\mathrm{a_1d_1bc})^2+4\mathrm{abcda_1b_1c_1d_1}\\ &=(bc,\ ad)^2\,(x-y)^2+4\mathrm{abcda_1b_1c_1d_1},\end{aligned}$$

this is

$$\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=4\,(a-b)^2\,(a-c)^2\,(a-d)^2\,\mathrm{abcda_1b_1c_1d_1},$$

or we have

$$\tfrac{1}{4}\frac{du}{dz}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{abcd}}\,\sqrt{\mathrm{a_1b_1c_1d_1}}.$$

Writing

$$a-z,\ b-z,\ c-z,\ d-z=\mathrm{a_2},\ \mathrm{b_2},\ \mathrm{c_2},\ \mathrm{d_2},$$

we have of course the like formulæ

$$\tfrac{1}{4}\frac{du}{dx}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{a_1b_1c_1d_1}}\,\sqrt{\mathrm{a_2b_2c_2d_2}},$$

$$\tfrac{1}{4}\frac{du}{dy}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{abcd}}\,\sqrt{\mathrm{a_3b_3c_3d_3}};$$

and the equation $du=0$ then gives

$$\frac{dx}{\sqrt{abcd}}+\frac{dy}{\sqrt{a_1b_1c_1d_1}}+\frac{dz}{\sqrt{a_2b_2c_2d_2}}=0,$$

as it should do. The differential equation might also have been verified directly from any one of the expressions for

$$\sqrt{\frac{a-z}{d-z}},\quad \sqrt{\frac{b-z}{d-z}} \text{ or } \sqrt{\frac{c-z}{d-z}}.$$

Writing for shortness

$$X=a-x\,.\,b-x\,.\,c-x\,.\,d-x, \text{ etc.},$$

then the general integral of the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$$

by Abel's theorem is

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix}=0,$$

where w is the constant of integration: and it is to be shown that the value of w which corresponds to the integral given in the present paper is $w=a$. Observe that writing in the determinant $w=a$, the determinant on putting therein $x=a$, would vanish whether z were or were not $=y$; but this is on account of an extraneous factor $a-w$, so that we do not thus prove the required theorem that (w being $=a$) we have $y=z$ when $x=a$.

An equivalent form of Abel's integral is that there exist values A, B, C such that

$$Ax^2+Bx+C=\sqrt{X},$$
$$Ay^2+By+C=\sqrt{Y},$$
$$Az^2+Bz+C=\sqrt{Z},$$
$$Aw^2+Bw+C=\sqrt{W},$$

or, what is the same thing, that we have identically

$$(A\theta^2+B\theta+C)^2-\Theta=(A^2-1)\,.\,\theta-x\,.\,\theta-y\,.\,\theta-z\,.\,\theta-w.$$

We have therefore

$$C^2-abcd=(A^2-1)\,xyzw,$$

or say

$$xyzw=\frac{C^2-abcd}{A^2-1};$$

which equation, regarding therein A, B, C as determined by the three equations

$$Ax^2 + Bx + C = \sqrt{X},$$
$$Ay^2 + By + C = \sqrt{Y},$$
$$Aw^2 + Bw + C = \sqrt{W},$$

is a form of Abel's integral, giving z rationally in terms of x, y, w.

Supposing that, when $x = a$, $z = y$: then the last-mentioned integral gives

$$ay^2w = \frac{C^2 - abcd}{A^2 - 1},$$

where A, C are now determined by the equations

$$Aa^2 + Ba + C = 0,$$
$$Ay^2 + By + C = \sqrt{Y},$$
$$Aw^2 + Bw + C = \sqrt{W},$$

and, imagining these values actually substituted, it is to be shown that the equation

$$ay^2w = \frac{C^2 - abcd}{A^2 - 1}$$

is satisfied by the value $w = a$.

We have

$$A\,.\,a-y\,.\,a-w\,.\,w-y = \qquad (a-w)\sqrt{Y} - \qquad (a-y)\sqrt{W},$$
$$B\,.\,a-y\,.\,a-w\,.\,w-y = (a-w)(a+w)\sqrt{Y} - (a-y)(a+y)\sqrt{W},$$
$$C\,.\,a-y\,.\,a-w\,.\,w-y = (a-w)\,aw \qquad \sqrt{Y} - (a-y)\,ay \qquad \sqrt{W},$$

or writing as before

$$a-y,\ b-y,\ c-y,\ d-y = \mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1,$$

and also

$$a-w,\ b-w,\ c-w,\ d-w = \mathrm{a}_3,\ \mathrm{b}_3,\ \mathrm{c}_3,\ \mathrm{d}_3,$$

then $Y = \mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1$, $W = \mathrm{a}_3\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3$, and the formulæ become

$$A = \frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1} - \sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\},$$

$$B = \frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{-(a+w)\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1} + (a+y)\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\},$$

$$C = \frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{aw\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1} - ay\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\}.$$

If in these formulæ w is indefinitely nearly $=a$, then a_3 is indefinitely small, so that $\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1}$ may be neglected in comparison with $\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}$: also $w-y$ may be put $=\mathrm{a}_1$; the formulæ thus become

$$A=-\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},\quad B=(a+y)\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},\quad C=-ay\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},$$

where the values of A, B, C are each of them indefinitely large on account of the factor $\sqrt{\mathrm{a}_3}$ in the denominator; the value of C is $C=ayA$, and substituting this value in the equation

$$ay^2w=\frac{C^2-abcd}{A^2-1},$$

and then considering A as indefinitely large, the equation becomes $ay^2w=a^2y^2$, that is, $w=a$; so that $w=a$ is a value of w satisfying this equation.

Cambridge, 3 *July*, 1878.

698.

ON A THEOREM RELATING TO COVARIANTS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 82, 83.]

THE theorem given by Prof. Sylvester, *Crelle*, vol. LXXXV., p. 109, may be stated as follows: If for a binary quantic of the order i in the variables, we consider the whole system of covariants of the degree j in the coefficients, then

$$\Sigma\theta(k+1) = \frac{\Pi(i+j)}{\Pi(i)\,\Pi(j)},$$

where θ denotes the number of asyzygetic covariants of the order θ in the variables, the values of θ being ij, $ij-2$, $ij-4, \ldots, 1$ or 0, according as ij is odd or even.

In the case of the binary quintic $(a, \ldots \between x, y)^5$, $(i=5)$, we have a series of verifications in the Table 88 of my "Ninth Memoir on Quantics," *Phil. Trans.* vol. CLXI. (1871), [462]: viz. writing the small letters $a, b, c, \ldots, u, v, w$ (instead of the capitals A, B, etc.) to denote the covariants of the quintic, a, the quintic itself, degree 1, order 5, or as I express it, deg-order 1.5: b, the covariant deg-order 2.2, etc., the whole series of deg-orders being

a,	b,	c,	d,	e,	f,	g,	h,	i,	j,	k,	l,
1.5,	2.2,	2.6,	3.3,	3.5,	3.9,	4.0,	4.4,	4.6,	5.1,	5.3,	5.7,
m,	n,	o,	p,	q,	r,	s,	t,	u,	v,	w,	
6.2,	6.4,	7.1,	7.5,	8.0,	8.2,	9.3,	11.1,	12.0,	13.1,	18.0,	

then the table shows for each deg-order, the several covariants of that deg-order, and

the number of them which are asyzygetic; for instance, $i=5$ as above, $j=6$, an extract from the table is

j	k	θ		$(k+1)\,\theta$
6	30	1	a^6	31
	28	0		0
	26	1	a^4c	27
	24	1	a^3f	25
	22	2	$a^4b,\ a^2c^2$	46
	20	2	$a^3e,\ acf$	42
	18	3	$a^3d,\ a^2bc,\ c^3,\ f^2$	57
	16	2	$a^2i,\ abf,\ ace$	34
	14	4	$a^2b^2,\ a^2h,\ acd,\ bc^2,\ ef$	60
	12	3	$abe,\ al,\ ce,\ df$	39
	10	4	$a^2g,\ abd,\ b^2c,\ ch,\ e^2$	44
	8	2	$ak,\ bi,\ de$	18
	6	4	$aj,\ b^3,\ bh,\ cg,\ d^2$	28
	4	1	n	5
	2	2	$bg,\ m$	6
	0	0		0

$$462 = \frac{\Pi\,(11)}{\Pi\,(5)\,\Pi\,(6)},$$

where, for instance deg-order 6 . 14, the covariants are a^2b^2, a^2h, acd, bc^2, ef, but the number against these in the third column being (not 5 but) 4, the meaning is that there exists between these five terms one syzygy, making the number of asyzygetic covariants of the deg-order 6 . 14 to be 4. The second column thus in fact contains the several values of k, and the third column the corresponding values of θ; whence, forming the several products $(k+1)$ as shown, the sum of these is as it should be $=462$.

Cambridge, 13 *July*, 1878.

699.

ON THE TRIPLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 134—138.]

THERE should be in all 64 functions proportional to irrational algebraical functions of three independent variables x, y, z; there is no difficulty in obtaining the expression of these 64 functions in the case of the system of differential equations connected with the integral

$$\int dx : \sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x\,.\,g-x\,.\,h-x}\,;$$

but this is *not the general form* of the system for the deficiency (Geschlecht) $p=3$; and I do not know how to deal with the general form: the present note relates therefore exclusively to the above-mentioned hyper-elliptic form.

I.

If in the Memoir, Weierstrass, "Theorie der Abel'schen Functionen," *Crelle*, t. LII. (1856), pp. 285—380, we take $\rho=3$, and write x, y, z; u, v, w; a, b, c, d, e, f, g instead of x_1, x_2, x_3; u_1, u_2, u_3; $a_1, a_2, a_3, a_4, a_5, a_6, a_7$; then, neglecting throughout mere constant factors, we have

$$X = a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x\,.\,g-x,$$

with the like values for Y and Z: the differential equations are

$$du = \frac{b-x\,.\,c-x\,.\,dx}{\sqrt{X}} + \frac{b-y\,.\,c-y\,.\,dy}{\sqrt{Y}} + \frac{b-z\,.\,c-z\,.\,dz}{\sqrt{Z}},$$

$$dv = \frac{c-x\,.\,a-x\,.\,dx}{\sqrt{X}} + \frac{c-y\,.\,a-y\,.\,dy}{\sqrt{Y}} + \frac{c-z\,.\,a-z\,.\,dz}{\sqrt{Z}},$$

$$dw = \frac{a-x\,.\,b-x\,.\,dx}{\sqrt{X}} + \frac{a-y\,.\,b-y\,.\,dy}{\sqrt{Y}} + \frac{a-z\,.\,b-z\,.\,dz}{\sqrt{Z}},$$

and if we write the single letters A, B, C, D, E, F, G for $\mathrm{al}\,(u, v, w)_1$, $\mathrm{al}\,(u, v, w)_2$, $\mathrm{al}\,(u, v, w)_3$, $\mathrm{al}\,(u, v, w)_4$, $\mathrm{al}\,(u, v, w)_5$, $\mathrm{al}\,(u, v, w)_6$, $\mathrm{al}\,(u, v, w)_7$ respectively, each of the capital letters thus denoting a function of (u, v, w), the expressions of these functions in terms of (x, y, z) are

$$A = \sqrt{a-x\,.\,b-x\,.\,c-x}, \quad \text{(seven equations).}$$
$$\vdots \qquad \vdots$$

Similarly, instead of the 21 functions $\mathrm{al}\,(u, v, w)_{12}, \ldots, \mathrm{al}\,(u, v, w)_{67}$ writing $AB, \ldots, FG$, each of these binary symbols denoting in like manner a function of (u, v, w), the definition of AB is

$$AB = A\nabla B - B\nabla A,$$

where

$$\nabla = \frac{d}{du} + \frac{d}{dv} + \frac{d}{dw};$$

we have

$$b-c\,.\,c-a\,.\,a-b\,.\,\frac{dx}{\sqrt{X}} = \frac{a-y\,.\,a-z}{x-y\,.\,x-z}(b-c)\,du + \frac{b-y\,.\,b-z}{x-y\,.\,x-z}(c-a)\,dv + \frac{c-y\,.\,c-z}{x-y\,.\,x-z}(a-b)\,dw,$$

$$\text{"} \qquad \frac{dy}{\sqrt{Y}} = \frac{a-z\,.\,a-x}{y-z\,.\,y-x}(b-c)\,du + \frac{b-z\,.\,b-x}{y-z\,.\,y-x}(c-a)\,dv + \frac{c-z\,.\,c-x}{y-z\,.\,y-x}(a-b)\,dw,$$

$$\text{"} \qquad \frac{dz}{\sqrt{Z}} = \frac{a-x\,.\,a-y}{z-x\,.\,z-y}(b-c)\,du + \frac{b-x\,.\,b-y}{z-x\,.\,z-y}(c-a)\,dv + \frac{c-x\,.\,c-y}{z-x\,.\,z-y}(a-b)\,dw;$$

hence

$$\frac{b-c\,.\,c-a\,.\,a-b}{\sqrt{X}}\nabla x = \frac{a-y\,.\,a-z}{x-y\,.\,x-z}(b-c) + \frac{b-y\,.\,b-z}{x-y\,.\,x-z}(c-a) + \frac{c-y\,.\,c-z}{x-y\,.\,x-z}(a-b),$$

$$= -\frac{b-c\,.\,c-a\,.\,a-b}{x-y\,.\,x-z},$$

that is,

$$\nabla x = \frac{-\sqrt{X}}{x-y\,.\,x-z};$$

and similarly

$$\nabla y = \frac{-\sqrt{Y}}{y-x\,.\,y-z}, \quad \nabla z = \frac{-\sqrt{Z}}{z-x\,.\,z-y}.$$

Hence from the equation

$$A = \sqrt{a-x\,.\,a-y\,.\,a-z}$$

we have

$$\nabla A = -\tfrac{1}{2}A\left(\frac{1}{a-x}\nabla x + \frac{1}{a-y}\nabla y + \frac{1}{a-z}\nabla z\right),$$

that is,

$$\nabla A = \frac{\tfrac{1}{2}A}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{y-z}{a-x}\sqrt{X} + \frac{z-x}{a-y}\sqrt{Y} + \frac{x-y}{a-z}\sqrt{Z}\right\}:$$

and similarly

$$\nabla B = \frac{\tfrac{1}{2}B}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{y-z}{b-x}\sqrt{X} + \frac{z-x}{b-y}\sqrt{Y} + \frac{x-y}{b-z}\sqrt{Z}\right\};$$

consequently

$$AB = \frac{\tfrac{1}{2}(a-b)\,AB}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{(y-z)\sqrt{X}}{a-x\,.\,b-x} + \frac{(z-x)\sqrt{Y}}{a-y\,.\,b-y} + \frac{(x-y)\sqrt{Z}}{a-z\,.\,b-z}\right\},$$

or substituting for A and B their values, and disregarding the constant factor $\frac{1}{2}(a-b)$, this is

$$AB = \frac{1}{y-z.z-x.x-y}\{(y-z)\sqrt{a-y.b-y.a-z.b-z.c-x.d-x.e-x.f-x.g-x}$$
$$+(z-x)\sqrt{a-z.b-z.a-x.b-x.c-y.d-y.e-y.f-y.g-y}$$
$$+(x-y)\sqrt{a-x.b-x.a-y.b-y.c-z.d-z.e-z.f-z.g-z}\}.$$

We have thus in all 21 equations, which exhibit the form of the Weierstrassian functions $\mathrm{al}\,(u, v, w)_{12}, \ldots, \mathrm{al}\,(u, v, w)_{67}$.

To complete the system, there should it is clear be 35 new functions $\mathrm{al}\,(u, v, w)_{123}, \ldots, \mathrm{al}\,(u, v, w)_{567}$, represented by $ABC, \ldots, EFG$, viz. the whole number of functions would then be

$$7 + \frac{7.6}{1.2} + \frac{7.6.5}{1.2.3} (= 7 + 21 + 35) = 63, \; = 64 - 1,$$

since the functions represent ratios of the ϑ-functions.

II.

Starting now with the radical

$$\sqrt{a-x.b-x.c-x.d-x.e-x.f-x.g-x.h-x}$$

composed of eight linear factors, and writing, as in my "Memoir on the double ϑ-functions," t. LXXXV. (1878), pp. 214—245, [665]; a, b, c, d, e, f, g, h to denote these factors, and similarly a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1 and a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2 to denote $a-y$, $b-y$, etc., and $a-z$, $b-z$, etc., so that $X = \mathrm{abcdefgh}$, $Y = \mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1\mathrm{g}_1\mathrm{h}_1$, $Z = \mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2\mathrm{g}_2\mathrm{h}_2$; then, instead of the Weierstrassian form, the differential equations may be taken to be

$$du = \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}},$$

$$dv = \frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} + \frac{z\,dz}{\sqrt{Z}},$$

$$dw = \frac{x^2\,dx}{\sqrt{X}} + \frac{y^2\,dy}{\sqrt{Y}} + \frac{z^2\,dz}{\sqrt{Z}}.$$

We then have 64 ϑ-functions and an ω-function, viz. writing

$$\theta = y - z.z - x.x - y,$$

and then

$$\sqrt{a} = \sqrt{\mathrm{aa}_1\mathrm{a}_2} \quad \text{(8 equations)}$$
$$\vdots \qquad \vdots$$
$$\sqrt{abc} = \frac{1}{\theta}\{(y-z)\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\mathrm{defgh}} + (z-x)\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\mathrm{abcd}_1\mathrm{e}_1\mathrm{f}_1\mathrm{g}_1\mathrm{h}_1} + (x-y)\sqrt{\mathrm{abca}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2\mathrm{g}_2\mathrm{h}_2}\}$$
$$\vdots \qquad \vdots \qquad \text{(56 equations)}$$

the equations, which define the ϑ-functions $A, B, \ldots, H, ABC, \ldots, FGH$, and the ω-function Ω, are

$$A = \Omega\sqrt{a} \qquad (8 \text{ equations})$$
$$\vdots \quad \vdots$$
$$ABC = \Omega\sqrt{abc} \quad (56 \text{ equations})$$
$$\vdots \quad \vdots$$

and one other relation which I have not as yet investigated.

As regards the algebraical relations between the 64 ϑ-functions, it is to be remarked that, selecting in a proper manner 8 of the functions, the square of any one of the other functions can be expressed as a linear function of the squares of the 8 selected functions. To explain this somewhat further, observe that, taking any 5 squares such as $(ABC)^2$, we can with these 5 squares form a linear combination which is rational in x, y, z. We have for instance, writing down the irrational part only,

$$(ABC)^2 = \frac{2}{\theta^2}\{\mathrm{abc}\,(z-x)(x-y)\sqrt{YZ} + \mathrm{a_1b_1c_1}\,(x-y)(y-z)\sqrt{ZX} + \mathrm{a_2b_2c_2}\,(y-z)(z-x)\sqrt{XY}\},$$

and forming in all five such equations, then inasmuch as the coefficients abc, ... of $(z-x)(x-y)\sqrt{YZ}$ are each of them a cubic function containing terms in x^0, x^1, x^2, x^3, we have a determinate set of constant factors such that the resulting term in $(z-x)(x-y)\sqrt{YZ}$ will be $=0$; but the coefficients $\mathrm{a_1b_1c_1}, \ldots$ of $(x-y)(y-z)\sqrt{ZX}$ only differ from the first set of coefficients by containing y instead of x, and the same set of constant factors will thus make the resulting term in $(x-y)(y-z)\sqrt{ZX}$ to be $=0$; and similarly the same set of constant factors will make the resulting term in $(y-z)(z-x)\sqrt{XY}$ to be $=0$; viz. we have thus a set of constant factors, such that the whole irrational part will disappear. *It seems to be in general true that the same set of constant factors will make the rational part integral;* viz. the rational part is a function of the form $\frac{1}{\theta^2}$ multiplied by a rational and integral function of x, y, z, and if this rational and integral function divide by θ^2, then the final result will be a rational and integral function, which, being symmetrical in x, y, z, is at once seen to be a linear function of the symmetrical combinations $1, x+y+z, yz+zx+xy, xyz$. Such a function is obviously a linear function of any four squares A^2, B^2, C^2, D^2; or the form is, linear function of five squares $(ABC)^2 =$ linear function of four squares A^2, that is, any one of the five squares is a linear function of 8 squares.

As an instance, consider the *three* squares $(ABC)^2, (ABD)^2, (ABE)^2$, which are such that we have a linear combination which is rational: in fact, we have here in each function the pair of factors ab, which unites itself with $(z-x)(x-y)\sqrt{XY}$, viz. it is only the coefficient of $\mathrm{ab}\,(z-x)(x-y)\sqrt{XY}$ which has to be made $=0$; the required combination is obviously

$$(d-e)(ABC)^2 + (e-c)(ABD)^2 + (c-d)(ABE)^2.$$

Here the irrational part vanishes and the rational part is found to be

$$= \frac{1}{\theta^2}\,[\mathrm{a_1b_1a_2b_2fgh}\,(y-z)^2 \left\{\begin{array}{l} (d-e)\,\mathrm{c_1c_2de} \\ +(e-c)\,\mathrm{d_1d_2ce} \\ +(c-d)\,\mathrm{e_1e_2dc} \end{array}\right\}$$

$$+\,\mathrm{a_2b_2abf_1g_1h_1}\,(z-x)^2 \left\{\begin{array}{l} (d-e)\,\mathrm{c_2cd_1e_1} \\ +(e-c)\,\mathrm{d_2dc_1e_1} \\ +(c-d)\,\mathrm{e_2ed_1c_1} \end{array}\right\}$$

$$+\,\mathrm{aba_1b_1f_2g_2h_2}\,(x-y)^2 \left\{\begin{array}{l} (d-e)\,\mathrm{cc_1d_2e_2} \\ +(e-c)\,\mathrm{dd_1c_2e_2} \\ +(c-d)\,\mathrm{ee_1d_2c_2} \end{array}\right\}].$$

The three terms in { } are here $=-(c-d)(d-e)(e-c)$ multiplied by $(z-x)(x-y)$, $(x-y)(y-z)$, $(y-z)(z-x)$ respectively; hence the term in [ ] divides by θ and the result is

$$= -\frac{(c-d)(d-e)(e-c)}{\theta}\,[\mathrm{a_1b_1a_2b_2f\,g\,h}\;(y-z)$$
$$+\,\mathrm{a_2b_2a\,b\,f_1g_1h_1}\,(z-x)$$
$$+\,\mathrm{a\,b\,a_1b_1f_2g_2h_2}\,(x-y)],$$

or finally this is

$$= -(c-d)(d-e)(e-c)$$

multiplied by

$$\begin{array}{lll} & & \{(a^2+ab+b^2)fgh-(a^2b+ab^2)(fg+fh+gh)+a^2b^2(f+g+h)\} \\ + & (x+y+z)\{ & -(a+b)fgh+ \quad ab(fg+fh+gh)-a^2b^2 \quad \} \\ +(yz+zx+xy)\{ & & fgh- \quad ab(f+g+h) \quad +a^2b+ab^2 \quad \} \\ + & xyz\{ & -(fg+fh+gh)+ \quad (a+b)(f+g+h)-(a^2+ab+b^2) \quad \}, \end{array}$$

that is, we have $(d-e)(ABC)^2+(e-c)(ABD)^2+(c-d)(ABE)^2=$ a sum of four squares, viz. we have here a linear relation between 7 squares.

I have not as yet investigated the forms of the relations between the products of pairs of ϑ-functions.

Cambridge, 30 *September*, 1878.

700.

ON THE TETRAHEDROID AS A PARTICULAR CASE OF THE 16-NODAL QUARTIC SURFACE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 161—164.]

IN the paper "Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers," *Crelle*, t. LXV. (1866), pp. 284—290, [356], I showed how the surface called the Tetrahedroid could be identified as a special form of Kummer's 16-nodal quartic surface; but I was not then in possession of the simplified form of the equation of the 16-nodal surface given in my paper "Note sur la surface du quatrième ordre douée de seize points singuliers et de seize plans singuliers," *Crelle*, t. LXXIII. (1871), pp. 292, 293, [442]; see also my paper, "A third memoir on Quartic surfaces," *Proc. Lond. Math. Soc.* t. III. (1871), p. 250, [454, this Collection, t. VII., p. 281]. Using the equation last referred to, I resume therefore the consideration of the question.

Taking the constants $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$, such that

$$\alpha+\beta+\gamma=0,\quad \alpha'+\beta'+\gamma'=0,\quad \alpha''+\beta''+\gamma''=0,$$

and writing also

$$\begin{aligned}M&=\alpha'\alpha''(\beta-\gamma)+\beta'\beta''(\gamma-\alpha)+\gamma'\gamma''(\alpha-\beta)\\&=\alpha''\alpha(\beta'-\gamma')+\beta''\beta(\gamma'-\alpha')+\gamma''\gamma(\alpha'-\beta')\\&=\alpha\alpha'(\beta''-\gamma'')+\beta\beta'(\gamma''-\alpha'')+\gamma\gamma'(\alpha''-\beta'')\\&=-\tfrac{1}{3}\{(\beta-\gamma)(\beta'-\gamma')(\beta''-\gamma'')+(\gamma-\alpha)(\gamma'-\alpha')(\gamma''-\alpha'')+(\alpha-\beta)(\alpha'-\beta')(\alpha''-\beta'')\},\end{aligned}$$

(the equivalence of which different expressions for M is verified without difficulty): writing also X, Y, Z, W as current coordinates, the equation of the 16-nodal surface is

$$0=\left\{\begin{aligned}&W^2(X^2+Y^2+Z^2-2YZ-2ZX-2XY)\\&+2W\{\alpha\alpha'\alpha''(Y^2Z-YZ^2)+\beta\beta'\beta''(Z^2X-ZX^2)+\gamma\gamma'\gamma''(X^2Y-XY^2)+MXYZ\}\\&+(\alpha\alpha'\alpha''YZ+\beta\beta'\beta''ZX+\gamma\gamma'\gamma''XY)^2,\end{aligned}\right.$$

where, α, β, γ, α', β', γ', α'', β'', γ'' being connected as above, the number of constants is $=6$.

The equations of the 16 singular planes are

$$\begin{array}{ll} X=0, & Y=0, \\ \alpha\ (\gamma'\gamma''Y-\beta'\beta''Z)-W=0, & \beta\ (\alpha'\alpha''Z-\gamma'\gamma''X)-W=0, \\ \alpha'\ (\gamma''\gamma\ Y-\beta''\beta\ Z)-W=0, & \beta'\ (\alpha''\alpha\ Z-\gamma''\gamma\ X)-W=0, \\ \alpha''\ (\gamma\gamma'\ Y-\beta\beta'\ Z)-W=0, & \beta''\ (\alpha\alpha'\ Z-\gamma\gamma'\ X)-W=0, \\ Z=0, & W=0, \\ \gamma\ (\beta'\beta''X-\alpha'\alpha''\ Y)-W=0, & \beta\gamma X+\ \gamma\alpha Y+\ \alpha\beta Z=0, \\ \gamma'\ (\beta''\beta\ X-\alpha''\alpha\ Y)-W=0, & \beta'\gamma'X+\ \gamma'\alpha'Y+\ \alpha'\beta'Z=0, \\ \gamma''\ (\beta\beta'\ X-\alpha\alpha'\ Y)-W=0, & \beta''\gamma''X+\gamma''\alpha''Y+\alpha''\beta''Z=0. \end{array}$$

Writing x, y, z, w as current coordinates, the equation of the Tetrahedroid is

$$\begin{aligned} & m^2n^2f^2x^4+n^2l^2g^2y^4+l^2m^2h^2z^4+f^2g^2h^2w^4 \\ & +(l^2f^2-m^2g^2-n^2h^2)(l^2y^2z^2+f^2x^2w^2)+(-l^2f^2+m^2g^2-n^2h^2)(m^2z^2x^2+g^2y^2w^2) \\ & \qquad +(-l^2f^2-m^2g^2+n^2h^2)(n^2x^2y^2+h^2z^2w^2)=0, \end{aligned}$$

where, inasmuch as f, g, h, l, m, n enter homogeneously, the number of constants is $=5$.

The equations of the 16 singular planes, written in an order corresponding to that used for the 16-nodal surface, are

$$\begin{array}{c|c|c|c} * \quad ny-mz+fw=0 & -nx\ *\ +lz\ +gw=0 & mx-ly\ *\ +hw=0 & -fx\ -gy-hz\ *\ =0 \\ fx\ -gy-hz\ *\ =0 & mx+ly\ *\ +hw=0 & -nx\ *\ -lz\ +gw=0 & *\quad ny-mz-fw=0 \\ -mx-ly\ *\ +hw=0 & -fx\ +gy-hz\ *\ =0 & *\quad ny+mz+fw=0 & -nx\ *\ +lz\ -gw=0 \\ nx\ *\ +lz\ +gw=0 & *\ -ny-mz+fw=0 & -fx\ -gy+hz\ *\ =0 & mx-ly\ *\ -hw=0. \end{array}$$

These equations can be made to agree each to each with those of the 16 singular planes of the 16-nodal surface, provided that we have

$$\frac{m}{\gamma}=\frac{n}{\beta},\quad \frac{n}{\alpha'}=\frac{l}{\gamma'},\quad \frac{l}{\beta''}=\frac{m}{\alpha''};\quad f=-l\alpha\alpha'\alpha'',\quad g=-m\beta\beta'\beta'',\quad h=-n\gamma\gamma'\gamma'',$$

where observe that the first three equations give $\alpha'\beta''\gamma=\alpha''\beta\gamma'$, which is the relation between the constants when the 16-nodal surface reduces itself to a tetrahedroid in the above manner. And if we then assume

$$X=ny-mz+fw,\quad Y=-nx+lz+gw,\quad Z=mx-ly+hw,\quad W=-fx-gy-hz,$$

the 16 linear functions of X, Y, Z, W will become mere constant multiples of the corresponding 16 linear functions of x, y, z, w; the constants, by which the several

functions of x, y, z, w have to be multiplied in order to reduce them each to the corresponding linear function of X, Y, Z, W, being given by the table

$$\begin{array}{cccc}
1, & 1, & 1, & 1, \\
\frac{1}{m\beta}(l\alpha - m\beta), & -\frac{\alpha'\alpha''}{m}(l\alpha - m\beta), & \frac{\alpha'\alpha''}{n}(l\alpha - m\beta), & \frac{\beta\gamma}{\alpha'\alpha''}(\beta'\gamma'' - \beta''\gamma'), \\
\frac{\beta''\beta}{l}(m\beta' - n\gamma'), & \frac{1}{n\gamma'}(m\beta' - n\gamma'), & -\frac{\beta''\beta}{n}(m\beta' - n\gamma'), & \frac{\gamma'\alpha'}{\beta''\beta}(\gamma''\alpha - \gamma\alpha''), \\
-\frac{\gamma\gamma'}{l}(n\gamma'' - l\alpha''), & \frac{\gamma\gamma'}{m}(n\gamma'' - l\alpha''), & \frac{1}{l\alpha''}(n\gamma'' - l\alpha''), & \frac{\alpha''\beta''}{\gamma\gamma'}(\alpha\beta' - \alpha'\beta).
\end{array}$$

For instance, we have

$$\alpha(\gamma'\gamma''Y - \beta'\beta''Z) - W = \frac{1}{m\beta}(l\alpha - m\beta)(fx - gy - hz),$$

viz. substituting for Y, Z, W their values, the relation is

$$\left.\begin{array}{l}
m\alpha\beta\,.\,\gamma'\gamma''\ (-nx \quad * \quad + lz + gw) \\
-m\alpha\beta\,.\,\beta'\beta''\ (\ mx - ly \quad * \quad + hw) \\
- \qquad m\beta\,(-fx - gy - hz \quad * \ )
\end{array}\right\} = (l\alpha - m\beta)(fx - gy - hz).$$

As regards the terms in y, z, and w, the identity is at once verified. As regards the term in x, we should have

$$m\alpha\beta(-n\gamma'\gamma'' - m\beta'\beta'') - (l\alpha - 2m\beta)f = 0,$$

viz. substituting for f its value, $-l\alpha\alpha'\alpha'' = -m\alpha\alpha'\beta''$, the equation divides by $m\alpha$ and we then have

$$\beta(-n\gamma'\gamma'' - m\beta'\beta'') + \alpha'\beta''(l\alpha - 2m\beta) = 0,$$

that is,

$$l\alpha\alpha'\beta'' - m\beta\beta''(\beta' + 2\alpha') - n\beta\gamma'\gamma'' = 0,$$

or writing herein $m\beta'' = l\alpha''$, $n\gamma' = l\alpha'$, and $\beta' + 2\alpha' = \alpha' - \gamma'$, the equation becomes $\alpha'\alpha\beta'' - \alpha''\beta(\alpha' - \gamma') - \alpha'\beta\gamma'' = 0$, that is, $\alpha'(\alpha\beta'' - \alpha''\beta) = \alpha'\beta\gamma'' - \alpha''\beta\gamma'$; or writing herein $\alpha''\beta\gamma' = \alpha'\beta''\gamma$, the equation divided by α' becomes $\alpha\beta'' - \alpha''\beta = \beta\gamma'' - \beta''\gamma$, which is true in virtue of $\alpha + \beta + \gamma = 0$ and $\alpha'' + \beta'' + \gamma'' = 0$. And in like manner the several other identities may be verified.

The equation $\alpha'\beta''\gamma = \alpha''\beta\gamma'$ might have been obtained as the condition of the intersection, in a common point, of four of the singular planes of the 16-nodal surface; and when this equation is satisfied, there are in fact four systems each of four planes, such that the four planes of a system meet in a common point: viz. we have

Planes

$$\begin{array}{llll}
X = 0, & \beta\gamma X + \gamma\alpha Y + \alpha\beta Z = 0, & \gamma'(\beta''\beta X - \alpha''\alpha Y) - W = 0, & \beta''(\alpha\alpha' Z - \gamma\gamma' X) - W = 0, \\
Y = 0, & \gamma(\beta'\beta'' X - \alpha'\alpha'' Y) - W = 0, & \beta'\gamma' X + \gamma'\alpha' Y + \alpha'\beta' Z = 0, & \alpha''(\gamma\gamma' Y - \beta\beta' Z) - W = 0, \\
Z = 0, & \beta(\alpha'\alpha'' Z - \gamma'\gamma'' X) - W = 0, & \alpha'(\gamma''\gamma Y - \beta''\beta Z) - W = 0, & \beta''\gamma'' X + \gamma''\alpha'' Y + \alpha''\beta'' Z = 0, \\
W = 0, & \alpha(\gamma'\gamma'' Y - \beta'\beta'' Z) - W = 0, & \beta'(\alpha''\alpha Z - \gamma''\gamma X) - W = 0, & \gamma''(\beta\beta' X - \alpha\alpha' Y) - W = 0,
\end{array}$$

meeting in points

$$\begin{array}{cccc} 0, & -\beta, & \gamma, & \alpha''\beta\gamma'.\alpha\ , \\ \alpha', & 0, & -\gamma', & \alpha''\beta\gamma'.\beta', \\ -\alpha'', & \beta'', & 0, & \alpha''\beta\gamma'.\gamma'', \\ \beta''.\alpha\alpha'\alpha'', & \alpha''.\beta\beta'\beta'', & \gamma''.\alpha''\beta\gamma', & 0, \end{array}$$

the four points being in fact the vertices of the tetrahedron formed by the four planes of the tetrahedroid. Observe that, if the singular planes of the 16-nodal surface in their original order are

$$\begin{array}{cccc} 1, & 2, & 3, & 4, \\ 5, & 6, & 7, & 8, \\ 9, & 10, & 11, & 12, \\ 13, & 14, & 15, & 16, \end{array}$$

then the planes forming the last-mentioned four systems of planes are

$$\begin{array}{cccc} (1, & 8, & 11, & 14), \\ (2, & 7, & 12, & 13), \\ (3, & 6, & 9, & 16), \\ (4, & 5, & 10, & 15), \end{array}$$

viz. they correspond each of them to a term which in the determinant formed with the 16 symbols would have the sign $+$.

The equation $\alpha'\beta''\gamma = \alpha''\beta\gamma'$ is evidently not unique. The triads (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ enter symmetrically into the equation of the 16-nodal surface; by taking the singular planes of one of the surfaces in a different order, the equation would present itself under one or other of the different forms

$$\alpha'\beta''\gamma = \alpha''\beta\gamma', \quad \alpha''\beta\gamma' = \alpha\beta'\gamma'', \quad \alpha\beta'\gamma'' = \alpha'\beta''\gamma,$$
$$\alpha'\beta\gamma'' = \alpha''\beta'\gamma, \quad \alpha''\beta'\gamma = \alpha\beta''\gamma', \quad \alpha\beta''\gamma' = \alpha'\beta\gamma''.$$

Cambridge, 9 *December*, 1878.

701.

ALGORITHM FOR THE CHARACTERISTICS OF THE TRIPLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 165—169.]

THE characteristics of the triple ϑ-functions may be represented, the 28 odd characteristics by the binary symbols or duads, 12, ..., 78, and the even ones $\left(\text{other than } \begin{matrix}000\\000\end{matrix}, =0\right)$, say the 35 even characteristics, by the ternary symbols or triads 123, ..., 567: which triads may be regarded as abbreviations for the double tetrads 1238.4567, ..., 5678.1234, the 8 being always attached to the expressed triad. The correspondence of the symbols is given by the diagram:

upper line of characteristic

lower line of characteristic	000	100	010	110	001	101	011	111
000	0	236	345	137	467	156	124	257
100	237	67	136	12	157	48	256	35
010	245	127	23	68	134	357	15	47
110	126	13	78	145	356	25	46	234
001	567	146	125	247	45	17	38	26
101	147	58	246	34	16	123	27	367
011	135	347	14	57	28	36	167	456
111	346	24	56	235	37	267	457	18

Or, what is the same thing, it is

upper line of characteristic

	000	100	010	110	001	101	011	111
12				100				
13		110						
14			011					
15							010	
16					101			
17						001		
18								111
23			010					
24		111						
25						110		
26								001
27							101	
28					011			
34				101				
35								100
36						011		
37					111			
38							001	
45					001			
46							110	
47								010
48						100		
56			111					
57				011				
58		101						
67		100						
68				010				
78			110					

lower line of characteristic

	000	100	010	110	001	101	011	111
123						101		
124							000	
125			001					
126	110							
127		010						
134					010			
135	011							
136			100					
137				000				
145				110				
146		001						
147	101							
156						000		
157					100			
167							011	
234								110
235				111				
236		000						
237	100							
245	010							
246			101					
247				001				
256							100	
257								000
267						111		
345			000					
346	111							
347		011						
356					110			
357						010		
367								101
456								011
457							111	
467					000			
567	001							

lower line of characteristic

by means of which the two-line-characteristic is at once found when the duad or triad is given.

The new algorithm renders unnecessary the Table I. of Weber's memoir "Theorie der Abel'schen Functionen vom Geschlecht 3" (Berlin, 1876). In fact, the system of six pairs corresponding to an odd characteristic such as 12 is

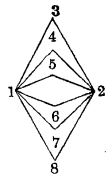
$$13.23,\quad 14.24,\quad 15.25,\quad 16.26,\quad 17.27,\quad 18.28,$$

and that corresponding to an even characteristic such as 123 (=1238.4567) is

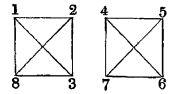
$$12.38,\quad 13.28,\quad 18.23,\quad 45.67,\quad 46.57,\quad 47.56:$$

so that all the (28 + 35 =) 63 systems can be at once formed.

The odd characteristics correspond to the bitangents of a quartic curve, and as regards these bitangents the notation is, in fact, the notation arising out of Hesse's investigations and explained Salmon's *Higher Plane Curves* (2nd Ed. 1873), pp. 222—225. It may be noticed that the geometrical symbols corresponding to the before-mentioned two systems are:



and



.

Hence, selecting out of the first system any two pairs, we have a symbol □: but selecting out of the second system any two pairs, we have a symbol which is either □ or ||||; so that in each case (Salmon, p. 224) the four bitangents are such that the eight points of contact lie on a conic.

The 28 bitangents of the general quartic curve

$$\sqrt{x_1\xi_1} + \sqrt{x_2\xi_2} + \sqrt{x_3\xi_3} = 0,$$

represented by the equations given by Weber, *l. c.*, pp. 100, 101, and taken in the order in which they are there written down, have for their duad-characteristics

$$18,\quad 28,\quad 38,\quad 23,\quad 13,\quad 12,\quad 48,\quad 14,\quad 58,\quad 15,\quad 68,\quad 16,\quad 78,\quad 17,\quad 24,\quad 34,\quad 25,\quad 35,$$
$$26,\quad 36,\quad 27,\quad 37,\quad 67,\quad 57,\quad 56,\quad 45,\quad 46,\quad 47$$

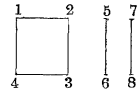
respectively. Taking out of any one of the 63 systems three pairs of bitangents at pleasure, these give rise to an equation of the curve of a form such as

$$\sqrt{x_1\xi_1} + \sqrt{x_2\xi_2} + \sqrt{x_3\xi_3} = 0,$$

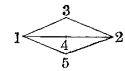
and the whole number of the forms of equation is thus $= 1260$. The triads of pairs which enter into the same equation may be

triads such as 12 . 34, 13 . 42, 14 . 23

No. = 70,

" " " 12 . 34, 13 . 42, 56 . 78 

" = 630,

" " " 13 . 23, 14 . 24, 15 . 25 

" = 560,

making the whole number $= 1260$, as already mentioned.

Cambridge, 7 *December*, 1878.

702.

ON THE TRIPLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 190—198.]

A QUARTIC curve has the deficiency 3, and depends therefore on the triple ϑ-functions: and these, as functions of 3 arguments, should be connected with functions of 3 points on the curve; but it is easy to understand that it is possible, and may be convenient, to introduce a fourth point, and so regard them as functions of 4 points on the curve: thus in the circle, the functions $\cos u$, $\sin u$ may be regarded as functions of one point $\cos u = x$, $\sin u = y$, or as functions of two points,

$$\cos u = xx_1 + yy_1, \quad \sin u = xy_1 - x_1y.$$

And accordingly in Weber's memoir "Theorie der Abel'schen Functionen vom Geschlecht 3," (1876), see p. 156, the triple ϑ-functions are regarded as functions of 4 points on the curve: viz. it is in effect shown that (disregarding constant factors) each of the 64 functions is proportional to a determinant, the four lines of which are algebraical functions of the coordinates of the four points respectively: the form of this determinant being different according as the characteristic of the ϑ-function is odd or even, or say according as the ϑ-function is odd or even. But the geometrical signification of these formulæ requires to be developed.

A quartic curve may be touched in six points by a cubic curve: but (Hesse, 1855*) there are two kinds of such tangent cubics, according as the six points of contact are on a conic, or are not on a conic; say we have a conic hexad of points on the quartic, and a cubic hexad of points on the quartic. In either case, three points of the hexad may be assumed at pleasure; we can then in 28 different ways determine the remaining three points of the conic hexad, and in 36 different

* See the two memoirs "Ueber Determinanten und ihre Anwendung in der Geometrie" and "Ueber die Doppeltangenten der Curven vierter Ordnung," *Crelle*, t. XLIX. (1855).

ways the remaining three points of the cubic hexad: or what is the same thing, there are 28 systems of cubics touching in a conic hexad, and 36 systems of cubics touching in a cubic hexad. The condition in order that four points of the quartic curve may belong to a hexad (conic or cubic) is given by an equation $\Omega = 0$, where Ω is a determinant the four lines of which are algebraical functions of the coordinates of the four points respectively: but the form of such determinant is different according as the condition belongs to a conic hexad, or to a cubic hexad: we have thus 28 conic determinants and 36 cubic determinants, Ω; and the 64 ϑ-functions are proportional to constant multiples of these determinants; viz. the odd functions correspond to the conic determinants, and the even functions to the cubic determinants.

First, as to the conic hexads: the points of a conic hexad lie in a conic with the two points of contact of some one of the bitangents of the quartic curve: so that, given any three points of the hexad, these together with the two points of contact of the bitangent determine a conic which meets the quartic in the remaining three points of the hexad. Suppose that a, b, c, f, g, h are linear functions of the coordinates such that the equation of the quartic curve is

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0;$$

then $a = 0$, $b = 0$, $c = 0$, $f = 0$, $g = 0$, $h = 0$ are six of the bitangents of the curve, and the bitangent $a = 0$ touches the curve at the two points of intersection of this line with the conic $bg - ch = 0$. The general equation of a conic through these two points $a = 0$, $bg - ch = 0$, may be written

$$bg - ch + a(Ax + By + Cz) = 0,$$

where for x, y, z we may if we please substitute any three of the six linear functions a, b, c, f, g, h, or any other linear functions of the coordinates (x, y, z): and the equation may also be written

$$af \pm (bg - ch) + a(Ax + By + Cz) = 0.$$

Adopting this latter form, and considering the intersections of the conic with the quartic, that is, considering the relation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0$$

as holding good, we have

$$af + bg - ch = -2\sqrt{afbg},$$
$$af - bg + ch = -2\sqrt{afch},$$

and we thus have at pleasure one or other of the two equations

$$-2\sqrt{afbg} + a(Ax + By + Cz) = 0,$$
$$-2\sqrt{afch} + a(Ax + By + Cz) = 0,$$

that is,

$$-2\sqrt{fbg} + \sqrt{a}(Ax + By + Cz) = 0,$$
$$-2\sqrt{fch} + \sqrt{a}(Ax + By + Cz) = 0.$$

Hence the condition in order that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), assumed to be points of the quartic, may belong to the conic hexad, may be written

$$\begin{vmatrix} \sqrt{f_1b_1g_1}, & x_1\sqrt{a_1}, & y_1\sqrt{a_1}, & z_1\sqrt{a_1} \\ \sqrt{f_2b_2g_2}, & x_2\sqrt{a_2}, & y_2\sqrt{a_2}, & z_2\sqrt{a_2} \\ \sqrt{f_3b_3g_3}, & x_3\sqrt{a_3}, & y_3\sqrt{a_3}, & z_3\sqrt{a_3} \\ \sqrt{f_4b_4g_4}, & x_4\sqrt{a_4}, & y_4\sqrt{a_4}, & z_4\sqrt{a_4} \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \sqrt{f_1c_1h_1}, & x_1\sqrt{a_1}, & y_1\sqrt{a_1}, & z_1\sqrt{a_1} \\ \sqrt{f_2c_2h_2}, & x_2\sqrt{a_2}, & y_2\sqrt{a_2}, & z_2\sqrt{a_2} \\ \sqrt{f_3c_3h_3}, & x_3\sqrt{a_3}, & y_3\sqrt{a_3}, & z_3\sqrt{a_3} \\ \sqrt{f_4c_4h_4}, & x_4\sqrt{a_4}, & y_4\sqrt{a_4}, & z_4\sqrt{a_4} \end{vmatrix} = 0,$$

where, as before, the x, y, z may be replaced by any three of the letters a, b, c, f, g, h, or by any other linear functions of (x, y, z): and, moreover, although in obtaining the condition we have used for the quartic the equation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

depending upon six bitangents, yet from the process itself it is clear that the condition can only depend upon the particular bitangent $a = 0$: calling the condition $\Omega = 0$, all the forms of condition which belong to the same bitangent $a = 0$, will be essentially identical, that is, the several determinants Ω will differ only by constant factors; or disregarding these constant factors, we have for the bitangent $a = 0$, a single determinant Ω, which may be taken to be any one of the determinants in question. And we have thus 28 determinants Ω, corresponding to the 28 bitangents respectively.

Coming now to the cubic hexads, Hesse showed that the equation of a quartic curve could be (and that in 36 different ways) expressed in the form, symmetrical determinant $= 0$, or say

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} = 0,$$

where $(a, b, c, d, f, g, h, l, m, n)$ are linear functions of the coordinates; and from each of these forms he obtains the equation of a cubic

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta & \end{vmatrix} = 0,$$

containing the four constants α, β, γ, δ, or say the 3 ratios of these constants, touching the quartic in a cubic hexad of points: that the cubic does touch the quartic in six points appears, in fact, from Hesse's identity

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta & \end{vmatrix} \begin{vmatrix} a, & h, & g, & l, & \alpha' \\ h, & b, & f, & m, & \beta' \\ g, & f, & c, & n, & \gamma' \\ l, & m, & n, & d, & \delta' \\ \alpha', & \beta', & \gamma', & \delta' & \end{vmatrix} - \begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha', & \beta', & \gamma', & \delta' & \end{vmatrix}^2 = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} U,$$

where U is an easily calculated function of the second order in $a, b, c, d, f, g, h, l, m, n$, and also of the second order in the determinants $\alpha\beta' - \alpha'\beta$, etc.

We can obtain such a form of the equation of the quartic, from the before-mentioned equation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

viz. this equation gives

$$\begin{vmatrix} *, & h, & g, & a \\ h, & *, & f, & b \\ g, & f, & *, & c \\ a, & b, & c, & * \end{vmatrix} = 0,$$

which is of the required form, symmetrical determinant $= 0$; the equation is, in fact,

$$a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0,$$

which is the rationalised form of

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

and we hence have the cubic

$$\begin{vmatrix} *, & h, & g, & a, & \alpha \\ h, & *, & f, & b, & \beta \\ g, & f, & *, & c, & \gamma \\ a, & b, & c, & *, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & * \end{vmatrix} = 0,$$

the developed form of which is

$$\begin{aligned} &\alpha^2bcf + \beta^2cag + \gamma^2abh + \delta^2fgh \\ &-(\alpha\beta\gamma + f\alpha\delta)(-af + bg + ch) \\ &-(b\gamma\alpha + g\beta\delta)(\ af - bg + ch) \\ &-(c\alpha\beta + h\gamma\delta)(\ af + bg - ch) = 0. \end{aligned}$$

Considering the intersections with the quartic

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

we have

$$-af + bg + ch,\ af - bg + ch,\ af + bg - ch = -2\sqrt{bcgh},\ -2\sqrt{cahf},\ -2\sqrt{abfg},$$

and the equation thus becomes

$$(\alpha\sqrt{bcf} + \beta\sqrt{cag} + \gamma\sqrt{abh} + \delta\sqrt{fgh})^2 = 0;$$

viz. for the points of the cubic hexad we have

$$\alpha\sqrt{bcf}+\beta\sqrt{cag}+\gamma\sqrt{abh}+\delta\sqrt{fgh}=0,$$

and hence the condition in order that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) may belong to the cubic hexad is

$$\begin{vmatrix} \sqrt{b_1c_1f_1}, & \sqrt{c_1a_1g_1}, & \sqrt{a_1b_1h_1}, & \sqrt{f_1g_1h_1} \\ \sqrt{b_2c_2f_2}, & \sqrt{c_2a_2g_2}, & \sqrt{a_2b_2h_2}, & \sqrt{f_2g_2h_2} \\ \sqrt{b_3c_3f_3}, & \sqrt{c_3a_3g_3}, & \sqrt{a_3b_3h_3}, & \sqrt{f_3g_3h_3} \\ \sqrt{b_4c_4f_4}, & \sqrt{c_4a_4g_4}, & \sqrt{a_4b_4h_4}, & \sqrt{f_4g_4h_4} \end{vmatrix}=0,$$

viz. we have thus the form of the determinant Ω which belongs to a cubic hexad.

It is to be observed that the equation

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0$$

remains unaltered by any of the interchanges a and f, b and g, c and h; but we thus obtain only two cubic hexads; those answering to the equations

$$\alpha\sqrt{bcf}+\beta\sqrt{cag}+\gamma\sqrt{abh}+\delta\sqrt{fgh}=0,$$

and

$$\alpha\sqrt{agh}+\beta\sqrt{bhf}+\gamma\sqrt{cfg}+\delta\sqrt{abc}=0,$$

which give distinct hexads. The whole number of ways in which the equation of the quartic can be expressed in a form such as

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0,$$

attending only to the pairs of bitangents, and disregarding the interchanges of the two bitangents of a pair, is $=1260$, and hence the number of forms for the determinant Ω of a cubic hexad is the double of this, $=2520$, which is $=36\times 70$: but the number of distinct hexads is $=36$, and thus there must be for each hexad, 70 equivalent forms.

To explain this, observe that every even characteristic except $\begin{smallmatrix}000\\000\end{smallmatrix}$, and every odd characteristic, can be (and that in 6 ways) expressed as a sum of two different odd characteristics; we have thus (see Weber's Table I.) a system of $(35+28=)$ 63 hexpairs; and selecting at pleasure any three pairs out of the same hexpair, we have a system of $(63\times 20=)$ 1260 tripairs; giving the 1260 representations of the quartic in a form such as

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0.$$

Each even characteristic $\left(\text{not excluding } \begin{smallmatrix}000\\000\end{smallmatrix}\right)$ can be in 56 different ways (Weber, p. 23) expressed as a sum of three different odd characteristics, and these are such that no two of them belong to the same pair, in any tripair; or we may say that each even characteristic gives rise to 56 hemi-tripairs. But a hemi-tripair can be in 5 different ways completed into a tripair; and we have thus, belonging to the same

even characteristic ($56 \times 5 =$) 280 tripairs, which are however 70 tripairs each taken 4 times. A tripair contains in all ($2^3 =$) 8 hemi-tripairs, but these divide themselves into two sets each of 4 hemi-tripairs such that for each hemi-tripair of the first set the three characteristics have a given sum, and for each hemi-tripair of the second set the three characteristics have a different given sum. Hence considering the 70 tripairs corresponding as above to a given even characteristic, in any one of the 70 tripairs, there is a set of 4 hemi-tripairs such that in each of them the sum of the three characteristics is equal to the given even characteristic; and taking the bitangents f, g, h to correspond to any one of these hemi-tripairs, the bitangents which correspond to the other three hemi-tripairs will be b, c, f; c, a, g and a, b, h respectively; and we thus obtain from any one of these one and the same representation

$$\alpha\sqrt{bcf} + \beta\sqrt{cag} + \gamma\sqrt{abh} + \delta\sqrt{fgh} = 0$$

of the cubic hexad. And the 70 tripairs give thus the 70 representations of the same cubic hexad.

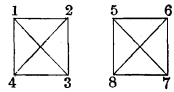
The whole number of hemi-tripairs is $36 \times 56 = 2016$: it may be remarked that there exists a system of 288 heptads, each of 7 odd characteristics such that selecting at pleasure any 3 characteristics out of the heptad, we obtain always a hemi-tripair: we have thus in all $288 \times 35 = 10080$ hemi-tripairs: this is $= 2016 \times 5$, or we have the 2016 hemi-tripairs each taken 5 times. Weber's Table II. exhibits 36 out of the 288 heptads.

I recall that in the algorithm derived from Hesse's theory the bitangents are represented by the duads 12, 13, ..., 78 formed with the eight figures 1, 2, 3, 4, 5, 6, 7, 8; these duads correspond to the odd characteristics as shown in the Table, and the table shows also triads corresponding to all the even characteristics except
000
000

Top line of characteristic.

Bottom line of characteristic.	000	100	010	110	001	101	011	111
000		236	345	137	467	156	124	257
100	237	67	136	12	157	48	256	35
010	245	127	23	68	134	357	15	47
110	126	13	78	145	356	25	46	234
001	567	146	125	247	45	17	38	26
101	147	58	246	34	16	123	27	367
011	135	347	14	57	17	36	167	456
111	346	24	56	235	37	267	457	18

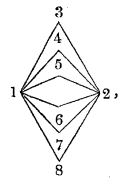
See my "Algorithm of the triple ϑ-functions," *Crelle*, t. LXXXVII. p. 165, [701]. The $(35+28=)$ 63 hexpairs then are

35 hexpairs such as 

, say this is 1234.5678 or for shortness 567 (the 8 going always with the expressed triad): that is, 567 denotes the hexpair

$$12.34;\ 13.24;\ 14.23;\ 56.78;\ 57.68;\ 58.67:$$

and

28 hexpairs such as 

, say this is 12; that is, 12 denotes the hexpair

$$13.32;\ 14.42;\ 15.52;\ 16.62;\ 17.72;\ 18.82.$$

It is to be noticed that the odd characteristics, as represented by their duad symbols, can be added by the formulæ

$$12+23=13, \text{ etc.},$$

or, what is the same thing,

$$12+13+23=0, \ =\begin{matrix}000\\000\end{matrix}, \text{ etc.},$$

and

$$12+34=13+24=14+23=56+78=57+68=58+67=567, \text{ etc.}$$

Thus, referring to the table,

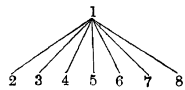
$$12+23=\ 13 \text{ means } \begin{matrix}110\\100\end{matrix}+\begin{matrix}010\\010\end{matrix}=\begin{matrix}100\\110\end{matrix},$$

and

$$12+34=567 \text{ means } \begin{matrix}110\\100\end{matrix}+\begin{matrix}110\\101\end{matrix}=\begin{matrix}000\\001\end{matrix},$$

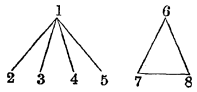
which are right.

The 288 heptads are

8 heptads such as 

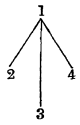
, say this is the heptad 1, denoting the seven duads 12, 13, 14, 15, 16, 17, 18:

and

280 heptads such as 

, say this is the heptad 1 . 678, denoting the seven duads 12, 13, 14, 15, 67, 68, 78.

We hence see that the 2016 hemi-tripairs are:

280 hemi-tripairs 

(I.), say this is 1 . 234, denoting the three duads 12, 13, 14:

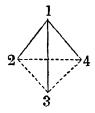
1680 hemi-tripairs

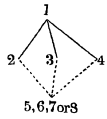
(II.), say this is 12 (6 . 78), denoting the three duads 12, 67, 68:

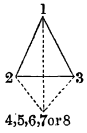
56 hemi-tripairs

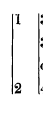
(III.), say this is 123, denoting the three duads 12, 13, 23:

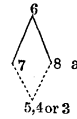
2016.

We further see how each hemi-tripair may be completed into a tripair in 5 different ways: thus (I.) gives the 5 tripairs 

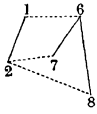


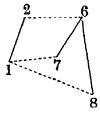
; (III.) gives the 5 tripairs 

; while (II.) gives the 3 tripairs 



and the 2 tripairs



and


To each even characteristic there belongs a system of 56 hemi-tripairs; thus for the characteristic $\begin{smallmatrix}000\\000\end{smallmatrix}$, the 56 hemi-tripairs are 123, that is, 12, 13, 23, etc.: whence the 70 tripairs are 1234, that is, 12.34; 13.24; 14.23, etc.; and in any such tripair, say in 1234, we have the set of four hemi-tripairs 123, 124, 134, 234, for each of which the sum of the three characteristics is

$$=\begin{smallmatrix}000\\000\end{smallmatrix}\left(12+23+13=\begin{smallmatrix}000\\000\end{smallmatrix}, \text{ etc.}\right):$$

and the other set 1.234, 2.134, 3.124, 4.123, for each of which the sum of the three characteristics is

$$=567\left(12+13+14,\ =23+14,\ =567,\ =\begin{smallmatrix}000\\001\end{smallmatrix}\right).$$

To find the hemi-tripairs that belong to any other even characteristic; for instance, $\begin{smallmatrix}000\\001\end{smallmatrix}$, corresponds to 567: we have 4 such as 1.234; 24 such as (5.12)34; 4 such as 5.678; and 24 such as (1.56)78; in all $4+24+4+24, =56$. The tripairs are the 2, 1234, 5678; 16 such as 54(123); 16 such as 15(678); 36 such as (5162)34.78; in all $2+16+16+36, =70$; and in each of these it is easy to select the hemi-tripairs for which the sum of the 3 duads is $=567$.

Cambridge, 27 *December*, 1878.

703.

ON THE ADDITION OF THE DOUBLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVIII. (1879), pp. 74—81.]

I ASSUME in general

$$\Theta = a-\theta\,.\,b-\theta\,.\,c-\theta\,.\,d-\theta\,.\,e-\theta\,.\,f-\theta,$$

and I consider the variables x, y, z, w, p, q, connected by the equations

$$\begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1 \\ x, & y, & z, & w, & p, & q \\ x^2, & y^2, & z^2, & w^2, & p^2, & q^2 \\ x^3, & y^3, & z^3, & w^3, & p^3, & q^3 \\ \sqrt{X}, & \sqrt{Y}, & \sqrt{Z}, & \sqrt{W}, & \sqrt{P}, & \sqrt{Q} \end{vmatrix} = 0,$$

equivalent to two independent equations, which in fact serve to determine p, q, or say the symmetrical functions $p+q$ and pq, in terms of x, y, z, w.

These equations, it is well known, constitute a particular integral of the differential equations

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}+\frac{dp}{\sqrt{P}}+\frac{dq}{\sqrt{Q}}=0,$$

$$\frac{x\,dx}{\sqrt{X}}+\frac{y\,dy}{\sqrt{Y}}+\frac{z\,dz}{\sqrt{Z}}+\frac{w\,dw}{\sqrt{W}}+\frac{p\,dp}{\sqrt{P}}+\frac{q\,dq}{\sqrt{Q}}=0,$$

or what is the same thing, regarding p, q as arbitrary constants, they constitute the general integral of the differential equations

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

$$\frac{x\,dx}{\sqrt{X}}+\frac{y\,dy}{\sqrt{Y}}+\frac{z\,dz}{\sqrt{Z}}+\frac{w\,dw}{\sqrt{W}}=0.$$

I attach the numbers 1, 2, 3, 4, 5, 6 to the variables x, y, z, w, p, q, respectively: and write

$$A_{12} = \sqrt{a-x\,.\,a-y}\,; \quad A_{34} = \sqrt{a-z\,.\,a-w}\,; \quad A_{56} = \sqrt{a-p\,.\,a-q}\,;$$

$$\vdots$$

(six equations),

$$AB_{12} = \frac{1}{x-y}\left\{\sqrt{a-x\,.\,b-x\,.\,f-x\,.\,c-y\,.\,d-y\,.\,e-y} - \sqrt{a-y\,.\,b-y\,.\,f-y\,.\,c-x\,.\,d-x\,.\,e-x}\right\}; \text{ etc.}$$

$$\vdots$$

(ten equations),

where it is to be borne in mind that AB is an abbreviation for $ABF\,.\,CDE$, and so in other cases, the letter F belonging always to the expressed duad: there are thus in all the sixteen functions A, B, C, D, E, F, AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, these being functions of x and y, of z and w, and of p and q, according as the suffix is 12, 34, or 56.

It is to be shown that the 16 functions A_{56}, AB_{56} of p and q can be by means of the given equations expressed as proportional to rational and integral functions of the 16 functions A_{12}, AB_{12}, A_{34}, AB_{34} of x and y, and of z and w respectively: and it is clear that in so expressing them we have in effect the solution of the problem of the addition of the double ϑ-functions.

I use when convenient the abbreviated notations

$$a-x = \mathrm{a}_1, \quad a-y = \mathrm{a}_2, \quad \text{etc.},$$

$$b-x = \mathrm{b}_1, \quad \text{etc.},$$

$$\theta_{12} = x-y, \quad \theta_{34} = z-w, \quad \theta_{56} = p-q\,;$$

we have of course

$$X = \mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1,$$

$$A_{12} = \sqrt{\mathrm{a}_1\mathrm{a}_2},$$

$$AB_{12} = \frac{1}{\theta_{12}}\left\{\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2} - \sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{f}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}\right\}, \text{ etc.}$$

Proceeding to the investigation, the equations between the variables are obviously those obtained by the elimination of the arbitrary multipliers α, β, γ, δ, ϵ from the six equations obtained from

$$\alpha\theta^3 + \beta\theta^2 + \gamma\theta + \delta = \epsilon\sqrt{\Theta},$$

by writing therein for θ the values x, y, z, w, p, q successively; we may consider the four equations

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = \epsilon\sqrt{X},$$

$$\alpha y^3 + \beta y^2 + \gamma y + \delta = \epsilon\sqrt{Y},$$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = \epsilon\sqrt{Z},$$

$$\alpha w^3 + \beta w^2 + \gamma w + \delta = \epsilon\sqrt{W},$$

as serving to determine the ratios $\alpha : \beta : \gamma : \delta : \epsilon$ in terms of x, y, z, w; and we have then for the determination of p, q the remaining two equations

$$\alpha p^3 + \beta p^2 + \gamma p + \delta = \epsilon\sqrt{P},$$
$$\alpha q^3 + \beta q^2 + \gamma q + \delta = \epsilon\sqrt{Q},$$

which two equations may be replaced by the identity

$$(\alpha\theta^3 + \beta\theta^2 + \gamma\theta + \delta)^2 - \epsilon^2\Theta = \alpha^2 - \epsilon^2 . \theta - x . \theta - y . \theta - z . \theta - w . \theta - p . \theta - q.$$

Writing herein $\theta =$ any one of the values a, b, c, d, e, f, for instance $\theta = a$, and taking the square root of each side, we have

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\alpha^2 - \epsilon^2}\sqrt{a - x . a - y}\sqrt{a - z . a - w}\sqrt{a - p . a - q},$$

or as this may be written

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\alpha^2 - \epsilon^2}\, A_{12} . A_{34} . A_{56},$$

which equation when reduced gives the proportional value of A_{56}.

For the reduction we require the value of $\alpha a^3 + \beta a^2 + \gamma a + \delta$: calling this for the moment Ω, we join to the four equations a fifth equation

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \Omega.$$

Eliminating $\alpha, \beta, \gamma, \delta$, we find

$$\begin{vmatrix} x^3, & x^2, & x, & 1, & \epsilon\sqrt{X} \\ y^3, & y^2, & y, & 1, & \epsilon\sqrt{Y} \\ z^3, & z^2, & z, & 1, & \epsilon\sqrt{Z} \\ w^3, & w^2, & w, & 1, & \epsilon\sqrt{W} \\ a^3, & a^2, & a, & 1, & \Omega \end{vmatrix} = 0,$$

or, what is the same thing,

$$\Omega \begin{vmatrix} x^3, & x^2, & x, & 1 \\ y^3, & y^2, & y, & 1 \\ z^3, & z^2, & z, & 1 \\ w^3, & w^2, & w, & 1 \end{vmatrix} + \epsilon \begin{vmatrix} \sqrt{X}, & x^3, & x^2, & x, & 1 \\ \sqrt{Y}, & y^3, & y^2, & y, & 1 \\ \sqrt{Z}, & z^3, & z^2, & z, & 1 \\ \sqrt{W}, & w^3, & w^2, & w, & 1 \\ & a^3, & a^2, & a, & 1 \end{vmatrix} = 0;$$

viz. this is

$$\begin{aligned} \Omega . x-y . x-z . x-w . y-z . y-w . z-w = -\epsilon\, \{ & \sqrt{X} . y-z . y-w . y-a . z-w . z-a . w-a \\ & + \sqrt{Y} . z-w . z-a . z-x . w-a . w-x . a-x \\ & + \sqrt{Z} . w-a . w-x . w-y . a-x . a-y . x-y \\ & + \sqrt{W} . a-x . a-y . a-z . x-y . x-z . y-z \}, \end{aligned}$$

or as it may be written

$$\Omega\,.\,x-z\,.\,x-w\,.\,y-z\,.\,y-w=\frac{\epsilon\,.\,a-z\,.\,a-w}{x-y}\{y-z\,.\,y-w\,.\,a-y\,.\,\sqrt{X}-x-z\,.\,x-w\,.\,a-x\,.\,\sqrt{Y}\}$$

$$+\frac{\epsilon\,.\,a-x\,.\,a-y}{z-w}\{w-x\,.\,w-y\,.\,a-w\,.\,\sqrt{Z}-z-x\,.\,z-y\,.\,a-z\,.\,\sqrt{W}\},$$

an equation for the determination of Ω.

Consider first the expression which multiplies $\epsilon\,.\,a-z\,.\,a-w$; this is

$$=\frac{1}{\theta_{12}}\{y-z\,.\,y-w\,.\,\mathrm{a}_2\sqrt{X}-x-z\,.\,x-w\,.\,\mathrm{a}_1\sqrt{Y}\};$$

we have

$$BE_{12}=\frac{1}{\theta_{12}}\{\sqrt{\mathrm{b}_1\mathrm{c}_1\mathrm{f}_1\mathrm{a}_2\mathrm{c}_2\mathrm{d}_2}-\sqrt{\mathrm{b}_2\mathrm{c}_2\mathrm{f}_2\mathrm{a}_1\mathrm{c}_1\mathrm{d}_1}\},$$

and multiplying this by

$$A_{12}\,.\,C_{12}\,.\,D_{12},\ =\sqrt{\mathrm{a}_1\mathrm{c}_1\mathrm{d}_1\mathrm{a}_2\mathrm{c}_2\mathrm{d}_2},$$

we derive

$$BE_{12}\,.\,C_{12}\,.\,D_{12}\,.\,A_{12}=\frac{1}{\theta_{12}}\{\mathrm{c}_2\mathrm{d}_2\mathrm{a}_2\sqrt{X}-\mathrm{c}_1\mathrm{d}_1\mathrm{a}_1\sqrt{Y}\},$$

and similarly two other equations; the system may be written

$$BE\,.\,C\,.\,D\,.\,A=\frac{1}{\theta_{12}}\{\mathrm{c}_2\mathrm{d}_2\mathrm{a}_2\sqrt{X}-\mathrm{c}_1\mathrm{d}_1\mathrm{a}_1\sqrt{Y}\},$$

$$CE\,.\,D\,.\,B\,.\,A=\ \text{,,}\ \{\mathrm{d}_2\mathrm{b}_2\ \text{,,}\ \ \text{,,}\ \ -\mathrm{d}_1\mathrm{b}_1\ \text{,,}\ \ \text{,,}\ \},$$

$$DE\,.\,B\,.\,C\,.\,A=\ \text{,,}\ \{\mathrm{b}_2\mathrm{c}_2\ \text{,,}\ \ \text{,,}\ \ -\mathrm{b}_1\mathrm{c}_1\ \text{,,}\ \ \text{,,}\ \},$$

the suffixes on the left-hand side being always 12. The letters b, c, d which enter cyclically into these equations are any three of the five letters other than a; the remaining two letters e and f enter symmetrically, for BE is a mere abbreviation for the double triad $BEF\,.\,ACD$; and the like for CE, and DE.

Multiplying these equations by

$$\frac{b-z\,.\,b-w}{b-c\,.\,b-d},\quad \frac{c-z\,.\,c-w}{c-d\,.\,c-b},\quad \frac{d-z\,.\,d-w}{d-b\,.\,d-c},$$

respectively, and then adding, the right-hand side becomes

$$=\frac{1}{\theta_{12}}\{y-z\,.\,y-w\,.\,\mathrm{a}_2\sqrt{X}-x-z\,.\,x-w\,.\,\mathrm{a}_1\sqrt{Y}\}.$$

Writing

$$\frac{b-z\,.\,b-w}{b-c\,.\,b-d}=\frac{-1}{c-d\,.\,d-b\,.\,b-c}\,.\,c-d\,.\,B_{34}^2,\ \text{etc.},$$

the left-hand side becomes

$$=\frac{-A_{12}}{c-d.d-b.b-c}\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}+d-b.C_{34}^2.CE_{12}.D_{12}.B_{12}+b-c.D_{34}^2.DE_{12}.B_{12}.C_{12}\},$$

which for shortness may be written

$$=\frac{-A_{12}}{c-d.d-b.b-c}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\},$$

the summation referring to the three terms obtained by the cyclical interchange of the letters b, c, d. The result thus is

$$\frac{1}{\theta_{12}}\{y-z.y-w.\mathrm{a}_2\sqrt{X}-x-z.x-w.\mathrm{a}_1\sqrt{Y}\}$$

$$=\frac{-A_{12}}{c-d.d-b.b-c}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\}.$$

Interchanging x, y with z, w respectively, we have of course to interchange the suffixes 1, 2 and 3, 4; we thus find

$$\frac{1}{\theta_{34}}\{w-x.w-y.\mathrm{a}_4\sqrt{Z}-z-x.z-y.\mathrm{a}_3\sqrt{W}\}$$

$$=\frac{-A_{34}}{c-d.d-b.b-c}\Sigma\{c-d.B_{12}^2.BE_{34}.C_{34}.D_{34}\},$$

and we hence find the value of $\Omega.x-z.x-w.y-z.y-w$. But Ω, $=\alpha a^3+\beta a^2+\gamma a+\delta$, is $=\sqrt{\alpha^2-\epsilon^2}.A_{12}.A_{34}.A_{56}$: the resulting equation divides by $A_{12}.A_{34}$: throwing out this factor, we have

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z.x-w.y-z.y-w)(c-d.d-b.b-c)A_{56}$$

$$=A_{34}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\}+A_{12}\Sigma\{c-d.B_{12}^2.BE_{34}.C_{34}.D_{34}\},$$

where, as before, the summations refer to the three terms obtained by the cyclical interchange of the letters b, c, d; these being any three of the five letters other than a; and the remaining two letters e, f enter into the formula symmetrically. The formula gives thus for A_{56} ten values which are of course equal to each other.

Writing for a each letter in succession, we obtain formulæ for each of the six single-letter functions A_{56} of p and q; and the factor

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z.x-w.y-z.y-w)$$

is the same in all the formulæ.

We require further the expressions for the double-letter functions of p, q. Considering for example the function DE_{56}, which is

$$=\frac{1}{\theta_{56}}\{\sqrt{\mathrm{d}_5\mathrm{e}_5\mathrm{f}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6}-\sqrt{\mathrm{d}_6\mathrm{e}_6\mathrm{f}_6\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5}\},$$

then multiplying by

$$A_{56}\,.\,B_{56}\,.\,C_{56},\ =\sqrt{\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6},$$

we have

$$DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56}=\frac{1}{\theta_{56}}\{\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6\sqrt{P}-\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\sqrt{Q}\},$$

$$=\frac{1}{p-q}\{a-q\,.\,b-q\,.\,c-q\,.\,\sqrt{P}-a-p\,.\,b-p\,.\,c-p\,.\,\sqrt{Q}\},$$

or recollecting that $\epsilon\sqrt{P}$, $\epsilon\sqrt{Q}$ are $=\alpha p^3+\beta p^2+\gamma p+\delta$ and $\alpha q^3+\beta q^2+\gamma q+\delta$ respectively, this is

$$\epsilon\,.\,DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56}$$

$$=\frac{1}{p-q}\{a-q\,.\,b-q\,.\,c-q\,.\,(\alpha p^3+\beta p^2+\gamma p+\delta)-a-p\,.\,b-p\,.\,c-p\,.\,(\alpha q^3+\beta q^2+\gamma q+\delta)\}.$$

Using the well-known identity

$$\begin{aligned}\alpha p^3+\beta p^2+\gamma p+\delta=\ &\alpha a^3+\beta a^2+\gamma a+\delta\,.\,\frac{b-p\,.\,c-p\,.\,d-p}{b-a\,.\,c-a\,.\,d-a}\\ &+\alpha b^3+\beta b^2+\gamma b+\delta\,.\,\frac{c-p\,.\,d-p\,.\,a-p}{c-b\,.\,d-b\,.\,a-b}\\ &+\alpha c^3+\beta c^2+\gamma c+\delta\,.\,\frac{d-p\,.\,a-p\,.\,b-p}{d-c\,.\,a-c\,.\,b-c}\\ &+\alpha d^3+\beta d^2+\gamma d+\delta\,.\,\frac{a-p\,.\,b-p\,.\,c-p}{a-d\,.\,b-d\,.\,c-d},\end{aligned}$$

and the like expression for $\alpha q^3+\beta q^2+\gamma q+\delta$, there will be on the right-hand side terms involving

$$\alpha a^3+\beta a^2+\gamma a+\delta,\quad \alpha b^3+\beta b^2+\gamma b+\delta,\quad \alpha c^3+\beta c^2+\gamma c+\delta:$$

but the term in $\alpha d^3+\beta d^2+\gamma d+\delta$ will disappear of itself.

The term in $\alpha a^3+\beta a^2+\gamma a+\delta$ is

$$\frac{1}{p-q}\,\frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a\,.\,d-a}\,.\,b-q\,.\,c-q\,.\,b-p\,.\,c-p\,.\,(a-q\,.\,d-p-a-p\,.\,d-q),$$

where the expression in ( ) is $=d-a\,.\,p-q$: hence the term is

$$=\frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,.\,b-q\,.\,c-q\,.\,b-p\,.\,c-p,$$

which is

$$=\frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,B_{56}^2\,.\,C_{56}^2.$$

Forming the two other like terms, the equation is

$$\begin{aligned}\epsilon\,.\,DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56}=\ &\frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,B_{56}^2\,.\,C_{56}^2\\ &+\frac{\alpha b^3+\beta b^2+\gamma b+\delta}{c-b\,.\,a-b}\,C_{56}^2\,.\,A_{56}^2\\ &+\frac{\alpha c^3+\beta c^2+\gamma c+\delta}{a-c\,.\,b-c}\,A_{56}^2\,.\,B_{56}^2.\end{aligned}$$

But the expressions

$$\alpha a^3+\beta a^2+\gamma a+\delta,\quad \alpha b^3+\beta b^2+\gamma b+\delta,\quad \alpha c^3+\beta c^2+\gamma c+\delta,$$

are

$$=\sqrt{\alpha^2-\epsilon^2}\,A_{12}\,.\,A_{34}\,.\,A_{56},\quad \sqrt{\alpha^2-\epsilon^2}\,B_{12}\,.\,B_{34}\,.\,B_{56},\quad \sqrt{\alpha^2-\epsilon^2}\,C_{12}\,.\,C_{34}\,.\,C_{56},$$

respectively: the whole equation thus divides by $A_{56}\,.\,B_{56}\,.\,C_{56}$; throwing out this factor, and then multiplying each side by $-\dfrac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}$, we find

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}DE_{56}=\frac{1}{b-c\,.\,c-a\,.\,a-b}\left(-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}\right)^2\{\ b-c\,.\,A_{12}\,.\,A_{34}\,.\,B_{56}\,.\,C_{56}$$
$$+c-a\,.\,B_{12}\,.\,B_{34}\,.\,C_{56}\,.\,A_{56}$$
$$+a-b\,.\,C_{12}\,.\,C_{34}\,.\,A_{56}\,.\,B_{56}\},$$

in which formula if we imagine

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}A_{56},\quad -\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}B_{56},\quad -\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}C_{56}$$

each replaced by its value in terms of the xy- and zw-functions, we have an equation of the form

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z\,.\,x-w\,.\,y-z\,.\,y-w)\,DE_{56}=\frac{1}{x-z\,.\,x-w\,.\,y-z\,.\,y-w}M,$$

where M is a given rational and integral function of the 16 and 16 functions A_{12}, AB_{12} and A_{34}, AB_{34} of x and y and of z and w respectively. The factor

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z\,.\,x-w\,.\,y-z\,.\,y-w)$$

is retained on the left-hand side as being the same factor which enters into the equations for A_{56}, etc.: but on the right-hand side $x-z\,.\,x-w\,.\,y-z\,.\,y-w$ should be expressed in terms of the xy- and zw-functions. This can be done by means of the identity

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=\Sigma\,\frac{\begin{vmatrix}1, & x+y, & xy\\ 1, & z+w, & zw\\ 1, & a+b, & ab\end{vmatrix}\begin{vmatrix}1, & x+y, & xy\\ 1, & z+w, & zw\\ 1, & a+c, & ac\end{vmatrix}}{a-b\,.\,a-c},$$

where the summation refers to the three terms obtained by the cyclical interchange of the letters a, b, c. The first determinant, multiplied by $a-b$, is in fact

$$=\begin{vmatrix}a-z\,.\,a-w, & a-x\,.\,a-y\\ b-z\,.\,b-w, & b-x\,.\,b-y\end{vmatrix},$$

and the second determinant, multiplied by $a-c$, is

$$=\begin{vmatrix}a-z\,.\,a-w, & a-x\,.\,a-y\\ c-z\,.\,c-w, & c-x\,.\,c-y\end{vmatrix},$$

so that the formula may also be written

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=\Sigma\,\frac{\begin{vmatrix} a-z\,.\,a-w, & a-x\,.\,a-y \\ b-z\,.\,b-w, & b-x\,.\,b-y \end{vmatrix}\,.\,\begin{vmatrix} a-z\,.\,a-w, & a-x\,.\,a-y \\ c-z\,.\,c-w, & c-x\,.\,c-y \end{vmatrix}}{(a-b)^2\,(a-c)^2};$$

or, what is the same thing, it is

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=\Sigma\,\frac{(A_{34}^2B_{12}^2-A_{12}^2B_{34}^2)\,(A_{34}^2C_{12}^2-A_{12}^2C_{34}^2)}{(a-b)^2\,(a-c)^2},$$

which is the required expression for $x-z\,.\,x-w\,.\,y-z\,.\,y-w$; the letters a, b, c, which enter into the formula, are any three of the six letters.

As regards the verification of the identity, observe that it may be written

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=\Sigma\,\frac{\{L+M\,(a+b)+Nab\}\,\{L+M\,(a+c)+Nac\}}{a-b\,.\,a-c},$$

where L, M, N are

$$=(x+y)\,zw-(z+w)\,xy,\quad xy-zw,\quad \text{and}\quad z+w-x-y:$$

this is readily reduced to

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=M^2-NL,$$

which can be at once verified.

Cambridge, 12th March, 1879.

I take the opportunity of remarking that, in the double-letter formulæ, the sign of the second term is, not as I have in general written it $-$, but is $+$,

$$AB=\frac{1}{x-y}\,\{\sqrt{\mathrm{abfc_1d_1e_1}}+\sqrt{\mathrm{a_1b_1f_1cde}}\},\ \text{etc.}$$

In fact, introducing a factor ω which is a function of x and y, the odd and even ϑ-functions are $=\omega\sqrt{\mathrm{aa_1}}$, etc., and

$$\frac{\omega}{x-y}\,\{\sqrt{\mathrm{abfc_1d_1e_1}}+\sqrt{\mathrm{a_1b_1f_1cde}}\},\ \text{etc.},$$

respectively; ω is a function which on the interchange of x, y changes only its sign; and this being so, then when x and y are interchanged, each single-letter function changes its sign, and each double-letter function remains unaltered.

Cambridge, 29th July, 1879.

704.

A MEMOIR ON THE SINGLE AND DOUBLE THETA-FUNCTIONS.

[From the *Philosophical Transactions of the Royal Society of London*, vol. 171, Part III., (1880), pp. 897—1002. Received November 14,—Read November 28, 1879.]

THE Theta-Functions, although arising historically from the Elliptic Functions, may be considered as in order of simplicity preceding these, and connecting themselves directly with the exponential function (e^x or) exp. x; viz. they may be defined each of them as a sum of a series of exponentials, singly infinite in the case of the single functions, doubly infinite in the case of the double functions; and so on. The number of the single functions is $=4$; and the quotients of these, or say three of them each divided by the fourth, are the elliptic functions sn, cn, dn; the number of the double functions is ($4^2=$) 16; and the quotients of these, or say fifteen of them each divided by the sixteenth, are the hyper-elliptic functions of two arguments depending on the square root of a sextic function. Generally, the number of the p-tuple theta-functions is $=4^p$; and the quotients of these, or say all but one of them each divided by the remaining function, are the Abelian functions of p arguments depending on the irrational function y defined by the equation $F(x, y)=0$ of a curve of deficiency p. If, instead of connecting the ratios of the functions with a plane curve, we consider the functions themselves as coordinates of a point in a space of (4^p-1) dimensions, then we have the single functions as the four coordinates of a point on a quadri-quadric curve (one-fold locus) in ordinary space; and the double functions as the sixteen coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space, the deficiency of this two-fold locus being of course $=2$.

The investigations contained in the First Part of the present Memoir, although for simplicity of notation exhibited only in regard to the double functions are, in fact, applicable to the general case of the p-tuple functions; but in the main the

Memoir relates only to the single and double functions, and the title has been given to it accordingly. The investigations just referred to extend to the single functions; and there is, it seems to me, an advantage in carrying on the two theories simultaneously up to and inclusive of the establishment of what I call the Product-theorem: this is a natural point of separation for the theories of the single and the double functions respectively. The ulterior developments of the two theories are indeed closely analogous to each other; but on the one hand the course of the single theory would be only with difficulty perceptible in the greater complexity of the double theory; and on the other hand we need the single theory as a guide for the course of the double theory.

I accordingly stop to point out in a general manner the course of the single theory, and, in connexion with it but more briefly, that of the double theory; and I then, in the Second and the Third Parts respectively, consider in detail the two theories separately; first, that of the single functions, and then that of the double functions. The paragraphs of the Memoir are numbered consecutively.

The definition adopted for the theta-functions differs somewhat from that which is ordinarily used.

The earlier memoirs on the double theta-functions are the well-known ones:—

Rosenhain, "Mémoire sur les fonctions de deux variables et à quatre périodes, qui sont les inverses des intégrales ultra-elliptiques de la première classe." [1846.] Paris: *Mém. Savans Étrang.*, t. XI. (1851), pp. 361—468.

Göpel, "Theoriæ transcendentium Abelianarum primi ordinis adumbratio levis," *Crelle*, t. XXXV. (1847), pp. 277—312.

My first paper—Cayley, "On the Double θ-Functions in connexion with a 16-nodal Surface," *Crelle-Borchardt*, t. LXXXIII. (1877), pp. 210—219, [662]—was founded directly upon these, and was immediately followed by Dr Borchardt's paper,

Borchardt, "Ueber die Darstellung der *Kummer*sche Fläche vierter Ordnung mit sechzehn Knotenpunkten durch die *Göpel*sche biquadratische Relation zwischen vier Thetafunctionen mit zwei Variabeln," *Ditto*, pp. 234—244.

My other later papers, [663, 664, 665, 697, 703], are contained in the same Journal.

FIRST PART.—INTRODUCTORY.

Definition of the theta-functions.

1. The p-tuple functions depend upon $\frac{1}{2}p(p-1)$ parameters which are the coefficients of a quadric function of p ultimately disappearing integers, upon p arguments, and upon $2p$ characters, each $=0$ or 1, which form the characteristic of the 4^p functions; but it will be sufficient to write down the formulæ in the case $p=2$.

As already mentioned, the adopted definition differs somewhat from that which is ordinarily used. I use, as will be seen, a quadric function $\frac{1}{4}(a, h, b \between m, n)^2$ with

even integer values of m, n, instead of $(a, h, b \between m, n)^2$ with even or odd values; and I write the other term $\frac{1}{2}\pi i(mu+nv)$, instead of $mu+nv$; this comes to affecting the arguments u, v with a factor πi, so that the quarter-periods (instead of being πi) are made to be $=1$.

2. We write

$$\begin{pmatrix} m, & n \\ u, & v \end{pmatrix} = \tfrac{1}{4}(a, h, b \between m, n)^2 + \tfrac{1}{2}\pi i(mu+nv),$$

and in like manner

$$\begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix} = \tfrac{1}{4}(a, h, b \between m+\alpha, n+\beta)^2 + \tfrac{1}{2}\pi i\,[(m+\alpha)(u+\gamma)+(n+\beta)(v+\delta)],$$

and prefixing to either of these the functional symbol exp. we have the exponential of the function in question, that is, e with the function as an exponent.

We then write, as the definition of the double theta-functions,

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u, v) = \Sigma \exp. \begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix},$$

where the summation extends to all positive and negative even integer values (zero included) of m and n respectively: α, β, γ, δ might denote any quantities whatever, but for the theta-functions they are regarded as denoting positive or negative integers; this being so, it will appear that the only effect of altering each or any of them by an even integer is to reverse (it may be) the sign of the function; and the distinct functions are consequently the $(4^2=)$ 16 functions obtained by giving to each of the quantities α, β, γ, δ the two values 0 and 1 successively.

3. We thus have the double theta-functions, depending on the parameters (a, h, b) which determine the quadric function $(a, h, b \between m, n)^2$ of the disappearing even integers (m, n), and on the two arguments (u, v): in the symbol $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, which is called the characteristic, the characters α, β, γ, δ are each of them $=0$ or 1; and we thus have the 16 functions.

The parameters (a, h, b) may be real or imaginary, but they must be such that reducing each of them to its real part the resulting function $(* \between m, n)^2$ is invariable in its sign, and negative for all real values of m and n: this is, in fact, the condition for the convergency of the series which give the values of the theta-functions.

4. The characteristic $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ is said to be even or odd according as the sum $\alpha\gamma+\beta\delta$ is even or odd.

Allied functions.

5. As already remarked, the definition of

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u, v)$$

is not restricted to the case where the α, β, γ, δ represent integers, and there is actually occasion to consider functions of this form where they are not integers: in particular, α, β may be either or each of them of the form, integer $+\frac{1}{2}$. But the functions thus obtained *are not regarded as theta-functions*, and the expression theta-function will consequently not extend to include them.

PROPERTIES OF THE THETA-FUNCTIONS: VARIOUS SUB-HEADINGS.

Even-integer alteration of characters.

6. If x, y be integers, then m, n having the several even integer values from $-\infty$ to $+\infty$ respectively, it is obvious that $m+\alpha+2x$, $n+\beta+2y$ will have the same series of values with $m+\alpha$, $n+\beta$ respectively; and it thence follows that

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma & , \delta\end{pmatrix}(u, v) = \vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v).$$

Similarly if z, w are integers, then in the function

$$\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v)$$

the argument of the exponential function contains the term

$$\tfrac{1}{2}\pi i\{m+\alpha\,.\,u+\gamma+2z+n+\beta\,.\,v+\delta+2w\};$$

this differs from its original value by

$$\begin{aligned}&\tfrac{1}{2}\pi i\,(m+\alpha\,.\,2z+n+\beta\,.\,2w),\\ &=\pi i\,(mz+nw)+\pi i\,(\alpha z+\beta w),\end{aligned}$$

and then, m and n being even integers, $mz+nw$ is also an even integer, and the term $\pi i\,(mz+nw)$ does not affect the value of the exponential: we thus introduce into each term of the series the factor exp. $\pi i\,(\alpha z+\beta w)$, which is, in fact, $=(-)^{\alpha z+\beta w}$; and we consequently have

$$\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v) = (-)^{\alpha z+\beta w}\,\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v);$$

or, uniting the two results,

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v) = (-)^{\alpha z+\beta w}\,\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v).$$

This sustains the before-mentioned conclusion that the only distinct functions are the 16 functions obtained by giving to the characters α, β, γ, δ the values 0 and 1 respectively.

Odd-integer alteration of characters.

7. The effect is obviously to interchange the different functions.

Even and odd functions.

8. It is clear that $-m-\alpha$, $-n-\beta$ have precisely the same series of values with $m+\alpha$, $n+\beta$ respectively: hence considering the function

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(-u, -v)$$

the linear term in the argument of the exponential may be taken to be

$$\tfrac{1}{2}\pi i\,\{-m-\alpha\,.\,-u+\gamma+-n-\beta\,.\,-v+\delta\},$$

which is

$$=\tfrac{1}{2}\pi i\,\{m+\alpha\,.\,u+\gamma+n+\beta\,.\,v+\delta\}-\pi i\,\{m+\alpha\,.\,\gamma+n+\beta\,.\,\delta\};$$

the second term is here

$$=-\pi i\,(m\gamma+n\delta)-\pi i\,(\alpha\gamma+\beta\delta),$$

where, $m\gamma+n\delta$ being an even integer, the part $-\pi i\,(m\gamma+n\delta)$ does not alter the value of the exponential: the effect of the remaining part $-\pi i\,(\alpha\gamma+\beta\delta)$ is to affect each term of the series with the factor exp. $-\pi i\,(\alpha\gamma+\beta\delta)$, or what is the same thing, exp. $\pi i\,(\alpha\gamma+\beta\delta)$, each of these being, in fact, $=(-)^{\alpha\gamma+\beta\delta}$.

We have thus

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(-u, -v)=(-)^{\alpha\gamma+\beta\delta}\,\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v),$$

viz. $\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v)$ is an even or odd function of the two arguments (u, v) conjointly, according as the characteristic $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ is even or odd.

The quarter-periods unity.

9. Taking z and w integers, we have from the definition

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+z, v+w)=\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+z, & \delta+w\end{pmatrix}(u, v),$$

viz. the effect of altering the arguments u, v into $u+z$, $v+w$ is simply to interchange the functions as shown by this formula.

If z and w are each of them even, then replacing them by $2z$, $2w$ respectively, we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+2z, v+2w)=\vartheta\begin{pmatrix}\alpha & , \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v),$$

which by a preceding formula is

$$=(-)^{\alpha z+\beta w}\,\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v),$$

or the function is altered at most in its sign. And again writing $2z$, $2w$ for z, w, we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+4z,\ v+4w)=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v).$$

In reference to the foregoing results we say that the theta-functions have the quarter-periods (1, 1), the half-periods (2, 2), and the whole periods (4, 4).

The conjoint quarter quasi-periods.

10. Taking x, y integers, we consider the effect of the change of u, v into

$$u+\frac{1}{\pi i}(ax+hy),\quad v+\frac{1}{\pi i}(hx+by).$$

It is convenient to start from the function

$$\vartheta\begin{pmatrix}\alpha-x, & \beta-y\\ \gamma\quad, & \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right);$$

the argument of the exponential is here

$$\tfrac{1}{4}(a,\ h,\ b\between m+\alpha-x,\ n+\beta-y)^2$$
$$+\tfrac{1}{2}\pi i\left\{m+\alpha-x\,.\,u+\gamma+\frac{1}{\pi i}(ax+hy)+n+\beta-y\,.\,v+\delta+\frac{1}{\pi i}(hx+by)\right\},$$

which is

$$=\tfrac{1}{4}(a,\ h,\ b\between m+\alpha,\ n+\beta)^2+\tfrac{1}{2}\pi i(m+\alpha\,.\,u+\gamma+n+\beta\,.\,v+\delta)$$

+ other terms which are as follows: viz. they are

$$\begin{array}{ll}-\tfrac{1}{2}(a,\ h,\ b\between m+\alpha,\ n+\beta\between x,\ y) & +\tfrac{1}{2}(m+\alpha\,.\,ax+hy+n+\beta\,.\,hx+by)\\ +\tfrac{1}{4}(a,\ h,\ b\between x,\ y)^2 & -\tfrac{1}{2}\pi i(x\,.\,u+\gamma+y\,.\,v+\delta)\\ & -\tfrac{1}{2}(x\,.\,ax+hy+y\,.\,hx+by),\end{array}$$

where the terms of the right-hand column are, in fact,

$$\begin{aligned}&=+\tfrac{1}{2}(a,\ h,\ b\between m+\alpha,\ n+\beta\between x,\ y)\\ &\quad.\ -\tfrac{1}{2}\pi i(x\,.\,u+\gamma+y\,.\,v+\delta)\\ &\quad\ -\tfrac{1}{2}(a,\ h,\ b\between x,\ y)^2,\end{aligned}$$

and the other terms in question thus reduce themselves to

$$-\tfrac{1}{4}(a,\ h,\ b\between x,\ y)^2\qquad-\tfrac{1}{2}\pi i(x\,.\,u+\gamma+y\,.\,v+\delta),$$

which are independent of m, n, and they thus affect each term of the series with the same exponential factor. The result is

$$\vartheta\begin{pmatrix}\alpha-x, & \beta-y\\ \gamma\quad, & \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right)$$
$$=\exp.\left\{-\tfrac{1}{4}(a,\ h,\ b\between x,\ y)^2-\tfrac{1}{2}\pi i(x\,.\,u+\gamma+y\,.\,v+\delta)\right\}\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v);$$

or (what is the same thing) for α, β, writing $\alpha+x$, $\beta+y$ respectively, we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right)$$

$$=\exp.\{-\tfrac{1}{4}(a,\ h,\ b\between x,\ y)^2-\tfrac{1}{2}\pi i(x\,.\,u+\gamma+y\,.\,v+\delta)\}\,\vartheta\begin{pmatrix}\alpha+x, & \beta+y\\ \gamma & ,\ \delta\end{pmatrix}(u,\ v).$$

Taking x, y even, or writing $2x$, $2y$ for x, y, then on the right-hand side we have

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma & ,\ \delta\end{pmatrix}(u,\ v),$$

which is

$$=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v):$$

but there is still the exponential factor.

11. The formulæ show that the effect of the change u, v into $u+\frac{1}{\pi i}(ax+hy)$, $v+\frac{1}{\pi i}(hx+by)$, where x, y are integers, is to interchange the functions, affecting them however with an exponential factor; and we hence say that $\frac{1}{\pi i}(a,\ h)$, $\frac{1}{\pi i}(h,\ b)$ are conjoint quarter quasi-periods.

The product-theorem.

12. We multiply two theta-functions

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+u',\ v+v'),\quad \vartheta\begin{pmatrix}\alpha', & \beta'\\ \gamma', & \delta'\end{pmatrix}(u-u',\ v-v');$$

it is found that the result is a sum of four products

$$\Theta\begin{pmatrix}\frac{1}{2}(\alpha+\alpha')+p, & \frac{1}{2}(\beta+\beta')+q\\ \gamma+\gamma' & ,\ \delta+\delta'\end{pmatrix}(2u,\ 2v)\,.\,\Theta\begin{pmatrix}\frac{1}{2}(\alpha-\alpha')+p, & \frac{1}{2}(\beta-\beta')+q\\ \gamma-\gamma' & ,\ \delta-\delta'\end{pmatrix}(2u',\ 2v'),$$

where p, q have in the four products respectively the values (0, 0), (1, 0), (0, 1), and (1, 1); Θ is written in place of ϑ to denote that the parameters $(a,\ h,\ b)$ are to be changed into $(2a,\ 2h,\ 2b)$. It is to be noticed that, if α, α' are both even or both odd, then $\frac{1}{2}(\alpha+\alpha')$, $\frac{1}{2}(\alpha-\alpha')$ are integers; and so, if β, β' are both even or both odd, then $\frac{1}{2}(\beta+\beta')$, $\frac{1}{2}(\beta-\beta')$ are integers; and these conditions being satisfied (and in particular they are so if $\alpha=\alpha'$, $\beta=\beta'$) then the functions on the right-hand side of the equation are theta-functions (with new parameters as already mentioned); but if the conditions are not satisfied, then the functions on the right-hand side are only allied functions. In the applications of the theorem the functions on the right-hand side are eliminated between the different equations, as will appear.

13. The proof is immediate: in the first of the theta-functions, the argument of the exponential is

$$\begin{pmatrix}m+\alpha & ,\ n+\beta\\ u+u'+\gamma, & v+v'+\delta\end{pmatrix},$$

and in the second, writing m', n' instead of m, n, the argument is

$$\begin{pmatrix} m'+\alpha' & , n'+\beta' \\ u-u'+\gamma', & v-v'+\delta' \end{pmatrix};$$

hence in the product, the argument of the exponential is the sum of these two functions, viz.

$$\begin{aligned} &= \tfrac{1}{4}(a,\ h,\ b \between m+\alpha,\ n+\beta)^2 + \tfrac{1}{2}\pi i\,(m+\alpha\,.\,u+u'+\gamma+n+\beta\,.\,v+v'+\delta) \\ &+ \tfrac{1}{4}(a,\ h,\ b \between m'+\alpha',\ n'+\beta')^2 + \tfrac{1}{2}\pi i\,(m'+\alpha'\,.\,u-u'+\gamma'+n'+\beta'\,.\,v-v'+\delta'). \end{aligned}$$

Comparing herewith the sum of the two functions

$$\begin{aligned} &\begin{pmatrix} \mu+\frac{1}{2}(\alpha+\alpha'), & \nu+\frac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' & , 2v+\delta+\delta' \end{pmatrix},\ \begin{pmatrix} \mu'+\frac{1}{2}(\alpha-\alpha'), & \nu'+\frac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' & , 2v'+\delta-\delta' \end{pmatrix}, \\ &= \tfrac{1}{4}(2a,\ 2h,\ 2b \between \mu+\tfrac{1}{2}(\alpha+\alpha'),\ \nu+\tfrac{1}{2}(\beta+\beta'))^2 \\ &\qquad + \tfrac{1}{2}\pi i\,\{\mu+\tfrac{1}{2}(\alpha+\alpha')\,.\,2u+\gamma+\gamma'+\nu+\tfrac{1}{2}(\beta+\beta')\,.\,2v+\delta+\delta'\} \\ &+ \tfrac{1}{4}(2a,\ 2h,\ 2b \between \mu'+\tfrac{1}{2}(\alpha-\alpha'),\ \nu'+\tfrac{1}{2}(\beta-\beta'))^2 \\ &\qquad + \tfrac{1}{2}\pi i\,\{\mu'+\tfrac{1}{2}(\alpha-\alpha')\,.\,2u'+\gamma-\gamma'+\nu'+\tfrac{1}{2}(\beta-\beta')\,.\,2v'+\delta-\delta'\}, \end{aligned}$$

the two sums are identical if only

$$\begin{aligned} m+m' &= 2\mu,\ \ n+n' = 2\nu, \\ m-m' &= 2\mu',\ \ n-n' = 2\nu', \end{aligned}$$

as may easily be verified by comparing the quadric and the linear terms separately. The product of the two theta-functions is thus

$$= \Sigma \exp. \begin{pmatrix} \mu+\frac{1}{2}(\alpha+\alpha'), & \nu+\frac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' & , 2v+\delta+\delta' \end{pmatrix}.\ \Sigma \exp. \begin{pmatrix} \mu'+\frac{1}{2}(\alpha-\alpha'), & \nu'+\frac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' & , 2v'+\delta-\delta' \end{pmatrix},$$

with the proper conditions as to the values of μ, ν and of μ', ν' in the two sums respectively. As to this, observe that m, m' are even integers; say for a moment that they are similar when they are both $\equiv 0$ or both $\equiv 2 \pmod 4$, but dissimilar when they are one of them $\equiv 0$ and the other of them $\equiv 2 \pmod 4$; and the like as regards n, n'. Hence if m, m' are similar, μ, μ' are both of them even; but if m, m' are dissimilar, then μ, μ' are both of them odd. And so if n, n' are similar, ν, ν' are both of them even; but if n, n' are dissimilar, then ν, ν' are both odd.

14. There are four cases:

m, m' similar, n, n' similar,
m, m' dissimilar, n, n' similar,
m, m' similar, n, n' dissimilar,
m, m' dissimilar, n, n' dissimilar.

In the first of these, μ, ν, μ', ν' are all of them even, and the product is

$$= \Theta \begin{pmatrix} \frac{1}{2}(\alpha+\alpha'), & \frac{1}{2}(\beta+\beta') \\ \gamma+\gamma', & \delta+\delta' \end{pmatrix} (2u,\ 2v)\,.\,\Theta \begin{pmatrix} \frac{1}{2}(\alpha-\alpha'), & \frac{1}{2}(\beta-\beta') \\ \gamma-\gamma', & \delta-\delta' \end{pmatrix} (2u',\ 2v').$$

In the second case, writing $\mu+1$, $\mu'+1$ for μ, μ', the new values of μ, μ' will be both even, and we have the like expression with only the characters $\frac{1}{2}(\alpha+\alpha')$, $\frac{1}{2}(\alpha-\alpha')$ each increased by 1; so in the third case we obtain the like expression with only the characters $\frac{1}{2}(\beta+\beta')$, $\frac{1}{2}(\beta-\beta')$ each increased by 1; and in the fourth case the like expression with the four upper characters each increased by 1. The product of the two theta-functions is thus equal to the sum of the four products, according to the theorem.

Résumé of the ulterior theory of the single functions.

15. For the single theta-functions the Product-theorem comprises 16 equations, and for the double theta-functions, 256 equations: these systems will be given in full in the sequel. But attending at present to the single functions, I write down here the first four of the 16 equations, viz. these are

$$\begin{array}{lllll}
0.0 & \vartheta\begin{pmatrix}0\\0\end{pmatrix}(u+u').\vartheta\begin{pmatrix}0\\0\end{pmatrix}(u-u') & = & XX'+YY', \\
1.0 & \vartheta\begin{matrix}1\\0\end{matrix} \quad ,, \quad \vartheta\begin{matrix}1\\0\end{matrix} & ,, & = YX'+XY', \\
0.1 & \vartheta\begin{matrix}0\\1\end{matrix} \quad ,, \quad \vartheta\begin{matrix}0\\1\end{matrix} & ,, & = XX'-YY', \\
1.1 & \vartheta\begin{matrix}1\\1\end{matrix} \quad ,, \quad \vartheta\begin{matrix}1\\1\end{matrix} & ,, & = -YX'+XY';
\end{array}$$

where X, Y denote $\Theta\begin{pmatrix}0\\0\end{pmatrix}(2u)$, $\Theta\begin{pmatrix}1\\0\end{pmatrix}(2u)$ respectively, and X', Y' the same functions of $2u'$ respectively. In the other equations we have on the left-hand the product of *different* theta-functions of $u+u'$, $u-u'$ respectively, and on the right-hand expressions involving other functions, X_1, Y_1, X_1', Y_1', &c., of $2u$ and $2u'$ respectively.

16. By writing $u'=0$, we have on the left-hand, squares or products of theta-functions of u, and on the right-hand expressions containing functions of $2u$: in particular, the above equations show that the squares of the four theta-functions are equal to linear functions of X, Y; that is, there exist between the squared functions two linear relations: or again, introducing a variable argument x, the four squared functions may be taken to be proportional to linear functions

$$\mathfrak{A}(a-x),\quad \mathfrak{B}(b-x),\quad \mathfrak{C}(c-x),\quad \mathfrak{D}(d-x),$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, a, b, c, d, are constants. This suggests a new notation for the four functions, viz. we write

$$\begin{array}{llll}
\vartheta\begin{pmatrix}0\\0\end{pmatrix}(u), & \vartheta\begin{pmatrix}1\\0\end{pmatrix}(u), & \vartheta\begin{pmatrix}0\\1\end{pmatrix}(u), & \vartheta\begin{pmatrix}1\\1\end{pmatrix}(u) \\
=Au, & Bu, & Cu, & Du;
\end{array}$$

and the result just mentioned then is

$$\begin{array}{l}
A^2u \quad : \quad B^2u \quad : \quad C^2u \quad : \quad D^2u \\
=\mathfrak{A}(a-x) : \mathfrak{B}(b-x) : \mathfrak{C}(c-x) : \mathfrak{D}(d-x),
\end{array}$$

which expresses that the four functions are the coordinates of a point on a quadri-quadric curve in ordinary space.

17. The remaining 12 of the 16 equations then contain on the left-hand products such as

$$A\,(u+u')\,.\,B\,(u-u');$$

and by suitably combining them we obtain equations such as

$$\frac{B^{u+u'}\,.\,A^{u-u'} - A^{u+u'}\,.\,B^{u-u'}}{C\,.\,D + D\,.\,C} = \text{function } (u'),$$

where for brevity the arguments are written above; viz. the numerator of the fraction is

$$B\,(u+u')\,A\,(u-u') - A\,(u+u')\,B\,(u-u'),$$

and its denominator is

$$C\,(u+u')\,D\,(u-u') + D\,(u+u')\,C\,(u-u').$$

Admitting the form of the equation, the value of the function of u' is at once found by writing in the equation $u=0$; it is, as it ought to be, a function vanishing for $u'=0$.

18. Take in this equation u' indefinitely small; each side divides by u', and the resulting equation is

$$\frac{AuB'u - BuA'u}{CuDu} = \text{const.},$$

where $A'u$, $B'u$ are the derived functions, or differential coefficients in regard to u. It thus appears that the combination $AuB'u - BuA'u$ is a constant multiple of $CuDu$: or, what is the same thing, that the differential coefficient of the quotient-function $\frac{Bu}{Au}$ is a constant multiple of the product of the two quotient-functions $\frac{Cu}{Au}$ and $\frac{Du}{Au}$.

19. And then substituting for the several quotient-functions their values in terms of x, we obtain a differential relation between x, u; viz. the form hereof is

$$du = \frac{Mdx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}},$$

and it thus appears that the quotient-functions are in fact elliptic-functions: the actual values as obtained in the sequel are

$$\operatorname{sn} Ku = -\frac{1}{\sqrt{k}}\,Du \div Cu,$$

$$\operatorname{cn} Ku = \sqrt{\frac{k'}{k}}\,Bu \div Cu,$$

$$\operatorname{dn} Ku = \sqrt{k'}\,Au \div Cu;$$

and we thus of course identify the functions Au, Bu, Cu, Du with the H and the Θ functions of Jacobi.

20. If in the above-mentioned four equations we write first $u=0$, and then $u'=0$, and by means of the results eliminate from the original equations the quantities X, Y, X', Y' which occur therein, we obtain expressions for the four products such as $A(u+u')A(u-u')$. One of these equations is

$$C^2 0 \,.\, C(u+u')\,C(u-u') = C^2u\,C^2u' - D^2u\,D^2u'.$$

Taking herein u' indefinitely small, we obtain

$$\frac{CuC''u-(C'u)^2}{C^2u} = \frac{C''0}{C0} - \left(\frac{D'0}{C0}\right)^2 \frac{D^2u}{C^2u},$$

where the left-hand side is in fact $\frac{d^2}{du^2}\log Cu$, or this second derived function of the theta-function Cu is given in terms of the quotient-function $\frac{Du}{Cu}$: hence, integrating twice and taking the exponential of each side, we obtain Cu as an exponential the argument of which contains the double integral $\iint \frac{D^2u}{C^2u}(du)^2$, of a squared quotient-function. This, in fact, corresponds to Jacobi's equation

$$\Theta u = \sqrt{\frac{2Kk'}{\pi}}\, e^{\frac{1}{2}u^2\left(1-\frac{E}{K}\right)-k^2\int_0 du \int_0 du\, \mathrm{sn}^2 u}.$$

21. From the same equation

$$C^2 0 \,.\, C(u+u')\,C(u-u') = C^2u\,C^2u' - D^2u\,D^2u',$$

differentiating logarithmically in regard to u' and integrating in regard to u, we obtain an equation containing on the left-hand side a term $\log\frac{C(u-u')}{C(u+u')}$, and on the right-hand an integral in regard to u; this, in fact, corresponds to Jacobi's equation

$$u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)} = \Pi(u,\, a)$$
$$= \int_0 \frac{k^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2 u\,du}{1-k^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u}.$$

22. It may further be noticed that if, in the equation in question and in the three other equations of the system, we introduce into the integral the variable x in place of u, and the corresponding quantity ξ in place of u', then the integral is that of an expression such as

$$\frac{dx}{T\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}},$$

where T is $=x-\xi$, or is $=$ any one of three forms such as

$$\begin{vmatrix} 1, & x+\xi, & x\xi \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}.$$

Résumé of the ulterior theory of the double functions.

23. The ulterior theory of the double functions is intended to be carried out on the like plan. As regards these, it is to be observed here that we have not only the 16 equations leading to linear relations between the squared functions, but that the remaining 240 equations lead also to linear relations between binary products of different functions. We have thus between the 16 functions a system of quadric relations, which in fact determine the ratios of the 16 functions in terms of two variable parameters x, y. (The 16 functions are thus the coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space.) The forms depend upon six constants, a, b, c, d, e, f: writing for shortness

$$\sqrt{a} = \sqrt{a-x\,.\,a-y},$$

$$\vdots$$

$$\sqrt{ab} = \frac{1}{x-y}\{\sqrt{a-x\,.\,b-x\,.\,f-x\,.\,c-y\,.\,d-y\,.\,e-y} + \sqrt{a-y\,.\,b-y\,.\,f-y\,.\,c-x\,.\,d-x\,.\,e-x}\},$$

$$\vdots$$

(observe that in the symbols $\sqrt{ab}$ it is always f that accompanies the two expressed letters a, b—or, what is the same thing, the duad ab is really an abbreviation for the double triad $abf\,.\,cde$): then the 16 functions are proportional to properly determined constant multiples of

$$\sqrt{a},\ \sqrt{b},\ \sqrt{c},\ \sqrt{d},\ \sqrt{e},\ \sqrt{f},\ \sqrt{ab},\ \sqrt{ac},\ \sqrt{ad},\ \sqrt{ae},\ \sqrt{bc},\ \sqrt{bd},\ \sqrt{be},\ \sqrt{cd},\ \sqrt{ce},\ \sqrt{de}:$$

and this suggests that the functions should be represented by the single and double letter notation $A(u, v), \ldots, AB(u, v), \ldots$; viz. if for shortness the arguments are omitted, then we have

$$A,\ B,\ C,\ D,\ E,\ F,\ AB,\ AC,\ AD,\ AE,\ BC,\ BD,\ BE,\ CD,\ CE,\ DE,$$

proportional to determinate constant multiples of the before-mentioned functions $\sqrt{a}, \ldots, \sqrt{ab}, \ldots$, of x and y.

24. It is interesting to notice why in the expressions for $\sqrt{ab}$, &c., the sign connecting the two radicals is $+$; the effect of the interchange of x, y is, in fact, to change (u, v) into $(-u, -v)$; consequently to change the sign of the odd functions, and to leave unaltered those of the even functions: the interchange does in fact leave $\sqrt{a}$, &c., unaltered, while it changes $\sqrt{ab}$, &c., into $-\sqrt{ab}$, &c.; and thus, since only the ratios are attended to, there is a change of sign as there should be.

25. The equations of the product-theorem lead to expressions for

$$\overset{u+u'}{A}\,.\,\overset{u-u'}{B} - \overset{u+u'}{B}\,.\,\overset{u-u'}{A},$$

where the arguments, written above, are used to denote the *two* arguments, viz. $u+u'$ to denote $(u+u', v+v')$ and $u-u'$ to denote $(u-u', v-v')$; and where the letters A, B denote each or either of them a single or double letter. These expressions

are found in terms of the functions of (u, v) and of (u', v'): in any such expression taking u', v' each of them indefinitely small, but with their ratio arbitrary, we obtain the value of

$$\overset{u}{A} \,.\, \partial \overset{u}{B} - \overset{u}{B} \,.\, \partial \overset{u}{A},$$

$\Big($viz. u here stands for the two arguments (u, v), and ∂ denotes total differentiation

$$\partial A = du \frac{d}{du} A(u, v) + dv \frac{d}{dv} A(u, v)\Big),$$

as a quadric function of the functions of (u, v): or dividing by A^2, the form is $\partial \dfrac{B}{A}$ equal to a function of the quotient-functions $\dfrac{B}{A}$, &c., that is, we have the differentials of the quotient-functions in terms of the quotient-functions themselves. Substituting for the quotient-functions their values in terms of x, y, we should obtain the differential relations between dx, dy, du, dv, viz. putting for shortness

$$X = a - x \,.\, b - x \,.\, c - x \,.\, d - x \,.\, e - x \,.\, f - x,$$

and

$$Y = a - y \,.\, b - y \,.\, c - y \,.\, d - y \,.\, e - y \,.\, f - y,$$

these are of the form

$$\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}}, \quad \frac{x\,dx}{\sqrt{X}} - \frac{y\,dy}{\sqrt{Y}},$$

each of them equal to a linear function of du and dv: so that the quotient-functions are in fact the 15 hyperelliptic functions belonging to the integrals $\int \dfrac{dx}{\sqrt{X}}$, $\int \dfrac{x\,dx}{\sqrt{X}}$; and there is thus an addition-theorem for them, in accordance with the theory of these integrals.

26. The first 16 equations of the product-theorem, putting therein first $u = 0$, $v = 0$, and then $u' = 0$, $v' = 0$, and using the results to eliminate the functions on the right-hand side, give expressions for

$$\overset{u+u'}{A} \,.\, \overset{u-u'}{B}, \text{ \&c.},$$

that is, they give $A(u+u', v+v') \,.\, B(u-u', v-v')$, &c., in terms of the functions of (u, v) and (u', v'): and we have thus an addition-with-subtraction theorem for the double theta-functions. And we have thence also consequences analogous to those which present themselves in the theory of the single functions.

Remark as to notation.

27. I remark, as regards the single theta-functions, that the characteristics

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix},$$

might for shortness be represented by a series of current numbers

$$0, \quad 1, \quad 2, \quad 3:$$

and the functions be accordingly called $\vartheta_0 u$, $\vartheta_1 u$, $\vartheta_2 u$, $\vartheta_3 u$; but that, instead of this, I prefer to use throughout the before-mentioned functional symbols

$$A, \quad B, \quad C, \quad D.$$

As regards the double functions, I do, however, denote the characteristics

$$\begin{array}{cccc|cccc|cccc|cccc|} 00 & 10 & 01 & 11 & 00 & 10 & 01 & 11 & 00 & 10 & 01 & 11 & 00 & 10 & 01 & 11 \\ 00' & 00' & 00' & 00 & 10' & 10' & 10' & 10 & 01' & 01' & 01' & 01 & 11' & 11' & 11' & 11 \end{array}$$

by a series of current numbers

$$0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 11, \quad 12, \quad 13, \quad 14, \quad 15,$$

and write the functions as $\vartheta_0, \vartheta_1, \ldots, \vartheta_{15}$ accordingly; and I use also, as and when it is convenient, the foregoing single and double letter notation A, $AB, \ldots$, which correspond to them in the order

$$BD, \; CE, \; CD, \; BE, \; AC, \; C, \; AB, \; B, \; BC, \; DE, \; F, \; A, \; AD, \; D, \; E, \; AE.$$

Moreover, I write down for the most part a single argument only: thus, $A(u+u')$ stands for $A(u+u', v+v')$, $A(0)$ for $A(0, 0)$: and so in other cases.

SECOND PART.—THE SINGLE THETA-FUNCTIONS.

Notation, &c.

28. Writing $\exp. a = q$, and converting the exponentials into circular functions, we have, directly from the definition,

$$\vartheta\,{}^{0}_{0}(u) = \vartheta u = Au = 1 + 2q\cos\pi u + 2q^4\cos 2\pi u + 2q^9\cos 3\pi u + \ldots,$$

$$\vartheta\,{}^{1}_{0}(u) = \vartheta_1 u = Bu = \quad 2q^{\frac{1}{4}}\cos\tfrac{1}{2}\pi u + 2q^{\frac{9}{4}}\cos\tfrac{3}{2}\pi u + 2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u + \ldots,$$

$$\vartheta\,{}^{0}_{1}(u) = \vartheta_2 u = Cu = 1 - 2q\cos\pi u + 2q^4\cos 2\pi u - 2q^9\cos 3\pi u + \ldots \; (= \Theta(Ku), \text{ Jacobi}),$$

$$\vartheta\,{}^{1}_{1}(u) = \vartheta_3 u = Du = \quad -2q^{\frac{1}{4}}\sin\tfrac{1}{2}\pi u + 2q^{\frac{9}{4}}\sin\tfrac{3}{2}\pi u - 2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u + \ldots \; (= -\mathrm{H}(Ku), \text{ Jacobi}),$$

where a is of the form $a = -\alpha + \beta i$, α being non-evanescent and positive: hence $q = \exp.(-\alpha+\beta i) = e^{-\alpha}(\cos\beta + i\sin\beta)$, where $e^{-\alpha}$, the modulus of q, is positive and less than 1; $\cos\beta$ may be either positive or negative, and $q^{\frac{1}{4}}$ is written to denote $\exp.\frac{1}{4}(-\alpha+\beta i)$, viz. this is $= e^{-\frac{1}{4}\alpha}\{\cos\frac{1}{4}\beta + i\sin\frac{1}{4}\beta\}$. But usually $\beta = 0$, viz. q is a real positive quantity less than 1, and $q^{\frac{1}{4}}$ denotes the real fourth root of q.

I have given above the three notations but, as already mentioned, I propose to employ for the four functions the notation Au, Bu, Cu, Du: it will be observed that Du is an odd function, but that Au, Bu, Cu are even functions, of u.

The constants of the theory.

29. We have

$$\begin{aligned}
A0 &= 1 + 2q + 2q^4 + 2q^9 + \dots,\\
B0 &= \quad 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \dots,\\
C0 &= 1 - 2q + 2q^4 - 2q^9 + \dots,\\
D0 &= 0,\\
D'0 &= -\pi\{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \dots\}.
\end{aligned}$$

If, as definitions of k, k', K, we assume

$$k = \frac{B^2 0}{A^2 0},\quad k' = \frac{C^2 0}{A^2 0},\quad K = -\frac{A0}{B0}\cdot\frac{D'0}{C0},$$

then we have

$$\begin{aligned}
k &= 4\sqrt{q}\left\{\frac{1+q^2+q^6+\dots}{1+2q+2q^4+\dots}\right\}^2, = 4\sqrt{q}\,(1-4q+14q^2+\dots),\\
k' &= \quad\left\{\frac{1-2q+2q^4-\dots}{1+2q+2q^4+\dots}\right\}^2, = 1-8q+32q^2-96q^3+\dots,\\
K &= \frac{\pi(1+2q+2q^4+\dots)(1-3q^2+5q^6-\dots)}{2(1-2q+2q^4-\dots)(1+q^2+q^6+\dots)}, = \tfrac{1}{2}\pi(1+4q+4q^2+0q^3+\dots),
\end{aligned}$$

where I have added the first few terms of the expansions of these quantities. We have identically

$$k^2 + k'^2 = 1.$$

It will be convenient to write also, as the definition of E,

$$K(K-E) = \frac{C''0}{C0}:$$

we have then

$$E = K - \frac{1}{K}\frac{C''0}{C0},\quad = \frac{1}{A0\,.\,B0\,.\,C0\,.\,D'0}\{-A^2 0\,(D'0)^2 + B^2 0\,.\,C0\,.\,C''0\};$$

moreover,

$$1 - \frac{E}{K} = \frac{1}{K^2}\cdot\frac{C''0}{C0},\quad = \frac{2\pi^2}{K^2}\cdot\frac{q-4q^4+9q^9-\dots}{1-2q+2q^4+\dots},$$

giving

$$\frac{E}{K} = \quad 1 - 8q + 48q^2 - 224q^3 + \dots,$$

and thence

$$E = \tfrac{1}{2}\pi\{1 - 4q + 20q^2 - \ 64q^3 + \dots\}.$$

30. Other formulæ are

$$\begin{aligned}
k &= 4\sqrt{q}\left\{\frac{1+q^2\,.\,1+q^4\dots}{1+q\,.\,1+q^3\dots}\right\}^4,\\
k' &= \quad\left\{\frac{1-q\,.\,1-q^3\dots}{1+q\,.\,1+q^3\dots}\right\}^4,\\
K &= \tfrac{1}{2}\pi\left\{\frac{1+q\,.\,1+q^3\dots\,1-q^2\,.\,1-q^4\dots}{1-q\,.\,1-q^3\dots\,1+q^2\,.\,1+q^4\dots}\right\}^2.
\end{aligned}$$

31. Jacobi's definition of q is from a different point of view altogether, viz. we have $q = \exp. -\frac{\pi K'}{K}$, where

$$K = \int_0^1 \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}};$$

and K' is the like function of k'; the equation gives $\log q = -\frac{\pi K'}{K}$, viz. we have

$$K' = -\frac{K}{\pi} \log q,$$

and, regarding herein K as a given function of q, this equation gives K' as a function of q.

The product-theorem.

32. The product-theorem is

$$\vartheta\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}(u+u').\vartheta\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix}(u-u')$$

$$= \Theta\begin{pmatrix}\frac{1}{2}(\alpha+\alpha')\\ \gamma+\gamma'\end{pmatrix}2u.\Theta\begin{pmatrix}\frac{1}{2}(\alpha-\alpha')\\ \gamma-\gamma'\end{pmatrix}2u' + \Theta\begin{pmatrix}\frac{1}{2}(\alpha+\alpha')+1\\ \gamma+\gamma'\end{pmatrix}2u.\Theta\begin{pmatrix}\frac{1}{2}(\alpha-\alpha')+1\\ \gamma-\gamma'\end{pmatrix}2u'.$$

Here giving to $\begin{matrix}\alpha\\ \gamma\end{matrix}, \begin{matrix}\alpha'\\ \gamma'\end{matrix}$ their different values, and introducing unaccented and accented capitals to denote the functions of $2u$ and $2u'$ respectively, the 16 equations are

$A.A$	$\vartheta\begin{matrix}0\\0\end{matrix}u+u'$	$\vartheta\begin{matrix}0\\0\end{matrix}u-u'$	$=$	$XX' + YY'$,	(square-set)
$B.B$	$\vartheta\begin{matrix}1\\0\end{matrix}$ „	$\vartheta\begin{matrix}1\\0\end{matrix}$ „	$=$	$YX' + XY'$,	
$C.C$	$\vartheta\begin{matrix}0\\1\end{matrix}$ „	$\vartheta\begin{matrix}0\\1\end{matrix}$ „	$=$	$XX' - YY'$,	
$D.D$	$\vartheta\begin{matrix}1\\1\end{matrix}$ „	$\vartheta\begin{matrix}1\\1\end{matrix}$ „	$=-$	$YX' + XY'$;	
$C.A$	$\vartheta\begin{matrix}0\\1\end{matrix}u+u'$	$\vartheta\begin{matrix}0\\0\end{matrix}u-u'$	$=$	$X_{,}X_{,}' + Y_{,}Y_{,}'$,	(first product-set)
$A.C$	$\vartheta\begin{matrix}0\\0\end{matrix}$ „	$\vartheta\begin{matrix}0\\1\end{matrix}$ „	$=$	$X_{,}X_{,}' - Y_{,}Y_{,}'$,	
$D.B$	$\vartheta\begin{matrix}1\\1\end{matrix}$ „	$\vartheta\begin{matrix}1\\0\end{matrix}$ „	$=$	$Y_{,}X_{,}' + X_{,}Y_{,}'$,	
$B.D$	$\vartheta\begin{matrix}1\\0\end{matrix}$ „	$\vartheta\begin{matrix}1\\1\end{matrix}$ „	$=$	$Y_{,}X_{,}' - X_{,}Y_{,}'$;	

$$B.A \quad \vartheta\begin{smallmatrix}1\\0\end{smallmatrix} u+u' \vartheta\begin{smallmatrix}0\\0\end{smallmatrix} u-u' = PP' + QQ', \text{ (second product-set)}$$

$$A.B \quad \vartheta\begin{smallmatrix}0\\0\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}1\\0\end{smallmatrix} \;\text{"} = PQ' + QP',$$

$$D.C \quad \vartheta\begin{smallmatrix}1\\1\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}0\\1\end{smallmatrix} \;\text{"} = iPP' - iQQ',$$

$$C.D \quad \vartheta\begin{smallmatrix}0\\1\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}1\\1\end{smallmatrix} \;\text{"} = iPQ' - iQP';$$

$$D.A \quad \vartheta\begin{smallmatrix}1\\1\end{smallmatrix} u+u' \vartheta\begin{smallmatrix}0\\0\end{smallmatrix} u-u' = P_{,}P'_{,} + Q_{,}Q'_{,}, \text{ (third product-set)}$$

$$A.D \quad \vartheta\begin{smallmatrix}0\\0\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}1\\1\end{smallmatrix} \;\text{"} = iP_{,}Q'_{,} - iQ_{,}P'_{,},$$

$$B.C \quad \vartheta\begin{smallmatrix}1\\0\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}0\\1\end{smallmatrix} \;\text{"} = -iP_{,}P'_{,} + iQ_{,}Q'_{,},$$

$$C.B \quad \vartheta\begin{smallmatrix}0\\1\end{smallmatrix} \;\text{"}\; \vartheta\begin{smallmatrix}1\\0\end{smallmatrix} \;\text{"} = P_{,}Q'_{,} + Q_{,}P'_{,}.$$

33. Here, and subsequently, we have

$$\begin{array}{l||l}
\Theta\begin{smallmatrix}0\\0\end{smallmatrix}, \Theta\begin{smallmatrix}1\\0\end{smallmatrix}, \Theta\begin{smallmatrix}0\\1\end{smallmatrix}, \Theta\begin{smallmatrix}1\\1\end{smallmatrix}(2u) = X,\ Y,\ X_{,},\ Y_{,} & \Theta\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}, \Theta\begin{smallmatrix}\frac{3}{2}\\0\end{smallmatrix}, \Theta\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix}, \Theta\begin{smallmatrix}\frac{3}{2}\\1\end{smallmatrix}(2u) = P,\ Q,\ P_{,},\ Q_{,}, \\
\text{"} \quad \text{"} \quad \text{"} \quad \text{"}\,(2u') = X',\ Y',\ X'_{,},\ Y'_{,} & \text{"} \quad \text{"} \quad \text{"} \quad \text{"}\,(2u') = P',\ Q',\ P'_{,},\ Q'_{,}, \\
\text{"} \quad \text{"} \quad \text{"} \quad \text{"}\,(0) = \alpha,\ \beta,\ \alpha_{,},\ \beta_{,} & \text{"} \quad \text{"} \quad \text{"} \quad \text{"}\,(0) = \mathrm{p},\ \mathrm{q},\ \mathrm{p}_{,},\ \mathrm{q}_{,};
\end{array}$$

viz. we use also α, β, $\alpha_{,}$, $\beta_{,}$ and p, q, $\mathrm{p}_{,}$, $\mathrm{q}_{,}$ to denote the zero-functions; $\beta_{,}$ is $=0$, but we use $\beta'_{,}$ to denote the zero-value of $\frac{d}{du} Y_{,}$.

34. In order to obtain the foregoing relations, it is necessary to observe that

$$\Theta\begin{smallmatrix}\alpha+2\\\gamma\end{smallmatrix} = \Theta\begin{smallmatrix}\alpha\\\gamma\end{smallmatrix};$$

by which the upper character is always reduced to 0, 1, $\frac{1}{2}$ or $\frac{3}{2}$; and that, for reducing the lower character, we have

$$\Theta\begin{smallmatrix}0\\\gamma+2\end{smallmatrix} = \Theta\begin{smallmatrix}0\\\gamma\end{smallmatrix};\ \Theta\begin{smallmatrix}1\\\gamma+2\end{smallmatrix} = -\Theta\begin{smallmatrix}1\\\gamma\end{smallmatrix};$$

$$\Theta\begin{smallmatrix}\frac{1}{2}\\\gamma+2\end{smallmatrix} = i\Theta\begin{smallmatrix}\frac{1}{2}\\\gamma\end{smallmatrix},\ \Theta\begin{smallmatrix}\frac{1}{2}\\\gamma-2\end{smallmatrix} = -i\Theta\begin{smallmatrix}\frac{1}{2}\\\gamma\end{smallmatrix};\ \Theta\begin{smallmatrix}\frac{3}{2}\\\gamma+2\end{smallmatrix} = -i\Theta\begin{smallmatrix}\frac{3}{2}\\\gamma\end{smallmatrix},\ \Theta\begin{smallmatrix}\frac{3}{2}\\\gamma-2\end{smallmatrix} = i\Theta\begin{smallmatrix}\frac{3}{2}\\\gamma\end{smallmatrix};$$

by means of which the lower character is always reduced to 0 or 1: in all these formulæ the argument is arbitrary, and it is thus $=2u$, or $2u'$ as the case requires. The formulæ are obtained without difficulty directly from the definition of the functions Θ.

35. As an instance, taking $\begin{matrix}\alpha & \alpha' \\ \gamma, & \gamma'\end{matrix} = \begin{matrix}1 & 0 \\ 1, & 1\end{matrix}$, the product-equation is

$$\begin{aligned}\vartheta\,{}^{1}_{1}(u+u')\,.\,\vartheta\,{}^{0}_{1}(u-u') &= \Theta\,{}^{\frac{1}{2}}_{2}(2u)\,.\,\Theta\,{}^{\frac{1}{2}}_{0}(2u') + \Theta\,{}^{\frac{3}{2}}_{2}(2u)\,.\,\Theta\,{}^{\frac{3}{2}}_{0}(2u'),\\ &= i\Theta\,{}^{\frac{1}{2}}_{0}(2u)\,.\,\Theta\,{}^{\frac{1}{2}}_{0}(2u') - i\Theta\,{}^{\frac{3}{2}}_{0}(2u)\,.\,\Theta\,{}^{\frac{3}{2}}_{0}(2u'),\\ &= iP\,.\,P' \qquad\qquad - iQ\,.\,Q',\end{aligned}$$

which agrees with the before-given value.

36. The following values are not actually required: but I give them to fix the ideas and to show the meaning of the quantities with which we work.

	$u=0$
$X = \Theta\,{}^{0}_{0}(2u) = 1 + 2q^2 \cos 2\pi u + 2q^8 \cos 4\pi u + \ldots,$	$\alpha = 1 + 2q^2 + 2q^8 + \ldots,$
$Y = \Theta\,{}^{1}_{0}(2u) = 2q^{\frac{1}{2}} \cos \pi u + 2q^{\frac{9}{2}} \cos 3\pi u + \ldots,$	$\beta = 2q^{\frac{1}{2}} + 2q^{\frac{9}{2}} + \ldots,$
$X_{,} = \Theta\,{}^{0}_{1}(2u) = 1 - 2q^2 \cos 2\pi u + 2q^8 \cos 4\pi u - \ldots,$	$\alpha_{,} = 1 - 2q^2 + 2q^8 - \ldots,$
$Y_{,} = \Theta\,{}^{1}_{1}(2u) = -2q^{\frac{1}{2}} \sin \pi u + 2q^{\frac{9}{2}} \sin 3\pi u - \ldots,$	$\beta_{,}' = 2\pi\,(-q^{\frac{1}{2}} + 3q^{\frac{9}{2}} - \ldots)$ $= \dfrac{d}{du} Y_{,}$ for $u=0$.

$$\begin{aligned}P &= \Theta\,{}^{\frac{1}{2}}_{0}(2u) = q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u + i\sin\tfrac{1}{2}\pi u) + q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u - i\sin\tfrac{3}{2}\pi u)\\ &\qquad + q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u + i\sin\tfrac{5}{2}\pi u) + \ldots,\\ Q &= \Theta\,{}^{\frac{3}{2}}_{0}(2u) = q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u - i\sin\tfrac{1}{2}\pi u) + q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u + i\sin\tfrac{3}{2}\pi u)\\ &\qquad + q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u - i\sin\tfrac{5}{2}\pi u) + \ldots,\\ P_{,} &= \Theta\,{}^{\frac{1}{2}}_{1}(2u) = \frac{1+i}{\sqrt{2}}\Big\{q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u + i\sin\tfrac{1}{2}\pi u) - q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u - i\sin\tfrac{3}{2}\pi u)\\ &\qquad - q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u + i\sin\tfrac{5}{2}\pi u) + \ldots\Big\},\\ Q_{,} &= \Theta\,{}^{\frac{3}{2}}_{1}(2u) = \frac{1-i}{\sqrt{2}}\Big\{q^{\frac{1}{8}}(\cos\tfrac{1}{2}\pi u - i\sin\tfrac{1}{2}\pi u) - q^{\frac{9}{8}}(\cos\tfrac{3}{2}\pi u + i\sin\tfrac{3}{2}\pi u)\\ &\qquad - q^{\frac{25}{8}}(\cos\tfrac{5}{2}\pi u - i\sin\tfrac{5}{2}\pi u) + \ldots\Big\};\end{aligned}$$

and therefore also

$$\begin{aligned}&\mathrm{p} = \mathrm{q} = q^{\frac{1}{8}} + q^{\frac{9}{8}} + q^{\frac{25}{8}} + \ldots\ldots,\\ &\mathrm{p}_{,} = \frac{1+i}{\sqrt{2}}\Big\{q^{\frac{1}{8}} - q^{\frac{9}{8}} - q^{\frac{25}{8}} + q^{\frac{49}{8}} + q^{\frac{81}{8}} - - \ldots\Big\},\ \mathrm{q}_{,} = \frac{1-i}{\sqrt{2}}\Big\{\text{Do.}\Big\};\ \mathrm{p}_{,} = i\mathrm{q}_{,}\end{aligned}$$

The square set, $u'=0$; and x-formulæ.

37. We use the square-set, in the first instance by writing therein $u'=0$; the equations become

$$\begin{aligned} A^2u &= \alpha X + \beta Y, &&= \omega^2\mathfrak{A}\,(a-x),\\ B^2u &= \beta X + \alpha Y, &&= \omega^2\mathfrak{B}\,(b-x),\\ C^2u &= \alpha X - \beta Y, &&= \omega^2\mathfrak{C}\,(c-x),\\ D^2u &= \beta X - \alpha Y, &&= \omega^2\mathfrak{D}\,(d-x), \end{aligned}$$

viz. the equations without their last members show that there exist functions ω^2 and $x\omega^2$, linear functions of X and Y, such that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{A}a$, $\mathfrak{B}b$, $\mathfrak{C}c$, $\mathfrak{D}d$, being constants, the squared functions may be assumed equal to $\mathfrak{A}a\,.\,\omega^2-\mathfrak{A}\,.\,\omega^2x$, &c., that is, $\omega^2\mathfrak{A}\,(a-x)$, &c., respectively: the squared functions are then *proportional* to the values $\mathfrak{A}\,(a-x)$, &c.

To show the meaning of the factor ω^2, observe that, from any two of the equations, for instance from the first and second, we have an equation without ω,

$$A^2u \div B^2u = \mathfrak{A}\,(a-x) \div \mathfrak{B}\,(b-x);$$

and using this to determine x, and then substituting in $\omega^2 = A^2u \div \mathfrak{A}\,(a-x)$, we find

$$\omega^2 = \frac{\mathfrak{B}A^2u - \mathfrak{A}B^2u}{(a-b)\,\mathfrak{A}\mathfrak{B}},$$

where the numerator is a function not in anywise more important than any other linear function of A^2u and B^2u.

38. The function Du vanishes for $u=0$, and we may assume that the corresponding value of x is $=d$. Writing in the other equations $u=0$, they become

$$\begin{aligned} A^20 &= (\alpha^2+\beta^2) &&= \omega_0^2\mathfrak{A}\,(a-d),\\ B^20 &= \quad 2\alpha\beta &&= \omega_0^2\mathfrak{B}\,(b-d),\\ C^20 &= \alpha^2-\beta^2 &&= \omega_0^2\mathfrak{C}\,(c-d), \end{aligned}$$

where ω_0^2 is what ω^2 becomes on writing therein $x=d$. It is convenient to omit altogether these factors ω^2 and ω_0^2; it being understood that, without them, the equations denote not absolute equalities but only equalities of ratios: thus, without the ω_0^2, the last-mentioned equations would denote

$$A^20 : B^20 : C^20 = \alpha^2+\beta^2 : 2\alpha\beta : \alpha^2-\beta^2, = \mathfrak{A}\,(a-d) : \mathfrak{B}\,(b-d) : \mathfrak{C}\,(c-d).$$

The quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ only present themselves in the products $\mathfrak{A}\omega^2$, &c., and their absolute magnitudes are therefore essentially indeterminate: but regarding ω^2 as containing a constant factor of properly determined value, the absolute values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ may be regarded as determinate, and this is accordingly done in the formulæ $\mathfrak{A}^2 = -agh$, &c., which follow.

Relations between the constants.

39. The formulæ contain the differences of the quantities a, b, c, d; denoting these differences

$$b-c,\quad c-a,\quad a-b,\quad a-d,\quad b-d,\quad c-d,$$

in the usual manner by

$$\mathrm{a},\quad \mathrm{b},\quad \mathrm{c},\quad \mathrm{f},\quad \mathrm{g},\quad \mathrm{h},$$

so that

$$\begin{aligned} .\ -\mathrm{h}\ +\mathrm{g}\ -\mathrm{a} &= 0,\\ \mathrm{h}\ .\ -\mathrm{f}\ -\mathrm{b} &= 0,\\ -\mathrm{g}\ +\mathrm{f}\ .\ -\mathrm{c} &= 0,\\ \mathrm{a}\ +\mathrm{b}\ +\mathrm{c}\ . &= 0, \end{aligned}$$

and also

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}\quad =0,$$

and then assuming the absolute value of one of the quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, we have the system of relations

$$\begin{array}{llll} \mathfrak{A}^2 = -\mathrm{agh}, & \mathfrak{B}\mathfrak{C}\mathrm{a} = \mathfrak{A}\mathfrak{D}\mathrm{f}, & \mathfrak{A}\mathrm{bcf} = -\mathfrak{B}\mathfrak{C}\mathfrak{D}, & \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} = \mathrm{abcfgh},\\ \mathfrak{B}^2 = \mathrm{bhf}, & \mathfrak{C}\mathfrak{A}\mathrm{b} = -\mathfrak{B}\mathfrak{D}\mathrm{g}, & \mathfrak{B}\mathrm{cag} = \mathfrak{C}\mathfrak{A}\mathfrak{D}, & \\ \mathfrak{C}^2 = \mathrm{cfg}, & \mathfrak{A}\mathfrak{B}\mathrm{c} = -\mathfrak{C}\mathfrak{D}\mathrm{h}, & \mathfrak{C}\mathrm{abh} = \mathfrak{A}\mathfrak{B}\mathfrak{D}, & \\ \mathfrak{D}^2 = -\mathrm{abc}, & & \mathfrak{D}\mathrm{fgh} = -\mathfrak{A}\mathfrak{B}\mathfrak{C}, & \end{array}$$

$$\begin{aligned} \mathrm{c}^2\mathfrak{B}^2 + \mathrm{b}^2\mathfrak{C}^2 - \mathrm{f}^2\mathfrak{D}^2 &= \mathrm{bcf}\ (\mathrm{af}+\mathrm{bg}+\mathrm{ch}), &&= 0,\\ -\mathrm{c}^2\mathfrak{A}^2\ .\ + \mathrm{a}^2\mathfrak{C}^2 - \mathrm{g}^2\mathfrak{D}^2 &= \mathrm{cag}\ (\quad ,, \quad), &&= 0,\\ -\mathrm{b}^2\mathfrak{A}^2 + \mathrm{a}^2\mathfrak{B}^2\ .\ - \mathrm{h}^2\mathfrak{D}^2 &= \mathrm{abh}\ (\quad ,, \quad), &&= 0,\\ -\mathrm{f}^2\mathfrak{A}^2 + \mathrm{g}^2\mathfrak{B}^2 + \mathrm{h}^2\mathfrak{C}^2\ . &= \mathrm{fgh}\ (\quad ,, \quad), &&= 0. \end{aligned}$$

It is to be remarked that, taking c, a, b, d in the order of decreasing magnitude, we have $-\mathrm{a}$, b, c, f, g, h all positive; hence $\mathfrak{A}^2$, $\mathfrak{B}^2$, $\mathfrak{C}^2$, $\mathfrak{D}^2$ all real; and taking as we may do, $\mathfrak{D}$ negative, then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ may be taken positive; that is, we have $-\mathrm{a}$, b, c, f, g, h, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $-\mathfrak{D}$ all of them positive.

40. We have

$$\begin{aligned} A^2 0 &= \alpha^2+\beta^2 = \mathfrak{A}\mathrm{f},\\ B^2 0 &= 2\alpha\beta\ = \mathfrak{B}\mathrm{g},\\ C^2 0 &= \alpha^2-\beta^2 = \mathfrak{C}\mathrm{h}. \end{aligned}$$

The foregoing equations

$$k=\frac{B^2 0}{A^2 0},\qquad k'=\frac{C^2 0}{A^2 0},$$

give

$$k=\frac{\mathfrak{B}\mathrm{g}}{\mathfrak{A}\mathrm{f}},\qquad k'=\frac{\mathfrak{C}\mathrm{h}}{\mathfrak{A}\mathrm{f}},$$

and we thence have

$$k^2 = \frac{\mathrm{bg}}{-\mathrm{af}}, \quad k'^2 = \frac{\mathrm{ch}}{-\mathrm{af}},$$

satisfying

$$k^2 + k'^2 = 1.$$

41. Observe further that, substituting for a, b, c, f, g, h their values, we have

$$\begin{aligned}
\mathfrak{A}^2 &= c - b.b - d.c - d, &&= c - d.d - b.b - c,\\
\mathfrak{B}^2 &= c - a.c - d.a - d, &&= d - a.a - c.c - d,\\
\mathfrak{C}^2 &= a - b.a - d.b - d, &&= -a - b.b - d.d - a,\\
\mathfrak{D}^2 &= c - b.c - a.a - b, &&= -b - c.c - a.a - b,
\end{aligned}$$

where in the first set of values all the differences are positive, but in the second set of values, we take the triads of *abcd*, in the cyclical order *bcd*, *cda*, *dab*, *abc*. There is in this last form an apparent want of symmetry as to the signs (viz. the order which might have been expected is $+-+-$), but taking the order of the letters to be $\mathfrak{C}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$ and *c*, *a*, *b*, *d*, then the cyclical arrangement is

$$\begin{aligned}
\mathfrak{C}^2 &= -b - d.d - a.a - b,\\
\mathfrak{A}^2 &= -d - c.c - b.b - d,\\
\mathfrak{B}^2 &= -c - a.a - d.d - c,\\
\mathfrak{D}^2 &= -a - b.b - c.c - a,
\end{aligned}$$

where the four outside signs are all $-$. Observe that the triads of *abcd*, and *abdc*, are

$$\begin{matrix} bcd, & cda, & dab, & abc, \\ bdc, & dca, & cab, & abd, \end{matrix}$$

and

where in the first and second columns the terms of the same column correspond to each other with a reversal of sign, whereas in the third and fourth columns the lower term of either column corresponds to the upper term of the other column, but without a reversal of sign.

The product-sets, $u \pm u'$: and u' indefinitely small, differential formulæ.

42. Coming now to the product-sets, these may be written

$$\begin{aligned}
&\tfrac{1}{2}\{C.A + A.C\} = X_{,}X'_{,}, &&\tfrac{1}{2}\{C.A - A.C\} = Y_{,}Y'_{,},\\
&\text{,,}\ \{D.B + B.D\} = Y_{,}X'_{,}, &&\text{,,}\ \{D.B - B.D\} = X_{,}Y'_{,},\\
&\tfrac{1}{2}\{B.A + A.B\} = (P + Q)(P' + Q'), &&\tfrac{1}{2}\{B.A - A.B\} = (P - Q)(P' - Q'),\\
&\text{,,}\ \{D.C + C.D\} = i(P - Q)(P' + Q'), &&\text{,,}\ \{D.C - C.D\} = i(P + Q)(P' - Q'),\\
&\tfrac{1}{2}\{D.A + A.D\} = (P_{,} - iQ_{,})(P'_{,} + iQ'_{,}), &&\tfrac{1}{2}\{D.A - A.D\} = (P_{,} + iQ_{,})(P'_{,} - iQ'_{,}),\\
&\text{,,}\ \{B.C + C.B\} = -i(P_{,} - iQ_{,})(P'_{,} + iQ'_{,}), &&\text{,,}\ \{B.C - C.B\} = -i(P_{,} - iQ_{,})(P'_{,} - iQ'_{,}).
\end{aligned}$$

43. We can from each set form two fractions (each of them a function of $u+u'$ and $u-u'$), which are equal to one and the same function of u' only: for instance, from the first set we have two fractions, each $\frac{Y'}{X'}$: putting in such equation $u=0$, we obtain a new expression for the function of u' involving only the theta-functions Au', &c., which new expression we may then substitute in the equations first obtained: we thus arrive at the six equations

$$\begin{aligned}
&\overset{u+u'}{\frac{C.A-A.C}{D.B+B.D}}\overset{u-u'}{=}\overset{u+u'}{\frac{D.B-B.D}{C.A+A.C}}\overset{u-u'}{=}\frac{Du'.Bu'}{Cu'.Au'},\\
&-\overset{u+u'}{\frac{B.A-A.B}{D.C+C.D}}\overset{u-u'}{=}\overset{u+u'}{\frac{D.C-C.D}{B.A+A.B}}\overset{u-u'}{=}\frac{Du'.Cu'}{Bu'.Au'},\\
&-\overset{u+u'}{\frac{B.C-C.B}{D.A+A.D}}\overset{u-u'}{=}\overset{u+u'}{\frac{D.A-A.D}{B.C+C.B}}\overset{u-u'}{=}\frac{Du'.Au'}{Bu'.Cu'},
\end{aligned}$$

where observe that the expressions all vanish for $u'=0$.

44. Taking herein u' indefinitely small, we obtain

$$\begin{aligned}
&\frac{Au.C'u-Cu.A'u}{Bu.Du}=\frac{Bu.D'u-Du.B'u}{Cu.Au}=\frac{D'0.B0}{C0.A0}=-K\frac{B^20}{A^20},\\
&-\frac{Au.B'u-Bu.A'u}{Cu.Du}=\frac{Cu.D'u-Du.C'u}{Au.Bu}=\frac{D'0.C0}{A0.B0}=-K\frac{C^20}{A^20},\\
&-\frac{Cu.B'u-Bu.C'u}{Au.Du}=\frac{Au.D'u-Du.A'u}{Bu.Cu}=\frac{D'0.A0}{B0.C0}=-K,
\end{aligned}$$

where the last column is added in order to introduce K in place of $D'0$.

45. These formulæ in effect give the derivatives of the quotient-functions in terms of quotient-functions: for instance, one of the equations is

$$\frac{d}{du}\frac{Du}{Au}=-K\frac{Bu}{Au}.\frac{Cu}{Au};$$

substituting herein for the quotient-fractions their values in terms of x, this becomes

$$\frac{d}{du}\sqrt{\frac{d-x}{a-x}}=-K\sqrt{\frac{\mathfrak{B}\mathfrak{C}}{\mathfrak{A}\mathfrak{D}}}\,\frac{\sqrt{b-x.c-x}}{a-x},\quad=-K\sqrt{\frac{\mathrm{f}}{\mathrm{a}}}\,\frac{\sqrt{b-x.c-x}}{a-x},$$

or the left-hand being

$$=\frac{-\frac{1}{2}\mathrm{f}}{(a-x)^{\frac{3}{2}}\sqrt{d-x}}\frac{dx}{du},$$

this is

$$Kdu=\frac{\frac{1}{2}\sqrt{\mathrm{af}}.dx}{\sqrt{a-x.b-x.c-x.d-x}},$$

where on the right-hand side it would be better to write $\sqrt{-\mathrm{af}}$ in the numerator and $x-d$ in place of $d-x$ in the denominator.

Comparison with Jacobi's formulæ.

46. The comparison of the formulæ with Jacobi's formulæ gives

$$\operatorname{sn} Ku = -\frac{1}{\sqrt{k}} Du \div Cu, \quad = \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{d-x}{c-x}} \left(\text{or better } \sqrt{\frac{-\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{x-d}{c-x}}\right),$$

$$\operatorname{cn} Ku = \sqrt{\frac{k'}{k}} Bu \div Cu, \quad = \sqrt{\frac{\mathrm{h}}{\mathrm{g}}}\sqrt{\frac{b-x}{c-x}},$$

$$\operatorname{dn} Ku = \sqrt{k'}\, Au \div Cu, \quad = \sqrt{\frac{\mathrm{h}}{\mathrm{f}}}\sqrt{\frac{a-x}{c-x}},$$

where it will be recollected that

$$k^2 = \frac{\mathrm{bg}}{-\mathrm{af}}, \quad k'^2 = \frac{\mathrm{ch}}{-\mathrm{af}}.$$

It may be remarked that we seek to determine everything in terms of a, b, c, d. The formula just written down, $k^2 = \mathrm{bg} \div -\mathrm{af}$, gives k in terms of these quantities; and k, K being each given in terms of q, we have virtually K as a function of k, that is, of a, b, c, d: but it would not be easy from the expressions of k, K, each in terms of q, to deduce the actual expression

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

of K as a function of k.

The square-set, $u \pm u'$.

47. Reverting to the square-set

$$\begin{aligned} A(u+u')\,A(u-u') &= XX' + YY', \\ B(u+u')\,B(u-u') &= YX' + XY', \\ C(u+u')\,C(u-u') &= XX' - YY', \\ D(u+u')\,D(u-u') &= -YX' + XY', \end{aligned}$$

if we first write herein $u' = 0$, and then $u = 0$, using the results to determine the values of X, Y, X', Y' we find

$$\begin{aligned} \alpha C^2u - \beta D^2u &= (\alpha^2 - \beta^2)\,X, \\ \beta C^2u - \alpha D^2u &= \quad ,, \quad Y, \end{aligned} \quad \Bigg| \quad \begin{aligned} \alpha C^2u' - \beta D^2u' &= (\alpha^2 - \beta^2)\,X', \\ \beta C^2u' - \alpha D^2u' &= \quad ,, \quad Y', \end{aligned}$$

and thence

$$\begin{aligned} (\alpha^2 - \beta^2)^2\,XX' &= \alpha^2 . C^2uC^2u' + \beta^2 . D^2uD^2u' - \alpha\beta\,(C^2uD^2u' + D^2uC^2u'), \\ ,, \quad YY' &= \beta^2 \quad ,, \quad + \alpha^2 \quad ,, \quad - \alpha\beta \quad ,, \quad , \end{aligned}$$

whence

$$\begin{aligned} (\alpha^2 - \beta^2)^2\,(XX' + YY') &= (\alpha^2 + \beta^2)\,(C^2uC^2u' + D^2uD^2u') - 2\alpha\beta\,(C^2uD^2u' + D^2uC^2u'), \\ (\alpha^2 - \beta^2)\,(XX' - YY') &= \qquad (C^2uC^2u' - D^2uD^2u'), \end{aligned}$$

where observe that, in taking the difference, the right-hand side becomes divisible by $\alpha^2-\beta^2$, and therefore in the final result we have on the left-hand side the simple factor $\alpha^2-\beta^2$ instead of $(\alpha^2-\beta^2)^2$.

Similarly

$$(\alpha^2-\beta^2)\,YX' = \alpha\beta\,(C^2uC^2u' + D^2uD^2u') - \alpha^2D^2uC^2u' - \beta^2C^2uD^2u',$$

$$\text{"}\quad XY' = \alpha\beta \quad \text{"} \quad -\beta^2 \quad \text{"} \quad -\alpha^2 \quad \text{"} \quad ,$$

and thence

$$(\alpha^2-\beta^2)^2(\ \ YX'+XY') = 2\alpha\beta\,(C^2uC^2u' + D^2uD^2u') - (\alpha^2+\beta^2)\,(C^2uD^2u' + D^2uC^2u'),$$

$$(\alpha^2-\beta^2)\,(-YX'+XY') = \quad D^2uC^2u' - C^2uD^2u'.$$

48. Hence recollecting that

$$A^20 = \alpha^2+\beta^2,$$

$$B^20 = 2\alpha\beta,$$

$$C^20 = \alpha^2-\beta^2,$$

the original equations become

$$C^40\,.\,A\,(u+u')\,A\,(u-u') = A^20\,(C^2uC^2u' + D^2uD^2u') - B^20\,(C^2uD^2u' + D^2uC^2u'),$$

$$C^40\,.\,B\,(u+u')\,B\,(u-u') = B^20\,(C^2uC^2u' + D^2uD^2u') - A^20\,(C^2uD^2u' + D^2uC^2u'),$$

$$C^20\,.\,C\,(u+u')\,C\,(u-u') = \quad C^2uC^2u' - D^2uD^2u',$$

$$C^20\,.\,D\,(u+u')\,D\,(u-u') = \quad D^2uC^2u' - C^2uD^2u'.$$

49. It will be observed that the four products $A\,(u+u')\,A\,(u-u')$, &c., are each of them expressed in terms of C^2u, D^2u, C^2u', D^2u'. Since each of the squared functions A^2u, B^2u, C^2u, D^2u is a linear function of any two of them, and the like as regards A^2u', B^2u', C^2u', D^2u', it is clear that in each equation we can on the right-hand side introduce any two at pleasure of the squared functions of u, and any two at pleasure of the squared functions of u'. But all the forms so obtained are of course identical if, taking x' the same function of u' that x is of u, we introduce on the right-hand side x, x' instead of u, u'; and the values of $A\,(u+u')\,.\,A\,(u-u')$, &c., are found to be equal to multiples of ∇, ∇_1, ∇_2, ∇_3, where

$$\nabla = x-x',\quad \nabla_1 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+d, & ad \\ 1, & b+c, & bc \end{vmatrix},\quad \nabla_2 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & b+d, & bd \\ 1, & c+a, & ca \end{vmatrix},\quad \nabla_3 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & c+d, & cd \\ 1, & a+b, & ab \end{vmatrix}.$$

50. In fact, from the equations

$$A^2u = \mathfrak{A}\,(a-x),\quad A^2u' = \mathfrak{A}\,(a-x'),$$

we have

$$\nabla = \frac{1}{\mathrm{a}\mathfrak{B}\mathfrak{C}}\,(B^2uC^2u' - C^2uB^2u'),\quad = \frac{1}{\mathrm{b}\mathfrak{C}\mathfrak{A}}\,(C^2uA^2u' - A^2uC^2u'),\quad = \frac{1}{\mathrm{c}\mathfrak{A}\mathfrak{B}}\,(A^2uB^2u' - B^2uA^2u'),$$

$$= \frac{1}{\mathrm{f}\mathfrak{A}\mathfrak{D}}\,(A^2uD^2u' - D^2uA^2u'),\quad = \frac{1}{\mathrm{g}\mathfrak{B}\mathfrak{D}}\,(B^2uD^2u' - D^2uB^2u'),\quad = \frac{1}{\mathrm{h}\mathfrak{C}\mathfrak{D}}\,(C^2uD^2u' - D^2uC^2u'),$$

where it will be recollected that

$$\mathfrak{f}\mathfrak{A}\mathfrak{D} = \mathfrak{a}\mathfrak{B}\mathfrak{C}, \quad -\mathfrak{g}\mathfrak{B}\mathfrak{D} = \mathfrak{b}\mathfrak{C}\mathfrak{A}, \quad -\mathfrak{h}\mathfrak{C}\mathfrak{D} = \mathfrak{c}\mathfrak{A}\mathfrak{B}.$$

Moreover

$$(b-c)\,\nabla_1 = -\begin{vmatrix} b-x\,.\,b-x', & c-x\,.\,c-x' \\ b-a\,.\,b-d, & c-a\,.\,c-d \end{vmatrix}, \quad (a-d)\,\nabla_1 = \begin{vmatrix} a-x\,.\,a-x', & d-x\,.\,d-x' \\ a-b\,.\,a-c, & d-b\,.\,d-c \end{vmatrix},$$

$$(c-a)\,\nabla_2 = -\begin{vmatrix} c-x\,.\,c-x', & a-x\,.\,a-x' \\ c-b\,.\,c-d, & a-b\,.\,a-d \end{vmatrix}, \quad (b-d)\,\nabla_2 = \begin{vmatrix} b-x\,.\,b-x', & d-x\,.\,d-x' \\ b-c\,.\,b-a, & d-c\,.\,d-a \end{vmatrix},$$

$$(a-b)\,\nabla_3 = -\begin{vmatrix} a-x\,.\,a-x', & b-x\,.\,b-x' \\ a-c\,.\,a-d, & b-c\,.\,b-d \end{vmatrix}, \quad (c-d)\,\nabla_3 = \begin{vmatrix} c-x\,.\,c-x', & d-x\,.\,d-x' \\ c-a\,.\,c-b, & d-a\,.\,d-b \end{vmatrix},$$

or as these may be written

$$\nabla_1 = -\frac{1}{\mathrm{a}}\{\mathrm{bh}\,.\,b-x\,.\,b-x'\,.+\mathrm{cg}\,.\,c-x\,.\,c-x'\}, \quad = \frac{1}{\mathrm{f}}\{\mathrm{gh}\,.\,a-x\,.\,a-x'\,.+\mathrm{bc}\,.\,d-x\,.\,d-x'\},$$

$$\nabla_2 = -\frac{1}{\mathrm{b}}\{\mathrm{cf}\,.\,c-x\,.\,c-x'\,.+\mathrm{ah}\,.\,a-x\,.\,a-x'\}, \quad = \frac{1}{\mathrm{g}}\{\mathrm{hf}\,.\,b-x\,.\,b-x'\,.+\mathrm{ca}\,.\,d-x\,.\,d-x'\},$$

$$\nabla_3 = -\frac{1}{\mathrm{c}}\{\mathrm{ag}\,.\,a-x\,.\,a-x'\,.+\mathrm{bf}\,.\,b-x\,.\,b-x'\}, \quad = \frac{1}{\mathrm{h}}\{\mathrm{fg}\,.\,c-x\,.\,c-x'\,.+\mathrm{ab}\,.\,d-x\,.\,d-x'\},$$

that is,

$$\nabla_1 = -\frac{1}{\mathrm{a}}\left\{\frac{\mathrm{bh}}{\mathfrak{B}^2}\,B^2uB^2u' + \frac{\mathrm{cg}}{\mathfrak{C}^2}\,C^2uC^2u'\right\}, \quad = \frac{1}{\mathrm{f}}\left\{\frac{\mathrm{gh}}{\mathfrak{A}^2}\,A^2uA^2u' + \frac{\mathrm{bc}}{\mathfrak{D}^2}\,D^2uD^2u'\right\},$$

$$\nabla_2 = -\frac{1}{\mathrm{b}}\left\{\frac{\mathrm{cf}}{\mathfrak{C}^2}\,C^2uC^2u' + \frac{\mathrm{ah}}{\mathfrak{A}^2}\,A^2uA^2u'\right\}, \quad = \frac{1}{\mathrm{g}}\left\{\frac{\mathrm{hf}}{\mathfrak{B}^2}\,B^2uB^2u' + \frac{\mathrm{ca}}{\mathfrak{D}^2}\,D^2uD^2u'\right\},$$

$$\nabla_3 = -\frac{1}{\mathrm{c}}\left\{\frac{\mathrm{ag}}{\mathfrak{A}^2}\,A^2uA^2u' + \frac{\mathrm{bf}}{\mathfrak{B}^2}\,B^2uB^2u'\right\}, \quad = \frac{1}{\mathrm{h}}\left\{\frac{\mathrm{fg}}{\mathfrak{C}^2}\,C^2uC^2u' + \frac{\mathrm{ab}}{\mathfrak{D}^2}\,D^2uD^2u'\right\},$$

or finally

$$\nabla_1 = -\frac{1}{\mathrm{af}}\,(\quad B^2uB^2u' + C^2uC^2u'), \quad = -\frac{1}{\mathrm{af}}\,(A^2uA^2u' + D^2uD^2u'),$$

$$\nabla_2 = -\frac{1}{\mathrm{bg}}\,(\quad C^2uC^2u' - A^2uA^2u'), \quad = \frac{1}{\mathrm{bg}}\,(B^2uB^2u' - D^2uD^2u'),$$

$$\nabla_3 = -\frac{1}{\mathrm{ch}}\,(-A^2uA^2u' + B^2uB^2u'), \quad = \frac{1}{\mathrm{ch}}\,(C^2uC^2u' - D^2uD^2u').$$

51. Hence ∇, ∇_1, ∇_2, ∇_3 denoting these functions of x, x' or of u, u', we have

$$A\,(u+u')\,A\,(u-u') = \frac{\mathfrak{A}}{\mathrm{gh}}\,\nabla_1,$$

$$B\,(u+u')\,B\,(u-u') = \frac{\mathfrak{B}}{\mathrm{hf}}\,\nabla_2,$$

$$C\,(u+u')\,C\,(u-u') = \frac{\mathfrak{C}}{\mathrm{fg}}\,\nabla_3,$$

$$D\,(u+u')\,D\,(u-u') = \mathfrak{D}\nabla.$$

The square-set $u \pm u'$, u' indefinitely small: differential formulæ of the second order.

52. I consider the original form

$$C^2 0\, C(u+u')\, C(u-u') = C^2 u C^2 u' - D^2 u D^2 u',$$

which is of course included in the last-mentioned equations.

Writing this in the form

$$C^2 0 \frac{C(u+u')\, C(u-u')}{C^2 u} = C^2 u' - \frac{D^2 u D^2 u'}{C^2 u},$$

and taking u' indefinitely small, whence

$$C(u+u') = Cu + u'C'u + \tfrac{1}{2}u'^2 C''u, \quad Cu' = C0,$$
$$C(u-u') = Cu - u'C'u + \tfrac{1}{2}u'^2 C''u, \quad Du' = u'D'0,$$
$$C(u+u')\, C(u-u') = C^2 u + u'^2 \{CuC''u - (C'u)^2\},$$

the equation becomes

$$C^2 0 \left(1 + u'^2 \left\{\frac{C''u}{Cu} - \left(\frac{C'u}{Cu}\right)^2\right\}\right) = C^2 0 + u'^2 \left\{C0C''0 - (D'0)^2 \frac{D^2 u}{C^2 u}\right\},$$

that is,

$$\frac{C''u}{Cu} - \left(\frac{C'u}{Cu}\right)^2 = \frac{C''0}{C0} - \left(\frac{D'0}{C0}\right)^2 \frac{D^2 u}{C^2 u},$$

viz. we have $\left(\frac{d}{du}\right)^2 \log Cu$ expressed in terms of the quotient-function $\frac{D^2 u}{C^2 u}$, and consequently Cu given as an exponential, the argument of which depends on the double integral $\int du \int du \frac{D^2 u}{C^2 u}$.

53. To complete the result, I write the equation in the form

$$\frac{d^2}{du^2} \log Cu = \frac{C''0}{C0} - \frac{1}{k}\left(\frac{D'0}{C0}\right)^2 + \frac{1}{k}\left(\frac{D'0}{C0}\right)^2 \left(1 - k\frac{D^2 u}{C^2 u}\right);$$

$\frac{D'0}{C0}$ is $= -\sqrt{k}K$, and $\frac{C''0}{C0}$ is $= K(K-E)$; hence the equation is

$$\frac{d^2}{du^2} \log Cu = K^2 \left(1 - \frac{E}{K} - k\frac{D^2 u}{C^2 u}\right), \quad = K^2 \left(1 - \frac{E}{K} - k^2 \operatorname{sn}^2 Ku\right),$$

or integrating twice, and observing that $\frac{d}{du} \log Cu$ and $\log Cu$, for $u = 0$, become $= 0$ and $\log C0$ respectively, we have

$$\log Cu = \log C0 + \tfrac{1}{2}\left(1 - \frac{E}{K}\right) K^2 u^2 - k^2 \int_0 du \int_0 du\, K^2 \operatorname{sn}^2 Ku,$$

which is in fact

$$\log \Theta(Ku) = \log C0 + \tfrac{1}{2}\left(1 - \frac{E}{K}\right) K^2 u^2 - k^2 \int_0 du \int_0 du\, K^2 \operatorname{sn}^2 Ku,$$

agreeing with Jacobi's formula

$$\log \Theta u = \log \Theta 0 + \tfrac{1}{2}\left(1 - \frac{E}{K}\right) u^2 - k^2 \int_0 du \int_0 du \operatorname{sn}^2 u.$$

Elliptic integrals of the third kind.

54. We may write

$$\frac{A(u+u')\,A(u-u')}{A^2uA^2u'} = \frac{1}{\mathfrak{A}\mathrm{gh}}\,\frac{\nabla_1}{a-x\,.\,a-x'},$$

$$\frac{B(u+u')\,B(u-u')}{B^2uB^2u'} = \frac{1}{\mathfrak{B}\mathrm{hf}}\,\frac{\nabla_2}{b-x\,.\,b-x'},$$

$$\frac{C(u+u')\,C(u-u')}{C^2uC^2u'} = \frac{1}{\mathfrak{C}\mathrm{fg}}\,\frac{\nabla_3}{c-x\,.\,c-x'},$$

$$\frac{D(u+u')\,D(u-u')}{D^2uD^2u'} = \frac{1}{\mathfrak{D}}\,\frac{x-x'}{d-x\,.\,d-x'}.$$

We differentiate logarithmically in regard to u'. Observing that

$$Kdu' = \frac{\tfrac{1}{2}\sqrt{\mathrm{af}}\,dx'}{\sqrt{a-x'\,.\,b-x'\,.\,c-x'\,.\,d-x'}}, \quad = \frac{\tfrac{1}{2}\sqrt{\mathrm{af}}}{\sqrt{X'}}\,dx',$$

suppose, the first equation gives

$$\frac{A'u}{Au} + \tfrac{1}{2}\frac{A'(u-u')}{A(u-u')} - \tfrac{1}{2}\frac{A'(u+u')}{A(u+u')} = -\frac{K\sqrt{X'}}{\sqrt{\mathrm{af}}}\,\frac{d}{dx'}\log\frac{\nabla_1}{a-x'},$$

and if for a moment

$$\nabla_1, = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+d, & ad \\ 1, & b+c, & bc \end{vmatrix},$$

is put

$$= P(a-x') + Q(d-x'),$$

then

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'}, \; = \frac{d}{dx'}\log\left(P + Q\,\frac{d-x'}{a-x'}\right) \text{ is } = \frac{Q(d-a)}{(a-x')\,\nabla_1}, \; = -\frac{Q\mathrm{f}}{(a-x')\,\nabla_1}.$$

But, writing $x' = a$, we have

$$Q(d-a), \; = -Q\mathrm{f} = \begin{vmatrix} 1, & a+x, & ax \\ 1, & a+d, & ad \end{vmatrix}, \; = (a-b)(a-c)(d-x), \; = -\mathrm{bc}\,(d-x),$$

that is,

$$Q\mathrm{f} = -\mathrm{bc}\,(d-x),$$

or

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'} = \frac{\mathrm{bc}\,(d-x)}{(a-x')\,\nabla_1}.$$

Hence the equation is

$$2\frac{A'(u')}{A(u')} + \frac{A'(u-u')}{A(u-u')} - \frac{A'(u+u')}{A(u+u')} = \frac{2K\mathrm{bc}}{\sqrt{\mathrm{af}}}\sqrt{X'}\,\frac{d-x}{(a-x')\,\nabla_1};$$

and similarly

$$2\frac{B'(u')}{B(u')}+\frac{B'(u-u')}{B(u-u')}-\frac{B'(u+u')}{B(u+u')}=\frac{2K\mathrm{ca}}{\sqrt{\mathrm{af}}}\sqrt{X'}\,\frac{d-x}{(b-x')\nabla_2},$$

$$2\frac{C'(u')}{C(u')}+\frac{C'(u-u')}{C(u-u')}-\frac{C'(u+u')}{C(u+u')}=\frac{2K\mathrm{ab}}{\sqrt{\mathrm{af}}}\sqrt{X'}\,\frac{d-x}{(c-x')\nabla_3},$$

$$2\frac{D'(u')}{D(u')}+\frac{D'(u-u')}{D(u-u')}-\frac{D'(u+u')}{D(u+u')}=\frac{2K}{\sqrt{\mathrm{af}}}\sqrt{X'}\,\frac{d-x}{(d-x')(x-x')}.$$

55. Multiply each of these equations by du, $=\frac{1}{2}\frac{\sqrt{\mathrm{af}}}{K}\frac{dx}{\sqrt{X}}$, and integrate. We have equations such as

$$2u\frac{A'(u')}{A(u')}+\log\frac{A(u-u')}{A(u+u')}=\text{const.}+\frac{\mathrm{bc}\sqrt{X'}}{\sqrt{\mathrm{af}}\,(a-x')}\int\frac{(d-x)\,dx}{\nabla_1\sqrt{X}},$$

showing how the integrals of the third kind

$$\int\frac{(d-x)\,dx}{\nabla_1\sqrt{X}},\quad \int\frac{(d-x)\,dx}{\nabla_2\sqrt{X}},\quad \int\frac{(d-x)\,dx}{\nabla_3\sqrt{X}},\quad \int\frac{(d-x)\,dx}{(x-x')\sqrt{X}}$$

depend on the theta-functions.

If, instead, we work with the original equation

$$C^2 0\frac{C(u+u')\,C(u-u')}{C^2u\,.\,C^2u'}=1-\frac{D^2u}{C^2u}\frac{D^2u'}{C^2u'},$$

we find in the same way

$$\begin{aligned}2\frac{C'(u')}{C(u')}+\frac{C'(u-u')}{C(u-u')}-\frac{C'(u+u')}{C(u+u')}&=-\frac{d}{du'}\log\left(1-\frac{D^2uD^2u'}{C^2uC^2u'}\right),\\&=-\frac{d}{du'}\log(1-k^2\,\mathrm{sn}^2\,Ku\,\mathrm{sn}^2\,Ku'),\\&=\frac{2k^2K\,\mathrm{sn}\,Ku'\,\mathrm{cn}\,Ku'\,\mathrm{dn}\,Ku'\,\mathrm{sn}^2\,Ku}{1-k^2\,\mathrm{sn}^2\,Ku'\,\mathrm{sn}^2\,Ku};\end{aligned}$$

or, multiplying by $\frac{1}{2}du$ and integrating, we have

$$u\frac{C'(u')}{C(u')}+\frac{1}{2}\log\frac{C(u-u')}{C(u+u')}=\int\frac{k^2\,\mathrm{sn}\,Ku'\,\mathrm{cn}\,Ku'\,\mathrm{dn}\,Ku'\,\mathrm{sn}^2\,Ku\,.\,Kdu}{1-k^2\,\mathrm{sn}^2\,Ku'\,\mathrm{sn}^2\,Ku},$$

which is in fact Jacobi's equation

$$u\frac{\Theta'a}{\Theta a}+\frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}=\int\frac{\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2\,u\,du}{1-k^2\,\mathrm{sn}^2\,a\,\mathrm{sn}^2\,u},\ =\Pi(u,\,a).$$

I do not effect the operation but consider the forms first obtained,

$$A(u+u')\,A(u-u')=\frac{2\mathfrak{l}}{\mathrm{gh}}\nabla_1,\ \&c.,$$

as the equivalent of Jacobi's last-mentioned equation.

Addition-formulæ.

56. The addition-theorem for the quotient-functions is of course given by means of the theorem for the elliptic functions: but more elegantly by the formulæ relating to the form $dx \div \sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}$ obtained in my paper "On the Double ϑ-Functions" (*Crelle*, t. LXXXVII. (1879), pp. 74—81, [697]); viz. for the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}-\frac{dz}{\sqrt{Z}}=0,$$

to obtain the particular integral which for $y=d$ reduces itself to $z=x$, we must, in the formulæ of the paper just referred to, interchange a and d: and writing for shortness $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}=a-x,\ b-x,\ c-x,\ d-x$, and similarly $\mathrm{a}_{,}, \mathrm{b}_{,}, \mathrm{c}_{,}, \mathrm{d}_{,}=a-y,\ b-y,\ c-y,\ d-y$, then when the interchange is made, the formulæ become

$$\sqrt{\frac{d-z}{a-z}}$$

$$=\frac{\sqrt{d-b\,.\,d-c}\,\{\sqrt{\mathrm{adb_{,}c_{,}}}+\sqrt{\mathrm{a_{,}d_{,}bc}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{d-b\,.\,d-c\,.\,x-y}}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{d-b\,.\,d-c}\,\{\sqrt{\mathrm{bdc_{,}a_{,}}}+\sqrt{\mathrm{b_{,}d_{,}ca}}\}}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{d-b\,.\,d-c}\,\{\sqrt{\mathrm{cda_{,}b_{,}}}+\sqrt{\mathrm{abc_{,}d_{,}}}\}}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}},$$

$$\sqrt{\frac{b-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\,\{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}+(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\,\{\sqrt{\mathrm{bda_{,}c_{,}}}-\sqrt{\mathrm{b_{,}d_{,}ac}}\}}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\,(ac,\ bd)}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\,\{(d-a)\sqrt{\mathrm{bcb_{,}c_{,}}}+(b-c)\sqrt{\mathrm{aba_{,}b_{,}}}\}}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}},$$

$$\sqrt{\frac{c-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\,\{(d-b)\sqrt{\mathrm{cac_{,}a_{,}}}+(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\,\{\sqrt{\mathrm{cda_{,}b_{,}}}-\sqrt{\mathrm{abc_{,}d_{,}}}\}}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\,\{(d-a)\sqrt{\mathrm{bcb_{,}c_{,}}}-(b-c)\sqrt{\mathrm{ada_{,}d_{,}}}\}}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\,(ab,\ cd)}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}}.$$

57. In the foregoing formulæ, (bc, ad), (ac, bd), and (ab, cd) denote respectively

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}, \quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & c+a, & ca \\ 1, & b+d, & bd \end{vmatrix}, \quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix};$$

and substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ their values, and for a, b, &c., writing again $a-x$, $b-x$, &c., we have moreover

$$\begin{array}{ll|ll} A^2u = \sqrt{c-b\,.\,b-d\,.\,c-d} & (a-x), & A^2v = \sqrt{\quad\text{"}\quad} & (a-y), \\ B^2u = \sqrt{c-a\,.\,c-d\,.\,a-d} & (b-x), & B^2v = \sqrt{\quad\text{"}\quad} & (b-y), \\ C^2u = \sqrt{a-b\,.\,a-d\,.\,b-d} & (c-x), & C^2v = \sqrt{\quad\text{"}\quad} & (c-y), \\ D^2u = \sqrt{c-b\,.\,c-a\,.\,a-b} & (d-x), & D^2v = \sqrt{\quad\text{"}\quad} & (d-y), \end{array}$$

$$\begin{array}{ll} A^2(u+v) = \sqrt{\quad\text{"}\quad} & (a-z), \\ B^2(u+v) = \sqrt{\quad\text{"}\quad} & (b-z), \\ C^2(u+v) = \sqrt{\quad\text{"}\quad} & (c-z), \\ D^2(u+v) = \sqrt{\quad\text{"}\quad} & (d-z), \end{array}$$

the constant multipliers being of course the same in the three columns respectively. According to what precedes, the radical of the fourth line should be taken with the sign $-$. The functions (bc, ad), &c., contained in the formulæ, require a transformation such as

$$(b-c)(bc, ad) = \begin{vmatrix} b-x\,.\,b-y, & c-x\,.\,c-y \\ b-a\,.\,b-d, & c-a\,.\,c-d \end{vmatrix},$$

in order to make them separately homogeneous in the differences $a-x$, $b-x$, $c-x$, $d-x$, and $a-y$, $b-y$, $c-y$, $d-y$, and therefore to make them expressible as linear functions of the squared functions A^2u, &c., and A^2v, &c., respectively: the formulæ then give the quotient-functions $A(u+v) \div D(u+v)$, &c., in terms of the quotient-functions of u and v respectively.

Doubly infinite product-forms.

58. The functions Au, Bu, Cu, Du may be expressed each as a doubly infinite product. Writing for shortness

$$m \quad + \quad n\,.\,\frac{a}{\pi i} = (m, n),$$

$$m+1 + \quad n\,.\,\frac{a}{\pi i} = (\overline{m}, n),$$

$$m \quad + (n+1)\frac{a}{\pi i} = (m, \overline{n}),$$

$$m+1 + (n+1)\frac{a}{\pi i} = (\overline{m}, \overline{n}),$$

then the formulæ are

$$Au = A0\,.\ \Pi\Pi\left\{1 + \frac{u}{(\overline{m},\ \overline{n})}\right\},$$

$$Bu = B0\,.\ \Pi\Pi\left\{1 + \frac{u}{(\overline{m},\ n)}\right\},$$

$$Cu = C0\,.\ \Pi\Pi\left\{1 + \frac{u}{(m,\ \overline{n})}\right\},$$

$$Du = D'0\,.\,u\Pi\Pi\left\{1 + \frac{u}{(m,\ n)}\right\},$$

where in all the formulæ, m and n denote even integers having all values whatever, zero included, from $-\infty$ to $+\infty$; only in the formula for Du, the term for which m and n are simultaneously $=0$, is to be omitted.

59. But a further definition in regard to the limits is required: first, we assume that m has the corresponding positive and negative values, and similarly that n has corresponding positive and negative values*; or say, in the four formulæ respectively, we consider m, n as extending

$$\begin{array}{llllllll} m & \text{from} & -\mu & \text{to} & \mu+2, & n & \text{from} & -\nu \text{ to } \nu+2, \\ \text{,,} & \text{,,} & -\mu & \text{,,} & \mu+2, & \text{,,} & \text{,,} & -\nu \text{ ,, } \nu, \\ \text{,,} & \text{,,} & -\mu & \text{,,} & \mu\ \ , & \text{,,} & \text{,,} & -\nu \text{ ,, } \nu+2, \\ \text{,,} & \text{,,} & -\mu & \text{,,} & \mu\ \ , & \text{,,} & \text{,,} & -\nu \text{ ,, } \nu, \end{array}$$

where μ and ν are each of them ultimately infinite. But, secondly, it is necessary that μ should be indefinitely larger than ν, or say that ultimately $\dfrac{\nu}{\mu} = 0$.

60. In fact, transforming the q-series into products as in the *Fundamenta Nova*, and neglecting for the moment mere constant factors, we have

$$\begin{aligned} Au &= \qquad\quad (1 + 2q\ \cos \pi u + q^2)(1 + 2q^3 \cos \pi u + q^6)\ldots, \\ Bu &= \cos \tfrac{1}{2}\pi u\,(1 + 2q^2 \cos \pi u + q^4)(1 + 2q^4 \cos \pi u + q^8)\ldots, \\ Cu &= \qquad\quad (1 - 2q\ \cos \pi u + q^2)(1 - 2q^3 \cos \pi u + q^6)\ldots, \\ Du &= \sin \tfrac{1}{2}\pi u\,(1 - 2q^2 \cos \pi u + q^4)(1 - 2q^4 \cos \pi u + q^8)\ldots, \end{aligned}$$

and writing for a moment $\alpha = \dfrac{a}{\pi i}$ and therefore $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, $= e^{\frac{1}{2}\pi a i} + e^{-\frac{1}{2}\pi a i}$, $= 2\cos \frac{1}{2}\pi\alpha$, &c., each of these expressions is readily converted into a singly infinite product of sines or cosines

$$Au = \Pi\,.\cos \tfrac{1}{2}\pi\,(u + \overline{n}\alpha),$$

$$Bu = \Pi\,.\cos \tfrac{1}{2}\pi\,(u + n\alpha),$$

$$Cu = \Pi\,.\sin \tfrac{1}{2}\pi\,(u + \overline{n}\alpha),$$

$$Du = \Pi\,.\sin \tfrac{1}{2}\pi\,(u + n\alpha),$$

* This is more than is necessary; it would be enough if the ultimate values of m were $-\mu$, μ', μ and μ' being in a ratio of equality; and the like as regards n. But it is convenient that the numbers should be absolutely equal.

where $\bar{n}$ is written to denote $n+1$, and n has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\nu$ to ν, if ν is ultimately infinite.

61. The sines and cosines are converted into infinite products by the ordinary formulæ, which neglecting constant factors may conveniently be written

$$\sin \tfrac{1}{2}\pi u = \Pi\,(u+m), \quad \cos \tfrac{1}{2}\pi u = \Pi\,(u+\overline{m}),$$

where $\overline{m}$ is written to denote $m+1$, and m has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\mu$ to μ, if μ be ultimately infinite. But in applying these formulæ to the case of a function such as

$$\sin \tfrac{1}{2}\pi\,(u+n\alpha),$$

it is assumed that the limiting values μ, $-\mu$ of m are indefinitely large in regard to $u+n\alpha$; and therefore, since n may approach to its limiting value $\pm\nu$, it is necessary that μ should be indefinitely large in comparison with ν, or that $\frac{\nu}{\mu}=0$.

62. It is on account of this unsymmetry as to the limits $\frac{\nu}{\mu}=0$, $\frac{\mu}{\nu}=\infty$, that we have 1 as a quarter-period, $\frac{a}{\pi i}$ only as a quarter-quasi-period of the single theta-functions.

The transformation q to r, $\log q \log r = \pi^2$.

63. In general, if we consider the ratio of two such infinite products, where for the first the limits are $(\pm\mu,\ \pm\nu)$, and for the second they are $(\pm\mu',\ \pm\nu')$, and where for convenience we take $\mu>\mu'$, $\nu>\nu'$, then the quotient, say $[\mu,\ \nu]\div[\mu',\ \nu']$ is $=\exp.\,(Mu^2)$, where

$$M = -\tfrac{1}{8}\iint \frac{dm\,dn}{(m,\ n)^2}$$

taken over the area included between the two rectangles. We have

$$(m,\ n) = m + \frac{a}{\pi i}\,n, \ = m + i\theta n$$

suppose, where (a being negative) $\theta = -\frac{a}{\pi}$ is positive: the integral is

$$\begin{aligned}\iint \frac{dm\,dn}{(m+i\theta n)^2}, &= \int dm\,.\,-\frac{1}{i\theta}\left(\frac{1}{m+i\theta n}\right)_{-\nu}^{\nu},\\ &= \frac{1}{i\theta}\int dm\left(\frac{1}{m-i\theta\nu}-\frac{1}{m+i\theta\nu}\right),\\ &= \frac{1}{i\theta}\log\frac{m-i\theta\nu}{m+i\theta\nu};\end{aligned}$$

or finally between the proper limits the value is

$$= \frac{2}{i\theta}\left\{\log\left(\frac{\mu - i\theta\nu}{\mu + i\theta\nu}\right) - \log\left(\frac{\mu' - i\theta\nu'}{\mu' + i\theta\nu'}\right)\right\},$$

where the logarithms are

$$\log(\mu - i\theta\nu) = \log\sqrt{\mu^2 + \nu^2} - i\tan^{-1}\frac{\theta\nu}{\mu}, \text{ \&c.},$$

and the $\tan^{-1}$ denotes always an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. Hence, if $\frac{\mu}{\nu} = \infty$, $\frac{\mu'}{\nu'} = 0$, the value is

$$\frac{2}{i\theta}(-0i - 0i + \tfrac{1}{2}\pi i + \tfrac{1}{2}\pi i) = \frac{2\pi}{\theta}, = -\frac{2\pi^2}{a}; \text{ or } K = \tfrac{1}{4}\frac{\pi^2}{a}.$$

Hence finally

$$[\mu \div \nu, = \infty] \div [\mu \div \nu, = 0] = \exp.\left(\tfrac{1}{4}\frac{\pi^2}{a}u^2\right).$$

64. We have $a, = \log q$, negative; hence taking r such that $\log q \log r = \pi^2$, we have $a' = \log r$, also negative; and r, like q, is positive and less than 1. We consider the theta-functions which depend on r in the same manner that the original functions did on q, say these are

$$A(u, r) = A(0, r) \quad \Pi\Pi\left\{1 + \frac{u}{\bar{m} + \bar{n}\frac{a'}{\pi i}}\right\},$$

$$B(u, r) = B(0, r) \quad \Pi\Pi\left\{1 + \frac{u}{\bar{m} + n\frac{a'}{\pi i}}\right\},$$

$$C(u, r) = C(0, r) \quad \Pi\Pi\left\{1 + \frac{u}{m + \bar{n}\frac{a'}{\pi i}}\right\},$$

$$D(u, r) = D'(0, r)\, u\, \Pi\Pi\left\{1 + \frac{u}{m + n\frac{a'}{\pi i}}\right\},$$

limits as before, and in particular $\frac{\mu}{\nu} = \infty$; it is at once seen that if in the original functions, which I now call $A(u, q)$, $B(u, q)$, $C(u, q)$, $D(u, q)$, we write $\frac{au}{\pi i}$ for u, we obtain the same infinite products which present themselves in the expressions of the new functions $A(u, r)$, &c., only with a different condition as to the limits; we have for instance

$$\Pi\Pi\left(1 + \frac{\frac{au}{\pi i}}{m + n\frac{a}{\pi i}}\right) = \Pi\Pi\left(1 + \frac{u}{n - m\frac{a'}{\pi i}}\right), = \Pi\Pi\left(1 + \frac{u}{n + m\frac{a'}{\pi i}}\right),$$

which, interchanging m, n, and of course also μ, ν, is

$$= \Pi\Pi\left(1 + \frac{u}{m + n\dfrac{a'}{\pi i}}\right),$$

with the condition $\dfrac{\mu}{\nu} = 0$ instead of $\dfrac{\mu}{\nu} = \infty$. Hence disregarding for the moment constant factors, and observing that a is replaced by a', we have

$$D(u, r) \div D\left(\frac{au}{\pi i}, q\right) = [\mu \div \nu, = \infty] \div [\mu \div \nu, = 0]$$

$$= \exp.\left(\tfrac{1}{4}\frac{\pi^2}{a'}u^2\right), = \exp.\left(\tfrac{1}{4}u^2 \log q\right).$$

65. We have equations of this form for the four functions, but with a proper constant multiplier in each equation: the equations, in fact, are

$$A(u, r) = \{A(0, r) \div A(0, q)\} \exp.\left(\tfrac{1}{4}u^2 \log q\right) A\left(\frac{au}{\pi i}, q\right),$$

$$B(u, r) = \{B(0, r) \div B(0, q)\} \quad \text{,,} \quad B\left(\frac{au}{\pi i}, q\right),$$

$$C(u, r) = \{C(0, r) \div C(0, q)\} \quad \text{,,} \quad C\left(\frac{au}{\pi i}, q\right),$$

$$D(u, r) = \{D'(0, r) \div D'(0, q)\}\frac{\pi i}{a} \quad \text{,,} \quad D\left(\frac{au}{\pi i}, q\right).$$

It is to be observed that r is the same function of k' that q is of k. This would at once follow from Jacobi's equation $\log q = -\dfrac{\pi K'}{K}$, for then $\log q \log r = \pi^2$ and therefore $\log r = -\dfrac{\pi K'}{K}$ $\left(\text{only we are not at liberty to use the relation in question } \log q = -\dfrac{\pi K'}{K}\right)$: assuming it to be true, we have

$$k = \frac{B^2(0, q)}{A^2(0, q)},\ k' = \frac{C^2(0, q)}{A^2(0, q)},\ K = -\frac{A(0, q)\,D'(0, q)}{B(0, q)\,C(0, q)},$$

$$k = \frac{C^2(0, r)}{A^2(0, r)},\ k' = \frac{B^2(0, r)}{A^2(0, r)},\ K' = -\frac{A(0, r)\,D'(0, r)}{B(0, r)\,C(0, r)},$$

$$\log q = -\frac{\pi K'}{K},\ \log r = -\frac{\pi K}{K'},$$

where, if the identity of the two values of k or of the two values of k' were proved independently (as might doubtless be done), the required theorem, viz. that r is the same function of k' that q is of k, would follow conversely: and thence the other equations of the system.

66. We have, in the *Fundamenta Nova*, p. 175, [Jacobi's *Ges. Werke*, t. I., p. 227], the equation

$$\frac{H(iu,\ k)}{\Theta(0,\ k)} = i\sqrt{\frac{k}{k'}}\, e^{\frac{\pi u^2}{4KK'}}\, \frac{H(u,\ k')}{\Theta(0,\ k')};$$

writing here $K'u$ instead of u the equation becomes

$$\frac{H(iK'u,\ k)}{\Theta(0,\ k)} = i\sqrt{\frac{k}{k'}}\, \exp.\left(\tfrac{1}{4}\frac{\pi K'}{K}u^2\right) \cdot \frac{H(K'u,\ k')}{\Theta(0,\ k')},$$

or, what is the same thing,

$$\frac{D\left(\frac{au}{\pi i},\ q\right)}{C(0,\ q)} = i\sqrt{\frac{k}{k'}}\, \exp.\left(-\tfrac{1}{4}u^2 \log q\right) \cdot \frac{D(u,\ r)}{C(0,\ r)}$$

which can be readily identified with the foregoing equation between $D\left(\frac{au}{\pi i},\ q\right)$ and $D(u,\ r)$. But the real meaning of the transformation is best seen by means of the double-product formulæ.

THIRD PART.—THE DOUBLE THETA-FUNCTIONS.

Notations, &c.

67. We have here 16 functions $\vartheta\begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix}(u,\ v)$: this notation by characteristics, containing each of them four numbers, is too cumbrous for ordinary use, and I therefore replace it by the current-number notation, in which the characteristics are denoted by the series of numbers 0, 1, 2, ..., 15: we cannot in place of this introduce the single-and-double-letter notation A, B, ..., AB, &c., for there is not here any correspondence of the two notations, nor is there anything in the definition of the functions which in anywise suggests the single-and-double-letter notation: this first presents itself in connexion with the relations between the functions given by the product-theorem: and as the product-theorem is based upon the notation by characteristics, it is proper to present the theorem in this notation, or in the equivalent current-number notation: and then to show how by the relations thus obtained between the functions we are led to the single-and-double-letter notation.

68. There are some other notations which may be referred to: and for showing the correspondence between them I annex the following table:—

THE DOUBLE THETA-FUNCTIONS.

Asterisk denotes the odd functions.	1. Current number.	2. Character.	3. Single-and-double-letter, Cayley.	4. Göpel.	5. Göpel-Cayley.	6. Rosenhain.	7. Weierstrass.	8. Kummer.
	ϑ_0	$\vartheta\,{}^{00}_{00}$	BD	P'''	P_3	ϑ_{22}	ϑ_5	12
	1	$\begin{matrix}10\\00\end{matrix}$	CE	R'''	R_3	32	4	8
	2	$\begin{matrix}01\\00\end{matrix}$	CD	Q'''	Q_3	23	01	10
	3	$\begin{matrix}11\\00\end{matrix}$	BE	S'''	S_3	33	23	6
	4	$\begin{matrix}00\\10\end{matrix}$	AC	P'	P_1	02	34	4
*	5	$\begin{matrix}10\\10\end{matrix}$	C	iR'	R_1	12	3	16
	6	$\begin{matrix}01\\10\end{matrix}$	AB	Q'	Q_1	03	2	2
*	7	$\begin{matrix}11\\10\end{matrix}$	B	iS'	S_1	13	24	14
	8	$\begin{matrix}00\\01\end{matrix}$	BC	P''	P_2	20	12	9
	9	$\begin{matrix}10\\01\end{matrix}$	DE	R''	R_2	30	03	5
*	10	$\begin{matrix}01\\01\end{matrix}$	F	iQ''	Q_2	21	02	11
*	11	$\begin{matrix}11\\01\end{matrix}$	A	iS''	S_2	31	13	7
	12	$\begin{matrix}00\\11\end{matrix}$	AD	P	P	00	0	1
*	13	$\begin{matrix}10\\11\end{matrix}$	D	iR	R	10	04	13
*	14	$\begin{matrix}01\\11\end{matrix}$	E	iQ	Q	01	1	3
	15	$\begin{matrix}11\\11\end{matrix}$	AE	S	S	11	14	15

69. These are the notations:—

1. By current-numbers. It may be remarked that the series was taken 0, 1, ..., 15, instead of 1, 2, ..., 16, in order that 0 might correspond to the characteristic $\begin{smallmatrix}00\\00\end{smallmatrix}$;

2. By characteristics;

3. By single-and-double letters;

4. Göpel's, in his paper above referred to, and

5. The same as used in my paper above referred to;

6. Rosenhain's, in his paper above referred to;

7. Weierstrass', as quoted by Königsberger in his paper "Ueber die Transformation der *Abel*schen Functionen erster Ordnung," *Crelle-Borchardt*, t. LXIV. (1865), p. 17, and by Borchardt in his paper above referred to;

8. Not a theta-notation, but the series of current-numbers used in Kummer's Memoir "Ueber die algebraischen Strahlen-systeme," *Berl. Abh.* 1866, for the nodes of his 16-nodal quartic surface, and connected with the double theta-functions in my paper above referred to.

But in the present memoir only the first three columns of the table need be attended to.

70. It will be convenient to introduce here some other remarks as to notation, &c.

The letter c is used in connexion with the zero values $u=0$, $v=0$ of the arguments, viz.:—

$$\vartheta_0,\ \vartheta_1,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4,\ \vartheta_6,\ \vartheta_8,\ \vartheta_9,\ \vartheta_{12},\ \vartheta_{15}$$

are even functions, and the corresponding zero-functions are denoted by

$$c_0,\ c_1,\ c_2,\ c_3,\ c_4,\ c_6,\ c_8,\ c_9,\ c_{12},\ c_{15}:$$

there are thus 10 constants c.

When (u, v) are indefinitely small each of these functions is of course equal to its zero-value *plus* a quadric term in (u, v), and we may write in general

$$\vartheta = c + \tfrac{1}{2}(c''',\ c^{\text{iv}},\ c^{\text{v}} \between u,\ v)^2:$$

there are thus 30 constants c''', c^{iv}, c^{v}.

The remaining functions

$$\vartheta_5,\ \vartheta_7,\ \vartheta_{10},\ \vartheta_{11},\ \vartheta_{13},\ \vartheta_{14}$$

are odd functions vanishing for $u=0$, $v=0$; when these arguments are indefinitely small, we may write in general

$$\vartheta = (c',\ c'' \between u,\ v):$$

there are thus 12 constants c', c''.

71. All these constants are in the first instance given as transcendental functions of the parameters, or say rather of exp. a, exp. h, exp. b, which exponentials correspond to the q of the single theory: viz., in a notation which will be readily understood, the constants c, c''', c^{iv}, c^{v} of the even functions are

$$\Sigma \exp. \begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix};$$

$$-\tfrac{1}{2}\pi^2\Sigma\,(m+\alpha)^2,\ 2\,(m+\alpha)\,(n+\beta),\ (n+\beta)^2,\ \exp. \begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix};$$

and the constants c', c'' of the odd functions are

$$\tfrac{1}{2}\pi i\Sigma\,(m+\alpha),\ (n+\beta),\ \exp. \begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix}.$$

72. It is convenient for the verification of results and otherwise, to have the values of the functions, belonging to small values of exp. $(-a)$, exp. $(-b)$; if to avoid fractional exponents we regard these and exp. $(-h)$ as fourth powers, and write

$$\exp.\,(-a) = Q^4,\ \exp.\,(-h) = R^4,\ \exp.\,(-b) = S^4,$$

also

$$QR^2S = \Lambda,\ QR^{-2}S = \Lambda',\ \text{and therefore } \Lambda\Lambda' = Q^2S^2,$$

then attending only to the lowest powers of Q, S we find without difficulty

$$\begin{aligned}
\vartheta_0\,(u) &= 1, && \text{and therefore also } c_0 = 1,\\
\vartheta_1 &= 2Q\cos\tfrac{1}{2}\pi u, && c_1 = 2Q,\\
\vartheta_2 &= 2S\cos\tfrac{1}{2}\pi v, && c_2 = 2S,\\
\vartheta_3 &= 2\Lambda\cos\tfrac{1}{2}\pi\,(u+v) + 2\Lambda'\cos\tfrac{1}{2}\pi\,(u-v), && c_3 = 2\Lambda + 2\Lambda',\\
\vartheta_4 &= 1, && c_4 = 1,\\
\vartheta_5 &= -2Q\sin\tfrac{1}{2}\pi u,\\
\vartheta_6 &= 2S\cos\tfrac{1}{2}\pi v, && c_6 = 2S,\\
\vartheta_7 &= -2\Lambda\sin\tfrac{1}{2}\pi\,(u+v) - 2\Lambda'\sin\tfrac{1}{2}\pi\,(u-v),\\
\vartheta_8 &= 1, && c_8 = 1,\\
\vartheta_9 &= 2Q\cos\tfrac{1}{2}\pi u, && c_9 = 2Q,\\
\vartheta_{10} &= -2S\sin\tfrac{1}{2}\pi v,\\
\vartheta_{11} &= -2\Lambda\sin\tfrac{1}{2}\pi\,(u+v) + 2\Lambda'\sin\tfrac{1}{2}\pi\,(u-v),\\
\vartheta_{12} &= 1, && c_{12} = 1,\\
\vartheta_{13} &= -2Q\sin\tfrac{1}{2}\pi u,\\
\vartheta_{14} &= -2S\sin\tfrac{1}{2}\pi v,\\
\vartheta_{15} &= -2\Lambda\cos\tfrac{1}{2}\pi\,(u+v) + 2\Lambda'\cos\tfrac{1}{2}\pi\,(u-v), && c_{15} = -2\Lambda + 2\Lambda'.
\end{aligned}$$

73. The expansions might be carried further; we have for instance

$$\vartheta_0(u) = 1 + 2Q^4 \cos \pi u + 2S^4 \cos \pi v, \qquad c_0 = 1 + 2Q^4 + 2S^4,$$

$$\vartheta_4 = 1 - 2Q^4 \quad " \quad + 2S^4 \quad " \quad , \qquad c_4 = 1 - 2Q^4 + 2S^4,$$

$$\vartheta_8 = 1 + 2Q^4 \quad " \quad - 2S^4 \quad " \quad , \qquad c_8 = 1 + 2Q^4 - 2S^4,$$

$$\vartheta_{12} = 1 - 2Q^4 \quad " \quad - 2S^4 \quad " \quad , \qquad c_{12} = 1 - 2Q^4 - 2S^4,$$

$$\vartheta_1 = 2Q \cos \tfrac{1}{2}\pi u + 2Q^9 \cos \tfrac{3}{2}\pi u + 2A \cos \tfrac{1}{2}\pi (u + 2v) + 2A' \cos \tfrac{1}{2}\pi (u - 2v),$$

$$c_1 = 2Q + 2Q^9 + 2A + 2A',$$

$$\vartheta_5 = -2Q \sin \tfrac{1}{2}\pi u + 2Q^9 \sin \tfrac{3}{2}\pi u - 2A \sin \tfrac{1}{2}\pi (u + 2v) - 2A' \sin \tfrac{1}{2}\pi (u - 2v),$$

$$\vartheta_9 = 2Q \cos \tfrac{1}{2}\pi u + 2Q^9 \cos \tfrac{3}{2}\pi u - 2A \cos \tfrac{1}{2}\pi (u + 2v) - 2A' \cos \tfrac{1}{2}\pi (u - 2v),$$

$$c_9 = 2Q + 2Q^9 - 2A - 2A',$$

$$\vartheta_{13} = -2Q \sin \tfrac{1}{2}\pi u + 2Q^9 \sin \tfrac{3}{2}\pi u + 2A \sin \tfrac{1}{2}\pi (u + 2v) + 2A' \sin \tfrac{1}{2}\pi (u - 2v),$$

in which last formulæ

$$A = QR^4S^4, = \frac{\Lambda^2 S^2}{Q}; \quad A' = QR^{-4}S^4, = \frac{\Lambda'^2 S^2}{Q}.$$

74. In the single-and-double-letter notation we have six letters A, B, C, D, E, F; and the duads $AB, AC, \ldots, DE$ are used as abbreviations for the double triads ABF, CDE, &c., the letter F always accompanying the expressed duad; there are thus in all six single-letter symbols, and 10 double-letter symbols, in all 16 symbols, corresponding to the double-theta functions, as already mentioned in the order

ϑ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	BD,	CE,	CD,	BE,	AC,	C,	AB,	B,	BC,	DE,	F,	A,	AD,	D,	E,	AE,

where observe that the single letters C, B, F, A, D, E correspond to the odd functions 5, 7, 10, 11, 13, 14 respectively.

75. The ground of the notation is as follows:—

The algebraical relations between the double theta-functions are such that, introducing six constant quantities a, b, c, d, e, f and two variable quantities x, y, it is allowable to express the 16 functions as proportional to given functions of these quantities $(a, b, c, d, e, f; x, y)$; viz. there are six functions the notations of which depend on the single letters a, b, c, d, e, f respectively, and 10 functions the notations of which depend on the pairs $ab, ac, ad, ae, bc, bd, be, cd, ce, de$ respectively: each of the 16 functions is, in fact, proportional to the product of two factors, viz. a constant factor depending only on (a, b, c, d, e, f), and a variable factor containing also x and y. Attending in the first instance to the variable factors, and writing for shortness

$$a - x,\ b - x,\ c - x,\ d - x,\ e - x,\ f - x = \mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e},\ \mathrm{f};\ x - y = \theta;$$

$$a - y,\ b - y,\ c - y,\ d - y,\ e - y,\ f - y = \mathrm{a}_{,},\ \mathrm{b}_{,},\ \mathrm{c}_{,},\ \mathrm{d}_{,},\ \mathrm{e}_{,},\ \mathrm{f}_{,};$$

these are of the forms

$$\sqrt{a} = \sqrt{\mathrm{aa}_{,}},\ \sqrt{ab} = \frac{1}{\theta}\{\sqrt{\mathrm{abfc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}} + \sqrt{\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{f}_{,}\mathrm{cde}}\}.$$

I remark that to avoid ambiguity the squares of these expressions are throughout written as $(\sqrt{a})^2$ and $(\sqrt{ab})^2$ respectively.

76. There is, for the constant factors, a like single-and-double-letter notation which will be mentioned presently; but in the first instance I use for the even functions the before-mentioned 10 letters c, and for the odd ones six letters k. I assume that the values $x,\ y = \infty,\ \infty$ (ratio not determined) correspond to the values $u = 0$, $v = 0$ of the arguments; and that ω is a function of $(x,\ y)$ which, when $(x,\ y)$ are interchanged, changes only its sign, and which for indefinitely large values of $(x,\ y)$ becomes $= \frac{x-y}{(xy)^{\frac{1}{2}}}$. This being so, we write (adding a second column which will be afterwards explained)

$$\begin{array}{ll}
\vartheta & \\
0 = BD = \omega c_0 \sqrt{bd}, & c_0 = \lambda \sqrt[4]{\overline{bd}}, \\
1 = CE = \,\text{,,}\, c_1 \sqrt{ce}, & c_1 = \,\text{,,}\, \sqrt[4]{\overline{ce}}, \\
2 = CD = \,\text{,,}\, c_2 \sqrt{cd}, & c_2 = \,\text{,,}\, \sqrt[4]{\overline{cd}}, \\
3 = BE = \,\text{,,}\, c_3 \sqrt{be}, & c_3 = \,\text{,,}\, \sqrt[4]{\overline{be}}, \\
4 = AC = \,\text{,,}\, c_4 \sqrt{ac}, & c_4 = \,\text{,,}\, \sqrt[4]{\overline{ac}}, \\
5 = \ \ C \ = \,\text{,,}\, k_5 \sqrt{c}, & k_5 \ = \,\text{,,}\, \sqrt[4]{\overline{c}}, \\
6 = AB = \,\text{,,}\, c_6 \sqrt{ab}, & c_6 = \,\text{,,}\, \sqrt[4]{\overline{ab}}, \\
7 = \ \ B \ = \,\text{,,}\, k_7 \sqrt{b}, & k_7 \ = \,\text{,,}\, \sqrt[4]{\overline{b}}, \\
8 = BC = \,\text{,,}\, c_8 \sqrt{bc}, & c_8 = \,\text{,,}\, \sqrt[4]{\overline{bc}}, \\
9 = DE = \,\text{,,}\, c_9 \sqrt{de}, & c_9 = \,\text{,,}\, \sqrt[4]{\overline{de}}, \\
10 = \ \ F \ = \,\text{,,}\, k_{10} \sqrt{f}, & k_{10} \ = \,\text{,,}\, \sqrt[4]{\overline{f}}, \\
11 = \ \ A \ = \,\text{,,}\, k_{11} \sqrt{a}, & k_{11} \ = \,\text{,,}\, \sqrt[4]{\overline{a}}, \\
12 = AD = \,\text{,,}\, c_{12} \sqrt{ad}, & c_{12} = \,\text{,,}\, \sqrt[4]{\overline{ad}}, \\
13 = \ \ D \ = \,\text{,,}\, k_{13} \sqrt{d}, & k_{13} \ = \,\text{,,}\, \sqrt[4]{\overline{d}}, \\
14 = \ \ E \ = \,\text{,,}\, k_{14} \sqrt{e}, & k_{14} \ = \,\text{,,}\, \sqrt[4]{\overline{e}}, \\
15 = AE = \,\text{,,}\, c_{15} \sqrt{ae}, & c_{15} = \,\text{,,}\, \sqrt[4]{\overline{ae}};
\end{array}$$

viz. here, on writing $x,\ y = \infty,\ \infty$, each of the functions $\sqrt{bd}$, &c. becomes $= 2\frac{(xy)^{\frac{1}{2}}}{x-y}$; and each of the functions $\sqrt{a}$, &c., becomes $= \sqrt{xy}$; hence by reason of the assumed

form of ω, the odd functions each vanish (their evanescent values being proportional to k_6, k_7, k_{10}, k_{11}, k_{13}, k_{14} respectively), while the even functions become equal to c_0, c_1, c_2, c_3, c_4, c_6, c_8, c_9, c_{12}, c_{15} respectively.

Observe further that on interchanging x, y, the even functions remain unaltered, while the odd functions change their sign; that is, the interchange of x, y corresponds to the change u, v into $-u$, $-v$.

77. As to the values of the 10 c's and the six k's in terms of a, b, c, d, e, f, these are proportional to fourth roots, $\sqrt[4]{a}$, &c., $\sqrt[4]{ab}$, &c.; in $\sqrt[4]{a}$, a is in the first instance regarded as standing for the pentad $bcdef$, and then this is used to denote a product of differences $bc.bd.be.bf.cd.ce.cf.de.df.ef$; similarly ab is in the first instance regarded as standing for the double triad $abf.cde$, and then each of these triads is used to denote a product of differences, $ab.af.bf$ and $cd.ce.de$ respectively. The order of the letters is always the alphabetical one, viz. the single letters and duads denote pentads and double triads, thus:

$$\begin{array}{ll}
a = bcdef, & ab = abf.cde, \\
b = acdef, & ac = acf.bde, \\
c = abdef, & ad = adf.bce, \\
d = abcef, & ae = aef.bcd, \\
e = abcdf, & bc = bcf.ade, \\
f = abcde, & bd = bdf.ace, \\
 & be = bef.acd, \\
 & cd = cdf.abe, \\
 & ce = cef.abd, \\
 & de = def.abc.
\end{array}$$

There is no fear of ambiguity in writing (and we accordingly write) the squares of $\sqrt[4]{a}$ and $\sqrt[4]{ab}$ as $\sqrt{a}$ and $\sqrt{ab}$ respectively; the fourth powers are written $(\sqrt{a})^2$ and $(\sqrt{ab})^2$; the double stroke of the radical symbol $\sqrt{}$ is in every case perfectly distinctive.

This being so we have as above $c_0 = \lambda\sqrt[4]{bd}$, &c., $k_5 = \lambda\sqrt[4]{c}$, &c.: it is, however, important to notice that the fourth roots in question do not denote positive values, but they are fourth roots each taken with its proper sign ($+$, $-$, $+i$, $-i$, as the case may be) so as to satisfy the identical relations which exist between the sixteen constants; and it is not easy to determine the signs.

The variables x, y are connected with u, v by the differential relations

$$\sigma\, du + \tau\, dv = -\tfrac{1}{2}\left\{\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}}\right\},$$

$$\varpi\, du + \rho\, dv = -\tfrac{1}{2}\left\{\frac{x\,dx}{\sqrt{X}} - \frac{y\,dy}{\sqrt{Y}}\right\},$$

where $X = \text{abcdef}$, $Y = \text{a,b,c,d,e,f,}$; which equations contain the constants ϖ, ρ, σ, τ, the values of which will be afterwards connected with the other constants.

78. The c's are expressed as functions of four quantities α, β, γ, δ, and connected with each other, and with the constants a, b, c, d, e, f, by the formulæ

$$\begin{aligned}
\overset{c^2}{\overline{0}} &= \alpha^2+\beta^2+\gamma^2+\delta^2 = \omega_0{}^2\sqrt{\overline{bd}},\\
1 &= 2(\alpha\beta+\gamma\delta) \quad = \;\text{,,}\; \sqrt{\overline{ce}},\\
2 &= 2(\alpha\gamma+\beta\delta) \quad = \;\text{,,}\; \sqrt{\overline{cd}},\\
3 &= 2(\alpha\delta+\beta\gamma) \quad = \;\text{,,}\; \sqrt{\overline{be}},\\
4 &= \alpha^2-\beta^2+\gamma^2-\delta^2 = \;\text{,,}\; \sqrt{\overline{ac}},\\
6 &= 2(\alpha\gamma-\beta\delta) \quad = \;\text{,,}\; \sqrt{\overline{ab}},\\
8 &= \alpha^2+\beta^2-\gamma^2-\delta^2 = \;\text{,,}\; \sqrt{\overline{bc}},\\
9 &= 2(\alpha\beta-\gamma\delta) \quad = \;\text{,,}\; \sqrt{\overline{de}},\\
12 &= \alpha^2-\beta^2-\gamma^2-\delta^2 = \;\text{,,}\; \sqrt{\overline{ad}},\\
15 &= 2(\alpha\delta-\beta\gamma) \quad = \;\text{,,}\; \sqrt{\overline{ae}}.
\end{aligned}$$

It hence appears that we can form an arrangement

$$\left|\begin{matrix} c_{12}{}^2, & c_1{}^2, & c_6{}^2 \\ c_9{}^2, & -c_4{}^2, & c_3{}^2 \\ c_2{}^2, & -c_{15}{}^2, & -c_8{}^2 \end{matrix}\right| \div c_0{}^2 \quad = \left|\begin{matrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{matrix}\right|,$$

a system of coefficients in the transformation between two sets of rectangular coordinates.

We have, between the squares of these coefficients of transformation, a system of $6+9$ equations

$$\begin{aligned}
a^2 + b^2 + c^2 &= 1,\\
a'^2 + b'^2 + c'^2 &= 1,\\
a''^2 + b''^2 + c''^2 &= 1,\\
a^2 + a'^2 + a''^2 &= 1,\\
b^2 + b'^2 + b''^2 &= 1,\\
c^2 + c'^2 + c''^2 &= 1,
\end{aligned}$$

$$\begin{aligned}
b^2+c^2 &= a'^2+a''^2, & b'^2+c'^2 &= a''^2+a^2, & b''^2+c''^2 &= a^2+a'^2,\\
c^2+a^2 &= b'^2+b''^2, & c'^2+a'^2 &= b''^2+b^2, & c''^2+a''^2 &= b^2+b'^2,\\
a^2+b^2 &= c'^2+c''^2, & a'^2+b'^2 &= c''^2+c^2, & a''^2+b''^2 &= c^2+c'^2:
\end{aligned}$$

that is,

$$\left|\begin{array}{cccc} c^4 & c^4 & c^4 & c^4 \\ \hline 12 & +1 & +6 & -0 \\ 9 & +4 & +3 & -0 \\ 2 & +15 & +8 & -0 \\ \hline 12 & +9 & +2 & -0 \\ 1 & +4 & +15 & -0 \\ 6 & +3 & +8 & -0 \\ \hline 1 & +6 & -9 & -2 \\ 6 & +12 & -4 & -15 \\ 12 & +1 & -3 & -8 \\ \hline 4 & +3 & -2 & -12 \\ 3 & +9 & -15 & -1 \\ 9 & +4 & -8 & -6 \\ \hline 15 & +8 & -12 & -9 \\ 8 & +2 & -1 & -4 \\ 2 & +15 & -6 & -3 \end{array}\right| = 0\,;$$

and we have, between their products, a system of $6+9$ equations

$$\begin{aligned} a'a'' + b'b'' + c'c'' &= 0, \\ a''a + b''b + c''c &= 0, \\ aa' + bb' + cc' &= 0, \\ bc + b'c' + b''c'' &= 0, \\ ca + c'a' + c''a'' &= 0, \\ ab + a'b' + a''b'' &= 0, \end{aligned}$$

$$\begin{array}{ccccccc} a\,, & b\,, & c & = b'c'' - b''c', & c'a'' - c''a', & a'b'' - a''b', \\ a'\,, & b'\,, & c' & b''c - bc'', & c''a - ca'', & ab'' - a''b\,, \\ a''\,, & b''\,, & c'' & bc' - b'c\,, & ca' - c'a\,, & ab' - a'b\,: \end{array}$$

that is,

c^2	c^2	c^2	c^2	c^2	c^2
9	2	+ 4	15	− 3	8
2	12	− 15	1	− 8	6
12	9	− 1	4	+ 6	3
1	6	− 4	3	+ 15	8
6	12	+ 3	9	− 8	2
12	1	− 9	4	− 2	15
− 0	12	+ 4	8	+ 3	15
− 0	1	+ 3	2	+ 8	9
− 0	6	− 9	15	+ 2	4
− 0	9	− 15	6	+ 8	1
+ 0	4	− 8	12	− 6	2
− 0	3	− 12	15	− 1	2
− 0	2	+ 1	3	+ 4	6
+ 0	15	+ 6	9	− 3	12
+ 0	8	− 12	4	− 9	1

$= 0$;

each of the first set of 15 giving a homogeneous linear relation between four terms c^4; and each of the second set giving a homogeneous linear relation between three terms $c^2 . c^2$, formed with the 10 constants c. Thus the first equation is

$$c_{12}^4 + c_1^4 + c_6^4 - c_0^4 = 0;$$

and so for the other lines of the two diagrams.

79. I form in the two notations the following tables:—

TABLE OF THE 16 KUMMER HEXADS.

A	*A*	*A*	*A*	*A*	*B*	*B*	*B*	*B*	*C*	*C*	*C*	*D*	*D*	*E*	*A*
B	*C*	*D*	*E*	*F*	*C*	*D*	*E*	*F*	*D*	*E*	*F*	*E*	*F*	*F*	*B*
AB	*AC*	*AD*	*AE*	*AB*	*BC*	*BD*	*BE*	*AB*	*CD*	*CE*	*AC*	*DE*	*AD*	*AE*	*C*
CD	*BD*	*BC*	*BC*	*AC*	*AD*	*AC*	*AC*	*BC*	*AB*	*AB*	*BC*	*AB*	*BD*	*BE*	*D*
CE	*BE*	*BE*	*BD*	*AD*	*AE*	*AE*	*AD*	*BD*	*AE*	*AD*	*CD*	*AC*	*CD*	*CE*	*E*
DE	*DE*	*CE*	*CD*	*AE*	*DE*	*CE*	*CD*	*BE*	*BE*	*BD*	*CE*	*BC*	*DE*	*DE*	*F*

11	11	11	11	11	7	7	7	7	5	5	5	13	13	14	11
7	5	13	14	10	5	13	14	10	13	14	10	14	10	10	7
6	4	14	12	6	8	0	3	6	2	1	4	9	12	15	5
2	0	8	8	4	12	4	4	8	6	6	8	6	0	3	13
1	3	3	3	12	15	15	12	0	15	12	2	4	2	1	14
9	9	1	2	15	9	1	2	3	3	0	1	8	9	9	10

80. TABLE OF THE 120 PAIRS.

$A.B$	$A.C$	$A.D$	$A.E$	$A.F$	$B.C$	$B.D$	$B.E$	$B.F$	$C.D$	$C.E$	$C.F$	$D.E$	$D.F$	$E.F$
$AC.BC$	$AB.BC$	$AB.BD$	$AB.BE$	$BC.DE$	$AB.AC$	$AB.AD$	$AB.AE$	$AC.DE$	$AC.AD$	$AC.AE$	$AB.DE$	$AD.AE$	$AB.CE$	$AB.CD$
$AD.BD$	$AD.CD$	$AC.CD$	$AC.CE$	$BD.CE$	$BD.CD$	$BC.CD$	$BC.CE$	$AD.CE$	$BC.BD$	$BC.BE$	$AD.BE$	$BD.BE$	$AC.BE$	$AC.BD$
$AE.BE$	$AE.CE$	$AE.DE$	$AD.DE$	$BE.CD$	$BE.CE$	$BE.DE$	$BD.DE$	$AE.CD$	$CE.DE$	$CD.DE$	$AE.BD$	$CD.CE$	$AE.BC$	$AD.BC$
$F.AB$	$F.AC$	$F.AD$	$F.AE$	$B.AB$	$F.BC$	$F.BD$	$F.BE$	$A.AB$	$F.CD$	$F.CE$	$A.AC$	$F.DE$	$A.AD$	$A.AE$
$C.DE$	$B.DE$	$B.CE$	$B.CD$	$C.AC$	$A.DE$	$A.CE$	$A.CD$	$C.BC$	$A.BE$	$A.BD$	$B.BC$	$A.BC$	$B.BD$	$B.BE$
$D.CE$	$D.BE$	$C.BE$	$C.BD$	$D.AD$	$D.AE$	$C.AE$	$C.AD$	$D.BD$	$B.AE$	$B.AD$	$D.CD$	$B.AC$	$C.CD$	$C.CE$
$E.CD$	$E.BD$	$E.BC$	$D.BC$	$E.AE$	$E.AD$	$E.AC$	$D.AC$	$E.BE$	$E.AB$	$D.AB$	$E.CE$	$C.AB$	$E.DE$	$D.DE$

11.7	11.5	11.13	11.14	11.10	7.5	7.13	7.14	7.10	5.13	5.14	5.10	13.14	13.10	14.10
4.8	6.8	6.0	6.3	8.9	6.4	6.12	6.15	4.9	4.12	4.15	6.9	12.15	6.1	6.2
12.0	12.2	4.2	4.1	0.1	0.2	8.2	8.1	12.1	8.0	8.3	12.3	0.3	4.3	4.0
15.3	15.1	15.9	12.9	3.2	3.1	3.9	0.9	15.2	1.9	2.9	15.0	2.1	15.8	12.8
10.6	10.4	10.12	10.15	7.6	10.8	10.0	10.3	11.6	10.2	10.3	11.4	10.9	11.12	11.15
5.9	7.9	7.1	7.2	5.4	11.9	11.1	11.2	5.8	11.3	11.0	7.8	11.8	7.0	7.3
13.1	13.3	5.3	5.0	13.12	13.15	5.15	5.12	13.0	7.15	7.12	13.2	7.4	5.2	5.1
14.2	14.0	14.8	13.8	14.15	14.12	14.4	13.4	14.3	14.6	13.6	14.1	5.6	14.9	13.9

81. Table of the 60 Göpel tetrads.

$A.B.AE.BE$	$C.D.CE.DE$	$E.F.AB.CD$	$AC.BD.AD.BC$
$A.B.AD.BD$	$C.E.CD.DE$	$D.F.AB.CE$	$AC.BE.AE.BC$
$A.B.AC.BC$	$C.F.AB.DE$	$D.E.CD.CE$	$AD.BE.AE.BD$
$A.C.AE.CE$	$B.D.BE.DE$	$E.F.AC.BD$	$AB.CD.AD.BC$
$A.C.AD.CD$	$B.E.BD.DE$	$D.F.AC.BE$	$AB.CE.AE.BC$
$A.C.AB.BC$	$B.F.AC.DE$	$D.E.BD.BE$	$AD.CE.AE.CD$
$A.D.AE.DE$	$B.C.BE.CE$	$E.F.AD.BC$	$AB.CD.AC.BD$
$A.D.AC.CD$	$B.E.BC.CE$	$C.F.AD.BE$	$AB.DE.AE.BD$
$A.D.AB.BD$	$B.F.AD.CE$	$C.E.CD.DE$	$AC.DE.AE.CD$
$A.E.AD.DE$	$B.C.BD.CD$	$D.F.AE.BC$	$AB.CE.AC.BE$
$A.E.AC.CE$	$B.D.BC.CD$	$C.F.AE.BD$	$AB.DE.AD.BE$
$A.E.AB.BE$	$B.F.AE.CD$	$C.D.BC.BD$	$AC.DE.AD.CE$
$A.F.BC.DE$	$B.C.AB.AC$	$D.E.AD.AE$	$BD.CE.BE.CD$
$A.F.BD.CE$	$B.D.AB.AD$	$C.E.AC.AE$	$BC.DE.BE.CD$
$A.F.BE.CD$	$B.E.AB.AE$	$C.D.AC.AD$	$BC.DE.BD.CE$

=

11 7 15 3	5 13 1 9	14 10 6 2	4 0 12 8
11 7 12 0	5 14 2 9	13 10 6 1	4 3 15 8
11 7 4 8	5 10 6 9	13 14 2 1	12 3 15 0
11 5 15 1	7 13 3 9	14 10 4 0	6 2 12 8
11 5 12 2	7 14 0 9	13 10 4 3	6 1 15 8
11 5 6 8	7 10 4 9	13 14 0 3	12 1 15 2
11 13 15 9	7 5 3 1	14 10 12 8	6 2 4 0
11 13 4 2	7 14 8 1	5 10 12 3	6 9 15 0
11 13 6 0	7 10 12 1	5 14 2 9	4 9 15 2
11 14 12 9	7 5 0 2	13 10 15 8	6 1 4 3
11 14 4 1	7 13 8 2	5 10 15 0	6 9 12 3
11 14 6 3	7 10 15 2	5 13 8 0	4 9 12 1
11 10 8 9	7 5 6 4	13 14 12 15	0 1 3 2
11 10 0 1	7 13 6 12	5 14 4 15	8 9 3 2
11 10 3 2	7 14 6 15	5 13 4 12	8 9 0 1

The product-theorem, and its results.

82. The product-theorem was

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+u').\vartheta\begin{pmatrix}\alpha', & \beta'\\ \gamma', & \delta'\end{pmatrix}(u-u')$$

$$=\Sigma\Theta\begin{matrix}\frac{1}{2}(\alpha+\alpha')+p, & \frac{1}{2}(\beta+\beta')+q\\ \gamma+\gamma' & \delta+\delta\end{matrix}(2u).\Theta\begin{matrix}\frac{1}{2}(\alpha-\alpha')+p, & \frac{1}{2}(\beta-\beta')+q\\ \gamma-\gamma' & \delta-\delta'\end{matrix}(2u'),$$

where only one argument is exhibited, viz. $u+u'$, $u-u'$, $2u$, $2u'$ are written in place of $(u+u', v+v')$, $(u-u', v-v')$, $(2u, 2v)$, $(2u', 2v')$ respectively. The expression on the right-hand side is always a sum of four terms, corresponding to the values (0, 0), (1, 0), (0, 1), and (1, 1) of (p, q). For the development of the results it was found convenient to use the following auxiliary diagram.

Upper half of characteristic.

α	β	α'	β'	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')+1$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')+1$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta')+1$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta')+1$	$\frac{1}{2}(\alpha+\alpha')+1$	$\frac{1}{2}(\beta+\beta')+1$	$\frac{1}{2}(\alpha-\alpha')+1$	$\frac{1}{2}(\beta-\beta')+1$
0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1
1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1
0	1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$
1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$	1
1	0	1	0	1	0	0	0	0	0	1	0	1	1	0	1	0	1	1	1
0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
1	1	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$
0	0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$
1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
0	1	0	1	0	1	0	0	1	1	1	0	0	0	0	1	1	0	1	1
1	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1
0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	1	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$
0	1	1	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	1
1	1	1	1	1	1	0	0	0	1	1	0	1	0	0	1	0	0	1	1

Lower half of characteristic.

γ	δ	γ'	δ'	$\gamma'+\gamma$	$\delta'+\delta$	$\gamma'-\gamma$	$\delta'-\delta$	$\gamma'+\gamma$	$\delta'+\delta$	$\gamma'-\gamma$	$\delta'-\delta$	$\gamma'+\gamma$	$\delta'+\delta$	$\gamma'-\gamma$	$\delta'-\delta$	$\gamma'+\gamma$	$\delta'+\delta$	$\gamma'-\gamma$	$\delta'-\delta$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0
1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0
0	1	1	0	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1
1	1	1	0	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0	1
0	0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1
1	0	0	1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1
0	1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0
1	1	0	1	1	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0
0	0	1	1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
1	0	1	1	2	1	0	−1	2	1	0	−1	2	1	0	−1	2	1	0	−1
0	1	1	1	1	2	−1	0	0	2	−1	0	0	2	−1	0	1	2	−1	0
1	1	1	1	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0

83. The upper characters of the Θ's have thus the values 0, 1, $\frac{1}{2}$, $\frac{3}{2}$; the lower characters are originally 2, 1, 0, or -1, and these have when necessary to be, by the addition or subtraction of 2, reduced to 0 or 1; the effect of this change is either to leave the Θ unaltered, or to multiply it by -1 or $\pm i$, as follows:

$$\Theta^{0}_{\gamma\pm 2} = \Theta^{0}_{\gamma}, \qquad \Theta^{\frac{1}{2}}_{\gamma+2} = i\Theta^{\frac{1}{2}}_{\gamma}, \quad \Theta^{\frac{1}{2}}_{\gamma-2} = -i\Theta^{\frac{1}{2}}_{\gamma},$$

$$\Theta^{1}_{\gamma\pm 2} = -\Theta^{1}_{\gamma}, \qquad \Theta^{\frac{3}{2}}_{\gamma+2} = -i\Theta^{\frac{3}{2}}_{\gamma}, \quad \Theta^{\frac{3}{2}}_{\gamma+2} = i\Theta^{\frac{3}{2}}_{\gamma},$$

where only the first column of characters is shown, but the same rule applies to the second column; and where we must of course combine the multipliers corresponding to the first and second columns respectively: for instance

$$\Theta^{\frac{3}{2}\ \frac{1}{2}}_{\gamma+2\ \delta+2} = (-i\,.\,-i\Theta^{\frac{3}{2}\ \frac{1}{2}}_{\gamma\ \delta}) = -\Theta^{\frac{3}{2}\ \frac{1}{2}}_{\gamma\ \delta}.$$

Thus taking the tenth line of the upper half, and the fifth line of the lower half, we have

10	01	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{3}{2}$	$\frac{3}{2}$ $\frac{1}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	$\frac{1}{2}$ $\frac{3}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{1}{2}$
00	10	1 0 −1 0	1 0 −1 0	1 0 −1 0	1 0 −1 0

giving the value of

$$\vartheta\begin{smallmatrix}1&0\\0&0\end{smallmatrix}(u+u').\vartheta\begin{smallmatrix}0&1\\1&0\end{smallmatrix}(u-u'):$$

viz. this is

$$\begin{aligned}
= \quad & \Theta\begin{smallmatrix}\frac{1}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(2u).\Theta\begin{smallmatrix}\frac{1}{2}&\frac{3}{2}\\-1&0\end{smallmatrix}(2u') = \quad i\Theta\begin{smallmatrix}\frac{1}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(2u).\Theta\begin{smallmatrix}\frac{1}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(2u') \\
& +\Theta\begin{smallmatrix}\frac{3}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{3}{2}&\frac{3}{2}\\-1&0\end{smallmatrix}(\,„\,) \quad -i\Theta\begin{smallmatrix}\frac{3}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{3}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(\,„\,) \\
& +\Theta\begin{smallmatrix}\frac{1}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{1}{2}&\frac{1}{2}\\-1&0\end{smallmatrix}(\,„\,) \quad +i\Theta\begin{smallmatrix}\frac{1}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{1}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(\,„\,) \\
& +\Theta\begin{smallmatrix}\frac{3}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{3}{2}&\frac{1}{2}\\-1&0\end{smallmatrix}(\,„\,) \quad -i\Theta\begin{smallmatrix}\frac{3}{2}&\frac{3}{2}\\1&0\end{smallmatrix}(\,„\,).\Theta\begin{smallmatrix}\frac{1}{2}&\frac{1}{2}\\1&0\end{smallmatrix}(\,„\,),
\end{aligned}$$

where the first column is the value given directly by the diagram: it is then reduced to that given by the second column.

84. But instead of the Θ's, we introduce single letters (X, Y, Z, W), (E, F, G, H), (I, J, K, L), (M, N, P, Q), with the suffixes (0, 1, 2, 3), in all 64 symbols, thus

Θ	00	10	01	11	$(2u) =$		X	Y	Z	W
00						0				
10						1				
01						2				
11						3				

that is,

$$\Theta\begin{smallmatrix}00\\00\end{smallmatrix}(2u) = X, \quad \Theta\begin{smallmatrix}10\\00\end{smallmatrix} = Y, \text{ \&c.},$$

$$\Theta\begin{smallmatrix}00\\10\end{smallmatrix}(2u) = X_1, \ldots$$

$$\vdots$$

Θ	$\frac{1}{2}0$	$\frac{1}{2}1$	$\frac{3}{2}0$	$\frac{3}{2}1$	$(2u) =$		E	F	G	H
00						0				
10						1				
01						2				
11						3				

that is,

$$\Theta\begin{smallmatrix}\frac{1}{2}0\\00\end{smallmatrix}(2u) = E, \text{ \&c.},$$

Θ	$0\frac{1}{2}$	$1\frac{1}{2}$	$0\frac{3}{2}$	$1\frac{3}{2}(2u)=$		I	J	K	L
00					0				
10					1				
01					2				
11					3				

The functions of $(2u')$ are denoted in like manner by accented letters

$$\Theta\begin{matrix}00\\00\end{matrix}(2u')=X',\ \&c.,$$

Θ	$\frac{11}{22}$	$\frac{31}{22}$	$\frac{13}{22}$	$\frac{33}{22}(2u)=$		M	N	P	Q
00					0				
10					1				
01					2				
11					3				

85. To simplify the expression of the results, instead of in each case writing down the suffixes, I have indicated them by means of the column headed "Suff."

Thus

$$|\,8-0\,| \qquad \vartheta\begin{matrix}01\\01\end{matrix}u+u'.\vartheta\begin{matrix}00\\00\end{matrix}u-u'=XX'+YY'+ZZ'+WW' \qquad \overset{\text{Suff.}}{|\,2\,|}$$

means that the equation is to be read

$$=X_2X_2'+Y_2Y_2'+Z_2Z_2'+W_2W_2'.$$

It is hardly necessary to mention that the $|\,8-0\,|$ of the left-hand column shows the current numbers of the theta-functions; viz. the left-hand side of the equation is

$$\vartheta_8(u+u').\vartheta_0(u-u').$$

And by a preceding remark the single arguments $u+u'$ and $u-u'$ are written in place of $(u+u',\ v+v')$ and $(u-u',\ v-v')$ respectively.

The 256 equations now are

86. First set, 64 equations.

				Suffixes.
$0-0$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u-u'$	$=$	$XX'+YY'+ZZ'+WW'$	0
$4-0$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$XX'+YY'+ZZ'+WW'$	1
$8-0$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$XX'+YY'+ZZ'+WW'$	2
$12-0$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$XX'+YY'+ZZ'+WW'$	3
$0-4$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\10\end{smallmatrix}u-u'$	$=$	$XX'-YY'+ZZ'-WW'$	1
$4-4$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$XX'-YY'+ZZ'-WW'$	0
$8-4$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$XX'-YY'+ZZ'-WW'$	3
$12-4$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$XX'-YY'+ZZ'-WW'$	2
$0-8$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\01\end{smallmatrix}u-u'$	$=$	$XX'+YY'-ZZ'-WW'$	2
$4-8$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$XX'+YY'-ZZ'-WW'$	3
$8-8$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$XX'+YY'-ZZ'-WW'$	0
$12-8$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$XX'+YY'-ZZ'-WW'$	1
$0-12$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\11\end{smallmatrix}u-u'$	$=$	$XX'-YY'-ZZ'+WW'$	3
$4-12$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$XX'-YY'-ZZ'+WW'$	2
$8-12$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$XX'-YY'-ZZ'+WW'$	1
$12-12$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$XX'-YY'-ZZ'+WW'$	0

First set, 64 equations (*continued*).

			Suffixes.
1 – 1	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u-u'$	$= YX' + XY' + WZ' + ZW'$	0
5 – 1	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= YX' + XY' + WZ' + ZW'$	1
9 – 1	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= YX' + XY' + WZ' + ZW'$	2
13 – 1	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= YX' + XY' + WZ' + ZW'$	3
1 – 5	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\10\end{smallmatrix}u-u'$	$= YX' - XY' + WZ' - ZW'$	1
5 – 5	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= - YX' + XY' - WZ' + ZW'$	0
9 – 5	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= YX' - XY' + WZ' - ZW'$	3
13 – 5	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= - YX' + XY' - WZ' + ZW'$	2
1 – 9	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\01\end{smallmatrix}u-u'$	$= YX' + XY' - WZ' - ZW'$	2
5 – 9	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= YX' + XY' - WZ' - ZW'$	3
9 – 9	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= YX' + XY' - WZ' - ZW'$	0
13 – 9	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= YX' + XY' - WZ' - ZW'$	1
1 – 13	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\11\end{smallmatrix}u-u'$	$= YX' - XY' - WZ' + ZW'$	3
5 – 13	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= - YX' + XY' + WZ' - ZW'$	2
9 – 13	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= YX' - XY' - WZ' + ZW'$	1
13 – 13	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= - YX' + XY' + WZ' - ZW'$	0

First set, 64 equations (*continued*).

				Suffixes.
$2-2$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u-u'$	$=$	$ZX' + WY' + XZ' + YW'$	0
$6-2$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= ZX' + WY' + XZ' + YW'$	1
$10-2$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= ZX' + WY' + XZ' + YW'$	2
$14-2$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= ZX' + WY' + XZ' + YW'$	3
$2-6$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\10\end{smallmatrix}u-u'$	$=$	$ZX' - WY' + XZ' - YW'$	1
$6-6$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= ZX' - WY' + XZ' - YW'$	0
$10-6$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= ZX' - WY' + XZ' - YW'$	3
$14-6$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= ZX' - WY' + XZ' - YW'$	2
$2-10$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\01\end{smallmatrix}u-u'$	$=$	$ZX' + WY' - XZ' - YW'$	2
$6-10$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= ZX' + WY' - XZ' - YW'$	3
$10-10$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= -ZX' - WY' + XZ' + YW'$	0
$14-10$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= -ZX' - WY' + XZ' + YW'$	1
$2-14$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\11\end{smallmatrix}u-u'$	$=$	$ZX' - WY' - XZ' + YW'$	3
$6-14$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= ZX' - WY' - XZ' + YW'$	2
$10-14$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -ZX' + WY' + XZ' - YW'$	1
$14-14$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -ZX' + WY' + XZ' - YW'$	0

First set, 64 equations (*concluded*).

								Suffixes.
3 – 3	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u-u'$	=	WX'	$+ZY'$	$+YZ'$	$+XW'$		0
7 – 3	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	=	WX'	$+ZY'$	$+YZ'$	$+XW'$		1
11 – 3	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	=	WX'	$+ZY'$	$+YZ'$	$+XW'$		2
15 – 3	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	=	WX'	$+ZY'$	$+YZ'$	$+XW'$		3
3 – 7	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\10\end{smallmatrix}u-u'$	=	WX'	$-ZY'$	$+YZ'$	$-XW'$		1
7 – 7	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	=	$-WX'$	$+ZY'$	$-YZ'$	$+XW'$		0
11 – 7	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	=	WX'	$-ZY'$	$+YZ'$	$-XW'$		3
15 – 7	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	=	$-WX'$	$+ZY'$	$-YZ'$	$+XW'$		2
3 – 11	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\01\end{smallmatrix}u-u'$	=	WX'	$+ZY'$	$-YZ'$	$-XW'$		2
7 – 11	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	=	WX'	$+ZY'$	$-YZ'$	$-XW'$		3
11 – 11	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	=	$-WX'$	$-ZY'$	$+YZ'$	$+XW'$		0
15 – 11	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	=	$-WX'$	$-ZY'$	$+YZ'$	$+XW'$		1
3 – 15	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\11\end{smallmatrix}u-u'$	=	WX'	$-ZY'$	$-YZ'$	$+XW'$		3
7 – 15	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	=	$-WX'$	$+ZY'$	$+YZ'$	$-XW'$		2
11 – 15	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	=	$-WX'$	$+ZY'$	$+YZ'$	$-XW'$		1
15 – 15	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	=	WX'	$-ZY'$	$-YZ'$	$+XW'$		0

87. Second set, 64 equations.

				Suffixes.
1 − 0	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u-u'$	$=$	$EE' + GG' + FF' + HH'$	0
5 − 0	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$EE' + GG' + FF' + HH'$	1
9 − 0	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$EE' + GG' + FF' + HH'$	2
13 − 0	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$=$	$EE' + GG' + FF' + HH'$	3
1 − 4	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}00\\10\end{smallmatrix}u-u'$	$=$	$-iEE' + iGG' - iFF' + iHH'$	1
5 − 4	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$iEE' - iGG' + iFF' - iHH'$	0
9 − 4	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$-iEE' + iGG' - iFF' + iHH'$	3
13 − 4	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$=$	$iEE' - iGG' + iFF' - iHH'$	2
1 − 8	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}00\\01\end{smallmatrix}u-u'$	$=$	$EE' + GG' - FF' - HH'$	2
5 − 8	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$EE' + GG' - FF' - HH'$	3
9 − 8	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$EE' + GG' - FF' - HH'$	0
13 − 8	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$=$	$EE' + GG' - FF' - HH'$	1
1 − 12	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}00\\11\end{smallmatrix}u-u'$	$=$	$-iEE' + iGG' + iFF' - iHH'$	3
5 − 12	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$iEE' - iGG' - iFF' + iHH'$	2
9 − 12	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$-iEE' + iGG' + iFF' - iHH'$	1
13 − 12	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$=$	$iEE' - iGG' - iFF' + iHH'$	0

Second set, 64 equations (*continued*).

				Suffixes.
0 − 1	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u-u'$	$=$	$EG' + GE' + FH' + HF'$	0
4 − 1	$\begin{smallmatrix}00\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$EG' + GE' + FH' + HF'$	1
8 − 1	$\begin{smallmatrix}00\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$EG' + GE' + FH' + HF'$	2
12 − 1	$\begin{smallmatrix}00\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$EG' + GE' + FH' + HF'$	3
0 − 5	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\10\end{smallmatrix}u-u'$	$=$	$iEG' - iGE' + iFH' - iHF'$	1
4 − 5	$\begin{smallmatrix}00\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iEG' - iGE' + iFH' - iHF'$	0
8 − 5	$\begin{smallmatrix}00\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iEG' - iGE' + iFH' - iHF'$	3
12 − 5	$\begin{smallmatrix}00\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iEG' - iGE' + iFH' - iHF'$	2
0 − 9	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\01\end{smallmatrix}u-u'$	$=$	$EG' + GE' - FH' - HF'$	2
4 − 9	$\begin{smallmatrix}00\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$EG' + GE' - FH' - HF'$	3
8 − 9	$\begin{smallmatrix}00\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$EG' + GE' - FH' - HF'$	0
12 − 9	$\begin{smallmatrix}00\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$EG' + GE' - FH' - HF'$	1
0 − 13	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\11\end{smallmatrix}u-u'$	$=$	$iEG' - iGE' - iFH' + iHF'$	3
4 − 13	$\begin{smallmatrix}00\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$iEG' - iGE' - iFH' + iHF'$	2
8 − 13	$\begin{smallmatrix}00\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$iEG' - iGE' - iFH' + iHF'$	1
12 − 13	$\begin{smallmatrix}00\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$iEG' - iGE' - iFH' + iHF'$	0

Second set, 64 equations (*continued*).

			Suffixes.
3 − 2	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u-u'$	$= FE' + HG' + EF' + GH'$	0
7 − 2	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= FE' + HG' + EF' + GH'$	1
11 − 2	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= FE' + HG' + EF' + GH'$	2
15 − 2	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= FE' + HG' + EF' + GH'$	3
3 − 6	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\10\end{smallmatrix}u-u'$	$= -iFE' + iHG' - iEF' + iGH'$	1
7 − 6	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= iFE' - iHG' + iEF' - iGH'$	0
11 − 6	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= -iFE' + iHG' - iEF' + iGH'$	3
15 − 6	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= iFE' - iHG' + iEF' - iGH'$	2
3 − 10	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\01\end{smallmatrix}u-u'$	$= FE' + HG' - EF' - GH'$	2
7 − 10	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= FE' + HG' - EF' - GH'$	3
11 − 10	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= -FE' - HG' + EF' + GH'$	0
15 − 10	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= -FE' - HG' + EF' + GH'$	1
3 − 14	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\11\end{smallmatrix}u-u'$	$= -iFE' + iHG' + iEF' - iGH'$	3
7 − 14	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= iFE' - iHG' - iEF' + iGH'$	2
11 − 14	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= iFE' - iHG' - iEF' + iGH'$	1
15 − 14	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -iFE' + iHG' + iEF' - iGH'$	0

Second set, 64 equations (*concluded*).

				Suffixes.
2 – 3	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u-u'$	$=$	$FG' + HE' + EH' + GF'$	0
6 – 3	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$FG' + HE' + EH' + GF'$	1
10 – 3	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$FG' + HE' + EH' + GF'$	2
14 – 3	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$FG' + HE' + EH' + GF'$	3
2 – 7	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\10\end{smallmatrix}u-u'$	$=$	$iFG' - iHE' + iEH' - iGF'$	1
6 – 7	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=$	$iFG' - iHE' + iEH' - iGF'$	0
10 – 7	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=$	$iFG' - iHE' + iEH' - iGF'$	3
14 – 7	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=$	$iFG' - iHE' + iEH' - iGF'$	2
2 – 11	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\01\end{smallmatrix}u-u'$	$=$	$FG' + HE' - EH' - GF'$	2
6 – 11	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=$	$FG' + HE' - EH' - GF'$	3
10 – 11	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=-$	$FG' - HE' + EH' + GF'$	0
14 – 11	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=-$	$FG' - HE' + EH' + GF'$	1
2 – 15	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}11\\11\end{smallmatrix}u-u'$	$=$	$iFG' - iHE' - iEH' + iGF'$	3
6 – 15	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=$	$iFG' - iHE' - iEH' + iGF'$	2
10 – 15	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=-$	$iFG' + iHE' + iEH' - iGF'$	1
14 – 15	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=-$	$iFG' + iHE' + iEH' - iGF'$	0

88. Third set, 64 equations.

				Suffixes.
$2-0$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u' \,.\, \vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u-u'$	$=$	$II' + JJ' + KK' + LL'$	0
$6-0$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= II' + JJ' + KK' + LL'$	1
$10-0$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= II' + JJ' + KK' + LL'$	2
$14-0$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\00\end{smallmatrix}$	$= II' + JJ' + KK' + LL'$	3
$2-4$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u' \,.\, \vartheta\begin{smallmatrix}00\\10\end{smallmatrix}u-u'$	$=$	$II' - JJ' + KK' - LL'$	1
$6-4$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= II' - JJ' + KK' - LL'$	0
$10-4$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= II' - JJ' + KK' - LL'$	3
$14-4$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$= II' - JJ' + KK' - LL'$	2
$2-8$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u' \,.\, \vartheta\begin{smallmatrix}00\\01\end{smallmatrix}u-u'$	$=$	$-iII' - iJJ' + iKK' + iLL'$	2
$6-8$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= -iII' - iJJ' + iKK' + iLL'$	3
$10-8$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= iII' + iJJ' - iKK' - iLL'$	0
$14-8$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$= iII' + iJJ' - iKK' - iLL'$	1
$2-12$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u' \,.\, \vartheta\begin{smallmatrix}00\\11\end{smallmatrix}u-u'$	$=$	$-iII' + iJJ' + iKK' - iLL'$	3
$6-12$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= -iII' + iJJ' + iKK' - iLL'$	2
$10-12$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= iII' - iJJ' - iKK' + iLL'$	1
$14-12$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$= iII' - iJJ' - iKK' + iLL'$	0

Third set, 64 equations (*continued*).

			Suffixes.
$3-1$	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u-u'$	$= JI' + IJ' + LK' + KL'$	0
$7-1$	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= JI' + IJ' + LK' + KL'$	1
$11-1$	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= JI' + IJ' + LK' + KL'$	2
$15-1$	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$= JI' + IJ' + LK' + KL'$	3
$3-5$	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\10\end{smallmatrix}u-u'$	$= JI' - IJ' + LK' - KL'$	1
$7-5$	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= - JI' + IJ' - LK' + KL'$	0
$11-5$	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= JI' - IJ' + LK' - KL'$	3
$15-5$	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$= - JI' + IJ' - LK' + KL'$	2
$3-9$	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\01\end{smallmatrix}u-u'$	$= -iJI' - iIJ' + iLK' + iKL'$	2
$7-9$	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= - iJI' - iIJ' + iLK' + iKL'$	3
$11-9$	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= iJI' + iIJ' - iLK' - iKL'$	0
$15-9$	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$= iJI' + iIJ' - iLK' - iKL'$	1
$3-13$	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}10\\11\end{smallmatrix}u-u'$	$= -iJI' + iIJ' + iLK' - iKL'$	3
$7-13$	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= iJI' - iIJ' - iLK' + iKL'$	2
$11-13$	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= iJI' - iIJ' - iLK' + iKL'$	1
$15-13$	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$= - iJI' + iIJ' + iLK' - iKL'$	0

Third set, 64 equations (*continued*).

				Suffixes.
$0-2$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u-u'$	$=$	$IK' + JL' + KI' + LJ'$	0
$4-2$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$IK' + JL' + KI' + LJ'$	1
$8-2$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$IK' + JL' + KI' + LJ'$	2
$12-2$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}01\\00\end{smallmatrix}$	$=$	$IK' + JL' + KI' + LJ'$	3
$0-6$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\10\end{smallmatrix}u-u'$	$=$	$IK' - JL' + KI' - LJ'$	1
$4-6$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$IK' - JL' + KI' - LJ'$	0
$8-6$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$IK' - JL' + KI' - LJ'$	3
$12-6$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}01\\10\end{smallmatrix}$	$=$	$IK' - JL' + KI' - LJ'$	2
$0-10$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\01\end{smallmatrix}u-u'$	$=$	$iIK' + iJL' - iKI' - iLJ'$	2
$4-10$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$iIK' + iJL' - iKI' - iLJ'$	3
$8-10$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$iIK' + iJL' - iKI' - iLJ'$	0
$12-10$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}01\\01\end{smallmatrix}$	$=$	$iIK' + iJL' - iKI' - iLJ'$	1
$0-14$	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'.\vartheta\begin{smallmatrix}01\\11\end{smallmatrix}u-u'$	$=$	$iIK' - iJL' - iKI' + iLJ'$	3
$4-14$	$\begin{smallmatrix}00\\10\end{smallmatrix}$ $\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$iIK' - iJL' - iKI' + iLJ'$	2
$8-14$	$\begin{smallmatrix}00\\01\end{smallmatrix}$ $\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$iIK' - iJL' - iKI' + iLJ'$	1
$12-14$	$\begin{smallmatrix}00\\11\end{smallmatrix}$ $\begin{smallmatrix}01\\11\end{smallmatrix}$	$=$	$iIK' - iJL' - iKI' + iLJ'$	0

Third set, 64 equations (*concluded*).

				Suffixes.
1 – 3	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u-u'$	$=$	$JK' + IL' + LI' + KJ'$	0
5 – 3	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$JK' + IL' + LI' + KJ'$	1
9 – 3	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$JK' + IL' + LI' + KJ'$	2
13 – 3	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\00\end{smallmatrix}$	$=$	$JK' + IL' + LI' + KJ'$	3
1 – 7	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\10\end{smallmatrix}u-u'$	$=$	$JK' - IL' + LI' - KJ'$	1
5 – 7	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=-$	$JK' + IL' - LI' + KJ'$	0
9 – 7	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=$	$JK' - IL' + LI' - KJ'$	3
13 – 7	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\10\end{smallmatrix}$	$=-$	$JK' + IL' - LI' + KJ'$	2
1 – 11	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\01\end{smallmatrix}u-u'$	$=$	$iJK' + iIL' - iLI' - iKJ'$	2
5 – 11	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=$	$iJK' + iIL' - iLI' - iKJ'$	3
9 – 11	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=$	$iJK' + iIL' - iLI' - iKJ'$	0
13 – 11	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\01\end{smallmatrix}$	$=$	$iJK' + iIL' - iLI' - iKJ'$	1
1 – 15	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\11\end{smallmatrix}u-u'$	$=$	$iJK' - iIL' - iLI' + iKJ'$	3
5 – 15	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=-$	$iJK' + iIL' + iLI' - iKJ'$	2
9 – 15	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=$	$iJK' - iIL' - iLI' + iKJ'$	1
13 – 15	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}11\\11\end{smallmatrix}$	$=-$	$iJK' + iIL' + iLI' - iKJ'$	0

89. Fourth set, 64 equations.

			Suffixes.
3 − 0	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u-u'$	$= MM' + NN' + PP' + QQ'$	0
7 − 0	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$= MM' + NN' + PP' + QQ'$	1
11 − 0	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$= MM' + NN' + PP' + QQ'$	2
15 − 0	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\00\end{smallmatrix}$	$= MM' + NN' + PP' + QQ'$	3
3 − 4	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\10\end{smallmatrix}u-u'$	$= -iMM' + iNN' - iPP' + iQQ'$	1
7 − 4	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$= +iMM' - iNN' + iPP' - iQQ'$	0
11 − 4	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$= -iMM' + iNN' - iPP' + iQQ'$	3
15 − 4	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\10\end{smallmatrix}$	$= +iMM' - iNN' + iPP' - iQQ'$	2
3 − 8	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\01\end{smallmatrix}u-u'$	$= -iMM' - iNN' + iPP' + iQQ'$	2
7 − 8	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$= -iMM' - iNN' + iPP' + iQQ'$	3
11 − 8	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$= iMM' + iNN' - iPP' - iQQ'$	0
15 − 8	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\01\end{smallmatrix}$	$= iMM' + iNN' - iPP' - iQQ'$	1
3 − 12	$\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}00\\11\end{smallmatrix}u-u'$	$= -MM' + NN' + PP' - QQ'$	3
7 − 12	$\begin{smallmatrix}11\\10\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$= +MM' - NN' - PP' + QQ'$	2
11 − 12	$\begin{smallmatrix}11\\01\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$= +MM' - NN' - PP' + QQ'$	1
15 − 12	$\begin{smallmatrix}11\\11\end{smallmatrix}\quad\begin{smallmatrix}00\\11\end{smallmatrix}$	$= -MM' + NN' + PP' - QQ'$	0

Fourth set, 64 equations (*continued*).

				Suffixes.
2 − 1	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u-u'$	$=$	$MN' + NM' + PQ' + QP'$	0
6 − 1	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$MN' + NM' + PQ' + QP'$	1
10 − 1	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$MN' + NM' + PQ' + QP'$	2
14 − 1	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$MN' + NM' + PQ' + QP'$	3
2 − 5	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\10\end{smallmatrix}u-u'$	$=$	$iMN' - iNM' + iPQ' - iQP'$	1
6 − 5	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iMN' - iNM' + iPQ' - iQP'$	0
10 − 5	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iMN' - iNM' + iPQ' - iQP'$	3
14 − 5	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iMN' - iNM' + iPQ' - iQP'$	2
2 − 9	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\01\end{smallmatrix}u-u'$	$=$	$-iMN' - iNM' + iPQ' + iQP'$	2
6 − 9	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$-iMN' - iNM' + iPQ' + iQP'$	3
10 − 9	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$iMN' + iNM' - iPQ' - iQP'$	0
14 − 9	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$iMN' + iNM' - iPQ' - iQP'$	1
2 − 13	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}10\\11\end{smallmatrix}u-u'$	$=$	$MN' - NM' - PQ' + QP'$	3
6 − 13	$\begin{smallmatrix}01\\10\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$MN' - NM' - PQ' + QP'$	2
10 − 13	$\begin{smallmatrix}01\\01\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$-MN' + NM' + PQ' - QP'$	1
14 − 13	$\begin{smallmatrix}01\\11\end{smallmatrix}\quad\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$-MN' + NM' + PQ' - QP'$	0

Fourth set, 64 equations (*continued*).

			Suffixes.
1 – 2	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}u-u'$	$= MP' + NQ' + PM' + QN'$	0
5 – 2	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= MP' + NQ' + PM' + QN'$	1
9 – 2	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= MP' + NQ' + PM' + QN'$	2
13 – 2	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\00\end{smallmatrix}$	$= MP' + NQ' + PM' + QN'$	3
1 – 6	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\10\end{smallmatrix}u-u'$	$= -iMP' + iNQ' - iPM' + iQN'$	1
5 – 6	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= iMP' - iNQ' + iPM' - iQN'$	0
9 – 6	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= -iMP' + iNQ' - iPM' + iQN'$	3
13 – 6	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\10\end{smallmatrix}$	$= iMP' - iNQ' + iPM' - iQN'$	2
1 – 10	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\01\end{smallmatrix}u-u'$	$= iMP' + iNQ' - iPM' - iQN'$	2
5 – 10	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= iMP' + iNQ' - iPM' - iQN'$	3
9 – 10	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= iMP' + iNQ' - iPM' - iQN'$	0
13 – 10	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\01\end{smallmatrix}$	$= iMP' + iNQ' - iPM' - iQN'$	1
1 – 14	$\vartheta\begin{smallmatrix}10\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}01\\11\end{smallmatrix}u-u'$	$= MP' - NQ' - PM' + QN'$	3
5 – 14	$\begin{smallmatrix}10\\10\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -MP' + NQ' + PM' - QN'$	2
9 – 14	$\begin{smallmatrix}10\\01\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= MP' - NQ' - PM' + QN'$	1
13 – 14	$\begin{smallmatrix}10\\11\end{smallmatrix}\quad\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -MP' + NQ' + PM' - QN'$	0

Fourth set, 64 equations (*concluded*).

				Suffixes.
0 – 3	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\00\end{smallmatrix}u-u'$	$=$	$MQ' + NP' + PN' + QM'$	0
4 – 3	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}11\\00\end{smallmatrix}$	$= MQ' + NP' + PN' + QM'$	1
8 – 3	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}11\\00\end{smallmatrix}$	$= MQ' + NP' + PN' + QM'$	2
12 – 3	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}11\\00\end{smallmatrix}$	$= MQ' + NP' + PN' + QM'$	3
0 – 7	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\10\end{smallmatrix}u-u'$	$=$	$iMQ' - iNP' + iPN' - iQM'$	1
4 – 7	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$= iMQ' - iNP' + iPN' - iQM'$	0
8 – 7	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$= iMQ' - iNP' + iPN' - iQM'$	3
12 – 7	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$= iMQ' - iNP' + iPN' - iQM'$	2
0 – 11	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\01\end{smallmatrix}u-u'$	$=$	$iMQ' + iNP' - iPN' - iQM'$	2
4 – 11	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$= iMQ' + iNP' - iPN' - iQM'$	3
8 – 11	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$= iMQ' + iNP' - iPN' - iQM'$	0
12 – 11	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$= iMQ' + iNP' - iPN' - iQM'$	1
0 – 15	$\vartheta\begin{smallmatrix}00\\00\end{smallmatrix}u+u'\,.\,\vartheta\begin{smallmatrix}11\\11\end{smallmatrix}u-u'$	$=$	$- MQ' + NP' + PN' - QM'$	3
4 – 15	$\begin{smallmatrix}00\\10\end{smallmatrix}$	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$= - MQ' + NP' + PN' - QM'$	2
8 – 15	$\begin{smallmatrix}00\\01\end{smallmatrix}$	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$= - MQ' + NP' + PN' - QM'$	1
12 – 15	$\begin{smallmatrix}00\\11\end{smallmatrix}$	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$= - MQ' + NP' + PN' - QM'$	0

90. I re-arrange these in sets of 16 equations, the equations of the first or square-set of 16 being taken as they stand, but those of the other sets being combined in pairs by addition and subtraction as will be seen. And I now drop altogether the characteristics, retaining only the current numbers: thus, in the set of equations next written down, the first equation is

$$\vartheta_0(u+u')\,\vartheta_0(u-u') = XX' + YY' + ZZ' + WW':$$

in the second set, the first equation is

$$\tfrac{1}{2}\{\vartheta_4(u+u')\,\vartheta_0(u-u') + \vartheta_0(u+u')\,\vartheta_4(u-u')\} = X_1X_1' + Z_1Z_1',$$

and so in other cases.

First or square-set of 16.

$u+u'$ ϑ	$u-u'$ ϑ	(Suffixes 0.)
0	0	$= XX' + YY' + ZZ' + WW'$
4	4	$= XX' - YY' + ZZ' - WW'$
8	8	$= XX' + YY' - ZZ' - WW'$
12	12	$= XX' - YY' - ZZ' + WW'$
1	1	$= YX' + XY' + WZ' + ZW'$
5	5	$= -YX' + XY' - WZ' + ZW'$
9	9	$= YX' + XY' - WZ' - ZW'$
13	13	$= -YX' + XY' + WZ' - ZW'$
2	2	$= ZX' + WY' + XZ' + YW'$
6	6	$= ZX' - WY' + XZ' - YW'$
10	10	$= -ZX' - WY' + XZ' + YW'$
14	14	$= -ZX' + WY' + XZ' - YW'$
3	3	$= WX' + ZY' + YZ' + XW'$
7	7	$= -WX' + ZY' - YZ' + XW'$
11	11	$= -WX' - ZY' + YZ' + XW'$
15	15	$= WX' - ZY' - YZ' + XW'$

91. Second set of 16.

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$	(Suffixes 1.)
4 0 0 4	$= XX' + ZZ'$
12 8 8 12	$XX' - ZZ'$
5 1 1 5	$YX' + WZ'$
13 9 9 13	$YX' - WZ'$
6 2 2 6	$ZX' + XZ'$
14 10 10 14	$-ZX' + XZ'$
7 3 3 7	$WX' + YZ'$
15 11 11 15	$-WX' + YZ'$

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$	(Suffixes 1.)
4 0 0 4	$= YY' + WW'$
12 8 8 12	$YY' - WW'$
5 1 1 5	$XY' + ZW'$
13 9 9 13	$XY' - ZW'$
6 2 2 6	$WY' + YW'$
14 10 10 14	$-WY' + YW'$
7 3 3 7	$ZY' + XW'$
15 11 11 15	$-ZY' + XW'$

92. Third set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 2.)
8	0	0	8	=	$XX' + YY'$
12	4	4	12		$XX' - YY'$
9	1	1	9		$YX' + XY'$
13	5	5	13		$-YX' + XY'$
10	2	2	10		$ZX' + WY'$
14	6	6	14		$ZX' - WY'$
11	3	3	11		$WX' + ZY'$
15	7	7	15		$-WX' + ZY'$

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 2.)
8	0	0	8	=	$ZZ' + WW'$
12	4	4	12		$ZZ' - WW'$
9	1	1	9		$WZ' + ZW'$
13	5	5	13		$-WZ' + ZW'$
10	2	2	10		$XZ' + YW'$
14	6	6	14		$XZ' - YW'$
11	3	3	11		$YZ' + XW'$
15	7	7	15		$-YZ' + XW'$

93. Fourth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 3.)
12	0	0	12	=	$XX' + WW'$
8	4	4	8		$XX' - WW'$
13	1	1	13		$YX' + ZW'$
9	5	5	9		$YX' - ZW'$
14	2	2	14		$ZX' + YW'$
10	6	6	10		$ZX' - YW'$
15	3	3	15		$WX' + XW'$
11	7	7	11		$WX' - XW'$

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 3.)
12	0	0	12	=	$YY' + ZZ'$
8	4	4	8		$-YY' + ZZ'$
13	1	1	13		$XY' + WZ'$
9	5	5	9		$-XY' + WZ'$
14	2	2	14		$WY' + XZ'$
10	6	6	10		$-WY' + XZ'$
15	3	3	15		$ZY' + YZ'$
11	7	7	11		$-ZY' + YZ'$

94. Fifth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$										(Suffixes 0.)
1	0	0	1	=	$E+G$	.	$E'+G'$	+	$F+H$.	$F'+H'$
5	4	4	5		$i.E-G$	„		+	$i.F-H$	„
9	8	8	9		$E+G$	„		−	$.F+H$	„
13	12	12	13		$i.E-G$	„		−	$i.F-H$	„
3	2	2	3		$F+H$	„		+	$E+G$	„
7	6	6	7		$i.F-H$	„		+	$i.E-G$	„
11	10	10	11	−	$.F+H$	„		+	$E+G$	„
15	14	14	15	−	$i.F-H$	„		+	$i.E-G$	„

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$										(Suffixes 0.)
1	0	0	1	=	$E-G$	.	$E'-G'$	+	$F-H$.	$F'-H'$
5	4	4	5		$i.E+G$	„		+	$i.F+H$	„
9	8	8	9		$E-G$	„		−	$.F-H$	„
13	12	12	13		$i.E+G$	„		−	$i.F+H$	„
3	2	2	3		$F-H$	„		+	$E-G$	„
7	6	6	7		$i.F+H$	„		+	$i.E+G$	„
11	10	10	11	−	$.F-H$	„		+	$E-G$	„
15	14	14	15	−	$i.F+H$	„		+	$i.E+G$	„

95. Sixth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$										(Suffixes 1.)
5	0	0	5	=	$E-iG$	.	$E'+iG'$	+	$F-iH$.	$F'+iH'$
1	4	4	1	−	$i.E+iG$	„		−	$i.F+iH$	„
8	13	13	8		$E-iG$	„		−	$.F-iH$	„
9	12	12	9	−	$i.E+iG$	„		+	$i.F+iH$	„
7	2	2	7		$F-iH$	„		+	$E-iG$	„
3	6	6	3	−	$i.F+iH$	„		−	$i.E+iG$	„
15	10	10	15	−	$.F-iH$	„		+	$E-iG$	„
11	14	14	11		$i.F+iH$	„		−	$i.E+iG$	„

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$										(Suffixes 1.)
5	0	0	5	=	$E+iG$	.	$E'-iG'$	+	$F+iH$.	$F'-iH'$
1	4	4	1	−	$i.E-iG$	„		−	$i.F-iH$	„
13	8	8	13		$E+iG$	„		−	$.F+iH$	„
9	12	12	9	−	$i.E-iG$	„		+	$i.F-iH$	„
7	2	2	7		$F+iH$	„		+	$E+iG$	„
3	6	6	3	−	$i.F-iH$	„		−	$i.E-iG$	„
15	10	10	15	−	$.F+iH$	„		+	$.E+iG$	„
11	14	14	11	+	$i.F-iH$	„		−	$i.E-iG$	„

96. Seventh set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 2.)				
9	0	0	9	=	$E+G$	. $E'+G'$	+	$F-H$	. $F'-H'$
13	4	4	13		$i.E-G$	„	+	$i.F+H$	„
1	8	8	1		$E+G$	„	−	$.F-H$	„
5	12	12	5		$i.E-G$	„	−	$i.F+H$	„
11	2	2	11		$F+H$	„	+	$E-G$	„
15	6	6	15		$i.F-H$	„	+	$i.E+G$	„
3	10	10	3		$F+H$	„	−	$.E-G$	„
7	14	14	7		$i.F-H$	„	−	$i.E+G$	„

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 2.)				
9	0	0	9	=	$E-G$	. $E'-G'$	+	$F+H$	. $F'+H'$
13	4	4	13		$i.E+G$	„	+	$i.F-H$	„
1	8	8	1		$E-G$	„	−	$.F+H$	„
5	12	12	5		$i.E+G$	„	−	$i.F-H$	„
11	2	2	11		$F-H$	„	+	$E+G$	„
15	6	6	15		$i.F+H$	„	+	$i.E-G$	„
3	10	10	3		$F-H$	„	−	$.E+G$	„
7	14	14	7		$i.F+H$	„	−	$i.E-G$	„

97. Eighth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 3.)				
13	0	0	13	=	$E-iG$	. $E'+iG'$	+	$F+iH$	. $F'-iH'$
9	4	4	9		$-i.E+iG$	„	−	$i.F-iH$	„
5	8	8	5		$E-iG$	„	−	$.F+iH$	„
1	12	12	1		$-i.E+iG$	„	+	$i.F-iH$	„
15	2	2	15		$F-iH$	„	+	$E+iG$	„
11	6	6	11		$-i.F+iH$	„	−	$i.E-iG$	„
7	10	10	7		$F-iH$	„	−	$.E+iG$	„
3	14	14	3		$-i.F+iH$	„	+	$i.E-iG$	„

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$					(Suffixes 3.)				
13	0	0	13	=	$E+iG$	. $E'-iG'$	+	$F-iH$	. $F'+iH'$
9	4	4	9		$-i.E-iG$	„	−	$i.F+iH$	„
5	8	8	5		$E+iG$	„	−	$.F-iH$	„
1	12	12	1		$-i.E-iG$	„	+	$i.F+iH$	„
15	2	2	15		$F+iH$	„	+	$E-iG$	„
11	6	6	11		$-i.F-iH$	„	−	$i.E+iG$	„
7	10	10	7		$F+iH$	„	−	$.E-iG$	„
3	14	14	3		$-i.F-iH$	„	+	$i.E+iG$	„

98. Ninth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} + \vartheta^{u+u'} . \vartheta^{u-u'}\}$										(Suffixes 0.)
2	0	0	2	=	$I+K$	.	$I'+K'$	+	$J+L$.	$J'+L'$
6	4	4	6		$I+K$	„		−	$.J+L$	„
10	8	8	10		$i.I-K$	„		+	$i.J-L$	„
14	12	12	14		$i.I-K$	„		−	$i.J-L$	„
3	1	1	3		$J+L$	„		+	$I+K$	„
7	5	5	7	−	$.J+L$	„		−	$.I+K$	„
11	9	9	11		$i.J-L$	„		+	$i.I-K$	„
15	13	13	15	−	$i.J-L$	„		−	$i.I-K$	„

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} - \vartheta^{u+u'} . \vartheta^{u-u'}\}$										(Suffixes 0.)
2	0	0	2	=	$I-K$	.	$I'-K'$	+	$J-L$.	$J'-L'$
6	4	4	6		$I-K$	„		−	$.J-L$	„
10	8	8	10		$i.I+K$	„		+	$i.J+L$	„
14	12	12	14		$i.I+K$	„		−	$i.J+L$	„
3	1	1	3		$J-L$	„		+	$I-K$	„
7	5	5	7	−	$.J-L$	„		+	$I-K$	„
11	9	9	11		$i.J+L$	„		+	$i.I+K$	„
15	13	13	15	−	$i.J+L$	„		+	$i.I+K$	„

99. Tenth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} + \vartheta^{u+u'} . \vartheta^{u-u'}\}$										(Suffixes 1.)
6	0	0	6	=	$I+K$	.	$I'+K'$	+	$J-L$.	$J'-L'$
2	4	4	2		$I+K$	„		−	$.J-L$	„
14	8	8	14		$i.I-K$	„		+	$i.J+L$	„
10	12	12	10		$i.I-K$	„		−	$i.J+L$	„
7	1	1	7		$J+L$	„		+	$I-K$	„
3	5	5	3		$J+L$	„		−	$.I-K$	„
15	9	9	15		$i.J-L$	„		+	$i.I+K$	„
11	13	13	11		$i.J-L$	„		−	$i.I+K$	„

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} - \vartheta^{u+u'} . \vartheta^{u-u'}\}$										(Suffixes 1.)
6	0	0	6	=	$I-K$	.	$I'-K'$	+	$J+L$.	$J'+L'$
2	4	4	2		$I-K$	„		−	$.J+L$	„
14	8	8	14		$i.I+K$	„		+	$i.J-L$	„
10	12	12	10		$i.I+K$	„		−	$i.J-L$	„
7	1	1	7		$J-L$	„		+	$I+K$	„
3	5	5	3		$J-L$	„		−	$.I+K$	„
15	9	9	15		$i.J+L$	„		+	$i.I-K$	„
11	13	13	11		$i.J+L$	„		−	$i.I-K$	„

100. Eleventh set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 2.)				
10	0	0	10 =	$I-iK$	.	$I'+iK'$	+	$.J-iL$. $J'+iL'$
14	4	4	14	$I-iK$	,,		−	$.J-iL$,,
2	8	8	2	$-i.I+iK$	,,		−	$i.J+iL$,,
6	12	12	6	$-i.I+iK$	,,		+	$i.J+iL$,,
11	1	1	11	$J-iL$	,,		+	$I-iK$,,
15	5	5	15	$-.J-iL$	,,		+	$I-iK$,,
3	9	9	3	$-i.J+iL$	,,		−	$i.I+iK$,,
7	13	13	7	$+i.J+iL$	,,		−	$i.I+iK$,,

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 2.)				
10	0	0	10 =	$I+iK$	.	$I'-iK'$	+	$J+iL$. $J'-iL'$
14	4	4	14	$I+iK$	,,		−	$J+iL$,,
2	8	8	2	$-i.I-iK$	,,		−	$i.J-iL$,,
6	12	12	6	$-i.I-iK$	,,		+	$i.J-iL$,,
11	1	1	11	$J+iL$	,,		+	$I+iK$,,
15	5	5	15	$-.J+iL$	,,		+	$I+iK$,,
3	9	9	3	$-i.J-iL$	,,		−	$i.I-iK$,,
7	13	13	7	$+i.J-iL$	,,		−	$i.I-iK$,,

101. Twelfth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 3.)				
14	0	0	14 =	$I-iK$	.	$I'+iK'$	+	$J+iL$. $J'-iL'$
10	4	4	10	$I-iK$	,,		−	$.J+iL$,,
6	8	8	6	$-i.I+iK$	,,		−	$i.J-iL$,,
2	12	12	2	$-i.I+iK$	,,		+	$i.J-iL$,,
15	1	1	15	$J-iL$	,,		+	$I+iK$,,
11	5	5	11	$J-iL$	,,		−	$.I+iK$,,
7	9	9	7	$-i.J+iL$	,,		−	$i.I-iK$,,
3	13	13	3	$-i.J+iL$	,,		+	$i.I-iK$,,

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 3.)				
14	0	0	14 =	$I+iK$	.	$I'-iK'$	+	$J-iL$. $J'+iL'$
10	4	4	10	$I+iK$	,,		−	$.J-iL$,,
6	8	8	6	$-i.I-iK$	,,		−	$i.J+iL$,,
2	12	12	2	$-i.I-iK$	,,		+	$i.J+iL$,,
15	1	1	15	$J+iL$	,,		+	$I-iK$,,
11	5	5	11	$J+iL$	,,		−	$.I-iK$,,
7	9	9	7	$-i.J-iL$	,,		−	$i.I+iK$,,
3	13	13	3	$-i.J-iL$	,,		+	$i.I+iK$,,

102. Thirteenth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} + \vartheta^{u+u'} . \vartheta^{u-u'}\}$				(Suffixes 0.)				
3	0	0	3	$=$	$M+Q$	$.\ M'+Q'$	$+$	$N+P\ .\ N'+P'$
7	4	4	7		$i.M-Q$	,,	$-$	$i.N-P$,,
11	8	8	11		$i.M-Q$	,,	$+$	$i.N-P$,,
15	12	12	15	$-$	$.M+Q$	,,	$+$	$N+P$,,
2	1	1	2		$N+P$	,,	$+$	$M+Q$,,
6	5	5	6		$-i.N-P$	,,	$+$	$i.M-Q$,,
10	9	9	10		$i.N-P$	,,	$+$	$i.M-Q$,,
14	13	13	14		$N+P$	,,	$-$	$.M+Q$,,

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} - \vartheta^{u+u'} . \vartheta^{u-u'}\}$				(Suffixes 0.)				
3	0	0	3	$=$	$M-Q$	$.\ M'-Q'$	$+$	$N-P\ .\ N'-P'$
7	4	4	7		$i.M+Q$	,,	$-$	$i.N+P$,,
11	8	8	11		$i.M+Q$	,,	$+$	$i.N+P$,,
15	12	12	15	$-$	$.M-Q$	,,	$+$	$N-P$,,
2	1	1	2		$N-P$	,,	$+$	$M-Q$,,
6	5	5	6		$-i.N+P$	,,	$+$	$i.M+Q$,,
10	9	9	10		$i.N+P$	,,	$+$	$i.M+Q$,,
14	13	13	14		$N-P$	,,	$-$	$.M-Q$,,

103. Fourteenth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} + \vartheta^{u+u'} . \vartheta^{u-u'}\}$				(Suffixes 1.)				
7	0	0	7	$=$	$M-iQ$	$.\ M'+iQ'$	$+$	$N+iP\ .\ N'-iP'$
3	4	4	3		$-i.M+iQ$	,,	$+$	$i.N-iP$,,
15	8	8	15		$i.M+iQ$	,,	$+$	$i.N-iP$,,
11	12	12	11		$M-iQ$	,,	$-$	$.N+iP$,,
6	1	1	6		$N-iP$	,,	$+$	$M+iQ$,,
2	5	5	2		$-i.N-iP$	,,	$+$	$i.M-iQ$,,
14	9	9	14		$+i.N+iP$	,,	$+$	$i.M-iQ$,,
10	13	13	10		$N-iP$	,,	$-$	$.M+iQ$,,

$\frac{1}{2}\{\vartheta^{u+u'} . \vartheta^{u-u'} - \vartheta^{u+u'} . \vartheta^{u-u'}\}$				(Suffixes 1.)				
7	0	0	7	$=$	$M+iQ$	$.\ M'-iQ'$	$+$	$N-iP\ .\ N'+iP$
3	4	4	3		$-i.M-iQ$	,,	$+$	$i.N+iP$,,
15	8	8	15		$-i.M-iQ$	,,	$+$	$i.N+iP$,,
11	12	12	11		$M+iQ$	,,	$-$	$.N+iP$,,
6	1	1	6		$N+iP$	,,	$+$	$M-iQ$,,
2	5	5	2		$-i.N-iP$	,,	$+$	$i.M+iQ$,,
14	9	9	14		$+i.N-iP$	,,	$+$	$i.M+iQ$,,
10	13	13	10		$N+iP$	,,	$-$	$.M-iQ$,,

104. Fifteenth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 2.)			
11	0	0	11 =	$M-iQ$.	$M'+iQ'$	+	$N-iP$. $N'+iP'$
15	4	4	15	$i.M+iQ$	,,	−	$i.N+iP$,,
3	8	8	3	$-i.M+iQ$	,,	−	$i.N+iP$,,
7	12	12	7	$M-iQ$	,,	−	$.N-iP$,,
10	1	1	10	$N-iP$	,,	+	$M-iQ$,,
14	5	5	14	$-i.N+iP$	,,	+	$i.M+iQ$,,
2	9	9	2	$-i.N+iP$	,,	−	$i.M+iQ$,,
6	13	13	6	$-.N-iP$	,,	+	$M-iQ$,,

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 2.)			
11	0	0	11 =	$M+iQ$.	$M'-iQ'$	+	$N+iP$. $N'-iP'$
15	4	4	15	$i.M-iQ$	,,	−	$i.N-iP$,,
3	8	8	3	$-i.M-iQ$	,,	−	$i.N-iP$,,
7	12	12	7	$M+iQ$	,,	−	$.N+iP$,,
10	1	1	10	$N+iP$	,,	+	$M+iQ$,,
14	5	5	14	$-i.N-iP$	,,	+	$i.M-iQ$,,
2	9	9	2	$-i.N-iP$	,,	−	$i.M-iQ$,,
6	13	13	6	$-.N+iP$	,,	+	$M+iQ$,,

105. Sixteenth set of 16.

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}+\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 3.)			
15	0	0	15 =	$M-Q$.	$M'-Q'$	+	$N+P$. $N'+P'$
11	4	4	11	$-i.M+Q$	,,	+	$i.N-P$,,
7	8	8	7	$-i.M+Q$	,,	−	$i.N-P$,,
3	12	12	3	$-.M-Q$	,,	+	$N+P$,,
14	1	1	14	$N-P$	,,	+	$M+Q$,,
10	5	5	10	$-i.N+P$	,,	+	$i.M-Q$,,
6	9	9	6	$-i.N+P$	,,	−	$i.M-Q$,,
2	13	13	2	$-.N-P$	,,	+	$M+Q$,,

$\frac{1}{2}\{\vartheta^{u+u'}.\vartheta^{u-u'}-\vartheta^{u+u'}.\vartheta^{u-u'}\}$				(Suffixes 3.)			
15	0	0	15 =	$M+Q$.	$M'+Q'$	+	$N-P$. $N'-P'$
11	4	4	11	$-i.M-Q$	,,	+	$i.N+P$,,
7	8	8	7	$-i.M-Q$	,,	−	$i.N+P$,,
3	12	12	3	$-.M+Q$	,,	+	$N-P$,,
14	1	1	14	$N+P$	,,	+	$M-Q$,,
10	5	5	10	$-i.N-P$	,,	+	$i.M+Q$,,
6	9	9	6	$-i.N-P$	,,	−	$i.M+Q$,,
2	13	13	2	$-.N+P$	,,	+	$M-Q$,,

106. In the square set, writing $u'=v'=0$, and α, β, γ, δ for X', Y', Z', W'; also slightly altering the arrangement,

the system becomes : and further writing herein $u=0$, $v=0$, it becomes

u			X	Y	Z	W				0				c^2
ϑ^2	0	=	α	β	γ	δ		ϑ^2	0	=	$\alpha^2+\beta^2+\gamma^2+\delta^2$	=	0
	4	=	α	$-\beta$	γ	$-\delta$			4	=	$\alpha^2-\beta^2+\gamma^2-\delta^2$	=	4
	8	=	α	β	$-\gamma$	$-\delta$			8	=	$\alpha^2-\beta^2-\gamma^2-\delta^2$	=	8
	12	=	α	$-\beta$	$-\gamma$	δ			12	=	$\alpha^2-\beta^2-\gamma^2+\delta^2$	=	12
	1	=	β	α	δ	γ			1	=	$2(\alpha\beta+\gamma\delta)$	=	1
	5	=	β	$-\alpha$	δ	$-\gamma$			5		0		
	9	=	β	α	$-\delta$	$-\gamma$			9	=	$2(\alpha\beta-\gamma\delta)$	=	9
	13	=	β	$-\alpha$	$-\delta$	γ			13		0		
	2	=	γ	δ	α	β			2	=	$2(\alpha\gamma+\beta\delta)$	=	2
	6	=	γ	$-\delta$	α	$-\beta$			6	=	$2(\alpha\gamma-\beta\delta)$	=	6
	10	=	γ	δ	$-\alpha$	$-\beta$			10		0		
	14	=	γ	$-\delta$	$-\alpha$	β			14		0		
	3	=	δ	γ	β	α			3	=	$2(\alpha\delta+\beta\gamma)$	=	3
	7	=	δ	$-\gamma$	β	$-\alpha$			7		0		
	11	=	δ	γ	$-\beta$	$-\alpha$			11		0		
	15	=	δ	$-\gamma$	$-\beta$	α			15	=	$2(\alpha\delta-\beta\gamma)$	=	15

viz. this last is the before-mentioned system of equations giving the values of the 10 zero-functions c in terms of the four constants α, β, γ, δ.

107. The system first obtained is a system of 16 equations

$$\vartheta_0^2(u, v)=\alpha X+\beta Y+\gamma Z+\delta W, \text{ \&c.},$$

showing that the squares of the theta-functions are each of them a linear function of the four quantities X, Y, Z, W. If the functions on the right-hand side were independent (asyzygetic) linear functions of X, Y, Z, W, it would follow that any four (selected at pleasure) of the squared theta-functions are linearly independent, and that we could in terms of these four express linearly each of the remaining 12 squared functions. But this is not so; the form of the linear functions of (X, Y, Z, W) is such that we can (and that in 16 different ways) select out of the 16 linear functions six functions, such that any four of them are connected by a linear equation; and there are consequently 16 hexads of squared theta-functions, such

that any four out of the same hexad are connected by a linear relation. The hexads are shown by the foregoing "Table of the 16 Kummer hexads."

108. The *à posteriori* verification is immediately effected; taking for instance the first column, the equations are

	$\vartheta^2 u$		X	Y	Z	W
A	11	=	δ	γ	$-\beta$	$-\alpha$,
B	7		δ	$-\gamma$	β	$-\alpha$,
AB	6		γ	$-\delta$	α	$-\beta$,
CD	2		γ	δ	α	β,
CE	1		β	α	δ	γ,
DE	9		β	α	δ	$-\gamma$;

viz. it should thence follow that there is a linear relation between any four of the six squared functions 11, 7, 6, 2, 1, 9: and it is accordingly seen that this is so. It further appears that, in the several linear relations, the coefficients (obtained in the first instance as functions of α, β, γ, δ) are in fact the 10 constants c: the 15 relations connecting the several systems of four out of the six squared functions are given in the following table.

109.

ϑ^2	11	7	6	2	1	9	$=0.$
c^2			6	− 2	1	− 9	
	6			+ 15	− 12	+ 4	
	− 2		− 15		+ 8	− 0	
	1		+ 12	− 8		+ 3	
	− 9		− 4	+ 0	− 3		
		6		3	− 0	+ 8	
		− 2	− 3		+ 4	− 12	
		1	+ 0	− 4		− 15	
		− 9	− 8	+ 12	+ 15		
	− 15	+ 3	+ 2	− 6			
	− 12	+ 0	+ 1		− 6		
	− 4	+ 8	+ 9			− 6	
	− 3	+ 15			+ 9	− 1	
	− 0	+ 12		+ 9		− 2	
	− 8	+ 4		+ 1	− 2		

Read

$$\cdot\quad c_6^2\vartheta_6^2 - c_2^2\vartheta_2^2 + c_1^2\vartheta_1^2 - c_9^2\vartheta_9^2 = 0,$$

$$c_6^2\vartheta_{11}^2\quad\cdot\quad + c_{15}^2\vartheta_2^2 - c_{12}^2\vartheta_1^2 + c_4^2\vartheta_9^2 = 0,\ \&c.$$

110. The first set of 16 equations is the square-set, which has been already considered. If in each of the other sets of 16 equations we write in like manner $u'=0$, each set in fact reduces itself to eight equations; sets 2, 3, 4 give thus $8+8+8$, $=24$ equations; sets 5 to 8, 9 to 12, and 13 to 15, give each $8+8+8+8$, $=32$ equations; or we have sets of 24, 32, 32, 32, together 120 equations, the number being of course one half of $256-16$, the number of equations after deducting the 16 equations of the square set.

111. The first set, 24 equations.

This is derived from the second, third, and fourth sets, each of 16 equations, by writing therein $u'=0$. Taking α_1, β_1, γ_1, δ_1 for the zero-functions corresponding to X_1, Y_1, Z_1, W_1, then on writing $u'=0$, X_1', Y_1', Z_1', W_1' become α_1, β_1, γ_1, δ_1. In the second set of 16 equations, the first equations thus are

$$\begin{array}{ll} \vartheta_4 u \,.\, \vartheta_0 u = \alpha_1 X_1 + \gamma_1 Z_1, & 0 = \beta_1 Y_1 + \delta_1 W_1, \\ \vartheta_{12} u \,.\, \vartheta_8 u = \alpha_1 X_1 - \gamma_1 Z_1, & 0 = \beta_1 Y_1 - \delta_1 W_1, \\ \quad\vdots & \quad\vdots \end{array}$$

viz. the equations of the column require that, and are all satisfied if, $\beta_1=0$, $\delta_1=0$: hence the zero-functions are α_1, 0, γ_1, 0; and this being so we have only the equations of the first column. And similarly as regards the third and fourth sets; the zero values corresponding to

	X_1,	Y_1,	Z_1,	W_1	X_2,	Y_2,	Z_2,	W_2	X_3,	Y_3,	Z_3,	W_3
are	α_1	0	γ_1	0	α_2	β_2	0	0	α_3	0	0	δ_3;

and we have in all $8+8+8$, $=24$ equations. These are

(Suffixes 1.)					(Suffixes 2.)					(Suffixes 3.)				
ϑu	ϑu		X	Z	ϑu	ϑu		X	Y	ϑu	ϑu		X	W
4	0	=	α	γ	8	0	=	α	β	12	0	=	α	δ
12	8	=	α	$-\gamma$	12	4	=	α	$-\beta$	8	4	=	α	$-\delta$
6	2	=	γ	α	9	1	=	β	α	15	3	=	δ	α
14	10	=	γ	$-\alpha$	13	5	=	β	$-\alpha$	11	7	=	$-\delta$	α
			Y	W				Z	W				Y	Z
5	1	=	α	γ	10	2	=	α	β	13	1	=	α	δ
13	9	=	α	$-\gamma$	14	6	=	α	$-\beta$	9	5	=	α	$-\delta$
7	3	=	γ	α	11	3	=	β	α	14	2	=	δ	α
15	11	=	γ	$-\alpha$	15	7	=	β	$-\alpha$	10	6	=	$-\delta$	α
$\vartheta\vartheta$	$\vartheta\vartheta$				$\vartheta\vartheta$	$\vartheta\vartheta$				$\vartheta\vartheta$	$\vartheta\vartheta$			
4	0	=	$\alpha^2+\gamma^2$		8	0	=	$\alpha^2+\beta^2$		12	0	=	$\alpha^2+\delta^2$	
12	8	=	$\alpha^2-\gamma^2$		12	4	=	$\alpha^2-\beta^2$		8	4	=	$\alpha^2-\delta^2$	
6	2	=	$2\alpha\gamma$		9	1	=	$2\alpha\beta$		15	3	=	$2\alpha\delta$.	

112. The second set, 32 equations.

To exhibit these in a convenient form, I alter the notation, viz. I write

$$\begin{array}{l|l} E+G,\ \ i(E-G),\ \ (F+H),\ \ i(F-H) & E_1+iG_1,\ \ E_1-iG_1,\ \ F_1+iH_1,\ \ F_1-iH_1 \\ = X,\ \ Y,\ \ Z,\ \ W & X_1,\ \ Y_1,\ \ Z_1,\ \ W_1 \end{array}$$

$$\begin{array}{l|l} (E_2+G_2),\ \ i(E_2-G_2),\ \ (F_2+H_2),\ \ i(F_2-H_2) & E_3+iG_3,\ \ E_3-iG_3,\ \ F_3+iH_3,\ \ F_3-iH_3 \\ = X_2,\ \ Y_2,\ \ Z_2,\ \ W_2 & X_3,\ \ Y_3,\ \ Z_3,\ \ W_3, \end{array}$$

so that as regards the present set of equations, X, Y, Z, W, signify as just mentioned. And, this being so, the corresponding zero-values are

$$\alpha,\ 0,\ \gamma,\ 0 \ \mid\ \alpha_1,\ 0,\ \gamma_1,\ 0 \ \mid\ \alpha_2,\ 0,\ 0,\ \delta_2 \ \mid\ \alpha_3,\ 0,\ 0,\ \delta_3.$$

The equations then are

(Suffixes 0.)	(Suffixes 1.)	(Suffixes 2.)	(Suffixes 3.)
$\vartheta u\ .\ \vartheta u$ $\quad X \quad Z$	$\vartheta u\ .\ \vartheta u$ $\quad X \quad Z$	$\vartheta u\ .\ \vartheta u$ $\quad X \quad W$	$\vartheta u\ .\ \vartheta u$ $\quad X \quad W$
$1 \quad 0 = \alpha \quad \gamma$	$1 \quad 4 = -i\alpha \quad -i\gamma$	$9 \quad 0 = \alpha \quad -\delta$	$9 \quad 4 = -i\alpha \quad -i\delta$
$9 \quad 8 = \alpha \quad -\gamma$	$9 \quad 12 = -i\alpha \quad +i\gamma$	$1 \quad 8 = \alpha \quad \delta$	$1 \quad 12 = -i\alpha \quad +i\delta$
$3 \quad 2 = \gamma \quad \alpha$	$3 \quad 6 = -i\gamma \quad -i\alpha$	$15 \quad 6 = \delta \quad \alpha$	$15 \quad 2 = \delta \quad \alpha$
$11 \quad 10 = \gamma \quad -\alpha$	$11 \quad 14 = -i\gamma \quad +i\alpha$	$7 \quad 14 = -\delta \quad \alpha$	$7 \quad 10 = -\delta \quad \alpha$
$Y \quad W$	$Y \quad W$	$Y \quad Z$	$Y \quad Z$
$5 \quad 4 = \alpha \quad \gamma$	$5 \quad 0 = \alpha \quad \gamma$	$13 \quad 4 = \alpha \quad \delta$	$13 \quad 0 = \alpha \quad \delta$
$13 \quad 12 = \alpha \quad -\gamma$	$13 \quad 8 = \alpha \quad -\gamma$	$5 \quad 12 = \alpha \quad -\delta$	$5 \quad 8 = \alpha \quad -\delta$
$7 \quad 6 = \gamma \quad \alpha$	$7 \quad 2 = \gamma \quad \alpha$	$11 \quad 2 = -\delta \quad \alpha$	$11 \quad 6 = -i\delta \quad -i\alpha$
$15 \quad 14 = \gamma \quad -\alpha$	$15 \quad 10 = \gamma \quad -\alpha$	$3 \quad 10 = \delta \quad \alpha$	$3 \quad 14 = i\delta \quad -i\alpha$
$\vartheta 0\ .\ \vartheta 0$	$\vartheta 0\ .\ \vartheta 0$	$\vartheta 0\ .\ \vartheta 0$	$\vartheta 0\ .\ \vartheta 0$
$1 \quad 0 = \alpha^2 + \gamma^2$	$1 \quad 4 = -i(\alpha^2+\gamma^2)$	$9 \quad 0 = \alpha^2 - \delta^2$	$9 \quad 4 = -i(\alpha^2+\delta^2)$
$9 \quad 8 = \alpha^2 - \gamma^2$	$9 \quad 12 = -i(\alpha^2-\gamma^2)$	$1 \quad 8 = \alpha^2 + \delta^2$	$1 \quad 12 = -i(\alpha^2-\delta^2)$
$3 \quad 2 = 2\alpha\gamma$	$3 \quad 6 = -2i\alpha\gamma$	$15 \quad 6 = 2\alpha\delta$	$15 \quad 2 = 2\alpha\delta.$

113. Third set, 32 equations.

We again change the notation, writing

$$\begin{array}{rllll|llll|}
 & I+K, & i(I-K), & J+L, & i(J-L) & I_1+K_1, & i(I_1-K_1), & (J_1+L_1), & i(J_1-L_1) \\
= & X, & Y, & Z, & W & X_1, & Y_1, & Z_1, & W_1 \\
 & I_2+iK_2, & I_2-iK_2, & J_2+iL_2, & J_2-iL_2 & I_3+iK_3, & I_3-iK_3, & J_3+iL_3, & J_3-iL_3 \\
= & X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3,
\end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \ \mid\ \alpha_1,\ 0,\ 0,\ \delta_1 \ \mid\ \alpha_2,\ 0,\ \gamma_2,\ 0 \ \mid\ \alpha_3,\ 0,\ 0,\ \delta_3.$$

Then equations are

(Suffixes 0.)	(Suffixes 1.)	(Suffixes 2.)	(Suffixes 3.)
$\vartheta u \,.\, \vartheta u \quad \overbrace{X \quad Z}$	$\vartheta u \,.\, \vartheta u \quad \overbrace{X \quad W}$	$\vartheta u \,.\, \vartheta u \quad \overbrace{X \quad Z}$	$\vartheta u \,.\, \vartheta u \quad \overbrace{X \quad W}$
$2 \quad 0 = \overbrace{\alpha \quad \gamma}$	$6 \quad 0 = \overbrace{\alpha \quad -\delta}$	$2 \quad 8 = \overbrace{-i\alpha \quad -i\gamma}$	$6 \quad 8 = \overbrace{-i\alpha \quad -i\delta}$
$6 \quad 4 = \alpha \quad -\gamma$	$2 \quad 4 = \alpha \quad \delta$	$6 \quad 12 = -i\alpha \quad +i\gamma$	$2 \quad 12 = -i\alpha \quad i\delta$
$3 \quad 1 = \gamma \quad \alpha$	$15 \quad 9 = \delta \quad \alpha$	$3 \quad 9 = -i\gamma \quad -i\alpha$	$15 \quad 1 = \delta \quad \alpha$
$7 \quad 5 = \gamma \quad -\alpha$	$11 \quad 13 = -\delta \quad \alpha$	$7 \quad 13 = -i\gamma \quad +i\alpha$	$11 \quad 5 = -\delta \quad \alpha$
$Y \quad W$	$Y \quad Z$	$Y \quad W$	$Y \quad Z$
$10 \quad 8 = \overbrace{\alpha \quad \gamma}$	$14 \quad 8 = \overbrace{\alpha \quad \delta}$	$10 \quad 0 = \overbrace{\alpha \quad \gamma}$	$14 \quad 0 = \overbrace{\alpha \quad \delta}$
$14 \quad 12 = \alpha \quad -\gamma$	$10 \quad 12 = \alpha \quad -\delta$	$14 \quad 4 = \alpha \quad -\gamma$	$10 \quad 4 = \alpha \quad -\delta$
$11 \quad 9 = \gamma \quad \alpha$	$7 \quad 1 = -\delta \quad \alpha$	$11 \quad 1 = \gamma \quad \alpha$	$7 \quad 9 = -i\delta \quad -i\alpha$
$15 \quad 13 = \gamma \quad -\alpha$	$3 \quad 5 = \delta \quad \alpha$	$15 \quad 5 = \gamma \quad -\alpha$	$3 \quad 13 = i\delta \quad -i\alpha$
$\vartheta 0 \,.\, \vartheta 0$	$\vartheta 0 \,.\, \vartheta 0$	$\vartheta 0 \,.\, \vartheta 0$	$\vartheta 0 \,.\, \vartheta 0$
$2 \quad 0 = \alpha^2 + \gamma^2$	$6 \quad 0 = \alpha^2 - \delta^2$	$2 \quad 8 = -i(\alpha^2+\gamma^2)$	$6 \quad 8 = -i(\alpha^2+\delta^2)$
$6 \quad 4 = \alpha^2 - \gamma^2$	$2 \quad 4 = \alpha^2 + \delta^2$	$6 \quad 12 = -i(\alpha^2-\gamma^2)$	$2 \quad 12 = -i(\alpha^2-\delta^2)$
$3 \quad 1 = 2\alpha\gamma$	$15 \quad 9 = 2\alpha\delta$	$3 \quad 9 = -2i\alpha\gamma$	$15 \quad 1 = 2\alpha\delta.$

114. Fourth set, 32 equations.

Again changing the notation, we write

$$\begin{array}{rllll|llll} & M_2+iQ_2, & M_2-iQ_2, & N_2+iP_2, & N_2-iP_2 & M_3+Q_3, & i(M_3-Q_3), & N_3+P_3, & i(N_3-P_3) \\ = & X, & Y, & Z, & W & X_1, & Y_1, & Z_1, & W_1, \\ & M+Q, & i(M-Q), & N+P, & i(N-P) & M_1+iQ_1, & M_1-iQ_1, & N_1+iP_1, & N_1-iP_1 \\ = & X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3, \end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \ \mid\ \alpha_1,\ 0,\ 0,\ \delta_1 \ \mid\ \alpha_2,\ 0,\ \gamma_2,\ 0 \ \mid\ 0,\ \beta_3,\ \gamma_3,\ 0.$$

The equations then are

(Suffixes 0.)				(Suffixes 1.)				(Suffixes 2.)				(Suffixes 3.)			
ϑu	ϑu	X	Z	ϑu	ϑu	X	W	ϑu	ϑu	X	Z	ϑu	ϑu	Y	Z
0	3 =	α	γ	3	4 =	$-i\alpha$	$i\delta$	15	4 =	$i\alpha$	$-i\gamma$	15	0 =	$-\beta$	γ
15	12 =	$-\alpha$	γ	15	8 =	$i\alpha$	$i\delta$	3	8 =	$-i\alpha$	$-i\gamma$	3	12 =	β	γ
2	1 =	γ	α	6	1 =	δ	α	14	5 =	$i\gamma$	$-i\alpha$	10	5 =	γ	$-\beta$
14	13 =	$-\gamma$	α	10	13 =	$-\delta$	α	2	9 =	$-i\gamma$	$-i\alpha$	6	9 =	$-\gamma$	$-\beta$
		Y	W			Y	Z			Y	W			X	W
4	7 =	α	$-\gamma$	7	0 =	α	δ	11	0 =	α	γ	11	4 =	$-\beta$	γ
8	11 =	α	γ	11	12 =	α	$-\delta$	7	12 =	α	$-\gamma$	7	8 =	$-\beta$	$-\gamma$
6	5 =	γ	$-\alpha$	2	5 =	$i\delta$	$-i\alpha$	10	1 =	γ	α	14	1 =	γ	$-\beta$
10	9 =	γ	α	14	9 =	$i\delta$	$i\alpha$	6	13 =	γ	$-\alpha$	2	13 =	γ	β
$\vartheta 0$	$\vartheta 0$			$\vartheta 0$	$\vartheta 0$			$\vartheta 0$	$\vartheta 0$			$\vartheta 0$	$\vartheta 0$		
0	3 =	$\alpha^2+\gamma^2$		3	4 =	$-i(\alpha^2-\delta^2)$		15	4 =	$i(\alpha^2-\gamma^2)$		15	0 =	$-(\beta^2-\gamma^2)$	
15	12 =	$-(\alpha^2-\gamma^2)$		15	8 =	$i(\alpha^2+\delta^2)$		3	8 =	$-i(\alpha^2+\gamma^2)$		3	12 =	$\beta^2+\gamma^2$	
2	1 =	$2\alpha\gamma$		6	1 =	$2\alpha\delta$		2	9 =	$-2i\alpha\gamma$		6	9 =	$-2\beta\gamma.$	

115. It will be noticed that the pairs of theta-functions which present themselves in these equations are the same as in the foregoing "Table of the 120 pairs." And the equations show that the four products, each of a pair of theta-functions, belonging to the upper half or to the lower half of any column of the table, are such that any three of the four products are connected by a linear equation. The coefficients of these linear relations are, in fact, functions such as the $\alpha^2+\delta^2$, $\alpha^2-\delta^2$, $2\alpha\delta$ written down at the foot of the several systems of eight equations, and they are consequently products each of two zero-functions c.

Thus (see "The first set, 24 equations") we have

		(Suffixes 3.)				(Suffixes 3.)				(Suffixes 3.)
ϑu .	ϑu	X	W	ϑu .	ϑu	Y	Z	$\vartheta 0$.	$\vartheta 0$	
4	$8 =$	α	$-\delta$	5	$9 =$	α	$-\delta$	4	$8 = \alpha^2 - \delta^2$	
0	$12 =$	α	δ	1	$13 =$	α	δ	0	$12 = \alpha^2 + \delta^2$	
3	$15 =$	δ	α	2	$14 =$	δ	α	15	$3 = 2\alpha\delta.$	
7	$11 =$	$-\delta$	α	6	$10 =$	$-\delta$	α			

116. In the left-hand four of these, omitting successively the first, second, third, and fourth equation, and from the remaining three eliminating the X_3 and W_3, we write down, almost mechanically,

$$\begin{array}{cc|cccc|}
\vartheta u & \vartheta u & & & & \\
4 & 8 & . & +2\alpha\delta, & -\delta^2-\alpha^2, & \alpha^2-\delta^2 \\
0 & 12 & -2\alpha\delta, & . & -\delta^2+\alpha^2, & \alpha^2+\delta^2 \\
3 & 15 & \alpha^2+\delta^2, & \delta^2-\alpha^2, & . & 2\alpha\delta \\
7 & 11 & -\alpha^2+\delta^2, & \delta^2+\alpha^2, & -2\alpha\delta &
\end{array},$$

and thence derive the first of the next following system of equations; read

$$\begin{aligned}
c_3c_{15}\vartheta_0\vartheta_{12} - c_0c_{12}\vartheta_3\vartheta_{15} + c_4c_8\,\vartheta_7\vartheta_{11} &= 0, \\
-c_3c_{15}\vartheta_4\vartheta_8 \qquad\qquad + c_4c_8\,\vartheta_3\vartheta_{15} - c_0c_{12}\vartheta_7\vartheta_{11} &= 0, \\
c_0c_{12}\vartheta_4\vartheta_8 - c_4c_8\,\vartheta_0\vartheta_{12} \qquad\qquad + c_3c_{15}\vartheta_7\vartheta_{11} &= 0, \\
-c_4c_8\,\vartheta_4\vartheta_8 + c_0c_{12}\vartheta_0\vartheta_{12} - c_3c_{15}\vartheta_3\vartheta_{15} \qquad\qquad &= 0,
\end{aligned}$$

where the theta-functions have the arguments u, v.

Observe that, on writing herein $u=0$, $v=0$, the first three equations become each of them identically $0=0$; the fourth equation becomes

$$-c_4^2c_8^2 + c_0^2c_{12}^2 - c_3^2c_{15}^2 = 0,$$

which is one of the relations between the c's and serves as a verification.

But in the right-hand system, on writing $u=v=0$, each of the four equations becomes identically $0=0$.

117. The equations are

ϑ	4.8	0.12	3.15	7.11	$=0,$
c		3.15	−0.12	4.8	
	−3.15		−4.8	−0.12	
	0.12	−4.8		3.15	
	−4.8	0.12	−3.15		

ϑ	5.9	1.13	2.14	6.10	$=0,$
c		3.15	−0.12	4.8	
	−3.15		4.8	−0.12	
	0.12	−4.8		3.15	
	−4.8	−0.12	−3.15		

ϑ	6.8	2.12	1.15	5.11	$=0,$
c		1.15	−2.12	6.8	
	−1.15		6.8	−2.11	
	2.12	−6.8		1.15	
	−6.8	2.11	−1.15		

ϑ	7.9	3.13	0.14	4.10	$=0,$
c		1.15	−2.12	6.8	
	−1.15		6.8	−2.12	
	2.12	−6.8		1.15	
	−6.8	2.12	−1.15		

ϑ	0.6	2.4	9.15	11.13	$=0,$
c		9.15	−2.4	0.6	
	−9.15		0.6	−2.4	
	2.4	−0.6		9.15	
	−0.6	2.4	−9.15		

ϑ	1.7	3.5	8.14	10.12	$=0,$
c		9.15	−2.4	0.6	
	−9.15		0.6	−2.4	
	2.4	−0.6		9.15	
	−0.6	2.4	−9.15		

ϑ	3.6	1.4	9.12	14.11	$=0,$
c		9.12	−1.4	3.6	
	−9.12		3.6	−1.4	
	1.4	−3.6		9.12	
	−3.6	1.4	−9.12		

ϑ	2.7	0.5	8.13	10.15	$=0,$
c		9.12	−1.4	3.6	
	−9.12		3.6	−1.4	
	1.4	−3.6		9.12	
	−3.6	1.4	−9.12		

ϑ	8.9	0.1	2.3	10.11	$=0,$
c		−2.3	0.1	8.9	
	2.3		−8.9	−0.1	
	−0.1	8.9		2.3	
	−8.9	0.1	−2.3		

ϑ	12.13	4.5	6.7	14.15	$=0,$
c		−2.3	0.1	8.9	
	2.3		−8.9	−0.1	
	−0.1	8.9		2.3	
	−8.9	0.1	−2.3		

ϑ	4.6	0.2	1.3	5.7	$=0,$
c		−1.3	0.2	4.6	
	1.3		−4.6	−0.2	
	−0.2	4.6		1.3	
	−4.6	0.2	−1.3		

ϑ	9.11	13.15	12.14	8.10	$=0,$
c		−1.3	0.2	4.6	
	1.3		−4.6	−0.2	
	−0.2	−4.6		1.3	
	4.6	0.2	−1.3		

ϑ	6.12	2.8	3.9	7.13	$=0,$
c		3.9	−2.8	−6.12	
	−3.9		6.12	2.8	
	2.8	−6.12		−3.9	
	6.12	−2.8	3.9		

ϑ	1.11	5.15	4.14	0.10	$=0,$
c		3.9	−2.8	6.12	
	−3.9		−6.12	2.8	
	2.8	6.12		−3.9	
	−6.12	−2.8	3.9		

ϑ	6.15	1.8	0.9	7.14	$=0,$
c		0.9	−1.8	−6.15	
	−0.9		6.15	1.8	
	1.8	−6.15		−0.9	
	6.15	−1.8	0.9		

ϑ	2.11	5.12	4.13	3.10	$=0,$
c		0.9	−1.8	6.15	
	−0.9		−6.15	1.8	
	1.8	6.15		−0.9	
	−6.15	−1.8	0.9		

ϑ	4.9	1.12	2.15	7.10	$=0,$
c		2.15	−1.12	4.9	
	−2.15		4.9	−1.12	
	1.12	−4.9		2.15	
	−4.9	1.12	−2.15		

ϑ	0.13	5.8	6.11	3.14	$=0,$
c		2.15	1.12	−4.9	
	−2.15		−4.9	1.12	
	−1.12	4.9		2.15	
	4.9	−1.12	−2.15		

ϑ	4.12	0.8	1.9	5.13	$=0,$
c		−1.9	0.8	4.12	
	1.9		−4.12	−0.8	
	−0.8	4.12		1.9	
	−4.12	0.8	−1.9		

ϑ	3.11	7.15	6.14	2.10	$=0,$
c		1.9	−0.8	4.12	
	−1.9		−4.12	0.8	
	0.8	4.12		−1.9	
	−4.12	−0.8	1.9		

ϑ	4.15	3.8	2.9	5.14	$=0,$
c		−2.9	3.8	4.15	
	2.9		−4.15	−3.8	
	−3.8	4.15		2.9	
	−4.15	3.8	−2.9		

ϑ	0.11	7.12	6.13	1.10	$=0,$
c		−2.9	3.8	−4.15	
	2.9		4.15	−3.8	
	−3.8	−4.15		2.9	
	4.15	3.8	−2.9		

ϑ	6.9	3.12	0.15	5.10	$=0,$
c		−0.15	3.12	−6.9	
	0.15		−6.9	3.12	
	−3.12	6.9		−0.15	
	6.9	−3.12	0.15		

ϑ	2.13	7.8	4.11	1.14	$=0,$
c		0.15	3.12	−6.9	
	−0.15		−6.9	3.12	
	−3.12	6.9		0.15	
	6.9	−3.12	−0.15		

ϑ	12.15	0.3	1.2	13.14	$=0,$
c		1.2	−0.3	−12.15	
	−1.2		12.15	0.3	
	0.3	−12.15		−1.2	
	12.15	−0.3	1.2		

ϑ	8.11	4.7	5.6	9.10	$=0,$
c		1.2	−0.3	12.15	
	−1.2		−12.15	0.3	
	0.3	12.15		−1.2	
	−12.15	−0.3	1.2		

ϑ	1.6	3.4	8.15	10.13	$=0,$
c		8.15	−3.4	1.6	
	−8.15		1.6	−3.4	
	3.4	−1.6		8.15	
	−1.6	3.4	−8.15		

ϑ	2.5	0.7	11.12	9.14	$=0,$
c		−8.15	−3.4	1.6	
	8.15		1.6	−3.4	
	3.4	−1.6		−8.15	
	−1.6	3.4	8.15		

ϑ	2.6	0.4	8.12	10.14	$=0,$
c		−8.12	0.4	−2.6	
	8.12		−2.6	0.4	
	−0.4	2.6		−8.12	
	2.6	−0.4	8.12		

ϑ	1.5	3.7	11.15	9.13	$=0.$
c		−8.12	−0.4	2.6	
	8.12		2.6	−0.4	
	0.4	−2.6		−8.12	
	−2.6	0.4	8.12		

118. The foregoing equations may be verified, and it is interesting to verify them, by means of the approximate values of the functions: thus, for one of the equations, we have

$$\begin{array}{l|l}
c_3c_{15}\vartheta_6\vartheta_{12}, \text{ i.e.,} & (2\Lambda+2\Lambda')(-2\Lambda+2\Lambda') \quad . \quad 1 \quad . \quad 1 \\
-c_0c_{12}\vartheta_3\vartheta_{15} & - \quad 1 \quad . \quad 1 \quad . \quad 2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad\qquad\qquad\qquad -2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v) \\
+c_4c_8\vartheta_7\vartheta_{11} & + \quad 1 \quad . \quad 1 \quad . -2\Lambda\sin\tfrac{1}{2}\pi(u+v)-2\Lambda'\sin\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad\qquad\qquad\qquad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v); \\
=0, & =0,
\end{array}$$

viz. the equation to be verified is here

$$\begin{aligned}
&-4\Lambda^2 \qquad\qquad\qquad\qquad +4\Lambda'^2 \\
&+4\Lambda^2\cos^2\tfrac{1}{2}\pi(u+v)-4\Lambda'^2\cos^2\tfrac{1}{2}\pi(u-v) \\
&+4\Lambda^2\sin^2\tfrac{1}{2}\pi(u+v)-4\Lambda'^2\sin^2\tfrac{1}{2}\pi(u-v) \\
&=0,
\end{aligned}$$

which is right.

119. In the equation

$$\begin{array}{l|l}
c_9c_{12}\vartheta_1\vartheta_4, \text{ i.e.,} & 2Q\,.\,1\,.\,2Q\cos\tfrac{1}{2}\pi u\,.\,1 \\
-c_1c_4\vartheta_9\vartheta_{12} & -2Q\,.\,1\,.\,2Q\cos\tfrac{1}{2}\pi u\,.\,1 \\
+c_3c_6\vartheta_{14}\vartheta_{11} & \\
=0, & =0;
\end{array}$$

this is right, but there is no verification as to the term $c_3c_6\vartheta_{14}\vartheta_{11}$; taking the more approximate values, the term in question taken negatively, that is, $-c_3c_6\vartheta_{14}\vartheta_{11}$ is

$$=-(2\Lambda+2\Lambda').\quad 2S.\quad -2S\sin\tfrac{1}{2}\pi v.\quad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v),$$

which is

$$=-8S^2(\Lambda+\Lambda')^2\cos\tfrac{1}{2}\pi u+8S^2(\Lambda+\Lambda')\Lambda\cos\tfrac{1}{2}\pi(u+2v)+8S^2(\Lambda+\Lambda')\Lambda'\cos\tfrac{1}{2}\pi(u-2v),$$

and this ought therefore to be the value of the first two terms, that is, of

$$\begin{aligned}
&(2Q+2Q^9-2A-2A')(1-2Q^4-2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u \\
&\qquad\qquad +2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u+2S^4\cos\pi v) \\
&-(2Q+2Q^9+2A+2A')(1-2Q^4+2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u \\
&\qquad\qquad -2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u-2S^4\cos\pi v),
\end{aligned}$$

which to the proper degree of approximation is

$$\begin{aligned}
&=(2Q-4Q^5-4QS^4+2Q^9-2A-2A')\{2Q\cos\tfrac{1}{2}\pi u-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u \\
&\qquad\qquad +4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u+2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\} \\
&-(2Q-4Q^5+4QS^4+2Q^9+2A+2A')\{2Q\cos\tfrac{1}{2}\pi u-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u \\
&\qquad\qquad -4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u-2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}.
\end{aligned}$$

This is

$$\begin{aligned}&(2M_0 - 2\Omega_0)(2M + 2\Omega)\\ &-(2M_0 + 2\Omega_0)(2M - 2\Omega), \quad = 8(M_0\Omega - M\Omega_0),\end{aligned}$$

if for a moment

$$M = Q\cos\tfrac{1}{2}\pi u - 2Q^5\cos\tfrac{1}{2}\pi u\cos\pi u + Q^9\cos\tfrac{3}{2}\pi u, \qquad M_0 = Q - 2Q^5 + Q^9,$$

$$\Omega = 2QS^4\cos\pi u\cos\pi v + A\cos\tfrac{1}{2}\pi(u+2v) + A'\cos\tfrac{1}{2}\pi(u-2v), \quad \Omega_0 = 2QS^4 \quad + A + A',$$

or substituting and reducing, the value of $8(M_0\Omega - M\Omega_0)$ to the proper degree of approximation is found to be

$$\begin{aligned}= &-8Q(2QS^4 + A + A')\cos\tfrac{1}{2}\pi u\\ &+8(Q^2S^4 + 8QA)\cos\tfrac{1}{2}\pi(u+2v) + 8(Q^2S^4 + 8QA')\cos\tfrac{1}{2}\pi(u-2v),\end{aligned}$$

which in virtue of the relations $QA = \Lambda^2S^3$, $QA' = \Lambda'^2S^3$, $Q^2S^2 = \Lambda\Lambda'$, is equal to the foregoing value of $c_3c_6\vartheta_{14}\vartheta_{11}$. I have thought it worth while to give this somewhat elaborate verification.

Résumé of the foregoing results.

120. In what precedes we have all the quadric relations between the 16 double theta-functions: or say we have the linear relations between squares (squared functions) and the linear relations between pairs (products of two functions): the number of the asyzygetic linear relations between squares is obviously $=12$; and that of the asyzygetic linear relations between pairs is $=60$ (since each of the 30 tetrads of pairs gives two asyzygetic relations): there are thus in all $12+60$, $=72$, asyzygetic linear relations. But these constitute only a 13-fold relation between the functions, viz. they are such as to give for the ratios of the 16 functions expressions depending upon two arbitrary parameters, x, y. Or taking the 16 functions as the coordinates of a point in 15-dimensional space, these coordinates are connected by a 13-fold relation (expressed by means of the foregoing system of 72 quadric equations), and the locus is thus a 13-fold, or two-dimensional, locus in 15-dimensional space.

Hence, taking any four of the functions, these are connected by a single equation; that is, regarding the four functions as the coordinates of a point in ordinary space, the locus of the point is a surface.

In particular, the four functions may be any four functions belonging to a hexad: by what precedes there is then a linear relation between the squares of the four functions: or the locus is a quadric surface. Each hexad gives 15 such surfaces, or the number of quadric surfaces is $(16\times 15=)$ 240.

The 16-*nodal quartic surfaces.*

121. If the four functions are those contained in any two pairs out of a tetrad of pairs (see the foregoing "Table of the 120 pairs"), then the locus is a quartic

surface, which is, in fact, a Kummer's 16-nodal quartic surface. For if for a moment $x.y$ and $z.w$ are two pairs out of a tetrad, and $r.s$ be either of the remaining pairs of the tetrad; then we have rs a linear function of xy and zw: squaring, r^2s^2 is a linear function of x^2y^2, $xyzw$, z^2w^2; but we then have r^2 and s^2, each of them a linear function of x^2, y^2, z^2, w^2; or substituting we have an equation of the fourth order, containing terms of the second order in (x^2, y^2, z^2, w^2), and also a term in $xyzw$. It is clear that, if instead of $r.s$ we had taken the remaining pair of the tetrad, we should have obtained the same quartic equation in (x, y, z, w). And moreover it appears by inspection that, if xy and zw are pairs in a tetrad, then xz and yw are pairs in a second tetrad, and xw and yz are pairs in a third tetrad: we obtain in each case the same quartic equation. We have from each tetrad of pairs six sets of four functions (x, y, z, w): and the number of such sets is thus $(\frac{1}{3}16 . 30 =)\, 60$: these are shown in the foregoing "Table of the 60 Göpel tetrads," viz. taking as coordinates of a point the four functions in any tetrad of this table, the locus is a 16-nodal quartic surface.

122. To exhibit the process I take a tetrad 4, 7, 8, 11 containing two odd functions; and representing these for convenience by x, y, z, w, viz. writing

$$\vartheta_4,\ \vartheta_7,\ \vartheta_8,\ \vartheta_{11}(u) = x,\ y,\ z,\ w,$$

we have then X, Y, Z, W linear functions of the four squares, viz. it is easy to obtain

$$\begin{aligned}
\alpha(x^2+z^2) - \delta(y^2+w^2) &= 2(\alpha^2-\delta^2)\,X,\\
\delta(\quad ,, \quad) - \alpha(\quad ,, \quad) &= 2(\quad ,, \quad)\,W,\\
-\beta(x^2-z^2) + \gamma(y^2-w^2) &= 2(\beta^2-\gamma^2)\,Y,\\
-\gamma(\quad ,, \quad) + \beta(\quad ,, \quad) &= 2(\quad ,, \quad)\,Z.
\end{aligned}$$

Also considering two other functions $\vartheta_0(u)$ and $\vartheta_{12}(u)$, or as for shortness I write them, ϑ_0 and ϑ_{12}, we have

$$\begin{aligned}
\vartheta_0^2 &= \alpha X + \beta Y + \gamma Z + \delta W,\\
\vartheta_{12}^2 &= \alpha X - \beta Y - \gamma Z + \delta W,
\end{aligned}$$

and substituting the foregoing values of X, Y, Z, W, we find

$$\begin{aligned}
M\vartheta_0^2 &= Ax^2 + By^2 + Cz^2 + Dw^2,\\
M\vartheta_{12}^2 &= Cx^2 + Dy^2 + Az^2 + Bw^2,
\end{aligned}$$

where, writing down the values first in terms of α, β, γ, δ and then in terms of the c's, we have

$$\begin{aligned}
M &= (\alpha^2-\delta^2)(\beta^2-\gamma^2) &&= \tfrac{1}{4}\ .\ c_8^4 - c_4^4,\\
A &= \beta^2\delta^2 - \alpha^2\gamma^2 &&= ,, \quad c_2^2c_6^2,\\
B &= -\alpha\delta(\beta^2-\gamma^2) + \beta\gamma(\alpha^2-\delta^2) &&= ,, \quad c_2^2c_4^2 - c_{15}^2c_8^2,\\
C &= \alpha^2\beta^2 - \gamma^2\delta^2 &&= ,, \quad c_1^2c_9^2,\\
D &= -\alpha\delta(\beta^2-\gamma^2) - \beta\gamma(\alpha^2-\delta^2) &&= ,, \quad c_{15}^2c_4^2 - c_2^2c_8^2;
\end{aligned}$$

and we then have further

$$c_4c_8\vartheta_0\vartheta_{12} = c_0c_{12}\vartheta_4\vartheta_8 + c_3c_{15}\vartheta_7\vartheta_{11},$$

that is,

$$c_4c_8\vartheta_0\vartheta_{12} = c_0c_{12}xz \quad + c_3c_{15}yw\,;$$

whence equating the two values of $\vartheta_0^2\vartheta_{12}^2$ we have the required quartic equation in x, y, z, w.

123. But the reduction is effected more simply if instead of the c's we introduce the rectangular coefficients a, b, c, &c. We then have

$$M = (c''^2 - b'^2),\quad A = -a''c,\quad C = a'b,$$
$$B = -b'c' - b''c'',\ = bc,\quad D = b'b'' + c'c'',\ = a'a''\,;$$

and the equations become

$$(c''^2 - b'^2)\vartheta_0^2 = -a''cx^2 + \quad bcy^2 + a'bz^2 - a'a''w^2,$$
$$(c''^2 - b'^2)\vartheta_{12}^2 = \quad a'bx^2 - a'a''y^2 - a''cz^2 + \quad bcw^2,$$
$$\sqrt{b'c''}\vartheta_0\vartheta_{12} = \sqrt{a}\,xz + \sqrt{-b''c'}\,yw,$$

so that the elimination gives

$$b'c''(-a''cx^2 + bcy^2 + a'bz^2 - a'a''w^2)(a'bx^2 - a'a''y^2 - a''cz^2 + bcw^2)$$
$$= (c''^2 - b'^2)^2\{ax^2z^2 - b''c'y^2w^2 + 2\sqrt{-ab''c'}\,xyzw\},$$

viz. this is

$$\begin{aligned}
&-a'a''bb'cc''(x^4 + y^4 + z^4 + w^4)\\
&+a'b'cc''(a''^2 + b^2)(x^2y^2 + z^2w^2)\\
&+\{b'c''(a'^2b^2 + a''^2c^2) - \quad a(b'^2 - c''^2)^2\}x^2z^2\\
&+\{b'c''(a'^2a''^2 + b^2c^2) + b''c'(b'^2 - c''^2)^2\}y^2w^2\\
&-a''bb'c''(a'^2 + c^2)(x^2w^2 + y^2z^2)\\
&-2(b'^2 - c''^2)^2\sqrt{-ab''c'}\,xyzw = 0.
\end{aligned}$$

124. In this equation the coefficients of x^2z^2 and y^2w^2 are each $= a'a''bc(b'^2 + c''^2)$, as at once appears from the identities

$$\begin{cases} a'b\,.\,b' - c''\,.\,a''c = a(b'^2 - c''^2),\\ a'b\,.\,c'' - b'\,.\,a''c = \quad (b'^2 - c''^2),\end{cases}$$
$$\begin{cases} a'a''\,.\,b' - c''\,.\,bc = -b''(b'^2 - c''^2),\\ a'a''\,.\,c - b'\,.\,bc = \quad c'(b'^2 - c''^2),\end{cases}$$

by multiplying together in each pair the left-hand and the right-hand sides respectively. Substituting and dividing by $-a'a''bb'cc''$, we have

$$\begin{aligned}
&x^4 + y^4 + z^4 + w^4\\
&-\frac{a''^2 + b^2}{a''b}(x^2y^2 + z^2w^2) - \frac{b'^2 + c''^2}{b'c''}(x^2z^2 + y^2w^2) + \frac{a'^2 + c^2}{a'c}(x^2w^2 + y^2z^2)\\
&+\frac{2(b'^2 - c''^2)^2\sqrt{-ab''c'}}{a'a''bb'cc''}\,xyzw = 0\,;
\end{aligned}$$

or, if we herein restore the c's in place of the rectangular coefficients, this is

$$\begin{aligned}&x^4+y^4+z^4+w^4\\&-\frac{c_1^4+c_2^4}{c_1^2c_2^2}(x^2y^2+z^2w^2)-\frac{c_4^4+c_3^4}{c_4^2c_3^2}(x^2z^2+y^2w^2)+\frac{c_6^4+c_9^4}{c_6^2c_9^2}(x^2w^2+y^2z^2)\\&+2\,\frac{c_0c_3c_{12}c_{15}(c_4^4-c_8^4)^2}{c_1^2c_2^2c_4^2c_6^2c_8^2c_9^2}\,xyzw=0,\end{aligned}$$

which is the equation of the 16-nodal quartic surface.

Substituting for x, y, z, w their values ϑ_4, ϑ_7, ϑ_8, $\vartheta_{11}(u)$, we have the equation connecting the four theta-functions 4, 7, 8, 11 of a Göpel tetrad. And there is an equation of the like form between the four functions of any other Göpel tetrad: for obtaining the actual equations some further investigation would be necessary.

The xy-expressions of the theta-functions.

125. The various quadric relations between the theta-functions, admitting that they constitute a 13-fold relation, show that the theta-functions may be expressed as proportional to functions of two arbitrary parameters x, y; and two of these functions being assumed at pleasure the others of them would be determinate; we have of course (though it would not be easy to arrive at it in this manner) such a system in the foregoing expressions of the 16 functions in terms of x, y; and conversely these expressions must satisfy identically the quadric relations between the theta-functions.

126. To show that this is so as to the general form of the equations, consider first the xy-factors $\sqrt{a}$, $\sqrt{ab}$, &c. As regards the squared functions $(\sqrt{ab})^2$, we have for instance

$$(\sqrt{ab})^2=\frac{1}{\theta^2}\{\mathrm{abfc_{,}d_{,}e_{,}}+\mathrm{a_{,}b_{,}f_{,}cde}+2\sqrt{XY}\},$$

$$(\sqrt{cd})^2=\frac{1}{\theta^2}\{\mathrm{cdfa_{,}b_{,}e_{,}}+\mathrm{c_{,}d_{,}f_{,}abe}+2\sqrt{XY}\};$$

each of these contains the same irrational part $\frac{2}{\theta^2}\sqrt{XY}$, and the difference is therefore rational: and it is moreover integral, for we have

$$(\sqrt{ab})^2-(\sqrt{cd})^2=\frac{1}{\theta^2}(\mathrm{abc_{,}d_{,}}-\mathrm{a_{,}b_{,}cd})(\mathrm{fe_{,}}-\mathrm{f_{,}e}),$$

where each factor divides by θ, and consequently the product by θ^2; the value is in fact

$$=(e-f)\begin{vmatrix}1, & x+y, & xy\\ 1, & a+b, & ab\\ 1, & c+d, & cd\end{vmatrix},$$

a linear function of 1, $x+y$, xy. This is the case as regards the difference of any two of the squares $(\sqrt{ab})^2$, $(\sqrt{ac})^2$, &c.; hence selecting any one of these squares, for instance $(\sqrt{de})^2$, any other of the squares is of the form

$$\lambda+\mu(x+y)+\nu xy+\rho(\sqrt{de})^2, \quad (\rho=1);$$

and obviously, the other squares $(\sqrt{a})^2$, &c., are of the like form, the last coefficient ρ being $=0$. We hence have the theorem that each square can be expressed as a linear function of any four (properly selected) squares.

127. But we have also the theorem of the 16 Kummer hexads.

Obviously the six squares

$$(\sqrt{a})^2,\quad (\sqrt{b})^2,\quad (\sqrt{c})^2,\quad (\sqrt{d})^2,\quad (\sqrt{e})^2,\quad (\sqrt{f})^2$$

are a hexad, viz. each of these is a linear function of 1, $x+y$, xy: and therefore selecting any three of them, each of the remaining three can be expressed as a linear function of these.

But further the squares $(\sqrt{a})^2$, $(\sqrt{b})^2$, $(\sqrt{ab})^2$, $(\sqrt{cd})^2$, $(\sqrt{ce})^2$, $(\sqrt{de})^2$ form a hexad. For reverting to the expression obtained for $(\sqrt{ab})^2-(\sqrt{cd})^2$, the determinant contained therein is a linear function of $\mathrm{aa}_{,}$ and $\mathrm{bb}_{,}$, that is, of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; we, in fact, have

$$(a-b)\begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix} = (b-c)(b-d)(a-x)(a-y)-(a-c)(a-d)(b-x)(b-y).$$

Hence $(\sqrt{ab})^2-(\sqrt{cd})^2$ is a linear function of $(\sqrt{a})^2$, $(\sqrt{b})^2$; and by a mere interchange of letters $(\sqrt{ab})^2-(\sqrt{ce})^2$, $(\sqrt{ab})^2-(\sqrt{de})^2$, are each of them also a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; whence the theorem. And we have thus all the remaining 15 hexads.

128. We have a like theory as regards the products of pairs of functions. A tetrad of pairs is of one of the two forms

$$\sqrt{a}\sqrt{b},\ \sqrt{ac}\sqrt{bc},\ \sqrt{ad}\sqrt{bd},\ \sqrt{ae}\sqrt{be},\ \text{and}\ \sqrt{f}\sqrt{ab},\ \sqrt{c}\sqrt{de},\ \sqrt{d}\sqrt{ce},\ \sqrt{e}\sqrt{cd};$$

in the first case the terms are

$$\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}},$$

$$\frac{1}{\theta^2}\{(\mathrm{ab}_{,}+\mathrm{a}_{,}\mathrm{b})\sqrt{\mathrm{cdefc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}}+(\mathrm{cfd}_{,}\mathrm{e}_{,}+\mathrm{c}_{,}\mathrm{f}_{,}\mathrm{de})\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}}\},$$

$$\frac{1}{\theta^2}\{\quad ,, \qquad ,, \qquad +(\mathrm{dfc}_{,}\mathrm{e}_{,}+\mathrm{d}_{,}\mathrm{f}_{,}\mathrm{ce}) \quad ,, \quad\},$$

$$\frac{1}{\theta^2}\{\quad ,, \qquad ,, \qquad +(\mathrm{efc}_{,}\mathrm{d}_{,}+\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{cd}) \quad ,, \quad\},$$

and as regards the last three terms the difference of any two of them is a mere constant multiple of $\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}}$; for instance, the second term − the third term is

$$=\frac{1}{\theta^2}(\mathrm{cd}_{,}-\mathrm{c}_{,}\mathrm{d})(\mathrm{fe}_{,}-\mathrm{f}_{,}\mathrm{e})\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}}, =(c-d)(f-e)\sqrt{\mathrm{aa}_{,}\mathrm{bb}_{,}};$$

we have thus a tetrad such that, selecting any two terms, each of the remaining terms is a linear function of these.

In the second case, the terms are

$$\frac{1}{\theta}\{\mathrm{f}\sqrt{\mathrm{abc,d,e,f,}}+\mathrm{f,}\sqrt{\mathrm{a,b,cdef}}\},$$

$$\frac{1}{\theta}\{\mathrm{c} \quad \text{,,} \quad +\mathrm{c,} \quad \text{,,} \quad \},$$

$$\frac{1}{\theta}\{\mathrm{d} \quad \text{,,} \quad +\mathrm{d,} \quad \text{,,} \quad \},$$

$$\frac{1}{\theta}\{\mathrm{e} \quad \text{,,} \quad +\mathrm{e,} \quad \text{,,} \quad \},$$

whence clearly the four terms are a tetrad as above. And it may be added that any linear function of the four terms is of the form

$$\frac{1}{\theta}\{(\lambda+\mu x)\sqrt{\mathrm{abc,d,e,f,}}+(\lambda+\mu y)\sqrt{\mathrm{a,b,cdef}}\}.$$

129. Considering next the actual equations between the squared theta-functions, take as a specimen

$$c_6^2\vartheta_6^2-c_2^2\vartheta_2^2+c_1^2\vartheta_1^2-c_9^2\vartheta_9^2=0,$$

that is,

$$c_6^4(\sqrt{ab})^2-c_2^4(\sqrt{cd})^2+c_1^4(\sqrt{ce})^2-c_9^4(\sqrt{de})^2=0,$$

where $c_6,\ c_2,\ c_1,\ c_9=\sqrt[4]{ab},\ \sqrt[4]{cd},\ \sqrt[4]{ce},\ \sqrt[4]{de}$ respectively. Since the functions $(\sqrt{ab})^2$, &c., contain the same irrational term $\frac{2}{\theta^2}\sqrt{XY}$, it is clear that the equation can only be true if

$$c_6^4-c_2^4+c_1^4-c_9^4=0:$$

and, this being so, it will be true if

$$c_2^4\{(\sqrt{ab})^2-(\sqrt{cd})^2\}-c_1^4\{(\sqrt{ab})^2-(\sqrt{ce})^2\}+c_9^4\{(\sqrt{ab})^2-(\sqrt{de})^2\}=0,$$

where, by what precedes, each of the terms in { } is a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$. Attending first to the term in $(\sqrt{a})^2$, the coefficient hereof is

$$ef.bc.bd.c_2^4-df.bc.be.c_1^4+cf.bd.be.c_9^4,$$

where for shortness bc, bd, &c., are written to denote the differences $b-c$, $b-d$, &c.: substituting for c_2^4 its value $(\sqrt{cd})^4$, $=cd.cf.df.ab.ae.be$, and similarly for c_1^4 and c_9^4 their values, $=ce.cf.ef.ab.ad.bd$, and $de.df.ef.ab.ac.bc$ respectively, the whole expression contains the factor $ab.bc.bd.be.cf.df.ef$, and throwing this out, the equation to be verified becomes

$$cd.ae-ce.ad+de.ac=0,$$

which is true identically. The verification is thus made to depend upon that of $c_6^4 - c_2^4 + c_1^4 - c_9^4 = 0$; and similarly for the other relations between the squared functions, the verification depends upon relations containing the fourth powers, or the products of squares, of the constants c and k.

130. Among these are included the before-mentioned system of equations involving the fourth powers or the products of squares of only the constants c; and it is interesting to show how these are satisfied identically by the values $c_0 = \sqrt[4]{bd}$, &c.

Thus one of these equations is $c_{12}^4 + c_1^4 + c_6^4 = c_9^4$; substituting the values, there is a factor ce which divides out, and the resulting equation is

$$ad\,.\,af\,.\,df\,.\,bc\,.\,be + cf\,.\,ef\,.\,ab\,.\,ad\,.\,bd + ab\,.\,af\,.\,bf\,.\,cd\,.\,de - ac\,.\,ae\,.\,bd\,.\,bf\,.\,df = 0.$$

There are here terms in a^2, a, a^0 which should separately vanish; for the terms in a^2, the equation becomes

$$df\,.\,bc\,.\,be + bd\,.\,cf\,.\,ef + bf\,.\,cd\,.\,de - bd\,.\,bf\,.\,df = 0,$$

which is easily verified; and the equations in a and a^0 may also be verified.

An equation involving products of the squares is $c_{12}^2c_9^2 - c_1^2c_4^2 + c_3^2c_6^2 = 0$. The term $c_{12}^2c_9^2$ is here $\sqrt{adf\,.\,bce}\,\sqrt{def\,.\,abc}$ which is $= \sqrt{(bc)^2(df)^2\,.\,ab\,.\,ac\,.\,ad\,.\,af\,.\,be\,.\,ce\,.\,de\,.\,ef}$, which is taken $= bc\,.\,df\sqrt{ab\,.\,ac\,.\,ad\,.\,af\,.\,be\,.\,ce\,.\,de\,.\,ef}$; similarly the values of $c_1^2c_4^2$ and $c_3^2c_6^2$ are $= bd\,.\,cf$ and $bf\,.\,cd$ each multiplied by the same radical, and the equation to be verified is

$$bc\,.\,df - bd\,.\,cf + bf\,.\,cd = 0,$$

which is right: the other equations may be verified in a similar manner.

131. Coming next to the equations connecting the pairs of theta-functions, for instance

$$c_3c_{15}\vartheta_0\vartheta_{12} - c_0c_{12}\vartheta_3\vartheta_{15} + c_4c_8\vartheta_7\vartheta_{11} = 0,$$

this is

$$c_3c_{15}c_0c_{12}\{\sqrt{bd}\,\sqrt{ad} - \sqrt{be}\,\sqrt{ae}\} + c_4c_8k_7k_{11}\,.\,\sqrt{b}\,\sqrt{a} = 0,$$

the products $\sqrt{bd}\,\sqrt{ad}$ and $\sqrt{be}\,\sqrt{ae}$ contain besides a common term the terms

$$\frac{1}{\theta^2}(\mathrm{dfc_{,}e_{,}} + \mathrm{d_{,}f_{,}ce})\sqrt{\mathrm{aa_{,}bb_{,}}},\ \text{and}\ \frac{1}{\theta^2}(\mathrm{efc_{,}d_{,}} + \mathrm{e_{,}f_{,}cd})\sqrt{\mathrm{aa_{,}bb_{,}}},$$

hence their difference contains $\frac{1}{\theta^2}(\mathrm{de_{,}} - \mathrm{d_{,}e})(\mathrm{fc_{,}} - \mathrm{f_{,}c})\sqrt{\mathrm{aa_{,}bb_{,}}}$ which is $= de\,.\,fc\sqrt{\mathrm{aa_{,}bb_{,}}}$, that is, $de\,.\,fc\sqrt{a}\,\sqrt{b}$: hence the equation to be verified is

$$de\,.\,fc\,.\,c_3c_{15}c_0c_{12} + c_4c_8k_7k_{11} = 0\,;$$

$c_3c_{15}c_0c_{12}$ is $= \sqrt[4]{bef\,.\,acd}\,\sqrt[4]{aef\,.\,bcd}\,\sqrt[4]{bdf\,.\,ace}\,\sqrt[4]{adf\,.\,bce}$, where under the fourth root we have 24 factors, which are, in fact, 12 factors twice repeated; and if we write Π, $= ab\,.\,ac\,.\,ad\,.\,ae\,.\,af\,.\,bc\,.\,bd\,.\,be\,.\,bf\,.\,cd\,.\,ce\,.\,cf\,.\,de\,.\,df\,.\,ef$, for the product of all the 15 factors, then the 12 factors are in fact all those of Π, except ab, cf, de; viz. we have

$$c_3c_{15}c_0c_{12} = \sqrt[4]{\Pi^2 \div (ab)^2(cf)^2(de)^2}.$$

Again, $c_4c_8k_7k_{11}, = \sqrt[4]{acf.bde}\,\sqrt[4]{bcf.ade}\,\sqrt[4]{acdef}\,\sqrt[4]{bcdef}$, is a fourth root of a product of 32 factors, which are in fact 16 factors twice repeated, and in the 16 factors, ab does not occur, cf and de occur each twice, and the other 12 factors each once: we thus have

$$c_4c_8k_7k_{11} = \sqrt[4]{11^2\,(cf)^2\,(de)^2 \div (ab)^2},$$

and the relation to be verified assumes the form

$$fc.de\,\sqrt[4]{1 \div (cf)^2\,(de)^2} + \sqrt[4]{(cf)^2\,(de)^2} = 0,$$

which, taking $fc.de = -\sqrt[4]{(cf)^4\,(de)^4}$, is right. And so for the other equations. It will be observed that, in the equation $de.fc.c_3c_{13}c_6c_{12} + c_4c_8k_7k_{11} = 0$, and in the other equations upon which the verifications depend, there is no ambiguity of sign: the signs of the radicals have to be determined consistently with all the equations which connect the c's and the k's: that this is possible appears evident *à priori*, but the actual verification presents some difficulty. I do not here enter further into the question.

Further results of the product-theorem, the $u \pm u'$ formulæ.

132. Recurring now to the equations in $u+u'$, $u-u'$, by putting therein $u'=0$, we can express X, Y, Z, W in terms of four of the squared functions of u, and by putting $u=0$ we can express X', Y', Z', W' in terms of four of the squared functions of u'; and, substituting in the original equations, we have the products

$$\vartheta(\ )u+u'.\vartheta(\ )u-u'$$

in terms of the squared functions of u and u'.

Selecting as in a former investigation the functions 4, 7, 8, 11, which were called x, y, z, w, it is more convenient to use single letters to represent the squared functions. I write

$\vartheta(u+u').\vartheta(u-u')$		ϑ^2u	ϑ^2u'	$\vartheta^2 0$
4	$4 = P,$	$4 = p,$	$4 = p',$	$4 = p_0\,(=c_4^2),$
7	$7 = Q,$	$7 = q,$	$7 = q',$	$7 = 0,$
8	$8 = R,$	$8 = r,$	$8 = r',$	$8 = r_0\,(=c_8^2),$
11	$11 = S,$	$11 = s,$	$11 = s',$	$11 = 0.$

Then

$X \quad Y \quad Z \quad W$	$X \quad Y \quad Z \quad W$	$X' \quad Y' \quad Z' \quad W'$
$P = X' - Y' + Z' - W',$	$p = \alpha - \beta + \gamma - \delta,$	$p' = \alpha - \beta + \gamma - \delta,$
$Q = W' - Z' + Y' - X',$	$q = \delta - \gamma + \beta - \alpha,$	$q' = \delta - \gamma + \beta - \alpha,$
$R = X' + Y' - Z' - W',$	$r = \alpha + \beta - \gamma - \delta,$	$r' = \alpha + \beta - \gamma - \delta,$
$S = W' + Z' - Y' - X',$	$s = \delta + \gamma - \beta - \alpha,$	$s' = \delta + \gamma - \beta - \alpha.$

Hence

$$\begin{aligned}
&\alpha(p+r) - \delta(q+s) = 2(\alpha^2-\delta^2)\,X, && \alpha(p'+r') - \delta(q'+s') = 2(\alpha^2-\delta^2)\,X',\\
&\delta \quad " \quad -\alpha \quad " \quad = 2 \quad " \quad W, && \delta \quad " \quad -\alpha \quad " \quad = 2 \quad " \quad W',\\
&-\beta(p-r) + \gamma(q-s) = 2(\beta^2-\gamma^2)\,Y, && -\beta(p'-r') + \gamma(q'-s') = 2(\beta^2-\gamma^2)\,Y',\\
&-\gamma \quad " \quad +\beta \quad " \quad = 2 \quad " \quad Z, && -\gamma \quad " \quad +\beta \quad " \quad = 2 \quad " \quad Z'.
\end{aligned}$$

By means of these values, we have

$$4(\alpha^2-\delta^2)^2 X'X = \alpha^2(p+r)(p'+r')+\delta^2(q+s)(q'+s')-\alpha\delta\,[(p+r)(q'+s')+(p'+r')(q+s)],$$

$$4 \quad ,, \quad W'W = \delta^2 \quad ,, \quad +\alpha^2 \quad ,, \quad -\alpha\delta\,[\quad ,, \quad ,, \quad],$$

$$4(\beta^2-\gamma^2)^2 Y'Y = \beta^2(p-r)(p'-r')+\gamma^2(q-s)(q'-s')-\beta\gamma\,[(p-r)(q'-s')+(p'-r')(q-s)],$$

$$4 \quad ,, \quad Z'Z = \gamma^2 \quad ,, \quad +\beta^2 \quad ,, \quad -\beta\gamma\,[\quad ,, \quad ,, \quad].$$

Hence

$$4(\alpha^2-\delta^2)(X'X-W'W) = (p+r)(p'+r')-(q+s)(q'+s'),$$

$$4(\beta^2-\gamma^2)(Y'Y-Z'\ Z) = (p-r)(p'-r')-(q-s)(q'-s'),$$

and substituting in the expressions for P and R,

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)P =$$

$$(\beta^2-\gamma^2)[(p+r)(p'+r')-(q+s)(q'+s')]-(\alpha^2-\delta^2)[(p-r)(p'-r')-(q-s)(q'-s')],$$

$$4 \quad ,, \quad R =$$

$$,, \quad [\quad ,, \quad ,, \quad]+ \quad ,, \quad [\quad ,, \quad ,, \quad].$$

Similarly

$$4(\alpha^2-\delta^2)^2 W'X =$$

$$\alpha\delta\,[(p+r)(p'+r')+(q+s)(q'+s')]-\alpha^2(p+r)(q'+s')-\delta^2(q+s)(p'+r'),$$

$$4 \quad ,, \quad X'W =$$

$$,, \quad [\quad ,, \quad ,, \quad]-\delta^2 \quad ,, \quad -\alpha^2 \quad ,, \quad ,$$

$$4(\beta^2-\gamma^2)^2 Z'Y =$$

$$\beta\gamma\,[(p-r)(p'-r')+(q-s)(q'-s')]-\beta^2(p-r)(q'-s')-\gamma^2(q-s)(p'-r'),$$

$$4 \quad ,, \quad Y'Z =$$

$$,, \quad [\quad ,, \quad ,, \quad]-\gamma^2 \quad ,, \quad -\beta^2 \quad ,, \quad ;$$

whence

$$4(\alpha^2-\delta^2)(W'X-X'W) = -[(p+r)(q'+s')-(p'+r')(q+s)],$$

$$4(\beta^2-\gamma^2)(Z'Y-Y'Z) = -[(p-r)(q'-s')-(p'-r')(q-s)],$$

and substituting in the expressions for Q and S

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)Q =$$

$$-(\beta^2-\gamma^2)[(p+r)(q'+s')-(p'+r')(q+s)]+(\alpha^2-\delta^2)[(p-r)(q'-s')-(p'-r')(q-s)],$$

$$4 \quad ,, \quad S =$$

$$- \quad ,, \quad [\quad ,, \quad ,, \quad]- \quad ,, \quad [\quad ,, \quad ,, \quad].$$

133. Hence, collecting and reducing,

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)P =$$
$$-(\alpha^2-\beta^2+\gamma^2-\delta^2)(pp'-qq'+rr'-ss')+(\alpha^2+\beta^2-\gamma^2-\delta^2)(pr'+p'r-qs'-q's),$$
$$4 \quad \text{,,} \quad R =$$
$$(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad)-(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad),$$
$$4 \quad \text{,,} \quad Q =$$
$$(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad)-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad),$$
$$4 \quad \text{,,} \quad S =$$
$$-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad)+(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad);$$

we have

$$p_0\,(=c_4^2)=\alpha^2-\beta^2+\gamma^2-\delta^2, \quad r_0\,(=c_6^2)=\alpha^2+\beta^2-\gamma^2-\delta^2,$$

and thence

$$r_0^2-p_0^2=4(\alpha^2-\delta^2)(\beta^2-\gamma^2);$$

the equations hence become

$$(r_0^2-p_0^2)P=-p_0(pp'-qq'+rr'-ss')+r_0(pr'+p'r-qs'-q's),$$
$$\text{,,} \quad R= \quad r_0(\quad \text{,,} \quad)-p_0(\quad \text{,,} \quad),$$
$$\text{,,} \quad Q= \quad p_0(pq'-p'q+rs'-r's)-r_0(\quad \text{,,} \quad),$$
$$\text{,,} \quad S=-r_0(\quad \text{,,} \quad)+p_0(\quad \text{,,} \quad).$$

On writing in the equations $u'=0$, then $P, Q, R, S, p', q', r', s'$ become $=p, q, r, s, p_0, 0, r_0, 0$; and the equations are (as they should be) true identically. The equations may be written

$(c^4\ c^4)$	$\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}$		$c^2(\overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2)$	$c^2(\overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2\ \overset{u}{\vartheta}^2.\overset{u'}{\vartheta}^2)$
$(8-4)$	4 . 4	$=$	$-4(4.4-7.7+8.8-11.11)$	$+8(4.8+8.4-7.11-11.7)$,
(,,)	8 8	$=$	$+8($,, $)$	$-4($,, $)$,
(,,)	7 7	$=$	$+4(4.7-7.4+8.11-11.8)$	$-8(4.11-11.4+8.7-7.8)$,
(,,)	11 11	$=$	$-8($,, $)$	$+4($,, $)$.

There is of course such a system for each of the 60 Göpel tetrads.

Differential relations connecting the theta-functions with the quotient-functions.

134. Imagine p, q, r, s, &c., changed into x^2, y^2, z^2, w^2, &c.; that is, let x, y, z, w represent the theta-functions 4, 7, 8, 11 of u, v; and similarly x', y', z', w' those of u', v', and $x_0, 0, z_0, 0$ those of 0, 0. Let u', v' be each of them indefinitely small; and take $\partial, =u'\dfrac{d}{du}+v'\dfrac{d}{dv}$, as the symbol of total differentiation in regard to u, v, the infinitesimals u' and v' being arbitrary: then, as far as the second order, we have in general

$$\vartheta(u+u', v+v')=\vartheta(u, v)+\partial\vartheta(u, v)+\tfrac{1}{2}\partial^2\vartheta(u, v),$$

and hence

$$P = (x + \partial x + \tfrac{1}{2}\partial^2 x)(x - \partial x + \tfrac{1}{2}\partial^2 x), = x^2 + \{x\partial^2 x - (\partial x)^2\},$$

and similarly for Q, R, S. Moreover, observing that x' and z' are even functions, y' and w' are odd functions, of u', v', we have

$$x',\ y',\ z',\ w' = x_0 + \tfrac{1}{2}\partial^2 x_0,\ \partial y_0,\ z_0 + \tfrac{1}{2}\partial^2 z_0,\ \partial w_0,$$

where $\partial^2 x_0$, ∂y_0, &c., are what $\partial^2 x$, ∂y, &c., become on writing therein $u = 0$, $v = 0$; ∂y_0, ∂w_0 are of course linear functions, $\partial^2 x_0$, $\partial^2 z_0$ quadric functions of u' and v'. The values of x'^2, y'^2, z'^2, w'^2 are thus $x_0^2 + x_0\partial^2 x_0$, $(\partial y_0)^2$, $z_0^2 + z_0\partial^2 z_0$, $(\partial w_0)^2$; and we have

$$\begin{array}{llllllllll}
 & & & & & & & x_0\partial^2 x_0 & (\partial y_0)^2 & z_0\partial^2 z_0 & (\partial w_0)^2 \\
\hline
x^2x'^2 & -y^2y'^2 & +z^2z'^2 & -w^2w'^2 = & x^2x_0^2 & +z^2z_0^2 & & +x^2 & -y^2 & +z^2 & -w^2, \\
x^2y'^2 & -y^2x'^2 & +z^2w'^2 & -w^2z'^2 = & -y^2x_0^2 & -w^2z_0^2 & & -y^2 & +x^2 & -w^2 & +z^2, \\
x^2z'^2 & -y^2w'^2 & +z^2x'^2 & -w^2y'^2 = & z^2x_0^2 & +x^2z_0^2 & & +z^2 & -w^2 & +x^2 & -y^2, \\
x^2w'^2 & -y^2z'^2 & +z^2y'^2 & -w^2x'^2 = & -w^2x_0^2 & -y^2z_0^2 & & -w^2 & +z^2 & -y^2 & +x^2.
\end{array}$$

135. On substituting these values, the constant terms (or terms independent of u', v') disappear of themselves; and the equations, transposing the second and third of them, become

$$\begin{array}{lllll}
 & & x_0\partial^2 x_0 & (\partial y_0)^2 & z_0\partial^2 z_0 & (\partial w_0)^2 \\
\hline
(z_0^4 - x_0^4)\{x\partial^2 x - (\partial x)^2\} = & & (-x_0^2x^2 + z_0^2z^2) & +(x_0^2y^2 - z_0^2w^2) & +(-x_0^2z^2 + z_0^2x^2) & +(x_0^2w^2 - z_0^2y^2), \\
\text{,,} & \{y\partial^2 y - (\partial y)^2\} = & -(x_0^2y^2 - z_0^2w^2) & -(-x_0^2x^2 + z_0^2z^2) & -(x_0^2w^2 - z_0^2y^2) & -(-x_0^2z^2 + z_0^2x^2), \\
\text{,,} & \{z\partial^2 z - (\partial z)^2\} = & (-x_0^2z^2 + z_0^2x^2) & +(x_0^2w^2 - z_0^2y^2) & +(-x_0^2x^2 + z_0^2z^2) & +(x_0^2y^2 - z_0^2w^2), \\
\text{,,} & \{w\partial^2 w - (\partial w)^2\} = & -(x_0^2w^2 - z_0^2y^2) & -(-x_0^2z^2 + z_0^2x^2) & -(x_0^2y^2 - z_0^2w^2) & -(-x_0^2x^2 + z_0^2z^2),
\end{array}$$

where it will be recollected that x, y, z, w mean ϑ_4, ϑ_7, ϑ_8, $\vartheta_{11}(u)$; x_0 is $\vartheta_4(0)$, that is, c_4, and z_0 is $\vartheta_8(0)$, that is, c_8. But the formulæ contain also

$$\partial^2 x_0 = (c_4''',\ c_4^{\text{iv}},\ c_4^{\text{v}}\between u',\ v')^2,\quad \partial y_0 = (c_7',\ c_7''\between u',\ v'),$$
$$\partial^2 z_0 = (c_8''',\ c_8^{\text{iv}},\ c_8^{\text{v}}\between u',\ v')^2,\quad \partial w_0 = (c_{11}',\ c_{11}''\between u',\ v').$$

The formulæ may be written

$$\begin{array}{lllll}
 & & c_4\partial^2 c_4 & (\partial c_7)^2 & c_8\partial^2 c_8 & (\partial c_{11})^2 \\
 & \{\overset{u}{\vartheta}.\partial^2\overset{u}{\vartheta} - (\partial\overset{u}{\vartheta})^2\} & c^2.\vartheta^2 \quad c^2.\vartheta^2 & c^2.\vartheta^2 \quad c^2.\vartheta^2 & c^2.\vartheta^2 \quad c^2.\vartheta^2 & c^2.\vartheta^2 \quad c^2.\vartheta^2 \\
\hline
(c_8^4 - c_4^4) & \{\ 4\ \ 4\ \ \ 4\ \} = & (-4\ \ 4 + 8\ \ 8) & +(\ 4\ \ 7 - 8\ 11) & +(-4\ \ 8 + 8\ \ 4) & +(\ 4\ 11 - 8\ \ 7), \\
\text{,,} & \{\ 7\ \ 7\ \ \ 7\ \} = & -(\ 4\ \ 7 - 8\ 11) & -(-4\ \ 4 + 8\ \ 8) & -(\ 4\ 11 - 8\ \ 7) & -(-4\ \ 8 + 8\ \ 4), \\
\text{,,} & \{\ 8\ \ 8\ \ \ 8\ \} = & (-4\ \ 8 + 8\ \ 4) & +(\ 4\ 11 - 8\ \ 7) & +(-4\ \ 4 + 8\ \ 8) & +(\ 4\ \ 7 - 8\ 11), \\
\text{,,} & \{11\ 11\ \ 11\ \} = & -(\ 4\ 11 - 8\ \ 7) & -(-4\ \ 8 + 8\ \ 4) & -(\ 4\ \ 7 - 8\ 11) & -(-4\ \ 4 + 8\ \ 8),
\end{array}$$

where $\partial^2 c_4$, $\partial^2 c_8$, ∂c_7, ∂c_{11} are written in place of $\partial^2 x_0$, $\partial^2 z_0$, ∂y_0, ∂w_0. There is of course a like system of equations for each of the Göpel tetrads.

136. Observe that, dividing the first equation by $\vartheta_4^2(u)$, or say by ϑ_4^2, the left-hand side is a mere constant multiple of $\partial^2 \log \vartheta_4$; and the right-hand side depends only on the quotient-functions $\vartheta_7 \div \vartheta_4$, $\vartheta_8 \div \vartheta_4$, $\vartheta_{11} \div \vartheta_4$; each side is a quadric function of u', v'. Equating the terms in u'^2, $u'v'$, v'^2 respectively, we have

$$\frac{d^2}{du^2} \log \vartheta_4, \quad \frac{d^2}{du\,dv} \log \vartheta_4, \quad \frac{d^2}{dv^2} \log \vartheta_4,$$

each of them expressed as a linear function of the squares of the quotient-functions $\vartheta_7 \div \vartheta_4$, $\vartheta_8 \div \vartheta_4$, $\vartheta_{11} \div \vartheta_4$. The formula is thus a second-derivative formula serving for the expression of a double theta-function by means of three quotient-functions.

Differential relations of the theta-functions.

137. In "The second set of 16," selecting the eight equations which contain Y_1 and W_1, these are

$$\begin{array}{rrrrrl}
 & u+u' & u-u' & u+u' & u-u' & \text{(Suffixes 1.)} \\
 & \vartheta\,. & \vartheta & \vartheta\,. & \vartheta & Y \quad W \\
\tfrac{1}{2}\{ & 4 & 0- & 0 & 4\} & = \overbrace{Y' + W'}, \\
 & 12 & 8- & 8 & 12 & = Y' - W', \\
 & 6 & 2- & 2 & 6 & = W' + Y', \\
 & 14 & 10- & 10 & 14 & = W' - Y', \\
\hline
\tfrac{1}{2}\{ & 5 & 1+ & 1 & 5\} & = X' + Z', \\
 & 13 & 9+ & 9 & 13 & = X' - Z', \\
 & 7 & 3+ & 3 & 7 & = Z' + X', \\
 & 15 & 11+ & 11 & 15 & = Z' - X'. \\
\hline
\end{array}$$

Then, considering any line in the upper half and any two lines in the lower half, we can from the three equations eliminate Y_1 and W_1, thus obtaining an equation such as

$$\begin{vmatrix} \vartheta_4\vartheta_0 - \vartheta_0\vartheta_4, & Y', & W' \\ \vartheta_5\vartheta_1 + \vartheta_1\vartheta_5, & X', & Z' \\ \vartheta_{13}\vartheta_0 + \vartheta_9\vartheta_{13}, & X', & -Z' \end{vmatrix} = 0,$$

viz. this is

$$\begin{aligned} & -2X'Z' && (\vartheta_4\vartheta_0 - \vartheta_0\vartheta_4) \\ & +(\quad X'W' + Y'Z') && (\vartheta_5\vartheta_1 + \vartheta_1\vartheta_5) \\ & +(-\ X'W' + Y'Z') && (\vartheta_{13}\vartheta_9 + \vartheta_9\vartheta_{13}) = 0, \end{aligned}$$

where the arguments of the theta-functions are as above, $u+u'$, $u-u'$, $u+u'$, $u-u'$; and the suffixes of the X', Y', Z', W' are all $=1$.

138. Suppose that in this equation u' becomes indefinitely small. If u' were $=0$, the values of X', Y', Z', W' would be α, 0, γ, 0: hence u' being indefinitely small, we take them to be α, $\partial\beta$, γ, $\partial\delta$, where

$$\partial\beta, \ =\left(u'\frac{d}{du}+v'\frac{d}{dv}\right)Y, \text{ and } \partial\delta, \ =\left(u'\frac{d}{du}+v'\frac{d}{dv}\right)W, \qquad (u=v=0),$$

are, in fact, linear functions of u' and v'.

We have $\vartheta_4\vartheta_0-\vartheta_0\vartheta_4$ standing for

$$\vartheta_4(u+u')\,\vartheta_0(u-u')-\vartheta_0(u+u')\,\vartheta_4(u-u'),$$

and here

$$\vartheta_4(u\pm u')=\vartheta_4\pm\partial\vartheta_4, \quad \vartheta_0(u\pm u')=\vartheta_0\pm\partial\vartheta_0;$$

the function in question is thus

$$(\vartheta_4+\partial\vartheta_4)(\vartheta_0-\partial\vartheta_0)-(\vartheta_4-\partial\vartheta_4)(\vartheta_0+\partial\vartheta_0)=2\{\vartheta_0\partial\vartheta_4-\vartheta_4\partial\vartheta_0\},$$

where the arguments are u, v, and the ∂ denotes $u'\dfrac{d}{du}+v'\dfrac{d}{dv}$.

Also $\vartheta_5\vartheta_1+\vartheta_1\vartheta_5$, that is, $\vartheta_5(u+u')\,\vartheta_1(u-u')+\vartheta_1(u+u')\,\vartheta_5(u-u')$, becomes simply $=2\vartheta_5\vartheta_1$, and similarly $\vartheta_{13}\vartheta_9+\vartheta_9\vartheta_{13}$ becomes $=2\vartheta_{13}\vartheta_9$; and the equation thus is

$$-2\alpha_1\gamma_1(\vartheta_0\partial\vartheta_4-\vartheta_4\partial\vartheta_0)+(\alpha_1\partial\delta_1+\gamma_1\partial\beta_1)\,\vartheta_5\vartheta_1+(-\alpha_1\partial\delta_1+\gamma_1\partial\beta_1)\,\vartheta_{13}\vartheta_9=0,$$

where the proper suffix 1 is restored to the α, $\partial\beta$, γ, and $\partial\delta$.

139. The equation shows that the differential combination $\vartheta_0\partial\vartheta_4-\vartheta_4\partial\vartheta_0$ is a linear function of $\vartheta_5\vartheta_1$ and $\vartheta_{13}\vartheta_9$, the coefficients of these products being of course linear functions of u' and v'. Writing the equation

$$\vartheta_0\partial\vartheta_4-\vartheta_4\partial\vartheta_0=A\vartheta_5\vartheta_1+B\vartheta_{13}\vartheta_9,$$

we can if we please determine the coefficients in terms of the constants c', c'', c''', c^{iv}, c^{v}; viz. taking u, v indefinitely small, we have

$$\begin{aligned}
&\vartheta_0=c_0, \quad \partial\vartheta_4=u'(c_4'''u+c_4^{\text{iv}}v)+v'(c_4^{\text{iv}}u+c_4^{\text{v}}v),\\
&\vartheta_4=c_4, \quad \partial\vartheta_0=u'(c_0'''u+c_0^{\text{iv}}v)+v'(c_0^{\text{iv}}u+c_0^{\text{v}}v),\\
&\vartheta_1=c_1, \quad \vartheta_5=\quad c_5'u+c_5''v,\\
&\vartheta_9=c_9, \quad \vartheta_{13}=\quad c_{13}'u+c_{13}''v,
\end{aligned}$$

or substituting, and equating the coefficients of u and v respectively, we have

$$\begin{aligned}
&c_0(c_4'''u'+c_4^{\text{iv}}v')-c_4(c_0'''u'+c_0^{\text{iv}}v')=Ac_1c_5'+Bc_9c_{13}',\\
&c_0(c_4^{\text{iv}}u'+c_4^{\text{v}}v')-c_4(c_0^{\text{iv}}u'+c_0^{\text{v}}v')=Ac_1c_5''+Bc_9c_{13}'',
\end{aligned}$$

which equations give the values of A, B.

140. Disregarding the values of the coefficients, and attending only to the form of the equation

$$\vartheta_0\partial\vartheta_4-\vartheta_4\partial\vartheta_0=A\vartheta_5\vartheta_1+B\vartheta_{13}\vartheta_9,$$

this is one of a system of 120 equations; viz. referring to the foregoing table of the 120 pairs, it in fact appears that taking any pair such as $\vartheta_0\vartheta_4$ out of the upper compartment or the lower compartment of any column of the table, the corresponding differential combination $\vartheta_0\partial\vartheta_4 - \vartheta_4\partial\vartheta_0$ is a linear function of any two of the four pairs in the other compartment of the same column.

Differential relation of u, v and x, y.

141. We have, as before, in the two notations, the pairs

$A\,.\,B$	$11\,.\,7$
$C\,.\,DE$	$5\,.\,9$
$D\,.\,CE$	$13\,.\,1$
$E\,.\,CD$	$14\,.\,2$
$F\,.\,AB$	$10\,.\,6.$

From the expressions given above for the four pairs below the line, it is clear that any linear function of these four pairs may be represented by

$$(a-b)\frac{1}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa,b,}}+(\lambda+\mu x)\sqrt{\mathrm{c,d,e,f,ab}}\},$$

where λ, μ are constant coefficients: the factor $(a-b)$ has been introduced for convenience, as will appear.

We have consequently a relation

$$\sqrt{\mathrm{aa,}}\,\partial\sqrt{\mathrm{bb,}}-\sqrt{\mathrm{bb,}}\,\partial\sqrt{\mathrm{aa,}}=\frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa,b,}}+(\lambda+\mu x)\sqrt{\mathrm{c,d,e,f,ab}}\},$$

where, as before, ∂ is used to denote $u'\dfrac{d}{du}+v'\dfrac{d}{dv}$, u' and v' being arbitrary multipliers; considering u, v as functions of x, y, we have

$$\frac{d}{du}=\frac{dx}{du}\frac{d}{dx}+\frac{dy}{du}\frac{d}{dy},$$

$$\frac{d}{dv}=\frac{dx}{dv}\frac{d}{dx}+\frac{dy}{dv}\frac{d}{dy},$$

and thence $\partial=P\dfrac{d}{dx}+Q\dfrac{d}{dy}$, if for shortness P and Q are written to denote $u'\dfrac{dx}{du}+v'\dfrac{dx}{dv}$ and $u'\dfrac{dy}{du}+v'\dfrac{dy}{dv}$ respectively.

142. The left-hand side then is

$$=P\left(\sqrt{\mathrm{aa,}}\frac{d}{dx}\sqrt{\mathrm{bb,}}-\sqrt{\mathrm{bb,}}\frac{d}{dx}\sqrt{\mathrm{aa,}}\right)+Q\left(\sqrt{\mathrm{aa,}}\frac{d}{dy}\sqrt{\mathrm{bb,}}-\sqrt{\mathrm{bb,}}\frac{d}{dy}\sqrt{\mathrm{aa,}}\right);$$

the coefficients of P and Q are at once found to be

$$= -\tfrac{1}{2}\frac{(\mathrm{a}-\mathrm{b})\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}}, \quad -\tfrac{1}{2}\frac{(\mathrm{a}_{,}-\mathrm{b}_{,})\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}},$$

respectively, or observing that $\mathrm{a}-\mathrm{b}, = \mathrm{a}_{,}-\mathrm{b}_{,}, = a-b$, the equation becomes

$$P\frac{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}} + Q\frac{\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}} = -\frac{2}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}} + (\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\};$$

or multiplying by $\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}$ and writing for shortness $\mathrm{abcdef} = X$, $\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,} = Y$, this becomes

$$\mathrm{a}_{,}\mathrm{b}_{,}\{P + \frac{2}{\theta}(\lambda+\mu y)\sqrt{X}\} + \mathrm{ab}\{Q + \frac{2}{\theta}(\lambda+\mu x)\sqrt{Y}\} = 0.$$

143. There are, it is clear, the like equations

$$\mathrm{b}_{,}\mathrm{c}_{,}\{P + \frac{2}{\theta}(\lambda'+\mu' y)\sqrt{X}\} + \mathrm{bc}\{Q + \frac{2}{\theta}(\lambda'+\mu' x)\sqrt{Y}\} = 0,$$

$$\mathrm{c}_{,}\mathrm{a}_{,}\{P + \frac{2}{\theta}(\lambda''+\mu'' y)\sqrt{X}\} + \mathrm{ca}\{Q + \frac{2}{\theta}(\lambda''+\mu'' x)\sqrt{Y}\} = 0,$$

and it is to be shown that $\lambda = \lambda' = \lambda''$ and $\mu = \mu' = \mu''$. For this purpose, recurring to the forms

$$\sqrt{\mathrm{aa}_{,}}\,\partial\sqrt{\mathrm{bb}_{,}} - \sqrt{\mathrm{bb}_{,}}\,\partial\sqrt{\mathrm{aa}_{,}} = \frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}} + (\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\},$$

$$\sqrt{\mathrm{bb}_{,}}\,\partial\sqrt{\mathrm{cc}_{,}} - \sqrt{\mathrm{cc}_{,}}\,\partial\sqrt{\mathrm{bb}_{,}} = \frac{b-c}{\theta}\{(\lambda'+\mu' y)\sqrt{\mathrm{adefb}_{,}\mathrm{c}_{,}} + (\lambda'+\mu' x)\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{bc}}\},$$

$$\sqrt{\mathrm{cc}_{,}}\,\partial\sqrt{\mathrm{aa}_{,}} - \sqrt{\mathrm{aa}_{,}}\,\partial\sqrt{\mathrm{cc}_{,}} = \frac{c-a}{\theta}\{(\lambda''+\mu'' y)\sqrt{\mathrm{bdefc}_{,}\mathrm{a}_{,}} + (\lambda''+\mu'' x)\sqrt{\mathrm{b}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ca}}\},$$

multiply the first equation by $\sqrt{\mathrm{cc}_{,}}$, the second by $\sqrt{\mathrm{aa}_{,}}$, and the third by $\sqrt{\mathrm{bb}_{,}}$, and add: the left-hand side vanishes, and therefore the right-hand side must also vanish identically.

144. But on the right-hand side we have the term $\frac{1}{\theta}\sqrt{\mathrm{defa}_{,}\mathrm{b}_{,}\mathrm{c}_{,}}$ multiplied by

$$(a-b)\,\mathrm{c}\,(\lambda+\mu y) + (b-c)\,\mathrm{a}\,(\lambda'+\mu' y) + (c-a)\,\mathrm{b}\,(\lambda''+\mu'' y),$$

and the term $-\frac{1}{\theta}\sqrt{\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{abc}}$ multiplied by

$$(a-b)\,\mathrm{c}_{,}\,(\lambda+\mu x) + (b-c)\,\mathrm{a}_{,}\,(\lambda'+\mu' x) + (c-a)\,\mathrm{b}_{,}\,(\lambda''+\mu'' x),$$

and it is clear that the whole can vanish only if these two coefficients separately vanish. This will be the case if we have for λ, λ', λ'' the equations

$$(a-b)\lambda + (b-c)\lambda' + (c-a)\lambda'' = 0,$$
$$c \quad " \quad +a \quad " \quad +b \quad " \quad = 0,$$

and the like equations for μ, μ', μ''. The equations written down give

$$(a-b)\lambda : (b-c)\lambda' : (c-a)\lambda'' = a-b : b-c : c-a,$$

that is, $\lambda = \lambda' = \lambda''$: and similarly $\mu = \mu' = \mu''$.

145. But this being so, the three equations in P, Q give

$$P + \frac{2}{\theta}(\lambda + \mu y)\sqrt{X} = 0, \quad Q + \frac{2}{\theta}(\lambda + \mu x)\sqrt{Y} = 0,$$

that is,

$$u'\frac{dx}{du} + v'\frac{dx}{dv} = -\frac{2}{x-y}(\lambda + \mu y)\sqrt{X},$$

$$u'\frac{dy}{du} + v'\frac{dy}{dv} = -\frac{2}{x-y}(\lambda + \mu x)\sqrt{Y}.$$

In these equations u' and v' are arbitrary; hence λ and μ must be linear functions of u' and v'; say their values are $= \varpi u' + \rho v'$, $\sigma u' + \tau v'$ respectively. We have therefore

$$\frac{dx}{du} = -\frac{2}{\theta}(\varpi + \sigma y)\sqrt{X}, \quad \frac{dx}{dv} = -\frac{2}{\theta}(\rho + \tau y)\sqrt{X},$$

$$\frac{dy}{du} = -\frac{2}{\theta}(\varpi + \sigma x)\sqrt{Y}, \quad \frac{dy}{dv} = -\frac{2}{\theta}(\rho + \tau x)\sqrt{Y},$$

or, what is the same thing,

$$-\tfrac{1}{2}\theta\frac{dx}{\sqrt{X}} = (\varpi + \sigma y)\,du + (\rho + \tau y)\,dv,$$

$$-\tfrac{1}{2}\theta\frac{dy}{\sqrt{Y}} = (\varpi + \sigma x)\,du + (\rho + \tau x)\,dv,$$

whence also

$$\sigma du + \tau dv = \tfrac{1}{2}\left(\frac{dx}{\sqrt{X}} - \frac{dy}{\sqrt{Y}}\right),$$

$$\varpi du + \rho dv = -\tfrac{1}{2}\left(\frac{x\,dx}{\sqrt{X}} - \frac{y\,dy}{\sqrt{Y}}\right),$$

which are the required relations, depending on the square roots of the sextic functions $X = \text{abcdef}$, and $Y = \text{a,b,c,d,e,f}$, of x and y respectively; but containing the constants ϖ, ρ, σ, τ, the values of which are not as yet ascertained.

146. I commence the integration of these equations on the assumption that the values $u = 0$, $v = 0$ correspond to indefinitely large values of x and y. We have

$$X = x^6\left(1 - \frac{S}{x} + \ldots\right), \quad Y = y^6\left(1 - \frac{S}{y} + \ldots\right),$$

where $S = a + b + c + d + e + f$; and thence the equations are

$$\sigma du + \tau dv = \tfrac{1}{2}\frac{dx}{x^3}\left(1 + \frac{\frac{1}{2}S}{x} + \ldots\right) - \tfrac{1}{2}\frac{dy}{y^3}\left(1 + \frac{\frac{1}{2}S}{y} + \ldots\right),$$

$$\varpi du + \rho dv = -\tfrac{1}{2}\frac{dx}{x^2}\left(1 + \frac{\frac{1}{2}S}{x} + \ldots\right) + \tfrac{1}{2}\frac{dy}{y^2}\left(1 + \frac{\frac{1}{2}S}{y} + \ldots\right).$$

Hence integrating, we have

$$\sigma u + \tau v = -\tfrac{1}{2}\left(\frac{1}{x^2} - \frac{1}{y^2}\right) + \ldots,$$

$$\varpi u + \rho v = \tfrac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \tfrac{1}{8}S\left(\frac{1}{x^2} - \frac{1}{y^2}\right) + \ldots,$$

and thence

$$\varpi u + \rho v + \tfrac{1}{4}S(\sigma u + \tau v) = \tfrac{1}{2}\left(\frac{1}{x} - \frac{1}{y}\right) + \ldots,$$

where the omitted terms depend on $\dfrac{1}{x^3}$, $\dfrac{1}{y^3}$ &c.

Hence, neglecting these terms, we have

$$\frac{\sigma u + \tau v}{\varpi u + \rho v + \frac{1}{4}S(\sigma u + \tau v)} = -\left(\frac{1}{x} + \frac{1}{y}\right),$$

an equation connecting the indefinitely small values of u, v, with the indefinitely large values of x, y.

147. From the equations $A = k_{11}\varpi\sqrt{a}$, $B = k_7\varpi\sqrt{b}$, taking (u, v) indefinitely small and therefore (x, y) indefinitely large, we deduce

$$\frac{c_{11}'u + c_{11}''v}{c_7'u + c_7''v} = \frac{k_{11}}{k_7}\,\frac{1 - \frac{1}{2}a\left(\frac{1}{x} + \frac{1}{y}\right)}{1 - \frac{1}{2}b\left(\frac{1}{x} + \frac{1}{y}\right)};$$

hence substituting for $\dfrac{1}{x} + \dfrac{1}{y}$ the foregoing value, and introducing an indeterminate multiplier M, we obtain

$$c_{11}'u + c_{11}''v = Mk_{11}\{\varpi u + \rho v + \tfrac{1}{4}S(\sigma u + \tau v) + \tfrac{1}{2}a(\sigma u + \tau v)\},$$

which breaks up into the two equations

$$c_{11}' = Mk_{11}\{\varpi + (\tfrac{1}{4}S + \tfrac{1}{2}a)\sigma\}, \quad c_{11}'' = Mk_{11}\{\rho + (\tfrac{1}{4}S + \tfrac{1}{2}a)\tau\}.$$

Similarly

$$\begin{aligned}
c_7' &= Mk_7\{\quad \text{,,} \quad b\ \}, & c_7'' &= Mk_7\{\quad \text{,,} \quad b\ \},\\
c_5' &= Mk_5\{\quad \text{,,} \quad c\ \}, & c_5'' &= Mk_5\{\quad \text{,,} \quad c\ \},\\
c_{13}' &= Mk_{13}\{\quad \text{,,} \quad d\ \}, & c_{13}'' &= Mk_{13}\{\quad \text{,,} \quad d\ \},\\
c_{14}' &= Mk_{14}\{\quad \text{,,} \quad e\ \}, & c_{14}'' &= Mk_{14}\{\quad \text{,,} \quad e\ \},\\
c_{10}' &= Mk_{10}\{\quad \text{,,} \quad f\ \}, & c_{10}'' &= Mk_{10}\{\quad \text{,,} \quad f\ \},
\end{aligned}$$

which twelve equations determine the coefficients ϖ, σ, ρ, τ in terms of the c', c'' of the odd functions 5, 7, 10, 11, 13, 14; and moreover give rise to relations connecting these c', c'' with each other and with the constants a, b, c, d, e, f.

148. It is observed that if, as before,

$$\partial = u'\frac{d}{du} + v'\frac{d}{dv}, \quad = P\frac{d}{dx} + Q\frac{d}{dy},$$

then, substituting for P and Q their values, we have

$$\partial = -\frac{2}{\theta}(\varpi u' + \rho v')\left(\sqrt{X}\frac{d}{dx} + \sqrt{Y}\frac{d}{dy}\right) - \frac{2}{\theta}(\sigma u' + \tau v')\left(y\sqrt{X}\frac{d}{dx} + x\sqrt{Y}\frac{d}{dy}\right),$$

$$= (\varpi u' + \rho v')\partial_1 + (\sigma u' + \tau v')\partial_2,$$

if for shortness

$$\partial_1 = -\frac{2}{\theta}\left(\sqrt{X}\frac{d}{dx} + \sqrt{Y}\frac{d}{dy}\right), \quad \partial_2 = -\frac{2}{\theta}\left(y\sqrt{X}\frac{d}{dx} + x\sqrt{Y}\frac{d}{dy}\right);$$

then operating with ∂ on the equations $A = \varpi k_{11}\sqrt{ab}$, &c., we have for instance

$$A\partial B - B\partial A = \varpi^2 k_{11} k_7 \{ \quad (\varpi u' + \rho v')(\sqrt{a}\,\partial_1\sqrt{b} - \sqrt{b}\,\partial_1\sqrt{a})$$
$$+ (\sigma u' + \tau v')(\sqrt{a}\,\partial_2\sqrt{b} - \sqrt{b}\,\partial_2\sqrt{a})\},$$

which is one of a system of 120 equations, the A, B being in fact any two of the 16 functions.

These are in fact nothing else than the foregoing system of 120 equations giving the values of the differential combinations $\vartheta_0\partial\vartheta_1 - \vartheta_1\partial\vartheta_0$, &c., each as a sum of products of pairs of functions, only on the right-hand sides we have expressions such as $\sqrt{a}\,\partial_1\sqrt{b} - \sqrt{b}\,\partial_1\sqrt{a}$, &c., which present themselves as perfectly determinate functions of x, y: so that, regarding $\varpi u' + \rho v'$, $\sigma u' + \tau v'$ as given linear functions of the arbitrary quantities u', v', there is no longer anything indeterminate in the form of the equations.

705.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. XIV. to LXI. (1871—1894).]

[Vol. XIV., July to December, 1870, pp. 17—19.]

3002. (PROPOSED by MATTHEW COLLINS, B.A.)—If every two of five circles A, B, C, D, E touch each other, except D and E, prove that the common tangent of D and E is just twice as long as it would be if D and E touched each other.

Solution by PROFESSOR CAYLEY.

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, foci R, S; the coordinates of a point U on the ellipse may be taken to be $(a\cos u,\ b\sin u)$, and then the distances of this point from the foci will be

$$r=a(1-e\cos u),\quad s=a(1+e\cos u).$$

Taking k arbitrarily, with centre R describe a circle radius $a-k$, with centre S a circle radius $a+k$, and with centre U a circle radius $k-ae\cos u$: say these are the circles R, S, U respectively; the circle U will touch each of the circles R, S (viz. assuming $ae<k<a$, so that the foregoing radii are all positive, it will touch the circle R externally and the circle S internally).

Considering next a point V, coordinates $(a\cos v,\ b\sin v)$, and the circle described about this point with the radius $k-ae\cos v$, say the circle V; this will touch in like manner the circles R, S respectively. And the circles U, V may be made to touch each other externally; viz. this will be the case if squared sum of radii = squared

distance of centres, or what is the same thing, squared difference of radii + 4 times the product of radii = squared distance of centres; that is,

$$a^2e^2(\cos u - \cos v)^2 + 4(k - ae\cos u)(k - ae\cos v) = a^2(\cos u - \cos v)^2 + b^2(\sin u - \sin v)^2,$$

or

$$2(k - ae\cos u)(k - ae\cos v) = b^2\{1 - \cos(u - v)\}.$$

If for a moment we write $\tan\frac{1}{2}u = x$, $\tan\frac{1}{2}v = y$, and therefore

$$\cos u = \frac{1 - x^2}{1 + x^2}, \quad \cos v = \frac{1 - y^2}{1 + y^2}, \quad \sin u = \frac{2x}{1 + x^2}, \quad \sin v = \frac{2y}{1 + y^2},$$

$$\cos(u - v) = \frac{(1 - x^2)(1 - y^2) + 4xy}{(1 + x^2)(1 + y^2)}, \quad 1 - \cos(u - v) = \frac{2(x - y)^2}{(1 + x^2)(1 + y^2)},$$

we have

$$\left\{k - \frac{ae(1 - x^2)}{1 + x^2}\right\}\left\{k - \frac{ae(1 - y^2)}{1 + y^2}\right\} = \frac{b^2(x - y)^2}{(1 + x^2)(1 + y^2)},$$

or

$$\{k - ae + (k + ae)x^2\}\{k - ae + (k + ae)y^2\} = b^2(x - y)^2,$$

which is readily identified with the circular relation

$$\tan^{-1} y\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}} - \tan^{-1} x\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}} = \tan^{-1}\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}}:$$

or, what is the same thing, in order that the circles U, V may touch, the relation between the parameters u, v must be

$$\tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}v\right\} - \tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\} = \tan^{-1}\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}}.$$

Considering in like manner a circle, centre the point W, coordinates $(a\cos w, b\sin w)$, and radius $k - ae\cos w$, say the circle W; this will, as before, touch the circles R, S; and we may make W touch each of the circles U, V; viz. we must have

$$\tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}w\right\} - \tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}v\right\} = \tan^{-1}\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}},$$

$$\tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\} - \tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}w\right\} = \tan^{-1}\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}},$$

where, in the last equation, $\tan^{-1}\left\{\left(\frac{k + ae}{k - ae}\right)^{\frac{1}{2}}\tan\frac{1}{2}u\right\}$ must be considered as denoting its value in the first equation increased by π. Hence, adding the three equations, we have

$$\pi = 3\tan^{-1}\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}},$$

that is,

$$\left(\frac{k^2 - a^2e^2}{a^2 - k^2}\right)^{\frac{1}{2}} = \tan\tfrac{1}{3}\pi = \sqrt{3};$$

or

$$k^2 - a^2e^2 = 3(a^2 - k^2),$$

that is,

$$3a^2 - 4k^2 + a^2e^2 = 0;$$

viz. this is the condition for the existence of the three circles U, V, W, each touching the two others, and also the circles R, S.

The circle R lies inside the circle S, and the tangential distance is thus imaginary; but defining it by the equation

squared tangential dist. = squared dist. of centres − squared sum of radii,

the squared tangential distance is

$$= 4a^2e^2 - 4a^2.$$

But if the circles were brought into contact, the distance of the centres would be $2k$, and the value of the squared tangential distance $= 4k^2 - 4a^2$; hence, if this be = one-fourth of the former value, we have

$$4(k^2 - a^2) = a^2e^2 - a^2,$$

that is,

$$3a^2 - 4k^2 + a^2e^2 = 0,$$

the same condition as above. The solution might easily be varied in such wise that the circles R, S should be external to each other, and therefore the tangential distance real; but the case here considered, where the locus of the centres of the circles U, V, W is an ellipse, is the more convenient, and may be regarded as the standard case.

[Vol. XIV., p. 19.]

3144. (Proposed by Professor CAYLEY.)—If the extremities A, A' of a given line AA' describe given lines respectively, show that there is a point rigidly connected with AA' which describes a circle.

[Vol. XIV., pp. 67, 68.]

3120. (Proposed by Professor CAYLEY.)—To find the equation of the Jacobian of the quadric surfaces through the six points

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), (\alpha, \beta, \gamma, \delta).$$

Solution by the PROPOSER.

Writing for shortness

$$a = \beta - \gamma, \quad b = \gamma - \alpha, \quad c = \alpha - \beta, \quad f = \alpha - \delta, \quad g = \beta - \delta, \quad h = \gamma - \delta,$$

(so that $a+h-g=0$, &c., $a+b+c=0$, $af+bg+ch=0$), the six points lie in each of the plane-pairs

$$x(hy-gz+aw)=0, \quad y(-hx+fz+bw)=0,$$
$$z(gx-fy+cw)=0, \quad w(-ax-by-cz)=0.$$

We cannot take these as the four quadrics, on account of the identical equation $0=0$, which is obtained by adding the four equations; but we may take the first three of them for three of the quadrics, and for the fourth quadric the cone, vertex (0, 0, 0, 1), which passes through the other five points; viz. this is

$$a\alpha yz+b\beta zx+c\gamma xy=0.$$

We write therefore

$$P=x(hy-gz+aw), \quad Q=y(-hx+fz+bw),$$
$$R=z(gx-fy+cw), \quad S=a\alpha yz+b\beta zx+c\gamma xy;$$

and we equate to zero the determinant formed with the derived functions of P, Q, R, S in regard to the coordinates (x, y, z, w) respectively. If, for a moment, we write A, B, C to denote $bg-ch$, $ch-af$, $af-bg$ respectively, it is easily found that the term containing d_xS is

$$(b\beta z+c\gamma y)\,x(-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -gB,\ hC,\ aA,\ b^2g,\ -c^2h\between x,\ y,\ z,\ w)^2:$$

the terms containing d_yS and d_zS are derived from this by a mere cyclical interchange of the letters (x, y, z), (A, B, C), (a, b, c), and (f, g, h). Collecting and reducing, it is found that the whole equation divides by $2abc$; and that, omitting this factor, the result is

$$\left.\begin{aligned} &ayz(\alpha w^2-\delta x^2)+fxw(\beta z^2-\gamma y^2)\\ &+bzx(\beta w^2-\delta y^2)+gyw(\gamma x^2-\alpha z^2)\\ &+cxy(\gamma w^2-\delta z^2)+hzw(\alpha y^2-\beta x^2)\end{aligned}\right\}=0,$$

which, substituting for a, b, c, f, g, h their values, is the required form.

If, in the equation, we write for instance $x=0$, the equation becomes

$$\alpha yzw(hy-gz+aw)=0;$$

or, the section by the plane is made up of four lines. Calling the given points 1, 2, 3, 4, 5, 6, it thus appears that the surface contains the fifteen lines 12, 13, ..., 56, and also the ten lines 123 . 456, &c.; in all twenty-five lines. Moreover, since the surface contains the lines 12, 13, 14, 15, 16, it is clear that the point 1 is a node (conical point) on the surface; and the like as to the points 2, 3, 4, 5, 6.

[Vol. XIV., pp. 104, 105.]

3249. (Proposed by Professor CAYLEY.)—Given on a given conic two quadrangles $PQRS$ and $pqrs$, having the same centres, and such that P, p; Q, q; R, r; S, s are the corresponding vertices (that is, the four lines PQ, RS, pq, rs all pass through

the same point; and similarly the lines PR, QS, pr, qs, and the lines PS, QR, ps, qr): it is required to show that a conic may be drawn, passing through the points p, q, r, s and touched at these points by the lines pP, qQ, rR, sS, respectively.

Solution by the PROPOSER.

Taking the centres for the vertices of the fundamental triangle, the equation of the given conic may be taken to be $x^2+y^2+z^2=0$; and then the coordinates of P, Q, R, S to be (A, B, C), $(A, -B, C)$, $(A, B, -C)$, $(A, -B, -C)$ respectively, where $A^2+B^2+C^2=0$; and those of p, q, r, s to be (α, β, γ), $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, -\gamma)$ respectively, where $\alpha^2+\beta^2+\gamma^2=0$. The required conic, assuming it to exist, will be given by an equation of the form $lx^2+my^2+nz^2=0$. This must pass through the point (α, β, γ), and the tangent at this point must be

$$x(B\gamma-C\beta)+y(C\alpha-A\gamma)+z(A\beta-B\alpha)=0;$$

that is, we must have $l\alpha^2+m\beta^2+n\gamma^2=0$, and

$$l\alpha : m\beta : n\gamma = B\gamma-C\beta : C\alpha-A\gamma : A\beta-B\alpha.$$

The first condition is obviously included in the second; and the second condition remains unaltered if we reverse the signs of B, β, or of C, γ, or of B, β and C, γ. Hence the conic passing through p, and touched at this point by pP, will also pass through the points q, r, s, and be touched at these points by the lines qQ, rR, sS, respectively; that is, the equation of the required conic is

$$\frac{B\gamma-C\beta}{\alpha}x^2+\frac{C\alpha-A\gamma}{\beta}y^2+\frac{A\beta-B\alpha}{\gamma}z^2=0;$$

or, what is the same thing,

$$\begin{vmatrix} \beta\gamma x^2, & \gamma\alpha y^2, & \alpha\beta z^2 \\ A\ , & B\ , & C \\ \alpha\ , & \beta\ , & \gamma \end{vmatrix}=0.$$

[Vol. XV., January to June, 1871, pp. 17—20.]

3206. (Proposed by Professor CAYLEY.)—In how many geometrically distinct ways can nine points lie in nine lines, each through three points?

3278. (Proposed by Professor CAYLEY.)—It is required, with nine numbers each taken three times, to form nine triads containing twenty-seven distinct duads (or, what is the same thing, no duad twice), and to find in how many essentially distinct ways this can be done.

Solution by the PROPOSER.

Let the numbers be 1, 2, 3, 4, 5, 6, 7, 8, 9. Any number, say 1, enters into three triads, no two of which have any number in common. We may take these triads to be 123, 145, 167. There remain the two numbers 8, 9; and these are, or are not, a duad of the system.

First Case.—8 and 9 a duad. In the triad which contains 89, the remaining number cannot be 1; it must therefore be one of the numbers 2, 3; 4, 5; 6, 7; and it is quite immaterial which; the triad may therefore be taken to be 289. There is one other triad containing 2, the remaining two numbers thereof being taken from the numbers 4, 5; 6, 7. They cannot be 4, 5 or 6, 7; and it is indifferent whether they are taken to be 4, 6; 4, 7; 5, 6, or 5, 7: the triad is taken to be 247. We have thus the triads

$$123,\ 145,\ 167,\ 289,\ 247;$$

and we require two triads containing 8 and two triads containing 9. These must be made up with the numbers 3, 4, 5, 6, 7: but as no one of them can contain 47, it follows that, of the two pairs which contain 8 and 9 respectively, one pair must be made up with 3, 5, 6, 7, and the other pair with 3, 5, 6, 4; say, the pairs which contain 8 are made up with 3, 5, 6, 7, and those which contain 9 are made up with 3, 5, 6, 4 (since obviously no distinct case would arise by the interchange of the numbers 8, 9). The triads which contain 8 must contain each of the numbers 3, 5, 6, 7, and they cannot be 835, 867, since we have 67 in the triad 167; similarly the triads which contain 9 must contain each of the numbers 3, 5, 6, 4, and they cannot be 845, 836, since we have 45 in 145. Hence the triads can only be

$$\begin{array}{cc|cc} 836, & 857 & 934, & 956, \\ 837, & 856 & 935, & 946; \end{array}$$

and clearly the top row of 8 must combine with the top row of 9, and the bottom row of 8 with the bottom row of 9; that is, the system of the nine triads is

$$123,\ 145,\ 167,\ 289,\ 247,$$

in combination with

$$836,\ 857,\ 934,\ 956,$$

or else in combination with

$$837,\ 856,\ 935,\ 946.$$

These are really systems of the same form, that is, each of them is of the form

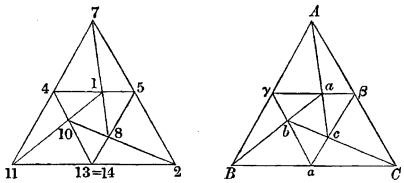
$$\begin{array}{c|c|c} BC\alpha & \beta\gamma a & bcC \\ CA\beta & \gamma ab & caA \\ AB\gamma & \alpha\beta c & abB; \end{array}$$

viz. in the first and second systems respectively we have

A	B	C	α	β	γ	a	b	c	
6	1	3	2	8	7	5	4	9	(First system),
5	1	3	2	9	4	6	7	8	(Second system),

as one out of many ways of effecting the identification. Observe that there is not in the system any triad of triads containing all the numbers. It thus appears that 8, 9, a duad, gives only a single form of the system.

Cor.—It is possible to find in a plane nine points such that the points belonging to the same triad lie *in lineâ*. The nine points are, in fact, on a cubic curve; and the figure is that belonging to a theorem of Prof. Sylvester's, according to which it is possible to find on a cubic curve a system of points 1, 2, 4, 5, 7, 8, &c., (a series of



numbers not divisible by 3), such that for any triad (such as 145) where the sum of the numbers, one taken negatively, $=0$, the three points are *in lineâ;* and so also that, if two of the points become identical, in the figure $13=14$, then there is not any new point, but the preceding points are indefinitely repeated; thus, 2, 14, 16 being *in lineâ*, and 14 being $=13$, 16 must be $=11$, and so on.

Second and Third Cases.—8 and 9 do not form a duad. There are thus three triads composed of 8 with (2, 3; 4, 5; 6, 7), and three triads composed of 9 with (2, 3; 4, 5; 6, 7). If with these numbers (2, 3; 4, 5; 6, 7) we form all the arrangements of three duads other than those which contain all or any of the duads 23, 45, 67, there are the eight arrangements

$$\begin{array}{ll} A=24,\ 37,\ 56, & E=26,\ 35,\ 47, \\ B=24,\ 36,\ 57, & F=26,\ 34,\ 57, \\ C=25,\ 36,\ 47, & G=27,\ 34,\ 56, \\ D=25,\ 37,\ 46, & H=27,\ 35,\ 46, \end{array}$$

where A has a duad in common with B, with D, and with G: but it has no duad in common with C, E, F, or H. We have thus the sixteen pairs

$$\begin{array}{llll} AC, & AE, & AF, & AH, \\ BD, & BE, & BG, & BH, \\ CF, & CG, & CH, & \\ DE, & DF, & DG, & \\ EG, & FH, & & \end{array}$$

where each pair contains six different duads.

Combining AC with 8, 9, we have the triads 8 (24, 37, 56) and 9 (24, 36, 57), that is, the triads

$$824,\ 837,\ 856:\ 924,\ 936,\ 957:$$

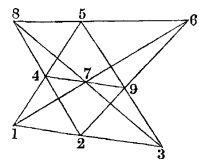
which, with the original three triads 123, 145, 167, form a system of nine triads; 8 and 9 might, of course, be interchanged, but no essentially distinct system would arise thereby. Hence we have a system of nine triads by combining the original three triads 123, 145, 167, with any one of the sixteen pairs AC, AE, &c. But it is sufficient to consider the combinations of the three triads with each of the pairs AC, AE, AF, AH; in fact, these are the only systems which contain the triad 824; and since there is no distinction between the two pairs 4, 5 and 6, 7, or between the two numbers of the same pair, it is allowable to take 824 as a triad of the system. Hence—

Second Case.—The system consists of the three triads combined with AE; viz. it is

$$123,\ 145,\ 167:\ 824,\ 837,\ 856:\ 926,\ 935,\ 947:$$

which, it is to be observed, consists of three triads of triads, each triad of triads containing all the nine numbers; viz. the system is

$$123,\ 479,\ 568:\ 145,\ 269,\ 378:\ 167,\ 248,\ 359.$$



Cor.—We may have nine points such that the points belonging to the same triad lie *in lineâ*, viz. the figure is that of Pascal's hexagon when the conic is a line-pair.

Third Case.—Combining the three triads with AC, AF, or AH, it is readily seen that we obtain in each case a system of the form

$$\begin{array}{lll} A\alpha\alpha', & A\beta\gamma, & A\beta'\gamma', \\ B\beta\beta', & B\gamma\alpha, & B\gamma'\alpha', \\ C\gamma\gamma', & C\alpha\beta, & C\alpha'\beta', \end{array}$$

viz. in the case where the pair is AC; that is, the system is

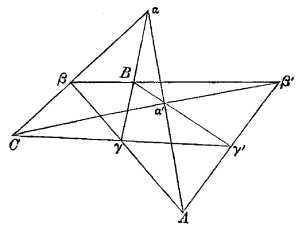
$$123,\ 145,\ 167:\ 824,\ 837,\ 856:\ 925,\ 936,\ 947;$$

and in the cases where the pair is AF or AH, the identifications may be taken to be

A	B	C	α	β	γ	α'	β'	γ'	
9,	8,	1;	4,	5,	2;	7,	6,	3	..... (AC),
9,	8,	1;	2,	3,	4;	6,	7,	5	..... (AF),
9,	8,	1;	5,	4,	6;	3,	2,	7	..... (AH).

Observe that there is in the system a single triad of triads $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$, containing all the numbers; viz. for the system with AC, this is 123, 856, 947; for the system with AF, it is 145, 837, 926; and for the system with AH, it is 167, 824, 935.

Cor.—It is possible to find a system of nine points such that the points belonging to the same triad lie *in lineâ*. Such a figure is this:—



The solution shows that these are the only systems of nine points satisfying the prescribed conditions.

[Vol. xv., pp. 66, 67.]

3329. (Proposed by Professor CAYLEY.)—It is required to show that every permutation of 12345 can be produced by means of the cyclical substitution (12345), and the interchange (12).

Solution by the PROPOSER.

It is sufficient to show that the interchanges (13), (14), (15) can be so produced; for then, with the interchanges (12), (13), (14), (15), we can, by at most two such interchanges, bring any number into any place.

Writing $P = (12345)$, $\alpha = (12)$, we have

$$(12) = \alpha,$$
$$(13) = \alpha P \alpha P^4 \alpha,$$
$$(14) = \alpha P \alpha P^4 \alpha P^2 \alpha P^3 \alpha P \alpha P^4 \alpha,$$
$$(15) = P^4 \alpha P,$$

as can be at once verified; and the theorem is thus proved.

I remark that, starting with any two or more substitutions, and combining them in every possible manner (each of them being repeatable an indefinite number of times), we obtain a "group"; viz. this is either (as in the problem proposed) the

system of all the substitutions (or say the entire group), or else it is a system the number of whose terms is a submultiple of the whole number of substitutions. The interesting question is, to determine those two or more substitutions, which, by their combination as above, do *not* give the entire group; for in this way we should arrive at all the forms of a submultiple group.

[Vol. XV., p. 80.]

3356. (Proposed by Professor CAYLEY.)—If the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $(a, b, c, d, e)(u, 1)^4 = 0$ are no two of them equal; and if there exist unequal magnitudes θ and ϕ, such that

$$(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4 = (\phi+\alpha)^4 : (\phi+\beta)^4 : (\phi+\gamma)^4 : (\phi+\delta)^4;$$

show that the cubinvariant

$$ace - ad^2 - b^2e - c^3 + 2bcd = 0;$$

and find the values of θ, ϕ.

[Vol. XVI., June to December, 1871, p. 65.]

3507. (Proposed by Professor CAYLEY.)—Show that, for the quadric cones which pass through six given points, the locus of the vertices is a quartic surface having upon it twenty-five right lines; and, thence or otherwise, that for the quadric cones passing through seven given points the locus of the vertices is a sextic curve.

[Vol. XVI., p. 90.]

3536. (Proposed by Professor CAYLEY.)—A particle describes an ellipse under the simultaneous action of given central forces, each varying as $(\text{distance})^{-2}$, at the two foci respectively: find the differential relation between the time and the excentric anomaly.

[Vol. XVII., January to June, 1872, p. 35.]

3591. (Proposed by Professor CAYLEY.)—If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation

$$\alpha . PAB + \beta . PBC + \gamma . PCD + \delta . PDA = 0,$$

which connects the areas of the triangles PAB, &c.

[Vol. XVII., p. 49.]

2652. (Proposed by Professor CAYLEY.)—Find the differential equation of the parallel surfaces of an ellipsoid.

[Vol. XVII., p. 60.]

3677. (Proposed by Professor CAYLEY.)—Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point.

[Vol. XVII., p. 72.]

3564. (Proposed by Professor CAYLEY.)—To determine the least circle enclosing three given points.

[Vol. XVIII., July to December, 1872, p. 68.]

3875. (Proposed by Professor CAYLEY.)—Given the constant a and the variables x, y, to construct mechanically $\dfrac{a^2-x^2}{y}$; or what is the same thing, given the fixed points A, B, and the moving point P, to mechanically connect therewith a point P' such that PP' shall be always at right angles to AB, and the point P' in the circle APB.

[Vol. XX., July to December, 1873, pp. 106, 107.]

3430. (Proposed by W. J. C. MILLER.)—Find the equation of the first negative focal pedal of (1) an ellipsoid, and (2) an ellipse.

Solution by PROFESSOR CAYLEY.

1. It is easily seen that if a sphere be drawn, passing through the centre of the given quadric and touching it at any point (x', y', z'), then the point (x, y, z) on the required surface, which corresponds to (x', y', z'), is the extremity of the diameter of this sphere which passes through the centre of the quadric. We thus easily find the expressions

$$x = x'\left(2 - \frac{t}{a^2}\right), \quad y = y'\left(2 - \frac{t}{b^2}\right), \quad z = z'\left(2 - \frac{t}{c^2}\right);$$

where

$$t = x'^2 + y'^2 + z'^2.$$

Solving these equations for x', y', z', and substituting in the two equations

$$xx' + yy' + zz' = x'^2 + y'^2 + z'^2, \quad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1,$$

we get

$$\frac{x^2}{\left(2-\frac{t}{a^2}\right)}+\frac{y^2}{\left(2-\frac{t}{b^2}\right)}+\frac{z^2}{\left(2-\frac{t}{c^2}\right)}=t \quad \text{.....(1)},$$

$$\frac{x^2}{a^2\left(2-\frac{t}{a^2}\right)^2}+\frac{y^2}{b^2\left(2-\frac{t}{b^2}\right)^2}+\frac{z^2}{c^2\left(2-\frac{t}{c^2}\right)^2}=1 \quad \text{.....(2)}.$$

Since (2) is the differential with respect to t of (1), the result of eliminating t between these two equations is the discriminant of (1). Hence the equation of the required surface is the discriminant of (1) with respect to t. Since (1) is only of the fourth degree, this discriminant is easily formed. If (1) be written in the form

$$At^4+4Bt^3+6Ct^2+4Dt+E=0,$$

it will be found that A and B do not contain x, y, z, while C, D, E contain them, each in the second degree. Now the discriminant is of the sixth degree in the coefficients, and of the form $A\phi+B^2\psi$ (see Salmon's *Higher Algebra*, § 107); hence it contains x, y, z only in the tenth degree. This is therefore the degree of the required surface.

The section of this derived surface by the principal plane z consists of the discriminant of

$$\frac{x^2}{2-\frac{t}{a^2}}+\frac{y^2}{2-\frac{t}{b^2}}=t \quad \text{..... (3)},$$

$\left(\text{which is of the sixth degree, and is the first negative pedal of } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$, together with the conic (taken twice), which is obtained by putting $t=2c^2$ in (3).

This conic, which is a double curve on the surface, touches the curve of the sixth degree in four points.

2. The formulæ for the conic are quite analogous to those for the ellipsoid, viz. we have

$$x=X\left\{2-\frac{1}{a^2}(X^2+Y^2)\right\},\quad y=Y\left\{2-\frac{1}{b^2}(X^2+Y^2)\right\},$$

leading to the equations

$$\theta=\frac{x^2}{2-\frac{\theta}{a^2}}+\frac{y^2}{2-\frac{\theta}{b^2}},$$

and its derived equation, from which to eliminate θ. The first is the cubic equation $(A, B, C, D)(\theta, 1)^3=0$, where

$$A=1,\quad B=-\tfrac{2}{3}(a^2+b^2),\quad C=\tfrac{1}{3}(a^2x^2+b^2y^2+4a^2b^2),\quad D=-2a^2b^2(x^2+y^2).$$

Equating the discriminant to zero, this is

$$0 = A^2 \nabla = 4(AC - B^2)^3 - (3ABC - A^2D - 2B^3)^2.$$

Or finally

$$(3a^2x^2 + 3b^2y^2 - 4a^4 + 4a^2b^2 - 4b^4)^3$$
$$+ \{9(a^2 - 2b^2)a^2x^2 + 9(b^2 - 2a^2)b^2y^2 - 8a^6 + 12a^4b^2 + 12a^2b^4 - b^6\}^2 = 0,$$

which is the required equation.

[Vol. XXI., January to June, 1874, pp. 29, 30.]

4298. (Proposed by J. W. L. GLAISHER, B.A.)—With four given straight lines to form a quadrilateral inscribable in a circle.

Solution by PROFESSOR CAYLEY.

Let the sides of the quadrilateral taken in order be a, b, c, d; and let its diagonals be x, y; viz. x the diagonal joining the intersection of the sides a, b with that of the sides c, d; y the diagonal joining the intersection of the sides a, d with that of the sides b, c; then, the quadrilateral being inscribed in a circle, the opposite angles are supplementary to each other. Suppose for a moment that the angles subtended by the diagonal x are θ, $\pi - \theta$, we have

$$x^2 = b^2 + c^2 + 2bc\cos\theta, \quad x^2 = a^2 + d^2 - 2ad\cos\theta;$$

and thence

$$(ad + bc)x^2 = ad(b^2 + c^2) + bc(a^2 + d^2) = (ac + bd)(ab + cd),$$

that is,

$$x^2 = (ac + bd)\frac{ab + cd}{ad + bc},$$

and similarly

$$y^2 = (ac + bd)\frac{ad + bc}{ab + cd},$$

agreeing as they should do with the known relation $xy = ac + bd$: the quadrilateral is thus determined by means of either of its diagonals. It is however interesting to treat the question in a different manner.

Considering a, b, c, d, x, y as the sides and diagonals of a quadrilateral, we have between these quantities a given relation, say

$$F(a, b, c, d, x, y) = 0,$$

and the quadrilateral being inscribed in a circle, we have also the relation $xy = ac + bd$; which two equations determine x, y; and thus give the solution of the problem.

The expression of the function F is in effect given in my paper, "Note on the value of certain determinants, &c.," *Quarterly Mathematical Journal*, t. III. (1860), pp. 275—277, [286]; viz. a, b, c being the edges of any face, and f, g, h the remaining edges of a tetrahedron, then

$$\begin{aligned}\text{volume} = \tfrac{1}{144}\ \{&b^2c^2\ (g^2+h^2)+c^2a^2(h^2+f^2)+a^2b^2(f^2+g^2)\\ &+\ g^2h^2\ (b^2+c^2)+h^2f^2(c^2+a^2)+f^2g^2(a^2+b^2)\\ &-\ a^2f^2\ (a^2+f^2)-b^2g^2(b^2+g^2)+c^2h^2(c^2+h^2)\\ &-\ a^2g^2h^2-b^2h^2f^2-c^2f^2g^2-a^2b^2c^2\},\end{aligned}$$

where, when the tetrahedron becomes a quadrilateral, the volume is $=0$.

In this formula, changing c, b, h, g, f, a into a, h, c, d, x, y, we have the required equation $F=0$; viz. this is found to be

$$\begin{aligned}a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2-b^2d^2(b^2+d^2)-a^2c^2(a^2+c^2)+x^2y^2(a^2+b^2+c^2+d^2-x^2-y^2)\\ +x^2(a^2c^2+b^2d^2-a^2d^2-b^2c^2)+y^2(a^2c^2+b^2d^2-a^2b^2-c^2d^2)=0,\end{aligned}$$

which, with $xy=ac+bd$, determines x, y. Substituting in the foregoing equation for xy its value, the equation becomes

$$(ad+bc)^2x^2+(ab+cd)^2y^2=2\{a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2+abcd(a^2+b^2+c^2+d^2)\},$$

or

$$(ad+bc)^2x^2+(ab+cd)^2y^2=2(ad+bc)(ab+cd)(ac+bd).$$

To show more clearly how this equation arises, I observe that we have identically

$$\begin{aligned}F-(a^2+b^2+c^2+d^2-x^2-y^2)(xy+ac+bd)(xy-ac-bd)-2(ad+bc)(ab+cd)(xy-ac-bd)\\ =\{(ad+bc)x-(ab+cd)y\}^2.\end{aligned}$$

The resulting equation $(ad+bc)x-(ab+cd)y=0$, together with $xy=ac+bd$, gives for x, y the foregoing values.

[Vol. XXI., pp. 81, 82.]

4392. (Proposed by S. ROBERTS, M.A.)—If N_p denotes the number of terms in a symmetrical determinant of p rows and columns, show that the successive numbers are given by the equation

$$N_k-N_{k-1}-(k-1)^2N_{k-2}+\tfrac{1}{2}(k-1)(k-2)\{N_{k-3}+(k-3)N_{k-4}\}=0,$$

k being positive and N_0 being taken equal to unity.

Solution by PROFESSOR CAYLEY.

It is a curious coincidence that the question of determining the number of distinct terms in a symmetrical determinant has been recently solved by Captain Allan

Cunningham in a paper in the last number of the *Quarterly Journal of Science**; and the question having been proposed to me by Mr Glaisher, I have also solved it in a paper [580] printed in the April Number of the *Monthly Notices of the Royal Astronomical Society*. I there obtain

$$N_k = 1 \,.\, 2 \ldots k \text{ coeff. } x^k \text{ in } \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}},$$

viz. writing

$$u = N_0 + N_1 \frac{x}{1} + N_2 \frac{x^2}{1\,.\,2} + \ldots,$$

I show that u satisfies the differential equation

$$2\frac{du}{dx} = \left\{1 + x + \frac{1}{1-x}\right\} u,$$

giving when the constant is determined

$$u = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}}.$$

Writing the differential equation in the form

$$2(1-x)\frac{du}{dx} = (2-x^2)\,u,$$

we at once obtain for N_k the equation of differences

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = 0,$$

which is in fact a particular first integral of Mr Roberts's equation; viz. from the above equation we have

$$N_{k-1} - (k-1)\,N_{k-2} + \tfrac{1}{2}(k-2)(k-3)\,N_{k-4} = 0,$$

and multiplying this last by $k-1$ and adding, we have

$$N_k - N_{k-1} - (k-1)^2 N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\{N_{k-3} + (k-3)\,N_{k-4}\} = 0,$$

which is the equation obtained by Mr Roberts. It thence appears that the *general* first integral of his equation is

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = (-)^k\,C1\,.\,2 \ldots (k-1).$$

The equation

$$N_k = kN_{k-1} - \tfrac{1}{2}(k-1)(k-2)\,N_{k-3}$$

gives very readily the numerical values, viz.

$1 = 1\,.\,1 - 0$	$17 = 4\,.\,5 - 3\,.\,1$	$2461 = 7\,.\,388 - 15\,.\,17$
$2 = 2\,.\,1 - 0$	$73 = 5\,.\,17 - 6\,.\,2$	$18155 = 8\,.\,2461 - 21\,.\,73.$
$5 = 3\,.\,2 - 1\,.\,1$	$388 = 6\,.\,73 - 10\,.\,5$	

* I have not the volume at hand to refer to, but he obtains an equation of differences, and gives the numbers 1, 2, 5, 73, 398 (should be 388), ...

[Vol. XXII., July to December, 1874, pp. 20, 21.]

4354. (Proposed by R. TUCKER, M.A.)—Solve the equations

$$-x^2+xy+xz=a=4 \quad\text{.....(1)},$$
$$-y^2+xy+yz=b=-20 \quad\text{.....(2)},$$
$$-z^2+xz+yz=c=-8 \quad\text{.....(3)}.$$

Note on Question 4354. *By* PROFESSOR CAYLEY.

A question of simple algebra such as this, becomes more interesting when interpreted geometrically: thus, writing the equations in the form

$$-x^2+xy+xz=aw^2,\quad yx-y^2+yz=bw^2,\quad zx+zy-z^2=cw^2,$$

and then putting for shortness

$$\alpha=-a+b+c,\quad \beta=a-b+c,\quad \gamma=a+b-c,$$

the solutions obtained are

$$x:y:z:w=a\alpha:b\beta:c\gamma:(\alpha\beta\gamma)^{\frac{1}{2}},$$
$$x:y:z:w=a\alpha:b\beta:c\gamma:-(\alpha\beta\gamma)^{\frac{1}{2}};$$

say these are

$$\{a\alpha,\ b\beta,\ c\gamma,\ (\alpha\beta\gamma)^{\frac{1}{2}}\}\text{ and }\{a\alpha,\ b\beta,\ c\gamma,\ -(\alpha\beta\gamma)^{\frac{1}{2}}\}.$$

But the equations are also satisfied by

$$(x=0,\ y=z,\ w=0),\quad (y=0,\ z=x,\ w=0),\quad (z=0,\ x=y,\ w=0),$$

or what is the same thing, (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0). The three equations represent quadric surfaces, each two of them intersecting in a proper quadric curve, and the three having in common 8 points; viz. these are made up of the first mentioned two points each once, and the last mentioned three points each twice: $2+3.2,=8$.

To verify this, observe that, at each of the three points, the tangent planes of the surfaces have a common line of intersection; this line is the tangent of the curve of intersection of any two of the surfaces, and the curve of intersection therefore touches the third surface; wherefore the point counts for two intersections. In fact, taking (X, Y, Z, W) as current coordinates, the equations of the tangent planes at the point (x, y, z, w) are

$$\begin{aligned}
&X(2x-y-z)-Yx && -Zx && +2aWw=0,\\
-Xy &&+Y(-x+2y-z)-Zy && +2bWw=0,\\
-Xz && -Yz && +Z(-x-y+2z)+2cWw=0:
\end{aligned}$$

hence at the point (0, 1, 1, 0) these equations are

$$-2X=0,\quad X+Y-Z=0,\quad -X-Y+Z=0,$$

which three planes meet in the line $X=0$, $Y-Z=0$; and similarly for the other two of the three points.

[Vol. XXII., pp. 60—64.]

4458. (Proposed by Professor CAYLEY.)—Find (1) the intersections of the two quartic curves

$$\lambda(ab-xy)^2 = abx(a-y)(b-y), \quad \mu(ab-xy)^2 = aby(a-x)(b-x);$$

and (2) trace the curves in some particular cases; for instance, when $a=1$, $b=2$, $\lambda=1$, $\mu=-2$.

Solution by the PROPOSER.

1. The 16 intersections are made up as follows: 5 points at infinity on the line $x=0$, 5 at infinity on the line $y=0$, the two points $(x=a,\ y=b)$, $(x=b,\ y=a)$, and 4 other points, $16=5+5+2+4$. As to the points at infinity, observe that, as regards the first curve, the point at infinity on the line $x=0$ is a flecnode having this line for a tangent to the flecnodal branch; and, as regards the second curve, the same point is a cusp, having this line for its tangent; hence the point in question counts as $2+3$, $=5$ intersections; and the like as to the point at infinity on the line $y=0$. It remains to find the coordinates of the 4 points of intersection. Assume $xy=ab\omega$, then the equations become

$$\lambda(1-\omega)^2 = x+\omega y-(a+b)\omega, \quad \mu(1-\omega)^2 = \omega x+y-(a+b)\omega;$$

hence, eliminating successively y and x, the factor $1-\omega$ divides out,—this factor belongs to the points $(x=a,\ y=b)$, $(x=b,\ y=a)$ for which obviously $\omega=1$—, and the equations become

$$(\lambda-\mu\omega)(1-\omega)+(a+b)\omega = (1+\omega)x, \quad (\mu-\lambda\omega)(1-\omega)+(a+b)\omega = (1+\omega)y.$$

Multiplying these two equations together, and substituting for xy its value $ab\omega$, we find

$$\{(\lambda-\mu\omega)(\mu-\lambda\omega)+(a+b)(\lambda+\mu)\omega\}(1-\omega)^2+(a+b)^2\omega^2-(1+\omega)^2\omega ab = 0.$$

Write, for shortness, $p=(\lambda+\mu)(a+b)-\lambda^2-\mu^2$, then, dividing by ω^2, and writing $\omega+\dfrac{1}{\omega}=\Omega$, the equation is

$$(\lambda\mu\Omega+p)(\Omega-2)+(a+b)^2-ab(\Omega+2)=0;$$

viz. this is a quadric equation for Ω. But, instead of Ω, it is convenient to introduce the quantity θ, $=\dfrac{\Omega-2}{\Omega+2}$, $=\left(\dfrac{\omega-1}{\omega+1}\right)^2$. The equation thus becomes

$$\left\{2\lambda\mu\frac{1+\theta}{1-\theta}+p\right\}\frac{4\theta}{1-\theta}+(a+b)^2-ab\frac{4}{1-\theta}=0,$$

or

$$\{2\lambda\mu(1+\theta)+p(1-\theta)\}4\theta+(a+b)^2(1-\theta)^2-4ab(1-\theta)=0,$$

or

$$\theta^2\{(a+b)^2-4(p-2\lambda\mu)\}+\theta\{-2a^2-2b^2+4(p+2\lambda\mu)\}+(a-b)^2=0;$$

viz. substituting for p its values, this is

$$\theta^2(a+b-2\lambda-2\mu)^2+2\theta\{-a^2-b^2+2(\lambda+\mu)(a+b)-2(\lambda-\mu)^2\}^2+(a-b)^2=0;$$

or if we write

$$A=a^2-2a(\lambda+\mu)+(\lambda-\mu)^2,\quad B=b^2-2b(\lambda+\mu)+(\lambda-\mu)^2,$$

this is

$$\theta^2(a+b-2\lambda-2\mu)^2-2(A+B)\theta+(a-b)^2=0,$$

whence

$$\{(a-b)^2-(A+B)\theta\}^2=\theta^2\{(A+B)^2-(a-b)^2(a+b-2\lambda-2\mu)^2\}$$
$$=\theta^2\{(A+B)^2-(A-B)^2\}=4AB\theta^2;$$

viz. taking for convenience the sign $-$ on the right-hand side, this is

$$(a-b)^2-(A+B)\theta=-2\theta\sqrt{AB};$$

and we have thus

$$\theta=\frac{(a-b)^2}{(\sqrt{A}-\sqrt{B})^2},$$

that is,

$$\theta,=\frac{\omega-1}{\omega+1}=\frac{a-b}{\sqrt{A}-\sqrt{B}};\quad \omega=\frac{\sqrt{A}-\sqrt{B}+a-b}{\sqrt{A}-\sqrt{B}-a+b}.$$

We may write

$$x=\mu(\omega-1)+\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1},$$

$$y=\lambda(\omega-1)+\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1};$$

whence also $x-y=(\mu-\lambda)(\omega-1)$, as is also seen at once from the original equations; then we have

$$\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1}=\frac{\tfrac{1}{2}(a-b)(a+b-2\lambda-2\mu)}{\sqrt{A}-\sqrt{B}}$$
$$=\frac{\tfrac{1}{2}(A-B)}{\sqrt{A}-\sqrt{B}},=\tfrac{1}{2}(\sqrt{A}+\sqrt{B});$$

and the values are

$$x=\frac{2\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b)$$
$$=\frac{(a-b)(\mu-\lambda)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},$$

$$y=\frac{2\lambda(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b)$$
$$=\frac{(a-b)(\lambda-\mu)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},$$

which may be expressed in the more simple form

$$x=\frac{1}{4\lambda}(a+\lambda-\mu+\sqrt{A})(b+\lambda-\mu+\sqrt{B}),$$

$$y=\frac{1}{4\mu}(a-\lambda+\mu+\sqrt{A})(b-\lambda+\mu+\sqrt{B}),$$

the transformations depending on the identity

$$\frac{8\lambda\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}=ab-(\lambda+\mu)(a+b)+(\lambda-\mu)^2+\sqrt{A}(b-\lambda-\mu)+\sqrt{B}(a-\lambda-\mu)+\sqrt{AB},$$

which is easily verified. Of course, since the signs of $\sqrt{A}$, $\sqrt{B}$ are arbitrary, we have 4 systems of values of (x, y), which is right.

In the original equations, for a, b, λ, μ, x, y, write 1, k^{-2}, λ^2, $-\mu^2$, x^2, $-y^2$; then the equations become

$$\lambda^2(1+k^2x^2y^2)^2=x^2(1+y^2)(1+k^2y^2),\quad \mu^2(1+k^2x^2y^2)^2=y^2(1-x^2)(1-k^2x^2),$$

and we thence have

$$\lambda+\mu i=\frac{x\sqrt{(1+y^2)(1+k^2y^2)}+iy\sqrt{(1-x^2)(1-k^2x^2)}}{1+k^2x^2y^2};$$

viz. assuming $x=\operatorname{sn}\alpha(\operatorname{sinam}\alpha)$, $iy=\operatorname{sn}i\beta$, this is $\lambda+\mu i=\operatorname{sn}(\alpha+\beta i)$; viz. the problem is (for a given modulus k, assumed as usual to be real, positive, and less than 1) to reduce a given imaginary quantity $\lambda+\mu i$ to the form $\operatorname{sn}(\alpha+\beta i)$. The proper solution is that in which the signs of the radicals are each $-$, viz. it may in this case be shown that the value of x^2 is positive and less than 1, that of y^2 positive. The values thus are

$$x^2=\frac{1}{4\lambda^2}(1+\lambda^2+\mu^2-\sqrt{A})\left(\frac{1}{k^2}+\lambda^2+\mu^2-\sqrt{B}\right),$$

$$y^2=\frac{1}{4\mu^2}(1-\lambda^2-\mu^2-\sqrt{A})\left(\frac{1}{k^2}-\lambda^2-\mu^2-\sqrt{B}\right),$$

where

$$A=1-2\lambda^2+2\mu^2+(\lambda^2+\mu^2),\quad B=\frac{1}{k^2}-\frac{2}{k^2}\lambda^2+\frac{2}{k^2}\mu^2+(\lambda^2+\mu^2)^2.$$

The solution is really equivalent to that given by Richelot (*Crelle*, t. XLV., 1853, p. 225). To verify this partially, observe that, writing σ, τ for Richelot's $\tan\frac{1}{2}\phi$, $\tan\frac{1}{2}\psi$, we have

$$\left(\sigma+\frac{1}{\sigma}\right)\lambda=1+\lambda^2+\mu^2,$$

giving

$$\left(\sigma-\frac{1}{\sigma}\right)\lambda=-\sqrt{A};$$

$$\left(\tau+\frac{1}{\tau}\right)\frac{\lambda}{k}=\frac{1}{k^2}+\lambda^2+\mu^2,$$

giving

$$\left(\tau-\frac{1}{\tau}\right)\frac{\lambda}{k}=-\sqrt{B};$$

whence

$$2\sigma\lambda=1+\lambda^2+\mu^2-\sqrt{A},\quad 2\tau\frac{\lambda}{k}=\frac{1}{k^2}+\lambda^2+\mu^2-\sqrt{B},$$

or the above value of x^2 is $=k^{-1}\sigma\tau$, agreeing with his; the value of y^2 is, however, presented under a somewhat different form.

2. The curves are

$$(2-xy)^2 = 2x(1-y)(2-y), \quad -(2-xy)^2 = y(1-x)(2-x) \ldots\ldots\ldots (1,\ 2),$$

each passing through the points (1, 2) and (2, 1); the four points of intersection found by the foregoing general theory are all real, viz. these are

$x = \frac{1}{2}(2+\sqrt{3})(5+\sqrt{17})$,	$y = -\frac{1}{8}(-1+\sqrt{3})(-1+\sqrt{17})$,	say	$+17{\cdot}00$	and	$-0{\cdot}57$,
$-\sqrt{3}$,	$+\sqrt{17}$	"	"	$+1{\cdot}65$	$+0{\cdot}94$,
$+\sqrt{3}$,	$-\sqrt{17}$	"	"	$+1{\cdot}22$	$+2{\cdot}13$,
$-\sqrt{3}$,	$-\sqrt{17}$	"	"	$-0{\cdot}12$	$-3{\cdot}49$.

The equation of the curve (1) may also be written in the forms

$$y^2(x^2-2x)+2yx-4x+4=0, \quad x^2y^2+x(-2y^2+2y-4)+4=0.$$

The original form shows that, if y is between 1 and 2, x is negative—(but by a further examination it appears that there is not in fact any branch of the curve between these limits of y)—but y being outside these limits, then x is positive; in fact, the whole curve lies on the positive side of the axis of y. And then the inspection of the first quadric equation shows that the lines $x=0$ and $x=2$ are each an asymptote.

The point at infinity on the axis of y is in fact a flecnode, the tangent to the flecnodal branch being $x=0$, and that of the ordinary branch $x=2$.

Similarly, from the second quadric equation, it appears that the line $y=0$ is an asymptote; the point at infinity on the axis of x is in fact a cusp, the axis in question $y=0$ being the cuspidal tangent.

The equation of the curve (2) may also be written in the forms

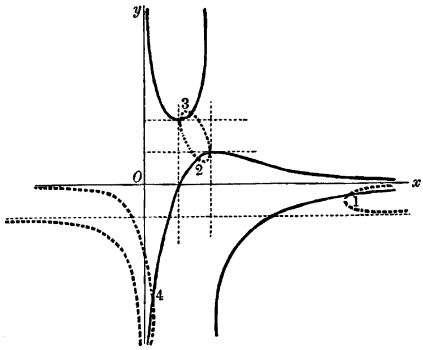
$$x^2y^2+(x^2-7x+2)y+4=0, \quad (y^2+y)x^2-7yx+2y+4=0.$$

The original form shows that, if x is between 1 and 2, y is positive; but that x being beyond these limits, y is negative; and as regards the first case, x between 1 and 2, we at once establish the existence of an oval, meeting the line $y=1$ in the points $x=2$ and $\frac{3}{2}$, and the line $y=2$ in the points $x=1$ and $\frac{4}{3}$; it is further easy to see that the horizontal tangents of the oval are $y=\frac{1}{16}(25\pm\sqrt{113})$, $=$ say $2{\cdot}2$ and $0{\cdot}9$.

The remainder of the curve lies wholly below the line $y=0$. The first quadric equation shows the asymptote $x=0$; the point at infinity on the axis of y is in fact a cusp, having the axis itself for the cuspidal tangent. The second quadric equation shows the asymptotes $y=0$, $y=-1$; the point at infinity on the axis of x is in fact a flecnode, having the line $y=0$ for the tangent to the flecnodal branch, and $y=-1$ for that of the other branch. It is further seen that there are two vertical tangents $x=\frac{1}{2}(11\pm\sqrt{113})=10{\cdot}8$ or $0{\cdot}2$; the former of these touches a branch

lying wholly between the two asymptotes $y=0$, $y=-1$; the latter one of the branches belonging to the cuspidal asymptote $x=0$; this last branch cuts the asymptote $x=0$ at $y=-2$, and then, cutting the asymptote $y=-1$ and $x=-\frac{2}{7}(=-0{\cdot}3)$, goes on to touch at infinity the asymptote $y=0$. It is now easy to trace the curve.

The figure shows the two curves. The curve (1) is shown by a continuous line, the curve (2) by a thick dotted line; the points 1, 2, 3, 4 show the above mentioned



four intersections of the curves; the point 1 and the dotted branch through it are of necessity drawn considerably out of their true positions; viz. as above appearing, the x-coordinate of 1 is $=17{\cdot}00$, and the equation of the vertical tangent to the branch is $x=10{\cdot}8$.

[Vol. XXII., pp. 78, 79.]

4520. (Proposed by A. B. EVANS, M.A.)—Find the least integral values of x and y that will satisfy the equation $x^2-953y^2=-1$.

Solution by PROFESSOR CAYLEY.

The values are given in Degen's Tables, viz.

$$x=2746864744,\quad y=88979677.$$

The work referred to is entitled "Canon Pellianus, sive Tabula simplicissimam æquationis celebratissimæ $y^2=ax^2+1$ solutionem pro singulis numeri dati valoribus ab 1 usque ad 1000 in numeris rationalibus iisdemque integris exhibens. Auctore C. F. Degen, Hafniæ (Copenhagen), 1817."

Table I., pp. 3—106 gives, for all numbers 1 to 1000, the denominators, (?) the quotients of the convergent fraction of $\sqrt{a}$, and also the least values of x, y which will satisfy the equation $x^2 - ay^2 = +1$. Thus

$$953 \left| \begin{array}{l} 30,\ 1\ ,\ 6,\ 1\ ,\ 2\ ,\ 1\ ,\ 3\ ,\ 8,\ 1\ ,\ 1\ ,\ (4\ ,\ 4\ ), \\ 1\ ,\ 53,\ 8,\ 41,\ 17,\ 37,\ 16,\ 7,\ 32,\ 29,\ (13,\ 13), \\ 48883027536761537 6,\ \ 15090531843660371073. \end{array} \right.$$

Table II., pp. 109—112, is described as giving for all those values of a between 1 and 1000, for which there exists a solution of the equation $x^2 - ay^2 = -1$, the least values of x and y which satisfy this equation: thus 953, x and y as above. It is, however, to be noticed that the values of $a = \beta^2 + 1$, for which there is the obvious solution $x = \beta$, $y = 1$, are omitted from the table. The reason for this appears, but the heading should have been different.

[Vol. XXIII., January to July, 1875, pp. 18, 19.]

4528. (Proposed by Professor CAYLEY.)—A lottery is arranged as follows:—There are n tickets representing a, b, c pounds respectively. A person draws once; looks at his ticket; and if he pleases, draws again (out of the remaining $n-1$ tickets); looks at his ticket, and if he pleases draws again (out of the remaining $n-2$ tickets); and so on, drawing in all not more than k times; and he receives the value of the last drawn ticket. Supposing that he regulates his drawings in the manner most advantageous to him according to the theory of probabilities, what is the value of his expectation?

Solution by the PROPOSER.

Let the expression "a or α" signify "a or α, whichever of the two is greatest," and let $M_1(a, b, c, \ldots)$ denote the mean of the quantities $(a, b, c, \ldots)$, viz. their sum, divided by the number of them.

To fix the ideas, consider five quantities a, b, c, d, e, and write

$$M_1(a, b, c, d, e) = M_1(a, b, c, d, e),$$

$$M_2(a, b, c, d, e) = M_1\{a \text{ or } M_1(b, c, d, e),\ b \text{ or } M_1(a, c, d, e), \ldots, e \text{ or } M_1(a, b, c, d)\},$$

$$M_3(a, b, c, d, e) = M_1\{a \text{ or } M_2(b, c, d, e),\ b \text{ or } M_2(a, c, d, e), \ldots, e \text{ or } M_2(a, b, c, d)\},$$

and so on. And the like in the case of any number of quantities a, b, c, ...

Then the value of the expectation is $= M_k(a, b, c, \ldots)$.

For, when $k = 1$, the value is obviously $= M_1(a, b, c, \ldots)$.

When $k=2$, if a is drawn, the adventurer will be satisfied or he will draw again, according as a or $M_1(b, c, \ldots)$ is greatest, viz. in this case the value of the expectation is "a or $M_1(b, c, \ldots)$."

So if b is drawn, the adventurer will be satisfied or he will draw again, according as b or $M_1(a, c, \ldots)$ is greatest; viz. in this case the value of the expectation is "b or $M_1(a, c, \ldots)$"; and so on: and the several cases being equally probable, the value of the total expectation is

$$= M_1\{a \text{ or } M_1(b, c, \ldots),\quad b \text{ or } M_1(a, c, \ldots), \ldots\} = M_2(a, b, c, \ldots):$$

and the like for $k=3$, $k=4$, &c.

For instance, $a, b, c, d = 1, 2, 3, 4$, $M_1(1, 2, 3, 4) = \frac{10}{4}$,

$$M_2(1, 2, 3, 4) = M_1(1 \text{ or } \tfrac{9}{3}, 2 \text{ or } \tfrac{8}{3}, 3 \text{ or } \tfrac{7}{3}, 4 \text{ or } \tfrac{6}{3}) = M_1(\tfrac{9}{3}, \tfrac{8}{3}, \tfrac{9}{3}, \tfrac{12}{3}) = \tfrac{38}{12},$$

$$M_2(2, 3, 4) = M_1(2 \text{ or } \tfrac{7}{2}, 3 \text{ or } \tfrac{6}{2}, 4 \text{ or } \tfrac{5}{2}) = M_1(\tfrac{7}{2}, \tfrac{6}{2}, \tfrac{8}{2}) = \tfrac{21}{6},$$

$$M_2(1, 3, 4) = M_1(1 \text{ or } \tfrac{7}{2}, 3 \text{ or } \tfrac{5}{2}, 4 \text{ or } \tfrac{4}{2}) = M_1(\tfrac{7}{2}, \tfrac{6}{2}, \tfrac{8}{2}) = \tfrac{21}{6},$$

$$M_2(1, 2, 4) = M_1(1 \text{ or } \tfrac{6}{2}, 2 \text{ or } \tfrac{5}{2}, 4 \text{ or } \tfrac{3}{2}) = M_1(\tfrac{6}{2}, \tfrac{5}{2}, \tfrac{8}{2}) = \tfrac{19}{6},$$

$$M_2(1, 2, 3) = M_1(1 \text{ or } \tfrac{5}{2}, 2 \text{ or } \tfrac{4}{2}, 3 \text{ or } \tfrac{3}{2}) = M_1(\tfrac{5}{2}, \tfrac{4}{2}, \tfrac{6}{2}) = \tfrac{15}{6},$$

$$M_3(1, 2, 3, 4) = M_1(1 \text{ or } \tfrac{21}{6}, 2 \text{ or } \tfrac{21}{6}, 3 \text{ or } \tfrac{19}{6}, 4 \text{ or } \tfrac{15}{6}) = M_1(\tfrac{21}{6}, \tfrac{21}{6}, \tfrac{19}{6}, \tfrac{24}{6}) = \tfrac{85}{24},$$

$$M_3(1, 2, 3) = 3 \text{ \&c.},$$

$$M_4(1, 2, 3, 4) = M_1(1 \text{ or } 4, 2 \text{ or } 4, 3 \text{ or } 4, 4 \text{ or } 3) = M_1(4, 4, 4, 4) = 4.$$

Or finally

$$M_1,\ M_2,\ M_3,\ M_4 = \tfrac{10}{4},\ \tfrac{38}{12},\ \tfrac{85}{24},\ 4 = \tfrac{60}{24},\ \tfrac{76}{24},\ \tfrac{85}{24},\ \tfrac{96}{24}.$$

COR. If the $a, b, c, \ldots$ denote penalties instead of prizes, then the solution is the same, except that "a or α" must now denote "a or α, whichever of them is least."

[Vol. XXIII., pp. 47, 48.]

4581. (Proposed by the Rev. M. M. U. WILKINSON.)—A witness, whose statement is what he opines once in m times, and whose opinion is correct once in n times, asserted that the number of a note, issued by a bank universally known to have issued notes numbered from B to $B+A-1$ inclusive, was $B+P$, where P is either 0, 1, 2, ..., or $A-1$. Prove (1) that the probability that the note in question was that note is

$$\frac{1}{mn}\left\{1 + \frac{(m-1)(n-1)}{A-1}\right\}.$$

The above witness also said that the note was signed by X, it being universally known that X has signed one note, and Y the remaining $A-1$ notes; find (2) the probability that this last statement was correct.

Remark by PROFESSOR CAYLEY.

There is a serious difficulty in the question, or the answer; I think, in the question. Try the answer in numbers $m=10$, $n=10$. The witness says what he opines once out of 10 times—he is in fact an atrocious liar; and he opines rightly once out of 10 times, that is, wrongly 9 times out of 10; he is therefore a blunderer—but a remarkably ingenious one, in that the chances are so greatly against his blundering upon a right result.

He says that the note was signed by X, and the chance of this being so is found to be $\frac{1}{100}+\frac{81}{100}=\frac{82}{100}$, or more than $\frac{8}{10}$; the larger part $\frac{81}{100}$ of this is obtained as follows:—the witness having said that the note was signed by X, the chances are 9 out of 10 that he thought the reverse; and, thinking the reverse, the chance is 9 out of 10 that he thought wrongly, viz. that the note was signed by X. But can the statement of such a witness create any probability in favour of the event?

The fallacy seems to consist in the assumption that n can have a determinate value irrespective of the nature of the opinion. Suppose there are 500 notes, and that the opinion is that the note was a definite number 99; it is quite conceivable that, in forming a series of such opinions, the witness may be wrong 9 times out of 10. But let the opinion be that the note was *not* 99; no amount of ingenuity of blundering can make him wrong 9 times out of 10 in a series of such opinions. If it could, a friend who knew the true opinion of the witness, would be able 9 times out of 10 to know the number of the note, from the mere fact that the witness opines that the note is not a named number.

[Vol. XXIII., p. 58.]

4638. (Proposed by Professor CAYLEY.)—Find the equation of the surface which is the envelope of the quadric surface $ax^2+by^2+cz^2+dw^2=0$, where a, b, c, d are variable parameters connected by the equation $Abc+Bca+Cab+Fad+Gbd+Hcd=0$; and consider in particular the case in which the constants A, B, C, F, G, H satisfy the condition

$$(AF)^{\frac{1}{2}}+(BG)^{\frac{1}{2}}+(CH)^{\frac{1}{2}}=0.$$

[Vol. XXIV., July to December, 1875, p. 41.]

4694. (Proposed by Professor CAYLEY.)—Taking F, F' a pair of reciprocal points in respect to a circle, centre O; then if F, F' are centres of force, each force varying as (distance)$^{-n}$, prove that (1) the resultant force upon any point P on the circle is in the direction of a fixed point S on the axis OFF'; and if, moreover, the forces at the unit of distance are as $(OF)^{\frac{1}{2}(n-1)}$ to $(OF')^{\frac{1}{2}(n-1)}$, then (2) the resultant force is proportional to

$$(SP)^{-\frac{1}{2}(n-1)}.(PV)^{-\frac{1}{2}(n+1)},$$

where PV is the chord through S.

[Vol. XXIV., pp. 72—74.]

4793. (Proposed by Professor WOLSTENHOLME, M.A.)—If $y = x^n (\log x)^r$, where n and r are integers, prove that

$$x^r \frac{d^{n+r}y}{dx^{n+r}} + \frac{r(r-1)}{2} x^{r-1} \frac{d^{n+2-1}y}{dx^{n+2-1}} + \frac{r(r-1)(r-2)(3r-5)}{24} x^{r-2} \frac{d^{n+r-2}y}{dx^{n+r-2}} + \dots$$

$$\dots + (2^{r-1}-1)\, x^2 \frac{d^{n+2}y}{dx^{n+2}} + x \frac{d^{n+1}y}{dx^{n+1}} = \underline{|r}\ \underline{|n},$$

the coefficients being

$$\frac{\Delta^{r-1} 1^{r-1}}{\underline{|r-1}},\ \frac{\Delta^{r-2} 1^{r-1}}{\underline{|r-2}},\ \frac{\Delta^{r-3} 1^{r-1}}{\underline{|r-3}},\ \dots\dots,\ \frac{\Delta 1^{r-1}}{\underline{|1}},\ \text{and } 1;$$

so that the result may be symbolically written

$$\epsilon^{xD\Delta}\left(1^{r-1} \frac{d^{n+1}y}{dx^{n+1}}\right) = \frac{\underline{|r}\ \underline{|n}}{x},$$

where D denotes $\frac{d}{dx}$ and operates on $\frac{d^n y}{dx^n}$ only, and Δ operates on 1^{r-1} only, the terms after the rth all vanishing since $\Delta^m x^n = 0$, when m is an integer $> n$. The calculations involved prove that, when $x = 1$,

$$\Delta^{n-1} x^n = \frac{\underline{|n+1}}{2},\quad \Delta^{n-2} x^n = \underline{|n+1} \cdot \frac{3n-2}{24},$$

$$\Delta^{n-3} x^n = \underline{|n+1} \cdot \frac{(n-1)(n-2)}{48}.$$

Solution by PROFESSOR CAYLEY.

Since $y = x^n (\log x)^r$, therefore $(xd_x - n)\, y = rx^n (\log x)^{r-1}$; by repeating the same operation, we have

$$(xd_x - n)^r y = [r]^r x^n;\ \text{whence}\ d_x^n (xd_x - n)^r y = [r]^r [n]^n.$$

Now, for any value whatever of the function y, we have

$$d_x^n (xd_x - n)^r y = Ax^r d_x^{r+n} y + Bx^{r-1} d_x^{r+n-1} y + Cx^{r-2} d_x^{r+n-2} y + \&c.,$$

the coefficients A, B, C, ... being functions, presumably of r, n, but independent of the form of the function y. It will, however, appear that A, B, C, ... are, in fact, functions of r only.

To see how this is, observe that $(xd_x - n)^r$ consists of a set of terms

$$(xd_x)^\theta,\quad (\theta = 0 \text{ to } r),$$

where $(xd_x)^\theta$ denotes θ repetitions of the operation xd_x; by a well-known theorem, this is $= [xd_x + \theta - 1]^\theta$, where, after expansion of the factorial, $(xd_x)^s$ is to be replaced by $x^s d_x^s$, thus

$$(xd_x)^2 = [xd_x + 1]^2 = x^2 d_x^2 + xd_x,\quad (xd_x)^3 = [xd_x + 2]^3 = x^3 d_x^3 + 3x^2 d_x^2 + 2xd_x,\ \&c.;$$

thus $(xd_x - n)^r$ consists of a series of terms $x^\theta d_x^\theta$, $(\theta = 0$ to $r)$, and, operating with d_x^n, this last, $= (d_x + d_x')^n$, consists of a series of terms such as $d_x^\alpha d_x'^{n-\alpha}$, where the unaccented symbol operates on the x^θ, and the accented symbol on the y; the term is thus $x^{\theta-\alpha} d_x^{n+\theta-\alpha}$, or observing that $\theta - \alpha$ is at most $= r$, and putting it $= r - k$, the term is $x^{r-k} d_x^{n+k}$, viz. $d_x^n (xd_x - n)^r$ consists of a series of terms of the form $x^{r-k} d_x^{n+k}$; or, what is the same thing, $d_x^n (xd_x - n)^r y$ is a series of the form in question.

To understand how it can be that the coefficients A, B, C, ... are independent of n, take the particular case $r = 2$; then we have here

$$d_x^n (xd_x - n)^2 y = Ax^2 d_x^{n+2} y + Bxd_x^{n+1} y + Cd_x^n y.$$

The right-hand side is

$$d_x^n \{x^2 d_x^2 - (2n-1)\, xd_x + n^2\}\, y,$$

which is

$$\begin{aligned} = \quad & \{x^2 d^{n+2} + 2nxd_x^{n+1} + (n^2 - n)\, d_x^n\}\, y \\ - (2n-1)\, & \{ \qquad\qquad xd_x^{n+1} + \qquad nd_x^n\}\, y \\ + n^2\, & \{ \qquad\qquad\qquad\qquad\quad d_x^n\}\, y; \end{aligned}$$

hence

$$A = 1, \quad B = 2n - (2n-1), = 1, \quad C = (n^2 - n) - n\,(2n-1) + n^2, = 0;$$

and we thus see also how in this particular case the last coefficient is $= 0$, viz. that we have

$$d_x^n (xd_x - n)^2 y = x^2 d_x^{n+2} y + xd_x^{n+1} y,$$

without any term in $d_x^n y$.

To find the coefficients A, B, C, ... generally, write $y = x^{r+n+\theta}$, then $xd_x - n = r + \theta$, and consequently

$$d_x^n (xd_x - n)^r y, \quad = (r+\theta)^r d_x^n x^{r+n+\theta}, \quad = (r+\theta)^r [r+n+\theta]^n x^{r+\theta};$$

whence

$$(r+\theta)^r [r+n+\theta]^n = A\,[r+\theta+n]^{n+r} + B\,[r+\theta+n]^{n+r-1} + \ldots\ldots;$$

or, what is the same thing,

$$(r+\theta)^r = A\,[r+\theta]^r + B\,[r+\theta]^{r-1} + \ldots\ldots.$$

Since the left-hand side, and every term $[r+\theta]^s$ on the right-hand side, contains the factor $r + \theta$, there is not on the right-hand side any term $[r+\theta]^0$; dividing the equation by $r + \theta$, it then becomes

$$(r+\theta)^{r-1} = A\,[r+\theta-1]^{r-1} + B\,[r+\theta-1]^{r-2} + \ldots\ldots,$$

and we thus have

$$A = \frac{\Delta^{r-1} 1^{r-1}}{[r-1]^{r-1}} (= 1), \quad B = \frac{\Delta^{r-2} 1^{r-1}}{[r-2]^{r-2}};$$

viz. writing $r + \theta = 1 + x$, $u_x = (1+x)^{r-1}$, and taking the terms in the reverse order, the series is the well-known one

$$u_x = u_0 + \frac{x}{1} \Delta u_0 + \frac{x\,.\,x-1}{1\,.\,2} \Delta^2 u_0 + \&c.$$

Hence, in general,

$$d_x^n (xd_x - n)^r y = \frac{\Delta^{r-1} 1^{r-1}}{[r-1]^{r-1}} x^r d_x^{r+n} y + \frac{\Delta^{r-2} 1^{r-1}}{[r-2]^{r-2}} x^{r-1} d_x^{r+n-1} y + \&c.,$$

where observe that the last term is $= xd_x^{n+1} y$.

For the function $y = x^n (\log x)^n$, the value of each side is $= [r]^r [n]^n$.

[Vol. XXIV., pp. 89—91.]

4752. (Proposed by Professor CAYLEY.)—Mr Wolstenholme's Question 3067 may evidently be stated as follows:—

If (a, b, c) are the coordinates of a point on the cubic curve

$$a^3 + b^3 + c^3 = (b+c)(c+a)(a+b),$$

and if

$$(b^2 + c^2 - a^2) x = (c^2 + a^2 - b^2) y = (a^2 + b^2 - c^2) z;$$

then (x, y, z) are the coordinates of a point on the same cubic curve.

This being so, it is required to find the geometrical relation of the two points to each other.

Solution by PROFESSOR CAYLEY.

1. On referring to Professor Wolstenholme's Solution of the original Question 3067 (*Reprint*, Vol. XIII., p. 70), it appears that the coordinates (x, y, z) of the point in question may be expressed in the more simple form

$$x : y : z = a(-a+b+c) : b(a-b+c) : c(a+b-c);$$

viz. the given relation between (a, b, c) being equivalent to

$$4abc + (-a+b+c)(a-b+c)(a+b-c) = 0,$$

we have

$$a^2 - (b-c)^2 = \frac{-4abc}{-a+b+c},$$

and thence

$$b^2 + c^2 - a^2 = 2bc\left(1 + \frac{2a}{-a+b+c}\right) = 2bc\left(\frac{a+b+c}{-a+b+c}\right);$$

and consequently

$$(b^2 + c^2 - a^2) x = \frac{2abc(a+b+c)x}{a(-a+b+c)},$$

whence the transformation in question.

2. Writing for greater symmetry (x, y, z) in place of (a, b, c), and (x', y', z') in place of (x, y, z), the coordinates (x, y, z) and (x', y', z') of the two points are connected by the relation

$$x' : y' : z' = x(-x+y+z) : y(x-y+z) : z(x+y-z),$$

and we thence at once deduce the converse relation

$$x : y : z = x'(-x'+y'+z') : y'(x'-y'+z') : z'(x'+y'-z').$$

Hence, writing

$$(-x+y+z,\quad x-y+z,\quad x+y-z) = (\xi,\ \eta,\ \zeta),$$

and similarly

$$(-x'+y'+z',\quad x'-y'+z',\quad x'+y'-z') = (\xi',\ \eta',\ \zeta'),$$

we have

$$x' : y' : z' = x\xi : y\eta : z\zeta,\quad x : y : z = x'\xi' : y'\eta' : z'\zeta',$$

and thence also $\xi\xi' = \eta\eta' = \zeta\zeta'$; so that, regarding $(\xi,\ \eta,\ \zeta)$, $(\xi',\ \eta',\ \zeta')$ as the coordinates of the two points, we see that these are inverse points one of the other in regard to the triangle $\xi = 0$, $\eta = 0$, $\zeta = 0$.

To complete the solution, we must introduce these new coordinates into the equation of the cubic curve. Writing this under the form

$$8xyz + 2(-x+y+z)(x-y+z)(x+y-z) = 0,$$

and observing that

$$(2x,\ 2y,\ 2z) = (\eta+\zeta,\ \zeta+\xi,\ \xi+\eta),$$

the equation is

$$(\eta+\zeta)(\zeta+\xi)(\xi+\eta) + 2\xi\eta\zeta = 0;$$

viz. this is a cubic curve inverting into itself. And the two points in question are thus any two inverse points on this cubic curve.

3. In regard to the original form, that the point $(x,\ y,\ z)$ defined by the equations

$$x(-a^2+b^2+c^2) = y(a^2-b^2+c^2) = z(a^2+b^2-c^2),$$

lies on the cubic curve

$$a^3+b^3+c^3-(b+c)(c+a)(a+b) = 0,$$

Professor Sylvester proceeds as follows:—Writing

$$(x,\ y,\ z) = \{a^4-(b^2-c^2)^2,\ b^4-(c^2-a^2)^2,\ c^4-(a^2-b^2)^2\},\ = (A,\ B,\ C),$$

suppose; and

$$F(a,\ b,\ c) = a^3+b^3+c^3-(b+c)(c+a)(a+b),$$

he observes that the truth of the theorem depends on the identity

$$F(A,\ B,\ C) + F(a,\ b,\ c)\,F(a,\ -b,\ c)\,F(a,\ b,\ -c)\,F(a,\ -b,\ -c) = 0,$$

and that, in order to prove the identity generally, it is sufficient to prove it for the three cases $a^2 = 0$, $a^2 = b^2+c^2$, $a^2 = b^2$, which may be effected without difficulty.

4. But, for a general proof of the identity, write

$$\lambda = b^2+c^2,\quad \mu = b^2-c^2,$$

so that

$$A = a^4-\mu^2,\quad B = (a^2+\mu)(-a^2+\lambda),\quad C = (-a^2+\lambda)(a^2-\mu),$$

whence

$$-F(A, B, C) = -(a^4-\mu^2)^3 + 2(a^2-\lambda)^3(a^6+3a^2\mu^2) - 8a^2b^2c^2(a^4-\mu^2)(a^2-\lambda),$$
$$= a^{12} - 6\lambda a^{10} + (6\lambda^2+9\mu^2-8b^2c^2)a^8 + \lambda(-2\lambda^2-18\mu^2+8b^2c^2)a^6$$
$$+ \mu^2(18\lambda^2-3\mu^2+8b^2c^2)a^4 + \lambda\mu^2(-6\lambda^2-8b^2c^2)a^2 + \mu^6.$$

Moreover

$$F(a, b, c) = a\{a^2-(b+c)^2\} - (b+c)\{a^2-(b-c)^2\},$$

therefore

$$F(a, -b, -c) = a\{a^2-(b+c)^2\} + (b+c)\{a^2-(b-c)^2\};$$

whence

$$F(a, b, c)F(a, -b, -c) = a^2\{a^2-(b+c)^2\}^2 - (b+c)^2\{a^2-(b-c)^2\}^2$$
$$= a^6 - 3\gamma^2a^4 + \gamma^2(\gamma^2+2\delta^2)a^2 - \gamma^2\delta^2,$$

if $\gamma = b+c$, $\delta = b-c$. By changing the sign of c, we interchange γ and δ, and we thus have

$$F(a, b, -c)F(a, -b, c) = a^6 - 3\delta^2a^4 + \delta^2(2\gamma^2+\delta^2)a^2 - \gamma^4\delta^2,$$

and the identity to be verified is thus

$$\{a^6 - 3\gamma^2a^4 + \gamma^2(\gamma^2+2\delta^2)a^2 - \gamma^2\delta^4\}\{a^6 - 3\delta^2a^4 + \delta^2(2\gamma^2+\delta^2)a^2 - \gamma^4\delta^2\}$$
$$= a^{12} - 6\lambda a^{10} + \ldots\ldots + \mu^6, \textit{ ut suprà};$$

the values of λ, μ in terms of γ, δ are $\lambda = \frac{1}{2}(\gamma^2+\delta^2)$, $\mu = \gamma\delta$; substituting these values on the right-hand side, the verification can be completed without difficulty.

[Vol. XXV., January to July, 1876, p. 82.]

4946. (Proposed by Professor CAYLEY.)—Show that the attraction of an indefinitely thin double convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.

[Vol. XXVI., July to December, 1876, pp. 41, 42.]

5020. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Let 1, δ_1, δ_2, δ_3, ..., δ_n be the first differences of the coefficients of the expansion of the binomial $(1+x)^{2n}$ taken as far as the central or maximum coefficient; also let

$$\nu = \tfrac{1}{2}(n+1)n, \quad \nu' = \tfrac{1}{2}n(n-1), \quad \nu'' = \tfrac{1}{2}(n-1)(n-2), \text{ \&c.};$$

then show that the algebraic function

$$x^\nu - \delta_1 x^{\nu'} + \delta_2 x^{\nu''} - \delta_3 x^{\nu'''} + \text{\&c.}$$

is divisible by $(x-1)^n$ without a remainder; and that the sum of the numerical coefficients of the quotient is equal to $1.3.5\ldots 2n-1$.

[See Solution to Question 1894, *Reprint*, vol. v., p. 113.]

Solution by PROFESSOR CAYLEY.

Mr Woolhouse's elegant theorem depends ultimately on the property of triangular numbers $\phi(n), = \frac{1}{2}(n^2 - n)$; then $\phi(n+1) = \phi(-n)$, so that, writing down the series of triangular numbers backwards and forwards,

$$\begin{array}{cccccccccccc} \ldots, & 10, & 6, & 3, & 1, & 0, & 0, & 1, & 3, & 6, & 10, & \ldots \\ \ldots & , & a, & b, & c, & d, & e, & f, & g, & h, & & \ldots \end{array}$$

we have, in fact, a continuous single series obtained by giving to n the different negative and positive integer values, zero included.

Thus a particular case is

$$(1+x)^6 - 5(1+x)^3 + 9(1+x) - 5 \equiv 0 \,(\text{mod. } x^2) = 1 \,.\, 3 \,.\, 5x^3 + \&\text{c. } x^4 + \ldots,$$

where, on the left-hand side, the exponents are the triangular numbers $\phi(n+1)$, $n = 0$ to 3; and the coefficients, after the first, are the differences of the binomial coefficients of the power $2n$ (in the particular case, $n = 3$); viz. the binomial coefficients being

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1,$$

the differences taken as far as they are positive are

$$5, \quad 9, \quad 5.$$

Expanding the several terms and writing down only the coefficients, we have a diagram

$$\begin{array}{r|llllllll} & 1, & 6, & 15, & 20, & 15, & 6, & 1, \\ -5 & 1, & 3, & 3, & 1, & & & \\ +9 & 1, & 1, & & & & & \\ -5 & 1, & & & & & & \end{array}$$

The theorem in the particular case depends on the identities

$$\begin{aligned} 1 - 5 + 9 - 5 &= 0, \\ 6 - 15 + 9 \quad &= 0, \\ 5 - 15 \quad\quad &= 0, \\ 20 - 5 \quad\quad\quad &= 1 \,.\, 3 \,.\, 5; \end{aligned}$$

or writing, as above, h, g, f, e, to denote the triangular numbers 6, 3, 1, 0, these may be replaced by

$$\begin{array}{lll} h^0 & -5g^0 & +9f^0 - 5e^0 = 0, \\ h & -5g & +9f - 5e = 0, \\ \frac{1}{2}h(h-1) & -5 \,.\, \frac{1}{2}g(g-1) & + \ldots \quad = 0, \\ \frac{1}{6}h(h-1)(h-2) & -5 \,.\, \frac{1}{6}g(g-1)(g-2) & + \ldots \quad = 1 \,.\, 3 \,.\, 5; \end{array}$$

or, reducing each equation by those which precede it, these become

$$h^0 - 5g^0 + 9f^0 - 5e^0 = 0,$$
$$h^1 - 5g^1 + 9f^1 - 5e^1 = 0,$$
$$h^2 - 5g^2 + 9f^2 - 5e^2 = 0,$$
$$h^3 - 5g^3 + 9f^3 - 5e^3 = 1.2.3.1.3.5.$$

Consider any one of these, for instance the third; the function on the left-hand is

$$1h^3 - (6-1)\,g^3 + (15-6)\,f^3 - (20-15)\,e^3,$$

or, introducing the values b, c, d as above,

$$1h^3 - 6g^3 + 15f^3 - 20e^3 + 15d^3 - 6c^3 + 1b^3,$$

which is, in fact, $=0$, if b, c, d, e, f, g, h are *any* successive triangular numbers; viz. this is an immediate consequence of the well-known theorem

$$1\,(\theta+6)^m - 6\,(\theta+5)^m + 15\,(\theta+4)^m - 20\,(\theta+3)^m + 15\,(\theta+2)^m - 6\,(\theta+1)^m + \theta^m$$
$$= \Delta^6\theta^m,\ =0 \text{ for any value of } m \text{ up to } m=5, \text{ and}$$
$$= 1.2.3.4.5.6 \text{ for } m=6.$$

We have thus all the equations except the last; and as regards the last equation, observe that the equation to be verified is

$$1\,[\tfrac{1}{2}(\theta+6)(\theta+5)]^3 - 6\,[\tfrac{1}{2}(\theta+5)(\theta+4)]^3 + \ldots = 1.2.3.1.3.5,$$

viz. this may be replaced by

$$1\,(\theta+6)^6 - 6\,(\theta+5)^6 + \ldots = 2^3.1.2.3.1.3.5 = 2.4.6.1.3.5 = 1.2.3.4.5.6,$$

which is right.

It is clear that the proof, although worked out on a particular case, is perfectly general; and Mr Woolhouse's theorem is thus proved.

[Vol. XXVI., pp. 77, 78.]

5079. (Proposed by Professor CAYLEY.)—Show that the curve

$$\{(\beta-\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x-\beta i)^2+y^2\}^{\frac{1}{2}} + q\,\{(\beta+\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x+\beta i)^2+y^2\}^{\frac{1}{2}}$$
$$= \left\{(1-q^2)\frac{\beta}{\delta}\right\}^{\frac{1}{2}}\{\beta^2-(\gamma-\delta i)^2\}^{\frac{1}{2}}\{(x-\gamma-\delta i)^2+y^2\}^{\frac{1}{2}},$$

where $i=(\sqrt{-1})$ as usual, is a real bicircular quartic having the axial foci

$$\beta i,\ -\beta i,\ \gamma+\delta i,\ \gamma-\delta i.$$

Solution by the PROPOSER.

Consider the equation

$$(l+mi)^{\frac{1}{2}}[(x-\beta i)^2+y^2]^{\frac{1}{2}}+q(l-mi)^{\frac{1}{2}}[(x+\beta i)^2+y^2]^{\frac{1}{2}}=(\lambda+\mu i)^{\frac{1}{2}}[(x-\gamma-\delta i)^2+y^2]^{\frac{1}{2}}.$$

This is

$$\begin{aligned}(l+mi)\{x^2+y^2-\beta^2-2\beta xi\}+q^2(l-mi)\{x^2+y^2-\beta^2+2\beta xi\}\\ -(\lambda+\mu i)\{x^2+y^2-\beta^2+\beta^2+\gamma^2-\delta^2-2\gamma x-2(x-\gamma)\delta i\}\\ +2q(l^2+m^2)^{\frac{1}{2}}[(x^2+y^2-\beta^2)^2+4\beta^2x^2]^{\frac{1}{2}}=0,\end{aligned}$$

where, putting the imaginary part equal to zero, we have

$$m(1-q^2)(x^2+y^2-\beta^2)-2l(1-q^2)\beta x-\mu\{x^2+y^2-\beta^2+(\beta^2+\gamma^2-\delta^2)-2\gamma x\}+2\lambda(x-\gamma)\delta=0,$$

which will be true identically if

$$\begin{aligned}m(1-q^2)-\mu&=0,\\ -l(1-q^2)\beta+\mu\gamma+\lambda\delta&=0,\\ -\mu(\beta^2+\gamma^2-\delta^2)-2\lambda\gamma\delta&=0.\end{aligned}$$

The last gives

$$\lambda=\theta(\beta^2+\gamma^2-\delta^2),\quad \mu=-2\theta\gamma\delta,\quad \theta \text{ arbitrary};$$

and then

$$l(1-q^2)\beta=\theta\delta(\beta^2+\gamma^2-\delta^2-2\gamma^2)=\theta\delta(\beta^2-\gamma^2-\delta^2),$$

$$m(1-q^2)=-2\theta\delta\gamma;$$

so that, putting

$$\theta\delta=(1-q^2)\beta,\ \text{or}\ \theta=(1-q^2)\frac{\beta}{\delta},$$

we have

$$l=\beta^2-\gamma^2-\delta^2,\quad m=-2\beta\gamma,$$

$$\lambda=(1-q^2)\frac{\beta}{\delta}(\beta^2+\gamma^2-\delta^2),\quad \mu=(1-q^2)\frac{\beta}{\delta}(-2\gamma\delta).$$

Therefore

$$l\pm mi=(\beta\mp\gamma i)^2-\delta^2,$$

$$\lambda\pm\mu i=(1-q^2)\frac{\beta}{\delta}[\beta^2+(\gamma\mp\delta i)^2];$$

and the equation is

$$\begin{aligned}\{(\beta-\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x-\beta i)^2+y^2\}^{\frac{1}{2}}+q\{(\beta+\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x+\beta i)^2+y^2\}^{\frac{1}{2}}\\ =\left\{(1-q^2)\frac{\beta}{\delta}\right\}^{\frac{1}{2}}\{\beta^2-(\gamma-\delta i)^2\}^{\frac{1}{2}}\{(x-\gamma-\delta i)^2+y^2\}^{\frac{1}{2}},\end{aligned}$$

which is a real curve having the axial foci $+\beta i$, $-\beta i$; $\gamma+\delta i$; $\gamma-\delta i$; viz. $\gamma+\delta i$ being a focus, and the curve being real, it is clear that $\gamma-\delta i$ is also a focus.

[Vol. XXVII., January to June, 1877, p. 20.]

5130. (Proposed by Professor CAYLEY.)—Show that the envelope of a variable circle, having its centre on a given conic and cutting at right angles a given circle, is a bicircular quartic; which, when the given conic and the circle have double contact, becomes a pair of circles; and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems—

(i) Given two conics, to find a polygon of n sides inscribed in the one and circumscribed about the other.

(ii) Given two circles, to find a closed series of n circles each touching the two circles and the two adjacent circles of the series.

[Vol. XXVII., pp. 81—83.]

5208. (Proposed by Professor SYLVESTER.)—Let the *magnitude* of any ramification signify the number of its branches, and let its partial magnitudes in respect to any node signify the magnitudes of the ramifications which come together at that node. If at any node the largest magnitude exceeds by k the sum of the other magnitudes, let the node be called superior by k, or be said to be of superiority k; but if no magnitude exceeds the sum of the other magnitudes, let the node be called subequal. Then the theorem is, in any ramification, *either* there is one and only one subequal node; *or else* there are two and only two nodes each superior by unity, these two nodes being contiguous.

Solution by PROFESSOR CAYLEY.

The proof consists in showing that (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity: and if there is one such node, then there is, contiguous to it, another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is either of smaller superiority, or else subequal; for, these theorems holding good, we can, by (3), always arrive at a node which is either subequal or else superior by unity; in the former case, by (1), the subequal node thus arrived at is unique; in the latter case, by (2), we have, contiguous to the node arrived at, a second node superior by unity; and we have thus a unique pair of nodes each superior by unity.

I will prove only (3), as it is easy to see that the like process applies to the proof of (1) and (2).

Let the whole magnitude be n; and suppose at a node P which is superior by k, the largest magnitude is α, and that the other magnitudes are, say, β, γ, δ. We

have $\alpha = \beta + \gamma + \delta + k$; and since $n = \alpha + \beta + \gamma + \delta$, we have thence $n = 2\alpha - k$, or $\alpha = \frac{1}{2}(n+k)$, $\beta + \gamma + \delta = \frac{1}{2}(n-k)$: clearly k is even or odd, according as n is even or odd.

Suppose now that we pass from P, along the branch of magnitude α, to a contiguous node Q; and let the magnitudes for Q be α', β', γ', δ', ϵ', where α' denotes the magnitude for the branch QP. We have $\alpha' = \beta + \gamma + \delta + 1$: for the ramification consists of the branch QP and of the ramifications of magnitudes β, γ, δ which meet in P. We have thus

$$\alpha' = \tfrac{1}{2}(n-k) + 1 = \tfrac{1}{2}n - \tfrac{1}{2}(k-2);$$

and thence

$$\beta' + \gamma' + \delta' + \epsilon' = \tfrac{1}{2}n + \tfrac{1}{2}(k-2).$$

Supposing here that k is greater than 1, viz. that it is $=$ or >2, $k-2$ is 0 or positive; and if α' be the greatest magnitude belonging to the node Q, this is a subequal node. But it may be that α' is not the greatest magnitude; supposing then that the greatest magnitude is β', we have

$$\beta' = \tfrac{1}{2}n + \tfrac{1}{2}(k-2) - \gamma' - \delta' - \epsilon',$$

$$\alpha' + \gamma' + \delta' + \epsilon' = \tfrac{1}{2}n - \tfrac{1}{2}(k-2) + \gamma' + \delta' + \epsilon',$$

and thence

$$\beta' - (\alpha' + \gamma' + \delta' + \epsilon') = k - 2 - 2(\gamma' + \delta' + \epsilon');$$

viz. either the node is subequal, or else, being superior, the superiority is at most $= k-2$; that is, if from the node P, of superiority $=$ or >2, we pass along the branch of greatest magnitude to the contiguous node Q, this is either subequal, or else of superiority less than that of P; which is the foregoing proposition (3).

The subequal node, and the two nodes of superiority 1, in the cases where they respectively exist, may be termed the centre and the bicentre respectively; and the theorem thus is, every ramification has either a centre or else a bicentre. But the centre and the bicentre here considered, due (as remarked by Professor Sylvester) to M. Camille Jordan, and which may for distinction be termed the centre and the bicentre of *magnitude*, are quite distinct from the centre and the bicentre discovered by Professor Sylvester, and considered in my researches upon trees, *British Association Report*, 1875, [610]. These last may for distinction be termed the centre and the bicentre of *distance:* viz. we here consider, not the magnitude, but the length of a ramification, such length being measured by the number of branches to be travelled over in order to reach the most distant terminal node. The ramification has either a centre or else a bicentre of distance: viz. the centre is a node such that, for two or more of the ramifications which proceed from it, the lengths are equal and superior to those of the other ramifications, if any; the bicentre a pair of contiguous nodes such that, disregarding the branch which unites the two nodes, there are from the two nodes respectively (one at least from each of them) two or more ramifications the lengths of which are equal to each other and superior to those of the other ramifications, if any.

It is very noticeable how close the agreement is between the proofs for the existence of the two kinds of centre or bicentre respectively. Say, first as regards distance, if at any node the length of the longest branch exceeds by k the length of the next longest branch or branches, then the node is superior by k, or is of the superiority k; but, if there are two or more longest branches, then the node is subequal. And say next, in regard to magnitude, if at any node the largest magnitude exceeds by k the sum of all the other magnitudes, the node is superior by k, or has a superiority k; but if the largest magnitude does not exceed the sum of the other magnitudes, then the node is subequal. Then, whether we attend to distance or to magnitude, the three propositions hold good: (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity: and if there is one such node, there is contiguous to it another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is of smaller superiority or else subequal; whence, as in the solution just referred to, there is always, as regards distance, a centre or a bicentre; and there is always, as regards magnitude, a centre or a bicentre.

[Vol. XXVII., pp. 89, 90.]

On Mr Artemas Martin's First Question in Probabilities. *By* PROFESSOR CAYLEY.

The question was, "A says that B says that a certain event took place: required the probability that the event did take place, p_1 and p_2 being A's and B's respective probabilities of speaking the truth."

The solutions, referred to or given on pp. 77—79 of volume XXVII. of the *Reprint*, give the following values for the probability in question:—

Todhunter's *Algebra* ..... $p_1p_2 + (1-p_1)(1-p_2)$.

Artemas Martin ..... $p_1[p_1p_2 + (1-p_1)(1-p_2)]$.

American Mathematicians and Woolhouse... p_1p_2.

It seems to me that the true answer cannot be expressed in terms of only p_1 and p_2, but that it involves two other constants β and k; and my value is—

Cayley ..... $p_1p_2 + \beta(1-p_1)(1-p_2) + k(1-\beta)(1-p_1)$.

In obtaining this I introduce, *but I think of necessity,* elements which Mr Woolhouse calls extraneous and imperfect.

B told A that the event happened, or he did not tell A this; the only evidence is A's statement that B told him that the event happened; and the chances are p_1 and $1-p_1$. But, in the latter case, either B told A that the event did not happen, or he did not tell him at all; the chances (on the supposition of the incorrectness of A's statement) are β and $1-\beta$; and the chances of the three cases

are thus p_1, $\beta(1-p_1)$, and $(1-\beta)(1-p_1)$. On the suppositions of the first and second cases respectively, the chances for the event having happened are p_2 and $1-p_2$; on the supposition of the third case (viz. here there is no information as to the event) the chance is k, the antecedent probability; and the whole chance in favour of the event is

$$p_1p_2+\beta(1-p_1)(1-p_2)+k(1-\beta)(1-p_1).$$

If $\beta=1$, we have Todhunter's solution; if $\beta=0$, and also $k=0$, we have the solution preferred by Woolhouse; but we do not (otherwise than by establishing between k and β a relation which is quite arbitrary) obtain Martin's solution. The error in this seems to be as follows:—A says that B told him as to the event, and says further that B told him that the event did happen; the probability of the truth of the compound statement is taken to be $=p_1^2$; whereas, in calling the probability of A's speaking the truth p_1, we mean that if A makes the statement, "B says that the event took place," this is to be regarded as a simple statement, and the probability of the truth of the statement is $=p_1$; viz. I think that Martin introduces into his solution a hypothesis contradictory to the assumptions of the question.

I remark further that in my solution I assume that the event is of such a nature that, when there is *any* testimony in regard to it, the probability is determined by that testimony, irrespectively of the antecedent probability. This is quite consistent with the antecedent probability being, not zero, but as small as we please; so that, if k is (as it may very well be) indefinitely small, the whole probability is the same as if k were $=0$. But there is absolutely no reason for assigning any determinate value to β; so that the solutions $p_1p_2+(1-p_1)(1-p_2)$ and p_1p_2, which assume respectively $\beta=1$ and $\beta=0$, seem to me on this ground erroneous.

[Vol. XXVIII., June to December, 1877, p. 17.]

5306. (Proposed by Professor CAYLEY.)—If α, β, γ, δ; α_1, β_1, γ_1, δ_1, are such that

$$(\alpha_1-\delta_1)(\beta_1-\gamma_1)=(\alpha-\delta)(\beta-\gamma),$$

$$(\beta_1-\delta_1)(\gamma_1-\alpha_1)=(\beta-\delta)(\gamma-\alpha),\quad (\gamma_1-\delta_1)(\alpha_1-\beta_1)=(\gamma-\delta)(\alpha-\beta);$$

show that the three equations

$$\frac{x_1-\alpha_1}{x_1-\delta_1}=\frac{1}{(\beta_1-\delta_1)(\gamma_1-\delta_1)}\{(x-\alpha)^{\frac12}(x-\delta)^{\frac12}-(x-\beta)^{\frac12}(x-\gamma)^{\frac12}\}^2,$$

$$\frac{x_1-\beta_1}{x_1-\delta_1}=\frac{1}{(\gamma_1-\delta_1)(\alpha_1-\delta_1)}\{(x-\beta)^{\frac12}(x-\delta)^{\frac12}-(x-\gamma)^{\frac12}(x-\alpha)^{\frac12}\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1}=\frac{1}{(\alpha_1-\delta_1)(\beta_1-\delta_1)}\{(x-\gamma)^{\frac12}(x-\delta)^{\frac12}-(x-\alpha)^{\frac12}(x-\beta)^{\frac12}\}^2,$$

are equivalent to each other; and show also that, consistently with the foregoing relations between the constants, the differences $\alpha_1-\delta_1$, $\beta_1-\delta_1$, $\gamma_1-\delta_1$ may be so determined that the equations in (x, x_1) constitute a particular integral of the differential equation

$$\frac{dx}{\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\}^{\frac{1}{2}}}=\frac{dx_1}{\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\}^{\frac{1}{2}}}.$$

[Vol. XXIX., January to June, 1878, p. 20.]

4870. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find on the first a point A', on the second a point B', and on the third a point C', such that the intersections of the lines

$A'B'$ and AC, $A'C'$ and AB, lie on the first conic;

$B'C'$ and BA, $B'A'$ and BC, lie on the second conic;

$C'A'$ and CB, $C'B'$ and CA, lie on the third conic.

[Vol. XXIX., pp. 96, 97.]

5625. (Proposed by Professor CAYLEY.)—The equation

$$\{q^2(x+y+z)^2-yz-zx-xy\}^2=4(2q+1)xyz(x+y+z)$$

represents a trinodal quartic curve having the lines $x=0$, $y=0$, $z=0$, $x+y+z=0$ for its four bitangents; it is required to transform to the coordinates X, Y, Z, where $X=0$, $Y=0$, $Z=0$ represent the sides of the triangle formed by the three nodes.

[Vol. XXXI., January to June, 1879, p. 38.]

5387. (Proposed by Professor CAYLEY.)—Show that a cubic surface has at most 4 conical points, and a quartic surface at most 16 conical points.

[Vol. XXXII., July to December, 1879, p. 35.]

5927. (Proposed by Professor CAYLEY.)—If $\{\alpha+\beta+\gamma+\ldots\}^p$, denote the expansion of $(\alpha+\beta+\gamma+\ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\delta^d\ldots$ only in which

$$b+c+d+\ldots \not> p-1,\quad c+d+\ldots \not> p-2, \text{ \&c. \&c.};$$

prove that

$$x^n=(x+\alpha)^n-(n)_1\{\alpha\}^1(x+\alpha+\beta)^{n-1}+\frac{n(n-1)}{1.2}\{\alpha+\beta\}^2(x+\alpha+\beta+\gamma)^{n-2}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}\{\alpha+\beta+\gamma\}^3(x+\alpha+\beta+\gamma+\delta)^{n-3}+\text{\&c.}\ldots(1).$$

[Vol. XXXIII., January to July, 1880, p. 17.]

6155. (Proposed by Professor CAYLEY.)—Given, by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance.

N.B.—If λ, μ, ν are the inclinations of a line to three rectangular axes, and α, β, γ the coordinates to the same axes of a point on the line, then the metrical coordinates of the line are

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h, \\ =\cos\lambda, & \cos\mu, & \cos\nu, & \beta\cos\nu-\gamma\cos\mu, & \gamma\cos\lambda-\alpha\cos\nu, & \alpha\cos\mu-\beta\cos\lambda, \end{array}$$

satisfying identically the relations

$$a^2+b^2+c^2=1, \quad af+bg+ch=0.$$

[Vol. XXXVI., 1881, p. 21.]

6470. (Proposed by Professor CAYLEY.)—It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form

$$xy-z^2=\{(z^2-x^2)(z^2-k^2x^2)\}^{\frac{1}{2}}.$$

[Vol. XXXVI., p. 64.]

6766. (Proposed by Professor CAYLEY.)—Find the stationary and the double tangents of the curve $x^4+y^4+z^4=0$.

Solution by the PROPOSER.

Take l a fourth root of -1; m and n fourth roots of $+1$; then the 28 double tangents are the lines $x=ly$, $x=lz$, $y=lz$, $(4+4+4=)$ 12 lines; and the lines $x+my+nz=0$, 16 lines; and the first 12 of these, each counted twice, are the 24 stationary tangents. In fact, any one of the 12 lines is an osculating tangent, or line meeting the curve in 4 coincident points; it counts therefore once as a double tangent, and twice as a stationary tangent. There should consequently be 16 other double tangents; and it only needs to be shown that these are the 16 lines $x+my+nz=0$. Consider any one line $x+my+nz=0$; for its intersections with the curve $x^4+y^4+z^4=0$, we have

$$(my+nz)^4+y^4+z^4=0,$$

or, as this may be written,

$$(my+nz)^4+m^4y^4+n^4z^4=0;$$

viz. this is

$$2\,(1,\ 2,\ 3,\ 2,\ 1\between my,\ nz)^4=0,$$

or, what is the same thing,

$$2\,[(1,\ 1,\ 1\between my,\ nz)^2]^2=0:$$

so that the line is a double tangent, the two points of contact being given by means of the equation $(1, 1, 1 \between my, nz)^2 = 0$; viz. ω being an imaginary cube root of unity, we have $nz = \omega my$ or $\omega^2 my$: and thence, for the points of contact,

$$x : y : z = 1 : \frac{\omega}{m} : \frac{\omega^2}{n}, \text{ or } = 1 : \frac{\omega^2}{m} : \frac{\omega}{n};$$

values which satisfy, as they should do, the two equations

$$x + my + nz = 0 \text{ and } x^4 + y^4 + z^4 = 0.$$

[Vol. XXXVI., pp. 106, 107.]

6800. (Proposed by W. J. C. MILLER, B.A.)—Prove that, if

$$\frac{ayz}{y^2+z^2} = \frac{bzx}{z^2+x^2} = \frac{cxy}{x^2+y^2} = 1,$$

then

$$a^2 + b^2 + c^2 = abc + 4.$$

Note on Question 6800. *By* PROFESSOR CAYLEY.

The identity given by the solution is a very interesting one. Instead of a, b, c, writing $(a, b, c) \div d$, we have

$$4d^3 - d(a^2 + b^2 + c^2) + abc = 0,$$

satisfied by

$$a : b : c : d = x(y^2 + z^2) : y(z^2 + x^2) : z(x^2 + y^2) : xyz;$$

or, considering (a, b, c, d) as the coordinates of a point in space, and (x, y, z) as the coordinates of a point in a plane, we have thus a correspondence between the points of the cubic surface $4d^3 - d(a^2 + b^2 + c^2) + abc = 0$, and the points of the plane. To a given system of values of (x, y, z) there corresponds, it is clear, a single system of values of (a, b, c, d); and it may be shown without difficulty that, to a given system of values of (a, b, c, d) satisfying the equation of the surface, there correspond two systems of values of (x, y, z); the plane and cubic surface have thus a (1, 2) correspondence with each other.

[Vol. XXXVII., 1882, p. 74.]

5244. (Proposed by Professor CAYLEY.)—Writing for shortness

$$L = \beta^2\gamma^2 - \alpha^2\delta^2, \quad M = \gamma^2\alpha^2 - \beta^2\delta^2, \quad N = \alpha^2\beta^2 - \gamma^2\delta^2,$$

$$F = \alpha^2 + \delta^2 - \beta^2 - \gamma^2, \quad G = \beta^2 + \delta^2 - \gamma^2 - \alpha^2, \quad H = \gamma^2 + \delta^2 - \alpha^2 - \beta^2,$$

$$\Delta = \alpha^2 + \beta^2 + \gamma^2 + \delta^2;$$

show that the equation

$$LMN(x^4+y^4+z^4+w^4)+MN(F\Delta+2L)(y^2z^2+x^2w^2)+NL(G\Delta+2M)(z^2x^2+y^2w^2)$$
$$+LM(H\Delta+2N)(x^2y^2+z^2w^2)-2\alpha\beta\gamma\delta\, FGH\Delta\, xyzw=0$$

belongs to a 16-nodal quartic surface, having the nodes

$x=\alpha$	α	α	α	β	β	β	β	γ	γ	γ	γ	δ	δ	δ	δ
$y=\beta$	$-\beta$	$-\beta$	β	α	$-\alpha$	$-\alpha$	α	δ	$-\delta$	$-\delta$	δ	γ	$-\gamma$	$-\gamma$	γ
$z=\gamma$	$-\gamma$	γ	$-\gamma$	δ	$-\delta$	δ	$-\delta$	α	$-\alpha$	α	$-\alpha$	β	$-\beta$	β	$-\beta$
$w=\delta$	δ	$-\delta$	$-\delta$	γ	γ	$-\gamma$	$-\gamma$	β	β	$-\beta$	$-\beta$	α	α	$-\alpha$	$-\alpha$

and the sixteen singular tangent planes represented by the equations

$$(\alpha, \beta, \gamma, \delta)(x, y, z, w)=0, \text{ \&c.}$$

[Vol. XXXVIII., 1883, pp. 87—89.]

7190. (Proposed by Professor WOLSTENHOLME, M.A.)—If x, y, z be three quantities satisfying the two symmetrical equations

$$yz+zx+xy=0, \quad x^3+y^3+z^3+4xyz=0;$$

prove that (1) they will also satisfy one of the two pairs of semi-symmetrical expressions

$$y^2z+z^2x+x^2y=(y-z)(z-x)(x-y), \ =+xyz,$$
$$yz^2+zx^2+xy^2=(y-z)(z-x)(x-y), \ =-xyz;$$

and (2) one set of the following equations will also be satisfied:—

$$(x^2+yz-y^2=0, \quad y^2+zx-z^2=0, \quad z^2+xy-x^2=0);$$
$$(x^2+yz-z^2=0, \quad z^2+zx-x^2=0, \quad z^2+xy-y^2=0).$$

Solution by PROFESSOR CAYLEY.

The two symmetrical equations represent a conic and a cubic respectively; they intersect therefore in 6 points, and if we denote by α a root of the equation

$$u^3+u^2-2u-1=0,$$

then the other two roots of this equation are

$$\beta, \ =-1-\frac{1}{\alpha}, \quad \gamma=\frac{-1}{\alpha+1};$$

viz. if $\alpha^3+\alpha^2-2\alpha-1=0$, then we have

$$(u-\alpha)\left(u+1+\frac{1}{\alpha}\right)\left(u+\frac{1}{\alpha+1}\right)=u^3+u^2-2u-1,$$

an identity which is easily verified. It may be remarked that, if

$$\phi\alpha = -1 - \frac{1}{\alpha},$$

then

$$\phi^2\alpha = \frac{-1}{\alpha+1}, \quad \phi^3\alpha = \alpha;$$

the left-hand side of the last mentioned equation thus is $(u-\alpha)(u-\phi\alpha)(u-\phi^2\alpha)$, which remains unaltered when α is changed into $\phi\alpha$ or $\phi^2\alpha$. Then the coordinates of the six points of intersection can be expressed indifferently in terms of any one of the roots (α, β, γ), viz. the coordinates are

$$(\alpha^2-1, -\alpha, -1), \quad (-1, \alpha^2-1, -\alpha), \quad (-\alpha, -1, \alpha^2-1), \ldots (1, 2, 3),$$
$$(\alpha^2-1, -1, -\alpha), \quad (-\alpha, \alpha^2-1, -1), \quad (-1, -\alpha, \alpha^2-1), \ldots (4, 5, 6);$$

or they are equal to the like expressions in β and in γ respectively; say these are the coordinates of the points 1, 2, 3, 4, 5, 6 respectively, as shown by the attached numbers. Thus, writing

$$x, y, z = \alpha^2-1, -\alpha, -1,$$

we find

$$yz + zx + xy = \alpha - \alpha^2 + 1 - \alpha^3 + \alpha = -(\alpha^3 + \alpha^2 - 2\alpha - 1) = 0,$$
$$x^3 + y^3 + z^3 + 4xyz = (\alpha^2-1)^3 - \alpha^3 - 1 + 4\alpha(\alpha^2-1)$$
$$= \alpha^6 - 3\alpha^4 + 3\alpha^3 + 3\alpha^2 - 4\alpha - 2 = (\alpha^3 + \alpha^2 - 2\alpha - 1)(\alpha^3 - \alpha^2 + 2) = 0,$$

which verifies the formulæ for the six points of intersection. Take, again,

$$x, y, z = \alpha^2-1, -\alpha, -1;$$

then we find

$$yz^2 + zx^2 + xy^2 = -\alpha - (\alpha^2-1)^2 + \alpha^2(\alpha^2-1) = \alpha^2 - \alpha - 1,$$
$$y^2z + z^2x + x^2y = -\alpha^2 + (\alpha^2-1) - \alpha(\alpha^2-1)^2 = -\alpha^5 + 2\alpha^3 - \alpha - 1.$$

Or, since $\alpha^3 = -\alpha^2 + 2\alpha + 1$, and thence

$$\alpha^4 = 3\alpha^2 - \alpha - 1, \quad \alpha^5 = -4\alpha^2 + 5\alpha + 3,$$

the last equation becomes

$$y^2z + z^2x + x^2y = 2\alpha^2 - 2\alpha - 2.$$

We have also

$$xyz = \alpha^3 - \alpha, = -\alpha^2 + \alpha + 1;$$

hence the point in question is situate on each of the cubics

$$yz^2 + zx^2 + xy^2 + xyz = 0, \quad y^2z + z^2x + x^2y + 2xyz = 0,$$
$$y^2z + z^2x + x^2y - 2(yz^2 + zx^2 + xy^2) = 0;$$

and this, of course, shows the points 1, 2, 3 are all three of them situate upon each of the three cubics; and in precisely the same manner it appears that the points 4, 5, 6 are all three of them situate on each of the three cubics

$$yz^2 + zx^2 + xy^2 + 2xyz = 0, \quad y^2z + z^2x + x^2y + xyz = 0,$$
$$yz^2 + zx^2 + xy^2 - 2(y^2z + z^2x + x^2y) = 0.$$

Again, from the values $x, y, z = \alpha^2 - 1, \ -\alpha, \ -1$, we have

$$x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0;$$

viz. the point 1 lies on each of these conics; similarly the point 2 lies on each of the same conics; and the point 3 lies on each of the same conics; that is, the conics in question have in common the points 1, 2, 3.

In like manner, the conics

$$x^2 + yz - z^2 = 0, \quad y^2 + zx - x^2 = 0, \quad z^2 + xy - y^2 = 0,$$

have in common the points 4, 5, 6.

The general result is that the given conic and the cubic meet in six points forming two groups of points (1, 2, 3) and (4, 5, 6); through the points (1, 2, 3) we have three cubics and three conics; and through the points (4, 5, 6) we have three cubics and three conics.

If in the equation $x^3 + x^2 - 2x - 1 = 0$, whose roots are α, $\phi(\alpha)$, $\phi^2(\alpha)$, we put $x = 2\cos\theta$, the equation becomes

$$2(3\cos\theta + \cos 3\theta) + 2(1 + \cos 2\theta) - 4\cos\theta - 2 = 0,$$

or

$$2\cos 3\theta + 2\cos 2\theta + 2\cos\theta = 0, \quad \text{or} \quad \frac{\sin\frac{7}{2}\theta}{\sin\frac{1}{2}\theta} = 0;$$

or the three roots are $2\cos\frac{2}{7}\pi$, $2\cos\frac{4}{7}\pi$, $2\cos\frac{6}{7}\pi$. The two equations

$$yz + zx + xy = 0, \quad x^3 + y^3 + z^3 + 3xyz = 0,$$

are satisfied if $x : y : z =$ these three roots in any order, giving the six solutions. The semi-symmetrical systems are satisfied, the one by

$$x : y : z, \quad \text{or} \quad y : z : x, \quad \text{or} \quad z : x : y, = \cos\tfrac{2}{7}\pi : \cos\tfrac{4}{7}\pi : \cos\tfrac{6}{7}\pi;$$

and the other by

$$z : y : x, \quad \text{or} \quad y : x : z, \quad \text{or} \quad x : z : y, = \cos\tfrac{2}{7}\pi : \cos\tfrac{4}{7}\pi : \cos\tfrac{6}{7}\pi.$$

[Vol. XXXIX., 1883, p. 31.]

5689. (Proposed by Professor CAYLEY.)—Show (1) that the apparent contour of a Steiner's surface $(2xyz + y^2z^2 + z^2x^2 + x^2y^2 = 0)$, as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm 1, 0)$ and $(0, \pm 1)$; and (2) find the surface-contour, or curve of contact, of the cone and surface.

[Vol. XXXIX., p. 49.]

4722. (Proposed by Professor CAYLEY.)—1. Show that the conditions in regard to the reality of the roots of the equation

$$(x^2-\alpha)^2+16A(x-m)=0,$$

are, if

$$(4m^2-3\alpha)^3-(8m^3-9m\alpha-27A)^2=-,$$

then the roots are two real, two imaginary; but if

$$(4m^2-3\alpha)^3-(8m^3-9m\alpha-27A)^2=+,$$

then, if simultaneously

$$\alpha=+,\quad A(m\alpha-9A)=+,$$

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y, the roots of the equation

$$(x^2+y^2-\alpha)^2+16A(x-m)=0$$

are all imaginary.

[Vol. XXXIX., pp. 69, 70.]

4387. (Proposed by Professor CAYLEY.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x=a$, $x=b$, $x=c$, $x=d$ on the axis of x, may be written in the four equivalent forms

$$\left(\begin{array}{cccc} \cdot\,, & \tau(d-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & \cdot\,, & \rho(d-a), & \sigma(a-c) \\ \sigma(d-b), & \rho(a-d), & \cdot\,, & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b), & \cdot \end{array}\right)(A^{\frac12}, B^{\frac12}, C^{\frac12}, D^{\frac12})=0,$$

that is,

$$\cdot\quad \tau(d-c)B^{\frac12}+\sigma(b-d)C^{\frac12}+\rho(c-b)D^{\frac12}=0,$$

$$\tau(c-d)A^{\frac12}\quad \cdot\quad +\rho(d-a)C^{\frac12}+\sigma(a-c)D^{\frac12}=0,$$

&c., &c.,

where $A^{\frac12}$, $B^{\frac12}$, $C^{\frac12}$, $D^{\frac12}$ are the distances from the four foci respectively, and the parameters ρ, σ, τ are connected by the equation

$$\rho^2(a-d)(b-c)+\sigma^2(b-d)(c-a)+\tau^2(c-d)(a-b)=0.$$

Show also that the curve has, at right angles to the axis of x, two double tangents, the equation whereof is any one of the three equivalent forms

$$(a+d-2x)(b+c-2x):(b+d-2x)(c+a-2x):(c+d-2x)(a+b-2x)=\rho^2:\sigma^2:\tau^2.$$

[Vol. XL., 1884, p. 32.]

7376. (Proposed by Professor CAYLEY.)—Show how the construction of a regular heptagon may be made to depend on the trisection of the angle $\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)$.

[Vol. XL., p. 110.]

7352. (Proposed by Professor CAYLEY.)—Denoting by $x, y, z, \xi, \eta, \zeta$ homogeneous linear functions of four coordinates, such that identically

$$x+y+z+\xi+\eta+\zeta=0, \quad ax+by+cz+f\xi+g\eta+h\zeta=0,$$

where $af=bg=ch=1$; show that

$$\sqrt{(x\xi)}+\sqrt{(y\eta)}+\sqrt{(z\zeta)}=0$$

is the equation of a quartic surface having the sixteen singular tangent planes (each touching it along a conic)

$$x=0, \quad y=0, \quad z=0, \quad \xi=0, \quad \eta=0, \quad \zeta=0,$$

$$x+y+z=0, \quad x+\eta+\zeta=0, \quad ax+by+cz=0, \quad ax+g\eta+cz=0,$$

$$\xi+y+z=0, \quad x+y+\zeta=0, \quad f\xi+by+cz=0, \quad ax+by+h\zeta=0,$$

$$\frac{x}{1-bc}+\frac{y}{1-ca}+\frac{z}{1-ab}=0, \quad \frac{\xi}{1-gh}+\frac{\eta}{1-hf}+\frac{\zeta}{1-fg}=0.$$

[Vol. XLI., 1884, p. 37.]

5421. (Proposed by Professor CAYLEY.)—Suppose

$$S_x=m_1(x-a_1),\ m_2(x-a_2),\ m_3(x-a_3),\ m_4(x-a_4);$$

where, for any given value of x, we write $+$, $-$, or 0, according as the linear function is positive, negative, or zero, and where the order of the terms is not attended to. If x is any one of the values a_1, a_2, a_3, a_4, the corresponding S is $0+++$, $0---$, $0++-$, or $0+--$: and if I denote indifferently the first or the second form, and R denote indifferently the third or the fourth form: then it is to be shown that the four S's are R, R, R, R, or else R, R, I, I.

[Vol. XLIV., 1886, p. 109.]

8340. (Proposed by F. MORLEY, B.A.)—Show that (1) on a chess-board the number of squares visible is 204, and the number of rectangles (including squares) visible is 1,296; and (2) on a similar board, with n squares in each side, the number of squares is the sum of the first n square numbers, and the number of rectangles (including squares) is the sum of the first n cube numbers.

Solution by PROFESSOR CAYLEY.

In a board of n^2 squares, the number of pairs of vertical lines at a distance from each other of $n-r+1$ squares is $=r$; and the number of pairs of horizontal

	4	3	2	1
4	1	2	3	4
3	2	4	6	8
2	3	6	9	12
1	4	8	12	16

lines at a distance from each other of $n-s$ squares is $=s$. Hence the number of rectangles, breadth $n-r+1$ and depth $n-s+1$, or say the number of

$$(n-r+1)(n-s+1)$$

rectangles, is $=rs$.

For instance, $n=4$, the number of rectangles 44, 43, 34, &c., is shown in the diagram; hence the whole number of rectangles is $(1+2+3+4)^2=1^3+2^3+3^3+4^3$, and so for any value of n.

The same diagram shows that the whole number of squares is $=1^2+2^2+3^2+4^2$; and so for any value of n.

[Vol. XLVI., 1887, pp. 49, 50.]

8636. (Proposed by Professor MAHENDRA NATH RAY, M.A., LL.B.)—Show that the following equations are satisfied by the *same* value of x, and find this value:—

$$ax(x^2-a^2)^{\frac{1}{2}}+bx(x^2-b^2)^{\frac{1}{2}}+cx(x^2-c^2)^{\frac{1}{2}}=2abc,$$

$$2(x^2-a^2)^{\frac{1}{2}}(x^2-b^2)^{\frac{1}{2}}(x^2-c^2)^{\frac{1}{2}}=x(a^2+b^2+c^2-2x^2).$$

Solution by PROFESSOR CAYLEY.

The second equation rationalised gives

$$4x^6-4x^4(a^2+b^2+c^2)+4x^2(b^2c^2+c^2a^2+a^2b^2)-4a^2b^2c^2=4x^6-4x^4(a^2+b^2+c^2)+x^2(a^2+b^2+c^2)^2;$$

that is,

$$\nabla x^2=4a^2b^2c^2,$$

if, for shortness,

$$\nabla = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2.$$

We thence find

$$\nabla(x^2 - a^2) = a^2(-a^2 + b^2 + c^2)^2,$$

$$\nabla(x^2 - b^2) = b^2(a^2 - b^2 + c^2)^2, \quad \nabla(x^2 - c^2) = c^2(a^2 + b^2 - c^2)^2,$$

and therefore also

$$\nabla^2 a^2x^2(x^2 - a^2) = 4a^6b^2c^2(-a^2 + b^2 + c^2)^2, \text{ \&c.}$$

Or, assuming the sign of the square roots,

$$\nabla ax(x^2 - a^2)^{\frac{1}{2}} = 2abc(-a^4 + a^2b^2 + a^2c^2), \quad \nabla bx(x^2 - b^2)^{\frac{1}{2}} = 2abc(+a^2b^2 - b^4 + b^2c^2),$$

$$\nabla cx(x^2 - c^2)^{\frac{1}{2}} = 2abc(a^2c^2 + b^2c^2 - c^4),$$

whence, adding, the whole divides by ∇ and we have

$$ax(x^2 - a^2)^{\frac{1}{2}} + bx(x^2 - b^2)^{\frac{1}{2}} + cx(x^2 - c^2)^{\frac{1}{2}} = 2abc,$$

the second equation. Observe that the second equation rationalised gives an equation of the form $(x^2, 1)^4 = 0$; the foregoing value $x^2 = 4a^2b^2c^2/\Delta$ is thus one of the four values of x^2.

[Vol. XLVII., 1887, p. 141.]

5271. (Proposed by Professor CAYLEY.)—If ω be an imaginary cube root of unity, show that, if

$$y = \frac{(\omega - \omega^2)x + \omega^2x^3}{1 - \omega^2(\omega - \omega^2)x^2},$$

then

$$\frac{dy}{(1 - y^2)^{\frac{1}{2}}(1 + \omega y^2)^{\frac{1}{2}}} = \frac{(\omega - \omega^2)\,dx}{(1 - x^2)^{\frac{1}{2}}(1 + \omega x^2)^{\frac{1}{2}}};$$

and explain the general theory.

[Vol. L., 1889, p. 189.]

3105. (Proposed by Professor CAYLEY.)—The following singular problem of literal partitions arises out of the geometrical theory given in Professor Cremona's Memoir, "Sulle trasformazioni geometriche delle figure piane," *Mem. di Bologna*, tom. v. (1865). It is best explained by an example:—A number is made up in any manner with the parts 2, 5, 8, 11, &c., viz. the parts are always the positive integers $\equiv 2$ (mod. 3); for instance, $27 = 1.11 + 8.2$. Forming, then, the product of 27 factors $a^{11}(bcdefghi)^2$, this may be partitioned on the same type $1.11 + 8.2$ as follows,

$$a^3bcdefghi, \quad ab, \quad ac, \quad ad, \quad ae, \quad af, \quad ag, \quad ah, \quad ai.$$

(Observe that the partitionment is to be symmetrical as regards the letters which have a common index.) But, to take another example,

$$37 = 0.11 + 3.8 + 1.5 + 4.2 = 1.11 + 0.8 + 4.5 + 3.2.$$

The first of these gives the product $(abc)^8 d^5 (efgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the second type; and the second gives the product $a^{11} (bcde)^8 (fgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the first type; viz. the partitions of the two products respectively are:

First product on second type,

$$(abc)^2\, defgh,\quad abcde,\quad abcdf,\quad abcdg,\quad abcdh,\quad ab,\quad ac,\quad bc;$$

Second product on first type,

$$a^2bcdefg,\quad a^2bcdefh,\quad a^2bcdegh,\quad abcde,\quad ab,\quad ac,\quad ad,\quad ae;$$

so that in the first example the type is sibi-reciprocal, but in the second example there are two conjugate types. Other examples are:

Parts	48	54	55	56	53		55		No.
2	14	3	1	0	3	6	0	2	Reciprocals.
5	0	2	3	0	6	0	5	0	
8	0	3	2	7	0	1	2	5	
11	0	0	2	0	0	3	0	1	
14	0	1	0	0	0	0	1	0	
17	0	0	0	0	1	0	0	0	
20	1	0	0	0	0	0	0	0	

viz. the first four columns give each of them a sibi-reciprocal type, but the last two double columns give conjugate types. It is required to investigate the general solution.

[Vol. L., p. 191.]

3304. (Proposed by Professor CAYLEY.)—The coordinates x, y, z being proportional to the perpendicular distances from the sides of an equilateral triangle, it is required to trace the curve

$$(y-z)\sqrt{x}+(z-x)\sqrt{y}+(x-y)\sqrt{z}=0.$$

[Prof. Cayley remarks that the curve in question is a particular case of that which presents itself in the following theorem, communicated to him (with a demonstration) several years ago by Mr J. Griffiths:—

The locus of a point (x, y, z) such that its pedal circle (that is, the circle which passes through the feet of the perpendiculars drawn from the point in question

upon the sides of the triangle of reference) touches the nine-point circle, is the sextic curve

$$\left\{x\cos A\,(y\cos B - z\cos C)\left(\frac{y}{\cos B}-\frac{z}{\cos C}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{y\cos B\,(z\cos C - x\cos A)\left(\frac{z}{\cos C}-\frac{x}{\cos A}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{z\cos C\,(x\cos A - y\cos B)\left(\frac{x}{\cos A}-\frac{y}{\cos B}\right)\right\}^{\frac{1}{2}}=0.$$

It would be an interesting problem to trace this more general curve.]

[Vol. L., p. 192.]

3481. (Proposed by Professor CAYLEY.)—Find, in the Hamiltonian form

$$\frac{d\eta}{dt}=\frac{dH}{d\varpi},\quad \frac{d\varpi}{dt}=-\frac{dH}{d\eta},\ \&c.,$$

the equations for the motion of a particle acted on by a central force.

[Vol. LV., 1891, p. 27.]

10716. (Proposed by Professor CAYLEY.)—In a hexahedron $ABCDA'B'C'D'$ the plane faces of which are $ABCD$, $A'B'C'D'$, $A'ADD'$, $D'DCC'$, $C'CBB'$, $B'BAA'$, the edges AA', BB', CC', DD' intersect in four points, say AA', DD' in α; BB', CC' in β; CC', DD' in γ; AA', BB' in δ: that is, starting with the duad of lines $\alpha\beta$, $\gamma\delta$, the four edges AA', BB', CC', DD' are the lines $\alpha\delta$, $\beta\delta$, $\beta\gamma$, $\alpha\gamma$ which join the extremities of these duads. Similarly, the four edges AB, CD, $A'B'$, $C'D'$ are the lines joining the extremities of a duad; and the four edges AD, BC, $A'D'$, $B'C'$ are the lines joining the extremities of a duad. The question arises, "Given two duads, is it possible to place them in space so that the two tetrads of joining lines may be eight of the twelve edges of a hexahedron?" The duad $\alpha\beta$, $\gamma\delta$ is considered to be given when there is given the tetrahedron $\alpha\beta\gamma\delta$, which determines the relative position of the two finite lines $\alpha\beta$ and $\gamma\delta$.

[Vol. LXI., 1894, pp. 122, 123.]

3162. (Proposed by Professor CAYLEY.)—By a proper determination of the coordinates, the skew cubic through any six given points may be taken to be

$$x : y : z = y : z : w;$$

or, what is the same thing, the coordinates of the six given points may be taken to be

$$(1,\ t_1,\ t_1^2,\ t_1^3),\ \ldots,\ (1,\ t_6,\ t_6^2,\ t_6^3).$$

Assuming this, it is required to show that if

$$p_1 = \Sigma t_1,\ p_2 = \Sigma t_1 t_2,\ \ldots,\ p_6 = t_1 t_2 t_3 t_4 t_5 t_6,$$

and if

$$\nabla = 6xyzw - 4xz^3 - 4y^3w + 3y^2z^2 - x^2w^2;$$

then the equation of the Jacobian surface of the six points is

$$\left.\begin{array}{rl} S(& xp_3 \quad\quad + zp_1 - 2w\,)\,\delta_x \nabla \\ + (& \quad\quad\quad\quad 2zp_2 - wp_1)\,\delta_y \nabla \\ + (& xp_5 - 2yp_4 \quad\quad\quad)\,\delta_z \nabla \\ + (& 2xp_6 - \ yp_5 \quad\quad - wp_5)\,\delta_w \nabla \end{array}\right\} = 0.$$

[Vol. LXI., p. 123.]

3185. (Proposed by Professor CAYLEY.)—An unclosed polygon of $(m+1)$ vertices is constructed as follows: viz. the abscissæ of the several vertices are $0, 1, 2, \ldots, m$, and corresponding to the abscissa k, the ordinate is equal to the chance of $m+k$ heads in $2m$ tosses of a coin; and m then continually increases up to any very large value; what information in regard to the successive polygons, and to the areas of any portions thereof, is afforded by the general results of the Theory of Probabilities?

[Vol. LXI., p. 124.]

3229. (Proposed by Professor CAYLEY.)—It is required to find the value of the elliptic integral $F(c, \theta)$ when c is very nearly $=1$ and θ very nearly $=\frac{1}{2}\pi$; that is, the value of

$$\int_0^{\frac{1}{2}\pi - \alpha} \frac{d\theta}{\{1-(1-b^2)\sin^2\theta\}^{\frac{1}{2}}},$$

where α, b are each of them indefinitely small.

N.B.—Observe that, for $\alpha = 0$, b small, the value is equal $\log 4/b$, and for $b=0$, α small, the value is $-\log\cot\frac{1}{2}\alpha$.

In the following Contents, the Problems are referred to, each by its number and the proposer's name; and the subject is briefly indicated. An asterisk shews that no solution was given. A line —— shews that there is no number.

			PAGE
3002	Collins	Systems of circles	566
*3144	Cayley	Locus *in plano*	568
3120	,,	Jacobian of quadric surfaces	ib.
3249	,,	Sets of four points on a conic	569
3206	,,	Arrangements of points and lines	570
3278	,,	Arrangements of numbers	ib.
3329	,,	Substitutions and permutations	574
*3356	,,	Roots of quartic equation	575
*3507	,,	Quadric cones through given points	ib.
*3536	,,	Particle under central forces	ib.
*3591	,,	Relation between areas of triangles	ib.
*2652	,,	Parallel surfaces of an ellipsoid	ib.
*3677	,,	Angle between normal and bisector of chord	576
*3564	,,	Minimum circle enclosing three points	ib.
3875	,,	Description of curves	ib.
3430	Miller	Negative pedals of ellipsoid and ellipse	ib.
4298	Glaisher	Quadrilateral inscribable in circle	578
4392	Roberts	Symmetrical determinant	579
4354	Tucker	Geometrical interpretation (note)	581
4458	Cayley	Two quartic curves	582
4520	Evans	Use of Degen's tables	586
4528	Cayley	Expectation	587
4581	Wilkinson	A question in chances	588
*4638	Cayley	Envelope of family of quadrics	589
*4694	,,	Resultant of forces	ib.
4793	Wolstenholme	Relation among derivatives of a function	590
4752	Cayley	Cubic curve	592
*4946	,,	Attraction of lens-shaped body	594
5020	Woolhouse	Algebraical theorem	ib.
5079	Cayley	Bicircular quartic	596
*5130	,,	Bicircular quartic	597
5208	Sylvester	Trees	598
——	Martin	A question in probabilities	600
*5306	Cayley	A system of equations	601
*4870	,,	Conics and lines	602
*5625	,,	A trinodal quartic	ib.
*5387	,,	Conical points	ib.
*5927	,,	Algebraical theorem	ib.
*6155	,,	Coordinates of a line	603
*6470	,,	Equation of a cubic curve	ib.
6766	,,	Singular tangents of a quartic	ib.
6800	Miller	Geometrical interpretation (note)	604
*5244	Cayley	16-nodal quartic surface	ib.

			PAGE
7190	Wolstenholme	Conic and cubic	605
*5689	Cayley	A Steiner's surface	607
*4722	,,	Reality of roots of a quartic	608
*4387	,,	Equation of a Cassinian	ib.
*7376	,,	Construction of a heptagon	609
*7352	,,	Equation of a quartic surface	ib.
*5421	,,	Algebraical theorem	ib.
8340	Morley	Topology of chess-board	ib.
8636	Nath Ray	Solution of equations	610
*5271	Cayley	Elliptic-function transformation	611
*3105	,,	Partitions	ib.
*3304	,,	Curve of sixth order	612
*3481	,,	Hamiltonian equations of central orbit	613
*10716	,,	Edges of a hexahedron	ib.
*3162	,,	Jacobian surface of six points	ib.
*3185	,,	Probability	614
*3229	,,	Elliptic integral	ib.

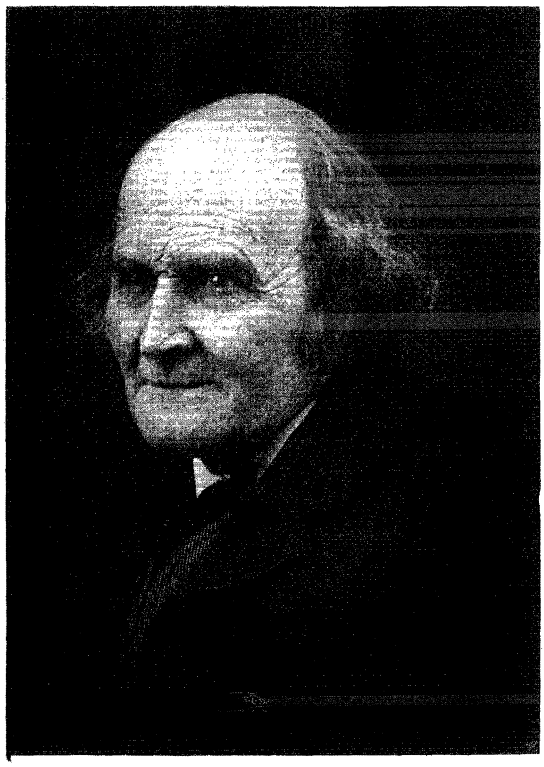
END OF VOL. X.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN CO.



THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. XI.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1896

[*All Rights reserved.*]

CAMBRIDGE:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 93 papers numbered 706 to 798, published, with the exception of one series, for the most part in the years 1878 to 1883. This series is constituted by the articles which Professor Cayley wrote for the *Encyclopædia Britannica* between the years 1878 and 1888; it seemed desirable to place these together in the same volume, in spite of the departure from the chronological arrangement which governs the sequence of the papers in the volumes generally. The Syndics of the University Press desire to acknowledge their obligation to Messrs Adam and Charles Black, Publishers of the ninth Edition of the *Encyclopædia Britannica*, for their courteous consent in allowing these articles to be included in the *Collected Mathematical Papers*. Exact references to the volumes, from which the articles are extracted, will be found in the Table of Contents.

The frontispiece to the present volume is a reproduction by Mr A. G. Dew-Smith, of Trinity College, of a photograph of Professor Cayley which he made in the year 1885. The Syndics of the Press desire to acknowledge their obligation to Mr Dew-Smith.

The Table for the eleven volumes is

Vol.	I.	Numbers	1	to	100,
,,	II.	,,	101	,,	158,
,,	III.	,,	159	,,	222,
,,	IV.	,,	223	,,	299,
,,	V.	,,	300	,,	383,
,,	VI.	,,	384	,,	416,
,,	VII.	,,	417	,,	485,
,,	VIII.	,,	486	,,	555,
,,	IX.	,,	556	,,	629,
,,	X.	,,	630	,,	705,
,,	XI.	,,	706	,,	798.

A. R. FORSYTH.

21 *November*, 1896.

CONTENTS.

[An Asterisk means that the paper is not printed in full.]

PAGE

706. *On the distribution of electricity on two spherical surfaces* . 1
Phil. Mag., Ser. 5, t. v. (1878), pp. 54—60

707. *On the colouring of maps* 7
Geogr. Soc. Proc., t. I. (1879), pp. 259—261

708. *Note sur la théorie des courbes de l'espace* 9
Assoc. Franç., Compt. Rend., t. IX. (1880), pp. 135—139

709. *On the number of constants in the equation of the surface* $PS-QR=0$ 14
Tidsskrift for Mathematik, Ser. 4, t. IV. (1880), pp. 145—148

710. *On a differential equation* 17
Collectanea Mathematica, in memoriam Dominici Chelini, (Milan, Hoepli, 1881), pp. 17—26

711. *On a diagram connected with the transformation of elliptic functions* 26
British Association Report, 1881, p. 534

712. *A partial differential equation connected with the simplest case of Abel's theorem* 27
British Association Report, 1881, pp. 534, 535

713. *Addition to Mr. Rowe's "Memoir on Abel's theorem"* . . 29
Phil. Trans., t. CLXXII. (1881), pp. 751—758

PAGE

714. *Various notes* 37

Messenger of Mathematics, t. VII. (1878), pp. 69 : 115 : 124 : 125

715. *Note on a system of algebraical equations* 39

Messenger of Mathematics, t. VII. (1878), pp. 17, 18

716. *An illustration of the theory of the* ϑ*-functions*. 41

Messenger of Mathematics, t. VII. (1878), pp. 27—32

717. *On the triple theta-functions* 47

Messenger of Mathematics, t. VII. (1878), pp. 48—50

718. *Addition to Mr. Genese's paper "On the theory of envelopes"*. 50

Messenger of Mathematics, t. VII. (1878), pp. 62, 63

719. *Suggestion of a mechanical integrator for the calculation of* $\int(Xdx+Ydy)$ *along an arbitrary path* 52

Messenger of Mathematics, t. VII. (1878), pp. 92—95; British Association Report, 1877, pp. 18—20

720. *Note on Arbogast's method of derivations* 55

Messenger of Mathematics, t. VII. (1878), p. 158

721. *Formulæ involving the seventh roots of unity* 56

Messenger of Mathematics, t. VII. (1878), pp. 177—182

722. *A problem in partitions* 61

Messenger of Mathematics, t. VII. (1878), pp. 187, 188

723. *Various notes* 63

Messenger of Mathematics, t. VIII. (1879), pp. 45, 46 : 126 : 127

724. *On the deformation of the model of a hyperboloid* 66

Messenger of Mathematics, t. VIII. (1879), pp. 51, 52

725. *New formulæ for the integration of* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$ 68

Messenger of Mathematics, t. VIII. (1879), pp. 60—62

726. *A formula by Gauss for the calculation of* $\log 2$ *and certain other logarithms* 70

Messenger of Mathematics, t. VIII. (1879), pp. 125, 126

PAGE

727. *Equation of the wave-surface in elliptic coordinates* . . . 71
Messenger of Mathematics, t. VIII. (1879), pp. 190, 191

728. *A theorem in elliptic functions* 73
Proc. Lond. Math. Soc., t. X. (1879), pp. 43—48

729. *On a theorem relating to conformable figures* 78
Proc. Lond. Math. Soc., t. X. (1879), pp. 143—146

730. [*Addition to Mr. Spottiswoode's paper "On the twenty-one coordinates of a conic in space"*] 82
Proc. Lond. Math. Soc., t. X. (1879), pp. 194—196

731. *On the binomial equation* $x^p - 1 = 0$; *trisection and quartisection* 84
Proc. Lond. Math. Soc., t. XI. (1880), pp. 4—17

732. *A theorem in spherical trigonometry* 97
Proc. Lond. Math. Soc., t. XI. (1880), pp. 48—50

733. *On a formula of elimination* 100
Proc. Lond. Math. Soc., t. XI. (1880), pp. 139—141

734. *On the kinematics of a plane* 103
Quart. Math. Journ., t. XVI. (1879), pp. 1—8

735. *Note on the theory of apsidal surfaces* 111
Quart. Math. Journ., t. XVI. (1879), pp. 109—112

736. *Application of the Newton-Fourier method to an imaginary root of an equation* 114
Quart. Math. Journ., t. XVI. (1879), pp. 179—185

737. *On a covariant formula* 122
Quart. Math. Journ., t. XVI. (1879), pp. 224—226

738. *Note on a hypergeometric series* 125
Quart. Math. Journ., t. XVI. (1879), pp. 268—270

739. *Note on the octahedron function* 128
Quart. Math. Journ., t. XVI. (1879), pp. 280, 281

PAGE

740. *On certain algebraical identities* 130
Quart. Math. Journ., t. XVI. (1879), pp. 281, 282

741. *On a theorem of Abel's relating to a quintic equation* . . 132
Camb. Phil. Soc. Proc., t. III. (1880), pp. 155—159

742. *On the transformation of coordinates* 136
Camb. Phil. Soc. Proc., t. III. (1880), pp. 178—184

743. *On the Newton-Fourier problem* 143
Camb. Phil. Soc. Proc., t. III. (1880), pp. 231, 232

744. *Table of* $\Delta^m 0^n \div \Pi(m)$ *up to* $m=n=20$ 144
Camb. Phil. Trans., t. XIII. (1883), pp. 1—4

745. *On the Schwarzian derivative, and the polyhedral functions* . 148
Camb. Phil. Trans., t. XIII. (1883), pp. 5—68

*746. *Higher Plane Curves* 217
Salmon's Higher Plane Curves, (3rd ed., 1879), Preface

747. *Note on the degenerate forms of curves* 218
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 383—385

748. *On the bitangents of a quartic* 221
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 387—389

*749. *Solid Geometry* 224
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), Preface

750. *On the theory of reciprocal surfaces* 225
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), pp. 539—550

751. *Note on Riemann's paper "Versuch einer allgemeinen Auffassung der Integration und Differentiation," Werke, pp.* 331—344 235
Mathematische Annalen, t. XVI. (1880), pp. 81, 82

752. *On the finite groups of linear transformations of a variable; with a correction* 237
Mathematische Annalen, t. XVI. (1880), pp. 260—263; 439, 440

753. *On a theorem relating to the multiple theta-functions* . . 242
Mathematische Annalen, t. XVII. (1880), pp. 115—122

PAGE

754. *On the connection of certain formulæ in elliptic functions* . 250
Messenger of Mathematics, t. IX. (1880), pp. 23—25

755. *On the matrix* $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, *and in connection therewith the function* $\frac{ax+b}{cx+d}$ 252
Messenger of Mathematics, t. IX. (1880), pp. 104—109

756. *A geometrical construction relating to imaginary quantities* . 258
Messenger of Mathematics, t. X. (1881), pp. 1—3

757. *On a Smith's Prize question, relating to potentials* . . . 261
Messenger of Mathematics, t. XI. (1882), pp. 15—18

758. *Solution of a Senate-House problem* 265
Messenger of Mathematics, t. XI. (1882), pp. 23—25

759. *Illustration of a theorem in the theory of equations* . . 268
Messenger of Mathematics, t. XI. (1882), pp. 111—113

760. *Reduction of* $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ *to elliptic integrals* 270
Messenger of Mathematics, t. XI. (1882), pp. 142, 143

761. *On the theorem of the finite number of the covariants of a binary quantic* 272
Quart. Math. Journ., t. XVII. (1881), pp. 137—147

762. *On Schubert's method for the contacts of a line with a surface* 281
Quart. Math. Journ., t. XVII. (1881), pp. 244—258

763. *On the theorems of the* 2, 4, 8, *and* 16 *squares* . . . 294
Quart. Math. Journ., t. XVII. (1881), pp. 258—276

764. *The binomial equation* $x^p - 1 = 0$; *quinquisection* . . . 314
Proc. Lond. Math. Soc., t. XII. (1881), pp. 15, 16

765. *On the flexure and equilibrium of a skew surface* . . . 317
Proc. Lond. Math. Soc., t. XII. (1881), pp. 103—108

766. *On the geodesic curvature of a curve on a surface* . . . 323
Proc. Lond. Math. Soc., t. XII. (1881), pp. 110—117

PAGE

767. *On the Gaussian theory of surfaces* 331
Proc. Lond. Math. Soc., t. XII. (1881), pp. 187—192

768. *Note on Landen's theorem* 337
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 47, 48

769. *On a formula relating to elliptic integrals of the third kind* . 340
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 175, 176

770. *On the* 34 *concomitants of the ternary cubic* 342
American Journal of Mathematics, t. IV. (1881), pp. 1—15

771. *Specimen of a literal table for binary quantics, otherwise a partition table* 357
American Journal of Mathematics, t. IV. (1881), pp. 248—255

772. *On the analytical forms called trees* 365
American Journal of Mathematics, t. IV. (1881), pp. 266—268

773. *On the* 8-*square imaginaries* 368
American Journal of Mathematics, t. IV. (1881), pp. 293—296

774. *Tables for the binary sextic* 372
American Journal of Mathematics, t. IV. (1881), pp. 379—384

775. *Tables of covariants of the binary sextic* 377
Written in 1894: now first published.

776. *On the Jacobian sextic equation* 389
Quart. Math. Journ., t. XVIII. (1882), pp. 52—65

777. *A solvable case of the quintic equation* 402
Quart. Math. Journ., t. XVIII. (1882), pp. 154—157

778. [*Addition to Mr. Hudson's paper "On equal roots of equations"*] 405
Quart. Math. Journ., t. XVIII. (1882), pp. 226—229

779. [*Note on Mr. Jeffery's paper "On certain quartic curves, which have a cusp at infinity, whereat the line at infinity is a tangent"*] 408
Proc. Lond. Math. Soc., t. XIV. (1883), p. 85

PAGE

780. [*Addition to Mr. Hammond's paper "Note on an exceptional case in which the fundamental postulate of Professor Sylvester's theory of tamisage fails"*] 409

Proc. Lond. Math. Soc., t. XIV. (1883), pp. 88—91

781. *On the automorphic transformation of the binary cubic function* 411

Proc. Lond. Math. Soc., t. XIV. (1883), pp. 103—108

782. *On Monge's "Mémoire sur la théorie des déblais et des remblais"* 417

Proc. Lond. Math. Soc., t. XIV. (1883), pp. 139—142

783. *On Mr. Wilkinson's rectangular transformation* . . . 421

Proc. Lond. Math. Soc., t. XIV. (1883), pp. 222—229

784. *Presidential Address to the British Association, Southport, September* 1883 429

British Association Report, 1883, pp. 3—37

785. *Curve* 460

Encyclopædia Britannica, 9th ed., t. VI. (1878), pp. 716—728

786. *Equation* 490

Encyclopædia Britannica, 9th ed., t. VIII. (1878), pp. 497—509

787. *Function* 522

Encyclopædia Britannica, 9th ed., t. IX. (1879), pp. 818—824

788. *Galois* 543

Encyclopædia Britannica, 9th ed., t. X. (1879), p. 48

789. *Gauss* 544

Encyclopædia Britannica, 9th ed., t. X. (1879), p. 116

790. *Geometry (analytical)* 546

Encyclopædia Britannica, 9th ed., t. X. (1879), pp. 408—420

791. *Landen* 583

Encyclopædia Britannica, 9th ed., t. XIV. (1882), p. 271

792. *Locus* 585

Encyclopædia Britannica, 9th ed., t. XIV. (1882), pp. 764, 765

PAGE

793. *Monge* 586

Encyclopædia Britannica, 9th ed., t. XVI. (1883), pp. 738, 739

794. *Numbers (partition of)* 589

Encyclopædia Britannica, 9th ed., t. XVII. (1884), p. 614

795. *Numbers (theory of)* 592

Encyclopædia Britannica, 9th ed., t. XVII. (1884), pp. 614—624

796. *Series* 617

Encyclopædia Britannica, 9th ed., t. XXI. (1886), pp. 677—682

797. *Surface* 628

Encyclopædia Britannica, 9th ed., t. XXII. (1887), pp. 668—672

798. *Wallis (John)* 640

Encyclopædia Britannica, 9th ed., t. XXIV. (1888), pp. 331, 332

Portrait . . . . . *To face Title.*

CLASSIFICATION.

ANALYSIS.

Calculation of log 2; 726.
Series, 796.
Prime roots of unity, 721.
8-square imaginaries, 773.
Squares, theorems of 2, 4, 8, 16; 763.
Difference-table for $\Delta^m 0^n \div \Pi(m)$; 744.
Equations, theory of, 736, 741, 743, 759, 776, 777, 778, 786.
Numbers, theory of, 731, 764, 795.
Partitions, 722, 771, 794.
Trees, 772.
Matrices, 755.
Elimination, 733.
Transformation of cubic function, 781.
Covariantive forms and tables, 737, 761, 770, 774, 775, 780.
Fractional differentiation, 751.
Mechanical integrator, 719.
Differential Equations, 725.
Schwarzian derivative, 745.
Hypergeometric series, 710, 738.
Finite groups, 752.
Polyhedral functions, 739, 745.
Elliptic functions, 728, 740, 754, 760, 768, 769.
Transformation of elliptic functions, 711.
Abel's theorem, 712, 713.
Theta Functions, 716, 717, 753.
Function, 787.
Conformal representation, 729.

GEOMETRY.

Analytical geometry in general, 790, 792, 797.
Plane Curves, 746*, 785.
Degenerate forms of curves, 747.
Quartic Curves, 748, 779.
Déblais et remblais, 782.
Tortuous curves, 708, 785.
Geodesic curvature, 766.
Theory of surfaces, general, 709, 749*, 767.
Transformation of coordinates, 742, 783.
Reciprocal surfaces, 750.
Wave-surface, 727.
Apsidal surfaces, 735.
Deformation and flexure of surfaces, 724, 765.
Hypergeometry, 730.
Schubert's numerative geometry, 762.

VARIOUS SUBJECTS.

Spherical Trigonometry, 732.
Kinematics of a plane, 734.
Maps, colouring of, 707.
Electricity, distribution of, on spherical surfaces, 706.
Potential, 757.
Presidential Address to the British Association, 784.
Biographical articles; Galois, 788.
Gauss, 789.
Landen, 791.
Monge, 793.
Wallis (John), 798.

MISCELLANEOUS

714, 715, 718, 720, 723, 756, 758.

706.

ON THE DISTRIBUTION OF ELECTRICITY ON TWO SPHERICAL SURFACES.

[From the *Philosophical Magazine*, vol. v. (1878), pp. 54—60.]

In the two memoirs "Sur la distribution de l'électricité à la surface des corps conducteurs," *Mém. de l'Inst.* 1811, Poisson considers the question of the distribution of electricity upon two spheres: viz. if the radii be a, b, and the distance of the centres be c (where $c > a + b$, the spheres being exterior to each other), and the potentials within the two spheres respectively have the constant values h and g, then—for Poisson's $f\left(\frac{x}{a}\right)$ writing $\phi(x)$, and for his $F\left(\frac{x}{b}\right)$ writing $\Phi(x)$—the question depends on the solution of the functional equations

$$a\phi(x) + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right) = h,$$

$$\frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right) + b\Phi(x) = g,$$

where of course the x of either equation may be replaced by a different variable.

It is proper to consider the meaning of these equations: for a point on the axis, at the distance x from the centre of the first sphere, or say from the point A, the potential of the electricity on this spherical surface is $a\phi x$ or $\frac{a^2}{x}\phi\left(\frac{a^2}{x}\right)$, according as the point is interior or exterior; and, similarly, if x now denote the distance from the centre of the second sphere (or, say, from the point B), then the potential of the electricity on this spherical surface is $b\Phi x$ or $\frac{b^2}{x}\Phi\left(\frac{b^2}{x}\right)$, according as the point is interior or exterior; $\phi(x)$ is thus the same function of (x, a, b) that $\Phi(x)$ is of

(x, b, a). Hence, first, for a point interior to the sphere A, if x denote the distance from A, and therefore $c-x$ the distance of the same point from B, the potential of the point in question is

$$= a\phi x + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right);$$

and, secondly, for a point interior to the sphere B, if x denote the distance from B and therefore $c-x$ the distance of the same point from A, the potential of the point is

$$= \frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right) + b\Phi(x).$$

The two equations thus express that the potentials of a point interior to A and of a point interior to B are $=h$ and g respectively.

It is to be added that the potential of an exterior point, distances from the points A and B $=x$ and $c-x$ respectively, is

$$= \frac{a^2}{x}\phi\left(\frac{a^2}{x}\right) + \frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right);$$

and that, by the known properties of Legendre's coefficients, when the potential upon an axial point is given, it is possible to pass at once to the expression for the potential of a point not on the axis, and also to the expression for the electrical density at a point on the two spherical surfaces respectively. The determination of the functions $\phi(x)$ and $\Phi(x)$ gives thus the complete solution of the question.

I obtain Poisson's solution by a different process as follows:—Consider the two functions

$$\frac{a^2(c-x)}{c^2-b^2-cx}, \quad = \frac{\mathrm{a}x+\mathrm{b}}{\mathrm{c}x+\mathrm{d}}, \text{ suppose,}$$

and

$$\frac{b^2(c-x)}{c^2-a^2-cx}, \quad = \frac{\alpha x+\beta}{\gamma x+\delta}, \text{ suppose;}$$

and let the nth functions be

$$\frac{\mathrm{a}_n x+\mathrm{b}_n}{\mathrm{c}_n x+\mathrm{d}_n} \text{ and } \frac{\alpha_n x+\beta_n}{\gamma_n x+\delta_n}$$

respectively.

Observing that the values of the coefficients are

$$\begin{pmatrix} \mathrm{a}, & \mathrm{b} \\ \mathrm{c}, & \mathrm{d} \end{pmatrix} = \begin{pmatrix} -a^2, & a^2c \\ -c, & c^2-b^2 \end{pmatrix}, \text{ and } \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} = \begin{pmatrix} -b^2, & b^2c \\ -c, & c^2-a^2 \end{pmatrix},$$

so that we have

$$\mathrm{a}+\mathrm{d} = \alpha+\delta, \quad = c^2-a^2-b^2, \qquad \mathrm{ad}-\mathrm{bc} = \alpha\delta-\beta\gamma, \quad = a^2b^2,$$

and consequently that the two equations

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(\mathrm{a}+\mathrm{d})^2}{\mathrm{ad}-\mathrm{bc}}, \quad \frac{(\lambda+1)^2}{\lambda} = \frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma},$$

are in fact one and the same equation

$$\frac{(\lambda+1)^2}{\lambda}=\frac{(c^2-a^2-b^2)^2}{a^2b^2}$$

for the determination of λ, then (by a theorem which [686, 687] I have recently obtained) we have the following equations for the coefficients

$$\begin{pmatrix} \mathrm{a}_n, & \mathrm{b}_n \\ \mathrm{c}_n, & \mathrm{d}_n \end{pmatrix}, \quad \begin{pmatrix} \alpha_n, & \beta_n \\ \gamma_n, & \delta_n \end{pmatrix}$$

of the nth functions; viz. these are:—

$$\mathrm{a}_n x+\mathrm{b}_n=\frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(\mathrm{a}x+\mathrm{b})+(\lambda^n-\lambda)(-\mathrm{d}x+\mathrm{b})\},$$

$$\mathrm{c}_n x+\mathrm{d}_n= \quad ,, \quad ,, \quad \{(\lambda^{n+1}-1)(\mathrm{c}x+\mathrm{d})+(\lambda^n-\lambda)(\ \mathrm{c}x-\mathrm{a})\};$$

and similarly

$$\alpha_n x+\beta_n=\frac{1}{\lambda^2-1}\left(\frac{\alpha+\delta}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)(\alpha x+\beta)+(\lambda^n-\lambda)(-\delta x+\beta)\},$$

$$\gamma_n x+\delta_n= \quad ,, \quad ,, \quad \{(\lambda^{n+1}-1)(\gamma x+\delta)+(\lambda^n-\lambda)(\ \gamma x-\alpha)\}.$$

Observe that these equations give, as they ought to do,

$$\mathrm{a}_0 x+\mathrm{b}_0=x, \quad \mathrm{c}_0 x+\mathrm{d}_0=1, \quad \mathrm{a}_1 x+\mathrm{b}_1=\mathrm{a}x+\mathrm{b}, \quad \mathrm{c}_1 x+\mathrm{d}_1=\mathrm{c}x+\mathrm{d};$$

and similarly

$$\alpha_0 x+\beta_0=x, \quad \gamma_0 x+\delta_0=1, \quad \alpha_1 x+\beta_1=\alpha x+\beta, \quad \gamma_1 x+\delta_1=\gamma x+\delta.$$

Substituting in the first two equations $\dfrac{a^2}{c-x}$ in place of x, and in the second two equations $\dfrac{b^2}{c-x}$ in place of x, we obtain the following results which will be useful:—

$$\mathrm{a}_n a^2+\mathrm{b}_n(c-x)=a^2(\gamma_n x+\delta_n),$$

$$\mathrm{c}_n a^2+\mathrm{d}_n(c-x)=\frac{1}{b^2}(\alpha_{n+1}x+\beta_{n+1}),$$

$$\alpha_n b^2+\beta_n(c-x)=b^2(\mathrm{c}_n x+\mathrm{d}_n),$$

$$\gamma_n b^2+\delta_n(c-x)=\frac{1}{a^2}(\mathrm{a}_{n+1}x+\mathrm{b}_{n+1}),$$

the last two of which are obtained from the first two by a mere interchange of letters; it will therefore be sufficient to prove the first and second equations.

For the first equation we have

$$\mathrm{a}_n a^2+\mathrm{b}_n(c-x)=\frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)[\mathrm{a}a^2+\mathrm{b}(c-x)]+(\lambda^n-\lambda)[-\mathrm{d}a^2+\mathrm{b}(c-x)]\},$$

where the term in { } is

$$=(\lambda^{n+1}-1)[-a^4+a^2c(c-x)]+(\lambda^n-\lambda)[a^2(b^2-c^2)+a^2c(c-x)];$$

viz. this is

$$=a^2\{(\lambda^{n+1}-1)(c^2-a^2-cx)+(\lambda^n-\lambda)(b^2-cx)\};$$

or it is

$$=a^2\{(\lambda^{n+1}-1)(\gamma x+\delta)\qquad+(\lambda^n-\lambda)(\gamma x-\alpha)\},$$

whence the relation in question.

The proof of the second equation is a little more complicated. We have

$$\mathrm{c}_n a^2+\mathrm{d}_n(c-x)=\frac{1}{\lambda^2-1}\left(\frac{\mathrm{a}+\mathrm{d}}{\lambda+1}\right)^{n-1}\{(\lambda^{n+1}-1)[\mathrm{c}a^2+\mathrm{d}(c-x)]+(\lambda^n-\lambda)[\mathrm{c}a^2-\mathrm{a}(c-x)]\},$$

where the term in { } is

$$=(\lambda^{n+1}-1)[-\mathrm{c}a^2+(c^2-b^2)(c-x)]+(\lambda^n-\lambda)[-\mathrm{c}a^2+a^2(c-x)].$$

Comparing this with

$$\alpha_{n+1}x+\beta_{n+1}=\frac{1}{\lambda^2+1}\left(\frac{\alpha+\delta}{\lambda+1}\right)^n\{(\lambda^{n+2}-1)(\alpha x+\beta)+(\lambda^{n+1}-\lambda)(-\delta x+\beta)\},$$

where the term in { } is

$$=(\lambda^{n+2}-1)[b^2(c-x)]+(\lambda^{n+1}-\lambda)[-c(c^2-a^2-b^2)+(c^2-a^2)(c-x)],$$

it is to be observed that the quotient of the two terms in { } is in fact a constant; this is most easily verified as follows. Dividing the first of them by the second, we have a quotient which when $x=c$ is

$$\frac{(\lambda^{n+1}-1)(-\mathrm{c}a^2)+(\lambda^n-\lambda)(-\mathrm{c}a^2)}{(\lambda^{n+1}-\lambda)\{-c(c^2-a^2-b^2)\}},\quad=\frac{a^2(\lambda^{n+1}-1+\lambda^n-\lambda)}{(\lambda^{n+1}-\lambda)(c^2-a^2-b^2)},\quad=\frac{a^2(\lambda+1)}{(c^2-a^2-b^2)\lambda},$$

and when $x=0$ is

$$\frac{(\lambda^{n+1}-1)c(c^2-a^2-b^2)}{(\lambda^{n+2}-1)b^2c+(\lambda^{n+1}-\lambda)b^2c},\quad=\frac{(\lambda^{n+1}-1)(c^2-a^2-b^2)}{(\lambda^{n+2}-1+\lambda^{n+1}-\lambda)b^2},\quad=\frac{c^2-a^2-b^2}{b^2(\lambda+1)}:$$

these two values are equal by virtue of the equation which defines λ; and hence the quotient of the two linear functions having equal values for $x=c$ and $x=0$, has always the same value; say it is $=\dfrac{c^2-a^2-b^2}{b^2(\lambda+1)}$. Hence, observing that $\mathrm{a}+\mathrm{d}=\alpha+\delta$, $=c^2-a^2-b^2$, the quotient, $\mathrm{c}_n a^2+\mathrm{d}_n(c-x)$ divided by $\alpha_{n+1}x+\beta_{n+1}$, is

$$=\frac{\lambda+1}{c^2-a^2-b^2}\cdot\frac{c^2-a^2-b^2}{b^2(\lambda+1)},\quad=\frac{1}{b^2};$$

or we have the required equation

$$\mathrm{c}_n a^2+\mathrm{d}_n(c-x)=\frac{1}{b^2}(\alpha_{n+1}x+\beta_{n+1}).$$

Considering now the functional equations, suppose for the moment that g is $=0$; the two equations may be satisfied by assuming

$$\phi(x) = h\left\{\frac{1}{c_0x+d_0}+\frac{\omega}{c_1x+d_1}+\ldots\right\}L,$$

$$\Phi(x) = -h\left\{\frac{\omega}{\alpha_1x+\beta_1}+\frac{\omega^2}{\alpha_2x+\beta_2}+\ldots\right\}M.$$

We in fact, from the foregoing relations, at once obtain

$$\frac{a^2}{c-x}\phi\frac{a^2}{c-x} = h\left\{\frac{\omega}{\alpha_1x+\beta_1}+\frac{\omega^2}{\alpha_2x+\beta_2}\ldots\right\}\frac{a^2b^2L}{\omega},$$

$$\frac{b^2}{c-x}\Phi\frac{b^2}{c-x} = -h\left\{\frac{\omega}{c_1x+d_1}+\frac{\omega^2}{c_2x+d_2}\ldots\right\}M.$$

To satisfy the first equation we must have $M=aL$; viz. this being so, the equation becomes

$$a\phi x+\frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right)=\frac{aLh}{c_0x+d_0};$$

or, since $c_0x+d_0=1$, the equation will be satisfied if only $aL=1$, whence also $M=1$. And the second equation will be satisfied if only $\frac{a^2b^2L}{\omega}=bM$; viz. substituting for L, M their value, we find $\omega=ab$.

Supposing, in like manner, that $h=0$, g retaining its proper value, we find a like solution for the two equations; and by simply adding the solutions thus obtained, we have a solution of the original two equations

$$a\phi(x)+\frac{b^2}{c-x}\Phi\left(\frac{b^2}{c-x}\right)=h,$$

$$\frac{a^2}{c-x}\phi\left(\frac{a^2}{c-x}\right)+b\Phi(x)=g;$$

viz. the solution is

$$\phi(x) = \frac{h}{a}\left\{\frac{1}{c_0x+d_0}+\frac{ab}{c_1x+d_1}+\ldots\right\}-g\left\{\frac{ab}{a_1x+b_1}+\frac{(ab)^2}{a_2x+b_2}+\ldots\right\}$$

$$\Phi(x) = -h\left\{\frac{ab}{\alpha_1x+\beta_1}+\frac{(ab)^2}{\alpha_2x+\beta_2}+\ldots\right\}+\frac{g}{b}\left\{\frac{1}{\gamma_0x+\delta_0}+\frac{ab}{\gamma_1x+\delta_1}+\ldots\right\}.$$

We have a general solution containing an arbitrary constant P by adding to the foregoing values for ϕx a term

$$=\frac{Pb(a-b)}{\sqrt{a^2(c-x)-x(c^2-b^2-cx)}},$$

and for Φx a term

$$=\frac{Pa(b-a)}{\sqrt{b^2(c-x)-x(c^2-a^2-cx)}},$$

as may be easily verified if we observe that the function

$$a^2(c-x) - x(c^2 - b^2 - cx),$$

writing therein $\dfrac{a^2}{c-x}$ for x, becomes

$$= \frac{a^2}{(c-x)^2}\{b^2(c-x) - x(c^2 - a^2 - cx)\}:$$

and similarly that

$$b^2(c-x) - x(c^2 - a^2 - cx),$$

writing therein $\dfrac{b^2}{c-x}$ for x, becomes

$$= \frac{b^2}{(c-x)^2}\{a^2(c-x) - x(c^2 - b^2 - cx)\}.$$

More generally, the terms to be added are for ϕx a term as above, where P denotes a function of x which remains unaltered when x is changed into $\dfrac{a^2(c-x)}{c^2-b^2-cx}$, and for Φx a term as above with P' instead of P, where P' denotes what P becomes when x is changed into $\dfrac{a^2}{c-x}$. But these additional terms vanish for the electrical problem, and the correct values of ϕx, Φx are the particular values given above.

It is to be remarked that the function

$$\frac{a^2(c-x)}{c^2-b^2-cx} \text{ is } = \frac{a^2}{c - \dfrac{b^2}{c-x}};$$

viz. considering x as the distance of a point X from A, then taking the image of X in regard to the sphere B, and again the image of this image in regard to the sphere A, the function in question is the distance of this second image from A. And similarly the function

$$\frac{b^2(c-x)}{c^2-a^2-cx} \text{ is } = \frac{b^2}{c - \dfrac{a^2}{c-x}};$$

viz. considering here x as the distance of the point X from B, then taking the image of X in regard to the sphere A, and again the image of this image in regard to the sphere B, the function in question is the distance of this second image from B. It thus appears that Poisson's solution depends upon the successive images of X in regard to the spheres B and A alternately, and also on the successive images of X in regard to the spheres A and B alternately. This method of images is in fact employed in Sir W. Thomson's paper "On the Mutual Attraction or Repulsion between two Electrified Spherical Conductors," *Phil. Mag.*, April and August, 1853.

707.

ON THE COLOURING OF MAPS.

[From the *Proceedings of the Royal Geographical Society*, vol. I., no. 4 (1879), pp. 259—261.]

THE theorem that four colours are sufficient for any map, is mentioned somewhere by the late Professor De Morgan, who refers to it as a theorem known to map-makers. To state the theorem in a precise form, let the term "area" be understood to mean a simply or multiply connected* area: and let two areas, if they touch along a line, be said to be "attached" to each other; but if they touch only at a point or points, let them be said to be "appointed" to each other. For instance, if a circular area be divided by radii into sectors, then each sector is attached to the two contiguous sectors, but it is appointed to the several other sectors. The theorem then is, that if an area be partitioned in any manner into areas, these can be, with four colours only, coloured in such wise that in every case two attached areas have distinct colours; appointed areas may have the same colour. Detached areas may in a map represent parts of the same country, but this relation is not in anywise attended to: the colours of such detached areas will be the same, or different, as the theorem may require.

It is easy to see that four colours are wanted; for instance, we have a circle divided into three sectors, the whole circle forming an *enclave* in another area; then we require three colours for the three sectors, and a fourth colour for the surrounding area: if the circle were divided into four sectors, then for these two colours would

* An area is "connected" when every two points of the area can be joined by a continuous line lying wholly within the area; the area within a non-intersecting closed curve, or say an area having a single boundary, is "simply connected"; but if besides the exterior boundary there is one or more than one interior boundary (that is, if there is within the exterior boundary one or more than one *enclave* not belonging to the area), then the area is "multiply connected." The theorem extends to multiply connected areas, but there is no real loss of generality in taking, and we may for convenience take the areas of the theorem to be each of them a simply connected area.

be sufficient, and taking a third colour for the surrounding area, three colours only would be wanted; and so in general according as the number of sectors is even or odd, three colours or four colours are wanted. And in any tolerably simple case it can be seen that four colours are sufficient. But I have not succeeded in obtaining a general proof: and it is worth while to explain wherein the difficulty consists. Supposing a system of n areas coloured according to the theorem with four colours only, if we add an $(n+1)$th area, it by no means follows that we can *without altering the original colouring* colour this with one of the four colours. For instance, if the original colouring be such that the four colours all present themselves in the exterior boundary of the n areas, and if the new area be an area enclosing the n areas, then there is not any one of the four colours available for the new area.

The theorem, if it is true at all, is true under more stringent conditions. For instance, if in any case the figure includes four or more areas meeting in a point (such as the sectors of a circle), then if (introducing a new area) we place at the point a small circular area, cut out from and attaching itself to each of the original sectorial areas, it must according to the theorem be possible with four colours only to colour the new figure; and this implies that it must be possible to colour the original figure so that only three colours (or it may be two) are used for the sectorial areas. And in precisely the same way (the theorem is in fact really the same) it must be possible to colour the original figure in such wise that only three colours (or it may be two) present themselves in the exterior boundary of the figure.

But now suppose that the theorem *under these more stringent conditions* is true for n areas: say that it is possible with four colours only, to colour the n areas in such wise that not more than three colours present themselves in the external boundary: then it might be easy to prove that the $n+1$ areas could be coloured with four colours only: but this would be insufficient for the purpose of a general proof; it would be necessary to show further that the $n+1$ areas could be with the four colours only coloured *in accordance with the foregoing boundary condition;* for without this we cannot from the case of the $n+1$ areas pass to the next case of $n+2$ areas. And so in general, whatever more stringent conditions we import into the theorem as regards the n areas, it is necessary to show not only that the $n+1$ areas can be coloured with four colours only, but that they can be coloured in accordance with the more stringent conditions. As already mentioned, I have failed to obtain a proof.

708.

NOTE SUR LA THÉORIE DES COURBES DE L'ESPACE.

[From the *Compte Rendu de l'Association Française pour l'Avancement des Sciences* (1880), pp. 135—139.]

En considérant dans l'espace une courbe d'espèce donnée, déterminée au moyen d'un nombre suffisant de points, la courbe n'est pas déterminée uniquement; mais on a par les points un certain nombre de telles courbes. Par exemple, la courbe unicursale d'ordre $2p$ dépend, comme on voit sans peine, de $8p$ constantes et sera ainsi déterminée par $4p$ points (le cas $p=1$ est une exception): on ne connaît pas, je pense, le nombre des courbes par les $4p$ points; mais pour le cas particulier $p=2$ (c'est-à-dire pour une courbe quartique de seconde espèce, ou autrement dit, une courbe excubo-quartique) ce nombre est $=4$: théorème démontré par moi depuis longtemps par des considérations géométriques. (Voir Salmon, *Geometry of three dimensions*, 3e éd. 1874, p. 319.) Ce n'est que dernièrement que j'ai considéré la question analytique, de trouver les équations d'une courbe excubo-quartique qui passe par 8 points donnés; et même j'ai pris pour les 8 points une disposition qui n'est pas tout à fait générale: l'investigation elle-même, et la forme du résultat, m'ont paru assez intéressantes pour que je les soumette à l'Association.

En considérant sur une courbe excubo-quartique 4 points donnés, le plan passant par 3 quelconques de ces points rencontre la courbe dans un seul point; et l'on obtient ainsi encore 4 points sur la courbe: voilà mon système de 8 points donnés, savoir en partant de 4 points quelconques, je prends un point quelconque dans chacun des plans qui passent par 3 de ces points, et j'obtiens ainsi les autres 4 points. Et par un tel système de 8 points, je cherche à faire passer une courbe de l'espèce dont il s'agit.

En prenant $x=0$, $y=0$, $z=0$, $w=0$, pour les équations des plans du tétraèdre formé par les 4 premiers points, les coordonnées de ces points seront $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$: et pour les coordonnées des 4 autres points, je prends $(0, y_1, z_1, w_1)$, $(x_2, 0, z_2, w_2)$, $(x_3, y_3, 0, w_3)$, $(x_4, y_4, z_4, 0)$.

Les équations de la courbe sont $x : y : z : w = P : Q : R : S$, où P, Q, R, S sont des fonctions $(*)(\theta, 1)^4$ d'un paramètre variable θ; il s'agit de faire passer une telle courbe par les 8 points.

Je prends $\alpha, \beta, \gamma, \delta, a, b, c, d$ pour les valeurs du paramètre θ qui correspondent aux 8 points respectivement.

Pour que la courbe passe par les premiers 4 points, il faut et il suffit que les équations soient de la forme

$$x : y : z : w = A\frac{\theta - a}{\theta - \alpha} : B\frac{\theta - b}{\theta - \beta} : C\frac{\theta - c}{\theta - \gamma} : D\frac{\theta - d}{\theta - \delta};$$

les conditions pour les autres 4 points seront alors

$$\begin{aligned}
\cdot\ y_1 : z_1 : w_1 &= \quad\cdot\quad B\frac{a-b}{a-\beta} : C\frac{a-c}{a-\gamma} : D\frac{a-d}{a-\delta},\\
x_2\ \cdot : z_2 : w_2 &= A\frac{b-a}{b-\alpha}\quad\cdot\quad : C\frac{b-c}{b-\gamma} : D\frac{b-d}{b-\delta},\\
x_3 : y_3\ \cdot : w_3 &= A\frac{c-a}{c-\alpha} : B\frac{c-b}{c-\beta}\quad\cdot\quad : D\frac{c-d}{c-\delta},\\
x_4 : y_4 : z_4\ \cdot &= A\frac{d-a}{d-\alpha} : B\frac{d-b}{d-\beta} : C\frac{d-c}{d-\gamma}\quad\cdot\ \cdot
\end{aligned}$$

Évidemment il y a deux équations qui donnent la valeur de $B : C$, et qui servent ainsi pour éliminer cette quantité. De cette manière on obtient six équations que j'écris comme voici:

$$\lambda = \frac{y_1 z_4}{y_4 z_1} = \frac{a-b.d-c}{a-c.d-b}\cdot\frac{a-\gamma.d-\beta}{a-\beta.d-\gamma},$$

$$\mu = \frac{w_1 y_3}{y_1 w_3} = \frac{a-d.c-b}{a-b.c-d}\cdot\frac{a-\beta.c-\delta}{a-\delta.c-\beta},$$

$$\nu = \frac{z_1 w_2}{z_2 w_1} = \frac{a-c.b-d}{a-d.b-c}\cdot\frac{a-\delta.b-\gamma}{a-\gamma.b-\delta},$$

$$\varpi = \frac{z_2 x_4}{z_4 x_2} = \frac{b-c.d-a}{b-a.d-c}\cdot\frac{b-\alpha.d-\gamma}{b-\gamma.d-\alpha},$$

$$\kappa = \frac{x_2 w_3}{x_3 w_2} = \frac{b-a.c-d}{b-d.c-a}\cdot\frac{b-\delta.c-\alpha}{b-\alpha.c-\delta},$$

$$\rho = \frac{x_3 y_4}{x_4 y_3} = \frac{c-a.d-b}{c-b.d-a}\cdot\frac{c-\beta.d-\alpha}{c-\alpha.d-\beta};$$

savoir $\lambda, \mu, \nu, \varpi, \kappa, \rho$ dénotent ici les quantités données $\lambda = \frac{y_1 z_4}{y_4 z_1}$, etc. Le nombre des équations indépendantes est 5, car l'on a identiquement $\lambda\mu\nu\varpi\kappa\rho = 1$. Je remarque que l'on peut faire sur le paramètre θ une transformation linéaire quelconque $(h\theta + i) : (j\theta + k)$, et introduire ainsi 3 constantes arbitraires; on peut donc prendre à

volonté 3 valeurs du paramètre θ, c'est-à-dire les valeurs de 3 quelconques des quantités α, β, γ, δ, a, b, c, d; et cela étant les 5 équations donneront les valeurs des autres 5 quantités. Si au moyen des équations on élimine α, β, γ, δ, on obtient entre a, b, c, d une équation qui sera, comme on va voir, de l'ordre 4 par rapport à chacune de ces quantités: en prenant comme données a, b, c il y aura donc 4 valeurs de d; et pour l'une quelconque de ces valeurs, celles de α, β, γ, δ seront déterminées uniquement: il y aura ainsi 4 courbes qui passent chacune par les 8 points; ce qui est le théorème dont il s'agit.

J'introduis, pour abréger, la notation

$$\begin{array}{cccccc} a-d, & b-d, & c-d, & b-c, & c-a, & a-b, \\ = \mathrm{f}, & \mathrm{g}, & \mathrm{h}, & \mathrm{a}, & \mathrm{b}, & \mathrm{c}: \end{array}$$

on a donc identiquement

$$\mathrm{a},\ \mathrm{b},\ \mathrm{c} = \mathrm{g}-\mathrm{h},\ \mathrm{h}-\mathrm{f},\ \mathrm{f}-\mathrm{g},$$
$$\mathrm{a}+\mathrm{b}+\mathrm{c}=0,$$
$$\mathrm{fa}+\mathrm{gb}+\mathrm{hc}=0.$$

Les équations prennent ainsi la forme

$$\lambda = -\frac{\mathrm{hc}}{\mathrm{gb}}\,\frac{a-\gamma.d-\beta}{a-\beta.d-\gamma},\ \text{etc.};$$

ou, en introduisant pour plus de commodité, les symboles

$$L, \qquad M, \qquad N, \qquad P, \qquad Q, \qquad R,$$

pour désigner respectivement

$$-\frac{\mathrm{gb}}{\mathrm{hc}}\lambda,\quad -\frac{\mathrm{hc}}{\mathrm{fa}}\mu,\quad -\frac{\mathrm{fa}}{\mathrm{gb}}\nu,\quad -\frac{\mathrm{hc}}{\mathrm{fa}}\varpi,\quad -\frac{\mathrm{gb}}{\mathrm{hc}}\kappa,\quad -\frac{\mathrm{fa}}{\mathrm{gb}}\rho,$$

les équations seront

$$L = \frac{a-\gamma.d-\beta}{a-\beta.d-\gamma},$$
$$M = \frac{a-\beta.c-\delta}{a-\delta.c-\beta},$$
$$N = \frac{a-\delta.b-\gamma}{a-\gamma.b-\delta},$$
$$P = \frac{b-\alpha.d-\gamma}{b-\gamma.d-\alpha},$$
$$Q = \frac{b-\delta.c-\alpha}{b-\alpha.c-\delta},$$
$$R = \frac{c-\beta.d-\alpha}{c-\alpha.d-\beta},$$

avec la relation identique $LMNPQR=1$; il s'agit entre ces 5 équations d'éliminer α, β, γ, δ.

J'écris $\alpha = a - \phi$, les facteurs $b-\alpha$, $c-\alpha$, $d-\alpha$ de P, Q, R deviennent ainsi respectivement $-c+\phi$, $g+\phi$, $-f+\phi$; cela étant, les valeurs de P, Q, R servent à exprimer β, γ, δ en fonction de ϕ: substituant ces valeurs de β, γ, δ dans celles de L, M, N, on obtient sans peine

$$L = -\frac{h}{gP}\,\frac{f(c-\phi)+cP(-f+\phi)}{b(-f+\phi)+fR(b+\phi)},$$

$$M = -\frac{a}{hR}\,\frac{b(-f+\phi)+fR(b+\phi)}{c(b+\phi)+bQ(-c+\phi)},$$

$$N = -\frac{g}{aQ}\,\frac{c(b+\phi)+bQ(-c+\phi)}{f(c-\phi)+cP(-f+\phi)},$$

valeurs qui donnent, comme cela doit être, $LMNPQR=1$: il faut entre ces équations éliminer ϕ.

En rétablissant λ, μ, ν, ϖ, κ, ρ au lieu de L, M, N, P, Q, R, ces équations deviennent

$$\xi = \frac{g}{h}\lambda\varpi = \frac{X+Y\phi}{X_1+Y_1\phi},$$

$$\eta = \frac{h}{a}\mu\rho = \frac{X_1+Y_1\phi}{X_2+Y_2\phi},$$

$$\zeta = \frac{a}{g}\kappa\nu = \frac{X_2+Y_2\phi}{X+Y\phi},$$

(évidemment $\xi\eta\zeta=1$), où j'écris ξ, η, ζ pour dénoter les expressions $\frac{b}{c}\lambda\varpi$, etc., et où les valeurs des coefficients X, Y, etc., sont

$$\begin{aligned} X &= fc\,(fa+\varpi hc), & Y &= -f^2a-\varpi hc^2,\\ X_1 &= fb\,(gb+\rho fa), & Y_1 &= gb^2+\rho f^2a,\\ X_2 &= bc\,(hc+\kappa gb), & Y_2 &= hc^2-\kappa gb^2. \end{aligned}$$

Les deux premières équations donnent

$$\xi\eta(X_1Y_2-X_2Y_1)+\eta(X_2Y-XY_2)+XY_1-XY_1=0,$$

ou, ce qui est la même chose,

$$\frac{1}{\zeta}(X_1Y_2-X_2Y_1)+\eta(X_2Y-XY_2)+XY_1-XY_1=0,$$

et l'on n'a qu'à substituer la valeur de ces coefficients.

On a

$$\begin{aligned} X_1Y_2-X_2Y_1 &= fb\,(gb+\rho fa)(hc^2-\kappa gb^2)-bc\,(hc+\kappa gb)(-gb^2+\rho f^2a)\\ &= fghb^3c^2-fg^2b^4\kappa+f^2habc^2\rho-f^2gab^3\kappa\rho+ghb^3c^2+g^2b^4c\kappa-f^2habc^2\rho-f^2gab^2c\kappa\rho\\ &= ghb^2c^2(f+b)+g^2b^4(-f+c)\rho-f^2gab^2(b+c)\kappa\rho\\ &= ghb^2c^2h+g^2b^4(-g)\rho+f^2gab^2a\kappa\rho\\ &= gb^2(h^2c^2-g^2b^2\kappa+a^2f^2\kappa\rho); \end{aligned}$$

et de même

$$XY_2 - X_2Y = hc^2(f^2a^2 - h^2c^2\varpi + g^2b^2\varpi\kappa),$$

$$XY_1 - X_1Y = f^2a(g^2b^2 - f^2a^2\rho + h^2c^2\varpi\rho).$$

Donc

$$\frac{g}{a}\frac{1}{\kappa\nu}gb^2(h^2c^2 - g^2b^2\kappa + f^2a^2\kappa\rho)$$

$$+\frac{h}{a}\mu\rho\, hc^2(f^2a^2 - h^2c^2\varpi + g^2b^2\varpi\kappa)$$

$$+ \quad f^2a(g^2b^2 - f^2a^2\rho + h^2c^2\varpi\rho) = 0,$$

ou enfin en multipliant par $-a\nu$, et dans un terme $-g^2b^2h^2c^2\mu\nu\rho\varpi\kappa$, au lieu de $\mu\nu\rho\varpi\kappa$ écrivant $\frac{1}{\lambda}$, l'équation devient

$$(fa)^4\nu\rho + (gb)^4 + (hc)^4\frac{1}{\lambda\kappa} - (gb)^2(hc)^2\left(\frac{1}{\kappa}+\frac{1}{\lambda}\right)$$

$$-(hc)^2(fa)^2\nu\rho(\varpi+\mu) - (fa)^2(gb)^2(\nu+\rho) = 0,$$

ou, comme on peut l'écrire,

$$\left(\nu\rho,\ 1,\ \frac{1}{\lambda\kappa},\ -\left(\frac{1}{\kappa}+\frac{1}{\lambda}\right),\ -\nu\rho(\varpi+\mu),\ -(\nu+\rho)\right)((fa)^2,\ (gb)^2,\ (hc)^2)^2 = 0.$$

C'est la deuxième d'un système de trois équations équivalentes; savoir, en multipliant par $\frac{1}{\nu\rho}$ et en réduisant par $\lambda\mu\nu\varpi\kappa\rho = 1$, on obtient la première forme: et, en multipliant par $\lambda\kappa$ et réduisant de même, on obtient la troisième forme: le système est

$$\left(1,\ \frac{1}{\nu\rho},\ \mu\varpi,\ -\mu\varpi(\lambda+\kappa),\ -(\mu+\varpi),\ -\left(\frac{1}{\nu}+\frac{1}{\rho}\right)\right)((fa)^2,\ (gb)^2,\ (hc)^2)^2 = 0,$$

$$\left(\nu\rho,\ 1,\ \frac{1}{\lambda\kappa},\ -\left(\frac{1}{\kappa}+\frac{1}{\lambda}\right),\ -\nu\rho(\mu+\varpi),\ -(\nu+\rho)\right)(\ \text{„}\quad\text{„}\quad\text{„}\ )^2 = 0,$$

$$\left(\frac{1}{\mu\varpi},\ \lambda\kappa,\ 1,\ -(\lambda+\kappa),\ -\left(\frac{1}{\mu}+\frac{1}{\varpi}\right),\ -\lambda\kappa(\nu+\rho)\right)(\ \text{„}\quad\text{„}\quad\text{„}\ )^2 = 0.$$

En écrivant $hc = -fa - gb$, on obtient une équation de la forme $(*)\,(fa,\ gb)^4 = 0$, savoir une équation quartique pour avoir $fa : gb$, c'est-à-dire, le rapport anharmonique $(a-d)(b-c) : (b-d)(c-a)$: en considérant a, b, c comme données, il y a donc 4 valeurs de d: et l'on a déjà vu que les valeurs α, β, γ, δ sont données rationnellement en fonctions de a, b, c, d: le théorème est donc démontré.

Cambridge, juillet, 1880.

709.

ON THE NUMBER OF CONSTANTS IN THE EQUATION OF A SURFACE $PS-QR=0$.

[From the *Tidsskrift for Mathematik*, Ser. 4, t. IV. (1880), pp. 145—148.]

THE very important results contained in Mr H. Valentiner's paper "Nogle Sætninger om fuldstændige Skjæringskurver mellem to Flader" may be considered from a somewhat different point of view, and established in a more simple manner, as follows*.

Assuming throughout $n \geqq p+q$, $p \geqq q$, and moreover that P, Q, R, S denote functions of the coordinates (x, y, z, w) of the orders p, q, $n-q$, $n-p$ respectively: then the equation of a surface of the order n containing the curve of intersection of two surfaces of the orders p and q respectively, is

$$\begin{vmatrix} P, & Q \\ R, & S \end{vmatrix} = 0,$$

so that the number of constants in the equation of a surface of the order n satisfying the condition in question is in fact the number of constants contained in an equation of the last-mentioned form. Writing for shortness

$$a_p = \tfrac{1}{6}(p+1)(p+2)(p+3)-1, \; = \tfrac{1}{6}p(p^2+6p+11),$$

the number of constants contained in a function of the order p is $=a_p+1$; or if we take one of the coefficients (for instance that of x^p) to be unity, then the number

* Idet vi med stor Glæde optage Prof. Cayley's simple Forklaring af den Reduktion af Konstanttallet i Ligningen $PS-QR=0$, som Hr. Valentiner havde paavist (*Tidsskr. f. Math.* 1879, S. 22), skulle vi dog bemærke, at Grunden til, at dennes Bevis er bleven saa vanskeligt, er den, at han tillige har villet bevise, at der ikke finder nogen *yderligere* Reduktion Sted.

of the remaining constants is $=a_p$; viz. a_p is the number of constants in the equation of a surface of the order p. As regards the surface in question

$$\begin{vmatrix} P, & Q \\ R, & S \end{vmatrix} = 0,$$

we may it is clear take P, Q, R each with a coefficient unity as above, but in the remaining function S, the coefficient must remain arbitrary: the apparent number of constants is thus $=a_p+a_q+a_{n-p}+a_{n-q}+1$; but there is a deduction from this number.

The equation may in fact be written in the form

$$\begin{vmatrix} P+\alpha Q, & Q \\ R+\alpha S+\beta P+\alpha\beta Q, & S+\beta Q \end{vmatrix} = 0,$$

where α represents an arbitrary function of the order $p-q$, and β an arbitrary function of the degree $n-p-q$: we thus introduce $(a_{p-q}+1)+(a_{n-p-q}+1)$, $=a_{p-q}+a_{n-p-q}+2$, constants, and by means of these we can impose the like number of arbitrary relations upon the constants originally contained in the functions P, Q, R, S respectively (say we can reduce to zero this number $a_{p-q}+a_{n-p-q}+2$ of the original constants): hence the real number of constants is

$$\begin{aligned} & a_p+a_q+a_{n-p}+a_{n-q}+1-(a_{p-q}+a_{n-p-q}+2), \\ &= a_p+a_q+a_{n-p}+a_{n-q}-a_{p-q}-a_{n-p-q}-1 \\ &= \omega \text{ suppose}; \end{aligned}$$

viz. this is the required number in the case $n>p+q$, $p>q$.

If however $n=p+q$, or $p=q$, or if these relations are both satisfied, then there is a further deduction of 1, 1, or 2: in fact, calling the last-mentioned determinant $\begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix}$, then the four cases are

$$n>p+q,\ p>q,\quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix}$$

$$n=p+q,\ p>q,\quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P'+kR', & Q'+kS' \\ R', & S' \end{vmatrix}$$

$$n>p+q,\ p=q,\quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P', & Q'+kP' \\ R', & S'+kR' \end{vmatrix}$$

$$n=p+q,\ p=q,\quad \begin{vmatrix} P', & Q' \\ R', & S' \end{vmatrix} = \begin{vmatrix} P'+kR', & Q'+lP'+kS'+klR' \\ R', & S'+\ lR' \end{vmatrix}$$

where k, l denote arbitrary constants: these, like the constants of α and β, may be used to impose arbitrary relations upon the original constants of P, Q, R, S; and hence the number of constants is $=\omega$, $\omega-1$, $\omega-1$, $\omega-2$ in the four cases respectively; where as above

$$\omega = a_p+a_q+a_{n-p}+a_{n-q}-a_{p-q}-a_{n-p-q}-1.$$

If $n=4$, there is in each of the four cases one system of values of p, q; viz. the cases are

$$\begin{array}{lll}
p,\ q = & & \\
2\ 1 & \text{No.} = a_2+a_1+a_2+a_3-a_1-a_1-1 = & 9+3+\ 9+19-3-3-1,\ =33, \\
3\ 1 & \quad\text{,,} \quad a_3+a_1+a_1+a_3-a_2-a_0-2 = & 19+3+\ 3+19-9-0-2,\ =33, \\
1\ 1 & \quad\text{,,} \quad a_1+a_1+a_3+a_3-a_0-a_3-2 = & 3+3+19+19-0-9-2,\ =33, \\
2\ 2 & \quad\text{,,} \quad a_2+a_2+a_2+a_2-a_0-a_0-3 = & 9+9+\ 9+\ 9-0-0-3,\ =33,
\end{array}$$

and the number of constants is in each case $=33$. This is easily verified: in the first case we have a quartic surface containing a conic, the plane of the conic is therefore a quadruple tangent plane; and the existence of such a plane is 1 condition. In the second case the surface contains a plane cubic; the plane of this cubic is a triple tangent plane, having the points of contact in a line; and this is 1 condition. In the third case the surface contains a line, which is 1 condition: hence in each of these cases the number of constants is $34-1$, $=33$. In the fourth case, where the surface contains a quadriquadric curve, we repeat in some measure the general reasoning: the quadriquadric curve contains 16 constants, and we have thus 16 as the number of constants really contained in the equations $P=0$, $Q=0$ of the quadriquadric curve: the equation $PS-QR=0$, contains in addition $9+10$, $=19$ constants, but writing it in the form $P(S+kQ)-Q(R+kP)=0$, we have a diminution $=1$, or the number apparently is $16+19-1$, $=34$. But the quadriquadric curve is one of a singly infinite series $P+lR=0$, $Q+lS=0$ of such curves, and we have on this account a diminution $=1$; the number of constants is thus $34-1$, $=33$ as above: the reasoning is, in fact, the same as for the case of a plane passing through a line; the line contains 4 constants, hence the plane, quà arbitrary plane through the line, would contain $1+4$, $=5$ constants; but the line being one of a doubly infinite system of lines on the plane the number is really $5-2$, $=3$, as it should be.

Cambridge, 2nd Sept., 1880.

710.

ON A DIFFERENTIAL EQUATION.

[From *Collectanea Mathematica: in memoriam Dominici Chelini*, (Milan, Hoepli, 1881), pp. 17—26.]

In the Memoir on hypergeometric series, *Crelle*, t. xv. (1836), Kummer in effect considers a differential equation

$$\frac{(a'z^2+2b'z+c')\,dz^2}{z^2(z-1)^2}=\frac{(ax^2+2bx+c)\,dx^2}{x^2(x-1)^2},$$

viz. he seeks for solutions of an equation of this form which also satisfy a certain differential equation of the third order. The coefficients a, b, c are either all arbitrary, or they are two or one of them, arbitrary; but this last case (or say the case where the function of x is the completely determinate function $x^2+2bx+c$) is scarcely considered: a', b', c' are regarded as determinable in terms of a, b, c; and z is to be found as a function of x independent of a, b, c: so that when these coefficients are arbitrary, the equation breaks up into three equations, and when two of the coefficients are arbitrary, it breaks up into two equations, satisfied in each case by the same value of z; and the value of z is thus determined without any integration: these cases will be considered in the sequel, but they are of course included in the general case where the coefficients a, b, c are regarded as having any given values whatever.

Writing for shortness $X=ax^2+2bx+c$, in general the integral

$$\int\frac{N\,dx}{D\sqrt{X}},$$

where D is the product of any number n of distinct linear factors $x-p$, and N is a rational and integral function of x of the order n at most, and therefore also the integral

$$\int\frac{N\sqrt{X}\,dx}{D}=\int\frac{NX\,dx}{D\sqrt{X}},$$

where N is now of the order $n-2$ at most, is expressible as the logarithm of a quasi-algebraical function, that is, a function containing powers the exponents of which are incommensurable (for instance, $x^{\sqrt{2}}$ is a quasi-algebraical function): in fact, the integral is of the form

$$\int\left(M+\frac{A}{x-p}+\frac{B}{x-q}+\ldots\right)\frac{dx}{\sqrt{X}},$$

where each term is separately integrable,

$$\int\frac{dx}{\sqrt{X}}=\frac{1}{\sqrt{a}}\log\{ax+b+\sqrt{a}\,.\,\sqrt{X}\},$$

$$\int\frac{dx}{(x-p)\sqrt{X}}=-\frac{1}{\sqrt{P}}\log\left\{\frac{(ap+b)\,x+(bp+c)+\sqrt{P}\,.\,\sqrt{X}}{x-p}\right\},$$

where P is written to denote $ap^2+2bp+c$: the integral is thus $=\log\Omega$, where Ω is a product of factors

$$ax+b+\sqrt{a}\,.\,\sqrt{X},\quad\frac{(ap+b)\,x+(bp+c)+\sqrt{P}\,.\,\sqrt{X}}{x-p},\text{ etc.},$$

raised to powers $\frac{M}{\sqrt{a}}$, $\frac{-A}{\sqrt{P}}$, etc.: hence, if we have a differential equation

$$\frac{N'dz}{D'\sqrt{Z}}=\frac{Ndx}{D\sqrt{X}},\quad\text{or}\quad\frac{N'\sqrt{Z}dz}{D'}=\frac{N\sqrt{X}dx}{D},$$

where $Z\,(=a'z^2+2b'z+c')$, and N', D' are functions of z such as X, N, D are of x; then, taking $\log C$ for the constant of integration, the general integral is

$$\log\Omega'=\log C+\log\Omega:$$

viz. we have the quasi-algebraical integral $\Omega'-C\Omega=0$.

The constants a, b, c, p, q, ... etc. may be such that the exponents are rational, and the integral is then algebraical: in particular, for the differential equation

$$\frac{\sqrt{z^2+14z+1}\,dz}{z\,(z-1)}=\frac{\sqrt{x^2+14x+1}\,dx}{x\,(x-1)},$$

the general integral is in the first instance obtained in the form

$$\frac{(z+1+\sqrt{Z})\,(z-1)^2}{\sqrt{z}\,(2z+2+\sqrt{Z})^2}=C\frac{(x+1+\sqrt{X})\,(x-1)^2}{\sqrt{x}\,(2x+2+\sqrt{X})^2}$$

which, observing that $(2x+2)^2-X=3\,(x-1)^2$, may also be written

$$\frac{(z+1)\,(z^2-34z+1)+Z\sqrt{Z}}{\sqrt{z}\,(z-1)^2}=C\frac{(x+1)\,(x^2-34x+1)+X\sqrt{X}}{\sqrt{x}\,(x-1)^2}.$$

I had previously obtained the solution

$$z=\left(\frac{1-\sqrt[4]{x}}{1+\sqrt[4]{x}}\right)^4,$$

and I wish to show that this is, in fact, the particular integral belonging to the value $C=1$ of the constant of integration: for this purpose I proceed to rationalise the general integral as regards z.

Writing for a moment

$$P=(z+1)(z^2-34z+1),$$

$$Q=(z^2+14z+1)\sqrt{z^2+14z+1},$$

$$R=M\sqrt{z}(z-1)^2,$$

where

$$M=C\frac{(x+1)(x^2-34x+1)+(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}(x-1)^2},$$

the integral is $P+Q+R=0$; or rationalising, it is

$$(P^2-Q^2)^2-2R^2(P^2+Q^2)+R^4=0;$$

we have

$$P^2=(1,\ -66,\ 1023,\ 2180,\ 1023,\ -66,\ 1\between z,\ 1)^6,$$

$$Q^2=(1,\ \ 42,\ \ 591,\ 2828,\ \ 591,\ \ 42,\ 1\between z,\ 1)^6,$$

and thence

$$P^2-Q^2=(0,\ -108,\ 432,\ -648,\ 432,\ -108,\ 0\between z,\ 1)^6,$$

$$=-108z(z-1)^4;$$

$$P^2+Q^2=2(1,\ -12,\ 807,\ 2504,\ 807,\ -\ 12,\ 1\between z,\ 1)^6.$$

Writing the equation in the form

$$\tfrac{1}{2}(P^2+Q^2)-\tfrac{1}{4}\left\{R^2+\frac{(P^2-Q^2)^2}{R^2}\right\}=0,$$

it thus becomes

$$(1,\ -12,\ 807,\ 2504,\ 807,\ -12,\ 1\between z,\ 1)^6-z(z-1)^4\left\{M^2+\frac{(108)^2}{M^2}\right\}=0,$$

where M has its above-mentioned value; and if we now assume $C=1$, then

$$M=\frac{(x+1)(x^2-34x+1)+(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}(x-1)^2},$$

$$\frac{108}{M}=\frac{(x+1)(x^2-34x+1)-(x^2+14x+1)\sqrt{x^2+14x+1}}{\sqrt{x}(x-1)^2},$$

and thence

$$M^2+\frac{(108)^2}{M^2},=\left(M-\frac{108}{M}\right)^2+216,=4\frac{(x+1)^2(x^2-34x+1)^2}{x(x-1)^4}+216,$$

$$=\frac{4}{x(x-1)^4}\cdot(1,\ -12,\ 807,\ 2504,\ 807,\ -12,\ 1\between x,\ 1)^6:$$

and the rationalised equation is

$$(1, -12, 807, 2504, 807, -12, 1 \between z, 1)^6$$
$$-\frac{z(z-1)^4}{x(x-1)^4}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6 = 0.$$

This is a sextic equation in z, of the form

$$z^3 + \frac{1}{z^3} + \lambda\left(z^2 + \frac{1}{z^2}\right) + \mu\left(z + \frac{1}{z}\right) + \nu = 0,$$

where

$$\lambda, \mu, \nu = -12 - \Omega, \quad 807 + 4\Omega, \quad 2504 - 6\Omega,$$

if Ω denote the function of x which enters into the equation; and writing $z + \frac{1}{z} = \theta$, this becomes

$$\theta^3 - 3\theta + \lambda(\theta^2 - 2) + \mu\theta + \nu = 0.$$

But the equation in z is satisfied by the value $z = x$, and therefore the equation in θ by the value $\theta = x + \frac{1}{x} = \alpha$ suppose, we have therefore

$$\alpha^3 - 3\alpha + \lambda(\alpha^2 - 2) + \mu\alpha + \nu = 0,$$

and thence subtracting, and throwing out the factor $\theta - \alpha$,

$$\theta^2 + \theta\alpha + \alpha^2 - 3 + \lambda(\theta + \alpha) + \mu = 0;$$

viz. writing for λ, μ, α their values, this is

$$\theta^2 + \theta\left(x + \frac{1}{x} - 12 - \Omega\right) + x^2 - 1 + \frac{1}{x^2} - \left(x + \frac{1}{x}\right)(12 + \Omega) + 807 + 4\Omega = 0,$$

or, what is the same thing,

$$\theta^2 + \theta\left(x - 12 + \frac{1}{x} - \Omega\right) + x^2 - 12x + 806 - \frac{12}{x} + \frac{1}{x^2} - \left(x - 4 - \frac{1}{x}\right)\Omega = 0,$$

where

$$\Omega = \frac{1}{x(x-1)^4}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6.$$

Hence in the quadric equation, the coefficients, each multiplied by $(x-1)^4$, are

$$(x-1)^4\left(x - 12 + \frac{1}{x}\right) - \frac{1}{x}(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6,$$

and

$$(x-1)^4\left(x^2 - 12x + 806 - \frac{12}{x} + \frac{1}{x^2}\right)$$
$$-\frac{1}{x}\left(x - 4 + \frac{1}{x}\right)(1, -12, 807, 2504, 807, -12, 1 \between x, 1)^6,$$

which are respectively rational and integral quartic functions of x; and, writing for θ its value, the equation finally is

$$\left(z + \frac{1}{z}\right)^2 - 4\left(z + \frac{1}{z}\right)\frac{(1, 188, 646, 188, 1 \between x, 1)^4}{(x-1)^4} + 4\frac{(1, -644, 3334, -644, 1 \between x, 1)^4}{(x-1)^4} = 0$$

Writing

$$\xi=\sqrt[4]{x},\quad A=\frac{1-\xi}{1+\xi},\quad B=\frac{1+\xi}{1-\xi},\quad C=\frac{1-i\xi}{1+i\xi},\quad D=\frac{1+i\xi}{1-i\xi},\qquad (i=\sqrt{-1}\text{ as usual}),$$

this is

$$(z-A^4)(z-B^4)(z-C^4)(z-D^4)=0,$$

or, what is the same thing,

$$\left\{z+\frac{1}{z}-(A^4+B^4)\right\}\left\{z+\frac{1}{z}-(C^4+D^4)\right\}=0,$$

that is,

$$\left(z+\frac{1}{z}\right)^2-\left(z+\frac{1}{z}\right)(A^4+B^4+C^4+D^4)+(A^4+B^4)(C^4+D^4)=0;$$

for we have

$$\tfrac{1}{2}(A^4+B^4)=\frac{(1,\quad 28,\ 70,\quad 28,\ 1\between\xi^2,\ 1)^4}{(\xi^2-1)^4},$$

$$\tfrac{1}{2}(C^4+D^4)=\frac{(1,\ -28,\ 70,\ -28,\ 1\between\xi^2,\ 1)^4}{(\xi^2+1)^4}.$$

And substituting these values, the coefficients will be rational functions of ξ^4, that is, of x, and it is easy to verify that they have in fact their foregoing values.

It thus appears that for $C=1$, besides the values x and $\frac{1}{x}$, we have for z only the values

$$\left(\frac{1-\xi}{1+\xi}\right)^4,\quad \left(\frac{1+\xi}{1-\xi}\right)^4,\quad \left(\frac{1-i\xi}{1+i\xi}\right)^4,\quad \left(\frac{1+i\xi}{1-i\xi}\right)^4;$$

viz. that the only solution is

$$z=\left(\frac{1-\sqrt[4]{x}}{1+\sqrt[4]{x}}\right)^4.$$

The example shows that although the differential equation

$$\frac{\sqrt{a'z^2+2b'z+c'}\,dz}{z(z-1)}=\frac{\sqrt{ax^2+2bx+c}\,dx}{x(x-1)}$$

can be integrated generally in a quasi-algebraical or algebraical form as above, yet we cannot from the general solution deduce, at once or easily, the various particular integrals comprised therein: nor can we find for what values of the constants a, b, c and a', b', c' the differential equation admits of a simple solution, or say of a solution where z is expressed as an explicit (irrational) function of x.

In the cases considered by Kummer there is a second (or it may be also a third) differential equation of the like form, the equations being each of them satisfied by the same value of z: hence eliminating the differentials dx, dz, the relation between x and z is of the form

$$\frac{P'}{Q'}=\frac{P}{Q},$$

where P, Q are quadric functions of x; P', Q' quadric functions of z. But P and Q may contain a common factor, and the integral is then expressible in the form $x=\frac{P'}{Q'}$, the quotient of two quadric functions of z; or P' and Q' may have a common factor, and the integral is then expressible in the form $z=\frac{P}{Q}$, the quotient of two quadric functions of x; or there may be a common factor of P, Q, and also a common factor of P' and Q', and the integral is then of the form $z=\frac{L}{M}$, the quotient of two linear functions of x.

In the general case the differential equation is

$$\frac{\lambda(aP'+bQ')\,dz^2}{z^2(z-1)^2}=\frac{(aP+bQ)\,dx^2}{x^2(x-1)^2},$$

where a, b are arbitrary constants, λ is a constant the value of which can in each particular case be at once determined; so when the integral is $z=\frac{P}{Q}$, the differential equation is

$$\frac{\lambda(az+b)\,dz^2}{z^2(z-1)^2}=\frac{(aP+bQ)\,dx^2}{x^2(x-1)^2},$$

where a, b are arbitrary constants, but λ is now a linear function of z the value of which can in each particular case be at once determined. When the integral is $z=\frac{L}{M}$, the differential equation is

$$\frac{\lambda(az^2+2bz+c)\,dz^2}{z^2(z-1)^2}=\frac{(aL^2+2bLM+cM^2)\,dx^2}{x^2(x-1)^2}$$

containing the three arbitrary constants a, b, c; λ is a constant the value of which can be at once determined.

There are in all 6 integrals of the form $z=\frac{L}{M}$, for which the differential equation contains three arbitrary constants: 18 integrals of the form $z=\frac{P}{Q}$ $\left(\text{and of course the same number of integrals of the form } x=\frac{P'}{Q'}\right)$, and 9 integrals of the form $\frac{P}{Q}=\frac{P'}{Q'}$, for all of which the differential equation contains two arbitrary constants. It is to be remarked that Kummer, considering the values of z as a function of x, obtains the 72 rational and irrational values mentioned in his equations (31), (35), (36), (37), (38), and (39): but the 72 values are made up as follows, viz. the 18 values of z as a rational function of x, the 36 irrational values obtained from the 18 expressions of x as a rational function of z, and the 18 irrational values of z obtained from the 9 integrals in which neither of the variables is a rational function of the other: $18+36+18=72$.

The several integrals together with the expressions of the functions

$$a'z^2 + 2b'z + c' \quad \text{and} \quad ax^2 + 2bx + c$$

which enter into the differential equation are as follows:

	$z =$	$a'z^2 + 2b'z + c' =$	$ax^2 + 2bx + c =$
1.	x	$az^2 + 2bz + c$	$ax^2 + 2bx + c$
	$1 - x$	,,	$a(x-1)^2 - 2b(x-1) + c$
	$\frac{1}{x}$	,,	$a + 2bx + cx^2$
	$\frac{1}{1-x}$	,,	$a - 2b(x-1) + c(x-1)^2$
	$\frac{x}{x-1}$	,,	$ax^2 + 2bx(x-1) + c(x-1)^2$
	$\frac{x-1}{x}$	,,	$a(x-1)^2 + 2bx(x-1) + cx^2$
2.	$\left(\frac{x+1}{x-1}\right)^2$	$az^2 + bz$	$a(x+1)^2 + b(x-1)^2$
	$(2x-1)^2$	,,	$a(2x-1)^2 + b$
	$\left(\frac{x-2}{x}\right)^2$	,,	$a(x-2)^2 + bx^2$
	$\frac{(x+1)^2}{4x}$	,,	$a(x+1)^2 + 4bx$
	$\frac{(2x-1)^2}{4x(x-1)}$	,,	$a(2x-1)^2 + 4bx(x-1)$
	$-\frac{(x-2)^2}{4(x-1)}$	,,	$a(x-2)^2 - 4b(x-1)$
3.	$\left(\frac{x-1}{x+1}\right)^2$	$bz + c$	$b(x-1)^2 + c(x+1)^2$
	$\left(\frac{1}{2x-1}\right)^2$	,,	$b + c(2x-1)^2$
	$\left(\frac{x}{x-2}\right)^2$	,,	$bx^2 + c(x-2)^2$
	$\frac{4x}{(x+1)^2}$	,,	$4bx + c(x+1)^2$
	$\frac{4x(x-1)}{(2x-1)^2}$	,,	$4bx(x-1) + c(2x-1)^2$
	$-\frac{4(x-1)}{(x-2)^2}$	,,	$-4b(x-1) + c(x-2)^2$

	$z =$	$a'z^2 + 2b'z + c' =$	$ax^2 + 2bx + c =$
4.	$-\dfrac{(x-1)^2}{4x}$	$az^2 - (a+c)z + c$	$a(x-1)^2 + 4cx$
	$-4x(x-1)$	,,	$4ax(x-1) + c$
	$\dfrac{4(x-1)}{x^2}$	,,	$-4a(x-1) + cx^2$
	$-\dfrac{4x}{(x-1)^2}$	,,	$4ax + c(x-1)^2$
	$\dfrac{-1}{4x(x-1)}$	,,	$a + 4cx(x-1)$
	$\dfrac{x^2}{4(x-1)}$	,,	$ax^2 - 4c(x-1)$

5.
6. } same as 2, 3, 4 interchanging x and z.
7.

	$z =$	$a'z^2 + 2b'z + c' =$	$ax^2 + 2bx + c =$
8.	$\dfrac{(z-1)^2}{4z} = \dfrac{4x}{(x-1)^2}$	$a(z-1)^2 + 4bz$	$4ax + b(x-1)^2$
	$\dfrac{z^2}{4(z-1)} = \dfrac{4(x-1)}{x^2}$	$az^2 + 4b(z-1)$	$-4a(x-1) - bx^2$
	$4z(z-1) = \dfrac{1}{4x(x-1)}$	$4az(z-1) + b$	$a + 4bx(x-1)$
9.	$\dfrac{(z-1)^2}{4z} = 4x(x-1)$	$a(z-1)^2 + 4bz$	$4ax(x-1) + b$
	$\dfrac{z^2}{4(z-1)} = -4x(x-1)$	$az^2 - 4b(z-1)$	$4ax(x-1) + b$
	$\dfrac{(z-1)^2}{4z} = -\dfrac{4(x-1)}{x^2}$	$a(z-1)^2 + 4bz$	$-4a(x-1) + bx^2$
10.	$4z(z-1) = \dfrac{(x-1)^2}{4x}$	$4az(z-1) + b$	$a(x-1)^2 + 4bx$
	$4z(z-1) = -\dfrac{x^2}{4(x-1)}$	$4az(z-1) + b$	$ax^2 - 4b(x-1)$
	$\dfrac{4(z-1)}{z^2} = -\dfrac{(x-1)^2}{4x}$	$-4a(z-1) + bz^2$	$a(x-1)^2 + 4bx$

The six functions of the set (1), that is,

$$x,\quad 1-x,\quad \frac{1}{x},\quad \frac{1}{1-x},\quad \frac{x}{x-1},\quad \frac{x-1}{x},$$

form a group; and by operating with the substitutions of this group, and of the like group

$$z,\quad 1-z,\quad \frac{1}{z},\quad \frac{1}{1-z},\quad \frac{z}{z-1},\quad \frac{z-1}{z},$$

upon any value of z in the sets (2), (3), (4), for instance upon $z=\left(\frac{x+1}{x-1}\right)^2$, we form all the 18 functions of these sets.

In any one of these sets (2), (3), and (4), comparing two forms (the same or different), for instance in the set (2), writing y for z and then in one form z for x,

$$y=\left(\frac{x+1}{x-1}\right)^2 \text{ and } y=\left(\frac{z+1}{z-1}\right)^2, \text{ whence } \left(\frac{x+1}{x-1}\right)^2=\left(\frac{z+1}{z-1}\right)^2,$$

or

$$y=\left(\frac{x+1}{x-1}\right)^2 \text{ and } y=\frac{(z+1)^2}{4z}, \text{ whence } \left(\frac{x+1}{x-1}\right)^2=\frac{(z+1)^2}{4z},$$

we obtain either the equations of the set (1) or those of the sets (8), (9) and (10); and whether we use the set (2), (3) or (4), the only new equations obtained are thus the 9 equations of the sets (8), (9) and (10). These several equations present themselves however in different forms: for instance, instead of the equation

$$\frac{(z-1)^2}{4z}=\frac{4x}{(x-1)^2},$$

we may obtain

$$\frac{(z+1)^2}{4z}=\left(\frac{x+1}{x-1}\right)^2.$$

If, to get rid of this variety of form, we multiply out the denominators, the 9 equations are

$$\begin{aligned}
0 &= x^2z^2-2x^2z-2xz^2+x^2-12xz+z^2-2x-2z+1,\\
0 &= x^2z^2-16xz+16x+16z-16,\\
0 &= 16x^2z^2-16x^2z-16xz^2+16xz-1,\\
0 &= x^2z^2-2x^2z+x^2+16xz-16z,\\
0 &= 16x^2z-16xz-z^2+2z-1,\\
0 &= 16x^2z-16x^2-16xz+z^2+16x,\\
0 &= x^2z^2-2xz^2+16xz+z^2-16x,\\
0 &= 16xz^2-x^2-16xz+2x+1,\\
0 &= 16xz^2+x^2-16xz-16z^2+16z.
\end{aligned}$$

These 9 equations are derivable all from any one of them by the changes of the set (1) upon x and z.

Cambridge, 3rd June, 1879.

711.

ON A DIAGRAM CONNECTED WITH THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Report of the British Association for the Advancement of Science*, (1881), p. 534.]

THE diagram relates to a known theorem, and is constructed as follows. Consider the infinite half-plane $y=+$; draw in it, centre the origin and radius unity, a semicircle; and draw the infinite half-lines $x=-\frac{1}{2}$, and $x=\frac{1}{2}$; then we have a region included between the lines, but exterior to the semicircle. The region in question may be regarded as a curvilinear triangle, with the angles 60°, 60°, and 0°. The region may be moved parallel to itself in the direction of the axis of x, through the distance 1; say this is a "displacement"; or we may take the "image" of the region in regard to the semicircle. Performing any number of times, and in any order, these two operations of making the displacement and of taking the image, we obtain a new region, which is always a curvilinear triangle (bounded by circular arcs) and having the angles 60°, 60°, 0°; and the theorem is that the whole series of the new regions thus obtained completely covers, without interstices or overlapping, the infinite half-plane. The number of regions is infinite, and the size of the successive regions diminishes very rapidly. The diagram was a coloured one, exhibiting the regions obtained by a few of the successive operations.

The analytical theorem is that the whole series of transformations, ω into $\dfrac{\alpha\omega+\beta}{\gamma\omega+\delta}$, where α, β, γ, δ are integers such that $\alpha\delta-\beta\gamma=1$, can be obtained by combination of the transformations ω into $\omega+1$ and ω into $-\dfrac{1}{\omega}$.

712.

A PARTIAL DIFFERENTIAL EQUATION CONNECTED WITH THE SIMPLEST CASE OF ABEL'S THEOREM.

[From the *Report of the British Association for the Advancement of Science*, (1881), pp. 534, 535.]

CONSIDER a given cubic curve cut by a line in the points (x_1, y_1), (x_2, y_2), (x_3, y_3); taking the first and second points at pleasure, these determine uniquely the third point. Analytically, the equation of the curve determines y_1 as a function of x_1, and y_2 as a function of x_2: writing in the equation

$$x_3 = \lambda x_1 + (1-\lambda) x_2, \quad y_3 = \lambda y_1 + (1-\lambda) y_2,$$

we have λ by a simple equation, and thence x_3; viz. x_3 is found as a function of x_1, x_2, and of the nine constants of the equation. Hence forming the derived equations (in regard to x_1, x_2) of the first, second, and third orders, we have $(1+2+3+4=)\,10$ equations from which to eliminate the 9 constants; x_3, considered as a function of x_1 and x_2, thus satisfies a partial differential equation of the third order, independent of the particular cubic curve.

To obtain this equation it is only necessary to observe that we have, by Abel's theorem,

$$\frac{dx_1}{X_1} + \frac{dx_2}{X_2} + \frac{dx_3}{X_3} = 0,$$

where X_1 is a given function of x_1 and y_1, that is, of x_1; X_2 and X_3 are the like functions of x_2 and x_3 respectively. Hence, considering x_3 as a function of x_1 and x_2, we have

$$\frac{dx_3}{dx_1} = -\frac{X_2}{X_1}, \quad \frac{dx_3}{dx_2} = -\frac{X_3}{X_2},$$

and consequently

$$\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2} = \frac{X_2}{X_1};$$

where X_2, X_1 are functions of x_2, x_1 respectively: hence taking the logarithm and differentiating successively with regard to x_1 and x_2, we have

$$\frac{d}{dx_1}\frac{d}{dx_2}\log\left(\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2}\right) = 0,$$

which is the required partial differential equation of the third order.

This differential equation has a simple geometrical signification. Consider three consecutive positions of the line meeting the cubic curve in the points 1, 2, 3; 1′, 2′, 3′; 1″, 2″, 3″ respectively: *quà* equation of the third order, the equation should in effect determine 3″ by means of the other points. And, in fact, the three positions of the line constitute a cubic curve; the nine points are thus the intersections of two cubic curves, or, say, they are an "ennead" of points; any eight of the points thus determine uniquely the ninth point.

713.

ADDITION TO MR ROWE'S MEMOIR ON ABEL'S THEOREM.

[From the *Philosophical Transactions of the Royal Society of London*, vol. 172, Part III. (1881), pp. 751—758. Received May 27,—Read June 10, 1880.]

In Abel's general theorem y is an irrational function of x determined by an equation $\chi(y)=0$, or say $\chi(x, y)=0$, of the order n as regards y: and it was shown by him that the sum of any number of the integrals considered may be reduced to a sum of γ integrals; where γ is a determinate number depending only on the form of the equation $\chi(x, y)=0$, and given in his equation (62), [*Œuvres Complètes*, (1881), t. I. p. 168]: viz. if, solving the equation so as to obtain from it developments of y in descending series of powers of x, we have*

$$\begin{array}{llll} n_1\mu_1 & \text{series each of the form} & & y = x^{\frac{m_1}{\mu_1}} + \ldots, \\ n_2\mu_2 & \quad\text{,,} & \quad\text{,,} & y = x^{\frac{m_2}{\mu_2}} + \ldots, \\ \vdots & & & \vdots \\ n_k\mu_k & \quad\text{,,} & \quad\text{,,} & y = x^{\frac{m_k}{\mu_k}} + \ldots, \end{array}$$

* The several powers of x have coefficients: the form really is $y = A_1 x^{\frac{m_1}{\mu_1}} + \ldots$, which is regarded as representing the μ_1 different values of y obtained by giving to the radical $x^{\frac{1}{\mu_1}}$ each of its μ_1 values, and the corresponding values to the radicals which enter into the coefficients of the series: and (so understanding it) the meaning is that there are n_1 such series each representing μ_1 values of y. It is assumed that the series contains *only* the radical $x^{\frac{1}{\mu_1}}$, that is, the indices after the leading index $\frac{m_1}{\mu_1}$ are $\frac{m_1-1}{\mu_1}$, $\frac{m_1-2}{\mu_1}$, ...; a series such as $y = A_1 x^{\frac{3}{5}} + B_1 x^{\frac{3}{9}} + \ldots$, depending on the two radicals $x^{\frac{1}{5}}$, $x^{\frac{1}{9}}$ represents 15 different values, and would be written $y = A_1 x^{\frac{27}{15}} + \ldots$, or the values of m_1 and μ_1 would be 20 and 15 respectively: in a case like this where $\frac{m_1}{\mu_1}$ is not in its least terms, the number of values of the leading coefficient A_1 is equal, not to μ_1, but to a submultiple of μ_1. But the case is excluded by Abel's assumption that $\frac{m_1}{\mu_1}$, $\frac{m_2}{\mu_2}$, ..., are fractions each of them in its least terms.

(so that $n = n_1\mu_1 + n_2\mu_2 + \ldots + n_k\mu_k$), then γ is a determinate function of n_1, m_1, μ_1; n_2, m_2, μ_2; ...; n_k, m_k, μ_k.

Mr Rowe has expressed Abel's γ in the following form, viz. assuming

$$\frac{m_1}{\mu_1} > \frac{m_2}{\mu_2} > \ldots > \frac{m_k}{\mu_k},$$

then this expression is

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1,$$

or, what is the same thing, for n writing its value $\Sigma n\mu$,

$$\gamma = \sum_{s>r} n_r m_r n_s \mu_s + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n\mu - \tfrac{1}{2}\Sigma n + 1,$$

where in the first sum r, s have each of them the values 1, 2, ..., k, subject to the condition $s > r$; in each of the other sums n, m, and μ are considered as having the suffix r, which has the values 1, 2, ..., k.

It is a leading result in Riemann's theory of the Abelian integrals that γ is the deficiency (Geschlecht) of the curve represented by the equation $\chi(x, y) = 0$: and it must consequently be demonstrable *à posteriori* that the foregoing expression for γ is in fact = deficiency of curve $\chi(x, y) = 0$. I propose to verify this by means of the formulæ given in my paper "On the Higher Singularities of a Plane Curve," *Quart. Math. Jour.*, vol. VII., (1866), pp. 212—223, [374].

It is necessary to distinguish between the values of $\frac{m}{\mu}$ which are $>$, $=$, and < 1; and to fix the ideas I assume $k = 7$, and

$$\frac{m_1}{\mu_1},\ \frac{m_2}{\mu_2},\ \frac{m_3}{\mu_3},\ \text{each} > 1,$$

$$\frac{m_4}{\mu_4} = 1;\ \text{say}\ m_4 = \mu_4 = \lambda,\ \text{and}\ n_4 = \theta;$$

$$\frac{m_5}{\mu_5},\ \frac{m_6}{\mu_6},\ \frac{m_7}{\mu_7},\ \text{each} < 1,$$

but it will be easily seen that the reasoning is quite general. I use Σ' to denote a sum in regard to the first set of suffixes 1, 2, 3, and Σ'' to denote a sum in regard to the second set of suffixes 5, 6, 7. The foregoing value of n is thus

$$n = \Sigma' n\mu + \lambda\theta + \Sigma'' n\mu.$$

Introducing a third coordinate z for homogeneity, the equation $\chi(x, y) = 0$ of the curve will be

$$0 = \left(yz^{\frac{m_1}{\mu_1}-1} - x^{\frac{m_1}{\mu_1}}\right)^{n_1\mu_1} \ldots \left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta} \left(y - x^{\frac{m_5}{\mu_5}} z^{1-\frac{m_5}{\mu_5}}\right) \ldots,$$

where it is to be observed that $(\ )^{n_1\mu_1}$ is written to denote the product of $n_1\mu_1$ different series each of the form $yz^{\frac{m_1}{\mu_1}-1} - A_1 x^{\frac{m_1}{\mu_1}} - \ldots$; these divide themselves into n_1

groups, each a product of μ_1 series; and in each such product the μ_1 coefficients A_1 are in general the μ_1 values of a function containing a radical $a^{\frac{1}{\mu_1}}$ and are thus different from each other: it is in what follows in effect assumed not only that this is so, but that all the $n_1\mu_1$ coefficients A_1 are different from each other*: the like remarks apply to the other factors. It applies in particular to the term $\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$, viz. it is assumed that the coefficients A in the $\lambda\theta$ series $y = Ax^{\frac{\lambda}{\lambda}} + \ldots$ are all of them different from each other. These assumptions as to the leading coefficients really imply Abel's assumption that $\frac{m_1}{\mu_1}, \ldots, \frac{m_k}{\mu_k}$ are all of them fractions in their least terms, and in particular that $\frac{\lambda}{\lambda}$ is a fraction in its least terms, viz. that $\lambda = 1$: I retain however for convenience the general value λ, putting it ultimately $= 1$.

In the product of the several infinite series, the terms containing negative powers all disappear of themselves; and the product is a rational and integral function $F(x, y, z)$ of the coordinates, which on putting therein $z = 1$ becomes $= \chi(x, y)$. The equation of the curve thus is $F(x, y, z) = 0$; and the order is

$$= \frac{m_1}{\mu_1} n_1\mu_1 + \ldots + \lambda\theta + n_5\mu_5 + \ldots, \; = m_1 n_1 + \ldots + \lambda\theta + n_5\mu_5 + \ldots;$$

viz. if K is the order of the curve $\chi(x, y) = 0$, then $K = \Sigma' nm + \lambda\theta + \Sigma'' n\mu$.

The curve has singularities (singular points) at infinity, that is, on the line $z = 0$: viz.—

First, a singularity at $(z = 0, x = 0)$, where the tangent is $x = 0$, and which, writing for convenience $y = 1$, is denoted by the function

$$\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{n_1(m_1 - \mu_1)} \ldots;$$

where observe that the expressed factor indicates n_1 branches $\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{m_1 - \mu_1}$, or say $n_1(m_1 - \mu_1)$ partial branches $z - x^{\frac{m_1}{m_1 - \mu_1}}$, that is, $n_1(m_1 - \mu_1)$ partial branches $z = A_1 x^{\frac{m_1}{m_1 - \mu_1}} + \ldots$, with in all $n_1(m_1 - \mu_1)$ distinct values of A_1: and the like as regards the unexpressed factors with the suffixes 2 and 3.

Secondly, a singularity at $(z = 0, y = 0)$, where the tangent is $y = 0$, and which, writing for convenience $x = 1$, is denoted by the function

$$\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5(\mu_5 - m_5)} \ldots;$$

* This assumption is virtually made by Abel, (*l. c.*) p. 162, in the expression "alors on aura en général, excepté quelques cas particuliers que je me dispense de considérer: $h(y' - y'') = hy'$, &c.": viz. the meaning is that the degree of y' being greater than or equal to that of y'', then the degree of $y' - y''$ is equal to that of y'': of course when the degrees are equal, this implies that the coefficients of the two leading terms must be unequal.

where observe that the expressed factor indicates n_5 branches $\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{\mu_5 - m_5}$, or say $n_5(\mu_5 - m_5)$ partial branches $z - y^{\frac{\mu_5}{\mu_5 - m_5}}$, that is, $n_5(\mu_5 - m_5)$ partial branches $z = A_5 y^{\frac{\mu_5}{\mu_5 - m_5}} + \ldots$, with in all $n_5(\mu_5 - m_5)$ distinct values of A_5: and the like as regards the unexpressed factors with the suffixes 6 and 7.

Thirdly, singularities at the θ points $(z = 0,\ y - Ax = 0)$, A having here θ distinct values, at any one of which the tangent is $y - Ax = 0$, and which are denoted by the function

$$\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}:$$

but in the case ultimately considered λ is $= 1$; and these are then the θ ordinary points at infinity, $(z = 0,\ y - Ax = 0)$.

According to the theory explained in my paper above referred to, these several singularities are together equivalent to a certain number $\delta' + \kappa'$ of nodes and cusps; viz. we have

$$\delta' = \tfrac{1}{2}M - \tfrac{3}{2}\Sigma(\alpha - 1),$$
$$\kappa' = \Sigma(\alpha - 1),$$

hence

$$\delta' + \kappa' = \tfrac{1}{2}M - \tfrac{1}{2}\Sigma(\alpha - 1).$$

Assuming that there are no other singularities, the deficiency

$$\tfrac{1}{2}(K-1)(K-2) - \delta' - \kappa'$$

is

$$= \tfrac{1}{2}(K-1)(K-2) - \tfrac{1}{2}M + \tfrac{1}{2}\Sigma(\alpha - 1).$$

This should be equal to the before-mentioned value of γ; viz. we ought to have

$$(K-1)(K-2) - M + \Sigma(\alpha - 1) = 2\sum_{s>r} n_r m_r n_s \mu_s + \Sigma n^2 m\mu - \Sigma nm - \Sigma n\mu - \Sigma n + 2,$$

or, as it will be convenient to write it,

$$M = K^2 - 3K + \Sigma(\alpha - 1) - 2\sum_{s>r} n_r m_r n_s \mu_s - \Sigma n^2 m\mu + \Sigma nm + \Sigma n\mu + \Sigma n,$$

which is the equation which ought to be satisfied by the values of M and $\Sigma(\alpha - 1)$ calculated, according to the method of my paper, for the foregoing singularities of the curve.

We have as before

$$K = \Sigma' nm + \Sigma'' n\mu + \theta\lambda.$$

The term $\sum_{s>r} n_r m_r n_s \mu_s$, written at length, is

$$\begin{aligned}
= \quad & n_1 m_1 (n_2\mu_2 + n_3\mu_3 + \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_2 m_2 (\qquad n_3\mu_3 + \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_3 m_3 (\qquad\qquad \theta\lambda + n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & \theta\lambda \; (\qquad\qquad\qquad n_5\mu_5 + n_6\mu_6 + n_7\mu_7) \\
+ \; & n_5 m_5 (\qquad\qquad\qquad\qquad n_6\mu_6 + n_7\mu_7) \\
+ \; & n_6 m_6 (\qquad\qquad\qquad\qquad\qquad n_7\mu_7),
\end{aligned}$$

which is

$$= \sum_{s>r}' n_r m_r n_s \mu_s + \theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) + \Sigma' nm\,.\,\Sigma'' n\mu + \sum_{s>r}'' n_r m_r n_s \mu_s.$$

We have moreover

$$\begin{aligned}
\Sigma n^2 m\mu &= \Sigma' n^2 m\mu + \theta^2\lambda^2 + \Sigma'' n^2 m\mu,\\
\Sigma nm &= \Sigma' nm + \theta\lambda + \Sigma'' nm,\\
\Sigma n\mu &= \Sigma' n\mu + \theta\lambda + \Sigma'' n\mu,\\
\Sigma n &= \Sigma' n + \theta + \Sigma'' n.
\end{aligned}$$

We next calculate $\Sigma\,(\alpha - 1)$.

For the singularity

$$\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{n_1 (m_1 - \mu_1)} \ldots,$$

each branch $\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{m_1 - \mu_1}$ gives $\alpha = m_1 - \mu_1$, and the value of $\Sigma\,(\alpha - 1)$ for this singularity is

$$n_1 (m_1 - \mu_1 - 1) + n_2 (m_2 - \mu_2 - 1) + n_3 (m_3 - \mu_3 - 1),$$

which is

$$= \Sigma' nm - \Sigma' n\mu - \Sigma' n.$$

For the singularity

$$\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5 (\mu_5 - m_5)} \ldots,$$

each branch $\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{\mu_5 - m_5}$ gives $\alpha = \mu_5 - m_5$, and the value of $\Sigma\,(\alpha - 1)$ for this singularity is

$$n_5 (\mu_5 - m_5 - 1) + n_6 (\mu_6 - m_6 - 1) + n_7 (\mu_7 - m_7 - 1),$$

which is

$$= \Sigma'' n\mu - \Sigma'' nm - \Sigma'' n.$$

For each of the θ singularities

$$\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta},$$

we have $\alpha = \lambda$ and the value of $\Sigma\,(\alpha - 1)$ is $= \theta\,(\lambda - 1)$: this is $= 0$ for the value $\lambda = 1$, which is ultimately attributed to λ.

The complete value of $\Sigma\,(\alpha - 1)$ is thus

$$= \Sigma' nm - \Sigma'' nm - \Sigma' n\mu + \Sigma'' n\mu - \Sigma' n - \Sigma'' n + \theta\lambda - \theta.$$

Substituting all these values, we have

$$\begin{aligned}
M = {} & (\Sigma' nm + \Sigma'' n\mu)^2 + 2\theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) + (\theta\lambda)^2\\
& - 3\,(\Sigma' nm + \Sigma'' n\mu) - 3\theta\lambda\\
& + \Sigma' nm - \Sigma'' nm - \Sigma' n\mu + \Sigma'' n\mu - \Sigma' n - \Sigma'' n + \theta\lambda - \theta\\
& - 2\sum_{s>r}' n_r m_r n_s \mu_s - 2\theta\lambda\,(\Sigma' nm + \Sigma'' n\mu) - 2\Sigma' nm\,.\,\Sigma'' n\mu - 2\sum_{s>r}'' n_r m_r n_s \mu_s\\
& - \Sigma' n^2 m\mu - (\theta\lambda)^2 - \Sigma'' n^2 m\mu\\
& + \Sigma' nm + \theta\lambda + \Sigma'' nm\\
& + \Sigma' n\mu + \theta\lambda + \Sigma'' n\mu\\
& + \Sigma' n + \theta + \Sigma'' n,
\end{aligned}$$

or, reducing,

$$M = (\Sigma' nm)^2 - \Sigma' nm - \Sigma' n^2 m\mu - 2\sum_{s>r}' n_r m_r n_s \mu_s$$
$$+ (\Sigma'' n\mu)^2 - \Sigma'' n\mu - \Sigma'' n^2 m\mu - 2\sum_{s>r}'' n_r m_r n_s \mu_s;$$

and it is to be shown that the two lines of this expression are in fact the values of M belonging to the singularities

$$\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)} \ldots, \text{ and } \left(z - y^{\frac{\mu_3}{\mu_3-m_3}}\right)^{n_3(\mu_3-m_3)} \ldots,$$

respectively. We assume $\lambda = 1$, and there is thus no singularity $\left(y - x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$.

I recall that, considering the several partial branches which meet at a singular point, M denotes the sum of the number of the intersections of each partial branch by every other partial branch: so that for each pair of partial branches the intersections are to be counted *twice*. Supposing that the tangent is $x=0$, and that for any two branches we have $z_1 = A_1 x^{p_1}$, $z_2 = A_2 x^{p_2}$ (where p_1, p_2 are each equal to or greater than 1), then if $p_2 = p_1$, and $z_1 - z_2 = (A_1 - A_2) x^{p_1}$ where $A_1 - A_2$ not $=0$ (an assumption which has been already made as regards the cases about to be considered), then the number of intersections is taken to be $= p_1$; and if p_1 and p_2 are unequal, then *taking p_2 to be the greater of them*, the leading term of $z_1 - z_2$ is $= A_1 x^{p_1}$, and the number of intersections is taken to be $= p_1$; viz. in the case of unequal exponents; it is equal to the smaller exponent.

Consider now the singularity $\left(z - x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)} \ldots$; and first the intersections of a partial branch $z - x^{\frac{m_1}{m_1-\mu_1}}$ by each of the remaining $n_1(m_1-\mu_1)-1$ partial branches of the same set: the number of intersections with any one of these is $= \frac{m_1}{m_1-\mu_1}$; and consequently the number with all of them is $= \frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1]$. But we obtain this same number from each of the $n_1(m_1-\mu_1)$ partial branches, and thus the whole number is

$$n_1(m_1-\mu_1)\frac{m_1}{m_1-\mu_1}[n_1(m_1-\mu_1)-1], = n_1 m_1 [n_1(m_1-\mu_1)-1].$$

Taking account of the other sets, each with itself, the whole number of such intersections is

$$n_1 m_1 [n_1(m_1-\mu_1)-1] + n_2 m_2 [n_2(m_2-\mu_2)-1] + n_3 m_3 [n_3(m_3-\mu_3)-1],$$

which is

$$= \Sigma' n^2 m^2 - \Sigma' n^2 m\mu - \Sigma' nm.$$

Observe now that $\frac{m_1}{\mu_1} > \frac{m_2}{\mu_2}$, that is, $\frac{\mu_1}{m_1} < \frac{\mu_2}{m_2}$, and that, these being each < 1, we thence have $1 - \frac{\mu_1}{m_1} > 1 - \frac{\mu_2}{m_2}$, that is, $\frac{m_1 - \mu_1}{m_1} > \frac{m_2 - \mu_2}{m_2}$: and we thus have

$$\frac{m_1}{m_1 - \mu_1} < \frac{m_2}{m_2 - \mu_2} < \frac{m_3}{m_3 - \mu_3}.$$

Considering now the intersections of partial branches of the two sets

$$\left(z - x^{\frac{m_1}{m_1 - \mu_1}}\right)^{n_1(m_1 - \mu_1)} \quad \text{and} \quad \left(z - x^{\frac{m_2}{m_2 - \mu_2}}\right)^{n_2(m_2 - \mu_2)}$$

respectively, a partial branch $z - x^{\frac{m_1}{m_1 - \mu_1}}$ gives with each partial branch of the other set a number $= \frac{m_1}{m_1 - \mu_1}$; and in this way taking each partial branch of each set, the number is

$$n_1(m_1 - \mu_1) \,.\, n_2(m_2 - \mu_2) \,.\, \frac{m_1}{m_1 - \mu_1}, \; = n_1 m_1 n_2 (m_2 - \mu_2);$$

and thus for all the sets the number is

$$= n_1 m_1 n_2 (m_2 - \mu_2) + n_1 m_1 n_3 (m_3 - \mu_3) + n_2 m_2 n_3 (m_3 - \mu_3),$$

which is

$$= \Sigma' n_r m_r n_s m_s - \sum_{s>r}' n_r m_r n_s \mu_s,$$

where in the first sum the Σ' refers to each pair of values of the suffixes. But the intersections are to be taken twice; the number thus is

$$= 2\Sigma' n_r m_r n_s m_s - 2\sum_{s>r}' n_r m_r n_s \mu_s.$$

Adding the foregoing number

$$\Sigma' n^2 m^2 - \Sigma' n^2 m \mu - \Sigma' n m,$$

the whole number for the singularity in question is

$$= (\Sigma' n m)^2 - \Sigma' n m - \Sigma' n^2 m \mu - 2\sum_{s>r}' n_r m_r n_s \mu_s.$$

Similarly for the singularity $\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5(\mu_5 - m_5)} \ldots$; taking each set with itself, the number of intersections is

$$n_5 \mu_5 [n_5 (\mu_5 - m_5) - 1] + n_6 \mu_6 [n_6 (\mu_6 - m_6) - 1] + n_7 \mu_7 [n_7 (\mu_7 - m_7) - 1],$$

which is

$$= \Sigma'' n^2 \mu^2 - \Sigma'' n^2 m \mu - \Sigma'' n \mu.$$

We have here $\frac{m_5}{\mu_5} > \frac{m_6}{\mu_6}$; each of these being less than 1, we have $1 - \frac{m_5}{\mu_5} < 1 - \frac{m_6}{\mu_6}$, that is, $\frac{\mu_5 - m_5}{\mu_5} < \frac{\mu_6 - m_6}{\mu_6}$, or $\frac{\mu_5}{\mu_5 - m_5} > \frac{\mu_6}{\mu_6 - m_6}$; and so

$$\frac{\mu_7}{\mu_7 - m_7} < \frac{\mu_6}{\mu_6 - m_6} < \frac{\mu_5}{\mu_5 - m_5}.$$

Hence considering the two sets

$$\left(z - y^{\frac{\mu_5}{\mu_5 - m_5}}\right)^{n_5(\mu_5 - m_5)} \text{ and } \left(z - y^{\frac{\mu_6}{\mu_6 - m_6}}\right)^{n_6(\mu_6 - m_6)},$$

a partial branch of the first set gives with a partial branch of the second set $\frac{\mu_6}{\mu_6 - m_6}$ intersections: and the number thus obtained is

$$n_5(\mu_5 - m_5) \,.\, n_6(\mu_6 - m_6) \,.\, \frac{\mu_6}{\mu_6 - m_6}, \; = n_5 n_6 \mu_6 (\mu_5 - m_5).$$

For all the sets the number is

$$n_5 n_6 \mu_6 (\mu_5 - m_5) + n_5 n_7 \mu_7 (\mu_5 - m_5) + n_6 n_7 \mu_7 (\mu_6 - m_6)$$

or taking this twice, the number is

$$= 2\Sigma'' n_r \mu_r n_s \mu_s - 2\sum_{s>r}'' n_r m_r n_s \mu_s$$

where in the first sum the Σ'' refers to each pair of suffixes. Adding the foregoing value

$$\Sigma'' n^2 \mu^2 - \Sigma'' n^2 m \mu - \Sigma'' n \mu,$$

the whole number for the singularity in question is

$$= (\Sigma'' n\mu)^2 - \Sigma'' n\mu - \Sigma'' n^2 m\mu - 2\sum_{s>r}'' n_r m_r n_s \mu_s;$$

and the proof is thus completed.

Referring to the foot-note (ante, p. 31), I remark that the theorem $\gamma =$ deficiency, is absolute, and applies to a curve with any singularities whatever: in a curve which has singularities not taken account of in Abel's theory, the "quelques cas particuliers que je me dispense de considérer," the singularities not taken account of give rise to a diminution in the deficiency of the curve, and also to an equal diminution of the value of γ as determined by Abel's formula; and the actual deficiency will be = Abel's γ − such diminution, that is, it will be = true value of γ.

714.

VARIOUS NOTES.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 69, 115, 124, 125.]

An Identity.

THE following remarkable identity is given under a slightly different form by Gauss, *Werke*, t. III., p. 424,

$$1+\left(\frac{\frac{1}{2}}{1}\right)^3 x+\left(\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\,.\,2}\right)^3 x^2+\left(\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\,.\,2\,.\,3}\right)^3 x^3+\&c.$$

$$=\left\{1+\left(\frac{\frac{1}{4}}{1}\right)^2 x+\left(\frac{\frac{1}{4}\cdot\frac{5}{4}}{1\,.\,2}\right)^2 x^2+\left(\frac{\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}}{1\,.\,2\,.\,3}\right)^2 x^3+\&c.\right\}^2.$$

On two related quadric functions.

Assume

$$\phi x=a^2(c-x)-x(c^2-b^2-cx),$$
$$\psi x=b^2(c-x)-x(c^2-a^2-cx):$$

then

$$\phi\left(\frac{a^2}{c-x}\right)=\frac{a^2}{(c-x)^2}\psi x,$$
$$\psi\left(\frac{b^2}{c-x}\right)=\frac{b^2}{(c-x)^2}\phi x.$$

In the first of these for x write $\dfrac{b^2}{c-x}$; then

$$\phi\left\{\frac{a^2(c-x)}{c^2-b^2-cx}\right\}=\frac{a^2(c-x)^2}{(c^2-b^2-cx)^2}\,\frac{b^2}{(c-x)^2}\,\phi x=\frac{a^2b}{(c^2-b^2-cx)^2}\,\phi(x).$$

A Trigonometrical Identity.

$$\begin{aligned}
&\cos(b-c)\cos(b+c+d)+\cos a\cos(a+d)\\
&=\cos(c-a)\cos(c+a+d)+\cos b\cos(b+d)\\
&=\cos(a-b)\cos(a+b+d)+\cos c\cos(c+d)\\
&=\cos a\cos(a+d)+\cos b\cos(b+d)+\cos c\cos(c+d)-\cos d.
\end{aligned}$$

Extract from a Letter.

"I wish to construct a correspondence such as

$$(x+iy)^3+(x+iy)=X+iY,$$

or, say, for greater convenience

$$4(x+iy)^3-3(x+iy)=X+iY;$$

viz. if

$$x+iy=\cos u,$$

then

$$X+iY=\cos 3u.$$

Suppose $3u_0$ is a value of $3u$ corresponding to a given value of $X+iY$, then the three values of $x+iy$ are of course $\cos u_0$, $\cos\left(u_0\pm\frac{2\pi}{3}\right)$; but I am afraid that the calculation of u_0, even with cosh and sinh tables, would be very laborious. Writing

$$X+iY=R(\cos\Theta+i\sin\Theta),$$

the intervals for Θ might be 5°, 10° or even 15°, those of R, say 0·1 from 0 to 2, and then 0·5 up to 4 or 5; and 2 places of decimals would be quite sufficient; but even this would probably involve a great mass of calculation.

It has occurred to me that perhaps a geometrical solution might be found for the equation $X+iY=\cos 3u$."

October 31, 1877.

715.

NOTE ON A SYSTEM OF ALGEBRAICAL EQUATIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 17, 18.]

ASSUME

$$\begin{aligned} x + y + z &= P, \\ yz + zx + xy &= Q, \\ xyz &= R, \\ A &= x(nyz + Q) - w^2(mx + P), \\ B &= y(nzx + Q) - w^2(my + P), \\ C &= z(nxy + Q) - w^2(mz + P), \\ \Theta &= -mnR + PQ. \end{aligned}$$

Then

$$\begin{aligned} &(mz + P)B - (my + P)C \\ &\quad = (myz + Py)(nzx + Q) - (myz + Pz)(nxy + Q) \\ &\quad = myz(nzx + Q - nxy - Q) + Pnxyz + PQy - Pnxyz - PQz \\ &\quad = mnxyz(z - y) - PQ(z - y) \\ &\quad = (z - y)\{mnxyz - PQ\} = (y - z)\Theta; \end{aligned}$$

whence, identically,

$$\begin{aligned} (mz + P)B - (my + P)C &= (y - z)\Theta, \\ (mx + P)C - (mz + P)A &= (z - x)\Theta, \\ (my + P)A - (mx + P)B &= (x - y)\Theta. \end{aligned}$$

Hence any two of the equations $A = 0$, $B = 0$, $C = 0$ imply the third equation.

We have

$$A = x\{(n+1)yz + zx + xy\} - w^2\{(m+1)x + (y+z)\}$$
$$= (x^2 - w^2)(y+z) - x[(m+1)w^2 - (n+1)yz],$$

and similarly for B and C. The three equations therefore are

$$\frac{x}{x^2 - w^2} = \frac{y+z}{(m+1)w^2 - (n+1)yz},$$

$$\frac{y}{y^2 - w^2} = \frac{z+x}{(m+1)w^2 - (n+1)zx},$$

$$\frac{z}{z^2 - w^2} = \frac{x+y}{(m+1)w^2 - (n+1)xy};$$

and any two of these equations imply the third equation.

716.

AN ILLUSTRATION OF THE THEORY OF THE ϑ-FUNCTIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 27—32.]

If X be a given quartic function of x, and if u, or for convenience a constant multiple αu, be the value of the integral $\int \frac{dx}{\sqrt{(X)}}$ taken from a given inferior limit to the superior limit x; then, conversely, x is expressible as a function of u, viz. it is expressible in terms of ϑ-functions of u, where ϑu, or say $\vartheta(u, \mathfrak{F})$ ($\mathfrak{F}$ a parameter upon which the function depends), is given by definition as the sum of a series of exponentials of u; and it is possible from the assumed equation $\alpha u = \int \frac{dx}{\sqrt{(X)}}$, and the definition of ϑu, to obtain by general theory the actual formulæ for the determination of x as such a function of u.

I propose here to obtain these formulæ, in the case where X is a product of real factors, in a less scientific manner, by connecting the function ϑu (as given by such definition) with Jacobi's function Θ, and by reducing the integral $\int \frac{dx}{\sqrt{(X)}}$ by a linear substitution to the form of an elliptic integral; the object being merely to obtain for the case in question the actual formulæ for the expression of x in terms of ϑ-functions of u.

The definition of ϑu or, when the parameter is expressed, $\vartheta(u, \mathfrak{F})$ is

$$\vartheta u = \Sigma\, (-)^s\, e^{-\mathfrak{F}s^2 + 2isu},$$

where s has all positive or negative integer values, zero included, from $-\infty$ to $+\infty$ (that is, from $-S$ to $+S$, $S = \infty$); the parameter $\mathfrak{F}$, or (if imaginary) its real part, must be positive.

Evidently ϑu is an even function: $\vartheta(-u) = \vartheta u$. Moreover, it is at once seen that we have

$$\vartheta(u+\pi) = \vartheta u,\ \vartheta(u+i\mathfrak{F}) = -e^{\mathfrak{F}-2iu}\,\vartheta u,$$

whence also

$$\vartheta(u+m\pi+ni\mathfrak{F}),$$

where m and n are any positive or negative integers, is the product of ϑu by an exponential factor, or say simply that it is a multiple of ϑu.

Writing $u = -\frac{1}{2}i\mathfrak{F}$, we have $\vartheta(-\frac{1}{2}i\mathfrak{F}) = \vartheta(\frac{1}{2}i\mathfrak{F})$, that is,

$$\vartheta(\tfrac{1}{2}i\mathfrak{F}) = 0,$$

and therefore also

$$\vartheta\{m\pi + (n+\tfrac{1}{2})i\mathfrak{F}\} = 0.$$

The above properties are general, but if $\mathfrak{F}$ be real, then k, K, K', q being as in Jacobi (consequently k being real, positive, and less than 1, and K and K' real and positive), and assuming $\mathfrak{F} = \dfrac{\pi K'}{K}$, or, what is the same thing,

$$q\left(=e^{-\frac{\pi K'}{K}}\right) = e^{-\mathfrak{F}},$$

the function ϑ is given in terms of Jacobi's Θ by the equation $\vartheta u = \Theta\left(\dfrac{2Ku}{\pi}\right)$; or, what is the same thing, $\Theta u = \vartheta\left(\dfrac{\pi u}{2K}\right)$.

We hence at once obtain expressions of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ in terms of ϑ, viz. these are

$$\operatorname{sn} u = \frac{-i}{\sqrt{k}}\quad e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{cn} u = \sqrt{\left(\frac{k'}{k}\right)}\,e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi+\tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{dn} u = \sqrt{k'}\quad \vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi\right) \div \vartheta\left(\frac{\pi u}{2K}\right).$$

Consider now the integral

$$\int_a \frac{dx}{\sqrt{\{(-)x-a\,.\,x-b\,.\,x-c\,.\,x-d\}}},\ = \int_a \frac{dx}{\sqrt{(X)}}\ \text{suppose},$$

where a, b, c, d are taken to be real, and in the order of increasing magnitude, viz. it is assumed that $b-a$, $c-a$, $d-a$, $c-b$, $d-b$, $d-c$ are all positive; x considered as the variable under the integral sign is always real; when it is between a and b or between c and d, X is positive, and we assume that $\sqrt{(X)}$ denotes the positive value of the radical; but if x is between b and c, X is negative, and we assume

that the sign of $\sqrt{(X)}$ is taken so that $\frac{1}{\sqrt{(X)}}$ is equal to a positive multiple of i, and this being so the integral is taken from the inferior limit a to the superior limit x, which is real.

Take x a linear function of y, such that for

$$x = a,\ b,\ c,\ d,$$

$$y = 0,\ 1,\ \frac{1}{k^2},\ \infty,\ \text{respectively},$$

so that, x increasing continuously from a to d, y will increase continuously from 0 to ∞. We have

$$k^2 = \frac{b-a}{d-b} \cdot \frac{d-c}{c-a},$$

$$y = \frac{b-d}{b-a}\,\frac{x-a}{x-d},$$

$$1-y = \frac{d-a}{b-a}\,\frac{x-b}{x-d},$$

$$1-k^2y = \frac{d-a}{c-a}\,\frac{x-c}{x-d};$$

and, thence,

$$\sqrt{(y \,.\, 1-y \,.\, 1-k^2y)} = \frac{d-a}{c-a}\sqrt{\left(\frac{d-b}{c-a}\right)} \cdot \frac{\sqrt{(X)}}{(x-d)^2},$$

where $\sqrt{\left(\frac{d-b}{c-a}\right)}$ is taken to be positive, and the sign of $\sqrt{(X)}$ is fixed as above. Then for y between 0 and 1 or $> \frac{1}{k^2}$, $y \,.\, 1-y \,.\, 1-k^2y$ will be positive, and $\sqrt{(y \,.\, 1-y \,.\, 1-k^2y)}$ will also be positive; but y being between 1 and $\frac{1}{k^2}$, $y \,.\, 1-y \,.\, 1-k^2y$ will be negative, and the sign of the radical is such that $\frac{1}{\sqrt{(y \,.\, 1-y \,.\, 1-k^2y)}}$ is a positive multiple of i.

We have moreover

$$dy = \frac{d-a}{b-a}(d-b)\frac{dx}{(x-d)^2};$$

and therefore

$$\frac{dy}{\sqrt{(y \,.\, 1-y \,.\, 1-k^2y)}} = \sqrt{(d-b \,.\, c-a)}\,\frac{dx}{\sqrt{(X)}},$$

where $\sqrt{(d-b \,.\, c-a)}$ is positive; or, say,

$$\int_0 \frac{dy}{\sqrt{(y \,.\, 1-y \,.\, 1-k^2y)}} = \sqrt{(d-b \,.\, c-a)}\int_a \frac{dx}{\sqrt{(X)}}.$$

Hence, writing $y = z^2 = \operatorname{sn}^2 u$, we have

$$2u = \sqrt{(d-b\,.\,c-a)}\int_a \frac{dx}{\sqrt{(X)}},$$

and it is to be further noticed that to

$$x = a,\ b,\ c,\ d,$$

correspond

$$\operatorname{sn} u = 0,\ 1,\ \frac{1}{k},\ \infty,$$

or we may say

$$u = 0,\quad K,\quad K+iK',\quad 2K+iK'.$$

Writing for shortness

$$\frac{2}{\sqrt{(d-b\,.\,c-a)}} = \alpha,$$

we have

$$\alpha u = \int_a \frac{dx}{\sqrt{(X)}};$$

and moreover

$$\alpha K = \int_a^b \frac{dx}{\sqrt{(X)}},$$

$$\alpha(K+iK') = \int_a^c \frac{dx}{\sqrt{(X)}},$$

$$\alpha(2K+iK') = \int_a^d \frac{dx}{\sqrt{(X)}},$$

or if for a moment we write

$$\int_0^a \frac{dx}{\sqrt{(X)}} = A,\ \&c.,$$

then these equations are

$$\alpha K = B - A,$$
$$\alpha(K+iK') = C - A,$$
$$\alpha(2K+iK') = D - A.$$

Hence $B+C-2A = D-A$, that is, $A-B-C+D=0$, or $B-A = D-C$, that is,

$$\int_a^b \frac{dx}{\sqrt{(X)}} = \int_c^d \frac{dx}{\sqrt{(X)}},$$

where observe as before that $x=a$ to $x=b$, or $x=c$ to $x=d$, X is positive, and the radical $\sqrt{(X)}$ is taken to be positive.

We have also

$$\alpha K = B - A = \int_a^b \frac{dx}{\sqrt{(X)}},$$

$$\alpha iK' = C - B = \int_b^c \frac{dx}{\sqrt{(X)}},$$

where, as before, from b to c, X is negative, and the sign of the radical is such that $\frac{1}{\sqrt{(X)}}$ is a positive multiple of i; the last formula may be more conveniently written

$$\alpha K' = \int_b^c \frac{dx}{\sqrt{(-X)}},$$

where, from b to c, $-X$ is positive, and $\sqrt{(-X)}$ is also taken to be positive.

Collecting the results, we have

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u, \quad \alpha = \frac{2}{\sqrt{(d-b\,.\,c-a)}}, \quad k^2 = \frac{b-a\,.\,d-c}{d-b\,.\,c-a},$$

and also

$$k'^2 = \frac{d-a\,.\,c-b}{d-b\,.\,c-a},$$

and then conversely

$$x = \frac{a(d-b)+d(b-a)\operatorname{sn}^2 u}{(d-b)+(b-a)\operatorname{sn}^2 u};$$

or, what is the same thing,

$$\operatorname{sn}^2 u = \frac{b-d\,.\,x-a}{b-a\,.\,x-d},$$

$$\operatorname{cn}^2 u = \frac{d-a\,.\,x-b}{b-a\,.\,x-d},$$

$$\operatorname{dn}^2 u = \frac{d-a\,.\,x-c}{c-a\,.\,x-d};$$

where, in place of the elliptic functions we are to substitute their ϑ-values; it will be recollected that $\mathfrak{F}$, the parameter of the ϑ-functions, has the value

$$\mathfrak{F}\left(=\frac{\pi K'}{K}\right) = \pi \int_b^c \frac{dx}{\sqrt{(-X)}} \div \int_a^b \frac{dx}{\sqrt{(X)}},$$

and, as before,

$$K = \frac{1}{\alpha}\int_a^b \frac{dx}{\sqrt{(X)}}.$$

Hence, finally, α, k, k', K, $\mathfrak{F}$ denoting given functions of a, b, c, d, if as above

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u,$$

we have conversely

$$\frac{b-d\,.\,x-a}{b-a\,.\,x-d} = -\frac{1}{k}\, e^{-\frac{1}{2}\mathfrak{F}+\frac{i\pi u}{2K}}\, \vartheta^2\left(\frac{\pi u}{2K}+\tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta^2\frac{\pi u}{2K},$$

$$\frac{d-a\,.\,x-b}{b-a\,.\,x-d} = \frac{k'}{k}\, e^{-\frac{1}{2}\mathfrak{F}+\frac{i\pi u}{2K}}\, \vartheta^2\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi+\tfrac{1}{2}i\mathfrak{F}\right) \div \vartheta^2\frac{\pi u}{2K},$$

$$\frac{d-a\,.\,x-c}{c-a\,.\,x-d} = k'\, \vartheta^2\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi\right) \div \vartheta^2\frac{\pi u}{2K},$$

which are the formulæ in question.

The problem is to obtain them (and that in the more general case where a, b, c, d have any given imaginary values) directly from the assumed equation

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u,$$

and from the foregoing definition of the function ϑ.

It may be recalled that the function ϑu is a doubly infinite product

$$\vartheta u = \Pi\Pi \left\{1 - \frac{u}{m\pi + (n + \frac{1}{2})\, i\vartheta}\right\};$$

m and n positive or negative integers from $-\infty$ to $+\infty$; I purposely omit all further explanations as to limits; or, what is the same thing,

$$\vartheta \frac{\pi u}{2K} = \Pi\Pi \left\{1 - \frac{u}{2mK + (2n+1)\, iK'}\right\};$$

and consequently that, disregarding constant and exponential factors, the foregoing expressions of

$$\frac{b-d\,.\,x-a}{b-a\,.\,x-d},\quad \frac{d-a\,.\,x-b}{b-a\,.\,x-d},\quad \frac{d-a\,.\,x-c}{c-a\,.\,x-d},$$

are the squares of the expressions $\frac{X}{W}$, $\frac{Y}{W}$, $\frac{Z}{W}$, where X, Y, Z, W are respectively of the form

$$u\Pi\Pi \left\{1 + \frac{u}{(m,\ n)}\right\},\quad \Pi\Pi \left\{1 + \frac{u}{(\bar{m},\ n)}\right\},$$

$$\Pi\Pi \left\{1 + \frac{u}{(\bar{m},\ \bar{n})}\right\},\quad \Pi\Pi \left\{1 + \frac{u}{(m,\ \bar{n})}\right\},$$

where $(m,\ n) = 2mK + 2niK'$, and the stroke over the m or the n denotes that the $2m$ or the $2n$ (as the case may be) is to be changed into $2m+1$ or $2n+1$. But this is a transformation which has apparently no application to the ϑ-functions of more than one variable.

717.

ON THE TRIPLE THETA-FUNCTIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 48—50.]

As a specimen of mathematical notation, viz. of the notation which appears to me the easiest to *read* and also to *print*, I give the definition and demonstration of the fundamental properties of the triple theta-functions.

Definition.

$$\vartheta(U, V, W) = \Sigma \exp. \Theta,$$

where

$$\Theta = (A, B, C, F, G, H)(l, m, n)^2 + 2(U, V, W)(l, m, n),$$

Σ denoting the sum in regard to all positive and negative integer values from $-\infty$ to $+\infty$ (zero included) of l, m, n respectively.

$\vartheta(U, V, W)$ is considered as a function of the arguments (U, V, W), and it depends also on the parameters (A, B, C, F, G, H).

First Property. $\vartheta(U, V, W) = 0$, for

$$U = \tfrac{1}{2}\{x\pi i + (A, H, G)(\alpha, \beta, \gamma)\},$$
$$V = \tfrac{1}{2}\{y\pi i + (H, B, F)(\alpha, \beta, \gamma)\},$$
$$W = \tfrac{1}{2}\{z\pi i + (G, F, C)(\alpha, \beta, \gamma)\},$$

$x, y, z, \alpha, \beta, \gamma$ being any positive or negative integer numbers, such that $\alpha x + \beta y + \gamma z$ = odd number.

Demonstration. It is only necessary to show that to each term of ϑ there corresponds a second term, such that the indices of the two exponentials differ by an odd multiple of πi.

Taking l, m, n as the integers which belong to the one term, those belonging to the other term are

$$-(l+\alpha),\quad -(m+\beta),\quad -(n+\gamma),$$

(where observe that one at least of the numbers α, β, γ being odd, this system of values is not in any case identical with l, m, n). The two exponents then are

$$\Theta, =(A, B, C, F, G, H)(l, m, n)^2+2(U, V, W)(l, m, n),$$

and

$$\Theta', =(A, B, C, F, G, H)(l+\alpha, m+\beta, n+\gamma)^2-2(U, V, W)(l+\alpha, m+\beta, n+\gamma);$$

viz. the value of Θ' is

$$\begin{aligned} = \quad & (A, B, C, F, G, H)(l, m, n)^2+(A, B, C, F, G, H)(\alpha, \beta, \gamma)^2 \\ & +2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma) \\ & -2(U, V, W)(l+\alpha, m+\beta, n+\gamma), \end{aligned}$$

and we then have

$$\begin{aligned} \Theta'-\Theta = & \, 2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma) \\ & +(A, B, C, F, G, H)(\alpha, \beta, \gamma)^2 \\ & -2(U, V, W)(2l+\alpha, 2m+\beta, 2n+\gamma). \end{aligned}$$

Substituting herein for U, V, W their values, the last term is

$$\begin{aligned} = & -\{(2l+\alpha)x+(2m+\beta)y+(2n+\gamma)z\} \\ & -2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma) \\ & - \quad (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2, \end{aligned}$$

and thence

$$\Theta'-\Theta=-\{(2l+\alpha)x+(2m+\beta)y+(2n+\gamma)z\}\pi i,$$

which proves the theorem.

As to the notation, remark that, after (A, B, C, F, G, H) has been once written out in full, we may instead of

$$(A, B, C, F, G, H)(l, m, n)^2, \text{ \&c., write } (A, \ldots)(l, m, n)^2, \text{ \&c.},$$

and that we may use the like abbreviations

$(A, \ldots)(l, m, n)$, to denote $(A, H, G)(l, m, n)$ respectively,
$(H, \ldots)(l, m, n)$, „ $(H, B, F)(l, m, n)$ „ ,
$(G, \ldots)(l, m, n)$, „ $(G, F, C)(l, m, n)$ „ .

These are not only abbreviations, but they make the formulæ actually clearer, as bringing them into a smaller compass; and I accordingly use them in the demonstration which follows.

Second Property. If U_1, V_1, W_1 denote

$$U + x\pi i + (A, H, G)(\alpha, \beta, \gamma),$$
$$V + y\pi i + (H, B, F)(\alpha, \beta, \gamma),$$
$$W + z\pi i + (G, F, C)(\alpha, \beta, \gamma),$$

respectively, where x, y, z, α, β, γ are any positive or negative integers (zero values admissible), then

$$\vartheta(U_1, V_1, W_1) = \exp.\{-(A, B, C, F, G, H)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W),$$

or say

$$= \exp.\{-(A, \ldots)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W).$$

Demonstration. Writing $\vartheta(U_1, V_1, W_1) = \Sigma.\exp.\Theta_1$, then in the expression of Θ_1 we may in place of l, m, n write $l-\alpha$, $m-\beta$, $n-\gamma$; we thus obtain

$$\begin{aligned}\Theta_1 = (A, \ldots)(l-\alpha, m-\beta, n-\gamma)^2 + \{&(l-\alpha)[U + x\pi i + (A, \ldots)(\alpha, \beta, \gamma)] \\ &+ (m-\beta)[V + y\pi i + (H, \ldots)(\alpha, \beta, \gamma)] \\ &+ (n-\gamma)[W + z\pi i + (G, \ldots)(\alpha, \beta, \gamma)]\},\end{aligned}$$

which is

$$\begin{aligned}= \quad & (A, \ldots)(l, m, n)^2 \\ &+ 2(lU + mV + nW) + 2(lx + my + nz)\pi i + 2(A, \ldots)(l, m, n)(\alpha, \beta, \gamma) \\ &\qquad - 2(A, \ldots)(l, m, n)(\alpha, \beta, \gamma) \\ &- 2(\alpha U + \beta V + \gamma W) - 2(\alpha x + \beta y + \gamma z)\pi i - 2(A, \ldots)(\alpha, \beta, \gamma)^2 \\ &\qquad + (A, \ldots)(\alpha, \beta, \gamma)^2,\end{aligned}$$

which is

$$\begin{aligned}= \quad & (A, \ldots)(l, m, n)^2 + 2(lU + mV + nW) \\ &- (A, \ldots)(\alpha, \beta, \gamma)^2 - 2(\alpha U + \beta V + \gamma W) \\ &+ 2[(l-\alpha)x + (m-\beta)y + (n-\gamma)z]\pi i.\end{aligned}$$

Hence, rejecting the last line, which (as an even multiple of πi) leaves the exponential unaltered, we see that $\vartheta(U_1, V_1, W_1)$ is $= \vartheta(U, V, W)$ multiplied by the factor

$$\exp.\{-(A, \ldots)(\alpha, \beta, \gamma)^2\}.\exp.\{-2(\alpha U + \beta V + \gamma W)\},$$

which is the theorem in question.

In many cases a formula, which belongs to an indefinite number s of letters, is most easily intelligible when written out for three letters, but it is sometimes convenient to speak of the s letters $l, m, \ldots, n$, or even the s letters $l, \ldots, n$, and to write out the formulæ accordingly.

718.

ADDITION TO MR GENESE'S NOTE ON THE THEORY OF ENVELOPES.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 62, 63.]

THE example, although simple, is an instructive one. Introducing z, μ for homogeneity, the equation is

$$\lambda^2 y(y-bz)+2\lambda\mu xy+\mu^2 x(x-az)=0,$$

giving the envelope

$$xy\left[(x-az)(y-bz)-xy\right]=0;$$

that is,

$$xy(bx+ay-abz)z=0;$$

viz. we have thus the four lines

$$x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}-z=0,\quad z=0.$$

Writing these values successively in the equation of the curve, we find respectively

$$\lambda^2 y(y-bz)=0,$$

$$\mu^2 x(x-az)=0,$$

$$(b\lambda-a\mu)^2\frac{xy}{ab}=0,$$

$$(\lambda y+\mu x)^2=0;$$

viz. in each case the equation in λ, μ has (as it should have) two equal roots; but in the first three cases the values are *constant;* viz. we find $\lambda=0$, $\mu=0$, $b\lambda-a\mu=0$, respectively; and the curves $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}-z=0$, are for this reason not proper envelopes.

It is to be remarked that writing in the equation of the parabola these values $\lambda = 0$, $\mu = 0$, $b\lambda - a\mu = 0$ successively, we find respectively

$$x(x - az) = 0,$$

$$y(y - bz) = 0,$$

$$(bx + ay)(bx + ay - abz) = 0;$$

viz. in each case the parabola reduces itself to a pair of lines, one of the given lines and a line parallel thereto through the intersection of the other two lines; the parabola thus becomes a curve having a dp on the line at infinity.

In the fourth case $z = 0$, the equation in λ, μ is $(\lambda y + \mu x)^2 = 0$, giving a variable value $\lambda \div \mu = -x \div y$; hence $z = 0$, the line at infinity is a proper envelope.

The true geometrical result is that the envelope consists of the three points A, B, C, and the line at infinity; a point *quâ* curve of the order 0 and class 1 is not representable by a single equation in point-coordinates, and hence the peculiarity in the form of the analytical result.

719.

SUGGESTION OF A MECHANICAL INTEGRATOR FOR THE CALCULATION OF $\int(Xdx+Ydy)$ ALONG AN ARBITRARY PATH*.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 92—95; *British Association Report*, 1877, pp. 18—20.]

I CONSIDER an integral $\int(Xdx+Ydy)$, where X, Y are each of them a given function of the variables (x, y); $Xdx+Ydy$ is thus not in general an exact differential; but assuming a relation between (x, y), that is, a path of the integral, there is in effect one variable only, and the integral becomes calculable. I wish to show how for any given values of the functions X, Y, but for an arbitrary path, it is possible to construct a mechanism for the calculation of the integral: viz. a mechanism such that, a point D thereof being moved in a plane along a path chosen at pleasure, the corresponding value of the integral shall be exhibited on a dial.

The mechanism (for convenience I speak of it as actually existing) consists of a square block or inverted box, the upper horizontal face whereof is taken as the plane of xy, the equations of its edges being $y=0$, $y=1$, $x=0$, $x=1$ respectively. In the wall faces represented by these equations, we have the endless bands A, A', B, B' respectively; and in the plane of xy, a driving point D, the coordinates of which are (x, y), and a regulating point R, mechanically connected with D, in suchwise that the coordinates of R are always the given functions X, Y of the coordinates of D†; the nature of the mechanical connexion will of course depend upon the particular functions X, Y.

This being so, D drives the bands A and B in such manner that, to the given motions dx, dy of D, correspond a motion dx of the band A and a motion dy of

* Read at the British Association Meeting at Plymouth, August 20, 1877.

† It might be convenient to have as the coordinates of R, not X, Y but ξ, η, determinate functions of X, Y respectively.

the band B; A drives A' with a velocity-ratio depending on the position of the regulator R in suchwise that, the coordinates of R being X, Y, then to the motion dx of A corresponds a motion Xdx of A'; and, similarly, B drives B' with a velocity-ratio depending on the position of R, in suchwise that to the motion dy of B corresponds a motion Ydy of B'. Hence, to the motions dx, dy of the driver D, there correspond the motions Xdx and Ydy of the bands A' and B' respectively; the band A' drives a hand or index, and the band B' drives in the contrary sense a graduated dial, the hand and dial rotating independently of each other about a common centre; the increased reading of the hand on the dial is thus $=Xdx+Ydy$; and supposing the original reading to be zero, and the driver D to be moved from its original position along an arbitrary path to any other position whatever, the reading on the dial will be the corresponding value of the integral $\int(Xdx+Ydy)$.

It is obvious that we might, by means of a combination of two such mechanisms, calculate the value of an integral $\int f(u)\,du$ along an arbitrary path of the complex variable u, $=x+iy$; in fact, writing $f(x+iy)=P+iQ$, the differential is

$$(P+iQ)(dx+idy), \quad =Pdx-Qdy+i(Qdx+Pdy);$$

and we thus require the calculation of the two integrals

$$\int(Pdx-Qdy) \text{ and } \int(Qdx+Pdy),$$

each of which is an integral of the above form. Taking for the path a closed curve, it would be very curious to see the machine giving a value zero or a value different from zero, according as the path did not include or included within it a critical point; it seems to me that this discontinuity would really exhibit itself without the necessity of any change in the setting of the machine.

The ordinary modes of establishing a continuously-variable velocity-ratio between two parts of a machine depend upon friction; and, in particular, this is the case in Prof. James Thomson's mechanical integrator—there is thus of course a limitation of the driving power. It seems to me that a variable velocity-ratio, the variation of which is practically although not strictly continuous, might be established by means of toothed wheels (and so with unlimited driving power) in the following manner.

Consider a revolving wheel A, which by means of a link BC, pivoted to a point B of the wheel A and a point C of a toothed wheel or arc D, communicates a reciprocating motion to D; the extent of this reciprocating motion depending on the distance of B from the centre of A, which distance, or say the half-throw, is assumed to be variable. Here during a half-revolution of A, D moves in one direction, say upwards; and during the other half-revolution of A, D moves in the other direction, say downwards; the extent of these equal and opposite motions varying with the throw. Suppose then that D works a pinion E, the centre of which is not absolutely fixed but is so connected with A that during the first half-revolution of A (or while D is moving upwards), E is in gear with D, and during the second half-revolution of A, or while

D is moving downwards, E is out of gear with D; the continuous rotation of A will communicate an intermittent rotation to E, in such manner nevertheless that, to each entire revolution of A or rotation through the angle 2π, there will (the throw remaining constant) correspond a rotation of E through the angle $n.2\pi$, where the coefficient n depends upon the throw*. And evidently if A be driven by a wheel A', the angular velocity of which is $\frac{1}{\lambda}$ times that of A, then to a rotation of A' through each angle $\frac{2\pi}{\lambda}$, there will correspond an entire revolution of A, and therefore, as before, a rotation of E through the determinate angle $n.2\pi$; hence, λ being sufficiently large to each increment of rotation of A', there corresponds in E an increment of rotation which is $n\lambda$ times the first-mentioned increment; viz. E moves (intermittently and possibly also with some "loss of time" on E coming successively in gear and out of gear with D, or in beats as explained) with an angular velocity which is $=n\lambda$ times the angular velocity of A'. And thus the throw (and therefore n) being variable, the velocity-ratio $n\lambda$ is also variable.

We may imagine the wheel A as carrying upon it a piece L sliding between guides, which piece L carries the pivot B of the link BC, and works by a rack on a toothed wheel α concentric with A, but capable of rotating independently thereof. Then if α rotates along with A, as if forming one piece therewith, it will act as a clamp upon L, keeping the distance of B from the centre of A, that is, the half-throw, constant; whereas, if α has given to it an angular velocity different from that of A, the effect will be to vary the distance in question; that is, to vary the half-throw, and consequently the velocity-ratio of A and E. And, in some such manner, substituting for A and E the bands A and A' of the foregoing description, it might be possible to establish between these bands the required variable velocity-ratio.

* If instead of the wheel or arc D with a reciprocating circular motion, we have a double rack D with a reciprocating rectilinear motion, such that the wheel E is placed between the two racks, and is in gear on the one side with one of them when the rack is moving upwards, and on the other side with the other of them when the rack is moving downwards; then the continuous circular motion of A will communicate to E a continuous circular motion, not of course uniform, but such that to each entire revolution of A or rotation through the angle 2π, there will correspond a rotation of E through an angle $n.2\pi$ as before. This is in fact a mechanical arrangement made use of in a mangle, the double rack being there the follower instead of the driver.

720.

NOTE ON ARBOGAST'S METHOD OF DERIVATIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), p. 158.]

IT is an injustice to Arbogast to speak of his *first* method, as Arbogast's method*. There is really nothing in this, it is the straightforward process of expanding

$$\phi\left(a+bx+\frac{1}{1.2}cx^2+\ldots\right)$$

by the differentiation of ϕu, writing $a, b, c, d, \ldots$ in place of $u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}$, &c. or say in place of u, u', u'', u''', &c. respectively; thus

$$\phi a,\ \phi' a.b,\ \tfrac{1}{2}\{\phi' a.c+\phi'' a.b^2\},\quad \tfrac{1}{6}\left\{\begin{array}{l}\phi' a.d+\phi'' a.\ bc\\ \qquad\quad +\phi'' a.2bc+\phi''' a.b^3\end{array}\right\}$$
$$=\tfrac{1}{6}\ \{\phi' a.d+\phi'' a.3bc+\phi''' a.b^3\},\ \&c.,$$

and in subsequent terms the number of additions necessary for obtaining the numerical coefficients increases with great rapidity.

That which is specifically Arbogast's method, is his *second* method, viz. here the coefficients of the successive powers of x in the expansion of $\phi\,(a+bx+cx^2+dx^3+\ldots)$, are obtained by the rule of the last and the last but one; thus we have

$$\phi a,\ \phi' a.b,\ \phi' a.c+\phi'' a.\tfrac{1}{2}b^2,\quad \phi' a.d+\phi'' a.bc+\phi''' a.\tfrac{1}{6}b^3,\ \&c.,$$

where each numerical coefficient is found directly, without an addition in any case.

* See *Messenger of Mathematics*, vol. VII. (1878), pp. 142, 143.

721.

FORMULÆ INVOLVING THE SEVENTH ROOTS OF UNITY.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 177—182.]

LET ω be an imaginary cube root of unity, $\omega^2+\omega+1=0$, or say $\omega=\frac{1}{2}\{-1+i\sqrt{(3)}\}$; $\alpha^3=-7(1+3\omega)$, $\beta^3=-7(1+3\omega^2)$, values giving $\alpha^3\beta^3=343$, and the cube roots α, β being such that $\alpha\beta=7$; then $\alpha+\beta$, $=\alpha+\frac{7}{\alpha}$, is a three-valued function $\left(\text{since changing the root } \omega \text{ we merely interchange } \alpha \text{ and } \frac{7}{\alpha}\right)$; and if r be an imaginary seventh root of unity, then

$$\begin{aligned}3(r+r^6)&=\alpha+\beta-1,\\3(r^2+r^5)&=\omega\alpha+\omega^2\beta-1,\\3(r^4+r^3)&=\omega^2\alpha+\omega\beta-1.\end{aligned}$$

Any one of these formulæ gives the other two; for observe that we have $\alpha^3=-\alpha\beta(1+3\omega)$, $\beta^3=-\alpha\beta(1+3\omega^2)$, that is, $\alpha^2=-\beta(1+3\omega)$, $\beta^2=-\alpha(1+3\omega^2)$; hence, starting for instance with the first formula, we deduce

$$\begin{aligned}9(r^2+r^5+2)&=\alpha^2+2\alpha\beta+\beta^2-2\alpha-2\beta+1,\\&=-\beta(1+3\omega)+14-\alpha(1+3\omega^2)-2\alpha-2\beta+1,\\&=-\alpha(3+3\omega^2)-\beta(3+3\omega)+15,\\&=3\omega\alpha+3\omega^2\beta+15,\end{aligned}$$

that is,

$$3(r^2+r^5)=\omega\alpha+\omega^2\beta-1;$$

and in like manner by squaring each side of this we have the third formula

$$3(r^4+r^3)=\omega^2\alpha+\omega\beta-1.$$

The foregoing formulæ apply to the combinations $r+r^6$, r^2+r^5, r^4+r^3 of the seventh roots of unity, but we may investigate the theory for the roots themselves $r, r^2, r^3, r^4, r^5, r^6$. These depend on the new radical $\sqrt{(-7)}$ or $i\sqrt{(7)}$; introducing instead hereof X, Y, where

$$X = \tfrac{1}{2}\{-1 + i\sqrt{(7)}\},$$
$$Y = \tfrac{1}{2}\{-1 - i\sqrt{(7)}\},$$

then if

$$A^3 = 6 + 3\omega X + (1 + 3\omega^2) Y,$$
$$B^3 = 6 + 3\omega^2 X + (1 + 3\omega) Y,$$

where

$$AB = i\sqrt{(7)},$$

we have (Lagrange, *Équations Numériques*, p. 294),

$$3r = X + A + B.$$

I found that, in order to bring this into connexion with the foregoing formula, $3(r+r^6) = \alpha + \beta - 1$, where as before $\alpha^3 = -7(1+3\omega)$, $\beta^3 = -7(1+3\omega^2)$, $\alpha\beta = 7$, it is necessary that B, A should be linear multiples of α, β respectively, the coefficients being rational functions of ω, X; and that the actual relations are

$$B = \frac{\alpha}{7}\{4 - \omega + X(1 - 2\omega)\},$$

$$A = \frac{\beta}{7}\{5 + \omega + X(3 + 2\omega)\};$$

in verification of which, it may be remarked that these equations give

$$AB = \frac{\alpha\beta}{49}\{(20 - \omega - \omega^2) + X(17 - 4\omega - 4\omega^2) + X^2(3 - 4\omega - 4\omega^2)\},$$

viz. in virtue of the equation $\omega^2 + \omega + 1 = 0$, the term in $\{\ \}$ is $= 21 + 21X + 7X^2$, $= 7(X^2 + 3X + 3)$, or since $X^2 + X + 2 = 0$, this is $= 7(2X+1)$, $= 7i\sqrt{(7)}$; the equation thus is $7AB = \alpha\beta . i\sqrt{(7)}$, which is true in virtue of $AB = i\sqrt{(7)}$ and $\alpha\beta = 7$. The same relations may also be written

$$-\alpha = B(\omega^2 + X),$$
$$-\beta = A(\omega + X).$$

I found in the first instance

$$3r = X + A + B,$$
$$3r^6 = -1 - X + A(\omega^2 - X) + B(\omega - X),$$
$$3r^2 = X + \omega^2 A + \omega B,$$
$$3r^5 = -1 - X + A(\omega - \omega^2 X) + B(\omega^2 - \omega X),$$
$$3r^4 = X + \omega A + \omega^2 B,$$
$$3r^3 = -1 - X + A(1 - \omega X) + B(1 - \omega^2 X),$$

which in fact gave the foregoing formulæ

$$3(r+r^6) = -1 + \alpha + \beta,$$
$$3(r^2+r^5) = -1 + \omega\alpha + \omega^2\beta,$$
$$3(r^4+r^3) = -1 + \omega^2\alpha + \omega\beta.$$

But there is a want of symmetry in these expressions for r, r^2, &c., inasmuch as the values of r, r^2, r^4 are of a different form from those of r^6, r^5, r^3; to obtain the proper forms, we must for A, B substitute their values in terms of α, β, and we thus obtain

$$3r = X + \frac{\alpha}{7}\{4 - \omega + X(1-2\omega)\} + \frac{\beta}{7}\{5 + \omega + X(3+2\omega)\},$$

$$3r^6 = -1 - X + \frac{\alpha}{7}\{3 + \omega + X(-1+2\omega)\} + \frac{\beta}{7}\{2 - \omega + X(-3-2\omega)\},$$

$$3r^2 = X + \frac{\alpha}{7}\{1 + 5\omega + X(2+3\omega)\} + \frac{\beta}{7}\{-4 - 5\omega + X(-1-3\omega)\},$$

$$3r^5 = -1 - X + \frac{\alpha}{7}\{-1 + 2\omega + X(-2-3\omega)\} + \frac{\beta}{7}\{-3 - 2\omega + X(1+3\omega)\},$$

$$3r^4 = X + \frac{\alpha}{7}\{-5 - 4\omega + X(-3-\omega)\} + \frac{\beta}{7}\{-1 + 4\omega + X(-2+\omega)\},$$

$$3r^3 = -1 - X + \frac{\alpha}{7}\{-2 - 3\omega + X(3+\omega)\} + \frac{\beta}{7}\{1 + 3\omega + X(2-\omega)\};$$

viz. each of the imaginary seventh roots is thus expressed as a linear function of the cubic radicals α, β (involving ω under the radical signs) with coefficients which are functions of ω, X.

Recollecting the equations $\alpha^2 = -\beta(1+3\omega)$, $\beta^2 = -\alpha(1+3\omega^2)$, $\alpha\beta = 7$; $\omega^2+\omega+1=0$, $X^2+X+2=0$; it is clear that, starting for instance from the equation for $3r$, and squaring each side of the equation, we should, after proper reductions, obtain for $9r^2$ an expression of the like form; viz. we thus in fact obtain the expression for $3r^2$; then from the expressions of $3r$ and $3r^2$, multiplying together and reducing, we should obtain the expression for $3r^3$; and so on; viz. from any one of the six equations we can in this manner obtain the remaining five equations.

At the time of writing what precedes I did not recollect Jacobi's paper "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Berliner Monatsber.*, (1837) and *Crelle*, t. XXX. (1846), pp. 166—182; [*Ges. Werke*, t. VI. pp. 254—274]. The starting-point is the following theorem: if x be a root of the equation $\frac{x^p-1}{x-1}=0$, p a prime number, and if g is a prime root of p, and

$$F(\alpha) = x + \alpha x^g + \alpha^2 x^{g^2} + \ldots + \alpha^{p-1} x^{g^{p-2}},$$

where α is any root of $\frac{\alpha^{p-1}-1}{\alpha-1}=0$, we have

$$F(\alpha^m)\,F(\alpha^n) = \psi(\alpha)\,F(\alpha^{m+n}),$$

where $\psi(\alpha)$ is a rational and integral function of α with integral coefficients; or, what is the same thing, if α and β be any two roots of the above-mentioned equation, then

$$F(\alpha)F(\beta)=\psi(\alpha,\ \beta)F(\alpha\beta),$$

where $\psi(\alpha,\ \beta)$ is a rational and integral function of α, β with integral coefficients. As regards the proof of this, it may be remarked that, writing x^3 for x, $F(\alpha)$, $F(\beta)$, and $F(\alpha\beta)$ become respectively $\alpha^{-1}F(\alpha)$, $\beta^{-1}F(\beta)$, $(\alpha\beta)^{-1}F(\alpha\beta)$; hence, $F(\alpha)F(\beta)\div F(\alpha\beta)$ remains unaltered, and it thus appears that the function in question is expressible rationally in terms of the *adjoint* quantities α and β. With this explanation the following extract will be easily intelligible:

"The true form (never yet given) of the roots of the equation $x^p-1=0$ is as follows: The roots, as is known, can easily be expressed by mere addition of the functions $F(\alpha)$. If λ is a factor of $p-1$ and $\alpha^\lambda=1$, then it is further known that $\{F(\alpha)\}^\lambda$ is a mere function of α. But it is only necessary to know those values of $F(\alpha)$ for which λ is *the power of a prime number*. For suppose $\lambda\lambda'\lambda''\ldots$ is a factor of $p-1$; further let λ, λ', λ'', ... be powers of different prime numbers, and α, α', α'', ... prime λth, λ'th, λ''th, ... roots of unity, then

$$F(\alpha\alpha'\alpha''\ldots)=\frac{F(\alpha)F(\alpha')F(\alpha'')\ldots}{\psi(\alpha,\ \alpha',\ \alpha'',\ldots)}$$

where $\psi(\alpha,\ \alpha',\ \alpha'',\ldots)$ denotes a rational and integral function of α, α', α'', ... with integral coefficients. Hence, considering always the $(p-1)$th roots of unity as given, there are contained in the expression for x only radicals, the exponents of which are powers of prime numbers, and products of such radicals. But if λ is a power of a prime number, $=\mu^n$, suppose, the corresponding function $F(\alpha)$ can be found as follows: Assume

$$F(\alpha)F(\alpha^i)=\psi_i(\alpha)F(\alpha^{i+1}):$$

then

$$F(\alpha)\ =\sqrt[\mu]{\{\psi_1(\alpha)\ \psi_2(\alpha)\ \ldots\ \psi_{\mu-1}(\alpha)\ \ F(\alpha^\mu)\}},$$

$$F(\alpha^\mu)=\sqrt[\mu]{\{\psi_1(\alpha^\mu)\psi_2(\alpha^\mu)\ldots\psi_{\mu-1}(\alpha^\mu)F(\alpha^{\mu^2})\}},$$

and so on, up to

$$F(\alpha^{\mu^{n-1}})=\sqrt[\mu]{\{\psi_1(\alpha^{\mu^{n-1}})\psi_2(\alpha^{\mu^{n-1}})\ldots\psi_{\mu-1}(\alpha^{\mu^{n-1}})(-)^{\frac{p-1}{\mu}}p\}},"$$

so that the formulæ contain ultimately μth roots only. It is remarked in a footnote that, when $n=1$, the $\mu-1$ functions can always be reduced to one-sixth part in number, and that by an induction continued as far as $\mu=31$, Jacobi had found that all the functions ψ could be expressed by means of the values of a single one of these functions.

"The $\mu-1$ functions determine, not only the values of all the magnitudes under the radical signs, but also the mutual dependence of the radicals themselves. For replacing α by the different powers of α, one can by means of the values so obtained for these functions rationally express all the μ^n-1 functions $F(\alpha^i)$ by means of the powers of $F(\alpha)$; since all the μ^n-1 magnitudes $\{F(\alpha)\}^i\div F(\alpha^i)$ are each of them

equal to a product of several of the functions $\psi(\alpha)$. Herein consists one of the great advantages of the method over that of Gauss, since in this the discovery of the mutual dependency of the different radicals requires a special investigation, which, on account of its laboriousness, is scarcely practicable for even small primes; whereas the introduction of the functions ψ gives simultaneously the quantities under the radical signs, and the mutual dependency of the radicals. The formation of the functions ψ is obtained by a very simple algorithm, which requires only that one should, from the table for the residues of g^m, form another table giving $g^{m'} = 1 + g^m$ (mod. p), [see Table IV. of the Memoir]. According to these rules one of my auditors [Rosenhain] in a Prize-Essay of the [Berlin] *Academy* has completely solved the equations $x^p - 1 = 0$ for all the prime numbers p up to 103."

I am endeavouring to procure the Prize-Essay just referred to. As an example—which however is too simple a one to fully bring out Jacobi's method, and its difference from that of Gauss—consider the equation for the fifth roots of unity, $x^4 + x^3 + x^2 + x + 1 = 0$. According to Gauss, we have $x + x^4$ and $x^2 + x^3$, the roots of the equation $u^2 + u - 1 = 0$; say $x + x^4 = \frac{1}{2}\{-1 + \sqrt{(5)}\}$, $x^2 + x^3 = \frac{1}{2}\{-1 - \sqrt{(5)}\}$. The first of these, combined with $x \,.\, x^4 = 1$, gives $x - x^4 = \sqrt{[-\frac{1}{2}\{5 + \sqrt{(5)}\}]}$; and thence $4x = -1 + \sqrt{(5)} + \sqrt{[-2\{5 + \sqrt{(5)}\}]}$; if from the second of them, combined with $x^2 \,.\, x^3 = 1$, we were in like manner to obtain the values of x^2 and x^3, it would be necessary to investigate the signs to be given to the radicals, in order that the values so obtained for x^2 and x^3 might be consistent with the value just found for x. For the Jacobian process, observing that a prime fourth root of unity is $\alpha = i$, and writing for shortness F_1, F_2, F_3, F_4 to denote $F(\alpha)$, $F(\alpha^2)$, $F(\alpha^3)$, $F(\alpha^4)$ respectively, these functions are

$$\begin{aligned} F_1 &= x - x^4 + i\,(x^2 - x^3), \\ F_2 &= x + x^4 - \,(x^2 + x^3), \\ F_3 &= x - x^4 - i\,(x^2 - x^3), \\ F_4 &= x + x^4 + \,(x^2 + x^3), \end{aligned}$$

viz. we have $F_4 = -1$, $F_2^2 = 5$, or say $F_2 = \sqrt{(5)}$, $F_1^2 = -(1 + 2i)\,F_2$, $= -(1 + 2i)\sqrt{(5)}$; and similarly $F_3^2 = -(1 - 2i)\,F_2$, $= -(1 - 2i)\sqrt{(5)}$; but also $F_1 F_3 = -5$, so that the values $F_1 = \sqrt{\{-(1 + 2i)\sqrt{(5)}\}}$, $F_3 = \sqrt{\{-(1 - 2i)\sqrt{(5)}\}}$, must be taken consistently with this last equation $F_1 F_3 = \sqrt{(5)}$. The values of F_1, F_2, F_3, F_4 being thus known, the four equations then give simultaneously x, x^4, x^2, x^3, these values being of course consistent with each other. It may be remarked that the form in which x presents itself is

$$4x = -1 + \sqrt{(5)} + \sqrt{\{-(1 + 2i)\sqrt{(5)}\}} + \sqrt{\{-(1 - 2i)\sqrt{(5)}\}},$$

with the before-mentioned condition as to the last two radicals; with this condition we, in fact, have

$$\sqrt{\{-(1 + 2i)\sqrt{(5)}\}} + \sqrt{\{-(1 - 2i)\sqrt{(5)}\}} = \sqrt{[-2\{5 + \sqrt{(5)}\}]},$$

as is at once verified by squaring the two sides.

722.

A PROBLEM IN PARTITIONS.

[From the *Messenger of Mathematics*, vol. VII. (1878), pp. 187, 188.]

TAKE for instance 6 letters; a partition into 3's, such as $abc\,.\,def$ contains the 6 duads ab, ac, bc, de, df, ef. A partition into 2's such as $ab\,.\,cd\,.\,ef$ contains the 3 duads ab, cd, ef. Hence if there are α partitions into 3's, and β partitions into 2's, and these contain all the duads each once and only once, $6\alpha+3\beta=15$, or $2\alpha+\beta=5$. The solutions of this last equation are ($\alpha=0$, $\beta=5$), ($\alpha=1$, $\beta=3$), ($\alpha=2$, $\beta=1$), and it is at once seen that the first two sets give solutions of the partition problem, but that the third set gives no solution; thus we have

$\alpha=0,\quad \beta=5$	$\alpha=1,\quad \beta=3$
$ab\,.\,cd\,.\,ef$	$abc\,.\,def$
$ac\,.\,be\,.\,df$	$ad\,.\,be\,.\,cf$
$ad\,.\,bf\,.\,ce$	$ae\,.\,bf\,.\,cd$
$ae\,.\,bd\,.\,cf$	$af\,.\,bd\,.\,ce.$
$af\,.\,bc\,.\,de$	

Similarly for any other number of letters, for instance 15; if we have α partitions into 5's and β partitions into 3's, then, if these contain all the duads, $4\alpha+2\beta=14$, or what is the same $2\alpha+\beta=7$; if $\alpha=0$, $\beta=7$, the partition problem can be solved (this is in fact the problem of the 15 school-girls): but can it be solved for any other values (and if so which values) of α, β? Or again for 30 letters; if we have α partitions into 5's, β partitions into 3's and γ partitions into 2's; then, if these contain all the duads, $4\alpha+2\beta+\gamma=29$; and the question is for what values of α, β, γ, does the partition-problem admit of solution.

The question is important from its connexion with the theory of groups, but it seems to be a very difficult one.

I take the opportunity of mentioning the following theorem: two non-commutative symbols α, β, which are such that $\beta\alpha = \alpha^2\beta^2$ cannot give rise to a group made up of symbols of the form $\alpha^p\beta^q$. In fact, the assumed relation gives $\beta\alpha^2 = \alpha^2\beta\alpha^2\beta^2$; and hence, if $\beta\alpha^2$ be of the form in question, $=\alpha^x\beta^y$ suppose, we have

$$\alpha^x\beta^y = \alpha^2 . \alpha^x\beta^y . \beta^2, \quad = \alpha^{x+2}\beta^{y+2};$$

that is, $1 = \alpha^2\beta^2$, and thence $\beta\alpha = 1$, that is, $\beta = \alpha^{-1}$, viz. the symbols are commutative, and the only group is that made up of the powers of α.

723.

VARIOUS NOTES.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 45—46, 126, 127.]

An Algebraical Identity: p. 45.

Let a, b, c, f, g, h be the differences of four quantities $\alpha, \beta, \gamma, \delta$, say

$$a,\ b,\ c,\ f,\ g,\ h = \beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta,\ \alpha-\delta,\ \beta-\delta,\ \gamma-\delta;$$

then

$$\begin{aligned} .\quad h-g+a&=0,\\ -h\quad .+f+b&=0,\\ g-f\quad .+c&=0,\\ -a-b-c\quad .&=0. \end{aligned}$$

Now Cauchy's identity

$$(a+b)^7-a^7-b^7=7ab\,(a+b)\,(a^2+ab+b^2)^2,$$

putting therein $a+b=-c$, becomes

$$a^7+b^7+c^7=\quad 7abc\,(\quad bc+ca+ab)^2;$$

hence we have

$$\begin{aligned} .\quad h^7-g^7+a^7&=-7agh\,(-ga+ah-hg)^2,\\ -h^7\quad .+f^7+b^7&=-7bhf\,(-hb+bf-fh)^2,\\ g^7-f^7\quad .+c^7&=-7cfg\,(-fc+cg-gf)^2,\\ -a^7-b^7-c^7\quad .&=-7abc\,(\quad bc+ca+ab)^2; \end{aligned}$$

whence, adding,

$$agh\,(-ga+ah-hg)^2+bhf\,(-hb+bf-fh)^2+cfg\,(-fc+cg-gf)^2+abc\,(bc+ca+ab)^2=0,$$

or, as this may also be written,

$$agh\,(g^2+h^2+a^2)^2+bhf\,(h^2+f^2+b^2)^2+cfg\,(f^2+g^2+c^2)^2+abc\,(a^2+b^2+c^2)^2=0,$$

an identity if a, b, c, f, g, h denote their values in terms of α, β, γ, δ.

Note on a Definite Integral: p. 126.

The integral

$$J=\int_0^1 \frac{k^2x^2dx}{\sqrt{(1-x^2\,.\,1-k^2x^2)}},$$

used by Weierstrass, is at once seen to be $=K-E$; but the proof that the other integral

$$J'=\int_1^{\frac{1}{k}} \frac{k^2x^2dx}{\sqrt{(x^2-1\,.\,1-k^2x^2)}}$$

is $=E'$ is not so immediate.

We have

$$\frac{d}{dy}\,\frac{y\sqrt{(1-y^2)}}{\sqrt{(1-k^2y^2)}}=\frac{1-2y^2+k^2y^4}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{2}}},$$

and thence

$$0=\int_0^1 \frac{(1-2y^2+k^2y^4)\,dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{2}}};$$

viz. replacing the numerator by

$$-\frac{k'^2}{k^2}+\frac{1}{k^2}(1-k^2y^2)^2,$$

this becomes

$$0=-\frac{k'^2}{k^2}\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{2}}}+\frac{1}{k^2}\int_0^1 \frac{(1-k^2y^2)^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{2}}},$$

that is,

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{2}}}=\frac{1}{k'^2}E;$$

or, writing k' for k,

$$\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k'^2y^2)^{\frac{3}{2}}}=\frac{1}{k^2}E'.$$

The integral J' writing therein $x=\dfrac{1}{\sqrt{(1-k'^2y^2)}}$ becomes

$$J'=k^2\int_0^1 \frac{dy}{(1-y^2)^{\frac{1}{2}}(1-k'^2y^2)^{\frac{3}{2}}},$$

viz. its value is thus $=E'$.

On a Formula in Elliptic Functions: p. 127.

Writing $\operatorname{en} u = \frac{\operatorname{cn} u}{\operatorname{dn} u}$, then the formulæ p. 63 of my *Elliptic Functions* give

$$\operatorname{sn}(u+v) = \frac{T-T'}{C-C'}, \quad \operatorname{en}(u+v) = \frac{B+B'}{C-C'};$$

and, substituting for T, T', B, B', and C, C' their values, we obtain

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{en} v + \operatorname{sn} v \operatorname{en} u}{1 + k^2 \operatorname{sn} u \operatorname{en} u \operatorname{sn} v \operatorname{en} v},$$

$$\operatorname{en}(u+v) = \frac{\operatorname{en} u \operatorname{en} v - \operatorname{sn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn} u \operatorname{en} u \operatorname{sn} v \operatorname{en} v},$$

formulæ which, as regards their numerators, correspond precisely with the formulæ,

$$\sin(u+v) = \sin u \cos v + \sin v \cos u$$

and

$$\cos(u+v) = \cos u \cos v - \sin u \sin v,$$

of the circular functions, and which in fact reduce themselves to these on putting $k=0$.

The foregoing formulæ, putting therein $k^2 = -1$, are the formulæ given by Gauss, *Werke*, t. III., p. 404, for the lemniscate functions $\sin \operatorname{lemn}(a \pm b)$ and $\cos \operatorname{lemn}(a \pm b)$; where it is to be observed that these notations do not represent a sine and a cosine, but they are related as the sn and en, viz. that

$$\cos \operatorname{lemn} a = \sqrt{(1 - \sin \operatorname{lemn}^2 a)} \div \sqrt{(1 + \sin \operatorname{lemn}^2 a)}.$$

724.

ON THE DEFORMATION OF A MODEL OF A HYPERBOLOID.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 51, 52.]

THE following is a solution of Mr Greenhill's problem set in the Senate-House Examination, January 14, 1878.

"Prove that, if a model of a hyperboloid of one sheet be constructed of rods representing the generating lines, jointed at the points of crossing; then if the model be deformed it will assume the form of a confocal hyperboloid, and prove that the trajectory of a point on the model will be orthogonal to the system of confocal hyperboloids."

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be points on the generating line of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1,$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-\frac{z_2^2}{c^2}=1,$$

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-\frac{z_1z_2}{c^2}=1;$$

or, what is the same thing, if

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c}=p_1,\ q_1,\ r_1;\quad \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c}=p_2,\ q_2,\ r_2;$$

then

$$p_1^2+q_1^2-r_1^2=1,$$
$$p_2^2+q_2^2-r_2^2=1,$$
$$p_1p_2+q_1q_2-r_1r_2=1.$$

Similarly, if (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) be points on generating line of

$$\frac{\xi^2}{\alpha^2}+\frac{\eta^2}{\beta^2}-\frac{\zeta^2}{\gamma^2}=1,$$

and if

$$\frac{\xi_1}{\alpha}, \frac{\eta_1}{\beta}, \frac{\zeta_1}{\gamma}=\mathrm{p}_1, \mathrm{q}_1, \mathrm{r}_1; \quad \frac{\xi_2}{\alpha}, \frac{\eta_2}{\beta}, \frac{\zeta_2}{\gamma}=\mathrm{p}_2, \mathrm{q}_2, \mathrm{r}_2;$$

then

$$\begin{aligned}\mathrm{p}_1^2+\mathrm{q}_1^2-\mathrm{r}_1^2&=1,\\ \mathrm{p}_2^2+\mathrm{q}_2^2-\mathrm{r}_2^2&=1,\\ \mathrm{p}_1\mathrm{p}_2+\mathrm{q}_1\mathrm{q}_2-\mathrm{r}_1\mathrm{r}_2&=1.\end{aligned}$$

Hence if (x_1, y_1, z_1), (ξ_1, η_1, ζ_1) be corresponding points on the two surfaces, that is, if

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}=\frac{\xi_1}{\alpha}, \frac{\eta_1}{\beta}, \frac{\zeta_1}{\gamma}, =p_1, q_1, r_1,$$

and similarly, if (x_2, y_2, z_2), (ξ_2, η_2, ζ_2) are corresponding points, that is, if

$$\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}=\frac{\xi_2}{\alpha}, \frac{\eta_2}{\beta}, \frac{\zeta_2}{\gamma}=p_2, q_2, r_2;$$

then we have, as before, the system of three equations

$$\begin{aligned}p_1^2+q_1^2-r_1^2&=1,\\ p_2^2+q_2^2-r_2^2&=1,\\ p_1p_2+q_1q_2-r_1r_2&=1.\end{aligned}$$

Then if the two surfaces are confocal, that is, if

$$\alpha^2, \beta^2, -\gamma^2=a^2+h, b^2+h, -c^2+h,$$

we shall have

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2.$$

For this equation is

$$a^2(p_1-p_2)^2+b^2(q_1-q_2)^2+c^2(r_1-r_2)^2=\alpha^2(p_1-p_2)^2+\beta^2(q_1-q_2)^2+\gamma^2(r_1-r_2)^2,$$

that is,

$$(p_1-p_2)^2+(q_1-q_2)^2-(r_1-r_2)^2=0,$$

an equation which is obviously true in virtue of the above system of three equations.

Hence, if on confocal surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad \frac{\xi^2}{a^2+h}+\frac{\eta^2}{b^2+h}-\frac{\zeta^2}{c^2-h}=1,$$

we take two points P_1, P_2 on the first, and Q_1, Q_2 the corresponding points on the second; then P_1, P_2 being on a generating line of the first surface, Q_1, Q_2 will be on a generating line of the second surface, and P_1P_2 will be $=Q_1Q_2$. The same is evidently true for the quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$, where P_1P_2, P_2P_3, P_3P_4, P_4P_1 are generating lines on the first surface: and therefore Q_1Q_2, Q_2Q_3, Q_3Q_4, Q_4Q_1 are generating lines on the second surface, which proves the theorem.

725.

NEW FORMULÆ FOR THE INTEGRATION OF $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 60—62.]

I HAVE found in regard to the differential equation

$$\frac{dx}{\sqrt{(a-x\,.\,b-x\,.\,c-x\,.\,d-x)}}+\frac{dy}{\sqrt{(a-y\,.\,b-y\,.\,c-y\,.\,d-y)}}=0,$$

a system of formulæ analogous to those given, p. 63, of my *Treatise on Elliptic Functions*, for the values of $\operatorname{sn}(u+v)$, $\operatorname{cn}(u+v)$, $\operatorname{dn}(u+v)$. Writing for shortness

$$\mathrm{a}\,,\ \mathrm{b}\,,\ \mathrm{c}\,,\ \mathrm{d}\ =a-x,\ b-x,\ c-x,\ d-x,$$

$$\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1=a-y,\ b-y,\ c-y,\ d-y,$$

and $(bc,\ ad)$ to denote the determinant

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix},$$

and $(cd,\ ab)$, $(bd,\ ac)$ to denote the like determinants; then the formulæ are

$$\sqrt{\left(\frac{a-z}{d-z}\right)}=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{adb_1c_1})}+\sqrt{(\mathrm{a_1d_1bc})}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{(a-b\,.\,a-c)}\,(x-y)}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{abc_1d_1})}+\sqrt{(\mathrm{a_1b_1cd})}\}}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{(a-b\,.\,a-c)}\,\{\sqrt{(\mathrm{acb_1d_1})}+\sqrt{(\mathrm{a_1c_1bd})}\}}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},$$

$$\sqrt{\left(\frac{b-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{(a-c)\sqrt{(\mathrm{bdb_1d_1})}+(b-d)\sqrt{(\mathrm{aca_1c_1})}\}}{(bc,\ ad)},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{\sqrt{(\mathrm{abc_1d_1})}-\sqrt{(\mathrm{a_1b_1cd})}\}}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}(cd,\ ab)}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{\left(\frac{a-b}{a-d}\right)}\{(a-d)\sqrt{(\mathrm{bcb_1c_1})}+(b-c)\sqrt{(\mathrm{ada_1d_1})}\}}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}},$$

$$\sqrt{\left(\frac{c-z}{d-z}\right)}=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{(a-b)\sqrt{(\mathrm{cdc_1d_1})}+(c-d)\sqrt{(\mathrm{aba_1b_1})}\}}{(bc,\ ad)}.$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{\sqrt{(\mathrm{acb_1d_1})}-\sqrt{(\mathrm{a_1c_1bd})}\}}{\sqrt{(\mathrm{adb_1c_1})}-\sqrt{(\mathrm{a_1d_1bc})}},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}\{(a-d)\sqrt{(\mathrm{bcb_1c_1})}-(b-c)\sqrt{(\mathrm{ada_1d_1})}\}}{(a-c)\sqrt{(\mathrm{bdb_1d_1})}-(b-d)\sqrt{(\mathrm{aca_1c_1})}},$$

$$=\frac{\sqrt{\left(\frac{a-c}{a-d}\right)}(bd,\ ac)}{(a-b)\sqrt{(\mathrm{cdc_1d_1})}-(c-d)\sqrt{(\mathrm{aba_1b_1})}}.$$

The twelve equations are equivalent to each other, each giving z as one and the same function of x, y; and regarding z as a constant of integration, any one of the equations is a form of the integral of the proposed differential equation.

Writing in the formulæ $x=a$, b, c, d successively, the formulæ become

$x=a$,	$x=b$,	$x=c$,	$x=d$,
$\frac{a-z}{d-z}=\frac{\mathrm{a_1}}{\mathrm{d_1}}$,	$-\frac{c-a}{d-b}\frac{\mathrm{b_1}}{\mathrm{c_1}}$,	$-\frac{b-a}{d-c}\frac{\mathrm{c_1}}{\mathrm{b_1}}$,	$\frac{a-b.a-c}{d-b.d-c}\frac{\mathrm{d_1}}{\mathrm{a_1}}$,
$\frac{b-z}{d-z}=\frac{\mathrm{b_1}}{\mathrm{d_1}}$,	$-\frac{c-b}{d-a}\frac{\mathrm{a_1}}{\mathrm{c_1}}$,	$\frac{b-a.b-c}{d-a.d-c}\frac{\mathrm{d_1}}{\mathrm{b_1}}$,	$-\frac{a-b}{d-c}\frac{\mathrm{c_1}}{\mathrm{a_1}}$,
$\frac{c-z}{d-z}=\frac{\mathrm{c_1}}{\mathrm{d_1}}$,	$\frac{c-a.c-b}{d-a.d-b}\frac{\mathrm{d_1}}{\mathrm{c_1}}$,	$-\frac{b-c}{d-a}\frac{\mathrm{a_1}}{\mathrm{b_1}}$,	$-\frac{a-c}{d-b}\frac{\mathrm{b_1}}{\mathrm{a_1}}$,

viz. in the first case we have $z=y$, and in each of the other cases z equal to a linear function $\frac{\alpha y+\beta}{\gamma y+\delta}$ of y.

Cambridge, July 3, 1878.

726.

A FORMULA BY GAUSS FOR THE CALCULATION OF LOG 2 AND CERTAIN OTHER LOGARITHMS.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 125, 126.]

GAUSS has given, *Werke*, t. II., p. 501, a formula which is in effect as follows:

$$2^{196} = 10^{59}\left(\frac{1025}{1024}\right)^5\left(\frac{1048576}{1048575}\right)^8\left(\frac{6560}{6561}\right)^3\left(\frac{15624}{15625}\right)^8\left(\frac{9801}{9800}\right)^4,$$

viz. this is

$$= 2^{59}\,.\,5^{59}\left(\frac{5^2\,.\,41}{2^{10}}\right)^5\left(\frac{2^{20}}{5^2\,.\,3\,.\,11\,.\,31\,.\,41}\right)^8\left(\frac{5\,.\,2^5\,.\,41}{3^8}\right)^3\left(\frac{2^3\,.\,3^2\,.\,7\,.\,31}{5^6}\right)^8\left(\frac{3^4\,.\,11^2}{2^3\,.\,5^2\,.\,7^2}\right)^4,$$

where on the right-hand side the several prime factors have the indices following, viz.

2,	index is	$(59+160+15+24-50-12)$	$=196,$
3	„	$(16+\ 16-\ 8-24\qquad\qquad)$	$=0,$
5	„	$(59+\ 10+\ 3-16-48-\ 8)$	$=0,$
7	„	$(\ 8-\ 8\qquad\qquad\qquad\qquad)$	$=0,$
11	„	$(\ 8-\ 8\qquad\qquad\qquad\qquad)$	$=0,$
31	„	$(\ 8-\ 8\qquad\qquad\qquad\qquad)$	$=0,$
41	„	$(\ 5+\ 3-\ 8\qquad\qquad\qquad)$	$=0,$

or the right-hand side is $=2^{196}$ as it should be. The value of $\log 2$ calculated from $2^{196}=10^{59}$ is $\log 2 = \frac{59}{196} = \cdot 301020$, viz. there is an error of a unit in fifth place of decimals. The actual value of 2^{196} has been given me by Mr Glaisher:

$$2^{196} = 10043\ 36277\ 66186\ 89222\ 13726\ 30771$$
$$32266\ 26576\ 37687\ 11142\ 45522\ 06336.*$$

Supposing $\log 2$ calculated by the form, we then have

$$41 = \left(\tfrac{1025}{1024}\right)2^{12} \div 10^2, \text{ giving } \log 41,$$

and

$$3^8 = 10\,.\,\tfrac{6561}{6560}\,.\,2^4\,.\,41, \text{ giving } \log 3;$$

and formulæ may be obtained proper for the calculation of the logarithms of $\frac{11}{7}$, $11\,.\,31$, and $7\,.\,31$.

* The value was deduced by Mr Glaisher from Mr Shanks's value of 2^{193} in his *Rectification of the Circle*, (1853), p. 90.

727.

EQUATION OF THE WAVE-SURFACE IN ELLIPTIC COORDINATES.

[From the *Messenger of Mathematics*, vol. VIII. (1879), pp. 190, 191.]

THE equation of the wave-surface

$$\frac{ax^2}{x^2+y^2+z^2-a}+\frac{by^2}{x^2+y^2+z^2-b}+\frac{cz^2}{x^2+y^2+z^2}=0,$$

when transformed to coordinates p, q, r, such that

$$\frac{x^2}{-a+p}+\frac{y^2}{-b+p}+\frac{z^2}{-c+p}=1,$$

$$\frac{x^2}{-a+q}+\frac{y^2}{-b+q}+\frac{z^2}{-c+q}=1,$$

$$\frac{x^2}{-a+r}+\frac{y^2}{-b+r}+\frac{z^2}{-c+r}=1;$$

(that is, to the elliptic coordinates belonging to the quadric surface $\frac{x^2}{-a}+\frac{y^2}{-b}+\frac{z^2}{-c}=1$), assumes the form

$$(q+r-a-b-c)(r+p-a-b-c)(p+q-a-b-c)=0,$$

(Senate-House Problem, January 14, 1879).

In fact, p, q, r are the roots of the equation

$$\frac{x^2}{-a+u}+\frac{y^2}{-b+u}+\frac{z^2}{-c+u}=1;$$

we have therefore

$$\begin{aligned}(u-p)(u-q)(u-r)=&(u-a)(u-b)(u-c)\\&-x^2(u-b)(u-c)-y^2(u-c)(u-a)-z^2(u-a)(u-b);\end{aligned}$$

whence, writing for shortness

$$\begin{aligned} A &= a+b+c \quad , & P &= p+q+r, \\ B &= bc+ca+ab, & Q &= qr+rp+pq, \\ C &= abc \quad , & R &= pqr, \end{aligned}$$

we have

$$\begin{aligned} x^2 + \quad y^2 + \quad z^2 &= P-A, \\ (b+c)\,x^2+(c+a)\,y^2+(a+b)\,z^2 &= Q-B, \\ bcx^2+ \quad cay^2+ \quad abz^2 &= R-C, \end{aligned}$$

and thence also

$$\begin{aligned} a\,(b+c)\,x^2+b\,(c+a)\,y^2+c\,(a+b)\,z^2 &= B\,(P-A)-(R-C), \\ ax^2+ \quad by^2+ \quad cz^2 &= A\,(P-A)-(Q-B). \end{aligned}$$

The equation of the wave-surface is

$$abc-\{a\,(b+c)\,x^2+b\,(c+a)\,y^2+c\,(a+b)\,z^2\}+(x^2+y^2+z^2)\,(ax^2+by^2+cz^2)=0.$$

By the formulæ just obtained, this is

$$C-[B\,(P-A)-(R-C)]+(P-A)\,[A\,(P-A)-(Q-B)]=0,$$

that is,

$$A^3-2A^2P+A\,(P^2+Q)-(PQ-R)=0,$$

that is,

$$\{A-(q+r)\}\,\{A-(r+p)\}\,\{A-(p+q)=0,$$

or, substituting for A its value $a+b+c$, and reversing the sign of each factor, we have the formula in question.

It is easy to see that, taking a, b, c to be each positive, $(a>b>c)$, and assuming also $p>q>r$, we obtain the different real points of space by giving to these coordinates respectively the different real values from ∞ to a, a to b, and b to c respectively. Hence

	greatest,	least value, is
$q+r$,	$a+b$,	$a+c$,
$r+p$,	∞ ,	$a+c$,
$p+q$,	∞ ,	$a+b$,

so that $r+p$, $p+q$, may be either of them $=a+b+c$, but $q+r$ cannot be $=a+b+c$, that is, $q+r=a+b+c$ does not belong to any real point on the wave-surface. We can only have $r+p$ and $p+q$ each $=a+b+c$, if $p=a+c$, $q=r=b$, and these values belong as is easily shown to the nodes on the wave-surface; hence, the equations $r+p=a+b+c$ and $p+q=a+b+c$ being satisfied simultaneously only at the nodes of the surface, must belong to the two sheets respectively. And it can be shown that $p+r=a+b+c$ belongs to the external sheet, and $p+q=a+b+c$ belongs to the internal sheet. In fact, for the point $(0, 0, \sqrt{a})$, which is on the external sheet, we have $p=a+c$, $q=a$, $r=b$, and therefore $p+r=a+b+c$: for the point $(0, 0, \sqrt{b})$, which is on the internal sheet, either

$$(p=b+c,\ q=a,\ r=b) \quad \text{or} \quad (p=a,\ q=b+c,\ r=c),$$

according as $b+c>a$ or $b+c<a$: but in each case

$$p+q=a+b+c.$$

728.

A THEOREM IN ELLIPTIC FUNCTIONS.

[From the *Proceedings of the London Mathematical Society*, vol. x. (1879), pp. 43—48. Read January 8, 1879.]

THE theorem is as follows:

If $u+v+r+s=0$, then

$$-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s + \operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s - \frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s = -\frac{k'^2}{k^2}.$$

It is easy to see that, if a linear relation exists between the three products, then it must be this relation: for the relation must be satisfied on writing therein $v=-u$, $s=-r$, and the only linear relation connecting $\operatorname{sn}^2 u \operatorname{sn}^2 r$, $\operatorname{cn}^2 u \operatorname{cn}^2 r$, $\operatorname{dn}^2 u \operatorname{dn}^2 r$ is the relation in question

$$-k'^2 \operatorname{sn}^2 u \operatorname{sn}^2 r + \operatorname{cn}^2 u \operatorname{cn}^2 r - \frac{1}{k^2} \operatorname{dn}^2 u \operatorname{dn}^2 r = -\frac{k'^2}{k^2}.$$

A demonstration of the theorem was recently communicated to me by Mr Glaisher; and this led me to the somewhat more general theorem

$$\begin{aligned} &-k'^2 \operatorname{sn}(\alpha+\beta) \operatorname{sn}(\alpha-\beta) \operatorname{sn}(\gamma+\delta) \operatorname{sn}(\gamma-\delta) \\ &+ \quad \operatorname{cn}(\alpha+\beta) \operatorname{cn}(\alpha-\beta) \operatorname{cn}(\gamma+\delta) \operatorname{cn}(\gamma-\delta) \\ &-\frac{1}{k^2} \operatorname{dn}(\alpha+\beta) \operatorname{dn}(\alpha-\beta) \operatorname{dn}(\gamma+\delta) \operatorname{dn}(\gamma-\delta) \\ &= -\frac{k'^2}{k^2} - \frac{2k'^2(\operatorname{sn}^2\alpha - \operatorname{sn}^2\gamma)(\operatorname{sn}^2\beta - \operatorname{sn}^2\delta)}{1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta \,.\, 1-k^2\operatorname{sn}^2\gamma\operatorname{sn}^2\delta}. \end{aligned}$$

In fact, writing herein $\alpha+\gamma=0$, that is, $\gamma=-\alpha$, the right-hand side becomes $=0$; and the arcs on the left-hand side are $\alpha+\beta$, $\alpha-\beta$, $-\alpha+\delta$, $-\alpha-\delta$, which represent any four arcs the sum of which is $=0$.

Writing in the last-mentioned equation x, y, z, w for the sn's of α, β, γ, δ respectively, also

$$\begin{aligned} P &= x^2-y^2, & P_1 &= z^2-w^2,\\ Q &= 1-x^2-y^2+k^2x^2y^2, & Q_1 &= 1-z^2-w^2+k^2z^2w^2,\\ R &= 1-k^2x^2-k^2y^2+k^2x^2y^2, & R_1 &= 1-k^2z^2-k^2w^2+k^2z^2w^2,\\ D &= 1-k^2x^2y^2, & D_1 &= 1-k^2z^2w^2, \end{aligned}$$

the equation is

$$-k'^2\frac{PP_1}{DD_1}+\frac{QQ_1}{DD_1}-\frac{1}{k^2}\frac{RR_1}{DD_1}=-\frac{k'^2}{k^2}-\frac{2k'^2(x^2-z^2)(y^2-w^2)}{DD_1},$$

that is,

$$-k'^2PP_1+QQ_1-\frac{1}{k^2}RR_1+\frac{k'^2}{k^2}DD_1+2k'^2(x^2-z^2)(y^2-w^2)=0.$$

It is easy to verify that the terms of the orders 0, 1, 2, 3 and 4 in x^2, y^2, z^2, w^2 separately destroy each other; for instance, for the terms of the order 2, we have

$$\begin{aligned} &-k'^2(x^2-y^2)(z^2-w^2)+\{(x^2+y^2)(z^2+w^2)+k^2(x^2y^2+z^2w^2)\}\\ &-\frac{1}{k^2}\{k^4(x^2+y^2)(z^2+w^2)+k^2(x^2y^2+z^2w^2)\}\\ &+\frac{k'^2}{k^2}\{-k^2(x^2y^2+z^2w^2)\}+2k'^2(x^2-z^2)(y^2-w^2)=0, \end{aligned}$$

that is,

$$\begin{aligned} -k'^2(x^2-y^2)(z^2-w^2)&+(1-k^2)(x^2+y^2)(z^2+w^2)\\ &+(k^2-1-k'^2)(x^2y^2+z^2w^2)+2k'^2(x^2-z^2)(y^2-w^2)=0; \end{aligned}$$

or, omitting the factor k'^2, this is

$$-(x^2-y^2)(z^2-w^2)+(x^2+y^2)(z^2+w^2)-2(x^2y^2+z^2w^2)+2(x^2-z^2)(y^2-w^2)=0,$$

as it should be.

The theorem in its original form was obtained by me as follows: using the elliptic coordinates p, q, r, such that

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}=1,$$

$$\frac{x^2}{a+q}+\frac{y^2}{b+q}+\frac{z^2}{c+q}=1,$$

$$\frac{x^2}{a+r}+\frac{y^2}{b+r}+\frac{z^2}{c+r}=1;$$

or, what is the same thing,

$$-\beta\gamma x^2 = a + p\,.\,a + q\,.\,a + r,$$
$$-\gamma\alpha y^2 = b + p\,.\,b + q\,.\,b + r,$$
$$-\alpha\beta z^2 = c + p\,.\,c + q\,.\,c + r,$$

where α, β, γ denote $b-c$, $c-a$, $a-b$ respectively; then, treating r as a constant, the coordinates x, y, z will belong to a point on the ellipsoid

$$\frac{x^2}{a+r} + \frac{y^2}{b+r} + \frac{z^2}{c+r} = 1,$$

and the differential equation of the right lines upon this surface is

$$\frac{dp}{\sqrt{a+p\,.\,b+p\,.\,c+p}} = \frac{dq}{\sqrt{a+q\,.\,b+q\,.\,c+q}}.$$

Take x_0, y_0, z_0 the coordinates of a point on the surface, and p_0, q_0 the corresponding values of p, q, so that

$$-\beta\gamma x_0^2 = a + p_0\,.\,a + q_0\,.\,a + r,$$
$$-\gamma\alpha y_0^2 = b + p_0\,.\,b + q_0\,.\,b + r,$$
$$-\alpha\beta z_0^2 = c + p_0\,.\,c + q_0\,.\,c + r,$$

then the equation of the tangent plane at the point (x_0, y_0, z_0) is

$$\frac{xx_0}{a+r} + \frac{yy_0}{b+r} + \frac{zz_0}{c+r} = 1,$$

or, substituting for x^2, x_0^2, &c., their values, we have

$$-\frac{\beta\gamma x x_0}{a+r} = \sqrt{a+p\,.\,a+q\,.\,a+p_0\,.\,a+q_0},\ \&c.,$$

and consequently the equation of the tangent plane is

$$\alpha\sqrt{a+p\,.\,a+q\,.\,a+p_0\,.\,a+q_0} + \beta\sqrt{b+p\,.\,b+q\,.\,b+p_0\,.\,b+q_0}$$
$$+\gamma\sqrt{c+p\,.\,c+q\,.\,c+p_0\,.\,c+q_0} = -\alpha\beta\gamma,$$

the equation of a plane intersecting the ellipsoid in a pair of lines; hence this equation (containing in appearance the two arbitrary constants p_0 and q_0) is the integral of the proposed differential equation.

Writing

$$\operatorname{sn}^2 u = A\,(a+p),\quad \operatorname{cn}^2 u = B\,(b+p),\quad \operatorname{dn}^2 u = C\,(c+p),$$

the values of A, B, C, k are determined; and, assuming for q, p_0, q_0 the like forms with the arguments v, u_0, v_0, the differential equation becomes $du = dv$, having the

integral $u-u_0=v-v_0$; while the foregoing integral equation, on reducing the constant coefficients contained therein, takes the form

$$\begin{aligned} &-k'^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn} u_0 \operatorname{sn} v_0 \\ &+ \quad \operatorname{cn} u \operatorname{cn} v \operatorname{cn} u_0 \operatorname{cn} v_0 \\ &-\frac{1}{k^2} \operatorname{dn} u \operatorname{dn} v \operatorname{dn} u_0 \operatorname{dn} v_0 \\ &= -\frac{k'^2}{k^2}; \end{aligned}$$

viz. this equation holds good if $u-u_0=v-v_0$. And by a change of signs we have the theorem.

If, as above, $u+v+r+s=0$, the theorem gives a linear relation between the three products $\operatorname{sn} u \operatorname{sn} v \operatorname{sn} r \operatorname{sn} s$, $\operatorname{cn} u \operatorname{cn} v \operatorname{cn} r \operatorname{cn} s$, $\operatorname{dn} u \operatorname{dn} v \operatorname{dn} r \operatorname{dn} s$, and regarding at pleasure the sn's, the cn's, or the dn's as rational, one of these products will be rational while the other two will be each of them a quadric radical; and hence, rationalising, we obtain an equation which contains the product in question linearly, and contains besides only the squares of the sn's, cn's, or dn's; that is, we have three such equations containing the three products respectively. Bringing to one side the terms which contain the product, and again squaring, we obtain an equation involving only the squares of the sn's, cn's, or dn's; but the three equations thus obtained represent, it is clear, one and the same rational equation, which may be expressed as an equation between the squares of the sn's, or of the cn's, or of the dn's, at pleasure. This equation may be obtained, as I will show, from the ordinary addition-equations of the elliptic functions, but it is not obvious how to obtain from them the three equations involving the products respectively, and these last have the advantage of being of a degree which is the half of the equation which involves only the squared functions.

Write x, y, z, w for $\operatorname{sn} u$, $\operatorname{sn} v$, $\operatorname{sn} r$, $\operatorname{sn} s$ respectively; then, writing

$$\begin{aligned} A &= x\sqrt{1-y^2 \,.\, 1-k^2y^2}, & \alpha &= z\sqrt{1-w^2 \,.\, 1-k^2w^2}, \\ A' &= y\sqrt{1-x^2 \,.\, 1-k^2x^2}, & \alpha' &= w\sqrt{1-z^2 \,.\, 1-k^2z^2}, \\ P &= x^2-y^2, & \varpi &= z^2-w^2, \\ D &= 1-k^2x^2y^2, & \delta &= 1-k^2z^2w^2, \end{aligned}$$

we have

$$\operatorname{sn}(u+v) = -\operatorname{sn}(r+s),$$

that is,

$$\frac{A+A'}{D} = \frac{P}{A-A'} = -\frac{\alpha+\alpha'}{\delta} = -\frac{\varpi}{\alpha-\alpha'},$$

and consequently

$$\begin{aligned} D\varpi &= -(\alpha-\alpha')(A+A'), \\ P\delta &= -(\alpha+\alpha')(A-A'); \end{aligned}$$

whence

$$D\varpi - P\delta = 2(A\alpha' - A'\alpha),$$

that is,
$$(z^2-w^2)(1-k^2x^2y^2)-(x^2-y^2)(1-k^2z^2w^2)$$
$$=2\{xw\sqrt{1-y^2.1-k^2y^2.1-z^2.1-k^2z^2}-yz\sqrt{1-x^2.1-k^2x^2.1-w^2.1-k^2w^2}\}.$$

Rationalising, we obtain, as mentioned above, an equation containing only the squares x^2, y^2, z^2, w^2; it therefore is of a degree twice that of the equation containing the product $xyzw$. I worked out in this way the equation in (x^2, y^2, z^2, w^2), but the calculation was lost, and the easier way of obtaining it is obviously by means of the equation involving $xyzw$.

We have, by the theorem,
$$-k'^2\,xyzw$$
$$+\sqrt{1-x^2.1-y^2.1-z^2.1-w^2}$$
$$-\frac{1}{k^2}\sqrt{1-k^2x^2.1-k^2y^2.1-k^2z^2.1-k^2w^2}=-\frac{k'^2}{k^2},$$
that is,
$$k'^2(1-k^2xyzw)=k^2\sqrt{1-x^2.1-y^2.1-z^2.1-w^2}$$
$$-\sqrt{1-k^2x^2.1-k^2y^2.1-k^2z^2.1-k^2w^2};$$
and then, writing
$$P=x^2+y^2+z^2+w^2,$$
$$Q=x^2y^2+x^2z^2+x^2w^2+y^2z^2+y^2w^2+z^2w^2,$$
$$R=x^2y^2z^2+x^2y^2w^2+x^2z^2w^2+y^2z^2w^2,$$
$$S=x^2y^2z^2w^2,$$
and using $\sqrt{S}$ to denote the rational function $xyzw$, we have
$$k'^4(1-2k^2\sqrt{S}+k^4S)$$
$$=k^4(1-P+Q-R+S)$$
$$+1-k^2P+k^4Q-k^6R+k^8S$$
$$-2k^2\sqrt{(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)};$$
or, if for a moment the radical is called $\sqrt{\Delta}$, then the factor k^2 divides out, and the equation becomes
$$2\sqrt{\Delta}=2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S+2k'^4\sqrt{S},$$
whence
$$4(1-P+Q-R+S)(1-k^2P+k^4Q-k^6R+k^8S)$$
$$-\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}^2-4k'^8S$$
$$=-2k^4\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\}.$$

The factor k'^4 divides out; omitting it, we have
$$4Q-P^2-4(1+k^2)R+16k^2S+2k^2PR-4(k^2+k^4)PS-k^4R^2+4k^4QS$$
$$=-2\sqrt{S}\{2-(1+k^2)P+2k^2Q-(k^2+k^4)R+2k^4S\},$$
or, as this may also be written,
$$\{(-P^2+4Q-4R)+k^2(-4R+2PR+16S-4PS)+k^4(-R^2+4QS-PS)\}$$
$$=-2\sqrt{S}\{2-P+k^2(-P+2Q-R)+k^4(-R+2S)\},$$
which is the required rational equation involving the product of the sn's.

729.

ON A THEOREM RELATING TO CONFORMABLE FIGURES.

[From the *Proceedings of the London Mathematical Society*, vol. x. (1879), pp. 143—146. Read May 8, 1879.]

CONSIDER two plane figures, say the figure of the points P referred to axes Ox, Oy, and that of the points P' referred to axes Ox', Oy'; and let x, y be the coordinates of P, and x', y' those of P'. If the figures correspond to each other in any manner whatever, P and P' being corresponding points, then we have x', y' each of them a function of x, y; and we may consider the second figure as derived from the first by altering the distance OP in the ratio $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and by rotating it through the angle $\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x}$; say by the Extension $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and by the Rotation $\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x}$; where the Extension and the Rotation are each of them a determinate function of x, y, the coordinates of P.

Passing from the point P to a consecutive point Q, the coordinates of which are $x+dx$, $y+dy$ (the ratio $dy \div dx$ being arbitrary), then the coordinates of the corresponding point Q' will be $x'+dx'$, $y'+dy'$, where

$$dx' = \frac{dx'}{dx}\,dx + \frac{dx'}{dy}\,dy, \quad dy' = \frac{dy'}{dx}\,dx + \frac{dy'}{dy}\,dy.$$

Writing $\frac{dy'}{dx'}$ and $\frac{dy}{dx}$ instead of $dy' \div dx'$ and $dy \div dx$, the expressions

$$\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}, \text{ and } \tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx},$$

will in general have values depending upon that of the arbitrary ratio $dy : dx$. But they may be independent of this ratio; viz. this is the case when x', y' are functions of x, y such that

$$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \frac{dy'}{dy} = \frac{dx'}{dx};$$

and the two figures are then conformable (or conjugate) figures; that is, figures similar as regards corresponding infinitesimal elements of area. We have, in this case,

$$\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}, \text{ and } \tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx},$$

each a determinate function of x, y, the coordinates of P; and we pass from the element PQ to the corresponding element $P'Q'$ by altering the length in the ratio $\sqrt{dx'^2+dy'^2} \div \sqrt{dx^2+dy^2}$, and rotating the element through the angle $\tan^{-1}\frac{dy'}{dx'} - \tan^{-1}\frac{dy}{dx}$; say, this ratio and this angle are the Auxesis and the Streblosis respectively, these being, as already mentioned, functions of x, y only.

Considering now any two conformable figures, say the figure of the points P, and that of the points P'; we have the theorem that we can from the first figure obtain a third conformable figure by means of an Auxesis and a Streblosis which are respectively equal to the Extension and the Rotation by which the second figure is derived from the first.

In fact, if in the three figures respectively we take x, y, x', y', and x'', y'', for the coordinates of the corresponding points P, P', P'', the first and second figures are conformable: and we have therefore

$$\frac{dx'}{dy} = -\frac{dy'}{dx}, \quad \frac{dy'}{dy} = \frac{dx'}{dx};$$

the third figure is to have the Auxesis $\sqrt{x'^2+y'^2} \div \sqrt{x^2+y^2}$, and the Streblosis

$$\tan^{-1}\frac{y'}{x'} - \tan^{-1}\frac{y}{x};$$

viz. writing r for $\sqrt{x^2+y^2}$, we ought to have

$$dx'' = \frac{xx'+yy'}{r^2}\,dx - \frac{xy'-x'y}{r^2}\,dy,$$

$$dy'' = \frac{xy'-x'y}{r^2}\,dx + \frac{xx'+yy'}{r^2}\,dy;$$

and it is therefore to be shown that there exist x'', y'' functions of x, y satisfying these relations; for, this being so, we have

$$\frac{dx''}{dy} = -\frac{dy''}{dx}, \quad \frac{dy''}{dy} = \frac{dx''}{dx};$$

and the third figure is thus conformable with the first.

Writing, for shortness,

$$A = \frac{xx'+yy'}{r^2}, \quad B = -\frac{xy'-x'y}{r^2},$$

the equations are

$$dx'' = Adx - Bdy,$$
$$dy'' = Bdx + Ady;$$

or the conditions for the existence of the functions x'', y'' are

$$\frac{dA}{dy} + \frac{dB}{dx} = 0, \quad \frac{dA}{dx} - \frac{dB}{dy} = 0.$$

We, in fact, have

$$\frac{dA}{dy} + \frac{dB}{dx} = \frac{1}{r^2}\left\{x\left(\frac{dx'}{dy} + \frac{dy'}{dx}\right) + y\left(\frac{dy'}{dy} - \frac{dx'}{dx}\right) + 2y'\right\} - \frac{2}{r^4}\{(xx' + yy')\,y + (xy' - x'y)\,x\}$$
$$= \frac{2y'}{r^2} - \frac{2}{r^4}(x^2 + y^2)\,y' = 0;$$

and similarly

$$\frac{dA}{dx} - \frac{dB}{dy} = \frac{1}{r^2}\left\{x\left(\frac{dx'}{dx} - \frac{dy'}{dy}\right) + y\left(\frac{dy'}{dx} + \frac{dx'}{dy}\right) + 2x'\right\} - \frac{2}{r^4}\{(xx' + yy')\,x - (xy' - x'y)\,y\}$$
$$= \frac{2x'}{r^2} - \frac{2}{r^4}(x^2 + y^2)\,x' = 0;$$

which proves the theorem.

The theorem is closely connected with the theory of the function of an imaginary variable; for, writing the conditions for the conformable figures in the form

$$\frac{dx'}{dx} = \frac{dy'}{dy} = F, \quad \frac{dx'}{dy} = -\frac{dy'}{dx} = -G,$$

we have

$$dx' = Fdx - Gdy,$$
$$dy' = Gdx - Fdy;$$

that is,

$$dx' + idy' = (F + iG)(dx + idy):$$

whence $F + iG$ is a function of $x + iy$, and then by integration $x' + iy'$ is also a function of $x + iy$. In one point of view, any function such as $\phi(x, y) + i\psi(x, y)$ is a function of $x + iy$, for the quantity $x + iy$ is only known by means of its real components x, y; that is, knowing $x + iy$, we know x, y, and therefore also

$$\phi(x, y) + i\psi(x, y);$$

and Cauchy, adopting this definition, introduced the expression "fonction monogène" of $x + iy$, to denote that which is in the more restricted (and the ordinary) sense termed a function of $x + iy$. And MM. Briot and Bouquet, in their "Théorie des fonctions elliptiques" (Paris, 1875), although not using Cauchy's expression *fonction monogène*, but the simple term *fonction*, do this under the qualification stated p. 3:—"Dans tout ce qui suit, nous ne nous occuperons que des fonctions qui admettent une dérivée." Now, a function admitting of a derivative (that is, in the ordinary

sense, a function) of the imaginary variable z, $=x+iy$, is a function such that, for a consecutive value z', $=x+iy+dx+idy$, we have

$$\frac{f(z')-f(z)}{z'-z}$$

$=$ a quantity independent of the ratio of the real components dx, dy of the increment $dx+idy$ of the imaginary variable. Or, what is the same thing, writing $f(z)=x'+iy'$, the condition in order that $x'+iy'$ may be a function of $x+iy$ is

$$dx'+idy'=(F+iG)(dx+idy),$$

where F and G are functions of x and y. It is not part of the condition that $F+iG$ shall be a function of $x+iy$, and it is only a long way further on that the authors prove that this is the case (see the definition of a "*fonction holomorphe*," p. 14; and the proof, p. 137). The last-mentioned equation

$$dx'+idy'=(F+iG)(dx+idy),$$

where F and G are only assumed to be functions of x and y, has, if we represent $x+iy$ by means of the point P with coordinates (x, y), and in like manner $x'+iy'$ by means of the point P' with coordinates (x', y'), the geometrical interpretation that the figures of the points P and P' are conformable figures, that is, figures similar as regards their infinitesimal elements. The foregoing theorem in regard to the Auxesis and the Streblosis is that we can, by means of F and G, construct a third conformable figure,—in fact, the Auxesis and the Streblosis are $=\sqrt{F^2+G^2}$ and $\tan^{-1}\frac{G}{F}$ respectively; and, using these as an Extension and a Rotation, we have the third conformable figure $x''+iy''=(F+iG)(x+iy)$; that is, $(F+iG)(x+iy)$, and therefore also $F+iG$, is a function of $x+iy$,—and we have thus the derivative of a function of $x+iy$ as itself a function of $x+iy$.

It is to be remarked that, although the theorem of the Auxesis and the Streblosis, considered as a property of conformable figures, is not by any means geometrically self-evident, yet the foregoing analytical proof is only a proof conducted by means of real quantities, of what (admitting the theory of imaginary quantities) is in fact self-evident; viz. the analytical conclusion really is that, F, G denoting functions of x, y, then, if $dx'+idy'=(F+iG)(dx+idy)$, that is, if $(F+iG)(dx+idy)$ be a complete differential, then $F+iG$ is a function of $x+iy$.

730.

[ADDITION TO MR SPOTTISWOODE'S PAPER "ON THE TWENTY-ONE COORDINATES OF A CONIC IN SPACE."]

[From the *Proceedings of the London Mathematical Society*, vol. x. (1879), pp. 194—196.]

WRITE

$$\begin{aligned}
U &= (a, b, c, d, f, g, h, l, m, n \between x, y, z, t)^2,\\
U_0 &= (\quad\text{,,}\quad \between \xi, \eta, \zeta, \omega)^2,\\
W &= (\quad\text{,,}\quad \between x, y, z, t \between \xi, \eta, \zeta, \omega),\\
P &= (\alpha, \beta, \gamma, \delta \between x, y, z, t),\\
P_0 &= (\alpha, \beta, \gamma, \delta \between \xi, \eta, \zeta, \omega).
\end{aligned}$$

Then the equation of the cone, having for its vertex the arbitrary point $(\xi, \eta, \zeta, \omega)$, and passing through the conic $U=0$, $P=0$, is

$$UP_0^2 - 2WPP_0 + U_0P^2 = 0.$$

Or if, to put the coefficients ξ, η, ζ, ω in evidence, we write for a moment

$$\begin{aligned}
A &= (a, h, g, l \between x, y, z, t),\\
B &= (h, b, f, m \between \quad\text{,,}\quad),\\
C &= (g, f, c, n \between \quad\text{,,}\quad),\\
D &= (l, m, n, d \between \quad\text{,,}\quad),
\end{aligned}$$

and therefore

$$W = A\xi + B\eta + C\zeta + D\omega;$$

then the equation is

$$\begin{aligned}
U(\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega)^2 - 2P(\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega)(A\xi + B\eta + C\zeta + D\omega)\\
+ P^2(a, b, c, d, f, g, h, l, m, n \between \xi, \eta, \zeta, \omega)^2 = 0.
\end{aligned}$$

And if we expand first in ξ, η, ζ, ω, and then in x, y, z, t, the final result is

	x^2	y^2	z^2	t^2	yz	zx	xy	xt	yt	zt	
ξ^2		C	B	F	$2A'$				$2L$	$2L'$	$=0.$
$+\eta^2$	C		A	G		$2B'$		$2M'$		$2M$	
$+\zeta^2$	B	A		H			$2C'$	$2N$	$2N'$		
$+\omega^2$	F	G	H		$2P'$	$2G'$	$2H'$				
$+\eta\zeta$	$2A'$			$2F'$	$-2A$	$-2C'$	$-2B'$	$2(Q-R)$	$-2M$	$-2N'$	
$+\zeta\xi$		$2B'$		$2G'$	$-2C'$	$-2B$	$-2A'$	$-2L'$	$2(R-P)$	$-2N$	
$+\xi\eta$			$2C'$	$2H'$	$-2B'$	$-2A'$	$-2C$	$-2L$	$-2M'$	$2(P-Q)$	
$+\xi\omega$		$2M'$	$2N$		$2(Q-R)$	$-2L'$	$-2L$	$-2F$	$-2H'$	$-2G'$	
$+\eta\omega$	$2L$		$2N'$		$-2M$	$-2(R-P)$	$-2M'$	$-2H'$	$-2G$	$-2F'$	
$+\zeta\omega$	$2L'$	$2M$			$-2N'$	$-2N$	$2(P-Q)$	$-2G'$	$-2F'$	$-2H$	

In particular, if $\eta=0$, $\zeta=0$, $\omega=0$, then we have the foregoing equation $X=0$; and the like for the equations $Y=0$, $Z=0$, and $W=0$ respectively.

Take a, b, c, f, g, h for the six coordinates of the line through the points

$$\begin{vmatrix} x, & y, & z, & t \\ \xi, & \eta, & \zeta, & \omega \end{vmatrix};$$

that is, write

$$\begin{aligned} \mathrm{a} &= y\zeta - z\eta, & \mathrm{f} &= x\omega - t\xi, \\ \mathrm{b} &= z\xi - x\zeta, & \mathrm{g} &= y\omega - t\eta, \\ \mathrm{c} &= x\eta - y\xi, & \mathrm{h} &= z\omega - t\zeta, \end{aligned}$$

where, of course,

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0.$$

Then the foregoing equation of the cone is

$$\left.\begin{aligned} & A\mathrm{a}^2 + B\mathrm{b}^2 + C\mathrm{c}^2 + F\mathrm{f}^2 + G\mathrm{g}^2 + H\mathrm{h}^2 \\ & - 2A'\mathrm{bc} - 2B'\mathrm{ca} - 2C'\mathrm{ab} + 2F'\mathrm{gh} + 2G'\mathrm{hf} + 2H'\mathrm{fg} \\ & + 2P\mathrm{af} + 2M\mathrm{ag} - 2N'\mathrm{ah} \\ & - 2L'\mathrm{bf} + 2Q\mathrm{bg} + 2N\mathrm{bh} \\ & + 2L\mathrm{cf} - 2M'\mathrm{cg} + 2R\mathrm{ch} \end{aligned}\right\} = 0.$$

And this may be regarded as the equation of the *conic* in terms of the twenty-one coordinates of the conic, and of the six coordinates of an arbitrary line meeting the conic. It is, in fact, the general form of the equation given in the paper—Cayley, "On a new Analytical Representation of a Curve in Space," *Quart. Math. Jour.*, vol. III. (1860), [284; this Collection, vol. IV. p. 453].

731.

ON THE BINOMIAL EQUATION $x^p - 1 = 0$; TRISECTION AND QUARTISECTION.

[From the *Proceedings of the London Mathematical Society*, vol. XI. (1880), pp. 4—17. Read November 13, 1879.]

THE solution of the binomial equation $x^p - 1 = 0$, p a prime number, or, say rather, the equation

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0,$$

depends upon the Jacobian function

$$F\alpha = x^1 + \alpha x^g + \ldots + \alpha^{p-2} x^{g^{p-2}},$$

where g is a prime root of p, α any root whatever of the equation $u^{p-1} - 1 = 0$. Taking e a factor of $p-1$, and f for the complementary factor (that is, $p - 1 = ef$), then, if for α we write α^f, or, what is the same thing, taking $\alpha^f, = \beta$, a root of $u^e - 1 = 0$, we have

$$F\beta = X_0 + \beta X_1 + \ldots + \beta^{e-1} X_{e-1},$$

where $X_0, X_1, \ldots, X_{e-1}$ denote each of them a period or sum of $f, = \dfrac{p-1}{e}$, roots, viz.

$$\begin{aligned} X_0 &= (1, \quad g^e, \quad \ldots, g^{(f-1)e}), \\ X_1 &= (g^e, \quad g^{e+1}, \ldots, g^{(f-1)e+1}), \\ &\vdots \\ X_{e-1} &= (g^{e-1}, \quad g^{2e-1}, \ldots, g^{fe-1}) \end{aligned}$$

(read $X_0 = x^1 + x^{g^e} + \ldots + x^{g^{(f-1)e}}$, and so for the other functions).

We have, of course, $F(1), = X_0 + X_1 + \ldots + X_{e-1}$, the sum of all the roots $= -1$; and, further, the general property that any rational and integral function of these periods is expressible as a sum

$$a_0 X_0 + a_1 X_1 + \ldots + a_{e-1} X_{e-1}$$

with known coefficients

$$a_0, \; a_1, \ldots, a_{e-1}.$$

The several cases $e = 2, 3, 4, \ldots$ may be termed those of the bisection, trisection, quartisection, &c., of the equation; viz.

$e = 2$, there are two periods, X, Y, and $F(-1) = X - Y$;

$e = 3$, three periods, X, Y, Z, and $F\gamma = X + \gamma Y + \gamma^2 Z$, if γ is a root of $u^3 - 1 = 0$;

$e = 4$, four periods, X, Y, Z, W, and $F\delta = X + \delta Y + \delta^2 Z + \delta^3 W$, if δ be a root of $u^4 - 1 = 0$.

It is sufficient to attend to the prime roots γ and δ of the equations

$$u^3 - 1 = 0, \quad u^4 - 1 = 0,$$

respectively; for, if γ or δ be $= 1$, we have simply $F(1)$, $= -1$; and if δ be $= -1$, then the function is $F(-1)$, $= X + Z - (Y + W)$, where $X + Z$ and $Y + W$ are the periods for the bisection. The prime roots δ are of course i and $-i$, and we have

$$F(i) \quad = X + iY - Z - iW,$$
$$F(-i) = X - iY - Z + iW,$$

respectively.

As regards the bisection, it is known that $(X - Y)^2 = (-)^{\frac{p-1}{4}} p$, which is $+p$ or $-p$, according as p is $\equiv 1$ or 3, mod. 4; and the values of X, Y are thus determined. In what follows, I consider the cases $e = 3$ and $e = 4$ of the trisection and the quartisection respectively.

It is to be remembered that, not the division into periods, but the order of the periods, depends on the choice of g, a prime root at pleasure of p; and, in what follows, I select the prime root used in Reuschle's *Tafeln complexer Primzahlen welche aus Wurzeln der Einheit gebildet sind* (4to, Berlin, 1875): viz. these are

$$p = 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,$$
$$59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 83,\ 89,\ 97,$$
$$g = 2,\ 2,\ 5,\ 2,\ 2,\ 3,\ 2,\ -2,\ 2,\ 3,\ 2,\ 6,\ 3,\ 10,\ 2,$$
$$2,\ 2,\ 2,\ 62,\ 5,\ 3,\ 2,\ 30,\ 10,$$

where I quote the whole series, although I am here only concerned with the values of p which are $\equiv 1$ (mod. 3), or $\equiv 1$ (mod. 4).

The periods are consequently those of Reuschle, viz. X, Y, Z are his η_0, η_1, η_2, and X, Y, Z, W his η_0, η_1, η_2, η_3: they can of course, without referring to his work, be easily recalculated, but it is, I think, convenient to have for his values of g the series of residues such as are given (for differently selected values of g) in Jacobi's *Canon Arithmeticus* (4to, Berlin, 1839); and I have accordingly taken out of Reuschle, and annex, such a table.

For instance, $p = 13$, the powers of g are 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7; and, by writing these down in order in columns of 3 or of 4,

1	8	12	5	1	3	9
2	3	11	10	2	6	5
4	6	9	7	4	12	10
				8	11	7

we have the periods X, Y, Z or X, Y, Z, W, belonging to the trisection and the quartisection of $p = 13$.

I further remark that the equations which I am concerned with are all given in Reuschle, but in a somewhat different form; thus, $p = 13$, quartisection (see p. 13), he has

$$\eta_0^2 = \eta_1 + 2\eta_2, \quad \eta_0\eta_1 = -1 - \eta_2, \quad \eta_0\eta_2 = 3 + \eta_1 + \eta_3, \quad \eta_0\eta_3 = -1 - \eta_1,$$

(where observe that here and in every case the value of $\eta_0\eta_3$ is at once obtained from that of $\eta_0\eta_1$ by a mere cyclical interchange of the suffixes, so that the last equation is in fact superfluous); the other equations, using $\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1$ to eliminate any constant term which occurs, give my values

$$\begin{aligned} X^2 &= (\ 0,\ \ 1,\ \ 2,\ \ 0)(X,\ Y,\ Z,\ W), \\ XY &= (\ 1,\ \ 1,\ \ 0,\ \ 1)(\quad ,, \quad), \\ XZ &= (-3,\ -2,\ -3,\ -2)(\quad ,, \quad). \end{aligned}$$

Similarly, in the case of a trisection, the equation for $\eta_0\eta_2$ is superfluous, and the other equations give my values of X^2 and XY.

Reuschle gives also, and I take from him, the cubic and the quartic equations (such as $p = 13$, $\eta^3 + \eta^2 - 4\eta + 1 = 0$, $\eta^4 + \eta^3 + 2\eta^2 - 4\eta + 3 = 0$), which determine the periods in the trisections and the quartisections respectively.

Many of the results obtained accord with, and furnish exemplifications of general theorems contained in Jacobi's memoir, "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie," *Crelle*, t. XXX. (1846), pp. 166—189; [*Ges. Werke*, t. VI. pp. 254—274].

Trisection, $e = 3$; $p \equiv 1$ (*mod.* 3).

We have three periods X, Y, Z; and we thence obtain

$$\begin{aligned} X^2 &= (a,\ b,\ c)(X,\ Y,\ Z), \\ XY &= (f,\ g,\ h)(\quad ,, \quad), \end{aligned}$$

the coefficients a, b, c, f, g, h being determinate integers. And, by cyclical interchanges, we obtain equations which may be written

$$\begin{aligned} X^2 &= a,\ b,\ c, \\ Y^2 &= c,\ a,\ b, \\ Z^2 &= b,\ c,\ a, \\ XY &= f,\ g,\ h, \\ YZ &= h,\ f,\ g, \\ ZX &= g,\ h,\ f; \end{aligned}$$

viz. here and elsewhere the coefficients a, b, c are written to denote the sum

$$aX + bY + cZ.$$

It is easy to see that

$$f + g + h = \tfrac{1}{3}(p - 1);$$

in fact, a period contains $\frac{1}{3}(p-1)$ terms, and in two consecutive periods X, Y, there are no terms the product of which is unity; hence XY contains $\frac{1}{9}(p-1)^2$ terms, each a power of x, and the sum $XY+YZ+ZX$ contains $\frac{1}{3}(p-1)^2$ such terms, being in fact the sum $X+Y+Z$ taken $\frac{1}{3}(p-1)$ times; whence the relation in question.

Hence also

$$YZ+ZX+XY=-\tfrac{1}{3}(p-1).$$

From the equation $X+Y+Z=-1$, multiplying by X, and for X^2, XY, XZ substituting their values, we obtain an expression

$$(a+f+g+1)X+(b+g+h)Y+(c+h+f)Z,$$

which must identically vanish; viz. the three coefficients must be each of them $=0$; or we must have

$$\begin{aligned} a &= -f-g-1, \\ b &= -g-h, \\ c &= -h-f; \end{aligned}$$

so that, taking f, g, h as known, the other coefficients a, b, c are given in terms of them. The equations give

$$a+b+c=-2(f+g+h)-1.$$

We have $X.YZ=Y.ZX$; that is, $X(h, f, g)=Y(g, h, f)$; or, substituting for X^2, XY, &c. their values,

$$\begin{aligned} h(a, b, c) = \;& g(f, g, h) \\ +f(f, g, h) \;& +h(c, a, b) \\ +g(g, h, f) \;& +f(h, f, g); \end{aligned}$$

that is,

$$\begin{aligned} ah+f^2+g^2 &= gf+ch+fh, \\ bh+fg+gh &= g^2+ah+f^2, \\ ch+fh+fg &= gh+bh+fg, \end{aligned}$$

equations which reduce themselves to the single equation

$$gh+hf+fg+h=f^2+g^2+h^2;$$

and this is the only relation obtainable by consideration of the three equal values

$$X.YZ, \quad Y.ZX, \quad Z.XY.$$

Moreover, this equation being satisfied, the six functions in the three equations become each of them $=fg-h^2$; or we have

$$XYZ=(fg-h^2,\ fg-h^2,\ fg-h^2);$$

that is,

$$XYZ=h^2-fg.$$

We have

$$\begin{aligned} F\gamma.F\gamma^2 = \;& X^2+Y^2+Z^2-YZ-ZX-XY \\ = \;& (a+b+c-f-g-h)(X+Y+Z) \\ = \;& -(a+b+c)+(f+g+h) \\ = \;& 3(f+g+h)+1; \end{aligned}$$

that is,

$$F\gamma \,.\, F\gamma^2 = p.$$

We have, moreover,

$$\begin{aligned}(F\gamma)^2 = \quad & X^2 + 2YZ + \gamma(Z^2 + 2XY) + \gamma^2(Y^2 + 2ZX) \\ = \quad & [(a,\ b,\ c) + 2(h,\ f,\ g)] \\ & + \gamma\,[(b,\ c,\ a) + 2(f,\ g,\ h)] \\ & + \gamma^2[(c,\ a,\ b) + 2(g,\ h,\ f)],\end{aligned}$$

which is

$$= \{(a + 2h) + \gamma(b + 2f) + \gamma^2(c + 2g)\}(X + \gamma^2 Y + \gamma Z),$$

as is at once seen by comparing the coefficients of X, Y, Z respectively.

Hence, writing

$$\begin{aligned}& a + 2h + \gamma(b + 2f) + \gamma^2(c + 2g) \\ & = a + 2h + \gamma(b + 2f) - (1 + \gamma)(c + 2g) \\ & = A + B\gamma,\end{aligned}$$

we have

$$\begin{aligned}A &= a + 2h - c - 2g = 3h - 3g - 1, \\ B &= b + 2f - c - 2g = 3f - 3g.\end{aligned}$$

We have

$$(F\gamma)^2 = (A + B\gamma)\,F\gamma^2,$$

and thence, writing γ^2 for γ,

$$(F\gamma^2)^2 = (A + B\gamma^2)\,F\gamma,$$

equations which give

$$F\gamma \,.\, F\gamma^2,\ = p,\ = (A + B\gamma)(A + B\gamma^2);$$

or, say $p = A^2 - AB + B^2$; viz. p has the complex factor

$$A + B\gamma,\ = 3h - 3g - 1 + \gamma(3f - 3g).$$

Hence also

$$\begin{aligned}(F\gamma)^2 &= p(A + B\gamma), \\ (F\gamma^2)^2 &= p(A + B\gamma^2),\end{aligned}$$

and, as before,

$$F\gamma \,.\, F\gamma^2 = p;$$

which equations determine $F\gamma$, $F\gamma^2$, and from these and $F(1) = -1$ we obtain the periods X, Y, Z; we have thus, in fact, the solution of the cubic equation which gives these periods. We have already found the coefficients of this cubic equation, viz.

$$X + Y + Z = -1, \quad YZ + ZX + XY = -\tfrac{1}{3}(p - 1), \quad XYZ = h^2 - fg;$$

the equation thus is

$$\eta^3 + \eta^2 - \tfrac{1}{3}(p - 1)\eta + (fg - h^2) = 0.$$

As already remarked, the values of a, b, c; f, g, h, and the equations in η, are in effect given in Reuschle; the complex factors of p, as given p. 1 ($7 = 2\gamma - 3\gamma^2$, &c.), when reduced to the form $A + B\gamma$, are not identical with the $A + B\gamma$ of the foregoing theory; viz. this $A + B\gamma$ is not Reuschle's selected primary form. I give, in the annexed table

for the primes 7, 13, ..., to 97, the values from Reuschle of a, b, c; f, g, h, and of the coefficients of the η-equation; also the values of A and B derived from f, g, h by the foregoing formulæ. It will be seen that all the values are consistent with the theory.

TABLE FOR THE TRISECTION.

p	a, f,	b, g,	c h	$\eta^3+\eta^2+$ η^1	η^0	A	B	Page in Reuschle
7	− 2	− 1	− 2	− 2	− 1	2	3	p. 6
	1	0	1					
13	− 4	− 3	− 2	− 4	− 1	− 4	− 3	p. 15
	1	2	1					
19	− 4	− 5	− 4	− 6	− 7	2	− 3	p. 26
	1	2	3					
31	− 7	− 6	− 8	− 10	− 8	5	6	p. 45
	4	2	4					
37	− 8	− 10	− 7	− 12	11	− 4	3	p. 54
	5	4	3					
43	− 11	− 8	− 10	− 14	8	− 1	6	p. 69
	6	4	4					
61	− 14	− 13	− 15	− 20	− 9	− 4	− 9	p. 97
	5	8	7					
67	− 16	− 13	− 16	− 22	5	2	9	p. 105
	9	6	7					
73	− 16	− 18	− 15	− 24	− 27	− 1	− 9	p. 128
	6	9	9					
79	− 20	− 17	− 16	− 26	41	− 10	− 3	p. 138
	9	10	7					
97	− 20	− 23	− 22	− 32	− 79	11	3	p. 168
	10	9	13					

Quartisection, $e = 4$; $p \equiv 1$ (*mod.* 4).

We have four periods X, Y, Z, W; and we obtain

$$\begin{aligned} X^2 &= (a, b, c, d)(X, Y, Z, W), \\ XY &= (f, g, h, k)(\quad \text{"} \quad), \\ XZ &= (l, m, l, m)(\quad \text{"} \quad), \end{aligned}$$

the coefficients being determinate integers. It can be shown that $l + m = \frac{1}{8}(p-1)$ or $-\frac{1}{8}(3p+1)$ according as $p \equiv 1$ or 5 (mod. 8). And then, by cyclical interchanges,

$$\begin{aligned} X^2 &= a, b, c, d, \\ Y^2 &= d, a, b, c, \\ Z^2 &= c, d, a, b, \\ W^2 &= b, c, d, a, \\ XY &= f, g, h, k, \\ YZ &= k, f, g, h, \\ ZX &= h, k, f, g, \\ XW &= g, h, k, f, \\ XZ &= l, m, l, m, \\ YW &= m, l, m, l. \end{aligned}$$

We have, in like manner as for the trisection,

$$f + g + h + k = \tfrac{1}{4}(p-1),$$

and so also the expression for

$$\Sigma XY, = XY + XZ + XW + YZ + YW + ZW$$

is

$$= -(f + g + h + k + l + m) = -\tfrac{1}{4}(p-1) - l - m;$$

and, in virtue of the foregoing value of $l + m$, this is $= -\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p \equiv 1$ or 5 (mod. 8).

Again, from the equation $X + Y + Z + W = -1$, multiplying by X and reducing,

$$\begin{aligned} a &= -1 - f - g - l, \\ b &= \quad\; - g - h - m, \\ c &= \quad\; - h - k - l, \\ d &= \quad\; - k - f - m, \end{aligned}$$

and thence

$$a + b + c + d = -1 - 2(f + g + h + k) - 2(l + m),$$

and

$$a - b + c - d = -1 + 2(m - l).$$

We have

$$X.YZ = Y.ZX = Z.XY,$$

that is,

$$X(k, f, g, h) = Y(l, m, l, m) = Z(f, g, h, k),$$

and thence

$$\begin{aligned} k(a, b, c, d) = {} & l(f, g, h, k) = f(l, m, l, m) \\ + f(f, g, h, k) & + m(d, a, b, c) + g(k, f, g, h) \\ + g(l, m, l, m) & + l(k, f, g, h) + h(c, d, a, b) \\ + h(g, h, k, f) & + m(m, l, m, l) + k(h, k, f, g), \end{aligned}$$

that is,

$$\begin{aligned} ka + f^2 + gl + gh &= lf + md + lk + m^2 = lf + gk + ch + kh, \\ kb + fg + gm + h^2 &= lg + am + lf + ml = fm + fg + hd + k^2, \\ kc + fh + gl + hk &= lh + mb + lg + m^2 = fl + g^2 + ah + kf, \\ kd + fk + gm + fh &= kl + mc + lh + lm = fm + gh + bh + gk, \end{aligned}$$

in which equations a, b, c, d may be regarded as having their foregoing values.

One of these equations is

$$kc + fh + gl + hk = lf + g^2 + ah + kf,$$

that is,

$$-k(h + k + l) + fh + gl + hk = lf + g^2 - h(f + g + l + 1) + kf,$$

or, reducing,

$$l(g + h - f - k) = g^2 + k^2 - 2hf - hg - h + kf,$$

which gives l.

Again, another equation is

$$kb + fg + gm + h^2 = fm + fg + hd + k^2,$$

that is,

$$-k(g + h + m) + fg + gm + h^2 = fm + fg - h(k + f + m) + k^2,$$

or, reducing,

$$m(g + h - f - k) = k^2 - h^2 + gk - hf,$$

which gives m.

And we have also

$$md + lk + m^2 = gk + ch + kh,$$

that is,

$$-m(k + f + m) + lk + m^2 = gk + kh - h(h + k + l),$$

or, reducing,

$$l(k + h) - m(f + k) = gk - h^2.$$

Substituting herein for l, m their values, we have

$$(k + h)[g^2 + k^2 - 2hf - hg + kf - h] - (f + k)[k^2 - h^2 + gk - hf] + (h^2 - gk)[g + h - f - k] = 0.$$

In this equation the only terms of the second order are $-h(h+k)$, which contain the factor h; the terms of the third order contain this same factor h, and throwing it out, and reducing, the equation is found to be

$$(g-k)^2 + (h-f)^2 = h + k,$$

or, as it may also be written,

$$g^2 + k^2 - 2hf - h + (h^2 + f^2 - 2gk - k) = 0;$$

and the foregoing values of l, m are

$$l = \frac{(g^2 + k^2 - 2hf - h) - (gh - kf)}{g + h - k - f},$$

$$m = \frac{k^2 - h^2 + gk - hf}{g + h - k - f};$$

and by means of these three equations all the foregoing equations are satisfied.

We have

$$\begin{aligned} FiFi^3 &= (X-Z)^2 + (Y-W)^2 \\ &= X^2 + Y^2 + Z^2 + W^2 - 2(XZ + YW) \\ &= -(a+b+c+d) + 2(l+m); \end{aligned}$$

or, substituting for a, b, c, d, this is

$$= 1 + 2(f+g+h+k) + 4(l+m),$$

viz. it is

$$= \tfrac{1}{2}(p+1) + 4(l+m);$$

or, substituting for $l+m$ its before-mentioned value, then, according as $p \equiv 1$ or 5 (mod. 8), the value is $=p$ or $-p$; that is, we have

$$FiFi^3 = (-)^{\frac{p-1}{4}} p.$$

Again, we have

$$\begin{aligned} (Fi)^2 &= (X + iY - Z - iW)^2 \\ &= X^2 - Y^2 + Z^2 - W^2 - 2XZ + 2YW + 2i(XY - YZ + ZW - WX) \\ &= \{a - b + c - d + 2(m-l) + 2(f-g+h-k)i\}(X - Y + Z - W) \\ &= (A + Bi)F(-1), \end{aligned}$$

where

$$A = a - b + c - d + 2(m-l), = -1 + 4(m-l),$$

$$B = 2(f - g + h - k);$$

or, since $X - Y + Z - W = F(-1)$, this equation is

$$(Fi)^2 = (A + Bi)F(-1):$$

and similarly

$$(Fi^3)^2 = (A - Bi)F(-1).$$

Moreover

$$[F(-1)]^2 = (-)^{\frac{p-1}{2}} p, = p;$$

and we have therefore

$$(\pm p)^2 = (A^2 + B^2) p,$$

that is,

$$A^2 + B^2 = p;$$

or the expression $A + Bi$ determined as above is a complex factor of p.

We may investigate the quartic equation for the determination of the periods X, Y, Z, W. The values of $X+Y+Z+W$ and $XY+XZ+XW+YZ+YW+ZW$ are already known: for the next coefficient $XYZ+XYW+XZW+YZW$, we have $XYZ = (\alpha, \beta, \gamma, \delta)$, where each of the coefficients α, β, γ, δ is given under three different forms: the values of YZW, ZWX, WXY are $(\delta, \alpha, \beta, \gamma)$, $(\gamma, \delta, \alpha, \beta)$, $(\beta, \gamma, \delta, \alpha)$; and the required sum therefore is

$$(\alpha+\beta+\gamma+\delta)(X+Y+Z+W), = -(\alpha+\beta+\gamma+\delta).$$

Taking the first expressions of these coefficients respectively, we have

$$\begin{aligned}
\alpha+\beta+\gamma+\delta = \quad & k(a+b+c+d) \\
& +f(f+g+h+k) \\
& +2g(l+m) \\
& +h(f+g+h+k), \\
= k\{-1-\tfrac{1}{2}(p-1)-2(l+m)\} & +(f+h)\{\tfrac{1}{4}(p-1)\}+2g(l+m), \\
= 2(g-k)(l+m) & +\tfrac{1}{4}(f+h)(p-1)-\tfrac{1}{2}k(p+1).
\end{aligned}$$

We find $XYZW$ most readily as the product of XZ and YW; we thus obtain

$$\begin{aligned}
XYZW &= lm(X^2+Y^2+Z^2+W^2+2XZ+2YW)+(l^2+m^2)(XY+XW+YZ+ZW), \\
&= lm(-a-b-c-d-2l-2m)-(l^2+m^2)(f+g+h+k), \\
&= lm\{1+2(f+g+h+k)\}-(l^2+m^2)(f+g+h+k);
\end{aligned}$$

or, substituting for $f+g+h+k$ its value $\frac{1}{4}(p-1)$, this is

$$lm - \tfrac{1}{4}(l-m)^2(p-1), = \tfrac{1}{4}\{(l+m)^2-(l-m)^2 p\}.$$

Hence the required equation, having roots X, Y, Z, W, is

$$\begin{aligned}
& \eta^4+\eta^3 \\
& -\eta^2\{\tfrac{1}{4}(p-1)+l+m\} \\
& +\eta\{\tfrac{1}{4}(f+h)(p-1)-\tfrac{1}{2}k(p+1)+2(g-k)(l+m)\} \\
& +lm-\tfrac{1}{4}(l-m)^2(p-1) \\
& =0,
\end{aligned}$$

where, for the sake of having a single formula, I have retained $l+m$ in place of its value $=-\frac{3}{8}(p-1)$ or $\frac{1}{8}(p+3)$ according as $p \equiv 1$ or 5 (mod. 8).

We thus have the following:—

TABLE FOR THE QUARTISECTION.

p	a	b	c	d	η^3	$\eta^4+\eta^2+\eta^1$	η^0	A	B	Page in Reuschle
	f	g	h	k						
	l	m								
5	0	1	0	0	1	1	1	−1	−2	p. 2
	0	0	0	1						
	−1	−1								
13	0	1	2	0	2	−4	3	3	−2	p. 13
	1	1	0	1						
	−3	−2								
17	−4	−2	−3	−4	−6	−1	1	−1	4	p. 19
	2	0	1	1						
	1	1								
29	2	3	0	2	4	20	23	−5	−2	p. 36
	1	1	2	3						
	−5	−6								
37	2	1	2	4	5	7	49	−1	6	p. 53
	2	2	4	1						
	−7	−7								
41	−10	−6	−7	−8	−15	18	4	−5	4	p. 61
	4	2	2	2						
	3	2								
53	2	3	6	2	7	−43	47	7	−2	p. 80
	4	4	2	3						
	−11	−9								
61	4	3	2	6	8	42	117	−5	6	p. 96
	3	3	6	3						
	−11	−12								
73	−16	−13	−12	−14	−27	−41	2	3	8	p. 126
	6	5	5	2						
	4	5								
89	−19	−18	−16	−14	−33	39	8	−5	−8	p. 152
	4	8	5	5						
	6	5								
97	−22	−16	−17	−18	−36	91	−61	−9	4	p. 167
	8	6	5	5						
	7	5								

TABLE OF THE POWERS OF REUSCHLE'S SELECTED PRIME ROOTS.

	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	2	2	5	2	2	3	2	21	2	3	2	6	3	10	2	2	2	2	62	5	3	2	30	10	1
2		4	4	4	4	9	4	4	4	9	4	36	9	6	4	4	4	4	10	25	9	4	10	3	2
3	...	3	6	8	8	10	8	15	8	27	8	11	27	13	8	8	8	8	52	52	27	8	33	30	3
4			2	5	3	13	16	16	16	19	16	25	38	36	16	16	16	16	29	41	2	16	11	9	4
5	...	...	3	10	6	5	13	14	3	26	32	27	28	31	32	32	32	32	23	59	6	32	63	90	5
6				9	12	15	7	18	6	16	27	39	41	28	11	5	3	64	6	3	18	64	21	27	6
7				7	11	11	14	10	12	17	17	29	37	45	22	10	6	61	17	15	54	45	7	76	7
8				3	9	16	9	3	24	20	34	10	25	27	44	20	12	55	60	2	4	7	32	81	8
9	...	...	...	6	5	14	18	17	19	29	31	19	32	35	35	40	24	43	28	10	12	14	70	34	9
10					10	8	17	12	9	25	25	32	10	21	17	21	48	19	32	50	36	28	53	49	10
11	...	...	...	...	7	7	15	22	18	13	13	28	30	22	34	42	35	38	67	31	29	56	77	5	11
12						4	11	2	7	8	26	4	4	32	15	25	9	9	36	9	8	29	85	50	12
13						12	3	19	14	24	15	24	12	38	30	50	18	18	31	45	24	58	58	15	13
14						2	6	8	28	10	30	21	36	4	7	41	36	36	5	6	72	33	49	53	14
15	...	...	...	...	...	...	6	12	7	27	30	23	3	22	40	14	23	11	5	26	30	58	66	46	45	15
16							5	9	25	28	9	18	23	24	28	46	22	10	50	4	16	49	45	62	16
17	...	...	...	...	...	...	10	5	21	22	18	26	26	5	3	33	44	20	47	20	48	15	15	88	17
18								13	13	4	36	33	35	3	6	7	27	40	3	27	65	30	5	89	18
19								20	26	12	35	34	19	30	12	14	54	13	44	62	37	60	61	17	19
20								6	23	5	33	40	14	18	24	28	47	26	30	18	32	37	50	73	20
21	...	...	...	...	...	...	...	11	17	15	29	35	42	39	48	56	33	52	14	17	17	74	76	51	21
22									5	14	21	5	40	14	43	53	5	37	16	12	51	65	55	25	22
23									10	11	5	30	34	46	33	47	10	7	69	60	74	47	48	56	23
24									20	2	10	16	16	37	13	35	20	14	18	8	64	11	16	75	24
25									11	6	20	14	5	41	26	11	40	28	51	40	34	22	35	71	25
26									22	18	3	2	15	34	52	22	19	56	38	54	23	44	71	31	26
27	...	...	...	...	...	...	...	...	15	23	6	12	2	11	51	44	38	45	13	51	69	5	83	19	27
28										7	12	31	6	16	49	29	15	23	25	36	48	10	87	93	28
29	...	...	...	...	...	...	...	...	...	21	24	22	18	19	45	58	30	46	59	34	68	20	29	57	29
30											11	9	11	2	37	57	60	25	37	24	46	40	69	85	30
31											22	13	33	20	21	55	59	50	22	47	59	80	23	74	31
32											7	37	13	12	42	51	57	33	15	16	19	77	67	61	32
33											14	17	39	26	31	43	53	66	7	7	57	71	52	28	33
34											28	20	31	25	9	27	45	65	8	35	13	59	47	86	34
35	...	...	...	...	...	...	...	...	...	...	19	38	7	15	18	54	29	63	70	29	39	35	75	84	35
36												23	21	9	36	49	58	59	9	72	38	70	25	64	36
37												15	20	43	19	39	55	51	61	68	35	57	38	58	37
38												8	17	7	38	19	49	35	19	48	26	31	72	95	38
39	...	...	...	...	...	...	...	...	...	...	...	7	8	23	23	38	37	3	42	21	78	62	24	77	39
40													24	42	46	17	13	6	48	32	76	41	8	91	40
41	...	...	...	...	...	...	...	...	...	...	...	...	29	44	39	34	26	12	65	14	70	82	62	37	41
42														11	25	9	52	24	45	70	52	81	80	79	42
43														29	50	18	43	28	11	58	77	79	86	14	43
44														8	47	36	25	29	43	71	73	75	88	43	44
45	...	...	...	...	...	...	...	...	...	...	...	...	...	33	41	13	50	58	39	63	61	67	59	42	45
46															29	26	39	49	4	23	25	51	79	32	46
47															5	52	17	31	35	42	75	19	56	29	47
48															10	45	34	62	40	64	67	38	78	96	48
49															20	31	7	57	66	28	43	76	26	87	49
50															40	3	14	47	45	67	50	69	68	94	50

TABLE (*continued*).

	53	59	61	67	71	73	79	83	89	97	
51	27	6	28	27	21	43	71	55	82	67	51
52		12	56	54	24	69	55	27	57	88	52
53		24	51	41	68	53	7	54	19	7	53
54		48	41	15	27	46	21	25	36	70	54
55		37	21	30	41	11	63	50	12	21	55
56		15	42	60	57	55	31	17	4	16	56
57		30	23	53	55	56	14	34	31	63	57
58			46	39	2	61	42	68	40	48	58
59			31	11	53	13	47	53	43	92	59
60				22	20	65	62	23	44	47	60
61				44	33	33	28	46	74	82	61
62				21	58	19	5	9	84	44	62
63				42	46	22	15	18	28	52	63
64				17	12	37	45	36	39	35	64
65				34	34	39	56	72	13	59	65
66					49	49	10	61	34	8	66
67					56	26	30	39	41	80	67
68					64	57	11	78	73	24	68
69					63	66	33	73	54	46	69
70						38	20	63	18	72	70
71						44	60	43	6	41	71
72							22	3	2	22	72
73							26	6	60	26	73
74							40	12	20	66	74
75							41	24	66	78	75
76							44	48	22	4	76
77							53	13	37	40	77
78								26	42	12	78
79								52	14	23	79
80								21	64	36	80
81								42	51	69	81
82									17	11	82
83									65	13	83
84									81	33	84
85									27	39	85
86									9	2	86
87									3	20	87
88										6	88
89										60	89
90										18	90
91										83	91
92										54	92
93										55	93
94										65	94
95										68	95

732.

A THEOREM IN SPHERICAL TRIGONOMETRY.

[From the *Proceedings of the London Mathematical Society*, vol. XI. (1880), pp. 48—50. Read January 8, 1880.]

IN a spherical triangle, where a, b, c are the sides, and A, B, C the opposite angles, we have

$$-\tan\tfrac{1}{2}c\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin(A-B)=\tan\tfrac{1}{2}b\sin A-\tan\tfrac{1}{2}a\sin B,$$

$$\tan\tfrac{1}{2}c\,\{1-\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\}=\tan\tfrac{1}{2}b\cos A+\tan\tfrac{1}{2}a\cos B;$$

which are both included in the form

$$\tan\tfrac{1}{2}a\,(\cos B-i\sin B)=\frac{\tan\frac{1}{2}c-\tan\frac{1}{2}b\,(\cos A+i\sin A)}{1+\tan\frac{1}{2}c\tan\frac{1}{2}b\,(\cos A+i\sin A)}.$$

For the first of the two identities: from

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C},$$

$$\cos b=\frac{\cos B+\cos A\cos C}{\sin A\sin C},$$

we deduce

$$\begin{aligned}\cos a-\cos b&=\frac{1}{\sin C}\left(\frac{\cos A}{\sin B}-\frac{\cos B}{\sin A}\right)+\frac{\cos C}{\sin C}\left(\frac{\cos B}{\sin B}-\frac{\cos A}{\sin A}\right)\\&=\frac{1}{\sin C}\,\frac{\frac{1}{2}(\sin 2A-\sin 2B)}{\sin A\sin B}+\frac{\cos C\sin(A-B)}{\sin C\sin A\sin B}\\&=\frac{\sin(A-B)}{\sin C\sin A\sin B}\{\cos(A+B)+\cos C\}\\&=\frac{\sin(A-B)}{\sin C}(\cos c-1);\end{aligned}$$

that is,

$$-\sin(A-B)=\frac{\sin C}{1-\cos c}(\cos a-\cos b)$$

$$=\frac{\sin C}{\sin c}\,\frac{\sin c}{1-\cos c}(\cos a-\cos b);$$

or, what is the same thing,

$$-\tan\tfrac{1}{2}c\sin(A-B)=\frac{\sin C}{\sin c}(\cos a-\cos b).$$

Here $\cos a-\cos b$ is $=(1+\cos a)-(1+\cos b)$; substituting for $\frac{\sin C}{\sin c}$ successively $\frac{\sin A}{\sin a}$ and $\frac{\sin B}{\sin b}$, the right-hand side is

$$=\frac{1+\cos a}{\sin a}\sin A-\frac{1+\cos b}{\sin b}\sin B,$$

$$=\cot\tfrac{1}{2}a\sin A-\cot\tfrac{1}{2}b\sin B;$$

whence, multiplying each side by $\tan\frac{1}{2}a\tan\frac{1}{2}b$, we have the relation in question.

For the second identity which is

$$\tan\tfrac{1}{2}c\,\{1-\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\}=\tan\tfrac{1}{2}b\cos A+\tan\tfrac{1}{2}a\cos B;$$

if on the right-hand side we substitute for $\cos A$, $\cos B$ their values

$$\frac{\cos a-\cos b\cos c}{\sin b\sin c}\quad\text{and}\quad\frac{\cos b-\cos a\cos c}{\sin a\sin c},$$

the right-hand side becomes

$$\frac{1}{\sin c}\left\{\frac{\cos a-\cos b\cos c}{1+\cos b}+\frac{\cos b-\cos a\cos c}{1+\cos a}\right\};$$

whence, multiplying the whole equation by $\sin c(1+\cos a)(1+\cos b)$, it becomes

$$(1-\cos c)\,\{(1+\cos a)(1+\cos b)-\sin a\sin b\cos(A-B)\}$$

$$=(1+\cos a)(\cos a-\cos b\cos c)+(1+\cos b)(\cos b-\cos c\cos a).$$

We have here

$$\cos(A-B)=\cos A\cos B+\sin A\sin B=\frac{(\cos a-\cos b\cos c)(\cos b-\cos c\cos a)+\square}{\sin^2 c\sin a\sin b},$$

by substituting for $\cos A$, $\cos B$ their foregoing values, and for $\sin A$, $\sin B$ their values $\frac{\sqrt{\square}}{\sin b\sin c}$, $\frac{\sqrt{\square}}{\sin a\sin c}$, where

$$\square=1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c.$$

The numerator is

$$\cos a \cos b - \cos c\,(\cos^2 a + \cos^2 b) + \cos a \cos b \cos^2 c$$
$$+ 1 - \cos^2 c - \qquad (\cos^2 a + \cos^2 b) + \cos a \cos b \,.\, 2 \cos c\,;$$

viz. this is

$$= \cos a \cos b\,(1 + \cos c)^2 - (\cos^2 a + \cos^2 b)\,(1 + \cos c) + 1 - \cos^2 c,$$

having the factor $1 + \cos c$, which is also a factor of $\sin^2 c$, $= 1 - \cos^2 c$, in the denominator. We have, therefore,

$$\cos(A - B) = \frac{\cos a \cos b\,(1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c}{(1 - \cos c) \sin a \sin b}\,;$$

and the equation thus is

$$(1 - \cos c)\,(1 + \cos a)\,(1 + \cos b) - \{\cos a \cos b\,(1 + \cos c) - (\cos^2 a + \cos^2 b) + 1 - \cos c\}$$
$$= (1 + \cos a)\,(\cos a - \cos b \cos c) + (1 + \cos b)\,(\cos b - \cos c \cos a),$$

where each side is in fact

$$= \cos a + \cos^2 a + \cos b + \cos^2 b - \cos c\,(\cos a + \cos b) - 2 \cos a \cos b \cos c\,;$$

and the second identity is thus proved.

733.

ON A FORMULA OF ELIMINATION.

[From the *Proceedings of the London Mathematical Society*, vol. XI. (1880), pp. 139—141. Read June 10, 1880.]

CONSIDER the equations

$$(a, \ldots \between \theta, 1)^n = 0,$$
$$(A, \ldots \between \theta, 1)^m = 0,$$

where $a, \ldots, A, \ldots$ are functions of coordinates. To fix the ideas, suppose that each of these coefficients is a linear function of the four coordinates x, y, z, w. Then, eliminating θ, we obtain $\nabla = 0$, the equation of a surface; and (as is known) this surface has a nodal curve.

It is easy to obtain the equations of the nodal curve in the case where one of the equations, say the second, is a quadric: the process is substantially the same whatever may be the order of the other equation, and I take it to be a cubic; the two equations therefore are

$$(a, b, c, d \between \theta, 1)^3 = 0,$$
$$(A, B, C \between \theta, 1)^2 = 0;$$

giving rise to an equation

$$\nabla, = (a, b, c, d)^2 (A, B, C)^3, = 0.$$

And it is required to perform the elimination so as to put in evidence the nodal line of this surface.

Take θ_1, θ_2 the roots of the second equation, or write

$$(A, B, C \between \theta, 1)^2 = A(\theta - \theta_1)(\theta - \theta_2);$$

that is,

$$\theta_1 + \theta_2 = -\frac{2B}{A}, \quad \theta_1\theta_2 = \frac{C}{A};$$

then, if

$$\Theta_1 = (a,\ b,\ c,\ d \between \theta_1,\ 1)^3,$$

$$\Theta_2 = (a,\ b,\ c,\ d \between \theta_2,\ 1)^3,$$

we have

$$\nabla = A^3\Theta_1\Theta_2;$$

viz. on the right-hand side, replacing the symmetrical functions of θ_1, θ_2 by their values in terms of A, B, C, we have the expression of ∇ in its known form

$$\nabla = a^2C^3 + \&c.$$

Form now the expressions

$$\Theta_1 - \Theta_2,\quad \theta_2\Theta_1 - \theta_1\Theta_2,\quad \theta_2^2\Theta_1 - \theta_1^2\Theta_2,\quad \theta_2^3\Theta_1 - \theta_1^3\Theta_2,$$

each divided by $\theta_1 - \theta_2$. These are evidently symmetrical functions of θ_1, θ_2, the values being given by the successive lines of the expression

$$\left(\begin{array}{cccc} 0, & 1, & \theta_1+\theta_2, & \theta_1^2+\theta_1\theta_2+\theta_2^2 \\ -1, & 0, & \theta_1\theta_2, & \theta_1\theta_2(\theta_1+\theta_2) \\ -(\theta_1+\theta_2), & -\theta_1\theta_2, & 0, & \theta_1^2\theta_2^2 \\ -(\theta_1^2+\theta_1\theta_2+\theta_2^2), & -\theta_1\theta_2(\theta_1+\theta_2), & -\theta_1^2\theta_2^2, & 0 \end{array}\right) \between d,\ 3c,\ 3b,\ a);$$

and, consequently, these same quantities, each multiplied by A^2, are given by the successive lines of

$$\left(\begin{array}{cccc} 0, & A^2, & -2AB, & -AC+4B^2 \\ -A^2, & 0, & AC, & -2BC \\ 2AB, & -AC, & 0, & C^2 \\ AC-4B^2, & 2BC, & -C^2, & 0 \end{array}\right) \between d,\ 3c,\ 3b,\ a).$$

Calling these X, Y, Z, W, that is, writing

$$X = 3A^2c - 6ABb + (-AC+4B^2)a,\ \&c.,$$

then X, Y, Z, W are the values of

$$\Theta_1 - \Theta_2,\quad \theta_2\Theta_1 - \theta_1\Theta_2,\quad \theta_2^2\Theta_1 - \theta_1^2\Theta_2,\quad \theta_2^3\Theta_1 - \theta_1^3\Theta_2,$$

each multiplied by $A^2 \div (\theta_1 - \theta_2)$; and the functions all four of them vanish if only $\Theta_1 = 0$, $\Theta_2 = 0$; or, what is the same thing, the equations $X=0$, $Y=0$, $Z=0$, $W=0$ constitute only a twofold system.

The functions

$$\left(\begin{array}{ccc} X, & Y, & Z \\ Y, & Z, & W \end{array}\right)$$

contain each of them the factor $\Theta_1\Theta_2$, that is, ∇; they, in fact, each of them vanish if $\Theta_1 = 0$, and they also vanish if $\Theta_2 = 0$; or, by a direct substitution, we have

$$\begin{aligned}
XZ - Y^2 &= \frac{A^4}{(\theta_1-\theta_2)^2} \cdot -(\theta_1-\theta_2)^2\,\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2,\\
XW - YZ &= \quad ,, \quad -(\theta_1-\theta_2)^2(\theta_1+\theta_2)\,\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2(\theta_1+\theta_2),\\
YW - Z^2 &= \quad ,, \quad -(\theta_1-\theta_2)^2\,\theta_1\theta_2\Theta_1\Theta_2, &&= -A^4\Theta_1\Theta_2\theta_1\theta_2.
\end{aligned}$$

Or, what is the same thing, these are $= -A\nabla$, $2B\nabla$, $-C\nabla$, respectively; thus the first equation is

$$\begin{aligned}
&\{3A^2c - 6ABb + (-AC + 4B^2)\,a\}\,\{2ABd - 3ACc + C^2a\}\\
&\qquad - (-A^2d + 3ACb - 2BCa)^2 = -A\,(A^3d^2 + \&c.),\ = -A\nabla;
\end{aligned}$$

and similarly for the other two equations. The nodal curve is thus given by the twofold system $X = 0$, $Y = 0$, $Z = 0$, $W = 0$.

The method may be extended to the case where, instead of the quadric equation $(A, B, C \between \theta, 1)^2 = 0$, we have an equation of any higher order, but the formulæ are less simple.

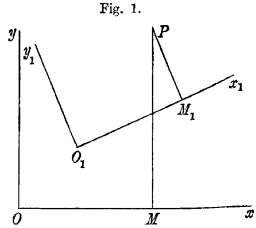
734.

ON THE KINEMATICS OF A PLANE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 1—8.]

It seems desirable to bring together under this title various questions which have been, or may be, proposed or discussed. We consider two planes in relative motion one upon the other, but, for convenience, they may be distinguished as a moving plane and a fixed plane, the first moving upon the second. Any point of the moving plane traces out on the fixed plane a curve, and any line of the moving plane envelopes on the fixed plane a curve; similarly, any point of the fixed plane traces out on the moving plane a curve, and any line of the fixed plane envelopes on the moving plane a curve. More generally, any curve of the moving plane envelopes on the fixed plane a curve, and any curve of the fixed plane envelopes on the moving plane a curve. There is, moreover, in the moving plane a curve which rolls upon a curve in the fixed plane, and these two curves (a single relative position being given) determine the motion.

Fig. 1.



The analytical theory presents no difficulty. Taking in the fixed plane the fixed axes Ox, Oy (fig. 1), and, fixed in the moveable plane so as to move with it, the axes O_1x_1, O_1y_1; then the position of the axes $O_1x_1y_1$ may be determined, say by

α, β, the coordinates of O_1 in regard to Oxy; and by θ, the inclination of O_1x_1 to Ox. And denoting by x, y, x_1, y_1 the coordinates of a point P in regard to the two sets of axes respectively, then

$$x = \alpha + x_1 \cos\theta - y_1 \sin\theta,$$
$$y = \beta + x_1 \sin\theta + y_1 \cos\theta;$$

or, what is the same thing,

$$x_1 = \quad (x-\alpha)\cos\theta + (y-\beta)\sin\theta,$$
$$y_1 = -(x-\alpha)\sin\theta + (y-\beta)\cos\theta;$$

or, as these last equations may be written,

$$x_1 = \alpha_1 + x\cos(-\theta) - y\sin(-\theta),$$
$$y_1 = \beta_1 + x\sin(-\theta) + y\cos(-\theta),$$

where α_1, β_1, $= -\alpha\cos\theta - \beta\sin\theta$, $\alpha\sin\theta - \beta\cos\theta$, are the coordinates of O referred to the axes $O_1x_1y_1$, and $-\theta$ is the inclination of Ox to O_1x_1.

When the motion is given, α, β, θ are given functions of a single variable parameter, say of t*; or, if we please, α, β are given functions of θ.

The velocities of a given point (x, y) are determined by the equations

$$x' = \alpha' - (x_1 \sin\theta + y_1 \cos\theta)\,\theta',$$
$$y' = \beta' + (x_1 \cos\theta - y_1 \sin\theta)\,\theta';$$

that is,

$$x' - \alpha' = -(y-\beta)\,\theta',$$
$$y' - \beta' = \quad (x-\alpha)\,\theta';$$

or, as these equations may also be written,

$$-(x'-\alpha')\sin\theta + (y'-\beta')\cos\theta = x_1\theta',$$
$$-(x'-\alpha')\cos\theta - (y'-\beta')\sin\theta = y_1\theta'.$$

Hence if $x' = 0$, $y' = 0$, we have

$$x_1\theta' = \alpha'\sin\theta - \beta'\cos\theta, \text{ or } \quad \alpha' = (y-\beta)\,\theta',$$
$$y_1\theta' = \alpha'\cos\theta + \beta'\sin\theta, \qquad -\beta' = (x-\alpha)\,\theta',$$

which equations determine in terms of t, x_1 and y_1 the coordinates in regard to the axes $O_1x_1y_1$, and x and y the coordinates in regard to the axes Oxy, of I, the centre of instantaneous rotation.

If from the expressions of x_1, y_1 we eliminate t, we obtain an equation between (x_1, y_1), which is that of the rolling curve in the moveable plane; and, similarly, if

* t may be regarded as denoting the time, and then the derived functions of x, y in regard to t will denote velocities; and, to simplify the expression of the theorems, it is convenient to do this.

from the expressions of x, y we eliminate t, we obtain a relation between (x, y), which is that of the rolled-on curve in the fixed plane.

The system may be written

$$x_1 = \frac{\alpha'}{\theta'}\sin\theta - \frac{\beta'}{\theta'}\cos\theta, \quad x = \alpha - \frac{\beta'}{\theta'},$$

$$y_1 = \frac{\alpha'}{\theta'}\cos\theta + \frac{\beta'}{\theta'}\sin\theta, \quad y = \beta + \frac{\alpha'}{\theta'};$$

or, if we take θ as the independent variable,

$$x_1 = \alpha'\sin\theta - \beta'\cos\theta, \quad x = \alpha - \beta',$$
$$y_1 = \alpha'\cos\theta + \beta'\sin\theta, \quad y = \beta + \alpha'.$$

To find the variations of I, we have

$$\begin{aligned}
x_1' &= \alpha''\sin\theta - \beta''\cos\theta + \alpha'\cos\theta + \beta'\sin\theta, &&= \alpha''\sin\theta - \beta''\cos\theta + y_1,\\
y_1' &= \alpha''\cos\theta + \beta''\sin\theta - \alpha'\sin\theta + \beta'\cos\theta, &&= \alpha''\cos\theta + \beta''\sin\theta - x_1,\\
y' &= \beta' + \alpha'',\\
x' &= \alpha' - \beta''.
\end{aligned}$$

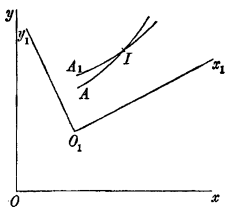
Hence

$$\begin{aligned}
x_1' &= x'\cos\theta + y'\sin\theta, \text{ or } x' = x_1'\cos\theta - y_1'\sin\theta,\\
y_1' &= -x'\sin\theta + y'\cos\theta, \qquad y' = x_1'\sin\theta + y_1'\cos\theta,
\end{aligned}$$

values which give $x'^2 + y'^2 = x_1'^2 + y_1'^2$, which equation expresses that the motion is in fact a rolling one.

Imagine the two curves, and the initial relative position given; say the two points A, A_1 (fig. 2) were originally in contact, then the arcs AI, A_1I are equal, and, calling each of these s, and X, Y, X_1, Y_1 the coordinates of I in regard to the two

Fig. 2.



sets of axes respectively, we have X, Y, X_1, Y_1 given functions of s, such that $X'^2 + Y'^2 = 1$, $X_1'^2 + Y_1'^2 = 1$, the accents now denoting differentiation in regard to s. We have, from the figure,

$$\theta = \tan^{-1}\frac{Y'}{X'} - \tan^{-1}\frac{Y_1'}{X_1'};$$

or, what is the same thing,

$$\tan\theta = (Y'X_1' - Y_1'X) \div (X'X_1' + Y'Y_1'),$$

say

$$\sin\theta,\ \cos\theta = Y'X_1' - Y_1'X,\ X'X_1' + Y'Y_1';$$

and then, as before,

$$x = \alpha + x_1\cos\theta - y_1\sin\theta,$$
$$y = \beta + x_1\sin\theta + y_1\cos\theta;$$

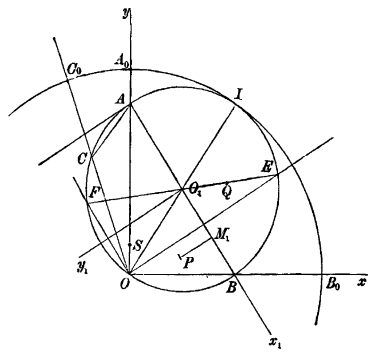
or, what is the same thing,

$$x - X = \cos\theta\,(x_1 - X_1) - \sin\theta\,(y_1 - Y_1),$$
$$y - Y = \sin\theta\,(x_1 - X_1) + \cos\theta\,(y_1 - Y_1),$$

where X, Y, X_1, Y_1, and therefore also θ, denote given functions of s. The formulæ will be of a like form if X, Y, X_1, Y_1 are given functions of a parameter t.

A well known but very interesting case is when two points of the moving plane describe right lines on the fixed plane. This may be discussed geometrically as follows: Suppose that we have the points A, C (fig. 3) describing the lines OA_0, OC_0, which meet in O; through A, C, O describe a circle, centre O_1, and with centre

Fig. 3.



O and radius $= 2OO_1$, describe a circle touching the first circle in a point I; and suppose that A_0, C_0 denote points on the second circle. Then it is at once seen that, considering the first or small circle as belonging to the moving plane, and the second or large circle as belonging to the fixed plane, the motion is in fact the rolling motion of the small upon the large circle; and, moreover, that each point of the small circle describes a right line, which is a diameter of the large circle. In fact, the angle IO_1C at the centre is the double of the angle IOC at the circumference; that is,

it is the double of the angle IOC_0; and therefore (the radius of the small circle being half that of the large circle) the arcs IC, IC_0 are equal, so that the rolling motion will carry the point C along the radius OC_0, and will, in like manner, carry the point A along the radius OA_0, or the motion will be as originally assumed. And, in like manner, for any other point B of the small circle the motion will be along the radius OB_0; in particular, taking AB a diameter, the angle A_0OB_0 will be a right angle; and the motion is determined by means of the two points A, B describing respectively the two lines OA_0, OB_0 *at right angles to each other*, viz. there is no loss of generality in assuming that the two fixed lines are at right angles to each other. It thence at once follows, as will presently appear, that each point of the moving plane describes an ellipse (but we have the special case already referred to, each point on the small circle describes a right line, and also the special case, the centre O_1 of the small circle describes a circle). Considering any point Q of the moving plane, let the line QO_1 meet the small circle in the points E, F (or, what is the same thing, let E, F be the extremities of the diameter which passes through Q); then the points E, F describe the lines OE, OF at right angles to each other, and Q is a point on EF or on this line produced; clearly the locus is an ellipse having the lines OE, OF for the directions of its axes, and having the lengths of the semi-axes $= QF$, QE respectively.

Taking the points to be A, B moving along the two lines OB_0, OA_0 at right angles to each other, these lines may be taken for the axes Ox, Oy; the point O_1 for the origin of the coordinates x_1, y_1, the axes O_1x_1 being in the direction O_1B and O_1y_1 at right angles to it; calling the length $AB = 2c$, we have $O_1A = O_1B = c$, and the angle ABO may be called θ (but this angle was previously taken with a contrary sign). We have then for the point P, having in regard to O_1x_1 and O_1y_1 the coordinates (x_1, y_1),

$$\left.\begin{aligned} x &= \alpha + x_1 \cos\theta - y_1 \sin\theta \\ y &= \beta - x_1 \sin\theta - y_1 \cos\theta \end{aligned}\right\},$$

where the sign of y_1 has been changed, and $\alpha = c\cos\theta$, $\beta = c\sin\theta$: the equations thus become

$$x = (c + x_1)\cos\theta - y_1 \sin\theta,$$

$$y = (c - x_1)\sin\theta - y_1 \cos\theta,$$

where observe that $c + x_1$, $c - x_1$ are the distances M_1A, M_1B respectively. And we have, conversely,

$$x_1 = \quad x\cos\theta - y\sin\theta - c\cos 2\theta,$$

$$y_1 = -x\sin\theta - y\cos\theta + c\sin 2\theta.$$

If, in particular, $y_1 = 0$, then

$$x,\ y = (c + x_1)\cos\theta,\ (c - x_1)\sin\theta;$$

or we have

$$\frac{x^2}{(c + x_1)^2} + \frac{y^2}{(c - x_1)^2} = 1;$$

viz. the curve on the first plane is an ellipse, the semi-axes of which are $\pm(c+x_1)$, $\pm(c-x_1)$, each taken positively; if $x_1^2+y_1^2=c^2$, viz. if P be on the circle having AB for its diameter, then $y_1^2=(c+x_1)(c-x_1)$, and we have

$$y \div x = -(c-x_1)\left(\sin\theta - \frac{y_1}{c-x_1}\cos\theta\right) \div y_1\left(\sin\theta - \frac{c+x_1}{y_1}\cos\theta\right), = -(c-x_1)\div y_1,$$

viz. as mentioned above, the curve on the fixed plane is a right line.

In the general case, we have

$$\begin{aligned} x(c-x_1) + \quad yy_1 \quad &= (c^2-x_1^2-y_1^2)\cos\theta, \\ xy_1 + y(c+x_1) &= (c^2-x_1^2-y_1^2)\sin\theta, \end{aligned}$$

and thence

$$\{x(c-x_1)+yy_1\}^2 + \{xy_1+y(c+x_1)\}^2 = (c^2-x_1^2-y_1^2)^2;$$

or, what is the same thing,

$$x^2\{(c-x_1)^2+y_1^2\} + 4xycy_1 + y^2\{(c+x_1)^2+y_1^2\} = (c^2-x_1^2-y_1^2)^2.$$

Considering (x_1, y_1) as given, the curve traced out by P on the fixed plane is of the second order; it would be easy to verify from the equation that it is an ellipse, and to obtain for the position and magnitude of the axes the construction already found geometrically.

The same equation, considering therein (x, y) as constant and (x_1, y_1) as current coordinates, gives the curve traced out on the moving plane; the curve is obviously of the fourth order. Transferring the origin to A, we must in place of x_1 write x_1-c_1; the equation thus becomes

$$x^2\{(x_1-2c)^2+y_1^2\} + 4cy_1xy + y^2(x_1^2+y_1^2) = (x_1^2+y_1^2-2cx_1)^2;$$

or, what is the same thing,

$$(x_1^2+y_1^2-2cx_1)^2 - (x^2+y^2)(x_1^2+y_1^2) + 4cx(xx_1-yy_1) - 4c^2x^2 = 0;$$

and if we suppose herein $x=0$, it becomes

$$(x_1^2+y_1^2-2cx_1)^2 - y^2(x_1^2+y_1^2) = 0;$$

or, writing $x_1 = r_1\cos\theta_1$, $y_1 = r_1\sin\theta_1$, where θ_1 = angle QAB, this is

$$(r_1-2c\cos\theta_1)^2 - y^2 = 0,$$

or say it is

$$r_1 = 2c\cos\theta_1 - y,$$

which is the polar equation of the curve described on the moveable plane by the point S, whose coordinates in respect to Ox and Oy are $(0, y)$.

There is no loss of generality in assuming $x=0$. In fact, starting with any point S whatever of the fixed plane, if we draw OS meeting the small circle in A, and

through O draw at right angles to this a line meeting the same circle in B, then, as before, the points A and B move along the fixed lines OA_0, OB_0; or as regards the relative motion, taking A, B as fixed points, we have the originally fixed plane now moving in such wise that the two lines OA_0, OB_0 thereof (at right angles to each other) pass always through the points A and B respectively, and the curve is that described by the point S on the line OA; the point O describes the circle on the diameter AB (the small circle), equation $r_1 = 2c\cos\theta_1$; and OQ having a given constant value $=y$, we have for the curve described by the point S the foregoing equation $r_1 = 2c\cos\theta_1 - y$; or writing $y = -f$, that is, taking S on the other side of O at a distance $OS = f$, the equation is $r_1 = 2c\cos\theta_1 + f$; viz. this is a nodal Cartesian or Limaçon, the origin being an acnode or a crunode according as $f >$ or $< 2c$; and if $f = 2c$, then we have the cuspidal curve or cardioid $r_1 = 2c(1+\cos\theta_1)$, $= 4c\cos^2\frac{1}{2}\theta_1$. The general conclusion is that the centre O of the large circle describes on the moving plane a small circle (centre O_1), and that every other point of the fixed plane describes on the moving plane a Limaçon having for its node a point of the small circle, and being, in fact, the curve obtained by measuring off along the radius vector of the small circle from its extremity a constant distance.

Considering in connexion with the point, coordinates (x_1, y_1), (x, y), a second point, coordinates (X_1, Y_1), (X, Y), in regard to the two sets of axes respectively, we have

$$x = (c + x_1)\cos\theta - y_1\sin\theta, \quad X = (c + X_1)\cos\theta - Y_1\sin\theta,$$
$$y = (c - x_1)\sin\theta - y_1\cos\theta, \quad Y = (c - X_1)\sin\theta - Y_1\cos\theta;$$

from the first two equations we have

$$\cos\theta : \sin\theta : 1 = x(c - x_1) + yy_1 : xy_1 + y(c + x_1) : c^2 - x_1^2 - y_1^2;$$

and substituting these values in the second set, we find

$$\begin{aligned} X : Y : 1 \\ = \; & x\{c^2 + c(X_1 - x_1) - X_1x_1 - Y_1y_1\} + y\{\quad c(y_1 - Y_1) + y_1X_1 - x_1Y_1\} \\ & : x\{\quad c(y_1 - Y_1) - y_1X_1 + x_1Y_1\} + y\{c^2 - c(X_1 - x_1) - X_1x_1 - Y_1y_1\} \\ & : c^2 - x_1^2 - y_1^2; \end{aligned}$$

or the points (x, y), (X, Y), considered as each of them moving on the fixed plane, are homographically related to each other.

To find the curve enveloped on the fixed plane by a given curve of the moving plane, we have only in the equation $f(x_1, y_1) = 0$ of the curve in the moving plane to substitute for x_1, y_1 their values in terms of x, y, θ, and then considering θ as a variable parameter, to find the envelope of the curve represented by this equation. And, similarly, we find the curve enveloped on the moving plane by a given curve of the fixed plane.

Thus, in the particular case of motion above considered, writing, as before,

$$x = (c + x_1)\cos\theta - y_1 \sin\theta,$$
$$y = (c - x_1)\sin\theta - y_1 \cos\theta;$$

or conversely

$$x_1 = \quad x\cos\theta - y\sin\theta - c\cos 2\theta,$$
$$y_1 = -x\sin\theta - y\cos\theta + c\sin 2\theta;$$

the envelope on the moving plane of the line

$$Ax + By + C = 0$$

of the fixed plane is given as the envelope of the line

$$\{A(c + x_1) - By_1\}\cos\theta + \{-A + B(c - x_1)\}\sin\theta + C = 0;$$

viz. this is

$$\{A(c + x_1) - By_1\}^2 + \{Ay_1 - B(c - x_1)\}^2 - C^2 = 0;$$

that is,

$$(A^2 + B^2)(x_1^2 + y_1^2 + c^2) + 2(A^2 - B^2)cx_1 - 4ABcy_1 = 0,$$

a circle.

But the envelope on the fixed plane of the line

$$Ax_1 + By_1 + C = 0$$

of the moving plane is given as the envelope of the line

$$C + (Ax + By)\cos\theta - (Ay + Bx)\sin\theta - AC\cos 2\theta + BC\sin 2\theta = 0,$$

which can be obtained by equating to zero the discriminant of a quartic function, and is apparently a sextic curve.

735.

NOTE ON THE THEORY OF APSIDAL SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 109—112.]

I OBTAIN in the present Note a system of formulæ which lead very simply to the known theorem, that the apsidals of reciprocal surfaces are reciprocal; or, what is the same thing, that the reciprocal of the apsidal of a given surface is the apsidal of its reciprocal; the surfaces are referred to the same axes, and by the reciprocal is meant the reciprocal surface in regard to a sphere radius unity, having for its centre a determinate point, say the origin; and it is this same point which is used in the construction of the apsidal surfaces. The apsidal of a given surface is constructed as follows; considering the section by any plane through the fixed point, and in this section the apsidal radii from the fixed point (that is, the radii which meet the curve at right angles), then drawing a line through the fixed point at right angles to the plane, and on this line measuring off from the fixed point distances equal to the apsidal radii respectively, the locus of the extremities of these distances is the apsidal surface. We have the surface, its reciprocal, the apsidal of the surface, the apsidal of the reciprocal; and I take

$$(x,\ y,\ z),\ (x',\ y',\ z'),\ (X,\ Y,\ Z),\ (X',\ Y',\ Z')$$

for the coordinates of corresponding points on the four surfaces respectively.

The condition of reciprocity gives $xx'+yy'+zz'-1=0$, and (the equations being $U=0$, $U'=0$) x', y', z' proportional to d_xU, d_yU, d_zU, and x, y, z proportional to $d_{x'}U'$, $d_{y'}U'$, $d_{z'}U'$; or, what is the same thing, we must have

$$x'dx+y'dy+z'dz=0 \text{ and } xdx'+ydy'+zdz'=0;$$

one of these is implied in the other, as appears at once by differentiating the equation $xx'+yy'+zz'-1=0$.

The other two surfaces will therefore be reciprocal if only we have the like relations between the coordinates (X, Y, Z) and (X', Y', Z'); that is, if

$$XX' + YY' + ZZ' - 1 = 0,$$
$$X'dX + Y'dY + Z'dZ = 0,$$
$$XdX' + YdY' + ZdZ' = 0.$$

To find the apsidal surface, we consider an arbitrary section $x\cos\alpha + y\cos\beta + z\cos\gamma = 0$ of the surface $U = 0$, and seek to determine the apsidal radii thereof, that is, the maximum or minimum values of $R^2 = x^2 + y^2 + z^2$ when x, y, z vary subject to these two conditions. Writing x', y', z' to denote functions proportional to d_xU, d_yU, d_zU, we thus have the set of equations

$$x + \lambda x' + \mu\cos\alpha = 0,$$
$$y + \lambda y' + \mu\cos\beta = 0,$$
$$z + \lambda z' + \mu\cos\gamma = 0,$$

where λ, μ are indeterminate coefficients; taking then X, Y, Z as the coordinates of the extremity of the line drawn at right angles to the plane, we have $R^2 = X^2 + Y^2 + Z^2$, and $\cos\alpha, \cos\beta, \cos\gamma = \frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$; substituting these values in the equation

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0,$$

we have $Xx + Yy + Zz = 0$, and substituting in the other equations, and instead of λ, μ introducing the new indeterminate coefficients ρ, σ, we obtain

$$X, Y, Z = \rho x + \sigma x', \ \rho y + \sigma y', \ \rho z + \sigma z'.$$

Hence these last equations, together with

$$R^2 = X^2 + Y^2 + Z^2 = x^2 + y^2 + z^2,$$

and

$$Xx + Yy + Zz = 1,$$

contain the solution of the problem. If for convenience we introduce R'^2 to denote $x'^2 + y'^2 + z'^2$, and imagine the absolute values of x', y', z' determined so that $xx' + yy' + zz' = 1$, then substituting for X, Y, Z their values in the equations $X^2 + Y^2 + Z^2 = R^2$ and $Xx + Yy + Zz = 1$, we find

$$R^2 = \rho^2R^2 + 2\rho\sigma + \sigma^2R'^2, \ 0 = \rho R^2 + \sigma,$$

and thence

$$\rho^2 = \frac{1}{R^2R'^2 - 1}, \ \sigma = -\rho R^2,$$

or, finally assuming

$$\rho = \frac{1}{\sqrt{(R^2R'^2 - 1)}},$$

we have

$$X, Y, Z = x - R^2x', \ y - R^2y', \ z - R^2z',$$

each divided by

$$\sqrt{(R^2R'^2 - 1)},$$

where I recall that x', y', z' are proportional to d_xU, d_yU, d_zU, and are such that $xx'+yy'+zz'=1$: they in fact denote

$$d_xU,\ d_yU,\ d_zU, \text{ each divided by } xd_xU+yd_yU+zd_zU;$$

and that R^2 and R'^2 denote $x^2+y^2+z^2$ and $x'^2+y'^2+z'^2$ respectively. The coordinates X, Y, Z of the point of the apsidal surface are thus determined as functions of x, y, z.

For the apsidal of the reciprocal surface, we have in like manner

$$X',\ Y',\ Z'=x'-R'^2x,\ y'-R'^2y,\ z'-R'^2z,$$

each divided by

$$-\sqrt{(R^2R'^2-1)},$$

and then the two sets of values give, not only

$$XX'+YY'+ZZ'=1,$$

as is obvious, but also

$$X'dX+Y'dY+Z'dZ=0, \text{ and } XdX'+YdY'+ZdZ'=0.$$

In fact, writing for a moment ρ, ρ' instead of R^2, R'^2, and $\sqrt{(R^2R'^2-1)}=\sqrt{(\rho\rho'-1)}, =\omega$, then

$$\begin{aligned}
&X'dX+Y'dY+Z'dZ\\
&\quad=\frac{x'-x\rho'}{\omega}\,d\,\frac{x-x'\rho}{\omega}+\&c.\\
&\quad=\frac{x'-x\rho'}{\omega}\left\{\frac{dx-\rho dx'-x'd\rho}{\omega}-\frac{(x-x'\rho)\,d\omega}{\omega^2}\right\}+\&c.\\
&\quad=\frac{1}{\omega^2}\{\quad x'dx+y'dy+z'dz\\
&\qquad-\rho\,(x'dx'+y'dy'+z'dz')\\
&\qquad-\quad(x'^2+y'^2+z'^2)\,d\rho\\
&\qquad-\rho'(xdx+ydy+zdz)\\
&\qquad+\rho\rho'(xdx'+ydy'+zdz')\\
&\qquad+\rho'(xx'+yy'+zz')\,d\rho\}\\
&\quad-\frac{d\omega}{\omega^3}\{\quad xx'+yy'+zz'\\
&\qquad-\rho\,(x'^2+y'^2+z'^2)\\
&\qquad-\rho'(x^2+y^2+z^2)\\
&\qquad+\rho\rho'(xx'+yy'+zz')\},
\end{aligned}$$

or, since the terms in { } are

$$0-\rho\,.\,\tfrac{1}{2}d\rho'-\rho'd\rho-\rho'\,.\,\tfrac{1}{2}d\rho+0+\rho'd\rho,\ =-\tfrac{1}{2}(\rho d\rho'+\rho'd\rho),$$

and

$$1-\rho\rho'-\rho\rho'+\rho\rho',\ =1-\rho\rho',\ =-\omega^2,$$

this is

$$=\frac{1}{\omega^2}\{-\tfrac{1}{2}(\rho d\rho'+\rho'd\rho)+\omega d\omega\},\ =0,$$

in virtue of $\omega^2=\rho\rho'-1$. And similarly the other equation $XdX'+YdY'+ZdZ'=0$ might be directly verified.

736.

APPLICATION OF THE NEWTON-FOURIER METHOD TO AN IMAGINARY ROOT OF AN EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 179—185.]

I CONSIDER only the most simple case, that of a quadric equation $x^2 = n^2$, where n^2 is a given imaginary quantity, having the square roots n, and $-n$; starting from an assumed approximate (imaginary) value $x = a$, we have $(a+h)^2 = n^2$, that is,

$$a^2 + 2ah = n^2, \quad h = -\frac{a^2 - n^2}{2a}, \quad \text{and} \quad a + h = \frac{a^2 + n^2}{2a};$$

that is, the successive values are

$$a_1 = \frac{a^2 + n^2}{2a}, \quad a_2 = \frac{a_1^2 + n^2}{2a_1}, \ \ldots,$$

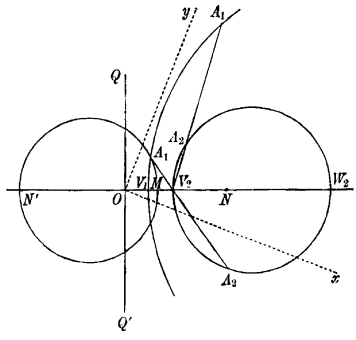
and the question is, under what conditions do we thus approximate to one determinate root (selected out of the two roots at pleasure), say n, of the given equation.

The nearness of two values is measured by the modulus of their difference; thus a nearer to n, than a_1 is to n, means $\text{mod.}(a-n) < \text{mod.}(a_1 - n)$, and so in other cases; in the course of the approximation $a, a_1, a_2, \ldots$ to n, any step, for instance a to a_1, is regular if a_1 is nearer to n than a is, but otherwise it is irregular; the approximation is regular if all the steps are regular, and if (after one or more irregular steps) all the subsequent steps are regular, then the approximation becomes regular at the step which is the first of the unbroken series of regular steps.

We do by an approximation, which is ultimately regular, obtain the value n, if only the assumed value a is nearer to n than it is to $-n$; or, say, if the condition $\text{mod.}(a-n) < \text{mod.}(a+n)$ is satisfied, and the approximation is regular from the beginning

if mod. $(a-n) < \frac{2}{3}$ mod. n, viz. this condition is a sufficient one*; the first step a to a_1 will moreover be regular under a less stringent condition imposed upon a; and it would seem that, without the condition mod. $(a-n) < \frac{2}{3}$ mod. n being satisfied, the subsequent steps will in some cases be also regular; that is, that the last-mentioned condition is not a necessary condition in order to the approximation being regular from the beginning; it is, however, the necessary and sufficient condition, *to be satisfied by the modulus* of $a-n$, in order that the approximation may be regular from the beginning. All this will clearly appear from the geometry.

Fig. 1.



We take N, N' (fig. 1) to represent the values n, $-n$; and similarly A, A_1, &c. to represent the quantities a, $a_1, \ldots$; we have then

$$AN = \text{mod.}\,(a-n), \quad A_1N = \text{mod.}\,(a_1-n)\ldots,$$

so that the approximation is measured by the approach of the points A, A_1 to N. The line NN' joining the points N, N' passes through, and is bisected at, the origin O; drawing then QQ' through O at right angles to NN' the condition

$$\text{mod.}\,(a-n) < \text{mod.}\,(a+n)$$

means that the point A, which represents the imaginary quantity a, lies on the N-side of QQ', and it will be assumed throughout that this is so. Take now on the line ON, $OM = \frac{1}{3}ON$, and on $N'M$ as diameter, describe a circle, which may be called the "circle of unfitness"; regarding as an area the segment hereof which lies on the N-side of QQ', say this is the "segment of unfitness." It will be shown that if according as A is situate inside, on the boundary of, or outside the segment of

* In the Smith's Prize Examination, Jan. 28, 1879, I gave the theorem under the following form: "If a, n are imaginary quantities, the latter of them given, and the former assumed at pleasure, subject only to the condition mod. $(a-n) < \frac{2}{3}$ mod. n; then if $a_1 = \frac{a^2+n^2}{2a}$, $a_2 = \frac{a_1^2+n^2}{2a_1}$, &c., show that the terms $a, a_1, a_2, \ldots$ will converge to the limit n." This is strictly true, but it would have been better to say "will converge regularly."

unfitness, A_1N will be greater than, equal to, or less than AN. It may be added that, if A be within or upon the boundary of the segment of unfitness, then A_1 will be outside it, but this by no means hinders that the next point A_2, or some later point, shall be within the segment of unfitness; and, further, that when A is outside the segment of unfitness, then the next point A_2, or some later point, may very well be within the segment of unfitness; the conclusion is, that A being inside the segment of unfitness, A_1N is less than AN, but that it does *not* thence follow that A_2N is less than A_1N, A_3N than $A_2N, \ldots$; the approximation although regular at the first step, may then, or afterwards, for a step or steps, cease to be regular.

If, however, AN be less than $\frac{2}{3}ON$, that is, if the condition $\text{mod.}(a-n) < \frac{2}{3}\,\text{mod.}\, n$ be satisfied, then the point A lies within the circle centre N and radius NM, and is consequently outside the segment of unfitness; A_1N being less than AN, the point A_1 is *a fortiori* outside the segment of unfitness, and the like for all the subsequent points $A_2, A_3, \ldots$, that is, in this case, the approximation is regular throughout. The circle, centre N, and radius NM, $= \frac{2}{3}\,\text{mod.}\, n$, may be called the "safe circle"; and the conclusion is that, if the point A or any subsequent point be within the safe circle, then every subsequent point will be within the safe circle, and the approximation will be regular.

The successive points $A, A_1, A_2, \ldots$ (or, as it will be convenient to call them, $A_1, A_2, \ldots$) may be obtained each from the preceding one by a simple geometrical construction.

I recall that any circle through the two (imaginary) antipoints of N, N' is a circle having its centre on the indefinite line NN'; it is such that the ratio of the distances of a point thereof from the points N, N' respectively has a certain constant value, viz. for the circles with which we are here alone concerned, those which lie on the N-side of QQ', the centres lie beyond the point N (further away, that is, from O), and the values of the ratio, distance from N to distance from N', are less unity.

Starting then from the given point A_1, for which this ratio $A_1N : A_1N'$ has a given value, suppose $A_1N = kA_1N'$, we describe a first circle (passing of course through A_1) for each point of which this ratio has the value k; let the diameter of this circle be V_1W_1, V_1 being the extremity between O and N, W_1 (not shown in the figure), that beyond N; we then describe a second circle, for which the ratio is $= k^2$; let its diameter be V_2W_2, V_2 being the extremity between O and N (or say between V_1 and N), W_2, that beyond N (or say between N and W_1); the point A_2 lies on this second circle, and is determined as the single intersection of the line V_2A_1 with the second circle. And of course drawing a third circle, for which the ratio is $= k^4$, on the diameter V_3W_3, then A_3 lies on the third circle, and is the intersection with it of the line V_3A_2, and so on; the radii of the successive circles diminish very rapidly, their centres, in like manner, continually approaching the point N; hence, the points $A_1, A_2, A_3, \ldots$, which lie on the several circles respectively approximate, and that very rapidly, to the point O. But by what precedes, if, for instance, the point A_1 be within the segment of unfitness, then also some of the subsequent points may be within the segment of unfitness, and for each point A_p,

for which this is the case, the next point A_{p+1} is at a greater distance, so that $NA_{p+1} > NA_p$; it is, however, clear that we always arrive at a point A_q, such that $NA_q < \frac{2}{3}ON$, and so soon as such a point is arrived at the approximation becomes regular.

The point A_2 determined from A_1, as above, is a point such that the subtended angle NA_2N' is = twice the subtended angle NA_1N'; or calling the latter angle ϕ, the former is $= 2\phi$. It is, in fact, this property which gives rise to the construction; for let the values of A_1N, A_1N', regarded as imaginary quantities, be called for a moment

$$\rho_1(\cos\theta_1 + i\sin\theta_1), \quad \rho_1'(\cos\theta_1' + i\sin\theta_1');$$

and, similarly, those of A_2N, A_2N' be called

$$\rho_2(\cos\theta_2 + i\sin\theta_2), \quad \rho_2'(\cos\theta_2' + i\sin\theta_2');$$

then these are the values of $a_1 - n$, $a_1 + n$, $a_2 - n$, $a_2 + n$ respectively, or we have

$$\frac{a_1 - n}{a_1 + n} = \frac{\rho_1}{\rho_1'}\{\cos(\theta_1 - \theta_1') + i\sin(\theta_1 - \theta_1')\} = k\ (\cos\ \phi + i\sin\ \phi),$$

$$\frac{a_2 - n}{a_2 + n} = \frac{\rho_2}{\rho_2'}\{\cos(\theta_2 - \theta_2') + i\sin(\theta_2 - \theta_2')\} = k^2(\cos 2\phi + i\sin 2\phi),$$

that is,

$$\frac{a_2 - n}{a_2 + n} = \left(\frac{a_1 - n}{a_1 + n}\right)^2,$$

which relation between a_2, a_1 is in fact the original relation

$$a_2 = \frac{a_1^2 + n^2}{2a_1};$$

and, conversely, a_1, a_2 being thus connected, then the representative A_2 is obtained from the representative point A_1 by the foregoing geometrical construction.

I give the analytical proofs; we may without loss of generality take, and it is convenient to do so, the axis of x as coinciding with the line ON, and to put also $ON = 1$. We then in place of the original coordinates x, y of any point take the new coordinates k, ϕ which are such that

$$\frac{x + iy - 1}{x + iy + 1} = ke^{i\phi},$$

$$\frac{x - iy - 1}{x - iy + 1} = ke^{-i\phi};$$

equations which may also be written

$$(x-1)^2 + y^2 = k^2[(x+1)^2 + y^2],$$

$$x^2 + (y-i)^2 = e^{-2i\phi}[x^2 + (y+i)^2];$$

or, what is the same thing,

$$x^2 + y^2 - 1 - 2y\cot\phi = 0,$$

where of course the equation with k shows that k is equal to the ratio of the distances of the point from the points N, N' respectively, and the equation in ϕ, taken in the second form, shows that ϕ is the angle subtended at the point by N, N'.

It is sometimes convenient to write $ke^{i\phi}$, $ke^{-2i\phi}=p$, q respectively; we then have

$$pq=k^2, \quad x+iy=\frac{1+p}{1-p}, \quad x-iy=\frac{1+q}{1-q}.$$

Suppose for a moment that we have (p_1, q_1), (p_2, q_2), (p_3, q_3) as the (p, q) coordinates of any three points, the condition that these three points may lie in a line, is given in the form, determinant $=0$, where each line of the determinant is of the form

$$\frac{1+p}{1-p}, \quad \frac{1+q}{1-q}, \quad 1,$$

or, what is the same thing, it is

$$1-pq+p-q, \quad 1-pq-p+q, \quad 1+pq-p-q,$$

or, again

$$pq-1, \quad p-q, \quad p+q-2,$$

viz. the condition is

$$\begin{vmatrix} p_1q_1-1, & p_1-q_1, & p_1+q_1-2 \\ p_2q_2-1, & p_2-q_2, & p_2+q_2-2 \\ p_3q_3-1, & p_3-q_3, & p_3+q_3-2 \end{vmatrix}=0.$$

Suppose the (k, ϕ) coordinates of the three points are (l, α), (m, β), (n, γ) respectively; then this equation is

$$\begin{vmatrix} l^2-1, & l\sin\alpha, & l\cos\alpha-1 \\ m^2-1, & m\sin\beta, & m\cos\beta-1 \\ n^2-1, & n\sin\gamma, & n\cos\gamma-1 \end{vmatrix}=0,$$

viz. it is

$$\begin{vmatrix} l^2-1, & l\sin\alpha, & l\cos\alpha \\ m^2-1, & m\sin\beta, & m\cos\beta \\ n^2-1, & n\sin\gamma, & n\cos\gamma \end{vmatrix}-\begin{vmatrix} l^2-1, & l\sin\alpha, & 1 \\ m^2-1, & m\sin\beta, & 1 \\ n^2-1, & n\sin\gamma, & 1 \end{vmatrix}=0,$$

or, what is the same thing, it is

$$\begin{aligned}&[(l^2-1)mn\sin(\beta-\gamma)+(m^2-1)nl\sin(\gamma-\alpha)+(n^2-1)lm\sin(\alpha-\beta)]\\ &\qquad+[(m^2-n^2)l\sin\alpha+(n^2-l^2)m\sin\beta+(l^2-m^2)n\sin\gamma]=0.\end{aligned}$$

If in this equation γ is put $=\pi$, and $\beta=2\alpha$, so that $\sin(\alpha-\beta)=-\sin\alpha$, the equation will contain only terms in $\sin\alpha$, and $\sin 2\alpha$, viz. it will be

$$\begin{aligned}&[\ (m^2-n^2)l+(m^2-1)nl-(n^2-1)lm]\sin\alpha\\ &+[-(l^2-1)mn+m(n^2-l^2)\qquad\qquad]\sin 2\alpha=0,\end{aligned}$$

that is,

$$l(m-1)(n+1)(m-n)\sin\alpha+m(m+1)(n-l^2)\sin 2\alpha=0,$$

or, what is the same thing,

$$(m+1)\sin\alpha\{l(n+1)(m-n)+2m(n-l^2)\cos\alpha\}=0,$$

which is satisfied for any values whatever of l, m, n, by a proper value of $\cos\alpha$; and is also satisfied irrespectively of the value of α if only $m=n=l^2$; or, writing k instead of l, say if $l=k$, $m=n=k^2$; that is, writing also ϕ in place of α, the three points

$$(k,\ \phi),\ (k^2,\ 2\phi) \text{ and } (k^2,\ \pi)$$

are in a right line; viz. the point A_1, circle k, subtended angle ϕ; the point A_2, circle k^2, subtended angle 2ϕ; and the point V_2, same circle, subtended angle π; are in a right line.

The equation of the circle of unfitness can be obtained more easily in a different manner; but I have thought it worth while to give the investigation by means of the foregoing $(p,\ q)$ coordinates.

Suppose that p_1, q_1 refer to the point A_1: then we have

$$(A_1N)^2=(x_1-1)^2+y_1^2=(x_1+iy_1-1)(x_1-iy_1-1),\quad =\left(\frac{1+p_1}{1-p_1}-1\right)\left(\frac{1+q_1}{1-q_1}-1\right),$$

that is,

$$(A_1N)^2=\frac{4p_1q_1}{1-p_1\,.\,1-q_1}.$$

Similarly, if p_2, q_2 refer to the point A_2, then

$$(A_2N)^2=\frac{4p_2q_2}{1-p_2\,.\,1-q_2},\quad =\frac{4p_1^2q_1^2}{1-p_1^2\,.\,1-q_1^2},$$

since $p_2,\ q_2=p_1^2,\ q_1^2$. The two are equal if

$$(1+p_1)(1+q_1)=p_1q_1,$$

that is,

$$p_1+q_1+1=0.$$

Writing for a moment $x_1+iy_1=\xi$, $x_1-iy_1=\eta$, we have

$$p_1,\ q_1=\frac{\xi-1}{\xi+1},\ \frac{\eta-1}{\eta+1},$$

and the equation is

$$\frac{\xi-1}{\xi+1}+\frac{\eta-1}{\eta+1}+1=0,$$

that is,

$$3\xi\eta+\xi+\eta-1=0;$$

or substituting for ξ, η their values, the equation is

$$3(x_1^2+y_1^2)+2x_1-1=0,$$

that is,

$$(x_1+1)(x_1-\tfrac{1}{3})+y_1^2=0,$$

the equation of a circle on the diameter $N'M$, which is, in fact, the before-mentioned circle of unfitness; viz. A_1 being on the circumference of this circle, or say on the boundary of the segment of unfitness, then $A_1N = A_2N$; whence also, according as A_1 is inside or outside the segment, $A_1N < A_2N$ or $> A_2N$.

Suppose A_1 to be on the circle, that is, $p_1 + q_1 + 1 = 0$; it is easy to show that the locus of A_2 is also a circle. We have in fact $(p_1 + q_1)^2 - 1 = 0$, that is,

$$p_2 + q_2 + 2k^2 - 1 = 0,$$

or say

$$\frac{\xi - 1}{\xi + 1} + \frac{\eta - 1}{\eta + 1} + 2k^2 - 1 = 0,$$

viz. this is

$$(2k^2 + 1)\,\xi\eta + (2k^2 - 1)(\xi + \eta) + 2k^2 - 3 = 0,$$

that is,

$$x_2^2 + y_2^2 + \frac{2k^2 - 1}{2k^2 + 1}\,2x_2 + \frac{2k^2 - 3}{2k^2 + 1} = 0,$$

or finally

$$(x_2 + 1)\left(x_2 - \frac{3 - 2k^2}{1 + 2k^2}\right) + y_2^2 = 0.$$

Measuring off from O in the direction of ON, a distance $OS = \dfrac{3 - 2k^2}{1 + 2k^2}$ (always $> \frac{1}{3}$, since $k^2 < 1$), the circle in question is that on the diameter $N'S$; this is a circle touching at N', and containing within it the circle of unfitness; if $k = 1$ (that is, for A_1 on the line QQ') it becomes identical with the circle of unfitness, but except in this limiting case it does not meet the circle of unfitness in any point on the N-side of QQ', that is, A_1 being on the boundary of the segment of unfitness A_2 is never on this boundary; and it thus appears that A_1 being inside the segment, A_2 is always outside the segment.

It is to be further noticed, that we have

$$\frac{(A_1N)^2}{(A_2N)^2} = \frac{1 + p_1 \,.\, 1 + q_1}{p_1 q_1},$$

or

$$\frac{(A_1N)^2}{(A_2N)^2} - 1 = \frac{1 + p_1 + q_1}{k^2} = \frac{3\xi\eta + \xi + \eta - 1}{k^2(\xi\eta + \xi + \eta + 1)}, \quad = \frac{3\,(x_1^2 + y_1^2 + \frac{2}{3}x_1 - \frac{1}{3})}{k^2\,[(x_1 + 1)^2 + y_1^2]},$$

that is,

$$\frac{(A_1N)^2}{(A_2N)^2} - 1 = \frac{3T^2}{k^2\,(A_1N)^2},$$

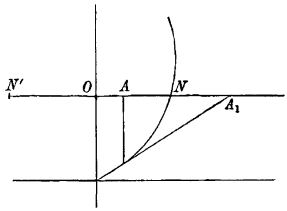
where T is the tangential distance of A_1 from the circle of unfitness; there should, it appears to me, be some more elegant formula for the ratio $A_1N \div A_2N$ which determines whether the step is regular or irregular.

It is worth noticing how the conditions

$$\text{mod.}\,(a-n) < \text{mod.}\,(a+n) \quad \text{and} \quad \text{mod.}\,(a-n) < \tfrac{2}{3}\,\text{mod.}\,n,$$

present themselves in the real theory. Making the usual construction by means of the parabola $y = x^2$, the first condition means that the point A must be taken on the N-side of O (fig. 2); the second that, in order to the regularity of the approxi-

Fig. 2.



mation, A must be taken at a distance from $O > \frac{1}{3}ON$; in fact, if (as in the figure) $OA = \frac{1}{3}ON$, then $AN = NA_1$, or the point A_1 is at an equal distance with A from N; and thence, according as OA is greater or less than $\frac{1}{3}ON$, the point A_1 is nearer or further than A to or from N.

737.

ON A COVARIANT FORMULA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 224—226.]

STARTING from the equation

$$x_1 = x - \frac{fx}{f'x},$$

which presents itself in the Newton-Fourier problem, it is easy to see that, if a be a root of the equation $fx = 0$, then

$$x_1 - a, \; = \frac{(x-a)f'x - fx}{f'x},$$

contains the factor $(x-a)^2$, that is, the equation $(x-x_1)f'x - fx = 0$, considered as an equation in x containing the parameter x_1, will have a twofold root, if x_1 is equal to any root a of the equation $fx = 0$; and, consequently, the discriminant in regard to x of the function $(x-x_1)f'x - fx$ will contain the factor fx_1. But if fx be of the order n, then the discriminant is of the order $2n-2$ in x_1, and there is consequently a remaining factor ϕx_1 of the order $n-2$.

The like theorem applies to the homogeneous form

$$(xy_1 - x_1y)\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy}\right)f(x, y) - (\alpha y_1 - \beta x_1)f(x, y),$$

which reduces itself to the foregoing on writing $\alpha = 1$, $\beta = 0$, $y = y_1 = 1$; or, changing the notation, say to the form

$$(\xi y - \eta x)\left(\alpha\frac{d}{d\xi} + \beta\frac{d}{d\eta}\right)f(\xi, \eta) - (\alpha y - \beta x)f(\xi, \eta),$$

viz. the discriminant hereof in regard to ξ, η, being a function, homogeneous of the order $2n-2$ in regard to x, y, to α, β, and to the coefficients of $f(\xi, \eta)$, will contain the factor $f(x, y)$, and there will be consequently a remaining factor of the order $n-2$ in (x, y), $2n-2$ in (α, β) and $2n-3$ in the coefficients of $f(\xi, \eta)$.

The most simple case is when $f(\xi, \eta)$ is the quadric function $(a, b, c \between \xi, \eta)^2$. The form here is

$$(\xi y - \eta x)\, 2\,\{(a\alpha + b\beta)\,\xi + (b\alpha + c\beta)\,\eta\} - (\alpha y - \beta x)(a, b, c \between \xi, \eta)^2 = (\mathrm{a}, \mathrm{b}, \mathrm{c} \between \xi, \eta)^2,$$

where the coefficients are

$$\begin{aligned}
\mathrm{a} &= 2y\,(a\alpha + b\beta) - a\,(\alpha y - \beta x), &&= a\beta x + (a\alpha + 2b\beta)\,y,\\
\mathrm{b} &= y\,(b\alpha + c\beta) - x\,(a\alpha + b\beta) - b\,(\alpha y - \beta x), &&\\
& &&= -\,a\alpha x + c\beta y\quad,\\
\mathrm{c} &= -\,2x\,(b\alpha + c\beta) - c\,(\alpha y - \beta x), &&= -\,(2b\alpha + c\beta)\,x - c\alpha y\quad;
\end{aligned}$$

and we then have

$$\begin{aligned}
\mathrm{ac} - \mathrm{b}^2 = &-(2b\alpha\beta + c\beta^2)\,ax^2\\
&-\{2ab\alpha^2 + (2ac + 4b^2)\,\alpha\beta + 2bc\beta^2\}\,xy - (a\alpha^2 + 2b\alpha\beta)\,cy^2\\
&-ax^2 \,.\, ax^2 - \{-2ac\alpha\beta\}\,xy - c\beta^2 \,.\, cy^2,
\end{aligned}$$

which is

$$= -\,(a\alpha^2 + 2b\alpha\beta + c\beta^2)\,(ax^2 + 2bxy + cy^2).$$

The discriminant is in this case

$$= -\,(a, b, c \between \alpha, \beta)^2 \,.\, (a, b, c \between x, y)^2.$$

In the case of the cubic function $(a, b, c, d \between \xi, \eta)^3$, the form is

$$\begin{aligned}
(\xi y - x\eta)\,\{3\,(a\alpha + b\beta,\ b\alpha + c\beta,\ c\alpha + d\beta \between \xi, \eta)^2\}&\\
-\,(\alpha y - \beta x)\,(a, b, c, d \between \xi, \eta)^3 &= (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \between \xi, \eta)^3,
\end{aligned}$$

the values of the coefficients being

$$\begin{aligned}
\mathrm{a} &= a\beta x + (2a\alpha + 3b\beta)\,y,\\
\mathrm{b} &= -\,a\alpha x + (\ b\alpha + 2c\beta)\,y,\\
\mathrm{c} &= -\,(2b\alpha + \ c\beta)\,x + d\beta\ y,\\
\mathrm{d} &= -\,(3c\alpha + 2d\beta)\,x - d\alpha\ y.
\end{aligned}$$

Attending only to the terms in x^2, we have

$$\begin{aligned}
\mathrm{ac} - \mathrm{b}^2 &= -\ (a\alpha^2 + 2b\alpha\beta + c\beta^2)\,ax^2,\\
\mathrm{ad} - \mathrm{bc} &= -\,2\,(b\alpha^2 + 2c\alpha\beta + d\beta^2)\,ax^2,\\
\mathrm{bd} - \mathrm{c}^2 &= \ \{(3ac - 4b^2)\,\alpha^2 + (2ad - 4bc)\,\alpha\beta - c^2\beta^2\}\,x^2.
\end{aligned}$$

And hence, in

$$a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd, = (ad - bc)^2 - 4(ac - b^2)(bd - c^2),$$

we have the term

$$4ax^3 . x\,[a\,(b\alpha^2 + 2c\alpha\beta + d\beta^2)^2 + (a\alpha^2 + 2b\alpha\beta + c\beta^2)\,\{(3ac - 4b^2)\,\alpha^2 + (2ad - 4bc)\,\alpha\beta - c^2\beta^2\}];$$

then, forming the analogous term in y^4, and assuming that the whole divides by $(a, b, c, d \between x, y)^3$, and also expanding the $\alpha\beta$-functions within the square brackets, we find

$$\text{Discriminant} = 4\,(a, b, c, d \between x, y)^3 \text{ multiplied by}$$

$$x \left| \begin{array}{l} 3a^2c - 3ab^2 \\ 2a^2d + 6abc - 8b^3 \\ 6abd + 6ac^2 - 12b^2c \\ 6acd - 6bc^2 \\ ad^2 - c^3 \end{array} \right| \between \alpha, \beta)^4 + y \left| \begin{array}{l} a^2d - b^3 \\ 6abd - 6b^2c \\ 6acd + 6b^2d - 12bc^2 \\ 2ad^2 + 6bcd - 8c^3 \\ 3bd^2 - 3c^2d \end{array} \right| \between \alpha, \beta)^4.$$

Writing down the Hessian of $(a, b, c, d \between \alpha, \beta)^3$,

$$H = (ac - b^2,\ ad - bc,\ bd - c^2 \between \alpha, \beta)^2,$$

and the cubicovariant

$$\Phi = \left\{ \begin{array}{c} a^2d - 3abc + 2b^3 \\ abd - 2ac^2 + b^2c \\ -acd + 2b^2d - bc^2 \\ -ad^2 + 3bcd - 2c^3 \end{array} \right\} (x, y)^2,$$

it is easy to see that the coefficient of x is

$$= 3\,(a, b, c \between \alpha, \beta)^2 . (H - \beta\Phi);$$

hence also that of y is

$$= 3\,(b, c, d \between \alpha, \beta)^2 . (H + \alpha\Phi),$$

and the final result is that the discriminant $= 4\,(a, b, c, d \between x, y)^3$ multiplied by

$$\{3\,(a, b, c, d \between \alpha, \beta)^2\,(x, y)\,H + (\alpha y - \beta x)\,\Phi\}.$$

It would be interesting to calculate the result for the quartic $(a, b, c, d, e \between \xi, \eta)^4$.

March 14, 1879.

738.

NOTE ON A HYPERGEOMETRIC SERIES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 268—270.]

IN the memoir on hypergeometric series, Schwarz, "Ueber diejenigen Fälle, &c.," *Crelle*, t. LXXV. (1873), pp. 292—335, the author shows, as part of his general theory, that the equation

$$\frac{d^2y}{dx^2} - \frac{\frac{2}{3} - \frac{7}{6}x}{x.1-x}\frac{dy}{dx} + \frac{\frac{1}{48}}{x.1-x}y = 0,$$

which belongs to the hypergeometric series $F(\frac{1}{2}, -\frac{1}{12}, \frac{2}{3}, x)$, is algebraically integrable, having in fact the two particular integrals

$$y^2 = \sqrt{(\alpha - \alpha^2 x^{\frac{1}{3}})} \pm \sqrt{(-\alpha^5 + \alpha x^{\frac{1}{3}})},$$

where α is a prime sixth root of -1, $\alpha^6 + 1 = 0$, or say $\alpha^4 - \alpha^2 + 1 = 0$ (see p. 326, α being for greater simplicity written instead of δ^2, and the form being somewhat simplified).

It is interesting to verify this directly; writing first $y = \sqrt{(Y)}$ and then $x = X^3$, the equation between Y, X is easily found to be

$$Y\frac{d^2Y}{dX^2} - \frac{\frac{3}{2}X^2}{1-X^3}Y\frac{dY}{dX} - \frac{1}{2}\left(\frac{dY}{dX}\right)^2 + \frac{\frac{3}{8}X}{1-X^3}Y^2 = 0,$$

and the theorem in effect is that that equation has the two particular integrals

$$Y = \sqrt{(P)} \pm \sqrt{(Q)},$$

P and Q being linear functions of X: in fact,

$$P = \alpha - \alpha^3 X,$$
$$Q = -\alpha^5 + \alpha X.$$

Starting say from the equation

$$Y = \sqrt{(P)} + \sqrt{(Q)},$$

or, as it is convenient to write it,

$$Y = P^{\frac{1}{2}} + Q^{\frac{1}{2}},$$

where P and Q are assumed to be linear functions of X, we have

$$\frac{dY}{dX} = \tfrac{1}{2}P^{-\frac{1}{2}}P' + \tfrac{1}{2}Q^{-\frac{1}{2}}Q',$$

$$\frac{d^2Y}{dX^2} = -\tfrac{1}{4}P^{-\frac{3}{2}}P'^2 - \tfrac{1}{4}Q^{-\frac{3}{2}}Q'^2,$$

and thence

$$Y\frac{d^2Y}{dX^2} = -\tfrac{1}{4}P^{-1}P'^2 - \tfrac{1}{4}Q^{-1}Q'^2 - \tfrac{1}{4}Q^{\frac{1}{2}}P^{-\frac{3}{2}}P'^2 - \tfrac{1}{4}P^{\frac{1}{2}}Q^{-\frac{3}{2}}Q'^2$$

$$Y\frac{dY}{dX} = \tfrac{1}{2}(P' + Q') \qquad + \tfrac{1}{2}P^{-\frac{1}{2}}Q^{\frac{1}{2}}P' + \tfrac{1}{2}P^{\frac{1}{2}}Q^{-\frac{1}{2}}Q',$$

$$\left(\frac{dY}{dX}\right)^2 = \tfrac{1}{4}P^{-1}P'^2 + \tfrac{1}{4}Q^{-1}Q'^2 + \tfrac{1}{2}P^{-\frac{1}{2}}Q^{-\frac{1}{2}}P'Q',$$

where P', Q' are written to denote the derived functions of P, Q respectively.

Substituting these values, the resulting equation contains on the left-hand side a rational part, and a part with the factor $P^{-\frac{3}{2}}Q^{-\frac{3}{2}}$, and it is clear the equation can only be true if these two parts are separately $= 0$. We have thus two equations which ought to be verified; viz. after a slight reduction these are found to be

$$\frac{1}{PQ}(QP'^2 + PQ'^2) + \frac{2X^2}{1-X^3}(P' + Q') - \frac{X}{1-X^3}(P + Q) = 0,$$

$$P^2Q'^2 + Q^2P'^2 + PQP'Q' + \frac{3X^2}{1-X^3}PQ(PQ' + P'Q) - \frac{3X}{1-X^3}P^2Q^2 = 0,$$

and it is very interesting to observe the manner in which these equations are, in fact, verified by the foregoing values of P, Q.

We have

$$P + Q = (\alpha - \alpha^5)(1 + X), \quad P' + Q' = \alpha - \alpha^5,$$

and hence

$$2X(P' + Q') - X(P + Q) = -(\alpha - \alpha^5)(1 - X),$$

or, in the first equation, the second part

$$\frac{2X^2}{1-X^3}(P' + Q') - \frac{X}{1-X^3}(P + Q)$$

is

$$= -(\alpha - \alpha^5)\frac{X(1-X)}{1-X^3};$$

viz. this is

$$=\frac{-(\alpha-\alpha^5)X}{1+X+X^2}.$$

We have

$$QP'^2+PQ'^2=\alpha^{10}(-\alpha^5+\alpha X)+\alpha^2(\alpha-\alpha^5X),$$
$$=\alpha^3-\alpha^{15}-(\alpha^7-\alpha^{11})X, \quad =(\alpha-\alpha^5)X;$$

and

$$PQ=-\alpha^6+(\alpha^2+\alpha^{10})X-\alpha^6X^2, \quad =1+X+X^2;$$

hence

$$\frac{1}{PQ}(QP'^2+PQ'^2)=\frac{(\alpha-\alpha^5)X}{1+X+X^2},$$

and the sum of the two parts is $=0$.

Similarly as regards the second equation, the second part

$$\frac{3X^2}{1-X^3}PQ(PQ'+P'Q)-\frac{3X}{1-X^3}P^2Q^2$$

is

$$=\frac{3PQX}{1-X^3}\{(PQ'+P'Q)X-PQ\}.$$

Here $PQ'+P'Q$ is $\alpha(\alpha-\alpha^5X)-\alpha^5(-\alpha^5+\alpha X)$, which is $=1+2X$; and PQ being $=1+X+X^2$, the term in $\{\ \}$ is

$$(1+2X)X-(1+X+X^2), \quad =-(1-X)(1+X);$$

hence, outside the $\{\ \}$ writing for PQ its value $=1+X+X^2$, the term is

$$=\frac{-3X(1+X+X^2)(1-X)(1+X)}{1-X^3}, \quad =-3X(1+X),$$

which is the value of the second part in question; the first part is

$$(PQ'+QP')^2-PQP'Q', \quad =(1+2X)^2-(1+X+X^2), \quad =3X(1+X);$$

and the sum of the two terms is thus $=0$.

739.

NOTE ON THE OCTAHEDRON FUNCTION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 280, 281.]

A SEXTIC function

$$U = (a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6,$$

such that its fourth derivative

$$\begin{aligned}(U,\ U)^4, = \quad & (ae - 4bd + 3c^2)\, x^4 \\ & + 2\,(af - 3be + 2cd)\, x^3y \\ & + \ (ag - 9ce + 8d^2)\, x^2y^2 \\ & + 2\,(bg - 3cf + 2de)\, xy^3 \\ & + \ (cg - 4df + 3e^2)\, y^4\end{aligned}$$

is identically $=0$, is considered by Dr Klein, and is called by him the octahedron function. Supposing that by a linear transformation the function is made to contain the factors x, y, or what is the same thing assuming $a=0$, $g=0$, then the equations to be satisfied become

$$-4bd + 3c^2 = 0,\quad -3be + 2cd = 0,\quad -9ce + 8d^2 = 0,\quad -3cf + 2de = 0,\quad -4df + 3e^2 = 0,$$

which are all satisfied if only $c = d = e = 0$; and then assuming, as is allowable,

$$b = -f = 1,$$

we have his canonical form $xy(x^4 - y^4)$ of the octahedron function.

But the equations may be satisfied in a different manner; viz. the first and last equations give

$$b = \frac{3c^2}{4d},\quad f = \frac{3e^2}{4d},$$

and, substituting these in the remaining equations, they become

$$\frac{c}{4d}(-9ce+8d^2)=0,\ -9ce+8d^2=0,\ \frac{e}{4d}(-9ce+8d^2)=0,$$

all satisfied if only $-9ce+8d^2=0$. Assuming $b=f=2$, the values are

$$b,\ c,\ d,\ e,\ f=2,\ 2\sqrt{(2)},\ 3,\ 2\sqrt{(2)},\ 2,$$

and the form is

$$xy\left(x^4+\frac{5}{\sqrt{(2)}}x^3y+5x^2y^2+\frac{5}{\sqrt{(2)}}xy^3+y^4\right),$$

$$=xy\left(x^2+\frac{3}{\sqrt{(2)}}xy+y^2\right)\{x^2+\sqrt{(2)}\,xy+y^2\},$$

$$=xy\left(x+\frac{1+i}{\sqrt{(2)}}y\right)\left(x+\frac{1-i}{\sqrt{(2)}}y\right)\{x+y\sqrt{(2)}\}\left(x+\frac{y}{\sqrt{(2)}}\right).$$

This is, in fact, a linear transformation of the foregoing form $XY(X^4-Y^4)$; for writing

$$X=\left(x+\frac{1+i}{\sqrt{(2)}}y\right),$$

$$Y=\left(x+\frac{1-i}{\sqrt{(2)}}y\right),$$

we have

$$X^2=x^2+(1+i)\sqrt{(2)}\,xy+iy^2,$$
$$Y^2=x^2+(1-i)\sqrt{(2)}\,xy-iy^2;$$

and therefore

$$X^2+Y^2=\quad 2x\{x+\sqrt{(2)}\,y\},$$

$$X^2-Y^2=2i\sqrt{(2)}\,y\left(x+\frac{y}{\sqrt{(2)}}\right),$$

or finally

$$XY(X^4-Y^4)=4i\sqrt{(2)}\,xy\left(x+\frac{1+i}{\sqrt{(2)}}y\right)\left(x+\frac{1-i}{\sqrt{(2)}}y\right)\{x+y\sqrt{(2)}\}\left(x+\frac{y}{\sqrt{(2)}}\right);$$

and the two forms are thus identical.

740.

ON CERTAIN ALGEBRAICAL IDENTITIES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVI. (1879), pp. 281, 282.]

IF P_0, P_1, P_2 are points on a circle, say the circle $x^2+y^2=1$, then it is possible to find functions of (P_0, P_1) and of (P_1, P_2) respectively, which are really independent of P_1, and consequently functions of only P_0 and P_2: the expression "function of a point or points" being here used to mean algebraical function of the coordinates of the point or points. Thus the functions of (P_0, P_1) and of (P_1, P_2) being $x_0x_1+y_0y_1$, $x_0y_1-x_1y_0$, and $x_1x_2+y_1y_2$, $x_1y_2-x_2y_1$, we have

$$(x_1x_2+y_1y_2)(x_0x_1+y_0y_1)+(x_1y_2-x_2y_1)(x_0y_1-x_1y_0)=x_0x_2+y_0y_2,$$

and another like equation. This depends obviously on the circumstance that the coordinates of a point of the circle are expressible by means of the functions sin, cos, $x=\cos u$, $y=\sin u$; and the identity written down is obtained by expressing the cosine of u_2-u_0, $=(u_2-u_1)+(u_1-u_0)$, in terms of the cosines and sines of u_2-u_1 and u_1-u_0.

Evidently the like property holds good for a curve, such that the coordinates of any point of it can be expressed by means of "additive" functions of a parameter u; where, by an additive function $f(u)$, is meant a function such that $f(u+v)$ is an algebraical function of $f(u), f(v)$; the sine and cosine are each of them an additive function, because

$$\sin(u+v)=\sin u\sqrt{(1-\sin^2 v)}+\sin v\sqrt{(1-\sin^2 u)},$$

and, similarly, for the cosine. But it is convenient to consider pairs or groups $f(u)$, $\phi(u), \ldots$, where $f(u+v)$, $\phi(u+v), \ldots$ are each of them an algebraical (rational) function of $f(u)$, $\phi(u), \ldots, f(v)$, $\phi(v), \ldots$; the sine and cosine are such a group, and so also are the elliptic functions sn, cn, dn; but the H and Θ, or say the ϑ-functions generally, are not additive.

In the case of the elliptic functions, we may consider the quadriquadric curve

$$y^2 = 1 - x^2, \quad z^2 = 1 - k^2x^2,$$

so that the coordinates of a point on the curve are $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$. Taking then P_0, P_1, P_2, points on the curve, and (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), the coordinates of these points respectively, we have in the same way, from $u_2 - u_0 = (u_2 - u_1) + (u_1 - u_0)$, three equations, of which the first is

$$\frac{x_2y_0z_0 - x_0y_2z_2}{1 - k^2x_0^2x_2^2} = \frac{\begin{array}{l}(1 - k^2x_1^2x_2^2)(x_2y_1z_1 - x_1y_2z_2)(y_0y_1 + x_0z_0x_1z_1)(z_0z_1 + k^2x_0y_0x_1y_1)\\ + (1 - k^2x_0^2x_1^2)(x_1y_0z_0 - x_0y_1z_1)(y_1y_2 + x_1z_1x_2z_2)(z_1z_2 + k^2x_1y_1x_2y_2)\end{array}}{(1 - k^2x_0^2x_1^2)^2(1 - k^2x_1^2x_2^2)^2 - k^2(x_1y_0z_0 - x_0y_1z_1)^2(x_2y_1z_1 - x_1y_2z_2)^2}.$$

The form of the right-hand side is

$$\frac{A + Bx_1y_1z_1}{C + Dx_1y_1z_1},$$

where A, B, C, D are each of them rational as regards x_1^2; and it is easy to see that the equation can only subsist under the condition that we have separately

$$\frac{x_2y_0z_0 - x_0y_2z_2}{1 - k^2x_0^2x_2^2} = \frac{A}{C} = \frac{B}{D},$$

implying of course the identity $AD - BC = 0$. The values of B and D are found without difficulty; we, in fact, have

$$B = 2k^2(x_2^2 \quad - x_0^2 \quad)(x_1^2y_0z_0y_2z_2 + x_0x_2y_1^2z_1^2),$$
$$D = 2k^2(x_2y_0z_0 + x_0y_2z_2)(x_1^2y_0z_0y_2z_2 + x_0x_2y_1^2z_1^2),$$

so that, comparing the left-hand side with $B \div D$, we have the identity

$$x_2^2y_0^2z_0^2 - x_0^2y_2^2z_2^2 = (x_2^2 - x_0^2)(1 - k^2x_2^2x_0^2),$$

which is right. The comparison with $A \div C$ would be somewhat more difficult to effect.

741.

ON A THEOREM OF ABEL'S RELATING TO A QUINTIC EQUATION.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. III. (1880), pp. 155—159.]

THE theorem in question is given, *Œuvres Complètes*, [Christiania, 1881], t. II., p. 266, as an extract from a letter to Crelle dated 14th March, 1826, as follows:

"Si une équation du cinquième degré dont les coefficients sont *des nombres rationnels* est résoluble algébriquement, on peut donner aux racines la forme suivante:

$$x = c + Aa^{\frac{1}{5}}a_1^{\frac{2}{5}}a_2^{\frac{4}{5}}a_3^{\frac{3}{5}} + A_1a_1^{\frac{1}{5}}a_2^{\frac{3}{5}}a_3^{\frac{4}{5}}a^{\frac{2}{5}} + A_2a_2^{\frac{1}{5}}a_3^{\frac{2}{5}}a^{\frac{4}{5}}a_1^{\frac{3}{5}} + A_3a_3^{\frac{1}{5}}a^{\frac{2}{5}}a_1^{\frac{4}{5}}a_2^{\frac{3}{5}},$$

où

$$\begin{aligned}
a &= m + n\sqrt{(1+e^2)} + \sqrt{[h(1+e^2+\sqrt{(1+e^2)})]},\\
a_1 &= m - n\sqrt{(1+e^2)} + \sqrt{[h(1+e^2-\sqrt{(1+e^2)})]},\\
a_2 &= m + n\sqrt{(1+e^2)} - \sqrt{[h(1+e^2+\sqrt{(1+e^2)})]},\\
a_3 &= m - n\sqrt{(1+e^2)} - \sqrt{[h(1+e^2-\sqrt{(1+e^2)})]},\\
A &= K + K'a + K''a_2 + K'''aa_2, \quad A_1 = K + K'a_1 + K''a_3 + K'''a_1a_3,\\
A_2 &= K + K'a_2 + K''a + K'''aa_2, \quad A_3 = K + K'a_3 + K''a_1 + K'''a_1a_3.
\end{aligned}$$

Les quantités c, h, e, m, n, K, K', K'', K''' sont des nombres *rationnels*. Mais de cette manière l'équation $x^5 + ax + b = 0$ n'est pas résoluble tant que a et b sont des quantités quelconques. J'ai trouvé de pareils théorèmes pour les équations du 7ème, 11ème, 13ème, etc. degré."

It is easy to see that x is the root of a quintic equation, the coefficients of which are rational and integral functions of a, a_1, a_2, a_3: these coefficients are not symmetrical functions of a, a_1, a_2, a_3, but they are functions which remain unaltered

by the cyclical change a into a_1, a_1 into a_2, a_2 into a_3, a_3 into a. But the coefficients of the quintic equation must be rational functions of $c, h, e, m, n, K, K', K'', K'''$: hence regarding a, a_1, a_2, a_3, as the roots of a quartic equation, the coefficients of this equation being rational functions of m, n, e, h, this equation must be such that every rational function of the roots, unchanged by the aforesaid cyclical change of the roots, shall be rationally expressible in terms of these quantities m, n, e, h: or, what is the same thing, the group of the quartic equation, using the term "group of the equation" in the sense assigned to it by Galois, must be $aa_1a_2a_3$, $a_1a_2a_3a$, $a_2a_3aa_1$, $a_3aa_1a_2$. And conversely, the quartic equation being of this form, x will be the root of a quintic equation, the coefficients whereof are rational and integral functions of $c, h, e, m, n, K, K', K'', K'''$.

To investigate the form of a quartic equation having the property just referred to, let it be proposed to find γ, γ' functions of e, h, such that $\gamma^2+\gamma'^2$ is a rational function of e, h, but that $\gamma^2-\gamma'^2$, $\gamma\gamma'$ are rational multiples of the same quadric radical $\sqrt{\theta}$. Assume that we have

$$\gamma^2-\gamma'^2=2p\sqrt{\theta}, \quad \gamma\gamma'=q\sqrt{\theta};$$

then

$$(\gamma^2+\gamma'^2)^2=4(p^2+q^2)\theta;$$

that $\gamma^2+\gamma'^2$ may be rational, we must have $p^2+q^2=\lambda^2\theta$, or say $p^2+q^2=h^2\theta$; hence, $\theta=\frac{p^2}{h^2}+\frac{q^2}{h^2}$ must be a sum of two squares, or, assuming one of these equal to unity and the other of them equal to e^2, say $\theta=1+e^2$, we satisfy the required equation by taking $p=h$, $q=he$: viz. we thus have

$$\gamma^2-\gamma'^2=2h\sqrt{1+e^2}, \quad \gamma\gamma'=he\sqrt{1+e^2}, \quad \gamma^2+\gamma'^2=2h(1+e^2);$$

and thence also

$$\gamma^2=h(1+e^2+\sqrt{1+e^2}), \quad \gamma'^2=h(1+e^2-\sqrt{1+e^2}),$$

the roots of these expressions, or values of γ, γ', being such that

$$\gamma\gamma'=he\sqrt{1+e^2}.$$

Taking now α rational, $=m$ suppose, and β a rational multiple of

$$\sqrt{1+e^2}, \ =h\sqrt{1+e^2},$$

suppose; it is easy to see that the quartic equation which has for its roots

$$a, a_1, a_2, a_3=\alpha+\beta+\gamma, \ \alpha-\beta+\gamma', \ \alpha+\beta-\gamma, \ \alpha-\beta-\gamma',$$

has the property in question, viz. that every rational function of the roots unchangeable by the cyclical change a into a_1, a_1 into a_2, a_2 into a_3, a_3 into a, is rationally expressible in terms of e, h, m, n.

It will be sufficient to give the proof in the case of a rational and integral function; such a function, unchangeable as aforesaid, is of the form

$$\phi(a, a_1, a_2, a_3)+\phi(a_1, a_2, a_3, a)+\phi(a_2, a_3, a, a_1)+\phi(a_3, a, a_1, a_2);$$

and if $\phi(a, a_1, a_2, a_3)$ contains a term $\alpha^m\beta^n\gamma^p\gamma'^q$, then the other three functions will contain respectively the terms

$$\alpha^m(-\beta)^n\gamma'^p(-\gamma)^q,\quad \alpha^m\beta^n(-\gamma)^p(-\gamma')^q,\quad \alpha^m(-\beta)^n(-\gamma')^p(\gamma)^q;$$

viz. the sum of the four terms is

$$=\alpha^m\beta^n[\{1+(-)^{p+q}1\}\gamma^p\gamma'^q+\{(-)^{n+p}1+(-)^{n+q}1\}\gamma'^q\gamma'^p].$$

This obviously vanishes unless p and q are both even, or both odd; and the cases to be considered are 1°, n even, p and q even; 2°, n odd, p and q even; 3°, n even, p and q odd; 4°, n odd, p and q odd. Writing, for greater distinctness, $2n$ or $2n+1$ for n, according as n is even or odd, and similarly for p and q, the term is, in the four cases respectively,

$$\begin{aligned}
&=2\alpha^m\beta^{2n}\quad(\gamma^{2p}\quad\gamma'^{2q}\quad+\gamma^{2q}\quad\gamma'^{2p}),\\
&=2\alpha^m\beta^{2n+1}(\gamma^{2p}\quad\gamma'^{2q}\quad-\gamma^{2q}\quad\gamma'^{2p}),\\
&=2\alpha^m\beta^{2n}\quad(\gamma^{2p+1}\gamma'^{2q+1}-\gamma^{2q+1}\gamma'^{2p+1}),\\
&=2\alpha^m\beta^{2n+1}(\gamma^{2p+1}\gamma'^{2q+1}+\gamma^{2q+1}\gamma'^{2p+1}).
\end{aligned}$$

The second, third, and fourth expressions contain the factors

$$\beta(\gamma^2-\gamma'^2),\quad \gamma\gamma'(\gamma^2-\gamma'^2),\quad \beta\gamma\gamma',$$

respectively; and the first expression as it stands, and the other three divested of these factors respectively are rational functions of α, β^2, γ^2, γ'^2, that is, they are rational functions of m, n, e, h. But the omitted factors $\beta(\gamma^2-\gamma'^2)$, $\gamma\gamma'(\gamma^2-\gamma'^2)$, $\beta\gamma\gamma'$, $=2nh(1+e^2)$, $2h^2e(1+e^2)$, $nhe(1+e^2)$ are rational functions of n, h, e; hence each of the original four expressions is a rational function of m, n, h, e; and the entire function

$$\phi(a, a_1, a_2, a_3)+\phi(a_1, a_2, a_3, a)+\phi(a_2, a_3, a, a_1)+\phi(a_3, a, a_1, a_2)$$

is a rational function of m, n, h, e.

Replacing α, β, γ, γ' by their values, the roots of the quartic equation are

$$\begin{aligned}
&m+n\sqrt{(1+e^2)}+\sqrt{[h(1+e^2+\sqrt{(1+e^2)})]},\\
&m-n\sqrt{(1+e^2)}+\sqrt{[h(1+e^2-\sqrt{(1+e^2)})]},\\
&m+n\sqrt{(1+e^2)}-\sqrt{[h(1+e^2+\sqrt{(1+e^2)})]},\\
&m-n\sqrt{(1+e^2)}-\sqrt{[h(1+e^2-\sqrt{(1+e^2)})]}.
\end{aligned}$$

And I stop to remark that taking $m, n, e, h=-\frac{1}{4}, +\frac{1}{4}, 2, -\frac{1}{8}$ respectively, the roots are

$$\begin{aligned}
&-\tfrac{1}{4}+\tfrac{1}{4}\sqrt{5}+\sqrt{[-\tfrac{1}{8}(5+\sqrt{5})]},\\
&-\tfrac{1}{4}-\tfrac{1}{4}\sqrt{5}+\sqrt{[-\tfrac{1}{8}(5-\sqrt{5})]},\\
&-\tfrac{1}{4}+\tfrac{1}{4}\sqrt{5}-\sqrt{[-\tfrac{1}{8}(5+\sqrt{5})]},\\
&-\tfrac{1}{4}-\tfrac{1}{4}\sqrt{5}-\sqrt{[-\tfrac{1}{8}(5-\sqrt{5})]},
\end{aligned}$$

viz. these are the imaginary fifth roots of unity, or roots r, r^2, r^4, r^3 of the quartic equation $x^4+x^3+x^2+x+1=0$; which equation, as is well known, has the group $rr^2r^4r^3$, $r^2r^4r^3r$, $r^4r^3rr^2$, $r^3rr^2r^4$.

Reverting to Abel's expression for x, and writing this for a moment in the form

$$x=c+p+s+r+q,$$

the quintic equation in x is

$$\begin{aligned}0=&(x-c)^5\\&+(x-c)^3.-5\,(pr+qs)\\&+(x-c)^2.-5\,(p^2s+q^2p+r^2q+s^2r)\\&+(x-c)\,.-5\,(p^3q+q^3r+r^3s+s^3p)+5\,(p^2r^2+q^2s^2)-5pqrs\\&+(x-c)^0.-\quad(p^5+q^5+r^5+s^5)\\&\qquad+5\,(p^2rs+q^3sp+r^2pq+s^3qr)\\&\qquad-5\,(p^2q^2r+q^2r^2s+r^2s^2p+s^2p^2q).\end{aligned}$$

If we substitute herein for p, q, r, s their values, then, altering the order of the terms, the final result is found to be

$$\begin{aligned}0=&(x-c)^5\\&+(x-c)^3.-5\,(AA_2+A_1A_3)\,aa_1a_2a_3\\&+(x-c)^2.-5\,(A^2A_1a_2a_3+A_1^2A_2a_3a+A_2^2A_3aa_1+A_3^2Aa_1a_2)\,aa_1a_2a_3\\&+(x-c)\,.-5\,(A^3A_3a_1a_2^2a_3+A_1^3Aa_2a_3^2a+A_2^3A_1a_3a^2a_1+A_3^3A_2aa_1^2a_2)\,aa_1a_2a_3\\&\qquad+5\,(A^2A_2^2+A_2^2A_3^2-A_1A_2A_3A)\,(aa_1a_2a_3)^2\\&+(x-c)^0.-\quad(A^5a_1a_2^3a_3^2+A_1^5a_2a_3^3a_1^2+A_2^5a_3a^3a_1^2+A_3^5aa_1^3a_2^2)\,aa_1a_2a_3\\&\qquad+5\,(A^3A_1A_2a_2a_3+A_1^3A_2A_3a_3a+A_2^3A_3Aaa_1+A_3^3AA_1a_1a_2)\,(aa_1a_2a_3)^2\\&\qquad-5\,(A^2A_3^2A_2a_1a_3+A_1^2A^2A_3a_2a_3+A_2^2A_1^2Aa_3a+A_3^2A_2^2A_1aa_1)\,(aa_1a_2a_3)^2;\end{aligned}$$

viz. considering herein A, A_1, A_2, A_3 as standing for their values

$$K+K'a+K''a_2+K'''aa_2,\ \&c.$$

respectively, each coefficient is a function of a, a_1, a_2, a_3, which is unaltered by the cyclical change of these values and therefore is a rational function of

$$m,\ n,\ e,\ h,\ K,\ K',\ K'',\ K'''.$$

742.

ON THE TRANSFORMATION OF COORDINATES.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. III. (1880), pp. 178—184.]

THE formulæ for the transformation between two sets of oblique coordinates assume a very elegant form when presented in the notation of matrices. I call to mind that a matrix denotes a system of quantities arranged in a square form

$$\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix},$$

see my "Memoir on the Theory of Matrices," *Phil. Trans.* t. CXLVIII. (1858), pp. 17—37, [152]; moreover $(\alpha, \beta, \gamma \between x, y, z)$ denotes $\alpha x + \beta y + \gamma z$, and so

$$\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} \between x, y, z)$$

denotes

$$(\alpha x + \beta y + \gamma z,\ \alpha' x + \beta' y + \gamma' z,\ \alpha'' x + \beta'' y + \gamma'' z),$$

and again

$$\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} \between x, y, z \between \xi, \eta, \zeta) \text{ denotes } \begin{aligned} &\xi(\alpha x + \beta y + \gamma z) \\ &+ \eta(\alpha' x + \beta' y + \gamma' z) \\ &+ \zeta(\alpha'' x + \beta'' y + \gamma'' z). \end{aligned}$$

Consequently

$$\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} \between x, y, z \between \xi, \eta, \zeta) = \begin{pmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{pmatrix} \between \xi, \eta, \zeta \between x, y, z).$$

In the case of a symmetrical matrix

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right),$$

the equal expressions

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right) \between x, y, z \between \xi, \eta, \zeta), \; = \left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right) \between \xi, \eta, \zeta \between x, y, z),$$

are also written

$$(a, b, c, f, g, h \between x, y, z \between \xi, \eta, \zeta), \text{ or } (a, \ldots \between \xi, \eta, \zeta \between x, y, z).$$

In particular, if

$$(\xi, \eta, \zeta) = (x, y, z),$$

then

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right) \between x, y, z)^2 \text{ is written } (a, b, c, f, g, h \between x, y, z)^2.$$

Two matrices are compounded together according to the law

$$\left(\begin{array}{ccc} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right) \between \left(\begin{array}{ccc} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right) = \left(\begin{array}{ccc} a\,, & b\,, & c\,) \\ (a', & b', & c') \\ (a'', & b'', & c'') \end{array}\right| \begin{array}{ccc} \underline{(\alpha, \alpha', \alpha''), & (\beta, \beta', \beta''), & (\gamma, \gamma', \gamma'')}; \\ \text{,,} & \text{,,} & \text{,,} \\ \text{,,} & \text{,,} & \text{,,} \end{array}$$

viz. in the compound matrix, the top-line is

$$(a, b, c \between \alpha, \alpha', \alpha''), \quad (a, b, c \between \beta, \beta', \beta''), \quad (a, b, c \between \gamma, \gamma', \gamma''),$$

and the other two lines are the like functions with (a', b', c'), and (a'', b'', c''), respectively, in the place of (a, b, c).

The inverse matrix is the matrix the terms of which are the minors of the determinant formed out of the original matrix, each minor being divided by this determinant, viz.

$$\left(\begin{array}{ccc} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)^{-1} = \frac{1}{\nabla} \left(\begin{array}{ccc} \beta'\gamma'' - \beta''\gamma', & \beta''\gamma - \beta\gamma'', & \beta\gamma' - \beta'\gamma \\ \gamma'\alpha'' - \gamma''\alpha', & \gamma''\alpha - \gamma\alpha'', & \gamma\alpha' - \gamma'\alpha \\ \alpha'\beta'' - \alpha''\beta', & \alpha''\beta - \alpha\beta'', & \alpha\beta' - \alpha'\beta \end{array}\right),$$

where ∇ is the determinant

$$\left|\begin{array}{ccc} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right|.$$

Coming now to the question of transformation, write

$$
\begin{array}{c|ccc|ccc|c|c|c}
 & x & y & z & x_1 & y_1 & z_1 & & x\ y\ z & x_1\ y_1\ z_1 \\
\hline
x & 1 & \nu & \mu & \alpha & \alpha' & \alpha'' & =x & & \\
y & \nu & 1 & \lambda & \beta & \beta' & \beta'' & y & \Omega & W \\
z & \mu & \lambda & 1 & \gamma & \gamma' & \gamma'' & z & & \\
\hline
x_1 & \alpha & \beta & \gamma & 1 & \nu_1 & \mu_1 & x_1 & & \\
y_1 & \alpha' & \beta' & \gamma' & \nu_1 & 1 & \lambda_1 & y_1 & V & \Omega_1 \\
z_1 & \alpha'' & \beta'' & \gamma'' & \mu_1 & \lambda_1 & 1 & z_1 & &
\end{array}
$$

viz. the axes of x, y, z are inclined to each other at angles the cosines whereof are λ, μ, ν: those of x_1, y_1, z_1 are inclined to each other at angles the cosines whereof are λ_1, μ_1, ν_1: and the cosines of the inclinations of the two sets of axes to each other are α, β, γ; α', β', γ'; α'', β'', γ'': as is more clearly indicated in the diagram, the top-line showing that cosine-inclinations of x to

$$x,\ y,\ z,\ x_1,\ y_1,\ z_1,$$

are

$$1,\ \nu,\ \mu,\ \alpha,\ \alpha',\ \alpha'',$$

respectively, and the like for the other lines of the diagram. The letters Ω, Ω_1, V, W are used to denote matrices, viz. as appearing by the diagram, these are

$$
\begin{pmatrix} 1, & \nu, & \mu \\ \nu, & 1, & \lambda \\ \mu, & \lambda, & 1 \end{pmatrix},\quad
\begin{pmatrix} 1, & \nu_1, & \mu_1 \\ \nu_1, & 1, & \lambda_1 \\ \mu_1, & \lambda_1, & 1 \end{pmatrix},\quad
\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix},\quad
\begin{pmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{pmatrix},
$$

respectively.

The coordinates (x, y, z) and (x_1, y_1, z_1) form each set a broken line extending from the origin to the point; hence projecting on the axes of x, y, z and on those of x_1, y_1, z_1 respectively, we have two sets, each of three equations, which may be written

$$(\Omega \between x, y, z) = (W \between x_1, y_1, z_1),$$
$$(V \between x, y, z) = (\Omega_1 \between x_1, y_1, z_1);$$

where of course each set implies the other set.

We have

$$(x, y, z) = (\Omega^{-1} W \between x_1, y_1, z_1), \ = (V^{-1} \Omega_1 \between x_1, y_1, z_1),$$
$$(x_1, y_1, z_1) = (W^{-1} \Omega \between x, y, z), \ = (\Omega_1^{-1} V \between x, y, z),$$

the first giving in two forms (x, y, z) as linear functions of (x_1, y_1, z_1), and the second giving in two forms (x_1, y_1, z_1) as linear functions of (x, y, z); comparing the two forms for each set, we have

$$\Omega^{-1} W = V^{-1} \Omega_1, \quad W^{-1} \Omega = \Omega_1^{-1} V,$$

or, what is the same thing,

$$V\Omega^{-1}W = \Omega_1, \qquad W\Omega_1^{-1}V = \Omega,$$

where in each equation the two sides are matrices which must be equal term by term to each other; but, the matrices being symmetrical, the equation thus gives (not nine but only) six equations. Writing

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}) = (1-\lambda^2,\ 1-\mu^2,\ 1-\nu^2,\ \mu\nu-\lambda,\ \nu\lambda-\mu,\ \lambda\mu-\nu),$$

and

$$K = 1-\lambda^2-\mu^2-\nu^2+2\lambda\mu\nu,$$

we have

$$\Omega^{-1} = \frac{1}{K}\begin{pmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{pmatrix}.$$

The first equation, written in the form

$$V\begin{pmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{pmatrix} W = K\Omega_1,$$

denotes the six equations

$$\begin{aligned}
(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h})(\alpha,\ \beta,\ \gamma)^2 &= K\ , \\
\text{,,}\quad (\alpha',\ \beta',\ \gamma')^2 &= K\ , \\
\text{,,}\quad (\alpha'',\ \beta'',\ \gamma'')^2 &= K\ , \\
\text{,,}\quad (\alpha',\ \beta',\ \gamma')(\alpha'',\ \beta'',\ \gamma'') &= K\lambda_1, \\
\text{,,}\quad (\alpha'',\ \beta'',\ \gamma'')(\alpha,\ \beta,\ \gamma) &= K\mu_1, \\
\text{,,}\quad (\alpha,\ \beta,\ \gamma)(\alpha',\ \beta',\ \gamma') &= K\nu_1.
\end{aligned}$$

And, similarly, writing

$$(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{f}_1,\ \mathrm{g}_1,\ \mathrm{h}_1) = (1-\lambda_1^2,\ 1-\mu_1^2,\ 1-\nu_1^2,\ \mu_1\nu_1-\lambda_1,\ \nu_1\lambda_1-\mu_1,\ \lambda_1\mu_1-\nu_1),$$

and

$$K_1 = 1-\lambda_1^2-\mu_1^2-\nu_1^2+2\lambda_1\mu_1\nu_1,$$

then

$$\Omega_1^{-1} = \frac{1}{K_1}\begin{pmatrix} \mathrm{a}_1, & \mathrm{h}_1, & \mathrm{g}_1 \\ \mathrm{h}_1, & \mathrm{b}_1, & \mathrm{f}_1 \\ \mathrm{g}_1, & \mathrm{f}_1, & \mathrm{c}_1 \end{pmatrix};$$

and the second equation, written in the form

$$W\begin{pmatrix} \mathrm{a}_1, & \mathrm{h}_1, & \mathrm{g}_1 \\ \mathrm{h}_1, & \mathrm{b}_1, & \mathrm{f}_1 \\ \mathrm{g}_1, & \mathrm{f}_1, & \mathrm{c}_1 \end{pmatrix} V = K_1\Omega,$$

denotes the six equations

$$\begin{aligned}
&(a_1, b_1, c_1, f_1, g_1, h_1 \between \alpha, \alpha', \alpha'')^2 &&= K_1 \;, \\
&\quad ,, \qquad (\beta, \beta', \beta'')^2 &&= K_1 \;, \\
&\quad ,, \qquad (\gamma, \gamma', \gamma'')^2 &&= K_1 \;, \\
&\quad ,, \qquad (\beta, \beta', \beta'' \between \gamma, \gamma', \gamma'') &&= K_1\lambda_1, \\
&\quad ,, \qquad (\gamma, \gamma', \gamma'' \between \alpha, \alpha', \alpha'') &&= K_1\mu_1, \\
&\quad ,, \qquad (\alpha, \alpha', \alpha'' \between \beta, \beta', \beta'') &&= K_1\nu_1.
\end{aligned}$$

The two sets each of six equations are, in fact, equivalent to a single set of six equations, and serve to express the relations between the nine cosines

$$(\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''),$$

and the cosines (λ, μ, ν) and $(\lambda_1, \mu_1, \nu_1)$. Observe that the nine cosines are *not* (as in the rectangular transformation) the coefficients of transformation between the two sets of coordinates.

From the original linear relations between the coordinates, multiplying the equations of the first set by x, y, z and adding, and again multiplying the equations of the second set by (x_1, y_1, z_1) and adding, we have

$$(\Omega \between x, y, z)^2 = (W \between x_1, y_1, z_1 \between x, y, z),$$
$$(\Omega_1 \between x_1, y_1, z_1)^2 = (V \between x, y, z \between x_1, y_1, z_1).$$

But

$$(W \between x_1, y_1, z_1 \between x, y, z)$$

and

$$(V \between x, y, z \between x_1, y_1, z_1)$$

denote one and the same function; hence

$$(\Omega \between x, y, z)^2 = (\Omega_1 \between x_1, y_1, z_1)^2,$$

that is,

$$(1, 1, 1, \lambda, \mu, \nu \between x, y, z)^2 = (1, 1, 1, \lambda_1, \mu_1, \nu_1 \between x_1, y_1, z_1)^2,$$

or the linear relations between (x, y, z) and (x_1, y_1, z_1) are such as to transform one of these quadric functions into the other: the two quadrics, in fact, denote the squared distance from the origin expressed in terms of the coordinates (x, y, z) and (x_1, y_1, z_1) respectively.

Since the nine cosines are connected by six equations, there should exist values containing three arbitrary constants, and satisfying these equations identically: but, by what just precedes, it appears that the problem of determining these values is, in fact, that of finding the linear transformation between two given quadric functions: the problem of the linear transformation of a quadric function into itself has an elegant solution; but it would seem that this is not the case for the transformation between two different functions.

The foregoing equation

$$K = (\mathrm{a,\ b,\ c,\ f,\ g,\ h} \between \alpha,\ \beta,\ \gamma)^2,$$

is a relation between λ, μ, ν, the cosines of the sides of a spherical triangle, and $(\alpha,\ \beta,\ \gamma)$ the cosines of the distances of a point P from the three vertices: it can be at once verified by means of the relation $A+B+C=2\pi$, and thence

$$1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C=0,$$

which connects the angles A, B, C which the sides subtend at P. Writing a, b, c for λ, μ, ν, and f, g, h for α, β, γ, the relation is

$$1-a^2-b^2-c^2+2abc=(1-a^2)f^2+(1-b^2)g^2+(1-c^2)h^2$$
$$+2(bc-a)gh+2(ca-b)hf+2(ab-c)fg,$$

viz. this is

$$1-a^2-b^2-c^2-f^2-g^2-h^2+2abc+2agh+2bhf+2cfg$$
$$-a^2f^2-b^2g^2-c^2h^2+2bcgh+2cahf+2abfg=0;$$

where $(a,\ b,\ c,\ f,\ g,\ h)$ are the cosines of the sides of a spherical quadrangle; $(a,\ b,\ c)$, $(a,\ h,\ g)$, $(h,\ b,\ f)$, $(g,\ f,\ c)$ belong respectively to sides forming a triangle, and the remaining sides $(f,\ g,\ h)$, $(b,\ c,\ f)$, $(c,\ a,\ g)$, $(a,\ b,\ h)$ are sides meeting in a vertex.

The equation

$$K\nu_1 = (\mathrm{a,\ b,\ c,\ f,\ g,\ h} \between \alpha,\ \beta,\ \gamma)(\alpha',\ \beta',\ \gamma')$$

is a relation between λ, μ, ν, the cosines of the sides of a spherical triangle; α, β, γ, the cosines of the distances of a point P from the three vertices; α', β', γ', the cosines of the distances of a point Q from the three vertices; and ν_1, the cosine of the distance PQ.

Drawing a figure, it is at once seen that

$$\nu_1 = \alpha\alpha' + \sqrt{1-\alpha^2}\sqrt{1-\alpha'^2}\cos(\theta-\theta'),$$

where

$$\cos\theta = \frac{\beta-\alpha\nu}{\sqrt{1-\alpha^2}\sqrt{1-\nu^2}},$$

and therefore

$$\sin\theta = \frac{\sqrt{\nabla}}{\sqrt{1-\alpha^2}\sqrt{1-\nu^2}};$$

also

$$\cos\theta' = \frac{\beta'-\alpha'\nu}{\sqrt{1-\alpha'^2}\sqrt{1-\nu^2}},$$

and therefore

$$\sin\theta' = \frac{\sqrt{\nabla'}}{\sqrt{1-\alpha'^2}\sqrt{1-\nu^2}},$$

the values of ∇, ∇' being

$$\nabla = 1 - \alpha^2 - \beta^2 - \nu^2 + 2\alpha\beta\nu,$$
$$\nabla' = 1 - \alpha'^2 - \beta'^2 - \nu^2 + 2\alpha'\beta'\nu;$$

the resulting value of ν_1 is therefore

$$\nu_1 = \alpha\alpha' + \frac{1}{1-\nu^2}\{(\beta - \alpha\nu)(\beta' - \alpha'\nu) + \sqrt{\nabla\nabla'}\}.$$

The equations

$$K = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(\alpha, \beta, \gamma)^2, \quad K = (\mathrm{a}, \ldots)(\alpha', \beta', \gamma')^2,$$

give

$$(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)^2 = K\nabla,$$
$$(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma')^2 = K\nabla':$$

and we therefore have

$$(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma') = K\sqrt{\nabla\nabla'};$$

recollecting that $1 - \nu^2 = \mathrm{c}$, the formula thus is

$$\nu_1 = \alpha\alpha' + \frac{1}{\mathrm{c}}\left\{(\beta - \alpha\nu)(\beta' - \alpha'\nu) + \frac{1}{K}(\mathrm{g}\alpha + \mathrm{f}\beta + \mathrm{c}\gamma)(\mathrm{g}\alpha' + \mathrm{f}\beta' + \mathrm{c}\gamma')\right\},$$

or say,

$$K\nu_1 = K\alpha\alpha' + \frac{1}{\mathrm{c}}\{K(\beta - \alpha\nu)(\beta' - \alpha'\nu) + (\mathrm{g}\alpha + \mathrm{f}\beta)(\mathrm{g}\alpha' + \mathrm{f}\beta')\} + \mathrm{g}(\alpha\gamma' + \alpha'\gamma) + \mathrm{f}(\beta\gamma' + \beta'\gamma) + \mathrm{c}\gamma\gamma'.$$

The sum of the first and second terms is readily found to be

$$= \mathrm{a}\alpha\alpha' + \mathrm{b}\beta\beta' + \mathrm{h}(\alpha\beta' + \alpha'\beta);$$

and the equation thus becomes

$$K\nu_1 = (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(\alpha, \beta, \gamma)(\alpha', \beta', \gamma'),$$

as it should do.

743.

ON THE NEWTON-FOURIER IMAGINARY PROBLEM.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. III. (1880), pp. 231, 232.]

THE Newtonian process of approximation to the root of a numerical equation $f(u)=0$, consists in deriving from an assumed approximate root ξ a new value $\xi_1=\xi-\frac{f(\xi)}{f'(\xi)}$, which should be a closer approximation to the root sought for: taking the coefficients of $f(u)$ to be real, and also the root sought for, and the assumed value ξ, to be each of them real, Fourier investigated the conditions under which ξ_1 is in fact a closer approximation. But the question may be looked at in a more general manner: ξ may be any real or imaginary value, and we have to inquire in what cases the series of derived values

$$\xi_1=\xi-\frac{f(\xi)}{f'(\xi)},\quad \xi_2=\xi_1-\frac{f(\xi_1)}{f'(\xi_1)},\ldots$$

converge to a root, real or imaginary, of the equation $f(u)=0$. Representing as usual the imaginary value ξ, $=x+iy$, by means of the point whose coordinates are x, y, and in like manner ξ_1, $=x_1+iy_1$, &c., then we have a problem relating to an infinite plane; the roots of the equation are represented by points A, B, $C,\ldots$; the value ξ is represented by an arbitrary point P; and from this by a determinate geometrical construction we obtain the point P_1, and thence in like manner the points P_2, $P_3,\ldots$ which represent the values ξ_1, ξ_2, $\xi_3,\ldots$ respectively. And the problem is to divide the plane into regions, such that, starting with a point P_1 anywhere in one region, we arrive ultimately at the root A; anywhere in another region we arrive ultimately at the root B; and so on for the several roots of the equation. The division into regions is made without difficulty in the case of a quadric equation; but in the next succeeding case, that of a cubic equation, it is anything but obvious what the division is: and the author had not succeeded in finding it.

744.

TABLE OF $\Delta^m 0^n \div \Pi(m)$ UP TO $m = n = 20$.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XIII. Part I. (1881), pp. 1—4. Read October 27, 1879.]

THE differences of the powers of zero, $\Delta^m 0^n$, present themselves in the Calculus of Finite Differences, and especially in the applications of Herschel's theorem,

$$f(e^t) = f(1+\Delta)\, e^{t \cdot 0},$$

for the expansion of the function of an exponential. A small Table up to $\Delta^{10} 0^{10}$ is given in Herschel's *Examples* (Camb. 1820), and is reproduced in the treatise on Finite Differences (1843) in the *Encyclopædia Metropolitana*. But, as is known, the successive differences $\Delta 0^n$, $\Delta^2 0^n$, $\Delta^3 0^n$, ... are divisible by 1, 1.2, 1.2.3, ... and generally $\Delta^m 0^n$ is divisible by $1.2.3 \ldots m, = \Pi(m)$; these quotients are much smaller numbers, and it is therefore desirable to tabulate them rather than the undivided differences $\Delta^m 0^n$: moreover, it is easier to calculate them. A table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 12$ is in fact given by Grunert, *Crelle*, t. XXV. (1843), p. 279, but without any explanation in the heading of the meaning of the tabulated numbers $C_n^k, = \Delta^n 0^k \div \Pi(n)$, and without using for their determination the convenient formula $C_n^{k+1} = nC_n^k + C_{n-1}^k$ given by Björling in a paper, *Crelle*, t. XXVIII. (1844), p. 284. The formula in question, say

$$\frac{\Delta^m 0^{n+1}}{\Pi(m)} = m\,\frac{\Delta^m 0^n}{\Pi(m)} + \frac{\Delta^{m-1} 0^n}{\Pi(m-1)},$$

is given in the second edition (by Moulton) of Boole's *Calculus of Finite Differences*, (London, 1872), p. 28, under the form

$$\Delta^m 0^n = m\,(\Delta^{m-1} 0^{n-1} + \Delta^m 0^{n-1}).$$

It occurred to me that it would be desirable to extend the table of the quotients $\Delta^m 0^n \div \Pi(m)$, up to $m = n = 20$. The calculation is effected very readily by means

of the foregoing theorem, which is used in the following form; viz. any column of the table for instance the fifth, being

A,	then the following column is	A,
B,	...	$2B+A$,
C,	...	$3C+B$,
D,	...	$4D+C$,
E,	...	$5E+D$,
		$+E$;

and then we obtain a good verification by taking the sum of the terms in the new column, and comparing it with the value as calculated from the formula,

$$\text{Sum} = 2A + 3B + 4C + 5D + 6E.$$

Observe that, in the two calculations, we take successive multiples such as $4D$ and $5D$ of each term of the preceding column, and that the verification is thus a safeguard against any error of multiplication or addition.

TABLE, No. 1, OF $\Delta^m 0^n \div \Pi(m)$.

Ind. Δ	0^1	0^2	0^3	0^4	0^5	0^6	0^7	0^8	0^9	0^{10}	0^{11}	0^{12}	0^{13}	0^{14}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1 023	2 047	4 095	8 191
3			1	6	25	90	301	966	3 025	9 330	28 501	86 526	261 625	788 970
4				1	10	65	350	1 701	7 770	34 105	145 750	611 501	2 532 530	10 391 745
5					1	15	140	1 050	6 951	42 525	246 730	1 379 400	7 508 501	40 075 035
6						1	21	266	2 646	22 827	179 487	1 323 652	9 321 312	63 436 373
7							1	28	462	5 880	63 987	627 396	5 715 424	49 329 280
8								1	36	750	11 880	159 027	1 899 612	20 912 320
9									1	45	1 155	22 275	359 502	5 135 130
10										1	55	1 705	39 325	752 752
11											1	66	2 431	66 066
12												1	78	3 367
13													1	91
14														1
15														
16														
17														
18														
19														
20														

Ind. Δ	0^{15}	0^{16}	0^{17}	0^{18}	0^{19}	0^{20}	
1	1	1	1	1	1	1	1
2	16 383	32 767	65 535	131 071	262 143	524 287	2
3	2 375 101	7 141 686	21 457 825	64 439 010	193 448 101	580 606 446	3
4	42 355 950	171 798 901	694 337 290	2 798 806 985	11 259 666 950	45 232 115 901	4
5	210 766 920	1 096 190 550	5 652 751 651	28 958 095 545	147 589 284 710	749 206 090 500	5
6	420 693 273	2 734 926 558	17 505 749 898	110 687 251 039	693 081 601 779	4 306 078 895 384	6
7	408 741 333	3 281 882 604	25 708 104 786	197 462 483 400	1 492 924 634 839	11 143 554 045 652	7
8	216 627 840	2 141 764 053	20 415 995 028	189 036 065 010	1 709 751 003 480	15 170 932 662 679	8
9	67 128 490	820 784 250	9 528 822 303	106 175 395 755	1 144 614 626 805	12 011 282 644 725	9
10	12 662 650	193 754 990	2 758 334 150	37 112 163 803	477 297 033 785	5 917 584 964 655	10
11	1 479 478	28 936 908	512 060 978	8 391 004 908	129 413 217 791	1 900 842 429 486	11
12	106 470	2 757 118	62 022 324	1 256 328 866	23 466 951 300	411 016 633 391	12
13	4 550	165 620	4 910 178	125 854 638	2 892 439 160	61 068 660 380	13
14	105	6 020	249 900	8 408 778	243 577 530	6 302 524 580	14
15	1	120	7 820	367 200	13 916 778	452 329 200	15
16		1	136	9 996	527 136	22 350 954	16
17			1	153	12 597	741 285	17
18				1	171	15 675	18
19					1	190	19
20						1	20

Writing down the sloping lines as columns thus:

1 (0)	2 (2)	3 (4)	4 (6)	5 (8)	6 (10)	7 (12)	8 etc. (14) etc.
1							
1	1						
1	3	1					
1	6	7	1				
1	10	25	15	1			
1	15	65	90	31	1		
1	21	140	350	301	63	1	
1	28	266	1 050	1 701	966	127	
1	36	462	2 646	6 951	7 770	3 025	
1	45	750	5 880	22 827	42 525	34 105	
1	55	1 155	11 880	63 987	179 487	246 730	
1	66	1 705	22 275	159 027	627 396	1 323 652	
1	78	2 431	39 325	359 502	1 899 612	5 715 424	
1	91	3 367	66 066	752 752	5 135 130	20 912 320	
1	105	4 550	106 470	1 479 478	12 662 650	67 128 490	
1	120	6 020	165 620	2 757 118	28 936 908	193 754 990	
1	136	7 820	249 900	4 910 178	62 022 324	512 060 978	
1	153	9 996	367 200	8 408 778	125 854 638	1 256 328 866	
1	171	12 597	527 136	13 916 778	243 577 530	2 892 439 160	
1	190	15 675	741 285	22 350 954	452 329 200	6 302 524 580	
20	19	18	17	16	15	14	13 etc.

it appears by inspection that, in the second column the second differences, are constant, in the third column the fourth differences, in the fourth column the sixth differences, and so on, are constant; and we thence deduce the law of the numbers in the successive columns: viz. this can be done up to column 7, in which we have 14 numbers in order to find the 12th differences: but in column 8 we have only 13 numbers, and therefore cannot find the 14th differences. The differences are given in the following

TABLE, No. 2 (*explanation infrà*).

Ind. Δ	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1		2	6	14	30	62	126
2		1	12	61	240	841	2 772
3			10	124	890	5 060	25 410
4			3	131	1 830	16 990	127 953
5				70	2 226	35 216	401 436
6				15	1 600	47 062	836 976
7					630	40 796	1 196 532
8					105	21 225	1 182 195
9						10 930	795 718
10						945	349 020
11							90 090
12							10 395

We have, by means of this Table, the general expressions of $\Delta^r 0^r$, $\Delta^{r-1}0^r$, $\Delta^{r-2}0^r$, ... up to $\Delta^{r-6}0^r$, viz. the formulæ are

$$\Delta^r 0^r \div \Pi(r) \quad = 1,$$

$$\Delta^{r-1}0^r \div \Pi(r-1) = 1 + 2\binom{r-2}{1}^1 + 1\binom{r-2}{2}^2,$$

$$\Delta^{r-2}0^r \div \Pi(r-2) = 1 + 6\binom{r-3}{1}^1 + 12\binom{r-3}{2}^2 + 10\binom{r-3}{3}^3 + 3\binom{r-3}{4}^4,$$

&c., &c.,

where the numerical coefficients are the numbers in the successive columns of the table; and where for shortness $\binom{r-m}{k}^k$ is written to denote the binomial coefficient $\frac{[r-m]^k}{[k]^k}$. For instance, $r=10$, we have

$$\Delta^8 0^{10} \div \Pi(8) = 1 + 6\,.\,7 + 12\,.\,21 + 10\,.\,35 + 3\,.\,35, = 750,$$

agreeing with the principal Table. It will be observed that, in the successive columns of the Table, the last terms are 1, 1, 1.3, 1.3.5, 1.3.5.7, 1.3.5.7.9, and 1.3.5.7.9.11. This is itself a good verification: I further verified the last column by calculating from it the value of $\Delta^{14}0^{20} \div \Pi(14)$, $= 6\,302\,524\,580$ as above. The Table shows that we have $\Delta^{r-m}0^r \div \Pi(r-m)$ given as an algebraical rational and integral function of r, of the degree $2m$. But the terms from the top of a column, $\Delta 0^r = 1$, $\Delta^2 0^r \div 1\,.\,2 = 2^{r-1} - 1$, &c., are not algebraical functions of r.

22 *October*, 1879.

745.

ON THE SCHWARZIAN DERIVATIVE, AND THE POLYHEDRAL FUNCTIONS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XIII. Part I. (1881), pp. 5—68. Read March 8, 1880.]

THE quotient s of any two solutions of a linear partial differential equation of the second order, $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$, is determined by a differential equation of the third order

$$\frac{\frac{d^3s}{dx^3}}{\frac{ds}{dx}}-\tfrac{3}{2}\left(\frac{\frac{d^2s}{dx^2}}{\frac{ds}{dx}}\right)^2=-\tfrac{1}{2}\left(p^2+2\frac{dp}{dx}-4q\right),$$

where the function on the left-hand is what I call the Schwarzian Derivative; or say this derivative is

$$\{s,\ x\},=\frac{s'''}{s'}-\tfrac{3}{2}\left(\frac{s''}{s'}\right)^2,$$

where the accents denote differentiations in regard to the second variable x of the symbol.

Writing in general $(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\between X,\ Y,\ Z)^2$ to denote a quadric function

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \tfrac{1}{2}(\mathrm{a}-\mathrm{b}-\mathrm{c}),\ \ \tfrac{1}{2}(-\mathrm{a}+\mathrm{b}-\mathrm{c}),\ \ \tfrac{1}{2}(-\mathrm{a}-\mathrm{b}+\mathrm{c})\between X,\ Y,\ Z)^2,$$

then, if the equation of the second order be that of the hypergeometric series, generalised by a homographic transformation upon the variable x, the resulting differential equation of the third order is of the form

$$\{s,\ x\}=(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore)\left(\frac{1}{x-a},\ \ \frac{1}{x-b},\ \ \frac{1}{x-c}\right)^2;$$

and, presenting themselves in connexion with the algebraically integrable cases of this equation, we have rational and integral functions of s, derived from the polygon, the double pyramid, and the five regular solids. They are called Polyhedral Functions.

The Schwarzian Derivative occurs implicitly in Jacobi's differential equation of the third order for the modulus in the transformation of an elliptic function (*Fund. Nova*, 1829, p. 79, [*Ges. Werke*, t. I., p. 133]) and in Kummer's fundamental equation for the transformation of a hypergeometric series (Kummer, 1836: see list of Memoirs): but it was first explicitly considered and brought into notice in the two Memoirs of Schwarz*, 1869 and 1873. The latter of these, relating to the algebraic integration of the differential equation for the hypergeometric series, is the fundamental Memoir upon the subject, but the theory is in some material points completed in the Memoirs by Klein and Brioschi.

The following list of Memoirs, relating as well to the Polyhedral Functions as to the Schwarzian Derivative, is arranged nearly in chronological order.

Kummer, Ueber die hypergeometrische Reihe $1+\frac{\alpha.\beta}{1.\gamma}x+\ldots$ *Crelle*, t. XV. (1836), pp. 39—83 and 127—172.

Schwarz, Ueber einige Abbildungsaufgaben. *Crelle-Borchardt*, t. LXX. (1869), pp. 105—120.

——— Ueber diejenigen Fälle in welchen die *Gauss*ische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. Do. t. LXXV. (1873), pp. 292—335.

Cayley, Notes on Polyhedra. *Quart. Math. Jour.* t. VII. (1866), pp. 304—316; [375].

——— On the Regular Solids. Do. t. XV. (1878), pp. 127—131; [679].

Fuchs, Ueber diejenigen Differentialgleichungen zweiter Ordnung welche algebraische Integralen besitzen, und eine Anwendung der Invariantentheorie. *Crelle-Borchardt*, t. LXXXI. (1875), pp. 97—142.

Klein, Ueber binäre Formen mit linearen Transformationen in sich selbst. *Math. Ann.* t. IX. (1875), pp. 183—209.

Brioschi, Extrait d'une lettre à M. Klein. *Math. Ann.* t. XI. (1877), pp. 111—114.

Klein, Ueber lineare Differentialgleichungen. *Math. Ann.* t. XI. (1877), pp. 115—118.

Brioschi, La théorie des formes dans l'intégration des équations différentielles linéaires du second ordre. *Math. Ann.* t. XI. (1877), pp. 401—411.

Gordan, Ueber endliche Gruppen linearer Transformationen einer Veränderlichen. *Math. Ann.* t. XII. (1877), pp. 23—46.

——— Binäre Formen mit verschwindenden Covarianten. *Math. Ann.* t. XII. (1877), pp. 147—166.

[* Schwarz, *Ges. Werke*, t. II., p. 351, remarks that the Derivative occurs implicitly in a memoir by Lagrange, "Sur la construction des cartes géographiques," (1779), *Œuvres*, t. IV., p. 651.]

Klein, Ueber lineare Differentialgleichungen. *Math. Ann.* t. XII. (1877), pp. 167—179.

——— Weitere Untersuchungen über das Icosaeder. *Math. Ann.* t. XII. (1877), pp. 503—560.

Cayley, On the Correspondence of Homographies and Rotations. *Math. Ann.* t. XV. (1879), pp. 238—240; [660].

——— On the finite Groups of linear transformations of a Variable. *Math. Ann.* t. XVI. (1880), pp. 260—263, and pp. 439—440; [752].

I propose in the present Memoir to consider the whole theory: and, in particular, to give some additional developments in regard to the Polyhedral Functions.

I remark that Schwarz starts with the foregoing differential equation of the third order

$$\{s, x\} = (\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore) \left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

and he shows (by very refined reasoning founded on the theory of conformable figures, which will be in part reproduced) that this equation is, in fact, algebraically integrable for 16 different sets of values of the coefficients a, b, c. It may I think be taken to be part of his theory, although not very clearly brought out by him, that these integrals are some of them of the form, $x =$ rational function of s: others of the form, rational function of $x =$ rational function of s; the rational functions of s being in fact the same in the last as in the first set of solutions: they are quotients of Polyhedral functions.

But as regards the second set of cases, the solution of these, introducing for convenience a new variable z in place of s, may be made to depend upon the solution in the form, $x =$ rational function of z, of an equation of a somewhat similar form, but involving two quadric functions of x and z respectively, viz. the equation

$$\{x, z\} + \left(\frac{dx}{dz}\right)^2 (\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore) \left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2 = (\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore) \left(\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-c_1}\right)^2;$$

and we have the theorem that the solution of this equation depends upon the determination of P, Q, R rational and integral functions of z, containing each of them multiple factors, which are such that $P + Q + R = 0$. Using accents to denote differentiation in regard to z, this implies $P' + Q' + R' = 0$, and consequently

$$QR' - Q'R = RP' - R'P = PQ' - P'Q.$$

Further, they are such that the equal functions $QR' - Q'R$, $RP' - R'P$, $PQ' - P'Q$ contain only factors which are factors of P, Q or R.

In fact, writing f, g, $h = b - c$, $c - a$, $a - b$, the required relation between x, z is then expressed in the symmetrical form $f(x-a) : g(x-b) : h(x-c) = P : Q : R$.

The last-mentioned differential equation is considered by Klein and Brioschi: the solutions in 13 cases, or such of them as had not been given by Schwarz, were obtained by Brioschi: and those of the remaining 3 cases, subject to a correction in one of them, were afterwards obtained by Klein.

The first part of the present Memoir relates, say to the foregoing equation

$$\{s, x\} = (\mathrm{a, b, c} \therefore) \left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

although the other form in $\{x, z\}$ may equally well be regarded as the fundamental form.

We consider in the theory:

A. The Derivative $\{s, x\}$, meaning as above explained.

B. Quadric functions of any three or more inverts $\frac{1}{x-l}$.

C. Rational and integral functions P, Q, R having a sum $=0$, and which are such that $QR'-Q'R, =RP'-R'P, =PQ'-P'Q$, contains only the factors of P, Q, R.

D. The differential equation of the third order.

E. The Schwarzian theory in regard to conformable figures and the corresponding values of the imaginary variables s and x.

F. Connexion with the differential equation for the hypergeometric series.

The second part of the Memoir relates to the Polyhedral Functions.

The paragraphs of the whole Memoir are numbered consecutively.

PART I.

The Derivative $\{s, x\}$. Art. Nos. 1 to 7.

1. If $p = \frac{s''}{s'} = \frac{d}{dx}\left(\log \frac{ds}{dx}\right)$, then $\{s, x\} = \frac{dp}{dx} - \frac{1}{2}p^2$.

2. The derivative $\{s, x\}$ may be transformed in regard to either or both of the variables.

Suppose, first, that s is a function of the new variable S, (hence also S is a function of x): using subscript numbers to denote differentiations in regard to S, and the accents as before for differentiations in regard to x, we have

$$s' = S's_1,$$

whence, differentiating the logarithms,

$$\frac{s''}{s'} = S'\frac{s_2}{s_1} + \frac{S''}{S'}.$$

Again differentiating, we have

$$\frac{s'''}{s'} - \left(\frac{s''}{s'}\right)^2 = S'^2\left[\frac{s_2}{s_1} - \left(\frac{s_2}{s_1}\right)^2\right] + S''\frac{s_2}{s_1} + \frac{S'''}{S'} - \left(\frac{S''}{S'}\right)^2.$$

But

$$-\tfrac{1}{2}\left(\frac{s''}{s'}\right)^2 = S'^2\left[-\tfrac{1}{2}\left(\frac{s_2}{s_1}\right)^2\right] - S''\frac{s_2}{s_1} \qquad -\tfrac{1}{2}\left(\frac{S''}{S'}\right)^2,$$

and consequently

$$\frac{s'''}{s'} - \tfrac{3}{2}\left(\frac{s''}{s'}\right)^2 = S'^2\left[\frac{s_2}{s_1} - \tfrac{3}{2}\left(\frac{s_2}{s_1}\right)^2\right] \qquad + \frac{S'''}{S'} - \tfrac{3}{2}\left(\frac{S''}{S'}\right)^2;$$

that is,

$$\{s,\ x\} = \left(\frac{dS}{dx}\right)^2 \{s,\ S\} + \{S,\ x\},$$

the required formula.

In a very similar manner, taking x a function of X, it is shown that

$$\{s,\ x\} = \left(\frac{dX}{dx}\right)^2 (\{s,\ X\} - \{x,\ X\}).$$

3. If in this formula we write S for s, and substitute the resulting value of $\{S,\ x\}$ in the former formula, we have

$$\{s,\ x\} = \left(\frac{dS}{dx}\right)^2 \{s,\ S\} - \left(\frac{dX}{dx}\right)^2 \{x,\ X\} + \left(\frac{dX}{dx}\right)^2 \{S,\ X\},$$

which is the formula for the change of both variables. It, in fact, includes the other two: viz. writing $X = x$, or $S = s$, and observing that $\{s,\ s\} = 0 = \{x,\ x\}$, we have the other two formulæ.

4. By putting in the first formula $X = s$, we obtain

$$\{s,\ x\} = -\left(\frac{ds}{dx}\right)^2 \{x,\ s\},$$

a formula for the interchange of the variables.

5. Writing $S = \dfrac{as+b}{cs+d}$, and using for a moment the accents to denote differentiation in regard to s, we have

$$S' = \frac{ad-bc}{(cs+d)^2}, \quad \frac{S''}{S'} = \frac{-2c}{cs+d},$$

and thence

$$\frac{S'''}{S'} - \left(\frac{S''}{S'}\right)^2 = \frac{2c^2}{(cs+d)^2},$$

$$-\tfrac{1}{2}\left(\frac{S''}{S'}\right)^2 = \frac{-2c^2}{(cs+d)^2}.$$

Consequently $\{S,\ s\} = 0$, whence also $\{s,\ S\} = 0$.

Hence in the first formula $\{S, x\} = \{s, x\}$, that is,

$$\left\{\frac{as+b}{cs+d}, x\right\} = \{s, x\};$$

viz. we may, in the derivative $\{s, x\}$, write for s any homographic function $(as+b) \div (cs+d)$ of s.

6. Again, if $X = \dfrac{\alpha x + \beta}{\gamma x + \delta}$, then from the second formula

$$\{s, x\} = \frac{(\alpha\delta - \beta\gamma)^2}{(\gamma x + \delta)^4} \{s, X\};$$

that is,

$$\left\{s, \frac{\alpha x + \beta}{\gamma x + \delta}\right\} = \frac{(\gamma x + \delta)^4}{(\alpha\delta - \beta\gamma)^2} \{s, x\};$$

and here, changing s into $(as+b) \div (cs+d)$, we have finally

$$\left\{\frac{as+b}{cs+d}, \frac{\alpha x + \beta}{\gamma x + \delta}\right\} = \frac{(\gamma x + \delta)^4}{(\alpha\delta - \beta\gamma)^2} \{s, x\},$$

which is the formula for the homographic transformation of the two variables s, x.

7. Let s be a given function of x, the equation $\{S, x\} = \{s, x\}$ is a differential equation of the third order in S, and by what precedes, its general integral is $S = \dfrac{as+b}{cs+d}$.

The direct process is as follows: we have a first integral $\dfrac{S''}{S'} = \dfrac{s''}{s'} - \dfrac{2cs'}{cs+d}$; a second integral $\log S' = \log s' - 2\log(cs+d) + \text{const.}$, that is, $S' = \dfrac{cAs'}{(cs+d)^2}$; and thence a final integral $S = B - \dfrac{A}{cs+d}$, which is equivalent to the foregoing value of S.

The Quadric Function of three or more Inverts. Art. Nos. 8 to 15.

8. We consider a quadric function of any number of inverts $\dfrac{1}{x-\alpha}, \dfrac{1}{x-\beta}, \ldots$, all of them different: it is assumed that the constant term is $=0$, and also that the sum of the coefficients of the linear terms is $=0$. We have therefore square terms $\dfrac{a}{(x-\alpha)^2}$, product terms $\dfrac{h}{x-\alpha \,.\, x-\beta}$, and linear terms $\dfrac{A}{x-\alpha}$, where the sum of the coefficients A is $=0$. Any product term $\dfrac{h}{x-\alpha \,.\, x-\beta}$ is expressible in the form of a difference $\dfrac{h}{\alpha-\beta}\dfrac{1}{x-\alpha} - \dfrac{h}{\alpha-\beta}\dfrac{1}{x-\beta}$ of two linear terms, and (the coefficients of these

being equal), after it is thus expressed, the sum of the coefficients of the linear terms is still $=0$. The function is thus always expressible in the form

$$\frac{a}{(x-\alpha)^2}+\frac{b}{(x-\beta)^2}+\ldots+\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\ldots$$

where the sum $A+B+\ldots$ is $=0$: this may be called the reduced form.

9. Observe that any particular invert $\frac{1}{x-\alpha}$ may disappear altogether from the reduced form: this will be the case if $a=0$, that is, if the original form contains no term in $\frac{1}{(x-\alpha)^2}$, and if also $A=0$. An invert thus disappearing from the reduced form is said to be non-essential: and the inverts which do not disappear are said to be essential. The original form contains in appearance the non-essential inverts, but it is really a quadric function of the essential inverts only.

10. Imagine the original function expressed as a rational fraction, the denominator being the product $(x-\alpha)^2(x-\beta)^2(x-\gamma)^2\ldots$ of the squared factors corresponding to all the inverts (non-essential as well as essential): the numerator will be in general of a degree less by 2 than that of the denominator, but the coefficients of any one or more of the higher powers of x may vanish, and the numerator will then be of a lower degree. But this numerator will for any non-essential invert $\frac{1}{x-\gamma}$ contain the factor $(x-\gamma)^2$, or, dividing the numerator and denominator each by this factor, the difference of the degrees of the numerator and denominator will remain unaltered; that is, the difference will have the same value whether we do or do not attend to the non-essential inverts; or say it will have the same value for the original form and for the reduced form.

11. It is to be remarked that the linear terms $\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{x-\gamma}+\ldots$, where $A+B+C+\ldots=0$, can be (and that in a variety of ways) expressed as a sum of differences $\frac{1}{x-\alpha}-\frac{1}{x-\beta}$, that is, as a sum of product-terms $\frac{1}{x-\alpha\,.\,x-\beta}$. Hence the quadric function can be (and that in a variety of ways) expressed as a homogeneous function $\left(\mathrm{a}, \ldots \between \frac{1}{x-\alpha}, \frac{1}{x-\beta}, \ldots\right)^2$; we must have in the form all the essential inverts, and we need have these only. Supposing that this is so, and that the number of the essential inverts is $=n$, then the number of constants is $=\frac{1}{2}n(n+1)$, whereas the number of constants in the reduced form is only $=2n-1$: hence the coefficients are not determinate; or, what is the same thing, we may have different quadric functions having each of them the same reduced function; these quadric functions, as having the same reduced function, can only differ by multiples of the evanescent expressions

$$\frac{\beta-\gamma}{x-\beta\,.\,x-\gamma}+\frac{\gamma-\alpha}{x-\gamma\,.\,x-\alpha}+\frac{\alpha-\beta}{x-\alpha\,.\,x-\beta}, \text{ \&c.}$$

In particular, if the number of essential inverts is $=3$, then the quadric function is of the form

$$\left(\mathrm{a, b, c, f, g, h}\between\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2,$$

which contains one superfluous constant, and equivalent functions differ only by a multiple of

$$\frac{\beta-\gamma}{x-\beta\,.\,x-\gamma}+\frac{\gamma-\alpha}{x-\gamma\,.\,x-\alpha}+\frac{\alpha-\beta}{x-\alpha\,.\,x-\beta}.$$

12. A quadric function such that the degree of the numerator is less by 4 than that of the denominator is said to be "curtate."

The conditions, in order that the function

$$\left(\mathrm{a, b, c, f, g, h}\between\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2$$

may be curtate, are easily found to be

$$\mathrm{a+b+c+2f+2g+2h=0},$$
$$\alpha(\mathrm{a+h+g})+\beta(\mathrm{h+b+f})+\gamma(\mathrm{g+f+c})=0;$$

and by reason of the superfluous constant we are at liberty to assume a third condition: the three conditions may be taken to be $\mathrm{a+h+g}$, $\mathrm{h+b+f}$, $\mathrm{g+f+c}$ each $=0$; and this being so the values of f, g, h are $=\frac{1}{2}(\mathrm{a-b-c})$, $\frac{1}{2}(\mathrm{-a+b-c})$, $\frac{1}{2}(\mathrm{-a-b+c})$ respectively. Hence the form is

$$\left(\mathrm{a, b, c}, \tfrac{1}{2}(\mathrm{a-b-c}), \tfrac{1}{2}(\mathrm{-a+b-c}), \tfrac{1}{2}(\mathrm{-a-b+c})\between\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2,$$

which, as already mentioned, we denote by

$$\left(\mathrm{a, b, c}\ \therefore\between\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2.$$

We have thus the theorem that a curtate function of any number of inverts, but with only the three essential inverts

$$\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma},$$

is always expressible in the foregoing form

$$\left(\mathrm{a, b, c}\ \therefore\between\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}\right)^2.$$

13. It may be remarked that the function $(\mathrm{a, b, c}\ \therefore\between X, Y, Z)^2$ is a function of the differences of the variables X, Y, Z; and similarly, in the case of four variables, a function $(\mathrm{a, b, c, d, f, g, h, l, m, n}\between X, Y, Z, W)^2$, for which

$$\mathrm{a+h+g+l,\quad h+b+f+m,\quad g+f+c+n,\quad l+m+n+d,}$$

are each $=0$, is a function of the differences of the variables X, Y, Z, W: and so in general. Any such function is said to be "diaphoric": and it is easy to see that, taking for the variables any inverts whatever, a diaphoric function is always curtate.

14. The function

$$\left\{-\frac{\mathrm{a}}{(x-\alpha)^2}-\frac{\mathrm{b}}{(x-\beta)^2}-\frac{\mathrm{c}}{(x-\gamma)^2}-\dots\right\}$$

$$-\tfrac{1}{2}\left\{\frac{\mathrm{a}}{x-\alpha}+\frac{\mathrm{b}}{x-\beta}+\frac{\mathrm{c}}{x-\gamma}+\dots\right\}^2,$$

where the coefficients a, b, c, ... satisfy the relation $\mathrm{a}+\mathrm{b}+\mathrm{c}+\dots=-2$, is diaphoric, and therefore curtate. In fact, forming the sum, coeff. $\dfrac{1}{(x-\alpha)^2}+\tfrac{1}{2}$ coeff. $\dfrac{1}{x-\alpha\,.\,x-\beta}+\dots$, this is $-\mathrm{a}-\tfrac{1}{2}\mathrm{a}^2-\tfrac{1}{2}\mathrm{ab}-\tfrac{1}{2}\mathrm{ac}-\dots,\ =-\tfrac{1}{2}\mathrm{a}\,(2+\mathrm{a}+\mathrm{b}+\mathrm{c}+\dots)$, which is $=0$; and similarly the other conditions are satisfied.

15. The function

$$\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \left(\frac{a}{x-\alpha}+\frac{a_1}{x-\alpha_1}+\dots,\quad \frac{b}{x-\beta}+\frac{b_1}{x-\beta_1}+\dots,\quad \frac{c}{x-\gamma}+\frac{c_1}{x-\gamma_1}+\dots\right)^2,$$

regarded as a function of the inverts

$$\frac{1}{x-\alpha},\ \frac{1}{x-\alpha_1},\ \dots,\ \frac{1}{x-\beta},\ \dots$$

where

$$a+a_1+\dots=b+b_1+\dots=c+c_1+\dots,\ =k \text{ suppose},$$

is diaphoric, and therefore curtate. In fact, the condition in regard to $\dfrac{1}{x-\alpha}$ is

$$\mathrm{a}\,(a^2+aa_1+aa_2+\dots)+\tfrac{1}{2}(-\mathrm{a}+\mathrm{b}-\mathrm{c})(ab+ab_1+\dots)+\tfrac{1}{2}(-\mathrm{a}-\mathrm{b}+\mathrm{c})(ac+ac_1+\dots)=0;$$

that is,

$$ak\,\{\mathrm{a}+\tfrac{1}{2}(-\mathrm{a}+\mathrm{b}-\mathrm{c})+\tfrac{1}{2}(-\mathrm{a}-\mathrm{b}+\mathrm{c})\}=0,$$

which is satisfied. And similarly the other conditions are satisfied.

The functions P, Q, R. Art. Nos. 16 to 20.

16. We consider P, Q, R, rational and integral functions of z, such that $P+Q+R=0$: hence, using the accent to denote differentiation in regard to z, we have also $P'+Q'+R'=0$; and therefore $QR'-Q'R=RP'-R'P=PQ'-P'Q,\ =\Theta$ suppose: and we require to find P, Q, R such that the function Θ contains only the factors of P, Q, R.

17. It is to be observed that, effecting upon a solution P, Q, R any linear substitution $(\alpha z+\beta)\div(\gamma z+\delta)$, and omitting the common denominator, we have a solution; but this is regarded as identical with the original solution. The three functions, if

not originally of the same order, can thus be made to be of the same order; or by taking account of the root $z=\infty$, we may in the original case regard them as being of the same order, and it is convenient so to regard them: say they are taken to be of the same order δ. And there is clearly no loss of generality in taking the three functions to be prime to each other; for any common factor of two of them would divide the third, and might therefore be struck out.

18. We may therefore write

$$P=F\Pi(z-l)^p,\quad Q=G\Pi(z-m)^q,\quad R=H\Pi(z-n)^r,$$

where $(z-l)^p$ is taken to denote the distinct simple or multiple factors of P, and the like as regards Q and R; the factors $z-l$, $z-m$, $z-n$ are thus all of them different. And we have $\delta=\Sigma p$, $=\Sigma q$, $=\Sigma r$.

19. It is at once seen that Θ is of the degree $2\delta-2$, and moreover that it contains the factors $\Pi(z-l)^{p-1}$, $\Pi(z-m)^{q-1}$, $\Pi(z-n)^{r-1}$; hence it contains the factor

$$\Pi(z-l)^{p-1}(z-m)^{q-1}(z-n)^{r-1}.$$

Suppose the number of distinct indices p is $=\sigma_1$, that of distinct indices q is σ_2, and that of distinct indices r is σ_3; then the degree of the factor is $=3\delta-\sigma_1-\sigma_2-\sigma_3$; and if this be $=2\delta-2$, then Θ can have no other variable factor: viz. if the numbers σ_1, σ_2, σ_3 of the distinct indices p, q, r respectively are such that $\sigma_1+\sigma_2+\sigma_3=\delta+2$, a relation which is henceforth taken to be satisfied, then we have

$$\Theta=K\Pi(z-l)^{p-1}(z-m)^{q-1}(z-n)^{r-1}.$$

As already in effect remarked, the conclusion extends to the case where P, Q, R are not of the same degree; the equation $P+Q+R=0$ here implies that two functions, say P, Q, are of the same degree, and the third function R of an inferior degree; but, this being so, we have only to regard R as containing the factor $\left(1-\frac{z}{\infty}\right)^t$ of the degree t proper for raising its degree up to that of P or Q.

20. Solutions are given in the following PQR-Table: in which, where required, the proper factor $\left(1-\frac{z}{\infty}\right)^t$ has been added; the first column headed Ref. No. (Reference Number) will be explained further on. The Annex to the same Table will also be explained.

Ref. No.	$P=$	$Q=$	$R=$	$\Theta=$
1	z^n	$-1\left(1-\frac{z}{\infty}\right)^n$	$-z^n+1$	$nz^{n-1}\left(1-\frac{z}{\infty}\right)^{n-1}$
2	$4z^n\left(1-\frac{z}{\infty}\right)^n$	$-(z^n+1)^2$	$(z^n-1)^2$	$-4n\,z^{n-1}(z^n+1)(z^n-1)\left(1-\frac{z}{\infty}\right)^{n-1}$
3	$(z^4+2\sqrt{-3}z^2+1)^3$	$-12\sqrt{-3}z^2(z^4-1)^2\left(1-\frac{z}{\infty}\right)^2$	$-(z^4-2\sqrt{-3}z^2+1)^3$	$24\sqrt{-3}(z^4+2\sqrt{-3}z^2+1)^2$ $(z^4-2\sqrt{-3}z^2+1)z(z^4-1)\left(1-\frac{z}{\infty}\right)$
4	$(z^8+14z^4+1)^3$	$-(z^{12}-33z^8-33z^4+1)^2$	$-108(z^5-z)^4\left(1-\frac{z}{\infty}\right)^4$	$576(z^8+\ldots)^2(z^{12}-\ldots)(z^5-z)^3\left(1-\frac{z}{\infty}\right)^3$
5	$(z^{20}-228z^{15}+494z^{10}+228z^5+1)^3$	$-(z^{30}-522z^{25}-10005z^{20}$ $+0z^{15}-10005z^{10}+522z^5+1)^2$	$-1728(z^{11}+11z^6-z)^5\left(1-\frac{z}{\infty}\right)^5$	$-8640(z^{20}-\ldots)^2(z^{11}+\ldots)^4\left(1-\frac{z}{\infty}\right)^4$
III, V, VII, VIII	$4z\left(1-\frac{z}{\infty}\right)$	$-(z+1)^2$	$(z-1)^2$	$-4(z+1)(z-1)$
IX	$(z-4)^3$	$-(z-1)(z+8)^2$	$27z^2\left(1-\frac{z}{\infty}\right)$	$27(z-4)^2(z+8)z$
X	$z(z+8)^3$	$-(z^2-20z-8)^2$	$-64(z-1)^3\left(1-\frac{z}{\infty}\right)$	$-64(z+8)^2(z^2-20z\ldots)(z-1)$
XI	$4(z^2-z+1)^3$	$-(2z^3-3z^2-3z+2)^2$	$-27z^2(z-1)^2\left(1-\frac{z}{\infty}\right)^2$	$-108(z^2-z\ldots)^2(2z^3-3z^2\ldots)$ $z(z-1)\left(1-\frac{z}{\infty}\right)$
XII	$z^3(z+5)^2(z+8)$	$-(z^3+9z^2+12z-8)^2$	$-64(3z-1)\left(1-\frac{z}{\infty}\right)^5$	$-960z^3(z+5)(z^3+9z^2\ldots)\left(1-\frac{z}{\infty}\right)^4$
XIII	$(z^2+14z+1)^3$	$-(z^3-33z^2-33z+1)^2$	$-108z(z-1)^4$	$-108(z^2+14z\ldots)^2(z^3-33z^2\ldots)(z-1)^3$
XIV	$(64z+189)(64z^2+133z+49)^2$	$-z(4096z^3+18816z^2+25725z+12005)^2$	$-27\,.\,7^7(z+1)^2\left(1-\frac{z}{\infty}\right)^5$	$-135\,.\,7^7(64z^2+\ldots)^2(4096z^3+\ldots)$ $(z+1)\left(1-\frac{z}{\infty}\right)^4$
XV	$-(5z-27)(125z^3-25z^2-265z-243)^3$	$-(-3125z^5+9375z^4+18750z^3$ $+8750z^2+30750z+19683)^2$	$+1382400000z^3(z+1)^2\left(1-\frac{z}{\infty}\right)^5$	$-1382400000(125z^3-\ldots)^2$ $(-3125z^5+\ldots)z^2(z+1)\left(1-\frac{z}{\infty}\right)^4$

In the second half of the table the functions P, Q, R, except in the lines XII, XIV and XV, were calculated by Brioschi: those for the lines XII and XIV were calculated by Klein, but as regards line XV there would seem to have been some error at the beginning of the calculation, and the values found by him are erroneous.

Ref. No.	(a, b, c)*	Calculation of (ap), (bq), (cr).				Ref. No.	
1	$\frac{1}{2}(1-n^{-2}),\ \frac{1}{2}(1-n^{-2}),\ 0$	$an=0$	$bn=0$	$c1=0$	$\frac{1}{2}(1-n^{-2}),\ \frac{1}{2}(1-n^{-2}),\ 0$		Polygon
2	$\frac{1}{2}(1-n^{-2}),\ \frac{3}{8},\ \frac{3}{8}$	$an=0$	$b2=0$	$c2=0$	$\frac{1}{2}(1-n^{-2}),\ \frac{3}{8},\ \frac{3}{8}$	I	Double Pyramid
3	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	$a3=0$	$b2=0$	$c3=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	II	Tetrahedron
4	$\frac{4}{9},\ \frac{3}{8},\ \frac{15}{32}$	$a3=0$	$b2=0$	$c4=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{15}{32}$	IV	Cube and Octahedron
5	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a3=0$	$b2=0$	$c5=0$	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	VI	Dodecahedron and Icosahedron
III	$\frac{4}{9},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{4}{9},\ a1=\frac{4}{9}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{4}{9},\ \frac{4}{9},\ \frac{5}{18}\therefore\mathfrak{X}\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$		Tetrahedron
V	$\frac{15}{32},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{15}{32},\ a1=\frac{15}{32}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{15}{32},\ \frac{15}{32},\ \frac{5}{18}\therefore\mathfrak{X}\ ,,\ ,,\ ,,\right)^2$		Cube and Octahedron
VII	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a1=\frac{4}{9},\ a1=\frac{4}{9}$	$b2=0$	$c2=\frac{21}{50}$	$\left(\frac{4}{9},\ \frac{4}{9},\ \frac{21}{50}\therefore\mathfrak{X}\ ,,\ ,,\ ,,\right)^2$		Dodecahedron and Icosahedron
VIII	$\frac{12}{25},\ \frac{3}{8},\ \frac{4}{9}$	$a1=\frac{12}{25},\ a1=\frac{12}{25}$	$b2=0$	$c2=\frac{5}{18}$	$\left(\frac{12}{25},\ \frac{12}{25},\ \frac{5}{18}\therefore\mathfrak{X}\ ,,\ ,,\ ,,\right)^2$	,,	,,
IX	$\frac{4}{9},\ \frac{3}{8},\ \frac{12}{25}$	$a3=0$	$b1=\frac{3}{8},\ b2=0$	$c1=\frac{12}{25},\ c2=\frac{21}{50}$	$\left(\frac{3}{8},\ \frac{12}{25},\ \frac{21}{50}\therefore\mathfrak{X}\frac{1}{z-1},\ \frac{1}{z-\infty},\ \frac{1}{z}\right)^2$	,,	,,
X	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b2=0$	$c1=\frac{12}{25},\ c3=\frac{8}{25}$	$\left(\frac{4}{9},\ \frac{12}{25},\ \frac{8}{25}\therefore\mathfrak{X}\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$	,,	,,
XI	,, ,, ,,	$a3=0$	$b2=0$	$c2=\frac{21}{50},\ c2=\frac{21}{50},\ c2=\frac{21}{50}$	$\left(\frac{21}{50},\ \frac{21}{50},\ \frac{21}{50}\therefore\mathfrak{X}\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$	,,	,,
XII	,, ,, ,,	$a1=\frac{4}{9},\ a2=\frac{5}{18},\ a3=0$	$b2=0$	$c1=\frac{12}{25},\ c5=0$	$\left(\frac{4}{9},\ \frac{5}{18},\ \frac{12}{25}\therefore\mathfrak{X}\frac{1}{z+8},\ \frac{1}{z+5},\ \frac{1}{z-\frac{1}{3}}\right)^2$	,,	,,
XIII	,, ,, ,,	$a3=0$	$b2=0$	$c1=\frac{12}{25},\ c1=\frac{12}{25},\ c4=\frac{9}{50}$	$\left(\frac{12}{25},\ \frac{12}{25},\ \frac{9}{50}\therefore\mathfrak{X}\ \frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2$	,,	,,
XIV	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b1=\frac{3}{8},\ b2=0$	$c2=\frac{21}{50},\ c5=0$	$\left(\frac{4}{9},\ \frac{3}{8},\ \frac{21}{50}\therefore\mathfrak{X}\frac{1}{z+\frac{189}{64}},\ \frac{1}{z},\ \frac{1}{z+1}\right)^2$	,,	,,
XV	,, ,, ,,	$a1=\frac{4}{9},\ a3=0$	$b2=0$	$c2=\frac{21}{50},\ c3=\frac{8}{25},\ c5=0$	$\left(\frac{4}{9},\ \frac{21}{50},\ \frac{8}{25}\therefore\mathfrak{X}\frac{1}{z-\frac{27}{5}},\ \frac{1}{z+1},\ \frac{1}{z}\right)^2$	,,	,,

* Observe, as regards the (a, b, c) column, that the line III agrees with 3; line V with 4 $\left(\text{only there is a transposition of } \frac{4}{9} \text{ and } \frac{15}{32}\right)$; lines VII and VIII agree each of them with 5 $\left(\text{only as regards VIII there is a transposition of } \frac{4}{9} \text{ and } \frac{12}{25}\right)$: the remaining lines IX to XV agree each of them with 5. Observe also as regards the Annex generally, the Roman numbering of the lines 2, 3, 4, 5; the lines after the first have thus the Roman numbers I to XV, corresponding to

The Differential Equations $\{x, z\}$ *and* $\{s, x\}$. Art. Nos. 21 to 45.

21. In reference to what follows, it is convenient to put $P = XP_0$, $P' = X_1 P_0$, where P_0 is written for $\Pi(z-l)^{p-1}$, the G.C.M. of P and P'; and X is consequently $= F$ multiplied by the product $\Pi(z-l)$ of the several factors taken each with the index unity; and so for Q and R: viz. we write

$$P,\ Q,\ R = XP_0,\ YQ_0,\ ZR_0,$$
$$P',\ Q',\ R' = X_1P_0,\ Y_1Q_0,\ Z_1R_0,$$

and the foregoing value of Θ then is

$$\Theta = KP_0Q_0R_0.$$

We come now to the investigation of the leading theorem. Take a, b, c arbitrary, f, g, $h = b-c$, $c-a$, $a-b$; P, Q, R functions of z as above; and write

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R,$$

equations, which are consistent with each other and determine x as a rational function of z. Using, as before, the accent to denote differentiation in regard to z, and taking the coefficients (a, b, c) arbitrary, it is required to find the value of

$$\{x, z\} + x'^2\left(\mathrm{a, b, c} \therefore \right)\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2.$$

22. Calculation of the first term $\{x, z\}$.

We have $x = \mathrm{a\ function}\left(\alpha\frac{P}{R}+\beta\right) \div \left(\gamma\frac{P}{R}+\delta\right)$, and thence $\{x, z\} = \left\{\frac{P}{R}, z\right\}$, $= \{\xi, z\}$ for a moment; then

$$\xi' = \left(\frac{P}{R}\right)' = \frac{RP'-R'P}{R^2}, \quad = \frac{P_0Q_0R_0}{Z^2R_0^2}, \quad = \frac{P_0Q_0}{Z^2R_0}.$$

Substituting the values

$$P_0 = \Pi(z-l)^{p-1}, \quad Q_0 = \Pi(z-m)^{q-1}, \quad R_0 = \Pi(z-n)^{r-1}, \quad Z = \Pi(z-n),$$

we have

$$\frac{\xi''}{\xi'} = \Sigma\frac{p-1}{z-l} + \Sigma\frac{q-1}{z-m} - \Sigma\frac{r+1}{z-n},$$

and thence

$$\{x, z\} = \left\{-\Sigma\frac{p-1}{(z-l)^2} - \Sigma\frac{q-1}{(z-m)^2} + \Sigma\frac{r+1}{(z-n)^2}\right\}$$
$$-\tfrac{1}{2}\left\{\Sigma\frac{p-1}{z-l} + \Sigma\frac{q-1}{z-m} - \Sigma\frac{r+1}{z-n}\right\}^2;$$

or say

$$= \left(-\frac{p-1}{(z-l)^2} - \frac{p_1-1}{(z-l_1)^2} - \dots - \frac{q-1}{(z-m)^2} - \frac{q_1-1}{(z-m_1)^2} - \dots + \frac{r+1}{(z-n)^2} + \frac{r_1+1}{(z-n_1)^2} + \dots\right)$$
$$-\tfrac{1}{2}\left(\frac{p-1}{z-l} + \frac{p_1-1}{z-l_1} + \dots + \frac{q-1}{z-m} + \frac{q_1-1}{z-m_1} + \dots - \frac{r+1}{z-n} - \frac{r_1+1}{z-n_1} - \dots\right)^2,$$

where it is to be observed that

$$\Sigma(p-1)+\Sigma(q-1)-\Sigma(r+1), \quad =\delta-\sigma_1+\delta-\sigma_2-(\delta+\sigma_3)=\delta-\sigma_1-\sigma_2-\sigma_3=-2;$$

consequently the function is diaphoric, and therefore curtate.

It is to be remarked that the function, although presenting itself in a form unsymmetric in regard to the factors of P and Q, and of R, is really symmetric as regards the three sets of factors; this is obvious *à priori*, and it will be presently verified.

23. For the calculation of the second term

$$x'^2\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,$$

we have

$$f(x-a),\ g(x-b),\ h(x-c)=\Omega P,\ \Omega Q,\ \Omega R,$$

where Ω is a determinate function of z; hence

$$\frac{x'}{x-a},\ \frac{x'}{x-b},\ \frac{x'}{x-c}=\frac{P'}{P}+\frac{\Omega'}{\Omega},\ \frac{Q'}{Q}+\frac{\Omega'}{\Omega},\ \frac{R'}{R}+\frac{\Omega'}{\Omega}.$$

Then substituting these values, by reason that the function is diaphoric, the terms in $\frac{\Omega'}{\Omega}$ disappear, and we have

$$x'^2\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2$$

$$=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \frac{P'}{P},\ \frac{Q'}{Q},\ \frac{R'}{R}\right)^2,$$

which is

$$=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \Sigma\frac{p}{z-l},\ \Sigma\frac{q}{z-m},\ \Sigma\frac{r}{z-n}\right)^2.$$

We have $\Sigma p=\Sigma q=\Sigma r,\ =\delta$: and hence by what precedes, this function, considered as a function of the inverts $\frac{1}{z-l}$, &c., is diaphoric, and therefore curtate.

24. We have therefore

$$\{x,\ z\}+x'^2\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2=$$

$$\left\{-\Sigma\frac{p-1}{(z-l)^2}-\Sigma\frac{q-1}{(z-m)^2}+\Sigma\frac{r+1}{(z-n)^2}\right\}$$

$$-\tfrac{1}{2}\left(\Sigma\frac{p-1}{z-l}+\Sigma\frac{q-1}{z-m}-\Sigma\frac{r+1}{z-n}\right)^2$$

$$+\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\ \Sigma\frac{p}{z-l},\ \Sigma\frac{q}{z-m},\ \Sigma\frac{r}{z-n}\right)^2,$$

where the whole function on the right-hand side is curtate.

25. We have to bring the function on the right-hand side into the reduced form

$$\frac{a}{(z-\alpha)^2}+\ldots+\frac{A}{z-\alpha}+\ldots$$

for the purpose of getting rid of the non-essential inverts (if any).

We write

$$\Sigma\frac{p-1}{(z-l)^2}=\frac{p-1}{(z-l)^2}+\frac{p_1-1}{(z-l_1)^2}+\ldots$$
$$=\frac{p-1}{(z-l)^2}+\Sigma'\frac{p_1-1}{(z-l_1)^2};$$

viz. $z-l$ here denotes any particular factor, and $z-l_1$ represents any other factor of the same set; and so in other like cases.

26. The whole coefficient of $\frac{1}{(z-l)^2}$ is

$$-(p-1)-\tfrac{1}{2}(p-1)^2+\mathrm{a}p^2,\quad =\tfrac{1}{2}(1-p^2)+\mathrm{a}p^2;$$

an expression which, regarded as a function of a and p, is represented by $(\mathrm{a}p)$: the parentheses are used only to avoid ambiguity, and are omitted when p is a number, thus $\mathrm{a}1=\mathrm{a}$, $\mathrm{a}2=-\frac{3}{2}+4\mathrm{a}$, and so in other cases.

27. The whole term in $\frac{1}{z-l}$ comes from

$$-\frac{p-1}{z-l}\left(\Sigma'\frac{p_1-1}{z-l_1}+\Sigma\frac{q-1}{z-m}-\Sigma\frac{r+1}{z-n}\right)$$
$$+\frac{p}{z-l}\left\{2\mathrm{a}\Sigma'\frac{p_1}{z-l}+(-\mathrm{a}-\mathrm{b}+\mathrm{c})\Sigma\frac{q}{z-m}+(-\mathrm{a}+\mathrm{b}-\mathrm{c})\Sigma\frac{r}{z-n}\right\},$$

viz. each term such as $\frac{1}{z-l\,.\,z-l_1}$ is to be replaced by $\frac{1}{l-l_1}\left(\frac{1}{z-l}-\frac{1}{z-l_1}\right)$, giving rise to the term $\frac{1}{l-l_1}\frac{1}{z-l}$, or contributing the term $\frac{1}{l-l_1}$ to the coefficient of $\frac{1}{z-l}$. The whole coefficient thus is

$$=-(p-1)\left(\Sigma'\frac{p_1-1}{l-l_1}+\Sigma\frac{q-1}{l-m}-\Sigma\frac{r+1}{l-n}\right)$$
$$+2\mathrm{a}p\Sigma'\frac{p_1}{l-l_1}+p(-\mathrm{a}-\mathrm{b}+\mathrm{c})\Sigma\frac{q}{l-m}+p(-\mathrm{a}+\mathrm{b}-\mathrm{c})\Sigma\frac{r}{l-n}.$$

28. Suppose first that $z-l$ is a multiple factor of P, viz. a factor with an index p greater than 1: then, for $z=l$, we have $Q+R=0$, $Q'+R'=0$, and thence $\frac{Q'}{Q}=\frac{R'}{R}$, that is, $\Sigma\frac{q}{l-m}=\Sigma\frac{r}{l-n}$. We have therefore

$$p(-\mathrm{a}-\mathrm{b}+\mathrm{c})\Sigma\frac{q}{l-m}+p(-\mathrm{a}+\mathrm{b}-\mathrm{c})\Sigma\frac{r}{l-n}$$
$$=-\mathrm{a}p\left(\Sigma\frac{q}{l-m}+\Sigma\frac{r}{l-n}\right);$$

moreover, in the top line, the terms $\Sigma \frac{q}{l-m}$ and $-\Sigma \frac{r}{l-n}$ destroy each other. The whole coefficient of $\frac{1}{z-l}$, when $z-l$ is a multiple factor of P, thus is

$$= -(p-1)\left(\Sigma' \frac{p_1-1}{l-l_1} - \Sigma \frac{1}{l-m} - \Sigma \frac{1}{l-n}\right)$$

$$+ 2ap\Sigma' \frac{p_1}{l-l_1} - ap\left(\Sigma \frac{q}{l-m} + \Sigma \frac{r}{l-n}\right),$$

a form which is now symmetrical in regard to the inverts $\frac{1}{l-m}$ and $\frac{1}{l-n}$.

29. The value just obtained must be equal to

$$(1-p^2+2ap^2)\left(\Sigma \frac{\frac{1}{2}q-1}{l-m} + \Sigma \frac{\frac{1}{2}r-1}{l-n} - \Sigma' \frac{1}{l-l_1}\right);$$

viz. comparing the two forms and reducing, they will be identical if only

$$(1-p+2ap)\left\{\Sigma' \frac{p+p_1}{l-l_1} - \Sigma \frac{\frac{1}{2}(1+p)q-p}{l-m} - \Sigma \frac{\frac{1}{2}(1+p)r-p}{l-n}\right\} = 0,$$

and it can be shown that the function inside the { } is in fact $=0$.

30. We have, as before, $\Sigma \frac{q}{l-m} = \Sigma \frac{r}{l-n}$; or writing each of these quantities $=\Phi$, the equation to be verified is

$$\Sigma' \frac{p+p_1}{l-l_1} = (p+1)\Phi - p\Sigma \frac{1}{l-m} - p\Sigma \frac{1}{l-n}.$$

We have

$$\frac{P'}{P} = \frac{p}{z-l} + \Sigma' \frac{p_1}{z-l_1}, \quad = \frac{X_1}{X},$$

that is,

$$\Sigma' \frac{p_1}{l-l_1} = \left[\frac{X_1}{X} - \frac{p}{z-l}\right] \text{ for } z=l,$$

$$= \left[\frac{X_1(z-l)-pX}{X(z-l)}\right].$$

The first derived function of the numerator is $X_1'(z-l)+X_1-pX'$, which for $z=l$ is X_1-pX', which is $=0$; and, for the denominator, it is $X'(z-l)+X$, which is also $=0$. Passing to the second derived functions, we find

$$\Sigma' \frac{p_1}{z-l_1} = \frac{2X_1'-pX''}{2X'}, \quad = \frac{X_1'-\frac{1}{2}pX''}{X'}.$$

From the equation

$$\frac{X'}{X} = \frac{1}{z-l} + \Sigma' \frac{1}{z-l}$$

we find in like manner

$$\Sigma' \frac{1}{l-l_1} = \frac{\frac{1}{2}X''}{X'},$$

and we thence obtain (z being always $=l$)

$$\Sigma' \frac{p+p_1}{z-l_1} = \frac{X_1'}{X'},$$

so that the equation to be verified becomes

$$\frac{X_1'}{X_1} = (p+1)\Phi - p\Sigma \frac{1}{l-m} - p\Sigma \frac{1}{l-n}.$$

31. But from the equation $\Theta, = PQ' - P'Q, = KP_0Q_0R_0$, we find $XY_1 - X_1Y = KR_0$, and then, differentiating, $XY_1' + X'Y_1 - X_1'Y - X_1Y' = KR_0'$: writing in these equations $z=l$, they become

$$-X_1Y = KR_0,$$

$$X'Y_1 - X_1'Y - X_1Y' = KR_0,$$

so that, dividing the second by the first,

$$-\frac{X'}{X_1}\frac{Y_1}{Y} + \frac{X_1'}{X_1} + \frac{Y'}{Y} = \frac{R_0'}{R_0};$$

or, recollecting that $X_1 = pX'$ and $\frac{Y_1}{Y} = \frac{Q'}{Q}$, we have

$$\frac{X_1'}{X'} = p\left(\frac{R_0'}{R_0} - \frac{Y'}{Y}\right) + \frac{Q'}{Q},$$

that is,

$$\frac{X_1'}{X'} = p\left(\Sigma \frac{r-1}{l-n} - \Sigma \frac{1}{l-m}\right) + \Sigma \frac{q}{l-m},$$

$$= (p+1)\Phi - p\Sigma \frac{1}{l-m} - p\Sigma \frac{1}{l-n},$$

the required relation.

32. The result is that, $z-l$ being a multiple factor of P, the coefficient of the term $\frac{1}{z-l}$ is

$$= (1-p^2+2ap^2)\left\{\Sigma \frac{\frac{1}{2}q-1}{l-m} + \Sigma \frac{\frac{1}{2}r-1}{l-n} - \Sigma' \frac{1}{l-l_1}\right\},$$

$$= 2(ap)\left[\frac{1}{2}\left(\frac{Q'}{Q} + \frac{R'}{R}\right) - \frac{Y'}{Y} - \frac{Z'}{Z} - \frac{1}{2}\frac{X''}{X'}\right].$$

33. In the case where $z-l$ is a simple factor of P we have $p=1$, and the coefficient is

$$= 2a\Sigma' \frac{p_1}{l-l_1} + (-a-b+c)\Sigma \frac{q}{l-m} + (-a+b-c)\Sigma \frac{r}{l-n},$$

$$= a\left(2\Sigma' \frac{p_1}{l-l_1} - \Sigma \frac{q}{l-m} - \Sigma \frac{r}{l-n}\right) - (b-c)\left(\Sigma \frac{q}{l-m} - \Sigma \frac{r}{l-n}\right).$$

34. Of course the formulæ for the coefficients of $\frac{1}{(z-l)^2}$ and $\frac{1}{z-l}$ give at once, by a mere change of letters, those for the coefficients of $\frac{1}{(z-m)^2}$, $\frac{1}{z-m}$, and $\frac{1}{(z-n)^2}$, $\frac{1}{z-n}$; and the function in question,

$$\{x, z\} + x'^2\left(\mathrm{a, b, c} \therefore \chi\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)\right)^2,$$

is now obtained in the required form

$$= \frac{(\mathrm{a}p)}{(z-l)^2}\cdots + \frac{(\mathrm{b}q)}{(z-m)^2}\cdots + \frac{(\mathrm{c}r)}{(z-n)^2}\cdots + \frac{A}{z-l}\cdots + \frac{B}{z-m}\cdots + \frac{C}{z-n}\cdots$$

where $(\mathrm{a}p)$ denotes $\frac{1}{2}(1-p^2)+\mathrm{a}p^2$, and the like for $(\mathrm{b}q)$ and $(\mathrm{c}r)$; and where, $z-l$ being a multiple factor of P, the coefficient A contains the factor $(\mathrm{a}p)$; and similarly for B and C.

35. Suppose that the coefficients a, b, c are no one of them $=0$; we have $\mathrm{a}1$, $=\mathrm{a}$, which does not vanish; that is, $z-l$ being a simple factor of P, the expression contains $\frac{1}{(z-l)^2}$, or the invert $\frac{1}{z-l}$ is essential: and similarly, $z-m$ being a simple factor of Q, or $z-n$ a simple factor of R, the inverts $\frac{1}{z-m}$ and $\frac{1}{z-n}$ are essential. But for $z-l$ a multiple factor of P, the coefficient $(\mathrm{a}p)$ of the term $\frac{1}{(z-l)^2}$ may vanish, viz. this will be the case if $\mathrm{a}=\frac{1}{2}\left(1-\frac{1}{p^2}\right)$; and, when this is so, the coefficient A of the corresponding term $\frac{1}{z-l}$ also vanishes; that is, $\frac{1}{z-l}$ is a non-essential invert. And similarly for any multiple factor $z-m$ of Q or $z-n$ of R, the invert $\frac{1}{z-m}$ or $\frac{1}{z-n}$ may be non-essential.

36. If P, Q, R contain each of them only multiple factors of the same index, say of the indices p, q, r for the three functions respectively, viz. if the functions are $F(\Pi(z-l))^p$, $G(\Pi(z-m))^q$, $H(\Pi(z-n))^r$, the result contains only the six terms written down: and then, if a, b, c are $=\frac{1}{2}\left(1-\frac{1}{p^2}\right)$, $\frac{1}{2}\left(1-\frac{1}{q^2}\right)$, $\frac{1}{2}\left(1-\frac{1}{r^2}\right)$ respectively the result is $=0$: viz. we then have

$$\{x, z\} + x'^2\left(\mathrm{a, b, c} \therefore \chi\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)\right) = 0,$$

or we in fact have, for the values in question of a, b, c, a solution

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R$$

of this differential equation of the third order.

37. The reasoning applies directly to lines 2, 3, 4, 5 of the PQR-Table: and with a slight variation to line 1; viz. here the factors of R $(=-1+z^n)$ are all simple factors, but in virtue of $c=0$ and $a=b$, the corresponding inverts disappear, and, the other inverts also disappearing, the value of the function is $=0$. Hence lines 1, 2, 3, 4, 5 of the PQR-Table give each of them a result $=0$, for the values of (a, b, c) appearing by the table itself, and shown explicitly in the corresponding line of the Annex.

Thus line 3 shows that the function x, determined by

$$f(x-a):g(x-b):h(x-c)=(z^4+2\sqrt{-3}z^2+1)^2:-12\sqrt{-3}(z^5-z)^2:-(z^4-2\sqrt{-3}z^2+1)^2,$$

satisfies

$$\{x, z\}+x'^2\left(\frac{4}{9}, \frac{3}{8}, \frac{4}{9}\therefore\right)\!\!\!

which are

$$=0;\ \frac{3}{8},\ 0;\ \frac{12}{25},\ \frac{21}{50} \text{ respectively.}$$

Hence writing $a_1, b_1, c_1 = \frac{3}{8}, \frac{12}{25}, \frac{21}{50}$, the corresponding inverts are $\frac{1}{z-1}, \frac{1}{z-\infty}, \frac{1}{z}$; and the result is

$$\{x, z\} + x'^2 \left(\frac{4}{9}, \frac{3}{8}, \frac{12}{25} \therefore \right)\!\!\!

and the foregoing is therefore a solution of

$$\{s, x\} = \left(\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore \left)\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

a differential equation of the third order. This is the Differential Equation $\{s, x\}$.

43. From the Roman lines, if we assume

$$f(x-a) : g(x-b) : h(x-c) = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R},$$

where $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ are functions of z, not the same functions that P, Q, R are of s, since they belong to a different line of the Table: we have, as before,

$$\{x, z\} + \left(\frac{dx}{dz}\right)^2 \left(\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore \left)\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2 = \left(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore \left)\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-c_1}\right)^2.$$

44. We may combine any such result with a properly selected result of the preceding system, the two results being such that (a, b, c) have the same values in each of them. (See as to this the foot-note referring to the Annex to the PQR-Table.) The last equation then becomes

$$\{x, z\} + \left(\frac{dx}{dz}\right)^2 \{s, x\} = \left(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore \left)\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-c_1}\right)^2,$$

or since

$$\{x, z\} + \left(\frac{dx}{dz}\right)^2 \{s, x\} = \{s, z\},$$

this is

$$\{s, z\} = \left(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore \left)\frac{1}{z-a_1}, \frac{1}{z-b_1}, \frac{1}{z-c_1}\right)^2,$$

the corresponding relation between s, z being of course obtained by the elimination of x from the two sets of equations

$$f(x-a) : g(x-b) : h(x-c) = P : Q : R, \text{ and } f(x-a) : g(x-b) : h(x-c) = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R};$$

viz. the required relation is

$$P : Q : R = \mathfrak{P} : \mathfrak{Q} : \mathfrak{R},$$

where P, Q, R are functions of s; $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ functions of z; and, in virtue of

$$P + Q + R = 0, \quad \mathfrak{P} + \mathfrak{Q} + \mathfrak{R} = 0,$$

the relations are equivalent to a single equation between z and s. And writing finally x in place of z, that is, now considering $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ as functions of x, we have

$$\mathfrak{P} : \mathfrak{Q} : \mathfrak{R} = P : Q : R$$

as a solution of

$$\{s, x\} = \left(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore \left)\frac{1}{x-a_1}, \frac{1}{x-b_1}, \frac{1}{x-c_1}\right)^2,$$

a differential equation of the third order of the foregoing form, $\{s, x\}$ = given function of x, but with different values of the coefficients, $(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1)$ instead of (a, b, c).

45. It thus appears that there are in all 16 sets of values of (a, b, c), for which the equation is solved, viz. the 16 sets of values are shown in the right-hand column of the Annex. For greater clearness I exhibit the integral equations as follows:

	Functions of x.		Functions of s.		
1	$f(x-a) : g(x-b) : h(x-c)$	$=$	$P : Q : R$	(1)	Polygon
I	,,	$=$	,,	(2)	Double Pyramid
II	,,	$=$	,,	(3)	Tetrahedron
III	$4x : -(x+1)^2 : (x-1)^2$	$=$	,,	(3)	,,
IV	$f(x-a) : g(x-b) : h(x-c)$	$=$	,,	(4)	Cube and Octahedron
V	$(x-1)^2 : -(x+1)^2 : 4x$	$=$	,,	(4)	,,
VI	$f(x-a) : g(x-b) : h(x-c)$	$=$	,,	(5)	Dodecahedron and Icosahedron
VII	$4x : -(x+1)^2 : (x-1)^2$	$=$	,,	(5)	,,
VIII	$(x-1)^2 : -(x+1)^2 : 4x$	$=$	,,	(5)	,,
IX	$P : Q : R$ (IX)	$=$	,,	(5)	,,
X	,, (X)	$=$	,,	(5)	,,
XI	,, (XI)	$=$	,,	(5)	,,
XII	,, (XII)	$=$	,,	(5)	,,
XIII	,, (XIII)	$=$	,,	(5)	,,
XIV	,, (XIV)	$=$	,,	(5)	,,
XV	,, (XV)	$=$	,,	(5)	,,

The values of the P, Q, R as functions of x, or of s, are taken out of the PQR-Table: only in the lines III, V, VII, VIII, where P, Q, R are given as

$$= 4z, \quad -(z+1)^2, \quad (z-1)^2,$$

and where, as regards V and VIII, there is a transposition of P and R, I have inserted the actual values of the x-functions. (See as to this the foot-note referring to the Annex.)

The Schwarzian Theory. Art. Nos. 46 to 62.

46. Considering the foregoing equation

$$\{s, x\} = \left(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1 \therefore \right)\left(\frac{1}{x-a_1}, \frac{1}{x-b_1}, \frac{1}{x-c_1}\right)^2$$

as a particular case of the equation $\{s, x\} =$ Rational function of x, $= R(x)$ suppose, then we have in 1, I, II, IV, VI solutions of the form $x =$ Rational function of s.

Consider, in general, a solution of this form, $x = F(s)$ a rational function of s: then s is an irrational function of x, and if s_1, s_2 are any two of its values, $\{s_1, x\} = R(x)$, $\{s_2, x\} = R(x)$; that is, $\{s_2, x\} = \{s_1, x\}$, and therefore (ante, No. 7) $s_2 = \dfrac{as_1 + b}{cs_1 + d}$. And then $x = F(s_2) = F\left(\dfrac{as_1 + b}{cs_1 + d}\right)$, $= F(s_1)$: viz. $F(s)$ is a rational function of s, transformable into itself by the transformation s into $\dfrac{as + b}{cs + d}$: and it is moreover clear that between any two roots s whatever of the equation $x = F(s)$ there exists a homographic relation of the form in question. Further, it is clear that these homographic transformations form a group; and consequently that $F(s)$ is a rational function of s, transformable into itself by the several homographic transformations of a group of such transformations: viz. taking x to be a rational function of s, it is *only* in the case $x = F(s)$, a function of the form in question, that $\{s, x\}$ can be equal to a rational function of x.

47. We may, in any equation between x and s, consider these as imaginary variables $p + qi$ and $u + vi$ respectively; considering then (p, q) and (u, v) as rectangular coordinates of points in different planes, we have a first plane the locus of the points x, and a second plane the locus of the points s: there is between the two planes a correspondence which is in fact the correspondence of conformable figures: to the infinitesimal element dx drawn from a point x of the first figure corresponds an infinitesimal element ds drawn from the corresponding point s of the second figure, these elements being in general connected by an equation of the form $ds = (a + bi)\,dx$, where a and b are functions of x or s; and this signifies that, to obtain the pencil of infinitesimal elements or radii ds proceeding in different directions from the point s, we alter in a determinate ratio the absolute lengths of the infinitesimal elements or radii proceeding from the corresponding point x, and rotate the pencil through a determinate angle: this ratio and angle of rotation, or say, the Auxesis and the Streblosis, being of course variable from point to point. Or, what comes to the same thing, if dx and d_1x be consecutive elements of the path of the point x, and ds, d_1s the corresponding consecutive elements of the path of the point s, then the ratio of the lengths of the elements dx, d_1x is equal to that of the lengths of the elements ds, d_1s; and the mutual inclination of the first pair of elements is equal to that of the second pair of elements. In particular, if at any point the path of x is a curved line without abrupt change of direction, then at the corresponding point the path of s is a curved line without abrupt change of direction. In what precedes, we have the relation at ordinary points; but there may be critical corresponding points (x, s), the relation at a critical point between the corresponding elements dx, ds being of the form $ds = (a + bi)(dx)^\lambda$, ($\lambda$ a positive integer or fraction): here the angle between two elements ds is $= \lambda$ times that between the two elements dx; or, if the path of the point x through the critical point is without abrupt change of direction, say if the angle between the two consecutive elements is the flat angle π, then the angle between the two consecutive elements ds is $= \lambda\pi$: viz. there may be in the path of the point s an abrupt change of direction.

48. I consider the foregoing equation $\{s, x\} = R(x)$, where $R(x)$ is a rational function, and is now taken to be a real function of x: we may assume $s' = i\rho'\theta' e^{i\theta}$, where the accents denote differentiation in regard to x, and where ρ', θ, and therefore also θ', are real functions of x. We have

$$\frac{s''}{s'} = \frac{\rho''}{\rho'} + \frac{\theta''}{\theta'} + i\theta',$$

and thence

$$\begin{aligned} \frac{s'''}{s'} - \left(\frac{s''}{s'}\right)^2 &= \frac{\rho'''}{\rho'} - \left(\frac{\rho''}{\rho'}\right)^2 + \frac{\theta'''}{\theta'} - \left(\frac{\theta''}{\theta'}\right)^2 \qquad\qquad + i\theta'' \\ -\tfrac{1}{2}\left(\frac{s''}{s'}\right)^2 &= \quad -\tfrac{1}{2}\left(\frac{\rho''}{\rho'}\right)^2 \quad -\tfrac{1}{2}\left(\frac{\theta''}{\theta'}\right)^2 + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} - i\theta'' - i\frac{\rho''\theta'}{\rho'}, \end{aligned}$$

and thence

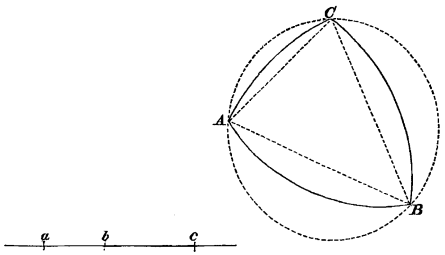
$$\{s, x\} = \{\rho, x\} + \{\theta, x\} + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} - i\frac{\rho''\theta'}{\rho'}.$$

Putting this $= R(x)$, and assuming that x is real, we have

$$\{\rho, x\} + \{\theta, x\} + \tfrac{1}{2}\theta'^2 - \frac{\rho''\theta''}{\rho'\theta'} = R(x); \quad 0 = i\frac{\rho''\theta'}{\rho'}.$$

The last equation gives $\rho''\theta' = 0$, that is, $\theta' = 0$, which gives $s' = 0$, and may be disregarded; or else $\rho'' = 0$, therefore ρ', a real constant, $= \gamma$ suppose, and $\{\rho, x\} = 0$: hence for the solution of the equation $\{s, x\} = R(x)$, we have $s' = i\gamma\theta' e^{i\theta}$, θ a real quantity determined by $\{\theta, x\} + \frac{1}{2}\theta'^2 = R(x)$: and then, integrating the equation for s', we have $s = \alpha + \beta i + \gamma e^{i\theta}$, α, β, γ real constants.

49. The conclusion is that, if $\{s, x\} = R(x)$, a real function of x, and if x be real, that is, if the point x move along a right line (say the x-line), then $s = \alpha + \beta i + \gamma e^{i\theta}$ (θ, and the constants α, β, γ, being real), that is, the point s moves in a circle, coordinates of the centre α, β, and radius $= \gamma$.



50. Suppose a, b, c are any real values of x representing points a, b, c on the x-line; and A, B, C any given imaginary values of s representing points A, B, C

in the s-plane: since $\{s, x\} = R(x)$ is a differential equation of the third order, the integral contains three arbitrary constants, and we may imagine these so determined that to the values $x = a, b, c$ shall correspond the values $s = A, B, C$ respectively.

If there is not on the x-line any critical point, as the point x moves continuously along this line the point s will move continuously along a circle, which (inasmuch as a, b, c and A, B, C are corresponding points) must be the circle through the three points A, B, C*.

51. If however the points a, b, c are critical points, such that the element ds at the corresponding points A, B, C are equal to multiples of $(dx)^\lambda$, $(dx)^\mu$, $(dx)^\nu$ respectively, then to the flat angles π at a, b, c correspond in the path of s the angles $\lambda\pi, \mu\pi, \nu\pi$ at the points A, B, C respectively: and, assuming that a, b, c are the only critical points on the x-line, the path of s is made up of the three circular arcs CA, AB, BC meeting at angles $\lambda\pi, \mu\pi, \nu\pi$ respectively. The arcs are completely determined by these conditions; for supposing the arc BC to make with the chord BC, at the points B and C, the angles f, f, and similarly the arcs CA and AB to make with the corresponding chords the angles g, g and h, h, then the conditions give $\lambda\pi, \mu\pi, \nu\pi = \angle A + g + h, \angle B + h + f, \angle C + f + g$, where the angles referred to are those of the rectilinear triangle ABC: we have thus the values of f, g, h; and the arc BC is the arc on the chord BC meeting it at angles f, f: and the like as regards the arcs CA and AB respectively.

52. The foregoing equation

$$\{s, x\} = \left(\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore \frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

where a, b, c have the values $\frac{1}{2}(1-\lambda^2)$, $\frac{1}{2}(1-\mu^2)$, $\frac{1}{2}(1-\nu^2)$, and λ, μ, ν are real and positive, has $x = a, b, c$ for critical points of the kind in question. In fact, writing $x - a = h$, the equation is of the form

$$\{s, h\} = \frac{\frac{1}{2}(1-\lambda^2)}{h^2} + \frac{a_0}{h} + a_1 + a_2 h + \ldots,$$

which is satisfied by

$$\frac{d}{dh}\log\frac{ds}{dh} = -\frac{1+\lambda}{h} + b_0 + b_1 h + b_2 h^2 + \ldots;$$

we thence obtain an integral of the form

$$s = kh^{-\lambda}(1 + k_1 h + k_2 h^2 + \ldots), \quad = k\phi \text{ for shortness.}$$

This is a particular integral, but we have from it the general integral

$$s = \frac{\alpha + \beta k\phi}{\gamma + \delta k\phi}.$$

* Since there is no critical point on the x-line there can be no abrupt change of direction in the path of s, that is, the path of s cannot consist of circular arcs meeting at an angle: but it is in the text further assumed that the path of s cannot consist of different arcs of circle, the one continuing the other without any abrupt change of direction.

If A be the value of s corresponding to $h=0$, then $\beta=\delta A$, and we find

$$s=\frac{\alpha+A\delta k\phi}{\gamma+\delta k\phi}, \quad =\left(A+\frac{\alpha}{\delta k\phi}\right)\left(1+\frac{\gamma}{\delta k\phi}\right)^{-1}, \quad =A+\frac{\alpha-\gamma A}{\delta k}\frac{1}{\phi}+\ldots;$$

viz. reducing $\frac{1}{\phi}$ to its principal term h^λ, and then writing ds, dx for $s-A$, and $h\,(=x-a)$ respectively, we have $ds=K\,(dx)^\lambda$, or $x=a$ is a critical point with the exponent λ; and similarly $x=b$ and $x=c$ are critical points with the exponents μ and ν respectively.

53. Hence in the equation

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\right)\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2,$$

as the point x, passing successively through a, b, c, describes the x-line, the point s, passing successively through A, B, C, describes the sides AB, BC, CA of the curvilinear triangle ABC. To points x indefinitely near the x-line correspond points s indefinitely near the boundary AB, BC, CA of the triangle, viz. to points x indefinitely near to and on one side, suppose the upper side, of the x-line, correspond the points s indefinitely near to and within the boundary of the triangle: and in like manner to whole series of the points x on the same upper side of the x-line, correspond the whole series of points s inside the triangle.

54. We have attended so far only to one of the points s which correspond to a given point x, but considering the set of points s which correspond to the same point x, we have in the s-plane entire circles forming by their intersections curvilinear triangles ABC, ABC', &c.; we have thus two systems, say ABC, &c., and ABC', &c., of triangles, such that to a point x on the upper side of the x-line correspond points s, one of them within each of the triangles ABC, &c., and to a point x on the lower side of the x-line correspond points s, one of them within each of the triangles ABC', &c.; and so consequently that, to the two half-planes on opposite sides of the x-line, correspond the two sets of triangles ABC, &c., and ABC', &c., respectively.

55. In order that the relation s and x may be an algebraical one, it is necessary that the two sets of triangles should completely cover, once or a finite number of times, the whole of the s-plane: and this implies that the angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ have certain determinate values; and, in fact, that dividing the surface of a sphere into triangles, each with these angles, the curvilinear triangles ABC, ABC', &c., are the stereographic projections of these triangles. It was by such considerations as these that Schwarz, in the Memoir of 1873, p. 323, obtained the series of values I to XV of λ, μ, ν, giving for a, b, c, $=\frac{1}{2}(1-\lambda^2)$, $\frac{1}{2}(1-\mu^2)$, $\frac{1}{2}(1-\nu^2)$, the series of values mentioned in the Annex of the PQR-Table: and thus showed *à priori* that the equation

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\right)\left(\frac{1}{x-a},\ \frac{1}{x-b},\ \frac{1}{x-c}\right)^2$$

is algebraically integrable for these values of a, b, c; and only for these values, or for values reducible to them.

56. As an instance, take the double pyramid form: the integral equation is

$$f(x-a) : g(x-b) : h(x-c) = 4s^n : -(s^n-1)^2 : (s^n+1)^2,$$

or say

$$\frac{(c-a)(x-b)}{(a-b)(x-c)} = -\frac{(s^n-1)^2}{(s^n+1)^2};$$

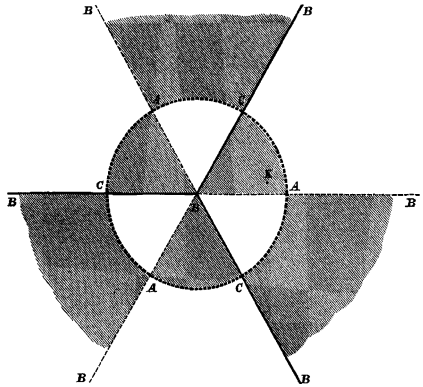
or if, for greater simplicity, we assume $a, b, c = 1, 0, \infty$, this is $x = \frac{(s^n-1)^2}{(s^n+1)^2}$, or say $-(s^n-1) = \sqrt{x}(s^n+1)$, that is, $s^n = \frac{1 \mp \sqrt{x}}{1 \pm \sqrt{x}}$, a solution of the differential equation

$$\{s, x\} = \left(\frac{3}{8}, \ \tfrac{1}{2}(1-n^{-2}), \ \frac{3}{8} \ \therefore \ \frac{1}{x}, \ \frac{1}{x-1}, \ \frac{1}{x-\infty}\right)^2.$$

In particular, if $n=3$, we have $x = \left(\frac{s^3-1}{s^3+1}\right)^2$ or $s^3 = \frac{1 \mp \sqrt{x}}{1 \pm \sqrt{x}}$, a solution of

$$\{s, x\} = \left(\frac{3}{8}, \ \frac{4}{9}, \ \frac{3}{8} \ \therefore \ \frac{1}{x}, \ \frac{1}{x-1}, \ \frac{1}{x-\infty}\right)^2.$$

57. We have here the spherical surface divided by the equator and three meridians into twelve triangles, each with the angles $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$: and then, projecting from the South pole on the plane of the equator, we have the annexed figure of the s-plane,



divided into 12 curvilinear triangles, each with these same angles 90°, 90°, 60°; the plane is divided by the shading into two systems, each of 6 triangles. The figure of the x-plane is by the x-line divided into two half-planes, one shaded, the other unshaded; and we have on the line the point c at ∞, a at the origin, and b at the distance unity.

58. Take x real; then, if x is positive and less than 1, s^3 is real and positive, and we have for s the infinite half-lines at the inclinations 0°, 120°, 240°, while if x is positive and greater than 1, s^3 is real and negative, and we have the infinite half-lines at the inclinations 60°, 180°, 300°. If x is real and negative, then s^3 is of the form $\dfrac{1-ki}{1+ki}$, $=\cos\theta + i\sin\theta$; whence s is of the same form, or the locus of the point s is a circle radius unity. Writing $s^3 = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, and supposing that the point x moves along the x-line from b through a to c at $-\infty$, and then from c at $+\infty$ to b, the point s describes the sides BA, AC, CB of the shaded triangle marked K.

59. Suppose that the point x is at k, in the shaded half-plane at an indefinitely small distance from a; say we have $x = -2\kappa^2 i$, (κ small), then taking for $\sqrt{x}$ the value $\kappa(1-i)$, we have $s^3 = \dfrac{1-\kappa(1-i)}{1+\kappa(1-i)}$, $=1-2\kappa(1-i)$ nearly, and hence a value of s is $=1-\frac{2}{3}\kappa+\frac{2}{3}\kappa i$, which belongs to a point K near A, and within the shaded triangle: we have thus, in respect of this value of s, the shaded half of the x-plane corresponding to this shaded triangle. To the same value $x=-2\kappa^2 i$, correspond in all six values of s, giving six points K each lying near a point A within one of the shaded triangles; and hence the shaded half-plane corresponds to the six shaded triangles, and the unshaded half-plane corresponds to the six unshaded triangles.

60. Suppose the equation is

$$\{s, x\} = \left(\mathrm{a}, \mathrm{b}, \mathrm{c} \therefore \right)\left(\frac{1}{x-a}, \frac{1}{x-b}, \frac{1}{x-c}\right)^2,$$

that is,

$$= \frac{-(b-c)(c-a)(a-b)}{x-a\,.\,x-b\,.\,x-c}\left(\frac{\mathrm{a}}{b-c\,.\,x-a} + \frac{\mathrm{b}}{c-a\,.\,x-b} + \frac{\mathrm{c}}{a-b\,.\,x-c}\right),$$

where a, b, c are real, but a, b, c are imaginary. It is to be shown that, if the path of x is the circle passing through the points a, b, c, then the path of s is a circle passing through the corresponding three points.

61. We may find α, β, γ, θ_0, θ_1, θ_2, such that a, b, c are $=\alpha+\beta i+\gamma e^{\theta_0 i}$, $\alpha+\beta i+\gamma e^{\theta_1 i}$, $\alpha+\beta i+\gamma e^{\theta_2 i}$ (this is, in fact, finding α and β the coordinates of the centre, and γ the radius of the circle through the three points a, b, c): we then have $x=\alpha+\beta i+\gamma e^{\theta i}$, θ a variable parameter, the equation which expresses that the point x is situate on the circle in question.

We have $x-a=\gamma(e^{\theta i}-e^{-\theta_0 i})$, $=\gamma e^{\frac{1}{2}(\theta+\theta_0)}\{e^{\frac{1}{2}(\theta-\theta_0)i}-e^{-\frac{1}{2}(\theta-\theta_0)i}\}$; the second factor is $i\sin\frac{1}{2}(\theta-\theta_0)$, $=iP$ suppose, or the equation is $x-a=iP\gamma e^{\frac{1}{2}(\theta+\theta_0)i}$, say

$$x-a = iP\gamma \operatorname{expi}\tfrac{1}{2}(\theta+\theta_0).$$

Similarly $x-b=iQ\gamma\operatorname{expi}\frac{1}{2}(\theta+\theta_1)$, and $x-c=iR\gamma\operatorname{expi}\frac{1}{2}(\theta+\theta_2)$; where P, Q, R denote $\sin\frac{1}{2}(\theta-\theta_0)$, $\sin\frac{1}{2}(\theta-\theta_1)$, $\sin\frac{1}{2}(\theta-\theta_2)$ respectively. In like manner, we have $b-c$, $c-a$, $a-b$, $=iF\gamma\operatorname{expi}\frac{1}{2}(\theta_1+\theta_2)$, $iG\gamma\operatorname{expi}\frac{1}{2}(\theta_2+\theta_0)$, $iH\gamma\operatorname{expi}\frac{1}{2}(\theta_0+\theta_1)$, where F, G, H denote $\sin\frac{1}{2}(\theta_1-\theta_2)$, $\sin\frac{1}{2}(\theta_2-\theta_0)$, $\sin\frac{1}{2}(\theta_0-\theta_1)$ respectively.

We have

$$-\frac{b-c\,.\,c-a\,.\,a-b}{x-a\,.\,x-b\,.\,x-c}=\frac{-FGH}{PQR}\operatorname{expi}\tfrac{1}{2}(\theta_0+\theta_1+\theta_2-3\theta),$$

$$\frac{1}{b-c\,.\,x-a}=\frac{-1}{\gamma^2 PF}\operatorname{expi}-\tfrac{1}{2}(\theta_0+\theta_1+\theta_2+\theta),$$

with the like values for $\dfrac{1}{c-a\,.\,x-b}$ and $\dfrac{1}{a-b\,.\,x-c}$. Hence the right-hand side of the equation is

$$=\frac{FGH}{PQR}\left(\frac{\mathrm{a}}{PF}+\frac{\mathrm{b}}{QG}+\frac{\mathrm{c}}{RH}\right)\operatorname{expi}(-2\theta).$$

62. Considering now the left-hand side of the equation, we have

$$\{s,\ x\}=\frac{1}{\left(\dfrac{dx}{d\theta}\right)^2}(\{s,\ \theta\}-\{x,\ \theta\});$$

substituting for x its value $=\alpha+\beta i+\gamma e^{\theta i}$, this becomes

$$\{s,\ x\}=-\frac{1}{\gamma^2}e^{-2\theta i}(\{s,\ \theta\}-\tfrac{1}{2}),$$

that is,

$$=-\frac{1}{\gamma^2}(\{s,\ \theta\}-\tfrac{1}{2})\operatorname{expi}(-2\theta).$$

Assume $s=L+Mi+Ne^{\Theta i}$, L, M, and N constants; then using the accent to denote differentiation in regard to θ, we find without difficulty $\{s,\ \theta\}=\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2$, and the value of $\{s,\ x\}$ becomes

$$=-\frac{1}{\gamma^2}\left(\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2-\tfrac{1}{2}\right)\operatorname{expi}(-2\theta).$$

Hence, substituting the values of the two sides of the equation, the imaginary factor $\operatorname{expi}(-2\theta)$ divides out, and the equation becomes

$$\{\Theta,\ \theta\}+\tfrac{1}{2}\Theta'^2-\tfrac{1}{2}=-\frac{FGH}{PQR}\left(\frac{\mathrm{a}}{PF}+\frac{\mathrm{b}}{QG}+\frac{\mathrm{c}}{RH}\right),$$

an equation, in which everything is real and which thus determines Θ as a real function of θ: and we have therefore the theorem in question.

Connexion with the differential equation for the hypergeometric series. Art. Nos. 63 to 68.

63. Take p, q given functions of x, and y a function of x determined by the equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0;$$

again P, Q given functions of z, and v a function of z determined by the equation

$$\frac{d^2v}{dz^2}+P\frac{dv}{dz}+Qv=0;$$

and assume

$$y=wv.$$

Substituting this value of y in the first equation, we obtain for v an equation of the second order (the coefficients of which contain w), and we may make this identical with the second equation; viz. comparing the coefficients of the two equations, we thus have two equations each containing w; and by eliminating w we obtain a differential equation of the third order between z and x. This is, in fact, the basis of Kummer's theory for the transformation of a hypergeometric series: the equation between z, x will be found presently in a different manner.

64. But if with Schwarz, instead of making the equation obtained for v as above identical with the given equation for v, we merely assume that the two equations are consistent, then there is nothing to determine the value of z, which may be regarded as an arbitrary function of x; y and v are then functions of x, and w denotes the quotient $y \div v$ of these two functions, and as such satisfies an equation the form of which will depend on the assumed relation between z and x. In particular, if P and Q denote the same functions of z that p and q are of x; and if we assume $z=x$, P, Q will become $=p$, q respectively: the given equation in v will be

$$\frac{d^2v}{dx^2}+p\frac{dv}{dx}+qv=0;$$

and w will thus denote the quotient of any two solutions of the equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0;$$

viz. writing $X=p^2+2\frac{dp}{dx}-4q$, then, by what precedes, the equation for w will be

$$\{w,\ x\}=-\tfrac{1}{2}X.$$

65. Returning now to Kummer's problem, and considering y, v as solutions of the two differential equations respectively, w is a function independent of the particular solutions denoted by these letters: we have $y=wv$, and taking any other two solutions we have $y_1=wv_1$, so that $\frac{y}{y_1}=\frac{v}{v_1}$; calling each of these equal quantities s, we have s denoting the quotient of two solutions of the equation in y, and also the quotient of two solutions of the equation in v; whence, writing as before $X=p^2+2\frac{dp}{dx}-4q$, and similarly $Z=P^2+2\frac{dP}{dz}-4Q$, we have

$$\{s,\ x\}=-\tfrac{1}{2}X,\quad \{s,\ z\}=-\tfrac{1}{2}Z,$$

and since in general

$$\{s,\ x\}=\left(\frac{dz}{dx}\right)^2\{s,\ z\}+\{z,\ x\},$$

we obtain

$$\{z,\ x\} = -\tfrac{1}{2}X + \tfrac{1}{2}Z\left(\frac{dz}{dx}\right)^2,$$

as the required equation for the determination of z as a function of x. The process does not give the value of w, but this can be found without difficulty, viz.

$$w^2 = Ce^{\int Pdz - \int pdx} \div \frac{dz}{dx}.$$

If z, x are regarded each of them as a function of the new independent variable θ, then the equation is

$$\{z,\ \theta\} - \tfrac{1}{2}\left(\frac{dz}{d\theta}\right)^2 Z = \{x,\ \theta\} - \tfrac{1}{2}\left(\frac{dx}{d\theta}\right)^2 X.$$

66. Jacobi's differential equation of the third order for the transformed modulus λ, *Fund. Nova*, p. 78, [*Ges. Werke*, t. I, p. 132], is

$$3(k'^2\lambda''^2 - \lambda'^2k''^2) - 2k'\lambda'(k'\lambda''' - \lambda'k''') + k'^2\lambda'^2\left\{\left(\frac{1+k^2}{k-k^3}\right)^2 k'^2 - \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 \lambda'^2\right\} = 0,$$

where the accents denote differentiations in regard to an independent variable θ: viz. dividing by $2k'^2\lambda'^2$, this becomes

$$\{k,\ \theta\} + \tfrac{1}{2}k'^2\left(\frac{1+k^2}{k-k^3}\right)^2 = \{\lambda,\ \theta\} + \tfrac{1}{2}\lambda'^2\left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2,$$

which is thus a particular case of Kummer's equation, k, λ corresponding to x, z respectively, and the values of X, Z being

$$X = -\left(\frac{1+k^2}{k-k^3}\right)^2,\quad Z = -\left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2.$$

67. In the case of the hypergeometric series, the two differential equations of the second order are

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha+\beta+1)x}{x\,.\,1-x}\frac{dy}{dx} - \frac{\alpha\beta y}{x\,.\,1-x} = 0,$$

$$\frac{d^2v}{dz^2} + \frac{\gamma' - (\alpha'+\beta'+1)z}{z\,.\,1-z}\frac{dv}{dz} - \frac{\alpha'\beta' v}{z\,.\,1-z} = 0.$$

Hence

$$p = \frac{\gamma(x+(1-x)) - (\alpha+\beta+1)x}{x\,.\,1-x} = \frac{\gamma}{x} + \frac{\gamma-\alpha-\beta-1}{1-x},\quad q = \frac{-\alpha\beta}{x\,.\,1-x},$$

and hence

$$p^2 + 2\frac{dp}{dx} - 4q = \frac{\gamma^2-2\gamma}{x^2} + \frac{(\gamma-\alpha-\beta-1)^2 + 2(\gamma-\alpha-\beta-1)}{(1-x)^2} + \frac{4\alpha\beta + 2\gamma(\gamma-\alpha-\beta-1)}{x\,.\,1-x};$$

viz. writing

$$\begin{aligned} \lambda^2 &= (1-\gamma)^2, & \mathrm{a} &= \tfrac{1}{2}(1-\lambda^2),\\ \mu^2 &= (\alpha-\beta)^2, & \mathrm{b} &= \tfrac{1}{2}(1-\mu^2),\\ \nu^2 &= (\gamma-\alpha-\beta)^2, & \mathrm{c} &= \tfrac{1}{2}(1-\nu^2), \end{aligned}$$

and putting in the formula $x-1, =-(1-x)$, we have

$$-\tfrac{1}{2}\left(p^2+2\frac{dp}{dx}-4q\right)=\frac{\frac{1}{2}(1-\lambda^2)}{x^2}+\frac{\frac{1}{2}(1-\nu^2)}{(x-1)^2}+\frac{\frac{1}{2}(\lambda^2-\mu^2+\nu^2-1)}{x\,.\,x-1},$$

$$=\frac{\mathrm{a}}{x^2}+\frac{\mathrm{c}}{(x-1)^2}+\frac{-\mathrm{a}+\mathrm{b}-\mathrm{c}}{x\,.\,x-1},$$

$$=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\left(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2,\right.$$

with a like formula for $\frac{1}{2}\left(P^2+2\dfrac{dP}{dz}-4Q\right)$. We then have

$$y=wv,$$

$$w^2=Cx^{-\gamma}(1-x)^{\gamma-\alpha-\beta-1}z^{\gamma'}(1-z)^{-\gamma'+\alpha'+\beta'+1}\frac{dx}{dz},$$

and the differential equation of the third order for the determination of z is

$$\{z,\ x\}+\left(\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1\ \therefore\left(\frac{1}{z},\ \frac{1}{z-\infty},\ \frac{1}{z-1}\right)^2\left(\frac{dz}{dx}\right)^2-\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\left(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2=0,\right.\right.$$

where a_1, b_1, c_1 are the same functions of α', β', γ' which a, b, c are of α, β, γ. This is, in effect, Kummer's equation for the transformation of the hypergeometric series.

68. And in like manner the Schwarzian equation for the determination of s, the quotient of two solutions, is

$$\{s,\ x\}=\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\ \therefore\left(\frac{1}{x},\ \frac{1}{x-\infty},\ \frac{1}{x-1}\right)^2.\right.$$

PART II. THE POLYHEDRAL FUNCTIONS.

Origin and Properties. Art. Nos. 69 to 80.

69. The functions in lines $1,\ldots,5$ of the PQR-Table are connected with the geometrical forms:

1. Polygon or
2. Double Pyramid *,
3. Tetrahedron,
4. Octahedron and Cube,
5. Dodecahedron and Icosahedron,

(these figures being regarded as situate on a spherical surface), and with the stereographic projections of these figures.

* Prof. Klein regards 1 as belonging to the polygon and 2 to the double pyramid: it seems to me that the fundamental figure, to which 1 and 2 each of them belong, is the polygon.

Consider a spherical surface and upon it any number of points: take at pleasure any point as South Pole, this determines the plane of the equator; and the stereographic projection of any point is the intersection with the plane of the equator of the line joining the point with the South Pole.

To fix the ideas take the radius of the sphere as unity: let the axes of x and y be drawn in the plane of the equator in longitudes 0° and 90° respectively, and the axis of z upwards through the North Pole: the position of a point on the sphere is determined by means of its N.P.D. θ and longitude f: moreover we take X, Y, Z for the coordinates of the point on the surface, and x, y for those of its projection; and we then have

$$X,\ Y,\ Z = \sin\theta\cos f,\ \sin\theta\sin f,\ \cos\theta;$$

$$x = \frac{X}{1+Z} = \tan\tfrac{1}{2}\theta\cos f,\quad y = \frac{Y}{1+Z} = \tan\tfrac{1}{2}\theta\sin f,$$

and conversely,

$$X,\ Y,\ Z = 2x,\ 2y,\ 1-x^2-y^2,\ \div(1+x^2+y^2).$$

We represent the point (X, Y, Z) on the spherical surface by means of the magnitude $x+iy$, $=\tan\frac{1}{2}\theta(\cos f + i\sin f)$, or say by the linear factor, $s-(x+iy)$: and similarly any system of points on the surface by means of the system of magnitudes $x+iy$, or say by the function $\Pi\{s-(x+iy)\}$, denoting in this manner the product of the linear factors which correspond to the different points respectively.

70. It will presently appear that, if (considering a different stereographic projection, that is, a different position of the South Pole) we take x', y' as the coordinates of the new projection of the point, then $x'+iy'$ is a homographic function

$$a(x+iy)+b \div \{c(x+iy)+d\}$$

of $x+iy$: and consequently that the functions of s, which belong to different projections, are linear transformations one of the other: but at present we consider a single projection.

It may be proper to remark that the figures in question are spherical figures having summits which are points on the spherical surface, edges (or sides) which are arcs of great circle joining two summits, and faces which are portions of the spherical surface: the mid-points of the sides, and the centres of the faces are of course points on the spherical surface.

71. (1), (2). Considering a regular polygon formed by n summits on the equator, the longitude of one of them being 0°, then the stereographic projections correspond with the points themselves, and the values of $x+iy$ are

$$1,\ \cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\ \ldots,\ \cos\frac{(n-1)2\pi}{n}+i\sin\frac{(n-1)2\pi}{n}.$$

The corresponding function of s is s^n-1.

The values of $x+iy$ for the mid-points of the sides are

$$\cos\frac{\pi}{n}+i\sin\frac{\pi}{n},\ \cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n},\ \ldots,\ \cos\frac{(2n-1)\pi}{n}+i\sin\frac{(2n-1)\pi}{n}.$$

The corresponding function of s is s^n+1.

The North and South Poles, which form with the n points a double pyramid of $n+2$ summits, correspond to the values $s=0$ and $s=\infty$. We have thus

$$s\left(1-\frac{s}{\infty}\right)(s^n-1)$$

as the function corresponding to the double pyramid.

72. (3). Considering for a moment the tetrahedron as a figure with rectilinear edges, this is so placed that two opposite edges are horizontal, and that the vertical planes passing through the centre and these two edges respectively are inclined at angles $\pm 45°$ to the meridian: viz. the upper edge has the longitudes 135°, 315°, and the lower edge the longitudes 45°, 225°. We thus explain the position of the spherical figure.

Corresponding to the summits we have the function $s^4-2i\sqrt{3}\,s^2+1$.

In fact, the equation $s^4-2i\sqrt{3}\,s^2+1=0$ gives $s^2=i(\sqrt{3}\pm 2)$, and hence the values of s are the four values of $x+iy$ shown in the annexed table for the values of X, Y, Z, and $x+iy$ for the summits of the tetrahedron,

long.	X	Y	Z	$x+iy$
45°	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1+i}{\sqrt{3}-1}$
135°	$-$	$+$	$+$	$\frac{-1+i}{\sqrt{3}+1}$
225°	$-$	$-$	$-$	$\frac{-1-i}{\sqrt{3}-1}$
315°	$+$	$-$	$+$	$\frac{1+i}{\sqrt{3}+1}$.

Corresponding to the centres of the faces, or summits of the opposite tetrahedron, we have the function $s^4+2i\sqrt{3}\,s^2+1$.

Corresponding to the mid-points of the sides, we have the function

$$s\left(1-\frac{s}{\infty}\right)(s^4-1);$$

viz. the points in question are the North Pole $s=0$, the South Pole $s=\infty$, and the four points $s=\pm 1$, $s=\pm i$ on the equator at longitudes 0°, 90°, 180°, 270° respectively.

73. (4). The octahedron is placed with two of its summits as poles, and the other four summits in the equator at longitudes 0°, 90°, 180°, 270° respectively: the values of s are, as in the last case, $0, \infty, \pm 1, \pm i$, and the function is

$$s\left(1-\frac{s}{\infty}\right)(s^4-1).$$

The function for the centres of the faces, or summits of the cube, is s^8+14s^4+1.

The function for the mid-points of the sides of the octahedron or of the cube is

$$s^{12}-33s^8-33s^4+1.$$

74. (5). The Icosahedron is placed with two of its summits for poles; five summits lying in a small circle above the plane of the equator at longitudes 0°, 72°, 144°, 288°, and the remaining five summits in the corresponding small circle below the equator at longitudes 36°, 108°, 180°, 252° and 324°.

The function for the summits of the Icosahedron is

$$s\left(1-\frac{s}{\infty}\right)(s^{10}+11s^5-1).$$

The function for the centres of the faces of the Icosahedron, or summits of the Dodecahedron, is $s^{20}-228s^{15}+494s^{10}+228s^5-1$.

The function for the mid-points of the sides of the Icosahedron or the Dodecahedron is

$$s^{30}-522s^{25}+10005s^{20}+0s^{15}-10005s^{10}+522s^5+1.$$

I give for the present these results without demonstration.

75. Writing $\frac{x}{y}$ for s so as to obtain homogeneous functions $(*\between x, y)^n$,—it will be recollected that the x, y of these functions have nothing to do with the x, y of the foregoing values $x+iy$—the forms which have thus presented themselves may be denoted as follows:

$$\begin{aligned}
(3):\quad & f3=(1,\ -2i\sqrt{3},\ 1\between x^2,\ y^2)^2,\\
& h3=(1,\ +2i\sqrt{3},\ 1\between x^2,\ y^2)^2,\\
& t3=xy\,(x^4-y^4),\\
(4):\quad & f4=xy\,(x^4-y^4),\\
& h4=(1,\ 14,\ 1\between x^4,\ y^4)^2,\\
& t4=(1,\ -33,\ -33,\ 1\between x^4,\ y^4)^3,\\
(5):\quad & f5=xy\,(1,\ 11,\ -1\between x^5,\ y^5)^2,\\
& h5=(1,\ -228,\ +494,\ +228,\ -1\between x^5,\ y^5)^4,\\
& t5=(1,\ -522,\ 10005,\ 0,\ -10005,\ 522,\ 1\between x^5,\ y^5)^6,
\end{aligned}$$

where observe that $f4$ is the same function as $t3$. In each set of functions f, h, t, we have h and t covariants of f, viz. disregarding numerical factors,

h is the Hessian, or derivative $(f, f)^2$, and t is the derivative (f, h).

76. Since $f4$ is the same function as $t3$, we have of course $f4$, $h4$ and $t4$ themselves covariants of $f3$: but it is convenient to separate the two systems.

77. It is to be observed that $f3$ is a quartic function having its quadrinvariant $(I)=0$; but independently of this, that is, quà quartic function, it has only the covariants $h3$ and $t3$ (the Hessian and the cubicovariant respectively), viz. every other covariant is a rational and integral function of $f3$, $h3$ and $t3$. In particular, $h4$ and $t4$ are rational and integral functions of $f3$, $h3$ and $t3$; but inasmuch as $f3$ and $h3$ are not covariants of $f4$, this is not a property of $h4$ and $t4$ considered as covariants of $f4$, and the relation in question need not be attended to.

78. It has just been stated that $f3$ quà quartic function has (in the sense explained) only the covariants $h3$ and $t3$: $f4$ quà *special* sextic function and $f5$ quà *special* dodecadic function have the like property, viz. $f4$ has only the covariants $h4$ and $t4$; $f5$ only the covariants $h5$ and $t5$. Hence $f3$, $f4$, $f5$ are "Prime-forms" in the sense defined in the paper by Fuchs, of 1875, viz. a Prime-form has no covariant of a lower order than itself, and also no covariant of a higher order which is a power of a form of a lower order.

79. The same functions have also the property that they are functions transformable into themselves by means of a group of linear transformations, and in this point of view they were considered in the nearly contemporaneous paper by Klein, of 1875; it is in this paper shown that the functions so transformable into themselves must be Polyhedral functions as above, the linear transformations in fact corresponding to the rotations whereby the spherical polyhedron can be brought into coincidence with its own original position. This theory will be presently given.

80. It is to be observed that, if U, V are functions $(*\between x, y)^n$ of the same order n, then using the accent to denote differentiation in regard to x, $UV'-U'V$ and (U, V) differ only by a numerical factor: and further that, writing as before $s=\frac{x}{y}$, and in the expression $UV'-U'V$ regarding U, V as functions $(*\between s, 1)^n$, and the accent as denoting differentiation in regard to s, we have $UV'-U'V$ and (U, V) differing by a numerical factor only. We have in the PQR-Table, lines 3, 4, 5, P, Q, R equal to given numerical multiples of h^β, t^γ, f^α, the indices α, β, γ being such as to make these to be functions of the same degree: hence, neglecting numerical multipliers, $PQ'-P'Q$ is equal to a function (h^β, t^γ), which is $=h^{\beta-1}t^{\gamma-1}(h, t)$: and the theorem that $PQ'-P'Q, =QR'-Q'R, =RP'-R'P$, contains only factors of P, Q, R is in fact the theorem that (h, t), (h, f), and (t, f) are each of them equal to a term or product of f, h, t: which is a result included in the theorem that f has only the covariants h and t. And by this last theorem we know already how from R, assumed to be known, we can derive P and Q: viz. R is a power of f; and we thence have $h=(f, f)^2$ and $t=(h, f)$, equations giving the functions h and t, upon which P and Q depend.

Covariantive Formulæ. Art. Nos. 81 to 84.

81. The various covariantive formulæ will be given with their proper numerical coefficients.

Tetrahedron function. $f,\ h,\ t$ stand for the before-mentioned values,

$$f3,\ h3,\ t3\ (P,\ Q,\ R = h^3,\ -12i\sqrt{3}\,.\,t^2,\ -f^3).$$

For $f3$.

$$(a,\ b,\ c,\ d,\ e) = 1,\ 0,\ \frac{-i}{\sqrt{3}},\ 0,\ 1.$$

$$\tfrac{1}{2}(f,\ f)^2 = -96i\sqrt{3}\,.\,h, \qquad \tfrac{1}{2}(h,\ h)^2 = 96i\sqrt{3}\,.\,f, \qquad \tfrac{1}{2}(t,\ t)^2 = -25fh,$$

$$(f,\ h) = 32i\sqrt{3}\,.\,t, \qquad (f,\ f)^4 = 576I = 0, \qquad (f,\ h)^4 = 1152J = 1152\,.\,\frac{-4i}{\sqrt{3}},$$

$$(f,\ t) = 4\,.\,h^2,$$

$$(h,\ t) = 4\,.\,f^2,$$

$$h^3 - f^3 - 12i\sqrt{3}\,t^2 = 0,$$

$$fh = (1,\ 14,\ 1\between x^4,\ y^4)^2\ (= f4).$$

It is convenient to remark that $t^2,\ f^3,\ h^3$ being of the same order we have

$$t^2(f^3,\ h^3) + f^3(h^3,\ t^2) + h^3(t^2,\ f^3) = 0,$$

that is,

$$t^2\,.\,3\,.\,3f^2h^2(f,\ h) + f^3\,.\,3\,.\,2h^2t(h,\ t) + h^3\,.\,2\,.\,3tf^2(t,\ f) = 0,$$

an equation which, substituting for $(f,\ h)$, $(h,\ t)$, $(t,\ f)$ their values, reduces itself to the before-mentioned relation $h^3 - f^3 - 12i\sqrt{3}\,t^2 = 0$; and we have thus a verification of the values of $(f,\ h)$, $(h,\ t)$ and $(t,\ f)$. The like remark applies to the other two cases, which follow.

82. Hexahedron function. $f,\ h,\ t$ stand for the before-mentioned values

$$f4,\ h4,\ t4\ (P,\ Q,\ R = h^3,\ -t^2,\ -108f^4).$$

For $f4$.

$$(a,\ b,\ c,\ d,\ e,\ f,\ g) = (0,\ \tfrac{1}{4},\ 0,\ 0,\ 0,\ -\tfrac{1}{4},\ 0).$$

$$\tfrac{1}{2}(f,\ f)^2 = -25h, \qquad \tfrac{1}{2}(f,\ f)^4 = 0, \qquad \tfrac{1}{2}(f,\ f)^6 = (720)^2\,.\,\tfrac{3}{8},$$

$$(f,\ h) = -8t, \qquad \tfrac{1}{2}(h,\ h)^2 = 3\,.\,2^6\,.\,7^2\,.\,f^2,$$

$$(f,\ t) = -12h^2, \qquad \tfrac{1}{2}(t,\ t)^2 = 2^4\,.\,3^3\,.\,11^2\,.\,f^2h,$$

$$(h,\ t) = -1728f^3,$$

$$h^3 - t^2 - 108f^4 = 0.$$

83. Dodecahedron function. f, h, t stand for the before-mentioned values

$$f5,\ h5,\ t5\ (P,\ Q,\ R = h^2,\ -t^2,\ -1728f^5).$$

For $f5$.

$$(a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m) = (0,\ \tfrac{1}{12},\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{84},\ 0,\ 0,\ 0,\ 0,\ -\tfrac{1}{12},\ 0).$$

$$\tfrac{1}{2}(f,f)^2 = -121h, \qquad \tfrac{1}{2}(f,f)^4 = 0, \qquad \tfrac{1}{2}(f,f)^6 = \tfrac{1}{2}(924)^2(720)^2 \cdot \tfrac{-5}{84} f^*,$$

$$\tfrac{1}{2}(f,f)^8 = 0, \qquad \tfrac{1}{2}(f,f)^{10} = 0, \qquad \tfrac{1}{2}(f,f)^{12} = \tfrac{1}{2}(924)^2(720)^4 \cdot \tfrac{25}{84}*,$$

$$(f,\ h) = -20t, \qquad \tfrac{1}{2}(h,\ h)^2 = 173280f^3,$$

$$(f,\ t) = -30h^2, \qquad \tfrac{1}{2}(t,\ t)^2 = 9082800f^3h,$$

$$(h,\ t) = -86400f^5,$$

$$h^3 - t^2 - 1728f^5 = 0.$$

84. We have

$$t = (x^{10} + y^{10})\,(1,\ 522,\ -10006,\ -522,\ 1 \between x^5,\ y^5)^4.$$

Write

$$\xi = (x^2 + y^2) \,.\, (1,\ 2,\ 6,\ -2,\ 1 \between x,\ y)^4,$$

then

$$t = \xi\,(1,\ -10,\ 45 \between \xi^2,\ f).$$

Or putting

$$p = \frac{\xi}{\sqrt{f}}, \quad = \frac{(x^2 + y^2)\,(1,\ 2,\ 6,\ -2,\ 1 \between x,\ y)^4}{\sqrt{xy\,(x^{10} + 11x^5y^5 - y^{10})}},$$

that is, $\xi = p\sqrt{f}$, then

$$p^5 - 10p^3 + 45p = \frac{t}{\sqrt{f^5}}. \quad \text{(Klein.)}$$

Investigation of the forms $f5$ and $h5$. Art. Nos. 85 and 86.

85. Writing for shortness† $k = \tan\alpha = \dfrac{\sqrt{5}-1}{2}$, and $g = \cos 36^\circ + i \sin 36^\circ$, then the values of $x + iy$ corresponding to the summits of the Icosahedron are

$$\begin{array}{llllll} 0, & & & & & \\ k, & kg^2, & kg^4, & kg^6, & kg^8, & \\ k^{-1}g, & k^{-1}g^3, & k^{-1}g^5, & k^{-1}g^7, & k^{-1}g^9, & \\ \infty\,; & & & & & \end{array}$$

and the function $f5$ is thus

$$= s\left(1 - \frac{s}{\infty}\right)(s^5 - k^5)\,(s - k^{-5}),$$

* The numerical coefficients $-\frac{5}{84}$ and $\frac{25}{84}$ are Klein's B and A: the latter of them is the ordinary quadrinvariant of a dodecadic function; the former is an invariant linear as regards the coefficients of f, and existing only for the special form f in question: viz. writing for a moment

$$f = \lambda\,(x^{11}y + 11x^6y^6 - xy^{11}),$$

then $(f,f)^6$ contains the factor λ^2, and (f containing the factor λ) the form is

$$\tfrac{1}{2}(f,f)^6 = \tfrac{1}{2}(924)^2(720)^2 \,.\, -\tfrac{5}{84}\lambda \,.\, f,$$

which is linear as regards λ. We have also

$$\tfrac{1}{2}(f,f)^{12} = \tfrac{1}{2}(924)^2(720)^4 \,.\, \tfrac{25}{84}\lambda^2:$$

say $A = \frac{25}{84}\lambda^2$, $B = -\frac{5}{84}\lambda$; or $84B^2 = A$. Of course in the case of a general dodecadic function f, we have $(f,f)^6$, an irreducible covariant, not breaking up into factors.

† α is the a, γ is the γ, and γ' the $\alpha - \beta$ of the Table, No. 99.

where the product of the last two factors is $s^{10}+(k^{-5}-k^5)\,s^5-1$. We have

$$k^{-5}=\tfrac{1}{32}(80\sqrt{5}+176),\quad =\tfrac{1}{2}(5\sqrt{5}+11),$$
$$k^{5}\;=\tfrac{1}{32}(80\sqrt{5}-176),\quad =\tfrac{1}{2}(5\sqrt{5}-11),$$

and consequently $k^{-5}-k^5=11$; or the function is

$$s\left(1-\frac{s}{\infty}\right)(s^{10}+11s^5-1).$$

86. Similarly, writing for shortness* $l=\tan\frac{1}{2}\gamma$, $l'=\tan\frac{1}{2}\gamma'$, where

$$\cos^2\gamma=\frac{5+2\sqrt{5}}{15},\quad \sin^2\gamma=\frac{10-2\sqrt{5}}{15};\ \text{and therefore}\ \frac{\cos\gamma}{\sin\gamma}=\frac{3+\sqrt{5}}{4};$$
$$\cos^2\gamma'=\frac{5-2\sqrt{5}}{15},\quad \sin^2\gamma'=\frac{10+2\sqrt{5}}{15};\qquad ,,\qquad \frac{\cos\gamma'}{\sin\gamma'}=\frac{3-\sqrt{5}}{4};$$

and $g=\cos 36^\circ+i\sin 36^\circ$ as before, then the values of $x+iy$ for the summits of the dodecahedron are

$$\begin{array}{lllll} lg, & lg^3, & lg^5, & lg^7, & lg^9,\\ l'g, & l'g^3, & l'g^5, & l'g^7, & l'g^9,\\ l'^{-1}, & l'^{-1}g^2, & l'^{-1}g^4, & l'^{-1}g^6, & l'^{-1}g^8,\\ l^{-1}, & l^{-1}g^2, & l^{-1}g^4, & l^{-1}g^6, & l^{-1}g^8. \end{array}$$

The function $h5$ is therefore

$$=s^{10}+s^5(l^5-l^{-5})+1\,.\,s^{10}+s^5(l'^5-l'^{-5})-1.$$

We have

$$l^{-5}-l^5=\frac{(1+\cos\gamma)^5-(1-\cos\gamma)^5}{\sin^5\gamma}=\frac{2\cos\gamma}{\sin^5\gamma}(5+10\cos^2\gamma+\cos^4\gamma)$$
$$=\frac{2\cos\gamma}{\sin^5\gamma}\cdot\frac{384+64\sqrt{5}}{45}=\frac{128}{45}\,\frac{\cos\gamma}{\sin^5\gamma}(6+\sqrt{5})=114+50\sqrt{5};$$

viz. this last identity depends on

$$\tfrac{32}{45}(3+\sqrt{5})(6+\sqrt{5})=(114+50\sqrt{5})\sin^4\gamma,$$

that is,

$$160(3+\sqrt{5})(6+\sqrt{5})=(114+50\sqrt{5})(120-40\sqrt{5}),$$

or

$$2(3+\sqrt{5})(6+\sqrt{5})=(57+25\sqrt{5})(3-\sqrt{5}),$$

or finally

$$(7+3\sqrt{5})(6+\sqrt{5})=\ 57+25\sqrt{5},$$

which is right.

Similarly

$$l'^{-5}-l'^5=114-50\sqrt{5},$$

and observing that the sum and product of $114+50\sqrt{5}$, $114-50\sqrt{5}$ are $=228$ and 496 respectively, the required function of s is

$$(s^{10}-1)^2-228(s^{15}-s^5)+496s^{10},$$
$$=s^{20}-228s^{15}+494s^{10}+228s^5+1,$$

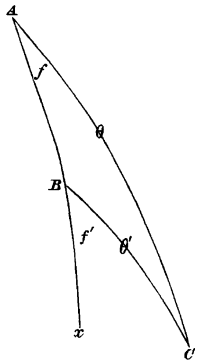
which is the required value of $h5$.

* α is the α, γ is the γ, and γ' the $\alpha-\beta$ of the Table, No. 99.

Invariantive property of the Stereographic Projection. Art. Nos. 87 to 93.

87. The before-mentioned theorem that the functions derived from two different stereographic projections of the same point are linear transformations one of the other, may be thus stated:

Considering on the surface of a sphere, two fixed points A and B; and determining the position of a point C, first in regard to A by its distance θ and azimuth f, and



next in regard to B by its distance θ' and azimuth f', the azimuths from the great circle ABx which joins the two points A and B, then we have

$$\tan\tfrac{1}{2}\theta(\cos f + i\sin f), \quad \text{and} \quad \tan\tfrac{1}{2}\theta'(\cos f' + i\sin f'),$$

homographic functions one of the other: calling them s, s', and putting the distance $AB = c$, the relation between them in fact is

$$s' = \frac{s - \tan\frac{1}{2}c}{1 + s\tan\frac{1}{2}c},$$

or, what is the same thing,

$$\tan\tfrac{1}{2}c(1 + ss') = s - s';$$

or, observing that

$$ss' = \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\{\cos(f+f') + i\sin(f+f')\},$$

we have the two equations

$$\tan\tfrac{1}{2}c\{1 + \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\cos(f+f')\} = \tan\tfrac{1}{2}\theta\cos f - \tan\tfrac{1}{2}\theta'\cos f',$$
$$\tan\tfrac{1}{2}c\{\quad \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta'\sin(f+f')\} = \tan\tfrac{1}{2}\theta\sin f - \tan\tfrac{1}{2}\theta'\sin f'.$$

88. If we denote the angles of the spherical triangle by C, A, B, and the opposite sides by c (as before), a, b, then θ, $\theta' = b$, a; f, $f' = A$, $\pi - B$, whence

$$s,\ s' = \tan\tfrac{1}{2}b(\cos A + i\sin A), \quad -\tan\tfrac{1}{2}a(\cos B - i\sin B):$$

or we have between the sides a, b, c and angles A, B of a spherical triangle the relations

$$\tan\tfrac{1}{2}c\{1 - \tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\cos(A-B)\} = \tan\tfrac{1}{2}b\cos A + \tan\tfrac{1}{2}a\cos B,$$
$$\tan\tfrac{1}{2}c\{\quad -\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\sin(A-B)\} = \tan\tfrac{1}{2}b\sin A - \tan\tfrac{1}{2}a\sin B;$$

equations which may be verified by means of the ordinary formulæ of Spherical Trigonometry.

89. But it is interesting to give the proof with rectangular coordinates.

Taking (X, Y, Z), (X_1, Y_1, Z_1) for the coordinates, referred to two different sets of axes, of a point on the spherical surface: also x, y, x_1, y_1 for the coordinates of the corresponding stereographic projections, we have

$$(X_1, Y_1, Z_1) = \begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} (X, Y, Z),$$

$$X : Y : Z : 1 = 2x : 2y : 1 - x^2 - y^2 : 1 + x^2 + y^2,$$

$$X_1 : Y_1 : Z_1 : 1 = 2x_1 : 2y_1 : 1 - x_1^2 - y_1^2 : 1 + x_1^2 + y_1^2,$$

and thence

$$\begin{aligned} x_1 : y_1 : 1 = \quad & 2\alpha x + 2\beta y + \gamma (1 - x^2 - y^2) \\ : \quad & 2\alpha' x + 2\beta' y + \gamma' (1 - x^2 - y^2) \\ : \quad & 1 + x^2 + y^2 + 2\alpha'' x + 2\beta'' y + \gamma'' (1 - x^2 - y^2). \end{aligned}$$

90. Introducing z, z_1 for homogeneity, or writing $\frac{x}{z}, \frac{y}{z}$ and $\frac{x_1}{z_1}, \frac{y_1}{z_1}$ in place of x, y and x_1, y_1, respectively, we have

$$\begin{aligned} x_1 &= 2\alpha x + 2\beta y + \gamma (z^2 - x^2 - y^2), &&= (-\gamma, -\gamma, \gamma, \beta, \alpha, 0 \between x, y, z)^2, \\ y_1 &= 2\alpha' x + 2\beta' y + \gamma' (z^2 - x^2 - y^2), &&= (-\gamma', -\gamma', \gamma', \beta', \alpha', 0 \between \quad ,, \quad)^2, \\ z_1 &= z^2 + x^2 + y^2 + 2\alpha'' x + 2\beta'' y + \gamma'' (z^2 - x^2 - y^2), &&= (1 - \gamma'', 1 - \gamma'', 1 + \gamma'', \beta'', \alpha'', 0 \between \quad ,, \quad)^2, \end{aligned}$$

and thence without difficulty

$$z_1 = \frac{1}{1 + \gamma''} \{(1 + \gamma'') z + (\alpha'' + i\beta'')(x - iy)\} \{(1 + \gamma'') z + (\alpha'' - i\beta'')(x + iy)\},$$

$$x_1 + iy_1 = \frac{1}{\gamma + \gamma'^2} \{(1 + \gamma'') z + (\alpha'' + i\beta'')(x - iy)\} \{(1 - \gamma'') z + (-\alpha'' + i\beta'')(x + iy)\},$$

$$x_1 - iy_1 = \frac{1}{\gamma - \gamma'^2} \{(1 - \gamma'') z - (\alpha'' + i\beta'')(x - iy)\} \{(1 + \gamma'') z + (\alpha'' - i\beta'')(x + iy)\},$$

viz. the form is $z_1 : x_1 + iy_1 : x_1 - iy_1 = MN : NL : LM$ (L, M, N linear functions of $z, x + iy, x - iy$): showing that the relation between two stereographic projections of the same spherical figure is in fact that of a quadric transformation, the fundamental points in each figure being an arbitrary point and the two circular points at infinity: or, what is the same thing, to any line in the one figure there corresponds a circle in the other figure, which is the "circular relation" of Möbius.

91. The actual values are

$$\frac{x_1 + iy_1}{z_1} = \frac{1 + \gamma''}{\gamma + \gamma' i} \cdot \frac{(1 - \gamma'') z - (\alpha'' - i\beta'')(x + iy)}{(1 + \gamma'') z + (\alpha'' - i\beta'')(x + iy)},$$

$$\frac{x_1 - iy_1}{z_1} = \frac{1 + \gamma''}{\gamma - \gamma' i} \cdot \frac{(1 - \gamma'') z - (\alpha'' + i\beta'')(x - iy)}{(1 + \gamma'') z + (\alpha'' + i\beta'')(x - iy)},$$

viz. attending only to the former of these, we have $\frac{x_1+iy_1}{z_1}$ a homographic function of $\frac{x+iy}{z}$, which is the before-mentioned theorem.

92. Supposing that the transformation from (X, Y, Z) to (X_1, Y_1, Z_1) is made by a rotation, the coordinates of which are λ, μ, ν: that is, if f, g, h are the inclinations of the resultant axis to the axes of x, y, z respectively, and θ the angle of rotation, putting $\lambda, \mu, \nu = \tan\frac{1}{2}\theta\cos f, \tan\frac{1}{2}\theta\cos g, \tan\frac{1}{2}\theta\cos h$: then the coefficients of transformation are

$$\begin{pmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{pmatrix} = \begin{pmatrix} 1+\lambda^2-\mu^2-\nu^2, & 2(\lambda\mu+\nu), & 2(\lambda\nu-\mu) \\ 2(\mu\lambda-\nu), & 1-\lambda^2+\mu^2-\nu^2, & 2(\mu\nu+\lambda) \\ 2(\nu\lambda+\mu), & 2(\mu\nu-\lambda), & 1-\lambda^2-\mu^2+\nu^2 \end{pmatrix} \div (1+\lambda^2+\mu^2+\nu^2).$$

Substituting these values, the formulæ become, after an easy reduction,

$$\frac{x_1+iy_1}{z_1} = \frac{-(\nu+i)(x+iy)+(\lambda+i\mu)z}{(\lambda-i\mu)(x+iy)+(\nu-i)z},$$

$$\frac{x_1-iy_1}{z_1} = \frac{-(\nu-i)(x-iy)+(\lambda-i\mu)z}{(\lambda+i\mu)(x-iy)+(\nu+i)z};$$

attending to the former of these, and writing for greater simplicity

$$\frac{x_1+iy_1}{z_1}, \quad \frac{x+iy}{z} = s_1, s$$

respectively, we have

$$s_1 = \frac{-(\nu+i)s+(\lambda+i\mu)}{(\lambda-i\mu)s+(\nu-i)},$$

or writing this

$$s_1 = \frac{As+B}{Cs+D},$$

then $$A : B : C : D = -\nu-i : \lambda+i\mu : \lambda-i\mu : \nu-i.$$

93. I call to mind that the condition, in order that the homographic transformation $s_1 = (As+B) \div (Cs+D)$ may be periodic of the order n, is

$$(A+D)^2 - 4(AD-BC)\cos^2\frac{m\pi}{n} = 0,$$

m being an integer different from zero and prime to n. In particular, when $n=2$, it is $A+D=0$: $n=3$, it is $A^2+AD+D^2+BC=0$: $n=4$, it is $A^2+D^2+2BC=0$: and $n=5$, it is $(A+D)^2 - \frac{1}{2}(3 \pm \sqrt{5})(AD-BC) = 0$.

Groups of homographic transformations. Art. Nos. 94 and 95.

94. The formulæ just obtained serve to connect the theory of the rotations of a polyhedron with that of the homographic transformations s into $(As+B) \div (Cs+D)$: and, corresponding to the rotations which leave the polyhedron unaltered, we have groups of homographic transformations. We have thus, corresponding to the cases of the tetrahedron, the cube and the octahedron, and the dodecahedron and icosahedron respectively, groups of 12, of 24, and of 60 homographic transformations s into

$(As+B)\div(Cs+D)$. The group of 60 and the group of 24 include each of them as part of itself the group of 12: it is further to be remarked that the group of 12 may be regarded as that of the positive substitutions upon four letters *abcd*, the group of 24 as that of all the substitutions upon the four letters, and the group of 60 as that of the positive substitutions upon five letters *abcde*.

95. I call to mind that a group of functional symbols 1, α, β, ... can always be expressed in the equivalent form 1, $\mathfrak{S}\alpha\mathfrak{S}^{-1}$, $\mathfrak{S}\beta\mathfrak{S}^{-1}$, ... where $\mathfrak{S}$ is any functional symbol whatever: clearly, α, β, ... being homographic transformations, then, $\mathfrak{S}$ being any homographic transformation whatever, the new symbols $\mathfrak{S}\alpha\mathfrak{S}^{-1}$, $\mathfrak{S}\beta\mathfrak{S}^{-1}$, ... will also be homographic transformations; and thus the group of homographic transformations can be expressed in various equivalent forms: these correspond to the different positions of the polyhedron in regard to the axes of coordinates: and there are in fact three cases which it is proper to consider, viz. attending for the moment to the dodecahedron, we may have the axis of z passing through the midpoint of a side, through the centre of a face, or through a summit; that is, in the language presently explained, the cases are 1°, Pole at a point Θ; 2°, Pole at a point A; 3°, Pole at a point B.

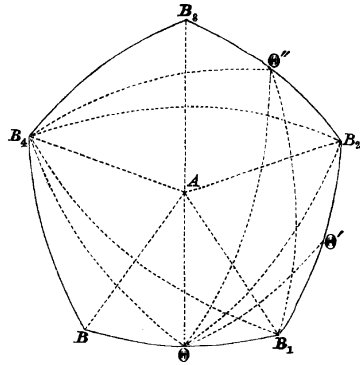
The regular Polyhedra. Art. Nos. 96 to 103.

96. We require a theory of the regular Polyhedra considered as systems of points on a sphere. I refer to my two papers [375] and [679]. In the latter paper, I remark that, considering the five regular figures drawn in proper relation to each other on the same spherical surface, the only points which have to be considered are 12 points A, 20 points B, 30 points Θ, and 60 points Φ. Describing these by reference to the dodecahedron, the points A are the centres of the faces, the points B are the summits, the points Θ are the midpoints of the sides, and the points Φ are the midpoints of the diagonals of the faces. Or describing them by reference to the icosahedron, the points A are the summits, the points B are the centres of the faces, the points Θ are the midpoints of the sides: viz. each point Θ is the common midpoint of a side of the dodecahedron and a side of the icosahedron, which there intersect at right angles: and the points Φ are points lying by threes on the faces of the icosahedron, each point Φ of the face being given as the intersection of a perpendicular $A\Theta$ of the face by a line BB joining the centres of two adjacent faces and which intersects $A\Theta$ at right angles.

97. The points Φ are comparatively unimportant, and it is proper in the first instance to attend only to the 12 points A, the 20 points B, and the 30 points Θ: these form 6 pairs of opposite points A, 10 pairs of opposite points B, and 15 pairs of opposite points Θ. Considering the diameters through each pair of opposite points Θ, we have thus a system of 15 axes, which in fact form 5 sets each of 3 rectangular axes: attending to any one of such sets, the diametral plane at right angles to one of the three axes contains of course the other two axes: it contains also two axes each through a pair of opposite points A, and two axes each through a pair of opposite points B. If instead of the plane we consider its intersection with the sphere, we have thus on the sphere 15 circles each containing 4 points Θ,

4 points A and 4 points B. The fifteen circles intersect by fives in the pairs of opposite points A, by threes in the pairs of opposite points B, and by twos in the pairs of opposite points Θ; the mutual inclinations of successive circles at the points A, B, Θ being $=36°$, $60°$ and $90°$ respectively. The whole number $15.14, =210$, of the intersections of the circles two and two together is thus made up of the 12 points A each counting 10 times, the 20 points B each counting 3 times, and the 30 points Θ each counting once; $210 = 120 + 60 + 30$.

98. The angular magnitudes which present themselves are all obtained from the dodecahedral pentagon, as shown in the annexed figure, in which the angle subtended by a side at the centre is $=72°$, and the angle between two adjacent sides is $=120°$.



We write $A\Theta = \alpha$, $B\Theta = \beta$, $AB = \gamma$, $B_1B_4 = x$, $\angle B_1B_4B = \theta$, $\Theta B_4 = g$, $\angle \Theta B_4 B = \phi$.

From the triangle $A\Theta B$, the angles of which are $36°$, $90°$, $60°$ and the opposite sides β, γ, α, we find the values of α, β, γ, and these are such that $\alpha + \beta + \gamma = \frac{1}{2}\pi$.

From the triangle B_4BB_1, where the sides B_4B, BB_1, and the included angle are 2β, 2β, $120°$, we have the opposite side x, and the other two angles each $=\theta$.

From the triangle $B_4B\Theta$, where the sides B_4B, $B\Theta$, and the included angle are 2β, β, $120°$, we find the opposite side g, the angle $BB_4\Theta$, $=\phi$, and the angle $B_4\Theta B$, $=45°$.

Hence each of the angles $B_4\Theta B$, $B_2\Theta B_1$, being $=45°$, the angle $B_4\Theta B_2$ is $=90°$: in this triangle the hypothenuse B_2B_4 is $=x$, and each of the other two sides is $=g$: whence we have $\cos x = \cos^2 g$, as is in fact the case, and moreover the values give $x + 2g = 180°$. Also each of the other angles is found to be $=60°$; that is, we have $\angle B_2B_4\Theta = 60°$, or the whole angle at B_4 being $=120°$, the sum of the remaining angles $B_3B_4B_2$ and $BB_4\Theta$ is $=60°$: that is, $\theta + \phi = 60°$.

From the triangle $\Theta B_1\Theta'$ where the two sides and the included angle are β, β, $120°$, we find $\Theta\Theta' = 36°$.

And from the triangle $\Theta B_1\Theta''$, where the two sides and the included angle are g, g and $(120° - 2\phi =) 2\theta$, we find $\Theta\Theta'' = 60°$.

99. We thus arrive at the following Table:

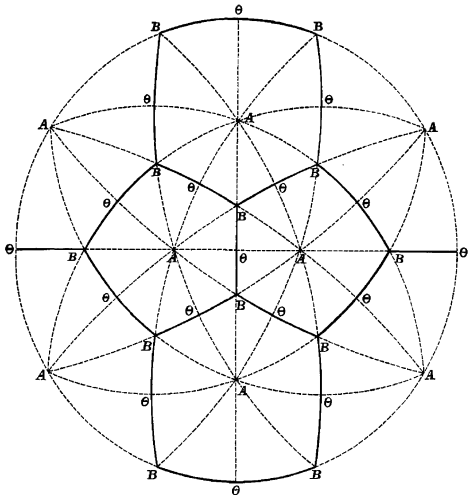
			sin	cos
$A\Theta$	α	31° 43′	$\sqrt{\frac{5-\sqrt{5}}{10}}$	$\sqrt{\frac{5+\sqrt{5}}{10}}$
$B\Theta$	β	20° 55′	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{\sqrt{5}+1}{2\sqrt{3}}$
AB	γ	37° 22′	$\sqrt{\frac{10-2\sqrt{5}}{15}}$	$\sqrt{\frac{5+2\sqrt{5}}{15}}$
(BB)	x	70° 32′	$\frac{2\sqrt{2}}{3}$	$\frac{1}{3}$
$(B\Theta)$	g	54 44	$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
BBB	θ	37° 46′	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{5}}{2\sqrt{2}}$
$B\Theta B$	ϕ	22 14	$\frac{\sqrt{3}(\sqrt{5}-1)}{4\sqrt{2}}$	$\frac{\sqrt{5}+3}{4\sqrt{2}}$
	2α	63 26	$\frac{2}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$
	2β	41 50	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$
	2γ	74 44	$\frac{2(\sqrt{5}+1)}{3\sqrt{5}}$	$\frac{4-\sqrt{5}}{3\sqrt{5}}$
	$\alpha-\beta$		$\sqrt{\frac{5-2\sqrt{5}}{15}}$	$\sqrt{\frac{10+2\sqrt{5}}{15}}$
		18°	$\frac{\sqrt{5}-1}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
$\Theta\Theta$		36°	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{\sqrt{5}+1}{4}$

where as above

$$\alpha + \beta + \gamma = 90°,$$
$$x + 2g = 180°,$$
$$\theta + \phi = 60°.$$

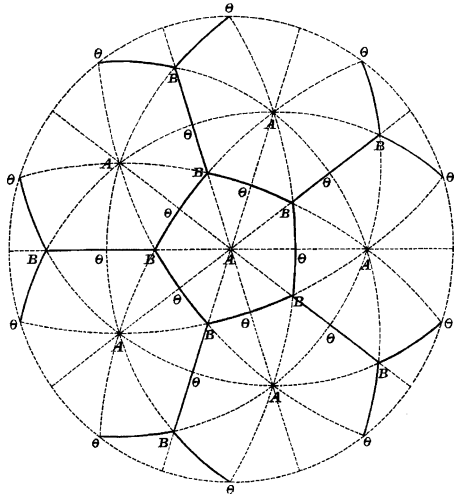
100. We now construct three figures of the points A, B, Θ; viz. these are stereographic projections, each showing the Northern hemisphere projected on the plane of the equator by lines drawn to the South Pole: hence, for any pair of opposite points not on the equator, only the point in the Northern hemisphere is shown: but for a pair of opposite points on the equator the two points are each of them shown. In fig. 1 the North Pole is taken to be a point Θ; in fig. 2 it is a point A; and in fig. 3 it is a point B. The position of any point on the sphere is determined by its N.P.D. and its longitude, measured from an arbitrary origin, say from the point E of the centre left-handedly: then, in the three figures, the positions are as follows.

101. Fig. 1. Pole at Θ.



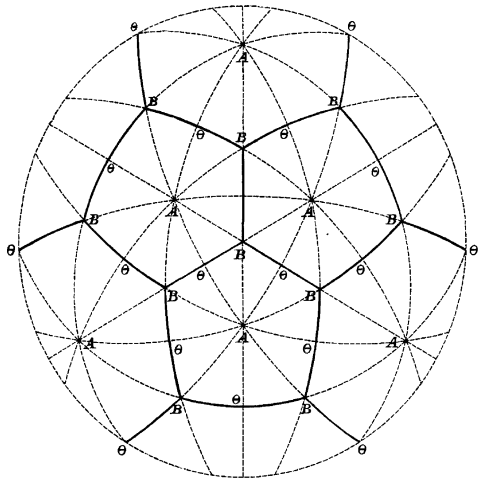
	N.P.D.'s	Longitudes.
$2A$	$\alpha = 31°\,43'$	0°, 180°
$2A$	$90° - \alpha = 58\ 17$	90 , 270
$4A$	90	(0 , 180) $\pm\ \alpha = 31°\,43'$
$2A$	$90° + \alpha = 121\ 43$	90 , 270
$2A$	$180° - \alpha = 148\ 17$	0 , 180
$2B$	$\beta = 20°\,55'$	90°, 270°
$4B$	$g = 54\ 44$	45 , 135, 225, 315
$2B$	$90° - \beta = 69\ 5$	0 , 180
$4B$	90	(90 , 270) $\pm\ \beta = 20°\ 55'$
$2B$	$90° + \beta = 110\ 55$	0 , 180
$4B$	$180° - g = 125\ 16$	45 , 135, 225, 315
$2B$	$180° - \beta = 159\ 5$	90 , 270
1Θ	0°	—
4Θ	36	(90°, 270°) $\pm\ \alpha = 31°\,43'$
4Θ	60	(0 , 180) $\pm\ \beta = 20\ 55$
4Θ	72	(90 , 270) $\pm\ \alpha = 31\ 43$
4Θ	90	0 , 90, 180, 270
4Θ	108	(90 , 270) $\pm\ \alpha = 31\ 43$
4Θ	120	(0 , 180) $\pm\ \beta = 20\ 55$
4Θ	144	(90 , 270) $\pm\ \alpha = 31\ 43$
1Θ	180	—

102. Fig. 2. Pole at A.



	N.P.D.'s	Longitudes.
A	0	—
$5A$	$2\alpha = 63° 26'$	0° 72° 144° 216° 288°
$5A$	$180° - 2\alpha = 116\ 34$	36 108 180 252 324
A	180	—
$5B$	$\gamma = 37\ 22$	36 108 180 252 324
$5B$	$90° - \alpha + \beta = 79\ 12$	36 108 180 252 324
$5B$	$90 + \alpha - \beta = 100\ 48$	0 72 144 216 288
$5B$	$180 - \gamma = 142\ 38$	0 72 144 216 288
5Θ	$\alpha = 31\ 43$	0 72 144 216 288
5Θ	$90° - \alpha = 58\ 17$	36 108 180 252 324
10Θ	90	(36 108 180 252 324) ± 18°
5Θ	$90 + \alpha = 121\ 43$	0 72 144 216 288
5Θ	$180 - \alpha = 144\ 17$	36 108 180 252 324

103. Fig. 3. Pole at B.



	N.P.D.'s	Longitudes.
$3A$	$\gamma = 37° 22'$	30° 150° 270°
$3A$	$90° - \alpha + \beta = 79\ 12$	90 210 330
$3A$	$90 + \alpha - \beta = 100\ 48$	30 150 270
$3A$	$180 \quad - \gamma = 142\ 38$	90 210 330
B	0	—
$3B$	$2\beta = 41\ 50$	90 210 330
$6B$	$x = 70\ 32$	$(30\ 150\ 270) \pm \vartheta = 37° 46'$
$6B$	$180° - x = 109\ 28$	$(90\ 210\ 330) \pm \vartheta = 37\ 46$
$3B$	$180 - 2\beta = 138\ 10$	30 150 270
B	180	—
3Θ	$\beta = 20\ 55$	90 210 330
6Θ	$g = 54\ 44$	$(90\ 210\ 330) \pm \phi = 22° 14'$
3Θ	$90° - \beta = 69\ 5$	30 150 270
6Θ	90	0 60 120 180° 240° 300°
3Θ	$90 + \beta = 110\ 55$	90 210 330
6Θ	$180 - g = 125\ 16$	$(30\ 150\ 270) \pm \phi = 22° 14'$
3Θ	$180 - \beta = 159\ 5$	30 150 270

The groups of homographic transformations, resumed. Art. Nos. 104 to 117.

104. The axes of rotation for the dodecahedron and the icosahedron are 15 axes each through a pair of opposite points Θ, 6 axes each through a pair of opposite points A, and 10 axes each through a pair of opposite points B; or say 15 Θ-axes, 10 B-axes and 6 A-axes: the corresponding angles of rotation are 180°, 72° and 120°; so that (excluding in each case the original position or that of a rotation 0) we have in respect of each Θ-axis 1 position, in respect of each A-axis 4 positions, and in respect of each B-axis 2 positions; in all, including the original position,

$$1+15+(6\times 4)+(10\times 2), =60 \text{ positions},$$

that is, a group of 60 rotations.

To find, in any one of the three forms, the group of homographic transformations, we can in each case obtain from the foregoing tables the values $\cos f$, $\cos g$, $\cos h$ of the cosine-inclination of an axis of rotation to the axes of coordinates, and thence calculate the values of

$$\lambda,\ \mu,\ \nu = \tan\tfrac{1}{2}\vartheta\cos f,\quad \tan\tfrac{1}{2}\vartheta\cos g,\quad \tan\tfrac{1}{2}\vartheta\cos h,$$

and thence the values of

$$A,\ B,\ C,\ D = -\nu-i,\quad \lambda+i\mu,\quad \lambda-i\mu,\quad \nu-i;$$

viz. in the case of a Θ-axis, ϑ is $=180°$, (so that here $\tan\frac{1}{2}\vartheta=\infty$, or the values of A, B, C, D are $=-\nu$, $\lambda+i\mu$, $\lambda-i\mu$, ν, that is, $-\cos h$, $\cos f+i\cos g$, $\cos f-i\cos g$, $\cos h$); in the case of a B-axis, the values are $\vartheta=120°$, 240°, and therefore $\tan\frac{1}{2}\vartheta=\pm\sqrt{3}$; and in the case of an A-axis, they are $\vartheta=72°$, 144°, 216°, 288°, and therefore

$$\tan\tfrac{1}{2}\vartheta=\pm\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1},\quad \pm\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}.$$

105. The Θ-form was first given in my paper of 1879, but in obtaining it I used results given in the paper of 1877. As regards the identification with the substitution-symbols, since there is nothing to distinguish *inter se* the letters a, b, c, d, e, any transformation A, B, C, D of the fifth order might have been taken for $abcde$, but No. 37 of the group having been taken for this substitution $abcde$, I do not recall in what manner I found that, consistently herewith, the transformation No. 2 (-1, 0, 0, 1, that is, s into $-s$) of the second order could be taken for $ab.cd$. But there is no sub-group of an order divisible by 5; and hence, these two transformations being identified with the two substitutions, the other transformations correspond each of them to a determinate substitution.

106. Homographic Transformations. The group of 60. Pole at Θ.

	$(Ax$	$+B)$	$\div$ $(Cx$	$+D)$	
1	1	0	0	1	1
2	-1	0	0	1	$ab\,.\,cd$
3	0	1	1	0	$ac\,.\,bd$
4	0	1	-1	0	$ad\,.\,bc$
5	2	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	$-3+\sqrt{5}+i(-1+\sqrt{5})$	-2	$bc\,.\,de$
6	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	-2	$ae\,.\,bc$
7	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	-2	$ad\,.\,ce$
8	2	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	$3-\sqrt{5}+i(-1+\sqrt{5})$	-2	$ad\,.\,be$
9	2	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	$-1-\sqrt{5}+i(-1+\sqrt{5})$	-2	$ae\,.\,cd$
10	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	-2	$ab\,.\,de$
11	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	-2	$be\,.\,cd$
12	2	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	$1+\sqrt{5}+i(-1+\sqrt{5})$	-2	$ab\,.\,ce$
13	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	-2	$ac\,.\,be$
14	2	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	$-1-\sqrt{5}+i(-3-\sqrt{5})$	-2	$bd\,.\,ce$
15	2	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	$1+\sqrt{5}+i(-3-\sqrt{5})$	-2	$ae\,.\,bd$
16	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	-2	$ac\,.\,de$
17	$-i$	i	1	1	abc
18	-1	i	1	i	acb
19	1	$-i$	1	i	adc
20	$-i$	$-i$	1	-1	acd
21	i	i	1	-1	adb
22	1	i	1	$-i$	abd
23	-1	$-i$	1	$-i$	bcd
24	i	$-i$	1	1	bdc
25	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	aec
26	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	2	-2	$1+\sqrt{5}+i(-3-\sqrt{5})$	ace
27	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	-2	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	bed
28	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	bde
29	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	2	2	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	bec
30	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	bce
31	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	aed
32	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	2	2	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	ade
33	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	cde
34	2	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	2	ced
35	2	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	2	aeb
36	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	abe
37	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	$abcde$
38	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	2	2	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	$acebd$
39	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	2	$1+\sqrt{5}+i(-1+\sqrt{5})$	$adbec$
40	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	2	2	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	$aedcb$
41	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	2	2	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	$adceb$
42	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	$acbde$
43	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	2	2	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	$aedbc$
44	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	$abecd$
45	$-1-\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	$acbed$

46	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	*abdce*
47	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	*aecdb*
48	$1+\sqrt{5}+i(-1+\sqrt{5})$	2	-2	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	*adebc*
49	$1+\sqrt{5}+i(\ 1-\sqrt{5})$	2	-2	$1+\sqrt{5}+i(-1+\sqrt{5})$	*aecbd*
50	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	2	-2	$3-\sqrt{5}+i(-1+\sqrt{5})$	*acdeb*
51	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	2	-2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	*abedc*
52	$-1-\sqrt{5}+i(\ 1-\sqrt{5})$	2	-2	$-1-\sqrt{5}+i(-1+\sqrt{5})$	*adbce*
53	2	$-3+\sqrt{5}+i(-1+\sqrt{5})$	$3-\sqrt{5}+i(-1+\sqrt{5})$	2	*acbdc*
54	2	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	2	*abced*
55	2	$1+\sqrt{5}+i(-3-\sqrt{5})$	$-1-\sqrt{5}+i(-3-\sqrt{5})$	2	*adecb*
56	2	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	2	*acdbe*
57	2	$-3+\sqrt{5}+i(\ 1-\sqrt{5})$	$3-\sqrt{5}+i(\ 1-\sqrt{5})$	2	*abdec*
58	2	$-1-\sqrt{5}+i(-3-\sqrt{5})$	$1+\sqrt{5}+i(-3-\sqrt{5})$	2	*adcbe*
59	2	$1+\sqrt{5}+i(\ 3+\sqrt{5})$	$-1-\sqrt{5}+i(\ 3+\sqrt{5})$	2	*aebcd*
60	2	$3-\sqrt{5}+i(-1+\sqrt{5})$	$-3+\sqrt{5}+i(-1+\sqrt{5})$	2	*acedb*

107. Taking out of the foregoing group of 60 a group of 12 contained in it, viz. that corresponding to the positive substitutions of the four letters *abcd*, it is easy to see, that there is a transformation $(i,\ 0,\ 0,\ 1)$, that is, s into is, which can be taken for the substitution *adbc*, and also to complete thence the group of 24. And we have thus the following Table.

Groups of 12 and 24. Pole at Θ.

	$(Ax$	$+B)$	$\div$ $(Cx$	$+D)$	
1	1	0	0	1	1
2	-1	0	0	1	*ab . cd*
3	0	1	1	0	*ac . bd*
4	0	1	-1	0	*ad . bc*
5	$-i$	i	1	1	*abc*
6	-1	i	1	i	*acb*
7		$-i$	1	i	*adc*
8	$-i$	$-i$	1	-1	*acd*
9	i	i	1	-1	*adb*
10	1	i	1	$-i$	*abd*
11	-1	$-i$	1	$-i$	*bcd*
12	i	$-i$	1	1	*bdc*
13	i	0	0	1	*adbc*
14	$-i$	0	0	1	*acbd*
15	0	i	1	0	*cd*
16	0	i	-1	0	*ab*
17	1	-1	1	1	*acdb*
18	$-i$	-1	1	i	*bd*
19	i	1	1	i	*abcd*
20	1	1	1	-1	*bc*
21	-1	-1	1	-1	*abdc*
22	i	-1	1	$-i$	*ac*
23	$-i$	1	1	$-i$	*adcb*
24	-1	1	1	1	*ad*

108. The group of 60 was obtained in the A-form by Gordan in his paper. The passage from the Θ-form to the A-form is made as follows: let X, Y, Z be the coordinates of a point when the axes are as in the Θ-form, X_1, Y_1, Z_1 the coordinates of the same point when the axes are as in the A-form: we may write

$$X,\ Y,\ Z = \mathrm{b}X_1 - \mathrm{a}Z_1 : Y_1 : \mathrm{a}X_1 + \mathrm{b}Z_1,$$

where

$$\mathrm{a},\ \mathrm{b} = \sqrt{\frac{5-\sqrt{5}}{10}},\ \sqrt{\frac{5+\sqrt{5}}{10}};$$

then, if the equations of an axis of rotation referred to the first set of coordinates are $X : Y : Z = L : M : N$, those of the same axis referred to the second set of coordinates are

$$\mathrm{b}X_1 + \mathrm{a}Z_1 : Y_1 : -\mathrm{a}X_1 + \mathrm{b}Z_1 = L : M : N;$$

or taking these to be

$$X_1 : Y_1 : Z_1 = L_1 : M_1 : N_1,$$

we may write

$$L_1,\ M_1,\ N_1 = \mathrm{b}L + \mathrm{a}N,\ M,\ -\mathrm{a}L + \mathrm{b}N:$$

these values are such that

$$L_1^2 + M_1^2 + N_1^2 = L^2 + M^2 + N^2,$$

and hence, λ, μ, ν and λ_1, μ_1, ν_1 being the rotations, we may write

$$L,\ M,\ N = \vartheta\lambda,\ \vartheta\mu,\ \vartheta\nu;\quad L_1,\ M_1,\ N_1 = \vartheta\lambda_1,\ \vartheta\mu_1,\ \vartheta\nu_1;$$

where ϑ has the same value in each set of equations. From the equations

$$A : B : C : D = -\nu - i : \lambda + i\mu : \lambda - i\mu : \nu - i,$$

we have

$$\begin{aligned} B + C : B - C : D - A : D + A &= \lambda : i\mu : \nu : -i \\ &= L : iM : N : -i\vartheta, \end{aligned}$$

and similarly

$$B_1 + C_1 : B_1 - C_1 : D_1 - A_1 : D_1 + A_1 = L_1 : iM_1 : N_1 : -i\vartheta.$$

Hence we may write

$$\begin{aligned} B_1 + C_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A), \\ B_1 - C_1 &= B - C, \\ D_1 - A_1 &= -\mathrm{a}(B + C) + \mathrm{b}(D - A), \\ D_1 + A_1 &= D + A; \end{aligned}$$

or say,

$$\begin{aligned} A_1 &= \mathrm{a}(B + C) - \mathrm{b}(D - A) + (D + A), \\ B_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A) + (B - C), \\ C_1 &= \mathrm{b}(B + C) + \mathrm{a}(D - A) - (B - C), \\ D_1 &= -\mathrm{a}(B + C) + \mathrm{b}(D - A) + (D + A), \end{aligned}$$

which are the values for a transformation (A_1, B_1, C_1, D_1) in the A-form: of course, as only the ratios are material, the values may be multiplied by any common factor.

109. The results are exhibited in terms of ϵ, an imaginary fifth root of unity: taking $\epsilon = \cos 72^\circ + i \sin 72^\circ$, we have

$$\epsilon,\ \epsilon^4 = \frac{\sqrt{5}-1}{4} \pm i\sqrt{\frac{5+\sqrt{5}}{8}},$$

$$\epsilon^2,\ \epsilon^3 = -\frac{\sqrt{5}+1}{4} \pm i\sqrt{\frac{5-\sqrt{5}}{8}};$$

where the upper signs belong to ϵ, ϵ^2 and the lower to ϵ^4, ϵ^3. It may be remarked that

$$\frac{1}{\mathrm{a}} = \sqrt{\frac{5+\sqrt{5}}{2}},\quad \frac{1}{\mathrm{b}} = \sqrt{\frac{5-\sqrt{5}}{2}},\quad \frac{\mathrm{b}}{\mathrm{a}} = \frac{\sqrt{5}+1}{2},\quad \frac{\mathrm{a}}{\mathrm{b}} = \frac{\sqrt{5}-1}{2}.$$

For instance, we have in the Θ-group $(A, B, C, D) = (-1, 0, 0, 1)$; $ab.cd$: and thence in the A-group $A_1, B_1, C_1, D_1 = (-2\mathrm{b}, 2\mathrm{a}, 2\mathrm{a}, 2\mathrm{b})$; $ab.cd$: or say this is

$$\left(-1, \frac{\mathrm{a}}{\mathrm{b}}, \frac{\mathrm{a}}{\mathrm{b}}, 1\right), = (-1, \epsilon+\epsilon^4, \epsilon+\epsilon^4, 1);$$

which in the Table is given as $(-\epsilon^3, \epsilon^2+\epsilon^4, \epsilon^2+\epsilon^4, \epsilon^3)$; $ab.cd$.

By effecting the passage to the A-group in this manner, we of course obtain the proper substitution corresponding to each transformation: but I found it easier starting from two transformations and the corresponding substitutions, to obtain thence by successive compositions the entire group.

110. Homographic Transformations. The group of 60. Pole at A.

	Θ No.	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	1			1	1
2	4	0	-1	1	0	$ad.bc$
3	13	0	$-\epsilon^4$	1	0	$ac.be$
4	9	0	$-\epsilon^3$	1	0	$ae.cd$
5	10	0	$-\epsilon^2$	1	0	$ab.de$
6	14	0	$-\epsilon$	1	0	$bd.ce$
7	6	$\epsilon+\epsilon^2$	ϵ^4	1	$-(\epsilon+\epsilon^3)$	$ae.bc$
8	5	$\epsilon+\epsilon^3$	1	ϵ^4	$-(\epsilon+\epsilon^3)$	$bc.de$
9	16	$\epsilon+\epsilon^3$	ϵ	ϵ^3	$-(\epsilon+\epsilon^3)$	$ac.de$
10	3	$\epsilon+\epsilon^3$	ϵ^2	ϵ^2	$-(\epsilon+\epsilon^3)$	$ac.bd$
11	15	$\epsilon+\epsilon^3$	ϵ^3	ϵ	$-(\epsilon+\epsilon^3)$	$ae.bd$
12	12	-1	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	1	$ab.ce$
13	11	$-\epsilon$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ	$be.cd$
14	7	$-\epsilon^2$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^2	$ad.ce$
15	2	$-\epsilon^3$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^3	$ab.cd$
16	8	$-\epsilon^4$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^4	$ad.be$

17	21	ϵ^3+1	ϵ	1	$-(\epsilon+\epsilon^3)$	*adb*
18	35	ϵ^3+1	ϵ^2	ϵ^4	$-(\epsilon+\epsilon^3)$	*aeb*
19	30	ϵ^3+1	ϵ^3	ϵ^3	$-(\epsilon+\epsilon^3)$	*bce*
20	34	ϵ^3+1	ϵ^4	ϵ^2	$-(\epsilon+\epsilon^3)$	*ced*
21	19	ϵ^3+1	1	ϵ	$-(\epsilon+\epsilon^3)$	*adc*
22	33	$\epsilon+\epsilon^4$	ϵ^2	1	$-(\epsilon+\epsilon^3)$	*cde*
23	20	$\epsilon+\epsilon^4$	ϵ^3	ϵ^4	$-(\epsilon+\epsilon^3)$	*acd*
24	22	$\epsilon+\epsilon^4$	ϵ^4	ϵ^3	$-(\epsilon+\epsilon^3)$	*abd*
25	36	$\epsilon+\epsilon^4$	1	ϵ^2	$-(\epsilon+\epsilon^3)$	*abe*
26	29	$\epsilon+\epsilon^4$	ϵ	ϵ	$-(\epsilon+\epsilon^3)$	*bec*
27	31	$-\epsilon$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	1	*aed*
28	17	$-\epsilon^2$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ	*abc*
29	27	$-\epsilon^3$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^2	*bed*
30	25	$-\epsilon^4$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^3	*aec*
31	23	-1	$1+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^4	*bcd*
32	24	$-\epsilon^4$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	1	*bdc*
33	32	-1	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ	*ade*
34	18	$-\epsilon$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^2	*acb*
35	28	$-\epsilon^2$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^3	*bde*
36	26	$-\epsilon^3$	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^4	*ace*
37	44	ϵ	0	0	1	*abecd*
38	43	ϵ^2	0	0	1	*aedbc*
39	42	ϵ^3	0	0	1	*acbde*
40	41	ϵ^4	0	0	1	*adceb*
41	38	$\epsilon^2+\epsilon^4$	1	1	$-(\epsilon+\epsilon^3)$	*acebd*
42	46	$\epsilon^2+\epsilon^4$	ϵ	ϵ^4	$-(\epsilon+\epsilon^3)$	*abdce*
43	58	$\epsilon^2+\epsilon^4$	ϵ^2	ϵ^3	$-(\epsilon+\epsilon^3)$	*adcbe*
44	55	$\epsilon^2+\epsilon^4$	ϵ^3	ϵ^2	$-(\epsilon+\epsilon^3)$	*adecb*
45	50	$\epsilon^2+\epsilon^4$	ϵ^4	ϵ	$-(\epsilon+\epsilon^3)$	*acdeb*
46	51	$1+\epsilon^2$	ϵ^3	1	$-(\epsilon+\epsilon^3)$	*abedc*
47	39	$1+\epsilon^2$	ϵ^4	ϵ^4	$-(\epsilon+\epsilon^3)$	*adbec*
48	47	$1+\epsilon^2$	1	ϵ^3	$-(\epsilon+\epsilon^3)$	*aecdb*
49	59	$1+\epsilon^2$	ϵ	ϵ^2	$-(\epsilon+\epsilon^3)$	*aebcd*
50	54	$1+\epsilon^2$	ϵ^2	ϵ	$-(\epsilon+\epsilon^3)$	*abced*
51	56	$-\epsilon^2$	ϵ^3+1	$\epsilon^2+\epsilon^4$	1	*acdbe*
52	49	$-\epsilon^3$	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ	*aecbd*
53	37	$-\epsilon^4$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^2	*abcde*
54	45	-1	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	ϵ^3	*acbed*
55	57	$-\epsilon$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^4	*abdec*
56	48	$-\epsilon^3$	$\epsilon^4+\epsilon$	$\epsilon^2+\epsilon^4$	1	*adebc*
57	60	$-\epsilon^4$	$\epsilon+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ	*acedb*
58	53	-1	ϵ^3+1	$\epsilon^2+\epsilon^4$	ϵ^2	*aebdc*
59	52	$-\epsilon$	$1+\epsilon^3$	$\epsilon^2+\epsilon^4$	ϵ^3	*adbce*
60	40	$-\epsilon^2$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^4	*aedcb*

111. Selecting the transformations which correspond to the positive substitutions *abcd*, and completing the group of 24 we have

Homographic Transformations. The groups of 12 and 24. Pole at A.

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	0	-1	1	0	$ad\,.\,bc$
3	$\epsilon+\epsilon^3$	ϵ^2	ϵ^2	$-(\epsilon+\epsilon^3)$	$ac\,.\,bd$
4	$-\epsilon^3$	$\epsilon^2+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^3	$ab\,.\,cd$
5	$-\epsilon^2$	$\epsilon+\epsilon^4$	$\epsilon^3+\epsilon^4$	ϵ	abc
6	$-\epsilon$	$\epsilon+\epsilon^4$	$\epsilon^2+\epsilon^4$	ϵ^2	acb
7	$\epsilon+\epsilon^4$	ϵ^3	ϵ^4	$-(\epsilon+\epsilon^3)$	acd
8	ϵ^3+1	1	ϵ	$-(\epsilon+\epsilon^3)$	adc
9	$\epsilon+\epsilon^4$	ϵ^4	ϵ^3	$-(\epsilon+\epsilon^3)$	abd
10	ϵ^3+1	ϵ	1	$-(\epsilon+\epsilon^3)$	adb
11	-1	$1+\epsilon^2$	$\epsilon^2+\epsilon^4$	ϵ^4	bcd
12	$-\epsilon^4$	$1+\epsilon^2$	$\epsilon^3+\epsilon^4$	1	bdc
13	1	$1+2\epsilon^4$	$1+2\epsilon$	-1	ab
14	$-\epsilon^2+\epsilon^3$	$1+\epsilon+3\epsilon^4$	$-1-3\epsilon-\epsilon^4$	$\epsilon^2-\epsilon^3$	cd
15	$\epsilon^3-\epsilon^4$	$3+\epsilon+\epsilon^3$	$-1-3\epsilon-\epsilon^3$	$-\epsilon^2+\epsilon^4$	ac
16	$-1+\epsilon^2$	$-1-\epsilon^2+2\epsilon^4$	$1+\epsilon^2-2\epsilon^3$	$1-\epsilon^2$	bd
17	$2+\epsilon^3+2\epsilon^4$	$-2-2\epsilon^2-\epsilon^3$	$2\epsilon+\epsilon^3+2\epsilon^4$	$2\epsilon+2\epsilon^3+\epsilon^3$	ad
18	$2+2\epsilon^2+\epsilon^3$	$2+\epsilon^3+2\epsilon^4$	$-2\epsilon-2\epsilon^2-\epsilon^3$	$2\epsilon+\epsilon^3+2\epsilon^4$	bc
19	$-2+\epsilon+\epsilon^3$	$-\epsilon+\epsilon^3$	$-\epsilon+\epsilon^3$	$\epsilon+\epsilon^3-2\epsilon^4$	$abcd$
20	1	-1	1	1	$abdc$
21	1	1	-1	1	$acdb$
22	$1+\epsilon+3\epsilon^4$	$\epsilon^2-\epsilon^3$	$\epsilon^2-\epsilon^2$	$1+3\epsilon+\epsilon^4$	$acbd$
23	$1+2\epsilon^4$	-1	-1	$-1-2\epsilon$	$adbc$
24	$3+\epsilon+\epsilon^3$	$-\epsilon^2+\epsilon^4$	$-\epsilon^2+\epsilon^4$	$1+3\epsilon+\epsilon^3$	$adcb$

As an example of the calculation we have $(A,\ B,\ C,\ D)=(0,\ i,\ -1,\ 0)$; ab. Hence

$$A_1,\ B_1,\ C_1,\ D_1=\Big(\mathrm{a}\,(i-1),\ \mathrm{b}\,(i-1)+i+1,\ \mathrm{b}\,(i-1)-(i+1),\ -\mathrm{a}\,(i+1)\Big),$$

$$=\left(1,\ \frac{\mathrm{b}-i}{\mathrm{a}},\ \frac{\mathrm{b}+i}{\mathrm{a}},\ -1\right).$$

The second and third coefficients are

$$\frac{\sqrt{5}+1}{2}-i\sqrt{\frac{5+\sqrt{5}}{2}},\ \frac{\sqrt{5}+1}{2}+i\sqrt{\frac{5+\sqrt{5}}{2}},$$

which, in virtue of the values of ϵ and ϵ^4, are $=1+2\epsilon^4$ and $1+2\epsilon$ respectively: or the result is as above $(1,\ 1+2\epsilon^4,\ 1+2\epsilon,\ -1)$.

112. In like manner for the passage from the Θ-form to the B-form, if X, Y, Z be the coordinates of a point on the spherical surface in regard to the Θ-axes, X_2, Y_2, Z_2 those of the same point in regard to the B-axes, we may write

$$X : Y : Z = X_2 : \mathrm{b}Y_2 + \mathrm{a}Z_2 : -\mathrm{a}Y_2 + \mathrm{b}Z_2,$$

where

$$\mathrm{a},\ \mathrm{b} = \frac{\sqrt{5}-1}{2\sqrt{3}},\ \frac{\sqrt{5}+1}{2\sqrt{3}}.$$

Hence $X : Y : Z = L : M : N$, being the equations of an axis of rotation in the first set of coordinates, those of the same axis in the second set of coordinates will be

$$X_2 : \mathrm{b}Y_2 + \mathrm{a}Z_2 : -\mathrm{a}Y_2 + \mathrm{b}Z_2 = L : M : N,$$

or calling these

$$X_2 : Y_2 : Z_2 = L_2 : M_2 : N_2,$$

we have

$$L_2,\ M_2,\ N_2 = L : \mathrm{b}M - \mathrm{a}N : \mathrm{a}M + \mathrm{b}N:$$

these values are such that

$$L_2^2 + M_2^2 + N_2^2 = L^2 + M^2 + N^2,$$

or λ, μ, ν, λ_2, μ_2, ν_2 being the rotations, we have

$$L,\ M,\ N = \vartheta\lambda,\ \vartheta\mu,\ \vartheta\nu;\ L_2,\ M_2,\ N_2 = \vartheta\lambda_2,\ \vartheta\mu_2,\ \vartheta\nu_2,$$

where ϑ has the same value in the two sets of equations. We have thus

$$B + C : B - C : D - A : D + A = L : 2M : N : -i\vartheta,$$

$$B_2 + C_2 : B_2 - C_2 : D_2 - A_2 : D_2 + A_2 = L_2 : 2M_2 : N_2 : -i\vartheta,$$

and hence

$$\begin{aligned} B_2 + C_2 &= B + C, \\ B_2 - C_2 &= \mathrm{b}(B - C) - \mathrm{a}i(D - A), \\ D_2 - A_2 &= -\mathrm{a}i(B - C) + \mathrm{b}(D - A), \\ D_2 + A_2 &= D + A; \end{aligned}$$

and thence

$$\begin{aligned} A_2 &= \mathrm{a}i(B - C) - \mathrm{b}(D - A) + (D + A), \\ B_2 &= \mathrm{b}(B - C) - \mathrm{a}i(D - A) + (B + C), \\ C_2 &= -\mathrm{b}(B - C) + \mathrm{a}i(D - A) + (B + C), \\ D_2 &= -\mathrm{a}i(B - C) + \mathrm{b}(D - A) + (D + A). \end{aligned}$$

113. As an example of the transformation, take

$$(A,\ B,\ C,\ D) = \left(2,\ -3 + \sqrt{5} + i(1 - \sqrt{5}),\ -3 + \sqrt{5} + i(-1 + \sqrt{5}),\ -2\right)\ [bc.de]:$$

then

$$B - C,\ B + C,\ D - A,\ D + A = i(1 - \sqrt{5}),\ -3 + \sqrt{5},\ -2,\ 0;$$

and thence

$$A_2 = \frac{1}{2\sqrt{3}}(\ \ 6 - 2\sqrt{5}) + \frac{1}{2\sqrt{3}}(\ \ 2 + 2\sqrt{5}),$$

$$B_2 = \frac{1}{2\sqrt{3}}(-4i) \qquad + \frac{1}{2\sqrt{3}}\Big(2i(1+\sqrt{5}) + (-3+\sqrt{5})\Big),$$

$$C_2 = \frac{1}{2\sqrt{3}}(\ \ 4i) \qquad + \frac{1}{2\sqrt{3}}\Big(2i(1-\sqrt{5}) + (-3+\sqrt{5})\Big),$$

$$D_2 = \frac{1}{2\sqrt{3}}(-6 + 2\sqrt{5}) + \frac{1}{2\sqrt{3}}(-2 - 2\sqrt{5});$$

viz. multiplying by $2\sqrt{3}$, these are

$$8,\quad i(-6+2\sqrt{5}) + 2\sqrt{3}(-3+\sqrt{5}),\quad i(6-2\sqrt{5}) + 2\sqrt{3}(-3+\sqrt{5}),\quad -8,$$

that is,

$$8,\quad (-6+2\sqrt{5})(i+\sqrt{3}),\quad (-6+2\sqrt{5})(-i+\sqrt{3}),\quad -8,$$

or since

$$2+\sqrt{3} = -2i\omega \text{ and } -2+\sqrt{3} = 2i\omega^2,$$

dividing by 4 these are

$$2,\quad i(3-\sqrt{5})\,\omega,\quad i(-3+\sqrt{5})\,\omega^2,\quad -2,$$

as in the table.

114. Homographic Transformations. The group of 60. Pole at B.

$$\omega = \tfrac{1}{2}(-1 + i\sqrt{3}).$$

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	0	1	1	0	$ac\,.\,bd$
3	0	ω	1	0	$ae\,.\,bd$
4	0	ω^2	1	0	$bd\,.\,ce$
5	2	$i(\ \ 3-\sqrt{5})$	$i(\ -3+\sqrt{5})$	-2	$ab\,.\,cd$
6	2	$i(-3-\sqrt{5})$	$i(\ \ 3+\sqrt{5})$	-2	$ad\,.\,bc$
7	2	$i(\ 3-\sqrt{5})\,\omega$	$i(\ -3+\sqrt{5})\,\omega^2$	-2	$bc\,.\,de$
8	2	$i(-3-\sqrt{5})\,\omega$	$i(\ \ 3+\sqrt{5})\,\omega^2$	-2	$be\,.\,cd$
9	2	$i(\ 3-\sqrt{5})\,\omega^2$	$i(\ -3+\sqrt{5})\,\omega$	-2	$ad\,.\,be$
10	2	$i(-3-\sqrt{5})\,\omega^2$	$i(\ \ 3+\sqrt{5})\,\omega$	-2	$ab\,.\,de$
11	2	$(-\sqrt{3}-i\sqrt{5})\,\omega$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	$ab\,.\,ce$
12	2	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2	$ac\,.\,be$
13	2	$(-\sqrt{3}-i\sqrt{5})\,\omega^2$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2	$ae\,.\,bc$
14	2	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2	$ac\,.\,de$
15	2	$(\ \sqrt{3}-i\sqrt{5})\,\omega$	$(\ \sqrt{3}+i\sqrt{5})\,\omega^2$	-2	$ad\,.\,ce$
16	2	$(\ \sqrt{3}-i\sqrt{5})\,\omega^2$	$(\ \sqrt{3}+i\sqrt{5})\,\omega$	-2	$ae\,.\,cd$

17	ω	0	0	1	ace
18	ω^2	0	0	1	aec
19	$\sqrt{3}-i\sqrt{5}$	2	-2	$\sqrt{3}+i\sqrt{5}$	bed
20	$-\sqrt{3}-i\sqrt{5}$	2	-2	$-\sqrt{3}+i\sqrt{5}$	bde
21	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$-\sqrt{3}+i\sqrt{5}$	bdc
22	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$\sqrt{3}+i\sqrt{5}$	bcd
23	$-\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$-\sqrt{3}+i\sqrt{5}$	abd
24	$\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$\sqrt{3}+i\sqrt{5}$	adb
25	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2ω	abc
26	2ω	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acb
27	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	abe
28	2	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$-2\omega^2$	aeb
29	2ω	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acd
30	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2ω	adc
31	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	-2	ade
32	2	$\sqrt{3}-i\sqrt{5}$	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	$-2\omega^2$	aed
33	2	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2ω	bce
34	2ω	$-\sqrt{3}-i\sqrt{5}$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	-2	bec
35	2ω	$\sqrt{3}-i\sqrt{5}$	$(\sqrt{3}+i\sqrt{5})\,\omega$	-2	cde
36	2	$\sqrt{3}-i\sqrt{5}$	$(\sqrt{3}+i\sqrt{5})\,\omega$	-2ω	ced
37	2	$i(3-\sqrt{5})\,\omega^2$	$i(-3+\sqrt{5})$	$-2\omega^2$	$adceb$
38	$-\sqrt{3}-i\sqrt{5}$	$+2\omega^2$	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$acbde$
39	$\sqrt{3}-i\sqrt{5}$	2	-2ω	$(\sqrt{3}+i\sqrt{5})\,\omega$	$aedbc$
40	2	$i(3-\sqrt{5})$	$i(-3+\sqrt{5})\,\omega$	-2ω	$abecd$
41	2	$i(3-\sqrt{5})\,\omega$	$i(-3+\sqrt{5})$	-2ω	$aedcb$
42	$-\sqrt{3}-i\sqrt{5}$	2ω	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega$	$adbec$
43	$\sqrt{3}-i\sqrt{5}$	2	$-2\omega^2$	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	$acebd$
44	2	$i(3-\sqrt{5})$	$i(-3+\sqrt{5})\,\omega^2$	$-2\omega^2$	$abcde$
45	2	$i(3-\sqrt{5})\,\omega^2$	$i(-3+\sqrt{5})\,\omega^2$	-2ω	$adebc$
46	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	$-2\omega^2$	$(\sqrt{3}+i\sqrt{5})\,\omega$	$aecdb$
47	$-\sqrt{3}-i\sqrt{5}$	2ω	-2ω	$(-\sqrt{3}+i\sqrt{5})\,\omega^2$	$abdce$
48	2	$i(3-\sqrt{5})\,\omega$	$i(-3+\sqrt{5})\,\omega$	$-2\omega^2$	$acbed$
49	2	$i(-3-\sqrt{5})\,\omega$	$i(3+\sqrt{5})\,\omega$	$-2\omega^2$	$acdeb$
50	$\sqrt{3}-i\sqrt{5}$	2ω	-2ω	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	$adbce$
51	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	$-2\omega^2$	$(-\sqrt{3}+i\sqrt{5})\,\omega$	$aecbd$
52	2	$i(-3-\sqrt{5})\,\omega^2$	$i(3+\sqrt{5})\,\omega^2$	-2ω	$abedc$
53	2	$i(-3-\sqrt{5})\,\omega$	$i(3+\sqrt{5})$	-2ω	$aebcd$
54	$-\sqrt{3}-i\sqrt{5}$	2ω	-2	$(-\sqrt{3}+i\sqrt{5})\,\omega$	$abdec$
55	$\sqrt{3}-i\sqrt{5}$	2	$-2\omega^2$	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	$acedb$
56	2	$i(-3-\sqrt{5})$	$i(3+\sqrt{5})\,\omega^2$	$-2\omega^2$	$adcbe$
57	2	$i(-3-\sqrt{5})$	$i(3+\sqrt{5})\,\omega$	-2ω	$adecb$
58	$-\sqrt{3}-i\sqrt{5}$	2	-2ω	$(-\sqrt{3}+i\sqrt{5})\,\omega$	$aebdc$
59	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2	$(\sqrt{3}+i\sqrt{5})\,\omega^2$	$acdbe$
60	2	$i(-3-\sqrt{5})\,\omega^2$	$i(3+\sqrt{5})$	$-2\omega^2$	$abced$

115. We hence derive

Homographic Transformations. The groups of 12 and 24. Pole at B.

	$(As$	$+B)$	$\div(Cs$	$+D)$	
1	1	0	0	1	1
2	2	$i(3-\sqrt{5})$	$i(-3+\sqrt{5})$	-2	$ab\,.\,cd$
3	0	1	1	0	$ac\,.\,bd$
4	2	$i(-3-\sqrt{5})$	$i(3+\sqrt{5})$	-2	$ad\,.\,bc$
5	$2\omega^2$	$-\sqrt{3}-i\sqrt{5}$	$-\sqrt{3}+i\sqrt{5}$	-2ω	abc
6	2ω	$-\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acb
7	$-\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$-\sqrt{3}+i\sqrt{5}$	abd
8	$\sqrt{3}-i\sqrt{5}$	2ω	$-2\omega^2$	$\sqrt{3}+i\sqrt{5}$	adb
9	2ω	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	$-2\omega^2$	acd
10	$2\omega^2$	$\sqrt{3}-i\sqrt{5}$	$\sqrt{3}+i\sqrt{5}$	-2ω	adc
11	$\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$\sqrt{3}+i\sqrt{5}$	bcd
12	$-\sqrt{3}-i\sqrt{5}$	$2\omega^2$	-2ω	$-\sqrt{3}+i\sqrt{5}$	bdc
13	2	$\sqrt{3}(1+\sqrt{5})+(-3-\sqrt{5})$	$\sqrt{3}(1+\sqrt{5})+i(3+\sqrt{5})$	-2	ab
14	2	$\sqrt{3}(-1-\sqrt{5})+(-3-\sqrt{5})$	$\sqrt{3}(-1-\sqrt{5})+i(3+\sqrt{5})$	-2	cd
15	$\sqrt{5}$	$-i$	i	$-\sqrt{5}$	ac
16	1	$i\sqrt{5}$	$-i\sqrt{5}$	-1	bd
17	2	$\sqrt{3}(-1+\sqrt{5})+i(3-\sqrt{5})$	$\sqrt{3}(-1+\sqrt{5})+i(-3+\sqrt{5})$	-2	ad
18	2	$\sqrt{3}(1-\sqrt{5})+i(3-\sqrt{5})$	$\sqrt{3}(1-\sqrt{5})+i(-3+\sqrt{5})$	-2	bc
19	1	i	i	1	$abcd$
20	1	$-i$	$-i$	1	$adcb$
21	$\sqrt{3}(1-\sqrt{5})+i(3+\sqrt{5})$	2	-2	$\sqrt{3}(1-\sqrt{5})+i(-3+\sqrt{5})$	$abdc$
22	$\sqrt{3}(1+\sqrt{5})+i(-3+\sqrt{5})$	2	-2	$\sqrt{3}(1+\sqrt{5})+i(3+\sqrt{5})$	$acbd$
23	$\sqrt{3}(-1+\sqrt{5})+i(3-\sqrt{5})$	2	-2	$\sqrt{3}(-1+\sqrt{5})+i(-3+\sqrt{5})$	$acdb$
24	$\sqrt{3}(-1-\sqrt{5})+i(-3-\sqrt{5})$	2	-2	$\sqrt{3}(-1-\sqrt{5})+i(3+\sqrt{5})$	$adbc$

116. I give also the group of 12, $(abce)$, slightly modifying the form: viz. I write first $\sqrt{3}+i\sqrt{5}=2\sqrt{2}k$, and therefore $\sqrt{3}-i\sqrt{5}=2\sqrt{2}:\frac{1}{k}$: then for x I write λx, and divide the A and B by λ: the A and B then contain $\frac{k}{\lambda}$, and the C and D contain $\frac{\lambda}{k}$, and assuming $\frac{k}{\lambda}=i$, we have $\frac{\lambda}{k}=-i$. For instance, in the transformation corresponding to abc, the $Ax+B$ and $Cx+D$,

$$=2\omega^2x-(\sqrt{3}+i\sqrt{5}) \text{ and } (-\sqrt{3}+2\sqrt{5})x-2\omega,$$

become first $2\omega^2x-2\sqrt{2}k$, and $-2\sqrt{2}\frac{1}{k}x-2\omega$, and then (omitting also the factor 2) $\omega^2x-\sqrt{2}\frac{k}{\lambda}$ and $-\sqrt{2}\frac{\lambda}{k}x-\omega$, viz. when $\frac{k}{\lambda}=i$, they are $\omega^2x-i\sqrt{2}$ and $x\,.\,i\sqrt{2}-\omega$; that is, the values of A, B, C, D are ω^2, $-i\sqrt{2}$, $i\sqrt{2}$, $-\omega$. The group is

Group of 12. Pole at B.

1	0	0	1	1
ω	0	0	1	ace
ω^2	0	0	1	aec
1	$-i\omega\sqrt{2}$	$i\omega\sqrt{2}$	$-\omega^2$	abc
1	$-i\omega^2\sqrt{2}$	$i\omega^2\sqrt{2}$	$-\omega$	acb
1	$-i-\omega\sqrt{2}$	$i\sqrt{2}$	$-\omega$	abe
1	$-i\sqrt{2}$	$i\omega^2\sqrt{2}$	$-\omega^2$	aeb
1	$-i\omega^2\sqrt{2}$	$i\sqrt{2}$	$-\omega^2$	bec
1	$-i\sqrt{2}$	$i\omega\sqrt{2}$	$-\omega$	bce
1	$-i\omega\sqrt{2}$	$i\omega^2\sqrt{2}$	-1	$ab\,.\,ce$
1	$-i\omega^2\sqrt{2}$	$i\omega\sqrt{2}$	-1	$ae\,.\,bc$
1	$-i\sqrt{2}$	$i\sqrt{2}$	-1	$ac\,.\,be$

117. From the Table of the Groups of 12 and 24, Θ-form, it appears that the group of 12 is

$$x,\ \frac{1}{x},\ -x,\ -\frac{1}{x},\ \frac{i(x-1)}{x+1},\ \frac{-i(x-1)}{x+1},\ \frac{i(x+1)}{x-1},\ \frac{-i(x+1)}{x-1},$$

$$\frac{x+i}{x-i},\ \frac{x-i}{x+i},\ \frac{-(x+i)}{x-i},\ \frac{-(x-i)}{x+i};$$

and if we proceed to form the product of the twelve factors $s-x$, $s-\frac{1}{x}$, $s+x$, &c., we have first the three products

$$s^2-x^2\,.\,s^2-\frac{1}{x^2};\quad s^2+\left(\frac{x-1}{x+1}\right)^2.\,s^2+\left(\frac{x+1}{x-1}\right)^2;\quad s^2-\left(\frac{x+i}{x-i}\right)^2.\,s^2-\left(\frac{x-i}{x+i}\right)^2$$

$$=s^4+\alpha s^2+1;\qquad s^4+\beta s^2+1;\qquad s^4+\gamma s^2+1;$$

if for shortness

$$\alpha,\ \beta,\ \gamma=-\left(x^2+\frac{1}{x^2}\right),\quad 2\,\frac{x^4+6x^2+1}{(x^2-1)^2},\quad -2\,\frac{x^4-6x^2+1}{(x^2+1)^2}.$$

The product of the three quartic functions is

$$=(s^4+1)^3+(s^4+1)^2s^2(\alpha+\beta+\gamma)+(s^4+1)\,s^4(\beta\gamma+\gamma\alpha+\alpha\beta)+s^6\,.\,\alpha^2\beta^2\gamma^2;$$

and we have

$$\beta+\gamma=\frac{32x^2(x^4+1)}{(x^4-1)^2},\quad \alpha+\beta+\gamma=\frac{-(x^{12}-33s^8-33s^4+1)}{x^2(x^4-1)^2},$$

$$\beta\gamma=\frac{-4(x^8-34x^4+1)}{(x^4-1)^2},\quad \alpha(\beta+\gamma)=\frac{-32x^2(x^4+1)^2}{x^2(x^4-1)^2},$$

$$\beta\gamma+\gamma\alpha+\alpha\beta,\ =\frac{-36x^2(x^4-1)^2}{x^2(x^4-1)^2},\ =-36,\quad \alpha\beta\gamma=\frac{4(x^{12}-33s^8-33s^4+1)}{x^2(x^4-1)^2}.$$

Hence the product is found to be

$$=(s^{12}-33s^8-33s^4+1)-s^2(s^4-1)^2.\,\frac{x^{12}-33x^8-33x^4+1}{x^2(x^4-1)^2},$$

which is

$$= s^2(s^4-1)^2\left\{\frac{s^{12}-33s^8-33s^4+1}{s^2(s^4-1)^2}-\frac{x^{12}-33x^8-33x^4+1}{x^2(x^4-1)^2}\right\}.$$

We thus verify that the twelve transformations x into x, into $\frac{1}{x}$, &c., give each of them a transformation of the function

$$\frac{x^{12}-33x^8-33x^4+1}{x^2(x^4-1)^2}$$

into itself.

The system of 15 *circles.* Art. Nos. 118 to 127.

118. It has been already remarked that we can from the coefficients (A, B, C, D) of the homographic transformation pass back to the position of the axis of rotation: viz. we have

$$A : B : C : D = -\nu - i : \lambda + i\mu : \lambda - i\mu : \nu - i,$$

and thence

$$\lambda : \mu : \nu : 1 = \quad B+C : -i(B-C) \quad : D-A \quad : i(D+A),$$

that is,

$$\lambda,\ \mu,\ \nu = -i(B+C),\ \ -(B-C),\ \ -i(D-A);\ \ \div(D+A).$$

The equations of the axis thus are

$$\frac{x}{B+C}=\frac{iy}{B-C}=\frac{z}{D-A},$$

and the equations of the central plane at right angles to the axis are

$$-(B+C)x+i(B-C)y+(A-D)z=0.$$

119. In particular, we may find the equations of the 15 planes at right angles to the Θ-axes: these are in fact the before-mentioned 15 planes, intersecting the sphere in great circles the projections of which are the circles in the three figures respectively. Taking the equation of the plane to be $Lx+My+Nz=0$, it is at once seen that the equation of the projecting cone (vertex at the South pole) is

$$N(x^2+y^2+z^2-1)-2(z+1)(Lx+My+Nz)=0,$$

and hence, writing $z=0$, we find

$$N(x^2+y^2-1)-2(Lx+My)=0$$

for the equation of the circle in the plane figure. We have thus the equations of a system of 15 circles related to each other in the manner before referred to.

120. Taking the Θ-form, the equations of the 15 planes are at once found: and we thence obtain the equations of the 15 circles: viz. writing for shortness

$$\Omega = x^2+y^2-1,$$

the equations are

$$\begin{array}{lll}
z=0, & (ab.cd) & \Omega=0, \\
x=0, & (ac.bd) & x=0, \\
y=0, & (ad.bc) & y=0, \\
(3-\sqrt{5})x+(\ \ 1-\sqrt{5})y+2z=0, & (ae.bc) & \Omega-[(\ \ 3-\sqrt{5})x+(\ \ 1-\sqrt{5})y]=0, \\
(-1-\sqrt{5})x+(-1+\sqrt{5})y+2z=0, & (ab.ce) & \Omega-[(-1-\sqrt{5})x+(-1+\sqrt{5})y]=0, \\
(1+\sqrt{5})x+(\ \ 3+\sqrt{5})y+2z=0, & (ac.be) & \Omega-[(\ \ 1+\sqrt{5})x+(\ \ 3+\sqrt{5})y]=0; \\
(-3+\sqrt{5})x+(-1+\sqrt{5})y+2z=0, & (ad.be) & \text{and similarly for the other circles.} \\
(1+\sqrt{5})x+(\ \ 1-\sqrt{5})y+2z=0, & (ab.de) & \\
(-1-\sqrt{5})x+(-3-\sqrt{5})y+2z=0, & (ae.bd) & \\
(-3+\sqrt{5})x+(\ \ 1-\sqrt{5})y+2z=0, & (ad.ce) & \\
(1+\sqrt{5})x+(-1+\sqrt{5})y+2z=0, & (ae.cd) & \\
(-1-\sqrt{5})x+(\ \ 3+\sqrt{5})y+2z=0, & (ac.de) & \\
(3-\sqrt{5})x+(-1+\sqrt{5})y+2z=0, & (bc.de) & \\
(-1-\sqrt{5})x+(\ \ 1-\sqrt{5})y+2z=0, & (be.cd) & \\
(1+\sqrt{5})x+(-3-\sqrt{5})y+2z=0, & (bd.ce). &
\end{array}$$

121. Observe that the arrangement is in sets of 3 planes, or circles, intersecting at right angles. One of the circles is the circle $\Omega, =x^2+y^2-1, =0$ corresponding to the equator, and two of them are the right lines $x=0$ and $y=0$. The equations of the remaining 12 circles may be written in the somewhat different form

$$\begin{array}{l}
\Omega+(\sqrt{5}-1)\,[y-\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega-(\sqrt{5}-1)\,[y-\tfrac{1}{2}(\sqrt{5}+3)\,x]=0, \\
\Omega-(\sqrt{5}+3)\,[y+\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega-(\sqrt{5}-1)\,[y-\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega+(\sqrt{5}-1)\,[y-\tfrac{1}{2}(\sqrt{5}+3)\,x]=0, \\
\Omega+(\sqrt{5}+3)\,[y+\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega+(\sqrt{5}-1)\,[y+\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega-(\sqrt{5}-1)\,[y+\tfrac{1}{2}(\sqrt{5}+3)\,x]=0, \\
\Omega-(\sqrt{5}+3)\,[y-\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega-(\sqrt{5}-1)\,[y+\tfrac{1}{2}(\sqrt{5}-1)\,x]=0, \\
\Omega+(\sqrt{5}-1)\,[y+\tfrac{1}{2}(\sqrt{5}+3)\,x]=0, \\
\Omega+(\sqrt{5}+3)\,[y-\tfrac{1}{2}(\sqrt{5}-1)\,x]=0.
\end{array}$$

It hence appears that 4 and 4 circles have with $\Omega=0$ the common chords $y+\frac{1}{2}(\sqrt{5}-1)x=0$, $y-\frac{1}{2}(\sqrt{5}-1)x=0$ respectively: and that 2 and 2 circles have with $\Omega=0$ the common chords $y+\frac{1}{2}(\sqrt{5}+3)x=0$, $y-\frac{1}{2}(\sqrt{5}+3)x=0$ respectively.

122. The equations of the 12 circles are, in fact,

$$\Omega \pm (\sqrt{5}-1)\left[y \pm \tfrac{1}{2}(\sqrt{5}-1)x\right]=0, \quad \Omega \pm (\sqrt{5}+3)\left[y \pm \tfrac{1}{2}(\sqrt{5}-1)x\right]=0,$$
$$\Omega \pm (\sqrt{5}-1)\left[y \pm \tfrac{1}{2}(\sqrt{5}+3)x\right]=0:$$

hence the radii are $=\sqrt{5}-1$, 2 and $\sqrt{5}+1$ respectively.

The construction of the 12 circles is as follows. Starting with a circle radius 1.

Lay down the diameters $y \pm \frac{1}{2}(\sqrt{5}-1)x=0$ (AA in the figure), and through the extremities of each describe 2 pairs of circles with the radii $\sqrt{5}-1$, $\sqrt{5}+1$ respectively.

Lay down the diameters $y \pm \frac{1}{2}(\sqrt{5}+3)x=0$ (BB in the figure), and through the extremities of each describe a pair of circles with the radius 2.

123. For the A-form, the equations of the fifteen planes are at once found to be

$$\begin{array}{rrrl}
 & y & =0, & ad\,.\,bc \\
-x & + & (\epsilon+\epsilon^4)z=0, & ac\,.\,bd \\
(\epsilon+\epsilon^4)x & + & z=0, & ab\,.\,cd \\
\hline
(\epsilon^2-\epsilon^3)x & -i(\epsilon^2+\epsilon^3)y & =0, & ac\,.\,be \\
-(\epsilon^2+\epsilon^3)x & +i(\epsilon^2-\epsilon^3)y+2(\epsilon+\epsilon^4)z=0, & & ae\,.\,bc \\
-x+i(\epsilon^2+\epsilon^4-\epsilon-\epsilon^3)y+ & & 2z=0, & ab\,.\,ce \\
\hline
(\epsilon-\epsilon^4)x & -i(\epsilon+\epsilon^4)y & =0, & ab\,.\,de \\
-(\epsilon+\epsilon^4)x & +i(\epsilon-\epsilon^4)y+2(\epsilon+\epsilon^4)z=0, & & ae\,.\,bd \\
+(\epsilon^2+\epsilon^3+2)x & -i(\epsilon^2-\epsilon^3)y+ & 2z=0, & ad\,.\,be \\
\hline
(\epsilon-\epsilon^4)x & +i(\epsilon+\epsilon^4)y & =0, & ae\,.\,cd \\
-(\epsilon+\epsilon^4)x & -i(\epsilon-\epsilon^4)y+2(\epsilon+\epsilon^4)z=0, & & ac\,.\,de \\
(\epsilon^2+\epsilon^3+2)x & +i(\epsilon^2-\epsilon^3)y+ & 2z=0, & ad\,.\,ce \\
\hline
(\epsilon^2-\epsilon^3)x & +i(\epsilon^2+\epsilon^3)y & =0, & bd\,.\,ce \\
-(\epsilon^2+\epsilon^3)x & -i(\epsilon^2-\epsilon^3)y+2(\epsilon+\epsilon^4)z=0, & & bc\,.\,de \\
-x-i(\epsilon^2+\epsilon^4-\epsilon-\epsilon^3)y+ & & 2z=0, & be\,.\,cd,
\end{array}$$

where, as before, the three planes of each set intersect at right angles.

124. Passing to the circles, the first plane of each set gives a right line, and we have thus five of the circles reducing themselves to right lines inclined to the axis of x at angles 0°, 36°, 72°, 108° and 144° respectively.

The remaining 10 circles form 5 pairs, the circles of a pair having different radii, but the two radii being the same for each pair, and so that for the several pairs the common chords with the circle $\Omega=0$, are the diameters inclined to the axis of x at the angles 18°, 54°, 90°, 126° and 162° respectively. Considering the two circles for which the inclination is 90°, these arise from the planes $-x+(\epsilon+\epsilon^4)z=0$, $(\epsilon+\epsilon^4)x+z=0$ respectively. The equations of the circles thus are $(\epsilon+\epsilon^4)\Omega+2x=0$,

$\Omega - 2(\epsilon + \epsilon^4)x = 0$, or recollecting that $2(\epsilon + \epsilon^4) = \sqrt{5} - 1$ and therefore $\dfrac{2}{\epsilon + \epsilon^4} = \sqrt{5} + 1$, the equations are

$$x^2 + y^2 - (\sqrt{5} - 1)x - 1 = 0, \quad x^2 + y^2 + (\sqrt{5} + 1)x = 0;$$

hence for the first circle the x-coordinate of the centre is $\frac{1}{2}(\sqrt{5} - 1)$ and the radius is $= \frac{1}{2}\sqrt{(10 - 2\sqrt{5})}$; for the second circle the x-coordinate of the centre is $= \frac{1}{2}(\sqrt{5} + 1)$, and the radius $= \frac{1}{2}\sqrt{(10 + 2\sqrt{5})}$. We have thus the construction of these two circles, and consequently the construction of all the 12 circles.

125. For the B-form, after some easy reductions and attending to the relation $\omega - \omega^2 = i\sqrt{3}$, the equations of the 15 planes become

$$\begin{array}{rrrl}
x & & = 0, & ac\,.\,bd \\
 & (-3 + \sqrt{5})\,y + & 2z = 0, & ab\,.\,cd \\
 & (3 + \sqrt{5})\,y + & 2z = 0, & ad\,.\,bc \\
\hline
\sqrt{3}x + & \sqrt{5}\;y + & 2z = 0, & ac\,.\,be \\
-(1 + \sqrt{5})\sqrt{3}x + & (\;3 - \sqrt{5})\,y + & 4z = 0, & ab\,.\,ce \\
(-1 + \sqrt{5})\sqrt{3}x + & (-3 - \sqrt{5})\,y + & 4z = 0, & ae\,.\,bc \\
\hline
x + & \sqrt{3}\;y & = 0, & ae\,.\,bd \\
-\sqrt{3}x + & y + (3 + \sqrt{5}) & z = 0, & ad\,.\,be \\
\sqrt{3}x - & y + (3 - \sqrt{5}) & z = 0, & ab\,.\,de \\
\hline
-\sqrt{3}x + & \sqrt{5}\;y + & 2z = 0, & ac\,.\,de \\
(1 - \sqrt{5})\sqrt{3}x + & (-3 - \sqrt{5})\,y + & 4z = 0, & ad\,.\,ce \\
(1 + \sqrt{5})\sqrt{3}x + & (3 - \sqrt{5})\,y + & 4z = 0, & ae\,.\,cd \\
\hline
x - & \sqrt{3}\;y & = 0, & bd\,.\,ce \\
\sqrt{3}x + & y + (3 + \sqrt{5}) & z = 0, & bc\,.\,de \\
-\sqrt{3}x - & y + (3 - \sqrt{5}) & z = 0, & be\,.\,cd.
\end{array}$$

126. Of the 15 circles, 3 are the lines $x - y\sqrt{3} = 0$, $x = 0$, $x + y\sqrt{3} = 0$, viz. these are lines at inclinations 30°, 90°, 150° to the axis of x. The equations of the remaining 12 circles are

$$\begin{aligned}
\Omega + \;(3 - \sqrt{5})\,y &= 0, \\
\Omega - \;(3 + \sqrt{5})\,y &= 0, \\
(3 + \sqrt{5})\,\Omega - 2(y - x\sqrt{3}) &= 0, \\
(3 - \sqrt{5})\,\Omega + 2(y - x\sqrt{3}) &= 0, \\
(3 + \sqrt{5})\,\Omega - 2(y + x\sqrt{3}) &= 0, \\
(3 - \sqrt{5})\,\Omega + 2(y + x\sqrt{3}) &= 0,
\end{aligned}$$

viz. these are pairs of circles having, for their common chords with $\Omega=0$, the diameters at inclinations 0°, 60°, 120° respectively. And, lastly, we have the circles

$$\begin{array}{l|l}
2\Omega-[(-1+\sqrt{5})\sqrt{3}x-(3+\sqrt{5})y]=0, & 2\Omega+[(-1+\sqrt{5})\sqrt{3}x+(3+\sqrt{5})y]=0, \\
\Omega-[\qquad\quad -\sqrt{3}x+\quad\sqrt{5}\ y]=0, & \Omega-[\qquad\quad \sqrt{3}x+\quad\sqrt{5}\ y]=0, \\
2\Omega+[(\ \ 1+\sqrt{5})\sqrt{3}x-(3-\sqrt{5})y]=0, & 2\Omega-[(\ \ 1+\sqrt{5})\sqrt{3}x+(3-\sqrt{5})y]=0.
\end{array}$$

127. The first three of these have, for common chords with $\Omega=0$, the diameters whose equations are

$$(-1+\sqrt{5})\sqrt{3}x-(3+\sqrt{5})y=0,\quad -\sqrt{3}x+\sqrt{5}y=0,\quad (1+\sqrt{5})\sqrt{3}x-(3-\sqrt{5})y=0:$$

viz. these equations are $y=(-2+\sqrt{5})x\sqrt{3}$, $y=\frac{\sqrt{3}}{\sqrt{5}}x$, $y=(2+\sqrt{5})x\sqrt{3}$. If, as in a foregoing table, $\theta=37°\,46'$, $\sin\theta=\frac{\sqrt{3}}{2\sqrt{2}}$, $\cos\theta=\frac{\sqrt{5}}{2\sqrt{2}}$, and therefore $\tan\theta=\frac{\sqrt{3}}{\sqrt{5}}$; then the inclinations of these diameters to the axis of x are respectively $60°-\theta$, θ and $120°-\theta$, or say $30°-(\theta-30°)$, $30°+(\theta-30°)$ and $90°-(\theta-30°)$, where $\theta-30°=7°\,46'$, i.e. the inclinations are $30°\pm7°\,46'$ and $90°-7°\,46'$. And for the other three circles the common chords are the diameters at the same inclinations taken negatively. The geometrical construction of the fifteen circles for the B-case in question is thus not so simple as in the Θ- and A-cases.

The Regular Polyhedra as Solid figures. Art. Nos. 128 to 134.

128. I annex some results relating to the polyhedra considered as solid figures bounded by plane faces; or say results relating to the regular solids: s is in each case taken for the length of the edge of the solid.

	Tetrahedron.	Cube.	Octahedron.	Dodecahedron.	Icosahedron.
Edge	s	s	s	s	s
Rad. of circum. sphere, R	$s\frac{\sqrt{3}}{2\sqrt{2}}$	$s\cdot\frac{1}{2}\sqrt{3}$	$s\frac{1}{\sqrt{2}}$	$s\frac{\sqrt{3}(\sqrt{5}+1)}{4}$	$s\sqrt{\frac{5+\sqrt{5}}{8}}$
Rad. of inters. sphere, ρ	$s\frac{1}{2\sqrt{2}}$	$s\frac{1}{\sqrt{2}}$	$s\cdot\frac{1}{2}$	$s\frac{3+\sqrt{5}}{4}$	$s\frac{1+\sqrt{5}}{4}$
Rad. of inscribed sphere, r	$s\frac{1}{2\sqrt{2}\sqrt{3}}$	$s\cdot\frac{1}{2}$	$s\frac{1}{\sqrt{2}\sqrt{3}}$	$s\sqrt{\frac{25+11\sqrt{5}}{40}}$	$s\frac{3+\sqrt{5}}{4\sqrt{3}}$
Rad. of circle circum. to face, R'	$s\cdot\frac{1}{\sqrt{3}}$	$s\frac{1}{\sqrt{2}}$	$s\frac{1}{\sqrt{3}}$	$s\sqrt{\frac{5+\sqrt{5}}{10}}$	$s\frac{1}{\sqrt{3}}$
Rad. of circle inscribed to face, r'	$s\cdot\frac{1}{2\sqrt{3}}$	$s\cdot\frac{1}{2}$	$s\frac{1}{2\sqrt{3}}$	$s\sqrt{\frac{5+2\sqrt{5}}{20}}$	$s\frac{1}{2\sqrt{3}}$
Incl. of adjacent faces	$\cos^{-1}\frac{1}{3}=70°\,28'$	90°	$\cos^{-1}-\frac{1}{3}=109°\,32'$		
Incl. of edge to adjacent face	$\cos^{-1}\frac{1}{\sqrt{3}}=54°\,46'$	90°	$\cos^{-1}-\frac{1}{\sqrt{3}}=125°\,44'$		

But we require further data in the cases of the dodecahedron and the icosahedron respectively.

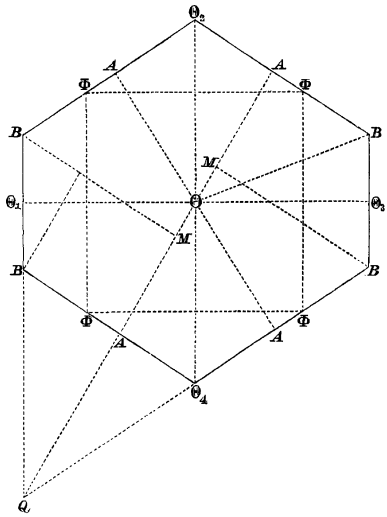
129. For the dodecahedron, taking the edge to be $=s$ as before, then in the pentagonal face

$$\begin{aligned} \text{diagonal, } g \text{ is } &= s\,.\,\tfrac{1}{2}(\sqrt{5}+1), \\ \text{altitude, } k \;\;\text{„}\; &= s\,.\,\tfrac{1}{2}\sqrt{(5+2\sqrt{5})}, \\ \text{segments of do., } e \;\;\text{„}\; &= s\,.\,\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}, \\ f \;\;\text{„}\; &= s\,.\,\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}, \end{aligned}$$

where

$$k = e + f = R' + r'.$$

130. The section through a pair of opposite edges is a hexagon, as shown in the figure, viz. this is constructed by taking the four equal distances $O\Theta$, $=\rho$, $=s\,.\,\frac{1}{4}(3+\sqrt{5})$, meeting at right angles in O; then drawing the double ordinates BB, each $=s$, through Θ_1 and Θ_3 respectively, and joining their extremities with Θ_2 and Θ_4: the sides $\Theta_2 B$ and $\Theta_4 B$ are then each $=k$, $=s\,.\,\frac{1}{2}\sqrt{(5+2\sqrt{5})}$; and inserting upon them the points A, Φ from the figure of the pentagon, we have several



geometrical relations; viz. the line AA cuts the parallel sides $B\Theta_2$, $B\Theta_4$ at right angles, and when produced passes through the intersection of $B\Theta_1$ and $B\Theta_4$: we have OA, OB, $O\Theta = r$, R, ρ respectively: the four points Φ form a square, the side of which is g, $=s\,.\,\frac{1}{2}(\sqrt{5}+1)$.

131. We find also

$$AM = s\sqrt{\frac{5+\sqrt{5}}{10}},$$

$$AQ = s\sqrt{\frac{5+2\sqrt{5}}{5}},$$

$$QM = s\sqrt{\frac{25+11\sqrt{5}}{5}}, = r\,.\,2\sqrt{2},$$

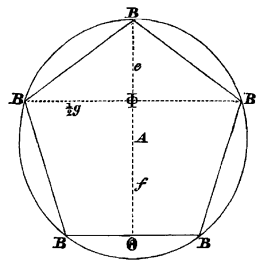
$$OQ = s\sqrt{\frac{25+11\sqrt{5}}{8}}, = r\,.\,\sqrt{5},$$

$$OM = s\sqrt{\frac{5-\sqrt{5}}{40}},$$

$$MB = s\sqrt{\frac{2(5+2\sqrt{5})}{B}}.$$

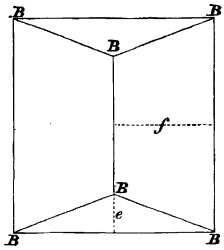
It may be remarked that in the figure $B\Theta_2$, $B\Theta_4$ are the projections of pentagonal faces, at right angles to the plane of the paper, having their centres at the points A, A, and the perpendicular distance between them $= AA$: the points Q, Q (only one of them shown in the figure) determine the directions of the $5+5$ sides which abut on these pentagonal faces respectively; and the $5+5$ points B which are the other extremities of these sides respectively form two pentagons, centres M, M in the planes MB and MB respectively: the remaining 10 sides of the dodecahedron are the skew decagon obtained by joining in order these 10 points B. We have thus the means of making the perspective delineation of the dodecahedron.

132. The dodecahedron is built up from the cube, by placing on each face a figure of two triangular and two quadrangular faces, the orthogonal projection of which on the face of the cube is as in the figure: the side of the square is g,



$= s\,.\,\frac{1}{2}(\sqrt{5}+1)$: the slope-breadths of the triangular faces are e, $= s\,.\,\frac{1}{4}\sqrt{(10-2\sqrt{5})}$, and those of the quadrangular faces are f, $= s\,.\,\frac{1}{4}\sqrt{(10+2\sqrt{5})}$; the lines represented by the other lines of the figure are in actual length each $= s$. We have thus a

section which is an isosceles triangle, base $=g$, other sides each $=f$; and the square of the altitude is thus $=f^2-\frac{1}{4}g^2=\frac{1}{4}s^2$, or the altitude $=\frac{1}{2}s$; viz. the altitude of the ridge-line BB, above the face of the cube is $=\frac{1}{2}s$, the half-side of the dodecahedron.



We have in this result the most simple means of forming the perspective delineation of the dodecahedron.

133. For the icosahedron the section through two opposite edges is a hexagon, as shown in the figure (p. 216): to construct it, we take the four distances $O\Theta$ each $=\rho=s.\frac{1}{4}(1+\sqrt{5})$ meeting at right angles; and then the distances $A\Theta_2$, $A\Theta_4$ each $=\frac{1}{2}s$; and complete the hexagon. This gives the sides $A\Theta_1$, $A\Theta_3$ each $=s.\frac{1}{2}\sqrt{3}$, the altitude of the triangular face, side $=s$; and then, taking $\Theta_1 B$ one-third of this, $=s\dfrac{1}{2\sqrt{3}}$, we have OB at right angles to $A\Theta_1$, and OA, OB, $O\Theta=R$, r, ρ respectively.

Moreover, joining $A_1\Theta_2$ and OA_2, we have these lines cutting at right angles in a point M: we find

$$A_1\Theta_3=s.\tfrac{1}{2}\sqrt{(5+2\sqrt{5})},$$

$$M\Theta_3=s\sqrt{\frac{5+2\sqrt{5}}{20}},$$

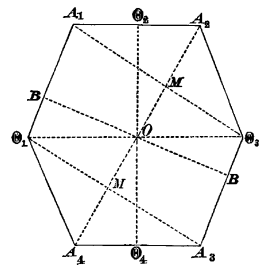
$$A_1M=s\sqrt{\frac{5+\sqrt{5}}{10}},$$

$$OM=s\sqrt{\frac{5+\sqrt{5}}{40}},\ =\tfrac{1}{2}A_1M,$$

$$A_2M=s\sqrt{\frac{5-\sqrt{5}}{10}}.$$

134. It may be remarked that $A_1\Theta_3$, $A_3\Theta_1$ are the projections of two pentagons in planes perpendicular to that of the paper, their centres being M, M: producing OM, OM to the points A_2, A_4 respectively, we have a pentagonal pyramid, summit A_2, standing on the first pentagon, and an opposite pyramid, summit A_4, standing on

the other pentagon: the $5+5$ triangular faces of the two pyramids are ten of the faces of the icosahedron, and the remaining ten faces are the triangles each having for its base a side of the one pentagon, and for its vertex a summit of the other



pentagon, viz. the sides are the sides of the skew decagon obtained by joining in order the angular points of the two pentagons. We have thus a convenient method of forming the perspective delineation of the icosahedron.

746.

HIGHER PLANE CURVES.

[From *Salmon's Higher Plane Curves*, (*3rd ed.*, 1879); *see the Preface.*]

ONE chapter and a large number of articles, in the second edition of Salmon's *Higher Plane Curves*, are due to Professor Cayley. Full reference to these is given by Dr Salmon in the preface.

747.

NOTE ON THE DEGENERATE FORMS OF CURVES.

[From *Salmon's Higher Plane Curves*, (*3rd ed.*, 1879), pp. 383—385.]

SOME remarks may be added as to the analytical theory of the degenerate forms of curves. As regards conics, a line-pair can be represented in point-coordinates by an equation of the form $xy=0$; and reciprocally a point-pair can be represented in line-coordinates by an equation $\xi\eta=0$, but we have to consider how the point-pair can be represented in point-coordinates: an equation $x^2=0$ is no adequate representation of the point-pair, but merely represents (as a two-fold or twice repeated line) the line joining the two points of the point-pair, all traces of the points themselves being lost in this representation: and it is to be noticed, that the conic, or two-fold line $x^2=0$, or say $(\alpha x+\beta y+\gamma z)^2=0$ is a conic which, analytically, and (in an improper sense) geometrically, satisfies the condition of touching *any line whatever;* whereas the only proper tangents of a point-pair are the lines which pass through one or other of the two points of the point-pair.

The solution arises out of the notion of a point-pair, considered as the limit of a conic, or say as an indefinitely flat conic; we have to consider conics certain of the coefficients whereof are infinitesimals, and which, when the infinitesimal coefficients actually vanish, reduce themselves to two-fold lines; and it is, moreover, necessary to consider the evanescent coefficients as infinitesimals of different orders. Thus consider the conics which pass through two given points, and touch two given lines (four conditions); take $y=0$, $z=0$ for the given lines, $x=0$ for the line joining the given points, and $(x=0,\ y-\alpha z=0)$, $(x=0,\ y-\beta z=0)$ for the given points; the equation of a conic satisfying the required conditions and containing one arbitrary parameter θ, is

$$x^2+2\theta xy+2\theta\sqrt{(\alpha\beta)}\,xz+\theta^2(y-\alpha z)(y-\beta z)=0\,;$$

or, what is the same thing,

$$\{x+\theta y+\theta\sqrt{(\alpha\beta)}\,z\}^2-\theta^2(\alpha+\beta)\,yz=0;$$

and this equation, considering therein θ as an infinitesimal, say of the first order, represents the flat conic or point-pair composed of the two given points. Comparing with the general equation

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2=0,$$

we have

$$a=1,\quad b=\theta^2,\quad c=\theta^2\alpha\beta,\quad f=-\tfrac{1}{2}\theta^2(\alpha+\beta),\quad g=\theta\sqrt{(\alpha\beta)},\quad h=\theta,$$

viz. a being taken to be finite, we have g and h infinitesimals of the first order; b, c, f infinitesimals of the second order; and the four ratios $\sqrt{(b)}:\sqrt{(c)}:\sqrt{(f)}:g:h$ are so determined as to satisfy the prescribed conditions.

Observe that the flat conic, considered as a conic passing through the two given points and touching the two given lines, is represented by a *determinate* equation, viz. considering the condition imposed upon θ ($\theta=$ infinitesimal) as a determination of θ, the equation is a completely determinate one; but considering the flat conic merely as a conic passing through the two given points, the equation would contain two arbitrary parameters, determinable if the flat conic was subjected to the condition of touching two given lines, or to any other two conditions.

Generally, we may consider the equation of a curve of the order n; such equation containing certain infinitesimal coefficients and, when these vanish, reducing itself to a composite equation $P^\alpha Q^\beta\ldots=0$; the equation in its original form represents a curve which may be called the penultimate curve. Consider the tangents from an arbitrary point to the penultimate curve; when this breaks up, the system of tangents reduces itself to (1) the tangents from the fixed point to the several component curves $P=0$, $Q=0$, &c. respectively; (2) the lines through the singular points of these same curves respectively; (3) the lines through the points of intersection $P=0$, $Q=0$, &c. of each two of the component curves; these points, each reckoned a proper number of times, are called "fixed summits"; (4) the lines from the fixed point to certain determinate points called "free summits" on the several component curves $P=0$, $Q=0$, &c. respectively. We have thus a degenerate form of the n-thic curve, which may be regarded as consisting of the component curves, each its proper number of times, and of the foregoing points called summits, and is consequently only inadequately represented by the ultimate equation $P^\alpha Q^\beta\ldots=0$; the number and distribution of the summits is not arbitrary, but is regulated by laws arising from the consideration of the penultimate curve, and there are of course for any given value of n various forms of degenerate curve, according to the different ultimate forms $P^\alpha Q^\beta\ldots=0$, and to the number and distribution of the summits on the different component curves. The case of a quartic curve having the ultimate form $x^2y^2=0$ has been considered by Cayley, *Comptes Rendus*, t. LXXIV. p. 708 (March, 1872), [515], who states his conclusion as follows:

"there exists a quartic curve the penultimate of $x^2y^2=0$, with nine free summits, three of them on one of the lines (say the line $y=0$), and which are three of the intersections of the quartic by this line (the fourth intersection being indefinitely near to the point $x=0$, $y=0$), six situate at pleasure on the other line $x=0$; and three fixed summits at the intersection of the two lines." Other forms have been considered by Dr Zeuthen, *Comptes Rendus*, t. LXXV. pp. 703 and 950 (September and October, 1872), and some other forms by Zeuthen; the whole question of the degenerate forms of curves is one well deserving further investigation.

The question of the number of cubic curves satisfying given elementary conditions (depending as it does on the consideration of the degenerate forms of these curves) has been solved by Maillard and Zeuthen; that of the number of quartic curves has been solved by Dr Zeuthen.

748.

ON THE BITANGENTS OF A QUARTIC.

[From *Salmon's Higher Plane Curves*, (*3rd ed.*, 1879), pp. 387—389.]

THE equations of the 28 bitangents of a quartic curve were obtained in a very elegant form by Riemann in the paper "Zur Theorie der Abel'schen Functionen für den Fall $p=3$," *Ges. Werke*, Leipzig, 1876, pp. 456—472; and see also Weber's *Theorie der Abel'schen Functionen vom Geschlecht* 3," Berlin, 1876. Riemann connects the several bitangents with the characteristics of the 28 odd functions, thus obtaining for them an algorithm which it is worth while to explain, but they will be given also with the algorithm employed p. 231 *et seq.* of the present work*, which is in fact the more simple one. The characteristic of a triple θ-function is a symbol of the form

$$\begin{matrix} \alpha\,\beta\,\gamma, \\ \alpha'\beta'\gamma', \end{matrix}$$

where each of the letters is $=0$ or 1; there are thus in all 64 such symbols, but they are considered as odd or even according as the sum $\alpha\alpha'+\beta\beta'+\gamma\gamma'$ is odd or even; and the numbers of the odd and even characteristics are 28 and 36 respectively; and, as already mentioned, the 28 odd characteristics correspond to the 28 bitangents respectively.

We have x, y, z trilinear coordinates, α, β, γ, α', β', γ' constants chosen at pleasure, and then α'', β'', γ'' determinate constants, such that the equations

$$\begin{aligned} x+\quad y+\quad z+\xi+\eta+\zeta&=0,\\ \alpha\,x+\beta\,y+\gamma\,z+\frac{\xi}{\alpha}+\frac{\eta}{\beta}+\frac{\zeta}{\gamma}&=0,\\ \alpha'x+\beta'y+\gamma'z+\frac{\xi}{\alpha'}+\frac{\eta}{\beta'}+\frac{\zeta}{\gamma'}&=0,\\ \alpha''x+\beta''y+\gamma''z+\frac{\xi}{\alpha''}+\frac{\eta}{\beta''}+\frac{\zeta}{\gamma''}&=0, \end{aligned}$$

[* That is, Salmon's *Higher Plane Curves*.]

are equivalent to three independent equations; this being so, they determine ξ, η, ζ, each of them as a linear function of (x, y, z); and the equations of the bitangents of the curve $\sqrt{(x\xi)}+\sqrt{(y\eta)}+\sqrt{(z\zeta)}=0$ (see Weber, p. 100) are

18	111 111	$x=0,$
28	001 011	$y=0,$
38	011 001	$z=0,$
23	010 010	$\xi=0,$
13	100 110	$\eta=0,$
12	110 100	$\zeta=0,$
48	101 100	$x+y+z=0,$
14	010 011	$\xi+y+z=0,$
58	100 101	$\alpha x+\beta y+\gamma z=0,$
15	011 010	$\frac{\xi}{\alpha}+\beta y+\gamma z=0,$
68	110 010	$\alpha' x+\beta' y+\gamma' z=0,$
16	001 101	$\frac{\xi}{\alpha'}+\beta' y+\gamma' z=0,$
78	010 110	$\alpha'' x+\beta'' y+\gamma'' z=0,$
17	101 001	$\frac{\xi}{\alpha''}+\beta'' y+\gamma'' z=0,$
24	100 111	$x+\eta+z=0,$
34	110 101	$x+y+\zeta=0,$
25	101 110	$\alpha x+\frac{\eta}{\beta}+\gamma z=0,$
35	111 100	$\alpha x+\beta y+\frac{\zeta}{\gamma}=0,$

26	111 001	$\alpha' x + \dfrac{\eta}{\beta'} + \gamma' z = 0,$
36	101 011	$\alpha' x + \beta' y + \dfrac{\zeta}{\gamma'} = 0,$
27	011 101	$\alpha'' x + \dfrac{\eta}{\beta''} + \gamma'' z = 0,$
37	001 111	$\alpha'' x + \beta'' y + \dfrac{\zeta}{\gamma''} = 0,$
67	100 100	$\dfrac{x}{1-\beta\gamma} + \dfrac{y}{1-\gamma\alpha} + \dfrac{z}{1-\alpha\beta} = 0,$
57	110 011	$\dfrac{x}{1-\beta'\gamma'} + \dfrac{y}{1-\gamma'\alpha'} + \dfrac{z}{1-\alpha'\beta'} = 0,$
56	010 111	$\dfrac{x}{1-\beta''\gamma''} + \dfrac{y}{1-\gamma''\alpha''} + \dfrac{z}{1-\alpha''\beta''} = 0,$
45	001 001	$\dfrac{\xi}{\alpha(1-\beta\gamma)} + \dfrac{\eta}{\beta(1-\gamma\alpha)} + \dfrac{\zeta}{\gamma(1-\alpha\beta)} = 0,$
46	011 110	$\dfrac{\xi}{\alpha'(1-\beta'\gamma')} + \dfrac{\eta}{\beta'(1-\gamma'\alpha')} + \dfrac{\zeta}{\gamma'(1-\alpha'\beta')} = 0,$
47	111 010	$\dfrac{\xi}{\alpha''(1-\beta''\gamma'')} + \dfrac{\eta}{\beta''(1-\gamma''\alpha'')} + \dfrac{\zeta}{\gamma''(1-\alpha''\beta'')} = 0.$

The whole number of ways in which the equation of the curve can be expressed in a form such as $\sqrt{(x\xi)} + \sqrt{(y\eta)} + \sqrt{(z\zeta)} = 0$ is 1260; viz. the three pairs of bitangents entering into the equation of the curve are of one of the types

12.34,	13.24,	14.23	⊠	No. is 70
12.34,	13.24,	56.78	□ ||	„ 630
13.23,	14.24,	15.25	◈	„ 560
				1260.

It may be remarked that, selecting at pleasure any two pairs out of a system of three pairs, the type is always □ or ||||, viz. (see p. 233) the four bitangents are such that their points of contact are situate on a conic.

749.

SOLID GEOMETRY.

[From *Salmon's Treatise on the analytic geometry of three dimensions,* (*3rd ed.,* 1874); *see the Preface.*]

A considerable number of articles in the third edition of Salmon's *Treatise* are due to Professor Cayley. Full reference to these is given by Dr Salmon in the preface.

750.

ON THE THEORY OF RECIPROCAL SURFACES.

[From *Salmon's Treatise on the analytic geometry of three dimensions*, (*3rd ed.*, 1874), pp. 539—550.]

600. In further developing the theory of reciprocal surfaces it has been found necessary to take account of other singularities, some of which are as yet only imperfectly understood. It will be convenient to give the following complete list of the quantities which present themselves:

n, order of the surface.

a, order of the tangent cone drawn from any point to the surface.

δ, number of nodal edges of the cone.

κ, number of its cuspidal edges.

ρ, class of nodal torse.

σ, class of cuspidal torse.

b, order of nodal curve.

k, number of its apparent double points.

f, number of its actual double points.

t, number of its triple points.

j, number of its pinch-points.

q, its class.

c, order of cuspidal curve.

h, number of its apparent double points.

θ, number of its points of an unexplained singularity.

χ, number of its close-points.

ω, number of its off-points.

r, its class.

β, number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve.

γ, number of intersections, stationary points on nodal curve.

i, number of intersections, not stationary points on either curve.

C, number of cnicnodes of surface.

B, number of binodes.

And corresponding reciprocally to these:

n', class of surface.

a', class of section by arbitrary plane.

δ', number of double tangents of section.

κ', number of its inflexions.

ρ', order of node-couple curve.

σ', order of spinode curve.

b', class of node-couple torse.

k', number of its apparent double planes.

f', number of its actual double planes.

t', number of its triple planes.

j', number of its pinch-planes.

q', its order.

c', class of spinode torse.

h', number of its apparent double planes.

θ', number of its planes of a certain unexplained singularity.

χ', number of its close-planes.

ω', number of its off-planes.

r', its order.

β', number of common planes of node-couple and spinode torse, stationary planes of spinode torse.

γ', number of common planes, stationary planes of node-couple torse.

ι', number of common planes, not stationary planes of either torse.

C', number of cnictropes of surface.

B', number of its bitropes.

In all, these are 46 quantities.

601. In part explanation, observe that the definitions of ρ and σ agree with those already given. The nodal torse is the torse enveloped by the tangent planes along the nodal curve; if the nodal curve meets the curve of contact a, then a tangent plane of the nodal torse passes through the arbitrary point, that is, ρ will be the number of these planes which pass through the arbitrary point, viz. the class of the torse. So also the cuspidal torse is the torse enveloped by the tangent planes along the cuspidal curve; and σ will be the number of these tangent planes which pass through the arbitrary point, viz. it will be the class of the torse. Again, as regards ρ' and σ': the node-couple torse is the envelope of the bitangent planes of the surface, and the node-couple curve is the locus of the points of contact of these planes. Similarly, the spinode torse is the envelope of the parabolic planes of the surface, and the spinode curve is the locus of the points of contact of these planes, viz. it is the curve UH of intersection of the surface and its Hessian; the two curves are the reciprocals of the nodal and the cuspidal torses respectively, and the definitions of ρ', σ' correspond to those of ρ and σ.

602. In regard to the nodal curve b, we consider k the number of its apparent double points (excluding actual double points); f the number of its actual double points (each of these is a point of contact of two sheets of the surface, and there is thus at the point a single tangent plane, viz. this is a plane f', and we thus have $f'=f$); t the number of its triple points; and j the number of its pinch-points—these last are not singular points of the nodal curve *per se*, but are singular in regard to the curve as nodal curve of the surface; viz. a pinch-point is a point at which the two tangent planes are coincident. The curve is considered as not having any stationary points other than the points γ, which lie also on the cuspidal curve; and the expression for the class consequently is $q=b^2-b-2k-2f-3\gamma-6t$.

603. In regard to the cuspidal curve c, we consider h the number of its apparent double points; and upon the curve, not singular points in regard to the curve *per se*, but only in regard to it as cuspidal curve of the surface, certain points in number θ, χ, ω respectively. The curve is considered as not having any actual double or other multiple points, and as not having any stationary points except the points β, which lie also on the nodal curve; and the expression for the class consequently is

$$r=c^2-c-2h-3\beta.$$

604. The points γ are points where the cuspidal curve with the two sheets (or say rather half-sheets) belonging to it are intersected by another sheet of the surface; the curve of intersection with such other sheet, belonging to the nodal curve of the surface, has evidently a stationary (cuspidal) point at the point of intersection.

As to the points β, to facilitate the conception, imagine the cuspidal curve to be a semi-cubical parabola, and the nodal curve a right line (not in the plane of the curve) passing through the cusp; then intersecting the two curves by a series of parallel planes, any plane which is, say, above the cusp, meets the parabola in two real points and the line in one real point, and the section of the surface is a curve with two real cusps and a real node; as the plane approaches the cusp, these approach

together, and, when the plane passes through the cusp, unite into a singular point in the nature of a triple point (= node + two cusps); and when the plane passes below the cusp, the two cusps of the section become imaginary, and the nodal line changes from crunodal to acnodal.

605. At a point i the nodal curve crosses the cuspidal curve, being on the side away from the two half-sheets of the surface acnodal, and on the side of the two half-sheets crunodal, viz. the two half-sheets intersect each other along this portion of the nodal curve. There is at the point a single tangent plane, which is a plane i'; and we thus have $i = i'$.

606. As already mentioned, a cnicnode C is a point where, instead of a tangent plane, we have a tangent quadri-cone; at a binode B, the quadri-cone degenerates into a pair of planes. A cnictrope C' is a plane touching the surface along a conic; in the case of a bitrope B', the conic degenerates into a flat conic or pair of points.

607. In the original formulæ for $a(n-2)$, $b(n-2)$, $c(n-2)$, we have to write $\kappa - B$ instead of κ, and the formulæ are further modified by reason of the singularities θ and ω. So, in the original formulæ, for $a(n-2)(n-3)$, $b(n-2)(n-3)$, $c(n-2)(n-3)$, we have instead of δ to write $\delta - C - 3\omega$, and to substitute new expressions for $[ab]$, $[ac]$, $[bc]$; viz. these are

$$[ab] = ab - 2\rho - j,$$
$$[ac] = ac - 3\sigma - \chi - \omega,$$
$$[bc] = bc - 3\beta - 2\gamma - i.$$

The whole series of equations thus is

(1) $a' = a.$

(2) $f' = f.$

(3) $i' = i.$

(4) $a = n(n-1) - 2b - 3c.$

(5) $\kappa' = 3n(n-2) - 6b - 8c.$

(6) $\delta' = \frac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \frac{9}{2}c(c-1).$

(7) $a(n-2) = \kappa - B + \rho + 2\sigma + 3\omega.$

(8) $b(n-2) = \rho + 2\beta + 3\gamma + 3t.$

(9) $c(n-2) = 2\sigma + 4\beta + \gamma + \theta + \omega.$

(10) $a(n-2)(n-3) = 2(\delta - C - 3\omega) + 3(ac - 3\sigma - \chi - 3\omega) + 2(ab - 2\rho - j).$

(11) $b(n-2)(n-3) = 4k + (ab - 2\rho - j) + 3(bc - 3\beta - 2\gamma - i).$

(12) $c(n-2)(n-3) = 6h + (ac - 3\sigma - \chi - 3\omega) + 2(bc - 3\beta - 2\gamma - i).$

(13) $q = b^2 - b - 2k - 2f - 3\gamma - 6t.$

(14) $r = c^2 - c - 2h - 3\beta.$

Also, reciprocal to these,

$$(15)\quad a' = n'(n'-1) - 2b' - 3c'.$$

$$(16)\quad \kappa = 3n'(n'-2) - 6b' - 8c'.$$

$$(17)\quad \delta = \tfrac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \tfrac{8}{2}c'(c'-1).$$

$$(18)\quad a'(n'-2) = \kappa - B' + \rho' + 2\sigma' + 3\omega'.$$

$$(19)\quad b'(n'-2) = \rho' + 2\beta' + 3\gamma' + 3t'.$$

$$(20)\quad c'(n'-2) = 2\sigma' + 4\beta' + \gamma' + \theta' + \omega'.$$

$$(21)\quad a'(n'-2)(n'-3) = 2(\delta' - C' - 3\omega') + 3(a'c' - 3\sigma' - \chi' - 3\omega') + 2(a'b' - 2\rho' - j').$$

$$(22)\quad b'(n'-2)(n'-3) = 4k' + (a'b' - 2\rho' - j') + 3(b'c' - 3\beta' - 2\gamma' - \iota').$$

$$(23)\quad c'(n'-2)(n'-3) = 6h' + (a'c' - 3\sigma' - \chi' - 3\omega') + 2(b'c' - 3\beta' - 2\gamma' - \iota').$$

$$(24)\quad q' = b'^2 - b' - 2k' - 2f'' - 3\gamma' - 6t'.$$

$$(25)\quad r' = c'^2 - c' - 2h' - 3\beta',$$

together with one other independent relation: in all 26 relations between the 46 quantities.

608. The new relation may be presented under several different forms, equivalent to each other in virtue of the foregoing 25 relations; these are

$$(26)\quad 2(n-1)(n-2)(n-3) - 12(n-3)(b+c) + 6q + 6r + 24t + 42\beta + 30\gamma - \tfrac{3}{2}\theta = \Sigma,$$

$$(27)\quad 26n - 12c - 4C - 10B + \beta - 7j - 8\chi + \tfrac{1}{2}\theta - 4\omega = \Sigma;$$

in each of which two equations Σ is used to denote the same function of the accented letters that the left-hand side is of the unaccented letters.

$$\begin{aligned}(28)\quad \beta' + \tfrac{1}{2}\theta' = {} & 2n(n-2)(11n-24) \\ & + (-66n+184)b \\ & + (-93n+252)c \\ & + 22(2\beta + 3\gamma + 3t) \\ & + 27(4\beta + \gamma + \theta) \\ & + \beta + \tfrac{1}{2}\theta \\ & - 24C - 28B - 27j - 38\chi - 73\omega \\ & + 4C' + 10B' + 7j' + 8\chi' - 4\omega'.\end{aligned}$$

Or, reciprocally,

$$\begin{aligned}(29)\quad \beta + \tfrac{1}{2}\theta = {} & 2n'(n'-2)(11n'-24) \\ & + (-66n'+184)b' \\ & + (-93n'+252)c' \\ & + 22(2\beta' + 3\gamma' + 3t') \\ & + 27(4\beta' + \gamma' + \theta') \\ & + \beta' + \tfrac{1}{2}\theta' \\ & - 24C' - 28B' - 27j' - 38\chi' - 73\omega' \\ & + 4C + 10B + 7j + 8\chi - 4\omega.\end{aligned}$$

The equation (26) expresses that the surface and its reciprocal have the same deficiency; viz. the expression for the deficiency is

$$(30)\quad \text{Deficiency} = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + i - \tfrac{1}{6}\theta,$$
$$= \tfrac{1}{6}(n'-1)(n'-2)(n'-3) - \&c.$$

609. The equation (28) (due to Prof. Cayley) is the correct form of an expression for β', first obtained by him (with some errors in the numerical coefficients) from independent considerations. But it is best obtained by means of the equation (26); and (27) is a relation presenting itself in the investigation. In fact, considering a as standing for its value $n(n-1) - 2b - 3c$, we have from the first 25 equations

$$\begin{array}{r|ll}
6 & a & = \Sigma, \\
+2 & 3n - c - \kappa & = \Sigma, \\
-2 & a(n-2) - \kappa + B - \rho - 2\sigma - 3\omega & = \Sigma, \\
-4 & b(n-2) - \rho - 2\beta - 3\gamma - 3t & = \Sigma, \\
-6 & c(n-2) - 2\sigma - 4\beta - \gamma - \theta - \omega & = \Sigma, \\
+2 & n + \kappa - \sigma - 2C - 4B - 2j - 3\chi - 3\omega & = \Sigma, \\
-3 & 2q - 2\rho + \beta + j & = \Sigma, \\
-2 & 3r + c - 5\sigma - \beta - 4\theta + \chi - \omega & = \Sigma;
\end{array}$$

multiplying these equations by the numbers set opposite to them respectively, and adding, we find

$$-2n^3 + 12n^2 + 4n + b(12n - 36) + c(12n - 48)$$
$$-6q - 6r - 4C - 10B - 41\beta - 30\gamma - 24t - 7j - 8\chi + 2\theta - 4\omega = \Sigma,$$

and adding hereto (26) we have the equation (27); and from this (28), or by a like process, (29), is obtained without much difficulty. As to the 8 Σ-equations or symmetries, observe that the first, third, fourth, and fifth are in fact included among the original equations (for an expression which vanishes is in fact $= \Sigma$); we have from them moreover $3n - c = 3a' - \kappa'$, and thence $3n - c - \kappa = 3a' - \kappa - \kappa'$, which is $= \Sigma$, or we have thus the second equation; but the sixth, seventh, and eighth equations have yet to be obtained.

610. The equations (15), (16), (17) give

$$n' = a(a-1) - 2\delta - 3\kappa,$$
$$c' = 3a(a-2) - 6\delta - 8\kappa,$$
$$b' = \tfrac{1}{2}a(a-2)(a^2-9) - (a^2 - a - 6)(2\delta + 3\kappa) + 2\delta(\delta - 1) + 6\delta\kappa + \tfrac{9}{2}\kappa(\kappa - 1).$$

From (7), (8), (9), we have

$$(a - b - c)(n-2) = \kappa - B - 6\beta - 4\gamma - 3t - \theta + 2\omega,$$
$$(a - 2b - 3c)(n-2)(n-3) = 2(\delta - C) - 8k - 18h - 6bc + 18\beta + 12\gamma + 6i - 6\omega;$$

substituting these values for κ and δ, and for a its value $=n(n-1)-2b-3c$, we obtain the values of n', c', b'; viz. the value of n' is

$$\begin{aligned} n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 8b + 9c^2 + 15c \\ -8k - 18h + 18\beta + 12\gamma + 12i - 9t \\ -2C - 3B - 3\theta. \end{aligned}$$

Observe that the effect of a cnicnode C is to reduce the class by 2, and that of a binode B to reduce it by 3.

611. We have

$$(n-2)(n-3) = n^2 - n + (-4n+6) = a + 2b + 3c + (-4n+6);$$

making this substitution in the equations (10), (11), (12), which contain $(n-2)(n-3)$, these become

$$\begin{aligned} a(-4n+6) &= 2(\delta - C) - a^2 - 4\rho - 9\sigma - 2j - 3\chi - 15\omega, \\ b(-4n+6) &= 4k - 2b^2 - 9\beta - 6\gamma - 3i - 2\rho - j, \\ c(-4n+6) &= 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma - \chi - 3\omega, \end{aligned}$$

which are the foregoing equations (C); adding to each equation four times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2 - 2a &= 2(\delta - C) + 4(\kappa - B) - \sigma - 2j - 3\chi - 3\omega, \\ 2b^2 - 2b &= 4k - \beta + 6\gamma + 12t - 3i + 2\rho - j, \\ 3c^2 - 2c &= 6h + 10\beta + 4\theta - 2i + 5\sigma - \chi + \omega. \end{aligned}$$

Writing in the first of these $a^2 - 2a = n' + 2\delta + 3\kappa - a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n' - a &= -2C - 4B + \kappa - \sigma - 2j - 3\chi - 3\omega, \\ 2q + \beta + 3i + j &= 2\rho, \\ 3r + c + 2i + \chi &= 5\sigma + \beta + 4\theta + \omega, \end{aligned}$$

which give at once the last three of the 8 Σ-equations.

The reciprocal of the first of these is

$$\sigma' = a - n + \kappa' - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

viz. writing herein

$$a = n(n-1) - 2b - 3c \quad \text{and} \quad \kappa' = 3n(n-2) - 6b - 8c,$$

this is

$$\sigma' = 4n(n-2) - 8b - 11c - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

giving the order of the spinode curve; viz. for a surface of the order n without singularities, this is $=4n(n-2)$, the product of the orders of the surface and its Hessian.

612. Instead of obtaining the second and third equations as above, we may to the value of $b(-4n+6)$ add twice the value of $b(n-2)$; and to twice the value of $c(-4n+6)$ add three times the value of $c(n-2)$, thus obtaining equations free from ρ and σ respectively; these equations are

$$b(-2n+2) = 4k - 2b^2 - 5\beta - 3i + 6t - j,$$
$$c(-5n+6) = 12h - 6c^2 - 5\gamma - 4i - 2\chi + 3\theta - 3\omega,$$

equations which, introducing therein the values of q and r, may also be written

$$b(2n-4) \qquad = 2q + 5\beta + 6\gamma + 6t + 3i + j + 4f,$$
$$c(5n-12) + 3\theta = 6r + 18\beta + 5\gamma \qquad + 4i + 2\chi + 3\omega.$$

Considering as given, n the order of the surface; the nodal curve, with its singularities b, k, f, t; the cuspidal curve, with its singularities c, h; and the quantities β, γ, i which relate to the intersections of the nodal and cuspidal curves; the first of the two equations gives j, the number of pinch-points, being singularities of the nodal curve, quoad the surface; and the second equation establishes a relation between θ, χ, ω, the numbers of singular points of the cuspidal curve quoad the surface.

In the case of a nodal curve only, if this be a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2 = 0$, and the first equation is

$$b(-2n+2) = 4k - 2b^2 + 6t - j;$$

or, assuming $t=0$, say $j = 2(n-1)b - 2b^2 + 4k$, which may be verified; and so in the case of a cuspidal curve only, when this is a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2 = 0$, where $AC - B^2 = MP + NQ$; and the second equation is

$$c(-5n+6) = 12h - 6c^2 - 2\chi + 3\theta - 3\omega,$$

or, say $2\chi + 3\omega = (5n-6)c - 6c^2 + 12h + 3\theta$, which may also be verified.

613. We may in the first instance out of the 46 quantities consider as given the 14 quantities

$$n : \qquad b, k, f, t : c, h, \theta, \chi \quad : \beta, \gamma, i : C, B,$$

then of the 26 relations, 17 determine the 17 quantities

$$\begin{array}{llll} & a, \delta, \kappa, \rho, \sigma : j, q & : r, \omega & \\ n' : a', \delta', \kappa' & : b', f' & : c' & : \quad i'; \end{array}$$

and there remain the 9 equations

$$(18), (19), (20), (21), (22), (23), (24), (25), (28),$$

connecting the 15 quantities

$$\rho', \sigma' : k', t', j', q' : h', \theta', \chi', \omega', r' : \beta', \gamma' : C', B'.$$

Taking then further as given the 5 quantities j', χ', ω', C', B',

equations (18) and (21) give ρ', σ',
equation (19) gives $2\beta' + 3\gamma' + 3t'$,
" (20) " $4\beta' + \gamma' + \theta'$,
" (28) " $\beta' + \frac{1}{2}\theta'$,

so that, taking also t' as given, these last three equations determine β', γ', θ'; and finally

equation (22) gives k',
" (23) " h',
" (24) " q',
" (25) " r',

viz. taking as given in all 20 quantities, the remaining 26 will be determined.

614. In the case of the general surface of the order n, without singularities, we have as follow:

$$n = n,$$
$$a = n(n-1),$$
$$\delta = \tfrac{1}{2}n(n-1)(n-2)(n-3),$$
$$\kappa = n(n-1)(n-2),$$
$$n' = n(n-1)^2,$$
$$a' = n(n-1),$$
$$\delta' = \tfrac{1}{2}n(n-2)(n^2-9),$$
$$\kappa' = 3n(n-2),$$
$$b' = \tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12),$$
$$k' = \tfrac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5 + 547n^4-1058n^3+1068n^2-1214n+1464),$$
$$t' = \tfrac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960),$$
$$q' = n(n-2)(n-3)(n^2+2n-4),$$
$$\rho' = n(n-2)(n^3-n^2+n-12),$$
$$c' = 4n(n-1)(n-2),$$
$$h' = \tfrac{1}{2}n(n-2)(16n^4-64n^3+80n^2-108n+156),$$
$$r' = 2n(n-2)(3n-4),$$
$$\sigma' = 4n(n-2),$$
$$\beta' = 2n(n-2)(11n-24),$$
$$\gamma' = 4n(n-2)(n-3)(n^3-3n+16),$$

the remaining quantities vanishing.

615. The question of singularities has been considered under a more general point of view by Zeuthen, in the memoir "Recherche des singularités qui ont rapport à une droite multiple d'une surface," *Math. Annalen*, t. IV. (1871), pp. 1—20. He attributes to the surface:

A number of singular points, viz. points at any one of which the tangents form a cone of the order μ, and class ν, with $y+\eta$ double lines, of which y are tangents to branches of the nodal curve through the point, and $z+\zeta$ stationary lines, whereof z are tangents to branches of the cuspidal curve through the point, and with u double planes and v stationary planes; moreover, these points have only the properties which are the most general in the case of a surface regarded as a locus of points; and Σ denotes a sum extending to all such points. (The foregoing general definition includes the cnicnodes $\mu=\nu=2$, $y=\eta=z=\zeta=u=v=0$, and the binodes $\mu=2$, $\eta=1$, $\nu=y=\&c.=0$.)

And, further, a number of singular planes, viz. planes any one of which touches along a curve of the class μ' and order ν', with $y'+\eta'$ double tangents, of which y' are generating lines of the node-couple torse, $z'+\zeta'$ stationary tangents, of which z' are generating lines of the spinode torse, u' double points and v' cusps; it is, moreover, supposed that these planes have only the properties which are the most general in the case of a surface regarded as an envelope of its tangent planes; and Σ' denotes a sum extending to all such planes. (The definition includes the cnictropes $\mu'=\nu'=2$, $y'=\eta'=z'=\zeta'=u'=v'=0$, and the bitropes $\mu'=2$, $\eta'=1$, $\nu'=y'=\&c.=0$.)

616. This being so, and writing

$$x=\nu+2\eta+3\zeta,\quad x'=\nu'+2\eta'+3\zeta',$$

the equations (7), (8), (9), (10), (11), (12), contain, in respect of the new singularities additional terms, viz. these are

$$\begin{aligned}
a(n-2)&=\ldots+\Sigma\,[x(\mu-2)-\eta-2\zeta],\\
b(n-2)&=\ldots+\Sigma\,[y(\mu-2)],\\
c(n-2)&=\ldots+\Sigma\,[z(\mu-2)],\\
a(n-2)(n-3)&=\ldots+\Sigma\,[x(-4\mu+7)+2\eta+4\zeta],\\
b(n-2)(n-3)&=\ldots+\Sigma\,[y(-4\mu+8)]-\Sigma'(4u'+3v'),\\
c(n-2)(n-3)&=\ldots+\Sigma\,[z(-4\mu+9)]-\Sigma'(2v'),
\end{aligned}$$

and there are of course the reciprocal terms in the reciprocal equations (18), (19), (20), (21), (22), (23). These formulæ are given without demonstration in the memoir just referred to: the principal object of the memoir, as shown by its title, is the consideration not of such singular points and planes, but of the multiple right lines of a surface; and in regard to these, the memoir should be consulted.

751.

NOTE ON RIEMANN'S PAPER "VERSUCH EINER ALLGEMEINEN AUFFASSUNG DER INTEGRATION UND DIFFERENTIATION*."

[From the *Mathematische Annalen*, t. XVI. (1880), pp. 81, 82.]

THE Editors of Riemann's works remark that the paper in question was contained in a MS. of his student time (dated 14 Jan. 1847) and was probably never intended for publication: indeed that he would not in later years have recognised the validity of the principles upon which it is founded. The idea is however a noticeable one: Riemann considers z_{x+h}, a function of $x+h$, expanded in a doubly infinite, necessarily divergent, series of integer or fractional powers of h, according to the law

$$z_{x+h} = \sum_{\nu=-\infty}^{\nu=+\infty} k_\nu \partial^\nu{}_x z \,.\, h^\nu, \tag{2}$$

where the meaning is explained to be that the exponents differ from each other by integer values, in effect, that ν has all the values $\alpha+p$, α a given integer or fractional value, and p any integer number from $-\infty$ to $+\infty$, zero included.

Riemann deduces a theory of fractional differentiation: but without considering the question which has always appeared to me to be the great difficulty in such a theory: what is the real meaning of a complementary function containing an infinity of arbitrary constants? or, in other words, what is the arbitrariness of the complementary function of this nature which presents itself in the theory?

I wish to point out the relation between the paper referred to, and a short paper of my own "On a doubly infinite Series," *Quart. Math. Journ.* t. VI. (1851), pp. 45—47, [102]: this commences with the remark "The following completely paradoxical investigation of the properties of the function Γ (which I have been in possession

* *Werke*, pp. 331—344.

of for some years) may perhaps be found interesting from its connexion with the theories of expansion and divergent series." And I then give the expansion

$$C_n e^x = \Sigma^r [n-r]^r x^{n-1-r},$$

where n is any integer or fractional number whatever, and the summation extends to all positive and negative integer values (zero included) of r. And I remark that, n being an integer, we have $C_n = \Gamma(n)$, and hence that assuming that this is so in general, or writing

$$\Gamma(n) . e^x = \Sigma^r [n-1]^r x^{n-1-r},$$

we have this equation as a definition of $\Gamma(n)$. The point of resemblance of course is that we have a doubly infinite expansion of e^x in a series of integer or fractional powers of x, corresponding to Riemann's like expansion of z_{x+h} in powers of h.

Cambridge, 10 *Sept.* 1879.

752.

ON THE FINITE GROUPS OF LINEAR TRANSFORMATIONS OF A VARIABLE; WITH A CORRECTION.

[From the *Mathematische Annalen*, t. XVI. (1880), pp. 260—263; 439, 440.]

IN the paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Math. Ann.* t. XII. (1877), pp. 23—46, Prof. Gordan gave in a very elegant form the groups of 12, 24 and 60 homographic transformations $\frac{ax+b}{cx+d}$. The groups of 12 and 24 are in the like form, the group of 24 thus containing as part of itself the group of 12; but the group of 60 is in a different form, not containing as part of itself the group of 12. It is, I think, desirable to present the group of 60 in the form in which it contains as part of itself Gordan's group of 12: and moreover to identify the group of 60 with the group of the 60 positive permutations of 5 letters: or (writing *abc* for the cyclical permutation *a* into *b*, *b* into *c*, *c* into *a*, and so in other cases) say with the group of the 60 positive permutations 1, *abc*, *ab.cd* and *abcde*.

Any two forms of a group are, it is well known, connected as follows, viz. if $1, \alpha, \beta, \ldots$ are the functional symbols of the one form, then those of the other form are $1, \vartheta\alpha\vartheta^{-1}, \vartheta\beta\vartheta^{-1}, \ldots$ $\left(\text{where in the case in question } \vartheta \text{ is a functional symbol of the like homographic form, } \vartheta x = \frac{Ax+B}{Cx+D}\right)$. But instead of obtaining the new form in this manner, I found it easier to use the values of the rotation-symbol

$$\cos\frac{\pi}{q} + \sin\frac{\pi}{q}(i\cos X + j\cos Y + k\cos Z)$$

for the axes of the icosahedron or dodecahedron, given in my paper "Notes on polyhedra," *Quart. Math. Jour.* t. VII. (1866), pp. 304—316, [375]; viz. if for any axes, λ, μ, ν denote the parameters of rotation $\tan\frac{\pi}{q}\cos X$, $\tan\frac{\pi}{q}\cos Y$, $\tan\frac{\pi}{q}\cos Z$, then,

by a formula which is in fact equivalent to that given in my note "On the correspondence of Homographies and Rotations," *Math. Annalen*, t. XV. (1879), pp. 238—240, [660], the corresponding homographic function of x is

$$\frac{(-\nu-i)\,x+\lambda+i\mu}{(\lambda-i\mu)\,x+\nu-i},$$

where i denotes $\sqrt{-1}$ as usual.

The new formulæ for the group of 60, or icosahedron group, of homographic functions $\dfrac{\alpha x+\beta}{\gamma x+\delta}$ are contained in the following table, where the four columns show the values of the coefficients α, β, γ, δ respectively: and where in the outside column, the substitution is represented as a permutation-symbol on the five letters *abcde*: moreover for shortness Θ is written to denote $\sqrt{5}$.

THE GROUP OF 60.

	α	β	γ	δ	
1	1	0	0	1	1
2	-1	0	0	1	*ab . cd*
3	0	1	1	0	*ac . bd*
4	0	-1	1	0	*ad . bc*
5	2	$-3+\Theta+i(1-\Theta)$	$-3+\Theta+i(-1+\Theta)$	-2	*bc . de*
6	2	$-3+\Theta+i(-1+\Theta)$	$-3+\Theta+i(1-\Theta)$	-2	*ae . bc*
7	2	$3-\Theta+i(-1+\Theta)$	$3-\Theta+i(1-\Theta)$	-2	*ad . ce*
8	2	$3-\Theta+i(1-\Theta)$	$3-\Theta+i(-1+\Theta)$	-2	*ad . be*
9	2	$-1-\Theta+i(1-\Theta)$	$-1-\Theta+i(-1+\Theta)$	-2	*ae . cd*
10	2	$-1-\Theta+i(-1+\Theta)$	$-1-\Theta+i(1-\Theta)$	-2	*ab . de*
11	2	$1+\Theta+i(-1+\Theta)$	$1+\Theta+i(1-\Theta)$	-2	*be . cd*
12	2	$1+\Theta+i(1-\Theta)$	$1+\Theta+i(-1+\Theta)$	-2	*ab . ce*
13	2	$-1-\Theta+i(-3-\Theta)$	$-1-\Theta+i(3+\Theta)$	-2	*ac . be*
14	2	$-1-\Theta+i(3+\Theta)$	$-1-\Theta+i(-3-\Theta)$	-2	*bd . ce*
15	2	$1+\Theta+i(3+\Theta)$	$1+\Theta+i(-3-\Theta)$	-2	*ae . bd*
16	2	$1+\Theta+i(-3-\Theta)$	$1+\Theta+i(3+\Theta)$	-2	*ac . de*
17	$-i$	i	1	1	*abc*
18	-1	i	1	i	*acb*
19	1	$-i$	1	i	*adc*
20	$-i$	$-i$	1	-1	*acd*
21	i	i	1	-1	*adb*
22	1	i	1	$-i$	*abd*
23	-1	$-i$	1	$-i$	*bcd*
24	i	$-i$	1	1	*bdc*

	α	β	γ	δ	
25	$-1-\theta+i(3+\theta)$	2	-2	$-1-\theta+i(-3-\theta)$	*aec*
26	$1+\theta+i(3+\theta)$	2	-2	$1+\theta+i(-3-\theta)$	*ace*
27	$1+\theta+i(-3-\theta)$	2	-2	$1+\theta+i(3+\theta)$	*bed*
28	$-1-\theta+i(-3-\theta)$	2	-2	$-1-\theta+i(3+\theta)$	*bde*
29	$-3+\theta+i(1-\theta)$	2	2	$3-\theta+i(1-\theta)$	*bec*
30	$-3+\theta+i(-1+\theta)$	2	2	$3-\theta+i(-1+\theta)$	*bce*
31	$3-\theta+i(-1+\theta)$	2	2	$-3+\theta+i(-1+\theta)$	*aed*
32	$3-\theta+i(1-\theta)$	2	2	$-3+\theta+i(1-\theta)$	*ade*
33	2	$-1-\theta+i(-1+\theta)$	$1+\theta+i(-1+\theta)$		*cde*
34	2	$1+\theta+i(1-\theta)$	$-1-\theta+i(1-\theta)$		*ced*
35	2	$-1-\theta+i(1-\theta)$	$1+\theta+i(1-\theta)$		*aeb*
36	2	$1+\theta+i(-1+\theta)$	$-1-\theta+i(-1+\theta)$		*abe*
37	$-1-\theta+i(-3-\theta)$	2	2	$1+\theta+i(-3-\theta)$	*abcde*
38	$-1-\theta+i(1-\theta)$	2	2	$1+\theta+i(1-\theta)$	*acebd*
39	$-1-\theta+i(-1+\theta)$	2	2	$1+\theta+i(-1+\theta)$	*adbec*
40	$-1-\theta+i(3+\theta)$	2	2	$1+\theta+i(3+\theta)$	*aedcb*
41	$1+\theta+i(3+\theta)$	2	2	$-1-\theta+i(3+\theta)$	*adceb*
42	$1+\theta+i(-1+\theta)$	2	2	$-1-\theta+i(-1+\theta)$	*acbde*
43	$1+\theta+i(1-\theta)$	2	2	$-1-\theta+i(1-\theta)$	*aedbc*
44	$1+\theta+i(-3-\theta)$	2	2	$-1-\theta+i(-3-\theta)$	*abecd*
45	$-1-\theta+i(-1+\theta)$	2	-2	$-1-\theta+i(1-\theta)$	*acbed*
46	$-3+\theta+i(-1+\theta)$	2	-2	$-3+\theta+i(1-\theta)$	*abdce*
47	$3-\theta+i(-1+\theta)$	2	-2	$3-\theta+i(1-\theta)$	*aecdb*
48	$1+\theta+i(-1+\theta)$	2	-2	$1+\theta+i(1-\theta)$	*adebc*
49	$1+\theta+i(1-\theta)$	2	-2	$1+\theta+i(-1+\theta)$	*aecbd*
50	$3-\theta+i(1-\theta)$	2	-2	$3-\theta+i(-1+\theta)$	*acdeb*
51	$-3+\theta+i(1-\theta)$	2	-2	$-3+\theta+i(-1+\theta)$	*abedc*
52	$-1-\theta+i(1-\theta)$	2	-2	$-1-\theta+i(-1+\theta)$	*adbce*
53	2	$-3+\theta+i(-1+\theta)$	$3-\theta+i(-1+\theta)$	2	*aebdc*
54	2	$-1-\theta+i(3+\theta)$	$1+\theta+i(3+\theta)$	2	*abced*
55	2	$1+\theta+i(-3-\theta)$	$-1-\theta+i(-3-\theta)$	2	*adecb*
56	2	$3-\theta+i(1-\theta)$	$-3+\theta+i(1-\theta)$	2	*acdbe*
57	2	$-3+\theta+i(1-\theta)$	$3-\theta+i(1-\theta)$	2	*abdec*
58	2	$-1-\theta+i(-3-\theta)$	$1+\theta+i(-3-\theta)$	2	*adcbe*
59	2	$1+\theta+i(3+\theta)$	$-1-\theta+i(3+\theta)$	2	*aebcd*
60	2	$3-\theta+i(-1+\theta)$	$-3+\theta+i(-1+\theta)$	2	*acedb*

This contains (as one of five groups of 12) the group of the positive permutations of *abcd*; and, completing this into a group of 24, we have

GROUPS OF 12 AND 24.

	α	β	γ	δ	
1	1	0	0	1	1
2	-1	0	0	1	$ab\,.\,cd$
3	0	1	1	0	$ac\,.\,bd$
4	0	-1	1	0	$ad\,.\,bc$
5	$-i$	i	1	1	abc
6	-1	i	1	i	acb
7	1	$-i$	1	i	adc
8	$-i$	$-i$	1	-1	acd
9	i	i	1	-1	adb
10	1	i	1	$-i$	abd
11	-1	$-i$	1	$-i$	bcd
12	i	$-i$	1	1	bdc
13	i	0	0	1	$adbc$
14	$-i$	0	0	1	$acbd$
15	0	i	1	0	cd
16	0	i	-1	0	ab
17	1	-1	1	1	$acdb$
18	$-i$	-1	1	i	bd
19	i	1	1	i	$abcd$
20	1	1	1	-1	bc
21	-1	-1	1	-1	$abdc$
22	i	-1	1	$-i$	ac
23	$-i$	1	1	$-i$	$adcb$
24	-1	1	1	1	ad

The groups of 60 and 24 thus each of them contain the group of 12,

$$\pm x,\quad \pm\frac{1}{x},\quad \pm i\frac{1-x}{1+x},\quad \pm i\frac{1+x}{1-x},\quad \pm\frac{x+i}{x-i},\quad \pm\frac{x-i}{x+i}.$$

It may be remarked that, to verify the periodicities of the forms contained in the group of 60, we have as the conditions that

$\dfrac{\alpha x+\beta}{\gamma x+\delta}$ may be periodic of the order 2, $\dfrac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}=0$, that is, $\alpha+\delta=0$,

" " " 3, " $=1$,

" " " 5, " $=\frac{1}{2}(3+\sqrt{5})$.

For instance, in the form

$$\frac{[-1-\Theta+i(-3-\Theta)]\,x+2}{2x+[1+\Theta+i(-3-\Theta)]},$$

we have

$$\alpha\delta=-(1+\Theta)^2-(3+\Theta)^2,\quad =-20-8\Theta,\quad \beta\gamma=4,$$

$$\alpha+\delta=-2i(3+\Theta):$$

and therefore

$$\frac{(\alpha+\delta)^2}{\alpha\delta-\beta\gamma}=\frac{-4(3+\Theta)^2}{-8(3+\Theta)},\quad =\frac{3+\Theta}{2}=\tfrac{1}{2}(3+\sqrt{5}),$$

as it should be.

Cambridge, 11 *Nov.* 1879.

CORRECTION*, pp. 439, 440.

I erroneously assumed that the symbol *adcb* could be taken as corresponding to the linear transformation ix: but this was obviously wrong, for it gave *bd* as corresponding to the transformation $-ix$, and these are not of the same order, but of the orders 4 and 2 respectively. The proper symbol is *adbc*, as given above, and the remaining eleven symbols are then at once obtained.

Cambridge, 17 *Feb.* 1880.

[* The correction in the Table of the Groups of 12 and 24 has been inserted in the Table as now printed on p. 240; it applies to the second half of the column of symbols on the extreme right-hand.]

753.

ON A THEOREM RELATING TO THE MULTIPLE THETA-FUNCTIONS.

[From the *Mathematische Annalen*, t. XVII. (1880), pp. 115—122.]

I PROPOSE—partly for the sake of the theorem itself, partly for that of the notation which will be employed—to demonstrate the general theorem (3′), p. 4, of Dr Schottky's *Abriss einer Theorie der Abel'schen Functionen von drei Variabeln*, (Leipzig, 1880), which theorem is there presented in the form:

$$e^{-\eta(u_1,\ldots;\,\mu',\,\nu')}\,\Theta\,(u_1+2\overline{\omega}_1{}',\ldots;\ \mu,\ \nu)=e^{-2\pi i\Sigma\mu_a\nu'_a}\,\Theta\,(u_1,\ldots;\ \mu+\mu',\ \nu+\nu'),\qquad(3')$$

but which I write in the slightly different form

$$\exp.\,[-H\,(u;\ \mu',\ \nu')]\,.\,\Theta\,(u+2\varpi';\ \mu,\ \nu)=\exp.\,[-2\pi i\mu\nu']\,.\,\Theta\,(u;\ \mu+\mu',\ \nu+\nu').$$

I remark that the theorem is given in the preliminary paragraphs the contents of which are, as mentioned by the Author, derived from Herr Weierstrass: and that the form of the theta-function is a very general one, depending on the general quadric function

$$G\,(u_1,\ldots,u_\rho;\ n_1,\ldots,n_\rho)$$

of 2ρ variables, ρ being the number of the arguments $u_1,\ldots,u_\rho$ (in fact, the periods are not reduced to the normal form, but are arbitrary); and the characters $\nu_1,\ldots,\nu_\rho$; $\mu_1,\ldots,\mu_\rho$, instead of having each of them the value 0 or 1, have each of them any integer or fractional value whatever. The meaning of the theorem (u denoting a set or row of ρ letters $u_1,\ldots,u_\rho$, and so in other cases), is that the function

$$\Theta\,(u;\ \mu+\mu',\ \nu+\nu')$$

with the new characters $\mu+\mu'$ and $\nu+\nu'$ is, save as to an exponential factor, equal to the function $\Theta(u+2\varpi';\ \mu,\ \nu)$ with the original characters μ, ν, but with the new arguments $u+2\varpi'$.

Notation.

This is in some measure a development of the notation employed in my "Memoir on the Theory of Matrices," *Phil. Trans.* t. CXLVIII. (1858), pp. 17—37, [152] I use certain single letters u, etc. to denote sets or rows each of ρ letters, $u=(u_1, \ldots, u_\rho)$: or if, to fix the ideas $\rho=3$, then $u=(u_1,\ u_2,\ u_3)$, and so in other cases.

But I use certain other letters a, etc. to denote squares or matrices each of ρ^2 letters; thus, if $\rho=3$ as before,

$$a=\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix},$$

and in any such case the transposed matrix is denoted by the same letter enclosed in parentheses

$$(a)=\begin{vmatrix} a_{11}, & a_{21}, & a_{31} \\ a_{12}, & a_{22}, & a_{32} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix}.$$

The sum $u+v$ of the row-letters $u,\ =(u_1,\ u_2,\ u_3)$ and $v,\ =(v_1,\ v_2,\ v_3)$ denotes the row $(u_1+v_1,\ u_2+v_2,\ u_3+v_3)$: and in like manner the sum $a+b$ of the two matrices, or square-letters a and b, denotes the matrix

$$\begin{vmatrix} a_{11}+b_{11}, & a_{12}+b_{12}, & a_{13}+b_{13} \\ a_{21}+b_{21}, & a_{22}+b_{22}, & a_{23}+b_{23} \\ a_{31}+b_{31}, & a_{32}+b_{32}, & a_{33}+b_{33} \end{vmatrix};$$

and similarly for a sum of three or more terms.

The product $uv,\ =(u_1,\ u_2,\ u_3)(v_1,\ v_2,\ v_3)$, of the two row-letters u, v denotes the single term $u_1v_1+u_2v_2+u_3v_3$. We have $uv=vu$.

The product

$$au,\ =\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix}(u_1,\ u_2,\ u_3),$$

of a preceding square-letter a and a succeeding row-letter u, denotes the set or row

$$(a_{11},\ a_{12},\ a_{13})(u_1,\ u_2,\ u_3),\quad (a_{21},\ a_{22},\ a_{23})(u_1,\ u_2,\ u_3),\quad (a_{31},\ a_{32},\ a_{33})(u_1,\ u_2,\ u_3);$$

the notation ua is not employed.

The product

$$auv = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} (u_1, u_2, u_3)(v_1, v_2, v_3),$$

of a preceding square-letter a followed by the two row-letters u and v, denotes the single term

$$(a_{11}, a_{12}, a_{13})(u_1, u_2, u_3)v_1 + (a_{21}, a_{22}, a_{23})(u_1, u_2, u_3)v_2 + (a_{31}, a_{32}, a_{33})(u_1, u_2, u_3)v_3.$$

Observe that auv is not in general $= avu$; but it is easy to verify that $auv = (a)vu$; and hence if $(a) = a$, that is, if the matrix a be symmetrical, then $auv = avu$.

A product of two matrices

$$ab, = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix},$$

denotes a matrix

$$\begin{array}{r|ccc} & (b_{11}, b_{21}, b_{31}), & (b_{12}, b_{22}, b_{32}), & (b_{13}, b_{23}, b_{33}) \\ \hline (a_{11}, a_{12}, a_{13}) & \text{,,} & \text{,,} & \text{,,} \\ (a_{21}, a_{22}, a_{23}) & \text{,,} & \text{,,} & \text{,,} \\ (a_{31}, a_{32}, a_{33}) & \text{,,} & \text{,,} & \text{,,} \end{array}$$

viz. the top-line of the compound matrix is

$$(a_{11}, a_{12}, a_{13})(b_{11}, b_{21}, b_{31}), \quad (a_{11}, a_{12}, a_{13})(b_{12}, b_{22}, b_{32}), \quad (a_{11}, a_{12}, a_{13})(b_{13}, b_{23}, b_{33}),$$

and so for the other lines: or expressing this in words, we say that any *line* of the compound matrix is obtained by compounding the corresponding *line* of the first or further component matrix with the several columns of the second or nearer component matrix.

Clearly ab is not in general $= ba$. We may easily verify that $(ab) = (b)(a)$, that is, the transposed matrix (ab) is that obtained by the composition of the transposed matrix (b) as first or further matrix, with the transposed matrix (a) as second or nearer matrix. Even if a and b are each symmetrical, we do not in general have $ab = ba$, but only $(ab) = ba$, or what is the same thing, $ab = (ba)$.

In a symbol such as $abuv$, we first combine a, b into a single matrix ab, and then regard the expression as a combination such as auv: the expression denotes therefore a single term. The theory might be explained in greater detail; but the mode of working with row- and square-letters will be readily understood from what precedes.

In all that follows, u, μ, ν, μ', ν', n, ϖ', ζ are row-letters; a, b, h, ω, ω', η, η' are square-letters: a and b are symmetrical, viz. $a = (a)$, $b = (b)$.

And I write

$$
\begin{aligned}
(*)(u,\ v)^2, &= (a,\ h,\ b)(u,\ v)^2 \\
&= au^2 + 2huv + bv^2 \\
&= \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} (u_1,\ u_2,\ u_3)^2 \\
&+ 2 \begin{vmatrix} h_{11}, & h_{12}, & h_{13} \\ h_{21}, & h_{22}, & h_{23} \\ h_{31}, & h_{32}, & h_{33} \end{vmatrix} (u_1,\ u_2,\ u_3)(v_1,\ v_2,\ v_3) \\
&+ \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} (v_1,\ v_2,\ v_3)^2
\end{aligned}
$$

to denote the general quadric function of the 2ρ letters u, v, with

$$\tfrac{1}{2}\rho(\rho+1) + \rho^2 + \tfrac{1}{2}\rho(\rho+1),\ = \rho(2\rho+1)$$

coefficients. It is assumed that the determinant formed with the $\frac{1}{2}\rho(\rho+1)$ coefficients b is negative: this is the necessary and sufficient condition for the convergence of the series.

Definition of $\Theta(u;\ \mu,\ \nu)$.

$\Theta(u;\ \mu,\ \nu)$, the general theta-function with ρ arguments u, and 2ρ characters μ, ν, is the sum of a ρ-tuple series of exponentials

$$\Theta(u;\ \mu,\ \nu) = \Sigma \exp.\left[(*)(u,\ n+\nu)^2 + 2\pi i\mu(n+\nu)\right],$$

where each of the letters n, $=(n_1, \ldots, n_\rho)$, has all integer values (zero included) from $-\infty$ to $+\infty$.

The general theorem in regard to $\Theta(u;\ \mu,\ \nu)$.

This is

$$\exp.\left[-H(u;\ \mu',\ \nu')\right].\Theta(u+2\varpi';\ \mu,\ \nu) = \exp.\left[-2\pi i\mu\nu'\right].\Theta(u;\ \mu+\mu',\ \nu+\nu'),$$

establishing a relation between the function $\Theta(u;\ \mu+\mu',\ \nu+\nu')$, with arbitrary character-increments μ', ν', and the function $\Theta(u+2\varpi';\ \mu,\ \nu)$ with the original characters, but with new arguments $u+2\varpi'$. Also $H(u;\ \mu',\ \nu')$ denotes a function, linear as regards the arguments u, but quadric as regards μ' and ν'; $-2\pi i\mu\nu'$ is a single term depending only on μ and ν'; and the theorem thus is that the two functions differ only by an exponential factor. The relations between the constants will be obtained in the course of the investigation.

Demonstration.

The truth of the theorem depends on the equality of corresponding exponentials on the two sides of the equation: viz. substituting for the theta-functions their values, and comparing the exponents or arguments of the exponentials: writing also for convenience

$$G(u+2\varpi',\ n+\nu),$$

to denote the quadric function $(*)(u+2\varpi',\ n+\nu)^2$; we ought to have

$$\begin{aligned}-H(u;\ \mu',\ \nu')+G(u+2\varpi',\ n+\nu)+2\pi i\mu(n+\nu)\\ =-2\pi i\mu\nu'+G(u,\ n+\nu+\nu')+2\pi i(\mu+\mu')(n+\nu+\nu'),\end{aligned}$$

or say

$$H(u;\ \mu',\ \nu')=G(u+2\varpi',\ n+\nu)-G(u,\ n+\nu+\nu')-2\pi i(n+\nu+\nu')\mu'.$$

In this equation, if true at all, the terms containing n must destroy each other; assuming that they do so, the equation becomes

$$H(u;\ \mu',\ \nu')=G(u+2\varpi',\ \nu)-G(u,\ \nu+\nu')-2\pi i(\nu+\nu')\mu'.$$

Consider first the terms in n: the right-hand side is

$$\begin{aligned}=\ & a(u+2\varpi')^2+2h(u+2\varpi')(n+\nu)+b(n+\nu)^2\\ -au^2\ & -2hu(n+\nu+\nu')-b(n+\nu+\nu')^2-2\pi in\mu';\end{aligned}$$

and the terms herein which contain n thus are

$$\begin{aligned}&2h(u+2\varpi')n+bn^2+2bn\nu\\ -2hun\ &-bn^2-2bn(\nu+\nu')-2\pi in\mu',\\ =4h\varpi'n\ &-2bn\nu'\ -2\pi in\mu',\end{aligned}$$

which, b being symmetrical, may be written

$$=2(2h\varpi'-b\nu'-\pi i\mu')n,$$

and these terms will vanish if, and only if

$$2h\varpi'-b\nu'-\pi i\mu'=0,$$

a system of ρ equations connecting $\varpi',\ \mu',\ \nu'$.

Assuming them to be satisfied, the remaining relation,

$$H(u;\ \mu',\ \nu')=\ G(u+2\varpi',\ \nu)-G(u,\ \nu+\nu')-2\pi i(\nu+\nu')\mu',$$

becomes

$$\begin{aligned}H(u;\ \mu',\ \nu')=\ & a(u+2\varpi')^2+2h(u+2\varpi')n+b\nu^2\\ -au^2\ & -2hu(\nu+\nu')-b(\nu+\nu')^2-2\pi i(\nu+\nu')\mu'.\end{aligned}$$

Here, a and b being symmetrical, we have

$$a(u+2\varpi')^2=au^2+4a\varpi'u+4a\varpi'^2,\quad b(\nu+\nu')^2=b\nu^2+2b\nu'\nu+b\nu'^2,$$

and the value therefore is

$$= 4a(\varpi' u + \varpi'^2) + 2h(2\varpi'\nu - u\nu') - b(2\nu'\nu + \nu'^2) - 2\pi i(\nu + \nu')\mu'.$$

On the right-hand side, putting the term in h under the form

$$-2h(u + \varpi')\nu' + 2h\varpi'(2\nu + \nu'), = -2(h)\nu'(u + \varpi') + 2h\varpi'(2\nu + \nu'),$$

and the last term under the form

$$-\pi i\mu'(2\nu + \nu') - \pi i\mu'\nu',$$

the equation becomes

$$\begin{aligned} H(u;\ \mu',\ \nu') = \ & (4a\varpi' - 2(h)\nu')(u + \varpi') - \pi i\mu'\nu' \\ & + (2h\varpi' - b\nu' - \pi i\mu')(2\nu + \nu'), \end{aligned}$$

where the second line vanishes in virtue of the foregoing equation

$$2h\varpi' - b\nu' - \pi i\mu' = 0;$$

the equation thus is

$$H(u;\ \mu',\ \nu') = (4a\varpi' - 2(h)\nu')(u + \varpi') - \pi i\mu'\nu',$$

which equation, regarding therein ϖ' as a linear function of μ' and ν', shows that $H(u;\ \mu',\ \nu')$ is a function linear as regards u (and containing this only through $u + \varpi'$), but quadric as regards μ', ν'.

Introducing the new row-letter ζ', we may write

$$H(u;\ \mu',\ \nu') = 2\zeta'(u + \varpi') - \pi i\mu'\nu',$$

viz. the expression on the right-hand side is here assumed as the value of the function

$$H(u;\ \mu',\ \nu'), = G(u + 2\varpi',\ \nu) - G(u,\ \nu + \nu') - 2\pi i(\nu + \nu')\mu';$$

and the theorem then is

$$\exp.[-H(u;\ \mu',\ \nu')].\Theta(u + 2\varpi';\ \mu,\ \nu) = \exp.[-2\pi i\mu\nu'].\Theta(u;\ \mu + \mu',\ \nu + \nu'),$$

where, by what precedes,

$$\begin{aligned} 2h\varpi' - b\nu' - \pi i\mu' &= 0, \\ 2a\varpi' - (h)\nu' - \zeta' &= 0, \end{aligned}$$

2ρ equations for determining the 2ρ functions ϖ', ζ' as linear functions of μ', ν': which equations depend on the $\rho(2\rho + 1)$ constants a, b, h.

Suppose that the resulting values of ϖ' and ζ' are

$$\varpi' = \omega\mu' + \omega'\nu',$$
$$\zeta' = \eta\mu' + \eta'\nu',$$

where ω, ω', η, η' are square-letters; then, regarding a, b, h as arbitrary, the $4\rho^2$ new constants ω, ω', η, η' cannot be all of them arbitrary, but must be connected by $4\rho^2 - \rho(2\rho + 1), = \rho(2\rho - 1)$ equations.

We may regard ω, ω', η, η' as satisfying these $\rho(2\rho-1)$ equations, but as being otherwise arbitrary; the foregoing equations then are

$$2h\varpi' - b\nu' - \pi i\mu' = 0,$$
$$2a\varpi' - (h)\nu' - \zeta' = 0,$$
$$\varpi' = \omega\mu' + \omega'\nu',$$
$$\zeta' = \eta\mu' + \eta'\nu',$$

which lead to the equations connecting a, b, h with ω, ω', η, η'.

The first and second equations, substituting for ϖ' and ζ' their values, become

$$(2h\omega - \pi i)\mu' + (2h\omega' - b)\nu' = 0,$$
$$(2a\omega - \eta)\mu' + (2a\omega' - (h) - \eta')\nu' = 0,$$

or μ', ν' being arbitrary, we thus obtain the $4\rho^2$ equations

$$2a\omega - \eta = 0,$$
$$2h\omega - \pi i = 0,$$
$$2a\omega' - \eta' - (h) = 0,$$
$$2h\omega' - b = 0,$$

which are the equations in question. It is to be observed that πi is, like the other symbols, a matrix, viz. it is regarded as containing the matrix unity; or, what is the same thing, it denotes

$$\pi i \left| \begin{matrix} 1, & 0, & 0, \ldots \\ 0, & 1, & 0, \\ \vdots & & \end{matrix} \right| .$$

We can eliminate a, b, h from these equations and thus obtain the $\rho(2\rho-1)$ equations before referred to, which connect the $4\rho^2$ constants ω, ω', η, η'. I give, but without a complete explanation, the steps of the elimination.

The equation $2a\omega - \eta = 0$, may be written in the form

$$2(a\omega) - (\eta) = 0,$$

that is,

$$2(\omega)(a) - (\eta) = 0,$$

or since $(a) = a$, this is

$$2(\omega)a - (\eta) = 0;$$

from the original form, and the new form respectively, we find

$$2(\omega)a\omega - (\omega)\eta = 0, \quad 2(\omega)a(\omega) - (\eta)\omega = 0;$$

and comparing these

$$(\omega)\eta - (\eta)\omega = 0, \quad \text{(first result)}.$$

The equation $2a\omega' - \eta' - (h) = 0$, or say $(h) = -\eta' + 2a\omega'$, may be written in the form

$$h = -(\eta') + 2(a\omega'),$$

that is, since $a=(a)$,

$$h=-(\eta')+2(\omega')a;$$

and we thence deduce

$$h\omega=-(\eta')\omega+2(\omega')a\omega.$$

But from the equation $2a\omega-\eta=0$, we have $2(\omega')a\omega-(\omega')\eta=0$, and the equation thus becomes $h\omega=-(\eta')\omega+(\omega')\eta$; which, in virtue of $2h\omega-\pi i=0$, becomes

$$\tfrac{1}{2}\pi i=-(\eta')\omega+(\omega')\eta, \quad \text{(second result).}$$

From the equation above obtained, $h=-(\eta')+2(\omega')a$, we have

$$h\omega'=-(\eta')\omega'+2(\omega')a\omega';$$

in virtue of $2h\omega'-b=0$, this becomes $-2(\eta')\omega'+4(\omega')a\omega'=b$; an equation which may also be written $-2((\eta')\omega')+4((\omega')a\omega')=(b)$, or, what is the same thing, $-2(\omega')\eta'+4(\omega')(a)\omega'=(b)$; or since $(a)=a$ and $(b)=b$, this is

$$-2(\omega')\eta'+4(\omega')a\omega'=b:$$

and comparing with the original equation

$$-2(\eta')\omega'+4(\omega')a\omega'=b,$$

we obtain

$$(\omega')\eta'-(\eta')\omega'=0, \quad \text{(third result).}$$

We have thus the three systems

$$\begin{array}{lll}
(\omega)\,\eta-(\eta)\,\omega=0 \;, & \tfrac{1}{2}\rho(\rho-1) & \text{equations,} \\
(\omega')\,\eta-(\eta')\,\omega=\tfrac{1}{2}\pi i, & \rho^2 & \text{,,} \\
(\omega')\,\eta'-(\eta')\,\omega'=0 \;, & \tfrac{1}{2}\rho(\rho-1) & \text{,,}
\end{array}$$

in all $\rho(2\rho-1)$ equations. As to these systems, observe that $(\omega)\eta$, $(\eta)\omega$, etc., are all of them matrices of ρ^2 terms; each of the three systems denotes therefore in the first instance ρ^2 equations, viz. the equations obtained by equating to zero the several terms of such a matrix: but in the first system each diagonal term so equated to zero gives the identity $0=0$; and equating to zero the terms which are symmetrical in regard to the diagonal we obtain twice over, in the forms $P=0$, and $-P=0$, one and the same equation; the number of equations is thus diminished from ρ^2 to $\tfrac{1}{2}\rho(\rho-1)$; and similarly in the third system the number of equations is $=\tfrac{1}{2}\rho(\rho-1)$: but for the second system the number of equations is really $=\rho^2$. It is hardly necessary to remark that in this second system $\tfrac{1}{2}\pi i$ is as before regarded as a matrix.

The foregoing three systems of equations are in fact the equations (6) p. 4 of Dr Schottky's work.

Cambridge, 12 *July*, 1880.

754.

ON THE CONNEXION OF CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. IX. (1880), pp. 23—25.]

IN reference to a like question in the theory of the double ϑ-functions, it is interesting to show that (if not completely, at least very nearly) the single formula

$$\Pi(u, a) = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

that is,

$$\int_0 \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} = u\frac{\Theta' a}{\Theta a} + \tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)},$$

leads not only to the relation

$$\log \Theta u = \tfrac{1}{2}\log\frac{2k'K}{\pi} + \tfrac{1}{2}\left(1 - \frac{E}{K}\right)u^2 - k^2\int_0 du \int_0 du \operatorname{sn}^2 u,$$

between the functions Θ, sn, but also to the addition-equation for the function sn.

Writing in the equation a indefinitely small, and assuming only that $\operatorname{sn} a$, $\operatorname{cn} a$, $\operatorname{dn} a$ then become a, 1, 1, respectively, the equation is

$$k^2 a \int_0 \operatorname{sn}^2 u\, du = u\frac{a\Theta''0}{\Theta 0} + \tfrac{1}{2}\log\frac{\Theta u - a\Theta' u}{\Theta u + a\Theta' u},$$

$$= ua\frac{\Theta''0}{\Theta 0} - a\frac{\Theta' u}{\Theta u},$$

that is,

$$\frac{\Theta' u}{\Theta u} = u\frac{\Theta''0}{\Theta 0} - k^2\int_0 du \operatorname{sn}^2 u,$$

or, integrating from $u = 0$, this is

$$\log \Theta u = C + \tfrac{1}{2}u^2\frac{\Theta''0}{\Theta 0} - k^2\int_0 du \int_0 du \operatorname{sn}^2 u,$$

which, except as regards the determination of the constants, is the required equation for $\log \Theta u$.

Next, differentiating twice the equation for $\Pi(u, a)$, and once the equation obtained for $\frac{\Theta' u}{\Theta u}$, we have

$$k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{d}{du}\left(\frac{\operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u}\right) = \tfrac{1}{2} \frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u-a) - \tfrac{1}{2}\frac{\Theta''\Theta - \Theta'^2}{\Theta^2}(u+a),$$

and

$$\frac{\Theta''\Theta - \Theta'^2}{\Theta^2} u = \frac{\Theta''0}{\Theta 0} - k^2 \operatorname{sn}^2 u,$$

where, for shortness, $\frac{\Theta''\Theta - \Theta'^2}{\Theta^2} u$ is written to denote $\frac{\Theta'' u\, \Theta u - (\Theta' u)^2}{\Theta^2 u}$, and the like in the first equation; the right-hand side of the first equation therefore is

$$-\tfrac{1}{2} k^2 \{\operatorname{sn}^2(u-a) - \operatorname{sn}^2(u+a)\},$$

or the equation becomes

$$2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \frac{d}{du} \frac{\operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a} = \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a),$$

that is,

$$\frac{4 \operatorname{sn} u \operatorname{sn}' u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} = \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a).$$

The numerator on the left-hand side must be a symmetrical function of u, a, and hence (even if the value of $\operatorname{sn}' u$ were unknown) it would appear that $\operatorname{sn}' u$ must be a mere constant multiple of $\operatorname{cn} u \operatorname{dn} u$; assuming, however, the actual value, $\operatorname{sn}' u = \operatorname{cn} u \operatorname{dn} u$, the formula is

$$\begin{aligned}\frac{4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 a)^2} \\ &= \operatorname{sn}^2(u+a) - \operatorname{sn}^2(u-a) \\ &= \{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}\{\operatorname{sn}(u+a) - \operatorname{sn}(u-a)\}.\end{aligned}$$

The factor $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}$ becomes $= 2 \operatorname{sn} u$ for $a = 0$, and this suggests that the factor $\operatorname{sn} u$ on the left-hand side is a factor of $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}$. That $\operatorname{cn} u$ is *not* a factor hereof would follow from the properties of the period K; viz. for $u = K$, $\operatorname{cn} u = 0$, but $\{\operatorname{sn}(u+a) + \operatorname{sn}(u-a)\}$, $= 2 \operatorname{sn}(K+a)$ is not $= 0$; and, similarly, that $\operatorname{dn} u$ is *not* a factor from the properties of the period iK; hence, $\operatorname{cn} u$, $\operatorname{dn} u$ belong to the other factor $\{\operatorname{sn}(u+a) - \operatorname{sn}(u-a)\}$, and by symmetry $\operatorname{cn} a$, $\operatorname{dn} a$ belong to the first-mentioned factor. And we are thus led to assume

$$\operatorname{sn}(u+a) + \operatorname{sn}(u-a) = 2M \operatorname{sn} u \operatorname{cn} a \operatorname{dn} a,$$

$$\operatorname{sn}(u+a) - \operatorname{sn}(u-a) = 2M' \operatorname{sn} a \operatorname{cn} u \operatorname{dn} u,$$

where

$$\text{denom.} = 1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u,$$

and $MM' = 1$. Some further investigation is wanting to show that M and M' are constants, but assuming that they are so and each $= 1$, the formulæ give at once the ordinary expression for $\operatorname{sn}(u+a)$; that is, we have the addition-equation for the function sn.

755.

ON THE MATRIX $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, AND IN CONNEXION THEREWITH THE FUNCTION $\dfrac{ax+b}{cx+d}$.

[From the *Messenger of Mathematics*, vol. IX. (1880), pp. 104—109.]

IN the preceding paper, [due to Prof. W. W. Johnson,] the theory of the symbolic powers and roots of the function $\dfrac{ax+b}{cx+d}$ is developed in a complete and satisfactory manner; the results in the main agreeing with those obtained in the original memoir, Babbage, "On Trigonometrical Series," *Memoirs of the Analytical Society* (1813), Note I. pp. 47—50, and which are to some extent reproduced in my "Memoir on the Theory of Matrices," *Phil. Trans.*, t. CXLVIII. (1858), pp. 17—37, [152]. I had recently occasion to reconsider the question, and have obtained for the nth function $\phi^n x$, where $\phi x = \dfrac{ax+b}{cx+d}$, a form which, although substantially identical with Babbage's, is a more compact and convenient one; viz. taking λ to be determined by the quadric equation

$$\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc},$$

the form is

$$\phi^n(x) = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(\ \ cx-a)}.$$

The question is, in effect, that of the determination of the nth power of the matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$; viz. in the notation of matrices

$$(x_1, y_1) = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y),$$

means the two equations $x_1 = ax + by$, $y_1 = cx + dy$; and then if x_2, y_2 are derived in like manner from x_1, y_1, that is, if $x_2 = ax_1 + by_1$, $y_2 = cx_1 + dy_1$, and so on, x_n, y_n will be linear functions of x, y; say we have $x_n = a_n x + b_n y$, $y_n = c_n x + d_n y$: and the nth power of $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ is, in fact, the matrix $\begin{pmatrix} a_n, & b_n \\ c_n, & d_n \end{pmatrix}$.

In particular, we have

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2, = \begin{pmatrix} a_2, & b_2 \\ c_2, & d_2 \end{pmatrix}, = \begin{pmatrix} a^2 + bc, & b(a+d) \\ c(a+d), & d^2 + bc \end{pmatrix},$$

and hence the identity

$$\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} = 0;$$

viz. this means that the matrix

$$\begin{pmatrix} a_2 - (a+d)a + ad - bc, & b_2 - (a+d)b \\ c_2 - (a+d)c, & d_2 - (a+d)d + ad - bc \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix},$$

or, what is the same thing, that each term of the left-hand matrix is $=0$; which is at once verified by substituting for a_2, b_2, c_2, d_2 their foregoing values.

The explanation just given will make the notation intelligible and show in a general way how a matrix may be worked in like manner with a single quantity: the theory is more fully developed in my Memoir above referred to. I proceed with the solution in the algorithm of matrices. Writing for shortness $M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, the identity is

$$M^2 - (a+d)M + (ad - bc) = 0,$$

the matrix $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ being in the theory regarded as $=1$; viz. M is determined by a quadric equation; and we have consequently $M^n =$ a linear function of M. Writing this in the form

$$M^n - AM + B = 0,$$

the unknown coefficients A, B can be at once obtained in terms of α, β, the roots of the equation

$$u^2 - (a+d)u + ad - bc = 0,$$

viz. we have

$$\alpha^n - A\alpha + B = 0,$$

$$\beta^n - A\beta + B = 0;$$

or more simply from these equations, and the equation for M^n, eliminating α, β, we have

$$\begin{vmatrix} M^n, & M, & 1 \\ \alpha^n, & \alpha, & 1 \\ \beta^n, & \beta, & 1 \end{vmatrix} = 0;$$

that is,

$$M^n(\alpha-\beta)-M(\alpha^n-\beta^n)+\alpha\beta(\alpha^{n-1}-\beta^{n-1})=0.$$

But instead of α, β, it is convenient to introduce the ratio λ of the two roots, say we have $\alpha=\lambda\beta$; we thence find

$$(\lambda+1)\beta=a+d,$$

$$\lambda\beta^2=ad-bc,$$

giving

$$\frac{(\lambda+1)^2}{\lambda}=\frac{(a+d)^2}{ad-bc}$$

for the determination of λ, and then

$$\beta=\frac{a+d}{\lambda+1},$$

$$\alpha=\frac{(a+d)\lambda}{\lambda+1}.$$

The equation thus becomes

$$M^n(\lambda-1)\beta-M(\lambda^n-1)\beta^n+(\lambda^n-\lambda)\beta^{n+1}=0,$$

or we have

$$M^n=\frac{\beta^{n-1}}{\lambda-1}\{(\lambda^n-1)M-(\lambda^n-\lambda)\beta\}.$$

It is convenient to multiply the numerator and denominator by $\lambda+1$, viz. we thus have

$$M^n=\frac{\beta^{n-1}}{\lambda^2-1}[(\lambda^{n+1}-1)M+(\lambda^n-\lambda)\{M-(\lambda+1)\beta\}].$$

The exterior factor is here

$$=\frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1},$$

moreover $(\lambda+1)\beta$ is $=a+d$: hence

$$M=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix},$$

and

$$M-(\lambda+1)\beta=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}-\begin{pmatrix} a+d, & 0 \\ 0, & a+d \end{pmatrix}, =\begin{pmatrix} -d, & b \\ c, & -a \end{pmatrix};$$

the formula thus is

$$M^n=\frac{1}{\lambda^2-1}\left(\frac{a+d}{\lambda+1}\right)^{n-1}\left\{(\lambda^{n+1}-1)\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}+(\lambda^n-\lambda)\begin{pmatrix} -d, & b \\ c, & -a \end{pmatrix}\right\},$$

viz. we have thus the values of the several terms of the nth matrix

$$M^n=\begin{pmatrix} a_n, & b_n \\ c_n, & d_n \end{pmatrix};$$

and, if instead of these we consider the combinations $a_n x + b_n$ and $c_n x + d_n$, we then obtain

$$a_n x + b_n = \frac{1}{\lambda^2 - 1}\left(\frac{a+d}{\lambda+1}\right)^{n-1} \{(\lambda^{n+1} - 1)(ax+b) + (\lambda^n - \lambda)(-dx+b)\},$$

$$c_n x + d_n = \quad ,, \quad ,, \quad \{(\lambda^{n+1} - 1)(cx+d) + (\lambda^n - \lambda)(\ \ cx - a)\};$$

and in dividing the first of these by the second, the exterior factor disappears.

It is to be remarked that, if $n = 0$, the formulæ become as they should do $a_0 x + b_0 = x$, $c_0 x + d_0 = 1$; and if $n = 1$, they become $a_1 x + b_1 = ax + b$, $c_1 x + d_1 = cx + d$.

If $\lambda^m - 1 = 0$, where m, the least exponent for which this equation is satisfied, is for the moment taken to be greater than 2, the terms in $\{\ \}$ are

$$(\lambda - 1)(ax+b) + (1-\lambda)(-dx+b),$$

and

$$(\lambda - 1)(cx+d) + (1-\lambda)(\ \ cx - a);$$

viz. these are $(\lambda - 1)(a+d)x$, and $(\lambda - 1)(a+d)$, or if for $(\lambda - 1)(a+d)$ we write $(\lambda^2 - 1)\dfrac{a+d}{\lambda+1}$, the formulæ become for $n = m$

$$a_m x + b_m = \left(\frac{a+d}{\lambda+1}\right)^m x,$$

$$c_m x + d_m = \left(\frac{a+d}{\lambda+1}\right)^m;$$

viz. we have here

$$\frac{a_m x + b_m}{c_m x + d_m} = x,$$

or the function is periodic of the mth order. Writing for shortness $\vartheta = \dfrac{s\pi}{n}$, s being any integer not $= 0$, and prime to n, we have $\lambda = \cos 2\vartheta + i \sin 2\vartheta$, hence

$$1 + \lambda = 2\cos\vartheta\,(\cos\vartheta + i\sin\vartheta),$$

or $\dfrac{(1+\lambda)^2}{\lambda} = 4\cos^2\vartheta$; consequently, in order to the function being periodic of the nth order, the relation between the coefficients is

$$4\cos^2\frac{s\pi}{n} = \frac{(a+d)^2}{ad - bc}.$$

The formula extends to the case $m = 2$, viz. $\cos\frac{1}{2}(s\pi) = 0$, or the condition is $a + d = 0$. But here $\lambda + 1 = 0$, and the case requires to be separately verified. Recurring to the original expression for M^2, we see that, for $a + d = 0$, this becomes

$$\begin{vmatrix} a^2 + bc, & 0 \\ 0 \ , & d^2 + bc \end{vmatrix}, \ = (a^2 + bc)\begin{vmatrix} 1, & 0 \\ 0, & 1 \end{vmatrix};$$

that is,

$$\frac{a_2 x + b_2}{c_2 x + d_2} = x,$$

or the result is thus verified.

But the case $m=1$ is a very remarkable one; we have here $\lambda=1$, and the relation between the coefficients is thus $(a+d)^2=4(ad-bc)$, or what is the same thing $(a-d)^2+4bc=0$. And then determining the values for $\lambda=1$ of the vanishing fractions which enter into the formulæ, we find

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(ax+b)+(n-1)(-dx+b)\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\{(n+1)(cx+d)+(n-1)(\quad cx-a)\},$$

or as these may also be written

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{x[n(a-d)+(a+d)]+2nb\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\{x\,.\,2nc+[-n(a-d)+a+d]\},$$

which for $n=0$, become as they should do $a_0x+b_0=x$, $c_0x+d_0=1$, and for $n=1$ they become $a_1x+b_1=ax+b$, $c_1x+d_1=cx+d$. We thus do *not* have $\dfrac{a_1x+b_1}{c_1x+d_1}=x$, and the function is *not* periodic of any order. This remarkable case is noticed by Mr Moulton in his edition (2nd edition, 1872) of Boole's *Finite Differences.*

If to satisfy the given relation $(a-d)^2+4bc=0$, we write $2b=k(a-d)$, $2c=-\dfrac{1}{k}(a-d)$. then the function of x is

$$\frac{ax+\frac{1}{2}k(a-d)}{-\frac{1}{2}k^{-1}(a-d)x+d},$$

and the formulæ for the nth function are

$$a_nx+b_n=\frac{1}{2^n}(a+d)^{n-1}\{(a+d)x+n(a-d)(x+k)\},$$

$$c_nx+d_n=\frac{1}{2^n}(a+d)^{n-1}\left\{(a+d)\quad-n(a-d)\left(\frac{x}{k}+1\right)\right\};$$

which may be verified successively for the different values of n.

Reverting to the general case, suppose $n=\infty$, and let u be the value of $\phi^\infty(x)$. Supposing that the modulus of λ is not $=1$, we have λ^n indefinitely large or indefinitely small. In the former case, we obtain

$$u=\frac{\lambda(ax+b)+(-dx+b)}{\lambda(cx+d)+(\quad cx-a)},\quad=\frac{(\lambda a-d)x+b(\lambda+1)}{c(\lambda+1)x+\lambda d-a};$$

which, observing that the equation in λ may be written

$$\frac{\lambda a-d}{c(\lambda+1)}=\frac{b(\lambda+1)}{\lambda d-a},$$

is independent of x, and equal to either of these equal quantities; and if from these two values of u we eliminate λ, we obtain for u the quadric equation

$$cu^2 - (a-d)u - b = 0,$$

that is,

$$u = \frac{au+b}{cu+d},$$

as is, in fact, obvious from the consideration that n being indefinitely large the nth and $(n+1)$th functions must be equal to each other. In the latter case, as λ^n is indefinitely small, we have the like formulæ, and we obtain for u the same quadric equation: the two values of u are however not the same, but (as is easily shown) their product is $= -b \div c$; u is therefore the other root of the quadric equation. Hence, as n increases, the function $\phi^n x$ continually approximates to one or the other of the roots of this quadric equation. The equation has equal roots if $(a-d)^2 + 4bc = 0$, which is the relation existing in the above-mentioned special case; and here $u = \frac{1}{2c}(a-d), = \frac{-2b}{a-d}$, which result is also given by the formulæ of the special case on writing therein $n = \infty$.

756.

A GEOMETRICAL CONSTRUCTION RELATING TO IMAGINARY QUANTITIES.

[From the *Messenger of Mathematics*, vol. x. (1881), pp. 1—3.]

LET A, B, C be given imaginary quantities, and let it be required to construct the roots of the quadric equation

$$\frac{1}{X-A}+\frac{1}{X-B}+\frac{1}{X-C}=0.$$

The equation is

$$(X-B)(X-C)+(X-C)(X-A)+(X-A)(X-B)=0,$$

that is,

$$3X^2-2(A+B+C)X+BC+CA+AB=0,$$

and we have therefore

$$3X-(A+B+C)=\pm\sqrt{\{(A+B+C)^2-3(BC+CA+AB)\}},$$
$$=\pm\sqrt{\{A^2+B^2+C^2-BC-CA-AB\}};$$

or as this may be written

$$X=\tfrac{1}{3}(A+B+C)\pm\sqrt{\{\tfrac{1}{3}(A+B\omega+C\omega^2).\tfrac{1}{3}(A+B\omega^2+C\omega)\}},$$

where ω is an imaginary cube root of unity,

$$=\cos 120^\circ+i\sin 120^\circ \text{ suppose.}$$

Taking an arbitrary point O as the origin, let the imaginary quantity A, $=\alpha+\alpha' i$ suppose, be represented by the point A, coordinates α and α'; and in like manner the imaginary quantities B and C by the points B and C respectively.

Then $B\omega$, $B\omega^2$ are represented by points B_1, B_2, obtained by rotating the point B about the origin through angles of 120° and 240° respectively; $C\omega^2$, $C\omega$ are repre-

sented by points C_1, C_2 obtained by rotating the point C about the origin through angles of 240° and 480° (=120°) respectively: and

$$\tfrac{1}{3}(A+B+C),\quad \tfrac{1}{3}(A+B\omega+C\omega^2),\quad \tfrac{1}{3}(A+B\omega^2+C\omega)$$

are represented by the points G, G_1, G_2 which are the C.G.'s of the triangles ABC, AB_1C_1, AB_2C_2 respectively. The formula therefore is

$$X = OG \pm \sqrt{(OG_1 . OG_2)},$$

where, if a, a′ are the coordinates of G, then OG is written to denote the imaginary quantity $a+a'i$; and the like as regards OG_1, OG_2. Taking $\sqrt{(OG_1 . OG_2)} = OH$, we then have H a point such, that the distance OH from the origin is = geometric mean of the distances OG_1, OG_2, and that the radial direction* of the distance OH bisects the radial directions of the distances OG_1, OG_2 respectively. Finally, measuring off from G in the radial direction OH, and in the opposite radial direction, the distances GX', GX'' each $= OH$; we have the two points X', X'' representing the two roots X.

The construction is somewhat simplified if we take for the origin the point G; for then $OG=0$, and we have $X = \pm\sqrt{(GG_1 . GG_2)}$, so that the points X', X'' are in fact the point H, and the opposite point in regard to G.

The theory of the more general equation

$$\frac{p}{X-A}+\frac{q}{X-B}+\frac{r}{X-C}=0,$$

(p, q, r real) is somewhat similar, but the construction is less simple; we have

$$(p+q+r)X^2-\{(q+r)A+(r+p)B+(p+q)C\}X+pBC+qCA+rAB=0.$$

Writing herein $q+r$, $r+p$, $p+q=l$, m, n, the equation becomes

$$(l+m+n)X^2-2(lA+mB+nC)X+(-l+m+n)BC+(l-m+n)CA+(l+m-n)AB=0,$$

that is,

$$\{(l+m+n)X-lA-mB-nC\}^2$$
$$=(lA+mB+nC)^2+\{l^2-(m+n)^2\}BC+\{m^2-(n+l)^2\}CA+\{n^2-(l+m)^2\}AB.$$

Here the right-hand side is

$$=l^2A^2+m^2B^2+n^2C^2+(l^2-m^2-n^2)BC+(-l^2+m^2-n^2)CA+(-l^2-m^2+n^2)AB,$$

which is

$$=-l^2(C-A)(A-B)-m^2(A-B)(B-C)-n^2(C-A)(A-B),$$

and consequently is a product of two linear factors; these, in fact, are

$$\frac{1}{l}\{l^2A+\tfrac{1}{2}(-l^2-m^2+n^2\pm\sqrt{\Delta})B+\tfrac{1}{2}(-l^2+m^2-n^2\mp\sqrt{\Delta})C\},$$

* Radial direction is, I think, a convenient expression for the direction of a line considered as drawn as a radius of a circle from the centre, and not as a diameter in two opposite radial directions.

where

$$\Delta = l^4 + m^4 + n^4 - 2m^2n^2 - 2n^2l^2 - 2l^2m^2.$$

It is to be observed that Δ, $=(l^2 - m^2 - n^2)^2 - 4m^2n^2$, is negative; hence, calling the factors $fA + gB + hC$, $f'A + g'B + h'C$ respectively, the coefficients f, g, h, and f', g', h' are imaginary; moreover $f + g + h = 0$, $f' + g' + h' = 0$.

The values of X thus are

$$(l + m + n)X = lA + mB + nC \pm \sqrt{\{(fA + gB + hC)(f'A + g'B + h'C)\}},$$

and then passing to the geometrical representation, we have $\dfrac{lA + mB + nC}{l + m + n}$ represented by the point which is the C.G. of weights l, m, n at the points A, B, C respectively; on account of the imaginary values of the coefficients the construction is not immediately applicable to the factors

$$fA + gB + hC, \quad f'A + g'B + h'C;$$

but a construction, such as was used for the factors

$$A + \omega B + \omega^2 C, \quad A + \omega^2 B + \omega C,$$

might be found without difficulty.

757.

ON A SMITH'S PRIZE QUESTION, RELATING TO POTENTIALS.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 15—18.]

A SPHERICAL *shell is divided by a plane into two segments A and B, one of them so small that it may be regarded as a plane disk: trace the curves which exhibit the potentials of the two segments and of the whole shell respectively, in regard to a point P moving along the axis of symmetry of the two segments.*

Criticise the following argument:

The potential of the segment A in regard to a point P, coordinates (x, y, z), is one and the same function of (x, y, z) whatever be the position of P; similarly the potential of the segment B in regard to the same point P is one and the same function of (x, y, z) whatever be the position of P: hence the potential of the whole shell in regard to the point P is one and the same function of (x, y, z) whatever be the position of P.

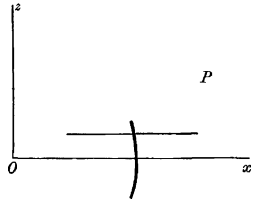
The question is taken from my memoir "On Prepotentials," *Phil. Trans.* vol. 165 (1875), pp. 675—774, [607]; and the figure of the curves is given p. 689*. There is no difficulty in tracing them by means of the expression for the potential of a plane circular disk in regard to a point on its axis of symmetry: it was in order that they might be so traced, that one of the segments was taken to be small; but I had overlooked the circumstance that the formula for the disk is in fact only a particular case of a similar and equally simple formula for the spherical segment: viz. (as was found in one of the papers) the potential of a spherical segment in regard to a point on the axis is $=\frac{2\pi a}{\rho}(\rho_1 \sim \rho_2)$, where ρ, ρ_1, ρ_2 are the distances of the attracted point from the centre of the sphere and from the centre and the circumference respectively of the segment. The segments might therefore just as well have been any two segments whatever, or (to take the most symmetrical case) they might have been hemispheres.

As to the argument: the assertion in regard to the potential of the segment

[* This Collection, vol. IX. p. 333.]

A is based upon the consideration of this segment *alone*; and, on the ground that we can without crossing the segment pass from any one position of P to any other position of P, it is inferred that the potential is one and the same function of the coordinates, whatever be the position of P: it is therefore unassailable by any considerations in relation to the non-existent segment B. Similarly the assertion in regard to the potential of the segment B is based upon the consideration of this segment alone, and it is unassailable upon any considerations in regard to the non-existent segment A: the potential of the whole sphere is certainly the sum of the potentials of the segments A and B: it is therefore altogether off the purpose to object that in the case of the whole sphere we cannot pass from a point outside the sphere to a point inside the sphere without crossing one or other of the segments A and B. I consider that the two assertions are each of them true, and that the conclusion is a legitimate one, but it is true only in the sense in which $a+x+\sqrt{[(a-x)^2]}$ is one and the same function of x whatever be the value of x: this is so, if $\sqrt{[(a-x)^2]}$ denotes indifferently or successively the two functions $\pm(a-x)$: but if, a and x being real, $\sqrt{[(a-x)^2]}$ is taken to mean the positive value, then the function $a+x+\sqrt{[(a-x)^2]}$ is $=2a$ or $=2x$ according as $a-x$ is positive or negative.

Fig. 1.



In further illustration, let the dark line of fig. 1 represent the intersection of an unclosed surface, or segment, by the plane of xz taken to be that of the paper, and consider the potential of the segment in regard to a point P in the plane of the paper, coordinates x, z. We have the potential V defined as a function of x, z by an equation $V=$ a definite integral, depending on the parameters x, z, and being in general a transcendental function of (x, z); V is a real, one-valued, finite, continuous function of x, z: in particular, if the point P, moving in any manner, traverses the dark line, there is not any discontinuity in the value of V. There is however in this case a discontinuity in the differential coefficients of V: if to fix the ideas we imagine P moving parallel to the axis of x, so that z is taken to be constant and V a function of x only, then when the path of P crosses the black line there is in general an abrupt change of value in $\frac{dV}{dx}$. Taking V as a coordinate y at right angles to the plane of the paper, a section by any plane parallel to that of xy is (when the trace of the plane upon that of xz does not meet the dark line) a continuous curve; but when the trace meets the dark line, then for this value of x there is an abrupt change of direction in the section.

If (as may very well happen in particular cases) V is algebraically determinable, then, *quâ* one-valued function of (x, z), V is not any root y at pleasure of an algebraical equation $\phi(x, y, z) = 0$, but it is for any given values of (x, z), some one determinate root y_1 of this equation: and we thus see how in this case the before-mentioned discontinuity in the value of $\frac{dV}{dx}$ must arise: viz. when the trace of the plane meets the dark line the section is a curve having a double point; and, for the positions of P on the two sides of the dark line, we have V the ordinate belonging to different branches of the curve of section. If the path of P passes through an extremity of the dark line, then the curve of section will, instead of a double point, have in general a cusp; and when the path of P does not cross the dark line, then the curve of section is a continuous line without singularity. It may be added that the surface $\phi(x, y, z) = 0$ must have a nodal line which as to a certain finite portion thereof is crunodal, giving the before-mentioned double points of the sections, but as to the residue thereof is acnodal or isolated.

It may happen that (the surface being algebraical) any particular section thereof, instead of being a single curve having a double point as above, breaks up into two distinct curves, so that for the two positions of P, we have V the ordinate of two distinct curves: and this is what really happens in the case of P a point on the axis of a circular disk or a spherical segment: thus in the case of the disk, taking c for the radius, and x for the distance from the centre of the disk, the formula is $V = 2\pi\{\sqrt{(c^2 + x^2)} \pm x\}$; or writing $V \div 2\pi = y$, the section is made up of the two distinct hyperbolas $y(y - 2x) = c^2$, and $y(y + 2x) = c^2$.

It may be remarked that in each case, it is only for P on the axis that the potential is algebraical.

In the case of the hemispheres, drawing OM a radius at right angles to the axis, the formula for the potential of an axial point P is of the form

$$V = \frac{2\pi \,.\, OM}{OP}(PM \sim PA),$$

or writing $V = 2\pi y$ we have for the hemisphere A, the curve (1) or (2) according as $(x - a)$ is positive or negative; and for the hemisphere B the curve (3) or (4) according as $x + a$ is positive or negative; viz. the equations are

$$(1) \quad y = \frac{a}{x}\{\sqrt{(a^2 + x^2)} - (x - a)\},$$

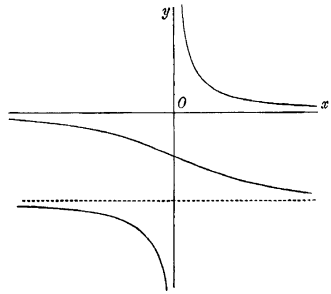
$$(2) \quad y = \frac{a}{x}\{\sqrt{(a^2 + x^2)} + (x - a)\},$$

$$(3) \quad y = \frac{a}{x}\{\sqrt{(a^2 + x^2)} - (x + a)\},$$

$$(4) \quad y = \frac{a}{x}\{\sqrt{(a^2 + x^2)} + (x + a)\},$$

being four cubic curves. The whole curve (1) is shown in fig. 2, and the others are

Fig. 2.



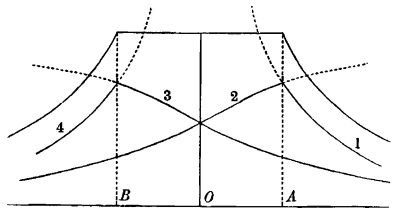
equal or opposite curves: the rationalised equation of (1) is in fact

$$x = \frac{2a^2(y+a)}{(y+a)^2 - a^2},$$

and by writing $-a$ for a, and in each equation $-x$ for x, we have the rational equations of the other three curves.

But, drawing only the required portions of the curves, we have fig. 3 exhibiting

Fig. 3.



the potentials of the two hemispheres A and B; and also the discontinuous potential of the whole shell, the ordinate for this last being the sum of the ordinates for the two hemispheres respectively.

758.

SOLUTION OF A SENATE-HOUSE PROBLEM.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 23—25.]

PROVE that, if $a+b+c=0$ and $x+y+z=0$, then

$$\begin{aligned}&4(ax+by+cz)^3\\&-3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2)\\&-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)\\&-54abcxyz=0.\end{aligned}$$

I do not know the origin of this identity, nor do I see any very simple way of proving it: that which seems the most straightforward way is to transform the third line, which, omitting the factor -2, is

$$\begin{vmatrix}1, & 1, & 1\\ a, & b, & c\\ a^2, & b^2, & c^2\end{vmatrix}\cdot\begin{vmatrix}1, & 1, & 1\\ x, & y, & z\\ x^2, & y^2, & z^2\end{vmatrix},$$

$$=\begin{vmatrix}3, & a+b+c, & a^2+b^2+c^2\\ x+y+z, & ax+by+cz, & a^2x+b^2y+c^2z\\ x^2+y^2+z^2, & ax^2+by^2+cz^2, & a^2x^2+b^2y^2+c^2z^2\end{vmatrix};$$

and therefore when $a+b+c=0$ and $x+y+z=0$, is

$$\begin{aligned}=\ &3(ax+by+cz)(a^2x^2+b^2y^2+c^2z^2)\\&-3(a^2x+b^2y+c^2z)(ax^2+by^2+cz^2)\\&-(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2);\end{aligned}$$

or, as this may be written,

$$\begin{aligned} &= 6\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\ &\quad - (ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\ &\quad - 3\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\ &\quad - 3\,(a^2x + b^2y + c^2z)\,(ax^2 + by^2 + cz^2). \end{aligned}$$

Here the third and fourth lines, omitting the factor -3, are

$$2\,(a^3x^3 + b^3y^3 + c^3z^3) + (ab^2 + a^2b)\,(xy^2 + x^2y) + (ac^2 + a^2c)\,(xz^2 + x^2z) + (bc^2 + b^2c)\,(yz^2 + y^2z),$$

where, in virtue of the two relations, each of the last three product-terms is $= abcxyz$, and the whole is thus

$$\begin{aligned} &= 2\,(a^3x^3 + b^3y^3 + c^3z^3) \\ &\quad + 3abcxyz. \end{aligned}$$

The product of the two determinants is thus

$$\begin{aligned} &= 6\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\ &\quad - (ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\ &\quad - 6\,(a^3x^3 + b^3y^3 + c^3z^3) \\ &\quad - 9\,abcxyz\,; \end{aligned}$$

and this being so the identity to be verified is

$$\begin{aligned} &4\,(ax + by + cz)^3 \\ +(-3 + 2 =)\,&-1\,(ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\ &- 12\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\ &+ 12\,(a^3x^3 + b^3y^3 + c^3z^3) \\ +(18 - 54 =)\,&- 36abcxyz \qquad = 0. \end{aligned}$$

We have here the terms

$$\begin{aligned} &12\,(a^3x^3 + b^3y^3 + c^3z^3 - 3abcxyz), \\ &= 12\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2 - bcyz - cazx - abxy), \end{aligned}$$

so that the left-hand side is now divisible by $ax + by + cz$, and throwing out this factor the equation becomes

$$\begin{aligned} &4\,(ax + by + cz)^2 \\ &- (a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\ &- 12\,(a^2x^2 + b^2y^2 + c^2z^2) \\ &+ 12\,(a^2x^2 + b^2y^2 + c^2z^2 - bcyz - cazx - abxy) = 0\,; \end{aligned}$$

or, as this may be written,

$$\begin{aligned} & 4(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy) \\ - \quad & (a^2+b^2+c^2)(x^2+y^2+z^2)=0, \end{aligned}$$

which under the assumed relations $a+b+c=0$, $x+y+z=0$ may be verified without difficulty. It may be remarked that we have identically

$$\begin{aligned} & 8(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy) \\ & -2(a^2+b^2+c^2)(x^2+y^2+z^2) \\ = & (x+y+z)\left\{\begin{array}{l} x(\ 3a^2-\ b^2-\ c^2+2bc-2ca-2ab) \\ +y(-\ a^2+3b^2-\ c^2-2bc+2ca-2ab) \\ +z(-\ a^2-\ b^2+3c^2-2bc-2ca+2ab) \end{array}\right\} \\ + & (a+b+c)\left\{\begin{array}{l} a(\ 3x^2-\ y^2-\ z^2+2yz-2zx-2xy) \\ +b(-\ x^2+3y^2-\ z^2-2yz+2zx-2xy) \\ +c(-\ x^2-\ y^2+3z^2-2yz-2zx+2xy) \end{array}\right\}, \end{aligned}$$

which is a more complete form of the last-mentioned theorem.

759.

ILLUSTRATION OF A THEOREM IN THE THEORY OF EQUATIONS.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 111—113.]

THE knowledge of the value of an unsymmetrical function of the roots of a numerical equation adds something to what is given by the equation itself; but it may or may not add anything to what is given by the equation itself in regard to each root separately. If, for instance, α, β, γ being the roots of a cubic equation, it is known that $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=$ a given value k, then α, β, γ must denote the roots, taken not in any order whatever, nor yet in a uniquely determinate order, but with a certain restriction as to order, viz. if the roots in a certain order are a, b, c, these roots being such that $a^2b+b^2c+c^2a=k$, then clearly the relation in question $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=k$, will be satisfied if α, β, $\gamma=a$, b, c, or $=b$, c, a, or $=c$, a, b (but not if α, β, $\gamma=b$, a, c, or $=$either of the remaining two arrangements); the relation thus allows α to be $=a$, or $=b$, or $=c$; that is, α is $=$any one at pleasure of the roots of the cubic equation, and it is thus determined by the cubic equation, and not by any inferior equation; but α being known, the other two roots β and γ will be uniquely, and therefore rationally, determined.

It is worth while to see how the result works out; suppose, for greater simplicity, the cubic equation is $x^3-7x+6=0$ having roots $(1, 2, -3)$, and that the given relation is $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=-1$, then the cubic equation gives

$$\alpha+\beta+\gamma=0,\quad \alpha\beta+\alpha\gamma+\beta\gamma=-7,\quad \alpha\beta\gamma=-6,$$

and we have, besides, the relation in question

$$\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=-1\,;$$

eliminating γ we have

$$\alpha^2+\alpha\beta+\beta^2=7,\quad \alpha\beta(\alpha+\beta)=6,\quad \alpha^3+3\alpha^2\beta-\beta^3+1=0\,;$$

or, as it is convenient to write these equations,

$$\beta^2+\alpha\beta+\alpha^2-7=0,$$

$$\beta^2+\alpha\beta-\frac{6}{\alpha}=0,$$

$$\beta^3-3\alpha^2\beta-\alpha^3-1=0.$$

If from these equations we eliminate β, we obtain two equations in α, which it might be supposed would determine α uniquely; but, by what precedes, α is any root at pleasure of the cubic equation and can thus be determined only by the cubic equation itself, and it follows that any equation obtained by the elimination of β must contain as a factor the cubic function $\alpha^3-7\alpha+6$, and be thus of the form $M(\alpha^3-7\alpha+6)=0$, where M is a function of α; one result of the elimination is $\alpha^3-7\alpha+6=0$, and every other result is of the form just referred to, $M(\alpha^3-7\alpha+6)=0$; hence we have definitely $\alpha^3-7\alpha+6=0$, viz. the roots of the equation $M=0$ do not apply to the question.

In verification, observe that the first and second equations give $\alpha^2-7=\frac{6}{\alpha}$, that is, $\alpha^2-6\alpha+7=0$. To eliminate β from the first and third equations we first find

$$\alpha\beta^2+(4\alpha^2-7)\,\beta+\alpha^3+1=0,$$

or say

$$\beta^2+\left(4\alpha-\frac{7}{\alpha}\right)\beta+\alpha^2+\frac{1}{\alpha}=0,$$

and combining herewith the first equation

$$\beta^2+\alpha\beta+\alpha^2-7=0,$$

we obtain

$$\beta\left(3\alpha-\frac{7}{\alpha}\right)+7+\frac{1}{\alpha}=0,$$

that is,

$$\beta=\frac{7\alpha+1}{-3\alpha^2+7};$$

substituting in the first equation,

$$\begin{aligned}&(7\alpha+1)^3\\&+\alpha(7\alpha+1)(-3\alpha^2+7)\\&+\quad(\alpha^2-7)(-3\alpha^2+7)^2=0,\end{aligned}$$

that is,

$$\begin{array}{rrrrrrr}
 & & & & 49 & 14 & 1\\
 & & -21 & -3 & +49 & +7 & \\
9 & 0 & -105 & & +343 & & -343\\
\hline
9 & 0 & -126 & -3 & +441 & +21 & -342,
\end{array}$$

or, dividing by 3,

$$3\alpha^6-42\alpha^4-\alpha^3+147\alpha^2+7\alpha-114=0,$$

which, in fact, is

$$(\alpha^3-7\alpha+6)(3\alpha^3-21\alpha-19)=0,$$

of the form in question $M(\alpha^3-7\alpha+6)=0$. Thus α has any one at pleasure of the three values 1, 2, -3, but α being known we have $\beta=\frac{7\alpha+1}{-3\alpha^2+7}$, and thence

$$\gamma=-\alpha+\frac{-7\alpha-1}{-3\alpha^2+7},\quad=\frac{3\alpha^3-14\alpha-1}{-3\alpha^2+7};$$

in particular, as $\alpha=1$, then $\beta=2$ and $\gamma=-3$.

760.

REDUCTION OF $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ TO ELLIPTIC INTEGRALS.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 142, 143.]

WRITING s, c, d for the sn, cn, and dn of u to a modulus k, which will be determined, and denoting by θ a constant which will also be determined, the formula of reduction is

$$x = \frac{-1+\theta scd}{1+\theta scd}.$$

To find from this the value of y, $=\sqrt[3]{(1-x^3)}$, putting for shortness $X=\theta scd$, the formula is $x=\frac{-1+X}{1+X}$, and we thence have

$$y^3,\ =1-x^3,\ =\frac{2(1+3X^2)}{(1+X)^3},$$

where

$$\begin{aligned} 1+3X^2 &= 1+3\theta^2 s^2(1-s^2)(1-k^2s^2), \\ &= 1+3\theta^2 s^2 - 3\theta^2(1+k^2)s^4 + 3\theta^2 k^2 s^6, \end{aligned}$$

may be put equal to $(1+\theta^2 s^2)^3$, that is,

$$= 1+3\theta^2 s^2 + 3\theta^4 s^4 \qquad + \theta^6 s^6;$$

viz. this will be the case if

$$3\theta^4 = -3\theta^2(1+k^2), \quad \theta^6 = 3\theta^2 k^2;$$

that is,

$$\theta^2 = -1-k^2, \qquad \theta^4 = 3k^2;$$

these give

$$k^4 - k^2 + 1 = 0;$$

that is, $k^2=\omega$, if $\omega = -\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, an imaginary cube root of unity; and then

$$\theta^2 = -1+\omega, \quad = \omega^2(\omega^2-\omega), \quad = -i\omega^2\sqrt{3};$$

that is,

$$\theta = \pm \frac{(1-\sqrt{3}) - i(1+\sqrt{3})}{2\sqrt{2}} \sqrt[4]{3},$$

as may be verified by squaring.

Hence finally, θ and k denoting the values just obtained,

$$x = \frac{-1+\theta scd}{1+\theta scd},$$

$$y = \sqrt[3]{(1-x^3)} = \frac{\sqrt[3]{2}(1+\theta^2 s^2)}{1+\theta scd};$$

or, writing as before, $X = \theta scd$, we have

$$dx = \frac{2dX}{(1+X)^2}, \quad y^2 = \frac{2^{\frac{2}{3}}(1+\theta^2 s^2)^2}{(1+X)^2};$$

whence

$$\frac{dx}{(1-x^3)^{\frac{2}{3}}}, \quad = \frac{dx}{y^2}, \quad = \frac{2^{\frac{1}{3}} dX}{(1+\theta^2 s^2)^2},$$

and then

$$dX = \theta\{1 - 2(1+k^2)s^2 + 3k^2 s^4\}\, du, \quad = \theta(1+\theta^2 s^2)^2\, du;$$

that is,

$$\frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta \,.\, du;$$

or say

$$\int \frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta u,$$

the required formula.

761.

ON THE THEOREM OF THE FINITE NUMBER OF THE COVARIANTS OF A BINARY QUANTIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVII. (1881), pp. 137—147.]

GORDAN'S proof, the only one hitherto given, is based upon the theory of derivatives (Uebereinanderschiebungen). It is shown that the irreducible covariants of the binary quantic f are included in the series

$$(f, f)^2,\ (f, f)^4,\ \ldots (f, h),\ (f, h)^2,\ \ldots$$

of the derivatives of the quantic upon itself or upon some other covariant, and that the number of the irreducible covariants thus obtained is finite. And not only so, but for the quintic and the sextic the complete systems were formed, and the numbers shown to be $=23$ and 26 respectively.

It would seem that there ought to be a more simple proof based upon the consideration of the fundamental covariants: for the cubic $(a, b, c, d \between x, y)^3$, these are the cubic itself $(a, \ldots \between x, y)^3$, the Hessian $(ac-b^2, \ldots \between x, y)^2$, and the cubicovariant $(a^2d-3abc+2b^3, \ldots \between x, y)^3$; and so in general for the quantic $(a, \ldots \between x, y)^n$, we have a series of fundamental covariants the leading coefficients whereof are the seminvariants

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ a^3e-4a^2bd+6ab^2c-3b^4,\ \&c.$$

It is known that every covariant can be expressed as a *rational* function of these, or more precisely that every covariant multiplied by a positive integral power of the quantic itself can be expressed as a *rational* and *integral* function of the fundamental covariants, and we may for the covariants substitute their leading coefficients, or say the seminvariants; hence, every seminvariant is a *rational* function of the fundamental seminvariants, and more precisely, every seminvariant multiplied by a positive integral

power of the first coefficient a is a *rational* function of the fundamental seminvariants. Thus, in the case of the cubic, we have the discriminant ∇,

$$= a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2,$$

obtained from

$$a,\ ac - b^2,\ a^2d - 3abc + 2b^3,$$

by the formula

$$a^2\nabla = (a^2d - 3abc + 2b^3)^2 + 4(ac - b^2)^3,$$

and it is easily shown that this invariant ∇ is the only new covariant thus obtainable, and that every other covariant is thus a *rational and integral* function of the irreducible covariants, the leading coefficients of which are

$$a,\ ac - b^2,\ a^2d - 3abc + 2b^3,$$

and ∇. It appears a truism, and it might be thought that it would be, if not easy, at least practicable, to show for a quantic of any given finite order n, that we can in this manner, as *rational* functions of the $n-1$ seminvariants, obtain only a *finite* number of new seminvariants, so that all the seminvariants would be expressible as *rational and integral* functions of a finite number of seminvariants; and, consequently, all the covariants be expressible as rational and integral functions of a finite number of irreducible covariants. But the large number, 23, of the covariants of the quintic is enough to show that the proof, even if it could be carried out, would involve algebraical operations of great complexity.

The theory may be considered from a different point of view, in connexion with the root-form $a(x-\alpha y)(x-\beta y)\ldots$, or say $(x-\alpha)(x-\beta)\ldots$ of the quantic; we have here what may be called the monomial form of covariant, viz. the general monomial form is

$$(\alpha-\beta)^m(\alpha-\gamma)^n(\beta-\gamma)^p\ldots(x-\alpha)^q(x-\beta)^r\ldots,$$

where in all the factors (whether $\alpha-\beta$ or $x-\alpha$) which contain α, in all the factors which contain $\beta,\ldots$, and so for each root in succession, the sum of the indices has one and the same value, $=\theta$ suppose. Thus, for the cubic

$$(x-\alpha)(x-\beta)(x-\gamma),$$

we have the monomial covariants

$$(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma),$$

$$(\beta-\gamma)(x-\alpha),\ (\alpha-\gamma)(x-\beta),\ (\alpha-\beta)(x-\gamma),$$

$$(x-\alpha)(x-\beta)(x-\gamma);$$

and so for the quartic

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta),$$

we have the monomial invariants

$$(\alpha-\beta)(\gamma-\delta),\ (\alpha-\gamma)(\beta-\delta),\ (\alpha-\delta)(\beta-\gamma).$$

Observe that the monomial form is considered as essential; a syzygetic function of

two or more monomials is not a monomial, and we are thus in no wise concerned with identities such as

$$(\beta-\gamma)(x-\alpha)-(\alpha-\gamma)(x-\beta)+(\alpha-\beta)(x-\gamma)=0,$$

or

$$(\alpha-\delta)(\beta-\gamma)-(\beta-\delta)(\alpha-\gamma)+(\gamma-\delta)(\alpha-\beta)=0\,;$$

notwithstanding these syzygies respectively,

$$(\beta-\gamma)(x-\alpha),\ (\alpha-\gamma)(x-\beta),\ \text{and}\ (\alpha-\beta)(x-\gamma)$$

are regarded as independent covariants of the cubic, and

$$(\alpha-\delta)(\beta-\gamma),\ (\beta-\delta)(\alpha-\gamma),\ \text{and}\ (\gamma-\delta)(\alpha-\beta),$$

as independent invariants of the quartic.

It is only when a monomial covariant is equal to a power or product of simple or other powers of lower monomial covariants that it is regarded as a function of these lower monomial covariants and therefore as not irreducible. Thus

$$(\alpha-\beta)(\alpha-\gamma)(\beta-\delta)(\gamma-\delta)=(\alpha-\beta)(\gamma-\delta).(\alpha-\gamma)(\beta-\delta),$$

is a reducible monomial covariant, expressible in terms of the lower irreducible monomial covariants

$$(\alpha-\beta)(\gamma-\delta)\ \text{and}\ (\alpha-\gamma)(\beta-\delta).$$

The theorem of the finite number of the irreducible monomial covariants (as just explained) of the root-quantic is a question of the same kind as, but entirely distinct from, that of the finite number of the covariants of the quantic in the ordinary form; and there are thus the two questions; (A), that of the finite number of the irreducible monomial covariants of the root-quantic; and (C), that of the finite number of the irreducible covariants of the ordinary quantic.

But we can pass from (A) to (C) by means of a lemma (B), which I have not proved, but which seems highly probable, and which I enunciate as follows: (B) The infinite system of terms X, rational and integral functions of a finite set of letters (a, b, c, ...) which remain unaltered by all the substitutions of a certain group G(a, b, c, ...) of substitutions upon these letters, includes always a finite set of terms P such that every term X whatever is a rational and integral function of these terms P.

In explanation of this lemma, observe that, if G(a, b, c, ...) denotes the entire group of substitutions upon these letters, so that the functions which remain unaltered by the substitutions of the group are in fact the symmetrical functions of (a, b, c, ...), then the theorem is "The infinite system of rational and integral symmetrical functions of (a, b, c, ...) includes always a finite set of terms P such that every such rational and integral symmetrical function is a rational and integral function of the terms P, viz. the terms P are here the several symmetrical functions

$$\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots,\ \mathrm{ab}+\mathrm{ac}+\mathrm{bc}+\ldots,\ \mathrm{abc}+\ldots,\ \&c.\text{''};$$

and so, if $G(\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots)$ is the group of all the positive substitutions, then we have the like theorem for the rational and integral two-valued functions of $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots)$, viz. the terms P are here the two-valued function $(\mathrm{a}-\mathrm{b})(\mathrm{a}-\mathrm{c})(\mathrm{b}-\mathrm{c})\ldots$, and the symmetrical functions

$$\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots,\ \mathrm{ab}+\mathrm{ac}+\mathrm{bc}+\ldots,\ \mathrm{abc}+\ldots,\ \&\mathrm{c.},$$

as before.

I return to the theorem (A), but instead of the covariants of a root-quantic of any order, I consider first the invariants of a root-quantic of any even order. The general form is

$$(\alpha-\beta)^m(\alpha-\gamma)^n(\beta-\gamma)^p\ldots,$$

where in all the factors which contain α, in all the factors which contain β, and so for each root in succession, the sum of the indices has one and the same value $=\theta$. Writing 12 for the index of $\alpha-\beta$, 13 for that of $\alpha-\gamma$, and so in other cases, then assuming always $12=21$, $13=31$, &c., the indices, taken each twice, form the square

0	12	13	...
21	0	23	...
31	32	0	
⋮			

the order of which, or number of its rows or columns, is equal to the order of the quantic; the terms of the dexter diagonal are each $=0$, and the square is symmetrical in regard to this dexter diagonal. Moreover, the square is such, that the sum of the terms in each row (or column) has one and the same value $=\theta$; and conversely, every such square, say R_θ, represents an invariant.

Thus, for the quartic $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, the square R_θ is a square of four rows (or columns) representing the invariant

$$(\alpha-\beta)^{12}(\alpha-\gamma)^{13}(\alpha-\delta)^{14},$$
$$(\beta-\gamma)^{23}(\beta-\delta)^{24},$$
$$(\gamma-\delta)^{34},$$

in which

$$\begin{aligned} &12+13+14=\theta,\\ 21\quad &\quad\ +23+24=\theta,\\ 31+32\quad &\quad\ \ \ +34=\theta,\\ 41+42+43\quad &\quad\ \ \ \ \ \ \ =\theta. \end{aligned}$$

There are three squares R_1, viz. these are the squares

	1		
1			
			1
		1	

,

		1	
			1
1			
	1		

,

			1
		1	
	1		
1			

representing the before-mentioned invariants $(\alpha-\beta)(\gamma-\delta)$, $(\alpha-\gamma)(\beta-\delta)$, $(\alpha-\delta)(\beta-\gamma)$ respectively: say these are a, b, c, and every other invariant is a *rational and integral* function of these; in fact, the θ-equations give easily $12=34$, $13=24$, $14=23$, so that the general form of the invariant is $=a^{12}b^{13}c^{14}$, where 12, 13, 14 are each of them a positive integer number (which may be $=0$). Or, what is the same thing, the square R_θ $(\theta=12+13+14)$ is a sum

$$=12\,.\,R_1+13\,.\,R_1'+14\,.\,R_1'',$$

with positive integer coefficients 12, 13, 14, say for shortness it is a *sum* of squares R_1. And so any like expression with a negative coefficient or coefficients may, for shortness, be called a difference of squares R_1.

Observe that, in general, two squares R_θ, R_ϕ are added together by adding their corresponding terms, the result being a square $R_{\theta+\phi}$; similarly, if each term of R_ϕ be less than or at most equal to the corresponding term of R_θ, then (but not otherwise) the square R_ϕ may be subtracted from R_θ, giving a square $R_{\theta-\phi}$.

In the case of the sextic

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon)(x-\zeta),$$

there are fifteen squares R_1, which may be represented as follows:

12.34.56	x_1
12.35.46	y_1
12.36.45	z_1
13.24.56	x_2
13.25.46	y_2
13.26.45	z_2
14.23.56	x_3
14.25.36	y_3
14.26.35	z_3
15.23.46	x_4
15.24.36	y_4
15.26.34	z_4
16.23.45	x_5
16.24.35	y_5
16.25.34	z_5;

viz. 12.34.56 here represents the square R_1, for which the terms 12, 34, 56 (and of course the symmetrical terms 21, 43, 65) are each $=1$, the other terms all vanishing; or, what is the same thing, it represents the invariant $(\alpha-\beta)^{12}(\gamma-\delta)^{34}(\epsilon-\zeta)^{56}$. But it is not true that *every* square R_θ is a sum of squares R_1; this is not the case, for the square R_2,

$$=12.13.23.45.46.56,$$

representing the invariant

$$(\alpha-\beta)^{12}(\alpha-\gamma)^{13}(\beta-\gamma)^{23}(\delta-\epsilon)^{45}(\delta-\zeta)^{46}(\epsilon-\zeta)^{56},$$

is not a sum of squares R_1.

But the square last referred to is a *difference* of squares R_1: it is in fact

$$=12.36.45+13.25.46+14.23.56-14.25.36,$$

or, what is the same thing, the corresponding invariant is the product of the invariants 12.36.45, 13.25.46, 14.23.56, divided by the invariant 14.25.36; viz. it is a rational function of invariants R_1.

It is required to show, first, that *every* square R_θ is a difference of squares R_1; and thence, secondly, that it is a sum of a finite number of squares R_k (being, in fact, squares R_1 and R_2).

For the first theorem we equate the general expression of R_θ with the assumed value

$$x_1.12.34.56+y_1.12.35.46+z_1.12.36.45+\ldots+z_5.16.25.34.$$

We thus obtain

fifteen equations	satisfied by
$12=y_1+x_1+z_1$	$x_1=34-26 \quad +r+s-t,$
$13=x_2+y_2+z_2$	$x_2=13-25 \quad +p \quad -r \quad +t,$
$14=x_3+y_3+z_3$	$x_3=14 \quad -p \quad -s \quad ,$
$15=x_4+y_4+z_4$	$x_4=15-26-36+p+q+r+s \quad ,$
$16=x_5+y_5+z_5$	$x_5=45 \quad -q-r \quad ,$
$23=x_3+x_4+x_5$	$y_1=12-34+26 \quad -q-r-s+t,$
$24=x_2+y_4+y_5$	$y_2=25 \quad -p \quad ,$
$25=y_2+y_3+z_5$	$y_3= \quad p \quad ,$
$26=z_2+z_3+z_4$	$y_4=36 \quad -p-q \quad ,$
$34=x_1+z_4+z_5$	$y_5=16-45 \quad +q+r \quad -t,$
$35=y_1+y_5+z_3$	$z_1= \quad q \quad ,$
$36=y_3+y_4+z_1$	$z_2= \quad r \quad ,$
$45=x_5+z_1+z_2$	$z_3= \quad s \quad ,$
$46=x_4+y_1+y_2$	$z_4=26 \quad -r-s \quad ,$
$56=x_1+x_2+x_3$	$z_5= \quad t,$

connecting $x_1, y_1, z_1, \ldots, z_5$ with the terms 12, 13, etc. of R_θ (or indices of the corresponding invariants). The fifteen equations are not independent, for regarding them as giving the values of 12, 13, ... in terms of the $x_1, y_1, z_1, \ldots, z_5$, these values satisfy identically the relations which ought to be satisfied by the terms 12, 13, etc., viz. the equations obtained by the elimination of θ from the equations

$$\begin{array}{l} 12+13+14+15+16=\theta, \\ 12 \qquad +23+24+25+26=\theta, \\ \vdots \\ 16+26+36+46+56 \qquad =\theta. \end{array}$$

The equations are thus insufficient to determine the values of $x_1, y_1, z_1, \ldots, z_5$, and the general values given by the equations will contain five indeterminate quantities which are taken to be p, q, r, s, t (these being in fact the values of y_3, z_1, z_2, z_3, z_5 respectively), and we then have the equations all of them satisfied by the above-mentioned values containing these indeterminate quantities; taking them to be positive or negative integers, then $x_1, y_1, z_1, \ldots, z_5$, will be all of them integers; but by what precedes, it appears that they cannot all of them be made to be *positive* integers, so that we have consequently R_θ,

$$=x_1 . 12.34.56+y_1 . 12.35.46+z_1 . 12.36.45+\ldots+z_5 . 16.25.34,$$

equal in general to a difference of squares R_1.

Suppose in such difference of squares R_1 we have any term, say $-12.34.56$, occurring with the coefficient -1. Since the expression represents a square R_θ, we must have among the positive terms, $12.35.46$ or $12.36.45$ to render possible the subtraction of the 12; $15.26.34$ or $16.25.34$ to render possible the subtraction of the 34; and $13.24.56$ or $14.23.56$ to render possible the subtraction of the 56; that is, the expression must contain one of the eight combinations

$$\begin{array}{l} 12.35.46+15.26.34+13.24.56-12.34.56, \\ 12.35.46+15.26.34+14.23.56-12.34.56, \\ 12.35.46+16.25.34+13.24.56-12.34.56, \\ 12.35.46+16.25.34+14.23.56-12.34.56, \\ 12.36.45+15.26.34+13.24.56-12.34.56, \\ 12.36.45+15.26.34+14.23.56-12.34.56, \\ 12.36.45+16.25.34+13.24.56-12.34.56, \\ 12.36.45+16.25.34+14.23.56-12.34.56. \end{array}$$

The first of these is $35.46.15.26.13.24$, viz. it is $13.15.35.24.26.46$ which is a square R_2 (of the form mentioned above); the second is $35.46.15.26.14.23$, which is $15.23.46+14.26.35$, a sum of squares R_1; and similarly each of the other combinations is either a square R_2 or a sum of squares R_1. We have thus got rid of the negative term $-12.34.56$, and in like manner if the negative term had been

$$-m.12.34.56, \; =-12.34.56-12.34.56-\&c.,$$

or, whatever the negative terms may be, we get rid one by one of each negative term; and thus ultimately express R_θ as a sum of squares R_1 and R_2. Or, what is the same thing, the invariant R_θ originally expressed as a *rational* function of invariants R_1, is finally expressed as a *rational and integral* function of invariants R_1 and R_2.

Similarly for a root-quantic of any even order n, we have the general square R_θ expressed, first as a difference of squares R_1, and then as a sum of squares R_1, R_2, or it may be higher squares R_3, &c., but certainly as a sum of a *finite* number of squares R_k. For a root-quantic of any odd order n, the investigation would be of a somewhat different form, since here there are no squares R_1, but the lowest squares are squares R_2 of a form such as $12.23.34.45.15$; but the general conclusion would still follow that every square R_θ is a sum of a *finite* number of squares R_k. And a like reasoning would apply to covariants instead of invariants: viz. the reasoning (although for simplicity it has been given for a very particular and special case) does, I think, really establish the theorem (A) in its generality, viz. the theorem that for a root-quantic of any given finite order, the number of irreducible monomial covariants is finite.

From any monomial covariant of the root-quantic, by taking the sum of the forms belonging to the different roots, so as to obtain a symmetrical function of the roots, that is, a rational and integral function of the coefficients, we obtain a covariant of the quartic in its ordinary form $(a, \ldots \between x, y)^n$. Consider for a moment the before-mentioned case of the invariants of the root-quartic

$$(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y),$$

now put

$$=\frac{1}{a}(a, b, c, d, e \between x, y)^4;$$

and to make the reasoning clearer, take a, b, c, f, g, h $=(\alpha-\delta)(\beta-\gamma)$, $(\beta-\delta)(\gamma-\alpha)$, $(\gamma-\delta)(\alpha-\beta)$, $(\alpha-\delta)(\gamma-\beta)$, $(\beta-\delta)(\alpha-\gamma)$, $(\gamma-\delta)(\beta-\alpha)$ respectively, these being, with the signs $\pm$, the before-mentioned three monomial invariants. In the root-theory, every monomial invariant is a rational and integral function of a, b, c, f, g, h. Every invariant of $(a, \ldots \between x, y)^4$, *quà* rational and integral function of the coefficients, is, when expressed in terms of the roots, a rational and integral function of the roots, and then *quà* invariant is a sum of monomial invariants, and as such a rational and integral function of a, b, c, f, g, h. But every such rational and integral function of a, b, c, f, g, h is not a symmetrical function of α, β, γ, δ, and consequently not in the present theory an invariant of $(a, \ldots \between x, y)^4$; the invariants are those rational and integral functions of a, b, c, f, g, h which are symmetrical functions of $(\alpha, \beta, \gamma, \delta)$, that is, which remain unaltered by every substitution whatever upon the roots $(\alpha, \beta, \gamma, \delta)$. Now each such substitution gives a substitution upon a, b, c, f, g, h, and the 24 substitutions upon α, β, γ, δ give a group of 6, $=\frac{1}{4}.24$ substitutions upon (a, b, c, f, g, h); the invariants are thus the rational and integral functions of (a, b, c, f, g, h) which are unaltered by each of the substitutions of a certain group G(a, b, c, d, e, f) of 6 substitutions. Theorem (B) asserts that, among the terms in

question, that is, among such rational and integral functions of (a, b, c, f, g, h), we have a finite number of terms P, such that every one of the terms is a rational and integral function of the terms P; and recollecting that $a+b+c=0$, these terms P are in fact two terms $bc+ca+ab$ and $(b-c)(c-a)(a-b)$; the conclusion being, that the invariants of the quartic $(a, b, c, d, e \between x, y)^4$ are all of them rational and integral functions of the last-mentioned two functions, that is, of

$$I, = ae - 4bd + 3c^2, \text{ and } J, = ace - ad^2 - b^2e + 2bcd - c^3.$$

As regards the group $G(a, b, c, f, g, h)$ of 6 substitutions upon a, b, c, f, g, h, observe that the 24 substitutions of $(\alpha, \beta, \gamma, \delta)$ operating upon a, b, c, f, g, h give 6 substitutions taken each four times; for instance, the substitutions 1, $\alpha\beta.\gamma\delta$, $\alpha\gamma.\beta\delta$, $\alpha\delta.\beta\gamma$ leave each of them a, b, c, f, g, h unaltered, that is, they each give the substitution 1. And we thus find for the group $G(a, b, c, f, g, h)$ the 6 substitutions

1,
abc.fgh,
acb.fhg,
af.bh.cg,
ah.bg.cf,
ag.bf.ch.

For the functions of a, b, c, f, g, h, which remain unaltered by the substitution of this group, observe that we have $f, g, h = -a, -b, -c$; so that any function of the six letters may be represented as a function of a, b, c. An odd symmetrical function, for instance abc, does *not* remain unaltered, for it is by any one of the last three substitutions changed into fgh, that is, into $-abc$; on the other hand, the two-valued function $(b-c)(c-a)(a-b)$ does remain unaltered: the functions which remain unaltered are therefore the even symmetrical functions of a, b, c (that is, the symmetric functions $a^2+b^2+c^2$, or $ab+ac+bc$, &c., which are of an even order in a, b, c conjointly), and the same even functions multiplied by $(b-c)(c-a)(a-b)$; and having regard to the relation $a+b+c=0$, all these can be expressed as already mentioned as rational and integral functions of $bc+ca+ab$ and $(b-c)(c-a)(a-b)$.

The proof applies to the general case of the theorem (C), viz. taking the theorem (A) to be proved, and putting the root-quantic

$$(x-\alpha y)(x-\beta y)\ldots = \frac{1}{a}(a, \ldots \between x, y)^n,$$

then we have a, b, c, d, ... a system of monomial covariants of the root-quantic; and all the covariants of $(a, \ldots \between x, y)$ are rational and integral functions of (a, b, c, d, ...) which remain unaltered by the substitutions of a certain group $G(a, b, c, d, \ldots)$; hence, assuming the theorem (B), they are rational and integral functions of a finite number of irreducible covariants. And the demonstration thus depends upon that of the theorem (B).

762.

ON SCHUBERT'S METHOD FOR THE CONTACTS OF A LINE WITH A SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVII. (1881), pp. 244—258.]

I WISH to reproduce in part § 33, "Coincidenz von Schnittpunkten einer Geraden mit einer Fläche" of Schubert's very interesting work *Calcul der abzählende Geometrie*, Leipzig, 1879, explaining in the first instance (but not altogether in the manner or from the point of view of the author) the general principles of the theory.

We have to do with *conditions* relating to a *subject;* the subject is a geometrical form or entity of any kind depending upon a certain number of constants; and the condition is onefold, twofold, &c., according as it imposes a onefold, twofold, &c., relation upon these constants. The number of constants is the Postulandum of the subject, and the manifoldness of the condition is called also its Postulation. A condition is incomplete when its postulation is less than postulandum of subject, complete when its postulation is equal to postulandum of subject; two or more incomplete conditions, making up a complete condition, are supplementary to each other. The case where the postulation exceeds the postulandum, or say that of a more than complete condition, is not in general considered; it may however sometimes present itself. For instance, the subject may be a line with n points upon it; the number of constants is here $=n+4$. A condition that the line shall meet a given line, or that a certain one of the n points shall lie on a given plane, is a onefold condition; the condition that such point shall lie upon a given line is a twofold condition; and so in other cases.

Conditions are denoted by letters, and simultaneous conditions by a product; for instance, the subject is a line carrying the n points 1, 2, ..., n; g is the condition that the line meets a given line; p_1 the condition that the point 1 lies on a given plane; then gp_1 is the twofold condition that the line meets a given line and that

the point 1 lies on a given plane; p_1^2 is the twofold condition that the point 1 lies on each of two given planes (in fact, on their line of intersection). The letters p, g, e are used as the initials of Punkt, Gerade, Ebene.

The letter or combination of letters denoting an incomplete condition, or, say, the incomplete condition itself, has no numerical value; but for a complete condition there exists a definite number of subjects satisfying the condition, and the condition is regarded as having this number as its value. A more than complete condition has the value 0.

Conditions of the same postulation may be connected by the sign $+$; for instance,

subject a line,

g_e the condition that it lies in a given plane,

g_p the condition that it passes through a given point,

then g_e+g_p is the condition that the line shall either lie in the given plane or else pass through the given point.

I abstain from attempting any definition in regard to the sign $-$.

Conditions of the same postulation may be connected by an equation or equations; for instance,

subject a point,

p the condition that the point shall lie in a given plane,

p_g the condition that the point shall lie in a given line,

then $p^2=p_g$.

This equation has (so far) no numerical signification; it has the logical signification that the condition that a point shall lie on each of two given planes is equivalent to the condition that the point shall lie on a given line.

Second example. Subject a line,

g the condition that the line meets a given line,

g_e the condition that it lies in a given plane,

g_p the condition that it passes through a given point,

then $g^2=g_e+g_p$.

This equation has (so far) no numerical signification, and I regard it as having no logical signification. Schubert, however, gives it a logical signification by means of his "Princip der speciellen Lage" (Principle of Special Situation), viz. the condition of the line meeting each of two given lines is, in the particular case where the two given lines meet, equivalent to the condition, that the line shall either lie in the plane of the two given lines or else pass through their point of intersection.

Third example. Subject a line bearing upon it the points 1 and 2,

ϵ the condition of the coincidence of the two points,

p " that the point 1 shall lie on a given plane,

q " " 2 " "

g " that the line shall meet a given line,

then $\epsilon = p + q - g$.

This equation has (so far) no numerical signification, and it does not appear to have any logical signification. In fact, in the actual form of the equation we have a sign $-$ which has not had given to it any logical interpretation; and if we write the equation in the form $\epsilon + g = p + q$, there seems to be no logical signification in the assertion, the condition that either the points shall coincide, or else the line meet a given line, is equivalent to the condition that either the first point, or else the second point, shall lie in a given plane.

Any equation connecting complete conditions is a numerical equation; and to render a condition complete, we have only to join to it a supplementary condition X of the proper postulation. Thus, in the last example the postulandum is $=6$; ϵ, p, q, g are onefold conditions, and joining to each of them one and the same fivefold condition X, we have $X\epsilon = Xp + Xq - Xg$. And, taking X to be an arbitrary fivefold condition, the original equation $\epsilon = p + q - g$ has in fact the meaning

$$X\epsilon = Xp + Xq - Xg.$$

For instance, the fivefold condition X may be that the line shall belong to a given regulus (scroll or developable surface), and that the points 1, 2 upon the line shall be the intersections of the line with given surfaces S_1, S_2 respectively. The subject is the line of the given regulus with its two points; and the meaning of the equation is that the number of subjects with two coincident points is equal to the number of subjects with the point 1 on a given plane, *plus* the number of subjects with the point 2 on a given plane, *minus* the number of subjects for which the line meets a given line. Although for the moment concerned only with the meaning of the theorem, not with its truth, I stop to show *à posteriori* that the theorem is in fact true: take k for the order of the regulus; m_1, m_2 for the orders of the surfaces S_1, S_2 respectively; then it is to be shown that $X\epsilon$, Xp, Xq, Xg are each $= km_1m_2$ (values which satisfy the equation). First $X\epsilon$: the points 1 and 2 here coincide at a point of the curve of the order m_1m_2, which is the intersection of S_1 and S_2; the regulus meets this curve in km_1m_2 points, and through each of these we have a line of the regulus having upon it the two coincident points; that is, $X\epsilon = km_1m_2$. Next Xp: the point 1 is here on the plane curve of the order m_1, which is the intersection of S_1 with the corresponding given plane; the regulus meets this plane curve in km_1 points; through each of these we have a line of the regulus intersecting S_2 in m_2 points, any one of which may be taken for the point 2; that is, the number of subjects is Xp, $= km_1 . m_2$. Then Xq: in precisely the same manner we have $Xq = km_2 . m_1$. Lastly Xg: the given line meets the regulus in k points, and

through each of these there is a line of the regulus meeting S_1 in m_1 points, any one of which may be taken for the point 1, and meeting S_2 in m_2 points, any one of which may be taken for the point 2; the number of the subjects Xg is thus $Xg, = k \,.\, m_1 \,.\, m_2$.

The general theorem $X\epsilon = Xp + Xq - Xg$ is proved by means of Chasles' theorem of united points as follows: the subject is a line, or say, for convenience, an *axis* ξ, bearing upon it the two points 1 and 2; we consider in conjunction therefore a given line λ, and through this draw the planes P_1, P_2 passing through the points 1 and 2 respectively.

Suppose that when 2 lies in a given plane there are α' positions of the axis, and on each of these β' positions of the point 1; and, similarly, that when 1 lies on a given plane there are α positions of the axis, and on each of these β positions of the point 2; then, 1 lying in a given plane, the number of subjects is $\alpha\beta$, or we have $Xp = \alpha\beta$; and, similarly, $Xq = \alpha'\beta'$. Take now for the point P_1 an arbitrary plane through λ; then, 1 lying on this plane, the number of the points 2 is $= \alpha\beta$, or, since each of these determines with λ a position of the plane P_2, the number of these planes is $= \alpha\beta$, that is, it is $= Xp$; and, similarly, taking P_2 an arbitrary plane through λ, the number of the planes P_1 is $\alpha'\beta'$, that is, it is $= Xq$; viz. the two planes P_1, P_2 through the line λ have an (Xp, Xq) correspondence; hence, by Chasles' theorem, the number of united planes is $= Xp + Xq$.

But we have a united plane, 1°, if the points 1 and 2 coincide, that is, if the condition $X\epsilon$ be satisfied, and the number of these united planes is $X\epsilon$; 2°, if the axis ξ meet the arbitrary line λ, that is, if the condition Xg be satisfied, and the number of these united planes is $= Xg$; hence the whole number is $= X\epsilon + Xg$; or we have $Xp + Xq = X\epsilon + Xg$, that is, $X\epsilon = Xp + Xq - Xg$, which is the theorem in question.

The conclusion is that the equation $\epsilon = p + q - g$, which in this, its original form, has neither a numerical nor a logical signification, is to be understood as meaning the numerical equation $X\epsilon = Xp + Xq - Xg$, the truth of which numerical equation has just been proved. Or we may, without explicit introduction of the condition X, understand the equation $\epsilon = p + q - g$ as a numerical equation as follows, viz. taking for the subject a line with two points, *which line and points are regarded as satisfying a given fivefold condition*, then

ϵ	is the	(additional onefold)	condition	that the two points shall coincide,
p	"	"	"	that the point 1 shall lie in given plane,
q	"	"	"	that the point 2 shall lie in given plane,
g	"	"	"	that line shall meet given line.

The conditions ϵ, p, q, g are thus in effect complete conditions, having values which may be connected by an equation; there, in fact, exists between them the relation

$$\epsilon = p + q - g.$$

The like remarks would apply to the before-mentioned equation (subject a point) $p^2 = p_g$: either adding to it a onefold condition X, and so taking it in the form $Xp^2 = Xp_g$, or understanding it in its original form $p^2 = p_g$ as belonging to a point which satisfies already a onefold condition, the equation is true as a numerical equation; and this in fact follows at once from its truth as a logical equation. But observe the difference: the equation in question $p^2 = p_g$ has, the equation $\epsilon = p + q - g$ has not, a logical signification.

I regard as the fundamental notion of the theory the existence of equations between conditions such as the foregoing equation $\epsilon = p + q - g$; equations which in their original form have not (of necessity) any logical signification, and have not any numerical signification; but which, when we adjoin to them a supplementary condition X of the proper postulation, become numerical equations, which are true, independently of the form of the supplementary condition X and whatever this condition may be. And this being so, it seems to follow at once that such equations may be treated and worked with as ordinary algebraical equations. For instance, let M be any condition of less postulation than X: then if from the equation $\epsilon = p + q - g$, assumed to be true, we deduce $M\epsilon = Mp + Mq - Mg$, this (like the original equation $\epsilon = p + q - g$) is in its actual form an equation without logical or numerical signification; but if we adjoin to it a supplementary condition K, such that postulation of $K +$ do. of $M =$ do. of X (or, what is the same thing, that the condition KM shall be supplementary to the several conditions contained in the original equation $\epsilon = p + q - g$), then the equation in question, $M\epsilon = Mp + Mq - Mg$, is to be interpreted as meaning

$$KM\epsilon = KMp + KMq - KMg,$$

that is,

$$X\epsilon = Xp + Xq - Xg,$$

which is numerically true. We thus see that the original equation $\epsilon = p + q - g$ implies the new equation

$$M\epsilon = Mp + Mq - Mg,$$

which is its algebraical consequence. And if we regard, for instance, $A + B$ as the condition that either the condition A shall be satisfied or else the condition B shall be satisfied, then $A + B$ is a condition, and as such we have

$$(A + B)\epsilon = (A + B)p + (A + B)q - (A + B)\epsilon.$$

It is going a step further to say that if we have, for instance, an equation $M = A + B - C$ between conditions M, A, B, C, then that, instead of

$$M\epsilon = Mp + Mq - M\epsilon,$$

we may write

$$(A + B - C)\epsilon = (A + B - C)p + (A + B - C)q - (A + B - C)\epsilon;$$

this is, in fact, treating $A + B - C$ as being to all intents and purposes a condition such as M, or an alternative condition $A + B$. It is, in fact, assumed that the step is permissible; and we thus make such deductions as

$$(\epsilon + p + q - g)(\epsilon - p - q + g) = 0;$$

that is,

$$\epsilon^2 - (p+q-g)^2 = 0,$$

or

$$\epsilon^2 = (p+q-g)^2, \; = p^2 + 2pq + q^2 - 2pg - 2qg + g^2;$$

viz. this is an equation such as the original equation $\epsilon = p + q - g$, acquiring a numerical signification when we adjoin to it a supplementary condition X of the proper postulation.

The section above referred to deals with the question to determine the number of lines which satisfy the several relations of contact in regard to a given surface F of the order n, without point-singularities, that is, the surface represented by the *general* equation $(*\between x, y, z, w)^n = 0$.

The chief results are contained in the following table, the notation of which will be explained:

1.	$\epsilon_2 g_s$	$=$	$n(n-1)$,
2.	$\epsilon_2 b_2 g_e$	$=$	n,
3.	$\epsilon_3 g_e$	$=$	$3n(n-2)$,
4.	$\epsilon_3 g_p$	$=$	$n(n-1)(n-2)$,
5.	$\epsilon_3 b_3^2$	$=$	$2n$,
6.	$\epsilon_{22} g_e$	$=$	$\frac{1}{2}n(n-2)(n-3)(n+3)$,
7.	$\epsilon_{22} g_p$	$=$	$\frac{1}{2}n(n-1)(n-2)(n-3)$,
8.	$\epsilon_{22} b_2^2$	$=$	$n(n-3)(n+2)$,
9.	$\epsilon_{22} b_2 c_2$	$=$	$n(n^3-2n^2+2n-6)$,
10.	$\epsilon_4 g$	$=$	$2n(n-3)(3n-2)$,
11.	$\epsilon_4 b_4$	$=$	$n(11n-24)$,
12.	$\epsilon_{32} g$	$=$	$n(n-3)(n-4)(n^2+6n-4)$,
13.	$\epsilon_{32} b_3$	$=$	$n(n-4)(3n^2+5n-24)$,
14.	$\epsilon_{32} b_2$	$=$	$n(n-2)(n-4)(n^2+2n+12)$,
15.	$\epsilon_{222} g$	$=$	$\frac{1}{3}n(n-3)(n-4)(n-5)(n^2+3n-2)$,
16.	$\epsilon_{222} b_2$	$=$	$\frac{1}{2}n(n-2)(n-4)(n-5)(n^2+5n+12)$,
17.	ϵ_5	$=$	$5n(n-4)(7n-12)$,
18.	ϵ_{42}	$=$	$2n(n-4)(n-5)(n+6)(3n-5)$,
19.	ϵ_{33}	$=$	$\frac{1}{2}n(n-4)(n-5)(n^3+3n^2+29n-60)$,
20.	ϵ_{322}	$=$	$\frac{1}{2}n(n-4)(n-5)(n-6)(n^3+9n^2+20n-60)$,
21.	ϵ_{2222}	$=$	$\frac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30)$,
22.	$\epsilon_{222} b_1$	$=$	$\frac{1}{3}n(n-4)(n-5)(n-6)(n^3+3n^2-2n-12)$,
23.	$\epsilon_2 b_1^2$	$=$	$(n-3)(n^2+2)$,
24.	$\epsilon_2 b_1 c_1 d_1$	$=$	$n^2(n-4)(2n^2-3n-3)$.

In the foregoing formulæ the suffixes of the ϵ refer to the contacts, viz. ϵ_2 denotes a 2-pointic intersection, ϵ_{22} a 3-pointic and a 2-pointic intersection. The letters b, c, d refer to the points of contact or intersection, thus $\epsilon_{32}b_3$, b_3 is the point of 3-pointic intersection; $\epsilon_{222}b_1$, b_1 is one of the points of simple intersection; b_1 is also the condition that the point in question lies on a given plane; g, g_s, g_e, g_p have their ordinary signification explained a little further on. Thus (15) $\epsilon_{222}g$ denotes the number of triple tangents which can be drawn to meet a given line; or, what is the same thing, it is the order of the regulus formed by the triple tangents.

The following are elementary formulæ used in the investigation of the foregoing results.

Subject a line having upon it a point,

		Postul.
p	the condition that point is in a given plane	1
p_g	" " line	2
g	" line meets a given line	1
g_e	" " is in a given plane	2
g_p	" " passes through a given point	2
g_s	" " lies in a given plane and passes through a given point of that plane	3
G	" " coincides with a given line	4

We have (p. 22 *et seq.*)

		Postul.	
p_g	$= p^2$	2	(*logical*)
p_g	$= p^2 + g_e$	2	
g^2	$= g_e + g_p$	2	
g_s	$= gg_e$	3	(*logical*)
g_s	$= gg_p$	3	(*logical*)
pg_p	$= p^3 + g_s$	3	(*demons. infrà*)
p^4	$= 0$	4	
$g_e g_p$	$= 0$	4	
g_e^2	$= G$	4	
g_p^2	$= G$	4	
$p^3 g$	$= p^2 g_e$	4	(*demons. infrà*)
pg_s	$= p^2 g_p$	4	"
pg_s	$= p^2 g_e + G$	4	"
$p^3 g_e$	$= 0$	5	"
$p^3 g_p$	$= pG$	5	"
$p^2 g_s$	$= pG$	5	"

$pg_p = p^3 + g_s$; we have $0 = g_e + p^2 - pg$, $0 = g_e + g_p - g^2$, and thence

$$\begin{aligned} 0 = & \quad (p+g)\{g_e + p(p-g)\} - p\{g_e + g_p - g^2\} \\ = & \quad pg_e + gg_e + p^3 - pg^2 \\ & - pg_e - pg_p \qquad + pg^2 \\ = & \quad gg_e - pg_p + p^3 \\ = & \quad g_s - pg_p + p^3. \end{aligned}$$

$p^2g = p^2g_e$: from $pg = p^2 + g_e$, we have $p^2g = p^4 + p^2g_e = p^2g_e$, since $p^4 = 0$,

$pg_s = p^2g_e + G$ „ $g_s = gg_e$ „ $pg_e = pgg_e = (p^2 + g_e)g_e = p^2g_e + G$,

$pg_s = p^2g_p$ „ $g_s = gg_p$ „ $pg_s = pgg_p = (p^2 + g_e)g_p = p^3g_p$, since $g_eg_p = 0$;

and in a similar manner we prove the last three equations.

For the demonstration of the formulæ of the table we take the subject to be a line bearing upon it the points 1, 2, ..., n, which are its intersections with a given surface of the order n. The symbols p_1, p_2, ... refer to these points respectively; thus, p_1 is the condition that the point 1 may lie on a given plane; and then, writing

$$\begin{aligned} \epsilon &= p_1 + p_2 - g, \\ \epsilon' &= p_1 + p_3 - g, \\ \epsilon'' &= p_3 + p_4 - g, \end{aligned}$$

it appears that ϵ will denote the condition of the coincidence of the points 1 and 2; ϵ' that of the points 1 and 3, &c. Hence also, $\epsilon\epsilon'$ will denote the twofold condition of the coincidence of the points 1, 2, 3; and so in other cases. But, according to the notation above explained, ϵ is also denoted by ϵ_2, $\epsilon\epsilon'$ by ϵ_3, $\epsilon\epsilon''$ by ϵ_{22}, &c.

We thus have

$$\begin{aligned} \epsilon_2 &= p_1 + p_2 - g, \\ \epsilon_3 &= (p_1 + p_2 - g)(p_1 + p_3 - g), \\ 2\epsilon_{22} &= (p_1 + p_2 - g)(p_3 + p_4 - g), \\ \epsilon_4 &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g), \\ \epsilon_{32} &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g), \\ 6\epsilon_{222} &= (p_1 + p_2 - g)(p_3 + p_4 - g)(p_5 + p_6 - g), \\ \epsilon_5 &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g)(p_1 + p_5 - g), \\ \epsilon_{42} &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_1 + p_4 - g)(p_5 + p_6 - g), \\ 2\epsilon_{33} &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g)(p_4 + p_6 - g), \\ 2\epsilon_{322} &= (p_1 + p_2 - g)(p_1 + p_3 - g)(p_4 + p_5 - g)(p_6 + p_7 - g), \\ 24\epsilon_{2222} &= (p_1 + p_2 - g)(p_3 + p_4 - g)(p_5 + p_6 - g)(p_7 + p_8 - g). \end{aligned}$$

We can now, by a mere analytical process of development and reduction, express each of the foregoing values as a linear function of

$$p_1^2p_2^2,\ p_1^2p_2p_3,\ p_1p_2p_3p_4,\ \text{and}\ G.$$

(Schubert says, as a linear function of these four symbols and $p_1p_2g_e$; but in fact $p_1p_2g_e$ is $=p_1^2p_2^2$.)

Observe, first, that we may, p. 287, in all the general equations instead of p write p_1, p_2, &c.; and, further, that any symbol containing for instance p_1^3 is $=0$. For the symbols now belong to the intersections of the line with a given surface; p_1^3 is the condition that a certain one of these intersections shall lie in three given planes, that is, that it shall coincide with a given arbitrary point; this cannot be the case, for the arbitrary point is not on the surface F; and therefore $p_1^3=0$.

We thus have $p_1g=p_1^3+g_e$, thence $p_1^2g=p_1^3+p_1g_e$, that is, $p_1^2g=p_1g_e$; and thence further $p_1^3g=p_1^2g_e$, that is, $p_1^2g_e=0$.

Again, from $p_2g=p_2^2+g_e$, $p_1g=p_1^2+g_e$, we have

$$p_1^2(p_2^2+g_e)=p_1p_2(p_1^2+g_e),$$

which, in virtue of $p_1^2g_e=0$ and $p_1^2p_2=0$, becomes

$$p_1^2p_2^2=p_1p_2g_e.$$

As a simple instance of the reductions, take

$$\epsilon_2g_s,\ =(p_1+p_2-g)g_s.$$

Here

$$p_1g_s,\ =p_2g_s,\ =p_1^2g_e+G,\ =G,\ \text{since}\ p_1^2g_e=0;$$

and

$$gg_s=g^2g_e=(g_e+g_p)g_e=g_e^2+g_eg_p=G,\ \text{since}\ g_e^2=0,\ g_eg_p=G;$$

whence the value is

$$\epsilon_2g_s=G+G-G,\ =G.$$

As a more complicated example, take

$$\epsilon_5,\ =(p_1+p_2-g)(p_1+p_3-g)(p_1+p_4-g)(p_1+p_5-g).$$

Observe that, after the multiplication is effected we may, in any way we please, interchange the suffixes, $p_1^2p_3p_4=p_1^2p_2p_3$, $p_3^2p_4^2=p_1^2p_2^2$, &c.; the suffixes serve only to distinguish from each other symbols in the same product (thus p_1^4 is different from $p_1p_2p_3p_4$), but there is nothing to distinguish one point of intersection from another. Thus the foregoing expression containing the terms $(p_2+p_3+p_4+p_5)(p_1-g)^3$, these may be combined into the single term $4p_2(p_1-g)^3$; expanding in powers of p_1-g and reducing in this manner, the value of ϵ_5 is, in fact, found to be

$$=(p_1-g)^4+4p_2(p_1-g)^3+6p_2p_3(p_1-g)^2+4p_2p_3p_4(p_1-g)+p_1p_2p_3p_4.$$

Developing this in powers of g, omitting the terms containing p_2^2 which vanish, and further reducing, the value is

$$6p_1^2p_2p_3+5p_1p_2p_3p_4+g(-12p_1^2p_2-16p_1p_2p_3)+g^2(6p_1^2+18p_1p_2)-8p_1g^3+g^4.$$

We have

$$g^4 = 2G,\ p_1g^3 = p_1g_s = p_1^2g_e + G,\ = G.$$

Next for the terms in g^2, from $p_1g = p_1^2 + g_e$ we have

$$p_1^2g = \qquad p_1g_e,$$

$$p_1p_2g = p_1^2p_2 \ + p_1g_e,$$

and thence

$$p_1^2g^2 = \qquad p_1g_s;$$

$$p_1p_2g^2 = p_1^2p_2g + p_1g_s,$$

or, since $p_1g_s = G$ as before, the whole term is $= 18p_1^2p_2g + 24G$. The terms in g thus become $= g(6p_1^2p_2 - 16p_1p_2p_3)$, and from the same equation $p_1g = p_1^2 + g_e$ we find

$$p_1^2p_2g = p_1^2p_2^2 \text{ and } p_1p_2p_3g = p_1^2p_2p_3 + p_1^2p_2^2.$$

The value is thus finally found to be

$$= -10p_1^2p_2^2 - 10p_1^2p_2p_3 + 5p_1p_2p_3p_4 + 10G.$$

The whole series of like results is

			$p_1^2p_2^2$	$p_1^2p_2p_3$	$p_1p_2p_3p_4$	G
1.	ϵ_2	g_s				+ 1
2.	,,	b_2g_e	+ 1			− 1
3.	ϵ_3	g_e	3			− 3
4.	,,	g_p				+ 1
5.	,,	b_3^2	− 2	+ 1		+ 1
6.	$2\epsilon_{22}$	g_e	+ 4			− 3
7.	2 ,,	g_p				+ 1
8.	,,	b_2^2	− 3	+ 2		+ 1
9.	ϵ_2	b_2c_2	− 2		+ 1	+ 1
10.	ϵ_4	g	− 2	+ 4		− 2
11.	,,	b_4	− 6		+ 1	+ 4
12.	ϵ_{32}	g	− 3	+ 6		− 2
13.	,,	b_3	− 7	− 1	+ 2	+ 4
14.	,,	b_2	− 6	− 3	+ 3	+ 4
15.	$6\epsilon_{222}$	g	− 4	+ 8		− 2
16.	2 ,,	b_2	− 7	− 4	+ 4	+ 4
17.	ϵ_5		− 10	− 10	+ 5	+ 10
18.	ϵ_{42}		− 10	− 16	+ 8	+ 10
19.	$2\epsilon_{33}$		− 9	− 18	+ 9	+ 10
20.	$2\epsilon_{322}$		− 9	− 24	+ 12	+ 10
21.	$24\epsilon_{2222}$		− 8	− 32	+ 16	+ 10
22.	$6\epsilon_{222}$	b_1	− 6	− 12	+ 8	+ 4
23.	$n\epsilon_3$	b_1^2	− 3	+ 3		+ 1
24.	ϵ_2	$b_1c_1d_1$	− 1	− 1	+ 2	

But in these formulæ $p_1^2p_2^2$, $p_1^2p_2p_3$, $p_1p_2p_3p_4$, G have numerical values which are different according to the number of points of intersection presenting themselves in the several formulæ; viz. this number being called i, we have for the formulæ in

	ϵ_2	ϵ_3	ϵ_{22}	ϵ_4	ϵ_{32}	ϵ_{222}	ϵ_5	ϵ_{42}	ϵ_{33}	ϵ_{322}	ϵ_{2222}	$\epsilon_{222}b_1$	$\epsilon_3b_1^2$	$\epsilon_2b_1c_1d_1$
$i=$	2	3	4	4	5	6	5	6	6	7	8	7	4	5,

and the values of the symbols are

$$\begin{aligned}
p_1^2p_2^2 &= n^2(n-2)(n-3) && \ldots(n-i+1),\\
p_1^2p_2p_3 &= n^2(n-1)(n-3) && \ldots(n-i+1),\\
p_1p_2p_3p_4 &= n^2(2n^2-6n+3)(n-4) && \ldots(n-i+1),\\
G &= n(n-1)(n-2) && \ldots(n-i+1).
\end{aligned}$$

Thus, suppose $i=4$, the subject is a line bearing the points 1, 2, 3, 4, which are intersections of the line with the surface F; we have then G as the condition in order that this line (or, say, the line of the subject) may coincide with a given line, which given line intersects the surface in n points; any four of these (their order being attended to) may be regarded as being the points 1, 2, 3, 4; or there are $n(n-1)(n-2)(n-3)$ subjects satisfying the prescribed condition (that the line of the subject may coincide with the given line). Hence here $G=n(n-1)(n-2)(n-3)$; and so in general $G=n(n-1)(n-2)\ldots(n-i+1)$.

Next, for $p_1^2p_2^2$. Here p_1^2 is the condition that the point 1 shall lie in each of two given planes, that is, in a given line, say L_1; and, similarly, p_2^2 is the condition that 2 may lie in a given line L_2. We take any one of the n intersections of L_1 with F for the point 1, and any one of the n intersections of L_2 with F for the point 2; this determines the line of the subject, but the $i-2$ points 3, 4, ..., i are then any $i-2$ of the remaining $n-2$ intersections of this line with F; that is, $p_1^2p_2^2=n^2(n-2)(n-3)\ldots(n-i+1)$ as above.

Again, for $p_1^2p_2p_3$. Here p_1^2 is the condition that 1 shall lie in a given line L_1; we therefore take for 1 any one of the n intersections of L_1 with F; p_2 is the condition that 2 may lie in a given plane P_2, it lies therefore in the curve of intersection of P_2 with F; and, similarly, 3 lies in the curve of intersection of a plane P_3 with F; the two planes intersect in a line meeting F in n points σ, and the two cones, vertex 1, which stand upon the plane curves respectively, intersect in the n lines joining 1 with the n points σ, and in n^2-n other lines. The line of the subject is then any one of these n^2-n lines, or, since the vertex is any one of n points, the line is any one of $n(n^2-n)$, $=n^2(n-1)$ lines; the remaining points 4, 5, ..., i are any $i-3$ of the remaining $n-3$ intersections of the line with F; hence the formula

$$p_1^2p_2p_3=n^2(n-1)(n-3)(n-4)\ldots(n-i+1).$$

For $p_1p_2p_3p_4$. We have here 1, 2, 3, 4 lying in given plane sections of the surface F, and we have consequently to find the number of lines which can be drawn to meet each of these four sections. Observing that any two of the sections meet in the n

intersections with F of the line of intersection of their planes, the order of the scroll generated by the lines which meet three of the sections is $2n^3-3n^2$; this scroll meets the fourth section in $n(2n^3-3n^2)$, $=2n^4-3n^3$ points; or we have this number of lines meeting each of the four sections. But among these are included $3n^2(n-1)$ lines which have to be rejected, viz. the sections 1 and 4 meet in n points, each of which is the vertex of cones through the sections 1 and 2 respectively; these cones meet in n lines, which are to be disregarded, and in n^2-n other lines, and we have thus $n(n^2-n)$, $=n^2(n-1)$ lines; and similarly from the intersections of 2 and 4, and from the intersections of 3 and 4, $n^2(n-1)$ and $n^2(n-1)$ lines, in all $3n^2(n-1)$ lines. Hence the number of lines meeting the four sections is

$$2n^4-3n^3-3n^3+3n^2,\ =2n^4-6n^3+3n^2;$$

taking any one of these for the line of the subject, the remaining points 5, 6, ..., i are any $i-4$ of the remaining $n-4$ intersections, or we have the required formula

$$p_1p_2p_3p_4=n^2(2n^2-6n+3)(n-4)\ldots(n-i+1).$$

The four numbers $p_1^2p_2^2$, $p_1^2p_2p_3$, $p_1p_2p_3p_4$, G for any line of the table being now known, we can at once calculate the required values $\epsilon_2 g_8$, &c., as the case may be; for instance,

$$\begin{aligned} i=5,\quad \epsilon_5 = &-10p_1^2p_2^2 &&= -10n^2(n-2)(n-3)(n-4)\\ &-10p_1^2p_2p_3 &&-10n^2(n-1)(n-3)(n-4)\\ &+\ 5p_1p_2p_3p_4 &&+\ 5n^2(2n^2-6n+3)(n-4)\\ &+10G &&+10n\ (n-1)(n-2)(n-3)(n-4)\\ &&&=\ \ 5n\ (n-4)(7n-12). \end{aligned}$$

In fact, throwing out $n(n-4)$, the remaining terms give

$$\begin{array}{l} -10n^3+50n^2-\ 60n\\ -10n^3+40n^2-\ 30n\\ +10n^3-30n^2+\ 15n\\ +10n^3-60n^2+110n-60\\ \hline \qquad\qquad 35n-60,\ =5(7n-12). \end{array}$$

And we obtain in like manner the other formulæ of the table.

The remainder of § 33 contains investigations of less systematically connected theorems, and I quote the results only.

25. If on the surface F_n there is a curve order r, then of the tangent planes of F_n along this curve there pass $r(n-1)$ through an arbitrary point of space; *aliter*, class of torse is $=r(n-1)$.

In particular, for curve of 4-pointic contact, $r=n(11n-24)$, class of torse is $=n(n-1)(11n-24)$.

No. of tangent planes through line, or class of surface, $=n(n-1)^2$.

26. $\epsilon_3 b_3 g = \epsilon_3 b_3^2 + \epsilon_3 g_e = 2n + 3n(n-2), = n(3n-4)$.

$\epsilon_3 b_3 g, = n(3n-4)$, is the order of curve of contact of the 3-pointic (chief) tangents which meet a given line.

Parabolic tangents are coincident chief tangents.

No. of 4-pointic parabolic tangents $= 2n(n-2)(11n-24)$.

27. Order of parabolic curve $= 4n(n-2)$.

Order of regulus formed by parabolic tangents

$$= 2n(n-2)(3n-4).$$

The parabolic curve and curve of contacts of an ϵ_4 tangent meet in

$$4n(n-2)(11n-24)$$

points, i.e., they *touch* in $2n(n-2)(11n-24)$ points.

28. Umbilici. No. is $= 2n(5n^2 - 14n + 11)$.

29. No. of points at which the chief tangents being distinct are each of them 4-pointic, or, what is the same thing, No. of actual double points of curve ϵ_4,

$$= 5n(7n^2 - 28n + 30),$$

$n = 3$, No. is $15(63 - 84 + 30), = 135$, viz. this is the number of points of intersection of two of the 27 lines; or, what is the same thing, the number of triple tangent planes is $= 45$.

30. No. of parabolic tangents which have besides a 2-pointic contact is

$$= 2n(n-2)(n-4)(3n^2 + 5n - 24).$$

31. No. of double tangent planes such that line through points of contact is at one of these points 3-pointic

$$= n(n-2)(n-4)(n^3 + 3n^2 + 13n - 48).$$

32. No. of points where one chief tangent is 4-pointic, the other 3-pointic and (at another point of the surface) 2-pointic is

$$= n(n-4)(27n^3 - 13n^2 - 264n + 396).$$

33. No. of points where chief tangents being distinct are each of them at another point of the surface 2-pointic is

$$= n(n-4)(4n^5 - 4n^4 - 95n^3 + 99n^2 + 544n - 840).$$

34. The curve of contacts b_3 of an ϵ_{22} tangent has with the parabolic curve 2-pointic intersections only, and these are at the points for which the chief tangent is (at another point of the surface) 2-pointic.

35. The curve of contacts b_2 of an ϵ_{32} tangent has, with the curve of contacts of an ϵ_4 tangent, 2-pointic intersections at the contacts of an ϵ_5 tangent; and has also simple intersections with the same curve, 1° at the contacts b_4 of an ϵ_{42} tangent, 2° at the points where the chief tangents are ϵ_4 and ϵ_{32}.

763.

ON THE THEOREMS OF THE 2, 4, 8, AND 16 SQUARES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVII. (1881), pp. 258—276.]

A SUM of 2 squares multiplied by a sum of 2 squares is a sum of 2 squares; a sum of 4 squares multiplied by a sum of 4 squares is a sum of 4 squares; a sum of 8 squares multiplied by a sum of 8 squares is a sum of 8 squares; but a sum of 16 squares multiplied by a sum of 16 squares is *not* a sum of 16 squares. These theorems were considered in the paper, Young, "On an extension of a theorem of Euler, with a determination of the limit beyond which it fails," *Trans. R. I. A.*, t. XXI. (1848), pp. 311—341; and the later history of the question is given in the paper by Mr S. Roberts, "On the Impossibility of the general Extension of Euler's Theorem &c.," *Quart. Math. Jour.* t. XVI. (1879), pp. 159—170; as regards the 16-question, it has been throughout assumed that there is only one type of synthematic arrangement (what this means will appear presently); but as regards this type, it is, I think, well shown that the signs cannot be determined. It will appear in the sequel, that there are in fact four types (the last three of them possibly equivalent) of synthematic arrangement; and for a complete proof, it is necessary to show in regard to each of these types that the signs cannot be determined. The existence of the four types has not (so far as I am aware) been hitherto noticed; and it hence follows, that no complete proof of the non-existence of the 16-square theorem has hitherto been given.

For the 2 squares the theorem is of course

$$(x_1^2+x_2^2)(y_1^2+y_2^2)=(x_1y_1+x_2y_2)^2+(x_1y_2-x_2y_1)^2.$$

For the 4 squares (for which the nature of the theorem is better seen) it is

$$\begin{aligned}(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)= \quad & (x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2\\ &+(x_1y_2-x_2y_1+x_3y_4-x_4y_3)^2\\ &+(x_1y_3-x_3y_1-x_2y_4+x_4y_2)^2\\ &+(x_1y_4-x_4y_1+x_2y_3-x_3y_2)^2;\end{aligned}$$

or, as this may be written,

$$\begin{aligned}(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2)-(x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2 \\ = \quad (12+34)^2 \\ +(13-24)^2 \\ +(14+23)^2;\end{aligned}$$

where 12 is used to denote $x_1y_2-x_2y_1$, &c., and the truth of the theorem depends on the identity $12.34-13.24+14.23=0$. Clearly, the first step for forming the equation is to arrange the duads in a synthematic form

$$\begin{array}{l}12.34\\13.24\\14.23,\end{array}$$

and then to determine the signs: such an arrangement exists in the case of 8, and the signs can be determined; it exists also in the case of 16, but the signs *cannot* be determined to satisfy all the necessary relations.

In the case of 8, we have the synthematic arrangement

$$\begin{array}{l}12.34.56.78\\13.24.57.68\\14.23.58.67\\15.26.37.48\\16.25.38.47\\17.28.35.46\\18.27.36.45,\end{array}$$

being the *only type* of synthematic arrangement. This is, in fact, important as regards the 16-question, and it will appear that the case is so; but in the 8-question, starting from this arrangement, we have to show that there exists an equation which, for convenience, I write as follows:

$$\begin{aligned}(x_1^2+\ldots+x_8^2)(y_1^2+\ldots+y_8^2)-(x_1y_1+\ldots+x_8y_8)^2 \\ = \quad (12+34+56+78)^2 \\ +(13+24+57+68)^2 \\ +(14+23+58+67)^2 \\ +(15+26+37+48)^2 \\ +(16+25+38+47)^2 \\ +(17+28+35+46)^2 \\ +(18+27+36+45)^2,\end{aligned}$$

but in which it is to be understood that each duad is affected by a factor ± 1 which is to be determined; say the factor of 12 is ϵ_{12}, that of 34, ϵ_{34}; and so in other cases. It is however assumed that ϵ_{12}, ϵ_{34}, ϵ_{56}, ϵ_{78}; ϵ_{13}, ϵ_{14}, ϵ_{15}, ϵ_{16}, ϵ_{17}, ϵ_{18} are each $=+1$.

We have then on the right-hand side triads of terms such as, 2 into

$$\epsilon_{12}\epsilon_{34}\,12.34+\epsilon_{13}\epsilon_{24}\,13.24+\epsilon_{14}\epsilon_{23}\,14.23,$$

which triad ought to vanish identically, as reducing itself to a multiple of

$$12.34-\ 13.24+14.23;$$

viz. we ought to have

$$\epsilon_{12}\epsilon_{34}=-\ \epsilon_{13}\epsilon_{24}\ =\ \epsilon_{14}\epsilon_{23};$$

or, *using now and henceforward when occasion requires*, 12, 34, &c. to denote ϵ_{12}, ϵ_{34}, &c. respectively, we have

$$12.34=+k,$$
$$13.24=-k,$$
$$14.23=+k,$$

where k, $=\pm 1$, has to be determined (in the actual case we have $12=+1$, $34=+1$, $13=1$, $14=1$; and therefore the first equation gives $k=1$, and the other two then give $24=-1$, $23=+1$).

We have in this way triads of values corresponding to the different tetrads

1234
1256
1278
1357
1368
1458
1467
2358
2367
2457
2468
3456
3478
5678,

which can be formed with the several lines of the formula. Thus we have from the first line 1234, 1256, 1278; then from the second line (not 1324 which in the form 1234 has been taken already) 1357, 1368, ...; and finally from the last line 5678.

We might consider each line as giving 6 tetrads, but the tetrads would then be obtained 3 times over; the number of tetrads is thus $6 \times 7 \div 3, = 14$ as above. And observe, that the systems of values for the coefficients $\epsilon = \pm 1$ are obtained directly from the tetrads, without the employment of any other formula.

We thus obtain the system of signs as follows:

12	$+1$	
13	$+1$	
14	$+1$	
15	$+1$	
16	$+1$	
17	$+1$	
18	$+1$	
23	$+1$	
24	-1	
25	$+1$	
26	-1	
27	$+1$	
28	-1	
34	$+1$	
35	a	$-\theta$
36	b	θ
37	$-a$	θ
38	$-b$	$-\theta$
45	c	θ
46	d	θ
47	$-d$	$-\theta$
48	$-c$	$-\theta$
56	$+1$	
57	a	$-\theta$
58	c	θ
67	d	θ
68	b	θ
78	$+1$	

viz. the original assumptions $12 = +1$, &c., and the tetrads 1234, 1256, 1278 give all the signs ± 1 up to $34 = +1$; from the tetrad 1357 we have

$$13.57 + 1 \quad a,$$

$$15.37 - 1 \quad a,$$

$$17.35 + 1 \quad a,$$

that is, $35 = a$, $37 = -a$, $57 = a$, where a, $= \pm 1$, is still undetermined; and similarly, the tetrads 1368, 1458, 1467 give the remaining signs b, c, d. The tetrad 2358 then gives

$$23.58 + 1 \quad c,$$

$$25.38 - 1 - b,$$

$$28.35 + -1 \quad a,$$

that is, $-a = b = c$; and similarly the tetrads 2367, 2457, 2468 give $-a = b = d$, $-a = c = d$, $b = c = d$ respectively; the four tetrads thus give $-a = b = c = d$, say each of these $= \theta$. But retaining for the moment a, b, c, d, the tetrad 3456 then gives

$$34.56 + 1 \quad 1,$$

$$35.46 - a \quad d,$$

$$36.45 + b \quad c,$$

that is, $1 = -ad = bc$, and similarly the last two tetrads 3478 and 5678 give $1 = -ac = bd$ and $1 = -ab = cd$ respectively; substituting the values in terms of θ, the several equations give only $\theta^2 = 1$, that is, $\theta = \pm 1$ at pleasure; and the series of signs for the 8-formula, containing this one arbitrary sign $\theta = \pm 1$, is thus determined.

Passing to the case of 16, we have in like manner to form a synthematic arrangement of the numbers 1, 2, ..., 16 in 15 lines, each containing the 16 numbers in 8 duads (no duad twice repeated), and this containing all the 120 duads. And, using for the moment letters instead of numbers, the necessary condition is, that $ab.cd$ occurring in one line, $ac.bd$ must occur in another line, and $ad.bc$ in a third line. Observe that as well the order of the letters in a duad as the order of the duads is thus far immaterial; so that a line containing $bd.ca$ may be considered as containing $ac.bd$.

Considering any such combination $ab.cd$, the line which contains it may be taken to be the first line; and the line which contains $ac.bd$ may be taken to be the second line. And then writing 1, 2, 3, 4 in place of a, b, c, d respectively, the first line will contain 12.34, and the second line will contain 13.24. Let e be any other symbol occurring in the first line, say in the duad ef, and in the second line say in the duad eg; then g must occur in the first line in some duad gh, or the first line will contain $ef.gh$, and then the second line as containing eg will contain

also fh, that is, it will contain $eg.fh$. And then writing 5, 6, 7, 8 in place of e, f, g, h respectively, the first line will contain 56.78 and the second line will contain 57.68. And continuing the like reasoning, it appears that the first line and the second line may be taken to be

$$1\ 2.\ 3\ 4.\ 5\ 6.\ 7\ 8.\ 9\ 10.\ 11\ 12.\ 13\ 14.\ 15\ 16,$$

and

$$1\ 3.\ 2\ 4.\ 5\ 7.\ 6\ 8.\ 9\ 11.\ 10\ 12.\ 13\ 15.\ 14\ 16,$$

respectively. There will then be a line containing 1 4 which may be taken for the third line, a line containing 1 5 which may be taken for the fourth line, and so on; viz. the successive lines may be taken to begin with 1 2, 1 3, 1 4, ..., 1 16 respectively.

Proceeding to form the synthematic arrangement, and starting with the first and second lines and first column as above, it appears that in each of the remaining lines there are three duads which occur of necessity, and putting these in the second, third, and fourth places (the order of the duads in any line being immaterial), it is seen that the second, third, and fourth columns can be filled up in one, and only one way; see the annexed first-half:

First-half common to all.

1	2	3	4	5	6	7	8
1	3	2	4	5	7	6	8
1	4	2	3	5	8	6	7
1	5	2	6	3	7	4	8
1	6	2	5	3	8	4	7
1	7	2	8	3	5	4	6
1	8	2	7	3	6	4	5
1	9	2	10	3	11	4	12
1	10	2	9	3	12	4	11
1	11	2	12	3	9	4	10
1	12	2	11	3	10	4	9
1	13	2	14	3	15	4	16
1	14	2	13	3	16	4	15
1	15	2	16	3	13	4	14
1	16	2	15	3	14	4	13

Four forms of second-half.

I.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	13	10	14	11	15	12	16
9	14	10	13	11	16	12	15
9	15	10	16	11	13	12	14
9	16	10	15	11	14	12	13
5	13	6	14	7	15	8	16
5	14	6	13	7	16	8	15
5	15	6	16	7	13	8	14
5	16	6	15	7	14	8	13
5	9	6	10	7	11	8	12
5	10	6	9	7	12	8	11
5	11	6	12	7	9	8	10
5	12	6	11	7	10	8	9

II.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	14	10	13	11	16	12	15
9	13	10	14	11	15	12	16
9	16	10	15	11	14	12	13
9	15	10	16	11	13	12	14
5	14	6	13	7	16	8	15
5	13	6	14	7	15	8	16
5	16	6	15	7	14	8	13
5	15	6	16	7	13	8	14
5	10	6	9	7	12	8	11
5	9	6	10	7	11	8	12
5	12	6	11	7	10	8	9
5	11	6	12	7	9	8	10

III.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	15	10	16	11	13	12	14
9	16	10	15	11	14	12	13
9	13	10	14	11	15	12	16
9	14	10	13	11	16	12	15
5	15	6	16	7	13	8	14
5	16	6	15	7	14	8	13
5	13	6	14	7	15	8	16
5	14	6	13	7	16	8	15
5	11	6	12	7	9	8	10
5	12	6	11	7	10	8	9
5	9	6	10	7	11	8	12
5	10	6	9	7	12	8	11

IV.

9	10	11	12	13	14	15	16
9	11	10	12	13	15	14	16
9	12	10	11	13	16	14	15
9	16	10	15	11	14	12	13
9	15	10	16	11	13	12	14
9	14	10	13	11	16	12	15
9	13	10	14	11	15	12	16
5	16	6	15	7	14	8	13
5	15	6	16	7	13	8	14
5	14	6	13	7	16	8	15
5	13	6	14	7	15	8	16
5	12	6	11	7	10	8	9
5	11	6	12	7	9	8	10
5	10	6	9	7	12	8	11
5	9	6	10	7	11	8	12

And it is to be noticed that in this first-half the upper part, or first seven lines, give in fact the synthematic arrangement for the 8-question; so that (as remarked above) in this 8-question there is but one form of synthematic arrangement.

Proceeding to fill up the remaining columns, the duad 59 cannot be placed in any line which contains a 5 or a 9; that is, it must be placed in some one of the

last 4 lines; and placing it successively in each of these, it appears that the columns can be filled up in one, and only one, way; we have thus the above "four forms of second-half," each of which, taken in conjunction with the common first-half, gives a synthematic arrangement of the 16 numbers.

Each of these synthematic arrangements may be converted into a square, the first line of which is formed with the numbers 1 to 16 in order, and the other fifteen lines of which are derived from the fifteen lines of the synthematic arrangement respectively: thus the line

$$1\ 2.\ 3\ 4.\ 5\ 6.\ 7\ 8.\ 9\ 10.\ 11\ 12.\ 13\ 14.\ 15\ 16$$

gives the second line of

$$1\ 2.\ 3\ 4.\ 5\ 6.\ 7\ 8.\ 9\ 10.\ 11\ 12.\ 13\ 14.\ 15\ 16,$$

$$2\ 1.\ 4\ 3.\ 6\ 5.\ 8\ 7.\ 10\ 9.\ 12\ 11.\ 14\ 13.\ 16\ 15,$$

and so in other cases. And conversely, by comparing with the first line of the square each of the other fifteen lines respectively, we have the fifteen lines of the synthematic arrangement; we thus obtain the four squares presently given. These squares are not required in the sequel, but they serve to put in a clearer light the construction of the synthematic arrangements; by converting in like manner into a square the formula p. 332 of Young's paper, it appears that his arrangement is in fact the first of the foregoing four arrangements. The squares are

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

II.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	14	13	16	15	10	9	12	11
6	5	8	7	2	1	4	3	13	14	15	16	9	10	11	12
7	8	5	6	3	4	1	2	16	15	14	13	12	11	10	9
8	7	6	5	4	3	2	1	15	16	13	14	11	12	9	10
9	10	11	12	14	13	16	15	1	2	3	4	6	5	8	7
10	9	12	11	13	14	15	16	2	1	4	3	5	6	7	8
11	12	9	10	16	15	14	13	3	4	1	2	8	7	6	5
12	11	10	9	15	16	13	14	4	3	2	1	7	8	5	6
13	14	15	16	10	9	12	11	6	5	8	7	1	2	3	4
14	13	16	15	9	10	11	12	5	6	7	8	2	1	4	3
15	16	13	14	12	11	10	9	8	7	6	5	3	4	1	2
16	15	14	13	11	12	9	10	7	8	5	6	4	3	2	1

III.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	13	14	15	16	9	10	11	12
8	7	6	5	4	3	2	1	14	13	16	15	10	9	12	11
9	10	11	12	15	16	13	14	1	2	3	4	7	8	5	6
10	9	12	11	16	15	14	13	2	1	4	3	8	7	6	5
11	12	9	10	13	14	15	16	3	4	1	2	5	6	7	8
12	11	10	9	14	13	16	15	4	3	2	1	6	5	8	7
13	14	15	16	11	12	9	10	7	8	5	6	1	2	3	4
14	13	16	15	12	11	10	9	8	7	6	5	2	1	4	3
15	16	13	14	9	10	11	12	5	6	7	8	3	4	1	2
16	15	14	13	10	9	12	11	6	5	8	7	4	3	2	1

IV.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
5	6	7	8	1	2	3	4	16	15	14	13	12	11	10	9
6	5	8	7	2	1	4	3	15	16	13	14	11	12	9	10
7	8	5	6	3	4	1	2	14	13	16	15	10	9	12	11
8	7	6	5	4	3	2	1	13	14	15	16	9	10	11	12
9	10	11	12	16	15	14	13	1	2	3	4	8	7	6	5
10	9	12	11	15	16	13	14	2	1	4	3	7	8	5	6
11	12	9	10	14	13	16	15	3	4	1	2	6	5	8	7
12	11	10	9	13	14	15	16	4	3	2	1	5	6	7	8
13	14	15	16	12	11	10	9	8	7	6	5	1	2	3	4
14	13	16	15	11	12	9	10	7	8	5	6	2	1	4	3
15	16	13	14	10	9	12	11	6	5	8	7	3	4	1	2
16	15	14	13	9	10	11	12	5	6	7	8	4	3	2	1

The foregoing investigation of the synthematic arrangements is exhaustive: it thereby appears that there are at most four types, viz. that every synthematic arrangement is of the type of one or other of the four arrangements above written down. The real nature of these is perhaps more clearly seen by means of the corresponding squares; and it will be observed, that there is in the first square a repetition of parts without transposition, which does not occur in the other three squares; this seems to suggest, that while the first square (and therefore the first synthematic arrangement) is really of a distinct type, the other three squares (or synthematic arrangements) *may possibly* belong to one and the same type. If this were so, it would be sufficient to prove the 16-theorem (viz. the non-existence of the 16-square formula) for the first and for any one of the other three synthematic arrangements; but I provisionally assume that the four types are really distinct, and propose therefore to prove the theorem for each of the four arrangements separately.

The process is the same as for the 8-theorem; we require the tetrads 1234, &c., contained in the synthematic arrangements. In any one of these, each line gives $\frac{1}{2}8 . 7, = 28$ tetrads, and the 15 lines give therefore $15 . 28, = 420$ tetrads: but we thus obtain each tetrad 3 times, or the number of the tetrads is $420 \div 3, = 140$.

For the four arrangements respectively, these are as follows: the word "same" means same as in column I.

I.	II.	III.	IV.
1 2 3 4	same	same	same
5 6			
7 8			
9 10			
11 12			
13 14			
15 16			
1 3 5 7			
6 8			
9 11			
10 12			
13 15			
14 16			
1 4 5 8			
6 7			
9 12			
10 11			
13 16			
14 15			
1 5 9 13	1 5 9 14	1 5 9 15	1 5 9 16
10 14	10 13	10 16	10 15
11 15	11 16	11 13	11 14
12 16	12 15	12 14	12 13
1 6 9 14	1 6 9 13	1 6 9 16	1 6 9 15
10 13	10 14	10 15	10 16
11 16	11 15	11 14	11 13
12 15	12 16	12 13	12 14
1 7 9 15	1 7 9 16	1 7 9 13	1 7 9 14
10 16	10 15	10 14	10 13
11 13	11 14	11 15	11 16
12 14	12 13	12 16	12 15
1 8 9 16	1 8 9 15	1 8 9 14	1 8 9 13
10 15	10 16	10 13	10 14
11 14	11 13	11 16	11 15
12 13	12 14	12 15	12 16

I.	II.	III.	IV.
2 3 5 8 6 7 9 12 10 11 13 16 14 15	same	same	same
2 4 5 7 6 8 9 11 10 12 13 15 14 16	same	same	same
2 5 9 14 10 13 11 16 12 15	2 5 9 13 10 14 11 15 12 16	2 5 9 16 10 15 11 14 12 13	2 5 9 15 10 16 11 13 12 14
2 6 9 13 10 14 11 15 12 16	2 6 9 14 10 13 11 16 12 15	2 6 9 15 10 16 11 13 12 14	2 6 9 16 10 15 11 14 12 13
2 7 9 16 10 15 11 14 12 13	2 7 9 15 10 16 11 13 12 14	2 7 9 14 10 13 11 16 12 15	2 7 9 13 10 14 11 15 12 16
2 8 9 15 10 16 11 13 12 14	2 8 9 16 10 15 11 14 12 13	2 8 9 13 10 14 11 15 12 16	2 8 9 14 10 13 11 16 12 15
3 4 5 6 7 8 9 10 11 12 13 14 15 16	same	same	same

I.	II.	III.	IV.
3 5 9 15	3 5 9 16	3 5 9 13	3 5 9 14
10 16	10 15	10 14	10 13
11 13	11 14	11 15	11 16
12 14	12 13	12 16	12 15
3 6 9 16	3 6 9 15	3 6 9 14	3 6 9 13
10 15	10 16	10 13	10 14
11 14	11 13	11 16	11 15
12 13	12 14	12 15	12 16
3 7 9 13	3 7 9 14	3 7 9 15	3 7 9 16
10 14	10 13	10 16	10 15
11 15	11 16	11 13	11 14
12 16	12 15	12 14	12 13
3 8 9 14	3 8 9 13	3 8 9 16	3 8 9 15
10 13	10 14	10 15	10 16
11 16	11 15	11 14	11 13
12 15	12 16	12 13	12 14
4 5 9 16	4 5 9 15	4 5 9 14	4 5 9 13
10 15	10 16	10 13	10 14
11 14	11 13	11 16	11 15
12 13	12 14	12 15	12 16
4 6 9 15	4 6 9 16	4 6 9 13	4 6 9 14
10 16	10 15	10 14	10 13
11 13	11 14	11 15	11 16
12 14	12 13	12 16	12 15
4 7 9 14	4 7 9 13	4 7 9 16	4 7 9 15
10 13	10 14	10 15	10 16
11 16	11 15	11 14	11 13
12 15	12 16	12 13	12 14
4 8 9 13	4 8 9 14	4 8 9 15	4 8 9 16
10 14	10 13	10 16	10 15
11 15	11 16	11 13	11 14
12 16	12 15	12 14	12 13
5 6 7 8	same	same	same
9 10			
11 12			
13 14			
15 16			

I.	II.	III.	IV.
5 7 9 11 10 12 13 15 14 16	same	same	same
5 8 9 12 10 11 13 16 14 15			
6 7 9 12 10 11 13 16 14 15			
6 8 9 11 10 12 13 15 14 16			
7 8 9 10 11 12 13 14 15 16			
9 10 11 12 13 14 15 16			
9 11 13 15 14 16			
9 12 13 16 14 15			
10 11 13 16 14 15			
10 12 13 15 14 16			
11 12 13 14 15 16			
13 14 15 16			

As regards the signs, observe that the first line may always be written

$$ab + cd + ef + \&c.,$$

with the signs all of them $+$; and then writing $a, b, c, \ldots = 1, 2, 3, \ldots, 16$ respectively, the first line will be

$$1\ 2 + 3\ 4 + 5\ 6 + 7\ 8 + 9\ 10 + 11\ 12 + 13\ 14 + 15\ 16,$$

with the signs all of them $+$; that is, we may assume ϵ_{12}, ϵ_{34}, &c., or say

$$1\ 2,\ 3\ 4,\ 5\ 6,\ 7\ 8,\ 9\ 10,\ 11\ 12,\ 13\ 14,\ 15\ 16,$$

all of them $= +1$. And in the other lines, the signs of all the terms of any line may be reversed at pleasure, that is, we may assume ϵ_{13}, ϵ_{14}, &c., or say 1 3, 1 4, 1 5, 1 6, 1 7, 1 8, 1 9, 1 10, 1 11, 1 12, 1 13, 1 14, 1 15, 1 16, all of them $= +1$.

Making these assumptions, then for any one of the synthematic arrangements the several tetrads give as before relations between the signs; among these are included the results already obtained for the 8-question, and taking as before

$$-a = b = c = d = \theta,$$

we have the signs of the several terms belonging to the 8-question given as $= \pm 1$ or $\pm\theta$ as before. The remaining tetrads up to 1 8 12 13 then serve to express all the remaining signs in terms of the as yet undetermined signs $e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z, \alpha, \beta$, for instance

$$\begin{array}{l} 1\ \ 3\,.\,9\ 11 +\ \ 1\ \ e, \\ 1\ \ 9\,.\,3\ 11 -\ \ 1\ \ e, \\ 1\ 11\,.\,3\ \ \ 9 +\ \ 1\ \ e, \end{array}$$

that is, $3\ 9 = e$, $3\ 11 = -e$, $9\ 11 = e$; and then the tetrads up to 2 8 9 15 serve to express these signs in terms of the undetermined signs $\lambda, \mu, \nu, \rho, \sigma, \tau$; for instance

$$\begin{array}{l} 2\ \ 3\,.\,9\ 12 +\ \ 1\ \ i, \\ 2\ \ 9\,.\,3\ 12 -\ \ 1 - f, \\ 2\ 12\,.\,3\ \ \ 9 + -1\ \ e, \end{array}$$

that is, $-e = f = i$; and in like manner 2 3 10 11, 2 4 9 11 and 2 4 10 12 give respectively $-e = f = j$, $-e = i = j$, $f = i = j$; that is, we have $-e = f = i = j, = \lambda$ suppose. And in this way we have, for each of the four synthematic arrangements the signs of all the terms expressed in terms of the undetermined signs $\theta, \lambda, \mu, \nu, \rho, \sigma, \tau$, as shown in the following table; where observe that the results apply to the four synthematic arrangements *separately*, viz. the e, f, g, &c., and the $\theta, \lambda, \mu, \nu, \rho, \sigma, \tau$ in each column are altogether independent of the like symbols in the other three columns.

Signs for the four synthematic arrangements:

	I.			II.	III.	IV.
1	2	$+1$		same	same	same
	3	$+1$				
	4	$+1$				
	5	$+1$				
	6	$+1$				
	7	$+1$				
	8	$+1$				
	9	$+1$				
	10	$+1$				
	11	$+1$				
	12	$+1$				
	13	$+1$				
	14	$+1$				
	15	$+1$				
	16	$+1$				
2	3	$+1$		same	same	same
	4	-1				
	5	$+1$				
	6	-1				
	7	$+1$				
	8	-1				
	9	$+1$				
	10	-1				
	11	$+1$				
	12	-1				
	13	$+1$				
	14	-1				
	15	$+1$				
	16	-1				
3	4	$+1$		same	same	same
	5	$-\theta$				
	6	θ				
	7	θ				
	8	$-\theta$				
	9	e	$-\lambda$			
	10	f	λ			
	11	$-e$	λ			
	12	$-f$	$-\lambda$			
	13	g	$-\mu$			
	14	h	μ			
	15	$-g$	μ			
	16	$-h$	$-\mu$			

		I.		II.		III.		IV.	
4	5	θ		same		same		same	
	6	θ							
	7	$-\theta$							
	8	$-\theta$							
	9	i	λ						
	10	j	λ						
	11	$-j$	$-\lambda$						
	12	$-i$	$-\lambda$						
	13	k	μ						
	14	l	μ						
	15	$-l$	$-\mu$						
	16	$-k$	$-\mu$						
5	6	$+1$		$+1$		$+1$		1	
	7	$-\theta$		$-\theta$		$-\theta$		$-\theta$	
	8	θ		θ		θ		θ	
	9	m	$-\nu$	m	ν	m	$-\nu$	m	ν
	10	n	ν	n	ν	n	ν	n	ν
	11	o	$-\rho$	o	ρ	o	$-\rho$	o	ρ
	12	p	ρ	p	ρ	p	ρ	p	ρ
	13	$-m$	ν	$-n$	$-\nu$	$-o$	ρ	$-p$	$-\rho$
	14	$-n$	$-\nu$	$-m$	$-\nu$	$-p$	$-\rho$	$-o$	$-\rho$
	15	$-o$	ρ	$-p$	$-\rho$	$-m$	ν	$-n$	$-\nu$
	16	$-p$	$-\rho$	$-o$	$-\rho$	$-n$	$-\nu$	$-m$	$-\nu$
6	7	θ		θ		θ		θ	
	8	θ		θ		θ		θ	
	9	q	ν	q	ν	q	ν	q	ν
	10	r	ν	r	$-\nu$	r	ν	r	$-\nu$
	11	s	ρ	s	ρ	s	ρ	s	ρ
	12	t	ρ	t	$-\rho$	t	ρ	t	$-\rho$
	13	$-r$	$-\nu$	$-q$	$-\nu$	$-t$	$-\rho$	$-s$	$-\rho$
	14	$-q$	$-\nu$	$-r$	ν	$-s$	$-\rho$	$-t$	ρ
	15	$-t$	$-\rho$	$-s$	$-\rho$	$-r$	$-\nu$	$-q$	$-\nu$
	16	$-s$	$-\rho$	$-t$	ρ	$-q$	$-\nu$	$-r$	ν
7	8	$+1$		$+1$		$+1$		$+1$	
	9	u	$-\sigma$	u	$-\sigma$	u	$-\sigma$	u	σ
	10	v	σ	v	σ	v	σ	v	σ
	11	w	$-\tau$	w	τ	w	$-\tau$	w	τ
	12	x	τ	x	τ	x	τ	x	τ
	13	$-w$	τ	$-x$	$-\tau$	$-u$	σ	$-v$	$-\sigma$
	14	$-x$	$-\tau$	$-w$	$-\tau$	$-v$	$-\sigma$	$-u$	$-\sigma$
	15	$-u$	σ	$-v$	$-\sigma$	$-w$	τ	$-x$	$-\tau$
	16	$-v$	$-\sigma$	$-u$	$-\sigma$	$-x$	$-\tau$	$-w$	$-\tau$

		I.		II.		III.		IV.	
8	9	y	σ	y	σ	y	σ	y	σ
	10	z	σ	z	$-\sigma$	z	σ	z	$-\sigma$
	11	α	τ	α	τ	α	τ	α	τ
	12	β	τ	β	$-\tau$	β	τ	β	$-\tau$
	13	$-\beta$	$-\tau$	$-\alpha$	$-\tau$	$-z$	$-\sigma$	$-y$	$-\sigma$
	14	$-\alpha$	$-\tau$	$-\beta$	τ	$-y$	$-\sigma$	$-z$	σ
	15	$-z$	$-\sigma$	$-y$	$-\sigma$	$-\beta$	$-\tau$	$-\alpha$	$-\tau$
	16	$-y$	$-\sigma$	$-z$	σ	$-\alpha$	$-\tau$	$-\beta$	τ
9	10	$+1$		$+1$		$+1$		$+1$	
	11	e	$-\lambda$	e	$-\lambda$	e	$-\lambda$	e	$-\lambda$
	12	i	λ	i	λ	i	λ	i	λ
	13	m	$-\nu$	q	ν	u	$-\sigma$	y	σ
	14	q	ν	m	ν	y	σ	u	σ
	15	u	$-\sigma$	y	σ	m	$-\nu$	q	ν
	16	y	σ	u	σ	q	ν	m	ν
10	11	j	λ	j	λ	j	λ	j	λ
	12	f	λ	f	λ	f	λ	f	λ
	13	r	ν	n	ν	z	σ	v	σ
	14	n	ν	r	$-\nu$	v	σ	z	$-\sigma$
	15	z	σ	v	σ	r	ν	n	ν
	16	v	σ	z	$-\sigma$	n	ν	r	$-\nu$
11	12	$+1$		$+1$		1		$+1$	
	13	w	$-\tau$	α	τ	o	$-\rho$	s	ρ
	14	α	τ	w	τ	s	ρ	o	ρ
	15	o	$-\rho$	s	ρ	w	$-\tau$	α	τ
	16	s	ρ	o	ρ	α	τ	w	τ
12	13	β	τ	x	τ	t	ρ	p	ρ
	14	x	τ	β	$-\tau$	p	ρ	t	$-\rho$
	15	t	ρ	p	ρ	β	τ	x	τ
	16	p	ρ	t	$-\rho$	x	τ	β	$-\tau$
13	14	$+1$		$+1$		$+1$		$+1$	
	15	g	$-\mu$	g	$-\mu$	g	$-\mu$	g	$-\mu$
	16	k	μ	k	μ	k	μ	k	μ
14	15	l	μ	l	μ	l	μ	l	μ
	16	h	μ	h	μ	h	μ	h	μ
15	16	$+1$		$+1$		$+1$		$+1$	

We have now for the four arrangements respectively, by means of hitherto unused tetrads, the following determinations of sign: these being in each case inconsistent with each other.

First arrangement.

3	5	9	15	+	−	θ .	−	σ
3	9	5	15	−	−	λ .		ρ
3	15	5	9	+		μ .	−	ν

that is,

$$\theta\sigma = \lambda\rho = -\mu\nu,$$

3	5	10	16	+	−	θ .		σ
3	10	5	16	−		λ .	−	ρ
3	16	5	10	+	−	μ .		ν

$$-\theta\sigma = \lambda\rho = -\mu\nu,$$

3	5	11	13	+	−	θ .	−	τ
3	11	5	13	−		λ .		ν
3	13	5	11	+	−	μ .	−	ρ

$$\theta\tau = -\lambda\nu = \mu\rho,$$

3	5	12	14	+	−	θ .		τ
3	12	5	14	−	−	λ .	−	ν
3	14	5	12	+		μ .		ρ

$$-\theta\tau = -\lambda\nu = \mu\rho.$$

Second arrangement.

3	5	9	16	+	−	θ .		σ
3	9	5	16	−	−	λ .	−	ρ
3	16	5	9	+	−	μ .		ν

$$\theta\sigma = \lambda\rho = \mu\nu,$$

3	5	10	15	+	−	θ .		σ
3	10	5	15	−		λ .	−	ρ
3	15	5	10	+		μ .		ν

$$-\theta\sigma = \lambda\rho = \mu\nu,$$

3	5	11	14	+	−	θ .		τ
3	11	5	14	−		λ .	−	ν
3	14	5	11	+		μ .		ρ

$$-\theta\tau = \lambda\nu = \mu\rho,$$

3	5	12	13	+	−	θ .		τ
3	12	5	13	−	−	λ .	−	ν
3	13	5	12	+	−	μ .		ρ

$$\theta\tau = \lambda\nu = \mu\rho.$$

Third arrangement.

$$\begin{array}{lll}
3\ \ 5\ \ 9\ 13 & +-\theta.-\sigma & \\
3\ \ 9\ \ 5\ 13 & --\lambda.\ \ \rho & \theta\sigma=-\lambda\rho=\ \ \mu\nu, \\
3\ 13\ \ 5\ \ 9 & +-\mu.-\nu & \\
\hline
3\ \ 5\ 10\ 14 & +-\theta.\ \ \sigma & \\
3\ 10\ \ 5\ 14 & -\ \ \lambda.-\rho & \theta\sigma=-\lambda\rho=-\mu\nu, \\
3\ 14\ \ 5\ 10 & +\ \ \mu.\ \ \nu & \\
\hline
3\ \ 5\ 11\ 15 & +-\theta.-\tau & \\
3\ 11\ \ 5\ 15 & -\ \ \lambda.\ \ \nu & \theta\tau=-\lambda\nu=-\mu\rho, \\
3\ 15\ \ 5\ 11 & +\ \ \mu.-\rho & \\
\hline
3\ \ 5\ 12\ 16 & +-\theta.\ \ \tau & \\
3\ 12\ \ 5\ 16 & --\lambda.-\nu & \theta\tau=-\lambda\nu=\ \ \mu\rho. \\
3\ 16\ \ 5\ 12 & +-\mu.\ \ \rho &
\end{array}$$

Fourth arrangement.

$$\begin{array}{lll}
3\ \ 5\ \ 9\ 14 & +-\theta.\ \ \sigma & \\
3\ \ 9\ \ 5\ 14 & --\lambda.-\rho & \theta\sigma=\ \ \lambda\rho=-\mu\nu, \\
3\ 14\ \ 5\ \ 9 & +\ \ \mu.\ \ \nu & \\
\hline
3\ \ 5\ 10\ 13 & +-\theta.\ \ \sigma & \\
3\ 10\ \ 5\ 13 & -\ \ \lambda.-\rho & \theta\sigma=-\lambda\rho=\ \ \mu\nu, \\
3\ 13\ \ 5\ 10 & +-\mu.\ \ \nu & \\
\hline
3\ \ 5\ 11\ 16 & +-\theta.\ \ \tau & \\
3\ 11\ \ 5\ 16 & -\ \ \lambda.-\nu & \theta\tau=-\lambda\nu=\ \ \mu\rho, \\
3\ 16\ \ 5\ 11 & +-\mu.\ \ \rho & \\
\hline
3\ \ 5\ 12\ 15 & +-\theta.-\tau & \\
3\ 12\ \ 5\ 15 & --\lambda.-\nu & \theta\tau=-\lambda\nu=-\mu\rho. \\
3\ 15\ \ 5\ 12 & +-\mu.\ \ \rho &
\end{array}$$

And it hence finally appears, that we cannot, in any one of the four arrangements, determine the signs so as to give rise to a 16-square theorem; that is, the product of a sum of 16 squares into a sum of 16 squares cannot be made equal to a sum of 16 squares.

764.

THE BINOMIAL EQUATION $x^p - 1 = 0$: QUINQUISECTION.

[From the *Proceedings of the London Mathematical Society*, vol. XII. (1881), pp. 15, 16. Read December 9, 1880.]

THE theory should be precisely analogous to those for the trisection and quartisection (see my paper, "The Binomial Equation $x^p - 1 = 0$, Trisection and Quartisection," *Proceedings of the London Mathematical Society*, vol. XI. (1879), pp. 4—17, [731]), only I have not been able to carry it so far. We have in the present case five periods X, Y, Z, W, T, the actual expressions for which, $X = \eta^1 + \ldots$, $Y = \eta^3 + \ldots$, etc., with Reuschle's selected prime root g, can be (for the primes $5n+1$ under 100) at once written down by means of the table given, pp. 16, 17, of that paper; [see this volume, pp. 95, 96]. The relations between the periods are of the form

$$\begin{array}{rccccc} & X & Y & Z & W & T \\ \hline X^2 = & a & b & c & d & e \\ XY = & f & g & h & i & j \\ XZ = & k & l & m & n & o; \end{array}$$

that is, we have

$$X^2 = (a, b, c, d, e \between X, Y, Z, W, T),$$

and thence, by cyclical permutations,

$$Y^2 = (e, a, b, c, d \between \quad ,, \quad), \text{ etc.};$$

viz. from the value of X^2 we have those of Y^2, Z^2, W^2, T^2; from the value of XY those of YZ, ZW, WT, TX; and from the value of XZ those of YW, ZT, WX, TY.

From the equation $X+Y+Z+W+T=-1$, multiplying by X and then substituting for X^2, XY, &c., their values, we obtain

$$\begin{aligned} -a &= 1+f+k+m+g, \\ -b &= \quad g+l+n+h, \\ -c &= \quad h+m+o+i, \\ -d &= \quad i+n+k+j, \\ -e &= \quad j+o+l+f, \end{aligned}$$

which determine (a, b, c, d, e) in terms of (f, g, h, i, j) and (k, l, m, n, o). It is, moreover, easy to prove that

$$\begin{aligned} f+g+h+i+j &= \tfrac{1}{5}(p-1), \\ k+l+m+n+o &= \tfrac{1}{5}(p-1), \end{aligned}$$

whence also

$$a+b+c+d+e = -1-\tfrac{4}{5}(p-1).$$

We obtain other relations between the coefficients by considering the two triple products XYZ and XYW: these are all that need be considered, since the other triple products are deducible from them by cyclical permutations. From the first of these we have

$$X.YZ = Y.XZ = Z.XY,$$

and from the second

$$X.YW = Y.XW = W.XY;$$

and if we herein substitute for YZ, XZ, &c., their values, and then in the resulting equations for X^2, XY, &c., their values as linear functions of X, Y, Z, W, T, we obtain in all $5.2.2=20$ quadric relations between the 15 coefficients; or if we substitute for (a, b, c, d, e) their foregoing values, in all 20 relations between the 10 coefficients (f, g, h, i, j) and (k, l, m, n, o). These are at most equivalent to 8 independent equations, since we have, besides, the sums $f+g+h+i+j$ and $k+l+m+n+o$ each $=\frac{1}{5}(p-1)$; but I have not succeeded in finding the connexions between them, or even in ascertaining to how many independent equations they are equivalent.

For any given prime $p=5n+1$, the values of the coefficients, and also the coefficients of the quintic equation for the periods, could of course be calculated directly from the expressions of the periods; but for the primes under 100, that is, for the values 11, 31, 41, 61, 71, they are at once obtained from Reuschle. We have thus the two Tables, the former giving the coefficients $a, b, \ldots, n, o$, and the latter the coefficients of the quintic equations.

TABLE 1.

p	a	b	c	d	e
	f	g	h	i	j
	k	l	m	n	o
11	− 2	− 1	− 2	− 2	− 2
	1	0	0	1	0
	0	0	0	1	1
31	− 4	− 6	− 6	− 4	− 5
	0	1	2	1	2
	0	2	2	1	1
41	− 8	− 5	− 6	− 6	− 8
	3	0	2	1	2
	2	2	2	1	1
61	− 10	− 9	− 12	− 8	− 10
	3	2	2	3	2
	0	2	4	3	3
71	− 14	− 10	− 12	− 9	− 12
	4	2	3	2	3
	2	3	5	2	2

TABLE 2 OF THE QUINTIC EQUATIONS.

COEFFICIENTS OF

p	η^5	η^4	η^3	η^2	η^1	1	$= 0.$
11	1	1	− 4	− 3	+ 3	+ 1	
31	1	1	− 12	− 2	+ 1	+ 5	
41	1	1	− 16	+ 5	+ 21	− 9	
61	1	1	− 24	− 17	+ 41	− 23	
71	1	1	− 28	+ 37	+ 25	+ 1	

765.

ON THE FLEXURE AND EQUILIBRIUM OF A SKEW SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. XII. (1881), pp. 103—108. Read March 10, 1881.]

THE skew surface is taken to be such that the strip between two consecutive generating lines is rigid, and that the flexure takes place by the rotation of the strips about the generating lines successively. The theory of the flexure is well known, but I am not aware that the theory of the equilibrium of such a surface, when acted upon by any given forces, has been considered; it is, however, a question which presents itself naturally in connexion with those relating to other continuous bodies treated of in the *Mécanique Analytique*, and forms a good example of the principles made use of.

To begin with the mechanical theory: we may regard the forces as acting on the generating lines regarded as material lines; and if for an element of mass dm, coordinates (x, y, z) of a particular generating line G, the forces parallel to the axes are X', Y', Z', then the corresponding term in the equation of equilibrium is

$$S(X'\delta x + Y'\delta y + Z'\delta z)\, dm;$$

and observing that there are (as will afterwards appear) *five* geometrical conditions, which I represent by $U_1 = 0$, $U_2 = 0, \ldots, U_5 = 0$, the equation of equilibrium is

$$S\{(X'\delta x + Y'\delta y + Z'\delta z)\, dm + T_1\delta U_1 + T_2\delta U_2 + T_3\delta U_3 + T_4\delta U_4 + T_5\delta U_5\} = 0,$$

where $T_1, T_2, \ldots, T_5$ are the indeterminate multipliers, representing colligation-forces which correspond to the five geometrical conditions respectively.

Taking (ξ, η, ζ) for the coordinates of a particular point P on the generating line; p, q, r for the cos-inclinations of the line (whence $U_1 = p^2 + q^2 + r^2 - 1 = 0$ is one of the geometrical relations), and ρ for the distance of dm from P, we have

$$x,\ y,\ z = \xi + \rho p,\ \eta + \rho q,\ \zeta + \rho r,$$
$$\delta x,\ \delta y,\ \delta z = \delta\xi + \rho\delta p,\ \delta\eta + \rho\delta q,\ \delta\zeta + \rho\delta r.$$

The summation S extends first to the different points of the generating line, and then to the different generating lines; applying it first to the particular generating line, we write

$$\begin{array}{rcccccc} & SX'dm, & SY'dm, & SZ'dm, & SX'\rho dm, & SY'\rho dm, & SZ'\rho dm \\ = & X, & Y, & Z, & L, & M, & N, \end{array}$$

where X, Y, Z are the whole forces, and L, M, N the whole moments about the point P, for the generating line G; retaining the same summatory symbol S, as now referring to the different generating lines, the equation becomes

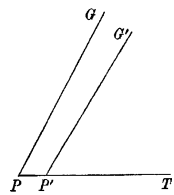
$$S\{X\delta\xi + Y\delta\eta + Z\delta\zeta + L\delta p + M\delta q + N\delta r + T_1\delta U_1 + \ldots + T_5\delta U_5\} = 0.$$

We have now to consider the geometrical theory of the flexure. Taking on the skew surface an arbitrary curve cutting each generating line G in a point P, coordinates (ξ, η, ζ), and taking σ for the distance along the curve of the point P from a fixed point of the curve; also p, q, r, as before, for the cos-inclinations of the generating line G, then when the surface is in a determinate state, ξ, η, ζ, p, q, r are given functions of σ; but these functions vary with the flexure of the surface, with, however, certain relations unaffected by the flexure; and the problem is to find first these relations. As already mentioned, one of them is $p^2 + q^2 + r^2 - 1 = 0$.

Taking P' as the consecutive point on the curve, so that the direction of the element PP' is that of the tangent PT at P, it is convenient to write l, m, n for the cosine-inclinations of the tangent; we have, it is clear,

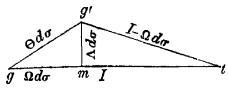
$$l,\ m,\ n = \frac{d\xi}{d\sigma},\ \frac{d\eta}{d\sigma},\ \frac{d\zeta}{d\sigma};\quad l^2 + m^2 + n^2 - 1 = 0.$$

The conditions in order to the rigidity of the strip, are that the angles GPP', $G'P'P$ $(= 180^\circ - G'P'T)$, and the inclination $G'P'$ to GP, shall have given values,



variable it may be from strip to strip—that is, these values must be given functions of σ. Taking $\angle GPT = I$, the value of $G'P'T$ can differ only infinitesimally from that of GPT, and we take it to be $G'P'T = I - \Omega d\sigma$; also the inclination GP to $G'P'$ is an infinitesimal, $= \Theta d\sigma$: we have I, Ω, Θ given functions of σ. It is to be remarked that these conditions imply, inclination of $G'P'$ to tangent plane GPT at P has a given value $\Lambda d\sigma$; in fact, if through P we draw a line $P\gamma$ parallel to $P'G'$, then, if P is regarded as the centre of a sphere which meets PG, $P\gamma$, PT in the

points g, g', t respectively, we have a spherical triangle $gg't$, the sides of which are $I-\Omega d\sigma$, I, and $\Theta d\sigma$, and of which the perpendicular $g'm$ is $=\Lambda d\sigma$; we have thus an infinitesimal right-angled triangle, the base and altitude of which are $\Omega d\sigma$, $\Lambda d\sigma$,



and the hypothenuse is $\Theta d\sigma$; whence $\Theta^2=\Omega^2+\Lambda^2$. In the case of the developable surface $\Lambda=0$ and $\Theta=\Omega$. It may be remarked that, when the curve on the skew surface is the line of striction, we have $\Omega=0$; in fact, taking P to be on the line of striction, the line

$$\frac{X-\xi}{qr'-q'r}=\frac{Y-\eta}{rp'-r'p}=\frac{Z-\zeta}{pq'-p'q},$$

through (ξ, η, ζ) at right angles to the two generating lines, meets the consecutive generating line X, Y, $Z=\xi'+\rho p'$, $\eta'+\rho q'$, $\zeta'+\rho r'$; and the condition that this may be so is easily found to be $\Omega=0$.

Take, for a moment, p', q', r' for the cos-inclinations of the consecutive generating line $P'G'$; we have

$$\begin{aligned} lp\ +mq\ +nr\ &=\cos I,\\ lp'+mq'+nr'&=\cos(I-\Omega d\sigma),\\ pp'+qq'\ +rr'&=\cos\Theta d\sigma;\end{aligned}$$

and then writing p', q', $r'=p+dp$, $q+dq$, $r+dr$, and observing that the equation $p'^2+q'^2+r'^2=1$ gives

$$pdp+qdq+rdr=-\tfrac{1}{2}(dp^2+dq^2+dr^2),$$

these equations and the before-mentioned two equations become

$$\begin{aligned} &(U_1)\quad p^2+q^2\ +r^2-1=0,\\ &(U_2)\quad l^2\ +m^2+n^2-1=0,\\ &(U_3)\quad lp+mq+nr-\cos I=0,\\ &(U_4)\quad ldp+mdq+ndr-\Omega\sin Id\sigma=0,\\ &(U_5)\quad dp^2+dq^2+dr^2-\Theta^2d\sigma^2=0,\end{aligned}$$

which equations, considering therein l, m, n as standing for their values $\frac{d\xi}{d\sigma}$, $\frac{d\eta}{d\sigma}$, $\frac{d\zeta}{d\sigma}$, are the geometrical relations which connect the six variables ξ, η, ζ, p, q, r, considered as functions of σ. And in these equations I, Ω, Θ denote given functions of σ, invariable by any flexure of the surface.

To complete the geometrical theory, it is to be observed that we can by flexure bring the generating lines of the surface to be parallel to those of any given cone

$C(p, q, r)=0$, where $C(p, q, r)$ denotes a homogeneous function of (p, q, r). Hence, joining to the foregoing five equations this new equation

$$C(p, q, r)=0,$$

these six equations determine $\xi, \eta, \zeta, p, q, r$ as functions of σ. To make the solution completely determinate, we have only to assume for the point P, which corresponds, say, to the value $\sigma=0$, a position in space at pleasure, and to take the corresponding generating line PG parallel to a generating line, at pleasure, of the cone.

As an example, writing γ to denote an arbitrary constant angle, if the invariable conditions are

$$I=\gamma, \quad \Theta=\sin\gamma, \quad \Omega=0,$$

then the five equations are

$$\begin{aligned} p^2+\ q^2+\ r^2-\ 1\ &=0, \\ l^2+\ m^2+\ n^2-\ 1\ &=0, \\ lp+\ mq+\ nr-\ \cos\gamma\ &=0, \\ dp^2+\ dq^2+dr^2-\sin^2\gamma\, d\sigma^2 &=0, \\ ldp+mdq+ndr\ &=0. \end{aligned}$$

We assume *first*

$$C(p, q, r)=p^2+q^2-r^2\tan^2\gamma, =0;$$

and *secondly*

$$C(p, q, r)=r, =0.$$

Then, in the former case, we find the solution

$$p, q, r=-\sin\gamma\sin\sigma, \ \sin\gamma\cos\sigma, \ \cos\gamma;$$

$$\xi, \eta, \zeta=\ \cos\sigma, \ \sin\sigma, \ 0;$$

giving

$$x, y, z=\cos\sigma-\rho\sin\gamma\sin\sigma, \ \sin\sigma+\rho\sin\gamma\cos\sigma, \ \cos\gamma;$$

and consequently

$$x^2+y^2-z^2\tan^2\gamma=0,$$

the hyperboloid of revolution. And, in the latter case,

$$p, q, r=\cos(\sigma\sin\gamma), \ \sin(\sigma\sin\gamma), \ 0,$$

$$\xi, \eta, \zeta=\cot\gamma\sin(\sigma\sin\gamma), \ -\cot\gamma\cos(\sigma\sin\gamma), \ \sigma\sin\gamma,$$

that is,

$$x, y=\cot\gamma\sin z+\rho\cos z, \ -\cot\gamma\cos z+\rho\sin z,$$

whence

$$x\sin z-y\cos z=\cot\gamma,$$

a skew helicoid generated by horizontal tangents of the cylinder $x^2+y^2=\cot^2\gamma$. This is a known deformation of the hyperboloid.

Returning now to the mechanical problem, we have to consider the terms

$$\begin{aligned} S\,.\; & T_1\delta\tfrac{1}{2}(p^2+q^2+r^2-1) \\ & + T_2\delta\tfrac{1}{2}(l^2+m^2+n^2-1) \\ & + T_3\delta(lp+mq+nr-\cos I) \\ & + T_4\delta\left(l\frac{dp}{d\sigma}+m\frac{dq}{d\sigma}+n\frac{dr}{d\sigma}\Omega\sin I\right) \\ & + T_5\delta\tfrac{1}{4}\left\{\left(\frac{dp}{d\sigma}\right)^2+\left(\frac{dq}{d\sigma}\right)^2+\left(\frac{dr}{d\sigma}\right)^2-\Theta^2\right\}. \end{aligned}$$

The first term gives, under the sign S,

$$T_1(p\delta p+q\delta q+r\delta r). \qquad (*)$$

The second term gives, in the first instance,

$$\frac{T_2}{d\sigma}(ld\delta\xi+md\delta\eta+nd\delta\zeta);$$

or, since in general

$$S\Omega d\delta\xi=\Omega''\delta\xi''-\Omega'\delta\xi'+S(-d\Omega\,.\,\delta\xi),$$

then, attending only to the terms under the sign S, these are

$$=-\frac{d}{d\sigma}T_2l\,.\,\delta\xi-\frac{d}{d\sigma}T_2m\,.\,\delta\eta-\frac{d}{d\sigma}T_2n\,.\,\delta\zeta. \qquad (*)$$

The third term gives

$$\begin{aligned} & T_3(l\delta p+m\delta q+n\delta r) \qquad (*) \\ & + T_3(p\delta l+q\delta m+r\delta n), \end{aligned}$$

where the second line,

$$=\frac{T_3}{d\sigma}(pd\delta\xi+qd\delta\eta+rd\delta\zeta),$$

attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_3p\,.\,\delta\xi-\frac{d}{d\sigma}T_3q\,.\,\delta\eta-\frac{d}{d\sigma}T_3r\,.\,\delta\zeta. \qquad (*)$$

The fourth term gives

$$\begin{aligned} & T_4\left(\frac{dp}{d\sigma}\delta l+\frac{dq}{d\sigma}\delta m+\frac{dr}{d\sigma}\delta n\right) \\ & + T_4\left(\frac{l}{d\sigma}d\delta p+\frac{m}{d\sigma}d\delta q+\frac{n}{d\sigma}d\delta r\right), \end{aligned}$$

where the first line, written under the form

$$\frac{T_4}{d\sigma}\left(\frac{dp}{d\sigma}d\delta\xi+\frac{dq}{d\sigma}d\delta\eta+\frac{dr}{d\sigma}d\delta\zeta\right),$$

and attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}\left(T_4\frac{dp}{d\sigma}\right).\,\delta\xi-\frac{d}{d\sigma}\left(T_4\frac{dq}{d\sigma}\right).\,\delta\eta-\frac{d}{d\sigma}\left(T_4\frac{dr}{d\sigma}\right).\,\delta\zeta; \qquad (*)$$

and the second line, attending in like manner only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_4 l\,.\,\delta p-\frac{d}{d\sigma}T_4 m\,.\,\delta q-\frac{d}{d\sigma}T_4 n\,.\,\delta r. \qquad (*)$$

The fifth term, written under the form

$$\frac{T_5}{d\sigma}\left(\frac{dp}{d\sigma}d\delta p+\frac{dq}{d\sigma}d\delta q+\frac{dr}{d\sigma}d\delta r\right), \qquad (*)$$

and attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_5\frac{dp}{d\sigma}\,.\,\delta p-\frac{d}{d\sigma}T_5\frac{dq}{d\sigma}\,.\,\delta q-\frac{d}{d\sigma}T_5\frac{dr}{d\sigma}\,.\,\delta r; \qquad (*)$$

where in each case I have marked with an asterisk the lines which present themselves in the final result.

Hence, joining to the foregoing the force-terms

$$X\delta\xi+Y\delta\eta+Z\delta\zeta+L\delta p+M\delta q+N\delta r, \qquad (*)$$

and equating to zero the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr respectively, we have

$$\left\{\begin{array}{llll}
0=X & -\frac{d}{d\sigma}T_2 l & -\frac{d}{d\sigma}T_3 p & -\frac{d}{d\sigma}T_4\frac{dp}{d\sigma}, \\
0=Y & -\frac{d}{d\sigma}T_2 m & -\frac{d}{d\sigma}T_3 q & -\frac{d}{d\sigma}T_4\frac{dq}{d\sigma}, \\
0=Z & -\frac{d}{d\sigma}T_2 n & -\frac{d}{d\sigma}T_3 r & -\frac{d}{d\sigma}T_4\frac{dr}{d\sigma}, \\
0=L+T_1 p & + & T_3 l & -\frac{d}{d\sigma}T_4 l\ -\frac{d}{d\sigma}T_5\frac{dp}{d\sigma}, \\
0=M+T_1 q & + & T_3 m & -\frac{d}{d\sigma}T_4 m-\frac{d}{d\sigma}T_5\frac{dq}{d\sigma}, \\
0=N+T_1 r & + & T_3 n & -\frac{d}{d\sigma}T_4 n\ -\frac{d}{d\sigma}T_5\frac{dr}{d\sigma},
\end{array}\right.$$

where it will be recollected that l, m, n stand for $\frac{d\xi}{d\sigma}$, $\frac{d\eta}{d\sigma}$, $\frac{d\zeta}{d\sigma}$, the variables being ξ, η, ζ, p, q, r, and σ. The elimination of T_1, T_2, ..., T_5 from the six equations should lead to a relation between ξ, η, ζ, p, q, r, which, with the foregoing five relations, would determine the six variables ξ, η, ζ, p, q, r in terms of σ.

In particular, the forces and moments X, Y, Z, L, M, N may all of them vanish; assuming that T_1, T_2, ..., T_5 do not all of them vanish, we still have the sixth relation, which (with the foregoing five relations) determines ξ, η, ζ, p, q, r in terms of σ; and it is to be remarked that the problem in question, of the figure of equilibrium of the skew surface not acted upon by any forces, is analogous to that of the geodesic line in space; only whilst here the solution is, curve a straight line, the solution for the case of the skew surface depends upon equations of a complex enough form; in the case of the developable surface, the required figure is of course the plane.

766.

ON THE GEODESIC CURVATURE OF A CURVE ON A SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. XII. (1881), pp. 110—117. Read April 14, 1881.]

THERE is contained in Liouville's Note II. to his edition of Monge's *Application de l'Analyse à la Géométrie* (Paris, 1850), see pp. 574 and 575, the following formula,

$$\frac{1}{\rho} = -\frac{di}{ds} + \frac{1}{2G\sqrt{E}}\frac{dG}{du}\cos i - \frac{1}{2E\sqrt{G}}\frac{dE}{dv}\sin i,$$

$$= -\frac{di}{ds} + \frac{\cos i}{\rho_2} + \frac{\sin i}{\rho_1},$$

which gives the radius of geodesic curvature of a curve upon a surface when the position of a point on the surface is defined by the parameters u, v, belonging to a system of orthotomic curves; or, what is the same thing, such that

$$ds^2 = Edu^2 + Gdv^2.$$

Writing with Gauss p, q instead of u, v, I propose to obtain the corresponding formula in the general case where the parameters p, q are such that

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

I call to mind that, if PQ, PQ' are equal infinitesimal arcs on the given curve and on its tangent geodesic, then the radius of geodesic curvature ρ is, by definition, a length ρ such that $2\rho . QQ' = \overline{PQ}^2$. More generally, if the curves on the surface are any two curves which touch each other, then ρ as thus determined is the radius of relative curvature of the two curves.

The notation is that of the Memoir, "Disquisitiones generales circa superficies curvas" (1827), Gauss, *Werke*, t. III.; see also my paper "On geodesic lines, in particular those of a quadric surface," *Proc. Lond. Math. Society*, t. IV. (1872), pp. 191—211, [508]; and Salmon's *Solid Geometry*, 3rd ed., 1874, pp. 251 *et seq.* The coordinates (x, y, z) of a point on the surface are taken to be functions of two independent parameters p, q; and we then write

$$dx + \tfrac{1}{2}d^2x = a\,dp + a'dq + \tfrac{1}{2}(\alpha dp^2 + 2\alpha' dpdq + \alpha'' dq^2),$$

$$dy + \tfrac{1}{2}d^2y = b\,dp + b'dq + \tfrac{1}{2}(\beta dp^2 + 2\beta' dpdq + \beta'' dq^2),$$

$$dz + \tfrac{1}{2}d^2z = c\,dp + c'dq + \tfrac{1}{2}(\gamma dp^2 + 2\gamma' dpdq + \gamma'' dq^2):$$

$$E,\ F,\ G = a^2 + b^2 + c^2,\quad aa' + bb' + cc',\quad a'^2 + b'^2 + c'^2;\quad V^2 = EG - F^2;$$

and therefore

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2,$$

where E, F, G are regarded as given functions of p and q.

To determine a curve on the surface, we establish a relation between the two parameters p, q, or, what is the same thing, take p, q to be functions of a single parameter θ; and we write as usual p', p'', q', etc., to denote the differential coefficients of p, q, etc., in regard to θ; we write also E_1, E_2, etc., to denote the differential coefficients $\dfrac{dE}{dp}, \dfrac{dE}{dq}$, etc. In the first instance, θ is taken to be an arbitrary parameter, but we afterwards take it to be the length s of the curve from a fixed point thereof.

First formula for the radius of relative curvature.

Consider any two curves touching at the point P, coordinates (x, y, z) which are regarded as given functions of (p, q); where (p, q) are for the one curve given functions, and for the other curve other given functions, of θ.

The coordinates of a consecutive point for the one curve are then

$$x + dx + \tfrac{1}{2}d^2x,\quad y + dy + \tfrac{1}{2}d^2y,\quad z + dz + \tfrac{1}{2}d^2z,$$

where

$$dp = p'd\theta + \tfrac{1}{2}p''d\theta^2,\quad dq = q'd\theta + \tfrac{1}{2}q''d\theta^2;$$

hence these coordinates are

$$x + (ap' + a'q')\,d\theta + \tfrac{1}{2}(\alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2)\,d\theta^2 + \tfrac{1}{2}(ap'' + a'q'')\,d\theta^2,$$
$$\vdots$$

and for the other curve they are in like manner

$$x + (ap' + a'q')\,d\theta + \tfrac{1}{2}(\alpha p'^2 + 2\alpha' p'q' + \alpha'' q'^2)\,d\theta^2 + \tfrac{1}{2}(aP'' + a'Q'')\,d\theta^2,$$
$$\vdots$$

the only difference being in the terms which contain the second differential coefficients, p'', q'' for the first curve, and P'', Q'' for the second curve. Hence the differences of the coordinates are

$$\tfrac{1}{2}\{a(p''-P'')+a'(q''-Q'')\}\,d\theta^2,\quad \tfrac{1}{2}\{b(p''-P'')+b'(q''-Q'')\}\,d\theta^2,$$
$$\tfrac{1}{2}\{c(p''-P'')+c'(q''-Q'')\}\,d\theta^2,$$

and consequently the distance QQ' of the two consecutive points Q, Q' is

$$=\tfrac{1}{2}\sqrt{(E,\ F,\ G \between p''-P'',\ q''-Q'')^2}\,d\theta^2.$$

The squared arc $\overline{PQ}^2$ is

$$=(E,\ F,\ G \between p',\ q')^2\,d\theta^2;$$

and hence, if as before $2\rho\,.\,QQ'=PQ^2$, that is, $\dfrac{1}{\rho}=2QQ'\div\overline{PQ}^2$, then

$$\frac{1}{\rho}=\frac{\sqrt{(E,\ F,\ G \between p''-P'',\ q''-Q'')^2}}{(E,\ F,\ G \between p',\ q')^2},$$

the required formula for ρ.

Second formula for the radius of relative curvature.

We now take the variable θ to be the length s of the curve measured from a fixed point thereof, so that p', p'', etc. denote $\dfrac{dp}{ds}$, $\dfrac{d^2p}{ds^2}$, etc. We have therefore

$$1=(E,\ F,\ G \between p',\ q')^2,$$

and the formula becomes

$$\frac{1}{\rho}=\sqrt{(E,\ F,\ G \between p''-P'',\ q''-Q'')^2}.$$

But, differentiating the above equation as regards the curve, we find

$$0=2(E,\ F,\ G \between p',\ q' \between p'',\ q'')+(\dot{E},\ \dot{F},\ \dot{G} \between p',\ q')^2,$$

where $\dot{E}$, $\dot{F}$, $\dot{G}$ are used to denote the complete differential coefficients $E_1p'+E_2q'$, etc. And similarly, differentiating in regard to the tangent geodesic, we obtain

$$0=2(E,\ F,\ G \between p',\ q' \between P'',\ Q'')+(\dot{E},\ \dot{F},\ \dot{G} \between p',\ q')^2;$$

and hence, taking the difference of the two equations,

$$0=(E,\ F,\ G \between p',\ q' \between p''-P'',\ q''-Q'').$$

Hence, in the equation for $\dfrac{1}{\rho}$, the function under the radical sign may be written

$$(E,\ F,\ G \between p',\ q')^2.(E,\ F,\ G \between p''-P'',\ q''-Q'')^2-\{(E,\ F,\ G \between p',\ q' \between p''-P'',\ q''-Q'')\}^2,$$

which is identically

$$=(EG-F^2)\{p'(q''-Q'')-q'(p''-P'')\}^2.$$

Hence, extracting the square root, and for $\sqrt{EG-F^2}$ writing V, we have

$$\frac{1}{\rho}=V\{p'(q''-Q'')-q'(p''-P'')\},$$

or say

$$\frac{1}{\rho}=V(p'q''-q'p'')-V(p'Q''-q'P''),$$

which is the new formula for the radius of relative curvature.

Formula for the radius of geodesic curvature.

In the paper "On Geodesic Lines, etc.," p. 195, [vol. VIII. of this Collection, p. 160], writing $EG-F^2=V^2$, and P'', Q'' in place of p'', q'', the differential equation of the geodesic line is obtained in the form

$$\begin{aligned}&(Ep'+Fq')\{(2F_1-E_2)p'^2+2G_1p'q'+G_2q'^2\}\\&-(Fp'+Gq')\{E_1p'^2+2E_2p'q'+(2F_2-G_1)q'^2\}\\&\qquad+2V^2(p'Q''-q'P'')=0;\end{aligned}$$

or, denoting by Ω the first two lines of this equation, we have

$$V(p'Q''-q'P'')=-\frac{\frac{1}{2}}{V}\Omega.$$

The foregoing equation gives therefore, for the radius of geodesic curvature,

$$\frac{1}{\rho}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}\Omega,$$

which is an expression depending only upon p', q', the first differential coefficients (common to the curve and geodesic), and on p'', q'', the second differential coefficients belonging to the curve.

Observe that Ω is a cubic function of p', q': we have

$$\Omega=(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}\between p',\ q')^3,$$

the values of the coefficients being

$$\begin{aligned}\mathfrak{A}&=2EF_1-\ EE_2-\ FE_1,\\\mathfrak{B}&=2EG_1+2FF_1-3FE_2-\ GE_1,\\\mathfrak{C}&=\ EG_2+3FG_1-2FF_2-2GE_2,\\\mathfrak{D}&=\ FG_2-2GF_2+\ GG_1.\end{aligned}$$

The Special Curves, $p=$ constant and $q=$ constant.

Consider the curve $p=\text{const.}$ For this curve $p'=0$, $p''=0$; therefore also $Gq'^2=1$, and, if R be the radius of geodesic curvature, then

$$\frac{1}{R}=\frac{\frac{1}{2}}{V}\mathfrak{D}q'^3, \quad =\frac{\frac{1}{2}}{V}\frac{\mathfrak{D}}{G\sqrt{G}}.$$

Similarly for the curve $q=\text{const.}$ Here $q'=0$, $q''=0$; therefore $Ep'^2=1$, and, if S be the radius of geodesic curvature, then

$$\frac{1}{S}=\frac{\frac{1}{2}}{V}\mathfrak{A}p'^3, \quad =\frac{\frac{1}{2}}{V}\frac{\mathfrak{A}}{E\sqrt{E}}.$$

These values of R and S are interesting for their own sakes, and they will be introduced into the expression for the radius of geodesic curvature ρ of the general curve.

Transformed Formula for the Radius of Geodesic Curvature.

From the values of $\frac{1}{R}$, $\frac{1}{S}$, we have

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}\left\{\Omega-\frac{\mathfrak{A}}{E}p'-\frac{\mathfrak{D}}{G}q'\right\},$$

where the term in { } is

$$=\mathfrak{A}p'^3-\frac{\mathfrak{A}}{E}p'+\mathfrak{B}p'^2q'+\mathfrak{C}p'q'^2+\mathfrak{D}q'^3-\frac{\mathfrak{D}}{G}q'.$$

The terms in $\mathfrak{A}$ are

$$=-\frac{\mathfrak{A}}{E}p'(1-Ep'^2), \quad =-\frac{\mathfrak{A}}{E}p'(2Fp'q'+Gq'^2),$$

and those in $\mathfrak{D}$ are

$$=-\frac{\mathfrak{D}}{G}q'(1-Gq'^2), \quad =-\frac{\mathfrak{D}}{G}q'(Ep'^2+2Fp'q').$$

Hence the whole expression contains the factor $p'q'$, and is, in fact,

$$=p'q'\left\{p'\left(\mathfrak{B}-\frac{2\mathfrak{A}F}{E}-\frac{\mathfrak{D}E}{G}\right)+q'\left(\mathfrak{B}-\frac{\mathfrak{A}G}{E}-\frac{2\mathfrak{D}F}{G}\right)\right\};$$

or substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ their values, this is

$$=p'q'\left\{p'\left(-GE_1+EG_1+\frac{2F^2E_1}{E}-2FF_1-FE_2+2EF_2-\frac{EFG_2}{G}\right)\right.$$
$$\left.+q'\left(-GE_2+EG_2-\frac{2F^2G_2}{G}+2FF_2+FG_1-2GF_1+\frac{GFE_1}{E}\right)\right\},$$

say this is

$$= p'q'(Lp' + Mq');$$

and the formula thus is

$$\frac{1}{\rho} - \frac{q'\sqrt{G}}{R} - \frac{p'\sqrt{E}}{S} = V(p'q'' - p''q') + \frac{\frac{1}{2}}{V} p'q'(Lp' + Mq').$$

Taking ϕ, θ to be the inclination of the curve to the curves $q = \text{const.}$, $p = \text{const.}$, respectively, and $\omega\,(= \phi + \theta)$ the inclination of these two curves to each other, then

$$\cos\phi = \frac{Fp' + Gq'}{\sqrt{G}}, \quad \cos\theta = \frac{Ep' + Fq'}{\sqrt{E}}, \quad \cos\omega = \frac{F}{\sqrt{EG}},$$

$$\sin\phi = \frac{Vp'}{\sqrt{G}}, \qquad \sin\theta = \frac{Vq'}{\sqrt{E}}, \qquad \sin\omega = \frac{V}{\sqrt{EG}};$$

hence $\dfrac{\sin\phi}{\sin\omega} = p'\sqrt{E}$, $\dfrac{\sin\theta}{\sin\omega} = q'\sqrt{G}$, and the formula may also be written

$$\frac{1}{\rho} - \frac{\sin\theta}{\sin\omega}\frac{1}{R} - \frac{\sin\phi}{\sin\omega}\frac{1}{S} = V(p'q'' - p''q') + \frac{\frac{1}{2}}{V} p'q'(Lp' + Mq').$$

The Orthotomic Case $F = 0$, or $ds^2 = Edp^2 + Gdq^2$.

The formula becomes in this case much more simple. We have

$$1 = Ep'^2 + Gq'^2, \quad V = \sqrt{EG}, \quad \omega = 90^\circ, \quad \sin\theta = \cos\phi;$$

and the term $Lp' + Mq'$ becomes $= E\dot{G} - \dot{E}G$, if, as before, $\dot{E}$, $\dot{G}$ denote the complete differential coefficients $E_1p' + E_2q'$ and $G_1p' + G_2q'$. The formula then is

$$\frac{1}{\rho} - \frac{\cos\phi}{R} - \frac{\sin\phi}{S} = V(p'q'' - p''q') + \frac{\frac{1}{2}}{V}(E\dot{G} - \dot{E}G),$$

where the values $\dfrac{1}{R}$ and $\dfrac{1}{S}$ are now $= \dfrac{\frac{1}{2}G_1}{G\sqrt{E}}$ and $\dfrac{-\frac{1}{2}E_2}{E\sqrt{G}}$, respectively. But we have moreover $\phi = \tan^{-1}\dfrac{p'\sqrt{E}}{q'\sqrt{G}}$, and thence

$$\phi' = q'\sqrt{G}\left(p''\sqrt{E} + \frac{\frac{1}{2}p'\dot{E}}{\sqrt{E}}\right) - p'\sqrt{E}\left(q''\sqrt{G} + \frac{\frac{1}{2}q'\dot{G}}{\sqrt{G}}\right),$$

$$= -V(p'q'' - p''q') - \frac{\frac{1}{2}}{V} p'q'(E\dot{G} - \dot{E}G);$$

or the formula finally is

$$\frac{1}{\rho} - \frac{\cos\phi}{R} - \frac{\sin\phi}{S} + \phi' = 0,$$

which is Liouville's formula referred to at the beginning of the present paper. It will be recollected that ϕ' is the differential coefficient $\dfrac{d\phi}{ds}$ with respect to the arc s of the curve.

ADDITION.—Since the foregoing paper was written, I have succeeded in obtaining a like interpretation of the term

$$V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq'),$$

which belongs to the general case. I find that these terms are, in fact, $=-\dot{\phi}+\omega_1 p'$; or, what is the same thing (since $\omega=\phi+\theta$ and therefore $\omega_1 p'+\omega_2 q'=\dot{\phi}+\dot{\theta}$), are $=\dot{\theta}-\omega_2 q'$. It will be recollected that ϕ is the inclination of the curve to the curve $q=c$, which passes through a given point of the curve, $\dot{\phi}$ is the variation of ϕ corresponding to the passage to the consecutive point of the curve, viz., $\phi+\dot{\phi}ds$ is the inclination at this consecutive point to the curve $q=c+dc$, which passes through the consecutive point; ω is the inclination to each other of the curves $p=b$, $q=c$, which pass through the given point of the curve, ω_1 the variation corresponding to the passage along the curve $q=c$, viz., $\omega+\omega_1 ds$ is the inclination to each other of the curves $p=b+db$, $q=c$; and the like as regards $\dot{\theta}$ and ω_2.

For the demonstration, we have, as above,

$$\phi=\tan^{-1}\frac{Vp'}{Fp'+Gq'},\quad \omega=\tan^{-1}\frac{V}{F},$$

where

$$V=\sqrt{EG-F^2};$$

and moreover $Ep'^2+2Fp'q'+Gq'^2=1$. In virtue of this last equation,

$$V^2p'^2+(Fp'+Gq')^2=G;$$

and we have

$$\dot{\phi}=-V(p'q''-p''q')+\frac{1}{G}\square,$$

where

$$\square=(Fp'+Gq')p'\dot{V}-Vp'(\dot{F}p'+\dot{G}q');$$

or, since $V^2=EG-F^2$, and thence $2V\dot{V}=G\dot{E}-2F\dot{F}+E\dot{G}$, we have

$$\square=\frac{\frac{1}{2}p'}{V}\{(Fp'+Gq')(G\dot{E}-2F\dot{F}+E\dot{G})-2(EG-F^2)(\dot{F}p'+\dot{G}q')\}.$$

Substituting herein for $\dot{E}$, $\dot{F}$, $\dot{G}$ their values $E_1p'+E_2q'$, $F_1p'+F_2q'$, $G_1p'+G_2q'$, the term in { } becomes

$$=Ip'^2+Jp'q'+Kq'^2,$$

where

$$I=FGE_1-2EGF_1+EFG_1,$$
$$J=G^2E_1-2FGF_1+(-EG+2F^2)G_1+FGE_2-2EGF_2+EFG_2,$$
$$K=G^2E_2-2FGF_2+(-EG+2F^2)G_2.$$

But from the equation $\omega=\tan^{-1}\frac{V}{F}$, differentiating in regard to p, we obtain

$$\omega_1=\frac{\frac{1}{2}}{EGV}(FG\dot{E}-2EG\dot{F}+EF\dot{G})=\frac{\frac{1}{2}}{EGV}I;$$

or, for p writing

$$p'(Ep'^2+2Fp'q'+Gq'^2), \quad = Ep'\left(p'^2+2\frac{F}{E}p'q'+\frac{G}{E}q'^2\right),$$

we have

$$\dot{\phi}-\omega_1 p' = -V(p'q''-p''q')+\frac{\frac{1}{2}p'}{GV}(Ip'^2+Jp'q'+Kq'^2)$$

$$-\frac{\frac{1}{2}p'}{GV}I\left(p'^2+2\frac{F}{E}p'q'+\frac{G}{E}q'^2\right).$$

The terms in p'^3 destroy each other, and the form thus is

$$\dot{\phi}-\omega_1 p' = -V(p'q''-p''q')-\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq'),$$

where

$$L = -\frac{J}{G}+\frac{2IF}{GE},$$

$$M = -\frac{K}{G}+\frac{I}{E};$$

and, upon substituting herein for I, J, K their values, we find

$$L = -GE_1+EG_1+\frac{2F^2E_1}{F}-2FF_1-FE_2+2EF_2-\frac{EFG_2}{G},$$

$$M = -GE_2+EG_2-\frac{2F^2G_2}{G}+2FF_2+FG_1-2GF_1+\frac{GFE_1}{E};$$

viz., these are the values denoted above by the same letters L, M. The final result thus is

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S} = -\dot{\phi}+\omega_1 p',$$

$$= \dot{\theta}-\omega_2 q',$$

where the meanings of the symbols have been already explained. A formula substantially equivalent to this, but in a different (and scarcely properly explained) notation, is given, Aoust, "Théorie des coordonnées curvilignes quelconques," *Annali di Matem.*, t. II. (1868), pp. 39—64; and I was, in fact, led thereby to the foregoing further investigation.

As to the definition of the radius of geodesic curvature, I remark that, for a curve on a given surface, if PQ be an infinitesimal arc of the curve, then if from Q we let fall the perpendicular QM on the tangent plane at P (the point M being thus a point on the tangent PT of the curve), and if from M, in the tangent plane and at right angles to the tangent, we draw MN to meet the osculating plane of the curve in N, then MN is in fact equal to the infinitesimal arc QQ' mentioned near the beginning of the present paper, and the radius of geodesic curvature ρ is thus a length such that $2\rho \,.\, MN = \overline{PQ}^2$.

767.

ON THE GAUSSIAN THEORY OF SURFACES.

[From the *Proceedings of the London Mathematical Society*, vol. XII. (1881), pp. 187—192. Read June 9, 1881.]

IN the Memoir, Bour, "Théorie de la déformation des surfaces" (*Jour. de l'Éc. Polyt.*, Cah. 39 (1862), pp. 1—148), the author, working with the form $ds^2 = dv^2 + g^2du^2$ as a special case of Gauss's formula $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$, obtains (p. 29) the following equations which he calls *fundamental*:—

$$[\text{IV.}] \ldots\ldots \begin{cases} \dfrac{1}{g}\dfrac{dg_1}{dv} = T^2 - HH_1, \\[2ex] \dfrac{dT}{du} + \dfrac{d\,.\,Hg}{dv} - H_1 g_1 = 0, \\[2ex] \dfrac{d\,.\,Tg^2}{dv} + g\dfrac{dH_1}{du} \quad = 0, \end{cases}$$

where g_1 is written to denote $\dfrac{dg}{dv}$, and where (see p. 26)

H is the curvature of the normal section containing the tangent to the curve $v = \text{constant}$,

H_1 is the curvature of the normal section at right angles to the preceding, containing the tangent to the (geodesic) curve $u = \text{constant}$,

T is the torsion of the same geodesic curve;

or, what is the same thing (see p. 25), the quadric equation for the determination of the principal radii of curvature at the point of the surface is

$$\left(\frac{1}{\rho} - H\right)\left(\frac{1}{\rho} - H_1\right) - T^2 = 0.$$

Writing for greater convenience K in place of the suffixed letter H_1, also V instead of g, so that the differential formula is $ds^2 = dv^2 + V^2du^2$, the equations become

$$\begin{cases} \dfrac{1}{V}\dfrac{d^2V}{dv^2} = T - HK, \\ \dfrac{dT}{du} + \dfrac{d.HV}{dv} - K\dfrac{dV}{dv} = 0, \\ \dfrac{d.TV^2}{dv} + V\dfrac{dK}{du} \qquad = 0; \end{cases}$$

or, if we use the suffix 1 to denote differentiation in regard to v, and the suffix 2 to denote differentiation in regard to u, then the equations are

$$\frac{V_{11}}{V} = T^2 - HK,$$

$$T_2 + (HV)_1 - KV_1 = 0,$$

$$(TV^2)_1 + K_2V = 0,$$

or, what is the same thing,

$$\begin{cases} V_{11} = V(T^2 - HK), \\ T_2 + H_1V + (H-K)\,V_1 = 0, \\ T_1V + 2TV_1 + K_2 \qquad = 0. \end{cases}$$

I wish to show how these formulæ connect themselves with formulæ belonging to the general form $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$. These involve not only Gauss's coefficients E, F, G, but also the coefficients E', F', G' belonging to the inflexional tangents; and, for convenience, I quote the system of definitions, Salmon's *Geometry of Three Dimensions*, 3rd ed., 1874, p. 251, viz.

$$dx,\ dy,\ dz = adp + a'dq,\quad bdp + b'dq,\quad cdp + c'dq;$$

$$d^2x = \alpha dp^2 + 2\alpha' dpdq + \alpha'' dq^2,$$

$$d^2y = \beta dp^2 + 2\beta' dpdq + \beta'' dq^2,$$

$$d^2z = \gamma dp^2 + 2\gamma' dpdq + \gamma'' dq^2;$$

$$A,\ B,\ C = bc' - b'c,\quad ca' - c'a,\quad ab' - a'b;\quad V^2 = EG - F^2;$$

$$E' = A\alpha + B\beta + C\gamma,\quad F' = A\alpha' + B\beta' + C\gamma',\quad G' = A\alpha'' + B\beta'' + C\gamma'',$$

so that E', F', G' are, in fact, the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

The equation for the determination of the principal radii of curvature is

$$(E'\rho - EV)(G'\rho - GV) - (F'\rho - FV)^2 = 0,$$

which, in the particular case $F=0$ (and therefore $V^2=EG$), becomes

$$(E'\rho - EV)(G'\rho - GV) - F'^2\rho^2 = 0,$$

or, as this may be written,

$$\left(\frac{1}{\rho} - \frac{E'}{EV}\right)\left(\frac{1}{\rho} - \frac{G'}{GV}\right) - \frac{F'^2}{EGV^2} = 0,$$

an equation which corresponds with Bour's form

$$\left(\frac{1}{\rho} - K\right)\left(\frac{1}{\rho} - H\right) - T^2 = 0,$$

and becomes identical with it, if

$$E' = EVK, \quad G' = GVH, \quad F' = -V^2T.$$

But, making p, q correspond to Bour's variables, p to v, and q to u, it is necessary to show that the foregoing values (and not the interchanged values $E'=GVH$, $G'=EVK$) are the correct ones. We have, Salmon, p. 254,

$$\left\| \begin{matrix} dq, & \rho E' - VE, & \rho F' - VF \\ -dp, & \rho F' - VF, & \rho G' - VG \end{matrix} \right\| = 0;$$

or, putting herein $F=0$, the equations may be written

$$\frac{dq}{-dp} = \frac{E'}{F'}\left(1 - \frac{VE}{\rho E'}\right) = \frac{F'}{G'} \div \left(1 - \frac{VG}{\rho G'}\right);$$

or, we see that to $dq=0$ corresponds the value $\frac{1}{\rho} = \frac{E'}{EV}$, and to $dp=0$ the value $\frac{1}{\rho} = \frac{G'}{GV}$. Hence the former of these values of $\frac{1}{\rho}$ corresponds to Bour's $du=0$, that is, to his $\frac{1}{\rho} = K$; and the latter to Bour's $dv=0$, that is, to his $\frac{1}{\rho} = H$; or the values are, as stated,

$$E' = EVK, \quad G' = GVH.$$

The formula $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$ agrees with Bour's $ds^2 = dv^2 + g^2du^2$, if $p=u$, $q=v$, $E=1$, $F=0$, $G=g^2$. With these values, $V^2 = EG - F^2 = g^2$, or say $g=V$, and Bour's equation is, as it was before written, $ds^2 = dv^2 + V^2du^2$. And we have to find the three equations which, putting therein $p=u$, $q=v$, $E=1$, $F=0$, $G=V^2$, $E'=VK$, $F'=-V^2T$, $G'=V^3H$, reduce themselves to Bour's equations.

The first of these is nothing else than the equation for the measure of curvature, viz. Salmon, p. 262 (but, using the suffixes 1 and 2 to denote differentiation in regard to p and q respectively), this is

$$\begin{aligned} 4(E'G' - F'^2) = \quad & E(E_2G_2 - 2F_1G_2 + G_1^2) \\ & + F(E_1G_2 - E_2G_1 - 2E_2F_1 + 4F_1F_2 - 2F_1G_1) \\ & + G(E_1G_1 - 2E_1F_2 + E_2^2) \\ & - 2(EG - F^2)(E_{22} - 2F_{12} + G_{11}). \end{aligned}$$

In fact, writing herein $E=1$, $F=0$, and therefore the differential coefficients of E and F each $=0$, the equation becomes

$$4(E'G'-F'^2)=G_1^2-2GG_{11},$$

which is

$$4V^4(HK-T^2)=(2VV_1)^2-2V^2(2V_1^2+2VV_{11}),\ =-4V^3V_{11};$$

or finally it is

$$V_{11}=V(T^2-HK).$$

The other two of Bour's equations are derived from equations which give respectively the values of $E_2'-F_1'$ and $F_2'-G_1'$; viz. starting from the equations

$$E'=A\alpha+B\beta+C\gamma,$$
$$F'=A\alpha'+B\beta'+C\gamma',$$
$$G'=A\alpha''+B\beta''+C\gamma'',$$

we see at once that E_2' and F_1' contain, E_2' the terms $A\alpha_2+B\beta_2+C\gamma_2$, and F_1' the terms $A\alpha_1'+B\beta_1'+C\gamma_1'$, which are equal to each other ($\alpha_2=\alpha_1'$ since α and α' are the differential coefficients x_{11}, x_{12} of x, and so $\beta_2=\beta_1'$ and $\gamma_2=\gamma_1'$). Hence

$$E_2'-F_1'=A_2\alpha+B_2\beta+C_2\gamma-A_1\alpha'-B_1\beta'-C_1\gamma';$$

and similarly

$$F_2'-G_1'=A_2\alpha'+B_2\beta'+C_2\gamma'-A_1\alpha''-B_1\beta''-C_1\gamma''.$$

Here, from the values of A, B, C, we have

$$A=bc'-cb';\quad A_1=\beta c'-\gamma b'+b\gamma'-c\beta';\quad A_2=\beta'c'-\gamma'b'+b\gamma''-c\beta'';$$
$$B=ca'-ac';\quad B_1=\gamma a'-\alpha c'+c\alpha'-a\gamma';\quad B_2=\gamma'a'-\alpha'c'+c\alpha''-a\gamma'';$$
$$C=ab'-ba';\quad C_1=\alpha b'-\beta a'+a\beta'-b\alpha';\quad C_2=\alpha'b'-\beta'a'+a\beta''-b\alpha'';$$

and, substituting, we find

$$E_2'-F_1'=\ \ 2a'\alpha\alpha'+a\alpha''\alpha,$$
$$F_2'-G_1'=-2a\alpha'\alpha''-a'\alpha''\alpha,$$

if, for shortness, $a'\alpha\alpha'$ denotes the determinant

$$\begin{vmatrix} a', & \alpha, & \alpha' \\ b', & \beta, & \beta' \\ c', & \gamma, & \gamma' \end{vmatrix},$$

and so for the other like symbols. Observe that, with

$$\begin{vmatrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \end{vmatrix},$$

we have in all 10 determinants, viz. these are $a\alpha'\alpha, =E'$; $a\alpha'\alpha', =F'$; $a\alpha'\alpha'', =G'$; $a\alpha'\alpha''$; and the six determinants $a\alpha\alpha'$, $a\alpha'\alpha''$, $a\alpha''\alpha$; $a'\alpha\alpha'$, $a'\alpha'\alpha''$, $a'\alpha''\alpha$. The foregoing expressions of $E_2'-F_1'$ and $F_2'-G_1'$ respectively, substituting therein for the determinants $a'\alpha\alpha'$, $a\alpha''\alpha$, $a\alpha'\alpha''$, $a'\alpha''\alpha$ their values as about to be obtained, are the required two equations. We have

$$\begin{aligned}
aa + bb + cc &= E, & a\alpha' + bb' + cc' &= F,\\
a'a + b'b + c'c &= F, & a'\alpha' + b'b' + c'c' &= G,\\
\alpha a + \beta b + \gamma c &= \tfrac{1}{2}E_1, & \alpha\alpha' + \beta b' + \gamma c' &= F_1 - \tfrac{1}{2}E_2,\\
\alpha' a + \beta' b + \gamma' c &= \tfrac{1}{2}E_2, & \alpha'\alpha' + \beta' b' + \gamma' c' &= \tfrac{1}{2}G_1,\\
\alpha'' a + \beta'' b + \gamma'' c &= F_2 - \tfrac{1}{2}G_1, & \alpha''\alpha' + \beta'' b' + \gamma'' c' &= \tfrac{1}{2}G_2;
\end{aligned}$$

and if from the first five equations, regarded as equations linear in (a, b, c), we eliminate these quantities, and from the second five equations, regarded as linear in (a', b', c'), we eliminate these quantities, we obtain two sets each of five equations,

$$\left\|\begin{matrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ E, & F, & \tfrac{1}{2}E_1, & \tfrac{1}{2}E_2, & F_2 - \tfrac{1}{2}G_1 \end{matrix}\right\| = 0, \text{ and } \left\|\begin{matrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ F, & G, & F_1 - \tfrac{1}{2}E_2, & \tfrac{1}{2}G_1, & \tfrac{1}{2}G_2 \end{matrix}\right\| = 0.$$

These may be written,

$$\begin{aligned}
F\alpha\alpha'\alpha'' - \tfrac{1}{2}E_1 a'\alpha'\alpha'' - \tfrac{1}{2}E_2 a'\alpha''\alpha - (F_2 - \tfrac{1}{2}G_1)\, a'\alpha\alpha' &= 0,\\
-E\alpha\alpha'\alpha'' + \tfrac{1}{2}E_1 a\alpha'\alpha'' + \tfrac{1}{2}E_2 a\alpha''\alpha + (F_2 - \tfrac{1}{2}G_1)\, a\alpha\alpha' &= 0,\\
Ea'\alpha'\alpha'' - F\, a\alpha'\alpha'' + \tfrac{1}{2}E_2 G' - (F_2 - \tfrac{1}{2}G_1)\, F' &= 0,\\
Ea'\alpha''\alpha - F\, a\alpha''\alpha - \tfrac{1}{2}E_1 G' + (F_2 - \tfrac{1}{2}G_1)\, E' &= 0,\\
Ea'\alpha\alpha' - F\, a\alpha\alpha' + \tfrac{1}{2}E_1 F' - \tfrac{1}{2}E_2 E' &= 0;
\end{aligned}$$

and

$$\begin{aligned}
G\alpha\alpha'\alpha'' - (F_1 - \tfrac{1}{2}E_2)\, a'\alpha'\alpha'' - \tfrac{1}{2}G_1 a'\alpha''\alpha - \tfrac{1}{2}G_2 a'\alpha\alpha' &= 0,\\
-F\alpha\alpha'\alpha'' + (F_1 - \tfrac{1}{2}E_2)\, a\alpha'\alpha'' + \tfrac{1}{2}G_1 a\alpha\alpha''\alpha + \tfrac{1}{2}G_2 a\alpha\alpha' &= 0,\\
Fa'\alpha'\alpha'' - G\, a\alpha'\alpha'' + \tfrac{1}{2}G_1 G - \tfrac{1}{2}G_2 F' &= 0,\\
Fa'\alpha''\alpha - G\, a\alpha''\alpha - (F_1 - \tfrac{1}{2}E_2)\, G' + \tfrac{1}{2}G_2 E' &= 0,\\
Fa'\alpha\alpha' - G\, a\alpha\alpha' + (F_1 - \tfrac{1}{2}E_2)\, F' - \tfrac{1}{2}G_1 E' &= 0.
\end{aligned}$$

Attending in each set only to the third, fourth, and fifth equations, and combining these in pairs, we obtain

$$\begin{aligned}
V^2 a\alpha'\alpha'' + (\tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)\, F' + (-\tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)\, G' &= 0,\\
V^2 a'\alpha'\alpha'' + (\tfrac{1}{2}GG_1 - GF_2 + \tfrac{1}{2}FG_2)\, F' + (-\tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)\, G' &= 0;\\
V^2 a\alpha''\alpha + (-\tfrac{1}{2}FE_1 + EF_1 - \tfrac{1}{2}EE_2)\, G' + (-\tfrac{1}{2}FG_1 + FF_2 - \tfrac{1}{2}EG_2)\, E' &= 0,\\
V^2 a'\alpha''\alpha + (-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FE_2)\, G' + (-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\, E' &= 0;\\
V^2 a\alpha\alpha' + (\tfrac{1}{2}EG_1 - \tfrac{1}{2}FE_2)\, E' + (\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)\, F' &= 0,\\
V^2 a'\alpha\alpha' + (\tfrac{1}{2}FG_1 - \tfrac{1}{2}GE_2)\, E' + (\tfrac{1}{2}GE_1 - FF_1 + \tfrac{1}{2}FE_2)\, F' &= 0.
\end{aligned}$$

We thus obtain

$$E_2' - F_1' = \frac{2}{V^2}\{(\quad - \tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)\,E' + (-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FF_2)\,F'\}$$

$$+ \frac{1}{V^2}\{(\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)\,G' + (\tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)\,E'\},$$

$$F_2' - G_1' = \frac{2}{V^2}\{(\tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)\,F' + (\quad - \tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)\,G'\}$$

$$+ \frac{1}{V^2}\{(-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FE_2)\,G' + (-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\,E'\};$$

or, finally,

$$E_2' - F_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}FG_1 + GE_2 - FF_2 + \tfrac{1}{2}EG_2)\,E'$$
$$+ (-GE_1 + 2FF_1 - FF_2)\,F' + (\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)\,G'\},$$

$$F_2' - G_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\,E'$$
$$+ (FG_1 - 2FF_2 + EG_2)\,F' + (-\tfrac{1}{2}GE_1 + FF_1 - EG_1 + \tfrac{1}{2}FE_2)\,G'\},$$

which are the required formulæ; and which may, I think, be regarded as new formulæ in the Gaussian theory of surfaces.

Writing herein as before, the first of these becomes

$$(VK)_2 + (V^2T)_1 = \frac{1}{V^2}\{\tfrac{1}{2}(V^2)_2\,VK\}, = V_2K,$$

that is,

$$V_2K + VK_2 + V^2T_1 + 2VV_1T = V_2K;$$

or finally

$$VT_1 + 2TV_1 + K_2 = 0,$$

which is Bour's third equation. And the second equation becomes

$$-(V^2T)_2 - (V^3H)_1 = \frac{1}{V^2}\{-\tfrac{1}{2}V^2(V^2)_1\,VK + (V^2)_2(-V^2T) - (V^2)_1\,V^3H\},$$
$$= -V^2V_1K - 2VV_2T - 2V^2V_1H,$$

that is,

$$-V^2T_2 - 2VV_2T - V^3H_1 - 3V^2V_1H = -V^2V_1K - 2VV_2T - 2V^2V_1H;$$

or finally

$$T_2 + VH_1 + (H - K)\,V_1 = 0,$$

which is Bour's second equation.

768.

NOTE ON LANDEN'S THEOREM.

[From the *Proceedings of the London Mathematical Society*, vol. XIII. (1882), pp. 47, 48. Read November 10, 1881.]

LANDEN'S theorem, as given in the paper "An Investigation of a General Theorem for finding the length of any Arc of any Conic Hyperbola by means of two Elliptic Arcs, with some other new and useful Theorems deduced therefrom," *Phil. Trans.*, t. LXV. (1775), pp. 283—289, is, as appears by the title, a theorem for finding the length of a hyperbolic arc in terms of the length of two elliptic arcs; this theorem being obtained by means of the following differential identity, viz., if

$$t = gx\sqrt{\frac{m^2 - x^2}{m^2 - gx^2}},$$

where

$$g = \frac{m^2 - n^2}{n^2},$$

then

$$\sqrt{\frac{m^2 - gx^2}{m^2 - x^2}}\,dx = \left\{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{\frac{(m-n)^2 - t^2}{(m+n)^2 - t^2}} + \tfrac{1}{4}\sqrt{\frac{(m+n)^2 - t^2}{(m-n)^2 - t^2}}\right\} dt,$$

(this is exactly Landen's form, except that he of course writes $\dot{x}$, $\dot{t}$ in place of dx, dt respectively): viz., integrating each side, and interpreting geometrically in a very ingenious and elegant manner the three integrals which present themselves, he arrives at his theorem for the hyperbolic arc; but with this I am not now concerned.

Writing for greater convenience $m = 1$, $n = k'$, and therefore $g = k^2$, if as usual $k^2 + k'^2 = 1$, the transformation is

$$t = k^2 x\sqrt{\frac{1 - x^2}{1 - k^2 x^2}},$$

leading to

$$\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx=\left\{\tfrac{1}{2}+\tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}}+\sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}}\right\}dt.$$

The form in which the transformation is usually employed (see my *Elliptic Functions*, pp. 177, 178) is

$$y=(1+k')\,x\sqrt{\frac{1-x^2}{1-k^2x^2}},$$

leading to

$$\frac{(1+k')\,dx}{\sqrt{1-x^2}.\sqrt{1-k^2x^2}}=\frac{dy}{\sqrt{1-y^2}.\sqrt{1-\lambda^2y^2}},$$

where

$$\lambda=\frac{1-k'}{1+k'}.$$

If, to identify the two forms, we write $y=\dfrac{t}{1-k'}$ and in the last equation introduce t in place of y, the last equation becomes

$$\frac{dx}{\sqrt{1-x^2}.\sqrt{1-k^2x^2}}=\frac{dt}{\sqrt{\{(1-k')^2-t^2\}\,\{(1+k')^2-t^2\}}}.$$

Comparing with Landen's form, in order that the two may be identical, we must have

$$1-k^2x^2=\left\{\tfrac{1}{2}+\tfrac{1}{4}\sqrt{\frac{(1-k')^2-t^2}{(1+k')^2-t^2}}+\tfrac{1}{4}\sqrt{\frac{(1+k')^2-t^2}{(1-k')^2-t^2}}\right\}$$

$$\times\sqrt{(1-k')^2-t^2}\,\sqrt{(1+k')^2-t^2},$$

viz., this is

$$1-k^2x^2=\tfrac{1}{4}\left\{\sqrt{(1-k')^2-t^2}+\sqrt{(1+k')^2-t^2}\right\}^2,$$

that is,

$$1-k^2x^2=\tfrac{1}{2}\left[1+k'^2-t^2+\sqrt{\{(1-k')^2-t^2\}\,\{(1+k')^2-t^2\}}\right],$$

where the function under the radical sign is

$$(1-k'^2)^2-2\,(1+k'^2)\,t^2+t^4\,(=T \text{ suppose});$$

and this must consequently be a form of the original integral equation

$$t=k^2x\sqrt{\frac{1-x^2}{1-k^2x^2}}.$$

In fact, squaring and solving in regard to x^2 with the assumed sign of the radical, we have

$$x^2=\frac{k^2+t^2-\sqrt{T}}{2k^2},$$

corresponding to an equation given by Landen. And we thence have

$$1 - k^2x^2 = \tfrac{1}{2}\{2 - k^2 - t^2 + \sqrt{T}\}, \quad = \tfrac{1}{2}\{1 + k'^2 - t^2 + \sqrt{T}\},$$

which is the required expression for $1 - k^2x^2$.

The trigonometrical form $\sin(2\phi' - \phi) = c \sin\phi$ of the relation between y and x does not occur in Landen; it is employed by Legendre, I believe, in an early paper, *Mém. de l'Acad. de Paris*, 1786, and in the *Exercices*, 1811, and also in the *Traité des Fonctions Elliptiques*, 1825, and by means of it he obtains an expression for the arc of a hyperbola in terms of two elliptic functions, $E(c, \phi)$, $E(c', \phi')$, showing that the arc of the hyperbola is expressible by means of two elliptic arcs,—this, he observes, "est le beau théorème dont Landen a enrichi la géométrie." We have, then (1828), Jacobi's proof, by two fixed circles, of the addition-theorem (see my *Elliptic Functions*, p. 28), and the application of this (p. 30) to Landen's theorem is also due to Jacobi, see the "Extrait d'une lettre adressée à M. Hermite," *Crelle*, t. XXXII. (1846), pp. 176—181; the connection of the demonstrations, by regarding the point, which is alone necessary for Landen's theorem as the limit of the smaller circle in the figure for the addition-theorem is due to Durège (see his *Theorie der elliptischen Functionen*, Leipzig, 1861, pp. 168, *et seq.*).

769.

ON A FORMULA RELATING TO ELLIPTIC INTEGRALS OF THE THIRD KIND.

[From the *Proceedings of the London Mathematical Society*, vol. XIII. (1882), pp. 175, 176. Presented May 11, 1882.]

THE formula for the differentiation of the integral of the third kind

$$\Pi = \int_0 \frac{d\phi}{(1 + n \sin^2 \phi)\,\Delta}$$

in regard to the parameter n, see my *Elliptic Functions*, Nos. 174 *et seq.*, may be presented under a very elegant form, by writing therein

$$\sin^2 \phi = x = \operatorname{sn}^2 u, \quad \sin\phi \cos\phi\, \Delta = y = \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u,$$

and thus connecting the formula with the cubic curve

$$y^2 = x(1-x)(1-k^2 x).$$

The parameter must, of course, be put under a corresponding form, say $n = -\dfrac{1}{a}$, where $a = \operatorname{sn}^2 \theta$, $b = \operatorname{sn}\theta \operatorname{cn}\theta \operatorname{dn}\theta$, and therefore (a, b) are the coordinates of the point corresponding to the argument θ. The steps of the substitution may be effected without difficulty, but it will be convenient to give at once the final result and then verify it directly. The result is

$$\frac{d}{d\theta}\frac{b}{a-x} - \frac{d}{du}\frac{y}{x-a} = k^2(a-x).$$

We, in fact, have

$$\frac{dx}{du} = 2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u = 2y,$$

and thence

$$y\frac{dx}{du}=2y^2,$$

that is,

$$y\frac{dx}{du}=2x\left[1-(1+k^2)x+k^2x^2\right].$$

Also

$$\frac{dy}{du}=\operatorname{cn}^2 u\operatorname{dn}^2 u-\operatorname{sn}^2 u\operatorname{dn}^2 u-k^2\operatorname{sn}^2 u\operatorname{cn}^2 u$$

$$=1-2(1+k^2)x+3k^2x^2,$$

and hence

$$\frac{d}{du}\frac{y}{x-a}=\frac{1}{(a-x)^2}\left\{(x-a)\frac{dy}{du}-y\frac{dx}{du}\right\}$$

$$=\frac{1}{(a-x)^2}\left\{-x-a+2(1+k^2)ax+k^2x^3-3k^2ax^2\right\}.$$

Interchanging the letters, we have

$$\frac{d}{d\theta}\frac{b}{a-x}=\frac{1}{(a-x)^2}\left\{-x-a+2(1+k^2)ax+k^2a^3-3k^2a^2x\right\},$$

and hence, subtracting,

$$\frac{d}{d\theta}\frac{b}{a-x}-\frac{d}{du}\frac{y}{x-a}=\frac{1}{(a-x)^2}\left\{k^2a^3-3k^2a^2x+3k^2ax^2-k^2x^3\right\}$$

$$=\frac{1}{(a-x)^2}k^2(a-x)^3$$

$$=k^2(a-x),$$

which is the required result.

770.

ON THE 34 CONCOMITANTS OF THE TERNARY CUBIC.

[From the *American Journal of Mathematics*, vol. IV. (1881), pp. 1—15.]

I HAVE, (by aid of Gundelfinger's formulæ, afterwards referred to), calculated, and I give in the present paper, the expressions of the 34 concomitants of the canonical ternary cubic $ax^3+by^3+cz^3+6lxyz$, or, what is the same thing, the 34 covariants of this cubic and the adjoint linear function $\xi x+\eta y+\zeta z$: this is the chief object of the paper. I prefix a list of memoirs, with short remarks upon some of them; and, after a few observations, proceed to the expressions for the 34 concomitants; and, in conclusion, exhibit the process of calculation of these concomitants other than such of them as are taken to be known forms. I insert a supplemental table of 6 derived forms.

The list of memoirs (not by any means a complete one) is as follows:

HESSE, Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln: *Crelle*, t. XXVIII. (1844), pp. 68—96. Although purporting to relate to a different subject, this is in fact the earliest, and a very important, memoir in regard to the general ternary cubic; and in it is established the canonical form, as Hesse writes it, $y_1^3+y_2^3+y_3^3+6\pi y_1y_2y_3$.

ARONHOLD, Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. XXXIX. (1850), pp. 140—159.

CAYLEY, A Third Memoir on Quantics: *Phil. Trans.*, t. CXLVI. (1856), pp. 627—647; [144].

ARONHOLD, Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. LV. (1858), pp. 97—191.

SALMON, *Lessons Introductory to the Modern Higher Algebra:* 8°, Dublin, 1859.

CAYLEY, A Seventh Memoir on Quantics: *Phil. Trans.*, t. CLI. (1861), pp. 277—292; [269].

BRIOSCHI, Sur la théorie des formes cubiques à trois indéterminées: *Comptes Rendus*, t. LVI. (1863), pp. 304—307.

HERMITE, Extrait d'une lettre à M. Brioschi: *Crelle*, t. LXIII. (1864), pp. 30—32, followed by a note by Brioschi, pp. 32—33.

The skew covariant of the ninth order, which is $y^3 - z^3 . z^3 - x^3 . x^3 - y^3$ for the canonical form $x^3 + y^3 + z^3 + 6lxyz$, and the corresponding contravariant $\eta^3 - \zeta^3 . \zeta^3 - \xi^3 . \xi^3 - \eta^3$, alluded to p. 116 of Salmon's *Lessons*, were obtained, the covariant by Brioschi and the contravariant by Hermite, in the last-mentioned papers.

CLEBSCH and GORDAN, Ueber die Theorie der ternären cubischen Formen: *Math. Annalen*, t. I. (1869), pp. 56—89.

The establishment of the complete system of the 34 covariants, contravariants and *Zwischenformen*, or, as I have here called them, the 34 concomitants, was first effected by Gordan in the next following memoir:

GORDAN, Ueber die ternären Formen dritten Grades: *Math. Annalen*, t. I. (1869), pp. 90—128.

And the theory is further considered:

GUNDELFINGER, Zur Theorie der ternären cubischen Formen: *Math. Annalen*, t. VI. (1871), pp. 144—163. The author speaks of the 34 forms as being "theils mit den von Gordan gewählten identisch, theils möglichst einfache Combinationen derselben." They are, in fact, the 34 forms given in the present paper for the canonical form of the cubic, and the meaning of the adopted combinations of Gordan's forms will presently clearly appear.

There is an advantage in using the form $ax^3 + by^3 + cz^3 + 6lxyz$ rather than the Hessian form $x^3 + y^3 + z^3 + 6lxyz$, employed in my Third and Seventh Memoirs on Quantics: for the form $ax^3 + by^3 + cz^3 + 6lxyz$ is what the general cubic

$$(a, b, c, f, g, h, i, j, k, l)(x, y, z)^3$$

becomes by no other change than the reduction to zero of certain of its coefficients; and thus any concomitant of the canonical form consists of terms which are leading terms of the same concomitant of the general form.

The concomitants are functions of the coefficients $(a, b, \ldots, l)$, of (ξ, η, ζ), and of (x, y, z): the dimensions in regard to the three sets respectively may be distinguished as the degree, class, and order; and we have thus to consider the deg-class-order of a concomitant.

Two or more concomitants of the same deg-class-order may be linearly combined together: viz., the linear combination is the sum of the concomitants each multiplied by a mere number. The question thus arises as to the selection of a representative concomitant. As already mentioned, I follow Gundelfinger, viz., my 34 concomitants of the canonical form correspond each to each (with only the difference of a numerical factor of the entire concomitant) to his 34 concomitants of the general form. The principle underlying the selection would, in regard to the general form, have to be explained altogether differently; but this principle exhibits itself in a very remarkable manner in regard to the canonical form $ax^3 + by^3 + cz^3 + 6lxyz$.

Each concomitant of the general form is an indecomposable function, not breaking up into rational factors; but this is not of necessity the case in regard to a canonical form: only a concomitant which *does* break up must be regarded as indecomposable, no factor of such concomitant being rejected, or separated. So far from it, there is, in regard to the canonical form in question, a frequent occurrence of $abc+8l^3$ or a power thereof, either as a factor of a unique concomitant, or when there are two or more concomitants of the same deg-class-order, then as a factor of a properly selected linear combination of such concomitants: and the principle referred to is, in fact, that of the selection of such combination for the representative concomitant; or (in other words) the representative concomitant is taken so as to contain as a factor the highest power that may be of $abc+8l^3$. As to the signification of this expression $abc+8l^3$, I call to mind that the discriminant of the form is $abc(abc+8l^3)^3$.

As to numerical factors: my principle has been, and is, to throw out any common numerical divisor of all the terms: thus I write $S=-abcl+l^4$, instead of Aronhold's $S=-4abcl+4l^4$. There is also the question of nomenclature: I retain that of my Seventh Memoir on Quantics, except that I use single letters H, P, &c., instead of the same letters with U, thus HU, PU, &c.; in particular, I use U, H, P, Q instead of Aronhold's f, Δ, S_f, T_f. It is thus at all events necessary to make some change in Gundelfinger's letters; and there is moreover a laxity in his use of accented letters; his B, B', B'', B''', and so in other cases E, E', E'', &c., are used to denote functions derived in a determinate manner each from the preceding one (by the δ-process explained further on); whereas his L, L'; M, M'; N, N' are functions having to each other an altogether different relation; also three of his functions are not denoted by any letters at all. Under the circumstances, I retain only a few of his letters; use the accent where it denotes the δ-process; and introduce barred letters $\bar{J}$, $\bar{K}$, &c., to denote a different correspondence with the unbarred letters J, K, &c. But I attach also to each concomitant a numerical symbol showing its deg-class-order, thus: 541 (degree = 5, class = 4, order = 1) or 1290, (there is no ambiguity in the two-digit numbers 10, 11, 12 which present themselves in the system of the 34 symbols); and it seems to me very desirable that the significations of these deg-class-order symbols should be considered as permanent and unalterable. Thus, in writing $S=400=-abcl+l^4$, I wish the 400 to be regarded as denoting its expressed value $-abcl+l^4$: if the same letter S is to be used in Aronhold's sense to denote $-4abcl+4l^4$, this would be completely expressed by the new definition $S=4.400$, the meaning of the symbol 400 being explained by reference to the present memoir, or by the actual quotation $400=-abcl+l^4$.

I proceed at once to the table: for shortness, I omit, in general, terms which can be derived from an expressed term by mere cyclical interchanges of the letters (a, b, c), (ξ, η, ζ), (x, y, z).

Table of the 34 *Covariants of the Canonical Cubic* $ax^3+by^3+cz^3+6lxyz$ *and the linear form* $\xi x+\eta y+\zeta z$.

First Part, 10 Forms. Class = Order.

Current No.

1 $S = 400 = -abcl+l^4.$

2 $T = 600 = a^2b^2c^2-20abcl^3-8l^6.$

3 $\Lambda = 011 = \xi x+\eta y+\zeta z.$

4 $\Theta = 222 = x^2[-l^2\xi^2-2al\eta\zeta]\ldots$
$+yz[bc\xi^2+2l^2\eta\zeta]\ldots.$

5 $\Theta' = 422 = x^2[l(abc+2l^3)\xi^2+a(abc-4l^3)\eta\zeta]\ldots$
$+yz[6bcl^2\xi^2-2l(abc+2l^3)\eta\zeta]\ldots.$

6 $\Theta'' = 622 = x^2[-(abc+2l^3)^2\xi^2+12al^2(abc+2l^3)\eta\zeta]\ldots$
$+yz[36bcl^4\xi^2+2(abc+2l^3)^2\eta\zeta]\ldots.$

7 $B = 333 = x^3[a^2(c\eta^3-b\zeta^3)]\ldots$
$+y^2z[(abc+8l^3)\eta^2\zeta+12bl^2\zeta^2\xi+6bcl\xi^2\eta]\ldots$
$+yz^2[-(abc+8l^3)\eta\zeta^2-6bcl\zeta\xi^2-12cl^2\xi\eta^2]\ldots.$

8 $B' = 533 = x^3[3a^2l^2(c\eta^3-b\zeta^3)]\ldots$
$+y^2z[-l^2(abc+8l^3)\eta^2\zeta+4bl(-abc+l^3)\zeta^2\xi-bc(abc-10l^3)\xi^2\eta]\ldots$
$+yz^2[l^2(abc+8l^3)\eta\zeta^2+bc(abc-10l^3)\zeta\xi^2-4cl(-abc+l^3)\xi\eta^2]\ldots.$

9 $B'' = 733 = x^3[9a^2l^4(c\eta^3-b\zeta^3)]\ldots$
$+y^2z[l(abc+8l^3)(2abc+l^3)\eta^2\zeta$
$+b(abc+2l^3)(abc-10l^3)\zeta^2\xi+6bcl^3(-abc+l^3)\xi^2\eta]\ldots$
$+yz^2[-l(abc+8l^3)(2abc+l^3)\eta\zeta^2$
$-6bcl^3(-abc+l^3)\zeta\xi^2-c(abc+2l^3)(abc-10l^3)\xi\eta^2]\ldots.$

10 $B''' = 933 = x^3[27a^2l^6(c\eta^3-b\zeta^3)]\ldots$
$+y^2z[-(abc+8l^3)(abc-l^3)^2\eta^2\zeta+9bl^2(abc+2l^3)^2\zeta^2\xi$
$-27bcl^4(abc+2l^3)\xi^2\eta]\ldots$
$+yz^2[(abc+8l^3)(abc-l^3)^2\eta\zeta^2+27bcl^4(abc+2l^3)\zeta\xi^2$
$-9cl^3(abc+2l^3)^2\xi\eta^2]\ldots.$

Second Part, $(4+4=)$ 8 forms. Class = 0, and Order = 0.

Class=0.

11 $U = 103 = ax^3+by^3+cz^3+6lxyz.$

12 $H = 303 = l^3(ax^3+by^3+cz^3)-(abc+2l^3)xyz.$

13 $\Psi = 806 = (abc+8l^3)^2\{a^2x^6+b^2y^6+c^2z^6-10(bcy^3z^3+caz^3x^3+abx^3y^3)\}.$

14 $\Omega = 1209 = (abc+8l^3)^2\{by^3-cz^3.cz^3-ax^3.ax^3-by^3\}.$

Current No. Order = 0.

15 $P = 330 = -l(bc\xi^3 + ca\eta^3 + ab\zeta^3) + (-abc + 4l^3)\xi\eta\zeta.$

16 $Q = 530 = (abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2(5abc + 4l^3)\xi\eta\zeta.$

17 $F = 460 = b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 - 2(abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3 + c\xi^3\eta^3)$
$- 24l^2(bc\xi^3 + ca\eta^3 + ab\zeta^3)\xi\eta\zeta - 24l(abc + 2l^3)\xi^2\eta^2\zeta^2.$

18 $\Pi = 1290 = (abc + 8l^3)^2\{c\eta^3 - b\zeta^3 . a\zeta^3 - c\xi^3 . b\xi^3 - a\eta^3\}.$

Third Part, (8 + 8 =) 16 forms. Class less or greater than Order.

Class less than Order.

19 $J = 414 = (abc + 8l^3)\{\xi x(by^3 - cz^3) + \eta y(cz^3 - ax^3) + \zeta z(ax^3 - by^3)\}.$

20 $K = 514 = (abc + 8l^3)\{\xi[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]\ldots\}.$

21 $K' = 714 = (abc + 8l^3)\{\xi[(abc + 2l^3)(ax^4 - 2bxy^3 - 2cxz^3) - 18bcl^2y^2z^2]\ldots\}.$

22 $E = 625 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[2l^2x^2 + bcyz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[4alx^2 + 2l^2yz]\ldots\}.$

23 $E' = 825 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[l(abc + 2l^3)x^2 - 3bcl^2yz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[a(abc - 4l^3)x^2 + l(abc + 2l^3)yz]\ldots\}.$

24 $E'' = 1025 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[(abc + 2l^3)^2x^2 + 18bcl^4yz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[-12al^2(abc + 2l^3)x^2 + (abc + 2l^3)^2yz]\ldots\}.$

25 $M = 917 = (abc + 8l^3)^2\{\xi(by^3 - cz^3)[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2]\ldots\}.$

26 $M' = 1117 = (abc + 8l^3)^2\{\xi(by^3 - cz^3)[(abc + 2l^3)(5alx^4 - bxy^3 - cxz^3)$
$+ 18bcl^2y^2z^2]\ldots\}.$

Order less than Class.

27 $\bar{J} = 841 = (abc + 8l^3)^2\{x\xi a(c\eta^3 - b\zeta^3) + y\eta b(a\zeta^3 - c\xi^3) + z\zeta c(b\xi^3 - a\eta^3)\}.$

28 $\bar{K} = 541 = (abc + 8l^3)\{x[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]\ldots\}.$

29 $\bar{K}' = 741 = (abc + 8l^3)\{x[l^3(bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3) + a(abc + 2l^3)\eta^2\zeta^2]\ldots\}.$

30 $\bar{E} = 652 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[2al\xi^2 + a^2\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[4l^2\xi^2 + 2al\eta\zeta]\ldots\}.$

31 $\bar{E}' = 852 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[a(abc - 4l^3)\xi^2 - 6a^2l^2\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[4l(abc + 2l^3)\xi^2 + a(abc - 4l^3)\eta\zeta]\ldots\}.$

32 $\bar{E}'' = 1052 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[-3al^2(abc + 2l^3)\xi^2 + 9a^2l^4\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[(abc + 2l^3)^2\xi^2 - 3al^2(abc - 4l^3)\eta\zeta]\ldots\}.$

33 $\bar{M} = 771 = (abc + 8l^3)\{x(c\eta^3 - b\zeta^3)[(abc - 8l^3)\xi^4 - a^2c\xi\eta^3 - a^2b\xi\zeta^3$
$- 12al^2\xi^2\eta\zeta - 6a^2l\eta^2\zeta^2]\ldots\}.$

34 $\bar{M}' = 971 = (abc + 8l^3)\{x(c\eta^3 - b\zeta^3)[l^2(7abc + 8l^3)\xi^4 - 3a^2cl^2\xi\eta^3 - 3a^2bl^2\xi\zeta^3$
$+ 4al(abc - l^3)\xi^2\eta\zeta + a^2(abc - 10l^3)\eta^2\zeta^2]\ldots\}.$

To this may be joined the following Supplemental Table of certain Derived Forms:

Current No.

35 $R = 1200 = 64S^3 - T^2 = -abc\,(abc + 8l^3)^3.$

36 $C = 703 = -TU + 24SH = (abc + 8l^3)\,\{(-abc + 4l^3)\,(ax^2 + by^2 + cz^2) + 18abclxyz\}.$

37 $D = 903 = 8S^2U - 3TH = (abc + 8l^3)\,\{l^2\,(5abc + 4l^3)\,(ax^2 + by^2 + cz^2) + 3abc\,(abc - 10l^3)\,xyz\}.$

38 $Y = 930 = 3TP - 4SQ = (abc + 8l^3)^2\,\{l\,(bc\xi^2 + ca\eta^2 + ab\zeta^2) - 3abc\xi\eta\zeta\}.$

39 $Z = 1130 = -48S^2P + TQ = (abc + 8l^3)^2\,\{(abc + 2l^3)\,(bc\xi^3 + ca\eta^3 + ab\zeta^3) + 18abcl^2\xi\eta\zeta\}.$

40 $\Phi = 1660 = 12\,(abc + 8l^3)^3\,F - 288STP^2 + 768S^2PQ - 8TQ^2$
$= (abc + 8l^3)^4\,\{b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 - 10abc\,(a\eta^3\zeta^3 + b\zeta^3\xi^3 + c\xi^3\eta^3)\},$

viz. these are derived forms characterized by having a power of $abc + 8l^3$ as a factor: R is the discriminant; C, D, Y, Z occur in Aronhold, and in my Seventh memoir on Quantics [269]: Φ in Clebsch and Gordan's memoir of 1869.

I regard as known forms Λ, U, H, P, Q, S, T, F, that is, the eight forms 3, 11, 12, 15, 16, 1, 2, 17; the remaining 26 forms are expressed in terms of these by formulæ involving notations which will be explained, viz. we have

13 $\Psi = 3\,(\mathrm{bc}' + \mathrm{b'c} - 2\mathrm{ff}', \ldots, \mathrm{gh}' + \mathrm{g'h} - \mathrm{af}' - \mathrm{a'f}, \ldots \between X,\ Y,\ Z \between X',\ Y',\ Z') + TU^2.$

14 $\Omega = \frac{1}{18}\,\mathrm{Jac}\,(U,\ H,\ \Psi).$

18 $\Pi = -\frac{1}{36}\,[\mathrm{Jac}]\,(P,\ Q,\ F).$

4 $\Theta = (\mathrm{bc} - \mathrm{f}^2, \ldots, \mathrm{gh} - \mathrm{af}, \ldots \between \xi,\ \eta,\ \zeta)^2.$

5 $\Theta' = \frac{1}{2}\,\delta\Theta.$

6 $\Theta'' = \frac{1}{2}\,\delta^2\Theta.$

7 $B = -\frac{1}{3}\,\mathrm{Jac}\,(U,\ \Theta,\ \Lambda).$

8 $B' = \frac{1}{6}\,\delta B.$

9 $B'' = \frac{1}{24}\,\delta^2 B.$

10 $B''' = \frac{1}{48}\,\delta^3 B.$

19 $J = -\frac{1}{3}\,\mathrm{Jac}\,(U,\ H,\ \Lambda).$

27 $\bar{J} = \frac{1}{3}\,[\mathrm{Jac}]\,(P,\ Q,\ \Lambda).$

20 $K = -\frac{3}{2}\,\{\partial_\xi\Theta\partial_x H + \partial_\eta\Theta\partial_y H + \partial_\zeta\Theta\partial_z H\} - SU\Lambda.$

21 $K' = -(\delta)\,K.$

28 $\bar{K} = 3\,\{\partial_x\Theta\partial_\xi P + \partial_y\Theta\partial_\eta P + \partial_z\Theta\partial_\zeta P\} + Q\Lambda.$

29 $\bar{K}' = \frac{1}{6}\,(\delta)\,\bar{K}.$

22 $E = -\frac{1}{18}\,\mathrm{Jac}\,(K,\ U,\ \Lambda).$

23 $E' = -\frac{1}{4}\,(\delta)\,E.$

24 $E'' = \frac{1}{4}(\delta^2)E.$

30 $\bar{E} = -\frac{1}{9}\,\mathrm{Jac}\,(\bar{\bar{K}},\ U,\ \Lambda).$

31 $\bar{E}' = -\frac{1}{2}(\delta)\bar{E}.$

32 $\bar{E}'' = -\frac{1}{8}(\delta^2)\bar{E}.$

25 $M = \frac{1}{36}\,\mathrm{Jac}\,(U,\ \Psi,\ \Lambda).$

26 $M' = -(\delta)M.$

33 $\bar{M} = -\frac{1}{6}[\mathrm{Jac}](P,\ F,\ \Lambda).$

34 $\bar{M}' = \frac{1}{6}(\delta)\bar{M}.$

In explanation of the notations, observe that

$$U = ax^3 + by^3 + cz^3 + 6lxyz,$$
$$H = l^2(ax^3 + by^3 + cz^3) - (abc + 2l^3)\,xyz.$$

Hence, writing

$$6H = a'x^3 + b'y^3 + c'z^3 + 6l'xyz,$$

we have

$$a',\ b',\ c',\ l' = 6al^2,\ 6bl^2,\ 6cl^2,\ -(abc + 2l^3).$$

And this being so, we write

$$X,\ Y,\ Z = ax^2 + 2lyz,\ by^2 + 2lzx,\ cz^2 + 2lxy,$$
$$\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h} = ax,\ by,\ cz,\ lx,\ ly,\ lz,$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of U; and in like manner

$$X',\ Y',\ Z' = a'x^2 + 2l'yz,\ b'y^2 + 2l'zx,\ c'z^2 + 2l'xy,$$
$$\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{f}',\ \mathrm{g}',\ \mathrm{h}' = a'x,\ b'y,\ c'z,\ l'x,\ l'y,\ l'z,$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of $6H$.

Jac is written to denote the Jacobian, viz.:

$$\mathrm{Jac}\,(U,\ H,\ \Psi) = \begin{vmatrix} \partial_x U, & \partial_y U, & \partial_z U \\ \partial_x H, & \partial_y H, & \partial_z H \\ \partial_x \Psi, & \partial_y \Psi, & \partial_z \Psi \end{vmatrix},$$

and in like manner [Jac] to denote the Jacobian, when the differentiations are in regard to $(\xi,\ \eta,\ \zeta)$ instead of $(x,\ y,\ z)$: δ is the symbol of the δ-process, or substitution of the coefficients $(a',\ b',\ c',\ l')$ in place of $(a,\ b,\ c,\ l)$; in fact,

$$\delta = a'\partial_a + b'\partial_b + c'\partial_c + l'\partial_l:$$

δ, δ^2, &c., each operate directly on a function of $(a,\ b,\ c,\ l)$, the $(a',\ b',\ c',\ l')$ of the symbol δ being in the first instance regarded as constants, and being replaced ultimately by their values; for instance,

$$\delta abc = a'bc + ab'c + abc',\ \delta^2 abc = 2\,(ab'c' + a'bc' + a'b'c),\ \delta^3 abc = 6a'b'c'.$$

In several of the formulæ, instead of δ or δ^2, the symbol used is (δ) or (δ^2); in these cases, the function operated upon contains the factor $(abc+8l^3)$ or $(abc+8l^3)^2$, and is of the form $(abc+8l^3)(aU+bV+cW)$ or $(abc+8l^3)^2\ (a^2U+abV+\&c.)$: the meaning is, that the δ or δ^2 is supposed to operate through the $(abc+8l^3)a$, or $(abc+8l^3)^2a^2$, &c., as if this were a constant, upon the U, V, &c., only; thus: $(\delta).(abc+8l^3)(aU+bV+cW)$ is used to denote $(abc+8l^3)(a\delta U+b\delta V+c\delta W)$. As to this, observe that, operating with δ instead of (δ), there would be the additional terms $U\delta(abc+8l^3)a+\&c.$; we have in this case

$$\delta(abc+8l^3)a,\ =a(2a'bc+ab'c+abc'+24l^3l')+8l^3a',$$
$$=24a^2bcl^2-24al^2(abc+2l^3)+48al^5,\ =0;$$

or the rejected terms in fact vanish. For $(\delta^2).(abc+8l^3)(aU+bV+cW)$, operating with δ^2, we should have, in like manner, terms $U\delta^2(abc+8l^3)a$, &c.; here

$$\delta^2(abc+8l^3)a=a'^2bc+2aba'c'+2aca'b'+a^2b'c'+24l^2a'l'+24all'^2,$$

which is found to be $=-24a(abc+8l^3)(-abcl+l^4)$, that is, $=-24S(abc+8l^3)a$; and the terms in question are thus $=-24S(abc+8l^3)(aU+bV+cW)$, viz.

$$(abc+8l^3)(aU+bV+cW)$$

being a covariant, this is also a covariant; that is, in using (δ^2) instead of δ^2, we in fact reject certain covariant terms; or say, for instance, δ^2E being a covariant, then $(\delta^2)E$ is also a covariant, but a different covariant. The calculation with (δ) or (δ^2) is more simple than it would have been with δ or δ^2. See *post*, the calculations of K, $\overline{K}'$, &c.

I give for each of the 26 covariants a calculation showing how at least a single term of the final result is arrived at, and, in the several cases for which there is a power of $abc+8l^3$ as a factor, showing how this factor presents itself.

Calculations for the 26 Covariants.

13. $$\Psi=3(\mathrm{bc'+b'c-2ff'},\ldots,\mathrm{gh'+g'h-af'-a'f},\ldots\between X,\ Y,\ Z\between X',\ Y',\ Z')+TU^2,$$
$$=3((bc'+b'c)yz-2ll'x^2,\ldots,2ll'yz-(al'+a'l)x^2,\ldots\between ax^2+2lyz,\ldots\between a'x^2+2l'yz,\ldots)$$
$$+T(a^2x^6+\ldots).$$

The whole coefficient of x^6 is

$$-6ll'aa'+Ta^2,\ =36a^2l^3(abc+2l^3)+Ta^2,$$

viz. the coefficient of a^2x^6 is

$$=36l^3(abc+2l^3)+a^2b^2c^2-20abcl^3-8l^6$$
$$=a^2b^2c^2+16abcl^3+64l^6$$
$$=(abc+8l^3)^2.$$

14. $$\Omega=\tfrac{1}{18}\mathrm{Jac}(U,\ H,\ \Psi),\ =\tfrac{1}{2}\begin{vmatrix} X, & X', & \tfrac{1}{6}\partial_x\Psi \\ Y, & Y', & \tfrac{1}{6}\partial_y\Psi \\ Z, & Z', & \tfrac{1}{6}\partial_z\Psi \end{vmatrix}.$$

Here

$$\begin{aligned} YZ' - Y'Z &= (by^2 + 2lzx)(c'z^2 + 2l'xy) - (cz^2 + 2lxy)(b'z^2 + 2l'xy) \\ &= (bc' - b'c)\, y^2z^2 + (2bl' - b'l)\, xy^3 - 2(cl' - c'l)\, xz^3 \\ &= -2(abc + 8l^3)\, x(by^3 - cz^3); \end{aligned}$$

$$\tfrac{1}{2} \cdot \tfrac{1}{6}\partial_x \Psi = \tfrac{1}{2}(a^2x^3 - 5abx^2y^3 - 5acx^2z^3).$$

Hence the whole is

$$\begin{aligned} &= -(abc + 8l^3)\{a^2x^6(by^3 - cz^3) + b^2y^6(cz^3 - ax^3) + c^2z^6(ax^3 - by^3)\}, \\ &= \quad (abc + 8l^3)(by^3 - cz^3)(cz^3 - ax^3)(ax^3 - by^3). \end{aligned}$$

18. $$\Pi = -\tfrac{1}{36}[\mathrm{Jac}](P,\ Q,\ F) = -\tfrac{1}{36}\begin{vmatrix} \partial_\xi P, & \partial_\xi Q, & \partial_\xi F \\ \partial_\eta P, & \partial_\eta Q, & \partial_\eta F \\ \partial_\zeta P, & \partial_\zeta Q, & \partial_\zeta F \end{vmatrix},$$

viz. if, in this calculation, we write

$$6P = \mathrm{a}\,\xi^3 + \mathrm{b}\,\eta^3 + \mathrm{c}\,\zeta^3 + 6\mathrm{l}\,\xi\eta\zeta, \text{ i.e. } \mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{l} = -6lbc,\ -6lca,\ -6lab,\ -abc + 4l^3,$$

$$Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\mathrm{l}'\xi\eta\zeta, \text{ ,, } \mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{l}' = (abc - 10l^3)(bc,\ ca,\ ab),\ -l^2(5abc + 4l^3),$$

then

$$\Pi = -\tfrac{1}{4}\begin{vmatrix} \mathrm{a}\xi^2 + 2\mathrm{l}\eta\zeta, & \mathrm{a}'\xi^2 + 2\mathrm{l}'\eta\zeta, & \tfrac{1}{6}\partial_\xi F \\ \mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi, & \mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi, & \tfrac{1}{6}\partial_\eta F \\ \mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta, & \mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta, & \tfrac{1}{6}\partial_\zeta F \end{vmatrix}.$$

Here

$$\begin{aligned} &(\mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi)(\mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi)(\mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta) \\ &= (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,\eta^2\zeta^2 + 2(\mathrm{bl}' - \mathrm{b}'\mathrm{l})\,\xi\eta^3 - 2(\mathrm{cl}' - \mathrm{c}'\mathrm{l})\,\xi\zeta^3, \end{aligned}$$

or since

$$\begin{aligned} \mathrm{bc}' - \mathrm{b}'\mathrm{c} &= 0, \\ \mathrm{bl}' - \mathrm{b}'\mathrm{l} &= -6lca\,.\,-l^2(5abc + 4l^3) - (abc - 10l^3)\,ca\,(-abc + 4l^3) \\ &= ca\,\{6l^3(5abc + 4l^3) + (abc - 4l^3)(abc - 10l^3)\} \\ &= ca\,(abc + 8l^3)^2, \end{aligned}$$

and the like for $\mathrm{cl}' - \mathrm{c}'\mathrm{l}$, the expression is

$$= 2(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\,\xi;$$

and the whole is thus

$$\begin{aligned} &= -\tfrac{1}{2}(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)\,\xi\,.\,\tfrac{1}{6}\partial_\xi F + \ldots\} \\ &= -\tfrac{1}{2}(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)[b^2c^2\xi^6 - (abc + 16l^3)(b\zeta^3\xi^3 + c\xi^3\eta^3) + \&c.] \\ &\qquad + (ab\zeta^3 - bc\xi^3)[c^2a^2\eta^6 - (abc + 16l^3)(c\xi^3\eta^3 + a\eta^3\zeta^3) + \&c.] \\ &\qquad + (bc\xi^3 - ca\eta^3)[a^2b^2\zeta^6 - (abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3) + \&c.]\}. \end{aligned}$$

Here the coefficient of $\xi^6\eta^3$, inside the $\{\}$, is

$$ab^2c^3 + bc^2(abc + 16l^3),\ = 2bc^2(abc + 8l^3),$$

and consequently the whole is

$$= -(abc + 8l^3)^3 (bc^2\xi^6\eta^3 - \ldots),$$
$$= \quad (abc + 8l^3)^3 \{(c\eta^3 - b\zeta^3)(a\zeta^3 - c\xi^3)(b\xi^3 - a\eta^3)\}.$$

4. $$\Theta = (\mathrm{bc} - \mathrm{f}^2, \ldots, \mathrm{gh} - \mathrm{af}, \ldots \between \xi, \eta, \zeta)^2$$
$$= (bcyz - l^2x^2)\xi^2 + \ldots + 2(l^2yz - alx^2)\eta\zeta + \ldots$$

which are the terms of the final result

$$\Theta = x^2[-l^2\xi^2 - 2al\eta\zeta] + yz[bc\xi^2 + 2l^2\eta\zeta].$$

5 and 6. The δ-process applied to the terms of Θ just written down gives

$$\Theta' = \tfrac{1}{2}\delta\Theta = x^2[-ll'\xi^2 - (al' + a'l)\eta\zeta] + yz[\tfrac{1}{2}(bc' + b'c)\xi^2 + 2ll'\eta\zeta],$$
$$\Theta'' = \tfrac{1}{2}\delta^2\Theta = x^2[-l'^2\xi^2 - 2a'l'\eta\zeta] + yz[b'c'\xi^2 + 2l'^2\eta\zeta];$$

substituting for a', b', c', l' their values, we have the corresponding terms of Θ' and Θ'' respectively.

7. $$B = -\tfrac{1}{3}\,\mathrm{Jac}\,(U, \Theta, \Lambda), = -\begin{vmatrix} X, & \partial_x\Theta, & \xi \\ Y, & \partial_y\Theta, & \eta \\ Z, & \partial_z\Theta, & \zeta \end{vmatrix}.$$

A term is $X(\eta\partial_z\Theta - \zeta\partial_y\Theta)$, and if, in this calculation, we write

$$\Theta = (A, B, C, F, G, H \between x, y, z)^2, \textit{ i.e. } A = -l^2\xi^2 - 2al\eta\zeta, \ldots, F = \tfrac{1}{2}bc\xi^2 + l^2\eta\zeta,$$

then the term is

$$= (ax^2 + 2lyz)\{x \,.\, 2(G\eta - H\zeta) + y \,.\, 2(F\eta - B\zeta) + z \,.\, 2(C\eta - F\zeta)\}.$$

Here

$$2(G\eta - H\zeta) = \eta(ca\eta^2 + l^2\zeta\xi) - \zeta(ab\zeta^2 + l^2\xi\eta), = a(c\eta^3 - b\zeta^3),$$

and hence the whole term in x^3 is $= a^2x^3(c\eta^3 - b\zeta^3)$.

8, 9, 10. The coefficient of $x^3\eta^3$ in B is a^2c, and hence in δB, $\delta^2 B$, $\delta^3 B$ the coefficients of this term are $2a'ac + a^2c'$, $2a'^2c + 4aa'c'$, $6a'^2c'$, whence in

$$B', B'', B''' = \tfrac{1}{6}\delta B, \tfrac{1}{24}\delta^2 B, \tfrac{1}{48}\delta^3 B \text{ respectively,}$$

the coefficients are

$$\tfrac{1}{6}(a^2c' + 2aa'c), \tfrac{1}{12}(a'^2c + 2aa'c'), \tfrac{1}{8}a'^2c',$$
$$= \quad 3l^2a^2c, \qquad 9l^2a^2c, \qquad 27l^2a^2c \text{ respectively.}$$

19 $$J = -\tfrac{1}{3}\,\mathrm{Jac}\,(U, H, \Lambda) = -\tfrac{1}{2}\begin{vmatrix} X, & X', & \xi \\ Y, & Y', & \eta \\ Z, & Z', & \zeta \end{vmatrix};$$

a term is $-\tfrac{1}{2}(YZ' - Y'Z)\xi$, where, as in a previous calculation,

$$YZ' - Y'Z = -2(abc + 8l^3)x(by^3 - cz^3).$$

Hence, the whole is

$$= (abc + 8l^3)\{\xi x(by^3 - cz^3) + \eta y(cz^3 - ax^3) + \zeta z(ax^3 - by^3)\}.$$

27. $$\bar{J} = \tfrac{1}{3}[\mathrm{Jac}](P, Q, \Lambda) = \tfrac{1}{2}\begin{vmatrix} \mathrm{a}\xi^2 + 2\mathrm{l}\eta\zeta, & \mathrm{a}'\xi^2 + 2\mathrm{l}'\eta\zeta, & x \\ \mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi, & \mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi, & y \\ \mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta, & \mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta, & z \end{vmatrix},$$

if, as in a previous calculation

$$6P = \mathrm{a}\xi^3 + \mathrm{b}\eta^3 + \mathrm{c}\zeta^3 + 6\mathrm{l}\xi\eta\zeta, \quad Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\mathrm{l}'\xi\eta\zeta.$$

Here, as before,

$$(\mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi)(\mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi)(\mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta) = 2(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\xi.$$

Hence, the whole is

$$= (abc + 8l^3)^2\{x\xi a(c\eta^3 - b\zeta^3) + y\eta b(a\zeta^3 - c\xi^3) + z\zeta c(b\xi^3 - a\eta^3)\}.$$

20. $$K = -\tfrac{3}{2}(\partial_\xi\Theta\partial_x H + \partial_\eta\Theta\partial_y H + \partial_\zeta\Theta\partial_z H) - SU\Lambda,$$

which, H being

$$= \tfrac{1}{6}(a'x^3 + b'y^3 + c'z^3 + 6l'xyz),$$

and putting

$$\Theta = (A, B, C, F, G, H \between \xi, \eta, \zeta)^2, \quad A = -l^2x^2 + bcyz, \ldots, F = -alx^2 + l^2yz, \ldots,$$

is

$$\begin{aligned} = -\tfrac{3}{2}\{&(a'x^2 + 2l'yz)(A\xi + H\eta + G\zeta) - (-abcl + l^4)\,U(\xi x + \eta y + \zeta z) \\ &+ (b'y^2 + 2l'zx)(H\xi + B\eta + F\zeta) \\ &+ (c'z^2 + 2l'xy)(G\xi + F\eta + C\zeta)\}. \end{aligned}$$

The whole coefficient of ξ is thus

$$\begin{aligned} &= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)A + (b'y^2 + 2l'zx)H + (c'z^2 + 2l'xy)G\} - (-abcl + l^4)\,Ux \\ &= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)(-l^2x^2 + bcyz) + (b'y^2 + 2l'zx)(-clz^2 + l^2xy) \\ &\qquad + (c'z^2 + 2l'xy)(-bly^2 + l^2zx)\} - (-abcl + l^4)\{ax^4 + bxy^3 + cxz^3 + 6lx^2yz\}, \end{aligned}$$

and herein the coefficient of x^4 is

$$= \tfrac{3}{2}a'l^2 - al(-abc + l^3), \ = 9al^4 - al(-abc + l^3), \ = (abc + 8l^3)\,al;$$

viz. we have thus the term $(abc + 8l^3)\,\xi\,.\,alx^4$ of the final result.

21. $K' = -(\delta)K$, where K is of the form $(abc + 8l^3)(aU + bV + cW)$; operating with (δ), we obtain $(abc + 8l^3)(a\delta U + b\delta V + c\delta W)$. Taking for instance the term of K, $(abc + 8l^3)\,\xi\,[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]$, then, in operating with (δ), the term bc may be considered indifferently as belonging to bV or cW, and the resulting term of K' is

$$\begin{aligned} K' = -(\delta)K &= -(abc + 8l^3)\,\xi\,[al'x^4 - 2bl'xy^3 - 2cl'xz^3 + 3bc'y^2z^2], \\ &= \quad (abc + 8l^3)\,\xi\,[(abc + 2l^3)(ax^4 - 2bxy^3 - 2cxz^3) - 18bcl^3y^2z^2]. \end{aligned}$$

28. $$\overline{K} = 3\,\{\partial_x\Theta\partial_\xi P + \partial_y\Theta\partial_\eta P + \partial_z\Theta\partial_\zeta P\} + Q\Lambda\,;\ \text{viz. writing}$$

$$\Theta = (A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2,\quad A = -l^2\xi^2 - 2al\eta\zeta,\ \ldots\ F = \tfrac{1}{2}\,bc\xi^2 + l^2\eta\zeta,\ \ldots,$$

then this is

$$\begin{aligned}
&= 3\,\{[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\,2\,(Ax + Hy + Gz)\\
&\quad + [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi]\,2\,(Hx + By + Fz)\\
&\quad + [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta]\,2\,(Gx + Fy + Cz)\}\\
&\quad + \{(abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2(5abc + 4l^3)\,\xi\eta\zeta\}\,(\xi x + \eta y + \zeta z).
\end{aligned}$$

The whole coefficient of x is thus

$$\begin{aligned}
&= 3\,\{[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\,(-2l^2\xi^2 - 4al\eta\zeta)\\
&\quad + [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi]\,(ab\zeta^2 + l^2\xi\eta)\\
&\quad + [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta]\,(ac\eta^2 + l^2\zeta\xi)\}\\
&\quad + \{(abc - 10l^3)(bc\xi^4 + ca\xi\eta^3 + ab\xi\zeta^3) - 6l^2(5abc + 4l^3)\,\xi^2\eta\zeta\}\,;
\end{aligned}$$

herein the coefficient of ξ^4 is $18bcl^3 + (abc - 10l^3)\,bc,\ = (abc + 8l^3)\,bc$, giving, in the final result, the term $(abc + 8l^3)\,\xi\,.\,bcx^4$.

29. $$\overline{K}' = \tfrac{1}{6}(\delta)\,\overline{K}.$$

Here $\overline{K}$ is of the form $(abc + 8l^3)(aU + bV + cW)$, and we have

$$\overline{K}' = \tfrac{1}{6}(abc + 8l^3)(a\delta U + b\delta V + c\delta W).$$

A term of $aU + bV + cW$ is $x\,[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]$, where $bc\xi^4$ may be considered as belonging indifferently to bV or cW; and so for the other terms. The resulting term in $\tfrac{1}{6}(a\delta U + b\delta V + c\delta W)$ is thus

$$\tfrac{1}{6}\,x\,[bc'\xi^4 - 2ca'\xi\eta^3 - 2ab'\xi\zeta^3 - 6al'\eta^2\zeta^2],$$

which is

$$= x\,[l^2(bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3) + a\,(abc + 2l^3)\,\eta^2\zeta^2],$$

and we have thus a term of $\overline{K}'$.

22. $$E = -\tfrac{1}{18}\,\mathrm{Jac}\,(K,\ U,\ \Lambda):$$

K contains the factor $abc + 8l^3$, and if, omitting this factor, the value of K is called $A\xi + B\eta + C\zeta$, then we have

$$\begin{aligned}
E = -\tfrac{1}{6}\,\{&(\xi\partial_x A + \eta\partial_x B + \zeta\partial_x C)(Y\zeta - Z\eta) + (\xi\partial_y A + \eta\partial_y B + \zeta\partial_y C)(Z\xi - X\zeta)\\
&+ (\xi\partial_z A + \eta\partial_z B + \zeta\partial_z C)(X\eta - Y\xi)\},
\end{aligned}$$

and the term herein in ξ^2 is $-\tfrac{1}{6}\,\xi^2(Z\partial_y A - Y\partial_z A)$, where A is

$$= alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2\,;$$

viz. the coefficient of ξ^2 is

$$\begin{aligned}&= -\tfrac{1}{6}\{(cz^2+2lxy)(-6blxy^2+6bcyz^2)-(by^2+2lzx)(-6clxz^2+6bcy^2z)\}\\&= b^2cy^4z-bc^2yz^4+2bl^2x^2y^3-2cl^2x^2z^3\\&= (2l^2x^2+bcyz)(by^3-cz^3).\end{aligned}$$

Hence, restoring the omitted factor $(abc+8l^3)$, we have in E the term

$$(abc+8l^3)\,\xi^2(by^3-cz^3)\,[2l^2x^2+bcyz].$$

23, 24. $$E' = -\tfrac{1}{4}(\delta)\,E,\ E'' = \tfrac{1}{4}(\delta^2)\,E:$$

E is of the form $(abc+8l^3)(aU+bV+cW)$, and, as before, in a term such as

$$(abc+8l^3)\,\xi^2(by^3-cz^3)(2l^2x^2+bcyz),$$

we operate with δ or δ^2 only on the factor $2l^2x^2+bcyz$; and in E' and E'' respectively, operating upon this factor, we obtain

$$-\tfrac{1}{4}\{4ll'x^2+(bc'+b'c)\,yz\},\ \text{and}\ \tfrac{1}{4}\{4l'^2x^2+2b'c'yz\},$$

viz. we thus obtain in E' the term

$$(abc+8l^3)\,\xi^2(by^3-cz^3)\,[l\,(abc+2l^3)\,x^2-3bcl^2yz],$$

and in E'' the term

$$(abc+8l^3)\,\xi^2(by^3-cz^3)\,[(abc+2l^3)^2x^2+18bcl^4yz].$$

30. $$\bar{E} = -\tfrac{1}{9}\,\mathrm{Jac}\,(\bar{K},\ U,\ \Lambda),\ = -\tfrac{1}{3}\begin{vmatrix}\partial_x\bar{K}, & X, & \xi\\ \partial_y\bar{K}, & Y, & \eta\\ \partial_z\bar{K}, & Z, & \zeta\end{vmatrix},$$

and, if omitting in $\bar{K}$ the factor $abc+8l^3$, we write $\bar{K} = Ax+By+Cz$, where

$$A = bc\xi^4-2ca\xi\eta^3-2ab\xi\zeta^3-6al\eta^2\zeta^2,\ \text{this is}\ = -\tfrac{1}{3}\begin{vmatrix}A, & X, & \xi\\ B, & Y, & \eta\\ C, & Z, & \zeta\end{vmatrix},$$

which contains the term

$$\begin{aligned}\tfrac{1}{3}X(B\zeta-C\eta),\ &= \tfrac{1}{3}(ax^2+2lyz)\{\zeta(ca\eta^4-2ab\eta\zeta^3-2bc\eta\xi^3-6bl\zeta^2\xi^2)\\&\qquad -\eta(ab\zeta^4-2bc\zeta\xi^3-2ca\zeta\eta^3-6cl\xi^2\eta^2)\},\\&= (ax^2+2lyz)(c\eta^3-b\zeta^3)(2l\xi^2+a\eta\zeta).\end{aligned}$$

Hence, restoring the factor $abc+8l^3$, we have the terms

$$\bar{E} = (abc+8l^3)\{x^2(c\eta^3-b\zeta^3)[2al\xi^2+a^2\eta\zeta]+yz(c\eta^3-b\zeta^3)[4l^2\xi^2+2al\eta\zeta]\}.$$

31 and 32. $$\bar{E}' = -\tfrac{1}{2}(\delta)\,\bar{E},\quad \bar{E}'' = -\tfrac{1}{8}(\delta^2)\,\bar{E}:$$

$\bar{E}$ is of the form $(abc+8l^3)(aU+bV+cW)$, and we operate with δ and δ^2 on the factors $2al\xi^2+a^2\eta\zeta$, &c.; viz.

$$\delta(2al\xi^2+a^2\eta\zeta) = 2(al'+a'l)\,\xi^2+2aa'\eta\zeta,\quad \delta^2(2al\xi^2+a^2\eta\zeta) = 4a'l'\xi^2+2a'^2\eta\zeta,$$

and we thus obtain in $\bar{E}'$ the term

$$(abc+8l^3)\,x^2\,(c\eta^3-b\zeta^3)\,[a\,(abc-4l^3)\,\xi^2-6a^2l^2\eta\zeta],$$

and in $\bar{E}''$ the term

$$(abc+8l^3)\,x^2\,(c\eta^3-b\zeta^3)\,[-3al^2\,(abc+2l^3)\,\xi^2+9a^3l^4\eta\zeta].$$

25. $M=\frac{1}{36}\,\mathrm{Jac}\,(U,\ \Psi,\ \Lambda)$: this, omitting the factor $(abc+8l^3)^2$ of Ψ, is

$$=\frac{1}{2}\begin{vmatrix} ax^2+2lyz, & ax^2\,(ax^3-5by^3-5cz^3), & \xi \\ by^2+2lzx, & by^2\,(by^3-5cz^3-5ax^3), & \eta \\ cz^2+2lxy, & cz^2\,(cz^3-5ax^3-5by^3), & \zeta \end{vmatrix};$$

the coefficient of ξ herein is

$$=\tfrac{1}{2}\,\{(bcy^2z^2+2clxz^3)\,(cz^3-5ax^3-5by^3)-(bcy^2z^2+2blxy^3)\,(by^3-5cz^3-5ax^3)\},$$

$$=\tfrac{1}{2}\,\{bcy^2z^2\,(-6by^3+6cz^3)+2lx\,[-b^2y^6+c^2z^6+5ax^3\,(by^3-cz^3)]\},$$

$$=\quad (by^3-cz^3)\,[5alx^4-blxy^3-clxz^3-3bcy^2z^2].$$

Hence, restoring the factor $(abc+8l^3)^2$, we have the term

$$(abc+8l^3)^2\,.\,\xi\,(by^3-cz^3)\,[5alx^4-blxy^3-clxz^3-3bcy^2z^2].$$

26. $M'=-(\delta)\,M$. Here M is of the form $(abc+8l^3)^2\,(a^2U+\&\text{c.})$; and the δ operates through the $(abc+8l^3)^2\,a^2$, &c.; we, in fact, have in M' the term

$$-(abc+8l^3)^2\,.\,\xi\,(by^3-cz^3)\,[5al'x^4-bl'xy^3-cl'xz^3-3bc'y^2z^2],$$

which is

$$=(abc+8l^3)^2\,.\,\xi\,(by^3-cz^3)\,[(abc+2l^3)\,(5ax^4-bxy^3-cxz^3)+18bcl^2y^2z^2].$$

33. $$\bar{M}=-\tfrac{1}{6}\,[\mathrm{Jac}]\,(P,\ F,\ \Lambda),\ =-\tfrac{1}{6}\begin{vmatrix} -3lbc\xi^2+(-abc+4l^3)\,\eta\zeta, & \partial_\xi F, & x \\ -3lca\eta^2+(-abc+4l^3)\,\zeta\xi, & \partial_\eta F, & y \\ -3lab\zeta^2+(-abc+4l^3)\,\xi\eta, & \partial_\zeta F, & z \end{vmatrix},$$

and the whole coefficient of x is thus

$$=\tfrac{1}{6}\,\{[3lca\eta^2+(abc-4l^3)\,\zeta\xi]\,\partial_\zeta F-[3lab\zeta^2+(abc-4l^3)\,\xi\eta]\,\partial_\eta F\},$$

or substituting for $\frac{1}{6}\,\partial_\zeta F$, $\frac{1}{6}\,\partial_\eta F$ their values, this is

$$=\{3lca\eta^2+(abc-4l^3)\,\zeta\xi\}\,[a^2b^2\zeta^3-(abc+16l^3)\,(b\zeta^3\xi^3+a\zeta^3\eta^3)$$
$$-4l^2\,(bc\xi^4\eta+ca\xi\eta^4+4ab\xi\eta\zeta^3)-8l\,(abc+2l^3)\,\xi^2\eta^2\zeta]$$

$$-\{3lab\zeta^2+(abc-4l^3)\,\xi\eta\}\,[a^2c^2\eta^3-(abc+16l^3)\,(a\eta^3\zeta^3+c\eta^3\xi^3)$$
$$-4l^2\,(bc\zeta\xi^4+4ca\xi\eta^3\zeta+ab\xi\zeta^4)-8l\,(abc+2l^3)\,\xi^2\eta\zeta^3].$$

Collecting, first, the terms independent of $abc-4l^3$, and, next, those which contain $abc-4l^3$, each set contains the factor $c\eta^3-b\zeta^3$, and the whole is $=c\eta^3-b\zeta^3$ multiplied by

$$-3la^3bc\eta^2\zeta^2-3a^2l\,(abc+8l^3)\,\eta^2\zeta^2-12l^3\,(abc\xi^4+a^2c\xi\eta^3+a^2b\xi\zeta^3)-24al^2\,(abc+2l^3)\,\xi^2\eta\zeta$$
$$+(abc-4l^3)\,\{a^2c\xi\eta^3+a^2b\xi\zeta^3-(abc+16l^3)\,\xi^4+12al^3\xi^2\eta\zeta\};$$

and here collecting the terms in ξ^4, $\xi\,(c\eta^3+b\zeta^3)$, $\xi^2\eta\zeta$, and $\eta^2\zeta^2$, each of these contains the factor $abc+8l^3$, and, finally, the term of $\overline{M}$ is

$$=(abc+8l^3)\,(c\eta^3-b\zeta^3)\,[(abc-8l^3)\,\xi^4-a^2c\xi\eta^3-a^2b\xi\zeta^3-12al^3\xi^2\eta\zeta-6a^2l\eta^2\zeta^2]\,x.$$

34. $$\overline{M}'=\tfrac{1}{6}\,(\delta)\,\overline{M}.$$

Here M is of the form $(abc+8l^3)\,(aU+bV+cW)$; and, operating with δ through the $(abc+8l^3)\,a$, &c., we obtain in $\overline{M}'$ the term

$$\tfrac{1}{6}\,(abc+8l^3)\,x\,(c\eta^3-b\zeta^3)\,[(a'bc+ab'c+abc'-24l^2l')\,\xi^4+\&c.],$$

where

$$a'bc+ab'c+abc'-24l^2l'=18abcl^2+24l^3\,(abc+2l^3),\;=6l^3\,(7abc+8l^3),$$

and the term thus is

$$=(abc+8l^3)\,x\,(c\eta^3-b\zeta^3)\,[(7abc+8l^3)\,l^3\xi^4+\ldots].$$

This concludes the series of calculations.

Cambridge, England, 17 *May*, 1881.

771.

SPECIMEN OF A LITERAL TABLE FOR BINARY QUANTICS, OTHERWISE A PARTITION TABLE.

[From the *American Journal of Mathematics*, vol. IV. (1881), pp. 248—255.]

THE Table, commencing 1; b; c, b^2; d, bc, b^3; ..., is in fact a Partition Table, viz. considering the letters b, c, d, ... as denoting 1, 2, 3, ... respectively, it is 1°; 1; 2, 11; 3, 12, 111; ... a table of the partitions of the numbers 0, 1, 2, 3, ..., expressed however in the literal form, in order to its giving the literal terms which enter into the coefficients of any covariant of a binary quantic. The table ought to have been made and published many years ago, before the calculation of the covariants of the quintic; and the present publication of it is, in some measure, an anachronism: but I in fact felt the need of it in some calculations in regard to the sextic; and I think the table may be found useful on other occasions. I have contented myself with calculating the table up to $s=18$, that is, so as to include in it all the partitions of 18: it would, I think, be desirable to extend it further, say to $z=26$; or even beyond this point, but perhaps without introducing any new letters, (that is, so as to give for the higher numbers only the partitions with a largest part not exceeding 26): the question of the space which such a table would occupy will be considered presently.

As to the employment of the table, observe that, in applying it to the case of a quantic $(a, b, c, d \between x, y)^3$, the terms containing the letters e, f, etc., posterior to the last coefficient d of the quantic are to be disregarded; and that the terms are to be rendered homogeneous by the introduction of the proper power of the first coefficient a, rejecting any term for which the exponent of a would be negative (or what is the same thing, any term of too high a degree in the coefficients b, c, d);

thus, for the cubicovariant, where the coefficients are of the degree 3, and of the weights 3, 4, 5, 6 respectively, from the portion of the table

d	e	f	g
bc	bd	be	bf
b^3	c^2	cd	ce
	b^2c	b^2d	d^2
	b^4	bc^2	b^2e
		b^3c	bcd
		b^5	c^3
			b^3d
			etc.

we at once copy out the terms

a^2d	abd	acd	ad^2
abc	ac^2	b^2d	bcd
b^3	b^2c	bc^2	c^3

which compose the coefficients in question.

As regards the formation of the table, this is at once effected, and the successive terms are obtained *currente calamo*, by Arbogast's rule of the last and the last but one: observing that each term is to be regarded as containing implicitly a power of a, so that operating on any term such as b^4, the operation on the last letter gives b^3c, and that on the last but one letter gives b^5. There is little risk of error except in the accidental omission of a term; but of course any one omission would occasion the omission of all the subsequent terms derivable from the omitted term, and would so be fatal: to remove this source of error, observe that for the successive numbers 0, 1, 2, 3, etc., the number of partitions should be

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 ...
1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385 ...

and we can thus, for each partible number successively, verify that the right number of partitions has been obtained.

But as the number of partitions becomes large, a further control is convenient, and even necessary—say we have the 176 partitions of 15, we have by the rule to derive thence the 231 partitions of 16, and it is not until the whole of this derivation is gone through, that we could by counting the number of the new terms ascertain that the right number of 231 terms has been obtained. To break up the verification, it is convenient to know that for the partitions of 16 into 1 part, 2 parts, 3 parts, 4 parts, etc., the numbers of partitions are 1, 8, 21, 34, etc., respectively: we can then as soon as the derivations giving the partitions into 1 part, 2 parts, 3 parts, etc., respectively, have been performed, verify that the right numbers 1, 8, 21, 34, etc., of terms have been obtained. The numbers are contained in the following table, each column of which is calculated from the preceding columns according to a rule which

is easily obtained, and which is itself verified by the condition that the sums of the numbers in the several columns give the before mentioned series of numbers 1, 1, 2, 3, 5, 7, etc.

No. of Parts.	Partible Number. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
3				1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27
4					1	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5						1	1	2	3	5	7	10	13	18	23	30	37	47	57
6							1	1	2	3	5	7	11	14	20	26	35	44	58
7								1	1	2	3	5	7	11	15	21	28	38	49
8									1	1	2	3	5	7	11	15	22	29	40
9										1	1	2	3	5	7	11	15	22	30
10											1	1	2	3	5	7	11	15	22
11												1	1	2	3	5	7	11	15
12													1	1	2	3	5	7	11
13														1	1	2	3	5	7
14															1	1	2	3	5
15																1	1	2	3
16																	1	1	2
17																		1	1
18																			1
	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385

The practical rule for the construction of the table thus is:—On a sheet of paper ruled in squares, and which is read as a continuous column from the bottom of one column to the top of the next column, form the terms by Arbogast's method as already explained; writing down in pencil a batch of terms, and counting them

to see that the right number has been obtained, then, at the same time verifying the derivations, mark these over in ink; and so on with another batch of terms, until the whole number of the partitions of any particular number is obtained.

The foregoing series 1, 1, 2, 3, ..., 385, for the number of the partitions of the successive numbers 0, 1, 2, 3, ..., 18 is carried by Euler up to the number of partitions of 59, $=831820$, see the paper "De Partitione Numerorum," *Op. Arith. Coll.* I., bottom line of the table pp. 97—101: the continuation from the number 385 and for the partible numbers 19 to 30 is as follows:

19	20	21	22	23	24	25	26	27	28	29	30
490	627	792	1002	1255	1575	1958	2436	3010	3718	4565	5604

the whole number of terms 1, 1, ..., 5604 amounts to 28629, which at the rate of 500 to a page would occupy somewhat under 60 pages; or, at the rate here employed of 369 to a page, somewhat under 78 pages.

THE PARTITION TABLE, 0 TO 18.

0.3	4.5	6.7	7.8	8.9	9	9.10	10	10.11
			cf	b^2g	bi	bc^4	bdg	c^5
$\boxed{0 \quad 1}$	$\boxed{4 \quad 5}$	$\boxed{6 \quad 11}$	de	bcf	ch	b^5e	bef	b^5f
			b^2f	bde	dg	b^4cd	c^2g	b^4ce
1			bce	c^2e	ef	b^3c^3	cdf	b^4d^2
	e	g	bd^2	cd^2	b^2h	b^6d	ce^2	b^3c^3d
$\boxed{1 \quad 1}$	bd	bf	c^2d	b^3f	bcg	b^5c^2	d^2e	b^2c^4
	c^2	ce	b^3e	b^2ce	bdf	b^7c	b^3h	b^6e
	b^2c	d^2	b^2cd	b^2d^2	be^2	b^9	b^2cg	b^5cd
b	b^4	b^2e	bc^2	bc^3d	c^2f		b^2df	b^4c^3
		bcd	b^4d	c^4	cde		b^2e^2	b^7d
$\boxed{2 \quad 2}$	$\boxed{5 \quad 7}$	c^3	b^3c^2	b^4e	d^3	$\boxed{10 \quad 42}$	bc^2f	b^6c^2
		b^3d	b^5c	b^3cd	b^3g		$bcde$	b^8c
c		b^2c^2	b^7	b^2c^3	b^2cf		bd^3	b^{10}
b^2	f	b^4c		b^5d	b^2de	k	c^3e	
	be	b^6	$\boxed{8 \quad 22}$	b^4c^2	bc^2e	bj	c^3d^2	$\boxed{11 \quad 56}$
$\boxed{3 \quad 3}$	cd			b^6c	bcd^2	ci	b^4g	
	b^3d	$\boxed{7 \quad 15}$	i	b^8	c^3d	dh	b^3cf	
	bc^2		bh		b^4f	eg	b^3de	
d	b^3c		cg	$\boxed{9 \quad 30}$	b^3ce	f^2	b^3c^2e	l
bc	b^5	h	df		b^3d^2	b^2i	b^2cd^2	bk
b^3		bg	e^2	j	b^2c^2d	bch	bc^3d	cj

THE PARTITION TABLE, 0 TO 18 (*continued*).

11	11.12	12	12.13	13	13.14	14	14	14
di	b^6f	$bcef$	b^5d^2	$bceg$	bc^3d^2	gi	b^4k	c^5e
eh	b^5ce	bd^2f	b^5c^3d	bcf^2	c^3d	h^2	b^3cj	c^4d^2
fg	b^5d^2	bde^2	b^4c^4	bd^2g	b^6h	b^2m	b^3di	b^6i
b^2j	b^4c^2d	c^3g	b^8e	$bdef$	b^5cg	bcl	b^3eh	b^6ch
bci	b^3c^4	c^2df	b^7cd	be^3	b^5df	bdk	b^3fg	b^5dg
bdh	b^7e	c^2e^2	b^6c^3	c^3h	b^5e^2	bej	b^3c^2i	b^5ef
beg	b^6cd	cd^2e	b^9d	c^3dg	b^4c^2f	bfi	b^3cdh	b^4c^2g
bf^2	b^5c^3	d^4	b^8c^2	c^2ef	b^4cde	bgh	b^3ceg	b^4cdf
c^2h	b^3d	b^4i	$b^{10}c$	cd^2f	b^4d^3	c^2k	b^3cf^2	b^4ce^2
cdg	b^7c^2	b^3ch	b^{12}	cde^2	b^3c^3e	cdj	b^3d^2g	b^4d^2e
cef	b^3c	b^2dg	13	d^3e	$b^3c^2d^2$	cei	b^2def	b^2c^3f
d^2f	b^{11}	b^3ef	101	b^4j	b^2c^4d	cfh	b^2e^3	b^3c^2de
de^2	12	b^2c^2g		b^3ci	bc^6	cg^2	bc^3h	b^3cd^3
b^3i	77	b^3cdf	n	b^3dh	b^7g	d^2i	bc^2dg	b^2c^4e
b^2ch		b^2ce^2	bm	b^3eg	b^6cf	deh	bc^2ef	$b^2c^3d^2$
b^2dg	m	b^2d^2e	cl	b^3f^2	b^6de	dfg	bcd^2f	bc^5d
b^2ef	bl	bc^2f	dk	b^2c^3h	b^5c^2e	e^2g	$bcde^2$	c^7
bc^2g	ck	bc^2de	ej	b^2cdg	b^5cd^2	ef^2	bd^2e	b^7h
$bcdf$	dj	bcd^3	fi	b^2cef	b^4c^2d	b^3l	c^4g	b^6cg
bce^2	ei	c^4e	gh	b^2d^2f	b^3c^5	b^2ck	c^3df	b^6df
bd^2e	fh	c^3d^2	b^2l	b^2de^2	b^3f	b^2dj	c^3e^2	b^6e^2
c^2f	g^2	b^3h	bck	bc^3g	b^7ce	b^2ei	c^2d^2e	b^5c^2f
c^2de	b^2k	b^4cg	bdj	bc^2df	b^7d^2	b^2fh	cd^4	b^5cde
cd^3	bcj	b^4df	bei	bc^2e^2	b^5c^2d	b^2g^2	b^5j	b^5d^3
b^4h	bdi	b^4e^2	bfh	bcd^2e	b^5c^4	bc^2j	b^4ci	b^4c^3e
b^3cg	beh	b^3c^2f	bg^2	bd^4	b^8e	$bcdi$	b^4dh	$b^4c^2d^2$
b^2df	bfg	b^3cde	c^2j	c^4f	b^8cd	$bceh$	b^4eg	b^3c^4d
b^3e^2	c^2i	b^3d^3	cdi	c^3de	b^7c^3	$bcfg$	b^4f^2	b^3c^6
b^9c^2f	cdh	b^2ce^2	ceh	c^2d^3	$b^{10}d$	bd^2h	b^3c^2h	b^6g
b^2cde	ceg	$b^2c^2d^2$	cfg	b^5i	b^9c^2	$bdeg$	b^3cdg	b^7cf
b^2d^3	cf^2	bc^4d	d^2h	b^4ch	$b^{11}c$	bdf^2	b^3cef	b^7de
bc^3e	d^2g	c^6	deg	b^4dg	b^{13}	be^2f	b^3d^2f	b^6c^2e
bc^2d^2	def	b^6g	df^2	b^4ef	14	c^3i	b^3de^2	b^5cd^2
c^4d	e^3	b^5cf	e^2f	b^5c^2g	135	c^2dh	b^2c^3g	b^5c^3d
b^5g	b^3j	b^5de	b^5k	b^3cdf		c^2eg	b^2c^2df	b^4c^5
b^4cf	b^2ci	b^4c^2e	b^2cj	b^3ce^2	o	c^3f^2	$b^2c^2e^2$	b^9f
b^4de	b^2dh	b^4cd^2	b^3di	b^3d^2e	bn	cd^2g	b^3cd^2e	b^8ce
b^3c^2e	b^2eg	b^3c^3d	b^2eh	b^2c^3f	cm	$cdef$	b^2d^4	b^9d^2
b^3cd^2	b^2f^2	b^2c^5	b^2fg	b^3c^2de	dl	ce^3	bc^4f	b^7c^2d
b^2c^3d	bc^2h	b^7f	bc^2i	b^2cd^3	ek	d^3f	bc^3de	b^6c^4
bc^5	$bcdg$	b^5ce	$bcdh$	bc^4e	fj	d^2e^2	bc^2d^3	$b^{10}e$

THE PARTITION TABLE, 0 TO 18 (*continued*).

14.15	15	15	15	15.16	16	16	16	16
b^9cd	b^2gh	$bcdef$	b^5f^2	b^4c^4d	cdl	cef^2	cd^2e^2	b^5fg
b^5c^3	bc^3k	bce^3	b^4c^2h	b^3c^6	cek	d^3h	d^4e	b^4c^2i
$b^{11}d$	$bcdj$	bd^3f	b^4cdg	b^8g	cfj	d^2eg	b^6l	b^4cdh
$b^{10}c^2$	$bcei$	bd^2e^2	b^4cef	b^8cf	cgi	d^2f^2	b^4ck	b^4ceg
$b^{12}c$	$bcfh$	c^4h	b^4d^2f	b^3de	ch^2	de^2f	b^4dj	b^4cf^2
b^{14}	bcg^2	c^3dg	b^4de^2	b^7c^2e	d^2k	e^4	b^4ei	b^4d^2g
15	bd^2i	c^3ef	b^3c^2g	b^7cd^2	dej	b^4m	b^4fh	b^4def
176	$bdeh$	c^2d^2f	b^3c^2df	b^8c^2d	dfi	b^3cl	b^4g^2	b^4e^2
	$bdfg$	c^2de^2	$b^3c^2e^2$	b^6c^5	dgh	b^3dk	b^3c^2j	b^3c^3h
p	be^2g	cd^3e	b^3cd^2e	$b^{10}f$	e^2i	b^3ej	b^3cdi	b^3c^2dg
bo	bef^2	d^5	b^3d^4	b^8ce	efh	b^3fi	b^3ceh	b^3c^2ef
cn	c^3j	b^5k	b^2c^4f	b^9d^2	eg^2	b^3gh	b^3cfg	b^3cd^2f
dm	c^2di	b^4cj	b^2c^3de	b^8c^2d	f^2g	b^2c^3k	b^3d^2h	b^3cde^2
el	c^2eh	b^4di	$b^2c^2d^3$	b^7c^4	b^3n	b^3cdj	b^3deg	b^3d^3e
fk	c^2fg	b^4eh	bc^5e	$b^{11}e$	b^2cm	b^2cei	b^3df^2	b^2c^4g
gj	cd^2h	b^4fg	bc^4d^2	$b^{10}cd$	b^2dl	b^2cfh	b^2e^2f	b^2c^3df
hi	$cdeg$	b^3c^2i	c^6d	b^9c^3	b^2ek	b^2cg^2	b^2c^3i	$b^2c^2e^2$
b^2n	cdf^2	b^3cdh	b^7i	$b^{12}d$	b^2fj	b^2d^2i	b^2c^2dh	$b^2c^2d^2e$
bcm	ce^2f	b^3ceg	b^8ch	$b^{11}c^2$	b^2gi	b^2deh	b^2c^2eg	b^2cd^4
bdl	d^3g	b^3cf^2	b^8dg	$b^{13}c$	b^2h^2	b^3dfg	$b^2c^2f^2$	bc^3f
bek	d^2ef	b^3d^2g	b^6ef	b^{15}	bc^2l	b^2e^2g	b^2cd^2g	bc^4de
bfj	de^3	b^3def	b^5c^2g	16	$bcdk$	b^2ef^2	b^2cdef	bc^3d^3
bgi	b^4l	b^3e^3	b^5cdf	231	$bcej$	bc^3j	b^2ce^3	c^6e
bh^2	b^3ck	b^3c^3h	b^5ce^2		$bcfi$	bc^2di	b^2d^3f	c^5d^2
c^2l	b^3dj	b^2c^2dg	b^5d^2e	q	$bcgh$	bc^2eh	$b^2d^2e^2$	b^7j
cdk	b^3ei	b^2c^2ef	b^4c^2f	bp	bd^2j	bc^2fg	bc^4h	b^6ci
cej	b^3fh	b^2cd^2f	b^4c^2de	co	$bdei$	bcd^2h	bc^3dg	b^6dh
cfi	b^3g^2	b^2cde^2	b^4cd^3	dn	$bdfh$	$bcdeg$	bc^2ef	b^6eg
cgh	b^2c^2j	b^2d^3e	b^3c^4e	em	bdg^2	$bcdf^2$	bc^2d^2f	b^5f^2
d^2j	b^2cdi	bc^4g	$b^3c^3d^2$	fl	be^2h	bce^2f	bc^2de^2	b^5c^2h
dei	b^2ceh	bc^3df	b^2c^3d	gk	$befg$	bd^3g	bcd^3e	b^5cdg
dfh	b^2cfg	bc^3e^2	bc^7	hj	bf^3	bd^2ef	bd^5	b^5cef
dg^2	b^2d^2h	bc^2d^2e	b^8h	i^2	c^3k	bde^3	c^5g	b^3d^2f
e^2h	b^2deg	bcd^4	b^7cg	b^2o	c^2dj	c^4i	c^4df	b^5de^2
efg	b^2df^2	c^3f	b^7df	bcn	c^2ei	c^3dh	c^4e^2	b^4c^3g
f^3	b^2e^2f	c^4de	b^7e^2	bdm	c^2fh	c^3eg	c^3d^2e	b^4c^2df
b^3m	bc^3i	c^3d^3	b^5c^3f	bel	c^2g^2	c^3f^2	c^2d^4	$b^4c^2e^2$
b^2cl	bc^2dh	b^6j	b^6cde	bfk	cd^2i	c^2d^2g	b^6k	b^4cd^2e
b^2dk	bc^2eg	b^5ci	b^6d^3	bgj	$cdeh$	c^2def	b^3cj	b^4d^4
b^2ej	bc^2f^2	b^5dh	b^6c^3e	bhi	$cdfg$	c^2e^3	b^5di	b^3c^4f
b^2fi	bcd^2g	b^5eg	$b^5c^2d^2$	c^2m	ce^2g	cd^3f	b^5eh	b^2c^3de

THE PARTITION TABLE, 0 TO 18 (*continued*).

16	16.17	17	17	17	17	17	17	17.18
$b^3c^5d^3$	b^6c^2d	efi	def^2	c^5deg	bc^2e^3	bc^5g	b^7f^2	$b^7c^3d^2$
b^2c^5e	b^8c^4	egh	e^3f	c^3df^2	bcd^3f	bc^4df	b^6c^3h	b^8c^4d
$b^2c^4d^2$	$b^{12}e$	f^3h	b^4n	c^2e^2f	bcd^2e^2	bc^4e^2	b^5cdg	b^5c^6
bc^6d	$b^{11}cd$	fg^2	b^3cm	cd^3g	bd^4e	bc^3d^2e	b^6cef	$b^{11}g$
c^8	$b^{10}c^3$	b^3o	b^3dl	cd^2ef	c^5h	bc^2d^4	b^6d^2f	$b^{10}cf$
b^8i	$b^{13}d$	b^2cn	b^3ek	cde^3	c^4dg	c^6f	b^6de^2	$b^{10}de$
b^7ch	$b^{12}c^2$	b^2dm	b^3fj	d^4f	c^4ef	c^5de	b^5c^3g	b^9c^2e
b^7dg	$b^{14}c$	b^2el	b^2gi	d^3e^2	c^3d^2f	c^4d^3	b^5c^3df	b^9cd^2
b^7ef	b^{16}	b^2fk	b^3h^2	b^5m	c^3de^2	b^7k	$b^5c^2e^2$	b^8c^2d
b^6c^2g	$\boxed{17}$	b^2gj	b^2c^2l	b^4cl	c^2d^3e	b^6cj	b^5cd^2e	b^7c^5
b^6cdf	$\boxed{297}$	b^2hi	b^2cdk	b^4dk	cd^5	b^6di	b^5d^4	$b^{12}f$
b^6ce^2		bc^2m	b^2cej	b^4ej	b^6l	b^6ek	b^4c^4f	$b^{11}ce$
b^5d^2e	r	$bcdl$	b^2cfi	b^4fi	b^5ck	b^6fg	b^4c^2de	$b^{11}d^2$
b^5c^3f	bq	$bcek$	b^2cgh	b^4gh	b^5dj	b^5c^2i	$b^4c^2d^2$	$b^{10}c^2d$
b^5c^2de	cp	$bcfj$	b^2d^2j	b^3c^2k	b^5ei	b^6cdh	b^3c^5e	b^9c^4
b^5cd^3	do	$bcgi$	b^2dei	b^2cdj	b^5fh	b^5ceg	$b^2c^4d^2$	$b^{13}e$
b^4c^4e	en	bch^2	b^2dfh	b^3cei	b^5g^2	b^5cf^2	b^2c^6d	$b^{10}cd$
$b^4c^3d^2$	fm	bd^2k	b^2dg^2	b^3cfh	b^4c^2j	b^5d^2g	bc^8	$b^{11}c^2$
b^3c^3d	gl	$bdej$	b^2e^2h	b^3cg^2	b^4cdi	b^5def	b^9i	$b^{14}d$
b^3c^7	hk	$bdfi$	b^2efg	b^3d^2i	b^4ceh	b^5e^3	b^8ch	$b^{13}c^2$
b^9h	ij	$bdgh$	b^2f^3	b^3deh	b^4cfg	b^4c^3h	b^8dg	$b^{15}c$
b^8cg	b^5p	be^2i	bc^3k	b^3dfg	b^4d^2h	b^4c^2dg	b^8ef	b^{17}
b^8df	bco	$befh$	bc^2dj	b^2e^2g	b^4deg	b^4c^2ef	b^7c^3g	$\boxed{18}$
b^8e^2	bdn	beg^2	bc^2ei	b^2ef^2	b^4df^2	b^4cd^2f	b^7cdf	$\boxed{385}$
b^7c^2f	bem	bf^2g	bc^2fh	b^2c^2j	b^4e^2f	b^4cde^2	b^7ce^2	
b^7cde	bfl	c^3l	bc^2g^2	b^2c^2di	b^3c^3i	b^4d^3e	b^7d^2e	s
b^7d^3	bgk	c^2dk	bcd^2i	b^2c^2eh	b^3c^2dh	b^3c^4g	b^6c^3f	br
b^6c^3e	bhj	c^2ej	$bcdeh$	b^2c^2fg	b^3c^2eg	b^3c^3df	b^6c^2de	cq
$b^6c^2d^2$	bi^2	c^2fi	$bcdfg$	b^2cd^2h	$b^3c^2f^2$	$b^3c^3e^2$	b^6cd^3	dp
b^6c^4d	c^2n	c^2gh	bce^2g	b^2cdeg	b^3cd^2g	$b^3c^2d^2e$	b^5c^4e	eo
b^4c^6	cdm	cd^2j	$bcef^2$	b^2cdf^2	b^3cdef	b^3cd^4	$b^5c^3d^2$	fn
$b^{10}g$	cel	$cdei$	bd^3h	b^2ce^2f	b^3ce^3	b^3c^5f	b^4c^5d	gm
b^9cf	cfk	$cdfh$	bd^2eg	b^2d^3g	b^2d^2f	b^3c^4de	b^3c^7	hl
b^9de	cgj	cdg^2	bd^2f^2	b^2d^2ef	$b^3d^2e^2$	$b^3c^3d^3$	$b^{10}h$	ik
b^8c^2e	chi	ce^2h	bde^2f	b^2de^3	b^2c^4h	bc^6e	b^9cg	j^2
b^8cd^2	d^2l	$cefg$	be^4	bc^4i	b^2c^3dg	bc^3d^2	b^9df	b^2q
b^7c^3d	dek	cf^3	c^4j	bc^3dh	b^2c^3ef	c^7d	b^9e^2	bcp
b^6c^5	dfj	d^3i	c^3di	bc^3eg	$b^2c^2d^2f$	b^8j	b^8c^2f	bdo
$b^{11}f$	dgi	d^2eh	c^3eh	bc^3f^2	$b^2c^2de^2$	b^7ci	b^8cde	ben
$b^{10}ce$	dh^2	d^2fg	c^3fg	bc^2d^2g	b^2cd^3e	b^7dh	b^8d^3	bfm
$b^{10}d^2$	e^2j	de^2g	c^2d^2h	bc^2def	b^2d^5	b^7eg	b^7c^2e	bgl

THE PARTITION TABLE, 0 TO 18 (*continued*).

18	18	18	18	18	18	18	18	18
bhk	$befi$	b^2dej	d^3ef	bc^2deg	b^3cdf^2	b^4c^2i	b^5cd^2f	b^8c^2g
bij	$begh$	b^3dfi	d^2e^3	bc^3df^2	b^5ce^2f	b^4c^2dh	b^5cde^2	b^8cdf
c^2o	bf^2h	b^2dgh	b^5n	bc^2e^2f	b^3d^2g	b^4c^2eg	b^5d^2e	b^8ce^2
cdn	bfg^2	b^2e^2i	b^4cm	bcd^2g	b^3d^2ef	b^4cf^2	b^4c^4g	b^8d^2e
cem	c^3m	b^2efh	b^4dl	bcd^2ef	b^3de^3	b^4cd^2g	b^4c^3df	b^7c^3f
cfl	c^2dl	b^2eg^2	b^4ek	$bcde^3$	b^3c^4i	b^4cdef	$b^4c^2e^2$	b^7c^2de
cgk	c^2ek	b^2f^2g	b^4fj	bd^4f	b^3c^3dh	b^4ce^3	$b^4c^2d^2e$	b^7cd^3
chj	c^2fj	bc^3l	b^4gi	bd^3e^2	b^3c^2eg	b^4d^3f	b^4cd^4	b^6c^4e
ci^2	c^2gi	bc^2dk	b^4h^2	c^5i	$b^3c^2f^2$	$b^4d^2e^2$	b^3c^5f	$b^6c^3d^2$
d^2m	c^2h^2	bc^2ej	b^3c^2l	c^4dh	$b^3c^2d^2g$	b^3c^4h	b^3c^4de	b^5c^5d
del	cd^2k	bc^2fi	b^3cdk	c^4eg	b^3c^2def	b^3c^3dg	$b^3c^3d^3$	b^4c^7
dfk	$cdej$	bc^2gh	b^3cej	c^4f^2	$b^3c^2e^3$	b^3c^3ef	b^3c^6e	$b^{11}h$
dgj	$cdfi$	bcd^2j	b^3cfi	c^3d^2g	b^3cd^2f	$b^3c^2d^2f$	$b^3c^3d^2$	$b^{10}cg$
dhi	$cdgh$	$bcdei$	b^3cgh	c^3def	$b^3cd^3e^2$	$b^3c^3de^2$	bc^7d	$b^{10}df$
e^2k	ce^2i	$bcdfh$	b^3d^2j	c^3e^3	b^3d^4e	b^3cd^3e	c^9	$b^{10}e^2$
efj	$cefh$	$bcdg^2$	b^3dei	c^3d^3f	bc^5h	b^3d^5	b^9j	b^9c^2f
egi	ceg^2	bce^2h	b^3dfh	$c^2d^2e^2$	bc^4dg	b^2c^6g	b^8ci	b^8cde
eh^2	cf^2g	$bcefg$	b^3dg^2	cd^4e	bc^4ef	b^2c^4df	b^9dh	b^9d^2
f^2i	d^3j	bcf^3	b^3e^2h	d^6	bc^3d^2f	$b^2c^4e^2$	b^8eg	b^8c^3e
fgh	d^2ei	bd^3i	b^3efg	b^6m	bc^3de^2	$b^2c^3d^2e$	b^8f^2	$b^8c^2d^2$
g^3	d^2fh	bd^2eh	b^3f^3	b^5cl	bc^2d^3e	$b^2c^2d^4$	b^7c^3h	b^7c^4d
b^3p	d^2g^2	bd^2fg	b^2c^3k	b^5dk	bcd^5	bc^6f	b^7cdg	b^6c^6
b^3co	de^2h	bde^2g	b^2c^2dj	b^5ej	c^6g	bc^5de	b^7cef	$b^{12}g$
b^3dn	$defg$	$bdef^2$	b^2c^2ei	b^5fi	c^5df	bc^4d^3	b^7d^2f	$b^{11}cf$
b^3em	df^3	be^3f	b^2e^2fh	b^5gh	c^5e^2	c^7e	b^7de^2	$b^{11}de$
b^3fl	e^3g	c^4k	$b^2c^2g^2$	b^4c^2k	c^4d^2e	c^6d^2	b^6c^3g	$b^{10}c^2e$
b^3gk	e^2f^2	c^3dj	b^2cd^2i	b^4cdj	c^3d^4	b^5k	b^6c^2df	$b^{10}cd^2$
b^3hj	b^4o	c^3ei	b^2cdeh	b^4cei	b^7l	b^7cj	$b^6c^2e^2$	b^9c^3d
b^3i^2	b^3cn	c^3fh	b^2cdfg	b^4cfh	b^6ck	b^7di	b^6cd^2e	b^8c^5
bc^2n	b^3dm	c^3g^2	b^2ce^2g	b^4cg^2	b^6dj	b^7eh	b^6d^4	$b^{13}f$
$bcdm$	b^3el	c^2d^2i	b^2cef^2	b^4d^2i	b^6ei	b^7fg	b^5c^4f	$b^{12}ce$
$bcel$	b^3fk	c^2deh	b^2d^3h	b^4deh	b^6fh	b^6c^2i	b^5c^3de	$b^{12}d^2$
$bcfk$	b^3gj	c^2dfg	b^2d^2eg	b^4dfg	b^5g^2	b^6cdh	$b^5c^2d^3$	$b^{11}c^2d$
$bcgj$	b^3hi	c^2e^2g	$b^2d^2f^2$	b^4e^2g	b^5c^2j	b^6ceg	b^4c^5e	$b^{10}c^4$
$bchi$	b^2c^2m	c^2ef^2	b^2de^2f	b^4ef^2	b^5cdi	b^6cf^2	$b^4c^4d^2$	$b^{14}e$
bd^2l	b^2cdl	cd^3h	b^2e^4	b^3c^3j	b^5ceh	b^6d^2g	b^3c^6d	$b^{13}cd$
$bdek$	b^2cek	cd^2eg	bc^4j	b^3c^2di	b^5cfg	b^6def	b^2c^8	$b^{12}c^3$
$bdfj$	b^2cfj	cd^2f^2	bc^3di	b^3c^2eh	b^5d^2h	b^6e^3	$b^{10}i$	$b^{15}d$
$bdgi$	b^2cgi	cde^2f	bc^3eh	b^3c^2fg	b^5deg	b^6c^3h	b^9ch	$b^{14}c^2$
bdh^2	b^2ch^2	ce^4	bc^3fg	b^3cd^2h	b^5df^2	b^6c^2dg	b^9dg	$b^{16}c$
be^2j	b^2d^2k	d^4g	bc^2d^2h	b^3cdeg	b^5e^2f	b^6c^2ef	b^9ef	b^{18}

772.

ON THE ANALYTICAL FORMS CALLED TREES.

[From the *American Journal of Mathematics*, vol. IV. (1881), pp. 266—268.]

IN a tree of N knots, selecting any knot at pleasure as a root, the tree may be regarded as springing from this root, and it is then called a root-tree. The same tree thus presents itself in various forms as a root-tree; and if we consider the different root-trees with N knots, these are not all of them distinct trees. We have thus the two questions, to find the number of root-trees with N knots; and, to find the number of distinct trees with N knots.

I have in my paper "On the Theory of the Analytical Forms called Trees," *Phil. Mag.*, t. XIII. (1857), pp. 172—176, [203] given the solution of the first question; viz. if ϕ_N denotes the number of the root-trees with N knots, then the successive numbers ϕ_1, ϕ_2, ϕ_3, etc., are given by the formula

$$\phi_1 + x\phi_2 + x^2\phi_3 + \ldots = (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3}\ldots,$$

viz. we thus find

suffix of ϕ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi =$	1	1	2	4	9	20	48	115	286	719	1842	4766	12486.

And I have, in the paper "On the analytical forms called Trees, with application to the theory of chemical combinations," *Brit. Assoc. Report*, 1875, pp. 257—305, [610] also shown how by the consideration of the centre or bicentre "of length" we can obtain formulæ for the number of central and bicentral trees, that is, for the number

of distinct trees, with N knots: the numerical result obtained for the total number of distinct trees with N knots is given as follows:

No. of Knots	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of Central Trees	1	0	1	1	2	3	7	12	27	55	127	284	682
„ Bicentral „	0	1	0	1	1	3	4	11	20	51	108	267	619
Total	1	1	1	2	3	6	11	23	47	106	235	551	1301.

But a more simple solution is obtained by the consideration of the centre or bicentre "of number." A tree of an odd number N of knots has a centre of number, and a tree of an even number N of knots has a centre or else a bicentre of number. To explain this notion (due to M. Camille Jordan) we consider the branches which proceed from any knot, and (excluding always this knot itself) we count the number of the knots upon the several branches; say these numbers are α, β, γ, δ, ϵ, etc., where of course $\alpha+\beta+\gamma+\delta+\epsilon+\text{etc.}=N-1$. If N is even we may have, say $\alpha=\frac{1}{2}N$; and then $\beta+\gamma+\delta+\epsilon+\text{etc.}=\frac{1}{2}N-1$, viz. α is larger by unity than the sum of the remaining numbers: the branch with α knots, or the number α, is said to be "merely dominant." If N be odd, we cannot of course have $\alpha=\frac{1}{2}N$, but we may have $\alpha>\frac{1}{2}N$; here α exceeds by 2 at least the sum of the other numbers; and the branch with α knots, or the number α, is said to be "predominant." In every other case, viz. in the case where each number α is less than $\frac{1}{2}N$, (and where consequently the largest number α does not exceed the sum of the remaining numbers), the several branches, or the numbers α, β, γ, etc., are said to be subequal. And we have the theorem. First, when N is odd, there is always one knot (and only one knot) for which the branches are subequal: such knot is called the centre of number. Secondly, when N is even, either there is one knot (and only one knot) for which the branches are subequal: and such knot is then called the centre of number; or else there is no such knot, but there are two adjacent knots (and no other knot) each having a merely-dominant branch: such two knots are called the bicentre of number, and each of them separately is a half-centre.

Considering now the trees with N knots as springing from a centre or a bicentre of number, and writing ψ_N for the whole number of distinct trees with N knots, we readily obtain these in terms of the foregoing numbers ϕ_1, ϕ_2, ϕ_3, etc., viz. we have

$$\begin{aligned}
\psi_1 &= 1,\\
\psi_2 &= \tfrac{1}{2}\phi_1(\phi_1+1),\\
\psi_3 &= \text{coeff. } x^2 \text{ in } (1-x)^{-\phi_1},\\
\psi_4 &= \tfrac{1}{2}\phi_2(\phi_2+1)+\text{coeff. } x^3 \text{ in } (1-x)^{-\phi_1},\\
\psi_5 &= \text{coeff. } x^4 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},\\
\psi_6 &= \tfrac{1}{2}\phi_3(\phi_3+1)+\text{coeff. } x^5 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},\\
\psi_7 &= \text{coeff. } x^6 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3},
\end{aligned}$$

and so on, the law being obvious. And the formulæ are at once seen to be true. Thus for $N=6$, the formula is

$$\psi_6 = \tfrac{1}{2}\phi_3(\phi_3+1) + \tfrac{1}{2}\phi_2(\phi_2+1).\phi_1 + \phi_2.\tfrac{1}{6}\phi_1(\phi_1+1)(\phi_1+2)$$
$$+ \tfrac{1}{120}\phi_1(\phi_1+1)(\phi_1+2)(\phi_1+3)(\phi_1+4).$$

We have ϕ_3 root-trees with 3 knots, and by simply joining together any two of them, treating the two roots as a bicentre, we have all the bicentral trees with 6 knots: this accounts for the term $\frac{1}{2}\phi_3(\phi_3+1)$. Again, we have ϕ_1 root-trees with 1 knot, ϕ_2 root-trees with 2 knots; and with a given knot as centre, and the partitions (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1) successively, we build up the central trees of 6 knots, viz. 1° we take as branches any two ϕ_2's and any one ϕ_1; 2° any one ϕ_2 and any three ϕ_1's; 3° any five ϕ_1's; the partitions in question being all the partitions of 5 with no part greater than 2, that is, all the partitions with sub-equal parts. We easily obtain

suffix of ψ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\psi =$	1	1	1	2	3	6	11	23	47	106	235	551	1301

agreeing with the results obtained by the much more complicated formulæ of the paper of 1875.

773.

ON THE 8-SQUARE IMAGINARIES.

[From the *American Journal of Mathematics*, vol. IV. (1881), pp. 293—296.]

I WRITE throughout **0** to denote positive unity, and uniting with it the seven imaginaries **1**, ..., **7**, form an octavic system **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7**, the laws of combination being

$$\mathbf{0}^2 = \mathbf{0}, \quad \mathbf{1}^2 = \mathbf{2}^2 = \mathbf{3}^2 = \mathbf{4}^2 = \mathbf{5}^2 = \mathbf{6}^2 = \mathbf{7}^2 = -\mathbf{0},$$

$$\mathbf{123} = \epsilon_1, \qquad \mathbf{145} = \epsilon_3, \qquad \mathbf{167} = \epsilon_3,$$

$$\mathbf{246} = \epsilon_4, \qquad \mathbf{257} = \epsilon_5,$$

$$\mathbf{347} = \epsilon_6, \qquad \mathbf{356} = \epsilon_7,$$

where $\epsilon = \pm$, viz. each ϵ has a determinate value $+$ or $-$ as the case may be; and where the formula, $\mathbf{123} = \epsilon_1$, denotes the six equations

$$\mathbf{23} = \epsilon_1\mathbf{1}, \qquad \mathbf{31} = \epsilon_1\mathbf{2}, \qquad \mathbf{12} = \epsilon_1\mathbf{3},$$

$$\mathbf{32} = -\epsilon_1\mathbf{1}, \qquad \mathbf{13} = -\epsilon_1\mathbf{2}, \qquad \mathbf{21} = -\epsilon_1\mathbf{3},$$

and so for the other formulæ. The multiplication table of the eight symbols thus is

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\epsilon_1 3$	$-\epsilon_1 2$	$\epsilon_2 5$	$-\epsilon_2 4$	$\epsilon_3 7$	$-\epsilon_3 6$
2	2	$-\epsilon_1 3$	-0	$\epsilon_1 1$	$\epsilon_4 6$	$\epsilon_5 7$	$-\epsilon_4 4$	$-\epsilon_5 5$
3	3	$\epsilon_1 2$	$-\epsilon_1 1$	-0	$\epsilon_6 7$	$\epsilon_7 6$	$-\epsilon_7 5$	$-\epsilon_6 4$
4	4	$-\epsilon_2 5$	$-\epsilon_4 6$	$-\epsilon_6 7$	-0	$\epsilon_2 1$	$\epsilon_4 2$	$\epsilon_6 3$
5	5	$\epsilon_2 4$	$-\epsilon_5 7$	$-\epsilon_7 6$	$-\epsilon_2 1$	-0	$\epsilon_7 3$	$\epsilon_5 2$
6	6	$-\epsilon_3 7$	$\epsilon_4 4$	$\epsilon_7 5$	$-\epsilon_4 2$	$-\epsilon_7 3$	-0	$\epsilon_3 1$
7	7	$\epsilon_3 6$	$\epsilon_5 5$	$\epsilon_6 4$	$-\epsilon_6 3$	$-\epsilon_5 2$	$-\epsilon_3 1$	-0

Hence if 0, 1, 2, 3, 4, 5, 6, 7 and 0′, 1′, 2′, 3′, 4′, 5′, 6′, 7′ denote ordinary algebraical magnitudes, and we form the product

$$(\mathbf{00}+\mathbf{11}+\mathbf{22}+\mathbf{33}+\mathbf{44}+\mathbf{55}+\mathbf{66}+\mathbf{77})(0'\mathbf{0}+1'\mathbf{1}+2'\mathbf{2}+3'\mathbf{3}+4'\mathbf{4}+5'\mathbf{5}+6'\mathbf{6}+7'\mathbf{7}),$$

this is at once found to be =

$$\begin{aligned}
&(00'-11'-22'-33'-44'-55'-66'-77')\,\mathbf{0}\\
+&(01'+0'1+\epsilon_1 23+\epsilon_2 45+\epsilon_3 67)\,\mathbf{1}\\
+&(02'+0'2+\epsilon_1 31+\epsilon_4 46+\epsilon_5 57)\,\mathbf{2}\\
+&(03'+0'3+\epsilon_1 12+\epsilon_6 47+\epsilon_7 56)\,\mathbf{3}\\
+&(04'+0'4+\epsilon_2 51+\epsilon_4 62+\epsilon_6 73)\,\mathbf{4}\\
+&(05'+0'5+\epsilon_3 14+\epsilon_5 72+\epsilon_7 63)\,\mathbf{5}\\
+&(06'+0'6+\epsilon_3 71+\epsilon_4 24+\epsilon_7 35)\,\mathbf{6}\\
+&(07'+0'7+\epsilon_3 16+\epsilon_5 25+\epsilon_6 34)\,\mathbf{7},
\end{aligned}$$

where 12 is written to denote $12'-1'2$, and so in other cases.

The sum of the squares of the eight coefficients of **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7** respectively will, if certain terms destroy each other, be

$$=(0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2)(0'^2+1'^2+2'^2+3'^2+4'^2+5'^2+6'^2+7'^2);$$

viz. the sum of the squares contains the several terms

$$\begin{array}{llllll}
\epsilon_1\epsilon_2 23.45, & \epsilon_1\epsilon_3 23.67, & \epsilon_1\epsilon_4 31.46, & \epsilon_1\epsilon_5 31.57, & \epsilon_1\epsilon_6 12.47, & \epsilon_1\epsilon_7 12.56, \\
\epsilon_2\epsilon_3 45.67, & \epsilon_4\epsilon_7 24.35, & \epsilon_4\epsilon_6 62.73, & \epsilon_2\epsilon_7 14.63, & \epsilon_2\epsilon_6 51.73, & \epsilon_2\epsilon_5 14.72, \\
\epsilon_3\epsilon_4 51.62, & \epsilon_4\epsilon_5 46.57, & \epsilon_5\epsilon_6 25.34, & \epsilon_5\epsilon_7 72.63, & \epsilon_3\epsilon_6 16.34, & \epsilon_3\epsilon_7 71.35, \\
\epsilon_3\epsilon_4 71.24, & \epsilon_3\epsilon_5 16.25, & \epsilon_6\epsilon_7 47.56, & & &
\end{array}$$

and observing that $\mathbf{21}=-\mathbf{12}$, etc., and that we have identically

$$\mathbf{23}.\mathbf{45}+\mathbf{24}.\mathbf{53}+\mathbf{25}.\mathbf{34}=\text{zero, etc.},$$

then the three terms of each column will vanish, provided a proper relation exists between the ϵ's: viz. the conditions which we thus obtain are

$$\begin{aligned}
\epsilon_1\epsilon_2 &= -\epsilon_4\epsilon_7 = \epsilon_5\epsilon_6,\\
\epsilon_1\epsilon_3 &= -\epsilon_4\epsilon_6 = \epsilon_5\epsilon_7,\\
\epsilon_1\epsilon_4 &= -\epsilon_3\epsilon_6 = -\epsilon_2\epsilon_7,\\
\epsilon_1\epsilon_5 &= \epsilon_3\epsilon_7 = \epsilon_2\epsilon_6,\\
\epsilon_1\epsilon_6 &= \epsilon_2\epsilon_5 = -\epsilon_3\epsilon_4,\\
\epsilon_1\epsilon_7 &= -\epsilon_2\epsilon_4 = \epsilon_3\epsilon_5,\\
\epsilon_2\epsilon_3 &= -\epsilon_4\epsilon_5 = \epsilon_6\epsilon_7.
\end{aligned}$$

We may without loss of generality assume $\epsilon_1=\epsilon_2=\epsilon_3=+$; the equations then become

$$\begin{aligned}
+ &= -\epsilon_4\epsilon_7 = \epsilon_5\epsilon_6,\\
+ &= -\epsilon_4\epsilon_6 = \epsilon_5\epsilon_7,\\
+ &= -\epsilon_4\epsilon_5 = \epsilon_6\epsilon_7,\\
\epsilon_4 &= -\epsilon_6 = -\epsilon_7,\\
\epsilon_5 &= \epsilon_7 = \epsilon_6,\\
\epsilon_6 &= \epsilon_5 = -\epsilon_4,\\
\epsilon_7 &= -\epsilon_4 = \epsilon_5;
\end{aligned}$$

and writing $\theta = \pm$ at pleasure, these are all satisfied if $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$. The terms written down all disappear, and the sum of the squares of the eight coefficients thus becomes equal to the product of two sums each of them of eight squares, viz. this is the case if $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$, $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$, θ being $= \pm$ at pleasure: the resulting system of imaginaries may be said to be an 8-square system.

We may inquire whether the system is associative; for this purpose, supposing in the first instance that the ϵ's remain arbitrary, we form the complete system of the values of the triplets **12.3**, **1.23**, etc., (read the top line $\mathbf{12.3} = -\epsilon_1\mathbf{0}$, $\mathbf{1.23} = -\epsilon_1\mathbf{0}$, the next line $\mathbf{12.4} = \epsilon_1\epsilon_6\mathbf{7}$, $\mathbf{1.24} = \epsilon_3\epsilon_4\mathbf{7}$, and so in other cases):

12.3 =	1.23 =	$-\epsilon_1$	$-\epsilon_1$	0
12.4 =	1.24 =	$\epsilon_1\epsilon_6$	$\epsilon_3\epsilon_4$	7
12.5 =	1.25 =	$\epsilon_1\epsilon_7$	$-\epsilon_3\epsilon_5$	6
12.6 =	1.26 =	$-\epsilon_1\epsilon_7$	$-\epsilon_3\epsilon_4$	5
12.7 =	1.27 =	$-\epsilon_1\epsilon_6$	$\epsilon_3\epsilon_5$	4
13.4 =	1.34 =	$-\epsilon_1\epsilon_4$	$-\epsilon_3\epsilon_6$	6
13.5 =	1.35 =	$-\epsilon_1\epsilon_5$	$\epsilon_3\epsilon_7$	7
13.6 =	1.36 =	$\epsilon_1\epsilon_4$	$\epsilon_3\epsilon_7$	4
13.7 =	1.37 =	$\epsilon_1\epsilon_5$	$-\epsilon_3\epsilon_6$	5
14.5 =	1.45 =	$-\epsilon_2$	$-\epsilon_2$	0
14.6 =	1.46 =	$\epsilon_2\epsilon_7$	$\epsilon_1\epsilon_4$	3
14.7 =	1.47 =	$\epsilon_2\epsilon_5$	$-\epsilon_1\epsilon_6$	2
15.6 =	1.56 =	$-\epsilon_2\epsilon_4$	$-\epsilon_1\epsilon_7$	2
15.7 =	1.57 =	$-\epsilon_2\epsilon_6$	$\epsilon_1\epsilon_5$	3
16.7 =	1.67 =	$-\epsilon_3$	$-\epsilon_3$	0
23.4 =	2.34 =	$\epsilon_1\epsilon_2$	$-\epsilon_5\epsilon_6$	5
23.5 =	2.35 =	$-\epsilon_1\epsilon_2$	$-\epsilon_4\epsilon_7$	4
23.6 =	2.36 =	$\epsilon_1\epsilon_3$	$-\epsilon_5\epsilon_7$	7
23.7 =	2.37 =	$-\epsilon_1\epsilon_3$	$-\epsilon_4\epsilon_6$	6
24.5 =	2.45 =	$-\epsilon_4\epsilon_7$	$-\epsilon_1\epsilon_2$	3
24.6 =	2.46 =	$-\epsilon_4$	$-\epsilon_4$	0
24.7 =	2.47 =	$\epsilon_3\epsilon_4$	$\epsilon_1\epsilon_6$	1
25.6 =	2.56 =	$-\epsilon_3\epsilon_5$	$\epsilon_1\epsilon_7$	1
25.7 =	2.57 =	$-\epsilon_5$	$-\epsilon_5$	0
26.7 =	2.67 =	$-\epsilon_4\epsilon_6$	$-\epsilon_1\epsilon_3$	3
34.5 =	3.45 =	$-\epsilon_5\epsilon_6$	$\epsilon_1\epsilon_2$	2
34.6 =	3.46 =	$-\epsilon_3\epsilon_6$	$-\epsilon_1\epsilon_4$	1
34.7 =	3.47 =	$-\epsilon_6$	$-\epsilon_6$	0
35.6 =	3.56 =	$-\epsilon_7$	$-\epsilon_7$	0
35.7 =	3.57 =	$\epsilon_3\epsilon_7$	$-\epsilon_1\epsilon_5$	1
36.7 =	3.67 =	$-\epsilon_5\epsilon_7$	$\epsilon_1\epsilon_3$	2
45.6 =	4.56 =	$\epsilon_2\epsilon_3$	$-\epsilon_6\epsilon_7$	7
45.7 =	4.57 =	$-\epsilon_2\epsilon_3$	$-\epsilon_4\epsilon_5$	6
46.7 =	4.67 =	$-\epsilon_4\epsilon_5$	$-\epsilon_2\epsilon_3$	5
56.7 =	5.67 =	$-\epsilon_6\epsilon_7$	$\epsilon_2\epsilon_3$	4.

Write as before $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$; then, disregarding the lines (such as the first line) which contain the symbol **0**, and writing down only the signs as given in the third and fourth columns, these are

ϵ_6	ϵ_4
ϵ_7	$-\epsilon_5$
$-\epsilon_7$	$-\epsilon_4$
$-\epsilon_6$	ϵ_5
$-\epsilon_4$	$-\epsilon_6$
$-\epsilon_5$	ϵ_7
ϵ_4	ϵ_7
ϵ_5	$-\epsilon_6$
ϵ_7	ϵ_4
ϵ_5	$-\epsilon_6$
$-\epsilon_4$	$-\epsilon_7$
$-\epsilon_6$	ϵ_5
$+$	$-\epsilon_5\epsilon_6$
$-$	$-\epsilon_4\epsilon_7$
$+$	$-\epsilon_5\epsilon_7$
$-$	$-\epsilon_4\epsilon_6$
$-\epsilon_4\epsilon_7$	$-$
ϵ_4	ϵ_6
$-\epsilon_5$	ϵ_7
$-\epsilon_4\epsilon_6$	$-$
$-\epsilon_5\epsilon_6$	$+$
$-\epsilon_6$	$-\epsilon_4$
ϵ_7	$-\epsilon_5$
$-\epsilon_5\epsilon_7$	$+$
$+$	$-\epsilon_6\epsilon_7$
$-$	$-\epsilon_4\epsilon_5$
$-\epsilon_4\epsilon_5$	$-$
$-\epsilon_6\epsilon_7$	$+$

We hence see at once that the pairs of signs in the two columns respectively cannot be made identical: to make them so, we should have $\epsilon_6 = \epsilon_4$, $\epsilon_7 = -\epsilon_5$, $\epsilon_7 = \epsilon_4$, that is, $\epsilon_4 = \epsilon_6 = \epsilon_7 = -\epsilon_5$, which is inconsistent with the last equation of the system $-\epsilon_5\epsilon_7 = +$. Hence the imaginaries **1**, **2**, **3**, **4**, **5**, **6**, **7**, as defined by the original conditions, are not in any case associative.

If we have $\epsilon_1 = \epsilon_2 = \epsilon_3 = +$ and also $-\epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \theta$, that is, if the imaginaries belong to the 8-square formula, then it is at once seen that each pair consists of two opposite signs; that is, for the several triads **123**, **145**, **167**, **246**, **257**, **347**, **356** used for the definition of the imaginaries, the associative property holds good, **12 . 3 = 1 . 23**, etc.; but for each of the remaining twenty-eight triads, *the two terms are equal but of opposite signs*, viz. **12 . 4 = − 1 . 24**, etc.; so that the product **124** of any such three symbols has no determinate meaning.

Baltimore, March 5th, 1882.

774.

TABLES FOR THE BINARY SEXTIC.

THE LEADING COEFFICIENTS OF THE FIRST 18 OF THE 26 COVARIANTS.

[From the *American Journal of Mathematics*, vol. IV. (1881), pp. 379—384.]

INCLUDING the sextic itself, the number of covariants of the binary sextic is $=26$, as shown in the table p. 296 of Clebsch's *Theorie der binären algebraischen Formen*, Leipzig, 1872; viz. this is

	Order						
Deg.	0	2	4	6	8	10	12
1				f			
2	A		i		H		
3		l		p	(f, i)		T
4	B		$(f, l)_2$	(f, l)		(H, i)	
5		$(i, l)_2$	(i, l)		(H, l)		
6	A_U			(p, l) $((f, i), l)_2$			
7		$(f, l^2)_4$	$(f, l^2)_3$				
8		$(i, l^2)_3$					
9			$((f, i), l^2)_4$				
10	$(f, l^3)_6$	$(f, l^3)_5$					
12		$((f, i), l^3)_6$					
15	$((f, i), l^4)_8$						

Or, using the capital letters $A, B, \ldots, Z$ to denote the 26 covariants in the same order, the table is

	0	2	4	6	8	10	12
1				A			
2	B		C		D		
3		E		F	$G, = (A, C)^1$		H
4	I		$J, = (A, E)^2$	$K, = (A, E)^1$		$L, = (D, C)^1$	
5		$M, = (C, E)^2$	$N, = (C, E)^1$		$O, = (D, E)^1$		
6	P			$Q, = (F, E)^1$ $R, = (G, E)^2$			
7		$S, = (A, E^2)^4$	$T, = (A, E^2)^3$				
8		$U, = (C, E^2)^3$					
9			$V, = (G, E^2)^4$				
10	$W, = (A, E^3)^6$	$X, = (A, E^3)^5$					
12		$Y, = (G, E^3)^6$					
15	$Z, = (G, E^4)^8$						

A is the sextic. P is Salmon's C, p. 204.
B is Salmon's A, p. 202. W ,, ,, D, p. 207.
I ,, ,, B, p. 203. Z ,, ,, E, p. 253.
The references are to Salmon's *Higher Algebra*, 2nd Ed., 1866.

In the present short paper I give the leading coefficients of the first 18 covariants, A to R (some of these are of course known values, but it is convenient to include them): for the next four covariants S, T, U, V, the leading coefficients depend upon the coefficients of A, C, G and E^2, viz. writing

$$\begin{aligned} A &= (a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6, \\ E^2 &= (\alpha,\ \tfrac{1}{4}\beta,\ \tfrac{1}{6}\gamma,\ \tfrac{1}{4}\delta,\ \epsilon \between x,\ y)^4, \\ C &= (\alpha',\ \tfrac{1}{4}\beta',\ \tfrac{1}{6}\gamma',\ \tfrac{1}{4}\delta',\ \epsilon' \between x,\ y)^4, \\ G &= (\alpha'',\ \tfrac{1}{8}\beta'',\ \tfrac{1}{28}\gamma'',\ \tfrac{1}{56}\delta'',\ \tfrac{1}{70}\epsilon'',\ \ldots \between x,\ y)^8, \end{aligned}$$

we have

$$\begin{aligned} S,\ \text{Coeff. } x^2 &= a\epsilon - b\delta + c\gamma - d\beta + e\alpha, \\ T,\ \quad ,, \quad x^4 &= a\delta - 2b\gamma + 3c\beta - 4d\alpha, \\ U,\ \quad ,, \quad x^2 &= 2\alpha'\delta - \beta'\gamma + \gamma'\beta - 2\delta'\alpha, \\ V,\ \quad ,, \quad x^4 &= 280\alpha''\epsilon - 35\beta''\delta + 10\gamma''\gamma - 20\delta''\beta + 24\epsilon''\alpha. \end{aligned}$$

Similarly the invariant W and the leading coefficients of X, Y depend on the coefficients of A, G and E^3; and the invariant Z depends on the coefficients of G and E^4.

But these two invariants W and Z have been already calculated; viz. as already mentioned, W is Salmon's invariant D, and Z his invariant E, given each of them in the second edition of his *Higher Algebra* (but not reproduced in the third edition): on account of the great length of these expressions, it has been thought that it was not expedient to give them here.

For the reason appearing above, I have added the expressions for the remaining coefficients of C, E, G.

A, x^6	B, x^0	C, x^4	D, x^8	E, x^2	F, x^6	G, x^8	H, x^{12}
$a + 1$	$ag + 1$	$ae + 1$	$ac + 1$	$acg + 1$	$ace + 1$	$a^2f + 1$	$a^2d + 1$
	$a^0bf - 6$	$a^0bd - 4$	$a^0b^2 - 1$	$df - 3$	$d^2 - 1$	$abe - 5$	$abc - 3$
	$ce + 15$	$c^2 + 3$		$e^2 + 2$	$a^0b^2e - 1$	$cd + 2$	$a^0b^3 + 2$
	$d^2 - 10$			$a^0b^2g - 1$	$bcd + 2$	$a^0b^2d + 8$	
				$bcf + 3$	$c^3 - 1$	$bc^2 - 6$	
				$bde - 1$			
				$c^2e - 3$			
				$cd^2 + 2$			

I, x^0	J, x^4	K, x^8	L, x^{10}	M, x^2	N, x^4	O, x^8
$aceg + 1$	$a^2f^2 + 1$	$a^3dg + 1$	$a^3cf + 1$	$a^2cg^2 + 1$	$a^2cfg - 1$	$a^2cdg 0$
$ef^2 - 1$	$abef - 10$	$ef - 1$	$de - 1$	$dfg - 6$	$deg + 1$	$cef - 1$
$d^2g - 1$	$cdf + 4$	$abcg - 3$	$ab^2f - 1$	$e^2g + 8$	$df^2 + 3$	$d^2f + 3$
$def + 2$	$ce^2 + 16$	$bdf - 2$	$bce - 2$	$ef^2 - 3$	$e^2f - 3$	$de^2 - 2$
$e^3 - 1$	$d^2e - 12$	$be^2 + 5$	$bd^2 + 4$	$ab^2g^2 - 1$	$ab^2fg + 1$	$ab^2dg 0$
$a^0b^2eg - 1$	$a^0b^2df + 16$	$c^2f + 9$	$c^2d - 1$	$bcfg + 6$	$bceg + 2$	$b^2ef + 1$
$b^2f^2 + 1$	$b^2e^2 + 9$	$cde - 17$	$a^0b^2e + 3$	$bdeg - 34$	$bef^2 - 3$	$bc^2g 0$
$bcdg + 2$	$bc^2f - 12$	$d^3 + 8$	$b^2cd - 6$	$bdf^2 + 48$	$bd^2g - 4$	$bcdf - 14$
$bcef - 2$	$bcde - 76$	$a^0b^2g + 2$	$bc^3 + 3$	$be^2f - 18$	$bdef - 12$	$bce^2 + 11$
$bd^2f - 2$	$bd^3 + 48$	$b^2cf - 6$		$c^2eg + 18$	$be^3 + 15$	$bd^2e + 1$
$bde^2 + 2$	$c^2e + 48$	$b^2de + 2$		$c^2f^2 - 45$	$c^2dg + 1$	$c^2f + 9$
$c^2g - 1$	$c^2d^2 - 32$	$bc^2e + 6$		$cd^2g + 4$	$c^2ef + 9$	$c^2de - 14$
$c^2df + 2$		$bcd^2 - 4$		$cdef + 78$	$cd^2f + 4$	$cd^3 + 6$
$c^2e^2 + 1$				$ce^3 - 36$	$cde^2 - 21$	$a^0b^3cg 0$
$cd^2e - 3$				$d^3f - 48$	$d^3e + 8$	$b^3df + 8$
$d^4 + 1$				$d^2e^2 + 28$	$a^0b^3eg - 3$	$b^3e^2 - 9$
				$a^0b^2ceg 0$	$b^2cdg + 6$	$b^2c^2f - 6$
				$b^2d^2g + 64$	$b^2cef + 9$	$b^2cde + 16$
				$b^2def - 144$	$b^2d^2f + 32$	$b^2d^3 - 8$
				$b^2e^3 + 81$	$b^2de^2 - 39$	$bc^2e - 3$
				$bc^2dg - 96$	$bc^3g - 3$	$bc^2d^2 + 2$
				$bc^2ef + 108$	$bc^2df - 66$	
				$bcd^2f + 96$	$bc^2e^2 + 18$	
				$bcde^2 - 126$	$bcd^2e + 76$	
				$bd^3e + 16$	$bd^4 - 32$	
				$c^4g + 36$	$c^4f + 27$	
				$c^3df - 72$	$c^3de - 45$	
				$c^3e^2 - 27$	$c^2d^3 + 20$	
				$c^2d^2e + 96$		
				$cd^4 - 32$		

P, x^0	
$a^2d^2g^2$	+ 1
$defg$	− 6
df^3	+ 4
e^3g	+ 4
e^2f^2	− 3
$a\,bcdg^2$	− 6
$bcefg$	+ 18
bef^3	− 12
bd^2fg	+ 12
bde^2g	− 18
be^3f	+ 6
c^3g^2	+ 4
c^2e^2y	− 24
c^2dfg	− 18
c^2ef^2	+ 30
cd^2eg	+ 54
cd^2f^2	− 12
cde^2f	− 42
ce^4	+ 12
d^4g	− 20
d^3ef	+ 24
d^2e^3	− 8
$a^0b^3dg^2$	+ 4
b^3efg	− 12
b^3f^3	+ 8
$b^2c^2g^2$	− 3
b^2ce^2g	+ 30
b^2cef^2	− 24
b^2d^2eg	− 12
$b^2d^2f^2$	− 24
b^2de^2f	+ 60
b^2e^4	− 27
bc^2fg	+ 6
bc^2deg	− 42
bc^2df^2	+ 60
bc^2e^2f	− 30
bcd^3g	+ 24
bcd^2ef	− 84
$bcde^3$	+ 66
bd^4f	+ 24
bd^2e^2	− 24
c^4eg	+ 12
c^4f^2	− 27
c^3d^2g	− 8
c^3def	+ 66
c^3e^3	− 8
c^2d^3f	− 24
$c^2d^2e^2$	− 39
cd^4e	+ 36
d^6	− 8

	Q, x^6	R, x^6
a^3dg^2		− 1
efg		+ 9
f^3		− 8
a^2bcg^2		+ 3
$bdfg$		− 24
be^2g		− 45
bef^2		+ 66
c^2fg	− 2	+ 3
$cdeg$	+ 5	+ 48
cdf^2	+ 6	− 12
ce^2f	− 7	− 51
d^3g	− 3	− 16
d^2ef	− 3	+ 36
de^3	+ 4	− 8
$a\,b^3g^2$	0	− 2
b^2efg	+ 4	+ 12
b^2deg	− 5	+ 192
b^2df^2	− 6	− 48
b^2e^2f	+ 7	− 144
bc^2eg	− 5	− 159
bc^2f^2	− 6	+ 18
bcd^2g	+ 7	− 48
$bcdef$	− 16	+ 24
bce^3	+ 23	+ 279
bd^3f	+ 30	− 48
bd^2e^2	− 33	− 84
c^3dg	− 1	+ 42
c^3ef	+ 36	+ 153
c^2d^2f	− 37	− 36
c^2de^2	− 53	− 399
cd^3e	+ 79	+ 312
d^5	− 24	− 64
a^0b^4fg	− 2	0
b^3ceg	+ 5	0
b^3cf^2	+ 6	0
b^3d^2g	+ 2	− 224
b^3def	+ 22	+ 144
b^3e^3	− 27	+ 54
b^2c^2dg	− 8	+ 336
b^2c^2ef	− 39	− 108
b^2cd^2f	− 50	+ 384
b^2cde^2	+ 107	− 684
b^2d^3e	− 22	+ 144
bc^4g	+ 3	− 126
bc^3df	+ 84	− 648
bc^3e^2	− 21	+ 432
bc^2d^2e	− 102	+ 564
bcd^4	+ 44	− 288
c^5f	− 27	+ 270
c^4de	+ 45	− 450
c^3d^3	− 20	+ 200

Remaining Coefficients of C, E, G.

C
x^3y
$af + 2$
$be - 6$
$cd + 4$
x^2y^2
$ag + 1$
$ce - 9$
$d^2 + 8$
xy^3
$bg + 2$
$cf - 6$
$de + 4$
y^4
$cg + 1$
$df - 4$
$e^2 + 3$

E
xy
$adg + 1$
$aef - 1$
$bcg - 1$
$bdf - 8$
$be^2 + 9$
$c^2f + 9$
$cde - 17$
$d^3 + 8$
y^2
$aeg + 1$
$af^2 - 1$
$bdg - 3$
$bef + 3$
$c^2g + 2$
$cdf - 1$
$ce^2 - 3$
$d^2e + 2$

G
x^7y
$a^2g + 1$
$abf + 2$
$ace - 19$
$ad^2 + 8$
$b^2e - 6$
$bcd + 44$
$c^3 - 30$
x^6y^2
$abg + 7$
$acf - 14$
$ade - 14$
$b^2f \quad 0$
$bce - 21$
$bd^2 + 112$
$c^2d - 70$
x^5y^3
$acg + 7$
$adf - 28$
$ae^2 - 14$
$b^2g + 14$
$bcf - 42$
$bde + 168$
$c^2e - 105$
x^4y^4
$adg \quad 0$
$aef - 35$
$bcg + 35$
$bdf \quad 0$
$be^2 + 105$
$c^2f - 105$

G
x^2y^5
$aeg - 7$
$af^2 - 14$
$bdg + 28$
$bef + 42$
$c^2g + 14$
$cdf - 168$
$ce^2 + 105$
x^2y^6
$afg - 7$
$beg + 14$
$bf^2 \quad 0$
$cdg + 14$
$cef + 21$
$d^2f - 112$
$de^2 + 70$
xy^7
$ag^2 - 1$
$bfg - 2$
$ceg + 19$
$cf^2 + 6$
$d^2g - 8$
$def - 44$
$e^3 + 30$
y^8
$bg^2 - 1$
$cfg + 5$
$deg - 2$
$df^2 - 8$
$e^2f + 6$

Note.—In the tables on this page, a has been treated like the other letters; on the preceding pages, the powers of a have been suppressed except in the first of every series of terms containing a common power of a.

The final result is that we have the values of the invariants B, I, P, W, Z and the leading coefficients of the covariants A, C, D, E, F, G, H, J, K, L, M, N, O, Q, R: also the means of calculating the leading coefficients of the remaining covariants S, T, U, V, X, Y.

775.

TABLES OF COVARIANTS OF THE BINARY SEXTIC.

[*Written in* 1894: *now first published.*]

THE binary sextic has in all (including the sextic itself and the invariants) 26 covariants which I have represented by the capital letters $A, B, C, \ldots, Z$. The leading coefficients of the covariants A to R (of course for an invariant this means the invariant itself) are given in my paper "Tables for the binary sextic," *Amer. Math. Jour.* vol. IV. (1881), pp. 379—384, [774]; the two invariants Z and W (Salmon's invariants D and E) had been already calculated. But I did not in my values of the leading coefficients, nor did Salmon in his values of the two invariants, insert the literal terms with zero coefficients: as remarked in my paper [143] "Tables of the covariants M to W of the binary quintic," it is very desirable to have in every case the complete series of literal terms, and I have accordingly in the expressions of the covariants A to R obtained for the leading coefficients, and in the expressions obtained from Salmon for the invariants W and Z, inserted in each case the complete series of literal terms.

I give a list of the 26 covariants nearly in the form of that given in the latter paper [143] for the covariants of the quintic, only instead of a separate column of deg-weights I insert these in the body of the symbol; thus

$$C = (3, 3, 4, 3, 3)^2\ 4 \text{ to } 8\ (x, y)^4,$$

the 5 coefficients of the quartic function contain respectively 3, 3, 4, 3, 3 terms (some of them it may be with zero coefficients), are of the degree 2, and of the weights 4, 5, 6, 7, 8 respectively.

The list is as follows:

$$A = (1, 1, 1, 1, 1, 1, 1)^1\ 0 \text{ to } 6\ (x, y)^6,$$

$$B = (4)^2\ 6\ (x, y)^6, \text{ Invt.},$$

$$C = (3, 3, 4, 3, 3)^2\ 4 \text{ to } 8\ (x, y)^4,$$

$$D = (2, 2, 3, 3, 4, 3, 3, 2, 2)^2\ 2 \text{ to } 10\ (x, y)^8,$$

$E = (8, 8, 8)^3$ 8 to 10 $(x, y)^2$,

$F = (7, 7, 8, 8, 8, 7, 7)^3$ 6 to 12 $(x, y)^6$,

$G = (5, 7, 7, 8, 8, 8, 7, 7, 5)^3$ 5 to 13 $(x, y)^8$,

$H = (3, 4, 5, 7, 7, 8, 8, 8, 7, 7, 5, 4, 3)^3$ 3 to 15 $(x, y)^{12}$,

$I = (18)^4$ 12 $(x, y)^0$, Invt.,

$J = (16, 16, 18, 16, 16)^4$ 10 to 14 $(x, y)^4$,

$K = (14, 16, 16, 18, 16, 16, 14)^4$ 9 to 15 $(x, y)^6$,

$L = (10, 13, 14, 16, 16, 18, 16, 16, 14, 13, 10)^4$ 7 to 17 $(x, y)^{10}$,

$M = (32, 32, 32)^5$ 14 to 16 $(x, y)^2$,

$N = (30, 32, 32, 32, 30)^5$ 13 to 17 $(x, y)^4$,

$O = (25, 29, 30, 32, 32, 32, 30, 29, 25)^5$ 11 to 19 $(x, y)^8$,

$P = (58)^6$ 18 $(x, y)^0$, Invt.,

$Q = (51, 55, 55, 58, 55, 55, 51)^6$ 15 to 21 $(x, y)^6$,

$R = (51, 55, 55, 58, 55, 55, 51)^6$ 15 to 21 $(x, y)^6$,

$S = (94, 94, 94)^7$ 20 to 22 $(x, y)^2$,

$T = (90, 94, 94, 94, 90)^7$ 19 to 23 $(x, y)^4$,

$U = (147, 151, 147)^8$ 23 to 25 $(x, y)^2$,

$V = (221, 227, 227, 227, 221)^9$ 25 to 29 $(x, y)^4$,

$W = (338)^{10}$ 30 $(x, y)^0$, Invt.,

$X = (332, 338, 332)^{10}$ 29 to 31 $(x, y)^2$,

$Y = (668, 676, 668)^{12}$ 35 to 37 $(x, y)^2$,

$Z = (1636)^{15}$ 45 $(x, y)^0$, Invt.

$A = (\ \ \ (x, y)^6$

x^6	x^5y	x^4y^2	x^3y^3	x^2y^4	xy^5	y^6
$a + 1$	$b + 6$	$c + 15$	$d + 20$	$e + 15$	$f + 6$	$g + 1$

$B = (\ \ \ (x, y)^0$, Invt.

$$\begin{vmatrix} ag + 1 \\ bf - 6 \\ ce + 15 \\ d^2 - 10 \end{vmatrix}$$

± 16

$C = (\mathfrak{X}x, y)^4$

x^4	x^3y	x^2y^2	xy^3	y^4
$ae + 1$	$af + 2$	$ag + 1$	$bg + 2$	$cg + 1$
$bd - 4$	$be - 6$	bf ...	$cf - 6$	$df - 4$
$c^2 + 3$	$cd + 4$	$ce - 9$	$de + 4$	$e^2 + 3$
		$d^2 + 8$		
± 4	± 6	± 9	± 6	± 4

$D = (\mathfrak{X}x, y)^6$

x^8	x^7y	x^6y^2	x^5y^3	x^4y^4	x^3y^5	x^2y^6	xy^7	y^8
$ac + 1$	$ad + 4$	$ae + 6$	$af + 4$	$ag + 1$	$bg + 4$	$cg + 6$	$dg + 4$	$eg + 1$
$b^2 - 1$	$bc - 4$	$bd + 4$	$be + 16$	$bf + 14$	$cf + 16$	$df + 4$	$ef - 4$	$f^2 - 1$
		$c^2 - 10$	$cd - 20$	$ce + 5$	$de - 20$	$e^2 - 10$		
				$d^2 - 20$				
± 1	± 4	± 10	± 20	± 20	± 20	± 10	± 4	± 1

$E = (\mathfrak{X}x, y)^2$

x^2	xy	y^2
$a\,cg + 1$	$a\,dg + 1$	$a\,eg + 1$
$df - 3$	$ef - 1$	$f^2 - 1$
$e^2 + 2$	$a^0bcg - 1$	$a^0bdg - 3$
$a^0b^2g - 1$	$bdf - 8$	$bef + 3$
$bcf + 3$	$be^2 + 9$	$c^2g + 2$
$bde - 1$	$c^2f + 9$	$cdf - 1$
$c^2e - 3$	$cde - 17$	$ce^2 - 3$
$cd^2 + 2$	$d^3 + 8$	$d^3 + 2$
± 3	± 1	± 1
± 5	± 26	± 7

$F = (\mathfrak{X}x, y)^6$

x^6	x^5y	x^4y^2	x^3y^3	x^2y^4	xy^5	y^6
a^2g ...	$a\,bg$ ...	$a\,cg + 1$	$a\,dg + 2$	$a\,eg + 1$	$a\,fg$ ...	$a\,g^2$ ...
$a\,bf$ ...	$cf + 2$	$df + 2$	$ef - 2$	$f^2 - 1$	$a^0beg + 2$	a^0bfg ...
$ce + 1$	$de - 2$	$e^2 - 3$	$a^0bcg - 2$	$a^0bdg + 2$	$bf^2 - 2$	$ceg + 1$
$d^3 - 1$	$a^0b^2f - 2$	$a^0b^2g - 1$	$bdf + 4$	$bef - 2$	$cdg - 2$	$cf^2 - 1$
$a^0b^2e - 1$	$bce + 2$	$bcf - 2$	$be^2 - 2$	$c^2g - 3$	$cef + 2$	$d^2g - 1$
$bcd + 2$	$bd^2 + 2$	$bde + 4$	$c^2f - 2$	$cdf + 4$	$d^2f + 2$	$def + 2$
$c^3 - 1$	$c^2d - 2$	$c^2e + 2$	$cde + 6$	$ce^2 + 2$	$de^2 - 2$	$e^3 - 1$
		$d^2 - 3$	$d^3 - 4$	$d^2e - 3$		
± 1	± 2	± 3	± 2	± 1	± 6	± 3
± 2	± 4	± 6	± 10	± 8		

$G = (\mathfrak{Q}x, y)^8$

x^8	x^7y	x^6y^2	x^5y^3	x^4y^4	x^3y^5	x^2y^6	xy^7	y^8
$a^2f + 1$	$a^2g + 1$	$a\,bg + 7$	$a\,cg + 7$	$a\,dg \;\ldots$	$a\,eg - 7$	$a\,fg - 7$	$a\,g^2 - 1$	$a^0bg^2 - 1$
$a\,be - 5$	$bf + 2$	$cf - 14$	$df - 28$	$ef - 35$	$f^2 - 14$	$a^0beg + 14$	$a^0bfg - 2$	$cfg + 5$
$cd + 2$	$ce - 19$	$de - 14$	$e^2 - 14$	$a^0bcg + 35$	$a^0bdg + 28$	$bf^2 \;\ldots$	$ceg + 19$	$deg - 2$
$a^0b^2d + 8$	$d^2 + 8$	$a^0b^2f \;\ldots$	$a^0b^2g + 14$	$bdf \;\ldots$	$bef + 42$	$cdg + 14$	$cf^2 + 6$	$df^2 - 8$
$bc^2 - 6$	$a^0b^2e - 6$	$bce - 21$	$bcf - 42$	$be^2 - 105$	$c^2g + 14$	$cef + 21$	$d^2g - 8$	$e^2f + 6$
	$bcd + 44$	$bd^2 + 112$	$bde + 168$	$c^2f + 105$	$cdf - 168$	$d^2f - 112$	$def - 44$	
	$c^3 - 30$	$c^2d - 70$	$c^2e - 105$	$cde \;\ldots$	$ce^2 + 105$	$de^2 + 70$	$e^3 + 30$	
			$d^2 \;\ldots$	$d^3 \;\ldots$	$d^2e \;\ldots$			
± 11	± 55	± 119	± 189	± 140	± 189	± 119	± 55	± 11

$H = (\mathfrak{Q}x, y)^{12}$

x^{12}	$x^{11}y$	$x^{10}y^2$	x^9y^3	x^8y^4	x^7y^5	x^6y^6	x^5y^7	x^4y^8	x^3y^9	x^2y^{10}	xy^{11}	y^{12}
$a^2d + 1$	$a^2e + 3$	$a^2f + 3$	$a^2g + 1$	$a\,bg + 9$	$a\,cg + 12$	$a\,dg \;\ldots$	$a\,eg - 12$	$a\,fg - 9$	$a\,g^2 - 1$	$a^0bg^2 - 3$	$a^0cg^2 - 3$	$a^0dg^2 - 1$
$abc - 3$	$a\,bd \;\ldots$	$a\,be + 18$	$a\,bf + 24$	$cf + 15$	$df - 48$	$ef - 84$	$f^2 - 24$	$a^0beg - 15$	$a^0bfg - 24$	$cfg - 18$	$dfg \;\ldots$	$efg + 3$
$b^3 + 2$	$c^2 - 15$	$cd - 60$	$ce - 30$	$de - 150$	$e^2 - 90$	$a^0bcg + 84$	$a^0bdg + 48$	$bf^2 - 66$	$ceg + 30$	$deg + 60$	$e^2g + 15$	$f^3 - 2$
	$a^0b^2c + 12$	$a^0b^2d + 24$	$d^2 - 80$	$a^0b^2f + 66$	$a^0b^2g + 24$	$bdf \;..$	$bef - 192$	$cdg + 150$	$cf^2 - 60$	$df^2 - 24$	$ef^2 - 12$	
		$bc^2 + 15$	$a^0b^2e + 60$	$bce + 105$	$bcf + 192$	$be^2 - 210$	$c^2g + 90$	$cef - 105$	$d^2g + 80$	$e^2f - 15$		
			$bcd \;\ldots$	$bd^2 - 120$	$bde - 240$	$c^2f + 210$	$cdf + 240$	$d^2f + 120$	$def \;\ldots$			
			$c^3 + 25$	$c^2d + 75$	$c^2e + 150$	$cde \;\ldots$	$de^2 - 150$	$de^2 - 75$	$e^3 - 25$			
					$cd^2 \;\ldots$	$d^3 \;\ldots$	$d^2e \;\ldots$					
± 3	± 15	± 60	± 110	± 270	± 378	± 294	± 378	± 270	± 110	± 60	± 15	± 3

$I = (\mathfrak{A}x, y)^6$, Invt.

a^2g^2	...
$a\,bfg$	...
ceg	$+1$
cf^2	-1
d^2g	-1
def	$+2$
e^3	-1
a^0b^2eg	-1
b^2f^2	$+1$
$bcdg$	$+2$
$bcef$	-2
bd^2f	-2
bde^2	$+2$
c^3g	-1
c^2df	$+2$
c^2e^2	$+1$
cd^2e	-3
d^4	$+1$
	± 3
	± 9

$J = (\mathfrak{A}x, y)^4$

x^4		x^3y		x^2y^2		xy^3		y^4	
a^2eg	...	a^2fg	$+2$	a^2g^2	$+1$	$a\,bg^2$	$+2$	$a\,cg^2$	...
f^2	$+1$	$a\,beg$	-10	$a\,bfg$	-6	cfg	-10	dfg	...
$a\,bdg$	...	bf^2	-8	ceg	-6	deg	$+4$	e^2g	...
bef	-10	cdg	$+4$	cf^2	$+6$	df^2	$+16$	ef^2	...
c^2g	...	cef	$+26$	d^2g	$+4$	e^2f	-12	$a^0b^2g^2$	$+1$
cdf	$+4$	d^2f	-8	def	$+12$	a^0b^2fg	-8	$bcfg$	-10
ce^2	$+16$	de^2	-8	e^3	-12	$bceg$	$+26$	$bdeg$	$+4$
d^2e	-12	a^0b^2dg	$+16$	a^0b^2eg	$+6$	bcf^2	$+24$	bdf^2	$+16$
a^0b^2cg	...	b^2ef	$+24$	b^2f^2	...	bd^2g	-8	be^2f	-12
b^2df	$+16$	bc^2g	-12	$bcdg$	$+12$	$bdef$	-64	c^3eg	$+16$
b^2e^2	$+9$	$bcdf$	-64	$bcef$	$+18$	be^3	$+36$	c^2f^2	$+9$
bc^2f	-12	bce^2	-42	bd^2f	-96	c^3dg	-8	cd^2g	-12
$bcde$	-76	bd^2e	$+56$	bde^2	$+60$	c^3ef	-42	$cdef$	-76
bd^3	$+48$	c^3f	$+36$	c^3g	-12	cd^2f	$+56$	ce^2	$+48$
c^2e	$+48$	c^2de	$+4$	c^2df	$+60$	cde^2	$+4$	d^3f	$+48$
c^2d^2	-32	cd^3	-16	c^2e^2	-99	d^3e	-16	d^2e^2	-32
				cd^2e	$+84$				
				d^4	-32				
	± 142		± 168		± 263		± 168		± 142

$K = (\mathfrak{A}x, y)^6$

x^6		x^5y		x^4y^2		x^3y^3		x^2y^4		xy^5		y^6	
a^2dg	$+1$	a^2eg	$+2$	a^2fg	...	a^2g^2	...	$a\,bg^2$	...	$a\,cg^2$	-2	$a\,dg^2$	-1
ef	-1	f^2	-2	$a\,beg$	$+10$	$a\,bfg$	...	cfg	-10	dfg	$+2$	efg	$+3$
$a\,bcg$	-3	$a\,bdg$	-2	bf^2	-10	ceg	...	deg	$+15$	e^2g	$+6$	f^3	-2
bdf	-2	bef	$+2$	cdg	-15	cf^2	-20	df^2	$+10$	ef^2	-6	a^0bcg^2	$+1$
be^2	$+5$	c^2g	-6	cef	-5	d^2g	...	e^2f	-15	$a^0b^2g^2$	$+2$	$bdfg$	$+2$
c^2f	$+9$	cdf	$+28$	d^2f	$+60$	def	$+60$	a^0b^2fg	$+10$	$bcfg$	-2	be^2g	-9
cde	-17	ce^2	-26	de^2	-40	e^3	-40	$bceg$	$+5$	$bdeg$	-28	bef^2	$+6$
d^3	$+8$	d^2e	$+4$	a^0b^2dg	-10	a^0b^2eg	$+20$	bcf^2	-30	bdf^2	$+32$	c^2fg	-5
a^0b^2g	$+2$	a^0b^2cg	$+6$	b^2ef	$+30$	b^2f^2	...	bd^2g	-60	be^2f	-6	$cdeg$	$+17$
b^2cf	-6	b^2df	-32	bc^2g	$+15$	$bcdg$	-60	$bdef$	$+110$	c^2eg	$+26$	cdf^2	-2
b^2de	$+2$	b^2e^2	$+36$	$bcdf$	-110	$bcef$	...	be^3	-45	c^2f^2	-36	ce^2f	-6
bc^2e	$+6$	bc^2f	$+6$	bce^2	$+15$	bd^2f	...	c^3dg	$+40$	cd^2g	-4	d^3g	-8
bcd^2	-4	$bcde$	-58	bd^2e	$+40$	bde^2	$+20$	c^3ef	-15	$cdef$	$+58$	d^2ef	$+4$
c^3d	...	bd^3	$+32$	c^3f	$+45$	c^3g	$+40$	cd^2f	-40	ce^3	-30	de^3	...
		c^3e	$+30$	c^2de	-25	c^2df	-20	cde^2	$+25$	d^2f	-32		
		c^2d^2	-20	cd^3	...	c^2e^2	...	d^3e	...	d^2e^2	$+20$		
						cd^2e	...						
						d^4	...						
	± 33		± 146		± 215		± 140		± 215		± 146		± 33

$L = (\ \ \mathfrak{Q}(x,\ y))^{10}$

x^{10}	x^9y	x^8y^2	x^7y^3	x^6y^4	x^5y^5	x^4y^6	x^3y^7	x^2y^8	xy^9	y^{10}
a^2bg ...	$a^2cg + 1$	$a^2dg + 3$	$a^2eg + 2$	a^2fg ...	a^2g^2 ...	$a\,bg^2$ ...	$a\,eg^2 - 2$	$a\,dg^2 - 3$	$a\,eg^2 - 1$	$a\,fg^2$ ...
$cf + 1$	$df + 2$	$ef - 3$	$f^2 - 2$	$a\,beg + 14$	$a\,bfg$ ...	$efg - 14$	$dfg - 16$	efg ...	$f^2g + 1$	$a^0beg^2 - 1$
$de - 1$	$e^2 - 3$	$a\,bcg$ ...	$a\,bdg + 16$	$bf^2 - 14$	ceg ...	deg ...	$e^2g + 6$	$f^3 + 3$	$a^0bdg^2 - 2$	$bf^2g + 1$
$a\,b^2f - 1$	$a\,b^2g - 1$	$bdf + 12$	$bef - 16$	cdg ...	$cf^2 - 42$	$df^2 - 28$	$ef^2 + 12$	$a^0bcg^2 + 3$	$befg + 2$	$cdg^2 + 1$
$bce - 2$	$bcf - 2$	$be^2 - 12$	$c^2g - 6$	$cef - 70$	d^2g ...	$e^2f + 42$	$a^0b^2g^2 + 2$	$bdfg - 12$	bf^3 ...	$cefg + 2$
$bd^2 + 4$	$bde + 4$	$c^2f - 18$	$cdf - 32$	d^2f ...	def ...	$a^0b^2fg + 14$	$bcfg + 16$	$be^2g + 18$	$c^2g^2 + 3$	$cf^3 - 3$
$c^2d - 1$	$c^2e - 13$	$cde - 6$	$ce^2 - 26$	$de^2 + 70$	$e^3 + 42$	$bceg + 70$	$bdeg + 32$	$bef^2 - 9$	$cdfg - 4$	$d^2fg - 4$
$a^0b^3e + 3$	$cd^2 + 12$	$d^3 + 24$	$d^3e + 64$	$a^0b^2dg + 28$	$a^0b^2eg + 42$	$bcf^2 - 42$	$bdf^2 - 64$	$c^2fg + 12$	$ce^2g + 13$	$de^2g + 1$
$b^2cd - 6$	a^0b^2f ...	$a^0b^3g - 3$	$a^0b^2cg - 12$	$b^2ef + 42$	b^2f^2 ...	bd^2g ...	$be^2f + 12$	$cdeg + 6$	$cef^2 - 15$	$def^2 + 6$
$bc^3 + 3$	$b^2ce + 15$	$b^2cf + 9$	$b^2df + 64$	$bc^2g - 42$	$bcdg$ ...	$bdef - 56$	$c^2eg + 26$	$cdf^2 - 42$	$d^2eg - 12$	$e^2f - 3$
	b^2d^2 ...	$b^2de + 42$	$b^2e^2 + 18$	$bcdf + 56$	$bcef$ ...	be^3 ...	$c^2f^2 - 18$	$ce^2f + 9$	d^2f^2 ...	
	$bc^2d - 30$	$bc^2e - 9$	$bc^2f - 12$	$bce^2 - 42$	bd^2f ...	$c^2dg - 70$	$cd^2g - 64$	$d^3g - 24$	$de^2f + 30$	
	$c^4 + 15$	$bcd^2 - 84$	$bcde - 64$	$bd^2e - 112$	$bde^2 - 84$	$c^2ef + 42$	$cdef + 64$	$d^2ef + 84$	$e^4 - 15$	
		$c^3d + 45$	$bd^3 - 64$	c^2f ...	$c^3g - 42$	$cd^2f + 112$	$ce^3 - 30$	$de^3 - 45$		
			$c^3e + 30$	$c^2de + 70$	$c^3df + 84$	$cde^2 - 70$	$d^2f + 64$			
			$c^2d^2 + 40$	cd^3 ...	c^2e^2 ...	d^2e ...	$d^2e^2 - 40$			
					cd^2e ...					
					d^4 ...					

$$M = (\quad \between (x, y)^2$$

x^2		xy		y^2	
a^2cg^2	+ 1	a^2dg^2	− 2	a^2eg^2	+ 1
dfg	− 6	efg	+ 8	f^2g	− 1
e^2g	+ 8	f^3	− 6	$a\,bdg^2$	− 6
ef^2	− 3	$a\,bcg^2$	+ 8	$befg$	+ 6
$a\,b^2g^2$	− 1	$bdfg$	− 20	bf^3	...
$bcfg$	+ 6	be^2g	− 24	c^2g^2	+ 8
$bdeg$	− 34	bef^2	+ 36	$cdfg$	− 34
bdf^2	+ 48	c^2fg	− 24	ce^2g	+ 18
be^2f	− 18	$cdeg$	+ 76	cef^2	...
c^2eg	+ 18	cdf^2	+ 36	d^2eg	+ 4
c^2f^2	− 45	ce^2f	− 72	d^2f^2	+ 64
cd^2g	+ 4	d^3g	− 32	de^2f	− 96
$cdef$	+ 78	d^2ef	− 8	e^4	+ 36
ce^3	− 36	de^3	+ 24	$a^0b^2cg^2$	− 3
d^3f	− 48	$a^0b^3g^2$	− 6	b^2dfg	+ 48
d^2e^2	+ 28	b^2cfg	+ 36	b^2e^2g	− 45
a^0b^2fg	...	b^2deg	+ 36	b^2ef^2	...
b^2ceg	...	b^2df^2	...	bc^2fg	− 18
b^2cf^2	...	b^2e^2f	− 54	$bcdeg$	+ 78
b^2d^2g	+ 64	bc^2eg	− 72	$bcdf^2$	− 144
b^2def	− 144	bc^2f^2	− 54	bce^2f	+ 108
b^2e^3	+ 81	bcd^2g	− 8	bd^2g	− 48
bc^2dg	− 96	$bcdef$	− 36	bd^2ef	+ 96
bc^2ef	+ 108	bce^3	+ 216	bde^3	− 72
bcd^2f	+ 96	bd^3f	+ 128	c^3eg	− 36
$bcde^2$	− 126	bd^2e^2	− 192	c^3f^2	+ 81
bd^3e	+ 16	c^3dg	+ 24	c^2d^2g	+ 28
c^4g	+ 36	c^2ef	+ 216	c^2def	− 126
c^3df	− 72	c^2d^2f	− 192	c^2e^3	− 27
c^3e^2	− 27	c^2de^2	− 378	cd^3f	+ 16
c^2d^2e	+ 96	cd^3e	+ 464	cd^2e^2	+ 96
cd^4	− 32	d^5	− 128	d^4e	− 32
	± 9		± 8		± 1
	182		180		± 136
	497		1120		± 551
	± 688		± 1308		± 688

$N = (\ \ \mathfrak{Y}(x, y)^4$

x^4		x^3y		x^2y^2		xy^3		y^4	
a^2bg^2	...	a^2cg^2	-1	a^2dg^2	...	a^2eg^2	$+1$	a^2fg^2	...
cfg	-1	dfg	$+4$	efg	$+3$	f^2g	-1	$a\,beg^2$	$+1$
deg	$+1$	e^2g	...	f^3	-3	$a\,bdg^2$	-4	bf^2g	-1
df^2	$+3$	ef^2	-3	$a\,bcg^2$	-3	$befg$	$+4$	cdg^2	-1
e^2f	-3	$a\,b^2g^2$	$+1$	$bdfg$	...	bf^3	...	$cefg$	-2
$a\,b^2fg$	$+1$	$bcfg$	-4	be^2g	-15	c^2g^2	...	cf^3	$+3$
$bceg$	$+2$	$bdeg$	-16	bef^2	$+18$	$cdfg$	$+16$	d^2fg	$+4$
bcf^2	-3	bdf^2	...	c^2fg	$+15$	ce^2g	-22	de^2g	-1
bd^2g	-4	be^2f	$+18$	$cdeg$	...	cef^2	$+6$	def^2	-6
$bdef$	-12	c^2eg	$+22$	cdf^2	-36	d^2eg	$+8$	e^2f	$+3$
be^3	$+15$	c^2f^2	$+3$	ce^2f	$+9$	d^2f^2	-32	$a^0b^2dg^2$	-3
c^2dg	$+1$	cd^2g	-8	d^3g	...	de^2f	$+36$	b^2efg	$+3$
c^2ef	$+9$	$cdef$	-48	d^2ef	$+24$	e^4	-12	b^2f^3	...
cd^2f	$+4$	ce^3	$+12$	de^3	-12	$a^0b^2cg^2$	$+3$	bc^2g^2	$+3$
cde^2	-21	d^3f	$+32$	$a^0b^2g^2$	$+3$	b^2dfg	...	$bcdfg$	$+12$
d^3e	$+8$	d^2e^2	-12	b^2cfg	-18	b^2e^2g	-3	bce^2g	-9
a^0b^2eg	-3	a^0b^2fg	...	b^2deg	$+36$	b^2ef^2	...	$bcef^2$	-9
b^2f^2	...	b^2ceg	-6	b^2df^2	...	bc^2fg	-18	bd^2eg	-4
b^2cdg	$+6$	b^2cf^2	...	b^2e^2f	-27	$bcdeg$	$+48$	bd^2f^2	-32
b^2cef	$+9$	b^2d^2g	$+32$	bc^2eg	-9	$bcdf^2$	...	bde^2f	$+66$
b^2d^2f	$+32$	b^2def	...	bc^2f^2	$+27$	bce^2f	-18	be^4	-27
b^2de^2	-39	b^2e^3	-27	bcd^2g	-24	bd^3g	-32	c^2fg	-15
bc^3g	-3	bc^2dg	-36	$bcdef$	...	bd^2ef	$+32$	c^2deg	$+21$
bc^2df	-66	bc^2ef	$+18$	bce^3	$+27$	bde^3	-12	c^2df^2	$+39$
bc^2e^2	$+18$	bcd^2f	-32	bd^3f	...	c^3eg	-12	c^2e^2f	-18
bcd^2e	$+76$	$bcde^2$	$+84$	bd^3e^2	-12	c^3f^2	$+27$	cd^3g	-8
bd^4	-32	bd^3e	-32	c^3dg	$+12$	c^2d^2g	$+12$	cd^2ef	-76
c^4f	$+27$	c^4g	$+12$	c^3ef	-27	c^2def	-84	cde^3	$+45$
c^3de	-45	c^3df	$+12$	c^3d^2f	$+12$	c^2e^3	$+45$	d^4f	$+32$
c^2d^3	$+20$	c^3e^2	-45	c^2de^2	...	cd^3f	$+32$	d^3e^2	-20
		c^2d^2e	$+20$	cd^3e	...	cd^2e^2	-20		
		cd^4	...	d^5	...	d^4e	...		

$$O = (\mathfrak{A}x, y)^8$$

x^8	x^7y	x^6y^2	x^5y^3	x^4y^4	x^3y^5	x^2y^6	xy^7	y^8
a^3fg ...	a^3g^2 ...	a^2bg ...	$a^2cg^2 - 1$	a^2dg^2 ...	$a^2eg^2 + 1$	a^2fg^2 ...	a^2g^3 ...	$a\,bg^3$ ...
a^2beg ...	a^2bfg ...	$cfg - 3$	$dfg + 4$	$efg + 5$	$f^2g - 1$	$a\,beg^2 + 3$	$a\,bfg^2$ ...	cfg^2 ...
cdg ...	$ceg - 1$	$deg + 10$	$e^3g + 7$	$f^3 - 5$	$a\,bdg^2 - 4$	$bf^2g - 3$	$ceg^2 + 1$	deg^2 ...
bf^2 ...	$cf^2 - 2$	$df^2 + 2$	$ef^2 - 10$	$a\,bcg^2 - 5$	$befg + 18$	$cdg^2 - 10$	$cf^2g - 1$	df^2g ...
$cef - 1$	$d^2g + 3$	$e^2f - 9$	$a\,b^2g^2 + 1$	$bdfg$ ...	$bf^3 - 14$	$cefg + 22$	$d^2g^2 - 3$	e^2fg ...
$d^2f + 3$	$def + 6$	$a\,b^2fg + 3$	$bcfg - 18$	$be^2g + 10$	$c^2g^2 - 7$	$cf^3 - 12$	$defg + 14$	ef^3 ...
$de^2 - 2$	$e^3 - 6$	$bceg - 22$	$bdeg - 16$	$bef^2 - 5$	$cdfg + 16$	$d^2fg + 19$	$df^3 - 8$	$a^0b^2fg^2$ ...
$a\,b^2dg$ ...	$a\,b^2eg + 1$	$bcf^2 - 2$	$bdf^2 + 28$	$c^2fg - 10$	$ce^2g - 1$	$de^2g - 24$	$e^3g - 9$	$bceg^2 + 1$
$b^2ef + 1$	$b^2f^2 + 2$	$bd^2g - 19$	$be^2f + 4$	$cdeg$ ...	$cef^2 - 1$	$def^2 - 4$	$e^2f + 6$	$bcf^2g - 1$
bc^2g ...	$bcdg - 14$	$bdef + 34$	$c^2eg + 1$	$cdf^2 + 80$	$d^2eg - 13$	$e^2f + 9$	$a^0b^2eg^2 + 2$	$bd^2g^2 - 3$
$bcdf - 14$	$bcef$ ...	$be^3 + 3$	$c^2f^2 + 24$	$ce^2f - 55$	$d^2f^2 + 38$	$a^0b^2dg^2 - 2$	$b^2f^2g - 2$	$bdefg + 14$
$bce^2 + 11$	$bd^2f - 18$	$c^2dg + 24$	$cd^2g + 13$	d^3g ...	$de^2f - 62$	$b^2efg + 2$	$bcdg^2 - 6$	$bdf^3 - 8$
$bd^2e + 1$	$bde^2 + 26$	$c^3ef + 4$	$cdef - 6$	$d^2ef - 65$	$e^4 + 30$	b^2f^3 ...	$bcefg$ ...	$be^2g - 9$
$c^3f + 9$	$c^3g + 9$	$cd^2f - 58$	$ce^3 - 37$	$de^3 + 50$	$a^0b^2cg^2 + 10$	$bc^2g^2 + 9$	$bcf^3 + 6$	$be^2f^2 + 6$
$c^2de - 14$	$c^2df + 10$	$cde^2 - 42$	$d^3f - 52$	$a^0b^3g^2 + 5$	$b^2dfg - 28$	$bcdfg - 34$	$bd^2fg + 18$	$c^2dg^2 + 2$
$cd^3 + 6$	$c^2e^2 + 13$	$d^3e + 38$	$d^2e^2 + 58$	$b^2cfg + 5$	$b^2e^2g - 24$	$bce^2g - 4$	$bde^2g - 10$	$c^2efg - 11$
a^0b^2cg ...	$cd^2e - 53$	$a^0b^3eg + 12$	$a^0b^3fg + 14$	$b^2deg - 80$	$b^2ef^2 + 42$	$bcef^2 + 57$	$bdef^2 - 20$	$c^2f^3 + 9$
$b^3df + 8$	$d^4 + 24$	b^3f^2 ...	$b^2ceg + 1$	b^2df^2 ...	$bc^2fg - 4$	$bd^2eg + 58$	$be^2f + 12$	$cd^2fg - 1$
$b^3e^2 - 9$	$a^0b^3dg + 8$	$b^2cdg + 4$	$b^2cf^2 - 42$	$b^2e^2f + 60$	$bcdeg + 6$	$bd^2f^2 + 16$	$c^3g^2 + 6$	$cde^2g + 14$
$b^2c^2f - 6$	$b^3ef - 6$	$b^2cef - 57$	$b^2d^2g - 38$	$bc^2eg + 55$	$bcdf^2$ ...	$bde^2f - 110$	$c^2dfg - 26$	$cdef^2 - 16$
$b^2cde + 16$	$b^2c^2g - 6$	$b^2d^2f - 16$	b^2def ...	$bc^2f^2 - 60$	$bce^2f - 18$	$be^4 + 45$	$c^2e^2g - 13$	$ce^2f + 3$
$b^2d^3 - 8$	$b^2cdf + 20$	$b^2de^2 + 30$	$b^2e^3 + 36$	$bcd^2g + 55$	$bd^3g + 52$	$c^3fg - 3$	$c^2ef^2 + 21$	$d^3eg - 6$
$bc^3e - 3$	$b^2ce^2 - 21$	$bc^3g - 9$	$bc^2dg + 62$	$bcdef$ ...	$bd^2ef - 66$	$c^2deg + 42$	$cd^2eg + 53$	$d^3f^2 + 8$
$bc^2d^2 + 2$	$b^2d^2e - 2$	$bc^2df + 110$	$bc^2ef + 18$	$bce^3 - 60$	$bde^3 + 30$	$c^2df^2 - 30$	$cd^2f^2 + 2$	$d^2e^2f - 2$
c^4d ...	$bc^3f - 12$	$bc^2e^2 + 12$	$bcd^2f + 66$	bd^2f ...	$c^3eg + 37$	$c^2e^2f - 12$	$cde^2f - 52$	de^4 ...
	$bc^2de + 52$	$bcd^2e - 87$	$bcde^2 - 126$	$bd^3e^2 + 60$	$c^3f^2 - 66$	$cd^3g - 38$	$ce^4 + 15$	
	$bcd^3 - 28$	$bd^4 + 16$	$bd^3e + 24$	$c^3dg - 50$	$c^3d^2g - 58$	$cd^3ef + 87$	$d^4g - 24$	
	$c^4e - 15$	$c^4f + 45$	$c^4g - 30$	$c^3ef + 60$	$c^3def + 126$	$cde^3 - 40$	$d^3ef + 28$	
	$c^3d^2 + 10$	$c^3de + 40$	$c^3df - 30$	$c^2d^2f - 15$	$c^3e^3 - 60$	$d^4f - 16$	$d^2e^3 - 10$	
		$c^2d^3 - 10$	$c^3e^2 + 60$	c^2de^2 ...	$cd^3f - 24$	$d^3e^2 + 10$		
			$c^2d^2e - 15$	cd^3e ...	$cd^2e^2 + 15$			
			cd^4 ...	d^5 ...	d^4e ...			

$P = (\ \ \mathfrak{A}x,\ y)^0$, Invt.

$a^5b^0g^3$	...	$a^0b^3dg^2$	+ 4
a^2bfg^2	...	efg	− 12
$a^2b^0ceg^2$	...	f^3	+ 8
cfg^2	...	$b^3c^2g^2$	− 3
d^2g^2	+ 1	$cdfg$	...
$defg$	− 6	ce^2g	+ 30
df^3	+ 4	cef^2	− 24
e^3g	+ 4	d^2eg	− 12
e^2f^2	− 3	d^2f^2	− 24
$a\,b^5eg^2$	...	de^2f	+ 60
f^2g	...	e^4	− 27
$a\,b\,cdg^2$	− 6	$b\,c^3fg$	+ 6
$cefg$	+ 18	c^3deg	− 42
cf^3	− 12	c^2df^2	+ 60
d^2fg	+ 12	c^2e^2f	− 30
de^2g	− 18	cd^3g	+ 24
def^2	...	cd^2ef	− 84
e^2f	+ 6	cde^3	+ 66
$b^0c^2g^2$	+ 4	d^4f	+ 24
c^2dfg	− 18	d^3e^2	− 24
c^2e^2g	− 24	b^0c^4eg	+ 12
c^2ef^2	+ 30	c^4f^2	− 27
cd^2eg	+ 54	c^3d^2g	− 8
cd^2f^2	− 12	c^3def	+ 66
cd^2ef	− 42	c^3e^3	− 8
ce^4	+ 12	c^2d^3f	− 24
d^4g	− 20	$c^2d^2e^2$	− 39
d^3ef	+ 24	cd^4e	+ 36
d^2e^3	− 8	d^6	− 8

$$Q = (\mathfrak{Q}x, y)^6$$

	51	55	55	58	55	55	51
	$a^2b^0dg^2$ ...	$a^3b^0eg^2$ ...	$a^3b^0fg^2$ ...	$a^3b^0g^3$ ...	$a^2b\,g^3$ ...	$a^2b^0cg^3$ ...	$a^2b^0dg^3$ ...
	efg ...	f^2g ...	$a^2b\,eg^2$ ...	$a^2b\,fg^2$ ...	b^0cfg^2 ...	dfg^2 ...	efg^2 ...
	f^3 ...	$a^2b\,dg^2$ ...	f^2g ...	b^0ceg^2 ...	$deg^2 + 5$	$e^2g^2 + 2$	f^3g ...
	$a^2b\,cg^2$ ...	efg ...	$b^0cdg^2 - 5$	cf^2g ...	$df^2g - 5$	$ef^2g - 4$	$a\,b\,cg^3$ ...
	dfg ...	f^3 ...	$cefg + 15$	d^2g^2 ...	$e^2fg - 5$	$f^4 + 2$	dfg^2 ...
	e^2g ...	$a^2b^0c^2g^2 - 2$	$cf^3 - 10$	$defg + 20$	$ef^3 + 5$	$a\,b^2g^3$ ...	$e^2g^2 + 2$
	ef^2 ...	$cdfg + 6$	$d^2fg + 20$	$df^3 - 20$	$a\,b^2fg^2$ ...	$b\,cfg^2$ ...	$ef^2g - 4$
	$b^0c^2fg - 2$	$ce^2g + 8$	$de^2g - 25$	$e^3g - 20$	$b\,ceg^2 - 15$	$deg^2 - 6$	$f^4 + 2$
	$cdeg + 5$	$cef^2 - 10$	$def^2 - 10$	$e^2f^2 + 20$	$cf^2g + 15$	$df^2g + 6$	$b^0c^2fg^2$ ...
10	$cdf^2 + 6$	$d^2eg - 10$	$e^3f + 15$	$a\,b^2eg^2$ ...	$d^2g^2 - 20$	$e^2fg + 6$	$cdeg^2 - 5$
	$ce^2f - 7$	$d^2f^2 + 18$	$a\,b^2dg^2 + 5$	f^2g ...	$defg + 90$	$ef^3 - 6$	$cdf^2g + 5$
	$d^3g - 3$	$de^2f - 22$	$efg - 15$	$b\,cdg^2 - 20$	$df^3 - 50$	$b^0c^2eg^2 - 8$	$ce^2fg + 5$
	$d^2ef - 3$	$e^4 + 12$	$f^3 + 10$	$cefg$ ...	$e^3g - 35$	$c^2fg + 8$	$cef^3 - 5$
	$de^3 + 4$	$a\,b^2cg^2 + 4$	$b\,c^2g^2 + 5$	$cf^3 + 20$	$e^2f^2 - 15$	$cd^2g^2 + 10$	$d^3g^2 + 3$
	$a\,b^3g^2$ ...	$dfg - 6$	$cdfg - 90$	d^2fg ...	$b^0c^2dg^2 + 25$	$cdefg + 4$	$d^2efg - 7$
	$b^2cfg + 4$	$e^2g - 8$	$ce^2g + 40$	$de^2g + 40$	$c^2efg - 40$	$cdf^3 - 24$	$d^2f^3 - 2$
	$deg - 5$	$ef^2 + 10$	$cef^2 + 40$	$def^2 - 20$	$c^2f^3 + 15$	$ce^3g - 8$	$de^3g + 1$
	$df^2 - 6$	$a\,b\,c^2fg - 6$	$d^2eg + 50$	$e^2f - 20$	$cd^2fg - 50$	$ce^2f^2 + 18$	$de^2f^2 + 8$
	$e^2f + 7$	$cdeg - 4$	$d^2f^2 + 10$	$b^0c^3g^2 + 20$	$cde^2g + 45$	$d^3fg - 22$	$e^4f - 3$
20	$b\,c^2eg - 5$	$cdf^2 - 68$	$de^2f - 40$	$c^2dfg - 40$	$cdef^2 + 5$	$d^2e^2g + 14$	$a^0b^3g^3$ ...
	$c^2f^2 - 6$	$ce^2f + 76$	$e^4 - 15$	c^2e^2g ...	ce^3f ...	$d^2ef^2 + 42$	b^2cfg^2 ...
	$cd^2g + 7$	$d^3g + 22$	$b^0c^3fg + 35$	$c^2ef^2 - 20$	$d^3eg - 5$	$de^2f - 46$	$deg^2 - 6$
	$cdef - 16$	$d^2ef + 38$	$c^2deg - 45$	cd^2eg ...	$d^2f^2 + 50$	$e^5 + 12$	$df^2g + 6$
	$ce^3 + 23$	$de^3 - 58$	c^2df^2 ...	$cd^2f^2 + 60$	$d^2e^2f - 65$	$a^0b^2fg^2$ ...	$e^2fg + 6$
	$d^2f + 30$	$b^0c^3eg + 8$	$c^2e^2f - 65$	$cde^2f - 20$	$de^4 + 20$	$b^2ceg^2 + 10$	$ef^3 - 6$
	$d^2e^2 - 33$	$c^3f^2 + 42$	$cd^3g + 5$	$ce^4 + 20$	$a^0b^3eg^2 + 10$	$cf^2g - 10$	$b\,c^2eg^2 + 7$
	$b^0c^3dg - 1$	$c^2d^2g - 14$	$cd^2ef + 65$	d^4g ...	$f^2g - 10$	$d^2g^2 - 18$	$c^2f^2g - 7$
	$c^2ef + 36$	$c^2def - 82$	$cde^3 - 45$	$d^3ef - 20$	$b^2cdg^2 + 10$	$defg + 68$	$cd^2g^2 + 3$
	$c^2d^2f - 37$	$c^2e^3 - 44$	$d^4f - 20$	d^2e^3 ...	$cefg - 40$	$df^3 - 32$	$cdefg + 16$
30	$c^2de^2 - 53$	$cd^3f + 12$	$d^3e^2 - 20$	$a^0b^2dg^2 + 20$	$cf^3 + 30$	$e^3g - 42$	$cdf^3 - 22$

$Q = (\mathfrak{Q}x, y)^6$ (*continued*).

51	55	55	58	55	55	51
$a\,b^0cd^3e + 79$	$a\,b^0cd^2e^2 + 122$	$a^0b^3cg^2 - 5$	$a^0b^3efg - 20$	$a^0b^2d^2fg - 10$	$a^0b^3e^2f^2 + 24$	$a^0b\,ce^3g - 86$
$d^5 - 24$	$d^4e - 44$	$dfg + 50$	$f^3 \quad \ldots$	$de^2g \quad \ldots$	$b\,c^3dg^2 + 22$	$ce^2f^2 + 39$
$a^0b^4fg - 2$	$a^0b^4g^2 - 2$	$e^2g - 15$	$b^2c^2g^2 - 20$	$def^2 + 10$	$c^2efg - 76$	$d^3fg - 30$
$b^3ceg + 5$	$b^3cfg + 6$	$ef^2 - 30$	$cdfg + 20$	$e^3f \quad \ldots$	$c^2f^3 + 54$	$d^2e^3g + 37$
$cf^2 + 6$	$deg + 24$	$b^2c^2fg - 15$	$ce^2g + 20$	$b\,c^3g^2 - 15$	$cd^2fg - 38$	$d^2ef^2 + 50$
$d^2g + 2$	$df^2 + 32$	$cdeg - 5$	$cef^2 \quad \ldots$	$c^2dfg + 40$	$cde^2g + 82$	$de^3f - 84$
$def + 22$	$e^2f - 54$	$cdf^2 - 10$	$d^2eg - 60$	$c^2e^2g + 65$	$cdef^2 - 50$	$e^5 + 27$
$e^3 - 27$	$b^2c^2eg - 18$	$ce^2f + 60$	$d^2f^2 \quad \ldots$	$c^2ef^2 + 60$	$ce^2f + 6$	$b^0c^2dg^2 - 4$
$b^2c^2dg - 8$	$c^2f^2 - 24$	$d^3g - 50$	$de^2f + 40$	$cd^2eg - 65$	$d^3eg - 12$	$c^3efg - 23$
$c^2ef - 39$	$cd^2g - 42$	$d^2ef - 10$	$e^4 \quad \ldots$	$cd^2f^2 + 10$	$d^3f^2 + 64$	$c^3f^2 + 27$
$cd^2f - 50$	$cdef + 50$	$de^3 + 30$	$b\,c^3fg + 20$	$cde^2f + 10$	$d^2e^2f - 82$	$c^2d^2fg + 33$
$cde^2 + 107$	$ce^3 + 54$	$b\,c^3eg \quad \ldots$	$c^2deg + 20$	$ce^4 \quad \ldots$	$de^4 + 30$	$c^3de^2g + 53$
$d^3e - 22$	$d^3f - 64$	$c^3f^2 \quad \ldots$	$c^2df^2 - 40$	$d^4g + 20$	$b^0c^4g^2 - 12$	$c^2def^2 - 107$
$b\,c^4g + 3$	$d^2e^2 + 32$	$c^2d^2g + 65$	$c^2e^2f \quad \ldots$	$d^3ef - 10$	$c^3dfg + 58$	$c^2e^3f + 21$
$c^3df + 84$	$b\,c^3dg + 46$	$c^2def - 10$	$cd^3g + 20$	$d^2e^3 + 5$	$c^3e^2g + 44$	$cd^3eg - 79$
$c^3e^2 - 21$	$c^3ef - 6$	$c^2e^3 - 30$	$cd^2ef \quad \ldots$	$b^0c^4fg + 15$	$c^2ef^2 - 54$	$cd^3f^2 + 22$
$c^3d^2e - 102$	$c^2d^2f + 82$	$cd^3f + 10$	$cde^3 - 40$	$c^3deg + 45$	$c^2d^2eg - 122$	$cd^2e^2f + 102$
$cd^4 + 44$	$c^2de^2 - 112$	$cd^2e^2 - 75$	$d^4f \quad \ldots$	$c^3df^2 - 30$	$c^3d^2f^2 - 32$	$cde^4 - 45$
$b^0c^5f - 27$	$cd^3e - 34$	$d^4e + 40$	$d^3e^2 + 20$	$c^3e^2f + 30$	$c^3de^2f + 112$	$d^5g + 24$
$c^4de + 45$	$d^5 + 32$	$b^0c^4dg - 20$	$b^0c^4eg - 20$	$c^2d^3g + 20$	$c^2e^4 - 30$	$d^4ef - 44$
$c^3d^3 - 20$	$b^0c^5g - 12$	$c^4ef \quad \ldots$	$c^4f^2 \quad \ldots$	$c^2d^2ef + 75$	$cd^4g + 44$	$d^3e^3 + 20$
	$c^4df - 30$	$c^3d^2f - 5$	$c^2d^2g \quad \ldots$	$c^2de^3 - 50$	$cd^2ef + 34$	
	$c^4e^2 + 30$	$c^3de^2 + 50$	$c^3def + 40$	$cd^4f - 40$	$cd^2e^3 - 30$	
	$c^3d^2e + 30$	$c^2d^3e - 25$	$c^3e^3 \quad \ldots$	$cd^3e^2 + 25$	$d^5f - 32$	
	$c^2d^4 - 20$	$cd^5 \quad \ldots$	$c^2d^3f - 20$	$d^5e \quad \ldots$	$d^4e^2 + 20$	
			$c^2d^2e^2 \quad \ldots$			
			$cd^4e \quad \ldots$			
			$d^6 \quad \ldots$			

776.

ON THE JACOBIAN SEXTIC EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVIII. (1882), pp. 52—65.]

THE Jacobian sextic equation has been discussed under the form

$$(z-a)^6 - 4a(z-a)^5 + 10b(z-a)^3 - 4c(z-a) + 5b^2 - 4ac = 0,$$

(see references at end of paper), but the connexion of this form with the general sextic equation has not, so far as I am aware, been considered. And although this is probably known, I do not find it to have been explicitly stated that the group of the equation is the positive half-group, or group of the 60 positive substitutions out of the 120 substitutions, which leave unaltered Serret's 6-valued function of six letters.

Invariantive Property of the Jacobian Sextic.

Taking $z-a$ as the variable, and comparing the equation with the general sextic equation

$$(\mathrm{a, b, c, d, e, f, g} \between z-a, 1)^6 = 0,$$

we have

$$\begin{aligned} &\mathrm{a,\ \ b,\ c,\ d,\ e,\ \ f\ ,\ g} \\ &= 1,\ -\tfrac{2}{3}a,\ 0,\ \tfrac{1}{2}b,\ 0,\ -\tfrac{2}{3}c,\ 5b^2-4ac; \end{aligned}$$

the Jacobian equation is thus an equation

$$(\mathrm{a, b, c, d, e, f, g} \between x, y)^6 = 0,$$

for which $\mathrm{c}=0$, $\mathrm{e}=0$, $\mathrm{ag}+9\mathrm{bf}-20\mathrm{d}^2=0$; but of course any equation, which can be by a linear transformation upon the variables brought into this form, may be regarded as a Jacobian equation.

Hence, using henceforward the small italic in place of the small roman letters, the Jacobian sextic may be regarded as an equation

$$(a, b, c, d, e, f, g \between x, y)^6 = 0,$$

linearly transformable into the form

$$(a, b, 0, d, 0, f, g \between x, y)^6 = 0,$$

where $ag+9bf-20d^2=0$. It is to be shown, that this implies a single relation between the four invariants A, B, C, and Δ of the sextic function.

I call to mind that the general sextic has five invariants A, B, C, D, E of the orders 2, 4, 6, 10, 15 respectively; the last of them E is not independent, but its square is equal to a rational and integral function of A, B, C, D; and instead of D, we consider the discriminant Δ which is an invariant of the same order 10. The values of A, B, C are given, Table Nos. 31, 34, and 35 of my Third Memoir on Quantics, *Phil. Trans.*, vol. CXLVI. (1856), pp. 627—647, [144]; those of D, Δ, E were obtained by Dr Salmon, see his *Higher Algebra*, second ed. 1866, where the values of A, B, C, D, Δ, E are all given; only those of A, B, C, Δ are reproduced in the third edition, 1876.

It may be remarked, that for the general form we have $A=ag-6bf+15ce-10d^2$, and that B is the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}:$$

C and Δ are complicated forms, the latter of them containing 246 terms. But writing $c=0$, $e=0$, there is a great reduction; we have

$A=$	$B=$	$C=$	$D=$
$ag+1$	ad^2g-1	$a^2d^2g^2+1$	a^5g^5+1
$bf-6$	b^2f^2+1	$,,\ df^3+4$	a^4bfg^4-30
d^2-10	bd^2f-2	$a\,bd^2fg+12$	$,,\ d^2g^2-300$
	d^4+1	$,,\ d^4g-20$	$,,\ df^2g-2500$
		$a^0b^3dg^2+4$	$,,\ f^5-3125$
		$,,\ b^3f^3+8$	$a^3b^2f^2g^3-15$
		$,,\ b^3d^2f^2-24$	$,,\ bd^2fg^2-4800$
		$,,\ bd^4f+24$	$,,\ bdf^4g-7500$
		$,,\ d^6-8$	$,,\ d^4g^3+30000$
			$,,\ d^3f^3g+50000$
			$a^2b^3dg^4-2500$
			$,,\ b^3f^3g^2-410$
			$,,\ b^3d^2f^2g^2-171300$
			$,,\ b^3df^5-240000$
			$,,\ bd^4fg^2+780000$
			$,,\ bd^2f^4+1200000$
			$,,\ d^8g^2-1000000$
			$,,\ d^5f^2-1600000$
			$a\,b^4dfg^3-7500$
			$,,\ b^4f^4g-11520$
			$,,\ b^3d^3g^3+50000$
			$,,\ b^3d^2f^3g+83200$
			$a^0b^6g^4-3125$
			$,,\ b^6df^2g^2-240000$
			$,,\ b^6f^5-331776$
			$,,\ b^4d^3fg^2+1200000$
			$,,\ b^4d^2f^4+1843200$
			$,,\ b^3d^5g^2-1600000$
			$,,\ b^3d^4f^3-2560000$

It is clear that these are all functions of ag, bf, d^2 and $a^2f^3+b^3g^2$, say of α, β, δ and ϕ. In fact, A and B are functions of α, β, δ; C contains two terms, coefficient 4, which are $=4\sqrt{\delta}\,.\,\phi$; Δ contains two terms

$$-3125\,(a^4f^3+b^5g^4),$$

which are $=-3125\,(\phi^2-2\alpha^2\beta^3)$; and also several pairs of terms, each which pair contains the factor ϕ. We thus have

$A=$	$B=$	$C=$	$\Delta=$
$\alpha+1$	$\alpha\delta-1$	$\alpha^2\delta+1$	α^5+1
$\beta-6$	β^2+1	$\alpha\beta\delta+12$	$\alpha^4\beta-30$
$\delta-10$	$\beta\delta-2$	$\alpha\delta^2-20$	$\alpha^4\delta-300$
	δ^2+1	β^3+8	$\alpha^2\beta^3+5840\,(=6250-410)$
		$\beta^2\delta-24$	$\alpha^3\beta^2-15$
		$\beta\delta^2+24$	$\alpha^3\beta\delta-4800$
		δ^3-8	$\alpha^3\delta^2+3000$
		$\phi\sqrt{\delta}+4$	$\alpha^2\beta^2\delta-171300$
			$\alpha^2\beta\delta^3+780000$
			$\alpha^2\delta^3-1000000$
			$\alpha\beta^4-11520$
			$\alpha\beta^3\delta+83200$
			$\beta^5-331776$
			$\beta^4\delta+1843200$
			$\beta^3\delta^2-2560000$
			$\phi\sqrt{\delta}\,.\,\alpha^3-2500$
			,, $\alpha\beta-7500$
			,, $\alpha\delta+50000$
			,, $\beta^2-240000$
			,, $\beta\delta+1200000$
			,, $\delta^2-1600000$
			$\phi^2.\quad-3125$

We have *ante*, the relation $\alpha+9\beta-20\delta=0$, and using this to eliminate α, we have A, B, C, Δ as functions of β, δ, ϕ (that is, of bf, d^2 and $a^2f^3+b^3g^2$). Effecting the substitution, we find the values of A, B, C without difficulty. As regards the value of Δ, this is

$$=-3125\phi^2+2\phi K\sqrt{\delta}+\text{terms without }\phi,$$

where

$$\begin{aligned} 2K = &- 2500\,(81\beta^2 - 360\beta\delta + 400\delta^2) \\ &- 7500\,(-9\beta^2 + 20\beta\delta) \\ &+ 50000\,(-9\beta\delta + 20\delta^2) \\ &- 240000\,(\beta^2) \\ &+ 1200000\,(\beta\delta) \\ &- 1600000\,(\delta^2), \end{aligned}$$

or, reducing and dividing by 2,

$$K = -3125\,(60\beta^2 - 240\beta\delta + 256\delta^2).$$

The calculation of the terms without ϕ is much more laborious, but they come out

$$= -3125\,(60\beta^2 - 240\beta\delta + 256\delta^2)^2\,\delta.$$

Hence the value of Δ is

$$\begin{aligned} \Delta = -3125\,\{ &\phi^2 \\ &+ 2\phi\,(60\beta^2 - 240\beta\delta + 256\delta^2)\sqrt{\delta} \\ &+ (60\beta^2 - 240\beta\delta + 256\delta^2)^2\,\delta\}, \end{aligned}$$

say this is

$$\Delta = -3125h^2,$$

where

$$h = \phi + (60\beta^2 - 240\beta\delta + 256\delta^2)\sqrt{\delta},$$

that is,

$$= a^2f^2 + b^2g^2 + (60b^2df^2 - 240bd^3f + 256d^5).$$

The values of A, B, C, and the foregoing value of h then are

$A =$	$B =$	$C =$	$h =$
$\beta - 15$	$\beta^2 + 1$	$\beta^3 + 8$	$\beta^2\sqrt{\delta} + 60$
$\delta + 10$	$\beta\delta + 7$	$\beta^2\delta + 51$	$\beta\delta\sqrt{\delta} - 240$
	$\delta^2 - 19$	$\beta\delta^2 + 84$	$\delta^2\sqrt{\delta} + 256$
		$\delta^3 - 8$	$\phi + 1$
		$\phi\sqrt{\delta} + 4$	

We may, if we please, regard β, δ, ϕ as irrational invariants of the sextic, viz. A, B, C being rational and integral functions of β, δ, ϕ, we have conversely β, δ, ϕ irrational functions of A, B, C; and then the equation for h, say

$$\frac{1}{25\sqrt{5}}\sqrt{(-\Delta)} = \phi + \sqrt{\delta}\,(60\beta^2 - 240\beta\delta + 256\delta^2)$$

is the invariantive relation which characterises the Jacobian sextic.

The expression for Δ in terms of A, B, C, D is

$$\Delta = A^5 - 375A^3B - 625A^2C + 3125D,$$

and it was in the foregoing investigation proper to use Δ in place of D. But I annex the value of D for the case in question $b=0$, $f=0$; and also its value in terms of α, β, δ, ϕ. These are

$D=$		$D=$	
a^4f^6	-1	ϕ^2	-1
a^3bdf^4g	-12	$\alpha^2\beta^3$	$+2$
,, d^4g^3	$+5$	$\alpha\beta^4$	-12
$a^2b^2d^2f^2g^2$	-90	β^5	-72
,, b^2df^5	-48	$\delta\,\alpha^2\beta^2$	-90
,, bd^3fg^2	$+246$	,, $\alpha\beta^3$	$+168$
,, bd^2f^4	$+480$	,, β^4	$+552$
,, d^6g^2	-258	$\delta^2\alpha^3$	$+5$
,, d^3f^3	-432	,, $\alpha^2\beta$	$+246$
$a\,b^4dfg^2$	-12	,, $\alpha\beta^2$	$+240$
,, b^4f^4g	-12	,, β^3	-976
,, $b^3d^2f^3g$	$+168$	$\delta^3\alpha^2$	-258
,, $b^2d^4f^2g$	$+240$	,, $\alpha\beta$	-168
,, bd^3fg	-168	,, β^2	$+336$
,, d^5g	-228	$\delta^4\alpha$	-228
$a^6b^3g^4$	-1	,, β	-408
,, $b^3df^2g^2$	-48	δ^5	-240
,, b^3f^5	-72	$\phi\sqrt{\delta}\,\alpha\beta$	-12
,, $b^4d^3fg^2$	$+480$	,, β^2	-48
,, $b^4d^2f^3$	$+552$	$\phi\delta\sqrt{\delta}\,\beta$	-480
,, $b^3d^3g^2$	-432	$\phi\delta^2\sqrt{\delta}$	-432
,, $b^3d^4f^3$	-976		
,, $b^2d^6f^2$	$+336$		
,, bd^8f	$+408$		
,, d^{10}	-248		

The Group of the Jacobian Sextic.

The solution of the Jacobian sextic equation depends upon that of a quintic; in fact, calling the roots z_∞, z_0, z_1, z_2, z_3, z_4, then there exists a quintic having the roots

$$\sqrt{(z_\infty - z_0 \,.\, z_2 - z_3 \,.\, z_4 - z_1)},$$
$$\sqrt{(z_\infty - z_1 \,.\, z_3 - z_4 \,.\, z_0 - z_2)},$$
$$\sqrt{(z_\infty - z_2 \,.\, z_4 - z_0 \,.\, z_1 - z_3)},$$
$$\sqrt{(z_\infty - z_3 \,.\, z_0 - z_1 \,.\, z_2 - z_4)},$$
$$\sqrt{(z_\infty - z_4 \,.\, z_1 - z_2 \,.\, z_3 - z_0)},$$

the coefficients of which are rational functions of the coefficients a, b, d, f, g, and of the fourth root of the discriminant, i.e., $\sqrt{h}$. But the meaning of this has not, so far as I am aware, been noticed. Passing to the quintic whose roots are the squares of the foregoing values, i.e., $z_\infty - z_0 . z_2 - z_3 . z_4 - z_1$, &c., the coefficients are here rational functions of a, b, d, f, g and h; that is, they are rational functions of a, b, d, f, g. The symmetrical functions of these roots $z_\infty - z_0 . z_2 - z_3 . z_4 - z_1$, &c., are thus rational functions of the coefficients of the sextic; each such rational function is a 12-valued function of $z_\infty, z_0, z_1, z_2, z_3, z_4$, invariable by all the substitutions of a group of 60 substitutions; and therefore also every like 12-valued function of the roots $z_\infty, z_0, z_1, z_2, z_3, z_4$ is invariable by the substitutions of this group of 60; or, in other words, this group of 60 is the group of the Jacobian sextic equation.

I write for convenience, in this section only,

$$z_\infty, z_0, z_1, z_2, z_3, z_4 = f, a, b, c, d, e;$$

and writing further ab for shortness instead of $a - b$, &c., (so that of course $ba = -ab$), and putting $B, C, D, E, F = -ab . cd . ef, -ac . bf . de, ad . bc . ef, ae . bd . cf, af . be . cd$, then the five functions are B, C, D, E, F, and the group of 60 which leaves unaltered every symmetrical function of these functions is made up of the substitutions

1.				1
ab . ce,	*ab . df,*	*ce . df,*		15
ac . bf,	*ac . de,*	*bf . de,*		
ad . bc,	*ad . ef,*	*bc . ef,*		
ae . bd,	*ae . cf,*	*bd . cf,*		
af . be,	*af . cd,*	*be . cd.*		
abcde,	*acebd,*	*adbec,*	*aedbc,*	24
afbce,	*abefc,*	*acfeb,*	*aecbf,*	
abdef,	*adfbe,*	*aebfd,*	*afedb,*	
afced,	*acdfe,*	*aefdc,*	*adecf,*	
afdbc,	*adcfb,*	*abfcd,*	*acbdf,*	
bdcef,	*bcfde,*	*bedfc,*	*bfecd.*	
abc . dfe,	*acb . def,*			20
abd . cfe,	*adb . cef,*			
abe . cfd,	*aeb . cdf,*			
abf . ced,	*afb . cde,*			
acd . bef,	*adc . bfe,*			
ace . bfd,	*aec . bdf,*			
acf . bed,	*afc . bde,*			
ade . bfc,	*aed . bcf,*			
adf . bce,	*afd . bec,*			
aef . bcd,	*afe . bdc,*			60

where the symbols, ab, $abcde$, abc, &c. denote cyclical substitutions. It is easy to verify that each of these substitutions does in fact merely permute B, C, D, E, F; thus

$$\begin{array}{lccccc}
 & B & C & D & E & F \\
abcde \text{ on } & -ab\,.\,ce\,.\,df, & -ac\,.\,bf\,.\,de, & ad\,.\,bc\,.\,ef, & ae\,.\,bd\,.\,cf, & af\,.\,be\,.\,cd \\
 & = -bc\,.\,da\,.\,ef, & -bd\,.\,cf\,.\,ea, & be\,.\,cd\,.\,af, & ba\,.\,ce\,.\,df, & bf\,.\,ca\,.\,de \\
 & = +ad\,.\,bc\,.\,ef, & ae\,.\,bd\,.\,cf, & af\,.\,be\,.\,cd, & -ab\,.\,ce\,.\,df, & -ac\,.\,bf\,.\,de \\
 & = \quad D & E & F & B & C,
\end{array}$$

which (expressed as a cyclical substitution) is $= BDFCE$, and so in other cases.

We may to the foregoing 60 substitutions join the 60 other substitutions:

$cdef$,	$cfed$,	30
$bdfe$,	$befd$,	
$becf$,	$bfce$,	
$bcdf$,	$bfdc$,	
$bced$,	$bdec$,	
$aedf$,	$afde$,	
$acef$,	$afec$,	
$acfd$,	$adfc$,	
$adce$,	$aecd$,	
$abfe$,	$aefb$,	
$adbf$,	$afbd$,	
$abed$,	$adeb$,	
$abcf$,	$afcb$,	
$acbe$,	$aebc$,	
$abdc$,	$acdb$.	
$ab\,.\,cd\,.\,ef$,	$ab\,.\,cf\,.\,de$,	10
$ac\,.\,bd\,.\,ef$,	$ac\,.\,be\,.\,df$,	
$ad\,.\,be\,.\,cf$,	$ad\,.\,bf\,.\,ce$,	
$ae\,.\,bc\,.\,df$,	$ae\,.\,bf\,.\,cd$,	
$af\,.\,bc\,.\,de$,	$af\,.\,bd\,.\,ce$.	
$abcefd$,	$adfecb$,	20
$abfdec$,	$acedfb$,	
$abecdf$,	$afdceb$,	
$abdfce$,	$aecfdb$,	
$acfbde$,	$aedbfc$,	
$acbfed$,	$adefbc$,	
$acdebf$,	$afbedc$,	
$adbcfe$,	$aefcbd$,	
$adcbef$,	$afebcd$,	
$aebdcf$,	$afcdbe$,	60

each of which changes B, C, D, E, F into a permutation of $-B, -C, -D, -E, -F$.

The 60 and 60 substitutions form together a group of 120 substitutions, which leave unaltered any even symmetrical function of B, C, D, E, F, or say any symmetrical function of B^2, C^2, D^2, E^2, F^2; such a function is thus a 6-valued function of a, b, c, d, e, f, viz. it is Serret's 6-valued function of 6 letters.

Transformation of the Jacobian Sextic into the Resolvent Sextic of a special quintic equation.

Starting from the Jacobian Sextic Equation

$$(a,\ b,\ 0,\ d,\ 0,\ f,\ g \between z,\ 1)^6 = 0,$$

$ag + 9bf - 20d^2 = 0$, I effect upon it the Tschirnhausen transformation

$$X = -az^3 - 6bz^2 - 10d;$$

which, it may be remarked, is a particular case of the Tschirnhausen-Hermite form

$$X(az+b)B + (az^2+6bz+5c)C + (az^3+6bz^2+15cz+10d)D$$
$$+ (az^4+6bz^3+15cz^2+20dz+10e)E + (az^5+6bz^4+15cz^3+20dz^2+15ez+5f)F.$$

Writing for convenience $Y = X + 10d$, $Z = X - 10d$, this is

$$\begin{array}{rrrrrrl} . & . & az^3 & +6bz^2 & - & . & +Y = 0, \end{array}$$

and we thence have

$$\begin{array}{rrrrrrl}
. & az^4 & +6bz^3 & . & +Yz & . & = 0, \\
az^5 & +6bz^4 & . & +Yz^2 & . & . & = 0, \\
 & & -Zz^3 & . & -6fz & -g & = 0, \\
 & -Zz^4 & . & -6fz^2 & -gz & . & = 0, \\
-Zz^5 & . & -6fz^3 & -gz^2 & . & . & = 0,
\end{array}$$

or, eliminating, the resulting equation is

$$\left|\begin{array}{rrrrrr}
. & . & a, & 6b, & . & Y \\
. & a, & 6b, & . & Y, & . \\
a, & 6b, & . & Y, & . & . \\
. & . & Z, & ., & 6f, & g \\
. & Z, & . & 6f, & g, & . \\
Z, & . & 6f, & g, & . & .
\end{array}\right| = 0.$$

The developed form is most easily obtained by expanding the determinant in the form

$$123 \,.\, \overline{456} - 456 \,.\, \overline{123}, \text{ \&c.},$$

where the terms 123, &c., belong to the matrix

$$\left|\begin{array}{rrrrrr}
. & . & a, & 6b, & . & Y \\
. & a, & 6b, & . & Y, & \\
a, & 6b, & . & Y, & &
\end{array}\right|,$$

and those of $\overline{123}$, &c., to the matrix

$$\left|\begin{matrix} . & . & Z, & . & 6f, & g \\ . & Z, & . & 6f, & g, & . \\ Z, & . & 6f, & g, & . & . \end{matrix}\right| .$$

The several terms are

$123 \,.\, \overline{456}$	$+ - a^3$	$. - g^3$
$- 124 \,.\, 356$	$- - 6a^2b$	$. - 6fg^2$
$+ 125 \,.\, 346$	$+ \quad 0$	$. - 36f^3g$
$- 126 \,.\, 345$	$- - a^2Y$	$. \quad g^2Z - 216f^3$
$+ 134 \,.\, 256$	$+ - 36ab^2$	$. \quad 0$
$- 135 \,.\, 246$	$- \quad a^2Y$	$. - g^2Z$
$+ 136 \,.\, 245$	$+ - 6ab\ Y$	$. - 6fgZ$
$+ 145 \,.\, 236$	$+ \quad 6ab\ Y$	$. - 6fgZ$
$- 146 \,.\, 235$	$- \quad 0$	$. - 36f^2Z$
$+ 156 \,.\, 234$	$+ - a\ Y^2$	$. - gZ^2$
$- 234 \,.\, 156$	$- - a^2Y - 216b^3$	$. \quad g^2Z$
$+ 235 \,.\, 146$	$+ \quad 6ab\ Y$	$. \quad 6fgZ$
$- 236 \,.\, 145$	$- - 36b^2Y$	$. \quad 36f^2Z$
$- 245 \,.\, 136$	$- \quad 36b^2Y$	$. \quad 0$
$+ 246 \,.\, 135$	$+ \quad aY^2$	$. \quad g\ Z^2$
$- 256 \,.\, 134$	$- - 6b\ Y^2$	$. \quad 6fZ^2$
$+ 345 \,.\, 126$	$+ - a\ Y^2$	$. - g\ Z^2$
$- 346 \,.\, 125$	$- \quad 6b\ Y^2$	$. - 6fZ^2$
$+ 356 \,.\, 124$	$+ \quad 0$	$. \quad 0$
$- 456 \,.\, 123$	$- - Y^3$	$. \quad Z^3.$

Hence, collecting and reducing, the equation is

$$\begin{aligned} 0 = \quad & Y^3Z^3 \,. \\ & + Y^2Z^2 \,. \quad (3ag + 72bf) \\ & + YZ \quad . \quad (3a^2g^2 + 36agbf + 1296b^2f^2) \\ & + Y \quad . \quad 216a^2f^3 \\ & + Z \quad . - 216b^3g^2 \\ & + \qquad\quad a^3g^3 - 36a^2g^2bf, \end{aligned}$$

where Y, Z denote $X+10d$, $X-10d$ respectively, and consequently $YZ=X^2-100d^2$. Hence, writing as before α, β, δ, ϕ to denote ag, bf, d^2 and $a^2f^3+b^3g^2$ respectively, the result finally is

$$\left(\begin{array}{c|c|c|c|c|c|c}
1 & 0 & \alpha+3 & 0 & \alpha^2+3 & a^2f^3+216 & \phi\sqrt{\delta}+2160 \\
 & & \beta+72 & & \alpha\beta+36 & b^3g^2-216 & \alpha^3+1 \\
 & & \delta-300 & & \alpha\delta-600 & & \alpha^2\beta-36 \\
 & & & & \beta^2+1296 & & \alpha^2\delta-30 \\
 & & & & \beta\delta-14400 & & \alpha\beta\delta-360 \\
 & & & & \delta^2+30000 & & \alpha\delta^2+30000 \\
 & & & & & & \beta^2\delta+12960 \\
 & & & & & & \beta\delta^2+720000 \\
 & & & & & & \delta^3-1000000
\end{array}\right.\ \between X, 1)^6=0,$$

where observe that the coefficient of the term in X is $216(a^2f^3-b^3g^2)$, $=216\sqrt{(\phi^2-4\alpha^2\beta^3)}$. We have as before $ag+9bf-20d^2=0$, that is, $\alpha+9\beta-20\delta=0$; and using this equation to eliminate α, also in the constant term writing its value for ϕ in terms of h,

$$\phi=h+(-60\beta^2+240\beta\delta-256\delta^2)\sqrt{\delta},$$

the new equation is

$$\left(\begin{array}{c|c|c|c|c|c|c}
1 & 0 & \overbrace{-5\times}^{} & 0 & \overbrace{5\times}^{} & -216\sqrt{\Lambda} & \overbrace{5\times}^{} \\
 & & \beta-9 & & \beta^2-243 & & h\sqrt{\delta}+432 \\
 & & \delta+48 & & \beta\delta-1872 & & \beta^3-729 \\
 & & & & \delta^2+3840 & & \beta^2\delta+4184 \\
 & & & & & & \beta\delta^2-11520 \\
 & & & & & & \delta^3+8292
\end{array}\right.\ \between X, 1)^6=0,$$

where

$$\begin{aligned}\Lambda &= \{h+(-60\beta^2+240\beta\delta-256\delta^2)\sqrt{\delta}\}^2-4(-9\beta+20\delta)^2\beta^3 \\ &= h^2+2h\sqrt{\delta}\,.\left|\begin{array}{l}\beta^2-60\\ \beta\delta+240\\ \delta^2-256\end{array}\right|-4(\beta-4\delta)^3(9\beta-16\delta)^2.\end{aligned}$$

It is to be shown that this Tschirnhausen-transformation of the Jacobian sextic is, in fact, the resolvent sextic of the quintic equation

$$(\mathrm{a},\ 0,\ \mathrm{c},\ 0,\ \mathrm{e},\ \mathrm{f}\between x,\ 1)^5=0,$$

where

$$\mathrm{a}=1,\quad \mathrm{c}=2d,\quad \mathrm{e}=-9bf+36d^2,\quad \mathrm{f}^2=216h.$$

I consider the general quintic $(a, b, c, d, e, f \between x, 1)^5 = 0$; taking the roots to be x_1, x_2, x_3, x_4, x_5, and writing

$$\phi_1 = 12345 - 24135,$$
$$\phi_2 = 13425 - 32145,$$
$$\phi_3 = 14235 - 43125,$$
$$\phi_4 = 21435 - 13245,$$
$$\phi_5 = 31245 - 14325,$$
$$\phi_6 = 41325 - 12435,$$

where 12345 is used to denote the function

$$= (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)\sqrt{(20)},$$

(this numerical factor $\sqrt{(20)}$ being inserted for greater convenience), then the equation whose roots are $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$, which equation may be regarded as the resolvent sextic of the given quintic equation, is

$$\left(\begin{array}{cccccc} a^6 \times & & -5a^4 \times & 5a^2 \times & -\sqrt{(\square)}\,.\,a^2 & +5 \\ 1 & 0 & ae & -2a^2df & +1 & +1a^3cf^2 \\ & & -4bd & +3a^2e^2 & & -2a^2def \\ & & +3c^2 & \&c. & & +\&c. \end{array}\right) \between \phi, 1)^6 = 0,$$

$\square = a^4f^4 + \&c.$, the discriminant of the quintic: see p. 274* of my paper "On a new auxiliary equation in the theory of equations of the fifth order," *Phil. Trans.* t. CLI. (1861), pp. 263—276, [268].

I now write $b = 0$, $d = 0$, but, to avoid confusion again, write roman instead of italic letters, viz. I consider the resolvent sextic of the quintic equation

$$(\mathrm{a}, 0, \mathrm{c}, 0, \mathrm{e}, \mathrm{f} \between x, 1)^5.$$

Many of the terms thus vanish, and the equation assumes the form

$$\left(\begin{array}{cccccc} \mathrm{a}^6 \times & & -5\mathrm{a}^4 & 5\mathrm{a}^2 & -\mathrm{a}^2\sqrt{\square} & +5 \\ 1 & 0 & \mathrm{ae}+1 & \mathrm{a}^2\mathrm{e}^2+3 & +1 & \mathrm{a}^3\mathrm{cf}^2+1 \\ & & \mathrm{c}^2+3 & \mathrm{ac}^2\mathrm{e}-2 & & \mathrm{a}^3\mathrm{e}^3+1 \\ & & & \mathrm{c}^4+15 & & \mathrm{a}^2\mathrm{c}^2\mathrm{e}^2-11 \\ & & & & & \mathrm{ac}^4\mathrm{e}+35 \\ & & & & & \mathrm{c}^6-25 \end{array}\right) \between X, 1)^6 = 0,$$

and then if, as before,

$$\mathrm{a} = 1,\ \mathrm{c} = 2d,\quad \mathrm{e} = -9bf + 36d^2,\ \mathrm{f}^2 = 216h,$$

or say

$$\mathrm{a} = 1,\ \mathrm{c} = 2\sqrt{\delta},\ \mathrm{e} = -9\beta + 36\delta,\ \mathrm{f}^2 = 216h,$$

* [This Collection, vol. IV., p. 321.]

this becomes identical with the foregoing Tschirnhausen-transformation equation; thus

$$\begin{aligned} ae + 3c^2 = -9\beta + 36\delta + 12\delta, &= \beta - 9 \\ &\quad\ \delta + 48; \end{aligned}$$

and similarly

$$\begin{aligned} 3a^2e^2 - 2ac^2e + 15c^4 = \beta^2 &+ 243, \\ \beta\delta &- 1872, \\ \delta^2 &+ 3840. \end{aligned}$$

So for the constant term, $+1a^3cf^2$ gives the term $432h\sqrt{\delta}$, and $+1a^3e^3$, &c., give the remaining terms $-729\beta^3$, &c. of the value in question.

It only remains to verify the equality of the coefficients of X,

$$216\sqrt{\Lambda} = \sqrt{\square} \text{ or } 46656\Lambda = \square.$$

Here $\square$, the discriminant of the quintic $(a, 0, c, 0, e, f \between x, 1)^5$, from the general form (see my Second Memoir on Quantics, [141], or Salmon's *Higher Algebra*, third edition, p. 209) putting therein $b = 0$, $d = 0$, is

$$\begin{aligned} \square = a^4f^4 &+ 1, \\ a^3ce^2f^2 &+ 160, \\ a^3e^5 &+ 256, \\ a^3c^3ef^2 &- 1440, \\ a^2c^2e^4 &- 2560, \\ ac^5f^2 &+ 3456, \\ ac^4e^3 &+ 6400, \end{aligned}$$

and writing for a, c, e, f their values 1, $2\sqrt{\delta}$, $9(-\beta + 4\delta)$, $216h$, the value becomes

$$\begin{aligned} \square = \ & (216)^2 . h^2 \ . \\ & + 432h\sqrt{\delta} \ . \ 12960\,(\beta - 4\delta)^2 \\ & \qquad\qquad + 34560\,(\beta - 4\delta)\,\delta \\ & \qquad\qquad + 55296\ \delta^2 \\ & - 256 \ . 9^5 . \ (\beta - 4\delta)^5 \\ & - 10240 \ . 9^4 . \ (\beta - 4\delta)^4\,\delta \\ & - 102400 . 9^3 . \ (\beta - 4\delta)^3\,\delta^3. \end{aligned}$$

The whole divides by $(216)^2$, and we thus obtain

$$\begin{aligned} \Lambda = h^2 + 2h\sqrt{\delta} . \ & 60\ (-\beta + 4\delta)^2 . + (-\beta - 4\delta)^3 . \quad 324\,(\beta - 4\delta)^2 \\ & - 240\,(-\beta + 4\delta)\,\delta \qquad\qquad + 1440\,(\beta - 4\delta)\,\delta \\ & + 256\ \ \delta^2 \qquad\qquad\qquad\qquad + 1600\,\delta^2, \end{aligned}$$

which is, in fact, equal to the foregoing value of Λ.

The conclusion is that, starting from the Jacobian sextic

$$(a,\ b,\ 0,\ d,\ 0,\ f,\ g \between z,\ 1)^6 = 0,$$

where $ag + 9bf - 20d^2 = 0$, and effecting upon it the Tschirnhausen-transformation

$$X = -az^3 - 6bz^2 - 10d,$$

so as to obtain from it a sextic equation in X, this sextic equation in X is the resolvent sextic of the quintic equation

$$(1,\ 0,\ \mathrm{c},\ 0,\ \mathrm{e},\ \mathrm{f} \between x,\ 1)^5 = 0,$$

where

$$\mathrm{c} = 2d,\ \mathrm{e} = -9bf + 36d^2,\ \mathrm{f} = \sqrt{(216h)},$$

and, Δ being the discriminant of the Jacobian sextic, then

$$h = \frac{1}{5^2\sqrt{5}}\sqrt{(-\Delta)},\ = a^2f^3 + b^3g^2 + 60b^2df^2 - 240bd^3f + 256d^5.$$

As to the subject of the present paper, see in particular Brioschi, "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Annalen*, t. XIII. (1878), pp. 109—160, and the third Appendix to his translation of my *Elliptic Functions*, Milan, 1880, each containing references to the earlier papers.

777.

A SOLVABLE CASE OF THE QUINTIC EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVIII. (1882), pp. 154—157.]

THE roots of the general quintic equation

$$(a,\ b,\ c,\ d,\ e,\ f \between x,\ 1)^5 = 0$$

may be taken to be

$$\begin{array}{l} -\dfrac{b}{a} + \ \ B + \ \ C + \ \ D + \ \ E \\ - \ \ \

have the above-mentioned values, omitting therefrom the terms $-\frac{b}{a}$; we find without difficulty

$$2\frac{c'}{a'} = -BE - CD,$$

$$2\frac{d'}{a'} = -B^2D - BC^2 - CE^2 - D^2E,$$

$$\frac{e'}{a'} = -B^3C - B^2E^2 + BCDE + BD^3 + C^3E + C^2D^2 - DE^2,$$

$$\frac{f'}{a'} = -B^5 + 5B^3DE - 5B^2C^2E - 5B^2CD^2 + 5BC^3D + 5BCE^3$$
$$- 5BD^2E^2 - C^5 + 5CD^3E - 5CD^2E^2 - D^5 - E^5,$$

and hence, when $E = 0$, we have

$$2\frac{c'}{a'} = -CD,$$

$$2\frac{d'}{a'} = -B^2D - BC^2,$$

$$\frac{e'}{a'} = -B^3C - BD^3 - C^2D^2,$$

$$\frac{f'}{a'} = -B^5 - 5B^2CD^2 + 5BC^3D - C^5 - D^5,$$

or, as these may be written,

$$-2\frac{c'}{a'} = CD,$$

$$-2\frac{d'}{a'} = B^2D + BC^2,$$

$$-\frac{e'}{a'} - 4\frac{c'^2}{a'^2} = B^3C - BD^3,$$

$$-\frac{f'}{a'} = B^5 + C^5 + D^5 - 10\frac{c'}{a'}(B^2D - BC^2),$$

equations which imply a single relation between the coefficients a', c', d', e', f'. Supposing this satisfied, we may attend only to the first three equations; or, writing for convenience,

$$\gamma = -2\frac{c'}{a'}, \quad = -\frac{2}{a^2}(ac - b^2),$$

$$\delta = -2\frac{d'}{a'}, \quad = -\frac{2}{a^3}(a^2d - 3abc + 2b^3),$$

$$\theta = -\frac{e'}{a'} - 4\frac{c'^2}{a'^2}, \quad = -\frac{1}{a^4}\{a^2(ae - 4bd + 3c^2) + (ac - b^2)^2\},$$

the equations are

$$\gamma = CD,$$
$$\delta = B(BD + C^2),$$
$$\theta = B(D^3 - B^2C).$$

The first equation gives $C=\frac{\gamma}{D}$, and substituting this value in the other two equations, we have

$$B^2D^3+B\gamma^4-\delta D^2=0,$$
$$B^3\gamma+BD^4+\theta D=0.$$

Eliminating B, the result is obtained in the form Det. $=0$. where in the last column of the determinant each term is divisible by D; and omitting this factor, the result is

$$\begin{vmatrix} & & D^3, & \gamma^2, & -\delta D \\ & D^3, & \gamma^2, & -\delta D^2, & \\ D^3, & \gamma^2, & -\delta D^2, & & \\ & \gamma, & 0, & -D^4, & \theta \\ \gamma, & 0, & -D^4, & \theta D, & \end{vmatrix}=0.$$

If, in order to develope the determinant, we consider it as a sum of products, each first factor being a minor composed out of columns 1 and 2, and the second factor being the complementary minor composed out of columns 3, 4, 5 (the several products being of course taken each with its proper sign), the expansion presents itself in the form

$$\begin{aligned} & D^3\gamma(-\theta\delta\gamma^2D^2+\delta^2D^7), \\ & -D^6(-\theta\gamma^2D^4+\delta D^9-\theta^2D^4) \\ & -\gamma D^3.-\delta D^2(\delta D^5-\theta\gamma^2) \\ & +\gamma^2(\gamma^2\delta D^5-\theta\delta D^5-\theta\gamma^4) \\ & -\gamma^2\delta^3D^5. \end{aligned}$$

Hence, collecting, and changing the sign of the whole expression, we obtain

$$\delta D^{15}-(2\gamma\delta^2+\gamma^2\theta+\theta^2)D^{10}+(-\gamma^3\delta+3\gamma\delta\theta+\delta^3)\gamma^2D^5+\gamma^7\theta=0,$$

a cubic equation for D^5. We have then as above $C=\frac{\gamma}{D}$, and B is given rationally as the common root of the foregoing quadric and cubic equations satisfied by B.

Substituting for γ, δ, θ their values in terms of the original coefficients, the equation for D^5 becomes

$$\begin{aligned} & 2(a^2d-3abc+2b^3)(aD)^{15} \\ & +\left\{\begin{array}{l} a^4(ae-4bd+3c^2)^2 \\ +a^2(ac-b^2)^2(ae-4bd+3c^2) \\ -16(ac-b^2)(a^2d-3abc+2b^3)^2 \end{array}\right\}(aD)^{10} \\ & +4(ac-b^2)^2\left\{\begin{array}{ll} 28 & (ac-b^2)^3(a^2d-3abc+2b^3) \\ +12a^2 & (ac-b^2)(a^2d-3abc+2b^3)(ae-4bd+3c^2) \\ +8 & (a^2d-3abc+2b^3)^2 \end{array}\right\}(aD)^5 \\ & -128(ac-b^2)^7\{a^2(ae-4bd+3c^2)+(ac^2-b^2)^2\}=0, \end{aligned}$$

and the solution of the given quintic equation thus ultimately depends upon that of this cubic equation.

778.

[ADDITION TO MR HUDSON'S PAPER "ON EQUAL ROOTS OF EQUATIONS."]

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XVIII. (1882), pp. 226—229.]

It seems desirable to present in a more developed form some of the results of the foregoing paper.

Thus, if the equation $(a_0, a_1, \ldots, a_n \between x, 1)^n = 0$ of the order n has $n-v$ equal roots, where v is not $> \frac{1}{2}n - 1$, then we have $\psi(r, v+1, m) = 0$, where m has any one of the values $0, 1, \ldots, n-2v-2$, and r any one of the values

$$2v+2,\ 2v+3,\ \ldots,\ n-m.$$

The signification is

$$\begin{array}{llllll}
\psi(r, v+1, m) = & r & . \quad 1 & . \dfrac{1}{[r]^{v+2}} & a_m & a_{r+m} \\
& -(r-2) & . \quad \dfrac{v+1}{1} & . \dfrac{1}{[r-1]^{v+2}} & a_{m+1} & a_{r+m-1} \\
& +(r-4) & . \dfrac{v+1\,.\,v+2}{1\,.\,2} & . \dfrac{1}{[r-2]^{v+2}} & a_{m+2} & a_{r+m-2} \\
& \vdots & & & & \\
& +(-)^s(r-2s) & . \quad \dfrac{[v+1]^s}{[s]^s} & . \dfrac{1}{[r-s]^{v+2}} & a_{m+s} & a_{r+m-s} \\
& \vdots & & & & \\
& +(-)^{v+1}(r-2v-2) & & . \dfrac{1}{[r-v-1]^{v+2}} & a_{m+v+1} & a_{r+m-v-1}.
\end{array}$$

Thus, when $v = 0$, the condition is

$$\left.\begin{array}{lll}
r & \dfrac{1}{r\,.\,r-1} & a_m \quad a_{r+m} \\
-(r-2) & \dfrac{1}{r-1\,.\,r-2} & a_{m+1}\, a_{r+m-1}
\end{array}\right\} = 0,$$

that is,

$$a_m a_{r+m} - a_{m+1} a_{r+m-1} = 0,$$

satisfied when the equation has all its roots equal.

The values of m are $0, 1, 2, \ldots, n-2$, and those of r are $2v+2, 2v+3, \ldots, n-m$; in particular, if $m=0$, the values of r are $2, 3, \ldots, n$, and the corresponding conditions are

$$\begin{aligned} a_0 a_2 - a_1^2 &= 0, \\ a_0 a_3 - a_1 a_2 &= 0, \\ &\vdots \\ a_0 a_n - a_1 a_{n-1} &= 0, \end{aligned}$$

and so for the different values of m up to the final value $n-2$, for which $r=2$, and the condition is

$$a_{n-2} a_n - a^2_{n-1} = 0;$$

we have thus, it is clear, the whole series of conditions included in

$$\left\| \begin{matrix} a_0, & a_1, & a_2, & \ldots, & a_{n-2}, & a_{n-1} \\ a_1, & a_2, & a_3, & \ldots, & a_{n-1}, & a_n \end{matrix} \right\| = 0,$$

which are obviously satisfied in the case in question of the roots being all equal.

Again, when $v=1$, the condition for $n-1$ equal roots is

$$\left.\begin{aligned} & r \quad .1 . \frac{1}{r.r-1.r-2} \quad a_m \quad a_{r+m} \\ & -(r-2).2 . \frac{1}{r-1.r-2.r-3} a_{m+1} a_{r+m-1} \\ & +(r-4).1 . \frac{1}{r-2.r-3.r-4} a_{m+2} a_{r+m-2} \end{aligned}\right\} = 0,$$

that is,

$$\frac{a_m a_{r+m}}{r-1.r-2} - \frac{2a_{m+1} a_{r+m-1}}{r-1.r-3} + \frac{a_{m+2} a_{r+m-2}}{r-2.r-3} = 0;$$

or, what is the same thing,

$$(r-3) a_m a_{r+m} - 2(r-2) a_{m+1} a_{r+m-1} + (r-1) a_{m+2} a_{r+m-2} = 0,$$

where $n=4$ at least, and m, r have the values

$m=$	$0, 1, 2, \ldots, n-4$
$r=$	$4, 4,$ $\quad 4$
	$5, 5$
	$\vdots \quad \vdots$
	$\vdots \quad n-1$
	n

thus, when $n=4$, the only values are $m=0$, $r=4$, and the condition is

$$a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 0.$$

Similarly, when $v=2$, the condition for $n-2$ equal roots is found to be

$$\frac{a_m a_{r+m}}{r-1\,.\,r-2\,.\,r-3}-\frac{3a_{m+1}a_{r+m-1}}{r-1\,.\,r-3\,.\,r-4}+\frac{3a_{m+2}a_{r+m-2}}{r-2\,.\,r-3\,.\,r-5}-\frac{a_{m+3}a_{r+m-3}}{r-3\,.\,r-4\,.\,r-5}=0;$$

or, what is the same thing,

$$\begin{aligned} & r-4\,.\,r-5\,.\,a_m\ \ a_{r+m} \\ & -3\,.\,r-2\,.\,r-5\,.\,a_{m+1}a_{r+m-1} \\ & +3\,.\,r-1\,.\,r-4\,.\,a_{m+2}a_{r+m-2} \\ & -\ \ .\,r-1\,.\,r-2\,.\,a_{m+3}a_{r+m-3}=0, \end{aligned}$$

where $n=6$ at least, and m, r have the values

$m=$	$0,\ 1,\ \ldots,\ n-6$
$r=$	$6,\ 6,\qquad 6$
	$7,\ 7$
	$\vdots\ \ \vdots$
	$\vdots\ \ n-1$
	n

Observe that the sum of the coefficients is $=0$, viz.

$$(r-4)(r-5)-3(r-2)(r-5)+3(r-1)(r-4)-(r-1)(r-2)=0,$$

this should obviously be the case, since the conditions for $n-2$ equal roots must be satisfied when the roots are all of them equal; and the property serves as a verification.

It is to be remarked that the equation $\psi(r,\ v+1,\ m)=0$ does not in all cases give all the conditions for the existence of $n-v$ equal roots in an equation of the order n; thus when $n=3$ and $v=1$, we cannot by means of it obtain the condition that a cubic equation may have 2 equal roots. The problem really considered is that of the determination of those *quadric* functions of the coefficients which vanish in the case of $n-v$ equal roots; and in the case in question ($n=3$, $v=1$) there is no quadric function which vanishes, but the condition depends on a cubic function.

The question of the quadric functions which vanish in the case of $n-v$ equal roots, and to a small extent that of the *cubic* functions which thus vanish, is considered in Dr Salmon's "Note on the conditions that an equation may have equal roots," *Camb. and Dublin Math. Jour.*, t. v. (1850), pp. 159—165, and in particular the equation there obtained p. 161 is the equation $\psi(0,\ v+1,\ n)=0$.

779.

[NOTE ON MR JEFFERY'S PAPER "ON CERTAIN QUARTIC CURVES WHICH HAVE A CUSP AT INFINITY, WHEREAT THE LINE AT INFINITY IS A TANGENT."]

[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), p. 85.]

THE assumed form $\kappa\alpha^3\beta = u_2$, or, as this is afterwards written,

$$2\kappa x^2 y = ax^2 + 2bxy + cy^2 + 2ex + 2dy + \lambda,$$

is, I think, introduced without a proper explanation. Say, the form is $x^2y = z^2(*\between x, y, z)^2$, it ought to be shown how for a cuspidal quartic we arrive at this form; viz. taking the cusp to be at the point $(x=0,\ z=0)$, $z=0$ for the tangent at the cusp, and $x=0$ an arbitrary line through the cusp; then the line $z=0$ besides intersects the curve in a single point, and, if $y=0$ is taken as the tangent at that point, the equation of the curve must, it can be seen, be of the form

$$(x^3 + \theta x^2 z)\, y = z^2 (a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2.$$

The conic $(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 0$ touches the quartic at each of the two intersections of the quartic with the arbitrary line $x=0$; and we cannot, so long as the line remains arbitrary, find a conic which shall osculate the quartic at the two points in question; but, for the particular line $x + \frac{1}{3}\theta z = 0$, there exists such a conic, viz. writing x instead of $x + \frac{1}{3}\theta z$, the form is $x^2y = z^2(a',\ b',\ c',\ f',\ g',\ h' \between x,\ y,\ z)^2$, and the new conic $(a', \ldots \between x,\ y,\ z)^2 = 0$ has the property in question. This is the adopted form, and it thus appears that in it the line $x=0$ is a determinate line, viz. the line passing through the cusp and the two points of osculation of the osculating conic. It thus appears that in the assumed form the lines $x=0$, $y=0$, $z=0$ are determinate lines.

780.

[ADDITION TO MR HAMMOND'S PAPER "NOTE ON AN EXCEPTIONAL CASE IN WHICH THE FUNDAMENTAL POSTULATE OF PROFESSOR SYLVESTER'S THEORY OF TAMISAGE FAILS."]

[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), pp. 88—91. Read Dec. 14, 1882.]

THE extreme importance of Mr Hammond's result, as regards the entire subject of Covariants, leads me to reproduce his investigation in the notation of my Memoirs on Quantics, and with a somewhat different arrangement of the formulæ. For the binary seventhic

$$(a, b, c, d, e, f, g, h \between x, y)^7,$$

the four composite seminvariants of the deg-weight 5 . 11 (sources of covariants of the deg-order 5 . 13) are

I.			II.		
1 . 7	4 . 6	... ...	2 . 10	3 . 3	Deg-order.
1 . 0	4 . 11	... ...	2 . 2	3 . 9	Deg-weight.
$a+1$	$a^2eh + 1$		$ac+1$	$ach + 2$	
	$fg - 1$		b^2-1	$dg - 7$	
	$abdh - 4$			$ef + 5$	
	$beg - 2$			$a^0b^2h - 2$	
	$bf^2 + 6$			$beg + 7$	
	$c^2h + 3$			$bdf + 22$	
	$cdg - 2$			$be^2 - 25$	
	$cef - 6$			$c^2f - 27$	
	$d^2f + 10$			$cde + 45$	
	$de^2 - 5$			$d^3 - 20$	
	$a^0b^2ch \quad 0$				
	$b^2dg + 20$				
	$b^2ef + 57$				
	$bc^2g - 15$				
	$bcdf - 24$				
	$bce^2 - 30$				
	$bd^2e - 10$				
	$c^3f + 27$				
	$c^2de - 45$				
	$cd^3 + 20$				

III.			IV.		
2 . 6	3 . 7	... ...	2 . 2	3 . 11	Deg-order.
2 . 4	3 . 7	... ...	2 . 6	3 . 5	Deg-weight.
$ae+1$	a^2h+1		$ag+1$	a^2f+1	
$bd-4$	$abg-7$		$bf-6$	$abe-5$	
c^2+3	$cf+9$		$ce+15$	$acd+2$	
	$de-5$		d^2-10	a^0b^2d+8	
	a^0b^2f+12			bc^2-6	
	$bce-30$				
	bd^2+20				

and it is here at once obvious that there exists a syzygy of the form I. = III. − IV.; in fact, if in III. and IV. we write $a=0$, then the values are each

$$= -2b\,(4bd-3c^2)\,(6bf-15ce+10d^2);$$

hence III. − IV. must divide by a, the quotient being a seminvariant of the deg-weight 4 . 11, which can only be a numerical multiple of the second factor of I., and is in fact = this second factor, that is, we have the syzygy I. = III. − IV.

Working out the values of the four products, and joining to them the expression for the irreducible seminvariant of the same deg-weight 5 . 11 (O, x^8 of my tables [774] for the binary sextic), we have the table:

5 . 10
a^3dh
eg
f
a^2bch
bdg
bef
c^2g
cdf
ce^2
d^2e
ab^3h
b^2cg
b^2df
b^2e^2
bc^2f
$bcde$
bd^3
c^3e
c^2d^2
a^0b^4g
b^3ef
b^2de
b^2c^2e
b^2cd^2
bc^3d
c^5

5 . 11	O	I.	III.	IV.	II.
a^3eh		+ 1	+ 1		
fg		− 1		+ 1	
a^2bdh		− 4	− 4		
beg		− 2	− 7	− 5	
bf^2		+ 6		− 6	
c^2h		+ 3	+ 3		+ 2
cdg		− 2		+ 2	− 7
cef	− 1	− 6	+ 9	+ 15	+ 5
d^2f	+ 3	+10		− 10	
de^2	− 2	− 5	− 5		
ab^3ch					− 4
b^2dg		+20	+ 28	+ 8	+ 7
b^2ef	+ 1	+57	+ 12	− 45	− 5
bc^2g		−15	− 21	− 6	+ 7
$bcdf$	−14	−24	− 36	− 12	+22
bce^2	+11	−30	− 30		−25
bd^2e	+ 1	−10	+ 40	+ 50	
c^3f	+ 9	+27	+ 27		−27
c^2de	−14	−45	− 15	+ 30	+45
cd^3	+ 6	+20		− 20	−20
a^0b^4h					− 2
b^3cg					− 7
b^3df	+ 8		− 48	− 48	−22
b^2e^2	− 9				+25
b^2c^2f	− 6		+ 36	+ 36	+27
b^2cde	+16		+120	+120	−45
b^2d^3	− 8		− 80	− 80	+20
bc^3e	− 3		− 90	− 90	
bc^2d^2	+ 2		+ 60	+ 60	
c^4d					

I have prefixed to the table the literal terms of the deg-weight 5 . 10; for the deg-weights 5 . 11 and 5 . 10, the numbers of terms are = 30 and 26 respectively; and it is the difference of these 30 − 26, = 4, which gives the number of asyzygetic seminvariants of the deg-weight 5 . 11.

781.

ON THE AUTOMORPHIC TRANSFORMATION OF THE BINARY CUBIC FUNCTION.

[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), pp. 103—108. Read Jan. 11, 1883.]

I CONSIDER the cubic equation $(a, b, c, d \between x, 1)^3 = 0$. It is shown (Serret, *Cours d'Algèbre supérieure*, 4th ed., Paris, 1879, t. II. pp. 466—471) how, given one root of the equation, the other two roots can be each of them expressed rationally in terms of this root and of the square root of the discriminant; viz. making the proper changes of notation, and writing

$$A,\ B,\ C = ac - b^2,\ ad - bc,\ bd - c^2,\ \lambda = \sqrt{-\tfrac{1}{3}},$$

$$\Omega = B^2 - 4AC,\ = a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd,$$

$$\alpha = \frac{-\lambda\sqrt{\Omega} + B}{2\lambda\sqrt{\Omega}},\ \beta = \frac{+2C}{2\lambda\sqrt{\Omega}},\ \gamma = \frac{-2A}{2\lambda\sqrt{\Omega}},\ \delta = \frac{-\lambda\sqrt{\Omega} - B}{2\lambda\sqrt{\Omega}},$$

(values which give $\alpha + \delta = -1$, $\alpha\delta - \beta\gamma = -1$, and therefore also

$$\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma = 0,$$

which is the condition in order that the function $\phi x, = \dfrac{\alpha x + \beta}{\gamma x + \delta}$, may be periodic of the third order, $\phi^3 x = x$), then, u being a root of the equation, say $(a, b, c, d \between u, 1)^3 = 0$, the other two roots are

$$\phi u, = \frac{\alpha u + \beta}{\gamma u + \delta},$$

and

$$\phi^2 u, = \phi^{-1} u, = \frac{(\alpha^2 + \beta\gamma)\, u + \beta(\alpha + \delta)}{\gamma(\alpha + \delta)\, u + \delta^2 + \beta\gamma},\ = \frac{\delta u - \beta}{-\gamma u + \alpha},$$

where observe that, by the change of $\sqrt{\Omega}$ into $-\sqrt{\Omega}$, α, β, γ, δ become δ, $-\beta$, $-\gamma$, α; so that the last-mentioned value $\phi^{-1} u$ is, in fact, the value obtained from ϕu by the mere change of sign of the radical.

It is to be observed that, if we have between two roots u, v of the equation

$$(a,\ b,\ c,\ d \between x,\ 1)^3 = 0,$$

a relation $v = \dfrac{\alpha u + \beta}{\gamma u + \delta}$, where α, β, γ, δ have given values, this implies in the first place a relation between a, b, c, d (and the given values of α, β, γ, δ), and it implies moreover that u, and consequently also v, are not any roots whatever, but two determinate roots of the equation; viz. u, v will be each of them expressible rationally in terms of a, b, c, d and α, β, γ, δ. And if, in order that (a, b, c, d) may remain arbitrary, we consider α, β, γ, δ as given quantities satisfying the relation which exists between these quantities and (a, b, c, d), then in general we still have u, v determinate roots of the cubic equation. But in the foregoing solution u is any root whatever of the cubic equation.

To examine how this is, starting from the equations

$$(a,\ b,\ c,\ d \between u,\ 1)^3 = 0,\quad (a,\ b,\ c,\ d \between v,\ 1)^3 = 0,\quad v = \frac{\alpha u + \beta}{\gamma u + \delta},$$

we have

$$a\,(u^3 - v^3) + 3b\,(u^2 - v^2) + 3c\,(u - v) = 0,$$

and therefore

$$a\,(u^2 + uv + v^2) + 3b\,(u + v) + 3c = 0,$$

that is,

$$av^2 + (au + 3b)\,v + au^2 + 3bu + 3c = 0\,;$$

or, writing herein for v its value,

$$a\,(\alpha u + \beta)^2 + (\alpha u + \beta)\,(\gamma u + \delta)\,(au + 3b) + (\gamma u + \delta)^2\,(au^2 + 3bu + 3) = 0\,;$$

that is,

$$\begin{aligned} a\,(\alpha u + \beta)^2 + \quad & \alpha\gamma\ \ (au^3 + 3bu^2) + \ \gamma^2\,(au^4 + 3bu^3 + 3cu^2) \\ & + (\alpha\delta + \beta\gamma)\,(au^2 + 3bu) + 2\gamma\delta\,(au^3 + 3bu^2 + 3cu) \\ & + \quad \beta\delta\ \ (au + 3b) \ + \ \delta^2\,(au^2 + 3bu + c) = 0\,; \end{aligned}$$

or, reducing by the equation $au^3 + 3bu^2 + 3cu + d = 0$, this is

$$\begin{aligned} a\,(\alpha u + \beta)^2 + \quad & \alpha\gamma\ \ (-3cu - d) + \ \gamma^2\,(-du) \\ & + (\alpha\delta + \beta\gamma)\,(au^2 + 3bu) + 2\gamma\delta\,(-d) \\ & + \quad \beta\delta\ \ (au + 3b) \ + \ \delta^2\,(au^2 + 3bu + 3c) = 0, \end{aligned}$$

and, collecting the terms, this is

$$\begin{aligned} & u^2 a\,(\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma) \\ & + u\,\{a\,(2\alpha\beta + \beta\delta) + 3b\,(\alpha\delta + \delta^2 + \beta\gamma) - 3c\alpha\gamma - d\gamma^2\} \\ & + \quad a\beta^2 + 3b\beta\delta + 3c\delta^2 + d\,(-\alpha\gamma - 2\gamma\delta) = 0. \end{aligned}$$

We can, from this equation, and the equation $au^3 + 3bu^2 + 3cu + d = 0$, eliminate u, thus obtaining a relation between a, b, c, d, α, β, γ, δ; and, this relation being satisfied, the two equations then determine u rationally in terms of these quantities.

We may without loss of generality assume $\alpha\delta - \beta\gamma = 1$; and, this being so, if we then further assume $\alpha + \delta = -1$, then we have

$$\alpha^2 + \alpha\delta + \delta^2 + \beta\gamma = 0,$$

which is, as above appearing, the condition for $\phi^2 x = 0$. The foregoing equation in u thus becomes

$$\begin{aligned} & u\,\{a\beta(\alpha - 1) - 3b\alpha^2 - 3c\alpha\gamma - d\gamma^2\} \\ & + \{a\beta^2 + 3b\beta\delta + 3c\delta^2 - d\gamma(\delta - 1)\} = 0; \end{aligned}$$

a linear equation giving (in a simplified form) the like results to those given by the quadric equation; viz. substituting in the cubic equation the value of u given by the linear equation, we have a relation between $a, b, c, d, \alpha, \beta, \gamma, \delta$; and, this relation being satisfied, u has the determinate value given by the linear equation.

The only way in which u can cease to have this determinate value, and so be capable of being any root whatever of the cubic equation, is when the linear equation becomes $0 = 0$; viz. if

$$\begin{aligned} & a\beta(\alpha - 1) - 3b\alpha^2 - 3c\alpha\gamma - d\gamma^2 \qquad = 0, \\ & a\beta^2 \qquad + 3b\beta\delta + 3c\delta^2 - d\gamma(\delta - 1) = 0, \end{aligned}$$

equations which are, in fact, satisfied by the foregoing values of $\alpha, \beta, \gamma, \delta$, as may be verified without difficulty.

It is to be remarked that if, instead of the root u and the equation $v = \dfrac{\alpha u + \beta}{\gamma u + \delta}$, we consider the root v and the equation $u = \dfrac{\delta v - \beta}{-\gamma v + \alpha}$; then, instead of $\alpha, \beta, \gamma, \delta$, we have $\delta, -\beta, -\gamma, \alpha$, and the corresponding equations are

$$\begin{aligned} & a\beta(\delta - 1) + 3b\delta^2 + 3c\gamma\delta + d\gamma^2 \qquad = 0, \\ & a\beta^2 \qquad + 3b\alpha\beta + 3c\alpha^2 + d\gamma(\alpha - 1) = 0, \end{aligned}$$

equations which are also satisfied by the foregoing values of $\alpha, \beta, \gamma, \delta$. And the four equations, together with $\alpha\delta - \beta\gamma = 1$ and $\alpha + \delta = 1$, are more than sufficient to determine the foregoing values of $\alpha, \beta, \gamma, \delta$.

But we further verify without difficulty that the foregoing values of $\alpha, \beta, \gamma, \delta$ give identically

$$(a, b, c, d \between \alpha x + \beta y, \gamma x + \delta y)^3 = (a, b, c, d \between x, y)^3;$$

or the formulæ lead to an automorphic transformation of the binary cubic $(a, b, c, d \between x, y)^3$. And conversely, starting from the notion of the automorphic transformation of the binary cubic, we ought to be able to obtain the foregoing formulæ.

For greater convenience, I write the equation of transformation in the form

$$(a, b, c, d \between \alpha x + \beta y, \gamma x + \delta y)^3 = -\theta\,(a, b, c, d \between x, y)^3;$$

the equations to be satisfied by α, β, γ, δ, θ then are

$$\begin{aligned}
&a\alpha^3 + b\,.\,3\alpha^2\gamma + c\,.\,3\alpha\gamma^2 + d\gamma^3 = -a\theta,\\
&a\alpha^2\beta + b(\alpha^2\delta + 2\alpha\beta\gamma) + c(2\alpha\gamma\delta + \beta\gamma^2) + d\gamma^2\delta = -b\theta,\\
&a\alpha\beta^2 + b(2\alpha\beta\delta + \beta^2\gamma) + c(\alpha\delta^2 + 2\beta\gamma\delta) + d\gamma\delta^2 = -c\theta,\\
&a\beta^3 + b\,.\,3\beta^2\delta + c\,.\,3\beta\delta^2 + d\delta^3 = -d\theta.
\end{aligned}$$

Writing $\alpha\delta - \beta\gamma = \nabla$, and as before Ω for the discriminant, the theory of invariants gives $\Omega\nabla^6 = \Omega\theta^4$. We are considering the case of the general cubic function $(a, b, c, d \between x, y)^3$, for which Ω is not $= 0$; and we have therefore $\nabla^6 - \theta^4 = 0$, or, what is the same thing, we may write $\nabla = q^2$, $\theta = q^3$, where q is arbitrary.

It is to be shown that $\alpha + \delta$ is $= q$ or $-2q$, the latter value giving the trivial solution $\alpha x + \beta y$, $\gamma x + \delta y = (x, y)$. The proper solution thus corresponds to $\nabla = q^2$, $\alpha + \delta = q$, that is,

$$(\alpha + \delta)^2 - (\alpha\delta - \beta\gamma) = 0, \text{ or } \alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0,$$

the condition for the periodic function $\phi^3 x - x = 0$.

For this purpose, from the foregoing equations eliminating a, b, c, d, we have

$$\begin{vmatrix}
\alpha^3 + \theta, & 3\alpha^2\gamma, & 3\alpha\gamma^2, & \gamma^3\\
\alpha^2\beta, & \alpha^2\delta + 2\alpha\beta\gamma + \theta, & 2\alpha\gamma\delta + \beta\gamma^2, & \gamma^2\delta\\
\alpha\beta^2, & 2\alpha\beta\delta + \beta^2\gamma, & \alpha\delta^2 + 2\beta\gamma\delta + \theta, & \gamma\delta^2\\
\beta^3, & 3\beta^2\delta, & 3\beta\delta^2, & \delta^3 + \theta
\end{vmatrix} = 0;$$

an equation which may be written

$$\square + \theta(123 + 234 + 341 + 412) + \theta^2(12 + 23 + 34 + 41 + 13 + 42) + \theta^3(1 + 2 + 3 + 4) + \theta^4 = 0,$$

where 123, &c., are the first diagonal minors, 12, &c., the second diagonal minors, 1, &c., the third diagonal minors, or diagonal terms of the foregoing determinant, writing therein $\theta = 0$. We find without difficulty

$$\begin{aligned}
1,\ 2,\ 3,\ 4 &= \alpha^3,\ \alpha^2\delta + 2\alpha\beta\gamma,\ \alpha\delta^2 + 2\beta\gamma\delta,\ \delta^3,\\
12,\ 13,\ 14,\ 23,\ 24,\ 34 &= \{\alpha^4,\ \alpha^2(\alpha\delta + 3\beta\gamma),\ (\alpha^2\delta^2 + \alpha\delta\beta\gamma + \beta^2\gamma^2),\\
&\qquad (\alpha^2\delta^2 + \alpha\delta\beta\gamma + \beta^2\gamma^2),\ \delta^2(\alpha\delta + 3\beta\gamma),\ \delta^4\}\,\nabla,\\
123,\ 124,\ 134,\ 234 &= \{\alpha^3,\ \alpha(\alpha\delta + 2\beta\gamma),\ \delta(\alpha\delta + 2\beta\gamma),\ \delta^3\}\,\nabla^3,\\
\square &= \nabla^6,
\end{aligned}$$

and the equation thus is

$$\begin{aligned}
&\theta^4\\
&+ \theta^3[\alpha^3 + \alpha^2\delta + \alpha\delta^2 + \delta^3 + 2\alpha\beta\gamma + 2\beta\gamma\delta]\\
&+ \theta^2\,\nabla\,[\alpha^4 + \alpha^3\delta + \alpha\delta^3 + \delta^4 + 3\alpha^2\beta\gamma + 3\beta\gamma\delta^2 + 2\alpha^2\delta^2 + 2\alpha\delta\beta\gamma + 2\beta^2\gamma^2]\\
&+ \theta\,\nabla^3[\alpha^3 + \delta^3 + (\alpha + \delta)(\alpha\delta + 2\beta\gamma)]\\
&+ \nabla^6 = 0.
\end{aligned}$$

Putting herein $\alpha+\delta=m$, $\alpha\delta=n$, $\beta\gamma=n-\nabla$, it is found that n disappears altogether from the equation; viz. the resulting form is

$$\theta^4+\theta^3 m(m^2-2\nabla)+\theta^2\nabla(m^4-3\nabla m^2+2\nabla^2)+\theta\nabla^3 m(m^2-2\nabla)+\nabla^6=0,$$

or, what is the same thing,

$$m^4 . \theta^2\nabla+m^3(\theta^3+\theta\nabla^3)-3\theta^2\nabla^2 m^2-2(\theta\nabla^4+\theta^3\nabla)m+\theta^4+2\theta^2\nabla^3+\nabla^6=0.$$

Putting for θ, ∇, their values q^3, q^2, the equation divides by q^6, and omitting this factor it becomes

$$m^4+2m^3q-3m^2q^2-4mq^3+4q^4=0;$$

viz. this is

$$\{(m-q)(m+2q)\}^2=0,$$

or we have

$$m=q \text{ or } -2q; \text{ that is, } \alpha+\delta=q, \text{ or } \alpha+\delta=-2q.$$

Writing, as before, A, B, $C=ac-b^2$, $ad-bc$, $bd-c^2$, we deduce from the foregoing equations

$$\begin{aligned}\theta^2 A &= \nabla^2[A\alpha^2 \quad +B . \alpha\gamma \quad +C\gamma^2],\\ \theta^2 B &= \nabla^2[A . 2\alpha\beta+B(\alpha\delta+\beta\gamma)+C . 2\gamma\delta],\\ \theta^2 C &= \nabla^2[A\beta^2 \quad +B . \beta\delta \quad +C\delta^2],\end{aligned}$$

which are, in fact, the equations for the automorphic transformation of the Hessian $(A, B, C \between x, y)^2$. And, writing herein θ, $\nabla=q^3$, q^2, the equations become

$$\begin{aligned}A(\alpha^2-q^2)+B\alpha\gamma \quad &+C . \gamma^2 &&=0,\\ A\,2\alpha\beta \quad +B(\alpha\delta+\beta\gamma-q^2)&+C . 2\gamma\delta &&=0,\\ A\beta^2 \quad +B\beta\delta \quad &+C(\delta^2-q^2) &&=0.\end{aligned}$$

From the first and second of these we have

$$A : B : C=\gamma^2(\nabla+q^2) : -2\gamma\alpha\nabla+2\gamma\delta q^3 : \alpha^2\nabla-(\alpha^2+\alpha\delta+\beta\gamma)q^2+q^4;$$

or, writing herein for ∇, q^4 the values q^2, $q^2(\alpha\delta-\beta\gamma)$, the three expressions divide by $2\gamma q^2$, and we have

$$A : B : C=\gamma : \delta-\alpha : -\beta.$$

Combining these values in the first place with the equation $\alpha+\delta=-2q$, we may write

$$\alpha,\ \beta,\ \gamma,\ \delta=-q-pB,\ -2pC,\ 2pA,\ -q+pB,$$

where p is to be determined. Substituting in the last of the three equations, we have

$$A . 4p^2C^2-2pBC(-q+pB)+C(-2pqB+p^2B^2)=0,$$

that is,

$$p^2C(4AC-B^2),\ =-p^2 . C\Omega,\ =0,$$

and the form $(a, b, c, d \between x, y)^2$ being arbitrary, neither C nor Ω is $=0$; whence $p=0$, and the values are α, β, γ, $\delta=-q$, 0, 0, $-q$, that is,

$$(a, b, c, d \between -qx, -qy)^2=-q^2 . (a, b, c, d \between x, y)^2,$$

a trivial result.

But, combining the same values with $\alpha+\delta=q$, we have

$$\alpha,\ \beta,\ \gamma,\ \delta=\tfrac{1}{2}q-pB,\ -2pC,\ 2pA,\ \tfrac{1}{2}q+pB;$$

and then, substituting in the third equation, we have

$$A\ .\ 4p^2C^2-2pBC\left(\tfrac{1}{2}q+pB\right)+C\left(-\tfrac{3}{4}q^2+pqB+p^2B^2\right)=0,$$

that is,

$$C\left\{(4AC-B^2)\,p^2-\tfrac{3}{4}q^2\right\}=0,$$

or, omitting the factor C, and introducing the foregoing notation $\lambda^2=-\frac{1}{3}$, this is

$$4\Omega\lambda^2p^2-q^2=0,\ \text{or say}\ p=\frac{1}{2\lambda\sqrt{\Omega}}\,q.$$

For the unimodular substitution $\alpha\delta-\beta\gamma=1$, we must have $q^2=1$: but, the transformation being

$$(a,\ b,\ c,\ d\between\alpha x+\beta y,\ \gamma x+\delta y)^3=-q^2\,.\,(a,\ b,\ c,\ d\between x,\ y)^3,$$

to make this strictly automorphic, we must have $q^3=-1$, and the two conditions are satisfied by $q=-1$; we then have $p=-\dfrac{1}{2\lambda\sqrt{\Omega}}$; and the coefficients are

$$\alpha,\ \beta,\ \gamma,\ \delta,\ =\frac{-\lambda\sqrt{\Omega}+B}{2\lambda\sqrt{\Omega}},\ \frac{2C}{2\lambda\sqrt{\Omega}},\ \frac{-2A}{2\lambda\sqrt{\Omega}},\ \frac{-\lambda\sqrt{\Omega}-B}{2\lambda\sqrt{\Omega}};$$

which are the values given at the beginning of the paper, and which belong to the automorphic transformation

$$(a,\ b,\ c,\ d\between\alpha x+\beta y,\ \gamma x+\delta y)^3=(a,\ b,\ c,\ d\between x,\ y)^3.$$

The *à priori* reason for the periodicity-equation $\phi^3x=x$, is best seen by expressing the cubic function as a product of factors

$$M\,(x-\mathrm{a}y)\,(x-\mathrm{b}y)\,(x-\mathrm{c}y).$$

The substitution must, it is clear, cyclically interchange these factors, and therefore, when performed three times in succession on any one of these factors, or consequently upon an arbitrary linear factor $x-\mathrm{f}y$, must leave the factor unaltered, and it must thus be a periodic substitution $\phi^3x=x$. But it was interesting to see how the condition for this, $\alpha^2+\delta^2+\alpha\delta+\beta\gamma=0$, comes out from the consideration of the equation

$$(a,\ b,\ c,\ d\between\alpha x+\beta y,\ \gamma x+\delta y)^3=(a,\ b,\ c,\ d\between x,\ y)^3.$$

782.

ON MONGE'S "MÉMOIRE SUR LA THÉORIE DES DÉBLAIS ET DES REMBLAIS."

[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), pp. 139—142. Read March 8, 1883.]

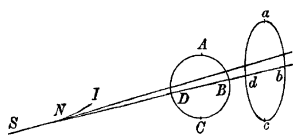
THE Memoir referred to, published in the *Mémoires de l'Académie*, 1781, pp. 666—704, is a very remarkable one, as well for the problem of earthwork there considered as because the author was led by it to his capital discovery of the curves of curvature of a surface. The problem is, from a given area, called technically the Déblai, to transport the earth to a given equal area, called the Remblai, with the least amount of carriage. Taking the earth to be of uniform infinitesimal thickness over the whole of each area (and therefore of the same thickness for both areas), the problem is a plane one; viz. stating it in a purely geometrical form, the problem is: Given two equal areas, to transfer the elements of the first area to the second area in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. And we have the corresponding solid problem: Given two equal volumes, to transfer the elements of the first volume to the second volume in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. The Memoir is divided into two parts: the first relating to the plane problem (and to some variations of it): the second part contains a theorem as to congruences, the general theory of the curvature of surfaces, and finally a solution of the solid problem; in regard to this, I find a difficulty which will be referred to further on.

I have said that Monge gives a theorem as to congruences. This is not stated quite in the best form,—viz. instead of speaking of a singly infinite system of lines, or even of the lines drawn according to a given law from the several points of a *surface*, he speaks of the lines drawn according to a given law from the several points

of a *plane* (but, of course, any congruence whatever of lines can be so represented); and he establishes the theorem that each line of the system is intersected by each of two consecutive lines,—viz. taking (x', y') as the coordinates of the point of intersection of any line with the plane of xy, he obtains, as the condition of intersection with the consecutive line a quadric equation in (dx', dy'). He then considers the normals of a surface, (which, as lines drawn according to a given law from any point of a *surface*, require a slightly different analytical investigation), establishes for them the like theorem, and shows moreover that the two directions of passage on the surface to a consecutive point are at right angles to each other; or, what is the same thing, that in the two sets of developable surfaces formed by the intersecting normals, each surface of the one set intersects each surface of the other set in a straight line, and at right angles. He speaks expressly of the lines of greatest and least curvature, and generally establishes the whole theory of the curvature of surfaces in a very complete and satisfactory manner; the particular case of surfaces of the second order is not considered. It may be remarked that, although not explicitly stating it, he must have seen that a congruence of lines is not, *in general*, a system of normals of a surface (that is, the lines of a congruence cannot be, in general, cut at right angles by any surface); he, in fact, assumes (quite correctly, but a proof should have been given) that a congruence of lines for which the two sets of developable surfaces intersect at right angles is a system of normals of a surface.

Reverting to the before-mentioned problem (plane or solid), I remark that this is a problem of minimum *sui generis*. Considering the first area or volume as divided in any manner into infinitesimal elements, we have to divide the second area or volume into corresponding equal elements, in such wise that the sum of the products of each element of the first area or volume into its distance from the corresponding element of the second area or volume may be a minimum; but, for doing this, we have no means of forming the analytical expression of any function which is to be, by the formulæ of the differential calculus or the calculus of variations, made a minimum.

For the plane problem, Monge obtains the solution by means of the very simple consideration that the routes of two elements must not cross each other; in fact, imagine an element A transferred to a, and an equal element B transferred to b: the lines Aa, Bb must not cross each other, for if they did, drawing the two lines



Ab and Ba, the sum $Aa+Bb$ would be greater than the sum $Ab+Ba$, contrary to the condition of the minimum. Imagine the areas intersected by two consecutive lines as shown in the figure: the filament between these two lines may be regarded as

a right line; and, assuming that some one element of the filament BD is transferred to a point of bd (that is, so as to coincide with an element of the filament bd), it follows that every other element of BD must be transferred so as to coincide with some other element of bd; and this obviously implies that the filaments BD and bd must be equal. Observe that, this being so, it is immaterial which element of BD is transferred to which element of bd; in whatever way this is done, the sum of the products will be the same*. The two lines may be regarded as the normals of a curve; and the problem thus is, to find a curve such that, drawing the normals thereof to intersect the two areas, then that the filaments BD and bd, cut off by consecutive normals on the two areas respectively, shall be equal. This leads to a differential equation of the second order for the normal curve; one of the constants of integration remains arbitrary, for the normal curve is any one of a system of parallel curves. It is to be observed that the filaments are the increments of the areas BCD and bcd; these increments are equal; a position of the line must be the common tangent Cc of the two areas (this, in fact, constitutes the condition for the determination of one of the arbitrary constants), and for this position the areas are each $=0$. Hence, in general, the areas must be equal; or the problem is, to find a curve such that any normal thereof cuts off equal areas BCD and bcd.

If, instead of the normal curve, we consider the curve which is the envelope of the several lines, or, what is the same thing, the locus of the point N, then we could, in like manner, obtain for this curve a differential equation of the first order: the constant of integration would be determined by the condition that Cc is a tangent. The curve in question is, of course, the evolute of the normal curve.

The several lines which intersect the two areas give rise to a finite arc IS of this evolute, and, as remarked by Monge, it is only when this arc IS lies (as in the figure) *outside* the two areas, that we have a true minimum.

Passing now to the solid problem, we may imagine a congruence of lines intersecting the two volumes; each line of the congruence is intersected by two consecutive lines, and the lines of the congruence thus form two sets of developable surfaces, each surface of the one set intersecting each surface of the other set. And, considering two consecutive surfaces of the one set, and two consecutive surfaces of the other set, these include between them a filament; and, treating the filament as a right line, it seems to follow (although it is more difficult to present the reasoning in a rigorous form) that, if any one element of the filament BD be transferred to any one element of the filament bd, then that every other element of the filament BD must be

* The most simple case is, take in the same straight line two equal segments AB, ab; it is immaterial how the elements of AB are transferred to ab, the sum of the products of each element into the traversed distance will be in every case the same. Analytically, if $dx=dx'$, then

$$\int (x'-x)\,dx = \int x'dx' - \int x\,dx,$$

the equation $dx'=dx$ meaning $x'=x+$a discontinuous constant. In the actual case of the filament, the formula is, if $rdr=r'dr'$, then

$$\int (r'-r)\,r\,dr = \int r'^2\,dr' - \int r^2\,dr.$$

transferred to some other element of the filament bd; and, this being so, the two filaments must be equal. But Monge goes on to argue that the condition of the minimum further requires that the developable surfaces shall cut at right angles, *and I cannot say that I see this.* He says (pp. 700, 701), "We know already that the routes must be the intersections of two sets of developable surfaces such that each surface of the first set intersects those of the second set in right lines; it remains to be found under what angles these surfaces must cut each other to satisfy the minimum. But it is evident that these angles must be right angles, for with these angles the elementary spaces comprised between four developable surfaces will be greater, and for equal distances the transported mass will be greater; therefore, in the case of a minimum, the routes must be the intersections of two sets of developable surfaces such that each surface of the one set cuts those of the second set in straight lines and at right angles." And, this being so, he infers, and it in fact follows, that the routes are the normals of a surface.

Admitting the conclusion, the problem becomes as follows:—Given two volumes, it is required to find a surface such that, drawing the normals thereof to intersect the two volumes, and considering the filament bounded by the developable surfaces which belong to two consecutive curves of curvature of the one set and those belonging to two consecutive curves of curvature of the other set, the portions cut off on the two volumes respectively may be equal. And we are thus led to a partial differential equation of the second order for determining the equation $z=f(x, y)$ of the required surface. As in the plane problem, it is immaterial how the elements of the one filament are transferred to the other filament.

783.

ON MR WILKINSON'S RECTANGULAR TRANSFORMATION.

[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), pp. 222—229. Read May 10, 1883.]

CONSIDERING the three cones,

$$(p+\lambda)X^2+(q+\lambda)Y^2+(r+\lambda)Z^2=0,$$
$$(p+\mu)X^2+(q+\mu)Y^2+(r+\mu)Z^2=0,$$
$$(p+\nu)X^2+(q+\nu)Y^2+(r+\nu)Z^2=0,$$

where

$$p+q+r+\lambda+\mu+\nu=0,$$

it is easy to see that these contain a singly infinite system of rectangular axes, viz. we have in each cone one axis of a rectangular system, and for one of the cones the axis may be any line at pleasure of the cone. In fact, taking for the three axes (x, y, z), (x', y', z'), (x'', y'', z'') respectively, that is, for the first axis $X : Y : Z = x : y : z$, and so for each of the other two axes, then (x, y, z) being an arbitrary line on the first cone, we can find (x', y', z') and (x'', y'', z'') such that

$$(p+\lambda)x^2+(q+\lambda)y^2+(r+\lambda)z^2=0,$$
$$(p+\mu)x'^2+(q+\mu)y'^2+(r+\mu)z'^2=0,$$
$$(p+\nu)x''^2+(q+\nu)y''^2+(r+\nu)z''^2=0,$$
$$x'x''+y'y''+z'z''=0,$$
$$x''x+y''y+z''z=0,$$
$$xx'+yy'+zz'=0.$$

For, eliminating (x'', y'', z'') from the third, fourth, and fifth equations, we have, first,

$$x'' : y'' : z'' = yz'-y'z : zx'-z'x : xy'-x'y,$$

and consequently

$$(p+\nu)(yz'-y'z)^2+(q+\nu)(zx'-z'x)^2+(r+\nu)(xy'-x'y)^2=0.$$

It is to be shown that this equation is implied in the remaining first, second, and third equations; for, this being so, (x, y, z), (x', y', z') satisfy only these equations; or (x, y, z) are any values whatever satisfying the first equation. The other two equations then determine (x', y', z'), and, these being known, (x'', y'', z'') are then determined as above.

In fact, attending to the sixth equation, the equation just obtained may be written in the form

$$(p+\nu)[(y^2+z^2)(y'^2+z'^2)-x^2x'^2]+(q+\nu)[(z^2+x^2)(z'^2+x'^2)-y^2y'^2]$$
$$+(r+\nu)[(x^2+y^2)(x'^2+y'^2)-z^2z'^2]=0,$$

or, what is the same thing, in the form

$$-(x^2+y^2+z^2)[(p+\mu)x'^2+(q+\mu)y'^2+(r+\mu)z'^2]$$
$$-(x'^2+y'^2+z'^2)[(p+\lambda)x^2+(q+\lambda)y^2+(r+\lambda)z^2]=0;$$

for, comparing in the two forms, first the coefficients of $x^2x'^2$, these are

$$(q+\nu)+(r+\nu)-(p+\nu) \quad \text{and} \quad -2p+\lambda+\mu,$$

which are equal in virtue of $p+q+r+\lambda+\mu+\nu=0$; and comparing next the coefficients of $y^2z'^2$, these are

$$p+\nu \quad \text{and} \quad -(r+\mu)-(q+\lambda),$$

which are equal in virtue of the same relation: and, similarly, the coefficients of the other terms $y^2y'^2$, &c., are equal in the two equations respectively.

Take now three arguments a_0, b_0, c_0, connected by the relation $a_0+b_0+c_0=0$, and write a, a, A for the sn, cn, and dn of a_0; and similarly b, b, B and c, c, C for those of b_0 and c_0 respectively: then we may write

$$p+\lambda,\ q+\lambda,\ r+\lambda=\left(1,\ \frac{\mathrm{a}}{\mathrm{bc}},\ \frac{A}{BC}\right),$$

$$p+\mu,\ q+\mu,\ r+\mu=\theta\left(1,\ \frac{\mathrm{b}}{\mathrm{ca}},\ \frac{B}{CA}\right),$$

$$p+\nu,\ q+\nu,\ r+\nu=\phi\left(1,\ \frac{\mathrm{c}}{\mathrm{ab}},\ \frac{C}{AB}\right);$$

for, starting from the first set of values, we have the second set if only

$$\mu-\lambda=\theta-1=\theta\frac{\mathrm{b}}{\mathrm{ca}}-\frac{\mathrm{a}}{\mathrm{bc}}=\theta\frac{B}{CA}-\frac{A}{BC}.$$

We thence obtain

$$\theta\left(1-\frac{\mathrm{b}}{\mathrm{ca}}\right)=1-\frac{\mathrm{a}}{\mathrm{bc}},\quad \theta\left(1-\frac{B}{CA}\right)=1-\frac{A}{BC};$$

and, in order to the identity of the two values of θ, we must have

$$\left(1-\frac{\mathrm{b}}{\mathrm{ca}}\right)\left(1-\frac{A}{BC}\right)-\left(1-\frac{\mathrm{a}}{\mathrm{bc}}\right)\left(1-\frac{B}{CA}\right)=0,$$

that is,

$$(\mathrm{abc}-\mathrm{b}^2)(ABC-A^2)-(\mathrm{abc}-\mathrm{a}^2)(ABC-B^2)=0,$$

or, reducing,

$$(\mathrm{a}^2-\mathrm{b}^2)ABC-(A^2-B^2)\,\mathrm{abc}+A^2\mathrm{b}^2-B^2\mathrm{a}^2=0.$$

But

$$\mathrm{a}^2-\mathrm{b}^2=-(a^2-b^2),\quad A^2-B^2=-k^2(a^2-b^2),\quad A^2\mathrm{b}^2-B^2\mathrm{a}^2=k'^2(a^2-b^2);$$

hence the whole equation divides by a^2-b^2, and, omitting this factor, it becomes

$$-ABC+k^2\mathrm{abc}+k'^2=0,$$

which is a known relation between the elliptic functions of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0=0$. Similarly, for ϕ, we have

$$\nu-\lambda=\phi-1=\phi\frac{\mathrm{c}}{\mathrm{ab}}-\frac{\mathrm{a}}{\mathrm{bc}}=\phi\frac{C}{AB}-\frac{A}{BC},$$

and, comparing the two values of ϕ, we have the same identical relation.

It thus appears that the three cones

$$X^2+\frac{\mathrm{a}}{\mathrm{bc}}Y^2+\frac{A}{BC}Z^2=0,$$

$$X^2+\frac{\mathrm{b}}{\mathrm{ca}}Y^2+\frac{B}{CA}Z^2=0,$$

$$X^2+\frac{\mathrm{c}}{\mathrm{ab}}Y^2+\frac{C}{AB}Z^2=0,$$

(the coefficients whereof depend on the elliptic functions sn, cn, and dn, of the arguments a_0, b_0, c_0 connected by the equation $a_0+b_0+c_0=0$) contain a singly infinite system of rectangular axes.

Considering an argument f_0, and denoting its sn, cn, dn by f, f, F respectively, we have, for an arbitrary line on the first cone, the values

$$x,\ y,\ z=M\sqrt{k'^2A\mathrm{a}},\quad M\sqrt{k^2A\mathrm{bc}}\,.\,\mathrm{f},\quad M\sqrt{-\mathrm{a}BC}\,.\,F.$$

In fact, substituting in the equation of the cone, we obtain the identity

$$k'^2+k^2\mathrm{f}^2-F^2=0;$$

and if we determine M by the condition that $x^2+y^2+z^2$ shall be $=1$, then we have

$$1=M^2\{k'^2A\mathrm{a}+k^2A\mathrm{bcf}^2-\mathrm{a}BCF^2\},$$

where the coefficient of M^2 is

$$=k'^2A\mathrm{a}+k^2A\mathrm{bc}\,(1-f^2)-\mathrm{a}BC(1-k^2f^2),$$

which is easily shown to be

$$= k^2k'^2bc\,(a^2-f^2),$$

so that the values of x, y, z are

$$= \{\sqrt{k'^2A\mathrm{a}},\quad \sqrt{k^2A\mathrm{bc}}\,.\,\mathrm{f},\quad \sqrt{-\mathrm{a}BC}\,.\,F\} \div \sqrt{k^2k'^2bc\,(a^2-f^2)},$$

and, similarly taking the arguments g_0, h_0, and denoting their elliptic functions by g, g, G, h, h, H, we have for a system of arbitrary lines in the three cones respectively, the values

$$\begin{array}{rcl|c|c|c|l}
x,\ y,\ z & = & & \sqrt{k'^2A\mathrm{a}} & \sqrt{k^2A\mathrm{bc}}\,.\,f & \sqrt{-\mathrm{a}BC}\,.\,F & \div\sqrt{k^2k'^2bc\,(a^2-f^2)} \\
x',\ y',\ z' & = & & \sqrt{k'^2B\mathrm{b}} & \sqrt{k^2B\mathrm{ca}}\,.\,g & \sqrt{-\mathrm{b}CA}\,.\,G & \div\sqrt{k^2k'^2ca\,(b^2-g^2)} \\
x'',\ y'',\ z'' & = & & \sqrt{k'^2C\mathrm{c}} & \sqrt{k^2C\mathrm{ab}}\,.\,h & \sqrt{-\mathrm{c}AB}\,.\,H & \div\sqrt{k^2k'^2ab\,(c^2-h^2)},
\end{array}$$

these values being such that $x^2+y^2+z^2$, $x'^2+y'^2+z'^2$, $x''^2+y''^2+z''^2$ are each $=1$. The radicals in the first line would be more correctly written, and may be understood as meaning $k'\sqrt{A}\sqrt{\mathrm{a}}$, $k\sqrt{A}\sqrt{\mathrm{b}}\sqrt{\mathrm{c}}$, $i\sqrt{\mathrm{a}}\sqrt{B}\sqrt{C}$, and similarly as regards the second and third lines respectively.

Taking now the arbitrary lines at right angles to each other, the condition for the second and third lines is

$$1+k^2\mathrm{agh}-k'^2AGH=0,$$

which is satisfied if $a_0=g_0-h_0$; similarly the condition for the third and first lines is satisfied if $b_0=h_0-f_0$; and we then have $a_0+b_0=g_0-f_0$; that is, $-c_0=g_0-f_0$ or $c_0=f_0-g_0$, which is the condition for the first and second lines; hence the arguments a_0, b_0, c_0, f_0, g_0, h_0 being such that

$$\begin{aligned}
\cdot\quad h_0-g_0+a_0&=0,\\
-h_0\quad\cdot\quad +f_0+b_0&=0,\\
g_0-f_0\quad\cdot\quad +c_0&=0,\\
-a_0-b_0-c_0\quad\cdot\quad &=0,
\end{aligned}$$

or, what is the same thing, a_0, b_0, c_0, f_0, g_0, h_0 being the differences of any four arguments α, β, γ, δ, the foregoing values of (x, y, z), (x', y', z'), (x'', y'', z'') will satisfy the equations

$$\begin{aligned}
x^2+y^2+z^2&=1,\\
x'^2+y'^2+z'^2&=1,\\
x''^2+y''^2+z''^2&=1,\\
x'x''+y'y''+z'z''&=0,\\
x''x+y''y+z''z&=0,\\
x\,x'+y\,y'+z\,z'&=0,
\end{aligned}$$

for the transformation of a set of rectangular axes. These are, in fact, Mr Wilkinson's expressions, the a_0, b_0, c_0, f_0, g_0, h_0 being his $t-p$, $p-q$, $q-t$, t, p, q respectively.

Returning to the three cones, it is to be remarked that, taking in the first of them a line 1 at pleasure, then we have in the second of them *two* lines 2, 2′ each at right angles to the line 1, and such that the line 3 at right angles to the plane 12, and the line 3′ at right angles to the plane 12′, lie each of them in the third cone; or, what is the same thing, we have in the two cones respectively the rectangular lines 1 and 2, and also the rectangular lines 1 and 2′, such that the planes 12 and 12′ each of them envelope one and the same cone, the reciprocal of the third cone; where by the reciprocal cone of a given cone is meant the cone generated by the lines through the vertex at right angles to the tangent planes of the given cone. Introducing the notion of the absolute cone $X^2+Y^2+Z^2=0$, a line and plane through the vertex at right angles to each other are, in fact, reciprocal polars in regard to this absolute cone; and two lines at right angles to each other are reciprocals (or harmonics) in regard to this absolute cone; that is, the reciprocal plane of either of them passes through the other. The two cones are cones intersecting each other in four lines lying on the absolute cone; and in virtue of this relation they have the property in question, viz. taking in the first cone a line 1 at pleasure, then the reciprocal plane hereof in regard to the absolute cone meets the second cone in a pair of lines 2 and 2′ such that the planes 12 and 12′ each of them envelope one and the same cone; the reciprocal of this cone is then the third cone of the system, and as such it passes through the four lines on the absolute cone.

In verification, observe that the coefficients $p+\lambda$, $q+\lambda$, &c. of the equations of the three cones satisfy the equations

$$\left\| \begin{matrix} 1, & p+\lambda, & p+\mu, & p+\nu, & (p+\lambda)(p+\mu)(p+\nu) \\ 1, & q+\lambda, & q+\mu, & q+\nu, & (q+\lambda)(q+\mu)(q+\nu) \\ 1, & r+\lambda, & r+\mu, & r+\nu, & (r+\lambda)(r+\mu)(r+\nu) \end{matrix} \right\| = 0.$$

This is obviously the case for each equation such as

$$|\,1,\ p+\lambda,\ p+\mu\,| = 0;$$

and any equation containing the fifth column is at once reducible to

$$|\,1,\ p,\ p^3+p^2(\lambda+\mu+\nu)\,| = 0,$$

that is,

$$|\,1,\ p,\ p^3\,| + (\lambda+\mu+\nu)\,|\,1,\ p,\ p^2\,| = 0;$$

or, dividing by $|\,1,\ p,\ p^2\,|$, this is $p+q+r+\lambda+\mu+\nu=0$, the equation connecting the coefficients.

Hence, representing the three cones by

$$p\,X^2+q\,Y^2+r\,Z^2=0,$$
$$p'\,X^2+q'\,Y^2+r'\,Z^2=0,$$
$$p''X^2+q''Y^2+r''Z^2=0,$$

and the absolute by

$$X^2+\ \ Y^2+\ \ Z^2=0,$$

the coefficients p, q, &c., are connected by the equations

$$\left\| \begin{matrix} 1, & p, & p', & p'', & pp'p'' \\ 1, & q, & q', & q'', & qq'q'' \\ 1, & r, & r', & r'', & rr'r'' \end{matrix} \right\| = 0;$$

among these are of course included the equation $|\,1,\ p,\ p'\,| = 0$, which expresses that the first and second cones intersect on the absolute; (p, q, r), (p', q', r') are any quantities satisfying this relation, and, regarding them as given, we have then two independent equations determining the ratios $p'' : q'' : r''$. The theorem is that the planes 12 and 12′ envelope one and the same quadric cone

$$\frac{X^2}{p''} + \frac{Y^2}{q''} + \frac{Z^2}{r''} = 0.$$

The equations $|\,1,\ p,\ p''\,| = 0$ and $|\,1,\ p,\ pp'p''\,| = 0$ give

$$\begin{aligned} (q-r)\,p'' \quad + (r-p)\,q'' \quad + (p-q)\,r'' \quad &= 0, \\ (q-r)\,pp'p'' + (r-p)\,qq'q'' + (p-q)\,rr'r'' &= 0, \end{aligned}$$

and thence

$$(q-r)\,p'' : (r-p)\,q'' : (p-q)\,r'' = qq' - rr' : rr' - pp' : pp' - qq';$$

or, observing that we have

$$q-r : r-p : p-q = qr' - q'r : rp' - r'p : pq' - p'q,$$

the equations may also be written

$$(qr' - q'r)\,p'' : (rp' - r'p)\,q'' : (pq' - p'q)\,r'' = qq' - rr' : rr' - pp' : pp' - qq'.$$

Starting with an arbitrary line (x, y, z) in the first cone, then the reciprocal plane thereof (in regard to the absolute cone) is the plane $Xx + Yy + Zz = 0$, which meets the second cone in two lines, say (2) and (2′), each of which is a line reciprocal to the line (1); and we have thus two planes (12) and (12′), each of which envelopes, as is to be shown, the same cone $q''r''X^2 + r''p''Y^2 + p''q''Z^2 = 0$.

Suppose, in general, that we have an arbitrary line (x, y, z) and an arbitrary plane $\alpha X + \beta Y + \gamma Z = 0$, and that it is required to find the equation of the two planes through the line (x, y, z), and the intersections of the plane $\alpha X + \beta Y + \gamma Z = 0$ with the cone $p'X^2 + q'Y^2 + r'Z^2 = 0$: the equation of the pair of planes is

$$\begin{aligned} &(\alpha X + \beta Y + \gamma Z)^2 (p'x^2 + q'y^2 + r'z^2) \\ &+ (\alpha x + \beta y + \gamma z)^2 (p'X^2 + q'Y^2 + r'Z^2) \\ &- 2\,(\alpha X + \beta Y + \gamma Z)\,(\alpha x + \beta y + \gamma z)\,(p'Xx + q'Yy + r'Zz) = 0. \end{aligned}$$

In the present case, the plane $\alpha X + \beta Y + \gamma Z = 0$ is the plane $xX + yY + zZ = 0$, which is the reciprocal of the line (x, y, z) in regard to the absolute cone, and the equation of the pair of planes is

$$\begin{aligned} &(xX + yY + zZ)^2 (p'x^2 + q'y^2 + r'z^2) \\ &+ (x^2 + y^2 + z^2)^2 (p'X^2 + q'Y^2 + r'Z^2) \\ &- 2\,(xX + yY + zZ)\,(x^2 + y^2 + z^2)\,(p'Xx + q'Yy + r'Zz) = 0, \end{aligned}$$

where the quantities (x, y, z), as belonging to a line on the first cone, satisfy the condition $px^2+qy^2+rz^2=0$. The equation may be written

$$(a, b, c, f, g, h \between yZ-zY, zX-xZ, xY-yX)^2=0,$$

where

$$a, b, c, f, g, h = q'z^2+r'y^2,\ r'x^2+p'z^2,\ p'y^2+q'x^2,\ -p'yz,\ -q'zx,\ -r'xy,$$

and, as before, $px^2+qy^2+rz^2=0$; viz. this is the equation of the pair of planes (12) and (12').

The equation of the pair of tangent planes through the line (x, y, z) to the cone $q''r''X^2+r''p''Y^2+p''q''Z^2=0$ is

$$(q''r''x^2+r''p''y^2+p''q''z^2)(q''r''X^2+r''p''Y^2+p''q''Z^2)-(q''r''xX+r''p''yY+p''q''zZ)^2=0;$$

viz. omitting a factor $p''q''r''$, this equation is

$$(p'', q'', r'', 0, 0, 0 \between yZ-zY, zX-xZ, xY-yX)^2=0.$$

And it is to be shown that this is equivalent to the former equation; viz. writing $yZ-zY, zX-xZ, xY-yX=\lambda, \mu, \nu$, then that the two equations

$$(q'z^2+r'y^2,\ r'x^2+p'z^2,\ p'y^2+q'x^2,\ -p'yz,\ -q'zx,\ -r'xy \between \lambda, \mu, \nu)^2=0,$$
$$p''\lambda^2+q''\mu^2+r''\nu^2=0,$$

are equivalent to each other.

We have identically $\lambda x+\mu y+\nu z=0$, and thence also

$$(\lambda x+\mu y+\nu z)[(p'-q'-r')\lambda x+(-p'+q'-r')\mu y+(-p'-q'+r')\nu z]=0,$$

where, on the left-hand side, the terms in $\mu\nu$, $\nu\lambda$, and $\lambda\mu$ are

$$=-2p'yz\mu\nu-2q'zx\nu\lambda-2r'xy\lambda\mu.$$

Hence the first equation may be written

$$[q'z^2+r'y^2+(p'-q'-r')x^2]\lambda^2+[r'x^2+p'z^2+(-p'+q'-r')y^2]\mu^2$$
$$+[p'y^2+q'x^2+(-p'-q'+r')z^2]\nu^2=0,$$

and it is to be shown that this is equivalent to

$$p''\lambda^2+q''\mu^2+r''\nu^2=0;$$

viz. that we have $p'' : q'' : r''=$

$$q'z^2+r'y^2-(\ \ p'-q'-r')x^2$$
$$: r'x^2+p'z^2-(-p'+q'-r')y^2$$
$$: p'y^2+q'x^2-(-p'-q'+r')z^2,$$

where $px^2+qy^2+rz^2=0$. Writing the equation in the form

$$p'' : q'' : r''=A : B : C,$$

we have

$$\begin{aligned} A &= q'z^2 + r'y^2 - p'x^2 + q'x^2 + r'x^2 \\ &= -p'x^2 + q'(x^2+z^2) + r'(x^2+y^2) \\ &= -p'x^2 + (q'+r')(x^2+y^2+z^2) - q'y^2 - r'z^2. \end{aligned}$$

By what precedes, we have an identity of the form

$$x^2+y^2+z^2 = \alpha(p'x^2+q'y^2+r'z^2) + \beta(px^2+qy^2+rz^2),$$

where, determining α from the equations $1 = q'\alpha + q\beta$, $1 = r'\alpha + r\beta$, we find

$$\alpha = (q-r) \div (qr' - q'r);$$

but $px^2+qy^2+rz^2 = 0$, and the relation thus is

$$x^2+y^2+z^2 = \alpha(p'x^2+q'y^2+r'z^2);$$

hence

$$A = \{(q'+r')\alpha - 1\}(p'x^2+q'y^2+r'z^2),$$

or, substituting for α its value, this is

$$\begin{aligned} A &= \left\{\frac{(q'+r')(q-r) - qr' + q'r}{qr'-q'r}\right\}(p'x^2+q'y^2+r'z^2), \\ &= \frac{qq'-rr'}{qr'-q'r}(p'x^2+q'y^2+r'z^2); \end{aligned}$$

and, forming the like values of B and C, the relations to be verified become

$$p'' : q'' : r'' = \frac{qq'-rr'}{qr'-q'r} : \frac{rr'-pp'}{rp'-r'p} : \frac{pp'-qq'}{pq'-p'q},$$

which are, in fact, the values of the ratios $p'' : q'' : r''$ obtained above; and the theorem is thus seen to be true. It may be remarked that, if the first and second cones, instead of intersecting in four lines on the absolute cone, had been arbitrary cones; then, taking in the first cone a line (1) and in the second cone a line (2), the reciprocal of (1) in regard to the absolute, the envelope of the plane (12) would have been (instead of a quadric cone) a cone of the class 8.

784.

PRESIDENTIAL ADDRESS TO THE BRITISH ASSOCIATION, SEPTEMBER 1883.

[From the *Report of the British Association for the Advancement of Science*, (1883), pp. 3—37.]

SINCE our last meeting we have been deprived of three of our most distinguished members. The loss by the death of Professor Henry John Stephen Smith is a very grievous one to those who knew and admired and loved him, to his University, and to mathematical science, which he cultivated with such ardour and success. I need hardly recall that the branch of mathematics to which he had specially devoted himself was that most interesting and difficult one, the Theory of Numbers. The immense range of this subject, connected with and ramifying into so many others, is nowhere so well seen as in the series of reports on the progress thereof, brought up unfortunately only to the year 1865, contributed by him to the Reports of the Association; but it will still better appear when to these are united (as will be done in the collected works in course of publication by the Clarendon Press) his other mathematical writings, many of them containing his own further developments of theories referred to in the reports. There have been recently or are being published many such collected editions—Abel, Cauchy, Clifford, Gauss, Green, Jacobi, Lagrange, Maxwell, Riemann, Steiner. Among these the works of Henry Smith will occupy a worthy position.

More recently, General Sir Edward Sabine, K.C.B., for twenty-one years general secretary of the Association, and a trustee, President of the meeting at Belfast in the year 1852, and for many years treasurer and afterwards President of the Royal Society, has been taken from us, at an age exceeding the ordinary age of man. Born October 1788, he entered the Royal Artillery in 1803, and commanded batteries at the siege of Fort Erie in 1814; made magnetic and other observations in Ross and Parry's North Polar exploration in 1818–19, and in a series of other voyages. He

contributed to the Association reports on Magnetic Forces in 1836–7–8, and about forty papers to the *Philosophical Transactions*; originated the system of Magnetic Observatories, and otherwise signally promoted the science of Terrestrial Magnetism.

There is yet a very great loss: another late President and trustee of the Association, one who has done for it so much, and has so often attended the meetings, whose presence among us at this meeting we might have hoped for—the President of the Royal Society, William Spottiswoode. It is unnecessary to say anything of his various merits: the place of his burial, the crowd of sorrowing friends who were present in the Abbey, bear witness to the esteem in which he was held.

I take the opportunity of mentioning the completion of a work promoted by the Association: the determination by Mr James Glaisher of the least factors of the missing three out of the first nine million numbers: the volume containing the sixth million is now published.

I wish to speak to you to-night upon Mathematics. I am quite aware of the difficulty arising from the abstract nature of my subject; and if, as I fear, many or some of you, recalling the Presidential Addresses at former meetings—for instance, the *résumé* and survey which we had at York of the progress, during the half century of the lifetime of the Association, of a whole circle of sciences—Biology, Palæontology, Geology, Astronomy, Chemistry—so much more familiar to you, and in which there was so much to tell of the fairy-tales of science; or at Southampton, the discourse of my friend who has in such kind terms introduced me to you, on the wondrous practical applications of science to electric lighting, telegraphy, the St Gothard Tunnel and the Suez Canal, gun-cotton, and a host of other purposes, and with the grand concluding speculation on the conservation of solar energy: if, I say, recalling these or any earlier Addresses, you should wish that you were now about to have, from a different President, a discourse on a different subject, I can very well sympathise with you in the feeling.

But be this as it may, I think it is more respectful to you that I should speak to you upon and do my best to interest you in the subject which has occupied me, and in which I am myself most interested. And in another point of view, I think it is right that the Address of a President should be on his own subject, and that different subjects should be thus brought in turn before the meetings. So much the worse, it may be, for a particular meeting; but the meeting is the individual, which on evolution principles must be sacrificed for the development of the race.

Mathematics connect themselves on the one side with common life and the physical sciences; on the other side with philosophy, in regard to our notions of space and time, and in the questions which have arisen as to the universality and necessity of the truths of mathematics, and the foundation of our knowledge of them. I would remark here that the connexion (if it exists) of arithmetic and algebra with the notion of time is far less obvious than that of geometry with the notion of space.

As to the former side, I am not making before you a defence of mathematics, but if I were I should desire to do it—in such manner as in the *Republic* Socrates was required to defend justice, quite irrespectively of the worldly advantages which

may accompany a life of virtue and justice, and to show that, independently of all these, justice was a thing desirable in itself and for its own sake—*not* by speaking to you of the utility of mathematics in any of the questions of common life or of physical science. Still less would I speak of this utility before, I trust, a friendly audience, interested or willing to appreciate an interest in mathematics in itself and for its own sake. I would, on the contrary, rather consider the obligations of mathematics to these different subjects as the sources of mathematical theories now as remote from them, and in as different a region of thought—for instance, geometry from the measurement of land, or the Theory of Numbers from arithmetic—as a river at its mouth is from its mountain source.

On the other side, the general opinion has been and is that it is indeed by experience that we arrive at the truths of mathematics, but that experience is not their proper foundation: the mind itself contributes something. This is involved in the Platonic theory of reminiscence; looking at two things, trees or stones or anything else, which seem to us more or less equal, we arrive at the idea of equality: but we must have had this idea of equality before the time when first seeing the two things we were led to regard them as coming up more or less perfectly to this idea of equality; and the like as regards our idea of the beautiful, and in other cases.

The same view is expressed in the answer of Leibnitz, the *nisi intellectus ipse*, to the scholastic dictum, *nihil in intellectu quod non prius in sensu*: there is nothing in the intellect which was not first in sensation, except (said Leibnitz) the intellect itself. And so again in the *Critick of Pure Reason*, Kant's view is that while there is no doubt but that all our cognition begins with experience, we are nevertheless in possession of cognitions *a priori*, independent, not of this or that experience, but absolutely so of all experience, and in particular that the axioms of mathematics furnish an example of such cognitions *a priori*. Kant holds further that space is no empirical conception which has been derived from external experiences, but that in order that sensations may be referred to something external, the representation of space must already lie at the foundation; and that the external experience is itself first only possible by this representation of space. And in like manner time is no empirical conception which can be deduced from an experience, but it is a necessary representation lying at the foundation of all intuitions.

And so in regard to mathematics, Sir W. R. Hamilton, in an Introductory Lecture on Astronomy (1836), observes: "These purely mathematical sciences of algebra and geometry are sciences of the pure reason, deriving no weight and no assistance from experiment, and isolated or at least isolable from all outward and accidental phenomena. The idea of order with its subordinate ideas of number and figure, we must not indeed call innate ideas, if that phrase be defined to imply that all men must possess them with equal clearness and fulness: they are, however, ideas which seem to be so far born with us that the possession of them in any conceivable degree is only the development of our original powers, the unfolding of our proper humanity."

The general question of the ideas of space and time, the axioms and definitions of geometry, the axioms relating to number, and the nature of mathematical reasoning, are

fully and ably discussed in Whewell's *Philosophy of the Inductive Sciences* (1840), which may be regarded as containing an exposition of the whole theory.

But it is maintained by John Stuart Mill that the truths of mathematics, in particular those of geometry, rest on experience; and as regards geometry, the same view is on very different grounds maintained by the mathematician Riemann.

It is not so easy as at first sight it appears to make out how far the views taken by Mill in his *System of Logic Ratiocinative and Inductive* (9th ed. 1879) are absolutely contradictory to those which have been spoken of; they profess to be so; there are most definite assertions (supported by argument), for instance, p. 263:—"It remains to enquire what is the ground of our belief in axioms, what is the evidence on which they rest. I answer, they are experimental truths, generalisations from experience. The proposition 'Two straight lines cannot enclose a space,' or, in other words, two straight lines which have once met cannot meet again, is an induction from the evidence of our senses." But I cannot help considering a previous argument (p. 259) as very materially modifying this absolute contradiction. After enquiring "Why are mathematics by almost all philosophers . . . considered to be independent of the evidence of experience and observation, and characterised as systems of necessary truth?" Mill proceeds (I quote the whole passage) as follows:—"The answer I conceive to be that this character of necessity ascribed to the truths of mathematics, and even (with some reservations to be hereafter made) the peculiar certainty ascribed to them, is a delusion, in order to sustain which it is necessary to suppose that those truths relate to and express the properties of purely imaginary objects. It is acknowledged that the conclusions of geometry are derived partly at least from the so-called definitions, and that these definitions are assumed to be correct representations, as far as they go, of the objects with which geometry is conversant. Now, we have pointed out that, from a definition as such, no proposition unless it be one concerning the meaning of a word can ever follow, and that what apparently follows from a definition, follows in reality from an implied assumption that there exists a real thing conformable thereto. This assumption in the case of the definitions of geometry is not strictly true: there exist no real things exactly conformable to the definitions. There exist no real points without magnitude, no lines without breadth, nor perfectly straight, no circles with all their radii exactly equal, nor squares with all their angles perfectly right. It will be said that the assumption does not extend to the actual but only to the possible existence of such things. I answer that according to every test we have of possibility they are not even possible. Their existence, so far as we can form any judgment, would seem to be inconsistent with the physical constitution of our planet at least, if not of the universal [*sic*]. To get rid of this difficulty and at the same time to save the credit of the supposed system of necessary truth, it is customary to say that the points, lines, circles and squares which are the subjects of geometry exist in our conceptions merely and are parts of our minds; which minds by working on their own materials construct an *a priori* science, the evidence of which is purely mental and has nothing to do with outward experience. By howsoever high authority this doctrine has been sanctioned, it appears to me psychologically incorrect. The points, lines and squares which anyone has in his mind are (as I apprehend) simply copies

of the points, lines and squares which he has known in his experience. Our idea of a point I apprehend to be simply our idea of the *minimum visibile,* the small portion of surface which we can see. We can reason about a line as if it had no breadth, because we have a power which we can exercise over the operations of our minds: the power, when a perception is present to our senses or a conception to our intellects, of *attending* to a part only of that perception or conception instead of the whole. But we cannot *conceive* a line without breadth: we can form no mental picture of such a line; all the lines which we have in our mind are lines possessing breadth. If anyone doubt this, we may refer him to his own experience. I much question if anyone who fancies that he can conceive of a mathematical line thinks so from the evidence of his own consciousness. I suspect it is rather because he supposes that, unless such a perception be possible, mathematics could not exist as a science: a supposition which there will be no difficulty in showing to be groundless."

I think it may be at once conceded that the truths of geometry are truths precisely because they relate to and express the properties of what Mill calls "purely imaginary objects"; that these objects do not exist in Mill's sense, that they do not exist in nature, may also be granted; that they are "not even possible," if this means not possible in an existing nature, may also be granted. That we cannot "conceive" them depends on the meaning which we attach to the word conceive. I would myself say that the purely imaginary objects are the only realities, the ὄντως ὄντα, in regard to which the corresponding physical objects are as the shadows in the cave; and it is only by means of them that we are able to deny the existence of a corresponding physical object; if there is no conception of straightness, then it is meaningless to deny the existence of a perfectly straight line.

But at any rate the objects of geometrical truth are the so-called imaginary objects of Mill, and the truths of geometry are only true, and *a fortiori* are only necessarily true, in regard to these so-called imaginary objects; and these objects, points, lines, circles, &c., in the mathematical sense of the terms, have a likeness to and are represented more or less imperfectly, and from a geometer's point of view no matter how imperfectly, by corresponding physical points, lines, circles, &c. I shall have to return to geometry, and will then speak of Riemann, but I will first refer to another passage of the Logic.

Speaking of the truths of arithmetic, Mill says (p. 297) that even here there is one hypothetical element: "In all propositions concerning numbers a condition is implied without which none of them would be true, and that condition is an assumption which may be false. The condition is that $1 = 1$: that all the numbers are numbers of the same or of equal units." Here at least the assumption may be absolutely true; one shilling = one shilling in purchasing power, although they may not be absolutely of the same weight and fineness: but it is hardly necessary; one coin + one coin = two coins, even if the one be a shilling and the other a half-crown. In fact, whatever difficulty be raisable as to geometry, it seems to me that no similar difficulty applies to arithmetic; mathematician or not, we have each of us, in its most abstract form, the idea of a number; we can each of us appreciate the truth of a proposition in regard to numbers; and we cannot but see that a truth in regard to numbers is something different in kind from an

experimental truth generalised from experience. Compare, for instance, the proposition that the sun, having already risen so many times, will rise to-morrow, and the next day, and the day after that, and so on; and the proposition that even and odd numbers succeed each other alternately *ad infinitum*: the latter at least seems to have the characters of universality and necessity. Or again, suppose a proposition observed to hold good for a long series of numbers, one thousand numbers, two thousand numbers, as the case may be: this is not only no proof, but it is absolutely no evidence, that the proposition is a true proposition, holding good for all numbers whatever; there are in the Theory of Numbers very remarkable instances of propositions observed to hold good for very long series of numbers and which are nevertheless untrue.

I pass in review certain mathematical theories.

In arithmetic and algebra, or say in analysis, the numbers or magnitudes which we represent by symbols are in the first instance ordinary (that is, positive) numbers or magnitudes. We have also in analysis and in analytical geometry *negative* magnitudes; there has been in regard to these plenty of philosophical discussion, and I might refer to Kant's paper, *Ueber die negativen Grössen in die Weltweisheit* (1763), but the notion of a negative magnitude has become quite a familiar one, and has extended itself into common phraseology. I may remark that it is used in a very refined manner in bookkeeping by double entry.

But it is far otherwise with the notion which is really the fundamental one (and I cannot too strongly emphasise the assertion) underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry: I use in each case the word imaginary as including real. This has not been, so far as I am aware, a subject of philosophical discussion or enquiry. As regards the older metaphysical writers this would be quite accounted for by saying that they knew nothing, and were not bound to know anything, about it; but at present, and, considering the prominent position which the notion occupies—say even that the conclusion were that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that (as a subject of philosophical discussion) the notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it.

Although in logical order I should perhaps now speak of the notion just referred to, it will be convenient to speak first of some other quasi-geometrical notions; those of more-than-three-dimensional space, and of non-Euclidian two- and three-dimensional space, and also of the generalised notion of distance. It is in connexion with these that Riemann considered that our notion of space is founded on experience, or rather that it is only by experience that we know that our space is Euclidian space.

It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidian plane geometry. There is a like system of non-Euclidian

solid geometry. My own view is that Euclid's twelfth axiom in Playfair's form of it does not need demonstration, but is part of our notion of space, of the physical space of our experience—the space, that is, which we become acquainted with by experience, but which is the representation lying at the foundation of all external experience. Riemann's view before referred to may I think be said to be that, having *in intellectu* a more general notion of space (in fact a notion of non-Euclidian space), we learn by experience that space (the physical space of our experience) is, if not exactly, at least to the highest degree of approximation, Euclidian space.

But suppose the physical space of our experience to be thus only approximately Euclidian space, what is the consequence which follows? *Not* that the propositions of geometry are only approximately true, but that they remain absolutely true in regard to that Euclidian space which has been so long regarded as being the physical space of our experience.

It is interesting to consider two different ways in which, without any modification at all of our notion of space, we can arrive at a system of non-Euclidian (plane or two-dimensional) geometry; and the doing so will, I think, throw some light on the whole question.

First, imagine the earth a perfectly smooth sphere; understand by a plane the surface of the earth, and by a line the apparently straight line (in fact, an arc of great circle) drawn on the surface; what experience would in the first instance teach would be Euclidian geometry; there would be intersecting lines which produced a few miles or so would seem to go on diverging: and apparently parallel lines which would exhibit no tendency to approach each other; and the inhabitants might very well conceive that they had by experience established the axiom that two straight lines cannot enclose a space, and the axiom as to parallel lines. A more extended experience and more accurate measurements would teach them that the axioms were each of them false; and that any two lines if produced far enough each way, would meet in two points: they would in fact arrive at a spherical geometry, accurately representing the properties of the two-dimensional space of their experience. But their original Euclidian geometry would not the less be a true system: only it would apply to an ideal space, not the space of their experience.

Secondly consider an ordinary, indefinitely extended plane; and let us modify only the notion of distance. We measure distance, say, by a yard measure or a foot rule, anything which is short enough to make the fractions of it of no consequence (in mathematical language, by an infinitesimal element of length); imagine, then, the length of this rule constantly changing (as it might do by an alteration of temperature), but under the condition that its actual length shall depend only on its situation on the plane and on its direction: viz. if for a given situation and direction it has a certain length, then whenever it comes back to the same situation and direction it must have the same length. The distance along a given straight or curved line between any two points could then be measured in the ordinary manner with this rule, and would have a perfectly determinate value: it could be measured over and over again, and would always be the same; but of course it would be the distance, not in the ordinary

acceptation of the term, but in quite a different acceptation. Or in a somewhat different way: if the rate of progress from a given point in a given direction be conceived as depending only on the configuration of the ground, and the distance along a given path between any two points thereof be measured by the time required for traversing it, then in this way also the distance would have a perfectly determinate value; but it would be a distance, not in the ordinary acceptation of the term, but in quite a different acceptation. And corresponding to the new notion of distance we should have a new non-Euclidian system of plane geometry; all theorems involving the notion of distance would be altered.

We may proceed further. Suppose that as the rule moves away from a fixed central point of the plane it becomes shorter and shorter; if this shortening takes place with sufficient rapidity, it may very well be that a distance which in the ordinary sense of the word is finite will in the new sense be infinite; no number of repetitions of the length of the ever-shortening rule will be sufficient to cover it. There will be surrounding the central point a certain finite area such that (in the new acceptation of the term distance) each point of the boundary thereof will be at an infinite distance from the central point; the points outside this area you cannot by any means arrive at with your rule; they will form a *terra incognita*, or rather an unknowable land: in mathematical language, an imaginary or impossible space: and the plane space of the theory will be that within the finite area—that is, it will be finite instead of infinite.

We thus with a proper law of shortening arrive at a system of non-Euclidian geometry which is essentially that of Lobatschewsky. But in so obtaining it we put out of sight its relation to spherical geometry: the three geometries (spherical, Euclidian, and Lobatschewsky's) should be regarded as members of a system: viz. they are the geometries of a plane (two-dimensional) space of constant positive curvature, zero curvature, and constant negative curvature respectively; or again, they are the plane geometries corresponding to three different notions of distance; in this point of view they are Klein's elliptic, parabolic, and hyperbolic geometries respectively.

Next as regards solid geometry: we can by a modification of the notion of distance (such as has just been explained in regard to Lobatschewsky's system) pass from our present system to a non-Euclidian system; for the other mode of passing to a non-Euclidian system, it would be necessary to regard our space as a flat three-dimensional space existing in a space of four dimensions (i.e., as the analogue of a plane existing in ordinary space); and to substitute for such flat three-dimensional space a curved three-dimensional space, say of constant positive or negative curvature. In regarding the physical space of our experience as possibly non-Euclidian, Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four-dimensional space.

I have just come to speak of four-dimensional space. What meaning do we attach to it? Or can we attach to it any meaning? It may be at once admitted that we cannot conceive of a fourth dimension of space; that space as we conceive of it, and the physical space of our experience, are alike three-dimensional; but we can, I think, conceive of space as being two- or even one-dimensional; we can imagine rational

beings living in a one-dimensional space (a line) or in a two-dimensional space (a surface), and conceiving of space accordingly, and to whom, therefore, a two-dimensional space, or (as the case may be) a three-dimensional space would be as inconceivable as a four-dimensional space is to us. And very curious speculative questions arise. Suppose the one-dimensional space a right line, and that it afterwards becomes a curved line: would there be any indication of the change? Or, if originally a curved line, would there be anything to suggest to them that it was not a right line? Probably not, for a one-dimensional geometry hardly exists. But let the space be two-dimensional, and imagine it originally a plane, and afterwards bent or converted into a curved surface (converted, that is, into some form of developable surface): or imagine it originally a developable or curved surface. In the former case there should be an indication of the change, for the geometry originally applicable to the space of their experience (our own Euclidian geometry) would cease to be applicable; but the change could not be apprehended by them as a bending or deformation of the plane, for this would imply the notion of a three-dimensional space in which this bending or deformation could take place. In the latter case their geometry would be that appropriate to the developable or curved surface which is their space: viz. this would be their Euclidian geometry: would they ever have arrived at our own more simple system? But take the case where the two-dimensional space is a plane, and imagine the beings of such a space familiar with our own Euclidian plane geometry; if, a third dimension being still inconceivable by them, they were by their geometry or otherwise led to the notion of it, there would be nothing to prevent them from forming a science such as our own science of three-dimensional geometry.

Evidently all the foregoing questions present themselves in regard to ourselves, and to three-dimensional space as we conceive of it, and as the physical space of our experience. And I need hardly say that the first step is the difficulty, and that granting a fourth dimension we may assume as many more dimensions as we please. But whatever answer be given to them, we have, as a branch of mathematics, potentially, if not actually, an analytical geometry of n-dimensional space. I shall have to speak again upon this.

Coming now to the fundamental notion already referred to, that of imaginary magnitude in analysis and imaginary space in geometry: I connect this with two great discoveries in mathematics made in the first half of the seventeenth century, Harriot's representation of an equation in the form $f(x)=0$, and the consequent notion of the roots of an equation as derived from the linear factors of $f(x)$, (Harriot, 1560—1621: his *Algebra*, published after his death, has the date 1631), and Descartes' method of coordinates, as given in the *Géométrie*, forming a short supplement to his *Traité de la Méthode, etc.*, (Leyden, 1637).

Taking the coefficients of an equation to be real magnitudes, it at once follows from Harriot's form of an equation that an equation of the order n ought to have n roots. But it is by no means true that there are always n real roots. In particular, an equation of the second order, or quadric equation, may have no real root; but if we assume the existence of a root i of the quadric equation $x^2+1=0$, then the

other root is $=-i$; and it is easily seen that every quadric equation (with real coefficients as before) has two roots, $a \pm bi$, where a and b are real magnitudes. We are thus led to the conception of an imaginary magnitude, $a+bi$, where a and b are real magnitudes, each susceptible of any positive or negative value, zero included. The general theorem is that, taking the coefficients of the equation to be imaginary magnitudes, then an equation of the order n has always n roots, each of them an imaginary magnitude, and it thus appears that the foregoing form $a+bi$ of imaginary magnitude is the only one that presents itself. Such imaginary magnitudes may be added or multiplied together or dealt with in any manner; the result is always a like imaginary magnitude. They are thus the magnitudes which are considered in analysis, and analysis is the science of such magnitudes. Observe the leading character that the imaginary magnitude $a+bi$ is a magnitude composed of the two real magnitudes a and b (in the case $b=0$ it is the real magnitude a, and in the case $a=0$ it is the pure imaginary magnitude bi). The idea is that of considering, in place of real magnitudes, these imaginary or complex magnitudes $a+bi$.

In the Cartesian geometry a curve is determined by means of the equation existing between the coordinates (x, y) of any point thereof. In the case of a right line, this equation is linear; in the case of a circle, or more generally of a conic, the equation is of the second order; and generally, when the equation is of the order n, the curve which it represents is said to be a curve of the order n. In the case of two given curves, there are thus two equations satisfied by the coordinates (x, y) of the several points of intersection, and these give rise to an equation of a certain order for the coordinate x or y of a point of intersection. In the case of a straight line and a circle, this is a quadric equation; it has two roots, real or imaginary. There are thus two values, say of x, and to each of these corresponds a single value of y. There are therefore two points of intersection—viz. a straight line and a circle intersect *always* in two points, real or imaginary. It is in this way that we are led analytically to the notion of imaginary points in geometry. The conclusion as to the two points of intersection cannot be contradicted by experience: take a sheet of paper and draw on it the straight line and circle, and try. But you might say, or at least be strongly tempted to say, that it is meaningless. The question of course arises, What is the meaning of an imaginary point? and further, In what manner can the notion be arrived at geometrically?

There is a well-known construction in perspective for drawing lines through the intersection of two lines, which are so nearly parallel as not to meet within the limits of the sheet of paper. You have two given lines which do not meet, and you draw a third line, which, when the lines are all of them produced, is found to pass through the intersection of the given lines. If instead of lines we have two circular arcs not meeting each other, then we can, by means of these arcs, construct a line; and if on completing the circles it is found that the circles intersect each other in two real points, then it will be found that the line passes through these two points: if the circles appear not to intersect, then the line will appear not to intersect either of the circles. But the geometrical construction being in each case the same, we say that in the second case also the line passes through the two intersections of the circles.

Of course it may be said in reply that the conclusion is a very natural one, provided we assume the existence of imaginary points; and that, this assumption not being made, then, if the circles do not intersect, it is meaningless to assert that the line passes through their points of intersection. The difficulty is not got over by the analytical method before referred to, for this introduces difficulties of its own: is there in a plane a point the coordinates of which have given imaginary values? As a matter of fact, we do consider in plane geometry imaginary points introduced into the theory analytically or geometrically as above.

The like considerations apply to solid geometry, and we thus arrive at the notion of imaginary space as a *locus in quo* of imaginary points and figures.

I have used the word imaginary rather than complex, and I repeat that the word has been used as including real. But, this once understood, the word becomes in many cases superfluous, and the use of it would even be misleading. Thus, "a problem has so many solutions": this means, so many imaginary (including real) solutions. But if it were said that the problem had "so many imaginary solutions," the word "imaginary" would here be understood to be used in opposition to real. I give this explanation the better to point out how wide the application of the notion of the imaginary is—viz. (unless expressly or by implication excluded), it is a notion implied and presupposed in all the conclusions of modern analysis and geometry. It is, as I have said, the fundamental notion underlying and pervading the whole of these branches of mathematical science.

I shall speak later on of the great extension which is thereby given to geometry, but I wish now to consider the effect as regards the theory of a function. In the original point of view, and for the original purposes, a function, algebraic or transcendental, such as $\sqrt{x}$, $\sin x$, or $\log x$, was considered as known, when the value was known for every real value (positive or negative) of the argument; or if for any such values the value of the function became imaginary, then it was enough to know that for such values of the argument there was no real value of the function. But now this is not enough, and to know the function means to know its value—of course, in general, an imaginary value $X+iY$,—for every imaginary value $x+iy$ whatever of the argument.

And this leads naturally to the question of the geometrical representation of an imaginary variable. We represent the imaginary variable $x+iy$ by means of a point in a plane, the coordinates of which are (x, y). This idea, due to Gauss, dates from about the year 1831. We thus picture to ourselves the succession of values of the imaginary variable $x+iy$ by means of the motion of the representative point: for instance, the succession of values corresponding to the motion of the point along a closed curve to its original position. The value $X+iY$ of the function can of course be represented by means of a point (taken for greater convenience in a different plane), the coordinates of which are X, Y.

We may consider in general two points, moving each in its own plane, so that the position of one of them determines the position of the other, and consequently

the motion of the one determines the motion of the other: for instance, the two points may be the tracing-point and the pencil of a pentagraph. You may with the first point draw any figure you please, there will be a corresponding figure drawn by the second point: for a good pentagraph, a copy on a different scale (it may be); for a badly-adjusted pentagraph, a distorted copy: but the one figure will always be a sort of copy of the first, so that to each point of the one figure there will correspond a point of the other figure.

In the case above referred to, where one point represents the value $x+iy$ of the imaginary variable and the other the value $X+iY$ of some function $\phi(x+iy)$ of that variable, there is a remarkable relation between the two figures: this is the relation of orthomorphic projection, the same which presents itself between a portion of the earth's surface, and the representation thereof by a map on the stereographic projection or on Mercator's projection—viz. any indefinitely small area of the one figure is represented in the other figure by an indefinitely small area of the same shape. There will possibly be for different parts of the figure great variations of scale, but the shape will be unaltered; if for the one area the boundary is a circle, then for the other area the boundary will be a circle; if for one it is an equilateral triangle, then for the other it will be an equilateral triangle.

I have for simplicity assumed that to each point of either figure there corresponds one, and only one, point of the other figure; but the general case is that to each point of either figure there corresponds a determinate number of points in the other figure; and we have thence arising new and very complicated relations which I must just refer to. Suppose that to each point of the first figure there correspond in the second figure two points: say one of them is a red point, the other a blue point; so that, speaking roughly, the second figure consists of two copies of the first figure, a red copy and a blue copy, the one superimposed on the other. But the difficulty is that the two copies cannot be kept distinct from each other. If we consider in the first figure a closed curve of any kind—say, for shortness, an oval—this will be in the second figure represented in some cases by a red oval and a blue oval, but in other cases by an oval half red and half blue; or, what comes to the same thing, if in the first figure we consider a point which moves continuously in any manner, at last returning to its original position, and attempt to follow the corresponding points in the second figure, then it may very well happen that, for the corresponding point of either colour, there will be abrupt changes of position, or say jumps, from one position to another; so that, to obtain in the second figure a continuous path, we must at intervals allow the point to change from red to blue, or from blue to red. There are in the first figure certain critical points called branch-points (*Verzweigungspunkte*), and a system of lines connecting these, by means of which the colours in the second figure are determined; but it is not possible for me to go further into the theory at present. The notion of colour has of course been introduced only for facility of expression; it may be proper to add that in speaking of the two figures I have been following Briot and Bouquet rather than Riemann, whose representation of the function of an imaginary variable is a different one.

I have been speaking of an imaginary variable $(x+iy)$, and of a function $\phi(x+iy)=X+iY$ of that variable, but the theory may equally well be stated in

regard to a plane curve: in fact, the $x+iy$ and the $X+iY$ are two imaginary variables connected by an equation; say their values are u and v, connected by an equation $F(u, v)=0$; then, regarding u, v as the coordinates of a point *in plano*, this will be a point on the curve represented by the equation. The curve, in the widest sense of the expression, is the whole series of points, real or imaginary, the coordinates of which satisfy the equation, and these are exhibited by the foregoing corresponding figures in two planes; but in the ordinary sense the curve is the series of real points, with coordinates u, v, which satisfy the equation.

In geometry it is the curve, whether defined by means of its equation, or in any other manner, which is the subject for contemplation and study. But we also use the curve as a representation of its equation—that is, of the relation existing between two magnitudes x, y, which are taken as the coordinates of a point on the curve. Such employment of a curve for all sorts of purposes—the fluctuations of the barometer, the Cambridge boat races, or the Funds—is familiar to most of you. It is in like manner convenient in analysis, for exhibiting the relations between any three magnitudes x, y, z, to regard them as the coordinates of a point in space; and, on the like ground, we should at least wish to regard any four or more magnitudes as the coordinates of a point in space of a corresponding number of dimensions. Starting with the hypothesis of such a space, and of points therein each determined by means of its coordinates, it is found possible to establish a system of n-dimensional geometry analogous in every respect to our two- and three-dimensional geometries, and to a very considerable extent serving to exhibit the relations of the variables. To quote from my memoir "On Abstract Geometry" (1869), [413]: "The science presents itself in two ways: as a legitimate extension of the ordinary two- and three-dimensional geometries, and as a need in these geometries and in analysis generally. In fact, whenever we are concerned with quantities connected in any manner, and which are considered as variable or determinable, then the nature of the connexion between the quantities is frequently rendered more intelligible by regarding them (if two or three in number) as the coordinates of a point in a plane or in space. For more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; but this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation we require a corresponding geometry. An important instance in plane geometry has already presented itself in the question of the number of curves which satisfy given conditions; the conditions imply relations between the coefficients in the equation of the curve; and for the better understanding of these relations it was expedient to consider the coefficients as the coordinates of a point in a space of the proper dimensionality."

It is to be borne in mind that the space, whatever its dimensionality may be, must always be regarded as an imaginary or complex space such as the two- or three-dimensional space of ordinary geometry; the advantages of the representation would otherwise altogether fail to be obtained.

I have spoken throughout of Cartesian coordinates; instead of these, it is in plane geometry not unusual to employ trilinear coordinates, and these may be regarded as absolutely undetermined in their magnitude—viz. we may take x, y, z to be, not equal,

but only proportional to the distances of a point from three given lines; the ratios of the coordinates (x, y, z) determine the point; and so in one-dimensional geometry, we may have a point determined by the ratio of its two coordinates x, y, these coordinates being proportional to the distances of the point from two fixed points; and generally in n-dimensional geometry a point will be determined by the ratios of the $(n+1)$ coordinates $(x, y, z, \ldots)$. The corresponding analytical change is in the expression of the original magnitudes as fractions with a common denominator; we thus, in place of rational and integral non-homogeneous functions of the original variables, introduce rational and integral homogeneous functions (quantics) of the next succeeding number of variables—viz. we have binary quantics corresponding to one-dimensional geometry, ternary to two-dimensional geometry, and so on.

It is a digression, but I wish to speak of the representation of points or figures in space upon a plane. In perspective, we represent a point in space by means of the intersection with the plane of the picture (suppose a pane of glass) of the line drawn from the point to the eye, and doing this for each point of the object we obtain a representation or picture of the object. But such representation is an imperfect one, as not determining the object: we cannot by means of the picture alone find out the form of the object; in fact, for a given point of the picture the corresponding point of the object is not a determinate point, but it is a point anywhere in the line joining the eye with the point of the picture. To determine the object we need two pictures, such as we have in a plan and elevation, or, what is the same thing, in a representation on the system of Monge's descriptive geometry. But it is theoretically more simple to consider two projections on the same plane, with different positions of the eye: the point in space is here represented on the plane by means of two points which are such that the line joining them passes through a fixed point of the plane (this point is in fact the intersection with the plane of the picture of the line joining the two positions of the eye); the figure in space is thus represented on the plane by two figures, which are such that the lines joining corresponding points of the two figures pass always through the fixed point. And such two figures completely replace the figure in space; we can by means of them perform on the plane any constructions which could be performed on the figure in space, and employ them in the demonstration of properties relating to such figure. A curious extension has recently been made: two figures in space such that the lines joining corresponding points pass through a fixed point have been regarded by the Italian geometer Veronese as representations of a figure in four-dimensional space, and have been used for the demonstration of properties of such figure.

I referred to the connexion of Mathematics with the notions of space and time, but I have hardly spoken of time. It is, I believe, usually considered that the notion of number is derived from that of time; thus Whewell in the work referred to, p. xx, says number is a modification of the conception of repetition, which belongs to that of *time*. I cannot recognise that this is so: it seems to me that we have (independently, I should say, of space or time, and in any case not more depending on time than on space) the notion of plurality; we think of, say, the letters a, b, c, &c., and thence in the case

of a finite set—for instance *a, b, c, d, e*—we arrive at the notion of number; coordinating them one by one with any other set of things, or, suppose, with the words first, second, &c., we find that the last of them goes with the word fifth, and we say that the number of things is = five: the notion of cardinal number would thus appear to be derived from that of ordinal number.

Questions of combination and arrangement present themselves, and it might be possible from the mere notion of plurality to develope a branch of mathematical science; this, however, would apparently be of a very limited extent, and it is difficult *not* to introduce into it the notion of number; in fact, in the case of a finite set of things, to avoid asking the question, How many? If we do this, we have a large enough subject, including the partition of numbers, which Sylvester has called Tactic.

From the notion thus arrived at of an integer number, we pass to that of a fractional number, and we see how by means of these the ratio of any two concrete magnitudes of the same kind can be expressed, not with absolute accuracy, but with any degree of accuracy we please: for instance, a length is so many feet, tenths of a foot, hundredths, thousandths, &c.; subdivide as you please, *non constat* that the length can be expressed accurately, we have in fact incommensurables; as to the part which these play in the Theory of Numbers, I shall have to speak presently: for the moment I am only concerned with them in so far as they show that we cannot from the notion of number pass to that which is required in analysis, the notion of an abstract (real and positive) magnitude susceptible of continuous variation. The difficulty is got over by a Postulate. We consider an abstract (real and positive) magnitude, and regard it as susceptible of continuous variation, without in anywise concerning ourselves about the actual expression of the magnitude by a numerical fraction or otherwise.

There is an interesting paper by Sir W. R. Hamilton, "Theory of Conjugate Functions, or Algebraical Couples: with a preliminary and elementary Essay on Algebra as the Science of Pure Time," 1833—35 (*Trans. R. I. Acad.* t. XVII.), in which, as appears by the title, he purposes to show that algebra is the science of pure time. He states there, in the General Introductory Remarks, his conclusions: first, that the notion of time is connected with existing algebra; second, that this notion or intuition of time may be unfolded into an independent pure science; and, third, that the science of pure time thus unfolded is coextensive and identical with algebra, so far as algebra itself is a science; and to sustain his first conclusion he remarks that "the history of algebraic science shows that the most remarkable discoveries in it have been made either expressly through the notion of *time*, or through the closely connected (and in some sort coincident) notion of continuous progression. It is the genius of algebra to consider what it reasons upon as *flowing*, as it was the genius of geometry to consider what it reasoned on as *fixed*. . . . And generally the revolution which Newton made in the higher parts of both pure and applied algebra was founded mainly on the notion of *fluxion*, which involves the notion of *time*." Hamilton uses the term algebra in a very wide sense, but whatever else he includes under it, he includes all that in contradistinction to the Differential Calculus would be called algebra. Using the word in this restricted sense, I cannot myself recognise the connexion of algebra with the notion of time: granting that the notion of continuous progression presents itself, and is of

importance, I do not see that it is in anywise the fundamental notion of the science. And still less can I appreciate the manner in which the author connects with the notion of time his algebraical couple, or imaginary magnitude $a+bi$ ($a+b\sqrt{-1}$, as written in the memoir).

I would go further: the notion of continuous variation is a very fundamental one, made a foundation in the Calculus of Fluxions (if not always so in the Differential Calculus) and presenting itself or implied throughout in mathematics: and it may be said that a change of any kind takes place only in time; it seems to me, however, that the changes which we consider in mathematics are for the most part considered quite irrespectively of time.

It appears to me that we do not have in Mathematics the notion of time until we bring it there: and that even in kinematics (the science of motion) we have very little to do with it; the motion is a hypothetical one; if the system be regarded as actually moving, the rate of motion is altogether undetermined and immaterial. The relative rates of motion of the different points of the system are nothing else than the ratios of purely geometrical quantities, the indefinitely short distances simultaneously described, or which might be simultaneously described, by these points respectively. But whether the notion of time does or does not sooner enter into mathematics, we at any rate have the notion in Mechanics, and along with it several other new notions.

Regarding Mechanics as divided into Statics and Dynamics, we have in dynamics the notion of time, and in connexion with it that of velocity: we have in statics and dynamics the notion of force; and also a notion which in its most general form I would call that of corpus: viz. this may be, the material point or particle, the flexible inextensible string or surface, or the rigid body, of ordinary mechanics; the incompressible perfect fluid of hydrostatics and hydrodynamics; the ether of any undulatory theory; or any other imaginable corpus; for instance, one really deserving of consideration in any general treatise of mechanics is a developable or skew surface with absolutely rigid generating lines, but which can be bent about these generating lines, so that the element of surface between two consecutive lines rotates as a whole about one of them. We have besides, in dynamics necessarily, the notion of mass or inertia.

We seem to be thus passing out of pure mathematics into physical science; but it is difficult to draw the line of separation, or to say of large portions of the *Principia*, and the *Mécanique céleste*, or of the whole of the *Mécanique analytique*, that they are not pure mathematics. It may be contended that we first come to physics when we attempt to make out the character of the corpus as it exists in nature. I do not at present speak of any physical theories which cannot be brought under the foregoing conception of mechanics.

I must return to the Theory of Numbers; the fundamental idea is here integer number: in the first instance positive integer number, but which may be extended to include negative integer number and zero. We have the notion of a product, and that of a prime number, which is not a product of other numbers; and thence also that of a number as the product of a determinate system of prime factors. We have here the

elements of a theory in many respects analogous to algebra: an equation is to be solved—that is, we have to find the integer values (if any) which satisfy the equation; and so in other cases: the congruence notation, although of the very highest importance, does not affect the character of the theory.

But as already noticed we have incommensurables, and the consideration of these gives rise to a new universe of theory. We may take into consideration any surd number such as $\sqrt{2}$, and so consider numbers of the form $a+b\sqrt{2}$, (a and b any positive or negative integer numbers not excluding zero); calling these integer numbers, every problem which before presented itself in regard to integer numbers in the original and ordinary sense of the word presents itself equally in regard to integer numbers in this new sense of the word; of course all definitions must be altered accordingly: an ordinary integer, which is in the ordinary sense of the word a prime number, may very well be the product of two integers of the form $a+b\sqrt{2}$, and consequently not a prime number in the new sense of the word. Among the incommensurables which can be thus introduced into the Theory of Numbers (and which was in fact *first* so introduced) we have the imaginary i of ordinary analysis: viz. we may consider numbers $a+bi$ (a and b ordinary positive or negative integers, not excluding zero), and, calling these integer numbers, establish in regard to them a theory analogous to that which exists for ordinary real integers. The point which I wish to bring out is that the imaginary i does not in the Theory of Numbers occupy a unique position, such as it does in analysis and geometry; it is in the Theory of Numbers one out of an indefinite multitude of incommensurables.

I said that I would speak to you, not of the utility of mathematics in any of the questions of common life or of physical science, but rather of the obligations of mathematics to these different subjects. The consideration which thus presents itself is in a great measure that of the history of the development of the different branches of mathematical science in connexion with the older physical sciences, Astronomy and Mechanics: the mathematical theory is in the first instance suggested by some question of common life or of physical science, is pursued and studied quite independently thereof, and perhaps after a long interval comes in contact with it, or with quite a different question. Geometry and algebra must, I think, be considered as each of them originating in connexion with objects or questions of common life—geometry, notwithstanding its name, hardly in the measurement of land, but rather from the contemplation of such forms as the straight line, the circle, the ball, the top (or sugar-loaf): the Greek geometers appropriated for the geometrical forms corresponding to the last two of these, the words σφαῖρα and κῶνος, our sphere and cone, and they extended the word cone to mean the complete figure obtained by producing the straight lines of the surface both ways indefinitely. And so algebra would seem to have arisen from the sort of easy puzzles in regard to numbers which may be made, either in the picturesque forms of the Bija-Ganita with its maiden with the beautiful locks, and its swarms of bees amid the fragrant blossoms, and the one queen-bee left humming around the lotus flower; or in the more prosaic form in which a student has presented to him in a modern text-book a problem leading to a simple equation.

The Greek geometry may be regarded as beginning with Plato (B.C. 430—347): the notions of geometrical analysis, loci, and the conic sections are attributed to him, and there are in his Dialogues many very interesting allusions to mathematical questions: in particular the passage in the *Theætetus*, where he affirms the incommensurability of the sides of certain squares. But the earliest extant writings are those of Euclid (B.C. 285): there is hardly anything in mathematics more beautiful than his wondrous fifth book; and he has also in the seventh, eighth, ninth and tenth books fully and ably developed the first principles of the Theory of Numbers, including the theory of incommensurables. We have next Apollonius (about B.C. 247), and Archimedes (B.C. 287—212), both geometers of the highest merit, and the latter of them the founder of the science of statics (including therein hydrostatics): his dictum about the lever, his "Εὕρηκα," and the story of the defence of Syracuse, are well known. Following these we have a worthy series of names, including the astronomers Hipparchus (B.C. 150) and Ptolemy (A.D. 125), and ending, say, with Pappus (A.D. 400), but continued by their Arabian commentators, and the Italian and other European geometers of the sixteenth century and later, who pursued the Greek geometry.

The Greek arithmetic was, from the want of a proper notation, singularly cumbrous and difficult; and it was for astronomical purposes superseded by the sexagesimal arithmetic, attributed to Ptolemy, but probably known before his time. The use of the present so-called Arabic figures became general among Arabian writers on arithmetic and astronomy about the middle of the tenth century, but was not introduced into Europe until about two centuries later. Algebra among the Greeks is represented almost exclusively by the treatise of Diophantus (A.D. 150), in fact a work on the Theory of Numbers containing questions relating to square and cube numbers, and other properties of numbers, with their solutions; this has no historical connexion with the later algebra, introduced into Italy from the East by Leonardi Bonacci of Pisa (A.D. 1202—1208) and successfully cultivated in the fifteenth and sixteenth centuries by Lucas Paciolus, or de Burgo, Tartaglia, Cardan, and Ferrari. Later on, we have Vieta (1540—1603), Harriot, already referred to, Wallis, and others.

Astronomy is of course intimately connected with geometry; the most simple facts of observation of the heavenly bodies can only be *stated* in geometrical language: for instance, that the stars describe circles about the pole-star, or that the different positions of the sun among the fixed stars in the course of the year form a circle. For astronomical calculations it was found necessary to determine the arc of a circle by means of its chord: the notion is as old as Hipparchus, a work of whom is referred to as consisting of twelve books on the chords of circular arcs; we have (A.D. 125) Ptolemy's *Almagest*, the first book of which contains a table of arcs and chords with the method of construction; and among other theorems on the subject he gives there the theorem afterwards inserted in Euclid (Book VI. Prop. D) relating to the rectangle contained by the diagonals of a quadrilateral inscribed in a circle. The Arabians made the improvement of using in place of the chord of an arc the sine, or half chord, of double the arc; and so brought the theory into the form in which it is used in modern trigonometry: the before-mentioned theorem of Ptolemy, or rather a particular case of it, translated into the notation of sines, gives the expression for the sine of the sum

of two arcs in terms of the sines and cosines of the component arcs; and it is thus the fundamental theorem on the subject. We have in the fifteenth and sixteenth centuries a series of mathematicians who with wonderful enthusiasm and perseverance calculated tables of the trigonometrical or circular functions, Purbach, Müller or Regiomontanus, Copernicus, Reinhold, Maurolycus, Vieta, and many others; the tabulations of the functions tangent and secant are due to Reinhold and Maurolycus respectively.

Logarithms were invented, not exclusively with reference to the calculation of trigonometrical tables, but in order to facilitate numerical calculations generally; the invention is due to John Napier of Merchiston, who died in 1618 at 67 years of age; the notion was based upon refined mathematical reasoning on the comparison of the spaces described by two points, the one moving with a uniform velocity, the other with a velocity varying according to a given law. It is to be observed that Napier's logarithms were nearly but not exactly those which are now called (sometimes Napierian, but more usually) hyperbolic logarithms—those to the base e; and that the change to the base 10 (the great step by which the invention was perfected for the object in view) was indicated by Napier but actually made by Henry Briggs, afterwards Savilian Professor at Oxford (d. 1630). But it is the hyperbolic logarithm which is mathematically important. The direct function e^x or $\exp. x$, which has for its inverse the hyperbolic logarithm, presented itself, but not in a prominent way. Tables were calculated of the logarithms of numbers, and of those of the trigonometrical functions.

The circular functions and the logarithm were thus invented each for a practical purpose, separately and without any proper connexion with each other. The functions are connected through the theory of imaginaries and form together a group of the utmost importance throughout mathematics: but this is mathematical theory; the obligation of mathematics is for the discovery of the functions.

Forms of spirals presented themselves in Greek architecture, and the curves were considered mathematically by Archimedes; the Greek geometers invented some other curves, more or less interesting, but recondite enough in their origin. A curve which might have presented itself to anybody, that described by a point in the circumference of a rolling carriage-wheel, was first noticed by Mersenne in 1615, and is the curve afterwards considered by Roberval, Pascal, and others under the name of the Roulette, otherwise the Cycloid. Pascal (1623—1662) wrote at the age of seventeen his *Essais pour les Coniques* in seven short pages, full of new views on these curves, and in which he gives, in a paragraph of eight lines, his theorem of the inscribed hexagon.

Kepler (1571—1630) by his empirical determination of the laws of planetary motion, brought into connexion with astronomy one of the forms of conic, the ellipse, and established a foundation for the theory of gravitation. Contemporary with him for most of his life, we have Galileo (1564—1642), the founder of the science of dynamics; and closely following upon Galileo we have Isaac Newton (1643—1727): the *Philosophiæ naturalis Principia Mathematica* known as the *Principia* was first published in 1687.

The physical, statical, or dynamical questions which presented themselves before the publication of the *Principia* were of no particular mathematical difficulty, but it

is quite otherwise with the crowd of interesting questions arising out of the theory of gravitation, and which, in becoming the subject of mathematical investigation, have contributed very much to the advance of mathematics. We have the problem of two bodies, or what is the same thing, that of the motion of a particle about a fixed centre of force, for any law of force; we have also the (mathematically very interesting) problem of the motion of a body attracted to two or more fixed centres of force; then, next preceding that of the actual solar system—the problem of three bodies; this has ever been and is far beyond the power of mathematics, and it is in the lunar and planetary theories replaced by what is mathematically a different problem, that of the motion of a body under the action of a principal central force and a disturbing force: or (in one mode of treatment) by the problem of disturbed elliptic motion. I would remark that we have here an instance in which an astronomical fact, the observed slow variation of the orbit of a planet, has directly suggested a mathematical method, applied to other dynamical problems, and which is the basis of very extensive modern investigations in regard to systems of differential equations. Again, immediately arising out of the theory of gravitation, we have the problem of finding the attraction of a solid body of any given form upon a particle, solved by Newton in the case of a homogeneous sphere, but which is far more difficult in the next succeeding cases of the spheroid of revolution (very ably treated by Maclaurin) and of the ellipsoid of three unequal axes: there is perhaps no problem of mathematics which has been treated by as great a variety of methods, or has given rise to so much interesting investigation as this last problem of the attraction of an ellipsoid upon an interior or exterior point. It was a dynamical problem, that of vibrating strings, by which Lagrange was led to the theory of the representation of a function as the sum of a series of multiple sines and cosines; and connected with this we have the expansions in terms of Legendre's functions P_n, suggested to him by the question just referred to of the attraction of an ellipsoid; the subsequent investigations of Laplace on the attractions of bodies differing slightly from the sphere led to the functions of two variables called Laplace's functions. I have been speaking of ellipsoids, but the general theory is that of attractions, which has become a very wide branch of modern mathematics; associated with it we have in particular the names of Gauss, Lejeune-Dirichlet, and Green; and I must not omit to mention that the theory is now one relating to n-dimensional space. Another great problem of celestial mechanics, that of the motion of the earth about its centre of gravity, in the most simple case, that of a body not acted upon by any forces, is a very interesting one in the mathematical point of view.

I may mention a few other instances where a practical or physical question has connected itself with the development of mathematical theory. I have spoken of two map projections—the stereographic, dating from Ptolemy; and Mercator's projection, invented by Edward Wright about the year 1600: each of these, as a particular case of the orthomorphic projection, belongs to the theory of the geometrical representation of an imaginary variable. I have spoken also of perspective, and of the representation of solid figures employed in Monge's descriptive geometry. Monge, it is well known, is the author of the geometrical theory of the curvature of surfaces and of curves of

curvature: he was led to this theory by a problem of earthwork; from a given area, covered with earth of uniform thickness, to carry the earth and distribute it over an equal given area, with the least amount of cartage. For the solution of the corresponding problem in solid geometry he had to consider the intersecting normals of a surface, and so arrived at the curves of curvature. (See his "Mémoire sur les Déblais et les Remblais," *Mém. de l'Acad.*, 1781.) The normals of a surface are, again, a particular case of a doubly infinite system of lines, and are so connected with the modern theories of congruences and complexes.

The undulatory theory of light led to Fresnel's wave-surface, a surface of the fourth order, by far the most interesting one which had then presented itself. A geometrical property of this surface, that of having tangent planes each touching it along a plane curve (in fact, a circle), gave to Sir W. R. Hamilton the theory of conical refraction. The wave-surface is now regarded in geometry as a particular case of Kummer's quartic surface, with sixteen conical points and sixteen singular tangent planes.

My imperfect acquaintance as well with the mathematics as the physics prevents me from speaking of the benefits which the theory of Partial Differential Equations has received from the hydrodynamical theory of vortex motion, and from the great physical theories of heat, electricity, magnetism, and energy.

It is difficult to give an idea of the vast extent of modern mathematics. This word "extent" is not the right one: I mean extent crowded with beautiful detail—not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for anything else, so for a mathematical theory—beauty can be perceived, but not explained. As for mere extent, I can perhaps best illustrate this by speaking of the dates at which some of the great extensions have been made in several branches of mathematical science.

As regards geometry, I have already spoken of the invention of the Cartesian coordinates (1637). This gave to geometers the whole series of geometric curves of higher order than the conic sections: curves of the third order, or cubic curves; curves of the fourth order, or quartic curves; and so on indefinitely. The first fruits of it were Newton's *Enumeratio linearum tertii ordinis*, and the extremely interesting investigations of Maclaurin as to corresponding points on a cubic curve. This was at once enough to show that the new theory of cubic curves was a theory quite as beautiful and far more extensive than that of conics. And I must here refer to Euler's remark in the paper "Sur une contradiction apparente dans la théorie des courbes planes" (Berlin Memoirs, 1748), in regard to the nine points of intersection of two cubic curves (viz. that when eight of the points are given the ninth point is thereby completely determined): this is not only a fundamental theorem in cubic curves (including in itself Pascal's theorem of the hexagon inscribed in a conic), but it introduces into plane geometry a new notion—that of the point-system, or system of the points of intersection of two curves.

A theory derived from the conic, that of polar reciprocals, led to the general notion of geometrical duality—viz. that in plane geometry the point and the line are correlative figures; and founded on this we have Plücker's great work, the *Theorie der algebraischen Curven* (Bonn, 1839), in which he establishes the relation which exists between the order and class of a curve and the number of its different point- and line-singularities (Plücker's six equations). It thus appears that the true division of curves is not a division according to order only, but according to order and class, and that the curves of a given order and class are again to be divided into families according to their singularities: this is not a mere subdivision, but is really a widening of the field of investigation; each such family of curves is in itself a subject as wide as the totality of the curves of a given order might previously have appeared.

We *unite* families by considering together the curves of a given *Geschlecht*, or deficiency; and in reference to what I shall have to say on the Abelian functions, I must speak of this notion introduced into geometry by Riemann in the memoir "Theorie der Abel'schen Functionen," *Crelle*, t. LIV. (1857). For a curve of a given order, reckoning cusps as double points, the deficiency is equal to the greatest number $\frac{1}{2}(n-1)(n-2)$ of the double points which a curve of that order can have, less the number of double points which the curve actually has. Thus a conic, a cubic with one double point, a quartic with three double points, &c., are all curves of the deficiency 0; the general cubic is a curve, and the most simple curve, of the deficiency 1; the general quartic is a curve of deficiency 3; and so on. The deficiency is usually represented by the letter p. Riemann considers the general question of the rational transformation of a plane curve: viz. here the coordinates, assumed to be homogeneous or trilinear, are replaced by any rational and integral functions, homogeneous of the same degree in the new coordinates; the transformed curve is in general a curve of a different order, with its own system of double points; but the deficiency p remains unaltered; and it is on this ground that he unites together and regards as a single class the whole system of curves of a given deficiency p. It must not be supposed that all such curves admit of rational transformation the one into the other: there is the further theorem that any curve of the class depends, in the case of a cubic, upon one parameter, but for $p>1$ upon $3p-3$ parameters, each such parameter being unaltered by the rational transformation; it is thus only the curves having the same one parameter, or $3p-3$ parameters, which can be rationally transformed the one into the other.

Solid geometry is a far wider subject: there are more theories, and each of them is of greater extent. The ratio is not that of the numbers of the dimensions of the spaces considered, or, what is the same thing, of the elementary figures—point and line in the one case; point, line and plane in the other case—belonging to these spaces respectively, but it is a very much higher one. For it is very inadequate to say that in plane geometry we have the curve, and in solid geometry the curve and surface: a more complete statement is required for the comparison. In plane geometry we have the curve, which may be regarded as a singly infinite system of points, and also as a singly infinite system of lines. In solid geometry we have, first, that which under one aspect is the curve, and under another aspect the developable, and which may be

regarded as a singly infinite system of points, of lines, or of planes; secondly, the surface, which may be regarded as a doubly infinite system of points or of planes, and also as a special triply infinite system of lines (viz. the tangent-lines of the surface are a special complex): as distinct particular cases of the former figure, we have the plane curve and the cone; and as a particular case of the latter figure, the ruled surface or singly infinite system of lines; we have *besides* the congruence, or doubly infinite system of lines, and the complex, or triply infinite system of lines. But, even if in solid geometry we attend only to the curve and the surface, there are crowds of theories which have scarcely any analogues in plane geometry. The relation of a curve to the various surfaces which can be drawn through it, or of a surface to the various curves that can be drawn upon it, is different in kind from that which in plane geometry most nearly corresponds to it, the relation of a system of points to the curves through them, or of a curve to the points upon it. In particular, there is nothing in plane geometry corresponding to the theory of the curves of curvature of a surface. To the single theorem of plane geometry, a right line is the shortest distance between two points, there correspond in solid geometry two extensive and difficult theories—that of the geodesic lines upon a given surface, and that of the surface of minimum area for any given boundary. Again, in solid geometry we have the interesting and difficult question of the representation of a curve by means of equations; it is not every curve, but only a curve which is the complete intersection of two surfaces, which can be properly represented by two equations $(x, y, z, w)^m = 0$, $(x, y, z, w)^n = 0$, in quadriplanar coordinates; and in regard to this question, which may also be regarded as that of the classification of curves in space, we have quite recently three elaborate memoirs by Nöther, Halphen, and Valentiner respectively.

In n-dimensional geometry, only isolated questions have been considered. The field is simply too wide; the comparison with each other of the two cases of plane geometry and solid geometry is enough to show how the complexity and difficulty of the theory would increase with each successive dimension.

In Transcendental Analysis, or the Theory of Functions, we have all that has been done in the present century with regard to the general theory of the function of an imaginary variable by Gauss, Cauchy, Puiseux, Briot, Bouquet, Liouville, Riemann, Fuchs, Weierstrass, and others. The fundamental idea of the geometrical representation of an imaginary variable $x+iy$, by means of the point having x, y for its coordinates, belongs, as I mentioned, to Gauss; of this I have already spoken at some length. The notion has been applied to differential equations; in the modern point of view, the problem in regard to a given differential equation is, not so much to reduce the differential equation to quadratures, as to determine from it directly the course of the integrals for all positions of the point representing the independent variable: in particular, the differential equation of the second order leading to the hypergeometric series $F(\alpha, \beta, \gamma, x)$ has been treated in this manner, with the most interesting results; the function so determined for all values of the parameters (α, β, γ) is thus becoming a known function. I would here also refer to the new notion in this part of analysis introduced by Weierstrass—that of the one-valued integer function, as defined by an

infinite series of ascending powers, convergent for all finite values, real or imaginary, of the variable x or $1/x-c$, and so having the one essential singular point $x=\infty$ or $x=c$, as the case may be: the memoir is published in the Berlin *Abhandlungen*, 1876.

But it is not only general theory: I have to speak of the various special functions to which the theory has been applied, or say the various known functions.

For a long time the only known transcendental functions were the circular functions sine, cosine, &c.; the logarithm—i.e. for analytical purposes the hyperbolic logarithm to the base e; and, as implied therein, the exponential function e^x. More completely stated, the group comprises the direct circular functions sin, cos, &c.; the inverse circular functions $\sin^{-1}$ or arc sin, &c.; the exponential function, exp.; and the inverse exponential, or logarithmic, function, log.

Passing over the very important Eulerian integral of the second kind or gamma-function, the theory of which has quite recently given rise to some very interesting developments—and omitting to mention at all various functions of minor importance,—we come (1811—1829) to the very wide groups, the elliptic functions and the single theta-functions. I give the interval of date so as to include Legendre's two systematic works, the *Exercices de Calcul Intégral* (1811—1816) and the *Théorie des Fonctions Elliptiques* (1825—1828); also Jacobi's *Fundamenta nova theoriæ Functionum Ellipticarum* (1829), calling to mind that many of Jacobi's results were obtained simultaneously by Abel. I remark that Legendre started from the consideration of the integrals depending on a radical $\sqrt{X}$, the square root of a rational and integral quartic function of a variable x; for this he substituted a radical $\Delta\phi$, $=\sqrt{1-k^2\sin^2\phi}$, and he arrived at his three kinds of elliptic integrals $F\phi$, $E\phi$, $\Pi\phi$, depending on the argument or amplitude ϕ, the modulus k, and also the last of them on a parameter n; the function F is properly an inverse function, and in place of it Abel and Jacobi each of them introduced the direct functions corresponding to the circular functions sine and cosine, Abel's functions called by him ϕ, f, F, and Jacobi's functions sinam, cosam, Δam, or as they are also written sn, cn, dn. Jacobi, moreover, in the development of his theory of transformation obtained a multitude of formulæ containing q, a transcendental function of the modulus defined by the equation $q=e^{-\pi K'/K}$, and he was also led by it to consider the two new functions H, Θ, which (taken each separately with two different arguments) are in fact the four functions called elsewhere by him Θ_1, Θ_2, Θ_3, Θ_4; these are the so-called theta-functions, or, when the distinction is necessary, the single theta-functions. Finally, Jacobi using the transformation $\sin\phi=\text{sinam}\,u$, expressed Legendre's integrals of the second and third kinds as integrals depending on the new variable u, denoting them by means of the letters Z, Π, and connecting them with his own functions H and Θ: and the elliptic functions sn, cn, dn are expressed with these, or say with Θ_1, Θ_2, Θ_3, Θ_4, as fractions having a common denominator.

It may be convenient to mention that Hermite in 1858, introducing into the theory in place of q the new variable ω connected with it by the equation $q=e^{i\pi\omega}$ (so that ω is in fact $=iK'/K$), was led to consider the three functions $\phi\omega$, $\psi\omega$, $\chi\omega$, which denote respectively the values of $\sqrt[4]{k}$, $\sqrt[4]{k'}$ and $\sqrt[12]{kk'}$ regarded as functions of ω.

A theta-function, putting the argument $=0$, and then regarding it as a function of ω, is what Professor Smith in a valuable memoir, left incomplete by his death, calls an omega-function, and the three functions $\phi\omega$, $\psi\omega$, $\chi\omega$ are his modular functions.

The proper elliptic functions sn, cn, dn form a system very analogous to the circular functions sine and cosine (say they are a sine and two separate cosines), having a like addition-theorem, viz. the form of this theorem is that the sn, cn and dn of $x+y$ are each of them expressible rationally in terms of the sn, cn and dn of x and of the sn, cn and dn of y; and, in fact, reducing itself to the system of the circular functions in the particular case $k=0$. But there is the important difference of form that the expressions for the sn, cn and dn of $x+y$ are fractional functions having a common denominator: this is a reason for regarding these functions as the ratios of four functions A, B, C, D, the absolute magnitudes of which are and remain indeterminate (the functions sn, cn, dn are in fact quotients $[\Theta_1, \Theta_2, \Theta_3] \div \Theta_4$ of the four theta-functions, but this is a further result in nowise deducible from the addition-equations, and which is intended to be for the moment disregarded; the remark has reference to what is said hereafter as to the Abelian functions). But there is in regard to the functions sn, cn, dn (what has no analogue for the circular functions), the whole theory of transformation of any order n prime or composite, and, as parts thereof, the whole theory of the modular and multiplier equations; and this theory of transformation spreads itself out in various directions, in geometry, in the Theory of Equations, and in the Theory of Numbers. Leaving the theta-functions out of consideration, the theory of the proper elliptic functions sn, cn, dn is at once seen to be a very wide one.

I assign to the Abelian functions the date 1826—1832. Abel gave what is called his theorem in various forms, but in its most general form in the *Mémoire sur une propriété générale d'une classe très-étendue de Fonctions Transcendantes* (1826), presented to the French Academy of Sciences, and crowned by them after the author's death, in the following year. This is in form a theorem of the integral calculus, relating to integrals depending on an irrational function y determined as a function of x by any algebraical equation $F(x, y)=0$ whatever: the theorem being that a sum of any number of such integrals is expressible by means of the sum of a determinate number p of like integrals, this number p depending on the form of the equation $F(x, y)=0$ which determines the irrational y (to fix the ideas, remark that considering this equation as representing a curve, then p is really the deficiency of the curve; but as already mentioned, the notion of deficiency dates only from 1857): thus in applying the theorem to the case where y is the square root of a function of the fourth order, we have in effect Legendre's theorem for elliptic integrals $F\phi + F\psi$ expressed by means of a single integral $F\mu$, and not a theorem applying in form to the elliptic functions sn, cn, dn. To be intelligible I must recall that the integrals belonging to the case where y is the square root of a rational and integral function of an order exceeding four are (in distinction from the general case) termed hyperelliptic integrals: viz. if the order be 5 or 6, then these are of the class $p=2$; if the order be 7 or 8, then they are of the class $p=3$, and so on; the *general* Abelian integral of the class $p=2$ is a hyperelliptic integral: but if $p=3$, or any greater

value, then the hyperelliptic integrals are only a particular case of the Abelian integrals of the same class. The further step was made by Jacobi in the short but very important memoir "Considerationes generales de transcendentibus Abelianis," *Crelle*, t. IX. (1832): viz. he there shows for the hyperelliptic integrals of any class (but the conclusion may be stated generally) that the direct functions to which Abel's theorem has reference are not functions of a single variable, such as the elliptic sn, cn, or dn, but functions of p variables. Thus, in the case $p=2$, which Jacobi specially considers, it is shown that Abel's theorem has reference to two functions $\lambda(u, v)$, $\lambda_1(u, v)$ each of two variables, and gives in effect an addition-theorem for the expression of the functions $\lambda(u+u', v+v')$, $\lambda_1(u+u', v+v')$ algebraically in terms of the functions $\lambda(u, v)$, $\lambda_1(u, v)$, $\lambda(u', v')$, $\lambda_1(u', v')$.

It is important to remark that Abel's theorem does not directly give, nor does Jacobi assert that it gives, the addition-theorem in a perfect form. Take the case $p=1$: the result from the theorem is that we have a function $\lambda(u)$, which is such that $\lambda(u+v)$ can be expressed algebraically in terms of $\lambda(u)$ and $\lambda(v)$. This is of course perfectly correct, $\operatorname{sn}(u+v)$ is expressible algebraically in terms of $\operatorname{sn} u$, $\operatorname{sn} v$, but the expression involves the radicals $\sqrt{1-\operatorname{sn}^2 u}$, $\sqrt{1-k^2\operatorname{sn}^2 u}$, $\sqrt{1-\operatorname{sn}^2 v}$, $\sqrt{1-k^2\operatorname{sn}^2 v}$; but it does not give the three functions sn, cn, dn, or in anywise amount to the statement that the sn, cn and $\operatorname{dn} u$ of $u+v$ are expressible rationally in terms of the sn, cn and dn of u and of v. In the case $p=1$, the right number of functions, each of one variable, is 3, but the three functions sn, cn and dn are properly considered as the ratios of 4 functions; and so, in general, the right number of functions, each of p variables, is 4^p-1, and these may be considered as the ratios of 4^p functions. But notwithstanding this last remark, it may be considered that the notion of the Abelian functions of p variables is established, and the addition-theorem for these functions in effect given by the memoirs (Abel 1826, Jacobi 1832) last referred to.

We have next for the case $p=2$, which is hyperelliptic, the two extremely valuable memoirs, Göpel, "Theoria transcendentium Abelianarum primi ordinis adumbratio læva," *Crelle*, t. XXXV. (1847), and Rosenhain, "Mémoire sur les fonctions de deux variables et à quatre périodes qui sont les inverses des intégrales ultra-elliptiques de la première classe" (1846), Paris, *Mém. Savans Étrang.* t. XI. (1851), each of them establishing on the analogy of the single theta-functions the corresponding functions of two variables, or double theta-functions, and in connexion with them the theory of the Abelian functions of two variables. It may be remarked that in order of simplicity the theta-functions certainly precede the Abelian functions.

Passing over some memoirs by Weierstrass which refer to the general hyperelliptic integrals, p any value whatever, we come to Riemann, who died 1866, at the age of forty: collected edition of his works, Leipzig, 1876. His great memoir on the Abelian and theta-functions is the memoir already incidentally referred to, "Theorie der Abel'schen Functionen," *Crelle*, t. LIV. (1857); but intimately connected therewith we have his Inaugural Dissertation (Göttingen, 1851), *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse:* his treatment of the problem of the Abelian functions, and establishment for the purpose of this theory of the multiple theta-functions, are alike founded on his general principles of the

theory of the functions of a variable complex magnitude $x+iy$, and it is this which would have to be gone into for any explanation of his method of dealing with the problem.

Riemann, starting with the integrals of the most general form, and considering the inverse functions corresponding to these integrals—that is, the Abelian functions of p variables—defines a theta-function of p variables, or p-tuple theta-function, as the sum of a p-tuply infinite series of exponentials, the general term of course depending on the p variables; and he shows that the Abelian functions are algebraically connected with theta-functions of the proper arguments. The theory is presented in the broadest form; in particular as regards the theta-functions, the 4^p functions are not even referred to, and there is no development as to the form of the algebraic relations between the two sets of functions.

In the Theory of Equations, the beginning of the century may be regarded as an epoch. Immediately preceding it, we have Lagrange's *Traité des Équations Numériques* (1st ed. 1798), the notes to which exhibit the then position of the theory. Immediately following it, the great work by Gauss, the *Disquisitiones Arithmeticæ* (1801), in which he establishes the theory for the case of a prime exponent n, of the binomial equation $x^n-1=0$: throwing out the factor $x-1$, the equation becomes an equation of the order $n-1$, and this is decomposed into equations the orders of which are the prime factors of $n-1$. In particular, Gauss was thereby led to the remarkable geometrical result that it was possible to construct geometrically—that is, with only the ruler and compass—the regular polygons of 17 sides and 257 sides respectively. We have then (1826—1829) Abel, who, besides his demonstration of the impossibility of the solution of a quintic equation by radicals, and his very important researches on the general question of the algebraic solution of equations, established the theory of the class of equations since called Abelian equations. He applied his methods to the problem of the division of the elliptic functions, to (what is a distinct question) the division of the complete functions, and to the very interesting special case of the lemniscate. But the theory of algebraic solutions in its most complete form was established by Galois (born 1811, killed in a duel 1832), who for this purpose introduced the notion of a group of substitutions; and to him also are due some most valuable results in relation to another set of equations presenting themselves in the theory of elliptic functions—viz. the modular equations. In 1835 we have Jerrard's transformation of the general quintic equation. In 1870 an elaborate work, Jordan's *Traité des Substitutions et des équations algébriques:* a mere inspection of the table of contents of this would serve to illustrate my proposition as to the great extension of this branch of mathematics.

The Theory of Numbers was, at the beginning of the century, represented by Legendre's *Théorie des Nombres* (1st ed. 1798), shortly followed by Gauss' *Disquisitiones Arithmeticæ* (1801). This work by Gauss is, throughout, a theory of ordinary real numbers. It establishes the notion of a congruence; gives a proof of the theorem of reciprocity in regard to quadratic residues; and contains a very complete theory of binary quadratic forms $(a, b, c)(x, y)^2$, of negative and positive determinant, including

the theory, there first given, of the composition of such forms. It gives also the commencement of a like theory of ternary quadratic forms. It contains also the theory already referred to, but which has since influenced in so remarkable a manner the whole theory of numbers—the theory of the solution of the binomial equation $x^n - 1 = 0$: it is, in fact, the roots or periods of roots derived from these equations which form the incommensurables, or unities, of the complex theories which have been chiefly worked at; thus, the i of ordinary analysis presents itself as a root of the equation $x^4 - 1 = 0$. It was Gauss himself who, for the development of a real theory—that of biquadratic residues—found it necessary to use complex numbers of the before-mentioned form, $a + bi$ (a and b positive or negative real integers, including zero), and the theory of these numbers was studied and cultivated by Lejeune-Dirichlet. We have thus a new theory of these complex numbers, side by side with the former theory of real numbers: everything in the real theory reproducing itself, prime numbers, congruences, theories of residues, reciprocity, quadratic forms, &c., but with greater variety and complexity, and increased difficulty of demonstration. But instead of the equation $x^4 - 1 = 0$, we may take the equation $x^3 - 1 = 0$: we have here the complex numbers $a + b\rho$ composed with an imaginary cube root of unity, the theory specially considered by Eisenstein: again a new theory, corresponding to but different from that of the numbers $a + bi$. The general case of any prime value of the exponent n, and with periods of roots, which here present themselves instead of single roots, was first considered by Kummer: viz. if $n - 1 = ef$, and $\eta_1, \eta_2, \ldots, \eta_e$ are the e periods, each of them a sum of f roots, of the equation $x^n - 1 = 0$, then the complex numbers considered are the numbers of the form $a_1\eta_1 + a_2\eta_2 + \ldots + a_e\eta_e$ ($a_1, a_2, \ldots, a_e$ positive or negative ordinary integers, including zero): f may be $= 1$, and the theory for the periods thus includes that for the single roots.

We have thus a new and very general theory, including within itself that of the complex numbers $a + bi$ and $a + b\rho$. But a new phenomenon presents itself; for these special forms the properties in regard to prime numbers corresponded precisely with those for real numbers; a non-prime number was in one way only a product of prime factors; the power of a prime number has only factors which are lower powers of the same prime number: for instance, if p be a prime number, then, excluding the obvious decomposition $p \,.\, p^2$, we cannot have $p^3 =$ a product of two factors A, B. In the general case this is not so, but the exception first presents itself for the number 23; in the theory of the numbers composed with the 23rd roots of unity, we have prime numbers p, such that $p^3 = AB$. To restore the theorem, it is necessary to establish the notion of ideal numbers; a prime number p is by definition not the product of two actual numbers, but in the example just referred to the number p is the product of two ideal numbers having for their cubes the two actual numbers A, B, respectively, and we thus have $p^3 = AB$. It is, I think, in this way that we most easily get some notion of the meaning of an ideal number, but the mode of treatment (in Kummer's great memoir, "Ueber die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren," *Crelle*, t. XXXV. 1847) is a much more refined one; an ideal number, without ever being isolated, is made to manifest itself in the properties of the prime number of which it is a factor, and without reference to the

theorem afterwards arrived at, that there is always some power of the ideal number which is an actual number. In the still later developments of the Theory of Numbers by Dedekind, the units, or incommensurables, are the roots of any irreducible equation having for its coefficients ordinary integer numbers, and with the coefficient unity for the highest power of x. The question arises, What is the analogue of a whole number? thus, for the very simple case of the equation $x^3+3=0$, we have as a whole number the apparently fractional form $\frac{1}{2}(1+i\sqrt{3})$ which is the imaginary cube root of unity, the ρ of Eisenstein's theory. We have, moreover, the (as far as appears) wholly distinct complex theory of the numbers composed with the congruence-imaginaries of Galois: viz. these are imaginary numbers assumed to satisfy a congruence which is not satisfied by any real number; for instance, the congruence $x^2-2=0$ (mod 5) has no real root, but we assume an imaginary root i, the other root is then $=-i$, and we then consider the system of complex numbers $a+bi$ (mod 5), viz. we have thus the 5^2 numbers obtained by giving to each of the numbers a, b, the values 0, 1, 2, 3, 4, successively. And so in general, the consideration of an irreducible congruence $F(x)=0$ (mod p) of the order n, to any prime modulus p, gives rise to an imaginary congruence root i, and to complex numbers of the form $a+bi+ci^2+\ldots+ki^{n-1}$, where a, b, k, ... &c., are ordinary integers each $=0, 1, 2, \ldots, p-1$.

As regards the theory of forms, we have in the ordinary theory, in addition to the binary and ternary quadratic forms, which have been very thoroughly studied, the quaternary and higher quadratic forms (to these last belong, as very particular cases, the theories of the representation of a number as a sum of four, five or more squares), and also binary cubic and quartic forms, and ternary cubic forms, in regard to all of which something has been done; the binary quadratic forms have been studied in the theory of the complex numbers $a+bi$.

A seemingly isolated question in the Theory of Numbers, the demonstration of Fermat's theorem of the impossibility for any exponent λ greater than 3, of the equation $x^\lambda+y^\lambda=z^\lambda$, has given rise to investigations of very great interest and difficulty.

Outside of ordinary mathematics, we have some theories which must be referred to: algebraical, geometrical, logical. It is, as in many other cases, difficult to draw the line; we do in ordinary mathematics use symbols not denoting quantities, which we nevertheless combine in the way of addition and multiplication, $a+b$, and ab, and which may be such as not to obey the commutative law $ab=ba$: in particular, this is or may be so in regard to symbols of operation; and it could hardly be said that any development whatever of the theory of such symbols of operation did not belong to ordinary algebra. But I do separate from ordinary mathematics the system of multiple algebra or linear associative algebra, developed in the valuable memoir by the late Benjamin Peirce, *Linear Associative Algebra* (1870, reprinted 1881 in the *American Journal of Mathematics*, vol. IV., with notes and addenda by his son, C. S. Peirce); we here consider symbols A, B, &c. which are linear functions of a determinate number of letters or units i, j, k, l, &c., with coefficients which are ordinary analytical magnitudes, real or imaginary, viz. the coefficients are in general of the form $x+iy$, where

i is the before-mentioned imaginary or $\sqrt{-1}$ of ordinary analysis. The letters i, j, &c., are such that every binary combination i^2, ij, ji, &c., (the ij being in general not $=ji$), is equal to a linear function of the letters, but under the restriction of satisfying the associative law: viz. for each combination of three letters $ij.k$ is $=i.jk$, so that there is a determinate and unique product of three or more letters; or, what is the same thing, the laws of combination of the units i, j, k, are defined by a multiplication table giving the values of i^2, ij, ji, &c.; the original units may be replaced by linear functions of these units, so as to give rise, for the units finally adopted, to a multiplication table of the most simple form; and it is very remarkable, how frequently in these simplified forms we have nilpotent or idempotent symbols ($i^2=0$, or $i^2=i$, as the case may be), and symbols i, j, such that $ij=ji=0$; and consequently how simple are the forms of the multiplication tables which define the several systems respectively.

I have spoken of this multiple algebra before referring to various geometrical theories of earlier date, because I consider it as the general analytical basis, and the true basis, of these theories. I do not realise to myself directly the notions of the addition or multiplication of two lines, areas, rotations, forces, or other geometrical, kinematical, or mechanical entities; and I would formulate a general theory as follows: consider any such entity as determined by the proper number of parameters a, b, c (for instance, in the case of a finite line given in magnitude and position, these might be the length, the coordinates of one end, and the direction-cosines of the line considered as drawn from this end); and represent it by or connect it with the linear function $ai+bj+ck+$ &c., formed with these parameters as coefficients, and with a given set of units, i, j, k, &c. Conversely, any such linear function represents an entity of the kind in question. Two given entities are represented by two linear functions; the sum of these is a like linear function representing an entity of the same kind, which may be regarded as the sum of the two entities; and the product of them (taken in a determined order, when the order is material) is an entity of the same kind, which may be regarded as the product (in the same order) of the two entities. We thus establish by definition the notion of the sum of the two entities, and that of the product (in a determinate order, when the order is material) of the two entities. The value of the theory in regard to any kind of entity would of course depend on the choice of a system of units, i, j, k, ..., with such laws of combination as would give a geometrical or kinematical or mechanical significance to the notions of the sum and product as thus defined.

Among the geometrical theories referred to, we have a theory (that of Argand, Warren, and Peacock) of imaginaries in plane geometry; Sir W. R. Hamilton's very valuable and important theory of Quaternions; the theories developed in Grassmann's *Ausdehnungslehre*, 1841 and 1862; Clifford's theory of Biquaternions; and recent extensions of Grassmann's theory to non-Euclidian space, by Mr Homersham Cox. These different theories have of course been developed, not in anywise from the point of view in which I have been considering them, but from the points of view of their several authors respectively.

The literal symbols x, y, &c., used in Boole's *Laws of Thought* (1854) to represent things as subjects of our conceptions, are symbols obeying the laws of algebraic com-

bination (the distributive, commutative, and associative laws) but which are such that for any one of them, say x, we have $x - x^2 = 0$, this equation not implying (as in ordinary algebra it would do) either $x = 0$ or else $x = 1$. In the latter part of the work relating to the Theory of Probabilities, there is a difficulty in making out the precise meaning of the symbols; and the remarkable theory there developed has, it seems to me, passed out of notice, without having been properly discussed. A paper by the same author, "Of Propositions numerically definite" (*Camb. Phil. Trans.* 1869), is also on the borderland of logic and mathematics. It would be out of place to consider other systems of mathematical logic, but I will just mention that Mr C. S. Peirce in his "Algebra of Logic," *American Math. Journal*, vol. III., establishes a notation for relative terms, and that these present themselves in connexion with the systems of units of the linear associative algebra.

Connected with logic, but primarily mathematical and of the highest importance, we have Schubert's *Abzählende Geometrie* (1878). The general question is, How many curves or other figures are there which satisfy given conditions? for example, How many conics are there which touch each of five given conics? The class of questions in regard to the conic was first considered by Chasles, and we have his beautiful theory of the characteristics μ, ν, of the conics which satisfy four given conditions; questions relating to cubics and quartics were afterwards considered by Maillard and Zeuthen; and in the work just referred to the theory has become a very wide one. The noticeable point is that the symbols used by Schubert are in the first instance, not numbers, but mere logical symbols: for example, a letter g denotes the condition that a line shall cut a given line; g^2 that it shall cut each of two given lines; and so in other cases; and these logical symbols are combined together by algebraical laws: they first acquire a numerical signification when the number of conditions becomes equal to the number of parameters upon which the figure in question depends.

In all that I have last said in regard to theories outside of ordinary mathematics, I have been still speaking on the text of the vast extent of modern mathematics. In conclusion I would say that mathematics have steadily advanced from the time of the Greek geometers. Nothing is lost or wasted; the achievements of Euclid, Archimedes, and Apollonius are as admirable now as they were in their own days. Descartes' method of coordinates is a possession for ever. But mathematics have never been cultivated more zealously and diligently, or with greater success, than in this century—in the last half of it, or at the present time: the advances made have been enormous, the actual field is boundless, the future full of hope. In regard to pure mathematics we may most confidently say:—

> Yet I doubt not through the ages one increasing purpose runs,
> And the thoughts of men are widened with the process of the suns.

785.

CURVE.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. VI. (1877), pp. 716—728.]

THIS subject is treated here from an historical point of view, for the purpose of showing how the different leading ideas in the theory were successively arrived at and developed.

A curve is a line, or continuous singly infinite system of points. We consider in the first instance, and chiefly, a plane curve described according to a law. Such a curve may be regarded geometrically as actually described, or kinematically as in course of description by the motion of a point; in the former point of view, it is the locus of all the points which satisfy a given condition; in the latter, it is the locus of a point moving subject to a given condition. Thus the most simple and earliest known curve, the circle, is the locus of all the points at a given distance from a fixed centre, or else the locus of a point moving so as to be always at a given distance from a fixed centre. (The straight line and the point are not for the moment regarded as curves.)

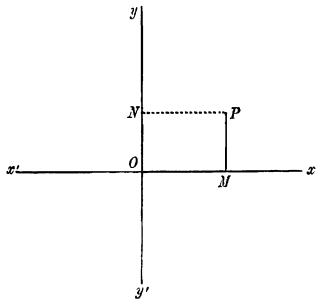
Next to the circle we have the conic sections, the invention of them attributed to Plato (who lived 430 to 347 B.C.); the original definition of them as the sections of a cone was by the Greek geometers who studied them soon replaced by a proper definition *in plano* like that for the circle, viz. a conic section (or as we now say a "conic") is the locus of a point such that its distance from a given point, the focus, is in a given ratio to its (perpendicular) distance from a given line, the directrix; or it is the locus of a point which moves so as always to satisfy the foregoing condition. Similarly any other property might be used as a definition; an ellipse is the locus of a point such that the sum of its distances from two fixed points (the foci) is constant, &c., &c.

The Greek geometers invented other curves; in particular, the "conchoid," which is the locus of a point such that its distance from a given line, measured along the

line drawn through it to a fixed point, is constant; and the "cissoid" which is the locus of a point such that its distance from a fixed point is always equal to the intercept (on the line through the fixed point) between a circle passing through the fixed point and the tangent to the circle at the point opposite to the fixed point. Obviously the number of such geometrical or kinematical definitions is infinite. In a machine of any kind, each point describes a curve; a simple but important instance is the "three-bar curve," or locus of a point in or rigidly connected with a bar pivotted on to two other bars which rotate about fixed centres respectively. Every curve thus arbitrarily defined has its own properties: and there was not any principle of classification.

The principle of classification first presented itself in the *Géométrie* of Descartes (1637). The idea was to represent any curve whatever by means of a relation between the coordinates (x, y) of a point of the curve, or say to represent the curve by means of its equation.

Descartes takes two lines xx', yy', called axes of coordinates, intersecting at a point O called the origin (the axes are usually at right angles to each other, and for the



present they are considered as being so); and he determines the position of a point P by means of its distances OM (or NP) $= x$, and MP (or ON) $= y$, from these two axes respectively; where x is regarded as positive or negative according as it is in the sense Ox or Ox' from O; and similarly y as positive or negative according as it is in the sense Oy or Oy' from O; or, what is the same thing,

				x	y
In the quadrant	xy,	or N.E.,	we have	+	+
„	$x'y$	„ N.W.	„	−	+
„	xy'	„ S.E.	„	+	−
„	$x'y'$	„ S.W.	„	−	−

Any relation whatever between (x, y) determines a curve, and conversely every curve whatever is determined by a relation between (x, y).

Observe that the distinctive feature is in the *exclusive* use of such determination of a curve by means of its equation. The Greek geometers were perfectly familiar with the property of an ellipse which in the Cartesian notation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of the curve; but it was as one of a number of properties, and in no wise selected out of the others for the characteristic property of the curve*.

We obtain from the equation the notion of an algebraical or geometrical as opposed to a transcendental curve, viz. an algebraical or geometrical curve is a curve having an equation $F(x, y)=0$, where $F(x, y)$ is a rational and integral algebraical function of the coordinates (x, y); and in what follows we attend throughout (unless the contrary is stated) only to such curves. The equation is sometimes given, and may conveniently be used, in an irrational form, but we always imagine it reduced to the foregoing rational and integral form, and regard this as the equation of the curve. And we have hence the notion of a curve of a *given order*, viz. the order of the curve is equal to that of the term or terms of highest order in the coordinates (x, y) conjointly in the equation of the curve; for instance, $xy-1=0$ is a curve of the second order.

It is to be noticed here that the axes of coordinates may be any two lines at right angles to each other whatever; and that the equation of a curve will be different according to the selection of the axes of coordinates; but the order is independent of the axes, and has a determinate value for any given curve.

We hence divide curves according to their order, viz. a curve is of the first order, second order, third order, &c., according as it is represented by an equation of the first order, $ax+by+c=0$, or say $(*\between x, y, 1)=0$; or by an equation of the second order, $ax^2+2hxy+by^2+2fy+2gx+c=0$, say $(*\between x, y, 1)^2=0$; or by an equation of the third order, &c.; or, what is the same thing, according as the equation is linear, quadric, cubic, &c.

A curve of the first order is a right line; and conversely every right line is a curve of the first order.

* There is no exercise more profitable for a student than that of tracing a curve from its equation, or say rather that of so tracing a considerable number of curves. And he should make the equations for himself. The equation should be in the first instance a purely numerical one, where y is given or can be found as an explicit function of x; here, by giving different numerical values to x, the corresponding values of y may be found; and a *sufficient* number of points being thus determined, the curve is traced by drawing a continuous line through these points. The next step should be to consider an equation involving literal coefficients; thus, after such curves as $y=x^3$, $y=x(x-1)(x-2)$, $y=(x-1)\sqrt{x-2}$, &c., he should proceed to trace such curves as $y=(x-a)(x-b)(x-c)$, $y=(x-a)\sqrt{x-b}$, &c., and endeavour to ascertain for what different relations of equality or inequality between the coefficients the curve will assume essentially or notably distinct forms. The purely numerical equations will present instances of nodes, cusps, inflexions, double tangents, asymptotes, &c.,—specialities which he should be familiar with before he has to consider their general theory. And he may then consider an equation such that neither coordinate can be expressed as an explicit function of the other of them (practically, an equation such as $x^3+y^3-3xy=0$, which requires the solution of a cubic equation, belongs to this class); the problem of tracing the curve here frequently requires special methods, and it may easily be such as to require and serve as an exercise for the powers of an advanced algebraist.

A curve of the second order is a conic, or as it is also called a quadric; and conversely every conic, or quadric, is a curve of the second order.

A curve of the third order is called a cubic; one of the fourth order a quartic; and so on.

A curve of the order m has for its equation $(*\between x, y, 1)^m = 0$; and when the coefficients of the function are arbitrary, the curve is said to be the general curve of the order m. The number of coefficients is $\frac{1}{2}(m+1)(m+2)$; but there is no loss of generality if the equation be divided by one coefficient so as to reduce the coefficient of the corresponding term to unity, hence the number of coefficients may be reckoned as $\frac{1}{2}(m+1)(m+2)-1$, that is, $\frac{1}{2}m(m+3)$; and a curve of the order m may be made to satisfy this number of conditions; for example, to pass through $\frac{1}{2}m(m+3)$ points.

It is to be remarked that an equation may *break up;* thus a quadric equation may be $(ax+by+c)(a'x+b'y+c')=0$, breaking up into the two equations $ax+by+c=0$, $a'x+b'y+c'=0$, viz. the original equation is satisfied if either of these is satisfied. Each of these last equations represents a curve of the first order, or right line; and the original equation represents this pair of lines, viz. the pair of lines is considered as a quadric curve. But it is an *improper* quadric curve; and in speaking of curves of the second or any other given order, we frequently imply that the curve is a proper curve represented by an equation which does not break up.

The intersections of two curves are obtained by combining their equations; viz. the elimination from the two equations of y (or x) gives for x (or y) an equation of a certain order, say the resultant equation; and then to each value of x (or y) satisfying this equation there corresponds in general a single value of y (or x), and consequently a single point of intersection; the number of intersections is thus equal to the order of the resultant equation in x (or y).

Supposing that the two curves are of the orders m, n, respectively, then the order of the resultant equation is in general and at most $=mn$; in particular, if the curve of the order n is an arbitrary line $(n=1)$, then the order of the resultant equation is $=m$; and the curve of the order m meets therefore the line in m points. But the resultant equation may have all or any of its roots imaginary, and it is thus not always that there are m real intersections.

The notion of imaginary intersections, thus presenting itself, through algebra, in geometry, must be accepted in geometry—and it in fact plays an all-important part in modern geometry. As in algebra we say that an equation of the mth order has m roots, viz. we state this generally without in the first instance, or it may be without ever, distinguishing whether these are real or imaginary; so in geometry we say that a curve of the mth order is met by an arbitrary line in m points, or rather we thus, through algebra, obtain the proper geometrical definition of a curve of the mth order, as a curve which is met by an arbitrary line in m points (that is, of course, in m, and not more than m, points).

The theorem of the m intersections has been stated in regard to an *arbitrary* line; in fact, for particular lines the resultant equation may be or appear to be of

an order less than m; for instance, taking $m=2$, if the hyperbola $xy-1=0$ be cut by the line $y=\beta$, the resultant equation in x is $\beta x-1=0$, and there is apparently only the intersection $\left(x=\frac{1}{\beta},\ y=\beta\right)$; but the theorem is, in fact, true for every line whatever: a curve of the order m meets every line whatever in precisely m points. We have, in the case just referred to, to take account of a point at infinity on the line $y=\beta$; the two intersections are the point $\left(x=\frac{1}{\beta},\ y=\beta\right)$, and the point at infinity on the line $y=\beta$.

It is moreover to be noticed that the points at infinity may be all or any of them imaginary, and that the points of intersection, whether finite or at infinity, real or imaginary, may coincide two or more of them together, and have to be counted accordingly; to support the theorem in its universality, it is necessary to take account of these various circumstances.

The foregoing notion of a point at infinity is a very important one in modern geometry; and we have also to consider the paradoxical statement that in plane geometry, or say as regards the plane, infinity is a right line. This admits of an easy illustration in solid geometry. If with a given centre of projection, by drawing from it lines to every point of a given line, we project the given line on a given plane, the projection is a line, i.e., this projection is the intersection of the given plane with the plane through the centre and the given line. Say the projection is *always* a line, then if the figure is such that the two planes are parallel, the projection is the intersection of the given plane by a parallel plane, or it is the system of points at infinity on the given plane, that is, these points at infinity are regarded as situate on a given line, the line infinity of the given plane*.

Reverting to the purely plane theory, infinity is a line, related like any other right line to the curve, and thus intersecting it in m points, real or imaginary, distinct or coincident.

Descartes in the *Géométrie* defined and considered the remarkable curves called after him ovals of Descartes, or simply Cartesians, which will be again referred to. The next important work, founded on the *Géométrie*, was Sir Isaac Newton's *Enumeratio linearum tertii ordinis* (1706), establishing a classification of cubic curves founded chiefly on the nature of their infinite branches, which was in some details completed by Stirling, Murdoch, and Cramer; the work contains also the remarkable theorem (to be again referred to), that there are five kinds of cubic curves giving by their projections every cubic curve whatever.

Various properties of curves in general, and of cubic curves, are established in Maclaurin's memoir, "De linearum geometricarum proprietatibus generalibus Tractatus" (posthumous, say 1746, published in the 6th edition of his *Algebra*). We have in it a particular kind of *correspondence* of two points on a cubic curve, viz. two points correspond to each other when the tangents at the two points again meet the cubic in the same point.

* More generally, in solid geometry infinity is a plane,—its intersection with any given plane being the right line which is the infinity of this given plane.

The *Géométrie Descriptive* by Monge was written in the year 1794 or 1795 (7th edition, Paris, 1847), and in it we find stated, *in plano* with regard to the circle, and in three dimensions with regard to a surface of the second order, the fundamental theorem of reciprocal polars, viz. "Given a surface of the second order and a circumscribed conic surface which touches it.... then if the conic surface moves so that its summit is always in the same plane, the plane of the curve of contact passes always through the same point." The theorem is here referred to partly on account of its bearing on the theory of imaginaries in geometry. It is, in Brianchon's memoir "Sur les surfaces du second degré" (*Jour. Polyt.*, t. VI., 1806), shown how for any given position of the summit the plane of contact is determined, or reciprocally; say the plane XY is determined when the point P is given, or reciprocally; and it is noticed that when P is situate in the interior of the surface the plane XY does not cut the surface; that is, we have a real plane XY intersecting the surface in the imaginary curve of contact of the imaginary circumscribed cone having for its summit a given real point P inside the surface.

Stating the theorem in regard to a conic, we have a real point P (called the pole) and a real line XY (called the polar), the line joining the two (real or imaginary) points of contact of the (real or imaginary) tangents drawn from the point to the conic; and the theorem is that when the point describes a line the line passes through a point, this line and point being polar and pole to each other. The term "pole" was first used by Servois, and "polar" by Gergonne (*Gerg.*, t. I. and III., 1810—13); and from the theorem we have the method of reciprocal polars for the transformation of geometrical theorems, used already by Brianchon (in the memoir above referred to) for the demonstration of the theorem called by his name, and in a similar manner by various writers in the earlier volumes of Gergonne. We are here concerned with the method less in itself than as leading to the general notion of duality. And, bearing in a somewhat similar manner also on the theory of imaginaries in geometry (but the notion presents itself in a more explicit form), there is the memoir by Gaultier, on the graphical construction of circles and spheres (*Jour. Polyt.*, t. IX., 1813). The well-known theorem as to radical axes may be stated as follows. Consider two circles partially drawn so that it does not appear whether the circles, if completed, would or would not intersect in real points, say two arcs of circles; then we can, by means of a third circle drawn so as to intersect in two real points each of the two arcs, determine a right line, which, if the complete circles intersect in two real points, passes through the points, and which is on this account regarded as a line passing through two (real or imaginary) points of intersection of the two circles. The construction in fact is, join the two points in which the third circle meets the first arc, and join also the two points in which the third circle meets the second arc, and from the point of intersection of the two joining lines, let fall a perpendicular on the line joining the centre of the two circles; this perpendicular (considered as an indefinite line) is what Gaultier terms the "radical axis of the two circles"; it is a line determined by a real construction and itself always real; and by what precedes it is the line joining two (real or imaginary, as the case may be) intersections of the given circles.

The intersections which lie on the radical axis are two out of the four intersections of the two circles. The question as to the remaining two intersections did not present itself to Gaultier, but it is answered in Poncelet's *Traité des propriétés projectives* (1822), where we find (p. 49) the statement, "deux circles placés arbitrairement sur un plan...ont idéalement deux points imaginaires communs à l'infini"; that is, a circle *qua* curve of the second order is met by the line infinity in two points; but, more than this, they are the same two points for any circle whatever. The points in question have since been called (it is believed first by Dr Salmon) the circular points at infinity, or they may be called the circular points; these are also frequently spoken of as the points I, J; and we have thus the circle characterized as a conic which passes through the two circular points at infinity; the number of conditions thus imposed upon the conic is $=2$, and there remain three arbitrary constants, which is the right number for the circle. Poncelet throughout his work makes continual use of the foregoing theories of imaginaries and infinity, and also of the before-mentioned theory of reciprocal polars.

Poncelet's two memoirs "Sur les centres des moyennes harmoniques," and "Sur la théorie générale des polaires réciproques," although presented to the Paris Academy in 1824 were only published (*Crelle*, t. III. and IV., 1828, 1829), subsequent to the memoir by Gergonne, "Considérations philosophiques sur les élémens de la science de l'étendue" (*Gerg.*, t. XVI., 1825—26). In this memoir by Gergonne, the theory of duality is very clearly and explicitly stated; for instance, we find "dans la géométrie plane, à chaque théorème il en répond nécessairement un autre qui s'en déduit en échangeant simplement entre eux les deux mots *points* et *droites*; tandis que dans la géométrie de l'espace ce sont les mots *points* et *plans* qu'il faut échanger entre eux pour passer d'un théorème à son corrélatif"; and the plan is introduced of printing correlative theorems, opposite to each other, in two columns. There was a reclamation as to priority by Poncelet in the *Bulletin Universel* reprinted with remarks by Gergonne (*Gerg.*, t. XIX., 1827), and followed by a short paper by Gergonne, "Rectifications de quelques théorèmes, &c.," which is important as first introducing the word *class*. We find in it explicitly the two correlative definitions:—"a plane curve is said to be of the mth degree (order) when it has with a line m real or ideal intersections," and "a plane curve is said to be of the mth class when from any point of its plane there can be drawn to it m real or ideal tangents."

It may be remarked that in Poncelet's memoir on reciprocal polars, above referred to, we have the theorem that the number of tangents from a point to a curve of the order m, or say the class of the curve, is in general and at most $=m(m-1)$, and that he mentions that this number is subject to reduction when the curve has double points or cusps.

The theorem of duality as regards plane figures may be thus stated:—two figures may correspond to each other in such manner that to each point and line in either figure there corresponds in the other figure a line and point respectively. It is to be understood that the theorem extends to all points or lines, drawn or not drawn; thus if in the first figure there are any number of points on a line drawn or not drawn, the corresponding lines in the second figure, produced if necessary, must meet

in a point. And we thus see how the theorem extends to curves, their points and tangents: if there is in the first figure a curve of the order m, any line meets it in m points; and hence from the corresponding point in the second figure there must be to the corresponding curve m tangents; that is, the corresponding curve must be of the class m.

Trilinear coordinates (to be again referred to) were first used by Bobillier in the memoir, "Essai sur un nouveau mode de recherche des propriétés de l'étendue" (*Gerg.*, t. XVIII., 1827—28). It is convenient to use these rather than Cartesian coordinates. We represent a curve of the order m by an equation $(*\between x, y, z)^m = 0$, the function on the left-hand being a homogeneous rational and integral function of the order m of the three coordinates (x, y, z); clearly the number of constants is the same as for the equation $(*\between x, y, 1)^m = 0$ in Cartesian coordinates.

The theory of duality is considered and developed, but chiefly in regard to its metrical applications, by Chasles in the "Mémoire de géométrie sur deux principes généraux de la science, la dualité et l'homographie," which forms a sequel to the "Aperçu historique sur l'origine et le développement des méthodes en géométrie" (*Mem. de Brux.*, t. XI., 1837).

We now come to Plücker; his "six equations" were given in a short memoir in *Crelle* (1842) preceding his great work, the *Theorie der algebraischen Curven* (1844).

Plücker first gave a scientific dual definition of a curve, viz. "A curve is a locus generated by a point, and enveloped by a line,—the point moving continuously along the line, while the line rotates continuously about the point"; the point is a point (ineunt) of the curve, the line is a tangent of the curve.

And, assuming the above theory of geometrical imaginaries, a curve such that m of its points are situate in an arbitrary line is said to be of the order m; a curve such that n of its tangents pass through an arbitrary point is said to be of the class n; as already appearing, this notion of the order and the class of a curve is, however, due to Gergonne. Thus the line is a curve of the order 1 and the class 0; and corresponding dually thereto, we have the point as a curve of the order 0 and the class 1.

Plücker moreover imagined a system of line-coordinates (tangential coordinates). The Cartesian coordinates (x, y) and trilinear coordinates (x, y, z) are point-coordinates for determining the position of a point; the new coordinates, say (ξ, η, ζ), are line-coordinates for determining the position of a line. It is possible, and (not so much for any application thereof as in order to more fully establish the analogy between the two kinds of coordinates) important, to give independent quantitative definitions of the two kinds of coordinates; but we may also derive the notion of line-coordinates from that of point-coordinates; viz. taking $\xi x + \eta y + \zeta z = 0$ to be the equation of a line, we say that (ξ, η, ζ) are the line-coordinates of this line. A linear relation $a\xi + b\eta + c\zeta = 0$ between these coordinates determines a point, viz. the point whose point-coordinates are (a, b, c); in fact, the equation in question $a\xi + b\eta + c\zeta = 0$ expresses that the equation $\xi x + \eta y + \zeta z = 0$, where (x, y, z) are current point-coordinates, is satisfied on writing therein $x, y, z = a, b, c$; or that the line in question passes through

the point (a, b, c). Thus (ξ, η, ζ) are the line-coordinates of any line whatever; but when these, instead of being absolutely arbitrary, are subject to the restriction $a\xi + b\eta + c\zeta = 0$, this obliges the line to pass through a point (a, b, c); and the last-mentioned equation $a\xi + b\eta + c\zeta = 0$ is considered as the line-equation of this point.

A line has only a point-equation, and a point has only a line-equation; but any other curve has a point-equation and also a line-equation; the point-equation $(*\between x, y, z)^m = 0$ is the relation which is satisfied by the point-coordinates (x, y, z) of each point of the curve; and similarly the line-equation $(*\between \xi, \eta, \zeta)^n = 0$ is the relation which is satisfied by the line-coordinates (ξ, η, ζ) of each line (tangent) of the curve.

There is in analytical geometry little occasion for any explicit use of line-coordinates; but the theory is very important; it serves to show that, in demonstrating by point-coordinates any purely descriptive theorem whatever, we demonstrate the correlative theorem; that is, we do not demonstrate the one theorem, and then (as by the method of reciprocal polars) deduce from it the other, but we do at one and the same time demonstrate the two theorems; our (x, y, z) instead of meaning point-coordinates may mean line-coordinates, and the demonstration is then in every step of it a demonstration of the correlative theorem.

The above dual generation explains the nature of the singularities of a plane curve. The ordinary singularities, arranged according to a cross division, are

	Proper.	*Improper.*
Point-singularities—	1. The stationary point, cusp, or spinode;	2. The double point, or node;
Line-singularities—	3. The stationary tangent, or inflexion;	4. The double tangent:—

arising as follows:—

1. The cusp: the point as it travels along the line may come to rest, and then reverse the direction of its motion.

3. The stationary tangent: the line may in the course of its rotation come to rest, and then reverse the direction of its rotation.

2. The node: the point may in the course of its motion come to coincide with a former position of the point, the two positions of the line not in general coinciding.

4. The double tangent: the line may in the course of its motion come to coincide with a former position of the line, the two positions of the point not in general coinciding.

It may be remarked that we cannot with a real point and line obtain the node with two imaginary tangents (conjugate or isolated point, or acnode), nor again the real double tangent with two imaginary points of contact; but this is of little consequence, since in the general theory the distinction between real and imaginary is not attended to.

The singularities (1) and (3) have been termed proper singularities, and (2) and (4) improper; in each of the first-mentioned cases there is a real singularity, or

peculiarity in the motion; in the other two cases there is not; in (2) there is not when the point is first at the node, or when it is secondly at the node, any peculiarity in the motion; the singularity consists in the point coming twice into the same position; and so in (4) the singularity is in the line coming twice into the same position. Moreover (1) and (2) are, the former a proper singularity, and the latter an improper singularity, *as regards the motion of the point;* and similarly (3) and (4) are, the former a proper singularity, and the latter an improper singularity, *as regards the motion of the line.*

But as regards the representation of a curve by an equation, the case is very different.

First, if the equation be in point-coordinates, (3) and (4) are in a sense not singularities at all. The curve $(*\between x, y, z)^m = 0$, or general curve of the order m, has double tangents and inflexions; (2) presents itself as a singularity, for the equations $d_x(*\between x, y, z)^m = 0$, $d_y(*\between x, y, z)^m = 0$, $d_z(*\between x, y, z)^m = 0$, implying $(*\between x, y, z)^m = 0$, are not in general satisfied by any values (a, b, c) whatever of (x, y, z), but if such values exist, then the point (a, b, c) is a node or double point; and (1) presents itself as a further singularity or sub-case of (2), a cusp being a double point for which the two tangents become coincident.

In line-coordinates all is reversed:—(1) and (2) are not singularities; (3) presents itself as a sub-case of (4).

The theory of compound singularities will be referred to further on.

In regard to the ordinary singularities, we have

m,	the order,		
n	„	class,	
δ	„	number of double points,	
ι	„	„	cusps,
τ	„	„	double tangents,
κ	„	„	inflexions;

and this being so, Plücker's "six equations" are

(1) $n = m(m-1) - 2\delta - 3\kappa,$

(2) $\iota = 3m(m-2) - 6\delta - 8\kappa,$

(3) $\tau = \frac{1}{2}m(m-2)(m^2-9) - (m^2-m-6)(2\delta+3\kappa) + 2\delta(\delta-1) + 6\delta\kappa + \frac{9}{2}\kappa(\kappa-1),$

(4) $m = n(n-1) - 2\tau - 3\iota,$

(5) $\kappa = 3n(n-2) - 6\tau - 8\iota,$

(6) $\delta = \frac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2\tau+3\iota) + 2\tau(\tau-1) + 6\tau\iota + \frac{9}{2}\iota(\iota-1).$

It is easy to derive the further forms—

$$
\begin{array}{lll}
(7) & \iota - \kappa & = 3(n-m), \\
(8) & 2(\tau - \delta) & = \quad (n-m)(n+m-9), \\
(9) & \frac{1}{2}m(m+3) - \delta - 2\kappa & = \frac{1}{2}n(n+3) - \tau - 2\iota, \\
(10) & \frac{1}{2}(m-1)(m-2) - \delta - \kappa & = \frac{1}{2}(n-1)(n-2) - \tau - \iota, \\
(11, 12) & m^2 - 2\delta - 3\kappa & = n^2 - 2\tau - 3\iota, \; = m + n,
\end{array}
$$

the whole system being equivalent to three equations only: and it may be added that, using α to denote the equal quantities $3m + \iota$ and $3n + \kappa$, everything may be expressed in terms of m, n, α. We have

$$
\begin{aligned}
\kappa &= \alpha - 3n, \\
\iota &= \alpha - 3m, \\
2\delta &= m^2 - m + 8n - 3\alpha, \\
2\tau &= n^2 - n + 8m - 3\alpha.
\end{aligned}
$$

It is implied in Plücker's theorem that, m, n, δ, κ, τ, ι signifying as above in regard to any curve, then in regard to the reciprocal curve n, m, τ, ι, δ, κ will have the same significations, viz. for the reciprocal curve these letters denote respectively the order, class, number of nodes, cusps, double tangents, and inflexions.

The expression $\frac{1}{2}m(m+3) - \delta - 2\kappa$ is that of the number of the disposable constants in a curve of the order m with δ nodes and κ cusps (in fact that there shall be a node is 1 condition, a cusp 2 conditions): and the equation (9) thus expresses that the curve and its reciprocal contain each of them the same number of disposable constants.

For a curve of the order m, the expression $\frac{1}{2}m(m-1) - \delta - \kappa$ is termed the "deficiency" (as to this more hereafter); the equation (10) expresses therefore that the curve and its reciprocal have each of them the same deficiency.

The relations $m^2 - 2\delta - 3\kappa = n^2 - 2\tau - 3\iota, \; = m + n$, present themselves in the theory of envelopes, as will appear further on.

With regard to the demonstration of Plücker's equations it is to be remarked that we are not able to write down the equation in point-coordinates of a curve of the order m, having the given numbers δ and κ of nodes and cusps. We can only use the general equation $(*\between x, y, z)^m = 0$, say for shortness $u = 0$, of a curve of the mth order, which equation, so long as the coefficients remain arbitrary, represents a curve without nodes or cusps. Seeking then, for this curve, the values n, ι, τ of the class, number of inflexions, and number of double tangents,—first, as regards the class, this is equal to the number of tangents which can be drawn to the curve from an arbitrary point, or what is the same thing, it is equal to the number of the points of contact of these tangents. The points of contact are found as the intersections of the curve $u = 0$ by a curve depending on the position of the arbitrary point, and called the "first polar" of this point; the order of the first polar is $= m - 1$, and

the number of intersections is thus $=m(m-1)$. But it can be shown, analytically or geometrically, that if the given curve has a node, the first polar passes through this node, which therefore counts as two intersections: and that if the curve has a cusp, the first polar passes through the cusp, touching the curve there, and hence the cusp counts as three intersections. But, as is evident, the node or cusp is not a point of contact of a proper tangent from the arbitrary point; we have, therefore, for a node a diminution 2, and for a cusp a diminution 3, in the number of the intersections; and thus, for a curve with δ nodes and κ cusps, there is a diminution $2\delta+3\kappa$, and the value of n is $n=m(m-1)-2\delta-3\kappa$.

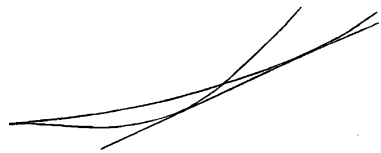
Secondly, as to the inflexions, the process is a similar one; it can be shown that the inflexions are the intersections of the curve by a derivative curve called (after Hesse, who first considered it) the Hessian, defined geometrically as the locus of a point such that its conic polar in regard to the curve breaks up into a pair of lines, and which has an equation $H=0$, where H is the determinant formed with the second differential coefficients of u in regard to the variables (x, y, z); $H=0$ is thus a curve of the order $3(m-2)$, and the number of inflexions is $=3m(m-2)$. But if the given curve has a node, then not only the Hessian passes through the node, but it has there a node the two branches at which touch respectively the two branches of the curve, and the node thus counts as six intersections; so if the curve has a cusp, then the Hessian not only passes through the cusp, but it has there a cusp through which it again passes, that is, there is a cuspidal branch touching the cuspidal branch of the curve, and besides a simple branch passing through the cusp, and hence the cusp counts as eight intersections. The node or cusp is not an inflexion, and we have thus for a node a diminution 6, and for a cusp a diminution 8, in the number of the intersections; hence for a curve with δ nodes and κ cusps, the diminution is $=6\delta+8\kappa$, and the number of inflexions is $\iota=3m(m-2)-6\delta-8\kappa$.

Thirdly, for the double tangents; the points of contact of these are obtained as the intersections of the curve by a curve $\Pi=0$, which has not as yet been geometrically defined, but which is found analytically to be of the order $(m-2)(m^2-9)$; the number of intersections is thus $=m(m-2)(m^2-9)$; but if the given curve has a node then there is a diminution $=4(m^2-m-6)$, and if it has a cusp then there is a diminution $=6(m^2-m-6)$, where, however, it is to be noticed that the factor (m^2-m-6) is in the case of a curve having only a node or only a cusp the number of the tangents which can be drawn from the node or cusp to the curve, and is used as denoting the number of these tangents, and ceases to be the correct expression if the number of nodes and cusps is greater than unity. Hence, in the case of a curve which has δ nodes and κ cusps, the apparent diminution $2(m^2-m-6)(2\delta+3\kappa)$ is too great, and it has in fact to be diminished by $2\{2\delta(\delta-1)+6\delta\kappa+\frac{9}{2}\kappa(\kappa-1)\}$, or the half thereof is 4 for each pair of nodes, 6 for each combination of a node and cusp, and 9 for each pair of cusps. We have thus finally an expression for 2τ, $=m(m-2)(m^2-9)-$&c.; or dividing the whole by 2, we have the expression for τ given by the third of Plücker's equations.

It is obvious that we cannot by consideration of the equation $u=0$ in point-coordinates obtain the remaining three of Plücker's equations; they might be obtained

in a precisely analogous manner by means of the equation $v=0$ in line-coordinates, but they follow at once from the principle of duality, viz. they are obtained by the mere interchange of m, δ, κ with n, τ, ι respectively.

To complete Plücker's theory it is necessary to take account of compound singularities; it might be possible, but it is at any rate difficult, to effect this by considering the curve as in course of description by the point moving along the rotating line; and it seems easier to consider the compound singularity as arising from the variation of an actually described curve with ordinary singularities. The most simple case is when three double points come into coincidence, thereby giving rise to a triple point; and a somewhat more complicated one is when we have a cusp of the second kind, or node-cusp arising from the coincidence of a node, a cusp, an inflexion, and a double tangent, as shown in the annexed figure, which represents the singularities as on the



point of coalescing. The general conclusion (see Cayley, *Quart. Math. Jour.* t. VII., 1866, [374], "On the higher singularities of a plane curve") is that every singularity whatever may be considered as compounded of ordinary singularities, say we have a singularity $=\delta'$ nodes, κ' cusps, τ' double tangents, and ι' inflexions. So that, in fact, Plücker's equations properly understood apply to a curve with any singularities whatever.

By means of Plücker's equations we may form a table—

m	n	δ	κ	τ	ι
0	1	—	—	0	0
1	0	0	0	—	—
2	2	0	0	0	0
3	6	0	0	0	9
,,	4	1	0	0	3
,,	3	0	1	0	1
4	12	0	0	28	24
,,	10	1	0	16	18
,,	9	0	1	10	16
,,	8	2	0	8	12
,,	7	1	1	4	10
,,	6	0	2	1	8
,,	6	3	0	4	6
,,	5	2	1	2	4
,,	4	1	2	1	2
,,	3	0	3	1	0

The table is arranged according to the value of m; and we have $m=0$, $n=1$, the point; $m=1$, $n=0$, the line; $m=2$, $n=2$, the conic; of $m=3$, the cubic, there are three cases, the class being 6, 4, or 3, according as the curve is without singularities, or as it has 1 node, or 1 cusp; and so of $m=4$, the quartic, there are nine cases, where observe that in *two* of them the class is $=6$,—the reduction of class arising from two cusps or else from three nodes. The nine cases may be also grouped together into four, according as the number of nodes and cusps $(\delta+\kappa)$ is $=0$, 1, 2, or 3.

The cases may be divided into sub-cases, by the consideration of compound singularities; thus when $m=4$, $n=6$, $=3$, the three nodes may be all distinct, which is the general case, or two of them may unite together into the singularity called a tacnode, or all three may unite together into a triple point, or else into an oscnode.

We may further consider the inflexions and double tangents, as well in general as in regard to cubic and quartic curves.

The expression for the number of inflexions $3m(m-2)$ for a curve of the order m was obtained analytically by Plücker, but the theory was first given in a complete form by Hesse in the two papers "Ueber die Elimination, u.s.w.," and "Ueber die Wendepuncte der Curven dritter Ordnung" (*Crelle*, t. XXVIII., 1844); in the latter of these the points of inflexion are obtained as the intersections of the curve $u=0$ with the Hessian, or curve $\Delta=0$, where Δ is the determinant formed with the second derived functions of u. We have in the Hessian the first instance of a covariant of a ternary form. The whole theory of the inflexions of a cubic curve is discussed in a very interesting manner by means of the canonical form of the equation $x^3+y^3+z^3+6lxyz=0$; and in particular a proof is given of Plücker's theorem that the nine points of inflexion of a cubic curve lie by threes in twelve lines.

It may be noticed that the nine inflexions of a cubic curve are three real, six imaginary; the three real inflexions lie in a line, as was known to Newton and Maclaurin. For an acnodal cubic the six imaginary inflexions disappear, and there remain three real inflexions lying in a line. For a crunodal cubic, the six inflexions which disappear are two of them real, the other four imaginary, and there remain two imaginary inflexions and one real inflexion. For a cuspidal cubic the six imaginary inflexions and two of the real inflexions disappear, and there remains one real inflexion.

A quartic curve has 24 inflexions; it was conjectured by Salmon, and has been verified recently by Zeuthen, that at most 8 of these are real.

The expression $\frac{1}{2}m(m-2)(m^2-9)$ for the number of double tangents of a curve of the order m was obtained by Plücker only as a consequence of his first, second, fourth, and fifth equations. An investigation by means of the curve $\Pi=0$, which by its intersections with the given curve determines the points of contact of the double tangents, is indicated by Cayley, "Recherches sur l'élimination et la théorie des courbes", (*Crelle*, t. XXXIV., 1847), [53]: and in part carried out by Hesse in the memoir "Ueber Curven dritter Ordnung" (*Crelle*, t. XXXVI., 1848). A better process was indicated by Salmon in the "Note on the double tangents to plane curves," *Phil. Mag.* 1858; considering the $m-2$ points in which any tangent to the curve again meets the

curve, he showed how to form the equation of a curve of the order $(m-2)$, giving by its intersection with the tangent the points in question; making the tangent touch this curve of the order $(m-2)$, it will be a double tangent of the original curve. See Cayley, "On the Double Tangents of a Plane Curve", (*Phil. Trans.* t. CXLVIII., 1859), [260], and Dersch (*Math. Ann.* t. VII., 1874). The solution is still in so far incomplete that we have no properties of the curve $\Pi=0$, to distinguish one such curve from the several other curves which pass through the points of contact of the double tangents.

A quartic curve has 28 double tangents, their points of contact determined as the intersections of the curve by a curve $\Pi=0$ of the order 14, the equation of which in a very elegant form was first obtained by Hesse (1849). Investigations in regard to them are given by Plücker in the *Theorie der algebraischen Curven*, and in two memoirs by Hesse and Steiner (*Crelle*, t. XLV., 1855), in respect to the triads of double tangents which have their points of contact on a conic, and other like relations. It was assumed by Plücker that the number of real double tangents might be 28, 16, 8, 4, or 0, but Zeuthen has recently found that the last case does not exist.

The Hessian Δ has just been spoken of as a covariant of the form u; the notion of invariants and covariants belongs rather to the form u than to the curve $u=0$ represented by means of this form; and the theory may be very briefly referred to. A curve $u=0$ may have some invariantive property, viz. a property independent of the particular axes of coordinates used in the representation of the curve by its equation; for instance, the curve may have a node, and in order to this, a relation, say $A=0$, must exist between the coefficients of the equation; supposing the axes of coordinates altered, so that the equation becomes $u'=0$, and writing $A'=0$ for the relation between the new coefficients, then the relations $A=0$, $A'=0$, as two different expressions of the same geometrical property, must each of them imply the other; this can only be the case when A, A' are functions differing only by a constant factor, or say, when A is an invariant of u. If, however, the geometrical property requires two or more relations between the coefficients, say $A=0$, $B=0$, &c., then we must have between the new coefficients the like relations, $A'=0$, $B'=0$, &c., and the two systems of equations must each of them imply the other; when this is so, the system of equations, $A=0$, $B=0$, &c., is said to be invariantive, but it does not follow that A, B, &c., are of necessity invariants of u. Similarly, if we have a curve $U=0$ derived from the curve $u=0$ in a manner independent of the particular axes of coordinates, then from the transformed equation $u'=0$ deriving in like manner the curve $U'=0$, the two equations $U=0$, $U'=0$ must each of them imply the other; and when this is so, U will be a covariant of u. The case is less frequent, but it may arise, that there are covariant systems $U=0$, $V=0$, &c., and $U'=0$, $V'=0$, &c., each implying the other, but where the functions U, V, &c., are not of necessity covariants of u.

The theory of the invariants and covariants of a ternary cubic function u has been studied in detail, and brought into connexion with the cubic curve $u=0$; but the theory of the invariants and covariants for the next succeeding case, the ternary quartic function, is still very incomplete.

In further illustration of the Plückerian dual generation of a curve, we may consider the question of the *envelope* of a variable curve. The notion is very probably older, but it is at any rate to be found in Lagrange's *Théorie des fonctions analytiques* (1798); it is there remarked that the equation obtained by the elimination of the parameter a from an equation $f(x, y, a)=0$ and the derived equation in respect to a is a curve, the envelope of the series of curves represented by the equation $f(x, y, a)=0$ in question. To develope the theory, consider the curve corresponding to any particular value of the parameter; this has with the consecutive curve (or curve belonging to the consecutive value of the parameter) a certain number of intersections, and of common tangents, which may be considered as the tangents at the intersections; and the so-called envelope is the curve which is at the same time generated by the points of intersection and enveloped by the common tangents; we have thus a dual generation. But the question needs to be further examined. Suppose that in general the variable curve is of the order m with δ nodes and κ cusps, and therefore of the class n with τ double tangents and ι inflexions, $m, n, \delta, \kappa, \tau, \iota$ being connected by the Plückerian equations,—the number of nodes or cusps may be greater for particular values of the parameter, but this is a speciality which may be here disregarded. Considering the variable curve corresponding to a given value of the parameter, or say simply the variable curve, the consecutive curve has then also δ and κ nodes and cusps, consecutive to those of the variable curve; and it is easy to see that among the intersections of the two curves we have the nodes each counting twice, and the cusps each counting three times; the number of the remaining intersections is $=m^2-2\delta-3\kappa$. Similarly among the common tangents of the two curves we have the double tangents each counting twice, and the stationary tangents each counting three times, and the number of the remaining common tangents is $=n^2-2\tau-3\iota$ $(=m^2-2\delta-3\kappa$, inasmuch as each of these numbers is as was seen $=m+n)$. At any one of the $m^2-2\delta-3\kappa$ points the variable curve and the consecutive curve have tangents distinct from yet infinitesimally near to each other, and each of these two tangents is also infinitesimally near to one of the $n^2-2\tau-3\iota$ common tangents of the two curves; whence, attending only to the variable curve, and considering the consecutive curve as coming into actual coincidence with it, the $n^2-2\tau-3\iota$ common tangents are the tangents to the variable curve at the $m^2-2\delta-3\kappa$ points respectively, and the envelope is at the same time generated by the $m^2-2\delta-3\kappa$ points, and enveloped by the $n^2-2\tau-3\iota$ tangents; we have thus a dual generation of the envelope, which only differs from Plücker's dual generation, in that in place of a single point and tangent we have the group of $m^2-2\delta-3\kappa$ points and $n^2-2\tau-3\iota$ tangents.

The parameter which determines the variable curve may be given as a point upon a given curve, or say as a parametric point; that is, to the different positions of the parametric point on the given curve correspond the different variable curves, and the nature of the envelope will thus depend on that of the given curve; we have thus the envelope as a derivative curve of the given curve. Many well-known derivative curves present themselves in this manner; thus the variable curve may be the normal (or line at right angles to the tangent) at any point of the given curve; the intersection of the consecutive normals is the centre of curvature; and we have the evolute

as at once the locus of the centre of curvature and the envelope of the normal. It may be added that the given curve is one of a series of curves, each cutting the several normals at right angles. Any one of these is a "parallel" of the given curve; and it can be obtained as the envelope of a circle of constant radius having its centre on the given curve. We have in like manner, as derivatives of a given curve, the caustic, catacaustic, or diacaustic, as the case may be, and the secondary caustic, or curve cutting at right angles the reflected or refracted rays.

We have in much that precedes disregarded, or at least been indifferent to, reality; it is only thus that the conception of a curve of the mth order, as one which is met by every right line in m points, is arrived at; and the curve itself, and the line which cuts it, although both are tacitly assumed to be real, may perfectly well be imaginary. For real figures we have the general theorem that imaginary intersections, &c., present themselves in conjugate pairs; hence, in particular, that a curve of an even order is met by a line in an even number (which may be $=0$) of points; a curve of an odd order in an odd number of points, hence in one point at least; it will be seen further on that the theorem may be generalized in a remarkable manner. Again, when there is in question only one pair of points or lines, these, if coincident, must be real; thus, a line meets a cubic curve in three points, one of them real, the other two real or imaginary; but if two of the intersections coincide they must be real, and we have a line cutting a cubic in one real point and touching it in another real point. It may be remarked that this is a limit separating the two cases where the intersections are all real, and where they are one real, two imaginary.

Considering always real curves, we obtain the notion of a branch; any portion capable of description by the continuous motion of a point is a branch; and a curve consists of one or more branches. Thus the curve of the first order or right line consists of one branch; but in curves of the second order, or conics, the ellipse and the parabola consist each of one branch, the hyperbola of two branches. A branch is either re-entrant, or it extends both ways to infinity, and in this case, we may regard it as consisting of two legs (*crura*, Newton), each extending one way to infinity, but without any definite separation. The branch, whether re-entrant or infinite, may have a cusp or cusps, or it may cut itself or another branch, thus having or giving rise to crunodes; an acnode is a branch by itself,—it may be considered as an indefinitely small re-entrant branch. A branch may have inflexions and double tangents, or there may be double tangents which touch two distinct branches; there are also double tangents with imaginary points of contact, which are thus lines having no visible connexion with the curve. A re-entrant branch not cutting itself may be everywhere convex, and it is then properly said to be an oval; but the term oval may be used more generally for any re-entrant branch not cutting itself; and we may thus speak of a once indented, twice indented oval, &c., or even of a cuspidate oval. Other descriptive names for ovals and re-entrant branches cutting themselves may be used when required; thus, in the last-mentioned case a simple form is that of a figure of eight; such a form may break up into two ovals, or into a doubly indented oval or hour-glass. A form which presents itself is when two ovals, one inside the other, unite, so as to give rise to a crunode—in default of a better name this may be called,

after the curve of that name, a limaçon. Names may also be used for the different forms of infinite branches, but we have first to consider the distinction of hyperbolic and parabolic. The leg of an infinite branch may have at the extremity a tangent; this is an asymptote of the curve, and the leg is then hyperbolic; or the leg may tend to a fixed direction, but so that the tangent goes further and further off to infinity, and the leg is then parabolic; a branch may thus be hyperbolic or parabolic as to its two legs; or it may be hyperbolic as to one leg, and parabolic as to the other. The epithets hyperbolic and parabolic are of course derived from the conics hyperbola and parabola respectively. The nature of the two kinds of branches is best understood by considering them as projections, in the same way as we in effect consider the hyperbola and the parabola as projections of the ellipse. If a line Ω cut an arc aa', so that the two segments ab, ba' lie on opposite sides of the line, then projecting the figure so that the line Ω goes off to infinity, the tangent at b is projected into the asymptote, and the arc ab is projected into a hyperbolic leg touching the asymptote at one extremity; the arc ba' will at the same time be projected into a hyperbolic leg touching the same asymptote at the other extremity (and on the opposite side), but so that the two hyperbolic legs may or may not belong to one and the same branch. And we thus see that the two hyperbolic legs belong to a simple intersection of the curve by the line infinity. Next, if the line Ω touch at b the arc aa' so that the two portions ab', ba lie on the same side of the line Ω, then projecting the figure as before, the tangent at b, that is, the line Ω itself, is projected to infinity; the arc ab is projected into a parabolic leg, and at the same time the arc ba' is projected into a parabolic leg, having at infinity the same direction as the other leg, but so that the two legs may or may not belong to the same branch. And we thus see that the two parabolic legs represent a contact of the line infinity with the curve,—the point of contact being of course the point at infinity determined by the common direction of the two legs. It will readily be understood how the like considerations apply to other cases,—for instance, if the line Ω is a tangent at an inflexion, passes through a crunode, or touches one of the branches of a crunode, &c.; thus, if the line Ω passes through a crunode we have pairs of hyperbolic legs belonging to two parallel asymptotes. The foregoing considerations also show (what is very important) how different branches are connected together at infinity, and lead to the notion of a complete branch, or circuit.

The two legs of a hyperbolic branch may belong to different asymptotes, and in this case we have the forms which Newton calls inscribed, circumscribed, ambigene, &c.; or they may belong to the same asymptote, and in this case we have the serpentine form, where the branch cuts the asymptote, so as to touch it at its two extremities on opposite sides, or the conchoidal form, where it touches the asymptote on the same side. The two legs of a parabolic branch may converge to ultimate parallelism, as in the conic parabola, or diverge to ultimate parallelism, as in the semi-cubical parabola $y^2 = x^3$, and the branch is said to be convergent, or divergent, accordingly; or they may tend to parallelism in opposite senses, as in the cubical parabola $y = x^3$. As mentioned with regard to a branch generally, an infinite branch of any kind may have cusps, or, by cutting itself or another branch, may have or give rise to a crunode, &c.

We may now consider the various forms of cubic curves, as appearing by Newton's *Enumeratio*, and by the figures belonging thereto. The species are reckoned as 72, which are numbered accordingly 1 to 72; but to these should be added 10^a, 13^a, 22^a, and 22^b. It is not intended here to consider the division into species, nor even completely that into genera, but only to explain the principle of classification. It may be remarked generally that there are at most three infinite branches, and that there may besides be a re-entrant branch or oval.

The genera may be arranged as follows:—

1, 2, 3, 4	redundant hyperbolas,
5, 6	defective hyperbolas,
7, 8	parabolic hyperbolas,
9	hyperbolisms of hyperbola,
10	" " ellipse,
11	" " parabola,
12	trident curve,
13	divergent parabolas,
14	cubic parabola;

and, thus arranged, they correspond to the different relations of the line infinity to the curve. First, if the three intersections by the line infinity are all distinct, we have the hyperbolas; if the points are real, the redundant hyperbolas, with three hyperbolic branches; but if only one of them is real, the defective hyperbolas, with one hyperbolic branch. Secondly, if two of the intersections coincide, say if the line infinity meets the curve in a onefold point and a twofold point, both of them real, then there is always one asymptote: the line infinity may at the twofold point touch the curve, and we have the parabolic hyperbolas; or the twofold point may be a singular point,—viz. a crunode giving the hyperbolisms of the hyperbola; an acnode, giving the hyperbolisms of the ellipse; or a cusp, giving the hyperbolisms of the parabola. As regards the so-called hyperbolisms, observe that (besides the single asymptote) we have in the case of those of the hyperbola two parallel asymptotes; in the case of those of the ellipse the two parallel asymptotes become imaginary, that is, they disappear, and in the case of those of the parabola they become coincident, that is, there is here an ordinary asymptote, and a special asymptote answering to a cusp at infinity. Thirdly, the three intersections by the line infinity may be coincident and real; or say we have a threefold point: this may be an inflexion, a crunode, or a cusp, that is, the line infinity may be a tangent at an inflexion, and we have the divergent parabolas; a tangent at a crunode to one branch, and we have the trident curve; or lastly, a tangent at a cusp, and we have the cubical parabola.

It is to be remarked that the classification mixes together non-singular and singular curves, in fact, the five kinds presently referred to: thus the hyperbolas and the divergent parabolas include curves of every kind, the separation being made in the

species; the hyperbolisms of the hyperbola and ellipse, and the trident curve, are nodal; the hyperbolisms of the parabola, and the cubical parabola, are cuspidal. The divergent parabolas are of five species which respectively belong to and determine the five kinds of cubic curves; Newton gives (in two short paragraphs without any development) the remarkable theorem that the five divergent parabolas by their shadows generate and exhibit all the cubic curves.

The five divergent parabolas are curves each of them symmetrical with regard to an axis. There are two non-singular kinds, the one with, the other without, an oval, but each of them has an infinite (as Newton describes it) *campaniform* branch; this cuts the axis at right angles, being at first convex, but ultimately concave, towards the axis, the two legs continually tending to become at right angles to the axis. The oval may unite itself with the infinite branch, or it may dwindle into a point, and we have the crunodal and the acnodal forms respectively; or if simultaneously the oval dwindles into a point and unites itself to the infinite branch, we have the cuspidal form. Drawing a line to cut any one of these curves and projecting the line to infinity, it would not be difficult to show how the line should be drawn in order to obtain a curve of any given species. We have herein a better principle of classification; considering cubic curves, in the first instance, according to singularities, the curves are non-singular, nodal (viz. crunodal or acnodal), or cuspidal; and we see further that there are two kinds of non-singular curves, the complex and the simplex. There is thus a complete division into the five kinds, the complex, simplex, crunodal, acnodal, and cuspidal. Each singular kind presents itself as a limit separating two kinds of inferior singularity; the cuspidal separates the crunodal and the acnodal, and these last separate from each other the complex and the simplex.

The whole question is discussed very fully and ably by Möbius in the memoir "Ueber die Grundformen der Linien dritter Ordnung" (*Abh. der K. Sachs. Ges. zu Leipzig*, t. I., 1852; *Ges. Werke*, t. I.). The author considers not only plane curves, but also cones, or, what is almost the same thing, the spherical curves which are their sections by a concentric sphere. Stated in regard to the cone, we have there the fundamental theorem that there are two different kinds of sheets: viz. the single sheet, not separated into two parts by the vertex (an instance is afforded by the plane considered as a cone of the first order generated by the motion of a line about a point), and the double or twin-pair sheet, separated into two parts by the vertex (as in the cone of the second order). And it then appears that there are two kinds of non-singular cubic cones, viz. the simplex, consisting of a single sheet, and the complex, consisting of a single sheet and a twin-pair sheet; and we thence obtain (as for cubic curves) the crunodal, the acnodal, and the cuspidal kinds of cubic cones. It may be mentioned that the single sheet is a sort of wavy form, having upon it three lines of inflexion, and which is met by any plane through the vertex in one or in three lines; the twin-pair sheet has no lines of inflexion, and resembles in its form a cone on an oval base.

In general a cone consists of one or more single or twin-pair sheets, and if we consider the section of the cone by a plane, the curve consists of one or more complete branches, or say circuits, each of them the section of one sheet of the cone;

thus, a cone of the second order is one twin-pair sheet, and any section of it is one circuit composed, it may be, of two branches. But although we thus arrive by projection at the notion of a circuit, it is not necessary to go out of the plane, and we may (with Zeuthen, using the shorter term *circuit* for his *complete branch*) define a circuit as any portion (of a curve) capable of description by the continuous motion of a point, it being understood that a passage through infinity is permitted. And we then say that a curve consists of one or more circuits; thus the right line, or curve of the first order, consists of one circuit; a curve of the second order consists of one circuit; a cubic curve consists of one circuit or else of two circuits.

A circuit is met by any right line always in an even number, or always in an odd number, of points, and it is said to be an even circuit or an odd circuit accordingly; the right line is an odd circuit, the conic an even circuit. And we have then the theorem, two odd circuits intersect in an odd number of points; an odd and an even circuit, or two even circuits, in an even number of points. An even circuit not cutting itself divides the plane into two parts, the one called the internal part, incapable of containing any odd circuit, the other called the external part, capable of containing an odd circuit.

We may now state in a more convenient form the fundamental distinction of the kinds of cubic curve. A non-singular cubic is simplex, consisting of one odd circuit, or it is complex, consisting of one odd circuit and one even circuit. It may be added that there are on the odd circuit three inflexions, but on the even circuit no inflexion; it hence also appears that from any point of the odd circuit there can be drawn to the odd circuit two tangents, and to the even circuit (if any) two tangents, but that from a point of the even circuit there cannot be drawn (either to the odd or the even circuit) any real tangent; consequently, in a simplex curve the number of tangents from any point is two; but in a complex curve the number is four, or none,—four if the point is on the odd circuit, none if it is on the even circuit. It at once appears from inspection of the figure of a non-singular cubic curve, which is the odd and which the even circuit. The singular kinds arise as before; in the crunodal and the cuspidal kinds the whole curve is an odd circuit, but in the acnodal kind the acnode must be regarded as an even circuit.

The analogous question of the classification of quartics (in particular non-singular quartics and nodal quartics) is considered in Zeuthen's memoir "Sur les différentes formes des courbes planes du quatrième ordre" (*Math. Ann.* t. VII., 1874). A non-singular quartic has only even circuits; it has at most four circuits external to each other, or two circuits one internal to the other, and in this last case the internal circuit has no double tangents or inflexions. A very remarkable theorem is established as to the double tangents of such a quartic:—distinguishing as a double tangent of the first kind a real double tangent which either twice touches the same circuit, or else touches the curve in two imaginary points, the number of the double tangents of the first kind of a non-singular quartic is $=4$; it follows that the quartic has at most 8 real inflexions. The forms of the non-singular quartics are very numerous, but it is not necessary to go further into the question.

We may consider in relation to a curve, not only the line infinity, but also the circular points at infinity; assuming the curve to be real, these present themselves always conjointly; thus a circle is a conic passing through the two circular points, and is thereby distinguished from other conics. Similarly a cubic through the two circular points is termed a circular cubic; a quartic through the two points is termed a circular quartic, and if it passes twice through each of them, that is, has each of them for a node, it is termed a bicircular quartic. Such a quartic is of course binodal ($m=4$, $\delta=2$, $\kappa=0$); it has not in general, but it may have, a third node, or a cusp. Or again, we may have a quartic curve having a cusp at each of the circular points: such a curve is a "Cartesian," it being a complete definition of the Cartesian to say that it is a bicuspidal quartic curve ($m=4$, $\delta=0$, $\kappa=2$), having a cusp at each of the circular points. The circular cubic and the bicircular quartic, together with the Cartesian (being in one point of view a particular case thereof), are interesting curves which have been much studied, generally, and in reference to their *focal* properties.

The points called *foci* presented themselves in the theory of the conic, and were well known to the Greek geometers, but the general notion of a focus was first established by Plücker, in the memoir "Ueber solche Puncte die bei Curven einer höheren Ordnung den Brennpuncten der Kegelschnitte entsprechen," (*Crelle*, t. x., 1833). We may from each of the circular points draw tangents to a given curve; the intersection of two such tangents (belonging of course to the two circular points respectively) is a focus. There will be from each circular point λ tangents (λ, a number depending on the class of the curve and its relation to the line infinity and the circular points, $=2$ for the general conic, 1 for the parabola, 2 for a circular cubic or a bicircular quartic, &c.); the λ tangents from the one circular point and those from the other circular point intersect in λ real foci (viz. each of these is the only real point on each of the tangents through it), and in $\lambda^2-\lambda$ imaginary foci; each pair of real foci determines a pair of imaginary foci (the so-called antipoints of the two real foci), and the $\frac{1}{2}\lambda(\lambda-1)$ pairs of real foci thus determine the $\lambda^2-\lambda$ imaginary foci. There are in some cases points termed centres, or singular or multiple foci (the nomenclature is unsettled), which are the intersections of improper tangents from the two circular points respectively; thus, in the circular cubic, the tangents to the curve at the two circular points respectively (or two imaginary asymptotes of the curve) meet in a centre.

The notions of *distance* and of lines *at right angles* are connected with the circular points; and almost every construction of a curve by means of lines of a determinate length, or at right angles to each other, and (as such) mechanical constructions by means of linkwork, give rise to curves passing the same definite number of times through the two circular points respectively, or say to circular curves, and in which the fixed centres of the construction present themselves as ordinary, or as singular, foci. Thus the general curve of three-bar motion (or locus of the vertex of a triangle, the other two vertices whereof move on fixed circles) is a tricircular sextic, having besides three nodes ($m=6$, $\delta=3+3+3$, $=9$), and having the centres of the fixed circles each for a singular focus; there is a third singular focus, and we have thus the remarkable theorem (due to Mr S. Roberts) of the triple generation of the curve by means of the three several pairs of singular foci.

Again, the normal, *qua* line at right angles to the tangent, is connected with the circular points, and these accordingly present themselves in the before-mentioned theories of evolutes and parallel curves.

We have several recent theories which depend on the notion of *correspondence:* two points whether in the same plane or in different planes, or on the same curve or in different curves, may determine each other in such wise that to any given position of the first point there correspond α' positions of the second point, and to any given position of the second point α positions of the first point; the two points have then an (α, α') correspondence; and if α, α' are each $=1$, then the two points have a (1, 1) or rational correspondence. Connecting with each theory the author's name, the theories in question are—Riemann, the rational transformation of a plane curve; Cremona, the rational transformation of a plane; and Chasles, correspondence of points on the same curve, and united points. The theory first referred to, with the resulting notion of Geschlecht, or *deficiency*, is more than the other two an essential part of the theory of curves, but they will all be considered.

Riemann's results are contained in the memoirs on "Theorie der Abel'schen Functionen," (*Crelle*, t. LIV., 1857); and we have next Clebsch, "Ueber die Singularitäten algebraischer Curven," (*Crelle*, t. LXV., 1865), and Cayley, "On the Transformation of Plane Curves," (*Proc. Lond. Math. Soc.* t. I., 1865, [384]). The fundamental notion of the rational transformation is as follows:—

Taking u, X, Y, Z to be rational and integral functions (X, Y, Z all of the same order) of the coordinates (x, y, z), and u', X', Y', Z' rational and integral functions (X', Y', Z' all of the same order) of the coordinates (x', y', z'), we transform a given curve $u=0$, by the equations $x' : y' : z' = X : Y : Z$, thereby obtaining a transformed curve $u'=0$, and a converse set of equations $x : y : z = X' : Y' : Z'$; viz. assuming that this is so, the point (x, y, z) on the curve $u=0$ and the point (x', y', z') on the curve $u'=0$ will be points having a (1, 1) correspondence. To show how this is, observe that to a given point (x, y, z) on the curve $u=0$ there corresponds a single point (x', y', z') determined by the equations $x' : y' : z' = X : Y : Z$; from these equations and the equation $u=0$ eliminating x, y, z we obtain the equation $u'=0$ of the transformed curve. To a given point (x', y', z') not on the curve $u'=0$ there corresponds, not a single point, but the system of points (x, y, z) given by the equations $x' : y' : z' = X : Y : Z$, viz. regarding x', y', z' as constants (and to fix the ideas, assuming that the curves $X=0$, $Y=0$, $Z=0$ have no common intersections), these are the points of intersection of the curves $X : Y : Z = x' : y' : z'$, but no one of these points is situate on the curve $u=0$. If, however, the point (x', y', z') is situate on the curve $u'=0$, then one point of the system of points in question is situate on the curve $u=0$, that is, to a given point of the curve $u'=0$ there corresponds a single point of the curve $u=0$; and hence also this point must be given by a system of equations such as $x : y : z = X' : Y' : Z'$.

It is an old and easily proved theorem that, for a curve of the order m, the number $\delta+\kappa$ of nodes and cusps is at most $=\frac{1}{2}(m-1)(m-2)$; for a given curve the deficiency of the actual number of nodes and cusps below this maximum number, viz.

$\frac{1}{2}(m-1)(m-2)-\delta-\kappa$, is the "Geschlecht," or "deficiency," of the curve, say this is $=D$. When $D=0$, the curve is said to be unicursal, when $=1$, bicursal, and so on.

The general theorem is that two curves corresponding rationally to each other have the same deficiency. In particular, a curve and its reciprocal have this rational or (1, 1) correspondence, and it has been already seen that a curve and its reciprocal have the same deficiency.

A curve of a given order can in general be rationally transformed into a curve of a lower order; thus a curve of any order for which $D=0$, that is, a unicursal curve, can be transformed into a line; a curve of any order having the deficiency 1 or 2 can be rationally transformed into a curve of the order $D+2$, deficiency D; and a curve of any order deficiency $=$ or >3 can be rationally transformed into a curve of the order $D+3$, deficiency D.

Taking x', y', z' as coordinates of a point of the transformed curve, and in its equation writing $x' : y' : z' = 1 : \theta : \phi$ we have ϕ a certain irrational function of θ, and the theorem is that the coordinates x, y, z of any point of the given curve can be expressed as proportional to rational and integral functions of θ, ϕ, that is, of θ and a certain irrational function of θ.

In particular, if $D=0$, that is, if the given curve be unicursal, the transformed curve is a line, ϕ is a mere linear function of θ, and the theorem is that the coordinates x, y, z of a point of the unicursal curve can be expressed as proportional to rational and integral functions of θ; it is easy to see that for a given curve of the order m, these functions of θ must be of the same order m.

If $D=1$, then the transformed curve is a cubic; it can be shown that in a cubic, the axes of coordinates being properly chosen, ϕ can be expressed as the square root of a quartic function of θ; and the theorem is that the coordinates x, y, z of a point of the bicursal curve can be expressed as proportional to rational and integral functions of θ, and of the square root of a quartic function of θ.

And so if $D=2$, then the transformed curve is a nodal quartic; ϕ can be expressed as the square root of a sextic function of θ, and the theorem is, that the coordinates x, y, z of a point of the tricursal curve can be expressed as proportional to rational and integral functions of θ, and of the square root of a sextic function of θ. But when $D=3$, we have no longer the like law, viz. ϕ is not expressible as the square root of an octic function of θ.

Observe that the radical, square root of a quartic function, is connected with the theory of elliptic functions, and the radical, square root of a sextic function, with that of the first kind of Abelian functions, but that the next kind of Abelian functions does not depend on the radical, square root of an octic function.

It is a form of the theorem for the case $D=1$, that the coordinates x, y, z of a point of the bicursal curve, or in particular the coordinates of a point of the cubic, can be expressed as proportional to rational and integral functions of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$; in fact, taking the radical to be $\sqrt{1-\theta^2 . 1-k^2\theta^2}$, and writing

$\theta = \operatorname{sn} u$, the radical becomes $= \operatorname{cn} u \,.\, \operatorname{dn} u$; and we have expressions of the form in question.

It will be observed that the equations $x' : y' : z' = X : Y : Z$ before-mentioned do not of themselves lead to the other system of equations $x : y : z = X' : Y' : Z'$, and thus that the theory does not in anywise establish a (1, 1) correspondence between the points (x, y, z) and (x', y', z') of two planes or of the same plane; this is the correspondence of Cremona's theory.

In this theory, given in the memoirs "Sulle trasformazioni geometriche delle figure piane," *Mem. di Bologna*, t. II. (1863), and t. V. (1865), we have a system of equations $x' : y' : z' = X : Y : Z$ which *does* lead to a system $x : y : z = X' : Y' : Z'$, where, as before, X, Y, Z denote rational and integral functions, all of the same order, of the coordinates x, y, z, and X', Y', Z' rational and integral functions, all of the same order, of the coordinates x', y', z', and there is thus a (1, 1) correspondence given by these equations between the two points (x, y, z) and (x', y', z'). To explain this, observe that starting from the equations $x' : y' : z' = X : Y : Z$, to a given point (x, y, z) there corresponds one point (x', y', z'), but that if n be the order of the functions X, Y, Z, then to a given point x', y', z' there would, if the curves $X = 0$, $Y = 0$, $Z = 0$ had no common intersections, correspond n^2 points (x, y, z). If, however, the functions are such that the curves $X = 0$, $Y = 0$, $Z = 0$ have k common intersections, then among the n^2 points are included these k points, which are fixed points independent of the point (x', y', z'); so that, disregarding these fixed points, the number of points (x, y, z) corresponding to the given point (x', y', z') is $= n^2 - k$; and in particular if $k = n^2 - 1$, then we have one corresponding point; and hence the original system of equations $x' : y' : z' = X : Y : Z$ must lead to the equivalent system $x : y : z = X' : Y' : Z'$; and in this system by the like reasoning the functions must be such that the curves $X' = 0$, $Y' = 0$, $Z' = 0$ have $n'^2 - 1$ common intersections. The most simple example is in the two systems of equations $x' : y' : z' = yz : zx : xy$ and $x : y : z = y'z' : z'x' : x'y'$; where $yz = 0$, $zx = 0$, $xy = 0$ are conics (pairs of lines) having three common intersections, and where obviously either system of equations leads to the other system. In the case where X, Y, Z are of an order exceeding 2, the required number $n^2 - 1$ of common intersections can only occur by reason of common multiple points on the three curves; and assuming that the curves $X = 0$, $Y = 0$, $Z = 0$ have $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1}$ common intersections, where the α_1 points are ordinary points, the α_2 points are double points, the α_3 points are triple points, &c., on each curve, we have the condition

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 + \ldots + (n-1)^2 \alpha_{n-1} = n^2 - 1\,;$$

but to this must be joined the condition

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha_{n-1} = \tfrac{1}{2}n(n+3) - 2,$$

(without which the transformation would be illusory); and the conclusion is that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ may be any numbers satisfying these two equations. It may be added that the two equations together give

$$\alpha_2 + 3\alpha_3 + \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

which expresses that the curves $X=0$, $Y=0$, $Z=0$ are unicursal. The transformation may be applied to any curve $u=0$, which is thus rationally transformed into a curve $u'=0$, by a rational transformation such as is considered in Riemann's theory; hence the two curves have the same deficiency.

Coming next to Chasles, the principle of correspondence is established and used by him in a series of memoirs relating to the conics which satisfy given conditions, and to other geometrical questions, contained in the *Comptes Rendus*, t. LVIII. *et seq.* (1864 to the present time). The theorem of united points in regard to points in a right line was given in a paper, June—July 1864, and it was extended to unicursal curves in a paper of the same series (March 1866), "Sur les courbes planes ou à double courbure dont les points peuvent se déterminer individuellement—application du principe de correspondance dans la théorie de ces courbes."

The theorem is as follows: if in a unicursal curve two points have an (α, β) correspondence, then the number of united points (or points each corresponding to itself) is $=\alpha+\beta$. In fact, in a unicursal curve the coordinates of a point are given as proportional to rational and integral functions of a parameter, so that any point of the curve is determined uniquely by means of this parameter; that is, to each point of the curve corresponds one value of the parameter, and to each value of the parameter one point on the curve; and the (α, β) correspondence between the two points is given by an equation of the form $(*\between\between\theta, 1)^\alpha(\phi, 1)^\beta=0$ between their parameters θ and ϕ; at a united point $\phi=\theta$, and the value of θ is given by an equation of the order $\alpha+\beta$. The extension to curves of any given deficiency D was made in the memoir of Cayley, "On the correspondence of two points on a curve,"—*Proc. Lond. Math. Soc.* t. I. (1866), [385],—viz. taking P, P' as the corresponding points in an (α, α') correspondence on a curve of deficiency D, and supposing that when P is given the corresponding points P' are found as the intersections of the curve by a curve Θ containing the coordinates of P as parameters, and having with the given curve k intersections at the point P, then the number of united points is $a=\alpha+\alpha'+2kD$; and more generally, if the curve Θ intersect the given curve in a set of points P' each p times, a set of points Q' each q times, &c., in such manner that the points (P, P'), the points (P, Q'), &c., are pairs of points corresponding to each other according to distinct laws; then if (P, P') are points having an (α, α') correspondence with a number $=a$ of united points, (P, Q') points having a (β, β') correspondence with a number $=b$ of united points, and so on, the theorem is that we have

$$p(a-\alpha-\alpha')+q(b-\beta-\beta')+\ldots=2kD.$$

The principle of correspondence, or say rather the theorem of united points, is a most powerful instrument of investigation, which may be used in place of analysis for the determination of the number of solutions of almost every geometrical problem. We can by means of it investigate the class of a curve, number of inflexions, &c.,—in fact, Plücker's equations; but it is necessary to take account of special solutions; thus, in one of the most simple instances, in finding the class of a curve, the cusps present themselves as special solutions.

Imagine a curve of order m, deficiency D, and let the corresponding points P, P' be such that the line joining them passes through a given point O; this is an $(m-1, m-1)$ correspondence, and the value of k is $=1$, hence the number of united points is $=2m-2+2D$; the united points are the points of contact of the tangents from O and (as special solutions) the cusps, and we have thus the relation $n+\kappa=2m-2+2D$; or, writing $D=\frac{1}{2}(m-1)(m-2)-\delta-\kappa$, this is $n=m(m-1)-2\delta-3\kappa$, which is right.

The principle in its original form as applying to a right line was used throughout by Chasles in the investigations on the number of the conics which satisfy given conditions, and on the number of solutions of very many other geometrical problems.

There is one application of the theory of the (α, α') correspondence between two planes which it is proper to notice.

Imagine a curve, real or imaginary, represented by an equation (involving, it may be, imaginary coefficients) between the Cartesian coordinates u, u'; then, writing $u=x+iy$, $u'=x'+iy'$, the equation determines real values of (x, y), and of (x', y'), corresponding to any given real values of (x', y') and (x, y) respectively; that is, it establishes a real correspondence (not of course a rational one) between the points (x, y) and (x', y'); for example in the imaginary circle $u^2+u'^2=(a+bi)^2$, the correspondence is given by the two equations $x^2-y^2+x'^2-y'^2=a^2-b^2$, $xy+x'y'=ab$. We have thus a means of geometrical representation for the portions, as well imaginary as real, of any real or imaginary curve. Considerations such as these have been used for determining the series of values of the independent variable, and the irrational functions thereof in the theory of Abelian integrals, but the theory seems to be worthy of further investigation.

The researches of Chasles (*Comptes Rendus*, t. LVIII., 1864, *et seq.*) refer to the conics which satisfy given conditions. There is an earlier paper by De Jonquières, "Théorèmes généraux concernant les courbes géométriques planes d'un ordre quelconque," *Liouv.* t. VI. (1861), which establishes the notion of a system of curves (of any order) of the index N, viz. considering the curves of the order n which satisfy $\frac{1}{2}n(n+3)-1$ conditions, then the index N is the number of these curves which pass through a given arbitrary point. But Chasles in the first of his papers (February 1864), considering the conics which satisfy four conditions, establishes the notion of the two characteristics (μ, ν) of such a system of conics, viz. μ is the number of the conics which pass through a given arbitrary point, and ν is the number of the conics which touch a given arbitrary line. And he gives the theorem, a system of conics satisfying four conditions, and having the characteristics (μ, ν) contains $2\nu-\mu$ line-pairs (that is, conics, each of them a pair of lines), and $2\mu-\nu$ point-pairs (that is, conics, each of them a pair of points,—coniques infiniment aplaties), which is a fundamental one in the theory. The characteristics of the system can be determined when it is known how many there are of these two kinds of degenerate conics in the system, and how often each is to be counted. It was thus that Zeuthen (in the paper *Nyt Bydrag*, "Contribution to the Theory of Systems of Conics which satisfy four Conditions," Copenhagen, 1865, translated with an addition in the *Nouvelles Annales*) solved the question of finding the characteristics of the systems of conics which satisfy four

conditions of contact with a given curve or curves; and this led to the solution of the further problem of finding the number of the conics which satisfy five conditions of contact with a given curve or curves (Cayley, *Comptes Rendus*, t. LXIII., 1866, [377]), and "On the Curves which satisfy given Conditions" (*Phil. Trans.* t. CLVIII., 1868, [406]).

It may be remarked that although, as a process of investigation, it is very convenient to seek for the characteristics of a system of conics satisfying 4 conditions, yet what is really determined is in every case the number of the conics which satisfy 5 conditions; the characteristics of the system $(4p)$ of the conics which pass through $4p$ points are $(5p)$, $(4p, 1l)$, the number of the conics which pass through 5 points, and which pass through 4 points and touch 1 line: and so in other cases. Similarly as regards cubics, or curves of any other order: a cubic depends on 9 constants, and the elementary problems are to find the number of the cubics $(9p)$, $(8p, 1l)$, &c., which pass through 9 points, pass through 8 points and touch 1 line, &c.; but it is in the investigation convenient to seek for the characteristics of the systems of cubics $(8p)$, &c., which satisfy 8 instead of 9 conditions.

The elementary problems in regard to cubics are solved very completely by Maillard in his *Thèse, Recherche des caractéristiques des systèmes élémentaires des courbes planes du troisième ordre* (Paris, 1871). Thus, considering the several cases of a cubic

		No. of consts.
1.	With a given cusp	5,
2.	„ cusp on given line	6,
3.	„ cusp	7,
4.	„ a given node	6,
5.	„ node on given line	7,
6.	„ node	8,
7.	non-singular	9,

he determines in every case the characteristics (μ, ν) of the corresponding systems of cubics $(4p)$, $(3p, 1l)$, &c. The same problems, or most of them, and also the elementary problems in regard to quartics are solved by Zeuthen, who in the elaborate memoir "Almindelige Egenskaber, &c.," *Danish Academy*, t. X. (1873), considers the problem in reference to curves of any order, and applies his results to cubic and quartic curves.

The methods of Maillard and Zeuthen are substantially identical; in each case the question considered is that of finding the characteristics (μ, ν) of a system of curves by consideration of the special or degenerate forms of the curves included in the system. The quantities which have to be considered are very numerous. Zeuthen in the case of curves of any given order establishes between the characteristics μ, ν', and 18 other quantities, in all 20 quantities, a set of 24 equations (equivalent to 23 independent equations), involving (besides the 20 quantities) other quantities relating to the various forms of the degenerate curves, which supplementary terms he determines, partially for curves of any order, but completely only for quartic curves. It is in the discussion and complete enumeration of the special or degenerate forms of the curves,

and of the supplementary terms to which they give rise, that the great difficulty of the question seems to consist; it would appear that the 24 equations are a complete system, and that (subject to a proper determination of the supplementary terms) they contain the solution of the general problem.

The remarks which follow have reference to the analytical theory of the degenerate curves which present themselves in the foregoing problem of the curves which satisfy given conditions.

A curve represented by an equation in point-coordinates may break up: thus if $P_1, P_2, \ldots$ be rational and integral functions of the coordinates (x, y, z) of the orders $m_1, m_2, \ldots$ respectively, we have the curve $P_1^{\alpha_1} P_2^{\alpha_2} \ldots = 0$, of the order $m, = \alpha_1 m_1 + \alpha_2 m_2 + \ldots$, composed of the curve $P_1 = 0$ taken α_1 times, the curve $P_2 = 0$ taken α_2 times, &c.

Instead of the equation $P_1^{\alpha_1} P_2^{\alpha_2} \ldots = 0$, we may start with an equation $u = 0$, where u is a function of the order m containing a parameter θ, and for a particular value say $\theta = 0$, of the parameter reducing itself to $P_1^{\alpha_1} P_2^{\alpha_2} \ldots$. Supposing θ indefinitely small, we have what may be called the penultimate curve, and when $\theta = 0$ the ultimate curve. Regarding the ultimate curve as derived from a given penultimate curve, we connect with the ultimate curve, and consider as belonging to it, certain points called "summits" on the component curves $P_1 = 0$, $P_2 = 0$, respectively; a summit Σ is a point such that, drawing from an arbitrary point O the tangents to the penultimate curve, we have $O\Sigma$ as the limit of one of these tangents. The ultimate curve together with its summits may be regarded as a degenerate form of the curve $u = 0$. Observe that the positions of the summits depend on the penultimate curve $u = 0$, viz. on the values of the coefficients in the terms multiplied by $\theta, \theta^2, \ldots$; they are thus in some measure arbitrary points as regards the ultimate curve $P_1^{\alpha_1} P_2^{\alpha_2} \ldots = 0$.

It may be added that we have summits only on the component curves $P_1 = 0$, of a multiplicity $\alpha_1 > 1$; the number of summits on such a curve is in general $= (\alpha_1^2 - \alpha_1) m_1^2$. Thus assuming that the penultimate curve is without nodes or cusps, the number of the tangents to it is $= m^2 - m, = (\alpha_1 m_1 + \alpha_2 m_2 + \ldots)^2 - (\alpha_1 m_1 + \alpha_2 m_2 + \ldots)$, taking $P_1 = 0$ to have δ_1 nodes and κ_1 cusps, and therefore its class n_1 to be $= m_1^2 - m_1 - 2\delta_1 - 3\kappa_1$, &c., the expression for the number of tangents to the penultimate curve is

$$= (\alpha_1^2 - \alpha_1) m_1^2 + (\alpha_2^2 - \alpha_2) m_2^2 + \ldots + 2\alpha_1 \alpha_2 m_1 m_2 + \ldots + \alpha_1 (n_1 + 2\delta_1 + 3\kappa_1) + \alpha_2 (n_2 + 2\delta_2 + 3\kappa_2) + \ldots$$

where a term $2\alpha_1 \alpha_2 m_1 m_2$ indicates tangents which are in the limit the lines drawn to the intersections of the curves $P_1 = 0$, $P_2 = 0$ each line $2\alpha_1 \alpha_2$ times; a term $\alpha_1 (n_1 + 2\delta_1 + 3\kappa_1)$ tangents which are in the limit the proper tangents to $P_1 = 0$ each α_1 times, the lines to its nodes each $2\alpha_1$ times, and the lines to its cusps each $3\alpha_1$ times; the remaining terms $(\alpha_1^2 - \alpha_1) m_1^2 + (\alpha_2^2 - \alpha_2) m_2^2 + \ldots$ indicate tangents which are in the limit the lines drawn to the several summits, that is, we have $(\alpha_1^2 - \alpha_1) m_1^2$ summits on the curve $P_1 = 0$, &c.

There is of course a precisely similar theory as regards line-coordinates; taking Π_1, Π_2, &c., to be rational and integral functions of the coordinates (ξ, η, ζ), we connect with the ultimate curve $\Pi_1^{\alpha_1} \Pi_2^{\alpha_2} \ldots = 0$, and consider as belonging to it certain lines, which for the moment may be called "axes," tangents to the component curves

$\Pi_1 = 0$, $\Pi_2 = 0$ respectively. Considering an equation in point-coordinates, we may have among the component curves right lines; and, if in order to put these in evidence, we take the equation to be $L_1^{\gamma_1} \ldots P_1^{\alpha_1} \ldots = 0$, where $L_1 = 0$ is a right line, $P_1 = 0$ a curve of the second or any higher order, then the curve will contain as part of itself summits not exhibited in this equation, but the corresponding line-equation will be $\Lambda_1^{\delta_1} \ldots \Pi_1^{\alpha_1} \ldots = 0$, where $\Lambda_1 = 0, \ldots$ are the equations of the summits in question, $\Pi_1 = 0$, &c., are the line-equations corresponding to the several point-equations $P_1 = 0$, &c.; and this curve will contain as part of itself axes not exhibited by this equation, but which are the lines $L_1 = 0, \ldots$ of the equation in point-coordinates.

In conclusion a little may be said as to curves of double curvature, otherwise twisted curves, or curves in space. The analytical theory by Cartesian coordinates was first considered by Clairaut, *Recherches sur les courbes à double courbure* (Paris, 1731). Such a curve may be considered as described by a point, moving in a line which at the same time rotates about the point in a plane which at the same time rotates about the line; the point is a point, the line a tangent, and the plane an osculating plane, of the curve; moreover the line is a generating line, and the plane a tangent plane, of a developable surface or torse, having the curve for its edge of regression. Analogous to the order and class of a plane curve we have the order, rank, and class, of the system (assumed to be a geometrical one), viz. if an arbitrary plane contains m points, an arbitrary line meets r lines, and an arbitrary point lies in n planes, of the system, then m, r, n are the order, rank, and class respectively. The system has singularities, and there exist between m, r, n and the numbers of the several singularities equations analogous to Plücker's equations for a plane curve.

It is a leading point in the theory that a curve in space cannot in general be represented by means of two equations $U = 0$, $V = 0$; the two equations represent surfaces, intersecting in a curve; but there are curves which are not the complete intersection of any two surfaces; thus we have the cubic in space, or skew cubic, which is the residual intersection of two quadric surfaces which have a line in common; the equations $U = 0$, $V = 0$ of the two quadric surfaces represent the cubic curve, not by itself, but together with the line.

786.

EQUATION.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. VIII. (1878), pp. 497—509.]

THE present article includes Determinant and Theory of Equations; and it may be proper to explain the relation to each other of the two subjects. Theory of Equations is used in its ordinary conventional sense to denote the theory of a single equation of any order in one unknown quantity; that is, it does not include the theory of a system or systems of equations of any order between any number of unknown quantities. Such systems occur very frequently in analytical geometry and other parts of mathematics, but they are hardly as yet the subject-matter of a distinct theory; and even Elimination, the transition-process for passing from a system of any number of equations involving the same number of unknown quantities to a single equation in one unknown quantity, hardly belongs to the Theory of Equations in the above restricted sense. But there is one case of a system of equations which precedes the Theory of Equations, and indeed presents itself at the outset of algebra, that of a system of simple (or linear) equations. Such a system gives rise to the function called a Determinant, and it is by means of these functions that the solution of the equations is effected. We have thus the subject Determinant as nearly equivalent to (but somewhat more extensive than) that of a system of linear equations; and we have the other subject, Theory of Equations, used in the restricted sense above referred to, and as not including Elimination.

Determinant.

1. A sketch of the history of determinants is given under [the Article] Algebra; it thereby appears that the algebraical function called a determinant presents itself in the solution of a system of simple equations, and we have herein a natural source of the theory. Thus, considering the equations

$$\begin{aligned} a\,x + b\,y + c\,z &= d\,, \\ a'x + b'y + c'z &= d'\,, \\ a''x + b''y + c''z &= d''\,, \end{aligned}$$

and proceeding to solve them by the so-called method of cross multiplication, we multiply the equations by factors selected in such a manner that, upon adding the results, the whole coefficient of y becomes $=0$ and the whole coefficient of z becomes $=0$; the factors in question are $b'c''-b''c'$, $b''c-bc''$, $bc'-b'c$ (values which, as at once seen, have the desired property); we thus obtain an equation which contains on the left-hand side only a multiple of x, and on the right-hand side a constant term; the coefficient of x has the value

$$a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c),$$

and this function, represented in the form

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

is said to be a determinant; or, the number of elements being 3^2, it is called a determinant of the third order. It is to be noticed that the resulting equation is

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} x = \begin{vmatrix} d, & b, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix},$$

where the expression on the right-hand side is the like function with d, d', d'' in place of a, a', a'' respectively, and is of course also a determinant. Moreover, the functions $b'c''-b''c'$, $b''c-bc''$, $bc'-b'c$ used in the process are themselves the determinants of the second order

$$\begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix}, \quad \begin{vmatrix} b'', & c'' \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}.$$

We have herein the suggestion of the rule for the derivation of the determinants of the orders 1, 2, 3, 4, &c., each from the preceding one, viz. we have

$$|a| = a,$$

$$\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = a|b'| - a'|b|,$$

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = a\begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + a'\begin{vmatrix} b'', & c'' \\ b, & c \end{vmatrix} + a''\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix} = a\begin{vmatrix} b', & c', & d' \\ b'', & c'', & d'' \\ b''', & c''', & d''' \end{vmatrix} - a'\begin{vmatrix} b'', & c'', & d'' \\ b''', & c''', & d''' \\ b, & c, & d \end{vmatrix} + a''\begin{vmatrix} b''', & c''', & d''' \\ b, & c, & d \\ b', & c', & d' \end{vmatrix} - a'''\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix},$$

and so on, the terms being all + for a determinant of an odd order, but alternately + and − for a determinant of an even order.

2. It is easy, by induction, to arrive at the general results:—

A determinant of the order n is the sum of the $1.2.3\ldots n$ products which can be formed with n elements out of n^2 elements arranged in the form of a square, no two of the n elements being in the same line or in the same column, and each such product having the coefficient $\pm$ unity.

The products in question may be obtained by permuting in every possible manner the columns (or the lines) of the determinant, and then taking for the factors the n elements in the dexter diagonal. And we thence derive the rule for the signs, viz. considering the primitive arrangement of the columns as positive, then an arrangement obtained therefrom by a single interchange (inversion, or derangement) of two columns is regarded as negative; and so in general an arrangement is positive or negative according as it is derived from the primitive arrangement by an even or an odd number of interchanges. This implies the theorem that a given arrangement can be derived from the primitive arrangement only by an odd number, or else only by an even number of interchanges,—a theorem the verification of which may be easily obtained from the theorem (in fact, a particular case of the general one), an arrangement can be derived from itself only by an even number of interchanges. And this being so, each product has the sign belonging to the corresponding arrangement of the columns; in particular, a determinant contains with the sign + the product of the elements in its dexter diagonal. It is to be observed that the rule gives as many positive as negative arrangements, the number of each being $=\frac{1}{2}.1.2\ldots n$.

The rule of signs may be expressed in a different form. Giving to the columns in the primitive arrangement the numbers 1, 2, 3, ..., n, to obtain the sign belonging to any other arrangement we take, as often as a lower number succeeds a higher one, the sign $-$, and, compounding together all these minus signs, obtain the proper sign, + or $-$ as the case may be.

Thus, for three columns, it appears by either rule that 123, 231, 312 are positive; 132, 321, 213 are negative; and the developed expression of the foregoing determinant of the third order is

$$=ab'c''-ab''c'+a'b''c-a'bc''+a''bc'-a''b'c.$$

3. It further appears that a determinant is a linear function* of the elements of each column thereof, and also a linear function of the elements of each line thereof; moreover, that the determinant retains the same value, only its sign being altered, when any two columns are interchanged, or when any two lines are interchanged; more generally, when the columns are permuted in any manner, or when the lines are permuted in any manner, the determinant retains its original value, with the sign + or $-$ according as the new arrangement (considered as derived from the primitive arrangement) is positive or negative according to the foregoing rule of signs.

* The expression, a linear function, is here used in its narrowest sense, a linear function without constant term; what is meant is, that the determinant is in regard to the elements a, a', a'', ... of any column or line thereof, a function of the form $Aa+A'a'+A''a''+\ldots$, without any term independent of a, a', a'', ...

It at once follows that, if two columns are identical, or if two lines are identical, the value of the determinant is $=0$. It may be added that, if the lines are converted into columns, and the columns into lines, in such a way as to leave the dexter diagonal unaltered, the value of the determinant is unaltered; the determinant is in this case said to be *transposed*.

4. By what precedes it appears that there exists a function of the n^2 elements, linear as regards the terms of each column (or say, for shortness, linear as to each column), and such that only the sign is altered when any two columns are interchanged; these properties completely determine the function, except as to a common factor which may multiply all the terms. If, to get rid of this arbitrary common factor, we assume that the product of the elements in the dexter diagonal has the coefficient $+1$, we have a complete definition of the determinant; and it is interesting to show how from these properties, assumed for the definition of the determinant, it at once appears that the determinant is a function serving for the solution of a system of linear equations. Observe that the properties show at once that if any column is $=0$ (that is, if the elements in the column are each $=0$), then the determinant is $=0$; and further that, if any two columns are identical, then the determinant is $=0$.

5. Reverting to the system of linear equations written down at the beginning of this article, consider the determinant

$$\begin{vmatrix} a\,x+b\,y+c\,z-d\,, & b\,, & c \\ a'x+b'y+c'z-d', & b', & c' \\ a''x+b''y+c''z-d'', & b'', & c'' \end{vmatrix};$$

it appears that this is

$$=x\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}+y\begin{vmatrix} b\,, & b\,, & c \\ b', & b', & c' \\ b'', & b'', & c'' \end{vmatrix}+z\begin{vmatrix} c\,, & b\,, & c \\ c', & b', & c' \\ c'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d\,, & b\,, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix},$$

viz. the second and the third terms each vanishing, it is

$$=x\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d\,, & b\,, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix}.$$

But if the linear equations hold good, then the first column of the original determinant is $=0$, and therefore the determinant itself is $=0$; that is, the linear equations give

$$x\begin{vmatrix} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d\,, & b\,, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix}=0;$$

which is the result obtained above.

We might in a similar way find the values of y and z, but there is a more symmetrical process. Join to the original equations the new equation

$$\alpha x + \beta y + \gamma z = \delta;$$

a like process shows that, the equations being satisfied, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix} = 0;$$

or, as this may be written,

$$\begin{vmatrix} \alpha, & \beta, & \gamma & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix} - \delta \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0;$$

which, considering δ as standing herein for its value $\alpha x + \beta y + \gamma z$, is a consequence of the original equations only. We have thus an expression for $\alpha x + \beta y + \gamma z$, an arbitrary linear function of the unknown quantities x, y, z; and by comparing the coefficients of α, β, γ on the two sides respectively, we have the values of x, y, z; in fact, these quantities, each multiplied by

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

are in the first instance obtained in the forms

$$\begin{vmatrix} 1 & & & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}, \quad \begin{vmatrix} & 1 & & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}, \quad \begin{vmatrix} & & 1 & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix};$$

but these are

$$= \begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}, \quad - \begin{vmatrix} c, & d, & a \\ c', & d', & a' \\ c'', & d'', & a'' \end{vmatrix}, \quad \begin{vmatrix} d, & a, & b \\ d', & a', & b' \\ d'', & a'', & b'' \end{vmatrix},$$

or, what is the same thing,

$$= \begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}, \quad \begin{vmatrix} c, & a, & d \\ c', & a', & d' \\ c'', & a'', & d'' \end{vmatrix}, \quad \begin{vmatrix} a, & b, & d \\ a', & b', & d' \\ a'', & b'', & d'' \end{vmatrix}$$

respectively.

6. *Multiplication of two determinants of the same order.*—The theorem is obtained very easily from the last preceding definition of a determinant. It is most simply expressed thus—

$$\begin{array}{c} \\ (a\,,\ b\,,\ c\,) \\ (a',\ b',\ c') \\ (a'',\ b'',\ c'') \end{array} \left| \begin{array}{ccc} (\alpha,\ \alpha',\ \alpha''), & (\beta,\ \beta',\ \beta''), & (\gamma,\ \gamma',\ \gamma'') \\ \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \end{array} \right| = \left| \begin{array}{ccc} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array} \right| . \left| \begin{array}{ccc} \alpha\,, & \beta\,, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array} \right|,$$

where the expression on the left side stands for a determinant, the terms of the first line being $(a, b, c)(\alpha, \alpha', \alpha'')$, that is, $a\alpha + b\alpha' + c\alpha''$, $(a, b, c)(\beta, \beta', \beta'')$, that is, $a\beta + b\beta' + c\beta''$, $(a, b, c)(\gamma, \gamma', \gamma'')$, that is, $a\gamma + b\gamma' + c\gamma''$; and similarly the terms in the second and third lines are the like functions with (a', b', c') and (a'', b'', c'') respectively.

There is an apparently arbitrary transposition of lines and columns; the result would hold good if on the left-hand side we had written (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, or what is the same thing, if on the right-hand side we had transposed the second determinant; and either of these changes would, it might be thought, increase the elegance of the form, but, for a reason which need not be explained*, the form actually adopted is the preferable one.

To indicate the method of proof, observe that the determinant on the left-hand side, *qua* linear function of its columns, may be broken up into a sum of $(3^3 =)\,27$ determinants, each of which is either of some such form as

$$\pm\,\alpha\beta\gamma' \left| \begin{array}{ccc} a\,, & a\,, & b \\ a', & a', & b' \\ a'', & a'', & b'' \end{array} \right|,$$

where the term $\alpha\beta\gamma'$ is not a term of the $\alpha\beta\gamma$-determinant, and its coefficient (as a determinant with two identical columns) vanishes; or else it is of a form such as

$$\pm\,\alpha\beta'\gamma'' \left| \begin{array}{ccc} a\,, & b\,, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array} \right|,$$

that is, every term which does not vanish contains as a factor the abc-determinant last written down; the sum of all other factors $\pm\,\alpha\beta'\gamma''$ is the $\alpha\beta\gamma$-determinant of the formula; and the final result then is, that the determinant on the left-hand side is equal to the product on the right-hand side of the formula.

7. *Decomposition of a determinant into complementary determinants.*—Consider, for simplicity, a determinant of the fifth order, $5 = 2 + 3$, and let the top two lines be

$$\begin{array}{ccccc} a\,, & b\,, & c\,, & d\,, & e\,, \\ a'\,, & b'\,, & c'\,, & d'\,, & e'; \end{array}$$

* The reason is the connexion with the corresponding theorem for the multiplication of two matrices.

then, if we consider how these elements enter into the determinant, it is at once seen that they enter only through the determinants of the second order $\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}$, &c., which can be formed by selecting any two columns at pleasure. Moreover, representing the remaining three lines by

$$\begin{matrix} a'', & b'', & c'', & d'', & e'', \\ a''', & b''', & c''', & d''', & e''', \\ a'''', & b'''', & c'''', & d'''', & e'''', \end{matrix}$$

it is further seen that the factor which multiplies the determinant formed with any two columns of the first set is the determinant of the third order formed with the complementary three columns of the second set; and it thus appears that the determinant of the fifth order is a sum of all the products of the form

$$\pm \begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} \begin{vmatrix} c'', & d'', & e'' \\ c''', & d''', & e''' \\ c'''', & d'''', & e'''' \end{vmatrix},$$

the sign $\pm$ being in each case such that the sign of the term $\pm ab'.c''d'''e''''$ obtained from the diagonal elements of the component determinants may be the actual sign of this term in the determinant of the fifth order; for the product written down the sign is obviously $+$.

Observe that for a determinant of the nth order, taking the decomposition to be $1+(n-1)$, we fall back upon the equations given at the commencement, in order to show the genesis of a determinant.

8. Any determinant $\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}$ formed out of the elements of the original determinant, by selecting the lines and columns at pleasure, is termed a *minor* of the original determinant; and when the number of lines and columns, or order of the determinant, is $n-1$, then such determinant is called a *first minor;* the number of the first minors is $=n^2$, the first minors, in fact, corresponding to the several elements of the determinant—that is, the coefficient therein of any term whatever is the corresponding first minor. The first minors, each divided by the determinant itself, form a system of elements *inverse* to the elements of the determinant.

A determinant is *symmetrical* when every two elements symmetrically situated in regard to the dexter diagonal are equal to each other; if they are equal and opposite (that is, if the sum of the two elements be $=0$), this relation not extending to the diagonal elements themselves, which remain arbitrary, then the determinant is *skew;* but if the relation does extend to the diagonal terms (that is, if these are each $=0$), then the determinant is *skew symmetrical;* thus the determinants

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}, \quad \begin{vmatrix} a, & \nu, & -\mu \\ -\nu, & b, & \lambda \\ \mu, & -\lambda, & c \end{vmatrix}, \quad \begin{vmatrix} 0, & \nu, & -\mu \\ -\nu, & 0, & \lambda \\ \mu, & -\lambda, & 0 \end{vmatrix},$$

are respectively symmetrical, skew, and skew symmetrical.

The theory admits of very extensive algebraic developments, and applications in algebraical geometry and other parts of mathematics; but the fundamental properties of the functions may fairly be considered as included in what precedes.

Theory of Equations.

9. In the subject "Theory of Equations," the term *equation* is used to denote an equation of the form $x^n - p_1 x^{n-1} + \ldots \pm p_n = 0$, where $p_1, p_2, \ldots, p_n$ are regarded as known, and x as a quantity to be determined; for shortness, the equation is written $f(x) = 0$.

The equation may be *numerical;* that is, the coefficients $p_1, p_2, \ldots, p_n$ are then numbers,—understanding by number a quantity of the form $\alpha + \beta i$ (α and β having any positive or negative real values whatever, or say each of these is regarded as susceptible of continuous variation from an indefinitely large negative to an indefinitely large positive value), and i denoting $\sqrt{-1}$.

Or the equation may be *algebraical;* that is, the coefficients are not then restricted to denote, or are not explicitly considered as denoting, numbers.

I. We consider first numerical equations. (Real theory, 10 to 14; Imaginary theory, 15 to 18.)

10. Postponing all consideration of imaginaries, we take in the first instance the coefficients to be real, and attend only to the real roots (if any); that is, $p_1, p_2, \ldots, p_n$ are real positive or negative quantities, and a root a, if it exists, is a positive or negative quantity such that $a^n - p_1 a^{n-1} + \ldots \pm p_n = 0$, or say, $f(a) = 0$. The fundamental theorems are given in the article Algebra, sections X., XIII., XIV.; but there are various points in the theory which require further development.

It is very useful to consider the curve $y = f(x)$,—or, what would come to the same, the curve $Ay = f(x)$,—but it is better to retain the first-mentioned form of equation, drawing, if need be, the ordinate y on a reduced scale. For instance, if the given equation be $x^3 - 6x^2 + 11x - 6{\cdot}06 = 0$,* then the curve $y = x^3 - 6x^2 + 11x - 6{\cdot}06$ is as shown in the figure at page 501, without any reduction of scale for the ordinate.

It is clear that, in general, y is a continuous one-valued function of x, finite for every finite value of x, but becoming infinite when x is infinite; i.e. assuming throughout that the coefficient of x^n is $+1$, then when $x = \infty$, $y = +\infty$; but when $x = -\infty$, then $y = +\infty$ or $-\infty$, according as n is even or odd; the curve cuts any line whatever, and in particular it cuts the axis of x, in at most n points; and the value of x, at any point of intersection with the axis, is a root of the equation $f(x) = 0$.

If β, α are any two values of x ($\alpha > \beta$, that is, α nearer $+\infty$), then if $f(\beta)$, $f(\alpha)$ have opposite signs, the curve cuts the axis an odd number of times, and therefore at least once, between the points $x = \beta$, $x = \alpha$; but if $f(\beta)$, $f(\alpha)$ have the same sign, then between these points the curve cuts the axis an even number of times, or it may be not at all. That is, $f(\beta)$, $f(\alpha)$ having opposite signs, there are between the limits β, α an odd number of real roots, and therefore at least one real

* The coefficients were selected so that the roots might be nearly 1, 2, 3.

root; but $f(\beta)$, $f(\alpha)$ having the same sign, there are between these limits an even number of real roots, or it may be there is no real root. In particular, by giving to β, α the values $-\infty$, $+\infty$ (or, what is the same thing, any two values sufficiently near to these values respectively) it appears that an equation of an odd order has always an odd number of real roots, and therefore at least one real root; but that an equation of an even order has an even number of real roots, or it may be no real root.

If α be such that for $x =$ or $> \alpha$ (that is, x nearer to $+\infty$) $f(x)$ is always $+$, and β be such that for $x =$ or $< \beta$ (that is, x nearer to $-\infty$) $f(x)$ is always $-$, then the real roots (if any) lie between these limits $x = \beta$, $x = \alpha$; and it is easy to find by trial such two limits including between them all the real roots (if any).

11. Suppose that the positive value δ is an inferior limit to the difference between two real roots of the equation; or rather (since the foregoing expression would imply the existence of real roots) suppose that there are not two real roots such that their difference taken positively is $=$ or $< \delta$; then, γ being any value whatever, there is clearly at most one real root between the limits γ and $\gamma + \delta$; and by what precedes there is such real root or there is not such real root, according as $f(\gamma)$, $f(\gamma + \delta)$ have opposite signs or have the same sign. And by dividing in this manner the interval β to α into intervals each of which is $=$ or $< \delta$, we should not only ascertain the number of the real roots (if any), but we should also *separate* the real roots, that is, find for each of them limits γ, $\gamma + \delta$ between which there lies this one, and only this one, real root.

In particular cases it is frequently possible to ascertain the number of the real roots, and to effect their separation by trial or otherwise, without much difficulty; but the foregoing was the general process as employed by Lagrange even in the second edition (1808) of the *Traité de la résolution des Équations Numériques**; the determination of the limit δ had to be effected by means of the "equation of differences" or equation of the order $\frac{1}{2}n(n-1)$, the roots of which are the squares of the differences of the roots of the given equation, and the process is a cumbrous and unsatisfactory one.

12. The great step was effected by Sturm's theorem (1835)—viz. here starting from the function $f(x)$, and its first derived function $f'(x)$, we have (by a process which is a slight modification of that for obtaining the greatest common measure of these two functions) to form a series of functions

$$f(x),\ f'(x),\ f_2(x), \ldots, f_n(x)$$

of the degrees n, $n-1$, $n-2, \ldots, 0$ respectively,—the last term $f_n(x)$ being thus an absolute constant. These lead to the immediate determination of the number of real roots (if any) between any two given limits β, α; viz. supposing $\alpha > \beta$ (that is, α nearer to $+\infty$), then substituting successively these two values in the series of functions, and attending only to the signs of the resulting values, the number of the changes of sign lost in passing from β to α is the required number of real roots

* The third edition (1826) is a reproduction of that of 1808; the first edition has the date 1798, but a large part of the contents is taken from memoirs of 1767—68 and 1770—71.

between the two limits. In particular, taking β, $\alpha = -\infty$, $+\infty$ respectively, the signs of the several functions depend merely on the signs of the terms which contain the highest powers of x, and are seen by inspection, and the theorem thus gives at once the whole number of real roots.

And although theoretically, in order to complete by a finite number of operations the separation of the real roots, we still need to know the value of the before-mentioned limit δ; yet in any given case the separation may be effected by a limited number of repetitions of the process. The practical difficulty is when two or more roots are very near to each other. Suppose, for instance, that the theorem shows that there are two roots between 0 and 10; by giving to x the values 1, 2, 3,... successively, it might appear that the two roots were between 5 and 6; then again that they were between 5·3 and 5·4, then between 5·34 and 5·35, and so on until we arrive at a separation; say it appears that between 5·346 and 5·347 there is one root, and between 5·348 and 5·349 the other root. But in the case in question δ would have a very small value, such as ·002, and even supposing this value known, the direct application of the first-mentioned process would be still more laborious.

13. Supposing the separation once effected, the determination of the single real root which lies between the two given limits may be effected to any required degree of approximation either by the processes of Horner and Lagrange (which are in principle a carrying out of the method of Sturm's theorem), or by the process of Newton, as perfected by Fourier (which requires to be separately considered).

First as to Horner and Lagrange. We know that between the limits β, α there lies one, and only one, real root of the equation; $f(\beta)$ and $f(\alpha)$ have therefore opposite signs. Suppose any intermediate value is θ; in order to determine by Sturm's theorem whether the root lies between β, θ, or between θ, α, it would be quite unnecessary to calculate the signs of $f(\theta)$, $f'(\theta)$, $f_2(\theta)$, ...; only the sign of $f(\theta)$ is required: for, if this has the same sign as $f(\beta)$, then the root is between β, θ; if the same sign as $f(\alpha)$, then the root is between θ, α. We want to make θ increase from the inferior limit β, at which $f(\theta)$ has the sign of $f(\beta)$, so long as $f(\theta)$ retains this sign, and then to a value for which it assumes the opposite sign; we have thus two nearer limits of the required root, and the process may be repeated indefinitely.

Horner's method (1819) gives the root as a decimal, figure by figure; thus, if the equation be known to have one real root between 0 and 10, it is in effect shown say that 5 is too small (that is, the root is between 5 and 6); next that 5·4 is too small (that is, the root is between 5·4 and 5·5); and so on to any number of decimals. Each figure is obtained, *not* by the successive trial of all the figures which precede it, but (as in the ordinary process of the extraction of a square root, which is in fact Horner's process applied to this particular case) it is given presumptively as the first figure of a quotient; such value may be too large, and then the next inferior integer must be tried instead of it, or it may require to be further diminished. And it is to be remarked that the process not only gives the approximate value α of the root, but (as in the extraction of a square root) it includes the calculation of the function $f(\alpha)$ which should be, and approximately is, $=0$. The arrangement of the

calculations is very elegant, and forms an integral part of the actual method. It is to be observed that after a certain number of decimal places have been obtained, a good many more can be found by a mere division. It is in the progress tacitly assumed that the roots have been first separated.

Lagrange's method (1767) gives the root as a continued fraction $a+\frac{1}{b}\,\frac{1}{c}+\dots$, where a is a positive or negative integer (which may be $=0$), but $b, c, \dots$ are positive integers. Suppose the roots have been separated; then (by trial if need be of consecutive integer values) the limits may be made to be consecutive integer numbers: say they are a, $a+1$; the value of x is therefore $=a+\frac{1}{y}$, where y is positive and greater than 1; from the given equation for x, writing therein $x=a+\frac{1}{y}$, we form an equation of the same order for y, and this equation will have one, and only one, positive root greater than 1; hence finding for it the limits b, $b+1$ (where b is $=$ or >1), we have $y=b+\frac{1}{z}$, where z is positive and greater than 1; and so on—that is, we thus obtain the successive denominators $b, c, d, \dots$ of the continued fraction. The method is theoretically very elegant, but the disadvantage is that it gives the result in the form of a continued fraction, which for the most part must ultimately be converted into a decimal. There is one advantage in the method, that a commensurable root (that is, a root equal to a rational fraction) is found accurately, since, when such root exists, the continued fraction terminates.

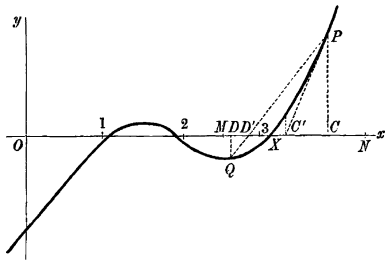
14. Newton's method (1711), as perfected by Fourier (1831), may be roughly stated as follows. If $x=\gamma$ be an approximate value of any root, and $\gamma+h$ the correct value, then $f(\gamma+h)=0$, that is,

$$f(\gamma)+\frac{h}{1}f'(\gamma)+\frac{h^2}{1.2}f''(\gamma)+\dots=0;$$

and then, if h be so small that the terms after the second may be neglected, $f(\gamma)+hf'(\gamma)=0$, that is, $h=-\frac{f(\gamma)}{f'(\gamma)}$, or the new approximate value is $x=\gamma-\frac{f(\gamma)}{f'(\gamma)}$; and so on, as often as we please. It will be observed that so far nothing has been assumed as to the separation of the roots, or even as to the existence of a real root; γ has been taken as the approximate value of a root, but no precise meaning has been attached to this expression. The question arises, what are the conditions to be satisfied by γ in order that the process may by successive repetitions actually lead to a certain real root of the equation; or say that, γ being an approximate value of a certain real root, the new value $\gamma-\frac{f(\gamma)}{f'(\gamma)}$ may be a more approximate value.

Referring to the figure, it is easy to see that, if OC represent the assumed value γ, then, drawing the ordinate CP to meet the curve in P, and the tangent PC' to meet the axis in C', we shall have OC' as the new approximate value of the root. But observe that there is here a real root OX, and that the curve beyond X

is convex to the axis; under these conditions the point C' is nearer to X than was C; and, starting with C' instead of C, and proceeding in like manner to draw a new ordinate and tangent, and so on as often as we please, we approximate continually, and that with great rapidity, to the true value OX. But if C had been taken on the other side of X, where the curve is concave to the axis, the new point C' might or might not be nearer to X than was the point C; and in this



case the method, if it succeeds at all, does so by accident only, i.e., it may happen that C' or some subsequent point comes to be a point C, such that OC is a *proper* approximate value of the root, and then the subsequent approximations proceed in the same manner as if this value had been assumed in the first instance, all the preceding work being wasted. It thus appears that for the proper application of the method we require *more* than the mere separation of the roots. In order to be able to approximate to a certain root $\alpha, = OX$, we require to know that, between OX and some value ON, the curve is always convex to the axis: analytically, between the two values, $f(x)$ and $f''(x)$ must have always the same sign. When this is so, the point C may be taken anywhere on the proper side of X, and within the portion XN of the axis; and the process is then the one already explained. The approximation is in general a very rapid one. If we know for the required root OX the two limits OM, ON such that from M to X the curve is always *concave* to the axis, while from X to N it is always convex to the axis,—then, taking D anywhere in the portion MX and (as before) C in the portion XN, drawing the ordinates DQ, CP, and joining the points P, Q by a line which meets the axis in D', also constructing the point C' by means of the tangent at P as before, we have for the required root the new limits OD', OC'; and proceeding in like manner with the points D', C', and so on as often as we please, we obtain at each step two limits approximating more and more nearly to the required root OX. The process as to the point D', translated into analysis, is the ordinate process of interpolation. Suppose $OD = \beta$, $OC = \alpha$, we have approximately

$$f(\beta + h) = f(\beta) + \frac{h\{f(\alpha) - f(\beta)\}}{\alpha - \beta},$$

whence, if the root is $\beta + h$, then

$$h = -\frac{(\alpha - \beta) f(\beta)}{f(\alpha) - f(\beta)}.$$

Returning for a moment to Horner's method, it may be remarked that the correction h, to an approximate value α, is therein found as a quotient, the same or such as the quotient $f(\alpha) \div f'(\alpha)$ which presents itself in Newton's method. The difference is that with Horner the integer part of this quotient, is taken as the presumptive value of h, and the figure is verified at each step. With Newton the quotient itself, developed to the proper number of decimal places, is taken as the value of h; if too many decimals are taken, there would be a waste of work; but the error would correct itself at the next step. Of course the calculation should be conducted without any such waste of work.

Next as to the theory of imaginaries.

15. It will be recollected that the expression *number* and the correlative epithet *numerical* were at the outset used in a wide sense, as extending to imaginaries. This extension arises out of the theory of equations by a process analogous to that by which number, in its original most restricted sense of positive integer number, was extended to have the meaning of a real positive or negative magnitude susceptible of continuous variation.

If for a moment number is understood in its most restricted sense as meaning positive integer number, the solution of a simple equation leads to an extension; $ax - b = 0$, gives $x = \frac{b}{a}$, a positive fraction, and we can in this manner represent, not accurately, but as nearly as we please, any positive magnitude whatever; so an equation $ax + b = 0$ gives $x = -\frac{b}{a}$, which (approximately as before) represents any negative magnitude. We thus arrive at the extended signification of number as a continuously varying positive or negative magnitude. Such numbers may be added or subtracted, multiplied or divided one by another, and the result is always a number. Now from a quadric equation we derive, in like manner, the notion of a complex or imaginary number such as is spoken of above. The equation $x^2 + 1 = 0$ is not (in the foregoing sense, number = real number) satisfied by any numerical value whatever of x; but we assume that there is a number which we call i, satisfying the equation $i^2 + 1 = 0$; and then taking a and b any real numbers, we form an expression such as $a + bi$, and use the expression number in this extended sense: any two such numbers may be added or subtracted, multiplied or divided one by the other, and the result is always a number. And if we consider first a quadric equation $x^2 + px + q = 0$ where p and q are real numbers, and next the like equation, where p and q are any numbers whatever, it can be shown that there exists for x a numerical value which satisfies the equation; or, in other words, it can be shown that the equation has a numerical root. The like theorem, in fact, holds good for an equation of any order whatever. But suppose for a moment that this was not the case: say that there was a cubic equation $x^3 + px^2 + qx + r = 0$, with numerical coefficients, not satisfied by any numerical value of x, we should have to establish a new imaginary j satisfying some such equation, and should then have to consider numbers of the form $a + bj$, or perhaps $a + bj + cj^2$ (a, b, c numbers $\alpha + \beta i$ of the kind heretofore considered),—first we should be thrown back on the quadric equation $x^2 + px + q = 0$, p and q being now numbers

of the last-mentioned extended form—*non constat* that every such equation has a numerical root—and if not, we might be led to *other* imaginaries k, l, &c., and so on *ad infinitum* in inextricable confusion.

But in fact a numerical equation of any order whatever has always a numerical root, and thus numbers (in the foregoing sense, number = quantity of the form $\alpha+\beta i$) form (*what real numbers do not*) a universe complete in itself, such that starting in it we are never led out of it. There may very well be, and perhaps are, numbers in a more general sense of the term (quaternions are not a case in point, as the ordinary laws of combination are not adhered to): but in order to have to do with such numbers (if any), we must start with them.

16. The capital theorem as regards numerical equations thus is, every numerical equation has a numerical root; or for shortness (the meaning being as before), every equation has a root. Of course the theorem is the reverse of self-evident, and it requires proof; but provisionally assuming it as true, we derive from it the general theory of numerical equations. As the term root was introduced in the course of an explanation, it will be convenient to give here the formal definition.

A number a such that substituted for x it makes the function $x^n-p_1x^{n-1}+\ldots\pm p_n$ to be $=0$, or say such that it satisfies the equation $f(x)=0$, is said to be a root of the equation; that is, a being a root, we have

$$a^n-p_1a^{n-1}+\ldots\pm p_n=0, \text{ or say } f(a)=0;$$

and it is then easily shown that $x-a$ is a factor of the function $f(x)$, viz. that we have $f(x)=(x-a)f_1(x)$, where $f_1(x)$ is a function $x^{n-1}-q_1x^{n-2}+\ldots\pm q_{n-1}$ of the order $n-1$, with numerical coefficients $q_1, q_2, \ldots, q_{n-1}$.

In general, a is not a root of the equation $f_1(x)=0$, but it may be so—i.e., $f_1(x)$ may contain the factor $x-a$; when this is so, $f(x)$ will contain the factor $(x-a)^2$; writing then $f(x)=(x-a)^2f_2(x)$, and assuming that a is not a root of the equation $f_2(x)=0$, $x=a$ is then said to be a double root of the equation $f(x)=0$; and similarly $f(x)$ may contain the factor $(x-a)^3$ and no higher power, and $x=a$ is then a triple root; and so on.

Supposing, in general, that $f(x)=(x-a)^\alpha F(x)$, α being a positive integer which may be $=1$, $(x-a)^\alpha$ the highest power of $x-a$ which divides $f(x)$, and $F(x)$ being of course of the order $n-\alpha$, then the equation $F(x)=0$ will have a root b which will be different from a; $x-b$ will be a factor, in general a simple one, but it may be a multiple one, of $F(x)$, and $f(x)$ will in this case be $=(x-a)^\alpha(x-b)^\beta\Phi(x)$, β a positive integer which may be $=1$, $(x-b)^\beta$ the highest power of $x-b$ in $F(x)$ or $f(x)$, and $\Phi(x)$ being of course of the order $n-\alpha-\beta$. The original equation $f(x)=0$ is in this case said to have α roots each $=a$, β roots each $=b$; and so on for any other factors $(x-c)^\gamma$, &c.

We have thus the *theorem*—A numerical equation of the order n has in every case n roots, viz. there exist n numbers $a, b, \ldots$, in general all distinct, but which may arrange themselves in any sets of equal values, such that $f(x)=(x-a)(x-b)(x-c)\ldots$ identically.

If the equation has equal roots, these can in general be determined: and the case is at any rate a special one which may be in the first instance excluded from consideration. It is therefore, in general, assumed that the equation $f(x)=0$ has all its roots unequal.

If the coefficients $p_1, p_2, \ldots$ are all or any one or more of them imaginary, then the equation $f(x)=0$, separating the real and imaginary parts thereof, may be written $F(x)+i\Phi(x)=0$, where $F(x)$, $\Phi(x)$ are each of them a function with real coefficients; and it thus appears that the equation $f(x)=0$, with imaginary coefficients, has not in general any real root; supposing it to have a real root a, this must be at once a root of each of the equations $F(x)=0$ and $\Phi(x)=0$.

But an equation with real coefficients may have as well imaginary as real roots, and we have further the theorem that for any such equation the imaginary roots enter in pairs, viz. $\alpha+\beta i$ being a root, then $\alpha-\beta i$ will be also a root. It follows that, if the order be odd, there is always an odd number of real roots, and therefore at least one real root.

17. In the case of an equation with real coefficients, the question of the existence of real roots, and of their separation, has been already considered. In the general case of an equation with imaginary (it may be real) coefficients, the like question arises as to the situation of the (real or imaginary) roots; thus if, for facility of conception, we regard the constituents α, β of a root $\alpha+\beta i$ as the coordinates of a point *in plano*, and accordingly represent the root by such point, then drawing in the plane any closed curve or "contour," the question is how many roots lie within such contour.

This is solved theoretically by means of a theorem of Cauchy's (1837), viz. writing in the original equation $x+iy$ in place of x, the function $f(x+iy)$ becomes $=P+iQ$, where P and Q are each of them a rational and integral function (with real coefficients) of (x, y). Imagining the point (x, y) to travel along the contour, and considering the number of changes of sign from $-$ to $+$ and from $+$ to $-$ of the fraction corresponding to passages of the fraction through zero, that is, to values for which P becomes $=0$, disregarding those for which Q becomes $=0$, the difference of these numbers gives the number of roots within the contour.

It is important to remark that the demonstration does not presuppose the existence of any root; the contour may be the infinity of the plane (such infinity regarded as a contour, or closed curve), and in this case it can be shown (and that very easily) that the difference of the numbers of changes of sign is $=n$; that is, there are within the infinite contour, or (what is the same thing) there are in all, n roots; thus Cauchy's theorem contains really the proof of the fundamental theorem that a numerical equation of the nth order (not only has a numerical root, but) has precisely n roots. It would appear that this proof of the fundamental theorem in its most complete form is in principle identical with Gauss's last proof (1849) of the theorem, in the form—A numerical equation of the nth order has always a root*.

* The earlier demonstrations by Euler, Lagrange, &c., relate to the case of a numerical equation with real coefficients; and they consist in showing that such equation has always a real quadratic divisor, furnishing two roots, which are either real or else conjugate imaginaries $\alpha+\beta i$: see Lagrange's *Équations Numériques*.

But in the case of a finite contour, the actual determination of the difference which gives the number of real roots can be effected only in the case of a rectangular contour, by applying to each of its sides separately a method such as that of Sturm's theorem; and thus the actual determination ultimately depends on a method such as that of Sturm's theorem.

Very little has been done in regard to the calculation of the imaginary roots of an equation by approximation; and the question is not here considered.

18. A class of numerical equations which needs to be considered is that of the binomial equations $x^n - a = 0$ ($a = \alpha + \beta i$, a complex number). The foregoing conclusions apply, viz. there are always n roots, which, it may be shown, are all unequal. And these can be found numerically by the extraction of the square root, and of an nth root, of *real* numbers, and by the aid of a table of natural sines and cosines*. For writing

$$\alpha + \beta i = \sqrt{\alpha^2 + \beta^2}\left\{\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} i\right\},$$

there is always a real angle λ (positive and less than 2π), such that its cosine and sine are $= \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$ and $\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$ respectively; that is, writing for shortness $\sqrt{\alpha^2 + \beta^2} = \rho$, we have $\alpha + \beta i = \rho(\cos\lambda + i\sin\lambda)$, or the equation is $x^n = \rho(\cos\lambda + i\sin\lambda)$; hence observing that $\left(\cos\frac{\lambda}{n} + i\sin\frac{\lambda}{n}\right)^n = \cos\lambda + i\sin\lambda$, a value of x is $= \sqrt[n]{\rho}\left(\cos\frac{\lambda}{n} + i\sin\frac{\lambda}{n}\right)$. The formula really gives all the roots, for instead of λ we may write $\lambda + 2s\pi$, s a positive or negative integer, and then we have

$$x = \sqrt[n]{\rho}\left(\cos\frac{\lambda + 2s\pi}{n} + i\sin\frac{\lambda + 2s\pi}{n}\right),$$

which has the n values obtained by giving to s the values $0, 1, 2, \ldots, n-1$ in succession; the roots are, it is clear, represented by points lying at equal intervals on a circle. But it is more convenient to proceed somewhat differently; taking one of the roots to be θ, so that $\theta^n = a$, then assuming $x = \theta y$, the equation becomes $y^n - 1 = 0$, which equation, like the original equation, has precisely n roots (one of them being of course $= 1$). And the original equation $x^n - a = 0$ is thus reduced to the more simple equation $x^n - 1 = 0$; and although the theory of this equation is included in the preceding one, yet it is proper to state it separately.

The equation $x^n - 1 = 0$ has its several roots expressed in the form $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where ω may be taken $= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$; in fact, ω having this value, any integer power ω^k is $= \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, and we thence have $(\omega^k)^n = \cos 2\pi k + i\sin 2\pi k, = 1$, that is, ω^k is a root of the equation. The theory will be resumed further on.

* The square root of $\alpha + \beta i$ can be determined by the extraction of square roots of positive real numbers, without the trigonometrical tables.

By what precedes, we are led to the notion (a numerical) of the radical $a^{\frac{1}{n}}$ regarded as an n-valued function; any one of these being denoted by $\sqrt[n]{a}$, then the series of values is $\sqrt[n]{a}$, $\omega\sqrt[n]{a}, \ldots, \omega^{n-1}\sqrt[n]{a}$; or we may, if we please, use $\sqrt[n]{a}$ instead of $a^{\frac{1}{n}}$ as a symbol to denote the n-valued function.

As the coefficients of an algebraical equation may be numerical, all which follows in regard to algebraical equations is (with, it may be, some few modifications) applicable to numerical equations; and hence, concluding for the present this subject, it will be convenient to pass on to algebraical equations.

II. We consider, secondly, algebraical equations (19 to 34).

19. The equation is

$$x^n - p_1 x^{n-1} + \ldots \pm p_n = 0,$$

and we here *assume* the existence of roots, viz. we assume that there are n quantities $a, b, c, \ldots$ (in general all of them different, but which in particular cases may become equal in sets in any manner), such that

$$x^n - p_1 x^{n-1} + \ldots \pm p_n = 0;$$

or looking at the question in a different point of view, and starting with the roots $a, b, c, \ldots$ as given, we express the product of the n factors $x-a$, $x-b, \ldots$ in the foregoing form, and thus arrive at an equation of the order n having the n roots $a, b, c, \ldots$. In either case we have

$$p_1 = \Sigma a, \quad p_2 = \Sigma ab, \ldots, \quad p_n = abc \ldots;$$

i.e., regarding the coefficients $p_1, p_2, \ldots, p_n$ as given, then we assume the existence of roots $a, b, c, \ldots$ such that $p_1 = \Sigma a$, &c.; or, regarding the roots as given, then we write p_1, p_2, &c., to denote the functions Σa, Σab, &c.

As already explained, the epithet algebraical is not used in opposition to numerical; an algebraical equation is merely an equation wherein the coefficients are not restricted to denote, or are not explicitly considered as denoting, numbers. That the abstraction is legitimate, appears by the simplest example; in saying that the equation $x^2 - px + q = 0$ has a root $x = \frac{1}{2}(p + \sqrt{p^2 - 4q})$, we mean that writing this value for x the equation becomes an identity, $\{\frac{1}{2}(p + \sqrt{p^2 - 4q})\}^2 - p\{\frac{1}{2}(p + \sqrt{p^2 - 4q})\} + q = 0$; and the verification of this identity in nowise depends upon p and q meaning numbers. But if it be asked what there is beyond numerical equations included in the term algebraical equation, or, again, what is the full extent of the meaning attributed to the term—the latter question at any rate it would be very difficult to answer; as to the former one, it may be said that the coefficients may, for instance, be symbols of operation. As regards such equations, there is certainly no proof that every equation has a root, or that an equation of the nth order has n roots; nor is it in any wise clear what the precise signification of the statement is. But it is found that the assumption of the existence of the n roots can be made without contradictory results; conclusions

derived from it, if they involve the roots, rest on the same ground as the original assumption; but the conclusion may be independent of the roots altogether, and in this case it is undoubtedly valid; the reasoning, although actually conducted by aid of the assumption (and, it may be, most easily and elegantly in this manner), is really independent of the assumption. In illustration, we observe that it is allowable to express a function of p and q as follows,—that is, by means of a rational symmetrical function of a and b; this can, as a fact, be expressed as a rational function of $a+b$ and ab; and if we prescribe that $a+b$ and ab shall then be changed into p and q respectively, we have the required function of p, q. That is, we have $F(\alpha, \beta)$ as a representation of $f(p, q)$, obtained as if we had $p=a+b$, $q=ab$, but without in any wise assuming the existence of the a, b of these equations.

20. Starting from the equation

$$x^n - p_1 x^{n-1} + \ldots = x - a \,.\, x - b \,.\, \&c.,$$

or the equivalent equations $p_1 = \Sigma a$, &c., we find

$$\begin{aligned} a^n - p_1 a^{n-1} + \ldots &= 0, \\ b^n - p_1 b^{n-1} + \ldots &= 0; \\ \vdots \qquad \vdots \qquad & \vdots \end{aligned}$$

(it is as satisfying these equations that $a, b, \ldots$ are said to be the roots of $x^n - p_1 x^{n-1} + \ldots = 0$); and conversely from the last-mentioned equations, assuming that $a, b, \ldots$ are all different, we deduce

$$p_1 = \Sigma a, \quad p_2 = \Sigma ab, \ \&c.,$$

and

$$x^n - p_1 x^{n-1} + \ldots = x - a \,.\, x - b \,.\, \&c.$$

Observe that if, for instance, $a=b$, then the equations $a^n - p_1 a^{n-1} + \ldots = 0$, $b^n - p_1 b^{n-1} + \ldots = 0$ would reduce themselves to a single relation, which would not of itself express that a was a double root,—that is, that $(x-a)^2$ was a factor of $x^n - p_1 x^{n-1} + \&c.$; but by considering b as the limit of $a+h$, h indefinitely small, we obtain a second equation

$$n a^{n-1} - (n-1)\, p_1 a^{n-2} + \ldots = 0,$$

which, with the first, expresses that a is a double root; and then the whole system of equations leads as before to the equations $p_1 = \Sigma a$, &c. But the existence of a double root implies a certain relation between the coefficients; the general case is when the roots are all unequal.

We have then the *theorem* that every rational symmetrical function of the roots is a rational function of the coefficients. This is an easy consequence from the less general theorem, every rational and integral symmetrical function of the roots is a rational and integral function of the coefficients.

In particular, the sums of the powers Σa^2, Σa^3, &c., are rational and integral functions of the coefficients.

The process originally employed for the expression of other functions $\Sigma a^\alpha b^\beta$, &c., in terms of the coefficients is to make them depend upon the sums of powers: for instance, $\Sigma a^\alpha b^\beta = \Sigma a^\alpha \Sigma a^\beta - \Sigma a^{\alpha+\beta}$; but this is very objectionable; the true theory consists in showing that we have systems of equations

$$p_1 = \Sigma a,$$

$$\begin{cases} p_2 = \Sigma ab, \\ p_1^2 = \Sigma a^2 + 2\Sigma ab, \end{cases}$$

$$\begin{cases} p_3 = \Sigma abc, \\ p_1 p_2 = \Sigma a^2 b + 3\Sigma abc, \\ p_1^3 = \Sigma a^3 + 3\Sigma a^2 b + 6\Sigma abc, \end{cases}$$

where in each system there are precisely as many equations as there are root-functions on the right-hand side—e.g. 3 equations and 3 functions Σabc, $\Sigma a^2 b$, Σa^3. Hence in each system the root-functions can be determined linearly in terms of the powers and products of the coefficients:

$$\begin{cases} \Sigma ab = p_2, \\ \Sigma a^2 = p_1^2 - 2p_2, \end{cases}$$

$$\begin{cases} \Sigma abc = p_3, \\ \Sigma a^2 b = p_1 p_2 - 3p_3, \\ \Sigma a^3 = p_1^3 - 3p_1 p_2 + 3p_3, \end{cases}$$

and so on. The older process, if applied consistently, would derive the originally assumed value $\Sigma ab, = p_2$, from the two equations $\Sigma a = p_1$, $\Sigma a^2 = p_1^2 - 2p_2$; i.e. we have $2\Sigma ab = \Sigma a \,.\, \Sigma a - \Sigma a^2, = p_1^2 - (p_1^2 - 2p_2), = 2p_2$.

21. It is convenient to mention here the theorem that, x being determined as above by an equation of the order n, any rational and integral function whatever of x, or more generally any rational function which does not become infinite in virtue of the equation itself, can be expressed as a rational and integral function of x, of the order $n-1$, the coefficients being rational functions of the coefficients of the equation. Thus the equation gives x^n a function of the form in question; multiplying each side by x, and on the right-hand side writing for x^n its foregoing value, we have x^{n+1}, a function of the form in question; and the like for any higher power of x, and therefore also for any rational and integral function of x. The proof in the case of a rational non-integral function is somewhat more complicated. The final result is of the form $\frac{\phi(x)}{\psi(x)} = I(x)$, or say $\phi(x) - \psi(x) I(x) = 0$, where ϕ, ψ, I are rational and integral functions; in other words, this equation, being true if only $f(x) = 0$, can only be so by reason that the left-hand side contains $f(x)$ as a factor, or we must have identically $\phi(x) - \psi(x) I(x) = M(x) f(x)$. And it is, moreover, clear that the equation $\frac{\phi(x)}{\psi(x)} = I(x)$, being satisfied if only $f(x) = 0$, must be satisfied by each root of the equation.

From the theorem that a rational symmetrical function of the roots is expressible in terms of the coefficients, it at once follows that it is possible to determine an equation (of an assignable order) having for its roots the several values of any given (unsymmetrical) function of the roots of the given equation. For example, in the case of a quartic equation, having the roots (a, b, c, d), it is possible to find an equation having the roots ab, ac, ad, bc, bd, cd, being therefore a sextic equation: viz. in the product

$$(y-ab)(y-ac)(y-ad)(y-bc)(y-bd)(y-cd),$$

the coefficients of the several powers of y will be symmetrical functions of a, b, c, d and therefore rational and integral functions of the coefficients of the quartic equation; hence, supposing the product so expressed, and equating it to zero, we have the required sextic equation. In the same manner can be found the sextic equation having the roots $(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2$, which is the equation of differences previously referred to; and similarly we obtain the equation of differences for a given equation of any order. Again, the equation sought for may be that having for its n roots the given rational functions $\phi(a), \phi(b), \ldots$ of the several roots of the given equation. Any such rational function can (as was shown) be expressed as a rational and integral function of the order $n-1$; and, retaining x in place of any one of the roots, the problem is to find y from the equations $x^n - p_1 x^{n-1} + \ldots = 0$, and $y = M_0 x^{n-1} + M_1 x^{n-2} + \ldots$, or, what is the same thing, from these two equations to eliminate x. This is, in fact, Tschirnhausen's transformation (1683).

22. In connexion with what precedes, the question arises as to the number of values (obtained by permutations of the roots) of given unsymmetrical functions of the roots, or say of a given set of letters: for instance, with roots or letters (a, b, c, d) as before, how many values are there of the function $ab+cd$, or better, how many functions are there of this form? The answer is 3, viz. $ab+cd$, $ac+bd$, $ad+bc$; or again we may ask whether, in the case of a given number of letters, there exist functions with a given number of values, 3-valued, 4-valued functions, &c.

It is at once seen that for any given number of letters there exist 2-valued functions; the product of the differences of the letters is such a function; however the letters are interchanged, it alters only its sign; or say the two values are $\Delta, -\Delta$. And if P, Q are symmetrical functions of the letters, then the general form of such a function is $P+Q\Delta$; this has only the two values $P+Q\Delta$, $P-Q\Delta$.

In the case of 4 letters there exist (as appears above) 3-valued functions: but in the case of 5 letters there does not exist any 3-valued or 4-valued function; and the only 5-valued functions are those which are symmetrical in regard to four of the letters, and can thus be expressed in terms of one letter and of symmetrical functions of all the letters. These last theorems present themselves in the demonstration of the non-existence of a solution of a quintic equation by radicals.

The theory is an extensive and important one, depending on the notions of *substitutions* and of *groups* *.

* A substitution is the operation by which we pass from the primitive arrangement of n letters to any other arrangement of the same letters: for instance, the substitution $\left(\frac{bcda}{abcd}\right)$ means that a is to be changed

23. Returning to equations, we have the very important theorem that, given the value of any unsymmetrical function of the roots, e.g. in the case of a quartic equation, the function $ab+cd$, it is in general possible to determine rationally the value of any similar function, such as $(a+b)^3+(c+d)^3$.

The *a priori* ground of this theorem may be illustrated by means of a numerical equation. Suppose that the roots of a quartic equation are 1, 2, 3, 4, then if it is given that $ab+cd=14$, this in effect determines a, b to be 1, 2 and c, d to be 3, 4 (viz. $a=1$, $b=2$ or $a=2$, $b=1$, and $c=3$, $d=4$ or $c=4$, $d=3$) or else a, b to be 3, 4 and c, d to be 1, 2; and it therefore in effect determines $(a+b)^3+(c+d)^3$ to be $=370$, and not any other value; that is, $(a+b)^3+(c+d)^3$, as having a single value, must be determinable rationally. And we can in the same way account for cases of failure as regards particular equations; thus, the roots being 1, 2, 3, 4 as before, $a^2b=2$ determines a to be $=1$ and b to be $=2$; but if the roots had been 1, 2, 4, 16 then $a^2b=16$ does not uniquely determine a, b but only makes them to be 1, 16 or 2, 4 respectively.

As to the *a posteriori* proof, assume, for instance,

$$t_1=ab+cd,\quad y_1=(a+b)^3+(c+d)^3,$$
$$t_2=ac+bd,\quad y_2=(a+c)^3+(b+d)^3,$$
$$t_3=ad+bc,\quad y_3=(a+d)^3+(b+c)^3:$$

then

$$y_1+y_2+y_3,\quad t_1y_1+t_2y_2+t_3y_3,\quad t_1^2y_1+t_2^2y_2+t_3^2y_3,$$

will be respectively symmetrical functions of the roots of the quartic, and therefore rational and integral functions of the coefficients; that is, they will be known.

Suppose for a moment that t_1, t_2, t_3 are *all* known; then the equations being linear in y_1, y_2, y_3 these can be expressed rationally in terms of the coefficients and of t_1, t_2, t_3; that is, y_1, y_2, y_3 will be known. But observe further that y_1 is obtained as a function of t_1, t_2, t_3 symmetrical as regards t_2, t_3; it can therefore be expressed

into b, b into c, c into d, d into a. Substitutions may, of course, be represented by single letters $\alpha, \beta, \ldots$; $\left(\frac{abcd}{abcd}\right)$, $=1$, is the substitution which leaves the letters unaltered. Two or more substitutions may be compounded together and give rise to a substitution; i.e., performing upon the primitive arrangement first the substitution β and then upon the result the substitution α, we have the substitution $\alpha\beta$. Substitutions are not commutative; thus, $\alpha\beta$ is not in general $=\beta\alpha$; but they are associative, $\alpha\beta.\gamma=\alpha.\beta\gamma$, so that $\alpha\beta\gamma$ has a determinate meaning. A substitution may be compounded any number of times with itself, and we thus have the powers $\alpha^2, \alpha^3, \ldots$, &c. Since the number of substitutions is limited, some power α^ν must be $=1$: or, as this may be expressed, every substitution is a root of unity. A group of substitutions is a set such that each two of them compounded together in either order gives a substitution belonging to the set; every group includes the substitution unity, so that we may in general speak of a group $1, \alpha, \beta, \ldots$ (the number of terms is the order of the group). The whole system of the $1.2.3\ldots n$ substitutions which can be performed upon the n letters is obviously a group: the order of every other group which can be formed out of these substitutions is a submultiple of this number; but it is not conversely true that a group exists the order of which is any given submultiple of this number. In the case of a determinant the substitutions which give rise to the positive terms form a group the order of which is $=\frac{1}{2}.1.2.3\ldots n$. For any function of the n letters, the whole series of substitutions which leave the value of the functions unaltered form a group; and thence also the number of values of the function is $=1.2.3\ldots n$ divided by the order of the group.

as a rational function of t_1 and of t_2+t_3, t_2t_3, and thence as a rational function of t_1 and of $t_1+t_2+t_3$, $t_1t_2+t_1t_3+t_2t_3$, $t_1t_2t_3$; but these last are symmetrical functions of the roots, and as such they are expressible rationally in terms of the coefficients; that is, y_1 will be expressed as a rational function of t_1 and of the coefficients; or t_1 (alone, not t_2 or t_3) being known, y_1 will be rationally determined.

24. We now consider the question of the algebraical solution of equations, or, more accurately, that of the *solution of equations by radicals.*

In the case of a quadric equation $x^2-px+q=0$, we can by the assistance of the sign $\sqrt{(\ )}$ or $(\ )^{\frac{1}{2}}$ find an expression for x as a two-valued function of the coefficients p, q such that, substituting this value in the equation, the equation is thereby identically satisfied; it has been found that this expression is

$$x=\tfrac{1}{2}\{p\pm\sqrt{p^2-4q}\},$$

and the equation is on this account said to be algebraically solvable, or more accurately solvable by radicals. Or we may by writing $x=-\frac{1}{2}p+z$, reduce the equation to $z^2=\frac{1}{4}(p^2-4q)$ viz. to an equation of the form $z^2=a$; and in virtue of its being thus reducible we say that the original equation is solvable by radicals. And the question for an equation of any higher order, say of the order n, is, can we by means of radicals, that is, by aid of the sign $\sqrt[m]{(\ )}$ or $(\ )^{\frac{1}{m}}$, using as many as we please of such signs and with any values of m, find an n-valued function (or any function) of the coefficients which substituted for x in the equation shall satisfy it identically.

It will be observed that the coefficients $p, q, \ldots$ are not explicitly considered as numbers, but even if they do denote numbers, the question whether a numerical equation admits of solution by radicals is wholly unconnected with the before-mentioned theorem of the existence of the n roots of such an equation. It does not even follow that in the case of a numerical equation solvable by radicals the algebraical solution gives the numerical solution, but this requires explanation. Consider first a numerical quadric equation with imaginary coefficients. In the formula $x=\frac{1}{2}(p\pm\sqrt{p^2-4q})$, substituting for p, q their given numerical values, we obtain for x an expression of the form $x=\alpha+\beta i\pm\sqrt{\gamma+\delta i}$, where α, β, γ, δ are real numbers. This expression substituted for x in the quadric equation would satisfy it identically, and it is thus an algebraical solution; but there is no obvious *a priori* reason why $\sqrt{\gamma+\delta i}$ should have a value $=c+di$, where c and d are real numbers calculable by the extraction of a root or roots of real numbers; however the case is (what there was no *a priori* right to expect) that $\sqrt{\gamma+\delta i}$ has such a value calculable by means of the radical expressions $\sqrt{\{\sqrt{\gamma^2+\delta^2}\pm\gamma\}}$: and hence the algebraical solution of a numerical quadric equation does in every case give the numerical solution. The case of a numerical cubic equation will be considered presently.

25. A cubic equation can be solved by radicals. Taking for greater simplicity the cubic in the reduced form $x^3+qx-r=0$, and assuming $x=a+b$, this will be a solution if only $3ab=q$ and $a^3+b^3=r$, equations which give $(a^3-b^3)^2=r^2-\frac{4}{27}q^3$, a

quadric equation solvable by radicals, and giving $a^3 - b^3 = \sqrt{r^2 - \frac{4}{27}q^3}$, a two-valued function of the coefficients: combining this with $a^3 + b^3 = r$, we have $a^3 = \frac{1}{2}(r + \sqrt{r^2 - \frac{4}{27}q^3})$, a two-valued function: we then have a by means of a cube root, viz.

$$a = \sqrt[3]{\{\tfrac{1}{2}(r + \sqrt{r^2 - \tfrac{4}{27}q^3})\}},$$

a six-valued function of the coefficients; but then, writing $q = \dfrac{b}{3a}$, we have, as may be shown, $a + b$ a three-valued function of the coefficients; and $x = a + b$ is the required solution by radicals. It would have been wrong to complete the solution by writing

$$b = \sqrt[3]{\{\tfrac{1}{2}(r - \sqrt{r^2 - \tfrac{4}{27}q^3})\}},$$

for then $a + b$ would have been given as a 9-valued function having only 3 of its values roots, and the other 6 values being irrelevant. Observe that in this last process we make no use of the equation $3ab = q$, in its original form, but use only the derived equation $27a^3b^3 = q^3$, implied in, but not implying, the original form.

An interesting variation of the solution is to write $x = ab(a + b)$, giving $a^3b^3(a^3 + b^3) = r$ and $3a^3b^3 = q$, or say $a^3 + b^3 = \dfrac{3r}{q}$, $a^3b^3 = \frac{1}{3}q$; and consequently

$$a^3 = \frac{\frac{3}{2}}{q}(r + \sqrt{r^2 - \tfrac{4}{27}q^3}),\ b^3 = \frac{\frac{3}{2}}{q}(r - \sqrt{r^2 - \tfrac{4}{27}q^3}),$$

i.e., here a^3, b^3 are each of them a two-valued function, but as the only effect of altering the sign of the quadric radical is to interchange a^3, b^3, they may be regarded as each of them one-valued; a and b are each of them 3-valued (for observe that here only a^3b^3, not ab, is given); and $ab(a + b)$ thus is in appearance a 9-valued function, but it can easily be shown that it is (as it ought to be) only 3-valued.

In the case of a numerical cubic, even when the coefficients are real, substituting their values in the expression

$$x = \sqrt[3]{\{\tfrac{1}{2}(r + \sqrt{r^2 - \tfrac{4}{27}q^3})\}} + \tfrac{1}{3}q \div \sqrt[3]{\{\tfrac{1}{2}(r + \sqrt{r^2 - \tfrac{4}{27}q^3})\}},$$

this may depend on an expression of the form $\sqrt[3]{\gamma + \delta i}$, where γ and δ are real numbers (it will do so if $r^2 - \frac{4}{27}q^3$ is a negative number), and then we *cannot* by the extraction of any root or roots of real positive numbers reduce $\sqrt[3]{\gamma + \delta i}$ to the form $c + di$, c and d real numbers; hence here the algebraical solution does *not* give the numerical solution, and we have here the so-called "irreducible case" of a cubic equation. By what precedes, there is nothing in this that might not have been expected; the algebraical solution makes the solution depend on the extraction of the cube root of a negative number, and there was no reason for expecting this to be a real number. It is well known that the case in question is that wherein the three roots of the numerical cubic equation are all real; if the roots are two imaginary, one real, then contrariwise the quantity under the cube root is real; and the algebraical solution gives the numerical one.

The irreducible case is solvable by a trigonometrical formula, but this is not a solution by radicals: it consists, in effect, in reducing the given numerical cubic (not to a cubic of the form $x^3 = a$, solvable by the extraction of a cube root, but) to a cubic of the form $4x^3 - 3x = a$, corresponding to the equation $4\cos^3\theta - 3\cos\theta = \cos 3\theta$ which serves to determine $\cos\theta$ when $\cos 3\theta$ is known. The theory is applicable to an algebraical cubic equation; say that such an equation, if it can be reduced to the form $4x^3 - 3x = a$, is solvable by "trisection"—then the general cubic equation is solvable by trisection.

26. A quartic equation is solvable by radicals: and it is to be remarked that the existence of such a solution depends on the existence of 3-valued functions such as $ab + cd$ of the four roots (a, b, c, d): by what precedes, $ab + cd$ is the root of a cubic equation, which equation is solvable by radicals: hence $ab + cd$ can be found by radicals; and since $abcd$ is a given function, ab and cd can then be found by radicals. But by what precedes, if ab be known then any similar function, say $a + b$, is obtainable rationally; and then from the values of $a + b$ and ab we may by radicals obtain the value of a or b, that is, an expression for the root of the given quartic equation: the expression ultimately obtained is 4-valued, corresponding to the different values of the several radicals which enter therein, and we have thus the expression by radicals of each of the four roots of the quartic equation. But when the quartic is numerical the same thing happens as in the cubic, and the algebraical solution does not in every case give the numerical one.

It will be understood, from the foregoing explanation as to the quartic, how in the next following case, that of the quintic, the question of the solvability by radicals depends on the existence or non-existence of k-valued functions of the five roots (a, b, c, d, e); the fundamental theorem is the one already stated, a rational function of five letters, if it has less than 5, cannot have more than 2 values, that is, there are no 3-valued or 4-valued functions of 5 letters: and by reasoning depending in part upon this theorem, Abel (1824) showed that a general quintic equation is not solvable by radicals; and *a fortiori* the general equation of any order higher than 5 is not solvable by radicals.

27. The general theory of the solvability of an equation by radicals depends fundamentally on Vandermonde's remark (1770) that, supposing an equation is solvable by radicals, and that we have therefore an algebraical expression of x in terms of the coefficients, then substituting for the coefficients their values in terms of the roots, the resulting expression must reduce itself to any one at pleasure of the roots $a, b, c, \ldots$; thus in the case of the quadric equation, in the expression $x = \frac{1}{2}(p + \sqrt{p^2 - 4q})$, substituting for p and q their values, and observing that $(a+b)^2 - 4ab = (a-b)^2$, this becomes $x = \frac{1}{2}\{a + b + \sqrt{(a-b)^2}\}$, the value being a or b according as the radical is taken to be $+(a-b)$ or $-(a-b)$.

So in the cubic equation $x^3 - px^2 + qx - r = 0$, if the roots are a, b, c, and if ω is used to denote an imaginary cube root of unity, $\omega^2 + \omega + 1 = 0$, then writing for shortness $p = a + b + c$, $L = a + \omega b + \omega^2 c$, $M = a + \omega^2 b + \omega c$, it is at once seen that LM,

L^3+M^3, and therefore also $(L^3-M^3)^2$ are symmetrical functions of the roots, and consequently rational functions of the coefficients: hence

$$\tfrac{1}{2}\{L^3+M^3+\sqrt{(L^3-M^3)^2}\}$$

is a rational function of the coefficients, which when these are replaced by their values as functions of the roots becomes, according to the sign given to the quadric radical, $=L^3$ or M^3: taking it $=L^3$, the cube root of the expression has the three values $L, \omega L, \omega^2 L$; and LM divided by the same cube root has therefore the values $M, \omega^2 M, \omega M$; whence finally the expression

$$\begin{aligned}\tfrac{1}{3}[p &+\sqrt[3]{\{\tfrac{1}{2}(L^3+M^3+\sqrt{(L^3-M^3)^2})\}}\\ &+LM\div\sqrt[3]{\{\tfrac{1}{2}(L^3+M^3+\sqrt{(L^3-M^3)^2})\}}]\end{aligned}$$

has the three values

$$\tfrac{1}{3}(p+L+M),\ \tfrac{1}{3}(p+\omega L+\omega^2 M),\ \tfrac{1}{3}(p+\omega^2 L+\omega M);$$

that is, these are $=a$, b, c respectively. If the value M^3 had been taken instead of L^3, then the expression would have had the same three values a, b, c. Comparing the solution given for the cubic $x^3+qx-r=0$, it will readily be seen that the two solutions are identical, and that the function $r^2-\frac{4}{27}q^3$ under the radical sign must (by aid of the relation $p=0$ which subsists in this case) reduce itself to $(L^3-M^3)^2$; it is only by each radical being equal to a rational function of the roots that the final expression *can* become equal to the roots a, b, c respectively.

28. The formulæ for the cubic were obtained by Lagrange (1770—71) from a different point of view. Upon examining and comparing the principal known methods for the solution of algebraical equations, he found that they all ultimately depended upon finding a "resolvent" equation of which the root is $a+\omega b+\omega^2 c+\omega^3 d+\ldots$, ω being an imaginary root of unity, of the same order as the equation; e.g., for the cubic the root is $a+\omega b+\omega^2 c$, ω an imaginary cube root of unity. Evidently the method gives for L^3 a quadric equation, which is the "resolvent" equation in this particular case.

For a quartic the formulæ present themselves in a somewhat different form, by reason that 4 is not a prime number. Attempting to apply it to a quintic, we seek for the equation of which the root is $(a+\omega b+\omega^2 c+\omega^3 d+\omega^4 e)$, ω an imaginary fifth root of unity, or rather the fifth power thereof $(a+\omega b+\omega^2 c+\omega^3 d+\omega^4 e)^5$; this is a 24-valued function, but if we consider the four values corresponding to the roots of unity ω, ω^2, ω^3, ω^4, viz. the values

$$\begin{aligned}&(a+\omega\, b+\omega^2 c+\omega^3 d+\omega^4 e)^5,\\ &(a+\omega^2 b+\omega^4 c+\omega\, d+\omega^3 e)^5,\\ &(a+\omega^3 b+\omega\, c+\omega^4 d+\omega^2 e)^5,\\ &(a+\omega^4 b+\omega^3 c+\omega^2 d+\omega\, e)^5,\end{aligned}$$

any symmetrical function of these, for instance their sum, is a six-valued function of the roots, and may therefore be determined by means of a sextic equation, the

coefficients whereof are rational functions of the coefficients of the original quintic equation; the conclusion being that the solution of an equation of the fifth order is made to depend upon that of an equation of the sixth order. This is, of course, useless for the solution of the quintic equation, which, as already mentioned, does not admit of solution by radicals; but the equation of the sixth order, Lagrange's resolvent sextic, is very important, and is intimately connected with all the later investigations in the theory.

29. It is to be remarked, in regard to the question of solvability by radicals, that not only the coefficients are taken to be arbitrary, but it is assumed that they are represented each by a single letter, or say rather that they are not so expressed in terms of other arbitrary quantities as to make a solution possible. If the coefficients are not all arbitrary, for instance, if some of them are zero, a sextic equation might be of the form $x^6+bx^4+cx^2+d=0$, and so be solvable as a cubic; or if the coefficients of the sextic are given functions of the six arbitrary quantities a, b, c, d, e, f, such that the sextic is really of the form

$$(x^2+ax+b)(x^4+cx^3+dx^2+ex+f)=0,$$

then it breaks up into the equations $x^2+ax+b=0$, $x^4+cx^3+dx^2+ex+f=0$, and is consequently solvable by radicals; so also if the form is

$$(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)=0,$$

then the equation is solvable by radicals,—in this extreme case rationally. Such cases of solvability are self-evident; but they are enough to show that the general theorem of the non-solvability by radicals of an equation of the fifth or any higher order does not in any wise exclude for such orders the existence of particular equations solvable by radicals, and there are, in fact, extensive classes of equations which are thus solvable; the binomial equations $x^n-1=0$ present an instance.

30. It has already been shown how the several roots of the equation $x^n-1=0$ can be expressed in the form $\cos\frac{2s\pi}{n}+i\sin\frac{2s\pi}{n}$, but the question is now that of the algebraical solution (or solution by radicals) of this equation. There is always a root $=1$; if ω be any other root, then obviously $\omega, \omega^2, \ldots, \omega^{n-1}$ are all of them roots; x^n-1 contains the factor $x-1$, and it thus appears that $\omega, \omega^2, \ldots, \omega^{n-1}$ are the $n-1$ roots of the equation

$$x^{n-1}+x^{n-2}+\ldots+x+1=0;$$

we have, of course,

$$\omega^{n-1}+\omega^{n-2}+\ldots+\omega+1=0.$$

It is proper to distinguish the cases n prime and n composite; and in the latter case there is a distinction according as the prime factors of n are simple or multiple. By way of illustration, suppose successively $n=15$ and $n=9$; in the former case, if α be an imaginary root of $x^3-1=0$ (or root of $x^2+x+1=0$), and β an imaginary root of $x^5-1=0$ (or root of $x^4+x^3+x^2+x+1=0$), then ω may be taken $=\alpha\beta$; the successive powers thereof, $\alpha\beta$, $\alpha^2\beta^2$, β^3, $\alpha\beta^4$, α^2, β, $\alpha\beta^2$, $\alpha^2\beta^3$, β^4, α,

$\alpha^2\beta$, β^2, $\alpha\beta^3$, $\alpha^2\beta^4$, are the roots of $x^{14}+x^{13}+\ldots+x+1=0$; the solution thus depends on the solution of the equations $x^3-1=0$ and $x^5-1=0$. In the latter case, if α be an imaginary root of $x^3-1=0$ (or root of $x^2+x+1=0$), then the equation $x^9-1=0$ gives $x^3=1$, α, or α^2; $x^3=1$ gives $x=1$, α, or α^2; and the solution thus depends on the solution of the equations $x^3-1=0$, $x^3-\alpha=0$, $x^3-\alpha^2=0$. The first equation has the roots 1, α, α^2; if β be a root of either of the others, say if $\beta^3=\alpha$, then assuming $\omega=\beta$, the successive powers are β, β^2, α, $\alpha\beta$, $\alpha\beta^2$, α^2, $\alpha^2\beta$, $\alpha^2\beta^2$, which are the roots of the equation $x^8+x^7+\ldots+x+1=0$.

It thus appears that the only case which need be considered is that of n a prime number, and writing (as is more usual) r in place of ω, we have $r, r^2, r^3, \ldots, r^{n-1}$ as the $(n-1)$ roots of the reduced equation

$$x^{n-1}+x^{n-2}+\ldots+x+1=0;$$

then not only $r^n-1=0$, but also $r^{n-1}+r^{n-2}+\ldots+r+1=0$.

31. The process of solution due to Gauss (1801) depends essentially on the arrangement of the roots in a certain order, viz. not as above, with the indices of r in arithmetical progression, but with their indices in geometrical progression; the prime number n has a certain number of prime roots g, which are such that g^{n-1} is the lowest power of g, which is $\equiv 1$ to the modulus n; or, what is the same thing, that the series of powers $1, g, g^2, \ldots, g^{n-2}$, each divided by n, leave (in a different order) the remainders $1, 2, 3, \ldots, n-1$; hence giving to r in succession the indices $1, g, g^2, \ldots, g^{n-2}$, we have, in a different order, the whole series of roots $r, r^2, r^3, \ldots, r^{n-1}$.

In the most simple case, $n=5$, the equation to be solved is $x^4+x^3+x^2+x+1=0$; here 2 is a prime root of 5, and the order of the roots is r, r^2, r^4, r^3. The Gaussian process consists in forming an equation for determining the periods P_1, P_2, $=r+r^4$ and r^2+r^3 respectively,—these being such that the symmetrical functions P_1+P_2, P_1P_2 are rationally determinable: in fact,

$$P_1+P_2=-1,\quad P_1P_2=(r+r^4)(r^2+r^3),\ =r^3+r^4+r^6+r^7,\ =r^3+r^4+r+r^2,\ =-1.$$

P_1, P_2 are thus the roots of $u^2+u-1=0$; and taking them to be known, they are themselves broken up into subperiods, in the present case single terms, r and r^4 for P_1, r^2 and r^3 for P_2; the symmetrical functions of these are then rationally determined in terms of P_1 and P_2; thus $r+r^4=P_1$, $r\,.\,r^4=1$, or r, r^4 are the roots of $u^2-P_1u+1=0$. The mode of division is more clearly seen for a larger value of n; thus, for $n=7$ a prime root is $=3$, and the arrangement of the roots is $r, r^3, r^2, r^6, r^4, r^5$. We may form either 3 periods each of 2 terms,

$$P_1, P_2, P_3,\ =r+r^6,\ r^3+r^4,\ r^2+r^5,$$

respectively; or else 2 periods each of 3 terms, P_1, $P_2=r+r^2+r^4$, $r^3+r^6+r^5$ respectively; in each case the symmetrical functions of the periods are rationally determinable; thus in the case of the two periods $P_1+P_2=-1$, $P_1P_2=3+r+r^2+r^3+r^4+r^5+r^6$, $=2$;

and, the periods being known, the symmetrical functions of the several terms of each period are rationally determined in terms of the periods, thus

$$r+r^2+r^4=P_1,\quad r\,.\,r^2+r\,.\,r^4+r^2\,.\,r^4=P_2,\quad r\,.\,r^2\,.\,r^4=1.$$

The theory was further developed by Lagrange (1808), who, applying his general process to the equation in question, $x^{n-1}+x^{n-2}+\ldots+x+1=0$, the roots $a, b, c, \ldots$ being the several powers of r, the indices in geometrical progression as above, showed that the function $(a+\omega b+\omega^2 c+\ldots)^{n-1}$ was in this case a given function of ω with integer coefficients. Reverting to the before-mentioned particular equation $x^4+x^3+x^2+x+1=0$, it is very interesting to compare the process of solution with that for the solution of the general quartic the roots whereof are a, b, c, d.

Take ω, a root of the equation $\omega^4-1=0$ (whence ω is $=1, -1, i$, or $-i$, at pleasure), and consider the expression

$$(a+\omega b+\omega^2 c+\omega^3 d)^4.$$

The developed value of this is

$$\begin{aligned}
= \quad & a^4+b^4+c^4+d^4+6\,(a^2c^2+b^2d^2)+12\,(a^2bd+b^2ca+c^2db+d^2ac)\\
+\,\omega\ & \{4\,(a^3b+b^3c+c^3d+d^3a)+12\,(a^2cd+b^2da+c^2ab+d^2bc)\}\\
+\,\omega^2\ & \{6\,(a^2b^2+b^2c^2+c^2d^2+d^2a^2)+4\,(a^3c+b^3d+c^3a+d^3b)+24abcd\}\\
+\,\omega^3\ & \{4\,(a^3d+b^3a+c^3b+d^3c)+12\,(a^2bc+b^2cd+c^2da+d^2ab)\}\,;
\end{aligned}$$

that is, this is a 6-valued function of a, b, c, d, the root of a sextic (which is, in fact, solvable by radicals; but this is not here material).

If, however, a, b, c, d denote the roots r, r^2, r^4, r^3 of the special equation, then the expression becomes

$$\begin{aligned}
& r^4+r^3+r+r^2+6\,(1+1)+12\,(r^2+r^4+r^3+r)\\
& +\,\omega\ \{4\,(1+1+1+1)+12\,(r^4+r^3+r+r^2)\}\\
& +\,\omega^2\,\{6\,(r+r^2+r^4+r^3)+\ 4\,(r^2+r^4+r^3+r)\}\\
& +\,\omega^3\,\{4\,(r+r^2+r^4+r^3)+12\,(r^3+r+r^2+r^4)\}\,;
\end{aligned}$$

viz. this is

$$=-1+4\omega+14\omega^2-16\omega^3,$$

a completely determined value. That is, we have

$$(r+\omega r^2+\omega^2 r^4+\omega^3 r^3)^4=-1+4\omega+14\omega^2-16\omega^3,$$

which result contains the solution of the equation. If $\omega=1$, we have $(r+r^2+r^4+r^3)^4=1$, which is right; if $\omega=-1$, then $(r+r^4-r^2-r^3)^4=25$; if $\omega=i$, then we have $\{r-r^4+i\,(r^2-r^3)\}^4=-15+20i$; and if $\omega=-i$, then $\{r-r^4-i\,(r^2-r^3)\}^4=-15-20i$; the solution may be completed without difficulty.

The result is perfectly general, thus:—n being a prime number, r a root of the equation $x^{n-1}+x^{n-2}+\ldots+x+1=0$, ω a root of $\omega^{n-1}-1=0$, and g a prime root of $g^{n-1}\equiv 1$ (mod. n), then

$$(r+\omega r^g+\ldots+\omega^{n-2}r^{g^{n-2}})^{n-1}$$

is a given function $M_0 + M_1\omega + \ldots + M_{n-2}\omega^{n-2}$ with integer coefficients, and by the extraction of $(n-1)$th roots of this and similar expressions we ultimately obtain r in terms of ω, which is taken to be known; the equation $x^n - 1 = 0$, n a prime number, is thus solvable by radicals. In particular, if $n-1$ be a power of 2, the solution (by either process) requires the extraction of square roots only; and it was thus that Gauss discovered that it was possible to construct geometrically the regular polygons of 17 sides and 257 sides respectively. Some interesting developments in regard to the theory were obtained by Jacobi (1837); see the memoir "Ueber die Kreistheilung, u.s.w.," *Crelle*, t. XXX. (1846).

The equation $x^{n-1} + \ldots + x + 1 = 0$ has been considered for its own sake, but it also serves as a specimen of a class of equations solvable by radicals, considered by Abel (1828), and since called Abelian equations, viz., for the Abelian equation of the order n, if x be any root, the roots are x, θx, $\theta^2 x, \ldots$, $\theta^{n-1}x$ (θx being a rational function of x, and $\theta^n x = x$); the theory is, in fact, very analogous to that of the above particular case. A more general theorem obtained by Abel is as follows:—If the roots of an equation of any order are connected together in such wise that *all* the roots can be expressed rationally in terms of any one of them, say x; if, moreover, θx, $\theta_1 x$ being any two of the roots, we have $\theta\theta_1 x = \theta_1\theta x$, the equation will be solvable algebraically. It is proper to refer also to Abel's definition of an *irreducible* equation:—an equation $\phi x = 0$, the coefficients of which are rational functions of a certain number of known quantities $a, b, c, \ldots$, is called irreducible when it is impossible to express its roots by an equation of an inferior degree, the coefficients of which are also rational functions of $a, b, c, \ldots$ (or, what is the same thing, when ϕx does not break up into factors which are rational functions of $a, b, c, \ldots$). Abel applied his theory to the equations which present themselves in the division of the elliptic functions, but not to the modular equations.

32. But the theory of the algebraical solution of equations in its most complete form was established by Galois (born October 1811, killed in a duel May 1832; see his collected works, *Liouville*, t. XI., 1846). The definition of an irreducible equation resembles Abel's,—an equation is reducible when it admits of a rational divisor, irreducible in the contrary case; only the word *rational* is used in this extended sense that, in connexion with the coefficients of the given equation, or with the irrational quantities (if any) whereof these are composed, he considers any number of other irrational quantities called "adjoint radicals," and he terms rational any rational function of the coefficients (or the irrationals whereof they are composed) and of these adjoint radicals; the epithet irreducible is thus taken either absolutely or in a relative sense, according to the system of adjoint radicals which are taken into account. For instance, the equation $x^4 + x^3 + x^2 + x + 1 = 0$; the left-hand side has here no rational divisor, and the equation is irreducible; but this function is $= (x^2 + \frac{1}{2}x + 1)^2 - \frac{5}{4}x^2$, and it has thus the irrational divisors $x^2 + \frac{1}{2}(1+\sqrt{5})x + 1$, $x^2 + \frac{1}{2}(1-\sqrt{5})x + 1$; and these, if we *adjoin* the radical $\sqrt{5}$, are rational, and the equation is no longer irreducible. In the case of a given equation, assumed to be irreducible, the problem to solve the equation is, in fact, that of finding radicals by the adjunction of which the equation

becomes reducible; for instance, the general quadric equation $x^2+px+q=0$ is irreducible, but it becomes reducible, breaking up into rational linear factors, when we adjoin the radical $\sqrt{\frac{1}{4}p^2-q}$.

The fundamental theorem is the Proposition I. of the "Mémoire sur les conditions de résolubilité des équations par radicaux"; viz. given an equation of which $a, b, c, \ldots$ are the m roots, there is always a group of permutations of the letters $a, b, c, \ldots$ possessed of the following properties:—

1. Every function of the roots invariable by the substitutions of the group is rationally known.

2. Reciprocally, every rationally determinable function of the roots is invariable by the substitutions of the group.

Here by an invariable function is meant not only a function of which the form is invariable by the substitutions of the group, but further, one of which the value is invariable by these substitutions: for instance, if the equation be $\phi x=0$, then ϕx is a function of the roots invariable by any substitution whatever. And in saying that a function is rationally known, it is meant that its value is expressible rationally in terms of the coefficients and of the adjoint quantities.

For instance, in the case of a general equation, the group is simply the system of the $1.2.3\ldots n$ permutations of all the roots, since, in this case, the only rationally determinable functions are the symmetric functions of the roots.

In the case of the equation $x^{n-1}+\ldots+x+1=0$, n a prime number,

$$a,\ b,\ c, \ldots, k=r,\ r^g,\ r^{g^2}, \ldots, r^{g^{n-2}},$$

where g is a prime root of n, then the group is the cyclical group $abc\ldots k$, $bc\ldots ka, \ldots, kab\ldots j$, that is, in this particular case the number of the permutations of the group is equal to the order of the equation.

This notion of the group of the original equation, or of the group of the equation as varied by the adjunction of a series of radicals, seems to be the fundamental one in Galois's theory. But the problem of solution by radicals, instead of being the sole object of the theory, appears as the first link of a long chain of questions relating to the transformation and classification of irrationals.

Returning to the question of solution by radicals, it will be readily understood that by the adjunction of a radical the group may be diminished; for instance, in the case of the general cubic, where the group is that of the six permutations, by the adjunction of the square root which enters into the solution, the group is reduced to abc, bca, cab; that is, it becomes possible to express rationally, in terms of the coefficients and of the adjoint square root, any function such as $a^2b+b^2c+c^2a$ which is not altered by the cyclical substitution a into b, b into c, c into a. And hence, to determine whether an equation of a given form is solvable by radicals, the course of investigation is to inquire whether, by the successive adjunction of radicals, it is

possible to reduce the original group of the equation so as to make it ultimately consist of a single permutation.

The condition in order that an equation of a given prime order n may be solvable by radicals was in this way obtained—in the first instance in the form, scarcely intelligible without further explanation, that every function of the roots $x_1, x_2, \ldots, x_n$, invariable by the substitutions x_{ak+b} for x_k, must be rationally known; and then in the equivalent form that the resolvent equation of the order $1.2\ldots\overline{n-2}$ must have a rational root. In particular, the condition in order that a quintic equation may be solvable is that Lagrange's resolvent of the order 6 may have a rational factor, a result obtained from a direct investigation in a valuable memoir by E. Luther, *Crelle*, t. XXXIV. (1847).

Among other results demonstrated or announced by Galois may be mentioned those relating to the modular equations in the theory of elliptic functions; for the transformations of the orders 5, 7, 11, the modular equations of the orders 6, 8, 12 are depressible to the orders 5, 7, 11 respectively; but for the transformation, n a prime number greater than 11, the depression is impossible.

The general theory of Galois in regard to the solution of equations was completed, and some of the demonstrations supplied, by Betti (1852). See also Serret's *Cours d'Algèbre supérieure*, 2nd ed. 1854; 4th ed. 1877—78.

33. Returning to quintic equations, Jerrard (1835) established the theorem that the general quintic equation is, by the extraction of only square and cubic roots, reducible to the form $x^5+ax+b=0$, or what is the same thing, to $x^5+x+b=0$. The actual reduction by means of Tschirnhausen's theorem was effected by Hermite in connexion with his elliptic-function solution of the quintic equation (1858) in a very elegant manner. It was shown by Cockle and Harley (1858—59) in connexion with the Jerrardian form, and by Cayley (1861), that Lagrange's resolvent equation of the sixth order can be replaced by a more simple sextic equation occupying a like place in the theory.

The theory of the modular equations, more particularly for the case $n=5$, has been studied by Hermite, Kronecker, and Brioschi. In the case $n=5$, the modular equation of the order 6 depends, as already mentioned, on an equation of the order 5; and conversely the general quintic equation may be made to depend upon this modular equation of the order 6; that is, assuming the solution of this modular equation, we can solve (not by radicals) the general quintic equation; this is Hermite's solution of the general quintic equation by elliptic functions (1858); it is analogous to the before-mentioned trigonometrical solution of the cubic equation. The theory is reproduced and developed in Brioschi's memoir, "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Annalen*, t. XIII. (1877—78).

34. The great modern work, reproducing the theories of Galois, and exhibiting the theory of algebraic equations as a whole, is Jordan's *Traité des Substitutions et des Équations Algébriques*, Paris, 1870. The work is divided into four books—book I.,

preliminary, relating to the theory of congruences; book II. is in two chapters, the first relating to substitutions in general, the second to substitutions defined analytically, and chiefly to linear substitutions; book III. has four chapters, the first discussing the principles of the general theory, the other three containing applications to algebra, geometry, and the theory of transcendents; lastly, book IV., divided into seven chapters, contains a determination of the general types of equations solvable by radicals, and a complete system of classification of these types. A glance through the index will show the vast extent which the theory has assumed, and the form of general conclusions arrived at; thus, in book III., the algebraical applications comprise Abelian equations, equations of Galois; the geometrical ones comprise Hesse's equation, Clebsch's equations, lines on a quartic surface having a nodal line, singular points of Kummer's surface, lines on a cubic surface, problems of contact; the applications to the theory of transcendents comprise circular functions, elliptic functions (including division and the modular equation), hyperelliptic functions, solution of equations by transcendents. And on this last subject, solution of equations by transcendents, we may quote the result,—"the solution of the general equation of an order superior to five cannot be made to depend upon that of the equations for the division of the circular or elliptic functions"; and again (but with a reference to a possible case of exception), "the general equation cannot be solved by aid of the equations which give the division of the hyperelliptic functions into an odd number of parts."

787.

FUNCTION.

[From the *Encyclopædia Britannica, Ninth Edition,* vol. IX. (1879), pp. 818—824.]

FUNCTIONALITY, in Analysis, is dependence on a variable or variables; in the case of a single variable u, it is the same thing to say that v depends upon u, or to say that v is a function of u, only in the latter form of expression the mode of dependence is embodied in the term "function." We have given or known functions such as u^2 or $\sin u$, and the general notation of the form ϕu, where the letter ϕ is used as a functional symbol to denote a function of u, known or unknown as the case may be: in each case u is the independent variable or argument of the function, but it is to be observed that, if v be a function of u, then v like u is a variable, the values of v regarded as known serve to determine those of u; that is, we may conversely regard u as a function of v. In the case of two or more independent variables, say when w depends on or is a function of u, v, &c., or $w=\phi(u, v, \ldots)$, then $u, v, \ldots$ are the independent variables or arguments of the function; frequently when one of these variables, say u, is principally or alone attended to, it is regarded as the independent variable or argument of the function, and the other variables v, &c., are regarded as parameters, the values of which serve to complete the definition of the function. We may have a set of quantities $w, t, \ldots$ each of them a function of the same variables $u, v, \ldots$; and this relation may be expressed by means of a single functional symbol ϕ, $(w, t, \ldots)=\phi(u, v, \ldots)$; but, as to this, more hereafter.

The notion of a function is applicable in geometry and mechanics as well as in analysis; for instance, a point Q, the position of which depends upon that of a variable point P, may be regarded as a function of the point P; but here, substituting for the points themselves the coordinates (of any kind whatever) which determine their positions, we may say that the coordinates of Q are each of them a function of the coordinates of P, and we thus return to the analytical notion of a function. And in what follows a function is regarded exclusively in this point of view,

viz. the variables are regarded as numbers; and we attend to the case of a function of one variable $v=fu$. But it has been remarked (see Equation) that it is not allowable to confine the attention to *real* numbers; a number u must in general be taken to be a complex number $u=x+iy$, x and y being real numbers, each susceptible of continuous variation between the limits $-\infty$, $+\infty$, and i denoting $\sqrt{-1}$. In regard to any particular function, fu, although it may for some purposes be sufficient to know the value of the function for any real value whatever of u, yet to attend only to the real values of u is an essentially incomplete view of the question; to properly know the function, it is necessary to consider u under the aforesaid imaginary or complex form $u=x+iy$.

To a given value $x+iy$ of u there corresponds in general for v a value or values of the like form $v=x'+iy'$, and we obtain a geometrical notion of the meaning of the functional relation $v=fu$ by regarding x, y as rectangular coordinates of a point P in a plane Π, and x', y' as rectangular coordinates of a point P' in a plane (for greater convenience a different plane) Π'; P, P' are thus the geometrical representations, or representative points, of the variables $u=x+iy$ and $u'=x'+iy'$ respectively; and, according to a locution above referred to, the point P' might be regarded as a function of the point P; a given value of $u=x+iy$ is thus represented by a point P in the plane Π, and corresponding hereto we have a point or points P' in the plane Π', representing (if more than one, each of them) a value of the variable $v=x'+iy'$. And, if we attend only to the values of u as corresponding to a series of positions of the representative point P, we have the notion of the "path" of a complex variable $u=x+iy$.

Known Functions.

1. The most simple kind of function is the rational and integral function. We have the series of powers u^2, u^3, ... each calculable not only for a real but also for a complex value of u, $(x+iy)^2=x^2-iy^2+2ixy$, $(x+iy)^3=x^3-3xy^2+i(3x^2y-y^3)$, &c., and thence, if a, b, ... be real or complex numbers, the general form $a+bu+cu^2+\ldots+ku^m$, of a rational and integral function of the order m. And taking two such functions, say of the orders m and n respectively, the quotient of one of these by the other represents the general form of a rational function of u.

The function which next presents itself is the algebraical function, and in particular the algebraical function expressible by radicals. To take the most simple case, suppose (m being a positive integer) that $v^m=u$; v is here the irrational function $=u^{\frac{1}{m}}$. Obviously, if u is real and positive, there is always a real and positive value of v, calculable to any extent of approximation from the equation $v^m=u$, which serves as the definition of $u^{\frac{1}{m}}$; but it is known (see Equation) that, as well in this case as in the general case where u is a complex number, there are in fact m values of the function $u^{\frac{1}{m}}$; and that for their determination we require the theory of the so-called

circular functions sine and cosine; and these depend on the exponential function $\exp u$, or, as it is commonly written, e^u, which has for its inverse the logarithmic function $\log u$; these are all of them transcendental functions.

2. In a rational and integral function $a + bu + cu^2 + \ldots + ku^m$, the number of terms is finite, and the coefficients $a, b, \ldots. k$ may have any values whatever, but if we imagine a like series $a + bu + cu^2 + \ldots$ extending to infinity, *non constat* that such an expression has any calculable value,—that is, any meaning at all; the coefficients $a, b, c, \ldots$ must be such as, either for every value whatever of u (that is, for every finite value) or for values included within certain limits, to make the series *convergent*. It is easy to see that the values of $a, b, c, \ldots$ may be such as to make the series always convergent; for instance, this is the case for the exponential function,

$$\exp u = 1 + \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \&c.;$$

taking for the moment u to be real and positive, then it is evident that however large u may be, the successive terms will become ultimately smaller and smaller, and the series will have a determinate calculable value. A function thus expressed by means of a convergent infinite series is not in general algebraical, and when it is not so, it is said to be transcendental (but observe that it is in nowise true that we have thus the most general form of a transcendental function); in particular, the exponential function above written down is not an algebraical function.

By forming the expression of $\exp v$, and multiplying together the two series, we derive the fundamental property

$$\exp u \exp v = \exp (u + v);$$

whence also

$$\exp x \exp iy = \exp (x + iy),$$

so that $\exp(x + iy)$ is given as the product of the two series $\exp x$ and $\exp iy$. As regards this last, if in place of u we actually write the value iy, we find

$$\exp iy = \left(1 - \frac{y^2}{1.2} + \frac{y^4}{1.2.3.4} - \ldots\right) + i\left(y - \frac{y^3}{1.2.3} + \ldots\right),$$

where obviously each series is convergent and actually calculable for any real value whatever of y. Calling the two series cosine y and sine y respectively, or in the ordinary abbreviated notation $\cos y$ and $\sin y$, the equation is

$$\exp iy = \cos y + i \sin y;$$

and if we herein for y write z, and multiply the two expressions together, observing that the product will be $= \exp i(y + z)$, we obtain the fundamental equations

$$\cos (y + z) = \cos y \cos z - \sin y \sin z,$$
$$\sin (y + z) = \sin y \cos z + \sin z \cos y,$$

for the functions sine and cosine.

Taking y as an angle, and defining as usual the sine and cosine as the ratios of the perpendicular and base respectively to the radius, the sine and cosine will be functions of y; and we obtain geometrically the foregoing fundamental equations for the sine and cosine; but in order to the truth of the foregoing equation $\exp iy = \cos y + i \sin y$, it is further necessary that the angle should be measured in circular measure, that is, by the ratio of the arc to the radius; so that π denoting as usual the number 3·14159..., the measure of a right angle is $= \frac{1}{2}\pi$. And this being so, the functions sine and cosine, obtained as above by consideration of the exponential function, have their ordinary geometrical significations.

3. The foregoing investigation was given in detail in order to the completion of the theory of the irrational function $u^{\frac{1}{m}}$. We henceforth take the theory of the circular functions as known, and speak of $\tan x$, &c., as the occasion may arise.

We have

$$x + iy = r(\cos\theta + i \sin\theta),$$

where, writing $\sqrt{x^2 + y^2}$ to denote the positive value of the square root, we have

$$r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}},$$

and therefore also

$$\tan\theta = \frac{y}{x}.$$

Treating x, y as the rectangular coordinates of a point P, r is the distance (regarded as positive) of this point from the origin, and θ is the inclination of r to the positive part of the axis of x; to fix the ideas θ may be regarded as lying within the limits $0, \pi$, or $0, -\pi$, according as y is positive or negative; θ is thus completely determinate, except in the case, x negative, $y = 0$, for which θ is $= \pi$ or $-\pi$ indifferently.

And if $u = x + iy$, we hence have

$$u^{\frac{1}{m}} = (x + iy)^{\frac{1}{m}} = r^{\frac{1}{m}}\left(\cos\frac{\theta + 2s\pi}{m} + i \sin\frac{\theta + 2s\pi}{m}\right),$$

where $r^{\frac{1}{m}}$ is real and positive and s has any positive or negative integer value whatever: but we thus obtain for $u^{\frac{1}{m}}$ only the m values corresponding to the values $0, 1, 2, \ldots, m-1$ of s. More generally we may, instead of the index $\frac{1}{m}$, take the index to be any rational fraction $\frac{n}{m}$. Supposing this to be in its least terms, and m to be positive, the number of distinct values is always $= m$. If instead of $\frac{n}{m}$ we take the index to be the general real or complex quantity m, we have u^m, no longer an algebraical function of u, and having in general an infinity of values.

4. The foregoing equation $\exp(x+y)=\exp x\,.\exp y$ is, in fact, the equation of indices, $a^{x+y}=a^x\,.\,a^y$; $\exp x$ is thus the same thing as e^x, where e denotes a properly determined number, and putting e^x equal to the series, and then writing $x=1$, we have $e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c., that is, $e=2{\cdot}7128\ldots$ But as well theoretically as for convenience of printing, there is considerable advantage in the use of the notation $\exp u$.

From the equation, $\exp iy=\cos y+i\sin y$, we deduce $\exp(-iy)=\cos y-i\sin y$, and thence

$$\cos y=\tfrac{1}{2}\{\exp(iy)+\exp(-iy)\},$$

$$\sin y=\frac{1}{2i}\{\exp(iy)-\exp(-iy)\};$$

if we write herein ix instead of y we have

$$\cos ix=\tfrac{1}{2}\{\exp x+\exp(-x)\},$$

$$\sin ix=\frac{i}{2}\{\exp x-\exp(-x)\},$$

viz. these values are

$$\cos ix=1+\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}+\ldots$$

$$\frac{1}{i}\sin ix=x+\frac{x^3}{1.2.3}+\ldots$$

each of them real when x is real. The functions in question $1+\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}+\ldots$ and $x+\frac{x^3}{1.2.3}+\ldots$, regarded as functions of x, are termed the hyperbolic cosine and sine, and are represented by the notations $\cosh x$ and $\sinh x$ respectively; and similarly we have the hyperbolic tangent $\tanh x$, &c.: although it is easy to remember that $\cos ix$, $\frac{1}{i}\sin ix$, are, in fact, real functions of x, and to understand accordingly the formulæ wherein they occur, yet the use of these notations of the hyperbolic functions is often convenient.

5. Writing $u=\exp v$, then v is conversely a function of u which is called the logarithm (hyperbolic logarithm, to distinguish it from the tabular or Briggian logarithm), and we write $v=\log u$, or what is the same thing, we have $u=\exp(\log u)$: and it is clear that if u be real and positive there is always a real and positive value of $\log u$, in particular the real logarithm of e is $=1$; it is however to be observed that the logarithm is not a one-valued function, but has an infinity of values corresponding to the different integer values of a constant s; in fact, if $\log u$ be any one of its values, then $\log u+2s\pi i$ is also a value, for we have $\exp(\log u+2s\pi i)=\exp\log u\exp 2s\pi i$, or since $\exp 2s\pi i$ is $=1$, this is $=u$; that is, $\log u+2s\pi i$ is a value of the logarithm of u.

We have

$$uv=\exp(\log uv)=\exp\log u\,.\exp\log v,$$

and hence the equation which is commonly written

$$\log uv = \log u + \log v,$$

but which requires the addition on one side of a term $2s\pi i$. And reverting to the equation $x+iy=r(\cos\theta+i\sin\theta)$, or as it is convenient to write it, $x+iy=r\exp i\theta$, we hence have

$$\log(x+iy)=\log r+i(\theta+2s\pi),$$

where $\log r$ may be taken to denote the real logarithm of the real positive quantity r, and θ the completely determinate angle defined as already mentioned.

Reverting to the function u^m, we have $u=\exp\log u$, and thence $u^m=\exp(m\log u)$, which, on account of the infinity of values of $\log u$, has in general (as before remarked) an infinity of values; if $u=e$, then e^m, $=\exp(m\log e)$, has in general in like manner an infinity of values, but in regarding e^m as identical with the one-valued function $\exp m$, we take $\log e$ to be $=$ its real value, 1.

The inverse functions $\cos^{-1}x$, $\sin^{-1}x$, $\tan^{-1}x$, are in fact logarithmic functions; thus in the equation $\exp ix=\cos x+i\sin x$, writing first $\cos x=u$, the equation becomes $\exp i\cos^{-1}u=u+i\sqrt{1-u^2}$, or we have $\cos^{-1}u=\frac{1}{i}\log(u+i\sqrt{1-u^2})$, and from the same equation, writing secondly $\sin x=u$, we have $\sin^{-1}u=\frac{1}{i}\log(\sqrt{1-u^2}+iu)$. But the formula for $\tan^{-1}u$ is a more elegant one, as not involving the radical $\sqrt{1-u^2}$; we have

$$i\tan x=\frac{\exp ix-\exp(-ix)}{\exp ix+\exp(-ix)},\ =\frac{\exp 2ix-1}{\exp 2ix+1},$$

and thence

$$\exp 2ix=\frac{1+i\tan x}{1-i\tan x},$$

that is,

$$x=\frac{1}{2i}\log\frac{1+i\tan x}{1-i\tan x},$$

or, if $\tan x=u$, then

$$\tan^{-1}u=\frac{1}{2i}\log\frac{1+iu}{1-iu}.$$

The logarithm (or inverse exponential function) and the inverse circular functions present themselves as the integrals of algebraic functions

$$\int\frac{dx}{x}=\log x,$$

whence also

$$\int\frac{dx}{1+x^2}=\frac{1}{2i}\log\frac{1+ix}{1-ix}=\tan^{-1}x,$$

and

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$$

6. Each of the functions $\exp u$, $\sin u$, $\cos u$, $\tan u$, &c., as a one-valued function of u, is in this respect analogous to a rational function of u; and there are further analogies of $\exp u$, $\sin u$, $\cos u$, to a rational and integral function; and of $\tan u$, $\sec u$, &c., to a rational non-integral function.

A rational and integral function has a certain number of roots, or zeros, each of a given multiplicity, and is completely determined (except as to a constant factor) when the several roots and the multiplicity of each of them is given; i.e., if $a, b, c, \ldots$ be the roots, $p, q, r, \ldots$ their multiplicities, then the form is $A\left(1-\frac{u}{a}\right)^p\left(1-\frac{u}{b}\right)^q\ldots$; a rational (non-integral) function has a certain number of infinities, or poles, each of them of a given multiplicity, viz. the infinities are the roots or zeros of the rational and integral function which is its denominator.

The function $\exp u$ has no finite roots, but an infinity of roots each $=-\infty$; this appears from the equation $\exp u=\left(1+\frac{u}{n}\right)^n$, where n is indefinitely large and positive. The function $\sin u$ has the roots $s\pi$ where s is any positive or negative integer, zero included; or, what is the same thing, its roots are 0 and $\pm s\pi$, s now denoting any positive integer from 1 to ∞; each of these is a simple root, and we in fact have $\sin u=u\Pi\left(1-\frac{u^2}{s^2\pi^2}\right)$. Similarly the roots of $\cos u$ are $(s+\frac{1}{2})\pi$, s denoting any positive or negative integer, zero included, or, what is the same thing, they are $\pm(s+\frac{1}{2})\pi$, s now denoting any positive integer from 0 to ∞; each root is simple, and we have $\cos u=\Pi\left(1+\frac{u^2}{(s+\frac{1}{2})^2\pi^2}\right)$. Obviously $\tan u$, as the quotient $\sin u\div\cos u$, has both roots and infinities, its roots being the roots of $\sin u$, its infinities the roots of $\cos u$; $\sec u$ as the reciprocal of $\cos u$ has infinities only, these being the roots of $\cos u$, &c.

In the foregoing expression $\sin u=u\Pi\left(1-\frac{u^2}{s^2\pi^2}\right)$, the product must be understood to mean the limit of $\Pi_1^n\left(1-\frac{u^2}{s^2\pi^2}\right)$ for an indefinitely large positive integer value of n, viz. the product is first to be formed for the values $s=1, 2, 3, \ldots$ up to a determinate number n, and then n is to be taken indefinitely large. If, separating the positive and the negative values of s, we consider the product $u\Pi_1^n\left(1+\frac{u}{s\pi}\right)\Pi_1^m\left(1-\frac{u}{s\pi}\right)$, (where in the first product s has all the positive integer values from 1 to n, and in the second product s has all the positive integer values from 1 to m), then by making m and n each of them indefinitely large, the function does *not* approximate to $\sin u$, unless $m:n$ be a ratio of equality*. And similarly as regards $\cos u$, the product $\Pi_0^n\left(1+\frac{u}{(s+\frac{1}{2})\pi}\right)\Pi\left(\frac{u}{(s+\frac{1}{2})\pi}\right)$, m and n indefinitely large, does not approximate to $\cos u$, unless $m:n$ be a ratio of equality.

* The value of the function in question $u\Pi_1^n\left(1+\frac{u}{s\pi}\right)\Pi_1^m\left(1-\frac{u}{s\pi}\right)$, when m, n are each indefinitely large, but $\frac{m}{n}$ not $=1$, is $=\left(\frac{n}{m}\right)^{\frac{u}{\pi}}\sin u$.

7. The functions $\sin u$, $\cos u$, are *periodic*, having the period 2π, $\frac{\sin}{\cos}(u+2\pi)=\frac{\sin}{\cos}(u)$; and the half-period π, $\frac{\sin}{\cos}(u+\pi)=-\frac{\sin}{\cos}u$; the periodicity may be verified by means of the foregoing fractional forms, but some attention is required; thus writing, as we may do, $\sin u=\frac{\Pi\,(u+s\pi)}{\Pi s\pi}$, where s extends from $-n$ to n, n ultimately infinite, if for u we write $u+\pi$, each factor of the numerator is changed into the following one and the numerator is unaltered, save only that there is an introduced factor $u+(n+1)\pi$ at the superior limit, and an omitted factor $u-n\pi$ at the inferior limit; the ratio of these, $(u+\overline{n+1}\pi)\div(u-n\pi)$, for n infinite is $=-1$, and we thus have, as we should have, $\sin(u+\pi)=-\sin u$.

The most general periodic function having no infinities, and each root a simple root, and having a given period a, has the form $A\sin\frac{2\pi u}{a}+B\cos\frac{2\pi u}{a}$, or, what is the same thing, $L\frac{\sin}{\cos}\left(\frac{2\pi u}{a}+\lambda\right)$.

8. We come now to the Elliptic Functions. These arose from the consideration of the integral $\int\frac{Rdx}{\sqrt{X}}$, where R is a rational function of x, and X is the general rational and integral quartic function

$$\alpha x^4+\beta x^3+\gamma x^2+\delta x+\epsilon;$$

a form arrived at was

$$\int\frac{dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},\ =\int\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

on putting therein $x=\sin\phi$, and this last integral was represented by $F\phi$, and called the elliptic integral of the first kind. In the particular case $k=0$, the integral is $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x$, and it thus appears that $F\phi$ is of the nature of an inverse function; for passing to the direct functions we write $F\phi=u$, and consider ϕ as hereby determined as a function of u, $\phi=$ amplitude of u, or for shortness $\operatorname{am}u$. And the functions $\sin\phi$, $\cos\phi$, and $\sqrt{1-k^2\sin^2\phi}$ were then considered as functions of the amplitude, and written $\sin\operatorname{am}u$, $\cos\operatorname{am}u$, $\Delta\operatorname{am}u$; these were afterwards written $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$, which may be regarded either as mere abbreviations of the former functional symbols, or (in a different point of view) as functions, no longer of $\operatorname{am}u$, but of u itself as the argument of the functions; sn is thus a function in some respects analogous to a sine, and cn and dn functions analogous to a cosine; they have the corresponding property that the three functions of $u+v$ are expressible in terms of the functions of u and of v. The following formulæ may be mentioned:

$$\operatorname{cn}^2u=1-\operatorname{sn}^2u,\quad \operatorname{dn}^2u=1-k^2\operatorname{sn}^2u,$$

$$\operatorname{sn}'u=\operatorname{cn}u\operatorname{dn}u,\quad \operatorname{cn}'u=-\operatorname{sn}u\operatorname{dn}u,\quad \operatorname{dn}'u=-k^2\operatorname{sn}u\operatorname{cn}u,$$

where the accent denotes differentiation in regard to u; and the addition-formulæ:

$$\begin{aligned} \operatorname{sn}(u+v) &= \operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u, && (\div), \\ \operatorname{cn}(u+v) &= \operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v, && (\div), \\ \operatorname{dn}(u+v) &= \operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{sn} v \operatorname{cn} v, && (\div), \end{aligned}$$

each of the expressions on the right-hand side being the numerator of a fraction of which

$$\text{Denom.} = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v.$$

It may be remarked that any one of the fractional expressions, differentiated in regard to u and to v respectively, gives the same result; such expression is therefore a function of $u+v$, and the addition-formulæ can be thus directly verified.

9. The existence of a denominator in the addition-formulæ suggests that sn, cn, dn are not, like the sine and cosine, functions having zeros only, without infinities; they are in fact functions, having each its own zeros, but having a common set of infinities; moreover, the zeros and the infinities are simple zeros and infinities respectively. And this further suggests, what in fact is the case, that the three functions are quotients having each its own numerator but a common denominator, say they are the quotients of four θ-functions, each of them having zeros only (and these simple zeros) but no infinities.

The functions sn, cn, dn, but not the θ-functions, are moreover *doubly periodic;* that is, there exist values 2ω, 2υ, $=4K$ and $4(K+iK')$ in the ordinary notation, such that the sn, cn, or dn of $u+2\omega$, and the sn, cn, and dn of $u+2\upsilon$ are equal to the sn, cn, and dn respectively of u; or say that $\phi(u+2\omega)=\phi(u+2\upsilon)=\phi u$, where ϕ is any one of the three functions.

As regards this double periodicity, it is to be observed that the equations $\phi(u+2\omega)=\phi u$, $\phi(u+2\upsilon)=\phi u$, imply $\phi(u+2m\omega+2n\upsilon)=\phi u$, and hence it easily follows that if ω, υ were commensurable, say if they were multiples of some quantity α, we should have $\phi(u+2\alpha)=\phi u$, an equation which would replace the original two equations $\phi(u+2\omega)=\phi u$, $\phi(u+2\upsilon)=\phi u$, or there would in this case be only the single period α; ω and υ must therefore be incommensurable. And this being so, they cannot have a real ratio, for if they had, the integer values m, n could be taken such as to make $2m\omega+2n\upsilon=k$ times a given real or imaginary value, k *as small as we please;* the ratio $\omega:\upsilon$ must be therefore imaginary, as is in fact the case when the values are $4K$ and $4(K+iK')$.

10. The function $\operatorname{sn} u$ has the zero 0 and the zeros $m\omega+n\upsilon$, m and n any positive or negative integers whatever; and this suggests that the numerator of $\operatorname{sn} u$ is equal to a doubly infinite product, (Cayley, "On the Inverse Elliptic Functions," *Camb. Math. Jour.* t. IV., 1845, [24]; and "Mémoire sur les fonctions doublement périodiques," *Liouville*, t. X., 1845, [25]). The numerator is equal to

$$u\Pi\Pi\left(1+\frac{u}{m\omega+n\upsilon}\right),$$

m and n having any positive or negative integer values whatever, including zero, except that m, n must not be simultaneously $=0$, these values being taken account of in the factor u outside the product. But until further defined, such a product has no definite value, and consequently no meaning whatever. Imagine m, n to be coordinates, and suppose that we have, surrounding the origin, a closed curve having the origin for its centre, i.e. the curve is such that, if α, β be the coordinates of a point thereof, then $-\alpha$, $-\beta$ are also the coordinates of a point thereof; suppose further that the form of the curve is given, but that its magnitude depends upon a parameter h, and that the curve is such that, when h is indefinitely large, each point of the curve is at an indefinitely large distance from the origin; for instance, the curve might be a circle or ellipse, or a parallelogram, the origin being in each case the centre. Then if in the double product, taking the value of h as given, we first give to m, n all the positive or negative integer values (the simultaneous values 0, 0 excluded) which correspond to points within the curve, and then make h indefinitely large, the product will thus have a definite value; *but this value will still be dependent on the form of the curve.* Moreover, varying in any manner the form of the curve, the ratio of the two values of the double product will be $=\exp\beta u^2$, where β is a determinate value depending only on the forms of the two curves; or, what is the same thing, if we first give to the curve a certain form, say we take it to be a circle, and then give it any other form, the product in the latter case is equal to its former value multiplied by $\exp\beta u^2$, where β depends only upon the form of the curve in the latter case.

Considering the form of the bounding curve as given, and writing the double product in the form

$$\Pi\Pi\left(\frac{u+m\omega+n\upsilon}{m\omega+n\upsilon}\right),$$

the simultaneous values $m=0$, $n=0$ being now admitted in the numerator, although still excluded from the denominator, then if we write for instance $u+2\omega$ instead of u, each factor in the numerator is changed into a contiguous factor, and the numerator remains unaltered, except that we introduce certain factors which lie outside the bounding curve, and omit certain factors which lie inside the bounding curve; we, in fact, affect the result by a singly infinite series of factors belonging to points adjacent to the bounding curve; and it appears on investigation that we thus introduce a constant factor $\exp\gamma(u+\omega)$. The final result thus is that the product

$$u\Pi\Pi\left(1+\frac{u}{m\omega+n\upsilon}\right)$$

does not remain unaltered when u is changed into $u+2\omega$, but that it becomes therefore affected with a constant factor, $\exp\gamma(u+\omega)$. And similarly the function does not remain unaltered when u is changed into $u+2\upsilon$, but it becomes affected with a factor, $\exp\delta(u+\upsilon)$. The bounding curve may however be taken such that the function is unaltered when u is changed into $u+2\omega$: this will be the case if the curve is a rectangle such that the length in the direction of the axis of m is infinitely great in comparison of that in the direction of the axis of n; or it may be taken such that the function is unaltered when u is changed into $u+2\upsilon$: this will be so if the curve

be a rectangle such that the length in the direction of the axis of n is indefinitely great in comparison with that in the direction of the axis of m; but the two conditions cannot be satisfied simultaneously.

11. We have three other like functions, viz. writing for shortness $\bar{m}$, $\bar{n}$ to denote $m+\frac{1}{2}$, $n+\frac{1}{2}$ respectively, and (m, n) to denote $m\omega+nv$, then the four functions are

$$u\Pi\Pi\left(1+\frac{u}{(m, n)}\right),\quad \Pi\Pi\left(1+\frac{u}{(\bar{m}, n)}\right),\quad \Pi\Pi\left(1+\frac{u}{(m, \bar{n})}\right),\quad \Pi\Pi\left(1+\frac{u}{(\bar{m}, \bar{n})}\right),$$

the bounding curve being in each case the same; and, dividing the first three of these each by the last, we have (except as to constant factors) the three functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$; writing in each of the four functions $u+2\omega$ or $u+2v$ in place of u, the functions acquire each of them the *same* exponential factor $\exp \gamma(u+\omega)$, or $\exp \delta(u+v)$, and the quotient of any two of them, and therefore each of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, remains unaltered.

It is easily seen that, disregarding constant factors, the four θ-functions are in fact one and the same function with different arguments, or they may be written θu, $\theta(u+\frac{1}{2}\omega)$, $\theta(u+\frac{1}{2}v)$, $\theta(u+\frac{1}{2}\omega+\frac{1}{2}v)$; by what precedes, the functions may be so determined that they shall remain unaltered when u is changed into $u+2\omega$, that is, be singly periodic, but that the change u into $u+2v$ shall affect them each with the same exponential factor $\exp \delta(u+v)$.

12. Taking the last-mentioned property as a definition of the function θ, it appears that θu may be expressed as a sum of exponentials

$$\theta u = A\Sigma \exp \frac{\pi i}{\omega}(vm^2+um),$$

where the summation extends to all positive and negative integer values of m, including zero. In fact, if we first write herein $u+2\omega$ instead of u, then in each term the index of the exponential is altered by $\frac{\pi i}{\omega}2\omega m, = 2m\pi i$, and the term itself thus remains unaltered; that is, $\theta(u+2\omega)=\theta u$. But writing $u+2v$ in place of u, each term is changed into the succeeding term, multiplied by the factor $\exp \frac{\pi i}{\omega}(u+v)$; in fact, making the change in question u into $u+2v$, and writing also $m-1$ in place of m, vm^2+um becomes $v(m-1)^2+(u+2v)(m-1), = vm^2+um-u-v$, and we thus have $\theta(u+2v)=\exp\left\{-\frac{\pi i}{\omega}(u+v)\right\}.\theta u$. In order to the convergency of the series it is necessary that $\exp \frac{\pi i v m^2}{\omega}$ should vanish for indefinitely large values of m, and this will be so if $\frac{iv}{\omega}$ be a complex quantity of the form $\alpha+\beta i$, α *negative;* for instance, this will be the case if ω be real and positive and v be $=i$ multiplied by a real and positive quantity. The original definition of θ as a double product seems to put more clearly in evidence the real nature of this function, but the new definition has the advantage that it admits of extension to the θ-functions of two or more variables.

The elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, have thus been expressed each of them as the quotient of two θ-functions, but the question arises to express conversely a θ-function by means of the elliptic functions; the form is found to be

$$\theta u = C \exp\left(Au^2 + B\int_0\int_0 \operatorname{sn}^2 u\, du^2\right),$$

viz. θu is expressible as an exponential, the index of which depends on the double integral

$$\int_0\int_0 \operatorname{sn}^2 u\, du^2.$$

The object has been to explain the general nature of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, and of the θ-functions with which they are thus intimately connected; it would be out of place to go into the theories of the multiplication, division, and transformation of the elliptic functions, or into the theory of the elliptic integrals, and of the application of the θ-functions to the representation of the elliptic integrals of the second and third kinds.

13. The reasoning which shows that for a doubly periodic function the ratio of the two periods 2ω, 2υ is imaginary shows that we cannot have a function *of a single variable*, which shall be *triply* periodic, or of any higher order of periodicity. For if the periods of a triply periodic function $\phi(u)$ were 2ω, 2υ, 2χ, then m, n, p being any positive or negative integer values, we should have $\phi(u + 2m\omega + 2n\upsilon + 2p\chi) = \phi u$; ω, υ, χ must be incommensurable, for if not, the three periods would really reduce themselves to two periods, or to a single period; and being incommensurable, it would be possible to determine the integers m, n, p in such manner that the real part and also the coefficient of i of the expression $m\omega + n\upsilon + p\chi$ shall be each of them as small as we please, say $\phi(u + \epsilon) = \phi u$, and thence $\phi(u + k\epsilon) = \phi u$ (k an integer), and $k\epsilon$ as near as we please to any given real or imaginary value whatever. We have thus the nugatory result $\phi u =$ a constant, or at least the function if not a constant is a function of an infinitely and perpetually discontinuous kind, a conception of which can hardly be formed. But a function of two variables may be triply or quadruply periodic—viz. we may have a function $\phi(u, v)$ having for u, v the simultaneous periods 2ω, $2\omega'$; 2υ, $2\upsilon'$; 2χ, $2\chi'$; 2ψ, $2\psi'$; or, what is the same thing, it may be such that, m, n, p, q being any integers whatever, we have

$$\phi(u + 2m\omega + 2n\upsilon + 2p\chi + 2q\psi,\ v + 2m\omega' + 2n\upsilon' + 2p\chi' + 2q\psi') = \phi(u, v);$$

and similarly a function of $2n$ variables may be $2n$-tuply periodic.

It is, in fact, in this manner that we pass from the elliptic functions and the single θ-functions to the hyperelliptic or Abelian functions and the multiple θ-functions; the case next succeeding the elliptic functions is when we have X, Y the same rational and integral sextic functions of x, y respectively, and then writing

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = du, \quad \frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} = dv,$$

we regard certain symmetrical functions of x, y, in fact, the ratios of $(2^4 =)$ 16 such symmetrical functions, as functions of (u, v); say we thus have 15 hyperelliptic functions $f(u, v)$, analogous to the 3 elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, and being quadruply periodic. And these are the quotients of 16 double θ-functions $\theta(u, v)$, the general form being

$$\theta(u, v) = A\Sigma\Sigma \exp\{\tfrac{1}{2}(a, h, b)(m, n)^2 + mu + nv\},$$

where the summations extend to all positive and negative integer values of (m, n); and we thus see the form of the θ-function for any number of variables whatever. The epithet "hyperelliptic" is used in the case where the differentials are of the form just mentioned $\frac{dx}{\sqrt{X}}$, where X is a rational and integral function of x; the epithet "Abelian" extends to the more general case where the differential involves the irrational function of x, determined by any rational and integral equation $\phi(x, y) = 0$ whatever.

As regards the literature of the subject, it may be noticed that the various memoirs by Riemann, 1851—1866, are republished in the collected edition of his works, Leipsic, 1876; and shortly after his death we have the *Theorie der Abel'schen Functionen*, by Clebsch and Gordan, Leipsic, 1866. Preceding this, we have by MM. Briot and Bouquet, the *Théorie des Fonctions doublement périodiques et en particulier des Fonctions Elliptiques*, Paris, 1859, the results of which are reproduced and developed in their larger work, *Théorie des Fonctions Elliptiques*, 2nd ed., Paris, 1875.

14. It is proper to mention the gamma (Γ) or Π function, $\Gamma(n+1) = \Pi n$, $= 1.2.3\ldots n$, when n is a positive integer. In the case just referred to, n a positive integer, this presents itself almost everywhere in analysis,—for instance, the binomial coefficients, and the coefficients of the exponential series are expressible by means of such functions of a number n. The definition for any real positive value of n is taken to be

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx;$$

it is then shown that, n being real and positive, $\Gamma(n+1) = n\Gamma n$, and by assuming that this equation holds good for positive or negative real values of n, the definition is extended to real negative values; the equation gives $\Gamma 1 = 0\Gamma 0$, that is, $\Gamma 0 = \infty$, and similarly $\Gamma(-n) = \infty$, where $-n$ is any negative integer. The definition by the definite integral has been or may be extended to imaginary values of n, but the theory is not an established one. A definition extending to all values of n is that of Gauss

$$\Pi n = \text{limit}\, \frac{1 \;.\; 2 \;.\; 3 \;\ldots\; k}{n+1 \,.\, n+2 \,.\, n+3 \ldots n+k}\, k^n,$$

the ultimate value of k being $= \infty$; but the function is chiefly considered for real values of the variable.

A formula for the calculation, when x has a large real and positive value, is

$$\Pi x = \sqrt{2\pi}\, x^{x+\frac{1}{2}} \exp\left(-x + \frac{1}{12x} + \ldots\right),$$

or as this may also be written, neglecting the negative powers of x,

$$\Pi x = \sqrt{2\pi}\, \exp\{(x+\tfrac{1}{2})\log x - x\}.$$

Another formula is $\Gamma x \Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$: or, as this may also be written,

$$\Pi(x-1)\,\Pi(-x) = \frac{\pi}{\sin \pi x}.$$

It is to be observed that the function Π serves to express the product of a set of factors in arithmetical progression; we have

$$(x+a)(x+2a)\ldots(x+ma) = a^m\left(\frac{x}{a}+1\right)\left(\frac{x}{a}+2\right)\ldots\left(\frac{x}{a}+m\right) = a^m \Pi\left(\frac{x}{a}+m\right) \div \Pi\frac{x}{a}.$$

We can consequently express by means of it the product of any number of the factors which present themselves in the factorial expression of $\sin u$. Starting from the form

$$u\Pi_1^m\left(1+\frac{u}{s\pi}\right)\Pi_1^n\left(1-\frac{u}{s\pi}\right),$$

where Π is here as before the sign of a product of factors corresponding to the different integer values of s, this is thus converted into

$$u\Pi\left(\frac{u}{\pi}+m\right)\Pi\left(-\frac{u}{\pi}+n\right) \div \Pi\left(\frac{u}{\pi}\right)\Pi\left(-\frac{u}{\pi}\right)\Pi m\,\Pi n,$$

or as this may also be written,

$$\pi\Pi\left(\frac{u}{\pi}+m\right)\Pi\left(-\frac{u}{\pi}+n\right) \div \Pi\left(\frac{u}{\pi}-1\right)\Pi\left(-\frac{u}{\pi}\right)\Pi m\,\Pi n,$$

which, in virtue of

$$\Pi\left(\frac{u}{\pi}-1\right)\Pi\left(-\frac{u}{\pi}\right) = \frac{\pi}{\sin u},$$

becomes

$$= \sin u\,\Pi\left(\frac{u}{\pi}+m\right)\Pi\left(-\frac{u}{\pi}+n\right) \div \Pi m\,\Pi n.$$

Here m and n are large and positive; calculating the second factor by means of the formula for Πx, in this case we have the before-mentioned formula

$$u\Pi_1^m\left(1+\frac{u}{s\pi}\right)\Pi_1^n\left(1-\frac{u}{s\pi}\right) = \left(\frac{n}{m}\right)^{\frac{u}{\pi}} \sin u.$$

The gamma or Π function is the so-called second Eulerian integral; the first Eulerian integral

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \;= \Gamma p \Gamma q \div \Gamma(p+q),$$

is at once expressible in terms of Γ, and is therefore not a new function to be considered.

15. We have the function defined by its expression as a hypergeometric series

$$F(\alpha,\ \beta,\ \gamma,\ u) = 1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}u + \frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,.\,2\,.\,\gamma\,.\,\gamma+1}u^2 + \&c.,$$

i.e., this expression of the function serves as a definition, if the series be finite or if, being infinite, it is convergent. The function may also be defined as a definite integral; in other words, if, in the integral

$$\int_0^1 x^{\alpha'-1}(1-x)^{\beta'-1}(1-ux)^{-\gamma'}\,dx,$$

we expand the factor $(1-ux)^{-\gamma'}$ in powers of ux, and then integrate each term separately by the formula for the second Eulerian integral, the result is

$$= \frac{\Gamma\alpha'\,.\,\Gamma\beta'}{\Gamma(\alpha'+\beta')} + \frac{\Gamma(\alpha'+1)\,.\,\Gamma\beta'}{\Gamma(\alpha'+\beta'+1)}\frac{\gamma'}{1}u + \&c.,$$

which is

$$= \frac{\Gamma\alpha'\,.\,\Gamma\beta'}{\Gamma(\alpha'+\beta')}\left\{1 + \frac{\alpha'\,.\,\gamma'}{\alpha'+\beta'\,.\,1}u + \frac{\alpha'\,.\ \ \alpha'+1\,.\ \ \gamma'\,.\,\gamma'+1}{\alpha'+\beta'\,.\,\alpha'+\beta'+1\,.\,1\,.\,2}u^2 + \ldots\right\}$$

or writing $\alpha',\ \beta',\ \gamma' = \alpha,\ \gamma-\alpha,\ \beta$ respectively, this is

$$= \frac{\Gamma\alpha\Gamma(\gamma-\alpha)}{\Gamma\gamma}F(\alpha,\ \beta,\ \gamma,\ u),$$

so that the new definition is

$$F(\alpha,\ \beta,\ \gamma,\ u) = \frac{\Gamma\alpha\Gamma(\gamma-\alpha)}{\Gamma\gamma}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-ux)^{-\beta}\,dx;$$

but this is in like manner only a definition under the proper limitations as to the values of $\alpha,\ \beta,\ \gamma,\ u$. It is not here considered how the definition is to be extended so as to give a meaning to the function $F(\alpha,\ \beta,\ \gamma,\ u)$ for all values, say of the parameters $\alpha,\ \beta,\ \gamma$, and of the variable u. There are included a large number of special forms which are either algebraic or circular or exponential, for instance $F(\alpha,\ \beta,\ \beta,\ u) = (1-u)^{-\alpha}$, &c.; or which are special transcendents which have been separately studied, for instance, Bessel's functions, the Legendrian functions X_n presently referred to, series occurring in the development of the reciprocal of the distance between two planets, &c.

16. There is a class of functions depending upon a variable or variables $x,\ y,\ldots$ and a parameter n, say the function for the parameter n is X_n such that the product of two functions having the same variables, multiplied it may be by a given function of the variables, and integrated between given limits, gives a result $=0$ or not $=0$, according as the parameters are unequal or equal; $\int UX_mX_n\,dx = 0$, but $\int UX_n^2dx$ not $=0$; the admissible values of the parameters being either any integer values, or it may be the roots of a determinate algebraical or transcendental equation; and the functions X_n may be either algebraical or transcendental. For instance, such a function is $\cos nx$; m and n being integers, we have $\int_0^\pi \cos mx\,.\cos nx\,dx = 0$, but $\int_0^\pi \cos^2 nx\,dx = \frac{1}{2}\pi$.

Assuming the existence of the expansion of a function fx, in a series of multiple cosines, we thus obtain at once the well-known Fourier series, wherein the coefficient of $\cos mx$ is $=\frac{1}{2}\pi\int_0^\pi \cos mx . fx dx$. The question whether the process is applicable is elaborately discussed in Riemann's memoir (1854), *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, No. XII. in the collected works. And again we have the Legendrian functions, which present themselves as the coefficients of the successive powers of α in the development of $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$, $X_0=1$, $X_1=x$, $X_2=\frac{3}{2}(x^2-\frac{1}{3})$, &c.: here m, n being any positive integers, $\int_{-1}^{1} X_m X_n\, dx=0$, but $\int_{-1}^{1} X_n^2\, dx=\dfrac{2}{2n+1}$. And we have also Laplace's functions, &c.

Functions in General.

17. In what precedes, a review has been given, not by any means an exhaustive one, but embracing the most important kinds of known functions; but there are questions to be considered in regard to functions in general.

A function of $x+iy$ has been built up by means of analytical operations performed upon $x+iy$, $(x+iy)^2=x^2-y^2+i.2xy$, &c., and the question next referred to has not arisen. But observe that, knowing $x+iy$, we know x and y, and therefore any two given functions $\phi(x, y)$, $\psi(x, y)$ of x and y: we therefore also know $\phi(x, y)+i\psi(x, y)$, and the question is, whether such a function of x, y (being known when $x+iy$ is known) is to be regarded as a function of $x+iy$; and if not, what is the condition to be satisfied in order that $\phi(x, y)+i\psi(x, y)$ may be a function of $x+iy$. Cauchy at one time considered that the general form was to be regarded as a function of $x+iy$, and he introduced the expression "fonction monogène," monogenous function, to denote the more restricted form which is the proper function of $x+iy$.

Consider for a moment the above general form, say $x'+iy'=\phi(x, y)+i\psi(x, y)$, where ϕ, ψ are any real functions of the real variables (x, y); or what is the same thing, assume $x'=\phi(x, y)$, $y'=\psi(x, y)$; if these functions have each or either of them more than one value, we attend only to one value of each of them. We may then as before take x, y to be the coordinates of a point P in a plane Π, and x', y' to be the coordinates of a point P' in a plane Π'. If, for any given values of x, y, the increments of $\phi(x, y)$, $\psi(x, y)$ corresponding to the indefinitely small real increments h, k of x, y be $Ah+Bk$, $Ch+Dk$, A, B, C, D being functions of x, y, then if the new coordinates of P are $x+h$, $y+k$, the new coordinates of P' will be $x'+Ah+Bk$, $y'+Ch+Dk$; or P, P' will respectively describe the indefinitely small straight paths at the inclinations $\tan^{-1}\dfrac{k}{h}$, $\tan^{-1}\dfrac{Ch+Dk}{Ah+Bk}$ to the axes of x, x' respectively; calling these angles θ, θ', we have therefore $\tan\theta'=\dfrac{C+D\tan\theta}{A+B\tan\theta}$. Now in order that $x'+iy'$ may be $=\phi(x+iy)$, a function of $x+iy$, the condition to be satisfied is that the increment of $x'+iy'$ shall be proportional to the increment $h+ik$ of $x+iy$, or say that it shall be $=(\lambda+i\mu)(h+ik)$, λ, μ being functions of x, y, but independent of

h, k; we must therefore have $Ah+Bk$, $Ch+Dk=\lambda h-\mu k$, $\mu h+\lambda k$ respectively, that is A, B, C, $D=\lambda$, $-\mu$, μ, λ respectively, and the equation for $\tan\theta'$ thus becomes $\tan\theta'=\dfrac{\mu+\lambda\tan\theta}{\lambda-\mu\tan\theta}$; hence writing $\dfrac{\mu}{\lambda}=\tan\alpha$, where α is a function of x, y, but independent of h, k, we have $\tan\theta'=\dfrac{\tan\alpha+\tan\theta}{1-\tan\alpha\tan\theta}$, that is, $\theta'=\alpha+\theta$; or for the given points (x, y), (x', y'), the path of P being at any inclination whatever θ to the axis of x, the path of P' is at the inclination $\theta+$constant angle α to the axis of x'; also $(\lambda h-\mu k)^2+(\mu h+\lambda k)^2=(\lambda^2+\mu^2)(h^2+k^2)$, i.e., the lengths of the paths are in a constant ratio.

The condition may be written $\delta(x'+iy')=(\lambda+i\mu)(\delta x+i\delta y)$; or what is the same thing

$$\left(\frac{dx'}{dx}+i\frac{dy'}{dx}\right)\delta x+\left(\frac{dx'}{dy}+i\frac{dy'}{dy}\right)\delta y=(\lambda+i\mu)(\delta x+i\delta y),$$

that is,

$$\frac{dx'}{dx}+i\frac{dy'}{dx}=(\lambda+i\mu),\quad \frac{dx'}{dy}+i\frac{dy'}{dy}=i(\lambda+i\mu);$$

consequently

$$\frac{dx'}{dy}+i\frac{dy'}{dy}=i\left(\frac{dx'}{dx}+i\frac{dy'}{dx}\right);$$

that is,

$$\frac{dx'}{dy}=-\frac{dy'}{dx},\quad \frac{dy'}{dy}=\frac{dx'}{dx},$$

as the analytical conditions in order that $x'+iy'$ may be a function of $x+iy$. They obviously imply

$$\frac{d^2x'}{dx^2}+\frac{d^2x'}{dy^2}=0,\quad \frac{d^2y'}{dx^2}+\frac{d^2y'}{dy^2}=0;$$

and if x' be a function of x, y, satisfying the first of these conditions, then

$$-\frac{dx'}{dy}dx+\frac{dx'}{dx}dy$$

is a complete differential, and

$$y'=\int\left(-\frac{dx'}{dy}dx+\frac{dx'}{dx}dy\right).$$

18. We have, in what just precedes, the ordinary behaviour of a function $\phi(x+iy)$ in the neighbourhood of the value $x+iy$ of the argument or point $x+iy$; or say the behaviour in regard to a point $x+iy$ such that the function is in the neighbourhood of this point a *continuous* function of $x+iy$ (or that the point is not a point of *discontinuity*): the correlative definition of continuity will be that the function $\phi(x+iy)$, assumed to have at the given point $x+iy$ a single finite value, is continuous in the neighbourhood of this point, when the point $x+iy$ describing continuously a straight infinitesimal element $h+ik$, the point $\phi(x+iy)$ describes continuously a straight infinitesimal element $(\lambda+i\mu)(h+ik)$; or what is really the same thing, when the function $(x+iy)$ has at the point $x+iy$ a differential coefficient.

19. It would doubtless be possible to give for the continuity of a function $\phi(x+iy)$ a less stringent definition not implying the existence of a differential coefficient; but we have this theory only in regard to the functions ϕx of a real variable in memoirs by Riemann, Hankel, du Bois Reymond, Schwarz, Gilbert, Klein, and Darboux. The last-mentioned geometer, in his "Mémoire sur les fonctions discontinues," *Jour. de l'École Normale*, t. IV. (1875), pp. 57—112, gives (after Bonnet) the following definition of a continuous function (observe that we are now dealing with real quantities only):—the function $f(x)$ is continuous for the value $x=x_0$ when, h and ϵ being positive quantities as small as we please and θ any positive quantity at pleasure between 0 and 1, we have, for all the values of θ, $f(x_0 \pm \theta h)-f(x)$ less in absolute magnitude than ϵ; and moreover $f(x)$ is continuous through the interval x_0, x_1 ($x_1 > x_0$, that is, nearer $+\infty$) when $f(x)$ is continuous for every value of x between x_0 and x_1, and, h tending to zero through positive values, $f(x_0+h)$ and $f(x_0-h)$ tend to the limits $f(x_0)$, $f(x_1)$ respectively. It is possible, consistently with this definition, to form continuous functions not having in any proper sense a differential coefficient, and having other anomalous properties; thus if $a_1, a_2, a_3, \ldots$ be an infinite series of real positive or negative quantities, such that the series Σa_n is absolutely convergent (i.e. the sum $\Sigma \pm a_n$, each term being made positive, is convergent), then the function $\Sigma a_n (\sin n\pi x)^{\frac{2}{3}}$ is a continuous function actually calculable for any assumed value of x; but it is shown in the memoir that, taking $x=$ any commensurable value $\frac{p}{q}$ whatever, and then writing $x=\frac{p}{q}+h$, h indefinitely small, the increment of the function is of the form $(k+\epsilon)h^{\frac{2}{3}}$, k finite, ϵ an indefinitely small quantity vanishing with h; there is thus no term varying with h, nor consequently any differential coefficient. See also Riemann's Memoir, *Ueber die Darstellbarkeit*, &c. (No. XII. in the collected works), already referred to.

20. It was necessary to allude to the foregoing theory of (as they may be termed) infinitely discontinuous functions; but the ordinary and most important functions of analysis are those which are continuous, except for a finite number (or it may be an infinite number) of points of discontinuity. It is to be observed that a point at which the function becomes infinite is *ipso facto* a point of discontinuity; a value of the variable for which the function becomes infinite is, as already mentioned, said to be an "infinity" (or a "pole") of the function; thus, in the case of a rational function expressed as a fraction in its least terms, if the denominator contains a factor $(x-a)^m$, a a real or imaginary value, m a positive integer, then a is said to be an infinity of the mth order (and in the particular case $m=1$, it is said to be a simple infinity). The circular functions $\tan x$, $\sec x$ are instances of a function having an infinite number of simple infinities.

A rational function is a one-valued function, and in regard to a rational function the infinities are the only points of discontinuity; but a one-valued function may have points of discontinuity of a character quite distinct from an infinity: for instance, in the exponential function $\exp\left(\frac{1}{u-a}\right)$ where a is real or imaginary, the value $u(=x+iy)=a$, is a point of discontinuity but not an infinity; taking $u=a+\rho e^{\alpha i}$, where ρ is an

indefinitely small real positive quantity, the value of the function is $\exp\left(\frac{1}{\rho}e^{-\alpha i}\right)$, $=\exp\frac{1}{\rho}(\cos\alpha - i\sin\alpha)$, which is indefinitely large or indefinitely small according as $\cos\alpha$ is positive or negative, and in the separating case $\cos\alpha = 0$, and therefore $\sin\alpha = \pm 1$, it is $=\cos\frac{1}{\rho} \pm i\sin\frac{1}{\rho}$ which is indeterminate. If, instead of $\exp\frac{1}{u-a}$, we consider a linear function

$$\left\{A + B\exp\frac{1}{u-a}\right\} \div \left\{C + D\exp\frac{1}{u-a}\right\},$$

then writing as before $u = a + \rho e^{\alpha i}$, the value is $= A \div C$, or $= B \div D$, according as $\cos\alpha$ is negative or positive. As regards the theory of one-valued functions in general, the memoir by Weierstrass, "Zur Theorie der eindeutigen analytischen Functionen," *Berl. Abh.* 1876, pp. 11—60, may be referred to.

21. A one-valued function *ex vi termini* cannot have a point of discontinuity of the kind next referred to; if the representative point P, moving in any manner whatever, returns to its original position, the corresponding point P' cannot but return to its original position. But consider a many-valued function, say an n-valued function $x' + iy'$, of $x + iy$; to each position of P there correspond n positions, in general all of them different, of P'. But the point P may be such that (to take the most simple case) two of the corresponding points P' coincide with each other, say such a position of P is at V, then (using for greater distinctness a different letter W' instead of V') corresponding thereto we have two coincident points (W'), and $n-2$ other points W'; V is then a branch-point (Verzweigungspunkt). Taking for P any point which is not a branch-point, then in the neighbourhood of this value each of the n functions $x' + iy'$ is a continuous function of $x + iy$, and by what precedes, if P describing an infinitely small closed curve (or oval) return to its original position, then each of the corresponding points P' describing a corresponding indefinitely small oval will return to its original position. But imagine the oval described by P to be gradually enlarged, so that it comes to pass through a branch-point V; the ovals described by two of the corresponding points P' will gradually approach each other, and will come to unite at the point (W'), each oval then sharpening itself out so that the two form together a figure of eight. And if we imagine the oval described by P to be still further enlarged so as to include within it the point V, then the figure of eight, losing the crossing, will be at first an hour-glass form, or twice-indented oval, and ultimately in form an ordinary oval, but having the character of a twofold oval; i.e. to the oval described by P (and which surrounds the branch-point V) there will correspond this twofold oval, and $n-2$ onefold ovals, in such wise that to a given position of P on its oval there correspond two points, say P_1', P_2', on the twofold oval, and $n-2$ points $P_3', \ldots, P_n'$, each on its own onefold oval. And then as P describing its oval returns to its original position, the point P_1' describing a portion only of the twofold oval, will pass to the original position of P_2', while the point P_2' describing the remaining portion of the twofold oval will pass to the original position of P_1'; the other points $P_3', \ldots, P_n'$, describing each of them its own onefold oval, will

return each of them to its original position. And it is easy to understand how, when the oval described by P surrounds two or more of the branch-points V, the corresponding curves for P' may be a system of manifold ovals, such that the sum of the manifoldness is always $=n$, and to conceive in a general way the behaviour of the corresponding points P and P'.

Writing for a moment $x+iy=u$, $x'+iy'=v$, the branch-points are the points of contact of parallel tangents to the curve $\phi(u, v)=0$, a line through a cusp (but not a line through a node), being reckoned as a tangent; that is, if this be a curve of the order n and class m, with δ nodes and κ cusps, the number of branch-points is $=m+\kappa$, that is, it is $=n^2-n-2\delta-2\kappa$, or if $p, =\frac{1}{2}(n-1)(n-2)-\delta-\kappa$, be the *deficiency*, then the number is $=2n-2+2p$.

To illustrate the theory of the n-valued algebraical function $x'+iy'$ of the complex variable $x+iy$, Riemann introduces the notion of a surface composed of n coincident planes or sheets, such that the transition from one sheet to another is made at the branch-points, and that the n sheets form together a multiply-connected surface, which can be by cross-cuts (Querschnitte) converted into a simply-connected surface; the n-valued function $x'+iy'$ becomes thus a one-valued function of $x+iy$, considered as belonging to a point on some determinate sheet of the surface: and upon such considerations he founds the whole theory of the functions which arise from the *integration* of the differential expressions depending on the n-valued algebraical function (that is, any irrational algebraical function whatever) of the independent variable, establishing as part of the investigation the theory of the multiple θ-functions. But it would be difficult to give a further account of these investigations.

The Calculus of Functions.

22. The so-called Calculus of Functions, as considered chiefly by Herschel and Babbage and De Morgan, is not so much a theory of functions as a theory of the solution of functional equations; or, as perhaps should rather be said, the solution of functional equations by means of known functions, or symbols,—the epithet known being here used in reference to the actual state of analysis. Thus for a functional equation $\phi x+\phi y=\phi(xy)$, taking the logarithm as a known function, the solution is $\phi x=c\log x$; or if the logarithm is not taken to be a known function, then a solution may be obtained by means of the sign of integration $\phi x=c\int\frac{dx}{x}$; but the establishment of the properties of the function logarithm (assumed to be previously unknown) would not be considered as coming within the theory. A class of equations specially considered is where αx, βx, ... being given functions of x, the unknown function ϕ is to be determined by means of a given relation between x, ϕx, $\phi\alpha x$, $\phi\beta x$, ...; in particular the given relation may be between x, ϕx, $\phi\alpha x$; this can be at once reduced to equations of finite differences; for writing $x=u_n$, $\alpha x=u_{n+1}$, we have $u_{n+1}=\alpha u_n$, giving u_n, and therefore also x, each of them as a function of n; and then writing $\phi x=v_n$, $\phi\alpha x$ will be the same function of $n+1$, $=v_{n+1}$, and the given relation is again an equation of finite differences in v_{n+1}, v_n, and n; we have thus $v_n, =\phi x$,

as a function of n, that is, of x. As regards the equation $u_{n+1} = \alpha u_n$, considered in itself apart from what precedes, observe that this is satisfied by writing $u_n = \alpha^n(x)$, or the question of solving this equation of finite differences is, in fact, identical with that of finding the nth function $\alpha^n(x)$, where $\alpha(x)$ is a given function of x. It of course depends on the form of $\alpha(x)$ whether this question admits of solution in any proper sense; thus, for a function such as $\log x$, the nth logarithm is expressible in its original function $\log^n x$, $(= \log \log \ldots x)$, and not in any other form. But there are forms, for instance $\alpha x = \dfrac{a + bx}{c + dx}$, where the nth function $\alpha^n x$ is a function of the like form $\alpha^n x = \dfrac{A + Bx}{C + Dx}$, in which the actual value can be expressed as a function of n; if α be such a form, then $\phi\alpha\phi^{-1}$, whatever ϕ may be, is a like form, for we obviously have $(\phi\alpha\phi^{-1})^n = \phi\alpha^n\phi^{-1}$. The determination of the nth function is, in fact, a leading question in the calculus of functions.

It is to be observed that considering the case of two variables, if for instance $\alpha(x, y)$ denote a given function of x, y, the notation $\alpha^2(x, y)$ is altogether meaningless; in order to generalize the question, we require an extended notation wherein a single functional symbol is used to denote two functions of the two variables. Thus $\phi(x, y) = \alpha(x, y)$, $\beta(x, y)$, α and β given functions; writing for shortness $x_1 = \alpha(x, y)$, $y_1 = \beta(x, y)$, then $\phi^2(x, y)$ will denote $\phi(x_1, y_1)$, that is, two functions $\alpha(x_1, y_1)$, $\beta(x_1, y_1)$, say these are x_2, y_2; $\phi^3(x, y)$ will denote $\phi(x_2, y_2)$, and so on, so that $\phi^n(x, y)$ will have a determinate meaning. And the like is obviously the case in regard to any number of variables, the single functional symbol denoting in each case a set of functions equal in number to the variables.

788.

GALOIS.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. x. (1879), p. 48.]

GALOIS, EVARISTE (1811—1832), an eminently original and profound French mathematician, born 26th October 1811, killed in a duel May 1832. A necrological notice by his friend M. Auguste Chevalier appeared in the *Revue Encyclopédique*, September 1832, p. 744; and his collected works are published, *Liouville*, t. XI. (1846), pp. 381—444, about fifty of these pages being occupied by researches on the resolubility of algebraic equations by radicals. But these researches, crowning as it were the previous labours of Lagrange, Gauss, and Abel, have in a signal manner advanced the theory, and it is not too much to say that they are the foundation of all that has since been done, or is doing, in the subject. The fundamental notion consists in the establishment of a group of permutations of the roots of an equation, such that every function of the roots invariable by the substitutions of the group is rationally known, and reciprocally that every rationally determinable function of the roots is invariable by the substitutions of the group; some further explanation of the theorem, and in connexion with it an explanation of the notion of an adjoint radical, is given under Equation, No. 32, [786]. As part of the theory (but the investigation has a very high independent value as regards the Theory of Numbers, to which it properly belongs), Galois introduces the notion of the imaginary roots of an irreducible congruence of a degree superior to unity; i.e., such a congruence, $F(x) \equiv 0$ (mod. a prime number p), has no integer root; but what is done is to introduce a quantity i subjected to the condition of verifying the congruence in question, $F(i) \equiv 1$ (mod. p), which quantity i is an imaginary of an entirely new kind, occupying in the theory of numbers a position analogous to that of $\sqrt{-1}$ in algebra.

789.

GAUSS.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. x. (1879), p. 116.]

GAUSS, CARL FRIEDRICH (1777—1855), an eminent German mathematician, was born of humble parents at Brunswick, April 23, 1777, and was indebted for a liberal education to the notice which his talents procured him from the reigning duke. His name became widely known by the publication, in his twenty-fifth year (1801), of the *Disquisitiones Arithmeticæ*. In 1807 he was appointed director of the Göttingen observatory, an office which he retained to his death: it is said that he never slept away from under the roof of his observatory, except on one occasion, when he accepted an invitation from Humboldt to attend a meeting of natural philosophers at Berlin. In 1809 he published at Hamburg his *Theoria Motus Corporum Cœlestium*, a work which gave a powerful impulse to the true methods of astronomical observation; and his astronomical workings, observations, calculations of orbits of planets and comets, &c., are very numerous and valuable. He continued his labours in the theory of numbers and other analytical subjects, and communicated a long series of memoirs to the Royal Society of Sciences at Göttingen. His first memoir on the theory of magnetism, *Intensitas vis magneticæ terrestris ad mensuram absolutam revocata*, was published in 1833, and he shortly afterwards proceeded, in conjunction with Professor Wilhelm Weber, to invent new apparatus for observing the earth's magnetism and its changes; the instruments devised by them were the declination instrument and the bifilar magnetometer. With Weber's assistance he erected in 1833 at Göttingen a magnetic observatory free from iron (as Humboldt and Arago had previously done on a smaller scale), where he made magnetic observations, and from this same observatory he sent telegraphic signals to the neighbouring town, thus showing the practicability of an electromagnetic telegraph. He further instituted an association (Magnetische Verein), composed at first almost entirely of Germans, whose continuous observations on fixed term-days extended from Holland to Sicily. The volumes of their publication, *Resultate aus den Beobachtungen des magnetischen Vereins*, extend from 1836 to 1839; and in those for 1838 and 1839 are contained the two important memoirs by Gauss, *Allgemeine Theorie des Erdmagnetismus*, and the *Allgemeine Lehrsätze*—on the theory of forces attracting according to the inverse square of the distance. The instruments and methods thus due to him are substantially those employed in the magnetic

observatories throughout the world. He co-operated in the Danish and Hanoverian measurements of an arc and trigonometrical operations (1821—48), and wrote (1843, 1846) the two memoirs *Ueber Gegenstände der höhern Geodäsie.* Connected with observations in general we have (1812—26) the memoir *Theoria combinationis observationum erroribus minimis obnoxia,* with a second part and a supplement. Another memoir of applied mathematics is the *Dioptrische Untersuchungen,* 1840. Gauss was well versed in general literature and the chief languages of modern Europe, and was a member of nearly all the leading scientific societies in Europe. He died at Göttingen early in the spring of 1855. The centenary of his birth was celebrated (1877) at his native place, Brunswick.

Gauss's collected works have been recently published by the Royal Society of Göttingen, in 7 vols. 4to, Gött., 1863—71, edited by E. J. Schering,—(1) the *Disquisitiones Arithmeticæ,* (2) *Theory of Numbers,* (3) *Analysis,* (4) *Geometry and Method of Least Squares,* (5) *Mathematical Physics,* (6) *Astronomy,* and (7) the *Theoria Motus Corporum Cœlestium.* They include, besides his various works and memoirs, notices by him of many of these, and of works of other authors in the *Göttingische gelehrte Anzeigen,* and a considerable amount of previously unpublished matter, *Nachlass.* Of the memoirs in pure mathematics, comprised for the most part in vols. II., III., and IV. (but to these must be added those on *Attractions* in vol. V.), it may be safely said there is not one which has not signally contributed to the progress of the branch of mathematics to which it belongs, or which would not require to be carefully analysed in a history of the subject. Running through these volumes in order, we have in the second the memoir, *Summatio quarundam serierum singularium,* the memoirs on the theory of biquadratic residues, in which the notion of complex numbers of the form $a + bi$ was first introduced into the theory of numbers; and included in the *Nachlass* are some valuable tables. That for the conversion of a fraction into decimals (giving the complete period for all the prime numbers up to 997) is a specimen of the extraordinary love which Gauss had for long arithmetical calculations; and the amount of work gone through in the construction of the table of the number of the classes of binary quadratic forms must also have been tremendous. In vol. III. we have memoirs relating to the proof of the theorem that every numerical equation has a real or imaginary root, the memoirs on the *Hypergeometric Series,* that on *Interpolation,* and the memoir *Determinatio Attractionis*—in which a planetary mass is considered as distributed over its orbit according to the time in which each portion of the orbit is described, and the question (having an implied reference to the theory of secular perturbations) is to find the attraction of such a ring. In the solution the value of an elliptic function is found by means of the *arithmetico-geometrical mean.* The *Nachlass* contains further researches on this subject, and also researches (unfortunately very fragmentary) on the lemniscate-function, &c., showing that Gauss was, even before 1800, in possession of many of the discoveries which have made the names of Abel and Jacobi illustrious. In vol. IV. we have the memoir *Allgemeine Auflösung...,* on the graphical representation of one surface upon another, and the *Disquisitiones generales circa superficies curvas.* And in vol. V. we have a memoir *On the Attraction of Homogeneous Ellipsoids,* and the already mentioned memoir *Allgemeine Lehrsätze...,* on the theory of forces attracting according to the inverse square of the distance.

790.

GEOMETRY (ANALYTICAL).

[From the *Encyclopædia Britannica, Ninth Edition*, vol. x. (1879), pp. 408—420.]

THIS will be here treated as a method. The science is Geometry; and it would be possible, analytically, or by the method of coordinates, to develope the truths of geometry in a systematic course. But it is proposed not in any way to attempt this, but simply to explain the method, giving such examples, interesting (it may be) in themselves, as are suitable for showing how the method is employed in the demonstration and solution of theorems or problems.

Geometry is one-, two-, or three-dimensional, or, what is the same thing, it is lineal, plane, or solid, according as the space dealt with is the line, the plane, or ordinary (three-dimensional) space. No more general view of the subject need here be taken:—but in a certain sense one-dimensional geometry does not exist, inasmuch as the geometrical constructions for points in a line can only be performed by travelling out of the line into other parts of a plane which contains it, and conformably to the usual practice Analytical Geometry will be treated under the two divisions, Plane and Solid.

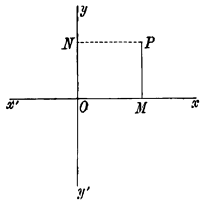
It is proposed to consider Cartesian coordinates almost exclusively; for the proper development of the science homogeneous coordinates (three and four in plane and solid geometry respectively) are required; and it is moreover necessary to have the correlative line- and plane-coordinates; and in solid geometry to have the *six* coordinates of the line. The most comprehensive English works are those of Dr Salmon, *Conic Sections* (5th edition, 1869), *Higher Plane Curves* (2nd edition, 1873), and *Geometry of Three Dimensions* (3rd edition, 1874); we have also, on plane geometry, Clebsch's *Vorlesungen über Geometrie*, posthumous, edited by Dr F. Lindemann, Leipsic, 1875, not yet complete.

I. PLANE ANALYTICAL GEOMETRY (§§ 1—25).

1. It is assumed that the points, lines, and figures considered exist in one and the same plane, which plane, therefore, need not be in any way referred to. The position of a point is determined by means of its (Cartesian) coordinates; i.e. as

explained under the article Curve, [785], we take the two lines $x'Ox$ and $y'Oy$, called the axes of x and y respectively, intersecting in a point O called the origin, and determine the position of any other point P by means of its coordinates $x = OM$ (or NP), and

Fig. 1.



$y = MP$ (or ON). The two axes are usually (as in fig. 1) at right angles to each other, and the lines PM, PN are then at right angles to the axes of x and y respectively. Assuming a scale at pleasure, the coordinates x, y of a point have numerical values.

It is necessary to attend to the signs: x has opposite signs according as the point is on one side or the other of the axis of y, and similarly y has opposite signs according as the point is on the one side or the other of the axis of x. Using the letters N., E., S., W. as in a map, and considering the plane as divided into four quadrants by the axes, the signs are usually taken to be—

x	y	for quadt.
+	+	N. E.
+	−	S. E.
−	+	N. W.
−	−	S. W.

A point is said to have the coordinates (a, b), and is referred to as the point (a, b), when its coordinates are $x = a$, $y = b$; the coordinates x, y of a variable point, or of a point which is for the time being regarded as variable, are said to be current coordinates.

2. It is sometimes convenient to use oblique coordinates; the only difference is that the axes are not at right angles to each other; the lines PM, PN are drawn parallel to the axes of y and x respectively, and the figure $OMPN$ is thus a parallelogram. But in all that follows, the Cartesian coordinates are taken to be rectangular; polar coordinates and other systems will be briefly referred to in the sequel.

3. If the coordinates (x, y) of a point are not given, but only a relation between them $f(x, y) = 0$, then we have a curve. For, if we consider x as a real quantity varying continuously from $-\infty$ to $+\infty$, then, for any given value of x, y has a value

or values. If these are all imaginary, there is not any real point; but if one or more of them be real, we have a real point or points, which (as the assumed value of x varies continuously) varies or vary continuously therewith; and the locus of all these real points is a curve. The equation completely defines the curve; to trace the curve directly from the equation, nothing else being known, we obtain as above a series of points sufficiently near to each other, and draw the curve through them. For instance, let this be done in a simple case. Suppose $y=2x-1$; it is quite easy to obtain and lay down a series of points as near to each other as we please, and the application of a ruler would show that these were in a line; that the curve is a line depends upon something more than the equation itself, viz. the theorem that every equation of the form $y=ax+b$ represents a line; supposing this known, it will be at once understood how the process of tracing the curve may be abbreviated; we have $x=0$, $y=-1$, and $x=\frac{1}{2}$, $y=0$; the curve is thus the line passing through these two points. But in the foregoing example the notion of a line is taken to be a known one, and such notion of a line does in fact precede the consideration of any equation of a curve whatever, since the notion of the coordinates themselves rests upon that of a line. In other cases it may very well be that the equation is the definition of the curve; the points laid down, although (as finite in number) they do not actually determine the curve, determine it to any degree of accuracy; and the equation thus enables us to construct the curve.

A curve may be determined in another way; viz. the coordinates x, y may be given each of them as a function of the same variable parameter θ; x, $y=f(\theta)$, $\phi(\theta)$ respectively. Here, giving to θ any number of values in succession, these equations determine the values of x, y, that is, the positions of a series of points on the curve. The ordinary form $y=\phi(x)$, where y is given explicitly as a function of x, is a particular case of each of the other two forms: we have $f(x, y)$, $=y-\phi(x)$, $=0$; and $x=\theta$, $y=\phi(\theta)$.

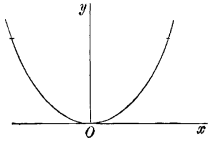
4. As remarked under Curve, [785], it is a useful exercise to trace a considerable number of curves, first taking equations which are purely numerical, and then equations which contain literal constants (representing numbers); the equations most easily dealt with are those wherein one coordinate is given as an explicit function of the other, say $y=\phi(x)$ as above. A few examples are here given, with such explanations as seem proper.

(i) $y=2x-1$, as before; it is at once seen that this is a line; and taking it to be so, any two points, for instance, $(0, -1)$ and $(\frac{1}{2}, 0)$, determine the line.

(ii) $y=x^2$. The equation shows that x may be positive or negative, but that y is always positive, and has the same values for equal positive and negative values of x: the curve passes through the origin, and through the points $(\pm 1, 1)$. It is already known that the curve lies wholly above the axis of x. To find its form in the neighbourhood of the origin, give x a small value, $x=\pm 0{\cdot}1$ or $\pm 0{\cdot}01$, then y is very much smaller, $=0{\cdot}01$ and $0{\cdot}0001$ in the two cases respectively; this shows that the curve touches the axis of x at the origin. Moreover, x may be as large as we please,

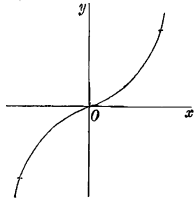
but when it is large, y is much larger; for instance, $x=10$, $y=100$. The curve is a parabola (fig. 2).

Fig. 2.



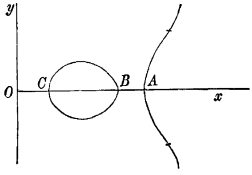
(iii) $y=x^3$. Here x being positive y is positive, but x being negative y is also negative: the curve passes through the origin, and also through the points (1, 1) and $(-1, -1)$. Moreover, when x is small, $=0{\cdot}1$ for example, then not only is y, $=0{\cdot}001$, very much smaller than x, but it is also very much smaller than y was for the last-mentioned curve $y=x^2$, that is, in the neighbourhood of the origin the present curve approaches more closely the axis of x. The axis of x is a tangent at the origin, but it is a tangent of a peculiar kind (a stationary or inflexional tangent), cutting the curve at the origin, which is an inflexion. The curve is the cubical parabola (fig. 3).

Fig. 3.



(iv) $y^2=x-1\,.\,x-3\,.\,x-4$. Here $y=0$ for $x=1, =3, =4$. Whenever $x-1.x-3.x-4$ is positive, y has two equal and opposite values; but when $x-1\,.\,x-3\,.\,x-4$ is negative, then y is imaginary. In particular, for x less than 1, or between 3 and 4, y is imaginary, but for x between 1 and 3, or greater than 4, y has two values. It

Fig. 4.



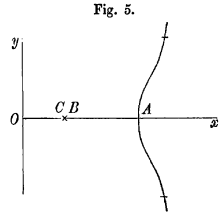
is clear that for x somewhere between 1 and 3, y will attain a maximum: the values of x and y may be found approximately by trial. The curve will consist of an oval and infinite branch, and it is easy to see that, as shown in fig. 4, the curve where it cuts the axis of x cuts it at right angles. It may be further remarked that, as

x increases from 4, the value of y will increase more and more rapidly; for instance, $x=5$, $y^2=8$, $x=10$, $y^2=378$, &c., and it is easy to see that this implies that the curve has on the infinite branch two inflexions as shown.

(v) $y^2=x-c\,.\,x-b\,.\,x-a$, where $a>b>c$ (that is, a nearer to $+\infty$, c to $-\infty$). The curve has the same general form as in the last figure, the oval extending between the limits $x=c$, $x=b$, the infinite branch commencing at the point $x=a$.

(vi) $y^2=(x-c)^2(x-a)$. Suppose that in the last-mentioned curve, $y^2=x-c.x-b.x-a$, b gradually diminishes, and becomes ultimately $=c$. The infinite branch (see fig. 5) changes its form, but not in a very marked manner, and it retains the two inflexions.

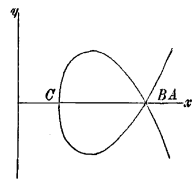
Fig. 5.



The oval lies always between the values $x=c$, $x=b$, and therefore its length continually diminishes; it is easy to see that its breadth will also continually diminish; ultimately it shrinks up into a mere point. The curve has thus a conjugate or isolated point, or acnode. For a direct verification observe that $x=c$, $y=0$, so that $(c, 0)$ is a point of the curve, but if x is either less than c, or between c and a, y^2 is negative, and y is imaginary.

(vii) $y^2=(x-c)(x-a)^2$. If in the same curve b gradually increases and becomes ultimately $=a$, the oval and the infinite branch change each of them its form, the oval extending always between the values $x=c$, $x=b$, and thus continually approaching the infinite branch, which begins at $x=a$. The consideration of a few numerical examples, *with careful drawing*, would show that the oval and the infinite branch as they approach sharpen out each towards the other, the two inflexions on the infinite branch coming always nearer to the point $(a, 0)$,—so that finally, when b becomes

Fig. 6.



$=a$, the curve has the form shown in fig. 6, there being now a double point or node (crunode) at A, and the inflexions on the infinite branch having disappeared.

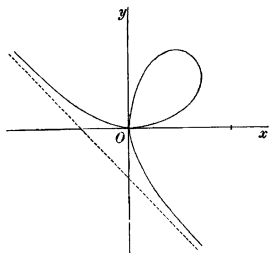
In the last four examples, the curve is one of the cubical curves called the divergent parabolas: (iv) is a mere numerical example of (v), and (vi), (vii), (viii) are in Newton's language the parabola *cum ovali*, *punctata*, and *nodata* respectively. When a, b, c are all equal, or the form is $y^2=(x-c)^3$, we have a cuspidal form, Newton's parabola *cuspidata*, otherwise the semicubical parabola.

(viii) As an example of a curve given by an implicit equation, suppose the equation is

$$x^3+y^3-3xy=0;$$

this is a nodal cubic curve, the node at the origin, and the axes touching the two branches respectively (fig. 7). An easy mode of tracing it is to express x, y each of them

Fig. 7.



in terms of a variable θ, $x=\dfrac{3\theta}{1+\theta^3}$, $y=\dfrac{3\theta^2}{1+\theta^3}$; but it is instructive to trace the curve directly from its equation.

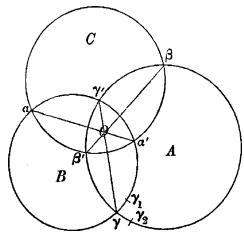
5. It may be remarked that the purely algebraical process, which is, in fact, that employed in finding a differential coefficient $\dfrac{dy}{dx}$, if applied directly to the equation of the curve, determines the point consecutive to any given point of the curve, that is, the direction of the curve at such given point, or, what is the same thing, the direction of the tangent at that point. In fact, if α, β are the coordinates of any point on a curve $f(x, y)=0$, then writing in the equation of the curve $x=\alpha+h$, $y=\beta+k$, and in the resulting equation $f(\alpha+h, \beta+k)=0$, developed in powers of h and k, omitting the term $f(\alpha, \beta)$, which vanishes, and the terms containing the second and higher powers of h, k, we have a linear equation $Ah+Bk=0$, which determines the ratio of the increments h, k. Of course, in the analytical development of the theory, we translate this into the notation of the differential calculus; but the question presents itself, and is thus seen to be solvable, as soon as it is attempted to trace a curve from its equation.

Geometry is Descriptive, or Metrical.

6. A geometrical proposition is either *descriptive* or *metrical*: in the former case it is altogether independent of the idea of magnitude (length, inclination, &c.); in the latter case it has reference to this idea. It is to be noticed that, although the method of coordinates seems to be by its inception essentially metrical, and we can hardly, except by metrical considerations, connect an equation with the curve which it represents (for instance, even assuming it to be known that an equation $Ax+By+C=0$ represents a line, yet if it be asked what line, the only form of answer is, that it is the line cutting the axes at *distances* from the origin $-C\div A$, $-C\div B$ respectively), yet in dealing by this method with descriptive propositions, we are, in fact, eminently free from all metrical considerations.

7. It is worth while to illustrate this by the instance of the well-known theorem of the radical centre of three circles. The theorem is that, given any three circles A, B, C (fig. 8), the common chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ of the three pairs of circles meet in a point.

Fig. 8.



The geometrical proof is metrical throughout:—

Take O the point of intersection of $\alpha\alpha'$, $\beta\beta'$, and joining this with γ', suppose that $\gamma' O$ does not pass through γ, but that it meets the circles A, B in two distinct points γ_1, γ_2 respectively. We have then the known metrical property of intersecting chords of a circle; viz. in the circle C, where $\alpha\alpha'$, $\beta\beta'$ are chords meeting at a point O,

$$O\alpha \,.\, O\alpha' = O\beta \,.\, O\beta',$$

where, as well as in what immediately follows $O\alpha$, &c., denote, of course, *lengths* or *distances*.

Similarly in the circle A,

$$O\beta \,.\, O\beta' = O\gamma_1 \,.\, O\gamma',$$

and in the circle B,

$$O\alpha \,.\, O\alpha' = O\gamma_2 \,.\, O\gamma'.$$

Consequently $O\gamma_1 \,.\, O\gamma' = O\gamma_2 \,.\, O\gamma'$, that is, $O\gamma_1 = O\gamma_2$, or the points γ_1 and γ_2 coincide; that is, they each coincide with γ.

We contrast this with the analytical method.

Here it only requires to be known that an equation $Ax+By+C=0$ represents a line, and an equation $x^2+y^2+Ax+By+C=0$ represents a circle. A, B, C have, in the two cases respectively, metrical significations; but these we are not concerned with. Using S to denote the function $x^2+y^2+Ax+By+C$, the equation of a circle is $S=0$, where S stands for its value; more briefly, we say the equation is $S, =x^2+y^2+Ax+By+C, =0$. Let the equation of any other circle be $S', =x^2+y^2+A'x+B'y+C'=0$; the equation $S-S'=0$ is a linear equation: $S-S'$ is, in fact, $=(A-A')x+(B-B')y+C-C'$: and it thus represents a line; this equation is satisfied by the coordinates of each of the points of intersection of the two circles (for at each of these points $S=0$ and $S'=0$, therefore also $S-S'=0$); hence the equation $S-S'=0$ is that of the line joining the two points of intersection of the two circles, or say it is the equation of the common chord of the two circles. Considering then a third circle $S'', =x^2+y^2+A''x+B''y+C''=0$, the equations of the common chords are $S-S'=0$, $S-S''=0$, $S'-S''=0$ (each of these a linear equation); at the intersection of the first and second of these lines $S=S'$ and $S=S''$, therefore also $S'=S''$, or the equation of the third line is satisfied by the coordinates of the point in question; that is, the three chords intersect in a point O, the coordinates of which are determined by the equations $S=S'=S''$.

It further appears that, if the two circles $S=0$, $S'=0$ do not intersect in any real points, they must be regarded as intersecting in two imaginary points, such that the line joining them is the real line represented by the equation $S-S'=0$; or that two circles, whether their intersections be real or imaginary, have always a real common chord (or radical axis), and that for *any* three circles the common chords intersect in a point (of course real) which is the radical centre. And by this very theorem, given two circles with imaginary intersections, we can, by drawing circles which meet each of them in real points, construct the radical axis of the first-mentioned two circles.

8. The principle employed in showing that the equation of the common chord of two circles is $S-S'=0$ is one of very extensive application, and some more illustrations of it may be given.

Suppose $S=0$, $S'=0$ are lines, that is, let S, S' now denote linear functions $Ax+By+C$, $A'x+B'y+C'$, then $S-kS'=0$ (k an arbitrary constant) is the equation of *any* line passing through the point of intersection of the two given lines. Such a line may be made to pass through any given point, say the point (x_0, y_0); i.e., if S_0, S_0' are what S, S' respectively become on writing for (x, y) the values (x_0, y_0), then the value of k is $k=S_0 \div S_0'$. The equation in fact is $SS_0'-S_0S'=0$; and starting from this equation we at once verify it *à posteriori;* the equation is a linear equation satisfied by the values of (x, y) which make $S=0$, $S'=0$; and satisfied also by the values (x_0, y_0); and it is thus the equation of the line in question.

If, as before, $S=0$, $S'=0$ represent circles, then (k being arbitrary) $S-kS'=0$ is the equation of any circle passing through the two points of intersection of the two circles; and to make this pass through a given point (x_0, y_0) we have again $k=S_0 \div S_0'$. In the particular case $k=1$, the circle becomes the common chord; more accurately,

it becomes the common chord together with the line infinity, but this is a question which is not here gone into.

If S denote the general quadric function,

$$S = ax^2 + 2hxy + by^2 + 2fy + 2gx + c, \quad = (a, b, c, f, g, h)(x, y, 1)^2,$$

then the equation $S = 0$ represents a conic; assuming this, then, if $S' = 0$ represents another conic, the equation $S - kS' = 0$ represents *any* conic through the four points of intersection of the two conics.

Returning to the equation $Ax + By + C = 0$ of a line, if this pass through two given points (x_1, y_1), (x_2, y_2), then we must have $Ax_1 + By_1 + C = 0$, $Ax_2 + By_2 + C = 0$, equations which determine the ratios $A : B : C$, and it thus appears that the equation of the line through the two given points is

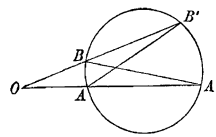
$$x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - x_2 y_1 = 0;$$

or, what is the same thing,

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0.$$

9. The object still being to illustrate the mode of working with coordinates, we consider the theorem of the polar of a point in regard to a circle. Given a circle and a point O (fig. 9), we draw through O any two lines meeting the circle in the points

Fig. 9.



A, A' and B, B' respectively, and then taking Q as the intersection of the lines AB' and $A'B$, the theorem is that the locus of the point Q is a right line depending only upon O and the circle, but independent of the particular lines OAA' and OBB'.

Taking O as the origin, and for the axes any two lines through O at right angles to each other, the equation of the circle will be

$$x^2 + y^2 + 2Ax + 2By + C = 0;$$

and if the equation of the line OAA' is taken to be $y = mx$, then the points A, A' are found as the intersections of the straight line with the circle; or to determine x we have

$$x^2(1 + m^2) + 2x(A + Bm) + C = 0.$$

If (x_1, y_1) are the coordinates of A, and (x_2, y_2) of A', then the roots of this equation are x_1, x_2, whence easily

$$\frac{1}{x_1}+\frac{1}{x_2}=-2\frac{A+Bm}{C}.$$

And similarly, if the equation of the line OBB' is taken to be $y=m'x$, and the coordinates of B, B' to be (x_3, y_3) and (x_4, y_4) respectively, then

$$\frac{1}{x_3}+\frac{1}{x_4}=-2\frac{A+Bm'}{C}.$$

We have then

$$x(y_1-y_4)-y(x_1-x_4)+x_1y_4-x_4y_1=0,$$
$$x(y_2-y_3)-y(x_2-x_3)+x_2y_3-x_3y_2=0,$$

as the equations of the lines AB' and $A'B$ respectively; for the first of these equations, being satisfied if we write therein (x_1, y_1) or (x_4, y_4) for (x, y), is the equation of the line AB': and similarly the second equation is that of the line $A'B$. Reducing by means of the relations $y_1-mx_1=0$, $y_2-mx_2=0$, $y_3-m'x_3=0$, $y_4-m'x_4=0$, the two equations become

$$x(mx_1-m'x_4)-y(x_1-x_4)+(m'-m)x_1x_4=0,$$
$$x(mx_2-m'x_3)-y(x_2-x_3)+(m'-m)x_2x_3=0;$$

and if we divide the first of these equations by m_1m_4, and the second by m_2m_3, and then add, we obtain

$$x\left\{m\left(\frac{1}{x_3}+\frac{1}{x_4}\right)-m'\left(\frac{1}{x_1}+\frac{1}{x_2}\right)\right\}-y\left\{\frac{1}{x_3}+\frac{1}{x_4}-\left(\frac{1}{x_1}+\frac{1}{x_2}\right)\right\}+2m'-2m=0,$$

or, what is the same thing,

$$\left(\frac{1}{x_1}+\frac{1}{x_2}\right)(y-m'x)-\left(\frac{1}{x_3}+\frac{1}{x_4}\right)(y-mx)+2m'-2m=0,$$

which by what precedes is the equation of a line through the point Q. Substituting herein for $\frac{1}{x_1}+\frac{1}{x_2}$, $\frac{1}{x_3}+\frac{1}{x_4}$ their foregoing values, the equation becomes

$$-(A+Bm)(y-m'x)+(A+Bm')(y-mx)+m'-m=0;$$

that is,

$$(m-m')(Ax+By+C)=0;$$

or finally it is $Ax+By+C=0$, showing that the point Q lies in a line the position of which is independent of the particular lines OAA', OBB' used in the construction. It is proper to notice that there is no correspondence to each other of the points A, A' and B, B'; the grouping might as well have been A, A' and B', B; and it thence appears that the line $Ax+By+C=0$ just obtained is in fact the line joining the point Q with the point R which is the intersection of AB and $A'B'$.

10. The equation $Ax+By+C=0$ of a line contains in appearance 3, but really only 2 constants (for one of the constants can be divided out), and a line depends

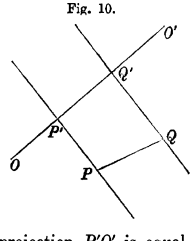
accordingly upon 2 parameters, or can be made to satisfy 2 conditions. Similarly, the equation $(a, b, c, f, g, h \between x, y, 1)^2 = 0$ of a conic contains really 5 constants, and the equation $(* \between x, y, 1)^3 = 0$ of a cubic contains really 9 constants. It thus appears that a cubic can be made to pass through 9 given points, and that the cubic so passing through 9 given points is completely determined. There is, however, a remarkable exception. Considering two given cubic curves $S = 0$, $S' = 0$, these intersect in 9 points, and through these 9 points we have the whole series of cubics $S - kS' = 0$, where k is an arbitrary constant: k may be determined so that the cubic shall pass through a given tenth point, viz. $k = S_0 \div S_0'$, if the coordinates are (x_0, y_0), and S_0, S_0' denote the corresponding values of S, S'. The resulting curve $SS_0' - S'S_0 = 0$ may be regarded as the cubic determined by the conditions of passing through 8 of the 9 points and through the given point (x_0, y_0); and from the equation it thence appears that the curve passes through the remaining one of the 9 points. In other words, we thus have the theorem, any cubic curve which passes through 8 of the 9 intersections of two given cubic curves passes through the 9th intersection.

The applications of this theorem are very numerous; for instance, we derive from it Pascal's theorem of the inscribed hexagon. Consider a hexagon inscribed in a conic. The three alternate sides constitute a cubic, and the other three alternate sides another cubic. The cubics intersect in 9 points, being the 6 vertices of the hexagon, and the 3 Pascalian points, or intersections of the pairs of opposite sides of the hexagon. Drawing a line through two of the Pascalian points, the conic and this line constitute a cubic passing through 8 of the 9 points of intersection, and it therefore passes through the remaining point of intersection—that is, the third Pascalian point; and since obviously this does not lie on the conic, it must lie on the line—that is, we have the theorem that the three Pascalian points (or points of intersection of the pairs of opposite sides) lie on a line.

Metrical Theory.

11. The foundation of the metrical theory consists in the simple theorem that if a finite line PQ (fig. 10) be projected upon any other line OO' by lines perpendicular

Fig. 10.

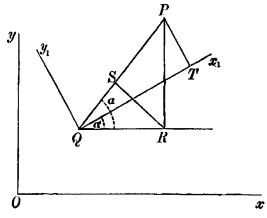


to OO', then the length of the projection $P'Q'$ is equal to the length of PQ multiplied by the cosine of its inclination to $P'Q'$; or, what is the same thing, that the perpendicular distance $P'Q'$ of any two parallel lines is equal to the inclined distance PQ

multiplied by the cosine of the inclination. It at once follows that the algebraical sum of the projections of the sides of a closed polygon upon any line is $=0$; or, reversing the signs of certain sides, and considering the polygon as consisting of two broken lines, each extending from the same initial to the same terminal point, the sum of the projections of the lines of the first set upon any line is equal to the sum of the projections of the lines of the second set. Observe that, if any line be perpendicular to the line on which the projection is made, then its projection is $=0$.

Thus, if we have a right-angled triangle PQR (fig. 11), where QR, RP, QP are

Fig. 11.



$=\xi$, η, ρ respectively, and whereof the base-angle is $=\alpha$, then projecting successively on the three sides, we have

$$\xi=\rho\cos\alpha,\quad \eta=\rho\sin\alpha,\quad \rho=\xi\cos\alpha+\eta\sin\alpha;$$

and we thence obtain

$$\rho^2=\xi^2+\eta^2;\ \cos^2\alpha+\sin^2\alpha=1.$$

And again, by projecting on a line Qx_1, inclined at the angle α' to QR, we have

$$\rho\cos(\alpha-\alpha')=\xi\cos\alpha'+\eta\sin\alpha';$$

and by substituting for ξ, η their foregoing values,

$$\cos(\alpha-\alpha')=\cos\alpha\cos\alpha'+\sin\alpha\sin\alpha'.$$

It is to be remarked that, assuming only the theory of similar triangles, we have herein a *proof* of Euclid, Book I., Prop. 47; in fact, the same as is given Book VI., Prop. 31; and also a *proof* of the trigonometrical formula for $\cos(\alpha-\alpha')$. The formulæ for $\cos(\alpha+\alpha')$ and $\sin(\alpha\pm\alpha')$ could be obtained in the same manner.

Draw PT at right angles to Qx_1, and suppose QT, $TP=\xi_1$, η_1 respectively, so that we have now the quadrilateral $QRPTQ$, or, what is the same thing, the two broken lines QRP and QTP, each extending from Q to P. Projecting on the four sides successively, we have

$$\begin{aligned}\xi &= \xi_1\cos\alpha'-\eta_1\sin\alpha',\\ \eta &= \xi_1\sin\alpha'+\eta_1\cos\alpha',\\ \xi_1 &= \xi\cos\alpha'+\eta\sin\alpha',\\ \eta_1 &= -\xi\sin\alpha'+\eta\cos\alpha',\end{aligned}$$

where the third equation is that previously written

$$\rho\cos(\alpha-\alpha')=\xi\cos\alpha+\eta\sin\alpha.$$

Equations of Right Line and Circle.—Transformation of Coordinates.

12. The required formulæ are really contained in the foregoing results. For, in fig. 11, supposing that the axis of x is parallel to QR, and taking a, b for the coordinates of Q, and (x, y) for those of P, then we have ξ, $\eta = x-a$, $y-b$ respectively; and therefore

$$x-a=\rho\cos\alpha,\quad y-b=\rho\sin\alpha,$$
$$\rho^2 \quad =(x-a)^2+(y-b)^2.$$

Writing the first two of these in the form

$$\frac{x-a}{\cos\alpha}=\frac{y-b}{\sin\alpha}\,(=\rho),$$

we may regard Q as a fixed point, but P as a point moving in the direction Q to P, so that α remains constant, and then, omitting the equation $(=\rho)$, we have a relation between the coordinates x, y of the point P thus moving in a right line,—that is, we have the equation of the line through the given point (a, b) at a given inclination α to the axis of x. And, moreover, if, using this equation $(=\rho)$, we write $x=a+\rho\cos\alpha$, $y=b+\rho\sin\alpha$, then we have expressions for the coordinates x, y of a point of this line, in terms of the variable parameter ρ.

Again, take the point T to be fixed, but consider the point P as moving in the line TP at right angles to QT. If instead of ξ_1 we take p for the distance QT, then the equation $\xi_1=\xi\cos\alpha'+\eta\sin\alpha'$ will be

$$(x-a)\cos\alpha'+(y-b)\sin\alpha'=p;$$

that is, this will be the equation of a line such that its perpendicular distance from the point (a, b) is $=p$, and that the inclination of this distance to the axis of x is $=\alpha'$.

From either form it appears that the equation of a line is, in fact, a linear equation of the form $Ax+By+C=0$. It is important to notice that, starting from this equation, we can determine conversely the α but not the (a, b) of the form of equation which contains these quantities; and in like manner the α' but not the (a, b) or p of the other form of equation. The reason is obvious. In each case (a, b) denote the coordinates of a point, fixed indeed, but which is in the first form any point of the line, and in the second form any point whatever. Thus, in the second form the point from which the perpendicular is let fall may be the origin. Here $(a, b)=(0, 0)$, and the equation is $x\cos\alpha'+y\sin\alpha'-p=0$. Comparing this with $Ax+By+C=0$, we have the values of $\cos\alpha'$, $\sin\alpha'$, and p.

13. The equation

$$\rho^2=(x-a)^2+(y-b)^2$$

is an expression for the squared distance of the two points (a, b) and (x, y). Taking as before the point Q, coordinates (a, b), as a fixed point, and writing c in the place of ρ, the equation

$$(x-a)^2+(y-b)^2=c^2$$

expresses that the point (x, y) is always at a given distance c from the given point (a, b); viz. this is the equation of a circle, having (a, b) for the coordinates of its centre, and c for its radius.

The equation is of the form

$$x^2+y^2+2Ax+2By+C=0,$$

and here, the number of constants being the same, we can identify the two equations; we find $a=-A$, $b=-B$, $c^2=A^2+B^2-C$, or the last equation is that of a circle having $-A$, $-B$ for the coordinates of its centre, and $\sqrt{A^2+B^2-C}$ for its radius.

14. Drawing (fig. 11) Qy_1 at right angles at Qx_1, and taking Qx_1, Qy_1 as a new set of rectangular axes, if instead of ξ_1, η_1 we write x_1, y_1, we have x_1, y_1 as the new coordinates of the point P; and writing also α in place of α', α now denoting the inclination of the axes Qx_1 and Ox, we have the formulæ for transformation between two sets of rectangular axes. These are

$$x-a=x_1\cos\alpha-y_1\sin\alpha,$$
$$y-b=x_1\sin\alpha+y_1\cos\alpha,$$

and

$$x_1 = (x-a)\cos\alpha+(y-b)\sin\alpha,$$
$$y_1 = -(x-a)\sin\alpha+(y-b)\cos\alpha,$$

each set being obviously at once deducible from the other one. In these formulæ (a, b) are the xy-coordinates of the new origin Q_1, and α is the inclination of Qx_1 to Ox. It is to be noticed that Qx_1, Qy_1 are so placed that, by moving O to Q, and then turning the axes Ox_1, Oy_1 round Q (through an angle α measured in the sense Ox to Oy), the original axes Ox, Oy will come to coincide with Qx_1, Qy_1 respectively. This could not have been done if Qy_1 had been drawn (at right angles always to Qx_1) in the reverse direction: we should then have had in the formulæ $-y_1$ instead of y_1. The new formulæ which would be thus obtained are of an essentially distinct form: the analytical test is that in the formulæ as written down we can, by giving to α a proper value (in fact, $\alpha=0$), make the $(x-a)$ and $(y-b)$ equal to x_1 and y_1 respectively; in the other system we could only make them equal to x_1, $-y_1$, or $-x_1$, y_1 respectively. But for the very reason that the second system can be so easily derived from the first, it is proper to attend exclusively to the first system,—that is, always to take the new axes so that the two sets admit of being brought into coincidence.

In the foregoing system of two pairs of equations, the first pair give the original coordinates x, y in terms of the new coordinates x_1, y_1; the second pair the new coordinates x_1, y_1 in terms of the original coordinates x, y. The formulæ involve (a, b), the original coordinates of the new origin; it would be easy, instead of these, to introduce (a_1, b_1), the new coordinates of the origin. Writing $(a, b)=(0, 0)$, we have, of course, the formulæ for transformation between two sets of rectangular axes *having the same origin*, and it is as well to write the formulæ in this more simple form; the subsequent transformation to a new origin, but with axes parallel to the original axes, can then be effected without any difficulty.

15. All questions in regard to the line may be solved by means of one or other of the foregoing forms—

$$Ax + By + C = 0,$$

$$y = Ax + B,$$

$$\frac{x-a}{\cos\alpha} = \frac{y-b}{\sin\alpha},$$

$$(x-a)\cos\alpha' + (y-b)\sin\alpha' - p = 0;$$

or it may be by a comparison of these different forms: thus, using the first form, it has been already shown that the equation of the line through two given points (x_1, y_1), (x_2, y_2) is

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1 = 0,$$

or, as this may be written,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

A particular case is the equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

representing the line through the points $(a, 0)$ and $(0, b)$, or, what is the same thing, the line meeting the axes of x and y at the distances from the origin a and b respectively. It may be noticed that, in the form $Ax + By + C = 0$, $-\frac{A}{B}$ denotes the tangent of the inclination to the axis of x, or we may say that $B \div \sqrt{A^2+B^2}$ and $-A \div \sqrt{A^2+B^2}$ denote respectively the cosine and the sine of the inclination to the axis of x. A better form is this: $A \div \sqrt{A^2+B^2}$ and $B \div \sqrt{A^2+B^2}$ denote respectively the cosine and the sine of inclination to the axis of x of the perpendicular upon the line. So, of course, in regard to the form $y = Ax + B$, A is here the tangent of the inclination to the axis of x; $1 \div \sqrt{A^2+1}$ and $A \div \sqrt{A^2+1}$ are the cosine and sine of this inclination, &c. It thus appears that the condition, in order that the lines $Ax + By + C = 0$ and $A'x + B'y - C' = 0$ may meet at right angles, is $AA' + BB' = 0$; so when the equations are $y = Ax + B$, $y = A'x + B'$, the condition is $AA' + 1 = 0$, or say the value of A' is $= -1 \div A$.

The perpendicular distance of the point (a, b) from the line $Ax + By + C = 0$ is $(Aa + Bb + C) \div \sqrt{A^2+B^2}$. In all the formulæ involving $\sqrt{A^2+B^2}$ or $\sqrt{A^2+1}$, the radical should be written with the sign $\pm$, which is essentially indeterminate: the like indeterminateness of sign presents itself in the expression for the distance of two points $\rho = \pm\sqrt{(x-a)^2 + (y-b)^2}$; if, as before, the points are Q, P, and the indefinite line through these is $z'QPz$, then it is the same thing whether we measure off from Q along this line, considered as drawn from z' towards z, a positive distance k, or along the line considered as drawn reversely from z towards z', the equal negative distance $-k$, and the expression for the distance ρ is thus properly of the form $\pm k$. It is interesting to compare expressions which do not involve a radical: thus, in

seeking for the expression for the perpendicular distance of the point (a, b) from a given line, let the equation of the given line be taken in the form, $x\cos\alpha + y\sin\alpha - p = 0$, p being the perpendicular distance from the origin, α its inclination to the axis of x: the equation of the line may also be written $(x-a)\cos\alpha + (y-b)\sin\alpha - p_1 = 0$, and we have thence $p_1 = p - a\cos\alpha - b\sin\alpha$, the required expression for the distance p_1: it is here assumed that p_1 is drawn from (a, b) in the same sense as p is drawn from the origin, and the indeterminateness of sign is thus removed.

16. As an instance of the mode of using the formulæ, take the problem of finding the locus of a point such that its distance from a given point is in a given ratio to its distance from a given line.

We take (a, b) as the coordinates of the given point, and it is convenient to take (x, y) as the coordinates of the variable point, the locus of which is required: it thus becomes necessary to use other letters, say (X, Y), for current coordinates in the equation of the given line. Suppose this is a line such that its perpendicular distance from the origin is $=p$, and that the inclination of p to the axis of x is $=\alpha$; the equation is $X\cos\alpha + Y\sin\alpha - p = 0$. In the result obtained in § 15, writing (x, y) in place of (a, b), it appears that the perpendicular distance of this line from the point (x, y) is

$$= p - x\cos\alpha - y\sin\alpha;$$

hence the equation of the locus is

$$\sqrt{(x-a)^2 + (y-b)^2} = e(p - x\cos\alpha - y\sin\alpha),$$

or say

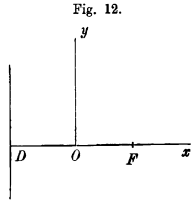
$$(x-a)^2 + (y-b)^2 - e^2(x\cos\alpha + y\sin\alpha - p)^2 = 0,$$

an equation of the second order.

The Conics (Parabola, Ellipse, Hyperbola).

17. The conics or, as they were called, conic sections were originally defined as the sections of a right circular cone; but Apollonius substituted a definition, which is, in fact, that of the last example: the curve is the locus of a point such that its

Fig. 12.



distance from a given point (called the focus) is in a given ratio to its distance from a given line (called the directrix); and taking the ratio as $e : 1$, then e is called the eccentricity.

Take FD for the perpendicular from the focus F upon the directrix, and the given ratio being that of $e : 1$ ($e >, =,$ or < 1, but positive), and let the distance FD

be divided at O in the given ratio, say we have $OD = m$, $OF = em$, where m is positive;—then the origin may be taken at O, the axis Ox being in the direction OF (that is, from O to F), and the axis Oy at right angles to it. The distance of the point (x, y) from F is $= \sqrt{(x - em)^2 + y^2}$, its distance from the directrix is $= x + m$; the equation therefore is

$$(x - em)^2 + y^2 = e^2 (x + m)^2;$$

or, what is the same thing, it is

$$(1 - e^2) x^2 - 2me (1 + e) x + y^2 = 0.$$

If $e^2 = 1$, or, since e is taken to be positive, if $e = 1$, this is

$$y^2 - 4mx = 0,$$

which is the parabola.

If e^2 not $= 1$, then the equation may be written

$$(1 - e^2) \left(x - \frac{me}{1 - e}\right)^2 + y^2 = \frac{m^2 e^2 (1 + e)}{1 - e}.$$

Supposing e positive and < 1, then, writing $m = \frac{a(1 - e)}{e}$, the equation becomes

$$(1 - e^2)(x - a)^2 + y^2 = a^2 (1 - e^2),$$

that is,

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1;$$

or, changing the origin and writing $b^2 = a^2 (1 - e^2)$, this is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the ellipse.

And similarly if e be positive and > 1, then writing $m = \frac{a(e - 1)}{e}$, the equation becomes

$$(1 - e^2)(x + a)^2 + y^2 = a^2 (1 - e^2),$$

that is,

$$\frac{(x + a)^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1,$$

or changing the origin and writing $b^2 = a^2 (e^2 - 1)$, this is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is the hyperbola.

18. The general equation $ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0$, or as it is written $(a, b, c, f, g, h)(x, y, 1)^2 = 0$, may be such that the quadric function breaks up into factors, $= (\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma')$; and in this case the equation represents a pair of lines, or (it may be) two coincident lines. When it does not so break up, the function can be put in the form $\lambda \{(x - a')^2 + (y - b')^2 - e^2 (x \cos \alpha + y \sin \alpha - p)^2\}$, or, equating the two expressions, there will be six equations for the determination of λ, a', b', e, p, α; and by what precedes, if a', b', e, p, α are real, the curve is either

a parabola, ellipse, or hyperbola. The original coefficients (a, b, c, f, g, h) may be such as not to give any system of real values for a', b', e, p, α; but when this is so the equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ does not represent a real curve*; the imaginary curve which it represents is, however, regarded as a conic. Disregarding the special cases of the pair of lines and the twice repeated line, it thus appears that the only real curves represented by the general equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ are the parabola, the ellipse, and the hyperbola. The circle is considered as a particular case of the ellipse.

The same result is obtained by transforming the equation $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ to new axes. If in the first place the origin be unaltered, then the directions of the new (rectangular) axes Ox_1, Oy_1 can be found so that h_1 (the coefficient of the term x_1y_1) shall be $=0$; when this is done, then either one of the coefficients of x_1^2, y_1^2 is $=0$, and the curve is then a parabola, or neither of these coefficients is $=0$, and the curve is then an ellipse or hyperbola, according as the two coefficients are of the same sign or of opposite signs.

19. The curves can be at once traced from their equations:—

$$y^2 = 4mx, \text{ for the parabola (fig. 13)},$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ for the ellipse (fig. 14)},$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ for the hyperbola (fig. 15)};$$

Fig. 13.

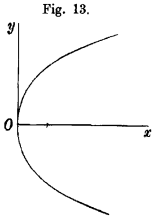


Fig. 14.

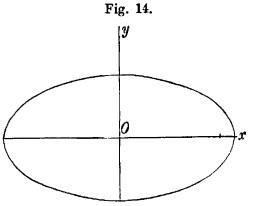
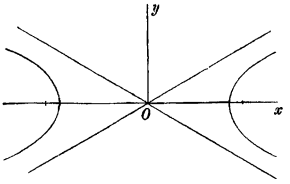


Fig. 15.



* It is proper to remark that, when $(a, b, c, f, g, h)(x, y, 1)^2 = 0$ *does* represent a real curve, there are, in fact, four systems of values of a', b', e, p, α, two real, the other two imaginary; we have thus two real equations and two imaginary equations, each of them of the form $(x-a')^2 + (y-b')^2 = e^2(x\cos\alpha + y\cos\beta - p)^2$, representing each of them one and the same real curve. This is consistent with the assertion of the text that the real curve is in every case represented by a real equation of this form.

and it will be noticed how the form of the last equation puts in evidence the two asymptotes $\frac{x}{a} = \pm \frac{y}{b}$ of the hyperbola. Referred to the asymptotes (as a set of oblique axes) the equation of the hyperbola takes the form $xy = c$; and in particular, if in this equation the axes are at right angles, then the equation represents the rectangular hyperbola referred to its asymptotes as axes.

Tangent, Normal, Circle and Radius of Curvature, &c.

20. There is great convenience in using the language and notation of the infinitesimal analysis; thus we consider on a curve a point with coordinates (x, y), and a consecutive point the coordinates of which are $(x + dx, y + dy)$, or again a second consecutive point with coordinates $(x + dx + \frac{1}{2}d^2x, y + dy + \frac{1}{2}d^2y)$, &c.; and in the final results the ratios of the infinitesimals must be replaced by differential coefficients in the proper manner; thus, if x, y are considered as given functions of a parameter θ, then dx, dy have in fact the values $\frac{dx}{d\theta}d\theta$, $\frac{dy}{d\theta}d\theta$, and (only the ratio being really material) they may in the result be replaced by $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$. This includes the case where the equation of the curve is given in the form $y = \phi(x)$; θ is here $= x$, and the increments dx, dy are in the result to be replaced by 1, $\frac{dy}{dx}$. So also with the infinitesimals of the higher orders d^2x, &c.

21. The tangent at the point (x, y) is the line through this point and the consecutive point $(x + dx, y + dy)$; hence, taking ξ, η as current coordinates, the equation is

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy},$$

an equation which is satisfied on writing therein $\xi, \eta = (x, y)$ or $= (x + dx, y + dy)$. The equation may be written

$$\eta - y = \frac{dy}{dx}(\xi - x),$$

$\frac{dy}{dx}$ being now the differential coefficient of y in regard to x; and this form is applicable whether y is given directly as a function of x, or in whatever way y is in effect given as a function of x: if as before x, y are given each of them as a function of θ, then the value of $\frac{dy}{dx}$ is $= \frac{dy}{d\theta} \div \frac{dx}{d\theta}$, which is the result obtained from the original form on writing therein $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, for dx, dy respectively.

So again, when the curve is given by an equation $u = 0$ between the coordinates (x, y), then $\frac{dy}{dx}$ is obtained from the equation $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0$. But here it is more

elegant, using the original form, to eliminate dx, dy by the formula $\frac{du}{dx}dx+\frac{du}{dy}dy$; we thus obtain the equation of the tangent in the form

$$\frac{du}{dx}(\xi-x)+\frac{du}{dy}(\eta-y)=0.$$

For example, in the case of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation is

$$\frac{x}{a^2}(\xi-x)+\frac{y}{b^2}(\eta-y)=0;$$

or reducing by means of the equation of the curve the equation of the tangent is

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}=1.$$

The normal is a line through the point at right angles to the tangent; the equation therefore is

$$(\xi-x)\,dx+(\eta-y)\,dy=0,$$

where dx, dy are to be replaced by their proportional values as before.

22. The circle of curvature is the circle through the point and two consecutive points of the curve. Taking the equation to be

$$(\xi-\alpha)^2+(\eta-\beta)^2=\gamma^2,$$

the values of α, β are given by

$$x-\alpha=\frac{dy\,(dx^2+dy^2)}{dx\,d^2y-dy\,d^2x},\quad y-\beta=\frac{-dx\,(dx^2+dy^2)}{dx\,d^2y-dy\,d^2x},$$

and we then have

$$\gamma^2,\ =(x-\alpha)^2+(y-\beta)^2,\ =\frac{(dx^2+dy^2)^3}{(dx\,d^2y-dy\,d^2x)^2}.$$

In the case where y is given directly as a function of x, then, writing for shortness $p=\frac{dy}{dx}$, $q=\frac{d^2y}{dx^2}$, this is $\gamma^2=\frac{(1+p^2)^3}{q^2}$, or, as the equation is usually written, $\gamma=\frac{(1+p^2)^{\frac{3}{2}}}{-q}$, the radius of curvature, considered to be positive or negative according as the curve is concave or convex to the axis of x.

It may be added that the centre of curvature is the intersection of the normal by the consecutive normal.

The locus of the centre of curvature is the evolute. If from the expressions of α, β regarded as functions of x we eliminate x, we have thus an equation between (α, β), which is the equation of the evolute.

Polar Coordinates.

23. The position of a point may be determined by means of its distance from a fixed point and the inclination of this distance to a fixed line through the fixed point. Say we have r the distance from the origin, and θ the inclination of r to the axis of x; r and θ are then the polar coordinates of the point, r the radius vector, and θ the inclination. These are immediately connected with the Cartesian coordinates x, y by the formulæ $x = r\cos\theta$, $y = r\sin\theta$; and the transition from either set of coordinates to the other can thus be made without difficulty. But the use of polar coordinates is very convenient, as well in reference to certain classes of questions relating to curves of any kind—for instance, in the dynamics of central forces—as in relation to curves having in regard to the origin the symmetry of the regular polygon (curves such as that represented by the equation $r = \cos m\theta$), and also in regard to the class of curves called spirals, where the radius vector r is given as an algebraical or exponential function of the inclination θ.

Trilinear Coordinates.

24. Consider a fixed triangle ABC, and (regarding the sides as indefinite lines) suppose for a moment that p, q, r denote the distances of a point P from the sides BC, CA, AB respectively,—these distances being measured either perpendicularly to the several sides, or each of them in a given direction. To fix the ideas each distance may be considered as positive for a point inside the triangle, and the sign is thus fixed for any point whatever. There is then an identical relation between p, q, r: if a, b, c are the lengths of the sides, and the distances are measured perpendicularly thereto, the relation is $ap + bq + cr =$ twice the area of triangle. But taking x, y, z proportional to p, q, r, or if we please proportional to given multiples of p, q, r, then only the ratios of x, y, z are determined; their absolute values remain arbitrary. But the ratios of p, q, r, and consequently also the ratios of x, y, z determine, and that uniquely, the point; and it being understood that only the ratios are attended to, we say that (x, y, z) are the coordinates of the point. The equation of a line has thus the form $ax + by + cz = 0$, and generally that of a curve of the nth order is a homogeneous equation of this order between the coordinates, $(*\between x, y, z)^n = 0$. The advantage over Cartesian coordinates is in the greater symmetry of the analytical forms, and in the more convenient treatment of the line infinity and of points at infinity. The method includes that of Cartesian coordinates, the homogeneous equation in x, y, z is, in fact, an equation in $\frac{x}{z}, \frac{y}{z}$, which two quantities may be regarded as denoting Cartesian coordinates; or, what is the same thing, we may in the equation write $z = 1$. It may be added that, if the trilinear coordinates (x, y, z) are regarded as the Cartesian coordinates of a point of space, then the equation is that of a cone having the origin for its vertex; and conversely that such equation of a cone may be regarded as the equation in trilinear coordinates of a plane curve.

General Point-Coordinates.—Line-Coordinates.

25. All the coordinates considered thus far are point-coordinates. More generally, any two quantities (or the ratios of three quantities) serving to determine the position

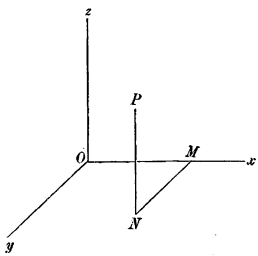
of a point in the plane may be regarded as the coordinates of the point; or, if instead of a single point they determine a system of two or more points, then as the coordinates of the system of points. But, as noticed under Curve, [785], there are also line-coordinates serving to determine the position of a line; the ordinary case is when the line is determined by means of the ratios of three quantities ξ, η, ζ (correlative to the trilinear coordinates x, y, z). A linear equation $a\xi + b\eta + c\zeta = 0$ represents then the system of lines such that the coordinates of each of them satisfy this relation, in fact, all the lines which pass through a given point; and it is thus regarded as the line-equation of this point; and generally a homogeneous equation $(*\between\xi, \eta, \zeta)^n = 0$ represents the curve which is the envelope of all the lines the coordinates of which satisfy this equation, and it is thus regarded as the line-equation of this curve.

II. Solid Analytical Geometry (§§ 26—40).

26. We are here concerned with points in space,—the position of a point being determined by its three coordinates x, y, z. We consider three coordinate planes, at right angles to each other, dividing the whole of space into eight portions called octants, the coordinates of a point being the perpendicular distances of the point from the three planes respectively, each distance being considered as positive or negative according as it lies on the one or the other side of the plane. Thus the coordinates in the eight octants have respectively the signs

x,	y,	z
+	+	+
+	−	+
−	+	+
−	−	+
+	+	−
+	−	−
−	+	−
−	−	−

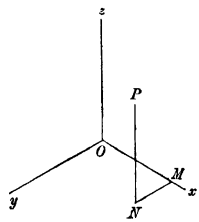
Fig. 16.



The positive parts of the axes are usually drawn as in fig. 16, which represents a point P, the coordinates of which have the positive values OM, MN, NP.

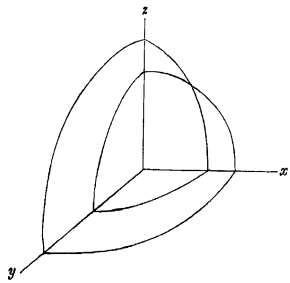
27. It may be remarked, as regards the delineation of such solid figures, that if we have in space three lines at right angles to each other, say Oa, Ob, Oc, of equal lengths, then it is possible to project these by parallel lines upon a plane in such wise that the projections Oa', Ob', Oc' shall be at given inclinations to each other, and that these lengths shall be to each other in given ratios: in particular, the two lines Oa', Oc' may be at right angles to each other, and their lengths equal, the direction of Ob', and its proportion to the two equal lengths Oa', Oc', being arbitrary. It thus appears that we may as in the figure draw Ox, Oz at right angles to each other, and Oy in an arbitrary direction; and moreover represent the coordinates x, z on equal scales, and the remaining coordinate y on an arbitrary scale (which may be that of the other two coordinates x, z, but is in practice usually smaller). The advantage, of course, is that a figure in *one* of the coordinate planes xz is represented in its proper form without distortion; but it may be in some cases preferable to employ the isometrical projection, wherein the three axes are represented by lines inclined to each other at angles of 120°, and the scales for the coordinates are equal (fig. 17).

Fig. 17.



For the delineation of a surface of a tolerably simple form, it is frequently sufficient to draw (according to the foregoing projection) the sections by the coordinate planes; and in particular, when the surface is symmetrical in regard to the

Fig. 18.



coordinate planes, it is sufficient to draw the quarter-sections belonging to a single octant of the surface; thus fig. 18 is a convenient representation of an octant of the

wave surface. Or a surface may be delineated by means of a series of parallel sections, or (taking these to be the sections by a series of horizontal planes) say by a series of contour lines. Of course, other sections may be drawn or indicated, if necessary. For the delineation of a curve, a convenient method is to represent, as above, a series of the points P thereof, each point P being accompanied by the ordinate PN, which serves to refer the point to the plane of xy; this is in effect a representation of each point P of the curve, by means of two points P, N such that the line PN has a fixed direction. Both as regards curves and surfaces, the employment of stereographic representations is very interesting.

28. In plane geometry, reckoning the line as a curve of the first order, we have only the point and the curve. In solid geometry, reckoning a line as a curve of the first order, and the plane as a surface of the first order, we have the point, the curve, and the surface; but the increase of complexity is far greater than would hence at first sight appear. In plane geometry a curve is considered in connexion with lines (its tangents); but in solid geometry the curve is considered in connexion with lines and planes (its tangents and osculating planes), and the surface also in connexion with lines and planes (its tangent lines and tangent planes); there are surfaces arising out of the line—cones, skew surfaces, developables, doubly and triply infinite systems of lines, and whole classes of theories which have nothing analogous to them in plane geometry: it is thus a very small part indeed of the subject which can be even referred to in the present article.

In the case of a surface, we have between the coordinates (x, y, z) a single, or say a onefold relation, which can be represented by a single relation $f(x, y, z)=0$; or we may consider the coordinates expressed each of them as a given function of two variable parameters p, q; the form $z=f(x, y)$ is a particular case of each of these modes of representation; in other words, we have in the first mode $f(x, y, z)=z-f(x, y)$, and in the second mode $x=p$, $y=q$ for the expression of two of the coordinates in terms of the parameters.

In the case of a curve, we have between the coordinates (x, y, z) a twofold relation: two equations $f(x, y, z)=0$, $\phi(x, y, z)=0$ give such a relation; i.e., the curve is here considered as the intersection of two surfaces (but the curve is not always the complete intersection of two surfaces, and there are hence difficulties); or, again, the coordinates may be given each of them as a function of a single variable parameter. The form $y=\phi x$, $z=\psi x$, where two of the coordinates are given in terms of the third, is a particular case of each of these modes of representation.

29. The remarks under plane geometry as to descriptive and metrical propositions, and as to the non-metrical character of the method of coordinates when used for the proof of a descriptive proposition, apply also to solid geometry; and they might be illustrated in like manner by the instance of the theorem of the radical centre of four spheres. The proof is obtained from the consideration that S and S' being each of them a function of the form $x^2+y^2+z^2+ax+by+cz+d$, the difference $S-S'$ is a mere linear function of the coordinates, and consequently that $S-S'=0$ is the equation of the plane containing the circle of intersection of the two spheres $S=0$ and $S'=0$.

Metrical Theory.

30. The foundation in solid geometry of the metrical theory is, in fact, the before-mentioned theorem that, if a finite right line PQ be projected upon any other line OO' by lines perpendicular to OO', then the length of the projection $P'Q'$ is equal to the length of PQ multiplied by the cosine of its inclination to $P'Q'$—or (in the form in which it is now convenient to state the theorem) the perpendicular distance $P'Q'$ of two parallel planes is equal to the inclined distance PQ into the cosine of the inclination. Hence also the algebraical sum of the projections of the sides of a closed polygon upon any line is $=0$; or, reversing the signs of certain sides and considering the polygon as made up of two broken lines each extending from the same initial to the same terminal point, the sum of the projections of the one set of lines upon any line is equal to the sum of the projections of the other set of lines upon the same line. When any of the lines are at right angles to the given line (or, what is the same thing, in a plane at right angles to the given line), the projections of these lines severally vanish.

31. Consider the skew quadrilateral $QMNP$, the sides QM, MN, NP being respectively parallel to the three rectangular axes Ox, Oy, Oz; let the lengths of these sides be ξ, η, ζ, and that of the side QP be $=\rho$; and let the cosines of the inclinations (or say the cosine-inclinations) of ρ to the three axes be α, β, γ; then projecting successively on the three sides and on QP, we have

$$\xi,\ \eta,\ \zeta = \rho\alpha,\ \rho\beta,\ \rho\gamma,$$

and

$$\rho = \alpha\xi + \beta\eta + \gamma\zeta,$$

whence $\rho^2 = \xi^2 + \eta^2 + \zeta^2$, which is the relation between a distance ρ and its projections ξ, η, ζ upon three rectangular axes. And from the same equations we obtain $\alpha^2 + \beta^2 + \gamma^2 = 1$, which is a relation connecting the cosine-inclinations of a line to three rectangular axes.

Suppose we have through Q any other line QT, and let the cosine-inclinations of this to the axes be α', β', γ', and δ be its cosine-inclination to QP; also let p be the length of the projection of QP upon QT; then projecting on QT, we have

$$p = \alpha'\xi + \beta'\eta + \gamma'\zeta, = \rho\delta.$$

And in the last equation substituting for ξ, η, ζ their values $\rho\alpha$, $\rho\beta$, $\rho\gamma$, we find

$$\delta = \alpha\alpha' + \beta\beta' + \gamma\gamma',$$

which is an expression for the mutual cosine-inclination of two lines, the cosine-inclinations of which to the axes are α, β, γ and α', β', γ' respectively. We have of course $\alpha^2 + \beta^2 + \gamma^2 = 1$, and $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$, and hence also

$$\begin{aligned} 1 - \delta^2 &= (\alpha^2 + \beta^2 + \gamma^2)(\alpha'^2 + \beta'^2 + \gamma'^2) - (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2 \\ &= (\beta\gamma' - \beta'\gamma)^2 + (\gamma\alpha' - \gamma'\alpha)^2 + (\alpha\beta' - \alpha'\beta)^2; \end{aligned}$$

so that the sine of the inclination can only be expressed as a square root. These formulæ are the foundation of spherical trigonometry.

The Line, Plane, and Sphere.

32. The foregoing formulæ give at once the equations of these loci.

For first, taking Q to be a fixed point, coordinates (a, b, c) and the cosine-inclinations (α, β, γ) to be constant, then P will be a point in the line through Q in the direction thus determined; or, taking (x, y, z) for its coordinates, these will be the current coordinates of a point in the line. The values of ξ, η, ζ then are $x-a$, $y-b$, $z-c$, and we thus have

$$\frac{x-a}{\alpha}=\frac{y-b}{\beta}=\frac{z-c}{\gamma}(=\rho),$$

which (omitting the last equation, $=\rho$) are the equations of the line through the point (a, b, c), the cosine-inclinations to the axes being α, β, γ, and these quantities being connected by the relation $\alpha^2+\beta^2+\gamma^2=1$. This equation may be omitted, and then α, β, γ, instead of being equal, will only be proportional to the cosine-inclinations.

Using the last equation, and writing

$$x,\ y,\ z=a+\alpha\rho,\ \ b+\beta\rho,\ \ c+\gamma\rho,$$

these are expressions for the current coordinates in terms of a parameter ρ, which is in fact the distance from the fixed point (a, b, c).

It is easy to see that, if the coordinates (x, y, z) are connected by any two linear equations, these equations can always be brought into the foregoing form, and hence that the two linear equations represent a line.

Secondly, taking for greater simplicity the point Q to be coincident with the origin, and $\alpha', \beta', \gamma', p$ to be constant, then p is the perpendicular distance of a plane from the origin, and α', β', γ' are the cosine-inclinations of this distance to the axes $(\alpha'^2+\beta'^2+\gamma'^2=1)$. Now P is any point in this plane; taking its coordinates to be (x, y, z), then (ξ, η, ζ) are $=(x, y, z)$, and the foregoing equation $p=\alpha'\xi+\beta'\eta+\gamma'\zeta$ becomes

$$\alpha'x+\beta'y+\gamma'z=p,$$

which is the equation of the plane in question.

If, more generally, Q is not coincident with the origin, then, taking its coordinates to be (a, b, c), and writing p_1 instead of p, the equation is

$$\alpha'(x-a)+\beta'(y-b)+\gamma'(z-c)=p_1;$$

and we thence have $p_1=p-(a\alpha'+b\beta'+c\gamma')$, which is an expression for the perpendicular distance of the point (a, b, c) from the plane in question.

It is obvious that any linear equation $Ax+By+Cz+D=0$ between the coordinates can always be brought into the foregoing form, and hence that such equation represents a plane.

Thirdly, supposing Q to be a fixed point, coordinates (a, b, c) and the distance QP, $=\rho$, to be constant, say this is $=d$, then, as before, the values of ξ, η, ζ are $x-a$, $y-b$, $z-c$, and the equation $\xi^2+\eta^2+\zeta^2=\rho^2$ becomes

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2,$$

which is the equation of the sphere, coordinates of the centre $=(a, b, c)$ and radius $=d$.

A quadric equation wherein the terms of the second order are $x^2+y^2+z^2$, viz. an equation

$$x^2+y^2+z^2+Ax+By+Cz+D=0,$$

can always, it is clear, be brought into the foregoing form; and it thus appears that this is the equation of a sphere, coordinates of the centre $-\frac{1}{2}A$, $-\frac{1}{2}B$, $-\frac{1}{2}C$, and squared radius $=\frac{1}{4}(A^2+B^2+C^2)-D$.

Cylinders, Cones, Ruled Surfaces.

33. A singly infinite system of lines, or a system of lines depending upon one variable parameter, forms a surface; and the equation of the surface is obtained by eliminating the parameter between the two equations of the line.

If the lines all pass through a given point, then the surface is a cone; and, in particular, if the lines are all parallel to a given line, then the surface is a cylinder.

Beginning with this last case, suppose the lines are parallel to the line $x=mz$, $y=nz$, the equations of a line of the system are $x=mz+a$, $y=nz+b$,—where a, b are supposed to be functions of the variable parameter, or, what is the same thing, there is between them a relation $f(a, b)=0$: we have $a=x-mz$, $b=y-nz$, and the result of the elimination of the parameter therefore is $f(x-mz, y-nz)=0$, which is thus the general equation of the cylinder the generating lines whereof are parallel to the line $x=mz$, $y=nz$. The equation of the section by the plane $z=0$ is $f(x, y)=0$, and conversely if the cylinder be determined by means of its curve of intersection with the plane $z=0$, then, taking the equation of this curve to be $f(x, y)=0$, the equation of the cylinder is $f(x-mz, y-nz)=0$. Thus, if the curve of intersection be the circle $(x-\alpha)^2+(y-\beta)^2=\gamma^2$, we have $(x-mz-\alpha)^2+(y-nz-\beta)^2=\gamma^2$ as the equation of an oblique cylinder on this base, and thus also $(x-\alpha)^2+(y-\beta)^2=\gamma^2$ as the equation of the right cylinder.

If the lines all pass through a given point (a, b, c), then the equations of a line are $x-a=\alpha(z-c)$, $y-b=\beta(z-c)$, where α, β are functions of the variable parameter, or, what is the same thing, there exists between them an equation $f(\alpha, \beta)=0$; the elimination of the parameter gives, therefore, $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)=0$; and this equation, or, what is the same thing, any homogeneous equation $f(x-a, y-b, z-c)=0$, or, taking f to be a rational and integral function of the order n, say $(*)(x-a, y-b, z-c)^n=0$, is the general equation of the cone having the point (a, b, c) for its vertex. Taking the vertex to be at the origin, the equation is $(*)(x, y, z)^n=0$; and, in particular, $(*)(x, y, z)^2=0$ is the equation of a cone of the second order, or quadricone, having the origin for its vertex.

34. In the general case of a singly infinite system of lines, the locus is a ruled surface (or *regulus*). If the system be such that a line does not intersect the consecutive line, then the surface is a skew surface, or scroll; but if it be such that each line intersects the consecutive line, then it is a developable, or torse.

Suppose, for instance, that the equations of a line (depending on the variable parameter θ) are

$$\frac{x}{a}+\frac{z}{c}=\theta\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\theta}\left(1-\frac{y}{b}\right),$$

then, eliminating θ, we have $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$, or say $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, the equation of a quadric surface, afterwards called the hyperboloid of one sheet; this surface is consequently a scroll. It is to be remarked that we have upon the surface a second singly infinite series of lines; the equations of a line of this second system (depending on the variable parameter ϕ) are

$$\frac{x}{a}+\frac{z}{c}=\phi\left(1-\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\phi}\left(1+\frac{y}{b}\right).$$

It is easily shown that any line of the one system intersects every line of the other system.

Considering any curve (of double curvature) whatever, the tangent lines of the curve form a singly infinite system of lines, each line intersecting the consecutive line of the system,—that is, they form a developable, or torse; the curve and torse are thus inseparably connected together, forming a single geometrical figure. A plane through three consecutive points of the curve (or osculating plane of the curve) contains two consecutive tangents, that is, two consecutive lines of the torse, and is thus a tangent plane of the torse along a generating line.

Transformation of Coordinates.

35. There is no difficulty in changing the origin, and it is for brevity assumed that the origin remains unaltered. We have, then, two sets of rectangular axes, Ox, Oy, Oz, and Ox_1, Oy_1, Oz_1, the mutual cosine-inclinations being shown by the diagram—

	x	y	z
x_1	α	β	γ
y_1	α'	β'	γ'
z_1	α''	β''	γ''

that is, α, β, γ are the cosine-inclinations of Ox_1 to Ox, Oy, Oz; α', β', γ' those of Oy_1, &c.

And this diagram gives also the linear expressions of the coordinates (x_1, y_1, z_1) or (x, y, z) of either set in terms of those of the other set; we thus have

$$\begin{aligned} x_1 &= \alpha\, x + \beta\, y + \gamma\, z, & x &= \alpha x_1 + \alpha' y_1 + \alpha'' z_1, \\ y_1 &= \alpha' x + \beta' y + \gamma' z, & y &= \beta x_1 + \beta' y_1 + \beta'' z_1, \\ z_1 &= \alpha'' x + \beta'' y + \gamma'' z, & z &= \gamma x_1 + \gamma' y_1 + \gamma'' z_1, \end{aligned}$$

which are obtained by projection, as above explained. Each of these equations is, in fact, nothing else than the before-mentioned equation $p = \alpha'\xi + \beta'\eta + \gamma'\zeta$, adapted to the problem in hand.

But we have to consider the relations between the nine coefficients. By what precedes, or by the consideration that we must have identically $x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2$, it appears that these satisfy the relations—

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1, & \alpha^2 + \alpha'^2 + \alpha''^2 &= 1, \\ \alpha'^2 + \beta'^2 + \gamma'^2 &= 1, & \beta^2 + \beta'^2 + \beta''^2 &= 1, \\ \alpha''^2 + \beta''^2 + \gamma''^2 &= 1, & \gamma^2 + \gamma'^2 + \gamma''^2 &= 1, \\ \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' &= 0, & \beta\gamma + \beta'\gamma' + \beta''\gamma'' &= 0, \\ \alpha''\alpha + \beta''\beta + \gamma''\gamma &= 0, & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' &= 0, \\ \alpha\alpha' + \beta\beta' + \gamma\gamma' &= 0, & \alpha\beta + \alpha'\beta' + \alpha''\beta'' &= 0, \end{aligned}$$

either set of six equations being implied in the other set.

It follows that the square of the determinant

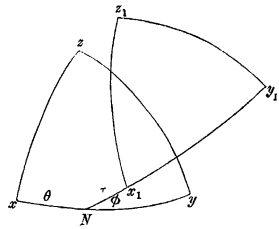
$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}$$

is $=1$; and hence that the determinant itself is $=\pm 1$. The distinction of the two cases is an important one: if the determinant is $=+1$, then the axes Ox_1, Oy_1, Oz_1 are such that they can by a rotation about O be brought to coincide with Ox, Oy, Oz respectively; if it is $=-1$, then they cannot. But in the latter case, by measuring x_1, y_1, z_1 in the opposite directions we change the signs of all the coefficients and so make the determinant to be $=+1$; hence this case need alone be considered, and it is accordingly assumed that the determinant is $=+1$. This being so, it is found that we have a further set of nine equations, $\alpha = \beta'\gamma'' - \beta''\gamma'$, &c.; that is, the coefficients arranged as in the diagram have the values

$\beta'\gamma'' - \beta''\gamma'$	$\gamma'\alpha'' - \gamma''\alpha'$	$\alpha'\beta'' - \alpha''\beta'$
$\beta''\gamma - \beta\gamma''$	$\gamma''\alpha - \gamma\alpha''$	$\alpha''\beta - \alpha\beta''$
$\beta\gamma' - \beta'\gamma$	$\gamma\alpha' - \gamma'\alpha$	$\alpha\beta' - \alpha'\beta$

36. It is important to express the nine coefficients in terms of three independent quantities. A solution which, although unsymmetrical, is very convenient in Astronomy and Dynamics is to use for the purpose the three angles θ, ϕ, τ of fig. 19; say $\theta =$ longitude of the node; $\phi =$ inclination; and $\tau =$ longitude of x_1 from node.

Fig. 19.



The diagram of transformation then is

	x	y	z
x_1	$\cos\tau\cos\theta - \sin\tau\sin\theta\cos\phi$	$\cos\tau\sin\theta + \sin\tau\cos\theta\cos\phi$	$\sin\tau\sin\phi$
y_1	$-\sin\tau\cos\theta - \cos\tau\sin\theta\cos\phi$	$-\sin\tau\sin\theta + \cos\tau\cos\theta\cos\phi$	$\cos\tau\sin\phi$
z_1	$\sin\theta\sin\phi$	$-\cos\theta\sin\phi$	$\cos\phi$

But a more elegant solution (due to Rodrigues) is that contained in the diagram

	x	y	z
x_1	$1+\lambda^2-\mu^2-\nu^2$	$2(\lambda\mu-\nu)$	$2(\lambda\nu+\mu)$
y_1	$2(\lambda\mu+\nu)$	$1-\lambda^2+\mu^2-\nu^2$	$2(\mu\nu-\lambda)$
z_1	$2(\nu\lambda-\mu)$	$2(\mu\nu+\lambda)$	$1-\lambda^2-\mu^2+\nu^2$

$$\div(1+\lambda^2+\mu^2+\nu^2).$$

The nine coefficients of transformation are the nine functions of the diagram, each divided by $1+\lambda^2+\mu^2+\nu^2$; the expressions contain as they should do the three arbitrary

quantities λ, μ, ν; and the identity $x_1^2+y_1^2+z_1^2=x^2+y^2+z^2$ can be at once verified. It may be added that the transformation can be expressed in the quaternion form

$$ix_1+jy_1+kz_1=(1+\Lambda)(ix+jy+kz)(1+\Lambda)^{-1},$$

where Λ denotes the vector $i\lambda+j\mu+k\nu$.

Quadric Surfaces (Paraboloids, Ellipsoid, Hyperboloids).

37. It appears, by a discussion of the general equation of the second order $(a,\ldots\between x, y, z, 1)^2=0$, that the proper quadric surfaces * represented by such an equation are the following five surfaces (a and b positive):—

(1) $z=\dfrac{x^2}{2a}+\dfrac{y^2}{2b}$, elliptic paraboloid.

(2) $z=\dfrac{x^2}{2a}-\dfrac{y^2}{2b}$, hyperbolic paraboloid.

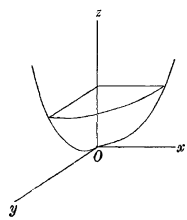
(3) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, ellipsoid.

(4) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, hyperboloid of one sheet.

(5) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=-1$, hyperboloid of two sheets.

It is at once seen that these are distinct surfaces; and the equations also show very readily the general form and mode of generation of the several surfaces.

Fig. 20.



In the elliptic paraboloid (fig. 20), the sections by the planes of zx and zy are the parabolas

$$z=\frac{x^2}{2a},\quad z=\frac{y^2}{2b},$$

* The improper quadric surfaces represented by the general equation of the second order are (1) the pair of planes or plane-pair, including as a special case the twice repeated plane, and (2) the cone, including as a special case the cylinder. There is but one form of cone; but the cylinder may be parabolic, elliptic, or hyperbolic.

having the common axis Oz; and the section by any plane $z=\gamma$ parallel to that of xy is the ellipse

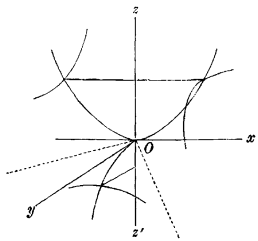
$$\gamma=\frac{x^2}{2a}+\frac{y^2}{2b};$$

so that the surface is generated by a variable ellipse moving parallel to itself along the parabolas as directrices.

In the hyperbolic paraboloid (fig. 21), the sections by the planes of zx, zy are the parabolas

$$z=\frac{x^2}{2a},\quad z=-\frac{y^2}{2b},$$

Fig. 21.

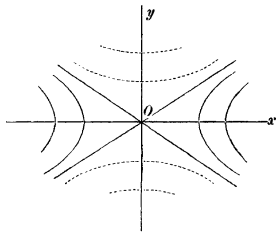


having the opposite axes Oz, Oz'; and the section by a plane $z=\gamma$ parallel to that of xy is the hyperbola

$$\gamma=\frac{x^2}{2a}-\frac{y^2}{2b},$$

which has its transverse axis parallel to Ox or Oy according as γ is positive or negative. The surface is thus generated by a variable hyperbola moving parallel to

Fig. 22.

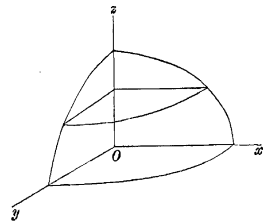


itself along the parabolas as directrices. The form is best seen from fig. 22, which represents the sections by planes parallel to the plane of xy, or say the contour lines;

the continuous lines are the sections above the plane of xy, and the dotted lines the sections below this plane. The form is, in fact, that of a saddle.

In the ellipsoid (fig. 23), the sections by the planes of zx, zy, and xy are each of them an ellipse, and the section by any parallel plane is also an ellipse. The

Fig. 23.



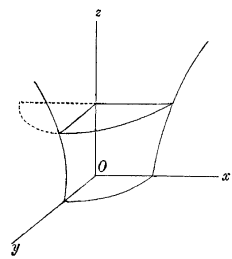
surface may be considered as generated by an ellipse moving parallel to itself along two ellipses as directrices.

In the hyperboloid of one sheet (fig. 24), the sections by the planes of zx, zy are the hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

having a common conjugate axis zOz'; the section by the plane of xy, and that by

Fig. 24.



any parallel plane, is an ellipse; and the surface may be considered as generated by a variable ellipse moving parallel to itself along the two hyperbolas as directrices.

In the hyperboloid of two sheets (fig. 25), the sections by the planes of zx and zy are the hyperbolas

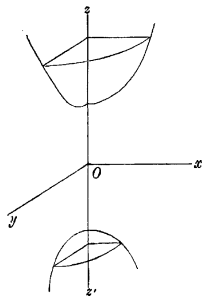
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1, \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1,$$

having the common transverse axis zOz'; the section by any plane $z = \pm\gamma$ parallel to that of xy, γ being in absolute magnitude $> c$, is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\gamma^2}{c^2} - 1;$$

and the surface, consisting of two distinct portions or sheets, may be considered as

Fig. 25.



generated by a variable ellipse moving parallel to itself along the hyperbolas as directrices.

The hyperbolic paraboloid is such (and it is easy from the figure to understand how this may be the case) that there exist upon it two singly infinite series of right lines. The same is the case with the hyperboloid of one sheet (ruled or skew hyperboloid, as with reference to this property it is termed). If we imagine two equal and parallel circular disks, their points connected by strings of equal length, so that these are the generating lines of a right circular cylinder, then by turning one of the disks about its centre through the same angle in one or the other direction, the strings will in each case generate one and the same hyperboloid, and will in regard to it be the two systems of lines on the surface, or say the two systems of generating lines; and the general configuration is the same when instead of circles we have ellipses. It has been already shown analytically that the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is satisfied by each of two pairs of linear relations between the coordinates.

Curves; Tangent, Osculating Plane, Curvature, &c.

38. It will be convenient to consider the coordinates (x, y, z) of the point on the curve as given in terms of a parameter θ, so that dx, dy, dz, d^2x, &c., will be proportional to $\frac{dx}{d\theta}, \frac{dy}{d\theta}, \frac{dz}{d\theta}, \frac{d^2x}{d\theta^2}$, &c. But only a part of the analytical formulæ will be given; in them ξ, η, ζ are used as current coordinates.

The tangent is the line through the point (x, y, z) and the consecutive point $(x+dx, y+dy, z+dz)$; its equations therefore are

$$\frac{\xi-x}{dx}=\frac{\eta-y}{dy}=\frac{\zeta-z}{dz}.$$

The osculating plane is the plane through the point and two consecutive points, and contains therefore the tangent; its equation is

$$\begin{vmatrix} \xi-x, & \eta-y, & \zeta-z \\ dx\ , & dy\ , & dz \\ d^2x\ , & d^2y\ , & d^2z \end{vmatrix}=0,$$

or, what is the same thing,

$$(\xi-x)(dyd^2z-dzd^2y)+(\eta-y)(dzd^2x-dxd^2z)+(\zeta-z)(dxd^2y-dyd^2x)=0.$$

The normal plane is the plane through the point at right angles to the tangent. It meets the osculating plane in a line called the principal normal; and drawing through the point a line at right angles to the osculating plane, this is called the binormal. We have thus at the point a set of three rectangular axes—the tangent, the principal normal, and the binormal.

We have through the point and three consecutive points a sphere of spherical curvature,—the centre and radius thereof being the centre, and radius, of spherical curvature. The sphere is met by the osculating plane in the circle of absolute curvature,—the centre and radius thereof being the centre, and radius, of absolute curvature. The centre of absolute curvature is also the intersection of the principal normal by the normal plane at the consecutive point.

Surfaces; Tangent Lines and Plane, Curvature, &c.

39. It will be convenient to consider the surface as given by an equation $f(x, y, z)=0$ between the coordinates; taking (x, y, z) for the coordinates of a given point, and $(x+dx, y+dy, z+dz)$ for those of a consecutive point, the increments dx, dy, dz satisfy the condition

$$\frac{df}{dx}dx+\frac{df}{dy}dy+\frac{df}{dz}dz=0,$$

but the ratio of two of the increments, suppose $dx : dy$, may be regarded as arbitrary. Only a part of the analytical formulæ will be given; in them ξ, η, ζ are used as current coordinates.

We have through the point a singly infinite series of right lines, each meeting the surface in a consecutive point, or say having each of them two-point intersection with the surface. These lines lie all of them in a plane which is the tangent plane; its equation is

$$\frac{df}{dx}(\xi-x)+\frac{df}{dy}(\eta-y)+\frac{df}{dz}(\zeta-z)=0,$$

as is at once verified by observing that this equation is satisfied (irrespectively of the value of $dx : dy$) on writing therein $\xi, \eta, \zeta=x+dx, y+dy, z+dz$.

The line through the point at right angles to the tangent plane is called the normal; its equations are

$$\frac{\xi - x}{\frac{df}{dx}} = \frac{\eta - y}{\frac{df}{dy}} = \frac{\zeta - z}{\frac{df}{dz}}.$$

In the series of tangent lines there are in general two (real or imaginary) lines, each of which meets the surface in a second consecutive point, or say it has three-point intersection with the surface; these are called the chief-tangents (Haupt-tangenten). The tangent-plane cuts the surface in a curve, having at the point of contact a node (double point), the tangents to the two branches being the chief-tangents.

In the case of a quadric surface the curve of intersection, *qua* curve of the second order, can only have a node by breaking up into a pair of lines; that is, every tangent-plane meets the surface in a pair of lines, or we have on the surface two singly infinite systems of lines; these are real for the hyperbolic paraboloid and the hyperboloid of one sheet, imaginary in other cases.

At each point of a surface the chief-tangents determine two directions; and passing along one of them to a consecutive point, and thence (without abrupt change of direction) along the new chief-tangent to a consecutive point, and so on, we have on the surface a chief-tangent curve; and there are, it is clear, two singly infinite series of such curves. In the case of a quadric surface, the curves are the right lines on the surface.

40. If at the point we draw in the tangent-plane two lines bisecting the angles between the chief-tangents, these lines (which are at right angles to each other) are called the principal tangents*. We have thus at each point of the surface a set of rectangular axes, the normal and the two principal tangents.

Proceeding from the point along a principal tangent to a consecutive point on the surface, and thence (without abrupt change of direction) along the new principal tangent to a consecutive point, and so on, we have on the surface a curve of curvature; there are, it is clear, two singly infinite series of such curves, cutting each other at right angles at each point of the surface.

Passing from the given point in an arbitrary direction to a consecutive point on the surface, the normal at the given point is not intersected by the normal at the consecutive point; but passing to the consecutive point along a curve of curvature (or, what is the same thing, along a principal tangent) the normal at the given point is intersected by the normal at the consecutive point; we have thus on the normal two centres of curvature, and the distances of these from the point on the surface are the two principal radii of curvature of the surface at that point; these are also the radii of curvature of the sections of the surface by planes through the normal and the two principal tangents respectively; or say they are the radii of curvature of the

* The point on the surface may be such that the directions of the principal tangents become arbitrary; the point is then an umbilicus. It is in the text assumed that the point on the surface is not an umbilicus.

normal sections through the two principal tangents respectively. Take at the point the axis of z in the direction of the normal, and those of x and y in the directions of the principal tangents respectively, then, if the radii of curvature be a, b (the signs being such that the coordinates of the two centres of curvature are $z=a$ and $z=b$ respectively), the surface has in the neighbourhood of the point the form of the paraboloid

$$z=\frac{x^2}{2a}+\frac{y^2}{2b},$$

and the chief-tangents are determined by the equation $0=\frac{x^2}{2a}+\frac{y^2}{2b}$. The two centres of curvature may be on the same side of the point or on opposite sides; in the former case a and b have the same sign, the paraboloid is elliptic, and the chief-tangents are imaginary; in the latter case a and b have opposite signs, the paraboloid is hyperbolic, and the chief-tangents are real.

The normal sections of the surface and the paraboloid by the same plane have the same radius of curvature; and it thence readily follows that the radius of curvature of a normal section of the surface by a plane inclined at an angle θ to that of zx is given by the equation

$$\frac{1}{\rho}=\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}.$$

The section in question is that by a plane through the normal and a line in the tangent plane inclined at an angle θ to the principal tangent along the axis of x. To complete the theory, consider the section by a plane having the same trace upon the tangent plane, but inclined to the normal at an angle ϕ; then it is shown without difficulty (Meunier's theorem) that the radius of curvature of this inclined section of the surface is $=\rho\cos\phi$.

791.

LANDEN.

[From the *Encyclopædia Britannica, Ninth Edition,* vol. XIV. (1882), p. 271.]

LANDEN, JOHN, a distinguished mathematician of the 18th century, was born at Peakirk near Peterborough in Northamptonshire in 1719, and died 15th January 1790 at Milton in the same county. Most of his time was spent in the pursuits of active life, but he early showed a strong talent for mathematical study, which he eagerly cultivated in his leisure hours. In 1762 he was appointed agent to the Earl Fitzwilliam, and held that office to within two years of his death. He lived a very retired life, and saw little or nothing of society; when he did mingle in it, his dogmatism and pugnacity caused him to be generally shunned. He was first known as a mathematician by his essays in the *Ladies' Diary* for 1744. In 1766 he was elected a Fellow of the Royal Society. He was well acquainted and *au courant* with the works of the mathematicians of his own time, and has been called the English D'Alembert. In his *Discourse* on the "Residual Analysis," in which he proposes to substitute for the method of fluxions a purely algebraical method, he says, "It is by means of the following theorem, viz.

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times 1 + \frac{v}{x} + \left(\frac{v}{x}\right)^2 + \ldots (m \text{ terms})$$

$$\div 1 + \left(\frac{v}{x}\right)^{\frac{m}{n}} + \left(\frac{v}{x}\right)^{\frac{2m}{n}} + \ldots (n \text{ terms})$$

(where m and n are integers), that we are able to perform all the principal operations in our said analysis; and I am not a little surprised that a theorem so obvious, and of such vast use, should so long escape the notice of algebraists." The idea is of course a perfectly legitimate one, and may be compared with that of Lagrange's *Calcul des Fonctions.* His memoir (1775) on the rotatory motion of a body contains (as the author was aware) conclusions at variance with those arrived at by D'Alembert and

Euler in their researches on the same subject. He reproduces and further develops and defends his own views in his *Mathematical Memoirs*, and in his paper in the *Philosophical Transactions* for 1785. But Landen's capital discovery is that of the theorem known by his name (obtained in its complete form in the memoir of 1775, and reproduced in the first volume of the *Mathematical Memoirs*) for the expression of the arc of an hyperbola in terms of two elliptic arcs. To find this, he integrates a differential equation derived from the equation

$$t = gx\sqrt{\frac{m^2 - x^2}{m^2 - gx^2}},$$

interpreting geometrically in an ingenious and elegant manner three integrals which present themselves. If in the foregoing equation we write $m = 1$, $g = k^2$, and instead of t consider the new variable $y = t \div (1 - k')$, then

$$y = (1 + k')\, x\sqrt{\frac{1 - x^2}{1 - k^2x^2}},$$

which is the form known as Landen's transformation in the theory of elliptic functions; but his investigation does not lead him to obtain the equivalent of the resulting differential equation

$$\frac{dy}{\sqrt{1 - y^2 \,.\, 1 - \lambda^2 y^2}} = \frac{(1 + k')\, dx}{\sqrt{1 - x^2 \,.\, 1 - k^2x^2}}, \text{ where } \lambda = \frac{1 - k'}{1 + k'},$$

due it would appear to Legendre, and which (over and above Landen's own beautiful result) gives importance to the theorem as leading directly to the quadric transformation of an elliptic integral in regard to the modulus.

The list of his writings is as follows:—*Ladies' Diary*, various communications, 1744—1760; papers in the *Phil. Trans.*, 1754, 1760, 1768, 1771, 1775, 1777, 1785; *Mathematical Lucubrations*, 1755; *A Discourse concerning the Residual Analysis*, 1758; *The Residual Analysis*, book I., 1764; *Animadversions on Dr Stewart's Method of computing the Sun's Distance from the Earth*, 1771; *Mathematical Memoirs*, 1780, 1789.

792.

LOCUS.

[From the *Encyclopædia Britannica, Ninth Edition,* vol. XIV. (1882), pp. 764, 765.]

LOCUS, in Greek τόπος, a geometrical term, the invention of the notion of which is attributed to Plato. It occurs in such statements as these:—the locus of the points which are at the same distance from a fixed point, or of a point which moves so as to be always at the same distance from a fixed point, is a circle; conversely a circle is the locus of the points at the same distance from a fixed point, or of a point moving so as to be always at the same distance from a fixed point; and so, in general, a curve of any given kind is the locus of the points which satisfy, or of a point moving so as always to satisfy, a given condition. The theory of loci is thus identical with that of curves; and it is in fact in this very point of view that a curve is considered in the article Curve, [785]; see that article, and also Geometry (Analytical), [790]. It is only necessary to add that the notion of a locus is useful as regards determinate problems or theorems: thus, to find the centre of the circle circumscribed about a given triangle ABC, we see that the circumscribed circle must pass through the two vertices A, B, and the locus of the centres of the circles which pass through these two points is the straight line at right angles to the side AB at its mid-point; similarly the circumscribed circle must pass through A, C, and the locus of the centres of the circles through these two points is the line at right angles to the side AC at its mid-point; thus we get the ordinary construction, and also the theorem that the lines at right angles to the sides, at their mid-points respectively, meet in a point. The notion of a locus applies, of course, not only to plane but also to solid geometry. Here the locus of the points satisfying a single (or onefold) condition is a surface; the locus of the points satisfying two conditions (or a twofold condition) is a curve in space, which is in general a twisted curve or curve of double curvature.

793.

MONGE.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. XVI. (1883), pp. 738, 739.]

MONGE, GASPARD (1746—1818), a French mathematician, the inventor of descriptive geometry, was born at Beaune on the 10th May 1746. He was educated first at the college of the Oratorians at Beaune, and then in their college at Lyons,—where, at sixteen, the year after he had been learning physics, he was made a teacher of it. Returning to Beaune for a vacation, he made, on a large scale, a plan of the town, inventing the methods of observation and constructing the necessary instruments; the plan was presented to the town, and preserved in their library. An officer of engineers seeing it wrote to recommend Monge to the commandant of the military school at Mézières, and he was received as draftsman and pupil in the practical school attached to that institution; the school itself was of too aristocratic a character to allow of his admission to it. His manual skill was duly appreciated: "I was a thousand times tempted," he said long afterwards, "to tear up my drawings in disgust at the esteem in which they were held, as if I had been good for nothing better." An opportunity, however, presented itself: being required to work out from data supplied to him the "défilement" of a proposed fortress (an operation then only performed by a long arithmetical process), Monge, substituting for this a geometrical method, obtained the result so quickly that the commandant at first refused to receive it—the time necessary for the work had not been taken; but upon examination the value of the discovery was recognized, and the method was adopted. And Monge, continuing his researches, arrived at that general method of the application of geometry to the arts of construction which is now called descriptive geometry. But such was the system in France before the Revolution that the officers instructed in the method were strictly forbidden to communicate it even to those engaged in other branches of the public service; and it was not until many years afterwards that an account of it was published. The method consists, as is well known, in the use of the two halves of a sheet of paper to represent say the planes of xy and xz at right angles to each other, and the

consequent representation of points, lines, and figures in space by means of their plan and elevation, placed in a determinate relative position.

In 1768 Monge became professor of mathematics, and in 1771 professor of physics, at Mézières; in 1778 he married Madame Horbon, a young widow whom he had previously defended in a very spirited manner from an unfounded charge; in 1780 he was appointed to a chair of hydraulics at the Lyceum in Paris (held by him together with his appointments at Mézières), and was received as a member of the Academy; his intimate friendship with Berthollet began at this time. In 1783, quitting Mézières, he was, on the death of Bezout, appointed examiner of naval candidates. Although pressed by the minister to prepare for them a complete course of mathematics, he declined to do so, on the ground that it would deprive Madame Bezout of her only income, arising from the sale of the works of her late husband; he wrote, however (1786), his *Traité élémentaire de la Statique.*

Monge contributed (1770—1790) to the *Memoirs* of the Academy of Turin, the *Mémoires des Savants Étrangers* of the Academy of Paris, the *Mémoires* of the same Academy, and the *Annales de Chimie,* various mathematical and physical papers. Among these may be noticed the memoir "Sur la théorie des déblais et des remblais" (*Mém. de l'Acad. de Paris,* 1781), which, while giving a remarkably elegant investigation in regard to the problem of earthwork referred to in the title, establishes in connexion with it his capital discovery of the curves of curvature of a surface. Euler, in his paper on curvature in the Berlin *Memoirs* for 1760, had considered, not the normals of the surface, but the normals of the plane sections through a particular normal, so that the question of the intersection of successive normals of the surface had never presented itself to him. Monge's memoir just referred to gives the ordinary differential equation of the curves of curvature, and establishes the general theory in a very satisfactory manner; but the application to the interesting particular case of the ellipsoid was first made by him in a later paper in 1795. A memoir in the volume for 1783 relates to the production of water by the combustion of hydrogen; but Monge's results in this matter had been anticipated by Watts and Cavendish.

In 1792, on the creation by the Legislative Assembly of an executive council, Monge accepted the office of minister of the marine, but retained it only until April 1793. When the Committee of Public Safety made an appeal to the savants to assist in producing the *matériel* required for the defence of the republic, he applied himself wholly to these operations, and distinguished himself by his indefatigable activity therein; he wrote at this time his *Description de l'art de fabriquer les canons,* and his *Avis aux ouvriers en fer sur la fabrication de l'acier.* He took a very active part in the measures for the establishment of the Normal School (which existed only during the first four months of the year 1795), and of the School for Public Works, afterwards the Polytechnic School, and was at each of them professor of descriptive geometry; his methods in that science were first published in the form in which the shorthand writers took down his lessons given at the Normal School in 1795, and again in 1798—99. In 1796 Monge was sent into Italy with Berthollet and some artists to receive the pictures and statues levied from several Italian towns, and made

there the acquaintance of General Bonaparte. Two years afterwards he was sent to Rome on a political mission, which terminated in the establishment, under Massena, of the shortlived Roman republic; and he thence joined the expedition to Egypt, taking part with his friend Berthollet as well in various operations of the war as in the scientific labours of the Egyptian Institute of Sciences and Arts; they accompanied Bonaparte to Syria, and returned with him in 1798 to France. Monge was appointed president of the Egyptian commission, and he resumed his connexion with the Polytechnic School. His later mathematical papers are published (1794—1816) in the *Journal* and the *Correspondance* of the Polytechnic School. On the formation of the Senate he was appointed a member of that body, with an ample provision and the title of count of Pelusium; but on the fall of Napoleon he was deprived of all his honours, and even excluded from the list of members of the reconstituted Institute. He died at Paris on the 28th July 1818.

For further information see B. Brisson, *Notice historique sur Gaspard Monge*; Dupin, *Essai historique sur les services et les travaux scientifiques de Gaspard Monge*, Paris, 1819, which contains (pp. 162—166) a list of Monge's memoirs and works; and the biography by Arago (*Œuvres*, t. II., 1854).

Monge's various mathematical papers are to a considerable extent reproduced in the *Application de l'Analyse à la Géométrie*, 4th edition (last revised by the author), Paris, 1819—the pure text of this is reproduced in the 5th edition (revue, corrigée et annotée par M. Liouville), Paris, 1850, which contains also Gauss's Memoir, "Disquisitiones generales circa superficies curvas," and some valuable notes by the editor. The other principal separate works are *Traité élémentaire de la Statique, 8e édition, conformée à la précédente, par M. Hachette, et suivie d'une Note etc., par M. Cauchy*, Paris, 1846; and the *Géométrie Descriptive* (originating, as mentioned above, in the lessons given at the Normal School). The 4th edition, published shortly after the author's death, seems to have been substantially the same as the 7th (*Géométrie Descriptive, par G. Monge, suivie d'une théorie des Ombres et de la Perspective, extraite des papiers de l'auteur, par M. Brisson*, Paris, 1847).

794.

NUMBERS, PARTITION OF.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. XVII. (1884), p. 614.]

THIS subject, created by Euler, though relating essentially to positive integer numbers, is scarcely regarded as a part of the Theory of Numbers. We consider in it a number as made up by the *addition* of other numbers: thus the partitions of the successive numbers 1, 2, 3, 4, 5, 6, &c., are as follows:—

1;
2, 11;
3, 21, 111;
4, 31, 22, 211, 1111;
5, 41, 32, 311, 221, 2111, 11111;
6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111.

These are formed each from the preceding ones; thus, to form the partitions of 6 we take first 6; secondly, 5 prefixed to each of the partitions of 1 (that is, 51); thirdly, 4 prefixed to each of the partitions of 2 (that is, 42, 411); fourthly, 3 prefixed to each of the partitions of 3 (that is, 33, 321, 3111); fifthly, 2 prefixed, not to each of the partitions of 4, but only to those partitions which begin with a number not exceeding 2 (that is, 222, 2211, 21111); and lastly, 1 prefixed to all the partitions of 5 which begin with a number not exceeding 1 (that is, 111111); and so in other cases.

The method gives *all* the partitions of a number, but we may consider different classes of partitions: the partitions into a given number of parts, or into not more than a given number of parts; or the partitions into given parts, either with repetitions or without repetitions, &c. It is possible, for any particular class of partitions, to obtain methods more or less easy for the formation of the partitions either of a given number or of the successive numbers 1, 2, 3, &c. And of course in any case, having obtained the partitions, we can count them and so obtain the number of partitions.

Another method is by Arbogast's rule of the last and the last but one; in fact, taking the value of a to be unity, and, understanding this letter in each term, the rule gives b; c, b^2; d, bc, b^3; e, bd, c^2, b^2c, b^4, &c., which, if b, c, d, e, &c., denote 1, 2, 3, 4, &c., respectively, are the partitions of 1, 2, 3, 4, &c., respectively.

An important notion is that of conjugate partitions. Thus a partition of 6 is 42; writing this in the form $\begin{cases}1111\\11\end{cases}$, and summing the columns instead of the lines, we obtain the conjugate partition 2211; evidently, starting from 2211, the conjugate partition is 42. If we form all the partitions of 6 into not more than three parts, these are

$$6, \quad 51, \quad 42, \quad 33, \quad 411, \quad 321, \quad 222,$$

and the conjugates are

$$111111, \quad 21111, \quad 2211, \quad 222, \quad 3111, \quad 321, \quad 33,$$

where no part is greater than 3; and so, in general, we have the theorem, the number of partitions of n into not more than k parts is equal to the number of partitions of n with no part greater than k.

We have for the number of partitions an analytical theory depending on generating functions; thus for the partitions of a number n with the parts 1, 2, 3, 4, 5, &c., without repetitions, writing down the product

$$1+x\,.\,1+x^2\,.\,1+x^3\,.\,1+x^4\ldots,\ =1+x+x^2+2x^3+\ldots+Nx^n+\ldots,$$

it is clear that, if x^α, x^β, $x^\gamma, \ldots$ are terms of the series x, x^2, $x^3, \ldots$ for which $\alpha+\beta+\gamma+\ldots=n$, then we have in the development of the product a term x^n, and hence that, in the term Nx^n of the product, the coefficient N is equal to the number of partitions of n with the parts 1, 2, 3, ..., without repetitions; or say that the product is the generating function (G. F.) for the number of such partitions. And so in other cases we obtain a generating function.

Thus for the function

$$\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3\ldots},\ =1+x+2x^2+\ldots+Nx^n+\ldots,$$

observing that any factor $1/1-x^l$ is $=1+x^l+x^{2l}+\ldots$, we see that, in the term Nx^n, the coefficient is equal to the number of partitions of n, with the parts 1, 2, 3, ..., with repetitions.

Introducing another letter z, and considering the function

$$1+xz\,.\,1+x^2z\,.\,1+x^3z\ldots,\ =1+z(x+x^2+\ldots)+\ldots+Nx^nz^k+\ldots,$$

we see that, in the term Nx^nz^k of the development, the coefficient N is equal to the number of partitions of n into k parts, with the parts 1, 2, 3, 4, ..., without repetitions.

And similarly, considering the function

$$\frac{1}{1-xz\,.\,1-x^2z\,.\,1-x^3z\ldots},\ =1+z(x+x^2+\ldots)+\ldots+Nx^nz^k+\ldots,$$

we see that, in the term Nx^nz^k of the development, the coefficient N is equal to the number of partitions of n into k parts, with the parts 1, 2, 3, 4, ..., with repetitions.

We have such analytical formulæ as

$$\frac{1}{1-xz\,.\,1-x^2z\,.\,1-x^3z\ldots}=1+\frac{zx}{1-x}+\frac{z^2x^2}{1-x\,.\,1-x^2}+\ldots,$$

which lead to theorems in the Partition of Numbers. A remarkable theorem is

$$1-x\,.\,1-x^2\,.\,1-x^3\,.\,1-x^4\ldots=1-x-x^3+x^5+x^7-x^{12}-x^{15}+\ldots,$$

where the only terms are those with an exponent $\frac{1}{2}(3n^2\pm n)$, and for each such pair of terms the coefficient is $(-)^n 1$. The formula shows that, except for numbers of the form $\frac{1}{2}(3n^2\pm n)$, the number of partitions without repetitions into an odd number of parts is equal to the number of partitions without repetitions into an even number of parts, whereas for the excepted numbers these numbers differ by unity. Thus for the number 11, which is not an excepted number, the two sets of partitions are

$$11,\quad 821,\ 731,\ 641,\ 632,\ 542,$$

$$10\,.\,1,\ 92,\ \ 83,\ \ 74,\ \ 65,\ \ 5321,$$

in each set 6.

We have

$$1-x\,.\,1+x\,.\,1+x^2\,.\,1+x^4\,.\,1+x^8\ldots=1;$$

or, as this may be written,

$$1+x\,.\,1+x^2\,.\,1+x^4\,.\,1+x^8\ldots=\frac{1}{1-x},\ =1+x+x^2+x^3+\ldots,$$

showing that a number n can always be made up, and in one way only, with the parts 1, 2, 4, 8, ... The product on the left-hand side may be taken to k terms only: thus if $k=4$, we have

$$1+x\,.\,1+x^2\,.\,1+x^4\,.\,1+x^8,\ =\frac{1-x^{16}}{1-x},\ =1+x+x^2+\ldots+x^{15},$$

that is, any number from 1 to 15 can be made up, and in one way only, with the parts 1, 2, 4, 8; and similarly any number from 1 to 2^k-1 can be made up, and in one way only, with the parts 1, 2, 4, ..., 2^{k-1}. A like formula is

$$\frac{1-x^3}{x\,.\,1-x}\,.\,\frac{1-x^9}{x^3\,.\,1-x^3}\,.\,\frac{1-x^{27}}{x^9\,.\,1-x^9}\,.\,\frac{1-x^{81}}{x^{27}\,.\,1-x^{27}}=\frac{1-x^{81}}{x^{40}\,.\,1-x};$$

that is,

$$x^{-1}+1+x\,.\,x^{-3}+1+x^3\,.\,x^{-9}+1+x^9\,.\,x^{-27}+1+x^{27}=x^{-40}+x^{-39}+\ldots+1+x+\ldots+x^{39}+x^{40},$$

showing that any number from -40 to $+40$ can be made up, and that in one way only, with the parts 1, 3, 9, 27 taken positively or negatively; and so in general any number from $-\frac{1}{2}(3^k-1)$ to $+\frac{1}{2}(3^k-1)$ can be made up, and that in one way only, with the parts 1, 3, 9, ..., 3^{k-1} taken positively or negatively.

795.

NUMBERS, THEORY OF.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. XVII. (1884), pp. 614—624.]

THE Theory of Numbers, or higher arithmetic, otherwise arithmology, is a subject which, originating with Euclid, has in modern times, in the hands of Legendre, Gauss, Lejeune-Dirichlet, Kummer, Kronecker, and others, been developed into a most extensive and interesting branch of mathematics. We distinguish between the ordinary (or say the simplex) theory and the various complex theories.

In the ordinary theory we have, in the first instance, positive integer numbers, the unit or unity 1, and the other numbers 2, 3, 4, 5, &c. We introduce the zero 0, which is a number *sui generis*, and the negative numbers -1, -2, -3, -4, &c., and we have thus the more general notion of integer numbers, 0, ± 1, ± 2, ± 3, &c.; $+1$ and -1 are units or unities. The sum of any two or more numbers is a number; conversely, any number is a sum of two or more parts; but even when the parts are positive a number cannot be, in a determinate manner, represented as a sum of parts. The product of two or more numbers is a number; but (disregarding the unities $+1$, -1, which may be introduced as factors at pleasure) it is not conversely true that every number is a product of numbers. A number such as 2, 3, 5, 7, 11, &c., which is not a product of numbers, is said to be a prime number; and a number which is not prime is said to be composite. A number other than zero is thus either prime or composite; and we have the theorem that every composite number is, in a determinate way, a product of prime factors.

We have complex theories in which all the foregoing notions (integer, unity, zero, prime, composite) occur; that which first presented itself was the theory with the unit i ($i^2 = -1$); we have here complex numbers, $a + bi$, where a and b are in the before-mentioned (ordinary) sense positive or negative integers, not excluding zero; we have the zero 0, $= 0 + 0i$, and the four units 1, -1, i, $-i$. A number other than zero is here either prime or else composite; for instance, 3, 7, 11, are prime numbers, and 5, $= (2+i)(2-i)$, 9, $= 3 . 3$, 13, $= (3+2i)(3-2i)$, are composite numbers (generally any

positive real prime of the form $4n+3$ is prime, but any positive real prime of the form $4n+1$ is a sum of two squares, and is thus composite). And disregarding unit factors we have, as in the ordinary theory, the theorem that every composite number is, in a determinate way, a product of prime factors.

There is, in like manner, a complex theory involving the cube roots of unity—if α be an imaginary cube root of unity ($\alpha^2+\alpha+1=0$), then the integers of this theory are $a+b\alpha$ (a and b real positive or negative integers, including zero); a complex theory with the fifth roots of unity—if α be an imaginary fifth root of unity ($\alpha^4+\alpha^3+\alpha^2+\alpha+1=0$), then the integers of the theory are $a+b\alpha+c\alpha^2+d\alpha^3$ (a, b, c, d, real positive or negative integers, including zero); and so on for the roots of the orders 7, 11, 13, 17, 19. In all these theories, or at any rate for the orders 3, 5, 7 (see No. 37, *post*), we have the foregoing theorem: disregarding unit factors, a number other than zero is either prime or composite, and every composite number is, in a determinate way, a product of prime factors. But coming to the 23rd roots of unity the theorem ceases to be true. Observe that it is a particular case of the theorem that, if N be a prime number, any integer power of N has for factors only the lower powers of N,—for instance, $N^3=N.N^2$; there is no other decomposition $N^3=AB$. This is obviously true in the ordinary theory, and it is true in the complex theories preceding those for the 3rd, 5th, and 7th roots of unity, and probably in those for the other roots preceding the 23rd roots; but it is not true in the theory for the 23rd roots of unity. We have, for instance, 47, a number not decomposable into factors, but 47^2, $=AB$, is a product of two numbers each of the form $a+b\alpha+\ldots+k\alpha^{21}$ (α a 23rd root). The theorem recovers its validity by the introduction into the theory of Kummer's notion of an ideal number.

The complex theories above referred to would be more accurately described as theories for the complex numbers involving the periods of the roots of unity: the units are the roots either of the equation $x^{p-1}+x^{p-2}+\ldots+x+1=0$ (p a prime number) or of any equation $x^{\frac{p-1}{e}}+\ldots\pm 1=0$ belonging to a factor of the function of the order $p-1$: in particular, this may be the quadric equation for the periods each of $\frac{1}{2}(p-1)$ roots; they are the theories which were first and have been most completely considered, and which led to the notion of an ideal number. But a yet higher generalization which has been made is to consider the complex theory, the units whereof are the roots of any given irreducible equation which has integer numbers for its coefficients.

There is another complex theory the relation of which to the foregoing is not very obvious, viz. Galois's theory of the numbers composed with the imaginary roots of an irreducible congruence, $F(x)\equiv 0$ (modulus a prime number p); the nature of this will be indicated in the sequel.

In any theory, ordinary or complex, we have a first part, which has been termed (but the name seems hardly wide enough) the theory of congruences; a second part, the theory of homogeneous forms: this includes in particular the theory of the binary quadratic forms $(a, b, c)(x, y)^2$; and a third part, comprising those miscellaneous investigations which do not come properly under either of the foregoing heads.

Ordinary Theory, First Part.

1. We are concerned with the integer numbers $0, \pm 1, \pm 2, \pm 3$, &c., or in the first place with the positive integer numbers 1, 2, 3, 4, 5, 6, &c. Some of these, 1, 2, 3, 5, 7, &c., are prime, others, $4, = 2^2$, $6, = 2 . 3$, &c., are composite; and we have the fundamental theorem that a composite number is expressible, and that in one way only, as a product of prime factors, $N = a^\alpha b^\beta c^\gamma \ldots$ ($a, b, c, \ldots$ primes other than 1; $\alpha, \beta, \gamma, \ldots$ positive integers).

Gauss makes the proof to depend on the following steps: (i) the product of two numbers each smaller than a given prime number is not divisible by this number; (ii) if neither of two numbers is divisible by a given prime number the product is not so divisible; (iii) the like as regards three or more numbers; (iv) a composite number cannot be resolved into factors in more than one way.

2. Proofs will in general be only indicated or be altogether omitted, but, as a specimen of the reasoning in regard to whole numbers, the proofs of these fundamental propositions are given at length. (i) Let p be the prime number, a a number less than p, and if possible let there be a number b less than p, and such that ab is divisible by p; it is further assumed that b is the only number, or, if there is more than one, then that b is the least number having the property in question; b is greater than 1, for a being less than p is not divisible by p. Now p as a prime number is not divisible by b, but must lie between two consecutive multiples mb and $(m+1)b$ of b. Hence, ab being divisible by p, mab is also divisible by p; moreover, ap is divisible by p, and hence the difference of these numbers, $= a(p - mb)$, must also be divisible by p, or, writing $p - mb = b'$, we have ab' divisible by p, where b' is less than b; so that b is not the least number for which ab is divisible by p. (ii) If a and b are neither of them divisible by p, then a divided by p leaves a remainder α which is less than p, say we have $a = mp + \alpha$; and similarly b divided by p leaves a remainder β which is less than p, say we have $b = np + \beta$; then

$$ab = (mp + \alpha)(np + \beta), \quad = (mnp + n\alpha + m\beta)p + \alpha\beta,$$

and $\alpha\beta$ is not divisible by p, therefore ab is not divisible by p. (iii) The like proof applies to the product of three or more factors $a, b, c, \ldots$ (iv) Suppose that the number $N, = a^\alpha b^\beta c^\gamma \ldots$ ($a, b, c, \ldots$ prime numbers other than 1), is decomposable in some other way into prime factors; we can have no prime factor p, other than $a, b, c, \ldots$, for no such number can divide $a^\alpha b^\beta c^\gamma \ldots$; and we must have each of the numbers $a, b, c, \ldots$, for if any one of them, suppose a, were wanting, the number N would not be divisible by a. Hence the new decomposition if it exists must be a decomposition $N = a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$; and here, if any two corresponding indices, say α, α', are different from each other, then one of them, suppose α', is the greater, and we have $N \div p^\alpha = b^\beta c^\gamma \ldots = a^{\alpha'-\alpha} b^{\beta'} c^{\gamma'} \ldots$ That is, we have the number $N \div p^\alpha$ expressed in two different ways as a product, the number a being a factor in the one case, but not a factor in the other case. Thus the two exponents cannot be unequal, that is, we must have $\alpha = \alpha'$, and similarly we have $\beta = \beta'$, $\gamma = \gamma', \ldots$; that is, there is *only* the original decomposition $N = a^\alpha b^\beta c^\gamma \ldots$

3. The only numbers divisible by a number $N = a^\alpha b^\beta c^\gamma \ldots$ are the numbers $a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$, where each exponent α' is equal to or greater than the corresponding exponent α. And conversely the only numbers which divide N are those of the form $a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$, where each index α' is at most equal to the corresponding index α; and in particular each or any of the indices α' may be $= 0$. Again, the least common multiple of two numbers $N = a^\alpha b^\beta c^\gamma \ldots$ and $N' = a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$ is $a^{\alpha''} b^{\beta''} c^{\gamma''} \ldots$, where each index α'' is equal to the largest of the corresponding indices α, α';—observe that any one or more of the indices $\alpha, \beta, \gamma, \ldots, \alpha', \beta', \gamma', \ldots$, may be $= 0$, so that the theorem extends to the case where either of the numbers N, N', has prime factors which are not factors of the other number. And so the greatest common measure of two numbers $N = a^\alpha b^\beta c^\gamma \ldots$ and $N' = a^{\alpha'} b^{\beta'} c^{\gamma'} \ldots$ is $a^{\alpha''} b^{\beta''} c^{\gamma''} \ldots$, where each index α'' is equal to the least of the corresponding indices α and α'.

4. The divisors of $N = a^\alpha b^\beta c^\gamma \ldots$ are the several terms of the product

$$(1 + a + \ldots + a^\alpha)(1 + b + \ldots + b^\beta)(1 + c + \ldots + c^\gamma),$$

where unity and the number N itself are reckoned each of them as a divisor. Hence the number of divisors is $= (\alpha + 1)(\beta + 1)(\gamma + 1) \ldots$, and the sum of the divisors is

$$= \frac{(a^{\alpha+1} - 1)(b^{\beta+1} - 1)(c^{\gamma+1} - 1) \ldots}{(a - 1)(b - 1)(c - 1) \ldots}.$$

5. In $N = a^\alpha b^\beta c^\gamma \ldots$ the number of integers less than N and prime to it is

$$\phi(N), = N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots$$

To find the numbers in question write down the series of numbers 1, 2, 3, ..., N; strike out all the numbers divisible by a, then those divisible by b, then those divisible by c, and so on; there will remain only the numbers prime to N. For actually finding the numbers we may of course in striking out those divisible by b disregard the numbers already struck out as divisible by a, and in striking out with respect to c disregard the numbers already struck out as divisible by a or b, and so on; but in order to count the remaining numbers it is more convenient to ignore the previous strikings out. Suppose, for a moment, there are only two prime factors a and b, then the number of terms struck out as divisible by a is $= N.\frac{1}{a}$, and the number of terms struck out as divisible by b is $= N.\frac{1}{b}$; but then each term divisible by ab will have been twice struck out; the number of these is $= N.\frac{1}{ab}$, and thus the number of the remaining terms is $N\left(1 - \frac{1}{a} - \frac{1}{b} + \frac{1}{ab}\right)$, which is $= N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)$. By treating in like manner the case of three or more prime factors $a, b, c, \ldots$ we arrive at the general theorem. The formula gives $\phi(1) = 1$ viz. when $N = 1$, there is no factor $1 - \frac{1}{a}$; and it is necessary to consider $\phi(1)$ as being $= 1$. The explanation is that $\phi(N)$

properly denotes the number of integers not greater than N and prime to it; so that, when $N=1$, we have 1 an integer not greater than N and prime to it; but in every other case the two definitions agree.

6. If N, N', are numbers prime to each other, then $\phi(NN')=\phi(N)\phi(N')$, and so also for any number of numbers having no common divisor; in particular,

$$\phi(a^\alpha b^\beta c^\gamma \ldots)=\phi(a^\alpha)\phi(b^\beta)\phi(c^\gamma)\ldots;\quad \phi(a^\alpha)=a^\alpha\left(1-\frac{1}{a}\right),$$

and the theorem is at once verified. We have $N=\Sigma\phi(N')$, where the summation extends to all the divisors N' of N, unity and the number N itself being included; thus $15=\phi(15)+\phi(5)+\phi(3)+\phi(1),\ =8+4+2+1$.

7. The prime factor of the binomial function x^N-1 is

$$=\frac{(x^N-1)(x^{N/ab}-1)\ldots}{(x^{N/a}-1)(x^{N/b}-1)\ldots},$$

a rational and integral function of the degree $\phi(N)$; say this is called $[x^N-1]$, and we have $x^N-1=\Pi[x^{N'}-1]$, where the product extends to all the divisors N' of N, unity and the number N included. For instance

$$[x^{15}-1]=\frac{(x^{15}-1)(x-1)}{(x^5-1)(x^3-1)},\ =x^8-x^7+x^5-x^4+x^3-x+1;$$

and we have

$$\begin{aligned}x^{15}-1&=[x^{15}-1][x^5-1][x^3-1][x-1],\\&=(x^8-x^7+\ldots-x+1)(x^4+x^3+x^2+x+1)(x^2+x+1)(x-1).\end{aligned}$$

8. Congruence to a given modulus. A number x is congruent to 0, to the modulus N, $x\equiv 0\,(\text{mod. } N)$, when x is divisible by N; two numbers x, y are congruent to the modulus N, $x\equiv y\,(\text{mod. } N)$, when their difference $x-y$ divides by N, or, what is the same thing, if $x-y\equiv 0\,(\text{mod. } N)$. Observe that, if $xy\equiv 0\,(\text{mod. } N)$, and x be prime to N, then $y\equiv 0\,(\text{mod. } N)$.

9. Residues to a given modulus. For a given modulus N we can always find, and that in an infinity of ways, a set of N numbers, say N residues, such that every number whatever is, to the modulus N, congruent to one and only one of these residues. For instance, the residues may be $0, 1, 2, 3, \ldots, N-1$ (the residue of a given number is here simply the positive remainder of the number when divided by N); or, N being odd, the system may be

$$0,\ \pm 1,\ \pm 2, \ldots, \pm\tfrac{1}{2}(N-1),$$

and N even,

$$0,\ \pm 1,\ \pm 2, \ldots, \pm\tfrac{1}{2}(N-2),\ +\tfrac{1}{2}N.$$

10. Prime residues to a given modulus. Considering only the numbers which are prime to a given modulus N, we have here a set of $\phi(N)$ numbers, say $\phi(N)$ prime residues, such that every number prime to N is, to the modulus N, congruent to

one and only one of these prime residues. For instance, the prime residues may be the numbers less than N and prime to it. In particular, if N is a prime number p, then the residues may be the $p-1$ numbers, $1, 2, 3, \ldots, p-1$.

In all that follows, the letter p, in the absence of any statement to the contrary, will be used to denote an *odd* prime other than unity. A theorem for p may hold good for the even prime 2, but it is in general easy to see whether this is so or not.

11. Fermat's theorem, $x^{p-1}-1\equiv 0$ (mod. p). The generalized theorem is $x^{\phi(N)}-1\equiv 0$ (mod. N). The proof of the generalized theorem is as easy as that of the original theorem. Consider the series of the $\phi(N)$ numbers $a, b, c, \ldots$, each less than N and prime to it; let x be any number prime to N, then each of the numbers $xa, xb, xc, \ldots$, is prime to N, and no two of them are congruent to the modulus N, that is, we cannot have $x(a-b)\equiv 0$ (mod. N); in fact, x is prime to N, and the difference $a-b$ of two positive numbers each less than N will be less than N. Hence the numbers $xa, xb, xc, \ldots$, are in a different order congruent to the numbers $a, b, c, \ldots$; and multiplying together the numbers of each set we have $x^{\phi(N)}\, abc \ldots \equiv abc \ldots$ (mod. N), or, since $a, b, c, \ldots$, are each prime to N, and therefore also the product $abc \ldots$ is prime to N, we have $x^{\phi(N)}\equiv 1$, or say $x^{\phi(N)}-1\equiv 0$ (mod. N).

In particular, if N be a prime number $=p$, then $\phi(N)$ is $=p-1$, and the theorem is $x^{p-1}-1\equiv 0$ (mod. p), x being now any number not divisible by p.

12. The general congruence $f(x)\equiv 0$ (mod. p). $f(x)$ is written to denote a rational and integral function with integer coefficients which may without loss of generality be taken to be each of them less than p; it is assumed that the coefficient A of the highest power of x is not $=0$. If there is for x an integer value a such that $f(a)\equiv 0$ (mod. p, throughout), then a is said to be a root of the congruence $f(x)\equiv 0$; we may, it is clear, for a substitute any value whatever $a'=a+kp$, or say any value a' which is $\equiv a$, but such value a' is considered not as a different root but as the same root of the congruence. We have thus $f(a)\equiv 0$; and therefore $f(x)\equiv f(x)-f(a)$, $=(x-a)f_1(x)$, where $f_1(x)$ is a function of like form with $f(x)$, that is, with integer coefficients, but of the next inferior order $n-1$. Suppose there is another root b of the congruence, that is, an integer value b such that $f(b)\equiv 0$; we have then $(b-a)f_1(b)\equiv 0$, and $b-a$ is not $\equiv 0$ (for then b would be the same root as a). Hence $f_1(b)\equiv 0$, and $f(x)=(x-a)\{f_1(x)-f_1(b)\}$, $=(x-a)(x-b)f_2(x)$, where $f_2(x)$ is an integral function such as $f(x)$, but of the order $n-2$; and so on, that is, if there exist n different (non-congruent) roots of the congruence $f(x)\equiv 0$, then $f(x)=A(x-a)(x-b)\ldots(x-k)$, and the congruence may be written $A(x-a)(x-b)\ldots(x-k)\equiv 0$. And this cannot be satisfied by any other value l; for if so we should have $A(l-a)(l-b)\ldots(l-k)\equiv 0$, that is, some one of the congruences $(l-a)\equiv 0$, &c., would have to be satisfied, and l would be the same as one of the roots $a, b, c, \ldots, k$. That is, a congruence of the order n cannot have more than n roots, and if it have precisely n roots $a, b, c, \ldots, k$, then the form is $f(x)\equiv A(x-a)(x-b)\ldots(x-k)$, $\equiv 0$.

Observe that a congruence may have equal roots, viz. if the form be

$$f(x)\equiv A(x-a)^\alpha(x-b)^\beta\ldots,\ \equiv 0,$$

then the roots $a, b, \ldots$ are to be counted α times, β times, ... respectively; but clearly the whole number of roots $\alpha+\beta+\ldots$ is at most $=n$.

It is hardly necessary to remark that this theory of a congruence of the order n is precisely analogous to that of an equation of the order n, when only real roots are attended to. The theory of the imaginary roots of a congruence will be considered further on (see No. 41).

13. The linear congruence $ax \equiv c$ (mod. b). This is equivalent to the indeterminate equation $ax+by=c$; if a and b are not prime to each other, but have a greatest common measure q, this must also divide c; supposing the division performed, the equation becomes $a'x+b'y=c'$, where a' and b' are prime to each other, or, what is the same thing, we have the congruence $a'x \equiv c'$ (mod. b'). This can always be solved, for, if we consider the b' numbers $0, 1, 2, \ldots, b'-1$, one and only one of these will be $\equiv c'$ (mod. b'). Multiplying these by any number a' prime to b', and taking the remainders in regard to b', we reproduce in a different order the same series of numbers $0, 1, 2, \ldots, b'-1$; that is, in the series $a', 2a', \ldots, (b'-1)a'$ there will be one and only one term $\equiv c'$ (mod. b'), or, calling the term in question α, we have $x=\alpha$ as the solution of the congruence $a'x \equiv c'$ (mod. b'); $a'\alpha - c'$ is then a multiple of b', say it is $=-b'\beta$, and the corresponding value of y is $y=\beta$. We may for α write $\alpha+mb'$, m being any positive or negative integer, not excluding zero (but, as already remarked, this is not considered as a distinct solution of the congruence); the corresponding value of y is clearly $=\beta-ma'$.

The value of x can be found by a process similar to that for finding the greatest common measure of the two numbers a' and c'; this is what is really done in the apparently tentative process which at once presents itself for small numbers, thus $6x \equiv 9$ (mod. 35), we have $36x \equiv 54$, or, rejecting multiples of 35, $x \equiv 19$, or, if we please, $x=35m+19$.

In particular, we can always find a number ξ such that $a'\xi \equiv 1$ (mod. b'); and we have then $x=c'\xi$ as the solution of the congruence $a'x \equiv c'$. The value of ξ may be written $\xi \equiv \frac{1}{a'}$ (mod. b'), where $\frac{1}{a'}$ stands for that integer value ξ which satisfies the original congruence $a'\xi \equiv 1$ (mod. b'); and the value of x may then be written $x \equiv \frac{c'}{a'}$ (mod. b'). Another solution of the linear congruence is given in No. 21.

14. Wilson's theorem, $1.2.3\ldots\overline{p-1}+1 \equiv 0$ (mod. p). It has been seen that, for any prime number p, the congruence $x^{p-1}-1 \equiv 0$ (mod. p) of the order $p-1$ has the $p-1$ roots $1, 2, \ldots, p-1$; we have therefore

$$x^{p-1}-1 \equiv (x-1)(x-2)\ldots(x-\overline{p-1}),$$

or, comparing the terms independent of x, it appears that $1.2.3\ldots p-1 \equiv -1$, that is, $1.2.3\ldots\overline{p-1}+1 \equiv 0$ (mod. p),—the required theorem. For instance, where $p=5$, then $1.2.3.4+1 \equiv 0$ (mod. 5), and where $p=7$, then $1.2.3.4.5.6+1 \equiv 0$ (mod. 7).

15. A proof on wholly different principles may be given. Suppose, to fix the ideas, $p=7$; consider on a circle 7 points, the summits of a regular heptagon, and join

these in any manner so as to form a heptagon; the whole number of heptagons is $\frac{1}{2}.1.2.3.4.5.6$. Now of these we have $\frac{1}{2}(7-1), =3$, which are regular heptagons (convex or stellated); the number of remaining heptagons must be divisible by 7, for with any one such heptagon we connect the 6 heptagons which can be obtained from it by making it rotate through successive angles of $\frac{1}{7}360°$. That is, $\frac{1}{2}.1.2.3.4.5.6-\frac{1}{2}(7-1)\equiv 0$ (mod. 7), whence $1.2.3.4.5.6-7+1\equiv 0$, or finally $1.2.3.4.5.6+1\equiv 0$ (mod. 7). It is clear that the proof applies without alteration to the case of any prime number p.

If p is not a prime number, then $1.2.3\ldots\overline{p-1}\equiv 0$ (mod. p); hence the theorem shows directly whether a number p is or is not a prime number; but it is not of any practical utility for this purpose.

16. Prime roots of a prime number—application to the binomial equation $x^p-1=0$. Take, for instance, $p=7$. By what precedes, we have

$$x^6-1=[x^6-1][x^3-1][x^2-1][x-1],\ =(x^2-x+1)(x^2+x+1)(x+1)(x-1);$$

and we have

$$x^6-1\equiv(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)\ (\text{mod. } 7);$$

whence also

$$(x^2-x+1)(x^2+x+1)(x+1)(x-1)\equiv(x-1)(x-2)(x-3)(x-4)(x-5)(x-6).$$

These two decompositions must agree together, and we in fact have

$$x^2-x+1\equiv(x-3)(x-5),\quad x^2+x+1\equiv(x-2)(x-3),\quad x+1\equiv x-6,\quad x-1\equiv x-1.$$

In particular, we thus have 3, 5, as the roots of the congruence $x^2-x+1\equiv 0$, that is, $[x^6-1]\equiv 0$, and these roots 3, 5, are not the roots of any other of the congruences $[x^3-1]\equiv 0$, $[x^2-1]\equiv 0$, $[x-1]\equiv 0$; that is, writing $a=3$ or 5 in the series of numbers $a, a^2, a^3, a^4, a^5, a^6$, we have a^6 as the first term which is $\equiv 1$ (mod. 7); the series in fact are

$$3,\ 9\ ,\ 27\ ,\ 81\ ,\ 243\ ,\ 729\quad \equiv 3,\ 2,\ 6,\ 4,\ 5,\ 1,$$

$$5,\ 25,\ 125,\ 625,\ 3125,\ 15625\equiv 5,\ 4,\ 6,\ 2,\ 3,\ 1.$$

And so, in general, the congruence $x^{p-1}-1\equiv 0$ (mod. p) has the $p-1$ real roots $1, 2, 3, \ldots, p-1$; hence the congruence $[x^{p-1}-1]\equiv 0$, which is of the order $\phi(p-1)$, has this number $\phi(p-1)$ of real roots; and, calling any one of these g, then in the series of powers $g, g^2, g^3, \ldots, g^{p-1}$, the first term which is $\equiv 1$ (mod. p) is g^{p-1}, that is, we have $g, g^2, g^3, \ldots, g^{p-1}=1, 2, 3, \ldots, p-1$ in a different order. Any such number g is said to be a prime root of p, and the number of prime roots is $\phi(p-1)$, the number of integers less than and prime to $p-1$.

The notion of a prime root was applied by Gauss to the solution of the binomial equation $x^p-1=0$, or, what is the same thing, to the question of the division of the circle (Kreistheilung), see Equation, Nos. 30 and 31, [786]; and, as remarked in the introduction to the present article, the roots or periods of roots of this equation present themselves as the units of a complex theory in the Theory of Numbers.

17. Any number x less than p is $\equiv g^m$, and, if m is not prime to $p-1$, but has with it a greatest common measure e, suppose $m = ke$, $p-1 = ef$, then

$$x \equiv g^{ke}, \quad x^f \equiv g^{kef} \equiv g^{k(p-1)} \equiv 1,$$

that is, $x^f \equiv 1$; and it is easily seen that in the series of powers $x, x^2, \ldots, x^f$, we have x^f as the first term which is $\equiv 1 \pmod{p}$. A number $\equiv g^m$, where m is not prime to $p-1$, is thus not a prime root; and it further appears that, g being any particular prime root, the $\phi(p-1)$ prime roots are $\equiv$ the numbers g^m, where m is any number less than $p-1$ and prime to it. Thus in the foregoing example $p=7$, where the prime roots were 3 and 5, the integers less than 6 and prime to it are 1, 5; and we, in fact, have $5 \equiv 3^5$ and $3 \equiv 5^5 \pmod{7}$.

18. Integers belonging to a given exponent; index of a number. If, as before, $p-1 = ef$, that is, if f be a submultiple of $p-1$, then any integer x such that x^f is the lowest power of x which is $\equiv 1 \pmod{p}$ is said to belong to the exponent f. The number of residues, or terms of the series $1, 2, 3, \ldots, p-1$, which belong to the exponent f is $\phi(f)$, the number of integers less than f and prime to it; these are the roots of the congruence $[x^f - 1] \equiv 0$ of the order $\phi(f)$. It is hardly necessary to remark that the prime roots belong to the exponent $p-1$.

A number $x \equiv g^m$ is said to have the index m; observe the distinction between the two terms exponent and index; and, further, that the index is dependent on the selected prime root g.

19. Special forms of composite modulus. If instead of a prime modulus p we have a modulus p^m which is the power of an odd prime, or a modulus $2p$ or $2p^m$ which is twice an odd prime or a power of an odd prime, then there is a theory analogous to that of prime roots, viz. the numbers less than the modulus and prime to it are congruent to successive powers of a prime root g; thus,

if $p^m = 3^2$, we have

$$2, 4, 8, 16, 32, 64 \equiv 2, 4, 8, 7, 5, 1 \pmod{9},$$

and if $2p^m = 2 \,.\, 3^3$, we have

$$5, 25, 125, 625, 3125, 15625 \equiv 5, 7, 11, 13, 17, 1 \pmod{18}.$$

As regards the even prime 2 and its powers—for the modulus 2 or 4 the theory of prime roots does not come into existence, and for the higher powers it is not applicable; thus with modulus $=8$ the numbers less than 8 and prime to it are 1, 3, 5, 7; and we have $3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod{8}$.

20. Composite modulus $N = a^\alpha b^\beta c^\gamma \ldots$—no prime roots—irregularity. In the general case of a composite modulus it has been seen that, if x is any number less than N and prime to it, then $x^{\phi(N)} - 1 \equiv 0 \pmod{N}$. But, except in the above-mentioned cases p^m, $2p^m$, 2 or 4, there is not any number a such that $a^{\phi(N)}$ is the first power of a which is $\equiv 1$; there is always some submultiple $i = \frac{1}{\theta}\phi(N)$ such that a^i is the first

power which is $\equiv 1$. For instance, say $N = 24$, $\phi(N) = 8$, then the numbers less than 24 and prime to it are 1, 5, 7, 11, 13, 17, 19, 23; and we have

$$1^1 \equiv 1,\ \ 5^2 \equiv 7^2 \equiv 13^2 \equiv 17^2 \equiv 19^2 \equiv 23^2 \equiv 1 \text{ (mod. 24)},$$

that is, 1 has the exponent 1, but all the other numbers have the exponent 2. So again where $N = 48$, the 16 numbers less than 48 and prime to it have, 1 the exponent 1, and 7, 13, 17, 23, 25, 31, 35, 41, 47 each the exponent 2, and the remaining numbers 5, 11, 19, 29, 37, 43 each the exponent 4. We cannot in this case by means of any single root or of any two roots express all the numbers, but we can by means of three roots, for instance, 5, 7, 13, express all the numbers less than 48 and prime to it; the numbers are in fact $\equiv 5^\alpha 7^\beta 13^\gamma$, where $\alpha = 0$, 1, 2, or 3, and β and γ each $= 0$ or 1.

Comparing with the theorem for a prime number p, where the several numbers $1, 2, 3, \ldots, p-1$, are expressed by means of a single prime root, $\equiv g^\alpha$, where $\alpha = 0, 1, 2, \ldots, p-1$, we have the analogue of a case presenting itself in the theory of quadratic forms,—the "irregularity" of a determinant (*post*, No. 31); the difference is that here (the law being known, $N =$ a composite number) the case is not regarded as an irregular one, while the irregular determinants do not present themselves according to any apparent law.

21. Maximum indicator—application to solution of a linear congruence. In the case $N = 48$ it was seen that the exponents were 1, 2, 4, the largest exponent 4 being divisible by each of the others, and this property is a general one, viz. if $N = a^\alpha b^\beta c^\gamma \ldots$ in the series of exponents (or, as Cauchy calls them, indicators) of the numbers less than N and prime to it, the largest exponent I is a multiple of each of the other exponents, and this largest exponent Cauchy calls the maximum indicator; the maximum indicator I is thus a submultiple of $\phi(N)$, and it is the smallest number such that for every number x less than N and prime to it we have $x^I - 1 \equiv 0$ (mod. N). The values of I have been tabulated from $N = 2$ to 1000.

Reverting to the linear congruence $ax \equiv c$ (mod. b), where a and b are prime to each other, then, if I is the maximum indicator for the modulus b, we have $a^I \equiv 1$, and hence it at once appears that the solution of the congruence is $x \equiv ca^{I-1}$.

22. Residues of powers for an odd prime modulus. For the modulus p, if g be a prime root, then every number not divisible by p is $\equiv$ one of the series of numbers $g, g^2, \ldots, g^{p-1}$; and, if k be any positive number prime to $p-1$, then raising each of these to the power k we reproduce in a different order the same series of numbers $g, g^2, \ldots, g^{p-1}$, which numbers are in a different order $\equiv 1, 2, \ldots, p-1$, that is, the residue of a kth power may be any number whatever of the series $1, 2, \ldots, p-1$.

But, if k is not prime to $p-1$, say their greatest common measure is e, and that we have $p-1 = ef$, $k = me$, then for any number not divisible by p the kth power is $\equiv$ one of the series of f numbers $g^e, g^{2e}, \ldots, g^{fe}$; there are thus only $f, = \frac{1}{e}(p-1)$, out of the $p-1$ numbers $1, 2, 3, \ldots, p-1$, which are residues of a kth power.

23. Quadratic residues for an odd prime modulus. In particular, if $k=2$, then $e=2$, $f=\frac{1}{2}(p-1)$, and the square of every number not divisible by p is $\equiv$ one of the $\frac{1}{2}(p-1)$ numbers $g^2, g^4, \ldots, g^{p-1}$; that is, there are only $\frac{1}{2}(p-1)$ numbers out of the series $1, 2, 3, \ldots, p-1$ which are residues of a square number, or say quadratic residues, and the remaining $\frac{1}{2}(p-1)$ numbers are said to be quadratic non-residues of the modulus p,—we may say simply, residues and non-residues. But this result can be obtained more easily without the aid of the theory of prime roots. Every number not divisible by p is, to the modulus p, $\equiv$ one of the series of numbers $\pm 1, \pm 2, \pm 3, \ldots, \pm\frac{1}{2}(p-1)$; hence every square number is $\equiv$ one of the series of numbers $1^2, 2^2, 3^2, \ldots, \frac{1}{4}(p-1)^2$; and thus the $p-1$ numbers $1, 2, 3, \ldots, p-1$, are one-half of them residues and the other half non-residues of p. Thus, in the case $p=11$, every number not divisible by 11 is, to this modulus, $\equiv$ one of the series $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$; whence the square of any such number is $\equiv$ one of the series 1, 4, 9, 16, 25, or say the series 1, 4, 9, 5, 3; that is, we have

$$\begin{array}{l|l|} \text{residues} & 1, .\ 3,\ 4,\ 5,\ .\ .\ .\ 9,\ . \\ \text{non-residues} & .\ 2,\ .\ .\ .\ 6,\ 7,\ 8,\ .\ 10 \end{array}.$$

Calling as usual the residues a and the non-residues b, we have in this case

$$\tfrac{1}{11}(\Sigma b - \Sigma a) = \tfrac{1}{11}(33-22),\ =1,$$

a positive integer; this is a property true for any prime number of the form $4n+3$, but for a prime number of the form $4n+1$ we have $\Sigma b - \Sigma a = 0$; the demonstration belongs to a higher part of the theory.

It is easily shown that the product of two residues or of two non-residues is a residue; but the product of a residue and a non-residue is a non-residue.

24. The law of reciprocity—Legendre's symbol. The question presents itself, given that P is a residue or a non-residue of Q, can we thence infer whether Q is a residue or a non-residue of P? In particular, if P, Q, are the odd primes p, q, for instance, given that $13 = R(17)$, can we thence infer that $17 = R(13)$, or that $17 = NR(13)$? The answer is contained in the following theorem: If p, q, are odd primes each or one of them of the form $4n+1$, then p, q, are each of them a residue or each of them a non-residue of the other; but, if p, q, are each of them of the form $4n+3$, then, according as p is a residue or a non-residue of q, we have q a non-residue or a residue of p.

The theorem is conveniently expressed by means of Legendre's symbol, viz. p being a positive odd prime, and Q any positive or negative number not divisible by p, then $\left(\frac{Q}{p}\right)$ denotes $+1$ or -1, according as Q is or is not a residue of p; if, as before, q is (as p) a positive odd prime, then the foregoing theorem is

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{1}{4}(p-1)(q-1)}.$$

The denominator symbol may be negative, say it is $-p$, we then have as a definition $\left(\frac{Q}{-p}\right)=\left(\frac{Q}{p}\right)$—observe that $\left(\frac{-Q}{p}\right)$ is not $=\left(\frac{Q}{-p}\right)$—and we have further the theorems

$$\left(\frac{-1}{p}\right)=(-1)^{\frac{1}{2}(p-1)},\quad \left(\frac{2}{p}\right)=(-1)^{\frac{1}{8}(p^2-1)},$$

viz. -1 is a residue or a non-residue of p according as $p\equiv 1$ or $\equiv 3$ (mod. 4), and 2 is a residue or a non-residue of p according as $p\equiv 1$ or 7, or $\equiv 3$ or 5 (mod. 8). If, as definitions, $\left(\frac{p}{-1}\right)=+1$ and $\left(\frac{p}{2}\right)=+1$, these may be written

$$\left(\frac{-1}{p}\right)\left(\frac{p}{-1}\right)=(-1)^{\frac{1}{2}(p-1)},\text{ and }\left(\frac{2}{p}\right)\left(\frac{p}{2}\right)=(-1)^{\frac{1}{8}(p^2-1)}.$$

We have also, what is in fact a theorem given at the end of No. 23,

$$\left(\frac{QQ'}{p}\right)=\left(\frac{Q}{p}\right)\left(\frac{Q'}{p}\right).$$

The further definition is sometimes convenient—

$$\left(\frac{Q}{p}\right)=0,\text{ when } p \text{ divides } Q.$$

The law of reciprocity, as contained in the theorem

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{1}{4}(p-1)(q-1)},$$

is a fundamental theorem in the whole theory; it was enunciated by Legendre, but first proved by Gauss, who gave no less than six demonstrations of it.

25. Jacobi's generalized symbol. Jacobi defined this as follows: The symbol $\left(\frac{Q}{\pm pp'p''\ldots}\right)$, where $p, p', p'', \ldots$ are positive odd primes equal or unequal, and Q is any positive or negative odd number prime to $pp'p''\ldots$, denotes $+1$ or -1 according to the definition

$$\left(\frac{Q}{\pm pp'p''\ldots}\right)=\left(\frac{Q}{p}\right)\left(\frac{Q}{p'}\right)\left(\frac{Q}{p''}\right)\ldots,$$

the symbols on the right-hand side being Legendre's symbols. But the definition may be regarded as extending to the case where Q is not prime to $pp'p''\ldots$: then we have Q divisible by some factor p, and by the definition of Legendre's symbol in this case we have $\left(\frac{Q}{p}\right)=0$; hence in the case in question of Q not being prime to $pp'p''\ldots$, the value of Jacobi's symbol is $=0$.

We may further extend the definition of the symbol to the case where the numerator and the denominator of the symbol are both or one of them even, and present the definition in the most general form, as follows: suppose that $p, p', p'', \ldots$

being positive or negative even or odd primes, equal or unequal, and similarly $q, q', q'', \ldots$ being positive or negative even or odd primes, equal or unequal, we have $P = pp'p''\ldots$ and $Q = qq'q''\ldots$, then the symbol $\left(\frac{Q}{P}\right)$ will denote $+1, -1$, or 0, according to the definition

$$\left(\frac{Q}{P}\right) = \left(\frac{q}{p}\right)\left(\frac{q}{p'}\right)\left(\frac{q}{p''}\right)\ldots\left(\frac{q'}{p}\right)\left(\frac{q'}{p'}\right)\left(\frac{q'}{p''}\right)\ldots,$$

the symbols on the right-hand being Legendre's symbols. If P and Q are not prime to each other, then for some pair of factors p and q we have $p = \pm q$, and the corresponding Legendrian symbol $\left(\frac{q}{p}\right)$ is $= 0$, whence in this case $\left(\frac{Q}{P}\right) = 0$.

It is important to remark that $\left(\frac{Q}{P}\right) = +1$ is not a sufficient condition in order that Q may be a residue of P; if $P = 2^a pp'p''\ldots$, $p, p', p'', \ldots$ being positive odd primes, then, in order that Q may be a residue of P, it must be a residue of each of the prime factors $p, p', p'', \ldots$, that is, we must have

$$\left(\frac{Q}{p}\right) = +1, \quad \left(\frac{Q}{p'}\right) = +1, \quad \left(\frac{Q}{p''}\right) = +1, \ldots,$$

as many equations as there are unequal factors $p, p', p'', \ldots$ of the modulus P.

Ordinary Theory, Second Part,—Theory of Forms.

26. Binary quadratic (or quadric) forms—transformation and equivalence. We consider a form

$$ax^2 + 2bxy + cy^2, \ = (a, b, c)(x, y)^2,$$

or when, as usual, only the coefficients are attended to, $= (a, b, c)$. The coefficients (a, b, c) and the variables (x, y) are taken to be positive or negative integers, not excluding zero. The discriminant $ac - b^2$ taken negatively, that is, $b^2 - ac$, is said to be the determinant of the form: and we thus distinguish between forms of a positive and of a negative determinant.

Considering new variables, $\alpha x + \beta y$, $\gamma x + \delta y$, where $\alpha, \beta, \gamma, \delta$, are positive or negative integers, not excluding zero, we have identically

$$(a, b, c)(\alpha x + \beta y, \gamma x + \delta y)^2 = (a', b', c')(x, y)^2,$$

where

$$\begin{aligned} a' &= (a, b, c)(\alpha, \gamma)^2, &&= a\alpha^2 + 2b\alpha\gamma + c\gamma^2, \\ b' &= (a, b, c)(\alpha, \gamma)(\beta, \delta), &&= a\alpha\beta + b(\alpha\delta + \beta\gamma) + c\gamma\delta, \\ c' &= (a, b, c)(\beta, \delta)^2, &&= a\beta^2 + 2b\beta\delta + c\delta^2; \end{aligned}$$

and thence

$$b'^2 - a'c' = (\alpha\delta - \beta\gamma)^2(b^2 - ac).$$

The form (a', b', c') is in this case said to be contained in the form (a, b, c); and a condition for this is obviously that the determinant D' of the contained form

shall be equal to the determinant D of the containing form multiplied by a square number; in particular, the determinants must be of the same sign. If the determinants are equal, then $(\alpha\delta-\beta\gamma)^2=1$, that is, $\alpha\delta-\beta\gamma=\pm 1$. Assuming in this case that the transformation exists, and writing $\alpha\delta-\beta\gamma=\epsilon$, and writing also

$$x'=\alpha x+\beta y,$$
$$y'=\gamma x+\delta y,$$

then conversely

$$x=\frac{1}{\epsilon}(\ \ \delta x'-\beta y'),\ =\alpha' x'+\beta' y',$$

$$y=\frac{1}{\epsilon}(-\gamma x'+\alpha y'),\ =\gamma' x'+\delta' y',$$

suppose, where α', β', γ', δ' are integers; and we have, moreover,

$$\alpha'\delta'-\beta'\gamma'=\frac{1}{\epsilon^2}(\alpha\delta-\beta\gamma),\ =\frac{1}{\epsilon},\ =\epsilon,$$

that is, $\alpha'\delta'-\beta'\gamma'=+1$ or -1, according as $\alpha\delta-\beta\gamma$ is $=+1$ or -1. The two forms (a, b, c), (a', b', c') are in this case said to be equivalent, and to be, in regard to the particular transformation, equivalent properly or improperly according as $\alpha\delta-\beta\gamma(=\alpha'\delta'-\beta'\gamma')$ is $=+1$ or $=-1$. We have, therefore, as a condition for the equivalence of two forms, that their determinants shall be equal; but this is not a sufficient condition. It is to be remarked also that two forms of the same determinant may be equivalent properly and also improperly; there may exist a transformation for which $\alpha\delta-\beta\gamma$ is $=+1$, and also a transformation for which $\alpha\delta-\beta\gamma$ is $=-1$. But this is only the case when each of the forms is improperly equivalent to itself; for instance, a form x^2-Dy^2, which remains unaltered by the change x, y, into x, $-y$ (that is, α, β, γ, $\delta=1$, 0, 0, -1, and therefore $\alpha\delta-\beta\gamma=-1$), is a form improperly equivalent to itself. A form improperly equivalent to itself is said to be an ambiguous form. In what follows, equivalent means always properly equivalent.

27. Forms for a given determinant—classes, &c. In the case where D, $=b^2-ac$, is a square, the form $(a, b, c)(x, y)^2$ is a product of two rational factors; this case may be excluded from consideration, and we thus assume that the determinant D is either negative, or, being positive, that it is not a square. The forms (a, b, c) of a given positive or negative determinant are each of them equivalent to some one out of a finite number of non-equivalent forms which may be considered as representing so many distinct classes. For instance, every form of the determinant -1 is equivalent to $(1, 0, 1)$, that is, given any form (a, b, c) for which $b^2-ac=-1$, it is possible to find integer values α, β, γ, δ, such that $\alpha\delta-\beta\gamma=+1$, and $(a, b, c)(\alpha x+\beta y, \gamma x+\delta y)^2=(1, 0, 1)(x, y)^2$, that is, $=x^2+y^2$. Or, to take a less simple example, every form of the determinant -35 is equivalent to one of the following forms: $(1, 0, 35)$, $(5, 0, 7)$, $(3, \pm 1, 12)$, $(4, \pm 1, 18)$,—$(2, 1, 8)$, $(6, 1, 6)$; for the first six forms, the numbers a, $2b$, c have no common factor, and these are said to be *properly primitive* forms, or to belong to the properly primitive order; for the last two forms, the numbers a, b, c have no common factor, but, a and c being each even, the numbers a, $2b$, c have a common factor 2,

and these are said to be *improperly primitive* forms, or to belong to the improperly primitive order. The properly primitive forms are thus the six forms (1, 0, 35), (5, 0, 7), (3, ± 1, 12), (4, ± 1, 18); or we may say that there are represented hereby six properly primitive classes. Derived forms, or forms which belong to a derived order, present themselves in the case of a determinant D having a square factor or factors, and it is not necessary to consider them here.

It is not proposed to give here the rules for the determination of the system of non-equivalent forms; it will be enough to state that this depends on the determination in the first instance of a system of *reduced* forms, that is, forms for which the coefficients a, b, c, taken positively satisfy certain numerical inequalities admitting only of a finite number of solutions. In the case of a negative determinant, the reduced forms are no two of them equivalent, and we thus have the required system of non-equivalent forms; in the case of a positive determinant, the reduced forms group themselves together in *periods* in such wise that the forms belonging to a period are equivalent to each other, and the required system of non-equivalent forms is obtained by selecting one form out of each such period. The principal difference in the theory of the two cases of a positive and a negative determinant consists in these periods; the system of non-equivalent forms once arrived at, the two theories are nearly identical.

28. Characters of a form or class—division into genera. Attending only to the properly primitive forms: for instance, those mentioned above for the determinant -35: the form (1, 0, 35) represents only numbers f which are residues of 5, and also residues of 7; we have, in fact, $f = x^2 + 35y^2$, $\equiv x^2 \pmod{5}$, and also $\equiv x^2 \pmod{7}$. Using the Legendrian symbols $\left(\frac{f}{5}\right)$ and $\left(\frac{f}{7}\right)$, we say that the form (1, 0, 35) has the characters $\left(\frac{f}{5}\right)$, $\left(\frac{f}{7}\right) = ++$. Each of the other forms has in like manner a determinate character $+$ or $-$ in regard to $\left(\frac{f}{5}\right)$ and also in regard to $\left(\frac{f}{7}\right)$; and it is found that for each of them the characters are $+\,+$ or else $-\,-$ (that is, they are never $+\,-$ or $-\,+$). We, in fact, have

$$\begin{array}{lcc}
 & \left(\frac{f}{5}\right) & \left(\frac{f}{7}\right) \\
\hline
(1,\ \ 0,\ \ 35) & + & + \\
(4,\ \pm 1,\ \ 9) & & \\
\hline
(5,\ \ 0,\ \ 7) & - & - \\
(3,\ \pm 1,\ 12) & &
\end{array},$$

and we thus arrange the six forms into genera, viz. we have three forms belonging to the genus $\left(\frac{f}{5}\right)$, $\left(\frac{f}{7}\right) = ++$, and three to the genus $\left(\frac{f}{5}\right)$, $\left(\frac{f}{7}\right) = --$, these characters $++$ and $--$ of genera being one-half of all the combinations $++$, $--$, $+-$, $-+$.

The like theory applies to any other negative or positive determinant; the several characters have reference in some cases not only to the odd prime factors of D but

also to the numbers 4 and 8, that is, there is occasion to consider also the Legendrian symbols $\left(\frac{-1}{f}\right)$, $=(-1)^{\frac{1}{2}(f-1)}$, and $\left(\frac{2}{f}\right)$, $=(-1)^{\frac{1}{8}(f^2-1)}$, and there are various cases to be considered according to the form of D in regard to its simple and squared factors respectively; but in every case there are certain combinations of characters (in number one-half of all the combinations) which correspond to genera, and the properly primitive forms belong to different genera accordingly, the number of forms being the same in each genus.

The form $(1, 0, -D)$ has the characters all $+$, and this is said to be the principal form, and the genus containing it the principal genus. For a given determinant, the characters of two genera may be compounded together according to the ordinary rule of signs, giving the characters of a new genus; in particular, if the characters of a genus are compounded with themselves, then we have the characters of the principal genus.

29. Composition of quadratic forms. Considering X, Y, as given lineo-linear functions of (x, y), (x', y'), defined by the equations

$$X = p_0xx' + p_1xy' + p_2yx' + p_3yy',$$
$$Y = q_0xx' + q_1xy' + q_2yx' + q_3yy',$$

the coefficients p_0, p_1, p_2, p_3, q_0, q_1, q_2, q_3, may be so connected with the coefficients (A, B, C), (a, b, c), (a', b', c'), of three quadratic forms as to give rise to the identity

$$(A, B, C)(X, Y)^2 = (a, b, c)(x, y)^2 . (a', b', c')(x', y')^2;$$

and, this being so, the form (A, B, C) is said to be compounded of the two forms (a, b, c) and (a', b', c'), the order of composition being indifferent.

The necessary and sufficient condition, in order that it may be possible to compound together two given forms (a, b, c), (a', b', c'), is that their determinants shall be to each other in the proportion of two square numbers; in particular, the two forms may have the same determinant D; and when this is so the compound form (A, B, C) will also have the same determinant D. The rules for this composition of two forms of the same determinant have been (as part of the general theory) investigated and established. The forms compounded of equivalent forms are equivalent to each other; we thus in effect compound *classes*, viz. considering any two classes, the composition of their representative forms gives a form which is the representative of a new class, and the composition of any two forms belonging to the two classes respectively gives a form belonging to the new class. But, this once understood, it is more simple to speak of the composition of forms, that is, of the forms belonging to the finite system of representative forms for a given determinant; and it will be enough to consider the properly primitive forms.

30. The principal form $(1, 0, D)$, compounded with any other form (a, b, c), gives rise to this same form (a, b, c); the principal form is on this account denoted by 1, viz. denoting the other form by ϕ, and expressing composition in like manner with

multiplication, we have $1 . \phi = \phi$. The form ϕ may be compounded with itself, giving a form denoted by ϕ^2; compounding this again with ϕ, we have a form denoted by ϕ^3; and so on. Since the whole number of forms is finite, we must in this manner arrive at the principal form, say we have $\phi^n = 1$, n being the least exponent for which this equation is satisfied. In particular, if the form ϕ belong to the principal genus, then the forms $\phi^2, \phi^3, \ldots, \phi^{n-1}$ will all belong to the principal genus, or the principal genus will include the forms $1, \phi, \phi^2, \ldots, \phi^{n-1}$, the powers of a form ϕ having the exponent n.

31. Regular and irregular determinants. The principal genus may consist of such a series of forms, and the determinant is then said to be *regular*; in particular, for a negative determinant D, $= -1$ to -1000, the determinant is always regular except in the thirteen cases $-D = 243$, 307, 339, 459, 576, 580, 675, 755, 820, 884, 891, 900, 974 (and, Perott, in *Crelle*, vol. XCV., 1883, except also for $-D = 468$, 931); the determinant is here said to be irregular. Thus for each of the values $-D = 576$, 580, 820, 900, the principal genus consists of four forms, not $1, \phi, \phi^2, \phi^3$, where $\phi^4 = 1$, but $1, \phi, \phi_1, \phi\phi_1$, where $\phi^2 = 1$, $\phi_1^2 = 1$, and therefore also $(\phi\phi_1)^2 = 1$.

Compounding together any two forms, we have a form with the characters compounded of the characters of the two forms; and in particular, combining a form with itself, we have a form with the characters of the principal form. Or, what is the same thing, any two genera compounded together give rise to a determinate genus, viz. the genus having the characters compounded of the characters of the two genera; and any genus compounded with itself gives rise to the principal genus.

Considering any regular determinant, suppose that there is more than one genus, and that the number of forms in each genus is $= n$; then, except in the case $n = 2$, it can be shown that there are always forms having the exponent $2n$. For instance, in the case $D = -35$, we have two genera each of three forms; there will be a form g having the exponent 6, or $g^6 = 1$; and the forms are $1, g, g^2, g^3, g^4, g^5$, where $1, g^2, g^4$, belong to the principal genus, and g, g^3, g^5, to the other genus. The characters refer to $\left(\frac{f}{5}\right)$, $\left(\frac{f}{7}\right)$, and the forms are

$$\begin{array}{rrrrl|rrrrl} + +, & (1, & 0, & 35) & 1 & - -, & (3, & -1, & 12) & g \\ & (4, & 1, & 9) & g^2 & & (5, & 0, & 7) & g^3 \\ & (4, & -1, & 9) & g^4 & & (3, & 1, & 12) & g^5. \end{array}$$

An instance of the case $n = 1$ is $D = -21$, there are here four genera each of a single form $1, c, c_1, cc_1$, where $c^2 = 1$, $c_1^2 = 1$; an instance of the case $n = 2$ is $D = -88$, there are here two genera each of two forms $1, c$, and c_1, cc_1, where $c^2 = 1$, $c_1^2 = 1$, thus there is here no form having the exponent $2n$. (See Cayley, *Tables, &c.*, in *Crelle*, t. LX., 1862, pp. 357—372, [335].) We may have 2^{k+1} genera, each of n forms, viz. such a system may be represented by $(1, \phi^2, \ldots, \phi^{2n-2}; \phi, \phi^3, \ldots, \phi^{2n-1})(1, c)(1, c_1)\ldots(1, c_{k-1})$, where $\phi^{2n} = 1$, $c^2 = 1$, $c_1^2 = 1, \ldots, c_{k-1}^2 = 1$; there is no peculiarity in the form ϕ: we may instead of it take any form such as $c\phi$, $cc_1\phi$, &c., for each of these is like ϕ, a form belonging to the exponent $2n$, and such that the even powers give the principal genus.

32. Ternary and higher quadratic forms—cubic forms, &c. The theory of the ternary quadratic forms

$$(a,\ b,\ c,\ a',\ b',\ c')(x,\ y,\ z)^2,\ = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy,$$

or when only the coefficients are attended to, $\begin{pmatrix} a, & b, & c \\ a', & b', & c' \end{pmatrix}$, has been studied in a very complete manner; and those of the quaternary and higher quadratic forms have also been studied; in particular, the forms $x^2 + y^2 + z^2$, $a^2 + y^2 + z^2 + w^2$ composed of three or four squares; and the like forms with five, six, seven, and eight squares. The binary cubic forms $(a,\ b,\ c,\ d)(x,\ y)^3,\ = ax^3 + 3bx^2y + 3cxy^2 + dy^3$, or when only the coefficients are attended to, $(a,\ b,\ c,\ d)$, have also been considered, though the higher binary forms have been scarcely considered at all. The special ternary cubic forms $ax^3 + by^3 + cz^3 + 6lxyz$ have been considered. Special forms of the degree n with n variables, the products of linear factors, present themselves in the theory of the division of the circle (the Kreistheilung) and of the complex numbers connected therewith; but it can hardly be said that these have been studied as a part of the general theory of forms.

Complex Theories.

33. The complex theory which first presented itself is that of the numbers $a + bi$ composed with the imaginary $i,\ = \sqrt{-1}$; here if a and b are ordinary, or say simplex positive or negative integers, including zero, we regard $a + bi$ as an integer number, or say simply as a number in this complex theory. We have here a zero 0 $(a = 0,\ b = 0)$ and the units $1,\ i,\ -1,\ -i$, or as these may be written, $1,\ i,\ i^2,\ i^3$ $(i^4 = 1)$; the numbers $a + bi,\ a - bi$, are said to be conjugate numbers, and their product $(a + bi)(a - bi)$, $= a^2 + b^2$, is the norm of each of them. And so the norm of the real number a is $= a^2$, and that of the pure imaginary number bi is $= b^2$. Denoting the norm by the letter N, $N(a \pm bi) = a^2 + b^2$.

Any simplex prime number, $\equiv 1 \pmod{4}$, is the sum of two squares $a^2 + b^2$, for instance $13 = 9 + 4$, and it is thus a product $(a + bi)(a - bi)$, that is, it is not a prime number in the present theory, but each of these factors (or say any number $a + bi$, where $a^2 + b^2$ is a prime number in the simplex theory) is a prime; and any simplex prime number, $\equiv 3 \pmod{4}$, is also a prime in the present theory. The number $2,\ = (1 + i)(1 - i)$, is not a prime, but the factors $1 + i,\ 1 - i$ are each of them prime; these last differ only by a unit factor i—$1 = i(1 + i)$—so that $2,\ = -i(1 + i)^2$, contains a square factor.

In the simplex theory we have numbers, for instance $5,\ -5$, differing from each other only by a unit factor, but we can out of these select one, say the positive number, and attend by preference to this number of the pair. It is in this way—viz. by restricting $a,\ b,\ c, \ldots$ to denote terms of the series $2,\ 3,\ 5,\ 7, \ldots$ of positive primes other than unity—that we are enabled to make the definite statement, a positive number N is, and that in one way only, $= a^\alpha b^\beta c^\gamma \ldots$; if N be a positive or negative number, then the theorem of course is, N is, and that in one way only, $= (-1)^m a^\alpha b^\beta c^\gamma \ldots$, where $m = 0$, or 1, and $a,\ b,\ c, \ldots,\ \alpha,\ \beta,\ \gamma, \ldots$ are as before. To

obtain a like definite statement in the present theory, we require to distinguish between the four numbers $a+bi$, $-a-bi$, $-b+ai$, $b-ai$, which differ from each other only by a unit factor -1, $\pm i$. Consider a number $a+bi$ where a and b are the one of them odd and the other even (a and b may be either of them $=0$, the other is then odd), every prime number $a+bi$ other than $\pm 1 \pm i$ is necessarily of this form: for if a and b were both even, the number would be divisible by 2, or say by $(1+i)^2$, and if a and b were both odd, it would be divisible by $1+i$; then of the four associated numbers $a+bi$, $-a-bi$, $-b+ai$, $b-ai$, there is one and only one, $a+bi$, such that b is even and $a+b-1$ is evenly even; or say one and only one which is $\equiv 1$ (mod. $2(1+i)$). We distinguish such one of the four numbers from the other three and call it a *primary* number; the units ± 1, $\pm i$, and the numbers $\pm 1 \pm i$, are none of them primary numbers. We have then the theorem, a number N is in one way only $= i^m (1+i)^n A^\alpha B^\beta \ldots$, where $m=0$, 1, 2, or 3, n is $=0$ or a positive integer, A, $B, \ldots$ are primary primes, α, $\beta, \ldots$ positive integers. Here i is a unit of the theory, $1+i$ is a special prime having reference to the number 2, but which might, by an extension of the definition, be called a primary prime, and so reckoned as one of the numbers A, $B, \ldots$; the theorem stated broadly still is that the number N is, and that in one way only, a product of prime factors, but the foregoing complete statement shows the precise sense in which this theorem must be understood. A like explanation is required in other complex theories; we have to select out of each set of primes differing only by unit factors some one number as a primary prime, and the general theorem then is that every number N is, and that in one way only, $= P.A^\alpha B^\beta C^\gamma \ldots$, where P is a product of unities, and A, B, $C, \ldots$ are primary primes.

34. We have in the simplex theory (*ante*, No. 10) the theorem that, p being an odd prime, there exists a system of $p-1$ residues, that is, that any number not divisible by p is, to the modulus p, congruent to one, and only one, of the $p-1$ numbers 1, 2, 3, ..., $p-1$. The analogous theorem in the complex theory is that, for any prime number p other than $\pm 1 \pm i$, there exists a system of $N(p)-1$ residues, that is, that every number not divisible by p is, to the modulus p, congruent to one of these $N(p)-1$ numbers.

But p may be a real prime such as 3, or a complex prime such as $3+2i$; and the system of residues presents itself naturally under very different forms in the two cases respectively. Thus in the case $p=3$, $N(3)=9$, the residues may be taken to be

$$\begin{array}{lll} 1 & , 2 & , \\ i, & 1+i\,, & 2+i\,, \\ 2i, & 1+2i, & 2+2i, \end{array}$$

being in number $N(3)-1=8$. And for $p=3+2i$, $N(3+2i)=13$, they may be taken to be the system of residues of 13 in the simplex theory, viz. the real numbers 1, 2, 3, ..., 12. We have in fact $5+i=(2+3i)(1-i)$, that is, $5+i \equiv 0$ (mod. $2+3i$), and consequently $a+bi \equiv a-5b$, a real number which, when $a+bi$ is not divisible by $3+2i$, may have any one of the foregoing values 1, 2, 3, ..., 12.

Taking then any number x not divisible by p, the $N(p)-1$ residues each multiplied by x are, to the modulus p, congruent to the series of residues in a different order; and we thus have,—say this is Fermat's theorem for the complex theory—$x^{N(p)-1}-1\equiv 0$ (mod. p), with all its consequences, in particular, the theory of prime roots.

In the case of a complex modulus such as $3+2i$, the theory is hardly to be distinguished from its analogue in the ordinary theorem; a prime root is $=2$, and the series of powers is 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, for the modulus $3+2i$ as for the modulus 13. But for a real prime such as 3 the prime root is a complex number; taking it to be $=2+i$, we have $(2+i)^8-1\equiv 0$ (mod. 3), and the series of powers in fact is $2+i$, i, $2+2i$, 2, $1+2i$, $2i$, $1+i$, 1, viz. we thus have the system of residues (mod. 3).

We have in like manner a theory of quadratic residues; a Legendrian symbol $\left[\frac{p}{q}\right]$ (which, if p, q, are uneven primes not necessarily primary but subject to the condition that their imaginary parts are even, denotes $+1$ or -1 according as $p^{\frac{1}{2}(Nq-1)}$ is $\equiv 1$ or $\equiv -1$ (mod. q), so that $\left[\frac{p}{q}\right]=+1$ or -1 according as p is or is not a residue of q), a law of reciprocity expressed by the very simple form of equation $\left[\frac{p}{q}\right]=\left[\frac{q}{p}\right]$, and generally a system of properties such as that which exists in the simplex theory.

The theory of quadratic forms (a, b, c) has been studied in this complex theory; the results correspond to those of the simplex theory.

35. The complex theory with the imaginary cube root of unity has also been studied; the imaginary element is here γ, $=\frac{1}{2}(-1+\sqrt{-3})$, a root of the equation $\gamma^2+\gamma+1=0$; the form of the complex number is thus $a+b\gamma$, where a and b are any positive or negative integers, including zero. The conjugate number is $a+b\gamma^2$, $=a-b-b\gamma$, and the product $(a+b\gamma)(a+b\gamma^2)$, $=a^2-ab+b^2$, is the norm of each of the factors $a+b\gamma$, $a+b\gamma^2$. The whole theory corresponds very closely to, but is somewhat more simple than, that of the complex numbers $a+bi$.

36. The last-mentioned theory is a particular case of the complex theory for the imaginary λth roots of unity, λ being an odd prime. Here α is determined by the equation $\frac{\alpha^\lambda-1}{\alpha-1}=0$, that is, $\alpha^{\lambda-1}+\alpha^{\lambda-2}+\ldots+\alpha+1=0$, and the form of the complex number is $f(\alpha)$, $=a+b\alpha+c\alpha^2+\ldots+k\alpha^{\lambda-2}$, where $a, b, c, \ldots, k$, are any positive or negative integers, including zero. We have $\lambda-1$ conjugate forms, viz. $f(\alpha), f(\alpha^2), \ldots, f(\alpha^{\lambda-1})$, and the product of these is the norm of each of the factors $Nf(\alpha)$, $=Nf(\alpha^2)$, $=\ldots$, $=Nf(\alpha^{\lambda-1})$. Taking g any prime root of λ, $g^{\lambda-1}-1\equiv 0$ (mod. λ), the roots $\alpha, \alpha^2, \ldots, \alpha^{\lambda-1}$, may be arranged in the order $\alpha, \alpha^g, \alpha^{g^2}, \ldots, \alpha^{g^{\lambda-2}}$; and we have thence a grouping of the roots in periods, viz. if $\lambda-1$ be in any manner whatever expressed as a product of two factors, $\lambda-1=ef$, we may with the $\lambda-1$ roots form e periods $\eta_0, \eta_1, \ldots, \eta_{e-1}$, each of

f roots. For instance, when $\lambda = 13$, a prime root is $g = 2$, and $\lambda - 1 = ef = 3 . 4$; then the three periods each of four roots are

$$\eta_0 = \alpha + \alpha^8 + \alpha^{12} + \alpha^5,$$
$$\eta_1 = \alpha^2 + \alpha^3 + \alpha^{11} + \alpha^{10},$$
$$\eta_2 = \alpha^4 + \alpha^6 + \alpha^9 + \alpha^7.$$

So also, if $ef = 2 . 6$, then the 2 periods each of 6 roots are

$$\eta_0 = \alpha + \alpha^4 + \alpha^3 + \alpha^{12} + \alpha^9 + \alpha^{10},$$
$$\eta_1 = \alpha^2 + \alpha^8 + \alpha^6 + \alpha^{11} + \alpha^5 + \alpha^7;$$

and so in other cases. In particular, if $f = 1$ and consequently $e = \lambda - 1$, the e periods each of f roots are, in fact, the single roots $\alpha, \alpha^g, \ldots, \alpha^{g^{\lambda-2}}$. We may, in place of the original form of the complex number

$$f(\alpha) = a + b\alpha + c\alpha^2 + \ldots + k\alpha^{\lambda-2},$$

consider the new form $f(\eta) = a\eta + b\eta_1 + \ldots + l\eta_{e-1}$, which when $f = 1$ is equivalent to the original form, but in any other case denotes a special form of complex number; instead of $\lambda - 1$ we have only e conjugate numbers, and the product of these e numbers may be regarded as the norm of $f(\eta)$.

37. The theory for the roots α includes as part of itself the theory for the periods corresponding to every decomposition whatever $\lambda - 1 = ef$ of $\lambda - 1$ into two factors, but each of these may be treated apart from the others as a theory complete in itself. In particular, a simple case is that of the half-periods $e = 2$, $f = \frac{1}{2}(\lambda - 1)$; and, inasmuch as the characteristic phenomenon of ideal numbers presents itself in this theory of the half-periods (first for the value $\lambda = 23$), it will be sufficient, by way of illustration of the general theory, to consider only this more special and far easier theory; we may even assume $\lambda = 23$.*

For the case in question, $\lambda - 1 = ef = 2 . \frac{1}{2}(\lambda - 1)$, we have the two periods η_0, η_1, each of $\frac{1}{2}(\lambda - 1)$ roots; from the expressions for η_0, η_1, in terms of the roots we obtain at once $\eta_0 + \eta_1 = -1$, and with a little more difficulty $\eta_0\eta_1 = -\frac{1}{4}(\lambda - 1)$ or $\frac{1}{4}(\lambda + 1)$, according as λ is $\equiv 1$ or $3 \pmod 4$, that is, in the two cases respectively η_0, η_1, are the roots of the equation $\eta^2 + \eta - \frac{1}{4}(\lambda - 1) = 0$, and $\eta^2 + \eta + \frac{1}{4}(\lambda + 1) = 0$. And this equation once obtained, there is no longer any occasion to consider the original equation of the order $\lambda - 1$, but the theory is that of the complex numbers $a\eta_0 + b\eta_1$, or

* In the theory of the roots α, ideal numbers do not present themselves for the values $\lambda = 3$, 5, or 7; they do for the value $\lambda = 23$. It is stated in Smith's "Report on the Theory of Numbers," *Brit. Assoc. Report* for 1860, p. 136, [Collected Works, vol. I. p. 114], that "for $\lambda = 11$, $\lambda = 13$, $\lambda = 17$, and $\lambda = 19$, it is not possible to say whether this is or is not the case for these values also." The writer is not aware whether this question has been settled; but in Reuschle's *Tafeln*, 1875, no ideal factors present themselves for these values of λ; and it is easy to see that, in the theory of the half-periods, the ideal factors first present themselves for the value $\lambda = 23$. It may be remarked that the solution of the question depends on the determination of a system of fundamental units for the values in question $\lambda = 11$, 13, 17, and 19; the theory of the units in the several complex theories is an important and difficult part of the theory, not presenting itself in the theory of the half-periods, which is alone attended to in the text.

if we please $a+b\eta$, composed with the roots of this quadric equation,—say the complex numbers $a+b\eta$, where a and b are any positive or negative integer numbers, including zero. In the case $\lambda=23$, the quadric equation is $\eta^2+\eta+6=0$. We have $N(a+b\eta)=(a+b\eta_0)(a+b\eta_1)=a^2-ab+\frac{1}{4}(\lambda+1)b^2$; and for $\lambda=23$, this is $N(a+b\eta)=a^2-ab+6b^2$. It may be remarked that there is a connexion with the theory of the quadratic forms of the determinant -23, viz. there are here the three improperly primitive forms (2, 1, 12), (4, 1, 6), (4, -1, 6), 23 being the smallest prime number for which there exists more than one improperly primitive form.

38. Considering then the case $\lambda=23$, we have η_0, η_1, the roots of the equation $\eta^2+\eta+6=0$; and a real number P is composite when it is $=(a+b\eta_0)(a+b\eta_1)$, $=a^2-ab+6b^2$, viz. if $4P=(2a-b)^2+23b^2$. Hence no number, and in particular no positive real prime P, can be composite unless it is a (quadratic) residue of 23; the residues of 23 are 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18; and we have thus, for instance, 5, 7, 11, as numbers which are *not* composite, while 2, 3, 13, are numbers which are not by the condition precluded from being composite: they are not, according to the foregoing signification of the word, composite (for 8, 12, 52, are none of them of the form x^2+23y^2), but *some* such numbers, residues that is of 23, are composite, for instance 59, $=(5-2\eta_0)(5-2\eta_1)$. And we have an indication, so to speak, of the composite nature of *all* such numbers; take for instance 13, we have $(\eta-4)(\eta+5)=-2.13$, where 13 does not divide either $\eta-4$ or $\eta+5$, and we are led to conceive it as the product of two ideal factors, one of them dividing $\eta-4$, the other dividing $\eta+5$. It appears, moreover, that a power 13^3 is in fact composite, viz. we have

$$13^3=(31-12\eta_0)(31-12\eta_1),\quad (2197=961+372+864);$$

and writing $13=\sqrt[3]{31-12\eta_0}.\sqrt[3]{31-12\eta_1}$ we have 13 as the product of two ideal numbers each represented as a cube root; it is to be observed that, 13 being in the simplex theory a prime number, these are regarded as prime ideal numbers. We have in like manner

$$2=\sqrt[3]{1-\eta_0}.\sqrt[3]{1-\eta_1},\quad 3=\sqrt[3]{1-2\eta_0}.\sqrt[3]{1-2\eta_1},\ \&c.;$$

every positive real prime which is a residue of 23 is thus a product of two factors ideal or actual. And, reverting to the equation $(\eta-4)(\eta+5)=-2.13$, or as this may be written

$$(\eta_1-4)(\eta_1+5)=-\sqrt[3]{1-\eta_0}\sqrt[3]{1-\eta_1}\sqrt[3]{31-12\eta_0}\sqrt[3]{31-12\eta_1},$$

we have $(\eta_1-4)^3$ and $(1-\eta_0)(31-12\eta_0)$ each $=14+55\eta_1$, or say

$$\eta_1-4=\sqrt[3]{1-\eta_0}\sqrt[3]{31-12\eta_0},$$

and similarly

$$\eta_1+5=-\sqrt[3]{1-\eta_1}\sqrt[3]{31-12\eta_1},$$

so that we verify that η_1-4, η_1+5, do thus in fact each of them contain an ideal factor of 13.

39. We have $2=\sqrt[3]{1-\eta_0}\sqrt[3]{1-\eta_1}$, viz. the ideal multiplier $\sqrt[3]{1-\eta_0}$ renders actual one of the ideal factors $\sqrt[3]{1-\eta_1}$ of 2, and it is found that this same ideal multiplier

$\sqrt[3]{1-\eta_0}$ renders actual one of the two ideal factors of any other decomposable number 3, 13, &c.,

$$\sqrt[3]{1-2\eta_0}\sqrt[3]{1-\eta_0}=1+\eta_0,\quad \sqrt[3]{31-12\eta_0}\sqrt[3]{1-\eta_0}=-5-\eta_0,\ \&c.$$

Similarly the conjugate multiplier $\sqrt[3]{1-\eta_1}$ renders actual the other ideal factor of any number 2, 3, 13, &c. We have thus two classes, or, reckoning also actual numbers, three classes of prime numbers, viz. (i) ideal primes rendered actual by the multiplier $\sqrt[3]{1-\eta_0}$, (ii) ideal primes rendered actual by the multiplier $\sqrt[3]{1-\eta_1}$, (iii) actual primes. This is a general property in the several complex theories; there is always a finite number of classes of ideal numbers, distinguished according to the multipliers by which they are rendered actual; the actual numbers form a "principal" class.

40. General theory of congruences—irreducible functions. In the complex theory relating to the roots of the equation $\eta^2+\eta+6=0$, there has just been occasion to consider the equation $(\eta-4)(\eta+5)=-2.13$, or say the congruence $(\eta-4)(\eta+5)\equiv 0\,(\text{mod. }13)$; in this form the relation $\eta^2+\eta+6=0$ is presupposed, but if, dropping this equation, η be regarded as arbitrary, then there is the congruence $\eta^2+\eta+6\equiv(\eta-4)(\eta+5)\,(\text{mod. }13)$. For a different modulus, for instance 11, there is not any such congruence exhibiting a decomposition of $\eta^2+\eta+6$ into factors. The function $\eta^2+\eta+6$ is irreducible, that is, it is not a product of factors with integer coefficients; in respect of the modulus 13 it becomes reducible, that is, it breaks up into factors having integer coefficients, while for the modulus 11 it continues irreducible. And there is a like general theory in regard to any rational and integral function $F(x)$ with integer coefficients; such function, assumed to be irreducible, may for a given prime modulus p continue irreducible, that is, it may not admit of any decomposition into factors with integer coefficients; or it may become reducible, that is, admit of a decomposition $F(x)\equiv\phi(x)\psi(x)\chi(x)\ldots(\text{mod. }p)$. And, when this is so, it is thus a product, in one way only, of factors $\phi(x)$, $\psi(x)$, $\chi(x)$, ..., which are each of them irreducible in regard to the same modulus p; any such factor may be a linear function of x, and as such irreducible; or it may be an irreducible function of the second or any higher degree. It is hardly necessary to remark that, in this theory, functions which are congruent to the modulus p are regarded as identical, and that in the expression of $F(x)$ an irreducible function $\phi(x)$ may present itself either as a simple factor, or as a multiple factor, with any exponent. The decomposition is analogous to that of a number into its prime factors; and the whole theory of the rational and integral function $F(x)$ in regard to the modulus p is in many respects analogous to that of a prime number regarded as a modulus. The theory has also been studied where the modulus is a power p^ν.

41. The congruence-imaginaries of Galois. If $F(x)$ be an irreducible function to a given prime modulus p, this implies that there is no integer value of x satisfying the congruence $F(x)\equiv 0\,(\text{mod. }p)$; we assume such a value and call it i, that is, we assume $F(i)\equiv 0\,(\text{mod. }p)$; the step is exactly analogous to that by which, starting from the notion of a real root, we introduce into algebra the ordinary imaginary $i=\sqrt{-1}$. For instance, x^2-x+3 is an irreducible function to the modulus 7: there is no integer solution of the congruence $x^2-x+3\equiv 0\,(\text{mod. }7)$. Assuming a solution i such that

$i^2 - i + 3 \equiv 0$ (mod. 7), we have, always to this modulus, $i^2 = i - 3$, and thence i^3, i^4, &c., each of them equal to a linear function of i. We consider the numbers of the form $a + bi$, where a and b are ordinary integers which may be regarded as having each of them the values 0, 1, 2, 3, 4, 5, or 6; there are thus 7^2, $= 49$, such numbers, or, excluding zero, 48 numbers; and it is easy to verify that these are, in fact, the numbers $i, i^2, \ldots, i^{47}, i^{48}, = 1$, that is, we have i a prime root of the congruence $x^{48} - 1 \equiv 0$ (mod. 7). The irreducible function may be of the third or any higher degree; thus for the same modulus 7 there is the cubic function $x^3 - x + 2$, giving rise to a theory of the numbers of the form $a + bi + ci^2$, where i is a congruence-imaginary such that $i^3 - i + 2 \equiv 0$ (mod. 7); and instead of 7 the modulus may be any other odd prime p.

Ordinary Theory, Third Part.

42. In what precedes, no mention has been made of the so-called Pellian equation $x^2 - Dy^2 = 1$ (where D is a given positive number), and of the allied equations $x^2 - Dy^2 = -1$, or $= \pm 4$. The equations with the sign + have always a series of solutions, those with the sign − only for certain values of D; in every case where the solutions exist, a least solution is obtainable by a process depending on the expression of $\sqrt{D}$ as a continued fraction, and from this least solution the whole series of solutions can be obtained without difficulty. The equations are very interesting, as well for their own sake as in connexion with the theory of the binary quadratic forms of a positive non-square determinant.

43. The theory of the expression of a number as a sum of squares or polygonal numbers has been developed apart from the general theory of the binary, ternary, and other quadratic forms to which it might be considered as belonging. The theorem for two squares, that every prime number of the form $4n + 1$ is, and that in one way only, a sum of two squares, is a fundamental theorem in relation to the complex numbers $a + bi$. A sum of two squares multiplied by a sum of two squares is always a sum of two squares, and hence it appears that every number of the form $2^a(4n + 1)$ is (in general, in a variety of ways) a sum of two squares.

Every number of the form $4n + 2$ or $8n + 3$ is a sum of three squares; even in the case of a prime number $8n + 3$ there is in general more than one decomposition, thus $59 = 25 + 25 + 9$ and $= 49 + 9 + 1$. Since a sum of three squares multiplied by a sum of three squares is not a sum of three squares, it is not enough to prove the theorem in regard to the primes of the form $8n + 3$.

Every prime number is (in general, in more than one way) a sum of four squares; and therefore every number is (in general, in more than one way) a sum of four squares, for a sum of four squares multiplied by a sum of four squares is always a sum of four squares.

Every number is (in general, in several ways) a sum of $m + 2$ $(m + 2)$gonal numbers, that is, of numbers of the form $\frac{1}{2}m(x^2 - x) + x$; and of these $m - 2$ may be at pleasure equal to 0 or 1; in particular, every number is a sum of three triangular numbers (a theorem of Fermat's).

The theorems in regard to three triangular numbers and to four square numbers are exhibited by certain remarkable identities in the Theory of Elliptic Functions; and generally there is in this subject a great mass of formulæ connected with the theory of the representation of numbers by quadratic forms. The various theorems in regard to the number of representations of a number as the sum of a definite number of squares cannot be here referred to.

44. The equation $x^\lambda + y^\lambda = z^\lambda$, where λ is any positive integer greater than 2, is not resoluble in whole numbers (a theorem of Fermat's). The general proof depends on the theory of the complex numbers composed of the λth roots of unity, and presents very great difficulty; in particular, distinctions arise according as the number λ does or does not divide certain of Bernoulli's numbers.

45. Lejeune-Dirichlet employs, for the determination of the number of quadratic forms of a given positive or negative determinant, a remarkable method depending on the summation of a series Σf^{-s}, where the index s is greater than but indefinitely near to unity.

46. Very remarkable formulæ have been given by Legendre, Tchebycheff, and Riemann for the approximate determination of the number of prime numbers less than a given large number x. Factor tables have been formed for the first nine million numbers, and the number of primes counted for successive intervals of 50,000; and these are found to agree very closely with the numbers calculated from the approximate formulæ. Legendre's expression is of the form $\dfrac{x}{\log x - A}$, where A is a constant not very different from unity; Tchebycheff's depends on the logarithm-integral $\mathrm{li}(x)$; and Riemann's, which is the most accurate, but is of a much more complicated form, contains a series of terms depending on the same integral.

The classical works on the Theory of Numbers are Legendre, *Théorie des Nombres*, 1st ed. 1798, 3rd ed. 1830; Gauss, *Disquisitiones Arithmeticæ*, Brunswick, 1801 (reprinted in the collected works, vol. I., Göttingen, 1863; French translation, under the title *Recherches Arithmétiques*, by Poullet-Delisle, Paris, 1807); and Lejeune-Dirichlet, *Vorlesungen über Zahlentheorie*, 3rd ed., with extensive and valuable additions by Dedekind, Brunswick, 1879—81. We have by the late Prof. H. J. S. Smith the extremely valuable series of "Reports on the Theory of Numbers," Parts I. to VI., *British Association Reports*, 1859—62, 1864—65, which, with his own original researches, [are] printed in the [first volume of the] collected works [published in 1894] by the Clarendon Press. See also Cayley, "Report of the Mathematical Tables Committee," *Brit. Assoc. Report*, 1875, pp. 305—336, [611], for a list of tables relating to the Theory of Numbers, and Mr J. W. L. Glaisher's introduction to the *Factor Table for the Sixth Million*, London, 1883, in regard to the approximate formulæ for the number of prime numbers.

796.

SERIES.

[From the *Encyclopædia Britannica, Ninth Edition,* vol. XXI. (1886), pp. 677—682.]

A SERIES is a set of terms considered as arranged in order. Usually the terms are or represent numerical magnitudes, and we are concerned with the sum of the series. The number of terms may be limited or without limit; and we have thus the two theories, finite series and infinite series. The notions of convergency and divergency present themselves only in the latter theory.

Finite Series.

1. Taking the terms to be numerical magnitudes, or say numbers, if there be a definite number of terms, then the sum of the series is nothing else than the number obtained by the addition of the terms; e.g. $4+9+10=23$, $1+2+4+8=15$. In the first example there is no apparent law for the successive terms; in the second example there is an apparent law. But it is important to notice that in neither case is there a determinate law: we can in an infinity of ways form series beginning with the apparently irregular succession of terms 4, 9, 10, or with the apparently regular succession of terms 1, 2, 4, 8. For instance, in the latter case we may have a series with the general term 2^n, when for $n=0, 1, 2, 3, 4, 5, \ldots$ the series will be $1, 2, 4, 8, 16, 32, \ldots$; or a series with the general term $\frac{1}{6}(n^3+5n+6)$, where for the same values of n the series will be $1, 2, 4, 8, 15, 26, \ldots$ The series may contain negative terms, and in forming the sum each term is of course to be taken with the proper sign.

2. But we may have a given law, such as either of those just mentioned, and the question then arises, to find the sum of an indefinite number of terms, or say of n terms (n standing for any positive integer number at pleasure) of the series. The expression for the sum cannot in this case be obtained by actual addition; the formation by addition of the sum of two terms, of three terms, &c., will, it may be, suggest (but it cannot do more than suggest) the expression for the sum of n terms

of the series. For instance, for the series of odd numbers $1+3+5+7+\ldots$, we have $1=1$, $1+3=4$, $1+3+5=9$, &c. These results at once suggest the law, $1+3+5+\ldots+(2n-1)=n^2$, which is in fact the true expression for the sum of n terms of the series; and this general expression, once obtained, can afterwards be verified.

3. We have here the theory of finite series: the general problem is, u_n being a given function of the positive integer n, to determine as a function of n the sum $u_0+u_1+u_2+\ldots+u_n$, or, in order to have n instead of $n+1$ terms, say the sum $u_0+u_1+u_2+\ldots+u_{n-1}$.

Simple cases are the three which follow.

(i) The arithmetic series,

$$a+(a+b)+(a+2b)+\ldots+(a+\overline{n-1}\,b);$$

writing here the terms in the reverse order, it at once appears that twice the sum is $=2a+\overline{n-1}\,b$ taken n times: that is, the sum $=na+\frac{1}{2}n(n-1)b$. In particular, we have an expression for the sum of the natural numbers

$$1+2+3+\ldots+n=\tfrac{1}{2}n(n+1),$$

and an expression for the sum of the odd numbers

$$1+3+5+\ldots+(2n-1)=n^2.$$

(ii) The geometric series,

$$a+ar+ar^2+\ldots+ar^{n-1};$$

here the difference between the sum and r times the sum is at once seen to be $=a-ar^n$, and the sum is thus $=a\dfrac{1-r^n}{1-r}$; in particular, the sum of the series

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}.$$

(iii) But the harmonic series,

$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\ldots+\frac{1}{a+(n-1)b},$$

or say $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\ldots+\dfrac{1}{n}$, does not admit of summation; there is no algebraical function of n which is equal to the sum of the series.

4. If the general term be a given function u_n, and we can find v_n a function of n such that $v_{n+1}-v_n=u_n$, then we have $u_0=v_1-v_0$, $u_1=v_2-v_1$, $u_2=v_3-v_2,\ldots$, $u_n=v_{n+1}-v_n$; and hence $u_0+u_1+u_2+\ldots+u_n=v_{n+1}-v_0$,—an expression for the required sum. This is in fact an application of the Calculus of Finite Differences. In the notation of this calculus $v_{n+1}-v_n$ is written Δv_n; and the general inverse problem, or problem of integration, is from the equation of differences $\Delta v_n=u_n$ (where u_n is a given function of n) to find v_n. The general solution contains an arbitrary constant, $v_n=V_n+C$; but this disappears in the difference $v_{n+1}-v_0$. As an example consider the series

$$u_0+u_1+\ldots+u_n=0+1+3+\ldots+\tfrac{1}{2}n(n+1);$$

here, observing that

$$n(n+1)(n+2)-(n-1)n(n+1)=n(n+1)(\overline{n+2}-\overline{n-1}),\ =3n(n+1),$$

we have

$$v_{n+1}=\tfrac{1}{6}n(n+1)(n+2);$$

and hence

$$1+3+6+\ldots+\tfrac{1}{2}n(n+1)=\tfrac{1}{6}n(n+1)(n+2),$$

as may be at once verified for any particular value of n.

Similarly, when the general term is a factorial of the order r, we have

$$1+\frac{r+1}{1}+\ldots+\frac{n(n+1)\ldots(n+r-1)}{1.2\ \ldots\ r}=\frac{n(n+1)\ldots(n+r)}{1.2\ \ldots(r+1)}.$$

5. If the general term u_n be any rational and integral function of n, we have

$$u_n=u_0+\frac{n}{1}\Delta u_0+\frac{n(n-1)}{1.2}\Delta^2 u_0+\ldots+\frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p}\Delta^p u_0,$$

where the series is continued only up to the term depending on p, the degree of the function u_n, for all the subsequent terms vanish. The series is thus decomposed into a set of series which have each a factorial for the general term, and which can be summed by the last formula; thus we obtain

$$u_0+u_1\ldots+u_n=(n+1)u_0+\frac{(n+1)n}{1.2}\Delta u_0+\ldots+\frac{(n+1)n(n-1)\ldots(n-p+1)}{1.2.3\ldots(p+1)}\Delta^p u_0,$$

which is a function of the degree $p+1$.

Thus for the before-mentioned series $1+2+4+8+\ldots$, if it be assumed that the general term u_n is a cubic function of n, and writing down the given terms and forming the differences, 1, 2, 4, 8; 1, 2, 4; 1, 2; 1, we have

$$u_n=1+\frac{n}{1}+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}\{=\tfrac{1}{6}(n^3+5n+6),\text{ as above}\};$$

and the sum

$$u_0+u_1+\ldots+u_n=n+1+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\frac{(n+1)n(n-1)(n-2)}{1.2.3.4},$$

$$=\tfrac{1}{24}(n^4+2n^3+11n^2+34n+24).$$

As particular cases we have expressions for the sums of the powers of the natural numbers—

$$1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1);\ \ 1^3+2^3+\ldots+n^3=\tfrac{1}{4}n^2(n+1)^2:$$

observe that the latter $=(1+2\ldots+n)^2$; and so on.

6. We may, from the expression for the sum of the geometric series, obtain by differentiation other results: thus

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}$$

gives

$$1+2r+3r^2+\ldots+(n-1)\,r^{n-2}=\frac{d}{dr}\,\frac{1-r^n}{1-r},\quad =\frac{1-nr^{n-1}+(n-1)\,r^n}{(1-r)^2};$$

and we might in this way find the sum $u_0+u_1r+\ldots+u_nr^n$, where u_n is any rational and integral function of n.

7. The expression for the sum $u_0+u_1+\ldots+u_n$ of an indefinite number of terms will in many cases lead to the sum of the infinite series $u_0+u_1+\ldots$; but the theory of infinite series requires to be considered separately. Often in dealing apparently with an infinite series $u_0+u_1+\ldots$ we consider rather an indefinite than an infinite series, and are not in any wise really concerned with the sum of the series or the question of its convergency: thus the equation

$$\left(1+mx+\frac{m\,(m-1)}{1\,.\,2}\,x^2+\ldots\right)\left(1+nx+\frac{n\,(n-1)}{1\,.\,2}\,x^2+\ldots\right)$$
$$=1+(m+n)\,x+\frac{(m+n)\,(m+n-1)}{1\,.\,2}\,x^2+\ldots$$

really means the series of identities

$$(m+n)=m+n,$$
$$\frac{(m+n)\,(m+n-1)}{1\,.\,2}=\frac{m\,(m-1)}{1\,.\,2}+2\,\frac{m}{1}\,\frac{n}{1}+\frac{n\,(n-1)}{1\,.\,2},\ \&c.,$$

obtained by multiplying together the two series of the left-hand side. Again, in the method of generating functions we are concerned with an equation $\phi(t)=A_0+A_1t+\ldots+A_nt^n+\ldots$, where the function $\phi(t)$ is used only to express the law of formation of the successive coefficients.

It is an obvious remark that, although according to the original definition of a series the terms are considered as arranged in a determinate order, yet in a finite series (whether the number of terms be definite or indefinite) the sum is independent of the order of arrangement.

Infinite Series.

8. We consider an infinite series $u_0+u_1+u_2+\ldots$ of terms proceeding according to a given law, that is, the general term u_n is given as a function of n. To fix the ideas the terms may be taken to be positive numerical magnitudes, or say numbers continually diminishing to zero; that is, $u_n>u_{n+1}$, and u_n is, moreover, such a function of n that, by taking n sufficiently large, u_n can be made as small as we please.

Forming the successive sums $S_0=u_0$, $S_1=u_0+u_1$, $S_2=u_0+u_1+u_2,\ldots$, these sums S_0, S_1, $S_2,\ldots$ will be a series of continually increasing terms, and if they increase up to a determinate finite limit S (that is, if there exists a determinate numerical magnitude S such that, by taking n sufficiently large, we can make $S-S_n$ as small as we please), S is said to be the sum of the infinite series. To show that we can

actually have an infinite series with a given sum S, take u_0 any number less than S, then $S-u_0$ is positive, and taking u_1 any numerical magnitude less than $S-u_0$, then $S-u_0-u_1$ is positive. And going on continually in this manner we obtain a series $u_0+u_1+u_2+\ldots$, such that for any value of n however large $S-u_0-u_1\ldots-u_n$ is positive; and if as n increases this difference diminishes to zero, we have $u_0+u_1+u_2+\ldots$, an infinite series having S for its sum. Thus, if $S=2$, and we take $u_0<2$, say $u_0=1$; $u_1<2-1$, say $u_1=\frac{1}{2}$; $u_2<2-1-\frac{1}{2}$, say $u_2=\frac{1}{4}$; and so on, we have $1+\frac{1}{2}+\frac{1}{4}+\ldots=2$; or, more generally, if r be any positive number less than 1, then $1+r+r^2+\ldots=\dfrac{1}{1-r}$, that is, the infinite geometric series with the first term $=1$, and with a ratio $r<1$, has the finite sum $\dfrac{1}{1-r}$. This, in fact, follows from the expression $1+r+r^2\ldots+r^{n-1}=\dfrac{1-r^n}{1-r}$ for the sum of the finite series; taking $r<1$, then as n increases r^n decreases to zero, and the sum becomes more and more nearly $=\dfrac{1}{1-r}$.

9. An infinite series of positive numbers can, it is clear, have a sum only if the terms continually diminish to zero; but it is not conversely true that, if this condition be satisfied, there will be a sum. For instance, in the case of the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$, it can be shown that by taking a sufficient number of terms the sum of the finite series may be made as large as we please. For, writing the series in the form $1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots$, the number of terms in the brackets being doubled at each successive step, it is clear that the sum of the terms in any bracket is always $>\frac{1}{2}$; hence by sufficiently increasing the number of brackets the sum may be made as large as we please. In the foregoing series, by grouping the terms in a different manner $1+(\frac{1}{2}+\frac{1}{3})+(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7})+\ldots$, the sum of the terms in any bracket is always <1; we thus arrive at the result that ($n=3$ at least) the sum of 2^n terms of the series is $>1+\frac{1}{2}n$ and $<n$.

10. An infinite series may contain negative terms; suppose in the first instance that the terms are alternately positive and negative. Here the absolute magnitudes of the terms must decrease down to zero, but this is a sufficient condition in order that the series may have a sum. The case in question is that of a series $v_0-v_1+v_2-\ldots$, where $v_0, v_1, v_2, \ldots$ are all positive and decrease down to zero. Here, forming the successive sums $S_0=v_0$, $S_1=v_0-v_1$, $S_2=v_0-v_1+v_2, \ldots$, $S_0, S_1, S_2, \ldots$ are all positive, and we have $S_0>S_1$, $S_1<S_2$, $S_2>S_3, \ldots$, and $S_{n+1}-S_n$ tends continually to zero. Hence the sums $S_0, S_1, S_2, \ldots$ tend continually to a positive limit S in such wise that $S_0, S_2, S_4, \ldots$ are each of them greater and $S_1, S_3, S_5, \ldots$ are each of them less than S; and we thus have S as the sum of the series. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ will serve as an example. The case just considered includes the apparently more general one where the series consists of alternate groups of positive and negative terms respectively; the terms of the same group may be united into a single term $\pm v_n$, and the original series will have a sum only if the resulting series $v_0-v_1+v_2\ldots$ has a sum, that is, if the positive partial sums $v_0, v_1, v_2, \ldots$ decrease down to zero.

The terms at the beginning of a series may be irregular as regards their signs; but, when this is so, all the terms in question (assumed to be finite in number) may

be united into a single term, which is of course finite, and instead of the original series only the remaining terms of the series need be considered. Every infinite series whatever is thus substantially included under the two forms,—terms all positive and terms alternately positive and negative.

11. In brief, the sum (if any) of the infinite series $u_0 + u_1 + u_2 + \ldots$ is the finite limit (if any) of the successive sums u_0, $u_0 + u_1$, $u_0 + u_1 + u_2, \ldots$; if there is no such limit, then there is no sum. Observe that the assumed order u_0, u_1, $u_2, \ldots$ of the terms is part of and essential to the definition; the terms in any other order may have a different sum, or may have no sum. A series having a sum is said to be "convergent"; a series which has no sum is "divergent."

If a series of positive terms be convergent, the terms cannot, it is clear, continually increase, nor can they tend to a fixed limit: the series $1 + 1 + 1 + \ldots$ is divergent. For the convergency of the series it is necessary (but, as has been shown, not sufficient) that the terms shall decrease to zero. So, if a series with alternately positive and negative terms be convergent, the absolute magnitudes cannot, it is clear, continually increase. In reference to such a series Abel remarks, "Peut-on imaginer rien de plus horrible que de débiter $0 = 1^n - 2^n + 3^n - 4^n +$, &c., où n est un nombre entier positif?" Neither is it allowable that the absolute magnitudes shall tend to a fixed limit. The so-called "neutral" series $1 - 1 + 1 - 1 \ldots$ is divergent: the successive sums do not tend to a determinate limit, but are alternately $+1$ and 0; it is necessary (and also sufficient) that the absolute magnitudes shall decrease to zero.

In the so-called semi-convergent series, we have an equation of the form

$$S = U_0 - U_1 + U_2 - \ldots,$$

where the positive values U_0, U_1, $U_2, \ldots$ decrease to a minimum value, suppose U_p, and afterwards increase; the series is divergent and has no sum, and thus S is not the sum of the series. S is only a number or function calculable approximately by means of the series regarded as a finite series terminating with the term $\pm U_p$. The successive sums U_0, $U_0 - U_1$, $U_0 - U_1 + U_2, \ldots$ up to that containing $\pm U_p$, give alternately superior and inferior limits of the number or function S.

12. The condition of convergency may be presented under a different form: let the series $u_0 + u_1 + u_2 + \ldots$ be convergent, then, taking m sufficiently large, the sum is the limit not only of $u_0 + u_1 + \ldots + u_m$ but also of $u_0 + u_1 + \ldots + u_{m+r}$, where r is any number as large as we please. The difference of these two expressions must therefore be indefinitely small; by taking m sufficiently large the sum $u_{m+1} + u_{m+2} + \ldots + u_{m+r}$ (where r is any number however large) can be made as small as we please; or, as this may also be stated, the sum of the infinite series $u_{m+1} + u_{m+2} + \ldots$ can be made as small as we please. If the terms are all positive (but not otherwise), we may take, instead of the entire series $u_{m+1} + u_{m+2} + \ldots$, any set of terms (not of necessity consecutive terms) subsequent to u_m; that is, for a convergent series of positive terms the sum of any set of terms subsequent to u_m can, by taking m sufficiently large, be made as small as we please.

13. It follows that, in a convergent series of positive terms, the terms may be grouped together in any manner so as to form a finite number of partial series which will be each of them convergent, and such that the sum of their sums will be the sum of the given series. For instance, if the given series be $u_0 + u_1 + u_2 + \dots$, then the two series $u_0 + u_2 + u_4 + \dots$ and $u_1 + u_3 + \dots$ will each be convergent and the sum of their sums will be the sum of the original series.

14. Obviously the conclusion does not hold good in general for series of positive and negative terms: for instance, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent, but the two series $1 + \frac{1}{3} + \frac{1}{5} + \dots$ and $-\frac{1}{2} - \frac{1}{4} - \dots$ are each divergent, and thus without a sum. In order that the conclusion may be applicable to a series of positive and negative terms the series must be "absolutely convergent," that is, it must be convergent when all the terms are made positive. This implies that the positive terms taken by themselves are a convergent series, and also that the negative terms taken by themselves are a convergent series. It is hardly necessary to remark that a convergent series of positive terms is absolutely convergent. The question of the convergency or divergency of a series of positive and negative terms is of less importance than the question whether it is or is not absolutely convergent. But in this latter question we regard the terms as all positive, and the question in effect relates to series containing positive terms only.

15. Consider, then, a series of positive terms $u_0 + u_1 + u_2 + \dots$; if they are increasing—that is, if in the limit u_{n+1}/u_n be greater than 1—the series is divergent, but if less than 1 the series is convergent. This may be called a first criterion; but there is the doubtful case where the limit $=1$. A second criterion was given by Cauchy and Raabe; but there is here again a doubtful case when the limit considered $=1$. A succession of criteria was established by De Morgan, which it seems proper to give in the original form; but the equivalent criteria established by Bertrand are somewhat more convenient. In what follows lx is for shortness written to denote the logarithm of x, no matter to what base. De Morgan's form is as follows:—Writing $u_n = \frac{1}{\phi(n)}$, put $p_0 = \frac{x\phi' x}{\phi x}$; if for $x = \infty$ the limit a_0 of p_0 be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit $a_0 = 1$, seek for the limit of p_1, $= (p_0 - 1)lx$; if this limit a_1 be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit $a_1 = 1$, seek for the limit p_2, $= (p_1 - 1)llx$; if this limit a_2 be greater than 1 the series is convergent, but if less than 1 it is divergent. And so on indefinitely.

16. Bertrand's form is:—If, in the limit for $n = \infty$, $l\frac{1}{u_n}\Big/ln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. If it $=1$, then if $l\frac{1}{nu_n}\Big/lln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. If it $=1$, then if $l\frac{1}{nu_n ln}\Big/llln$ be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. And so on indefinitely.

The last-mentioned criteria follow at once from the theorem that the several series having the general terms $\frac{1}{n^\alpha}$, $\frac{1}{n\,(ln)^\alpha}$, $\frac{1}{nln\,(lln)^\alpha}$, $\frac{1}{nlnlln\,(llln)^\alpha}$, ... respectively are each of them convergent if α be greater than 1, but divergent if α be negative or less than 1 or $=1$. In the simplest case, the series having the general term $\frac{1}{n^\alpha}$, the theorem may be proved nearly in the manner in which it is shown above (cf. § 9) that the harmonic series is divergent.

17. Two or more absolutely convergent series may be added together,

$$\{u_0+u_1+u_2+\ldots\}+\{v_0+v_1+v_2+\ldots\}=(u_0+v_0)+(u_1+v_1)+\ldots;$$

that is, the resulting series is absolutely convergent and has for its sum the sum of the two sums. And similarly two or more absolutely convergent series may be multiplied together

$$\{u_0+u_1+u_2+\ldots\}\times\{v_0+v_1+v_2+\ldots\}=u_0v_0+(u_0v_1+u_1v_0)+(u_0v_2+u_1v_1+u_2v_0)+\ldots;$$

that is, the resulting series is absolutely convergent and has for its sum the product of the two sums. But more properly the multiplication gives rise to a doubly infinite series—

$$\begin{array}{lll} u_0v_0, & u_0v_1, & u_0v_2\ldots \\ u_1v_0, & u_1v_1, & u_1v_2 \\ \vdots & & \end{array}$$

—which is a kind of series which will be presently considered.

18. But it is, in the first instance, proper to consider a single series extending backwards and forwards to infinity, or say a back-and-forwards infinite series $\ldots+u_{-2}+u_{-1}+u_0+u_1+u_2+\ldots$; such a series may be absolutely convergent, and the sum is then independent of the order of the terms, and in fact equal to the sum of the sums of the two series $u_0+u_1+u_2+\ldots$ and $u_{-1}+u_{-2}+u_{-3}+\ldots$ respectively. But, if not absolutely convergent, the expression has no definite meaning until it is explained in what manner the terms are intended to be grouped together; for instance, the expression may be used to denote the foregoing sum of two series, or to denote the series $u_0+(u_1+u_{-1})+(u_2+u_{-2})+\ldots$ and the sum may have different values, or there may be no sum, accordingly. Thus, if the series be $\ldots-\frac{1}{2}-\frac{1}{1}+0+\frac{1}{1}+\frac{1}{2}+\ldots$, in the former meaning the two series $0+\frac{1}{1}+\frac{1}{2}+\ldots$ and $-\frac{1}{1}-\frac{1}{2}-\ldots$ are each divergent, and there is not any sum. But in the latter meaning the series is $0+0+0+\ldots$, which has a sum $=0$. So, if the series be taken to denote the limit of

$$(u_0+u_1+u_2+\ldots+u_m)+(u_{-1}+u_{-2}+\ldots+u_{-m'}),$$

where m, m' are each of them ultimately infinite, there may be a sum depending on the ratio $m:m'$, which sum consequently acquires a determinate value only when this ratio is given.

19. In a singly infinite series we have a general term u_n, where n is an integer positive in the case of an ordinary series, and positive or negative in the case of a

back-and-forwards series. Similarly for a doubly infinite series, we have a general term $u_{m,n}$, where m, n are integers which may be each of them positive, and the form of the series is then

$$\begin{matrix} u_{0,0}, & u_{0,1}, & u_{0,2} \cdots \\ u_{1,0}, & u_{1,1}, & u_{1,2} \\ \vdots & & \end{matrix} ;$$

or m, n may be each of them positive or negative. The latter is the more general supposition, and includes the former, since $u_{m,n}$ may $=0$ for m or n each or either of them negative. To put a definite meaning on the notion of a sum, we may regard m, n as the rectangular coordinates of a point in a plane; that is, if m, n are each of them positive, we attend only to the positive quadrant of the plane, but otherwise to the whole plane; and we have thus a doubly infinite system or lattice-work of points. We may imagine a boundary depending on a parameter T which for $T=\infty$ is at every point thereof at an infinite distance from the origin; for instance, the boundary may be the circle $x^2+y^2=T$, or the four sides of a rectangle, $x=\pm\alpha T$, $y=\pm\beta T$. Suppose the form is given and the value of T, and let the sum $\Sigma u_{m,n}$ be understood to denote the sum of those terms $u_{m,n}$ which correspond to points within the boundary, then, if as T increases without limit the sum in question continually approaches a determinate limit (dependent, it may be, on the form of the boundary), *for such form of boundary* the series is said to be convergent, and the sum of the doubly infinite series is the aforesaid limit of the sum $\Sigma u_{m,n}$. The condition of convergency may be otherwise stated: it must be possible to take T so large that the sum $\Sigma u_{m,n}$ for all terms $u_{m,n}$ which correspond to points outside the boundary shall be as small as we please.

It is easy to see that, if the terms $u_{m,n}$ be all of them positive, and the series be convergent for any particular form of boundary, it will be convergent for any other form of boundary, and the sum will be the same in each case. Thus, let the boundary be in the first instance the circle $x^2+y^2=T$; by taking T sufficiently large the sum $\Sigma u_{m,n}$ for points outside the circle may be made as small as we please. Consider any other form of boundary—for instance, an ellipse of given eccentricity,—and let such an ellipse be drawn including within it the circle $x^2+y^2=T$. Then the sum $\Sigma u_{m,n}$ for terms $u_{m,n}$ corresponding to points outside the ellipse will be smaller than the sum for points outside the circle, and the difference of the two sums—that is, the sum for points outside the circle and inside the ellipse—will also be less than that for points outside the circle, and can thus be made as small as we please. Hence finally the sum $\Sigma u_{m,n}$, whether restricted to terms $u_{m,n}$ corresponding to points inside the circle or to terms corresponding to points inside the ellipse, will have the same value, or the sum of the series is independent of the form of the boundary. Such a series, viz. a doubly infinite convergent series of positive terms, is said to be absolutely convergent; and similarly a doubly infinite series of positive and negative terms which is convergent when the terms are all taken as positive is absolutely convergent.

20. We have in the preceding theory the foundation of the theorem (§ 17) as to the product of two absolutely convergent series. The product is in the first instance

expressed as a doubly infinite series; and, if we sum this for the boundary $x+y=T$, this is in effect a summation of the series $u_0v_0+(u_0v_1+u_1v_0)+\ldots$, which is the product of the two series. It may be further remarked that, starting with the doubly infinite series and summing for the rectangular boundary $x=\alpha T$, $y=\beta T$, we obtain the sum as the product of the sums of the two single series. For series not absolutely convergent, the theorem is not true. A striking instance is given by Cauchy: the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ is convergent and has a calculable sum, but it can be shown without difficulty that its square, viz. the series $1-\frac{2}{\sqrt{2}}+\left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)-\ldots$, is divergent.

21. The case where the terms of a series are imaginary comes under that where they are real. Suppose the general term is p_n+q_ni, then the series will have a sum, or will be convergent, if and only if the series having for its general term p_n and the series having for its general term q_n be each convergent; then the sum = sum of first series + i multiplied by sum of second series. The notion of absolute convergence will of course apply to each of the series separately; further, if the series having for its general term the modulus $\sqrt{p_n^2+q_n^2}$ be convergent (that is, absolutely convergent, since the terms are all positive), each of the component series will be absolutely convergent; but the condition is not necessary for the convergence, or the absolute convergence, of the two component series respectively.

22. In the series thus far considered, the terms are actual numbers, or are at least regarded as constant; but we may have a series $u_0+u_1+u_2+\ldots$, where the successive terms are functions of a parameter z; in particular, we may have a series $a_0+a_1z+a_2z^2+\ldots$ arranged in powers of z. It is in view of a complete theory *necessary* to consider z as having the imaginary value $x+iy=r(\cos\phi+i\sin\phi)$. The two component series will then have the general terms $a_nr^n\cos n\phi$ and $a_nr^n\sin n\phi$ respectively; accordingly each of these series will be absolutely convergent for any value whatever of ϕ, provided the series with the general term a_nr^n be absolutely convergent. Moreover, the series, if thus absolutely convergent for any particular value R of r, will be absolutely convergent for any smaller value of r, that is, for any value of $x+iy$ having a modulus not exceeding R; or, representing as usual $x+iy$ by the point whose rectangular coordinates are x, y, the series will be absolutely convergent for any point whatever inside or on the circumference of the circle having the origin for centre and its radius $=R$. The origin is of course an arbitrary point: or, what is the same thing, instead of a series in powers of z, we may consider a series in powers of $z-c$ (where c is a given imaginary value $=\alpha+\beta i$). Starting from the series, we may within the aforesaid limit of absolute convergency consider the series as the definition of a function of the variable z; in particular, the series may be absolutely convergent for every finite value of the modulus, and we have then a function defined for every finite value whatever $x+iy$ of the variable. Conversely, starting from a given function of the variable, we may inquire under what conditions it admits of expansion in a series of powers of z (or $z-c$), and seek to determine the expansion of the function in a series of this form. But in all this, however, we are travelling out of the theory of series into the general theory of functions.

23. Considering the modulus r as a given quantity and the several powers of r as included in the coefficients, the component series are of the forms $a_0+a_1\cos\phi+a_2\cos 2\phi+\ldots$ and $a_1\sin\phi+a_2\sin 2\phi+\ldots$ respectively. The theory of these trigonometrical or multiple sine and cosine series, and of the development, under proper conditions, of an arbitrary function in series of these forms, constitutes an important and interesting branch of analysis.

24. In the case of a real variable z, we may have a series $a_0+a_1z+a_2z^2+\ldots$, where the series $a_0+a_1+a_2+\ldots$ is a divergent series of decreasing positive terms (or as a limiting case where this series is $1+1+1+\ldots$). For a value of z inferior but indefinitely near to ± 1, say $z=\pm(1-\epsilon)$, where ϵ is indefinitely small and positive, the series will be convergent and have a determinate sum $\phi(z)$, and we may write $\phi(\pm 1)$ to denote the limit of $\phi(\pm(1-\epsilon))$ as ϵ diminishes to zero; but unless the series be convergent for the value $z=\pm 1$, it cannot for this value have a sum, nor consequently a sum $=\phi(\pm 1)$. For instance, let the series be $z+\frac{z^2}{2}+\frac{z^3}{3}+\ldots$, which for values of z between the limits ± 1 (both limits excluded) $=-\log(1-z)$. For $z=+1$ the series is divergent and has no sum; but for $z=1-\epsilon$, as ϵ diminishes to zero, we have $-\log\epsilon$ and $(1-\epsilon)+\frac{1}{2}(1-\epsilon)^2+\ldots$, each positive and increasing without limit; for $z=-1$, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent, and we have *at the limit* $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$. As a second example, consider the series $1+z+z^2+\ldots$, which for values of z between the limits ± 1 (both limits excluded) $=\frac{1}{1-z}$. For $z=+1$, the series is divergent and has no sum; but for $z=1-\epsilon$, as ϵ diminishes to zero, we have $\frac{1}{\epsilon}$ and $1+(1-\epsilon)+(1-\epsilon)^2+\ldots$, each positive and increasing without limit; for $z=-1$ the series is divergent and has no sum; the equation $\frac{1}{2-\epsilon}=1-(1-\epsilon)+(1-\epsilon)^2-\ldots$ is true for any positive value of ϵ however small, but *not* for the value $\epsilon=0$.

The following memoirs and works may be consulted:—Cauchy, *Cours d'Analyse de l'École Polytechnique*—part I., *Analyse Algébrique*, 8vo. Paris, 1821; Abel, "Untersuchungen über die Reihe $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots$," in Crelle's *Journ. de Math.*, vol. I. (1826), pp. 211—239, and *Œuvres* (French trans.), vol. I.; De Morgan, *Treatise on the Differential and Integral Calculus*, 8vo. London, 1842; Id., "On Divergent Series and various Points of Analysis connected with them" (1844), in *Camb. Phil. Trans.*, vol. VIII. (1849), and other memoirs in *Camb. Phil. Trans.*; Bertrand, "Règles sur la Convergence des Séries," in *Liouv. Journ. de Math.*, vol. VII. (1842), pp. 35—54; Cayley, "On the Inverse Elliptic Functions," *Camb. Math. Journ.*, vol. IV. (1845), pp. 257—277, [24], and "Mémoire sur les Fonctions doublement périodiques," in *Liouv. Journ. de Math.*, vol. X. (1845), pp. 385—420, [25], (as to the boundary for a doubly infinite series); Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," in *Gött. Abh.*, vol. XIII. (1854), and *Werke*, Leipsic, 1876, pp. 213—253 (contains an account of preceding researches by Euler, D'Alembert, Fourier, Lejeune-Dirichlet, &c.); Catalan, *Traité Élémentaire des Séries*, 8vo. Paris, 1860; Boole, *Treatise on the Calculus of Finite Differences*, 2nd ed. by Moulton, 8vo. London, 1872.

797.

SURFACE, CONGRUENCE, COMPLEX.

[From the *Encyclopædia Britannica, Ninth Edition*, vol. XXII. (1887), pp. 668—672.]

IN the article Curve [785], the subject was treated from an historical point of view for the purpose of showing how the leading ideas of the theory were successively arrived at. These leading ideas apply to surfaces, but the ideas peculiar to surfaces are scarcely of the like fundamental nature, being rather developments of the former set in their application to a more advanced portion of geometry; there is consequently less occasion for the historical mode of treatment. Curves in space were briefly considered in the same article, and they will not be discussed here; but it is proper to refer to them in connexion with the other notions of solid geometry. In plane geometry the elementary figures are the point and the line; and we then have the curve, which may be regarded as a singly infinite system of points, and also as a singly infinite system of lines. In solid geometry the elementary figures are the point, the line, and the plane; we have, moreover, first, that which under one aspect is the curve and under another aspect the developable (or torse), and which may be regarded as a singly infinite system of points, of lines, or of planes; and secondly, the surface, which may be regarded as a doubly infinite system of points or of planes, and also as a special triply infinite system of lines. (The tangent lines of a surface are a special complex.) As distinct particular cases of the first figure, we have the plane curve and the cone: and as a particular case of the second figure, the ruled surface, regulus, or singly infinite system of lines; we have, besides, the congruence or doubly infinite system of lines, and the complex or triply infinite system of lines. And thus crowds of theories arise which have hardly any analogues in plane geometry; the relation of a curve to the various surfaces which can be drawn through it, and that of a surface to the various curves which can be drawn upon it, are different in kind from those which in plane geometry most nearly correspond to them,—the relation of a system of points to the different curves through them and that of a curve to the systems of points upon it. In particular, there is nothing in plane geometry to correspond to the theory of the curves of curvature of a surface. Again, to the single

theorem of plane geometry, that a line is the shortest distance between two points, there correspond in solid geometry two extensive and difficult theories,—that of the geodesic lines on a surface and that of the minimal surface, or surface of minimum area, for a given boundary. And it would be easy to say more in illustration of the great extent and complexity of the subject.

Surfaces in General; Torses, &c.

1. A surface may be regarded as the locus of a doubly infinite system of points,—that is, the locus of the system of points determined by a single equation $U=(*\between x, y, z, 1)^n, =0$, between the Cartesian coordinates (to fix the ideas, say rectangular coordinates) x, y, z; or, if we please, by a single homogeneous relation $U=(*\between x, y, z, w)^n, =0$, between the quadriplanar coordinates x, y, z, w. The degree n of the equation is the order of the surface; and this definition of the order agrees with the geometrical one, that the order of the surface is equal to the number of the intersections of the surface by an arbitrary line. Starting from the foregoing point definition of the surface, we might develop the notions of the tangent line and the tangent plane; but it will be more convenient to consider the surface *ab initio* from the more general point of view in its relation to the point, the line, and the plane.

2. Mention has been made of the plane curve and the cone; it is proper to recall that the *order* of a plane curve is equal to the number of its intersections by an arbitrary line (in the plane of the curve), and that its *class* is equal to the number of tangents to the curve which pass through an arbitrary point (in the plane of the curve). The cone is a figure correlative to the plane curve: corresponding to the plane of the curve we have the vertex of the cone, to its tangents the generating lines of the cone, and to its points the tangent planes of the cone. But from a different point of view, we may consider the generating lines of the cone as corresponding to the points of the curve and its tangent planes as corresponding to the tangents of the curve. From this point of view, we define the order of the cone as equal to the number of its intersections (generating lines) by an arbitrary plane through the vertex, and its class as equal to the number of the tangent planes which pass through an arbitrary line through the vertex. And in the same way that a plane curve has singularities (singular points and singular tangents), so a cone has singularities (singular generating lines and singular tangent planes).

3. Consider now a surface in connexion with an arbitrary line. The line meets the surface in a certain number of points, and, as already mentioned, the *order* of the surface is equal to the number of these intersections. We have through the line a certain number of tangent planes of the surface, and the *class* of the surface is equal to the number of these tangent planes.

But, further, through the line imagine a plane; this meets the surface in a curve the order of which is equal (as is at once seen) to the order of the surface. Again, on the line imagine a point; this is the vertex of a cone circumscribing the surface, and the class of this cone is equal (as is at once seen) to the class of the surface.

The tangent lines of the surface, which lie in the plane, are nothing else than the tangents of the plane section, and thus form a singly infinite series of lines; similarly, the tangent lines of the surface, which pass through the point, are nothing else than the generating lines of the circumscribed cone, and thus form a singly infinite series of lines. But, if we consider those tangent lines of the surface which are at once in the plane and through the point, we see that they are finite in number; and we define the *rank* of a surface as equal to the number of tangent lines which lie in a given plane and pass through a given point in that plane. It at once follows that the class of the plane section and the order of the circumscribed cone are each equal to the rank of the surface, and are thus equal to each other. It may be noticed that for a general surface $(*\between x, y, z, w)^n, = 0$, of order n without point singularities the rank is $a, = n(n-1)$, and the class is $n', = n(n-1)^2$; this implies (what is, in fact, the case) that the circumscribed cone has line singularities, for otherwise its class, that is, the class of the surface, would be $a(a-1)$, which is not $= n(n-1)^2$.

4. In the last preceding number, the notions of the tangent line and the tangent plane have been assumed as known, but they require to be further explained in reference to the original point definition of the surface. Speaking generally, we may say that the points of the surface consecutive to a given point on it lie in a plane which is the tangent plane at the given point, and conversely the given point is the point of contact of this tangent plane, and that any line through the point of contact and in the tangent plane is a tangent line touching the surface at the point of contact. Hence we see at once that the tangent line is any line meeting the surface in two consecutive points, or—what is the same thing—a line meeting the surface in the point of contact, counting as two intersections, and in $\overline{n-2}$ other points. But, from the foregoing notion of the tangent plane as a plane containing the point of contact and the consecutive points of the surface, the passage to the true definition of the tangent plane is not equally obvious. A plane in general meets the surface of the order n in a curve of that order without double points; but the plane may be such that the curve has a double point, and when this is so the plane is a tangent plane having the double point for its point of contact. The double point is either an acnode (isolated point), then the surface at the point in question is convex towards (that is, concave away from) the tangent plane; or else it is a crunode, and the surface at the point in question is then concavo-convex, that is, it has its two curvatures in opposite senses (see *infra*, No. 16). Observe that, in either case, any line whatever in the plane and through the point meets the surface in the points in which it meets the plane curve, namely, in the point of contact, which *qua* double point counts as two intersections, and in $\overline{n-2}$ other points; that is, we have the preceding definition of the tangent line.

5. The complete enumeration and discussion of the singularities of a surface is a question of extreme difficulty which has not yet been solved*. A plane curve has

* In a plane curve, the only singularities which need to be considered are those that present themselves in Plücker's equations: for every higher singularity whatever is equivalent to a certain number of nodes, cusps, inflexions, and double tangents. As regards a surface, no such reduction of the higher singularities has as yet been made.

point singularities and line singularities; corresponding to these, we have for the surface isolated point singularities and isolated plane singularities, but there are besides continuous singularities applying to curves on or torses circumscribed to the surface, and it is among these that we have the non-special singularities which play the most important part in the theory. Thus the plane curve represented by the general equation $(*\between x, y, z)^n = 0$, of any given order n, has the non-special line singularities of inflexions and double tangents; corresponding to this, the surface represented by the general equation $(*\between x, y, z, w)^n = 0$, of any given order n, has, not the isolated plane singularities, but the continuous singularities of the spinode curve or torse and the node-couple curve or torse. A plane may meet the surface in a curve having (1) a cusp (spinode) or (2) a pair of double points; in each case, there is a singly infinite system of such singular tangent planes, and the locus of the points of contact is the curve, the envelope of the tangent planes the torse. The reciprocal singularities to these are the nodal curve and the cuspidal curve: the surface may intersect or touch itself along a curve in such wise that, cutting the surface by an arbitrary plane, the curve of intersection has, at each intersection of the plane with the curve on the surface, (1) a double point (node) or (2) a cusp. Observe that these are singularities not occurring in the surface represented by the general equation $(*\between x, y, z, w)^n = 0$ of any order; observe further that, in the case of both or either of these singularities, the definition of the tangent plane must be modified. A tangent plane is a plane such that there is in the plane section a double point in addition to the nodes or cusps at the intersections with the singular lines on the surface.

6. As regards isolated singularities, it will be sufficient to mention the point singularity of the conical point (or cnicnode) and the corresponding plane singularity of the conic of contact (or cnictrope). In the former case, we have a point such that the consecutive points, instead of lying in a tangent plane, lie on a quadric cone, having the point for its vertex; in the latter case, we have a plane touching the surface along a conic, that is, the complete intersection of the surface by the plane is made up of the conic taken twice and of a residual curve of the order $n-4$.

7. We may, in the general theory of surfaces, consider either a surface and its reciprocal surface, the reciprocal surface being taken to be the surface enveloped by the polar planes (in regard to a given quadric surface) of the points of the original surface; or—what is better—we may consider a given surface in reference to the reciprocal relations of its order, rank, class, and singularities. In either case, we have a series of unaccented letters and a corresponding series of accented letters, and the relations between them are such that we may in any equation interchange the accented and the unaccented letters; in some cases, an unaccented letter may be equal to the corresponding accented letter. Thus, let n, n' be as before the order and the class of the surface, but, instead of immediately defining the rank, let a be used to denote the class of the plane section and a' the order of the circumscribed cone; also let S, S' be numbers referring to the singularities. The form of the relations is $a = a'$ (= rank of surface); $a' = n(n-1) - S$; $n' = n(n-1)^2 - S$; $a = n'(n'-1) - S'$; $n = n'(n'-1)^2 - S'$. In these last equations S, S' are merely written down to denote proper corresponding combinations of the several numbers referring to the singularities collectively denoted

by S, S' respectively. The theory, as already mentioned, is a complex and difficult one, and it is not the intention to further develop it here.

8. A developable or torse corresponds to a curve in space, in the same manner as a cone corresponds to a plane curve: although capable of representation by an equation $U = (*\between x, y, z, w)^n, = 0$, and so of coming under the foregoing point definition of a surface, it is an entirely distinct geometrical conception. We may indeed, *qua* surface, regard it as a surface characterized by the property that each of its tangent planes touches it, not at a single point, but along a line; this is equivalent to saying that it is the envelope, not of a doubly infinite series of planes, as is a proper surface, but of a singly infinite system of planes. But it is perhaps easier to regard it as the locus of a singly infinite system of lines, each line meeting the consecutive line, or, what is the same thing, the lines being tangent lines of a curve in space. The tangent plane is then the plane through two consecutive lines, or, what is the same thing, an osculating plane of the curve, whence also the tangent plane intersects the surface in the generating line counting twice, and in a residual curve of the order $n-2$. The curve is said to be the edge of regression of the developable, and it is a cuspidal curve thereof; that is to say, any plane section of the developable has at each point of intersection with the edge of regression a cusp. A sheet of paper bent in any manner without crumpling gives a developable; but we cannot with a single sheet of paper properly exhibit the form in the neighbourhood of the edge of regression: we need two sheets connected along a plane curve, which, when the paper is bent, becomes the edge of regression and appears as a cuspidal curve on the surface.

It may be mentioned that the condition which must be satisfied in order that the equation $U=0$ shall represent a developable is $H(U)=0$; that is, the Hessian or functional determinant formed with the second differential coefficients of U must vanish in virtue of the equation $U=0$, or—what is the same thing—$H(U)$ must contain U as a factor. If in Cartesian coordinates the equation is taken in the form $z-f(x, y)=0$, then the condition is $rt-s^2=0$ identically, where r, s, t denote as usual the second differential coefficients of z in regard to x, y respectively.

9. A ruled surface or regulus is the locus of a singly infinite system of lines, where the consecutive lines do not intersect; this is a true surface, for there is a doubly infinite series of tangent planes,—in fact, any plane through any one of the lines is a tangent plane of the surface, touching it at a point on the line, and in such wise that, as the tangent plane turns about the line, the point of contact moves along the line. The complete intersection of the surface by the tangent plane is made up of the line counting once and of a residual curve of the order $n-1$. A quadric surface is a regulus in a twofold manner, for there are on the surface two systems of lines each of which is a regulus. A cubic surface may be a regulus (see No. 11 *infra*).

Surfaces of the Orders 2, 3, *and* 4.

10. A surface of the second order or a quadric surface is a surface such that every line meets it in two points, or—what comes to the same thing—such that every plane section thereof is a conic or quadric curve. Such surfaces have been studied

from every point of view. The only singular forms are when there is (i) a conical point (cnicnode), when the surface is a cone of the second order or quadricone; (ii) a conic of contact (cnictrope), when the surface is this conic; from a different point of view it is a *surface aplatie* or flattened surface. Excluding these degenerate forms, the surface is of the order, rank, and class each $=2$, and it has no singularities. Distinguishing the forms according to *reality*, we have the ellipsoid, the hyperboloid of two sheets, the hyperboloid of one sheet, the elliptic paraboloid, and the hyperbolic paraboloid (see Geometry, Analytical, [790]). A particular case of the ellipsoid is the sphere; in abstract geometry, this is a quadric surface passing through a given quadric curve, the circle at infinity. The tangent plane of a quadric surface meets it in a quadric curve having a node, that is, in a pair of lines; hence there are on the surface two singly infinite sets of lines. Two lines of the same set do not meet, but each line of the one set meets each line of the other set; the surface is thus a regulus in a twofold manner. The lines are real for the hyperboloid of one sheet and for the hyperbolic paraboloid; for the other forms of surface they are imaginary.

11. We have next the surface of the third order or cubic surface, which has also been very completely studied. Such a surface may have isolated point singularities (cnicnodes or points of higher singularity), or it may have a nodal line; we have thus $21+2, =23$ cases. In the general case of a surface without any singularities, the order, rank, and class are $=3$, 6, 12 respectively. The surface has upon it 27 lines, lying by threes in 45 planes, which are triple tangent planes. Observe that the tangent plane is a plane meeting the surface in a curve having a node. For a surface of any given order n there will be a certain number of planes each meeting the surface in a curve with 3 nodes, that is, triple tangent planes; and, in the particular case where $n=3$, the cubic curve with 3 nodes is of course a set of 3 lines; it is found that the number of triple tangent planes is, as just mentioned, $=45$. This would give 135 lines, but through each line we have 5 such planes, and the number of lines is thus $=27$. The theory of the 27 lines is an extensive and interesting one; in particular, it may be noticed that we can, in thirty-six ways, select a system of 6×6 lines, or "double sixer," such that no 2 lines of the same set intersect each other, but that each line of the one set intersects each line of the other set.

A cubic surface having a nodal line is a ruled surface or regulus; in fact, any plane through the nodal line meets the surface in this line counting twice and in a residual line, and there is thus on the surface a singly infinite set of lines. There are two forms; but the distinction between them need not be referred to here.

12. As regards quartic surfaces, only particular forms have been much studied. A quartic surface can have at most 16 conical points (cnicnodes); an instance of such a surface is Fresnel's wave surface, which has 4 real cnicnodes in one of the principal planes, 4×2 imaginary ones in the other two principal planes, and 4 imaginary ones at infinity,—in all 16 cnicnodes; the same surface has also 4 real + 12 imaginary planes each touching the surface along a circle (cnictropes),—in all 16 cnictropes. It was easy by a mere homographic transformation to pass to the more general surface called the tetrahedroid; but this was itself only a particular form of the general surface

with 16 cnicnodes and 16 cnictropes first studied by Kummer. Quartic surfaces with a smaller number of cnicnodes have also been considered.

Another very important form is the quartic surface having a nodal conic; the nodal conic may be the circle at infinity, and we have then the so-called anallagmatic surface, otherwise the cyclide (which includes the particular form called Dupin's cyclide). These correspond to the bicircular quartic curve of plane geometry. Other forms of quartic surface might be referred to.

Congruences and Complexes.

13. A congruence is a doubly infinite system of lines. A line depends on four parameters and can therefore be determined so as to satisfy four conditions; if only two conditions are imposed on the line, we have a doubly infinite system of lines or a congruence. For instance, the lines meeting each of two given lines form a congruence. It is hardly necessary to remark that, imposing on the line one more condition, we have a ruled surface or regulus; thus we can in an infinity of ways separate the congruence into a singly infinite system of reguli or of torses (see *infra*, No. 16).

Considering in connexion with the congruence two arbitrary lines, there will be in the congruence a determinate number of lines which meet each of these two lines; and the number of lines thus meeting the two lines is said to be the *order-class* of the congruence. If the two arbitrary lines are taken to intersect each other, the congruence lines which meet each of the two lines separate themselves into two sets,—those which lie in the plane of the two lines and those which pass through their intersection. There will be in the former set a determinate number of congruence lines which is the *order* of the congruence, and in the latter set a determinate number of congruence lines which is the *class* of the congruence. In other words, the order of the congruence is equal to the number of congruence lines lying in an arbitrary plane, and its class to the number of congruence lines passing through an arbitrary point.

The following systems of lines form each of them a congruence:—(A) lines meeting each of two given curves; (B) lines meeting a given curve twice; (C) lines meeting a given curve and touching a given surface; (D) lines touching each of two given surfaces; (E) lines touching a given surface twice, or, say, the bitangents of a given surface.

The last case is the most general one; and conversely, for a given congruence, there will be in general a surface having the congruence lines for bitangents. This surface is said to be the *focal surface* of the congruence; the general surface with 16 cnicnodes first presented itself in this manner as the focal surface of a congruence. But the focal surface may degenerate into the forms belonging to the other cases A, B, C, D.

14. A complex is a triply infinite system of lines,—for instance, the tangent lines of a surface. Considering an arbitrary point in connexion with the complex, the com-

plex lines which pass through the point form a cone; considering a plane in connexion with it, the complex lines which lie in the plane envelope a curve. It is easy to see that the class of the curve is equal to the order of the cone; in fact, each of these numbers is equal to the number of complex lines which lie in an arbitrary plane and pass through an arbitrary point of that plane; and we then say *order* of complex = order of curve; *rank* of complex = class of curve = order of cone; *class* of complex = class of cone. It is to be observed that, while for a congruence there is in general a surface having the congruence lines for bitangents, for a complex there is not in general any surface having the complex lines for tangents; the tangent lines of a surface are thus only a special form of complex. The theory of complexes first presented itself in the researches of Malus on systems of rays of light in connexion with double refraction.

15. The analytical theory as well of congruences as of complexes is most easily carried out by means of the six coordinates of a line; viz. there are coordinates (a, b, c, f, g, h) connected by the equation $af+bg+ch=0$, and therefore such that the ratios $a : b : c : f : g : h$ constitute a system of four arbitrary parameters. We have thus a congruence of the order n represented by a single homogeneous equation of that order $(*\between a, b, c, f, g, h)^n=0$ between the six coordinates; two such relations determine a congruence. But we have in regard to congruences the same difficulty as that which presents itself in regard to curves in space: it is not every congruence which can be represented completely and precisely by *two* such equations.

The linear equation $(*\between a, b, c, f, g, h)=0$ represents a congruence of the first order or linear congruence; such congruences are interesting both in geometry and in connexion with the theory of forces acting on a rigid body.

Curves of Curvature; Asymptotic Lines.

16. The normals of a surface form a congruence. In any congruence, the lines consecutive to a given congruence line do not in general meet this line; but there is a determinate number of consecutive lines which do meet it; or, attending for the moment to only one of these, say the congruence line is met by a consecutive congruence line. In particular, each normal is met by a consecutive normal; this again is met by a consecutive normal, and so on. That is, we have a singly infinite system of normals each meeting the consecutive normal, and so forming a torse; starting from different normals successively, we obtain a singly infinite system of such torses. But each normal is in fact met by two consecutive normals, and, using in the construction first the one and then the other of these, we obtain two singly infinite systems of torses each intersecting the given surface at right angles. In other words, if in place of the normal we consider the point on the surface, we obtain on the surface two singly infinite systems of curves such that for any curve of either system the normals at consecutive points intersect each other; moreover, for each normal the torses of the two systems intersect each other at right angles; and therefore for each point of the surface the curves of the two systems intersect each other at right angles. The two systems of curves are said to be the curves of curvature of the surface.

The normal is met by the two consecutive normals in two points which are the centres of curvature for the point on the surface; these lie either on the same side of the point or on opposite sides, and the surface has at the point in question like curvatures or opposite curvatures in the two cases respectively (see *supra*, No. 4).

17. In immediate connexion with the curves of curvature, we have the so-called asymptotic curves (Haupt-tangenten-linien). The tangent plane at a point of the surface cuts the surface in a curve having at that point a node. Thus we have at the point of the surface two directions of passage to a consecutive point, or, say, two elements of arc; and, passing along one of these to the consecutive point, and thence to a consecutive point, and so on, we obtain on the surface a curve. Starting successively from different points of the surface we thus obtain a singly infinite system of curves; or, using first one and then the other of the two directions, we obtain two singly infinite systems of curves, which are the curves above referred to. The two curves at any point are equally inclined to the two curves of curvature at that point, or—what is the same thing—the supplementary angles formed by the two asymptotic lines are bisected by the two curves of curvature. In the case of a quadric surface, the asymptotic curves are the two systems of lines on the surface.

Geodesic Lines.

18. A geodesic line (or curve) is a shortest curve on a surface; more accurately, the element of arc between two consecutive points of a geodesic line is a shortest arc on the surface. We are thus led to the fundamental property that, at each point of the curve, the osculating plane of the curve passes through the normal of the surface; in other words, any two consecutive arcs PP', $P'P''$ are *in plano* with the normal at P'. Starting from a given point P on the surface, we have a singly infinite system of geodesics proceeding along the surface in the direction of the several tangent lines at the point P; and, if the direction PP' is given, the property gives a construction by successive elements of arc for the required geodesic line.

Considering the geodesic lines which proceed from a given point P of the surface, any particular geodesic line is or is not again intersected by the consecutive generating line; if it is thus intersected, the generating line is a shortest line on the surface up to, but not beyond, the point at which it is first intersected by the consecutive generating line; if it is not intersected, it continues a shortest line for the whole course.

In the analytical theory both of geodesic lines and of the curves of curvature, and in other parts of the theory of surfaces, it is very convenient to consider the rectangular coordinates x, y, z of a point of the surface as given functions of two independent parameters p, q; the form of these functions of course determines the surface, since by the elimination of p, q from the three equations we obtain the equation in the coordinates x, y, z. We have for the geodesic lines a differential equation of the second order between p and q; the general solution contains two arbitrary constants,

and is thus capable of representing the geodesic line which can be drawn from a given point in a given direction on the surface. In the case of a quadric surface, the solution involves hyperelliptic integrals of the first kind, depending on the square root of a sextic function.

Curvilinear Coordinates.

19. The expressions of the coordinates x, y, z in terms of p, q may contain a parameter r, and, if this is regarded as a given constant, these expressions will as before refer to a point on a given surface. But, if p, q, r are regarded as three independent parameters, x, y, z will be the coordinates of a point in space, determined by means of the three parameters p, q, r; these parameters are said to be the curvilinear coordinates, or (in a generalized sense of the term) simply the coordinates of the point. We arrive otherwise at the notion by taking p, q, r each as a given function of x, y, z; say we have $p=f_1(x, y, z)$, $q=f_2(x, y, z)$, $r=f_3(x, y, z)$, which equations of course lead to expressions for p, q, r each as a function of x, y, z. The first equation determines a singly infinite set of surfaces: for any given value of p we have a surface; and similarly the second and third equations determine each a singly infinite set of surfaces. If, to fix the ideas, f_1, f_2, f_3 are taken to denote each a rational and integral function of x, y, z, then two surfaces of the same set will not intersect each other, and through a given point of space there will pass one surface of each set; that is, the point will be determined as a point of intersection of three surfaces belonging to the three sets respectively; moreover, the whole of space will be divided by the three sets of surfaces into a triply infinite system of elements, each of them being a parallelepiped.

Orthotomic Surfaces; Parallel Surfaces.

20. The three sets of surfaces may be such that the three surfaces through any point of space whatever intersect each other at right angles; and they are in this case said to be orthotomic. The term curvilinear coordinates was almost appropriated by Lamé, to whom this theory is chiefly due, to the case in question: assuming that the equations $p=f_1(x, y, z)$, $q=f_2(x, y, z)$, $r=f_3(x, y, z)$ refer to a system of orthotomic surfaces, we have in the restricted sense p, q, r as the curvilinear coordinates of the point.

An interesting special case is that of confocal quadric surfaces. The general equation of a surface confocal with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1;$$

and, if in this equation we consider x, y, z as given, we have for θ a cubic equation with three real roots p, q, r, and thus we have through the point three real surfaces, one an ellipsoid, one a hyperboloid of one sheet, and one a hyperboloid of two sheets.

21. The theory is connected with that of curves of curvature by Dupin's theorem. Thus in any system of orthotomic surfaces, each surface of any one of the three sets is intersected by the surfaces of the other two sets in its curves of curvature.

22. No one of the three sets of surfaces is altogether arbitrary: in the equation $p=f_1(x, y, z)$, p is not an arbitrary function of x, y, z, but it must satisfy a certain partial differential equation of the third order. Assuming that p has this value, we have $q=f_2(x, y, z)$ and $r=f_3(x, y, z)$ determinate functions of x, y, z, such that the three sets of surfaces form an orthotomic system.

23. Starting from a given surface, it has been seen (No. 16) that the normals along the curves of curvature form two systems of torses intersecting each other, and also the given surface, at right angles. But there are, intersecting the two systems of torses at right angles, not only the given surface, but a singly infinite system of surfaces. If at each point of the given surface we measure off along the normal one and the same distance at pleasure, then the locus of the points thus obtained is a surface cutting all the normals of the given surface at right angles, or, in other words, having the same normals as the given surface; and it is therefore a parallel surface to the given surface. Hence the singly infinite system of parallel surfaces and the two singly infinite systems of torses form together a set of orthotomic surfaces.

The Minimal Surface.

24. This is the surface of minimum area—more accurately, a surface such that, for any indefinitely small closed curve which can be drawn on it round any point, the area of the surface is less than it is for any other surface whatever through the closed curve. It at once follows that the surface at every point is concavo-convex; for, if at any point this was not the case, we could, by cutting the surface by a plane, describe round the point an indefinitely small closed plane curve, and the plane area within the closed curve would then be less than the area of the element of surface within the same curve. The condition leads to a partial differential equation of the second order for the determination of the minimal surface: considering z as a function of x, y, and writing as usual p, q, r, s, t for the first and second differential coefficients of z in regard to x, y respectively, the equation (as first shown by Lagrange) is $(1+q^2)r-2pqs+(1+p^2)t=0$, or, as this may also be written,

$$\frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}+\frac{d}{dx}\frac{p}{\sqrt{1+p^2+q^2}}=0.$$

The general integral contains of course arbitrary functions, and, if we imagine these so determined that the surface may pass through a given closed curve, and if, moreover, there is but one minimal surface passing through that curve, we have the solution of the problem of finding the surface of minimum area within the same curve. The surface continued beyond the closed curve is a minimal surface, but it is not of necessity or in general a surface of minimum area for an arbitrary bounding curve not wholly included within the given closed curve. It is hardly necessary to

remark that the plane is a minimal surface, and that, if the given closed curve is a plane curve, the plane is the proper solution; that is, the plane area within the given closed curve is less than the area for any other surface through the same curve. The given closed curve is not of necessity a single curve: it may be, for instance, a skew polygon of four or more sides.

The partial differential equation was dealt with in a very remarkable manner by Riemann. From the second form given above it appears that we have $\frac{q\,dx - p\,dy}{\sqrt{1+p^2+q^2}} =$ a complete differential, or, putting this $= d\zeta$, we introduce into the solution a variable ζ, which combines with z in the forms $z \pm i\zeta$ ($i = \sqrt{-1}$ as usual). The boundary conditions have to be satisfied by the determination of the conjugate variables η, η' as functions of $z + i\zeta$, $z - i\zeta$, or, say, of Z, Z' respectively. By writing S, S' to denote $x + iy$, $x - iy$ respectively, Riemann obtains finally two ordinary differential equations of the first order in S, S', η, η', Z, Z', and the results are completely worked out in some very interesting special cases.

The memoirs on various parts of the general subject are very numerous; references to many of them will be found in Salmon's *Treatise on the Analytic Geometry of Three Dimensions*, 4th ed., Dublin, 1882 (the most comprehensive work on solid geometry); for the minimal surface (which is not considered there) see Memoirs XVII. and XXVI. in Riemann's *Gesammelte mathematische Werke*, Leipsic, 1876; the former—"Ueber die Fläche vom kleinsten Inhalt bei gegebener Begrenzung," as published in *Gött. Abhandl.*, vol. XIII. (1866—67)—contains an introduction by Hattendorff giving the history of the question.

798.

WALLIS (JOHN).

[From the *Encyclopædia Britannica, Ninth Edition,* vol. XXIV. (1888), pp. 331, 332.]

WALLIS, JOHN (1616—1703), an eminent English mathematician, logician, and grammarian, was born on the 23rd November 1616 at Ashford, in Kent, of which parish his father was then incumbent. Having been previously instructed in Latin, Greek, and Hebrew, he was in 1632 sent to Emmanuel College, Cambridge, and afterwards was chosen fellow of Queens' College. Having been admitted to holy orders, he left the university in 1641 to act as chaplain to Sir William Darley, and in the following year accepted a similar appointment from the widow of Sir Horatio Vere. It was about this period that he displayed surprising talents in deciphering the intercepted letters and papers of the Royalists. His adherence to the Parliamentary party was in 1643 rewarded by the living of St Gabriel, Fenchurch Street, London. In 1644 he was appointed one of the scribes or secretaries of the Assembly of Divines at Westminster. During the same year he married Susanna Glyde, and thus vacated his fellowship; but the death of his mother had left him in possession of a handsome fortune. In 1645 he attended those scientific meetings which led to the establishment of the Royal Society. When the Independents obtained the superiority, Wallis adhered to the Solemn League and Covenant. The living of St Gabriel he exchanged for that of St Martin, Ironmonger Lane; and, as rector of that parish, he in 1648 subscribed the Remonstrance against putting Charles I. to death. Notwithstanding this act of opposition, he was in June 1649 appointed Savilian professor of geometry at Oxford. In 1654 he there took the degree of D.D., and four years later succeeded Dr Langbaine as keeper of the archives. After the Restoration, he was named one of the king's chaplains in ordinary. While complying with the terms of the Act of Uniformity, Wallis seems always to have retained moderate and rational notions of ecclesiastical polity. He died at Oxford on the 28th of October 1703, in the eighty-seventh year of his age.

The works of Wallis are numerous, and relate to a multiplicity of subjects. His *Institutio Logicæ*, published in 1687, was very popular, and in his *Grammatica Linguæ Anglicanæ* we find indications of an acute and philosophic intellect. The mathematical works are published some of them in a small 4to volume, Oxford, 1657, and a complete collection in three thick folio volumes, Oxford, 1695–93–99. The third volume includes, however, some theological treatises, and the first part of it is occupied with editions of treatises on harmonics and other works of Greek geometers, some of them first editions from the MSS., and in general with Latin versions and notes (Ptolemy, Porphyrius, Briennius, Archimedes, Eutocius, Aristarchus, and Pappus). The second and third volumes include also two collections of letters to and from Brouncker, Frenicle, Leibnitz, Newton, Oldenburg, Schooten, and others; and there is a tract on trigonometry by Caswell. Excluding all these, the mathematical works contained in the first and second volumes occupy about 1800 pages. The titles in the order adopted, but with date of publication, are as follows:—"Oratio Inauguralis," on his appointment (1649) as Savilian professor, 1657; "Mathesis Universalis, seu Opus Arithmeticum Philologice et Mathematice Traditum, Arithmeticam Numerosam et Speciosam Aliaque Continens," 1657; "Adversus Meibomium, de Proportionibus Dialogus," 1657; "De Sectionibus Conicis Nova Methodo Expositis," 1655; "Arithmetica Infinitorum, sive Nova Methodus Inquirendi in Curvilineorum Quadraturam Aliaque Difficiliora Matheseos Problemata," 1655; "Eclipsis Solaris Observatio Oxonii Habita 2nd Aug. 1654," 1655; "Tractatus Duo, prior de Cycloide, posterior de Cissoide et de Curvarum tum Linearum Εὐθύνσει tum Superficierum Πλατυσμῶ," 1659; "Mechanica, sive de Motu Tractatus Geometricus," three parts, 1669–70–71; "De Algebra Tractatus Historicus et Practicus, ejusdem originem et progressus varios ostendens," English, 1685; "De Combinationibus Alternationibus et Partibus Aliquotis Tractatus," English, 1685; "De Sectionibus Angularibus Tractatus," English, 1685; "De Angulo Contactus et Semicirculi Tractatus," 1656; "Ejusdem Tractatus Defensio," 1685; "De Postulato Quinto, et Quinta Definitione, Lib. VI. Euclidis, Disceptatio Geometrica," ?1663; "Cuno-Cuneus, seu Corpus partim Conum partim Cuneum Representans Geometrice Consideratum," English, 1685; "De Gravitate et Gravitatione Disquisitio Geometrica," 1662 (English, 1674); "De Æstu Maris Hypothesis Nova," 1666—69.

The *Arithmetica Infinitorum* relates chiefly to the quadrature of curves by the so-called method of indivisibles established by Cavalieri, 1629, and cultivated in the interval by him, Fermat, Descartes, and Roberval. The method is substantially that of the integral calculus; thus, e.g., for the curve $y = x^2$ to find the area from $x = 0$ to $x = 1$, the base is divided into n equal parts, and the area is obtained as

$$= \frac{1}{n^3}(1^2 + 2^2 + \ldots + n^2), \ = \frac{1}{6n^2} n(n+1)(2n+1),$$

which, taking n indefinitely large, is $= \frac{1}{3}$. The case of the general parabola $y = x^m$ (m a positive integer or fraction), where the area is $\frac{1}{m+1}$, had been previously solved. Wallis made the important remark that the reciprocal of such a power of x could be regarded as a power with a negative exponent $\left(\frac{1}{x^m} = x^{-m}\right)$, and he was thus enabled

to extend the theorem to certain hyperbolic curves, but the case m a negative value larger than 1 presented a difficulty which he did not succeed in overcoming. It should be noticed that Wallis, although not using the notation x^m in the case of a positive or negative fractional value, nor indeed in the case of a negative integer value of m, deals continually with such powers, and speaks of the positive or negative integer or fractional value of m as the index of the power. The area of a curve, $y = \text{sum of}$ a finite number of terms Ax^m, was at once obtained from that for the case of a single term; and Wallis, after thus establishing the several results which would now be written $\int_0^1 (x-x^2)^0\,dx = 1$, $\int_0^1 (x-x^2)^1\,dx = \frac{1}{6}$, $\int_0^1 (x-x^2)^2\,dx = \frac{1}{30}$, $\int_0^1 (x-x^2)^3\,dx = \frac{1}{140}$, &c., proposed to himself to interpolate from these the value of $\int_0^1 (x-x^2)^{\frac{1}{2}}\,dx$, which is the expression for the area $(=\frac{1}{8}\pi)$ of a semicircle, diameter $=1$; making a slight transformation, the actual problem was to find the value of $\square\left(=\frac{4}{\pi}\right)$, the term halfway between 1 and 2, in the series of terms 1, 2, 6, 20, 70, ...; and he thus obtained the remarkable expression $\pi = \frac{2.4.4.6.6.8.8\ldots}{3.3.5.5.7.7.9\ldots}$, together with a succession of superior and inferior limits for the number π.

In the same work, Wallis obtained the expression which would now be written $ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ for the length of the element of a curve, thus reducing the problem of rectification to that of quadrature. An application of this formula to an algebraical curve was first made a few years later by W. Neil; the investigation is reproduced in the "Tractatus de Cissoide, &c." (1659, as above), and Wallis adds the remark that the curve thus rectified is in fact the semicubical parabola.

The *Mathesis Universalis* is a more elementary work intended for learners. It contains copious dissertations on fundamental points of algebra, arithmetic, and geometry, and critical remarks.

The *De Algebra Tractatus* contains (chapters 66—69) the idea of the interpretation of imaginary quantities in geometry. This is given somewhat as follows: the distance represented by the square root of a negative quantity cannot be measured in the line backwards or forwards, but can be measured in the same plane above the line, or (as appears elsewhere) at right angles to the line either in the plane, or in the plane at right angles thereto. Considered as a history of algebra, this work is strongly objected to by Montucla, on the ground of its unfairness as against the early Italian algebraists and also Vieta and Descartes, and in favour of Harriot; but De Morgan, while admitting this, attributes to it considerable merit.

The two treatises on the cycloid and on the cissoid, &c., and the *Mechanica* contain many results which were then new and valuable. The latter work contains elaborate investigations in regard to the centre of gravity, and it is remarkable also for the employment of the principle of virtual velocities. The cuno-cuneus is a highly

interesting surface; it is a ruled quartic surface, the equation of which may be written $c^2y^2 = (c - z)^2 (a^2 - x^2)$.

Among the letters in volume III., there is one to the editor of the Leipsic *Acts*, giving the decipherment of two letters in secret characters. The ciphers are different, but on the same principle: the characters in each are either single digits or combinations of two or three digits, standing some of them for letters, others for syllables or words,—the number of distinct characters which had to be deciphered being thus very considerable.

For the prolonged conflict between Hobbes and Wallis, see the article Hobbes, [*Encyclopædia Britannica, ninth edition,*] vol. XII. pp. 36—38.

END OF VOL. XI.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. XII.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1897.

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY J. AND C. F. CLAY,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 89 papers, numbered 799 to 887, published for the most part in the years 1883 to 1889.

The Table for the twelve volumes is

Vol.	I.	Numbers	1	to	100,
,,	II.	,,	101	,,	158,
,,	III.	,,	159	,,	222,
,,	IV.	,,	223	,,	299,
,,	V.	,,	300	,,	383,
,,	VI.	,,	384	,,	416,
,,	VII.	,,	417	,,	485,
,,	VIII.	,,	486	,,	555,
,,	IX.	,,	556	,,	629,
,,	X.	,,	630	,,	705,
,,	XI.	,,	706	,,	798,
,,	XII.	,,	799	,,	887.

A. R. FORSYTH.

19 *May*, 1897.

CONTENTS.

[An Asterisk means that the paper is not printed in full.]

PAGE

799. *On curvilinear coordinates* 1
Quart. Math. Journ., t. XIX. (1883), pp. 1—22

800. *Note on the standard solutions of a system of linear equations* 19
Quart. Math. Journ., t. XIX. (1883), pp. 38—40

801. *On seminvariants* 22
Quart. Math. Journ., t. XIX. (1883), pp. 131—138

802. *Note on Captain MacMahon's paper "On the differential equation* $X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0$*"* 30
Quart. Math. Journ., t. XIX. (1883), pp. 182—184

803. *On Mr Anglin's formula for the successive powers of the root of an algebraical equation* 33
Quart. Math. Journ., t. XIX. (1883), pp. 223, 224

804. *On the elliptic-function solution of the equation* $x^3 + y^3 - 1 = 0$. 35
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 106—109

805. *Note on Abel's theorem* 38
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 119—122

806. *Determination of the order of a surface* 42
Messenger of Mathematics, t. XII. (1883), pp. 29—32

807. *A proof of Wilson's theorem* 45
Messenger of Mathematics, t. XII. (1883), p. 41

PAGE

808. *Note on a form of the modular equation in the transformation of the third order* 46
Messenger of Mathematics, t. XII. (1883), pp. 173, 174

809. *Schröter's construction of the regular pentagon* 47
Messenger of Mathematics, t. XII. (1883), p. 177

810. *Note on a system of equations* 48
Messenger of Mathematics, t. XII. (1883), pp. 191, 192

811. *On the linear transformation of the theta-functions* . . . 50
Messenger of Mathematics, t. XIII. (1884), pp. 54—60

812. *On Archimedes' theorem for the surface of a cylinder* . . 56
Messenger of Mathematics, t. XIII. (1884), pp. 107, 108

813. [*Note on Mr Griffiths' paper "On a deduction from the elliptic-integral formula* $y = \sin(A + B + C + \ldots)$*"*] . . 58
Proc. Lond. Math. Soc., t. XV. (1884), p. 81

814. *On double algebra* 60
Proc. Lond. Math. Soc., t. XV. (1884), pp. 185—197

815. *The binomial equation* $x^p - 1 = 0$; *quinquisection. Second part* 72
Proc. Lond. Math. Soc., t. XVI. (1885), pp. 61—63

816. *On the bitangents of a plane quartic* 74
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 93—115; Camb. Phil. Soc. Proc., t. IV. (1883), p. 321

817. *On the sixteen-nodal quartic surface* 95
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 270—272

818. *Note on hyperelliptic integrals of the first order* . . . 98
Crelle's Journal der Mathem., t. XCVIII. (1885), pp. 95, 96

819. *On two cases of the quadric transformation between two planes* 100
Johns Hopkins University Circulars, No. 13 (1882), pp. 178, 179

820. *On a problem of analytical geometry* 102
Johns Hopkins University Circulars, No. 15 (1882), p. 209

PAGE

821. *On the geometrical representation of an equation between two variables* 104

Johns Hopkins University Circulars, No. 15 (1882), p. 210

822. *On associative imaginaries* 105

Johns Hopkins University Circulars, No. 15 (1882), pp. 211, 212

823. *On the geometrical interpretation of certain formulæ in elliptic functions* 107

Johns Hopkins University Circulars, No. 17 (1882), p. 238

824. *Note on the formulæ of trigonometry* 108

Johns Hopkins University Circulars, No. 17 (1882), p. 241

825. *A memoir on the Abelian and Theta Functions* 109

Chapters I to III, American Journal of Mathematics, t. v. (1882), pp. 137—179; Chapters IV to VII, *ib.*, t. vii. (1885), pp. 101—167

826. *Note on a partition series* 217

American Journal of Mathematics, t. vi. (1884), pp. 63, 64

827. *On the non-Euclidian plane geometry* 220

Proc. Roy. Soc., t. xxxvii. (1884), pp. 82—102

828. *A memoir on seminvariants* 239

American Journal of Mathematics, t. vii. (1885), pp. 1—25

829. *Tables of the symmetric functions of the roots, to the degree* 10, *for the form* $1+bx+\frac{cx^2}{1.2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$. 263

American Journal of Mathematics, t. vii. (1885), pp. 47—56

830. *Non-unitary partition tables* 273

American Journal of Mathematics, t. vii. (1885), pp. 57, 58

831. *Seminvariant tables* 275

American Journal of Mathematics, t. vii. (1885), pp. 59—73

832. *Note on an apparent difficulty in the theory of curves, when the coordinates of a point are given as functions of a variable parameter* 290

Messenger of Mathematics, t. xiv. (1885), pp. 12—14

PAGE

833. *On a formula in elliptic functions* 292
Messenger of Mathematics, t. XIV. (1885), pp. 21, 22

834. *On the addition of the elliptic functions* 294
Messenger of Mathematics, t. XIV. (1885), pp. 56—61

835. *On Cardan's solution of a cubic equation* 299
Messenger of Mathematics, t. XIV. (1885), pp. 96, 97

836. *On the quaternion equation* $qQ-Qq'=0$ 300
Messenger of Mathematics, t. XIV. (1885), pp. 108—112

837. *On the so-called D'Alembert Carnot geometrical paradox* . 305
Messenger of Mathematics, t. XIV. (1885), pp. 113, 114

838. *On the twisted cubics upon a quadric surface* 307
Messenger of Mathematics, t. XIV. (1885), pp. 129—132

839. *On the matrical equation* $qQ-Qq'=0$ 311
Messenger of Mathematics, t. XIV. (1885), pp. 176—178

840. *On Mascheroni's geometry of the compass* 314
Messenger of Mathematics, t. XIV. (1885), pp. 179—181

841. *On a differential operator* 318
Messenger of Mathematics, t. XIV. (1885), pp. 190, 191

842. *On the value of* $\tan(\sin\theta)-\sin(\tan\theta)$ 319
Messenger of Mathematics, t. XIV. (1885), pp. 191, 192

843. *On the quadri-quadric curve in connexion with the theory of elliptic functions* 321
Mathematische Annalen, t. XXV. (1885), pp. 152—156

844. *On a theorem relating to seminvariants* 326
Quart. Math. Journ., t. XX. (1885), pp. 212, 213

845. *On the orthomorphosis of the circle into the parabola* . . 328
Quart. Math. Journ., t. XX. (1885), pp. 213—220

846. *A verification in regard to the linear transformation of the theta-functions* 337
Quart. Math. Journ., t. XXI. (1886), pp. 77—84

PAGE

847. *On the theory of seminvariants* 344

Quart. Math. Journ., t. XXI. (1886), pp. 92—107

848. *On the transformation of the double theta-functions* . . 358

Quart. Math. Journ., t. XXI. (1886), pp. 142—178

849. *On the invariants of a linear differential equation* . . 390

Quart. Math. Journ., t. XXI. (1886), pp. 257—261

850. *On linear differential equations* 394

Quart. Math. Journ., t. XXI. (1886), pp. 321—331

851. *On linear differential equations: the theory of decomposition* . 403

Quart. Math. Journ., t. XXI. (1886), pp. 331—335

852. *Note sur le mémoire de M. Picard "Sur les intégrales de différentielles totales algébriques de première espèce"* . . 408

Bull. des Sciences Math., 2me Sér., t. X. (1886), pp. 75—78

853. *Note on a formula for* $\Delta^n 0^i/n^i$ *when* n, i *are very large numbers* 412

Proc. Roy. Soc. Edin., t. XIV. (1887), pp. 149—153

854. *An algebraical transformation* 416

Messenger of Mathematics, t. XV. (1886), pp. 58, 59

855. *Solution of* $(a, b, c, d) = (a^2, b^2, c^2, d^2)$ 418

Messenger of Mathematics, t. XV. (1886), pp. 59—61

856. *Note on a cubic equation* 421

Messenger of Mathematics, t. XV. (1886), pp. 62—64

857. *Analytical geometrical note on the conic* 424

Messenger of Mathematics, t. XV. (1886), p. 192

858. *Comparison of the Weierstrassian and Jacobian elliptic functions* 425

Messenger of Mathematics, t. XVI. (1887), pp. 129—132

859. *On the complex of lines which meet a unicursal quartic curve* 428

Proc. Lond. Math. Soc., t. XVII. (1886), pp. 232—238

PAGE

860. *On Briot and Bouquet's theory of the differential equation* $F\left(u, \frac{du}{dz}\right)=0$ 432

Proc. Lond. Math. Soc., t. XVIII. (1887), pp. 314—324

861. *Note on a formula relating to the zero-value of a theta-function* 442

Crelle's Journal der Mathem., t. C. (1887), pp. 87, 88

862. *Note on the theory of linear differential equations* . . . 444

Crelle's Journal der Mathem., t. C. (1887), pp. 286—295

863. *Note on the theory of linear differential equations* . . . 453

Crelle's Journal der Mathem., t. CI. (1887), pp. 209—213

864. *On Rudio's inverse centro-surface* 457

Quart. Math. Journ., t. XXII. (1887), pp. 156—158

865. *On multiple algebra* 459

Quart. Math. Journ., t. XXII. (1887), pp. 270—308

866. *Note on Kiepert's L-equations, in the transformation of elliptic functions* 490

Mathematische Annalen, t. XXX. (1887), pp. 75—77

867. *Note on the Jacobian sextic equation* 493

Mathematische Annalen, t. XXX. (1887), pp. 78—84

868. *On the intersection of curves* 500

Mathematische Annalen, t. XXX. (1887), pp. 85—90

869. *On the transformation of elliptic functions* 505

American Journal of Mathematics, t. IX. (1887), pp. 193—224

870. *On the transformation of elliptic functions (sequel)* . . . 535

American Journal of Mathematics, t. X. (1888), pp. 71—93

871. *A case of complex multiplication with imaginary modulus arising out of the cubic transformation in elliptic functions* 556

Proc. Lond. Math. Soc., t. XIX. (1888), pp. 300, 301

*872. *On the finite number of the covariants of a binary quantic* . 558

Mathematische Annalen, t. XXXIV. (1889), pp. 319, 320

PAGE

873. *System of equations for three circles which cut each other at given angles* 559

Messenger of Mathematics, t. XVII. (1888), pp. 18—21

874. *Note on the Legendrian coefficients of the second kind* . . 562

Messenger of Mathematics, t. XVII. (1888), pp. 21—23

875. *On the system of three circles which cut each other at given angles and have their centres in a line* 564

Messenger of Mathematics, t. XVII. (1888), pp. 60—69

876. *On systems of rays* 571

Messenger of Mathematics, t. XVII. (1888), pp. 73—78

877. *Note on the two relations connecting the distances of four points on a circle* 576

Messenger of Mathematics, t. XVII. (1888), pp. 94, 95

878. *Note on the anharmonic ratio equation* 578

Messenger of Mathematics, t. XVII. (1888), pp. 95, 96

879. *Note on the differential equation* $\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0$. 580

Messenger of Mathematics, t. XVIII. (1889), p. 90

880. *Note on the relation between the distance of five points in space* 581

Messenger of Mathematics, t. XVIII. (1889), pp. 100—102

881. *On Hermite's H-product theorem* 584

Messenger of Mathematics, t. XVIII. (1889), pp. 104—107

882. *A correspondence of confocal Cartesians with the right lines of a hyperboloid* 587

Messenger of Mathematics, t. XVIII. (1889), pp. 128—130

883. *Analytical formulæ in regard to an octad of points* . . 590

Messenger of Mathematics, t. XVIII. (1889), pp. 149—152

884. *Note sur les surfaces minima et le théorème de Joachimsthal* 594

Comptes Rendus, t. CVI. (1888), pp. 995, 996

PAGE

885. *On the Diophantine relation,* $y^2+y'^2=$ *Square* 596

Proc. Lond. Math. Soc., t. xx. (1889), pp. 122—127

886. *On the surfaces with plane or spherical curves of curvature* . 601

American Journal of Mathematics, t. xi. (1889), pp. 71—98; pp. 293—306

887. *On the theory of groups* 639

American Journal of Mathematics, t. xi. (1889), pp. 139—157

CLASSIFICATION.

ANALYSIS.

Equations, 801, 803, 835, 855, 856, 867, 878.

Theory of numbers, 807, 815.

Groups, 887.

Quaternions, 836.

Matrices, 839.

Trigonometry, 824, 842.

Symmetric functions, 829, 841.

Partitions, 826, 830.

Invariants and covariants, 800, 801, 828, 831, 844, 847, *872.

Multiple Algebra, 814, 822, 865.

Integral calculus, 852.

Legendre's coefficients, 874.

Finite differences, 853.

Differential equations (general), 802, 860, 879, 885.

Differential equations (linear), 849, 850, 851, 862, 863.

Elliptic functions, 804, 813, 823, 833, 834, 843, 858, 881.

Transformation of elliptic functions, 808, 854, 866, 869, 870, 871.

Abel's theorem, 805.

Abelian functions, 818, 825.

Theta functions, 811, 825, 846, 848, 861.

GEOMETRY.

Properties of circles, 873, 875, 877.

Non-Euclidian geometry, 827.

General theory of curves, 820, 832, 868.

Correspondence, 819, 821.

Orthomorphosis, 821, 845.

Plane Quartics, 816.

Groups of points, 880, 883.

General theory of surfaces, 799, 806.

Quadrics, 882.

Quartic surfaces, 817.

Centro-surfaces, 864.

Minimal surfaces, 884.

Rays, 876.

Curves of curvature, 886.

Twisted cubic curves, 838.

Twisted quartic curves, 843, 859.

MISCELLANEOUS, 809, 810, 812, 837, 840, 857.

799.

ON CURVILINEAR COORDINATES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 1—22.]

THE present memoir is based upon Mr Warren's "Exercises in Curvilinear and Normal Coordinates," *Camb. Phil. Trans.* t. XII. (1877), pp. 455—502, and has for a principal object the establishment of the six differential equations of the second order corresponding to his six equations for normal coordinates: but the notation is different; the results are more general, inasmuch as I use throughout general curvilinear coordinates instead of his normal coordinates; and as regards my six equations for general curvilinear coordinates, the terms containing differential coefficients of the first order are presented under a different form.

If the position of a point in space is determined by the rectangular coordinates x, y, z; then p, q, r being each of them a given function of x, y, z, we have conversely x, y, z, each of them a given function of p, q, r, which are thus in effect coordinates serving to determine the position of the point, and are called curvilinear coordinates.

But it is not in the first instance necessary to regard x, y, z as rectangular coordinates, or even as Cartesian coordinates at all; we are simply concerned with the two sets of variables x, y, z and p, q, r, each variable of the one set being a given function of the variables of the other set; and, in particular, the x, y, z are regarded as being each of them a given function of the p, q, r.

Except as regards the symbols ξ, η, ζ presently mentioned, the suffixes 1, 2, 3 refer to the variables p, q, r respectively, and are used to denote differentiations in regard to these variables, viz.

$$x_1,\ x_2,\ x_3,\ x_{11},\ x_{12},\ x_{22}, \ldots$$

are written to denote

$$\frac{dx}{dp},\ \frac{dx}{dq},\ \frac{dx}{dr},\ \frac{d^2x}{dp^2},\ \frac{d^2x}{dp\,dq},\ \frac{d^2x}{dq^2},\ \&c.,$$

and so in other cases; in particular,

$$\begin{array}{llll} x_1,\ x_2,\ x_3 & \text{denote} & \dfrac{dx}{dp},\ \dfrac{dx}{dq},\ \dfrac{dx}{dr}, \\ y_1,\ y_2,\ y_3 & \quad\text{,,} & \dfrac{dy}{dp},\ \dfrac{dy}{dq},\ \dfrac{dy}{dr}, \\ z_1,\ z_2,\ z_3 & \quad\text{,,} & \dfrac{dz}{dp},\ \dfrac{dz}{dq},\ \dfrac{dz}{dr}. \end{array}$$

The minors formed with these differential coefficients are denoted by suffixed letters ξ, η, ζ, thus

$$\begin{array}{llll} \xi_1,\ \xi_2,\ \xi_3 & \text{denote} & y_2z_3 - y_3z_2, & y_3z_1 - y_1z_3,\quad y_1z_2 - y_2z_1, \\ \eta_1,\ \eta_2,\ \eta_3 & \quad\text{,,} & z_2x_3 - z_3x_2, & z_3x_1 - z_1x_3,\quad z_1x_2 - z_2x_1, \\ \zeta_1,\ \zeta_2,\ \zeta_3 & \quad\text{,,} & x_2y_3 - x_3y_2, & x_3y_1 - x_1y_3,\quad x_1y_2 - x_2y_1, \end{array}$$

so that, as regards these letters ξ, η, ζ, the suffixes do *not* denote differentiations.

The determinant $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$ is put $= L$:

and the symbols (a, b, c, f, g, h), (A, B, C, F, G, H) are defined as follows:

$$\begin{array}{ll} \mathrm{a} = x_1^2 + y_1^2 + z_1^2, & A = \xi_1^2 + \eta_1^2 + \zeta_1^2, \\ \mathrm{b} = x_2^2 + y_2^2 + z_2^2, & B = \xi_2^2 + \eta_2^2 + \zeta_2^2, \\ \mathrm{c} = x_3^2 + y_3^2 + z_3^2, & C = \xi_3^2 + \eta_3^2 + \zeta_3^2, \\ \mathrm{f} = x_2x_3 + y_2y_3 + z_2z_3, & F = \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3, \\ \mathrm{g} = x_3x_1 + y_3y_1 + z_3z_1, & G = \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1, \\ \mathrm{h} = x_1x_2 + y_1y_2 + z_1z_2, & H = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2. \end{array}$$

We have then, further,

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = L, \qquad \begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = L^2,$$

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix} = L^2, \qquad \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \frac{1}{L^2},$$

$$(A, B, C, F, G, H) = (\mathrm{bc} - \mathrm{f}^2,\ \mathrm{ca} - \mathrm{g}^2,\ \mathrm{ab} - \mathrm{h}^2,\ \mathrm{gh} - \mathrm{af},\ \mathrm{hf} - \mathrm{bg},\ \mathrm{fg} - \mathrm{ch}),$$

$$L^2(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h}) = (BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH),$$

which equations are at once proved, and are fundamental ones in the theory.

It is convenient to add that we have

$$\left(\begin{array}{c|ccc} \frac{dp}{dx}, & \frac{dp}{dy}, & \frac{dp}{dz} \\ \frac{dq}{dx}, & \frac{dq}{dy}, & \frac{dq}{dz} \\ \frac{dr}{dx}, & \frac{dr}{dy}, & \frac{dr}{dz} \end{array}\right) = \frac{1}{L}\left(\begin{array}{c|ccc} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{array}\right),$$

that is, $\frac{dp}{dx} = \frac{1}{L}\xi_1$, &c.; and, further,

$$dx^2 + dy^2 + dz^2 = \quad (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between dp, dq, dr)^2,$$

$$dp^2 + dq^2 + dr^2 = \frac{1}{L^2}(A, B, C, F, G, H \between dx, dy, dz)^2.$$

Differentiating the values of a, b, c, f, g, h, we have $\frac{1}{2}\mathrm{a}_1 = x_1 x_{11} + y_1 y_{11} + z_1 z_{11}$, which may be written in the abbreviated form $\frac{1}{2}\mathrm{a}_1 = 1.11$; similarly

$$\mathrm{f}_1 = x_2 x_{13} + y_2 y_{13} + z_2 z_{13} + x_3 x_{12} + y_3 y_{12} + z_3 z_{12},$$

which in like manner may be written $\mathrm{f}_1 = 2.13 + 3.12$, and so in other cases; observe that, in the duad part of any symbol, the order of the numbers is immaterial, $2.13 = 2.31$. The whole system of equations is

$$\begin{array}{lll} \frac{1}{2}\mathrm{a}_1 = 1.11 \quad , & \frac{1}{2}\mathrm{a}_2 = 1.12 \quad , & \frac{1}{2}\mathrm{a}_3 = 1.13 \quad , \\ \frac{1}{2}\mathrm{b}_1 = 2.12 \quad , & \frac{1}{2}\mathrm{b}_2 = 2.22 \quad , & \frac{1}{2}\mathrm{b}_3 = 2.23 \quad , \\ \frac{1}{2}\mathrm{c}_1 = 3.13 \quad , & \frac{1}{2}\mathrm{c}_2 = 3.23 \quad , & \frac{1}{2}\mathrm{c}_3 = 3.33 \quad , \end{array}$$

$$\begin{array}{lll} \mathrm{f}_1 = 2.13 + 3.12, & \mathrm{f}_2 = 2.23 + 3.22, & \mathrm{f}_3 = 2.33 + 3.23, \\ \mathrm{g}_1 = 3.11 + 1.13, & \mathrm{g}_2 = 3.12 + 1.23, & \mathrm{g}_3 = 3.13 + 1.33, \\ \mathrm{h}_1 = 1.12 + 2.11, & \mathrm{h}_2 = 1.22 + 2.12, & \mathrm{h}_3 = 1.23 + 2.13. \end{array}$$

These may also be written

$$\begin{array}{lll} 1.11 = \frac{1}{2}\mathrm{a}_1 \quad , & 1.22 = \mathrm{h}_2 - \frac{1}{2}\mathrm{b}_1 \quad , & 1.33 = \mathrm{g}_3 - \frac{1}{2}\mathrm{c}_1 \quad , \\ 2.11 = \mathrm{h}_1 - \frac{1}{2}\mathrm{a}_2 \quad , & 2.22 = \frac{1}{2}\mathrm{b}_2 \quad , & 2.33 = \mathrm{f}_3 - \frac{1}{2}\mathrm{c}_2 \quad , \\ 3.11 = \mathrm{g}_1 - \frac{1}{2}\mathrm{a}_3 \quad , & 3.22 = \mathrm{f}_2 - \frac{1}{2}\mathrm{b}_3 \quad , & 3.33 = \frac{1}{2}\mathrm{c}_3 \quad , \\ 1.23 = \frac{1}{2}(-\mathrm{f}_1 + \mathrm{g}_2 + \mathrm{h}_3), & 1.31 = \frac{1}{2}\mathrm{a}_3 \quad , & 1.12 = \frac{1}{2}\mathrm{a}_2 \quad , \\ 2.23 = \frac{1}{2}\mathrm{b}_3 \quad , & 2.31 = \frac{1}{2}(\mathrm{f}_1 - \mathrm{g}_2 + \mathrm{h}_3), & 2.12 = \frac{1}{2}\mathrm{b}_1 \quad , \\ 3.23 = \frac{1}{2}\mathrm{c}_2 \quad , & 3.31 = \frac{1}{2}\mathrm{c}_1 \quad , & 3.12 = \frac{1}{2}(\mathrm{f}_1 + \mathrm{g}_2 - \mathrm{h}_3). \end{array}$$

It is to be observed that we can, from each system of three equations, express a set of second differential coefficients of the x, y, z in terms of the first differential

coefficients of the a, b, c, f, g, h; thus the three equations containing 11, written at length, are

$$\begin{aligned} x_1 . x_{11} + y_1 . y_{11} + z_1 . z_{11} &= \tfrac{1}{2}\mathrm{a}_1, \\ x_2 \;\;,, + y_2 \;\;,, + z_2 \;\;,, &= \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2, \\ x_3 \;\;,, + y_3 \;\;,, + z_3 \;\;,, &= \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3, \end{aligned}$$

three linear equations for the determination of x_{11}, y_{11}, z_{11}; hence for x_{11} we have

$$x_{11} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \tfrac{1}{2}\mathrm{a}_1 (y_3 z_2 - y_2 z_3) + (\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2)(y_3 z_1 - y_1 z_3) + (\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3)(y_1 z_2 - y_2 z_1),$$

or, what is the same thing,

$$L x_{11} = \tfrac{1}{2}\mathrm{a}_1 . \xi_1 + (\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2)\,\xi_2 + (\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3)\,\xi_3,$$

and so for y_{11}, z_{11}; or if (as in the sequel) we desire the value of a linear function $\alpha x_{11} + \beta y_{11} + \gamma z_{11}$, calling this for a moment $\square$, we join to the foregoing a new equation

$$\alpha x_{11} + \beta y_{11} + \gamma z_{11} = \square,$$

and then, eliminating the three quantities, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}\mathrm{a}_1 \\ x_2, & y_2, & z_2, & \mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2 \\ x_3, & y_3, & z_3, & \mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3 \end{vmatrix} = 0,$$

giving $L\square$ as a linear function of $\tfrac{1}{2}\mathrm{a}_1$, $\mathrm{h}_1 - \tfrac{1}{2}\mathrm{a}_2$, $\mathrm{g}_1 - \tfrac{1}{2}\mathrm{a}_3$.

We can in like manner form the expressions for the second differential coefficients of the a, b, c, f, g, h: these will of course contain third differential coefficients of the x, y, z.

Writing down only what is wanted, we have

$$\begin{aligned} \tfrac{1}{2}\mathrm{a}_{22} &= 12 . 12 + 1 . 122, \\ \tfrac{1}{2}\mathrm{b}_{11} &= 12 . 12 + 2 . 112, \\ \mathrm{h}_{12} &= 12 . 12 + 11 . 22 + 1 . 122 + 2 . 112, \end{aligned}$$

where of course $12 . 12$ denotes $x^2_{12} + y^2_{12} + z^2_{12}$, $1 . 122$ denotes $x_1 . x_{122} + y_1 . y_{122} + z_1 . z_{122}$, and so in other cases: it follows that

$$\tfrac{1}{2}(\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12}) = 12 . 12 - 11 . 22,$$

so that the third differential coefficients of x, y, z, which enter into the expression of $\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12}$ destroy each other, and this combination contains really only second differential coefficients of x, y, z.

Similarly

$$\begin{aligned}
f_{31} &= 31\,.\,23 + 33\,.\,12 + 3\,.\,123 + 2\,.\,133,\\
g_{23} &= 31\,.\,23 + 33\,.\,12 + 3\,.\,123 + 1\,.\,233\\
c_{12} &= 2\,.\,31\,.\,23 + 2\,.\,3\,.\,123\\
h_{33} &= 2\,.\,31\,.\,23 + 1\,.\,233 + 2\,.\,133,
\end{aligned}$$

and thence

$$f_{31} + g_{23} - c_{12} - h_{33} = -2\,(31\,.\,23 - 33\,.\,12),$$

so that here again we have a combination containing only second differential coefficients of x, y, z.

There are thus, in all, the six combinations

$$\begin{aligned}
&b_{33} + c_{22} - 2f_{23} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{A}),\\
&c_{11} + a_{33} - 2g_{31} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{B}),\\
&a_{22} + b_{11} - 2h_{12} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{C}),\\
&g_{12} + h_{31} - a_{23} - f_{11} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{F}),\\
&h_{23} + f_{12} - b_{31} - g_{22} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{G}),\\
&f_{31} + g_{23} - c_{12} - h_{33} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathfrak{H}),
\end{aligned}$$

each really containing only second differential coefficients of x, y, z; and we thus understand how each of these combinations may be expressible in terms of the first differential coefficients of (a, b, c, f, g, h). We have, in fact, for thus expressing these combinations the six equations called $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$, $(\mathfrak{F})$, $(\mathfrak{G})$, $(\mathfrak{H})$ about to be obtained, and which are the generalisations of Warren's six equations for normal coordinates.

I consider the several determinants of the form

$$\begin{vmatrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

in all 18, since the suffixes for the top row may be 11, 22, 33, 23, 31, 12, and those for the second and third rows 2, 3; 3, 1; or 1, 2. Each determinant is a linear function of second differential coefficients of x, y, z (thus the determinant written down is $= \xi_3 x_{11} + \eta_3 y_{11} + \zeta_3 z_{11}$), and as such it can, by what precedes, be expressed by means of the first differential coefficients of the a, b, c, f, g, h. Thus, if the determinant above written down be called $\square$, writing ξ_3, η_3, ζ_3 for α, β, γ, we have

$$\begin{vmatrix} \xi_3, & \eta_3, & \zeta_3, & \square \\ x_1, & y_1, & z_1, & \frac{1}{2}a_1 \\ x_2, & y_2, & z_2, & h_1 - \frac{1}{2}a_2 \\ x_3, & y_3, & z_3, & g_1 - \frac{1}{2}a_3 \end{vmatrix} = 0,$$

that is,

$$L\,.\,\square = -\begin{vmatrix} & & \xi_3, & \eta_3, & \zeta_3 \\ & \frac{1}{2}a_1, & x_1, & y_1, & z_1 \\ h_1 - \frac{1}{2}a_2, & & x_2, & y_2, & z_2 \\ g_1 - \frac{1}{2}a_3, & & x_3, & y_3, & z_3 \end{vmatrix},$$

where the right-hand side is

$$\begin{aligned} = \quad & \tfrac{1}{2}a_3 \,.\, \xi_1\xi_3 + \eta_1\eta_3 + \zeta_1\zeta_3 \\ & - (h_1 - \tfrac{1}{2}a_2) \,.\, -(\xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3) \\ & + (g_1 - \tfrac{1}{2}a_3) \,.\, \xi_3^2 + \eta_3^2 + \zeta_3^2, \end{aligned}$$

which is

$$= G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

or, as this might be written,

$$= (G,\ F,\ C \between a_1,\ h_1,\ g_1) - \tfrac{1}{2}(G,\ F,\ C \between a_1,\ a_2,\ a_3).$$

Retaining the former form, the result is

$$L \,.\, \square = G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

and it is now very easy to write down the complete system of the 18 equations; viz. if the determinant above written down be called 11.1.2, and so in other cases, then we have

	2.3	A	H	G
	3.1	H	B	F
	1.2	G	F	C
L.11	„ =	$\tfrac{1}{2}a_1$	$h_1 - \tfrac{1}{2}a_2$	$g_1 - \tfrac{1}{2}a_3$
„ 22	„ =	$h_2 - \tfrac{1}{2}b_1$	$\tfrac{1}{2}b_2$	$f_2 - \tfrac{1}{2}b_3$
„ 33	„ =	$g_3 - \tfrac{1}{2}c_1$	$f_3 - \tfrac{1}{2}c_2$	$\tfrac{1}{2}c_3$
„ 23	„ =	$\tfrac{1}{2}(-f_1 + g_3 + h_3)$	$\tfrac{1}{2}b_3$	$\tfrac{1}{2}c_2$
„ 31	„ =	$\tfrac{1}{2}a_3$	$\tfrac{1}{2}(f_1 - g_3 + h_3)$	$\tfrac{1}{2}c_1$
„ 12	„ =	$\tfrac{1}{2}a_2$	$\tfrac{1}{2}b_1$	$\tfrac{1}{2}(f_1 + g_2 - h_3)$:

read for instance

$$L \,.\, 11 \,.\, 1 \,.\, 2 = G \,.\, \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

the equation obtained above.

There are eighteen functions as shown by the following diagram:

	2.3	3.1	1.2
$(22)(33) - (23)^2$	2.3		
$(33)(11) - (31)^2$	3.1		
$(11)(22) - (12)^2$	1.2		
$(31)(12) - (11)\ (23)$	2.3		
$(12)(23) - (22)\ (31)$	3.1		
$(23)(31) - (33)\ (12)$	1.2,		

viz. in any line of the diagram the bracketed duads may belong to any one at pleasure, but all to the same, of the three pairs 2.3, 3.1, 1.2; thus the first line might be

$$(22.3.1)(33.3.1)-(23.3.1)^2,$$

or, instead of the 3.1, we might have 2.3 or 1.2. But of the 18 functions I distinguish 6, viz. those in which for the six lines respectively the pairs are 2.3, 3.1, 1.2, 2.3, 3.1, 1.2, as shown in the diagram. Each of these six functions can be obtained under two different forms, and by equating these we have the equations ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$), ($\mathfrak{F}$), ($\mathfrak{G}$), ($\mathfrak{H}$), before referred to; thus ($\mathfrak{C}$) is obtained by equating two different forms of the function $(11.1.2)(22.1.2)-(12.1.2)^2$; and ($\mathfrak{H}$) is obtained by equating two different forms of the function $(23.1.2)(31.1.2)-(33.1.2)(12.1.2)$.

The determinants 11.1.2, &c., may be denoted by accented letters a, b, c, f, g, h, as follows:

$$\left|\begin{matrix} x_{11}, & x_{22}, & x_{33}, & x_{23}, & x_{31}, & x_{12} \\ y_{11}, & y_{22}, & y_{33}, & y_{23}, & y_{31}, & y_{12} \\ z_{11}, & z_{22}, & z_{33}, & z_{23}, & z_{31}, & z_{12} \end{matrix}\right| \begin{matrix} x_2, x_3 = \mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}', \\ y_2, y_3 \\ z_2, z_3 \end{matrix}$$

$$\left|\begin{matrix} & \text{"} & & \text{"} & & \end{matrix}\right| \begin{matrix} x_3, x_1 = \mathrm{a}'', \mathrm{b}'', \mathrm{c}'', \mathrm{f}'', \mathrm{g}'', \mathrm{h}'', \\ y_3, y_1 \\ z_3, z_1 \end{matrix}$$

$$\left|\begin{matrix} & \text{"} & & \text{"} & & \end{matrix}\right| \begin{matrix} x_1, x_2 = \mathrm{a}''', \mathrm{b}''', \mathrm{c}''', \mathrm{f}''', \mathrm{g}''', \mathrm{h}''; \\ y_1, y_2 \\ z_1, z_2 \end{matrix}$$

viz.

$$\left|\begin{matrix} x_{11}, & x_2, & x_3 \\ y_{11}, & y_2, & y_3 \\ z_{11}, & z_2, & z_3 \end{matrix}\right| = \mathrm{a}', \text{ \&c.}$$

In this notation, ($\mathfrak{C}$) is obtained by equating two values of $\mathrm{a}'''\mathrm{b}'''-(\mathrm{h}''')^2$, and ($\mathfrak{H}$) by equating two values of $\mathrm{f}'''\mathrm{g}'''-\mathrm{c}'''\mathrm{h}'''$.

The forms which I call the second are those given by the immediate substitution of the foregoing values of (11.1.2), &c.; to obtain the first forms, I proceed to calculate those of ($\mathfrak{C}$) and ($\mathfrak{H}$).

Forming by the ordinary rule the product of the determinants (11.1.2) and (22.1.2), which are

$$\left|\begin{matrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right| \text{ and } \left|\begin{matrix} x_{22}, & y_{22}, & z_{22} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{matrix}\right|,$$

this is

$$\begin{vmatrix} 11.22, & 1.11, & 2.11 \\ 1.22, & 1.1\ , & 1.2 \\ 2.22, & 1.2\ , & 2.2 \end{vmatrix},$$

where 11.22 denotes $x_{11}x_{22}+y_{11}y_{22}+z_{11}z_{22}$, and the like for the other symbols. In like manner, the square of the determinant $(12.1.2)$, that is, of

$$\begin{vmatrix} x_{12}, & y_{12}, & z_{12} \\ x_1\ , & y_1\ , & z_1 \\ x_2\ , & y_2\ , & z_2 \end{vmatrix},$$

is

$$\begin{vmatrix} 12.12, & 1.12, & 2.12 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 2.1\ , & 2.2 \end{vmatrix},$$

or, observing that in the two resulting determinants the terms 11.22 and 12.12 are multiplied by the same factor, the expression for the difference gives

$$(11.1.2)(22.1.2)-(12.1.2)^2$$

$$=(11.22-12.12)\begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix}+\begin{vmatrix} 0\ , & 1.11, & 2.11 \\ 1.22, & 1.1\ , & 1.2 \\ 2.22, & 1.2\ , & 2.2 \end{vmatrix}-\begin{vmatrix} 0\ , & 1.12, & 2.12 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 1.2\ , & 2.2 \end{vmatrix},$$

containing $11.22-12.12$, which, by what precedes, is

$$=-\tfrac{1}{2}(\mathrm{a}_{22}+\mathrm{b}_{11}-2\mathrm{h}_{12});$$

the other terms are also known, viz. the whole value is

$$=-\tfrac{1}{2}(\mathrm{a}_{22}+\mathrm{b}_{11}-2\mathrm{h}_{12})(\mathrm{ab}-\mathrm{h}^2)+\begin{vmatrix} 0\ , & \tfrac{1}{2}\mathrm{a}, & \mathrm{h}_1-\tfrac{1}{2}\mathrm{a}_2 \\ \mathrm{h}_2-\tfrac{1}{2}\mathrm{b}_1, & \mathrm{a}, & \mathrm{h} \\ \tfrac{1}{2}\mathrm{b}_2, & \mathrm{h}, & \mathrm{b} \end{vmatrix}-\begin{vmatrix} 0\ , & \tfrac{1}{2}\mathrm{a}_2, & \tfrac{1}{2}\mathrm{b}_1 \\ \tfrac{1}{2}\mathrm{a}_2, & \mathrm{a}\ , & \mathrm{h} \\ \tfrac{1}{2}\mathrm{b}_1, & \mathrm{h}\ , & \mathrm{b} \end{vmatrix},$$

which is

$$\begin{aligned} =&-\tfrac{1}{2}(\mathrm{a}_{22}+\mathrm{b}_{11}-2\mathrm{h}_{12})(\mathrm{ab}-\mathrm{h}^2) \\ &+\mathrm{a}\,(\tfrac{1}{4}\mathrm{b}_1^2-\tfrac{1}{2}\mathrm{b}_2\mathrm{h}_1+\tfrac{1}{4}\mathrm{a}_2\mathrm{b}_2) \\ &+\mathrm{b}\,(\tfrac{1}{4}\mathrm{a}_2^2-\tfrac{1}{2}\mathrm{a}_1\mathrm{h}_2+\tfrac{1}{4}\mathrm{a}_1\mathrm{b}_1) \\ &+\mathrm{h}\,(\tfrac{1}{4}\mathrm{a}_1\mathrm{b}_2-\tfrac{1}{4}\mathrm{a}_2\mathrm{b}_1+\mathrm{h}_1\mathrm{h}_2-\tfrac{1}{2}\mathrm{a}_2\mathrm{h}_2-\tfrac{1}{2}\mathrm{b}_1\mathrm{h}_1), \end{aligned}$$

$=(\mathrm{ab}-\mathrm{h}^2)k$, where k is the measure of curvature.

In the same way, we have

$$(23.1.2)(31.1.2)-(33.1.2)(12.1.2)$$

$$=\begin{vmatrix} 23.31, & 1.31, & 2.31 \\ 1.23, & 1.1\ , & 1.2 \\ 2.23, & 1.2\ , & 2.2 \end{vmatrix}-\begin{vmatrix} 12.33, & 1.33, & 2.33 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 1.2\ , & 2.2 \end{vmatrix},$$

which is

$$=(23.31-12.33)\begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix}+\begin{vmatrix} 0\ , & 1.13, & 2.13 \\ 1.23, & 1.1\ , & 1.2 \\ 2.23, & 1.2\ , & 2.2 \end{vmatrix}-\begin{vmatrix} 0\ , & 1.33, & 2.33 \\ 1.12, & 1.1\ , & 1.2 \\ 2.12, & 1.2\ , & 2.2 \end{vmatrix},$$

containing $23.31-12.33$, which by what precedes is

$$=-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33});$$

the other terms are also known, and the value is

$$=-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33})(ab-h^2)$$
$$+\begin{vmatrix} 0 & , & \tfrac{1}{2}a_3, & \tfrac{1}{2}(f_1-g_2+h_3) \\ \tfrac{1}{2}(-f_1+g_2+h_3), & & a\ , & h \\ \tfrac{1}{2}b_3 & , & h\ , & b \end{vmatrix}-\begin{vmatrix} 0\ , & g_3-\tfrac{1}{2}c_1, & f_3-\tfrac{1}{2}c_2 \\ \tfrac{1}{2}a_2, & a\ , & h \\ \tfrac{1}{2}b_1, & h\ , & b \end{vmatrix},$$

or finally this is

$$\begin{aligned} &=-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33})(ab-h^2) \\ &\quad+a(\tfrac{1}{2}b_1f_3-\tfrac{1}{4}b_3f_1-\tfrac{1}{4}b_1c_2+\tfrac{1}{4}b_3g_2-\tfrac{1}{4}b_3h_3) \\ &\quad+b(\tfrac{1}{2}a_2g_3-\tfrac{1}{4}a_3g_2-\tfrac{1}{4}a_2c_1+\tfrac{1}{4}a_3f_1-\tfrac{1}{4}a_3h_3) \\ &\quad+h\{\tfrac{1}{4}a_3b_3+\tfrac{1}{4}b_1c_1+\tfrac{1}{4}a_2c_2-\tfrac{1}{2}b_1g_3-\tfrac{1}{2}a_2f_3+\tfrac{1}{4}h_3^2-\tfrac{1}{4}(f_1-g_2)^2\}. \end{aligned}$$

The remaining four of the six functions can of course be obtained from the two just found by a cyclical interchange of the letters and suffix-numbers 1, 2, 3, and it is not worth while to write down the values.

The two values of $(11.1.2)(22.1.2)-(12.1.2)^2$ are

$$\begin{aligned} &-(a_{22}+b_{11}-2h_{12})(ab-h^2)+a(\tfrac{1}{4}b_1^2-\tfrac{1}{2}b_2h_1+\tfrac{1}{4}a_2b_2) \\ &\qquad+b(\tfrac{1}{4}a_2^2-\tfrac{1}{2}a_1h_2+\tfrac{1}{4}a_1b_1) \\ &\qquad+h\{\tfrac{1}{4}(a_1b_2-a_2b_1)+h_1h_2-\tfrac{1}{2}b_1h_1-\tfrac{1}{2}a_2h_2), \end{aligned}$$

and

$$\begin{aligned} &\{G.\tfrac{1}{2}a_1+F(h_1-\tfrac{1}{2}a_2)+C(g_1-\tfrac{1}{2}a_3)\}.\{G(h_2-\tfrac{1}{2}b_1)+F.\tfrac{1}{2}b_2+C(f_2-\tfrac{1}{2}b_3)\} \\ &\quad-\{G.\tfrac{1}{2}a_2+F.\tfrac{1}{2}b_1+C.\tfrac{1}{2}(f_1+g_3-h_3)\}^2, \end{aligned}$$

where, in the first value, for a, b, h, $ab-h^2$ we must write $L^{-2}(BC-F^2)$, $L^{-2}(CA-G^2)$, $L^{-2}(FG-CH)$, and L^2C; making this change, multiplying by 4 and equating, we obtain

$$\begin{aligned} &-2L^2C(a_{22}+b_{11}-2h_{12}) \\ &\quad+(BC-F^2)(b_1^2-2b_2h_1+a_2b_2) \\ &\quad+(CA-G^2)(a_2^2-2a_1h_2+a_1b_1) \\ &\quad+(FG-CH)(a_1b_2-a_2b_1+4h_1h_2-2b_1h_1-2a_2h_3) \\ &\quad-\{Ga_1+F(2h_1-a_2)+C(2g_1-a_3)\}\{G(2h_2-b_1)+Fb_2+C(2f_2-b_3)\} \\ &\quad+\{Ga_2+Fb_1+C(f_1+g_2-h_3)\}^2 \qquad =0. \end{aligned}$$

Developing the fourth and fifth lines, it appears that in this expression the coefficients of F^2, G^2 and FG each of them vanish; the whole equation is thus divisible by C, and omitting this factor throughout, the equation becomes

$$\begin{aligned}
0 = &- 2L^2(a_{22} + b_{11} - 2h_{12}) \\
&+ A(a_2^2 - 2a_1h_2 + a_1b_1) \\
&+ B(b_1^2 - 2b_2h_1 + a_2b_2) \\
&+ C\{-a_3b_3 + 2a_3f_3 + 2b_3g_1 - 4f_2g_1 + (f_1 + g_2 - h_3)^2\} \\
&+ F\{(a_3b_2 - a_2b_3) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) + 2(a_3f_2 + b_1f_1 - 2h_1f_2)\} \\
&+ G\{a_1b_3 - a_3b_1 + 2(a_2f_1 - a_1f_2) + 2(a_3h_2 - a_2h_3) + 2(a_2g_2 + b_1g_1 - 2g_1h_2)\} \\
&+ H\{\qquad\qquad\qquad\qquad 2(a_2h_2 + b_1h_1 - 2h_1h_2)\};
\end{aligned}$$

this is substantially the required equation ($\mathfrak{E}$), but the form of it may be greatly simplified.

Forming the identity,

$$\begin{aligned}
0 = 2L[(a_2 - h_1)L_2 + (b_1 - h_2)L_1] = &- A(a_1b_1 - a_1h_2 - a_2h_1 + a_2^2) \\
&- B(b_1^2 - b_1h_2 - b_2h_1 + a_2b_2) \\
&- C(b_1c_1 + a_2c_2 - c_1h_2 - c_2h_1) \\
&- 2F(b_1f_1 + a_2f_2 - f_1h_2 - f_2h_1) \\
&- 2G(b_1g_1 + a_2g_2 - g_1h_2 - g_2h_1) \\
&- 2H(a_2h_2 + b_1h_1 - 2h_1h_2),
\end{aligned}$$

we add hereto the last preceding equation; the coefficients of A, B, F, G, H thus assume new and simple forms, but the coefficient of C requires a further transformation.

Assume

$$\begin{aligned}
\Omega = &(b_1c_1 - f_1^2) + (c_2a_2 - g_2^2) + (a_3b_3 - h_3^2) \\
&+ (g_2h_3 + g_3h_2 - a_2f_3 - a_3f_2) + (h_3f_1 + h_1f_3 - b_1g_3 - b_3g_1) + (f_1g_2 + f_2g_1 - c_1h_2 - c_2h_1);
\end{aligned}$$

then, if we add to the equation C multiplied by this value and subtract $C\Omega$, the coefficient of C takes its proper form, and the equation is

$$\begin{aligned}
&- 2L^2(a_{22} + b_{11} - 2h_{12}) \\
&+ 2L[(b_1 - h_2)L_1 + (a_2 - h_1)L_2] - C\Omega \\
&+ A\{-(ah_2 - a_2h)\} \\
&+ B\{\ (b_1h_2 - b_2h_1)\} \\
&+ C\{-(a_2f_3 - a_3f_2) + (b_3g_1 - b_1g_3) - (g_2h_3 - g_3h_2) - (h_3f_1 - h_1f_3) + 3(f_1g_2 - f_2g_1)\} \\
&+ F\{-(a_3b_3 - a_3b_2) + 2(b_1g_2 - b_2g_1) + 2(b_3h_1 - b_1h_3) - 2(h_1f_2 - h_2f_1)\} \\
&+ G\{-(a_3b_1 - a_1b_3) - 2(a_1f_2 - a_2f_1) - 2(a_2h_3 - a_3h_2) - 2(g_1h_2 - g_2h_1)\} \\
&+ H\{-(a_1b_2 - a_2b_1)\} \\
&= 0;
\end{aligned}$$

where, as a verification, observe that the letters (a, b), (f, g) may be interchanged if at the same time we interchange the suffixes 1 and 2.

The two values of $(23.1.2)(31.1.2)-(33.1.2)(12.1.2)$ are

$$\begin{aligned}
&-\tfrac{1}{2}(g_{23}+f_{31}-c_{12}-h_{33})(ab-h^2)\\
&+a\,(\tfrac{1}{2}b_1f_3-\tfrac{1}{4}b_3f_1-\tfrac{1}{4}b_1c_2+\tfrac{1}{4}b_3g_2-\tfrac{1}{4}b_3h_3)\\
&+b\,(\tfrac{1}{2}a_2g_3-\tfrac{1}{4}a_3g_2-\tfrac{1}{4}a_2c_1+\tfrac{1}{4}a_3f_1-\tfrac{1}{4}a_3h_3)\\
&+h\,\{\tfrac{1}{4}a_3b_3+\tfrac{1}{4}b_1c_1+\tfrac{1}{4}a_2c_2-\tfrac{1}{2}b_1g_3-\tfrac{1}{2}a_2f_3+\tfrac{1}{4}h_3^2-\tfrac{1}{4}(f_1-g_2)^2\},
\end{aligned}$$

and

$$\begin{aligned}
&\{G\,.\,\tfrac{1}{2}(-f_1+g_2+h_3)+F\,.\,\tfrac{1}{2}b_3+C\,.\,\tfrac{1}{2}c_2\}\,\{G\,.\,\tfrac{1}{2}a_3+F\,.\,\tfrac{1}{2}(f_1-g_2+h_3)+C\,.\,\tfrac{1}{2}c_1\}\\
&-\{G\,(g_3-\tfrac{1}{2}c_1)+F\,(f_3-\tfrac{1}{2}c_2)+C\,.\,\tfrac{1}{2}c_3\}\,\{G\,.\,\tfrac{1}{2}a_2+F\,.\,\tfrac{1}{2}b_1+C\,.\,\tfrac{1}{2}(f_1+g_2-h_3)\},
\end{aligned}$$

and here for a, b, h, $ab-h^2$, substituting their values, multiplying by 4, and equating, we have

$$\begin{aligned}
&-2L^2C\,(g_{23}+f_{31}-c_{12}-h_{33})\\
&+(BC-F^2)\,(2b_1f_3-b_3f_1-b_1c_2+b_3g_2-b_3h_3)\\
&+(CA-G^2)\,(2a_2g_3-a_3g_2-a_2c_1+a_3f_1-a_3h_3)\\
&+(FG-CH)\,(a_3b_3+b_1c_1+a_2c_2-2b_1g_3-2a_2f_3+h_3^2-f_1^2+2f_1g_2-g_2^2)\\
&-\{G\,(-f_1+g_2+h_3)+Fb_3+Cc_2\}\,\{Ga_3+F\,(f_1-g_2+h_3)+Cc_1\}\\
&+\{G\,(2g_3-c_1)+F\,(2f_3-c_2)+Cc_3\}\,\{Ga_2+Fb_1+C\,(f_1+g_2-h_3)\}=0.
\end{aligned}$$

Here again the terms in F^2, FG, G^2 all vanish, hence the whole equation divides by C; throwing this factor out, we have

$$\begin{aligned}
&-2L^2\,(g_{23}+f_{31}-c_{12}-h_{33})\\
&+A\,(\;2a_2g_3-a_3g_2-a_2c_1+a_3f_1-a_3h_3)\\
&+B\,(\;2b_1f_3-b_3f_1-b_1c_2+b_3g_2-b_3h_3)\\
&+C\,(-c_1c_2+c_3f_1+c_3g_2-c_3h_3)\\
&+F\,\{-(b_3c_1-b_1c_3)-2c_2f_1+2f_1f_3+2g_2f_3-2f_2h_3\}\\
&+G\,(\;a_2c_3-a_2c_2-2c_1g_2+2f_1g_3+2g_2g_3-2g_3h_3)\\
&+H\,(-a_3b_3-b_1c_1-a_2c_2+2b_1g_3+2a_2f_3-h_3^2+f_1^2-2f_1g_2+g_2^2)=0,
\end{aligned}$$

which is substantially the required equation ($\mathfrak{H}$); but the form has to be altered. First, multiply by 2, then forming the expression

$$\begin{aligned}
&2L\,[(f_3-c_2)\,L_1+(g_3-c_1)\,L_2+(f_1+g_2-2h_3)\,L_3]\\
&\quad=\quad(f_3-c_2)\,(Aa_1+Bb_1+Cc_1+2Ff_1+2Gg_1+2Hh_1)\\
&\qquad+(g_3-c_1)\,(Aa_2+Bb_2+Cc_2+2Ff_2+2Gg_2+2Hh_2)\\
&\qquad+(f_1+g_2-2h_3)\,(Aa_3+Bb_3+Cc_3+2Ff_3+2Gg_3+2Hh_3),
\end{aligned}$$

this is

$$
\begin{aligned}
= \quad & A\,(-\,a_1c_2 - a_3c_1 + a_1f_3 + a_3f_1 + a_2g_3 + a_3g_2 - 2a_3h_3) \\
+ \quad & B\,(-\,b_1c_2 - b_2c_1 + b_1f_3 + b_3f_1 + b_2g_3 + b_3g_2 - 2b_3h_3) \\
+ \quad & C\,(-2c_1c_2 \qquad + c_1f_3 + c_3f_1 + c_2g_3 + c_3g_2 - 2c_3h_3) \\
+ \quad & 2F\,(-\,c_1f_2 - c_2f_1 + 2f_1f_3 + \qquad f_2g_3 + f_3g_2 - 2f_3h_3) \\
+ \quad & 2G\,(-\,c_1g_2 - c_2g_1 + f_1g_3 + f_3g_1 + 2g_2g_3 - 2g_3h_3) \\
+ \quad & 2H\,(-\,c_1h_2 - c_2h_1 + g_2h_3 + g_3h_2 + h_3f_1 + h_1f_3 - 2h_3^2);
\end{aligned}
$$

then adding this expression

$$(f_3 - c_3)\,L_1 + (g_3 - c_1)\,L_2 + (f_1 + g_2 - 2h_3)\,L_3$$

and subtracting its foregoing value, we have an equation with new coefficients of A, B, C, F, G, H, all of which, except that of H, are in the proper form, and for the coefficient of H, recurring to the foregoing value of Ω, we must add to the equation $2H$ multiplied by the value of Ω and subtract $2H\Omega$. The final result is

$$
\begin{aligned}
& -4L^2\,(g_{23} + f_{31} - c_{12} - h_{33}) \\
& + 2L\,[(f_3 - c_3)\,L_1 + (g_3 - c_1)\,L_2 + (f_1 + g_2 - 2h_3)\,L_3] - 2H\Omega \\
& \qquad + A\,\{\ 3\,(a_2g_3 - a_3g_2) - (c_1a_2 - c_2a_1) + (a_3f_1 - a_1f_3\ )\} \\
& \qquad + B\,\{-3\,(b_3f_1 - b_1f_3) - (b_1c_2 - b_2c_1) - (b_2g_3 - b_3g_2)\} \\
& \qquad + C\,\{\ \ (c_3f_1 - c_1f_3\ ) - (c_2g_3 - c_3g_2) \qquad\qquad\ \} \\
& \qquad + 2F\,\{-\ \ (b_3c_1 - b_1c_3) + (c_1f_2 - c_2f_1\ ) - (f_2g_3 - f_3g_2\ )\} \\
& \qquad + 2G\,\{-\ \ (c_2a_3 - c_3a_2) - (c_1g_2 - c_2g_1) - (f_2g_1 - f_1g_3)\} \\
& \qquad + 2H\,\{-\ \ (b_3g_1 - b_1g_3) + (a_2f_3 - a_3f_2) \qquad\qquad\ \} = 0,
\end{aligned}
$$

which, it will be observed, remains unaltered by the interchange of a and b, f and g A and B, F and G, the suffix numbers 1 and 2 being at the same time interchanged.

We have thus the required six equations, in which

$$L^2 = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

$$\Omega = b_1c_1 - f_1^2 + c_2a_2 - g_3^2 + a_3b_3 - h_3^2$$

$$+ (g_3h_3 + g_3h_2 - a_2f_3 - a_3f_2) + (h_3f_1 + h_1f_3 - b_3g_1 - b_1g_3) + (f_1g_2 + f_2g_1 - a_1f_2 - a_2f_1),$$

and where I write also, for shortness, ab12 for $a_1b_2 - a_2b_1$, &c.; viz. the equations are

$$
\begin{aligned}
0 = & -2L^2\,(b_{33} + c_{22} - 2f_{23}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2\mathfrak{l}). \\
& + 2L\,[\ \ .\ \ \ (c_2 - f_3)\,L_2 + (b_3 - f_2)\,L_3] \qquad\qquad\qquad\qquad - A\Omega \\
& + A\,\{\ .\ \ - \ \ bg31 + \ \ ch12 + 3\,.\,gh23 - hf31 - \ \ fg12\} \\
& + B\,\{-\,bf23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \} \\
& + C\,\{+\,cf23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \} \\
& + F\,\{-\,bc23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \} \\
& + G\,\{-\,bc31 + 2\,.\,cf12 \qquad . \qquad + 2\,.\,ch23 \qquad . \qquad - 2\,.\,fg23\} \\
& + H\,\{-\,bc12 - 2\,.\,bf31 - 2\,.\,bg23 \qquad . \qquad . \qquad - 2\,.\,hf23\}
\end{aligned}
$$

$$
\begin{aligned}
0 = &- 2L^2(c_{11} + a_{33} - 2g_{31}) \\
&+ 2L\,[(c_1 - g_3)\,L_1 \quad . \quad + (a_3 - g_1)\,L_3] \quad - B\Omega \\
&+ A\,\{\quad . \quad ag31 \quad\quad\quad\

$$
\begin{aligned}
0 = &- 4L^2 (\mathrm{f}_{31} + \mathrm{g}_{23} - \mathrm{c}_{12} - \mathrm{h}_{33}) \qquad (\mathfrak{H}).\\
&+ 2L\,[(\mathrm{f}_3 - \mathrm{c}_2)\,L_1 + (\mathrm{g}_3 - \mathrm{c}_1)\,L_2 + (\mathrm{f}_1 + \mathrm{g}_2 - 2\mathrm{h}_3)\,L_3] \;\; - 2H\Omega\\
&+ \;\; A\,\{- \mathrm{ca12} + \;\; \mathrm{af31} + 3\,.\,\mathrm{ag23} \qquad\}\\
&+ \;\; B\,\{- \mathrm{bc12} - 3\,.\,\mathrm{bf31} - \;\; \mathrm{bg23} \qquad\}\\
&+ \;\; C\,\{ \;\; . \qquad \mathrm{cf31} - \;\; \mathrm{cg23} \qquad\}\\
&+ \;\; F\,\{- \mathrm{bc31} + \;\; \mathrm{cf12} \qquad . \;\; - \mathrm{fg23}\}\\
&+ \;\; G\,\{- \mathrm{ca23} \qquad . \;\; - \;\; \mathrm{cg12} - \mathrm{fg31}\}\\
&+ \;\; H\,\{ \;\; . \;\; + \;\; \mathrm{af23} - \;\; \mathrm{bg31} \;\; . \;\;\}
\end{aligned}
$$

It would be possible in these equations to introduce the symbols $AB12$, &c., in place of ab12, &c., and then writing $A = B = C = 1$, all these symbols other than those where the letters are GH, HF or FG would vanish, and we should obtain Mr Warren's six equations for normal coordinates. But in the general case it would seem that there is not any advantage in the introduction of the new symbols $AB12$, &c., and I retain by preference the equations in the form in which I have given them.

To the foregoing may be joined a symmetrical equation obtained (as by Mr Warren) by multiplying the several equations by a, b, c, f, g, h respectively, and adding; the result is in the first instance obtained in the form

$$-2L^2\Theta + 2L\Psi - 3L^2\Omega + \square = 0 \qquad (\mathfrak{M}),$$

where

$$
\begin{aligned}
\Theta = \;\; & \mathrm{a}\,(\mathrm{b}_{33} + \mathrm{c}_{22} - 2\mathrm{f}_{23})\\
&+ \;\mathrm{b}\,(\mathrm{c}_{11} + \mathrm{a}_{33} - 2\mathrm{g}_{31})\\
&+ \;\mathrm{c}\,(\mathrm{a}_{22} + \mathrm{b}_{11} - 2\mathrm{h}_{12})\\
&+ 2\mathrm{f}\,(\mathrm{g}_{12} + \mathrm{h}_{31} - \mathrm{a}_{23} - \mathrm{f}_{11})\\
&+ 2\mathrm{g}\,(\mathrm{h}_{23} + \mathrm{f}_{12} - \mathrm{b}_{31} - \mathrm{g}_{22})\\
&+ 2\mathrm{h}\,(\mathrm{f}_{31} + \mathrm{g}_{23} - \mathrm{c}_{12} - \mathrm{h}_{33}).
\end{aligned}
$$

For Ψ, collecting the terms which contain L_1, L_2, L_3 respectively, and attending to the values of (A, B, C, F, G, H),

$$= (\mathrm{bc} - \mathrm{f}^2,\ \mathrm{ca} - \mathrm{g}^2,\ \mathrm{ab} - \mathrm{h}^2,\ \mathrm{gh} - \mathrm{af},\ \mathrm{hf} - \mathrm{bg},\ \mathrm{fg} - \mathrm{ch}),$$

this is easily reduced to the form

$$\Psi = (A_1 + H_2 + G_3)\,L_1 + (H_1 + B_2 + F_3)\,L_2 + (G_1 + F_2 + C_3)\,L_3.$$

The term in Ω is

$$
\begin{aligned}
&= -(A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h})\,\Omega,\\
&= -3L^2\Omega, \text{ as above.}
\end{aligned}
$$

For the calculation of $\square$, collecting the terms, we have $\square =$

23	A	B	C	F	G	H	31	A	B	C	F	G	H	12	A	B	C	F	G	H
bc				− a			bc			− g	− 2h	− a		bc		− h		− 2g		− a
ca			− f	− b	− 2h		ca					− b		ca	− h				− 2f	− b
ab		− f		− c		− 2g	ab	− g				− c	− 2f	ab						− c
af		b	− c		− 2g	2h	af	h					2b	af	− g				− 2c	
bf		− a					bf		− 3h		− 2g		− 2a	bf		− g				
cf			a				cf			h				cf			3g	2h	2a	
ag	3h				2f	2b	ag	b						ag	f					
bg		− h				− 2a	bg	− a		+ c	2f		− 2h	bg		f		2c		
cg			− h				cg			− b				cg			− 3f	− 2b	− 2h	
ah	− 3g				− 2c	− 2f	ah	− f						ah	− c					
bh		g					bh		3f		2c		2g	bh		c				
ch			g		2a		ch			− f	− 2b			ch	a	− b		− 2f	+ 2g	
gh	− 3a	− b	− c				gh					− 2f	− 2b	gh					− 2c	− 2f
hf				− 2g		− 2a	hf	− a	3b	− c				hf				− 2c		− 2g
fg				− 2h	− 2a		fg				− 2b	− 2h		fg	− a	− b	+ 3c			

viz. this is

$$\square = \mathrm{bc}23 \,.\, -\mathrm{a}F + \mathrm{bc}31\,(-\mathrm{g}C - 2\mathrm{h}F - \mathrm{a}G) + \mathrm{bc}12\,(-\mathrm{h}B - 2\mathrm{g}F - \mathrm{a}H) + \&\mathrm{c}.$$

Some however of the coefficients require reduction; for instance, that of bc31 is $= -(\mathrm{a}, \mathrm{h}, \mathrm{g} \between G, F, C) - \mathrm{h}F, = -\mathrm{h}F$; and so $-\mathrm{h}B - 2\mathrm{g}F - \mathrm{a}H$ is $= -(\mathrm{a}, \mathrm{h}, \mathrm{g} \between H, B, F) - \mathrm{g}F$, $= -\mathrm{g}F$. After these reductions, the value is found to be $\square =$

$\mathrm{bc}23 \,.\, -\mathrm{a}F$	$+\ \mathrm{bc}31 \,.\, -\mathrm{h}F$	$+\ \mathrm{bc}12 \,.\, -\mathrm{g}F$
$+\ \mathrm{ca}23 \,.\, -\mathrm{h}G$	$+\ \mathrm{ca}31 \,.\, -\mathrm{b}G$	$+\ \mathrm{ca}12 \,.\, -\mathrm{f}G$
$+\ \mathrm{ab}23 \,.\, -\mathrm{g}H$	$+\ \mathrm{ab}31 \,.\, -\mathrm{f}H$	$+\ \mathrm{ab}12 \,.\, -\mathrm{c}H$
$+\ \mathrm{af}23\,(-\mathrm{g}G + \mathrm{h}H)$	$+\ \mathrm{af}31\,(\mathrm{b}H - \mathrm{f}G)$	$+\ \mathrm{af}12\,(\mathrm{f}H - \mathrm{c}G)$
$+\ \mathrm{bf}23 \,.\, -\mathrm{a}B$	$+\ \mathrm{bf}31 \,.\, -\mathrm{h}B$	$+\ \mathrm{bf}12 \,.\, -\mathrm{g}B$
$+\ \mathrm{cf}23 \,.\, \mathrm{a}C$	$+\ \mathrm{cf}31 \,.\, \mathrm{h}C$	$+\ \mathrm{cf}12 \,.\, \mathrm{g}C$
$+\ \mathrm{ag}23 \,.\, \mathrm{h}A$	$+\ \mathrm{ag}31 \,.\, \mathrm{b}A$	$+\ \mathrm{ag}12 \,.\, \mathrm{f}A$
$+\ \mathrm{bg}23\,(\mathrm{g}F - \mathrm{a}H)$	$+\ \mathrm{bg}31\,(-\mathrm{h}H + \mathrm{f}F)$	$+\ \mathrm{bg}12\,(\mathrm{c}F - \mathrm{g}H)$
$+\ \mathrm{cg}23 \,.\, -\mathrm{h}C$	$+\ \mathrm{cg}31 \,.\, -\mathrm{b}C$	$+\ \mathrm{cg}12 \,.\, -\mathrm{f}C$
$+\ \mathrm{ah}23 \,.\, -\mathrm{g}A$	$+\ \mathrm{ah}31 \,.\, -\mathrm{f}A$	$+\ \mathrm{ah}12 \,.\, -\mathrm{c}A$
$+\ \mathrm{bh}23 \,.\, \mathrm{g}B$	$+\ \mathrm{bh}31 \,.\, \mathrm{f}B$	$+\ \mathrm{bh}12 \,.\, \mathrm{c}B$
$+\ \mathrm{ch}23\,(\mathrm{a}G - \mathrm{h}F)$	$+\ \mathrm{ch}31\,(\mathrm{h}G - \mathrm{b}F)$	$+\ \mathrm{ch}12\,(-\mathrm{f}F + \mathrm{g}G)$
$+\ \mathrm{gh}23\,(3\mathrm{a}A - \mathrm{b}B - \mathrm{c}C)$	$+\ \mathrm{gh}31 \,.\, 2\mathrm{h}A$	$+\ \mathrm{gh}12 \,.\, 2\mathrm{g}A$
$+\ \mathrm{hf}23 \,.\, 2\mathrm{h}B$	$+\ \mathrm{hf}31\,(-\mathrm{a}A + 3\mathrm{b}B - \mathrm{c}C)$	$+\ \mathrm{hf}12 \,.\, 2\mathrm{f}B$
$+\ \mathrm{fg}23 \,.\, 2\mathrm{g}C$	$+\ \mathrm{fg}31 \,.\, 2\mathrm{f}C$	$+\ \mathrm{fg}12\,(-\mathrm{a}A - \mathrm{b}B + 3\mathrm{c}C)$;

or, arranging the terms in a different order, this may be written

$$\square =$$

$$\begin{array}{ll}
A\,\{ & \mathrm{a}\,(3\,.\,\mathrm{gh23}-\mathrm{hf31}-\mathrm{fg12})+\mathrm{b}\,.\,\mathrm{ag31}-\mathrm{c}\,.\,\mathrm{ah12} \qquad -\mathrm{c}\,.\,\mathrm{ah12} \\
 & \qquad +\mathrm{f}(\mathrm{ag12}-\mathrm{ah31}) \quad +\mathrm{g}\,(2\,.\,\mathrm{gh12}-\mathrm{ah23})+\mathrm{h}\,(2\,.\,\mathrm{gh31}+\mathrm{ag23})\} \\
+B\,\{ & -\mathrm{a}\,.\,\mathrm{bf23} \qquad +\mathrm{b}\,(-\mathrm{gh23}+3\,.\,\mathrm{hf31}-\mathrm{fg12})+\mathrm{c}\,.\,\mathrm{bh12} \\
 & \qquad +\mathrm{f}\,(2\,.\,\mathrm{hf12}+\mathrm{bh31})+\mathrm{g}\,(\mathrm{bh23}-\mathrm{bf12}) \quad +\mathrm{h}\,(2\,.\,\mathrm{hf23}-\mathrm{bf31})\} \\
+C\,\{ & \mathrm{a}\,.\,\mathrm{cf23} \qquad -\mathrm{b}\,.\,\mathrm{cg31} \qquad +\mathrm{c}\,(-\mathrm{gh23}-\mathrm{hf31}+3\,.\,\mathrm{fg12}) \\
 & \qquad +\mathrm{f}\,(2\,.\,\mathrm{fg31}-\mathrm{cg12}) \quad +\mathrm{g}\,(2\,.\,\mathrm{fg23}+\mathrm{cf12}) \quad +\mathrm{h}\,(\mathrm{cf31}-\mathrm{cg23})\} \\
+F\,\{ & -\mathrm{a}\,.\,\mathrm{bc23} \qquad -\mathrm{b}\,.\,\mathrm{ch31} \qquad +\mathrm{c}\,.\,\mathrm{bg12} \\
 & \qquad +\mathrm{f}\,(\mathrm{bg31}-\mathrm{ch12}) \quad +\mathrm{g}\,(\mathrm{bg23}-\mathrm{bc12}) \quad -\mathrm{h}\,(\mathrm{ch23}+\mathrm{bc31})\} \\
+G\,\{ & +\mathrm{a}\,.\,\mathrm{ch23} \qquad -\mathrm{b}\,.\,\mathrm{ca31} \qquad -\mathrm{c}\,.\,\mathrm{af12} \\
 & \qquad -\mathrm{f}\,(\mathrm{af31}+\mathrm{ca12}) \quad +\mathrm{g}\,(\mathrm{ch12}-\mathrm{af23}) \quad +\mathrm{h}\,(\mathrm{ch31}-\mathrm{ca23})\} \\
+H\,\{ & -\mathrm{a}\,.\,\mathrm{bg23} \qquad +\mathrm{b}\,.\,\mathrm{af31} \qquad -\mathrm{c}\,.\,\mathrm{ab12} \\
 & \qquad +\mathrm{f}\,(\mathrm{af12}-\mathrm{ab31}) \quad -\mathrm{g}\,(\mathrm{bg12}+\mathrm{ab23}) \quad +\mathrm{h}\,(\mathrm{af23}-\mathrm{bg31})\}.
\end{array}$$

Attending to the values of A, B, C, F, G, H, we have

$$\begin{aligned}
A_{11}+B_{22}+C_{33}+2F_{23}+2G_{31}+2H_{12} & \\
= \quad & \mathrm{bc}_{11}+\mathrm{cb}_{11}-2\mathrm{ff}_{11} \qquad +2\,(\mathrm{b}_1\mathrm{c}_1-\mathrm{f}_1^2) \\
+ \quad & \mathrm{ca}_{22}+\mathrm{ac}_{22}-2\mathrm{gg}_{22} \qquad +2\,(\mathrm{c}_2\mathrm{a}_2-\mathrm{g}_2^2) \\
+ \quad & \mathrm{ab}_{33}+\mathrm{ba}_{33}-2\mathrm{hh}_{33} \qquad +2\,(\mathrm{a}_3\mathrm{b}_3-\mathrm{h}_3^2) \\
& +2\,\{\mathrm{gh}_{23}+\mathrm{hg}_{23}-\mathrm{af}_{23}-\mathrm{fa}_{23}+\quad \mathrm{g}_2\mathrm{h}_3+\mathrm{g}_3\mathrm{h}_2-\mathrm{a}_2\mathrm{f}_3-\mathrm{a}_3\mathrm{f}_2\} \\
& +2\,\{\mathrm{hf}_{31}+\mathrm{fh}_{31}-\mathrm{bg}_{31}-\mathrm{gb}_{31}+\quad \mathrm{h}_3\mathrm{f}_1+\mathrm{h}_1\mathrm{f}_3-\mathrm{b}_3\mathrm{g}_1-\mathrm{b}_1\mathrm{g}_3\} \\
& +2\,\{\mathrm{fg}_{12}+\mathrm{gf}_{12}-\mathrm{ch}_{12}-\mathrm{hc}_{12}+\quad \mathrm{f}_1\mathrm{g}_2+\mathrm{f}_2\mathrm{g}_1-\mathrm{c}_1\mathrm{h}_2-\mathrm{c}_2\mathrm{h}_1\},
\end{aligned}$$

which is, in fact, $=2\Theta+2\Omega$.

Hence the foregoing equation ($\mathfrak{M}$) may also be written

$$-L^2\,(A_{11}+B_{22}+C_{33}+2F_{23}+2G_{31}+2H_{12})+2L\Psi-L^2\Omega+\square=0,$$

where Ψ, Ω, $\square$ have their before-mentioned values.

In the particular case where $\mathrm{f}=0$, $\mathrm{g}=0$, $\mathrm{h}=0$, we have

$$A,\ B,\ C,\ F,\ G,\ H=\mathrm{bc},\ \mathrm{ca},\ \mathrm{ab},\ 0,\ 0,\ 0;\ L^2=\mathrm{abc};$$

$$\Omega=\mathrm{b}_1\mathrm{c}_1+\mathrm{c}_2\mathrm{a}_2+\mathrm{a}_3\mathrm{b}_3;$$

the equation ($\mathfrak{L}$) becomes

$$-2\mathrm{abc}\,(\mathrm{b}_{33}+\mathrm{c}_{22})+[\mathrm{c}_2\,(\mathrm{a}_2\mathrm{bc}+\mathrm{b}_2\mathrm{ca}+\mathrm{c}_2\mathrm{ab})+\mathrm{b}_3\,(\mathrm{a}_3\mathrm{bc}+\mathrm{b}_3\mathrm{ca}+\mathrm{c}_3\mathrm{ab})]-\mathrm{bc}\,(\mathrm{b}_1\mathrm{c}_1+\mathrm{c}_2\mathrm{a}_2+\mathrm{a}_3\mathrm{b}_3)=0,$$

that is,

$$-2\mathrm{abc}\,(\mathrm{b}_{33}+\mathrm{c}_{22})-\mathrm{bcb}_1\mathrm{c}_1+\mathrm{ca}\,(\mathrm{b}_2\mathrm{c}_2+\mathrm{b}_2^2)+\mathrm{ab}\,(\mathrm{b}_3\mathrm{c}_3+\mathrm{c}_2^2)=0,$$

and the equation ($\mathfrak{F}$) becomes

$$4\mathrm{abc}\,.\,\mathrm{a}_{23} - [\mathrm{a}_3(\mathrm{a}_2\mathrm{bc} + \mathrm{b}_2\mathrm{ca} + \mathrm{c}_2\mathrm{ab}) + \mathrm{a}_2(\mathrm{a}_3\mathrm{bc} + \mathrm{b}_3\mathrm{ca} + \mathrm{c}_3\mathrm{ab})]$$
$$- \mathrm{ca}(\mathrm{a}_2\mathrm{b}_3 - \mathrm{a}_3\mathrm{b}_2) - \mathrm{ab}(\mathrm{c}_2\mathrm{a}_3 - \mathrm{c}_3\mathrm{a}_2) = 0,$$

that is,

$$4\mathrm{abc}\,.\,\mathrm{a}_{23} - 2\mathrm{bc}\,.\,\mathrm{a}_2\mathrm{a}_3 - 2\mathrm{ca}\,.\,\mathrm{a}_2\mathrm{b}_3 - 2\mathrm{ab}\,.\,\mathrm{a}_3\mathrm{b}_2 = 0.$$

Dividing by $-2\mathrm{abc}$ and $4\mathrm{abc}$ respectively, and completing the system, the six equations are

$$\mathrm{b}_{33} + \mathrm{c}_{22} - \frac{\mathrm{b}_3^2}{2\mathrm{b}} - \frac{\mathrm{c}_2^2}{2\mathrm{c}} + \frac{\mathrm{b}_1\mathrm{c}_1}{2\mathrm{a}} - \frac{\mathrm{b}_2\mathrm{c}_2}{2\mathrm{b}} - \frac{\mathrm{b}_3\mathrm{c}_3}{2\mathrm{c}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{A}),$$

$$\mathrm{c}_{11} + \mathrm{a}_{33} - \frac{\mathrm{c}_1^2}{2\mathrm{c}} - \frac{\mathrm{a}_3^2}{2\mathrm{a}} - \frac{\mathrm{c}_1\mathrm{a}_1}{2\mathrm{a}} + \frac{\mathrm{c}_2\mathrm{a}_2}{2\mathrm{b}} - \frac{\mathrm{c}_3\mathrm{a}_3}{2\mathrm{c}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{B}),$$

$$\mathrm{a}_{22} + \mathrm{b}_{11} - \frac{\mathrm{a}_2^2}{2\mathrm{a}} - \frac{\mathrm{b}_1^2}{2\mathrm{b}} - \frac{\mathrm{a}_1\mathrm{b}_1}{2\mathrm{a}} - \frac{\mathrm{a}_2\mathrm{b}_2}{2\mathrm{b}} + \frac{\mathrm{a}_3\mathrm{b}_3}{2\mathrm{c}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{C}),$$

$$\mathrm{a}_{23} \quad - \frac{\mathrm{a}_2\mathrm{a}_3}{2\mathrm{a}} - \frac{\mathrm{a}_2\mathrm{b}_3}{2\mathrm{b}} - \frac{\mathrm{a}_3\mathrm{c}_2}{2\mathrm{c}} \qquad = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{F}),$$

$$\mathrm{b}_{31} \quad - \frac{\mathrm{b}_1\mathrm{a}_3}{2\mathrm{a}} - \frac{\mathrm{b}_3\mathrm{b}_1}{2\mathrm{b}} - \frac{\mathrm{b}_3\mathrm{c}_1}{2\mathrm{c}} \qquad = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{G}),$$

$$\mathrm{c}_{12} \quad - \frac{\mathrm{c}_1\mathrm{a}_2}{2\mathrm{a}} - \frac{\mathrm{c}_2\mathrm{b}_1}{2\mathrm{b}} - \frac{\mathrm{c}_1\mathrm{c}_2}{2\mathrm{c}} \qquad = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{H}).$$

These are in fact Lamé's equations, *Leçons sur les coordonnées curvilignes, etc.*, Paris (1859), pp. 76, 78, viz. the first of the equations (8), p. 76, is

$$\frac{d^2H}{d\rho_1 d\rho_2} = \frac{1}{H_1}\frac{dH}{d\rho_1}\frac{dH_1}{d\rho_2} + \frac{1}{H_2}\frac{dH}{d\rho_2}\frac{dH_2}{d\rho_1},$$

which, in the notation of the present paper, is

$$(\sqrt{\mathrm{a}})_{23} = \frac{1}{\sqrt{\mathrm{b}}}(\sqrt{\mathrm{a}})_2(\sqrt{\mathrm{b}})_3 + \frac{1}{\sqrt{\mathrm{c}}}(\sqrt{\mathrm{a}})_3(\sqrt{\mathrm{c}})_2.$$

Here

$$(\sqrt{\mathrm{a}})_2 = \tfrac{1}{2}\,\frac{\mathrm{a}_2}{\sqrt{\mathrm{a}}},$$

$$(\sqrt{\mathrm{a}})_{23} = \tfrac{1}{2}\left(\frac{\mathrm{a}_2}{\sqrt{\mathrm{a}}}\right)_3 = \tfrac{1}{2}\left\{\frac{\mathrm{a}_{23}}{\sqrt{\mathrm{a}}} - \frac{\frac{1}{2}\mathrm{a}_2\mathrm{a}_3}{\mathrm{a}\sqrt{\mathrm{a}}}\right\};$$

and the equation therefore is

$$\frac{\mathrm{a}_{23}}{\sqrt{\mathrm{a}}} - \frac{\frac{1}{2}\mathrm{a}_2\mathrm{a}_3}{\mathrm{a}\sqrt{\mathrm{a}}} = \tfrac{1}{2}\frac{1}{\sqrt{\mathrm{b}}}\frac{\mathrm{a}_2}{\sqrt{\mathrm{a}}}\frac{\mathrm{b}_3}{\sqrt{\mathrm{b}}} + \tfrac{1}{2}\frac{1}{\sqrt{\mathrm{c}}}\frac{\mathrm{a}_3}{\sqrt{\mathrm{a}}}\frac{\mathrm{c}_2}{\sqrt{\mathrm{c}}},$$

which, multiplying by $\sqrt{\mathrm{a}}$, gives the foregoing equation

$$\mathrm{a}_{23} - \frac{\mathrm{a}_2\mathrm{a}_3}{\mathrm{a}} - \frac{\mathrm{a}_2\mathrm{b}_3}{\mathrm{b}} - \frac{\mathrm{a}_3\mathrm{c}_2}{\mathrm{c}} = 0.$$

The first of the equations (9) p. 78 is

$$\frac{d}{d\rho_1}\left(\frac{1}{H_1}\frac{dH}{d\rho_1}\right)+\frac{d}{d\rho}\left(\frac{1}{H}\frac{dH_1}{d\rho}\right)+\frac{1}{H_2^2}\frac{dH}{d\rho_2}\frac{dH_1}{d\rho_2}=0,$$

which, in the notation of the present paper, is

$$\left\{\frac{(\sqrt{a})_2}{\sqrt{b}}\right\}_2+\left\{\frac{(\sqrt{b})_1}{\sqrt{a}}\right\}_1+\frac{1}{c}(\sqrt{a})_3(\sqrt{b})_3=0.$$

This gives first

$$\tfrac{1}{2}\left(\frac{a_2}{\sqrt{a}\sqrt{b}}\right)_2+\tfrac{1}{2}\left(\frac{b_1}{\sqrt{a}\sqrt{b}}\right)_1+\tfrac{1}{4}\frac{a_3b_3}{c\sqrt{a}\sqrt{b}}=0,$$

and then

$$\begin{aligned}&\tfrac{1}{2}\frac{a_{22}}{\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_2^2}{a\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_2b_2}{b\sqrt{a}\sqrt{b}}\\&+\tfrac{1}{2}\frac{b_{11}}{\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{b_1^2}{b\sqrt{a}\sqrt{b}}-\tfrac{1}{4}\frac{a_1b_1}{a\sqrt{a}\sqrt{b}}\end{aligned}+\tfrac{1}{4}\frac{a_3b_3}{c\sqrt{a}\sqrt{b}}=0,$$

which, on multiplying by $\sqrt{a}\sqrt{b}$ and reducing, gives the foregoing equation

$$a_{22}+b_{11}-\frac{a_2^2}{a}-\frac{b_1^2}{b}-\frac{a_1b_1}{a}-\frac{a_2b_2}{b}+\frac{a_3b_3}{c}=0.$$

800.

NOTE ON THE STANDARD SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 38—40.]

To fix the ideas, the equations are assumed to be without constant terms. Supposing the system to be insufficient for the determination of the ratios of the unknown quantities, then regarding the unknown quantities as having a definite order of arrangement, there are certain solutions which may be regarded as standard solutions. Take the unknown quantities to be A, B, C, D, E, F, G, &c.; then assuming $A=0$, or else A, B, each $=0$, or else A, B, C, each $=0$, as many equations as possible, the system as thus modified will have a definite solution; for instance, the assumed equations A, B, C, D, E, each $=0$, may give a definite solution in which F is not $=0$, and it may then for convenience be put $=1$. We have thus a solution beginning with $F=1$. This being so, there will be a solution or solutions with $F=0$; we cannot then have A, B, C, D, E, each $=0$, but we again assume $A=0$, or else A, B, each $=0$, as many equations as possible; suppose A, B, C, each $=0$ give a definite solution, with D not $=0$; and then taking it for convenience to be $=1$, we have a solution beginning $D=1$, and for which $F=0$. Going on in this manner we obtain, it may be, a solution beginning $B=1$, and for which $D=0$, $F=0$; and so on, the process stopping, if not sooner, with a solution beginning $A=1$, and with the initial letters of the preceding solutions, each $=0$. We have in this manner the system of standard solutions, of a form such as

	A	B	C	D	E	F	G ...
$=$	0	0	0	0	0	1	*
$=$	0	0	0	1	*	0	*
$=$	0	1	0	0	*	0	*
$=$	1	0	*	0	*	0	*

where the $*$ denotes a value which is not in general $=0$, but which may in any particular case happen to be so.

For instance, let it be required for the binary quartic $(a, b, c, d, e \between x, y)^4$, to find the asyzygetic seminvariants of the degree 4 and weight 6. Assuming for the seminvariant the value in the left-hand column of the diagram, the unknown coefficients being A, B, C, D, E, F, G, this must be reduced to zero by the operation

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e;$$

and we thus obtain as many equations as there are terms of the degree 4 and weight 5, as appearing by the second column

		$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e$	
A a^2ce	a^2be	$2C + 2A$	$=0,$
B „ d^2	„ cd	$D \quad + 6B + 4A$	$=0,$
C ab^2e	ab^2d	$3F + 2D + 4C$	$=0,$
D „ bcd	„ bc^2	$2G + 6E + 3D$	$=0,$
E „ c^3	a^0b^3c	$4G + 3F$	$=0;$
F a^0b^2d			
G „ b^3c^2			

viz. the equations are

A	B	C	D	E	F	G	
2		+ 2					$=0,$
4 + 6			+ 1				$=0,$
		4 + 2			+ 3		$=0,$
			3 + 6			+ 2	$=0,$
					3 + 4		$=0.$

We have first a solution beginning $B=1$, and secondly a solution beginning $A=1$, with $B=0$: the resulting two seminvariants, say P and Q, are

		$P=$	$Q=$	I	II
A	a^2ce	0	1	1	1
B	„ d^2	1	0	− 1	0
C	ab^2e	0	− 1	− 1	− 1
D	„ bcd	− 6	− 4	+ 2	− 4
E	„ c^3	+ 4	3	− 1	+ 3
F	a^0b^2d	+ 4	4		+ 4
G	„ b^3c^2	− 3	− 3		+ 3

.

As is known, there is no irreducible solution, but only the composite forms

$$\begin{aligned} \mathrm{I} &= a\,(ace - ad^2 - b^2e + 2bcd - c^3), \\ \mathrm{II} &= (ac - b^2)\,(ae - 4bd + 3c^2), \end{aligned}$$

the developed values of which are given above: II (as a form beginning $A = 1$ and with $B = 0$) can be nothing else than, and is in fact $= Q$: and so I (as a form beginning with $A = 1$, $B = -1$) can be nothing else than, and is in fact $= Q - P$; that is, we have

$$\begin{aligned} P &= -\mathrm{I} + \mathrm{II}, \quad \text{or} \quad \mathrm{I} = -P + Q, \\ Q &= \qquad \mathrm{II}, \qquad\qquad \mathrm{II} = \qquad Q; \end{aligned}$$

and so, in general, we have a standard set of values for the asyzygetic seminvariants of a given degree and weight; or, what is the same thing, for the covariants of a given deg-order.

801.

ON SEMINVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 131—138.]

THE present paper is a somewhat fragmentary one, but it contains some results which seem to me to be worth putting on record.

I consider here not any binary quantic in particular, but the whole series $(a, b, c \between x, y)^2$, $(a, b, c, d \between x, y)^3$, &c.; or in a somewhat different point of view, I consider the indefinite series of coefficients $(a, b, c, d, e, \ldots)$; here, instead of covariants and invariants, we have only seminvariants; viz. a seminvariant is a function reduced to zero by the operator

$$\Delta = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots;$$

for instance, seminvariants are

$$a, \quad ac - b^2, \quad a^2d - 3abc + 2b^3, \quad a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$
$$ae - 4bd + 3c^2, \quad ace - ad^2 - b^2e + 2bcd - c^3, \text{ \&c.}$$

A seminvariant is of a certain degree θ in the coefficients, and of a certain weight w (viz. the coefficients $a, b, c, d, \ldots$ are reckoned as being of the weights $0, 1, 2, 3, \ldots$ respectively); it is, moreover, of a certain rank ρ; viz. according as the highest letter therein is $a, c, d, e, \ldots$ (it is never b), the rank is taken to be $0, 2, 3, 4, \ldots$, and we have $w =$ or $< \frac{1}{2}\rho\theta$. The seminvariant may be regarded as belonging to a quantic $(a, \ldots \between x, y)^n$, the order of which, n, is equal to or greater than ρ; viz. in regard to such quantic the seminvariant, say A, is the leading coefficient of a covariant

$$(A, B, \ldots, K \between x, y)^\mu,$$

where the weights of the successive coefficients are $w, w+1, \ldots$ up to $n\theta - w$; hence number of terms less unity, that is, μ, is $= n\theta - 2w$; the least value of μ is thus $= \rho\theta - 2w$, which is either zero, or positive; in the former case, $w = \frac{1}{2}\rho\theta$, the seminvariant is an invariant of the quantic $(a, \ldots \between x, y)^\rho$, the order of which is equal to the rank of the seminvariant; but if $w < \frac{1}{2}\rho\theta$, then it is the leading coefficient of a covariant $(A, B, \ldots, K \between x, y)^{\rho\theta - 2w}$ of the same quantic $(a, \ldots \between x, y)^\rho$; and in every case, taking $n > \rho$, the seminvariant is the leading coefficient A of a covariant

$$(A, B, \ldots, K \between x, y)^{n\theta - 2w}$$

of a quantic $(a, \ldots \between x, y)^n$.

Take A as belonging to the quantic $(a, \ldots \between x, y)^n$; corresponding to such quantic, we have an operator Λ of the same rank n, viz.

$$\begin{aligned} \Lambda &= 2b\partial_a + c\partial_b && \text{for } n = 2, \\ &= 3b\partial_a + 2c\partial_b + d\partial_c && \text{,, } 3, \\ &= 4b\partial_a + 3c\partial_b + 2d\partial_c + e\partial_d && \text{,, } 4, \\ &\vdots && \vdots \end{aligned}$$

Operating with Λ on A, we have a series of terms

$$A,\ \Lambda A,\ \Lambda^2 A,\ \ldots,\ \Lambda^{n\theta-2w} A,$$

but the next term $\Lambda^{n\theta-2w+1}A$, and of course every succeeding term, is $=0$, and this being so, the coefficients of the covariant $(A, B, \ldots, K \between x, y)^{n\theta-2w}$ are

$$(1,\ \tfrac{1}{1}\Lambda,\ \tfrac{1}{1\,.\,2}\Lambda^2, \ldots)\, A,$$

or what is the same thing, each coefficient is obtained from the next preceding one by the formulæ

$$B = \tfrac{1}{1}\Lambda A, \quad C = \tfrac{1}{2}\Lambda B, \quad D = \tfrac{1}{3}\Lambda C, \ldots.$$

The coefficients A and K, B and J, ... are derived one from the other by reversal of the order of the coefficients of $(a, b, \ldots \between x, y)^n$, with or without a change of sign, and thus it is only necessary to calculate up to the middle coefficient, or pair of coefficients; and we obtain, moreover, a verification.

Calculating in this manner the covariant

$$(A, B, \ldots, K)^{\rho\theta-2w},$$

which belongs to the quantic $(a, \ldots \between x, y)^\rho$, if we herein change $a, b, c, \ldots$ into $ax+by$, $bx+cy$, $cx+dy, \ldots$ we obtain the covariant belonging to the quantic $(a, \ldots \between x, y)^{\rho+1}$; and in this covariant making the like change, or what is the same thing, in the first-mentioned covariant changing $a, b, c, \ldots$ into $(a, b, c \between x, y)^2$, $(b, c, d \between x, y)^2$, $(c, d, e \between x, y)^2, \ldots$ we have the covariant belonging to $(a, \ldots \between x, y)^{\rho+2}$; and in like manner we obtain the covariant belonging to the quantic $(a, \ldots \between x, y)^n$ of any given order n.

In particular, if $w = \frac{1}{2}\rho\theta$, that is, if the given seminvariant be an invariant of $(a, \ldots \between x, y)^\rho$, then we obtain the series of covariants directly from A by therein changing $a, b, c, \ldots$ into $ax+by$, $bx+cy$, $cx+dy, \ldots$ and in the result making the like change; or what is the same thing, in A changing $a, b, c, \ldots$ into $(a, b, c \between x, y)^2$, $(b, c, d \between x, y)^2$, $(c, d, e \between x, y)^2, \ldots$: and so on until we obtain the covariant for the quantic $(a, \ldots \between x, y)^n$ of the given order n.

A seminvariant which cannot be expressed as a rational and integral function of lower seminvariants is said to be irreducible. The theory is distinct from that of the irreducible covariants of a quantic of a given order; for instance, as regards the cubic $(a, b, c, d \between x, y)^3$, we have the irreducible covariant (invariant)

$$a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$

but this is not an irreducible seminvariant; it is

$$\begin{aligned} &= (ac - b^2)(ae - 4bd + 3c^2) \\ &\quad - a\,.\,(ace - ad^2 - b^2e - c^3 + 2bcd), \end{aligned}$$

or, what is the same thing, there is not for the quartic $(a, b, c, d, e \between x, y)^4$, or for the higher quantics, any *irreducible* covariant having this for the leading coefficient.

We may consider the question to determine the number of asyzygetic seminvariants of a given degree and weight. For instance, taking the weights up to 12, so that the series of letters extends as far as m, then for the degrees 1, 2, 3 we have as follows:

$W=$	0	1	2	3	4	5	6	7	8	9	10	11	12
Deg. 1	a	b	c	d	e	f	g	h	i	j	k	l	m
Nos.	1	1	1	1	1	1	1	1	1	1	1	1	1
Diff.	0	0	0	0	0	0	0	0	0	0	0	0	0
Deg. 2	a^2	ab	ac	ad	ae	af	ag	ah	ai	aj	ak	al	am
			b^2	bc	bd	be	bf	bg	bh	bi	bj	bk	bl
					c^2	cd	ce	cf	cg	ch	ci	cj	ck
							d^2	de	df	dg	dh	di	dj
									e^2	ef	eg	eh	ei
											f^2	fg	fh
													g^2
Nos.	1	1	2	2	3	3	4	4	5	5	6	6	7
Diff.	1	0	1	0	1	0	1	0	1	0	1	0	1
Deg. 3	a^3	a^2b	a^2c	a^2d	a^2e	a^2f	a^2g	a^2h	a^2i	a^2j	a^2k	a^2l	a^2m
			ab^2	abc	abd	abe	abf	abg	abh	abi	abj	abk	abl
					ac^2	acd	ace	acf	acg	ach	aci	acj	ack
							ad^2	ade	adf	adg	adh	adi	adj
									ae^2	aef	aeg	aeh	aei
											af^2	afg	afh
													ag^2
				b^3	b^2c	b^2d	b^2e	b^2f	b^2g	b^2h	b^2i	b^2j	b^2k
						bc^2	bcd	bce	bcf	bcg	bch	bci	bcj
								bd^2	bde	bdf	bdg	bdh	bdi
										be^2	bef	beg	beh
												bf^2	bfg
							c^3	c^2d	c^2e	c^2f	c^2g	c^2h	c^2i
									cd^2	cde	cdf	cdg	cdh
											ce^2	cef	ceg
													cf^2
										d^3	d^2e	d^2f	d^2g
												de^2	def
													e^3
Nos.	1	1	2	3	4	5	7	8	10	12	14	16	19
Diff.	1	0	1	1	1	2	1	2	2	2	2	2	3

For the degree 1, the line of differences shows that the only seminvariant is $(W=0)$, the seminvariant a.

For the degree 2, the line of differences 1, 0, 1, 0, ..., shows that the number of seminvariants is $=1$ for each even degree, $=0$ for each odd degree; thus for the weight 0 there is a seminvariant $=a^2$, which of course is not irreducible; while for each of the other even weights we have a single irreducible seminvariant; as is well known, the forms are

$W=$	2	4	6	8	10	12
	$ac+1$	$ae+1$	$ag+1$	$ai+1$	$ak+1$	$am+1$
	b^2-1	$bd-4$	$bf-6$	$bh-8$	$bj-10$	$bl-12$
		c^2+3	$ce+15$	$cg+28$	$ci+45$	$ck+66$
			d^2-10	$df-56$	$dh-120$	$dj-220$
				e^2+35	$eg+210$	$ei+495$
					f^2-126	$fh-792$
						g^2+462

For degree 3, the line of differences shows that for

W	$=0$	1	2	3	4	5	6	7	8	9	10	11	12
Nos. are	$=1$	0	1	1	1	1	2	1	2	2	2	2	3;

but inasmuch as for each even weight there is a quadric seminvariant, which multiplied by a gives a cubic seminvariant, to obtain the number of irreducible cubic seminvariants we subtract

1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	0	1	0	1	1	1	1	2	1	2	2,

or the numbers of irreducible cubic seminvariants are as in the line last written down.

There is a convenience however in giving, for each even weight, as well the rejected reducible covariant; and the entire series of results is found to be

$W=$ 0	2	3	4	5	6		7	8	
a^3+1	a^2c+1	a^2d+1	a^2e+1	a^2f+1	a^2g+1		a^2h+1	a^2i+1	
	ab^2-1	$abc-3$	$abd-4$	$abe-5$	$abf-6$		$abg-7$	$abh-8$	
		b^3+2	ac^2+3	$acd+2$	$ace+15$	$+1$	$acf+9$	$acg+28$	$+1$
				b^2d+8	ad^2-10	-1	$ade-5$	$adf-56$	-3
				bc^2-6	b^2e	-1	b^2f+12	ae^2+35	$+2$
					bcd	$+2$	$bce-30$	b^2g	-1
					c^3	-1	bd^2+20	bcf	$+3$
							c^2d	bde	-1
								c^2e	-3
								cd^2	$+2$

9		10		11		12			
a^2j+1		a^2k+1		a^2l+1		a^2m	$+1$		
$abi-9$		$abj-10$		$abk-11$		abl	-12		
ach	$+2$	$aci+45$	$+1$	$acj+35$	$+2$	ack	$+66$	$+3$	
$adg+42$	-7	$adh-120$	-4	$adi-75$	-9	adj	-220	-15	
$aef-36$	$+5$	$aeg+210$	$+8$	$aeh+90$	$+14$	aei	$+495$	$+40$	$+1$
b^2h+36	-2	af^2-126	-5	$afg-42$	-7	afh	-792	-70	-4
$bcg-126$	$+7$	b^2i	-1	b^2j+20	-2	ag^2	$+462$	$+42$	$+3$
$bdf-108$	$+22$	bch	$+4$	$bci-90$	$+9$	b^2k		-3	
be^2+180	-25	bdg	-4	$bdh+240$	$+16$	bcj		$+15$	
c^2f+270	-27	bef	$+2$	$beg-420$	-63	bdi		-25	-4
$cde-450$	$+45$	c^2g	-4	bf^2+252	$+42$	beh		$+30$	$+12$
d^3+200	-20	cdf	$+8$	c^2h	-30	bfg		-14	-8
		ce^2	-5	cdg	$+70$	c^2i		-15	$+3$
		d^2e		cef	-21	cdh		$+40$	-8
				d^2f	-56	ceg		-70	-22
				de^2	$+35$	cf^2		$+42$	$+24$
						d^2g			$+24$
						def			-36
						e^3			$+15$

The canonical form given for the quintic in my Tenth Memoir on Quantics [693] belongs to a series, viz. writing now the small roman letters (instead of the italic letters) for the series of coefficients, and using the italic letters $a, c, d, e, f, \ldots$ to denote seminvariants, they are as follows:

$$\begin{array}{ll}
0 & a = \mathrm{a}, \\
2 & c = \mathrm{ac} - \mathrm{b}^2, \\
3 & d = \mathrm{a}^2\mathrm{d} - 3\mathrm{abc} + 2\mathrm{b}^3 \ (= f \text{ in the tenth memoir}), \\
4 & e = \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2 \ (= b \text{ in the tenth memoir}), \\
5 & f = \mathrm{a}^2\mathrm{f} - 5\mathrm{abe} + 2\mathrm{acd} + 8\mathrm{b}^2\mathrm{d} - 6\mathrm{bc}^2, \\
6 & g = \mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2, \\
7 & h = \mathrm{a}^2\mathrm{h} - 7\mathrm{abg} + 9\mathrm{acf} - 5\mathrm{ade} + 12\mathrm{b}^2\mathrm{f} - 30\mathrm{bce} + 20\mathrm{bd}^2, \\
8 & i = \mathrm{ai} - 8\mathrm{bh} + 28\mathrm{cg} - 56\mathrm{df} + 35\mathrm{e}^2, \\
 & \&\mathrm{c.}
\end{array}$$

Writing also (instead of d in the tenth memoir)

$$\epsilon = \mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} + 2\mathrm{bcd} - \mathrm{c}^3,$$

so that the equation $a^3d - a^2bc + 4c^3 - f^2 = 0$ of the tenth memoir, is in the present notation $a^3\epsilon - a^2ec + 4c^3 - d^2 = 0$, then the series of canonical forms is

$$\begin{array}{ll}
\text{Quadric} & (1,\ 0,\ c \between x,\ y)^2, \\
\text{Cubic} & (1,\ 0,\ c,\ d \between x,\ y)^3, \\
\text{Quartic} & (1,\ 0,\ c,\ d,\ a^2e - 3c^2 \between x,\ y)^4, \\
\text{Quintic} & (1,\ 0,\ c,\ d,\ a^2e - 3c^2,\ a^2f - 2cd \between x,\ y)^5, \\
 & \&\mathrm{c.}
\end{array}$$

the series of coefficients being

$$\begin{array}{lllllll}
1,\ 0,\ c,\ d, & a^2e + 1, & a^2f + 1, & a^4g + 1, & a^4h + 1, & a^6i + 1, \\
 & c^2 - 3 & cd - 2 & a^2ce - 15 & a^2cf - 9 & a^4cg - 28 \\
 & & & c^3 + 45 & a^2de + 5 & a^4e^2 - 35 \\
 & & & d^2 + 10 & c^2d + 3 & a^2c^2e + 630 \\
 & & & & & a^2df + 56 \\
 & & & & & c^4 - 1575 \\
 & & & & & cd^2 - 392
\end{array}$$

these values being, in fact, the expressions in terms of the seminvariants $a, c, d, \&\mathrm{c.}$ of

$$\begin{array}{lllllll}
1,\ 0, & \mathrm{ac}, & \mathrm{a}^2\mathrm{d}, & \mathrm{a}^3\mathrm{e}, & \mathrm{a}^4\mathrm{f}, & \mathrm{a}^5\mathrm{g}, & \mathrm{a}^6\mathrm{h}, & \mathrm{a}^7\mathrm{i} \\
 & -\mathrm{b}^2, & -3\mathrm{abc}, & -4\mathrm{a}^2\mathrm{bd}, & -5\mathrm{a}^3\mathrm{be}, & -6\mathrm{a}^4\mathrm{bf}, & -7\mathrm{a}^5\mathrm{bg}, & -8\mathrm{a}^6\mathrm{bh} \\
 & & +2\,\mathrm{b}^3, & +6\mathrm{ab}^2\mathrm{c}, & +10\mathrm{a}^2\mathrm{b}^2\mathrm{d}, & +15\mathrm{a}^3\mathrm{b}^2\mathrm{e}, & +21\mathrm{a}^4\mathrm{b}^2\mathrm{f}, & +28\mathrm{a}^5\mathrm{b}^2\mathrm{g} \\
 & & & -3\,\mathrm{b}^4, & -10\mathrm{ab}^3\mathrm{c}, & -20\mathrm{a}^3\mathrm{b}^3\mathrm{d}, & -35\mathrm{a}^3\mathrm{b}^3\mathrm{e}, & -56\mathrm{a}^4\mathrm{b}^3\mathrm{f} \\
 & & & & +4\,\mathrm{b}^5, & +15\mathrm{ab}^4\mathrm{c}, & +35\mathrm{a}^2\mathrm{b}^4\mathrm{d}, & +70\mathrm{a}^3\mathrm{b}^4\mathrm{e} \\
 & & & & & -5\,\mathrm{b}^6, & -21\mathrm{ab}^5\mathrm{c}, & -56\mathrm{a}^2\mathrm{b}^5\mathrm{d} \\
 & & & & & & +6\,\mathrm{b}^7, & +28\mathrm{ab}^6\mathrm{c} \\
 & & & & & & & -7\,\mathrm{b}^8
\end{array}$$

I annex verifications of the foregoing values:

	$a^2e =$	$-3c^2$	
a^3e	1		+1
a^2bd	−4		−4
a^2c^2	+3	−3	0
ab^2c		+6	+6
b^4		−3	−3

	$a^2f =$	$-2cd$	
a^4f	+1		+1
a^3be	−5		−5
a^3cd	+2	−2	0
a^2b^2d	+8	+2	+10
a^2bc^2	−6	+6	0
ab^3c		−10	−10
b^5		+4	+4

	$a^4g =$	$-15a^2ce$	$+45c^3$	$+10d^2$	
a^5g	+1				1
a^4bf	−6				−6
a^4ce	+15	−15			0
a^4d^2	−10			+10	0
a^3b^2e		+15			+15
a^3bcd		+60		−60	0
a^3c^3		−45	+45		0
a^2b^3d		−60		+40	−20
$a^2b^2c^2$		+45	−135	+90	0
ab^4c			+135	−120	+15
b^6			−45	+40	−5

	$a^4h =$	$-9a^3cf$	$+5a^3de$	$+3c^2d$	
a^6h	+1				+1
a^5bg	−7				−7
a^5cf	+9	−9			0
a^5de	−5		+5		0
a^4b^2f	+12	+9			+21
a^4bce	−30	+45	−15		0
a^4bd^2	+20		−20		0
a^4c^2d		−18	+15	+3	0
a^3b^3e		−45	+10		−35
a^3b^2cd		−54	+60	−6	0
a^3bc^3		+54	−45	−9	0
a^2b^4d		+72	−40	+3	+35
$a^2b^3c^2$		−54	+30	+24	0
ab^5c				−21	−21
b^7				+6	+6

	$a^6i =$	$-28a^4cg$	$-35a^4e^2$	$+56a^2df$	$+630a^2c^2e^2$	$-1575c^4$	$-392c^3d$	
a^7i	+ 1							+ 1
a^6bh	− 8							− 8
cg	+ 28	− 28						0
df	− 56				+ 56			0
e^2	+ 35		− 35					0
a^5b^2g		+ 28						+ 28
bcf		+ 168			− 168			0
bde			+ 280		− 280			0
c^2e		− 420	− 210	+ 630				0
cd^2		+ 280			+ 112		− 392	0
a^4b^3f		− 168			+ 112			− 56
b^2ce		+ 420		− 1260	+ 840			0
b^3d^2		− 280	− 560		+ 448		+ 392	0
bc^2d			+ 840	− 2520	− 672		+ 2352	0
c^4			− 315	+ 1890		− 1575		0
a^3b^4e				+ 630	− 560			+ 70
b^3cd				+ 5040	− 1120		− 3920	0
b^2c^3				− 3780	+ 1008	+ 6300	− 3528	0
a^2b^5d				− 2520	+ 896		+ 1568	− 56
b^4c^2				+ 1890	− 672	− 9450	+ 8232	0
$a\,b^6d$						+ 6300	− 6272	+ 28
a^0b^8						− 1575	+ 1568	− 7

It would be interesting to obtain the general law for the expressions of the canonical coefficients in terms of the seminvariants.

802.

NOTE ON CAPTAIN MACMAHON'S PAPER, "ON THE DIFFERENTIAL EQUATION $X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0$."

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 182—184.]

IN general, if $f, = (x, y, 1)^3, = 0$ be the equation of a cubic curve, and if

$$d\omega = \frac{dx}{\frac{df}{dy}}, \; = \frac{-dy}{\frac{df}{dx}},$$

then if 1, 2, 3 are the intersections of the curve by an arbitrary right line, the coordinates of these points being (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, we have, by Abel's theorem,

$$d\omega_1 + d\omega_2 + d\omega_3 = 0,$$

viz. this is the differential relation corresponding to the integral relation which expresses that the three points are the intersections of the cubic curve by a right line, or say to the integral equation

$$\nabla, \; = \begin{vmatrix} y_1, & y_2, & y_3 \\ x_1, & x_2, & x_3 \\ 1\,, & 1\,, & 1 \end{vmatrix}, \; = 0,$$

in which equation y_1, y_2, y_3 are regarded as functions of x_1, x_2, x_3 respectively, given by means of the equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ which express that the points are on the cubic curve. See my "Memoir on the Abelian and Theta Functions," [819].

In particular, if the equation of the curve is

$$f, \; = \tfrac{1}{3}\{y^3 - (A + 3Bx + 3Cx^2 + Dx^3)\}, \; = \tfrac{1}{3}(y^3 - X), \; = 0,$$

then

$$d\omega = \frac{dx}{y^2}, \quad = \frac{dx}{X^{\frac{2}{3}}}:$$

and corresponding to the differential relation

$$X_1^{-\frac{2}{3}}\,dx_1 + X_2^{-\frac{2}{3}}\,dx_2 + X_3^{-\frac{2}{3}}\,dx_3 = 0,$$

we have the integral relation

$$\nabla, \quad = \begin{vmatrix} X_1^{\frac{1}{3}}, & X_2^{\frac{1}{3}}, & X_3^{\frac{1}{3}} \\ x_1\ , & x_2\ , & x_3 \\ 1\ , & 1\ , & 1 \end{vmatrix} = 0,$$

viz. this last equation, as containing no arbitrary constant, is a particular integral of the differential equation.

If instead of x_1, x_2, x_3 we write x, y, z, then we have

$$\nabla = \begin{vmatrix} X^{\frac{1}{3}}, & Y^{\frac{1}{3}}, & Z^{\frac{1}{3}} \\ x\ , & y\ , & z \\ 1\ , & 1\ , & 1 \end{vmatrix}, \quad = 0,$$

as a particular integral of the differential equation

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0.$$

To rationalize the integral equation, write

$$\alpha,\ \beta,\ \gamma = y - z,\ z - x,\ x - y \ \text{(so that } \alpha + \beta + \gamma = 0\text{)},$$

the equation is

$$\alpha X^{\frac{1}{3}} + \beta Y^{\frac{1}{3}} + \gamma Z^{\frac{1}{3}} = 0\,;$$

and we thence have

$$\alpha^3 X + \beta^3 Y + \gamma^3 Z = 3\alpha\beta\gamma X^{\frac{1}{3}} Y^{\frac{1}{3}} Z^{\frac{1}{3}}.$$

The left-hand side is

$$\begin{aligned} &\alpha^3\,(A + 3Bx + 3Cx^2 + Dx^3) \\ &+ \beta^3\,(A + 3By + 3Cy^2 + Dy^3) \\ &+ \gamma^3\,(A + 3Bz + 3Cz^2 + Dz^3)\,; \end{aligned}$$

or assuming

$$\alpha',\ \beta',\ \gamma' = x\,(y - z),\ y\,(z - x),\ z\,(x - y) \ \text{(so that } \alpha' + \beta' + \gamma' = 0\text{)},$$

this is

$$= A\,(\alpha^3 + \beta^3 + \gamma^3) + 3B\,(x^2\alpha' + \beta^2\beta' + \gamma^2\gamma') + 3C\,(\alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2) + D\,(\alpha'^3 + \beta'^3 + \gamma'^3).$$

But taking λ arbitrary, and

$$a,\ b,\ c = \alpha + \lambda\alpha',\quad \beta + \lambda\beta',\quad \gamma + \lambda\gamma',$$

then

$$a + b + c = 0,$$

whence

$$a^3 + b^3 + c^3 = 3abc\,;$$

or substituting for a, b, c their values, and comparing the coefficients of the several powers of λ,

$$\begin{aligned}
\alpha^3 + \beta^3 + \gamma^3 &= 3\alpha\beta\gamma, \\
\alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma' &= \alpha'\beta\gamma + \beta'\gamma\alpha + \gamma'\alpha\beta = \alpha\beta\gamma(x + y + z), \\
\alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2 &= \alpha\beta'\gamma' + \beta\gamma'\alpha' + \gamma\alpha'\beta' = \alpha\beta\gamma(yz + zx + xy), \\
\alpha'^3 + \beta'^3 + \gamma'^3 &= 3\alpha'\beta'\gamma' \qquad\qquad = 3\alpha\beta\gamma \,.\, xyz.
\end{aligned}$$

Hence we have

$$\alpha^3 X + \beta^3 Y + \gamma^3 Z = 3\alpha\beta\gamma\,\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\},$$

or the integral equation is

$$\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\} = X^{\frac{1}{3}} Y^{\frac{1}{3}} Z^{\frac{1}{3}},$$

that is,

$$\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\}^3 = XYZ,$$

the elegant result given by Capt. MacMahon at the beginning of his paper.

The author in a letter to me, dated Jan. 13, 1883, remarks that the particular integral of the equation in question

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0,$$

is expressible as a determinant in a rational form as follows. Writing it $XYZ = P^3$, where

$$P = A + B(x+y+z) + C(yz+zx+xy) + Dxyz,$$

then the form is

$$\nabla, = \begin{vmatrix} 1, & \left(\frac{1}{3}P\dfrac{dX}{dx} - X\dfrac{dP}{dx}\right)^3, & X \\ 1, & \left(\frac{1}{3}P\dfrac{dY}{dy} - Y\dfrac{dP}{dy}\right)^3, & Y \\ 1, & \left(\frac{1}{3}P\dfrac{dZ}{dz} - Z\dfrac{dP}{dz}\right)^3, & Z \end{vmatrix} = 0,$$

for, as shown by Captain MacMahon in his paper, each of the three terms such as

$$Z\left(\frac{1}{3}P\frac{dY}{dy} - Y\frac{dP}{dy}\right)^3 - Y\left(\frac{1}{3}P\frac{dZ}{dz} - Z\frac{dP}{dz}\right)^3,$$

which compose the determinant, is divisible by $XYZ - P^3$.

It may be added that we have identically

$$\frac{1}{3}P\frac{dX}{dx} - X\frac{dP}{dx} = (AC - B^2)(2x - y - z) + (AD - BC)(x^2 - yz) + (BD - C^2)(x^2y + x^2z - 2xyz),$$

and of course like values for the other two expressions in the determinant.

803.

ON MR ANGLIN'S FORMULA FOR THE SUCCESSIVE POWERS OF THE ROOT OF AN ALGEBRAICAL EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 223, 224.]

SUPPOSE $x^m - px^{m-1} + qx^{m-2} - \ldots = 0$, then the successive powers x^m, x^{m+1}, x^{m+2}, &c. of x can be expressed in the form $Px^{m-1} - Qx^{m-2} + Rx^{m-2} -$ &c. Mr Anglin has obtained for this purpose a very elegant formula, with a demonstration which (it occurred to me) might be presented under a somewhat simplified form; and he has permitted me to draw up the present Note.

Take, for greater convenience, the equation to be

$$x^4 - px^3 + qx^2 - rx + s = 0,$$

and let $h_1, h_2, h_3, \ldots$ be the sums of the homogeneous products of the roots, of the orders 1, 2, 3, &c. respectively; then, writing also $h_0 = 1$, we have

$$\begin{aligned}
h_1 &= h_0 p, \\
h_2 &= h_1 p - h_0 q, \\
h_3 &= h_2 p - h_1 q + h_0 r, \\
h_4 &= h_3 p - h_2 q + h_1 r - h_0 s, \\
h_5 &= h_4 p - h_3 q + h_2 r - h_1 s, \\
&\vdots
\end{aligned}$$

And this being so, starting from the equation

$$x^4 = px^3 - \quad qx^2 + \quad rx - s,$$

that is,

$$= h_1 x^3 - h_0 q x^2 + h_0 r x - h_0 s,$$

we obtain successively

$$\begin{aligned}
x^5 = \quad & h_1(px^3 - qx^2 + rx - s) \\
& - h_0 qx^3 + h_0 rx^2 - h_0 sx \\
= \quad & h_2 x^3 - (h_1 q - h_0 r)\, x^2 + (h_1 r - h_0 s)\, x - h_1 s, \\
x^6 = \quad & h_2(px^3 - qx^2 + rx - s) \\
& - (h_1 q - h_0 r)\, x^3 + (h_1 r - h_0 s)\, x^2 - h_1 sx \\
= \quad & h_3 x^3 - (h_2 q - h_1 r + h_0 s)\, x^2 + (h_2 r - h_1 s)\, x - h_2 s, \\
x^7 = \quad & h_3(px^3 - qx^2 + rx - s) \\
& - (h_2 q - h_1 r + h_0 s)\, x^3 + (h_2 r - h_1 s)\, x^2 - h_2 sx \\
= \quad & h_4 x^3 - (h_3 q - h_2 r + h_1 s)\, x^2 + (h_3 r - h_2 s)\, x - h_3 s,
\end{aligned}$$

and so on, the characteristic feature being that by the introduction of the symbols h, the coefficient of x^3 presents itself at each step as a monomial, and the coefficients of the lower powers require no reduction. It is obvious that the process is a perfectly general one, and that for the equation

$$x^m - p_1 x^{m-1} + p_2 x^{m-2} - \ldots + (-)^m p_m = 0,$$

the formula is

$$\begin{aligned}
x^{m+\theta} = \quad & h_{\theta+1}\, x^{m-1} \\
- \quad & (h_\theta p_2 - h_{\theta-1} p_3 + \ldots\ )\, x^{m-2} \\
+ \quad & (h_\theta p_2 - h_{\theta-1} p_4 + \ldots\ )\, x^{m-3} \\
& \vdots \\
+ (-)^{s-1} \quad & (h_\theta p_s - h_{\theta-1} p_{s+1} + \ldots)\, x^{m-s} \\
& \vdots \\
+ (-)^{m-1} . \quad & h_\theta p_m \qquad\qquad x^0,
\end{aligned}$$

where, as regards each power of x, the series forming the coefficient thereof is continued as far as possible, that is, up to the term which contains p_m or h_0 as the case may be.

804.

ON THE ELLIPTIC-FUNCTION SOLUTION OF THE EQUATION $x^3+y^3-1=0$.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. IV. (1883), pp. 106—109.]

I HAD occasion to find elliptic-function expressions for the coordinates (x, y) of a point on the cubic curve $x^3+y^3=1$. These are derivable from the formulæ given, Legendre, *Fonctions Elliptiques*, t. I. pp. 185, 186, for the reduction to elliptic integrals of the integral $R=\int\frac{dr}{(1-z^3)^{\frac{2}{3}}}$. Legendre, writing

$$z=\frac{\sqrt{4y^3-1}-\sqrt{3}}{\sqrt{4y^3-1}+\sqrt{3}},$$

and then

$$m^3=2 \text{ and } m^2y=1+x^2,$$

finds first

$$R=m\sqrt{3}\int\frac{dx}{\sqrt{x^4+3x^2+3}};$$

and then writing $r=\sqrt[4]{3}$, $x=\tan\frac{1}{2}\phi$, and $c^2=\frac{1}{4}(2-r^2)$, finds

$$R=\tfrac{1}{2}mr\int\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}};$$

we have therefore only to write $\sin\phi=\operatorname{sn}u$, to modulus

$$c, =\tfrac{1}{2}\sqrt{2-\sqrt{3}},$$

and we thence obtain an expression for z in terms of the elliptic functions $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$.

Writing x instead of z, and k for c, then

$$m=\sqrt[3]{2},\quad r=\sqrt[4]{3};\quad k=\tfrac{1}{2}\sqrt{2-r^2},\quad k'=\tfrac{1}{2}\sqrt{2+r^2}.$$

Working out the substitutions, the resulting formulæ are

$$x = \frac{2r \operatorname{sn} u \operatorname{dn} u - (1 + \operatorname{cn} u)^2}{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2},$$

$$y = \frac{m(1 + \operatorname{cn} u)\{1 + r^2 + (1 + r^2) \operatorname{cn} u\}}{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2},$$

where the modulus is k as above; and these values give

$$x^3 + y^3 = 1,$$

$$\frac{dx}{(1 - x^3)^{\frac{2}{3}}}, \quad = \frac{-dy}{(1 - y^3)^{\frac{2}{3}}} = \tfrac{1}{2} mr\, du.$$

The verification is interesting enough; starting from the expression for x, and for shortness representing it by

$$x = \frac{A - B}{A + B},$$

we have

$$1 - x^3 = \frac{2B(3A^2 + B^2)}{(A + B)^3}, \quad = \frac{m^3(1 + \operatorname{cn} u)^3 (3A^2 + B^2)}{\{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2\}^3}.$$

We find

$$3A^2 + B^2 = 12r^2 \operatorname{cn}^2 u \operatorname{dn}^2 u + (1 + \operatorname{cn} u)^4,$$

$$= (1 + \operatorname{cn} u)\{12r^2(1 - \operatorname{cn} u)(k'^2 + k^2 \operatorname{cn}^2 u) + (1 + \operatorname{cn} u)^3\},$$

where the term in { } is a perfect cube

$$= [1 + \operatorname{cn} u + r^2(1 - \operatorname{cn} u)]^3.$$

The last-mentioned expression is, in fact,

$$= (1 + \operatorname{cn} u)^3 + r^2(1 - \operatorname{cn} u)[3(1 + \operatorname{cn} u)^2 + 3r^2(1 + \operatorname{cn} u)(1 - \operatorname{cn} u) + r^4(1 - \operatorname{cn} u)^2],$$

where the second term is

$$= 12r^2(1 - \operatorname{cn} u)[\tfrac{1}{2}(1 + \operatorname{cn}^2 u) + \tfrac{1}{4} r^2(1 - \operatorname{cn}^2 u)],$$

that is, it is

$$= 12r^2(1 - \operatorname{cn} u)(k'^2 + k^2 \operatorname{cn}^2 u)).$$

We have consequently

$$1 - x^3 = \frac{m^3(1 + \operatorname{cn} u)^3 \{1 + r^2 + (1 - r^2) \operatorname{cn} u\}^3}{\{2r \operatorname{sn} u \operatorname{dn} u + (1 + \operatorname{cn} u)^2\}^3},$$

or extracting the cube root y, $= \sqrt{1 - x^3}$, has its foregoing value: and the differential expressions are then verified.

Suppose $y = 1$, we have

$$(m - 1)(1 + \operatorname{cn} u)^2 + mr^2(1 - \operatorname{cn}^2 u) = 2r \operatorname{sn} u \operatorname{dn} u,$$

that is,

$$(m-1)^2(1+\operatorname{cn} u)^3+2m(m-1)r^2(1+\operatorname{cn} u)^2(1-\operatorname{cn} u)$$
$$+3m^2(1+\operatorname{cn} u)(1-\operatorname{cn} u)^2=r^2(1-\operatorname{cn} u)\{4-4k^2(1-\operatorname{cn}^2 u)\},$$

or observing that the right-hand side is

$$=r^2(1-\operatorname{cn} u)\{(1+\operatorname{cn} u)^2+(1-\operatorname{cn} u)^2+r^2(1+\operatorname{cn} u)(1-\operatorname{cn} u)\},$$

and multiplying by $\frac{1}{3}r^3$, the equation becomes

$$0=\tfrac{1}{3}(m-1)^2r^2(1+\operatorname{cn} u)^3+(2m^2-2m+1)(1+\operatorname{cn} u)^2(1-\operatorname{cn} u)$$
$$+(m^2-1)r^2(1+\operatorname{cn} u)(1-\operatorname{cn} u)^2-(1-\operatorname{cn} u)^3;$$

viz. this is

$$0=\{\tfrac{1}{3}r^2(m^2-1)(1+\operatorname{cn} u)-(1-\operatorname{cn} u)\}^3,$$

as is immediately verified: hence writing $\frac{1}{3}r^2=\frac{1}{r^2}$, we have for the value in question, $y=1$,

$$(m^2-1)(1+\operatorname{cn} u)-r^2(1-\operatorname{cn} u)=0,$$

or say

$$m^2(1+\operatorname{cn} u)=(1+\operatorname{cn} u)+r^2(1-\operatorname{cn} u),$$

that is,

$$\operatorname{cn} u=\frac{r^2+1-m^2}{r^2-1+m^2},$$

which is one of the values of $\operatorname{cn} u$ derived from the equation $x=0$; but this equation $x=0$ gives, not the foregoing equation, but

$$m^6(1+\operatorname{cn} u)^3=\{(1+\operatorname{cn} u)+r^2(1-\operatorname{cn} u)\}^3,$$

viz. the three values of $\operatorname{cn} u$ are the foregoing value and the two values obtained therefrom by changing m into ωm and $\omega^2 m$ respectively, ω being an imaginary cube root of unity. In fact, the curve $x^3+y^3=1$ has at the point $x=0$, $y=1$ an inflexion, the tangent being $y=1$, so that this line meets the curve in the point counting three times; but the line $x=0$ meets the curve in the point, and besides in two imaginary points.

805.

NOTE ON ABEL'S THEOREM.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. IV. (1883), pp. 119—122.]

CONSIDERING Abel's theorem in so far as it relates to the first kind of integrals, and as a differential instead of an integral theorem, the theorem may be stated as follows:

We have a fixed curve $f(x, y, 1) = 0$ of the order m; this implies a relation $f'(x)\,dx + f'(y)\,dy = 0$, between the differentials dx, dy of the coordinates of a point on the curve; and we may therefore write

$$d\omega = \frac{dx}{f'(y)} = -\frac{dy}{f'(x)},$$

and, instead of dx or dy, use $d\omega$ to denote the displacement of a point (x, y) on the curve.

Taking for greater simplicity the fixed curve to be a curve without nodes or cusps, and therefore of the deficiency $\frac{1}{2}(m-1)(m-2)$, we consider its mn intersections by a variable curve $\phi(x, y, 1) = 0$ of the order n. And then, if $(x, y, 1)^{m-3}$ denote an arbitrary rational and integral function of (x, y) of the order $m-3$, the theorem is that we have between the displacements $d\omega_1, d\omega_2, \ldots, d\omega_{mn}$ of the mn points of intersection, the relation

$$\Sigma\,(x, y, 1)^{m-3}\,d\omega = 0,$$

where the left-hand side is the sum of the values of $(x, y, 1)^{m-3}\,d\omega$, belonging to the mn points of intersection respectively.

For the proof, observe that, varying in any manner the curve ϕ, we obtain

$$\frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy + \delta\phi = 0,$$

where $\delta\phi$ is that part which depends on the variation of the coefficients, of the whole variation of ϕ; viz. if $\phi = ax^n + bx^{n-1}y + \ldots$, then $\delta\phi = x^n da + x^{n-1}y\,db + \ldots$; $\delta\phi$ is thus, in regard to the coordinates (x, y), a rational and integral function of the order n. Writing in this equation

$$dx,\ dy = \frac{df}{dy}\,d\omega,\quad -\frac{df}{dx}\,d\omega,$$

the equation becomes

$$\left(\frac{d\phi}{dx}\frac{df}{dy} - \frac{d\phi}{dy}\frac{df}{dx}\right)d\omega + \delta\phi = 0,$$

or say

$$-J(f,\ \phi)\,d\omega + \delta\phi = 0,$$

that is,

$$d\omega = \frac{\delta\phi}{J(f,\ \phi)};$$

and then multiplying each side by the arbitrary function $(x, y, 1)^{m-3}$, we have

$$\Sigma(x,\ y,\ 1)^{m-3}\,d\omega = \Sigma\frac{(x,\ y,\ 1)^{m-3}}{J(f,\ \phi)}\,\delta\phi,$$

where $\delta\phi$ being of the order n in the variables, the numerator is a rational and integral function of (x, y) of the order $m+n-3$: hence by a theorem contained in Jacobi's paper "Theoremata nova algebraica circa systema duarum æquationum inter duas variabiles," *Crelle*, t. XIV. (1835), pp. 281—288, [*Ges. Werke*, t. III., pp. 285—294], the sum on the right-hand side is $=0$: hence the required result $\Sigma(x,\ y,\ 1)^{m-3}\,d\omega = 0$.

Observing that $(x, y, 1)^{m-3}$ is an arbitrary function, the equation just obtained breaks up into the equations

$$\Sigma\,d\omega = 0,\quad \Sigma x\,d\omega = 0,\quad \Sigma y\,d\omega = 0, \ldots,\quad \Sigma x^{m-3}\,d\omega = 0, \ldots,\quad \Sigma y^{m-3}\,d\omega = 0,$$

viz. the number of equations is

$$1 + 2 + \ldots + (m-2),\quad = \tfrac{1}{2}(m-1)(m-2),$$

which is $=p$, the deficiency of the curve.

Suppose the fixed curve $f(x, y, 1) = 0$ is a cubic, $m = 3$, and we have the single relation $\Sigma\,d\omega = 0$, where the summation refers to the $3n$ points of intersection of the cubic and of the variable curve of the order n, $\phi(x, y, 1) = 0$.

In particular, if this curve be a line, $n = 1$, and the equation is $d\omega_1 + d\omega_2 + d\omega_3 = 0$; here the two points (x_1, y_1), (x_2, y_2) taken at pleasure on the cubic, determine the line, and they consequently determine uniquely the third point of intersection (x_3, y_3); there should thus be a single equation giving the displacement $d\omega_3$ in terms of the displacements $d\omega_1$, $d\omega_2$; viz. this is the equation just found

$$d\omega_1 + d\omega_2 + d\omega_3 = 0.$$

So if the variable curve be a conic, $n=2$; and we have between the displacements of the six points the relation

$$d\omega_1 + d\omega_2 + \ldots + d\omega_6 = 0:$$

here five of the points determine the conic, and they therefore determine uniquely the sixth point; and there should be between the displacements a single relation as just found.

If the variable curve be a cubic, $n=3$, and we have between the displacements of the nine points the relation

$$d\omega_1 + d\omega_2 + \ldots + d\omega_9 = 0:$$

here eight of the points do *not* determine the cubic ϕ, but they nevertheless determine the ninth point, viz. (reproducing the reasoning which establishes this well-known and fundamental theorem as to cubic curves) if $\phi_0=0$ be a particular cubic through the 8 points, then the general cubic is $\phi_0 + kf = 0$, and the intersections with $f=0$ are given by the equations $\phi_0=0$, $f=0$; whence the ninth point is independent of k, and is determined uniquely by the 8 points. There should thus be a single relation between the displacements, viz. this is the relation just found.

And so if the variable curve be a quartic, or curve of any higher order, it appears in like manner that there should be a single relation between the displacements; this relation being in fact the foregoing relation $\Sigma d\omega = 0$.

But take the fixed curve to be a quartic, $m=4$: then we have between the displacements $d\omega$ the relation

$$\Sigma(x,\ y,\ 1)\, d\omega = 0,$$

that is, the three equations

$$\Sigma x\, d\omega = 0, \quad \Sigma y\, d\omega = 0, \quad \Sigma\, d\omega = 0.$$

If the variable curve is a conic, $n=2$, then there are 8 points of intersection; 5 of these taken at pleasure determine the conic, and they consequently determine the remaining 3 points of intersection: hence there should be 3 equations. And so if the variable curve be a curve of any higher order, then by considerations similar to those made use of in the case where the first curve is a cubic it appears that the number of equations between the displacements $d\omega$ should always be $=3$.

But if the variable curve be a line, $n=1$, then the number of the points of intersection is $=4$: 2 of these taken at pleasure determine the line, and they consequently determine the remaining 2 points of intersection; and the number of equations between the displacements $d\omega$ should thus be $=2$. But by what precedes, we have the 3 equations

$$\begin{aligned} d\omega_1 + \ \ d\omega_2 + \ \ d\omega_3 + \ \ d\omega_4 &= 0, \\ x_1 d\omega_1 + x_2 d\omega_2 + x_3 d\omega_3 + x_4 d\omega_4 &= 0, \\ y_1 d\omega_1 + y_2 d\omega_2 + y_3 d\omega_3 + y_4 d\omega_4 &= 0; \end{aligned}$$

here the 4 points of intersection are on a line $y = ax + b$; we have therefore $y_1 = ax_1 + b, \ldots, y_4 = ax_4 + b$; the equations between the $d\omega$'s give

$$(y_1 - ax_1 - b)\, d\omega_1 + \ldots + (y_4 - ax_4 - b)\, d\omega_4 = 0,$$

that is, is a single relation $0 = 0$; or the 3 equations thus reduce themselves to 2 independent equations.

Again, if the fixed curve be a quintic, $m = 5$, there are here between the displacements the 6 equations

$$\Sigma x^2\, d\omega = 0, \quad \Sigma xy\, d\omega = 0, \quad \Sigma y^2\, d\omega = 0,$$
$$\Sigma x\, d\omega = 0, \quad \Sigma y\, d\omega = 0, \quad \Sigma\, d\omega = 0;$$

the two cases in which the number of independent equations is less than 6 are (i) when the variable curve is a line, and (ii) when the variable curve is a conic. For the line $n = 1$, and the number should be $= 3$. We have the above 6 equations; but the equation of the line is $ax + by + c = 0$, that is, we have $ax_1 + by_1 + c = 0$, &c.; we deduce the 3 identical equations

$$\Sigma x\,(ax + by + c) = 0, \quad \Sigma y\,(ax + by + c) = 0, \quad \Sigma\,(ax + by + c) = 0,$$

and the number of independent equations is thus $6 - 3, = 3$ as it should be.

So when the variable curve is a conic, $n = 2$; the number of independent equations should be $= 5$. The points of intersection lie on a conic $(a, b, c, f, g, h \between x, y, 1)^2 = 0$; we have therefore the several equations $(a, b, c, f, g, h \between x_1, y_1, 1)^2 = 0$, &c.: we have therefore the single identical equation

$$\Sigma\,(a, b, c, f, g, h \between x, y, 1)^2\, d\omega = 0,$$

and the number of independent equations is $6 - 1, = 5$ as it should be.

Obviously the like considerations apply to the case where the fixed curve is a curve of any given order whatever.

806.

DETERMINATION OF THE ORDER OF A SURFACE.

[From the *Messenger of Mathematics*, vol. XII. (1883), pp. 29—32.]

[ON p. lxx of the Prolegomena to C. Taylor's *Introduction to the Geometry of Conics* (1881) occurs the following passage:

"*Proof and extension of Newton's Descriptio Organica.*

"Let two angles AOB and $A\omega B$ of given magnitudes turn about O and ω respectively, and let the intersection A trace a curve of the nth order. For a given position of the arm OB there are n positions of A and therefore n of B. When OB is in the position $O\omega$ the n B's coincide with ω, which is therefore an n-fold point on the locus of B, as is also the point O; and since any line through O (or ω) meets the locus of B in n other points, the locus is of the order $2n$. Its order is the same when $A\omega B$ is a zero-angle or straight line.

"Let a given trihedral angle $O(ABC)$—or a plane OBC and a line OA rigidly attached to it—turn about O, and let a variable plane through a fixed point ω meet OA in A and the plane OBC in BC; then if the line BC describes a ruled surface of the order n the point A describes a surface of the order $4n$."

And in a foot-note it is stated that the author is indebted to Professor Cayley for the determination of the order of this surface. The following paper contains Professor Cayley's determination, which was communicated by him to Mr Taylor.]*

LEMMA. Take (a, b, c, f, g, h) the six coordinates of a line; these are connected by the equation

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0.$$

[* *l.c.*, p. 29.]

In order that the line may belong to a ruled surface, we must have between the coordinates three more equations, say these are

$$F\ (\mathrm{a, b, c, f, g, h}) = 0,$$
$$G\ (\mathrm{a} \quad \text{,,} \quad) = 0,$$
$$H\ (\mathrm{a} \quad \text{,,} \quad) = 0,$$

of the orders p, q, r respectively; then the scroll is of the order $n = 2pqr$.

For expressing that the line meets an arbitrary line (a′, b′, c′, f′, g′, h′), we have the linear relation

$$\mathrm{f'a + g'b + h'c + a'f + b'g + c'h} = 0;$$

the five relations determine the ratios a : b : c : f : g : h, and the number of systems of values is = product of orders $= 2 . p . q . r . 1$, $= 2pqr$, viz. this is the number of lines meeting the arbitrary line; or, what is the same thing, it is = order of ruled surface.

Consider now a trihedral angle $OABC$ rotating about a fixed point O which may be taken for the origin, and consider a fixed point ω. Let the lines OB, OC each meet a line L, and the plane ωL intersect OA in a point P; then it is to be shown that, if L is a line of a ruled surface of the order n, the locus of P is a surface of the order $4n$.

Observe that, for a given position of the line L, the position of the lines OB, OC is not determinate, but that the angle BOC has any position at pleasure in the plane OL, or say it rotates round the line ON which is the normal at O to the plane OL; the line OA therefore also rotates about ON, being always inclined to it at a determinate angle θ, or the locus of OA is a right cone axis ON and semi-aperture $= \theta$. The points P are the intersections of the several lines of the cone with the fixed plane ωL, or say that (for the given position of L) the locus of P is the conic C which is the intersection of the cone in question by the plane ωL. And then varying the position of L, the required surface is the locus of the corresponding conics C.

Take now

$$Ax + By + Cz + D = 0,\ \ A'x + B'y + C'z + D' = 0,$$

for the equations of a particular line L; then writing

$$AD' - A'D,\ BD' - B'D,\ CD' - C'D,\ BC' - B'C,\ CA' - C'A,\ AB' - A'B = (\mathrm{a, b, c, f, g, h})$$

respectively, these are the six coordinates of the line L.

The equation of the plane OL is

$$(AD' - A'D)\,x + (BD' - B'D)\,y + (CD' - C'D)\,z = 0,$$

that is,

$$\mathrm{a}x + \mathrm{b}y + \mathrm{c}z = 0;$$

and the equations of the normal ON are therefore

$$\frac{x}{\mathrm{a}}=\frac{y}{\mathrm{b}}=\frac{z}{\mathrm{c}};$$

we have therefore for the cone

$$\frac{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z}{\sqrt{(x^2+y^2+z^2)}\sqrt{(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2)}}=\cos\theta;$$

or if, for convenience, $\cos^2\theta=k$, then the equation of the cone is

$$(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)^2-k(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2)(x^2+y^2+z^2)=0;$$

say this is

$$\{\mathrm{a}^2-k(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2),\ \mathrm{b}^2-k(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2),\ \mathrm{c}^2-k(\mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2),\ \mathrm{bc},\ \mathrm{ca},\ \mathrm{ab}\}(x,\ y,\ z)^2=0\ldots(1).$$

Taking then $(x_0,\ y_0,\ z_0)$ for the coordinates of the point ω, the equation of the plane ωL is

$$\frac{Ax+By+Cz+D}{Ax_0+By_0+Cz_0+D}-\frac{A'x+B'y+C'z+D'}{A'x_0+B'y_0+C'z_0+D'}=0,$$

viz. it is

$$(BC'-B'C)(yz_0-y_0z)+\ldots+(AD'-A'D)(x-x_0)+\ldots=0,$$

that is,

$$\mathrm{f}(yz_0-y_0z)+\mathrm{g}(zx_0-z_0x)+\mathrm{h}(xy_0-x_0y)+\mathrm{a}(x-x_0)+\mathrm{b}(y-y_0)+\mathrm{c}(z-z_0)=0,$$

or say

$$x(\mathrm{h}y_0-\mathrm{g}z_0+\mathrm{a})+y(-\mathrm{h}x_0+\mathrm{f}z_0+\mathrm{b})+z(\mathrm{g}x_0-\mathrm{f}y_0+\mathrm{c})+(-\mathrm{a}x_0-\mathrm{b}y_0-\mathrm{c}z_0)=0\ldots(2),$$

viz. (1) and (2) are the equations of the conic C, and the coordinates (a, b, c, f, g, h) satisfy of course the equation

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Considering now the line L as belonging to a ruled surface, the coordinates (a, ...) satisfy as before three equations

$$F\ (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h})=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$G\ (\qquad\quad ,,\qquad\quad)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$H\ (\qquad\quad ,,\qquad\quad)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

of the orders p, q, r respectively, and we can from the six equations eliminate a, b, c, f, g, h. The resulting equation $\nabla=0$ contains the coefficients of (1) in the order $1.2.p.q.r=2pqr$ (which is the product of the orders of the other 5 equations), and the coefficients of (2) in the order $2.2.p.q.r=4pqr$, (which is the product of the orders of the other 5 equations). But the coefficients of (1) being quadric functions of $(x,\ y,\ z)$, and those of (2) being linear functions of $(x,\ y,\ z)$, the aggregate order in $(x,\ y,\ z)$ is

$$2.2pqr+1.4pqr,\ =8pqr;$$

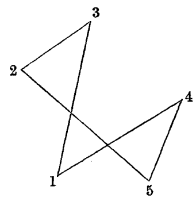
or, since the order of the ruled surface is n, $=2pqr$, the order of the locus is $=4n$; which is the above-stated theorem.

807.

A PROOF OF WILSON'S THEOREM.

[From the *Messenger of Mathematics*, vol. XII. (1883), p. 41.]

LET n be a prime number; and imagine n points, the vertices of a regular polygon; any polygon which can be formed with these n points as vertices is either regular or else it is one of a set of n equal and similar polygons. For instance, $n=5$, the polygon as shown in the figure is one of a set of 5 equal and similar



polygons; in fact, if the points taken in their cyclical order, but beginning at pleasure with any one of the 5 points are called 1, 2, 3, 4, 5, then we have 5 such polygons 13254; and so in general. The whole number of polygons is $\frac{1}{2}.1.2.3\ldots(n-1)$; and the number of the regular polygons is $\frac{1}{2}(n-1)$; hence the number of the remaining polygons is $=\frac{1}{2}(n-1)\{1.2\ldots(n-2)-1\}$; and this number must therefore be divisible by n; that is, $1.2\ldots(n-1)-n+1$ is divisible by n; or, what is the same thing, $1.2\ldots(n-1)+1$ is divisible by n, which is the theorem in question.

808.

NOTE ON A FORM OF THE MODULAR EQUATION IN THE TRANSFORMATION OF THE THIRD ORDER.

[From the *Messenger of Mathematics*, vol. XII. (1883), pp. 173, 174.]

In my *Treatise on Elliptic Functions*, pp. 214—216, writing only $\frac{1}{J}$, $\frac{1}{J'}$ instead of Ω, Ω', and α, β instead of α', β', I have shown as follows: viz. if k, λ denote as usual the original modulus, and the transformed modulus, and if

$$J=\frac{(k^4+14k^2+1)^3}{108k^2(1-k^2)^4},\quad J'=\frac{(\lambda^4+14\lambda^2+1)^3}{108\lambda^2(1-\lambda^2)^4},$$

then the relation between J and J' can be found by the elimination of α, β from the equations

$$\alpha+\beta=1,$$

$$J=\frac{(1+8\alpha)^3}{64\alpha(1-\alpha)^3},\quad J'=\frac{(1+8\beta)^3}{64\beta(1-\beta)^3}.$$

By a very slight change we obtain the result given by Prof. Klein in his paper, "Ueber die Transformation der elliptischen Functionen, &c.," *Math. Ann.* t. XIV. (1879), pp. 111—172; viz., see p. 143, the relation is to be obtained by the elimination of τ, τ' from the equation $\tau\tau'=1$, and the equations

$$J\ :\ J-1\ :\ 1=(\tau-1)(9\tau-1)^3\ :\ (27\tau^2-18\tau-1)^2\ :\ -64\tau;$$
$$J'\ :\ J'-1\ :\ 1=(\tau'-1)(9\tau'-1)^3\ :\ (27\tau'^2-18\tau'-1)^2\ :\ -64\tau';$$

these last equations being equivalent to two equations only in virtue of the identity

$$(\tau-1)(9\tau-1)^3+64\tau=(27\tau^2-18\tau-1)^2,$$

and the like identity in τ'.

In fact, writing $\alpha=\frac{\tau}{\tau-1}$, $\beta=\frac{\tau'}{\tau'-1}$, the equation $\alpha+\beta=1$ becomes $\tau\tau'=1$; and then for α, β substituting their values, we have

$$J=\frac{(9\tau-1)^3(\tau-1)}{-64\tau},\quad J'=\frac{(9\tau'-1)^3(\tau'-1)}{-64\tau'},$$

which are the formulæ in question.

809.

SCHRÖTER'S CONSTRUCTION OF THE REGULAR PENTAGON.

[From the *Messenger of Mathematics*, vol. XII. (1883), p. 177.]

THE following construction of the regular pentagon, analogous to the more complicated one for the polygon of 17 sides, is given in the paper, H. Schröter, Zur *v. Staudt*schen "Construction des regulären Siebenzehnecks," *Crelle*, t. LXXV. (1873), pp. 13—24.

Take in a circle AB, CD diameters at right angles to each other, and at C, D draw in the directions A to B and B to A respectively, the lines Cc, $=4$ radii, and Dd, $=1$ radius; draw cd meeting the circle in the points E and F; draw CE and CF cutting AB in the points e and f respectively, and through these at right angles to AB draw the chords 34 and 25 respectively, then we have $A2345$ a regular pentagon.

810.

NOTE ON A SYSTEM OF EQUATIONS.

[From the *Messenger of Mathematics*, vol. XII. (1883), pp. 191, 192.]

THE equations are

$$x^2 = ax + by, \quad xy = cx + dy, \quad y^2 = ex + fy,$$

where

$$\frac{b}{d} = \frac{a-d}{c-f} = \frac{c}{e};$$

or, what is the same thing, if a, b, c, d are given, then

$$e = \frac{cd}{b}, \quad f = c - \frac{d(a-d)}{b};$$

and this being so, the equations are equivalent to two independent equations; viz. starting from the first and the second equations, we have

$$dx^2 - bxy = (ad - bc)\,x,$$

that is,

$$dx - by = (ad - bc):$$

and thence

$$dxy - by^2 = (ad - bc)\,y,$$

or

$$d(cx + dy) - by^2 = (ad - bc)\,y;$$

which, attending to the values of e and f, is the third equation

$$y^2 = ex + fy.$$

We have

$$\frac{x}{y} = \frac{ax + by}{cx + dy}, \quad \frac{y}{x} = \frac{ex + fy}{cx + dy};$$

that is,

$$cx^2 - (a-d)\,xy - by^2 = 0,$$
$$ex^2 - (c-f)\,xy - dy^2 = 0,$$

and eliminating the (x, y) from these equations we have an equation $\Omega = 0$ as the condition that the original three equations may have a single common root; the before-mentioned equations $\frac{b}{d} = \frac{a-d}{c-f} = \frac{c}{e}$, are the conditions in order that the three equations may have two common roots, that is, that there may be *two* systems of (x, y) satisfying the three equations.

We have, moreover, $y(x-d) = cx$, $x(y-c) = dy$, and substituting these values, say $y = \frac{cx}{x-d}$ and $x = \frac{dy}{y-c}$, in the first and third equations respectively, they become

$$x - a = \frac{bc}{x-d}, \quad y - f = \frac{de}{y-c},$$

that is,

$$x^2 - (a+d)\,x + ad - bc = 0,$$
$$y^2 - (c+f)\,y + cf - de = 0,$$

which are quadric equations for x and y respectively; it is easy to express the second equation (like the first) in terms of (a, b, c, d), and the first equation (like the second) in terms of (c, d, e, f), but the forms are less simple.

Suppose $(a, b, c, d) = (-1, -1, 1, 0)$, then we have $(e, f) = (0, 1)$, the two equations in $x : y$ become $x^2 + xy + y^2 = 0$, and $0 = 0$ respectively; those in x and y become $x^2 + x + 1 = 0$, $y^2 - 2y + 1 = 0$ respectively; this is right, for the three equations are

$$x^2 = -x - y, \quad xy = x, \quad y^2 = y;$$

viz. from the third equation we have $y = 1$, a value satisfying the second equation, and then the first equation becomes $x^2 + x + 1 = 0$; or, if we please, $x^2 + xy + y^2 = 0$, the values in fact being $x = \omega$, an imaginary cube root of unity, and $y = 1$.

In the general case, the values (x, y) may be regarded as units in a complex numerical theory, viz. if (a, b, c, d, e, f) are integers, and p, q, p', q', P, Q are also integers, then the product of the two complex integers $px + qy$ and $p'x + q'y$ will be a complex integer $Px + Qy$.

811.

ON THE LINEAR TRANSFORMATION OF THE THETA FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XIII. (1884), pp. 54—60.]

THE functions referred to are the single Theta Functions; these may be defined as doubly infinite products, as was in fact done in my "Mémoire sur les fonctions doublement périodiques," *Liouv.* t. x. (1845), pp. 385—420, [25]; and it is interesting to consider from this point of view the theory of their linear transformation: this I propose to do in the present paper, adopting throughout the notation of Smith's* "Memoir on the Theta and Omega Functions."

The periods K, iK' are, in general, imaginary quantities

$$K = A + Bi,$$
$$iK' = C + Di,$$

where $AD - BC$ is positive; writing then $\omega = \frac{iK'}{K}$, and $q = e^{i\pi\omega}$, also for shortness

$$(q1) = 2q^{\frac{1}{4}} \Pi_1^\infty (1 - q^{2n})^3,$$

where $q^{\frac{1}{4}}$ denotes $e^{\frac{1}{4}i\pi\omega}$, the expression of the odd theta-function $\vartheta_1(x, \omega)$ as a doubly infinite product is

$$\vartheta_1(x, \omega) = (q1)\, x \Pi\Pi \left(1 + \frac{x}{m\pi + n\omega\pi}\right), \quad \left(\frac{\mu}{\nu} = \infty\right),$$

where (m, n) have any positive or negative integer values (the combination $m = 0$, $n = 0$ excluded) from $m = -\mu$ to μ, and $n = -\nu$ to ν, μ and ν being each ultimately infinite but so that μ is infinite in comparison with ν; this condition in regard to the limits is indicated by $\mu/\nu = \infty$; and similarly $\nu/\mu = \infty$ would indicate that ν was infinite in comparison with μ.

[* Smith's *Collected Mathematical Papers*, vol. II., pp. 415—621.]

The condition as to the limits might be that (m, n) have any positive or negative values (excluding as before) such that the modulus of $m+n\omega$ does not exceed a positive value T, which is ultimately taken to be infinite; this condition may be indicated by $\text{mod}=\infty$.

The values of the double product corresponding to the different conditions as to the limits are not equal, but they differ only by an exponential factor, the exponent being a multiple of x^2; we thus have

$$x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)\left(\frac{\mu}{\nu}=\infty\right)=\exp\left(\nabla x^2\right)x\Pi\Pi\left(1+\frac{x}{mK+niK'}\right)(\text{mod}=\infty),$$

where ∇ is a determinate value, depending on K and K'; and similarly

$$x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right)\left(\frac{\mu}{\nu}=\infty\right)=\exp\left(\square x^2\right)x\Pi\Pi\left(1+\frac{x}{m\Lambda+ni\Lambda'}\right)(\text{mod}=\infty),$$

where $\square$ is a determinate value depending in like manner on Λ, Λ'.

We have, then, as above

$$\begin{aligned}\vartheta_1(x, \omega)&=(q1)\,x\Pi\Pi\left(1+\frac{x}{m\pi+n\omega\pi}\right)\left(\frac{\mu}{\nu}=\infty\right)\\&=(q1)\frac{\pi}{K}\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right)\\&=(q1)\frac{\pi}{K}\exp\left(\nabla\frac{K^2x^2}{\pi^2}\right)\frac{Kx}{\pi}\Pi\Pi\left(1+\frac{\frac{Kx}{\pi}}{mK+niK'}\right)(\text{mod}=\infty),\end{aligned}$$

viz. we have thus defined $\vartheta_1(x, \omega)$ as a doubly infinite product with the limiting condition $(\text{mod}=\infty)$; if for x we write $\frac{\pi x}{h}$, h arbitrary, we have

$$\vartheta_1\left(\frac{\pi x}{h}, \omega\right)=(q1)\frac{\pi}{K}\exp\left(\nabla\frac{K^2x^2}{h^2}\right)\frac{Kx}{h}\Pi\Pi\left(1+\frac{\frac{Kx}{h}}{mK+niK'}\right)(\text{mod}=\infty),$$

and similarly, if $\Omega=\frac{i\Lambda'}{\Lambda}$, $Q=e^{i\pi\Omega}$, then

$$\begin{aligned}\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h}, \Omega\right\}=(Q1)\frac{\pi}{\Lambda}\exp\left\{(a+b\Omega)^2\square\frac{\Lambda^2x^2}{h^2}\right\}\\\times(a+b\Omega)\frac{\Lambda x}{h}\Pi\Pi\left(1+\frac{(a+b\Omega)\frac{\Lambda x}{h}}{m\Lambda+ni\Lambda'}\right)(\text{mod}=\infty).\end{aligned}$$

In the case of a linear transformation, we have

$$\omega=\begin{vmatrix}a, & b\\c, & d\end{vmatrix}\times\Omega,\ \text{that is,}\ \omega=\frac{c+d\Omega}{a+b\Omega},$$

where a, b, c, d are positive or negative integers such that $ad-bc=+1$; it is to be shown that the two infinite products are in this case identical; this being so, we have

$$\frac{\vartheta_1\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}}{\vartheta_1\left\{\frac{\pi x}{h},\ \omega\right\}}=\frac{(Q1)}{(q1)}\exp\left\{\left\{(a+b\Omega)^2\,\square\Lambda^2-\nabla K^2\right\}\frac{x^2}{h^2}\right\},$$

viz. the two functions differ only by a constant factor and by an exponential factor, the exponent being a multiple of x^2; after all reductions, this factor is found to be

$$=\exp\left(-i\pi b\,(a+b\Omega)\frac{x^2}{h^2}\right).$$

We have

$$\omega=\frac{c+d\Omega}{a+b\Omega},$$

or since

$$\omega=\frac{iK'}{K},\quad \Omega=\frac{i\Lambda'}{\Lambda},$$

this is

$$\frac{iK'}{K}=\frac{c\Lambda+di\Lambda'}{a\Lambda+bi\Lambda'},$$

or say

$$\frac{1}{M}K=a\Lambda+bi\Lambda',$$

$$\frac{1}{M}iK'=c\Lambda+di\Lambda',$$

either of which equations may be taken as a definition of the multiplier M. We have

$$\frac{K}{M\Lambda}=a+b\Omega,$$

$$\frac{1}{M}(mK+niK')=(am+cn)\,\Lambda+(bm+dn)\,i\Lambda'$$
$$=m'\Lambda+n'i\Lambda',$$

if

$$m'=am+cn,$$
$$n'=bm+dn.$$

Here to any integer values of $(m,\ n)$ there correspond integer values of m', n'; and conversely, in virtue of the equation $ad-bc=1$, to any integer values of m', n' there correspond integer values of m, n. The two products are

$$\Pi\Pi\left(1+\frac{\frac{Kx}{Mh}}{m'\Lambda+n'i\Lambda'}\right),\ (\text{mod}=\infty),$$

$$\Pi\Pi\left(1+\frac{\frac{(a+b\Omega)\,\Lambda x}{h}}{m\Lambda+ni\Lambda'}\right),\ (\text{mod}=\infty).$$

But, as above, we have $\frac{K}{M} = (a + b\Omega)\Lambda$: and then, observing that in the first of the two products we may for m', n' write m, n, it at once appears that the two products are identical.

The exponential factor, writing therein $(a+b\Omega)\Lambda = \frac{K}{M}$, becomes

$$\exp\left\{\left(\frac{\square}{M^2} - \nabla\right)\frac{K^2x^2}{h^2}\right\}.$$

The values of ∇, $\square$ are at once obtained by means of a formula* given in my Memoir, viz. we have

$$\nabla = -\tfrac{1}{2}(B + \beta),$$

where

$$B = \frac{\pi(\omega\upsilon + \omega'\upsilon')}{\Omega\Upsilon \bmod(\omega\upsilon' - \omega'\upsilon)},$$

$$\beta = \frac{\pi i}{\Omega\Upsilon}\frac{(\omega\upsilon' - \omega'\upsilon)}{\bmod(\omega\upsilon' - \omega'\upsilon)}.$$

Comparing with the present notation

$$\Omega = \omega + \omega' i, \quad = A + Bi = K,$$

$$\Upsilon = \upsilon + \upsilon' i, \quad = C + Di = K'i,$$

so that Ω, Υ denote K, $K'i$, and ω, ω', υ, υ' denote A, B, C, D respectively: $\omega\upsilon' - \omega'\upsilon$ is thus $= AD - BC$, which has been assumed to be positive; hence also $\bmod(\omega\upsilon' - \omega'\upsilon) = AD - BC$, and the formula becomes

$$\nabla = -\tfrac{1}{2}\pi\left\{\frac{AC + BD}{i(AD - BC)} + 1\right\}\frac{1}{KK'}.$$

Now writing

$$\Lambda = A_1 + B_1 i,$$

$$i\Lambda' = C_1 + D_1 i,$$

then we have

$$\frac{1}{M}(A + Bi) = a\Lambda + bi\Lambda' = a(A_1 + B_1 i) + b(C_1 + D_1 i),$$

$$\frac{1}{M}(C + Di) = c\Lambda + di\Lambda' = c(A_1 + B_1 i) + d(C_1 + D_1 i);$$

consequently, if

$$M = \rho(\cos\theta + i\sin\theta),$$

[* *Collected Mathematical Papers*, t. I., p. 164. The denominator factor $\Omega\Upsilon$ has been omitted (p. 165) by mistake.]

we have

$$\frac{1}{\rho}(\quad A\cos\theta + B\sin\theta) = aA_1 + bC_1,$$

$$\frac{1}{\rho}(-A\sin\theta + B\cos\theta) = aB_1 + bD_1,$$

$$\frac{1}{\rho}(\quad C\cos\theta + D\sin\theta) = cA_1 + dC_1,$$

$$\frac{1}{\rho}(-C\sin\theta + D\cos\theta) = cB_1 + dD_1,$$

and thence

$$\frac{1}{\rho^2}(AD - BC) = (ad - bc)(A_1D_1 - B_1C_1), \quad = (A_1D_1 - B_1C_1).$$

Hence $A_1D_1 - B_1C_1$ is positive, and we have

$$\square = -\tfrac{1}{2}\pi\left\{\frac{A_1C_1 + B_1D_1}{i(A_1D_1 - B_1C_1)} + 1\right\}\frac{1}{\Lambda\Lambda'}.$$

Take K_1 the conjugate of K, Λ_1 the conjugate of Λ, then

$$K_1 = A - Bi, \quad \Lambda_1 = A_1 - B_1 i,$$

$$iK' = C + Di, \quad i\Lambda' = C_1 + D_1 i.$$

We have

$$iK_1K' = AC + BD + i(AD - BC),$$

and therefore

$$\frac{K_1K'}{AD - BC} = \frac{AC + BD}{i(AD - BC)} + 1, \quad \nabla = \frac{-\tfrac{1}{2}\pi}{AD - BC}\frac{K_1}{K},$$

and similarly

$$\square = \frac{-\tfrac{1}{2}\pi}{A_1D_1 - B_1C_1}\frac{\Lambda_1}{\Lambda}.$$

The exponential is

$$\left(\frac{\square}{M^2} - \nabla\right)\frac{K^2x^2}{h^2};$$

and we have

$$\frac{\square}{M^2} - \nabla = \frac{-\tfrac{1}{2}\pi}{M^2(A_1D_1 - B_1C_1)}\frac{\Lambda_1}{\Lambda} + \frac{\tfrac{1}{2}\pi}{AD - BC}\frac{K_1}{K},$$

which is

$$= \frac{-\tfrac{1}{2}\pi}{M^2(A_1D_1 - B_1C_1)}\frac{\Lambda_1}{\Lambda} + \frac{\tfrac{1}{2}\pi}{\rho^2(A_1D_1 - B_1C_1)}\frac{K_1}{K},$$

$$= \frac{-\tfrac{1}{2}\pi}{A_1D_1 - B_1C_1}\left(\frac{1}{M^2}\frac{\Lambda_1}{\Lambda} - \frac{1}{\rho^2}\frac{K_1}{K}\right).$$

But $\rho/M = \cos\theta - i\sin\theta$, or calling this for a moment P, then $1/M^2 = P^2/\rho^2$, and the formula may be written

$$\frac{\square}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}\left(P\frac{\Lambda_1}{\Lambda} - P^{-1}\frac{K_1}{K}\right)$$

$$= \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}\{(\cos\theta - i\sin\theta)\Lambda_1 K - (\cos\theta + i\sin\theta)\Lambda K_1\}\frac{1}{K\Lambda}.$$

The term in { } is

$$\begin{aligned}
&(\cos\theta - i\sin\theta)(A + Bi)(A_1 - B_1 i) - (\cos\theta + i\sin\theta)(A - Bi)(A_1 + B_1 i),\\
&= \quad 2\cos\theta\,[-(AB_1 - A_1B)i] - 2i\sin\theta\,(AA_1 + BB_1),\\
&= -2i\,\{(AB_1 - A_1B)\cos\theta + (AA_1 + BB_1)\sin\theta\},\\
&= -2i\,\{B_1(A\cos\theta + B\sin\theta) - A_1(-A\sin\theta + B\cos\theta)\},\\
&= -2i\rho\,\{B_1(aA_1 + bC_1) - A_1(aB_1 + bD_1)\},\\
&= +2i\rho b\,(A_1D_1 - B_1C_1).
\end{aligned}$$

Hence

$$\frac{\square}{M^2} - \nabla = \frac{-\frac{1}{2}\pi P}{\rho^2(A_1D_1 - B_1C_1)}2ib\rho\,(A_1D_1 - B_1C_1)\frac{1}{K\Lambda}$$

$$= -\frac{i\pi bP}{\rho}\frac{1}{K\Lambda} = -\frac{i\pi b}{M}\frac{1}{K\Lambda},$$

and the exponential thus is

$$= \exp\left(-\frac{i\pi b}{M}\frac{1}{K\Lambda}\frac{K^2x^2}{h^2}\right), \quad = \exp\left(-i\pi b\frac{K}{M\Lambda}\frac{x^2}{h^2}\right);$$

or, since $\dfrac{K}{M\Lambda} = (a + b\Omega)$, this is

$$= \exp\left(-i\pi b\,(a + b\Omega)\frac{x^2}{h^2}\right);$$

and we have thus the required formula

$$\frac{\vartheta_1\left\{(a + b\Omega)\dfrac{\pi x}{h},\ \Omega\right\}}{\vartheta_1\left\{\dfrac{\pi x}{h},\ \omega\right\}} = \frac{(Q1)}{(q1)}(a + b\Omega)\exp\left(-i\pi b\,(a + b\Omega)\frac{x^2}{h^2}\right).$$

812.

ON ARCHIMEDES' THEOREM FOR THE SURFACE OF A CYLINDER.

[From the *Messenger of Mathematics*, vol. XIII. (1884), pp. 107, 108.]

THE measure of the surface of a cylinder was first obtained by Archimedes in his Treatise on the Sphere and Cylinder (Book I., Prop. XIV.), *Œuvres d'Archimède*, par F. Peyrard, 4° Paris, 1807, pp. 26—31; viz. Archimedes showed that the surface of the cylinder was equal to the area of a circle, having its radius a mean proportional between the height and the diameter of the circular base $[S = 2\pi ah, = \pi\{\sqrt{(2a \,.\, h)}\}^2]$.

The following is *in effect* his demonstration:

He considers regular polygons (with the same number of sides) inscribed in and circumscribed about a circle; and, as regards the cylinder, the prisms standing on these polygons.

Say for the circular base of the cylinder we have

$S^\times$ surface of circumscribed prism,

S „ „ cylinder,

S° „ „ inscribed prism;

and for the circle, having its radius a mean proportional between the height and the diameter of the circular base,

$B^\times$ area of circumscribed polygon,

B „ „ circle,

B° „ „ inscribed polygon,

where the four polygons referred to by $S^\times$, S°, $B^\times$, B° have all of them the same number of sides.

It is in the preceding propositions (by means of an axiom as to curve lines) shown that

$$S^{\times} > S > S^{\circ}, \quad B^{\times} > B > B^{\circ};$$

and it is further shown that

$$S^{\times} = B^{\times}, \quad S^{\circ} = B^{\circ}.$$

It is moreover shown that, by taking the number of sides sufficiently large, the ratio $B^{\times} : B^{\circ}$, or say the fraction $B^{\times}/B^{\circ}$ (which is greater than 1) may be made less than any given quantity $1+\epsilon$.

It is then to be shown that $S = B$.

If not, then

either

$$B < S.$$

This being so, it is possible to make

$$B^{\times}/B^{\circ} < S/B,$$

that is,

$$S^{\times}/B^{\circ} < S/B,$$

or

$$S^{\times}/S < B^{\circ}/B,$$

which is absurd, since

$$S^{\times}/S > 1; \; B^{\circ}/B < 1;$$

or else

$$B > S.$$

This being so, it is possible to make

$$B^{\times}/B^{\circ} < B/S,$$

that is,

$$B^{\times}/S^{\circ} < B/S,$$

or

$$B^{\times}/B < S^{\circ}/S,$$

which is absurd, since

$$B^{\times}/B > 1; \; S^{\circ}/S < 1;$$

and consequently $S = B$, the theorem in question.

I take the opportunity of referring to two theorems by Archimedes, Lemmas, Prop. V. and VI., Peyrard, pp. 429—435, which relate to the contacts of circles. We have in each of them the figure which he calls the Arbelon, viz. if A, C, B are points in this order on the same straight line, then the figure consists of the three semicircles on the diameters AC, CB, and AB respectively, and the Arbelon is the space included between the three semi-circumferences.

In Prop. V., we have also the common tangent at C to the two semicircles AC, CB; this divides the Arbelon into two mixtilinear triangles (each bounded by the common tangent, one of the smaller semicircles, and a portion of the larger semicircle), and inscribing each of these a circle, the theorem is that the two inscribed circles are of equal magnitude.

In Prop. VI., the theorem is that the radii of the smaller semicircles being as 3 : 2, then the radius of the circle inscribed in the Arbelon is to the diameter of the larger semicircle as 6 to 19. But it is noticed that the demonstration would apply to any other value of the ratio; and, in fact, if the radii of the two smaller circles are as $a : b$, then the radius of the inscribed circle is to the diameter of the larger semicircle as ab to $a^2 + ab + b^2$, which is the general form of the theorem.

813.

[NOTE ON MR GRIFFITHS' PAPER "ON A DEDUCTION FROM THE ELLIPTIC-INTEGRAL FORMULA $y = \sin(A + B + C + \ldots)$".]

[From the *Proceedings of the London Mathematical Society*, vol. XV. (1884), p. 81.]

CONSIDER, for instance,

the cubic transformation

$$y = \frac{x[1 + 2\alpha' - (1+\alpha')^2 x^2]}{1 - \alpha^2 x^2},$$

where $\alpha^2 + \alpha'^2 = 1$.

This implies

$$\sqrt{1-y^2} = \frac{\sqrt{1-x^2}\,[1 - (1+\alpha')^2 x^2]}{1 - \alpha^2 x^2},$$

viz., $\sqrt{1-y^2}$ = a rational multiple of $\sqrt{1-x^2}$.

Also the quadric transformation

$$z = \frac{1 - (1 + \beta'^2) x^2}{1 - \beta^2 x^2};$$

where $\beta^2 + \beta'^2 = 1$.

This implies

$$\sqrt{1-z^2} = \frac{\sqrt{1-x^2}\,.\,2\beta' x}{1 - \beta^2 x^2},$$

viz., $\sqrt{1-z^2}$ = a rational multiple of $\sqrt{1-x^2}$.

Hence, assuming

$$u = yz - \sqrt{1-y^2}\,\sqrt{1-z^2},$$

which is a rational function

$$= \frac{x\,(a_0 - a_2 x^2 + a_4 x^4)}{1 - \alpha^2 x^2\,.\,1 - \beta^2 y^2},$$

we have

$$\sqrt{1-u^2} = y\sqrt{1-z^2} + z\sqrt{1-y^2},$$

which is $= \sqrt{1-x^2}$ multiplied by a like rational function.

That is, in defining the a_0, a_2, a_4, functions of the two arbitrary coefficients α, β, as above, we have in effect so determined them that $\sqrt{1-u^2}$ shall be $=\sqrt{1-x^2}$ multiplied by a rational function of x.

We can then further determine a_0, a_2, a_4 in such wise that the change of x into $\frac{1}{kx}$ shall change u into $\frac{1}{\lambda u}$; and, this being so, making the change in $\sqrt{1-u^2}$, we obtain $\sqrt{1-\lambda^2u^2}$ in the form, $\sqrt{1-k^2x^2}$ multiplied by a rational function of x; viz. u is a function of x such that

$$\frac{du}{\sqrt{1-u^2 \,.\, 1-\lambda^2u^2}}=\frac{Mdu}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}.$$

The theory is thus in effect Jacobi's—with the *novelty* of combining two lower transformations in such wise that the assumed expression for u as a rational function of x shall give

$$\sqrt{1-u^2}=\sqrt{1-x^2} \text{ multiplied by a rational function of } x.$$

It is not necessary that the equations

$$y=\text{rational function of } x \quad \text{and} \quad z=\text{rational function of } x$$

should be elliptic-function transformations. All that is required is that they should be such as to give $\sqrt{1-y^2}$ and $\sqrt{1-z^2}$ each $=\sqrt{1-x^2}$ multiplied by a rational function of x.

814.

ON DOUBLE ALGEBRA.

[From the *Proceedings of the London Mathematical Society*, vol. XV. (1884), pp. 185—197. Read April 3, 1884.]

1. I CONSIDER the Double Algebra formed with the extraordinary symbols, or "extraordinaries" x, y, which are such that

$$\begin{aligned} x^2 &= ax + by, \\ xy &= cx + dy, \\ yx &= ex + fy, \\ y^2 &= gx + hy, \end{aligned}$$

or, as these equations may also be written,

$$\begin{array}{c|c|c|} & x & y \\ \hline x & (a,\ b) & (c,\ d) \\ \hline y & (e,\ f) & (g,\ h) \\ \hline \end{array},$$

where a, b, c, d, e, f, g, h are ordinary symbols, or say coefficients; all coefficients being commutative and associative *inter se* and with the extraordinaries x, y.

The system depends in the first instance on the eight parameters a, b, c, d, e, f, g, h; but we may, instead of the extraordinaries x, y, consider the new extraordinaries connected therewith by the linear relations $\xi = \alpha x + \beta y$, $\eta = \gamma x + \delta y$, where the coefficients α, β, γ, δ may be determined so as to establish between the eight parameters any four relations at pleasure (or, what is the same thing, α, β, γ, δ are what I call "apoclastic" constants): and the number of parameters is thus properly $8 - 4, = 4$.

2. The extraordinaries here considered are not in general associative; differing herein from the imaginaries of Peirce's Memoir, "Linear Associative Algebra" (1870),

reprinted in the *American Mathematical Journal*, t. IV. (1881), pp. 97—227, which, as appears by the title, refers only to associative imaginaries. I recall some definitions and results. The symbol x is said to be *idempotent* if $x^2 = x$, *nilpotent* if $x^2 = 0$; and the systems of associative symbols are expressed as much as may be by means of such idempotent and nilpotent symbols: thus the linear systems are $(a_1)\, x^2 = x$, $(b_1)\, x^2 = 0$. A double system composed of independent symbols, that is, symbols x, y each belonging to its own linear system, and moreover such that $xy = yx = 0$, is said to be "mixed"; thus the mixed double systems are

$$\begin{array}{c|c|c|}
 & x & y \\ \hline
x & x & 0 \\ \hline
y & 0 & y \\ \hline
\end{array}, \qquad
\begin{array}{c|c|c|}
 & x & y \\ \hline
x & x & 0 \\ \hline
y & 0 & 0 \\ \hline
\end{array}, \qquad
\begin{array}{c|c|c|}
 & x & y \\ \hline
x & 0 & 0 \\ \hline
y & 0 & 0 \\ \hline
\end{array}.$$

But Peirce excludes these from consideration, attending only to the pure systems, which he finds to be

$$\begin{array}{c|c|c|}
 & x & y \\ \hline
(a_2)\ x & x & y \\ \hline
y & y & 0 \\ \hline
\end{array}, \qquad
\begin{array}{c|c|c|}
 & x & y \\ \hline
(b_2)\ x & x & y \\ \hline
y & 0 & 0 \\ \hline
\end{array}, \qquad
\begin{array}{c|c|c|}
 & x & y \\ \hline
(c_2)\ x & y & 0 \\ \hline
y & 0 & 0 \\ \hline
\end{array}.$$

To these, however, should be added the system

$$\begin{array}{c|c|c|}
 & x & y \\ \hline
(d_2)\ x & x & 0 \\ \hline
y & y & 0 \\ \hline
\end{array};$$

see *post*, No. 19.

3. In the general theory, where the symbols are not in the first instance taken to be associative, we may of course establish between the coefficients such relations as will make the symbols associative; and the question presents itself to show how in this case the system reduces itself to one of Peirce's systems. This I considered in my note "On Associative Imaginaries," *Johns Hopkins University Circular*, No. 15 (1882), p. 211 [822]; I there obtained, as the general form of the commutative and associative system,

$$x^2 = ax + by,$$

$$xy = yx = cx + dy,$$

$$y^2 = \frac{cd}{b}\, x + \frac{d^2 + bc - ad}{b}\, y,$$

the relation of which to Peirce's system was, as I there remarked, pointed out to me by Mr C. S. Peirce: this will be considered in the sequel, Nos. 13 to 19.

4. Starting now with the general equations

$$\begin{aligned} x^2 &= ax + by, \\ xy &= cx + dy, \\ yx &= ex + fy, \\ y^2 &= gx + hy, \end{aligned}$$

we may attempt to find an extraordinary ξ, $= \alpha x + \beta y$ (α, β coefficients), such that $\xi^2 = K\xi$ (K a coefficient). In general, K is not $=0$, and, when it is not $=0$, it may without loss of generality be taken to be $=1$; we have then $\xi^2 = \xi$, ξ an *idempotent* symbol. But K may be $=0$; and then $\xi^2 = 0$, ξ a *nilpotent* symbol. To include the two cases, I retain K, it being understood that, when K is not $=0$, it may be taken to be $=1$. We have

$$\begin{aligned} \xi^2 &= \alpha^2 (ax + by) + \alpha\beta (\overline{c+e}\, x + \overline{d+f}\, y) + \beta^2 (gx + hy) \\ &= \{a\alpha^2 + (c+e)\alpha\beta + g\beta^2\} x + \{b\alpha^2 + (d+f)\alpha\beta + h\beta^2\} y. \end{aligned}$$

Hence, when this is $= K\xi$, that is,

$$= K(\alpha x + \beta y),$$

we have

$$\frac{\alpha}{\beta} = \frac{a\alpha^2 + (c+e)\alpha\beta + g\beta^2}{\alpha^2 + (d+f)\alpha\beta + h\beta^2},$$

a cubic equation for the determination of the ratio $\alpha : \beta$; and, for any particular value of the ratio, we can in general determine the absolute magnitudes, so that

$$K, = \frac{1}{\alpha}\{a\alpha^2 + (c+e)\alpha\beta + g\beta^2\}, \quad = \frac{1}{\beta}\{b\alpha^2 + (d+f)\alpha\beta + h\beta^2\},$$

shall be $=1$. If, however, for the given value of the ratio we have

$$a\alpha^2 + (c+e)\alpha\beta + g\beta^2 = 0, \quad b\alpha^2 + (d+f)\alpha\beta + h\beta^2 = 0,$$

one of these equations, of course, implying the other, then the value of K is $=0$.

5. It follows that there are in general three idempotent symbols ξ, η, ζ, that is, extraordinaries such that $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$. The cubic equation may, however, have two equal roots, or three equal roots, or it may vanish identically; in this last case, any linear function $\alpha x + \beta y$ is in general idempotent. But (as will be considered in detail further on) we may, instead of an idempotent symbol or symbols, have a nilpotent symbol or symbols. It might be convenient to use the term Potency for a symbol which is in general idempotent, but which may be nilpotent. Writing $\frac{\alpha}{\beta} = \frac{-y}{x}$, we obtain a cubic equation $\Omega = (x, y)^3 = 0$, where obviously the linear factors of Ω are the just-mentioned functions ξ, η, ζ; that is, we have ξ, η, ζ as the linear factors of the cubic function

$$\Omega, = gx^3 + (h - c - e)x^2y + (a - d - f)xy^2 + by^3;$$

each such factor, except in the case where it is nilpotent, being determined so that it shall be idempotent. The cubic function of course vanishes identically if

$$g=0,\quad h-c-e=0,\quad a-d-f=0,\quad b=0.$$

6. Two extraordinaries $\xi, =\alpha x+\beta y$; $\eta, =\gamma x+\delta y$ (α, β, γ, δ coefficients); may be such that $\xi\eta=0$: this, of course, does not imply $\eta\xi=0$; we have, in fact,

$$\begin{aligned}\xi\eta &= (\alpha x+\beta y)(\gamma x+\delta y)\\ &= \alpha\gamma x^2+\alpha\delta xy+\beta\gamma yx+\beta\delta y^2\\ &= \alpha\gamma(ax+by)+\alpha\delta(cx+dy)+\beta\gamma(ex+fy)+\beta\delta(gx+hy)\\ &= (a\alpha\gamma+c\alpha\delta+e\beta\gamma+g\beta\delta)x+(b\alpha\gamma+d\alpha\delta+f\beta\gamma+h\beta\delta)y;\end{aligned}$$

and the required condition is satisfied if

$$\begin{aligned}a\alpha\gamma+c\alpha\delta+e\beta\gamma+g\beta\delta&=0,\\ b\alpha\gamma+d\alpha\delta+f\beta\gamma+h\beta\delta&=0.\end{aligned}$$

Writing these equations first under the form

$$\begin{aligned}\gamma(a\alpha+e\beta)+\delta(c\alpha+g\beta)&=0,\\ \gamma(b\alpha+f\beta)+\delta(d\alpha+h\beta)&=0,\end{aligned}$$

and then under the form

$$\begin{aligned}\alpha(a\gamma+c\delta)+\beta(e\gamma+g\delta)&=0,\\ \alpha(b\gamma+d\delta)+\beta(f\gamma+h\delta)&=0,\end{aligned}$$

we have

$$(a\alpha+e\beta)(d\alpha+h\beta)-(b\alpha+f\beta)(c\alpha+g\beta)=0,$$

a quadric equation for the determination of $\alpha:\beta$; and then

$$(a\gamma+c\delta)(f\gamma+h\delta)-(b\gamma+d\delta)(e\gamma+g\delta)=0,$$

a quadric equation for the determination of $\gamma:\delta$; that is, there are two values of the left-hand factor ξ, and two values of the right-hand factor η. But, of course, these correspond each to each, viz. either factor being given, the other factor is determined uniquely.

Writing successively $\frac{\alpha}{\beta}=\frac{-y}{x}$, and $\frac{\gamma}{\delta}=\frac{-y}{x}$, we have the quadric functions $(x, y)^2$,

$$\begin{aligned}\Phi &= (eh-fg)x^2+(-ah-de+bg+cf)xy+(ad-bc)y^2,\\ \Phi_1 &= (ch-dg)x^2+(-ah-cf+bg+de)xy+(af-be)y^2,\end{aligned}$$

where the linear factors of Φ are the two values ξ_1, ξ_2 of the left-hand factor ξ, and the linear factors of Φ_1 are the two values η_1, η_2 of the right-hand factor η.

7. In the commutative case, $c=e$, and $d=f$, we have

$$\Phi=\Phi_1=(ch-dg)x^2+(-ah+bg)xy+(ad-bc)y^2;$$

here $\xi\eta = 0$, $\eta\xi = 0$, and the values may be taken to be $(\xi_1, \eta_1) = (\xi, \eta)$, $(\xi_2, \eta_2) = (\eta, \xi)$, so that $\xi_1\xi_2 = \xi\eta$, $= \eta_1\eta_2$; that is, $\Phi = \Phi_1$, as above. The value of Ω is

$$\Omega = gx^3 + (h - 2c)\, x^2y + (a - 2d)\, xy^2 + by^3.$$

8. In the commutative and associative case, taking ξ, η, ζ to be the three idempotent symbols, $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$, we have

$$\xi(\eta - \xi\eta) = \xi\eta - \xi^2\eta = \xi\eta - \xi\eta = 0;$$

and in this manner we have the six equations

$$\xi(\eta - \xi\eta) = 0,\quad \xi(\zeta - \xi\zeta) = 0;\quad \eta(\xi - \eta\xi) = 0,\quad \eta(\zeta - \eta\zeta) = 0;\quad \zeta(\xi - \zeta\xi) = 0,\quad \zeta(\eta - \zeta\eta) = 0;$$

viz. regarding the right-hand factor as being in each case expressed as a linear function of x, y, we have apparently six products of two linear factors, each $= 0$. There is only one such product $\Phi = 0$; hence, disregarding coefficients, each of the six products must be $= \Phi$, or it must be identically $= 0$, viz. this will be the case if the second factor be $= 0$. We hence conclude that two of the symbols ξ, η, ζ, suppose ξ and η, must be factors of Φ, viz. Φ must be $= \xi\eta$. We have $\Omega = \xi\eta\zeta$; consequently $\Omega = \zeta\Phi$, that is, two of the three linear factors of Ω are the symbols ξ, η, which are such that $\xi\eta\,(= \eta\xi) = 0$. To complete the theory, observe that ζ must be a linear function of ξ, η, $= a\xi + b\eta$ suppose (a, b coefficients, neither of them $= 0$); we thence have

$$\zeta^2,\ = a^2\xi^2 + b^2\eta^2,\ = a^2\xi + b^2\eta,\ = \zeta,\ = a\xi + b\eta;$$

that is, $(a^2 - a)\,\xi + (b^2 - b)\,\eta = 0$; whence $a = 1$, $b = 1$, and therefore $\zeta = \xi + \eta$; hence also $\xi\zeta = \xi$ and $\eta\zeta = \eta$; $\zeta - \xi\zeta = \eta$, $\zeta - \eta\zeta = \xi$. The six products consequently are $\xi\eta$, $\xi\eta$, $\eta\xi$, $\eta\xi$, $\zeta 0$, $\zeta 0$, each $= \Phi$ or identically $= 0$.

9. In verification of the theorem that for the commutative and associative system the cubic function Ω contains the quadric function Φ as a factor, we may write, as above,

$$g = \frac{cd}{b},\quad h = \frac{d^2 + bc - ad}{b},$$

values which give

$$\begin{aligned} b\Phi &= (bc^2 - acd)\, x^2 + (-ad^2 - abc + a^2d + bcd)\, xy + (abd - b^2c)\, y^2 \\ &= -(ad - bc)\,\{cx^2 + (d - a)\, xy - by^2\}, \\ b\Omega &= cdx^3 + (d^2 - bc - ad)\, x^2y + (ab - 2bd)\, xy^2 + b^2y^3, \\ &= (dx - by)\,\{cx^2 + (d - a)\, xy - by^2\}, \end{aligned}$$

which gives the theorem in question. And observe further that

$$\begin{aligned} (dx - by)^2 &= d^2(ax + by) - 2bd\,(cx + dy) + b^2\left(\frac{cd}{b}\,x + \frac{d^2 + bc - ad}{b}\,y\right), \\ &= (ad - bc)\,(dx - by); \end{aligned}$$

that is, disregarding coefficients, the two idempotent symbols ξ, η are the linear factors of $cx^2 + (d - a)\, xy - by^2$, and the third idempotent symbol ζ is $= dx - by$.

10. Introducing coefficients in order to make the symbols ξ, η, ζ idempotent, and writing accordingly

$$\xi = \frac{1}{K}\{cx + \tfrac{1}{2}(d - a + \sqrt{\nabla})\,y\}, \quad \nabla = (d-a)^2 + 4bc,$$

so that

$$\eta = \frac{1}{L}\{cx + \tfrac{1}{2}(d - a - \sqrt{\nabla})\,y\}, \quad \xi\eta = \frac{c}{KL}\{cx^2 + (d-a)\,xy - by^2\},$$

$$\zeta = \frac{1}{P}(dx - by),$$

we have to verify that it is possible to determine K, L, P so that $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$, $\zeta = \xi + \eta$. The last equation gives

$$\frac{d}{P} = \frac{c}{K} + \frac{c}{L},$$

$$-\frac{2b}{P} = \frac{d - a + \sqrt{\nabla}}{K} + \frac{d - a - \sqrt{\nabla}}{L},$$

and we thence have

$$\frac{d\,(d-a) + 2bc - d\sqrt{\nabla}}{P} = -\frac{2c\sqrt{\nabla}}{K},$$

$$\frac{d\,(d-a) + 2bc + d\sqrt{\nabla}}{P} = \frac{2c\sqrt{\nabla}}{L},$$

and we can from the equation $\zeta^2 = \zeta$ find P; viz. comparing the coefficients of x, we have

$$\frac{d}{P} = \frac{1}{P^2}\left(ad^2 - 2b\,dc + b^2\frac{cd}{b}\right), \quad = \frac{d}{P^2}(ad - bc), \text{ that is, } P = (ad - bc),$$

or the values of K and L are

$$\frac{\{d\,(d-a) + 2bc - d\sqrt{\nabla}\}}{ad - bc} = -\frac{2c\sqrt{\nabla}}{K},$$

$$\frac{\{d\,(d-a) + 2bc + d\sqrt{\nabla}\}}{ad - bc} = +\frac{2c\sqrt{\nabla}}{L},$$

which should agree with the values of K and L found from the equations $\xi^2 = \xi$, $\eta^2 = \eta$, respectively. Comparing the coefficients of x, the first of these equations gives

$$\frac{c}{K} = \frac{1}{K^2}\left\{c^2a + c\,(d - a + \sqrt{\nabla})\,c + \tfrac{1}{4}(d - a + \sqrt{\nabla})^2\frac{cd}{b}\right\}$$

$$= \frac{c}{4bK^2}\left\{4abc + 4bc\,(d-a) + d\,\{(d-a)^2 + \nabla\} + 2\,\{2bc + d\,(d-a)\}\sqrt{\nabla}\right\}$$

$$= \frac{c}{2bK^2}\left\{d\nabla + \{d\,(d-a) + 2bc\}\sqrt{\nabla}\right\},$$

that is,

$$K = \frac{1}{2b}\sqrt{\nabla}\,\{d\,(d - a + 2bc) + d\sqrt{\nabla}\},$$

and the equation for K becomes

$$\frac{d(d-a)+2bc-d\sqrt{\nabla}}{ad-bc}=\frac{-4bc}{d(d-a)+2bc+d\sqrt{\nabla}};$$

that is,

$$\{d(d-a)+2bc\}^2-d^2\{(d-a)^2+4bc\}=-4bc(ad-bc),$$

which is right; and similarly the equation for L leads to this same equation.

11. We may now establish, on the principles appearing in No. 5, the different forms of the system. Using *idem* and *nil* as abbreviations for idempotent and nilpotent respectively, there are in all 11 cases.

(i) 3 idems. Taking two of these to be x and y, the system is

$$x^2=x,\quad xy=cx+dy,\quad yx=ex+fy,\quad y^2=y.$$

Hence $\Omega=(1-c-e)x^2y+(1-d-f)xy^2$, so that the third factor is

$$(1-c-e)x+(1-d-f)y.$$

This must not reduce itself to x or y, for, if so, there would be a twofold idem; viz. as negative conditions we must have $c+e\neq 1$, $d+f\neq 1$.

And we have

$$\{(1-c-e)x+(1-d-f)y\}^2=\{1-(c+e)(d+f)\}[(1-c-e)x+(1-d-f)y],$$

which must be an idem: viz. we have the further negative condition $(c+e)(d+f)\neq 1$.

(ii) 2 idems and 1 nil. This arises from (i) by assuming therein

$$(c+e)(d+f)=1, \text{ say } d+f=\frac{1}{c+e}:$$

for then, writing $z=-(c+e)x+y$, we have

$$\begin{aligned} z^2&=(c+e)^2x-(c+e)\{(c+e)x+(d+f)y\}+y,\\ &=\{1-(c+e)(d+f)\}y,\ =0; \end{aligned}$$

viz. z is a nil. And, if in the equations instead of the idem y we introduce the nil z, then the equations assume the form

$$x^2=x,\quad xz=[(c+e)d-e]x+dz,\quad zx=[(c+e)f-c]x+fz,\quad z^2=0;$$

with the idem $y=(c+e)x+z$: hence the negative conditions $c+e\neq 1$ or 0, implying $d+f\neq 1$.

But the equations are obtained in a more simple form by taking x for the idem and y for the nil; viz. we then have $x^2=x$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$: we must then have z, $=-(c+e)x+(1-d-f)y$, for an idem; this gives $z^2=-(c+e)(d+f)z$, and we have the negative conditions $c+e\neq 0$, $d+f\neq 0$ or 1.

(iii) 1 idem and 2 nils. This may be deduced from (ii) by writing therein $d+f=0$; for then $z, =-(c+e)x+y$, is a nil. The equations are $x^2=x$, $xy=cx+dy$, $yx=ex-dy$, $y^2=0$: and if, instead of x, we introduce therein z by the equation $z=-(c+e)x+y$, the equations become $y^2=0$, $yz=[-e+d(c+e)]y-ez$, $zy=[-c-d(c+e)]y+cz$, $z^2=0$, with the negative condition $c+e\neq 0$.

But it is more simple to take x, y as the nils: the equations then are $x^2=0$, $xy=cx+dy$, $yx=ex+fy$, $y^2=0$. We must have $z, =(c+e)x+(d+f)y$, an idem: this gives $z^2=(c+e)(d+f)z$; and we have the negative conditions $c+e\neq 0$, $d+f\neq 0$.

(*) We cannot have three nils. For in (iii), to make z a nil, we must have $c+e=0$ or $d+f=0$, and in the two cases respectively $z, =(c+e)x+(d+f)y$, becomes $=x$ and $=y$; so that x or y is a twofold nil. Or, what comes to the same thing, we have

$$\Omega=-(c+e)x^2y-(d+f)xy^2,$$

and Ω has a twofold factor if $c+e=0$ or $d+f=0$.

(iv) A twofold idem and a onefold idem. Taking x for the twofold idem and y for the onefold idem, Ω must reduce itself to $(1-c-e)x^2y$, viz. we must have $d+f=1$, or say $f=1-d$. The equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=y$; and we have the negative condition $c+e\neq 1$, for otherwise Ω would vanish identically.

(v) A twofold idem and a onefold nil. Taking x for the twofold idem and y for the onefold nil, then the equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=0$; and we have the negative condition $c+e\neq 0$.

(vi) A twofold nil and a onefold idem. Taking these to be x and y, then $d+f=0$, and the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=y$; and we have the negative condition $c+e\neq 1$.

(vii) A twofold nil and a onefold nil. Taking these to be x and y, we have $d+f=0$, and the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=0$; with the negative condition $c+e\neq 0$.

(viii) A threefold idem. Taking this to be x, then Ω must reduce itself to gx^3, viz. we must have $h=c+e$, $1=d+f$; and the equations are $x^2=x$, $xy=cx+dy$, $yx=ex+(1-d)y$, $y^2=gx+(c+e)y$; we have the negative condition $g\neq 0$, for otherwise Ω would vanish identically.

(ix) A threefold nil. Taking this to be x, then we must have $h=c+e$, $0=d+f$; the equations are $x^2=0$, $xy=cx+dy$, $yx=ex-dy$, $y^2=gx+(c+e)y$; and there is again the negative condition $g\neq 0$.

(x) $\Omega=0$ identically: infinity of idems, 1 nil. Ω will vanish identically if $g=0$, $h=c+e$, $a=d+f$, $b=0$. If there is 1 idem, there will be an infinity of idems, and 1 nil. For, assume an idem x, $x^2=x$; and, if possible, let there be no other idem; then there will be a nil y, $y^2=0$. We have $c+e=0$, $d+f=1$; and the equations

are $x^2 = x$, $xy = cx + dy$, $yx = -cx + (1-d)y$, $y^2 = 0$; whence $xy + yx = y$. Taking α, β arbitrary coefficients, we have

$$(\alpha x + \beta y)^2 = \alpha^2 x + \alpha\beta y, \; = \alpha(\alpha x + \beta y);$$

hence $\alpha x + \beta y$ is an idem, except in the case $\alpha = 0$, when it is the original nil y.

If besides the idem x we have an idem y, then the conditions are $c + e = 1$, $d + f = 1$: the equations are

$$x^2 = x, \quad xy = cx + dy, \quad yx = (1-c)x + (1-d)y, \quad y^2 = y;$$

whence $xy + yx = x + y$. Considering the combination $\alpha x + \beta y$, we have

$$(\alpha x + \beta y)^2 = \alpha^2 x + \alpha\beta(x+y) + \beta^2 y, \quad = (\alpha + \beta)(\alpha x + \beta y).$$

This is an idem, except in the case $\alpha + \beta = 0$, when it is a nil; or, say we have the single nil $x - y$. We have thus again an infinity of idems, 1 nil.

(xi) $\Omega = 0$ identically; an infinity of nils. Taking the two nils x and y, the conditions are $c + e = 0$, $d + f = 0$; the equations are $x^2 = 0$, $xy = cx + dy$, $yx = -cx - dy$, $y^2 = 0$; whence $xy + yx = 0$. Considering the arbitrary combination $\alpha x + \beta y$, we have

$$(\alpha x + \beta y)^2 = \alpha\beta(xy + yx), \; = 0,$$

viz. $\alpha x + \beta y$ is a nil; or there are an infinity of nils.

12. The different cases may be grouped together as follows:—

A. 2 idems, (i), (ii), (iv), (x).

Equations $x^2 = x$, $xy = cx + dy$, $yx = ex + fy$, $y^2 = y$.

B. 1 idem and 1 nil, (ii), (iii), (v), (vi), (x).

Equations $x^2 = x$, $xy = cx + dy$, $yx = ex + fy$, $y^2 = 0$.

C. 2 nils, (iii), (vii), (xi).

Equations $x^2 = 0$, $xy = cx + dy$, $yx = ex + fy$, $y^2 = 0$.

D. Threefold idem, (viii).

Equations $x^2 = x$, $xy = cx + dy$, $yx = ex - (1-d)y$, $y^2 = gx + (c+e)y$.

E. Threefold nil, (ix).

Equations $x^2 = 0$, $xy = cx + dy$, $yx = ex - dy$, $y^2 = gx + (c+e)y$.

The several cases of A, B, C respectively are distinguished by negative conditions which need not be here repeated.

13. I consider, as in my Note before referred to, the conditions in order that the system may be associative. We have the 8 products, x^3, x^2y, xyx, xy^2, yx^2, y^2x, yxy, y^3, giving rise to equations $x.x^2 = x^2.x$, $x.xy = x^2.y$, ..., $y.y^2 = y^2.y$, which, on putting therein for x^2, xy, yx, y^2 their values, must be satisfied identically. We thus obtain in

the first instance 16 relations, but some of these are repeated, and we have actually only 12 relations; viz. the relations are

$$\begin{array}{ll}
\text{(twice)} & b\,(c-e)=0,\\
 & b\,(f-d)=0,\\
 & g\,(c-e)=0,\\
\text{(twice)} & g\,(f-d)=0,\\
\text{(twice)} & bg-cd=0,\\
\text{(twice)} & bg-ef=0,\\
 & c\,(c-h)+g\,(d-a)=0,\\
 & d\,(d-a)+b\,(c-h)=0,\\
 & e\,(e-h)+g\,(f-a)=0,\\
 & b\,(e-h)+f\,(f-a)=0,\\
 & a\,(c-e)-cf+de=0,\\
 & h\,(f-d)-cf+de=0.
\end{array}$$

14. From the first four equations it appears that either $b=0$ and $g=0$, or else $c=e$ and $d=f$. I attend first to the latter case, viz. we have here the commutative system

$$x^2=ax+by,\quad xy=yx=cx+dy,\quad y^2=gx+hy.$$

In order that this may be associative, we must still have the relations

$$\begin{aligned}
bg-cd&=0,\\
c\,(c-h)+g\,(d-a)&=0,\\
d\,(d-a)+b\,(c-h)&=0,
\end{aligned}$$

or, as they may be written,

$$\begin{vmatrix} b, & -c, & d-a \\ -d, & g, & c-h \end{vmatrix}=0.$$

These are satisfied by $g=\dfrac{cd}{b}$, $h=\dfrac{d^2+bc-ad}{b}$, and we have thus the commutative and associative system of the Note.

Every system is of the form A, B, C, D, or E; and it can be shown that the commutative and associative system is not of the form D. For, if D were commutative, we should have $e=c$, $d=\frac{1}{2}$, viz. the equations will be $x^2=x$, $xy=yx=cx+\frac{1}{2}y$, $y^2=gx+2cy$, that is,

$$a,\ b,\ c,\ d,\ g,\ h=1,\ 0,\ c,\ \tfrac{1}{2},\ g,\ 2c;$$

and the last of the three relations, viz. $d\,(d-a)+b\,(c-h)=0$, would thus be $\frac{1}{2}(\frac{1}{2}-1)=0$, which is not satisfied. Hence the commutative and associative system can only be of one of the forms A, B, C, and E.

15. First, if the form be A, B, or C; there will be the two idem-or-nil symbols x and y, that is, we may assume $b=0$, $g=0$; and the associative conditions then become $cd=0$, $c(c-h)=0$, $d(d-a)=0$, viz. for the forms A, B, C,

$$\text{A.}\quad x^2=x,\quad y^2=y,$$
$$\text{B.}\quad x^2=x,\quad y^2=0,$$
$$\text{C.}\quad x^2=0,\quad y^2=0,$$

$$\begin{array}{llllll}
\text{these are} & cd=0, & c(c-1)=0, & d(d-1)=0; & c=0 \text{ or } 1, & d=0 \text{ or } 1, \\
\text{,,} & cd=0, & c^2=0, & d(d-1)=0; & c=0, & d=0 \text{ or } 1, \\
\text{,,} & cd=0, & c^2=0, & d^2=0; & c=0, & d=0.
\end{array}$$

But for the form A, if $c=0$, $d=1$, that is, $xy=yx=y$, then, writing $z=x-y$, we have $z^2=z$, $yz=zy=0$, $y^2=y$. And similarly, if $c=1$, $d=0$, that is, $xy=yx=x$, then, writing $z=-x+y$, we have $z^2=z$, $zx=xz=0$, $x^2=x$. That is, each of these is reduced to the first case $c=0$, $d=0$; that is, $x^2=x$, $xy=yx=0$, $y^2=y$.

For the form B, if $c=0$, $d=1$, then the system is $x^2=x$, $xy=yx=y$, $y^2=0$; and this cannot be reduced to the first case $x^2=x$, $xy=yx=0$, $y^2=0$.

For the form C, there is only one case, as above.

For the form E, we have $a=0$, $b=0$, ($c=e$, $d=0$, in order that the system may be commutative), $h=2c$, viz. the equations must be $x^2=0$, $xy=yx=cx$, $y^2=gx+2cy$. The associative conditions then give $c=0$; or, the system is $x^2=0$, $xy=yx=0$, $y^2=gx$. Writing $\frac{x}{g}$ instead of x, and for convenience interchanging x and y, the equations are $x^2=y$, $xy=yx=0$, $y^2=0$.

16. The commutative associative system is thus seen to be reducible as follows:—

A. system is $x^2=x$, $xy=yx=0$, $y^2=y$, first mixed system, see No. 2.

B. ,, $x^2=x$, $xy=yx=y$, $y^2=0$, Peirce's system (a_2),

or else

B. ,, $x^2=x$, $xy=yx=0$, $y^2=0$, second mixed system.

C. ,, $x^2=0$, $xy=yx=0$, $y^2=0$, third mixed system.

E. ,, $x^2=y$, $xy=yx=0$, $y^2=0$, Peirce's system (c_2).

I said, at the end of my Note before referred to, that it had been pointed out to me "that my system [the commutative associative system], in the general case $ad-bc$ not $=0$, is expressible as a mixture of two algebras of the form (a_1), see *American Journal of Mathematics*, vol. IV., p. 120; whereas, if $ad-bc=0$, it is reducible to the form (c_2), see p. 122 (*l.c.*)." The accurate conclusion is as above, that the commutative associative system is either a mixed system of one of the three forms, or else a system (a_2), or (c_2).

17. Considering next the non-commutative associative systems, we have here, *ante*, No. 14, $b=0$, $g=0$; and the relations which remain to be satisfied then are

$$cd=0,\ ef=0,\ c(c-h)=0,\ d(d-a)=0,\ e(e-h)=0,\ f(f-a)=0,$$
$$a(c-e)-cf+de=0,\ h(f-d)-cf+de=0.$$

The first equation gives $cd=0$, that is, $c=0$ or $d=0$; but we may attend exclusively to the case $c=0$, for the case $d=0$ may be deduced from this by the interchange of x, y. We have then $ef=0$; and it will be convenient to separate the cases

I. $c=0,\ e=0,\ f=0$, giving $d(d-a)=0,\quad dh=0,$

II. $c=0,\ e=0,\ f\neq 0$, „ $d(d-a)=0,\ f-a=0,\ h(f-d)=0,$

III. $c=0,\ e\neq 0,\ f=0$, „ $d(d-a)=0,\ e-h=0,\quad (d-a)=0,\ d(e-h)=0,$

that is, $d-a=0,\ e-h=0.$

18. We have thus five cases;

I. (*a*) $d=0$: $x^2=ax,\ xy=yx=0,\ y^2=hy$: commutative, and so included in what precedes.

I. (*b*) $d=a,\ h=0$: $x^2=ax,\ xy=ay,\ yx=0,\ y^2=0$: or, writing as we may do $a=1$, this is $x^2=x,\ xy=y,\ yx=0,\ y^2=0$; which is Peirce's system (b_2).

II. (*c*) $d=f=a$: $x^2=ax,\ xy=yx=ay,\ y^2=hy$: commutative, and so included in what precedes.

II. (*d*) $d=0,\ f=a,\ h=0$: $x^2=ax,\ xy=0,\ yx=ay,\ y^2=0$; or, writing as we may do $a=1$, this is $x^2=x,\ xy=0,\ yx=y,\ y^2=0$; which is the system (d_2).

III. (*e*) $d=a,\ e=h$: $x^2=ax,\ xy=ay,\ yx=hx,\ y^2=hy$; or, writing as we may do $a=1,\ h=1$, this is $x^2=x,\ xy=y;\ yx=x;\ y^2=y$. Introducing here the new symbol $z,\ =x-y$, we have $z^2=0,\ xz=z,\ zx=0,\ yz=z,\ zy=0$. Thus x, z form the system $x^2=x,\ xz=z,\ zx=0,\ z^2=0$ (or, what is the same thing, y, z form a system $y^2=y,\ yz=z,\ zy=0,\ z^2=0$); each of these is Peirce's system (b_2).

The conclusion is that every non-commutative associative system is either Peirce's system (b_2), or else the omitted system (d_2). Hence, disregarding the mixed systems, every associative system is either (a_2), (b_2), (c_2), or (d_2).

19. It may be proper to show that the systems (b_2), $x^2=x,\ xy=y,\ yx=0,\ y^2=0$, and (d_2), $x^2=x,\ xy=0,\ yx=y,\ y^2=0$, or say

	a	b	c	d	e	f	g	h
(b_2)	1	0	0	1	0	0	0	0
(d_2)	1	0	0	0	0	1	0	0,

are really distinct from each other. Observe that they each belong to the case (x), $\Omega=0$, an infinity of idems and 1 nil; viz. in each of them writing $z=x+\beta y$, β an arbitrary coefficient, we have $z^2=x^2+\beta(xy+yx)+\beta^2y^2,\ =x+\beta y,\ =z$, we have z an idem, and y is the only nil. And, this being so, we have in the first system $zy=y,\ yz=0$, viz. the system is $z^2=z,\ zy=y,\ yz=0,\ y^2=0$, retaining, when we write z for x, its original form. And similarly, in the second system, $zy=0,\ yz=y$; viz. the system is $z^2=z,\ zy=0,\ yz=y,\ y^2=0$, retaining, when we write therein z for x, its original form. The two are thus distinct systems, in no wise transformable the one into the other.

815.

THE BINOMIAL EQUATION $x^p - 1 = 0$; QUINQUISECTION. SECOND PART.

[From the *Proceedings of the London Mathematical Society*, vol. XVI. (1885), pp. 61—63.]

IN the paper, "The binomial equation $x^p - 1 = 0$; quinquisection," *Proc. Lond. Math. Soc.*, t. XII. (1881), pp. 15, 16, [764], I considered for an exponent $p = 5n + 1$, the five periods X, Y, Z, W, T connected by the equations

$$\begin{array}{rcccccc} & X, & Y, & Z, & W, & T \\ \hline X^2 = & a, & b, & c, & d, & e \\ XY = & f, & g, & h, & i, & j \\ XZ = & k, & l, & m, & n, & o, \end{array}$$

and the equations deduced from these by cyclical permutations of the periods and of the coefficients of each set; but I did not obtain completely even the linear relations connecting the coefficients. I since found, by induction from the examples given in the Table 1, that the coefficients could be expressed linearly in terms of the linearly independent integer numbers α, β, f, k as follows: viz. introducing for convenience the new number θ, such that

$$\alpha + \beta + \theta = \tfrac{1}{5}(p-1),$$

then the expressions in question are

$$\begin{array}{lccccc} a, b, c, d, e = & -1-2\theta+\alpha+\beta, & -\theta-\alpha-\beta+f, & -\theta-\alpha-\beta+k, & -\alpha-2\beta-k, & -2\alpha-\beta-f, \\ f, g, h, i, j = & f, & \theta-\alpha-f, & \alpha, & \beta, & \alpha, \\ k, l, m, n, o = & k, & \alpha, & \theta-\beta-k, & \beta, & \beta, \end{array}$$

and I found further that, substituting these values of the coefficients in the 20 quadric relations referred to in the former paper, the 20 relations reduced themselves to two equations only, viz. these were

$$\theta(-2\alpha+\beta+k) + 3\alpha^2 - \beta^2 + \alpha(f-k-1) - \beta f + f^2 - 2fk = 0,$$
$$-\theta^2 + \theta(3\beta+2k+f) + \alpha^2 - \alpha\beta - 3\beta^2 - \alpha k + \beta(1-f-k) - k^2 - 2fk = 0.$$

The final result thus is that the coefficients are expressed as functions of the five numbers α, β, f, k, θ, connected by the linear equation $\alpha + \beta + \theta = \frac{1}{5}(p-1)$, and the two quadric equations. I remark that formulæ equivalent to these were obtained and proved by Mr F. S. Carey in his Trinity Fellowship Dissertation, 1884; viz. writing $n = \frac{1}{5}(p-1)$, his formulæ were

$$\begin{array}{llllll} a, b, c, d, e = & \alpha - n, & \beta - n, & \gamma - n, & \delta - n, & \epsilon - n, \\ f, g, h, i, j = & \beta, & \epsilon, & \rho, & \sigma, & \rho, \\ k, l, m, n, o = & \gamma, & \rho, & \delta, & \sigma, & \sigma, \end{array}$$

with the three linear relations

$$\begin{aligned} \alpha + \beta + \gamma + \delta + \epsilon &= n - 1, \\ \beta + \epsilon + 2\rho + \sigma &= n, \\ \gamma + \delta + \rho + 2\sigma &= n, \end{aligned}$$

and the two quadric relations

$$\begin{aligned} \delta^2 + \gamma^2 + 2\sigma\alpha + (\rho - \sigma)(\delta + \gamma) - 2\rho(\rho + \sigma) &= (\delta - \gamma)(\beta - \epsilon), \\ \beta^2 + \epsilon^2 + 2\rho\alpha + (\sigma - \rho)(\beta + \epsilon) - 2\sigma(\rho + \sigma) &= (\gamma - \delta)(\beta - \epsilon), \end{aligned}$$

the coefficients being thus expressed in terms of the seven numbers α, β, γ, δ, ϵ, ρ, σ connected by five equations. The equivalence of the two sets of formulæ may be shown without difficulty.

To the Table 2 of the Quintic Equations, given in the paper, may be added the following result from Legendre's *Théorie des Nombres*, Ed. 3, t. II., p. 213,

p	η^5	η^4	η^3	η^2	η	1
641	1	+ 1	− 256	− 564	+ 5238	− 5120

$= 0$,

calculated by him for the isolated case $p = 641$.

816.

ON THE BITANGENTS OF A PLANE QUARTIC.

[From *Crelle's Journal der Mathem.*, t. XCIV. (1883), pp. 93—115; *Camb. Phil. Soc. Proc.*, t. IV. (1883), p. 321.]

RIEMANN in the paper "Zur Theorie der *Abel*schen Functionen für den Fall $p=3$," *Werke*, pp. 456—479, has given a remarkably elegant solution of the problem of the bitangents of a plane quartic. But his formulæ may be improved by a slight change; viz. we may in his first equation $x+y+z+\xi+\eta+\zeta=0$ introduce coefficients so as to bring this into the same form as the other three equations. It thus appears that, instead of his $3+3$ equations of the forms

$$\frac{x}{1-\beta\gamma}+\frac{y}{1-\gamma\alpha}+\frac{z}{1-\alpha\beta}=0 \text{ and } \frac{\xi}{\alpha(\gamma-\beta)}+\frac{\eta}{\beta(\gamma-\alpha)}+\frac{\zeta}{\gamma(\beta-\alpha)}=0$$

respectively, we have $6+6$ equations of like forms; but these two systems each of 6 equations are equivalent to each other, so that instead of $6+16+3+3=28$, we have $6+16+6(=6)=28$ equations for the 28 bitangents*. I make another slight change of notation by introducing the single letters f, g, h to denote the reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$; and I consider the whole question as follows. The theory is based on the equations

$$\begin{aligned}
a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta&=0,\\
a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\\
a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta&=0,\\
a_3x+b_3y+c_3z+f_3\xi+g_3\eta+h_3\zeta&=0,
\end{aligned}$$

where $af=bg=ch=a_1f_1=\&c.=c_3h_3=1$: and the coefficients a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 are arbitrary. The equations $x=0$, $y=0$, $z=0$ represent any three given lines: and

* It will appear further on that the equation of each of the last-mentioned 6 bitangents can be expressed in 8 different forms.

considering ξ, η, ζ to be determined as linear functions of x, y, z by means of the first three equations, then (the coefficients a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 being determined accordingly) the equations $\xi=0$, $\eta=0$, $\zeta=0$ will also represent any three given lines. But observe that, if the equation of the first of these lines is $x+my+nz=0$, ξ is not = an arbitrary multiple $\theta(x+my+nz)$ of the linear function $x+my+nz$, but the constant factor θ has a completely determinate value: and the like as regards η and ζ respectively.

The coefficients a_3, b_3, c_3 of the fourth equation are such that this fourth equation is a mere consequence of the other three: viz. we must have

$$\begin{aligned} a_3 &= \lambda a + \lambda_1 a_1 + \lambda_2 a_2, \\ b_3 &= \lambda b + \lambda_1 b_1 + \lambda_2 b_2, \\ c_3 &= \lambda c + \lambda_1 c_1 + \lambda_2 c_2, \\ f_3 &= \lambda f + \lambda_1 f_1 + \lambda_2 f_2, \\ g_3 &= \lambda g + \lambda_1 g_1 + \lambda_2 g_2, \\ h_3 &= \lambda h + \lambda_1 h_1 + \lambda_2 h_2, \end{aligned}$$

or, what is the same thing, we must have

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \end{matrix} \right\| = 0,$$

viz. each of the determinants formed with four columns out of this matrix is equal to 0.

Using the equations in λ, λ_1, λ_2, and writing for shortness

$$\begin{aligned} a_1 f_2 + a_2 f_1, \quad & a_2 f + a f_2, \quad a f_1 + a_1 f = F, \ F_1, \ F_2, \\ b_1 g_2 + b_2 g_1, \quad & b_2 g + b g_2, \quad b g_1 + b_1 g = G, \ G_1, \ G_2, \\ c_1 h_2 + c_2 h_1, \quad & c_2 h + c h_2, \quad c h_1 + c_1 h = H, \ H_1, \ H_2; \end{aligned}$$

then, forming the products $a_3 f_3$, $b_3 g_3$, $c_3 h_3$, we find

$$\begin{aligned} 1-\lambda^2-\lambda_1^2-\lambda_2^2 &= F\lambda_1\lambda_2 + F_1\lambda_2\lambda + F_2\lambda\lambda_1, \\ \text{,,} \qquad &= G\lambda_1\lambda_2 + G_1\lambda_2\lambda + G_2\lambda\lambda_1, \\ \text{,,} \qquad &= H\lambda_1\lambda_2 + H_1\lambda_2\lambda + H_2\lambda\lambda_1, \end{aligned}$$

or, as these equations may be written,

$$\begin{aligned} & 1-\lambda^2-\lambda_1^2-\lambda_2^2 \ : \ \lambda_1\lambda_2 \ : \ \lambda_2\lambda \ : \ \lambda\lambda_1 \\ = & \begin{vmatrix} F, & F_1, & F_2 \\ G, & G_1, & G_2 \\ H, & H_1, & H_2 \end{vmatrix} : \begin{vmatrix} 1, & F_1, & F_2 \\ 1, & G_1, & G_2 \\ 1, & H_1, & H_2 \end{vmatrix} : \begin{vmatrix} 1, & F_2, & F \\ 1, & G_2, & G \\ 1, & H_2, & H \end{vmatrix} : \begin{vmatrix} 1, & F, & F_1 \\ 1, & G, & G_1 \\ 1, & H, & H_1 \end{vmatrix}, \end{aligned}$$

say for shortness

$$= \Pi : \Lambda : \Lambda_1 : \Lambda_2.$$

We have thus

$$\frac{\lambda^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda_1\Lambda_2}{\Pi\Lambda},$$

$$\frac{\lambda_1^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda_2\Lambda}{\Pi\Lambda_1},$$

$$\frac{\lambda_2^2}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{\Lambda\Lambda_1}{\Pi\Lambda_2},$$

and thence

$$\frac{1}{1-\lambda^2-\lambda_1^2-\lambda_2^2} = \frac{1}{\Pi}\left(\frac{\Lambda_1\Lambda_2}{\Lambda} + \frac{\Lambda_2\Lambda}{\Lambda_1} + \frac{\Lambda\Lambda_1}{\Lambda_2}\right),$$

equations which give λ, λ_1, λ_2, and consequently a_3, b_3, c_3, f_3, g_3, h_3 in terms of known quantities, the several expressions depending on the single radical

$$\sqrt{\frac{1}{\Pi}\left(\frac{\Lambda_1\Lambda_2}{\Lambda} + \frac{\Lambda_2\Lambda}{\Lambda_1} + \frac{\Lambda\Lambda_1}{\Lambda_2}\right)}.$$

Observe that we have rationally

$$\lambda : \lambda_1 : \lambda_2 = \frac{1}{\Lambda} : \frac{1}{\Lambda_1} : \frac{1}{\Lambda_2}.$$

I consider now the quartic curve

$$\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0,$$

and I write down the equations of the 28 bitangents, each with its triple-theta characteristic as given by Riemann, and also with its duad symbol, derived from Hesse's method. I assume that these characteristics and duad symbols are properly attached to the several lines: and I insert also the current nos. 1 to 28. The equations are:

Current No.	Duad Symbol	Characteristic	
1	18	111 111	$x=0$,
2	28	001 011	$y=0$,
3	38	011 001	$z=0$,
4	23	010 010	$\xi=0$,
5	13	100 110	$\eta=0$,
6	12	110 100	$\zeta=0$,

Current No.	Duad Symbol	Characteristic	
7	48	101 100	$ax+by+cz=0,\quad f\xi+g\eta+h\zeta=0,$
8	14	010 011	$f\xi+by+cz=0,\quad ax+g\eta+h\zeta=0,$
9	24	100 111	$ax+g\eta+cz=0,\quad f\xi+by+h\zeta=0,$
10	34	110 101	$ax+by+h\zeta=0,\quad f\xi+g\eta+cz=0,$
11	58	100 101	$a_1x+b_1y+c_1z=0,$
12	15	011 010	$f_1\xi+b_1y+c_1z=0,$
13	25	101 110	$a_1x+g_1\eta+c_1z=0,$
14	35	111 100	$a_1x+b_1y+h_1\zeta=0,$
15	68	110 010	$a_2x+b_2y+c_2z=0,$
16	16	001 101	$f_2\xi+b_2y+c_2z=0,$
17	26	111 001	$a_2x+g_2\eta+c_2z=0,$
18	36	101 011	$a_2x+b_2y+h_2\zeta=0,$
19	78	010 110	$a_3x+b_3y+c_3z=0,$
20	17	101 001	$f_3\xi+b_3y+c_3z=0,$
21	27	011 101	$a_3x+g_3\eta+c_3z=0,$
22	37	000 111	$a_3x+b_3y+h_3\zeta=0,$
23	67	100 100	$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0,\quad \frac{\xi}{g_2h_2-g_3h_3}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{\zeta}{f_2g_2-f_3g_3}=0,$ $\frac{x}{bc-b_1c_1}+\frac{\eta}{bb_1(ca_1-c_1a)}-\frac{\zeta}{cc_1(ab_1-a_1b)}=0,\quad \frac{\xi}{g_2h_2-g_3h_3}+\frac{y}{g_2g_3(h_2f_3-h_3f_2)}-\frac{z}{h_2h_3(f_2g_3-f_3g_2)}=0,$ $\frac{-\xi}{aa_1(bc_1-b_1c)}+\frac{y}{ca-c_1a_1}+\frac{\zeta}{cc_1(ab_1-a_1b)}=0,\quad \frac{-x}{f_2f_3(g_2h_3-g_3h_2)}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{z}{h_2h_3(f_2g_3-f_3g_2)}=0,$ $\frac{\xi}{aa_1(bc_1-b_1c)}-\frac{\eta}{bb_1(ca_1-c_1a)}+\frac{z}{ab-a_1b_1}=0,\quad \frac{x}{f_2f_3(g_2h_3-g_3h_2)}-\frac{y}{g_2g_3(h_2f_3-h_3f_2)}+\frac{\zeta}{f_2g_2-f_3g_3}=0,$
24	57	110 011	$\frac{x}{bc-b_2c_2}+\frac{y}{ca-c_2a_2}+\frac{z}{ab-a_2b_2}=0,$
25	56	010 111	$\frac{x}{bc-b_3c_3}+\frac{y}{ca-c_3a_3}+\frac{z}{ab-a_3b_3}=0,$
26	45	001 001	$\frac{x}{b_3c_3-b_3c_3}+\frac{y}{c_2a_2-c_3a_3}+\frac{z}{a_2b_2-a_3b_3}=0,$
27	46	011 110	$\frac{x}{b_3c_3-b_1c_1}+\frac{y}{c_3a_3-c_1a_1}+\frac{z}{a_3b_3-a_1b_1}=0,$
28	47	111 010	$\frac{x}{b_1c_1-b_2c_2}+\frac{y}{c_1a_1-c_2a_2}+\frac{z}{a_1b_1-a_2b_2}=0.$

As regards each of the bitangents 7 to 22, the equivalence of the two forms of equation is at once seen by means of the fundamental equations $ax+by+cz+f\xi+g\eta+h\zeta=0$: as regards the remaining bitangents 23 to 28, the equation of each of these is expressible in eight different forms, which are written down in full for the bitangent 23; the equivalence of the eight forms of equation must of course be proved.

I proceed to show that each of the 28 lines is, in fact, a bitangent to the quartic curve

$$\sqrt{x\xi}+\sqrt{y\eta}+\sqrt{z\zeta}=0,$$

or, what is the same thing, that writing this equation in the form

$$\Omega=x^2\xi^2+y^2\eta^2+z^2\zeta^2-2yz\eta\zeta-2zx\zeta\xi-2xy\xi\eta=0,$$

then that by means of any one of these equations Ω becomes a perfect square.

This is obviously the case for each of the equations 1, 2, 3, 4, 5, 6.

For the equations 7, 8, 9, 10, observe that the equation of the quartic may be written:

$$\sqrt{axf\xi}+\sqrt{byg\eta}+\sqrt{czh\zeta}=0,$$

or, what is the same thing,

$$\Omega=a^2x^2.f^2\xi^2+\ldots=0.$$

Hence writing x, y, z, ξ, η, ζ in place of ax, by, cz, $f\xi$, $g\eta$, $h\zeta$, so that now

$$x+y+z+\xi+\eta+\zeta=0,$$

the equation of the line 7 is expressed in the two equivalent forms $x+y+z=0$, $\xi+\eta+\zeta=0$. We have for Ω its original value

$$\begin{aligned}\Omega&=x^2\xi^2+y^2\eta^2+z^2\zeta^2-2yz\eta\zeta-2zx\zeta\xi-2xy\xi\eta,\\&=(z\zeta-x\xi-y\eta)^2-4xy\xi\eta;\end{aligned}$$

and writing herein $z=-x-y$, $\zeta=-\xi-\eta$, we have $z\zeta-x\xi-y\eta=x\eta+y\xi$ and consequently $\Omega=(x\eta+y\xi)^2-4xy\xi\eta$, $=(x\eta-y\xi)^2$, a perfect square. The like proof applies to the equations 8, 9, 10: and then clearly the result applies to the equations 11, 12, 13, 14; 15, 16, 17, 18; 19, 20, 21, 22.

We come now to the equation of the line 23, say this is in the first instance taken to be

$$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0;$$

if from this equation, by means of the two fundamental equations

$$\begin{aligned}ax+by+cz+f\xi+g\eta+h\zeta&=0,\\a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\end{aligned}$$

we eliminate first (y, z), secondly (z, x), and thirdly (x, y), it is found that ξ, η, ζ will also disappear in the three cases respectively, and that the resulting equations are

$$\frac{x}{bc-b_1c_1}+\frac{\eta}{bb_1(ca_1-c_1a)}-\frac{\zeta}{cc_1(ab_1-a_1b)}=0,$$

$$-\frac{\xi}{aa_1(bc_1-b_1c)}+\frac{y}{ca-c_1a_1}+\frac{\zeta}{cc_1(ab_1-a_1b)}=0,$$

$$\frac{\xi}{aa_1(bc_1-b_1c)}-\frac{\eta}{bb_1(ca_1-c_1a)}+\frac{z}{ab-a_1b_1}=0,$$

so that each of these is, in fact, equivalent to the original equation in (x, y, z): and observe that, adding together the three equations, we reproduce the original equation in (x, y, z).

Write for shortness

$$P=bc-b_1c_1,\quad \alpha=bc_1-b_1c,$$
$$Q=ca-c_1a_1,\quad \beta=ca_1-c_1a,$$
$$R=ab-a_1b_1,\quad \gamma=ab_1-a_1b,$$

then the equation in (x, y, z) is

$$QRx+RPy+PQz=0,$$

so that, eliminating y and z, we find

$$\begin{vmatrix} b\,, & c\,, & ax+f\xi+g\eta+h\zeta \\ b_1\,, & c_1\,, & a_1x+f_1\xi+g_1\eta+h_1\zeta \\ RP, & PQ, & QRx \end{vmatrix}=0.$$

In this equation, the coefficient of x is

$$\begin{aligned}
&(bc_1-b_1c)QR+(ca_1-c_1a)RP+(ab_1-a_1b)PQ,\\
=\;&(bc_1-b_1c)\{a^2bc+a_1^2b_1c_1-aa_1(bc_1+b_1c)\}\\
&+(ca_1-c_1a)\{ab^2c+a_1b_1^2c_1-bb_1(ca_1+c_1a)\}\\
&+(ab_1-a_1b)\{abc^2+a_1b_1c_1^2-cc_1(ab_1+a_1b)\},\\
=&-aa_1(b^2c_1^2-b_1^2c^2)-bb_1(c^2a_1^2-c_1^2a^2)-cc_1(a^2b_1^2-a_1^2b^2),\\
=&-(bc_1-b_1c)(ca_1-c_1a)(ab_1-a_1b),\\
=&-\alpha\beta\gamma.
\end{aligned}$$

The coefficient of ξ is

$$=RP(cf_1-c_1f)-PQ(bf_1-b_1f),$$

which $\left(\text{observing that } cf_1-c_1f, =\frac{c}{a_1}-\frac{c_1}{a}, \text{ is } =\frac{1}{aa_1}Q, \text{ and } bf_1-b_1f, =\frac{b}{a_1}-\frac{b_1}{a}, \text{ is } =\frac{1}{aa_1}R\right)$ is $=0$.

The coefficient of η is

$$RP(cg_1-c_1g)-PQ(bg_1-b_1g),$$

which is

$$\begin{aligned}
&= RP\frac{1}{bb_1}(bc-b_1c_1)-PQ\frac{1}{bb_1}(b^2-b_1^2),\\
&= \frac{P}{bb_1}\{(bc-b_1c_1)(ab-a_1b_1)-(b^2-b_1^2)(ca-c_1a_1)\},\\
&= \frac{P}{bb_1}\{b_1^2ca+b^2c_1a_1-bb_1(ac_1+a_1c)\}\\
&= -\frac{P}{bb_1}\gamma\alpha;
\end{aligned}$$

and similarly the coefficient of ζ is

$$= \frac{P}{cc_1}\alpha\beta.$$

We have thus in x, η, ζ the equation

$$-\alpha\beta\gamma x-\frac{P}{bb_1}\gamma\alpha\eta+\frac{P}{cc_1}\alpha\beta\zeta=0,$$

or, what is the same thing,

$$\frac{x}{P}+\frac{\eta}{bb_1\beta}-\frac{\zeta}{cc_1\gamma}=0:$$

and in like manner, by the elimination of (z, x) and (x, y) respectively,

$$-\frac{\xi}{aa_1\alpha}+\frac{y}{Q}+\frac{\zeta}{cc_1\gamma}=0,$$

$$\frac{\xi}{aa_1\alpha}-\frac{\eta}{bb_1\beta}+\frac{z}{R}=0,$$

which are the required three equations.

We have next to show that the line is a bitangent—write for a moment $aa_1\alpha$, $bb_1\beta$, $cc_1\gamma=\lambda$, μ, ν, then the three equations give

$$x,\ y,\ z=P\left(-\frac{\eta}{\mu}+\frac{\zeta}{\nu}\right),\quad Q\left(-\frac{\zeta}{\nu}+\frac{\xi}{\lambda}\right),\quad R\left(-\frac{\xi}{\lambda}+\frac{\eta}{\mu}\right),$$

and we thence have

$$\begin{aligned}
\Omega = \quad & P^2\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right)^2\\
&+Q^2\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right)^2\\
&+R^2\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right)^2\\
&-2QR\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right)\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right)\\
&-2RP\left(-\frac{\zeta\xi}{\lambda}+\frac{\eta\zeta}{\mu}\right)\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right)\\
&-2PQ\left(-\frac{\xi\eta}{\mu}+\frac{\zeta\xi}{\nu}\right)\left(-\frac{\eta\zeta}{\nu}+\frac{\xi\eta}{\lambda}\right);
\end{aligned}$$

which, putting further $P\lambda$, $Q\mu$, $R\nu = \mathrm{a}$, b, c, becomes

$$\begin{aligned}\Omega = \ & \frac{\eta^2\zeta^2}{\mu^2\nu^2}(\mathrm{b}+\mathrm{c})^2 \\ & + \frac{\zeta^2\xi^2}{\nu^2\lambda^2}(\mathrm{c}+\mathrm{a})^2 \\ & + \frac{\xi^2\eta^2}{\lambda^2\mu^2}(\mathrm{a}+\mathrm{b})^2 \\ & + \frac{2\xi^2\eta\zeta}{\lambda^2\mu\nu}\{-\mathrm{a}(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{bc}\} \\ & + \frac{2\xi\eta^2\zeta}{\lambda\mu^2\nu}\{-\mathrm{b}(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{ca}\} \\ & + \frac{2\xi\eta\zeta^2}{\lambda\mu\nu^2}\{-\mathrm{c}(\mathrm{a}+\mathrm{b}+\mathrm{c})+\mathrm{ab}\}.\end{aligned}$$

But here

$$\begin{aligned}\mathrm{a} &= aa_1(bc_1-b_1c)(bc-b_1c_1) = aa_1\{(b^2+b_1^2)cc_1-(c^2+c_1^2)bb_1\}, \\ \mathrm{b} &= bb_1(ca_1-c_1a)(ca-c_1a_1) = bb_1\{(c^2+c_1^2)aa_1-(a^2+a_1^2)cc_1\}, \\ \mathrm{c} &= cc_1(ab_1-a_1b)(ab-a_1b_1) = cc_1\{(a^2+a_1^2)bb_1-(b^2+b_1^2)aa_1\}.\end{aligned}$$

Hence

$$\mathrm{a}+\mathrm{b}+\mathrm{c}=0,$$

and we obtain

$$\begin{aligned}\Omega &= \left(\frac{\mathrm{a}\eta\zeta}{\mu\nu}+\frac{\mathrm{b}\zeta\xi}{\nu\lambda}+\frac{\mathrm{c}\xi\eta}{\lambda\mu}\right)^2, \\ &= \left\{\frac{aa_1\alpha P}{bb_1cc_1\beta\gamma}\eta\zeta+\frac{bb_1\beta Q}{cc_1aa_1\gamma\alpha}\zeta\xi+\frac{cc_1\gamma R}{aa_1bb_1\alpha\beta}\xi\eta\right\}^2, \\ &= \frac{1}{(aa_1bb_1cc_1\alpha\beta\gamma)^2}(a^2a_1^2\alpha^2P\eta\zeta+b^2b_1^2\beta^2Q\zeta\xi+c^2c_1^2\gamma^2R\xi\eta)^2,\end{aligned}$$

a perfect square, and the line 23 is thus a bitangent.

We have next to prove the equivalence of the two equations

$$\frac{x}{bc-b_1c_1}+\frac{y}{ca-c_1a_1}+\frac{z}{ab-a_1b_1}=0,$$

and

$$\frac{\xi}{g_2h_2-g_3h_3}+\frac{\eta}{h_2f_2-h_3f_3}+\frac{\zeta}{f_2g_2-f_3g_3}=0.$$

Starting from the former equation written in the form

$$-QRx-RPy-PQz=0,$$

and combining herewith the three fundamental equations

$$\begin{aligned}a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta&=0,\\ a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\\ a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta&=0,\end{aligned}$$

if we eliminate from these the (x, y, z), we obtain the following equation in (ξ, η, ζ)

$$\begin{vmatrix} a, & b, & c, & f\xi + g\eta + h\zeta \\ a_1, & b_1, & c_1, & f_1\xi + g_1\eta + h_1\zeta \\ a_2, & b_2, & c_2, & f_2\xi + g_2\eta + h_2\zeta \\ -QR, & -RP, & -PQ & \end{vmatrix} = 0,$$

which, observing that the values of $-QR$, $-RP$, $-PQ$ are

$$-a^2bc - a_1^2b_1c_1 + aa_1(bc_1 + b_1c), \quad -ab^2c - a_1b_1^2c_1 + bb_1(ca_1 + c_1a), \quad -abc^2 - a_1b_1c_1^2 + cc_1(ab_1 + a_1b),$$

is immediately transformed into

$$\begin{vmatrix} a, & b, & c, & f\xi + g\eta + h\zeta \\ a_1, & b_1, & c_1, & f_1\xi + g_1\eta + h_1\zeta \\ a_2, & b_2, & c_2, & f_2\xi + g_2\eta + h_2\zeta \\ aa_1(bc_1 + b_1c), & bb_1(ca_1 + c_1a), & cc_1(ab_1 + a_1b), & \begin{cases} abc(f\xi + g\eta + h\zeta) \\ + a_1b_1c_1(f_1\xi + g_1\eta + h_1\zeta) \end{cases} \end{vmatrix} = 0$$

Here the coefficient of ξ is

$$\begin{vmatrix} a, & b, & c, & f \\ a_1, & b_1, & c_1, & f_1 \\ a_2, & b_2, & c_2, & f_2 \\ aa_1(bc_1 + b_1c), & bb_1(ca_1 + c_1a), & cc_1(ab_1 + a_1b), & abcf + a_1b_1c_1f_1 \end{vmatrix}$$

which is of the form $lf + l_1f_1 + l_2f_2$; and we find without difficulty

$$\begin{aligned} l = {} & a\,a_2(bc - b_1c_1)(bc_1 - b_1c), & = {} & a\,a_2P\alpha \\ & + b\,b_2(ca - c_1a_1)(ca_1 - c_1a) & & + b\,b_2Q\beta \\ & + c\,c_2(ab - a_1b_1)(ab_1 - a_1b) & & + c\,c_2R\gamma; \end{aligned}$$

$$\begin{aligned} l_1 = {} & -a_1a_2(bc - b_1c_1)(bc_1 - b_1c), & = {} & -a_1a_2P\alpha \\ & - b_1b_2(ca - c_1a_1)(ca_1 - c_1a) & & - b_1b_2Q\beta \\ & - c_1c_2(ab - a_1b_1)(ab_1 - a_1b) & & - c_1c_2R\gamma; \end{aligned}$$

$$l_2 = -(bc_1 - b_1c)(ca_1 - c_1a)(ab_1 - a_1b) = -\alpha\beta\gamma.$$

Hence

$$lf + l_1f_1 + l_2f_2 = (af - a_1f_1)\,a_2P\alpha + (bf - b_1f_1)\,b_2Q\beta + (cf - c_1f_1)\,c_2R\gamma - \alpha\beta\gamma f_2,$$

which, observing that

$$af - a_1f_1 = 0, \quad bf - b_1f_1 = -\frac{\gamma}{aa_1}, \quad cf - c_1f_1 = \frac{\beta}{aa_1},$$

becomes

$$= -b_2Q\frac{\beta\gamma}{aa_1} + c_2R\frac{\beta\gamma}{aa_1} - \alpha\beta\gamma f_2,$$

or, in the last term writing $f_2 = \frac{1}{a_2}$, this is

$$= \frac{\beta\gamma}{aa_1a_2}\{-a_2b_2Q + a_2c_2R - aa_1\alpha\},$$

$$= \frac{\beta\gamma}{aa_1a_2}\{-a_2b_2(ca - c_1a_1) + a_2c_2(ab - a_1b_1) - aa_1(bc_1 - b_1c)\},$$

$$= \frac{\beta\gamma}{aa_1a_2}\begin{vmatrix} ab\ , & ac\ , & 1 \\ a_1b_1, & a_1c_1, & 1 \\ a_2b_2, & a_2c_2, & 1 \end{vmatrix}, \quad = \beta\gamma\begin{vmatrix} b\ , & c\ , & f \\ b_1, & c_1, & f_1 \\ b_2, & c_2, & f_2 \end{vmatrix};$$

viz. representing for shortness the last-mentioned determinant by (bcf), the coefficient of ξ is $= \beta\gamma(bcf)$. Similarly the coefficients of η, ζ are equal $\gamma\alpha(cag)$, and $\alpha\beta(abh)$, respectively. Hence, for convenience dividing by $\alpha\beta\gamma$, the original equation

$$\frac{x}{bc - b_1c_1} + \frac{y}{ca - c_1a_1} + \frac{z}{ab - a_1b_1} = 0,$$

expressed in terms of (ξ, η, ζ), becomes

$$\frac{1}{\alpha}(bcf)\xi + \frac{1}{\beta}(cag)\eta + \frac{1}{\gamma}(abh)\zeta = 0;$$

and it is to be shown that this is, in fact, equivalent to

$$\frac{\xi}{g_2h_2 - g_3h_3} + \frac{\eta}{h_2f_2 - h_3f_3} + \frac{\zeta}{f_2g_2 - f_3g_3} = 0.$$

We ought therefore to have

$$(g_2h_2 - g_3h_3)\frac{1}{\alpha}(bcf) = (h_2f_2 - h_3f_3)\frac{1}{\beta}(cag) = (f_2g_2 - f_3g_3)\frac{1}{\gamma}(abh),$$

and it will be sufficient to prove the first of these equalities, say

$$(g_2h_2 - g_3h_3)\beta(bcf) = (h_2f_2 - h_3f_3)\alpha(cag).$$

Multiplying by c_2c_3, this becomes

$$\begin{aligned} &c_3[g_2\beta(bcf) - f_2\alpha(cag)] \\ &- c_2g_3\beta(bcf) + c_2f_3\alpha(cag) = 0, \end{aligned}$$

viz. this is

$$\begin{aligned} &(\lambda c + \lambda_1c_1 + \lambda_2c_2)[g_2\beta(bcf) - f_2\alpha(cag)] \\ &- (\lambda g + \lambda_1g_1 + \lambda_2g_2)c_2\beta(bcf) \\ &+ (\lambda f + \lambda_1f_1 + \lambda_2f_2)c_2\alpha(cag) = 0. \end{aligned}$$

The term in λ_2 disappears, and the equation is

$$\lambda\,[(c\,g_2-c_2g\,)\,\beta\,(bcf)-(c\,f_2-c_2f\,)\,\alpha\,(cag)]$$
$$+\,\lambda_1\,[(c_1g_2-c_2g_1)\,\beta\,(bcf)-(c_1f_2-c_2f_1)\,\alpha\,(cag)]=0.$$

But we have $\lambda:\lambda_1=\dfrac{1}{\Lambda}:\dfrac{1}{\Lambda_1}$, that is, $\Lambda\lambda-\Lambda_1\lambda_1=0$, or if we write for Λ and Λ_1 their values $(1F_1F_2)$, $(=-(1F_2F_1))$, and $(1F_2F)$ respectively, then the equation connecting λ, λ_1 is

$$\lambda\,(1F_2F_1)+\lambda_1\,(1F_2F)=0,$$

where $(1F_2F_1)$ and $(1F_2F)$ denote the determinants

$$\begin{vmatrix} 1, & af_1+a_1f, & af_2+a_2f \\ 1, & bg_1+b_1g\,, & bg_2+b_2g \\ 1, & ch_1+c_1h\,, & ch_2+c_2h \end{vmatrix},\quad \begin{vmatrix} 1, & af_1+a_1f, & a_1f_2+a_2f_1 \\ 1, & bg_1+b_1g\,, & b_1g_2+b_2g_1 \\ 1, & ch_1+c_1h\,, & c_1h_2+c_2h_1 \end{vmatrix}.$$

Multiplying by cc_1c_2, the equation may be written

$$\lambda\begin{vmatrix} af, & af_1+a_1f, & af_2+a_2f \\ bg, & bg_1+b_1g, & bg_2+b_2g \\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2) \end{vmatrix}+\lambda_1\begin{vmatrix} a_1f_1, & af_1+a_1f, & a_1f_2+a_2f_1 \\ b_1g_1, & bg_1+b_1g, & b_1g_2+b_2g_1 \\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2) \end{vmatrix}=0.$$

This should agree with the equation in (λ, λ_1) obtained above: and it can, in fact, be shown that the coefficients

$$(c\,g_2-c_2g\,)\,\beta\,(bcf)-(c\,f_2-c_2f\,)\,\alpha\,(cag),$$

and

$$(c_1g_2-c_2g_1)\,\beta\,(bcf)-(c_1f_2-c_2f_1)\,\alpha\,(cag),$$

are equal to the two determinants respectively; it will be sufficient to prove one of the two relations, say

$$-(cg_2-c_2g)\,(ca_1-c_1a)\,(bcf)+(cf_2-c_2f)\,(bc_1-b_1c)\,(cag)$$
$$+\begin{vmatrix} af, & af_1+a_1f, & af_2+a_2f \\ bg, & bg_1+b_1g, & bg_2+b_2g \\ cc_1c_2, & c_2(c^2+c_1^2), & c_1(c^2+c_2^2) \end{vmatrix}=0;$$

where it will be recollected that (bcf) and (cag) stand for the determinants

$$\begin{vmatrix} b\,, & c\,, & f \\ b_1, & c_1, & f_1 \\ b_2, & c_2, & f_2 \end{vmatrix},\quad \begin{vmatrix} c\,, & a\,, & g \\ c_1, & a_1, & g_1 \\ c_2, & a_2, & g_2 \end{vmatrix},$$

respectively: each term is thus of the seventh order in all the letters conjointly, but is quadric in f, f_1, f_2, g, g_1, g_2. Instead of α, β, γ used hitherto to denote bc_1-b_1c,

$ca_1 - c_1 a$, $ab_1 - a_1 b$, it will be convenient to call these α_2, β_2, γ_2, and to consider the complete system of coefficients

$$\alpha,\ \alpha_1,\ \alpha_2 = b_1 c_2 - b_2 c_1,\ b_2 c - bc_2,\ bc_1 - b_1 c;\quad \beta,\ \beta_1,\ \beta_2 = c_1 a_2 - c_2 a_1,\ c_2 a - ca_2,\ ca_1 - c_1 a;$$
$$\gamma,\ \gamma_1,\ \gamma_2 = a_1 b_2 - a_2 b_1,\ a_2 b - ab_2,\ ab_1 - a_1 b;$$

and to denote by ∇ the determinant formed with α, β, γ; α_1, β_1, γ_1; α_2, β_2, γ_2. If we then expand in terms of the products $(f, f_1, f_2)(g, g_1, g_2)$, we find first

$$\begin{aligned}
&-(cg_2 - c_2 g)(ca_1 - c_1 a)(bcf) + (cf_2 - c_2 f)(bc_1 - b_1 c)(cag)\\
&= -(cg_2 - c_2 g)\beta_2(\alpha f + \alpha_1 f_1 + \alpha_2 f_2) + (cf_2 - c_2 f)\alpha_2(\beta g + \beta_1 g_1 + \beta_2 g_2),\\
&= -\alpha_2\beta + \alpha\beta_2 (= -c_1\nabla)\, c_2 fg\\
&\quad -\alpha_2\beta_1 c_2 fg_1\\
&\quad +\alpha_1\beta_2 c_2 f_1 g\\
&\quad -\beta_2(\ \ \alpha c + \alpha_2 c_2 = -\alpha_1 c_1) fg_2\\
&\quad -\alpha_2(-\beta c - \beta_2 c_2 = \ \ \beta_1 c_1) f_2 g\\
&\quad -\alpha_1\beta_2 c f_1 g_2\\
&\quad +\alpha_2\beta_1 c f_2 g_1;
\end{aligned}$$

and next

$$\begin{aligned}
&\begin{vmatrix} af, & af_1 + a_1 f, & af_2 + a_2 f\\ bg, & bg_1 + b_1 g, & bg_2 + b_2 g\\ cc_1 c_2, & c_2(c^2 + c_1^2), & c_1(c^2 + c_2^2)\end{vmatrix}.\\
&= cc_1 c_2\gamma + c_2(c^2 + c_1^2)\gamma_1 + c_1(c^2 + c_2^2)\gamma_2 (= c_1 c_2\nabla + c^2(c_1\gamma_2 + c_2\gamma_1))fg\\
&\quad + c_1(\beta_1 c_2 b + c^2 ab) fg_1\\
&\quad + c_1(\alpha_1 c_2 a - c^2 ab) f_1 g\\
&\quad + c_2(\beta_2 c_1 b - c^2 ab) fg_2\\
&\quad + c_2(\alpha_2 c_1 a + c^2 ab) f_2 g\\
&\quad + c_1 c_2 abc f_1 g_2\\
&\quad - c_1 c_2 abc f_2 g_1.
\end{aligned}$$

Uniting the two parts, we ought to have

$$\begin{aligned}
0 &= c^2(c_2\gamma_1 + c_1\gamma_2) fg\\
&\quad + [\beta_1 c_2(c_1 b - \alpha_2) + c_1 c^2 ab] fg_1\\
&\quad + [\alpha_1 c_2(c_1 a + \beta_2) - c_1 c^2 ab] f_1 g\\
&\quad + [c_1\beta_2(c_2 b + \alpha_1) - c_2 c^2 ab] fg_2\\
&\quad + [c_1\alpha_2(c_2 a - \beta_1) + c_2 c^2 ab] f_2 g\\
&\quad + c(c_1 c_2 ab - \alpha_1\beta_2) f_1 g_2\\
&\quad - c(c_1 c_2 ab - \alpha_2\beta_1) f_2 g_1.
\end{aligned}$$

Substituting for α_1, β_1, γ_1, α_2, β_2, γ_2, then after a slight reduction it is found that the whole equation divides by c, and omitting this factor, we have

$$\begin{aligned}
0 = c\,[(c_2a_2 - c_1a_1)\,b - (c_2b_2 - c_1b_1)\,a]\,fg \\
+[\quad cc_1ab + b_1c_2\,(c_2a - ca_2)\quad]\,fg_1 \\
+[-\,cc_1ab + a_1c_2\,(b_2c - bc_2)\quad]\,f_1g \\
+[-\,cc_2ab + c_1b_2\,(ca_1 - c_1a)\quad]\,fg_2 \\
+[\quad cc_2ab + c_1a_2\,(bc_1 - b_1c)\quad]\,f_2g \\
+[-\,c^2a_1b_2 + ca\,b_2c_1 + bc\,a_1c_2\quad]\,f_1g_2 \\
+[\quad c^2a_2b_1 - bc\,a_2c_1 - ca\,b_1c_2\quad]\,f_2g_1.
\end{aligned}$$

Writing here $af = bg = \ldots = 1$, the equation becomes

$$\begin{aligned}
0 = c\,(c_2a_2 - c_1a_1)\,f - c\,(c_2b_2 - c_1b_1)\,g \\
+ cc_1bg_1 + c_2\,(c_2a - ca_2)\,f \\
- cc_1af_1 + c_2\,(b_2c - bc_2)\,g \\
- cc_2bg_2 + c_1\,(ca_1 - c_1a)\,f \\
+ cc_2af_2 + c_1\,(bc_1 - b_1c)\,g \\
- c^2 + acc_1f_1 + bcc_2g_2 \\
+ c^2 - bcc_1g_1 - cac_2f_2,
\end{aligned}$$

where all the terms in f_1, g_1 destroy each other; the equation thus becomes

$$\begin{aligned}
0 = \quad (\quad cc_2a_2 - cc_1a_1 + c_2^2a - cc_2a_2 + cc_1a_1 - c_1^2a)\,f \\
+ (-\,cc_2b_2 + cc_1b_1 + cc_2b_2 - c_2^2b + c_1^2b - cc_1b_1)\,g
\end{aligned}$$

that is,

$$0 = (c_2^2 - c_1^2)\,af + (-\,c_2^2 + c_1^2)\,bg.$$

Or again writing $af = bg = 1$, we have the identity $0 = 0$. This completes the proof of the equivalence of the two equations

$$\frac{x}{bc - b_1c_1} + \frac{y}{ca - c_1a_1} + \frac{z}{ab - a_1b_1} = 0, \quad \frac{\xi}{g_2h_3 - g_3h_3} + \frac{\eta}{h_2f_2 - h_3f_3} + \frac{\zeta}{f_2g_2 - f_3g_3} = 0.$$

We have obviously three new forms derived from the equation in (ξ, η, ζ) in like manner as we had previously three new forms derived from the equation in (x, y, z); the equivalence of the 8 forms to each other is thus shown, and the proof is now complete for each of the 28 bitangents.

Cambridge, 15 *Dec.* 1882.

In what follows, instead of the four equations of the form

$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

I take the first equation in Riemann's form with the coefficients unity. The equations thus are

$$\begin{aligned} x+\ \ y+\ \ z+\ \ \xi+\ \ \eta+\ \ \zeta&=0,\\ a\,x+b\,y+c\,z+f\,\xi+g\,\eta+h\,\zeta&=0,\\ a_1x+b_1y+c_1z+f_1\xi+g_1\eta+h_1\zeta&=0,\\ a_2x+b_2y+c_2z+f_2\xi+g_2\eta+h_2\zeta&=0, \end{aligned}$$

and if we assume

$$a_2=\lambda+\lambda_1 a+\lambda_2 a_1,\ \&c.$$

then we have

$$\begin{aligned} 1-\lambda^2-\lambda_1^2-\lambda_2^2&=\lambda_1\lambda_2(af_1+a_1f)+\lambda_2\lambda(a_1+f_1)+\lambda\lambda_1(a+f),\\ \text{,,}\qquad &=\lambda_1\lambda_2(bg_1+b_1g)+\lambda_2\lambda(b_1+g_1)+\lambda\lambda_1(b+g),\\ \text{,,}\qquad &=\lambda_1\lambda_2(ch_1+c_1h)+\lambda_2\lambda(c_1+h_1)+\lambda\lambda_1(c+h), \end{aligned}$$

and consequently

$$\lambda_1\lambda_2:\lambda_2\lambda:\lambda\lambda_1=P:Q:R,$$

or say

$$\lambda:\lambda_1:\lambda_2=QR:RP:PQ,$$

where

$$P,\ Q,\ R=\begin{vmatrix}1, & a+f_1, & a+f\\ 1, & b_1+g_1, & b+g\\ 1, & c_1+h_1, & c+h\end{vmatrix},\ \begin{vmatrix}1, & a+f, & af_1+a_1f\\ 1, & b+g, & bg_1+b_1g\\ 1, & c+h, & ch_1+c_1h\end{vmatrix},\ \begin{vmatrix}1, & af_1+a_1f, & a_1+f_1\\ 1, & bg_1+b_1g, & b_1+g_1\\ 1, & ch_1+c_1h, & c_1+h_1\end{vmatrix}.$$

Suppose that one of these determinants, to fix the ideas, say Q, is $=0$; we have $\lambda:\lambda_1:\lambda_2=0:RP:0$; that is, λ and λ_2 each $=0$; and consequently $\lambda_1=1$: that is, we have $a_2,\ b_2,\ c_2,\ f_2,\ g_2,\ h_2=a,\ b,\ c,\ f,\ g,\ h$, the fourth equation identical with the second, or say the second equation,

$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

is a double equation: as just seen, the condition for this is

$$Q=\begin{vmatrix}1, & a+f, & af_1+a_1f\\ 1, & b+g, & bg_1+b_1g\\ 1, & c+h, & ch_1+c_1h\end{vmatrix}=0,$$

say this is

$$L(af_1+a_1f)+M(bg_1+b_1g)+N(ch_1+c_1h)=0,$$

if for shortness

$$L,\ M,\ N=b+g-c-h,\quad c+h-a-f,\quad a+f-b-g.$$

A special solution is if $a_1, b_1, c_1 = a^2, b^2, c^2$; for then

$$a_1, b_1, c_1, f_1, g_1, h_1 = a^2, b^2, c^2, f^2, g^2, h^2;$$

consequently $af_1 + a_1 f,\ bg_1 + b_1 g,\ ch_1 + c_1 h = a + f,\ b + g,\ c + h$: and the condition in question $Q = 0$ is thus satisfied.

It is to be shown that, in the general case of the equation $Q = 0$, the curve is a nodal quartic having the node at the point

$$x : y : z : \xi : \eta : \zeta = fL : gM : hN : aL : bM : cN,$$

and that in the special case $a_1, b_1, c_1 = a^2, b^2, c^2$ the node is a fleflecnode, viz. a node which is a point of inflexion on each of its two branches. In the former case, the six tangents from the node are

$$\begin{aligned} ax + by + cz &= 0 \quad \text{or} \quad f\xi + g\eta + h\zeta = 0, \\ f\xi + by + cz &= 0 \quad \text{,,} \quad ax + g\eta + h\zeta = 0, \\ ax + g\eta + cz &= 0 \quad \text{,,} \quad f\xi + by + h\zeta = 0, \\ ax + by + h\zeta &= 0 \quad \text{,,} \quad f\xi + g\eta + cz = 0, \end{aligned}$$

$$\frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab} = 0, \quad \text{or} \quad \frac{\xi}{gh - g_1 h_1} + \frac{\eta}{hf - h_1 f_1} + \frac{\zeta}{fg - f_1 g_1} = 0,$$

$$\frac{x}{bc - b_1 c_1} + \frac{y}{ca - c_1 a_1} + \frac{z}{ab - a_1 b_1} = 0, \quad \text{,,} \quad \frac{\xi}{1-gh} + \frac{\eta}{1-hf} + \frac{\zeta}{1-fg} = 0.$$

In the latter case, the same equations represent the four tangents from the node and the two tangents at the node respectively.

For the proof, we assume that the four lines

$$\begin{aligned} ax + by + cz &= 0, \\ f\xi + by + cz &= 0, \\ ax + g\eta + cz &= 0, \\ ax + by + h\zeta &= 0, \end{aligned}$$

have a common intersection: this gives $ax = f\xi,\ by = g\eta,\ cz = h\zeta$; or say

$$x, y, z, \xi, \eta, \zeta = fL, gM, hN, aL, bM, cN,$$

where for the moment L, M, N are indeterminate quantities: but substituting these values in the first, second and third equations, we find

$$\begin{aligned} (a+f)L + (b+g)M + (c+h)N &= 0, \\ L + M + N &= 0, \\ (af_1 + a_1 f)L + (bg_1 + b_1 g)M + (ch_1 + c_1 h)N &= 0, \end{aligned}$$

giving for L, M, N the values $b + g - c - h,\ c + h - a - f,\ a + f - b - g$ already attributed to these letters.

And we see further that the point in question

$$x,\ y,\ z,\ \xi,\ \eta,\ \zeta, = fL,\ gM,\ hN,\ aL,\ bM,\ cN,$$

is a point on each of the lines

$$\frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab} = 0,$$

$$\frac{x}{bc-b_1c_1} + \frac{y}{ca-c_1a_1} + \frac{z}{ab-a_1b_1} = 0.$$

In fact, for the first of these lines, we have only to verify the equation

$$fL(1-ca)(1-ab) + gM(1-ab)(1-bc) + hN(1-bc)(1-ca) = 0,$$

that is,

$$L(f-b-c+abc) + M(g-c-a+abc) + N(h-a-b+abc) = 0,$$

viz. this is

$$L(a+f) + M(b+g) + N(c+h) + (-a-b-c+abc)(L+M+N) = 0,$$

which is right. And similarly, for the second line, we have to verify that

$$fL(ca-c_1a_1)(ab-a_1b_1) + gM(ab-a_1b_1)(bc-b_1c_1) + hN(bc-b_1c_1)(ca-c_1a_1) = 0,$$

that is,

$$L(abc - bc_1a_1 - ca_1b_1 + fa_1^2b_1c_1) + \&c. = 0,$$

or, multiplying by $f_1g_1h_1$, this is

$$L(abcf_1g_1h_1 - bg_1 - ch_1 + a_1f) + M(abcf_1g_1h_1 - ch_1 - af_1 + b_1g) + N(abcf_1g_1h_1 - af_1 - bg_1 + c_1h) = 0,$$

viz. this is

$$(abcf_1g_1h_1 - af_1 - bg_1 - ch_1)(L+M+N) + (af_1 + a_1f)L + (bg_1 + b_1g)M + (ch_1 + c_1h)N = 0.$$

We have thus six of the double tangents meeting at the point

$$x,\ y,\ z,\ \xi,\ \eta,\ \zeta, = fL,\ gM,\ hN,\ aL,\ bM,\ cN,$$

which implies that this point is a node on the curve: and we at once see that the values satisfy the equation $\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0$ of the curve; viz. the equation for the values in question becomes $\sqrt{L^2} + \sqrt{M^2} + \sqrt{N^2} = 0$, and the rationalised equation is satisfied as containing the factor $L+M+N$ which is $=0$.

But to show *à posteriori* with greater distinctness that the point is, in fact, a node, observe that, the rationalised equation being

$$\Omega = x^2\xi^2 + y^2\eta^2 + z^2\zeta^2 - 2yz\eta\zeta - 2zx\zeta\xi - 2xy\xi\eta,$$

the differential $d\Omega$ is

$$= (xd\xi + \xi dx)(x\xi - y\eta - z\zeta) + (yd\eta + \eta dy)(-x\xi + y\eta - z\zeta) + (zd\zeta + \zeta dz)(-x\xi - y\eta + z\zeta),$$

which for the values in question becomes

$$= L(f d\xi + a dx)(L^2 - M^2 - N^2) + M(g d\eta + b dy)(-L^2 + M^2 - N^2) + N(h d\zeta + c dz)(-L^2 - M^2 + N^2);$$

and this, in virtue of $L + M + N = 0$, and therefore $L^2 - M^2 - N^2 = 2MN$, &c., is

$$= 2LMN(a\,dx + b\,dy + c\,dz + f\,d\xi + g\,d\eta + h\,d\zeta),$$

which is $= 0$ in virtue of the second equation.

Consider now in the special case $a_1, b_1, c_1 = a^2, b^2, c^2$, the line

$$\frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab} = 0;$$

combining this with the first and second equations

$$x + y + z + \xi + \eta + \zeta = 0,$$
$$ax + by + cz + f\xi + g\eta + h\zeta = 0,$$

we deduce

$$x = -F\left(\frac{\eta}{b\beta} - \frac{\zeta}{c\gamma}\right),$$

$$y = -G\left(\frac{\zeta}{c\gamma} - \frac{\xi}{a\alpha}\right),$$

$$z = -H\left(\frac{\xi}{a\alpha} - \frac{\eta}{b\beta}\right),$$

if for shortness $F, G, H, \alpha, \beta, \gamma = bc - 1,\ ca - 1,\ ab - 1,\ b - c,\ c - a,\ a - b$. Observe that we have $L = b + g - c - h = (b - c)\left(1 - \frac{1}{bc}\right)$, that is, $bcL = \alpha F$; and similarly $caM = \beta G$, and $abN = \gamma H$. And if for a moment we write further

$$\xi, \eta, \zeta, I, J, K, = a\alpha X,\ b\beta Y,\ c\gamma Z,\ a\alpha F,\ b\beta G,\ c\gamma H,$$

values which give

$$I + J + K = (abc - a)(b - c) + (abc - b)(c - a) + (abc - c)(a - b), \quad = 0,$$

then the equation $\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0$ of the curve becomes

$$\sqrt{IX(Y - Z)} + \sqrt{JY(Z - X)} + \sqrt{KZ(X - Y)} = 0,$$

which in the rationalised form is

$$I^2X^2(Y - Z)^2 + J^2Y^2(Z - X)^2 + K^2Z^2(X - Y)^2$$
$$- 2JKYZ(Z - X)(X - Y) - 2KIZX(X - Y)(Y - Z) - 2IJXY(Y - Z)(Z - X) = 0,$$

viz. this is

$$Y^2Z^2(J^2 + 2JK + K^2) + \ldots + 2YZ(JK - I^2 - IJ - IK) + \ldots = 0,$$

which, in virtue of $I + J + K = 0$, becomes

$$(IYZ + JZX + KXY)^2 = 0,$$

that is, the quartic is met by the line in question at the intersections (each taken twice) of the line with the conic

$$IYZ + JZX + KXY = 0,$$

that is,

$$Fa^2\alpha^2 . \eta\zeta + Gb^2\beta^2 . \zeta\xi + Hc^2\gamma^2 . \xi\eta = 0.$$

But the equation of the line in terms of ξ, η, ζ is

$$\frac{\xi}{gh - g_1h_1} + \frac{\eta}{hf - h_1f_1} + \frac{\zeta}{fg - f_1g_1} = 0,$$

that is,

$$\frac{\xi}{gh - g^2h^2} + \frac{\eta}{hf - h^2f^2} + \frac{\zeta}{fg - f^2g^2} = 0,$$

or, what is the same thing,

$$\frac{b^2c^2\xi}{F} + \frac{c^2a^2\eta}{G} + \frac{a^2b^2\zeta}{H} = 0,$$

and this is clearly a tangent to the conic; viz. the rationalised form of

$$\sqrt{Fa^2\alpha^2 . \frac{b^2c^2}{F}} + \sqrt{Gb^2\beta^2 . \frac{c^2a^2}{G}} + \sqrt{Hc^2\gamma^2 . \frac{a^2b^2}{H}} = 0,$$

that is, $\sqrt{\alpha^2} + \sqrt{\beta^2} + \sqrt{\gamma^2} = 0$ is satisfied in virtue of $\alpha + \beta + \gamma = 0$. Hence the four intersections coincide together, or the line has with the curve a quadruple intersection at the node, that is, it is a tangent at the node to a branch having an inflexion. And similarly the other line

$$\frac{x}{bc - b^2c^2} + \frac{y}{ca - c^2a^2} + \frac{z}{ab - a^2b^2} = 0, \text{ that is, } \frac{\xi}{1 - gh} + \frac{\eta}{1 - hf} + \frac{\zeta}{1 - fg} = 0,$$

is a tangent at the node to the other branch having also an inflexion; thus the node is a fleflecnode, and the lines are the tangents to the two branches.

I continue the investigation of the special case $a_1, b_1, c_1, = a^2, b^2, c^2$. The three equations give

$$\begin{aligned}
&-(h-f)(f-g)\xi + (a-g)(a-h)x + (b-g)(b-h)y + (c-g)(c-h)z = 0,\\
&-(f-g)(g-h)\eta + (a-h)(a-f)x + (b-h)(b-f)y + (c-h)(c-f)z = 0,\\
&-(g-h)(h-f)\zeta + (a-f)(a-g)x + (b-f)(b-g)y + (c-f)(c-g)z = 0;
\end{aligned}$$

writing

$$A, B, C, F, G, H;\ \alpha, \beta, \gamma, = a^2 - 1, b^2 - 1, c^2 - 1, bc - 1, ca - 1, ab - 1;\ b - c, c - a, a - b,$$

and also

$$u = \frac{1}{\alpha\beta\gamma}(Ax + By + Cz),$$

then we find

$$\begin{aligned}
\xi &= a^2(\alpha Fu - x),\\
\eta &= b^2(\beta Gu - y),\\
\zeta &= c^2(\gamma Hu - z),
\end{aligned}$$

say for shortness these values are

$$\xi,\ \eta,\ \zeta = pu - a^2x,\quad qu - b^2y,\quad ru - c^2z,$$

where

$$p,\ q,\ r = a^2\alpha F,\ b^2\beta G,\ c^2\gamma H.$$

We have moreover $f^2\beta\gamma\,.\,\xi = \beta\gamma(\alpha Fu - x),\ = F(Ax + By + Cz) - \beta\gamma x$; or observing that $GH - AF = -\beta\gamma$, &c., we have

$$\begin{aligned} f^2\beta\gamma\xi &= GHx + BFy + CFz,\\ g^2\gamma\alpha\eta &= AGx + HFy + CGz,\\ h^2\alpha\beta\zeta &= AHx + BHy + FGz. \end{aligned}$$

It is to be shown that the tangents from the fleflecnode

$$\begin{aligned} ax + by + cz &= 0,\\ f\xi + by + cz &= 0,\\ ax + g\eta + cz &= 0,\\ ax + by + h\zeta &= 0, \end{aligned}$$

touch the quartic at its intersections with the line $u = 0$.

To prove this, for the first line we have $ax + by + cz = 0$, $f\xi + g\eta + h\zeta = 0$, and consequently $x\xi - y\eta - z\zeta = bhy\zeta + cgz\eta$. But the equation of the curve is

$$\Omega = (x\xi - y\eta - z\zeta)^2 - 4yz\eta\zeta = 0\,;$$

and this becomes thus

$$\Omega = (bhy\zeta + cgz\eta)^2 - 4bcghyz\eta\zeta = (bhy\zeta - cgz\eta)^2 = 0\,;$$

and we thus find that, for the four lines respectively, the equations for the points of contact are

$$\begin{aligned} &ax + by + cz = 0, && bhy\zeta - cgz\eta = 0,\\ &f\xi + by + cz = 0, && \quad\text{,,}\\ &ax + g\eta + cz = 0, && ghz\eta\zeta - bcyz = 0,\\ &ax + by + h\zeta = 0, && \quad\text{,,}\quad . \end{aligned}$$

But for the first line we have

$$\begin{aligned} bhy\zeta - cgz\eta &= bhyc^2(\gamma Hu - z) - cgzb^2(\beta Gu - y),\\ &= bcy(\gamma Hu - z) - bcz(\beta Gu - y),\\ &= bcu(\gamma Hy - \beta Gz), \end{aligned}$$

so that the intersections with the conic are given by the equations $\gamma Hy - \beta Gz = 0$, and $u = 0$ respectively: the first of these corresponds to the fleflecnode (in fact, we have $\gamma H\,.\,gM - \beta G\,.\,hN = 0$, viz. multiplying by abc, this is

$$\gamma H\,.\,caM - \beta G\,.\,abN = 0,$$

which is right), hence the second corresponds to the point of contact, that is, the point of contact lies on the line $u = 0$. And similarly, for each of the other three tangents, the point of contact lies on the same line $u = 0$.

It is clear that writing $T_1=0$, $T_2=0$, $T_3=0$, $T_4=0$ for the four tangents from the flecnode and $T_5=0$, $T_6=0$ for the tangents at this point, the equation of the curve must be expressible in the form

$$\lambda u^2 T_5 T_6 + \mu T_1 T_2 T_3 T_4 = 0.$$

The reduction to this form is, I think, effected by means of the formulæ which give ξ, η, ζ in terms of x, y, z, more readily than by means of simpler formulæ for ξ, η, ζ in terms of x, y, z and u. We have

$$\Omega = x^2\xi^2 + y^2\eta^2 + z^2\zeta^2 - 2yz\eta\zeta - 2zx\zeta\xi - 2xy\xi\eta = 0,$$

and hence

$$\begin{aligned}\Omega' = f^4g^4h^4\alpha^2\beta^2\gamma^2\Omega = \quad & g^4h^4\alpha^2x^2\ (GHx + BFy + CFz)^2 \\ & + h^4f^4\beta^2y^2\,(AGx + HFy + CGz)^2 \\ & + f^4g^4\gamma^2z^2\ (AHx + BHy + FGz)^2 \\ & - 2f^4g^2h^2\beta\gamma yz\,(AGx + HFy + CGz)\,(AHx + BHy + FGz) \\ & - 2g^4h^2f^2\gamma\alpha zx\,(AHx + BHy + FGz)\,(GHx + BFy + CFz) \\ & - 2h^4f^2g^2\alpha\beta xy\,(GHx + BFy + CFz)\,(AGx + HFy + CGz) = 0.\end{aligned}$$

The result after some reductions, and putting for shortness

$$\begin{aligned} h^2\beta CG - g^2\gamma BH &= \mathfrak{A}, \\ f^2\gamma AH - h^2\alpha CF &= \mathfrak{B}, \\ g^2\alpha BF - f^2\beta AG &= \mathfrak{C}, \end{aligned}$$

is

$$\begin{aligned}\Omega' = \quad & g^4h^4G^2H^2\alpha^2x^4 + h^4f^4H^2F^2\beta^2y^4 + f^4g^4F^2G^2\gamma^2z^4 \\ & + 2h^2f^4HF\beta\,.\,\mathfrak{A}y^3z + 2g^2f^4FG\gamma\,.\,\mathfrak{A}yz^3 \\ & + 2f^2g^4FG\gamma\,.\,\mathfrak{B}z^3x + 2h^2g^4GH\alpha\,.\,\mathfrak{B}zx^3 \\ & + 2g^2h^4GH\alpha\,.\,\mathfrak{C}x^3y + 2f^2h^4HF\beta\,.\,\mathfrak{C}xy^3 \\ & + f^4y^2z^2\ (\mathfrak{A}^2 - 2g^2h^2\beta\gamma F^2G\,H\ ) \\ & + g^4z^2x^2\ (\mathfrak{B}^2 - 2h^2f^2\gamma\alpha F\,G^2H\ ) \\ & + h^4x^2y^2\ (\mathfrak{C}^2 - 2f^2g^2\alpha\beta F\,G\,H^2) \\ & + 2g^2h^2x^2yz\ \{-\mathfrak{B}\mathfrak{C} - f^2GH\,(2f^2\beta\gamma A^2 + g^2\gamma\alpha BH + h^2\alpha\beta CG)\} \\ & + 2h^2f^2xy^2z\,\{-\mathfrak{C}\mathfrak{A} - g^2HF\,(\,f^2\beta\gamma AH + 2g^2\gamma\alpha B^2 + h^2\alpha\beta CF)\} \\ & + 2f^2g^2xyz^2\,\{-\mathfrak{A}\mathfrak{B} - h^2FG\,(\,f^2\beta\gamma AG + g^2\gamma\alpha BF + 2h^2\alpha\beta C^2\,)\} = 0.\end{aligned}$$

We have

$$u = \frac{1}{\alpha\beta\gamma}(Ax + By + Cz),$$

$$T_5 = \frac{x}{1-bc} + \frac{y}{1-ca} + \frac{z}{1-ab},$$

$$T_6 = \frac{x}{bc-b^2c^2} + \frac{y}{ca-c^2a^2} + \frac{z}{ab-a^2b^2},$$

or, as these equations may be written,

$$\begin{aligned} \alpha\beta\gamma u &= Ax + By + Cz, \\ -FGHT_5 &= GHx + HFy + FGz, \\ -abcFGHT_6 &= aGHx + bHFy + cFGz; \end{aligned}$$

whence

$$abcF^2G^2H^2 \,.\, \alpha\beta\gamma \,.\, uT_5T_6 = (Ax + By + Cz)^2\,(GHx + HFy + FGz)\,(aGHx + bHFy + cFGz),$$

where the coefficient of x^4 is $= aA^2G^2H^2$.

Again $T_1 = ax + by + cz$. For T_2 we have

$$\begin{aligned} bc\beta\gamma T_2 &= abcf\beta\gamma\,(f\xi + by + cz) \\ &= abc\,(GHx + BFy + CFz) + bc\beta\gamma\,(by + cz), \end{aligned}$$

where the coefficient of x is $abcGH$. The coefficient of y is

$$\begin{aligned} abc\,(b^2-1)(bc-1) + b^2c\,(c-a)(a-b), \quad &= bc\,(ab^3c - b^2c - a^2b + a), \\ &= bc\,(ab-1)(b^2c-a), \\ &= bcH\,(b^2c-a), \end{aligned}$$

and similarly the coefficient of z is $= bcG\,(bc^2 - a)$. We have thus the expression for T_2; and forming in like manner the expressions for T_3 and T_4, we may write

$$\begin{aligned} T_1 &= ax + by + cz, \\ bc\beta\gamma T_2 &= abcGHx + bcH\,(b^2c-a)\,y + bcG\,(bc^2-a)\,z, \\ ca\gamma\alpha T_3 &= caH\,(ca^2-b)\,x + abcHFy + caF\,(c^2a-b)\,z, \\ ab\alpha\beta T_4 &= abG\,(a^2b-c)\,x + abF\,(ab^2-c)\,y + abcFGz, \end{aligned}$$

whence in $a^2b^2c^2\alpha^2\beta^2\gamma^2T_1T_2T_3T_4$ the coefficient of x^4 is

$$= a^4b^3c^2 \,.\, G^2H^2 \,.\, (ca^2-b)(a^2b-c) = a^4b^3c^2 \,.\, G^2H^2\,(bcA^2 - a^2\alpha^2).$$

We hence obtain the required identity: viz. this is

$$\begin{aligned} a^6b^6c^6\Omega' = a^2b^2c^2 \,.\, abcF^2G^2H^2\alpha\beta\gamma \,.\, uT_5T_6 \\ -1 \,.\quad a^2b^2c^2\alpha^2\beta^2\gamma^2 \,.\, T_1T_2T_3T_4; \end{aligned}$$

for here comparing the terms in x^4, the factor G^2H^2 divides out, and omitting it, we have

$$a^6b^2c^2\alpha^2 = a^3b^2c^3 \,.\, aA^2 - a^4b^2c^2 \,.\, (bcA^2 - a^2\alpha^2),$$

which is right. If for Ω' we substitute its value, $= f^4g^4h^4\alpha^2\beta^2\gamma^2\Omega$, then the identity divides by $a^2b^2c^2\alpha\beta\gamma$, and omitting this factor, we obtain the equation of the curve in the form

$$\alpha\beta\gamma\Omega = a^2b^2c^2FGH \,.\, uT_5T_6 - \alpha\beta\gamma \,.\, T_1T_2T_3T_4 = 0;$$

the required form, putting in evidence the two tangents at the fleflecnode, and the four tangents from this point.

Cambridge, 3rd January, 1883.

817.

ON THE SIXTEEN-NODAL QUARTIC SURFACE.

[From *Crelle's Journal der Mathem.*, t. XCIV. (1883), pp. 270—272.]

RIEMANN'S theory of the bitangents of a plane quartic leads at once to a very simple form of the equation of the sixteen-nodal surface: viz. if ξ, η, ζ denote linear functions of the coordinates (x, y, z, w) such that identically

$$x+y+z+\xi+\eta+\zeta=0,$$
$$ax+by+cz+f\xi+g\eta+h\zeta=0,$$

(where $af=bg=ch=1$), then the quartic surface

$$\sqrt{x\xi}+\sqrt{y\eta}+\sqrt{z\zeta}=0$$

has the sixteen singular tangent planes (each touching it along a conic)

$$x=0,\quad y=0,\quad z=0,\quad \xi=0,\quad \eta=0,\quad \zeta=0,$$
$$x+y+z=0,\quad ax+by+cz=0,$$
$$\xi+y+z=0,\quad f\xi+by+cz=0,$$
$$x+\eta+z=0,\quad ax+g\eta+cz=0,$$
$$x+y+\zeta=0,\quad ax+by+h\zeta=0,$$
$$\frac{x}{1-bc}+\frac{y}{1-ca}+\frac{z}{1-ab}=0,\quad \frac{\xi}{1-gh}+\frac{\eta}{1-hf}+\frac{\zeta}{1-fg}=0:$$

and it is thus a sixteen-nodal surface.

I have formerly given the equation of this surface under the form

$$\sqrt{x(X-w)}+\sqrt{y(Y-w)}+\sqrt{z(Z-w)}=0,$$

where

$$\begin{aligned}
\alpha + \beta + \gamma &= 0, & X &= \alpha(\gamma'\gamma''y - \beta'\beta''z),\\
\alpha' + \beta' + \gamma' &= 0, & Y &= \beta(\alpha'\alpha''z - \gamma'\gamma''x),\\
\alpha'' + \beta'' + \gamma'' &= 0, & Z &= \gamma(\beta'\beta''x - \alpha'\alpha''y),\\
P &= \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}, & X' &= \alpha'(\gamma''\gamma y - \beta''\beta z),\\
P' &= \frac{x}{\alpha'} + \frac{y}{\beta'} + \frac{z}{\gamma'}, & Y' &= \beta'(\alpha''\alpha z - \gamma''\gamma x),\\
P'' &= \frac{x}{\alpha''} + \frac{y}{\beta''} + \frac{z}{\gamma''}, & Z' &= \gamma'(\beta''\beta x - \alpha''\alpha y),\\
& & X'' &= \alpha''(\gamma\gamma' y - \beta\beta' z),\\
& & Y'' &= \beta''(\alpha\alpha' z - \gamma\gamma' x),\\
& & Z'' &= \gamma''(\beta\beta' x - \alpha\alpha' y),
\end{aligned}$$

and where the equations of the sixteen singular tangent planes are

$$\begin{array}{llll}
x = 0, & y = 0, & z = 0, & w = 0,\\
X - w = 0, & Y - w = 0, & Z - w = 0, & P = 0,\\
X' - w = 0, & Y' - w = 0, & Z' - w = 0, & P' = 0,\\
X'' - w = 0, & Y'' - w = 0, & Z'' - w = 0, & P'' = 0;
\end{array}$$

see *Crelle's Journal*, vol. LXXIII. (1871), pp. 292, 293, [442], and also *Proc. Lond. Math. Soc.*, vol. III. (1871), p. 251*, [454].

To identify the two forms, using $x', y', z', \xi', \eta', \zeta'$ for the new form, I assume

$$x', y', z', \xi', \eta', \zeta' = lx,\ my,\ nz,\ p(X - w),\ q(Y - w),\ r(Z - w),$$

where $lp = mq = nr = 1$; and so convert the equation

$$\sqrt{x(X - w)} + \sqrt{y(Y - w)} + \sqrt{z(Z - w)} = 0$$

into

$$\sqrt{x'\xi'} + \sqrt{y'\eta'} + \sqrt{z'\zeta'} = 0.$$

The constants (l, m, n, p, q, r) and (a, b, c, f, g, h), where $af = bg = ch = 1$, are then to be determined so that we may have identically

$$\begin{aligned}
x' + y' + z' + \xi' + \eta' + \zeta' &= 0,\\
ax' + by' + cz' + f\xi' + g\eta' + h\zeta' &= 0,
\end{aligned}$$

and we thus obtain 8 new equations to be satisfied by the 12 constants, viz. these are

$$\begin{aligned}
l &+ r\,.\,\gamma\beta'\beta'' - q\,.\,\beta\gamma'\gamma'' = 0,\\
m &+ p\,.\,\alpha\gamma'\gamma'' - r\,.\,\gamma\alpha'\alpha'' = 0,\\
n &+ q\,.\,\beta\alpha'\alpha'' - p\,.\,\alpha\beta'\beta'' = 0,\\
p &+ q + r = 0,\\
al &+ hr\,.\,\gamma\beta'\beta'' - gq\,.\,\beta\gamma'\gamma'' = 0,\\
bm &+ fp\,.\,\alpha\gamma'\gamma'' - hr\,.\,\gamma\alpha'\alpha'' = 0,\\
cn &+ gq\,.\,\beta\alpha'\alpha'' - fp\,.\,\alpha\beta'\beta'' = 0,\\
fp &+ gq + hr = 0.
\end{aligned}$$

[* This Collection, vol. VII., p. 282.]

But substituting for a, b, c, l, m, n their values $\frac{1}{f}, \frac{1}{g}, \frac{1}{h}, \frac{1}{p}, \frac{1}{q}, \frac{1}{r}$, we have in all 8 equations for the determination of qr, rp, pq, gh, hf, fg; viz. if for greater convenience we introduce the new symbols $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} = qr\alpha'\alpha'', rp\beta'\beta'', pq\gamma'\gamma''$, then the 8 equations are

$$\frac{1}{\beta\gamma} + \frac{\mathfrak{B}}{\beta} - \frac{\mathfrak{C}}{\gamma} = 0,$$

$$\frac{1}{\gamma\alpha} + \frac{\mathfrak{C}}{\gamma} - \frac{\mathfrak{A}}{\alpha} = 0,$$

$$\frac{1}{\alpha\beta} + \frac{\mathfrak{A}}{\alpha} - \frac{\mathfrak{B}}{\beta} = 0,$$

$$\frac{\alpha'\alpha''}{\mathfrak{A}} + \frac{\beta'\beta''}{\mathfrak{B}} + \frac{\gamma'\gamma''}{\mathfrak{C}} = 0,$$

$$\frac{1}{\beta\gamma} + hf.\frac{\mathfrak{B}}{\beta} - fg.\frac{\mathfrak{C}}{\gamma} = 0,$$

$$\frac{1}{\gamma\alpha} + fg.\frac{\mathfrak{C}}{\gamma} - gh.\frac{\mathfrak{A}}{\alpha} = 0,$$

$$\frac{1}{\alpha\beta} + gh.\frac{\mathfrak{A}}{\alpha} - hf.\frac{\mathfrak{B}}{\beta} = 0,$$

$$\frac{\alpha'\alpha''}{\mathfrak{A}gh} + \frac{\beta'\beta''}{\mathfrak{B}hf} + \frac{\gamma'\gamma''}{\mathfrak{C}fg} = 0.$$

But in virtue of the equation $\alpha + \beta + \gamma = 0$ the first four equations are equivalent to three equations only, and they determine $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, that is, p, q, r, which give at once l, m, n; and similarly the second four equations are equivalent to three equations only, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ being known they determine gh, hf, fg, that is, f, g, h, which give at once a, b, c: the identification of the two forms is thus completed.

Cambridge, 11th January, 1883.

818.

NOTE IN CONNEXION WITH THE HYPERELLIPTIC INTEGRALS OF THE FIRST ORDER.

[From *Crelle's Journal der Mathem.*, t. XCVIII. (1885), pp. 95, 96.]

IN the early paper by Mr Weierstrass "Zur Theorie der *Abel*schen Functionen," *Crelle's Journal*, t. XLVII. (1854), pp. 289—306, we have pp. 302, 303, certain equations (43), and (stated to be deduced from them) an equation (49). Taking for greater simplicity $n=2$, the equations (43) written at full length are

$$(43)\quad \begin{cases} K_{11}J_{12}-K_{12}J_{11}+K_{21}J_{22}-K_{22}J_{21}=0, & K'_{11}J'_{12}-K'_{12}J'_{11}+K'_{21}J'_{22}-K'_{22}J'_{21}=0,\\ K_{11}J'_{12}-K'_{12}J_{11}+K_{21}J'_{22}-K'_{22}J_{21}=0, & K_{12}J'_{11}-K'_{11}J_{12}+K_{22}J'_{21}-K'_{21}J_{22}=0,\\ K_{11}J'_{11}-K'_{11}J_{11}+K_{21}J'_{21}-K'_{21}J_{21}=\tfrac{1}{2}\pi, & K_{12}J'_{12}-K'_{12}J_{12}+K_{22}J'_{22}-K'_{22}J_{22}=\tfrac{1}{2}\pi; \end{cases}$$

viz. in the theory of the hyperelliptic functions depending on the radical

$$\sqrt{x-a_0\,.\,x-a_1\,.\,x-a_2\,.\,x-a_3\,.\,x-a_4},$$

these are relations between the eight integrals K of the first kind, and the eight integrals J of the second kind. Each equation contains both K's and J's, and there is not in the paper any express mention of a relation between the K's only, which occurs in Rosenhain's Memoir, and is a leading equation in the theory. But taking as before $n=2$, and for the G's which occur in (49) substituting their values as obtained from the preceding equations (46) and (47), the equation becomes

$$(49)\quad K_{11}K'_{21}-K_{21}K'_{11}+K_{12}K'_{22}-K_{22}K'_{12}=0,$$

which is the equation in question: it is the equation $\omega_0 v_3-\omega_3 v_0+\omega_1 v_2-\omega_2 v_1=0$ of Hermite's Memoir "Sur la théorie de la transformation des fonctions Abéliennes," *Comptes Rendus*, t. XL. (1855).

It is interesting to see how the equation (49) is derived from the equations (43). I write for greater convenience

$$\begin{array}{cccccccccccccccc} K_{11}, & K_{12}, & K_{21}, & K_{22}, & K'_{11}, & K'_{12}, & K'_{21}, & K'_{22}, & J_{11}, & J_{12}, & J_{21}, & J_{22}, & J'_{11}, & J'_{12}, & J'_{21}, & J'_{22} \\ =A\ , & B\ , & C\ , & D\ , & A'\ , & B'\ , & C'\ , & D'\ , & \alpha\ , & \beta\ , & \gamma\ , & \delta\ , & \alpha'\ , & \beta'\ , & \gamma'\ , & \delta'. \end{array}$$

The given equations then are

$$(43)\quad \begin{cases} A\beta - B\alpha + C\delta - D\gamma = 0, & A'\beta' - B'\alpha' + C'\delta' - D'\gamma' = 0, \\ A\beta' - B'\alpha + C\delta' - D'\gamma = 0, & A'\beta - B\alpha' + C'\delta - D\gamma' = 0, \\ A\alpha' - A'\alpha + C\gamma' - C'\gamma = \frac{1}{2}\pi, & B\beta' - B'\beta + D\delta' - D'\delta = \frac{1}{2}\pi; \end{cases}$$

and it is required to show that these lead to the relation

$$(49)\quad AC' - A'C + BD' - B'D = 0.$$

From the first and fourth equations, and from the second and third equations of (43), we deduce

$$(AC' - A'C)\beta + (C\alpha' - C'\alpha)B + (C\gamma' - C'\gamma)D = 0,$$
$$(AC' - A'C)\beta' + (C\alpha' - C'\alpha)B' + (C\gamma' - C'\gamma)D' = 0;$$

and again from the first and third equations, and from the second and fourth equations of (43), we deduce

$$(BD' - B'D)\alpha + (D\beta' - D'\beta)A + (D\delta' - D'\delta)C = 0,$$
$$(BD' - B'D)\alpha' + (D\beta' - D'\beta)A' + (D\delta' - D'\delta)C' = 0.$$

These pairs of equations give respectively

$$AC' - A'C : C\alpha' - C'\alpha : C\gamma' - C'\gamma = BD' - B'D : D\beta' - D'\beta : -(B\beta' - B'\beta),$$

and

$$AC' - A'C : C\alpha' - C'\alpha : -(A\alpha' - A'\alpha) = BD' - B'D : D\beta' - D'\beta : D\delta' - D'\delta;$$

whence putting for shortness $A\alpha' - A'\alpha$, $B\beta' - B'\beta$, $C\gamma' - C'\gamma$, $D\delta' - D'\delta = \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$, we have

$$\frac{AC' - A'C}{BD' - B'D} = -\frac{\mathrm{c}}{\mathrm{b}} = -\frac{\mathrm{a}}{\mathrm{d}};\ \text{whence } \mathrm{ab} = \mathrm{cd}.$$

But the last two of the equations (43) are

$$\mathrm{a} + \mathrm{c} = \tfrac{1}{2}\pi,\quad \mathrm{b} + \mathrm{d} = \tfrac{1}{2}\pi;$$

we have thus $\mathrm{a} + \mathrm{c} = \mathrm{b} + \mathrm{d}$, $= \mathrm{b} + \frac{\mathrm{ab}}{\mathrm{c}}$, $= \frac{\mathrm{b}}{\mathrm{c}}(\mathrm{a} + \mathrm{c})$; or, since $\mathrm{a} + \mathrm{c}$, $= \frac{1}{2}\pi$, is not $= 0$, this gives $\mathrm{b} = \mathrm{c}$, whence also $\mathrm{a} = \mathrm{d}$, and we have

$$\frac{AC' - A'C}{BD' - B'D} = -1,$$

that is,

$$AC' - A'C + BD' - B'D = 0,$$

the required equation.

Cambridge, 10*th September*, 1884.

819.

ON TWO CASES OF THE QUADRIC TRANSFORMATION BETWEEN TWO PLANES.

[From the *Johns Hopkins University Circulars*, No. 13 (1882), pp. 178, 179.]

SEEKING for the coordinates x_3, y_3, z_3 of the third point of intersection of the cubic curve $x^3+y^3+z^3+6lxyz=0$ by the line through any two points (x_1, y_1, z_1), (x_2, y_2, z_2) on the curve, the expressions present themselves in the form

$$x_3 : y_3 : z_3 = P+2lA : Q+2lB : R+2lC,$$

where

$$P = x_1y_1y_2^2 + z_1x_1z_2^2 - y_1^2x_2y_2 - z_1^2z_2x_2, \quad A = x_1^2y_2z_2 - y_1z_1x_2^2,$$
$$Q = y_1z_1z_2^2 + x_1y_1x_2^2 - z_1^2y_2z_2 - x_1^2x_2y_2, \quad B = y_1^2z_2x_2 - z_1x_1y_2^2,$$
$$R = z_1x_1x_2^2 + y_1z_1y_2^2 - x_1^2z_2x_2 - y_1^2y_2z_2, \quad C = z_1^2x_2y_2 - x_1y_1z_2^2;$$

but it is known that, in virtue of

$$U_1 = x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = 0, \quad U_2 = x_2^3+y_2^3+z_2^3+6lx_2y_2z_2 = 0,$$

which connect the coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), we have $P:Q:R = A:B:C$*, so that the coordinates (x_3, y_3, z_3) of the third point of intersection may be expressed indifferently in the two forms

$$x_3 : y_3 : z_3 = P : Q : R, \quad \text{and} \quad x_3 : y_3 : z_3 = A : B : C.$$

But these considered irrespectively of the equations $U_1=0$, $U_2=0$, are distinct formulæ, each of them separately establishing a correspondence between the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), or if we regard one of these points as a fixed point, then a correspondence between the remaining two points, or if we consider these as belonging each to its own plane, then a correspondence between two planes.

* See Sylvester on Rational Derivation of Points on Cubic Curves, *Amer. Jour. of Math.* vol. III. p. 62.

Writing for convenience (a, b, c) for the coordinates of the fixed point, and (x_1, y_1, z_1), (x_2, y_2, z_2) for those of the other two points, the formulæ with A, B, C give thus the correspondence

$$x_2 : y_2 : z_2 = bcx_1^2 - a^2y_1z_1 : cay_1^2 - b^2z_1x_1 : abz_1^2 - c^2x_1y_1,$$

which is the first of the two cases in question. These equations give reciprocally

$$x_1 : y_1 : z_1 = bcx_2^2 - a^2y_2z_2 : cay_2^2 - b^2z_2x_2 : abz_2^2 - c^2x_2y_2,$$

or the correspondence is a (1, 1) quadric correspondence.

The formulæ with P, Q, R give in like manner

$$x_2 : y_2 : z_2 = a(ax_1^2 + by_1^2 + cz_1^2) - x_1(a^2x_1 + b^2y_1 + c^2z_1), \text{ \&c.},$$

or if for shortness

$$\Omega_1 = ax_1^2 + by_1^2 + cz_1^2, \quad \Theta_1 = a^2x_1 + b^2y_1 + c^2z_1,$$

then

$$x_2 : y_2 : z_2 = a\Omega_1 - x_1\Theta_1 : b\Omega_1 - y_1\Theta_1 : c\Omega_1 - z_1\Theta_1,$$

which is the second of the two cases. We have reciprocally

$$x_1 : y_1 : z_1 = a\Omega_2 - x_2\Theta_2 : b\Omega_2 - y_2\Theta_2 : c\Omega_2 - z_2\Theta_2,$$

where

$$\Omega_2 = ax_2^2 + by_2^2 + cz_2^2, \quad \Theta_2 = a^2x_2 + b^2y_2 + c^2z_2,$$

and the correspondence is thus in this case also a (1, 1) quadric correspondence.

820.

ON A PROBLEM OF ANALYTICAL GEOMETRY.

[From the *Johns Hopkins University Circulars*, No. 15 (1882), p. 209.]

THE object of the present note is only to call attention to a problem of Analytical Geometry which presents itself in connexion with the reduction of an algebraical integral, and which is solved, pp. 21, 22 of Clebsch and Gordan's *Theorie der Abel'schen Functionen* (Leipzig, 1866); viz. the problem is, considering a line drawn through two given points of a curve $f=0$ of the order n, to find the equation of a curve $\Omega=0$ of the order $n-2$ passing through the remaining $n-2$ points of intersection of the line with the curve f, through the double points of f, and through as many other given points as are required for the determination of the curve. If, for instance, f is a quartic curve without double points, then Ω is the quadric curve which passes through the remaining two intersections of the line with Ω, and through three given points. Take (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) the coordinates of the two given points on the curve f; (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) for the coordinates of the three given points: and write $\Omega, =(a, b, c, f, g, h \between x, y, z)^2=0$ for the equation of the required curve. In the equation $f, =(x, y, z)^4=0$, write $x, y, z=\lambda\xi_1+\mu\xi_2,\ \lambda\eta_1+\mu\eta_2,\ \lambda\zeta_1+\mu\zeta_2$: we obtain an equation originally of the fourth order in (λ, μ), but which divides by $\lambda\mu$, and which when this factor is thrown out becomes

$$\alpha\lambda^2+\beta\lambda\mu+\gamma\mu^2=0\,;$$

where

$$\alpha=(\xi_1, \eta_1, \zeta_1)^3(\xi_2, \eta_2, \zeta_2)\,, \quad =(\xi_2\partial_{\xi_1}+\eta_2\partial_{\eta_1}+\zeta_2\partial_{\zeta_1})f_1,$$

$$\beta=(\xi_1, \eta_1, \zeta_1)^2(\xi_2, \eta_2, \zeta_2)^2,$$

$$\gamma=(\xi_1, \eta_1, \zeta_1)\,(\xi_2, \eta_2, \zeta_2)^3, \quad =(\xi_1\partial_{\xi_2}+\eta_1\partial_{\eta_2}+\zeta_1\partial_{\zeta_2})f_2,$$

where for shortness f_1, f_2 are written to denote $f(\xi_1, \eta_1, \zeta_1)$, $f(\xi_2, \eta_2, \zeta_2)$ respectively.

The condition as to the two points obviously is that, making the same substitution $x, y, z = \lambda\xi_1 + \mu\xi_2, \lambda\eta_1 + \mu\eta_2, \lambda\zeta_1 + \mu\zeta_2$ in the equation $\Omega = (a, \ldots \between x, y, z)^2, = 0$, we must obtain the same quadric equation in (λ, μ). We have thus two conditions, which, introducing an indeterminate multiplier θ, are expressed by the three equations

$$\begin{aligned}
&(a, \ldots \between \xi_1, \eta_1, \zeta_1)^2 &&= \theta\alpha,\\
&(a, \ldots \between \xi_1, \eta_1, \zeta_1)(\xi_2, \eta_2, \zeta_2) &&= \theta\beta,\\
&(a, \ldots \between \xi_2, \eta_2, \zeta_2)^2 &&= \theta\gamma.
\end{aligned}$$

The conditions as to the three points are obviously

$$\begin{aligned}
(a, \ldots \between x_1, y_1, z_1)^2 &= 0,\\
(a, \ldots \between x_2, y_2, z_2)^2 &= 0,\\
(a, \ldots \between x_3, y_3, z_3)^2 &= 0,
\end{aligned}$$

and these equations determine the ratios of a, b, c, f, g, h. But to complete the solution the convenient course is to regard the function $\Omega, = (a, \ldots \between x, y, z)^2$ as a quantity to be determined, and consequently to join to the foregoing the equation

$$(a, \ldots \between x, y, z)^2 = \Omega;$$

we have thus seven equations from which (a, b, c, f, g, h) may be eliminated, the result being expressed by means of a determinant of the seventh order

$$\begin{vmatrix}
(x, y, z)^2, & \Omega\\
(\xi_1, \eta_1, \zeta_1)^2, & \theta\alpha\\
(\xi_1, \eta_1, \zeta_1 \between \xi_2, \eta_2, \zeta_2), & \theta\beta\\
(\xi_2, \eta_2, \zeta_2)^2, & \theta\gamma\\
(x_1, y_1, z_1)^2, & 0\\
(x_2, y_2, z_2)^2, & 0\\
(x_3, y_3, z_3)^2, & 0
\end{vmatrix} = 0,$$

viz. this is an equation of the form $A\Omega = \theta\nabla$, where A is a constant determinant of the sixth order (i.e. a determinant not involving x, y, z), ∇ a determinant of the seventh order, a quadric function of (x, y, z), obtained from the foregoing determinant by writing therein $\Omega = 0$ and $\theta = 1$: the multiplier θ is and remains arbitrary: but it is convenient to take it to be $= 1$, viz. we thus not only find the equation $\Omega = 0$, of the required conic, but we put a determinate value on the quadric function Ω itself. And this being so, it is to be remarked that, for $(x, y, z) = (\xi_1, \eta_1, \zeta_1)$, we have $\Omega = \alpha, = (\xi_2\partial_{\xi_1} + \eta_2\partial_{\eta_1} + \zeta_2\partial_{\zeta_1})f_1$: and so for $(x, y, z) = (\xi_2, \eta_2, \zeta_2)$, we have

$$\Omega = \gamma, = (\xi_1\partial_{\xi_2} + \eta_1\partial_{\eta_2} + \zeta_1\partial_{\zeta_2})f_2.$$

821.

ON THE GEOMETRICAL REPRESENTATION OF AN EQUATION BETWEEN TWO VARIABLES.

[From the *Johns Hopkins University Circulars*, No. 15 (1882), p. 210.]

AN equation between two variables cannot be represented in a satisfactory manner by a curve, for this serves only to represent the corresponding *real* values of the variables: to represent the imaginary values the natural course is to represent each variable by a point in a plane, viz. the variable $z, = x + iy$, will be represented by a point the coordinates of which are the components x and y of the variable, and similarly the variable $z', = x' + iy'$, by a point the coordinates of which are the components x' and y' of the variable: the equation between the two variables then establishes a correspondence between the two variable points, or say a correspondence between the planes which contain the two points respectively: and it is this correspondence of two planes which is the proper geometrical representation of the equation between the two variables: to exhibit the correspondence we may in either of the planes draw a network of curves at pleasure, and then draw in the other plane the network of corresponding curves. This well-known theory [can be] illustrated for the case $z' = \sqrt{z^4 - 1}$; taking in the infinite half-plane y positive about the origin as centre a system of semicircles, to these correspond in the infinite plane of $x'y'$ a series of lemniscate-shaped curves: and by means of these it is easy to show in the second plane the path corresponding to a given path of the point $z, = x + iy$, in the first half-plane.

822.

ON ASSOCIATIVE IMAGINARIES.

[From the *Johns Hopkins University Circulars*, No. 15 (1882), pp. 211, 212.]

THE imaginaries (x, y) defined by the equations

$$x^2 = ax + by,$$
$$xy = cx + dy,$$
$$yx = ex + fy,$$
$$y^2 = gx + hy,$$

will not be in general associative: to make them so, we must have 8 double relations corresponding to the combinations x^3, x^2y, xyx, xy^2, yx^2, yxy, y^2x, y^3 respectively, viz. the first of these gives $x.x^2 - x^2.x = 0$, that is, $0 = x(ax+by) - (ax+by)x$, $= b(xy - yx)$, $= b[(c-e)x + (d-f)y]$; that is, $0 = b(c-e)$ and $0 = b(d-f)$: and similarly for each of the other terms. We thus obtain apparently 16, but really only 12, relations between the 8 coefficients a, b, c, d, e, f, g, h, viz. the relations so obtained are

$$b(e-c) = 0 \text{ (twice)}, \quad b(d-f) = 0, \quad g(c-e) = 0, \quad g(f-d) = 0 \text{ (twice)},$$
$$bg - ed = 0 \text{ (twice)}, \quad bg - ef = 0 \text{ (twice)},$$
$$c^2 + dg - ag - ch = 0, \quad d^2 + bc - ad - bh = 0, \quad e^2 + fg - ag - eh = 0, \quad f^2 + be - af - bh = 0,$$
$$a(c-e) - cf + de = 0, \quad h(f-d) - cf + de = 0.$$

From the first four equations it appears that either $b = 0$ and $g = 0$, or else $c = e$ and $d = f$: for brevity, I attend only to the latter case, giving the commutative system

$$x^2 = ax + by,$$
$$xy = yx = cx + dy,$$
$$y^2 = gx + hy.$$

In order that this may be associative, we must still have the relations

$$bg = cd,$$
$$c^2 + dg - ag - ch = 0,$$
$$d^2 + bc - ad - bh = 0,$$

which are all three of them satisfied by $g = \frac{cd}{b}$, $h = \frac{d^2 + bc - ad}{b}$, viz. we thus have the associative and commutative system

$$x^2 = ax + by,$$
$$xy = yx = cx + dy,$$
$$y^2 = \frac{cd}{b}x + \frac{d^2 + bc - ad}{b}y.$$

I did not perceive how to identify this system with any of the double algebras of B. Peirce's Linear Associative Algebra, see pp. 120—122 of the Reprint, *American Journal of Mathematics*, t. IV. (1881); but it has been pointed out to me by Mr C. S. Peirce that my system, in the general case $ad - bc$ not $= 0$, is expressible as a mixture of two algebras of the form (a_1), see p. 120 (*l.c.*); whereas if $ad - bc = 0$, it is reducible to the form (c_2), see p. 122 (*l.c.*). The object of the present Note is to exhibit in the simple case of two imaginaries the whole system of relations which must subsist between the coefficients in order that the imaginaries may be associative; that is, the system of equations which are solved implicitly by the establishment of the several multiplication tables of the memoir just referred to.

823.

ON THE GEOMETRICAL INTERPRETATION OF CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

[From the *Johns Hopkins University Circulars*, No. 17 (1882), p. 238.]

I HAVE given in my *Elliptic Functions* expressions for the sn^2 of $u+\frac{1}{2}K$, $u+\frac{1}{2}iK'$, $u+\frac{1}{2}K+\frac{1}{2}iK'$; but it is better to consider the dn^2, sn^2, cn^2 of these combinations respectively, and to write the formulæ thus:

$$\mathrm{dn}^2(u+\tfrac{1}{2}K) = k'\frac{\mathrm{dn}\,u-(1-k')\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u+(1-k')\,\mathrm{sn}\,u\,\mathrm{cn}\,u}, \quad = k'\frac{1-k^2x-(1-k')\,y}{1-k^2x+(1-k')\,y};$$

$$\mathrm{sn}^2(u+\tfrac{1}{2}iK') = \frac{1}{k}\frac{(1+k)\,\mathrm{sn}\,u+i\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{(1+k)\,\mathrm{sn}\,u-i\,\mathrm{cn}\,u\,\mathrm{dn}\,u}, \quad = \frac{1}{k}\frac{(1+k)\,x+iy}{(1+k)\,x-iy};$$

$$\mathrm{cn}^2(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{-ik'}{k}\frac{\mathrm{cn}\,u-(k+ik')\,\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u+(k+ik')\,\mathrm{sn}\,u\,\mathrm{dn}\,u}, \quad = \frac{-ik'}{k}\frac{1-x-(k+ik')\,y}{1-x+(k+ik')\,y};$$

where in the last set of values x, y are used to denote $\mathrm{sn}^2 u$ and $\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u$ respectively; and the formulæ are thus brought into connexion with the cubic curve $y^2 = x(1-x)(1-k^2x)$. The curve has an inflexion at infinity on the line $x=0$; and the three tangents from the inflexion are $x=0$, $x=1$, $x=\frac{1}{k^2}$, touching the curve at the points $x, y = (0, 0), (1, 0), \left(\frac{1}{k^2}, 0\right)$ respectively: hence these points are sextactic points. We may from any one of them, for instance the point $(0, 0)$, draw four tangents to the curve, $(1+k)\,x+iy=0$, $(1+k)\,x-iy=0$; $(1-k)\,x+iy=0$, $(1-k)\,x-iy=0$; where the first and second of these lines form a pair, and the third and fourth of them form a pair, viz. the two tangents of a pair touch in points such that the line joining them passes through the point of inflexion: in particular, for the first-mentioned pair, the equation of the line joining the points of contact is $1+kx=0$. The linear functions belonging to a pair of tangents are precisely those which present themselves in the formulæ; thus if $T_1=(1+k)\,x+iy$, $T_2=(1+k)\,x-iy$, the second of the three formulæ is $\mathrm{sn}^2(u+\frac{1}{2}K)=\frac{1}{k}\frac{T_1}{T_2}$; and the other two formulæ correspond in like manner to pairs of tangents from the sextactic points $\left(\frac{1}{k^2}, 0\right)$, and $(1, 0)$ respectively. The formulæ are connected with the fundamental equations expressing the functions sn, cn, dn as quotients of theta functions.

824.

NOTE ON THE FORMULÆ OF TRIGONOMETRY.

[From the *Johns Hopkins University Circulars*, No. 17 (1882), p. 241.]

THE equations $a = c\cos B + b\cos C$, $b = a\cos C + c\cos A$, $c = b\cos A + a\cos B$, which connect together the sides a, b, c and the angles A, B, C of a plane triangle, may be presented in an algebraical rational form, by introducing in place of the angles A, B, C the functions $\cos A + i\sin A$, $\cos B + i\sin B$, $\cos C + i\sin C$, viz. calling these $\frac{x}{w}$, $\frac{y}{w}$, $\frac{z}{w}$ respectively, or, what is the same thing, writing $2\cos A = \frac{x}{w} + \frac{w}{x}$, $2\cos B = \frac{y}{w} + \frac{w}{y}$, $2\cos C = \frac{z}{w} + \frac{w}{z}$, then the foregoing equations may be written

$$\begin{aligned}
&(-2yzw\ ,\ y(z^2+w^2),\ z(y^2+w^2)\between a,\ b,\ c) = 0,\\
&(x(z^2+w^2),\ -2zxw\ ,\ z(x^2+w^2)\between\ \ ,,\ \ ) = 0,\\
&(x(y^2+w^2),\ y(x^2+w^2),\ -2xyw\ \between\ \ ,,\ \ ) = 0,
\end{aligned}$$

that is, as a system of bipartite equations linear in (a, b, c) and cubic in (x, y, z, w) respectively.

Similarly in Spherical Trigonometry, writing as above for the angles, and for the sides writing in like manner $2\cos a = \frac{\alpha}{\delta} + \frac{\delta}{\alpha}$, $2\cos b = \frac{\beta}{\delta} + \frac{\delta}{\beta}$, $2\cos c = \frac{\gamma}{\delta} + \frac{\delta}{\gamma}$, we have a system of bipartite equations separately homogeneous in regard to (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$ respectively.

825.

A MEMOIR ON THE ABELIAN AND THETA FUNCTIONS.

[Chapters I to III, *American Journal of Mathematics*, t. V. (1882), pp. 137—179; Chapters IV to VII, *ib.*, t. VII. (1885), pp. 101—167.]

THE present memoir is based upon Clebsch and Gordan's *Theorie der Abel'schen Functionen*, Leipzig, 1866 (here cited as C. and G.); the employment of differential rather than of integral equations is a novelty; but the chief addition to the theory consists in the determination which I have made for the cubic curve, and also (but not as yet in a perfect form) for the quartic curve, of the differential expression $d\Pi_{\xi\eta}$ (or as I write it $d\Pi_{12}$) in the integral of the third kind $\int_\alpha^\beta d\Pi_{\xi\eta}$ in the final normal form (endliche Normalform) for which we have (p. 117) $\int_\xi^\eta d\Pi_{\alpha\beta} = \int_\alpha^\beta d\Pi_{\xi\eta}$, the limits and parametric points interchangeable. The want of this determination presented itself to me as a *lacuna* in the theory during the course of lectures on the subject which I had the pleasure of giving at the Johns Hopkins University, Baltimore, U.S.A., in the months January to June, 1882, and I succeeded in effecting it for the cubic curve; but it was not until shortly after my return to England that I was able partially to effect the like determination in the far more difficult case of the quartic curve. The memoir contains, with additional developments, a reproduction of the course of lectures just referred to. I have endeavoured to simplify as much as possible the notations and demonstrations of Clebsch and Gordan's admirable treatise; to bring some of the geometrical results into greater prominence; and to illustrate the theory by examples in regard to the cubic, the nodal quartic, and the general quartic curves respectively. The various chapters are: I, Abel's Theorem; II, Proof of Abel's Theorem; III, The Major Function; IV, The Major Function (continued); V, Miscellaneous Investigations; VI, The Nodal Quartic; VII, The Functions T, U, V, Θ. The paragraphs of the whole memoir will be numbered continuously.

CHAPTER I. ABEL'S THEOREM.

The Differential Pure and Affected Theorems. Art. Nos. 1 to 5.

1. We have a fixed curve and a variable curve, and the differential pure theorem consists in a set of linear relations between the displacements of the intersections of the two curves; in the affected theorem, a linear function of the displacements is equated to another differential expression. I state the two theorems, giving afterwards the necessary explanations.

The pure theorem is

$$\Sigma (x,\ y,\ z)^{n-3}\, d\omega = 0.$$

The affected theorem is

$$\Sigma \frac{(x,\ y,\ z)_{12}^{n-2}\, d\omega}{012} = -\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2}.\,*$$

2. We have a fixed curve $f=0$, or say the curve f, or simply the fixed curve, of the order n, with δ dps, and therefore of the deficiency $\frac{1}{2}(n-1)(n-2)-\delta,\ =p$. The expression "the dps" means always the δ dps of f.

And we have a variable curve $\phi=0$, or say the curve ϕ, or simply the variable curve, of the order m, passing through the dps and besides meeting the fixed curve in $mn-2\delta$ variable points.

Moreover, $d\omega$ is the displacement of the current point 0, coordinates $(x,\ y,\ z)$, on the fixed curve, viz. the equation $f=0$ gives

$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0,$$

$$\frac{df}{dx}\ x + \frac{df}{dy}\ y + \frac{df}{dz}\ z = 0,$$

and we thence have

$$\frac{df}{dx} : \frac{df}{dy} : \frac{df}{dz} = y\,dz - z\,dy\ :\ z\,dx - x\,dz\ :\ x\,dy - y\,dx,$$

so that we have three equal values each of which is put $= d\omega$, viz. we write

$$\frac{y\,dz - z\,dy}{\dfrac{df}{dx}} = \frac{z\,dx - x\,dz}{\dfrac{df}{dy}} = \frac{x\,dy - y\,dx}{\dfrac{df}{dz}},\ = d\omega,$$

and $d\omega$ as thus defined is the displacement.

* For comparison with C. and G. observe that in the equation, p. 47, $V = \log\dfrac{\psi(\eta)\,\phi(\xi)}{\phi(\eta)\,\psi(\xi)},\ = \log\dfrac{\psi_2\phi_1}{\phi_2\psi_1}$ suppose, their ψ belongs to the upper limit and corresponds to my ϕ: the equation gives therefore $dV = -\dfrac{\delta\psi_1}{\psi_1} + \dfrac{\delta\psi_2}{\psi_2}$, agreeing with the formula in the text.

$(x, y, z)^{n-3}=0$ is the minor curve, viz. the general curve of the order $n-3$, which passes through the dps*; and the function $(x, y, z)^{n-3}$ is the minor function.

1 and 2 are fixed points on f, called the parametric points, coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; and 012 denotes the determinant

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

so that $012=0$ is the equation of the line joining the points 1 and 2: this line meets the fixed curve in $n-2$ other points, called the residues of 1, 2.

$(x, y, z)_{12}^{n-2}=0$ is the major curve *quoad* the points 1 and 2; viz. this is the general curve of the order $n-2$, passing through the dps and also through the residues of 1, 2.

But further, the function $(x, y, z)_{12}^{n-2}$ is the proper major function; viz. the implicit factor of the function is so determined that, taking $0=1$, the current point at 1, that is, writing (x_1, y_1, z_1) for (x, y, z), the function $(x, y, z)_{12}^{n-2}$ reduces itself to the polar function $\left(x_2\frac{d}{dx_1}+y_2\frac{d}{dy_1}+z_2\frac{d}{dz_1}\right)f_1$, afterwards written $n\,.\,1^{n-1}\,2$, of f: this implies that taking $0=2$, the current point at 2, the function reduces itself to the polar function $n\,.\,12^{n-1}$.

ϕ_1 is what ϕ becomes on writing therein (x_1, y_1, z_1) for (x, y, z): and similarly ϕ_2 is what ϕ becomes on writing therein (x_2, y_2, z_2) for (x, y, z).

δ denotes differentiation in regard only to the coefficients of ϕ; viz. writing $\phi=(\mathrm{a},\ldots\between x, y, z)^m$ we have $\delta\phi=(d\mathrm{a},\ldots\between x, y, z)^m$, and similarly $\delta\phi_1$ and $\delta\phi_2=(d\mathrm{a},\ldots\between x_1, y_1, z_1)^m$ and $(d\mathrm{a},\ldots\between x_2, y_2, z_2)^m$ respectively.

The sum Σ extends to all the variable intersections of the two curves.

3. As to the meaning of the theorems, consider first the pure theorem. The variable intersections are not all of them arbitrary points on the fixed curve: a certain number of them taken at pleasure on the fixed curve will determine the remaining variable intersections; and there are thus a certain number of integral relations between the coordinates of the variable intersections; to each such integral relation there corresponds a linear relation between the displacements $d\omega$ of these points, or say a displacement-relation. It is precisely these displacement-relations which are given by the theorem, viz. the equation

$$\Sigma\,(x, y, z)^{n-3}\,d\omega=0$$

breaks up into as many linear relations as there are arbitrary constants in the function $(x, y, z)^{n-3}$, which equated to zero gives a curve of the order $n-3$ passing through the dps; for instance $n=3$, $\delta=0$, the equation gives the single relation $\Sigma d\omega=0$; but $n=4$, $\delta=0$, the equation gives the three relations $\Sigma x\,d\omega=0$, $\Sigma y\,d\omega=0$, $\Sigma z\,d\omega=0$.

* This definition implies that the number of dps is at most $=\frac{1}{2}(n-1)(n-2)-1$, that is, that the fixed curve is not unicursal. But see *post*, No. 21.

4. It is of course important to show, and it will be shown, that the number of independent displacement-relations given by the theorem is equal to the number of independent integral relations between the variable intersections.

5. Observe that the pure theorem gives *all* the displacement-relations between the variable intersections; we are hereby led to see the nature of the affected theorem. Taking at pleasure on the fixed curve the sufficient number of variable intersections, the coefficients of ϕ are thereby determined in terms of the coordinates of the assumed variable intersections*, and hence the value of $-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}$ is given as a linear function of the corresponding displacements $d\omega$; and, substituting this value, the affected theorem gives a linear relation between the displacements $d\omega$ of the several variable intersections. But any such linear relation must clearly be a mere linear combination of the displacement-relations $\Sigma(x, y, z)^{n-3}\,d\omega=0$ given by the pure theorem.

Examples of the Pure Theorem—The Fixed Curve a Cubic. Art. Nos. 6 to 12.

6. The pure theorem is not applicable to the case $n=2$, the fixed curve a conic: it in fact gives no displacement-relation; and this is as it should be, for the variable intersections are all of them arbitrary.

The next case is $n=3$, $\delta=0$, the fixed curve a cubic. For greater simplicity the equation is taken in Cartesian coordinates. In general for such an equation, writing in the homogeneous formulæ $z=1$, we have

$$d\omega=\frac{dx}{\frac{df}{dy}}=-\frac{dy}{\frac{df}{dx}},$$

the two values being of course equal in virtue of $\frac{df}{dx}dx+\frac{df}{dy}dy=0$; taking the former value and considering $\frac{df}{dy}$ as expressed in terms of x, let this be called P (of course, P is an irrational function of x): then we have $d\omega=\frac{dx}{P}$; and similarly $d\omega_1=\frac{dx_1}{P_1}$, &c.

The fixed curve being then a cubic, let the variable curve be a line; this meets the cubic in three points, say 1, 2, 3; and any two of these determine the line, and therefore the third point; there should therefore be one integral relation, and consequently one displacement-relation; and this is what is given by the theorem, viz. we have $\Sigma d\omega=0$, that is, $d\omega_1+d\omega_2+d\omega_3=0$, or, what is the same thing,

$$\frac{dx_1}{P_1}+\frac{dx_2}{P_2}+\frac{dx_3}{P_3}=0.$$

* The coefficients are determined, except it may be as to some constants which remain arbitrary but which disappear from the difference $-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}$; this will be explained further on in the text.

The corresponding integral equation is the equation which expresses that the points 1, 2, 3 are in a line, viz. considering y_1, y_2, y_3 as given functions of x_1, x_2, x_3 respectively, this is

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0,$$

or, in the notation already made use of for such a determinant, $123 = 0$.

7. This equation $d\omega_1 + d\omega_2 + d\omega_3 = 0$, where $d\omega$ denotes $\frac{dx}{P}$, has a peculiar interpretation when we consider the coefficients of the cubic as arbitrary constants, and therefore the cubic as a curve depending upon nine arbitrary constants*. In taking 1 a point on the curve, we in effect determine y_1 as a function of x_1 and the nine constants; and similarly in taking 2 a point on the curve, we determine y_2 as a function of x_2 and the nine constants; the points 1 and 2 determine the third intersection 3, and we have thus x_3 determined as a function of x_1, x_2 and the nine constants.

Considering x_3 as thus expressed, we have $dx_3 = \frac{dx_3}{dx_1}dx_1 + \frac{dx_3}{dx_2}dx_2$, an equation which must agree with $d\omega_1 + d\omega_2 + d\omega_3 = 0$, that is, with $dx_3 = -\frac{P_3}{P_1}dx_1 - \frac{P_3}{P_2}dx_2$. It follows that we have $\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2} = \frac{P_2}{P_1}$, and taking the logarithms and differentiating with regard to x_1 and x_2, we find $\frac{d}{dx_1}\frac{d}{dx_2}\log\left(\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2}\right) = 0$, a partial differential equation of the third order, independent of any particular cubic curve, and satisfied by x_3 considered as a function of x_1, x_2 and the nine constants. Observe that starting from the expression for x_3, and proceeding to the differential coefficients of the third order, we have ten equations from which the nine constants can be eliminated, that is, we ought to have a partial differential equation of the third order: and conversely that the equation for x_3, as containing nine arbitrary constants, is a complete solution of the partial differential equation: the complete solution of the partial differential equation in question is thus the equation which expresses that 3 is the remaining intersection of the line through 1 and 2 with a cubic.

8. The partial differential equation has a geometrical interpretation, or is at least very closely connected with a geometrical property. Consider three consecutive positions of the line, meeting the cubic in the points 1, 2, 3; 1′, 2′, 3′ and 1″, 2″, 3″ respectively: the three lines constitute a cubic curve: the nine points are thus the intersections of two cubic curves, or say they are an "ennead" of points: and any eight of the points thus determine uniquely the ninth point.

* This theory was communicated by me to Section A of the British Association at the York meeting. See *B. A. Report*, 1881, pp. 534, 535, [712], "A Partial Differential Equation connected with the Simplest Case of Abel's Theorem."

9. As a particular example, let the cubic be $x^3+y^3-1=0$; then $y=\sqrt[3]{1-x^3}$, and $d\omega=\dfrac{dx}{y^2}$, $=\dfrac{dx}{(1-x^3)^{\frac{2}{3}}}$ *; and with these values we have as before the differential relation $d\omega_1+d\omega_2+d\omega_3=0$, and the integral relation $123=0$. I give a direct verification. To find x_3, y_3 the coordinates of the third intersection, we may in the equation of the cubic write x_3, y_3, $1=\lambda x_1+\mu x_2$, $\lambda y_1+\mu y_2$, $\lambda+\mu$ respectively, and then writing for shortness $1^2 2=x_1^2x_2+y_1^2y_2-1$, $12^2=x_1x_2^2+y_1y_2^2-1$, we obtain for the determination of λ, μ the equation $\lambda\,.\,1^2 2+\mu\,.\,12^2=0$.

This being so, from the equation $123=0$ we obtain by differentiation

$$\Sigma\{dx_1(y_2-y_3)-dy_1(x_2-x_3)\}=0,$$

the sum consisting of three terms, the second and third of them being obtained from the one written down by the cyclical interchange of the numbers 1, 2, 3. But we have $x_1^2dx_1+y_1^2dy_1=0$, and the equation thus is

$$\Sigma\frac{dx_1}{y_1^2}\{y_1^2(y_2-y_3)+x_1^2(x_2-x_3)\}=0:$$

this will reduce itself to $\Sigma\dfrac{dx_1}{y_1^2}=0$, if only the three coefficients in { } are equal, that is, we ought to have

$$y_1^2(y_2-y_3)+x_1^2(x_2-x_3)=y_2^2(y_3-y_1)+x_2^2(x_3-x_1)=y_3^2(y_1-y_2)+x_3^2(x_1-x_2).$$

Comparing for instance the first and second terms, the equation is

$$-y_3(y_1^2+y_2^2)-x_3(x_1^2+x_2^2)+(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

or, as this may be written,

$$-(\lambda y_1+\mu y_2)(y_1^2+y_2^2)-(\lambda x_1+\mu x_2)(x_1^2+x_2^2)+(\lambda+\mu)(x_1^2x_2+y_1^2y_2+x_1x_2^2+y_1y_2^2)=0,$$

where the whole coefficient of λ is $-x_1^3-y_1^3+x_1^2x_2+y_1^2y_2$, which in fact is $x_1^2x_2+y_1^2y_2-1$, $=1^2 2$; and similarly the whole coefficient of μ is 12^2; the equation is thus $\lambda\,.\,1^2 2+\mu\,.\,12^2=0$, which is right. The first and second coefficients are thus equal, and in like manner the first and third coefficients are equal; we have thus the required result,

$$\frac{dx_1}{y_1^2}+\frac{dx_2}{y_2^2}+\frac{dx_3}{y_3^2}=0.$$

10. In all that follows, the cubic might be any cubic whatever, but to fix the ideas I take a particular form.

Let the cubic be $y^2-X=0$, X a cubic function $(x,1)^3$, or say even $X=x\,.\,1-x\,.\,1-k^2x$, then $y=\sqrt{X}$, $d\omega=\dfrac{dx}{y}$, $=\dfrac{dx}{\sqrt{X}}$; and with these values we have the differential relation

* Writing $f=x^3+y^3-1$ we should have $\dfrac{df}{dy}=3y^2$, and therefore $d\omega=\dfrac{dx}{3y^2}$; but the $\frac{1}{3}$ enters as a common factor in all the $d\omega$'s, and it may clearly be disregarded: the value in the text, $d\omega=\dfrac{dx}{y^2}$ could of course be obtained by writing, as we may do, $f=\frac{1}{3}(x^3+y^3-1)$, and so in other cases.

$d\omega_1 + d\omega_2 + d\omega_3 = 0$, and the integral relation $123 = 0$. This last equation is an integral of the differential equation $d\omega_1 + d\omega_2 + d\omega_3 = 0$; as not containing any arbitrary constant, it is a particular integral.

But regard one of the three points, say 3, as a fixed point, that is, let the line pass through the fixed point 3 of the cubic, and besides meet it in the points 1 and 2. We write $d\omega_3 = 0$, and the differential equation thus is $d\omega_1 + d\omega_2 = 0$, while the integral equation is as before $123 = 0$; this equation, as containing one arbitrary constant, is the general integral of $d\omega_1 + d\omega_2 = 0$.

Let the variable curve be a conic; say the intersections with the cubic are 1, 2, 3, 4, 5, 6. Any five of these points determine the conic, and therefore the sixth point; there is thus one integral relation, the equation $123456 = 0$, which expresses that the six points are in a conic, and there should therefore be one displacement-relation, viz. this is the equation $\Sigma d\omega = 0$, that is, $d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 + d\omega_5 + d\omega_6 = 0$.

We have thus $123456 = 0$, as a particular integral of

$$d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 + d\omega_5 + d\omega_6 = 0.$$

If, however, we take 6 a fixed point on the cubic, then we have the same equation as the general integral of $d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 + d\omega_5 = 0$.

But taking also 5 a fixed point of the cubic we have as an integral of $d\omega_1 + d\omega_2 + d\omega_3 + d\omega_4 = 0$, the foregoing equation $123456 = 0$, which contains apparently two arbitrary constants; and so if we also fix the point 4, or the points 4 and 3, we have for the differential equations $d\omega_1 + d\omega_2 + d\omega_3 = 0$ and $d\omega_1 + d\omega_2 = 0$, integrals with apparently three arbitrary constants and four arbitrary constants respectively.

11. The explanation is contained in the theory of *Residuation* on a cubic curve. Take the case $d\omega_1 + d\omega_2 + d\omega_3 = 0$, with the integral $123456 = 0$, containing apparently three arbitrary constants, viz. the relation between the variable points 1, 2, 3, is here given by a construction depending on the three fixed points 4, 5, 6 on the cubic; it is to be shown that two of these points can always be regarded as no-matter-what* points. To see that this is so, take on the cubic any two no-matter-what points 4′, 5′: then according to the theory referred to, we can find on the cubic a determinate point 6′ such that the points 4′, 5′ and 6′ establish between the variable points 1, 2, 3, the same relation which is established between them by means of the points 4, 5 and 6; viz. whether in order to determine the point 3 we draw a conic through 1, 2, 4, 5 and 6, or a conic through 1, 2, 4′, 5′ and 6′, we obtain as the remaining intersection of the conic with the cubic one and the same point 3. The construction of 6′ is, through 4, 5 and 6 draw a conic meeting the cubic in any three points 1, 2, 3; through these points and 4′, 5′ draw a conic, the remaining intersection of this with the cubic will be the required point 6′, and the point 6′ thus obtained will be a

* The epithet explains, I think, itself; the point may be any point at pleasure, but it is quite immaterial what point, and for this reason it is not counted as an arbitrary point. The most simple instance is that of two constants presenting themselves in a combination such as $c + c'$: either of them may be regarded as a no-matter-what constant.

determinate point, independent of the particular conic through 4, 5 and 6 used for the construction. Thus 4 and 5 are replaceable by the no-matter-what points 4′ and 5′, or, what is the same thing, two of the points 4, 5 and 6 may be regarded as no-matter-what points, and the number of arbitrary constants is thus reduced to one. And so in other cases, all but one of the fixed points may be regarded as no-matter-what points, and the integral as containing in each case only one arbitrary constant.

But conversely, it being known that the integral of the differential equation contains but one arbitrary constant, we can thence arrive at the theory of residuation.

12. We might go on to the case where the variable curve is a cubic; there are here nine intersections; any eight of these do *not* determine the variable cubic, but they *do* determine the ninth intersection; and there is between the nine intersections one integral relation, and corresponding to it one displacement-relation $\Sigma d\omega = 0$, that is, $d\omega_1 + d\omega_2 + \ldots + d\omega_9 = 0$, given by the pure theorem. But as to this see further on, where it is shown in general that the number of independent integral relations is equal to the number of independent displacement-relations given by the theorem.

Example of the Affected Theorem—Fixed Curve a Circle. Art. Nos. 13 and 14.

13. The fixed curve is taken to be the circle $x^2 + y^2 - 1 = 0$, and the parametric points 1 and 2 to be the points (1, 0) and (0, 1) on this circle. The variable curve is taken to be a line, say the line $ax + by - 1 = 0$, meeting the circle in the points 3 and 4, coordinates (x_3, y_3) and (x_4, y_4) respectively.

Starting from the formula

$$\Sigma \frac{(x, y, 1)^0_{12}\, d\omega}{012} = -\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2},$$

where the summation extends to the points 3 and 4, $(x, y, 1)^0_{12}$ is here a constant, $= 2 \,.\, 12$, that is, $2(x_1x_2 + y_1y_2 - 1)$, which for the points 1, 2 in question is $= -2$. We have 012 denoting the determinant

$$\begin{vmatrix} x, & y, & 1 \\ 1, & 0, & 1 \\ 0, & 1, & 1 \end{vmatrix},$$

which is $= -x - y + 1$; and $d\omega = \dfrac{dx}{2y}$. Also $\dfrac{\delta\phi_1}{\phi_1}$, $= \dfrac{x\,da + y\,db}{ax + by - 1}$, is $= \dfrac{da}{a-1}$, and similarly $\dfrac{\delta\phi_2}{\phi_2}$ is $= \dfrac{db}{b-1}$. The formula thus is

$$\Sigma \frac{dx}{y(x+y-1)} = -\frac{da}{a-1} + \frac{db}{b-1}.$$

The coefficients a and b are determined by means of the points 3 and 4, that is, they are functions of x_3, x_4; and considering them as thus expressed, then (inasmuch

as there is no linear relation between the displacements $\frac{dx_3}{y_3}$ and $\frac{dx_4}{y_4}$ of the two arbitrary points 3 and 4 on the circle) the equation must become an identity in regard to the terms in dx_3 and dx_4 respectively. It only remains to verify that this is so.

14. Writing P, Q, $R = -y_3 + y_4$, $x_3 - x_4$, $x_3y_4 - x_4y_3$; also L_3 and $L_4 = x_3 + y_3 - 1$ and $x_4 + y_4 - 1$ respectively, we have $a = P \div R$, $b = Q \div R$, and the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{1}{(Q-R)(R-P)}\{(Q-R)\,dP + (R-P)\,dQ + (P-Q)\,dR\},$$

where, substituting for dy_3, dy_4 their values in terms of dx_3, dx_4, we have

$$dP,\ dQ,\ dR = \frac{1}{y_3}x_3dx_3 - \frac{1}{y_4}x_4dx_4,\quad \frac{1}{y_3}y_3dx_3 - \frac{1}{y_4}y_4dx_4,\quad \frac{1}{y_3}(x_3x_4 + y_3y_4)\,dx_3 - \frac{1}{y_4}(x_3x_4 + y_3y_4)\,dx_4,$$

and with these values, and by aid of the relations

$$Q - R,\quad R - P,\quad P - Q = x_4L_3 - x_3L_4,\quad y_4L_3 - y_3L_4,\quad -L_3 + L_4,$$

the equation is found to be

$$\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4} = \frac{L_3L_4(x_3x_4 + y_3y_4 - 1)}{(x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4)}\left(\frac{dx_3}{y_3L_3} + \frac{dx_4}{y_4L_4}\right);$$

viz. this will be true if only

$$L_3L_4(x_3x_4 + y_3y_4 - 1) - (x_4L_3 - x_3L_4)(y_4L_3 - y_3L_4) = 0,$$

that is,

$$-x_4y_4L_3^2 - x_3y_3L_4^2 + L_3L_4(x_3x_4 + y_3y_4 + x_3y_4 + x_4y_3 - 1) = 0.$$

But from the values of L_3, L_4 we have $x_4y_4 = \frac{1}{2}L_4^2 + L_4$, $x_3y_3 = \frac{1}{2}L_3^2 + L_3$, and the coefficient of L_3L_4 is $= L_3L_4 + L_3 + L_4$; the equation is thus verified.

The example would perhaps have been more instructive if the points 1 and 2 had been left arbitrary points on the circle, but the working out would have been more difficult.

The Variable Intersections of the Two Curves—Number of Independent Integral Relations.
Art. Nos. 15 to 19.

15. Suppose $n = 3$, $\delta = 0\ (p = 1)$, the fixed curve a cubic; and suppose successively $m = 1, 2, 3, \ldots$, the variable curve a line, conic, cubic, &c.

If $m = 1$, then two points on the cubic determine the line, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m = 2$, then five points on the cubic determine the conic, and consequently the remaining intersection with the cubic; hence there is one integral relation.

If $m=3$, then eight points on the fixed cubic do not determine the variable cubic, but they do determine the ninth intersection. For draw through the eight points a no-matter-what cubic $\chi=0$, the general cubic through the eight points is $\chi+\alpha f=0$, and this meets the fixed cubic in the points $\chi=0$, $f=0$, that is, in the eight points and in one other point independent of the constant α and therefore completely determinate. Hence in this case also there is one integral relation.

So if $m=4$, then eleven points on the cubic do not determine the quartic, but they do determine the remaining intersection. For draw through the eleven points a no-matter-what quartic $\chi=0$, the general quartic through the eleven points is $\chi+(x, y, z)^1 f=0$, and this meets the cubic in the points $\chi=0, f=0$, that is, in the eleven points and in one other point independent of the constants of the linear function $(x, y, z)^1$, and therefore completely determinate. Hence there is one integral relation.

And so in general, the fixed curve being a cubic, then, whatever be the order of the variable curve, there is always one integral relation.

16. Suppose next $n=4$, $\delta=0\,(p=3)$, the fixed curve a general quartic; and as before suppose successively $m=1, 2, 3, \ldots$, the variable curve a line, conic, cubic, &c.

If $m=1$, then two points on the quartic determine the line, and therefore the remaining two intersections; the number of integral relations is $=2$.

If $m=2$, then five points on the quartic determine the conic, and therefore the remaining three intersections; the number of integral relations is $=3$, and similarly if $m=3$, the number of integral relations is $=3$.

If $m=4$, then thirteen points on the fixed quartic do not determine the variable quartic, but they do determine the remaining three intersections. For draw through the thirteen points a no-matter-what quartic $\chi=0$; the general quartic through the thirteen points is $\chi+\alpha f=0$, and this meets the fixed quartic in the points $\chi=0$, $f=0$, that is, in the thirteen points and in three other points, independent of α and thus completely determinate, and the number of integral relations is $=0$; and so in general for any higher value of m, the number is still $=3$.

17. Suppose $n=5$, $\delta=0\,(p=6)$, the fixed curve a general quintic; and as before suppose $m=1, 2, 3, \ldots$ successively.

If $m=1$, then two points on the quintic determine the line, and therefore the remaining three intersections; the number of integral relations is $=3$.

If $m=2$, then five points on the quintic determine the conic, and therefore the remaining five intersections; the number of integral relations is $=5$.

If $m=3$, then 9 points on the quintic determine the cubic, and therefore the remaining six intersections; the number of integral relations is $=6$, and so if $m=4$, or if $m=5$ or any greater number, in the first case directly, and in the other cases by consideration of the quintic $\chi+\alpha f=0$, &c., we find that the number of integral relations is always $=6$.

18. The reasoning is scarcely altered in the case where the fixed curve has dps; thus considering the general case of a fixed curve n, δ, p:

If $m=1$, then $2-\delta$ points on the fixed curve (this implies $\delta \not> 2$) determine the line, and therefore the remaining $n-2\delta-(2-\delta)$, $=n-2-\delta$ intersections; the number of integral relations is $=n-2-\delta$.

If $m=2$, then $5-\delta$ points on the fixed curve (this implies $\delta \not> 5$) determine the conic, and therefore the remaining $2n-2\delta-(5-\delta)$, $=2n-5-\delta$ intersections; the number of integral relations is $=2n-5-\delta$, and so for any value of $m \not> n-3$, and indeed for the values $n-2$ and $n-1$; here $\frac{1}{2}m(m+3)-\delta$ points on the fixed curve (this implies $\delta \not> \frac{1}{2}m(m+3)$) determine the variable curve, and therefore the remaining $mn-2\delta-\{\frac{1}{2}m(m+3)-\delta\}$, $=mn-\frac{1}{2}m(m+3)-\delta$ intersections. Hence the number of integral relations is $=mn-\frac{1}{2}m(m+3)-\delta$, that is, $=p-\frac{1}{2}(n-m-1)(n-m-2)$. And thus in the cases $m=n-2$ and $n-1$ the number is $=p$.

If $m=n$, then $\frac{1}{2}n(n+3)-1-\delta$ points on the fixed curve do not determine the variable curve, but they do determine the remaining $n^2-2\delta-\{\frac{1}{2}n(n+3)-1-\delta\}$, $=\frac{1}{2}(n-1)(n-2)-\delta$, that is, p intersections, and the number of integral relations is thus $=p$; and so, for any higher value of m, the number is still $=p$.

19. The conclusion thus is

$m \not> n-3$, the number of integral relations $=p-\frac{1}{2}(n-m-1)(n-m-2)$,

$m=$ or $>n-2$, " " $=p$.

The integral equations spoken of throughout are of course independent relations.

The Variable Intersections of the Two Curves. Number of Independent Displacement Relations given by the Pure Theorem. Art. No. 20.

20. The number of displacement-relations given by the pure theorem is = number of constants in minor function $(x, y, z)^{n-3}$, which equated to zero represents a curve through the dps, viz. this is always

$$\tfrac{1}{2}(n-1)(n-2)-\delta, =p.$$

But for $m>n-2$, these relations are not independent. For instance, for $n=4$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^1 d\omega=0, \text{ that is, } \Sigma x\, d\omega=0, \quad \Sigma y\, d\omega=0, \quad \Sigma z\, d\omega=0,$$

and conversely from these last equations we have $\Sigma(x, y, z)^1 d\omega=0$. But in this case the variable curve is a line $ax+by+cz=0$; hence writing $(x, y, z)^1=ax+by+cz$, the equation $(x, y, z)^1=0$ is satisfied for each of the intersections of the line with the quartic, and the corresponding equation $\Sigma(x, y, z)^1 d\omega=0$ is an identity. Hence the number of independent displacement-relations is $3-1$, $=2$.

So for $n=5$, $\delta=0$, $m=1$, the displacement-relations are

$$\Sigma(x, y, z)^2 d\omega=0, \text{ that is, } \Sigma(x^2, y^2, z^2, yz, zx, xy)\, d\omega=0,$$

and these six equations give conversely $\Sigma(x, y, z)^2 d\omega = 0$, and in particular they give $\Sigma x(x, y, z)^1 d\omega = 0$, $\Sigma y(x, y, z)^1 d\omega = 0$, $\Sigma z(x, y, z)^1 d\omega = 0$. But if $(x, y, z)^1$ denote $ax + by + cz$, then as before we have $(x, y, z)^1 = 0$, for each of the intersections of the two curves, and the last-mentioned three equations are identities. The number of independent displacement-relations is thus $6 - 3, = 3$.

So for $n = 5$, $\delta = 0$, $m = 2$. Here if the variable curve is $\phi = (a, \ldots \between x, y, z)^2 = 0$, then taking $(x, y, z)^2 = (a, \ldots \between x, y, z)^2$, the equation $(x, y, z)^2 = 0$ is satisfied for each of the intersections of the two curves, and the corresponding equation $\Sigma(x, y, z)^2 d\omega = 0$ is an identity; the number of independent displacement-relations is $6 - 1, = 5$.

The reasoning is the same when δ is not $= 0$, and we see generally that for $m < n - 2$, or say

$m \not> n - 3$, the number of independent displacement-relations

$$= p - \tfrac{1}{2}(n - m - 1)(n - m - 2);$$

while for

$m =$ or $> n - 2$, the number is $= p$;

since in this case the relations given by the theorem are all of them independent. It thus appears *à posteriori*, that in every case the number of independent displacement-relations given by the pure theorem is equal to the number of independent integral relations.

As to the dps of the Fixed Curve. Art. No. 21.

21. I conclude with a general remark applicable to the whole of the three chapters. There is no necessity to attend to all or indeed to any of the dps of the fixed curve. Suppose that the fixed curve has $\delta + \delta'$ dps, where δ, δ' may be either of them or each $= 0$, but attend only to the δ dps, the δ' dps being wholly disregarded, and accordingly let the expression the dps mean as before the δ dps of the fixed curve. No alteration at all is required: only, if δ' be not $= 0$, then $p = \frac{1}{2}(n-1)(n-2) - \delta$ will no longer be the deficiency. To obtain the best theorems we use all the $\delta + \delta'$ dps: but disregarding the δ' dps, we obtain theorems, as for a curve with δ dps, which are true, and may frequently be useful either in their original form or with simplifications introduced therein by afterwards taking account of the δ' dps.

Chapter II. Proof of Abel's Theorem.

Preparation. Art. Nos. 22 and 23.

22. Starting from the equation $\phi = (a, \ldots \between x, y, z)^m = 0$ of the variable curve, we have

$$\frac{d\phi}{dx} dx + \frac{d\phi}{dy} dy + \frac{d\phi}{dz} dz + \delta\phi = 0,$$

$$\frac{d\phi}{dx} x + \frac{d\phi}{dy} y + \frac{d\phi}{dz} z \qquad = 0,$$

where $\delta\phi = (da, \ldots \between x, y, z)^m$. Let τ denote an arbitrary linear function, $= ax + by + cz$; multiply the two equations by $ax + by + cz$, $= \tau$, and $-(a\,dx + b\,dy + c\,dz)$, $= -d\tau$ respectively, and add: we obtain

$$(y\,dz - z\,dy)\left(b\frac{d\phi}{dz} - c\frac{d\phi}{dy}\right) + (z\,dx - x\,dz)\left(c\frac{d\phi}{dx} - a\frac{d\phi}{dz}\right)$$
$$+ (x\,dy - y\,dx)\left(a\frac{d\phi}{dy} - b\frac{d\phi}{dx}\right) + \tau\delta\phi = 0;$$

introducing $d\omega$, this becomes

$$d\omega\left[\frac{df}{dx}\left(b\frac{d\phi}{dz} - c\frac{d\phi}{dy}\right) + \frac{df}{dy}\left(c\frac{d\phi}{dx} - a\frac{d\phi}{dz}\right) + \frac{df}{dz}\left(a\frac{d\phi}{dy} - b\frac{d\phi}{dx}\right)\right] + \tau\delta\phi = 0,$$

or observing that a, b, c are the differential coefficients $\frac{d\tau}{dx}, \frac{d\tau}{dy}, \frac{d\tau}{dz}$, the term in [ ] is $J(f, \tau, \phi)$, or say $J(\phi, f, \tau)$, and the equation is

$$d\omega J(\phi, f, \tau) + \tau\delta\phi = 0,$$

where $J(\phi, f, \tau)$ is the Jacobian, or functional determinant

$$\begin{vmatrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \\ \frac{df}{dx}, & \frac{df}{dy}, & \frac{df}{dz} \\ \frac{d\tau}{dx}, & \frac{d\tau}{dy}, & \frac{d\tau}{dz} \end{vmatrix}, = \frac{d(\phi, f, \tau)}{d(x, y, z)};$$

and we hence have

$$d\omega = \frac{-\tau\delta\phi}{J(\phi, f, \tau)}.$$

23. The two theorems thus become

$$\Sigma(x, y, z)^{n-3}\,\frac{\tau\delta\phi}{J(\phi, f, \tau)} = 0,$$

$$\Sigma\frac{(x, y, z)_{12}^{n-2}}{012}\,\frac{-\tau\delta\phi}{J(\phi, f, \tau)} = -\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2}.$$

But further, if in the first equation we write $\tau = z$, and in the second equation we retain τ, using it to denote the linear function 012, the equations become

$$\Sigma(x, y, z)^{n-3}\,\frac{z\delta\phi}{J(\phi, f)} = 0;$$

$$\Sigma(x, y, z)_{12}^{n-2}\,\frac{-\delta\phi}{J(\phi, f, \tau)} = -\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2};$$

where, in the first equation, $J(\phi, f)$ denotes the Jacobian

$$\begin{vmatrix} \dfrac{d\phi}{dx}, & \dfrac{d\phi}{dy} \\ \dfrac{df}{dx}, & \dfrac{df}{dy} \end{vmatrix}, = \frac{d(\phi, f)}{d(x, y)}.$$

In these equations the only differential symbol is the δ affecting the coefficients a, b, ... of ϕ, ϕ_1, ϕ_2; the equations are, in respect to the coordinates (x, y, z) of the several variable intersections of the two curves, purely algebraical equations, which are in fact given by Jacobi's Fraction-theorem about to be explained. But for the further reduction of the affected theorem I interpose the next three articles.

The Coordinates (ρ, σ, τ). Art. Nos. 24 to 26.

24. The letter τ has just been used to denote the determinant 012: there is often occasion to consider three points 1, 2, 3 coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively; and then writing ρ, σ, τ to denote the determinants 023, 031, 012 respectively, and Δ the determinant 123, we have identically

$$\begin{aligned} \Delta x &= x_1\rho + x_2\sigma + x_3\tau, \\ \Delta y &= y_1\rho + y_2\sigma + y_3\tau, \\ \Delta z &= z_1\rho + z_2\sigma + z_3\tau, \end{aligned}$$

which equations, regarding therein the point 0, coordinates (x, y, z), as a current point, are in fact equations for transformation from the coordinates (x, y, z) to the coordinates ρ, σ, τ belonging to the triangle of reference 123. The points 1 and 2 have been already taken to be on the fixed curve, and it will be assumed that 3 is also a point on this curve.

25. If the function f, which equated to zero gives the equation of the fixed curve, be transformed to the new coordinates (ρ, σ, τ), the coefficients of the transformed function are polar functions, each divided by ∇^n, viz. the coefficient of ρ^n is $\frac{1}{\nabla^n} 1^n$, which by reason that 1 is a point on the curve is $=0$ (and similarly the coefficients of σ^n and of τ^n are each $=0$), the coefficient of $\rho^{n-1}\sigma$ is $=\frac{1}{\nabla^n} n \,.\, 1^{n-1} 2$; that of $\rho^{n-1}\tau$ is $=\frac{1}{\nabla^n} n \,.\, 1^{n-1} 3$; that of $\rho^{n-2}\sigma^2$ is $=\frac{1}{\nabla^n} \frac{1}{2} n(n-1) 1^{n-2} 2^2$; and so in other cases. I write this in the form

$$f = \frac{1}{\nabla^n} (1^n = 0, \ldots \between \rho, \sigma, \tau)^n;$$

or we might also use the symbolic form

$$f = \frac{1}{\Delta^n} (\rho 1 + \sigma 2 + \tau 3)^n.$$

The terms independent of τ contain, it is clear, the factor $\rho\sigma$, and separating these terms from the others, the equation may be written

$$f = \frac{1}{\nabla^n}\rho\sigma(n\,.\,1^{n-1}2, \ldots \dagger\between\rho, \sigma)^{n-2} + \text{terms involving } \tau.$$

26. The equations $\tau = 0$, $(\ldots\dagger\between\rho, \sigma)^{n-2} = 0$, determine the residues of the points 1, 2, and hence the major function $(x, y, z)_{12}{}^{n-2}$, expressed in terms of ρ, σ, τ, and writing therein $\tau = 0$, must reduce itself to $(\ldots\dagger\between\rho, \sigma)^{n-2}$ multiplied by a constant factor which is at once found to be $= \frac{1}{\nabla^{n-2}}$; for taking the current point at 1 we have $(\rho, \sigma, \tau) = (\Delta, 0, 0)$, and the corresponding value of the major function is thus $\frac{1}{\nabla^{n-2}} n\,.\,1^{n-1}2\,.\,\Delta^{n-2}, = n\,.\,1^{n-2}2$, as it ought to be. We have thus

$$(x, y, z)_{12}{}^{n-2} = \frac{1}{\nabla^{n-2}}(n\,.\,1^{n-1}2, \ldots\dagger\between\rho, \sigma)^{n-2} + \text{terms involving } \tau;$$

and we hence see that, for $\tau = 0$,

$$(x, y, z)_{12}{}^{n-2} = \frac{\Delta^2 f}{\rho\sigma},$$

an equation which will be useful.

The Preparation for the Affected Theorem resumed. Art. No. 27.

27. In the affected theorem instead of (x, y, z) we introduce the new coordinates (ρ, σ, τ). We have

$$J(\phi, f, \tau) = \frac{d(\phi, f, \tau)}{d(\rho, \sigma, \tau)}\frac{d(\rho, \sigma, \tau)}{d(x, y, z)},$$

where the first factor is $= \frac{d(\phi, f)}{d(\rho, \sigma)}$, say that this is $\bar{J}(\phi, f)$, the Jacobian in regard to ρ, σ: and the second factor is at once found to be $= \Delta^2$. We have consequently

$$\frac{1}{J(\phi, f, \tau)} = \frac{1}{\Delta^2\bar{J}(\phi, f)},$$

and the equation for the affected theorem becomes

$$\Sigma(x, y, z)_{12}{}^{n-2}\frac{\delta\phi}{\bar{J}(\phi, f)} = -\Delta^2\left(-\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2}\right),$$

where $(x, y, z)_{12}{}^{n-2}$ is to be regarded as standing for its value in terms of (ρ, σ, τ).

Jacobi's Fraction-Theorem. Art. Nos. 28 to 31.

28. This is the extension of a well-known theorem, which, in a somewhat disguised form, may be thus written: viz. if U be any rational and integral function $(x, 1)^m$, then we have

$$\frac{1}{U} = \Sigma\frac{1}{x - x'\,.\,J(U')},$$

or, introducing an arbitrary constant A by the equation $AU=X$, say this is

$$\frac{A}{X}\left(=\frac{1}{U}\right)=\Sigma\frac{1}{x-x'.J(U')},$$

where U' is the same function $(x', 1)^m$ of x' that U is of x: $J(U')$, $=\frac{dU'}{dx'}$ is the Jacobian of U', and the summation extends to all roots x' of the equation $U'=0$: obviously this is nothing else than the formula for the decomposition of $\frac{1}{U}$ into simple fractions.

29. Take now $U=(x, y, 1)^m$, $V=(x, y, 1)^n$, functions of x, y of the degrees m and n respectively, and assume

$$AU+BV=X, \text{ a function } (x, 1)^{mn},$$
$$CU+DV=Y, \quad \text{"} \quad (y, 1)^{mn},$$

viz. let $X=0$ and $Y=0$ be the equations obtained by elimination from $U=0$ and $V=0$ of the y and the x respectively. The forms are

$$A=(x, y^{n-1}, 1)^{mn-m}, \qquad B=(x, y^{m-1}, 1)^{mn-n},$$
$$C=(x^{n-1}, y, 1)^{mn-m}, \qquad D=(x^{m-1}, y, 1)^{mn-n},$$

where these equations denote the first of them that A is a rational and integral function of the degree $mn-m$ in x and y jointly, but only of the degree $n-1$ in y: and so for the other equations. It follows that

$$AD-BC=(x^{mn-1}, y^{mn-1}, 1)^{2mn-m-n}.$$

The theorem now is

$$\frac{AD-BC}{XY}=\Sigma\frac{1}{x-x'.y-y'.J(U', V')},$$

where U', V' are the same functions of (x', y') that U, V are of (x, y); $J(U', V')$ is the Jacobian $\frac{d(U', V')}{d(x', y')}$; and the summation extends to all the simultaneous roots x', y' of the equations $U=0$, $V=0$.

30. For the proof, observe that $AD-BC$ is a sum of terms of the form $x^\alpha y^\beta$ where α and β are each of them at most $=mn-1$; hence X being of the degree mn we have $\frac{x^\alpha}{X}=$ a sum of fractions $\frac{L}{x-x'}$, where x' is any root of $X=0$; and similarly $\frac{y^\beta}{Y}=$ a sum of fractions $\frac{M}{y-y'}$, where y' is any root of $Y=0$; multiplying the two expressions and taking the sum for the several terms $\lambda x^\alpha y^\beta$ of $AD-BC$ we obtain a formula

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'.y-y'},$$

where the summation extends to all the combinations of the mn values of x' with the mn values of y. But such a formula existing, the coefficients K may be determined

in the usual manner, viz. multiplying by XY and then writing $x=x'$, $y=y'$, there is on the right-hand only one term which does not vanish, and we find

$$(AD-BC)_{x'y'}=K\left(\frac{X}{x-x'}\right)_{x'}\left(\frac{Y}{y-y'}\right)_{y'},\ =K\left(\frac{dX}{dx}\,\frac{dY}{dy}\right)_{x'y'},$$

where the factor which multiplies K does not vanish.

We distinguish the cases where (x', y') are corresponding or non-corresponding roots of $X=0$, $Y=0$; viz. corresponding roots are those for which $U=0$, $V=0$, but for non-corresponding roots these equations do not hold good; there are obviously mn pairs of corresponding roots.

In the latter case $(AD-BC)\,U=DX-BY$; $(AD-BC)\,V=-CX+AY$, and since for the values in question X, Y each vanish, but U, V do not each of them vanish, we must for these values have $AD-BC=0$, and the foregoing equation for K gives then $K=0$.

31. The formula thus is

$$\frac{AD-BC}{XY}=\Sigma\frac{K}{x-x'\,.\,y-y'},$$

where the summation now extends only to corresponding roots x', y', for which we have $U=0$, $V=0$. We have for K the foregoing expression, which, to complete the determination, we write under the form

$$AD-BC=KJ(X,\ Y)_{x'y'};$$

this is allowable, for $J(X,\ Y),\ =\dfrac{d\,(X,\ Y)}{d\,(x,\ y)}$, differs from $\dfrac{dX}{dx}\,\dfrac{dY}{dy}$ only by the zero term $-\dfrac{dX}{dy}\,\dfrac{dY}{dx}$. Moreover, differentiating the expressions for X, Y, and considering (x, y) as therein standing for a pair of corresponding roots (x', y'), the terms containing U, V will all vanish; we thus in effect differentiate as if A, B, C, D were constants, and the result is $(AD-BC)\,J(U,\ V)$, or say this is $(AD-BC)_{x'y'}\,J(U',\ V')$: hence, in the equation for K, the factor $(AD-BC)_{x'y'}$ divides out, and we have $1=KJ(U',\ V')$; hence the required formula is

$$\frac{AD-BC}{XY}=\Sigma\frac{1}{x-x'\,.\,y-y'}\,.\,J(U',\ V')$$

the summation extending to all the simultaneous roots (x', y') of $U=0$, $V=0$.

Homogeneous Form of the Fraction-Theorem. Art. No. 32.

32. For x, y, x', y' we write $\dfrac{x}{z}$, $\dfrac{y}{z}$, $\dfrac{x'}{z'}$, $\dfrac{y'}{z'}$; supposing that U, V now denote homogeneous functions $(x,\ y,\ z)^m$, $(x,\ y,\ z)^n$, and that we have

$$AU+BV=X,\ =(x,\ z)^{mn},\ =\alpha x^{mn}+\ldots,$$
$$CU+DV=Y,\ =(y,\ z)^{mn},\ =\beta y^{mn}+\ldots,$$

where the forms are

$$A = (x,\ y^{n-1},\ z)^{mn-m},\quad B = (x,\ y^{m-1},\ z)^{mn-n},$$
$$C = (x^{n-1},\ y,\ z)^{mn-m},\quad D = (x^{m-1},\ y,\ z)^{mn-n},$$
$$AD - BC = (x^{mn-1},\ y^{mn-1},\ z)^{2mn-m-n},$$

viz. the degree of A in (x, y, z) is $= mn - m$, but y rises only to the degree $n-1$, and so in other cases; then the theorem becomes

$$\frac{z^{m+n-2}(AD-BC)}{XY} = \Sigma \frac{z'^{m+n}}{xz' - x'z\,.\,yz' - y'z}\,.\,J(U',\ V'),$$

where $J(U', V')$ denotes the Jacobian $\dfrac{d(U',\ V')}{d(x',\ y')}$, and the summation extends to the simultaneous roots (x', y', z') of $U = 0$, $V = 0$.

It is proper to introduce into the formula τ', an arbitrary linear function $ax' + by' + cz'$ of (x', y', z'): observe that, in the Jacobian, (x', y', z') have always values for which $U' = 0$, $V' = 0$: we have therefore

$$x'\frac{dU'}{dx'} + y'\frac{dU'}{dy'} + z'\frac{dU'}{dz'} = 0,$$
$$x'\frac{dV'}{dx'} + y'\frac{dV'}{dy'} + z'\frac{dV'}{dz'} = 0,$$

and thence

$$x' : y' : z' = \frac{d(U',\ V')}{d(y',\ z')} : \frac{d(U',\ V')}{d(z',\ x')} : \frac{d(U',\ V')}{d(x',\ y')},$$

and if the expressions on the right-hand are for a moment called A', B', C', then writing $\tau' = ax' + by' + cz'$, we have $J(U',\ V',\ \tau') = aA' + bB' + cC'$, $= \dfrac{\tau'}{z'}C'$, $= \dfrac{\tau'}{z'}J(U',\ V')$, that is,

$$\frac{1}{J(U',\ V')} = \frac{\tau'}{z'J(U',\ V',\ \tau')},$$

or the equation becomes

$$\frac{z^{m+n-2}(AD-BC)}{XY} = \Sigma \frac{z'^{m+n-1}\,\tau'}{xz' - x'z\,.\,yz' - y'z}\,.\,J(U',\ V',\ \tau'),$$

the summation being as before.

Resulting Special Theorems. Art. Nos. 33 to 35.

33. Reverting to the Cartesian form, we have

$$\frac{xy(AD-BC)}{XY} = \Sigma \frac{1}{J(U',\ V')}\left(1 + \frac{x'}{x} + \ldots\right)\left(1 + \frac{y'}{y} + \ldots\right),$$
$$= \Sigma \frac{1}{J(U',\ V')}\left\{1 + H_1\left(\frac{x'}{x},\ \frac{y'}{y}\right) + H_2\left(\frac{x'}{x},\ \frac{y'}{y}\right) + \ldots\right\}$$

where H_m is the homogeneous sum of the order m,

$$H_1(u,\ v) = u + v,\ H_2(u,\ v) = u^2 + uv + v^2,\ \&c.$$

The left-hand side is

$$(AD-BC)\left(\frac{1}{\alpha x^{mn-1}}+\frac{\&c.}{x^{mn}}\cdots\right)\left(\frac{1}{\beta y^{mn-1}}+\frac{\&c.}{y^{mn}}+\ldots\right)$$

and in $AD-BC$ the terms of the highest order in (x, y), say $(AD-BC)_0$ are

$$(AD-BC)_0=(xy)^{mn-m-n+1}(a, b,\ldots, k\dagger\between x, y)^{m+n-2}.$$

There is thus on the left-hand no term which is in (x, y) of a higher degree than $-(m+n-2)$; hence on the right-hand every term of a higher degree than this in (x, y) must vanish, viz. we must have

$$0=\Sigma\frac{x'^{\alpha}y'^{\beta}}{J(U', V')} \text{ so long as } \alpha+\beta\not> m+n-3,$$

or, what is the same thing, we must have

$$0=\Sigma\frac{(x', y', 1)^{m+n-3}}{J(U', V')}, \text{ say the } (m+n-3) \text{ theorem,}$$

where $(x', y', 1)^{m+n-3}$ is the arbitrary function of the degree $m+n-3$.

34. Passing to the next lower degree $-(m+n-2)$, we have

$$\frac{1}{\alpha\beta(xy)^{m+n-2}}(a, b,\ldots, k\dagger\between x, y)^{m+n-2}=\Sigma\frac{1}{J(U', V')}H_{m+n-2}\left(\frac{x'}{x}, \frac{y'}{y}\right)$$

and if in $(a, b,\ldots, k\dagger\between x, y)^{m+n-2}$ we consider any term $gx^{m+n-2-p}y^{m+n-2-q}$, where $p+q=m+n-2$, then we have on the left-hand the term $\frac{g}{\alpha\beta x^p y^q}$, and the corresponding term on the right-hand must be

$$\Sigma\frac{1}{J(U', V')}\frac{x'^p y'^q}{x^p y^q};$$

that is, we have

$$\frac{g}{\alpha\beta}=\Sigma\frac{x'^p y'^q}{J(U', V')}.$$

But from the foregoing expression for $(AD-BC)_0$, it appears that $(AD-BC)_0$ contains the term $gx^{mn-1-p}y^{mn-1-q}$, and it hence appears that g is the constant term of the quotient $(AD-BC)_0$ divided by $x^{mn-1-p}y^{mn-1-q}$, or as this may be written

$$g=\text{const. of } \frac{(AD-BC)_0 x^p y^q}{(xy)^{mn-1}};$$

comparing the two values of g, we obtain

$$\text{const. of } \frac{(AD-BC)_0 x^p y^q}{\alpha\beta(xy)^{mn-1}}=\Sigma\frac{x'^p y'^q}{J(U', V')}, \quad (p+q=m+n-2),$$

and we hence derive

$$\text{Const. of } \frac{(AD-BC)_0(x, y)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}=\Sigma\frac{(x', y')^{m+n-2}}{J(U', V')},$$

where $(x, y)^{m+n-2}$ is the general function of the degree $m+n-2$, and, of course, $(x', y')^{m+n-2}$ is the same function of x', y'. The two functions may be written $(x, y, 0)^{m+n-2}$ and $(x', y', 0)^{m+n-2}$; and this being so we may on the right-hand write instead $(x', y', 1)^{m+n-2}$, for, by so doing we introduce in the numerator of the fraction new terms of an order not exceeding $m+n-3$, and by the $(m+n-3)$ theorem already obtained the sum Σ of the quotient of such terms by $J(U', V')$ is $=0$. We thus have

$$\text{Const. of } \frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}} = \Sigma\frac{(x', y', 1)^{m+n-2}}{J(U', V')}, \text{ say the } (m+n-2) \text{ theorem,}$$

where $(x', y', 1)^{m+n-2}$ is the general non-homogeneous function of the degree $m+n-2$, and $(x, y, 0)^{m+n-2}$ is obtained from it by attending only to the terms of the highest degree $m+n-2$, and therein substituting x, y for x', y'.

35. We may, it is clear, in the equations for the $(m+n-3)$ and for the $(m+n-2)$ theorems respectively, omit the accents on the right-hand sides; doing this, and moreover in each equation transposing the two sides, the two special theorems are

$$\Sigma\frac{(x, y, 1)^{m+n-3}}{J(U, V)} = 0, \qquad (m+n-3) \text{ theorem,}$$

$$\Sigma\frac{(x, y, 1)^{m+n-2}}{J(U, V)} = \text{const. of } \frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}, \quad (m+n-2) \text{ theorem,}$$

Homogeneous Form of the Special Theorems. Art. No. 36.

36. Writing $\frac{x}{z}, \frac{y}{z}$ for x, y, and introducing as before the arbitrary linear function $\tau = ax+by+cz$, we at once obtain, U, V being now homogeneous functions $(x, y, z)^m$ and $(x, y, z)^n$ respectively, and the A, B, C, D being also homogeneous functions accordingly,

$$\Sigma\frac{z(x, y, z)^{m+n-3}}{J(U, V)} = 0, \qquad (m+n-3) \text{ theorem,}$$

$$\Sigma\frac{\tau(x, y, z)^{m+n-2}}{zJ(U, V, \tau)} = \text{const. of } \frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}, \quad (m+n-2) \text{ theorem,}$$

where the suffix 0 denotes that we are in $AD-BC$ to write $z=0$.

If in the last formula we change throughout the letters x, y, z into ρ, σ, τ (that is, consider U, V as given functions of ρ, σ, τ), but retain τ as standing for the particular function $0\rho+0\sigma+1\tau$, then the formula becomes

$$\Sigma\frac{(\rho, \sigma, \tau)^{m+n-2}}{\bar{J}(U, V)} = \text{const. of } \frac{(AD-BC)_0(\rho, \sigma, \tau)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}}, \quad (m+n-2) \text{ theorem,}$$

where $\bar{J}(U, V)$ denotes $\frac{d(U, V)}{d(\rho, \sigma)}$, the Jacobian in regard to ρ, σ.

The effect of dps of the Curves $U=0$, $V=0$. Art. Nos. 37 and 38.

37. We must, in regard to the foregoing special theorems, consider the effect of any dps of the curves $U=0$, $V=0$.

Suppose one of the curves, say V, has a dp, but that the other curve U does not pass through it; the dp is not an intersection of U, V, and the theorems are in nowise affected.

If U passes through the dp, then the dp counts twice among the intersections of U, V; at the dp we have $J(U', V')=0$, and (to fix the ideas, attending to the $(m+n-3)$ theorem) the sum $\Sigma\frac{(x, y, z)^{m+n-3}}{J(U, V)}$ will contain two infinite terms; these may very well, and indeed (assuming that the theorem remains true) must have a finite sum, but except by the theorem itself, this finite sum is not calculable, and the theorem thus becomes nugatory.

If, however, the curve $(x, y, z)^{m+n-3}=0$ be a curve passing through the dp, then considering, instead, the case where the last-mentioned curve and U each approach indefinitely near to the dp of V; there are two intersections of U, V indefinitely near to each other and to the dp; at either intersection, the numerator $(x', y', z')^{m+n-3}$ and the denominator $J(U, V)$ are infinitesimals of the same order, say the first, and the fraction has a finite value; the finite values for the two intersections have not in general a zero sum, and consequently in the limit it would not be allowable to disregard the intersections belonging to the dp.

38. But if the numerator curve $(x, y, z)^{m+n-3}=0$ passes twice through the dp (that is, has there a dp), then reverting to the two consecutive intersections, at either of these the denominator $J(U, V)$ is as before an infinitesimal of the first order, but the numerator $(x, y, z)^{m+n-3}$ is an infinitesimal of the second order, and in the limit the value of the fraction is $=0$; we may in this case disregard the intersections belonging to the dp; and so in general, the curve $(x, y, z)^{m+n-3}=0$ passing twice through each dp of U which lies upon V, we have

$$\Sigma\frac{z(x, y, z)^{m+n-3}}{J(U, V)}=0,$$

the summation now extending to all the intersections of U, V other than the dps in question, which are to be disregarded. And the like in regard to the other theorem

$$\Sigma\frac{(x, y, z)^{m+n-2}}{J(U, V)}=\text{const. of }\frac{(AD-BC)_0(x, y, 0)^{m+n-2}}{\alpha\beta(xy)^{mn-1}}.$$

The Pure Theorem.—Completion of the Proof. Art. No. 39.

39. The theorem was reduced to

$$\Sigma\frac{z(x, y, z)^{n-3}\,\delta\phi}{J(\phi, f)}=0,$$

which is therefore the equation to be proved.

The $(m+n-3)$ theorem, writing therein ϕ, f in place of U, V respectively (the degrees being as before m and n), is

$$\Sigma \frac{z(x, y, z)^{m+n-3}}{J(U, V)} = 0.$$

Here $(x, y, z)^{m+n-3}$ is an arbitrary function of the degree $m+n-3$, and this may therefore be put $=(x, y, z)^{n-3}\delta\phi$, where $\delta\phi, = (da, \ldots \delta x, y, z)^m$, is a function of the degree m; and since the curve $\phi = 0$ passes always through the dps of f, and varies subject to this condition, the curve $\delta\phi = 0$ will also pass through the dps; hence taking $(x, y, z)^{n-2} = 0$ a curve through the dps, the curve $(x, y, z)^{n-3}\delta\phi = 0$ is a curve passing twice through each of dps, and the $(m+n-3)$ theorem thus gives the equation which was to be proved. This completes the proof of the pure theorem

$$\Sigma (x, y, z)^{n-3} d\omega = 0.$$

The Affected Theorem.—Completion of the Proof. Art. Nos. 40 and 41.

40. The theorem was reduced to

$$\Sigma \frac{(x, y, z)_{12}^{n-2}\delta\phi}{\bar{J}(\phi, f)} = -\Delta^2\left(-\frac{\delta\phi_1}{\phi_1} + \frac{\delta\phi_2}{\phi_2}\right),$$

which is therefore the equation to be proved.

The $(m+n-2)$ theorem, written with (ρ, σ, τ) in place of (x, y, z), and putting therein ϕ, f for U, V, is

$$\Sigma \frac{(\rho, \sigma, \tau)^{m+n-2}}{\bar{J}(\phi, f)} = \text{const. of } \frac{(AD-BC)_0(\rho, \sigma, 0)^{m+n-2}}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where it will be recollected that the suffix (0) denotes that τ is to be put $=0$. Here $(\rho, \sigma, \tau)^{m+n-2}$ is an arbitrary function of the degree $m+n-2$, and this may therefore be put $=(x, y, z)_{12}^{n-2}\delta\phi$, the two factors being each of them considered as expressed in terms of (ρ, σ, τ); and since each of the curves $(x, y, z)_{12}^{n-2} = 0$ and $\delta\phi = 0$ passes through the dps of f, the curve $(x, y, z)_{12}^{n-2}\delta\phi = 0$ is a curve passing twice through each of the dps. We have therefore

$$\Sigma \frac{(x, y, z)_{12}^{n-2}\delta\phi}{\bar{J}(\phi, f)} = \text{const. of } \frac{(AD-BC)_0(x, y, z)_{12}^{n-2}\delta\phi_0}{\alpha\beta(\rho\sigma)^{mn-1}},$$

where on the right-hand side $(x, y, z)_{12}^{n-2}$ is considered as a function of ρ, σ, τ, and we are to put therein $\tau = 0$; it has been seen (No. 26) that the value is $=\frac{f_0\Delta^2}{\rho\sigma}$, where f_0 is what f, considered as a function of ρ, σ, τ, becomes on writing therein $\tau = 0$; the right-hand side thus becomes

$$= \text{const. of } \frac{(AD-BC)_0 f_0 \Delta^2 \delta\phi_0}{\alpha\beta(\rho\sigma)^{mn}}.$$

41. But for $\tau=0$, we have

$$A_0\phi_0+B_0f_0=\alpha\rho^{mn},$$
$$C_0\phi_0+D_0f_0=\beta\sigma^{mn},$$

and hence

$$(AD-BC)_0f_0=A_0\beta\sigma^{mn}-C_0\alpha\rho^{mn},$$

and the right-hand side thus becomes, say

$$-\Delta^2 \text{ const. of } \left(-\frac{A_0}{\alpha\rho^{mn}}+\frac{C_0}{\beta\sigma^{mn}}\right)\delta\phi_0.$$

But in calculating the constant of $\frac{A_0}{\alpha\rho^{mn}}\delta\phi_0$, we may suppose not only $\tau=0$, but also $\sigma=0$: we then have $\phi_0=(x,\ y,\ z)^m,\ =\left(\frac{\rho}{\Delta}\right)^m(x_1,\ y_1,\ z_1)^m,\ =\left(\frac{\rho}{\Delta}\right)^m\phi_1$, and hence also $\delta\phi_0=\left(\frac{\rho}{\Delta}\right)^m\delta\phi_1$.

Similarly, in calculating the constant of $\frac{B_0}{\beta\sigma^{mn}}\delta\phi_0$, we may suppose not only $\tau=0$, but also $\rho=0$: we then have $\phi_0=(x,\ y,\ z)^m,\ =\left(\frac{\sigma}{\Delta}\right)^m(x_2,\ y_2,\ z_2)^m,\ =\left(\frac{\sigma}{\Delta}\right)^m\phi_2$, and hence $\delta\phi_0=\left(\frac{\sigma}{\Delta}\right)^m\delta\phi_2$.

Moreover, in the equations

$$A_0\phi_0+B_0f_0=\alpha\rho^{mn},$$
$$C_0\phi_0+D_0f_0=\beta\sigma^{mn},$$

writing in the first equation $\sigma=0$, we find $A_0\left(\frac{\rho}{\Delta}\right)^m\phi_1=\alpha\rho^{mn}$, that is, $\frac{A_0}{\alpha\rho^{mn}}=\left(\frac{\Delta}{\rho}\right)^m\frac{1}{\phi_1}$; and similarly writing in the second equation $\rho=0$, we find $C_0\left(\frac{\sigma}{\Delta}\right)^m\phi_2=\beta\sigma^{mn}$, that is, $\frac{C_0}{\beta\sigma^{mn}}=\left(\frac{\Delta}{\sigma}\right)^m\frac{1}{\phi_2}$: and the expression thus becomes

$$=-\Delta^2\left(-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}\right),$$

giving the equation which was to be proved. This completes the proof of the affected theorem

$$\Sigma\frac{(x,\ y,\ z)_{12}^{n-2}\,d\omega}{012}=-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}.$$

CHAPTER III. THE MAJOR FUNCTION $(x, y, z)_{12}{}^{n-2}$.

Analytical Expression of the Function. Art. Nos. 42 to 49.

42. The function has been defined by the conditions that the curve $(x, y, z)_{12}{}^{n-2}=0$ shall pass through the dps, and also through the $n-2$ residues of the parametric points 1, 2: and moreover, that on writing therein (x_1, y_1, z_1) for (x, y, z), the function shall become $=n\,.\,1^{n-1}2$. Obviously the function is not completely determined: calling it Ω (or when required Ω_{12}), then if Ω' be any particular form of it, the general form is $\Omega=\Omega'+(x, y, z)^{n-3}.012$, where $(x, y, z)^{n-3}$ is the general minor function, viz. $(x, y, z)^{n-3}=0$ is a curve passing through the dps: the major function thus contains $\frac{1}{2}(n-1)(n-2)-\delta$, $=p$, arbitrary constants.

Agreeing with the definition, we have the before-mentioned equation

$$\Omega=\frac{1}{\Delta^{n-2}}(n\,.\,1^{n-1}2, \ldots \dagger\between\rho, \sigma)^{n-2}+\text{terms involving } \tau,$$

viz. from this expression for Ω it appears that the curve $\Omega=0$ meets the line through 1, 2 in the $n-2$ residues of these points, and moreover, for $(x, y, z)=(x_1, y_1, z_1)$ and therefore $(\rho, \sigma, \tau)=(\Delta, 0, 0)$, the value of Ω is $=n\,.\,1^{n-1}2$.

43. We can without difficulty write down an equation determining Ω' as a function $(x, y, z)^{n-2}$, which, on putting therein $\tau=0$, becomes equal to the foregoing expression $\frac{1}{\Delta^{n-2}}(n\,.\,1^{n-1}2, \ldots \dagger\between\rho, \sigma)^{n-2}$, and which is moreover such that the curve $\Omega'=0$ passes through the dps; which being so, we have as before, $\Omega=\Omega'+(x, y, z)^{n-3}.012$, for the general value of Ω.

To fix the ideas, consider the particular case $n=4$, the fixed curve a quartic: Ω', on putting therein $\tau=0$, should become

$$=\frac{1}{\Delta^2}(4\,.\,1^32,\ 6\,.\,1^22^2,\ 4\,.\,12^3\dagger\between\rho, \sigma)^2;$$

and it is to be shown that this will be the case if we determine Ω', a quadric function of (x, y, z), by the equation

$$\begin{vmatrix} (x\,,\ y\,,\ z)^2 & , & \Omega' \\ 1\,(x_1, y_1, z_1)^2 & , & 4\,.\,1^32 \\ 2\,(x_1, y_1, z_1\between x_2, y_2, z_2), & & 6\,.\,1^22^2 \\ 1\,(x_2, y_2, z_2)^2 & , & 4\,.\,12^3 \\ a\,,\ b\,,\ c,\ f\,,\ g\,,\ h\,, & & 0 \\ \vdots & & \end{vmatrix}=0,$$

where the left-hand side is a determinant of seven lines and columns, the top line being x^2, y^2, z^2, $2yz$, $2zx$, $2xy$, Ω', and similarly for the second line; the third line is

$2x_1x_2$, $2y_1y_2$, $2z_1z_2$, $2(y_1z_2+y_2z_1)$, $2(z_2x_1+z_1x_2)$, $2(x_1y_2+x_2y_1)$, $6\,.\,1^2 2^2$, and in each of the last three lines we have six arbitrary constants followed by a 0. The equation is of the form $\square + M\Omega' = 0$, where $\square$ is a quadric function $(x, y, z)^2$, and M is a constant factor.

44. If the quartic curve has a dp, suppose at the point α, coordinates $(x_\alpha, y_\alpha, z_\alpha)$, then in order that the curve $\Omega' = 0$ may pass through the dp, we must for one of the last three lines substitute $(x_\alpha, y_\alpha, z_\alpha)^2$, 0; and so for any other dp or dps of the quartic curve. And the conditions as to the dp or dps (if any) being satisfied in this manner, we may if we please, taking $(x_\beta, y_\beta, z_\beta)$ as the coordinates of an arbitrary point β (not of necessity on the fixed curve), write any line not already so expressed, of the last three lines, in the form $(x_\beta, y_\beta, z_\beta)^2$, 0; the effect being to make the curve $\Omega' = 0$ pass through the arbitrary point β.

45. To show that the equation on putting therein $\tau = 0$ does in fact give the required value, $\Omega' = \frac{1}{\Delta^2}(4\,.\,1^3 2,\ 6\,.\,1^2 2^2,\ 4\,.\,12^3 \dagger \between \rho, \sigma)^2, = \Phi$ suppose, it is to be observed that, effecting a linear substitution upon the first six columns, the equation may be written

$$\begin{vmatrix} (\rho\,,\ \sigma\,,\ \tau)^2 & , & \Omega' \\ 1\,(\rho_1,\ \sigma_1,\ \tau_1)^2 & , & 4\,.\,1^3 2 \\ 2\,(\rho_1,\ \sigma_1,\ \tau_1 \between \rho_2,\ \sigma_2,\ \tau_2), & & 6\,.\,1^2 2^2 \\ 1\,(\rho_2,\ \sigma_2,\ \tau_2)^2 & , & 4\,.\,12^3 \\ a',\ b',\ c',\ f',\ g',\ h' & , & 0 \\ \vdots & & \end{vmatrix} = 0,$$

where $(\rho_1, \sigma_1, \tau_1)$, $(\rho_2, \sigma_2, \tau_2)$ are what (ρ, σ, τ) become on writing therein for (x, y, z) the values (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively; viz. we have $(\rho_1, \sigma_1, \tau_1) = (\Delta, 0, 0)$; $(\rho_2, \sigma_2, \tau_2) = (0, \Delta, 0)$; the equation thus is

$$\begin{vmatrix} \rho^2, & \sigma^2, & \tau^2, & 2\sigma\tau, & 2\tau\rho, & 2\rho\sigma, & \Omega' \\ \Delta^2, & 0\,, & 0\,, & 0\,, & 0\,, & 0\,, & 4\,.\,1^3 2 \\ 0\,, & 0\,, & 0\,, & 0\,, & 0\,, & 2\Delta^2, & 6\,.\,1^2 2^2 \\ 0\,, & \Delta^2, & 0\,, & 0\,, & 0\,, & 0\,, & 4\,.\,12^3 \\ a', & b', & c', & f', & g', & h', & 0 \\ \vdots & & & & & & \end{vmatrix} = 0,$$

and then by another linear substitution upon the columns, the last column can be changed into $\Omega' - \Phi$, 0, 0, 0, 0, 0, 0; whence writing $\tau = 0$, the equation becomes

$$\begin{vmatrix} \rho^2, & \sigma^2, & 0, & 0, & 0, & 2\rho\sigma, & \Omega' - \Phi \\ \Delta^2, & 0\,, & 0, & 0, & 0, & 0\,, & 0 \\ 0\,, & 0\,, & 0, & 0, & 0, & 2\Delta^2, & 0 \\ 0\,, & \Delta^2, & 0, & 0, & 0, & 0\,, & 0 \\ a', & b', & c', & f', & g', & h', & 0 \\ \vdots & & & & & & \end{vmatrix} = 0,$$

or, omitting a constant factor, it is

$$\begin{vmatrix} \rho^2, & \sigma^2, & 2\rho\sigma, & \Omega'-\Phi \\ \Delta^2, & 0, & 0, & 0 \\ 0, & 0, & 2\Delta^2, & 0 \\ 0, & \Delta^2, & 0, & 0 \end{vmatrix} = 0,$$

that is, $\Omega'-\Phi=0$, or $\Omega'=\Phi$, $=\frac{1}{\Delta^2}(4.1^2 2,\ 6.1^2 2^2,\ 4.12^2 \between (\rho,\ \sigma)^2$, the required value.

46. Considering the equation for Ω' as expressed in the before-mentioned form $\square + M\Omega' = 0$, the value of the constant factor M is

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ a,\ b,\ c,\ f,\ g,\ h, \\ \vdots \end{vmatrix};$$

or if, instead of each line such as a, b, c, f, g, h, we have a line $(x_\alpha,\ y_\alpha,\ z_\alpha)^2$, then we have

$$M = \begin{vmatrix} (x_1,\ y_1,\ z_1)^2 \\ (x_1,\ y_1,\ z_1 \between x_2,\ y_2,\ z_2) \\ (x_2,\ y_2,\ z_2)^2 \\ (x_\alpha,\ y_\alpha,\ z_\alpha)^2 \\ (x_\beta,\ y_\beta,\ z_\beta)^2 \\ (x_\gamma,\ y_\gamma,\ z_\gamma)^2 \end{vmatrix},$$

a value which is

$$= \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\alpha, & y_\alpha, & z_\alpha \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\beta, & y_\beta, & z_\beta \end{vmatrix} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix} \begin{vmatrix} x_\alpha, & y_\alpha, & z_\alpha \\ x_\beta, & y_\beta, & z_\beta \\ x_\gamma, & y_\gamma, & z_\gamma \end{vmatrix},$$

or say this is $=12\alpha.12\beta.12\gamma.\alpha\beta\gamma$.

47. It is obvious that the foregoing process is applicable to the general case of the fixed curve of the order n with δ dps, and gives always Ω', by an equation of the foregoing form $\square + M\Omega' = 0$, where $\square$ is a function $(x,\ y,\ z)^{n-2}$ of the coordinates, and M is a constant factor. Supposing that in the determinant for Ω', each of the lower lines is written in the form $(x_\alpha,\ y_\alpha,\ z_\alpha)^{n-2}$, 0, the number of the points α is $=\frac{1}{2}(n-1)(n-2)$, viz. these are the δ dps, and $\frac{1}{2}(n-1)(n-2)-\delta$, $=p$, other points α. The general expression of M is $M=12\alpha.12\beta\ldots(\alpha^{n-3}\beta^{n-3}\ldots)$, viz. equating to zero a factor such as 12α, this expresses that the point α is on the line 12; but equating to zero the last factor $(\alpha^{n-3}\beta^{n-3}\ldots)$, this expresses that the several points α, viz. the dps and the p other points α, are on a curve of the order $n-3$.

48. Preceding the case $n=4$, above considered, we have, of course, the case $n=3$, $\delta=0$, the fixed curve a cubic; the equation for Ω' is here

$$\begin{vmatrix} x, & y, & z, & \Omega' \\ x_1, & y_1, & z_1, & 3\,.\,1^2 2 \\ x_2, & y_2, & z_2, & 3\,.\,12^2 \\ x_\alpha, & y_\alpha, & z_\alpha, & \end{vmatrix} = 0,$$

giving

$$\tfrac{1}{3}\Omega' = \frac{1^2 2\,.\,02\alpha + 12^2\,.\,0\alpha 1}{12\alpha},$$

or if we write herein 3 for α, this is

$$\tfrac{1}{3}\Omega' = \frac{1^2 2\,.\,023 + 12^2\,.\,031}{123};$$

and we have hence the general form

$$\tfrac{1}{3}\Omega = \frac{1^2 2\,.\,023 + 12^2\,.\,031}{123} + K\,.\,012,$$

where K is an arbitrary constant.

49. There is, however, a more simple particular solution $\frac{1}{3}\Omega' =$ polar function 012 ($f = x^3 + y^3 + z^3$, then $012 = xx_1x_2 + yy_1y_2 + zz_1z_2$), which, to avoid a confusion of notation, we may write $= \widetilde{01}\widetilde{2}$. We at once verify this; for, expressing the coordinates (x, y, z) in terms of (ρ, σ, τ), we have $\frac{1}{3}\Omega' = \widetilde{01}\widetilde{2}$, $= \frac{1}{\Delta}(1^2 2\,.\,\rho + 12^2\,.\,\sigma + \widetilde{12}\widetilde{3}\,.\,\tau)$, which, for $\tau = 0$ becomes $= \frac{1}{\Delta}\{1^2 2\,.\,\rho + 12^2\,.\,\sigma\}$.

We must, of course, have an identity of the form

$$\widetilde{01}\widetilde{2} = \frac{1^2 2\,.\,023 + 12^2\,.\,031}{123} + K\,.\,012,$$

and to find K, writing here $0 = 3$, we have $K = \frac{\widetilde{12}\widetilde{3}}{\widetilde{12}\widetilde{3}}$, or we have the identity

$$123\,\widetilde{01}\widetilde{2} - \widetilde{12}\widetilde{3}\,012 = 1^2 2\,.\,023 + 12^2\,.\,031.$$

Single Letter Notation for the Polar Functions of the Cubic. Art. Nos. 50 and 51.

50. The notation of single letters for the polar functions is not much required in the case of the cubic, but, in the next following case of the quartic it can hardly be dispensed with, and I therefore establish it in the case of the cubic: viz. I write

$$2^2 3,\ 3^2 1,\ 1^2 2 = f,\ g,\ h,\quad 23^2,\ 31^2,\ 12^2 = i,\ j,\ k;\quad \widetilde{12}\widetilde{3} = l,$$

or, what is the same thing, the expression for the cubic function f in terms of ρ, σ, τ is

$$\Delta^3\,.\,f = 3h\rho^2\sigma + 3j\rho^2\tau + 3k\rho\sigma^2 + 6l\rho\sigma\tau + 3g\rho\tau^2 + 3f\sigma^2\tau + 3i\sigma\tau^2;$$

an equation which, writing 0^3 instead of f, may also be written

$$\Delta^3 . 0^3 = (3h,\ 3j,\ 3k,\ 6l,\ 3g,\ 3f,\ 3i \between \rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^2\tau,\ \sigma\tau^2);$$

and I join to it the series of equations

$$\begin{aligned}
\Delta^2 . 0^2 1 &= (0,\ 2h,\ 2j,\ k,\ 2l,\ g \between \rho^2,\ \rho\sigma,\ \rho\tau,\ \sigma^2,\ \sigma\tau,\ \tau^2),\\
\text{,,}\quad 0^2 2 &= (h,\ 2k,\ 2l,\ 0,\ 2f,\ i \between \quad \text{,,} \quad),\\
\text{,,}\quad 0^2 3 &= (j,\ 2l,\ 2g,\ f,\ 2i,\ 0 \between \quad \text{,,} \quad),\\
\Delta . 01^2 &= (0,\ h,\ j \between \rho,\ \sigma,\ \tau),\\
\Delta\ \widetilde{012} &= (h,\ k,\ l \between \quad \text{,,} \quad),\\
\text{,,}\ \widetilde{013} &= (j,\ l,\ g \between \quad \text{,,} \quad),\\
\text{,,}\ 02^2 &= (k,\ 0,\ f \between \quad \text{,,} \quad),\\
\text{,,}\ \widetilde{023} &= (l,\ f,\ i \between \quad \text{,,} \quad),\\
\text{,,}\ 03^2 &= (g,\ i,\ 0 \between \quad \text{,,} \quad).
\end{aligned}$$

51. In particular, we have $\Delta . \widetilde{012} = h\rho + k\sigma + l\tau$, and the above-mentioned identity $123\,\widetilde{012} - \widetilde{123}\,012 = 1^2 2 . 023 + 12^2 . 031$ is simply $h\rho + k\sigma + l\tau - l\tau = h\rho + k\sigma$.

Single Letter Notation for the Polar Functions of the Quartic. Art. No. 52.

52. I write here

$$2^2 3,\ 3^2 1,\ 1^2 2 = f,\ g,\ h;\ 23^2,\ 31^3,\ 12^2 = i,\ j,\ k;$$

$$1^2 23,\ 12^2 3,\ 123^2 = l,\ m,\ n;\ 2^2 3^2,\ 3^2 1^2,\ 1^2 2^2 = p,\ q,\ r;$$

so that the expression for the quartic function f in terms of ρ, σ, τ is

$$\begin{aligned}
\Delta^4 . f = {} & 4h\rho^3\sigma + 4j\rho^3\tau + 6p\rho^2\sigma^2 + 12l\rho^2\sigma\tau + 6q\rho^2\tau^2\\
& + 4k\rho\sigma^3 + 12m\rho\sigma^2\tau + 12n\rho\sigma\tau^2 + 4g\rho\tau^3 + 4f\sigma^3\tau + 6r\sigma^2\tau^2 + 4i\sigma\tau^3,
\end{aligned}$$

which, putting 0^4 for f, may also be written

$$\begin{aligned}
\Delta^4 . 0^4 = {} & (4h,\ 4j;\ 6p,\ 12l,\ 6q:\ 4k,\ 12m,\ 12n,\ 4g:\ 4f,\ 6r,\ 4i\dagger)\\
& (\rho^3\sigma,\ \rho^3\tau;\ \rho^2\sigma^2,\ \rho^2\sigma\tau,\ \rho^2\tau^2,\ \rho\sigma^3,\ \rho\sigma^2\tau,\ \rho\sigma\tau^2,\ \rho\tau^3,\ \sigma^3\tau,\ \sigma^2\tau^2,\ \sigma\tau^3);
\end{aligned}$$

and I join to it the series of equations

$$\begin{aligned}
\Delta^3 . 0^3 1 &= (0;\ 3h,\ 3j;\ 3r,\ 6l,\ 3q;\ k,\ 3m,\ 3n,\ g \between \rho^3,\ \rho^2\sigma,\ \rho^2\tau,\ \rho\sigma^2,\ \rho\sigma\tau,\ \rho\tau^2,\ \sigma^3,\ \sigma^2\tau,\ \sigma\tau^2,\ \tau^3),\\
\text{,,}\quad 0^3 2 &= (h;\ 3r,\ 3l;\ 3k,\ 6m,\ 3n;\ 0,\ 3f,\ 3p,\ i \between \quad \text{,,} \quad),\\
\text{,,}\quad 0^3 3 &= (j;\ 3l,\ 3q;\ 3n,\ 6n,\ 3g;\ f,\ 3p,\ 3i,\ 0 \between \quad \text{,,} \quad),
\end{aligned}$$

$$\begin{aligned}
\Delta^2 . 0^2 1^2 &= (0;\ 2h,\ 2j;\ r,\ 2l,\ q \between \rho^2;\ \rho\sigma,\ \rho\tau;\ \sigma^2,\ \sigma\tau,\ \tau^2),\\
\text{,,}\quad 0^2 12 &= (h;\ 2r,\ 2l;\ k,\ 2m,\ n \between \qquad \text{,,} \qquad),\\
\text{,,}\quad 0^2 13 &= (j;\ 2l,\ 2q;\ m,\ 2n,\ g \between \qquad \text{,,} \qquad),\\
\text{,,}\quad 0^2 2^2 &= (r;\ 2k,\ 2m;\ 0,\ 2f,\ p \between \qquad \text{,,} \qquad),\\
\text{,,}\quad 0^2 23 &= (l;\ 2m,\ 2n;\ f,\ 2p,\ i \between \qquad \text{,,} \qquad),\\
\text{,,}\quad 0^2 3^2 &= (q;\ 2n,\ 2g;\ p,\ 2i,\ 0 \between \qquad \text{,,} \qquad),\\
\Delta . 01^3 &= (0,\ h,\ j \between \rho,\ \sigma,\ \tau),\\
\text{,,}\quad 01^2 2 &= (h,\ r,\ l \between \quad \text{,,} \quad),\\
\text{,,}\quad 01^2 3 &= (j,\ l,\ q \between \quad \text{,,} \quad),\\
\text{,,}\quad 012^2 &= (r,\ k,\ m \between \quad \text{,,} \quad),\\
\text{,,}\quad 0123 &= (l,\ m,\ n \between \quad \text{,,} \quad),\\
\text{,,}\quad 013^2 &= (q,\ n,\ g \between \quad \text{,,} \quad),\\
\text{,,}\quad 02^3 &= (k,\ 0,\ f \between \quad \text{,,} \quad),\\
\text{,,}\quad 02^2 3 &= (m,\ f,\ p \between \quad \text{,,} \quad),\\
\text{,,}\quad 023^2 &= (n,\ p,\ i \between \quad \text{,,} \quad),\\
\text{,,}\quad 03^3 &= (g,\ i,\ 0 \between \quad \text{,,} \quad),
\end{aligned}$$

which will be convenient in the sequel.

Major Function—The Fixed Curve a Cubic. Art. No. 53.

53. It has been already seen that a simple particular form is $\frac{1}{3}\Omega' = \widetilde{012}$: and that the general form is $\Omega = \Omega' + K . 012$.

Major Function—The Fixed Curve a Quartic. Art. No. 54.

54. It is to be shown that a particular form is

$$\tfrac{1}{2}\Omega' = \frac{-01^3 . 02^3 + 01^2 2 . 012^2 + 0^2 12 . 1^2 2^2}{1^2 2^2}.$$

In fact, by the foregoing values of $\Delta . 01^3$, &c., the numerator of this expression, multiplied by Δ^2, is =

$$\begin{aligned}
&-(h\sigma + j\tau)(k\rho + f\tau)\\
&+(h\rho + r\sigma + l\tau)(r\rho + k\sigma + m\tau)\\
&+r(h\rho^2 + 2r\rho\sigma + 2l\rho\tau + k\sigma^2 + 2m\sigma\tau + n\tau^2),
\end{aligned}$$

which is

$$= 2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau + 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2;$$

and this, for $\tau = 0$, becomes

$$= r(2h\rho^2 + 3r\rho\sigma + 2k\sigma^2).$$

Hence for $\tau = 0$, we have

$$\Omega' = \frac{1}{\Delta^2}(4h\rho^2 + 6r\rho\sigma + 4k\sigma^2),$$

that is,

$$= \frac{1}{\Delta^2}\{4 \,.\, 1^2 2 \,.\, \rho^2 + 6 \,.\, 1^2 2^2 \,.\, \rho\sigma + 4 \,.\, 12^3 \,.\, \sigma^2\};$$

and Ω' is thus a form of the major function $(x, y, z)_{12}^2$. Of course the general form is $\Omega = \Omega' + (x, y, z)^1 \,.\, 012$.

Syzygy of the Major Function. Art. No. 55.

55. Writing now $(x, y, z)_{12}^{n-2} = \Omega_{12}$; and taking on the fixed curve a new point 3, consider the like functions Ω_{23} and Ω_{31}: it is to be shown that we have identically

$$\Omega_{23} \,.\, 031 \,.\, 012 + \Omega_{31} \,.\, 012 \,.\, 023 + \Omega_{12} \,.\, 023 \,.\, 031 - (123)^2 f = 023 \,.\, 031 \,.\, 012\,(x, y, z)^{n-3},$$

where $(x, y, z)^{n-3}$ is a *properly determined* minor function; or, considering herein 0 as a point on the fixed curve and writing therefore $f = 0$, the equation is

$$\frac{\Omega_{23}}{023} + \frac{\Omega_{31}}{031} + \frac{\Omega_{12}}{012} = (x, y, z)^{n-3}.*$$

56. Write for a moment $X = \Omega_{23} \,.\, 031 \,.\, 012 + \Omega_{31} \,.\, 012 \,.\, 023 + \Omega_{12} \,.\, 023 \,.\, 031$; then k being an arbitrary coefficient, we have $X - kf = 0$, a curve of the order n, passing through the points 1, 2, 3, and also through the residues of 2, 3, the residues of 3, 1, and the residues of 1, 2; in fact, at the point 1 we have $012 = 0$, $031 = 0$, and therefore $X = 0$; also $f = 0$; and therefore 1 is a point on the curve. Again, at any residue of 2, 3, we have $\Omega_{23} = 0$, $023 = 0$, and therefore $X = 0$; also $f = 0$; and hence the residue of 2, 3 is a point on the curve.

It is next to be shown that k can be so determined that the curve $X - kf = 0$ shall have a dp at each of the points 1, 2, 3. Supposing this to be so, we have the line 23 meeting the curve $X - kf = 0$ in the points 2 and 3, each counting twice, and in the $n-2$ residues of 2, 3, that is, in $n+2$ points; hence the curve $X - kf = 0$ must contain as part of itself the line 23, and similarly it must contain as part of itself each of the other lines 31 and 12, viz. we shall then have $X - kf = 023.031.012.(x, y, z)^{n-3}$; and from this equation observing that the curves $\Omega_{23} = 0$, $\Omega_{31} = 0$, $\Omega_{12} = 0$ each pass through the dps, it follows that the curve $(x, y, z)^{n-3} = 0$ also passes through the dps; hence, k being found to be $= (123)^2$, the theorem will be proved.

57. Taking an arbitrary point α coordinates $(x_\alpha, y_\alpha, z_\alpha)$, and writing

$$D = x_\alpha \frac{d}{dx} + y_\alpha \frac{d}{dy} + z_\alpha \frac{d}{dz},$$

* This is the differential theorem corresponding to C. and G.'s integral theorem, p. 26, viz. this is $S\xi\eta + S\eta\zeta + S\zeta\xi = I$, a sum of three integrals of the third kind = an integral of the first kind.

we have to find k, so that the curve $D(X-kf)=0$ shall pass through the point 1. Observing that $D023=\alpha 23$, &c., we have

$$\begin{aligned} D(X-kf) = {} & D\Omega_{23}\,.\,031\,.\,012 + \Omega_{23}(031\,.\,\alpha 12 + \alpha 31\,.\,012) \\ & + \alpha 23\,(\Omega_{31}\,.\,012 + \Omega_{12}\,.\,031) \\ & + 023\,\{\Omega_{31}\,.\,\alpha 12 + \Omega_{12}\,.\,\alpha 31 + D\Omega_{31}\,.\,012 + D\Omega_{12}\,.\,031\} \\ & - kDf, \end{aligned}$$

and, to make the curve pass through 1, writing herein $0=1$, we have

$$0 = 123\,(\Omega_{31}{}^{1}\,.\,\alpha 12 + \Omega_{12}{}^{1}\,.\,\alpha 31) - k\,(Df)^{1},$$

where the superfix (1) denotes that we are in Ω_{31}, Ω_{12} and Df respectively to write $0=1$. We have $\Omega_{31}{}^{1}=n\,.\,1^{n-1}\,3$, $\Omega_{12}{}^{1}=n\,.\,1^{n-1}\,2$, $(Df)^{1}=n\,.\,1^{n-1}\,\alpha$, and the equation thus is

$$n\,.\,123\,(1^{n-1}\,3\,.\,\alpha 12 + 1^{n-1}\,2\,.\,\alpha 31) - kn\,.\,1^{n-1}\,\alpha = 0.$$

But we have identically $1^{n-1}\,1\,.\,\alpha 23 + 1^{n-1}\,2\,.\,\alpha 31 + 1^{n-1}\,3\,.\,\alpha 12 = 1^{n-1}\,\alpha\,.\,123$, where $1^{n-1}\,1$, $=1^{n}$ is, in fact, $=0$; the factor $1^{n-1}\,\alpha$ thus divides out, and the equation becomes $k=(123)^{2}$; viz. k having this value, the curve $X-kf=0$ will have a dp at 1; and clearly by symmetry, it will also have a dp at 2, and at 3; the theorem is thus proved.

The Syzygy, Fixed Curve a Cubic. Art. No. 58.

58. The syzygy may be verified independently in the case where the fixed curve is a cubic. Observe that the syzygy, if satisfied for any particular form of Ω, will be generally satisfied; we may therefore take $\frac{1}{3}\Omega_{12}=\widetilde{012}$. Writing then

$$\frac{\frac{1}{3}\Omega_{12}}{012}=\frac{\widetilde{012}}{012},\ =\{012\}\ \text{suppose},$$

and taking 0 to be a point on the cubic curve, we ought to have $\{023\}+\{031\}+\{012\}=$ a constant; the value of this constant comes out to be $=\{123\}$, and the syzygy in its complete form thus is

$$\{023\}+\{031\}+\{012\}=\{123\}.$$

We have

$$\Delta\,\widetilde{023},\ \Delta\,\widetilde{031},\ \Delta\,\widetilde{012} = l\rho + f\sigma + i\tau,\ j\rho + l\sigma + g\tau,\ h\rho + k\sigma + l\tau,$$

and the equation thus is

$$\frac{l\rho + f\sigma + i\tau}{\rho} + \frac{j\rho + l\sigma + g\tau}{\sigma} + \frac{h\rho + k\sigma + l\tau}{\tau} - l = 0;$$

this, multiplied by $\rho\sigma\tau$, becomes

$$h\rho^{3}\sigma + j\rho^{2}\tau + k\rho\sigma^{2} + 2l\rho\sigma\tau + g\rho\tau^{2} + f\sigma^{2}\tau + i\sigma\tau^{2} = 0,$$

which is, in fact, $\frac{1}{3}f=0$, the equation of the cubic curve.

Observe that the new symbol $\{012\}$ is, in virtue of its determinant denominator, an alternate function, $\{012\}=-\{102\}$, $\{012\}=\{120\}=\{201\}$. The syzygy is a relation between any four points 1, 2, 3, 0 of the curve, and it may be also expressed in the form

$$\{123\}-\{230\}+\{301\}-\{012\}=0.$$

The Syzygy, Fixed Curve a Quartic. Art. No. 59.

59. Taking Ω_{12} as before, we have

$$\frac{\frac{1}{2}\Omega_{12}}{012} = \frac{-01^3 . 02^3 + 01^2 2 . 012^2 + 0^2 12 . 1^2 2^2}{012 . 1^2 2^2} = \{0^2 12\} \text{ suppose:}$$

and then, taking 0 to be a point on the quartic curve, we ought to have

$$\{0^2 23\} + \{0^2 31\} + \{0^2 12\} = (x, y, z)^1, \text{ a linear function of } (x, y, z),$$

or, what is the same thing, considering the left-hand side as expressed in terms of ρ, σ, τ, the sum should be

$$= (\rho, \sigma, \tau)^1, \text{ a linear function of } (\rho, \sigma, \tau).$$

By a preceding formula we have

$$\{0^2 12\} = \frac{1}{\Delta^2 r\tau} \{2hr\rho^2 + 3r^2\rho\sigma + (hm - jk + 3lr)\rho\tau$$
$$+ 2kr\sigma^2 + (-fh + kl + 3mr)\sigma\tau + (-fj + lm + nr)\tau^2\},$$

which is

$$= \frac{1}{\Delta^2}\left\{\left(3l + \frac{hm - jk}{r}\right)\rho + \left(3m + \frac{-fh + kl}{r}\right)\sigma + \left(n + \frac{-fj + lm}{r}\right)\tau\right\} + \frac{1}{\Delta^2}\frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}.$$

And hence, forming the sum $\{0^2 23\} + \{0^2 31\} + \{0^2 12\}$, we have first a fractional part which is found to be integral, viz. this is

$$\frac{1}{\Delta^2}\left\{\frac{2f\sigma^2 + 3p\sigma\tau + 2i\tau^2}{\rho} + \frac{2g\tau^2 + 2q\tau\rho + 2j\rho^2}{\sigma} + \frac{2h\rho^2 + 3r\rho\sigma + 2k\sigma^2}{\tau}\right\},$$

$$= \frac{1}{\Delta^2\rho\sigma\tau}\{2h\rho^3\sigma + 2j\rho^3\tau + 3r\rho^3\sigma^2 + 3q\rho^2\tau^2 + 2k\rho\sigma^3 + 2g\rho\tau^3 + 2f\sigma^3\tau + 3p\sigma^2\tau^2 + 2i\sigma\tau^3\},$$

$$= \frac{1}{\Delta^2\rho\sigma\tau}\{\tfrac{1}{2}\Delta^4 f - 6l\rho^3\sigma\tau - 6m\rho\sigma^2\tau - 6n\rho\sigma\tau^2\},$$

or since $f = 0$, this is

$$= \frac{1}{\Delta^2}(-6l\rho - 6m\sigma - 6n\tau).$$

We then have integral terms which are at once deduced from the above integral terms of $0^2 12$; collecting the several terms, we find

$$\{0^2 23\} + \{0^2 31\} + \{0^2 12\} =$$

$$\frac{1}{\Delta^2}\left\{\rho\left(l + \frac{mn - gk}{p} + \frac{jn - gh}{q} + \frac{hm - jk}{r}\right) + \sigma\left(m + \frac{fn - ik}{p} + \frac{ln - hi}{q} + \frac{kl - fh}{r}\right)\right.$$
$$\left. + \tau\left(n + \frac{lm - fg}{p} + \frac{gl - ij}{q} + \frac{lm - fj}{r}\right)\right\},$$

which is the required result.

Preparation for the Conversion—The Symbol ∂. Art. Nos. 60 to 63.

60. I use ∂ as the symbol of a quasi-differentiation, viz. U being any function of (x, y, z), ∂U denotes $\frac{1}{d\omega}$ multiplied by the differential $\frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dz}dz$; in such a differential, the increments dx, dy, dz do not in general present themselves in the combinations $ydz - zdy$, $zdx - xdz$, $xdy - ydx$; but they will do so if U is a function of the degree zero in the coordinates x, y, z (that is, if U be the quotient of two homogeneous functions of the same degree); and this being so, we can by the equations

$$\frac{ydz - zdy}{\frac{df}{dx}} = \frac{zdx - xdz}{\frac{df}{dy}} = \frac{xdy - ydx}{\frac{df}{dz}}, = d\omega$$

get rid of the increments, and ∂U will denote a function of (x, y, z) derived in a definite manner from the function U. The symbol ∂ will be used only in the case in question of a function of the degree zero. Of course ∂_1 will denote the like operation in regard to (x_1, y_1, z_1); and so ∂_2, &c.; and we may for greater clearness write ∂_0 in place of ∂.

61. Consider then $\partial \frac{P}{Q}$, where P, Q are functions $(x, y, z)^m$ of the same degree: we have

$$\partial \frac{P}{Q} = \frac{1}{Q^2 d\omega}(QdP - PdQ),$$

and then

$$dP = \frac{dP}{dx}dx + \frac{dP}{dy}dy + \frac{dP}{dz}dz, \quad \frac{1}{m}P = \frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z,$$

with the like formulæ for Q. Substituting, we find

$$\partial \frac{P}{Q} = \frac{1}{mQ^2 d\omega}\left\{\frac{d(Q, P)}{d(y, z)}(ydz - zdy) + \frac{d(Q, P)}{d(z, x)}(zdx - xdz) + \frac{d(Q, P)}{d(x, y)}(xdy - ydx)\right\},$$

that is,

$$\partial \frac{P}{Q} = \frac{1}{mQ^2}\left\{\frac{df}{dx}\frac{d(Q, P)}{d(y, z)} + \&c.\right\} = \frac{1}{mQ^2}\frac{d(f, Q, P)}{d(x, y, z)}, \quad = \frac{1}{mQ^3}J(f, Q, P),$$

or say

$$\partial \frac{P}{Q} = -\frac{1}{mQ^2}J(P, Q, f).$$

62. As an example, consider

$$\partial\{012\}, = -\frac{1}{(012)^2}J(\widetilde{012}, 012, f).$$

The determinant is

$$\begin{vmatrix} \frac{d}{dx}\widetilde{012}, & y_1z_2 - y_2z_1, & \frac{df}{dx} \\ \frac{d}{dy}\widetilde{012}, & z_1x_2 - z_2x_1, & \frac{df}{dy} \\ \frac{d}{dz}\widetilde{012}, & x_1y_2 - x_2y_1, & \frac{df}{dz} \end{vmatrix},$$

and the coefficient herein of $\frac{d}{dx}\widetilde{012}$ is $(z_1x_2 - z_2x_1)\frac{df}{dz} - (x_1y_2 - x_2y_1)\frac{df}{dy}$, which is

$$= x_2\left(x_1\frac{df}{dx} + y_1\frac{df}{dy} + z_1\frac{df}{dz}\right) - x_1\left(x_2\frac{df}{dx} + y_2\frac{df}{dy} + z_2\frac{df}{dz}\right), = 3\,(0^21\,.\,x_2 - 0^22\,.\,x_1);$$

and so for the other terms.

The determinant is thus

$$= 3\left[0^21\left(x_2\frac{d}{dx} + y_2\frac{d}{dy} + z_2\frac{d}{dz}\right) - 0^22\left(x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz}\right)\right]\widetilde{012}$$

say this is

$$= 3\,[0^21\,.\,\mathbb{D} - 0^22\,\mathbb{D}]\,\widetilde{012}.$$

But we have $\mathbb{D}\widetilde{012} = 12^2$, $\mathbb{D}\widetilde{012} = 1^22$, and the determinant is then $= 3(0^21\,.\,12^2 - 0^22.1^22)$; whence finally, writing ∂_0 instead of ∂,

$$\partial_0\,\{012\} = -3\,.\,\frac{0^21\,.\,12^2 - 0^22\,.\,1^22}{(012)^2}.$$

63. By cyclical interchange of the 0, 1, 2, we have

$$\partial_1\,\{012\} = -3\,.\,\frac{1^22\,.\,0^22 - 01^2\,.\,02^2}{(012)^2},$$

$$\partial_2\,\{012\} = -3\,.\,\frac{02^2\,.\,01^2 - 12^2\,.\,0^21}{(012)^2};$$

and thence adding, we find

$$(\partial_0 + \partial_1 + \partial_2)\,\{012\} = 0,$$

an important property which, joined to the equation before obtained,

$$\{023\} + \{031\} + \{012\} = \{123\},$$

completes the theory of the function $\{012\}$.

Conversion of the Major Function (Interchange of Limits and Parametric Points).

Art. No. 64.

64. Write in general

$$\frac{(x,\,y,\,z)_{12}^{\,n-2}}{012} = Q_{0,\,12},$$

$Q_{0,\,12}$ is an alternate function in regard to the points 1, 2 ($Q_{0,\,12} = -Q_{0,\,21}$); and in regard to the coordinates of the points 0, 1, 2, it is rational, but not integral, of the degrees $n-3$, 0, 0 respectively: it can therefore be operated upon with ∂_1 or ∂_2, but (except in the case $n = 3$) not with ∂_0.

The conversion relates not to the general major function $(x, y, z)_{12}^{n-2}$, but to this function *with the arbitrary constants properly determined*, and consists in a relation between two functions $Q_{4,12}$ and $Q_{1,45}$ (each of them a function of three out of four arbitrary points 1, 2, 4, 5 on the fixed curve), viz. the conversion is

$$\partial_1 Q_{4,12} = \partial_4 Q_{1,45},$$

an equation which may be written in four different forms, viz. we may in the form written down interchange 1, 2 and also 4, 5.*

The determination of the constants is a very peculiar one, inasmuch as it is not algebraical; viz. in the case of the cubic curve, about to be considered, it appears that $Q_{0,12}$ contains the term $\int_2^1 d\omega \partial_3 \{036\}$, which is a transcendental function of the coordinates of the parametric points 1 and 2.

The Conversion, Fixed Curve a Cubic. Art. No. 65.

65. We may write $Q_{0,12} = \{012\} + K$, where K is a constant, that is, it is independent of the point 0, but depends on the parametric points 1 and 2. I assume K to be properly determined, and give an *à posteriori* verification of the equation $\partial_1 Q_{4,12} = \partial_4 Q_{1,45}$. The value is $K = \int_2^1 d\omega \partial_3 \{036\} - \{123\}$, where 3, 6 are arbitrary points on the cubic curve, and where in the definite integral, regarded as an integral $\int U\,du$ with a current variable u, the meaning is that this variable has at the limits the values u_1, u_2 which belong to the points 1 and 2 respectively: a fuller explanation might be proper, but the investigation will presently be given in a form not depending on any integral at all.

Substituting for K its value, we have

$$Q_{0,12} = \{012\} + \left[\int_2^1 d\omega\, \partial_2 \{036\} - \{123\}\right],$$

or, as this may also be written,

$$= -\{023\} - \{031\} + \int_2^1 d\omega\, \partial_3 \{036\}.$$

We have thence

$$\partial_3 Q_{0,12} = -\partial_1 \{031\} + \partial_3 \{136\},$$

The meaning of the property is better seen from the integral form: $Q_{0,12}$ is a function of the points 0, 1, 2 and $Q_{0,45}$ the like function of the points 0, 4, 5 such that $\int_5^4 d\omega\, Q_{0,12} = \int_2^1 d\omega\, Q_{0,45}$; which equation operated upon with $\partial_1 \partial_4$ gives the formula of the text. And there is thus the meaning (alluded to in the heading) that there exists for the integral of the third kind a canonical form (C. and G.'s endliche Normalform), such that the integral is not altered by the interchange of the limits and the parametric points. The expression for $Q_{0,12}$ mentioned further on in the text for the case, fixed curve a cubic, shows that in this case the canonical form of the integral of the third kind is $\int_5^4 d\omega \left[\{012\} + \left(\int_2^1 d\omega\, \partial_3 \{036\} - \{123\}\right)\right]$.

and consequently

$$\partial_1 Q_{4,\,12} = -\partial_1 \{431\} + \partial_3 \{136\},$$
$$\partial_4 Q_{1,\,45} = -\partial_4 \{134\} + \partial_3 \{436\};$$

hence, observing that $\{431\} = -\{134\}$, &c., we have

$$\partial_1 Q_{4,\,12} - \partial_4 Q_{1,\,45} = \quad (\partial_1 + \partial_4)\{134\} + \partial_3 \{136\} - \partial_3 \{436\},$$
$$= -\partial_3 \{134\} + \partial_3 \{136\} - \partial_3 \{436\},$$

which, observing that we have $\partial_3 \{641\} = 0$, is

$$= \partial_3 (\{136\} - \{364\} + \{641\} - \{413\}), = 0,$$

the required theorem.

To avoid, in the proof, the use of the integral sign, we have only to consider the required function $Q_{0,\,12}$ as given by the foregoing differential formula

$$\partial_1 Q_{0,\,12} = -\partial_1 \{031\} + \partial_3 \{136\},$$

for we have then the values of $\partial_1 Q_{4,\,12}$ and $\partial_4 Q_{1,\,45}$: the rest of the proof the same as before.

The Conversion, Fixed Curve a Quartic. Art. Nos. 66 to 73.

66. We have

$$Q_{0,\,12} = \{0^2 12\} + (x,\ y,\ z)^1,$$

where $(x,\ y,\ z)^1$ is a linear function of $(x,\ y,\ z)$, but depending also on the parametric points 1 and 2, which is to be determined so as to satisfy the conversion equation

$$\partial_1 Q_{4,\,12} = \partial_4 Q_{1,\,45}.$$

Observing that we have $\{0^2 23\} + \{0^2 31\} + \{0^2 12\} =$ a linear function of $(x,\ y,\ z)$, the linear function $(x,\ y,\ z)^1$ of $Q_{0,\,12}$ may be taken to be $= \Theta_{0,\,12} - \{0^2 23\} - \{0^2 31\} - \{0^2 12\}$; that is, we may assume

$$Q_{0,\,12} = \{0^2 12\} + \Theta_{0,\,12} - (\{0^2 23\} + \{0^2 31\} + \{0^2 12\}),$$
$$= -\{0^2 23\} - \{0^2 31\} + \Theta_{0,\,12},$$

where $\Theta_{0,\,12}$ is a linear function of $(x,\ y,\ z)$, but depending also on the points 1 and 2, which has to be determined. We have

$$\partial_1 Q_{0,\,12} = -\partial_1 \{0^2 31\} + \partial_1 \Theta_{0,\,12},$$

and thence

$$\partial_1 Q_{4,\,12} = -\partial_1 \{4^2 31\} + \partial_1 \Theta_{4,\,12},$$
$$\partial_4 Q_{1,\,45} = -\partial_4 \{1^2 34\} + \partial_4 \Theta_{1,\,45},$$

giving an equation for Θ,

$$\partial_1 \Theta_{4,\,12} - \partial_4 \Theta_{1,\,45} = \partial_1 \{4^2 31\} - \partial_4 \{1^2 34\};$$

here 4 is an arbitrary point of the quartic, and we may instead of it write 0, the equation thus becoming

$$\partial_1 \Theta_{0,\,12} - \partial_0 \Theta_{1,\,05} = \partial_1 \{0^2 31\} - \partial_0 \{1^2 30\}.$$

67. Of the terms on the left-hand side, the first is a linear function of (x, y, z), or say it is an integral function 0^1, and the second is a linear function of (x_1, y_1, z_1), or say it is an integral function 1^1: the given function on the right-hand side must therefore admit of expression in the form $\phi(0^1, 1, 3) - \phi(1^1, 0, 3)$, where $\phi(0^1, 1, 3)$ is a known function, integral and linear as regards the coordinates (x, y, z) of the point 0, but depending also on the points 1, 3; and $\phi(1^1, 0, 3)$ is the like known function, integral and linear as regards the coordinates (x_1, y_1, z_1) of the point 1, but depending also on the points 0, 3. Moreover, since 2 and 5 are arbitrary points entering only on the left-hand side, it is clear that $\partial_1\Theta_{0,12}$ must be independent of 2, and $\partial_0\Theta_{1,05}$ independent of 5; reverting to the cubic case, observe that here $\Theta_{0,12} = \int_2^1 d\omega\, \partial_3\{036\}$, whence $\partial_1\Theta_{0,12} = \partial_3\{136\}$, and so $\partial_0\Theta_{1,05} = \partial_3\{036\}$, and that the corresponding equation thus is $\partial_3\{136\} - \partial_3\{036\} = \partial_1\{031\} - \partial_0\{130\}$, where the left-hand side is $= \partial_3\{013\}$, and the equation itself $(\partial_0 + \partial_3 + \partial_1)\{031\} = 0$. We then have

$$\partial_1\Theta_{0,12} - \phi(0^1, 1, 3) = \partial_0\Theta_{1,02} - \phi(1^1, 0, 3),$$

where the one side is derived from the other by the interchange of the 0, 1. The solution therefore is

$$\partial_1\Theta_{0,12} - \phi(0^1, 1, 3) = X(\overline{0, 1}, 3),$$

a function which is symmetrical in regard to the points 0 and 1, and, inasmuch as the left-hand is an integral function 0^1, must itself be an integral function $(0^1, 1^1)$, that is, integral and linear as regards the coordinates (x, y, z) and (x_1, y_1, z_1) of the points 0 and 1 respectively. We thus have

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3),$$

and thence

$$\partial_2\Theta_{0,12} = -\phi(0^1, 2, 3) - X(\overline{0, 2}, 3),$$

viz. the second of these expressions is, with its sign reversed, the same function of 2 that the first is of 1.

68. It follows that, taking a new symbol 7 for the variable of the definite integral (in the cubic case $\Theta_{0,12}$ was independent of 0, and there was nothing to prevent the use of 0 for the current point of the definite integral), we may write $\Theta_{0,12} = \int_2^1 d\omega_7 P(7, 0, 3)$, where $\partial_1 P(1, 0, 3) = \phi(0^1, 1, 3) + X(\overline{0, 1}, 3)$, an equation which implies $\partial_2 P(0, 2, 3) = \phi(0^1, 2, 3) + X(\overline{0, 2}, 3)$. But the first of these equations in P is nothing else than the first of the equations in $\Theta_{0,12}$.

69. I have succeeded in finding $\phi(0^1, 1, 3)$, but the calculation is a very tedious one, and I give only the principal steps, omitting all details. We have to bring $\partial_1 0^2 13 - \partial_0 1^2 03$ into the form $\phi(0^1, 1, 3) - \phi(1^1, 0, 3)$. From the value of

$$\{0^2 13\}, = \frac{-01^3 . 03^3 + 01^2 3 . 013^2 - 0^2 13 . 1^2 3^2}{013 . 1^2 3^2},$$

we find, by a process such as that of No. 62,

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\sigma^2}\Big\{&-0^2 3^2 . 01^3 - 0^2 3 . j \\ &+\frac{1}{q}\Big(2 . 01^3 [03^3 . 01^2 3 - (013^2)^2] \\ &\qquad + [2 . 0^2 13 . 013^2 + 1 . 0^2 3^2 . 01^2 3 - 3 . 0^2 1^2 . 0^2 3^2] j\Big) \\ &+\frac{1}{q^2}\Big(\begin{matrix} 2 . 01^3 (-01^3 . 03^3 + 013^2 . 01^2 3) g \\ -2 . 013^2 (-01^2 . 03^3 + 013^2 . 01^2 3) j \end{matrix}\Big)\Big\}.\end{aligned}$$

Substituting herein the values $01^3 = \frac{1}{\Delta}(h\sigma + j\tau)$, &c., we have $\frac{1}{\sigma^2}$ multiplied by a cubic function $(\rho, \sigma, \tau)^3$; writing down first the integral terms, and then the others, we have

$$\begin{aligned}\partial_1\{0^2 13\} = \frac{1}{\Delta^3}\Big\{&\rho\Big[(-6hn + 5jm) + \frac{1}{q}(4ghl - 3gjr - 2hij + j^2 p + 2jnl) + \frac{1}{q^2}(-2g^2 h^2 + 4ghjn - 2j^2 n^2)\Big] \\ &+\sigma\Big[fj - hp + \frac{1}{q}(2hil - 2hn^2 - 3ijr + jlp + 2jmn) + \frac{1}{q^2}(2ghln - 2gh^2 i + 2hijn - 2jln^2)\Big] \\ &+\tau\Big[3jp + \frac{1}{q}(-2ghn + 2gjm - 2ijl - 2jn^2) + \frac{1}{q^2}(2g^2 hl - 2ghij - 2gjln + 2ij^2 n)\Big]\Big\}\end{aligned}$$

(say this linear function of ρ, σ, τ is $= \square$)

$$+\frac{1}{\Delta^3\sigma^2}\{\rho^3 . 2j^2 + \rho^2\sigma(6jl - 3hg) + \rho^2\tau . 3jq + \rho\sigma\tau(-2gh + 6jn) + \rho\tau^2 . 2gj + \sigma\tau^2 . 2ij\}.$$

70. The expression of $\partial_0\{1^2 03\}$ is deduced from this by the interchange of 0, 1: and I write

$$\begin{aligned}\partial_1\{0^2 13\} - \partial_0\{1^2 03\} = \square - *\\ +\frac{1}{\sigma^2\Delta^3\rho^3}[\rho^3\{\rho^3 . 2j^2 + \rho^2\sigma(-3hq + 6jl) + \rho^2\tau . 3jq + \rho\sigma\tau(-2gh + 6jn) + \rho\tau^2 . 2gj + \sigma\tau^2 . 2ij\} \\ - \Delta^3\{\Delta^3 . 2(0^2 3^2)^2 - \Delta^2\sigma(-3 . 0^3 2 . 0^2 3^2 + 6 . 0^2 3 . 0^2 23) - \Delta^2\tau . 3 . 0^2 3 . 0^2 3^2 \\ + \Delta\sigma\tau(-2 . 03^3 . 0^3 2 + 6 . 0^3 3 . 023^2) + \Delta\tau^2 . 2 . 03^3 . 0^3 3 - \sigma\tau^2 . 2i . 0^3 3\}],\end{aligned}$$

where, and in what follows, the $*$ denotes the function immediately to the left of it, interchanging therein the 0, 1. It will be observed that the $\square$, *quâ* linear function of (ρ, σ, τ), that is, of (x, y, z), is a term of the required function $\phi(0^3, 1, 3)$: the remaining portion has to be reduced by means of the expressions for $\Delta^2(0^2 3^2)$, &c., in terms of ρ, σ, τ.

71. We obtain

$$\begin{aligned}\partial_1\{0^2 13\} - \partial_0\{1^2 03\} = \square - *\\ +\frac{1}{\Delta^3}\{\sigma(2fj - 3hp - 3kq + 18lm - 9nr) + \tau(-3gr - 3jp + 9mq)\} \\ +\frac{1}{\Delta^3\rho}\{\sigma^2(12fl - 12kn + 18m^2 - 9pr) + \sigma\tau(6fg - 6gk - 6ir + 18mn) + \tau^2 . 6gm\} \\ +\frac{1}{\Delta^3\rho^2}\{\sigma^3(18fm - 9kp) + \sigma^2\tau(12fn - 8ik + 9mp) + \sigma\tau^2(4fg + 6im)\} \\ +\frac{1}{\Delta^3\rho^3}\{\sigma^4 . 4f^2 + \sigma^3\tau . 6fp + \sigma^2\tau^2 . 4fi\}.\end{aligned}$$

The terms of the second line may be transformed as follows:

$$\frac{\sigma}{\Delta^3}(2fj-3hp-3kq+18lm-9nr)$$

$$=\tfrac{1}{2}\frac{\sigma}{\Delta^3}(2fj-3hp-3kq+18lm-9nr)-*$$

$$+\frac{1}{\Delta^2\rho}\{\sigma^2(-12fl+12kn-18m^2+9pr)+\sigma\tau(-\tfrac{3}{2}fq+3gk+\tfrac{9}{2}ir-\tfrac{3}{2}lp-9mn)\}$$

$$+\frac{1}{\Delta^2\rho^2}\{\sigma^3(-30fm+15kp)+\sigma^2\tau(-12fn+12ik-18mp)+\sigma\tau^2.-9np\}$$

$$+\frac{1}{\Delta^3\rho^3}\{\sigma^4.-10f^2+\sigma^3\tau.-15fp+\sigma^2\tau^2.-9p^2+\sigma\tau^3.-3ip\},$$

and

$$\frac{\tau}{\Delta^3}(-3gr-3jp+9mq)$$

$$=\tfrac{1}{2}\frac{\tau}{\Delta^3}(-3gr-3jp+9mq)-*$$

$$+\frac{1}{\Delta^2\rho}\{\sigma\tau(-\tfrac{9}{2}fq+3gk+\tfrac{3}{2}ir+\tfrac{3}{2}lp-9mn)+\tau^2.-6$$

$$+\frac{1}{\Delta^2\rho^2}\{\sigma^2\tau(-9fn+3ik)+\sigma\tau^2(-6fg-6im)+\tau^3.-$$

$$+\frac{1}{\Delta^2\rho^3}\{\sigma^2\tau.-3fp+\sigma^2\tau^2.-6fi+\sigma\tau^3.-3ip\};$$

substituting these values, the whole third line is destroyed, and we find

$$\partial_1\{0^2 13\}-\partial_0\{1^2 03\}=\square-*$$

$$+\tfrac{1}{2}\cdot\frac{1}{\Delta^3}\{\sigma(2fj-3hq-3kj+18lm-9nr)+\tau(-3gr-3jp+9mq)\}-*$$

$$+\frac{1}{\Delta^3\rho^2}\{\sigma^3(-12fm+6kp)+\sigma^2\tau(-9fn+7ik-9mp)+\sigma\tau^2(-2fg-9np)+\tau^3.-3gp\}$$

$$+\frac{1}{\Delta^3\rho^3}\{\sigma^4.-6f^2+\sigma^3\tau.-12fp+\sigma^2\tau^2(-2fi-9p^2)+\sigma\tau^3.-6ip\}.$$

Ultimately the last two lines of this expression are found to be

$$=\frac{1}{\Delta^3}\{\rho(-2hn+4jm+2l^2-2qr)+\sigma(-2fj+2hp+2kq-10lm+5nr)$$
$$+\tau(2gr+hi-jp+7ln-4mq)\}-*$$

so that the whole is now a sum of three linear functions of $(\rho,\ \sigma,\ \tau).-*$.

72. Collecting the terms, we have

$$\partial_1\{0^2 13\} - \partial_0\{1^2 03\} =$$

$$\frac{1}{\Delta^2}\Big[\rho\Big\{(-8hn+9jm+2l^2-2qr)+\frac{1}{q}(4ghl-3gjr-2hij+j^2p+2jln)$$
$$+\frac{1}{q^2}(-2g^2h^2+4ghjn-2j^2n^2)\Big\}$$
$$+\sigma\Big\{\tfrac{1}{2}(-hp+kq-2lm+nr)+\frac{1}{q}(2hil-2hn^2-3ijr+jlp+2jmn)$$
$$+\frac{1}{q^2}(-2gh^2i+2ghln+2hijn-2jln^2)\Big\}$$
$$+\tau\Big\{\tfrac{1}{2}(gr+2hi+jp+mq+14ln)+\frac{1}{q}(-2ghn+2gjm-2ijl-2jn^2)$$
$$+\frac{1}{q^2}(2g^2hl-2ghij-2gjln+2ij^2n)\Big\}\Big]$$
$$-\ *.$$

73. The right-hand side depends on the points 0, 1, 3 and 2: viz. we have therein $\rho = 023$, $\Delta = 123$, &c., but the left-hand side depending on only the points 0, 1 and 3, the right-hand side cannot really contain 2, and it must thus remain unaltered, if for 2 we substitute any other point on the quartic, say 6: the right-hand side may therefore be understood as a function of 0, 1, 3 and 6, viz. ρ, Δ, f, &c., will mean 063, 163, $6^2 3$, &c.: we have thus $\phi(0^1, 1, 3) =$ the above linear function with 2 thus replaced by 6; say

$$\phi(0^1, 1, 3) = \frac{1}{\Delta^2}[\rho(\ \ )+\sigma(\ \ )+\tau(\ \ )],$$

a given function of the points 0, 1, 3 and the arbitrary point 6, on the quartic curve; we therefore write it $\phi(0^1, 1, 3, 6)$. There is no obvious value for $X(\overline{0, 1}, 3)$ which will produce any simplification: I therefore take this function to be $= 0$; and the final result is

$$Q_{0,12} = \{0^2 12\} + \Theta_{0,12} - (\{0^2 23\} + \{0^2 31\} + \{0^2 12\}),$$

where $\Theta_{0,12}$ is a function integral and linear as regards the coordinates (x, y, z) of the point 0, but transcendental as regards the parametric points 1, 2; and containing besides the arbitrary points 3, 6, of the quartic curve, its value being determined by the differential formulæ

$$\partial_1\Theta_{0,12} = \phi(0^1, 1, 3, 6), \quad \partial_2\Theta_{0,12} = -\phi(0^1, 2, 3, 6),$$

where $\phi(0^1, 1, 3, 6)$ is a given function as above. I do not see the meaning of the very complicated linear function of (ρ, σ, τ), nor how to reduce it to any form such as the simple one $\partial_3\{036\}$, which presents itself in the case of the cubic curve.

Cambridge, England, October 5, 1882.

CHAPTER IV. THE MAJOR FUNCTION $(x, y, z)_{12}^{n-2}$, CONTINUED.

The Conversion, Fixed Curve a Quartic, continued. Art. Nos. 74 to 82.

74. I resume the question considered *ante* Nos. 66 to 73. The general problem, where the fixed curve is any given curve whatever, has recently been solved in a very complete and elegant form by Dr Nöther, in the two notes "Zur Reduction algebraischer Differentialausdrücke auf die Normalformen" and "Ueber die algebraischen Differentialausdrücke, 2e Note," *Sitzungsb. der phys.-med. Soc. zu Erlangen*, 10 Dec. 1883 and 14 Jan. 1884. I consider here the case of the quartic curve, $n=4$, and connect his result with my former investigations.

We have the differential

$$\frac{(x, y, z)_{12}^{n-2}\, d\omega}{012}, \; = \frac{\Omega_{12} d\omega}{012},$$

where Ω_{12}, or as I also write it $\Omega\,(0;\ 1,\ 2;\ 3,\ 4,\ 5)$, is a rational and integral function of the degree $(n-2=)\,2$ in the current coordinates (x, y, z): it depends also on the parametric points 1, 2, which are points on the quartic, coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively; and on $(p=)\,3$ other points 3, 4, 5 on the quartic, coordinates (x_3, y_3, z_3), (x_4, y_4, z_4), (x_5, y_5, z_5) respectively. The curve $\Omega=0$ is a conic, which is taken to pass through the dps (none in the present case) and through the $(n-2=)\,2$ residues of the parametric points; and the function Ω is such that on writing therein (x_1, y_1, z_1) for (x, y, z) it becomes $=(n.1^{n-1}2^{*}=)\,_4.1^{3}2$: viz. we have $\Omega\,(1;\,1,\,2;\,3,\,4,\,5)=\,_4.1^{3}2$, which implies also $\Omega\,(2;\ 1,\ 2;\ 3,\ 4,\ 5)=\,_4.12^{3}$: so defined, the function would contain $(p=)\,3$ arbitrary constants, but these are determined so that the curve $\Omega=0$ passes through the 3 points 3, 4, 5 on the quartic: and the function Ω, $=\Omega\,(0:\ 1,\ 2:\ 3,\ 4,\ 5)$ is thus a completely determinate function, rational and integral of the degree 2 in the coordinates (x, y, z) of the current point, and rational in the coordinates of the other five points respectively. I call to mind that 012 denotes the determinant formed with the coordinates (x, y, z), &c., of the points 0, 1, 2 respectively: the like notation is used throughout.

75. The function $\Omega\,(0;\ 1,\ 2;\ 3,\ 4,\ 5)$ is, in fact, the function Ω' of No. 43 with only the further condition in regard to the points 3, 4, 5 of the quartic; viz. Ω is the function determined by the equation

$$\left|\begin{array}{ll} (x\,,\,y\,,\,z)^2 & ,\ \Omega \\ 1\,(x_1,\ y_1,\ z_1)^2 & ,\ 4\,.\,1^3 2 \\ 2\,(x_1,\ y_1,\ z_1)(x_2,\ y_2,\ z_2), & 6\,.\,1^2 2^2 \\ 1\,(x_2,\ y_2,\ z_2)^2 & ,\ 4\,.\,12^3 \\ (x_3,\ y_3,\ z_3)^2 & ,\ 0 \\ (x_4,\ y_4,\ z_4)^2 & ,\ 0 \\ (x_5,\ y_5,\ z_5)^2 & ,\ 0 \end{array}\right| = 0:$$

* $\left(x_2\dfrac{d}{dx_1}+y_2\dfrac{d}{dy_1}+z_2\dfrac{d}{dz_1}\right)f(x_1,\ y_1,\ z_1)=n\,.\,1^{n-1}2$; see No. 2.

this is of the form

$$M\Omega + \square = 0,$$

and as appears in No. 46, M is $= 123.124.125.345$. Hence writing $\square(0;\ 1, 2;\ 3, 4, 5)$ for $\square$, we have

$$\Omega, = \Omega(0;\ 1, 2;\ 3, 4, 5), = \frac{-\square(0;\ 1, 2;\ 3, 4, 5)}{123.124.125.345}.$$

Hence further writing

$$Q, = Q(0;\ 1, 2;\ 3, 4, 5), = \frac{\Omega(0;\ 1, 2;\ 3, 4, 5)}{012},$$

so that the differential is $Qd\omega, = Q(0;\ 1, 2;\ 3, 4, 5)\,d\omega$, we have

$$Q = Q(0;\ 1, 2;\ 3, 4, 5), = \frac{-\square(0;\ 1, 2;\ 3, 4, 5)}{012.123.124.125.345},$$

which is of the form

$$Q = \frac{0^2\overline{12}^4\overline{345}^2}{0^1\overline{12}^3\overline{345}^2},\ = 0^1\overline{12345}^0,$$

viz. Q is a rational fraction where the numerator is of the degree 2, and the denominator of the degree 1 as regards the coordinates (x, y, z) of the current point: but the numerator and denominator are each of the degree 4 as regards the coordinates of the points 1, 2 separately, and of the degree 2 as regards the coordinates of the points 3, 4, 5 separately: that is, Q is of the degree 1 as regards the coordinates (x, y, z), but of the degree 0 as regards the coordinates of the points 1, 2, 3, 4, 5 separately.

76. The signification of the symbol of quasi-differentiation ∂ (applicable only to a function of the degree 0 in the coordinates to which the differentiations have reference) is explained *ante* No. 60. The function Q just mentioned is of the degree 0 in regard to the coordinates of each of the points 1, 2, 3, 4, 5; and it can thus be operated upon by the symbols $\partial_1, \partial_2, \partial_3, \partial_4, \partial_5$ respectively. Observe, in particular, that we have $\partial_1 Q(0;\ 1, 2;\ 3, 4, 5) = \overline{01}^1\overline{2345}^0$, viz. it is of the degree 1 in the coordinates of the points 0 and 1 respectively, but of the degree 0 in regard to the coordinates of the points 2, 3, 4, 5 respectively.

77. This being so, we may consider the function

$$\begin{aligned} H(0;\ 1, 2;\ 3, 4, 5;\ 6, 7, 8) = {} & \partial_1 Q(0;\ 1, 2;\ 3, 4, 5) \\ & + \partial_3 Q(1;\ 3, 2;\ 6, 4, 5).\frac{045}{\overline{345}} \\ & + \partial_4 Q(1;\ 4, 2;\ 3, 7, 5).\frac{053}{\overline{453}} \\ & + \partial_5 Q(1;\ 5, 2;\ 3, 4, 8).\frac{034}{\overline{534}}, \end{aligned}$$

where 6, 7, 8 are arbitrary points on the quartic; the functions

$$\partial_3 Q(1;\ 3, 2;\ 6, 4, 5),\quad \partial_4 Q(1;\ 4, 2;\ 3, 7, 5),\quad \partial_5 Q(1;\ 5, 2;\ 3, 4, 8),$$

are functions of the same form as $\partial_1 Q(0;\ 1,\ 2;\ 3,\ 4,\ 5)$, and derived from it by changing in each case the current point 0 into the parametric point 1, and by further changing in the three cases this parametric point into the points 3, 4, 5 respectively, and replacing the corresponding point 3, 4 or 5 by the new arbitrary point 6, 7 or 8. Further 045, &c., denote determinants as above; so that in H each of the last three terms is, in fact, as regards the point 0, a mere linear function of the coordinates (x, y, z) of this point.

We have $\partial_3 Q(1;\ 3,\ 2;\ 6,\ 4,\ 5) = \overline{13}^1\overline{2645}^0$, and hence this function multiplied by $\dfrac{045}{\overline{345}}$ is $= \overline{01}^1\overline{23456}^0$; and so for the third and fourth terms of H: thus each of the four terms of H is $= \overline{01}^1\overline{2345678}^0$, of the degree 1 in the coordinates of the points 0 and 1 respectively, but of the degree 0 in the coordinates of the other points 2, 3, 4, 5, 6, 7, 8 respectively.

Nöther's conversion-theorem consists herein, that the function

$$H(0;\ 1,\ 2;\ 3,\ 4,\ 5;\ 6,\ 7,\ 8)$$

is unaltered by the interchange of the two points 0, 1; or putting for shortness

$$H(0;\ 1,\ 2;\ 3,\ 4,\ 5;\ 6,\ 7,\ 8) = H_1(0),$$

the theorem is

$$H_1(0) = H_0(1).$$

78. We have, No. 59,

$$\frac{\frac{1}{2}\Omega_{12}}{012} = \frac{-01^3 . 02^2 + 01^2 2 . 012^2 + 0^2 12 . 1^2 2^2}{012 . 1^2 2^2}, = \{0^2 12\},$$

or as for greater simplicity I write it $= 0^2 12$,

viz. $0^2 12$ is now written instead of $\{0^2 12\}$ to denote the function just given as the value of $\dfrac{\frac{1}{2}\Omega_{12}}{012}$, Ω_{12} is thus $= 2 . 012 . 0^2 12$, viz. this is a particular form of Ω_{12} satisfying the conditions that $\Omega_{12} = 0$ is a conic passing through the residues of the points 1, 2, and such that Ω_{12} on writing therein (x_1, y_1, z_1) for (x, y, z) becomes $= 4 . 1^2 2$: hence the general form of the function satisfying these conditions is $= 2 . 012\,\{0^2 12 +$ arbitrary linear function of $(x, y, z)\}$. The before-mentioned function $\Omega(0;\ 1,\ 2;\ 3,\ 4,\ 5)$ is a function satisfying these conditions and the further conditions that the conic $\Omega = 0$ shall pass through the three points 3, 4, 5 on the quartic: these further conditions serve to determine the linear function: and we at once obtain

$$\frac{\frac{1}{2}\Omega(0;\ 1,\ 2;\ 3,\ 4,\ 5)}{012} = 0^2 12 - 3^2 12\,\frac{045}{\overline{345}} - 4^2 12\,\frac{053}{\overline{453}} - 5^2 12 . \frac{034}{\overline{534}},$$

viz. the value of Ω given by this equation, on writing therein $0 = 3$, 4, or 5, becomes $= 0$ as it should do.

79. We thus have

$$Q(0;\ 1,\ 2;\ 3,\ 4,\ 5) = 0^2 12 - 3^2 12 \frac{045}{345} - 4^2 12 \frac{053}{453} - 5^2 12 \frac{034}{534};$$

and Nöther's conversion-equation becomes

$$\begin{aligned}
&\partial_1 \left\{0^2 12 - 3^2 12 \frac{045}{345} - 4^2 12 \frac{053}{453} - 5^2 12 \frac{034}{534}\right\} \\
&+ \partial_3 \left\{1^2 32 - 6^2 32 \frac{145}{645} - 4^2 32 \frac{156}{456} - 5^2 32 \frac{164}{564}\right\} \cdot \frac{045}{345} \\
&+ \partial_4 \left\{1^2 42 - 3^2 42 \frac{175}{375} - 7^2 42 \frac{153}{753} - 5^2 42 \frac{137}{537}\right\} \cdot \frac{053}{453} \\
&+ \partial_5 \left\{1^2 52 - 3^2 52 \frac{148}{348} - 4^2 52 \frac{183}{483} - 5^2 48 \frac{134}{834}\right\} \cdot \frac{034}{534} \\
= \ &\partial_0 \left\{1^2 02 - 3^2 02 \frac{145}{345} - 4^2 02 \frac{153}{453} - 5^2 02 \frac{134}{534}\right\} \\
&+ \partial_3 \left\{0^2 32 - 6^2 32 \frac{045}{645} - 4^2 32 \frac{056}{456} - 5^2 32 \frac{064}{564}\right\} \cdot \frac{145}{345} \\
&+ \partial_4 \left\{0^2 42 - 3^2 42 \frac{075}{375} - 7^2 42 \frac{053}{753} - 5^2 42 \frac{037}{537}\right\} \cdot \frac{153}{453} \\
&+ \partial_5 \left\{0^2 52 - 3^2 52 \frac{048}{348} - 4^2 52 \frac{083}{483} - 5^2 48 \frac{034}{834}\right\} \cdot \frac{134}{534},
\end{aligned}$$

an equation where the functions operated on with the ∂'s are only functions such as $0^2 12$; for there is not any determinant operated upon containing the number which is the suffix of the ∂ operating upon it.

80. Taking all the terms over to the left-hand side, there are in all 32 terms: but of these $3+3$ destroy each other, and $6+6$ unite in pairs into 6 terms: there are thus in all $7+7+6, =20$ terms: viz. multiplying the whole equation by 345, it is found that the equation becomes

	or as this may be written	where
$345\,(\ \partial_1 0^2 12 - \partial_0 1^2 02)$	$345\ \boxed{012}$	$\boxed{012} = \partial_1 0^2 12 - \partial_0 1^2 02$, &c.
$-045\,(-\partial_3 1^2 32 + \partial_1 3^2 12)$	$-045\ \boxed{312}$	
$-053\,(-\partial_4 1^2 42 + \partial_1 4^2 12)$	$-305\ \boxed{412}$	
$-034\,(-\partial_5 1^2 52 + \partial_1 5^2 12)$	$-340\ \boxed{512}$	

$$
\begin{array}{ll}
-145\,(\ \partial_3\, 0^2 32 - \partial_0\, 3^2 02) & -145\ \boxed{032} \\[2ex]
-153\,(\ \partial_4\, 0^2 42 - \partial_0\, 4^2 02) & -315\ \boxed{042} \\[2ex]
-134\,(\ \partial_5\, 0^2 52 - \partial_0\, 5^2 02) & -341\ \boxed{052} \\[2ex]
+013\,(-\partial_4\, 5^2 42 + \partial_5\, 4^2 52) & +301\ \boxed{452} \\[2ex]
+014\,(-\partial_5\, 3^2 52 + \partial_3\, 5^2 32) & +140\ \boxed{532} \\[2ex]
+015\,(-\partial_3\, 4^2 32 + \partial_4\, 3^2 42) & +015\ \boxed{342} \\[2ex]
= 0, & = 0,
\end{array}
$$

viz. the equation is

$$\Sigma \pm 345\ \boxed{012} = 0,$$

the nine terms which follow the first term $345\ \boxed{012}$ of the sum being obtained by the interchanges of 0, 1 (one or each) with the 3, 4, 5, each interchange giving rise to a sign $-$.

81. In obtaining the foregoing result, we have, for instance, a pair of terms

$$\partial_4\, 5^2 42\, \frac{-137\,.\,053 + 037\,.\,153}{537}, \quad = \partial_4\, 5^2 42\, \frac{-013\,.\,537}{537}, \quad = \partial_4\, 5^2 42\,(-013),$$

viz. this depends on the equation

$$137\,.\,053 - 037\,.\,153 - 537\,.\,013 = 0,$$

or say

$$-137\,.\,035 + 037\,.\,135 + 013\,.\,357 = 0,$$

an identity which, in a form which will be readily understood, may be written

$$\det.\ \begin{vmatrix} 0137 \\ 013735 \end{vmatrix} = 0.$$

Similarly, the two terms which contain $\partial_5\, 4^2 52$ combine into the single term $\partial_5\, 4^2 52\,(013)$: and the two new terms taken together are

$$013\,(-\partial_4\, 5^2 42 + \partial_5\, 4^2 52). = 301\ \boxed{452}.$$

82. The proof of the identity,

$$\Sigma \pm 345 \boxed{012} = 0,$$

depends on the property of the function

$$\boxed{012}, \quad = \partial_1 0^2 12 - \partial_0 1^2 02,$$

enunciated No. 67, and proved *à posteriori* by the tedious calculation Nos. 69 to 73, viz. in No. 67, writing 2 in place of 3, this is:—$\partial_1 0^2 12 - \partial_0 1^2 02$ is equal to the difference of two functions, the first of them linear in the coordinates (x, y, z) of the point 0, but depending also on the coordinates of the points 1 and 2; the second of them linear in the coordinates (x_1, y_1, z_1) of the point 1 but depending also on the coordinates of the points 0 and 2. Or, what is the same thing, the property is

$$\boxed{012} = A_{12}x + B_{12}y + C_{12}z - (A_{02}x_1 + B_{02}y_1 + C_{02}z_1),$$

where A_{12}, B_{12}, C_{12} are functions of (x_1, y_1, z_1), (x_2, y_2, z_2), and A_{02}, B_{02}, C_{02} are the like functions of (x, y, z), (x_2, y_2, z_2).

Substituting such values in the sum $\Sigma \pm 345 \boxed{012}$, but writing down only the terms which contain x, these are

$$\begin{array}{lll}
 & 345\,(A_{12}x - A_{02}x_1) & \\
-045\,(A_{12}x_3 - A_{32}x_1) & -145\,(A_{32}x - A_{02}x_3) & +301\,(A_{52}x_4 - A_{42}x_5) \\
-305\,(A_{12}x_4 - A_{42}x_1) & -315\,(A_{42}x - A_{02}x_4) & +140\,(A_{32}x_5 - A_{52}x_3) \\
-340\,(A_{12}x_5 - A_{52}x_1) & -341\,(A_{52}x - A_{02}x_5) & +015\,(A_{42}x_3 - A_{32}x_4).
\end{array}$$

This is

$$\begin{aligned}
= \quad & A_{02}(-x_1 345 + x_3 145 + x_4 315 + x_5 341) \\
& + A_{12}(\ \ x\,345 - x_3 045 - x_4 305 - x_5 340) \\
& + A_{32}(\ \ x_1 045 - x\,145 + x_5 140 - x_4 015) \\
& + A_{42}(-x\,315 + x_3 015 + x_1 305 - x_5 301) \\
& + A_{52}(-x\,341 + x_4 301 + x_1 340 - x_3 140),
\end{aligned}$$

where the coefficient of each of the A's is identically $=0$: and similarly, the terms in y and the terms in z are each $=0$. We have thus the proof of the identity

$$\Sigma \pm 345 \boxed{012} = 0,$$

that is, of the conversion-equation $H_1(0) = H_0(1)$.

The Syzygy—Fixed Curve a Quartic. Art. No. 83.

I revert to the theory of the Syzygy, *ante* No. 59.

83. We have

$$Q(0;\ 1,\ 2;\ 3,\ 4,\ 5) = 0^2 12 - 3^2 12\,\frac{045}{345} - 4^2 12\,\frac{053}{453} - 5^2 12\,\frac{034}{534},$$

or if for convenience we take instead of 1, 2, the parametric points to be α, β coordinates $(x_\alpha,\ y_\alpha,\ z_\alpha)$ and $(x_\beta,\ y_\beta,\ z_\beta)$ respectively, then this equation is

$$Q_{\alpha\beta} = Q(0;\ \alpha,\ \beta;\ 3,\ 4,\ 5) = 0^2\alpha\beta - 3^2\alpha\beta\,\frac{045}{345} - 4^2\alpha\beta\,\frac{053}{453} - 5^2\alpha\beta\,\frac{034}{534}.$$

Considering a new parametric point γ, and forming the like functions $Q_{\beta\gamma}$ and $Q_{\gamma\alpha}$, it is to be shown that we have identically

$$Q_{\alpha\beta} + Q_{\beta\gamma} + Q_{\gamma\alpha} = 0.$$

To prove this, observe that, in the equation at the end of No. 59, Δ, ρ, σ, τ denote 123, 023, 031, 012 respectively. Hence writing therein α, β, γ in place of 1, 2, 3 respectively, and putting A, B, C for the coefficients $\left(\text{including therein the factor } \frac{1}{\Delta^2}\right)$ of ρ, σ, τ respectively, the equation is

$$0^2\beta\gamma + 0^2\gamma\alpha + 0^2\alpha\beta = A\,.\,0\beta\gamma + B\,.\,0\gamma\alpha + C\,.\,0\alpha\beta,$$

where A, B, C are absolute constants (functions, that is, of the coefficients of the quartic) each divided by $(\alpha\beta\gamma)^2$. We hence obtain

$$\begin{aligned}(Q_{\beta\gamma} + Q_{\gamma\alpha} + Q_{\alpha\beta})\,.\,345 = \quad & 345\,(A\,.\,0\beta\gamma + B\,.\,0\gamma\alpha + C\,.\,0\alpha\beta)\\ & - 045\,(A\,.\,3\beta\gamma + B\,.\,3\gamma\alpha + C\,.\,3\alpha\beta)\\ & - 053\,(A\,.\,4\beta\gamma + B\,.\,4\gamma\alpha + C\,.\,4\alpha\beta)\\ & - 034\,(A\,.\,5\beta\gamma + B\,.\,5\gamma\alpha + C\,.\,5\alpha\beta).\end{aligned}$$

On the left-hand side the whole coefficient of A is $=0$; viz. the coefficient has the value det. $\begin{vmatrix} 0345 & \\ & 0345\beta\gamma \end{vmatrix}$, which is $=0$. Similarly, the whole coefficient of B is $=0$, and the whole coefficient of C is $=0$: and we have thus the required result

$$Q_{\beta\gamma} + Q_{\gamma\alpha} + Q_{\alpha\beta} = 0.$$

The syzygy is thus obtained in a more perfect form than in No. 59; viz. by considering (instead of $0^2\alpha\beta$) the new form $Q_{\alpha\beta}$, then, instead of a sum which is a linear function of the coordinates $(x,\ y,\ z)$, we obtain a sum $=0$.

The Fixed Curve a Cubic—Syzygy and Conversion. Art. Nos. 84 and 85.

84. In the case when the fixed curve is a cubic (see Nos. 58 and 64), the analogous formulæ are

$$\tfrac{1}{3}\Omega(0;\ 1,\ 2;\ 3) = \frac{1^2.023 + 12^2.031}{123},\ = \widetilde{012} - \frac{\widetilde{123}}{123}012 \text{ (see No. 49)},$$

that is,

$$Q(0;\ 1,\ 2;\ 3),\ = \frac{\tfrac{1}{3}\Omega}{012},\ = \frac{\widetilde{012}}{012} - \frac{\widetilde{123}}{123},\ = \{012\} - \{123\},$$

where 1, 2 are the parametric points: 3 any other point on the cubic: the brackets { } are of course here necessary in order to distinguish {012} from the determinant 012. It will be remembered that {012} is an alternate function

$$\{012\},\ = -\{102\},\ = \{120\},\ \&c.$$

If instead of 1, 2 we take the parametric points to be α, β, coordinates $(x_\alpha,\ y_\alpha,\ z_\alpha)$ and $(x_\beta,\ y_\beta,\ z_\beta)$ respectively, then the formula is

$$Q_{\alpha\beta} = Q(0;\ \alpha,\ \beta;\ 3) = \{0\alpha\beta\} - \{3\alpha\beta\}.$$

Hence taking on the cubic a new point γ, coordinates $(x_\gamma,\ y_\gamma,\ z_\gamma)$ and forming the functions $Q_{\beta\gamma}$ and $Q_{\gamma\alpha}$ we have

$$\begin{aligned} Q_{\beta\gamma} + Q_{\gamma\alpha} + Q_{\alpha\beta} = {} & \{0\beta\gamma\} + \{0\gamma\alpha\} + \{0\alpha\beta\} \\ & - \{3\beta\gamma\} - \{3\gamma\alpha\} - \{3\alpha\beta\}. \end{aligned}$$

But by the formula No. 58,

$$\{0\beta\gamma\} + \{0\gamma\alpha\} + \{0\alpha\beta\} = \{\alpha\beta\gamma\};$$

hence also

$$\{3\beta\gamma\} + \{3\gamma\alpha\} + \{3\alpha\beta\} = \{\alpha\beta\gamma\}:$$

and we have thus

$$Q_{\beta\gamma} + Q_{\gamma\alpha} + Q_{\alpha\beta} = 0,$$

the syzygy for the cubic.

85. For the conversion, the definition of H is

$$\begin{aligned} H(0;\ 1,\ 2;\ 3,\ 6) = {} & \partial_1 Q(0;\ 1,\ 2;\ 3) \\ & + \partial_3 Q(1;\ 3,\ 2;\ 6), \end{aligned}$$

viz. this is

$$\begin{aligned} H_0(1) = H(0;\ 1,\ 2;\ 3,\ 6) = {} & \partial_1(\{012\} - \{123\}) \\ & + \partial_3(\{132\} - \{326\}), \\ = {} & \partial_1\ \{012\} - (\partial_1 + \partial_3)\{123\} - \partial_3\{326\}, \end{aligned}$$

which, in virtue of $(\partial_1 + \partial_2 + \partial_3)\{123\} = 0$ (see No. 63), becomes

$$H_0(1) = \partial_1\{012\} + \partial_2\{123\} - \partial_3\{326\}.$$

Interchanging the 0 and 1, we thence have

$$H_1(0) = \partial_0 \{102\} + \partial_2 \{023\} - \partial_3 \{326\}.$$

Hence the difference $H_0(1) - H_1(0)$ is

$$= \partial_1 \{012\} - \partial_0 \{102\} + \partial_3 \{123\} - \partial_2 \{023\},$$

viz. this is

$$= (\partial_1 + \partial_0) \{012\} + \partial_3 (\{123\} - \{230\}),$$

where the first term is

$$= -\partial_2 \{012\},$$

and the whole therefore is

$$= \partial_2 (\{123\} - \{230\} - \{012\})$$
$$= -\partial_2 \{301\},$$

in virtue of

$$\{123\} - \{230\} + \{301\} - \{012\} = 0;$$

the whole is consequently $= 0$.

We have thus

$$H_0(1) - H_1(0) = 0,$$

the conversion-equation in the case of the cubic.

Chapter V. Miscellaneous Investigations.

The Differential Symbol $d\omega$. Art. Nos. 86 and 87.

86. The definition is

$$\frac{y\,dz - z\,dy}{\dfrac{df}{dx}} = \frac{z\,dx - x\,dz}{\dfrac{df}{dy}} = \frac{x\,dy - y\,dx}{\dfrac{df}{dz}} = d\omega,$$

and it hence follows that we have

$$d\omega = \frac{\begin{vmatrix} dx, & dy, & dz \\ x, & y, & z \\ \lambda, & \mu, & \nu \end{vmatrix}}{\lambda \dfrac{df}{dx} + \mu \dfrac{df}{dy} + \nu \dfrac{df}{dz}},$$

where (λ, μ, ν) are arbitrary constants or, if we please, arbitrary functions of (x, y, z): viz. the expression just written down is altogether independent of the values of λ, μ, ν: and is consequently equal to the value obtained by writing any two of these symbols $= 0$, that is, the expression is equal to any one of the foregoing three equal values of $d\omega$. The expression was first given by Aronhold (1863), in the memoir presently referred to.

It is to be remarked that, considering (λ, μ, ν) as the coordinates of a point, the denominator $\lambda \frac{df}{dx} + \mu \frac{df}{dy} + \nu \frac{df}{dz}$ equated to 0 is the polar $(n-1)$thic of the point λ, μ, ν in regard to the fixed curve.

If instead of λ, μ, ν we write $bc'-b'c$, $ca'-c'a$, $ab'-a'b$, where $(a, b, c)(a', b', c')$ are constants, then the numerator is

$$=(ax+by+cz)(a'dx+b'dy+c'dz)-(a'x+b'y+c'z)(adx+bdy+cdz),$$

or introducing ρ, σ to denote the arbitrary linear functions $ax+by+cz$ and $a'x+b'y+c'z$ respectively, the numerator is $=\rho d\sigma - \sigma d\rho$: moreover, observing that a, b, c and a', b', c' are the differential coefficients of ρ, σ in regard to the coordinates (x, y, z), the denominator is $=J(f, \rho, \sigma)$; and the value of $d\omega$ is

$$d\omega = \frac{\rho d\sigma - \sigma d\rho}{J(f, \rho, \sigma)},$$

where, in accordance with a previous remark, the denominator equated to 0 is the polar $(n-1)$thic of the intersection of the lines $\rho=0$, $\sigma=0$ in regard to the fixed curve.

Obviously, by taking for ρ, σ any two of the three coordinates x, y, z, we reproduce the original three forms of $d\omega$.

87. The last-mentioned form of $d\omega$ suggests the expression for this symbol in the case where the fixed curve, instead of being a plane curve, is a curve of double curvature defined by two equations $f=0$, $g=0$ between the four coordinates (x, y, z, w): viz. ρ, σ being now arbitrary linear functions

$$ax+by+cz+dw, \text{ and } a'x+b'y+c'z+d'w$$

of the four coordinates, the expression is

$$d\omega = \frac{\rho d\sigma - \sigma d\rho}{J(f, g, \rho, \sigma)}:$$

and by taking for ρ, σ any two of the four coordinates x, y, z, w, we have for $d\omega$ six values which must of course be equal to each other; it is easy to verify *à posteriori* that this is so.

In the case where the curve of double curvature is not the complete intersection of two surfaces, the denominator (regarded as the Jacobian of the curve and of the arbitrary planes ρ, σ) will have a definite meaning, but what this is I do not at present consider.

The last-mentioned expression for $d\omega$ will be applied further on to the case of the quadri-quadric curve $y^2+x^2=1$, $z^2+k^2x^2=1$.

Integral Formulæ. Art. Nos. 88 to 90.

88. In what precedes, $d\omega$ has been used as a single symbol to denote any one of the equal differential expressions

$$\frac{y\,dz - z\,dy}{\frac{df}{dx}},\ =\frac{z\,dx - x\,dz}{\frac{df}{dy}},\ =\frac{x\,dy - y\,dx}{\frac{df}{dz}};$$

there is no quantity ω. The expressions are of the order $-(n-3)$ in the coordinates (x, y, z), and since (x, y, z) are as to their absolute magnitudes altogether arbitrary (only their ratios being determinate), a symbol such as

$$\omega, = \int d\omega, = \int \frac{y\,dz - z\,dy}{\frac{df}{dy}},$$

would, except in the case $n=3$, be altogether meaningless. In fact, the integral would be

$$\int \frac{z^3 d\left(\frac{y}{z}\right)}{z^{n-1}\,\phi\left(\frac{x}{z}, \frac{y}{z}\right)},\ = \int \frac{d\left(\frac{y}{z}\right)}{z^{n-3}\,\phi\left(\frac{x}{z}, \frac{y}{z}\right)},$$

where $\frac{x}{z}$ is, by the equation of the fixed curve, given as a function of $\frac{y}{z}$; but the other factor z^{n-3} is an absolutely indeterminate variable value, and the expression is meaningless.

But we have integrals $\int Qd\omega$, where Q is a homogeneous function of the order $n-3$ in the coordinates (x, y, z); and, in particular, we have such integrals where (corresponding to the forms which present themselves in the differential pure and affected theorems respectively) Q is either a rational and integral function $(x, y, z)^{n-3}$, or a rational and integral function $(x, y, z)^{n-2}$ divided by a linear function $(x, y, z)^1$: for in every such case, the form of integral is

$$\int \frac{d\left(\frac{y}{z}\right)}{\phi\left(\frac{x}{z}, \frac{y}{z}\right)},$$

where $\frac{x}{z}$ is a given function of $\frac{y}{z}$, and the factor of $d\left(\frac{y}{z}\right)$ is thus a mere function of $\frac{y}{z}$. More definitely, in the integrals $\int Qd\omega$ which are considered, Q is either a minor function $(x, y, z)^{n-3}$, or it is the quotient of a major function $(x, y, z)_{12}^{n-2}$ by the linear function 012.

In the case $n=2$, there is no rational and integral function $(x, y, z)^{n-3}$, but the function may be of the form belonging to the affected theorem, viz. it is unity divided by a linear function $(x, y, z)^1$; or say the integral is $\int \frac{d\omega}{\alpha x+\beta y+\gamma z}$, where the (x, y, z) are connected by a quadric equation $(a, \ldots \between x, y, z)^2=0$: it will be shown presently that this integral is obtainable as a logarithmic function.

In the case $n=3$, we have the rational and integral function $(x, y, z)^{n-3}$, $=$ a constant, or say $=1$, so that there is here an integral $\int d\omega$: we do not call this ω, but introducing a new letter, say u, and fixing at pleasure the inferior limit of the integral, we write $u=\int d\omega$.

89. In the foregoing form $\int Qd\omega$, so long as we retain the symbol $d\omega$, there is nothing to show what is the variable in regard to which the integration is to be performed; we may, for instance, writing

$$d\omega=\frac{y^2 d\left(\frac{z}{y}\right)}{\frac{df}{dx}},$$

make it to be $\frac{y}{z}$, or in like manner to be any other of the six quotients. We thus cannot attribute a *value* to the inferior or superior limit of such an integral, but we may take the limits to be each of them a point on the fixed curve: for instance, if 1, 0 be points on the fixed curve, then the integral $\int_0^1 Qd\omega$ means the integral taken from the value at the point 0 to the value at the point 1 of the variable in regard to which the integration is performed; or when there is no expressed superior limit, then the integral is to be taken from the value for the expressed or known inferior limit to the value at the current point (x, y, z) of the variable in regard to which the integration is performed. The actual value of the integral will of course depend upon the *path* of the variable; but this is a question which is not here entered upon.

If using Cartesian Coordinates x, y, we write for instance

$$d\omega=\frac{dx}{\frac{df}{dy}}, \text{ then } \int \frac{Qdx}{\frac{df}{dy}}$$

will denote an integral $\int \phi x\, dx$ in regard to the variable x, and the inferior and superior limits will be as usual values of x; or if there is no expressed superior limit, then the integral $\int_0 \phi x\, dx$ will be the integral taken from the inferior limit x_0 to the current value x.

We may, if we please, consider the coordinates (x, y, z) as depending upon a parameter s, viz. the ratios $x : y : z$ may be regarded as given functions of s, and the integral $\int Q d\omega$, is then an integral $\int \Omega ds$, which taken from a constant inferior limit up to the value s, belonging to a given point 1 of the curve, is a given function of s_1, or say of the point 1. But except in the case of the cubic (or generally if $p=1$), we do not have the coordinates actually given as known functions of a parameter s (say they are potentially known functions of s), and it is further to be noticed the functions which present themselves are functions not of a single point, but of p or more points: thus in the case of the quartic, $n=4$, $p=3$; we have $\int^1 x\, d\omega$, $\int^1 y\, d\omega$, $\int^1 z\, d\omega$, each standing for a given function of the parameter s_1, but these integrals do not present themselves singly, but in combinations such as

$$\left(\int^1 + \int^2 + \int^3 + \int^\xi\right)(x\, d\omega,\ y\, d\omega,\ z\, d\omega),$$

say these sums of integrals are u, v, w: each of the functions u, v, w is a potentially known function of the parameters s_1, s_2, s_3, s_ξ which belong to the points 1, 2, 3, ξ respectively, and is consequently regarded as a given function of these four points.

90. Consider as before, in the case of a cubic curve, the integral $u = \int d\omega$: it will presently be seen that for the general curve as given by a cubic equation $f=0$ of any form whatever, we arrive at a form of elliptic function: but the ordinary elliptic functions sn, cn, dn connect themselves most readily with the cubic curve $y^2 = x \,.\, 1 - x \,.\, 1 - k^2 x$. We have here

$$d\omega = \frac{\frac{1}{2} dx}{y}, \quad = \frac{\frac{1}{2} dx}{\sqrt{x \,.\, 1 - x \,.\, 1 - k^2 x}},$$

or, in the equation $u = \int d\omega$, taking the inferior limit to be 0, say

$$u = \int_0 \frac{\frac{1}{2} dx}{\sqrt{x \,.\, 1 - x \,.\, 1 - k^2 x}},$$

an equation which determines u as a function of x, or conversely, x as a function of u. We might thence, by means of Abel's theorem as applied to the curve in question, investigate the properties of the function $x = \lambda(u)$ thus arising, and so establish the theory of elliptic functions: but it is more convenient, treating the elliptic functions as known functions, to write for λu its value; viz. to take for x as given by this equation, the value $x = \mathrm{sn}^2 u$: we thence have $y = \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u$; viz. these values $x = \mathrm{sn}^2 u$, $y = \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u$, satisfy the equation $y^2 = x \,.\, 1 - x \,.\, 1 - k^2 x$ of the curve, and give, moreover, $d\omega = du = \frac{\frac{1}{2} dx}{y}$: and we can with these values, and the formulæ for elliptic functions, verify any results given by Abel's theorem. This will be done in considerable detail: but at present I wish only to remark that the formulæ give the coordinates x, y of a point on the cubic curve expressed as one-valued functions of

a parameter or argument u: but that this argument u is not a one-valued function of the coordinate x, or even of the coordinates x, y of the given point on the curve: say the argument u has not a unique value for a given point (x, y) of the curve. There are, in fact, an infinity of values $u = u_0 + 2mK + 2m'iK'$, where m, m' are any positive or negative integers: that this is so, depends on the multiplicity of values, according to the different paths of the variable, of the integral

$$u = \int_0 \frac{\frac{1}{2}\,dx}{\sqrt{x\,.\,1-x\,.\,1-k^2x}};$$

or, regarding the elliptic functions as known functions, it depends upon the double periodicity of these functions.

Aronhold's Quadric Integral. Art. Nos. 91 to 93.

91. I reproduce the investigation contained in Aronhold's paper "Ueber eine neue algebraische Behandlungsweise u.s.w.," *Crelle*, t. LXI. (1863), pp. 95—145. We take f the general quadric function $(a, b, c, f, g, h \between x, y, z)^2$; $\alpha x + \beta y + \gamma z$ an arbitrary linear function of x, y, z: the theorem is $\dfrac{d\omega}{\alpha x + \beta y + \gamma z}$ = differential of logarithm of an algebraic function of (x, y, z); viz. taking (ξ, η, ζ) for the coordinates of either of the points of intersection of the line $\alpha x + \beta y + \gamma z = 0$ with the quadric $(a, \ldots \between x, y, z)^2 = 0$, and writing also

$$\Omega^2 = -(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch \between \alpha,\ \beta,\ \gamma)^2,$$

then the theorem is

$$\frac{d\omega}{\alpha x + \beta y + \gamma z} = \frac{1}{\Omega}\, d\,.\log \frac{(a, \ldots \between x,\ y,\ z \between \xi,\ \eta,\ \zeta)}{\alpha x + \beta y + \gamma z},$$

or, what is the same thing,

$$\int \frac{d\omega}{\alpha x + \beta y + \gamma z} = \frac{1}{\Omega} \log \frac{(a, \ldots \between x,\ y,\ z \between \xi,\ \eta,\ \zeta)}{\alpha x + \beta y + \gamma z} + \text{const.}$$

It is to be observed, in reference to this equation, that the two sides respectively are in regard to (α, β, γ) homogeneous functions of the degree -1, and in regard to (ξ, η, ζ) homogeneous of the degree 0; viz. on the right-hand side the effect of a change in the absolute magnitudes of ξ, η, ζ, say the change into $k\xi$, $k\eta$, $k\zeta$, is merely to change by $\log k$ the constant of integration.

It is to be remarked also that the equation $(a, \ldots \between \xi, \eta, \zeta \between x, y, z) = 0$ represents the tangent to the conic at the point (ξ, η, ζ) of intersection with the line $\alpha x + \beta y + \gamma z = 0$; calling the linear function in question T, the value of the integral is $\dfrac{1}{\Omega} \log \dfrac{T}{\alpha x + \beta y + \gamma z}$; if (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are the coordinates of the two points of intersection respectively, then in passing from one of these to the other we change the sign of the radical Ω, and the two values thus are $\dfrac{1}{\Omega} \log \dfrac{T_1}{\alpha x + \beta y + \gamma z}$ and $-\dfrac{1}{\Omega} \log \dfrac{T_2}{\alpha x + \beta y + \gamma z}$. These must

differ by a constant only; viz. we should have $\log \frac{T_1 T_2}{(\alpha x+\beta y+\gamma z)^2} = \text{a const.}$ And, in fact, T_1 and T_2 being the tangents to the conic f at its intersections with the line $\alpha x+\beta y+\gamma z=0$, we have it is clear $f=\lambda T_1 T_2+\mu(\alpha x+\beta y+\gamma z)^2$, that is, (x, y, z) referring to a point of the conic $f=0$, we have $\frac{T_1 T_2}{(\alpha x+\beta y+\gamma z)^2} = \text{a constant}$, which is right.

92. We require the coordinates (ξ, η, ζ) of an intersection: these are determined by the equations $\alpha\xi+\beta\eta+\gamma\zeta=0$, $(a, \ldots \between \xi, \eta, \zeta)^2=0$, or as these may be written

$$\begin{aligned} &\alpha\xi \qquad\qquad +\beta\eta \qquad\qquad +\gamma\zeta=0,\\ &(a\xi+h\eta+g\zeta)\,\xi+(h\xi+b\eta+f\zeta)\,\eta+(g\xi+f\eta+c\zeta)\,\zeta=0; \end{aligned}$$

we have thence ξ, η, ζ proportional to the determinants

$$\begin{vmatrix} a\xi+h\eta+g\zeta, & h\xi+b\eta+f\zeta, & g\xi+f\eta+c\zeta \\ \alpha\quad, & \beta\quad, & \gamma \end{vmatrix},$$

say these determinants are $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$, where Ω is a value as yet undetermined. The equations are $\gamma(h\xi+b\eta+f\zeta)-\beta(g\xi+f\eta+c\zeta)-\Omega\xi=0$, &c., viz. these are

$$\begin{aligned} &(\gamma h-\beta g-\Omega)\,\xi+(\gamma b-\beta f\quad)\,\eta+(\gamma f-\beta c\quad)\,\zeta=0\\ &(\alpha g-\gamma a\quad)\,\xi+(\alpha f-\gamma h-\Omega)\,\eta+(\alpha c-\gamma g\quad)\,\zeta=0\\ &(\beta a-\alpha h\quad)\,\xi+(\beta h-\alpha b\quad)\,\eta+(\beta g-\alpha f-\Omega)\,\zeta=0; \end{aligned}$$

eliminating (ξ, η, ζ), we have an equation which may be written

$$\begin{vmatrix} A-\Omega, & B\quad, & C \\ A'\quad, & B'-\Omega, & C' \\ A''\quad, & B''\quad, & C''-\Omega \end{vmatrix}=0,$$

that is,

$$\begin{vmatrix} A\ , & B\ , & C \\ A'\ ; & B'\ , & C' \\ A'', & B'', & C'' \end{vmatrix} -\Omega(B'C''-B''C'+C''A-CA''+AB'-A'B)+\Omega^2(A+B'+C'')-\Omega^3=0.$$

We find very easily that the determinant and $A+B'+C''$ are each $=0$; and the equation thus reduces itself to

$$\Omega^2=B'C''-B''C'+C''A-CA''+AB'-A'B,$$

or substituting for A, B, &c., their values,

$$\Omega^2=-(bc-f^2, \ldots \between \alpha, \beta, \gamma)^2;$$

this being so, the ratios of ξ, η, ζ are determined by means of any two of the foregoing linear equations.

93. We may now verify the theorem; in the general expression for $d\omega$ writing for λ, μ, ν the values ξ, η, ζ, the equation to be verified becomes

$$\frac{\begin{vmatrix} dx, & dy, & dz \\ x\ , & y\ , & z \\ \xi\ , & \eta\ , & \zeta \end{vmatrix}}{(\alpha x+\beta y+\gamma z)\,(a,\ldots\between x,\ y,\ z\between\xi,\ \eta,\ \zeta)} = \frac{1}{\Omega}\left\{\frac{(a,\ldots\between\xi,\ \eta,\ \zeta\between dx,\ dy,\ dz)}{(a,\ldots\between\xi,\ \eta,\ \zeta\between x,\ y,\ z)} - \frac{\alpha dx+\beta dy+\gamma dz}{\alpha x+\beta y+\gamma z}\right\},$$

viz. this is

$$\Omega\begin{vmatrix} dx, & dy, & dz \\ x\ , & y\ , & z \\ \xi\ , & \eta\ , & \zeta \end{vmatrix} = (a,\ldots\between\xi,\ \eta,\ \zeta\between dx,\ dy,\ dz)\,.\,(\alpha x+\beta y+\gamma z) - (a,\ldots\between\xi,\ \eta,\ \zeta\between x,\ y,\ z)\,.\,(\alpha dx+\beta dy+\gamma dz).$$

Here, on the right-hand side, the coefficient of dx is

$$(a\xi+h\eta+g\zeta)\,(\alpha x+\beta y+\gamma z)$$
$$-\alpha\,\{(a\xi+h\eta+g\zeta)\,x+(h\xi+b\eta+f\zeta)\,y+(g\xi+f\eta+c\zeta)\,z\},$$

which is

$$= y\,\{\beta\,(a\xi+h\eta+g\zeta)-\alpha\,(h\xi+b\eta+f\zeta)\}$$
$$-z\,\{\alpha\,(g\xi+f\eta+c\zeta)-\gamma\,(a\xi+h\eta+g\zeta)\},$$
$$= y\,.\,\Omega\zeta - z\,.\,\Omega\eta,$$
$$= \Omega\,(y\zeta - z\eta),$$

which is right; and similarly, the coefficients of dy and dz have the same values on the two sides of the equation respectively.

Aronhold's Quadric Integral deduced from the Affected Theorem. Art. Nos. 94 to 98.

94. Let the fixed curve be a conic, say $f=\frac{1}{2}(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2,=0$: and let the variable curve be a line meeting the conic in the points 3 and 4. The affected theorem is

$$\Sigma\,\frac{12d\omega}{012} = -\frac{\delta 134}{134}+\frac{\delta 234}{234},$$

where $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ being the coordinates of the points 1 and 2 respectively, 12 denotes the constant $(a,\ldots\between x_1,\ y_1,\ z_1\between x_2,\ y_2,\ z_2)$: and 012, &c., denote determinants as usual.

The left-hand side is here

$$12\left\{\frac{d\omega_3}{312}+\frac{d\omega_4}{412}\right\}:$$

on the right-hand side, δ refers to the variation of the constants of ϕ, that is, to the variations of the points 3 and 4; or we may write $\delta = d_3+d_4$; the points 3, 4 are independent, and the equation, being satisfied at all, must be satisfied separately

in regard to the variations of 3, and in regard to the variations of 4: we must therefore have

$$12\frac{d\omega_3}{312} = -\frac{d_3 134}{134} + \frac{d_3 234}{234},$$

and the like equation obtained herefrom by the interchange of the numbers 3 and 4.

95. The equation just written down relates to any four points 1, 2, 3, 4 of the conic; and if for 3, 4 we write 0, 3 respectively, it becomes

$$12\frac{d\omega}{012} = -\frac{d.031}{031} + \frac{d.023}{023},$$

which relates to the points 0, 1, 2, 3 of the conic: writing, as before, 023, 031, 012 $= \rho, \sigma, \tau$, this equation is

$$12\frac{d\omega}{\tau} = -\frac{d\sigma}{\sigma} + \frac{d\rho}{\rho},$$

which may be verified as follows: the equation of the conic is $f = 23.\sigma\tau + 31.\tau\rho + 12.\rho\sigma, = 0$: we have $d\omega = \dfrac{\rho\, d\sigma - \sigma\, d\rho}{\dfrac{df}{d\tau}}$, where $\dfrac{df}{d\tau} = 23.\sigma + 31\rho, = -\dfrac{12.\rho\sigma}{\tau}$, that is, $d\omega = \dfrac{12}{\tau}\left(-\dfrac{d\sigma}{\sigma} + \dfrac{d\rho}{\rho}\right)$, the equation in question.

96. We have, as a property of any four points 0, 1, 2, 3 of a conic,

$$\frac{23}{123.023} = \frac{-01}{012.031}, \text{ or say } \frac{23}{\Delta.\rho} = \frac{-01}{\sigma\tau}, \text{ that is, } \frac{23}{\Delta}\frac{\sigma}{\rho} = -\frac{01}{\tau};$$

hence considering 0 as a variable point, and differentiating the logarithms,

$$-d\log\frac{01}{\tau} = -\frac{d\sigma}{\sigma} + \frac{d\rho}{\rho},$$

and the foregoing equation $12\dfrac{d\omega}{\tau} = -\dfrac{d\sigma}{\sigma} + \dfrac{d\rho}{\rho}$ thus becomes $12\dfrac{d\omega}{\tau} = -d\log\dfrac{01}{\tau}$, or restoring for τ its value 012,

$$12\frac{d\omega}{012} = -d\log\frac{01}{012}.$$

Taking now $\alpha x + \beta y + \gamma z = 0$ for the equation of the line 012; this meets the conic in the points 1, 2, coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively: and we have

$$\alpha, \beta, \gamma = y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_2 y_1 - x_1 y_2,$$
$$12 = (a, \ldots \between x_1, y_1, z_1 \between x_2, y_2, z_2),$$

and from this last value

$$\overline{12}^2 = \{(a, \ldots \between x_1, y_1, z_1 \between x_2, y_2, z_2)\}^2 - (a, \ldots \between x_1, y_1, z_1)^2 . (a, \ldots \between x_2, y_2, z_2)^2$$

(the second term being of course $=0$), viz. this is

$$\begin{aligned}\overline{12}^2 &= -(bc - f^2, \ldots \between y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1)^2 \\ &= -(bc - f^2, \ldots \between \alpha, \beta, \gamma)^2,\end{aligned}$$

or say $12 = -\Omega$, if $\Omega^2 = -(bc - f^2, \ldots \between \alpha, \beta, \gamma)^2$ as before: and the equation thus is

$$\frac{\Omega d\omega}{\alpha x + \beta y + \gamma z} = d \log \frac{(a, \ldots \between x_1, y_1, z_1 \between x, y, z)}{\alpha x + \beta y + \gamma z},$$

or finally, writing (ξ, η, ζ) instead of (x_1, y_1, z_1) to denote the coordinates of one or other of the intersections of the line $\alpha x + \beta y + \gamma z = 0$ with the conic, the equation becomes

$$\frac{\Omega d\omega}{\alpha x + \beta y + \gamma z} = d \log \frac{(a, \ldots \between \xi, \eta, \zeta \between x, y, z)}{\alpha x + \beta y + \gamma z},$$

which is Aronhold's quadric integral.

97. (The foregoing property, which may also be written

$$\frac{23}{023 \,.\, 123} = \frac{01}{201 \,.\, 301},$$

is verified very simply in the case of four points 0, 1, 2, 3 of a circle: in fact

$$23 = x_2 x_3 + y_2 y_3 - 1, \; = \cos 23 - 1, \; = -2 \sin^2 \tfrac{1}{2} 23,$$
$$023 = 2 \sin \tfrac{1}{2} 23 \sin \tfrac{1}{2} 30 \sin \tfrac{1}{2} 02;$$

and so for the other like expressions; each side of the equation is thus reduced to $1 \div \sin \frac{1}{2} 02 \sin \frac{1}{2} 03 \sin \frac{1}{2} 12 \sin \frac{1}{2} 13$.)

98. In particular, if the conic is taken to be the circle $x^2 + y^2 - 1 = 0$, then for the coordinates $\left(\frac{\xi}{\zeta}, \frac{\eta}{\zeta}\right)$ of the intersections with the line $\alpha x + \beta y + \gamma = 0$, we have

$$\begin{aligned} \Omega\xi + \gamma\eta + \beta\zeta &= 0, \\ \gamma\xi - \Omega\eta + \alpha\zeta &= 0, \\ \beta\xi - \alpha\eta + \Omega\zeta &= 0, \end{aligned}$$

giving $\Omega^2 = \alpha^2 + \beta^2 + \gamma^2$; and then

$$\begin{aligned} \xi : \eta : \zeta &= -\beta^2 + \gamma^2 : \alpha\beta - \gamma\Omega : \alpha\gamma + \beta\Omega \\ &= -\alpha\beta - \gamma\Omega : \alpha^2 - \gamma^2 : \beta\gamma + \alpha\Omega \\ &= \alpha\gamma + \beta\Omega : \beta\gamma - \alpha\Omega : -\alpha^2 - \beta^2. \end{aligned}$$

The formula then becomes

$$\int \frac{dx}{(\alpha x + \beta\sqrt{1 - x^2} + \gamma)\sqrt{1 - x^2}} = \frac{1}{\sqrt{\alpha^2 + \beta^2 - \gamma^2}} \log \frac{\xi x + \eta\sqrt{1 - x^2} - \zeta}{\alpha x + \beta\sqrt{1 - x^2} + \gamma}:$$

or, retaining Ω, y for the values $\sqrt{\alpha^2 + \beta^2 - \gamma^2}$, and $\sqrt{1 - x^2}$, this may also be written

$$= \frac{1}{\Omega} \log \frac{\gamma(\alpha x + \beta y + \gamma) + \Omega(\beta x - \alpha y + \Omega)}{\alpha x + \beta y + \gamma}.$$

The form of the integral is still such that the value is not very readily obtainable by ordinary methods: the value just written down can of course be verified, but the verification is scarcely easier than for the original more general form.

In the very particular case $\alpha=0$, $\beta=0$, $\gamma=1$, we have $\Omega=i$; $\xi:\eta:\zeta=1:-i:0$, and the formula becomes

$$\int\frac{dx}{\sqrt{1-x^2}}=\frac{1}{i}\log(x-iy):$$

viz. this is

$$\sin^{-1}x=\tfrac{1}{2}\pi+\frac{1}{i}\log(x-i\sqrt{1-x^2}),$$

which is right; for putting $\sin^{-1}x=u$, and therefore $x=\sin u$, the equation becomes $i(u-\frac{1}{2}\pi)=\log(\sin u-i\cos u)$: that is, $\cos(u-\frac{1}{2}\pi)+i\sin(u-\frac{1}{2}\pi)=\sin u-i\cos u$.

Fixed Curve a Cubic: the Parametric Points 1, 2 *consecutive points on the Curve.*
Art. Nos. 99 to 106.

99. The major function $(x, y, z)^1_{12}$ is taken to be $=\dfrac{1^2.023-12^2.013}{123}$, so that, calling the differential $Qd\omega$, we have

$$Q=\frac{1^2.023-12^2.013}{123.012};$$

it is required to find what this becomes when 1, 2 are consecutive points on the curve, or what is the same thing when the line 012 is a tangent at the point 1.

I take for convenience the cubic to be $f, =\frac{1}{3}(x^3+y^3+z^3), =0$. The coordinates of 1 are (x_1, y_1, z_1), those of 2 are $(x_1+\delta x_1, y_1+\delta y_1, z_1+\delta z_1)$, or as for shortness I write them $(x_1+\alpha, y_1+\beta, z_1+\gamma)$, where α, β, γ are considered as infinitesimals of the first order: this being so, the denominator of Q is at once seen to be of the second order; it will appear that the numerator is of the third order; whence Q is of the first order.

100. We have

$$d\omega=\frac{y\,dz-z\,dy}{x^2},\ =\frac{z\,dx-x\,dz}{y^2}=\frac{x\,dy-y\,dx}{z^2},$$

and in analogy herewith we may write

$$\delta\omega_1=\frac{y_1\gamma-z_1\beta}{x_1^2},\ =\frac{z_1\alpha-x_1\gamma}{y_1^2},\ =\frac{x_1\beta-y_1\alpha}{z_1^2};$$

this being so, we have

$$012=\begin{vmatrix} x, & y, & z\\ x_1 & y_1 & z_1\\ \alpha & \beta & \gamma\end{vmatrix}=(xx_1^2+yy_1^2+zz_1^2)\,\delta\omega_1=01^2.\,\delta\omega_1,$$

and similarly $312=31^2.\,\delta\omega_1$.

Moreover

$$023=\begin{vmatrix} x, & y, & z\\ x_1 & y_1 & z_1\\ x_3 & y_3 & z_3\end{vmatrix}+\begin{vmatrix} x, & y, & z\\ \alpha, & \beta, & \gamma\\ x_3, & y_3, & z_3\end{vmatrix}=013+0\delta 13,$$

as the second term may be written; moreover

$$1^2 2 = x_1^2(x_1+\alpha) + y_1^2(y_1+\beta) + z_1^2(z_1+\gamma), \quad = \quad \alpha x_1^2 + \beta y_1^2 + \gamma z_1^2,$$
$$12^2 = x_1\,(x_1+\alpha)^2 + y_1\,(y_1+\beta)^2 + z_1\,(z_1+\gamma)^2, \quad = 2\,(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + \alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1,$$

and hence

$$\begin{aligned} 1^2 2\,.\,023 - 12^2\,.\,013 = & \quad (\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2)\quad (013 + 0\delta 13) \\ & - [2\,(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)]\,013 \\ = & - \quad [(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)]\,013 \\ & + \quad (\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2)\,.\,0\delta 13, \end{aligned}$$

or reducing by

$$3\,(\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2) + 3\,(\alpha^2 x + \beta^2 y + \gamma^2 z) + (\alpha^3 + \beta^3 + \gamma^3) = 0,$$

this is

$$= + \tfrac{1}{3}\,(\alpha^3 + \beta^3 + \gamma^3)\,013 - (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1)\,0\delta 13,$$

which is of the third order.

101. We may show that each of the terms contains the factor $(\delta\omega_1)^2$: we have, in fact,

$$y_1 z_1\,(\delta\omega_1)^2 = \alpha\beta\,\frac{x_1}{y_1} - \alpha^2 - \beta\gamma\,\frac{x_1^2}{y_1 z_1} + \gamma\alpha\,\frac{x_1}{z_1},$$

$$z_1 x_1\,(\delta\omega_1)^2 = \beta\gamma\,\frac{y_1}{z_1} - \beta^2 - \gamma\alpha\,\frac{y_1^2}{z_1 x_1} + \alpha\beta\,\frac{y_1}{x_1},$$

$$x_1 y_1\,(\delta\omega_1)^2 = \gamma\alpha\,\frac{z_1}{x_1} - \gamma^2 - \alpha\beta\,\frac{z_1^2}{x_1 y_1} + \beta\gamma\,\frac{z_1}{y_1};$$

hence, first multiplying by α, β, γ and adding, we have

$$\begin{aligned} (\alpha y_1 z_1 + \beta z_1 x_1 + \gamma x_1 y_1)\,(\delta\omega_1)^2 = & \frac{\beta\gamma}{y_1 z_1}\,(\beta y_1^2 + \gamma z_1^2) + \frac{\gamma\alpha}{z_1 x_1}\,(\gamma z_1^2 + \alpha x_1^2) + \frac{\alpha\beta}{x_1 y_1}\,(\alpha x_1^2 + \beta y_1^2) \\ & - \alpha\beta\gamma\left(\frac{x_1^2}{y_1 z_1} + \frac{y_1^2}{z_1 x_1} + \frac{z_1^2}{x_1 y_1}\right) \\ & - (\alpha^3 + \beta^3 + \gamma^3). \end{aligned}$$

But in virtue of $\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2 = 0$, the first line becomes = the second line, or the two together are

$$= - 2\alpha\beta\gamma\left(\frac{x_1^2}{y_1 z_1} + \frac{y_1^2}{z_1 x_1} + \frac{z_1^2}{x_1 y_1}\right),$$

which is $= 0$ in virtue of $x_1^3 + y_1^3 + z_1^3 = 0$; hence the equation is

$$(\alpha y_1 z_1 + \beta z_1 x_1 + \gamma x_1 y_1)\,(\delta\omega_1)^2 = -\,(\alpha^3 + \beta^3 + \gamma^3),$$

the required expression for the first term.

102. Again, multiplying by x_1, y_1, z_1, and adding, we have

$$3 x_1 y_1 z_1\,(\delta\omega_1)^2 = \frac{\beta\gamma}{y_1 z_1}\,(y_1^3 + z_1^3 - x_1^3) + \frac{\gamma\alpha}{z_1 x_1}\,(z_1^3 + x_1^3 - y_1^3) + \frac{\alpha\beta}{x_1 y_1}\,(x_1^3 + y_1^3 - z_1^3) - (\alpha^2 x_1 + \beta^2 y_1 + \gamma^2 z_1),$$

where, in virtue of $x_1^2+y_1^2+z_1^2=0$, the first line is

$$=-\frac{2}{x_1y_1z_1}(\beta\gamma x_1^4+\gamma\alpha y_1^4+\alpha\beta z_1^4),$$

and this again is $=-2(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)$: in fact, we have identically

$$\begin{aligned}x_1y_1z_1(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)=&(\alpha x_1^2+\beta y_1^2+\gamma z_1^2)(\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1)\\&-(\beta\gamma x_1+\gamma\alpha y_1+\alpha\beta z_1)(x_1^3+y_1^3+z_1^3)\\&+(\beta\gamma x_1^4+\gamma\alpha y_1^4+\alpha\beta z_1^4),\end{aligned}$$

which, in virtue of $\alpha x_1^2+\beta y_1^2+\gamma z_1^2=0$, and $x_1^3+y_1^3+z_1^3=0$, becomes

$$x_1y_1z_1(\alpha^2x_1+\beta^2y_1+\gamma^2z_1)=(\beta\gamma x_1^4+\gamma\alpha y_1^4+\alpha\beta z_1^4).$$

Hence the equation is

$$3x_1y_1z_1(\delta\omega_1)^2=-3(\alpha^2x_1+\beta^2y_1+\gamma^2z_1),$$

or finally

$$x_1y_1z_1(\delta\omega_1)^2=-\quad(\alpha^2x_1+\beta^2y_1+\gamma^2z_1),$$

the required expression for the second term.

103. Writing for shortness

$$\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1=\delta(x_1y_1z_1),$$

we have

$$1^2.023-12^2.013=\{-\tfrac{1}{3}\delta(x_1y_1z_1)013+x_1y_1z_1.0\delta13\}(\delta\omega_1)^2,$$

and hence dividing by

$$012.123,\quad=01^2.31^2.(\delta\omega_1)^2,$$

we have

$$Q=\frac{1^2.023-12^2.013}{012.123}=\frac{-\frac{1}{3}\delta(x_1,\ y_1,\ z_1).013+x_1y_1z_1.0\delta13}{01^2.31^2}.$$

But this can be further reduced: the numerator, multiplied by 3, is

$$=-(\alpha y_1z_1+\beta z_1x_1+\gamma x_1y_1)\begin{vmatrix}x, & y, & z\\ x_1, & y_1, & z_1\\ x_3, & y_3, & z_3\end{vmatrix}+3x_1y_1z_1\begin{vmatrix}x, & y, & z\\ \alpha, & \beta, & \gamma\\ x_3, & y_3, & z_3\end{vmatrix},$$

which is

$$=\begin{vmatrix}x, & y, & z\\ x_1(y_1^3-z_1^3), & y_1(z_1^3-x_1^3), & z_1(x_1^3-y_1^3)\\ x_3, & y_3, & z_3\end{vmatrix}\delta\omega_1,$$

where $x_1(y_1^3-z_1^3)$, $y_1(z_1^3-x_1^3)$, $z_1(x_1^3-y_1^3)$ are the coordinates of the tangential of the point 1 in regard to the cubic, viz. the point of intersection of the tangent at 1 with the cubic. The determinant may for shortness be called $0t13$; and we thus have

$$Q=\frac{1^2.023-12^2.013}{012.123}=\frac{\frac{1}{3}\delta\omega_1}{31^2}\frac{0t13}{01^2},$$

where observe that $01^2=0$ is the equation of the tangent at the point 0: and $0t13=0$ is the equation of the line joining the tangential of 1 with the arbitrary point 3.

104. The identity just referred to is proved very easily. Comparing on each side the coefficient of $yz_3 - y_3 z$, the factor x_1 divides out and we ought to have

$$-(\alpha y_1 z_1 + \beta z_1 x_1 + \gamma x_1 y_1) + 3 y_1 z_1 \alpha = (y_1^3 - z_1^3)\,\delta\omega_1,$$

that is,

$$(y_1^3 - z_1^3)\,\delta\omega_1 = 2\alpha y_1 z_1 - \beta z_1 x_1 - \gamma x_1 y_1:$$

and, in fact, from $y_1^2 \delta\omega_1 = z_1\alpha - x_1\gamma$, $z_1^2\delta\omega_1 = x_1\beta - y_1\alpha$, we have

$$(y_1^3 - z_1^3)\,\delta\omega_1 = y_1(z_1\alpha - x_1\gamma) - z_1(x_1\beta - y_1\alpha),$$

which is the value in question. Similarly the coefficients of $zx_3 - z_3 x$, $xy_3 - x_3 y$ are equal on the two sides; and the equation is thus verified.

105. The proof has been given in regard to the particular cubic $x^3 + y^3 + z^3 = 0$; but it might have been given for the canonical form $x^3 + y^3 + z^3 + 6lxyz = 0$: and from the invariantive form it is clear that the result in fact applies to any cubic whatever. The result is an important one: we see by it that when the points 1 and 2 are consecutive points on the curve we must, in place of the differential $Qd\omega$, which is evanescent, consider a new form $\dfrac{0t13}{01^2}\,\delta\omega_1$, where, as already remarked, the denominator represents the tangent at the point 1, and the numerator the line joining the tangential of this point with the point 3.

106. We have

$$\{023\} + \{031\} + \{012\} = \{123\},$$

or writing this in the form

$$\{012\} - \{312\} + \{023\} - \{013\} = 0,$$

suppose 2 is here the consecutive point $1 + \delta 1$; then

$$\{012\} - \{312\}, = \frac{1^2 2 \,.\, 023 - 12^2 \,.\, 013}{012 \,.\, 312}$$

becomes $= \{0t13\}\,\delta\omega_1$: we have also

$$\{023\} = \{013\} + \partial_1 \{013\}\,\delta\omega_1,$$

and the result is

$$-\{0t13\} + \partial_1 013 = 0,$$

that is, $\partial_1\{013\} = \{0t13\}$. The form in the case of the cubic $x^3 + y^3 + z^3 + 6lxyz = 0$, is $=$

$$-\frac{\begin{vmatrix} x\,, & y\,, & z \\ x_1(y_1^3 - z_1^3), & y_1(z_1^3 - x_1^3), & z_1(x_1^3 - y_1^3) \\ x_3\,, & y_3\,, & z_3 \end{vmatrix}}{3(x_1^2 x + y_1^2 y + z_1^2 z)(x_1^2 x_3 + y_1^2 y_3 + z_1^2 z_3)},$$

i.e. the differential coefficient of $\{013\}$ in regard to the parametric point 1 is $= \{0t13\}$, the symbol for the case where the parametric line is the tangent at 1.

Fixed Curve a Cubic: the Parametric Points corresponding points. Art. Nos. 107 to 110.

107. The parametric points 1, 2 are taken to be corresponding points, that is, such that the tangents at these points meet at a point, say 3, on the cubic. We may from 3 draw two other tangents, touching the cubic, say at the points 1′ and 2′. The four points 1, 2, 1′, 2′ are then such that the lines 12, 1′2′ meet in a point, say 4, of the cubic; and moreover 3, 4 are corresponding points.

We may take $(x, y, z), =(1, 0, 0), (0, 1, 0), (0, 0, 1)$ for the coordinates of the points 1, 2, 3 respectively: $x=0$, $y=0$ are thus the equations of the lines 32, 31 respectively, and $z=0$ is the equation of the line 12, viz. we have $z=012$. Taking $x-M_1y=0$, $x-M_2y=0$, for the equations of the tangents 31′, 32′ respectively, and $\zeta=0$ for the equation of the line 1′2′ joining their points of contact, where ζ is a properly determined linear function of (x, y, z), it is to be shown that the differential $Qd\omega$ may be taken to be $\frac{\zeta d\omega}{z}$, and that this is $=\frac{1}{2}\left(\frac{dx}{x}-\frac{dy}{y}\right)$: the affected theorem thus assumes a special form, which will be noticed.

108. The cubic passes through the points $(x=0, z=0)$ and $(y=0, z=0)$, the tangents at these points being $x=0$, and $y=0$ respectively: also through the point $x=0$, $y=0$: its equation thus is

$$f, = gz^2x+2lzxy+iz^2y+hx^2y+kxy^2, =0,$$

and writing

$$d\omega=\frac{x\,dy-y\,dx}{\frac{df}{dz}},$$

we have

$$\frac{df}{dz}=2\,(gzx+lxy+izy),$$

which, from the equation of the curve written in the form

$$z\,(gzx+lxy+izy)+xy\,(hx+ky+lz)=0,$$

or say

$$z\,(gzx+lxy+izy)+xy\zeta=0,$$

becomes

$$=\frac{-2xy\zeta}{z};$$

and we thus have

$$d\omega=\frac{-z}{2xy\zeta}(x\,dy-y\,dx), \;=\frac{1}{2}\frac{z}{\zeta}\left(\frac{dx}{x}-\frac{dy}{y}\right),$$

where $\zeta=hx+ky+lz$. To find the meaning of ζ, observe that the line $x-My=0$ meets the curve in the point $(x=0, y=0)$, and in two other points determined by the equation

$$z^2\,(gm+i)+2zylM+y^2\,(hM^2+kM)=0;$$

this line will be a tangent if

$$(gM+i)\,(hM+k)-l^2M=0,$$

and we then have at the point of contact $(hM+k)y+lz=0$; and writing this in the form $hx+ky+lz=0$, we see that the equation $\zeta=0$ is satisfied at the point of contact of each of the two tangents $x-M_1y=0$, $x-M_2y=0$; viz. $\zeta=0$ is the equation of the line joining the two points of contact. Moreover, from the equation of the curve written in the foregoing form

$$z(gzx+lxy+izy)+xy\zeta=0,$$

it appears that the lines $z=0$, $\zeta=0$, meet on the curve; or, what is the same thing, that the line $\zeta=0$ passes through the residue of the parametric points 1, 2.

109. The function ζ at 1 becomes $=h$, and this is the value of $3\,.\,1^2 2$; in fact,

$$\begin{aligned}3\,.\,1^2 2 &= \left(x_2\frac{d}{dx_1}+y_2\frac{d}{dy_1}+z_2\frac{d}{dz_1}\right)f_1, \quad (x_2,\ y_2,\ z_2)=(0,\ 1,\ 0),\\ &= \frac{df_1}{dy_1},\\ &= 2lz_1x_1+iz_1^2+hx_1^2+2kx_1y_1, \quad (x_1,\ y_1,\ z_1)=(1,\ 0,\ 0),\\ &= h.\end{aligned}$$

We have thus ζ, satisfying the required conditions for the major function: and the differential $Qd\omega$ may therefore be taken to be $=\frac{\zeta}{z}d\omega$, that is, we have

$$Qd\omega=\tfrac{1}{2}\left(\frac{dx}{x}-\frac{dy}{y}\right).$$

The affected theorem thus becomes

$$\Sigma\tfrac{1}{2}\left(\frac{dx}{x}-\frac{dy}{y}\right)=-\frac{\delta\phi_1}{\phi_1}+\frac{\delta\phi_2}{\phi_2}.$$

110. The meaning of this will be better understood from the integral form. Integrating each side, and assuming that the superior limits are given by a line ϕ which cuts the cubic in the points 4, 5, 6, and the inferior limits by a line ψ which cuts the cubic in the points 7, 8, 9, we find

$$\log\frac{x_4x_5x_6}{x_7x_8x_9}-\log\frac{y_4y_5y_6}{y_7y_8y_9}=2\log\frac{\phi_2}{\psi_2}\,\frac{\psi_1}{\phi_1},$$

that is,

$$\frac{x_4x_5x_6\,y_7y_8y_9}{x_7x_8x_9\,y_4y_5y_6}=\left(\frac{\phi_2\psi_1}{\psi_2\phi_1}\right)^2,$$

where ϕ_1, ψ_1, ϕ_2, ψ_2 denote the values of the linear functions ϕ, ψ at the points 1 and 2 respectively. We have a cubic cut by the lines ϕ, ψ, x, y in the points 4, 5, 6; 7, 8, 9; 2, 2′, 3 and 1, 1′, 3 respectively: where for the moment 1′, 2′ are written to denote the points on the curve consecutive to 1 and 2 respectively. Hence, by a known theorem in transversals,

$$\left(\frac{x}{y}\right)_{456}\div\left(\frac{x}{y}\right)_{789}=\left(\frac{\phi}{\psi}\right)_{22'3}\div\left(\frac{\phi}{\psi}\right)_{11'3},$$

that is,

$$\frac{x_4x_5x_6\,y_7y_8y_9}{x_7x_8x_9\,y_4y_5y_6}=\frac{\psi_1\psi_{1'}\psi_3\,.\,\phi_2\phi_{2'}\phi_3}{\psi_2\psi_{2'}\psi_3\,.\,\phi_1\phi_{1'}\phi_3},$$

which, dividing out the $\phi_3\psi_3$, and writing 1, 2 in place of 1′, 2′, becomes

$$=\left(\frac{\phi_3\psi_1}{\phi_1\psi_2}\right)^2,$$

agreeing with the result just obtained.

Aronhold's Cubic Transformation. Art. Nos. 111 to 119.

111. This was obtained in the paper "Algebraische Reduction des Integrals $\int F(x, y)\,dx$, u. s. w.," *Berl. Monatsb.*, April, 1861, pp. 462—468. I give in the first place the analytical results, independently of the general theory, with the values for the canonical form $f, =\frac{1}{3}(x^3+y^3+z^3+6lxyz), =0$, of the cubic.

T	sextic invariant,	$=1-20l^3-8l^6,$
S	quartic „ (Aronhold's)	$=-4(l-l^4),$
R	discriminant	$=(1+8l^3)^2,$
$P=3h\alpha^2 0$		$=\{-3l^2\alpha^2+(1+2l^3)\beta\gamma\}\,x+\&c.,$
$Q=f\alpha^2 0$		$=(\alpha^2+2l\beta\gamma)\,x+\&c.,$
$B=f\mathrm{a}^2 0$		$=(\mathrm{a}^2+2l\mathrm{bc})\,x+\&c.,$
$C=f\mathrm{a}0^2$		$=\mathrm{a}\,(x^2+2lyz)+\&c.,$
$D=f0^3$		$=x^3+y^3+z^3+6lxyz,$

where $\mathrm{a},\ \mathrm{b},\ \mathrm{c}=\alpha(\beta^3-\gamma^3),\ \beta(\gamma^3-\alpha^3),\ \gamma(\alpha^3-\beta^3)$.

Then we have

$$2TQ^4+6SPQ^3+8P^3Q=-R^{\frac{2}{3}}(6C^2-8BD),$$

viz. this equation, where each side is a quartic function $(x, y, z)^4$, is an identity when (α, β, γ) are connected by the equation, $f\alpha^3, =\alpha^3+\beta^3+\gamma^3+6l\alpha\beta\gamma, =0$; and further,

$$Q\,dP-P\,dQ=-R^{\frac{1}{3}}\{\mathrm{a}\,(y\,dz-z\,dy)+\mathrm{b}\,(z\,dx-x\,dz)+\mathrm{c}\,(x\,dy-y\,dx)\}.$$

Hence writing

$$\lambda=\frac{6h\alpha^2 0}{f\alpha^2 0},\ =\frac{2P}{Q},\ =\frac{2\,(\{-3l^2\alpha^2+(1+2l^3)\beta\gamma\}\,x+\&c.)}{(\alpha^2+2l\beta\gamma)\,x+\&c.},$$

we have

$$Q^4(\lambda^3-3S\lambda-2T)=R^{\frac{2}{3}}(6C^2-8BD),$$

and

$$Q^2\,d\lambda,\ =2\,(Q\,dP-P\,dQ)=-2R^{\frac{1}{3}}\{\mathrm{a}\,(y\,dz-z\,dy)+\&c.\}.$$

112. Supposing now that (x, y, z) are the coordinates of a point on the cubic, then $D=0$; and taking the square root of each side of the first equation, we may write

$$Q^2\sqrt{\lambda^3-3S\lambda-2T} = -R^{\frac{1}{2}}\sqrt{6}\,.\,C,$$

$$Q^2 d\lambda \qquad\qquad = -2R^{\frac{1}{2}}\{\mathrm{a}\,(y\,dz - z\,dy) + \&\mathrm{c.}\}.$$

We have

$$d\omega = \frac{y\,dz - z\,dy}{x^2+2lyz} = \frac{z\,dx - x\,dz}{y^2+2lzx} = \frac{x\,dy - y\,dx}{z^2+2lxy},$$

whence

$$d\omega = \frac{\mathrm{a}\,(y\,dz - z\,dy) + \&\mathrm{c.}}{C};$$

and we consequently have

$$\frac{d\lambda}{\sqrt{\lambda^3-3S\lambda-2T}} = \frac{2}{\sqrt{6}},$$

or, as this may also be written,

$$\frac{d\lambda}{\sqrt{4\lambda^3-12S\lambda-8T}} = \frac{1}{\sqrt{6}}\,d\omega,$$

which, if $12S$, $8T$ are put $=g_2$, g_3 respectively, takes the Weierstrassian form

$$\frac{d\lambda}{\sqrt{4\lambda^3-g_2\lambda-g_3}} = \frac{1}{\sqrt{6}}\,d\omega.$$

The conclusion is that for the cubic curve, taking λ a quotient of two linear functions of (x, y, z), the differential $d\omega$ is transformed into $d\lambda \div$ square root of a cubic function of λ: viz. we have thus a form of differential, not the same, but such as that which belongs to the ordinary theory of elliptic functions, and which has been adopted by Weierstrass as a canonical form.

113. The transformation depends on the arbitrary point (α, β, γ) of the cubic: the point (a, b, c) is the tangential of this point, viz. the point of intersection of the tangent at (α, β, γ) with the cubic: we can from (a, b, c) draw four tangents to the cubic, viz. the tangent at (α, β, γ) and three other tangents: the equations of the four tangents being $\dfrac{2P}{Q}$, $=\dfrac{6h\alpha^2 0}{f\alpha^2 0}$, $=\infty$, λ_1, λ_2, λ_3 respectively; where λ_1, λ_2, λ_3 are the roots of the equation $\lambda^3-3S\lambda-2T=0$.

Suppose for a moment that (α, β, γ) is a point not on the cubic curve, and write $A=\alpha^3+\beta^3+\gamma^3+6l\alpha\beta\gamma$. We have

$$A^2D^2+4AC^3+4B^3D-3B^2C^2-6ABCD=0,$$

for the equation of the six tangents which can be drawn from the point (α, β, γ) to the cubic: when (α, β, γ) is on the cubic, $A=0$, and the equation becomes $B^2(4BD-3C^2)=0$, where $B=0$ is the equation of the tangent at the point (α, β, γ): throwing out the factor B^2, we have $4BD-3C^2=0$ for the equation of the four tangents from (α, β, γ) to the curve; viz. the equation of the four tangents is

$$2TQ^4+6SPQ^2+8P^3Q=0,$$

or, as this may be written,

$$Q(2P-\lambda_1 Q)(2P-\lambda_2 Q)(2P-\lambda_3 Q)=0,$$

viz. the equations of the four tangents are as is mentioned above; it was, in fact, by these geometrical considerations that Aronhold obtained his results.

114. The foregoing expression for $Q\,dP-P\,dQ$, say

$$Q\,dP-P\,dQ=(1+8l^3)\{\mathrm{a}(y\,dz-z\,dy)+\mathrm{b}(z\,dx-x\,dz)+\mathrm{c}(x\,dy-y\,dx)\},$$

may be verified without difficulty. Writing for a moment

$$\begin{aligned} Q\,dP-P\,dQ= &\quad (Ax+By+Cz)(Ldx+Mdy+Ndz)\\ &-(Adx+Bdy+Cdz)(Lx+My+Nz)\\ = &\quad (BN-CM)(y\,dz-z\,dy)-\&\text{c.};\end{aligned}$$

we have

$$\begin{aligned} BN-CM= &\quad \{-3l^2\beta^2+(1+2l^3)\gamma\alpha\}(\gamma^2+2l\alpha\beta)\\ &-\{-3l^2\gamma^2+(1+2l^3)\alpha\beta\}(\beta^2+2l\gamma\alpha)\\ &=-6l^3\alpha\beta^3+(1+2l^3)\alpha\gamma^3+6l^3\alpha\gamma^3-(1+2l^3)\alpha\beta^3,\\ &=-(1+8l^3)\alpha(\beta^3-\gamma^3)\\ &=-(1+8l^3)\,\mathrm{a},\end{aligned}$$

which proves the theorem.

115. I content myself with a partial verification of the identity

$$2TQ^4+6SPQ^3-8P^3Q=-(1+8l^3)^2(6C^2-8BD).$$

Writing herein $x, y, z=1, -1, 0$, we have $D=0$, and the equation becomes

$$2TQ^4+6SPQ^3-8P^3Q+6(1+8l^3)^2C^2=0,$$

where now

$$\begin{aligned} &Q=(\alpha-\beta)(\alpha+\beta-2l\gamma),\ P=(\alpha-\beta)\{-3l^2(\alpha+\beta)-(1+2l^3)\gamma\},\\ &C=\mathrm{a}+\mathrm{b}-2l\mathrm{c},\ =(\alpha-\beta)\{-\alpha\beta^2-\alpha^2\beta-\gamma^3-2l\gamma(\alpha^2+\alpha\beta+\beta^2)\},\end{aligned}$$

which, putting therein $-\gamma^3=\alpha^3+\beta^3+6l\alpha\beta\gamma$, becomes

$$=(\alpha-\beta)^2(\alpha+\beta-2l\gamma).$$

Hence writing

$$X=\alpha+\beta-2l\gamma,\quad Y=-3l^2(\alpha+\beta)-(1+2l^3)\gamma,$$

we have

$$Q,\ P,\ C=(\alpha-\beta)X,\quad (\alpha-\beta)Y,\quad (\alpha-\beta)^3X:$$

substituting these values, the factor $(\alpha-\beta)^4X$ divides out, and the equation becomes

$$2TX^3+6SX^2Y-8Y^3+6(1+8l^3)^2(\alpha-\beta)^2X=0.$$

To complete the verification, observe that we have $Y+3l^2X=-(1+8l^3)\gamma$, whence

$$-Y^3=(1+8l^3)^3\gamma^3+9(1+8l^3)^2l^2\gamma^2X+27(1+8l^3)l^4\gamma X^2+27l^6X^3,$$

and herein $-\gamma^3 = \alpha^3 + \beta^3 + 6l\alpha\beta\gamma$, whence

$$-\gamma^3 + \alpha\beta X = (\alpha+\beta)^3 = (X + 2l\gamma)^3 = X^3 + 6l\gamma X^2 + 12l^2\gamma^2 X + 8l^3\gamma^3,$$

that is,

$$-(1+8l^3)\gamma^3 = X^3 + 6l\gamma X^2 + (12l^2\gamma^2 - 3\alpha\beta) X.$$

Hence the equation to be verified becomes

$$2TX^3 + 6SX^2Y - 8 \left\{\begin{array}{l} (1+8l^3)^2[X^3 + 6l\gamma X^2 + (12l^2\gamma^2 - 3\alpha\beta) X] \\ -(1+8l^3)^2 9l^2\gamma^2 X \\ -(1+8l^3) 27l^4\gamma X^2 \\ -27l^6 X^3 \end{array}\right\}$$
$$+ 6(1+8l^3)^2(\alpha-\beta)^2 X = 0;$$

viz. throwing out the factor X, this is

$$\{2T - 8(1+8l^3)^2 + 216l^6\} X^2 + 6SXY - 48(1+8l^3)^2 l\gamma X + 216(1+8l^3) l^4\gamma X$$
$$-(1+8l^3)^2\{96l^2\gamma^2 - 24\alpha\beta - 72l^2\gamma^2 - 6(\alpha-\beta)^2\} = 0,$$

where the last term is

$$= +6(1+8l^3)^2\{(\alpha+\beta)^2 - 4l^2\gamma^2\},$$

viz. this is

$$= 6(1+8l^3)^2(\alpha+\beta+2l\gamma) X,$$

and there is again the factor X which can be thrown out: the equation thus becomes

$$[2T - 8(1+8l^3)^2 + 216l^6] X + 6SY - 48(1+8l^3)^2 l\gamma + 216(1+8l^3) l^4\gamma$$
$$+ 6(1+8l^3)^2(\alpha+\beta+2l\gamma) = 0.$$

This may be written

$$[2T - 8(1+8l^3)^2 + 216l^6] X + 6S[-3l^2X - (1+8l^3)\gamma] - 48(1+8l^3)^2 l\gamma$$
$$+ 216(1+8l^3) l^4\gamma + 6(1+8l^3)^2(X + 4l\gamma) = 0,$$

or, finally, it is

$$[2T - 8(1+8l^3)^2 + 216l^6 - 18l^2S + 6(1+8l^3)^2] X$$
$$+ [-6l^{-1}S - 48(1+8l^3) + 216l^3 + 24(1+8l^3)](1+8l^3) l\gamma = 0;$$

substituting for T, S their values $1 - 20l^3 - 8l^6$ and $-4l + 4l^4$ respectively, the coefficients of X and $(1+8l^3) l\gamma$ are separately $= 0$, and the equation is thus verified.

116. The foregoing equation $\lambda = \dfrac{6h\alpha^2 0}{f\alpha^2 0}$, regarding therein λ as an arbitrary parameter and (x, y, z) as current coordinates, is the equation of an arbitrary line through the point (a, b, c) of the cubic: it meets the cubic in two other points depending, of course, on the value of λ; and the coordinates of either of these is thus expressible, irrationally, in terms of λ, the expressions involving the radical $\sqrt{\lambda^3 - 3S\lambda - 2T}$: from the values of x, y, z in terms of λ, we should be able to deduce the foregoing equation

$\frac{2}{\sqrt{6}}d\omega = \frac{d\lambda}{\sqrt{\lambda^3 - 3S\lambda - 2T}}$. The expressions assume a peculiarly simple form when (α, β, γ), instead of being an arbitrary point of the cubic, is a point of inflexion of the cubic; and it is easy to see *à priori* why this is so: in fact, if we assume

$$x : y : z = u + \alpha\sqrt{\Lambda} : v + \beta\sqrt{\Lambda} : w + \gamma\sqrt{\Lambda},$$

where u, v, w are linear functions and Λ a cubic function of λ; then the locus is a cubic curve, and corresponding to the value $\lambda = \infty$, we have $x : y : z = \alpha : \beta : \gamma$, viz. the curve passes through the point (α, β, γ): moreover, it can be shown that this point is an inflexion of the curve; expressions of the foregoing simple form thus only exist in the case where the point (α, β, γ) is an inflexion; and the formulæ referring to an arbitrary point (α, β, γ) of the curve are necessarily of a more complex form.

117. To work this out, we start from the foregoing equation

$$\lambda = \frac{6h\alpha^2 0}{f\alpha^2 0} = \frac{2\left(\{-3l^2\alpha^2 + (1+2l^3)\beta\gamma\}x + \&c.\right)}{(\alpha^2 + 2l\beta\gamma)x + \&c.}$$

which, putting therein

$$L = \lambda + 6l^2,\quad M = l\lambda - (1+2l^3),$$

and

$$A,\ B,\ C = L\alpha^2 + 2M\beta\gamma,\quad L\beta^2 + 2M\gamma\alpha,\quad L\gamma^2 + 2M\alpha\beta,$$

becomes $Ax + By + Cz = 0$, the equation of a line through the point

$$\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ = \alpha(\beta^3 - \gamma^3),\quad \beta(\gamma^3 - \alpha^3),\quad \gamma(\alpha^3 - \beta^3),$$

as before: and we have to find the intersections of this line with the cubic

$$x^3 + y^3 + z^3 + 6lxyz = 0.$$

We have

$$C^3(x^3 + y^3) - (Ax + By)^3 - 6lC^2(Ax + By)xy = 0:$$

the cubic function contains as we know the factor $\mathrm{b}x - \mathrm{a}y$, and in the remaining quadric factor it is easy to calculate the coefficients of x^2 and y^2: we thus obtain the identity

$$\begin{aligned} &C^3(x^3 + y^3) - (Ax + By)^3 - 6lC^2(Ax + By)xy \\ &= (\mathrm{b}x - \mathrm{a}y)\{[(-\beta^2 - 6l\gamma\alpha)L^3 + 6\alpha\gamma L^2M - 8\beta^2M^3]x^2 \\ &\qquad\qquad + 2H\quad xy \\ &\qquad + [(-\alpha^2 - 6l\alpha\beta)L^3 + 6\beta\gamma L^2M - 8\alpha^2M^3]y^2\}, \end{aligned}$$

from which the as yet unknown coefficient $2H$ is to be obtained. This is most easily effected by assuming $x, y = \alpha, -\beta$; values which give

$$x^3 + y^3 = \alpha^3 - \beta^3,\quad Ax + By = L(\alpha^3 - \beta^3),\quad \mathrm{b}x - \mathrm{a}y = -\alpha\beta(\alpha^3 - \beta^3):$$

the whole equation becomes divisible by $\alpha^3 - \beta^3$ and, omitting this factor, we have

$$\begin{aligned} &C^2 - L^3(\alpha^3 - \beta^3)^2 + 6lC^2L\alpha\beta \\ &\qquad = \alpha\beta\{[2\alpha^2\beta^2 + 6l\gamma(\alpha^3 + \beta^3)]L^3 - 6\gamma(\alpha^3 + \beta^3)L^2M + 16\alpha^2\beta^2M^3\} + 2H\alpha^2\beta^2, \end{aligned}$$

where for C is to be substituted its value $L\gamma^2 + 2M\alpha\beta$. We may also reduce by $\alpha^3 + \beta^3 + \gamma^3 + 6l\alpha\beta\gamma = 0$. The left-hand side is

$$C^3 - L^3(\alpha^3 + \beta^3)^2 - 4\alpha^3\beta^3 L^3 + 6lC^2 L\alpha\beta,$$

which after reduction is found to contain the factor $\alpha\beta$; and omitting this factor and reducing also the right-hand side, the whole equation becomes

$$\begin{aligned}
&L^3(-6l\gamma^4 - 36l^2\gamma^2\alpha\beta + 4\alpha^2\beta^2) && = L^3(-6l\gamma^4 - 36l^2\gamma^2\alpha\beta + 2\alpha^2\beta^2)\\
&+ L^2M(6\gamma^4 + 24l\gamma^2\alpha\beta) && + L^2M(6\gamma^4 + 36l\gamma^2\alpha\beta)\\
&+ LM^2(12\gamma^2\alpha\beta + 24l\alpha^2\beta^2) && \\
&+ M^3(8\alpha^2\beta^2) && + M^3(16\alpha^2\beta^2)\\
& && + 2H\alpha\beta;
\end{aligned}$$

omitting here the terms which destroy each other, the equation again divides by $\alpha\beta$ and we thus obtain the value of H; and the required identity is

$$\begin{aligned}
C^3(x^3 + y^3) - (Ax + By)^3 - 6lC^2(Ax + By)xy\\
= (\mathrm{b}x - \mathrm{a}y)\{&[(-\beta^2 - 6l\gamma\alpha)L^3 + 6\alpha\gamma L^2M - 8\beta^2M^3]x^2\\
&+ [\alpha\beta L^3 - 6l\gamma^2L^2M + (6\gamma^2 + 12l\alpha\beta)LM^2 - 4\alpha\beta M^3]2xy\\
&+ [(-\alpha^2 - 6l\beta\gamma)L^3 + 6\beta\gamma L^2M - 8\alpha^2M^3]y^2\}.
\end{aligned}$$

Hence putting for shortness

$$\begin{aligned}
\mathfrak{A} &= (\alpha^2 + 6l\beta\gamma)L^3 - 6\beta\gamma L^2M && + 8\alpha^2M^3,\\
\mathfrak{B} &= (\beta^2 + 6l\gamma\alpha)L^3 - 6\alpha\gamma L^2M && + 8\beta^2M^3,\\
\mathfrak{H} &= \alpha\beta L^3 - 6l\gamma^2L^2M + (6\gamma^2 + 12l\alpha\beta)LM^2 && - 4\alpha\beta M^3,
\end{aligned}$$

the equation in (x, y) is

$$\mathfrak{B}x^2 - 2\mathfrak{H}xy + \mathfrak{A}y^2 = 0,$$

giving

$$\mathfrak{B}x - \mathfrak{H}y = \sqrt{\mathfrak{H}^2 - \mathfrak{A}\mathfrak{B}}\,y,$$

or say

$$x : y = \mathfrak{H} + \sqrt{\mathfrak{H}^2 - \mathfrak{A}\mathfrak{B}} : \mathfrak{B}.$$

We find without difficulty, reducing always by $\alpha^3 + \beta^3 + \gamma^3 + 6l\alpha\beta\gamma = 0$,

$$\begin{aligned}
\tfrac{1}{6}(\mathfrak{H}^2 - \mathfrak{A}\mathfrak{B}) = \quad & l\gamma^4 && && L^6\\
&+ (-\gamma^4 && + 4l\alpha\beta\gamma^2 &&)L^5M\\
&+ (6l^2\gamma^4 && - 4\alpha\beta\gamma^2 && + 4l\alpha^2\beta^2)L^4M^2\\
&+ (-4l\gamma^4 && + 24l^2\alpha\beta\gamma^2 && - 4\alpha^2\beta^2)L^3M^3\\
&+ (-2\gamma^4 && - 16l\alpha\beta\gamma^2 && + 24l\alpha^2\beta^2)L^2M^4\\
&+ (&& - 8\alpha\beta\gamma^2 && - 16l\alpha^2\beta^2)LM^5\\
&+ (&& && - 8\alpha^2\beta^2)M^6,
\end{aligned}$$

which is

$$= (lL - M)(L^3 + 6lLM^2 + 2M^3)(\gamma^2 L + 2\alpha\beta M)^2.$$

But we have

$$lL - M = 1 + 8l^3,$$

$$L^3 + 6lLM^2 + 2M^3 = (1 + 8l^3)(\lambda^3 - 3S\lambda - 2T),$$

and the equation thus is

$$\tfrac{1}{8}(\mathfrak{H}^2 - 2\mathfrak{B}) = (1 + 8l^3)^2(\lambda^3 - 3S\lambda - 2T)[(\gamma^2 + 2l\alpha\beta)\lambda + \{6l^2\gamma^2 - 2(1 + 2l^3)\alpha\beta\}]^2,$$

showing that the solution involves the radical $\sqrt{\lambda^3 - 3S\lambda - 2T}$.

118. If (α, β, γ) is the inflexion $(1, -1, 0)$, the expression for λ is here

$$\lambda = \frac{-6l^2x - 6l^2y - 2(1 + 2l^3)z}{x + y - 2lz};$$

the equation in (x, y) is

$$(L^3 + 8M^3)x^2 + (2L^3 + 24lLM^2 - 8M^3)xy + (L^3 + 8M^3)y^2 = 0,$$

or, as this may be written,

$$(L^3 + 6lLM^2 + 2M^3)(x + y)^2 + (-6lLM^2 + 6M^3)(x - y)^2 = 0,$$

say

$$(L^3 + 6lLM^2 + 2M^3)(x + y)^2 = 6M^3(lL - M)(x - y)^2;$$

viz. we thus have

$$\sqrt{\lambda^3 - 3S\lambda - 2T}\,(x + y) = M\sqrt{6}\,(x - y),$$

or, substituting for M its value,

$$x = \sqrt{6}\,\{l\lambda - (1 + 2l^3)\} + \sqrt{\lambda^3 - 3S\lambda - 2T},$$

$$y = \sqrt{6}\,\{l\lambda - (1 + 2l^3)\} - \sqrt{\lambda^3 - 3S\lambda + 2T},$$

whence also

$$z = \sqrt{6}\,(\lambda + 6l^2),$$

these values satisfying identically

$$(\lambda + 6l^2)(x + y) - 2[l\lambda - (1 + 2l^3)]z = 0,$$

and

$$x^3 + y^3 + z^3 + 6lxyz = 0.$$

119. Starting from these values we, in fact, easily obtain

$$x\,dy - y\,dx = \frac{-\sqrt{6}\,d\lambda}{\sqrt{\lambda^3 - 3S\lambda - 2T}} \times$$

$$\{l\lambda^3 + (-3 - 6l^3)\lambda^2 + (-12l^2 + 12l^5)\lambda + (-8l - 92l^4 - 8l^7)\},$$

$$z^2 + 2lxy = -2\,\{\text{Do.}\},$$

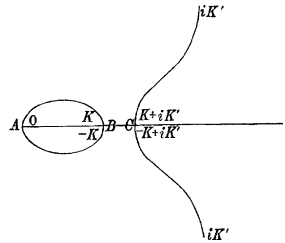
and hence

$$d\omega = \frac{x\,dy - y\,dx}{z^2 + 2lxy} = \frac{\frac{1}{2}\sqrt{6}\,d\lambda}{\sqrt{\lambda^3 - 3S\lambda - 2T}}.$$

The same result might of course have been obtained from the values of x, z or y, z, the factor which divides out being in each of these cases irrational.

The Cubic $y^2 = x(1-x)(1-k^2x)$. Art. Nos. 120 to 130 (*several sub-headings*).

120. The curve is a semi-cubical parabola, symmetrical in regard to the axis of x; and if, as usual, k^2 is taken to be real, positive and less than 1, then the curve consists of an oval, and an infinite portion which may be called the anguis. (See Figure.)



The equation is satisfied by

$$x = \mathrm{sn}^2 u,$$

$$y = \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u.$$

Observe that the periods for these combinations of the elliptic functions are $2K$, $2iK'$; in fact,

$$\begin{array}{llll} \mathrm{sn}\,(u+2K) = -\mathrm{sn}\, u, & \mathrm{sn}\,(u+2iK') = & \mathrm{sn}\, u, \\ \mathrm{cn} \quad ,, \quad = -\mathrm{cn}\, u, & \mathrm{cn} \quad ,, & = -\mathrm{cn}\, u, \\ \mathrm{dn} \quad ,, \quad = \mathrm{dn}\, u, & \mathrm{dn} \quad ,, & = -\mathrm{dn}\, u, \end{array}$$

whence the sn^2 and the $\mathrm{sn}\,.\,\mathrm{cn}\,.\,\mathrm{dn}$ are each unaltered by the change of u into $u+2K$ or $u+2iK'$. Hence to a given point (x, y) on the curve, the argument u is not = a determinate value u_0, for it may be equally well taken to be $= u_0 + 2mK + 2m'iK'$, where m, m' are any positive or negative integers whatever: we express this by $u \equiv u_0$, or say u is congruent to u_0. But when u is thus given by a congruence $u \equiv u_0$ where u_0 has a determinate value, the point on the curve is uniquely determined. It is, however, to be noticed that a congruence $2u \equiv u_0$ does not uniquely determine the point on the curve: there are, in fact, four incongruent values of u, viz.

$$\tfrac{1}{2}u_0,\ \tfrac{1}{2}u_0 + K,\ \tfrac{1}{2}u_0 + iK',\ \tfrac{1}{2}u_0 + K + iK',$$

and the point on the curve is thus one of the four points belonging to these values of u respectively.

121. If, to fix the ideas, we select for each point of the curve one of the congruent values of the argument, we may assume for the oval, u real: at A, $u=0$; from A to B above the axis u positive and at B, $=K$; below the axis u negative and at B, $=-K$; there is thus a discontinuity, K, $-K$ at B, but the two values are congruent. For the anguis, $u = iK' +$ real value v: above the axis v positive, viz.

at infinity $v=0$, and at $C, =K$; below the axis v negative, viz. at infinity $v=0$ and at $C, =-K$: there thus is a discontinuity $iK'+K$, $iK'-K$ at C, but the two values are congruent. Observe that for points opposite to each other in regard to the axis, the arguments are, for points on the oval u, $-u$; for points on the anguis $iK'+v$, $iK'-v$: but that we have $iK'-v\equiv-(iK'-v)$.

122. The pure theorem gives for three points u_1, u_2, u_3 in a line

$$du_1+du_2+du_3=0;$$

and thence $u_1+u_2+u_3=C$. The constant C cannot have a determinate value (for if it had, then assuming the values of u_1 and u_2 at pleasure u_3 would have the determinate value $=C-u_1-u_2$), but must be given by a congruence: or, what is the same thing, assigning to C any admissible value, we must instead of $u_1+u_2+u_3=C$, write $u_1+u_2+u_3\equiv C$. Taking any particular line, for instance the tangent at A, we have $u_1, u_2, u_3=0, 0, iK'$; whence $C=iK'$; and we have $u_1+u_2+u_3\equiv iK'$, viz. this is the relation between the arguments u_1, u_2, u_3 belonging to the points of intersection of the cubic with a line: in particular, for a line at right angles to the axis, we have $u_1, u_2, u_3=\alpha, -\alpha, iK'$ or $=iK'+\beta, iK'-\beta, iK'$ (according as the line cuts the oval or the anguis): and the congruence is in each case satisfied. But I shall in general instead of $\equiv$ use the sign $=$, understanding it as in general meaning $\equiv$, and only replacing it by this sign when for clearness it seems necessary to do so.

Writing $\operatorname{sn} u_1$, $\operatorname{cn} u_1$, $\operatorname{dn} u_1=s_1$, c_1, d_1, and so in other cases, the condition in order that the three points may be in a line is

$$\begin{vmatrix} s_1^2, & s_1c_1d_1, & 1 \\ s_2^2, & s_2c_2d_2, & 1 \\ s_3^2, & s_3c_3d_3, & 1 \end{vmatrix}=0,$$

a relation which must be satisfied when the arguments are connected by the foregoing relation $u_1+u_2+u_3\equiv iK'$.

We can show from this equation alone that s_3^2 and $s_3c_3d_3$ are expressible rationally in terms of s_1^2, $s_1c_1d_1$, s_2^2, $s_2c_2d_2$; in fact, writing therein x_1, y_1 in place of s_1^2, $s_1c_1d_1$, &c., we thence have x_3, y_3, $1=\lambda x_1+\mu x_2$, $\lambda y_1+\mu y_2$, $\lambda+\mu$, and substituting in

$$y_3^2=x_3(1-x_3)(1-k^2x_3),$$

we obtain an equation $\lambda\mu(F\lambda+G\mu)=0$, that is, $F\lambda+G\mu=0$, or say $\lambda, \mu=G, -F$, and thence

$$x_3=s_3^2=\frac{Gx_1-Fx_2}{G-F},\quad y_3=s_3c_3d_3=\frac{Gy_1-Fy_2}{G-F}.$$

The values of F, G are easily found to be

$$F=y_1^2+2y_1y_2-x_1(1-x_1)(1-k^2x_2)-x_1(1-k^2x_1)(1-x_2)-(1-x_1)(1-k^2x_1)x_2,$$

$$G=2y_1y_2+y_2^2-x_2(1-x_2)(1-k^2x_1)-x_2(1-k^2x_2)(1-x_1)-(1-x_2)(1-k^2x_2)x_1,$$

or, as these may also be written,

$$F = -(y_1 - y_2)^2 + (x_1 - x_2)^2 \{1 + k^2 + k^2(x_1 + x_2)\} + k^2(x_1 - x_2)^2 x_1,$$
$$G = -(y_1 - y_2)^2 + (x_1 - x_2)^2 \{1 + k^2 + k^2(x_1 + x_2)\} + k^2(x_1 - x_2)^2 x_2,$$

where of course x_1, y_1, x_2, y_2 should be replaced by their values s_1^2, $s_1c_1d_1$, s_2^2, $s_2c_2d_2$. This is, in fact, the ordinary process of finding the third point of intersection of a cubic by a line which meets it in two given points.

Writing $iK' - u_3 = u$, and s, c, d for the sn, cn, dn of u, we have

$$s_3,\ c_3,\ d_3 = -\frac{1}{ks},\ \frac{id}{ks},\ \frac{ic}{s},$$

whence

$$s_3^2 = \frac{1}{k^2s^2},\quad s_3c_3d_3 = \frac{cd}{k^2s^3},$$

and the determinant equation becomes

$$\begin{vmatrix} s_1^2, & s_1c_1d_1, & 1 \\ s_2^2, & s_2c_2d_2, & 1 \\ 1, & \dfrac{cd}{s}, & k^2s^2 \end{vmatrix} = 0,$$

that is,

$$(1 - k^2s^2s_2^2)\, s_1c_1d_1 - (1 - k^2s^2s_1^2)\, s_2c_2d_2 - (s_1^2 - s_2^2)\frac{cd}{s} = 0,$$

corresponding to the relation $u = u_1 + u_2$ of the arguments. This is easily verified: we have

$$s_1 = \frac{sc_2d_2 - s_2cd}{1 - k^2s^2s_2^2},\quad s_2 = \frac{sc_1d_1 - s_1cd}{1 - k^2s^2s_1^2},\quad s = \frac{s_1^2 - s_2^2}{s_1c_2d_2 - s_2c_1d_1};$$

the equation thus becomes

$$(sc_2d_2 - s_2cd)\, c_1d_1 - (sc_1d_1 - s_1cd)\, c_2d_2 - (s_1^2 - s_2^2)\frac{cd}{s} = 0,$$

that is,

$$cd\left\{-s_2c_1d_1 + s_1c_2d_2 - \frac{s_1^2 - s_2^2}{s}\right\} = 0,$$

which is right.

The Four Tangents from a Point of the Cubic.

123. Suppose that the line is a tangent to the cubic, say the line touches the cubic at the point u, and again meets it at the point w: then instead of u_1, u_2, u_3 we have u, u, w: and the relation becomes $2u + w \equiv iK'$.

Here u being given, w is uniquely determined: viz. given the argument u of the point of contact, we have a unique value for the argument w of the tangential. But given w, we have $2u = iK' - w$; and we have thus for u the four values

$$\tfrac{1}{2}(iK' - w),\quad \text{Do.} + K,\quad \text{Do.} + iK',\quad \text{Do.} + K + iK',$$

corresponding to the four tangents which can be drawn from the point w to the cubic.

The tangents are real for a point of the anguis, and for such a point we may write $w = iK' + v$, where v is real and included between the values $\pm K$; the corresponding values of u are

$$u_1 = -\tfrac{1}{2}v, \quad u_2 = -\tfrac{1}{2}v + K, \quad u_3 = -\tfrac{1}{2}v + iK', \quad u_4 = -\tfrac{1}{2}v + K + iK':$$

the first and second of these belong to tangents to the oval, the third and fourth to tangents to the anguis. We may further distinguish a tangent according as it passes between or does not pass between the vertices B and C: say in the former case it is intermediate, and in the latter case extramediate: and we then see that, for the tangents from the point $iK' + v$ of anguis,

$$\begin{array}{ll} u = -\tfrac{1}{2}v & \text{for intermediate to oval,} \\ u = -\tfrac{1}{2}v + K & \text{„ extramediate to oval,} \\ u = -\tfrac{1}{2}v + iK' & \text{„ extramediate to anguis,} \\ u = -\tfrac{1}{2}v + K + iK' & \text{„ intermediate to anguis.} \end{array}$$

124. We may make a corresponding division of the real lines which meet the curve in three real points: any such line meets the oval twice (and then of course the anguis once), or else it meets the anguis three times: and taking the arguments to be u_1, u_2, u_3, we have

$$\begin{array}{lll} \tfrac{1}{2}(u_1 + u_2 + u_3) = & \tfrac{1}{2}iK' & \text{for intermediate line meeting oval twice,} \\ \text{„} \quad = & \tfrac{1}{2}iK' + K & \text{„ extramediate line, Do.,} \\ \text{„} \quad = & -\tfrac{1}{2}iK' & \text{„ extramediate line meeting anguis three times,} \\ \text{„} \quad = & -\tfrac{1}{2}iK' + K & \text{„ intermediate line, Do.} \end{array}$$

125. Returning to the tangents, the point $iK' + v$ may be an inflexion: we have then the point of contact of the intermediate tangent to the anguis coinciding with the point $iK' + v$; viz. $iK' + v \equiv -\tfrac{1}{2}v + K + iK'$, or say $= -\tfrac{1}{2}v \pm K + iK'$: that is, $v = \pm\tfrac{2}{3}K$; or $iK' + \tfrac{2}{3}K$ and $iK' - \tfrac{2}{3}K$ are the arguments for the real points of inflexion, above and below the axis respectively.

126. Write for a moment the equation in the form $y^2 = Bx + Cx^2 + Dx^3$; then if (α, β) be a point on the curve $(\beta^2 = B\alpha + C\alpha^2 + D\alpha^3)$, and we consider the intersections of the curve with the line $y - \beta = m(x - \alpha)$, we find for the remaining two intersections

$$B + C(x + \alpha) + D(x^2 + \alpha x + \alpha^2) = 2m\beta + m^2(x - \alpha).$$

If the line be a tangent, this will be satisfied by $x - \alpha$; the condition for this is $2m\beta = B + 2C\alpha + 3D\alpha^2$, and supposing this satisfied, then throwing out the factor $x - \alpha$ we obtain $C + D(x + 2\alpha) = m^2$, giving $Dx = m^2 - C - 2D\alpha$ for the coordinate x of the tangential of the point (α, β).

In the case of an inflexion, $x = \alpha$; and we have

$$m^2 = C + 3D\alpha, \; = \frac{(B + 2C\alpha + 3D\alpha^2)^2}{4(B\alpha + C\alpha^2 + D\alpha^3)},$$

giving for α the equation

$$3D^2\alpha^4 + 4CD\alpha^3 + 6BD\alpha - B^2 = 0,$$

or for B, C, D writing their values 1, $-(1+k^2)$, k^2, this is

$$3k^4\alpha^4 - 4k^2(1+k^2)\alpha^3 + 6k^2\alpha^2 - 1 = 0$$

for the x-coordinate of the inflexion. There is one negative root, and one or three positive roots; but only one positive root giving a real value of β, and corresponding hereto we have the two real inflexions on the anguis.

It should be possible, from the formulæ of No. 122, writing therein $(x_2, y_2) = (x_1, y_1)$, to arrive at the foregoing result, say $Dx_3 = m^2 - C - 2Dx_1$; but the functions F and G present themselves in vanishing forms, and the reduction is not immediate.

127. The general condition for an inflexion is $3u \equiv iK'$: the nine inflexions thus are $u = iK'$, inflexion at infinity, $u = iK' \pm \frac{2}{3}K$, the two real inflexions, and besides $u = \pm\frac{1}{3}iK'$, $u = \pm\frac{1}{3}iK' \pm \frac{2}{3}K$, six imaginary inflexions.

The Sextactic Points.

128. The vertices A, B, C are each of them a sextactic point: in fact, writing the equation in the form $y^2 = x - (1+k^2)x^2 + k^2x^3$, we see at once that the conic $y^2 = x - (1+k^2)x^2$ meets the curve in the point A counting six times: and there is obviously a like proof for the vertices B and C respectively. Hence, for the six intersections with any conic whatever, we have the condition

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \equiv 0:$$

and for the sextactic points we have the condition $6u \equiv 0$. This gives the 36 points $u = \frac{1}{6}(2mK + 2m'iK')$ or say $= \frac{1}{3}(mK + m'iK')$, $m = 0, \pm 1, \pm 2, 3$, $m' = 0, \pm 1, \pm 2, 3$: but among these are included the 9 inflexions (each of these being an improper sextactic point, the conic becoming the tangent taken twice) and there remain 27 points: among these are included the three vertices $(u = 0, K, iK' + K)$, points of contact with the tangents from the inflexion at infinity; and of the remaining 24 points 6 are real, viz. these are the points $u = \pm\frac{1}{3}K, \pm\frac{2}{3}K$ on the oval, and the points $iK' \pm \frac{1}{3}K$ on the anguis: these are, in fact, the points of contact of the tangents from the real inflexions, viz. the three tangents from the inflexion $iK' + \frac{2}{3}K$ touch the oval in the points $\frac{2}{3}K$, $-\frac{1}{3}K$, and the anguis in the point $iK' - \frac{1}{3}K$; the three tangents from the inflexion $iK' - \frac{2}{3}K$ touch the oval in the points $-\frac{2}{3}K$, $\frac{1}{3}K$, and the anguis in the point $iK' + \frac{1}{3}K$.

Formulæ Relating to the Tangents from the Vertices.

129. I annex some formulæ relating to the tangents to the curve from the vertices A, B, C respectively. We have from each vertex four tangents say $\rho = 0$, $\sigma = 0$, symmetrically situate in regard to the axis, and $\rho' = 0$, $\sigma' = 0$, symmetrically situate in regard to the axis: the line joining the points of contact of ρ, σ is a line $\tau = 0$ at

right angles to the axis, and that joining the points of contact of ρ', σ' is a line $\tau' = 0$ at right angles to the axis.

Vertex A, Tangents Imaginary.	Coordinates of Point of Contact.
$\rho,\ \sigma,\ \tau = y - i(1+k)x,\ y + i(1+k)x,\ kx+1;$	$x = -\dfrac{1}{k},\ y = \mp\dfrac{i(1+k)}{k},$
$\rho',\ \sigma',\ \tau' = y - i(1-k)x,\ y + i(1-k)x,\ -kx+1;$	$x = \dfrac{1}{k},\ y = \pm\dfrac{i(1-k)}{k};$

the equation of the curve is

$$f, = y^2 - x(1-x)(1-k^2x),\ = \rho\sigma - x\tau^2,\ = \rho'\sigma' - x\tau'^2,\ = 0.$$

Vertex B, Tangents Imaginary.	Contacts.
$\rho,\sigma,\tau = y-(k-ik')(1-x),\ y+(k+ik')(1-x),\ kx-(k-ik');$	$x = 1 - \dfrac{ik'}{k},\ y = \mp\dfrac{ik'}{k}(k+ik'),$
$\rho',\sigma',\tau' = y-(k-ik')(1-x),\ y+(k-ik')(1-x),\ kx-(k+ik');$	$x = 1 + \dfrac{ik'}{k},\ y = \pm\dfrac{ik'}{k}(k-ik');$

the equation of the curve is

$$f = \rho\sigma + (1-x)\tau^2,\ = \rho'\sigma' + (1-x)\tau'^2,\ = 0.$$

Vertex C, Tangents Real.	Contacts.
$\rho,\ \sigma,\ \tau = y - \dfrac{1}{1+k'}(1-k^2x),\ y + \dfrac{1}{1+k'}(1-k^2x),\ x - \dfrac{1}{1+k'};$	$x = \dfrac{1}{1+k'},\ y = \dfrac{\pm k'}{1+k'},$
$\rho',\ \sigma',\ \tau' = y - \dfrac{1}{1-k'}(1-k^2x),\ y + \dfrac{1}{1-k'}(1-k^2x),\ x - \dfrac{1}{1-k'};$	$x = \dfrac{1}{1-k'},\ y = \dfrac{\pm k'}{1-k'};$

the equation of the curve is

$$f, = \rho\sigma - (1-k^2x)\tau,\ = \rho'\sigma' - (1-k^2x)\tau',\ = 0.$$

130. These linear functions ρ, σ, &c., considering therein x, y as denoting $\operatorname{sn}^2 u$, $\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u$, respectively present themselves as the numerators and the denominators of some formulæ given No. 105 of my *Elliptic Functions* (1876), see p. 76: viz. we have

$$\operatorname{sn}^2(u + \tfrac{1}{2}K) = \frac{1}{1+k'}\,\frac{1-k^2x+(1+k')y}{1-k^2x+(1-k')y},$$

which is

$$= \frac{1}{1-k'}\,\frac{y + \dfrac{1}{1+k'}(1-k^2x)}{y + \dfrac{1}{1-k'}(1-k^2x)},\quad = \frac{1}{1-k'}\,\frac{\sigma}{\sigma'},\quad \text{(vertex C)};$$

$$\operatorname{sn}^2(u + \tfrac{1}{2}iK') = \frac{1}{k}\,\frac{(1+k)x+iy}{(1+k)x-iy},$$

which is

$$= -\frac{1}{k}\,\frac{y - i(1+k)x}{y + i(1+k)x},\qquad = -\frac{1}{k}\,\frac{\rho}{\sigma},\qquad \text{(vertex A)};$$

and

$$\operatorname{sn}^2(u + \tfrac{1}{2}K + \tfrac{1}{2}iK') = \frac{k+ik'}{k}\,\frac{1-x+(k-ik)y}{1-x+(k+ik)y},$$

which is

$$=\frac{k-ik'}{k}\,\frac{y+(k+ik')(1-x)}{y+(k-ik')(1-x)}, \qquad =\frac{k-ik'}{k}\,\frac{\sigma}{\sigma'}, \text{ (vertex B).}$$

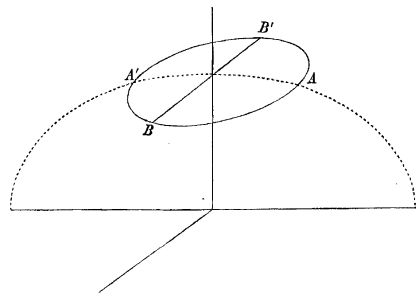
Observe here that, in the second formula, we have a pair of tangents ρ, σ which belong to a chord τ through the inflexion at ∞; but in the first and third formulæ we have tangents σ, σ' not forming such a pair. This is as it should be, for the zero and infinity of $\mathrm{sn}^2(u+\frac{1}{2}iK')$ are $u=-\frac{1}{2}iK'$, $u=\frac{1}{2}iK'$, which belong to points in lineâ with the inflexion at infinity: but for $\mathrm{sn}^2(u+\frac{1}{2}K)$, the zero is $u=-\frac{1}{2}K$, and the infinity is $u=iK'-\frac{1}{2}K$, which do not belong to points in lineâ with the inflexion at infinity: and the like for $\mathrm{sn}^2(u+\frac{1}{2}K+\frac{1}{2}iK')$.

Fixed Curve a Quartic in Space, the Quadri-quadric Curve $y^2=1-x^2$, $z^2=1-k^2x^2$.
Art. Nos. 131 to 135.

131. It is assumed that k^2 is real, positive, and less than unity: the curve may be regarded as the intersection of the two cylinders

$$x^2+y^2=1, \quad k^2x^2+z^2=1\,;$$

but there is through it a third cylinder $y^2-k^2z^2=k'^2$. The cylinder $k^2x^2+z^2=1$, or say the horizontal cylinder, has for its section an ellipse axes $\frac{1}{k}$ and 1 respectively: and it is pierced by the cylinder $x^2+y^2=1$, or say the vertical cylinder, in two detached ovals (of double curvature) lying on opposite sides of the plane of xy: only the upper oval $ABA'B'$ is shown in the figure.



Each of the vertices A, A', B, B' is an inflexion*, viz. a point such that the osculating plane at that point meets the curve in the point counting four times. We may consider two generating lines of the horizontal cylinder, each meeting the oval in two points; the plane through the generating lines meets the curve in these

* There are in all 16 inflexions: 4 in each of the planes $x=0$, $y=0$, $z=0$, and 4 in the plane infinity.

four points, and when the generating lines come each of them to coincide with the tangent at A, we have the osculating plane meeting the curve in the point A counting four times. The like reasoning, with two generating lines of the third cylinder, shows that the vertex B is an inflexion.

132. The equations are satisfied by writing therein $x, y, z = \operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$: the periods are here $4K$, $4iK'$: hence, at a given point on the curve, the argument is not $u =$ a determinate value u_0, but it may equally well be taken to be $= u_0 + 4mK + 4m'iK'$, where m and m' are any positive or negative integers: we express this by $u \equiv u_0$, or say u congruent to u_0. For the upper oval, u may be taken to be real, and to be $= 0$ at B, positive for the half oval BAB', and negative for the half oval $BA'B'$; having the values $+K$, $-K$ at A and A' respectively, and the discontinuity $2K$, $-2K$ at B', these two values being congruent. For the lower oval, we have $u = 2iK' +$ real value v.

For the intersections of the curve with a plane, we have

$$du_1 + du_2 + du_3 + du_4 = 0\,; \text{ whence } u_1 + u_2 + u_3 + u_4 = C\,;$$

and by taking the plane to be the osculating plane at B, we find 0 as a value of the constant, and the relation thus is $u_1 + u_2 + u_3 + u_4 \equiv 0$. Writing as before $\operatorname{sn} u_1$, $\operatorname{cn} u_1$, $\operatorname{dn} u_1$, $= s_1, c_1, d_1$ and so in other cases, the relation between the elliptic functions is

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix} = 0.$$

It is important to remark that, given three of the points, the fourth point is determined uniquely: that is, the equation really gives s_4, c_4, d_4, each as a rational function of the $s_1, c_1, d_1, s_2, c_2, d_2, s_3, c_3, d_3$.

In fact, we may write $s_4 = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3$, and similarly for c_4 and d_4, and $1 = \lambda_1 + \lambda_2 + \lambda_3$: substituting in $s_4^2 + c_4^2 - 1 = 0$, $k^2 s_4^2 + d_4^2 - 1 = 0$, we have

$$X_{23}\lambda_2\lambda_3 + X_{31}\lambda_3\lambda_1 + X_{12}\lambda_1\lambda_2 = 0,$$
$$Y_{23} \;\;,, \;\; + Y_{31} \;\;,, \;\; + Y_{12} \;\;,, \;\; = 0,$$

where

$$X_{12} = s_1 s_2 + c_1 c_2 - 1, \quad Y_{12} = k^2 s_1 s_2 + d_1 d_2 - 1, \text{ \&c.};$$

we thence have

$$\begin{aligned} \lambda_2\lambda_3 : \lambda_3\lambda_1 : \lambda_1\lambda_2 &= X_{31}Y_{12} - X_{12}Y_{31} : X_{12}Y_{23} - X_{23}Y_{12} : X_{23}Y_{31} - X_{31}Y_{23} \\ &= \quad A_1 \quad : \quad A_2 \quad : \quad A_3, \text{ suppose}; \end{aligned}$$

that is,

$$\lambda_1 : \lambda_2 : \lambda_3 = A_2 A_3 : A_3 A_1 : A_1 A_2,$$

and consequently

$$\begin{aligned} s_4, &= -\operatorname{sn}(u_1 + u_2 + u_3), &&= A_2A_3s_1 + A_3A_1s_2 + A_1A_2s_3 \div (A_2A_3 + A_3A_1 + A_1A_2), \\ c_4, &= \operatorname{cn}(\quad ,, \quad) &&= ,, \; c_1 + ,, \; c_2 + ,, \; c_3 \div (\quad ,, \quad), \\ d_4, &= \operatorname{dn}(\quad ,, \quad) &&= ,, \; d_1 + ,, \; d_2 + ,, \; d_3 \div (\quad ,, \quad), \end{aligned}$$

which are the required expressions. If $u_3 = 0$, and consequently s_3, c_3, $d_3 = 0$, 1, 1, the resulting expressions give the sn, cn, and dn of $u_1 + u_2$; but the expressions are in a very complicated form, not easily identifiable with the ordinary ones.

133. The determinant equation may be written

$$\begin{aligned}&(s_1 - s_2)(c_3 d_4 - c_4 d_3) + (s_3 - s_4)(c_1 d_2 - c_2 d_1)\\ &+ (c_1 - c_2)(d_3 s_4 - d_4 s_3) + (c_3 - c_4)(d_1 s_2 - d_2 s_1)\\ &+ (d_1 - d_2)(s_3 c_4 - s_4 c_3) + (d_3 - d_4)(s_1 c_2 - s_2 c_1) = 0,\end{aligned}$$

and, in fact, each of the three lines is separately $= 0$. This appears from the following three formulæ:—

$$\frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{cn}(u_1 + u_2) - \operatorname{dn}(u_1 + u_2)} = \frac{s_1 - s_2}{c_1 d_2 - c_2 d_1},$$

$$\frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{cn}(u_1 + u_2) + 1} = \frac{c_1 - c_2}{d_1 s_2 - d_2 s_1},$$

$$\frac{\operatorname{sn}(u_1 + u_2)}{\operatorname{dn}(u_1 + u_2) + 1} = \frac{-\frac{1}{k^2}(d_1 - d_2)}{s_1 c_2 - s_2 c_1},$$

which are themselves at once deducible from the formulæ

$$\begin{aligned}&\operatorname{sn}(u_1 + u_2) = s_1^2 - s_2^2, \ = -(c_1^2 - c_2^2), \ = -\frac{1}{k^2}(d_1^2 - d_2^2), && \div (s_1 c_2 d_2 - s_2 c_1 d_1),\\ &\operatorname{cn}(u_1 + u_2) = s_1 c_1 d_2 - s_2 c_2 d_1 && \div \quad \text{Do.}\\ &\operatorname{dn}(u_1 + u_2) = s_1 d_1 c_2 - s_2 d_2 c_1 && \div \quad \text{Do.}\end{aligned}$$

In fact, the numerators of $\operatorname{cn}(u_1 + u_2) + \operatorname{dn}(u_1 + u_2)$, $\operatorname{cn}(u_1 + u_2) + 1$, $\operatorname{dn}(u_1 + u_2) + 1$, thus become $= (s_1 + s_2)(c_1 d_2 - c_2 d_1)$, $-(c_1 + c_2)(d_1 s_2 - d_2 s_1)$, $(d_1 + d_2)(s_1 c_2 - s_2 c_1)$, respectively: so that, taking the numerator of $\operatorname{sn}(u_1 + u_2)$ under its three forms successively, we have by division the formulæ in question.

134. The above three equations, putting therein $u_1 + u_2 = -2u_3$ and reducing the functions of $2u_3$, become

$$\frac{1}{k'^2}\frac{c_3 d_3}{s_3} = \frac{s_1 - s_2}{c_1 d_2 - c_2 d_1}, \text{ giving } -k^2 s_3^2 = \frac{(c_1 - c_2)(d_1 - d_2)}{(d_1 s_2 - d_2 s_1)(s_1 c_2 - s_2 c_1)},$$

$$-\frac{s_3 d_3}{c_3} = \frac{c_1 - c_2}{d_1 s_2 - d_2 s_1}, \quad \text{,,} \quad \frac{k^2}{k'^2} c_3^2 = \frac{(d_1 - d_2)(s_1 - s_2)}{(s_1 c_2 - s_2 c_1)(c_1 d_2 - c_2 d_1)},$$

$$\frac{k^2 s_3 c_3}{d_3} = \frac{d_1 - d_2}{s_1 c_2 - s_2 c_1}, \quad \text{,,} \quad -\frac{1}{k'^2} d_3^2 = \frac{(s_1 - s_2)(c_1 - c_2)}{(c_1 d_2 - c_2 d_1)(d_1 s_2 - d_2 s_1)},$$

equations which must of course give each of them the same value for s_3^2: the equations belong to the relation $u_1 + u_2 + 2u_3 = 0$, viz. (s_3, c_3, d_3) are the coordinates of a point of contact of the tangent plane drawn through the two given points (s_1, c_1, d_1) and (s_2, c_2, d_2) of the curve.

135. Write

$$a,\ b,\ c,\ f,\ g,\ h = s_1 - s_2,\ c_1 - c_2,\ d_1 - d_2,\ c_1 d_2 - c_2 d_1,\ d_1 s_2 - d_2 s_1,\ s_1 c_2 - s_2 c_1,$$
$$a',\ b',\ c',\ f',\ g',\ h' = s_3 - s_4,\ c_3 - c_4,\ d_3 - d_4,\ c_3 d_4 - c_4 d_3,\ d_3 s_4 - d_4 s_3,\ s_3 c_4 - s_4 c_3,$$

so that (a, b, c, f, g, h) are the six coordinates of the line 12, and a', b', c', f', g', h' are the six coordinates of the line 34. The determinant equation is nothing else than the condition of the intersection of the two lines, viz. this is

$$af' + a'f + bg' + b'g + ch' + c'h = 0.$$

By what precedes, it appears that not only is this so: but that we have separately $af' + a'f = 0$, $bg' + b'g = 0$, $ch' + c'h = 0$, viz. these three equations are satisfied by the coordinates of the lines 12 and 34, which join in pairs the intersections 1, 2 and 3, 4 of the quadri-quadric curve by a plane. But this is a geometrical property depending only on the four points being in a plane: and it is thus a result of Abel's theorem that, when the arguments are such that

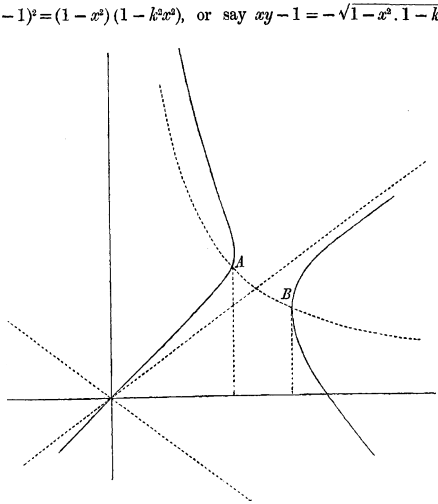
$$u_1 + u_2 + u_3 + u_4 = 0,$$

then not only the original equation, but each of the three equations, holds good.

The Cubic Curve $xy^2 - 2y + (1 + k^2)x - k^2x^3 = 0$. Art. Nos. 136 and 137.

136. Writing the equation in the form

$$(xy - 1)^2 = (1 - x^2)(1 - k^2x^2), \text{ or say } xy - 1 = -\sqrt{1 - x^2 . 1 - k^2x^2},$$



we see that the general form is as shown in the figure: the real portions of the

curve lie between the values $x = -\infty, -\frac{1}{k}; -1, +1$; and $\frac{1}{k}, \infty$. The curve may be made to depend on elliptic functions in two different ways: we may write

$$x = \operatorname{sn} u \quad , \quad = \frac{2i}{1+k} \frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v},$$

$$y = \frac{1 - \operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}, \quad = \frac{i}{1+k} \frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v} \{1 + k^2 - (1-k)^2 \operatorname{sn}_1^2 v\},$$

where the functions sn, cn, dn belong to the modulus k, and the functions sn_1, cn_1, dn_1 to the modulus $\theta, = \frac{1-k}{1+k}$. The first mode is obvious; as to the second, observe that the formulæ give

$$y - kx = \frac{i}{1+k} \frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v} (1-k)^2 \operatorname{cn}_1^2 v, \quad = \frac{i(1-k)^2}{1+k} \frac{\operatorname{sn}_1 v \operatorname{cn}_1 v}{\operatorname{dn}_1 v},$$

$$y + kx = \frac{i}{1+k} \frac{\operatorname{sn}_1 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v} (1+k)^2 \operatorname{dn}_1^2 v, \quad = i(1+k) \frac{\operatorname{sn}_1 v \operatorname{dn}_1 v}{\operatorname{cn}_1 v},$$

whence

$$y^2 - k^2 x^2 = -(1-k)^2 \operatorname{sn}_1^2 v;$$

and therefore

$$x(y^2 - k^2 x^2) = \frac{-2i(1-k)^2}{1+k} \frac{\operatorname{sn}_1^3 v}{\operatorname{cn}_1 v \operatorname{dn}_1 v},$$

which is also the value of $2y - (1+k^2)x$, as it should be.

We find, moreover, $d\omega, = du, = \frac{2i dv}{1+k}$; and thence, u, v each vanishing together, $u = \frac{2iv}{1+k}$. Writing for shortness s_1, c_1, d_1 to denote $\operatorname{sn}_1 v$, $\operatorname{cn}_1 v$, $\operatorname{dn}_1 v$: that is, s_1, c_1, d_1 are the elliptic functions of v to the modulus θ: we have

$$\operatorname{sn}\left(\frac{2iv}{1+k}, k\right) = \frac{2i}{1+k} \frac{s_1}{c_1 d_1},$$

$$\operatorname{cn}\left(\frac{2iv}{1+k}, k\right) = \frac{1}{(1+k) c_1 d_1} \{1 + k + (1-k) s_1^2\},$$

$$\operatorname{dn}\left(\frac{2iv}{1+k}, k\right) = \frac{1}{(1+k) c_1 d_1} \{1 + k - (1-k) s_1^2\}.$$

137. To bring these into a known form, for k write $\frac{1-k}{1+k}$, then θ is changed into k, and the formulæ become

$$\operatorname{sn}_1 (1+k) iv = (1+k) i \frac{s}{cd},$$

$$\operatorname{cn}_1 (1+k) iv = \frac{1}{cd} (1 + ks^2),$$

$$\operatorname{dn}_1 (1+k) iv = \frac{1}{cd} (1 - ks^2),$$

where the sn_1, cn_1, dn_1 refer to the modulus θ, and s, c, d denote $\mathrm{sn}\,v$, $\mathrm{cn}\,v$, $\mathrm{dn}\,v$, modulus k.

But from the formulæ, p. 63, of my *Elliptic Functions*,

$$\mathrm{sn}\,(iu,\ k),\ \mathrm{cn}\,(iu,\ k),\ \mathrm{dn}\,(iu,\ k) = \frac{i\,\mathrm{sn}\,(u,\ k')}{\mathrm{cn}\,(u,\ k')},\ \frac{1}{\mathrm{cn}\,(u,\ k')},\ \frac{\mathrm{dn}\,(u,\ k')}{\mathrm{cn}\,(u,\ k')},$$

and herein for u writing $(1+k)\,v$, and for k writing θ, $=\dfrac{1-k}{1+k}$, whence k' becomes $\dfrac{2\sqrt{k}}{1+k}$, $=\gamma$, and $\gamma' = \dfrac{1-k}{1+k}$, we have

$$\mathrm{sn}_1\,(1+k)\,iv,\ \mathrm{cn}_1\,(1+k)\,iv,\ \mathrm{dn}_1\,(1+k)\,iv = \frac{i\,\mathrm{sn}\,(\overline{1+k}v,\ \gamma)}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)},\ \frac{1}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)},\ \frac{\mathrm{dn}\,(\overline{1+k}v,\ \gamma)}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)};$$

and the formulæ above obtained are

$$\frac{\mathrm{sn}\,(\overline{1+k}v,\ \gamma)}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)} = (1+k)\frac{s}{cd},\quad \text{giving}\quad \mathrm{sn}\,(\overline{1+k}v,\ \gamma) = \frac{(1+k)\,s}{1+ks^2},$$

$$\frac{1}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)} = \frac{1}{cd}(1+ks^2),\quad ,,\quad \mathrm{cn}\,(\overline{1+k}v,\ \gamma) = \frac{cd}{1+ks^2},$$

$$\frac{\mathrm{dn}\,(\overline{1+k}v,\ \gamma)}{\mathrm{cn}\,(\overline{1+k}v,\ \gamma)} = \frac{1}{cd}(1-ks^2),\quad ,,\quad \mathrm{dn}\,(\overline{1+k}v,\ \gamma) = \frac{1-ks^2}{1+ks^2},$$

where, as before, s, c, d denote $\mathrm{sn}\,(v,\ k)$, $\mathrm{cn}\,(v,\ k)$, $\mathrm{dn}\,(v,\ k)$: these last are, in fact, the formulæ of the second line of the table, *Elliptic Functions*, p. 183.

Fixed Curve the Cubic $y^2 = x\,(1-x)\,(1-k^2x)$: *the Function* $\{01'\theta\}$. Art. Nos. 138 to 142.

138. It was shown, No. 65, how for the affected theorem, when the fixed curve is a cubic, the form of differential is $d\omega$ multiplied by

$$\{012\} + \left[\int_2^1 d\omega\,\partial_3\,\{036\} - \{123\}\right],$$

the last term being the properly determined constant K, attached to the variable term $\{012\}$, in order to obtain a standard form of integral. The object of the present articles is to show what these formulæ become for the before-mentioned form of cubic curve $y^2 = x\,(1-x)\,(1-k^2x)$, which is most directly connected with elliptic functions: and to exhibit the connexion of the formulæ with the ordinary formulæ for elliptic integrals of the second and the third kinds respectively.

139. We have, in general,

$$\{012\},\ = \frac{\widetilde{012}}{\overline{012}},\ = \left\{\begin{array}{l} (1+k^2)\,x_1x_2 - (x_1+x_2) + y_1y_2 \\ + x\,[-1 + (1+k^2)\,(x_1+x_2) \\ + y\,(y_1+y_2) \end{array}\right\} \div \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix}.$$

Taking here 2, $=\theta$, for the point, coordinates x_2, $y_2=0$, 0, we have

$$\{01\theta\}=\frac{-x_1-x+(1+k^2)\,xx_1+yy_1}{xy_1-x_1y};$$

and if, retaining 1 to denote the point coordinates (x_1, y_1) belonging to the argument u_1, we write 1′ for the point belonging to the argument $u+iK'$, then the coordinates of 1′ are $\frac{1}{k^2x_1}$, $\frac{-y_1}{k^2x_1^2}$, and the formula becomes

$$\{01'\theta\}=\frac{x_1+k^2xx_1^2-(1+k^2)\,xx_1}{xy_1+x_1y};$$

the numerator hereof multiplied by x is $=y(xy_1+x_1y)-k^2x^2x_1(x-x_1)$, and we thence have

$$\{01'\theta\}=\frac{y}{x}-\frac{k^2xx_1(x-x_1)}{xy_1-x_1y},$$

which, substituting for x, y, x_1, y_1 their values in terms of u, u_1, is

$$=\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}-k^2\operatorname{sn}u\operatorname{sn}u_1\operatorname{sn}(u-u_1).$$

Operating on each side with $\frac{d}{du_1}$, $=\partial_1$, we obtain

$$\partial_1\{01'\theta\}=k^2\operatorname{sn}^2u_1-k^2\operatorname{sn}^2(u-u_1),$$

the differentiation being, in fact, that which occurs in establishing the fundamental property of the elliptic integral of the second kind

$$Zu=u\left(1-\frac{E}{K}\right)+k^2\int_0\operatorname{sn}^2u\,du:$$

viz. we have

$$Zu-Zu_1-Z(u-u_1)=-k^2\operatorname{sn}u\operatorname{sn}u_1\operatorname{sn}(u-u_1),$$

and thence

$$\partial_1[-k^2\operatorname{sn}u\operatorname{sn}u_1\operatorname{sn}(u-u_1)]=-Z'u_1+Z'(u-u_1),\ =k^2\operatorname{sn}^2u_1-k^2\operatorname{sn}^2(u-u_1).$$

Observe that, 1′ referring to u_1+ik', the subscript 1 might be written 1′.

The same result should of course be obtainable by the differentiation of the expression of $\{01'\theta\}$ in terms of x, y, x_1, y_1. We have

$$\partial_1x_1=2y_1,\ \partial_1y_1=1-2(1+k^2)\,x_1+3k^2x_1^2,\ =\Omega_1;$$

and we thence obtain

$$\partial_1\{01'\theta\}=\frac{k^2}{(xy_1+x_1y)^2}[-2(xy_1+x_1y)(x-2x_1)\,xy_1+xx_1(x-x_1)(x\Omega_1+2yy_1)],$$

where the term in [ ] is found to be $=x(xy_1+x_1y)^2-xx_1(x-x_1)^2$; whence the equation is

$$\partial_1\{01'\theta\}=k^2x-\frac{k^2xx_1(x-x_1)^2}{(xy_1+x_1y)^2},$$

giving the foregoing result.

140. To introduce into the formulæ 1 instead of 1′, we have only to write $u_1 - iK'$ instead of u_1; putting also for shortness s, c, d, s_1, c_1, d_1 for the functions of u and u_1 respectively, we thus obtain

$$\{01\theta\} = -\frac{cd}{s} + \frac{s}{s_1 \operatorname{sn}(u-u_1)},$$

where observe that, interchanging u and u_1, we have

$$\{10\theta\} = -\frac{c_1 d_1}{s_1} - \frac{s_1}{s \operatorname{sn}(u-u_1)},$$

that is, $\{10\theta\} = -\{01\theta\}$, as it should be. The formulæ may be written

$$\{01\theta\}, = -\{10\theta\}, = \frac{s^3 c_1 d_1 + s_1^3 cd}{s c_1 d_1 + s_1 cd} \frac{1}{s s_1 \operatorname{sn}(u-u_1)}:$$

and

$$\partial_1 \{01\theta\} = \frac{1}{s_1^2} - \frac{1}{\operatorname{sn}^2(u-u_1)},$$

whence

$$\partial_0 \{01\theta\} = -\frac{1}{s^2} + \frac{1}{\operatorname{sn}^2(u-u_1)};$$

we have, moreover,

$$\{012\} = \{12\theta\} + \frac{s}{s_1 \operatorname{sn}(u-u_1)} - \frac{s}{s_2 \operatorname{sn}(u-u_1)},$$

and

$$\partial_0 \{012\} = \frac{1}{\operatorname{sn}^2(u-u_1)} - \frac{1}{\operatorname{sn}^2(u-u_2)},$$

which last equation gives $(\partial_0 + \partial_1 + \partial_2)\{012\} = 0$, as it should do.

141. Supposing that the differential $d\Pi_{12}$ is defined by the equation

$$d\Pi_{12} = du\,\{012\} + du\left[\int_2^1 du\,\partial_3\{036\} - \{123\}\right],$$

we have

$$\int_5^4 d\Pi_{12} = \int_5^4 du\,\{012\} + \int_5^4 du\left[\int_2^1 du\,\partial_3\{036\} - \{123\}\right],$$

and thence

$$\begin{aligned}
\partial_1\partial_4 \int_5^4 d\Pi_{12} &= \partial_1\{124\} + \partial_3\{136\} - \partial_1\{123\},\\
&= \partial_1\{134\} - \partial_3\{316\},\\
&= \left[\frac{1}{\operatorname{sn}^2(u_1-u_3)} - \frac{1}{\operatorname{sn}^2(u_1-u_4)}\right] - \left[\frac{1}{\operatorname{sn}^2(u_3-u_1)} - \frac{1}{\operatorname{sn}^2(u_3-u_6)}\right],\\
&= \frac{1}{\operatorname{sn}^2(u_3-u_6)} - \frac{1}{\operatorname{sn}^2(u_1-u_4)},
\end{aligned}$$

or establishing between the constants u_3, u_6 the relation $\operatorname{sn}^2(u_3-u_6)=\dfrac{K}{K-E}$, this becomes

$$=1-\frac{E}{K}-k^2\operatorname{sn}^2(u_4-u_1+iK'),$$

which is

$$=\partial_2\partial_4\log\frac{\Theta(u_4-u_2+iK')\,\Theta(u_5-u_1+iK')}{\Theta(u_4-u_1+iK')\,\Theta(u_5-u_2+iK')},$$

where Θ is Jacobi's theta-function, see my *Elliptic Functions*, p. 144. The expression is, in fact, $=-\partial_1\partial_4\log\theta(u_4-u_1+iK')$, $=\phi(u_4-u_1+iK')$, if for a moment $\phi v=\partial_v^2\log\Theta v$. But we have

$$\log\Theta v=\log\sqrt{\frac{2k'K}{\pi}}+\tfrac{1}{2}v^2\left(1-\frac{E}{K}\right)-k^2\int_0\int_0\operatorname{sn}^2 v\,dv^2,$$

that is, $\phi v=1-\dfrac{E}{K}-k^2\operatorname{sn}^2 v$, and consequently we have $\phi(u_4-u_1+iK')$

$$=1-\frac{E}{K}-k^2\operatorname{sn}^2(u_4-u_1+iK').$$

142. In connexion with the same curve $y^2=x(1-x)(1-k^2x)$, we may establish the identity

$$\frac{d}{du_1}\frac{y_1}{x_1-x}+\frac{d}{du}\frac{y}{x_1-x}=k^2(x_1-x),$$

where as before x, $y=s^2$, scd, and x_1, $y_1=s_1^2$, $s_1c_1d_1$. We have

$$(x_1-x)\frac{dy_1}{du_1}-y_1\frac{dx_1}{du_1}=(s_1^2-s^2)\{1-2(1+k^2)s_1^2+3k^2s_1^4\}-2s_1^2(1-s_1^2)(1-k^2s_1^2)$$
$$=-s^2-s_1^2+2(1+k^2)s^2s_1^2-3k^2s^2s_1^4+k^2s_1^6;$$

and similarly

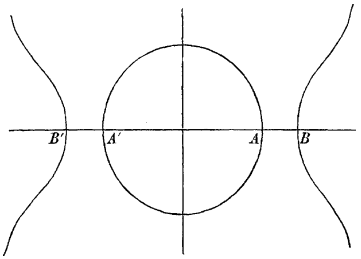
$$(x-x_1)\frac{dy}{du}-y\frac{dx}{du}=-s^2-s_1^2+2(1+k^2)s^2s_1^2-3k^2s^4s_1^2+k^2s^6.$$

The difference of the two functions on the right-hand side is $=k^2(s_1^2-s^2)^2$; which is $=k^2(x_1-x)^3$, and this divided by $(x_1-x)^2$ is $=k^2(x_1-x)$; the identity is thus verified.

Fixed Curve the Quartic $y^2=(1-x^2)(1-k^2x^2)$. Art. Nos. 143 to 145.

143. This is a curve having a tacnode at infinity on the line $x=0$, as may be seen by writing the equation in the homogeneous form $y^2z^2=(z^2-x^2)(z^2-k^2x^2)$; we have as it were two branches having the line infinity for a common tangent at the point in question. The equation is satisfied by $x=\operatorname{sn}u$, $y=\operatorname{cn}u\operatorname{dn}u$, values which are unaltered by the change of u into $u+4mK+2m'iK'$, m and m' any positive or negative integers; in regard to this curve, the sign $\equiv$ is to be understood accordingly. I consider with

reference to this curve only the affected theorem, in the particular form in which it most readily connects itself with the ordinary theory of the integral of the third kind.



144. I consider the differential $\dfrac{(x, y, z)^2_{12}\, d\omega}{012}$ in the particular case where the line 012 is a line parallel to the axis of y: taking its equation to be $x - x_1 = 0$, and putting for shortness $X = \sqrt{1 - x^2 . 1 - k^2x^2}$, $X_1 = \sqrt{1 - x_1^2 . 1 - k^2x_1^2}$, the parametric points are taken to be $(x_1, \sqrt{X_1})$, $(x_1, -\sqrt{X_1})$, and the residues are the intersections with the two branches at the tacnode. The conic $(x, y, z)^2_{12} = 0$ is to satisfy the conditions of passing through the two nodes of the tacnode, and through the two residues, that is, again through the tacnode twice—in all, four conditions; and we have thus the form $z(x - \theta z) = 0$, containing the arbitrary constant θ. The major function itself is then easily determined, and putting again $z = 1$ we arrive at the form

$$\frac{\sqrt{X_1}}{x_1 - \theta}\, \frac{x - \theta}{x - x_1}\, \frac{dx}{\sqrt{X}}.$$

If the limits are taken to be two points on a line parallel to the axis of x, or what is the same thing, if the limits in regard to x are x, $-x$, we have the integral

$$\int_{-x}^{x} \frac{\sqrt{X_1}}{x_1 - \theta}\, \frac{x - \theta}{x - x_1}\, \frac{dx}{\sqrt{X}}, \quad = \int_0^x \frac{\sqrt{X_1}}{x_1 - \theta} \left(\frac{x - \theta}{x - x_1} + \frac{x + \theta}{x + x_1}\right) \frac{dx}{\sqrt{X}},$$

$$= 2\int_0^x \frac{\sqrt{X_1}}{x_1 - \theta} \cdot \frac{x^2 - x_1\theta}{x^2 - x_1^2}\, \frac{dx}{\sqrt{X}}.$$

We have

$$\frac{1}{x_1}\left(\frac{x^2}{x_1^2 - x^2} - \frac{\theta}{x_1 - \theta}\right) = \frac{x^2 - x_1\theta}{x_1^2 - x^2 . x_1 - \theta},$$

and the integral thus becomes

$$= -2\int_0^x \frac{\sqrt{X_1}\, x^2 dx}{x_1(x_1^2 - x^2)\sqrt{X}} + 2\, \frac{\theta\sqrt{X_1}}{x_1(x_1 - \theta)} \int_0^x \frac{dx}{\sqrt{X}}.$$

Taking here $x = \operatorname{sn} u$, $x_1 = \operatorname{sn}(a + iK')$, $= \dfrac{1}{k \operatorname{sn} a}$, we have $dx = \operatorname{cn} u \operatorname{dn} u\, du = \sqrt{X}\, du$, and the result is

$$\int_{-x}^{x} \frac{\sqrt{X_1}}{x_1 - \theta} \frac{x - \theta}{x - x_1} \frac{dx}{\sqrt{X}} = -2 \int_0 \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u\, du}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} + \frac{2k \operatorname{sn} a \operatorname{cn} a}{1 - k\theta \operatorname{sn} a} u,$$

where on the right-hand side the first term is $= -2\Pi(u, a)$, if $\Pi(u, a)$ be Jacobi's form of the integral of the third kind, see my *Elliptic Functions*, p. 143.

145. It is to be observed that the proper normal form is not $\Pi(u, a)$, but $\Pi(u, a) - uZa$; say this is $\overline{\Pi}(u, a)$. We then have

$$\overline{\Pi}(u, a) = \Pi(u, a) - u\left[a\left(1 - \frac{E}{K}\right) - k^2 \int \operatorname{sn}^2 a\, da\right],$$

and thence

$$\partial_u \overline{\Pi}(u, a) = \frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 u}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} - a\left(1 - \frac{E}{K}\right) + k^2 \int \operatorname{sn}^2 a\, da,$$

$$\partial_a \partial_u \overline{\Pi}(u, a) = k^2 \operatorname{sn}^2 u \partial_a \frac{\operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u} - \left(1 - \frac{E}{K}\right) + k^2 \operatorname{sn}^2 a;$$

or, if for shortness we write $\operatorname{sn} u$, $\operatorname{sn} a = s$, σ, this is

$$\partial_a \partial_u \overline{\Pi}(u, a) = \frac{k^2 s^2 [1 - 2(1 + k^2)\sigma^2 + 3k^2\sigma^4] + k^4 s^4 \sigma^2}{(1 - k^2 s^2 \sigma^2)^2} - \left(1 - \frac{E}{K}\right) + k^2 \sigma^2,$$

which is

$$= \frac{k^2 \{(s^2 + \sigma^2)(1 + k^2 s^2 \sigma^2) - 2(1 + k^2) s^2 \sigma^2\}}{(1 - k^2 s^2 \sigma^2)^2} - \left(1 - \frac{E}{K}\right);$$

or, this being symmetrical in regard to s, σ, we have

$$\partial_a \partial_u \overline{\Pi}(u, a) = \partial_a \partial_u \overline{\Pi}(a, u),$$

and thence by integration, and a proper determination of the constants,

$$\overline{\Pi}(u, a) = \overline{\Pi}(a, u).$$

Chapter VI. The Nodal Quartic.

Nodal Quartic; the General and Fleflecnodal Forms. Art. Nos. 146 to 148.

146. For a cubic, or other curve of deficiency 1, we are concerned with single points on the curve, and corresponding thereto with functions of a single argument (elliptic functions): but for a curve of deficiency 2, we have to consider pairs of points on the curve, and functions of two arguments: there is thus a marked change in the form of the results.

The most simple curve of deficiency 2 is the nodal quartic, $n=4$, $p=2$. Using homogeneous coordinates, the general form is $Az^2+2Bz+C=0$, where

$$\begin{aligned} A &= \quad (i, j, k \between x, y)^2, \\ B &= \quad (l, m, n, o \between x, y)^3, \\ C &= (p, q, r, s, t \between x, y)^4, \end{aligned}$$

and where we write also

$$B^2-AC=(l^2-ip)(x-ay)(x-by)(x-cy)(x-dy)(x-ey)(x-fy).$$

Clearly the equation of the two tangents at the node is $A=0$; and the equations of the six tangents from the node are $x-ay=0, \ldots, x-fy=0$: at the points of contact we have $Az+B=0$, viz. this is the equation of a nodal cubic, the node and the two tangents there being the same with the node and two tangents of the quartic. Hence the node counts as 6 intersections, and there are besides 6 intersections which are the points of contact of the 6 tangents respectively: say these are the points a, b, c, d, e, f: the coordinates of the point a are given by the equations

$$x : y : z = a : 1 : -\frac{B_a}{A_a}\left(=-\frac{C_a}{B_a}\right),$$

where A_a, B_a, C_a are what A, B, C become on writing therein a, 1 for x, y: and similarly for the other points.

147. An important special case occurs when $B=0$; say we have here

$$\begin{aligned} A &= i(x-ey)(x-fy), \\ B &= 0, \\ C &= p(x-ay)(x-by)(x-cy)(x-dy), \end{aligned}$$

or, omitting the factors i and p,

$$(x-ey)(x-fy)z^2=(x-ay)(x-by)(x-cy)(x-dy).$$

The origin is here a fleflecnode; the tangents $x-ey=0$, $x-fy=0$ count as two of the six tangents from the node, and there remain the four tangents

$$x-cy=0, \quad x-dy=0, \quad x-ay=0, \quad x-by=0;$$

the four points of contact are the intersections of the curve with the line $z=0$.

148. The general nodal form depends on 11 constants, but by writing $\alpha x+\beta y$, $\gamma x+\delta y$, ϵz in place of x, y, z, we introduce 5 apoclastic constants, and so reduce the number to $11+1-5, =7$. Similarly the fleflecnodal form depends on 7 constants, but we reduce the number in like manner to $7+1-5, =3$: the final form might here be taken to be

$$z^2xy=(x-y)(x-by)(x-cy)(x-dy),$$

but it is more convenient to retain the original form

$$z^2(x-ey)(x-fy)=(x-ay)(x-by)(x-cy)(x-dy),$$

bearing in mind that this is reducible to the form just referred to, and thus depends virtually upon only 3 constants.

It is a general property that a curve of deficiency p greater than 1 can be by a rational transformation reduced to a curve of that deficiency depending upon $3p-3$ parameters: in particular, if $p=2$, then the form depending upon 3 parameters may be taken to be the fleflecnodal quartic as above: and I proceed to show how the general nodal quartic can, in fact, be reduced to this fleflecnodal form.

Reduction to the Fleflecnodal Form. Art. Nos. 149 to 152.

149. Consider the general nodal quartic $Az^2+2Bz+C=0$: take $\zeta=0$ for the equation of the line joining the points of contact of the tangents $x-ey=0$, $x-fy=0$; and then writing $x=\xi$, $y=\eta$, let the curve be transformed in the first instance from x, y, z to the new coordinates ξ, η, ζ.

Writing A_e for the value $(i, j, k \between e, 1)^2$, which A assumes on putting therein $(e, 1)$ for (x, y) respectively, and similarly A_f, B_e, B_f for the other like values, we may take

$$A_eA_f(e-f)\zeta=\begin{vmatrix} x\ , & y\ , & z \\ eA_e, & A_e, & -B_e \\ fA_f, & A_f, & -B_f \end{vmatrix},$$

$$=-x(A_eB_f-A_fB_e)+y(eA_eB_f-fB_eA_f)+z(e-f)A_eA_f,$$

say this equation is $\zeta=-\lambda x-\mu y+z$, the values of λ, μ being

$$\lambda=\frac{A_eB_f-A_fB_e}{(e-f)A_eA_f}, \qquad \mu=\frac{-eA_eB_f+fB_eA_f}{(e-f)A_eA_f},$$

and therefore

$$\lambda e+\mu=-\frac{B_e}{A_e}, \qquad \lambda f+\mu=-\frac{B_f}{A_f}.$$

150. From the values $\zeta, \xi, \eta=-\lambda x-\mu y+z, x, y$, we obtain $z, x, y=\zeta+\lambda\xi+\mu\eta, \xi, \eta$; and the transformed equation is

$$A'(\zeta+\lambda\xi+\mu\eta)^2+2B'(\zeta+\lambda\xi+\mu\eta)+C'=0,$$

where

$$\begin{aligned} A' &= \quad (i, j, k \between \xi, \eta)^2, \\ B' &= (l, m, n, o \between \xi, \eta)^3, \\ C' &= (p, q, r, s, t \between \xi, \eta)^4, \end{aligned}$$

say this equation is $\mathfrak{A}\zeta^2+2\mathfrak{B}\zeta+\mathfrak{C}=0$, where

$$\begin{aligned} \mathfrak{A} &= A', \\ \mathfrak{B} &= A'(\lambda\xi+\mu\eta)+B', \\ \mathfrak{C} &= A'(\lambda\xi+\mu\eta)^2+2B'(\lambda\xi+\mu\eta)+C', \end{aligned}$$

and thence

$$\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} = B'^2 - A'C', \; = (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta).$$

We have here $\mathfrak{B}, = A'(\lambda\xi + \mu\eta) + B'$, a cubic function $(\xi, \eta)^3$ containing the factors $\xi - e\eta$, and $\xi - f\eta$: in fact, writing $\xi, \eta = e, 1$, it becomes $A_e(\lambda e + \mu) + B_e$, which is $= 0$; and similarly writing $\xi, \eta = f, 1$, it becomes $A_f(\lambda f + \mu) + B_f$, which is $= 0$. Calling the other factor $L\xi + M\eta$, we have thus

$$\mathfrak{B} = (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta),$$

and thence

$$\begin{aligned}\mathfrak{A}\mathfrak{C} &= (\xi - e\eta)^2(\xi - f\eta)^2(L\xi + M\eta)^2 - (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta),\\ &= (\xi - e\eta)(\xi - f\eta)[(\xi - e\eta)(\xi - f\eta)(L\xi + M\eta)^2 - (l^2 - ip)(\xi - c\eta)(\xi - d\eta)(\xi - e\eta)(\xi - f\eta)].\end{aligned}$$

Hence $\mathfrak{C}$ contains the factor $(\xi - e\eta)(\xi - f\eta)$, say we have

$$\mathfrak{C} = \theta(\xi - e\eta)(\xi - f\eta)(\xi - \epsilon\eta)(\xi - \phi\eta).$$

151. In the equation $\mathfrak{A}\zeta^2 + 2\mathfrak{B}\zeta + \mathfrak{C} = 0$ of the quartic curve, writing $\zeta = 0$, we find $\mathfrak{C} = 0$, that is, $(\xi - e\eta)(\xi - f\eta)(\xi - \epsilon\eta)(\xi - \phi\eta) = 0$: but $\zeta = 0$ is the equation of the line joining the points of contact of the tangents $\xi - e\eta = 0$, $\xi - f\eta = 0$; hence $\xi - \epsilon\eta = 0$, $\xi - \phi\eta = 0$ are the lines drawn from the node to the two points ϵ, ϕ which are the residues of these two points of contact. We now have

$$\theta\mathfrak{A}(\xi - \epsilon\eta)(\xi - \phi\eta) = (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta)^2 - (l^2 - ip)(\xi - a\eta)(\xi - b\eta)(\xi - c\eta)(\xi - d\eta),$$

and thence

$$\begin{aligned}0 &= (\epsilon - e)(\epsilon - f)(L\epsilon + M)^2 - (l^2 - ip)(\epsilon - a)(\epsilon - b)(\epsilon - c)(\epsilon - d),\\ 0 &= (\phi - e)(\phi - f)(L\phi + M)^2 - (l^2 - ip)(\phi - a)(\phi - b)(\phi - c)(\phi - d),\end{aligned}$$

which equations determine L and M; and then with these values of L, M, and for $\mathfrak{A}$ substituting its value $(i, j, k\between\xi, \eta)^2$, the equation must become an identity.

We have in what precedes, by the transformation $z = \zeta + \lambda\xi + \mu\eta$, $x = \xi$, $y = \eta$, passed from the form $Az^2 + 2Bz + C = 0$ to the form

$$\mathfrak{A}\zeta^2 + 2\mathfrak{B}\zeta + \mathfrak{C} = 0,$$

where

$$\begin{aligned}\mathfrak{A} &= (i, j, k)(\xi, \eta)^2,\\ \mathfrak{B} &= (\xi - e\eta)(\xi - f\eta)(L\xi + M\eta),\\ \mathfrak{C} &= \theta(\xi - e\eta)(\xi - f\eta)(\xi - \epsilon\eta)(\xi - \phi\eta),\end{aligned}$$

viz. $\mathfrak{B}$ and $\mathfrak{C}$ have here the common factor $(\xi - e\eta)(\xi - f\eta)$.

152. Assume now

$$\xi, \eta, \zeta = X, Y, \theta\frac{(X - \epsilon Y)(X - \phi Y)}{Z - LX - MY},$$

and therefore conversely

$$X, Y, Z = \xi, \eta, L\xi + M\eta + \theta\frac{(\xi - \epsilon\eta)(\xi - \phi\eta)}{\zeta};$$

then in the new coordinates (X, Y, Z) we have the equation

$$(i, j, k)(X, Y)^2\theta^2\left\{\frac{(X-\epsilon Y)(X-\phi Y)}{Z-LX-MY}\right\}^2$$

$$+2(X-eY)(X-fY)(LX+MY)\theta\frac{(X-\epsilon Y)(X-\phi Y)}{Z-LX-MY}$$

$$+\theta(X-eY)(X-fY)(X-\epsilon Y)(X-\phi Y)=0,$$

that is,

$$(i, j, k)(X, Y)^2\theta(X-\epsilon Y)(X-\phi Y)$$

$$+2(X-eY)(X-fY)(LX+MY)(Z-LX-MY)$$

$$+\ (X-eY)(X-fY)(Z-LX-MY)^2=0,$$

where the second and third lines together are

$$=(X-eY)(X-fY)\{Z^2-(LX+MY)^2\},$$

and the equation thus is

$$(X-eY)(X-fY)Z^2+\{\theta(i, j, k)(X, Y)^2(X-\epsilon Y)(X-\phi Y)$$
$$-(X-eY)(X-fY)(LX+MY)^2\}=0.$$

But the term in { } is identically

$$=-(l^2-ip)(X-aY)(X-bY)(X-cY)(X-dY),$$

and the equation thus becomes

$$(X-eY)(X-fY)Z^2-(l^2-ip)(X-aY)(X-bY)(X-cY)(X-dY)=0;$$

viz. the original equation $Az^2+2Bz+C=0$ of the general nodal quartic is, by the equations

$$x\ ,\ y\ ,\ z=X,\ Y,\ \lambda X+\mu Y+\theta\frac{(X-\epsilon Y)(X-\phi Y)}{Z-LX-MY},$$

or conversely

$$X,\ Y,\ Z=x\ ,\ y\ ,\ Lx+My+\frac{\theta(x-\epsilon y)(x-\phi y)}{z-\lambda x-\mu y},$$

transformed into the fleflecnodal form as above.

It originally appeared to me that the fleflecnodal form was more easily dealt with than the general form; and I effected the transformation for this reason: there is, however, the disadvantage that the six points a, b, c, d, e, f enter into the equation unsymmetrically; and I afterwards found that the general form could be dealt with nearly as easily, and in what follows I use therefore the general form. The transformation is given as interesting for its own sake, and as an illustration of the theorem in regard to the number of constants in a curve of deficiency p.

Application of Abel's Theorem. Art. Nos. 153 to 157.

153. Taking the fixed curve to be $f, = \frac{1}{2}(Az^2 + 2Bz + C), = 0$, we have

$$\frac{df}{dz} = Az + B = \sqrt{(x, y)^6},$$

if for shortness we write

$$(x, y)^6 = B^2 - AC, = (l^2 - ip)(x - ay)(x - by)(x - cy)(x - dy)(x - ey)(x - fy),$$

and we thence have

$$d\omega = \frac{x\,dy - y\,dx}{\sqrt{(x, y)^6}}.$$

The minor curve $(x, y, z)^{n-3} = 0$ is an arbitrary line passing through the node, that is, the point $x = 0$, $y = 0$; and the pure theorem thus gives the two relations $\Sigma x\,d\omega = 0$, $\Sigma y\,d\omega = 0$; where the summation extends to the intersections of the fixed curve $Az^2 + 2Bz + C = 0$ with the variable curve ϕ.

The variable curve is taken to be a cubic $Az + B = (\alpha, \beta, \gamma, \delta \between x, y)^3$, or say $Az + B = \Omega$, where Ω is a given cubic function $(x, y)^3$: viz. this is a nodal cubic, the node and the two tangents there being the same with the node and the two tangents of the quartic: hence it meets the quartic in the node counting 6 times, and in 6 other points, say these are the points 1, 2, 3, 4, 5, 6: hence the differential relations are

$$x_1 d\omega_1 + x_2 d\omega_2 + x_3 d\omega_3 + x_4 d\omega_4 + x_5 d\omega_5 + x_6 d\omega_6 = 0,$$

$$y_1 d\omega_1 + y_2 d\omega_2 + y_3 d\omega_3 + y_4 d\omega_4 + y_5 d\omega_5 + y_6 d\omega_6 = 0.$$

154. Observe that the intersections of the cubic with the fixed curve are given by the equation $\Omega^2 = B^2 - AC$, or say $\Omega^2 = (x, y)^6$, an equation which determines the ratio $x : y$ for the six points respectively; and the ratio $z : x$ is then determined rationally by the original equation $Az + B = \Omega$. Instead of regarding Ω as a given function, we may, if we please, take 1, 2, 3, 4 given points on the quartic: we then have four equations for the determination of the coefficients $(\alpha, \beta, \gamma, \delta)$ of the function Ω; viz. these equations may be taken to be

$$(\alpha, \beta, \gamma, \delta)(x_1, y_1)^3 = \sqrt{(x_1, y_1)^6},$$

$$(\quad ,, \quad)(x_2, y_2)^3 = \sqrt{(x_2, y_2)^6},$$

$$(\quad ,, \quad)(x_3, y_3)^3 = \sqrt{(x_3, y_3)^6},$$

$$(\quad ,, \quad)(x_4, y_4)^3 = \sqrt{(x_4, y_4)^6}.$$

Ω is hereby completely determined: and this being so, the remaining intersections 5 and 6 are also completely determined: there are thus between the six points 2 integral relations, which agrees with the number, $= 2$, of the differential relations obtained above.

155. If we now assume

$$du = x_1 d\omega_1 + x_2 d\omega_2, \quad du' = x_3 d\omega_3 + x_4 d\omega_4, \quad du'' = x_5 d\omega_5 + x_6 d\omega_6,$$
$$dv = y_1 d\omega_1 + y_2 d\omega_2, \quad dv' = y_3 d\omega_3 + y_4 d\omega_4, \quad dv'' = y_5 d\omega_5 + y_6 d\omega_6,$$

or say

$$u,\ u',\ u'' = \int_{\alpha\beta}^{12},\ \int_{\alpha\beta}^{34},\ \int_{\alpha\beta}^{56} \frac{x\,(x\,dy - y\,dx)}{\sqrt{(x,\ y)^6}},$$

$$v,\ v',\ v'' = \int_{\alpha\beta}^{12},\ \int_{\alpha\beta}^{34},\ \int_{\alpha\beta}^{56} \frac{y\,(x\,dy - y\,dx)}{\sqrt{(x,\ y)^6}};$$

that is,

$$u = \left(\int_\alpha^1 + \int_\beta^2\right) \frac{x\,(x\,dy - y\,dx)}{\sqrt{(x,\ y)^6}}, \quad v = \left(\int_\alpha^1 + \int_\beta^2\right) \frac{y\,(x\,dy - y\,dx)}{\sqrt{(x,\ y)^6}},$$

where α, β are points assumed at pleasure on the quartic: and similarly for u', v': u'', v'': then u, v are hereby determined as functions of the points 1, 2: and we may conversely regard the points 1, 2 as determined in terms of the two arguments u, v. We might, selecting any two symmetrical functions of the degree zero, for instance, $\frac{x_1}{y_1} + \frac{x_2}{y_2}$, $\frac{x_1 x_2}{y_1 y_2}$, represent them as functions $\phi(u, v)$, $\psi(u, v)$; and then $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ will be functions of $\phi(u, v)$, $\psi(u, v)$. But instead of this selection, it is proper to consider the ratios of six functions depending on the points a, b, c, d, e, f respectively: viz. we assume

$$\begin{aligned} &\sqrt{(x_1 - ay_1)(x_2 - ay_2)} : \sqrt{(x_1 - by_1)(x_2 - by_2)} :\ldots: \sqrt{(x_1 - fy_1)(x_2 - fy_2)} \\ = \quad & A(u, v) \quad : \quad B(u, v) \quad :\ldots: \quad F(u, v), \end{aligned}$$

and of course 3, 4 will be in like manner determined by means of the corresponding functions of u', v', and 5, 6 by means of the corresponding functions of u'', v''. The squared functions $A^2, B^2, C^2, D^2, E^2, F^2$ are proportional to given linear functions of x_1x_2, $x_1y_2 + x_2y_1$, y_1y_2, and are thus connected by three independent linear relations.

156. The differential relations then become

$$du + du' + du'' = 0, \quad dv + dv' + dv'' = 0,$$

and we have consequently

$$u + u' + u'' = I, \quad v + v' + v'' = J,$$

where I, J are constants which are determinable as definite integrals by the consideration that, when the cubic is taken to be $Az + B = 0$, the six points 1, 2, 3, 4, 5, 6 coincide with the points of contact a, b, c, d, e, f. I do not at present see my way to a proper development of this point of the theory: but in explanation of the nature of the result, I assume for the moment that by a proper determination of the inferior limits α, β, or otherwise, we may take $I = 0$, $J = 0$. We then have $u'' = -u - u'$, $v'' = -v - v'$; and the integral equations, which determine the points 5, 6 in terms of the points 1, 2 and the points 3, 4, then in effect determine the functions A, B, &c., of $-u - u'$, $-v - v'$, or say those of $u + u'$, $v + v'$ in terms of the like functions of (u, v) and of (u', v'): viz. these equations give the addition-theory of the functions $A(u, v)$, &c.

157. We may, in the first instance, disregarding altogether the consideration of the arguments u, v, &c., attend only to the algebraic functions such as $\sqrt{(x_1-ay_1)(x_2-ay_2)}$, &c., of the coordinates of the pairs of points 1, 2; 3, 4, and 5, 6; and we can in regard to these develope a proper theory. This depends only on the equation $\Omega=\sqrt{(x,\ y)^6}$; it will be convenient to assume herein $y=1$, and slightly modifying the form, to write it

$$(\alpha,\ \beta,\ \gamma,\ \delta)(x,\ 1)^3=\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x};$$

and accordingly to consider the functions $\sqrt{a-x_1\,.\,a-x_2}$, &c. These are called the single-letter functions A, &c., but there are certain double-letter functions AB, &c., which have also to be considered; and I will, in the first instance, show how these present themselves in connexion with the cubic curve.

Origin of the Double-Letter Functions. Art. Nos. 158 and 159.

158. The cubic curve $Az+B=\Omega$ may be taken to be a curve through two of the points of contact, say the points a, b; these will then be two out of the six points, and taking the remaining four points to be the pairs 1, 2 and 3, 4, we have single-letter functions of 3, 4 presenting themselves as double-letter functions of 1, 2. In fact, the equation of the curve is

$$Az+B=\lambda(x-ay)(x-by)(x-ky);$$

for the intersections with the quartic we have $\lambda^2(x-ay)^2(x-by)^2(x-ky)^2=\Omega^2$, or throwing out the factor $(x-ay)(x-by)$ and changing the constant λ, this is

$$(x-ay)(x-by)(x-ky)^2-\lambda(x-cy)(x-dy)(x-ey)(x-fy)=0;$$

and the quartic function must be a multiple of

$$(xy_1-x_1y)(xy_2-x_2y)(xy_3-x_3y)(xy_4-x_4y).$$

Putting each of the y's equal 1, we have the identity

$$(a-x)(b-x)(k-x)^2-\lambda(c-x)(d-x)(e-x)(f-x)=\mu(x_1-x)(x_2-x)(x_3-x)(x_4-x);$$

and hence, introducing a notation which will be convenient, $a-x=\mathrm{a}$, $a-x_1=\mathrm{a}_1$, and so in other cases, we have by giving different values to x the equations

$$\begin{aligned}
\mathrm{a}_1\mathrm{b}_1\mathrm{k}_1^2&=\lambda\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1, & (a-c)(b-c)(k-c)^2&=\mu\mathrm{c}_1\mathrm{c}_2\mathrm{c}_2\mathrm{c}_4,\\
\mathrm{a}_2\mathrm{b}_2\mathrm{k}_2^2&=\lambda\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2\mathrm{f}_2, & (a-d)(b-d)(k-d)^2&=\mu\mathrm{d}_1\mathrm{d}_2\mathrm{d}_3\mathrm{d}_4,\\
\mathrm{a}_3\mathrm{b}_3\mathrm{k}_3^2&=\lambda\mathrm{c}_3\mathrm{d}_3\mathrm{e}_3\mathrm{f}_3, & (a-e)(b-e)(k-e)^2&=\mu\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_4,\\
\mathrm{a}_4\mathrm{b}_4\mathrm{k}_4^2&=\lambda\mathrm{c}_4\mathrm{d}_4\mathrm{e}_4\mathrm{f}_4, & (a-f)(b-f)(k-f)^2&=\mu\mathrm{f}_1\mathrm{f}_2\mathrm{f}_3\mathrm{f}_4,\\
& & -\lambda(c-a)(d-a)(e-a)(f-a)&=\mu\mathrm{a}_1\mathrm{a}_2\mathrm{a}_3\mathrm{a}_4,\\
& & -\lambda(c-b)(d-b)(e-b)(f-b)&=\mu\mathrm{b}_1\mathrm{b}_2\mathrm{b}_3\mathrm{b}_4.
\end{aligned}$$

We have thus

$$\frac{(a-c)(b-c)}{(a-d)(b-d)}\left(\frac{c-k}{d-k}\right)^2=\frac{c_1c_2c_3c_4}{d_1d_2d_3d_4},$$

and

$$\frac{k_1^2}{k_2^2}=\frac{a_2b_2c_1d_1e_1f_1}{a_1b_1c_2d_2e_2f_2},$$

which last equation, writing for a moment γ, $\delta=\sqrt{a_2b_2c_1d_1e_1f_1}$, $\sqrt{a_1b_1c_2d_2e_2f_2}$, gives $\frac{k-x_1}{k-x_2}=\frac{\gamma}{\delta}$, whence $k(\gamma-\delta)=x_2\gamma-x_1\delta$, and thence

$$\frac{c-k}{d-k}=\frac{\gamma c_2-\delta c_1}{\gamma d_2-\delta d_1}=\frac{\sqrt{c_1c_3}\,\{\sqrt{a_2b_2c_2d_1e_1f_1}-\sqrt{a_1b_1c_1d_2e_2f_2}\}}{\sqrt{d_1d_2}\,\{\sqrt{a_2b_2d_2c_1e_1f_1}-\sqrt{a_1b_1d_1c_2e_2f_2}\}},$$

or, substituting in the first equation, we have

$$\frac{\sqrt{(a-c)(b-c)}}{\sqrt{(a-d)(b-d)}}\cdot\frac{\sqrt{a_2b_2c_2d_1e_1f_1}-\sqrt{a_1b_1c_1d_2e_2f_2}}{\sqrt{a_2b_2d_2c_1e_1f_1}-\sqrt{a_1b_1d_1c_2e_2f_2}}=\frac{\sqrt{c_3c_4}}{\sqrt{d_3d_4}}.$$

159. Considering the duad DE as an abbreviation for the double triad $ABC.DEF$, the expressed duad being always accompanied by the letter F, we are thus led to the consideration of the double-letter functions

$$AB_{12}=\frac{1}{x_1-x_2}\{\sqrt{a_1b_1f_1c_2d_2e_2}-\sqrt{a_2b_2f_2c_1d_1e_1}\},\ \&c.,$$

in connexion with the already mentioned single-letter functions $A_{12}=\sqrt{a_1a_2}$, &c., viz. in this notation the equation just obtained is

$$\frac{C_{34}}{D_{34}}=\sqrt{\frac{(a-c)(b-c)}{(a-d)(b-d)}}\,\frac{DE_{12}}{CE_{12}},$$

and it thus appears that, the points 3, 4 being obtained as above from the given points 1, 2, then the quotient of two of the single-letter functions of 3, 4 is a constant multiple of the quotient of two of the double-letter functions of 1, 2. Observe that the points 3, 4 are derived from 1, 2 by means of the two points a, b: we have DE standing for $ABC.DEF$, CE for $ABD.CEF$, and if the two functions were represented by ABC, ABD respectively, then the form would have been

$$\frac{C_{34}}{D_{34}}=\sqrt{\frac{(a-c)(b-c)}{(a-d)(b-d)}}\cdot\frac{ABC_{12}}{ABD_{12}},$$

which is a clearer expression of the theorem; the apparent want of symmetry of the first form arises only from the arbitrary selection of the letter F to accompany the expressed duad, and is at once removed by substituting for a duad DE the triad $ABC.DEF$ which is thereby signified. The denominator factor x_1-x_2 is introduced in order to make the degree in x_1 or x_2 equal to that of the single-letter functions.

The Addition Theory. Art. Nos. 160 to 163.

160. We have the six single-letter symbols A, B, C, D, E, F; viz. $A_{12}=\sqrt{a_1 a_2}$, &c.: and the ten double-letter symbols $AB, AC, AD, AE, BC, BD, BE, CD, CE, DE$, viz.

$$AB_{12}=\frac{1}{x_1-x_2}\{\sqrt{a_1 b_1 f_1 c_2 d_2 e_2}-\sqrt{a_2 b_2 f_2 c_1 d_1 e_1}\},\ \&c.,$$

these 16 functions being connected by algebraical relations which are immediately deducible from these expressions of the functions in terms of x_1, x_2. The problem is to express the functions of 5, 6 in terms of those of 1, 2 and of those of 3, 4. The relation between the variables $x_1, x_2, x_3, x_4, x_5, x_6$ consists herein that we have $x_1, x_2, x_3, x_4, x_5, x_6$ as the roots of the equation

$$(\alpha x^3+\beta x^2+\gamma x+\delta)^2-\lambda(a-x)(b-x)(c-x)(d-x)(e-x)(f-x)=0;$$

or, what is the same thing, it consists in the identity

$$(\alpha x^3+\beta x^2+\gamma x+\delta)^2-\lambda(a-x)(b-x)(c-x)(d-x)(e-x)(f-x)$$
$$-\mu(x_1-x)(x_2-x)(x_3-x)(x_4-x)(x_5-x)(x_6-x)=0.$$

Again, it may be expressed by the plexus of equations

$$\left\|\begin{matrix} 1, & 1, & 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ x_1^2, & x_2^2, & x_3^2, & x_4^2, & x_5^2, & x_6^2 \\ x_1^3, & x_2^3, & x_3^3, & x_4^3, & x_5^3, & x_6^3 \\ \sqrt{X_1}, & \sqrt{X_2}, & \sqrt{X_3}, & \sqrt{X_4}, & \sqrt{X_5}, & \sqrt{X_6} \end{matrix}\right\|=0,$$

where $X_1=(a-x_1)(b-x_1)\ldots(f-x_1)$, &c.; these are equivalent of course to two equations, and serve to determine x_5, x_6 in terms of x_1, x_2, x_3, x_4.

161. The solution is, in fact, as is given in my paper "On the addition of the double ϑ-functions," *Crelle*, t. LXXXVIII. (1880), pp. 74—81, [703]. Writing successively $x=x_1, x_2, x_3, x_4$, we have

$$\alpha x_1^3+\beta x_1^2+\gamma x_1+\delta=\sqrt{\lambda}\sqrt{X_1},$$
$$\alpha x_2^3+\beta x_2^2+\gamma x_2+\delta=\sqrt{\lambda}\sqrt{X_2},$$
$$\alpha x_3^3+\beta x_3^2+\gamma x_3+\delta=\sqrt{\lambda}\sqrt{X_3},$$
$$\alpha x_4^3+\beta x_4^2+\gamma x_4+\delta=\sqrt{\lambda}\sqrt{X_4},$$

which equations serve to determine the ratios $\alpha:\beta:\gamma:\delta$ in terms of x_1, x_2, x_3, x_4; and we have then the two like equations

$$\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta=\sqrt{\lambda}\sqrt{X_5},$$
$$\alpha x_6^3+\beta x_6^2+\gamma x_6+\delta=\sqrt{\lambda}\sqrt{X_6},$$

which determine the symmetric functions of x_5, x_6.

If, reverting to the identity, we write therein for instance $x = a$, we find

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\mu}\, A_{12} A_{34} A_{56},$$

which equation when properly reduced gives the proportional value of A_{56}.

162. Calling for a moment the function on the left-hand side Ω, we have

$$\begin{vmatrix} x_1^3, & x_1^2, & x_1, & 1, & \sqrt{\lambda}\sqrt{X_1} \\ x_2^3, & x_2^2, & x_2, & 1, & \sqrt{\lambda}\sqrt{X_2} \\ x_3^3, & x_3^2, & x_3, & 1, & \sqrt{\lambda}\sqrt{X_3} \\ x_4^3, & x_4^2, & x_4, & 1, & \sqrt{\lambda}\sqrt{X_4} \\ a^3, & a^2, & a, & 1, & \Omega \end{vmatrix} = 0,$$

that is,

$$\Omega \begin{vmatrix} x_1^3, & x_1^2, & x_1, & 1 \\ x_2^3, & x_2^2, & x_2, & 1 \\ x_3^3, & x_3^2, & x_3, & 1 \\ x_4^3, & x_4^2, & x_4, & 1 \end{vmatrix} + \sqrt{\lambda} \begin{vmatrix} x_1^3, & x_1^2, & x_1, & 1, & \sqrt{X_1} \\ x_2^3, & x_2^2, & x_2, & 1, & \sqrt{X_2} \\ x_3^3, & x_3^2, & x_3, & 1, & \sqrt{X_3} \\ x_4^3, & x_4^2, & x_4, & 1, & \sqrt{X_4} \\ a^3, & a^2, & a, & 1, & 0 \end{vmatrix} = 0,$$

viz. this is

$$\begin{aligned} \Omega\,(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\ = -\sqrt{\lambda}\,\{&\sqrt{X_1}\,.\,x_2 - x_3\,.\,x_2 - x_4\,.\,x_2 - a\,.\,x_3 - x_4\,.\,x_3 - a\,.\,x_4 - a \\ &+ \sqrt{X_2}\,.\,x_3 - x_4\,.\,x_3 - a\,.\,x_3 - x_1\,.\,x_4 - a\,.\,x_4 - x_1\,.\,a - x_1 \\ &+ \sqrt{X_3}\,.\,x_4 - a\,.\,x_4 - x_1\,.\,x_4 - x_2\,.\,a - x_1\,.\,a - x_2\,.\,x_1 - x_2 \\ &+ \sqrt{X_4}\,.\,a - x_1\,.\,a - x_2\,.\,a - x_3\,.\,x_2 - x_3\,.\,x_2 - x_4\,.\,x_3 - x_4\}, \end{aligned}$$

or, as this may be written,

$$\begin{aligned} &\Omega\,.\,x_1 - x_3\,.\,x_1 - x_4\,.\,x_2 - x_3\,.\,x_2 - x_4 \\ &= \frac{\sqrt{\lambda}\,.\,a - x_3\,.\,a - x_4}{x_1 - x_2}\{x_2 - x_3\,.\,x_2 - x_4\,.\,a - x_2\,.\,\sqrt{X_1} - (x_1 - x_3\,.\,x_1 - x_4\,.\,a - x_1\,.\,\sqrt{X_2})\} \\ &\quad + \frac{\sqrt{\lambda}\,.\,a - x_1\,.\,a - x_2}{x_3 - x_4}\{x_4 - x_1\,.\,x_4 - x_2\,.\,a - x_4\,.\,\sqrt{X_3} - (x_3 - x_1\,.\,x_3 - x_2\,.\,a - x_3\,.\,\sqrt{X_4})\}. \end{aligned}$$

We have here $\sqrt{\lambda}\,.\,a - x_3\,.\,a - x_4 = \sqrt{\lambda}\,A^2_{34}$, and the function

$$\frac{1}{x_1 - x_2}\{x_2 - x_3\,.\,x_2 - x_4\,.\,\mathrm{a}_2\sqrt{X_1} - (x_1 - x_3\,.\,x_1 - x_4\,.\,\mathrm{a}_1\sqrt{X_2})\},$$

which multiplies this, is without difficulty found to be

$$= \frac{-A_{12}}{c - d\,.\,d - b\,.\,b - c}\,\Sigma\,\{c - d\,.\,B^2_{34} BE_{12}\,.\,C_{12}\,.\,D_{12}\},$$

where the summation extends to the three terms obtained by the cyclical interchange of the letters b, c, d: these being a set of three out of the five letters other than a. Similarly $\sqrt{\lambda}\,.\,a-x_1\,.\,a-x_2$ is $=\sqrt{\lambda}\,A^2_{12}$, and the function which multiplies this is

$$=\frac{-A_{34}}{c-d\,.\,d-b\,.\,b-c}\,\Sigma\,\{c-d\,.\,B^2_{12}\,.\,BE_{34}C_{14}D_{14}\}.$$

The expression for Ω thus contains the factor $A_{12}A_{34}$: but we have

$$\Omega,\ =a\alpha^3+b\alpha^2+c\alpha+d,\ =\sqrt{\mu}\,A_{12}A_{34}A_{56};$$

this equation contains therefore the factor $A_{12}A_{34}$, and omitting it we find

$$-\frac{\sqrt{\mu}}{\sqrt{\lambda}}(x_1-x_3\,.\,x_1-x_4\,.\,x_2-x_3\,.\,x_2-x_4)\,(c-d\,.\,d-b\,.\,b-c)\,A_{56}$$
$$=A_{34}\,\Sigma\,\{c-d\,.\,B^2_{34}BE_{12}C_{12}D_{12}\}+A_{12}\,\Sigma\,\{c-d\,.\,B^2_{12}BE_{34}C_{34}D_{14}\},$$

where, as before, the summations refer each to the three terms obtained by the cyclical interchange of the letters b, c, d; these being any three of the five letters other than a: and the remaining two letters e, f enter into the formulæ symmetrically. The formula thus gives for A_{56} ten values which are of course equal to each other.

By reason of the undetermined factor $\frac{\sqrt{\mu}}{\sqrt{\lambda}}$, the formula gives only the proportional value of A_{56}; viz. combining it with the like formulæ for B_{56}, &c., we have determinate values of the ratios $A_{56}:B_{56}:\ldots:F_{56}$. But this being understood, we regard the formula as a formula for each single-letter function of x_5, x_6 in terms of the single and double-letter functions of x_1, x_2 and of x_3, x_4 respectively.

163. We require further the expressions for the double-letter functions of x_5, x_6. Consider for example the function DE_{56}, which is

$$=\frac{1}{x_5-x_6}\{\sqrt{\mathrm{d}_5\mathrm{c}_5\mathrm{f}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6}-\sqrt{\mathrm{d}_6\mathrm{e}_6\mathrm{f}_6\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5}\};$$

then multiplying by $A_{56}B_{56}C_{56}$, $=\sqrt{\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6}$, we have

$$DE_{56}A_{56}B_{56}C_{56}=\frac{1}{x_5-x_6}\{\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6\sqrt{X_5}-\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\sqrt{X_6}\},$$

or recollecting that $\sqrt{\lambda}\sqrt{X_5}$ and $\sqrt{\lambda}\sqrt{X_6}$ are $=\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta$ and $\alpha x_6^3+\beta x_6^2+\gamma x_6+\delta$ respectively, this may be written

$$\sqrt{\lambda}\,DE_{56}A_{56}B_{56}C_{56}=\frac{1}{x_5-x_6}\{a-x_6\,.\,b-x_6\,.\,c-x_6\,.\,(\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta)$$
$$-(a-x_5\,.\,b-x_5\,.\,c-x_5)\,(\alpha x_6^3+\beta x_6^2+\gamma x_6+\delta)\}.$$

Using the well-known identity

$$\alpha x_5^3+\beta x_5^2+\gamma x_5+\delta=\Sigma\,.\,\alpha a^3+\beta a^2+\gamma a+\delta\,.\,\frac{b-x_5\,.\,c-x_5\,.\,d-x_5}{b-a\,.\,c-a\,.\,d-a},$$

where the summation extends to the four terms obtained by the cyclical interchanges of the letters a, b, c, d: and the like identity for $\alpha x_6^3 + \beta x_6^2 + \gamma x_6 + \delta$: there will be terms in $\alpha a^3 + \beta a^2 + \gamma a + \delta$, $\alpha b^3 + \beta b^2 + \gamma b + \delta$, $\alpha c^3 + \beta c^2 + \gamma c + \delta$, but the term in $\alpha d^3 + \beta d^2 + \gamma d + \delta$ will disappear of itself. After some easy reductions, the result is

$$\sqrt{\lambda}\, DE_{56} A_{56} B_{56} C_{56} = \Sigma \frac{\alpha a^3 + \beta a^2 + \gamma a + \delta}{b - a \,.\, c - a} B^2_{56} C^2_{56},$$

where the summation extends to the three terms obtained by the cyclical interchanges of the letters a, b, c. We have $\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\mu} \,.\, A_{12} A_{34} A_{56}$, and similarly for the other two terms; the whole equation thus divides by $A_{56} B_{56} C_{56}$, and we find

$$-\frac{\sqrt{\mu}}{\sqrt{\lambda}} DE_{56} = \frac{1}{b - c \,.\, c - a \,.\, a - b} \left(\frac{\sqrt{\mu}}{\sqrt{\lambda}}\right)^2 . \Sigma\, (b - c \,.\, A_{12} A_{34} B_{56} C_{56}),$$

in which equation, if we imagine $\frac{\sqrt{\mu}}{\sqrt{\lambda}} A_{56}$, $\frac{\sqrt{\mu}}{\sqrt{\lambda}} B_{56}$, $\frac{\sqrt{\mu}}{\sqrt{\lambda}} C_{56}$, each replaced by its value in terms of the single and double-letter functions of x_1, x_2 and x_3, x_4, we have an equation of the form

$$-\frac{\sqrt{\mu}}{\sqrt{\lambda}} (x_1 - x_3 \,.\, x_1 - x_4 \,.\, x_2 - x_3 \,.\, x_2 - x_4)\, DE_{56} = \frac{1}{x_1 - x_3 \,.\, x_1 - x_4 \,.\, x_2 - x_3 \,.\, x_2 - x_4} M,$$

where M is a given rational and integral function of the single and double-letter functions of x_1, x_2 and x_3, x_4. The factor on the left-hand side has been made the same as in the formula for the single-letter functions A_{56}, &c., and to do this it was necessary to bring in on the right-hand side the factor

$$\frac{1}{x_1 - x_3 \,.\, x_1 - x_4 \,.\, x_2 - x_3 \,.\, x_2 - x_4};$$

this disappears in the expression for the ratio of two double-letter functions; but it enters into the expression for the ratio of a single-letter to a double-letter function, and it then requires to be itself expressed in terms of the functions of x_1, x_2 and x_3, x_4: it is easy to see that we have

$$x_1 - x_3 \,.\, x_1 - x_4 \,.\, x_2 - x_3 \,.\, x_2 - x_4 = \Sigma \frac{(A^2_{34} B^2_{12} - A^2_{12} B^2_{34})\,(A^2_{34} C^2_{12} - A^2_{12} C^2_{34})}{(a - b)^2 (a - c)^2},$$

where the summation extends to the three terms obtained by the cyclical interchanges of the letters a, b, c: these being a set of any three out of the six letters.

We have, in what precedes, obtained the expressions for the ratios of the 16 functions $A_{56}, \ldots, F_{56}$, $AB_{56}, \ldots, DE_{56}$ in terms of the ratios of the like functions of x_1, x_2 and x_3, x_4.

CHAPTER VII. THE FUNCTIONS T, U, V, Θ.

The present chapter is substantially a reproduction of C. and G.'s seventh section, "Die Function $T_{\xi\eta}$", borrowing only from the next section the definition of the theta-function; but for greater simplicity I consider for the most part, the case, fixed curve a quartic, $n=4$, $p=3$.

Integral Form of the Affected Theorem. Art. Nos. 164 to 169.

164. Writing for shortness $\dfrac{(x, y, z)_{12}{}^{n-2}d\omega}{012}=d\Pi_{12}$, we are concerned with the integrals $\int_{a'}^{a} d\Pi_{12}$ which present themselves in connexion with the affected theorem: the notation is explained, Chap. V.; a, a' are points on the curve f; the variable may be any parameter serving for the determination of the current point, and the integral, taken from the value which belongs to the point a' to the value which belongs to the point a, is represented as above by means of the two points a, a' as limits of the integral. It is assumed that the integral is a canonical integral having the limits and the parametric points interchangeable, $\int_{a'}^{a} d\Pi_{12}=\int_{2}^{1} d\Pi_{aa'}$: see Chapter IV.

165. Writing for shortness

$$\left(\int_{a'}^{a}+\int_{b'}^{b}+\int_{c'}^{c}+\ldots\right) d\Pi_{12}=\int\begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} d\Pi_{12},$$

then if ϕ, ψ are curves each of the order m, the former of them intersecting the fixed curve f in the points $a, b, c, \ldots$, and the latter of them intersecting the same curve in the points $a', b', c', \ldots$, and if ϕ_1, ψ_1, ϕ_2, ψ_2 are what the functions ϕ, ψ become on substituting therein in place of the current coordinates the values which belong to the parametric points 1, 2 respectively; the theorem becomes

$$\int\begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} d\Pi_{12}=\log\frac{\phi_2\psi_1}{\phi_1\psi_2}.$$

The superior limits may be interchanged in any manner, and so also the inferior limits may be interchanged in any manner. If a superior limit coincide with an inferior limit, the two may thus be considered as belonging to an integral which will then have the value 0, and the coincident points may therefore be omitted from the expression on the left-hand side: and so in the case of any number of coincidences.

166. If the intersections of the curves ϕ, ψ and the parametric points are situate on a curve of the order m; then taking the equation of this curve to be $\phi+\lambda\psi=0$, we have simultaneously $\phi_1+\lambda\psi_1=0$, $\phi_2+\lambda\psi_2=0$; whence $\phi_2\psi_1=\psi_1\phi_2$, and the logarithmic term disappears: viz. the theorem becomes

$$\int\begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} d\Pi_{12}=0.$$

167. Suppose that the curves ϕ, ψ are each of them a major curve, that is, a curve of the order $n-2$ passing through the δ dps, and consequently besides meeting the curve f in $n(n-2)-2\delta, =2p+n-2$ points: the theorem is

$$\int \begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} d\Pi_{12} = \log \frac{\phi_2 \psi_1}{\phi_1 \psi_2},$$

where the numbers of the superior and of the inferior points are each $=2p+n-2$.

168. Suppose further that the curves ϕ, ψ, being major curves as above, pass each of them through the $n-2$ residues of 1, 2; they besides meet in $(n-2)(n-3)$ points, viz. these are the δ dps and $(n-2)(n-3)-\delta$ variable points: these $(n-2)(n-3)$ points lie on a minor curve, that is, a curve of the order $n-3$ passing through the dps; and the minor curve together with the parametric line 12 make together a major curve, passing through the intersections of ϕ, ψ and also through the parametric points 1, 2: viz. these points and the intersections of ϕ, ψ are situate on a curve of the order $n-2$; the logarithmic term thus vanishes. The intersections of ϕ with the fixed curve are the δ dps, the $n-2$ residues and $2p$ other points, say these are $a, b, c, \ldots, a^\times, b^\times, c^\times, \ldots$; similarly the curve ψ meets the fixed curve in the δ dps, the $n-2$ residues, and in $2p$ other points, say these are $d, e, f, \ldots, d^\times, e^\times, f^\times, \ldots$: the theorem is

$$\int \begin{pmatrix} a, & b, & c, \ldots & a^\times, & b^\times, & c^\times, \ldots \\ d, & e, & f, \ldots & d^\times, & e^\times, & f^\times, \ldots \end{pmatrix} d\Pi_{12} = 0,$$

where there are $2p$ superior and inferior points respectively.

169. I introduce the definitions: a minor curve meets the fixed curve in the dps and in $2p-2$ other points, called "cominors": a major curve passing through the $n-2$ residues of the points 1, 2, meets the fixed curve in the δ dps, the $n-2$ residues and in $2p$ other points, called "comajors in regard to the points 1, 2." Observe that $p-1$ of the cominors determine uniquely the remaining $p-1$ cominors; and similarly p of the comajors determine uniquely the remaining p comajors.

The foregoing theorem thus is that the sum $\int \left(\quad \right) d\Pi_{12}$ is $=0$, when the superior points and the inferior points are each of them a system of comajors in regard to the parametric points 1, 2.

Fixed Curve a Quartic. Art. No. 170.

170. It would be easy to go on with the general form; but as already mentioned, I prefer to consider the case, fixed curve a quartic, $n=4$, $p=3$. A minor curve is here a line meeting the quartic in 4 points, which are "cominors"; the major curve is a conic, and if this passes through the residues of 1, 2 it besides meets the quartic in 6 points, which are "comajors in regard to the points 1, 2." Two points and their residues are cominors, but this is only by reason that $n-3=1$.

The Function T. Art. No. 171.

171. In conformity with C. and G., I introduce the functional symbol

$$T_{12} \begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} = \int \begin{pmatrix} a, & b, & c, & \ldots \\ a', & b', & c', & \ldots \end{pmatrix} d\Pi_{12},$$

so that T denotes a function of the parametric points, and of the sets of superior and inferior points respectively. The foregoing theorem for the quartic thus is

$$T_{12}\begin{pmatrix} a, & b, & c, & a^\times, & b^\times, & c^\times \\ d, & e, & f, & d^\times, & e^\times, & f^\times \end{pmatrix} = 0.$$

Observing that

$$\int_d^a + \int_{d^\times}^{a^\times} = 2\int_d^a - \int_{a^\times}^a + \int_{d^\times}^d,$$

and so in other cases, this may be written

$$2T_{12}\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix} = T_{12}\begin{pmatrix} a, & b, & c \\ a^\times, & b^\times, & c^\times \end{pmatrix} - T_{12}\begin{pmatrix} d, & e, & f \\ d^\times, & e^\times, & f^\times \end{pmatrix};$$

and if, as a definition of $T_{12}(a, b, c)$, we write

$$T_{12}(a, b, c) = T_{12}\begin{pmatrix} a, & b, & c \\ a^\times, & b^\times, & c^\times \end{pmatrix},$$

where $a^\times$, $b^\times$, $c^\times$ are the comajors of a, b, c in regard to 1, 2, then the equation is

$$2T_{12}\begin{pmatrix} a, & b, & c \\ d, & e, & f \end{pmatrix} = T_{12}(a, b, c) - T_{12}(d, e, f),$$

viz. the function of the $(2p+2=)8$ points 1, 2, a, b, c, d, e, f is here expressed as a difference of two functions each of $(p+2=)5$ points: $T_{12}(a, b, c)$ is regarded as a function of the 5 points 1, 2, a, b, c, because the remaining points $a^\times$, $b^\times$, $c^\times$ depend only on these 5 points.

The Function U. Art. Nos. 172 to 175.

172. We consider on the quartic the points ξ, μ; 1, 2, 3; and take f, f' for the cominors of 2, 3; g, g' for the cominors of 3, 1; and h, h' for the cominors of 1, 2. We write

$$T = T_{\xi\mu}(1, 2, 3),$$

$$T_1 = T_{1\mu}(\xi, f, f'),\quad T_2 = T_{2\mu}(\xi, g, g'),\quad T_3 = T_{3\mu}(\xi, h, h');$$

it is to be shown that there exists a function $U(1, 2, 3; \xi)$, such that

$$\delta U = \tfrac{1}{2}\{\delta_\xi T + \delta_1 T_1 + \delta_2 T_2 + \delta_3 T_3\},$$

viz. considering ξ, 1, 2, 3 as variable points on the quartic, the whole infinitesimal variation of U is the sum of these parts, where $\delta_\xi T$ is the variation of T when only ξ is varied, $\delta_1 T_1$ the variation of T_1 when only 1 is varied, and similarly for $\delta_2 T_2$ and $\delta_3 T_3$. We consider in the proof three other points 4, 5, 6 on the quartic; and taking l, l' for the cominors of 5, 6; m, m' for those of 6, 4; and n, n' for those of 4, 5, we write further

$$X_1 = T_{1\mu}(4, l, l'),\quad X_2 = T_{2\mu}(5, m, m'),\quad X_3 = T_{3\mu}(6, n, n');$$

and it then requires to be shown that

$$\tfrac{1}{2}T \qquad = \int \begin{pmatrix} 123 \\ 456 \end{pmatrix} d\Pi_{\xi\mu} + \tfrac{1}{2}T_{\xi\mu}(4,\ 5,\ 6),$$

$$\tfrac{1}{2}(T_1 - X_1) = \int \begin{pmatrix} \xi 56 \\ 423 \end{pmatrix} d\Pi_{1\mu} + \log \frac{\mu 23 \,.\, 156}{\mu 56 \,.\, 123},$$

$$\tfrac{1}{2}(T_2 - X_2) = \int \begin{pmatrix} \xi 64 \\ 531 \end{pmatrix} d\Pi_{2\mu} + \log \frac{\mu 31 \,.\, 264}{\mu 64 \,.\, 123},$$

$$\tfrac{1}{2}(T_3 - X_3) = \int \begin{pmatrix} \xi 45 \\ 612 \end{pmatrix} d\Pi_{3\mu} + \log \frac{\mu 12 \,.\, 345}{\mu 45 \,.\, 123},$$

where $\mu 23$ is the determinant formed with the coordinates of the points μ, 2, 3 respectively: and so in other cases.

173. We have

$$T_{\xi\mu}\begin{pmatrix} 1, & 2, & 3 \\ 4, & 5, & 6 \end{pmatrix} = \tfrac{1}{2}T_{\xi\mu}(1,\ 2,\ 3) - \tfrac{1}{2}T_{\xi\mu}(4,\ 5,\ 6),$$

that is,

$$= \tfrac{1}{2}T \qquad\qquad - \tfrac{1}{2}T_{\xi\mu}(4,\ 5,\ 6),$$

and thence the above value of $\tfrac{1}{2}T$.

The affected theorem gives

$$\int \begin{pmatrix} f, & f', & 2, & 3 \\ l, & l', & 5, & 6 \end{pmatrix} d\Pi_{1\mu} = \log \frac{F_\mu L_1}{F_1 L_\mu},$$

where $F = 0$ is the equation of the line through f, f', 2, 3; and F_1, F_μ are what the function F becomes, on substituting therein for the current coordinates the coordinates of the points 1, μ respectively. And similarly $L = 0$ is the equation of the line through l, l', 5, 6; and L_1, L_μ are what the function L becomes by the same substitutions respectively. The values of F_1, F_μ are 123, $\mu 23$: those of L_1, L_μ are 156, $\mu 56$; and the logarithmic term is thus

$$= \log \frac{\mu 23 \,.\, 156}{\mu 56 \,.\, 123}.$$

We then have

$$\tfrac{1}{2}(T_1 - X_1) = \int \begin{pmatrix} \xi, & f, & f' \\ 4, & l, & l' \end{pmatrix} d\Pi_{1\mu}, \ = \int \begin{pmatrix} \xi, & 5, & 6 \\ 4, & 2, & 3 \end{pmatrix} d\Pi_{1\mu} + \int \begin{pmatrix} f, & f', & 2, & 3 \\ l, & l', & 5, & 6 \end{pmatrix} d\Pi_{1\mu},$$

and in this last expression for $\tfrac{1}{2}(T_1 - X_1)$ substituting for the second term the logarithmic value just obtained, we have the required expression for $\tfrac{1}{2}(T_1 - X_1)$: and those for $\tfrac{1}{2}(T_2 - X_2)$ and $\tfrac{1}{2}(T_3 - X_3)$ are deduced by mere cyclical permutations of the letters.

174. Returning to the assumed relation $\delta U = \tfrac{1}{2}\{\delta_\xi T + \delta_1 T_1 + \delta_2 T_2 + \delta_3 T_3\}$; in order to the existence of the function U, it is only necessary to show that $T - T_1$ contains no term in 1, ξ, that is, no term depending on both these points, and that $T_1 - T_2$ contains no term in 1, 2: for then, by symmetry, the like properties hold in regard to $T - T_2$, $T - T_3$, $T_1 - T_3$, $T_2 - T_3$ respectively, and the assumed expression is a complete differential, from which the function U may be obtained by integration.

175. To show that $T - T_1$ contains no term in 1, ξ.

For T, the only term in 1, ξ is $\int_4^1 d\Pi_{\xi\mu}$,

" T_1 " " $\int_4^\xi d\Pi_{1\mu}$,

and it is to be shown that the difference of the two integrals contains no term in 1, ξ. Considering on the quartic the two new points δ, ϵ, the first integral is

$$\int \begin{pmatrix} 1, & \delta \\ \delta, & 4 \end{pmatrix} (d\Pi_{\xi\epsilon} + d\Pi_{\epsilon\mu}), \; = \int_\delta^1 d\Pi_{\xi\epsilon} + \int_4^\delta d\Pi_{\xi\epsilon} + \int_4^1 d\Pi_{\epsilon\mu},$$

and the second is

$$\int \begin{pmatrix} \xi, & \epsilon \\ \epsilon, & 4 \end{pmatrix} (d\Pi_{1\delta} + d\Pi_{\delta\mu}), \; = \int_\epsilon^\xi d\Pi_{1\delta} + \int_4^\epsilon d\Pi_{\delta\mu} + \int_\mu^\xi d\Pi_{\delta\mu}.$$

Hence in the difference the only terms which can contain 1, ξ are

$$\int_\delta^1 d\Pi_{\xi\epsilon} - \int_\epsilon^\xi d\Pi_{1\delta}$$

and this is $=0$: wherefore there is not in the difference any term in 1, ξ. This proves the property for $T - T_1$. The property for $T_1 - T_2$ is proved in a similar manner.

Theorems in regard to the Function U. Art. Nos. 176 to 179.

176. Theorem (A). To prove

$$U(1, 2, 3;\ \xi) - U(1, 2, 3;\ \mu) = \tfrac{1}{2} T_{\xi\mu}(1, 2, 3), \tag{A}$$

we have

$$U(1, 2, 3;\ \xi) - U(1, 2, 3;\ \mu) = \int_\mu^\xi d_\xi U = \tfrac{1}{2} \int_\mu^\xi d_\xi T$$
$$= \tfrac{1}{2} T_{\xi\mu}(1, 2, 3) - \tfrac{1}{2} T_{\mu\mu}(1, 2, 3),$$

and $T_{\mu\mu}(1, 2, 3) = \int \begin{pmatrix} 1, & 2, & 3 \\ 1^\times & 2^\times & 3^\times \end{pmatrix} d\Pi_{\mu\mu}$, where $d\Pi_{\mu\mu} = 0$, viz. considering this as derived from $d\Pi_{\xi\mu}, = Q_{\xi\mu}\, d\omega$, by making the point ξ coincide with μ, then when ξ is indefinitely near to μ, the numerator and denominator of $Q_{\xi\mu}$ are each of them infinitesimal of the orders 3 and 2 respectively, and thus the function $Q_{\xi\mu}$ ultimately vanishes (see as to this, Chap. V. Art. Nos. 99 to 106). We have therefore $T_{\mu\mu}(1, 2, 3) = 0$, and the required theorem is proved.

177. Theorem (B). To prove

$$U(1, 2, 3;\ \xi) - U(4, 2, 3;\ \xi) = \tfrac{1}{2} T_{14}(\xi, f, f'), \tag{B}$$

where, as before, f, f' are the cominors of 2, 3, that is, 2, 3, f, f' lie on a line, we have

$$U(1, 2, 3;\ \xi) - U(4, 2, 3;\ \xi) = \int_4^1 d_1 U = \tfrac{1}{2} \int_4^1 d_1 T_1$$
$$= \tfrac{1}{2} T_{1\mu}(\xi, f, f') - \tfrac{1}{2} T_{4\mu}(\xi, f, f'),$$

the point μ is arbitrary, and it may be taken to coincide with 4; but we then have $T_{44}(\xi, f, f')=0$, and the theorem is thus proved.

178. Theorem (C). We have

$$T_{11^\times}(\xi, f, f')+T_{11^\times}(\eta, k, k')=0; \qquad \text{(C)}$$

where ξ, η, 1, 2, 3 are arbitrary points on the quartic; $1^\times$, $2^\times$, $3^\times$ are the comajors of 1, 2, 3 in regard to ξ, η, viz. the points 1, 2, 3, $1^\times$, $2^\times$, $3^\times$ lie on a conic which passes through ξ', η' the residues (or cominors) of ξ, η: f, f' are the cominors of 2, 3; and k, k' are the cominors of $2^\times$, $3^\times$.

Taking θ, θ' for the cominors of 1, $1^\times$, the four lines $\xi\eta\xi'\eta'$, $11^\times\theta\theta'$, $23ff'$ and $2^\times 3^\times kk'$ form a quartic cutting the fixed quartic in the 16 points: but of these, ξ', η', 1, 2, 3, $1^\times$, $2^\times$, $3^\times$ lie in a conic: hence the remaining 8 points θ, θ', ξ, η, f, f', k, k' lie in a conic; that is, ξ, η, f, f', k, k' lie on a conic through θ, θ', the residues of 1, $1^\times$, or they are comajors in regard to 1, $1^\times$; whence the theorem.

179. We have

$$\begin{array}{lll} & \text{From A.} & \text{From B.} \\ \tfrac{1}{2}T_{11^\times}(\xi, f, f') & = U(\xi, f, f';\ 1)-U(\xi, f, f';\ 1^\times), & = U(1, 2, 3;\ \xi)-U(1^\times, 2, 3;\ \xi), \\ \tfrac{1}{2}T_{11^\times}(\eta, k, k') & = U(\eta, k, k';\ 1)-U(\eta, k, k';\ 1^\times), & = U(1, 2^\times, 3^\times;\ \eta)-U(1^\times, 2, 3;\ \eta); \end{array}$$

viz. we have thus two expressions for each term of the equation (C),

$$T_{11^\times}(\xi, f, f')+T_{11^\times}(\eta, k, k')=0.$$

In particular, we have Theorem (D)

$$U(1, 2, 3;\ \xi)-U(1^\times, 2, 3;\ \xi)=-U(1, 2^\times, 3^\times;\ \eta)+U(1^\times, 2^\times, 3^\times;\ \eta). \qquad \text{(D)}$$

Again, we have $T_{\xi\eta}(1, 2, 3)+T_{\xi\eta}(1^\times, 2^\times, 3^\times)=0$; where 1, 2, 3, $1^\times$, $2^\times$, $3^\times$ are comajors in regard to ξ, η: and

$$\tfrac{1}{2}T_{\xi\eta}(1, 2, 3)=U(1, 2, 3;\ \xi)-U(1, 2, 3;\ \eta),$$
$$\tfrac{1}{2}T_{\xi\eta}(1^\times, 2^\times, 3^\times)=U(1^\times, 2^\times, 3^\times, \xi)-U(1^\times, 2^\times, 3^\times, \eta);$$

whence Theorem (E),

$$U(1, 2, 3;\ \xi)-U(1, 2, 3;\ \eta)=-U(1^\times, 2^\times, 3^\times;\ \xi)+U(1^\times, 2^\times, 3^\times, \eta). \qquad \text{(E)}$$

The Function V. Art. Nos. 180 to 182.

180. It is convenient to consider U as a logarithm, say

$$-U(1, 2, 3;\ \xi)=\log V(1, 2, 3;\ \xi), \text{ or } V(1, 2, 3;\ \xi)=\exp.-U(1, 2, 3;\ \xi);$$

V, like U, is a function of the $(p+1=)\,4$ points 1, 2, 3, ξ, on the quartic.

The equation (D) thus becomes

$$\frac{V(1, 2, 3;\ \xi)}{V(1^\times, 2^\times, 3^\times;\ \eta)}=\frac{V(1^\times, 2, 3;\ \xi)}{V(1, 2^\times, 3^\times;\ \eta)},$$

where 1, 2, 3, $1^\times$, $2^\times$, $3^\times$ are comajors in regard to ξ, η: the equation shows that, in the function $\dfrac{V(1,\ 2,\ 3;\ \xi)}{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}$, we can without alteration of the value interchange a pair of points 1, $1^\times$ out of the system of comajor points; and it of course follows that we can in any manner whatever interchange these points, so as to have any three of them in the numerator-function and the remaining three in the denominator-function. In particular, we have

$$\frac{V(1,\ 2,\ 3;\ \xi)}{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}=\frac{V(1^\times,\ 2^\times,\ 3^\times;\ \xi)}{V(1,\ 2,\ 3;\ \eta)}.$$

The equation (E) becomes

$$\frac{V(1,\ 2,\ 3;\ \xi)}{V(1,\ 2,\ 3;\ \eta)}=\frac{V(1^\times,\ 2^\times,\ 3^\times;\ \eta)}{V(1^\times,\ 2^\times,\ 3^\times;\ \xi)},$$

and multiplying we find

$$V^2(1,\ 2,\ 3;\ \xi)=V^2(1^\times,\ 2^\times,\ 3^\times;\ \eta),$$

that is, $V(1,\ 2,\ 3;\ \xi)=\pm V(1^\times,\ 2^\times,\ 3^\times;\ \eta)$, the sign being determinately + or determinately −, according to the precise definition of the function V.

181. Considering η and also $1^\times$, $2^\times$, $3^\times$ as fixed points on the curve; but ξ as a variable point (that is, the parametric line $\xi\eta$ as rotating about the fixed point η), the points 1, 2, 3 are then determined as the remaining intersections with the quartic of the conic which passes through the points $1^\times$, $2^\times$, $3^\times$ and through the points ξ', η', which are the residues of ξ, η. And by the theorem just obtained it appears that, ξ, 1, 2, 3 thus varying, the function $V(1,\ 2,\ 3;\ \xi)$ remains constant. This comes to saying that V, considered as a function of the points 1, 2, 3, ξ, satisfies a certain linear partial differential equation of the first order, having a solution $V=F(u,\ v,\ w)$, an arbitrary function of u, v, w, determinate functions of the points 1, 2, 3, ξ. And if we can find u, v, w functions of these points such that they each of them remain constant when the points 1, 2, 3, ξ vary as above, then the arbitrary function of u, v, w will remain constant for the variation in question and will thus be a value of the function V.

182. It is easily seen that such functions are

$$u,\ v,\ w=\left(\int^1+\int^2+\int^3-\int^\xi\right)x\,d\omega,\ y\,d\omega,\ z\,d\omega,$$

the inferior limits being given points which are regarded as absolute constants. For by the pure theorem, we have

$$\Sigma\,(x,\ y,\ z)^1\,d\omega=0,$$

where $(x,\ y,\ z)^1$ is an arbitrary linear function, and where the summation extends to all the intersections of the quartic with any given curve. Writing

$$p=\int x\,d\omega,\ \int y\,d\omega\ \text{ or }\ \int z\,d\omega,\ \text{ that is, }\ p_1=\int^1 x\,d\omega,\ \int^1 y\,d\omega\ \text{ or }\ \int^1 z\,d\omega,$$

the inferior limits being any absolutely fixed point on the curve, and similarly p_2, &c.; the integral form of the theorem is $\Sigma p=$ constant. And applying the theorem

successively to the parametric line, and to the conic which determines the points 1, 2, 3, we have

$$p_\xi + p_\eta + p_{\xi'} + p_{\eta'} = \text{const.},$$

$$p_{\xi'} + p_{\eta'} + p_1 + p_2 + p_3 + p_{1^\times} + p_{2^\times} + p_{3^\times} = \text{const.}$$

Taking the difference of these equations, we have

$$p_1 + p_2 + p_3 - p_\xi + p_{1^\times} + p_{2^\times} + p_{3^\times} - p_\eta = \text{const.},$$

viz. the points η, $1^\times$, $2^\times$, $3^\times$ being fixed points, this is

$$p_1 + p_2 + p_3 - p_\xi = \text{const.},$$

that is, the functions u, v, w defined as above are each of them constant under the variation in question.

The Function Θ. Art. Nos. 183 and 184.

183. The function $V(1, 2, 3; \xi)$ of the $(p+1=)$ 4 points 1, 2, 3, ξ, is thus a function of the $(p=)$ 3 arguments

$$u, v, w, = \left(\int^1 + \int^2 + \int^3 - \int^\xi\right) x\, d\omega,\ y\, d\omega,\ z\, d\omega.$$

Disregarding a constant and exponential factor, we say that it is a theta-function of these arguments, and we write the result provisionally in the form

$$V(1, 2, 3; \xi) = \Theta(u, v, w),$$

the more precise definition of the theta-function being reserved for further consideration.

184. It appears by what precedes that a sum of $(p=)3$ integrals $\int \begin{pmatrix} abc \\ def \end{pmatrix} d\Pi_{12}$, otherwise called $T_{12}\begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix}$, is in the first place expressed (see No. 171) as a difference, $= \frac{1}{2}T_{12}(a, b, c) - \frac{1}{2}T_{12}(d, e, f)$, of two functions T. Each of these is by Theorem (A) (No. 176) expressed as a difference of two functions U, that is, as the difference of the logarithms, or logarithm of the quotient, of two functions V: such function V is according to its original definition a function of $(p+1=)$ 4 points, but in such wise that the function is expressible as a function of $(p=)$ 3 arguments, and so expressed it is a Θ-function of these arguments: and the final result thus is that the sum of $(p=)$ 3 integrals $\int \begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix} d\Pi_{12}$ is equal to the logarithm of a fraction, whereof the numerator and the denominator are each of them a product of two Θ-functions.

826.

NOTE ON A PARTITION-SERIES.

[From the *American Journal of Mathematics*, vol. VI. (1884), pp. 63, 64.]

PROF. SYLVESTER, in his paper, "A Constructive theory of Partitions, &c.," *American Journal of Mathematics*, vol. V. (1883), p. 282, has given the following very beautiful formula

$$(1+ax)(1+ax^2)(1+ax^3)\ldots = 1+\frac{1}{1-x}(1+ax^2)\,xa+\frac{1}{1-x\,.\,1-x^2}(1+ax)(1+ax^4)\,x^5a^2$$

$$+\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3}(1+ax)(1+ax^2)(1+ax^6)\,x^{12}a^3+\ldots,$$

or, as this may be written,

$$\Omega = 1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3)+\ldots \quad ,$$

where

$$P=\frac{(1+ax^2)\,xa}{\mathbf{1}},\quad Q=\frac{(1+ax^4)\,x^5a^2}{\mathbf{1.2}},\quad R=\frac{(1+ax^6)\,x^{12}a^3}{\mathbf{1.2.3}},\quad S=\frac{(1+ax^8)\,x^{22}a^4}{\mathbf{1.2.3.4}},\ \&c.,$$

the heavy figures **1**, **2**, **3**, **4**,... of the denominators being, for shortness, written to denote $1-x$, $1-x^2$, $1-x^3$, $1-x^4$,... respectively. The x-exponents 1, 5, 12, 22,... are the pentagonal numbers $\frac{1}{2}(3n^2-n)$.

To prove this, writing

$$P'=\frac{ax^2}{\mathbf{1}},\quad Q'=\frac{ax^5}{\mathbf{1}}+\frac{a^2x^7}{\mathbf{1.2}},\quad R'=\frac{ax^4}{\mathbf{1}}+\frac{a^2x^9}{\mathbf{1.2}}+\frac{a^3x^{15}}{\mathbf{1.2.3}},\quad S'=\frac{ax^5}{\mathbf{1}}+\frac{a^2x^{11}}{\mathbf{1.2}}+\frac{a^3x^{18}}{\mathbf{1.2.3}}+\frac{a^4x^{26}}{\mathbf{1.2.3.4}},\ \&c.,$$

where the x-exponents are

$$2;\ 3,\ 3+4;\ 4,\ 4+5,\ 4+5+6;\ 5,\ 5+6,\ 5+6+7,\ 5+6+7+8;\ \&c.,$$

we find without difficulty (see infrà) that

$$\begin{aligned} 1+P &= (1+ax)(1+P'), \\ 1+P'+Q &= (1+ax^2)(1+Q'), \\ 1+Q'+R &= (1+ax^3)(1+R'), \\ 1+R'+S &= (1+ax^4)(1+S'), \text{ \&c.}; \end{aligned}$$

and hence, using Ω to denote the sum

$$\Omega = 1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3)+\ldots,$$

we obtain successively

$$\begin{aligned} \Omega \div (1+ax) &= 1+P'+Q+R(1+ax^3)+S(1+ax^2)(1+ax^3)+\ldots, \\ \Omega \div (1+ax)(1+ax^2) &= 1+Q'+R+S(1+ax^3)+T(1+ax^3)(1+ax^4)+\ldots, \\ \Omega \div (1+ax)(1+ax^2)(1+ax^3) &= 1+R'+S+T(1+ax^4)+\ldots, \end{aligned}$$

and so on. In these equations, on the right-hand sides, the lowest exponent of x is 2, 3, 4, &c., respectively, so that in the limit the right-hand side becomes $=1$, or the final equation is $\Omega = (1+ax)(1+ax^2)(1+ax^3)\ldots$; viz. we have the series represented by Ω equal to this infinite product, which is the theorem in question.

One of the foregoing identities is

$$1+R'+S = (1+ax^4)(1+S'),$$

viz. substituting for R', S, S' their values, this is

$$1+\frac{ax^4}{1}+\frac{a^2x^9}{1.2}+\frac{a^3x^{15}}{1.2.3}+\frac{(1+ax^3)\,a^4x^{22}}{1.2.3.4} = (1+ax^4)\left\{1+\frac{ax^5}{1}+\frac{a^2x^{11}}{1.2}+\frac{a^3x^{18}}{1.2.3}+\frac{a^4x^{26}}{1.2.3.4}\right\},$$

viz. this equation is

$$\begin{aligned} -ax^4+\frac{ax^4-ax^5(1+ax^4)}{1}+\frac{a^2x^9-a^2x^{11}(1+ax^4)}{1.2} \\ +\frac{a^3x^{15}-a^3x^{18}(1+ax^4)}{1.2.3}+\frac{(1+ax^3)\,a^4x^{22}-a^4x^{26}(1+ax^4)}{1.2.3.4}, \end{aligned}$$

that is,

$$0 = -ax^4+ax^4-\frac{a^2x^9}{1}+\frac{a^2x^9}{1}-\frac{a^3x^{15}}{1.2}+\frac{a^3x^{15}}{1.2}-\frac{a^4x^{22}}{1.2.3}+\frac{a^4x^{22}}{1.2.3}.$$

In the same way each of the other identities is proved.

Writing $a=-1$, we have $\Omega, = 1.2.3.4.\ldots,$

$$= 1+P+Q.1+R.1.2+S.1.2.3+\ldots,$$

where

$$P = -(1+x)\,x, \quad Q = \frac{(1+x^2)\,x^5}{1}, \quad R = -\frac{(1+x^3)\,x^{12}}{1.2}, \ldots$$

and therefore

$$\mathbf{1.2.3.4}\ldots = 1-(1+x)x+(1+x^2)x^5-(1+x^3)x^{12}+\ldots,$$

which is Euler's theorem.

It might appear that the identities used in the proof would also, for this particular value $a=-1$, lead to interesting theorems; but this is found *not* to be the case: we have

$$P'=\frac{-x^2}{\mathbf{1}},\quad Q'=\frac{-x^3}{\mathbf{1}}+\frac{x^7}{\mathbf{1.2}},\quad R'=\frac{-x^4}{\mathbf{1}}+\frac{x^9}{\mathbf{1.2}}-\frac{x^5}{\mathbf{1.2.3}},\ \&c.,$$

but the expressions in terms of these quantities for the products $\mathbf{2.3.4}\ldots$, $\mathbf{3.4}\ldots$, &c., contain denominator factors, and are thus altogether without interest; we have, for example,

$$\mathbf{2.3.4}\ldots = 1+\frac{-x^2+x^5+x^7}{\mathbf{1}}-\frac{(1+x^3)x^{12}}{\mathbf{1}}+\&c.,$$

which is, with scarcely a change of form, the expression obtained from that of the original product $\mathbf{1.2.3.4}\ldots$, by division by $\mathbf{1}$, $=1-x$. And similarly as regards the products $\mathbf{3.4}\ldots$, &c.

Cambridge, June, 1883.

827.

ON THE NON-EUCLIDIAN PLANE GEOMETRY.

[From the *Proceedings of the Royal Society of London*, vol. XXXVII. (1884), pp. 82—102. Received May 27, 1884.]

1. I CONSIDER the hyperbolic or Lobatschewskian geometry: this is a geometry such as that of the imaginary spherical surface $x^2+y^2+z^2=-1$; and the imaginary surface may be *bent* (without extension or contraction) into the real surface considered by Beltrami, which I will call the Pseudosphere, viz. this is the surface of revolution defined by the equations $x=\log\cot\frac{1}{2}\theta-\cos\theta$, $\sqrt{y^2+z^2}=\sin\theta$. We have on the imaginary spherical surface imaginary points corresponding to real points of the pseudosphere, and imaginary lines (arcs of great circle) corresponding to real lines (geodesics) of the pseudosphere, and, moreover, any two such imaginary points or lines of the imaginary spherical surface have a real distance or inclination equal to the corresponding distance or inclination on the pseudosphere. Thus the geometry of the pseudosphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface; for in regard to the descriptive geometry, the statement requires (as will presently appear) some qualification.

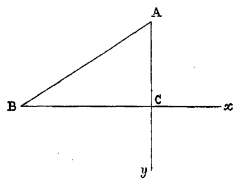
2. I would remark that this realisation of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable. We may imagine rational beings living in a two-dimensional space and conceiving of space accordingly, that is, having no conception of a third dimension of space; this two-dimensional space need not however be a plane, and taking it to be the pseudospherical surface, the geometry to which their experience would lead them would be the geometry of this surface, that is, the Lobatschewskian geometry. With regard to our own two-dimensional space, the plane, I have, in my Presidential Address (B.A., Southport, 1883), [784], expressed the opinion that Euclid's twelfth axiom in Playfair's form of it does not need demonstration, but is part of our notion of space, of the physical space of our experience;

the space, that is, which we become acquainted with by experience, but which is the representation lying at the foundation of all physical experience.

3. I propose in the present paper to develope further the geometry of the pseudosphere. In regard to the name, and the subject generally, I refer to two memoirs by Beltrami, "Teoria fondamentale degli spazii di curvatura costante," *Annali di Matem.*, t. II. (1868—69), pp. 232—255, and "Saggio di interpretazione della geometria non-Euclidea," *Battaglini, Giorn. di Matem.*, t. VI. (1868), pp. 284—312, both translated, *Ann. de l'École Normale*, t. VI. (1869); in the last of these, he speaks of surfaces of constant negative curvature as "pseudospherical," and in a later paper, "Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche," *Battaglini, Giorn. di Matem.*, t. X. (1872), pp. 147—160, he treats of the particular surface which I have called the pseudosphere. The surface is mentioned, Note IV. of Liouville's edition of Monge's *Application de l'Analyse à la Géométrie* (1850), and the generating curve is there spoken of as "bien connue des géomètres."

4. In ordinary plane geometry, take (fig. 1) a line Bx, and on it a point B; from B, in any direction, draw the line BA; take upon it a point A, and from

Fig. 1.



this point, at right angles to Bx, draw Ay, cutting it at C. We have thus a triangle ACB, right-angled at C; and we may denote the other angles, and the lengths of the sides, by A, B, c, a, b, respectively. In the construction of the figure, the length c and the angle B are arbitrary.

The plane is a surface which is homogeneous, isotropic, and palintropic, that is, whatever be the position of B, the direction of Bx, and the sense in which the angle B is measured, we have the same expressions for a, b as functions of c, B; these expressions, of course, are

$$a = c \cos B, \quad b = c \sin B.$$

But considering Ay as the initial line and AB, $= c$, as a line drawn from A at an inclination thereto $= A$, we have in like manner

$$b = c \cos A, \quad a = c \sin A,$$

and consequently $\cos A = \sin B$, $\sin A = \cos B$; whence $\sin(A+B) = 1$, $\cos(A+B) = 0$, and thence $A + B =$ a right angle, or $A + B + C =$ two right angles.

Hence also in any triangle ABC, drawing a perpendicular, say AD, from A to the side BC, and so dividing the triangle into two right-angled triangles, we prove that the sum $A+B+C$ of the angles is = two right angles, and we further establish the relations

$$a = b\cos C + c\cos B,\quad b = c\cos A + a\cos C,\quad c = a\cos B + b\cos A,$$

which are the fundamental formulæ of plane trigonometry; that is, we derive the metrical geometry or trigonometry of the plane from the two original equations $a = c\cos B$, $b = c\sin B$.

5. Supposing the plane bent in any manner, that is, converted into a developable surface or torse, and using the term straight line to denote a geodesic of the surface, then the straight line of the surface is in fact the form assumed, in consequence of the bending, by a straight line of the plane. The sides and angles of the rectilinear triangle ABC on the surface are equal to those of the rectilinear triangle ABC on the plane, and the metrical relations hold good without variation. But it is not *simpliciter* true that the descriptive properties of the torse are identical with those of the plane. This will be the case if the points of the plane and torse have with each other a (1, 1) correspondence, but not otherwise. For instance, consider a plane curve (such as the parabola or one branch of the hyperbola) extending from infinity to infinity, and let the torse be the cylinder having this curve for a plane section; then to each point of the plane there corresponds a single point of the cylinder; and conversely to each point of the cylinder there corresponds a single point of the plane; and the descriptive geometries are identical. In particular, two straight lines (geodesics) on the cylinder cannot inclose a space; and Euclid's twelfth axiom holds good in regard to the straight lines (geodesics) of the cylinder. But take the plane curve to be a closed curve, or (to fix the ideas) a circle; the infinite plane is bent into a cylinder considered as composed of an infinity of convolutions; to each point of the plane there corresponds a single point of the cylinder, but to each point of the cylinder an infinity of points of the plane; and the descriptive properties are in this case altered; the straight lines (geodesics) of the cylinder are helices; and we can through two given points of the cylinder draw, not only one, but an infinity of helices; any two of these will inclose a space. And even if instead of the geodesics we consider only the shortest lines, or helices of greatest inclination; yet even here for a pair of points on opposite generating lines of the cylinder, there are two helices of equal inclination, that is, two shortest lines inclosing a space. We have, in what precedes, an illustration in regard to the descriptive geometry of the pseudosphere; this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

6. The surface of the sphere is, like the plane, homogeneous, isotropic, and palintropic. We may on the spherical surface construct, as above, a right-angled triangle ABC, wherein the side c and the angle B are arbitrary; and (corresponding to the before-mentioned formulæ for the plane) we then have

$$\tan a = \tan c\cos B,\quad \sin b = \sin c\sin B,$$

whence also

$$\tan b = \tan c \cos A, \quad \sin a = \sin c \sin A.$$

We deduce from these

$$\frac{\tan^2 a}{\tan^2 c} + \frac{\sin^2 b}{\sin^2 c} = 1,$$

leading to $\cos^2 c = \cos^2 a \cos^2 b$; and then

$$\frac{\sin b}{\tan a} = \cos c \tan B, \quad \frac{\sin a}{\tan b} = \cos c \tan A,$$

giving

$$\cos a \cos b = \cos^2 c \tan A \tan B;$$

that is,

$$\tan A \tan B = \frac{1}{\cos a \cos b}, \text{ which is } > 1.$$

Hence $A + B >$ a right angle, or in the right-angled triangle ACB, the sum $A + B + C$ of the angles is $>$ two right angles. Whence also in any triangle ABC whatever, dividing it into two right-angled triangles by means of a perpendicular let fall from an angle on the opposite side, we have the sum $A + B + C$ of the angles $>$ two right angles. And we obtain, moreover,

$$a = \tan^{-1}(\tan c \cos B) + \tan^{-1}(\tan b \cos C),$$
$$b = \tan^{-1}(\tan a \cos C) + \tan^{-1}(\tan c \cos A),$$
$$c = \tan^{-1}(\tan b \cos A) + \tan^{-1}(\tan a \cos B),$$

which lead to all the formulæ of spherical trigonometry.

7. Suppose the radius of the sphere to be $1/\lambda$: then a, b, c being the lengths of the sides, the lengths in spherical measure are λa, λb, λc; and we must in the formulæ instead of a, b, c write λa, λb, λc respectively. In particular, for the imaginary sphere $x^2 + y^2 + z^2 = -1$, we have $\lambda = i$, and we must instead of a, b, c write ai, bi, ci respectively. The fundamental formulæ for the right-angled triangle thus become

$$\tanh a = \tanh c \cos B, \quad \sinh b = \sinh c \sin B,$$

and these lead to all the trigonometrical formulæ, viz. any one of these is deduced from the corresponding formula of spherical trigonometry by writing therein ai, bi, ci for a, b, c respectively; or, what is the same thing, by changing the circular functions of the sides into the corresponding hyperbolic functions.

In particular, for the right-angled triangle ACB, we have

$$\tan A \tan B = \frac{1}{\cosh a \cosh b},$$

which for a and b real is < 1, that is, $A + B <$ a right angle, or $A + B + C <$ two right angles, and thence also in any triangle whatever $A + B + C <$ two right angles. But the points A, B, C of any such triangle ABC on the imaginary sphere, and the lines BC, CA, AB which connect them, are imaginary: the meaning of the proof will better appear on passing to the pseudosphere.

8. We have to consider the imaginary spherical surface as *bent* into a real surface. This is, of course, an imaginary process, as any process must be which gives a transformation of imaginary points and lines into real points and lines; but the notion is not more difficult than that of the transformation of imaginary similarity, consisting in the substitution of ix, iy, iz for x, y, z respectively. We thus pass from imaginary points of the imaginary sphere $x^2+y^2+z^2=-1$ to real points of the real sphere $x^2+y^2+z^2=1$; or, again, from imaginary points of either of the real hyperboloids $x^2+y^2-z^2=-1$, $x^2+y^2-z^2=1$, to real points of the other of the same two real hyperboloids.

9. I consider the formulæ for the flexure of the imaginary sphere $X^2+Y^2+Z^2=-1$, into the pseudosphere $x=\log\cot\frac{1}{2}\theta-\cos\theta$, $\sqrt{y^2+z^2}=\sin\theta$: it would be allowable to dispense with Beltrami's subsidiary variables u, v, but I prefer to collect here all the formulæ. We have

$$X=\frac{-i}{\sqrt{1-u^2-v^2}},\quad Y=\frac{u}{\sqrt{1-u^2-v^2}},\quad Z=\frac{v}{\sqrt{1-u^2-v^2}},$$

values which give $X^2+Y^2+Z^2=-1$. And observe that, taking u, v to be real magnitudes such that $u^2+v^2<1$, we have X a pure imaginary, but Y and Z each of them real. We consider on the imaginary sphere points having such coordinates X, Y, Z; any such point corresponds as will immediately appear to a real point on the pseudosphere, and (the distances and angles being the same for the pseudosphere as for the original imaginary spherical surface) it hence appears that (notwithstanding that the points on the imaginary spherical surface, and the lines joining such points, are imaginary) the distances and angles on the imaginary spherical surface are real. Also

$$\sin\theta=\frac{1-u}{\sqrt{1-u^2-v^2}},\quad \phi=\frac{v}{1-u},$$

and thence

$$iX-Y=\sin\theta,\quad iX+Y=\sin\theta\,(\phi^2+\operatorname{cosec}^2\theta),\quad Z=\sin\theta\,.\,\phi.$$

Further

$$u=\frac{\phi^2-1+\operatorname{cosec}^2\theta}{\phi^2+1+\operatorname{cosec}^2\theta},\quad v=\frac{2\phi}{\phi^2+1+\operatorname{cosec}^2\theta},$$

$$x=\log\cot\tfrac{1}{2}\theta+\cos\theta,\quad y=\sin\theta\cos\phi,\quad z=\sin\theta\sin\phi.$$

10. We have $dX^2+dY^2+dZ^2$ and $dx^2+dy^2+dz^2$ each $=\cot^2\theta\,d\theta^2+\sin^2\theta\,d\phi^2$. Writing P, $Q=iX-Y$, $iX+Y$ respectively, we in fact have

$$dX^2+dY^2+dZ^2=-dP\,dQ+dZ^2,$$

where P, Q, $Z=\sin\theta$, $\phi^2\sin\theta+\operatorname{cosec}\theta$, $\phi\sin\theta$ respectively; and thence

$$\begin{aligned}dZ&=\sin\theta\,d\phi+\phi\cos\theta\,d\theta,\\ dP&=\qquad\qquad\cos\theta\,d\theta,\\ dQ&=2\sin\theta\phi\,d\phi+(\phi^2\cos\theta-\operatorname{cosec}\theta\cot\theta)\,d\theta,\end{aligned}$$

giving the formula

$$dX^2 + dY^2 + dZ^2 = \cot^2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2;$$

and then also

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dx^2 + (d\sin\theta)^2 + \sin^2\theta\, d\phi^2 \\ &= (\cos^2\theta\cot^2\theta + \cos^2\theta)\, d\theta^2 + \sin^2\theta\, d\phi^2 = \cot^2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2. \end{aligned}$$

Joining to these the differential expression in u, v, we have

$$\begin{aligned} dX^2 + dY^2 + dZ^2 &= \frac{(1-u^2-v^2)(du^2+dv^2) + (u\,du + v\,dv)^2}{(1-u^2-v^2)^2}, \\ &= \cot^2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2, \\ &= dx^2 + dy^2 + dz^2, \end{aligned}$$

where the final equation $dX^2 + dY^2 + dZ^2 = dx^2 + dy^2 + dz^2$ shows that the imaginary sphere $X^2 + Y^2 + Z^2 = -1$ can be bent into the pseudosphere.

Observe that to given values of θ, ϕ there corresponds a single point on the pseudosphere, but not conversely; for if θ, ϕ be values corresponding to a given point, then corresponding to the same point we have θ, $\phi + n\pi$, where n is an arbitrary integer.

11. The geodesics of the imaginary spherical surface are, of course, its plane sections, any such section being determined by a linear equation $\alpha X + \beta Y + \gamma Z = 0$ between the coordinates X, Y, Z. Since for a point corresponding to a real point of the pseudosphere, X is a pure imaginary while Y and Z are real, we see that for a geodesic corresponding to a real geodesic of the pseudosphere we must have α a pure imaginary, β and γ real; and, in fact, writing as above, $P = iX - Y$, $Q = iX + Y$, and therefore conversely $X = \frac{1}{2}i(-P-Q)$, $Y = \frac{1}{2}(-P+Q)$, the equation $\alpha X + \beta Y + \gamma Z = 0$ becomes $(-\frac{1}{2}i\alpha - \frac{1}{2}\beta)P + (-\frac{1}{2}i\alpha + \frac{1}{2}\beta)Q + \gamma Z = 0$, which will then be of the form $AP + BQ + CZ = 0$, with real coefficients A, B, C: viz. we have

$$P,\ Q,\ Z = \sin\theta,\quad \sin\theta(\phi^2 + \operatorname{cosec}^2\theta),\quad \sin\theta\,.\,\phi;$$

and the equation thus is

$$A + B(\phi^2 + \operatorname{cosec}^2\theta) + C\phi = 0,$$

which is the equation for a geodesic (or straight line) on the pseudosphere. The equation $A + C\phi = 0$, that is, $\phi = \text{const.}$, is obviously that of a meridian.

12. If the geodesic pass through a given point θ_1, ϕ_1, we have, of course,

$$A + B(\phi_1^2 + \operatorname{cosec}^2\theta_1) + C\phi_1 = 0,$$

and hence also the equation of a geodesic through the two points (θ_1, ϕ_1), (θ_2, ϕ_2) is

$$\begin{vmatrix} 1, & \phi^2 + \operatorname{cosec}^2\theta, & \phi \\ 1, & \phi_1^2 + \operatorname{cosec}^2\theta_1, & \phi_1 \\ 1, & \phi_2^2 + \operatorname{cosec}^2\theta_2, & \phi_2 \end{vmatrix} = 0.$$

We may for ϕ_1, ϕ_2 write $\phi_1+2n_1\pi$, $\phi_2+2n_2\pi$ respectively, n_1, n_2 being arbitrary integers; and it would thus at first sight appear that there could be drawn through the two points a doubly infinite series of geodesics. There is, in fact, a singly infinite system of geodesics: to show how this is, write for shortness Λ, Λ_1, Λ_2, α, α_1, α_2 for $\operatorname{cosec}^2\theta$, $\operatorname{cosec}^2\theta_1$, $\operatorname{cosec}^2\theta_2$, $2n\pi$, $2n_1\pi$, $2n_2\pi$ respectively; then the equation of the geodesic through the two points may be written

$$\begin{vmatrix} 1, & (\phi+\alpha)^2+\Lambda, & \phi+\alpha \\ 1, & (\phi_1+\alpha_1)^2+\Lambda_1, & \phi_1+\alpha_1 \\ 1, & (\phi_2+\alpha_2)^2+\Lambda_2, & \phi_2+\alpha_2 \end{vmatrix}=0,$$

where the constant $\alpha=2n\pi$ may be disposed of so as to simplify the formula as much as may be: it is what I have called an apoclastic constant. Taking β an arbitrary value, this may be transformed into

$$\begin{vmatrix} 1, & (\phi+\alpha+\beta)^2+\Lambda, & \phi+\alpha+\beta \\ 1, & (\phi_1+\alpha_1+\beta)^2+\Lambda_1, & \phi_1+\alpha_1+\beta \\ 1, & (\phi_2+\alpha_2+\beta)^2+\Lambda_2, & \phi_2+\alpha_2+\beta \end{vmatrix}=0,$$

and then assuming $\alpha=\alpha_1$, $\beta=-\alpha_1$, this becomes

$$\begin{vmatrix} 1, & \phi^2+\Lambda & , & \phi \\ 1, & \phi_1^2+\Lambda_1 & , & \phi_1 \\ 1, & (\phi_2+\alpha_2-\alpha_1)^2+\Lambda_2, & & \phi_2+\alpha_2-\alpha_1 \end{vmatrix}=0,$$

which is what the equation

$$\begin{vmatrix} 1, & \phi^2+\Lambda, & \phi \\ 1, & \phi_1^2+\Lambda_1, & \phi_1 \\ 1, & \phi_2^2+\Lambda_2, & \phi_2 \end{vmatrix}=0,$$

becomes on changing only ϕ_2 into $\phi_2+\alpha_2-\alpha_1$, that is, $\phi_2+2k_2\pi$, where k_2 is an arbitrary integer. We have thus through the two points a singly infinite series of geodesic lines; in general, only one of these is a shortest line, but for points on opposite meridians there are two equal shortest lines.

13. For the distance between two points (θ_1, ϕ_1) and (θ_2, ϕ_2) on the pseudosphere, taking (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) for the corresponding points on the imaginary sphere, and writing as above P_1, $Q_1=iX_1-Y_1$, iX_1+Y_1; P_2, $Q_2=iX_2-Y_2$, iX_2+Y_2, we have

$$\begin{aligned}\cosh\delta &= -X_1X_2-Y_1Y_2-Z_1Z_2,\\ &= \tfrac{1}{2}(P_1Q_2+P_2Q_1)-Z_1Z_2,\\ &= \sin\theta_1\sin\theta_2\{\tfrac{1}{2}(\phi_1^2+\operatorname{cosec}^2\theta_1)+\tfrac{1}{2}(\phi_2^2+\operatorname{cosec}^2\theta_2)-\phi_1\phi_2\},\\ &= \tfrac{1}{2}\sin\theta_1\sin\theta_2(\phi_2-\phi_1)^2+1+\frac{\frac{1}{2}(\sin\theta_2-\sin\theta_1)^2}{\sin\theta_1\sin\theta_2}.\end{aligned}$$

Observe here that, writing θ_2, $\phi_2 = \theta_1 + d\theta_1$, $\phi_1 + d\phi_1$, and therefore δ small so that $\cosh\delta = 1 + \frac{1}{2}\delta^2$, we obtain

$$\delta^2 = \sin^2\theta_1 d\phi_1^2 + \cot^2\theta_1 d\theta_1^2,$$

agreeing with the expression for $dx^2 + dy^2 + dz^2$. If in the form first obtained we write $\Lambda_1 = \operatorname{cosec}^2\theta_1$, $\Lambda_2 = \operatorname{cosec}^2\theta_2$, we find

$$\cosh\delta = \frac{\frac{1}{2}}{\sqrt{\Lambda_1\Lambda_2}}\{\phi_1^2 + \Lambda_1 + \phi_2^2 + \Lambda_2 - 2\phi_1\phi_2\},$$

which is a convenient form.

In like manner, to find the mutual inclination of the two geodesics

$$A_1 + B_1(\phi^2 + \operatorname{cosec}^2\theta) + C_1\phi = 0,$$
$$A_2 + B_2(\phi^2 + \operatorname{cosec}^2\theta) + C_2\phi = 0,$$

these correspond to the plane sections

$$A_1P + B_1Q + C_1Z = 0, \quad A_2P + B_2Q + C_2Z = 0,$$

that is,

$$(A_1 + B_1)iX + (-A_1 + B_1)Y + C_1Z = 0, \quad (A_2 + B_2)iX + (-A_2 + B_2)Y + C_2Z = 0,$$

of the imaginary sphere: and we thence find

$$\cos\Omega = \frac{C_1C_2 - 2(A_1B_2 + A_2B_1)}{\sqrt{C_1^2 - 4A_1B_1}\sqrt{C_2^2 - 4A_2B_2}}.$$

14. Suppose that the two geodesics meet in the point θ_0, ϕ_0: then writing for shortness $\operatorname{cosec}^2\theta = \Lambda$, and therefore $\operatorname{cosec}^2\theta_0 = \Lambda_0$, we have

$$A_1 + B_1(\phi_0^2 + \Lambda_0) + C_1\phi_0 = 0,$$
$$A_2 + B_2(\phi_0^2 + \Lambda_0) + C_2\phi_0 = 0.$$

Suppose that the meridian through this point is

$$A_3 + B_3(\phi^2 + \Lambda) + C_3\phi = 0;$$

then $B_3 = 0$, $A_3 + C_3\phi_0 = 0$. Take Ω_1, Ω_2, for the inclinations to this meridian of the two geodesics respectively; then

$$\cos\Omega_1 = \frac{C_1C_3 - 2A_3B_1}{\sqrt{C_1^2 - 4A_1B_1 \cdot C_3}} = \frac{C_1 + 2B_1\phi_0}{\sqrt{C_1^2 - 4A_1B_1}},$$

whence

$$\sin\Omega_1 = \frac{2B_1\sqrt{\Lambda_0}}{\sqrt{C_1^2 - 4A_1B_1}},$$

and similarly

$$\cos\Omega_2 = \frac{C_2 + 2B_2\phi_0}{\sqrt{C_2^2 - 4A_2B_2}},$$

whence

$$\sin \Omega_2 = \frac{2B_2 \sqrt{\Lambda_0}}{\sqrt{C_2^2 - 4A_2B_2}}.$$

We thence obtain

$$\cos(\Omega_1 - \Omega_2) = \frac{C_1C_2 + 2\phi_0(B_1C_2 + B_2C_1) + 4B_1B_2(\phi_0^2 + \Lambda_0)}{\sqrt{C_1^2 - 4A_1B_1}\sqrt{C_2^2 - 4A_2B_2}},$$

which is

$$= \frac{C_1C_2 - 2(A_1B_2 + A_2B_1)}{\sqrt{C_1^2 - 4A_1B_1}\sqrt{C_2^2 - 4A_2B_2}} = \cos \Omega,$$

as above, the equality of the two numerators depending on the identity

$$\{A_1 + B_1(\phi_0^2 + \Lambda_0) + C_1\phi_0\} B_2 + \{A_2 + B_2(\phi_0^2 + \Lambda_0) + C_2\phi_0\} B_1 = 0.$$

In particular, if we consider the two geodesics

$$\phi^2 + \operatorname{cosec}^2\theta - \operatorname{cosec}^2\theta_1 + C_1\phi = 0, \quad \phi = 0,$$

the second of which may be considered as representing any meridian section of the pseudosphere, and the first is an arbitrary geodesic meeting this at the point $\theta = \theta_1$, $\phi = 0$, then the formula for the inclination is

$$\cos \Omega = \frac{C_1}{\sqrt{C_1^2 + 4 \operatorname{cosec}^2\theta_1}}.$$

Hence also, $\cos \Omega = 0$, or $\Omega = 90°$, if $C_1 = 0$: viz. we have $\phi^2 + \operatorname{cosec}^2\theta - \operatorname{cosec}^2\theta_1 = 0$ for the equation of the geodesic through the point $\theta = \theta_1$, $\phi = 0$, at right angles to the meridian section $\phi = 0$.

15. Consider a right-angled triangle ACB, where the points A, C are on the meridian $\phi = 0$, and write $(\theta_1, 0;\ \Lambda_1 = \operatorname{cosec}^2\theta_1)$, $(\theta_2, \phi_2;\ \Lambda_2 = \operatorname{cosec}^2\theta_2)$, $(\theta_3, 0;\ \Lambda_3 = \operatorname{cosec}^2\theta_3)$, for the points A, B, C respectively. Then if the equations are—

for the side BC, $A_1 + B_1(\phi^2 + \Lambda) + C_1\phi = 0$, we have $C_1 = 0$,

$A_1 + B_1(\phi_2^2 + \Lambda_2) = 0$, $A_1 + B_1\Lambda_3 = 0$, whence $\phi_2^2 + \Lambda_2 = \Lambda_3$;

for the side CA, $A_2 + B_2(\phi^2 + \Lambda) + C_2\phi = 0$, we have $A_2 = 0$, $B_2 = 0$;

for the side AB, $A_3 + B_3(\phi^2 + \Lambda) + C_3\phi = 0$,

we have $A_3 + B_3(\phi_2^2 + \Lambda_2) + C_3\phi_2 = 0$, $A_3 + B_3\Lambda_1 = 0$.

Observing that $\phi_1 = \phi_2 = 0$, we have

$$\cosh a = \frac{1}{2\sqrt{\Lambda_2\Lambda_3}}(\phi_2^2 + \Lambda_2 + \Lambda_3),$$

$$\cosh b = \frac{1}{2\sqrt{\Lambda_1\Lambda_3}}(\Lambda_3 + \Lambda_1),$$

$$\cosh c = \frac{1}{2\sqrt{\Lambda_1\Lambda_2}}(\phi_2^2 + \Lambda_1 + \Lambda_2);$$

or, reducing these by the relation $\phi_2^2 + \Lambda_2 = \Lambda_3$, they become

$$\cosh a = \frac{\sqrt{\Lambda_3}}{\sqrt{\Lambda_2}}, \text{ whence } \sinh a = \frac{\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_2}}, \ \tanh a = \frac{\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_3}};$$

$$\cosh b = \frac{\Lambda_1 + \Lambda_3}{2\sqrt{\Lambda_1\Lambda_3}}, \quad \text{,,} \quad \sinh b = \frac{\Lambda_1 - \Lambda_3}{2\sqrt{\Lambda_1\Lambda_3}}, \ \tanh b = \frac{\Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_3};$$

$$\cosh c = \frac{\Lambda_1 + \Lambda_3}{2\sqrt{\Lambda_1\Lambda_2}}, \quad \text{,,} \quad \sinh c = \frac{\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1\Lambda_2}}{2\sqrt{\Lambda_1\Lambda_2}}, \ \tanh c = \frac{\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1\Lambda_2}}{\Lambda_1 + \Lambda_3}.$$

We have, moreover,

$$\cos B = \frac{C_1C_3 - 2(A_1B_3 + A_3B_1)}{\sqrt{C_1^2 - 4A_1B_1}\sqrt{C_3^2 - 4A_3B_3}} = \frac{-2(A_1B_3 + A_3B_1)}{\sqrt{-4A_1B_1}\sqrt{C_3^2 - 4A_3B_3}},$$

which, writing $A_3 = -B_3\Lambda_1$ and $A_1 = -B_1\Lambda_3$, becomes

$$\cos B = \frac{B_3(\Lambda_1 + \Lambda_3)}{\sqrt{\Lambda_3}\sqrt{C_3^2 - 4A_3B_3}};$$

or, further reducing by means of

$$\begin{aligned}\phi_2^2(C_3^2 - 4A_3B_3) &= B_3^2\,(\phi_2^2 + \Lambda_2 - \Lambda_1)^2 + 4\phi_2^2B_3^2\Lambda_1^2 \\ &= B_3^2\,\{(\phi_2^2 + \Lambda_2 - \Lambda_1)^2 + 4\phi_2^2\Lambda_1\} \\ &= B_3^2\,\{(\Lambda_3 + \Lambda_1)^2 + 4\Lambda_1(\Lambda_3 - \Lambda_2)\} \\ &= B_3^2\,\{(\Lambda_3 + \Lambda_1)^2 - 4\Lambda_1\Lambda_2\},\end{aligned}$$

this becomes

$$\cos B = \frac{(\Lambda_1 + \Lambda_3)\sqrt{\Lambda_3 - \Lambda_2}}{\sqrt{\Lambda_3}\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1\Lambda_2}},$$

whence

$$\sin B = \frac{\sqrt{\Lambda_2}(\Lambda_1 - \Lambda_3)}{\sqrt{\Lambda_3}\sqrt{(\Lambda_1 + \Lambda_3)^2 - 4\Lambda_1\Lambda_2}};$$

and with these values we verify

$$\tanh a = \tanh c \,.\, \cos B,$$
$$\sinh b = \sinh c \,.\, \sin B,$$

which are the expressions for the sides BC, CA, in terms of the length BA, $= c$ and angle B, which are arbitrary. I have not thought it necessary to give the direct verification of these equations for a more general position of the right-angled triangle: we already know, and it appears *à posteriori* by the following number, that the verification really extends to any right-angled triangle whatever on the surface.

16. The pseudosphere is homogeneous, isotropic, and palintropic, viz. this is the case when bending is allowed; in other words, the surface is applicable upon itself, with three degrees of freedom. Considering any infinitesimal linear element Ax, the point A may be brought to coincide with an arbitrary point A' of the surface, and

the element Ax to lie in an arbitrary direction $A'x'$ through A'; the area about A will then coincide with the area about A'. The analytical theory is at once derived from that for the sphere, viz. we have a rectangular transformation

	X	Y	Z
X_1	α	β	γ
Y_1	α'	β'	γ'
Z_1	α''	β''	γ''

where the coefficients are such that identically

$$X_1^2 + Y_1^2 + Z_1^2 = X^2 + Y^2 + Z^2;$$

in fact, the coefficients are connected by six equations only, the system thus depending on three arbitrary parameters. If, as before, we write P_1, Q_1, P, Q, for $iX_1 - Y_1$, $iX_1 + Y_1$, $iX - Y$, $iX + Y$ respectively, then the relation is readily found to be

	P	Q	Z
P_1	$\frac{1}{2}(\alpha + i\alpha' - i\beta + \beta')$	$\frac{1}{2}(\alpha + i\alpha' + i\beta - \beta')$	$i\gamma - \gamma'$
Q_1	$\frac{1}{2}(\alpha - i\alpha' - i\beta - \beta')$	$\frac{1}{2}(\alpha - i\alpha' + i\beta + \beta')$	$i\gamma + \gamma'$
Z_1	$\frac{1}{2}(-i\alpha'' - \beta'')$	$\frac{1}{2}(-i\alpha'' + \beta'')$	γ''

this being read according to the lines only $P_1 = \frac{1}{2}(\alpha + i\alpha' - i\beta + \beta')P + \&c.$, not according to the columns: in order that the coefficients may be real, we must have $\alpha, \beta', \gamma', \beta'', \gamma''$, real, $\beta, \gamma, \alpha', \alpha''$ pure imaginaries.

Writing the equations in the form

	P	Q	Z
P_1	A	B	C
Q_1	A'	B'	C'
Z_1	A''	B''	C''

viz.

$$P_1 = AP + BQ + CZ, \ \&c.,$$

it would be possible to deduce the equations which connect the new coefficients; but these are more easily obtained from the consideration that we must have identically $P_1Q_1 - Z_1^2 = PQ - Z^2$; the equations are thus found to be

$$A''^2 - AA' = 0, \quad B''^2 - BB' = 0, \quad C''^2 - CC' = 1,$$
$$2A''B'' - AB' - A'B = -1, \quad 2A''C'' - AC' - A'C = 0, \quad 2B''C'' - BC' - B'C = 0.$$

17. The general theory of the transformation of a quadric function into itself enables us to express the coefficients in terms of three arbitrary parameters. There is no difficulty in working out the formulæ, and we finally obtain

$$\begin{aligned}
\Omega P_1 &= -(\nu+1)^2 P - \lambda^2 Q + 2\lambda(\nu+1) Z,\\
\Omega Q_1 &= -\mu^2 P - (\nu-1)^2 Q + 2\mu(\nu-1) Z,\\
\Omega Z_1 &= -\mu(\nu+1) P - \lambda(\nu-1) Q + (-1+\nu^2+\lambda\mu) Z;
\end{aligned}$$

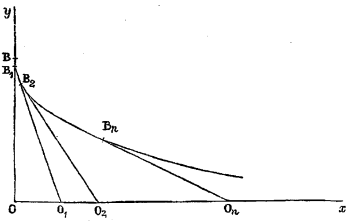
and conversely

$$\begin{aligned}
\Omega P &= -(\nu-1)^2 P_1 - \lambda^2 Q_1 + 2\lambda(\nu-1) Z_1,\\
\Omega Q &= -\mu^2 P_1 - (\nu+1)^2 Q_1 + 2\mu(\nu+1) Z_1,\\
\Omega Z &= -\mu(\nu-1) P_1 - \lambda(\nu+1) Q_1 + (1+\nu^2+\lambda\mu) Z_1,
\end{aligned}$$

where $\Omega = -1+\nu^2-\lambda\mu$: it can be at once verified that each of the two sets of formulæ does, in fact, give $P_1Q_1 - Z_1^2 = PQ - Z^2$.

18. The pseudosphere is a surface of revolution having for its meridian section the curve $x = \log\cot\frac{1}{2}\theta - \cos\theta$, $y = \sin\theta$. This is a curve symmetrical in regard to the axis of y; and we obtain the portion of it lying on the positive side of this axis, by giving to θ the series of values $\theta = 0$ to $\theta = 90°$; for $\theta = 0$, we have $y = 0$, $x = \infty$, or the axis of x is an asymptote; for $\theta = 90°$, $x = 0$, $y = 1$, the point being a cusp of the curve. The geometrical definition is that the portion of the tangent included between the curve and the axis of x has the constant length $= 1$; the inclination of

Fig. 2.



the tangent is in fact $= \theta$. We have $dx = \dfrac{\cos^2\theta\, d\theta}{\sin\theta}$, $dy = \cos\theta\, d\theta$; and thence $ds = \cot\theta\, d\theta$, and the length in question is $\dfrac{y\,ds}{dy} = 1$. The curve may be constructed graphically: take (fig. 2) the distance $BO = 1$, on OB, B_1 very near to B, and then $B_1O_1 = 1$; on

O_1B_1, B_2 very near to B_1, and then $B_2O_2=1$, and so on; the curve is shown on a larger scale in fig. 3, p. 235.

But the curve may also be laid down numerically; writing $\alpha=\frac{1}{2}\pi-\theta$, so that α is the inclination of the tangent to the axis of y, we have $x=\log\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)-\sin\alpha$, $y=\cos\alpha$, where $\log\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$, the hyperbolic logarithm (which has been the signification of log throughout), is the function tabulated Tab. IV., Legendre's *Traité des Fonctions Elliptiques*, t. II. pp. 256—259.

We may hence obtain the values of the coordinates as follows:—

$\alpha=90°-\theta$	$\log\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$	$-\sin\alpha$	x	$y=\cos\alpha$
0	0·0000000	−0·0000000	0·0000000	1·0000000
10°	0·1754258	0·1736482	0·0017776	0·9848078
20°	0·3563785	0·3420201	0·0143584	0·9396926
30°	0·5493061	0·5000000	0·0493061	0·8660254
40°	0·7629096	0·6427876	0·1201220	0·7660444
50°	1·0106831	0·7660444	0·2446387	0·6427876
60°	1·3169578	0·8660254	0·4509324	0·5000000
70°	1·7354151	0·9396926	0·7957225	0·3563785
80°	2·4362460	0·9848078	1·4514382	0·1754258
85°	3·1313013	0·9961947	2·1351066	0·0871557
86°	3·3546735	0·9975641	2·3571019	0·0697565
87°	3·6425333	0·9986295	2·6439038	0·0523360
88°	4·0481254	0·9993908	3·0487346	0·0348995
89°	4·7413487	0·9998477	3·7415010	0·0174524
90°	∞	−1·0000000	∞	0·0000000

Attending only to one-half of the surface, we may regard the surface as standing on the circular base $y^2+z^2=1$: say this circle is the equator, or the unit-circle: the horizontal section being always a circle, the radius diminishing at first rapidly and then more and more slowly from 1 to 0 as the height increases from 0 to ∞. It is hardly necessary to remark that the radius of the equator is any given length whatever, taken as unity: the equations might, of course, have been written

$$x=c\{\log\cot\tfrac{1}{2}\theta-\cos\theta\},\quad \sqrt{y^2+z^2}=c\sin\theta,$$

but there would have been no gain of generality in this.

19. The geodesics are as already seen given by an equation

$$A+B(\phi^2+\operatorname{cosec}^2\theta)+C\phi=0.$$

If $B=0$, we have $A+C\phi=0$, that is, $\phi=$ const., which belongs to the meridians; if B be not $=0$, we may by a mere change of ϕ, that is, of the initial meridian, reduce the form to $A+B(\phi^2+\operatorname{cosec}^2\theta)=0$, which is the equation of a geodesic cutting at right angles the meridian $\phi=0$; writing herein $\sin\theta=\frac{1}{r}$, we have $A+B\left(\phi^2+\frac{1}{r^2}\right)=0$, which is the equation in the polar coordinates r, ϕ of the projection of the geodesic on the equatorial plane $x=0$: putting herein for greater convenience $B=-Ak^2$, we have $r^2=\frac{k^2}{1-k^2\phi^2}$: we require only such portions of the curves as lie within the unit-circle, and need therefore attend only to those for which k is not greater than 1, and in any such curve consider ϕ as extending from $\phi=0$ to $\phi=\pm\frac{\sqrt{1-k^2}}{k}$: writing this last value $=\pm\gamma$, we have $k=\frac{1}{\sqrt{1+\gamma^2}}$; if $\gamma<\pi$, that is, $k<\frac{1}{\sqrt{1+\pi^2}}$, the curve is a mere arc cutting at right angles (at the distance $r=k$ from the centre) the meridian $\phi=0$, and extending itself out on each side to meet the unit-circle in the points $\phi=\gamma$, $\phi=-\gamma$ respectively; in the case $\gamma=\pi$, that is, $k=\frac{1}{\sqrt{1+\pi^2}}$, the two points $\phi=\pm\gamma$ come together at the point $\phi=\pi$, or the curve becomes a loop; and for larger values, $k=\frac{1}{\sqrt{1+\pi^2}}$ to $\frac{1}{\sqrt{1+4\pi^2}}$, we have the two branches crossing each other on the meridian $\phi=\pi$ at the distance $r=\frac{k}{\sqrt{1-k^2\pi^2}}$ from the centre and then extending themselves in the opposite semicircles, so as to meet the unit-circle at the points $\phi=\pm\gamma$. And we have thus another critical value $k=\frac{1}{\sqrt{1+4\pi^2}}$, for which the two branches having thus crossed each other come to unite themselves at the point $\phi(=2\pi)=0$ of the unit-circle; and in like manner the critical values $\frac{1}{\sqrt{1+9\pi^2}}$, $\frac{1}{\sqrt{1+16\pi^2}}$, &c.: for a value of k between such limits, the branch is a spiral having a determinate number of convolutions, and the two branches cross each other always on the radii $\phi=0$ and $\phi=\pi$ respectively.

20. Let ψ denote the inclination of the radius vector to the normal, or, what is the same thing, that of the element of the circular arc to the tangent; we have $\tan\psi=\frac{dr}{rd\phi}$, and $\frac{dr}{rd\phi}=\frac{k^2\phi}{1-k^2\phi^2}$, $=r^2\phi$, that is, $\tan\psi=r^2$. At the intersection with the unit section $r=1$, and therefore $\tan\psi=\phi$; moreover putting $k=\cos\kappa$, so that the equation of the curve now is $r^2=\frac{\cos^2\kappa}{1-\phi^2\cos^2\kappa}$, then for $r=1$ we have $\phi=\tan\kappa$; and hence at the intersection with the unit-circle $\psi=\kappa$, that is, as k decreases from $k=1$, or k increases from $k=0$, the angle at which each curve cuts the unit-circle is always $=\kappa$, and thus this angle continually increases from $\kappa=0$; for $k=\frac{1}{\sqrt{1+\pi^2}}=\cos\kappa$,

and therefore $\tan\kappa = \pi$, we have $\kappa = 72° 20'$ nearly: the complement hereof $17° 40'$ is thus the angle at which each branch of the loop cuts the meridian $\phi = \pi$.

21. To obtain another datum convenient in tracing the curve, I write $\phi = \phi_0 = \tan\kappa$ for the value of ϕ at the unit-circle; and introducing for the moment the rectangular coordinates $X = r\sin\phi$, $Y = 1 - r\cos\phi$, then we easily find

$$\frac{dY}{dX} = \frac{r\sin\phi - r^2\phi\cos\phi}{r\cos\phi + r^2\phi\sin\phi};$$

and thence, for the equation of the tangent at the point on the unit-circle,

$$(y - 1 + \cos\phi_0) = \frac{\sin\phi_0 - \phi_0\cos\phi_0}{\cos\phi_0 + \phi_0\sin\phi_0}(x - \sin\phi_0).$$

For the tangent at the point of intersection with the radius $\phi = 0$, or say the apse, we have $y = 1 - \cos\kappa$; and hence, at the intersection of the two tangents,

$$x = \sin\phi_0 + \frac{\cos\phi_0 + \phi_0\sin\phi_0}{\sin\phi_0 - \phi_0\cos\phi_0}(\cos\phi_0 - \cos\kappa)$$

$$= \frac{1 - \cos\kappa\,(\cos\phi_0 + \phi_0\sin\phi_0)}{\sin\phi_0 - \phi_0\cos\phi_0},$$

which, putting therein $\phi_0 = \tan\kappa$, becomes

$$= \frac{\cos\kappa\,\{1 - \cos(\phi_0 - \kappa)\}}{\sin(\phi_0 - \kappa)} = \cos\kappa\tan\tfrac{1}{2}(\phi_0 - \kappa),$$

where ϕ_0 is given in terms of κ by the just-mentioned equation $\phi_0 = \tan\kappa$. We have $y = 1 - \cos\kappa$, $x = \cos\kappa\tan\frac{1}{2}(\phi_0 - \kappa)$, for the locus of the intersection of the two tangents; this is easily seen to be a curve having a cusp at the unit-circle.

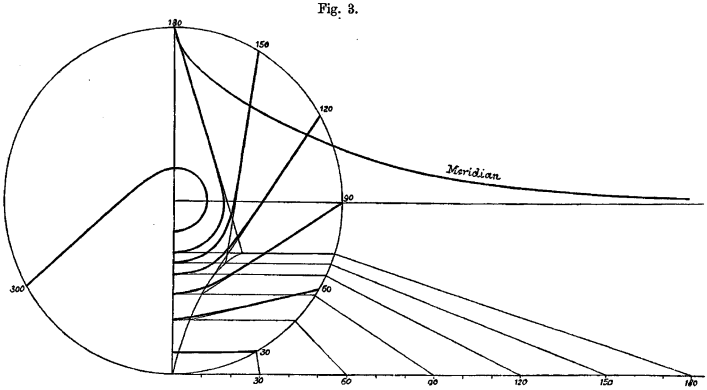
22. Fig. 3 shows the curves for the values

$\phi_0 =$	$\tan\kappa$	$\kappa =$
$30° = \frac{1}{6}\pi$	0·5235988	27° 38′
60	1·0471976	46 19
90	1·5707963	57 31
120	2·0943941	64 29
150	2·6179939	69 5
$180 = \pi$	3·1415926	72 20

We construct and graduate the unit-circle; draw to it a tangent at 0°, and measuring off from 0 a distance equal to the semi-circumference, graduate this in like manner in equal parts 0° to 180°; then to find the curve belonging, for instance, to

$\phi_0 = 90°$, we join with the centre of the circle the point 90° of the tangent, thus determining on the unit-circle a point belonging to the angle $\kappa = 57° 31'$; at this point we draw parallel to the tangent a line which is the tangent at the lowest

Fig. 3.



point; the curve passes through the point 90° on the unit-circle, and there cuts the circle at the angle $\kappa = 57° 31'$ (or, what is the same thing, the radius at the complementary angle), and we have thus the tangent at the point 90° of the unit-circle; it will be noticed that this meets the tangent at the apse at a point near to this apse, so that the arc as determined by the two tangents is for a large part of its course nearly a right line; this is still more the case for smaller values of ϕ_0 or κ, while for larger values the deviation increases, but in the neighbourhood of the unit-circle the form is always nearly rectilinear.

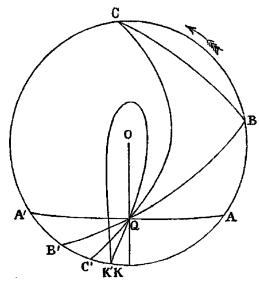
I show in the same figure the form of the curve for $\phi_0 = 300°$, $= 5{\cdot}2359877$, $= \tan\kappa$, that is, $\kappa = 79° 11'$, $r = \cos\kappa = 0{\cdot}1876670$, the value at the apse: the construction for the tangent at the unit-circle is the same as before, but in order to lay down the curve with tolerable accuracy we require also the value of r at the node on the meridian $\phi = 180°$; this is, of course, given by $r = \dfrac{\cos\kappa}{\sqrt{1 - \pi^2 \cos^2\kappa}}$, that is, $r = \dfrac{\cos\kappa}{\cos\alpha}$, if $\pi \cos\kappa = \sin\alpha$; whence without difficulty $r = 0{\cdot}23236$, the value at the node.

23. The curves shown in the figure are projections upon the plane of the unit-circle, viz. they are the projections on this plane of the geodesics, which cut at right angles a given meridian; but bearing in mind the form of the meridian, it is easy, by means of the projection, to understand the actual forms on the surface of the pseudosphere. A point near the centre of the figure represents a point high up on the surface; and in any radius the portions near the centre are the more foreshortened in the figure, and represent greater distances on the surface. Each geodesic

cutting the meridian at right angles at the apse descends symmetrically on the two sides, reaches ultimately—it may be after many convolutions—the unit-circle; the meridian itself is a limiting or special form of geodesic. The unit-circle is not properly a geodesic, but it is an envelope of geodesics.

24. To obtain all the geodesics, we have to consider the geodesics which cut at right angles a given meridian; and then to imagine this meridian (with the geodesics which belong to it) turned round so as to occupy successively the positions of all the other meridians. The same remark applies of course to the projections; the figure shows the projections cutting at right angles a given radius of the circle; and this radius (with the projections belonging to it) is then to be turned round so as to occupy successively the positions of all the other radii. We may imagine the several geodesics turned round separately, each through a different angle, so as to bring each of them to pass through one and the same point of the surface; we have then the geodesics drawn in all directions through this point of the surface; doing the same thing with the projections, we have, it is clear, the projections of the geodesics drawn in all directions through the point. It is easy, by drawing the projections each on a separate circle of paper, and passing a pin through the centres, to form a model by means of which an accurate figure of the projection may be constructed. But I content myself with a mere diagram (fig. 4).

Fig. 4.



25. Taking a point Q so low down on the surface that the geodesic at right angles to the meridian through Q is a simple arc $A'A$, then imagine the two extremities A, A' each moving in the same sense round the circle, but A faster than A', so as to assume the positions B, B'; C, C'; and so on to K, K' coinciding with each other. We have the arcs $B'B$, $C'C$, and so on until we come to the loop form $K'K$: after which we have L' in advance of L, and so on to curves of any number of convolutions. Considering any two arcs—$B'B$, $C'C$—and drawing the geodesic BC which joins their extremities B and C, then any geodesic through Q intermediate to $B'B$, $C'C$, or, say, to QB, QC, will meet the arc BC; while the geodesics through Q extramediate to QB, QC *will not meet, or will only after a convolution or convolutions meet,*

the arc BC. This of course corresponds to the Lobatschewskian theory, according to which we have through a point Q to the extremities at infinity of a line BC, two distinct lines QB, QC, said to be the parallels through Q of the line BC; and which are such that any line through Q intermediate to QB, QC meets the line BC; while any line through Q extramediate to QB, QC *does not meet* the line BC.

26. It is interesting to connect the theory of the geodesics of the pseudosphere with the general theory of geodesics. Starting with the form

$$ds^2 = \cot^2\theta\, d\theta^2 + \sin^2\theta\, d\phi^2, \quad = E\, d\theta^2 + 2F\, d\theta\, d\phi + G\, d\phi^2,$$

we have $E = \cot^2\theta$, $F = 0$, $G = \sin^2\theta$; and therefore $E + 1 = \dfrac{1}{G}$, or $E = \dfrac{1}{G} - 1$, and the differential equation of the geodesic becomes

$$E\theta' \,.\, 2G_1\theta'\phi' - G\phi'(E_1\theta'^2 - G_1\phi'^2) + 2EG(\theta'\phi'' - \theta''\phi') = 0,$$

that is,

$$\phi'[(2EG_1 - GE_1)\,\theta'^2 + GG_1\phi'^2] + 2EG(\theta'\phi'' - \theta''\phi') = 0,$$

where

$$E_1 = \frac{dE}{d\theta}, \quad G_1 = \frac{dG}{d\theta};$$

and writing here

$$E = \frac{1}{G} - 1,$$

we have

$$E_1 = -\frac{G_1}{G^2}, \quad 2EG_1 - E_1G = G_1\left(\frac{3}{G} - 2\right).$$

Moreover, from $G = \sin^2\theta$, we find $G_1 = 2\sqrt{G\,.\,1-G}$; and the equation becomes

$$\frac{\sqrt{G}}{\sqrt{1-G}}\left[\left(\frac{3}{G} - 2\right)\theta'^2 + G\phi'^2\right]\phi' + \theta'\phi'' - \theta''\phi' = 0.$$

Introducing here G in place of θ by the equation $G = \sin^2\theta$, we have

$$\theta' = \frac{G'}{2\sqrt{G\,.\,1-G}},$$

$$\theta'' = \frac{1}{4G^{\frac{3}{2}}(1-G)^{\frac{3}{2}}}\{2G(1-G)\,G'' - G'^2(1-2G)\},$$

and the equation thus becomes

$$(3-2G)\,G'^2\phi' + 4G^2(1-G)\,\phi'^3 + 2G(1-G)\,G'\phi'' + \{-2G(1-G)\,G'' + (1-2G)\,G'^2\}\,\phi' = 0.$$

The whole term in ϕ' is thus $\phi'\{-2G(1-G)\,G'' + (4-4G)\,G'^2\}$, which divides by $2(1-G)$; the whole equation thus divides by $2(1-G)$, and omitting this factor, the equation becomes

$$\phi'(2G'^2 - GG'') + 2\phi'^3G^2 + \phi''GG' = 0,$$

which is simplified by introducing $H = \sin^2\theta = \frac{1}{G}$, instead of G. We, in fact, have

$$G' = \frac{-H'}{H^2}, \quad G'' = \frac{-H''}{H^2} + \frac{2H'^2}{H^3},$$

and substituting these values, the equation becomes

$$\phi'\left(\frac{2H'^2}{H^4} + \frac{H''}{H^3} - \frac{2H'^2}{H^4}\right) + \frac{2}{H^3}\phi'^3 - \frac{H'}{H^3}\phi'' = 0;$$

viz. this is

$$H''\phi' + 2\phi'^3 - H'\phi'' = 0.$$

Writing herein $\phi + \alpha = \sqrt{K}$ (α an arbitrary constant), we have

$$\phi' = \frac{\frac{1}{2}K'}{\sqrt{K}}, \quad \phi'' = \frac{\frac{1}{2}K''}{\sqrt{K}} - \frac{\frac{1}{4}K'^2}{K\sqrt{K}},$$

and the equation becomes

$$H''\frac{\frac{1}{2}K'}{\sqrt{K}} + \frac{\frac{1}{4}K'^3}{K\sqrt{K}} - \frac{\frac{1}{2}H'K''}{\sqrt{K}} + \frac{\frac{1}{4}H'K'^2}{K\sqrt{K}} = 0;$$

viz. this is

$$2(H''K' - H'K'')K + (K' + H')K'^2 = 0,$$

which is satisfied by $K' + H' = 0$ or $K + H = \beta$, β an arbitrary constant. Substituting for K, H their values, this is

$$(\phi + \alpha)^2 + \sin^2\theta = \beta,$$

that is,

$$\alpha^2 - \beta + (\phi^2 + \operatorname{cosec}^2\theta) + 2\alpha\phi = 0,$$

or, what is the same thing,

$$A + B(\phi^2 + \operatorname{cosec}^2\theta) + C\phi = 0,$$

where the ratios $A : B : C$ are arbitrary.

828.

A MEMOIR ON SEMINVARIANTS.

[From the *American Journal of Mathematics*, t. VII. (1885), pp. 1—25.]

Introductory. Art. Nos. 1 to 3.

1. A VERY remarkable discovery in the Theory of Seminvariants has been recently made by Capt. MacMahon, viz. considering the equation

$$0 = 1 + b\frac{x}{1} + c\frac{x^2}{1.2} + d\frac{x^3}{1.2.3} + \&c.,$$

and its roots $\alpha, \beta, \gamma, \ldots$, as defined by the identity

$$1 + b\frac{x}{1} + c\frac{x^2}{1.2} + \&c. = 1 - \alpha x . 1 - \beta x . 1 - \gamma x \ldots,$$

then, any symmetrical function of the roots being represented by a partition symbol in the usual manner, $1 = \Sigma\alpha$, $2 = \Sigma\alpha^2$, 11 or $1^2 = \Sigma\alpha\beta$, &c., the theorem is that any symmetric function represented by a non-unitary symbol (or symbol not containing a 1), say for shortness any non-unitary symmetric function, is a seminvariant in regard to the coefficients 1, b, c, d, e, &c.

We have for instance

$$2 = - \quad (c - b^2),$$

$$3 = -\tfrac{1}{2}(d - 3bc + 2b^3),$$

$$4 = \quad \tfrac{1}{6}(-e + 4bd + 3c^2 - 12bc + 6b^4),$$

where to verify that this is a seminvariant, observe that the value may be written

$$= \tfrac{1}{6}\{-(e-4bd+3c^2)+6(c-b^2)^2\},$$

$$22 = \tfrac{1}{12}(e-4bd+3c^2),$$

&c.;

and observe further that the forms 2, 4 and 22, are connected by the identical relation

$$2\,.\,2 = 4 + {}_2\,.\,22.$$

2. We conclude that the theory of seminvariants is a part of that of symmetric functions. I take the opportunity of remarking that (the subject of the memoir being seminvariants) I use in general non-unitary symbols, even in cases where the restriction is unnecessary, and the symbols might have contained 1's: thus instead of $2\,.\,2 = 4 + {}_2\,.\,22$, the equation $1\,.\,1 = 2 + {}_2\,.\,11$ would have served equally well as an instance of an identical relation between symmetric functions; and so in general, in formulæ relating to symmetric functions, the symbols are not restricted to be non-unitary. I remark also that, for instance, instead of 4443322222, or $4^3 3^3 2^5$, I usually write 444332^5, introducing the index only for the 2; the reason is only that the 2 is often repeated a large number of times, so that the abbreviation, which I dispense with for the higher numbers, becomes convenient for the 2.

3. Reckoning the coefficients $1, b, c, d, e, \ldots$ as being each of them of the degree 1, and of the weights $0, 1, 2, 3, 4, \ldots$ respectively, then any symmetric function is of a degree which is equal to the highest number, and of a weight which is equal to the sum of all the numbers, in the partition-symbol. And we frequently speak of the deg. weight: thus for the function 22, the deg. weight is $= 2\,.\,4$.

Multiplication of Two Symmetric Functions. Art. Nos. 4 to 17.

4. We require a theory for the multiplication of two symmetric functions. We have for instance $3\,.\,2 = 5 + 32$: for 3 denoting $\Sigma\alpha^3$, and 2 denoting $\Sigma\alpha^2$, the product contains the term α^5, and the term $\alpha^3\beta^2$, and it is thus $= \Sigma\alpha^5 + \Sigma\alpha^3\beta^2$, which is $= 5 + 32$. But multiplying for instance 2 by itself, the product contains the term α^4, and the term $\alpha^2\beta^2$ twice, and it is thus $= \Sigma\alpha^4 + 2\Sigma\alpha^2\beta^2$, and we thus have the before-mentioned formula $2\,.\,2 = 4 + {}_2\,.\,22$.

And so, l, m being different

$$l\,.\,m = (l+m) + lm;$$

but when $m = l$,

$$l\,.\,l = (2l) \quad + {}_2\,.\,ll.$$

And in general, for any symmetric function $lmnpqr\ldots$, where the numbers are all of them different, if any two of the m's become equal, we must multiply the term by 2; if any three of them become equal, we must multiply the term by 6; and so in other cases, viz. if the term becomes $l^\alpha m^\beta n^\gamma \ldots$, we must multiply it by $[\alpha]^\alpha\,[\beta]^\beta\,[\gamma]^\gamma \ldots$

5. We may, taking in the first instance the numbers $l, m, n, p, q, \ldots$ to be all of them different, develope an algorithm as follows:

$$\begin{array}{l|l} l & m \\ \hline m & \\ & \\ & m \end{array} = \begin{array}{l} (l+m), \\ \\ lm \end{array} \quad \text{that is,} \quad \begin{array}{l} l\,.\,m = (l+m) \\ \\ \qquad\qquad + \; lm, \end{array}$$

$$\begin{array}{l|l} lm & n \\ \hline n & \\ & \\ n & \\ & n \end{array} = \begin{array}{l} (l+n)\,m, \\ \\ l\,(m+n) \\ \\ lmn \end{array} \quad \text{that is,} \quad \begin{array}{l} lm\,.\,n = \quad (l+n)\,m \\ \qquad\qquad + \, l\,(m+n) \\ \qquad\qquad + \, lmn, \end{array}$$

and so in other cases;

$$\begin{array}{l|l} lmn & p \\ \hline p & \\ & \\ p & \\ & \\ & p \end{array} = \begin{array}{l} (l+p)\,mn \\ \\ l\,(m+p)\,n \\ \\ lm\,(n+p) \\ \\ lmnp, \end{array}$$

$$\begin{array}{l|l|l} lm & np & \\ \hline np & & = (l+n)\,(m+p) \\ pn & & \quad (l+p)\,(m+n) \\ n & p & \quad (l+n)\,mp \\ p & n & \quad (l+p)\,mn \\ n & p & \quad l\,(m+n)\,p \\ p & n & \quad l\,(m+p)\,n \\ & np & \quad lmnp, \end{array}$$

$$\begin{array}{l|l|l} lmn & pq & \\ \hline pq & & = (l+p)\,(m+q)\,n \\ qp & & \quad (l+q)\,(m+p)\,n \\ p\;\;q & & \quad (l+p)\,m\,(n+q) \\ q\;\;p & & \quad (l+q)\,m\,(n+p) \\ pq & & \quad l\,(m+p)\,(n+q) \\ qp & & \quad l\,(m+q)\,(n+p) \\ p & q & \quad (l+p)\,mnq \\ q & p & \quad (l+q)\,mnp \\ p & q & \quad l\,(m+p)\,q \\ q & p & \quad l\,(m+q)\,p \\ p & q & \quad lm\,(n+p)\,q \\ q & p & \quad lm\,(n+q)\,p \\ & pq & \quad lmnpq\,; \end{array}$$

and so on.

6. Observe that, if the two factors contain i numbers and j numbers respectively, $i >$ or $= j$, then in the product we have

$$\begin{array}{lll}
[i]^j & \text{terms each containing } i \text{ numbers,} & \\
\frac{[j]^1}{[1]^1}[i]^{j-1} & \quad \text{,,} \quad \text{,,} \quad i+1 & \text{,,} \\
\frac{[j]^2}{[2]^2}[i]^{j-2} & \quad \text{,,} \quad \text{,,} \quad i+2 & \text{,,} \\
\vdots & & \\
\frac{[j]^j}{[j]^j}[i]^0, = 1, & \text{term containing } i+j & \text{,,}
\end{array}$$

so that the whole number of terms in the product is

$$\{i, j\}, = [i]^j + \frac{j}{1}[i]^{j-1} + \ldots + 1.$$

We may, if we please, take the smaller number first, and we then have

$$\{j, i\} = \{i, j\};$$

the $\{j, i\}$ series in fact begins with zero terms, but following these we have terms which are identical each to each with those of the $\{i, j\}$ series. Thus

$$\{5, 3\} = \quad [5]^3 + 3[5]^2 + 3[5]^1 + [5]^0 = \quad 60 + 3.20 + 3.5 + 1, = 136,$$
$$\{3, 5\} = [3]^5 + 5[3]^4 + 10[3]^3 + 10[3]^2 + 5[3]^1 + [3]^0 = 0 + 0 + 10.6 + 10.6 + 5.3 + 1, = 136$$

and it is easy to see that the general theorem can be verified in like manner.

In particular, we have		Putting $i = 1$, these give
$\{i, 1\} = [i]^1 + 1$	$= i + 1$	, $\{1, 1\} = 2$,
$\{i, 2\} = [i]^2 + 2[i]^1 + 1$	$= i^2 + i + 1$	, $\{1, 2\} = 3$,
$\{i, 3\} = [i]^3 + 3[i]^2 + 3[i]^1 + 1$	$= i^3 + 2i + 1$	, $\{1, 3\} = 4$,
$\{i, 4\} = [i]^4 + 4[i]^3 + 6[i]^2 + 4[i]^1 + 1$	$= i^4 - 2i^3 + 5i^2 + 1$	, $\{1, 4\} = 5$,
$\{i, 5\} = [i]^5 + 5[i]^4 + 10[i]^3 + 10[i]^2 + 5[i]^1 + 1$	$= i^5 - 5i^4 + 15i^3 - 15i^2 + 9i + 1$,	$\{1, 5\} = 6$,

which agree with

$$\{i, 1\} = i + 1.$$

Hence also the values of

$\{1, 1\}$,	are	2,
$\{1, 2\}$, $\{2, 2\}$,		3, 7,
$\{1, 3\}$, $\{2, 3\}$, $\{3, 3\}$,		4, 13, 34,
$\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$, $\{4, 4\}$,		5, 21, 73, 209,
$\{1, 5\}$, $\{2, 5\}$, $\{3, 5\}$, $\{4, 5\}$, $\{5, 5\}$,		6, 31, 136, 501, 1546.

7. In forming a product $lmn \ldots pqr \ldots$, we may have equalities among the numbers $l, m, n, \ldots$ of the first symbol, and also between the numbers $p, q, r, \ldots$ of the second symbol: moreover (whether there are or are not any such equalities), we may have equalities presenting themselves between the numbers $l+p$, $m+q, \ldots$ of any symbol $(l+p)(m+q)\ldots$ on the right-hand side of the equation: and the process must be performed so as to take account of all these equalities. The actual process is best shown by an example: say we require the product 3332 . 322, which is of the deg. weight 6 . 18.

3332	322			÷12	
322		6	6552	2	1
32 2		12	6543	1	1
22 3		6	5553	6	3
32	2	12	65322	2	2
3 2	2	6	64332	2	1
2 3	2	6	55332	4	2 }
22	3	6	55332	4	2 }
2 2	3	6	54333	6	3
3	22	3	633222	12	3
3	22	1	533322	12	1 }
2	32	6	533322	12	6 }
2	32	2	433332	24	4
	322	1	3333222	144	12
		73			

8. Observe first that $\{4, 3\} = 24+36+12+1, = 73$. In placing all or any of the numbers of the 322 under those of the 3332, we do this in all the really distinct ways, inserting a numerical coefficient for the frequency of each way. Thus when the whole of the 322 is thus placed, the 3 may be under a 3, and the two 2's may then be under 33 or under 32: or else the 3 may be under a 2, and the two 2's must then be under 33: there are thus three ways, and these have the frequencies 6, 12, 6 respectively. For as to the first way, the 3 may be under any one of the three 3's, and for each such position of the 3, the 22 can be placed in either of two orders under the other two 3's; the frequency is thus 3 . 2, $=6$. And similarly the other two frequencies are 12 and 6; the sum $6+12+6, = 24$, is, the first term of $\{4, 3\}$. In like manner, when two of the numbers of the 322 are placed under the 3332, there are five ways having the frequencies 12, 6, 6, 6, 6 respectively, the sum of these frequencies is 36, which is the second term of $\{4, 3\}$. And so when one number of the 322 is placed under the 3332, there are four ways having the frequencies 3, 1, 6, 2 respectively: the sum of these frequencies is 12, which is the

third term of $\{4, 3\}$. Finally, when no number of the 322 is placed under the 3332, there is a single way, having the frequency 1, which is the last term of $\{4, 3\}$. These agreements are a useful verification for the frequencies: the sum of all the frequencies is of course $=73$, the value of $\{4, 3\}$.

9. We next form the column of symmetric functions by adding to each line the 3332: to avoid accidental errors of addition, it is proper to verify for each of these that the weight is the sum, $=18$, of the weights 11 and 7 of the two factors respectively. It is to be observed that the same symmetric function may present itself more than once: thus we have the functions 55332, and 533322, each of them twice. We then form a column of multiplicities: in 6552, the two 5's give the multiplicity 2; in 5553, the three 5's give 6: in 633222, the two 3's and the three 2's give $2 . 6, =12$, and so on. There is in like manner for the factor 3332 a multiplicity 6, and for the factor 322 a multiplicity 2; and these combined together give $6 . 2, = 12$, viz. this is the heading $\div 12$ of the column. And then forming the products $6 . 2$, $12 . 1$, $6 . 6$, &c., of the corresponding frequencies and multiplicities and dividing in each case by 12, we have the right-hand column of numerical coefficients.

10. The result is to be read

$$\begin{aligned} 3332 . 322 = \; & 1 . 6552 \\ & + 1 . 6543 \\ & + 3 . 5553 \\ & + \&c., \end{aligned}$$

the coefficients 2, 2 and 1, 6 of the repeated terms being of course united together, so that we have

$$\begin{aligned} & + 4 . 55332 \\ & + 7 . 533322. \end{aligned}$$

With a little practice, the operation is performed without difficulty and with very small risk of error.

11. We may apply the process to obtain analytical formulæ for certain forms of products. Consider, for instance, the product $2^\alpha . 2^\beta$, where $\alpha >$ or $= \beta$: and in the coefficients write for shortness α, instead of $[\alpha]^\alpha$ or $\Pi\alpha$, to denote $1 . 2 . 3 \ldots \alpha$. We have

$$\begin{array}{c|c} 2^\alpha & 2^\beta \\ \hline 2^A & 2^{\beta-A} \end{array} \quad \begin{array}{c} \div \alpha . \beta \\ \hline \dfrac{\alpha}{A . \alpha - A} \dfrac{\beta}{\beta - A} . 4^A 2^{\alpha+\beta-2A} . A . \alpha + \beta - 2A \end{array} \Bigg| \; \frac{\alpha + \beta - 2A}{\alpha - A . \beta - A},$$

viz. the formula is

$$2^\alpha . 2^\beta = \Sigma \frac{\alpha + \beta - 2A}{\alpha - A . \beta - A} . 4^A 2^{\alpha+\beta-2A},$$

where A has any integer value from β to 0; the first term is $= 1 . 4^\beta 2^{\alpha-\beta}$, and the last term is $\dfrac{\alpha+\beta}{\alpha . \beta} 2^{\alpha+\beta}$. The weight of any term is $4A + 2(\alpha + \beta - 2A), = 2\alpha + 2\beta$, as it should be.

In explanation as to the frequency, observe that out of the β 2's we take any A 2's and place them in any order under any A of the α 2's. The number of combinations taken is $\dfrac{\beta}{A\,.\,\beta-A}$, which (in A orders) gives $\dfrac{\beta}{\beta-A}$ sets to be placed under $\dfrac{\alpha}{A\,.\,\alpha-A}$ sets out of the α 2's: we thus have the foregoing coefficient of frequency $\dfrac{\alpha}{A\,.\,\alpha-A}\cdot\dfrac{\beta}{\beta-A}$, where as mentioned above α, &c. are written to denote $[\alpha]^{\alpha}$, &c.

12. We may in like manner find a formula for the product $3^{\alpha}2^{\beta}\,.\,3^{\gamma}2^{\delta}$. Taking $\gamma, =x+y+z$, any partition of γ, and $\delta, =p+q+r$, any partition of δ, and writing also $A=x$, $B=y+p$, $C=q$, we have

$$\frac{3^{\alpha}2^{\beta}\;|\;3^{\gamma}2^{\delta}}{3^{x}2^{p}3^{y}2^{q}\;|\;3^{z}2^{r}\,M\,.\,6^{A}5^{B}4^{C}3^{\alpha+\gamma-2A-B}2^{\beta+\delta-B-2C}\,.\,N\,|}\div\alpha\,.\,\beta\,.\,\gamma\,.\,\delta$$

where for the frequency, M, we have

$$\text{No. of terms } 3^{x},=\frac{\gamma}{x\,.\,\gamma-x};\qquad 3^{y},=\frac{\gamma-x}{y\,.\,z},$$

$$\text{"}\quad\text{"}\quad 2^{p},=\frac{\delta}{p\,.\,\delta-p};\qquad 2^{q},=\frac{\delta-p}{q\,.\,r},$$

to be placed under

$$\text{sets of terms } 3^{\alpha},=\frac{\alpha}{x+p\,.\,\alpha-x-p};\quad 2^{\beta},=\frac{\beta}{y+q\,.\,\beta-y-q},$$

$$\text{in orders, }=x+p;\qquad\qquad =y+q,$$

whence, multiplying, we have for the frequency

$$M=\frac{\alpha\,.\,\beta\,.\,\gamma\,.\,\delta}{x\,.\,y\,.\,z\,.\,p\,.\,q\,.\,r\,.\,\alpha-x-p\,.\,\beta-y-q};$$

and for the multiplicity, N, we have

$$\begin{aligned}N&=A\,.\,B\,.\,C\,.\,\alpha+\gamma-2A-B\,.\,\beta+\delta-B-2C,\\&=x\,.\,B\,.\,q\,.\,\alpha+\gamma-2A-B\,.\,\beta+\delta-B-2C.\end{aligned}$$

13. The coefficient is thus

$$\begin{aligned}\frac{M\,.\,N}{\alpha\,.\,\beta\,.\,\gamma\,.\,\delta},&=\frac{B\,.\,\alpha+\gamma-2A-B\,.\,\beta+\delta-B-2C}{y\,.\,z\,.\,p\,.\,r\,.\,\alpha-x-p\,.\,\beta-y-q},\\&=\frac{B\,.\,\alpha+\gamma-2A-B\,.\,\beta+\delta-B-2C}{y\,.\,p\,.\,\gamma-A-y\,.\,\alpha-A-p\,.\,\beta-C-y\,.\,\delta-C-p},\end{aligned}$$

or, putting also

$$\begin{aligned}D&=\alpha+\gamma-2A-B,\\E&=\beta+\delta\qquad\;\;-B-2C,\end{aligned}$$

whence

$$6A+5B+4C+3D+2E=3(\alpha+\gamma)+2(\beta+\delta),$$

the formula finally is

$$3^\alpha 2^\beta . 3^\gamma 2^\delta = \Sigma\Lambda . 6^A 5^B 4^C 3^D 2^E,$$

where

$$\Lambda = \frac{B}{y.p.\gamma-A-y.\alpha-A-p.\beta-C-y.\delta-C-p} \cdot \frac{D}{} \cdot \frac{E}{},$$

or, as this may also be written,

$$= \frac{B}{y.B-y} \cdot \frac{D}{z.D-z} \cdot \frac{E}{r.E-r},$$

so that the coefficient Λ is in fact the product of three binomial coefficients: it must, however, be recollected that the same term $6^A 5^B 4^C 3^D 2^E$ may occur more than once, with different coefficients Λ, so that in the final result, when these terms are united together, the numerical coefficients are not each of them of the form in question.

14. The limits of the summation are conveniently defined by means of the diagram

A	p	$\alpha-A-p$	α	$D=\alpha+\gamma-2A-B,$
y	C	$\beta-C-y$	β	$E=\beta+\delta \quad -B-2C,$
z	r			*ut suprà,*
γ	δ			

viz. here the sums of the first and second lines are α and β respectively, and the sums of the first and second columns are γ and δ respectively; we have $B=y+p$, a partition of B; but the values of y, p must be such as not to render negative any one of the four terms z, r, $\alpha-A-p$, $\beta-C-y$ of the diagram: for if any one of these numbers were negative, the corresponding factorial in the denominator of Λ would be infinite and we should have $\Lambda=0$. For any given term $6^A 5^B 4^C 3^D 2^E$ of the proper weight $3(\alpha+\gamma)+2(\beta+\delta)$, there may be no suitable values of y, p, and the coefficient is then $=0$: there may be a single pair of values, and the coefficient is then (as remarked above) = a product of three binomial coefficients: or there may be more than a single pair of values, and the entire coefficient of the term has not in this case a like simple form.

15. To exhibit the working of the formula, I apply it to the recalculation of the foregoing product 3332.322 (α, β, γ, $\delta=3$, 1, 1, 2). Properly the whole series of symbols 666, 6642, 6633, &c., of the symmetric functions of the weight 18 should be written down, and the coefficient be calculated for each of them: but I write down only those which have coefficients not $=0$: for each of these, I take *all* the partitions

$B = y + p$, several of these giving, as will be seen, zero values, and the others giving the values already obtained for the coefficients of the several terms.

						1			2			3						
	A	B	C	D	E	B	y	p	D	$\gamma - A - y$	$\alpha - A - p$	E	$\beta - C - y$	$\delta - C - p$	1	2	3	
6552	1	2	0	0	1	2	2 1 0	0 1 2	0	−2 −1 0	2 1 0	1	−1 0 1	2 1 0	1 2 1	0 0 1	0 1 1	1.6552
6543	1	1	1	1	0	1	1 0	0 1	1	−1 0	2 1	0	−1 0	1 0	1 1	0 1	0 1	1.6543
6532^2	1	1	0	1	2	1	1 0	0 1	1	−1 0	2 1	2	0 1	2 1	1 1	0 1	1 2	2.6532^2
64332	1	0	1	2	1	0	0	0	2	0	2	1	0	1	1	1	1	1.64332
6332^3	1	0	0	2	3	0	0	0	2	0	2	3	1	2	1	1	3	3.6332^3
5553	0	3	0	1	0	3	3 2 1 0	0 1 2 3	1	−2 −1 0 1	3 2 1 0	0	−2 −1 0 1	2 1 0 −1	1 3 3 1	0 0 1 1	0 0 1 0	3.5553
55332	0	2	0	2	1	2	2 1 0	0 1 2	2	−1 0 1	3 2 1	1	−1 0 1	2 1 0	1 2 1	0 1 2	0 1 1	2) 4.55332 2)
54333	0	1	1	3	0	1	1 0	0 1	3	0 1	3 2	0	−1 0	1 0	1 1	1 3	0 1	3.54333
53332^2	0	1	0	3	2	1	1 0	0 1	3	0 1	3 2	2	0 2	1 1	1 1	1 3	1 2	1) 7.53332^2 6)
433332	0	0	1	4	1	0	0	0	4	1	3	1	0	1	1	4	1	4.433332
33332^3	0	0	0	4	3	0	0	0	4	1	3	3	1	2	1	4	3	12.33332^3

It is quite possible that abbreviations and verifications might be introduced, but the process as it stands seems to be at once less expeditious and less safe than the one first made use of.

16. Particular cases of the general formula are

$$2^\beta . 2^\delta = \Sigma\Lambda 4^C 2^E;\ E = \beta + \delta - 2C, \text{ and thence } 4C + 2E = 2(\beta + \delta),$$

$$\Lambda = \frac{E}{\beta - C . \delta - C},$$

which agrees with a result before obtained. Again,

$$3^\alpha 2^\beta . 2^\delta = \Sigma\Lambda . 5^B 4^C 3^D 2^E;\ D = \alpha - B,$$
$$E = \beta + \delta - B - 2C,$$

and thence

$$5B + 4C + 3D + 2E = 3\alpha + 2\beta + 2\delta,$$

$$\Lambda = \frac{E}{\beta - C . \delta - B - C}.$$

17. We may, in like manner with the formula for $3^\alpha 2^\beta . 3^\gamma 2^\delta$, obtain the following:

$$4^\theta 3^\alpha 2^\beta . 2^\delta = \Sigma\Lambda . 6^A 5^B 4^C 3^D 2^E,$$

where

$$D = \alpha - B,$$
$$E = 2\theta + \beta + \delta - 3A - B - 2C,$$

and thence

$$6A + 5B + 4C + 3D + 2E = 4\theta + 3\alpha + 2\beta + 2\delta,$$

and the value of the coefficient Λ is

$$\Lambda = \frac{C}{\theta - A . C - \theta + A} . \frac{E}{\theta + \beta - A - C . \theta + \delta - 2A - B - C},$$

viz. Λ is the product of two binomial coefficients: and since here a given term $6^A 5^B 4^C 3^D 2^E$ occurs once only, each numerical coefficient is actually of this form.

In the particular case $\theta = 0$, we must have $A = 0$ (this appears *à posteriori* from the denominator factor $-A$, a factorial which is infinite for any positive value of A); and we thus obtain the first given formula for $3^\alpha 2^\beta . 2^\delta$.

Capitation and Decapitation. Art. Nos. 18 to 21.

18. I explain the converse processes of Decapitation and Capitation. In any symmetric function, for instance 6552 of the degree 6, the whole coefficient of α^6 is 552, this symbol referring in the first instance to the series of remaining roots $\beta, \gamma, \delta, \ldots$; but as the series of roots is unlimited, we may ultimately replace these by $\alpha, \beta, \gamma, \ldots$, and so use 552 in its original sense. Similarly, in 6652, the whole coefficient

of α^6 is $=652$, the only difference being that, while in the former case the degree is reduced to 5, in the present case it remains $=6$. In every case we decapitate the symbol by striking out the highest number—in the case of two or more equal numbers, one only of these being struck out. Observe that by decapitation we always diminish the weight, but we do or do not diminish the degree. In a product such as 3332.322 we obtain in like manner the whole coefficient of α^6 by the decapitation of each factor, viz. the coefficient is $=332.22$: and in any equation such as that obtained above for the product 3332.322, the whole coefficient of the highest power of α must be $=0$, viz. we can by decapitation obtain a new equation of lower weight: thus from the equation in question of weight 18, we obtain the new equation of weight 12,

$$\begin{aligned} 332.22 = \; & 1.552 \\ & +1.543 \\ & +2.5322 \\ & +1.4332 \\ & +3.33222, \end{aligned}$$

where observe that the terms of a degree lower than 6 in the original equation give no term in the new equation. The new equation of the deg. weight 5.12 might of course be obtained independently in like manner with the original equation.

19. We capitate a symbol by prefixing to it a number which is not less than the highest number contained in it: thus 552 may be capitated into 5552, 6552, &c.: and so a product 332.22 may be capitated into 3332.222, 3332.322, &c.; moreover a single symbol may be capitated into a product, 552 into 5552.4: in fact, the capitation may be any operation such that by decapitation we reproduce the original symbol. The increase of weight may be any number not less than the degree of the original symbol: but it is usually taken to be a given number: thus for any symbol of a degree not exceeding 6, we may capitate so as to increase the weight by 6. The capitation does or does not increase the degree.

20. An identical equation may be capitated in a variety of ways, but instead of an identity we obtain only a congruence, that is, an equation which requires to be completed by the adjunction to it of proper terms of lower degrees. Thus from the above equation of the deg. weight 5.12 we may obtain

$$\begin{aligned} 3332.322 \equiv \; & 1.6552 \\ & +1.6543 \\ & +2.65322 \\ & +1.64332 \\ & +3.633222. \end{aligned}$$

Imagine here all the terms brought to the same side so that the form is $\Omega \equiv 0$: Ω is a function not containing α^6, for the whole coefficient of α^6 therein is precisely

that function which by the equation of the deg. weight 5.12 is expressed to be $=0$: and hence Ω, *quâ* symmetric function cannot contain $\beta^6, \gamma^8, \ldots$; viz. Ω is a symmetric function of the degree 5 at most: the congruence $\Omega \equiv 0$ thus means, $\Omega =$ a properly determined function of the degree 5 at most. Obviously by development of the term 3332.322, that is, by substituting for this term its value as given by the equation of the deg. weight 6.18, the function of the degree 5 at most would be found to be $=$ the sum of those terms in the expression of 3332.322, which are of a degree inferior to 6: and the congruence $\Omega \equiv 0$ thus completed would be nothing else than the equation of the deg. weight 6.18.

21. We might have capitated in a different manner: for instance

$$\begin{aligned} 4332.222 = \; & {}_1.6552 \\ & + {}_1.6543 \\ & + {}_2.65322 \\ & + {}_1.44332.2 \\ & + {}_3.33222.3, \end{aligned}$$

there being here three products requiring to be developed: on replacing them by their values, there would be found for Ω a value which would be a determinate function of a degree less than 6: and putting $\Omega =$ this value, the congruence $\Omega \equiv 0$ would be completed into an equation.

Seminvariants of a given Degree; Perpetuants, &c. Art. Nos. 22 to 38.

22. We consider now seminvariants according to their degrees; in particular, those of the degrees 2, 3, 4, 5 and 6, or say quadric, cubic, quartic, quintic and sextic seminvariants: the forms of these are $2^{\alpha+1}$, $3^{\alpha+1}2^{\beta}$, $4^{\alpha+1}3^{\beta}2^{\gamma}$, $5^{\alpha+1}4^{\beta}3^{\gamma}2^{\delta}$, $6^{\alpha+1}5^{\beta}4^{\gamma}3^{\delta}2^{\epsilon}$, where each exponent α, β, γ, δ, ϵ is a positive integer not excluding zero: the exponent of the highest number is in each case written $\alpha+1$, and has thus the value 1 at least, for otherwise the form would not be of the proper degree. The several weights are $2(\alpha+1)$, $3(\alpha+1)+2\beta$, $4(\alpha+1)+3\beta+2\gamma$, &c., or, what is the same thing, we have for the several degrees respectively

$$\begin{aligned} w-2 &= 2\alpha, \\ w-3 &= 3\alpha+2\beta, \\ w-4 &= 4\alpha+3\beta+2\gamma, \\ w-5 &= 5\alpha+4\beta+3\gamma+2\delta, \\ w-6 &= 6\alpha+5\beta+4\gamma+3\delta+2\epsilon, \end{aligned}$$

and we have for a given degree and weight as many seminvariants as there are systems of exponents satisfying the corresponding equation.

23. These numbers are at once expressible by means of a series of Generating Functions (G. F.), viz. writing for shortness **2**, **3**, **4**, **5**, **6** to denote $1-x^2$, $1-x^3$, $1-x^4$, $1-x^5$, $1-x^6$, the G. F.'s are

$$\frac{x^2}{\mathbf{2}},\ \frac{x^3}{\mathbf{2.3}},\ \frac{x^4}{\mathbf{2.3.4}},\ \frac{x^5}{\mathbf{2.3.4.5}},\ \frac{x^6}{\mathbf{2.3.4.5.6}}.$$

In fact, the number of seminvariants of a given weight is = coefficient of x^w in the corresponding G. F.; for the quadric seminvariants (or those of deg. weight $2.w$) in $\frac{x^2}{\mathbf{2}}$; for the cubic seminvariants (or those of deg. weight $3.w$) in $\frac{x^3}{\mathbf{2.3}}$; and so for the others.

24. A seminvariant of a given degree may be a sum of products (of that degree) of seminvariants of lower degrees, and of seminvariants of lower degrees: and it is in this case said to be reducible: a seminvariant which is not reducible is said to be irreducible, or otherwise to be a perpetuant. This notion of a perpetuant is due to Sylvester, see his Memoir "On Subinvariants, i.e. Semi-Invariants to Binary Quantics of an Unlimited Order," *American Journal of Mathematics*, vol. v. (1882—83), pp. 78—137 (§ 4 Perpetuants, pp. 105—118). In speaking of the number of perpetuants of a given deg. weight, we assume throughout that these are independent perpetuants, not connected by any linear relation.

25. Since the seminvariants used for the reduction of a given reducible seminvariant can themselves be expressed in terms of perpetuants, we may say more definitely that a seminvariant of a given degree, which is a sum of products (of that degree) of perpetuants of lower degrees, and of perpetuants of lower degrees, is reducible. The words "of that degree" are essential to the definition: a seminvariant may be expressible as a sum of products (of a higher degree) of perpetuants of lower degrees, and of perpetuants of lower degrees, and it is not on this account reducible: a seminvariant so expressible is said to be a "syzygant"; but as to this, see No. 49.

26. Every quadric or cubic seminvariant is obviously a perpetuant: the quadric and cubic perpetuants have thus the before-mentioned G. F.'s $\frac{x^2}{\mathbf{2}}$ and $\frac{x^3}{\mathbf{2.3}}$ respectively.

27. A reducible quartic seminvariant can only be a sum of products (2.2) of two quadric perpetuants, and of quadric perpetuants, and it is clear that no quartic seminvariant the symbol of which contains a 3 is thus expressible. If the symbol does not contain a 3, viz. when the form is $4^{\alpha+1}\,2^{\gamma}$, the seminvariant is reducible: we have for instance

$$4 = 2.2 - {}_2.22,$$

$$42 = 22.2 - {}_3.222, \text{ \&c.}$$

To show that this is so in general, observe that any symmetric function $2^{\alpha+1}\,1^{\gamma}$, *quâ* symmetric function can be expressed as a rational and integral function of the

degree $2(\alpha+1)+\gamma$, of the coefficients 1, 1^2, 1^3, &c.: instead of the roots considering their squares, we have thence an expression for the quartic seminvariant $4^{\alpha+1}2^{\gamma}$ in terms of the quadric perpetuants 2, 2^2, 2^3, &c., and such expression will be of the same degree $4(\alpha+1)+2\gamma$, as the quartic seminvariant.

It thus appears, as regards the quartic seminvariants, that whenever the symbol contains a 3, and in this case only, the seminvariant is a perpetuant: or, what is the same thing, the form of a quartic perpetuant is $4^{\alpha+1}3^{\beta+1}2^{\gamma}$: for the weight w, the number is equal to that of the sets of values α, β, γ, such that $w-7=4\alpha+3\beta+2\gamma$: or, what is the same thing, the G. F. of the quartic perpetuants is $=\dfrac{x^7}{2.3.4}$.

28. Sylvester, in the memoir referred to, obtained this result in a different manner: the quartic seminvariants of a given weight are the quartic perpetuants of that weight and also the products (of that weight) of two quadric perpetuants, the same or different: say (4) is the G. F. for the perpetuants, and (2, 2) for the products: then the G. F. for the quartic seminvariants being as already mentioned $\dfrac{x^4}{2.3.4}$, we have his equation

$$(4)+(2,\ 2)=\frac{x^4}{2.3.4}.$$

He deduces (2, 2) from the G. F. $=\dfrac{x^2}{2}$ of the quadric perpetuants and thence obtains (4), $=\dfrac{x^7}{2.3.4}$, as above.

29. Write for a moment ϕx to represent the G. F. $=\dfrac{x^2}{2}$ of the quadric perpetuants, and $A, B, C, \ldots$ to represent these quadric perpetuants: we have, in an algorithm which will be readily understood,

$$\begin{aligned}
\phi x &= (A+B+C\ldots), \\
(\phi x)^2 &= (A+B+C\ldots)^2, &&= A^2+2AB+\ldots, \\
\phi x^2 &= &&\quad A^2+\&c.,
\end{aligned}$$

and thence

$$\tfrac{1}{2}\{(\phi x)^2+\phi x^2\} = \qquad A^2+AB+\&c.;$$

viz. the G. F. (2, 2), is

$$\tfrac{1}{2}((\phi x)^2+\phi x^2),\ =\tfrac{1}{2}\left(\frac{x^4}{2.2}+\frac{x^4}{4}\right),\ =\tfrac{1}{2}\left(\frac{x^4(1+x^2)}{2.4}+\frac{x^4(1-x^2)}{2.4}\right),\ =\frac{x^4}{2.4},$$

and we thence have

$$(4)+\frac{x^4}{2.4}=\frac{x^4}{2.3.4},$$

that is, $(4)=\dfrac{x^7}{2.3.4}$, the same result as was found above by independent considerations.

30. Sylvester established in like manner (but without the terms S which will be presently explained) the equations

$$(5)+(3,\ 2)=\frac{x^5}{2.3.4.5}+S_5,$$

$$(6)+(4,\ 2)+(3,\ 3)+(2,\ 2,\ 2)=\frac{x^6}{2.3.4.5.6}+S_6,$$

viz. here (5) is the G. F. for the quintic perpetuants, (2, 3) that for the products of the quadric and the cubic perpetuants: and similarly (6) is the G. F. for the sextic perpetuants, (4, 2) that for the products of the quadric and the quartic perpetuants, (3, 3) for the products of two cubic perpetuants, the same or different: and (2, 2, 2) for the products of three quadric perpetuants, the same or different. We have at once $(3,\ 2)=(3).(2)$; $(4,\ 2)=(4).(2)$; $(3,\ 3)$ is found by the same process as was used for finding $(2,\ 2)$, substituting only $\frac{x^3}{3}$ for $\frac{x^2}{2}$; and $(2,\ 2,\ 2)$ is found by a like process, viz. the G. F. for A^3+A^2B+ABC is $\frac{1}{6}\{(\phi x)^3+3\phi x\,.\,\phi x^2+2\phi x^3\}$, where $\phi x=\frac{x^2}{2}$ as before, viz. this is $\frac{1}{6}\left\{\frac{x^6}{2.2.2}+\frac{3x^6}{2.4}+\frac{2x^6}{6}\right\}$: reducing to the common denominator $2.4.6$, the numerator is

$$=\tfrac{1}{6}x^6\{(1+x^2)(1+x^2+x^4)+3(1-x^6)+2(1-x^2)(1-x^4)\},$$

viz. this is $=x^6$. The several functions thus are

$$(3,\ 2)=\frac{x^5}{2.2.3},\qquad (4,\ 2)=\frac{x^9}{2.2.3.4},$$

$$(3,\ 3)=\frac{x^6}{3.6},\qquad (2,\ 2,\ 2)=\frac{x^6}{2.4.6}.$$

31. Mr Hammond, in regard to the equation for the quintic perpetuants, made the very important observation—see his paper "On the Solution of the Differential Equation of Sources," *American Journal of Mathematics*, t. v. (1882), pp. 218—227—that the products (3, 2) of a cubic perpetuant and a quadric perpetuant are not independent: we have between them syzygies such as $32.2-3.22\equiv 0$ (viz. the difference of the two products contains no term of the degree 5; the actual value is $=43+322$: Hammond's equation (12), p. 222), hence the necessity in the equation of a term S_5 referring to these syzygies; and he moreover obtained the expression, $S_5=\frac{x^7}{2.4}$ of the G. F. for these syzygies.

The equation gives

$$(5)=\frac{-x^7+x^{10}+x^{12}}{2.3.4.5}+S_5,$$

where of course the first term is the value of (5) which would be given by the equation without the term S_5; and substituting herein for S_5 the foregoing value, viz.

$$S_5=\frac{x^7}{2.4},\quad =\frac{x^7(1-x^3)(1-x^5)}{2.3.4.5},$$

we find

$$(5) = \frac{x^{15}}{2.3.4.5},$$

which is the correct value of the G. F. for the quintic perpetuants: the lowest quintic perpetuant is thus of the weight 15.

32. The equation for the sextic perpetuants gives

$$(6) = \frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{2.3.4.5.6} + S_6,$$

which is an equation connecting (6) and S_6, the G. F.'s for the sextic perpetuants, and the sextic syzygies respectively. I have, in the investigation of the value of S_6, met with a difficulty which I have not been able to overcome: but I find that

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18} + \omega(x)}{2.3.4.5.6},$$

where $\omega(x)$ is possibly the monomial function x^{31}, but this result (which Capt. MacMahon believes to be true) is not yet completely established; it is a function containing no term lower than x^{31}. We have therefore

$$(6) = \frac{\omega(x)}{2.3.4.5.6},$$

and there is, it would appear, no sextic perpetuant of a weight lower than 31.

33. But before entering on the investigation, it is proper to further develope the theory of the quintic perpetuants. We have quintic seminvariants: 5 for weight 5; 52 for weight 7; 53 for weight 8; 54, 522 for weight 9; 55, 532 for weight 10; 542, 533, 5222 for weight 11; and so on. These are reduced by means of the products 3.2 of a cubic perpetuant and a quadric perpetuant; viz. for any given weight, we have all the products $3^\alpha 2^\beta . 2^\gamma$ of that weight. Thus for weight 5 there is only the product 3.2, and this in fact serves to reduce the seminvariant 5: we have $3.2 = 5 + 32$, and therefore $5 = 3.2 - 32$.

For the weight 7; there is only the seminvariant 52, and there are the two products 32.2 and 3.22: either of these would serve for the reduction: we have

$$\begin{aligned} 32.2 = \quad & 52\ , & 3.22 = \quad & 52\ , \\ + \quad & 43\ , & & + 322, \\ + 2 . & 322, & & \end{aligned}$$

and these two equations imply the before-mentioned syzygy, $32.2 - 3.22 \equiv 0$, in virtue of which the two reductions become equivalent; or say there remains a single equation serving for the reduction: the most simple form is $52 = 3.22 - 322$.

Weight 8: there is only the seminvariant 53, and the product 33.2: this gives the reduction.

Weight 9: seminvariants 54, 522: products $32^2.2$, 32.2^2, 3.2^3; there is between the first and last of these a syzygy; and this being satisfied, there remain two equations for the reductions.

Weight 10: seminvariants 55, 532: products 332.2, 33.22: these give the reductions.

Weight 11: seminvariants 533; 542, 5222: products 333.2; $32^3.2$, $32^2.2^2$, 32.2^3, 32^4.

34. Observe that the seminvariants, and in like manner the products, form two classes, according as the symbols contain three odd numbers or a single odd number: these correspond separately to each other, for the development of any product will contain only seminvariants having each of them as many odd numbers as there are odd numbers in the product. Hence for the weight 11 just referred to, 333.2 serves for the reduction of 533; $32^3.2$, $32^2.2^2$, 32.2^3, 3.2^4 are connected the first and fourth of them by a syzygy, and the second and third by a syzygy; and there remain two equations serving for the reduction of the two seminvariants 542, 5222.

It is easy to show in this manner that there is no quintic perpetuant for any weight under 15; and that there is a single quintic perpetuant for the weight 15.

35. Generally the syzygies exist only for an odd weight $w=2\beta+3$ between the products $32^{\beta-1}.2$, $32^{\beta-2}.2^2, \ldots, 32.2^{\beta-1}$, 3.2^β; viz. there is a syzygy between the first and last terms: a syzygy between the second and last but one terms; and so on. The existence of these syzygies at once appears from the principle of decapitation: decapitating $32^{\beta-1}.2-3.2^\beta$, we have $2^{\beta-1}-2^{\beta-1}$, which is identically $=0$, hence the function contains no term of the degree 5, that is, $32^{\beta-1}.2-3.2^\beta \equiv 0$; and similarly for the other pairs of terms.

There are β terms: hence in the case β even, we have $\frac{1}{2}\beta$ pairs of terms and therefore $\frac{1}{2}\beta$ syzygies: in the case β odd, there is a middle term, not connected by a syzygy with any other term, and the number of syzygies is thus $=\frac{1}{2}(\beta-1)$: writing for β its value $\frac{1}{2}(w-3)$, the number is $=\frac{1}{4}(w-3)$, and $\frac{1}{4}(w-5)$ in the two cases respectively: and it thus appears that the G. F. is $=\dfrac{x^7}{2.4}$ as already mentioned.

36. In the case w an odd number, we have seminvariants, and in like manner products, containing respectively one odd number, three odd numbers, five odd numbers, and so on: thus $w=15$, we have

Seminvariants.	Products.	
555	$3332^2.2$	3 equations,
5532	3332.2^2	
5433	333.2^3	
5332^2		
5442	$32^5.2$	$\frac{1}{2}6$, $=3$ equations;
542^3	$32^4.2^2$	
52^5	$32^3.2^3$	
	$32^2.2^4$	
	32.2^5	
	3.2^6	

hence the seminvariants 555, 5532, 5433, 5332^2 with three odd numbers are not reducible, but they can be linearly expressed in terms of the 3 like products $3332^2 . 2$, $3332 . 2^2$, $333 . 2^3$, and of $4-3$, $=1$ arbitrary quantity (observe, however, that this must not be the seminvariant 5332^2, for this is in fact reducible): the seminvariants 5442, 542^2, 52^5, with one odd number, are reducible. And the like as regards any other odd value of w.

37. In the case w an even number, we have seminvariants, and in like manner products, containing respectively two odd numbers, four odd numbers, six odd numbers, and so on. Thus $w=16$, we have

Seminvariants.	Products.	
5533	$33332 . 2$	2 equations,
53332	$3333 . 2^2$	
5542	$332^4 . 2$	5 equations;
552^3	$332^3 . 2^2$	
5443	$332^2 . 2^3$	
5432^2	$332 . 2^4$	
532^4	$33 . 2^5$	

and thus the seminvariants with four odd numbers, and those with two odd numbers, are each set reducible.

38. I give in the case $w=19$ the following results: the expression for the G. F. shows that there are 2 quintic perpetuants: viz. two forms X, Y such that every quintic seminvariant of the weight 19 is expressible as a linear function of these, of products $(3 . 2)$ of a cubic and a quadric perpetuant, and of forms of a degree inferior to 5, that is, quartic, cubic and quadric perpetuants. Attending only to the terms in X, Y, the actual values are:

$$
\begin{aligned}
5554 &= X, \\
5552^2 &= Y, \\
55432 &= -X, \\
55333 &= 0, \\
5532^3 &= -Y, \\
54442 &= 0, \\
54433 &= X, \\
5442^3 &= 0, \\
54332^2 &= Y, \\
542^5 &= 0, \\
533332 &= 0, \\
5332^4 &= 0, \\
52^7 &= 0,
\end{aligned}
$$

viz. of the 13 quintic seminvariants of the weight in question there are 7, which as not containing either X or Y are each of them reducible; while the remaining 6 can only be expressed as linear functions of X and Y. It would be allowable to select $5554\,(=X)$ and $5552^2\,(=Y)$ as the two representative perpetuants, but there is no particular advantage in this.

Sextic Perpetuants and Sextic Syzygies: Syzygants. Art. Nos. 39 to 51.

39. Returning now to the sextic seminvariants, these are weight 6; 6: weight 8; 62: weight 9; 63: weight 10; 64, 62^2: and so on. And they are reducible by means of the products (4, 2), (3, 3) and (2, 2, 2), that is, of a quartic perpetuant and a quadric perpetuant, of two cubic perpetuants, and of three quadric perpetuants: this last form of product existing only in the case of an even weight.

40. For weight 6, we have seminvariant 6, and the two products 3.3 and $2.2.2$; this implies a syzygy $3.3-2.2.2\equiv 0$; and there then remains a single equation for the reduction of the seminvariant. The formulæ are

$$\begin{aligned} 3.3 &= 6 \\ &\quad +_2.33, \end{aligned} \qquad \begin{aligned} 2.2.2 &= 6 \\ &\quad +_3.22.2 \\ &\quad -_3.222, \end{aligned}$$

so that the complete syzygy, and the most simple reduction, are

$$\begin{aligned} &\ 3.3 \\ &-\ 2.2.2 \\ &-_2.33 \\ &+_3.22.2 \\ &-_3.222 = 0, \end{aligned} \qquad \begin{aligned} 6 = &3.3 \\ &-_2.33, \end{aligned}$$

and we might in this way verify that, for the successive weights 8, 9, 10, &c., there are no sextic perpetuants; and find for these weights respectively, the number of the sextic syzygies. But such direct investigation becomes soon impracticable.

41. I endeavour to determine the number of sextic syzygies for the weight w; and for this purpose I establish the following relation:

$$(S_6) = ((0)) + ((2))' + ((3))' + ((2,\ 2))' + ((3,\ 2))' + (S_5)' + (S_6)' - ((5))' + ((\theta))',$$

where (S_6) is the number of sextic syzygies for the weight w, or, what is the same thing, it is the coefficient of x^w in the function S_6, which is the G. F. for these syzygies: $((0))$ has the value 1 for $w=6$, and the value 0 in all other cases. The accented symbols refer to the weight $w-6$; $(S_6)'$ is thus the number of sextic syzygies for this weight: and for the same weight $w-6$, $((2))'$ denotes the number of quadric perpetuants, or coefficient of x^{w-6} in the function (2) which is the G. F. for these perpetuants: $((3))'$ the number of cubic perpetuants, $((2,\ 2))'$ the number of

products of two quadric perpetuants, the same or different, $((3,\ 2))'$ the number of products of a cubic perpetuant and a quadric perpetuant, $(S_5)'$ the number of quintic syzygies, $((5))'$ the number of quintic perpetuants, and $((\theta))'$ a term of unascertained form which will be explained further on. Transposing the term $(S_6)'$ to the left-hand side, and passing to the generating functions, we have

$$(1-x^6)\,S_6 = (0) + (2)' + (3)' + (2,\ 2)' + (3,\ 2)' + S_5' - (5)' + (\theta)',$$

or, as this may be written,

$$= x^6\,\{1 + (2) + (3) + (2,\ 2) + (3,\ 2) + (S_5) - (5) + (\theta)\},$$

where (2), (3), &c., have the values already obtained for these G. F.'s respectively: viz. writing $x^6\,(\theta) = \dfrac{\omega x}{2.3.4.5}$, the equation is

$$6\,.\,S_6 = x^6 + \frac{x^8}{2} + \frac{x^9}{3} + \frac{x^4}{2.4} + \frac{x^{11}}{2.2.3} + \frac{x^{13}}{2.4} - \frac{x^{21}}{2.3.4.5} + \frac{\omega\,(x)}{2.3.4.5}.$$

42. Reducing on the right-hand side the known terms to the common denominator **2.3.4.5**, the numerator is

$$\begin{array}{lllllllllll}
x^6\,.\,1-x^2\,.\,1-x^3\,.\,1-x^4\,.\,1-x^5 = & x^6 - x^8 - x^9 - x^{10} & & & + x^{12} + 2x^{13} + x^{14} & & - x^{16} - x^{17} - x^{18} + x^{20} & & \\
+x^8\,. \quad 1-x^3\,.\,1-x^4\,.\,1-x^5 & + x^8 & & - x^{11} - x^{12} - x^{13} & & + x^{15} + x^{16} + x^{17} & & - x^{20} & \\
+x^9\,. \quad 1-x^4\,.\,1-x^5 & & + x^9 & & - x^{13} - x^{14} & & & + x^{18} & \\
+x^{10}\,. \quad 1-x^3\,.\,1-x^5 & & & + x^{10} & - x^{13} & - x^{15} & & + x^{18} & \\
+x^{11}\,. \quad 1+x^2\,.\,1-x^5 & & & + x^{11} & + x^{13} & & - x^{16} & - x^{18} & \\
+x^{13}\,. \quad 1-x^3\,.\,1-x^5 & & & & + x^{13} & & - x^{16} & - x^{18} & + x^{21} \\
-x^{21}\,. & & & & & & & & - x^{21} \\
\hline
 & x^6 & & & + x^{13} & & - 2x^{16} & - x^{18} & ;
\end{array}$$

whence, dividing by 6, we have the before-mentioned formula

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18} + \omega\,(x)}{2.3.4.5.6}.$$

43. We have to prove the formula for (S_6): this symbol denotes, for the weight w, the number of syzygies between the products (4, 2), (3, 3) and (2, 2, 2): we have to consider separately the cases w odd, and w even. First, if w be odd, there are no products (2, 2, 2) and the only forms are (4, 2) and (3, 3).

I consider a particular value of w, say $w = 15$. The whole series of seminvariants weight 15 is

663	555	4443	33333
654	5532	4432^2	3332^3
652^2	5442	43332	32^6;
6432	5433	432^4,	
6333	542^3		
632^3,	5332^3		
	52^5,		

and from the quartic and cubic forms we obtain the forms of the products (4, 2) and (3, 3), viz. these are

$4432 \,.\, 2$	$3333 \,.\, 3$
$443 \,.\, 2^2$	$333 \,.\, 33$
$4333 \,.\, 2$	$332^3 \,.\, 3$
$432^3 \,.\, 2$	$332^2 \,.\, 32$
$432^2 \,.\, 2^2$	$332 \,.\, 32^2$
$432 \,.\, 2^3$	$33 \,.\, 32^3$;
$43 \,.\, 2^4$,	

and (S_6) will denote the number of syzygies between these products. Now from any such syzygy, we obtain by decapitation, it may be an identity, but if not an identity, then a syzygy, of the weight $15-6, =9$: and from these lower identities or syzygies we can pass back to the syzygies of the weight 15. To show how this is, I decapitate the several products, thus obtaining the forms

432	333 ; viz. the distinct		432 ; and of these	333	occur each of
$43 \,.\, 2$	$33 \,.\, 3$ forms are		$43 \,.\, 2$	32^3	them twice.
333	32^3		333	$32^2 \,.\, 2$	
32^3	$32^2 \,.\, 2$	$33 \,.\, 3$	32^3	$32 \,.\, 2^2$	
$32^2 \,.\, 2$	$32 \,.\, 2^2$		$32^2 \,.\, 2$	$3 \,.\, 2^3$	
$32 \,.\, 2^2$	$3 \,.\, 2^3$		$32 \,.\, 2^2$		
$3 \,.\, 2^3$			$3 \,.\, 2^3$		

The forms occurring each twice are 333, 32^3, viz. these are the forms (3), or cubic perpetuants of the weight 9: and $32^2 \,.\, 2$, $32 \,.\, 2^2$, $3 \,.\, 2^3$, viz. these are the forms (3, 2) or products of a cubic perpetuant and a quadric perpetuant for the weight 9; and any form thus occurring twice gives a syzygy of the weight 15: thus 333, we capitate it with $4 \,.\, 2$ or with $3 \,.\, 3$, and so obtain the syzygy $4333 \,.\, 2 - 3333 \,.\, 3 \equiv 0$; and in like manner for each of the other forms 32^3, $32^2 \,.\, 2$, $32 \,.\, 2^2$, $3 \,.\, 2^3$. And so for any other odd weight: (S_6) contains the terms $((3))'$ and $((3, 2))'$, and for an odd value of w we may assume that S_6 contains also the terms $((2))'$ and $((2, 2))'$: for these, it is clear, vanish for any odd value of w.

44. When w is even, it appears by a similar investigation that (S_6) contains the terms $((2))'$, and $((2, 2))'$, which in this case do not vanish, and also the before-mentioned terms $((3))'$ and $((3, 2))'$: so that whether w be even or odd, (S_6) contains the terms $((2))'$, $((3))'$, $((2, 2))'$, $((3, 2))'$.

In the particular case $w=6$, there is the sextic syzygy $3 \,.\, 3 - 2 \,.\, 2 \,.\, 2 \equiv 0$, obtained by capitation from the identity $1-1=0$; and by reason hereof, we introduce into the formula the term $((0))$, $=1$ for $w=6$, and $=0$ in every other case.

In what immediately follows, I revert to the instance $w=19$, but this now represents indifferently an odd or an even value of w, there being no distinction between the two cases.

45. Attending next to the remaining distinct terms, these are 333, 32^2 of the degree 3; 432 of the degree 4; $32^2.2$, 32.2^2, 3.2^3 of the degree 5; and 43.2, 33.3 of the degree 6. For the degrees 3 and 4, there are no syzygies: but for the degree 5 we have a syzygy: this, written as a congruence, is $32^2.2-3.2^3\equiv 0$, and *quâ* quintic syzygy, it will, when completed, not contain any 5; the completed form in fact is

$$\begin{array}{rl} & 32^2.2 \\ - & 3.2^3 \\ - & 432 \\ -2 & .32^3=0. \end{array}$$

We can capitate this, each term with 4.2 or else 3.3, or it may be indifferently either with 4.2 or 3.3; and so obtain therefrom a syzygy of the weight 15; such a syzygy (in the congruence form) is

$$\begin{array}{rl} & 432^2.2^2 \\ - & 43.2^4 \\ - & 4432.2 \\ -2 & .432^3.2\equiv 0; \end{array}$$

and it is to be observed that it is quite indifferent how the capitation is performed: if for instance the first term had been capitated into $332^2.32$, then in virtue of the before-obtained syzygy $432^2.2^2-332^2.32\equiv 0$, the new form would be equivalent to the old one. (It is I think convenient to capitate, when this can be done, with 4.2; and only the other terms with 3.3.) Clearly the case is the same with any other odd weight; and we thus see that (S_8) contains the term $(S_5)Y$.

46. But further we have, between the terms 43.2 and 33.3 of the degree 6, a syzygy $43.2-33.3\equiv 0$. Completing this, there will be a term containing a 5, viz. the syzygy is

$$\begin{array}{rl} & 43.2 \\ - & 33.3 \\ - & 54 \\ - & 432 \\ -3 & .333=0; \end{array}$$

and in this form we cannot capitate it, for the quintic term 54 is not to be capitated either with 4.2 or with 3.3. But 54 is not a perpetuant: we have

$$\begin{array}{rl} 54= & 32.2^2, \\ -2 & .3.2^3 \\ - & 432 \\ - & 32^3 \end{array}$$

and thence the syzygy is

$$\begin{array}{rl} & 43.2 \\ - & 33.3 \\ - & 32.2^2 \\ +2 & .3.2^3 \\ +3 & .333 \\ + & 32^3=0; \end{array}$$

and it can be capitated, for instance, into

$$\begin{aligned} & 443\,.\,2^2 \\ - \quad & 333\,.\,33 \\ - \quad & 432\,.\,2^3 \\ + {}_2 & .\,43\,.\,2^4 \\ + {}_3 & .\,4333\,.\,2 \\ + \quad & 432^3\,.\,2 \equiv 0, \end{aligned}$$

the form of capitation being (for the reason mentioned above) quite immaterial. Observe that in every case where the sextic syzygy contains in the first instance any quintic seminvariants, it is assumed that each of these is expressed in terms of quintic perpetuants, as shown in No. 38; and this being done, the sextic syzygy exhibits itself as a syzygy containing, or else not containing, a quintic perpetuant or perpetuants.

47. The conclusion is that from any sextic syzygy of the weight $w-6$, *which does not contain a quintic perpetuant,* we can obtain by capitation a sextic syzygy of the weight w. The number of sextic syzygies of the weight $w-6$ is $(S_6)'$, and the number of quintic perpetuants of the same weight is $((5))'$: the former of these is (for not too large values of w) the greater; and at first sight it would appear that we can, by elimination of the quintic perpetuants, obtain from the $(S_6)'$ syzygies, $(S_6)'-((5))'$ syzygies which do not contain a quintic perpetuant: if this was always the case, we should have in (S_6) the term $(S_6)'-((5))'$, completing the series of terms, and the formula would be

$$(S_6) = ((0)) + ((2))' + ((3))' + ((2,\ 2))' + ((3,\ 2))' + (S_5)' + (S_6)' - ((5))',$$

leading to

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18}}{2\,.\,3\,.\,4\,.\,5\,.\,6}.$$

48. But this result is on the face of it wrong, for as remarked by Sylvester in the memoir referred to, from the mere fact that the sum $1+1-2-1$ of the numerator coefficients is negative, it follows that the coefficients of the development ultimately become negative; and the actual calculation showing when this happens is given by him. And it is further to be noticed that not only the formula cannot be correct beyond the point at which the coefficients become negative, but it cannot be correct beyond the point for which $(S_6)'-((5))'$ becomes negative: the sextic syzygies of the weight $w-6$ may add nothing to, but they cannot take anything away from, the number of the sextic syzygies of the weight w.

49. If for a moment we further consider these syzygies of the weight $w-6$; so long as the number of these is greater than the number of quintic perpetuants of the same weight, we can by means of them presumably express each of the quintic perpetuants in terms of sextic products, viz. in the language of Capt. MacMahon, express each quintic perpetuant as a "Sextic Syzygant." The syzygy of the weight 9, above

obtained, will serve as an example: 54 is not a quintic perpetuant, but ignoring this, it is by the syzygy in question expressible in the form

$$\begin{aligned} 54 = \quad & 43.2 \\ - \quad & 33.3 \\ - \quad & 432 \\ - 3 & .333, \end{aligned}$$

viz. as a Sextic Syzygant, inasmuch as on the right-hand side we have terms 43.2 and 33.3, of the degree 6, which exceeds the degree 5 of the seminvariant 54 in question. Referring back to the definition of reduction, No. 25, observe that this is *not* a reduction of the seminvariant 54. It may be remarked that for the weight 19 we have 15 sextic syzygies: the number of quintic perpetuants is $=3$: so that while it is conceivable that the 15 equations might be such that they would fail to determine the 3 perpetuants, it is *primâ facie* very unlikely that this should be so. I have in fact ascertained that the equations are sufficient for the determination; that is, that (weight 19) each of the three quintic perpetuants is a sextic syzygant. So in the case $w=23$, the number of the sextic syzygies is $=28$, and that of the quintic perpetuants is $=5$; here also the 28 equations are sufficient to determine the 5 perpetuants, viz. (weight 23) each of the 5 quintic perpetuants is a sextic syzygant.

50. Supposing that for any given weight $w-6$, each of the quintic perpetuants *is* a sextic syzygant: this implies that the number of sextic syzygies $(S_6)'$ is at least equal to the number $((5))'$ of quintic perpetuants (for each expression of a quintic perpetuant as a sextic syzygant is in fact a sextic syzygy): and not only so, but it further implies that the number of the sextic syzygies, which do not contain a quintic perpetuant, is precisely equal $(S_6)'-((5))'$: for if besides the equations which serve to express the perpetuants as syzygants, we have any other sextic syzygy, then either this does not contain a quintic perpetuant, or it can (by substituting therein for every quintic perpetuant its value as a sextic syzygant) be reduced to a syzygy which does not contain any quintic perpetuant.

51. In the general case, we have $(S_6)'$ sextic syzygies of the weight $w-6$, and $((5))'$ quintic perpetuants of this weight: but it may happen that certain of the quintic perpetuants do not enter into any of the sextic syzygies; and those which enter, may do so in definite combinations: by elimination of these combinations of perpetuants we obtain (it may be) α sextic syzygies not containing any quintic perpetuant; and the remaining $(S_6)'-\alpha$ equations will then serve to express each of them a quintic perpetuant, or combination of quintic perpetuants, as a sextic syzygant. The number α is at most $=((5))'$, or taking it to be $=((5))'-((\theta))'$, the number of sextic syzygies not containing any quintic perpetuant will be $=(S_6)'-((5))'+((\theta))'$, that is, the number of sextic syzygies not containing any quintic perpetuant will be equal to the whole number $(S_6)'$ of sextic syzygies diminished by some number $((5))'-((\theta))'$, which is less than or at most equal to the whole number $((5))'$ of quintic perpetuants of the weight in question $w-6$. But as already mentioned, I have not been able to obtain the expression of the function (θ), $=\dfrac{\omega(x)}{2.3.4.5}$, which is the G. F. of the number $((\theta))'$.

Cambridge, England, 17*th March,* 1884.

829.

TABLES OF THE SYMMETRIC FUNCTIONS OF THE ROOTS, TO THE DEGREE 10, FOR THE FORM

$$1+bx+\frac{cx^2}{1\,.\,2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$$

[From the *American Journal of Mathematics*, t. VII. (1885), pp. 47—56.]

THE tables are derived from the tables (b) of my "Memoir on the Symmetric Functions of the Roots of an Equation," *Phil. Trans.*, vol. CXLVII. (1857), pp. 489—496, [147]. These refer in effect to the form $1+bx+cx^2+\ldots$, and we have consequently to change $b, c, d, \ldots$ into $\frac{b}{1}, \frac{c}{1\,.\,2}, \frac{d}{1\,.\,2\,.\,3}, \ldots$ respectively. Thus in the heading of the original table V(b), we must instead of

$$f\,,\quad be,\quad cd,\quad b^2d,\quad bc^2,\quad b^3c,\quad b^5,$$

write

$$\frac{f}{120},\quad \frac{be}{24},\quad \frac{cd}{12},\quad \frac{b^2d}{6},\quad \frac{bc^2}{4},\quad \frac{b^3c}{2},\quad \frac{b^5}{1}$$

$$=\frac{1}{120}(f,\quad 5be,\quad 10cd,\quad 20b^2d,\quad 30bc^2,\quad 60b^3c,\quad 120b^5);$$

the several columns of the original table are then multiplied by 1, 5, 10, 20, 30, 60, 120, and we thus obtain the new table with the heading

$$\frac{1}{120}(f,\quad be,\quad cd,\quad b^2d,\quad bc^2,\quad b^3c,\quad b^5).$$

In the original tables, there is a remarkable property (very easily proved) in regard to the sums of the numbers in a *column*. Thus for the table V(b) these sums are

$$-1,\quad +2,\quad +2,\quad -3,\quad -3,\quad +4,\quad -1,$$

where the sign is + or − according as the heading is the product of an even or an odd number of letters; and the numerical value depends only on the indices in the heading: these indices are

$$1,\quad 11,\quad 11,\quad 21,\quad 21,\quad 31,\quad 5,$$

and they give the foregoing values

$$1,\quad 2,\quad 2,\quad 3,\quad 3,\quad 4,\quad 1,$$

viz. b^3c, $=31$ gives the value $\Pi 4 \div \Pi 3\,.\,\Pi 1$, $=4$; b^2d, bc^2, each $=21$, give the value $\Pi 3 \div \Pi 2\,.\,\Pi 1$, $=3$; and so in other cases.

In the new tables we have a property in regard to the sums of the numbers in a *line:* viz. except for the last line of each table, where there is only a single number $+1$ or -1, this sum is always $=0$. I have given in the several tables on the right-hand of each line, the sums for the positive and the negative coefficients separately: thus V (b), line 1, the number ± 375 means that these sums are $+375$ and -375 respectively, the sum of all the coefficients being of course $=0$. The property is an important verification as well of the original tables (b) as of the new tables derived from them; and I had the pleasure of thus ascertaining that there was not a single inaccuracy in the original tables (b).

The symbols in the left-hand outside column of each table denote symmetric functions of the roots $\alpha, \beta, \gamma, \ldots$; $5 = \Sigma\alpha^5$, $41 = \Sigma\alpha^4\beta$, &c.: and the tables are read according to the lines: thus in table V (b),

$$5\,(=\Sigma\alpha^5\;) = \tfrac{1}{120}(5f + 25be + 50cd - 100b^2d - 150bc^2 + 300b^3c - 120b^5),$$

$$41\,(=\Sigma\alpha^4\beta) = \tfrac{1}{120}(5f - 5be - 50cd + 20b^3d + 90bc^2 - 60b^3c), \text{ \&c.}$$

I (b)

	$=$ b	
1	-1	-1

II (b)

$\div 2$

	$=$ c	b^2	
2	-2	$+2$	± 2
1^2	$+1$	$+1$	

III (b)

$\div 6$

	$=$ d	bc	b^3	
3	-3	$+9$	-6	± 9
21	$+3$	-3	± 3	
1^3	-1	-1		

IV (b)

$\div 24$

	$=$ e	bd	c^2	b^3c	b^4	
4	-4	$+16$	$+12$	-48	$+24$	± 52
31	$+4$	-4	-12	$+12$	± 16	
2^2	$+2$	-8	$+6$	± 8		
21^2	-4	$+4$	± 4			
1^4	$+1$	$+1$				

V (b)

$\div 120$

	f	be	cd	b^2d	bc^2	b^3c	b^5	
5	-5	$+25$	$+50$	-100	-150	$+300$	-120	± 375
41	$+5$	-5	-50	$+20$	$+90$	-60	± 115	
32	$+5$	-25	$+10$	$+40$	-30	± 55		
31^2	-5	$+5$	$+20$	-20	± 25			
2^21	-5	$+15$	-10	± 15				
21^3	$+5$	-5	± 5					
15	-1	-1						

VI (b)

	÷720 g	bf	ce	b^2e	d^2	bcd	b^3d	c^3	b^2c^2	b^4c	b^6	
6	− 6	+36	+90	−180	+60	−720	+720	−180	+1620	−2160	+720	±3246
51	+ 6	− 6	−90	+ 30	−60	+420	−120	+180	− 720	+ 360		±996
42	+ 6	−36	+30	+ 60	−60	+240	−240	−180	+ 180			±516
3^2	+ 3	−18	−45	+ 90	+60	−180	0	+ 90				± 243
41^2	− 6	+ 6	+30	− 30	+60	−180	+120					±216
321	−12	+42	+60	− 90	−60	+ 60						±162
2^3	− 2	+12	−30	0	+20							± 32
31^3	+ 6	− 6	−30	+ 30								±36
2^21^2	+ 9	−24	+15									± 15
21^4	− 6	+ 6										± 6
1^6	+ 1											± 1

VII (b)

	÷5040 h	bg	cf	b^2f	de	bce	b^3e	bd^2	c^2d	b^2cd	b^4d	bc^3	b^3c^2	b^5c	b^7	
7	− 7	+49	+147	−294	+245	+1470	+1470	−980	−1470	+8820	−5880	+4410	−17640	+17640	−5040	±32781
61	+ 7	− 7	−147	+ 42	−245	+ 840	− 210	+560	+1470	−3780	+ 840	−3150	+ 6300	− 2520		±10059
52	+ 7	−49	+ 63	+ 84	−245	+ 420	− 420	+980	− 630	−2520	+1680	+1890	− 1200			±5124
43	+ 7	−49	−147	+294	+175	+ 210	− 630	−700	+ 210	+1260	0	− 630				±2156
51^2	− 7	+ 7	+ 42	− 42	+245	− 315	+ 210	−560	− 420	+1680	− 840					±2184
421	−14	+56	+ 84	−126	+ 70	− 840	+ 630	+140	+ 420	− 420						±1400
3^21	− 7	+28	+147	−168	−175	+ 105	0	+280	− 210							±560
32^2	− 7	+49	− 63	− 84	+ 35	+ 210	0	−140								±294
41^3	+ 7	− 7	− 42	+ 42	−105	+ 315	− 210									±364
321^2	+21	−63	−126	+168	+105	− 105										± 294
2^31	+ 7	−35	+ 63	0	− 35											± 70
31^4	− 7	+ 7	+ 42	− 42												± 49
2^21^3	−14	+35	− 21													± 35
21^5	+ 7	− 7														± 7
1^7	− 1															− 1

(*Concluded infrà.*)

VIII (*b*)

÷ 40320

	i	bh	cg	b^2g	df	bcf	b^2f	e^2	bde	c^2e	b^2ce	b^4e	cd^2
8	− 8	+ 64	+224	−448	+448	−2688	+2688	+280	−4480	−3360	+20160	−13440	−4480
71	+ 8	− 8	−224	+ 56	−448	+1512	− 336	−280	+2520	+3360	− 8400	+ 1680	+4480
62	+ 8	− 64	+112	+112	−448	+ 672	− 672	−280	+4480	−1680	− 5040	+ 3360	+1120
53	+ 8	− 64	−224	+448	+392	+ 168	−1008	−280	+ 280	+3360	− 7560	+ 5040	−3920
4^2	+ 4	− 32	−112	+224	−224	+1344	−1344	+420	−2240	−1680	+ 3360	0	+2240
61^2	− 8	+ 8	+ 56	− 56	+448	− 504	+ 336	+280	−2520	− 840	+ 3360	− 1680	−2800
521	−16	+ 72	+112	−168	+ 56	−1354	+1008	+560	−2800	−1680	+ 9240	− 5040	+2800
431	−16	+ 72	+448	−504	+ 56	−1680	+1344	− 560	+2800	0	− 840	0	− 560
42^2	− 8	+ 64	−112	−112	+448	− 672	+ 672	−280	0	+1680	− 1680	0	−1120
3^22	− 8	+ 64	+ 56	−280	−392	+ 840	0	+280	− 280	− 840	0	0	+ 560
51^3	+ 8	− 8	− 56	+ 56	−168	+ 504	− 336	−280	+1120	+ 840	− 3360	+ 1680	±4208
421^2	+24	− 80	−168	+224	−504	+1848	−1344	+280	− 280	− 840	+ 840	± 3216	
3^21^2	+12	− 40	−252	+280	+168	− 168	0	+140	− 560	+ 420	± 1020		
32^21	+24	−136	0	+280	+336	− 504	0	−280	+ 280	± 920			
2^4	+ 2	− 16	+ 56	0	−112	0	0	+ 70	± 128				
41	− 8	+ 8	+ 56	56	+168	− 504	+ 336	±568					
321^3	32	+ 88	+224	−280	−168	+ 168	± 480						
2^31^2	−16	+ 72	−112	0	+ 56	± 128							
31^5	+ 8	− 8	− 56	+ 56	± 64								
2^21^4	+20	− 48	+ 28	± 48									
21^6	8	+ 8	± 8										
1^8	+ 1	+ 1											

÷ 40320

	b^2d^2	bc^2d	b^3cd	b^5d	c^4	b^2c^3	b^4c^2	b^6c	b^8
8	+13440	+40320	−107520	+53760	+ 5040	−80640	+201600	−161280	+ 40320	±377344
71	− 5600	−28560	+ 36960	− 6720	− 5040	+45360	− 60480	+ 20160	±116096
62	−10080	0	+ 26880	−13440	+ 5040	− 20160	+ 10080	± 51864
53	+ 3360	+10080	− 10080	0	− 5040	+ 5040	± 28176
4^2	+ 2240	− 6720	0	0	+ 2520	±12352
61^2	+ 5600	+ 8400	− 16800	+ 6720	±25208
521	− 1120	− 5040	+ 3360	±17208
431	− 2240	+ 680	± 6400
42^2	+ 1120	± 3984
	± 1800								

IX (*b*)

÷ 362880

	j	bi	ch	b^2h	dg	bcg	b^3g	ef	bdf	c^2f	b^2ef	b^4f	be^2
9	− 9	+ 81	+324	−648	+ 756	−4536	+4536	+1134	−9072	−6804	+40824	−27216	−5670
81	+ 9	− 9	−324	+ 72	− 756	+2520	− 504	−1134	+5040	+6804	−16632	+ 3024	+3150
72	+ 9	− 81	+180	+144	− 756	+1008	−1008	−1134	+9072	−3780	− 9072	+ 6048	+5670
63	+ 9	− 81	−324	+648	+ 756	0	−1512	−1134	0	+6804	−13608	+ 9072	+5670
54	+ 9	− 81	−324	+648	− 756	+4536	−4536	+1386	−1008	− 756	−10584	+12096	−6930
71^2	− 9	+ 9	+ 72	− 72	+ 756	− 756	+ 504	+1134	−5040	−1512	+ 6048	− 3024	−3150
621	−18	+ 90	+144	−216	0	−2016	+1512	+2268	−5040	−3024	+16632	− 9072	−8820
531	−18	+ 90	+648	−720	0	−2520	+2016	− 252	−2520	−6048	+22680	−12096	+3780
4^21	− 9	+ 45	+324	−360	+ 756	−3528	+2520	−1386	+3024	+ 756	− 1512	0	+3150
52^2	− 9	+ 81	−180	−144	+ 756	−1008	+1008	− 126	−4032	0	+ 9072	− 6048	+ 630
432	−18	+162	+144	−792	0	−1008	+2520	− 252	+1008	+4536	− 7560	0	+1260
3^3	− 3	+ 27	+108	−216	− 504	+ 756	0	+ 378	+1512	−2268	0	0	−1890
61^3	+ 9	− 9	− 72	+ 72	− 252	+ 756	− 504	−1124	+2016	+1512	− 6048	+ 3024	+3150
521^2	+27	− 99	−216	+288	− 756	+2772	−2016	− 882	+7560	+4536	−22680	+12096	− 630
431^2	+27	− 99	−720	+792	− 756	+3276	−2520	+1638	−2520	0	+ 1512	0	−3150
42^21	+27	−171	+ 36	+360	− 756	+3024	−2520	+ 378	−1008	−4536	+ 4536	0	+ 630
3^221	+27	−171	−468	+864	+1512	−1764	0	− 882	−2016	+2268	0	0	+1890
32^3	+ 9	− 81	+180	+144	− 252	− 504	0	+ 126	+1008	0	0	0	− 630
51^4	− 9	+ 9	+ 72	− 72	+ 252	− 756	+ 504	+ 504	−2016	−1512	+ 6048	− 3024	±7389
421^3	−36	+108	+288	−360	+1008	−3528	+2520	− 504	+ 504	+1512	− 1512	± 5940	
3^21^3	−18	+ 54	+396	−432	− 252	+ 252	0	− 252	+1008	− 756	± 1710		
32^21^2	−54	+270	+180	−648	− 756	+1008	0	+ 504	− 504	±1962			
2^41	− 9	+ 63	−180	0	+ 252	0	0	− 126	± 315				
41^5	+ 9	− 9	− 72	+ 72	− 252	+ 756	− 504	± 837					
321^4	+45	−117	−360	+432	+ 252	− 252	± 729						
2^21^3	+30	−126	+180	0	− 84	± 210							
31^6	− 9	+ 9	+ 72	− 72	± 81								
2^21^5	−27	+ 63	− 36	± 63									
21^7	+ 9	− 9	± 9										
1^9	− 1	− 1											

(*Continued next page.*)

(Concluded infrà.)

IX (*b*)

	cde	e^2de	bc^2e	b^2ce	b^5e	d^3	bcd^2	b^2d^2	c^3d	b^2c^3d	
9	−22680	+68040	+102060	−272160	+136080	− 5040	+136080	−181440	+68040	−816480	
81	+22680	−27720	− 71820	+ 90720	− 15120	+ 5040	− 95760	+ 60480	−68040	+453600	
72	+ 5040	−50400	+ 3780	+ 60480	− 30240	+ 5040	− 65520	+110880	+37800	+ 75600	
63	0	−22680	− 34020	+ 90720	− 45360	−10080	+ 90720	− 30240	−22680	−136080	
54	− 2520	+32760	+ 11340	− 30240	0	+ 5040	− 35280	− 20160	+ 7560	+ 60480	
71^2	−13860	+27720	+ 18900	− 37800	+ 15120	− 5040	+ 60480	− 60480	+15120	−136080	
621	− 5040	+32760	+ 45360	−105840	+ 45360	+ 5040	− 35280	+ 10080	−15120	+ 60480	
531	+ 2520	−12600	− 7560	+ 7560	0	+ 5040	− 20160	+ 20160	+15120	− 15120	
4^21	+ 2520	−12600	+ 3780	0	0	− 5040	+ 15120	0	− 7560	± 31995	
52^2	+ 7560	0	− 22680	+ 15120	0	− 5040	+ 15120	− 10080	±29347		
432	−10080	+ 2520	+ 7560	0	0	+ 5040	− 5040	± 24750			
3^3	+ 3780	0	0	0	0	− 1680	± 6561				
61^3	+ 6300	−12600	− 18900	+ 37800	− 15120	±54369					
521^2	− 6300	+ 2520	+ 11340	− 7560	± 41139						
431^2	+ 1260	+ 5040	− 3780	± 13545							
42^21	+ 2520	− 2520	± 11511								
3^221	− 1260	± 6561									
	± 1467										

÷ 40320

	b^4cd	b^6d	bc^4	b^3c^3	b^5c^2	b^7c	b^9	
9	+1360800	−544320	−204120	+1360800	−2449440	+1632960	− 362880	±4912515
81	− 393120	+ 60480	+158760	− 635040	+ 635040	− 181440	±1507419	
72	− 302400	+120960	−113400	+ 226800	− 90720	± 668511		
63	+ 90720	0	+ 68040	− 45360	± 363159			
54	0	0	− 22680	± 135855				
71^2	+ 181440	− 60480	±327303					
621	− 30240	±219726						
	± 79164							

X (*b*)

÷ 3628800

		k	bj	ci	b^2i	dh	bch	b^3h	eg	bdg	c^2g	b^2cg	b^4g
1	10	−10	+100	+450	−900	+1200	−7200	+7200	+2100	−16800	−12600	+75600	−50400
2	91	+10	−10	−450	+90	−1200	+3960	−720	−2100	+9240	+12600	−30240	+5040
3	82	+10	−100	+270	+180	−1200	+1440	−1440	−2100	+16800	−7560	−15120	+10080
4	73	+10	−100	−450	+900	+1320	−360	−2160	−2100	−840	+12600	−22680	+15120
5	64	+10	−100	−450	+900	−1200	+7200	−7200	+2940	−3360	−2520	−15120	+20160
6	5^2	+5	−50	−225	+450	−600	+3600	−3600	−1050	+8400	+6300	−37800	+25200
7	81^2	−10	+10	+90	−90	+1200	−1080	+720	+2100	−9240	−2520	+10080	−5040
8	721	−20	+110	+180	−270	−120	−2880	+2160	+4200	−8400	−5040	+27720	−15120
9	631	−20	+110	+900	−990	−120	−3600	+2880	−840	−3360	−10080	+37800	−20160
10	541	−20	+110	+900	−990	+2400	−11160	+7920	−840	−5880	−10080	+45360	−25200
11	62^2	−10	+100	−270	−180	+1200	−1440	+1440	−420	−6720	.	+15120	−10080
12	532	−20	+200	+180	−1080	−120	−1080	+3600	+4200	−15960	−5040	+37800	−25200
13	$4^2 2$	−10	+100	+90	−540	+1200	−4320	+4320	−2940	+3360	+12600	−15120	.
14	43^2	−10	+100	+450	−900	−1320	+360	+2160	−420	+10920	−5040	−7560	.
15	71^3	+10	−10	−90	+90	−360	+1080	−720	−2100	+3360	+2520	−10080	+5040
16	621^2	+30	−120	−270	+360	−1080	+3960	−2880	−1260	+12600	+7560	−37800	+20160
17	531^2	+30	−120	−990	+1080	−1080	+4680	−3600	−1260	+12600	+12600	−47880	+25200
18	$4^3 1^2$	+15	−60	−495	+540	−1800	+6120	−4320	+1890	−2520	−1260	+2520	.
19	$52^2 1$	+30	−210	+90	+450	−1080	+4320	−3600	−3780	+15120	+5040	−42840	+25200
20	4321	+60	−420	−1260	+2340	+360	+7560	−8640	+2520	−12600	−10080	+17640	.
21	$3^3 1$	+10	−70	−450	+630	+1320	−1440	.	+420	−5880	+5040	.	.
22	42^3	+10	−100	+270	+180	−1200	+1440	−1440	+2100	.	−5040	+5040	.
23	$3^2 2^2$	+15	−150	+45	+630	+720	−2520	.	−1890	+2520	+2520	.	.
24	61^4	−10	+10	+90	−90	+360	−1080	+720	+840	−3360	−2520	+10080	−5040
25	521^3	−40	+130	+360	−450	+1440	−5040	+3600	+3360	−15960	−10080	+47880	−25200
26	431^3	−40	+130	+1080	−1170	+1440	−5760	+4320	−1680	+4200	.	−2520	.
27	$42^2 1^2$	−60	+330	+180	−810	+2160	−8280	+6480	−2520	+2520	+10080	−10080	.
28	$3^2 21^2$	−60	+330	+1260	−1890	−2880	+3240	.	.	+5040	−5040	.	.
29	$32^3 1$	−40	+310	−360	−630	−240	+1800	.	+1680	−2520	.	.	.
30	2^5	−2	+20	−90	.	+240	.	.	−420	.	.	.	.

(*Continued next page.*)

X (*b*)

		f^2	bef	cdf	b^2df	bc^2f	b^3ef	b^5f	ce^2	b^2e^2	d^2e
1	10	+1260	−25200	−50400	+151200	+226800	−604800	+302400	− 31500	+ 94500	−42000
2	91	−1260	+13860	+50400	− 60480	−158760	+196560	− 30240	+ 31500	− 37800	+42000
3	82	−1260	+25200	+10080	−110880	+ 15120	+120960	− 60480	+ 6300	− 69300	+42000
4	73	−1260	+25200	− 2520	− 45360	− 68040	+181440	− 90720	+ 31500	− 94500	−46200
5	64	−1260	− 5040	+50400	− 30240	−136080	+241920	−120960	− 44100	+ 56700	− 8400
6	5^2	+2520	−18900	−37800	+ 50400	+ 75600	− 75600	.	+ 15750	+ 31500	+21000
7	81^2	+1260	−13860	−30240	+ 60480	+ 37800	− 75600	+ 30240	− 18900	+ 37800	−42000
8	721	+2520	−39060	− 7560	+ 65520	+ 90720	−211680	+ 90720	− 37800	+107100	+ 4200
9	631	+2520	− 8820	−47880	+ 75600	+136080	−287280	+120960	+ 12600	− 18900	+54600
10	541	−3780	+28980	+25200	− 60480	− 7560	+ 15120	.	+ 12600	− 50400	−33600
11	62^2	+1260	−10080	−10080	+ 50400	+ 30240	−120960	+ 60480	+ 31500	− 6300	−16800
12	532	−3780	+12600	+42840	− 20160	− 98280	+ 75600	.	− 37800	+ 6300	+ 4200
13	4^22	+1260	+ 5040	−30240	+ 10080	+ 15120	.	.	+ 6300	− 18900	+ 8400
14	43^2	+1260	−10080	+ 2520	− 15120	+ 22680	.	.	+ 6300	+ 18900	− 4200
15	71^3	−1250	+13860	+12600	− 25200	− 37800	+ 75600	− 30240	+ 18900	− 37800	+12600
16	621^2	−3780	+22680	+37800	− 95760	−128520	+287280	−120960	− 18900	+ 6300	−12600
17	531^2	+2520	−15120	−10080	+ 15120	+ 15120	− 15120	.	+ 6300	+ 12600	−12600
18	4^21^2	+1260	−10080	+ 2520	+ 10080	− 7560	.	.	− 9450	+ 18900	+12600
19	52^21	+2520	− 1260	−32760	+ 10080	+ 68040	− 45360	.	+ 6300	− 6300	+12600
20	4321	−1260	.	+12600	+ 20160	− 22680	.	.	+ 12600	− 18900	−12600
21	3^31	−1260	+ 6300	− 2520	.	.	.	.	− 6300	.	+ 4200
22	42^3	−1260	.	+10080	− 10080	.	.	.	− 6300	+ 6300	±25420
23	3^22^2	+1260	− 1260	− 5040	.	.	.	.	+ 3150	± 10860	
24	61^4	+1260	− 6300	−12600	+ 25200	+ 37800	− 75600	+ 30240	±106600		
25	521^3	−1260	+ 1260	+12600	− 5040	− 22680	+ 15120	± 85750			
26	431^3	−1260	+ 6300	− 2520	− 10080	+ 7560	± 25030				
27	42^21^2	+1260	− 1260	− 5040	+ 5040	± 28050					
28	3^221^2	+1260	− 3780	+ 2520	± 13650						
29	32^31	−1260	+ 1260	± 5050							
30	2^5	+ 252	± 512								

(*Continued next page.*)

X (*b*)

		$bcde$	b^3de	c^3e	b^2c^2e	b^4ce	b^6e	bd^3	c^2d^2	b^2cd^2	b^4d^2
1	10	+756000	−1008000	+189000	−2268000	+3780000	−1512000	+168000	+378000	−3024000	+2520000
2	91	−529200	+ 327600	−189000	+1247400	−1058400	+ 151200	−117600	−378000	+1663200	− 705600
3	82	−352800	+ 604800	+113400	+ 151200	− 756000	+ 302400	−168000	+ 25200	+1411200	+1310400
4	73	+ 37800	+ 302400	−189000	+ 680400	−1134000	+ 453600	+184800	+151200	−1209600	+ 302400
5	64	+151200	− 201600	+189000	− 453600	+ 302400	.	+ 33600	−226800	.	+ 201600
6	5^2	− 63000	− 126000	− 94500	+ 189000	.	.	− 84000	+126000	+ 252000	.
7	81^2	+327600	− 327600	+ 37800	− 340200	+ 453600	− 151200	+117600	+176400	−1058400	+ 705600
8	721	+264600	− 403200	+ 75600	− 869400	+1285200	− 453600	− 67200	−176400	+ 453600	− 100800
9	631	−189000	+ 126000	.	+ 113400	− 75600	.	−117600	+ 75600	+ 302400	− 201600
10	541	.	+ 126000	.	− 37800	.	.	+ 84000	− 25200	− 151200	.
11	62^2	−100800	.	− 75600	+ 302400	− 151200	.	+ 67200	+ 50400	− 201600	+ 100800
12	532	+ 63000	− 25200	+ 75600	− 75600	.	.	− 16800	− 50400	+ 50400	± 376520
13	4^22	+ 50400	.	− 37800	.	.	.	− 33600	+ 25200	± 143470	
14	43^2	− 37800	.	.	.	.	.	+ 16800	± 82450		
15	71^3	−151200	+ 151200	− 37800	+ 340200	− 453600	+ 151200	± 788260			
16	621^2	+ 88200	− 25200	+ 37800	− 151200	+ 75600	± 600330				
17	531^2	+ 50400	− 50400	− 37800	+ 37800	± 196050					
18	4^21^2	− 37800	.	+ 18900	± 75345						
19	52^21	− 37800	+ 25200	± 174990							
20	4321	+ 12600	± 88440								
		± 17920									

(*Continued next page.*)

X (*b*)

		bc^3d	b^3c^2d	b^5cd	b^7d	c^5	b^2c^4	b^4c^3	b^6c^2	b^8c	b^{10}	
1	10	−3024000	+15120000	−18144000	+6048000	−226800	+5670000	−22680000	+31752000	−18144000	+3628800	±70872610
2	91	+2343600	−6955200	+4536000	−604800	+226800	−3628800	+9072000	−7257600	+1814400	±21747460	
3	82	−604800	−1814400	+3628800	−1209600	−226800	+2041200	−2721600	+907200	±9433840		
4	73	−151200	+1814400	−907200	.	+226800	−907200	+453600	±4875490			
5	64	+604800	−604800	.	.	−226800	+226800	±2089630				
6	5^2	−378000	.	.	.	+113400	±921125					
7	81^2	−529200	+2116800	−2116800	+604800	±4721980						
8	721	+378000	−756000	+302400	±3154550							
9	631	−226800	+151200	±1212650								
10	541	+75600	±424190									
		±712540										

		k	bj	ci	b^2i	dh	bch	b^3h	eg	bdg	c^2g	b^3cg	b^4g	
31	51^5	+10	−10	−90	+90	−360	+1080	−720	−840	+3360	+2520	−10080	+5040	
32	421^4	+50	−140	−450	+540	−1800	+6120	−4320	+840	−840	−2520	+2520	±10070	
33	3^21^4	+25	−70	−585	+630	+360	−360	.	+420	−1680	+1260	±2695		
34	32^21^3	+100	−460	−540	+1260	+1440	−1800	.	−840	+840	±3640			
35	241^2	+25	−160	+405	.	−480	.	.	+210	±640				
36	41^6	−10	+10	+90	−90	+360	−1080	+720	±1180					
37	321^5	−60	+150	+540	−630	−360	+360	±1050						
38	2^31^4	−50	+200	−270	.	+120	±320							
39	31^7	+10	−10	−90	+90	±100								
40	2^21^6	+35	−80	+45	±80									
41	21^8	−10	+10	±10										
42	1^{10}	+1	+1											

830.

NON-UNITARY PARTITION TABLES.

[From the *American Journal of Mathematics*, t. VII. (1885), pp. 57, 58.]

In the theory of Seminvariants we are concerned with the non-unitary partitions of a number, that is, the number of ways of making up the number with the parts 2, 3, 4, ...; or what is the same thing, writing $\mathbf{2} = 1 - x^2$, $\mathbf{3} = 1 - x^3$, &c., with the Generating Functions having in their denominators the factors **2**, **3**, **4**, &c. In the present short paper, I give the developments up to x^{100} of the functions $1 \div \mathbf{2}$, $\mathbf{2.3}$, $\mathbf{2.3.4}$, $\mathbf{2.3.4.5}$, $\mathbf{2.3.4.5.6}$, respectively: and also of the function

$$x^6 + x^{13} - 2x^{16} - x^{18} + x^{31} \div \mathbf{2.3.4.5.6},$$

which function is (there is strong reason to believe) the G. F. for the number of sextic syzygies of a given weight: the same function without the term x^{31} occurs (p. 115) in Professor Sylvester's paper "On Subinvariants, i.e. Seminvariants to Binary Quantics of an Unlimited Order," *American Journal of Mathematics*, t. V. (1882), pp. 79—136.

In the tables, X is written to denote $x^6 + x^{13} - 2x^{16} - x^{18} + x^{31}$.

Ind. x	$1\div$ 2.3	2.3.4	2.3.4.5	2.3.4.5.6	$X\div$ 2.3.4.5.6	Ind. x	$1\div$ 2.3	2.3.4	2.3.4.5	2.3.4.5.6	$X\div$ 2.3.4.5.6
0	1	1	1	1		50	9	65	258	750	186
1	0	0	0	0		51	9	61	268	783	226
2	1	1	1	1		52	9	70	286	854	203
3	1	1	1	1		53	9	65	297	891	248
4	1	2	2	2		54	10	75	316	972	223
5	1	1	2	2		55	9	70	328	1010	270
6	2	3	3	4	1	56	10	80	348	1098	242
7	1	2	3	3	0	57	10	75	361	1144	294
8	2	4	5	6	1	58	10	85	382	1236	262
9	2	3	5	6	1	59	10	80	396	1287	319
10	2	5	7	9	2	60	11	91	419	1391	284
11	2	4	7	9	2	61	10	85	433	1443	344
12	3	7	10	14	4	62	11	96	457	1555	306
13	2	5	10	13	4	63	11	91	473	1617	371
14	3	8	13	19	6	64	11	102	598	1734	328
15	3	7	14	20	7	65	11	96	515	1802	399
16	3	10	17	26	8	66	12	108	541	1932	353
17	3	8	18	27	11	67	11	102	559	2002	427
18	4	12	22	36	13	68	12	114	587	2142	377
19	3	10	23	36	15	69	12	108	606	2223	457
20	4	14	28	47	17	70	12	120	635	2369	402
21	4	12	29	49	21	71	12	114	655	2457	490
22	4	16	34	60	22	72	13	127	686	2618	429
23	4	14	36	63	28	73	12	120	707	2709	519
24	5	19	42	78	29	74	13	133	739	2881	456
25	4	16	44	80	35	75	13	127	762	2985	552
26	5	21	50	97	36	76	13	140	795	3164	483
27	5	19	53	102	44	77	13	133	819	3276	586
28	5	24	60	120	43	78	14	147	854	3472	513
29	5	21	63	126	54	79	13	140	879	3588	620
30	6	27	71	149	53	80	14	154	916	3797	542
31	5	24	74	154	64	81	14	147	942	3927	656
32	6	30	83	180	62	82	14	161	980	4144	572
33	6	27	87	189	78	83	14	154	1008	4284	693
34	6	33	96	216	72	84	15	169	1048	4520	604
35	6	30	101	227	89	85	14	161	1077	4665	730
36	7	37	111	260	84	86	15	176	1118	4915	636
37	6	33	116	270	102	87	15	169	1149	5076	769
38	7	40	127	307	96	88	15	184	1192	5336	568
39	7	37	133	322	117	89	15	176	1224	5508	809
40	7	44	145	361	108	90	16	192	1269	5789	703
41	7	40	151	378	133	91	15	184	1302	5967	849
42	8	48	164	424	123	92	16	200	1349	6264	736
43	7	44	171	441	149	93	16	192	1384	6460	891
44	8	52	185	492	137	94	16	208	1432	6768	772
45	8	48	193	515	167	95	16	200	1469	6977	934
46	8	56	207	568	152	96	17	217	1519	7308	809
47	8	52	216	594	186	97	16	208	1557	7524	977
48	9	61	232	656	169	98	17	225	1609	7873	846
49	8	56	241	682	205	99	17	217	1649	8109	1022
						100	17	234	1883	8651	883

831.

SEMINVARIANT TABLES.

[From the *American Journal of Mathematics*, t. VII. (1885), pp. 59—73.]

THE present tables are not, I think, superseded by the tables A, pp. 149—163, contained in Capt. MacMahon's paper, "Seminvariants and Symmetric Functions," *American Journal of Mathematics*, t. VI. (1883), pp. 131—163. His order of the terms, though a very ingenious one, and giving rise to a most remarkable symmetry in the form of the tables, seems to me too artificial—and I cannot satisfy myself that it ought to be adopted in preference to the more simple one which I use: I attach also considerable importance to the employment of the simple letters b, c, d, e, &c. in place of the suffixed ones a_1, a_2, a_3, a_4, &c. There is, moreover, the question of the identification of the seminvariants with their expressions as non-unitary symmetric functions of the roots of the equation $1 + bx + \frac{cx^2}{1.2} + \&c. = 0$, which requires to be considered.

As to the form in which the tables present themselves, I remark that every seminvariant is a rational and integral function of the fundamental seminvariants

$$\mathrm{c} = (1,\ b,\ c \between -b,\ 1)^2,$$
$$\mathrm{d} = (1,\ b,\ c,\ d \between -b,\ 1)^3,$$
$$\mathrm{e} = (1,\ b,\ c,\ d,\ e \between -b,\ 1)^4,\ \&c.,$$

viz. up to g, these are

c =	d =	e =	f =	g =	ce =	d^2 =	c^3 =
$c + 1$	$d + 1$	$e + 1$	$f + 1$	$g + 1$			
$b^2 - 1$	$bc - 3$	$bd - 4$	$be - 5$	$bf - 6$			
	$b^3 + 2$	c^2	cd	ce	$+ 1$		
		$b^2c + 6$	$b^2d + 10$	d^2		$+ 1$	
		$b^4 - 3$	bc^2	$b^2e + 15$	$- 1$		
			$b^3c - 10$	bcd	$- 4$	$- 6$	
			$b^5 + 4$	c^3			$+ 1$
				$b^3d - 20$	$+ 4$	$+ 4$	
				b^2c^2	$+ 6$	$+ 9$	$- 3$
				$b^4c + 15$	$- 9$	$- 12$	$+ 3$
				$b^6 - 5$	$+ 3$	$+ 4$	$- 1$

and if to the value of g, which is of the weight 6, we join those of the products ce, d^2, c^3 of this same weight 6, as just written down, we have what is in effect a table of the asyzygetic seminvariants of the weight 6 and which I call the Crude Table. But we do not, in this way, obtain immediately the seminvariants of the lowest degrees: in fact, the only seminvariant containing g is given by the first column as a function $g - \ldots - 5b^6$ of the degree 6, whereas there is the seminvariant $g - 6bf + 15ce - 10d^2$ of the degree 2: to obtain this, we have to form a linear combination of the columns: the proper combination is $g + 15ce - 10d^2$, giving rise to the column g of the table (p. 278, $g = 6$). And similarly each other column of the same table is a linear combination of columns of the Crude Table: and so in every case. The process would be a very laborious one, and the tables were not, in fact, thus calculated; but we see very clearly in this manner the origin and meaning of the tables.

The mere inspection of the tables gives rise to several remarks. We see that each column begins with a non-unitary term (term without the letter b), and that it ends with a power-ending term (product wherein the last letter enters as a power)—thus, weight 8, the initial terms are

$$i, \quad cg, \quad df, \quad e^2, \quad c^2e, \quad cd^2, \quad c^4,$$

the finals are

$$e^2, \quad cd^2, \quad b^3d^2, \quad c^4, \quad b^2c^3, \quad b^4c^2, \quad b^8;$$

and it will be observed further that in this case the initial terms are all the non-unitary terms taken in order, and the corresponding final terms are all the power-ending terms taken also in order. The arrangement of the columns *inter se* is of course arbitrary, and they are, in fact, arranged so that the initial terms are the non-unitary terms taken in order—and this being so, then for each weight up to the weight 9 the final terms are the power-ending terms taken in order: but for each of the weights 10 and 11, there is a single deviation from this order; and for the weight 12, there are a great many deviations from the order.

The initial terms being in order, the broken line which bounds the tops of the columns forms a series of continually descending steps; and when the final terms are also in order, the case is the same with the broken line bounding the bottom of the columns: any deviation in the order of the final terms is shown by an ascending step or steps in the broken line bounding the bottom of the columns: thus in the table (p. 281, $k = 10$) the column cdf is longer than the next following one ce^2, and there is an ascending step accordingly.

It is to be remarked that any ascending step gives rise to a certain indeterminateness in a preceding column or columns: thus in the case just referred to, the column cdf might be replaced by any linear combination of itself with the column ce^2, it would still have the original initial and final terms cdf and b^4d^2 respectively. It would be possible to fix a standard form; we might, for instance, say that the column cdf should be that combination $cdf + 2ce^2$, which does not contain the leading term ce^2 of the ce^2-column: but I have not thought it worth while to attend to this.

It will be observed that, except in the case of an ascending step or steps, each column is completely determinate: we cannot with any column combine a preceding column, for this would give it a higher initial term: nor can we with it combine a succeeding column, for this would give it a lower final term. The numbers in the column may be taken to be without any common divisor, for any such divisor, if it existed, might be divided out: and the leading coefficient of the column may be taken to be positive.

I add certain subsidiary tables to enable the expression of any column in terms of the non-unitary symmetric functions of the roots of the equation $0 = 1 + bx + \frac{cx^2}{1.2} + \&c.$ These consist of left-hand tables and right-hand tables: the left-hand table for any weight is the original table for that weight, writing therein $b = 0$ and converting the columns into lines: thus weight $= 6$, we have

$$\begin{array}{llll}
\text{col. } g = & g + 15ce - 10d^2, & & \\
\text{col. } ce = & & ce - d^2 - c^3, & \\
\text{col. } d^2 = & & d^2 + 4c^3, & \\
\text{col. } c^3 = & & & c^3;
\end{array}$$

viz. these are the values of the original columns writing therein $b = 0$.

The right-hand table is the table for the same weight taken from my paper "Tables of the Symmetric Functions of the Roots, to the Degree 10, for the form $1 + bx + \frac{cx^2}{1.2} + \ldots = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)\ldots$," [829], writing therein $b = 0$, and giving only those lines of the table which relate to the non-unitary symmetric functions. Thus for weight 6, we have

$$\begin{array}{ll}
6\ (= \Sigma\alpha^6) & = \frac{1}{720}(-6g + 90ce + 60d^2 - 180c^3), \\
42\ (= \Sigma\alpha^4\beta^2) & = \frac{1}{720}(\ \ 6g + 30ce - 60d^2 - 180c^3), \\
3^2\ (= \Sigma\alpha^3\beta^3) & = \frac{1}{720}(\ \ 3g - 45ce + 60d^2 + \ \ 90c^3), \\
2^3\ (= \Sigma\alpha^2\beta^2\gamma^2) & = \frac{1}{720}(-2g - 30ce + 20d^2),
\end{array}$$

we thus have on the one side col. g, col. ce, col. d^2 and col. c^3, and on the other side the symmetric functions 6, 42, 3^2 and 2^3, each of them expressed as a linear function of g, ce, d^2 and c^3. It follows that each of the columns can be expressed as a linear function of the symmetric functions: and conversely each of the symmetric functions as a linear function of the columns: and this being done, each of the columns is to be regarded as having its complete value as a function of b and the other letters: for the columns *quâ* seminvariants are linear functions of the symmetric functions: and assuming them to be so, they can only be the linear functions determined by the foregoing process of writing $b = 0$.

The left-hand tables are carried up to $m = 12$; the right-hand only up to $k = 10$, the limit of the tables in the memoir last referred to.

Seminvariant Tables up to $(m=12)$.

$(1=0)$

1	+1

$(b=1)$

b

$(c=2)$ c

c	1
b^2	−1

$(d=3)$ d

d	1
bc	−3
b^3	+2

$(e=4)$ e c^2

e	1	
bd	−4	
c^2	+3	1
b^2c		−2
b^4		+1

$(f=5)$ f cd

f	1	
be	−5	
cd	+2	+1
b^2d	+8	−1
bc^2	−6	−3
b^3c		+5
b^5		−2

$(g=6)$ g ce d^2 c^3

g	+1			
bf	−6			
ce	+15	+1		
d^2	−10	−1	+1	
b^3e		−1		
bcd		+2	−6	
c^3		−1	+4	+1
b^3d			+4	
b^2c^2			−3	−3
b^4c				+3
b^6				−1

$(h=7)$	h	cf	de	c^2d
h	+ 1			
bg	− 7			
cf	+ 9	+1		
de	− 5	−1	+ 1	
b^2f	+12	−1		
bce	−30	−2	− 3	
bd^2	+20	+4	− 4	
c^2d		−1	+ 3	+1
b^2e		+3	+ 2	
b^2cd		−6	+12	−2
bc^3		+3	− 9	−3
b^4d			− 8	+1
b^3c^2			+ 6	+8
b^5c				−7
b^7				+2

$(i=8)$	i	cg	df	e^2	c^2e	cd^2	c^4
i	+ 1						
bh	− 8						
cg	+28	+1					
df	−56	−3	+ 3				
e^2	+35	+2	− 2	+ 1			
b^2g		−1					
bcf		+3	− 9				
bde		−1	+ 1	− 8			
c^2e		−3	+18	+ 6	+1		
cd^2		+2	−12		−1	+ 1	
b^3f			+ 6				
b^2ce			−15		−2		
b^2d^2			+10	+16	+1	− 1	
bc^2d				−24	+2	− 6	
c^4				+ 9	−1	+ 4	+1
b^4e					+1		
b^3cd					−2	+10	
b^2c^3					+1	− 7	−4
b^5d						− 4	
b^4c^2						+ 3	+6
b^6c							−4
b^8							+1

$(j=9)$	j	ch	dg	ef	c^2f	cde	d^3	c^3d
j	+ 1							
bi	− 9							
ch	+ 20	+ 2						
dg	− 28	− 7	+ 1					
ef	+ 14	+ 5	− 1	+ 1				
b^2h	+ 16	− 2						
bcg	− 56	+ 7	− 3					
bdf	+112	+22	− 2	− 4				
be^2	− 70	−25	+ 5	− 5				
c^2f		−27	+ 9	− 9	+2			
cde		+45	−17	+32	−5	+1		
d^3		−20	+ 8	−18	+3	−1	+ 1	
b^3g			+ 2					
b^2cf			− 6	+24	−4			
b^2de			+ 2	− 2	+5	−1		
bc^2e			+ 6	−51	+5	−3		
bcd^2			− 4	+34	−7	+5	− 9	
c^3d					+1	−1	+ 4	+ 1
b^4f				−12	+2			
b^3ce				+30	−5	+5		
b^3d^2				−20	−2	−2	+ 6	
b^2c^2d					+8	−6	+15	− 3
bc^4					−3	+3	−12	− 3
b^5e						−2		
b^4cd						+4	−24	+ 3
b^3c^3						−2	+17	+11
b^6d							+ 8	− 1
b^5c^2							− 6	−15
b^7c								+ 9
b^9								− 2

$(k=10)$	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
k	+ 1											
bj	− 10											
ci	+ 45	+1										
dh	−120	−4	+ 4									
eg	+210	+8	− 8	+ 16								
f^2	−126	−5	+ 5	+ 15	+ 1							
b^2i		−1										
bch		+4	−12									
bdg		−4	+ 4	− 64								
bef		+2	− 2	+ 54	−10							
c^2g		−4	+32	+ 48		+1						
cdf		+8	−64	− 60	+ 4	−3	+ 3					
ce^2		−5	+40		+16	+2	− 2	+ 1				
d^2e				+ 20	−12			− 1	+ 1			
b^2h			+ 8									
b^2cg			−28			−2						
b^2df			+56	+144	+16	+3	− 3					
b^2e^2			−35	−135	+ 9	−2	+ 2	− 1				
bc^2f				−108	−12	+3	− 9					
$bcde$				+180	−76	−1	+ 1	− 2	− 6			
bd^3				− 80	+48			+ 4	− 4			
c^3e					+48	−3	+18	+ 2	+ 4	+1		
c^2d^2					−32	+2	−12	− 3	+ 3	−1	+ 1	
b^4g						+1						
b^3cf						−3	+15					
b^3de						+1	− 1	+ 4	+ 4			
b^2c^2e						+3	−33	− 3	− 3	− 3		
b^2cd^2						−2	+22	− 8	+24	+2	− 2	
bc^3d								+10	−34	+2	− 6	
c^5								− 3	+ 12	−1	+ 4	+ 1
b^5f							− 6					
b^4ce							+15			+3		
b^4d^2							−10		−16	−1	+ 1	
b^3c^2d									+24	−4	+16	
b^2c^4									− 9	+2	−11	
b^6e										−1		− 5
b^5cd										+2	−14	
b^4c^3										−1	+10	+10
b^7d											+ 4	
b^6c^2											− 3	−10
b^8c												+ 5
b^{10}												− 1

$(l=11)$	l	cj	di	eh	c^2h	fg	cdg	cef	d^2f	c^3f	de^2	c^2de	cd^3	c^4d
l	+ 1													
bk	− 11													
cj	+ 35	+ 2												
di	− 75	− 9	+ 1											
eh	+ 90	+14	− 2	+ 1										
fg	− 42	− 7	+ 1	− 1	+ 1	+ 1								
b^2j	+ 20	− 2												
bci	− 90	+ 9	− 3											
bdh	+240	+16		− 4										
beg	−420	−63	+ 9	− 2	− 5	− 5								
bf^2	+252	+42	− 6	+ 6	− 6	− 6								
c^2h		−30	+10	+ 3	− 16									
cdg		+70	−26	− 2	+ 58	+ 2	+ 1							
cef		−21	+ 7	− 6	+ 5	− 35	− 3	+ 1						
d^2f		−56	+24	+10	−100	−100	+ 6	− 3	+ 1					
de^2		+35	−15	− 5	+ 60	+ 60	− 4	+ 2	− 1	+ 1	+ 1			
b^2i			+ 2											
b^2ch			− 8		+ 32									
b^2dg			+ 8	+20	− 48	+ 8	− 1							
b^2ef			− 4	−18	+ 40		+ 3	− 1						
bc^2g			+ 8	−15	− 62	− 6	− 3							
$bcdf$			−16	−24	+232	+408	−30	+14	− 6					
bce^2			+10	+45	−205	−405	+27	−11	+ 3	− 3	− 3			
bd^2e				−10	+ 20	+ 20	+ 2	− 1	+ 3	− 8	− 8			
c^3f				+27	− 54	−270	+27	− 9	+ 4	−12				
c^2de				−45	+ 90	+450	−45	+14	− 6	+36	+ 6	+ 1		
cd^3				+20	− 40	−200	+20	− 6	+ 2	−18		− 1	+ 1	
b^4h					− 16									

(*Continued next page.*)

	c^3h	fg	cdg	cef	d^2f	c^3f	de^2	c^2de	cd^3	c^4d
b^3cg	+ 56		+ 5							
b^3df	− 112	− 288	+ 18	− 8	+ 4					
b^3e^2	+ 70	+ 270	− 23	+ 9	− 2	+ 2				
b^2c^2f		+ 216	− 27	+ 6	− 3	+ 36				
b^2cde		− 360	+ 51	− 16	+ 6	− 36	+ 96	− 2		
b^2d^3		+ 160	− 24	+ 8	− 8	+ 34	+ 16	+ 1	− 1	
bc^3e				+ 3	− 2	− 48	− 18	− 3		
bc^2d^2				− 2	+ 6	+ 18	− 24	+ 5	− 9	
c^4d					− 1	+ 3	+ 9	− 1	+ 4	+ 1
b^2g			− 2							
b^4cf			+ 6			− 36				
b^4de			− 2		− 4	+ 14	− 16	+ 1		
b^3c^2e			− 6		+ 3	+ 72	+ 12	+ 8		
b^3cd^2			+ 4		+ 8	− 78	− 48	− 7	+ 15	
b^2c^3d					− 10	+ 30	+ 72	− 5	+ 11	− 4
bc^5					+ 3	− 9	− 27	+ 3	− 12	− 3
b^6f						+ 12				
b^5ce						− 30		− 7		
b^5d^2						+ 20	+ 32	+ 2	− 6	
b^4c^2d							− 48	+ 10	− 39	+ 6
b^3c^4	. . . . .						+ 18	− 5	+ 29	+ 14
b^7e								+ 2		
b^6cd								− 4	+ 32	− 4
b^5c^3								+ 2	− 23	− 26
b^8d									− 8	+ 1
b^7c^2									+ 6	+ 24
b^9c										− 11
b^{11}										+ 2

$(m=12)$	m	ck	dj	ei	fh	g^2	c^2i	cdh	ceg	d^2g	$\div 2$ cf^2	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2	c^6
m	+ 1																				
bl	− 12																				
ck	+ 66	+ 3																			
dj	− 220	− 15	+ 15																		
ei	+ 495	+ 40	− 40	+ 1																	
fh	− 792	− 70	+ 70	− 4	+ 25																
g^2	+ 462	+ 42	− 42	+ 3	− 24	+ 1															
b^2k		− 3																			
bcj		+ 15	− 45																		
bdi		− 25	+ 25	− 4																	
beh		+ 30	− 30	+ 12	− 125																
bfg		− 14	+ 18	− 8	+ 113	− 12															
c^2i		− 15	+ 150	+ 3			+ 1														
cdh		+ 40	− 400	− 8	+ 50		− 4	+ 4													
ceg		− 70	+ 700	− 22	+ 680	− 70	+ 8	− 8	+ 1												
cf^2		+ 42	− 420	+ 24	− 675	+ 100	− 5	+ 5	− 1	+ 2	+ 2										
d^2g				+ 24	− 570	+ 80			− 1	+ 5											
def				− 36	+ 925	− 200			+ 2	− 19	− 4	+ 18									
e^3				+ 15	− 400	+ 100			− 1	+ 12	+ 2	− 17	+ 1								
b^3j	. . . .		+ 30																		
b^2ci			− 135				− 2														
b^2dh			+ 360		+ 200		+ 4	− 4													
b^2eg			− 630		− 525	+ 100	− 8	+ 8	− 1												
b^2f^2			+ 378		+ 336	− 64	+ 5	− 5	+ 1	− 2	− 2										
bc^2h					− 150		+ 4	− 12													
$bcdg$					+ 350	− 200	− 4	+ 4	+ 2	− 30											
$bcef$					− 105	+ 20	+ 2	− 2	− 2	+ 37	− 8	− 54									
bd^2f					− 280	+ 320			− 2	+ 46	+ 16	− 72									
bde^2					+ 175	− 200			+ 2	− 49	− 4	+ 114	− 12								
c^3g						+ 100	− 4	+ 32	− 1	+ 20				+ 1							
c^2df						− 200	+ 8	− 64	+ 2	− 49	− 4	+ 54		− 3	+ 3						
c^2e^2						+ 125	− 5	+ 40	+ 1	− 32	+ 28	+ 162	− 18			+ 1					
cd^2e									− 3	+ 91	− 44	− 342	+ 54	+ 4		− 2	+ 1				
d^4									+ 1	− 32	+ 18	+ 135	− 27	− 2	− 2	+ 1	− 1	+ 1			
b^4i	. . . . .						+ 1														
b^3ch							− 4	+ 20													
b^3dg							+ 4	− 4		+ 20											
b^3ef							− 2	+ 2		− 18	+ 12	+ 36									
b^2c^2g							+ 4	− 60		− 15				− 3							

(*Continued next page.*)

	c^2i	cdh	ceg	d^2g	$\div 2$ cf^2	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2	c^6
ϑ^2cdf	−8	+120		−24	−24	+216		+6							
ϑ^2ce^2	+5	−75		+45	−30	−360	+54	+2		−2					
ϑ^2d^2e				−10	+20	+66	−6	−6		+2	−1				
ϑ^2cf				+27	+12	−162		+3	−9						
ϑ^2de				−45	+60	−486	−180	−8	−15	+4	−6				
ϑ^2cd^2				+20	−40	−252	+108	+7	+24	−4	+8	−12			
c^4e					−30	−81	+81	+1	+30	−2	+4		+1		
c^3d^2					+20	+54	−54	−2	−28	+2	−5	+8	−1	+1	
ϑ^3h		−8													
ϑ^2cg		+28						+3							
ϑ^2df		−56				−144		−3	+3						
ϑ^2e^2		+35				+135	−27			+1					
ϑ^2c^2f						+108		−6	+24						
ϑ^2cde						−180	+108	+9	+30	−4	+10				
ϑ^2d^3						+80	−64	+1	−16		−4	+8			
ϑ^2c^2e							−54	+2	−75	+2	−7		−4		
$\vartheta^2c^2d^2$							+36	−12	+6	+4	−9	+30	+3	−3	
ϑ^2c^3d								+8	+48	−4	+14	−48	+2	−6	
c^6								−2	+2	+1		+16	−1	+4	+1
ϑ^3g								−1	−16		−4				
ϑ^3cf								+3	−21						
ϑ^3de								−1	−15		−4				
ϑ^3c^2e								−3	+60		+3		+6		
ϑ^3cd^2								+2			+8	−48	−3	+3	
ϑ^3c^2d									−40		−10	+68	−6	+22	
ϑ^3c^3									+12		+3	−24	+3	−15	−6
ϑ^4f									+6						
ϑ^4ce									−15				−4		
ϑ^4d^2									+10			+16	−23	−1	
ϑ^4c^2d												−24	+30	−30	
ϑ^4c^3												+9	−3	+21	+15
ϑ^5e													+1		
ϑ^5cd													+2	+14	−20
ϑ^5c^2													−3	−9	
ϑ^6d														−4	
ϑ^6c^2														+3	+15
ϑ^7c															−6
ϑ^8															+1

SUBSIDIARY TABLES: $b = 0$.

Left-hand: up to $(m = 12)$.

Col.	c
c	+ 1

Col.	d
d	+ 1

Col.	e	c^2
e	+ 1	+ 3
c^2	.	+ 1

Col.	f	cd
f	+ 1	+ 2
cd		+ 1

Col.	g	ce	d^2	c^3
g	+ 1	+15	−10	
ce		+ 1	− 1	− 1
d^2			+ 1	+ 4
c^3				+ 1

Col.	h	cf	de	c^2d
h	+ 1	+ 9	− 5	
cf		+ 1	− 1	− 1
de			+ 1	+ 3
c^2d				+ 1

Col.	l	cg	df	e^2	c^2e	cd^2	c^4
l	+1	+28	−56	+35			
cg		+ 1	− 3	+ 2	− 3	+ 2	
df			+ 3	− 2	+18	−12	
e^2				+ 1	+ 6	.	+9
c^2e					+ 1	− 1	−1
cd^2						+ 1	+4
c^4							+1

Col.	j	ch	dg	ef	c^2f	cde	d^3	c^3d
j	+1	+20	−28	+14				
ch		+ 2	− 7	+ 5	−27	+45	−20	
dg			+ 1	− 1	+ 9	−17	+ 8	
ef				+ 1	− 9	+32	−18	
c^2f					+ 2	− 5	+ 3	+1
cde						+ 1	− 1	−1
d^3							+ 1	+4
c^3d								+1

Right-hand: up to ($k = 10$).

÷2

	c
2	−2

÷6

	d
3	−3

÷24

	e	c^2
4	−4	+12
2^2	+2	+6

÷120

	f	cd
5	−5	+50
32	+5	+10

÷720

	g	ce	d^2	c^3
6	−6	+90	+60	−180
42	+6	+30	−60	−180
3^2	+3	−45	+60	+90
2^3	−2	−30	+20	.

÷5040

	h	cf	de	c^2d
7	−7	+147	+245	−1470
52	+7	+63	−245	−630
43	+7	−147	+175	+210
32^2	−7	−63	+35	.

÷40320

	i	cg	df	e^2	c^2e	cd^2	c^4
8	−8	+224	+448	+280	−3360	−4480	+5040
62	+8	+112	−448	−280	−1680	+1120	+5040
53	+8	−224	+392	−280	+3360	−3920	−5040
4^2	+4	−112	−224	+420	−1680	+2240	+2520
42^2	−8	−112	+448	−280	+1680	−1120	.
3^22	−8	+56	−392	+280	−840	+560	.
2^4	+2	+56	−112	+70			

÷362880

	j	ch	dg	ef	c^2f	cde	d^3	c^3d
9	−9	+324	+756	+1134	−6804	−22680	−5040	+68040
72	+9	+180	−756	−1134	−3780	+5040	+5040	+37800
63	+9	−324	+756	−1134	+6804	.	−10080	−22680
54	+9	−324	−756	+1386	−756	−2520	+5040	+7560
52^2	−9	−180	+756	−126	.	+7560	−5040	.
432	−18	+144	.	−252	+4536	−10080	+5040	.
3^3	−3	+108	−504	+378	−2268	+3780	−1680	.
32^3	+9	+180	−252	+126	.	.	.	.

Left-hand. Col.	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
k	+1	+45	−120	+210	−126							
ci		+1	−4	+8	−5	−4	+8	−5				
dh			+4	−8	+5	+32	−64	+40				
eg				+16	−15	+48	−60	·	+20			
f^2					+1	·	+4	+16	−12	+48	−32	
c^2g						+1	−3	+2	·	−3	+2	
cdf							+3	−2	·	+18	−12	
ce^2								+1	−1	+2	−3	−3
d^2e									+1	+4	+3	+12
c^3e										+1	−1	−1
c^2d^2											+1	+4
c^5												+1

÷3628800

Right-hand.	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
10	−10	+450	+1320	+2100	+1260	−12600	−50400	−31500	−42000	+189000	+378000	−226800
82	+10	+270	−1320	−2100	−1260	−7560	+10080	+6300	+42000	+113400	+25200	−226800
73	+10	−450	+1320	−2100	−1260	+12600	−2520	+31500	−46200	−189000	+151200	+226800
64	+10	−450	−1200	+2940	−1260	−2520	+50400	−44100	−8400	+189000	−226800	−226800
5^2	+5	−225	−600	−1050	+2520	+6300	−37800	+15700	+21000	−94500	+126000	+113400
62^2	−10	−270	+1200	−420	+1260	·	−10080	+31500	−16800	−75600	−50400	
532	−20	+180	−120	+4200	−3780	−5040	+42840	−37800	+4200	+75600	−50400	
4^22	−10	+90	+1200	−2940	+1260	+12600	−30240	+6300	+8400	−37800	+25200	
43^2	−10	+450	−1320	−420	+1260	−5040	+2520	+6300	−4200			
42^3	+10	+270	−1200	+2100	−1260	−5040	+10080	−6300				
3^22^2	+15	+45	+720	−1890	+1260	+2520	−5040	+3150				
2^5	−2	−90	+240	−420	+252							

Left-hand, no right-hand.

Col.	l	cj	di	eh	fg	c^3h	cdg	cef	d^2f	de^2	c^3f	c^2de	cd^3	c^4d
l	+1	+35	−75	+90	−42									
cj		+2	−9	+14	−7	−30	+70	−21	−56	+35				
di			+1	−2	+1	+10	−26	+7	+24	−15				
eh				+1	−1	+3	−2	−6	+10	−5	+27	−45	+20	
fg					+1	−16	+58	+5	−100	+60	−54	+90	−40	
c^3h						+2	−7	+5	·	·	−27	+45	−20	
cdg							+1	−3	+6	−4	+27	−45	+20	
cef								+1	−3	+2	−9	+14	−6	
d^2f									+1	−1	+4	−6	+2	−1
de^2										+1	−12	+36	−18	+3
c^3f											+2	−5	+3	+1
c^2de												+1	−1	−1
cd^3													+1	+4
c^4d														+1

Left-hand, no right-hand.

Col.	m	ck	dj	ei	fh	g^2	c^2i	cdh	ceg	cf^2	d^2g	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2	c^6
m	+1	+66	−220	+495	−792	+462															
ck		+3	−15	+40	−70	+42	−15	+40	−70	+42											
dj			+15	−40	+70	−42	+150	−400	+700	−420											
ei				+1	−4	+3	+3	−8	−22	+24	+24	−36	+15								
fh					+25	−24		+50	+680	−675	−570	+925	−400								
g^2						+1			−70	+100	+80	−200	+100	−4	+8	−5					
c^2i							+1	−4	+8	−5				−4	+8	−5					
cdh								+4	−8	+5				+32	−64	+40					
ceg									+1	−1	−1	+2	−1	−1	+2	+1	−3	+1			
cf^2										+2	+5	−19	+12	+20	−49	−32	+91	−32			
d^2g											+1	−3	+2	+4	−9	−12	+27	−10	+6	−4	
def												+18	−17		+54	+162	−342	+135	−81	+54	
e^3													+1			−18	+54	−27	+81	−54	
c^3g														+1	−3				+1	−2	−2
c^2df															+3			−2	+30	−28	+2
c^2e^2																+1	−2	+1	−2	+2	+1
cd^2e																	+1	−1	+4	−5	·
d^4																		+1		+8	+16
c^4e																			+1	−1	−1
c^3d^2																				+1	+4
c^6																					+1

832.

NOTE ON AN APPARENT DIFFICULTY IN THE THEORY OF CURVES, WHEN THE COORDINATES OF A POINT ARE GIVEN AS FUNCTIONS OF A VARIABLE PARAMETER.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 12—14.]

SUPPOSE that the homogeneous coordinates x, y, z are given as proportional to the following functions of a parameter λ,

$$x : y : z = u + \alpha\sqrt{(\Omega)},\quad v + \beta\sqrt{(\Omega)},\quad w + \gamma\sqrt{(\Omega)},$$

where u, v, w are linear functions, Ω a cubic function, of the parameter. For the intersections of the curve with the arbitrary line $Ax + By + Cz = 0$, we have

$$Au + Bv + Cw + (A\alpha + B\beta + C\gamma)\sqrt{(\Omega)} = 0,$$

that is,

$$(Au + Bv + Cw)^2 - (A\alpha + B\beta + C\gamma)^2\,\Omega = 0,$$

a cubic equation in λ; and the curve is thus a cubic. For the value $\lambda = \infty$ we have $x : y : z = \alpha : \beta : \gamma$, or the point (α, β, γ) is a point of the curve.

Suppose now that the line $Ax + By + Cz = 0$ is an arbitrary line through the point (α, β, γ); viz. let the coefficients A, B, C satisfy the relation $A\alpha + B\beta + C\gamma = 0$; the equation for the determination of λ becomes

$$(Au + Bv + Cw)^2 = 0,$$

which equation has two equal roots, suppose $\lambda = \lambda_0$; and the meaning of this is not at once obvious.

Observe that more properly there is a root $\lambda = \infty$ which has dropped out, and that the roots are $\lambda = \infty$, $\lambda = \lambda_0$, $\lambda = \lambda_0$. The root $\lambda = \infty$ gives the point (α, β, γ), which is of course one of the intersections of the line with the curve. The two roots λ_0 give *not the same intersection* but two different intersections of the line with the curve; the line being in fact a line through the point (α, β, γ) of the curve, and which besides meets the curve in two distinct points.

To see how this is, observe that, in the general case where $A\alpha + B\beta + C\gamma$ is not $= 0$, we have λ determined by a cubic equation as above; and then taking λ equal to any root of this equation, we have further

$$Au + Bv + Cw + (A\alpha + B\beta + C\gamma)\sqrt{(\Omega)} = 0,$$

viz. the value of $\sqrt{(\Omega)}$ is hereby uniquely determined; and to each of the three values of λ, $\sqrt{(\Omega)}$, there corresponds a determinate point (x, y, z).

But suppose now $A\alpha + B\beta + C\gamma = 0$, and λ determined by the equation

$$(Au + Bv + Cw)^2 = 0,$$

giving $\lambda = \lambda_0$, as above. There is no longer an equation for the unique determination of $\sqrt{(\Omega)}$, and to the value $\lambda = \lambda_0$, there correspond the two values $\sqrt{(\Omega_0)}$, $-\sqrt{(\Omega_0)}$ of the radical: and thus to the two roots $\lambda = \lambda_0$, $\lambda = \lambda_0$ correspond the two different points

$$x : y : z = u_0 + \alpha\sqrt{(\Omega_0)} : v_0 + \beta\sqrt{(\Omega_0)} : w_0 + \gamma\sqrt{(\Omega_0)};$$

and

$$x : y : z = u_0 - \alpha\sqrt{(\Omega_0)} : v_0 - \beta\sqrt{(\Omega_0)} : w_0 - \gamma\sqrt{(\Omega_0)}.$$

It is to be added that the point (α, β, γ) is an inflexion on the curve. Write for a moment

$$u,\ v,\ w = a\lambda + f,\ b\lambda + g,\ c\lambda + h,$$

and let A, B, C be determined by the conditions

$$A\alpha + B\beta + C\gamma = 0,$$
$$Aa + Bb + Cc = 0.$$

Then the equation for the determination of λ becomes $(Af + Bg + Ch)^2 = 0$, viz. the left-hand is a mere constant, or there are the three equal roots $\lambda = \infty$; the intersections with the curve are thus the point (α, β, γ) three times; hence this point is an inflexion, the tangent being $Ax + By + Cz = 0$. The second of the two equations may be written

$$Au_\infty + Bv_\infty + Cw_\infty = 0.$$

Let λ_1 be one of the roots of the equation $\Omega = 0$; u_1, v_1, w_1 the corresponding values of u, v, w, and let A, B, C, be determined by the conditions

$$A\alpha + B\beta + C\gamma = 0,$$
$$Au_1 + Bv_1 + Cw_1 = 0.$$

The equation $(Au + Bv + Cw)^2 = 0$ for the intersections with the curve has the two equal roots $\lambda = \lambda_1$; and to each of these, since now $\sqrt{(\Omega_1)} = 0$, there corresponds the same point $x : y : z = u_1 : v_1 : w_1$; hence the line $Ax + By + Cz = 0$, or say

$$A_1x + B_1y + C_1z = 0,$$

is a tangent from the inflexion. Similarly, if λ_2, λ_3 are the other two roots of the equation $\Omega = 0$, we have $A_2x + B_2y + C_2z = 0$, $A_3x + B_3y + C_3z = 0$ for the other two tangents from the inflexion.

It would have been to some extent clearer to have represented the parameter λ as a quotient, say $\lambda = p/q$; the equations for x, y, z would then have been

$$x : y : z = (ap + fq)\sqrt{(q)} + \alpha\sqrt{(\Omega)} : (bp + gq)\sqrt{(q)} + \beta\sqrt{(\Omega)} : (cp + hq)\sqrt{(q)} + \gamma\sqrt{(\Omega)},$$

where Ω is now a homogeneous function $(p, q)^3$.

833.

ON A FORMULA IN ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 21, 22.]

WRITING s, c, d for the sn, cn, and dn of an argument u, and so in other cases: we have s, c, d for the coordinates of a point on the quadriquadric curve $x^2+y^2=1$, $z^2+k^2x^2=1$. Applying Abel's theorem to this curve, it appears that, if $u_1+u_2+u_3+u_4=0$, the corresponding points are in a plane; that is, the elliptic functions satisfy the relation

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix} = 0.$$

This may be written

$$\begin{aligned} &(s_2-s_1)(c_3d_4-c_4d_3)+(s_4-s_3)(c_1d_2-c_2d_1) \\ &+(c_2-c_1)(d_3s_4-d_4s_3)+(c_4-c_3)(d_1s_2-d_2s_1) \\ &+(d_2-d_1)(s_3c_4-s_4c_3)+(d_4-d_3)(s_1c_2-s_2c_1)=0; \end{aligned}$$

and it may be shown that each of the three lines is, in fact, separately $=0$.

This appears from the following three formulæ:

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{cn}(u_1+u_2)-\operatorname{dn}(u_1+u_2)} = \frac{s_1-s_2}{c_1d_2-c_2d_1},$$

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{cn}(u_1+u_2)+1} = \frac{c_1-c_2}{d_1s_2-d_2s_1},$$

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{dn}(u_1+u_2)+1} = \frac{-\frac{1}{k^2}(d_1-d_2)}{s_1c_2-s_2c_1},$$

which are themselves at once deducible from formulæ given, p. 63, of my *Elliptic Functions*, and which may be written

$$\operatorname{sn}(u_1+u_2) = s_1^2 - s_2^2 = -(c_1^2 - c_2^2) = -\frac{1}{k^2}(d_1^2 - d_2^2), \div (s_1c_2d_2 - s_2c_1d_1),$$

$$\operatorname{cn}(u_1+u_2) = s_1c_1d_2 - s_2c_2d_1, \qquad \div \quad ,,$$

$$\operatorname{dn}(u_1+u_2) = s_1d_1c_2 - s_2d_2c_1, \qquad \div \quad ,,$$

In fact, the numerators of $\operatorname{cn}(u_1+u_2) - \operatorname{dn}(u_1+u_2)$, $\operatorname{cn}(u_1+u_2)+1$, $\operatorname{dn}(u_1+u_2)+1$ thus become $=(s_1+s_2)(c_1d_2-c_2d_1)$, $-(c_1+c_2)(d_1s_2-d_2s_1)$, $(d_1+d_2)(s_1c_2-s_2c_1)$ respectively: so that, taking the numerator of $\operatorname{sn}(u_1+u_2)$ successively under its three forms, we have by division the formulæ in question. And then, if $u_1+u_2=-(u_3+u_4)$, the functions on the left-hand side become, with only a change of sign, the like functions of u_3+u_4; and we thence have the required equations

$$\frac{s_1-s_2}{c_1d_2-c_2d_1} = -\frac{s_3-s_4}{c_3d_4-c_4d_3}, \text{ \&c.}$$

834.

ON THE ADDITION OF THE ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 56—61.]

MR FORSYTH'S Note [*l. c.*, p. 23] on my "Formula in Elliptic Functions" has supplied a missing link, and I am now able to obtain the addition formulæ very simply from the application of Abel's theorem to the Quadriquadric Curve.

I remark that, instead of coplanar points 1, 2, 3, 4, it is advantageous to consider coresidual points 1, 2 and 3, 4; that is, pairs 1, 2 and 3, 4, which are each of them coplanar with one and the same pair of points 5, 6. The difference is as follows: for the coplanar points 1, 2, 3, 4, we have

$$du_1 + du_2 + du_3 + du_4 = 0,$$

giving

$$u_1 + u_2 + u_3 + u_4 = C,$$

and for the addition theory it is necessary to have $C = 0$; for the coresidual points, we have

$$u_1 + u_2 + u_5 + u_6 = C, \quad u_3 + u_4 + u_5 + u_6 = C;$$

and thence $u_1 + u_2 = u_3 + u_4$, irrespectively of the value of C.

As to the general theory of a curve in space, observe that, when this is a complete intersection of two surfaces

$$f(x, y, z, w) = 0, \quad g(x, y, z, w) = 0,$$

then at the point (x, y, z, w), if

$$(x + dx, \; y + dy, \; z + dz, \; w + dw)$$

are the coordinates of the consecutive point, the six coordinates of the tangent line are

$$y\,dz - z\,dy, \quad z\,dx - x\,dz, \quad x\,dy - y\,dx, \quad x\,dw - w\,dx, \quad y\,dw - w\,dy, \quad z\,dw - w\,dz.$$

But considering the line as the intersection of the two tangent planes

$$\frac{df}{dx}X + \frac{df}{dy}Y + \frac{df}{dz}Z + \frac{df}{dw}W = 0,$$

and

$$\frac{dg}{dx}X + \frac{dg}{dy}Y + \frac{dg}{dz}Z + \frac{dg}{dw}W = 0,$$

the six coordinates are

$$\frac{d(f,g)}{d(x,w)},\quad \frac{d(f,g)}{d(y,w)},\quad \frac{d(f,g)}{d(z,w)},\quad \frac{d(f,g)}{d(y,z)},\quad \frac{d(f,g)}{d(z,x)},\quad \frac{d(f,g)}{d(x,y)},$$

so that the six quotients

$$(y\,dz - z\,dy)\Big/\frac{d(f,g)}{d(x,w)},\ \&c.,$$

are equal to each other, and may be put $= d\omega$.

Considering any two quadric surfaces, there is in general a system of four conjugate points, or points such that in regard to each of the quadrics the polar plane of any one of the points is the plane through the other three points. And then taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the faces of the tetrahedron formed by the four points, the equations of the quadric surfaces will be of the form

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}w^2 = 0,$$
$$\mathrm{a}'x^2 + \mathrm{b}'y^2 + \mathrm{c}'z^2 + \mathrm{d}'w^2 = 0;$$

we then have the six quotients

$$(y\,dz - z\,dy)/(\mathrm{ad}' - \mathrm{a}'\mathrm{d})\,xw,\ \&c.,$$

equal to each other, and each $= d\omega$. Here $d\omega$ is homogeneous of the degree zero in the coordinates (x, y, z, w), or, what is the same thing, it is a differential $F\left(\frac{x}{w}\right)d\frac{x}{w}$, say it is $= du$; and taking the integrals always from one and the same fixed point on the curve, we have each point of the curve corresponding to a determinate value of a parameter u.

Supposing that u_1, u_2, u_5, u_6 are the values of u, belonging to any four coplanar points 1, 2, 5, 6; then, by Abel's theorem, $du_1 + du_2 + du_5 + du_6 = 0$; that is, we have

$$u_1 + u_2 + u_5 + u_6 = C,$$

as the condition in order that the four points 1, 2, 5, 6 may be coplanar; similarly, we have

$$u_3 + u_4 + u_5 + u_6 = C,$$

as the condition in order that the four points 3, 4, 5, 6 may be coplanar; and we have therefore

$$u_1 + u_2 = u_3 + u_4,$$

as the condition that the two pairs of points 1, 2 and 3, 4 may be coresidual.

The points 1, 2, 5, 6 are coplanar, hence the line 56 meets the line 12, say in the point A; and the points 3, 4, 5, 6 are coplanar, hence the line 56 meets the line 34, say in the point B. We can, through the curve and any arbitrary point in space, draw a quadric surface

$$(\mathrm{a}+\lambda\mathrm{a}')x^2+(\mathrm{b}+\lambda\mathrm{b}')y^2+(\mathrm{c}+\lambda\mathrm{c}')z^2+(\mathrm{d}+\lambda\mathrm{d}')w^2=0.$$

Hence we have such a quadric surface through the point A; and this surface, passing through 5 and 6, will contain the line 56, and therefore also the point B; hence, passing through 3 and 4, it will contain the line 34; viz. we have the lines 12, 34 as generating lines, obviously of the same kind, on the last-mentioned quadric surface. I say that if, on such a surface, that is, on any surface

$$Ax^2+By^2+Cz^2+Dw^2=0,$$

we have

$$(a,\ b,\ c,\ f,\ g,\ h),\quad (a',\ b',\ c',\ f',\ g',\ h'),$$

the coordinates of two generating lines of the same kind, then

$$\frac{a}{f}=\frac{a'}{f'},\quad \frac{b}{g}=\frac{b'}{g'},\quad \frac{c}{h}=\frac{c'}{h'}.$$

This is at once seen to be the case; for, taking θ an arbitrary parameter, we have for the equations of a generating line

$$\{x\sqrt{(A)}+iy\sqrt{(B)}\}+\theta\{z\sqrt{(C)}+iw\sqrt{(D)}\}=0,$$
$$\theta\{x\sqrt{(A)}-iy\sqrt{(B)}\}-\ \{z\sqrt{(C)}-iw\sqrt{(D)}\}=0,$$

and the coordinates $(a,\ b,\ c,\ f,\ g,\ h)$ of this line are

$$i\sqrt{(AD)}(1-\theta^2),\quad \sqrt{(BD)}(-1-\theta^2),\quad i\sqrt{(CD)}\,2\theta,$$
$$i\sqrt{(BC)}(-1+\theta^2),\quad \sqrt{(CA)}(1+\theta^2),\quad i\sqrt{(AB)}(-2\theta);$$

that is, the quotients $\frac{a}{f}$, $\frac{b}{g}$, $\frac{c}{h}$ are each of them independent of θ; and they have consequently their values unaltered when for the original line we substitute any other generating line of the same kind. Or, to prove the statement in a different manner, the equation of the quadric surface through the line $(a,\ b,\ c,\ f,\ g,\ h)$ is

$$aghx^2+bhfy^2+cfgz^2+abcw^2=0;$$

hence, if this contains the line $(a',\ b',\ c',\ f',\ g',\ h')$, we must have

$$agh : bhf : cfg : abc = a'g'h' : b'h'f' : c'f'g' : a'b'c',$$

equations which give either

$$af'+a'f=0,\quad bg'+b'g=0,\quad ch'+c'h=0,$$

or else

$$af'-a'f=0,\quad bg'-b'g=0,\quad ch'-c'h=0.$$

In the former case, the two lines are generating lines of different kinds; in the latter, they are generating lines of the same kind.

Now, considering (a, b, c, f, g, h) as the coordinates of the line 12 and (a', b', c', f', g', h') as those of the line 34, the equations just obtained are

$$\frac{y_1z_2 - y_2z_1}{x_1w_2 - x_2w_1} = \frac{y_3z_4 - y_4z_3}{x_3w_4 - x_4w_3}, \quad \frac{z_1x_2 - z_2x_1}{y_1w_2 - y_2w_1} = \frac{z_3x_4 - z_4x_3}{y_3w_4 - y_4w_3}, \quad \frac{x_1y_2 - x_2y_1}{z_1w_2 - z_2w_1} = \frac{x_3y_4 - x_4y_3}{z_3w_4 - z_4w_3}.$$

Of course the equations hold good if, instead of the two lines, we have one and the same line; the equations

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}w^2 = 0, \quad \mathrm{a}'x^2 + \mathrm{b}'y^2 + \mathrm{c}'z^2 + \mathrm{d}'w^2 = 0,$$

considering therein x^2, y^2, z^2, w^2 as coordinates, may be regarded as the equations of a line; and thus the points $(x_1^2, y_1^2, z_1^2, w_1^2)$, &c., will be four points on a line. And we have thus

$$\frac{y_1^2z_2^2 - y_2^2z_1^2}{x_1^2w_2^2 - x_2^2w_1^2} = \frac{y_3^2z_4^2 - y_4^2z_3^2}{x_3^2w_4^2 - x_4^2w_3^2}, \text{ \&c.},$$

equations which are, by means of the foregoing set, converted into

$$\frac{y_1z_2 + y_2z_1}{x_1w_2 + x_2w_1} = \frac{y_3z_4 + y_4z_3}{x_3w_4 + x_4w_3}, \quad \frac{z_1x_2 + z_2x_1}{y_1w_2 + y_2w_1} = \frac{z_3x_4 + z_4x_3}{y_3w_4 + y_4w_3}, \quad \frac{x_1y_2 + x_2y_1}{z_1w_2 + z_2w_1} = \frac{x_3y_4 + x_4y_3}{z_3w_4 + z_4w_3}.$$

If for x, y, z, w we write s, c, d, 1, then the equations are

$$\frac{c_1d_2 - c_2d_1}{s_1 - s_2} = \frac{c_3d_4 - c_4d_3}{s_3 - s_4}, \quad \frac{d_1s_2 - d_2s_1}{c_1 - c_2} = \frac{d_3s_4 - d_4s_3}{c_3 - c_4}, \quad \frac{s_1c_2 - s_2c_1}{d_1 - d_2} = \frac{s_3c_4 - s_4c_3}{d_3 - d_4},$$

$$\frac{c_1d_2 + c_2d_1}{s_1 + s_2} = \frac{c_3d_4 + c_4d_3}{s_3 + s_4}, \quad \frac{d_1s_2 + d_2s_1}{c_1 + c_2} = \frac{d_3s_4 + d_4s_3}{c_3 + c_4}, \quad \frac{s_1c_2 + s_2c_1}{d_1 + d_2} = \frac{s_3c_4 + s_4c_3}{d_3 + d_4},$$

where s_1, c_1, d_1 are the sn, cn and dn of u_1, &c.; and where the relation between the arguments is $u_1 + u_2 = u_3 + u_4$.

In particular, if $u_4 = 0$, we have s_4, c_4, $d_4 = 0$, 1, 1; and then writing u for u_3, and consequently s, c, d for s_3, c_3, d_3, the relation between the arguments is $u = u_1 + u_2$; and we have

$$\frac{c_1d_2 - c_2d_1}{s_1 - s_2} = \frac{c - d}{s}, \quad \frac{d_1s_2 - d_2s_1}{c_1 - c_2} = \frac{s}{1 - c}, \quad \frac{s_1c_2 - s_2c_1}{d_1 - d_2} = \frac{s}{d - 1},$$

$$\frac{c_1d_2 + c_2d_1}{s_1 + s_2} = \frac{c + d}{s}, \quad \frac{d_1s_2 + d_2s_1}{c_1 + c_2} = \frac{s}{1 + c}, \quad \frac{s_1c_2 + s_2c_1}{d_1 + d_2} = \frac{s}{d + 1}.$$

The last two pairs give

$$\frac{1 + c}{1 - c} = \frac{(d_1s_2 - d_2s_1)(c_1 + c_2)}{(d_1s_2 + d_2s_1)(c_1 - c_2)}, \quad \frac{d + 1}{d - 1} = \frac{(s_1c_2 - s_2c_1)(d_1 + d_2)}{(s_1c_2 + s_2c_1)(d_1 - d_2)},$$

that is,

$$c = \frac{s_1c_1d_2 - s_2c_2d_1}{s_1c_2d_2 - s_2c_1d_1}, \qquad d = \frac{s_1d_1c_2 - s_2d_2c_1}{s_1c_2d_2 - s_2c_1d_1}.$$

These last values give

$$c-d=\frac{(s_1+s_2)(c_1d_2-c_2d_1)}{s_1c_2d_2-s_2c_1d_1},\quad c+d=\frac{(s_1-s_2)(c_1d_2+c_2d_1)}{s_1c_2d_2-s_2c_1d_1};$$

and then, from the given value of either $\frac{c-d}{s}$ or $\frac{c+d}{s}$, we obtain

$$s=\frac{s_1^2-s_2^2}{s_1c_2d_2-s_2c_1d_1};$$

viz. the resulting equations thus are

$$s=\frac{s_1^2-s_2^2}{s_1c_2d_2-s_2c_1d_1},\quad c=\frac{s_1c_1d_2-s_2c_2d_1}{s_1c_2d_2-s_2c_1d_1},\quad d=\frac{s_1d_1c_2-s_2d_2c_1}{s_1c_2d_2-s_2c_1d_1},$$

which are one of the four sets given (p. 63) of my *Elliptic Functions*; it may be noticed that they have the advantage of not containing k explicitly, and the disadvantage of becoming vanishing fractions for $u_1=u_2$. To obtain the ordinary forms, we have only to multiply the numerators and denominators each by $s_1c_2d_2+s_2c_1d_1$; the denominator thus becomes $=(s_1^2-s_2^2)(1-k^2s_1^2s_2^2)$, and each of the numerators has, or acquires, the factor $s_1^2-s_2^2$, so that this factor divides out.

835.

ON CARDAN'S SOLUTION OF A CUBIC EQUATION.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 96, 97.]

It is interesting to see how the solution comes out when one root of the equation is known. Say the equation is $x^3+qx-r=0$, where $a^3-qa-r=0$, that is, $r=a^3+qa$.

Solving in the usual manner, we have

$$x=y+z,\quad y^3+z^3-r+(y+z)(3yz+q)=0,$$

whence

$$y^3+z^3=r,$$
$$yz=-\tfrac{1}{3}q;$$

and thence

$$(y^3-z^3)^2=r^2+\tfrac{4}{27}q^3,\quad =a^6+2qa^4+q^2a^2-\tfrac{4}{27}q^3,\quad =(a^2+\tfrac{4}{3}q)(a^2+\tfrac{1}{3}q)^2;$$

or say

$$y^3-z^3=(a^2+\tfrac{1}{3}q)\sqrt{(a^2+\tfrac{4}{3}q)};$$

and therefore

$$8y^3=4a^3+4qa+(4a^2+\tfrac{4}{3}q)\sqrt{(a^2+\tfrac{4}{3}q)},\quad =\{a+\sqrt{(a^2+\tfrac{4}{3}q)}\}^3,$$
$$8z^3=4a^3+4qa-(4a^2+\tfrac{4}{3}q)\sqrt{(a^2+\tfrac{4}{3}q)},\quad =\{a-\sqrt{(a^2+\tfrac{4}{3}q)}\}^3;$$

where observe that the essential step is the expression of the irrational functions as perfect cubes: that the functions are the cubes of $\alpha\pm\sqrt{(a^2+\frac{4}{3}q)}$ respectively is seen to be true; but if we were to attempt to find a cube root $\alpha+\beta\sqrt{(a^2+\frac{4}{3}q)}$ by an algebraical process, we should be thrown back upon the original cubic equation.

Writing then ω for an imaginary cube root of unity, we have

$$2y=(1,\ \omega\ \text{ or }\ \omega^2)\{a+\sqrt{(a^2+\tfrac{4}{3}q)}\},$$
$$2z=(1,\ \omega^2\ \text{ or }\ \omega\ )\{a-\sqrt{(a^2+\tfrac{4}{3}q)}\};$$

and then

$$x=y+z=a,\ \text{or}\ =-\tfrac{1}{2}a\pm\tfrac{1}{2}(\omega-\omega^2)\sqrt{(a^2+\tfrac{4}{3}q)},$$

where $\omega-\omega^2=i\sqrt{3}$; the last two roots are of course the roots of the quadric equation $x^2+ax+a^2+q=0$, which is obtained by throwing out the factor $x-a$ from the given equation $x^3+qx-r=0$.

Cambridge, Sep. 17, 1884.

836.

ON THE QUATERNION EQUATION $qQ - Qq' = 0$.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 108—112.]

I CONSIDER the equation $qQ - Qq' = 0$, where q, q' are given quaternions, and Q is a quaternion to be determined. Obviously a condition must be satisfied by the given quaternions; for, substituting in the given equation for q, q', Q their values, say $w + ix + jy + kz$, $w' + ix' + jy' + kz'$, and $W + iX + jY + kZ$ respectively, and equating to zero the scalar part and the coefficients of i, j, k, we have four equations linear in W, X, Y, Z, and then eliminating these quantities, we have the condition in question. Supposing the condition satisfied, the ratios of W, X, Y, Z are then completely determined, and the required quaternion Q is thus determinate except as to a scalar factor, or say Q is = product of an arbitrary scalar into a determinate quaternion expression.

It might, at first sight, appear that the condition is that the given quaternions shall have their tensors equal, $Tq = Tq'$; for the equation gives $Tq \,.\, TQ - TQ \,.\, Tq' = 0$, that is, $TQ(Tq - Tq') = 0$. But we cannot thence infer, and it is not true, that the condition is $Tq - Tq' = 0$; the formula does not give the required condition at all, but the conclusion to which it leads is that, when the condition is satisfied, then in general (that is, unless $Tq - Tq' = 0$) the required quaternion is an imaginary quaternion (or, as Hamilton calls it, a biquaternion) having its tensor $TQ = 0$. In the particular case where the given quaternions are such that $Tq - Tq' = 0$, then the required quaternion Q is determined less definitely, viz. it becomes = product of an arbitrary scalar into a not completely determined quaternion expression; and it is thus in general such that TQ is not $= 0$. In explanation, observe that, for the particular case in question, the four linear equations for W, X, Y, Z reduce themselves to two independent relations, and they give therefore for the ratios of W, X, Y, Z expressions involving an arbitrary parameter A; these expressions cannot, it is clear, be deduced from the determinate expressions which belong to the general case.

Instead of directly working out the condition in the manner indicated above, I present the investigation in a synthetic form as follows:

Taking v, v' for the vector parts of the two given quaternions, so that $q = w + v$, $q' = w' + v'$, I write for shortness

$$\begin{aligned}
\theta &= w - w', \\
\alpha &= v^2 + v'^2, &&= -x^2 - y^2 - z^2 - x'^2 - y'^2 - z'^2, \\
\beta &= v^2 - v'^2, &&= -x^2 - y^2 - z^2 + x'^2 + y'^2 + z'^2, \\
D &= -\theta(\alpha - \theta^2), \\
A &= \beta - \theta^2, \\
B &= \beta + \theta^2;
\end{aligned}$$

so that θ, α, β, D, A, B are all of them scalars. With these I form a quaternion $Q = (D + Av)(D + Bv')$; I say that we have identically

$$qQ - Qq' = \{D - v \,.\, v'^2 + v' \,.\, v^2 + vv' \,.\, \theta\}(\theta^4 - 2\alpha\theta^2 + \beta^2).$$

It of course follows that, if $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, then $qQ - Qq' = 0$, viz. $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$ is the condition $\Omega = 0$, for the existence of the required quaternion; and this condition being satisfied, then (omitting the arbitrary scalar factor) the value of the quaternion is $Q = (D + Av)(D + Bv')$, a value giving $T(Q) = 0$, that is, $W^2 + X^2 + Y^2 + Z^2 = 0$. If $\theta = 0$ (that is, $w = w'$), then the condition becomes $\beta = 0$, that is, $x^2 + y^2 + z^2 - x'^2 - y'^2 - z'^2 = 0$; and these two conditions being satisfied, Q ceases to have the determinate value given by the foregoing formula: it has a value involving an arbitrary parameter, and is no longer such that $W^2 + X^2 + Y^2 + Z^2 = 0$.

The identical equation is at once verified; we have

$$\begin{aligned}
qQ - Qq' &= (w + v)Q - Q(w' - v') \\
&= \theta\ (D^2 + DAv + DBv' + ABvv') \\
&\quad + v\ (D^2 + DAv + DBv' + ABvv') \\
&\quad - \ (D^2 + DAv + DBv' + ABvv')\, v' \\
&= \theta D^2 + DAv^2 - DBv'^2 \\
&\quad + v\ (DA\theta + D^2 - ABv'^2) \\
&\quad + v'\ (DB\theta + ABv^2 - D^2) \\
&\quad + vv'\,(\theta AB + DB - DA).
\end{aligned}$$

The first line is here $= D\{D\theta + Av^2 + Bv'^2\}$, viz. the term in $\{\ \}$ is

$$\begin{aligned}
&-(\alpha - \theta^2)\,\theta^2 + (\beta - \theta^2)\,v^2 - (\beta + \theta^2)\,v'^2, \\
&= -\alpha\theta^2 + \theta^4 + \beta \,.\, \beta - \theta^2 \,.\, \alpha, \\
&= \theta^4 - 2\alpha\theta^2 + \beta^2;
\end{aligned}$$

and similarly each of the other lines contains the factor $\theta^4 - 2\alpha\theta^2 + \beta^2$, and the equation is thus seen to hold good.

The tensor of Q is $= (D^2 - A^2v^2)(D^2 - B^2v'^2)$; and we have

$$D^2 - A^2v^2 = (\alpha - \theta^2)^2\theta^2 - v^2(\beta - \theta^2),$$
$$= \theta^6 - (2\alpha + v^2)\theta^4 + (\alpha^2 + 2\beta v^2)\theta^2 - v^2\beta^2,$$

which, observing that

$$\alpha^2 + 2\beta v^2 = \beta^2 + 2\alpha v^2 \quad \text{is} \quad = (\theta^2 - v^2)(\theta^4 - 2\alpha\theta^2 + \beta^2);$$

and similarly

$$D^2 - B^2v'^2 \qquad \text{is} \quad = (\theta^2 - v'^2)(\theta^4 - 2\alpha\theta^2 + \beta^2);$$

hence the tensor is

$$TQ = (\theta^2 - v^2)(\theta^2 - v'^2)(\theta^4 - 2\alpha\theta^2 + \beta^2)^2,$$

which, in virtue of $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, is $= 0$.

The particular case is when $Tq - Tq' = 0$, that is, $w^2 - v^2 - w'^2 + v'^2 = 0$, or say $w^2 - w'^2 = v^2 - v'^2$, that is, $\theta(w + w') = \beta$. Combining with this the general condition $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, we find $\theta^2\{\theta^2 - 2\alpha + (w + w')^2\} = 0$, or in the second factor, for θ and α substituting their values, we have $2\theta^2(w^2 - v^2 + w'^2 - v'^2) = 0$, that is, $2\theta^2(Tq + Tq') = 0$. Attending to the assumed relation $Tq - Tq' = 0$, the second factor can only vanish if $Tq = 0$, $Tq' = 0$; hence, disregarding this more special case, the factor which vanishes must be the first factor, that is, $\theta = 0$; or the equations $Tq - Tq' = 0$ and $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$ give $\beta = 0$ and $\theta = 0$, that is, we have as already mentioned $w - w' = 0$, and $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$, viz. the given quaternions have their scalars equal, and the squares of their vectors also equal. The equation here is $vQ - Qv' = 0$; and writing $v^2 = v'^2 = -p^2$, we see at once that a solution is

$$Q = (-p^2 + vv') + A(v + v'),$$

where A is an arbitrary scalar; in fact, with this value of Q, we have at once vQ and Qv' each

$$= -p^2(v + v') + A(-p^2 + vv');$$

and the equation $vQ - Qv' = 0$ is thus satisfied. The value of the tensor is easily found to be

$$TQ = 2(A^2 + p^2)(p^2 + xx' + yy' + zz'),$$

which is not $= 0$.

In accordance with a remark in the introductory paragraphs, the solution

$$Q = -p^2 + vv' + A(v + v')$$

is not comprised in the general solution. As to this, observe that, in the case in question $\theta = 0$, $\beta = 0$, we have from the general theorem the form

$$Q = \left(-\frac{\alpha\theta}{\beta} + v\right)\left(-\frac{\alpha\theta}{\beta} + v'\right);$$

that is,

$$Q = \frac{\alpha^2\theta^2}{\beta^2} + vv' - \frac{\alpha\theta}{\beta}(v + v');$$

in the condition $\theta^4 - 2\alpha\theta^2 + \beta^2 = 0$, writing $\theta = 0$, we have $\frac{\beta^2}{\theta^2} = 2\alpha$, or for α writing its value $= -2p^2$, we have $\alpha^2 = 4p^4$, $\frac{\theta^2}{\beta^2} = -\frac{1}{4p^2}$, and thence $\frac{\alpha^2\theta^2}{\beta^2} = -p^2$, and $\frac{\alpha\theta}{\beta} = \lambda p$, if λ denote the $\sqrt{(-1)}$ of ordinary algebra. The resulting formula is thus

$$Q = -p^2 + vv' - \lambda p\,(v + v'),$$

which corresponds to the determinate value $-\lambda p$ of the constant A.

The foregoing solution $Q = -p^2 + vv' + A\,(v + v')$ may be easily identified with that given pp. 124, 125 of Tait's *Elementary Treatise on Quaternions*; the case there considered is that of a real quaternion, and it was therefore assumed that the two conditions $w = w'$, and $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$, were each of them satisfied.

The theory of quaternions is, as is well known, identical with that of matrices of the second order; the identity is, in effect, established by the remark and footnote "Linear Associative Algebra," *American Journal of Mathematics*, t. IV. (1881), p. 132. Writing x, y, z, w for Peirce's imaginaries i, j, k, l, these have the multiplication table

	x	y	z	w
x	x	y	0	0
y	0	0	x	y
z	z	w	0	0
w	0	0	z	w

Then if λ, $= \sqrt{(-1)}$, be the imaginary of ordinary algebra, and i, j, k the quaternion imaginaries, the relations between i, j, k and x, y, z, w are

$$\begin{aligned}
x &= \tfrac{1}{2}(\ \ 1 - \lambda i), && \text{or conversely} && 1 = \ \ x + w\,,\\
y &= \tfrac{1}{2}(\ \ j - \lambda k), && \quad ,, && i = \lambda\,(x - w),\\
z &= \tfrac{1}{2}(-\,j - \lambda k), && \quad ,, && j = \ \ (y - z),\\
w &= \tfrac{1}{2}(\ \ 1 + \lambda i), && \quad ,, && k = \lambda\,(y + z);
\end{aligned}$$

and we can thus at once express a quaternion as a linear function of the x, y, z, w, or a linear function of the x, y, z, w as a quaternion. And we then consider

$ax + by + cz + dw$ as denoting the matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, we obtain for the product of two matrices the ordinary formula

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \cdot \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = \begin{array}{c} \underline{(a',\ c'),\ \ (b',\ d')} \\ (a,\ b) \quad \text{,,} \quad\ \ \text{,,} \\ (c,\ d) \quad \text{,,} \quad\ \ \text{,,} \end{array};$$

viz. we have

$$\begin{aligned} (ax + by + cz + dw)(a'x + b'y + c'z + d'w) &= aa'x^2 + bc'yz + \&c., \\ &= (aa' + bc')\,x + (ab' + bd')\,y + (ca' + dc')\,z + (cb' + dd')\,w, \end{aligned}$$

in accordance with the formula for the product of the two matrices. Observe that, writing

$$\begin{aligned} A + Bi + Cj + Dk &= A\,(x + w) + B\lambda\,(x - w) + C\,(y - z) + D\lambda\,(y + z) \\ &= ax + by + cz + dw, \end{aligned}$$

we have

$$a,\ d,\ b,\ c = A + B\lambda,\quad A - B\lambda,\quad C + D\lambda,\quad -C + D\lambda,$$

and thence

$$ad - bc = A^2 + B^2 + C^2 + D^2,$$

so that the determinant of the matrix corresponds to the tensor of the quaternion.

837.

ON THE SO-CALLED D'ALEMBERT CARNOT GEOMETRICAL PARADOX.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 113, 114.]

THE present note has reference to Prof. Sylvester's paper on this subject [*l.c.*, pp. 92—96]. I cannot admit that D'Alembert and Carnot raised a well-founded objection "to the then and even now too prevalent interpretation of the meaning of the geometrical positive and negative": it appears to me that the objection was not a well-founded one.

Consider through the origin K an indefinite line $t'Kt$, and measure off from K in the sense Kt a distance equal to the positive quantity α, and let m be the extremity of the distance thus measured off. There is not in the ordinary theory any reason why the distance Km should be $=+\alpha$ rather than $=-\alpha$; it is $=+\alpha$, if Kt be the positive sense of the line through K, and it is $=-\alpha$ if Kt' be the positive sense of the line through K; if it be undetermined which of the two is the positive sense, then the distance Km is $=\pm\alpha$, the sign being essentially indeterminate.

The problem is from a point K outside a given circle to draw a line Kmm' such that the intercepted portion mm' within the circle has a given value c.

Supposing that the line from K to the centre meets the circle in the points A, B at the distances $KA=a$, $KB=b$; then if $Km=r$, we have $ab=r(c+r)$, or $r=-\frac{1}{2}c\pm\sqrt{(\frac{1}{4}c^2+ab)}$; viz. we have for r, not simultaneously but alternatively, the positive value $-\frac{1}{2}c+\sqrt{(\frac{1}{4}c^2+ab)}$, and the negative value $-\frac{1}{2}c-\sqrt{(\frac{1}{4}c^2+ab)}$, the latter of these being the greatest in absolute magnitude; say the values are $+\rho_1$ and $-\rho_2$. We may with either of these values construct the point m; viz. we obtain m as one of the intersections of the given circle with the circle centre K and radius ρ_1, or else with the circle centre K and radius $-\rho_2$ (that is, radius ρ_2); and attending to the intersections on the same side of the line from K to the centre, it happens that

the two points m thus determined are on one and the same line $t'Kt$; but there is no *à priori* reason why the positive senses should be the same, and they are in fact opposite to each other, in the two cases respectively; in the one case we measure off the distance ρ_1 in the sense Kt, in the other case the distance $-\rho_2$ in the sense Kt'; that is, we in fact measure off the positive distances $+\rho_1$, and $+\rho_2$, in one and the same sense Kt; thus obtaining for the point m one or the other extremity of a determinate secant through K.

The best illustration is I think in the elementary problem of finding the perpendicular distance of a given line from the origin. Let $Ax+By+C=0$ be the equation of the given line: and first let a line be drawn *in a determinate sense*, say at the inclination θ to the positive part of the axis of x, to meet the given line. Taking r for the distance from the origin of the point of intersection, we have, for the coordinates of the point of intersection, x, $y=r\cos\theta$, $r\sin\theta$; and thence

$$r(A\cos\theta+B\sin\theta)+C=0,$$

that is,

$$r=\frac{-C}{A\cos\theta+B\sin\theta},$$

a perfectly determinate value. But the perpendicular on the given line may be considered as drawn in one or the other of two opposite senses; that is, we have at pleasure

$$\cos\theta,\ \sin\theta=\frac{A}{\sqrt{(A^2+B^2)}},\ \frac{B}{\sqrt{(A^2+B^2)}},$$

or else

$$=\frac{-A}{\sqrt{(A^2+B^2)}},\ \frac{-B}{\sqrt{(A^2+B^2)}};$$

and thence $r=\dfrac{-C}{\sqrt{(A^2+B^2)}}$, or else $r=\dfrac{+C}{\sqrt{(A^2+B^2)}}$; that is, the perpendicular distance is $=\dfrac{\pm C}{\sqrt{(A^2+B^2)}}$, with the essentially indeterminate sign $\pm$, because the distance may be considered as drawn from the origin in one or the other of the two opposite senses.

838.

ON THE TWISTED CUBICS UPON A QUADRIC SURFACE.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 129—132.]

A CUBIC (twisted cubic) on a quadric surface meets each generator of the one kind twice, and each generator of the other kind once (Salmon, *Solid Geometry*, Ed. 4, p. 301). There are thus on the surface two kinds of cubics, viz. distinguishing for convenience the generating lines as generators and directors, a cubic may meet each generator twice and each director once; or it may meet each generator once and each director twice. And two cubic curves are accordingly of the same kind or of different kinds.

Consider for example the quadric surface $xw - yz = 0$, and the cubic

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

If we call the line $(x = ky,\ kw = z)$ a generator, and the line $(x = kz,\ kw = y)$ a director, for the intersections with a generator we have $1 = k\phi,\ k\phi^2 = \phi^2$; equations with the common root $k\phi = 1$, or there is a single intersection; for the intersections with a director, $1 = k\phi^2,\ k\phi^3 = \phi$, equations with the common roots $k\phi^2 = 1$, or there are two intersections.

Consider on the same quadric surface the cubic

$$x : y : z : w = \theta - \alpha . \theta - \beta . \theta - \gamma : \theta - \alpha . \theta - \epsilon . \theta - \zeta : \theta - \delta . \theta - \beta . \theta - \gamma : \theta - \delta . \theta - \epsilon . \theta - \zeta;$$

or as, for shortness, I write these,

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

viz. α is written to denote $\theta - \alpha$, and so in other cases; for the intersections with a generator, we have $\alpha\beta\gamma = k\alpha\epsilon\zeta,\ k\delta\epsilon\zeta = \delta\beta\gamma$, equations having the two common roots

$\beta\gamma = k\epsilon\zeta$; or there are two intersections with the generator. And in like manner there is one intersection with the director. The cubic

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta$$

is thus a cubic of different kind from

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

And in the same manner it appears that the cubic

$$x : y : z : w = \alpha\beta\gamma : \delta\beta\gamma : \alpha\epsilon\zeta : \delta\epsilon\zeta$$

is of the same kind with

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3.$$

The two cubics of different kinds intersect in 5 points; the two cubics of the same kind in only 4 points. In fact, for the intersections with the cubic

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3,$$

we have $xz - y^2 = 0$, $yw - z^2 = 0$, and substituting herein the θ-values,

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

we find

$$\alpha\delta\beta^2\gamma^2 - \alpha^2\epsilon^2\zeta^2 = 0, \quad \alpha\delta\epsilon^2\zeta^2 - \delta^2\beta^2\gamma^2 = 0,$$

equations with the common five roots $\delta\beta^2\gamma^2 - \alpha\epsilon^2\zeta^2 = 0$; whereas for the θ-values $x : y : z : w = \alpha\beta\gamma : \delta\beta\gamma : \alpha\epsilon\zeta : \delta\epsilon\zeta$, we have $\alpha^2\beta\gamma\epsilon\zeta - \delta^2\beta^2\gamma^2 = 0$, $\delta^2\beta\gamma\epsilon\zeta - \alpha^2\epsilon^2\zeta^2 = 0$, equations with the common four roots $\alpha^2\epsilon\zeta - \delta^2\beta\gamma = 0$.

I remark in passing that the equations just obtained have each of them a root $\theta = \infty$, but this would have been avoided if the θ-equations had been taken in the slightly altered form

$$x : y : z : w = \alpha\beta\gamma : b\alpha\epsilon\zeta : c\delta\beta\gamma : bc\delta\epsilon\zeta,$$

and

$$x : y : z : w = \alpha\beta\gamma : c\delta\beta\gamma : b\alpha\epsilon\zeta : bc\delta\epsilon\zeta,$$

b, c being arbitrary constants; and the special value is thus of no importance.

The two cubics intersecting in 5 points constitute the complete intersection of a quadric surface and a cubic surface; and conversely when a quadric surface and a cubic surface intersect in two cubics, then the cubics must have 5 intersections, and are thus by what precedes cubics of different kinds in regard to the quadric surface. In fact, considering two cubics meeting in 5 points, we can, through these 5 points, through 2 points assumed at pleasure on the first cubic, and through 2 points assumed at pleasure on the second cubic, draw a determinate quadric surface: this meets each of the two cubics in $5 + 2$ points, and consequently it entirely contains the cubic: viz. we have through the two cubics a determinate quadric surface. Again, through the 5 points, through 5 points at pleasure on the first cubic and through 5 points at pleasure on the second cubic, we may draw a cubic surface; this meets each cubic

in $5+5$ points, and therefore it entirely contains the cubic; we have thus a cubic surface through the two cubics. The cubic surface has been subjected only to $5+5+5=15$ conditions, and the equation thus contains homogeneously $20-15, =5$ constants; viz. if $\Theta=0$ be the quadric surface through the two cubics the general form is $(\alpha x+\beta y+\gamma z+\delta w)\,\Theta+\lambda U=0$; disregarding the term in Θ, we have thus, in fact, a determinate cubic $U=0$.

Taking as above the first cubic as given by the equations

$$x : y : z : w = 1 : \phi : \phi^2 : \phi^3,$$

and the second cubic as given by the equations

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

the equation of the cubic surface which contains the two cubics will be of the form

$$(Ax+By+Cz+Dw)\,(yw-z^2)+(A''x+B''y+C''z+D''w)\,(xz-y^2)=0.$$

Substituting herein the θ-values and omitting the common factor $\alpha\epsilon^2\zeta^2-\delta\beta^2\gamma^2$, we have

$$\delta\,(A\alpha\beta\gamma+B\alpha\epsilon\zeta+C\delta\beta\gamma+D\delta\epsilon\zeta)-\alpha\,(A''\alpha\beta\gamma+B''\alpha\epsilon\zeta+C''\delta\beta\gamma+D''\delta\epsilon\zeta)=0,$$

which is of the fourth order in θ. We may assume $C''=0$, and $D''=0$; in fact, the terms in C'' and D'' might be got rid of by the substitutions

$$z\,(xz-y^2)=y\,(xw-yz)-x\,(yw-z^2),$$

$$w\,(xz-y^2)=z\,(xw-yz)-y\,(yw-z^2);$$

we then have 6 coefficients A, B, C, D, A'', B'', and equating to zero the terms in θ^0, θ^1, θ^2, θ^3, θ^4, the ratios of these are determined by the five equations: and we have thus the required cubic surface.

Starting from either cubic considered as a curve on the given quadric surface; if through the centre we describe a cubic surface, this meets the quadric surface in a second cubic and the two cubics intersect each other in 5 points. In particular, if the one cubic is $x : y : z : w=1 : \phi : \phi^2 : \phi^3$, it appears by what precedes that the other cubic may be taken to be

$$x : y : z : w = \alpha\beta\gamma : \alpha\epsilon\zeta : \delta\beta\gamma : \delta\epsilon\zeta,$$

intersecting the former in the 5 points given by the equation $\delta\beta^2\gamma^2-\alpha\epsilon^2\zeta^2=0$. The five points are of course points of contact of the quadric surface and the cubic surface.

A very simple instance is the following. The quadric surface $xw-yz=0$ and the cubic surface

$$y\,(xz-y^2)+z\,(yw-z^2)=0$$

intersect in the two cubics

$$x : y : z : w=1 : \phi : \phi^2 : \phi^3 \quad\text{and}\quad x : y : z : w=1 : \theta^2 : \theta : \theta^3.$$

The two cubics meet in the 5 points

$$\theta = (0,\ \infty,\ 1,\ \omega,\ \omega^2);\quad \phi = (0,\ \infty,\ 1,\ \omega^2,\ \omega),$$

or, what is the same thing, in the 5 points

$$(x,\ y,\ z,\ w) = (1,\ 0,\ 0,\ 0),\quad (0,\ 0,\ 0,\ 1),\quad (1,\ 1,\ 1,\ 1),\quad (1,\ \omega,\ \omega^2,\ 1),\quad (1,\ \omega^2,\ \omega,\ 1),$$

where ω is an imaginary cube root of unity.

I was anxious to work out this result not for its own sake, but as an illustration of a point which first arises as to octics: a unicursal octic curve is not in general situate on any surface lower than a quartic surface; if on the octic we take at pleasure 33 points, we may through these draw two quartic surfaces $U=0$, $V=0$, each of which will entirely contain the curve: the two surfaces meet in a second octic also unicursal, and the two curves intersect in 34 points, which are points of contact of the two quartic surfaces. We cannot, by adjoining to the given octic any curve of an inferior to its own, obtain a complete intersection of two surfaces: the most simple case is when we adjoin to the given octic as above another unicursal octic, and thus obtain a complete intersection of two quartic surfaces. It is to be observed that the second curve is determined completely and uniquely by the given curve: considering the former curve as given by the expressions of the coordinates in terms of a parameter ϕ, the determination of the like expressions for the latter curve would appear to be a problem of very great difficulty.

839.

ON THE MATRICAL EQUATION $qQ-Qq'=0$.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 176—178.]

I CONSIDER the matrical equation $qQ-Qq'=0$, where q, q' are given matrices $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, $\begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}$ and $Q, = \begin{vmatrix} A, & B \\ C, & D \end{vmatrix}$ is a matrix which has to be determined. As remarked in the paper "On the Quaternion Equation $qQ-Qq'=0$," *Messenger*, t. XIV. (1885), pp. 108—112, [836] the question for matrices is equivalent to that for quaternions: for a matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ may be regarded as a quaternion

$$\tfrac{1}{2}a(1-\lambda i)+\tfrac{1}{2}b(j+\lambda k)+\tfrac{1}{2}c(-j-\lambda k)+\tfrac{1}{2}d(1+\lambda i),$$

or (omitting the factor $\frac{1}{2}$) as the quaternion

$$(a+d)-\lambda(a-d)i+(b-c)j-\lambda(b+c)k,$$

where λ, $=\sqrt{(-1)}$, is the imaginary of ordinary algebra. Hence considering q, q' as denoting the quaternions

$$(a+d)-\lambda(a-d)i+(b-c)j-\lambda(b+c)k,$$
$$(a'+d')-\lambda(a'-d')i+(b'-c')j-\lambda(b'+c')k,$$

we can, if a certain condition is satisfied, find a quaternion Q such that $qQ-Qq'=0$; say this is $Q=W+iX+jY+kZ$; putting this

$$=\tfrac{1}{2}\{(A+D)-\lambda(A-D)i+(B-C)j-\lambda(B+C)k\},$$

we find

$$Q=\begin{vmatrix} A, & B \\ C, & D \end{vmatrix}, \quad =\begin{vmatrix} W+\lambda X, & Y+\lambda Z \\ -Y+\lambda Z, & W-\lambda X \end{vmatrix},$$

for the required matrix Q; this being an indeterminate matrix, such that $AD-BC=0$.

But it is better to solve directly the matrical equation

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \begin{vmatrix} A, & B \\ C, & D \end{vmatrix} - \begin{vmatrix} A, & B \\ C, & D \end{vmatrix} \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = 0,$$

viz.

$$\begin{array}{c} \\ (a,\ b) \\ (c,\ d) \end{array} \begin{array}{|cc|} (A,\ C), & (B,\ D) \\ \text{"} & \text{"} \\ \text{"} & \text{"} \end{array} - \begin{array}{c} \\ (A,\ B) \\ (C,\ D) \end{array} \begin{array}{|cc|} (a',\ c'), & (b',\ d') \\ \text{"} & \text{"} \\ \text{"} & \text{"} \end{array} = 0,$$

that is,

$$\begin{aligned} Aa + Cb - (Aa' + Bc') &= 0, \\ Ba + Db - (Ab' + Bd') &= 0, \\ Ac + Cd - (Ca' + Dc') &= 0, \\ Bc + Dd - (Cb' + Dd') &= 0, \end{aligned}$$

or, what is the same thing,

$$\begin{vmatrix} a-a', & -c', & b, & 0, \\ -b', & a-d', & 0, & b, \\ c, & 0, & d-a', & -c', \\ 0, & c, & -b', & d-d', \end{vmatrix} \between (A,\ B,\ C,\ D) = 0,$$

viz. we have (A, B, C, D) connected by these four linear equations: viz. the necessary condition is that the determinant formed out of the matrix which here presents itself shall be $=0$.

After some reductions, and putting for shortness

$$\nabla = ad - bc, \quad \nabla' = a'd' - b'c',$$

this is found to be

$$(\nabla - \nabla')^2 + \{\nabla (a' + d') - \nabla' (a + d)\} (a' + d' - a - d) = 0;$$

which is the condition for the existence of a solution. This condition being satisfied, the four equations will be equivalent to three independent equations, which serve to determine A, B, C, D; and, assuming the absolute value of A, we find

$$\begin{aligned} A &= \qquad -(a' + d' - a - d), \\ c'B &= \nabla - \nabla' + a' (a' + d' - a - d), \\ bC &= \nabla - \nabla' + a (a' + d' - a - d), \\ bc'D &= (d' - a) \nabla - (d - a') \nabla' - aa' (a' + d' - a - d), \end{aligned}$$

values which give

$$-bc' (AD - BC) = (\nabla - \nabla')^2 + \{\nabla (a' + d') - \nabla' (a + d)\} (a' + d' - a - d), = 0,$$

viz. in the case where the given matrices satisfy the above-mentioned condition, the components A, B, C, D of the required matrix have determinate values which are such that $AD - BC = 0$.

If we have $\nabla - \nabla' = 0$, that is, $ad - bc = a'd' - b'c'$, then the condition becomes $a' + d' = a + d$; and these two conditions being satisfied, the four equations reduce themselves to two independent equations, and the ratios $A : B : C : D$ have no longer determinate values; the linear equations may in fact be written

$$\left(\begin{array}{rrrr} 0, & b, & -c', & a-a' \\ -b, & 0, & -a+d', & b' \\ c', & a'-d, & 0, & -c \\ d-d', & -b', & c, & 0 \end{array}\right) \between D, C, B, A) = 0,$$

which are of the form

$$\left(\begin{array}{rrrr} 0, & \mathrm{h}, & -\mathrm{g}, & \mathrm{a} \\ -\mathrm{h}, & 0, & \mathrm{f}, & \mathrm{b} \\ \mathrm{g}, & -\mathrm{f}, & 0, & \mathrm{c} \\ -\mathrm{a}, & -\mathrm{b}, & -\mathrm{c}, & 0 \end{array}\right) \between D, C, B, A) = 0,$$

where the coefficients a, b, c, f, g, h are such that $\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0$; and the four equations are thus equivalent to two independent equations. To obtain a symmetrical solution, assume a relation

$$D\xi + C\eta + B\zeta + A\omega = 0,$$

with arbitrary coefficients ξ, η, ζ, ω; we then find

$$\begin{aligned} D &= \quad . \quad \mathrm{c}\eta - \mathrm{b}\zeta + \mathrm{f}\omega, \\ C &= -\mathrm{c}\xi \quad . \quad + \mathrm{a}\zeta + \mathrm{g}\omega, \\ B &= \mathrm{b}\xi - \mathrm{a}\eta \quad . \quad + \mathrm{h}\omega, \\ A &= -\mathrm{f}\xi - \mathrm{g}\eta - \mathrm{h}\zeta \quad . \ , \end{aligned}$$

viz. the complete theorem is that, if the matrices

$$q = \left|\begin{array}{rr} a, & b \\ c, & d \end{array}\right|, \quad q' = \left|\begin{array}{rr} a', & b' \\ c', & d' \end{array}\right|$$

are such that $ad - bc = a'd' - b'c'$, $a + d = a' + d'$, then writing

$$\begin{aligned} \mathrm{a} &= \quad a - a', = -d + d', & \mathrm{f} &= -a + d', = -a' + d, \\ \mathrm{b} &= \quad b', & \mathrm{g} &= -c', \\ \mathrm{c} &= -c, & \mathrm{h} &= \quad b, \end{aligned}$$

values which satisfy

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0,$$

and taking ξ, η, ζ, ω arbitrary, we have for the matrix Q, which is such that $Qq - Qq' = 0$, the expression

$$Q = \left|\begin{array}{ll} -\mathrm{f}\xi - \mathrm{g}\eta - \mathrm{h}\zeta, & \mathrm{b}\xi - \mathrm{a}\eta \quad . \quad + \mathrm{h}\omega \\ -\mathrm{c}\xi \quad . \quad + \mathrm{a}\zeta + \mathrm{g}\omega, & \mathrm{c}\eta - \mathrm{b}\zeta + \mathrm{f}\omega \end{array}\right|,$$

depending really on one arbitrary parameter, viz. we may without loss of generality put any two of the coefficients ξ, η, ζ, ω equal to 0.

840.

ON MASCHERONI'S GEOMETRY OF THE COMPASS.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 179—181.]

I HAVE not seen the original of Mascheroni's work, *La Geometria del Compasso*, 8°, Pavia (1797), but only the French translation, *Géométrie du Compas*, par L. Mascheroni, traduite de l'Italien par A. M. Carette, 2 ed. 8° Paris, 1828 (Author's Preface, pp. 5—24, and pp. 25—328, with 14 plates containing 108 figures). The title expresses the notion of the work: a straight line is given by means of two points thereof, but is not allowed to be actually drawn; and the problem is, with the compass alone to perform all the constructions of Geometry. Observe that, for the purpose of demonstration, any lines may be imagined to be drawn, and such lines are in fact shown as dotted lines in the figures; but this does not in any wise interfere with the fundamental postulate that the constructions are to be performed with the compass only.

Assuming, then, that a line is in every case given by means of two points thereof, the leading questions are those considered Book VII. "on the intersections of straight lines with circular arcs and with each other," viz. they are (1) to find the intersections of a given line and circle; (2) to find the intersection of two given lines. But in the first problem it is necessary to consider two cases; (*A*) the general case, where the line does not pass through the centre of the circle, and (*B*) the particular, but actually more difficult, case, where the line passes through the centre of the circle. It is assumed that the centre of the circle is given: if it is not given, it can be found as afterwards mentioned.

It will be convenient to establish the definition of counter-points in regard to a given line: two points, which are such that the line joining them is bisected at right angles by the given line, are said to be counter-points in regard to that line.

(1), *A*. Consider a line given by means of two points P, Q; and a given circle, centre C. With centre P and radius PC describe a circle, and with centre Q and

radius QC a circle; these meet in C and in a second point D which will be the counter-point of C in regard to the line PQ; hence with centre D and radius = that of given circle, describing a circle, this meets the given circle in two points which are the intersections of the given circle by the line PQ.

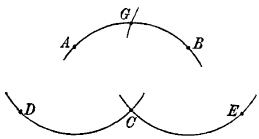
The construction fails if the line PQ passes through the centre of the given circle, for then the two auxiliary circles touch at C, or we have D coincident with C. We have therefore considered this case; Q may be taken to be the centre C of the circle; so that we have:—

(1), B. The problem is, given a point P, and a circle, centre C: to find the intersections of the circle by the line CP. With centre P, and an arbitrary radius, describe a circle meeting the given circle in points A, B (which are of course counter-points in regard to line CP); the circle is divided into two arcs AB and $360° - AB$, and the problem is to bisect each of these two arcs; for the points of bisection are obviously the required intersections of CP with the circle.

Hence we have:—

(1), C. To bisect a given arc AB of a circle, centre C (fig. 1). With centre A and radius AC describe a circle; and with centre B, and equal radius BC describe a circle; and on these circles respectively, find the points D, E such that $CD = CE = AB$.

Fig. 1.



With centre E and radius EA, describe a circle; and with centre D, and equal radius DB, describe a circle; and let these two circles meet in F; then with centre E (or D) and radius CF describe a circle: this will meet the given circle in the required middle point G of the arc AB.

The proof depends on the theorem that in a rhombus the sum of the squares of the diagonals is = sum of the squares of the four side. We have a rhombus $ACEB$, and therefore

$$(AE)^2 + (BC)^2 = 2(AB)^2 + 2(AC)^2,$$

that is,

$$(AE)^2 = 2(AB)^2 + \text{rad.}^2.$$

But by construction, $EF^2 = (AE)^2$; therefore

$$EF^2 = 2(AB)^2 + \text{rad.}^2, \ = 2(CE)^2 + \text{rad.}^2.$$

Hence in the right-angled triangle FCE, we have

$$(CF)^2 = (EF)^2 - CE^2, \ = (CE)^2 + \text{rad.}^2;$$

and by construction, $(EG)^2 = (CF)^2$; that is,

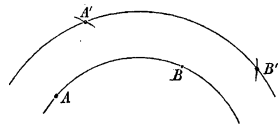
$$(EG)^2 = (CE)^2 + \text{rad.}^2, \ = (CE)^2 + (CG)^2;$$

viz. CEG is a right-angled triangle; that is, CG is at right angles to BCE, or the line CGF bisects the arc AB in G.

For the second problem, (2), to find the intersection of two given lines, we require the solution of the problem to find a fourth proportional to three given distances, and this immediately depends on the following problem.

Problem. In two concentric circles to place chords subtending equal angles. If AB (fig. 2) be a chord of the one circle, then with centres A and B respectively, and

Fig. 2.



C.

an arbitrary radius $AA' = BB'$, describing circles to cut the larger circle in the points A' and B' (A' is either of the two intersections and B' is the intersection lying in regard to CB as A' lies in regard to CA) then clearly $\angle ACB = \angle A'CB'$. And hence also we have the following problem.

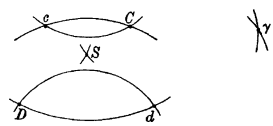
Problem. To find a fourth proportional to three given distances. We have only to take as the given distances the two radii CA, CA' and the chord AB; and then from the similar triangles ACB, $A'CB'$, we have $CA : CA' :: AB : A'B'$, viz. $A'B'$ is a fourth proportional to CA, CA', AB.

We have now the solution of

(2) To construct the intersection of two given lines AB and CD (fig. 3). Find

Fig. 3.

A.



B.

c the counter-point of C, and d the counter-point of D, in regard to the line AB:

and find γ such that the distances $C\gamma$, $d\gamma$ are $= Dd$, DC respectively; γ is in a line with cC, and we have $C\gamma dD$ a parallelogram. Find cS a fourth proportional to $c\gamma$, cd, cC, and with centres c, C respectively and radii each $= cS$ describe circles cutting in the point S; this will be the required intersection of the two lines. In fact, the required point S will be the intersection of the two lines CD, cd: supposing these lines each of them drawn, and also the lines $cC\gamma$ and $d\gamma$, we have DC parallel to $d\gamma$, that is, the triangles $cd\gamma$, cSC are similar to each other or $c\gamma : cd :: cC : cS$: viz. the distance cS having been found by this proportion, and the point S found as the intersection of the two circles, centres c and C respectively, the point S so determined is the required point of intersection of the given lines AB and CD.

If a circle be given without its centre being known, then taking any three points A, B, C on the circle, and a pair of counter-points D, E of the line AB, and also a pair of counter-points F, G of the line AC, we have obviously the centre of the given circle as the intersection of the lines DE and FG; and the centre can thus be found with the compass only.

It is proper to remark that the problems considered in the present paper are those connecting the theory with ordinary geometry, not the problems which are most readily and elegantly solved with the compass only: a large collection of these are contained in the work, and in particular the twelfth book contains some interesting *approximate* solutions of the problems of the quadrature of the circle, the duplication of the cube, and other problems not solvable by ordinary geometry.

Cambridge, March 19, 1885.

841.

ON A DIFFERENTIAL OPERATOR.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 190, 191.]

WRITE $X = 1 + bx + cx^2 + \ldots, = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$; then by Capt. MacMahon's theorem, any non-unitary function of the roots $\alpha, \beta, \gamma, \ldots$ is reduced to zero by the operation

$$\Delta, = \partial_b + b\partial_c + c\partial_d + \ldots;$$

for instance, if

$$(2), = \Sigma\alpha^2 = b^2 - 2c,$$

we have

$$\Delta(b^2 - 2c) = 2b + b(-2), = 0.$$

We have

$$\Delta X = x + bx^2 + cx^3 + \ldots = xX;$$

and writing, moreover, $X', = b + 2cx + 3dx^2 + \&c.$, for the derived function of X, then

$$\Delta X' = 1 + 2bx + 3cx^2 + \ldots = (xX)'.$$

We can hence shew that $\Delta\left(\frac{X'}{X} - b\right) = 0$; the value is, in fact,

$$\frac{\Delta X'}{X} - \frac{X'\Delta X}{X^2} - \Delta b, \text{ that is, } \frac{(xX)'}{X} - \frac{X'xX}{X^2} - 1,$$

which is

$$= \frac{X + xX'}{X} - \frac{xX'}{X} - 1, = 0.$$

This is right, for $\frac{X'}{X}$ is a sum of non-unitary symmetric functions of the roots; in fact,

$$\frac{X'}{X} = \Sigma\frac{-\alpha}{1-\alpha x} = -(1) - (2)x - (3)x^2 - \&c.,$$

or since $b = -(1)$, this is

$$\frac{X'}{X} - b = -(2)x - (3)x^2 - \&c.,$$

a sum of non-unitary functions of the roots.

842.

ON THE VALUE OF $\tan(\sin\theta) - \sin(\tan\theta)$.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 191, 192.]

THE following equation is given p. 59 of the *Lady's and Gentleman's Diary* for 1853:

$$\tan(\sin\theta) - \sin\tan\theta = \tfrac{1}{30}\theta^7 + \tfrac{29}{756}\theta^9 + \&c.$$

Write in general

$$X = \theta + A\theta^3 + B\theta^5 + C\theta^7 + D\theta^9 + \ldots,$$

$$Y = \theta + A'\theta^3 + B'\theta^5 + C'\theta^7 + D'\theta^9 + \ldots.$$

Then, as far as θ^9, we have

$$\begin{aligned}
& X + A'X^3 + B'X^5 + C'X^7 + D'X^9 \\
= {} & \theta + A\theta^3 + B\theta^5 + C\theta^7 + D\theta^9 \\
& + A'\{\theta^3 + 3A\theta^5 + 3(A^2+B)\theta^7 + (A^3 + 6AB + 3C)\theta^9\} \\
& + B'\{\theta^5 + 5A\theta^7 + (10A^2 + 5B)\theta^9\} \\
& + C'\{\theta^7 + 7A\theta^9\} \\
& + D'\{\theta^9\} \\
= {} & + \theta \\
& + \theta^3(A + A') \\
& + \theta^5(B + 3AA' + B') \\
& + \theta^7(C + 3A^2A' + 3A'B + 5AB' + C') \\
& + \theta^9(D + A^3A' + 6AA'B + 3A'C + 10A^2B' + 5BB' + 7AC' + D'),
\end{aligned}$$

and hence

$$\begin{aligned} & X + A'X^3 + B'X^5 + C'X^7 + D'X^9 \\ & - Y - AY^3 - BY^5 - CY^7 - DY^9 \\ = & \ \theta^7 \{3AA'(A - A') + 2(AB' - A'B)\} \\ + & \ \theta^9 \{AA'(A^2 - A'^2) + 6AA'(B - B') + 4(AC' - A'C) + 10(A^2B' - A'^2B)\}. \end{aligned}$$

Now let

$$\begin{aligned} X &= \sin\theta = \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 - \tfrac{1}{5040}\theta^7 + \ldots; \quad A = -\tfrac{1}{6},\ B = \tfrac{1}{120},\ C = -\tfrac{1}{5040}, \\ Y &= \tan\theta = \tfrac{1}{3}\theta^3 + \tfrac{2}{15}\theta^5 + \tfrac{17}{315}\theta^7 - \ldots; \quad A' = \tfrac{1}{3},\ B' = \tfrac{2}{15},\ C' = \tfrac{17}{315}; \end{aligned}$$

we have therefore

$$\begin{aligned} & AA' = -\tfrac{1}{18}, \quad A - A' = -\tfrac{1}{2}, \quad A + A' = \tfrac{1}{6}, \quad B - B' = -\tfrac{1}{8}, \\ & AB' - A'B = -\tfrac{1}{40}, \quad AC' - A'C = -\tfrac{1}{112}, \quad A^2B' - A'^2B = \tfrac{1}{360}. \end{aligned}$$

Hence

$$\begin{aligned} & \text{coeff. } \theta^7 = \tfrac{1}{12} - \tfrac{1}{20}, = \tfrac{1}{30}, \\ & \text{coeff. } \theta^9 = \tfrac{1}{216} + \tfrac{1}{24} - \tfrac{1}{28} + \tfrac{1}{36}, = \tfrac{29}{756}; \end{aligned}$$

and the required equation is thus verified.

843.

ON THE QUADRIQUADRIC CURVE IN CONNEXION WITH THE THEORY OF ELLIPTIC FUNCTIONS.

[From the *Mathematische Annalen*, t. XXV. (1885), pp. 152—156.]

I CALL to mind that, if we have on a line two points $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, and through the line two planes

$$Ax + By + Cz + Dw = 0, \quad A'x + B'y + C'z + D'w = 0,$$

then

$$\begin{aligned} &\beta\gamma' - \beta'\gamma \ : \ \gamma\alpha' - \gamma'\alpha \ : \ \alpha\beta' - \alpha'\beta \ : \ \alpha\delta' - \alpha'\delta \ : \ \beta\delta' - \beta'\delta \ : \ \gamma\delta' - \gamma'\delta \\ &= AD' - A'D : BD' - B'D : CD' - C'D : BC' - B'C : CA' - C'A : AB' - A'B; \end{aligned}$$

and that, putting each of these two equal sets of ratios

$$= a : b : c : f : g : h,$$

then the quantities (a, b, c, f, g, h), which it is easy to see satisfy the relation $af + bg + ch = 0$, are said to be the "six coordinates" of the line: as only the ratios of the six quantities are material, and as the last-mentioned equation establishes a single relation between these ratios, the system of the six coordinates contains four arbitrary ratios or parameters, for the determination of the particular line. See my paper "On the six coordinates of a line," *Camb. Phil. Trans.* t. XI. (1869), pp. 290—323, [435].

I consider for a moment the quadric surface

$$x^2 + y^2 + z^2 + w^2 = 0,$$

and I proceed to show that, if (a, b, c, f, g, h) are the coordinates of a generating line on the surface, then either $a = f$, $b = g$, $c = h$, or else $a = -f$, $b = -g$, $c = -h$; the one or other system of equations according as the line belongs to the one or other system of generating lines.

We satisfy the equation by

$$x+iy+\theta z+\theta iw=0,$$

$$x-iy-\frac{1}{\theta}z+\frac{1}{\theta}iw=0,$$

where θ is an arbitrary parameter: hence these equations determine a generating line of the surface: and the coordinates of this line are

$$\begin{matrix} a, & b, & c, & f, & g, & h \\ =-i\left(\theta-\frac{1}{\theta}\right), & -\left(\theta+\frac{1}{\theta}\right), & zi, & i\left(\theta-\frac{1}{\theta}\right), & \theta+\frac{1}{\theta}, & -zi; \end{matrix}$$

viz. these values give $a=-f$, $b=-g$, $c=-h$.

Similarly we satisfy the equation by

$$x+iy+\phi z-\phi iw=0,$$

$$x-iy-\frac{1}{\phi}z-\frac{1}{\phi}iw=0,$$

where ϕ is an arbitrary parameter: hence these equations also determine a generating line of the surface: and the coordinates of this line are

$$\begin{matrix} a, & b, & c, & f, & g, & h \\ =i\left(\phi-\frac{1}{\phi}\right), & \phi+\frac{1}{\phi}, & -zi, & i\left(\phi+\frac{1}{\phi}\right), & \phi+\frac{1}{\phi}, & -zi; \end{matrix}$$

viz. these values give $a=f$, $b=g$, $c=h$.

If for x, y, z, w we write $x\sqrt{p}$, $y\sqrt{q}$, $z\sqrt{r}$, $w\sqrt{s}$ respectively, then we have the theorem that, for the quadric surface

$$px^2+qy^2+rz^2+sw^2=0,$$

the coordinates (a, b, c, f, g, h) of a generating line are such that

$$\frac{a}{f}=\pm\sqrt{\frac{ps}{qr}},\quad \frac{b}{g}=\pm\sqrt{\frac{qs}{rp}},\quad \frac{c}{h}=\pm\sqrt{\frac{rs}{pq}},$$

the signs being all + or all −, according as the line belongs to one or other of the systems of generating lines.

Take (a', b', c', f', g', h') for the coordinates of an arbitrary line, and write $p, q, r, s=a'g'h', b'h'f', c'f'g', a'b'c'$; the quadric surface is

$$a'g'h'x^2+b'h'f'y^2+c'f'g'z^2+a'b'c'w^2=0,$$

which is a surface having the line (a', b', c', f', g', h') for a generating line. To verify this, observe that for the line in question we have

$$\begin{aligned} h'y - g'z + a'w &= 0, \\ -h'x \quad . \quad + f'z + b'w &= 0, \\ g'x + f'y \quad . \quad + c'w &= 0, \\ -a'x - b'y - c'z \quad . \quad &= 0, \end{aligned}$$

equivalent of course to two independent equations: these give $h'x = f'z + b'w$, $h'y = g'z - a'w$, values which substituted in the quadric equation satisfy it identically. And for the last-mentioned quadric surface we have the theorem that, if (a, b, c, f, g, h) are the coordinates of a generating line, then

$$\frac{a}{f} = \pm\frac{a'}{f'}, \quad \frac{b}{g} = \pm\frac{b'}{g'}, \quad \frac{c}{h} = \pm\frac{c'}{h'},$$

where obviously the sign + belongs to a generating line of the same system with the line (a', b', c', f', g', h'), and the sign − to a line of the other system.

Taking the sign −, we thus see that if (a, b, c, f, g, h), (a', b', c', f', g', h') are the coordinates of lines of the two systems respectively, then

$$af' + a'f = 0, \quad bg' + b'g = 0, \quad ch' + c'h = 0;$$

where observe that the resulting equation

$$af' + a'f + bg' + b'g + ch' + c'h = 0,$$

is the condition which expresses that the two lines meet each other.

Consider now the quadriquadric curve

$$\begin{aligned} U_1 &= Ax^2 + By^2 + Cz^2 + Dw^2 = 0, \\ U_1' &= A'x^2 + B'y^2 + C'z^2 + D'w^2 = 0; \end{aligned}$$

and let (a, b, c, f, g, h) and (a', b', c', f', g', h') be the coordinates of two lines meeting each other, and each meeting the quadric curve twice: or, again, let these be lines joining in pairs the four intersections of the curve by an arbitrary plane: or, again, let them be the nodal lines of the binodal quartic cone having an arbitrary vertex and passing through the curve. The two lines are generating lines, belonging to the two systems respectively, of a properly determined quadric surface $U + \lambda U' = 0$ passing through the curve: and by what precedes, we have

$$af' + a'f = 0, \quad bg' + b'g = 0, \quad ch' + c'h = 0,$$

the fundamental theorem which I wished to establish.

Writing in the equations $w = 1$, we have in particular the quadriquadric curve $y^2 = 1 - x^2$, $z^2 = 1 - k^2x^2$, equations which are satisfied by $x = \operatorname{sn} u$, $y = \operatorname{cn} u$, $z = \operatorname{dn} u$. Consider on the curve four points belonging to the arguments u_1, u_2, u_3, u_4 respectively;

and write for shortness s_1, c_1, d_1 for the sn, cn, and dn of u_1; and similarly for u_2, u_3 and u_4. It appears by Abel's theorem that the condition in order that the four points may be in a plane is $u_1+u_2+u_3+u_4=0$; viz. when this equation is satisfied we have

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix},$$

or writing

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h \\ = c_1d_2-c_2d_1, & d_1s_2-d_2s_1, & s_1c_2-s_2c_1, & s_1-s_2, & c_1-c_2, & d_1-d_2, \\ a', & b', & c', & f', & g', & h' \\ = c_3d_4-c_4d_3, & d_3s_4-d_4s_3, & s_3c_4-s_4c_3, & s_3-s_4, & c_3-c_4, & d_3-d_4, \end{array}$$

this equation is

$$af'+a'f+bg'+b'g+ch'+c'h=0;$$

or by what precedes, it appears that not only is this so, but that we have separately

$$af'+a'f=0,\quad bg'+b'g=0,\quad ch'+c'h=0,$$

viz. it follows from Abel's theorem that, when the arguments u_1, u_2, u_3, u_4 are connected by the equation $u_1+u_2+u_3+u_4=0$, then each of these three equations holds good.

I assume in particular $u_4=0$, $u_3=-u$, so that $u=u_1+u_2$; we have

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h \\ = c_1d_2-c_2d_1, & d_1s_2-d_2s_1, & s_1c_2-s_2c_1, & s_1-s_2, & c_1-c_2, & d_1-d_2, \\ a', & b', & c', & f', & g', & h' \\ = \quad c-d, & s, & -s, & -s, & c-1, & d-1, \end{array}$$

and the three equations become

$$\frac{c-d}{s}=\frac{c_1d_2-c_2d_1}{s_1-s_2},\quad \frac{-s}{c-1}=\frac{d_1s_2-d_2s_1}{c_1-c_2},\quad \frac{s}{d-1}=\frac{s_1c_2-s_2c_1}{d_1-d_2},$$

equations which may also be written

$$\frac{c-d}{s}=\frac{c_1d_2-c_2d_1}{s_1-s_2},\quad \frac{-c+1}{s}=\frac{c_1-c_2}{d_1s_2-d_2s_1},\quad \frac{d-1}{s}=\frac{d_1-d_2}{s_1c_2-s_2c_1},$$

so that, adding these three equations, we must have an identity.

Representing the second and third of the last-mentioned equations by

$$\frac{-c+1}{s}=C,\quad \frac{d-1}{s}=D,$$

we have

$$c = 1 - Cs, \quad d = 1 + Ds,$$

from either of which we can obtain s rationally; viz. the first equation gives

$$1 - s^2 = 1 - 2Cs + C^2s^2,$$

that is, $s = \dfrac{2C}{1 + C^2}$, whence

$$c = \frac{1 - C^2}{1 + C^2}, \quad d = \frac{1 + C^2 + 2CD}{1 + C^2};$$

and similarly from the second equation $s = -\dfrac{2D}{k^2 + D^2}$, whence also

$$c = \frac{k^2 + D^2 - 2CD}{k^2 + D^2}, \quad d = \frac{k^2 - D^2}{k^2 + D^2}.$$

Substituting for C its value, the first-mentioned value of s becomes

$$s = \frac{(c_1 - c_2)(d_1s_2 - d_2s_1)}{1 - k^2s_1^2s_2^2 - c_1c_2 - d_1s_1d_2s_2},$$

which is a form that can be easily verified: we in fact have

$$s_1 = \operatorname{sn}(u_1 + u_2) = \frac{s_1c_2d_2 + s_2c_1d_1}{1 - k^2s_1^2s_2^2}, \quad = \frac{s_1^2 - s_2^2}{s_1c_2d_2 - s_2c_1d_1}, \quad = \frac{s_1c_1d_2 + s_2c_2d_1}{c_1c_2 + s_1d_1s_2d_2}, \quad = \frac{s_1d_1c_2 + s_2d_2c_1}{d_1d_2 + k^2s_1c_1s_2c_2},$$

see my *Elliptic Functions*, p. 63: or calling these values

$$= \frac{N_1}{D_1} = \frac{N_2}{D_2} = \frac{N_3}{D_3} = \frac{N_4}{D_4},$$

the above value is

$$s = \frac{N_1 - N_3}{D_1 - D_3},$$

which is right.

Cambridge, 28 *June*, 1884.

844.

ON A THEOREM RELATING TO SEMINVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XX. (1885), pp. 212, 213.]

I FIND among my papers the following example of a general theorem relating to seminvariants.

Write

$$\Theta = b\partial_a + c\partial_b + d\partial_c + e\partial_d + f\partial_e + g\partial_f + h\partial_g + i\partial_h,$$
$$\Omega = \quad c\partial_b + 2d\partial_c + 3e\partial_d + 4f\partial_e + 5g\partial_f + 6h\partial_g + 7i\partial_h,$$

then from a seminvariant S of the degree δ and weight w, operating upon it with $\Theta - \delta b$ and $\Omega - 2wb$, we derive two seminvariants, each of them of the degree $\delta + 1$ and weight $w + 1$; we obtain a combination of these by operating on S with $2w(\Theta - \delta b) - \delta(\Omega - 2wb)$, that is, with $2w\Theta - \delta\Omega$, and this combination is of the weight $w + 1$, but only of the degree δ; viz. operating with $2w\Theta - \delta\Omega$, we obtain a new seminvariant of the same degree but of a weight increased by unity.

The example relates to the under-mentioned seminvariant S of the degree 3 and weight 7:

	$S=$
h	+ 1
bg	− 7
cf	+ 9
de	− 5
b^2f	+ 12
bce	− 30
bd^2	+ 20
	± 42

	ΘS	$-3bS$	Sum	ΩS	$-14bS$	Sum	$14\Theta S$	$-3\Omega S$			$\div 110$
i	1		1	+ 7		+ 7	+ 14	− 21	+ 7	0	
bh	− 5	− 3	− 8	− 42	− 14	− 56	− 70	+ 126	− 56	0	
cg	+ 2		+ 2	+ 38		+ 38	+ 28	− 114	+ 196	+ 110	+ 1
df	+ 4		+ 4	− 2		− 2	+ 56	+ 6	− 392	− 330	− 3
e^3	− 5		− 5	− 15		− 15	− 70	+ 45	+ 245	+ 220	+ 2
b^2g	+ 5	+ 21	+ 26	+ 60	+ 98	+ 158	+ 70	− 180		− 110	− 1
bcf	+ 3	− 27	− 24	− 96	− 126	− 222	+ 42	+ 288		+ 330	+ 3
bde	+ 5	+ 15	+ 20	+ 60	+ 70	+ 130	+ 70	− 180		− 110	− 1
c^2e	− 30		− 30	− 30		− 30	− 420	+ 90		− 330	− 3
cd^2	+ 20		+ 20	+ 20		+ 20	+ 280	− 60		+ 220	+ 2
b^3f		− 36	− 36		− 168	− 168					
bce		+ 90	+ 90		+ 420	+ 420					
b^2d^2		− 60	− 60		− 280	− 280					
	± 40	± 126	± 163	± 185	± 588	± 773	± 560	± 555	± 448	± 880	± 8

The columns show ΘS (where observe that, to operate with the $b\partial_a$ of Θ, we restore the proper power of a, reading S as being $= +1a^2h - 7abg + \&c.$, and putting ultimately $a=1$) and $-3bS$, and the sum of these, which is a seminvariant, degree 4, weight 8; also ΩS and $-14bS$, and the sum of these, which is a seminvariant, degree 4, weight 8. They also show $14\Theta S$ and $-3\Omega S$, the sum of which would be a seminvariant, degree 3, and weight 8; instead of giving this sum, I have added a column equal to $+7(i - 8bh + 28cg - 56df + 35e^2)$, and given the sum of the three columns which will of course be a seminvariant of the same degree and weight; the coefficients contain all of them the factor 110, and, throwing this out, we have in the last column the seminvariant $cg - 3df + \ldots + 2cd^2$ of the degree 3 and weight 8, derived by the foregoing direct process from the given seminvariant S of the degree 3 and weight 7.

845.

ON THE ORTHOMORPHOSIS OF THE CIRCLE INTO THE PARABOLA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XX. (1885), pp. 213—220.]

IT is remarked by Schwarz (see his Memoir "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120; p. 115), that the circle $x'^2 + y'^2 - 1 = 0$ can be orthomorphosed into the parabola $y^2 = 4(1-x)$ by the equation

$$\sqrt{(x' + iy')} = \tan \tfrac{1}{4}\pi \sqrt{(x + iy)}:$$

viz. this equation establishes a (1, 1) relation between the points interior to the circle and those interior to the parabola, which, quà relation between $x' + iy'$ and $x + iy$, will be orthomorphic, that is, infinitesimal elements of the one area will correspond to similar infinitesimal elements of the other area. The diameter $y' = 0$ of the circle is trans-

Fig. 1.

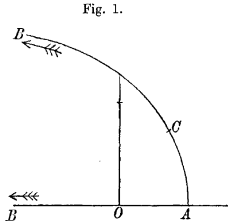
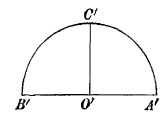


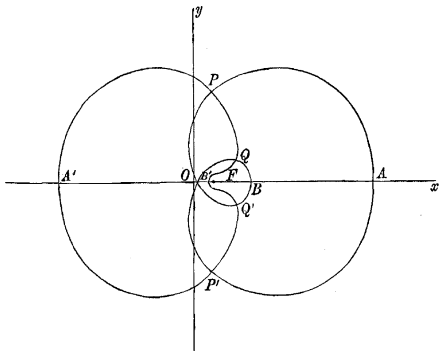
Fig. 2.



formed into the axis $y = 0$ of the parabola, and figs. 1, 2 are symmetrical on the two sides of these lines respectively; we may therefore consider only the transformation

of the upper semicircle into the upper half of the parabola; we have (see figs. 1 and 2) A' corresponding to the vertex A, O' to the focus O, and B' to the point at infinity ($x = -\infty$) on the axis of the parabola; the semicircular arc $A'C'B'$ corresponds to the infinite half-arc ACB of the parabola, the highest point C' corresponding to a point C between the vertex and the semi-latus rectum.

Fig. 3.



We may divide the circle by concentric circles and by radii; the corresponding curves for the parabola will be ovals and radials from the focus, the curves of each system being transcendental curves. Or we may divide the parabola by means of two systems of confocal parabolas; the corresponding curves for the circle will be (see fig. 3) two systems of orthotomic limaçons, those of the one system having B' for a crunode, and those of the other system having B' for an acnode.

To show that the circle thus corresponds to the parabola, it is only necessary to write $\sqrt{(x+iy)} = 1 + qi$, that is, $x = 1 - q^2$, $y = 2q$, implying $y^2 = 4(1-x)$, or (x, y) is a point on the parabola; and we then obtain for $x' + iy'$ a value of the form $\cos\theta' + i\sin\theta'$, that is, (x', y') is a point on the circle $x'^2 + y'^2 = 1$; but in reference to what follows, I give the proof in a somewhat more artificial form.

Writing log tan to denote the hyperbolic logarithm of the tangent, then if ϕ, ϕ' are such that

$$\phi = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right),$$

this equation, as is known, may also be presented in the forms

$$i\phi' = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}i\phi\right),$$
$$i\tan\tfrac{1}{2}\phi' = \tan\tfrac{1}{2}i\phi,$$

or, what is the same thing,

$$\tan\tfrac{1}{2}\phi' = \tanh\tfrac{1}{2}\phi,$$

where observe that, as ϕ' increases from 0 to $\frac{1}{2}\pi$, ϕ increases from 0 to ∞, and that always $\phi > \phi'$.

We can now establish as follows the three correspondences AO with $A'O'$, OB with $O'B'$, and ACB with $A'C'B'$.

(1) $$\sqrt{(x+iy)} = \frac{2\phi}{\pi}; \quad \sqrt{(x'+iy')} = \tan\tfrac{1}{2}\phi';$$

that is,

$$x = \frac{4\phi^2}{\pi^2},\ y = 0; \quad x' = \tan^2\tfrac{1}{2}\phi',\ y' = 0;$$

$\phi = 0$ gives A, A'; $\phi' = \frac{1}{2}\pi$ gives O, O'.

(2) $$\sqrt{(x+iy)} = \frac{2i\phi}{\pi}; \quad \sqrt{(x'+iy')} = i\tan\tfrac{1}{2}\phi';$$

that is,

$$x = -\frac{4\phi^2}{\pi^2},\ y = 0; \quad x' = -\tan^2\tfrac{1}{2}\phi',\ y' = 0;$$

$\phi = \phi' = 0$ gives O, O';

$\phi = \infty$, $\phi' = \frac{1}{2}\pi$ give B, B'.

(3) $$\sqrt{(x+iy)} = 1 + \frac{2i\phi}{\pi}; \quad \sqrt{(x'+iy')} = e^{i\phi'};$$

that is,

$$x = 1 - \frac{4\phi^2}{\pi^2},\ y = \frac{4\phi}{\pi},$$

and therefore

$$y^2 = 4(1-x),$$

the parabola; and

$$x' = \cos 2\phi',\ y' = \sin 2\phi',$$

and therefore

$$x'^2 + y'^2 = 1,$$

the circle;

$\phi = \phi' = 0$ gives A, A'; $\phi = \infty$, $\phi' = \frac{1}{2}\pi$ give B, B'.

Observe that for points in $O'A'$, $O'B'$, equidistant from O', we have $x' = \tan^2\frac{1}{2}\phi'$, $x' = -\tan^2\frac{1}{2}\phi'$; and corresponding hereto we have points in OA, OB on the axis of the parabola, the values of x being $x = \frac{4\phi'^2}{\pi^2}$, $x = -\frac{4\phi^2}{\pi^2}$, viz. the negative value is always the greater.

Observe further that, to the points $(x', y') = (\cos 2\phi', \sin 2\phi')$ and $(x', y') = (-\tan^2\frac{1}{2}\phi', 0)$ on the circle, and on the radius OB', correspond the points

$$(x, y) = \left(1 - \frac{4\phi^2}{\pi^2},\ \frac{2\phi}{\pi}\right), \text{ and } (x, y) = \left(-\frac{4\phi^2}{\pi^2},\ 0\right),$$

on the parabola and on the axis OB respectively; the axial distance of these two points is $\left(1 - \frac{4\phi^2}{\pi^2}\right) + \frac{4\phi^2}{\pi^2} = 1$, the radius of the circle, or the distance OA of the vertex and focus of the parabola; this is a rather curious theorem.

The function $\log\tan(45^\circ + \frac{1}{2}\text{arg.})$ is tabulated, Legendre, *Théorie des Fonctions Elliptiques*, t. II. table iv. pp. 256—259, viz. writing as above ϕ' for the argument, we have hereby the value of ϕ, $= \log\tan(\frac{1}{4}\pi + \frac{1}{2}\phi')$, for every value of ϕ' from 0° to 90° at intervals of 30′ and to 12 decimals, and with fifth differences. Observe that ϕ' is thus given in degrees and minutes as a circular arc, ϕ as a number; it is convenient to have ϕ and ϕ' each as arcs, or each as numbers, the conversion being of course at once made by means of a table of the lengths of circular arcs, and I have calculated the Table which I give at the end of the present paper.

I had previously calculated for the correspondence of $A'O'$, $O'B'$ and $A'C'B'$ with AO, OB and ACB, the following values, at irregular intervals suited to the construction of a figure.

Circle $\angle A'O'P'$	Parabola Ordinate PM
0°	0
30	·3372
60	·6994
90	1·1220
120	1·6768
150	2·5817
160	3·1018
170	3·9869
172	4·2713
174	4·6378
176	5·1542
178	6·0258
179	6·9195
180°	∞

Circle $O'A'$ $x'=$	Circle $O'B'$ $x'=$	Parabola OA $x=$	Parabola OB $x=$
0	0	0	0
·1	− ·1	·152	− ·173
·2	− ·2	·287	− ·375
·3	− ·3	·407	− ·611
·4	− ·4	·515	− ·943
·5	− ·5	·615	− 1·257
·6	− ·6	·704	− 1·724
·7	− ·7	·787	− 2·368
·8	− ·8	·853	− 3·386
·9	− ·9	·935	− 5·368
1·0	− 1·0	1·000	$-\infty$

Write in general

$$\sqrt{(x+iy)} = p + qi, = \frac{2}{\pi}(\psi + i\phi);$$

viz. considering x, y as given, find thence p, q by the equations $x = p^2 - q^2$, $y = 2pq$: and then writing $\frac{1}{2}\pi p = \psi$, $\frac{1}{2}\pi q = \phi$, we have

$$\sqrt{(x'+iy')} = \tan\tfrac{1}{4}\pi\,(p+qi) = \tan(\tfrac{1}{2}\psi + \tfrac{1}{2}i\phi) = \frac{P+iQ}{1-iPQ},$$

where

$$P = \tan \tfrac{1}{4}\pi p, \quad = \tan \tfrac{1}{2}\psi,$$

$$Q = \frac{1}{i} \tan \tfrac{1}{4}\pi q i, \quad = \tanh \tfrac{1}{4}\pi q, \quad = \frac{1}{i} \tan \tfrac{1}{2}\phi i, \quad = \tanh \tfrac{1}{2}\phi;$$

whence, if we introduce ϕ' connected with ϕ by the equation $\phi = \log \tan (\frac{1}{4}\pi + \frac{1}{2}\phi')$ as before, we have $Q = \tan \frac{1}{2}\phi'$, and the formula is

$$\sqrt{(x' + iy')} = \frac{P + iQ}{1 - iPQ}, \quad P = \tan \tfrac{1}{2}\psi, \quad Q = \tan \tfrac{1}{2}\phi',$$

giving the values of x', y'.

It is clear that we have

$$\sqrt{(x' + iy')} = \frac{P - iQ}{1 + iPQ},$$

and thence

$$\sqrt{(x'^2 + y'^2)} = \frac{P^2 + Q^2}{1 + P^2Q^2}.$$

Hence, to the circle $x'^2 + y'^2 = c'^2$, corresponds in the parabola the curve

$$P^2 + Q^2 = c'(1 + P^2Q^2),$$

where $p + iq = \sqrt{(x + iy)}$, $P = \tan \frac{1}{4}\pi p$, $Q = \tanh \frac{1}{4}\pi q$. This is a complicated transcendental equation, and I do not see any convenient way of tracing the curve. The set of curves satisfy the differential equation

$$\frac{P\,dP + Q\,dQ}{P^2 + Q^2} = \frac{PQ(Q\,dP + P\,dQ)}{1 + P^2Q^2},$$

that is,

$$P\,dP(1 - Q^4) + Q\,dQ(1 - P^4) = 0,$$

where dP, dQ are given in terms of dp, dq by the equations

$$\frac{dP}{1 + P^2} = \tfrac{1}{4}\pi\,dp, \quad \frac{dQ}{1 - Q^2} = \tfrac{1}{4}\pi\,dq.$$

We have $y = 2pq$, and thence at the highest points, or summits of the several curves, $q\,dp + p\,dq = 0$. Combining these equations, we have

$$dP : dQ = p(1 + P^2) : -q(1 - Q^2);$$

and thence

$$pP(1 + Q^2) - qQ(1 - P^2) = 0,$$

as an equation to the locus of the summits in question. If p and q are small, then putting for a moment $\frac{1}{4}\pi = m$ for shortness, we have

$$P = mp + \tfrac{1}{3}m^3p^3, \quad Q = mq - \tfrac{1}{3}m^3q^3,$$

and the equation becomes

$$p^2(1 + \tfrac{1}{3}m^3p^2)(1 + m^2q^2) - q^2(1 - \tfrac{1}{3}m^2q^2)(1 - m^2p^2) = 0,$$

that is,

$$p^2 - q^2 + \tfrac{1}{3}m^2(p^4 + 6p^2q^2 + q^4) = 0;$$

writing this in the form

$$p^2 - q^2 + \frac{\pi^2}{48}[(p^2 - q^2)^2 + 8p^2q^2] = 0,$$

we find $x + \frac{\pi^2}{48}(x^2 + 2y^2) = 0$, or say $y^2 = -\frac{24}{\pi^2}x$, as the locus of the summits in the neighbourhood of the focus O, viz. the summits lie all of them, as might have been expected, on the left-hand (or negative side) of the focus.

I have constructed for the parabola, to the scale $1 = 1\frac{1}{2}$ inch, as accurately as the data enable, the figure corresponding to the concentric circles and the radii of the circle.

Resuming the equation $\sqrt{(x+iy)} = p + iq$, that is, $x = p^2 - q^2$ and $y = 2pq$, we have ($p = \text{const.}$), the curves $y^2 = 4p^2(p^2 - x)$, and ($q = \text{const.}$), $y^2 = 4q^2(q^2 + x)$, which are two systems of confocal parabolas, cutting each other at right angles; the curves of the former set all of them interior to the given parabola, those of the latter set of course cutting it at right angles.

Corresponding hereto in the circle, writing for a moment

$$\sqrt{(x' + iy')} = p' + iq',$$

we have

$$p' + iq' = \frac{P + iQ}{1 - iPQ},$$

whence

$$p' = P(1 - q'Q), \quad q' = Q(1 + p'P);$$

whence eliminating P and Q successively, we find

$$p'^2 + q'^2 - p'\left(P - \frac{1}{P}\right) - 1 = 0,$$

$$p'^2 + q'^2 - q'\left(Q + \frac{1}{Q}\right) + 1 = 0;$$

say, for shortness, these are $p'^2 + q'^2 - 2mp' + 1 = 0$, and $p'^2 + q'^2 - 2nq' - 1 = 0$. Introducing the polar coordinates r', θ', we have

$$x' + iy' = r'(\cos\theta' + i\sin\theta'),$$

and thence

$$p',\ q' = \sqrt{(r')}\cos\tfrac{1}{2}\theta', \quad \sqrt{(r')}\sin\tfrac{1}{2}\theta';$$

the equations thus become

$$r' - 1 - 2m\sqrt{(r')}\cos\tfrac{1}{2}\theta' = 0,$$

$$r' + 1 - 2n\sqrt{(r')}\sin\tfrac{1}{2}\theta' = 0,$$

which belong to two limaçons having each of them B' for a node and O for a focus, and which of course cut each other at right angles; see fig. 4, which is a mere

Fig. 4.

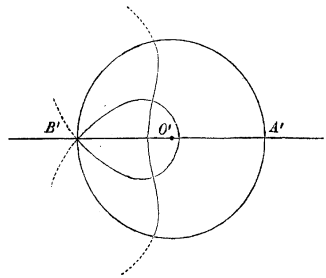


diagram. In fact, omitting for convenience the accents, but recollecting always that the curves belong to the figure of the circle, the first equation gives

$$(r-1)^2 = 4m^2 r \cos^2 \tfrac{1}{2}\theta', \quad = 2m^2 r\,(1+\cos\theta) = 2m^2(r+x),$$

that is,

$$(r^2+1-2m^2x)^2 = 4\,(m^2+1)^2 r^2.$$

Transforming to the point B' as origin, we must write $x+1$ for x, and then

$$r^2 = (x-1)^2 + y^2;$$

the equation thus becomes

$$\{x^2+y^2-2\,(m^2+1)\,x+2\,(m^2+1)\}^2 = 4\,(m^2+1)^2(x^2+y^2+2x+1),$$

that is,

$$\{x^2+y^2-2\,(m^2+1)\,x\}^2 \qquad\qquad = 4m^2\,(m^2+1)\,(x^2+y^2),$$

or say

$$(x^2+y^2)^2 - 4\,(m^2+1)\,(x^2+y^2)\,x + 4\,(m^2+1)\,(x^2-m^2y^2) = 0,$$

showing that the point B' is a crunode, the tangents there being $x = \pm my$.

Writing in the equation $y=0$, we have $x^2=0$, the crunode, and

$$x = 2\sqrt{(m^2+1)}\,\{\sqrt{(m^2+1)} \pm m\};$$

the product of these two values is $=4\,(m^2+1)$, that is, $=\left(P+\dfrac{1}{P}\right)^2$, viz. one value is greater, the other less, than 2. We see also that

$$2\sqrt{(m^2+1)}\,\{\sqrt{(m^2+1)} - m\},$$

the smaller root, is greater than 1. The curve corresponding to a parabola $y^2 = 4p^2(p^2 - x)$ is thus a crunodal limaçon, the crunode at B', and the loop lying wholly within the circle. Moreover, the loop includes within itself the centre O' of the circle.

The other curve is in like manner

$$\{r^2 + 1 + 2n^2x\}^2 = 4(n^2 - 1)r^2,$$

viz. transforming to the origin B', and therefore putting $x - 1$ for x, and

$$r^2 = (x-1)^2 + y^2,$$

the equation is

$$\{x^2 + y^2 + 2(n^2 - 1)x - 2(n^2 - 1)\}^2 = 4(n^2 - 1)^2(x^2 + y^2 - 2x + 1),$$

that is,

$$\{x^2 + y^2 + 2(n^2 - 1)x\}^2 \qquad = 4n^2(n^2 - 1)(x^2 + y^2),$$

or say

$$(x^2 + y^2)^2 + 4(n^2 - 1)(x^2 + y^2)x - 4(n^2 - 1)(x^2 + n^2y^2) = 0,$$

viz. the point B' is an acnode with the imaginary tangents $x = \pm iny$.

Writing in the equation $y = 0$, we have $x^2 = 0$ the acnode, and

$$x = -\sqrt{(n^2 - 1)}\{\sqrt{(n^2 - 1)} \pm n\},$$

where

$$\sqrt{(n^2 - 1)} = \tfrac{1}{2}\sqrt{\left(Q - \frac{1}{Q}\right)^2}$$

is real and may be taken to be positive; there is thus one root

$$-\sqrt{(n^2 - 1)}\{\sqrt{(n^2 - 1)} + n\},$$

which is negative; the other root, say

$$\sqrt{(n^2 - 1)}\{n - \sqrt{(n^2 - 1)}\},$$

is positive and less than 1; the curve is thus an acnodal limaçon, having B' for the acnode and cutting the axis outside the circle on the negative side of B', and inside the circle between B' and O'.

Taking B' as origin, the equation of the circle is $x^2 + y^2 - 2x = 0$, and we hence find for the intersection of the limaçon with the circle $n^2x = 2(n^2 - 1)$, that is, $x = 2\left(1 - \frac{1}{n^2}\right)$; whence also $y^2 = \frac{4}{n^2}\left(1 - \frac{1}{n^2}\right)$, values which belong to a real intersection; for $q = 0$, we have $Q = 0$, $n = \infty$, and therefore $(x, y) = (2, 0)$, viz. the intersection is at B'; for $q = \infty$, we have $Q = 1$, $n = 1$, and therefore $(x, y) = (0, 0)$, viz. the intersection is at A', results which are obviously right. Observe that for the other limaçon we have $x = 2\left(1 + \frac{1}{m^2}\right)$, $y = -\frac{4}{m^2}\left(1 + \frac{1}{m^2}\right)$, viz. there is no real intersection with the circle.

The Table above referred to.

$\phi = \log\tan(45^\circ + \frac{1}{2}\phi')$.				$\phi = \log\tan(45^\circ + \frac{1}{2}\phi')$.		
ϕ'	ϕ			ϕ'	ϕ	
0°	·0	·0	0°	46°	·8028515	·9062755	51°55′·546
1	0174533	0174542	1 0′·003	47	·8203047	·9316316	53 22 ·714
2	0349066	0349137	2 0 ·024	48	·8377580	·9574669	54 51 ·529
3	0523599	0523838	3 0 ·082	49	·8552113	·9838079	56 22 ·082
4	0698132	0698699	4 0 ·195	50	·8726646	1·0106832	57 54 ·473
5	0872665	0873774	5 0 ·381	51	·8901179	1·0381235	59 28 ·806
6	1047198	1049117	6 0 ·657	52	·9075712	1·0661617	61 5 ·194
7	1221730	1224781	7 1 ·152	53	·9250245	1·0948335	62 43 ·761
8	1396263	1400822	8 1 ·568	54	·9424778	1·1241772	64 24 ·637
9	1570796	1577296	9 2 ·269	55	·9509311	1·1542346	66 7 ·967
10	1745329	1754258	10 3 ·069	56	·9773844	1·1850507	67 53 ·904
11	1919862	1931766	11 4 ·092	57	·9948377	1·2166748	69 42 ·620
12	2094395	2109867	12 5 ·323	58	1·0122910	1·2491606	71 34 ·298
13	2268928	2288650	13 6 ·670	59	1·0297443	1·2825668	73 29 ·140
14	2443461	2468145	14 8 ·486	60	1·0471976	1·3169579	75 27 ·368
15	2617994	2648422	15 10 ·460	61	1·0646508	1·3524048	77 29 ·225
16	2792527	2829545	16 12 ·726	62	1·0821041	1·3889860	79 34 ·639
17	2967060	3011577	17 15 ·304	63	1·0995574	1·4267882	81 44 ·937
18	3141593	3194583	18 18 ·217	64	1·1170107	1·4659083	83 59 ·421
19	3316126	3378629	19 21 ·487	65	1·1344640	1·5064542	86 18 ·808
20	3490659	3563785	20 25 ·139	66	1·1519173	1·5485472	88 43 ·513
21	3665191	3750121	21 29 ·197	67	1·1693706	1·5923237	91 14 ·006
22	3839724	3937710	22 33 ·685	68	1·1868239	1·6379387	93 50 ·819
23	4014257	4126626	23 39 ·630	69	1·2042772	1·6855685	96 34 ·558
24	4188790	4316947	24 44 ·057	70	1·2217305	1·7354152	99 25 ·918
25	4363323	4508753	25 49 ·995	71	1·2391838	1·7877120	102 25 ·701
26	4537856	4702127	26 56 ·472	72	1·2566371	1·8427300	105 34 ·839
27	4712389	4897154	28 3 ·518	73	1·2740904	1·9007867	108 54 ·423
28	4886922	5093923	29 11 ·162	74	1·2915436	1·9622572	112 25 ·744
29	5061455	5292527	30 19 ·437	75	1·3089969	2·0275894	116 10 ·339
30	5235988	5493061	31 28 ·375	76	1·3264502	2·0973240	120 10 ·069
31	5410521	5695627	32 38 ·012	77	1·3439035	2·1721218	124 27 ·205
32	5585054	5900329	33 48 ·383	78	1·3613568	2·2528027	129 4 ·566
33	5759587	6107275	34 59 ·527	79	1·3788101	2·3404007	134 5 ·705
34	5934119	6316581	36 11 ·480	80	1·3962634	2·4362460	139 35 ·197
35	6108652	6528366	37 24 ·287	81	1·4137167	2·5420904	145 39 ·063
36	6283185	6742755	38 37 ·988	82	1·4311700	2·6603061	152 25 ·459
37	6457718	6959880	39 52 ·631	83	1·4486233	2·7942190	160 5 ·818
38	6632251	7179880	41 8 ·261	84	1·4660766	2·9487002	168 56 ·885
39	6806784	7402901	42 24 ·930	85	1·4835299	3·1313013	179 24 ·621
40	6981317	7629095	43 42 ·690	86	1·5009832	3·3546735	192 12 ·518
41	7155850	7858630	45 1 ·597	87	1·5184364	3·6425334	208 42 ·107
42	7330383	8091672	46 21 ·712	88	1·5358897	4·0481254	231 56 ·416
43	7504916	8328406	47 43 ·095	89	1·5533430	4·7413488	271 39 ·557
44	7679449	8569026	49 5 ·814	90	1·5707963	∞	∞
45	·7853982	·8813736	50 29 ·939				

846.

A VERIFICATION IN REGARD TO THE LINEAR TRANSFORMATION OF THE THETA-FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 77—84.]

THE notation is that of Smith's* "Memoir on the Theta and Omega Functions," differing from Hermite's only in a factor, $\vartheta_{m,n}$ (Smith) $= i^{mn}\vartheta_{m,n}$ (Hermite); and it is in consequence of this that the factor $i^{\mu\nu-mn}$ occurs in the expression presently given for δ. Hermite's Memoir "Sur quelques formules relatives à la transformation des fonctions elliptiques" appeared in *Liouville*, t. III. (1858), pp. 27—36.

Writing

$$\omega = \frac{c + d\Omega}{a + b\Omega},$$

where $ad - bc = 1$, the formula of transformation is

$$\vartheta_{\mu,\nu}\left\{(a + b\Omega)\frac{\pi x}{h},\ \Omega\right\} = C\ \exp\left\{-i\pi b\,(a + b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,n}\left(\frac{\pi x}{h},\ \omega\right),$$

where

$$m = a\mu + b\nu + ab,$$
$$n = c\mu + d\nu + cd.$$

We have, according as b is positive or negative,

$$C = \frac{\delta H}{\sqrt{\{-i\,(a + b\Omega)\}}},\ \text{or}\ C = \frac{\delta H}{\sqrt{\{i\,(a + b\Omega)\}}},$$

where for each case the square root is to be taken in such wise that the real part is positive; δ, H are eighth roots of unity; viz. in each case

$$\delta = \exp\left\{-\tfrac{1}{4}\pi i\,(ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2abc\mu + 2abd\nu + ab^2c)\right\} i^{\mu\nu-mn},$$

[* H. J. S. Smith, *Collected Mathematical Papers*, vol. II., pp. 415—621.]

and, according as b is positive or negative,

$$H = \left(\frac{b}{a}\right) i^{-\frac{1}{2}a}, \quad a \text{ odd}, \qquad\Bigg|\qquad H = \left(\frac{-b}{-a}\right) i^{\frac{1}{2}a}, \quad a \text{ odd},$$

$$H = \left(\frac{a}{b}\right) i^{-\frac{1}{2}a-\frac{1}{2}(a-1)(b-1)}, \quad b \text{ odd}, \qquad\Bigg|\qquad H = \left(\frac{-a}{-b}\right) i^{\frac{1}{2}a-\frac{1}{2}(a+1)(b+1)}, \quad b \text{ odd},$$

where $\left(\frac{a}{b}\right)$, &c., are the Legendrian symbols as generalised by Jacobi; if a and b are each of them odd, the formulæ for the cases a odd and b odd respectively are equivalent to each other, and either may be used indifferently.

To fix the ideas, I assume throughout that b is positive and odd; the proper formulæ thus are

$$C = \frac{\delta H}{\sqrt{\{-i(a+b\Omega)\}}},$$

δ as above, and

$$H = \left(\frac{a}{b}\right) i^{-\frac{1}{2}a-\frac{1}{2}(a-1)(b-1)}.$$

I will also, for greater convenience, assume that c is odd; ad is consequently even, viz. the numbers a and d are each of them even, or else one is odd and the other even.

The equation $\omega = \dfrac{c+d\Omega}{a+b\Omega}$ gives $\Omega = \dfrac{c-a\omega}{-d+b\omega}$, and thence

$$a + b\Omega = \frac{-1}{-d+b\omega}.$$

Comparing the expressions for ω, Ω, it appears that we may interchange ω and Ω, writing also $-d$, b, c, $-a$ for a, b, c, d; and changing the other letters, we may for

$$a,\ b,\ c,\quad d,\ \omega,\ \Omega,\ \mu,\ \nu,\ m,\ n,$$

write

$$-d,\ b,\ c,\ -a,\ \Omega,\ \omega,\ m,\ n,\ m',\ n',$$

where

$$m' = -dm + bn - bd,$$

$$n' = \quad cm - an - ac.$$

The formula thus becomes

$$\vartheta_{m,n}\left\{(-d+b\omega)\frac{\pi x}{h},\ \omega\right\} = C' \exp\left\{-i\pi b(-d+b\omega)\frac{x^2}{h^2}\right\} \vartheta_{m',n'}\left(\frac{\pi x}{h},\ \Omega\right),$$

or, for x writing $(a+b\Omega)x$, this is

$$\vartheta_{m,n}\left\{-\frac{\pi x}{h},\ \omega\right\} = C' \exp\left\{i\pi(a+b\Omega)\frac{x^2}{h^2}\right\} \vartheta_{m',n'}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

or, observing that the left-hand side is $=(-)^{mn}\vartheta\left(\frac{\pi x}{h},\ \omega\right)$, we have

$$\vartheta_{m',n'}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=\frac{(-)^{mn}}{C'}\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,n}\left(\frac{\pi x}{h},\ \omega\right),$$

an equation which is of the same form as the original equation of transformation, and is to be identified with it.

We have

$$\begin{aligned} m' &= -dm+bn-bd = -d(a\mu+b\nu+ab) && =-\mu+bd(-a+c-1),\\ &\qquad\qquad +b(c\mu+d\nu+cd)-bd, && \\ n' &= cm-am-ac = c(a\mu+b\nu+ab) && =-\nu+ac(b-d-1);\\ &\qquad\qquad -a(c\mu+d\nu+cd)-ac, && \end{aligned}$$

values which may be written

$$\begin{aligned} m' &= 2P-\mu, \text{ where } P=\tfrac{1}{2}bd(-a+c-1),\\ n' &= 2Q-\nu, \quad \text{,,} \quad Q=\tfrac{1}{2}ac(b-d-1), \end{aligned}$$

P and Q being integers. In fact, P is an integer if b or d is even, and if they are each of them odd, then from the equation $ad-bc=1$, a and c will be one of them odd, the other even, and $-a+c-1$ will be even; and similarly for Q.

The left-hand side is

$$\vartheta_{2P-\mu,\,2Q-\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

which is

$$=(-)^{\mu(Q-\nu)}\vartheta_{\mu,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\},$$

and the equation finally is

$$(-)^{\mu(Q-\nu)}\vartheta_{\mu,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=\frac{(-)^{mn}}{C'}\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,n}\left(\frac{\pi x}{h},\ \omega\right).$$

Comparing this with the original equation

$$\vartheta_{\mu,\nu}\left\{(a+b\Omega)\frac{\pi x}{h},\ \Omega\right\}=C\exp\left\{-i\pi(a+b\Omega)\frac{x^2}{h^2}\right\}\vartheta_{m,n}\left(\frac{\pi x}{h},\ \omega\right),$$

the two equations will be identical if only

$$CC'=(-)^{-\mu Q+\mu\nu-mn}.$$

We have

$$C=\frac{\delta H}{\sqrt{\{-i(a+b\Omega)\}}},\quad C'=\frac{\delta' H'}{\sqrt{\{-i(-d+b\omega)\}}},$$

where the square roots are taken in such wise that the real part is positive; hence

$$CC'=\frac{\delta\delta' HH'}{\sqrt{(+1)}},$$

where $\sqrt{(+1)}$ means

$$\sqrt{\{-i(a+b\Omega)\}}\,.\,\sqrt{\{-i(-d+b\omega)\}},$$

the last-mentioned two square roots being as just explained; and we have moreover

$$\delta = \exp\{-\tfrac{1}{4}i\pi(\quad ac\mu^2 + 2bc\mu\nu + bd\nu^2 + 2abc\mu + 2abd\nu + ab^2c)\}\, i^{\mu\nu-mn},$$
$$\delta' = \exp\{-\tfrac{1}{4}i\pi(-dcm^2 + 2bcmn - abn^2 - 2dbcm + 2abdn - db^2c)\}\, i^{mn-m'n'},$$

viz. the value of δ' is obtained from that of δ by the change $a, b, c, d, \mu, \nu, m, n$ into $-d, b, c, -a, m, n, m', n'$.

Representing these by

$$\delta = \exp\{-\tfrac{1}{4}(i\pi)\Delta\}\, i^{\mu\nu-mn}, \quad \delta' = \exp\{-\tfrac{1}{4}(i\pi)\Delta'\}\, i^{mn-m'n'},$$

we have

$$\delta\delta' = \exp\{-\tfrac{1}{4}i\pi(\Delta+\Delta')\}\, i^{\mu\nu-m'n'}.$$

But

$$m'n' = (2P-\mu)(2Q-\nu), \quad = 4PQ - 2P\nu - 2Q\mu + \mu\nu,$$

that is,

$$\mu\nu - m'n' = -4PQ + 2P\nu + 2Q\mu;$$

or, omitting the term divisible by 4,

$$i^{\mu\nu-m'n'} = i^{2P\nu+2Q\mu}, \quad = (-)^{P\nu+Q\mu}.$$

To calculate $\Delta+\Delta'$, we have

$$dm - bn = \mu + bd(\quad a-c),$$
$$-cm + an = \nu + ac(-b+d),$$

and thence

$$-cdm^2 + (ad+bc)mn - abn^2 = \quad \mu\nu + \mu ac(-b+d) + \nu bd(a-c) + abcd(d-b)(a-c)$$
$$(-ad+bc)mn \qquad = -ac\mu^2 - bd\nu^2 - (ad+bc)\mu\nu - \mu ac(b+d) - \nu bd(a+c) - abcd,$$

consequently

$$-cdm^2 + 2bcmn - abn^2 = -ac\mu^2 - bd\nu^2 - 2bc\mu\nu - 2abc\mu - 2bcd\nu + abcd(2bc - ab - cd);$$

also

$$\begin{aligned} -2bcdm &= -2abcd\mu - 2b^2cd\nu - 2ab^2cd, \\ 2abdn &= 2abcd\mu + 2abd^2\nu + 2abcd^2, \\ -db^2c &= -db^2c, \end{aligned}$$

whence, adding, we obtain

$$\Delta' = -ac\mu^2 - 2bc\mu\nu - bd\nu^2 - 2abc\mu + (-2bcd - 2b^2cd + 2abd^2)\nu$$
$$+ abcd(2bc - ab - cd - 2b + 2d) - db^2c,$$

and, adding to this the expression of Δ, we find

$$\Delta + \Delta' = (2abd - 2bcd - 2b^2cd + 2abd^2)\nu + ab^2c - db^2c + abcd(2bc - ab - cd - 2b + 2d).$$

The coefficient of ν is $=2bd(a-c-bc+ad)=2bd(a-c+1)$, which is $=-4P\nu$. Hence writing

$$\Theta=\tfrac{1}{4}abcd(2bc-ab-cd-2b+2d)+\tfrac{1}{4}b^2c(a-d),$$

where observe that 4Θ, but not in every case Θ itself, is an integer, we have $\Delta+\Delta'=-4P\nu+4\Theta$, and consequently

$$\delta\delta'=(-)^{\Theta+P\nu}(-)^{P\nu+Q\mu},\ =(-)^{\Theta+2P\nu+Q\mu},$$

or, omitting the even number $2P\nu$,

$$\delta\delta'=(-)^{\Theta+Q\mu}.$$

Observe that $(-)^\Theta$ denotes, and it might properly have been written, $\exp i\pi\Theta$. The foregoing equation

$$\frac{\delta\delta' HH'}{\sqrt{(+1)}}=(-)^{-Q\mu+\mu\nu-mn}$$

becomes thus

$$(-)^{\Theta+Q\mu}\frac{HH'}{\sqrt{(+1)}}=(-)^{-Q\mu+\mu\nu-mn},$$

that is,

$$\frac{HH'}{\sqrt{(+1)}}=(-)^{-\Theta-2Q\mu+\mu\nu-mn},$$

where the even term $-2Q\mu$ may be omitted. We have moreover

$$\begin{aligned} mn &= ac\mu^2+(ad+bc)\mu\nu+bd\nu^2+ac(b+d)\mu+bd(a+c)\nu+abcd,\\ -\mu\nu &= \qquad(-ad+bc)\mu\nu,\end{aligned}$$

and thence

$$mn-\mu\nu=ac(\mu^2-\mu)+2bc\mu\nu+bd(\nu^2-\nu)+ac(b+d+1)\mu+bd(a+c+1)\nu+abcd,$$

where each term is even; hence $mn-\mu\nu$ is even, and we have simply

$$\frac{HH'}{\sqrt{(+1)}}=(-)^{-\Theta},$$

where

$$\Theta=\tfrac{1}{4}abcd(ab+cd-2bc+2b-2d)-\tfrac{1}{4}b^2c(a-d).$$

I write $M=\tfrac{1}{4}b(a-d)$, then

$$\Theta-M=\tfrac{1}{4}abcd(ab+cd-2bc+2b-2d)-\tfrac{1}{4}(bc+1)b(a-d),$$

where the second term is $=-\tfrac{1}{4}abd(a-d)$, and we have therefore

$$\Theta-M=\tfrac{1}{4}abd(abc+c^2d-2bc^2+2bc-2cd-a+d).$$

I assume, as above, that b and c are each of them odd; therefore ad is even. I suppose, first, that ad divides by 4, then $\tfrac{1}{4}abd$ is an integer, and in the expression of $\Theta-M$, omitting even numbers, we have

$$\Theta-M=\tfrac{1}{4}abd(abc+c^2d-a+d),$$

which, putting therein $bc = ad - 1$, becomes

$$= \tfrac{1}{4}abd\,(a^2d + c^2d - 2a + d),$$
$$= \tfrac{1}{4}abd\,\{a^2d + (c^2 - c)\,d + (c+1)\,d - 2a\},$$

where inside the { } each term is even; hence $\Theta - M$ is even.

Next, if ad is even but not divisible by 4, then $bc = ad - 1$, which is $\equiv 1$ (mod. 4), thus b and c are

$$= 4\sigma + 1 \text{ and } 4\tau + 1,$$

or else

$$= 4\sigma - 1 \text{ and } 4\tau - 1,$$

and, moreover, $bc = 4\theta + 1$, and $c^2 = 4\phi + 1$; hence

$$\Theta - M = \tfrac{1}{2}b \,.\, \tfrac{1}{2}ad\,\{a\,(4\theta + 1) + d\,(4\phi + 1) - 2b\,(4\phi + 1) + 2\,(4\theta + 1) - 2cd - a + d\},$$

or, omitting even numbers, that is, inside the { } numbers which contain the factor 4, this is

$$= \tfrac{1}{4}abd\,(a + d - 2b + 2 - a + d),$$
$$= \tfrac{1}{2}abd\,(d - b + 1 - cd),$$
$$= \tfrac{1}{2}abd\,\{d\,(1 - c) + 1 - b\},$$

or, since each term within the { } is even, we have in this case also $\Theta - M$ even. And this being so, the foregoing equation for HH' becomes

$$\frac{HH'}{\sqrt{(+1)}} = (-)^{-M}, \;= (-)^{\frac{1}{2}b\,(d-a)}, \text{ or say } = i^{\frac{1}{2}b\,(d-a)}.$$

The values of H, H', refer each of them to a positive odd value of b, and they thus are

$$H = \left(\frac{a}{b}\right) \quad i^{-\frac{1}{2}a - \frac{1}{2}(a-1)\,(b-1)},$$

$$H' = \left(-\frac{d}{b}\right) i^{\frac{1}{2}d - \frac{1}{2}(-d-1)\,(b-1)};$$

hence

$$HH' = \left(\frac{a}{b}\right)\left(\frac{-d}{b}\right) i^{-\frac{1}{2}(a-d) - \frac{1}{2}(a-d-2)\,(b-1)} = \left(\frac{a}{b}\right)\left(\frac{-d}{b}\right) i^{b-1+\frac{1}{2}b\,(d-a)},$$

or, since

$$\left(\frac{-d}{b}\right) = (-)^{\frac{1}{2}(b-1)}\left(\frac{d}{b}\right) = i^{b-1}\left(\frac{d}{b}\right),$$

and $2\,(b-1)$ divides by 4, this is

$$HH' = \left(\frac{a}{b}\right)\left(\frac{d}{b}\right) i^{\frac{1}{2}b\,(d-a)}.$$

Also $\left(\frac{a}{b}\right)\left(\frac{d}{b}\right) = \left(\frac{ad}{b}\right)$, but from the equation $ad - bc = 1$, or $\frac{ad}{b} = c + \frac{1}{b}$, we have $\left(\frac{ad}{b}\right) = \left(\frac{1}{b}\right) = 1$; whence

$$HH' = i^{\frac{1}{2}b\,(d-a)}.$$

We have $\omega = x + iy$, where y is positive; hence

$$-d + b\omega = -d + bx + iby = \alpha + 2by, \text{ if } \alpha = -d + bx;$$

hence

$$a + b\Omega = \frac{-1}{\alpha + iby} = \frac{-\alpha + iby}{\alpha^2 + \beta^2 y^2},$$

$$-i(-d - b\Omega) = by - i\alpha = R(\cos\theta + i\sin\theta),$$

$$-i(\quad a + b\Omega) = \frac{by + i\alpha}{\alpha^2 + \beta^2 y^2} = \frac{R(\cos\theta - i\sin\theta)}{\alpha^2 + \beta^2 y^2},$$

where R is positive; and $\cos\theta$ is positive since y is positive, and thus θ lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. Hence

$$\sqrt{\{-i(-d + b\Omega)\}} = \sqrt{R}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta),$$

$\cos\frac{1}{2}\theta$ being positive,

$$\sqrt{\{-i(\quad a + b\Omega)\}} = \frac{\sqrt{R}(\cos\frac{1}{2}\theta - i\sin\frac{1}{2}\theta)}{\sqrt{(\alpha^2 + \beta^2 y^2)}},$$

and thus

$$\sqrt{\{-i(a + b\Omega)\}}\sqrt{\{-i(-d + b\Omega)\}} = \frac{R}{\sqrt{(\alpha^2 + \beta^2 y^2)}} = +1,$$

that is, $\sqrt{(+1)} = +1$, and we thus have, as we should do,

$$\frac{HH'}{\sqrt{(+1)}} = i^{\frac{1}{2}b(d-a)},$$

the equation which was to be verified.

847.

ON THE THEORY OF SEMINVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 92—107.]

IN my "Mémoire sur les Hyperdéterminants," *Crelle*, t. XXX. (1846), pp. 1—37, [13, 14, 16*], I gave an investigation for the number of and relations between the quartic invariants of a given binary quantic: the very same formulæ apply to the cubic covariants of a given binary quantic, or what is the same thing to the cubic seminvariants of a given weight, and I propose at present (considering the formulæ in this point of view) to further develope the theory in regard to the solution of the systems of linear equations obtained in the memoir in question. But I first reproduce the investigation as it stands: I recall that the notation is the ordinary hyperdeterminant notation: for instance,

$$U = \tfrac{1}{2}(a,\ b,\ c \between x_1,\ y_1)^2, \quad V = \tfrac{1}{2}(a,\ b,\ c \between x_2,\ y_2)^2.$$

$$\begin{aligned} T\overline{12}^2 UV &= (\partial_{x_1}\partial_{y_2} - \partial_{y_1}\partial_{x_2})^2\, UV \\ &= (\xi_1\eta_2 - \eta_1\xi_2)^2\, UV \\ &= \xi_1^2 U\,.\,\eta_2^2 V - 2\xi_1\eta_1 U\,.\,\xi_2\eta_2 V + \eta_1^2 U\,.\,\xi_2^2 V \\ &= \quad a\,.\,c \quad - \quad 2b\,.\,b \quad + \quad c\,.\,a \\ &= 2\,(ac - b^2); \end{aligned}$$

and that, when the variables do not disappear, each set $(x_1,\ y_1)$, $(x_2,\ y_2)$, ... is ultimately replaced by $(x,\ y)$, so that these are the variables of any covariant.

The investigation* is as follows:

"...We pass to the derivatives of the fourth degree, considering the forms in which all the differential coefficients are of the same order. It is easy to see that we may write

$$\begin{aligned} \square\, UVWX &= (\overline{12}\,.\,\overline{34})^\alpha\, (\overline{13}\,.\,\overline{42})^\beta\, (\overline{14}\,.\,\overline{23})^\gamma\, UVWX \\ &= D_{\alpha,\beta,\gamma} UVWX \text{ or } D_{\alpha,\beta,\gamma}. \end{aligned}$$

[* This Collection, vol. I., p. 104: see also, vol. I., p. 117.]

Writing for shortness

$$\overline{12}.\overline{34}=\mathfrak{A},\quad \overline{13}.\overline{42}=\mathfrak{B},\quad \overline{14}.\overline{23}=\mathfrak{C},$$

we have

$$D_{\alpha,\beta,\gamma}=\mathfrak{A}^{\alpha}\mathfrak{B}^{\beta}\mathfrak{C}^{\gamma}UVWX.$$

Suppose $U=V=W=X$, and consider the derivatives which correspond to the same value f of α, β, γ: we have to find how many of these derivatives are independent, and to express the others in terms of these. Since the functions are equal after the differentiations, we may before the differentiations interchange in any manner the symbolic numbers 1, 2, 3, 4: this gives

$$D_{\alpha,\beta,\gamma}=D_{\beta,\gamma,\alpha}=D_{\gamma,\alpha,\beta}=(-)^f D_{\alpha,\gamma,\beta}=(-)^f D_{\gamma,\beta,\alpha}=(-)^f D_{\beta,\alpha,\gamma}.$$

But we have identically

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}=0.$$

Multiplying by $\mathfrak{A}^a\mathfrak{B}^b\mathfrak{C}^c$, and applying it to the product $UVWX$, we have

$$D_{a+1,b,c}+D_{a,b+1,c}+D_{a,b,c+1}=0,$$

which, putting $a+b+c=f-1$, gives a system of equations between the derivatives $D_{\alpha,\beta,\gamma}$ for which $\alpha+\beta+\gamma=f$. Reducing these by the conditions just obtained, suppose that Θf denotes the number of partitions of f into three parts, zero included, and the permutations of the parts being disregarded, we have a number Θf of derivatives $D_{\alpha,\beta,\gamma}$, and a number $\Theta(f-1)$ of linear relations between these derivatives. There remains therefore a number $\Theta f-\Theta(f-1)$ of independent derivatives: but when f is an even number, we have included among these the functions $D_{f,0,0}$, that is, $\overline{12}^f.\overline{34}^f UVWX$, which evidently reduces itself to

$$\overline{12}^f UV.\overline{34}^f WX,\ \text{ or }\ B_f(U,V).B_f(W,X),$$

that is to say, to the square of $B_f(U,V)$. We must therefore diminish by unity this number $\Theta f-\Theta(f-1)$, when f is even. Let $E\left(\frac{a}{b}\right)$ denote the integer part of the fraction $\frac{a}{b}$: it can be shown that the required number is equal to

$$E\left(\frac{f}{6}\right)\ \text{ or }\ E\left(\frac{f+3}{6}\right)$$

according as f is even or odd. Giving to f the six forms

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

we obtain for the independent derivatives of the fourth degree the corresponding numbers

$$g,\quad g,\quad g,\quad g+1,\quad g,\quad g+1;$$

there is for example a single derivative for each of the orders 3, 5, 6, 7, 8, 10, two for each of the orders 9, 11, 12, 13, 14, 16, &c.

We may take for independent derivatives, when f is even, the terms of the series $D_{f-3,\,3,\,0}$, $D_{f-6,\,6,\,0}$, ..., and when f is odd, those of the series $D_{f-1,\,1,\,0}$, $D_{f-4,\,4,\,0}$, $D_{f-7,\,7,\,0}$...: continuing each series until the last term in which the first suffix exceeds or is equal to the second suffix. We have in each case the right number of terms; for example, in the case $f=9$, we take for independent derivatives the two functions D_{810} and D_{540}, and we form the equations

$$\begin{aligned}
D_{900} + D_{810} + D_{801} &= 0, & D_{621} + D_{531} + D_{522} &= 0,\\
D_{810} + D_{720} + D_{711} &= 0, & D_{540} + D_{450} + D_{441} &= 0,\\
D_{720} + D_{630} + D_{621} &= 0, & D_{531} + D_{441} + D_{432} &= 0,\\
D_{711} + D_{621} + D_{612} &= 0, & D_{522} + D_{432} + D_{423} &= 0,\\
D_{630} + D_{540} + D_{531} &= 0, & D_{432} + D_{342} + D_{333} &= 0,
\end{aligned}$$

which are to be reduced by means of the formulæ

$$D_{900} = -D_{900} = 0, \qquad D_{810} = -D_{801}, \text{ \&c.}$$

We will presently give the solution of these equations; but beginning with the second order and going successively to the ninth order, we form easily the following table:

$$\begin{aligned}
D_{200} &= B_2^2,\\
D_{110} &= -\tfrac{1}{2} B_2^2.
\end{aligned}$$

$$\begin{aligned}
D_{300} &= 0,\\
D_{210}&,\\
D_{111} &= 0.
\end{aligned}$$

$$\begin{aligned}
D_{400} &= B_4^2,\\
D_{310} &= -\tfrac{1}{2} B_4^2,\\
D_{220} &= \tfrac{1}{2} B_4^2,\\
D_{211} &= 0.
\end{aligned}$$

$$\begin{aligned}
D_{500} &= 0,\\
D_{410}&,\\
D_{320} &= -D_{410},\\
D_{311} &= 0,\\
D_{221} &= 0,
\end{aligned}$$

$$\begin{aligned}
D_{600} &= B_6^2,\\
D_{510} &= -\tfrac{1}{2} B_6^2,\\
D_{420} &= \tfrac{1}{6} B_6^2 - \tfrac{2}{3} D_{330},\\
D_{411} &= \tfrac{1}{3} B_6^2 + \tfrac{2}{3} D_{330},\\
D_{330}&,\\
D_{321} &= -\tfrac{1}{6} B_6^2 - \tfrac{1}{3} D_{330},\\
D_{222} &= \tfrac{1}{3} B_6^2 + \tfrac{2}{3} D_{330}.
\end{aligned}$$

$$\begin{aligned}
D_{700} &= 0,\\
D_{610}&,\\
D_{520} &= -D_{610},\\
D_{511} &= 0,\\
D_{430} &= D_{610},\\
D_{421} &= 0,\\
D_{331} &= 0,\\
D_{322} &= 0.
\end{aligned}$$

$$\begin{aligned}
D_{800} &= B_8^2,\\
D_{710} &= -\tfrac{1}{2} B_8^2,\\
D_{620} &= \tfrac{1}{6} B_8^2 - \tfrac{2}{3} D_{530},\\
D_{611} &= -\tfrac{1}{3} B_8^2 - \tfrac{2}{3} D_{530},\\
D_{530}&,\\
D_{521} &= -\tfrac{1}{12} B_8^2 - \tfrac{1}{3} D_{530},\\
D_{440} &= -\tfrac{1}{30} B_8^2 - \tfrac{16}{15} D_{530},\\
D_{431} &= \tfrac{1}{30} B_8^2 + \tfrac{1}{15} D_{530},\\
D_{422} &= \tfrac{2}{15} B_8^2 + \tfrac{4}{15} D_{530},\\
D_{332} &= -\tfrac{1}{15} B_8^2 - \tfrac{2}{15} D_{530}.
\end{aligned}$$

$$\begin{aligned}
D_{900} &= 0,\\
D_{810}&,\\
D_{720} &= -D_{810},\\
D_{711} &= 0,\\
D_{630} &= \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540},\\
D_{621} &= \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540},\\
D_{540}&,\\
D_{531} &= -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540},\\
D_{522} &= 0,\\
D_{441} &= 0,\\
D_{432} &= \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540},\\
D_{333} &= 0.
\end{aligned}$$

For any value of f except $f=2$, 3, or 4, the table commences, for f even and f odd respectively, in the following manner:

$$\begin{aligned}
D_{f,\,0,\,6} &= B_f^2, & D_{f,\,0,\,0} &= 0,\\
D_{f-1,\,1,\,0} &= -\tfrac{1}{2} B_f^2, & D_{f-1,\,1,\,0} &\\
D_{f-2,\,2,\,0} &= \tfrac{1}{6} B_f^2 - \tfrac{2}{3} D_{f-3,\,3,\,0}, & D_{f-2,\,2,\,0} &= -D_{f-1,\,1,\,0},\\
D_{f-2,\,1,\,1} &= \tfrac{2}{3} B_f^2 + \tfrac{3}{2} D_{f-3,\,3,\,0}, & D_{f-2,\,1,\,1} &= 0,
\end{aligned}$$

but beyond this I do not know the law of the series."

Before going further, I remark that if $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, instead of the foregoing values, denote respectively

$$\begin{aligned}
\mathfrak{A} &= (x\partial_{x_1} + y\partial_{y_1})\,\overline{23}, &&= (x\xi_1 + y\eta_1)(\xi_2\eta_3 - \xi_3\eta_2),\\
\mathfrak{B} &= (x\partial_{x_2} + y\partial_{y_2})\,\overline{31}, &&= (x\xi_2 + y\eta_2)(\xi_3\eta_1 - \xi_1\eta_3),\\
\mathfrak{C} &= (x\partial_{x_3} + y\partial_{y_3})\,\overline{12}, &&= (x\xi_3 + y\eta_3)(\xi_1\eta_2 - \xi_2\eta_1),
\end{aligned}$$

then identically

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

and the same theory applies to the cubic derivatives $\mathfrak{A}^\alpha\mathfrak{B}^\beta\mathfrak{C}^\gamma . UVW$, that is, to the cubic covariants, or, attending only to the coefficients of the highest powers of x, to the cubic seminvariants.

Or, we may use the Clebschian notation, for instance

$$(a,\ b,\ c \between x,\ y)^2 = (a_0x + a_1y)^2 = (b_0x + b_1y)^2 = \&c.,$$

that is,

$$\begin{aligned}
a_0^2 &= b_0^2 = \ldots = a,\\
a_0a_1 &= b_0b_1 = \ldots = b,\\
a_1^2 &= b_1^2 = \ldots = c,
\end{aligned}$$

viz. in the language of Prof. Sylvester, $a_0, a_1, b_0, b_1, \ldots$ are here *umbræ*. The invariant is

$$\begin{aligned}
(a_0b_1 - a_1b_0)^2 &= a_0^2b_1^2 - 2a_0b_0a_1b_1 + b_0^2a_1^2,\\
&= ac \quad - 2b\,.\,b \quad + ac,\\
&= 2\,(ac - b^2),
\end{aligned}$$

and so in other cases. To apply this directly to the theory of seminvariants, it is more convenient to write

$$(1,\ b,\ c \between x,\ y)^2 = (x + \alpha y)^2 = (x + \beta y)^2 = \&c.,$$

that is,

$$\begin{aligned}
\alpha &= \beta = \ldots = b,\\
\alpha^2 &= \beta^2 = \ldots = c,
\end{aligned}$$

where $\alpha, \beta, \ldots$ are *umbræ*.

The invariant is

$$\begin{aligned}
(\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2\\
&= c - 2b\,.\,b + c\\
&= 2\,(c - b^2);
\end{aligned}$$

or, to take the case of a cubic function

$$(1,\ b,\ c,\ d) = (x+\alpha y)^3 = (x+\beta y)^3 = (x+\gamma y)^3,$$

that is,

$$\begin{aligned} \alpha = \beta = \gamma &= b, \\ \alpha^2 = \beta^2 = \gamma^2 &= c, \\ \alpha^3 = \beta^3 = \gamma^3 &= d, \end{aligned}$$

a seminvariant is

$$\begin{aligned} (\alpha-\beta)^2(\alpha-\gamma) &= \alpha^3 - 2\alpha^2\beta + \alpha\beta^2 - \alpha^2\gamma + 2\alpha\beta\gamma - \beta^2\gamma, \\ &= d - 2bc + bc - bc + 2b^3 - bc, \\ &= d - 3bc + 2b^3, \end{aligned}$$

and so in other cases. Writing here

$$\mathfrak{A} = \beta - \gamma, \quad \mathfrak{B} = \gamma - \alpha, \quad \mathfrak{C} = \alpha - \beta,$$

the general form of a cubic seminvariant of the weight ω is now $\mathfrak{A}^p\mathfrak{B}^q\mathfrak{C}^r$, or say $D_{p,q,r}$, where $p+q+r=\omega$ (the new letters p, q, r being used for the exponents, since α, β, γ are now employed in a different sense; and since f will be required as a coefficient, I have also written ω instead of f for the weight). We have, as before,

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

and we again see that the theory as to the number of and relations between the cubic seminvariants is identical with that of the quartic invariants. Observe also that for ω odd, $D_{\omega,0,0}$ is $=0$, while for ω even, $D_{\omega,0,0}$, $=\mathfrak{A}^\omega$, is a quadric seminvariant, which is of course not to be reckoned among the proper cubic seminvariants; this exactly corresponds to the B_f^2 which is not a proper quartic invariant.

The choice of the functions $D_{f,0,0}$, $D_{f-3,3,0}, \ldots$, or $D_{f-1,1,0}$, $D_{f-4,4,0}, \ldots$, for expressing in terms of them the other functions has its advantages, but also its disadvantages; in what follows, I in effect replace them by $D_{f,0,0}$, $D_{f-2,2,0}, \ldots$ and $D_{f-1,1,0}$, $D_{f-3,3,0}, \ldots$ respectively. And instead of considering the whole series of the functions $D_{p,q,r}$ for a given weight ω, I consider only those of the form $D_{p,q,0}$; these form a single series $D_{\omega,0,0}$, $D_{\omega-1,1,0}$, $D_{\omega-2,2,0}, \ldots$; and for shortness, I call them A, B, C, D, &c. respectively, viz. I write

$$A,\ B,\ C,\ D \ldots = (\mathfrak{A}^\omega,\ \mathfrak{A}^{\omega-1}\mathfrak{B},\ \mathfrak{A}^{\omega-2}\mathfrak{B}^2,\ \mathfrak{A}^{\omega-3}\mathfrak{B}^3, \ldots).$$

Suppose first that ω is odd; we have

$$D_{\omega-q-r,\,q,\,r} = (-)^\omega D_{\omega-q-r,\,r,\,q};$$

and, in particular,

$$D_{\omega-2q,\,q,\,q} = - D_{\omega-2q,\,q,\,q};$$

that is,

$$D_{\omega-2q,\,q,\,q} = 0;$$

that is,

$$\mathfrak{A}^\omega = 0, \quad \mathfrak{A}^{\omega-2}\mathfrak{B}\mathfrak{C} = 0, \quad \mathfrak{A}^{\omega-4}\mathfrak{B}^2\mathfrak{C}^2 = 0, \ldots.$$

Hence

$$A = 0;$$

moreover

$$B+C=\mathfrak{A}^{\omega-2}\mathfrak{B}(\mathfrak{A}+\mathfrak{B})=-\mathfrak{A}^{\omega-2}\mathfrak{B}\mathfrak{C}=0\,;$$

and similarly $C+2D+E=0$, &c.; that is, we have

$$\begin{array}{lll} A & & =0, \\ & B+C & =0, \\ & C+2D+E & =0, \\ & D+3E+3F+G & =0, \\ \text{\&c.} & & \end{array}$$

The readiest way of solving these is to express the other functions in terms of B, D, F, H, &c.; viz. we thus have

	A	B	C	D	E	F	G	H	I	J	K	L	M
B	0	1	−1	0	1	0	−3	0	17	0	−155	0	2073
D				1	−2	0	5	0	−28	0	255	0	−3410
F						1	−3	0	14	0	−126	0	1683
H								1	−4	0	30	0	−396
J										1	−5	0	55
L												1	−6

read according to the columns

$$\begin{array}{lll} A= & 0, & \\ B= & B, & \\ C= & -B, & \\ D= & & D, \\ E= & B-2D, & \\ F= & & F, \\ G= & -3B+5D-3F, & \\ \vdots & & \end{array}$$

and, by way of verification of the numbers, observe that the sum of the numbers in a column is 1 and −1 alternately.

The formulæ are true for any odd value of ω whatever; but they require an explanation, viz. for any finite value of ω, the terms B, D, F,... are not all of them arbitrary. For any given value of ω the number of arbitrary terms is in fact given by what precedes, and it is also known by Captain MacMahon's theorem, viz. the number of cubic seminvariants is = that of the non-unitary symmetric functions $3^\alpha 2^\beta$ of the proper weight $3\alpha+2\beta=\omega$; thus for $\omega=9$, the forms are 3222 and 333; so that there should be only two arbitrary terms.

We have here, writing for shortness 900, 810, &c., instead of D_{900}, D_{810}, &c. respectively,

$$\begin{aligned}
900 &= 0,\\
810 &= B,\\
720 &= -B,\\
630 &= D,\\
540 &= B-2D,\\
450 &= F,\\
360 &= -3B+5D-3F,\\
270 &= H,\\
180 &= 17B-28D+14F-4H,\\
090 &= L.
\end{aligned}$$

But we have $900=0$, $810+180=0$, $720+270=0$, $630+360=0$, $540+450=0$; that is,

$$\begin{aligned}
L &= 0,\\
18B-28D+14F-4H &= 0,\\
-B \qquad\qquad +H &= 0,\\
-3B+6D-3F \qquad &= 0,\\
B-2D+F \qquad &= 0,
\end{aligned}$$

all satisfied by $F=-B+2D$, $H=B$, $L=0$. The proper course is to stop at the equation for 540, viz. the system is

$$\begin{aligned}
900 &= 0,\\
810 &= B,\\
720 &= -B,\\
630 &= D,\\
540 &= B-2D,
\end{aligned}$$

equations which may be considered as expressing the several functions 900, 810, 720, 630, 540 in terms of $B=810$ and $D=630$. They agree, as they should do, with a foregoing result,

$$\begin{aligned}
&900, = 0,\\
&810,\\
&720, = -810,\\
&630, = \tfrac{1}{2}810-\tfrac{1}{2}540,\\
&540,
\end{aligned}$$

and it would of course be possible to express the remaining forms 711, 621, 522, 441, 432, and 333 in terms of $B=810$ and $D=630$. But observe that the speciality of

the present process is that, instead of the whole series of forms 900, 810, 720, 711, 630, 621, 540, 531, 522, 441, 432, 330, we work only with the forms 900, 810, 720, 630, 540, for which the last element is $=0$.

A more simple solution of the system of linear equations in $A, B, C, \ldots$ is the following:

	A	B	C	D	E	F	G	H	I	J	K	L	M
x	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
y				1	-2	3	-4	5	-6	7	-8	9	-10
z						1	-3	6	-10	15	-21	28	-36
w								1	-4	10	-20	35	-56
t										1	-5	15	-35
u												1	-6

read

$$\begin{aligned}
A &= 0,\\
B &= x,\\
C &= -x,\\
D &= x + y,\\
E &= -x - 2y,\\
F &= x + 3y + z,\\
G &= -x - 4y - 3z,
\end{aligned}$$

where $x, y, z, \ldots$ are arbitrary; the numbers in the table are the binomial coefficients with the signs $+$ and $-$ alternately.

We may, it is clear, express $x, y, z, \ldots$ in terms of $B, D, F, \ldots$, viz. we thus have

$$\begin{aligned}
x &= B,\\
y &= -B + D,\\
z &= 2B - 3D + F,\\
&\vdots
\end{aligned}$$

and substituting these values, we reproduce the foregoing expressions of $A, B, C, \ldots$ in terms of $B, D, F, \ldots$.

For a given finite value of ω, we have of course the same number of arbitrary coefficients $x, y, z, \ldots$ as there were of arbitrary coefficients $B, D, F, \ldots$; thus for $\omega = 9$, only x and y are arbitrary, and the remaining coefficients z, &c., are determined in terms of these. Or, what is the same thing, we stop with the equation $E = -x - 2y$, next preceding an equation with the non-arbitrary coefficient z.

For the case ω even, I write

$$\begin{aligned} A + 2B &= A', \\ B + 2C &= B', \\ C + 2D &= C', \\ &\&c.; \end{aligned}$$

and I say that the new functions $A', B', C', \ldots$ are related together precisely in the same manner as are the functions $A, B, C, \ldots$ belonging to an odd weight ω; viz. we have

$$\begin{aligned} A' \qquad\qquad\qquad &= 0, \\ B' + C' \qquad\quad &= 0, \\ C' + 2D' + E' &= 0, \\ \vdots \end{aligned}$$

which being so, the theory is included in what precedes, and there is no occasion to consider it separately.

To prove this, observe that ω being even, we have

$$D_{\omega-q-r,q,r} = D_{\omega-q-r,r,q},$$

that is,

$$\mathfrak{A}^{\omega-q-r}\mathfrak{B}^q\mathfrak{C}^r = \mathfrak{A}^{\omega-q-r}\mathfrak{B}^r\mathfrak{C}^q,$$

whence, in particular,

$$\mathfrak{A}^{\omega-1}\mathfrak{B} = \mathfrak{A}^{\omega-1}\mathfrak{C},\ \mathfrak{A}^{\omega-3}\mathfrak{B}^2\mathfrak{C} = \mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}^2,\ \&c.;$$

we thus have

$$A' = \mathfrak{A}^\omega + 2\mathfrak{A}^{\omega-1}\mathfrak{B},\ = \mathfrak{A}^\omega + \mathfrak{A}^{\omega-1}\mathfrak{B} + \mathfrak{A}^{\omega-1}\mathfrak{C},\ = \mathfrak{A}^{\omega-1}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}),$$

which is $= 0$. Similarly

$$\begin{aligned} B' + C' = B + 3C + 2D &= \mathfrak{A}^{\omega-1}\mathfrak{B} + 3\mathfrak{A}^{\omega-2}\mathfrak{B}^2 + 2\mathfrak{A}^{\omega-3}\mathfrak{B}^3, \\ &= \mathfrak{A}^{\omega-3}\mathfrak{B}(\mathfrak{A}^2 + 3\mathfrak{A}\mathfrak{B} + 2\mathfrak{B}^2),\ = \mathfrak{A}^{\omega-3}\mathfrak{B}(\mathfrak{A} + \mathfrak{B})(\mathfrak{A} + 2\mathfrak{B}), \end{aligned}$$

or, since

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

this is

$$= -\mathfrak{A}^{\omega-2}\mathfrak{B}\mathfrak{C}(\mathfrak{A} + 2\mathfrak{B}),$$

which, in virtue of

$$\mathfrak{A}^{\omega-3}\mathfrak{B}^2\mathfrak{C} = \mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}^2,$$

is

$$= -\mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}),$$

which is $= 0$. And similarly

$$C' + 2D' + E' = 0,\ \&c.$$

I come to a new part of the theory: to fix the ideas, consider the weight $\omega = 27$; and write also

$$a_0, a_1, a_2, a_3, \ldots = 1, b, c, d, e, f, g, h, i, j, k, l, m, \ldots,$$

using, if we please, the suffixed a for the higher terms. The cubic seminvariants are $3^2 2^{12}$, $3^3 2^9$, $3^5 2^6$, $3^7 2^3$, 3^9, viz. the number of them is $=5$; we have thus the 5 terms B, D, F, H, J. Here $B=(\beta-\gamma)^{28}(\gamma-\alpha)=(\alpha-\beta)^{26}(\beta-\gamma)$, or, interchanging the α and β, $=(\alpha-\beta)^{26}(\alpha-\gamma)$, there is an initial term $\alpha^{27}=a_{27}$, and a final term $\alpha^{13}\beta^{13}\gamma=bn^2$, or we may write

$$B=(\alpha-\beta)^{26}(\alpha-\gamma)=a_{27}+bn^2;$$

and similarly for the other forms. We have thus

$$\begin{aligned}
B&=(\alpha-\beta)^{26}(\alpha-\gamma)^1=a_{27}+bn^2,\\
D&=(\alpha-\beta)^{24}(\alpha-\gamma)^3=a_{27}+dm^2,\\
F&=(\alpha-\beta)^{22}(\alpha-\gamma)^5=a_{27}+fl^2,\\
H&=(\alpha-\beta)^{20}(\alpha-\gamma)^7=a_{27}+hk^2,\\
J&=(\alpha-\beta)^{18}(\alpha-\gamma)^9=a_{27}+j^3,
\end{aligned}$$

where the initial term $\alpha^{27}=a_{27}$ has in each case the coefficient unity, but the final terms have each of them the proper numerical coefficient.

It must be possible to form linear combinations

$$\begin{aligned}
&B &&=a_{27} &&+bn^2,\\
&(B,\ D) &&=a_{25}(c+b^2)&&+dm^2,\\
&(B,\ D,\ F) &&=a_{23}(e+c^2)&&+fl^2,\\
&(B,\ D,\ F,\ H) &&=a_{21}(g+d^2)&&+hk^2,\\
&(B,\ D,\ F,\ H,\ J) &&=a_{19}(i+e^2)&&+j^3,
\end{aligned}$$

where $c+b^2$, $e+c^2$, $g+d^2$, $i+e^2$ denote the seminvariants

$$c-b^2,\ e-4bd+3c^2,\ g-6bf+15ce-10d^2,$$
$$i-8bh+28cg-56df+35e^2,$$

respectively. The expressions indicated by their initial and final terms $a_{27}+bn^2$, $a_{25}c+dm^2$, &c., are what I call "columns;" see my paper "Seminvariant Tables," *American Journal of Mathematics*, t. VII. (1885), pp. 59—73, [831], where, however, the theory is not by any means completely developed.

As to this, observe that B, D are each of them of the form $a_{27}+a_{25}b+a_{25}(c, b^2)+\dots$, the proper combination in order to eliminate a_{27} is obviously $B-D$, the term in $a_{25}b$ will then disappear of itself, and the terms in $a_{25}(c, b^2)$ will combine into a term $a_{25}(c-b^2)$, viz. in the function *quâ* seminvariant the coefficient of the highest letter a_{25} will be the seminvariant $c-b^2$. Similarly, in the combination B, D, F, the two coefficients will be determinable so that there are no terms in a_{27}, a_{25}, a_{25} or a_{24}, and this being so, the term in a_{23} will be $a_{23}(e-4bd+3c^2)$, viz. in the function, *quâ* seminvariant, the coefficient of the highest letter a_{23} will be the seminvariant $e-4bd+3c^2$. And similarly for the remaining two linear combinations.

As a partial verification, I write

$$\begin{aligned} B &= (\alpha-\beta)^{26}(\alpha-\gamma) \\ &= \{\alpha^{26}+\beta^{26}-26(\alpha^{25}\beta+\alpha\beta^{25})+325(\alpha^{24}\beta^2+\alpha^2\beta^{24})\ldots\}(\alpha-\gamma), \end{aligned}$$

or, retaining only the terms which have an index at least $=25$, this is

$$\begin{aligned} B = {} & \alpha^{27}+\alpha\beta^{26} - 26(\alpha^{26}\beta+\alpha^2\beta^{25})+325\alpha^{25}\beta^2 \\ & -(\alpha^{26}+\beta^{26})\gamma - 26(\alpha^{25}\beta+\alpha\beta^{25})\gamma, \\ = {} & a_{27}+a_{26}b \\ & -26a_{26}b - 26a_{25}c \\ & +325a_{25}c \\ & -2a_{26}b + 52a_{25}b^2 \\ = {} & a_{27}-27a_{26}b + a_{25}(299c+52b^2); \end{aligned}$$

and similarly

$$\begin{aligned} D &= (\alpha-\beta)^{24}(\alpha-\gamma)^3 \\ &= \{\alpha^{24}+\beta^{24}-24(\alpha^{23}\beta+\alpha\beta^{23})+276(\alpha^{22}\beta^2+\alpha^2\beta^{22})\ldots\}(\alpha^3-3\alpha^2\gamma+3\alpha\gamma^2-\gamma^3), \end{aligned}$$

or, retaining only the terms which have an index at least $=25$, this is

$$\begin{aligned} D = {} & \alpha^{27} - 24\alpha^{26}\beta + 276\alpha^{25}\beta^2 \\ & -3\alpha^{26}\gamma + 72\alpha^{25}\beta\gamma \\ & +3\alpha^{25}\gamma^2 \\ = {} & a_{27} - 24a_{26}b + 276a_{25}c \\ & -3a_{26}b + 72a_{25}b^2 \\ & +3a_{25}c \\ = {} & a_{27} - 27a_{26}b + a_{25}(279c+72b^2), \end{aligned}$$

and thence

$$B-D = a_{25}(20c-20b^2),$$

viz. the coefficient of a_{25} is $=20(c-b^2)$.

But instead of the foregoing forms B, D, F, H, J, we may start with five other forms which are in fact linear combinations of these, viz. these are the forms

$$\begin{aligned} X &= (\alpha-\beta)^{26}(\alpha+\beta-2\gamma) &&= a_{27} &&+bn^2, \\ Y &= (\alpha-\beta)^{24}(\alpha+\beta-2\gamma)(\alpha-\gamma)(\beta-\gamma) &&= a_{26}(c+b^2) &&+dm^2, \\ Z &= (\alpha-\beta)^{22}(\alpha+\beta-2\gamma)(\alpha-\gamma)^2(\beta-\gamma)^2 &&= a_{25}(c+b^2) &&+fl^2, \\ W &= (\alpha-\beta)^{20}(\alpha+\beta-2\gamma)(\alpha-\gamma)^3(\beta-\gamma)^3 &&= a_{24}(e+c^2) &&+hk^2, \\ T &= (\alpha-\beta)^{18}(\alpha+\beta-2\gamma)(\alpha-\gamma)^4(\beta-\gamma)^4 &&= a_{23}(e+c^2) &&+j^3, \end{aligned}$$

where as well the initial as the final terms should have their proper numerical coefficients.

As to this, observe that in Y we have a term $\alpha^{25}(\beta-\gamma)$ which is $=0$, and similarly a term $\beta^{25}(\alpha-\gamma)$ which is $=0$; the highest powers are thus α^{25} and β^{25}, giving a term in a_{25} which can only be $a_{25}(c-b^2)$. In Z we have the terms $\alpha^{25}(\beta-\gamma)^2$, and $\beta^{25}(\alpha-\gamma)^2$, giving the term $a_{25}(c-b^2)$; in W we have the terms $\alpha^{24}(\beta-\gamma)^3$ and $\beta^{24}(\alpha-\gamma)^3$ which are each $=0$, the highest powers thus are α^{23}, β^{23} giving a term in a_{23} which can only be $a_{23}(e-4bd+3c^2)$; and in T we have the terms $\alpha^{23}(\beta-\gamma)^4$ and $\beta^{23}(\alpha-\gamma)^4$, which give the term $a_{23}(e-4bd+3c^2)$.

The combinations which give the columns thus are

$$\begin{aligned}
&X && = a_{27} \qquad\qquad + bn^2,\\
&(Y) && = a_{25}(c+b^2)+dm^2,\\
&(Y,\ Z) && = a_{23}(e+c^2)+fl^2,\\
&(Y,\ Z,\ W) && = a_{21}(g+d^2)+hk^2,\\
&(Y,\ Z,\ W,\ T) && = a_{19}(i+e^2)+j^3.
\end{aligned}$$

I start with the form

$$u=(Y,\ Z,\ W,\ T)=Y+\lambda Z+\mu W+\nu T.$$

We know that it is possible to determine the coefficients λ, μ, ν, in suchwise that the linear function shall be of the proper form $a_{19}(i+e^2)+j^3$; or, what is the same thing, that there shall be no term with an index exceeding 19; the conditions to be satisfied are apparently more than three, but of course the number of independent conditions must be $=3$. We have, in particular, to get rid of all the terms of the second degree which precede $a_{19}i$, viz. the coefficients of $a_{26}c$, $a_{24}d$, $a_{22}e$, $a_{23}f$, $a_{21}g$, $a_{20}h$ must all of them vanish. Now the terms of u which produce terms of the second degree are the terms containing any two but not all three of the symbolic letters α, β, γ, say the biliteral terms; we have for instance $\alpha^{25}\beta^2$, $\alpha^{25}\gamma^2$, $\beta^{25}\alpha^2$, $\beta^{25}\gamma^2$ (there are no terms $\gamma^{25}\alpha^2$ or $\gamma^{25}\beta^2$, since the highest power of γ is γ^3), each of which is $=a_{25}c$. But in the development of u, we may in any biliteral term by an interchange of the letters α, β, γ make the letter of highest index to be α, and the other letter to be β; thus the several terms $\alpha^{25}\beta^2$, $\alpha^{25}\gamma^2$, $\beta^{25}\alpha^2$, $\beta^{25}\gamma^2$ may be each of them converted into $\alpha^{25}\beta^2$, and so in other cases. Imagining this to be done, the term in $a_{25}c$ is simply the term in $\alpha^{25}\beta^2$, and in like manner the term in $a_{24}d$ is the term in $\alpha^{24}\beta^3$, and so in other cases; the condition as to the disappearance of the terms $a_{26}c, \ldots, a_{20}h$ is the condition that the terms in $\alpha^{25}\beta^2$, $\alpha^{24}\beta^3$, $\alpha^{23}\beta^4$, $\alpha^{22}\beta^5$, $\alpha^{21}\beta^6$, $\alpha^{20}\beta^7$ shall all of them disappear. And if, in the function u transformed in the foregoing manner, we write for convenience $\alpha=1$, so that u has become a function of β only (and observe that u will contain the factor β^2), then the condition is that the terms in β^2, β^3, β^4, β^5, β^6, β^7 shall all of them disappear. The conditions may in fact be written $u=0$; viz. it is assumed that u is in the first instance transformed into a function of β as just explained, and the equation is then to be understood as denoting that the coefficients of u are to be so determined that as many as possible of the terms (beginning with that which contains the lowest power of β) shall each of them vanish.

Consider any term of u, for instance the term

$$Y=(\alpha-\beta)^{24}(\alpha+\beta-2\gamma)(\alpha-\gamma)(\beta-\gamma).$$

To obtain the biliteral terms in the proper form 1° we write therein $1=0$, 2° we write $\beta=0$, changing γ into β, 3° we write $\alpha=0$, changing β, γ into α, β; we thus obtain

$$(\alpha-\beta)^{24}(\alpha+\beta)\alpha\beta,$$
$$\alpha^{24}(\alpha-2\beta)(\alpha-\beta)(\quad-\beta),$$
$$\alpha^{24}(\alpha-2\beta)(\quad-\beta)(\alpha-\beta);$$

viz. the second and third terms are identical; the first term requires, however, the factor 2 (for any term $\alpha^p\beta^q$ is accompanied by a corresponding term $\alpha^q\beta^p$), so that, omitting this factor throughout, the terms are

$$(\alpha-\beta)^{24}(\alpha+\beta)\alpha\beta-\alpha^{24}(\alpha-2\beta)(\alpha-\beta)\beta,$$

or, putting therein $\alpha=1$, the terms are

$$(1-\beta)^{24}(1+\beta)\beta-(1-2\beta)(1-\beta)\beta.$$

Similarly from

$$Z=(\alpha-\beta)^{22}(\alpha+\beta-2\gamma)(\alpha-\gamma)^3(\beta-\gamma)^2,$$

we have

$$(1-\beta)^{22}(1+\beta)\beta^3+(1-2\beta)(1-\beta)^3\beta^2,$$

and the like as regards W and T. The result thus is

$$(1+\beta)\beta(1-\beta)^{24}+\lambda(1-\beta)^{22}\beta^2+\mu(1-\beta)^{20}\beta^3+\nu(1-\beta)^{18}\beta^4$$
$$-(1-2\beta)(1-\beta)\beta\{1-\lambda\beta(1-\beta)+\mu\beta^2(1-\beta)^2-\nu\beta^3(1-\beta)^3\}=0,$$

or, as this may be written,

$$\frac{(1+\beta)(1-\beta)^{23}}{1-2\beta}\left\{1+\lambda\frac{\beta}{(1-\beta)^2}+\mu\frac{\beta^2}{(1-\beta)^4}+\nu\frac{\beta^3}{(1-\beta)^6}\right\}$$
$$-1+\lambda(\beta-\beta^2)-\mu(\beta-\beta^2)^2+\nu(\beta-\beta^2)^3=0,$$

viz. each side is to be expanded in ascending powers of β, and the coefficients λ, μ, ν are then to be determined so that as many terms as possible shall vanish.

Expanding, we have

$$(1-20\beta+190\beta^2-1138\beta^3+4808\beta^4-15178\beta^5+\ldots)$$
$$\times\{1+\beta\lambda+\beta^2(2\lambda+\mu)+\beta^3(3\lambda+4\mu+\nu)+\beta^4(4\lambda+10\mu+6\nu)+\beta^5(5\lambda+20\mu+21\nu)+\ldots\}$$
$$-1+\beta\lambda+\beta^2(-\lambda-\mu)+\beta^3(2\mu+\nu)+\beta^4(-3\nu)+\beta^5.3\nu+\ldots=0,$$

or finally, as far as β^5, the equation is

$$0=0+\beta(2\lambda-20)+\beta^2(-19\lambda+190)+\beta^3(153\lambda-14\mu+2\nu-1138)$$
$$+\beta^4(-814\lambda+119\mu-17\nu+4808)+\beta^5(3027\lambda-558\mu+94\nu-15178).$$

We have in the first place $\lambda=10$, which makes the coefficient of β^2 to be $=0$; and we then have

$$-14\mu+2\nu+392=0, \text{ or say } 7\mu-\nu-196=0,$$
$$119\mu-17\nu-3332=0, \qquad 119\mu-17\nu-3332=0,$$
$$-558\mu+94\nu+15092=0, \qquad 272\mu-47\nu-7546=0,$$

where the second equation is equivalent to the first: the three equations give $\mu = \frac{833}{25}$, $\nu = \frac{231}{25}$. I have since found that proceeding one step further, that is, to β^6, the coefficient is $-8310\lambda + 1791\mu - 363\nu + 36942$, viz. putting this $=0$, and for λ substituting its value $=10$, we have $597\mu - 121\nu = 15386$, an equation which is in fact satisfied by the values just obtained for λ and μ. For the function (Y, Z, W) we have the same equations with $\nu = 0$; and therefore $\lambda = 10$, $\mu = 28$; and for the function (Y, Z) the same equations with $\mu = 0$, $\nu = 0$; and therefore $\lambda = 10$. The linear combinations thus are

$$\begin{aligned}
X &= a_{27} + bn^2,\\
Y &= a_{25}(c + b^2) + dm^2,\\
Y + 10Z &= a_{23}(e + c^2) + fl^2,\\
Y + 10Z + 28W &= a_{21}(g + d^2) + hk^2,\\
25Y + 250Z + 823W + 931T &= a_{19}(i + e^2) + j^3,
\end{aligned}$$

where remark that in $Y + 10Z$, by means of the coefficient 10, we make to disappear the two terms $a_{25}c$, $a_{24}d$; in the next function, by means of the coefficients 10 and 28, the four terms $a_{25}c$, $a_{24}d$, $a_{23}e$, $a_{22}f$; and in the last function, by means of the three coefficients, the six terms $a_{25}c$, $a_{24}d$, $a_{23}e$, $a_{22}f$, $a_{21}g$ and $a_{20}h$.

It is possible that a larger number of terms will disappear; but if this is so, the general form shown by the combinations on p. 353 will require modification.

The investigation applies without alteration to any odd weight ω whatever; the condition for the determination of the coefficients $\lambda, \mu, \nu, \ldots$ is obviously

$$\frac{(1+\beta)(1-\beta)^{\omega-4}}{1-2\beta}\left\{1 + \lambda\frac{\beta}{(1-\beta)^2} + \mu\frac{\beta^2}{(1-\beta)^4} + \nu\frac{\beta^3}{(1-\beta)^6} + \ldots\right\}$$
$$-1 + \lambda(\beta - \beta^2) - \mu(\beta - \beta^2)^2 + \nu(\beta - \beta^2)^4 - \ldots = 0.$$

For the case of an even weight ω, the series of functions is

$$\begin{aligned}
X &= (\alpha - \beta)^\omega,\\
Y &= (\alpha - \beta)^{\omega-2}(\alpha - \gamma)(\beta - \gamma),\\
Z &= (\alpha - \beta)^{\omega-4}(\alpha - \gamma)^2(\beta - \gamma)^2,
\end{aligned}$$

and the condition for the determination of the coefficients $\lambda, \mu, \nu, \ldots$ is in like manner found to be

$$(1-\beta)^{\omega-3}\left\{1 + \lambda\frac{\beta}{(1-\beta)^2} + \mu\frac{\beta^2}{(1-\beta)^4} + \nu\frac{\beta^3}{(1-\beta)^6} + \ldots\right\}$$
$$-1 + \lambda(\beta - \beta^2) - \mu(\beta - \beta^2)^2 + \nu(\beta - \beta^2)^2 + \ldots = 0,$$

which is to be understood as explained above.

848.

ON THE TRANSFORMATION OF THE DOUBLE-THETA FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 142—178.]

I PROPOSE to reproduce Hermite's Memoir "Sur la théorie de la transformation des fonctions Abéliennes," *Comptes Rendus*, t. XL. (1855), pp. 249, ..., 784, with some changes of notation and developments. Hermite's functions are even or odd according as we have $\mu q + \nu p$ even or odd; viz. his characteristic is $\begin{pmatrix}\mu, & \nu\\ q, & p\end{pmatrix}$, or the letters p, q, are misplaced; I write, therefore, r instead of p, so as to have the characteristic $\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}$; and then for symmetry it is necessary to interchange the suffixes 2, 3 and the letters c, d; the invariant function of the periods, instead of being as with him

$$\omega_0 v_3 - \omega_3 v_0 + \omega_1 v_2 - \omega_2 v_1,$$

must be taken to be

$$\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1.$$

Moreover, I write A, B for his G, G', so as, instead of

$$(G, H, G' \between x, y)^2,$$

to have in the expressions of the theta-functions the quadric function $(A, H, B \between x, y)^2$; and I alter the arrangement of the memoir so as to separate more completely the preliminary theory from the theory of the transformation.

GENERAL THEORY. Art. Nos. 1 to 21 (*several sub-headings*).

The functions $\Pi\,\{K$, indef. or def.$\}$.

1. Consider a function

$$\Pi\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(x, y)(A, H, B)\{K, \text{indef. or def.}\},$$

having: the characteristic $\begin{pmatrix} \mu, & \nu \\ q, & r \end{pmatrix}$, where the characters μ, ν, q, r are positive or negative integers, which may be taken to have each of them only the values (0, 1) at pleasure, so that the number of functions is 2^4, $=16$;

the arguments (x, y);

the parameters, or conjoint quarter-periods, (A, H, B);

and the potency K, a positive integer;

and which is either indefinite or definite, as will be explained; the function, moreover, contains linearly certain arbitrary constants, the number of them depending on the value of K, as will be explained.

The function may be written $\Pi(x, y)$ or in any other less abbreviated form which may be convenient.

2. The function $\Pi(x, y)\{K \text{ indef.}\}$ is defined by the following four equations:

$$\begin{aligned}
\Pi(x+1, y) &= (-)^{\mu}\,\Pi(x, y),\\
\Pi(x, y+1) &= (-)^{\nu}\,\Pi(x, y),\\
\Pi(x+A, y+H) &= (-)^{q}\,\Pi(x, y)\exp.-2\pi K(2x+A),\\
\Pi(x+H, y+B) &= (-)^{r}\,\Pi(x, y)\exp.-2\pi K(2y+B),
\end{aligned}$$

and the function $\Pi(x, y)\{K \text{ def.}\}$ by the same equations, together with the following fifth equation,

$$\Pi(-x, -y) = (-)^{\mu q+\nu r}\,\Pi(x, y);$$

viz. the definite function is an even function or else an odd function of the arguments according as $\mu q+\nu r$ is even or odd. We may call $\mu q+\nu r$ the index; and the function is then even or odd according as the index is even or odd.

It is perhaps worth noticing that it would be allowable to define a function $\Pi(x, y)\{K, \text{ skew def.}\}$, by the corresponding relation

$$\Pi(-x, -y) = -(-)^{\mu q+\nu r}\,\Pi(x, y),$$

but I do not propose here to develope this notion.

3. The four equations give rise to the following one,

$$\begin{aligned}
&\Pi(x+a_0+Aa_2+Ha_3,\ y+a_1+Ha_2+Ba_3)\\
&\quad = (-)^{\mu a_0+\nu a_1+q a_2+r a_3}\,\Pi(x, y)\times\exp.-2\pi K\{2a_2x+2a_3y+(A, H, B)(a_2, a_3)^2\},
\end{aligned}$$

where a_0, a_1, a_2, a_3 are any positive or negative integers (zero not excluded), and which single equation, in fact, includes the preceding four equations.

4. In regard to the parameters it is to be observed that, if $A, H, B = A_0+i\alpha$, $H_0+i\eta$, $B_0+i\beta$, we must have (α, η, β) a determinate positive quadratic form; viz. this is the necessary and sufficient condition for the convergence of the series for the development of the function.

5. The number of arbitrary constants is for the indefinite function $=K^2$; but for the definite function it is, when K is odd, $=\frac{1}{2}(K^2+1)$; and when K is even, it is $=\frac{1}{2}K^2$, except in the case of a characteristic $\begin{pmatrix}0, & 0\\ q, & r\end{pmatrix}$, when it is $=\frac{1}{2}(K^2+4)$.

In particular, for $K=1$, there is only a single arbitrary constant, which is a mere factor of the function; taking it to be $=1$, as presently explained, we have the 16 theta-functions.

6. The function $\Pi(x, y)$ is developed in a series of exponentials, in the form

$$\Pi(x, y)=\Sigma(-)^{mq+nr}A_{m,n}\exp.\, i\pi\left\{(2m+\mu)x+(2n+\nu)y+\frac{1}{4K}(A, H, B\between 2m+\mu, 2n+\nu)^2\right\},$$

where m and n have each of them all positive and negative integer values (zero not excluded) from $-\infty$ to ∞. In fact, substituting this series in the four equations, they are all of them satisfied if only

$$A_{m+K,n}=A_{m,n}\,;\quad A_{m,n+K}=A_{m,n}.$$

Consequently the following K^2 coefficients remain arbitrary, viz. those with the suffixes

$$\begin{array}{c|ccc|}
 & 0, & 1, \ldots & K-1\,; \\
\hline
0 & ,, & ,, & ,, \\
1 & & & \\
\vdots & & & \\
K-1 & ,, & ,, & ,, \\
\hline
\end{array}$$

and we have for $\Pi(x, y)$ a sum of K^2 terms, each a determinate series multiplied into one of the arbitrary coefficients $A_{0,0}$, $A_{0,1}$, &c. The indefinite function thus contains, as already mentioned, K^2 arbitrary constants.

7. Substituting in the fifth equation, we have for the definite function the further condition

$$A_{-m-\mu,\,-n-\nu}=A_{m,n},$$

which it is clear will be satisfied generally if only it is satisfied by the coefficients in the foregoing set of K^2 coefficients.

8. In the case K odd, we thus reduce the number of arbitrary coefficients to $\frac{1}{2}(K^2+1)$; the mode in which this takes place is best seen by an example. Suppose $K=3$, so that $A_{m+3,n}=A_{m,n}$; $A_{m,n+3}=A_{m,n}$. For the coefficients of the indefinite function, the suffixes are

$$\begin{array}{ccc}
00, & 01, & 02, \\
10, & 11, & 12, \\
20, & 21, & 22.
\end{array}$$

And if we suppose $\mu=0$, $\nu=1$, then the new condition is $A_{-m,\,-n-1}=A_{m,\,n}$, viz. writing down only the suffixes, we thus obtain

$$\begin{array}{lll|lll|lll}
0, & 0=0, & -1 & 1, & 0=-1, & -1 & 2, & 0=-2, & -1, \\
0, & 1=0, & -2 & 1, & 1=-1, & -2 & 2, & 1=-2, & -2, \\
0, & 2=0, & -3 & 1, & 2=-1, & -3 & 2, & 2=-2, & -3,
\end{array}$$

that is,

$$\begin{array}{lll|lll|lll}
0, & 0=0, & 2 & 1, & 0=2, & 2 & 2, & 0=1, & 2, \\
0, & 1=0, & 1 & 1, & 1=2, & 1 & 2, & 1=1, & 1, \\
0, & 2=0, & 0 & 1, & 2=2, & 0 & 2, & 2=1, & 0,
\end{array}$$

viz. one of these equations $0,\ 1=0,\ 1$ is an identity, but the other equations occur each twice; or we have four equations, each of them an equality between two out of the remaining 8 coefficients; the number of arbitrary coefficients is thus $1+\frac{1}{2}(9-1), =5$; and so in general the number is

$$1+\tfrac{1}{2}(K^2-1), \quad =\tfrac{1}{2}(K^2+1).$$

9. When K is even, it is necessary to distinguish between the case $(\mu,\ \nu)=(0,\ 0)$ and the remaining three cases $(\mu,\ \nu)=(1,\ 0)$, $(0,\ 1)$ or $(1,\ 1)$. In the former case, the relation between the coefficients is $A_{-m,\,-n}=A_{m,\,n}$; there are four identities, $0,\ 0=0,\ 0$; $0,\ \frac{1}{2}K=0,\ \frac{1}{2}K$; $\frac{1}{2}K,\ 0=\frac{1}{2}K,\ 0$; $\frac{1}{2}K,\ \frac{1}{2}K=\frac{1}{2}K,\ \frac{1}{2}K$; and the remaining K^2-4 equations occur each twice, that is, we have $\frac{1}{2}(K^2-4)$ equations, each of them an equality between two of the remaining K^2-4 coefficients; the number of arbitrary coefficients is thus $4+\frac{1}{2}(K^2-4), =\frac{1}{2}(K^2+4)$.

In the latter case, there are no identities and the K^2 equations occur each twice, that is, we have $\frac{1}{2}K^2$ equations, each of them an equality between two of the K^2 coefficients; and we thus have $\frac{1}{2}K^2$ arbitrary coefficients.

10. Recapitulating, it thus appears that

for an indefinite function, the number of coefficients $=K^2$;
for a definite function, the number $=\frac{1}{2}(K^2+1)$, K odd;
$=\frac{1}{2}K^2$, K even, and
$(\mu,\ \nu)=(1,\ 0)$, $(0,\ 1)$ or $(1,\ 1)$;
$=\frac{1}{2}(K^2+4)$, K even, and
$(\mu,\ \nu)=(0,\ 0)$.

The Theta-functions.

11. In the particular case $K=1$, the distinction between the indefinite function and the definite function disappears, and we have instead of $\Pi(x,\ y)$, the theta-functions $\Theta(x,\ y)$, satisfying the four equations

$$\begin{aligned}
\Theta(x+1,\ y) &=(-)^{\mu}\,\Theta(x,\ y),\\
\Theta(x,\ y+1) &=(-)^{\nu}\,\Theta(x,\ y),\\
\Theta(x+A,\ y+H) &=(-)^{\varrho}\,\Theta(x,\ y)\exp.-i\pi(2x+A),\\
\Theta(x+H,\ y+B) &=(-)^{\tau}\,\Theta(x,\ y)\exp.-i\pi(2y+B),
\end{aligned}$$

and the fifth equation

$$\Theta(-x, -y) \qquad = (-)^{\mu q+\nu r}\,\Theta(x, y);$$

as before, $\mu q+\nu r$ is the index of the function.

The four equations are all of them included in the following one:

$$\Theta(x+a_0+Aa_2+Ha_3,\ y+a_1+Ha_2+Ba_3) = (-)^{\mu a_0+\nu a_1+q a_2+r a_3}\,\Theta(x, y)$$
$$\times \exp. -i\pi\,\{2a_2x+2a_3y+(A,\ H,\ B\between a_2,\ a_3)^2\},$$

where a_0, a_1, a_2, a_3 are each of them any positive or negative integer, zero not excluded.

Moreover, we take $A_{0,0}=1$, and the value of the function thus is

$$\Theta(x, y) = \Sigma\,(-)^{mq+nr}\exp. i\pi\,\{(2m+\mu)\,x+(2n+\nu)\,y+\tfrac{1}{4}\,(A,\ H,\ B\between 2m+\mu,\ 2n+\nu)^2\}.$$

12. The sum of two characteristics is the characteristic obtained by taking the sums of the component terms or characters,

$$\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}+\begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix}=\begin{pmatrix}\mu+\mu', & \nu+\nu'\\ q+q', & r+r'\end{pmatrix};$$

and similarly for any number of characteristics. I use the sign $=$, but this properly denotes a congruence, mod. 2; and the like as regards the indices.

The sum of two identical characteristics, or generally of any number of characteristics taken each of them any even number of times, is $=\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$. And this characteristic $\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ may be called the characteristic 0.

It should be observed, that the index of the sum is not in general equal to the sum of the indices. To make it so, we must have, for two characteristics,

$$(\mu+\mu')(q+q')+(\nu+\nu')(r+r')=\mu q+\nu r+\mu' q'+\nu' r',$$

that is,

$$\mu q'+\mu' q+\nu r'+\nu' r=0;$$

and there is obviously a like formula for the case of more than two characteristics.

Two or more characteristics, such that they have the sum of the indices equal to the index of the sum, are said to be "in direct relation" or "directly related" to each other. The sum of the indices and the index of the sum may differ by unity and we then have the inverse relation; but I do not propose to consider this.

13. Consider any number K of theta-functions, of the same arguments and parameters, but with the same or different characteristics. The product of these functions is in general a function $\Pi(x, y)\,\{K \text{ indef.}\}$, having a characteristic which is $=$ the sum of the characteristics of the theta-functions. In fact, from the four

equations of the theta-functions, we at once obtain for their products $\Pi(x, y)$ the four equations

$$\begin{aligned}
\Pi(x+1, y) &= (-)^{\Sigma\mu}\,\Pi(x, y),\\
\Pi(x, y+1) &= (-)^{\Sigma\nu}\,\Pi(x, y),\\
\Pi(x+A, y+H) &= (-)^{\Sigma q}\,\Pi(x, y)\exp. - i\pi K(2x+A),\\
\Pi(x+H, y+B) &= (-)^{\Sigma r}\,\Pi(x, y)\exp. - i\pi K(2y+B),
\end{aligned}$$

which proves the theorem.

14. But if the indices are in direct relation to each other, then we have further

$$\Pi(-x, -y) = (-)^{\Sigma\mu\Sigma q+\Sigma\nu\Sigma r}\,\Pi(x, y),$$

and the product is thus a function $\Pi(x, y)$ $\{K, \text{def.}\}$.

15. Take the square of a theta-function, the characteristic is $=\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ or 0, and we have also twice the index $=0$; viz. the theta-function is in direct relation with itself. Hence the squared function is a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}(x, y)$ $\{2, \text{def.}\}$, and as such it contains linearly $\frac{1}{2}(2^2+4), =4$ arbitrary constants. Hence, taking any five squares, since each of them is a function of the form in question, it follows that the squares of the 5 theta-functions are connected by a linear relation.

Göpel's relation between 4 theta-functions.

16. We may in a variety of ways (in fact, in 60 ways, as will presently be shown) select four theta-functions, all of them even, or else two of them even and two odd (that is, having the sum of their indices $=0$), such that the sum of their characteristics is $=0$; for instance, the functions may be

$$P' = \begin{pmatrix}0, & 0\\ 1, & 1\end{pmatrix},\quad P'' = \begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix},\quad S' = \begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix},\quad S'' = \begin{pmatrix}1, & 1\\ 1, & 0\end{pmatrix},$$

indices

$$0\ ,\qquad 0\ ,\qquad 1\ ,\qquad 1\ .$$

The functions are thus in direct relation, and the product of the four functions is a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ $\{4, \text{def.}\}$. But obviously any one of the functions taken four times, or any two of them taken each twice, are in like manner four functions in direct relation, or the fourth powers P'^4, P''^4, S'^4, S''^4, and the squared products $P'^2P''^2$, $S'^2S''^2$, $P'^2S'^2$, $P'^2S''^2$, $P''^2S'^2$, $P''^2S''^2$, are in like manner each of them a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ $\{4, \text{def.}\}$, viz. we have thus in all $1+4+6, =11$ such functions. But the Π function contains only $\frac{1}{2}(4^2+4), =10$ arbitrary constants; hence there must be a linear relation between the 11 powers and products, and this is Göpel's relation.

17. Starting with any two characteristics a, b at pleasure, the remaining characteristics form seven pairs, such that

$$a+b=c+d=e+f=g+h=i+j=k+l=m+n=o+p;$$

but among the seven pairs we have only three, suppose $c+d$, $e+f$, $g+h$, which are such that (a, b, c, d), (a, b, e, f), (a, b, g, h) are each of them either all even or else two even and two odd; that is, starting with any pair (a, b), we have these three tetrads having each of them the required property. The number of pairs (a, b) is $\frac{1}{2}16.15, =120$; and we thence derive $120\times 3, =360$ tetrads; but each such tetrad is of course derivable from any one of the six pairs contained in it; or the number of distinct tetrads is $\frac{1}{6}360, =60$, viz. we have, as mentioned above, 60 Göpel-tetrads.

The four functions Π_0, Π_1, Π_2, Π_3.

18. We consider four theta-functions θ_0, θ_1, θ_2, θ_3, which are such that to the modulus 2, the sum of the characters is $\equiv 0$, and also the sum of the indices is $\equiv 0$; taking the characters to be

$$\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix},\quad \begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix},\quad \begin{pmatrix}\mu'', & \nu''\\ q'', & r''\end{pmatrix},\quad \begin{pmatrix}\mu''', & \nu'''\\ q''', & r'''\end{pmatrix},$$

and writing throughout $=$ for $\equiv$ (mod. 2), we have

$$\begin{aligned}\mu+\mu'+\mu''+\mu'''&=0,\\ \nu+\nu'+\nu''+\nu'''&=0,\\ q+q'+q''+q'''&=0,\\ r+r'+r''+r'''&=0,\\ \mu q+\nu r+\mu' q'+\nu' r'+\mu'' q''+\nu'' r''+\mu''' q'''+\nu''' r'''&=0.\end{aligned}$$

Writing for shortness $(01)=\mu q'+\mu' q+\nu r'+\nu' r$, and so in other cases; and further $(01)+(02)+(12)=(012)$, &c., then substituting for μ''', ν''', q''', r''' their values from the first four equations, we deduce $(012)=0$; and similarly $(013)=0$, $(023)=0$, $(123)=0$.

19. Consider now a product $\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}}$, where $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}$ is $=$ a given odd number k; the characteristic is

$$\begin{pmatrix}\mu\mathrm{a}+\mu'\mathrm{b}+\mu''\mathrm{c}+\mu'''\mathrm{d}, & \nu\mathrm{a}+\nu'\mathrm{b}+\nu''\mathrm{c}+\nu'''\mathrm{d}\\ q\mathrm{a}+q'\mathrm{b}+q''\mathrm{c}+q'''\mathrm{d}, & r\mathrm{a}+r'\mathrm{b}+r''\mathrm{c}+r'''\mathrm{d}\end{pmatrix},$$

and it hence follows that the index is

$$=\mathrm{a}(\mu q+\nu r)+\mathrm{b}(\mu' q'+\nu' r')+\mathrm{c}(\mu'' q''+\nu'' r'')+\mathrm{d}(\mu''' q'''+\nu''' r''').$$

In fact, forming the index in question, we have first terms in a^2, b^2, c^2, d^2, which upon writing therein a, b, c, d for these values respectively ($\mathrm{a}^2=\mathrm{a}$, &c.) give the

required value; we have therefore only to show that the sum of the remaining terms in ab, &c., is $=0$. These terms are

$$\mathrm{ab}\,(01)+\mathrm{ac}\,(02)+\mathrm{ad}\,(03)+\mathrm{bc}\,(12)+\mathrm{bd}\,(13)+\mathrm{cd}\,(23),$$

and writing herein $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=1$, and thence $\mathrm{ad}=\mathrm{a}(1-\mathrm{a}-\mathrm{b}-\mathrm{c})$, $=\mathrm{ab}+\mathrm{ac}$, and similarly $\mathrm{bd}=\mathrm{ab}+\mathrm{bc}$, $\mathrm{cd}=\mathrm{ac}+\mathrm{bc}$, the terms in question become

$$=\mathrm{ab}\,(013)+\mathrm{ac}\,(023)+\mathrm{bc}\,(123),$$

that is, they become $=0$.

20. We thus see that the function $\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}}$ has an index which is

$$=\mathrm{a}\,\mathrm{ind}\,\theta_0+\mathrm{b}\,\mathrm{ind}\,\theta_1+\mathrm{c}\,\mathrm{ind}\,\theta_2+\mathrm{d}\,\mathrm{ind}\,\theta_3.$$

Consider separately four products $\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}}$, in which the exponents a, b, c, d satisfy successively the relations (always to modulus 2)

$$\begin{array}{ll} \mathrm{b}+\mathrm{d}=0, & \mathrm{c}+\mathrm{d}=0, \\ \mathrm{b}+\mathrm{d}=1, & \mathrm{c}+\mathrm{d}=0, \\ \mathrm{b}+\mathrm{d}=0, & \mathrm{c}+\mathrm{d}=1, \\ \mathrm{b}+\mathrm{d}=1, & \mathrm{c}+\mathrm{d}=1. \end{array}$$

Combining herewith the relation $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=1$, it follows that the exponents a, b, c, d are

$$\begin{array}{llll} =\mathrm{d}+1, & \mathrm{d}\ , & \mathrm{d}\ , & \mathrm{d}\ , \\ \mathrm{d}\ , & \mathrm{d}+1, & \mathrm{d}\ , & \mathrm{d}\ , \\ \mathrm{d}\ , & \mathrm{d}\ , & \mathrm{d}+1, & \mathrm{d}\ , \\ \mathrm{d}\ , & \mathrm{d}\ , & \mathrm{d}\ , & \mathrm{d}+1, \end{array}$$

in the four cases respectively. Then substituting these values, the characteristics become

$$\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}, \quad \begin{vmatrix}\mu', & \nu' \\ q', & r'\end{vmatrix}, \quad \begin{vmatrix}\mu'', & \nu'' \\ q'', & r''\end{vmatrix}, \quad \begin{vmatrix}\mu''', & \nu''' \\ q''', & r'''\end{vmatrix},$$

viz. the four products have the same characters as θ_0, θ_1, θ_2, θ_3 respectively; and in like manner recollecting that

$$\mathrm{ind}\,\theta_0+\mathrm{ind}\,\theta_1+\mathrm{ind}\,\theta_2+\mathrm{ind}\,\theta_3=0,$$

we see that the four products have the same indices as θ_0, θ_1, θ_2, θ_3 respectively.

More generally write Π_0, Π_1, Π_2, $\Pi_3=\Sigma\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}}$, where for the four cases respectively the exponents a, b, c, d satisfy the conditions already referred to; then Π_0, Π_1, Π_2, Π_3 have the same characteristics, and the same indices, as θ_0, θ_1, θ_2, θ_3 respectively.

21. It can be shown that each of the functions Π contains $\frac{1}{2}(k^2+1)$ constants. It will be recollected that we have between θ_0, θ_1, θ_2, θ_3 an equation of the form

$$0=(\theta_0^4,\ \theta_1^4,\ \theta_2^4,\ \theta_3^4,\ \theta_0^2\theta_1^2,\ \theta_0^2\theta_2^2,\ \theta_0^2\theta_3^2,\ \theta_1^2\theta_2^2,\ \theta_1^2\theta_3^2,\ \theta_2^2\theta_3^2,\ \theta_0\theta_1\theta_2\theta_3);$$

this serves to express, say θ_3^4, in lower powers of θ_3; and by successive applications of this equation, we can reduce $\Sigma\theta_0^a\theta_1^b\theta_2^c\theta_3^d$ to a form in which d has only one of the values 0, 1, 2 or 3; we do not by this transformation alter the suffix of Π, viz. a term originally of the form Π_0, Π_1, Π_2 or Π_3, will by the transformation give rise only to terms which are of the same form Π_0, Π_1, Π_2, Π_3 (as the case may be). The number of constants in Π_0 is thus

$$= \text{number of partitions of } k \text{ into four parts a, b, c, d,}$$

under the conditions

$$\mathrm{d} = 0 \text{ or } 2;\ \mathrm{a} \text{ odd},\ \mathrm{b}, \mathrm{c} \text{ each even},$$
$$\mathrm{d} = 1 \text{ or } 3;\ \mathrm{a} \text{ even},\ \mathrm{b}, \mathrm{c} \text{ each odd},$$

where, in reckoning the partitions, the order of the parts is taken into account: the partitions are thus as follows

$$\begin{aligned}
\mathrm{d} = 0,\quad & (\mathrm{a}-1) + \mathrm{b} + \mathrm{c} = k-1,\\
\mathrm{d} = 1,\quad & \mathrm{a} + (\mathrm{b}-1) + (\mathrm{c}-1) = k-3,\\
\mathrm{d} = 2,\quad & (\mathrm{a}-1) + \mathrm{b} + \mathrm{c} = k-3,\\
\mathrm{d} = 3,\quad & \mathrm{a} + (\mathrm{b}-1) + (\mathrm{c}-1) = k-5,
\end{aligned}$$

where the parts a or $(\mathrm{a}-1)$, b or $(\mathrm{b}-1)$, c or $(\mathrm{c}-1)$, as the case may be, are all of them even; hence, writing $k' = \frac{1}{2}(k-1)$, the cases are

$$\mathrm{a}' + \mathrm{b}' + \mathrm{c}' = k',\ k'-1,\ k'-1, \text{ or } k'-2,$$

where the a′, b′, c′ are odd or even (zero not excluded) at pleasure; as already mentioned, the order of the parts is taken into account: thus the particulars of 3 would be

$$\begin{array}{llll}
300, & 210, & 120, & 030, \quad \text{No. is } 10, = \tfrac{1}{2}4\,.\,5.\\
 & 201, & 111, & 021,\\
 & & 102, & 012,\\
 & & & 033.
\end{array}$$

Hence, in the four cases respectively, the numbers are

$$\begin{aligned}
&\tfrac{1}{2}(k'^2 + 3k' + 2),\\
&\tfrac{1}{2}(k'^2 + k'),\\
&\tfrac{1}{2}(k'^2 + k'),\\
&\tfrac{1}{2}(k'^2 - k'),
\end{aligned}$$

giving a total

$$= \tfrac{1}{2}(4k'^2 + 4k' + 2),\ = \tfrac{1}{2}\{(2k'+1)^2 + 1\},$$

that is, $= \frac{1}{2}(k^2+1)$. And similarly the number is $= \frac{1}{2}(k^2+1)$ in the other three cases respectively.

PREPARATION FOR THE TRANSFORMATION. Art. Nos. 22 to 44 (*several sub-headings*).

The Hermitian quartic matrix.

22. Observing that, for the adopted form of the Π- or Θ-functions, the periods

$$\begin{array}{lll} \omega_0,\ v_0 & \text{are} & 1,\ 0, \\ \omega_1,\ v_1 & & 0,\ 1, \\ \omega_2,\ v_2 & & A,\ H, \\ \omega_3,\ v_3 & & H,\ B, \end{array}$$

so that we have

$$\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1, \quad = H - 0 + 0 - H, = 0,$$

we have to consider the automorphic transformation of the bilinear form

$$\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1.$$

23. We write

$$(\omega_0,\ \omega_1,\ \omega_2,\ \omega_3) = \left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array} \right) (\Omega_0,\ \Omega_1,\ \Omega_2,\ \Omega_3),$$

$$(v_0,\ v_1,\ v_2,\ v_3) = \left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array} \right) (\Upsilon_0,\ \Upsilon_1,\ \Upsilon_2,\ \Upsilon_3),$$

and the coefficients are assumed to be such that we have identically

$$\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1 = k\,(\Omega_0 \Upsilon_2 - \Omega_2 \Upsilon_0 + \Omega_1 \Upsilon_3 - \Omega_3 \Upsilon_1),$$

where k is in the sequel taken to be a positive integer. We obtain by direct substitution the value of $\omega_0 v_2 - \omega_3 v_0 + \omega_1 v_3 - \omega_3 v_1$ in the following form:

	Ω_0	Ω_1	Ω_2	Ω_3
Υ_0	$a_0c_0 - c_0a_0 + b_0d_0 - d_0b_0$	$a_1c_0 - c_1a_0 + b_1d_0 - d_1b_0$	$a_2c_0 - c_2a_0 + b_2d_0 - d_2b_0$	$a_3c_0 - c_3a_0 + b_3d_0 - d_3b_0$
Υ_1	$a_0c_1 - c_0a_1 + b_0d_1 - d_0b_1$	$a_1c_1 - c_1a_1 + b_1d_1 - d_1b_1$	$a_2c_1 - c_2a_1 + b_2d_1 - d_2b_1$	$a_3c_1 - c_3a_1 + b_3d_1 - d_3b_1$
Υ_2	$a_0c_2 - c_0a_2 + b_0d_2 - d_0b_2$	$a_1c_2 - c_1a_2 + b_1d_2 - d_1b_2$	$a_2c_2 - c_2a_2 + b_2d_2 - d_2b_2$	$a_3c_2 - c_3a_2 + b_3d_2 - d_3b_2$
Υ_3	$a_0c_3 - c_0a_3 + b_0d_3 - d_0b_3$	$a_1c_3 - c_1a_3 + b_1d_3 - d_1b_3$	$a_2c_3 - c_2a_3 + b_2d_3 - d_2b_3$	$a_3c_3 - c_3a_3 + b_3d_3 - d_3b_3$

viz. equating this to its value $k(\Omega_0\Upsilon_2-\Omega_2\Upsilon_0+\Omega_1\Upsilon_3-\Omega_3\Upsilon_1)$, we have 4 identities and 12 equations which are, in fact, 6 equations occurring each twice. We have thus six equations, which are the conditions in order that the matrix may be the matrix of automorphic transformation of the bilinear form $\omega_0 v_2-\omega_2 v_0+\omega_1 v_3-\omega_3 v_1$. The six equations may be written

$$\begin{aligned}(ac+bd)_{01}&=0,\\(ac+bd)_{02}&=k,\\(ac+bd)_{03}&=0,\\(ac+bd)_{12}&=0,\\(ac+bd)_{13}&=k,\\(ac+bd)_{23}&=0,\end{aligned}$$

viz. the first of these equations is $a_0c_1-a_1c_0+b_0d_1-b_1d_0=0$, and so in other cases. It is convenient to remark that each interchange of two letters a and c, b and d, also interchange of the suffixes 0 and 2, produces a change of sign; thus the second equation may be written $(ca+db)_{02}=-k$, or $(ca+db)_{20}=k$.

24. The inverse matrix is found to be

$$\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3\\ b_0, & b_1, & b_2, & b_3\\ c_0, & c_1, & c_2, & c_3\\ d_0, & d_1, & d_2, & d_3\end{array}\right)^{-1}=\frac{1}{k}\left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2\\ c_3, & d_3, & -a_3, & -b_3\\ -c_0, & -d_0, & a_0, & b_0\\ -c_1, & -d_1, & a_1, & b_1\end{array}\right);$$

and the determinant of each of the matrices in this formula is $=k^2$.

We have thus

$$k(\Omega_0,\ \Omega_1,\ \Omega_2,\ \Omega_3)=\left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2\\ c_3, & d_3, & -a_3, & -b_3\\ -c_0, & -d_0, & a_0, & b_0\\ -c_1, & -d_1, & a_1, & b_1\end{array}\right)(\omega_0,\ \omega_1,\ \omega_2,\ \omega_3);$$

and the like formula for the Υ, v. Substituting these values in the equation

$$k(\Omega_0\Upsilon_2-\Omega_2\Upsilon_0+\Omega_1\Upsilon_3-\Omega_3\Upsilon_1)=\omega_0 v_2-\omega_2 v_0+\omega_1 v_3-\omega_3 v_1,$$

we obtain 6 new equations, which are in a different form the conditions for the automorphic transformation.

The 6 new equations may be written

$$\begin{aligned}(02+13)_{cd}&=0,\\(02+13)_{ac}&=k,\\(02+13)_{bc}&=0,\\(02+13)_{ad}&=0,\\(02+13)_{bd}&=k,\\(02+13)_{ab}&=0,\end{aligned}$$

viz. these equations are

$$c_0d_2-c_2d_0+c_1d_3-c_3d_1=0,\ \&c.$$

25. It is worth while to show how the foregoing formula for the inverse matrix comes out. Take for instance the diagonal minor

$$\begin{vmatrix} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix},$$

this is

$$= c_1 (db)_{23} + c_2 (db)_{31} + c_3 (db)_{12},$$

which is

$$= c_1 (ac)_{23} + c_2 \{k + (ac)_{31}\} + c_3 (ac)_{12},$$
$$= c_2 k,$$

since the remaining terms destroy each other. And dividing by the determinant, which is $= k^2$, we have the term $c_2 \div k$ of the inverse matrix.

The Symmetrical Hermitian Matrix.

26. We may consider a symmetrical Hermitian matrix, say the matrix

$$\left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array} \right),$$

viz. we have

$$\begin{aligned} \mathfrak{A}\mathfrak{C} - \mathfrak{G}^2 + \mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} &= \phi, \\ \mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} + \mathfrak{B}\mathfrak{D} - \mathfrak{M}^2 &= \phi, \\ \mathfrak{A}\mathfrak{F} - \mathfrak{G}\mathfrak{H} + \mathfrak{H}\mathfrak{M} - \mathfrak{B}\mathfrak{L} &= 0, \\ \mathfrak{A}\mathfrak{N} - \mathfrak{L}\mathfrak{G} + \mathfrak{H}\mathfrak{D} - \mathfrak{L}\mathfrak{M} &= 0, \\ \mathfrak{C}\mathfrak{H} - \mathfrak{F}\mathfrak{G} + \mathfrak{B}\mathfrak{N} - \mathfrak{M}\mathfrak{F} &= 0, \\ \mathfrak{G}\mathfrak{N} - \mathfrak{C}\mathfrak{L} + \mathfrak{F}\mathfrak{D} - \mathfrak{M}\mathfrak{N} &= 0. \end{aligned}$$

The characteristic property is that, effecting a Hermitian transformation, we have a new symmetrical Hermitian matrix

$$\left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array} \right) (a_0 x + a_1 y + a_2 z + a_3 w,\ b_0 x + \ldots,\ c_0 x + \ldots,\ d_0 x + \ldots)^2$$

$$= \left(\begin{array}{cccc} \mathfrak{A}', & \mathfrak{H}', & \mathfrak{G}', & \mathfrak{L}' \\ \mathfrak{H}', & \mathfrak{B}', & \mathfrak{F}', & \mathfrak{M}' \\ \mathfrak{G}', & \mathfrak{F}', & \mathfrak{C}', & \mathfrak{N}' \\ \mathfrak{L}', & \mathfrak{M}', & \mathfrak{N}', & \mathfrak{D}' \end{array} \right) (x, y, z, w)^2.$$

In fact, the new matrix is

$$(\mathfrak{A}'\ldots)=\left(\begin{array}{cccc} a_0, & b_0, & c_0, & d_0 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_3, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{array}\right)(\mathfrak{A}\ldots)\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right),$$

where the first factor, *quà* transposed matrix, is Hermitian. Hence the product is a symmetrical $k^2\phi$ matrix.

27. Write

$$\Delta_0 = H_0^2 - A_0B_0 - (\eta^2 - \alpha\beta),\quad \theta = \alpha\beta - \eta^2,$$
$$\delta = 2H_0\eta - \alpha B_0 - \beta A_0,$$

and consider the following matrix

$$\left(\begin{array}{cccc} \beta, & -\eta, & -\eta H_0 + \beta A_0, & -\eta B_0 + \beta H_0 \\ -\eta, & \alpha, & -\eta A_0 + \alpha H_0, & -\eta H_0 + \alpha B_0 \\ -\eta H_0 + \beta A_0, & -\eta A_0 + \alpha H_0, & +\alpha\Delta_0 - \delta A_0, & -\delta H_0 + \eta\Delta_0 \\ -\eta B_0 + \beta H_0, & -\eta H_0 + \alpha B_0, & -\delta H_0 + \eta\Delta_0, & +\beta\Delta_0 - \delta B_0 \end{array}\right)\between(x, y, z, w)^2$$

$$= (\beta, -\eta, \alpha\between x + A_0z + H_0\omega_1 y + H_0z + B_0w)^2 - (\eta^2 - \alpha\beta)(\alpha z^2 + 2\eta zw + \beta w^2);$$

it is easily shown that this is a symmetrical θ^2-matrix. In fact, representing it for a moment by

$$\left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array}\right),$$

so that $\mathfrak{A} = \beta$, $\mathfrak{B} = \alpha$, $\mathfrak{H} = -\eta$, &c., we have

$$\mathfrak{A}\mathfrak{C} - \mathfrak{G}^2 + \mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} = \beta(\alpha\Delta_0 - \delta A_0) - (-\eta H_0 + \beta A_0)^2$$
$$- \eta(-\delta H_0 + \eta\Delta_0) - (-\eta B_0 + \beta H_0)(-\eta A_0 + \alpha H_0),$$

which is

$$= \alpha\beta\Delta_0 - \beta\delta A_0 - \eta^2 H_0^2 + 2\beta\eta A_0H_0 - \beta^2A_0^2 + \eta\delta H_0 - \eta^2\Delta_0 - \eta^2A_0B_0 + \eta\alpha B_0H_0 + \eta\beta A_0H_0 - \alpha\beta H_0^2;$$

or, for the two terms $-\beta\delta A_0 + \eta\delta H_0$, $= (\eta H_0 - \beta A_0)\delta$, substituting the value

$$(\eta H_0 - \beta A_0)(2H_0\eta - \alpha B_0 - \beta A_0),$$

the whole is found to be

$$= (\alpha\beta - \eta^2)(\Delta_0 - H_0 + A_0B_0), \text{ that is, } = (\alpha\beta - \eta^2)^2 = \theta^2.$$

And, similarly, $\mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} + \mathfrak{B}\mathfrak{D} - \mathfrak{M}^2$ is found to be $= \theta^2$; and the remaining four combinations of terms to be each of them $= 0$; the matrix is thus a symmetrical θ^2-matrix.

28. It is to be added that the diagonal minors and the determinant

$$\mathfrak{A},\quad \mathfrak{A}\mathfrak{B} - \mathfrak{H}^2,\quad \mathfrak{A}\mathfrak{B}\mathfrak{C} + \&c.,\quad \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} + \&c.,$$

have respectively the values β, θ, $\alpha\theta^2$, θ^4; viz. if α, β, θ are positive, then these are all positive; or the last-mentioned quadric function is a definite positive form.

The general matrix resumed: Arithmetical theory.

29. The matrix containing, as before, the parameter k, may be called a k-matrix; if k be $=1$, it is a unit-matrix. From the fundamental equation

$$k(\Omega_0\Upsilon_2-\Omega_2\Upsilon_0+\Omega_1\Upsilon_3-\Omega_3\Upsilon_1)=\omega_0\upsilon_2-\omega_2\upsilon_0+\omega_1\upsilon_3-\omega_3\upsilon_1,$$

it at once follows that, compounding a k-matrix with a k'-matrix, we have a kk'-matrix; and in particular, compounding a k-matrix with a unit-matrix, we have a k-matrix.

30. The symbols a_0, a_1, b_0, &c., and k, have thus far been arbitrary magnitudes, but we now take them to be integers; and we consider in particular the case where k is a positive odd prime. The number of k-matrices is of course infinite, but if we regard as equivalent any two such matrices which are derivable one from the other by post-multiplication by a unit-matrix (viz. U being a unit-matrix, the matrices M and $M.U$ are regarded as equivalent), then the number of distinct k-matrices is finite, and $=1+k+k^2+k^3$.

The first step is to show that we can by post-multiplication, by a properly determined unit-matrix, reduce the k-matrix to the form

$$\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ 0, & b_1, & b_2, & b_3 \\ 0, & 0, & c_2, & 0 \\ 0, & 0, & d_2, & d_3 \end{array}\right),$$

these values being such as to satisfy identically two out of the six conditions; the remaining conditions present themselves under the two equivalent forms

$$a_0c_2=k,\ b_1d_3=k,\ a_0b_2+a_1b_3-a_3b_1=0,\ a_0d_2+a_1d_3=0,$$

and

$$a_0c_2=k,\ b_1d_3=k,\ a_1c_2+b_1d_2 \qquad =0,\ -a_3c_2+b_2d_3-b_2d_2=0.$$

Hence a_0, $c_2=1$, k, or k, 1; and b_1, $d_3=1$, k, or k, 1; so that, combining these pairs of values, we have four different types of matrix, each type depending on the coefficients a_1, d_2, a_2, b_2, a_3, b_3, connected together by two equations. But the forms of the same type are not distinct from each other, and we have to determine for each of the four types a system of non-equivalent forms comprised therein, and such that from these, by post-multiplication by a unit-matrix as before, the other forms of the type can be obtained. This final system is:—

$$\text{I.}\ \left|\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & k \end{array}\right|,\qquad \text{II.}\ \left|\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & k, & 0, & i \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right|,$$

$$\text{III.}\ \left|\begin{array}{cccc} k, & i, & i', & 0 \\ 0, & 1, & 1, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & -i, & k \end{array}\right|,\qquad \text{IV.}\ \left|\begin{array}{cccc} k, & 0, & i', & i \\ 0, & k, & i, & i'' \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right|,$$

where i, i', i'' are integers, having each of them any one of the values $0, 1, 2, \ldots, k-1$, viz. there are in the four types 1, k, k^2, k^3 forms respectively, and the number of forms is thus $=1+k+k^2+k^3$, as already mentioned. I abstain from the further details of the proof.

There is obviously a like theory for which, in place of post-multiplication, we have pre-multiplication; viz. here the matrices M and $U.M$ are regarded as equivalent.

31. Any two k-matrices are reducible one to the other by a combined pre- and post-multiplication; viz. we have always $M' = U.M.U'$, where M, M' are any two k-matrices, and U, U' properly determined unit-matrices; and in particular, M' being any given k-matrix, this is expressible in the foregoing form, where M denotes the principal matrix

$$\left(\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & k \end{array}\right).$$

Congruence theorems, k an odd number.

32. Taking k an odd number, and using throughout $=$ instead of $\equiv$ (mod. 2), we have the following congruences:

$$\begin{aligned}&(a_0a_2+a_1a_3,\ b_0b_2+b_1b_3,\ c_0c_2+c_1c_3,\ d_0d_2+d_1d_3)\\ &\quad=\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right)(a_2c_2+b_2d_2,\ a_3c_3+b_3d_3,\ a_0c_0+b_0d_0,\ a_1c_1+b_1d_1);\end{aligned}$$

or, conversely,

$$\begin{aligned}&(a_2c_2+b_2d_2,\ a_3c_3+b_3d_3,\ a_0c_0+b_0d_0,\ a_1c_1+b_1d_1)\\ &\quad=\left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2 \\ c_3, & d_3, & -a_3, & -b_3 \\ -c_0, & -d_0, & a_0, & b_0 \\ -c_1, & -d_1, & a_1, & b_1 \end{array}\right)(a_0a_2+a_1a_3,\ b_0b_2+b_1b_3,\ c_0c_2+c_1c_3,\ d_0d_2+d_1d_3),\end{aligned}$$

where observe that on the right-hand side the signs $-$ may be changed into $+$; in fact, to the modulus 2, we have for any integer value whatever $-p \equiv +p$.

33. The first congruence is

$$a_0a_2+a_1a_3 = a_0(a_2c_2+b_2d_2)+a_1(a_3c_3+b_3d_3)+a_2(a_0c_0+b_0d_0)+a_3(a_1c_1+b_1d_1),\ \text{say}\ X=Y,$$

and to verify this, I see no other method than that of considering separately all the combinations of even and odd values of a_0, a_1, a_2, a_3; viz. we have the 16 cases

Case	a_0,	a_2,	a_1,	a_3	X,	Y
1	0	0	0	0	does not exist	
2	0	0	0	1	0	0
3	0	0	1	0	0	0
4	0	0	1	1	1	1
5	0	1	0	0	0	0
6	0	1	0	1	0	0
7	0	1	1	0	0	0
8	0	1	1	1	1	1
9	1	0	0	0	0	0
10	1	0	0	1	0	0
11	1	0	1	0	0	0
12	1	0	1	1	1	1
13	1	1	0	0	1	1
14	1	1	0	1	1	1
15	1	1	1	0	1	1
16	1	1	1	1	0	0

viz. the X column gives in each case the value of $X, = a_0a_2 + a_1a_3$, and then it has to be shown that Y has the same value. The coefficients satisfy the conditions

$$\begin{aligned}
a_0c_2 - a_2c_0 + a_1c_3 - a_3c_1 &= 1,\\
b_0d_2 - b_2d_0 + b_1d_3 - b_3d_1 &= 1,\\
a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1 &= 0,\\
a_0d_2 - a_2d_0 + a_1d_3 - a_3d_1 &= 0,\\
b_0c_2 - b_2c_0 + b_1c_3 - b_3c_1 &= 0,\\
c_0d_2 - c_2d_0 + c_1d_3 - c_3d_1 &= 0,
\end{aligned}$$

so that from the first equation we cannot have a_0, a_1, a_2, a_3 each $= 0$, or the first case does not exist. As to the remaining cases, it is easy to see that they group themselves as follows: 2, 3, 5, 9: 4, 13: 6, 7, 10, 11: 8, 12, 14, 15: 16: the proof being substantially the same for the several cases in the same group.

34. Case 2. We have $Y = b_1 d_1$ which should be $=0$, and in fact, the six equations give $b_1 = 0$, $d_1 = 0$, whence $Y = 0$.

Case 4. We have $Y = c_1 + c_3 + b_1 d_1 + b_3 d_3$ which should $=1$, and in fact, the six equations give $c_3 - c_1 = 1$, $b_3 - b_1 = 0$, $d_3 - d_1 = 0$; whence $c_1 + c_3 = 1$, $b_1 d_1 + b_3 d_3 = 0$, and therefore $Y = 1$.

Case 6. Here $Y = b_0 d_0 + b_1 d_1$ which should $=0$; and in fact, the six equations give $b_0 + b_1 = 0$, $d_0 + d_1 = 0$, whence $Y = 0$.

Case 16. Here $Y = c_0 + c_1 + c_2 + c_3 + b_0 d_0 + b_1 d_1 + b_2 d_2 + b_3 d_3$, which should be $=0$; the six equations give

$$-c_0 - c_1 + c_2 + c_3 = 1, \quad -b_0 - b_1 + b_2 + b_3 = 0,$$

$$-d_0 - d_1 + d_2 + d_3 = 0, \text{ and } b_0 d_2 - b_2 d_0 + b_1 d_3 - b_3 d_1 = 1.$$

Writing

$$b_0 + b_2 = b_1 + b_3 \text{ and } d_0 + d_2 = d_1 + d_3,$$

we find

$$b_0 d_0 + b_2 d_2 + b_0 d_2 + b_2 d_0 = b_1 d_1 + b_3 d_3 + b_1 d_3 + b_3 d_1,$$

that is,

$$b_0 d_0 + b_1 d_1 + b_2 d_2 + b_3 d_3 = b_0 d_2 + b_2 d_0 + b_1 d_3 + b_3 d_1, \; = 1;$$

and $c_0 + c_1 + c_2 + c_3 = 1$, whence $Y = 0$.

35. Case 8. This is the only case of any difficulty: we have

$$Y = c_1 + c_3 + b_0 d_0 + b_1 d_1 + b_3 d_3,$$

which should be $=1$. The six equations give

$$-c_0 + c_3 - c_1 = 1, \quad -b_0 + b_3 - b_1 = 0, \quad -d_0 + d_3 - d_1 = 0,$$

or, say

$$c_0 = -1 - c_1 + c_3, \quad b_0 = -b_1 + b_3, \quad d_0 = -d_1 + d_3;$$

or, substituting these values and omitting even terms

$$Y = c_1 + c_3 - b_1 d_3 - b_3 d_1.$$

The remaining three of the six equations are

$$b_0 d_2 - b_2 d_0 + b_1 d_3 - b_3 d_1 = 1, \quad b_0 c_2 - b_2 c_0 + b_1 c_3 - b_3 c_1 = 0, \quad c_0 d_2 - c_2 d_0 + c_1 d_3 - c_3 d_1 = 0;$$

or, substituting for b_0, c_0, d_0 their values, these become

$$\begin{aligned} (1 + c_1 - c_3)\, b_2 + (b_3 - b_1)\, c_2 \qquad\qquad &= -b_1 c_3 + b_3 c_1, \\ (d_1 - d_3)\, b_2 + \qquad\qquad (b_3 - b_1)\, d_2 &= 1 - b_1 d_3 + b_3 d_1, \\ (d_1 - d_3)\, c_2 + (-1 - c_1 + c_3)\, d_2 &= -c_1 d_3 + c_3 d_1; \end{aligned}$$

we can from these equations eliminate b_2, c_2, d_2; viz. from the first and third equations eliminating c_2, we have

$$(1 + c_1 - c_3)\{(d_3 - d_1)\, b_2 + (b_1 - b_3)\, d_3\} = (d_3 - d_1)(-b_1 c_3 + b_3 c_1) + (b_3 - b_1)(-c_1 d_3 + c_3 d_1),$$

and this, by means of the second equation, becomes

$$(1+c_1-c_3)(-1+b_1d_3-b_3d_1)=(d_3-d_1)(-b_1c_3+b_3c_1)+(b_3-b_1)(-c_1d_3+c_3d_1),$$

viz. reducing, this is

$$-c_1+c_3-1+b_1d_3-b_3d_1=0;$$

and in virtue of it we have $Y=c_1+c_3-b_1d_3-b_3d_1=1$.

It can be further shown that, to the modulus 2 (k being, as before, odd), we have

$$(a_0a_2+a_1a_3)(c_0c_2+c_1c_3)+(b_0b_2+b_1b_3)(d_0d_2+d_1d_3)=0,$$
$$(a_0c_0+b_0d_0)(a_2c_2+b_2d_2)+(a_1c_1+b_1d_1)(a_3c_3+b_3d_3)=0.$$

To prove the first equation, write for a moment

$$\Omega=(a_0a_2+a_1a_3)(c_0c_2+c_1c_3),\quad \Omega'=(b_0b_2+b_1b_3)(d_0d_2+d_1d_3),$$
$$X=(a_0c_1-a_1c_0)(a_2c_3-a_3c_2),\quad X'=(b_0d_1-b_1d_0)(b_2d_3-b_3d_2);$$

then in virtue of the equations

$$a_0c_1-a_1c_0+b_0d_1-b_1d_0,$$
$$a_2c_3-a_3c_2+b_2d_3-b_3d_2,$$

we have $X=X'$. But we have identically

$$\Omega-X=(a_0c_2+a_1c_3)(a_2c_0+a_3c_1),$$

and from the equation

$$a_0c_2-a_2c_0+a_1c_3-a_3c_1=1,$$

that is, $a_2c_0+a_3c_1=-1+a_0c_2+a_1c_3$, we have $\Omega-X=0$; and similarly $\Omega'-X'=0$, that is, $\Omega-\Omega'=X-X'=0$; we have thus the required equation $\Omega+\Omega'=0$. In a similar manner the second equation may be verified.

36. Write

$$\mu'=\mu a_0+\nu a_1+qa_2+ra_3+a_0a_2+a_1a_3,$$
$$\nu'=\mu b_0+\nu b_1+qb_2+rb_3+b_0b_2+b_1b_3,$$
$$q'=\mu c_0+\nu c_1+qc_2+rc_3+c_0c_2+c_1c_3,$$
$$r'=\mu d_0+\nu d_1+qd_2+rd_3+d_0d_2+d_1d_3.$$

It is to be shown that to the modulus 2 we have

$$\mu'q'+\nu'r'=\mu q+\nu r.$$

In fact, forming the value of $\mu'q'+\nu'r'$, we have first a constant term (term without μ, ν, q, r) which vanishes; next writing $\mu^2=\mu$, the whole term in μ is

$$a_0c_0+b_0d_0+a_0(c_0c_2+c_1c_3)+b_0(d_0d_2+d_1d_3)+c_0(a_0a_2+a_1a_3)+d_0(b_0b_2+b_1b_3),$$

which also vanishes; and similarly the terms in ν, q, r each of them vanish; there remain only the terms in $\mu\nu$, μq, &c. The coefficient of μq is $a_0c_2+a_2c_0+b_0d_2+b_2d_0$,

which (always to the modulus 2) is $=a_0c_2-a_2c_0+b_0d_2-b_2d_0$, that is, it is $=1$; and similarly the coefficient of νr is $=1$; and in like manner the coefficients of the other terms are each $=0$; we have thus the required congruence $\mu'q'+\nu'r'=\mu q+\nu r$.

The quintic matrix.

37. I consider a quintic matrix composed of the foregoing coefficients $(a, b, c, d)_{0,1,2,3}$, viz. the matrix contained in a linear transformation which I write as follows:

$$(T', P', Q', R', S') = \begin{array}{c} \\ ab \\ cb \\ ac \\ ad \\ dc \end{array} \begin{array}{c} \begin{array}{ccccc} 01 & 21 & 2.02 & 03 & 32 \end{array} \\ \left|\begin{array}{ccccc} . & . & . & . & . \\ . & . & . & . & . \\ . & . & -k & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array}\right| \end{array} (T, P, Q, R, S),$$

read

$$T' = (ab)_{01}\,T + (ab)_{21}\,P + 2\,(ab)_{02}\,Q + (ab)_{03}\,R + (ab)_{32}\,S,$$

where as before $(ab)_{01}=a_0b_1-a_1b_0$, &c., and so in other cases; in particular, observe that, in the expression of Q', the term involving Q is $\{-k+2\,(ac)_{02}\}\,Q$ which, in virtue of the relation $(ac+bd)_{02}=k$, may also be written $\{(ac)_{02}-(bd)_{02}\}\,Q$. I notice that in Hermite's paper, p. 366, the term is in effect written without the $-k$, $=2\,(ac)_{02}\,Q$; the correction of this erratum and of a corresponding one, p. 366, is made p. 787 at the conclusion of the memoir.

38. The matrix is automorphic for the form $T^2-PR-TS$, viz. we have identically

$$Q'^2-P'R'-T'S' = k^2\,(Q^2-PR-TS).$$

As a partial verification, consider in $Q'^2-P'R'-T'S'$ the term containing Q^2. The coefficient of Q^2 is

$$\{-k+2\,(ac)_{02}\}^2-4\,(cb)_{02}\,(ad)_{02}-4\,(ab)_{02}\,(dc)_{02},$$

where the first term is

$$\{(ac)_{02}-(bd)_{02}\}^2, \text{ which is } =\{(ac)_{02}+(bd)_{02}\}^2-4\,(ac)_{02}\,(bd)_{02}=k^2-4\,(ac)_{02}\,(bd)_{02}.$$

Hence, observing that we have

$$(ac)_{02}\,(bd)_{02}+(cb)_{02}\,(ad)_{02}+(ab)_{02}\,(dc)_{02}=0,$$

the whole coefficient is $=k^2$, as it should be; and in like manner the verification may be effected for any other term.

39. We require the following formulæ:

$$\left(\begin{array}{ccc} -c_0, & a_0, & b_0 \\ -c_1, & a_1, & b_1 \\ -c_2, & a_2, & b_2 \\ -c_3, & a_3, & b_3 \end{array}\right)(T', P', Q') = k\left(\begin{array}{ccccc} ., & b_1, & -b_0, & ., & -b_3 \\ ., & ., & b_1, & -b_0, & -b_2 \\ -b_1, & ., & -b_2, & -b_3, & . \\ b_0, & b_2, & b_3, & ., & . \end{array}\right)(T, P, Q, R, S),$$

that is,

$$-c_0T'' + a_0P' + b_0Q' = k(b_1P - b_0Q - b_2S), \text{ \&c.},$$

and

$$\left(\begin{array}{ccc} -d_0, & a_0, & b_0 \\ -d_1, & a_1, & b_1 \\ -d_2, & a_2, & b_2 \\ -d_3, & a_3, & b_3 \end{array}\right)(T', Q', R') = k\left(\begin{array}{ccccc} ., & -a_1, & a_0, & ., & a_3 \\ ., & ., & -a_1, & a_0, & -a_2 \\ a_1, & ., & a_2, & a_3, & . \\ -a_0, & -a_2, & -a_3, & ., & . \end{array}\right)(T, P, Q, R, S).$$

From the first set, multiplying the first, third and fourth equations by T, P, Q respectively and adding, and again multiplying the second, third and fourth equations by T, Q, R, and adding, we obtain

$$-(c_0T + c_2P + c_3Q)\,T' + (a_0T + a_2P + a_3Q)\,P' + (b_0T + b_2P + b_3Q)\,Q' = \quad kb_3(Q^2 - PR - TS),$$
$$-(c_1T + c_2Q + c_3R)\,T' + (a_1T + a_2Q + a_3R)\,P' + (b_1T + b_2Q + b_3R)\,Q' = -kb_2(Q^2 - PR - TS);$$

and similarly, from the second set of equations, we obtain

$$-(d_0T + d_2P + d_3Q)\,T' + (a_0T + a_2P + a_3Q)\,Q' + (b_0T + b_2P + b_3Q)\,R' = -ka_3(Q^2 - PR - TS),$$
$$-(d_1T + d_2Q + d_3R)\,T' + (a_1T + a_2Q + a_3R)\,Q' + (b_1T + b_2Q + b_3R)\,R' = \quad ka_2(Q^2 - PR - TS).$$

40. We have the inverse system

$$(T, P, Q, R, S) = \begin{array}{r} \\ 32 \\ 03 \\ 13 \\ 21 \\ 01 \end{array} \begin{array}{|ccccc|} dc & ad & 2bd & cb & ab \\ \hline . & . & . & . & . \\ . & . & . & . & . \\ . & . & -k & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array} (T', P', Q', R', S'),$$

read

$$T = (dc)_{32}\,T' + (ad)_{32}\,P' + 2\,(bd)_{32}\,Q' + (cb)_{32}\,R' + (ab)_{32}\,S';$$

and in particular observe that, in the expression for Q, the term containing Q' is $\{-k + 2\,(bd)_{13}\}\,Q'$, where, in virtue of $(ac + bd)_{13} = k$, the coefficient of Q' is also $= (bd)_{13} - (ac)_{13}$.

41. These equations give

$$\left(\begin{array}{ccc} a_0, & a_2, & a_3 \\ b_0, & b_2, & b_3 \\ c_0, & c_2, & c_3 \\ d_0, & d_2, & d_3 \end{array}\right)(T, P, Q) = k\left(\begin{array}{ccccc} d_3, & ., & -a_3, & -b_3, & . \\ -c_3, & a_3, & b_3, & ., & . \\ ., & d_3, & -c_3, & ., & b_3 \\ ., & ., & d_3, & -c_3, & -a_3 \end{array}\right)(T', P', Q', R', S'),$$

and

$$\left(\begin{array}{ccc|} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{array}\right)(T, Q, R) = k\left(\begin{array}{ccccc|} -d_2, & ., & a_2, & b_2, & . \\ c_2, & -a_2, & -b_2, & ., & . \\ ., & -d_2, & c_2, & ., & -b_2 \\ ., & ., & -d_2, & c_2, & a_2 \end{array}\right)(T', P', Q', R', S').$$

From the first set of equations, multiplying the first, third and fourth equations by $-S'$, $-R'$, $+Q'$ respectively, and adding, and again multiplying the second, third and fourth equations by $-S'$, Q', $-P'$ respectively, and adding, we deduce

$$T(-a_0S' - c_0R' + d_0Q') + P(-a_2S' - c_2R' + d_2Q') \\ + Q(-a_3S' - c_3R' + d_3Q') = -kc_3(Q'^2 - P'R' - T'S'),$$

$$T(-b_0S' + c_0Q' - d_0P') + P(-b_2S' + c_2Q' - d_2P') \\ + Q(-b_3S' + c_3Q' - d_3P') = \quad kd_3(Q'^2 - P'R' - T'S');$$

and in like manner, from the second set of equations,

$$T(-a_1S' - c_1R' + d_1Q') + Q(-a_2S' - c_2R' + d_2Q') \\ + R(-a_3S' - c_3R' + d_3Q') = -kd_2(Q'^2 - P'R' - T'S'),$$

$$T(-b_1S' + c_1Q' - d_1P') + Q(-b_2S' + c_2Q' - d_2P') \\ + R(-b_3S' + c_3Q' - d_3P') = \quad kc_2(Q'^2 - P'R' - T'S').$$

42. Assume that T, P, Q, R, S and T', P', Q', R', S' are linearly connected as above; and write

$$T : P : Q : R : S = 1 : A : H : B : H^2 - AB,$$
$$T' : P' : Q' : R' : S' = 1 : A' : H' : B' : H'^2 - A'B',$$

equations which establish a like relation between 1, A, H, B, $H^2 - AB$, and 1, A', H', B', $H'^2 - A'B'$. Observe that these forms are admissible since, if

$$T^2 - PR - QS = 0,$$

then also

$$Q'^2 - P'R' - T'S' = 0.$$

The quantities A, H, B were taken as the parameters of a theta-function; viz. taking

$$A,\ H,\ B = A_0 + i\alpha,\ H_0 + i\eta,\ B_0 + i\beta,$$

then $(\alpha, \eta, \beta \between x, y)^2$ must be a positive form (or what is the same thing, α and $\alpha\beta - \eta^2$ must be positive). If A', H', B' are also the parameters of a theta-function, then writing

$$A',\ H',\ B' = A_0' + i\alpha',\ H_0' + i\eta',\ B_0' + i\beta',$$

$(\alpha', \eta', \beta' \between x, y)^2$ must also be a positive form. It can be shown that, A', H', B' being determined as above, the former condition implies the latter one; viz. if the form $(\alpha, \eta, \beta \between x, y)^2$ be positive, then also the form $(\alpha', \eta', \beta' \between x, y)^2$ will be positive.

Write

$$
\begin{aligned}
A &= A_0 + i\alpha, \quad A' = A_0' + i\alpha', \\
B &= B_0 + i\beta, \quad B' = B_0' + i\beta', \\
H &= H_0 + i\eta, \quad H' = H_0' + i\eta'. \\
\Delta &= \Delta_0 + i\delta, \\
\Delta_0 &= H_0^2 - A_0 B_0 - (\eta^2 - \alpha\beta), \\
\delta &= 2H_0\eta - B_0\alpha - A_0\beta,
\end{aligned}
$$

and let the quintic matrix in the foregoing transformation

$$(T', P', Q', R', S') = (M \between T, P, Q, R, S)$$

be represented by

$$
\begin{array}{c|ccccc|}
 & t & p & q & r & s \\
\hline
0 & & & & & \\
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
\hline
\end{array}
$$

43. It is to be shown that $\alpha' x^2 + 2\eta' xy + \beta' y^2$ is definite and positive.

We require α', β', η'; we have

$$A_0' + i\alpha' = \frac{(t_1, p_1, q_1, r_1, s_1)(1, A_0 + i\alpha, H_0 + i\eta, B_0 + i\beta, \Delta_0 + i\delta)}{(t_0, p_0, q_0, r_0, s_0)(\quad \text{"} \quad)} = \frac{M_1 + N_1 i}{M_0 + N_0 i},$$

and thence

$$\alpha' = \frac{M_0 N_1 - M_1 N_0}{M_0^2 + N_0^2} = \frac{1}{\square}(M_0 N_1 - M_1 N_0).$$

Now

$$
\begin{aligned}
M_0 N_1 - M_1 N_0 = \; & (t_0 p_1 - t_1 p_0)\,\alpha \\
& + (t_0 q_1 - t_1 q_0)\,\eta \\
& + (t_0 r_1 - t_1 r_0)\,\beta \\
& + (t_0 s_1 - t_1 s_0)\,\delta \\
& + (p_0 q_1 - p_1 q_0)(A_0\eta - H_0\alpha) \\
& + (p_0 r_1 - p_1 r_0)(A_0\beta - B_0\alpha) \\
& + (p_0 s_1 - p_1 s_0)(A_0\delta - \Delta_0\alpha) \\
& + (q_0 r_1 - q_1 r_0)(H_0\beta - B_0\eta) \\
& + (q_0 r_1 - q_1 r_0)(H_0\beta - B_0\eta) \\
& + (q_0 s_1 - q_1 s_0)(H_0\delta - \Delta_0\eta) \\
& + (r_0 s_1 - r_1 s_0)(B_0\delta - \Delta_0\beta);
\end{aligned}
$$

and so for η', β' with only the change of t_1, p_1, q_1, r_1, s_1 into t_2, p_2, q_2, r_2, s_2 and t_3, p_3, q_3, r_3, s_3 respectively; we find, omitting a factor $\frac{k}{\square}$ throughout,

	$\alpha' =$	$\eta' =$	$\beta' =$
α	b_1^2,	$-a_1b_1$,	a_1^2,
η	$-2b_0b_1$,	$a_0b_1+a_1b_0$,	$-2a_0a_1$,
β	b_0^2,	$-a_0b_0$,	a_0^2,
δ	$-(b_0b_2+b_1b_3)$,	$\frac{1}{2}(a_0b_2+a_2b_0+a_1b_3+a_3b_1)$,	$-(a_0a_2+a_1a_3)$,
$A_0\eta-H_0\alpha$	$-2b_1b_2$,	$a_1b_2+a_2b_1$,	$-2a_1a_2$,
$H_0\beta-B_0\eta$	$+2b_0b_3$,	$-(a_0b_3+a_3b_0)$,	$+2a_0a_3$,
$A_0\beta-B_0\alpha$	$b_0b_2-b_1b_3$,	$-\frac{1}{2}(a_0b_2+a_2b_0-a_1b_3-a_3b_1)$,	$a_0a_2-a_1a_3$,
$A_0\delta-\Delta_0\alpha$	$-b_2^2$,	a_2b_2,	$-a_2^2$,
$H_0\delta-\Delta_0\eta$	$-2b_2b_3$,	$+a_2b_3+a_3b_2$,	$-2a_2a_3$,
$B_0\delta-\Delta_0\beta$	$-b_3^2$,	a_3b_3,	$-a_3^2$,

and hence

$$\alpha'x^2+2\eta'xy+\beta'y^2$$

$$=\frac{k}{\square}\left(\begin{array}{cccc} \beta, & -\eta, & -\eta H_0+\beta A_0, & -\eta B_0+\beta H_0 \\ -\eta, & \alpha, & -\eta A_0+\alpha H_0, & -\eta H_0+\alpha B_0 \\ -\eta H_0+\beta A_0, & -\eta A_0+\alpha H_0, & +\alpha\Delta_0-\delta A_0, & -\delta H_0+\eta\Delta_0 \\ -\eta B_0+\beta H_0, & -\eta H_0+\alpha B_0, & \delta H_0-\eta\Delta_0, & +\beta\Delta_0-\delta B_0 \end{array}\right)\{b_0x-a_0y,\ b_1x-a_1y,\ b_2x-a_2y,\ b_3x-a_3y\}^2.$$

The right-hand is here definite and positive (*suprà* No. 28), hence also the left-hand is definite and positive.

44. As a specimen of the work, observe that we have

$$\begin{aligned} t_0p_1-t_1p_0 &= (ab)_{01}(cb)_{21}-(ab)_{21}(cb)_{01} \\ &= b_1\{b_0(ac)_{21}+b_2(ac)_{10}+b_1(ac)_{02}\} \\ &= b_1[-b_0(bd)_{21}-b_2(bd)_{10}+b_1\{-(bd)_{02}+k\}] \\ &= b_1^2k, \end{aligned}$$

since the remaining terms destroy each other. Dividing by $\square$, and then omitting the factor $\frac{k}{\square}$, we have thus the term αb_1^2 in the foregoing expression for α'.

The Transformation. Art. Nos. 45 to 53.

45. Consider $(a, b, c, d)_{0,1,2,3}$, the components of a quartic matrix as above, and also T, P, Q, R, S; T', P', Q', R', S' connected as already mentioned; and write for shortness

$$z_0=a_0x+b_0y,\quad z_1=a_1x+b_1y,$$
$$z_2=a_2x+b_2y,\quad z_3=a_3x+b_3y,$$

and consider the function

$$\Pi(x, y) = \Theta(z_0 + Az_2 + Hz_3,\ z_1 + Hz_2 + Bz_3) \exp. i\pi\{(z_0 z_2 + z_1 z_3) + (A, H, B)(z_2, z_3)^2\},$$

where Θ denotes the function

$$\Theta\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(A, H, B).$$

It is to be shown that Π is a function

$$\Pi\begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix}(x, y)(A', H', B')(K, \text{def.}),$$

where the new parameters A', H', B' are given in terms of the original parameters A, H, B by the equations

$$1 : A : H : B : H^2 - AB = T : P : Q : R : S,$$
$$1 : A' : H' : B' : H'^2 - A'B' = T' : P' : Q' : R' : S',$$

and where μ', ν', q', r' have the values

$$\mu' = \mu a_0 + \nu a_1 + q a_2 + r a_3 + a_0 a_2 + a_1 a_3,$$
$$\nu' = \mu b_0 + \nu b_1 + q b_2 + r b_3 + b_0 b_2 + b_1 b_3,$$
$$q' = \mu c_0 + \nu c_1 + q c_2 + r c_3 + c_0 c_2 + c_1 c_3,$$
$$r' = \mu d_0 + \nu d_1 + q d_2 + r d_3 + d_0 d_2 + d_1 d_3;$$

viz. it is to be shown that the function Π, as above defined, satisfies the fundamental equations

$$\Pi(x+1, y) = (-)^{\mu'}\Pi(x, y),$$
$$\Pi(x, y+1) = (-)^{\nu}\Pi(x, y),$$
$$\Pi(x+A', y+H') = (-)^{q'}\Pi(x, y)\exp. -2\pi k(2x+A'),$$
$$\Pi(x+H', y+B') = (-)^{r'}\Pi(x, y)\exp. -2\pi k(2y+B'),$$

and

$$\Pi(-x, -y) = (-)^{\mu'\nu'+q'r'}\Pi(x, y).$$

46. It is proper in the first place to show how it is that A', H', B' are capable of being the parameters of the new function.

Write for shortness

$$X = z_0 + Az_2 + Hz_3,$$
$$Y = z_1 + Hz_2 + Bz_3,$$

and suppose x changed into $x+1$: we have z_0, z_1, z_2, z_3 increased by a_0, a_1, a_2, a_3 respectively; and thence X, Y increased by $a_0 + Aa_2 + Ha_3$, $a_1 + Ha_2 + Ba_3$; the theta-function is thus changed into

$$\Theta(X + a_0 + Aa_2 + Ha_3,\quad Y + a_1 + Ha_2 + Ba_3),$$

viz. the arguments are increased by multiples of the quarter-periods $(1, 0, A, H)$ and $(0, 1, H, B)$.

And, similarly, by the change of y into $y+1$, the theta-function is changed into

$$\Theta(X+b_0+Ab_2+Hb_3,\quad Y+b_1+Hb_2+Bb_3),$$

or the arguments are increased by integer multiples of the quarter-periods.

47. But suppose next that x, y are changed into $x+A'$, $y+B'$: we have z_0, z_1, z_2, z_3 increased by

$$A'a_0+H'b_0,\quad A'a_1+H'b_1,\quad A'a_2+H'b_2,\quad A'a_3+H'b_3,$$

and thence

$$\begin{aligned} X &\text{ increased by } A'(a_0+Aa_2+Ha_3)+H'(b_0+Ab_3+Hb_3), = c_0+Ac_2+Hc_3,\\ Y &\quad\text{"}\quad\text{"}\quad A'(a_1+Ha_2+Ba_3)+H'(b_1+Hb_2+Bb_3), = c_1+Hc_2+Bc_3,\end{aligned}$$

since these equalities are the before-mentioned equations

$$\begin{aligned} &P'(Ta_0+Pa_2+Qa_3)+Q'(Tb_0+Pb_2+Qb_3)-T'(Tc_0+Pc_2+Qc_3)=0,\\ &P'(Ta_1+Qa_2+Ra_3)+Q'(Tb_1+Qb_2+Rb_3)-T'(Tc_1+Qc_2+Rc_3)=0.\end{aligned}$$

It thus appears that, by the change of x, y into $x+A'$, $y+H'$, the theta-function is changed into

$$\Theta(X+c_0+Ac_2+Hc_3,\quad Y+c_1+Hc_2+Bc_3),$$

viz. we have again the arguments increased by integer multiples of the quarter-periods; and in like manner, by the change of x, y into $x+H'$, $y+B'$, the theta-function is changed into

$$\Theta(X+d_0+Ad_2+Hd_3,\quad Y+d_1+Hd_2+Bd_3),$$

or the arguments are again increased by integer multiples of the quarter-periods.

48. We have to complete the verifications, first for the equation

$$\Pi(x+1,\ y)=(-)^{\mu'}\Pi(x,\ y).$$

Reverting to the definition of Π, we have

$$\begin{aligned} &(-)^{\mu'}\Pi(x+1,\ y)\div\Pi(x,\ y)\\ &\quad=(-)^{-\mu a_2-\nu a_1-\varrho a_3-\tau a_2}\exp.-i\pi\{a_0a_2+a_1a_3\};\\ &\Theta(X+a_0+Aa_2+Ha_3,\ Y+a_1+Ha_2+Ba_3)\div\Theta(X,\ Y)\\ &\quad=\exp.i\pi\{a_0z_2+a_2z_0+a_1z_3+a_3z_1+a_0a_2+a_1a_3\}\times\\ &\qquad\exp.i\pi\{2(Aa_2+Ha_3)z_2+2(Ha_2+Ba_3)z_3+(A,\ H,\ B)(a_2,\ a_3)^2\};\end{aligned}$$

the second line of this is

$$=(-)^{\mu a_0+\nu a_1+\varrho a_2+\tau a_3}\exp.i\pi\{-2a_2X-2a_3Y-(A,\ H,\ B)(a_2,\ a_3)^2\},$$

and the right-hand side of the equation will thus be $=1$ if only the whole argument of the exponential be $=0$; that is, omitting the terms which destroy each other and the common factor $i\pi$, if only

$$\begin{aligned} &-2a_2X-2a_3Y\\ &+a_0z_2+a_2z_0+a_1z_3+a_3z_1\\ &+2(Aa_2+Ha_3)z_2+2(Ha_2+Ba_3)z_3=0.\end{aligned}$$

Substituting herein for X, Y their values

$$a_0 + Aa_2 + Ha_3, \quad a_1 + Ha_2 + Ba_3,$$

the equation becomes

$$a_0 z_2 - a_2 z_0 + a_1 z_3 - a_3 z_1 = 0,$$

and finally substituting herein for z_0, z_1, z_2, z_3 their values, the coefficient of x is identically $= 0$, and the coefficient of y is

$$a_0 b_2 - a_2 b_0 + a_1 b_3 - a_3 b_1,$$

which is $(02 + 13)_{ab}$, and is thus $= 0$. This completes the proof of the equation

$$\Pi(x+1, y) = (-)^{\mu'} \Pi(x, y);$$

and we have of course a precisely similar proof for the equation

$$\Pi(x, y+1) = (-)^{\nu'} \Pi(x, y).$$

49. For the next equation, writing for shortness α_0, α_1, α_2, α_3 for $A'a_0 + H'b_0$, $A'a_1 + H'b_1$, $A'a_2 + H'b_2$, $A'a_3 + H'b_3$, so that z_0, z_1, z_2, z_3 are increased by α_0, α_1, α_2, α_3 respectively, we have

$$\begin{aligned}
&(-)^{-q'} \Pi(x + A', y + H') \div \Pi(x, y) \\
&\quad = (-)^{-\mu c_0 - \nu c_1 - q c_2 - r c_3} \exp. - i\pi (c_0 c_2 + c_1 c_3); \\
&\Theta(X + c_0 + Ac_2 + Hc_3, Y + c_1 + Hc_2 + Bc_3) \div \Theta(x, y) \\
&\quad = \exp. i\pi \{\alpha_0 z_2 + \alpha_2 z_0 + \alpha_1 z_3 + \alpha_3 z_1 + \alpha_0 \alpha_2 + \alpha_1 \alpha_3\} \times \\
&\qquad \exp. i\pi \{2(A\alpha_2 + H\alpha_3) z_2 + 2(H\alpha_2 + B\alpha_3) z_3 + (A, H, B)(\alpha_2, \alpha_3)^2\}.
\end{aligned}$$

The second line is here

$$= (-)^{\mu c_0 + \nu c_1 + q c_2 + r c_3} \exp. i\pi \{-2c_2 X - 2c_3 Y - (A, H, B)(c_2, c_3)^2\},$$

and the whole expression should be

$$= \exp. i\pi \{-2kx - kA'\}.$$

Hence bringing these terms over to the left-hand side, the equation will be satisfied if only the whole argument of the exponential be $= 0$; viz. omitting the factor $i\pi$, the equation will be satisfied if only

$$\begin{aligned}
&\alpha_0 z_2 + \alpha_2 z_0 + \alpha_1 z_3 + \alpha_3 z_1 + \alpha_0 \alpha_2 + \alpha_1 \alpha_3 - c_0 c_2 - c_1 c_3 \\
&\quad + 2(A, H, B)(\alpha_2, \alpha_3)(z_2, z_3) + (A, H, B)(\alpha_2, \alpha_3)^2 \\
&\quad - 2c_2 X - 2c_3 Y \qquad - (A, H, B)(c_2, c_3)^2 + k(2x + A') = 0.
\end{aligned}$$

Substituting here for X, Y their values

$$z_0 + Az_2 + Hz_3, \quad z_1 + Hz_2 + Bz_3,$$

and attending to the values

$$a_0 x + b_0 y, \quad a_1 x + b_1 y, \quad a_2 x + b_2 y, \quad a_3 x + b_3 y$$

of z_0, z_1, z_2, z_3, the equation contains a term in x, a term in y, and a constant term; and we may consider these terms separately.

50. The term in x will vanish if

$$\alpha_0 a_2 + \alpha_2 a_0 + \alpha_3 a_1 + \alpha_1 a_3 + 2(A\alpha_2 + H\alpha_3)a_2 + 2(H\alpha_2 + B\alpha_3)a_3$$
$$- 2c_2(a_0 + Aa_2 + Ha_3) - 2c_3(a_1 + Ha_2 + Ba_3) + 2k = 0,$$

and substituting herein for α_0, α_1, α_2, α_3 their values

$$A'a_0 + H'b_0, \quad A'a_1 + H'b_1, \quad A'a_2 + H'b_2, \quad A'a_3 + H'b_3,$$

the equation becomes

$$A'[2a_0a_2 + 2a_1a_3 + 2(Aa_2 + Ha_3)a_2 + 2(Ha_2 + Ba_3)a_3]$$
$$+ H'[a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1 + 2(Aa_2 + Ha_3)b_2 + 2(Ha_2 + Ba_3)b_3]$$
$$- 2c_2(a_0 + Aa_2 + Ha_3) - 2c_3(a_1 + Ha_2 + Ba_3) + 2k = 0,$$

or observing that the first and second lines may be expressed in the form

$$A'[2a_2(a_0 + Aa_2 + Ha_3) + 2a_3(a_1 + Ha_2 + Ba_3)]$$
$$+ H'[2b_2(a_0 + Aa_2 + Ha_3) + 2b_3(a_1 + Ha_2 + Ba_3)] - (a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1),$$

where the term $a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1$, that is, $(02+13)_{ab}$, is $=0$, and may therefore be omitted, the whole equation, omitting the factor 2, which divides out, is

$$(A'a_2 + H'b_2 - c_2)(a_0 + Aa_2 + Ha_3) + (A'a_3 + H'b_3 - c_3)(a_1 + Ha_2 + Ba_3) + k = 0,$$

an equation which, in the form

$$(-c_2T' + a_2P' + b_2Q')(a_0T + a_2P + a_3Q) + (-c_3T' + a_3P' + b_3Q')(a_1T + a_2Q + a_3R) + kTT' = 0,$$

has been above shown to be true. The term in x thus vanishes.

51. The term in y will vanish if

$$\alpha_0 b_2 + \alpha_2 b_0 + \alpha_1 b_3 + \alpha_3 b_1 + 2(A\alpha_2 + H\alpha_3)b_2 + 2(H\alpha_2 + B\alpha_3)b_3$$
$$- 2c_2(b_0 + Ab_2 + Hb_3) - 2c_3(b_1 + Hb_2 + Bb_3) = 0;$$

and this is in a similar manner reduced to

$$(A'a_2 + H'b_2 - c_2)(b_0 + Ab_2 + Hb_3) + (A'a_3 + H'b_3 - c_3)(b_1 + Hb_2 + Bb_3) = 0,$$

an equation which, in the form

$$(-c_2T' + a_2P' + b_2Q')(b_0T + b_2P + b_3Q) + (-c_3T' + a_3P' + b_3Q')(b_1T + b_2Q + b_3R) = 0,$$

has been above shown to be true. The term in y thus vanishes.

52. It only remains to show that the constant term also vanishes, viz. that we have

$$\alpha_0\alpha_2 + \alpha_1\alpha_3 + (A, H, B)(\alpha_2, \alpha_3)^2$$
$$- c_0c_2 - c_1c_3 - (A, H, B)(c_2, c_3)^2 + kA' = 0.$$

Substituting here for α_0, α_1, α_2, α_3, their values

$$= A'a_0 + H'b_0, \quad A'a_1 + H'b_1, \quad A'a_2 + H'b_2, \quad A'a_3 + H'b_3,$$

the equation becomes

$$\begin{aligned} & A'^2(a_0a_2 + a_1a_3) + A'H'(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + H'^2(b_0b_2 + b_1b_3) \\ & \quad + \quad A\,[A'^2a_2^2 \;\; + 2A'H'a_2b_2 \qquad\qquad + H'^2b_2^2\;] \\ & \quad + 2H\,[A'^2a_2a_3 + \;\; A'H'(a_2b_3 + a_3b_2) \qquad + H'^2b_2b_3] \\ & \quad + \quad B\,[A'^2a_3^2 \;\; + 2A'H'a_3b_3 \qquad\qquad + H'^2b_3^2\;] \\ & \quad - c_0c_2 - c_1c_3 - (A,\ H,\ B \between c_2,\ c_3)^2 + kA' = 0. \end{aligned}$$

This may be written

$$\begin{aligned} & A'^2\{a_2(a_0 + Aa_2 + Ha_3) + a_3(a_1 + Ha_2 + Ba_3)\} \\ + \; & A'H'\{b_2(a_0 + Aa_2 + Ha_3) + b_3(a_1 + Ha_2 + Ba_3) \\ & \qquad + a_2(b_0 + Ab_2 + Hb_3) + a_3(b_1 + Hb_2 + Bb_3)\} \\ + \; & H'^2\{b_2(b_0 + Ab_2 + Hb_3) + b_3(b_1 + Hb_2 + Bb_3)\} \\ & \qquad - c_2(c_0 + Ac_2 + Hc_3) - c_3(c_1 + Hc_2 + Bc_3) + kA' = 0. \end{aligned}$$

Writing herein

$$\begin{aligned} c_0 + Ac_2 + Hc_3 &= (a_0 + Aa_2 + Ha_3)\,A' + (b_0 + Ab_2 + Hb_3)\,H', \\ c_1 + Hc_2 + Bc_3 &= (a_1 + Ha_2 + Ba_3)\,A' + (b_1 + Hb_2 + Bb_3)\,H', \end{aligned}$$

equations which are true in virtue of

$$\begin{aligned} -(c_0T + c_2P + c_3Q)\,T' + (a_0T + a_2P + a_3Q)\,P' + (b_0T + b_2P + b_3Q)\,Q' &= 0, \\ -(c_1T + c_2Q + c_3R)\,T' + (a_1T + a_2Q + a_3R)\,P' + (b_1T + b_2Q + b_3R)\,Q' &= 0, \end{aligned}$$

the equation becomes

$$\begin{aligned} & (a_0 + Aa_2 + Ha_3)\,A'\,(A'a_2 + H'b_2 - c_2) \\ & + (a_1 + Ha_2 + Ba_3)\,A'\,(A'a_3 + H'b_3 - c_3) \\ & + (b_0 + Ab_2 + Hb_3)\,H'\,(A'a_2 + H'b_2 - c_2) \\ & + (b_1 + Hb_2 + Bb_3)\,H'\,(A'a_3 + H'b_3 - c_3) + kA' = 0, \end{aligned}$$

that is,

$$\begin{aligned} & A'\,[(a_0 + Aa_2 + Ha_3)(-c_2 + A'a_2 + H'b_2) + (a_1 + Ha_2 + Ba_3)(-c_3 + A'a_3 + H'b_3) + k] \\ & + H'\,[(b_0 + Ab_2 + Hb_3)(-c_2 + A'a_2 + H'b_2) + (b_1 + Hb_2 + Bb_3)(-c_3 + A'a_3 + H'b_3)] = 0\,; \end{aligned}$$

and we have the coefficients of A' and H' each $=0$, in virtue of

$$\begin{aligned} (a_0T + a_2P + a_3Q)(-c_2T' + a_2P' + b_2Q') + (a_1T + a_2Q + a_3R)(-c_3T' + a_3P' + b_3Q') + kTT' &= 0, \\ (b_0T + b_2P + b_3Q)(-c_2T' + a_2P' + b_2Q') + (b_1T + b_2Q + b_3R)(-c_3T' + a_3P' + b_3Q') &= 0. \end{aligned}$$

53. This completes the proof of the equation

$$\Pi(x+A',\ y+H') = (-)^{q'}\,\Pi(x,\ y)\exp.-i\pi k(2x+A');$$

the proof of the remaining equation

$$\Pi(x+H',\ y+B') = (-)^{r'}\,\Pi(x,\ y)\exp.-i\pi k(2y+B'),$$

is of course precisely similar.

It has already been seen that $\mu'q'+\nu'r' = \mu q+\nu r$; we have

$$\Theta(-X,\ -Y) = (-)^{\mu q+\nu r}\,\Theta(X,\ Y),$$

and thence

$$\Pi(-x,\ -y) = (-)^{\mu q+\nu r}\,\Pi(x,\ y),$$

that is,

$$\Pi(-x,\ -y) = (-)^{\mu' q'+\nu' r'}\,\Pi(x,\ y),$$

which is the last of the equations which should be satisfied by the function $\Pi(x,\ y)$.

Recapitulation, and Final Form. Art. Nos. 54 to 58.

54. Recapitulating, we have a Hermitian k-matrix (k an odd prime)

$$\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right),$$

susceptible of $(1+k+k^2+k^3)$ forms; further, writing for shortness

$$\begin{aligned} z_0 &= a_0x+b_0y, \\ z_1 &= a_1x+b_1y, \\ z_2 &= a_2x+b_2y, \\ z_3 &= a_3x+b_3y, \end{aligned}$$

and then

$$\begin{aligned} X &= z_0+Az_2+Hz_3, \\ Y &= z_1+Hz_2+Bz_3, \end{aligned}$$

and assuming

$$\Pi(x,\ y) = \Theta\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(X,\ Y)(A,\ H,\ B)\exp.\ i\pi\{(z_0z_2+z_1z_3)+(A,\ H,\ B)(z_2,\ z_3)^2\},$$

then $\Pi(x,\ y)$ satisfies the equations

$$\begin{aligned} \Pi(x+1,\ \ y\ \ ) &= (-)^{\mu'}\,\Pi(x,\ y), \\ \Pi(\ \ x\ \ ,\ y+1) &= (-)^{\nu'}\,\Pi(x,\ y), \\ \Pi(x+A',\ y+H') &= (-)^{q'}\,\Pi(x,\ y)\exp.-i\pi k(2x+A'), \\ \Pi(x+H',\ y+B') &= (-)^{r'}\,\Pi(x,\ y)\exp.-i\pi k(2y+B'), \\ \Pi(-x,\ -y) &= (-)^{\mu' q'+\nu' r'}\,\Pi(x,\ y), \end{aligned}$$

and it is consequently a function

$$\Pi\begin{pmatrix}\mu', & \nu' \\ q', & r'\end{pmatrix}(x, y)(A', H', B')\{K, \text{def.}\},$$

and as such, contains linearly $\frac{1}{2}(k^2+1)$ constants.

55. The values of the new parameters are given as above, viz. T', P', Q', R', S' being linear functions of T, P, Q, R, S (the coefficients in these relations being given functions of the coefficients $(a, b, c, d)_{0,1,2,3}$ of the Hermitian matrix), then

$$1 : A : H : B : H^2 - AB = T : P : Q : R : S,$$

$$1 : A' : H' : B' : H'^2 - A'B' = T' : P' : Q' : R' : S',$$

which represent the required relations.

56. We consider four such functions $\Pi(x, y)$, derived from Θ-functions having respectively the characteristics

$$\begin{pmatrix}\mu_0, & \nu_0 \\ q_0, & r_0\end{pmatrix}, \quad \begin{pmatrix}\mu_1, & \nu_1 \\ q_1, & r_1\end{pmatrix}, \quad \begin{pmatrix}\mu_2, & \nu_2 \\ q_2, & r_2\end{pmatrix}, \quad \begin{pmatrix}\mu_3, & \nu_3 \\ q_3, & r_3\end{pmatrix},$$

being a (0123) system; and having consequently the characteristics

$$\begin{pmatrix}\mu_0', & \nu_0' \\ q_0', & r_0'\end{pmatrix}, \quad \begin{pmatrix}\mu_1', & \nu_1' \\ q_1', & r_1'\end{pmatrix}, \quad \begin{pmatrix}\mu_2', & \nu_2' \\ q_2', & r_2'\end{pmatrix}, \quad \begin{pmatrix}\mu_3', & \nu_3' \\ q_3', & r_3'\end{pmatrix},$$

which also form a (0123) system; say the four are Π_0, Π_1, Π_2, Π_3.

This being so, consider four theta-functions

$$\theta(x, y)(A', H', B'),$$

having respectively the characteristics

$$\begin{pmatrix}\mu_0', & \nu_0' \\ q_0', & r_0'\end{pmatrix}, \quad \begin{pmatrix}\mu_1', & \nu_1' \\ q_1', & r_1'\end{pmatrix}, \quad \begin{pmatrix}\mu_2', & \nu_2' \\ q_2', & r_2'\end{pmatrix}, \quad \begin{pmatrix}\mu_3', & \nu_3' \\ q_3', & r_3'\end{pmatrix},$$

which as already mentioned form a (0123) system; form with these the four sums

$$\Sigma\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}},$$

where in each case $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=k$, but in the first sum a, in the second sum b, in the third sum c, and in the fourth sum d is of contrary parity to the other three letters (even, if they are odd; odd, if they are even), say the four sums are Σ_0, Σ_1, Σ_2, Σ_3 respectively; each sum depends linearly on $\frac{1}{2}(k^2+1)$ constants.

The general transformation theorem then is

$$\Sigma_0 = \Pi_0, \quad \Sigma_1 = \Pi_1, \quad \Sigma_2 = \Pi_2, \quad \Sigma_3 = \Pi_3;$$

each of which equations in fact represents $\frac{1}{2}(k^2+1)$ equations; viz. on the left-hand side we assign to the constants $\frac{1}{2}(k^2+1)$ systems of values at pleasure, then to each such system there corresponds on the right-hand side a determinate system of values of the $\frac{1}{2}(k^2+1)$ constants.

57. The results may be presented in a more symmetrical form. Writing ξ, η, ζ, ω for the foregoing x, y, z, w, we may consider

$$\Theta\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(\xi+A'\zeta+H'\omega,\ \eta+H'\zeta+B'\omega)(A',\ H',\ B')$$

as a function of the four arguments ξ, η, ζ, ω, and express it by

$$\Phi\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(\xi,\ \eta,\ \zeta,\ \omega)(A',\ H',\ B').$$

Writing then

$$\begin{aligned} x &= \xi + A'\zeta + H'\omega,\\ y &= \eta + H'\zeta + B'\omega,\\ X,\ Y,\ Z,\ W &= \left(\begin{array}{cccc} a_0, & b_0, & c_0, & d_0\\ a_1, & b_1, & c_1, & d_1\\ a_2, & b_2, & c_2, & d_2\\ a_3, & b_3, & c_3, & d_3\end{array}\right)(\xi,\ \eta,\ \zeta,\ \omega),\end{aligned}$$

we have

$$\begin{aligned} z_1 + Az_2 + Hz_3 &= a_0x + b_0y + A\,(a_2x + b_2y) + H\,(a_3x + b_3y)\\ &= (a_0 + Aa_2 + Ha_3)\,(\xi + A'\zeta + H'\omega)\\ &\quad + (b_0 + Bb_2 + Hb_3)\,(\eta + H'\zeta + B'\omega),\end{aligned}$$

and also $X + AZ + HW =$

$$\begin{aligned} a_0\xi + b_0\eta + c_0\zeta + d_0\omega \quad &= \quad (a_0 + Aa_2 + Ha_3)\,\xi\\ + A\,(a_2\xi + b_2\eta + c_2\zeta + d_2\omega) \quad &\quad + (b_0 + Ab_2 + Hb_3)\,\eta\\ + H\,(a_3\xi + b_3\eta + c_3\zeta + d_3\omega) \quad &\quad + (c_0 + Ac_2 + Hc_3)\,\zeta\\ &\quad + (d_0 + Ad_2 + Hd_3)\,\omega,\end{aligned}$$

or since, as above,

$$\begin{aligned} A'\,(a_0 + Aa_2 + Ha_3) + H'\,(b_0 + Ab_2 + Hb_3) &= c_0 + Ac_2 + Hc_3,\\ H'\,(a_0 + Aa_2 + Ha_3) + B'\,(b_0 + Ab_2 + Hb_3) &= d_0 + Ad_2 + Hd_3,\end{aligned}$$

we find

$$z_0 + Az_2 + Hz_3 = X + AZ + HW;$$

and in like manner another equation, viz. we have

$$\begin{aligned} z_0 + Az_2 + Hz_3 &= X + AZ + HW,\\ z_1 + Hz_2 + Bz_3 &= Y + HZ + BW.\end{aligned}$$

Moreover

$$\begin{aligned}
z_0 z_2 + z_1 z_3 + (A,\ H,\ B \between z_2,\ z_3)^2 & \\
= \quad z_2 (z_0 + A z_2 + H z_3) = & \quad (a_2 x + b_2 y)(X + AZ + HW) \\
+ \ z_3 (z_1 + H z_2 + B z_3) & \quad + (a_3 x + b_3 y)(Y + HZ + BW) \\
= & \quad a_2 (\xi + A'\zeta + H'\omega)(X + AZ + HW) \\
& + b_2 (\eta + H'\zeta + B'\omega)(\qquad \text{,,} \qquad) \\
& + a_3 (\xi + A'\zeta + H'\omega)(Y + HZ + BW) \\
& + b_3 (\eta + H'\zeta + B'\omega)(\qquad \text{,,} \qquad),
\end{aligned}$$

which *quà* function of $X,\ Y,\ Z,\ W$ may be called χ.

58. And we then have the final result in the following form, viz.

$$\exp.\ i\pi\chi \cdot \Phi\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(X,\ Y,\ Z,\ W)(A,\ H,\ B),$$

in each of the four forms, is a homogeneous function of the order k of the corresponding four forms

$$\Phi\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(\xi,\ \eta,\ \zeta,\ \omega)(A',\ H',\ B').$$

849.

ON THE INVARIANTS OF A LINEAR DIFFERENTIAL EQUATION.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 257—261.]

CONSIDER the equation of the second order

$$\frac{d^2y}{dx^2}+2p\frac{dy}{dx}+qy=0;$$

if we effect a transformation of the dependent variable y, say $y=f.Y$, where f is an arbitrary function of x, we obtain a new equation

$$\frac{d^2Y}{dx^2}+2P\frac{dY}{dx}+QY=0,$$

where

$$P=p+\frac{f'}{f},$$

$$Q=q+2p\frac{f'}{f}+\frac{f''}{f},$$

(the accents denoting differentiation in regard to x); and we thence establish the identity

$$Q-P^2-P'=q-p^2-p',$$

viz. we have $q-p^2-p'$ a function possessing this invariantive property, but, as remarked by Mr Harley, it is the analogue rather of a seminvariant than of an invariant; I will call it an α-seminvariant of the differential equation. The class of function was considered long ago by Sir James Cockle, and more recently by Mr Malet; see Mr Harley's paper "Professor Malet's Classes of Invariants identified with Sir James Cockle's Criticoids," *Proc. R. Soc.*, vol. XXXVIII. (1884), pp. 45—57.

Effecting for the same equation a transformation of the independent variable, say $x=\phi$, an arbitrary function of X, we obtain a new equation

$$\frac{d^2y}{dX^2}+2P\frac{dy}{dX}+Qy=0,$$

where

$$P=p\phi_1-\tfrac{1}{2}\frac{\phi_2}{\phi_1},$$
$$Q=q\phi_1^2,$$

(the subscript numbers denoting differentiation in regard to X). There is not in the present case any function possessing a like invariantive property, say there is not any β-seminvariant; but such functions exist for differential equations of the third and higher orders, and have been considered by Sir James Cockle and Mr Malet. It is to be noticed that, attempting to obtain a β-seminvariant, we are led to the equation

$$Q-P^2-P_1=\phi_1^2(q-p^2-p')+\tfrac{1}{2}\left\{\frac{\phi_3}{\phi_1}-\tfrac{3}{2}\left(\frac{\phi_2}{\phi_1}\right)^2\right\},$$

which but for the last term would be invariantive. It is curious to find this last term presenting itself in the form of a Schwarzian derivative.

Passing to the differential equation of the third order

$$\frac{d^3y}{dx^3}+3p\frac{d^2y}{dx^2}+3q\frac{dy}{dx}+r=0,$$

here the transformation $y=f.Y$ of the dependent variable gives the new equation

$$\frac{d^3Y}{dx^3}+3P\frac{d^2Y}{dx^2}+3Q\frac{dY}{dx}+R=0,$$

where

$$P=p+\frac{f'}{f},$$
$$Q=q+2p\frac{f'}{f}+\frac{f''}{f},$$
$$R=r+3q\frac{f'}{f}+3p\frac{f''}{f}+\frac{f'''}{f};$$

and we thence derive as well the before-mentioned α-seminvariant $q-p^2-p'$, as also a new α-seminvariant $r-3pq+2p^3-p''$.

Again, effecting the transformation $x=\phi$ of the independent variable, we have the new equation

$$\frac{d^3y}{dX^3}+3P\frac{d^2y}{dX^2}+3Q\frac{dy}{dX}+R=0,$$

where

$$P=p\phi_1-\frac{\phi_2}{\phi_1},$$
$$Q=q\phi_1^2-p\phi_2+\left(\frac{\phi_2}{\phi_1}\right)^2-\tfrac{1}{3}\frac{\phi_3}{\phi_1},$$
$$R=r\phi_1^3.$$

We thence obtain the identities

$$\frac{P_1 + 2P^2 - 3Q}{R^{\frac{2}{3}}} = \frac{p' + 2p^2 - 3q}{r^{\frac{2}{3}}},$$

$$\frac{R_1 + 3PR}{R^{\frac{4}{3}}} = \frac{r' + 3pr}{r^{\frac{4}{3}}};$$

so that we have $\frac{p' + 2p^2 - 3q}{r^{\frac{2}{3}}}$ and $\frac{r' + 3pr}{r^{\frac{4}{3}}}$ as β-seminvariants of the differential equation of the third order.

But these are by no means the best conclusions; it is shown by M. Halphen in his great Memoir, "Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables," *Sav. Etrang.* t. XXVIII. (1884), pp. 1—297, see p. 127, that there is a function invariantive in regard to each of the two transformations, and which is thus an invariant, viz. this is the function

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3);$$

this is for the first transformation unaltered, viz. it is

$$= P'' - 3(Q' - 2PP') + 2(R - 3PQ + 2P^3),$$

and for the second transformation it is only altered by the factor ϕ_1^{-3}, viz. it is

$$= \phi_1^{-3}\{P_2 - 3(Q_1 - 2PP_1) + 2(R - 3PQ + 2P^3)\}.$$

It is interesting to directly verify this last result. Performing the differentiations in regard to X, we find without difficulty

$$P_2 = p_2\phi_1 + 2p_1\phi_2 + p\phi_3 - \frac{\phi_4}{\phi_1} + \frac{3\phi_2\phi_3}{\phi_1^2} - \frac{2\phi_2^3}{\phi_1^3},$$

$$Q_1 - 2PP_1 = (q_1 - 2pp_1)\phi_1^2 + p_1\phi_2 - 2p^2\phi_1\phi_2 + 2q\phi_1\phi_2 + p\phi_3 - \frac{\phi_4}{\phi_1} + \frac{\phi_2\phi_3}{\phi_1^2},$$

$$R - 3PQ + 2Q^3 = (r - 3pq + 2p^3)\phi_1^3 + p\phi_3 - 3p^3\phi_1\phi_2 + 3q\phi_1\phi_2 - \frac{\phi_2\phi_3}{\phi_1^2} + \frac{\phi_2^3}{\phi_1^3};$$

and thence

$$P_2 - 3(Q_1 - 2PP_1) + 2(R - 3PQ + 2Q^3) = p_2\phi_1 - 3(q_1 - 2pp_1)\phi_1^2 + 2(r - 3pq + 2p^3)\phi_1^3 - p_1\phi_2.$$

But, introducing herein the derived functions in regard to x, we have

$$p_1 = p'\phi_1, \quad q_1 = q'\phi_1, \quad p_2 = p'\phi_2 + p''\phi_1^2,$$

whence

$$p_2\phi_1 - p_1\phi_2 = p'''\phi_1^3;$$

and the right-hand side becomes

$$= \phi_1^3\{p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3)\},$$

which is the required result.

We have thus

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3)$$

as an "invariant" of the differential equation

$$\frac{d^3y}{dx^3} + 3p\frac{d^2y}{dx^2} + 3q\frac{dy}{dx} + r = 0.$$

It is to be remarked that this is *not* what M. Halphen calls a "differential invariant;" he uses this expression not in regard to a differential equation, but in regard to a curve defined by an equation between the coordinates (x, y), and the differential invariant is a function of the derivatives $y'', y''', \ldots$ which is either $=0$ in virtue of the equation of the curve, or else, being put $=0$, it determines certain singularities of the curve. Thus y'' is a differential invariant; the equation $y''=0$ determines the points of inflexion. Again

$$\tfrac{1}{3}\left(\frac{y'''}{y''}\right)'' - \tfrac{2}{3}\frac{y'''}{y''}\left(\frac{y'''}{y''}\right)' + \tfrac{4}{27}\left(\frac{y'''}{y''}\right)^3$$

is a differential invariant, vanishing identically if the variables (x, y) are connected by any quadric equation whatever; it is thus the differential invariant of a conic.

This last differential invariant is intimately connected with the above-mentioned invariant

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3),$$

viz. writing with M. Halphen

$$p = -\tfrac{1}{3}\frac{y'''}{y''},\ q = 0,\ r = 0,$$

we have

$$\begin{aligned} p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3) &= p'' + 6pp' + 4p^3 \\ &= -\left\{\tfrac{1}{3}\left(\frac{y'''}{y''}\right)'' - \tfrac{2}{3}\frac{y'''}{y''}\left(\frac{y'''}{y''}\right)' + \tfrac{4}{27}\left(\frac{y'''}{y''}\right)^3\right\}. \end{aligned}$$

It is moreover noticed by him that, writing

$$y'', y''', y'''' = 2a,\ 6b,\ 24c,\ 20d,$$

respectively, the function in { } becomes

$$= \frac{20}{a^2}(a^2d - 3abc + 2b^3);$$

the form under which he had previously obtained the differential invariant of the conic. As remarked by Sylvester, it is mentioned pp. 19 and 20 in Boole's *Differential Equations* (Cambridge, 1859), that the general differential equation of a conic was obtained by Monge in the form

$$9y''^2y'''' - 45y''y'''y'''' + 40(y''')^3 = 0,$$

which (representing the differential coefficients as just mentioned) becomes $a^2d - 3abc + 2b^3 = 0$, but putting them $= a, b, c, d$ respectively, it becomes $9a^2d - 45abc + 40b^3 = 0$; the last-mentioned form presented itself to Sylvester in his theory of Reciprocants.

850.

ON LINEAR DIFFERENTIAL EQUATIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 321—331.]

1. THE researches of Fuchs, Thomé, Frobenius, Tannery, Floquet, and others, relate to linear differential equations of the form

$$p_0 \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y = 0,$$

or say

$$(p_0, p_1, \ldots, p_m \between \frac{d}{dx}, 1)^m y = 0,$$

where $p_0, p_1, \ldots, p_m$ are rational and integral functions of the independent variable x; any common factor of all the functions could of course be thrown out, and it is therefore assumed that the functions have no common factor. It is to be throughout understood that x and y denote complex magnitudes, which may be regarded as points in the infinite plane; viz. $x, = \xi + i\eta$, is the point the coordinates whereof are ξ, η: and similarly for y.

2. Suppose $x - a$ is *not* a factor of p_0, the point $x = a$ is in this case said to be an ordinary point in regard to the differential equation; and let $y_0, y_1, y_2, \ldots, y_{m-1}$ be arbitrary constants denoting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{m-1}y}{dx^{m-1}}$ for the point $x = a$. We can from the differential equation, and the equations derived therefrom by successive differentiations in regard to x, obtain the values for $x = a$ of the subsequent differential coefficients $\frac{d^m y}{dx^m}, \frac{d^{m+1}y}{dx^{m+1}}, \ldots$, say these are $y_m, y_{m+1}, \ldots$; viz. the value of each of these quantities will be determined as a linear function of $y_0, y_1, \ldots, y_{m-1}$; and we thus have a development of y in positive integer powers of $x - a$, viz. this is

$$y = y_0 + y_1 (x - a) + \frac{1}{1.2} y_2 (x - a)^2 + \ldots,$$

which will in fact be a sum of m determinate series multiplied by $y_0, y_1, y_2, \ldots, y_{m-1}$ respectively; say the form is

$$y = y_0 X_0 + y_1 X_1 + \ldots + y_{m-1} X_{m-1},$$

where $X_0, X_1, \ldots, X_{m-1}$ are each of them a series of positive integer powers of $x-a$. Each of these series will be convergent for values of x sufficiently near to a, or say for points x within the domain of the point a; and since $y_0, y_1, \ldots, y_{m-1}$ are arbitrary, each series separately will satisfy the differential equation; and we have thus m particular integrals of the differential equation.

3. Suppose next that $x-a$ is a factor of p_0, the point $x=a$ is in this case said to be a singular point in regard to the differential equation. The foregoing process of development fails, as leading to infinite values of $\frac{d^m y}{dx^m}, \frac{d^{m+1} y}{dx^{m+1}}, \ldots$; and we have to consider the developments of y which belong to the neighbourhood of the point $x=a$, or say to the domain of the singular point $x=a$. This has to be done separately in regard to each of the singular points, and among these we have, it may be, the point ∞. To decide whether ∞ is or is not a singular point, we may in the differential equation write $x = 1/t$; and then transforming to this new variable, and throwing out any common factor, the coefficients of the transformed equation will be rational and integral functions of t, without any common factor; say these are $P_0, P_1, \ldots, P_m$; if t is not a factor of P_0, then ∞ will be an ordinary point of the original equation; but if t is a factor of P_0, then ∞ will be a singular point of the original equation.

4. In considering the singular point $x=a$, we may, it is clear, transform the equation to this point as origin, and it is convenient to do this; supposing it done, we have an equation wherein $x=0$ is a singular point, viz. p_0 contains x as a factor, and we have to consider the developments of y which belong to the domain of this singular point $x=0$.

It is convenient to change the form of the differential equation by dividing the whole equation by the first coefficient p_0, and then expanding each of the quotients $\frac{p_1}{p_0}, \frac{p_2}{p_0}, \ldots, \frac{p_m}{p_0}$ in a series of ascending powers of x. The new form is

$$\frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y = 0,$$

or say

$$P(y) = (1, p_1, \ldots, p_m \between \frac{d}{dx}, 1)^m y = 0,$$

where $p_1, p_2, \ldots, p_m$ now denote each of them a series (finite or infinite) of integer powers of x, but containing only a finite number of negative powers of x.

5. Such an equation frequently admits of a "regular integral" of the form $y = x^\rho E(x)$, where $E(x)$ is a series of positive integer powers $E_0 + E_1 x + E_2 x^2 + \ldots$, ($E_0$ not $=0$, for this would imply a different value of ρ)*. To determine whether

* The expression regular integral is afterwards used in a more general sense, see *post*, No. 10.

this is so, we substitute in the differential equation for y the value in question $x^\rho E(x)$, thus obtaining a series

$$\Omega_0 x^{\rho-\theta} + \Omega_1 x^{\rho-\theta+1} + \ldots,$$

(where θ is a determinate positive integer depending on the negative powers of x in the equation); the coefficients $\Omega_0, \Omega_1, \ldots$ are functions of ρ of an order not exceeding m, and contain also the coefficients $E_0, E_1, E_2, \ldots$ linearly; in particular, Ω_0 contains E_0 as a factor, say its value is $= E_0 \Pi_0$. The series should vanish identically. Supposing that Π_0 contains ρ, then we have $\Pi_0 = 0$, an equation of an order not exceeding m for the determination of ρ. For any root $\rho = \rho_0$ of this equation, E_0 remains arbitrary and may be taken $= 1$; the equations $\Omega_1 = 0$, $\Omega_2 = 0, \ldots$ then serve to determine the ratios to E_0 of the remaining coefficients $E_1, E_2, \ldots$; and we thus have the solution $y = x^{\rho_0}(1 + E_1 x + E_2 x^2 + \ldots)$, where ρ_0 and the coefficients have determinate values.

6. I stop to notice a curious form of illusory solution; the assumed form of solution is

$$y = x^\rho (\ldots + E_{-2}x^{-2} + E_{-1}x^{-1} + E_0 + E_1 x + \ldots),$$

the series being a double series extending both ways to infinity, or say a back-and-forward series; we have here a series of equations

$$\ldots \Omega_{-2} = 0, \quad \Omega_{-1} = 0, \quad \Omega_0 = 0, \quad \Omega_1 = 0, \ldots,$$

which leave ρ undetermined, but determine the ratios of the several coefficients to one of these coefficients, say E_0; or taking this $= 1$, we have a solution

$$y = x^\rho (\ldots + E_{-2}x^{-2} + E_{-1}x^{-1} + 1 + E_1 x + E_2 x^2 + \ldots)$$

where the coefficients are determinate functions of the arbitrary symbol ρ. Such a series is in general divergent for all values of the variable, and thus is altogether without meaning. As a simple instance, take the differential equation $\frac{dy}{dx} - y = 0$, which is satisfied by

$$y = \left\{\ldots (\rho - 1)\rho x^{\rho-2} + \rho x^{\rho-1} + x^\rho + \frac{x^{\rho+1}}{\rho+1} + \ldots\right\};$$

see my paper, Cayley, Note on Riemann's paper, "Versuch einer allgemeinen Auffassung der Integration und Differentiation," *Werke*, pp. 331—344; *Math. Ann.* t. XVI. (1880), pp. 81, 82), [751].

7. A more general form of integral is Thomé's "normal elementary integral," $y = e^w x^\rho E(x)$, where w is $=$ a finite series $\frac{c_{\alpha-1}}{x^{\alpha-1}} + \ldots + \frac{c_1}{x}$ of negative powers of x (α a positive integer, $= 2$ at least). To discover whether such a form exists, observe that, writing for a moment $\frac{dy}{dx} = y'$, and so for the other symbols, we have $\frac{y'}{y} = w' + \frac{\rho}{x} + \frac{E'(x)}{E(x)}$,

where the last term $\frac{E'(x)}{E(x)}$ may be expanded as a series of positive integer powers of x, so that, writing $\frac{y'}{y}=z$, we have

$$z=-\frac{(\alpha-1)\,c_{\alpha-1}}{x^{\alpha}}-\ldots-\frac{c_1}{x^2}+\frac{\rho}{x}+G_0+G_1x+\ldots;$$

we can, by introducing into the differential equation the new variable $z\left(=\frac{y'}{y}\right)$ in place of y, obtain for z a differential equation (not a linear one) of the order $m-1$, and this should be satisfied by the series in question, viz. a series containing a finite number of negative powers of x; we endeavour to determine the first term $\frac{-(\alpha-1)c_{\alpha-1}}{x^{\alpha}}$, or say $\frac{A}{x^{\alpha}}$, where the exponent α is a positive integer which is to be determined, and A is not $=0$; this is done by a well-known process; we write in the equation $z=\frac{A}{x^{\alpha}}$, and (if possible to do so) determine α (a positive integer $=2$ at least) by the condition that two or more of the terms shall have the same negative index $-p$ preceding in order all the other indices, viz. the absolute value of p must be greater than the absolute value of any other negative index; we then equate to zero the whole coefficient of the term in question x^{-p}, and thus obtain a value not $=0$ of A. And having thus obtained the first term $\frac{A}{x^{\alpha}}$, we can, by assuming the form

$$z=\frac{A}{x^{\alpha}}+\frac{B}{x^{\alpha-1}}+\ldots+\frac{K}{x^2}+\frac{L}{x}+\ldots$$

with indeterminate coefficients, and substituting in the equation, find the remaining coefficients $B,\ldots,K,L$; we then have

$$c_{\alpha-1}=\frac{-A}{\alpha-1},\ \ldots,\ c_1=-K,$$

giving the value $\frac{c_{\alpha-1}}{x^{\alpha-1}}+\ldots+\frac{c_1}{x}$ of w; instead of determining the subsequent coefficients from the z-equation, we may, from the original equation, writing therein $y=e^{w}Y$, obtain an equation for $Y(=e^{-w}y)$ which will be linear of the order m, and will admit of a regular integral $Y=x^{\rho}E(x)$. Or we may, going on a step further with the z-equation, find therefrom the coefficient L which is $=\rho$.

8. As an example, take the equation

$$\frac{d^2y}{dx^2}+\left(\frac{3}{x}-\frac{1}{x^2}\right)\frac{dy}{dx}+\frac{1}{x^2}y=0,$$

which has the elementary normal integral

$$y=e^{-\frac{1}{x}}\frac{1}{x}.$$

To investigate this, writing $\frac{y'}{y}=z$, we have $\frac{y''}{y}=z'+z^2$, and thence the z-equation

$$\frac{dz}{dx}+z^2+\left(\frac{3}{x}-\frac{1}{x^3}\right)z+\frac{1}{x^2}=0.$$

Substituting herein the value $z=\frac{A}{x^\alpha}$, we obtain the function

$$-\frac{A\alpha}{x^{\alpha+1}}+\frac{A^2}{x^{2\alpha}}-\frac{A}{x^{\alpha+2}}+\frac{1}{x^2},$$

and here the value $\alpha=2$ gives the two terms $\frac{A^2}{x^4}$ and $\frac{-A}{x^4}$, each with a negative index -4, the absolute value whereof is greater than the absolute values of the other indices; we then have $A^2-A=0$, giving a value $A=1$, which is not $=0$; and we have thus a first term $z=\frac{1}{x^2}$; the complete value is, as it happens, the finite series $z=\frac{1}{x^2}-\frac{1}{x}$; and z being $=\frac{y'}{y}$, we thence obtain the integral in question, $y=e^{-\frac{1}{x}}\frac{1}{x}$.

9. Returning to the question of the regular integrals, we may consider the function

$$P(x^\rho),\ =(1,\ p_1,\ \ldots,\ p_m\between\frac{d}{dx},\ 1)^m\, x^\rho,$$

say this is the so-called "determinirende Function," but I will call it the Indicial function; this is a function of $(x,\ \rho)$; we have therein terms arising from

$$\left(\frac{d}{dx}\right)^m x^\rho,\quad \left(\frac{d}{dx}\right)^{m-1} x^\rho,\ \ldots,$$

viz. these are equal to $[\rho]^m x^{\rho-m}$, $[\rho]^{m-1}x^{\rho-m-1}, \ldots$ respectively, but these will be multiplied by powers of x contained in the coefficients $p_1,\ p_2,\ \ldots,\ p_m$ respectively; the function is thus of the degree m as regards ρ, but the coefficient of any power of x is of the degree m, or of some inferior degree, according to the terms $[\rho]^m$, $[\rho]^{m-1}, \ldots$ which enter into the coefficient. The coefficient of the lowest power of x in the indicial function may be termed the indicial coefficient, or simply the indicial; and the equation obtained by equating this coefficient to zero the indicial equation. By what precedes, the indicial is a function of ρ, which is of the degree m at most, but which may be of any inferior degree, or even be an absolute constant. Hence, considering a regular integral $x^\rho E(x)$, the index ρ is determined by the indicial equation, and to each root of this equation there corresponds in general a regular integral $x^\rho E(x)$; the number of regular integrals is thus at most $=m$.

10. Suppose, first, that the indicial equation is of the degree m, and that the roots of this equation are all unequal; there are, in this case, m values, say $\rho_1,\ \rho_2,\ \ldots,\ \rho_m$ of ρ, and each of these gives rise to a regular integral $x^\rho E(x)$, viz. there will be corresponding to each value of ρ a series of positive integer powers $E(x)$, which will be

a convergent series for sufficiently small values of x. Some of these series may be finite series, viz. this will be the case if the indicial equation has roots differing from each other by integer values (but a finite series may of course be regarded as convergent); and in the case of two or more equal roots, it is necessary to extend the notion of a regular integral by including under it series multiplied into positive integer powers of $\log x$; but, with this modification, the conclusion holds good; if the indicial equation be of the degree m, the number of regular integrals will always be $=m$, viz. there will be m integrals involving series $E(x)$, or, it may be, $E(x)(\log x)^k$, where each series $E(x)$ is a convergent series for sufficiently small values of x.

11. But suppose the indicial equation is of an order $m-\gamma$ inferior to m, and to fix the ideas suppose that the roots are all unequal. There are in this case $m-\gamma$ values, say $\rho_1, \rho_2, \ldots, \rho_{m-\gamma}$ of ρ, and each of these will apparently give rise to a regular integral $x^\rho E(x)$; but it may be that in any such case the series $E(x)$ is a series divergent for all values, however small, of the variable x, and that we thus have in appearance only, but not in reality, a regular integral; or say the integral is illusory. The conclusion is that the indicial equation being of a degree $m-\gamma$ inferior to m, the number of regular integrals is at most $=m-\gamma$; but it may have any less value than $m-\gamma$, or even there may be no regular integral. It is hardly necessary to remark that, if the indicial be an absolute constant, or say if the degree of the indicial equation be $=0$, then there is no value of ρ, and consequently no regular integral.

12. By way of illustration, consider first the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x}y=0,$$

for which the degree of the indicial equation is equal to the order of the equation $(=2)$; viz. the indicial function is

$$=\rho(\rho-1)x^{\rho-2}+\rho x^{\rho-2}+x^{\rho-1},$$

and the indicial equation is thus $\rho(\rho-1)+\rho=0$, that is, $\rho^2=0$, viz. there are here two roots, each $=0$.

It is convenient not in the first instance to write $\rho=0$, but to substitute in the equation the value

$$y=Ax^\rho+Bx^{\rho+1}+Cx^{\rho+2}+\ldots.$$

We thus find

$x^{\rho-2}$,	$x^{\rho-1}$,	$x^\rho, \ldots$
$0=\rho.\rho-1.A$,	$\rho+1.\rho.B$,	$\rho+2.\rho+1.C$
$+\quad\rho A$,	$+\rho+1.B$,	$+\rho+2.C$
	$+A$,	$+B$,

viz. we have the equations

$$\begin{aligned}\rho^2A \quad &=0,\\ (\rho+1)^2B+A&=0,\\ (\rho+2)^2C+B&=0,\end{aligned}$$

where observe that in each equation the function of ρ, which multiplies the posterior coefficient, is of a degree superior to that which multiplies the other coefficient. The first equation gives $\rho=0$; A arbitrary, but say $A=1$; the values of $B, C, \ldots$ are then

$$-\frac{1}{(\rho+1)^2},\quad +\frac{1}{(\rho+1)^2(\rho+2)^2},\quad -\frac{1}{(\rho+1)^2(\rho+2)^2(\rho+3)^2},\ldots$$

giving rise to a series which in this form, and when ρ is put equal to its value, $\rho=0$, is at once seen to be convergent (even for indefinitely large values of x, but this is an accident; it would have been enough if the series had been convergent for sufficiently small values), viz. the series is

$$y=1-\frac{x}{1^2}+\frac{x^2}{1^2\,.\,2^2}-\frac{x^3}{1^2\,.\,2^2\,.\,3^2}+\ldots.$$

There is another integral

$$y=\left(1-\frac{x}{1^2}+\frac{x^2}{1^2\,.\,2^2}-\ldots\right)\log x+2x-\tfrac{3}{4}x^2+\tfrac{11}{108}x^3-\tfrac{25}{3456}x^4+\ldots,$$

involving $\log x$ and a second series which is also convergent.

13. But consider next the before-mentioned equation

$$\frac{d^2y}{dx^2}+\left(\frac{3}{x}-\frac{1}{x^2}\right)\frac{dy}{dx}+\frac{1}{x^2}y=0,$$

for which the degree of the indicial equation is equal to order $-1\,(=1)$; in fact, here the indicial function is

$$=\rho\,(\rho-1)\,x^{\rho-2}+3\rho x^{\rho-2}-\rho x^{\rho-3}+x^{\rho-2},$$

and the indicial is thus $=-\rho$, giving the index $\rho=0$. There is thus at most one regular integral $E(x)$, but this is in fact an illusory one. Substituting in the equation the value

$$y=Ax^{\rho}+Bx^{\rho+1}+Cx^{\rho+2}+\ldots,$$

we find

	$x^{\rho-3}$,	$x^{\rho-2}$,	$x^{\rho-1}$,	x^{ρ}.
$0=$		$\rho\,(\rho-1)\,A$,	$(\rho+1)\,\rho B$,	$(\rho+2)\,(\rho+1)\,C$,
	$-\rho A$,	$-(\rho+1)\,B$,	$-\ (\rho+2)\,C$,	$-\ (\rho+3)\,D$,
		$+3\rho A$,	$+3\,(\rho+1)\,B$,	$+3\,(\rho+2)\,C$,
		$+\ A$,	$+\qquad B$,	$+\qquad C$,

and we thus have the equations

$$\begin{aligned}\rho A\qquad\qquad\qquad &=0,\\ (\rho+1)^2A-(\rho+1)\,B&=0,\\ (\rho+2)^2B-(\rho+2)\,C&=0,\\ (\rho+3)^2C-(\rho+3)\,D&=0,\end{aligned}$$

where observe that in each equation the function of ρ, which multiplies the posterior coefficient, is of a degree inferior to that which multiplies the other coefficient. The

first equation then gives $\rho=0$; A arbitrary, or say $=1$: the other coefficients $B, C, \ldots$ are then found to be

$$\rho+1, \quad (\rho+1)(\rho+2), \quad (\rho+1)(\rho+2)(\rho+3), \ldots$$

or substituting for ρ its value $=0$, the series is

$$1+1x+1.2.x^2+1.2.3x^3+\ldots,$$

which is divergent for any value whatever, however small, of x; or the integral is as mentioned an illusory one.

14. The last-mentioned differential equation is *reducible*, viz. we have

$$P(y)=\left\{\frac{d^2}{dx^2}+\left(\frac{3}{x}-\frac{1}{x^2}\right)\frac{d}{dx}+\frac{1}{x^2}\right\}y=\left(\frac{d}{dx}+\frac{2}{x}\right)\left(\frac{d}{dx}+\frac{1}{x}-\frac{1}{x^2}\right)y, \text{ say } =QDy,$$

where of course the order of the factors Q, D is material.

It is a general theorem that the indicial, belonging to the product P, is of a degree equal to the sum of the degrees of the indicials, belonging to the factors Q, D respectively; and, in fact, the indicial of P is (as was seen) of the degree 1; and the indicials of Q, D are of the degrees 1, 0 respectively. But, moreover, if the indicial of P is of a degree $m-\gamma$, less than m the order of the equation (in the present case $m=2$, $m-\gamma=1$), then, in order that the equation may have $m-\gamma$ regular integrals, it is a necessary and sufficient condition that P shall be decomposable into a product QD, where the factors Q, D (being functions of the same form as P) are of the orders γ and $m-\gamma$ respectively, and where, moreover, the *second* factor D shall have an indicial of the degree $m-\gamma$, and consequently the *first* factor Q an indicial of the degree zero. In the present case, it is the second factor which has an indicial of degree zero; and thus the condition is not satisfied. And accordingly the equation ought not to have $(m-\gamma=)$ 1 regular integral; and we have seen that there is in fact no regular integral. The investigation serves to show in what sense it is that there is no regular integral, or generally in what sense it is that the number of regular integrals may be less than $m-\gamma$.

15. The theorem may, it appears to me, be stated in the more general form: the necessary and sufficient condition that the differential equation $P(y)=0$ of the order m, but having an indicial of the degree $m-\gamma$, shall have $m-\gamma-\delta$ regular integrals is that the function P shall be a product of the form $P=QMD$, where the orders of Q, M, D are $\gamma+\delta-\theta$, θ, $m-\gamma-\delta$ respectively, and the degrees of their indicials δ, 0, $m-\gamma-\delta$ respectively; or, to denote this in a compendious manner, say

$$\underset{m-\gamma}{\overset{m}{P}}=\underset{\delta}{\overset{\gamma+\delta-\theta}{Q}}\;\underset{0}{\overset{\theta}{M}}\;\underset{m-\gamma-\delta}{\overset{m-\gamma-\delta}{D}};$$

θ is an integer which is not $=0$, and which is at most $=\gamma$, for otherwise the function

Q of the order $\gamma+\delta-\theta$ could not have an indicial of the degree δ. If there is no regular integral, then $m=\gamma+\delta$, D is a mere constant or say unity, and the formula is

$$\underset{m-\gamma}{\overset{m}{P}} = \underset{m-\gamma}{\overset{m-\theta}{Q}}\ \underset{0}{\overset{\theta}{M}},$$

where θ is not $=0$, and is at most $=m-\gamma$.

16. For differential equations of the form $P(y)=0$ above considered, Floquet gave a theorem, which as afterwards remarked by him is not generally true, viz. this was $P=ABC\ldots$, a product of linear factors of the form $\frac{d}{dx}+\Sigma_{-\infty}^{\infty}C_ix^i$ (i an integer). The question of decomposition is considered in my next following paper "On Linear Differential Equations (the Theory of Decomposition)," [851], and by means of the formulæ there given (and by the process indicated *ante* No. 7), it could be decided whether any particular differential equation admits of a decomposition $P=ABC\ldots$, where the linear factors are functions *not* of the form just referred to, but of the form $\frac{d}{dx}+\Sigma C_ix^i$, in which the series contains only a finite number of negative powers, say this is a decomposition into linear regular factors.

851.

ON LINEAR DIFFERENTIAL EQUATIONS (THE THEORY OF DECOMPOSITION).

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXI. (1886), pp. 331—335.]

1. In the theory of linear differential equations the question arises, to decompose a quantic $\left(*\between\frac{d}{dx}, 1\right)^n$ into linear factors: we have, for instance, a differential equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0,$$

which is to be expressed in the form

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right)y=0,$$

or say we have to express $\frac{d^2}{dx^2}+p\frac{d}{dx}+q$ as a product of linear factors

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right).$$

The problem is analogous to, but wholly distinct from, that of the resolution of an algebraic equation: using accents to denote differentiation in regard to x, the relation (in the simple case just referred to) between the coefficients (p, q) and the roots (α, β) is (not $p=\alpha+\beta$, $q=\alpha\beta$, but in place thereof) $p=\alpha+\beta$, $q=\alpha\beta+\beta'$; and it thus appears that there is the important distinction that, in the present problem, the order of the factors is not indifferent.

2. The problem may be solved when the general solution of the differential equation is known, or, what is the same thing, by means of n particular solutions

of the equation; it is not here considered in this point of view, but the intention is to treat it directly by means of the relations between the coefficients and the roots: thus in the above case, $p=\alpha+\beta$, $q=\alpha\beta+\beta'$, we may (1) find first β and then α, viz. eliminating α we have $\beta'+\beta(p-\beta)-q=0$, β determined by a differential equation (not linear) of the first order; and then $\alpha=p-\beta$; or we may (2) find first α and then β, viz. eliminating β, we have $\alpha'-p'+\alpha(\alpha-p)+q=0$, α determined by a differential equation (not linear) of the first order, and then $\beta=p-\alpha$.

In the case of a product

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right)\left(\frac{d}{dx}+\gamma\right)\ldots$$

of more than two factors, the roots might be determined in any order; but the two orders which naturally present themselves and which will be alone considered are, say, the reverse order $(\ldots, \gamma, \beta, \alpha)$ and the direct order $(\alpha, \beta, \gamma, \ldots)$.

3. *The Reverse Order.* Writing D for $\frac{d}{dx}$, we assume

$$\begin{aligned} D+\alpha &= (1,\ p_1 \between D,\ 1), \\ (D+\alpha)(D+\beta) &= (1,\ p_2,\ q_2 \between D,\ 1)^2, \\ (D+\alpha)(D+\beta)(D+\gamma) &= (1,\ p_3,\ q_3,\ r_3 \between D,\ 1)^3, \end{aligned}$$

and so on. Then using accents to denote differentiation, we find

$$\begin{aligned} (1)\quad & p_1 = \alpha. \\ (2)\quad & p_2 = \beta + p_1, \\ & q_2 = \beta' + p_1\beta. \\ (3)\quad & p_3 = \gamma + p_2, \\ & q_3 = 2\gamma' + p_2\gamma + q_2, \\ & r_3 = \gamma'' + p_2\gamma' + q_2\gamma. \\ (4)\quad & p_4 = \delta + p_3, \\ & q_4 = 3\delta' + p_3\delta + q_3, \\ & r_4 = 3\delta'' + 2p_3\delta' + q_3\delta + r_3, \\ & s_4 = \delta''' + p_3\delta'' + q_3\delta' + r_3\delta, \end{aligned}$$

where the law is obvious: thus in the last set of equations, the several columns (after the first) contain the factors p_3, q_3, r_3, s_3 respectively; and there is in each column a head term with the coefficient unity: omitting these head terms, we have in the several columns the sets $(\delta, 3\delta', 3\delta'', \delta''')$, $(\delta, 2\delta', \delta'')$, (δ, δ'), δ, of derivatives of the root δ.

4. We may from each set of equations determine in order the coefficients of lower rank contained in the equations, and the last equation of each set then gives an equation independent of these coefficients of lower rank. Thus set (2) gives

$$\begin{aligned} p_1 = -\beta \quad & 0 = \beta' - \beta^2 \\ + p_2; \quad & \quad + p_2\beta \\ & \quad - p_2. \end{aligned}$$

Set (3) gives

$$\begin{array}{lll} p_2 = -\gamma & q_2 = -2\gamma' + \gamma^2 & 0 = \gamma'' - 3\gamma\gamma' + \gamma^2 \\ \quad + p_3; & \quad - p_3\gamma & \quad + p_3(\gamma' - \gamma^2) \\ & \quad + q_3; & \quad + q_3\gamma \\ & & \quad - r_3. \end{array}$$

Set (4) gives

$$\begin{array}{lll} p_3 = -\delta & q_3 = -3\delta' + \delta^2 & r_3 = -3\delta'' + 5\delta\delta' - \delta^3 \\ \quad + p_4; & \quad - p_4\delta & \quad + p_4(-2\delta' + \delta^2) \\ & \quad + q_4; & \quad - q_4\delta \\ & & \quad + r_4; \end{array}$$

$$\begin{aligned} 0 = \ & \delta''' - 4\delta\delta'' + 6\delta^2\delta' - 3\delta'^2 - \delta^4 \\ & + p_4(\delta'' - 3\delta\delta' + \delta^3) \\ & + q_4(\delta' - \delta^2) \\ & + r_4\delta \\ & - s_4; \end{aligned}$$

and so on.

5. Thus suppose (p_4, q_4, r_4, s_4) are given, we have δ determined by a differential equation (not linear) of the third order; and δ being known, p_3, q_3, r_3 are also known; then γ is determined by a differential equation (not linear) of the second order, and γ being known, p_2, q_2 are also known; then β is determined by a differential equation (not linear) of the first order, and β being known, p_1, that is, α is also known.

Comparing the last equations of each set, that is, the equations for the determination of β, γ, δ, ... respectively, it will be observed that they depend on the single series of derivatives

$$\begin{aligned} & -1, \\ & \phi, \\ & \phi' - \phi^2, \\ & \phi'' - 3\phi\phi' + \phi^3, \\ & \phi''' - 4\phi\phi'' + 6\phi^2\phi' - 3\phi'^2 - \phi^4, \end{aligned}$$

where, calling the successive terms A, B, C, D, E, ..., we have

$$B = A' - \phi A,\ C = B' - \phi B,\ D = C' - \phi C,\ E = D' - \phi D.$$

6. *Direct Order.* Considering the product

$$\left(\frac{d}{dx} + \alpha\right)\left(\frac{d}{dx} + \beta\right)\left(\frac{d}{dx} + \gamma\right)\left(\frac{d}{dx} + \delta\right)$$

of four factors, and assuming

$$D+\delta = (1,\ p_1 \between D,\ 1),$$
$$(D+\gamma)(D+\delta) = (1,\ p_2,\ q_2 \between D,\ 1)^2,$$
$$(D+\beta)(D+\gamma)(D+\delta) = (1,\ p_3,\ q_3,\ r_3 \between D,\ 1)^3,$$
$$(D+\alpha)(D+\beta)(D+\gamma)(D+\delta) = (1,\ p_4,\ q_4,\ r_4,\ s_4 \between D,\ 1)^4,$$

we have the sets of equations

$$\begin{aligned}
(1)\quad & p_1 = \delta. \\
(2)\quad & p_2 = \gamma + p_1, \\
& q_2 = \quad \gamma p_1 + p_1'. \\
(3)\quad & p_3 = \beta + p_2, \\
& q_3 = \quad \beta p_2 + p_2' + q_2, \\
& r_3 = \qquad\qquad \beta q_2 + q_3'. \\
(4)\quad & p_4 = \alpha + p_3, \\
& q_4 = \quad \alpha p_3 + p_3' + q_3, \\
& r_4 = \qquad\qquad \alpha q_3 + q_3' + r_3, \\
& s_4 = \qquad\qquad\qquad\qquad \alpha r_3 + r_3'.
\end{aligned}$$

7. I stop to remark that there would have been a convenience in considering the product

$$\ldots\left(\frac{d}{dx}+\delta\right)\left(\frac{d}{dx}+\gamma\right)\left(\frac{d}{dx}+\beta\right)\left(\frac{d}{dx}+\alpha\right);$$

for the first, second, third, &c., sets of equations would then have contained α, β, γ, &c., respectively; but for better comparison with the equations of the reverse order, I have preferred not to make this change of notation.

8. The set (1) gives

$$\delta - p_1 = 0.$$

Set (2) gives

$$\begin{aligned}
p_1 = -\gamma + p_2, \quad 0 = \ & \gamma' + \gamma^2 \\
& -p_2\gamma - p_2' \\
& +q_2.
\end{aligned}$$

Set (3) gives

$$\begin{aligned}
p_2 = -\beta + p_3, \quad q_2 = \ & \beta' + \beta^2 \quad & 0 = \ & \beta'' + 3\beta\beta' + \beta^3 \\
& -p_3\beta - p_3' & & -p_3(\beta' + \beta^2) - 2p\beta_3 - p_3'' \\
& +q_3, & & +q_3\beta + q_3' \\
& & & -r_3.
\end{aligned}$$

Set (4) gives

$$\begin{aligned} p_3 = -\alpha + p_4, \quad q_3 = \; & \alpha' + \alpha^2 & r_3 = -(\alpha'' + 3\alpha\alpha' + \alpha^3) \\ & -p_4\alpha - p' & \quad + p_4(\alpha^2 + \alpha') + 2p_4'\alpha + p_4'' \\ & + p_4, & \quad - q_4\alpha - q_4' \\ & & \quad + r_4, \end{aligned}$$

$$\begin{aligned} 0 = \; & \alpha''' + 4\alpha\alpha'' + 6\alpha^2\alpha' + 3\alpha'^2 + \alpha^4 \\ & - p_4(\alpha'' + 3\alpha\alpha' + \alpha^3) - p_4'(3\alpha' + 3\alpha^2) - 3p_4''\alpha - p_4''' \\ & + q_4(\alpha' + \alpha^2) + 2q_4'\alpha + q_4'' \\ & - r_4\alpha - r_4' \\ & + s_4, \end{aligned}$$

which differ in form from the equations belonging to the reverse order in containing the derived functions p_4', p_4'', p_4''', q_4', ... of the coefficients.

9. Taking p_4, q_4, r_4, s_4 as known, we have α determined by a differential equation (not linear) of the third order; and α being known, we know p_3, q_3, r_3. We then have β determined by a differential equation (not linear) of the second order; and β being known, we know p_2, q_2. We then have γ determined by a differential equation (not linear) of the first order; and γ being known, we know p_1, that is, δ.

852.

NOTE SUR LE MÉMOIRE DE M. PICARD "SUR LES INTÉGRALES DE DIFFÉRENTIELLES TOTALES ALGÉBRIQUES DE PREMIÈRE ESPÈCE."

[From the *Bulletin des Sciences Mathématiques*, 2me Sér., t. x. (1886), pp. 75—78.]

On peut présenter l'analyse sur laquelle est fondé le Mémoire sous une forme plus symétrique en introduisant dès le commencement les fonctions homogènes.

Soit $f=(*)(x, y, z, t)^m$ une fonction du degré m des variables x, y, z, t, lesquelles seront toujours liées par l'équation $f=0$: écrivons aussi $\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}, \frac{df}{dt}=X, Y, Z, T$, de manière que X, Y, Z, T sont des fonctions du degré $m-1$: et soient A, B, C, D des fonctions chacune du degré $m-3$ et Q une fonction du degré $m-4$, telles que $AX+BY+CZ+DT=Qf$ identiquement; donc, en supposant $f=0$, on aura

$$AX+BY+CZ+DT=0.$$

On vérifie sans peine que l'expression

$$d\Omega=\begin{vmatrix} \lambda, & \mu, & \nu, & \rho \\ A, & B, & C, & D \\ x, & y, & z, & t \\ dx, & dy, & dz, & dt \end{vmatrix} \div (\lambda X+\mu Y+\nu Z+\rho T)$$

est indépendante des valeurs de λ, μ, ν, ρ, et ainsi égale à chacune des quatre expressions $d\Omega_x, d\Omega_y, d\Omega_z, d\Omega_t$,

$$=\frac{1}{X}\begin{vmatrix} B, & C, & D \\ y, & z, & t \\ dy, & dz, & dt \end{vmatrix}, \quad -\frac{1}{Y}\begin{vmatrix} C, & D, & A \\ z, & t, & x \\ dz, & dt, & dx \end{vmatrix},$$

$$\frac{1}{Z}\begin{vmatrix} D, & A, & B \\ t, & x, & y \\ dt, & dx, & dy \end{vmatrix}, \quad -\frac{1}{T}\begin{vmatrix} A, & B, & C \\ x, & y, & z \\ dx, & dy, & dz \end{vmatrix},$$

respectivement.

Cela étant, soit

$$d\Omega_t = -\frac{1}{T}\begin{vmatrix} A, & B, & C \\ x, & y, & z \\ dx, & dy, & dz \end{vmatrix} = \text{une différentielle totale};$$

en écrivant pour un moment $x, y, z = x't, y't, z't$, et en dénotant par f' la fonction $(*)(x', y', z', 1)^m$, et de même par $X', Y', Z', T', A', B', C'$ les valeurs correspondantes de X, Y, Z, T, A, B, C, les variables x', y', z' seront liées par l'équation $f' = 0$, ce qui donne

$$X'dx' + Y'dy' + Z'dz' = 0;$$

et l'on voit sans peine que l'expression de $d\Omega_t$ se réduit à

$$-\frac{1}{T'}\begin{vmatrix} A', & B', & C' \\ x', & y', & z' \\ dx', & dy', & dz' \end{vmatrix},$$

fonction de la forme

$$F'dx' + G'dy' + H'dz',$$

qui ne contient que les variables x', y', z'. Donc, en omettant les accents, il est permis de prendre $t = \text{const.}$, ce qui donne

$$X\,dx + Y\,dy + Z\,dz = 0;$$

et avec cette relation entre les différentielles dx, dy, dz, de faire que $d\Omega_t = F\,dx + G\,dy + H\,dz$ soit une différentielle totale: cela donne la condition

$$X\left(\frac{dG}{dz} - \frac{dH}{dy}\right) + Y\left(\frac{dH}{dx} - \frac{dF}{dz}\right) + Z\left(\frac{dF}{dy} - \frac{dG}{dx}\right) = 0,$$

ou enfin

$$\begin{aligned} &X\left(\frac{d}{dz}\,\frac{Cx - Az}{T} - \frac{d}{dy}\,\frac{Ay - Bx}{T}\right) \\ &+ Y\left(\frac{d}{dx}\,\frac{Ay - Bx}{T} - \frac{d}{dz}\,\frac{Bz - Cy}{T}\right) \\ &+ Z\left(\frac{d}{dy}\,\frac{Bz - Cy}{T} - \frac{d}{dx}\,\frac{Cx - Az}{T}\right) = 0. \end{aligned}$$

On a d'abord un terme $\mathfrak{A} \div T$, où

$$\begin{aligned}\mathfrak{A} = \; & X\left[-2A + x\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)-\left(x\frac{dA}{dx}+y\frac{dA}{dy}+z\frac{dA}{dz}\right)\right]\\ & + Y\left[-2B + y\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)-\left(x\frac{dB}{dx}+y\frac{dB}{dy}+z\frac{dB}{dz}\right)\right]\\ & + Z\left[-2C + z\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)-\left(x\frac{dC}{dx}+y\frac{dC}{dy}+z\frac{dC}{dz}\right)\right],\end{aligned}$$

ou, en réduisant,

$$\begin{aligned}\mathfrak{A} = & -2(AX+BY+CZ)\\ & + (Xx+Yy+Zz)\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)\\ & - (m-3)(AX+BY+CZ)\\ & + t\left(X\frac{dA}{dt}+Y\frac{dB}{dt}+Z\frac{dC}{dt}\right)\\ = & \; (m-1)DT\\ & - Tt\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}+\frac{dD}{dt}\right)\\ & + t\left(X\frac{dA}{dt}+Y\frac{dB}{dt}+Z\frac{dC}{dt}+T\frac{dD}{dt}\right);\end{aligned}$$

puis un terme $\mathfrak{B} \div T^2$, où

$$\begin{aligned}\mathfrak{B} = & -X\left[(Cx-Az)\frac{dT}{dz}-(Ay-Bx)\frac{dT}{dy}\right]\\ & - Y\left[(Ay-Bx)\frac{dT}{dx}-(Bz-Cy)\frac{dT}{dz}\right]\\ & - Z\left[(Bz-Cy)\frac{dT}{dy}-(Cx-Az)\frac{dT}{dx}\right],\end{aligned}$$

ou, en réduisant,

$$\begin{aligned}\mathfrak{B} = & \frac{dT}{dx}[(AX+BY+CZ)x - A(xX+yY+zZ)]\\ & + \frac{dT}{dy}[(AX+BY+CZ)y - B(xX+yY+zZ)]\\ & + \frac{dT}{dz}[(AX+BY+CZ)z - C(xX+yY+zZ)]\\ = & -DT\left(x\frac{dT}{dx}+y\frac{dT}{dy}+z\frac{dT}{dz}\right)\\ & + tT\left(A\frac{dT}{dx}+B\frac{dT}{dy}+C\frac{dT}{dz}\right)\\ = & -(m-1)DT^2\\ & + Tt\left(A\frac{dX}{dt}+B\frac{dY}{dt}+C\frac{dZ}{dt}+D\frac{dT}{dt}\right).\end{aligned}$$

Donc, en réunissant les deux parties, $0 = \mathfrak{A} + \frac{\mathfrak{B}}{T}$, c'est-à-dire,

$$\begin{aligned} 0 &= -Tt\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} + \frac{dD}{dt}\right) \\ &\quad + t\frac{d}{dt}(AX + BY + CZ + DT) \\ &= -Tt\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} + \frac{dD}{dt}\right) \\ &\quad + tQT, \end{aligned}$$

ou, en omettant le facteur tT, on obtient enfin

$$0 = Q - \left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} + \frac{dD}{dt}\right),$$

c'est-à-dire que les fonctions A, B, C, D sont telles que

$$AX + BY + CZ + DT = \left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} + \frac{dD}{dt}\right)f;$$

et, cela étant, l'expression générale $d\Omega$, et de même chacune des expressions $d\Omega_x$, $d\Omega_y$, $d\Omega_z$, $d\Omega_t$, sera égale à une différentielle totale.

Cambridge, le 8 *janvier* 1886.

853.

NOTE ON A FORMULA FOR $\Delta^n 0^i/n^i$ WHEN n, i ARE VERY LARGE NUMBERS.

[From the *Proceedings of the Royal Society of Edinburgh*, t. XIV. (1887), pp. 149—153.]

THE following formula

$$\frac{\Delta^n 0^i}{n^i} = e^{-nq}\left[1 + \left(\frac{i+1-2n}{2n}\right)q - \left(\frac{n+i+2}{2}\right)q^2\right],\ q = e^{-(i+1)/n},$$

is given by Laplace (*Théorie Analytique des Probabilités*, 2nd ed., Paris, 1814, p. 195) as an approximate value of $\Delta^n 0^i/n^i$, when n and i are very large numbers, and is applied immediately afterwards to the case where i is of the order $n \log n$. As remarked by Professor Tait, it is certainly not applicable to the case where i is of the order n; for taking $i = An$, where A is a given number however large, then q is indefinitely near to the very small value e^{-A}, but nevertheless the last term $-\frac{1}{2}(n+i+2)q^2$, by taking n sufficiently large, may be made as large as we please, and the value would thus come out negative. It is thus necessary that i should be at least of the order $n \log n$; but it may be of any higher order.

Writing for greater convenience $r = ne^{-i/n}$ (where r is not very large), then $nq = re^{-1/n} = r(1-X)$, if $X = 1 - e^{-1/n}$; and the formula becomes

$$\frac{\Delta^n 0^i}{n^i} = e^{-r(1-X)}\left[1 + \frac{i+1-2n}{2n}\, e^{-1/n}\frac{r}{n} - \frac{n+i+2}{2}\, e^{-2/n}\frac{r^2}{n^2}\right].$$

Here $X = \frac{1}{n} - \frac{1}{1.2}\frac{1}{n^2} + \frac{1}{1.2.3}\frac{1}{n^3} + \&c.$, and the exponential $e^{rX} = 1 + rX + \frac{r^2X^2}{1.2} + \ldots$ is thus also expansible in negative powers of n; the formula becomes

$$\frac{\Delta^n 0^i}{n^i} = e^{-r}\left(1 + rX + \frac{r^2X^2}{1.2} + \ldots\right)\left[1 + \frac{i+1-2n}{2n}\,.\,e^{-1/n}\frac{r}{n} - \frac{n+i+2}{2}\, e^{-2/n}\frac{r^2}{n^2}\right];$$

viz. putting for X its value,

$$= e^{-r}\Big\{1$$

$$+ r\left(\frac{i+1-2n}{2n^2}e^{-1/n}+1-e^{-1/n}\right)$$

$$+ r^2\left(\frac{-n-i-2}{2n^2}e^{-2/n}+\frac{i+1-2n}{2n^2}(1-e^{-1/n})\,e^{-1/n}+\tfrac{1}{2}(1-e^{-1/n})^2\right)$$

$$+ \&c.\Big\};$$

or finally, expanding $e^{-1/n}$ and taking the whole result as far as $\frac{1}{n^2}$, the coefficient of r is

$$\left(-\frac{1}{n}+\frac{i+1}{2n^2}\right)\left(1-\frac{1}{n}\right)+\frac{1}{n}-\frac{1}{2n^2},\quad =\frac{1+\frac{1}{2}i}{n^2};$$

the coefficient of r^2 is

$$\left(-\frac{1}{n}+\frac{-2-i}{n^2}\right)\left(1-\frac{2}{n}\right)+\left(-\frac{1}{n}+\frac{4i}{2n^2}\right)\frac{1}{n}+\frac{1}{2n^2},\quad =-\frac{\frac{1}{2}}{n}+\frac{-\frac{1}{2}-\frac{1}{2}i}{n^2};$$

whence the formula becomes

$$\frac{\Delta^n 0^i}{n^2}=e^{-r}\left\{1+r\frac{1+\frac{1}{2}i}{n^2}+\frac{r^2}{1.2}\left(-\frac{1}{n}+\frac{-1-i}{n^2}\right)+\ldots\right\}.$$

It seems to me that the correct result up to this order of approximation is

$$\frac{\Delta^n 0^i}{n^2}=e^{-r}\left\{1+r\frac{\frac{1}{2}i}{n^2}+\frac{r^2}{1.2}\left(-\frac{1}{n}+\frac{-i}{n^2}\right)\right\}.$$

My investigation is as follows: we have

$$\frac{\Delta^n 0^i}{n^i}=1-\frac{n}{1}\left(1-\frac{1}{n}\right)^i+\frac{n.n-1}{1.2}\left(1-\frac{2}{n}\right)^i+\ldots,$$

the series being a finite one; but the number of terms is very large. But observe that, however large n is, we can take i so large that the second term $n\left(1-\frac{1}{n}\right)^i$ may be as small as we please; taking this term to be of moderate amount, say $=r_1$, the subsequent terms will be not very different from $\frac{r_1^2}{1.2}, \frac{r_1^3}{1.2.3}, \ldots$, and the approximate value is $1-r_1+\frac{r_1^2}{1.2}-\&c.$, which is a convergent series having its

sum $= e^{-r_1}$. To work this properly out, I represent the successive terms by r_1, $\frac{r_2}{1.2}$, $\frac{r_3}{1.2.3}$, ..., so that the series is

$$= 1 - r_1 + \frac{r_2}{1.2} - \frac{r_3}{1.2.3} + \dots$$

Taking r a value at pleasure not very different from r_1, and multiplying by

$$(1 =)\, e^{-r} . e^r = e^{-r} . \left(1 + r + \frac{r^2}{1.2} + \dots\right),$$

the sum is

$$= e^{-r} . \left\{1 + (r - r_1) + \frac{1}{1.2}(r^2 - 2rr_1 + r_2) + \frac{1}{1.2.3}(r^3 - 3r^2 r_1 + 3rr_2 - r_3) + \dots\right\}.$$

Assuming now $r = ne^{-i/n}$, we have

$$r_1 = n\left(1 - \frac{1}{n}\right)^i = ne^{i\log\left(1-\frac{1}{n}\right)} = r(1 + X_1),$$

where $X_1 = e^{-\frac{1}{2}\frac{i}{n^2} - \frac{1}{3}\frac{i}{n^3} - \dots}$; and similarly

$$r_2 = n . n - 1 . \left(1 - \frac{2}{n}\right)^i = n^2\left(1 - \frac{1}{n}\right) e^{i\log\left(1-\frac{2}{n}\right)}$$

$$= \left(1 - \frac{1}{n}\right) r^2 (1 + X_2),$$

where $X_2 = e^{-\frac{14i}{2n^2} - \frac{18i}{3n^3} - \dots}$; also

$$r_3 = n . n - 1 . \left(1 - \frac{2}{n}\right)^i = n^3\left(1 - \frac{1}{n}\right) . e^{i\log\left(1-\frac{2}{n}\right)}$$

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) r^3 (1 + X_3),$$

where $X_3 = e^{-\frac{19i}{2n^2} - \frac{127i}{3n^3} - \dots}$, and so on. It is now easy to calculate the successive terms $r - r_1$, $r^2 - 2rr_1 + r_2$, &c.; and it is to be observed that, in the parts independent of the X's, we have only terms divided by n, n^2, or higher powers of n: thus in $r^4 - 4r^3 r_1 + 6r^2 r_2 - 4r^3 r_3 + r_4$, we have r^4 multiplied by

$$1 - 4 + 6\left(1 - \frac{1}{n}\right) - 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)$$

$$= \frac{3}{n^2} - \frac{6}{n^3}.$$

We thus obtain the formula

$$\begin{aligned}\frac{\Delta^n 0^i}{n^i} = e^{-r}\Big\{1 & \\ &+ r \quad (\qquad -1X_1) \\ &+ \frac{r^2}{1.2}\left\{-\frac{1}{n} \qquad -2X_1 + \left(1-\frac{1}{n}\right)X_2\right\} \\ &+ \frac{r^3}{1.2.3}\left\{-\frac{2}{n^2} \qquad -3X_1 + 3\left(1-\frac{1}{n}\right)X_2 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)X_3\right\} \\ &+ \frac{r^4}{1.2.3.4}\left\{-\frac{3}{n^2} - \frac{6}{n^3} - 4X_1 + 6\left(1-\frac{1}{n}\right)X_2 - 4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)X_3\right. \\ &\qquad\qquad \left. + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)X_4\right\} + \ldots, \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \Big\},\end{aligned}$$

where $r = ne^{-i/n}$ as above, and X_1, X_2, ... have the above-mentioned values, the exponentials being expanded in negative powers of n.

Writing

$$X_1 = \frac{-\frac{1}{2}i}{n^2}, \quad X_2 = \frac{-2i}{n^2},$$

we have

$$\frac{\Delta^n 0^i}{n^2} = e^{-r}\left\{1 + r\frac{\frac{1}{2}i}{n^2} + \frac{r^2}{2}\left(-\frac{1}{n} + \frac{-i}{n^2}\right)\right\},$$

which is the foregoing approximate value.

854.

AN ALGEBRAICAL TRANSFORMATION.

[From the *Messenger of Mathematics*, vol. XV. (1886), pp. 58, 59.]

THE following algebraical transformation occurs in a paper by Hermite "On the theory of the Modular Equations," *Comptes Rendus*, t. XLVIII. (1859), p. 1100.

Writing $q = 1 - 2u^8$, $l = 1 - 2v^8$, then in the transformation of the fifth order, the modular equation was expressed by Jacobi in the form

$$\Omega, = (q-l)^6 - 256(1-q^2)(1-l^2)\{16ql(9-ql)^2 + 9(45-ql)(q-l)^2\}, = 0;$$

and if we write herein $q = 1 - 2x$, $l = \dfrac{x+1}{x-1}$, or, what is the same thing, establish between q, l the relation $q - l = 3 + ql$, that is, between u, v the relation $v^8 = 1 \div (1 - u^8)$, then the function Ω becomes

$$\Omega = \frac{64}{(1-x)^6}\{(x^2-x+1)^3 + 2^7(x^2-x)^2\}\{(x^2-x+1)^3 + 2^7 . 3^3(x^2-x)^2\};$$

or, what is the same thing, the equation $\Omega = 0$ gives for $\dfrac{(x^2-x+1)^3}{(x^2-x)^2}$ the values -2^7 and $-2^7 . 3^3$.

We, in fact, have

$$q - l = 3 + ql = \frac{2(x^2-x+1)}{1-x},$$

$$1 - q^2 = 4x(1-x), \quad 1 - l^2 = \frac{-4x}{(1-x)^2},$$

and therefore

$$(1-q^2)(1-l^2) = \frac{-16x^2}{1-x}.$$

Hence

$$\Omega = \frac{64}{(1-x)^6}\left[(x^2-x+1)^6 + 64\,(1-x)^5 \times x^2\,\{16ql\,(9-ql)^2 + 9\,(45-ql)\,(3+ql)^2\}\right];$$

and, putting for a moment $ql = \theta - 3$, the term in { } is found to be

$$= 7\theta^3 + 3456\theta - 6912\,;$$

viz. this is

$$= \frac{56\,(x^2-x+1)^2}{(1-x)^3} + \frac{6912\,(x^2-x+1)}{1-x} - 6912,$$

$$= \frac{8}{(1-x)^3}\{7\,(x^2-x+1)^3 + 864\,(x-1)^2\,(x^2-x+1) + 864\,(x-1)^3\},$$

$$= \frac{8}{(1-x)^3}\{7\,(x^2-x+1)^3 + 864\,(x^2-x)^2\}.$$

Hence

$$\Omega = \frac{64}{(1-x)^6}\left[(x^2-x+1)^6 + 512\,(x^2-x)^2\,\{7\,(x^2-x+1)^3 + 864\,(x^2-x)^2\}\right],$$

which is

$$= \frac{64}{(1-x)^6}\{(x^2-x+1)^3 + 2^7\,(x^2-x)^2\}\,\{(x^2-x+1)^3 + 2^7 . 3^3\,(x^2-x)^2\}.$$

855.

SOLUTION OF $(a, b, c, d) = (a^2, b^2, c^2, d^2)$.

[From the *Messenger of Mathematics*, vol. XV. (1886), pp. 59—61.]

It is required to find four quantities (no one of them zero) which are in some order or other equal to their squares, say

$$(a, b, c, d) = (a^2, b^2, c^2, d^2).$$

Supposing that the required quantities (a, b, c, d) are the roots of the biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

(s not $= 0$), then the function

$$(x^4 + qx^2 + s)^2 - (px^3 + rx)^2 \text{ must be } = x^8 + px^6 + qx^4 + rx^2 + s;$$

and we have thus the conditions

$$2q - p^2 = p, \quad 2s + q^2 - 2pr = q, \quad 2qs - r^2 = r, \quad s^2 = s,$$

the last of which (since s is not $= 0$) gives $s = 1$, and the others then become

$$2q = p^2 + p, \quad 2(pr - 1) = q^2 - q, \quad 2q = r^2 + r;$$

viz. regarding p, q, r as the coordinates of a point in space, this is determined as the intersection of three quadric surfaces, and the number of solutions is thus $= 8$.

We, in fact, have $2q = p^2 + p = r^2 + r$; that is, $p^2 + p = r^2 + r$, or $(p - r)(p + r + 1) = 0$; hence $r = p$ or $r = -1 - p$.

First, if $r = p$; here $2q = p^2 + p$, $2(p^2 - 1) = q^2 - q$: the last equation multiplied by 4 gives

$$8(p^2 - 1) = (p^2 + p)(p^2 + p - 2), \quad = p(p^2 - 1)(p + 2),$$

that is, $p^2 - 1 = 0$ or $p^2 + 2p - 8 = 0$.

If $p^2 - 1 = 0$, then either $p = 1$, giving $q = 1$, $r = 1$, and hence the equation is $x^4 + x^3 + x^2 + x + 1 = 0$; or else $p = -1$, giving $q = 0$, $r = -1$, and hence the equation is $x^4 - x^3 - x + 1 = 0$, that is, $(x-1)^2(x^2 + x + 1) = 0$.

If $p^2 + 2p - 8 = 0$, then either $p = 2$, giving $q = 3$, $r = 2$, and hence the equation is $x^4 + 2x^3 + 3x^2 + 2x + 1$, that is,

$$(x^2 + x + 1)^2 = 0;$$

or else $p = -4$, giving $q = 6$, $r = -4$, and hence the equation is $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$, that is, $(x-1)^4 = 0$.

Secondly, if $r = -1 - p$; here

$$2q = p^2 + p, \quad 2(-p^2 - p - 1) = q^2 - q;$$

the last equation multiplied by 4 gives

$$8(-p^2 - p - 1) = (p^2 + p)(p^2 + p - 2),$$

that is,

$$p^4 + 2p^3 + 7p^2 + 6p + 8 = 0, \text{ or } (p^2 + p + 4)(p^2 + p + 2) = 0.$$

If $p^2 + p + 4 = 0$, then $p = \frac{1}{2}\{-1 \pm i\sqrt{(15)}\}$, whence

$$r = \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}, \quad 2q = p^2 + p, \ = -4, \text{ or } q = -2;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x^3 - 2x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x + 1 = 0.$$

If $p^2 + p + 2 = 0$, then $p = \frac{1}{2}\{-1 \pm i\sqrt{(7)}\}$; whence

$$r = \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}, \quad 2q = p^2 + p, \ = -2, \text{ or } q = -1;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x^3 - x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x + 1 = 0,$$

that is,

$$(x-1)[x^3 + \tfrac{1}{2}\{1 \pm i\sqrt{(7)}\}x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x - 1] = 0.$$

We thus see that the eight equations are

$$\begin{array}{ll}
1 & (x-1)^4 = 0, \\
1 & (x^2 + x + 1)^2 = 0, \\
1 & (x-1)^2(x^2 + x + 1) = 0, \\
1 & x^4 + x^3 + x^2 + x + 1 = 0, \\
2 & (x-1)\{x^3 + \tfrac{1}{2}(1 \pm i\sqrt{7})x^2 + \tfrac{1}{2}(-1 \pm i\sqrt{7})x - 1\} = 0, \\
2 & x^4 + \tfrac{1}{2}(-1 \pm i\sqrt{15})x^3 - 2x^2 + \tfrac{1}{2}(-1 \mp i\sqrt{15})x + 1 = 0, \\
\hline
8 &
\end{array}$$

and it hence appears that, writing γ, ϵ, θ to denote respectively an imaginary cube root, fifth root, and seventh root of unity, then the values of (a, b, c, d) are

$$\begin{array}{llll}
1, & 1, & 1, & 1; \\
\gamma, & \gamma, & \gamma^2, & \gamma^2; \\
1, & 1, & \gamma, & \gamma^2; \\
\epsilon, & \epsilon^2, & \epsilon^3, & \epsilon^4; \\
\epsilon\gamma, & \epsilon^3\gamma^2, & \epsilon^4\gamma, & \epsilon^2\gamma^2; \\
\epsilon^2\gamma, & \epsilon^4\gamma^2, & \epsilon^3\gamma, & \epsilon\gamma^2; \\
1, & \theta, & \theta^2, & \theta^4; \\
1, & \theta^3, & \theta^6, & \theta^5;
\end{array}$$

viz. for each of these systems we have the required relation

$$(a, b, c, d) = (a^2, b^2, c^2, d^2).$$

It may be noticed that out of the eight equations we have the following three which are irreducible:—

$$x^4 + x^3 + x^2 + x + 1 = 0,$$
$$x^4 + \tfrac{1}{2}(-1 + i\sqrt{15})x^3 - 2x^2 + \tfrac{1}{2}(-1 - i\sqrt{15})x + 1 = 0,$$
$$x^4 + \tfrac{1}{2}(-1 - i\sqrt{15})x^3 - 2x^2 + \tfrac{1}{2}(-1 + i\sqrt{15})x + 1 = 0.$$

Each of these is an Abelian equation, viz. the roots are of the form

$$a,\ \theta(a),\ \theta^2(a),\ \theta^3(a),\ (= a,\ a^3,\ a^4,\ a^8),$$

where $\theta^4(a) = a$, not identically but in virtue of the value of a, viz. we have $\theta^4(a) = a^{16} = a$, in virtue of $a^{15} = 1$: (in the first equation $a^5 = 1$, and therefore $a^{15} = 1$; in each of the other two, a^{15} is the lowest power which is $= 1$).

In the first equation, we have evidently

$$x^4 + x^3 + x^2 + x + 1$$

as the irreducible factor of $x^5 - 1$.

The second and third equations combined together give

$$(x^4 - \tfrac{1}{2}x^3 - 2x^2 - \tfrac{1}{2}x + 1)^2 + \tfrac{15}{4}(x^3 - x)^2 = 0;$$

that is,

$$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = 0,$$

where the left-hand side is the irreducible factor of $x^{15} - 1$.

856.

NOTE ON A CUBIC EQUATION.

[From the *Messenger of Mathematics*, vol. XV. (1886), pp. 62—64.]

CONSIDER the cubic equation

$$x^3 + 3cx + d = 0;$$

then effecting upon this the Tschirnhausen-Hermite transformation

$$y = xT_1 + (x^2 + 2c)\,T_2,$$

the resulting equation in y is

$$\begin{aligned} y^3 &+ 3y\,(cT_1^2 + dT_1T_2 - c^2T_2^2) \\ &+ dT_1^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3 = 0, \end{aligned}$$

and this will be

$$y^3 + 3cy + d = 0,$$

if only

$$\begin{aligned} c &= cT_1^2 + dT_1T_2 - c^2T_2^2, \\ d &= dT_1^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3, \end{aligned}$$

equations which give

$$(d^2 + 4c^3) = (d^2 + 4c^3)\,(T_1^3 + 3cT_1T_2^2 + dT_2^3)^2,$$

viz. assuming that $d^2 + 4c^3$ not $= 0$, this is

$$1 = T_1^3 + 3cT_1T_2^2 + dT_2^3.$$

Hence the coefficients T_1, T_2 being such as to satisfy these relations, the equation in z is identical with the equation in x; or, what is the same thing, if α, β, γ are the roots of the equation in x, then we have between these roots the relations

$$\begin{aligned} \beta &= \alpha T_1 + (\alpha^2 + 2c)\,T_2, \\ \gamma &= \beta T_1 + (\beta^2 + 2c)\,T_2, \\ \alpha &= \gamma T_1 + (\gamma^2 + 2c)\,T_2, \end{aligned}$$

viz. the general cubic equation $x^3+3cx+d=0$, adjoining thereto the radicals T_1, T_2 may be regarded as an Abelian equation.

In particular, if c, $d=-1$, 1, then we may write $T_1=0$, $T_2=1$; the cubic equation is here

$$x^3+3x-1=0,$$

and the roots α, β, γ are such that $\beta=\alpha^2-2$, $\gamma=\beta^2-2$, $\alpha=\gamma^2-2$; in fact, taking θ a primitive ninth root of unity, $\theta^6+\theta^3+1=0$; we have α, β, $\gamma=\theta+\theta^8$, $\theta^2+\theta^7$, $\theta^4+\theta^5$; values which satisfy $x^3+3x-1=0$, and the relations in question.

The same question may be considered from a different point of view. Take the transforming equation to be

$$y=A+Bx+Cx^2,$$

then assuming that the values of y corresponding to the values $x=\alpha$, β, γ are β, γ, α respectively, we have

$$\beta=A+B\alpha+C\alpha^2,$$

$$\gamma=A+B\beta+C\beta^2,$$

$$\alpha=A+B\gamma+C\gamma^2,$$

and the transforming equation thus is

$$\begin{vmatrix} y, & 1, & x, & x^2 \\ \beta, & 1, & \alpha, & \alpha^2 \\ \gamma, & 1, & \beta, & \beta^2 \\ \alpha, & 1, & \gamma, & \gamma^2 \end{vmatrix}=0.$$

This may also be written

$$\begin{aligned} &(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{y-\tfrac{1}{2}(\alpha+\beta+\gamma+x)\} \\ = \quad & \beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2-\tfrac{1}{2}(\beta^3\gamma+\beta\gamma^3+\gamma^3\alpha+\gamma\alpha^3+\alpha^3\beta+\alpha\beta^3) \\ &+x\{\alpha^3+\beta^3+\gamma^3-\tfrac{1}{2}(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2)\} \\ &+x^2\{\beta\gamma+\gamma\alpha+\alpha\beta-(\alpha^2+\beta^2+\gamma^2)\}. \end{aligned}$$

We have

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=\frac{-27}{a^4}(a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd),$$

$$=\frac{-27}{a^4}\Delta,$$

or, say

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=\frac{3(\omega-\omega^2)}{a^2}\sqrt{(\Delta)},$$

if Δ be the discriminant, and ω an imaginary cube root of unity, $\{(\omega-\omega^2)^2=-3\}$.

The remaining functions of α, β, γ are of course expressible rationally in terms of the coefficients: we have

$$\Sigma\beta^3\gamma^2 = \frac{1}{a^3}(-6bd + 9c^2),$$

$$\Sigma\beta^3\gamma = \frac{1}{a^3}(-3abd - 18ac^2 + 27b^2c),$$

$$\Sigma\alpha^3 = \frac{1}{a^3}(-3a^2d + 27abc - 27b^3),$$

$$\Sigma\beta^2\gamma = \frac{1}{a^2}(3ad - 9bc),$$

$$\Sigma\beta\gamma = \frac{3c}{a},$$

$$\Sigma\alpha^2 = \frac{1}{a^2}(9b^2 - 6ac),$$

and the final result is

$$\begin{aligned}\tfrac{1}{3}(\omega - \omega^2)\sqrt{(\Delta)}\,\{2a(y + x) + 3b\} = \quad & -abd + 4ac^2 - 3b^3c \\ & + x\,(-a^2d + 7abc - 6b^3) \\ & + x^2\,(\ 2a^2c - 2ab^2);\end{aligned}$$

viz. we have thus an automorphic transformation of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0.$$

857.

ANALYTICAL GEOMETRICAL NOTE ON THE CONIC.

[From the *Messenger of Mathematics*, vol. XV. (1886), p. 192.]

TAKE (X, Y, Z) the coordinates of a point on the conic $yz+zx+xy=0$, so that $YZ+ZX+XY=0$; clearly (Y, Z, X) and (Z, X, Y) are the coordinates of two other points on the same conic; I say that the three points are the vertices of a triangle circumscribed about the conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

In fact, the equation of one of the sides is

$$\begin{vmatrix} x, & y, & z \\ X, & Y, & Z \\ Y, & Z, & X \end{vmatrix} = 0,$$

say this is $AX+BY+CZ=0$, where $A, B, C = XY-Z^2, YZ-X^2, XZ-Y^2$; and the condition in order that this side may touch the conic

$$x^2+y^2+z^2-2yz-2zx-2xy=0$$

is

$$BC+CA+AB=0.$$

But we have

$$\begin{aligned} BC+CA+AB &= Y^2Z^2+Z^2X^2+X^2Y^2-X(Y^3+Z^3)-Y(Z^3+X^3)-Z(X^3+Y^3) \\ &\qquad\qquad\qquad

858.

COMPARISON OF THE WEIERSTRASSIAN AND JACOBIAN ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XVI. (1887), pp. 129—132.]

THE Weierstrassian function $\sigma(u)$ corresponds of course with Jacobi's $H(u)$, but it is worth while to establish the actual formulæ of transformation.

Writing, for a moment,

$$\omega = \omega_1 + i\upsilon_1,$$
$$\omega' = \omega_2 + i\upsilon_2,$$

it is convenient to assume $\omega_1\upsilon_2 - \omega_2\upsilon_1$ positive; we then have

$$2(\eta\omega' - \eta'\omega) = +\pi i;$$

in particular, this will be the case if $\omega = \omega_1$, $\omega' = i\upsilon_2$, where ω_1, υ_2 are each positive.

To reduce the periods into the Jacobian form, we may assume

$$\omega = \lambda K,$$
$$\omega' = \lambda i K',$$

(where observe that, if as usual k, K, K' are each real and positive, and if as above $\omega = \omega_1$, $\omega' = i\upsilon_2$, ω_1 and υ_2 positive, then also λ will be real and positive). We have

$$\frac{iK'}{K} = \frac{\omega'}{\omega}, \text{ or say } q = e^{-\frac{\pi K'}{K}} = e^{\frac{i\pi\omega'}{\omega}},$$

which determines first q, and thence k, K, K', as functions of $\frac{\omega'}{\omega}$, and we then have $\lambda = \frac{\omega}{K} = \frac{-i\omega'}{K'}$, either of which equations gives λ as a function of ω, ω'. Conversely, starting with k, λ, the original equations give the values of ω, ω'; those of η, η' will be determined presently.

The form of relation is at once seen to be

$$H(u) = Ae^{Bu^2}\sigma(\lambda u),$$

and observing that, for u small, we have $H(u) = \sqrt{\left(\frac{2kk'K}{\pi}\right)}u$ and $\sigma(\lambda u) = \lambda u$, we have $A = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}$; I first write down and afterwards verify the value of B, viz. this is $= -\frac{1}{2}\frac{\lambda^2\eta}{\omega}$; and the formula thus is

$$H(u) = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}e^{-\frac{1}{2}\frac{\lambda^2\eta}{\omega}u^2}\sigma(\lambda u).$$

In fact, for u writing successively $u + 2K$, and $u + 2iK'$, we obtain

$$\frac{H(u+2K)}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2K(u+K)}\frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)},$$

$$\frac{H(u+2iK')}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2iK'(u+iK')}\frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)},$$

which should be satisfied in virtue of

$$\frac{H(u+2K)}{H(u)} = -1, \qquad \frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)} = -e^{2\eta(\lambda u+\omega)},$$

$$\frac{H(u+2iK')}{H(u)} = -e^{-\frac{i\pi}{K}(u+iK')}, \quad \frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)} = -e^{2\eta'(\lambda u+\omega')};$$

viz. we ought to have

$$0 = -\frac{\lambda^2\eta}{\omega}2K\ (u + K) + 2\eta\ (\lambda u + \omega)$$

$$-\frac{i\pi}{K}(u+iK') = -\frac{\lambda^2\eta}{\omega}2iK'(u+iK') + 2\eta'(\lambda u + \omega').$$

The first of these is

$$0 = -\frac{\lambda K}{\omega}(u+K) + \left(4 + \frac{\omega}{\lambda}\right),$$

that is,

$$0 = (-1+1)(u+K);$$

and the second is

$$0 = \left(\frac{\frac{1}{2}iK}{\lambda K} - \frac{\lambda iK'\eta}{\omega}\right)(u+iK') + \eta'\left(u + \frac{\omega'}{\lambda}\right).$$

viz. for $i\pi$ writing $2(\eta\omega' - \eta'\omega)$, this is

$$0 = \left(\frac{\eta\omega' - \eta'\omega}{\omega} - \frac{\eta\omega'}{\omega} + \eta'\right)(u+iK'),$$

and the two equations are thus each of them an identity.

The Weierstrassian function $\wp(u)$ is defined as

$$= -\frac{d^2}{du^2}\log\sigma(u);$$

or, what is the same thing, we have

$$\wp(u) = -\frac{d}{du}\frac{\sigma'(u)}{\sigma(u)}.$$

Hence

$$\wp(\lambda u) = -\frac{1}{\lambda}\frac{d}{du}\frac{\sigma'(\lambda u)}{\sigma(\lambda u)}.$$

But

$$\frac{H'(u)}{H(u)} = -\frac{\lambda^2\eta u}{\omega} + \frac{\lambda\sigma'(\lambda u)}{\sigma(\lambda u)},$$

or

$$\wp(\lambda u) = -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\frac{d}{du}\frac{H'(u)}{H(u)}.$$

But from the equation $\sqrt{k}\,\mathrm{sn}\,u = \frac{H(u)}{\Theta(u)}$, we obtain

$$\frac{d}{du}\frac{H'(u)}{H(u)} = Z'(u) - \frac{1}{\mathrm{sn}^2 u} + k^2\,\mathrm{sn}^2 u, \quad = 1 - \frac{E}{K} - \frac{1}{\mathrm{sn}^2 u},$$

and consequently

$$\wp(\lambda u) = -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

where, on the right-hand side, expanding in ascending powers of u, the constant term is

$$= -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2}\tfrac{1}{3}(1 + k^2).$$

But in the function $\wp(\lambda u)$ this constant term is $= 0$, and we thus have

$$\frac{\eta}{\omega} = \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

and then, since

$$\frac{\eta}{\omega} - \frac{\eta'}{\omega'} = \frac{\frac{1}{2}\pi i}{\omega\omega'} = \frac{\frac{1}{2}\pi}{\lambda^2 KK'},$$

we have

$$\frac{\eta'}{\omega'} = -\frac{\frac{1}{2}\pi}{\lambda^2 KK'} + \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

or, as these equations may also be written,

$$\eta = \frac{K}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

$$\eta' = \frac{-\frac{1}{2}\pi i}{\lambda K} + \frac{2K'}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

which are the values of η, η'. And we then have

$$\wp(\lambda u) = -\frac{1}{\lambda^2}\tfrac{1}{3}(1 + k^2) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

the equation connecting $\wp(\lambda u)$ and $\mathrm{sn}\,u$.

859.

ON THE COMPLEX OF LINES WHICH MEET A UNICURSAL QUARTIC CURVE.

[From the *Proceedings of the London Mathematical Society*, vol. XVII. (1886), pp. 232—238.]

THE curve is taken to be that determined by the equations

$$x : y : z : w = 1 : \theta : \theta^2 : \theta^4,$$

viz. it is the common intersection of the quadric surface $\Theta = 0$, and the cubic surfaces $P = 0$, $Q = 0$, $R = 0$, where

$$\Theta = xw - yz, \quad P = x^2z - y^3, \quad Q = xz^2 - y^2w, \quad R = z^3 - yw^2.$$

Writing (a, b, c, f, g, h) as the six coordinates of a line, viz.

$$(a, b, c, f, g, h) = (\beta z - \gamma y, \gamma x - \alpha z, \alpha y - \beta x, \alpha w - \delta x, \beta w - \delta y, \gamma w - \delta z),$$

if $(\alpha, \beta, \gamma, \delta)$, (x, y, z, w) are the coordinates of any two points on the line; then, if the line meet the curve, we have

$$\begin{aligned} . \quad h\theta - g\theta^3 + a\theta^4 &= 0, \\ -h \quad . \quad + f\theta^3 + b\theta^4 &= 0, \\ g - f\theta \quad . \quad + c\theta^4 &= 0, \\ -a - b\theta - c\theta^3 \quad . \quad &= 0, \end{aligned}$$

from which four equations (equivalent, in virtue of the identity $af + bg + ch = 0$, to two independent equations), eliminating θ, we have the equation of the complex. The form may, of course, be modified at pleasure by means of the identity just referred to, but one form is

$$\Omega, = a^4 - b^3h + bf^3g + cg^3 - acfh + 2c^2h^2 - 4a^2ch + af^3 - a^3f = 0,$$

as may be verified by substituting therein the values $a = -b\theta - c\theta^3$, $g = f\theta - c\theta^4$, $h = f\theta^3 + b\theta^4$. The last-mentioned equation is thus the equation of the complex in question, in terms of the six coordinates (a, b, c, f, g, h).

If for the six coordinates we substitute their values, $\beta z-\gamma y$, &c., we obtain $\Omega, =(x, y, z, w)^4(\alpha, \beta, \gamma, \delta)^4=0$, which, regarded as an equation in (x, y, z, w), is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, passing through the quartic curve; this equation should evidently be satisfied if only Θ, P, Q, R are each $=0$, viz. Ω must be a linear function of (Θ, P, Q, R); and by symmetry, it must be also a linear function of $(\Theta_0, P_0, Q_0, R_0)$, where

$$\Theta_0=\alpha\delta-\beta\gamma,\quad P_0=\alpha^2\gamma-\beta^3,\quad Q_0=\alpha\gamma^2-\beta^2\delta,\quad R_0=\gamma^3-\beta\delta^2,$$

viz. the form is $\Omega, =(\Theta, P, Q, R)(\Theta_0, P_0, Q_0, R_0)$, an expression with coefficients which are of the first or second degree in (x, y, z, w) and also of the first or second degree in $(\alpha, \beta, \gamma, \delta)$.

To work this out, I first arrange in powers and products of (α, δ), (β, γ), expressing the quartic functions of (x, y, z, w) in terms of (Θ, P, Q, R), as follows:

$\Omega=$

	a^4	$-b^3h$	$+bf^2g$	$+cg^3$	$-acfh$	$+2c^2h^2$	$-4a^2ch$	$+af^3$	$-a^3f$		
α^4										0	
$\alpha^3\delta$		$-z^4$	$+yzw^2$							$-z^4+yzw^2$	$-zR$
$\alpha^2\delta^2$			$-2xyzw$			$+2y^2z^2$				$-2xyzw+2y^2z^2$	$-2yz\Theta$
$\alpha\delta^3$			$+x^2yz$	$-y^4$						$+x^2yz-y^4$	$+yP$
δ^4										0	
$\alpha^3\beta$			$-zw^3$					$+zw^3$		0	
$\alpha^2\beta\delta$			$+2xzw^2$		$+yz^2w$			$-3xzw^2$		$-xzw^2+yz^2w$	$-zw\Theta$
$\alpha\beta\delta^2$			$-x^2zw$	$+3y^2w$	$-xyz^2$	$-4xyz^2$		$+3x^2zw$		$+2x^2zw+3y^2w-5xyz^2$	$+2xz\Theta-3yQ$
$\beta\delta^3$				$+xy^3$				$-x^3z$		$+xy^3-x^3z$	$-xP$
$\alpha^2\gamma$		$+z^3w$						$-yw^3$		$+z^3w-yw^3$	$+wR$
$\alpha^2\gamma\delta$		$+3xz^3$	$-xyw^2$		$-y^2zw$	$-4y^2zw$		$+3xyw^2$		$+3xz^3+2xyw^2-5y^2zw$	$+2yw\Theta+3zQ$
$\alpha\gamma\delta^2$			$+2x^2yw$		$+xyz^2$			$-3x^2yw$		$-x^2yw+xyz^2$	$-xy\Theta$
$\gamma\delta^3$				$-x^3y$				$+x^3y$		0	
$\alpha^2\beta^2$										0	
$\alpha^2\beta\gamma$			$+xw^3$		$-yzw^2$					$+xw^3-yzw^2$	$+w^2\Theta$
$\alpha^2\gamma^2$		$-3xz^2w$			$+y^2w^2$	$+2y^2w^2$				$-3xz^2w+3y^2w^2$	$-3wQ$
$\alpha\beta^2\delta$				$-3y^2w^2$	$-xz^2w$		$+4yz^3$			$-3y^2w^2-xz^2w+4yz^3$	$-4z^2\Theta+3wQ$
$\alpha\beta\gamma\delta$			$-2x^2w^2$		$+2xyzw$	$+8xyzw$	$-8y^2z^2$			$-2x^2w^2+10xyzw-8y^2z^2$	$+(-2xw+8yz)\Theta$
$\alpha\gamma^2\delta$		$-3x^2z^2$			$-xy^2w$		$+4y^3z$			$-3x^2z^2-xy^2w+4y^3z$	$-4y^2\Theta-3xQ$
$\beta^2\delta^2$				$-3xy^2w$	$+x^2z^2$	$+2x^2z^2$				$-3xy^2w+3x^2z^2$	$+3xQ$
$\beta\gamma\delta^2$			$+x^3w$		$-x^2yz$					$+x^3w-x^2yz$	$+x^2\Theta$
$\gamma^2\delta^2$										0	
$\alpha\beta^3$				$+yw^3$					$-z^3w$	$+yw^3-z^3w$	$-wR$
$\alpha\beta^2\gamma$					$+xzw^2$		$-4yz^2w$		$+3yz^2w$	$+xzw^2-yz^2w$	$+zw\Theta$
$\alpha\beta\gamma^2$					$-xyw^2$	$-4xyw^2$	$+8y^2zw$		$-3y^2zw$	$-5xyw^2+5y^2zw$	$-5ywe\Theta$
$\alpha\gamma^3$		$+3x^2zw$					$-4y^3w$		$+y^3w$	$+3x^2zw-3y^3w$	$+3wP$
$\beta^3\delta$				$+3xyw^2$			$-4xz^3$		$+xz^3$	$+3xyw^2-3xz^3$	$-3xR$
$\beta^2\gamma\delta$					$-x^2zw$	$-4x^2zw$	$+8xyz^2$		$-3xyz^2$	$-5x^2zw+5xyz^2$	$-5xz\Theta$
$\beta\gamma^2\delta$					$+x^2yw$		$-4xy^2z$		$+3xy^2z$	$+x^2yw-xy^2z$	$+xy\Theta$
$\gamma^3\delta$		$+x^3z$							$-xy^3$	$+x^3z-xy^3$	$+xP$
β^4	$+z^4$			$-xw^3$						$+z^4-xw^3$	$+zR-w^2\Theta$
$\beta^3\gamma$	$-4yz^3$						$+4xz^2w$			$-4yz^3+4xz^2w$	$+4z^2\Theta$
$\beta^2\gamma^2$	$+6y^2z^2$					$+2x^2w^2$	$-8xyzw$			$+2x^2w^2-8xyzw+6y^2z^2$	$+(2xw-6yz)\Theta$
$\beta\gamma^3$	$-4y^3z$						$+4xy^2w$			$-4y^3z+4xy^2w$	$+4y^2\Theta$
γ^4	$+y^4$	$-x^3w$								$+y^4-x^3w$	$-yP-x^2\Theta$

Collecting the terms multiplied by P, Q, R, Θ, respectively, we have

$$\begin{aligned}\Omega = \; & P\,\{y\alpha\delta^3 - x\beta\delta^3 + 3w\alpha\gamma^3 + x\gamma^3\delta - y\gamma^4\} \\ & + Q\,\{-3y\alpha\beta\delta^2 + 3z\alpha^2\gamma\delta - 3w\alpha^2\gamma^2 + 3w\alpha\beta^2\delta - 3x\alpha\gamma^2\delta + 3x\beta^2\delta^2\} \\ & + R\,\{-z\alpha^3\delta + w\alpha^3\gamma - w\alpha\beta^3 - 3x\beta^2\delta + z\beta^4\} \\ & + \Theta\,\{-2yz\alpha^2\delta^2 - zw\alpha^2\beta\delta + 2xz\alpha\beta\delta^2 + 2yw\alpha^2\gamma\delta - xy\alpha\gamma\delta^2 \\ & \quad + w^2\alpha^2\beta\gamma - 4z^2\alpha\beta^2\delta + (-2xw + 8yz)\,\alpha\beta\gamma\delta - 4y^2\alpha\gamma^2 + x^2\beta\gamma\delta^2 \\ & \quad + zw\alpha\beta^2\gamma - 5yw\alpha\beta\gamma\delta - 5xz\beta^2\gamma\delta + xy\beta\gamma^3\delta \\ & \quad - w^2\beta^4 + 4z^2\beta^3\gamma + (2xw - 6yz)\,\beta^2\gamma^2 + 4y^2\beta\gamma^3 - x^2\gamma^4\},\end{aligned}$$

which may be written as follows:—

$$\begin{aligned}\Omega = \; & P\,\{y\,(\alpha\delta^3 - \gamma^4) + x\,(\gamma^3\delta - \beta\delta^3)\} && + P\,(3w\alpha\gamma^3) \\ & + Q\,\{3x\,(\beta^2\delta^2 - \alpha\gamma^2\delta) + 3w\,(\alpha\beta^2\delta - \alpha^2\gamma^2)\} && + Q\,(3z\alpha^2\gamma\delta - 3y\alpha\beta\delta^2) \\ & + R\,\{-z\,(\alpha^2\delta - \beta^4) + w\,(\alpha^3\gamma - \alpha\beta^3)\} && + R\,(-3x\beta^3\delta) \\ & + \Theta\,\{zw\,(-\alpha^2\beta\delta + \alpha\beta^2\gamma) \\ & \quad + xz\,2\,(\alpha\beta\delta^2 - \beta^2\gamma\delta) && + \Theta\,(-3xz\beta^2\gamma\delta) \\ & \quad + yw\,2\,(\alpha^2\gamma\delta - \alpha\beta\gamma^2) && + \Theta\,(-3yw\alpha\beta\gamma^2) \\ & \quad + xy\,(-\alpha\gamma\delta^2 + \beta\gamma^2\delta) \\ & \quad + xw\,2\,(-\alpha\beta\gamma\delta + \beta^2\gamma^2) \\ & \quad + yz\,(-2\alpha^2\delta^2 + 8\alpha\beta\gamma\delta - 6\beta^2\gamma^2) \\ & \quad + x^2\,(\beta\gamma\delta^2 - \gamma^4) \\ & \quad + y^2\,4\,(-\alpha\gamma^2\delta + \beta\gamma^3) \\ & \quad + z^2\,4\,(-\alpha\beta^2\delta + \beta^3\gamma) \\ & \quad + w^2\,(\alpha^2\beta\gamma - \beta^4) && \},\end{aligned}$$

in which all the terms contained in the { } admit of expression in terms of P_0, Q_0, R_0, Θ_0; the remaining six terms not included within { } may be written

$$\begin{aligned} & 3wP\alpha\,(\gamma^3 - \beta\delta^2) + 3\,(wP - yQ)\,\alpha\beta\delta^2 - 3\Theta xz\beta^2\gamma\delta, \\ & -3xR\delta\,(\beta^3 - \alpha^2\gamma) + 3\,(-xR + zQ)\,\alpha^2\gamma\delta - 3\Theta yw\alpha\beta\gamma^2\,; \end{aligned}$$

which, observing that $wP - yQ = xz\Theta$, and $-xR + zQ = yw\Theta$, are

$$\begin{aligned} & -3wP\alpha\,(\gamma^3 - \beta\delta^2) + 3xz\Theta\,(\alpha\beta\delta^2 - \beta^2\gamma\delta), \\ & -3xR\delta\,(\beta^3 - \alpha^2\gamma) + 3yw\Theta\,(\alpha^2\gamma\delta - \alpha\beta\gamma^2). \end{aligned}$$

The expression thus becomes

$$
\begin{aligned}
\Omega = \quad P\,. \quad & x(\gamma^3\delta - \beta\delta^3) && = \quad x\delta R_0 \\
& + \; y(\alpha\delta^3 - \gamma^4) && = \quad y(-\gamma R_0 + \delta^2\Theta) \\
& + 3w\alpha(\gamma^3 - \beta\delta^3) && = \quad 3w\alpha R_0 \\
+ Q\,. & - 3x(\beta^3\delta^3 - \alpha\gamma^2\delta) && = - 3x\delta Q_0 \\
& + 3w(\alpha\beta^2\delta - \alpha^2\gamma^2) && = - 3w\alpha Q_0 \\
+ R\,. & - 3x\delta(\beta^3 - \alpha^2\gamma) && = \quad 3x\delta P_0 \\
& - \; z(\alpha^3\delta - \beta^4) && = \quad z(-\beta P_0 - \alpha^2\Theta_0) \\
& + \; w(\alpha^2\gamma - \alpha\beta^3) && = \quad w\alpha P_0 \\
+ \Theta\,. & \quad zw(-\alpha^2\beta\delta + \alpha\beta^2\gamma) && = - \; zw\alpha\beta\Theta_0 \\
& + 5xz\,(\alpha\beta\delta^2 - \beta^3\gamma\delta) && = \quad 5xz\beta\delta\Theta_0 \\
& + 5yw\,(\alpha^2\gamma\delta - \alpha\beta\gamma^2) && = \quad 5yw\alpha\gamma\Theta_0 \\
& + \; xy\,(-\alpha\gamma\delta^2 + \beta\gamma^2\delta) && = - \; xy\gamma\delta\Theta_0 \\
& + 2xw\,(-\alpha\beta\gamma\delta + \beta^2\gamma^2) && = - 2xw\beta\gamma\Theta_0 \\
& + \; yz\,(-2\alpha^2\delta^2 + 8\alpha\beta\gamma\delta - 6\beta^3\gamma^2) && = - 2yz\,(\alpha\delta - 3\beta\gamma)\,\Theta_0 \\
& + \; x^2\,(\beta\gamma\delta^2 - \gamma^4) && = - \; x^2\gamma R_0 \\
& + 4y^2\,(-\alpha\gamma^2\delta + \beta\gamma^3) && = - 4y^2\gamma^2\Theta_0 \\
& + 4z^2\,(-\alpha\beta^2\delta + \beta^3\gamma) && = - 4z^2\beta^2\Theta_0 \\
& + \; w^2(\alpha^2\beta\gamma - \beta^4) && = \quad w^2\beta P\,;
\end{aligned}
$$

and we thus finally obtain

$$
\begin{aligned}
\Omega = \quad & PR_0(3\alpha w - \gamma y + \delta x) \\
& + RP_0(3\delta x - \beta z + \alpha w) \\
& + P\Theta_0\,.\,\delta^2 y \\
& + R\Theta_0\,.\,-\alpha^2 z \\
& + P_0\Theta\,.\,\beta w^2 \\
& + R_0\Theta\,.\,-\gamma x^2 \\
& - QQ_0\,.\,-3(\alpha w + \delta x) \\
& + \Theta\Theta_0\,\{-\alpha\beta zw - \gamma\delta xy + 5\beta\delta xz + 5\alpha\gamma yw - 2\beta\gamma xw - 2\alpha\delta yz \\
& \qquad\qquad - 4\gamma^2 y^2 + 6\beta\gamma yz - 4\beta^2 z^2\},
\end{aligned}
$$

viz. $\Omega = 0$ is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the quartic curve $x : y : z : w = 1 : \theta : \theta^3 : \theta^4$. As regards the symmetry of this expression, it is to be remarked that, changing (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$ into (w, z, y, x) and $(\delta, \gamma, \beta, \alpha)$ respectively, we change (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$ into $(\Theta, -R, -Q, -P)$ and $(\Theta_0, -R_0, -Q_0, -P_0)$, respectively, and so leave Ω unaltered. Again, interchanging (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$, we interchange (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$, and so leave Ω unaltered.

860.

ON BRIOT AND BOUQUET'S THEORY OF THE DIFFERENTIAL EQUATION $F\left(u, \frac{du}{dz}\right)=0$.

[From the *Proceedings of the London Mathematical Society*, vol. XVIII. (1887), pp. 314—324.]

I CONSIDER the theory of a differential equation of the first order as developed in Briot and Bouquet's *Théorie des fonctions doublement périodiques et en particulier des fonctions elliptiques* (8°, 1st ed., Paris, 1859), but I make some substantial variations in the mode of treatment.

I remark that, writing $u=x$, $\frac{du}{dz}=y$, I make the theory to depend altogether upon that of the curve $F(x, y)=0$; viz. the form of my result is, in order that a differential equation of the first order $F\left(u, \frac{du}{dz}\right)=0$ may have a monotropic integral, the curve $F(x, y)=0$ must satisfy certain conditions. Briot and Bouquet give their result (Theorem IV., p. 296) under a somewhat different form, as follows; viz. changing only the word "monodrome" into "monotrope," their theorem is:—

"Pour qu'une équation différentielle du premier ordre de la forme

$$\left(\frac{du}{dz}\right)^m + f_1(u)\left(\frac{du}{dz}\right)^{m-1} + \ldots\ldots + f_m(u) = 0$$

admette une intégrale monotrope: 1° les coefficients $f_1(u)$, $f_2(u)$, ..., $f_m(u)$ doivent être des polynômes entiers en u et, au plus, le premier du second degré, le second du quatrième degré, ..., le dernier du degré $2m$; 2° quand pour une certaine valeur de u l'équation a une racine multiple différente de zéro, $\frac{du}{dz}$ doit rester monotrope par rapport à u; 3° quand pour une certaine valeur u_1 de u, l'équation a une racine

multiple égale à zéro, le premier terme du développement de $\frac{du}{dz}$ suivant les puissances croissantes de $(u-u_1)^{1/n}$ doit avoir l'exposant $1-\frac{1}{n}$, si cet exposant est plus petit que l'unité; 4° enfin l'équation différentielle, que l'on déduit de la première en posant $u=\frac{1}{v}$, doit offrir pour $v=0$ les mêmes caractères.—Ces conditions sont suffisantes."

I notice that this may be regarded as a statement referring to the curve

$$F(x, y), = y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

but that in 2° a notion of monotropy is assumed, for y must be a monotropic function of x; and that 4° in effect introduces the new curve $F\left(\frac{1}{x}, -\frac{y}{x^2}\right)=0$.

1. We have between the variables u and z a differential equation of the first order $F\left(u, \frac{du}{dz}\right)=0$, F a rational and integral function of u and $\frac{du}{dz}$. The equation determines u as a function of z, and we wish to know whether u is a monotropic function of z. It will not be so if we have a tropical point, viz. a point $z=c$, such that, in the neighbourhood thereof, the value of u is given by a series $u-a=B(z-c)^\beta+\ldots$, in ascending powers of $z-c$, involving fractional powers of $z-c$. Conversely, if there is no tropical point, then u will be a monotropic function of z; but we must consider and exclude not only tropical points at a finite distance, but also the point at infinity, viz. there must not be any development $u=Bz^\beta+\ldots$, in descending powers, involving fractional powers of z.

2. A one-valued function (*eindeutige Function*, or *fonction uniforme* or *bien déterminée*) is monotropic; since there is only one value, the function must, after a circuit described by the point z, recover its original value. But, conversely, a monotropic function is one-valued, that is, no two or more valued function can be monotropic. For, suppose a two-valued function of z, having the values Z_1 and Z_2, is monotropic; there is by hypothesis no tropical point, that is, no point $z=c$, such that the function Z_1, after a circuit described by the point z, instead of recovering its original value, assumes the value Z_2; and the like as to Z_2. But, this being so, it is meaningless to regard Z_1 and Z_2 as two values of the same function; they must be considered as two distinct and separate functions, each of them a one-valued monotropic function. And it thus appears that the two notions, monotropic and one-valued, are in fact equivalent.

3. In what precedes, u and z are regarded as complex variables geometrically represented by means of two real points in their own planes respectively. But we now write $\frac{du}{dz}=y$, $u=x$, whence $F(x, y)=0$, regarding x, y as the coordinates, real or imaginary, of a point of the curve represented by this equation. And this curve has, moreover, to be considered from a non-projective point of view; we have to distinguish between, and separately consider, finite points (x and y each of them

finite), and the points at infinity (x and y either or each of them infinite). And it is moreover necessary to distinguish between, and separately consider, points on the axis of x (or say axial points), and points off the axis of x (say non-axial points). Taking, for convenience, a to denote a finite value which may be $=0$, b a finite value which is not $=0$ (or say, a not $=\infty$, b not $=\infty$ or 0), we have thus the six cases, 1° (a, b); 2° $(a, 0)$; 3° (a, ∞); 4° (∞, b); 5° $(\infty, 0)$; 6° (∞, ∞). I regard also the axes of x and y as horizontal and vertical respectively, and speak of the tangent or element of the curve as horizontal, vertical, or inclined, accordingly.

4. In the neighbourhood of any given point of the curve, we have $y=$ a series of powers of x, the form of the series being different for different classes of points. I write $P(x)$ to denote a series (finite or infinite) $A+Bx+Cx^2+\ldots$, in ascending powers of x; it is to be throughout understood that the leading coefficient A is not $=0$. Of course, $P(x-a)$, $P(x-a)^{1/m}$, $P(x^{-1})$, &c., will denote the like series $A+B(x-a)+\ldots$, $A+B(x-a)^{1/m}+\ldots$, $A+Bx^{-1}+\ldots$, &c. The coefficients after the leading coefficient A may any or all of them vanish; thus $P(x-a)^{1/m}$ extends to denote the series $P(x-a)$ of integer powers, but naturally the former notation is not used unless the series contains fractional powers. We can, by means of the symbol P, express for any given point of the curve the form of the expansion in the neighbourhood thereof; and, attaching to the point the expansion which belongs to it, we may, for instance, speak of a point $y=P(x-a)$; viz. if b is the constant term of the series, then this is a non-axial point (a, b), which is an ordinary (non-singular) point of the curve, and for which the element is not vertical; a like point with the element vertical would be a point $y=b+(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$. Observe that, in the case of a multiple point, there is a separate expansion for each branch through the point, and in thus attaching an expansion to the point we regard the point as belonging to one of these branches; in dealing with a multiple point, it is necessary to consider separately all the different expansions, that is, all the branches through the point. It is clear that for a point (a, b), $(a, 0)$, or (a, ∞) the expression for y depends on $P(x-a)$, or $P(x-a)^{1/m}$; while, for a point (∞, b), $(\infty, 0)$, or (∞, ∞), it depends on $P(x^{-1})$ or $P(x^{-1/m})$; viz. in the two cases respectively, we have a series in ascending powers of $x-a$, or in descending powers of x.

5. In regard to any given point of the curve, substituting for y, x their values $\frac{du}{dz}$, u, and thence forming the reciprocal of $\frac{du}{dz}$, we obtain $\frac{dz}{du}=$ a series in u; viz. this is a series in ascending powers of $(u-a)$, or in descending powers of u. Integrating, we have $z-c=$ a series in u, which series may or may not contain a logarithmic term $\log u$ or $\log(u-a)$; and, reverting the series, we obtain $u=$ a series in $(z-c)$. This result, applicable to the neighbourhood of the point $u=a$, may contain only integer powers of $z-c$, and be accordingly of the form $u=$ a one-valued series in z; and we then say that the point on the curve is a "permissive" point. Or it may contain fractional powers of $z-c$, and be accordingly of the form $u=$ a more-than-one-valued series in z; and we then say that the point on the curve is a "prohibitive" point. The necessary and sufficient condition in order that u may be

a monotropic function of z is:—the points of the curve must be all of them permissive points; or, what is the same thing, there must not be any prohibitive point.

6. I form a complete table of permissive points, as follows:—

		n a positive integer $=2$ at least. $\epsilon=0$, 1, or -1,
(a, b),	$y=P(x-a)$,	
$(a, 0)$,	$y=(x-a)P(x-a)$ or $(x-a)^2P(x-a)$,	$y=(x-a)^{1+\epsilon/n}P(x-a)^{1/n}$,
(a, ∞),		
(∞, b),	$y=P(x^{-1})$,	
$(\infty, 0)$,		
(∞, ∞),	$y=xP(x^{-1})$ or $x^2P(x^{-1})$,	$y=x^{1+\epsilon/n}P\left(\dfrac{1}{x^{1/n}}\right)$,

viz. it is only a point (a, b), $(a, 0)$, (∞, b) or (∞, ∞) which may be permissive; a point (a, ∞) or $(\infty, 0)$ is prohibitive.

7. A point $y=P(x-a)$ is permissive. We have

$$\frac{du}{dz}=P(u-a),$$

$$\frac{dz}{du}=P(u-a),$$

$$z-c=(u-a)P(u-a),$$

$$u-a=(z-c)P(z-c),$$

the required result. The several steps will be readily understood; the reciprocal of a series $P(u-a)$ is a series of the like form $P(u-a)$; integrating this in regard to u as a series in $u-a$ (no constant being added on the right-hand side), we obtain a series $(u-a)P(u-a)$; and finally, $z-c$ being equal to this expression, we obtain, by reversion, a like expression $u-a=(z-c)P(z-c)$.

We might, for convenience, have written x, u, z in place of $x-a$, $u-a$, $z-c$, respectively. The proof would have run

$$y=P(x),\ \frac{du}{dz}=P(u),\ \frac{dz}{du}=P(u),\ z=uP(u),\ u=zP(z),$$

meaning $u-a=(z-c)P(z-c)$, as above; and, in what follows, this is done accordingly.

It is worth while to make an instance in which the integration can be performed—say we consider the point (0, 1) of the curve $y=\dfrac{1+x+x^2}{1+2x}$, which point is $y=\dfrac{1+x+x^2}{1+2x}=P(x)$. We have

$$\frac{du}{dz}=\frac{1+u+u^2}{1+2u},\ \frac{dz}{du}=\frac{1+2u}{1+u+u^2},$$

$$z=\log(1+u+u^2),\ \text{or}\ u+u^2=(e^z-1),$$

giving a series $u = zP(z)$; or the point is a permissive point, as in the general case. We have here $u + u^2 = (e^z - 1)$, viz. u is not a monotropic function of z; but this is by reason of a prohibitive point $(-\frac{1}{2}, \infty)$.

8. A point $y = (x-a)P(x-a)$ is permissive.

Writing as above u, z in place of $u-a$, $z-c$, we have

$$\frac{du}{dz} = uP(u), \quad \frac{dz}{du} = \frac{1}{u}P(u), \text{ or say } m\frac{dz}{du} = \frac{1}{u} + Pu,$$

whence

$$mz = \log u + uPu,$$

that is,

$$ue^{uPu} = e^{mz}, \text{ or say } uP(u) = e^{mz},$$

whence

$$u = e^{mz}P(e^{mz}), \text{ a one-valued series.}$$

I take a particular instance, the point (0, 0) for the curve $y = \dfrac{x - \frac{1}{3}x^3}{1 - x^2}$; here

$$\frac{du}{dz} = \frac{u - \frac{1}{3}u^3}{1 - u^2}, \text{ or } \frac{dz}{du} = \frac{1 - u^2}{u - \frac{1}{3}u^3},$$

$$z = \log(u - \tfrac{1}{3}u^3), \text{ or } u - \tfrac{1}{3}u^3 = e^z;$$

the point is permissive. The equation just obtained shows that u is not a monotropic function of z; but this is by reason of the prohibitive points $(\pm 1, \infty)$.

9. A point $y = (x-a)^2 P(x-a)$ is permissive. Here

$$\frac{du}{dz} = u^2 P(u), \quad \frac{dz}{du} = \frac{1}{u^2}P(u), \quad = \frac{A}{u^2} + \frac{B}{u} + P(u),$$

whence

$$z = -\frac{A}{u} + B\log u + uP(u),$$

where the logarithmic term occurs or does not occur, according as B is not or is $= 0$. In the former case, we have

$$e^{z/B} = ue^{-\frac{A}{B}u + uP(u)};$$

in the latter case,

$$z = -\frac{A}{u} + uP(u);$$

and in each case we can, by reversion, express u as a one-valued series in z. We may take as examples the two curves $y = \dfrac{x^2}{1-x}$ and $y = \dfrac{x^2}{1-x^2}$; in each case the point (0, 0) is a permissive point, but, by reason of the prohibitive points $(1, \infty)$ and $(\pm 1, \infty)$, u is not a monotropic function of z.

10. As to the other forms of permissive points, the proofs depend upon those for the forms already considered. Thus for

$$y = (x-a)^{1+\epsilon/n} P(x-a)^{1/n}, \text{ or } \frac{du}{dz} = u^{1+\epsilon/n} P(u^{1/n}),$$

writing $v = u^{1/n}$, we find

$$\frac{dv}{dz} = u^{(1+\epsilon)/n} P(u^{1/n}), = v^{1+\epsilon} P(v),$$

that is,

$$\frac{dv}{dz} = P(v),\ vP(v), \text{ or } v^2 P(v);$$

in each case $v =$ one-valued series in z, and thence $u = v^n =$ one-valued series in z.

Similarly for

$$y = P(x^{-1}),\ y = xP(x^{-1}),\ y = x^2 P(x^{-1}),$$

or say

$$y = x^{1+\epsilon} P(x^{-1});$$

here

$$\frac{du}{dz} = u^{1+\epsilon} P(u^{-1}),$$

or, writing $v = u^{-1}$, we have

$$\frac{dv}{dz} = u^{\epsilon-1} P(u^{-1}), = v^{1-\epsilon} P(v);$$

viz. this is

$$\frac{dv}{dz} = P(v),\ vP(v), \text{ and } v^2 P(v);$$

in each case $v =$ one-valued series in z, and thence $u = v^{-1} =$ one-valued series in z.

And finally, for

$$y = x^{1+\epsilon/n} P\left(\frac{1}{x^{1/n}}\right);$$

here

$$\frac{du}{dz} = u^{1+\epsilon/n} P\left(\frac{1}{u^{1/n}}\right),$$

or, writing $v = u^{-\epsilon/n}$, we have

$$\frac{dv}{dz} = u^{(\epsilon-1)/n} P\left(\frac{1}{u^{1/n}}\right), = v^{1-\epsilon} P(v);$$

that is,

$$\frac{dv}{dz} = P(v),\ vP(v), \text{ or } v^2 P(v),$$

as before; $v =$ one-valued series in z, and thence $u = v^{-n} =$ one-valued series in z.

11. There is no difficulty in showing that every point of the curve, not being a permissive point as above, is a prohibitive point. Thus for a point (a, b), if the series for $y - b$ contain any fractional power, say, if we have $y = b + (x-a)^\alpha$, then

$$\frac{du}{dz} = b + u^\alpha,\ \frac{dz}{du} = \frac{1}{b} - \frac{1}{b^2} u^\alpha + \ldots,\ z = \frac{u}{b} - \frac{1}{(\alpha+1) b^2} u^{\alpha+1} + \ldots,$$

whence, reverting, $u = bz +$ series containing the fractional power $z^{\alpha+1}$.

Again, a point $y=(x-a)^m P(x-a)$, $m=3$ or any higher integer value, is prohibitive. Attending only to the leading term, we have $\frac{du}{dz}=u^m$, whence $\frac{dz}{du}=u^{-m}$ or $z=u^{1-m}$, whence, reverting, $u=z^{1/(1-m)}$, which is a fractional power; and this is also the case if m be a negative integer. And similarly, in other cases, the series for u will always contain fractional powers of z.

12. In order that u may be a monotropic function of z, the points of the curve must be all of them permissive, or, what is the same thing, there must be no prohibitive point; we have to consider the geometrical signification of this condition. If we attend first to the ordinary (or non-singular) points of the curve, a non-axial point (a, b) must be a point $y=P(x-a)$, viz. there must be no such point with a vertical element.

An axial point $(a, 0)$ may be $y=(x-a)P(x-a)$, viz. the element may be inclined; or the point may be $y=(x-a)^2P(x-a)$, viz. the element may be horizontal (but in this case there must be an ordinary or two-pointic contact only); or the point may be $y=(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$, viz. the element may be vertical.

The point must not be (a, ∞), viz. there must be no asymptote parallel to (or coincident with) the axis of y.

The point may be (∞, b), $y=P(x^{-1})$, that is, there may be an ordinary, or osculating, asymptote parallel to the axis of x. But there is no point $(\infty, 0)$, that is, there must be no asymptote coincident with the axis of x.

There may be points (∞, ∞), $y=xP(x^{-1})$, that is, ordinary or osculating asymptotes inclined to the axes; and there may also be points (∞, ∞), $y=x^2P(x^{-1})$, that is, asymptotic parabolas of vertical axis.

13. We have, moreover, conditions as to the singular points; thus, every non-axial multiple point (a, b) must be a point with each branch of the form $y=P(x-a)$, that is, each branch must be an ordinary (non-cuspidal) branch, and must be non-vertical. The axial multiple or singular points $(a, 0)$ may have ordinary (non-cuspidal) branches $y=(x-a)P(x-a)$ and $y=(x-a)^2P(x-a)$, inclined or horizontal, or else $y=(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$, vertical; and there may also be cuspidal branches $y=(x-a)P(x-a)^{1/n}$, $y=(x-a)^{1+1/n}P(x-a)^{1/n}$, and $y=(x-a)^{1-1/n}P(x-a)^{1/n}$, inclined, horizontal, or vertical.

There must be no singular points (a, ∞); any such point is, in fact, excluded by the condition, no asymptote parallel to or coincident with the axis of y.

There may be multiple points (∞, b), but not $(\infty, 0)$; viz. as above, there may be asymptotes parallel to, but not coincident with, the axis of x; but in this case, the several branches must be each of them an ordinary branch $y=P(x^{-1})$.

Finally, there may be multiple or singular points (∞, ∞), but each branch must be either an ordinary branch $y=xP(x^{-1})$ or $y=x^2P(x^{-1})$, or a cuspidal branch

$$y=xP\left(\frac{1}{x^{1/n}}\right),\quad y=x^{1+1/n}P\left(\frac{1}{x^{1/n}}\right),\quad \text{or}\quad y=x^{1-1/n}P\left(\frac{1}{x^{1/n}}\right).$$

14. The enumeration seems long and difficult. But for a given curve we have, in fact, only to see that there are no ordinary non-axial points (a, b) of vertical element, and to further examine the finite singular points (a, b) and $(a, 0)$, and also the several infinite branches as giving ordinary or else singular points (a, ∞), (∞, b), $(\infty, 0)$, or (∞, ∞), viz. it has to be seen for each infinite branch that the expansion is of the proper form.

15. Writing the equation of the curve in the form

$$f_0(x)\, y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

where $f_0, f_1, \ldots, f_m$ denote rational and integral functions of x, then, if a be a root of the equation $f_0(x) = 0$, there will be on the curve a point (a, ∞), which is prohibitive; $f_0(x)$ is therefore a constant, or, taking it to be $= 1$, the form must be

$$y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

where $f_1, f_2, \ldots, f_m$ denote rational and integral functions of x.

It is to be shown that the degrees of these functions are equal respectively to $2, 4, \ldots, m$ at most. Supposing that the degrees are $2, 4, \ldots, m$; then for the points at infinity we find an expansion $y = Ax^\omega + \ldots$, in descending powers of x, by substitution in an equation

$$y^m + \alpha_1 x^2 y^{m-1} + \alpha_2 x^4 y^{m-2} + \ldots + \alpha_n x^{2n} = 0;$$

whence $\omega = 2$, and A is determined by an equation of the order m; there are thus m branches $y = x^2 P(x^{-1})$, corresponding to permissive points. If, however, any one of the functions f be of a higher degree, then it may be shown (but the formal proof is not easily given) that there will be at any rate a branch $y = Ax^\omega + \&c.$, where ω has a value > 2, and we have thus a prohibitive point; hence the degrees must be as stated.

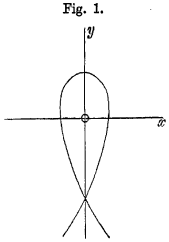
16. As an example of a differential equation with a monotropic solution, take

$$\left(\frac{du}{dz}\right)^3 + 3\left(\frac{du}{dz}\right)^2 + 27u^2 - 4 = 0;$$

we have here the curve

$$y^3 + 3y^2 + 27x^2 - 4 = 0,$$

Fig. 1.



which is a nodal cubic, as shown in the annexed figure. To put the node in evidence, the equation may be written in the form $(y+2)^2(y-1) + 27x^2 = 0$; viz. the node is

the point $(0, -2)$, and the equation of the tangents is $(y+2)^2 - 27x^2 = 0$. The curve besides meets the line $x=0$ in the point $y=1$, where the tangent is horizontal, and it meets the axis $y=0$ in the points $x = \pm \frac{2}{3\sqrt{3}}$, at each of which the tangent is vertical; these two points, as lying on the axis $y=0$, are thus each of them permissive; and they are the *only* points where the tangent is vertical (in fact, the tangents from the point at infinity on the line $x=0$ are the two lines $x = \pm \frac{2}{3\sqrt{3}}$, and the line infinity counting twice, as a tangent at an inflexion).

For the points at infinity, we have $y = -3x^{\frac{2}{3}} - 1 - \frac{1}{3}x^{-\frac{2}{3}} + \&c.$, which is of the form $y = x^{\frac{2}{3}} P\left(\frac{1}{x^{\frac{1}{3}}}\right)$, included in $y = x^{1-1/m} P\left(\frac{1}{x^{1/m}}\right)$, and the point is thus permissive; there is no prohibitive point, and the differential equation has a monotropic solution. This is, in fact, the rational solution $u = (z-c) - (z-c)^3$, or say $u = z - z^3$; the curve is thus given by the two equations $x = z - z^3$, $y = 1 - 3z^2$.

17. As a second example, take the differential equation

$$\left(\frac{du}{dz}\right)^3 - \left(\frac{du}{dz}\right)^2 + 4u^2 - 27u^4 = 0.$$

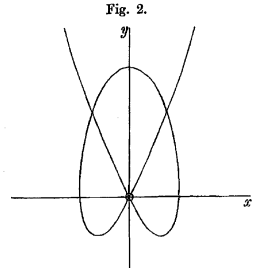
We have here the curve

$$y^3 - y^2 + 4x^2 - 27x^4 = 0,$$

which is a trinodal quartic curve, as shown in the figure. There is a node at the origin with the tangents $y^2 - 4x^2 = 0$, and, writing the equation in the form

$$(3y+1)(3y-2)^2 - (27x^2 - 2)^2 = 0,$$

Fig. 2.



we have for the other two nodes $x = \pm \frac{2}{3\sqrt{3}}$, $y = \frac{2}{3}$. The lowest points are given by $x = \pm \frac{2}{3\sqrt{3}}$, $y = -\frac{1}{3}$, viz. the line $y = -\frac{1}{3}$ is a horizontal tangent. Writing $x=0$, we have $y^3 - y^2 = 0$, that is, $y^2 = 0$, the node at the origin, and $y=1$, the height of the

loop; and, writing $y=0$, we have $4x^2-27x^4=0$, that is, $x^2=0$, the node at the origin, and $x=\pm\frac{2}{3\sqrt{3}}$, the other two intersections with the axis of x. The tangents at these two points are vertical; as being on the axis, they are thus permissive points. And they are the *only* points with a vertical tangent; in fact, the point $(x=0,\ y=\infty)$ is a point of the curve, with the line ∞ as an osculating tangent (4-pointic intersection); hence the tangents from the point $x=0,\ y=\infty$ are the line ∞ counting four times, and the lines $x=\pm\frac{2}{3\sqrt{3}}$ touching at the points $\left(\pm\frac{2}{3\sqrt{3}},\ 0\right)$ as above. For the infinite branches, we have $y=3x+\ldots$, which is of the right form $y=x^{1+1/n}P\left(\frac{1}{x^{1/n}}\right)$. We thus see that there is no prohibitive point; and the differential equation has a monotropic solution accordingly.

Writing z in place of $z-c$, the solution in fact is

$$u=\frac{e^z-e^{-z}}{e^z+e^{-z}}-\left(\frac{e^z-e^{-z}}{e^z+e^{-z}}\right)^3;$$

hence, putting $\theta=\frac{e^z-e^{-z}}{e^z+e^{-z}}$, the curve is given by the two equations

$$x=\theta-\theta^3,\quad y=(1-3\theta^2)(1-\theta^2),\quad =1-4\theta^2+3\theta^4:$$

the monotropic function u satisfies the differential equation.

18. $F\left(u,\ \frac{du}{dz}\right)=0$ must be either a rational function, a singly periodic function, or a doubly periodic function of z; say the forms are

$$u=P(z),\quad u=P(e^{gz}),$$

and

$$u=P\{\operatorname{sn}(gz,\ k)\}+\operatorname{cn}(gz,\ k)\operatorname{dn}(gz,\ k)\,Q\{\operatorname{sn}(gz,\ k)\},$$

where P, Q denote rational functions. I do not at present consider the criteria (such as are given in Briot and Bouquet's Theorem v., p. 301) for determining by means of the curve which is the form of the integral. I remark, however, that in the first and second cases the curve is unicursal, while in the third case it is bicursal; or say that, according as the deficiency is $=0$ or 1, the integral is rational or simply periodic, or else it is doubly periodic. Moreover, the curve being unicursal, we can express the coordinates as equal to rational functions $P(\theta)$, $Q(\theta)$ of a parameter θ; and, being bicursal, we can express them as functions of the form

$$P(\theta)+P_1(\theta)\sqrt{1-\theta^2\,.\,1-k^2\theta^2},\quad Q(\theta)+Q_1(\theta)\sqrt{1-\theta^2\,.\,1-k^2\theta^2},$$

of the parameter θ; and supposing the coordinates $(x,\ y)$, that is, u and $\frac{du}{dz}$, thus expressed, it should be easy to establish the relation of θ with z, e^{gz} or $\operatorname{sn}(gz,\ k)$ in the three cases respectively.

861.

NOTE ON A FORMULA RELATING TO THE ZERO-VALUE OF A THETA-FUNCTION.

[From *Crelle's Journal der Mathem.*, t. C. (1887), pp. 87, 88.]

I HAD some difficulty in verifying for the case of a single theta-function, a formula given in Herr Thomae's paper "Beitrag zur Theorie der ϑ-Functionen," *Crelle's Journal*, vol. LXXI. (1870), pp. 201—222. The formula in question (see p. 216) is given as follows:

$$(11) \qquad \vartheta(0, 0, \ldots 0) = \sqrt{\frac{|A_\lambda^{(\lambda')}|}{(2\pi i)^p}} \sqrt[4]{\text{Discr.}(0, 0, \ldots 0)\,\text{Discr.}(0, 0, \ldots 0)},$$

but the denominator factor should I think be $(\pi i)^p$ instead of $(2\pi i)^p$. Making this alteration, then in the case of a single theta-function, $p = 1$, and the function belongs to the radical

$$\sqrt{x - k_1 \,.\, x - k_2 \,.\, x - k_3 \,.\, x - k_4},$$

where

$$(k_1, k_2, k_3, k_4) = \left(-\frac{1}{k}, -1, +1, +\frac{1}{k}\right).$$

The determinant $|A_\lambda^{(\lambda')}|$ is a single term $= A$, and the formula becomes

$$\vartheta(0) = \sqrt{\frac{A}{\pi i}} \sqrt[4]{(k_3 - k_1)(k_4 - k_2)},$$

where $k_3 - k_1$, $k_4 - k_2$ are each $= 1 + \frac{1}{k}$, and we have therefore

$$\vartheta(0) = \sqrt{\frac{A}{\pi i}\left(1 + \frac{1}{k}\right)};$$

also A denotes the integral

$$\int_{k_1}^{k_2} \frac{dx}{\sqrt{x - k_1 \,.\, x - k_2 \,.\, x - k_3 \,.\, x - k_4}}, \quad = \int_{-\frac{1}{k}}^{-1} \frac{k\,dx}{\sqrt{1 - x^2 \,.\, 1 - k^2x^2}}$$

$$= \int_1^{\frac{1}{k}} \frac{k\,dx}{\sqrt{1 - x^2 \,.\, 1 - k^2x^2}} = ikK',$$

and the formula thus is

$$\vartheta(0) = \sqrt{\frac{K'(1+k)}{\pi}}.$$

But observe that, in the theta-function as defined by the equation

$$\vartheta(x) = \Sigma e^{am^2+2mx},$$

a is used to denote the value

$$a = a_1'B, \quad = \frac{2\pi}{A} B,$$

where A is the above-mentioned integral, and B is the integral

$$B = \int_{k_2}^{k_3} \frac{dx}{\sqrt{x-k_1 \,.\, x-k_2 \,.\, x-k_3 \,.\, x-k_4}}, \quad = \int_{-1}^{1} \frac{k\,dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}} = 2kK,$$

which value must however be taken negatively, viz. we must write $B = -2kK$, and we then have

$$a = -\frac{2\pi K}{K'},$$

viz. writing as usual

$$q = e^{-\frac{\pi K'}{K}}, \quad r = e^{-\frac{\pi K}{K'}},$$

the e^a of the theta-function is not $= q$, but it is $= r^2$; and the zero-value $\vartheta(0)$ is $= 1 + 2r^2 + 2r^8 + 2r^{18} + \ldots$. The equation thus is

$$1 + 2r^2 + 2r^8 + 2r^{18} + \ldots = \sqrt{\frac{K'(1+k)}{\pi}},$$

which is right; in fact, writing k' in place of k, and consequently K, q in place of K', r respectively, the equation becomes

$$1 + 2q^2 + 2q^8 + 2q^{18} + \ldots = \sqrt{\frac{K(1+k')}{\pi}};$$

we have

$$1 + 2q + 2q^4 + 2q^9 + \ldots = \sqrt{\frac{2K}{\pi}},$$

and changing q into q^2, then (*Fund. Nova*, p. 92) K is changed into $\frac{K(1+k')}{2}$, and we have the formula in question. As a verification for small values of q, observe that we have

$$\frac{2K}{\pi} = 1 + 4q + 4q^2, \quad \frac{1+k'}{2} = 1 - 4q + 16q^2,$$

and thence

$$\frac{K(1+k')}{\pi} = 1 + 4q^2 \text{ or } \sqrt{\frac{K(1+k')}{\pi}} = 1 + 2q^2.$$

Cambridge, 12 *February*, 1886.

862.

NOTE ON THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.

[From *Crelle's Journal der Mathem.*, t. C. (1887), pp. 286—295.]

1. I CONSIDER a linear differential equation

$$p_0 \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y = 0,$$

where $p_0, p_1, \ldots, p_m$ are rational and integral functions of x, having no common factor: as usual, x is a complex magnitude represented by a point, and we consider the integrals belonging to a singular point $x = a$ of the differential equation. An integral may be a regular integral, or it may be what Thomé calls a normal elementary integral: the theory of these integrals (which I would rather call subregular integrals) requires, I think, further examination.

2. I retain x as the independent variable, but for shortness use t to denote $\frac{1}{x-a}$: and I take as the dependent variable z, $= \frac{1}{y}\frac{dy}{dx}$: we have then z determined by a non-linear differential equation of the order $m-1$, and from any value of z we derive a corresponding value $e^{\int z dx}$ of y.

3. To obtain the z-equation, using for shortness accents to denote differentiation in regard to x, we have $z = \frac{y'}{y}$, and thence $z' = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2$, that is, $z^2 + z' = \frac{y''}{y}$: hence also $2zz' + z'' = \frac{y'''}{y} - \frac{y'y''}{y^2}$, that is,

$$z(z^2 + z') + 2zz' + z'' = \frac{y'''}{y}, \text{ or finally } z^3 + 3zz' + z'' = \frac{y'''}{y},$$

and so on; viz. the values of $\frac{y'}{y}, \frac{y''}{y}, \frac{y'''}{y}, \ldots$ are $z, z^2 + z', z^3 + 3zz' + z'', \ldots$; generally for

$\frac{y^{(n)}}{y}$ the first term is z^n and the last term is $z^{(n-1)}$. If, to fix the ideas, the y-equation is

$$p_0 y''' + p_1 y'' + p_2 y' + p_3 = 0,$$

then, dividing by y, the z-equation is

$$p_0 (z^3 + 3zz' + z'') + p_1 (z^2 + z') + p_2 z + p_3 = 0;$$

and similarly, from the y-equation of the order m, we derive a z-equation of the order $m-1$,

$$p_0 \{z^m + \ldots + z^{(m-1)}\} + \ldots + p_{m-2} (z^2 + z') + p_{m-1} z + p_m = 0.$$

4. In this equation, each coefficient is in the first instance a rational and integral function of x, or, what is the same thing, of $x-a$; writing t to denote $\frac{1}{x-a}$, each coefficient is thus the sum of a finite number of negative integer powers of t; hence multiplying the whole equation by a proper positive power of t, the several coefficients become each of them the sum of a finite number of positive powers of t, or arranging in descending powers of t, say the general form is $p_r = \alpha_r t^\lambda + \beta_r t^{\lambda-1} + \ldots + \kappa_r$, where λ is a positive integer which may be $=0$, and which has for each coefficient its proper value. We wish to satisfy the z-equation by a value $z =$ descending series in t, viz. the form is $z = At^\alpha + A't^{\alpha-\alpha'} + \ldots$, with powers which are integral or fractional, but where the number of positive powers is finite. The theory is almost identical with that of the solution in like form of the algebraical equation

$$p_0 z^m + p_1 z^{m-1} + \ldots + p_{m-1} z + p_m = 0,$$

or say the equation $U = 0$, where U is a rational and integral function of z and t, of the degree m as regards z.

5. Considering z and t as Cartesian coordinates, the equation in question, $U = 0$, represents a geometrical curve such for any given value of the abscissa t, the ordinate z has m values: the curve is of the order m at least, but it may be of a higher order $m + \kappa$; when this is so, there is at infinity on the axis $t = 0$, a κ-tuple point K, and thus any line parallel to the axis $t = 0$, intersects the curve in the point K counting as κ intersections, and in m other points, which are the points belonging to the m values of the ordinate z. Taking $s = 1$, and for z, t writing $\frac{z}{s}$, $\frac{t}{s}$ respectively, we have between the trilinear coordinates z, t, s an equation $U' = 0$ of the order $m + \kappa$, where $s = 0$ is the equation of the line infinity: and the curve has a κ-tuple point at the point K for which $t = 0$, $s = 0$.

6. Starting from the equation $U = 0$, for an arbitrary value of t we have m values of z all of them different. These may be developed, each of them as a descending series in t, and the development is effected in the usual manner; viz. considering for the moment t as very large, then z is in general very large and it has a leading term At^α, which is determined by writing in the equation $z = At^\alpha$, and then finding α in such wise that there are two or more exponents equal to each other and greater

(that is, nearer $+\infty$) than any other exponent: we have thus a highest power of t, the whole coefficient for which must vanish, viz. we obtain an equation of two or more terms giving for A a value or values not $=0$; in the term At^{α} in question, α is in general positive, but it may be $=0$, or be negative. The leading term At^{α} being found in this manner, the law for the exponents of the subsequent terms is frequently at once apparent; but, if necessary, we write $z=At^{\alpha}+A't^{\alpha-\alpha'}$, and determine in like manner the exponent $\alpha-\alpha'$ and the coefficient A'. Proceeding in this manner until the law of the exponent becomes apparent, and then by the method of indeterminate coefficients, we finally arrive at a series

$$z=At^{\alpha}+A't^{\alpha-\alpha'}+A''t^{\alpha-\alpha'-\alpha''}+\ldots,$$

in descending powers of t: the exponents are integral or fractional, but the number of positive exponents is always finite. There is either a single series or it may be two or more series; but the number of series is at most $=m$, and when it is less than m, then the coefficients in the several series or some of them will contain radicals, and by giving to each radical its different values, the system of series will determine m different values of z.

7. The curve meets the line infinity ($s=0$) in the point K counting as κ intersections, and in m other points, some or all of which may coincide with K; the forms of the series depend on the configuration of the m points, or say on the relation of the curve to the line infinity. Thus suppose $\kappa=0$, and further that the m points are all of them distinct, that is, let the curve be a curve of the order m meeting the line infinity in m distinct points, or, what is the same thing, having m asymptotes, no two of them parallel; we have in this case m series, each of the form

$$z=At+B+\frac{C}{t}+\frac{D}{t^2}+\ldots,$$

where the several coefficients A have distinct values.

8. In the foregoing case A is determined by an equation of the order m, having unequal roots; and taking for A any root at pleasure, the remaining coefficients $B, C, \ldots$ are each of them linearly determined. If the equation has two equal roots, this may correspond to the case of two parallel asymptotes (the curve has here a node at infinity); the coefficients $B, C, \ldots$ will in this case depend on a quadric radical, and by giving to this radical its two values, we obtain the two series

$$z=At+B+\frac{C}{t}+\ldots,$$

$$z=At+B'+\frac{C'}{t}+\ldots,$$

corresponding to the two equal roots of the equation. But if instead of a node at infinity we have the line infinity a tangent to the curve, then instead of the two parallel asymptotes, we have an asymptotic parabola; the series assumes a new form, viz. it contains terms in $t^{\frac{1}{2}}, t^{-\frac{1}{2}}, \ldots$; the coefficients of the integer powers have

determinate values, but those of the powers $t^{\frac{1}{2}}$, $t^{-\frac{1}{2}}, \ldots$ contain as factor a quadric radical, and giving to this radical its two values, we have two series

$$z = At + C + \ldots + t^{\frac{1}{2}}\left(B + \frac{D}{t} + \ldots\right),$$

$$z = At + C + \ldots - t^{\frac{1}{2}}\left(B + \frac{D}{t} + \ldots\right).$$

9. The like considerations apply in the case where the line infinity passes through higher multiple points of the curve, or has with it a contact or contacts other than a single ordinary contact: in every case, reckoning as distinct series those obtained by attaching to each radical its different values, the number of distinct series will be $= m$ precisely.

10. Reverting now to the differential equation

$$p_0\{z^m + \ldots + z^{(m-1)}\} + \ldots + p_{m-2}(z^2 + z') + p_{m-1}z + p_m = 0,$$

it is to be observed that the very same process is applicable to the determination of z in the form of a descending series $z = At^\alpha + A't^{\alpha-\alpha'} + \ldots$; moreover, it frequently happens that the determination of the leading term At^α is made by means of the algebraical equation

$$p_0 z^m + \ldots + p_{m-2}z^2 + p_{m-1}z + p_m = 0,$$

viz. this is so whenever the value of α is greater than 1: in fact, writing $z = At^\alpha\,(\alpha > 1)$, then in the sets of terms $\{z^m + \ldots + z^{(m-1)}\}, \ldots, (z^3 + 3zz' + z''), (z^2 + z')$, the first terms $z^m, \ldots, z^3, z^2$ are each of them of a higher degree than any other term in the same brackets, so that attending only to the terms of highest degree the sets may be reduced to $z^m, \ldots, z^3, z^2$. But in the determination of the subsequent exponents and coefficients, the omitted terms or some of them would of course come into play: and it is at least not obvious that we can, for the forms of the series which satisfy the differential equation, employ geometrical considerations such as those which were used in regard to the series satisfying the algebraical equation.

11. Considering as above the coefficients $p_0, p_1, \ldots, p_m$ as rational and integral functions of t, it is easy to determine the degrees of these coefficients in such wise that the exponent α of the leading term At^α shall have a given value. Suppose first, this given value is $\alpha = \mathrm{a}$, a positive integer greater than 1: the degrees may be $\theta, \theta + \mathrm{a}, \theta + 2\mathrm{a}, \ldots, \theta + m\mathrm{a}$, ($\theta = 0$ or any positive integer at pleasure); and not only so, but if we write

$$p_0 = L_0 t^\theta + \ldots, \quad p_1 = L_1 t^{\theta+\mathrm{a}} + \ldots, \quad p_m = L_m t^{\theta+m\mathrm{a}} + \ldots,$$

then writing $z = At^{\mathrm{a}}$, clearly the highest power in the equation is $t^{\theta+m\mathrm{a}}$, and equating the coefficient hereof to zero, we have

$$L_0 A^m + L_1 A^{m-1} + \ldots + L_{m-1}A + L_m = 0,$$

an equation of the degree m for the determination of the coefficient of the leading term At^{a} $(\mathrm{a}>1)$. Assuming that the roots are all unequal, we have m series each of the form

$$z = At^{\mathrm{a}} + Bt^{\mathrm{a}-1} + \ldots + K + \frac{L}{t} + \ldots.$$

12. Suppose $\mathrm{a}=1$, that is, let the leading term be At; here the degrees may be θ, $\theta+1$, $\theta+2, \ldots$, $\theta+m$, ($\theta=0$ or any positive integer as before), but we have a different form for the A-equation. In fact, here $(z=At)$ in the sets of terms $\{z^m + \ldots + z^{(m-1)}\}, \ldots$, $(z^3+3zz'+z'')$, (z^2+z'), the terms in each brackets are of the same degree, viz. the degrees are $m, \ldots, 3, 2$; and not only so, but we have $z^2+z'=(A^2+A)\,t^2$, that is, $=[A]^2\,t^2$, $z^3+3zz'+z''=[A]^3\,t^3, \ldots$. Hence if

$$p_0 = L_0 t^\theta + \ldots, \quad p_1 = L_1 t^{\theta+1} + \ldots, \quad p_m = L_m t^{\theta+m} + \ldots,$$

the A-equation is

$$L_0\,[A]^m + L_1\,[A]^{m-1} + \ldots + L_{m-1}\,[A]^1 + L_m = 0,$$

an equation of the degree m as before; assuming that the roots are unequal, we have here m series each of the form

$$z = At + B + \frac{C}{t} + \frac{D}{t^2} + \ldots,$$

being, as will presently appear, the case of m regular integrals for the y-equation.

13. In either of the last-mentioned cases, the formula for the A-equation holds good if any two or more of the coefficients $p_0, p_1, \ldots, p_m$ are of the degrees aforesaid, the others of them being of inferior degrees; we have only to put $=0$ the L's which belong to the coefficients of inferior degrees. If neither L_0 nor L_m be $=0$, the equation is still an equation of the order m, with m effective roots; but if any of the coefficients L at the beginning of the equation vanish we thus introduce roots $=\infty$: and if any of the coefficients L at the end of the equation vanish, we thus introduce roots $=0$: the number of effective roots is $=m-$the number of roots ∞ or 0; the case requires further consideration. If the equation has equal roots, then from what precedes, it is easy to infer that we may have distinct series containing as in the general case only integer powers of t, or we may have series beginning with At^{a} or At, but containing also fractional powers of t.

14. If the leading term is $z=At^{\alpha}$, $\alpha=\mathrm{a}$, a value <1, then in the sets of terms $\{z^m + \ldots + z^{(m-1)}\}, \ldots$, $(z^3+3zz'+z'')$, (z^2+z') the last terms $z^{(m-1)}, \ldots, z'', z'$ are each of them of a higher degree than any other term in the same brackets, so that attending only to the terms of the highest degree the sets may be reduced to $z^{(m-1)}, \ldots, z'', z'$, or we may consider the equation

$$p_0 z^{(m-1)} + p_1 z^{(m-2)} + \ldots + p_{m-2} z' + p_{m-1} z + p_m = 0.$$

In particular, this is the case for $z=At^{-\mathrm{a}}$, a being a positive integer; in this case, the degrees of $p_0, p_1, \ldots, p^{m-1}, p_m$ may be $\theta+\mathrm{a}-m+1$, $\theta+\mathrm{a}-m+2, \ldots$, $\theta+\mathrm{a}$, θ; and this being so, the leading coefficient A will be determined by a linear equation. But the case is, in fact, that of a non-singular point $x=a$: and I do not here further consider it.

15. Reverting to the case where the series is

$$z = At + B + \frac{C}{t} + \frac{D}{t^2} + \ldots,$$

and substituting for t its value $\frac{1}{x-a}$, or for greater convenience $\frac{1}{x}$, this is

$$\frac{1}{y}\frac{dy}{dx} = \frac{A}{x} + B + Cx + Dx^2 + \ldots,$$

whence

$$\log y = A \log x + Bx + \tfrac{1}{2}Cx^2 + \ldots,$$

or passing to the value of y, but expanding the exponential of $Bx + Cx^2 + \ldots$, this is

$$y = x^A (1 + B'x + C'x^2 + \ldots),$$

viz. we have here a regular integral. By what precedes, it appears that, for the values $p_0 = L_0 x^{-\theta} + \ldots$, $p_1 = L_1 x^{-\theta-1} + \ldots$, $p_m = L_m x^{-\theta-m} + \ldots$, the exponent A is determined by the equation

$$L_0 [A]^m + L_1 [A]^{m-1} + \ldots + L_{m-1} [A]^1 + L_m = 0,$$

which is the equation for that purpose considered by Fuchs, and is what I would call the indicial equation.

The logarithmic forms which belong to the case of equal roots may be deduced from the case of unequal roots, by supposing two or more of the roots to approach each other continuously and become ultimately equal.

16. Similarly, if the series be

$$z = Ft^{\mathrm{a}} + \ldots + Kt^2 + At + B + \frac{C}{t} + \ldots,$$

(a being a positive integer $= 2$ at least), then writing as above $t = \frac{1}{x}$, this is

$$\frac{1}{y}\frac{dy}{dx} = \frac{F}{x^{\mathrm{a}}} + \ldots + \frac{K}{x^2} + \frac{A}{x} + B + Cx + \ldots,$$

whence, if for shortness $w = -\frac{F}{\mathrm{a}-1}\frac{1}{x^{\mathrm{a}-1}} - \ldots - \frac{K}{x}$, we have

$$\log y = w + A \log x + Bx + \tfrac{1}{2}Cx^2 + \ldots,$$

or passing to the value of y, but expanding as before the exponential of $Bx + \frac{1}{2}Cx^2 + \ldots$, this is

$$y = e^w x^A (1 + B'x + C'x^2 + \ldots),$$

a subregular integral. By what precedes, it appears that, if

$$p_0 = L_0 x^{-\theta} + \ldots,\ p_1 = L_1 x^{-\theta-\mathrm{a}} + \ldots,\ p_m = L_m x^{-\theta-m\mathrm{a}} + \ldots,$$

then the equation for the leading coefficient F (previously represented by A) is

$$L_0 F^m + L_1 F^{m-1} + \ldots + L_{m-1} F + L_m = 0;$$

but I do not see any direct or simple way of obtaining the corresponding equation for the exponent A of this subregular integral.

17. To illustrate the case of a series with fractional powers, and also in order to work out a simple example with some completeness, I consider the differential equation of the second order $\frac{d^2y}{dx^2} + p_2 y = 0$, giving the z-equation $z^2 + z' + p_2 = 0$; p_2 is a rational and integral function of $t\left(=\frac{1}{x}\right)$, and we have two cases according as the order of the function is even or odd.

18. Suppose first that the order is even, and for convenience taking it to be $= 4$, and assuming the value of p_2 accordingly, I consider the equation

$$z^2 + z' = \alpha t^4 + \beta t^3 + \gamma t^2 + \delta t + \epsilon.$$

Here the form is at once seen to be

$$z = At^2 + Bt + C + \frac{D}{t} + \frac{E}{t^2} + \ldots,$$

and observing that we have

$$z' = -t^2 \frac{dz}{dt} = -2At^3 - Bt^2 + D + \frac{2E}{t} + \ldots,$$

we find

$$\begin{aligned}
A^2 \qquad\qquad\qquad &= \alpha, \\
2AB \qquad\qquad &- 2A = \beta, \\
2AC + B^2 \qquad &- B = \gamma, \\
2AD + 2BC \qquad &+ 0C = \delta, \\
2AE + 2BD + C^2 &+ D = \epsilon, \\
2AF + 2BE + 2CD &+ 2E = 0, \\
&\vdots
\end{aligned}$$

equations which give

$$\begin{aligned}
A &= \sqrt{\alpha}, \\
B &= 1 + \frac{\beta}{2\sqrt{\alpha}}, \\
C &= \frac{\gamma}{2\sqrt{\alpha}} - \frac{\beta}{4\alpha} - \frac{\beta^2}{8\alpha\sqrt{\alpha}}, \\
D &= \frac{\delta}{2\sqrt{\alpha}} - \frac{\gamma}{2\alpha} + \frac{1}{4\alpha\sqrt{\alpha}}(\beta - \beta\gamma) + \frac{\beta^2}{4\alpha^2} + \frac{\beta^3}{16\alpha^2\sqrt{\alpha}}, \\
E &= \frac{\epsilon}{2\sqrt{\alpha}} - \frac{3\delta}{4\alpha} + \frac{1}{4\alpha\sqrt{\alpha}}(3\gamma - \beta\delta - \tfrac{1}{2}\gamma^2) + \frac{1}{4\alpha^2}(-\tfrac{3}{2}\beta + 3\beta\gamma) \\
&\quad + \frac{1}{16\alpha^2\sqrt{\alpha}}(-\tfrac{17}{2}\beta^2 + 3\beta^2\gamma) - \frac{\beta^3}{4\alpha^3} - \tfrac{5}{128}\frac{\beta^4}{\alpha^3\sqrt{\alpha}}; \text{ \&c.}
\end{aligned}$$

We then have

$$\frac{1}{y}\frac{dy}{dx} = \frac{A}{x^2} + \frac{B}{x} + C + Dx + Ex^2 + \ldots,$$

and consequently

$$y = e^{-\frac{A}{x}} x^B e^{Cx + \frac{1}{2}Dx^2 + \frac{1}{3}Ex^3 + \ldots},$$

where the second exponential is to be expanded in the form $1 + C'x + D'x^2 + \ldots$; there are of course two values, say y_1 and y_2, corresponding to the values $+\sqrt{\alpha}$ and $-\sqrt{\alpha}$ of the radical respectively.

19. Secondly, if the order be odd, and for convenience taking it to be $= 3$, and assuming the value of p_2 accordingly, I consider the equation

$$z^2 + z' = \beta t^3 + \gamma t^2 + \delta t + \epsilon.$$

Here the series is at once seen to be

$$z = \quad At^{\frac{3}{2}} + Bt + \quad Ct^{\frac{1}{2}} + D + Et^{-\frac{1}{2}} + \ldots:$$

we have

$$z' = -\tfrac{3}{2}At^{\frac{1}{2}} - Bt^2 - \tfrac{1}{2}Ct^{\frac{3}{2}} + \quad \tfrac{1}{2}Et^{\frac{1}{2}} \quad + \ldots,$$

and we thence derive the equations

$$\begin{aligned}
A^2 \qquad\qquad\qquad\qquad &= \beta,\\
2AB \qquad\qquad\qquad - \tfrac{3}{2}A &= 0,\\
2AC + B^2 \qquad\qquad - B &= \gamma,\\
2AD + 2BC \qquad\quad - \tfrac{1}{2}C &= 0,\\
2AE + 2BD + C^2 \qquad\quad &= \delta,\\
2AF + 2BE + 2CD + \tfrac{1}{2}E &= 0,\\
\vdots
\end{aligned}$$

(where in the next equation we have on the right-hand side ϵ, and in each of the subsequent equations we have on the right-hand side 0). We find

$$\begin{aligned}
A &= \sqrt{\beta},\\
B &= \tfrac{3}{4},\\
C &= \frac{1}{2\sqrt{\beta}}(\gamma + \tfrac{3}{16}),\\
D &= -\frac{1}{4\beta}(\gamma + \tfrac{5}{16}),\\
E &= \frac{1}{2\beta\sqrt{\beta}}(\delta + \tfrac{9}{32}\gamma - \tfrac{1}{4}\gamma^2 + \tfrac{63}{1024}), \text{ \&c.},
\end{aligned}$$

where observe that the coefficients $A, C, E, \ldots$ contain as a factor $\frac{1}{\sqrt{\beta}}$, but the coefficients $B, D, \ldots$ are rational.

We have

$$z = \frac{1}{y}\frac{dy}{dx} = \frac{A}{x^{\frac{3}{2}}} + \frac{B}{x} \qquad + \frac{C}{x^{\frac{1}{2}}} + D + Ex^{\frac{1}{2}} + Fx + \ldots,$$

giving

$$\log y = -\frac{2A}{x^{\frac{1}{2}}} + B\log x + 2Cx^{\frac{1}{2}} + Dx + \tfrac{2}{3}Ex^{\frac{3}{2}} + \tfrac{1}{2}Fx^2 + \ldots;$$

and thence passing to the value of y, but expanding the exponential of $2Cx^{\frac{1}{2}} + Dx + \ldots$, we have say the integral

$$y_1 = e^{-\frac{2A}{x^{\frac{1}{2}}}} x^B (1 + C'x^{\frac{1}{2}} + D'x + E'x^{\frac{3}{2}} + \ldots),$$

and then also, changing the sign of $\sqrt{\beta}$, the integral

$$y_2 = e^{\frac{2A}{x^{\frac{1}{2}}}} x^B (1 - C'x^{\frac{1}{2}} + D'x - E'x^{\frac{3}{2}} + \ldots).$$

Writing for a moment

$$Q = e^{-\frac{2A}{x^{\frac{1}{2}}}},$$
$$M = 1 + D'x + F'x^2 + \ldots,$$
$$N = C' + E'x + G'x^2 + \ldots,$$

the two integrals are $x^B Q(M + Nx^{\frac{1}{2}})$, and $x^B Q^{-1}(M - Nx^{\frac{1}{2}})$: these may be replaced by their sum and difference, viz. these are

$$y_1 + y_2 = Mx^B(Q + Q^{-1}) + Nx^{B+\frac{1}{2}}(Q - Q^{-1}),$$
$$y_1 - y_2 = Mx^B(Q - Q^{-1}) + Nx^{B+\frac{1}{2}}(Q + Q^{-1}),$$

where $Q + Q^{-1}$ is a rational function of x, and $Q - Q^{-1}$ is $= x^{\frac{1}{2}}$ multiplied by a rational function of x; hence $y_1 + y_2$ is $= x^B$ multiplied by a rational function of x, but $y_1 - y_2$ is $= x^{B+\frac{1}{2}}$ multiplied by a rational function of x.

Cambridge, 28 *April* 1886.

863.

NOTE ON THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.

[From *Crelle's Journal der Mathem.*, t. CI. (1887), pp. 209—213.]

THE theorem V. given by Fuchs in the memoir "Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten," *Crelle's Journal*, t. LXVIII. (1868), pp. 354—385 (see p. 374) for the purpose of deciding whether the integrals belonging to a group of roots of the "determinirenden Fundamentalgleichung" (or as I call it, the Indicial equation) do or do not involve logarithms, may I think be exhibited in a clearer form.

Starting from the differential equation

$$P(y), \quad = p_0 \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y, \quad = 0,$$

of the order m, then if X be any function of x not satisfying the differential equation, we can at once form a differential equation of the order $m+1$, satisfied by all the solutions of the differential equation, and having also the solution $y = X$; the required equation is in fact

$$\partial_x P(y) . P(X) - P(y) . \partial_x P(X) = 0.$$

This I call the augmented equation.

I recall that the equation $P(y) = 0$, considered by Fuchs, is an equation having for each singular point $x = a$, m regular integrals, viz. the coefficients $p_0, p_1, \ldots, p_m$ have the forms $q_0 (x-a)^m$, $q_1 (x-a)^{m-1}, \ldots, q_m$, where $q_0, q_1, \ldots, q_m$ are rational and integral functions of $x-a$, q_0 not vanishing for $x = a$, and the other functions $q_1, q_2, \ldots, q_m$ not in general vanishing for $x = a$. Writing $y = (x-a)^\theta$, we obtain

$$P(x-a)^\theta = I(\theta)(x-a)^\theta + \text{higher powers of } (x-a),$$

where $I(\theta)$, the coefficient of the lowest power of $(x-a)$, is a function of θ of the order m, which I call the indicial coefficient; and equating it to zero, we have $I(\theta) = 0$, the determinirende Fundamentalgleichung, or Indicial equation, being an equation of

the order m. If the roots of this equation are such that no two of them are equal or differ only by an integer number, then we have m particular integrals each of them of the form

$$y = (x-a)^r + \text{higher powers of } (x-a),$$

where r is any root of the indicial equation: but if we have in the indicial equation a group of λ roots $r_1, r_2, \ldots, r_\lambda$, such that the difference of each two of them is either zero or an integer, then the integrals which correspond to these roots involve or may involve logarithms; in particular, if any two of the roots are equal, the integrals for the group will involve logarithms.

Consider now the differential equation $P(y)=0$ in reference to the singular point $x=a$ as above, and writing $X=(x-a)^\epsilon f$ where ϵ is in the first instance arbitrary, and f is a rational and integral function of $x-a$ not vanishing for $x=a$, we form the augmented equation which, observing that we have in general $P(X)=(x-a)^\epsilon Q$, Q a rational and integral function of $x-a$ not vanishing for $x=a$, and dividing the whole equation by $(x-a)^{\epsilon-1}$, may be written

$$\partial_x P(y) \,.\, (x-a)\, Q - P(y)\, \{\epsilon Q + (x-a)\, \partial_x Q\} = 0,$$

an equation of the same form as the original equation (but of the order $m+1$ instead of m), and having an indicial equation

$$(\theta - \epsilon)\, I(\theta) = 0.$$

In fact, writing as before $y=(x-a)^\theta$, we have in $\partial_x P(y) \,.\, (x-a)\, Q$ the term of lowest order $\theta I(\theta)\, Q_0 (x-a)^\theta$ and in $P(y) \,.\, \epsilon Q$ the term of lowest order $\epsilon I(\theta)\, Q_0 (x-a)^\theta$, whereas in $P(y)(x-a)\partial_x Q$ the term of lowest order is $(x-a)^{\theta+1}$; the indicial equation is thus as just found.

If however ϵ be equal to a root of the indicial equation $I(\theta)=0$, then instead of $P(X)=(x-a)^\epsilon Q$, we have $P(X)=(x-a)^\mu Q$, where the index μ is $=\epsilon+$ a positive integer, and where the value of the difference $\mu-\epsilon$ may depend upon the determination of the function f in the expression $(x-a)^\epsilon f$. The indicial equation for the augmented equation is in this case $(\theta-\mu)\, I(\theta)=0$.

If the indicial equation $I(\theta)=0$ of the given differential equation has a group of roots $r_1, r_2, \ldots, r_\lambda$, the difference of any two of these roots being zero or an integer, then taking $\epsilon=$ any one of these roots, the augmented equation will have a group of roots $(\mu, r_1, r_2, \ldots, r_\lambda)$.

If any two of the roots $r_1, r_2, \ldots, r_\lambda$ are equal, the group of integrals $u_1, u_2, \ldots, u_\lambda$ will involve logarithms: the question only arises when these roots are unequal, and taking them to be so, the theorem v. is in effect as follows: "If by taking $\epsilon=$ some one of the roots $r_1, r_2, \ldots, r_\lambda$, and by a proper determination of the function f we can make μ to be $=$ one of the same roots $r_1, r_2, \ldots, r_\lambda$, then the group of integrals $u_1, u_2, \ldots, u_\lambda$ will involve logarithms; but if μ cannot be made $=$ one of the roots $r_1, r_2, \ldots, r_\lambda$, then the group of integrals will be free from logarithms."

As an example, I consider the equation

$$P(y) = (x^2 - x^4)\frac{d^2y}{dx^2} - 2x^3\frac{dy}{dx} - (n^2+n)\, y = 0.$$

This is Legendre's equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(n^2+n)y=0$, with $\frac{1}{x}$ substituted for x, so that, instead of a singular point $x=\infty$, there may be a singular point $x=0$. Attending to the singular point $x=0$, we have $P(x^\theta)=(\theta^2-\theta-n^2-n)x^\theta+$ higher powers, so that the indicial equation $I(\theta)=0$ is $\theta^2-\theta-n^2-n=0$, that is, $(\theta+n)(\theta-n-1)=0$, or we have the roots $-n$, $n+1$, which differ by an integer, and thus form a group, if n be $=$ an integer, or be $=$ an integer $-\frac{1}{2}$; to fix the ideas, say that the roots are $-p$, $p+1$ or else $-p+\frac{1}{2}$, $p+\frac{1}{2}$ where p is a positive integer.

Writing for greater convenience $x^\epsilon f=x^\epsilon+F$, where F is a sum of powers of x higher than ϵ, we find without difficulty

$$P(x^\epsilon f)=x^\epsilon\{(\epsilon+n)(\epsilon-n-1)-(\epsilon^2+\epsilon)x^2+(x^2-x^4)x^{-\epsilon}F''-(n^2+n)x^{-\epsilon}F\}$$

which, so long as ϵ remains arbitrary, is of the form $x^\epsilon Q$, $Q=(\epsilon+n)(\epsilon-n-1)+$ powers of x; if however ϵ be a root of the indicial equation, for instance, if $\epsilon=-n$, then the expression in brackets $\{\ \}$ contains at any rate the factor x, so that the form is $P(x^{-n}f)=x^\mu Q$, where μ is $=-n+1$ at least; we can however, by a proper determination of the function f, make μ acquire a larger value.

For instance, suppose $-n$, $n+1=-2$, 3; $\epsilon=-n=-2$, and assume

$$x^\epsilon f=x^{-2}+Bx^{-1}+Cx^0+Dx^1+Ex^2+Fx^3+Gx^4+\ldots$$

To calculate $P(x^\epsilon f)$, we have

	x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4	x^5 ...
	6	$2B$	$0C$	$0D$	$2E$	$6F$	$12G$	$20H$ ...
			-6	$-2B$	$-0C$	$-0D$	$-2E$	$-6F$
			$+4$	$+2B$	$-0C$	$-2D$	$-4E$	$-6F$
	-6	$-6B$	$-6C$	$-6D$	$-6E$	$-6F$	$-6G$	$-6H$.
$P(x^\epsilon f)=$	0	$-4B$	$-6C$	$-6D$	$-4E$	$0F$	$6G$	$14H$ ...
			-2			$-2D$	$-6E$	$-12F$ ...

Hence if B not $=0$, we have $\mu=-1$; if $B=0$, $-6C-2$ not $=0$, we have $\mu=0$; if $B=0$, $-6C-2=0$, D not $=0$, we have $\mu=1$; if $B=0$, $-6C-2=0$, $D=0$, but E not $=0$, we have $\mu=2$; if $B=0$, $-6C-2=0$, $D=0$, $E=0$, then the coefficient of x^3, $=0F-2D$, is $=0$, and we have not $\mu=3$, but $\mu=4$ at least, viz. μ will be $=4$, if $6G-6E=0$, that is, if $G=0$; but leaving F arbitrary, we can by giving proper values to the subsequent coefficients H, I, &c., make μ to be $=5$ or any larger integer value. The values of μ are thus $=-1$, 0, 1, 2, 4, 5, ..., and we see that the group $(\mu, -n, n+1)$, that is, $(\mu, -2, 3)$, does not in any case contain two equal indices. Starting from the value $\epsilon=3$, the value of μ is >3, and thus here also the group $(\mu, -2, 3)$ does not contain two equal indices.

The conclusion from the theorem thus is that the integrals u_1, u_2, belonging to the roots -2, 3, do not involve logarithms: and in precisely the same manner, it appears that the integrals, belonging to the two roots $-p$, $p+1$ (p any positive

integer), do not involve logarithms: this is right, for the integrals are, in fact, the Legendrian functions of the first and second kinds P_p and Q_p, with only $\frac{1}{x}$ written therein instead of x.

Similarly, if for instance $-n$, $n+1=-\frac{1}{2}$, $\frac{3}{2}$, then, if $\epsilon=-n=-\frac{1}{2}$, assuming

$$x^{\epsilon}f = x^{-\frac{1}{2}} + Bx^{\frac{1}{2}} + Cx^{\frac{3}{2}} + Dx^{\frac{5}{2}} + Ex^{\frac{7}{2}} + \ldots,$$

we have

$$\begin{array}{rrrrrr}
 & x^{-\frac{1}{2}} & x^{\frac{1}{2}} & x^{\frac{3}{2}} & x^{\frac{5}{2}} & x^{\frac{7}{2}} \ldots \\
\hline
 & \frac{3}{4} & -\frac{1}{4}B & \frac{3}{4}C & \frac{15}{4}D & \frac{35}{4}E \\
 & & & -\frac{3}{4} & +\frac{1}{4}B & -\frac{3}{4}C \\
 & & & +1 & -B & -3C \\
 & -\frac{3}{4} & -\frac{3}{4}B & -\frac{3}{4}C & -\frac{3}{4}D & -\frac{3}{4}E \\
\hline
P(x^{\epsilon}f) = & 0 & -B & 0C & 3D & 8E \ldots \\
 & & & +\frac{1}{4} & -\frac{3}{4}B & -\frac{15}{4}C \ldots
\end{array}$$

We have here if B not $=0$, $\mu=\frac{1}{2}$; but if $B=0$, then we cannot in any way make the coefficient of $x^{\frac{3}{2}}$ to vanish, and consequently $\mu=\frac{3}{2}$. With this last value of μ, the group $(\mu, -n, n+1)$, that is, $(\mu, -\frac{1}{2}, \frac{3}{2})$, becomes $(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2})$ which contains two equal roots, and the conclusion from the theorem thus is that the integrals u_1, u_2, corresponding to the roots $-\frac{1}{2}$, $\frac{3}{2}$, involve logarithmic values. And similarly in general the integrals u_1, u_2, corresponding to the roots $-p+\frac{1}{2}$, $p+\frac{1}{2}$ (p any positive integer), involve logarithmic values: this also is right.

The examples exhibit the true character of the theorem, and show I think that it is a less remarkable one than would at first sight appear: in fact, in working them out, we really ascertain by an actual substitution whether the differential equation can be satisfied by series of powers only, without logarithms. Thus for $n=2$ as above, it appears that the equation is satisfied by the series

$$y = x^{-2} + Bx^{-1} + Cx^{0} + Dx^{1} + Ex^{2} + Fx^{3} + Gx^{4} + Hx^{5} + \ldots,$$

where

$$B=0,\ C=-\tfrac{1}{3},\ D=0,\ E=0,\ F=F,\ G=0,\ H=-\tfrac{6}{7}F, \ldots,$$

that is, by

$$y = x^{-2} + \tfrac{1}{3} + F(x^{3} - \tfrac{6}{7}x^{5} + \ldots),$$

in other words, that we have the two particular integrals $y=x^{-2}+\frac{1}{3}$, and $y=x^{3}-\frac{6}{7}x+\ldots$, belonging to the two roots -2, 3 respectively.

Similarly, when $n=\frac{1}{2}$, we cannot satisfy the equation by a series

$$y = x^{-\frac{1}{2}} + Bx^{\frac{1}{2}} + Cx^{\frac{3}{2}} + Dx^{\frac{5}{2}} + \ldots;$$

for in order to satisfy the equation, we must have $B=0$, $C=\infty$; there is thus no series of powers $y=x^{-\frac{1}{2}}+Cx^{\frac{3}{2}}+\ldots$, corresponding to the root $-\frac{1}{2}$: but there is a series $y=x^{\frac{3}{2}}+kx^{\frac{7}{2}}+\ldots$ corresponding to the root $\frac{3}{2}$; and thus the integrals u_1, u_2, corresponding to these roots $-\frac{1}{2}$, $\frac{3}{2}$, involve logarithms.

Cambridge, 23 *March* 1887.

864.

ON RUDIO'S INVERSE CENTRO-SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXII. (1887), pp. 156—158.]

DR F. RUDIO, in an inaugural dissertation "Ueber diejenigen Flächen deren Krümmungsmittelpunktsflächen confokale Flächen zweiten Grades sind," Berlin, 1880, and *Crelle's Journal*, t. XCV., p. 240, has determined the surfaces having for their centro-surface (i.e., the locus of centres of curvature) the aggregate of the confocal quadric surfaces

$$\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}+\frac{z^2}{c-\lambda}=1,$$

$$\frac{x^2}{a-\mu}+\frac{y^2}{b-\mu}+\frac{z^2}{c-\mu}=1,$$

or, what is the same thing, the surfaces orthotomic to the common tangents of these two surfaces. He obtains, as the final result of an elegant analytical investigation, the following formulæ:

$$x=\sqrt{(a-\lambda)}\sqrt{\left(\frac{a-u\,.\,a-v}{a-b\,.\,a-c}\right)},$$

$$y=\sqrt{(b-\lambda)}\sqrt{\left(\frac{b-u\,.\,b-v}{b-c\,.\,b-a}\right)},$$

$$z=\sqrt{(c-\lambda)}\sqrt{\left(\frac{c-u\,.\,c-v}{c-a\,.\,c-b}\right)},$$

$$U=\sqrt{\left(\frac{a-u\,.\,b-u\,.\,c-u}{\lambda-u\,.\,\mu-u}\right)},$$

$$V=\sqrt{\left(\frac{a-v\,.\,b-v\,.\,c-v}{\lambda-v\,.\,\mu-v}\right)},$$

$$\xi\frac{U-V}{x}=1-\frac{b-\lambda\,.\,c-\lambda}{\lambda-u\,.\,\lambda-v}-\frac{a-\mu}{a-u\,.\,a-v}\,UV,$$

$$\eta\frac{U-V}{y}=1-\frac{c-\lambda\,.\,a-\lambda}{\lambda-u\,.\,\lambda-v}-\frac{b-\mu}{b-u\,.\,b-v}\,UV,$$

$$\zeta\frac{U-V}{z}=1-\frac{a-\lambda\,.\,b-\lambda}{\lambda-u\,.\,\lambda-v}-\frac{c-\mu}{c-u\,.\,c-v}\,UV,$$

(values which are such that $\xi^2+\eta^2+\zeta^2=1$),

$$\rho=\tfrac{1}{2}\left(\int\frac{du}{U}+\int\frac{dv}{V}\right)+C.$$

And then

$$x'=x+\rho\xi,\quad y'=y+\rho\eta,\quad z'=z+\rho\zeta,$$

viz. the equations give x, y, z, U, V, ξ, η, ζ, each of them as a function of two independent parameters u, v; ρ is a function of u, v and of the arbitrary constant C; hence, giving to C any assumed value, we have x', y', z' each of them a function of the two arbitrary parameters u, v; that is, x', y', z' are the coordinates of a point on a surface, one of a system of parallel surfaces (corresponding to the different values of C) which are the surfaces in question.

Observe that u, v are the elliptic coordinates of the point (x, y, z) on the first of the two confocal surfaces, and that ξ, η, ζ are the cosine-inclinations of one of the tangents from this point to the other confocal surface; so that, if ρ were left arbitrary, the equations $x'=x+\rho\xi$, $y'=y+\rho\eta$, $z'=z+\rho\zeta$ would be the equations of a common tangent of the two confocal surfaces; but ρ, as determined, is the distance of the point x, y, z from the point x', y', z' on the required surface. The expression for ρ involves hyper-elliptic integrals of the first species, which are the same as those which present themselves in the determination of the geodesic lines upon either of the two confocal surfaces.

I have, in quoting these remarkable results, written for greater simplicity a, b, c instead of the author's a^2, b^2, c^2.

Cambridge, Dec. 20, 1886.

865.

ON MULTIPLE ALGEBRA.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXII. (1887), pp. 270—308.]

1. I REPRODUCE a passage from my Presidential Address, British Association, Southport, 1883.

"Outside of ordinary mathematics we have some theories which must be referred to: algebraical, geometrical, logical. It is, as in many other cases, difficult to draw the line: we do in ordinary mathematics use symbols not denoting quantities, which we nevertheless combine in the way of addition and multiplication, $a+b$ and ab, and which may be such as not to obey the commutative law $ab=ba$; in particular, this is or may be so in regard to symbols of operation; and it could hardly be said that any development whatever of the theory of such symbols of operation did not belong to ordinary algebra. But I do separate from ordinary algebra the system of multiple algebra or linear associative algebra developed in the valuable memoir by the late Benjamin Peirce, "Linear Associative Algebra" (1870, reprinted 1881 in the *American Journal of Mathematics*, vol. IV., with notes and addenda by his son, C. S. Peirce): we here consider symbols A, B, &c., which are linear functions of a determinate number of letters or units i, j, k, l, &c., with coefficients which are ordinary analytical magnitudes real or imaginary, viz. the coefficients are in general of the form $x+iy$, where i is the before-mentioned imaginary, or $\sqrt{(-1)}$ of ordinary analysis. The letters i, j, k, &c., are such that every binary combination i^2, ij, ji, &c., (ij not in general $=ji$), is equal to a linear function of the letters, but under the restriction of satisfying the associative law; viz. for each combination of three letters $ij.k$ is $=i.jk$, so that there is a determinate and unique product of three or more letters; or, what is the same thing, the laws of combination of the units $i, j, k, \ldots$ are defined by a multiplication table giving the values of i^2, ij, ji, &c.; the original units may be replaced by linear functions of these units, so as to give rise for the units finally adopted to a multiplication table of the most simple form; and it is very remarkable how

frequently in these simplified forms we have nilpotent or idempotent symbols ($i^2=0$ or $i^2=i$ as the case may be), and symbols i, j, such that $ij=ji=0$; and consequently how simple are the forms of the multiplication tables which define the several systems respectively.

"I have spoken of this multiple algebra before referring to various geometrical theories of earlier date, because I consider it as the general analytical basis, and the true basis, of these theories. I do not realise to myself directly the notions of the addition or multiplication of two lines, areas, rotations, or other geometrical, kinematical, or mechanical entities; and I would formulate a general theory as follows: consider any such entity as determined by the proper number of parameters a, b, c, ... (for instance, in the case of a finite line given in magnitude and position, these might be the length, the coordinates of one end, and the direction-cosines of the line considered as drawn from this end); and represent it by or connect it with the linear function $ai+bj+ck+$ &c., formed with these parameters as coefficients and with a given set of units i, j, k, &c. Conversely, any such linear function represents an entity of the kind in question. Two given entities are represented by two linear functions; the sum of these is a like function representing an entity of the same kind, which may be regarded as the sum of the two entities; and the product of them (taken in a determined order, when the order is material) is an entity of the same kind, which may be regarded as the product (in the same order) of the two entities. We thus establish by definition the notion of the sum of the two entities, and that of the product (in a determinate order, when the order is material) of the two entities. The value of the theory in regard to any kind of entity would of course depend on the choice of a system of units i, j, k, ..., with such laws of combination as would give a geometrical or kinematical or mechanical significance to the notions of the sum and product as thus defined.

"Among the geometrical theories referred to, we have a theory (that of Argand, Warren, and Peacock) of imaginaries in plane geometry; Sir W. R. Hamilton's very valuable and important theory of quaternions; the theories developed in Grassmann's *Ausdehnungslehre*, 1844 and 1862; Clifford's theory of biquaternions, and recent extensions of Grassmann's theory to non-Euclidian space by Mr Homersham Cox. These different theories have of course been developed, not in anywise from the point of view from which I have been considering them, but from the points of view of their several authors respectively."

2. The present paper is in a great measure the development of the views contained in the foregoing extract; but, instead of establishing *ab initio* a linear function $ai+bj+ck+\ldots$ as above, I deduce this, as will be seen from the notion of addition.

3. If x, y, ... denote ordinary (real or imaginary) analytical magnitudes, which (as such) are susceptible of addition and multiplication, and for each of these operations are commutative and associative, then we may consider a multiple symbol $(x, y, \ldots)$ involving any given number of letters, to fix the ideas say (x, y), susceptible of addition and multiplication according to determinate laws

$$(x, y)+(x', y')=(P, Q), \quad (x, y)(x', y')=(X, Y),$$

where P, Q, X, Y are given functions of x, y, x', y'. For greater simplicity the law of addition is taken to be

$$(x,\ y)+(x',\ y')=(x+x',\ y+y'),$$

so that as regards addition the multiple symbols are commutative and associative. But this is or is not the case for multiplication, according to the form of the given functions X, Y; for instance, if

$$(x,\ y)(x',\ y')=(xx'-yy',\ xy'+yx'),$$

then in regard to multiplication the symbols will be commutative and associative. But if

$$(x,\ y,\ z)(x',\ y',\ z')=(yz'-y'z,\ zx'-z'x,\ xy'-x'y),$$

then the symbols will be associative, but not commutative.

4. I remark here that we are in general concerned with symbols of a given multiplicity, double symbols (x, y), triple symbols (x, y, z), n-tuple symbols $(x_1, x_2, \ldots, x_n)$, as the case may be, and that as well the product as the sum is a symbol *ejusdem generis*, and consequently of the same multiplicity, with the component symbols; this is to be assumed throughout in the absence of an express statement to the contrary. It is, moreover, proper to narrow the notion of multiplication by restricting it to the case where the terms $(X, Y, \ldots)$ of the product are linear functions of the terms $(x, y, \ldots)$ and $(x', y', \ldots)$ of the component symbols respectively; any other form $(X, Y, \ldots)$ is better designated *not as a product*, but as a combination (or by some other name) of the component symbols $(x, y, \ldots)$ and $(x', y', \ldots)$.

5. I assume, moreover, that, if m be any ordinary analytical magnitude, this may be multiplied into a multiple symbol $(x, y, \ldots)$, according to the law

$$m(x,\ y, \ldots)=(mx,\ my, \ldots).$$

6. As a consequence of this last assumption and of the assumed law of addition, we have for instance

$$\begin{aligned}(x,\ y,\ z)&=(x,\ 0,\ 0)+(0,\ y,\ 0)+(0,\ 0,\ z)\\&=x(1,\ 0,\ 0)+y(0,\ 1,\ 0)+z(0,\ 0,\ 1);\end{aligned}$$

that is, using single letters i, j, k for the multiple symbols $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, we have

$$(x,\ y,\ z)=xi+yj+zk,$$

where the letters i, j, k, thus standing for determinate multiple symbols, may be termed "extraordinaries." Each extraordinary may be multiplied into any ordinary symbol x, and is commutative therewith, $xi=ix$; moreover, each extraordinary may be multiplied into itself, or into another extraordinary, according to laws which are, in fact, determined by means of the assumed law of multiplication of the original multiple symbols; and, conversely, the law of multiplication of the extraordinaries determines that of the original multiple symbols; thus, if

$$(x,\ y)(x',\ y')=(xx'-yy',\ xy'+yx'),$$

then

$$(ix+jy)(ix'+jy')=i(xx'-yy')+j(xy'+yx'),$$

and also

$$=i^2xx'+ijxy'+jixx'+j^2yy',$$

which expressions will agree together if, and only if, $i^2=i$, $ij=j$, $ji=j$, $j^2=-i$, or, as these equations may be written,

$$\begin{array}{c|c|c|}
 & i & j \\
\hline
i & i & j \\
\hline
j & j & -i \\
\hline
\end{array}.$$

And so in general we have a multiplication table giving each square or product as a homogeneous linear function of all or any of the extraordinaries.

7. And, conversely, from this multiplication table of the extraordinaries i, j, we have

$$(ix+jy)(ix'+jy')=i(xx'-yy')+j(xy'+yx'),$$

that is,

$$(x,\ y)(x',\ y')=(xx'-yy',\ xy'+yx')$$

the originally assumed formula of multiplication.

In the example just given, we have $i^2=i$, $ij=ji=j$, viz. the symbol i comports itself like unity, and may be put $=1$; we have $(x,\ y)=x+jy$, with the multiplication table

$$\begin{array}{c|c|c|}
 & 1 & j \\
\hline
1 & 1 & j \\
\hline
j & j & -1 \\
\hline
\end{array},$$

or simply the equation $j^2=-1$; and then

$$(x+jy)(x'+jy')=xx'-yy'+j(xy'+yx');$$

and it is convenient to regard 1 as an extraordinary, and speak of the system of extraordinaries 1, j.

8. The separate terms $x, y, \ldots$, whatever be their number, may always without loss of generality be arranged in a line; but it may be convenient to arrange them in a different form, for instance, in that of a square; we may have for instance symbols $\begin{vmatrix} x, & y \\ z, & w \end{vmatrix}$ with the laws of combination

$$\begin{vmatrix} x, & y \\ z, & w \end{vmatrix}+\begin{vmatrix} x', & y' \\ z', & w' \end{vmatrix}=\begin{vmatrix} x+x', & y+y' \\ z+z', & w+w' \end{vmatrix},$$

$$\begin{vmatrix} x, & y \\ z, & w \end{vmatrix}\cdot\begin{vmatrix} x', & y' \\ z', & w' \end{vmatrix}=\begin{vmatrix} xx'+yz', & xy'+yw' \\ zx'+wz', & zy'+ww' \end{vmatrix},$$

where observe that the multiplication is not commutative.

9. A multiple symbol may be connected with a geometrical or physical entity of any kind: viz. any such entity, depending on a number of parameters susceptible of analytical magnitude, to fix the ideas say on two parameters (x, y), may be connected with the multiple symbol (x, y). We cannot in general directly conceive the addition or multiplication of such entities, but the assumed laws of combination of the multiple symbols in effect serve as definitions of the operations. Thus

$$(x, y)+(x', y')=(x+x', y+y'); \quad (x, y).(x', y')=(xx'-yy', xy'+yx'):$$

or the sum of the entities whose parameters are x, y and x', y' is by definition the entity whose parameters are $x+x'$, $y+y'$; and the product of the same entities is by definition the entity whose parameters are $xx'-yy'$, $xy'+yx'$. The entities are thus, as regards addition, commutative and associative; but, as regards multiplication they are or are not commutative, or associative, according to the assumed law of multiplication of the multiple symbols.

10. If, as above, the multiple symbol be represented by means of extraordinaries, $(x, y)=ix+jy$, then the extraordinaries i, j, *quà* multiple symbols (1, 0), (0, 1), are themselves special entities of the kind in question, their laws of combination (as such special entities) being included in the general laws for the combination of such entities; or, if we please, these general laws being derived from the assumed laws for the special entities. Thus, if $ix+jy$ represent the point whose coordinates are x, y, then i represents the point whose coordinates are (1, 0), and j the point whose coordinates are (0, 1).

11. It has been assumed that the multiple symbols combined together, and their sum and product, are symbols of one and the same multiplicity, say the types of combination are

$$(x, y, z)+(x', y', z')=(x+x', y+y', z+z'); \quad (x, y, z)(x', y', z')=(X, Y, Z);$$

if this were not so, if for instance we had

$$(x, y, z)(x', y', z')=\text{a double symbol } (X, Y),$$

then this given equation for the multiplication of two triple symbols would not serve as a definition for the multiplication of double symbols, or of a double and a triple symbol, and new definitions would be required. We might, however, have symbols of indefinite multiplicity $(x, y, z, w, \ldots)$, including within them all finite multiplicities, viz. (x, y) meaning $(x, y, 0, 0, \ldots)$, and so in other cases, these being combined into symbols of like indefinite multiplicity

$$(x, y, z, \ldots)+(x', y', z', \ldots)=(x+x', y+y', z+z', \ldots);$$
$$(x, y, z, \ldots)(x', y', z', \ldots)=(X, Y, Z, \ldots);$$

for instance, the law of multiplication might be

$$(x, y, z, \ldots)(x', y', z', \ldots)=(xx', xy'+yx', xz'+yy'+zx', \ldots),$$

and we could hereby combine symbols of any finite multiplicities (the same or different) whatever

12. Another peculiarity may be noticed: suppose that, in general,

$$(x, y, z, w)+(x', y', z', w')=(x+x', y+y', z+z', w+w');$$
$$(x, y, z, w)(x', y', z', w')=(X, Y, Z, W);$$

then, as a particular case hereof, we have the addition-equation

$$(x, y, 0, 0)+(x', y', 0, 0)=(x+x', y+y', 0, 0)$$

and it may very well be that for the same two symbols the multiplication-equation may take the form

$$(x, y, 0, 0)(x', y', 0, 0)=(0, 0, Z, W),$$

(Z, W each of them a function of x, y, x', y'); viz. here, in a different point of view, the component symbols $(x, y, 0, 0)$ and $(x', y', 0, 0)$, are double systems of a certain kind $\{x, y\}$, $\{x', y'\}$, and the product is a double system of a different kind $[Z, W]$, or it might have been $(0, Y, Z, W)$, = a triple system $[Y, Z, W]$. Of course, all peculiarity disappears when we revert from the particular symbols $(x, y, 0, 0)$ to the general symbols (x, y, z, w), from which we may regard them as derived.

13. The peculiarity just referred to presents itself naturally when we regard the symbols as representing geometrical or physical entities; it may very well be that the product of two entities of a certain kind is taken to be an entity of a different kind (for instance, the product of two points to be a line), and that the analytical theory of the multiple symbol is constructed in order to such a relation; and it may further be that only the symbol $(x, y, 0, 0)$ or $\{x, y\}$ is interpreted with reference to an entity of the one kind, and only the symbol $(0, 0, Z, W)$, or $[Z, W]$ is interpreted by reference to an entity of the other kind, without any interpretation at all being given to the general symbol (x, y, z, w); thus the two forms $(x, y, 0, 0)$, and $(0, 0, Z, W)$, naturally present themselves as double symbols $\{x, y\}$ and $[Z, W]$ of different kinds.

14. In further illustration, suppose the symbols represented as linear functions of extraordinaries,

$$(x, y, z, w)=x\epsilon_1+y\epsilon_2+z\eta_1+w\eta_2,$$

with a multiplication table for the four extraordinaries ϵ_1, ϵ_2, η_1, η_2. We then have $(x, y, 0, 0)$, $=\{x, y\}=x\epsilon_1+y\epsilon_2$; the ϵ_1, ϵ_2 have no proper multiplication table of their own, but the squares and products ϵ_1^2, $\epsilon_1\epsilon_2$, $\epsilon_2\epsilon_1$, ϵ_2^2 are each given as a linear function of η_1, η_2; hence we have

$$(x\epsilon_1+y\epsilon_2)(x'\epsilon_1+y'\epsilon_2)=Z\eta_1+W\eta_2,$$

which is a symbol $(0, 0, Z, W)$, or $[Z, W]$, of a different kind; it may, in completion of the theory, be assumed that in like manner η_1, η_2 have no proper multiplication table of their own, but that the squares and products η_1^2, $\eta_1\eta_2$, $\eta_2\eta_1$, η_2^2 are each given as a linear function of ϵ_1, ϵ_2, leading to a relation

$$(z\eta_1+w\eta_2)(z'\eta_1+w'\eta_2)=X\epsilon_1+Y\epsilon_2,$$

which is a symbol $(X, Y, 0, 0)$ or $\{X, Y\}$ of the first kind; it may further happen that each of the several products $\epsilon\eta$, $\eta\epsilon$ is $=0$, in which case

$$(x\epsilon_1 + y\epsilon_2)(z\eta_1 + w\eta_2) = 0, \quad (z\eta_1 + w\eta_2)(x\epsilon_1 + y\epsilon_2) = 0.$$

For the complete theory, we require the multiplication table of the four extraordinaries ϵ_1, ϵ_2, η_1, η_2. The geometrical interpretation may be that $x\epsilon_1 + y\epsilon_2$ represents a point, $z\eta_1 + w\eta_2$ a line (with or without any geometrical interpretation of a symbol $x\epsilon_1 + y\epsilon_2 + z\eta_1 + w\eta_2$), the product of two points is a line, the product of two lines is a point; that of a line and point is $=0$, (see *post* Grassmann, where however the system is a somewhat different one).

15. I do not propose in the present paper to consider the subject of multiple algebra from an analytical point of view; and I will make only a few remarks and give some references. The general theory of associative linear forms is treated in a very satisfactory manner in Peirce's Memoir (1870) above referred to, only it is assumed throughout *ab initio* that the forms are associative. In a Note "On Associative Imaginaries," *Johns Hopkins Circulars*, No. 15 (1882), [822], and more fully in the paper "On Double Algebra," *Proc. Lond. Math. Soc.*, t. XV. (1884), pp. 185—197, [814], starting from the assumed equations $x^2 = ax + by$, $xy = cx + dy$, $yx = ex + fy$, $y^2 = gx + hy$, between the extraordinaries x, y, I considered under what conditions the algebra of these symbols was in fact associative. And in a paper "On the 8-square Imaginaries," *American Journal of Mathematics*, t. IV. (1881), pp. 293—296, [773], I showed that the extraordinaries 0, 1, 2, 3, 4, 5, 6, 7, connected with Euler's theorem of the 8 squares, are of necessity *non-associative*. I do not know that anything else has been done in regard to non-associative algebras: and it thus appears that the question has hardly been discussed at all. Matrices are associative: I shall have again to refer to them.

16. The object of the present paper is to discuss in detail, from the point of view explained in the extract from my British Association Address, the different theories in regard to geometrical or other entities. I do this under various headings.

The $i = \sqrt{(-1)}$ of Analysis and Analytical Geometry. Art. Nos. 17—26.

17. We may, of course, consider the i as an extraordinary. We have a double symbol (x, y), combining according to the laws

$$(x, y) + (x', y') = (x + x', y + y'),$$
$$(x, y)(x', y') = (xx' - yy', xy' + yx');$$

and then, introducing the extraordinaries k, i, and writing

$$(x, y) = kx + iy,$$

we have

$$(kx + iy)(kx' + iy') = k^2xx' + kixy' + ikyx' + i^2yy';$$

whence

$$k^2 = k, \quad ki = ik = i, \quad i^2 = -k,$$

or the multiplication table is

	k	i
k	k	i
i	i	$-k$

But the conditions are satisfied by, and we accordingly assume, $k=1$; and there then remains only the condition $i^2=-1$. Compare herewith Sir W. R. Hamilton, "Theory of Conjugate Functions, or Algebraical Couples; with a Preliminary and Elementary Essay on Algebra as a Science of pure time" (*Trans. R. I. Acad.*, t. XVII., 1833—35). I refer to this paper in my Address, and remark upon it that I cannot appreciate the manner in which the author connects, with the notion of time, his algebraical couple or imaginary magnitude $a+bi$, $=a+b\sqrt{(-1)}$ as written in the memoir.

18. But we have, in fact, passed out of this view, and have come to regard $a+bi$ as an ordinary analytical magnitude; viz. in every case an ordinary symbol represents or may represent such a magnitude, and the magnitude (and as a particular case thereof, the symbol i) is commutable with the extraordinaries of any system of multiple algebra. And similarly in analytical geometry, without seeking for any real representation, we deal with imaginary points, lines, &c., that is, with points, lines, &c., depending on parameters of the foregoing form $a+bi$.

$\sqrt{(-1)}$ *denotes Perpendicularity.* Art. Nos. 19—26.

19. I give a list of the earlier works and memoirs, marking with an asterisk those which I have not seen.

Wallis. *De Algebrâ Tractatus*, 1685 Anglice editus; 1693. Chapters 66, 67, 68, and 69 have respectively the titles: De quadratis negativis eorumque radicibus dictis Imaginariis;—Eorundem exemplificatio in Geometriâ;—Effectiones geometricæ his accommodatæ; Aliæ quæ huc spectant constructiones geometricæ. I quote a single sentence from Chap. 67: "Prout igitur cum æquationis quadraticæ radix prodit negativa, dicendum verbi gratiâ punctum B pro eo statu haberi non posse ut supponitur in expositâ AC prorsum; posse tamen retrorsum ab A in eâdem rectâ—hic vero (de radice quadrati negativi) dicendum, non haberi quidem posse punctum B ut erat suppositum in AC rectâ, vel ante vel retro: posse tamen (in eodem plano) suprà rectam illam": viz. the distance represented by the square root of a negative quantity cannot be measured in the line, forwards or backwards; but can be measured (in the same plane) above the line—or, as appears elsewhere, at right angles to the line, either in the plane, or in a plane at right angles thereto.

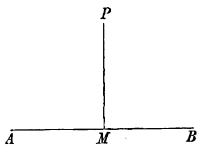
Forcenex. "Réflexions sur les quantités imaginaires," *Mel. de Turin*, t. I. (1759), pp. 113—146. The author, after stating that there was no geometrical construction

for imaginary quantities, proceeds "Cependant, pour conserver une certaine analogie avec les quantités négatives, un auteur dont nous avons un cours d'algèbre d'ailleurs fort estimable a prétendu les devoir prendre sur une ligne perpendiculaire à celle où l'on les avait supposées, si par exemple &c.," the example which he considers being as follows: On a given line AB to find a point P, such that the product of the distances AP, BP is $=\frac{1}{2}(AB)^2$. Taking $AB=2a$ and $AP=x$, then $x(2a-x)=2a^2$, that is, $(x-a)^2=-a^2$, or, $x=a+a\sqrt{(-1)}$, an imaginary value; the condition is, however, satisfied by a real point P *off* the line AB

$$(AM=MP=\tfrac{1}{2}AB=a),\ \text{(fig. 1)},$$

and hence the author in question infers that the imaginary value $a\sqrt{(-1)}$ is represented by the line MP at right angles to AB. Forcenex objects to this: if the

Fig. 1.



point P may be taken off the axis of x, then the condition $AP.BP=2a^2$ gives for the locus of P a certain curve; and there is no reason why the foregoing solution $x=a+a\sqrt{(-1)}$ should represent one point rather than another of this curve. As to this see *post* (Peacock).

***Truel,** Henri Dominique, 1786.

***Suremain-de-Missery.** *Théorie purement algébrique des quantités imaginaires,* Paris, 1801; it is referred to in the letter by Servois, *infra.*

20. **Buée.** "Mémoire sur les quantités imaginaires," *Phil. Trans.*, 1806, pp. 23—88.

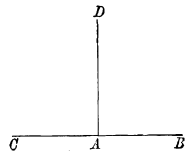
I give an extract: "Du signe $\sqrt{(-1)}$. Je mets en titre, du signe $\sqrt{(-1)}$ et non de la quantité imaginaire ou de l'unité imaginaire, parce que $\sqrt{(-1)}$ est un signe particulier joint à l'unité réelle, non une quantité particulière: c'est un nouvel adjectif joint au substantif ordinaire et non un nouveau substantif.

"Mais que veut dire ce signe? il n'indique ni une addition ni une soustraction, ni une suppression ni une opposition par rapport aux signes + et −: une quantité accompagnée par $\sqrt{(-1)}$ n'est ni additive ni subtractive, ni égale à zéro. La qualité marquée par $\sqrt{(-1)}$ n'est opposée ni à celle qu'indique +, ni à celle marquée par −; qu'est-elle donc?

"Pour la découvrir supposons trois lignes égales AB, AC, AD (fig. 2) qui partent toutes du point A. Si je désigne la ligne AB par $+1$, la ligne AC sera -1, et la ligne AD qui est une moyenne proportionnelle entre $+1$ et -1, sera nécessairement $\sqrt{(-1^2)}$ ou plus simplement $\sqrt{(-1)}$. Ainsi $\sqrt{(-1)}$ est le signe de la PERPENDICULARITÉ: donc la propriété caractéristique est *que tous les points de la perpendiculaire sont*

également éloignés de points placés à égale distance de part et d'autre de son pied. Le signe $\sqrt{(-1)}$ signifie tout cela et il est le seul qui l'exprime.

Fig. 2.



"*Ce signe mis devant a (a signifiant une ligne ou une surface) veut donc dire qu'il faut donner à a une situation perpendiculaire à celle qu'on lui donnerait si l'on avait simplement a ou* $-a$."

And he, in fact, gives (Prob. VI.) the problem before referred to, and finds no difficulty in assigning to P a position off the line AB. I notice also that Buée considers a curve as having branches in a perpendicular plane; thus for the circle $x^2+y^2=a^2$ in the plane of the paper, putting $-x^2$ in place of x^2, there are branches, or, as he calls it, "une appendice," $y^2=a^2+x^2$ (or say $y^2=a^2+z^2$), being a rectangular hyperbola, in a plane at right angles to the plane of the paper.

21. Other writings are:

Argand. *Essai sur une manière de représenter les quantités imaginaires.* Paris, 1806. Partly reproduced in Argand's memoir under the same title presently referred to.

Francais, J. F. "Nouveaux principes de géométrie de position et interprétation géométrique des symboles imaginaires." *Gergonne Annales*, t. IV. 1813—14, pp. 61—72.

Argand. "Essai sur une manière, &c." *Ibid.* pp. 133—148.

Francais, J. F. "Lettre au Rédacteur." *Ibid.* pp. 222—227.

Servois. "Lettre au Rédacteur." *Ibid.* pp. 228—235.

This last is *against* the theory, with a running commentary of notes *in favour* by Gergonne.

Francais, J. F. "Autre lettre." *Ibid.* pp. 364—366.

Lacroix. "Note transmise à M. Vecten"; it calls attention to Buée's memoir. *Ibid.* p. 367.

***Mourey.** *Vraie théorie des quantités négatives et des quantités prétendues imaginaires*, 1828, reprinted 1861.

22. **Warren, J.** *A treatise on the geometrical representation of the square roots of negative quantities.* 8vo. Cambridge, 1828, pp. 1—154.

Peacock, Preface to *Algebra*, 1830, speaks of Warren's work as "distinguished for great originality and for extreme boldness in the use of definitions." There is no Preface or Introduction; the first Chapter is entitled Definitions, Addition, Subtraction, Proportion, Multiplication, Division, Fractions, and Raising of Powers. I quote certain articles as follows:

(1) All straight lines drawn in a given direction from a given point are represented in *length* and *direction* by algebraic quantities; and in the following treatise whenever the word *quantity* is used it is to be understood as signifying *a line*.

(2) DEF. The given point from which the straight lines are measured is called the *origin*.

(3) DEF. The *sum* of two quantities is the diagonal of the parallelogram whose sides are the two quantities.

(12) DEF. The first of four quantities is said to have to the second the same ratio which the third has to the fourth: when the first has in *length* to the second the same ratio which the third has in *length* to the fourth, according to Euclid's definition; and also the angle at which the fourth is inclined to the third is equal to the angle at which the second is inclined to the first, and is measured in the same direction.

(17) DEF. Unity is a positive quantity arbitrarily assumed, from a comparison of which the values of other quantities are obtained.

(18) DEF. If there be three quantities such that unity is to the first as the second is to the third, then the third is called the *product* which arises from the *multiplication* of the second by the first.

23. The signification of these definitions may be thus expressed. Let the line, drawn from an origin O to the point whose rectangular coordinates are x, y, be called the line $x+iy$; of course the line, length unity in the direction of the axis of x, is the line 1. Then, observing that, putting x, $y=r\cos\theta$, $r\sin\theta$, and x', $y'=r'\cos\theta'$, $r'\sin\theta'$, we have $(x+iy)(x'+iy')=rr'\{\cos(\theta+\theta')+i\sin(\theta+\theta')\}$, the two definitions are:

The *sum* of the lines $x+iy$ and $x'+iy'$ is the line

$$x+x'+i(y+y');$$

and the *product* of the same lines is the line

$$(x+iy)(x'+iy');$$

viz. Warren in effect establishes by definition the notions of the sum and product of two lines, and that by connecting the line with a linear symbol $x+iy$.

It may be right to remark that the general notion of a proportion, used as above to define multiplication, is geometrically the more simple of the two, and that the condition for the proportion $OA : OB = OC : OD$ may be thus expressed: the lengths of the lines are proportional, and the inclination OA to OB is equal to the inclination OC to OD, these inclinations being in the same sense.

24. **Peacock, G.** *A Treatise on Algebra.* 8vo. Cambridge, 1830 (one Vol., pp. 5—38 and 1—685).

—— "Report on the Recent Progress and Present State of Certain Branches of Analysis." *B. A. Report*, (1833), pp. 185—352.

—— *A Treatise on Algebra.* Vol. I. Arithmetical Algebra; Vol. II. On Symbolical Algebra and its Application to the Geometry of Position. 8vo. Cambridge, 1842 and 1845.

The statement of the theory is substantially identical with that of the earlier writers, thus *Algebra*, (1830), p. 362, No. 439, is:—"We have shown on a former occasion that, if a designated a line in one direction, $-a$ must designate a line in the direction opposite to it, or making an angle with the former equal to two right angles or 180 degrees; but inasmuch as $a\{\sqrt{(-1)}\}^2 = -a$, it follows that the *double affectation* of the line a with the sign $\sqrt{(-1)}$ produces a result represented by $-a$, and is consequently equivalent to its transfer through 180°: it follows therefore that its single affectation with the sign $\sqrt{(-1)}$ is equivalent to its transfer through half that angle, or through 90°; or, in other words, if a represents a line, $a\sqrt{(-1)}$ will represent a line at right angles to it." But further, pp. 666, 667, the Author considers the before-mentioned problem (for convenience, I write $2a$ instead of a) in the form "to divide a line $2a$ into two such parts that the rectangle contained by them may be $=b^2$"; he takes x, y for the two parts, and in the case where $b^2 > a^2$, finds the two parts $x = a \pm \sqrt{(-1)}\sqrt{(b^2-a^2)}$, $y = a \mp \sqrt{(-1)}\sqrt{(b^2-a^2)}$, which he interprets as meaning the two parts are AC, CB, where C is a point off the line AB.

25. It seems to me that, except in Warren, the defect in the exposition of the theory is that it is not made clear, that the theory is in fact a theory of lines in a plane through a given point, with assumed definitions for the addition and for the multiplication of lines; thus in Peacock's problem "to divide a line a into two such parts that the product of them is $=b^2$," the real meaning is "given a line OA, $=a$, and a line OB, $=b^2$ (a and b^2 real and positive, so that each of these lines is in the given direction Ox), to find lines OP, OQ such that their sum may be equal to the given line OA, and their product equal to the given line OB." To solve it, take the two lines to be $x+iy=z$, and $x'+iy'$, $=z'$; then, working with z and z', we have $z+z'=2a$, $zz'=b^2$, giving a solution which may be written in the two forms

$$\{z = a + \sqrt{(a^2-b^2)},\quad z' = a - \sqrt{(a^2-b^2)}\},$$

and

$$\{z = a + i\sqrt{(b^2-a^2)},\quad z' = a - i\sqrt{(b^2-a^2)}\},$$

viz. for $a > b$, the two lines lie each of them in the line Ox, while for $b > a$, they are inclined at equal angles on opposite sides of the line Ox. Or if we work with the real values x, y, x', y', then the same two equations give $x+x'=2a$, $y+y'=0$, $xx'-yy'=b^2$, $xy'+yx'=0$; viz. writing $y'=-y$, then we have $x+x'=2a$, $xx'+y^2=b^2$, $y(x'-x)=0$, whence either $x=x'$ or else $y=0$, and we have the same two solutions as before. Observe that the second condition is $zz'=b^2$, not (as Forcenex understood it) the product of the lengths OP, $OQ=b^2$; and his objection is thus answered. But I

cannot but consider that the form of enunciation, "to divide the given line into two parts, such that their product shall be $= b^2$," does not express the meaning with sufficient clearness.

26. I have referred to Buée's notion of a curve as having branches in a perpendicular plane—this notion is generalised and developed in the papers:—

Gregory, D. F. "On the existence of branches of curves in several planes." *Camb. Math. Journ.*, t. I. (1839), pp. 259—266 (2nd Ed., pp. 284—292).

Walton. "On the general interpretation of equations between two variables in analytical geometry." *Do.* t. II. (1840), pp. 103—113.

—— "On the general theory of multiple points." *Do.* pp. 155—167.

And see also:—

Gregory's *Examples of the Processes of the Differential and Integral Calculus.* 8vo. Cambridge, 1841, pp. 172 et seq.;

and the criticism on these papers:—

Salmon. *A Treatise on the Higher Plane Curves.* 8vo. Dublin, 1852, The Note on Imaginary Points and Curves, pp. 301—306.

Geometrical representation of $x+iy$: Gauss. Art. Nos. 27—29.

27. The theory is that established by **Gauss** in the paper "Theoria Residuorum biquadraticorum Commentatio Secunda," *Comm. Gott. Recent.*, t. VII. (1832); *Werke*, t. II. pp. 95—148; viz. Gauss remarks that, in the same way that a real quantity x may be represented by means of a point on a line, so an imaginary (or complex) quantity $x+iy$ may be represented by means of a point the abscissa of which is $=x$, and its ordinate is $=y$. Taking m, m' as given complex quantities, he speaks of the points answering to the complex quantities mm', m, m', 1 as forming a *proportion* (thus in effect defining the product of two points M, M'). But he defers to another occasion all further discussion of the theory.

28. The direction in which the theory has been developed is as follows: Considering two complex values z, $=x+iy$, and Z, $=X+iY$, connected by an equation $\phi(Z, z)=0$, then we have z represented as above by means of a point in a plane, and Z represented in like manner by means of another point in a different plane; to any given value of z or Z there corresponds a determinate number of values of Z or z; that is, to any given real point of either plane, there corresponds a determinate number of real points in the other plane: the equation thus establishes a correspondence between the points of the two planes; and it is known that this is an orthomorphic correspondence, viz. to any indefinitely small area of the one plane, there correspond in the other plane indefinitely small areas each of them of the same shape as the area in the first plane. We have thus a solution of the geometrical problem of orthomorphic projection; and in the analytical point of view, the theory is of great importance as exhibiting the relation to each other of complex variables connected by an equation.

29. It is to be remarked that, although the correspondence is primarily a correspondence between real points of the two planes respectively, yet in the ulterior development of the theory of this correspondence it may become necessary to consider imaginary points in the two planes. In view hereto, there would be some propriety in replacing the i of the z, Z by a like symbol I; say we have $z = x + Iy$, $Z = X + IY$ (where the x, y, X, Y may now be ordinary complex magnitudes $u + vi$; $i^2 = -1$ as before: $I^2 = -1$, and $Ii = iI$, viz. I is a square root of -1, commutable with the ordinary imaginary i). But this in passing.

The Barycentric Calculus; Möbius. Art. Nos. 30, 31.

30. **Möbius,** *Der barycentrische Calcul,* 8vo. Leipzig, 1827, reprinted in Vol. I. of the *Gesammelte Werke,* Leipzig, 1885.

The theory applies to points in a line, in a plane, or in space; but it will be sufficient to consider the case of points in a plane. The idea is that, taking in the plane any three points A, B, C as fundamental points, then regarding these as loaded with the proper positive or negative weights p, q, r, any other point P of the plane may be regarded as the centre of gravity of these weights; say we have $P = pA + qB + rC$, as a representation of the point P; p, q, r are, in the first instance, given positive or negative real values (but they may be taken to be imaginary values), P, A, B, C are symbols denoting points. Observe that p, q, r are, in fact, trilinear coordinates: if for a moment ξ, η, ζ are the perpendicular distances of P from the sides of the triangle, and α, β, γ the perpendicular distances of the opposite vertices from the same sides respectively, then p, q, r are proportional to ξ/α, η/β, ζ/γ, or, what is the same thing, they are proportional to PBC/Δ, PCA/Δ, PAB/Δ, where PBC, &c., denote the three triangles and Δ the triangle ABC. It would be allowable to take p, q, r *equal* to these values, which would give $p + q + r = 1$. Möbius does this very nearly, for he writes $p + q + r + s = 0$, and then writing $-(p + q + r) D$ instead of P he has

$$pA + qB + rC + sD = 0;$$

that is,

$$D = -\frac{p}{s} A - \frac{q}{s} B - \frac{r}{s} C,$$

where

$$-\frac{p}{s} - \frac{q}{s} - \frac{r}{s} = 1:$$

writing x, y, z, instead of these quantities, I take therefore as the representation of the point, $P = xA + yB + zC$, where $x + y + z = 1$.

31. The symbols A, B, C may be regarded as extraordinaries; but it can hardly be said that Möbius so regards them, for he does not establish the notion of the multiplication of points, nor consequently any multiplication table for these symbols A, B, C: he does however deal with the *addition* of points, viz. if

$$P = xA + yB + zC, \quad P' = x'A + y'B + z'C,$$

then

$$P + \lambda P', \; = (x + \lambda x') A + (y + \lambda y') B + (z + \lambda z') C,$$

(where λ is indeterminate) denotes any point whatever in the line PP', (so also $P = xA + yB + zC$, where x, y, z are regarded as functions of a variable parameter θ, denotes any point whatever in the curve denoted by writing x, y, z proportional to such functions of θ). And he also considers (not quite generally) the question of the transformation of the fundamental points: this is really a treatment of the A, B, C as extraordinaries, the complete theory being as follows; assuming

$$A = \alpha A_1 + \beta B_1 + \gamma C_1, \quad B = \alpha' A_1 + \beta' B_1 + \gamma' C_1, \quad C = \alpha'' A_1 + \beta'' B_1 + \gamma'' C_1,$$

(if $\alpha + \beta + \gamma = 1$, $\alpha' + \beta' + \gamma' = 1$, $\alpha'' + \beta'' + \gamma'' = 1$, then these equations give conversely

$$A_1 = aA + a'B + a''C, \quad B_1 = bA + b'B + b''C, \quad C_1 = cA + c'B + c''C,$$

where $a + a' + a'' = 1$, $b + b' + b'' = 1$, $c + c' + c'' = 1$), then we have

$$P_1 = x\alpha + yB + zC_1 = (x\alpha + y\alpha' + z\alpha'')\, A_1 + (x\beta + y\beta' + z\beta'')\, B_1 + (x\gamma + y\gamma' + z\gamma'')\, C_1,$$

where the sum of the coefficients is

$$x(\alpha + \beta + \gamma) + y(\alpha' + \beta' + \gamma') + z(\alpha'' + \beta'' + \gamma''),$$

which is $= x + y + z$, $= 1$.

Equipollences: Bellavitis. Art. Nos. 32, 33.

32. There are earlier papers in the years 1832 and 1833, but the method is explained in the Memoir:—

Bellavitis. "Saggio di applicazioni di un nuovo metodo di geometria analitica (Calcolo delle Equipollenze)," *Ann. Lomb. Veneto*, t. v. (1835), pp. 244—259,

and more fully in Memoirs, t. VII. (1837), and t. VIII. (1838); the author notices that his results were obtained independently of those of Möbius but that, on studying the *Barycentrische Calcul*, he had recognised that the two methods started from the same principles, and could easily be reduced to identity, but that they speedily separated from each other as well in regard to object as to form. But, in fact, the principles are rather those of Warren than of Möbius. They may be thus stated: 1°. two lines, equal, parallel and in the same sense, are said to be *Equipollent*, $AB \equiv CD$ (where the sign $\equiv$ is used instead of a special sign employed by Bellavitis); this first assumption in effect reduces the whole theory to that of lines drawn through a fixed point; 2°. $AB + BC \equiv AC$ (addition); and 3°. $AB \equiv \dfrac{CD \,.\, EF}{GH}$, means not only that the lengths are connected by this relation, but that the inclination of AB to any fixed axis is $=$ inc. $CD +$ inc. $EF -$ inc. GH (proportion); or in particular, if GH be a line, length unity, in the direction of the axis, then $AB \equiv CD \,.\, EF$, viz. the length $AB =$ product of lengths CD and EF; and, further,

$$\text{inc. } AB = \text{inc. } CD + \text{inc. } EF \text{ (multiplication)};$$

the agreement with Warren is thus complete.

33. The method is applied very elegantly to the solution of problems, for instance, t. v. p. 248; given two lines AB and CD, to find a point H, such that the triangles ABH, CDH may be directly similar, that is, such that they can be, by a rotation of one of them, brought to be similarly situate. The condition for this (a figure is easily supplied) is

$$\frac{AH}{AB} \equiv \frac{CH}{CD}, \text{ that is, } AH \,.\, CD \equiv AB \,.\, CH, \; = AB\,(AH - AC),$$

whence $AB \,.\, AC = AH\,(AB - CD)$: or constructing a point E such that $BE \equiv DC$, viz. the line BE is drawn from B equal and parallel to and in the same sense with DC, then $AB - CD \equiv AE$, and the foregoing equation becomes

$$AH \,.\, AE \equiv AB \,.\, AC, \text{ or say } \frac{AH}{AC} \equiv \frac{AB}{AE},$$

viz. the required point H is such that the triangle ACH is directly similar to AEB. A solution of the like problem with the triangles ABH, CDH inversely similar is given, t. VIII., p. 27. I notice an expression t. VIII., p. 19, for the area of a triangle ABC,

$$= \frac{1}{4i}(BC \,.\, \text{cj.}\, AB - AB \,.\, \text{cj.}\, BC),$$

where cj. AB, conjugate of AB, is an equal line through A with an inclination $= -$ incl. AB; also a proof, t. VIII., p. 86, of the general theorem that an algebraical equation of any order has a root (this is, in a geometrical form, equivalent to the proofs given by Gauss and Cauchy); also t. VIII., p. 111, the notion of the "punti fittizj-conjugati," or real antipoints of a pair of conjugate imaginary points. The section "Delle figure a tre dimensioni," t. VIII., pp. 115—121, is confessedly quite incomplete, and without any anticipation of the quaternion or other three-dimensional imaginaries.

We have subsequently by the author

"Sposizione del Metodo delle Equipollenze," *Mem. Soc. Ital.*, t. XXV. (1855), pp. 225—309: published separately, Modena, 1854;

and in French under the title

Exposition de la Méthode des Equipollences, par G. Bellavitis, traduit de l'italien par C. A. Laisant: 8vo. Paris, 1874.

Quaternions: Sir W. R. Hamilton. Art. Nos. 34—42.

34. I give the following references to early papers and systematic works:

Sir W. R. Hamilton. "Abstract of a paper On Quaternions or on a new system of imaginaries in Algebra, with some geometrical illustrations." *Proc. R. I. Acad.*, vol. III., Nov. 11, 1844, pp. 1—16. A note added during the printing refers to Cayley *infra*.

—— "On Quaternions; or a new system of imaginaries in Algebra." *Phil. Mag.*, vol. XXV. (1844), pp. 489—495.

Cayley. "On certain results relating to Quaternions." *Phil. Mag.*, vol. XXVI. (1845), pp. 141—144, [20].

Sir W. R. Hamilton. *Lectures on Quaternions.* 8vo. Dublin, 1853.

—— *Elements of Quaternions.* 8vo. Dublin, 1866.

Tait, P. G. *An Elementary Treatise on Quaternions.* 8vo. Oxford, 1859.

Kelland, P. and **Tait, P. G.** *Introduction to Quaternions.* 8vo. London, 1873.

35. A quaternion is the sum of a scalar or mere magnitude w, plus a vector $ix+jy+kz$, which represents a line drawn from an origin O to the point whose rectangular coordinates are x, y, z; we have thus a quaternion $Q, =w+ix+jy+kz$, representing a line, and in connexion therewith a scalar w. Hamilton refers to the theory of Warren and Peacock as having in part suggested his investigations, but the contrast is very striking; neither the form $x+iy$, where the directed magnitude x is unaccompanied by an extraordinary, nor the form $kx+iy$ with any multiplication table for k, i, is in any wise analogous to the expression $ix+jy+kz$ of a vector. We have, for addition, the ordinary formula $Q+Q'=w+w'+i(x+x')+j(y+y')+k(z+z')$; and then we have the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

,

or, what is the same thing, we have

$$i^2=j^2=k^2=-1,\ \ i=jk=-kj,\ \ j=ki=-ik,\ \ k=ij=-ji,$$

and thence for the product of two quaternions

$$\begin{aligned}(w+ix+jy+kz)(w'+ix'+jy'+kz') \\ = \quad ww'-xx'-yy'-zz' \\ +i\,(wx'+xw'+yz'-y'z) \\ +j\,(wy'+yw'+zx'-z'x) \\ +k\,(wz'+zw'+xy'-x'y);\end{aligned}$$

these, of course, include the forms for the addition and multiplication of two vectors $ix+jy+kz$ and $ix'+jy'+kz'$.

But for the multiplication of two vectors, the form is simplified: viz. we have

$$\begin{aligned} VV' &= (ix+jy+kz)(ix'+jy'+kz') \\ &= -(xx'+yy'+zz')+i(yz'-y'z)+j(zx'-z'x)+k(xy'-x'y). \end{aligned}$$

36. We may consider unit-vectors,

$$U = ix+jy+kz, \text{ where } x^2+y^2+z^2=1,$$

and unit-quaternions

$$O = w+ix+jy+kz, \text{ where } w^2+x^2+y^2+z^2=1;$$

obviously the general form of a vector is $V=rU$, or, say the length is r, and the cosine-inclinations are x, y, z; similarly, the general form of a unit-quaternion is

$$O = \cos\delta + \sin\delta \,.\, U,$$

and that of a quaternion is

$$Q = TO = T(\cos\delta + \sin\delta \,.\, U),$$

T is the tensor, δ the amplitude, and x, y, z the cosine-inclinations of the unit-vector.

For the product of two unit-vectors, we have

$$UU' = -(xx'+yy'+zz')+i(yz'-y'z)+j(zx'-z'x)+k(xy'-x'y),$$

which is a unit-quaternion. Let δ, considered as a positive angle less than π, so that

$$\cos\delta = \pm, \quad \sin\delta = +,$$

be the inclination of the two vectors; we have $\cos\delta = xx'+yy'+zz'$. We can at right angles to the plane of U, U' draw in either of two opposite senses a unit-vector U''; and we may choose the sense in such wise that the rotation from U through angle δ to U' about U'' shall be in the sense Ox through angle $\frac{1}{2}\pi$ to Oy about Oz, say each of these is a right-handed rotation. This being so, we may write

$$\begin{aligned} &i(yz'-y'z)+j(zx'-z'x)+k(xy'-x'y), \\ &= \sin\delta\,(ix''+jy''+kz'') = \sin\delta \,.\, U'', \end{aligned}$$

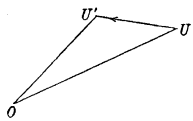
and we have

$$UU' = -\cos\delta + \sin\delta \,.\, U'';$$

viz. the product is a unit-quaternion, the scalar part $=-\cos\delta$, and the vector part having a length $=\sin\delta$, and being at right angles to the plane of the two vectors U, U'; δ being, as above, the inclination of the two vectors. And similarly

$$U'U = -\cos\delta - \sin\delta \,.\, U''.$$

Fig. 3.



Representing the vectors as lines OU, OU', the product UU' will be represented, as in fig. 3, by means of an arrow drawn in the angle δ from U to U'; and of

course, for the product $U'U$, there would be the same figure with only the arrow drawn in the opposite sense from U' to U. The figure serves also to represent the unit-quaternion $-\cos\delta+\sin\delta\,.\,U''$, which is the product of the two vectors. Observe that, in thus expressing a quaternion as a product of two vectors, only the plane of the vectors and the angle between them are determinate; we may rotate the angle UOU' in its own plane at pleasure.

In particular, if the two vectors are at right angles, then $\delta=90°$, and the product is $UU'=U''$, viz. the product of the two unit-vectors at right angles to each other is a unit-vector at right angles to the plane of the two unit-vectors. If $\delta=0$, then the two unit-vectors coincide, and we have $U^2=-1$, viz. the square of a unit-vector is the scalar -1.

The product of any two vectors is given by the like formulæ; if V, $V'=rU$, $r'U'$ respectively, then $VV'=rr'\,.\,UU'$.

37. The product of a quaternion by a vector is given by the formula

$$\begin{aligned}QV'&=(w+ix+jy+kz)(ix'+jy'+kz'),\\&=-(xx'+yy'+zz')+i(wx'+yz'-y'z)+j(wy'+zx'-z'x)+k(wz'+xy'-x'y).\end{aligned}$$

In particular, if the quaternion and the vector are at right angles (that is, if the vector part of Q and V' are at right angles), $xx'+yy'+zz'=0$, and the product is a vector. We may use this formula to express a given quaternion as the product of two vectors. Writing, for convenience,

$$U=i\xi+j\eta+k\zeta,$$

a unit-vector at right angles to Q, so that $\xi^2+\eta^2+\zeta^2=1$ and $\xi x+\eta y+\zeta z=0$, then

$$\begin{aligned}QU&=(w+ix+jy+kz)(i\xi+j\eta+k\zeta)\\&=i(w\xi+y\zeta-z\eta)+j(w\eta+z\xi-x\zeta)+k(w\zeta+x\eta-y\xi),\end{aligned}$$

which is a vector, say it is

$$V=iX+jY+kZ;$$

and then

$$Q=-QU^2=-VU=-(iX+jY+kZ)(i\xi+j\eta+k\zeta),$$

that is,

$$w+ix+jy+kz=\xi X+\eta Y+\zeta Z+i(\eta Z-\zeta Y)+j(\zeta X-\xi Z)+k(\xi Y-\eta X),$$

as is at once verified by substituting for X, Y, Z their values

$$w\xi+y\zeta-z\eta,\quad w\eta+z\xi-x\zeta,\quad w\zeta+x\eta-y\xi.$$

Observe that

$$Xx+Yy+Zz=w(\xi x+\eta y+\zeta z)=0.$$

38. And we hence reduce the multiplication of quaternions to that of vectors. For, considering another quaternion Q', take $U=i\xi+j\eta+k\zeta$ as before, with the further condition that it is at right angles to Q', so that $\xi^2+\eta^2+\zeta^2=1$, $\xi x+\eta y+\zeta z=0$, $\xi x'+\eta y'+\zeta z'=0$; say U is the unit-vector at right angles to the vectors of Q, Q': then we have

$$\begin{aligned}-UQ'&=-(i\xi+j\eta+k\zeta)(w'+ix'+jy'+kz')\\&=i(-w'\xi+y'\zeta-z'\eta)+j(-w'\eta+z'\xi-x'\zeta)+k(-w'\zeta+x'\eta-y'\xi),\end{aligned}$$

which is a vector, say it is

$$V' = iX' + jY' + kZ';$$

and then

$$Q' = -U^2Q' = UV' = (i\xi + j\eta + k\zeta)(iX' + jY' + kZ'),$$

that is,

$$w' + ix' + jy' + kz' = -\xi X' - \eta Y' - \zeta Z' + i(\eta Z' - \zeta Y') + j(\zeta X' - \xi Z') + k(\xi Y' - \eta X'),$$

as may be verified by substituting for X', Y', Z' their values

$$-w'\xi + y'\zeta - z'\eta, \quad -w'\eta + z'\xi - x'\zeta, \quad -w'\zeta + x'\eta - y'\xi;$$

and it may be observed that

$$X'x' + Y'y' + Z'z', \quad = -w(\xi x' + \eta y' + \zeta z') = 0.$$

We hence find

$$\begin{aligned} QQ' = QU.-UQ' = VV' &= (iX + jY + kZ)(iX' + jY' + kZ'), \\ &= -XX' - YY' - ZZ' + i(YZ' - Y'Z) + j(ZX' - Z'X) + k(XY' - X'Y). \end{aligned}$$

39. This should, of course, be identically equal to

$$\begin{aligned} & ww' - xx' - yy' - zz' \\ & + i(wx' + xw' + yz' - y'z) \\ & + j(wy' + yw' + zx' - z'x) \\ & + k(wz' + zw' + xy' - x'y), \end{aligned}$$

on substituting for X, Y, Z, X', Y', Z' their values as above.

Thus, forming the value of $XX' + YY' + ZZ'$, and adding thereto the expression $(\xi x + \eta y + \zeta z)(\xi x' + \eta y' + \zeta z')$, which is identically $=0$, we obtain

$$\begin{aligned} XX' + YY' + ZZ' &= (-ww' + xx' + yy' + zz')(\xi^2 + \eta^2 + \zeta^2), \\ &= -ww' + xx' + yy' + zz'; \end{aligned}$$

and similarly the other three relations may be verified.

40. The steps are, taking U the unit-vector at right angles to Q, Q', then the vectors V, V' are defined by $V = QU$, $V' = -UQ'$; and we then have $Q = -VU$, $Q' = UV'$, and thence $QQ' = VV'$.

We may look at the process as follows: taking first $Q = -VU$, we have V, U vectors in a determinate plane and at a determinate inclination to each other; taking then $Q' = U'V'$, we have U', V' vectors in a determinate plane and at a determinate inclination to each other; we then rotate the vector-pairs (V, U) and (U', V') each in its own plane and in the senses V to U and U' to V' respectively, in such wise as to bring the vectors U, U' into coincidence (along one or other of the opposite vectors which form the intersection of the two planes), and this being so, and U, U' being unit-vectors, we have $U = U'$, and consequently $QQ' = -VU.U'V' = VV'$ as above.

41. There is a kind of quaternion operator $Q^{-1}(\ )Q$, or, what is the same thing, if $\Lambda = i\lambda + j\mu + k\nu$, $(1-\Lambda)(\ )(1+\Lambda)$, which is the symbol of a rotation; viz. the operand, say the vector $ix+jy+kz$, is to be placed within the vacant ( ), say

$$Q^{-1}(\ )Q.(ix+jy+kz) \text{ means } Q^{-1}(ix+jy+kz)Q;$$

and this being so, the result is a vector $ix_1+jy_1+kz_1$, where the x_1, y_1, z_1 are what the x, y, z become by a rotation of the vector $ix+jy+kz$ about an axis and through an angle determined by the quaternion Q; viz. if as above

$$Q = 1 + i\lambda + j\mu + k\nu,$$

then writing

$$\lambda,\ \mu,\ \nu = \tan\tfrac{1}{2}\theta\cos f,\quad \tan\tfrac{1}{2}\theta\cos g,\quad \tan\tfrac{1}{2}\theta\cos h,$$

where $\cos^2 f + \cos^2 g + \cos^2 h = 1$, and therefore $\lambda^2+\mu^2+\nu^2 = \tan^2\frac{1}{2}\theta$, then f, g, h are the inclinations of the axis, and θ is the angle of rotation. The actual formula is

$$(1-i\lambda-j\mu-k\nu)(ix+jy+kz)(1+i\lambda+j\mu+k\nu) = ix_1+jy_1+kz_1,$$

where

$$ix_1+jy_1+kz_1 = \frac{1}{1+\lambda^2+\mu^2+\nu^2}\left\{\begin{array}{l} i\,[(1+\lambda^2-\mu^2-\nu^2)x+2(\lambda\mu+\nu)y+2(\lambda\nu-\mu)z] \\ +j\,[2(\lambda\mu-\nu)x+(1-\lambda^2+\mu^2-\nu^2)y+2(\mu\nu+\lambda)z] \\ +k\,[2(\nu\lambda+\mu)x+2(\mu\nu-\lambda)y+(1-\lambda^2-\mu^2+\nu^2)z] \end{array}\right\},$$

viz. the form on the right-hand side is

$$\begin{array}{l} i\,(\alpha x+\alpha' y+\alpha'' z) \\ +j\,(\beta x+\beta' y+\beta'' z) \\ +k\,(\gamma x+\gamma' y+\gamma'' z), \end{array}$$

where the coefficients are those of a rectangular transformation.

42. I call to mind that a binary matrix may be regarded as a quaternion; viz. writing

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = (a+d) - \lambda(a-d)i + (b-c)j - \lambda(b+c)k,$$

where i, j, k are the quaternion imaginaries, and λ is written to denote the $i=\sqrt{(-1)}$ of ordinary analysis, then we at once deduce the equation

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} . \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = (a,\ b)\left|\begin{array}{c} (a',\ c'),\ (b',\ d') \\ \hline (c,\ d) \end{array}\right| = \begin{vmatrix} aa'+bc', & ab'+bd' \\ ca'+dc', & cb'+dd' \end{vmatrix}$$

for the product of two matrices.

The Ausdehnungslehre: Grassmann. Art. Nos. 43—63.

43. **Grassmann.** *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik,* Leipzig, 1844; 2nd edition, with the title *Die Ausdehnungslehre von* 1844, Leipzig, 1878.

—— *Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet,* Berlin, 1862, referred to as *Die Ausdehnungslehre von* 1862.

44. Plane Geometry. The representation of a point is that employed by Möbius in the *Barycentrische Calcul* (1827), viz. considering in the plane three fixed points A_1, A_2, A_3, and in regard to the triangle formed by these points, taking x_1, x_2, x_3 as the areal coordinates of a point x (x_1, x_2, x_3 are equal to the areas of the triangles xA_2A_3, xA_3A_1, xA_1A_2, each divided by the area of the triangle $A_1A_2A_3$; whence $x_1+x_2+x_3=1$), we write

$$x, \ =x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3$$

as the representation of the point x. Here ϵ_1, ϵ_2, ϵ_3 are extraordinaries, or they may also be regarded as points, viz. as the points (1, 0, 0), (0, 1, 0), (0, 0, 1), or say the points A_1, A_2, A_3 respectively. If the sum $x_1+x_2+x_3$, instead of being $=1$ has any other value $=w$, then $x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3$ represents not a point simpliciter, but a point of the weight w; a point simpliciter thus means a point of the weight 1.

45. Considering then

$$\lambda x, \ =\lambda(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3), \quad (x_1+x_2+x_3=1),$$

a point of the weight λ; and similarly

$$\mu y, \ =\mu(y_1\epsilon_1+y_2\epsilon_2+y_3\epsilon_3), \quad (y_1+y_2+y_3=1),$$

a point of the weight μ; we form herewith a sum $\lambda x+\mu y$,

$$=(\lambda x_1+\mu y_1)\,\epsilon_1+(\lambda x_2+\mu y_2)\,\epsilon_2+(\lambda x_3+\mu y_3)\,\epsilon_3,$$

which is a point of the weight $\lambda+\mu$, at the C. G. of the two given points, considered as being of the weights λ and μ respectively; if the weights λ and μ are regarded as indeterminate, then the ratio $\lambda:\mu$ is a variable parameter, and the sum will be any point in the line through the given points. It may be remarked that, if $\lambda+\mu$ be $=1$, then the sum will be a point simpliciter; in particular,

$$\tfrac{1}{2}(x+y), \ =\tfrac{1}{2}(x_1+y_1)\,\epsilon_1+\tfrac{1}{2}(x_2+y_2)\,\epsilon_2+\tfrac{1}{2}(x_3+y_3)\,\epsilon_3,$$

will be the point M, midway between the given points x and y.

Observe that $x-y$ is not properly a point; it may be regarded as a point, weight zero, at an infinite distance.

46. The extraordinaries ϵ_1, ϵ_2, ϵ_3 are assumed to be such that

$$\epsilon_1^2=0, \quad \epsilon_2^2=0, \quad \epsilon_3^2=0, \quad \epsilon_2\epsilon_1=-\epsilon_1\epsilon_2, \quad \epsilon_3\epsilon_2=-\epsilon_2\epsilon_3, \quad \epsilon_1\epsilon_3=-\epsilon_3\epsilon_1;$$

the product of the two points x, y is thus

$$x\,.\,y=(x_2y_3-x_3y_2)\,\epsilon_2\epsilon_3+(x_3y_1-x_1y_3)\,\epsilon_3\epsilon_1+(x_1y_2-x_2y_1)\,\epsilon_1\epsilon_2;$$

depending on the new extraordinaries $\epsilon_2\epsilon_3$, $\epsilon_3\epsilon_1$, $\epsilon_1\epsilon_2$. Grassmann further assumes $\epsilon_1\epsilon_2\epsilon_3 = 1$, and he calls the foregoing combinations the complements (Ergänzungen) of ϵ_1, ϵ_2, ϵ_3, denoting them by $/\epsilon_1$, $/\epsilon_2$, $/\epsilon_3$ respectively; it is better to use new letters, and I call them η_1, η_2, η_3 respectively; we thus have

$$\epsilon_1^2 = 0, \quad \epsilon_2^2 = 0, \quad \epsilon_3^2 = 0,$$

$$\epsilon_2\epsilon_3 = -\epsilon_3\epsilon_2 = \eta_1, \quad \epsilon_3\epsilon_1 = -\epsilon_1\epsilon_3 = \eta_2, \quad \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1 = \eta_3.$$

But we require further assumptions for the combinations of the η's *inter se* and with the ϵ's; the forms are

$$\eta_1^2 = 0, \quad \eta_2^2 = 0, \quad \eta_3^2 = 0,$$

$$-\eta_2\eta_3 = \eta_3\eta_2 = \epsilon_1; \quad -\eta_3\eta_1 = \eta_1\eta_3 = \epsilon_2, \quad -\eta_1\eta_2 = \eta_2\eta_1 = \epsilon_3;$$

$$\epsilon_1\eta_1 = \eta_1\epsilon_1 = 1, \quad \epsilon_2\eta_2 = \eta_2\epsilon_2 = 1, \quad \epsilon_3\eta_3 = \eta_3\epsilon_3 = 1, \quad \epsilon_1\eta_2 = \eta_2\epsilon_1 = 0, \text{ \&c.},$$

(viz. each such combination is $= 0$); or, what is the same thing, we have for 1, ϵ_1, ϵ_2, ϵ_3, η_1, η_2, η_3 the multiplication table

	1	ϵ_1	ϵ_2	ϵ_3	η_1	η_2	η_3
1	1	ϵ_1	ϵ_2	ϵ_3	η_1	η_2	η_3
ϵ_1	ϵ_1	0	η_3	$-\eta_2$	1	0	0
ϵ_2	ϵ_2	$-\eta_3$	0	η_1	0	1	0
ϵ_3	ϵ_3	η_2	$-\eta_1$	0	0	0	1
η_1	η_1	1	0	0	0	$-\epsilon_3$	ϵ_2
η_2	η_2	0	1	0	ϵ_3	0	$-\epsilon_1$
η_3	η_3	0	0	1	$-\epsilon_2$	ϵ_1	0

It is proper to remark that in Grassmann's theory there is no interpretation for the general linear symbol

$$\omega + x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + y_1\eta_1 + y_2\eta_2 + y_3\eta_3,$$

which presents itself in connexion with the seven extraordinaries (1, ϵ_1, ϵ_2, ϵ_3, η_1, η_2, η_3).

47. It is to be noticed that the symbols are not associative; assuming them to be so, we should have $\epsilon_2\epsilon_3 . \epsilon_3\epsilon_1 = \epsilon_2 . \epsilon_3^2 . \epsilon_1 = 0$, which is inconsistent with $\eta_1\eta_2 = -\epsilon_3$; and so in other cases. But any *three* ϵ's are associative, and hence a product

$$x . y . z, = (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3)(y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3)(z_1\epsilon_1 + z_2\epsilon_2 + z_3\epsilon_3),$$

is associative; in fact, if this be regarded as standing for $(x\,.\,y)\,z$, the value is

$$=\{(x_2y_3-x_3y_2)\,\eta_1+(x_3y_1-x_1y_3)\,\eta_2+(x_1y_2-x_2y_1)\,\eta_3\}\,(z_1\epsilon_1+z_2\epsilon_2+z_3\epsilon_3),$$

which is

$$=(x_2y_3-x_3y_2)\,z_1+(x_3y_1-x_1y_3)\,z_2+(x_1y_2-x_2y_1)\,z_3,\ =\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix};$$

and similarly, regarding it as standing for $x\,(y\,.\,z)$, the value is

$$=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)\,\{(y_2z_3-y_3z_2)\,\eta_1+(y_3z_1-y_1z_3)\,\eta_2+(y_1z_2-y_2z_1)\,\eta_3\},$$

which is

$$=x_1\,(y_2z_3-y_3z_2)+x_2\,(y_3z_1-y_1z_3)+x_3\,(y_1z_2-y_2z_1),\ =\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

as before. The product of the three points is thus a scalar, viz. it is equal to the area of the triangle formed by the three points divided by that of the triangle $A_1A_2A_3$.

48. But a product $x\,.\,Y\,.\,z$,

$$=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)\,(Y_1\eta_1+Y_2\eta_2+Y_3\eta_3)\,(z_1\epsilon_1+z_2\epsilon_2+z_3\epsilon_3),$$

is not associative, and has thus no meaning until the grouping of the factors is determined. In fact, $(x\,.\,Y)\,z$ will be

$$=(x_1Y_1+x_2Y_2+x_3Y_3)\,(z_1\epsilon_1+z_2\epsilon_2+z_3\epsilon_3),$$

which denotes the point z_1 with a weight $=x_1Y_1+x_2Y_2+x_3Y_3$; whereas $x\,(Y\,.\,z)$ will be

$$=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)\,(y_1Z_1+y_2Z_2+y_3Z_3),$$

which denotes the point x_1 with a weight $=y_1Z_1+y_2Z_2+y_3Z_3$.

Hence also a product $x\,.\,y\,.\,z\,.\,w$,

$$=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)\,(y_1\epsilon_1+y_2\epsilon_2+y_3\epsilon_3)\,(z_1\epsilon_1+z_2\epsilon_2+z_3\epsilon_3)\,(w_1\epsilon_1+w_2\epsilon_2+w_3\epsilon_3),$$

of four factors is not associative, and has thus no meaning until the grouping of the factors is determined. Grassmann attributes to such a product the value $=0$; but this is not a value corresponding to any grouping of the factors, and the equation $x\,.\,y\,.\,z\,.\,w=0$ can only be regarded as an independent assumption.

For the product of a point into itself, or say the square of a point, we have

$$x^2=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)^2,\quad =(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)\,(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3),\ =0,$$

viz. this is $=$ zero.

49. The product of two points x and y is defined to be the "Strecke", the finite line or stroke (xy), considered as drawn from x to y, say we have $x\,.\,y=(xy)$; observe

that $y.x=(yx)$: by what precedes $y.x+x.y=0$, whence also $(yx)+(xy)=0$, which is an equation in the addition of strokes. Hence, if as before, the points x, y are $x=x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3$, and $y=y_1\epsilon_1+y_2\epsilon_2+y_3\epsilon_3$, then for the stroke (xy) we have

$$(xy)=(x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3)(y_1\epsilon_1+y_2\epsilon_2+y_3\epsilon_3),$$
$$=(x_2y_3-x_3y_2)\eta_1+(x_3y_1-x_1y_3)\eta_2+(x_1y_2-x_2y_1)\eta_3,$$

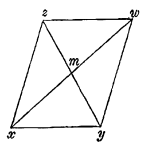
which is, in fact,

$$=(A_1xy.\eta_1+A_2xy.\eta_2+A_3xy.\eta_3)\div A_1A_2A_3,$$

if A_1xy, A_2xy, A_3xy, $A_1A_2A_3$ denote the areas of the triangles formed by the points (A_1, x, y), (A_2, x, y), (A_3, x, y), (A_1, A_2, A_3) respectively. These triangles remain unaltered if the stroke (xy) is slid along the indefinite line joining the original points x, y, the absolute distance xy being unaltered; that is, strokes of equal length upon the same line are regarded as identical. Hence also strokes on the same line may be added together by adding their lengths.

50. Consider two strokes (xy), (xz), (fig. 4), having a common point x, then completing the parallelogram, we prove that the sum $(xy)+(xz)$ is equal to the diagonal (xw).

Fig. 4.



We have, in fact,

$$x+w=y+z,$$

and thence

$$x(x+w)=x(y+z),$$

that is,

$$x^2+x.w=x.y+x.z,$$

or, since $x^2=0$ and $x.y=(xy)$, &c., this is

$$(xw)=(xy)+(xz),$$

the required formula for the addition of the strokes.

51. The same thing appears thus: writing

$$x=x_1\epsilon_1+x_2\epsilon_2+x_3\epsilon_3,$$
$$y=y_1\epsilon_1+y_2\epsilon_2+y_3\epsilon_3,$$
$$z=z_1\epsilon_1+z_2\epsilon_2+z_3\epsilon_3,$$

we have

$$(xy)=(x_2y_3-x_3y_2)\eta_1+(x_3y_1-x_1y_3)\eta_2+(x_1y_2-x_2y_1)\eta_3,$$
$$(xz)=(x_2z_3-x_3z_2)\eta_1+(x_3z_1-x_1z_3)\eta_2+(x_1z_2-x_2z_1)\eta_3.$$

Or, putting for a moment

$$m_1,\ m_2,\ m_3 = \tfrac{1}{2}(y_1 + z_1),\quad \tfrac{1}{2}(y_2 + z_2),\quad \tfrac{1}{2}(y_3 + z_3),$$

we have for the point m, midway between y and z, the equation $m = m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3$; and then

$$\begin{aligned}(xy) + (xz) &= 2\{(x_2m_3 - x_3m_2)\,\eta_1 + (x_3m_1 - x_1m_3)\,\eta_2 + (x_1m_2 - x_2m_1)\,\eta_3\}\\ &= 2\,(xm),\quad = xw,\end{aligned}$$

which is the theorem in question.

52. Recurring to the foregoing equation $x + w = y + z$, we deduce

$$xz - yw = zw - xy = (yzw) = (zwx) = -(wxy) = -(xyz),$$

if for shortness $(yzw) = yz + zw + wy$, and similarly for (zwx), (wxy), and (xyz). Observe here that $xz - yw$ and $zw - xy$ are each of them equal to the difference between two equal and parallel strokes; the strokes yz, zw, wy are the sides (in order) of a triangle; *quà* forces, the expressions in question represent each of them a couple.

53. We have thus arrived at the representation of a stroke

$$(xy) = (x_2y_3 - x_3y_2)\,\eta_1 + (x_3y_1 - x_1y_3)\,\eta_2 + (x_1y_2 - x_2y_1)\,\eta_3,$$

or say

$$\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3,$$

where the meaning of the coordinates λ_1, λ_2, λ_3 is such that, if x, y are the two extremities of the stroke, then λ_1, λ_2, λ_3 are equal to the areas A_1xy, A_2xy, A_3xy, each divided by the area $A_1A_2A_3$. And we have also established the rule for the addition of strokes: it will be noticed that this has been done without the aid of any expression for the length of the stroke.

54. *Metrical relations.* For the expression of the distance of two points or length of a stroke, and of the inclinations of a stroke to the sides of the fundamental triangle, &c., we require the values of the angles and sides of this triangle $A_1A_2A_3$, say the angles are A_1, A_2, A_3, and the sides are $= R\sin A_1$, $R\sin A_2$, $R\sin A_3$ respectively; R is thus equal to the diameter of the circumscribed circle. Moreover, considering a stroke xy, if through x we draw lines $x\theta_1$, $x\theta_2$, $x\theta_3$ in the senses A_2A_3, A_3A_1, A_1A_2 respectively, then taking the angles $yx\theta_1$, $yx\theta_2$, $yx\theta_3$ each of them in the same sense, or say each right-handedly, we call these angles θ_1, θ_2, θ_3 respectively, and take also ρ, $=(xy)$ for the length of the stroke. This being so, and if the two extremities are

$$x = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3,\ \text{and}\ y = y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3,$$

then we have, first,

$$\theta_2 - \theta_3 = \pi - A_1,\quad \theta_3 - \theta_1 = \pi - A_2,\quad \theta_1 - \theta_2 = -\pi - A_3,$$

giving

$$A_1 + A_2 + A_3 = \pi,$$

and

$$\sin A_1,\quad \sin A_2,\quad \sin A_3 = \sin(\theta_2 - \theta_3),\quad \sin(\theta_3 - \theta_1),\quad \sin(\theta_1 - \theta_2).$$

And then

$$y_1 = x_1 + \frac{\rho}{R}\frac{\sin\theta_1}{\sin A_2 \sin A_3}, \quad y_2 = x_2 + \frac{\rho}{R}\frac{\sin\theta_2}{\sin A_3 \sin A_1}, \quad y_3 = x_3 + \frac{\rho}{R}\frac{\sin\theta_3}{\sin A_1 \sin A_2}.$$

As regards these last equations, writing for a moment p_1, p_2, p_3 for the perpendicular distances of x, and q_1, q_2, q_3 for those of y from the sides of the triangle, then we have $q_1, q_2, q_3 = p_1 + \rho\sin\theta_1,\ p_2 + \rho\sin\theta_2,\ p_3 + \rho\sin\theta_3$, and we thence easily derive the equations in question.

55. *Expression for length of a stroke.* We hence, by a somewhat complicated analytical process, find an expression for ρ in terms of the coordinates (x_1, x_2, x_3), and (y_1, y_2, y_3), which enter into it through the combinations

$$x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_1y_2 - x_2y_1.$$

In fact, writing for shortness

$$\frac{x_1}{\sin A_1}, \ \frac{x_2}{\sin A_2}, \ \frac{x_3}{\sin A_3} = K_1, K_2, K_3,$$

we have

$$(x_2y_3 - x_3y_2)\sin A_1 = \frac{\rho}{R}(K_2\sin\theta_3 - K_3\sin\theta_2),$$

$$(x_3y_1 - x_1y_3)\sin A_2 = \frac{\rho}{R}(K_3\sin\theta_1 - K_1\sin\theta_3),$$

$$(x_1y_2 - x_2y_1)\sin A_3 = \frac{\rho}{R}(K_1\sin\theta_2 - K_2\sin\theta_1);$$

and hence also, if $(*\between\xi, \eta, \zeta)^2$ denote the quadric function

$$(1, 1, 1, -\cos A_1, -\cos A_2, -\cos A_3 \between \xi, \eta, \zeta)^2,$$

we have

$$(*\between(x_2y_3 - x_3y_2)\sin A_1, \ldots)^2$$
$$= \frac{\rho^2}{R^2}(*\between K_2\sin\theta_3 - K_3\sin\theta_2,\ K_3\sin\theta_1 - K_1\sin\theta_3,\ K_1\sin\theta_2 - K_2\sin\theta_1)^2,$$

where the quadric function on the right-hand side is in fact

$$= (K_1\sin A_1 + K_2\sin A_2 + K_3\sin A_3)^2,$$

that is,

$$= (x_1 + x_2 + x_3)^2, = 1;$$

and we thus finally obtain

$$\rho^2 = R^2(*\between(x_2y_3 - x_3y_2)\sin A_1,\ (x_3y_1 - x_1y_3)\sin A_2,\ (x_1y_2 - x_2y_1)\sin A_3)^2,$$

the required formula for the length of the stroke xy.

56. To effect the foregoing reduction of the quadric function

$$(*\between K_2\sin\theta_3 - K_3\sin\theta_2,\ K_3\sin\theta_1 - K_1\sin\theta_3,\ K_1\sin\theta_2 - K_2\sin\theta_1)^2,$$

observe that the coefficient herein of K_1^2 is

$$= \sin^2\theta_2 + \sin^2\theta_3 + 2\cos A_1 \sin\theta_2 \sin\theta_3,$$

where $\theta_2 - \theta_3 = \pi - A_1$, and thence

$$\cos\theta_2\cos\theta_3 + \sin\theta_2\sin\theta_3 = -\cos A_1;$$

that is, we have

$$\cos A_1 + \sin\theta_2\sin\theta_3 = -\cos\theta_2\cos\theta_3,$$

and thence

$$1 - \sin^2\theta_2 - \sin^2\theta_3 + \sin^2\theta_2\sin^2\theta_3 = \cos^2 A_1 + 2\cos A_1\sin\theta_2\sin\theta_3 + \sin^2\theta_2\sin^2\theta_3;$$

whence the coefficient in question is $=\sin^2 A_1$. And finding in like manner the coefficients of K_2^2, K_3^2, K_2K_3, &c., the whole function is, as already mentioned,

$$= (K_1\sin A_1 + K_2\sin A_2 + K_3\sin A_3)^2, \text{ that is, } = 1.$$

57. *New representation of a stroke.* Representing the stroke in the form

$$\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3,$$

we have only to replace $x_2y_3 - x_3y_2$, $x_3y_1 - x_1y_3$, $x_1y_2 - x_2y_1$ by their values $=\lambda_1$, λ_2, λ_3 respectively, viz. for the length ρ of the stroke, we have

$$\rho^2 = R^2(*\between\lambda_1\sin A_1,\ \lambda_2\sin A_2,\ \lambda_3\sin A_3)^2.$$

Say this equation is

$$\rho^2 = R^2.\Omega, \text{ then } \rho = R\sqrt{(\Omega)}, \text{ or } 1 = \frac{\rho}{R\sqrt{(\Omega)}},$$

and the stroke may be represented in the form

$$\lambda = \frac{\rho}{R\sqrt{(\Omega)}}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3);$$

or, what is the same thing, if the absolute magnitudes of the coordinates λ_1, λ_2, λ_3 are such that

$$(*\between\lambda_1\sin A_1,\ \lambda_2\sin A_2,\ \lambda_3\sin A_3)^2 = 1,$$

then the expression is

$$\lambda = \frac{\rho}{R}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3),$$

where ρ is the length of the stroke.

58. It has been seen that the coefficients of η_1, η_2, η_3 have the values A_1xy, A_2xy, A_3xy, each divided by $A_1A_2A_3$; or, what is the same thing, if p_1, p_2, p_3 are the lengths of the perpendiculars on the stroke from the points A_1, A_2, A_3 respectively, then these coefficients are $(\frac{1}{2}\rho p_1, \frac{1}{2}\rho p_2, \frac{1}{2}\rho p_3)$, each divided by $A_1A_2A_3$; or, what is the same thing, the perpendicular distances p_1, p_2, p_3 are

$$= \frac{2A_1A_2A_3}{R\sqrt{\Omega}}(\lambda_1,\ \lambda_2,\ \lambda_3)$$

respectively, where $A_1A_2A_3$ is the area of the fundamental triangle.

59. *Expression for the mutual inclination of two strokes.* In connexion with the stroke $\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3$, considering a stroke $\mu = \mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3$, suppose the inclinations hereof to the axes are ϕ_1, ϕ_2, ϕ_3, then

$$\phi_2 - \phi_3 = \pi - A_1, \quad \phi_3 - \phi_1 = \pi - A_2, \quad \phi_1 - \phi_2 = -\pi - A_3,$$

and we have

$$\theta_1 - \phi_1 = \theta_2 - \phi_2 = \theta_3 - \phi_3,$$

viz. each of these equal angles is the inclination of the two strokes to each other, say this angle is $= \delta$. The expression for this angle is

$$\cos\delta = (*\between \lambda_1 \sin A_1,\ \lambda_2 \sin A_2,\ \lambda_3 \sin A_3 \between \mu_1 \sin A_1,\ \mu_2 \sin A_2,\ \mu_3 \sin A_3)$$
$$\div \sqrt{\{(1, \ldots \between \lambda_1 \sin A_1,\ \lambda_2 \sin A_2,\ \lambda_3 \sin A_3)^2\}} \cdot \sqrt{\{(1, \ldots \between \mu_1 \sin A_1,\ \mu_2 \sin A_2,\ \mu_3 \sin A_3)^2\}}.$$

In fact, considering (as we may do) the two strokes as proceeding from a common point $x = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3$, then the function in question is

$$= (*\between K_2 \sin\theta_3 - K_3 \sin\theta_2,\ K_3 \sin\theta_1 - K_1 \sin\theta_3,\ K_1 \sin\theta_2 - K_2 \sin\theta_1 \between K_2 \sin\phi_3 - K_3 \sin\phi_2, \ldots),$$

which is

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2 \cos\delta, \quad = (x_1 + x_2 + x_3)^2 \cos\delta, \quad = \cos\delta.$$

60. In verification, observe that the whole coefficient of K_1^2 is

$$= \sin\theta_2 \sin\phi_2 + \sin\theta_3 \sin\phi_3 + \cos A_1 (\sin\theta_2 \sin\phi_3 + \sin\theta_3 \sin\phi_2),$$

where $\theta_2 - \theta_3 = \pi - A_1$, $\phi_2 - \phi_3 = \pi - A_1$. Hence

$$\sin\theta_2 = -\sin(\theta_3 - A_1), \quad \sin\phi_2 = -\sin(\phi_3 - A_1);$$

and the expression in question is

$$\begin{aligned} = \ & \sin(\theta_3 - A_1)\sin(\phi_3 - A_1) \\ & - \cos A_1 \sin(\theta_3 - A_1)\sin\phi_3, \\ & - \cos A_1 \sin(\phi_3 - A_1)\sin\theta_3, \\ & + \sin\theta_3 \sin\phi_3, \end{aligned}$$

which, expanding the sines, and writing the last term in the form

$$-\sin\theta_3 \sin\phi_3 (\cos^2 A_1 + \sin^2 A_1)$$

is

$$\begin{aligned} = \ & \cos^2 A_1 (\sin\theta_3 \sin\phi_3 - \sin\theta_3 \sin\phi_3 - \sin\theta_3 \sin\phi_3 + \sin\theta_3 \sin\phi_3) \\ & + \sin^2 A_1 (\cos\theta_3 \cos\phi_3 + \sin\theta_3 \sin\phi_3) \\ & + \cos A_1 \sin A_1 (-\sin\theta_3 \cos\phi_3 - \sin\phi_3 \cos\theta_3 + \sin\phi_3 \cos\theta_3 + \sin\theta_3 \cos\phi_3), \\ = \ & \sin^2 A_1 \cos(\theta_3 - \phi_3), \quad = \sin^2 A_1 \cos\delta, \end{aligned}$$

and reducing in like manner the coefficients of K_2^2, K_3^2, K_2K_3, &c., the whole expression becomes

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2 \cos\delta, \ = \cos\delta \text{ as above.}$$

61. The foregoing formulæ agree with the theorem, No. 50, that the sum of the strokes

$$\lambda, = \frac{\rho}{R\sqrt{(\Omega)}}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3),$$

and

$$\mu, = \frac{\sigma}{R\sqrt{(\Upsilon)}}(\mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3),$$

is a stroke

$$= \frac{\tau}{R\sqrt{(\Pi)}}(\nu_1\eta_1 + \nu_2\eta_2 + \nu_3\eta_3),$$

which is the diagonal of the parallelogram constructed with the given strokes λ and μ; the proof is a little simplified by assuming (as is allowable) $\Omega = 1$ and $\Upsilon = 1$, in which case we have

$$\lambda + \mu = \frac{1}{R}\{(\rho\lambda_1 + \sigma\mu_1)\eta_1 + (\rho\lambda_2 + \sigma\mu_2)\eta_2 + (\rho\lambda_3 + \sigma\mu_3)\eta_3\},$$

from which the length and inclination may be calculated.

As already appearing, the product x, y, z of any three points is the scalar

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

which is equal to the area of the triangle xyz, divided by that of the triangle $A_1A_2A_3$. We have in like manner the product of a point x into a stroke λ, viz. this is

$$= (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3)(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3),$$

which is $= x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3$, the two factors being in this case commutative; the value is equal to the area of the triangle formed by the point and the stroke, divided by the area $A_1A_2A_3$. Of course, if in the one case the three points are in a line, or if in the other the point is in the line of the stroke, then the product is $= 0$.

62. We have yet to consider the product of two strokes; say these are

$$\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3, \text{ and } \mu = \mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3,$$

then we have

$$\begin{aligned}\lambda \,.\, \mu &= (\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3)(\mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3) \\ &= -\{(\lambda_2\mu_3 - \lambda_3\mu_2)\epsilon_1 + (\lambda_3\mu_1 - \lambda_1\mu_3)\epsilon_2 + (\lambda_1\mu_2 - \lambda_2\mu_1)\epsilon_3\},\end{aligned}$$

which is a point regarded as having weight. If we take the two strokes to be

$$\lambda = (xy) = (x_2y_3 - x_3y_2)\eta_1 + (x_3y_1 - x_1y_3)\eta_2 + (x_1y_2 - x_2y_1)\eta_3,$$

and

$$\mu = (xz) = (x_2z_3 - x_3z_2)\eta_1 + (x_3z_1 - x_1z_3)\eta_2 + (x_1z_2 - x_2z_1)\eta_3,$$

then the product is

$$\lambda . \mu = -(x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3) \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

viz. this is the common point x, with a weight = area of triangle $xyz \div$ area $A_1A_2A_3$, or say = the area of triangle formed by the two strokes $\div$ area $A_1A_2A_3$. The product is non-commutative: we, in fact, have $\lambda . \mu = -\mu . \lambda$.

63. The product of three strokes is associative; and it is a scalar. If the three strokes are the sides of a triangle LMN, then the value is $=(LMN)^2 \div (A_1A_2A_3)^2$, where LMN and $A_1A_2A_3$ denote the areas of the triangle LMN and the triangle $A_1A_2A_3$ respectively; and if any stroke instead of being a side of the triangle LMN be a part only of this side, then the value is diminished proportionally; hence the value is $=\dfrac{\rho\sigma\tau}{MN . NL . LM}[(LMN)^2 \div (A_1A_2A_3)^2]$, where ρ, σ, τ are the lengths of the three strokes, MN, NL, LM the lengths of the sides of the triangle LMN, and LMN, $A_1A_2A_3$ are the areas as above.

64. It is to be remarked that the form

$$(*\between \xi, \eta, \zeta)^2, \; = (1, 1, 1, -\cos A_1, -\cos A_2, -\cos A_3 \between \xi, \eta, \zeta)^2$$

is the product of two linear factors; it is

$$= (\xi - \eta \cos A_3 - \zeta \cos A_2)^2 + (\eta \sin A_3 - \zeta \sin A_2)^2,$$

and thus

$$= (\xi - \eta e^{iA_3} - \zeta e^{-iA_2})(\xi - \eta e^{-iA_3} - \zeta e^{iA_2}).$$

This corresponds to the theorem that the distance of two points is $=0$, when the line joining them passes through one of the circular points at infinity; the coordinates of one of these points may be written under the three equivalent forms

$$\begin{aligned} x_1 : x_2 : x_3 = \; & \sin A_1 & : -\sin A_2 e^{-iA_3} : -\sin A_3 e^{-iA_2}, \\ = \; & -\sin A_1 e^{-iA_3} : & \sin A_2 \quad : -\sin A_3 e^{iA_1}, \\ = \; & -\sin A_1 e^{iA_2} : & -\sin A_2 e^{-iA_1} : \quad \sin A_3, \end{aligned}$$

and those of the other are of course obtained therefrom by the change of i into $-i$.

(*To be continued*)*.

[* This paper remains incomplete; no continuation appears to have been prepared by Professor Cayley.]

866.

NOTE ON KIEPERT'S L-EQUATIONS, IN THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 75—77.]

IT appears, by comparison with Klein's paper "Ueber die Transformation u. s. w.," *Math. Annalen*, t. XIV. (1878), see p. 144, that Kiepert's L made use of in the Memoir "Ueber Theilung und Transformation der elliptischen Functionen," *Math. Annalen*, t. XXVI. (1886), pp. 369—454, is, in fact, the square of the multiplier, "für das durch $\sqrt[12]{\Delta}$ normirte Integral," viz. considering the general quartic function $(a, \ldots)(x, 1)^4 = (a, b, c, d, e)(x, 1)^4$, and the transformed function $(a_1, \ldots)(y, 1)^4$, then we have

$$\frac{L^2 \sqrt[12]{\Delta} dx}{\sqrt{(a, \ldots)(x, 1)^4}} = \frac{\sqrt[12]{\Delta_1} dy}{\sqrt{(a_1, \ldots)(y, 1)^4}},$$

where if

$$I = ae - 4bd + 3c^2,$$

$$J = ace - ad^2 - b^2e + 2bcd - c^3,$$

and similarly I_1, J_1, are the invariants of the two functions, then Δ, Δ_1 are the discriminants

$$\Delta = I^3 - 27J^2, \quad \Delta_1 = I_1^3 - 27J_1^2,$$

and the γ_2, γ_3 of Kiepert's equations are

$$\gamma_2 = I \div \sqrt[3]{\Delta}, \quad \gamma_3 = J \div \sqrt{\Delta},$$

whence

$$\gamma_2^3 - 27\gamma_3^2 = 1.$$

In particular, if the forms are

$$1 - x^2 . 1 - k^2x^2, \text{ and } 1 - y^2 . 1 - \lambda^2y^2,$$

and if as usual $k = u^4$, $\lambda = v^4$, and M is the multiplier for the form

$$\frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}} = \frac{M dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}}$$

then we have

$$I = \tfrac{1}{12}(1 + 14u^8 + u^{16}),$$

$$J = \tfrac{1}{216}(1 + u^8)(1 - 34u^8 + u^{16}),$$

$$\Delta = \tfrac{1}{16}u^8(1 - u^8)^4, \quad \Delta_1 = \tfrac{1}{16}v^8(1 - v^8)^4,$$

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2}\,\frac{1 + 14u^8 + u^{16}}{u^{\frac{8}{3}}(1 - u^8)^{\frac{4}{3}}}, \quad \gamma_3 = \tfrac{1}{54}\,\frac{(1 + u^8)(1 - 34u^8 + u^{16})}{u^4(1 - u^8)^2},$$

and thence

$$L^2 = \frac{v^{\frac{8}{3}}(1 - v^8)^{\frac{1}{3}}}{u^{\frac{8}{3}}(1 - u^8)^{\frac{1}{3}}}\,\frac{1}{M},$$

which last equation is the expression for L^2 in terms of the Jacobian symbols u, v, M.

As an easy verification in a particular case, suppose $n = 5$. We have here

$$L^2 = \frac{v^{\frac{8}{3}}(1 - v^8)^{\frac{1}{3}}}{u^{\frac{8}{3}}(1 - u^8)^{\frac{1}{3}}} \cdot \frac{1}{M}, \quad M = \frac{v(1 - uv^3)}{v - u^5}, \quad \left(= \frac{v + u^5}{5u(1 + u^3v)}\right),$$

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0,$$

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2}\,\frac{1 + 14u^8 + u^{16}}{u^{\frac{8}{3}}(1 - u^8)^{\frac{4}{3}}};$$

and it should be possible, by eliminating u, v, M, to deduce hence the L-equation

$$L^{12} + 10L^6 - 12\gamma_2 L^2 + 5 = 0. \quad \text{(Kiepert, p. 428.)}$$

It does not seem in any wise easy to do this in the case of an arbitrary modulus; but writing the modular equation in the form

$$(u^2 - v^2)(u^4 + 6u^2v^2 + v^4) + 4uv(1 - u^4v^4) = 0,$$

we satisfy this by

$$uv - 1 = 0, \quad u^4 + 6u^2v^2 + v^4 = 0,$$

or say by

$$v = \frac{1}{u}, \quad u^8 + 6u^4 + 1 = 0,$$

and the equation may be verified for this particular modulus.

We have

$$1 + 14u^8 + u^{16} = 48u^8, \quad (1 - u^8)^2 = 32u^8,$$

and consequently

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2} \,.\, \frac{48u^8}{u^{\frac{8}{3}}(32u^8)^{\frac{4}{3}}}, \quad = \frac{1}{2\,.\,3}\,2^{\frac{1}{3}}\,\frac{2^4\,.\,3u^8}{u^{\frac{8}{3}}\,.\,2^{\frac{16}{3}}\,.\,u^{\frac{16}{3}}} = 1, \text{ (whence also } \gamma_3 = 0).$$

Moreover

$$M = \frac{\dfrac{1}{u} - \dfrac{u}{u^4}}{\dfrac{1}{u} - u^5} = -\frac{1}{u^2}\frac{1-u^2}{1-u^6} = \frac{-1}{u^2(1+u^2+u^4)},$$

and thence

$$L^2 = \frac{\dfrac{1}{u^{\frac{2}{3}}}\left(1-\dfrac{1}{u^8}\right)^{\frac{1}{3}}}{u^{\frac{2}{3}}(1-u^8)^{\frac{1}{3}}}\frac{1}{M} = \frac{(u^8-1)^{\frac{1}{3}}}{u^4(1-u^8)^{\frac{1}{3}}}\frac{1}{M} = -\frac{1}{u^4 M},$$

that is,

$$L^2 = 1 + u^2 + u^{-2}.$$

But we have

$$u^4 = -3 + 2\sqrt{2},$$

and thence

$$u^2 = \quad i(1-\sqrt{2}), \quad u^{-2} = i(1+\sqrt{2}),$$

and

$$L^2 = \quad 1 + 2i,$$

whence

$$L^{12} + 10L^6 - 12\gamma_2 L^2 + 5 = (117 + 44i) + 10(-11 - 2i) - 12(1 + 2i) + 5 = 0,$$

or the L-equation is satisfied.

Cambridge, 14 *March* 1887.

867.

NOTE ON THE JACOBIAN SEXTIC EQUATION.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 78—84.]

In the Jacobian sextic equation

$$(z-a)^6 - 4a(z-a)^5 + 16b(z-a)^3 - 4c(z-a) + 5b^2 - 4ac = 0,$$

if A, B, C, D, E, F are the square roots (each with a determinate sign) of the roots $z_0, z_1, z_2, z_3, z_4, z_\infty$ of the equation, and if ϵ be an imaginary fifth root of unity, such that $\epsilon + \epsilon^4 - \epsilon^2 - \epsilon^3 = \sqrt{5}$, then we have between the square roots a system of three linear equations

$$\begin{aligned} 0 &= -A\sqrt{5} + B + C + D + E + F, \\ 0 &= B + \epsilon^3 C + \epsilon D + \epsilon^4 E + \epsilon^2 F, \\ 0 &= B + \epsilon^2 C + \epsilon^4 D + \epsilon E + \epsilon^3 F, \end{aligned}$$

see Brioschi's *Funzioni ellittiche* (Milan, 1880), third appendix, p. 402.

The sextic equation is irreducible, and there is thus nothing to distinguish any one square root from any other square root. But in the foregoing system of equations, A is distinguished from the other square roots; hence the system must be one of 6 systems, wherein the place occupied by A is occupied by A, B, C, D, E, F respectively. Observe however that the letters being square roots, it may very well happen, and (as will appear) it does in fact happen, that the passage from the first to another system implies a change of sign in certain of the letters A, B, C, D, E, F.

The selection of one of the square roots as $=A$, does not determine which of the other square roots shall be $=B, C, D, E, F$ respectively: and in fact each system of equations may be represented in 10 different forms: viz. by multiplying the second and third equations by $\epsilon, \epsilon^3, \epsilon^3, \epsilon^4$ respectively, we obtain four new forms: we have thus five forms: and then in each of them transposing the second and third equations, we have five other forms: we have thus in all 10 forms.

Write $ABCDEF$ to denote the foregoing system of three equations

$$\begin{aligned} 0 &= -A\sqrt{5} + B + C + D + E + F, \\ 0 &= \qquad\quad B + \epsilon^3 C + \epsilon D + \epsilon^4 E + \epsilon^3 F, \\ 0 &= \qquad\quad B + \epsilon^2 C + \epsilon^4 D + \epsilon E + \epsilon^3 F; \end{aligned}$$

the 10 forms of the system are

$A\,B\,C\,D\,E\,F$	$A\,B\,F\,E\,D\,C$
$A\,C\,D\,E\,F\,B$	$A\,C\,B\,F\,E\,D$
$A\,D\,E\,F\,B\,C$	$A\,D\,C\,B\,F\,E$
$A\,E\,F\,B\,C\,D$	$A\,E\,D\,C\,B\,F$
$A\,F\,B\,C\,D\,E$	$A\,F\,E\,D\,C\,B,$

where observe that the five forms in the same column are connected by cyclical substitutions on the last five letters; and that the forms in each line are connected by two inversions of the last four letters.

It is hardly necessary to remark that $ABFEDC$ means the system of three equations

$$\begin{aligned} 0 &= -A\sqrt{5} + B + F + E + D + C, \\ 0 &= \qquad\quad B + \epsilon^3 F + \epsilon E + \epsilon^4 D + \epsilon^3 C, \\ 0 &= \qquad\quad B + \epsilon^2 F + \epsilon^4 E + \epsilon D + \epsilon^3 C, \end{aligned}$$

which are the same with the equations of the original form; and similarly that $ACDEFB$ means the system of three equations

$$\begin{aligned} 0 &= -A\sqrt{5} + C + D + E + F + B, \\ 0 &= \qquad\quad C + \epsilon^3 D + \epsilon E + \epsilon^4 F + \epsilon^3 B, \\ 0 &= \qquad\quad C + \epsilon^2 D + \epsilon^4 E + \epsilon F + \epsilon^3 B, \end{aligned}$$

where the first equation is the same with the first equation, and the second and third equations only differ by a factor from the second and third equations respectively of the original form. And similarly as regards the others of the 10 forms.

I say that we have a second system $BCAF\bar{D}\bar{E}$ (where $\bar{D}$, $\bar{E}$ mean $-D$, $-E$ respectively): in fact, this denotes the system of three equations

$$\begin{aligned} 0 &= -B\sqrt{5} + C + A + F - D - E, \\ 0 &= \qquad\quad C + \epsilon^3 A + \epsilon F - \epsilon^4 D - \epsilon^2 E, \\ 0 &= \qquad\quad C + \epsilon^2 A + \epsilon^4 F - \epsilon D - \epsilon^3 E, \end{aligned}$$

which are deducible from the original three equations.

We have, in fact, the identities

$$
\begin{aligned}
-\sqrt{5}\,\{-B\sqrt{5} + C + A + F - D - E\} \\
= 1\,(B + C - A\sqrt{5} + F + D + E) \\
+ 2\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
+ 2\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}
$$

$$
\begin{aligned}
(1 - \epsilon - \epsilon^3 + \epsilon^4)\,\{ C + \epsilon^3 A + \epsilon F - \epsilon^4 D - \epsilon^2 E\} \\
= 1\,(B + C - A\sqrt{5} + F + D + E) \\
+ (\epsilon + \epsilon^4)\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
+ (\epsilon^2 + \epsilon^3)\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}
$$

and

$$
\begin{aligned}
(1 + \epsilon - \epsilon^2 - \epsilon^4)\,\{ C + \epsilon^2 A + \epsilon^4 F - \epsilon D - \epsilon^3 E\} \\
= 1\,(B + C - A\sqrt{5} + F + D + E) \\
+ (\epsilon^2 + \epsilon^3)\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
+ (\epsilon + \epsilon^4)\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}
$$

all which identities are at once verified on recollecting that

$$1 = -\epsilon - \epsilon^2 - \epsilon^3 - \epsilon^4 \text{ and } \sqrt{5} = \epsilon - \epsilon^2 - \epsilon^3 + \epsilon^4.$$

We can now write down the 6 systems, each of them under its 10 forms: these, in fact, are

$A\,B\,C\,D\,E\,F$	$A\,B\,F\,E\,D\,C$
$A\,C\,D\,E\,F\,B$	$A\,C\,B\,F\,E\,D$
$A\,D\,E\,F\,B\,C$	$A\,D\,C\,B\,F\,E$
$A\,E\,F\,B\,C\,D$	$A\,E\,D\,C\,B\,F$
$A\,F\,B\,C\,D\,E$	$A\,F\,E\,D\,C\,B$
$B\,C\,A\,F\,\bar{D}\,\bar{E}$	$B\,C\,\bar{E}\,\bar{D}\,F\,A$
$B\,A\,F\,\bar{D}\,\bar{E}\,C$	$B\,A\,C\,\bar{E}\,\bar{D}\,F$
$B\,F\,\bar{D}\,\bar{E}\,C\,A$	$B\,F\,A\,C\,\bar{E}\,\bar{D}$
$B\,\bar{D}\,\bar{E}\,C\,A\,F$	$B\,\bar{D}\,F\,A\,C\,\bar{E}$
$B\,\bar{E}\,C\,A\,F\,\bar{D}$	$B\,\bar{E}\,\bar{D}\,F\,A\,C$
$C\,D\,A\,B\,\bar{E}\,\bar{F}$	$C\,D\,\bar{F}\,\bar{E}\,B\,A$
$C\,A\,B\,\bar{E}\,\bar{F}\,D$	$C\,A\,D\,\bar{F}\,\bar{E}\,B$
$C\,B\,\bar{E}\,\bar{F}\,D\,A$	$C\,B\,A\,D\,\bar{F}\,\bar{E}$
$C\,\bar{E}\,\bar{F}\,D\,A\,B$	$C\,\bar{E}\,B\,A\,D\,\bar{F}$
$C\,\bar{F}\,D\,A\,B\,\bar{E}$	$C\,\bar{F}\,\bar{E}\,B\,A\,D$

$D\,E\,A\,C\,\bar{F}\,\bar{B}$	$D\,E\,\bar{B}\,\bar{F}\,C\,A$
$D\,A\,C\,\bar{F}\,\bar{B}\,E$	$D\,A\,E\,\bar{B}\,\bar{F}\,C$
$D\,C\,\bar{F}\,\bar{B}\,E\,A$	$D\,C\,A\,E\,\bar{B}\,\bar{F}$
$D\,\bar{F}\,\bar{B}\,E\,A\,C$	$D\,\bar{F}\,C\,A\,E\,\bar{B}$
$D\,\bar{B}\,E\,A\,C\,\bar{F}$	$D\,\bar{B}\,\bar{F}\,C\,A\,E$
$E\,F\,A\,D\,\bar{B}\,\bar{C}$	$E\,F\,\bar{C}\,\bar{B}\,D\,A$
$E\,A\,D\,\bar{B}\,\bar{C}\,F$	$E\,A\,F\,\bar{C}\,\bar{B}\,D$
$E\,D\,\bar{B}\,\bar{C}\,F\,A$	$E\,D\,A\,F\,\bar{C}\,\bar{B}$
$E\,\bar{B}\,\bar{C}\,F\,A\,D$	$E\,\bar{B}\,D\,A\,F\,\bar{C}$
$E\,\bar{C}\,F\,A\,D\,\bar{B}$	$E\,\bar{C}\,\bar{B}\,D\,A\,F$
$F\,A\,E\,\bar{C}\,\bar{D}\,B$	$F\,A\,B\,\bar{D}\,\bar{C}\,E$
$F\,E\,\bar{C}\,\bar{D}\,B\,A$	$F\,E\,A\,B\,\bar{D}\,\bar{C}$
$F\,\bar{C}\,\bar{D}\,B\,A\,E$	$F\,\bar{C}\,E\,A\,B\,\bar{D}$
$F\,\bar{D}\,B\,A\,E\,\bar{C}$	$F\,\bar{D}\,\bar{C}\,E\,A\,B$
$F\,B\,A\,E\,\bar{C}\,\bar{D}$	$F\,B\,\bar{D}\,\bar{C}\,E\,A,$

where, as before, the superscript line denotes a change of sign, $\bar{A} = -A$, &c.

As to the formation of this table, observe that we have $ABCDEF$ and $BCAF\bar{D}\bar{E}$; repeating the substitution, we have

$$\begin{array}{c} A\,B\,C\,D\,E\,F \\ B\,C\,A\,F\,\bar{D}\,\bar{E} \\ C\,A\,B\,\bar{E}\,\bar{F}\,D \\ \hline A\,B\,C\,D\,\bar{E}\,\bar{F}; \end{array}$$

viz. we have thus the $CAB\bar{E}\bar{F}D$ which belongs to the third system; but nothing further, for the substitution is periodic of the third order, and the three forms are merely repeated. But if, upon

$$\begin{array}{c} A\,B\,C\,D\,E\,F \\ B\,C\,A\,F\,\bar{D}\,\bar{E}, \end{array}$$

we operate successively with the substitutions which change the upper line into the three forms of the first system

$$A\,D\,E\,F\,B\,C,\quad A\,E\,F\,B\,C\,D,\quad A\,F\,B\,C\,D\,E,$$

then the lower line is changed into the three forms

$$D\,E\,A\,C\,\bar{F}\,\bar{B},\quad E\,F\,A\,D\,\bar{B}\,\bar{C},\quad F\,B\,A\,E\,\bar{C}\,\bar{D},$$

which are forms belonging to the fourth, fifth, and sixth systems respectively. By way of verification, observe that, for instance repeating upon the second line the substitution

$$A\,B\,C\,D\,E\,F,$$

in place of

$$D\,E\,A\,C\,\bar{F}\,B,$$

we obtain

$$C\,\bar{F}\,D\,A\,B\,\bar{E},$$

which is one of the forms of the third system, assumed to have been previously found; and so in other instances.

Reverting to the three equations belonging to the form $ABCDEF$, by subtracting the third from the second equation, we obtain a linear relation between the four square roots C, D, E, F, viz. this is

$$0 = (\epsilon^3 - \epsilon^2)(C - F) + (\epsilon - \epsilon^4)(D - E);$$

and the same equation is obtained by means of the form $ABFEDC$. We thus obtain from the thirty lines of either column of the preceding table, thirty such equations, but obviously the number of such equations should be fifteen, for there can be but one relation between any four square roots C, D, E, F; consequently each equation will be obtained twice; and, in fact, it is clear that the forms $ABCDEF$ and $BAF\bar{D}\bar{E}C$, and so in general any two forms which begin with the same pair of letters, give the same equation. But for greater symmetry, I write down the thirty equations in the order in which they are given by the left-hand column of the table: the equations are

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+(\epsilon - \epsilon^4)$ multiplied by
$C - F$	$D - E$
$D - B$	$E - F$
$E - C$	$F - B$
$F - D$	$B - C$
$B - E$	$C - D$
$A + E$	$F + D$
$F - C$	$-D + E$
$-D - A$	$-E - C$
$-E - F$	$C - A$
$C + D$	$A - F$
$A + F$	$B + E$
$B - D$	$-E + F$
$-E - A$	$-F - D$
$-F - B$	$D - A$
$D + E$	$A - B$

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+ (\epsilon - \epsilon^4)$ multiplied by
$A + B$	$C + F$
$C - E$	$-F + B$
$-F - A$	$-B - E$
$-B - C$	$E - A$
$E + F$	$A - C$
$A + C$	$D + B$
$D - F$	$-B + C$
$-B - A$	$-C - F$
$-C - D$	$F - A$
$F + B$	$A - D$
$E - B$	$-C + D$
$-C - A$	$-D - B$
$-D - E$	$B - A$
$B + C$	$A - E$
$A + D$	$E + C$

where it is to be observed that, adding together the five equations given by any one of the systems, we obtain the identical result $0 = 0$.

I write down the 15 equations in a different order, in some cases changing the sign of the whole equation, as follows

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+ (\epsilon - \epsilon^4)$ multiplied by
$A + C$	$B + D$
$B + C$	$A - E$
$A + B$	$C + F$
$D + E$	$A - B$
$B + F$	$A - D$
$A + F$	$B + E$
$A + D$	$C + E$
$C + D$	$A - F$
$E + F$	$A - C$
$A + E$	$D + F$
$B - E$	$C - D$
$F - D$	$B - C$
$E - C$	$F - B$
$D - B$	$E - F$
$C - F$	$D - E$

The first three of these equations, or writing $\lambda = \epsilon^3 - \epsilon^2$, $\mu = \epsilon - \epsilon^4$, say

$$\lambda(A + C) + \mu(B + D) = 0,$$
$$\lambda(B + C) + \mu(A - E) = 0,$$
$$\lambda(A + B) + \mu(C + F) = 0,$$

constitute the entire system of independent linear relations between the square roots A, B, C, D, E, F. The coefficients λ, μ are such that

$$\lambda^2 - \mu^2 = \lambda\mu\,(= \epsilon - \epsilon^2 - \epsilon^3 + \epsilon^4, = \sqrt{5}),$$

and it is hence easy to verify that the remaining twelve equations can be deduced from these by the elimination of one or two of the square roots A, B, C. For instance, to eliminate A from the first and second equations, multiplying by $-\mu$, λ and adding, we obtain

$$(-\lambda\mu + \lambda^2)\,C + (-\mu^2 + \lambda^2)\,B - \mu^2 D - \lambda\mu E = 0,$$

that is,

$$\mu^2 C + \qquad \lambda\mu B - \mu^2 D - \lambda\mu E = 0,$$

or finally

$$\lambda(B - E) + \mu(C - D) = 0,$$

which is one of the equations. And so again eliminating A from the first and third equations, we find

$$\lambda(B - C) + \mu(C + F - B - D) = 0,$$

that is,

$$(\lambda - \mu)(B - C) + \mu(F - D) = 0,$$

or multiplying by λ,

$$\mu^2(B - C) + \lambda\mu(F - D) = 0,$$

that is, finally

$$\lambda\,(F - D) + \mu(B - C) = 0,$$

which is one of the equations.

Cambridge, 21 *March* 1887.

868.

ON THE INTERSECTION OF CURVES.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 85—90.]

It is only recently that I have studied Bacharach's paper "Ueber den Cayley'schen Schnittpunktsatz," *Math. Ann.*, t. XXVI. (1885), pp. 275—299: his theorem in regard to the case where the δ points lie on a curve of the order $\gamma-3$ is a very interesting and valuable one, but I consider it rather as an addition than as a correction to my original theorem; and I cannot by any means agree that the method by counting of constants is to be rejected as not trustworthy; on the contrary, it seems to me to be the proper foundation of the whole theory; it must of course be employed with due consideration of special cases. I reproduce the theorem in what appears to me the complete form.

Writing with Bacharach

$$r \gtreqless m,\ r \gtreqless n,\ r \lesseqgtr m+n-3,\ \gamma=m+n-r,\ \delta=\tfrac{1}{2}(\gamma-1)(\gamma-2),$$

and assuming $n \gtreqless m$, (these equations and inequalities are to be attended to throughout the present paper), I consider two curves of the orders m, n respectively meeting in δ points B, and in $(mn-\delta)$ points A; and I state the theorem as follows:

1°. The $mn-\delta$ points A are in general such that a curve of the order r, which passes through $mn-\delta-1$ of these points, does not of necessity pass through the remaining point; and in this case the general curve of the order r, which passes through the $mn-\delta$ points A, has for its form of equation

$$0=L_{r-m}P_m+M_{r-n}Q_n,$$

where $P_m=0$, $Q_n=0$ are the equations of the given curves and L_{r-m}, M_{r-n} denote functions of the orders $r-m$, $r-n$ respectively; and it thus appears that the curve of the order r through the $mn-\delta$ points A passes also through the δ points B.

2°. If however the δ points B are on a curve of the order $\gamma-3$, then the $mn-\delta$ points A are a system such that every curve of the order r passing through $mn-\delta-1$ of these points passes through the remaining point; and in this case the general curve of the order r, which passes through the $mn-\delta$ points A, has for its form of equation

$$0=\Omega_r+L_{r-m}P_m+M_{r-n}Q_n,$$

$\Omega_r=0$ is a particular curve through the $mn-\delta$ points A, which does not go through any of the points B; and consequently the curve of the order r does not pass through any of the points B.

For the proof of the theorem I premise as follows:

A curve of the order r depends upon $\frac{1}{2}r(r+3)$ constants, or to use a convenient expression, its Postulandum is $=\frac{1}{2}r(r+3)$: if the curve has to pass through a given point, this imposes a single relation upon the constants, or say the Postulation is $=1$; similarly, if the curve has to pass through k given points, this imposes k relations, or say the Postulation is $=k$. The points may be however a special system, for instance, they may be such that every curve of the order r which passes through $k-1$ of the points, will pass through the remaining point; the Postulation is in this case $=k-1$; and so in other cases. Assuming the Postulation of the k points to be $=k$, then the Postulandum of a curve of the order r through the k points is $=\frac{1}{2}r(r+3)-k$. I stop to remark that the Postulation has reference to the particular curve or other entity in question; thus in the case of a curve passing through k points, the Postulation for a curve of a certain order may be $=k$, and for a curve of a different order it may be less than k.

Considering now, as above, two given curves of the orders m and n intersecting in the $mn-\delta$ points A and the δ points B, then assuming that the $mn-\delta$ points are not a special system, viz. that their Postulation in regard to a curve of the order r is $=mn-\delta$, the Postulandum of a curve of the order r through the $mn-\delta$ points is

$$=\tfrac{1}{2}r(r+3)-mn+\delta,$$

which is

$$=\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2)-1,$$

viz. this is identically true when for δ we write its value

$$=\tfrac{1}{2}(m+n-r-1)(m+n-r-2).$$

But we have through the $mn-\delta$ points A, a curve

$$L_{r-m}P_m+M_{r-n}Q_n=0$$

with the proper Postulandum: viz. L_{r-m} contains $\frac{1}{2}(r-m+1)(r-m+2)$ constants, M_{r-n} contains $\frac{1}{2}(r-n+1)(r-n+2)$ constants, and there is a diminution -1 for the constant which divides out; hence this is the general equation of the curve of the order r through the $mn-\delta$ points A; and the curve passes through the remaining δ points B.

In the case where the δ points B are on a curve of the order $\gamma-3$, (observe that this is a single condition imposed on the δ, $=\frac{1}{2}(\gamma-1)(\gamma-2)$ points, for a curve of the order $(\gamma-3)$ can be drawn through $\frac{1}{2}\gamma(\gamma-3)$ points), it is to be shown that the Postulation of the $mn-\delta$ points A is $=mn-\delta-1$; for, this being so, the Postulandum of the curve of the order r through the $(mn-\delta)$ points A will be

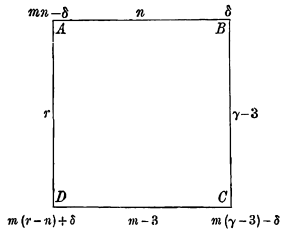
$$=\tfrac{1}{2}r(r+3)-mn+\delta+1,$$

and the equation of the curve will no longer be of the foregoing form, but it will be of the form

$$\Omega_r+L_{r-m}P_m+M_{r-n}Q_n=0,$$

$\Omega_r=0$ being a particular curve through the $mn-\delta$ points A, which does not pass through any of the points B. The proof depends on the theory of Residuation: which for the present purpose may be presented under the following form.

Let A, B, ... denote systems of points upon a given Basis-curve, for instance the foregoing curve $P_m=0$, of the order m. And let $A=ci$ denote that the points A are the complete intersection of the basis-curve by some other curve; (this implies that the number of the points is $=km$, a multiple of m, and the intersecting curve is then of course a curve of the order k). It is clear that, if $A=ci$, and $B=ci$, then also $A+B=ci$. But conversely we have the theorem that, if $A+B=ci$ and $A=ci$, then also $B=ci$. And we at once deduce the further theorem: if $A+B=ci$, $B+C=ci$, $C+D=ci$, then also $A+D=ci$. For the first and third relations give $A+B+C+D=ci$, and the second relation then gives $A+D=ci$.



Starting now (see the diagram) with the δ points B on a curve of the order $\gamma-3$, suppose that we have through these points the basis-curve $P_m=0$ of the order m, and another given curve $Q_n=0$, of the order n; and let these besides meet in the $mn-\delta$ points A. Let the curve of the order $\gamma-3$ besides meet the basis-curve in the $m(\gamma-3)-\delta$ points C; and through these let there be drawn a curve of the order $m-3$, which besides meets the basis-curve in the $m(r-n)+\delta$ points D. We have here $A+B=ci$, $B+C=ci$, $C+D=ci$; consequently $A+D=ci$, that is, the $mn-\delta$ points A and the $m(r-n)+\delta$ points D lie on a curve of the order r. The curve of the order $m-3$ passes through the $m(\gamma-3)-\delta$ points C; its Postulandum is thus

$$=\tfrac{1}{2}m(m-3)-m(\gamma-3)+\delta,$$

which is

$$=\tfrac{1}{2}(r-n+1)(r-n+2).$$

In fact, substituting for γ, δ their values $=m+n-r$, and $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ respectively, this equation is satisfied identically. The system of the $m(r-n)+\delta$ points D thus depends upon $\frac{1}{2}(r-n+1)(r-n+2)$ constants, or say the Postulandum of the points D is $=\frac{1}{2}(r-n+1)(r-n+2)$. It follows that the curve of the order r through the $(mn-\delta)$ points A and the $m(r-n)+\delta$ points D *cannot* have an equation of the form

$$L_{r-m}P_m+M_{r-n}Q_n=0;$$

for the intersections of this curve with the basis-curve $P_m=0$ are given by the equation $M_{r-n}Q_n=0$, which contains only the

$$\tfrac{1}{2}(r-n+1)(r-n+2)-1$$

constants of M_{r-n} (one constant of course divides out, giving the diminution -1). The equation must have the more general form

$$\Omega_r+L_{r-m}P_m+M_{r-n}Q_n=0;$$

and it thus appears that the Postulation of the $mn-\delta$ points A, instead of being $=mn-\delta$, must be $=mn-\delta-1$. This completes the proof.

I notice that, combining the last-mentioned identity

$$\tfrac{1}{2}m(m-3)-m(\gamma-3)+\delta=\tfrac{1}{2}(r-n+1)(r-n+2)$$

with the like identity

$$\tfrac{1}{2}n(n-3)-n(\gamma-3)+\delta=\tfrac{1}{2}(r-m+1)(r-m+2),$$

we obtain

$$\tfrac{1}{2}m(m-3)+\tfrac{1}{2}n(n-3)-(m+n)(\gamma-3)+2\delta$$
$$=\tfrac{1}{2}(r-n+1)(r-n+2)+\tfrac{1}{2}(r-m+1)(r-m+2),$$

and consequently, referring to a former result, the left-hand side should be

$$=\tfrac{1}{2}r(r+3)-mn+\delta+1;$$

substituting for γ, δ their values, this is at once verified.

As appears by what precedes, Bacharach's special case is that in which the δ, $=\frac{1}{2}(\gamma-1)(\gamma-2)$ points B satisfy the single condition of lying on a curve of the order $\gamma-3$. We may have between the points B more than a single relation; in particular, the points B may be such as to include among themselves the complete intersection of two curves of the orders a, b respectively $(ab \gtreqless \delta)$: this will be the case, if the given curves are of the form

$$P_m=\lambda_a S_{m-a}-\mu_b R_{m-b},$$
$$Q_n=\lambda_a V_{n-a}-\mu_b U_{n-b},$$

it being understood, here and in what follows, that the values of a, b are such that the suffixes are none of them negative.

The two curves here intersect in the ab points ($\lambda_a = 0$, $\mu_b = 0$), and in the $mn - ab$ points

$$\left\| \begin{matrix} \lambda_a, & R_{m-b}, & U_{n-b} \\ \mu_b, & S_{m-a}, & V_{n-a} \end{matrix} \right\| = 0,$$

say the $(mn - \delta)$ points A are $mn - ab - \Theta$ of the last-mentioned points, and the δ points B are the remaining Θ points together with the ab points. Here the general form of the curve of the order r passing through the $mn - ab$ points, and therefore through the $mn - \delta$ points A, is

$$\left| \begin{matrix} L_{r-m-n+a+b}, & M_{r-n}, & N_{r-m} \\ \lambda_a \quad , & R_{m-b}, & U_{n-b} \\ \mu_b \quad , & S_{m-a}, & V_{n-a} \end{matrix} \right| = 0,$$

where $L_{r-m-n+a+b}$, M_{r-n}, N_{r-m} are arbitrary functions of the orders indicated by the respective suffixes. The theory in regard to the number of constants is of course altogether different from that which belongs to the case of the general functions P_m, Q_n; and it is probable that much interesting theory would present itself in the consideration of particular cases.

Cambridge, 22 *March* 1887.

869.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *American Journal of Mathematics*, vol. IX. (1887), pp. 193—224.]

THE algebraical theory of the Transformation of Elliptic Functions was established by Jacobi in a remarkably simple and elegant form, but it has not hitherto been developed with much completeness or success. The cases $n=3$ and $n=5$ are worked out very completely in the *Fundamenta Nova* (1829); viz. considering the equation

$$\frac{Mdy}{\sqrt{1-y^2 . 1-\lambda^2 y^2}} = \frac{dx}{\sqrt{1-x^2 . 1-k^2 x^2}},$$

($k=u^4$, $\lambda=v^4$; say this is the $Mk\lambda$- or Muv-form), Jacobi finds, in the two cases respectively, a modular equation between the fourth roots u, v, say the uv-modular equation, and, as rational functions of u, v, the value of M and the values of the coefficients of the several powers of x in the numerator and denominator of the fraction which gives the value of y; but there is no attempt at a like development of the general case. I shall have occasion to speak of other researches by Jacobi, Brioschi and myself; but I will first mention that my original idea in the present memoir was to develop the following mode of treatment of the theory:

In place of the $Mk\lambda$-form, using the $\rho\alpha\beta$-form

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}}$$

(I write for greater convenience 2α, 2β in place of the α of Jacobi and Brioschi and the β of Brioschi), we can, by expanding each side in a series, integrating, and reverting the resulting series for y, obtain y in the form

$$y = \rho x (1 + \Pi_1 x^2 + \Pi_2 x^4 + \ldots),$$

where $\Pi_1, \Pi_2, \Pi_3, \ldots$ denote given functions of ρ, α, β. Taking n odd and $=2s+1$ we assume for y an expression

$$y = \frac{x(A_s + A_{s-1}x^2 + \ldots + A_1 x^{2s-2} + x^{2s})}{1 + A_1 x^2 + \ldots + A_{s-1}x^{2s-2} + A_s x^{2s}},$$

where the last coefficient A_s is at once seen to be $=\rho$. Comparing with the series-value $y = \rho x(1 + \Pi_1 x^2 + \Pi_2 x^4 + \ldots)$, we have an infinite series of equations. The first of these is, in fact, $A_s = \rho$; the next $(s-1)$ equations give linearly $A_1, A_2, \ldots, A_{s-1}$ in terms of the coefficients Π; that is, of ρ, α, β: the two which follow serve in effect to determine ρ, β as functions of α: and then, ρ and β having these values, all the remaining equations will be satisfied identically.

The process is an eminently practical one, so far as regards the determination of the coefficients $A_1, A_2, \ldots, A_{s-1}$ as functions of ρ, α, β; it is less so, and requires eliminations more or less complicated, as regards the determination of the relations between ρ, α, β. As to this, it may be remarked that the problem is not so much the determination of the equation between ρ and α (or say the $\rho\alpha$-multiplier equation, or simply the $\rho\alpha$-equation), and of the equation between β, α (or say the $\alpha\beta$-modular equation, or simply the $\alpha\beta$-equation), as it is to determine the complete system of relations between ρ, α, β; treating these as coordinates, we have what may be called the multiplier-modular-curve, or say the MM-curve, and the relations in question are those which determine this curve.

In the absence of special exceptions, it follows from general principles that the coefficients $A_1, A_2, \ldots, A_{s-1}$, *quà* rational functions of ρ, α, β, must also be rational functions of α, β or of α, ρ; and I think it may be assumed that this is the case; the method, however, affords but little assistance towards thus expressing them.

In connexion with the foregoing theory, I consider the solutions of the problem of transformation given by Jacobi's partial differential equation ("Suite de Notices sur les Fonctions elliptiques," *Crelle*, t. IV. (1829), pp. 185—193), and by what I call the Jacobi-Brioschi differential equations. The first and third of these were obtained by Jacobi in the memoir*, "De functionibus ellipticis Commentatio," *Crelle*, t. IV. (1829), pp. 371—390 (see p. 376); but the second equation, which completes the system, was, I believe, first given by Brioschi in the second appendix to his translation of my *Elliptic Functions: Trattato elementare delle Funzioni ellittiche:* Milan, 1880. I had, strangely enough, overlooked the great importance of these equations. I shall have occasion also to refer to results, and further develop the theory contained in my memoir, "On the Transformation of Elliptic Functions," *Phil. Trans.*, t. CLXIV. (1874), pp. 397—456, [578], and the addition thereto, *Phil. Trans.*, t. CLXXXIX. (1878), pp. 419—424, [692].

I remark that, while I have only worked out the formulæ for the cases $n=3$ and $n=5$, and a few formulæ for the case $n=7$, the memoir is intended to be a contribution to the general theory of the $\rho\alpha\beta$-transformation; I hope to be able to complete the theory for the case $n=7$.

[* *Ges. Werke*, bd. I., pp. 295—318; in particular, p. 303.]

Comparison of the Mkλ- and ραβ-Forms. The Modular and Multiplier Equations.
Art. Nos. 1 to 12.

1. The equation

$$\frac{Mdy}{\sqrt{1-y^2 \,.\, 1-\lambda^2 y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2 x^2}},$$

if we write therein

$$x = \frac{x}{\sqrt{k}}, \quad = \frac{x}{u^2}; \quad y = \frac{y}{\sqrt{\lambda}}, \quad = \frac{y}{v^2},$$

becomes

$$\frac{Mdy}{v^2\sqrt{1-(v^4+v^{-4})\,y^2+y^4}} = \frac{dx}{u^2\sqrt{1-(u^4+u^{-4})\,x^2+x^4}};$$

viz. this is

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$$

if only

$$2\alpha = u^4 + \frac{1}{u^4}, \quad 2\beta = v^4 + \frac{1}{v^4}, \quad \rho = \frac{v^2}{u^2 M}.$$

2. We have a uv-modular equation, and, as shown in my Transformation Memoir*, p. 450, this may be converted into a u^4v^4-modular equation; in particular, $n=3$, the equation is

$$y^4 + 6x^2y^2 + x^4 - 4xy\,(4x^2y^2 - 3x^2 - 3y^2 + 4) = 0,$$

where x, y denote u^4, v^4 respectively; say the equation is

$$F(x,\, y),\ = x^4 + x^3\,(-16y^3 + 12y) + x^2\,(6y^2) + x\,(12y^3 - 16y) + y^4,\ = 0.$$

From the equation $F(x,\, y) = 0$, we derive

$$x^{-2}F(x,\, y)\,.\,x^{-2}F(x,\, y^{-1}) = 0;$$

say this is

$$(Ax^2 + Bx + C + Dx^{-1} + Ex^{-2})\,(A'x^2 + B'x + C' + D'x^{-1} + E'x^{-2}) = 0,$$

viz. the equation is

$$AA'x^4 + (AB' + A'B)\,x^3 + \ldots + EE'x^{-4} = 0,$$

where, by reason of the symmetry of $F(x,\, y)$, the coefficients AA', EE' of x^4, x^{-4}, those of x^3, x^{-3}, &c., have equal values; the form thus is

$$\mathfrak{A}\,(x^4 + x^{-4}) + \mathfrak{B}\,(x^3 + x^{-3}) + \mathfrak{C}\,(x^2 + x^{-2}) + \mathfrak{D}\,(x + x^{-1}) + \mathfrak{E} = 0,$$

where $x^4 + x^{-4}$, $x^3 + x^{-3}$, $x^2 + x^{-2}$, are given functions of $x + x^{-1}$, $= 2\alpha$; viz. we have

$$\begin{aligned} x + x^{-1} &= 2\alpha, \\ x^2 + x^{-2} &= 4\alpha^2 - 2, \\ x^3 + x^{-3} &= 8\alpha^3 - 6\alpha, \\ x^4 + x^{-4} &= 16\alpha^4 - 16\alpha^2 + 2. \end{aligned}$$

[* This Collection, vol. IX., p. 170.]

The coefficients $\mathfrak{A}, \mathfrak{B}, \ldots$ are in like manner expressible as functions of $y+y^{-1}, =2\beta$; thus we have $\mathfrak{A}=1$,

$$\begin{aligned}\mathfrak{B} &= AB'+A'B \\ &= -16(y^3+y^{-3})+12(y+y^{-1}), \ =-16(8\beta^3-6\beta)+12\,.\,2\beta;\end{aligned}$$

or, finally, $\mathfrak{B}=-128\beta^3+120\beta$; and so for the other coefficients. The numerical coefficients contain, all of them, the factor 16; and, throwing this out, we obtain, for $n=3$, the $\alpha\beta$-modular equation in the form

	α^4	α^3	α^2	α	1	
β^4					+ 1	
β^3		− 64		+ 60		
β^2			− 186		+ 192	= 0,
β		+ 60		− 64		
1	+ 1		+ 192		− 192	
	+ 1	− 4	+ 6	− 4	+ 1	

where observe that the form is symmetrical as regards α, β; and, further, that the sums of the numerical coefficients in the lines or columns are the binomial coefficients $1, -4, +6, -4, +1$. Observe, further, that the sums in the direction of the sinister diagonal are $-64, -64, +320, -192$; viz. dividing by -64, it thus appears that, writing $\beta=\alpha$, the equation becomes

$$\alpha^6+\alpha^4-5\alpha^2+3=0;$$

that is, $(\alpha^2-1)^2(\alpha^2+3)=0$.

Again, writing $\beta=-\alpha$, then dividing by 16, the equation becomes

$$4\alpha^6-19\alpha^4+28\alpha^2-12=0;$$

that is,

$$(4\alpha^2-3)(\alpha^2-2)^2=0.$$

3. So also, for $n=5$, we have the u^4v^4-modular equation in the form

$$\left.\begin{aligned}&x^6+655x^4y^2+655x^2y^4+y^6-640x^2y^2-640x^4y^4\\ &\quad+xy(-256+320x^2+320y^2-70x^4-660x^2y^2-70y^4\\ &\qquad+320x^4y^2+320x^2y^4-256x^4y^4)\end{aligned}\right\}=0;$$

and in precisely the same manner, we obtain the $\alpha\beta$-modular equation; viz. casting out a factor 64, this is

	β^6	β^5	β^4	β^3	β^2	β	1	
α^6							$+1$	
α^5		-4096		$+6400$		-2310		
α^4			$+69120$		-172785		$+103680$	
α^3		$+6400$		-133140		$+126720$		$=0,$
α^2			-172785		$+276480$		-103680	
α		-2310		$+126720$		-124416		
1	$+1$		$+103680$		-103680			
	$+1$	-6	$+15$	-20	$+15$	-6	$+1$	

where the form is symmetrical as regards α, β; the sums of the numerical coefficients in the lines or columns are $1, -6, +15, -20, +15, -6, +1$. The sums in the direction of the sinister diagonal all divide by -4096; viz. throwing out this factor, we have, for $\beta=\alpha$, the equation

$$\alpha^{10}-20\alpha^8+118\alpha^6-180\alpha^4+81\alpha^2=0\,;$$

that is,

$$\alpha^2(\alpha^2-1)^2(\alpha^2-9)^2=0.$$

If $\beta=-\alpha$, the coefficients divide by 64; and throwing out this factor, the equation is

$$64\alpha^{10}+880\alpha^8-3247\alpha^6+3600\alpha^4-1296\alpha^2=0\,;$$

that is,

$$\alpha^2(\alpha^4+16\alpha^2-16)(8\alpha^2-9)^2=0.$$

4. We have a Mu-multiplier equation of the form $F\left(\frac{1}{M},\ 2u^3-1\right)=0$ (see Memoir*, pp. 420—422), but we cannot, by the preceding formulæ, deduce thence a $\rho\alpha$-multiplier equation; in fact, writing therein $\frac{1}{M}=\frac{u^2\rho}{v^2}$, the resulting equation is $F\left(\frac{u^2\rho}{v^2},\ 1-2u^3\right)=0$, which is a $\rho\alpha$-multiplier equation only on the assumption that $1-2u^3$, u^2 and v^2 are therein regarded as given functions of α. But it is very

[* This Collection, vol. x., pp. 334, 335.]

remarkable that the $\rho\alpha$-equation, in fact, is $F(\rho, \alpha) = 0$. To prove this, assume that the equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho\, dx}{\sqrt{1-2\alpha x^2+x^4}}$$

has a $\rho\alpha$-multiplier equation $F(\rho, \alpha) = 0$. Starting from the equation

$$\frac{M\mathrm{dy}}{\sqrt{1-\mathrm{y}^2 . 1-\lambda^2\mathrm{y}^2}} = \frac{\mathrm{dx}}{\sqrt{1-\mathrm{x}^2 . 1-k^2\mathrm{x}^2}},$$

we may, by effecting on each side a quadric transformation, convert this into

$$\frac{dy}{\sqrt{1-2(2v^3-1)y^2+y^4}} = \frac{M^{-1}dx}{\sqrt{1-2(2u^3-1)x^2+x^4}};$$

and this being so, we have, between M^{-1} and $2u^3-1$, the relation

$$F\left(\frac{1}{M},\ 2u^3-1\right) = 0;$$

or, conversely, if this be the form of the Mu-multiplier equation, then the $\rho\alpha$-multiplier equation is $F(\rho, \alpha) = 0$.

5. The quadric transformations are

$$x = \frac{\sqrt{1-\mathrm{x}^2}}{\mathrm{x}\sqrt{1-k^2\mathrm{x}^2}},\quad y = \frac{\sqrt{1-\mathrm{y}^2}}{\mathrm{y}\sqrt{1-\lambda^2\mathrm{y}^2}}.$$

We have then only to show that

$$\frac{dx}{\sqrt{1-2(2u^3-1)x^2+x^4}} = \frac{\mathrm{dx}}{\sqrt{1-\mathrm{x}^2 . 1-k^2\mathrm{x}^2}};$$

for then, in like manner,

$$\frac{dy}{\sqrt{1-2(2v^3-1)y^2+y^4}} = \frac{\mathrm{dy}}{\sqrt{1-\mathrm{y}^2 . 1-\lambda^2\mathrm{y}^2}},$$

and we pass from the assumed differential relation between x, y to the above-mentioned differential equation between x, y.

6. For the quadric transformation between x, x, write

$$\theta^{\frac{1}{2}} = k - ik',\quad \theta^{-\frac{1}{2}} = k + ik',$$

$\left(\text{whence also } \theta = \dfrac{k-ik'}{k+ik'}\right)$, and therefore

$$\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} = 2k,\quad \theta + \theta^{-1} = 2k^2 - 2k'^2 = 2(2k^2-1),\quad = 2(2u^3-1);$$

we have

$$1 - \theta x^2 = 1 - \theta\frac{1-\mathrm{x}^2}{\mathrm{x}^2(1-k^2\mathrm{x}^2)},\quad = \frac{1}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}\{-\theta + (\theta+1)\mathrm{x}^2 - k^2\mathrm{x}^4\},$$

$$= \frac{-\theta}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}(1-\theta^{-\frac{1}{2}}k\mathrm{x}^2)^2;$$

and similarly,

$$1-\theta^{-1}x^2=\frac{-\theta^{-1}}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}(1-\theta^{\frac{1}{2}}k\mathrm{x}^2)^2.$$

Consequently,

$$(1-\theta x^2)(1-\theta^{-1}x^2)=1-2(2w^8-1)x^2+x^4=\frac{1}{\mathrm{x}^4(1-k^2\mathrm{x}^2)^2}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4)^2;$$

or say

$$\sqrt{1-2(2w^8-1)x^2+x^4}=\frac{1}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4).$$

Moreover,

$$dx=\frac{d\mathrm{x}}{\mathrm{x}^2(1-\mathrm{x}^2)^{\frac{1}{2}}(1-k^2\mathrm{x}^2)^{\frac{3}{2}}}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4),$$

and thence the required equation

$$\frac{dx}{\sqrt{1-2(1-2w^8)x^2+x^4}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2}.\sqrt{1-k^2\mathrm{x}^2}};$$

this completes the proof.

7. Thus, referring to the Mu-equations given in the place referred to, we obtain the following $\rho\alpha$-multiplier equations. When $n=3$, we have

$$\rho^4-6\rho^2-8\alpha\rho-3=0.$$

This may be written in the forms

$$\begin{aligned}8\alpha\rho&=\rho^4-6\rho^2-3,\\8(\alpha+1)\rho&=(\rho-1)^3(\rho+3),\\8(\alpha-1)\rho&=(\rho+1)^3(\rho-3).\end{aligned}$$

Next, for $n=5$, we have $\rho^6-10\rho^5+35\rho^4-60\rho^3+55\rho^2+(38-64\alpha^2)\rho+5=0$.

This may be written in the two forms

$$64\alpha^2\rho=(\rho^2-4\rho-1)^2(\rho^2-2\rho+5)$$

and

$$64(\alpha^2-1)\rho=(\rho-1)^5(\rho-5).$$

And, for $n=7$, we have

$$\rho^8-28\rho^6-112\alpha\rho^5-210\rho^4-224\alpha\rho^3+(-1484+1344\alpha^2)\rho^2+(-560\alpha+512\alpha^3)\rho+7=0.$$

8. The relation between ρ and β, or say the $\rho\beta$-multiplier equation, may be obtained by a known property of elliptic functions; viz. writing $\rho\sigma=\pm n$ (the sign is $-$ for $n=3$, $n=7$, or generally for any prime value $4p+3$: and it is $+$ for $n=5$ and generally for any prime value $=4p+1$), then we have between σ, β the same relation as between ρ, α. Thus, if $n=3$, $\sigma=-\frac{3}{\rho}$, for ρ, α writing σ, β, the equation is $\sigma^4-6\sigma^2-8\beta\sigma-3=0$; or, as this may be written,

$$\rho^4+8\beta\rho^3+18\rho^2-27=0;$$

and so for the other cases; but it is perhaps more convenient to retain the σ; thus, if $n=5$, $\sigma=\frac{5}{\rho}$, we have

$$\sigma^6-10\sigma^5+35\sigma^4-60\sigma^3+55\sigma^2+(38-64\beta^2)\sigma+5=0.$$

9. We are hence able to express β as a rational function of ρ, α. We, in fact, have

$$8\alpha = \frac{1}{\sqrt{\rho}}(\rho^2 - 4\rho - 1)\sqrt{\rho^2 - 2\rho + 5}, \quad 8\beta = -\frac{1}{\sqrt{\sigma}}(\sigma^2 - 4\sigma - 1)\sqrt{\sigma^2 - 2\sigma + 5},$$

(the signs must be opposite), and then for σ, substituting its value $= \frac{1}{5}\rho$, and observing that $\sigma^2 - 2\sigma + 5$ is thus $= \frac{5}{\rho^2}(\rho^2 - 2\rho + 5)$, we find

$$\frac{\beta}{\alpha} = \frac{\rho^2 + 20\rho - 25}{\rho^2(\rho^2 - 4\rho - 1)},$$

which is the required formula.

Observe that, for $\rho = \sigma = \sqrt{5}$, the formulæ with the sign $-$, as above, give $\beta = -\alpha$, whereas with the sign $+$ they would have given $\beta = \alpha$. For the value in question, $\rho = \sqrt{5}$, the equation

$$64\alpha^2 = \frac{1}{\rho}(\rho^2 - 4\rho - 1)^2(\rho^2 - 2\rho + 5),$$

gives

$$64\alpha^2 = \frac{1}{\sqrt{5}}\,16\,(1 - \sqrt{5})^2(10 - 2\sqrt{5});$$

that is,

$$\alpha^2 = \frac{1}{\sqrt{5}}(3 - \sqrt{5})(5 - \sqrt{5}), \quad = (3 - \sqrt{5})(\sqrt{5} - 1);$$

that is, $\alpha^2 = -8 + 4\sqrt{5}$, or $\alpha^4 + 16\alpha^2 - 16 = 0$; it appears, *ante* No. 3, that this value belongs to the case $\beta = -\alpha$ and not to $\beta = \alpha$.

10. But there is another way of arriving at a formula containing β. Starting from Jacobi's equation

$$nM^2 = \frac{\lambda\lambda'^2}{kk'^2} \cdot \frac{dk}{d\lambda},$$

and introducing for λ, λ', k, k', M their values in terms of u, v, we have

$$\frac{nv^4}{u^4\rho^2} = \frac{v^4(1 - v^8)}{u^4(1 - u^8)}\frac{u^3du}{v^3dv},$$

that is,

$$\frac{dv}{du} = \frac{\rho^2}{n}\frac{u^3(1 - v^8)}{v^3(1 - u^8)};$$

but, from the values of α, β, we find

$$\frac{dv}{du} = \frac{v^5(1 - u^8)}{u^5(1 - v^8)}\frac{d\beta}{d\alpha},$$

and, combining these results,

$$\frac{d\beta}{d\alpha} = \frac{\rho^2}{n}\frac{u^8}{v^8} \cdot \frac{(1 - v^8)^2}{(1 - u^8)^2}, \quad = \frac{\rho^2}{n}\frac{(v^4 - v^{-4})^2}{(u^4 - u^{-4})^2};$$

that is,

$$\frac{d\beta}{d\alpha} = \frac{\rho^2}{n}\frac{\beta^2-1}{\alpha^2-1}.$$

We have, consequently,

$$\frac{d\beta}{\beta^2-1} = \frac{\rho^2 d\alpha}{n(\alpha^2-1)},$$

and therefore

$$\tfrac{1}{2}\log\frac{\beta-1}{\beta+1} = \frac{1}{n}\int\frac{\rho^2 d\alpha}{\alpha^2-1},$$

where ρ^2 must be regarded as a function of α, or α of ρ; and from the form of the equation, it appears that the integral must be expressible as the logarithm of an algebraic function of ρ, α.

11. Thus, when $n=3$, we have

$$8\alpha = \rho^3 - 6\rho - \frac{3}{\rho};$$

whence

$$8\frac{d\alpha}{d\rho} = 3\rho^2 - 6 + \frac{3}{\rho^2}, \quad = \frac{3}{\rho^2}(\rho^2-1)^2,$$

and thence easily

$$\tfrac{1}{2}\log\frac{\beta-1}{\beta+1} = \int\frac{8\rho^2 d\rho}{\rho^2-1\,.\,\rho^2-9}, \quad = -\int\frac{d\rho}{\rho^2-1} + 9\int\frac{d\rho}{\rho^2-9},$$

$$= -\tfrac{1}{2}\log\frac{\rho-1}{\rho+1} + \tfrac{3}{2}\log\frac{\rho-3}{\rho+3};$$

that is,

$$\frac{\beta-1}{\beta+1} = \frac{(\rho-3)^3(\rho+1)}{(\rho+3)^3(\rho-1)},$$

as may be at once verified.

12. In the case $n=5$, I verify the equation under the form

$$\frac{d\beta}{\beta^2-1} = \frac{\rho^2}{5}\cdot\frac{d\alpha}{\alpha^2-1}.$$

From the equations

$$64(\alpha^2-1) = \frac{1}{\rho}(\rho-1)^5(\rho-5), \text{ and } 8\alpha = \frac{1}{\sqrt{\rho}}(\rho^2-4\rho-1)\sqrt{\rho^2+2\rho-5},$$

we have

$$\frac{128\alpha\, d\alpha}{\alpha^2-1} = \frac{5(\rho^2-4\rho-1)\,d\rho}{\rho(\rho-1)(\rho-5)},$$

and thence

$$\frac{16 d\alpha}{\alpha^2-1} = \frac{5 d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}}.$$

Similarly, observing the $-$ sign of 8β, we have

$$\frac{16 d\beta}{\beta^2-1} = \frac{-5 d\sigma}{(\sigma-1)(\sigma-5)\sqrt{\sigma(\sigma^2-2\sigma+5)}},$$

whence, substituting for σ its value $=\frac{5}{\rho}$, we have

$$\frac{16d\beta}{\beta^2-1}=\frac{\rho^2\,d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}},\quad =\frac{\rho^2}{5}\cdot\frac{16d\alpha}{\alpha^2-1},$$

which is right.

Connexion of the Mkλ- and ραβ-Theories. Order of Modular Equation.
Art. Nos. 13 to 18.

13. In the Transformation Memoir [578], starting from the equation

$$\frac{1-\mathrm{y}}{1+\mathrm{y}}=\frac{1-\mathrm{x}}{1+\mathrm{x}}\left(\frac{P-Q\mathrm{x}}{P+Q\mathrm{x}}\right)^2,$$

I sought to determine the coefficients of P, Q by the consideration that the relation between x, y remains unaltered when x, y are changed into $\frac{1}{k\mathrm{x}}$, $\frac{1}{\lambda\mathrm{y}}$ respectively. This comes to saying that, when for x, y we write $\frac{x}{u^2}$, $\frac{y}{v^2}$ respectively, the relation between x and y presents itself in the form

$$y=\frac{x(A_s+A_{s-1}x^2+\ldots+A_0x^{2s})}{A_0+A_1x^2+\ldots+A_sx^{2s}},$$

where $s=\frac{1}{2}(n-1)$, as before. For instance, when $n=7$, $P=\alpha+\gamma\mathrm{x}^2$, $Q=\beta+\delta\mathrm{x}^2$.

If, solving for y, we then for x, y write $\frac{x}{u^2}$, $\frac{y}{v^2}$, we find

$$y=\frac{v^2u^{-2}x\left\{(\alpha^2+2\alpha\beta)+(2\alpha\gamma+\beta^2+2\alpha\delta+2\beta\gamma)x^2u^{-4}+(\gamma^2+2\beta\delta+2\gamma\delta)x^4u^{-8}+\delta^2x^6u^{-12}\right\}}{\alpha^2+(2\alpha\gamma+\beta^2+2\alpha\beta)x^2u^{-4}+(\gamma^2+2\beta\delta+2\alpha\delta+2\beta\gamma)x^4u^{-8}+(\delta^2+2\gamma\delta)x^6u^{-12}};$$

and comparing this with

$$y=\frac{x(A_3+A_2x^2+A_1x^4+A_0x^6)}{A_0+A_1x^2+A_2x^4+A_3x^6},$$

we have for each of the coefficients A two different expressions. Equating these and making a slight change of form, we obtain the relations between u, v, α, β, γ, δ used in the Memoir: thus,

$$A_0=\alpha^2=v^2u^{-14}\delta^2,\ A_1=v^2u^{-10}(\gamma^2+2\beta\delta+2\gamma\delta)=u^{-4}(2\alpha\gamma+\beta^2+2\alpha\beta),\ \&c.;$$

in the Memoir, $k\,(=u^4)$ is used instead of u, and $\Omega\,(=v^2u^{-2})$ instead of v, and the equations thus are

$$k^3\alpha^2=\Omega\delta^2,$$
$$k(2\alpha\gamma+2\alpha\beta+\beta^2)=\Omega(\gamma^2+2\gamma\delta+2\beta\delta),$$
$$\gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta=\Omega k(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),$$
$$\delta^2+2\gamma\delta=\Omega k^3(\alpha^2+2\alpha\beta);$$

viz. these are the equations [*Coll. Math. Papers*, vol. IX., p. 119]. The idea in the present Memoir is that of considering the coefficients A in the stead of α, β, ...

14. We have here, and in general for any odd value of n, equations of the form

$$(\Omega =)\ \frac{U}{U'} = \frac{V}{V'} = \ldots,$$

where $U, V, \ldots, U', V', \ldots$ are quadric functions of the coefficients $\alpha, \beta, \gamma, \ldots$; and these equations serve to establish between Ω and k a relation called the Ωk-modular equation, and which in regard to Ω is of the same degree as the uv-modular equation is in regard to v. Leaving out the equation $(\Omega =)$, we have

$$\begin{vmatrix} U, & V, & W, \ldots \\ U', & V', & W', \ldots \end{vmatrix} = 0;$$

and to each system of values of $\alpha, \beta, \gamma, \delta, \ldots$ (or say of their ratios) given by these equations, there corresponds a single value of Ω; the number of values of Ω, or the degree in Ω, of the Ωk-equation is thus found as $=(n+1)\,2^{\frac{1}{2}(n-3)}$. This is far too high; for $n=3, 5, 7, \ldots$, the degrees are $4, 12, 32, \ldots$; those of the proper Ωk-equations are $4, 6, 8, \ldots$.

15. I showed, or endeavoured to show, that in the case $n=5$, the extraneous factor was $(\Omega-1)^6$, $(\Omega-1=0$, the Ωk-modular equation belonging to $n=1$, for which the transformation is the trivial one $y=x)$, and that in the case $n=7$, the extraneous factor was $\{(\Omega, 1)^4\}^6$, $((\Omega, 1)^4=0$, the Ωk-modular equation for the case $n=3)$; generally the extraneous factors seem to depend on the Ωk-functions for the values $n-4$, $n-8$, &c. The ground for this is that, in the assumed formula for any given value n, we may take P, Q to contain a common factor $1 \pm k\mathrm{x}^2$ $\Big($observe that, to a factor près, this is unaltered by the change x into $\frac{1}{k\mathrm{x}}$, viz. it becomes $\frac{1}{k\mathrm{x}^2}(1 \pm k\mathrm{x}^2)$, a condition which is necessary$\Big)$, and we thereby reduce the equation to

$$\frac{1-\mathrm{y}}{1+\mathrm{y}} = \frac{1-\mathrm{x}}{1+\mathrm{x}}\left(\frac{P'-Q'\mathrm{x}}{P'+Q'\mathrm{x}}\right)^2,$$

in which equation the degrees of the numerator and the denominator are each diminished by 4, and the equation thus belongs to the value $n-4$.

16. I remark here that, in the case of n an odd prime, the degree of the modular equation is $=n+1$; but for any other odd value, the degree is

$$\sigma'(n),\ = n\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\ldots,$$

where $a, b, \ldots$ are all the unequal prime factors of n; thus, if $n=a^a$, the degree is

$$a^a\left(1+\frac{1}{a}\right),\ = a^{a-1}(a+1).$$

In the case of a number $n=abc\ldots$, without any squared factor, the degree is

$$abc\ldots\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\ldots,\ = (a+1)(b+1)(c+1)\ldots,$$

the sum of the factors of n. We have

$$\sigma'(n) = \text{coeff. } x^n \text{ in } \Sigma\phi(x^N),$$

where

$$\phi x = x + 3x^3 + 5x^5 + \ldots, = \frac{x(1+x^2)}{(1-x^2)^2},$$

and the summation extends to all odd values of N having no squared factor; thus,

$$\begin{array}{lllllllllll}
\phi(x) & = x + 3x^3 + 5x^5 + 7x^7 + & 9x^9 + 11x^{11} + 13x^{13} + 15x^{15} \ldots \\
\phi(x^3) = & 1x^3 & + 3x^9 & + 5x^{15}, \\
\phi(x^5) = & 1x^5 & + 3x^{15}, \\
\phi(x^7) = & 1x^7 \\
\phi(x^{11}) = & 1x^{11} \\
\phi(x^{13}) = & 1x^{13} \\
\phi(x^{15}) = & 1x^{15}, \\
\ldots\ldots & \ldots\ldots \\
\ldots\ldots & \ldots\ldots \\
\hline
\Sigma\phi(x^N) = & x + 4x^3 + 6x^5 + 8x^7 + 12x^9 + 12x^{11} + 14x^{13} + 24x^{15} \ldots
\end{array}$$

17. Supposing that the reduction is completely accounted for as above, then, to obtain the numerical relations, the numbers 1, 4, 12, 32, ..., $(n+1)2^{\frac{1}{2}(n-3)}$ have to be expressed linearly in terms of 1, 4, 6, 8, ..., $\sigma'(n)$, viz. $(n+1)2^{\frac{1}{2}(n-3)}$ as a linear function of $\sigma'(n)$, $\sigma'(n-4)$, $\sigma'(n-8)$, ..., and we have

$$\begin{aligned}
1 &= 1, \\
4 &= 4, \\
12 &= 6 + 6 \,.\, 1, \\
32 &= 8 + 6 \,.\, 4, \\
80 &= 12 + 6 \,.\, 6 + 32 \,.\, 1, \\
192 &= 12 + 6 \,.\, 8 + 33 \,.\, 4, \\
448 &= 14 + 6 \,.\, 12 + 33 \,.\, 6 + 164 \,.\, 1, \\
1024 &= 24 + 6 \,.\, 12 + 33 \,.\, 8 + 166 \,.\, 4, \\
2304 &= 18 + 6 \,.\, 14 + 33 \,.\, 12 + 166 \,.\, 6 + 810 \,.\, 1, \\
5120 &= 20 + 6 \,.\, 24 + 33 \,.\, 12 + 166 \,.\, 8 + 817 \,.\, 4, \\
11264 &= 32 + 6 \,.\, 18 + 33 \,.\, 14 + 166 \,.\, 12 + 817 \,.\, 6 + 3768 \,.\, 1, \\
24576 &= 24 + 6 \,.\, 20 + 33 \,.\, 24 + 166 \,.\, 12 + 817 \,.\, 8 + 3778 \,.\, 4, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

but it is of course very doubtful whether these relations have any value in regard to the present theory.

18. In the same way that, by assuming a common factor, $1+kx^2$, in the values of P and Q, we pass from the case n to the case $n-4$, so, by assuming a common factor, $1+x^2$, in the numerator and denominator of the expression for y in terms of x and the coefficients B, we pass from the case n to the case $n-2$. Contrariwise, in the solutions given by the Jacobi-Brioschi differential equations and by the Jacobi partial differential equation, the solution for a given value of n does not thus contain the solution for an inferior value of n; see *post* Nos. 36 and 43.

I pass now to the theory before referred to.

The Development $y=\rho x(1+\Pi_1 x^2+\Pi_2 x^4+\ldots)$. Art. Nos. 19 and 20.

19. Starting from the equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho\,dx}{\sqrt{1-2\alpha x^2+x^4}},$$

and writing for shortness

$$\begin{array}{ll}
R_1=\tfrac{1}{3}\alpha, & S_1=\tfrac{1}{3}\beta,\\
R_2=\tfrac{1}{5}(\tfrac{3}{2}\alpha^2-\tfrac{1}{2}), & S_2=\tfrac{1}{5}(\tfrac{3}{2}\beta^2-\tfrac{1}{2}),\\
R_3=\tfrac{1}{7}(\tfrac{5}{2}\alpha^3-\tfrac{3}{2}\alpha), & S_3=\tfrac{1}{7}(\tfrac{5}{2}\beta^3-\tfrac{3}{2}\beta),\\
R_4=\tfrac{1}{9}(\tfrac{35}{8}\alpha^4-\tfrac{15}{4}\alpha^2+\tfrac{3}{8}), & S_4=\tfrac{1}{9}(\tfrac{35}{8}\beta^4-\tfrac{15}{4}\beta^2+\tfrac{3}{8}),\\
\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots &
\end{array}$$

(viz. save as to the exterior factors $\frac{1}{3}, \frac{1}{5}, \ldots, 3R_1, 5R_2, \ldots$ are the Legendrian functions of α, and $3S_1, 5S_2, \ldots$ the Legendrian functions of β), we have

$$dy(1+3S_1y^2+5S_2y^4+\ldots)=\rho\,dx(1+3R_1x^2+5R_2x^4+\ldots),$$

whence integrating, so that y may vanish with x, we have

$$y+S_1y^3+S_2y^5+\ldots=\rho(x+R_1x^3+R_2x^5+\ldots),$$

say this is

$$=u.$$

20. We then have $y=u-fy$, where $fy=S_1y^3+S_2y^5+\ldots$; and thence, expanding by Lagrange's theorem,

$$y=u-fu+\frac{1}{2}(f^2u)'-\frac{1}{2.3}(f^3u)''+\frac{1}{2.3.4}(f^4u)'''-\ldots,$$

we have

$$fu=S_1u^3+S_2u^5+S_3u^7+S_4u^9+\ldots,$$

and thence

$$\begin{aligned}
f^2u&=S_1^2u^6+2S_1S_2u^8+(2S_1S_3+S_2^2)u^{10}+\ldots,\\
f^3u&=S_1^3u^9+3S_1^2S_2u^{11}+\ldots,\\
f^4u&=S_1^4u^{12}+\ldots;
\end{aligned}$$

consequently,

$$\begin{aligned} y = \; & u, \\ & + u^3(-S_1), \\ & + u^5(-S_2 + 3S_1^2), \\ & + u^7(-S_3 + 8S_1S_2 - 12S_1^3), \\ & + u^9(-S_4 + 10S_1S_3 + 5S_2^2 - 55S_1^2S_2 + 55S_1^4) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and writing herein

$$\begin{aligned} u &= \rho\,\{x + R_1x^3 + R_2x^5 + R_3x^7 + R_4x^9 + \ldots\}, \\ u^3 &= \rho^3\,\{x^3 + 3R_1x^5 + (3R_2 + 3R_1^2)\,x^7 + (3R_3 + 6R_1R_2 + R_1^3)\,x^9 + \ldots\}, \\ u^5 &= \rho^5\,\{x^5 + 5R_1x^7 + (5R_2 + 10R_1^2)\,x^9 + \ldots\}, \\ u^7 &= \rho^7\,\{x^7 + 7R_1x^9 + \ldots\}, \\ u^9 &= \rho^9\,\{x^9 + \ldots\}, \end{aligned}$$

we have the required series

$$y = \rho x\,\{1 + \Pi_1x^2 + \Pi_2x^4 + \Pi_3x^6 + \ldots\},$$

where the values of the coefficients are

$$\begin{aligned} \Pi_1 &= R_1 + (-S_1)\,\rho^2, \\ \Pi_2 &= R_2 + (-S_1)\,3R_1\rho^2 + (-S_2 + 3S_1^2)\,\rho^4, \\ \Pi_3 &= R_3 + (-S_1)\,(3R_2 + 3R_1^2)\,\rho^2 + (-S_2 + 3S_1^2)\,5R_1\rho^4 + (-S_3 + 8S_1S_2 - 12S_1^3)\,\rho^6, \\ \Pi_4 &= R_4 + (-S_1)\,(3R_3 + 6R_1R_2 + R_1^3)\,\rho^2 + (-S_2 + 3S_1^2)\,(5R_2 + 10R_1^2)\,\rho^4 \\ &\qquad + (-S_3 + 8S_1S_2 - 12S_1^3)\,7R_1\rho^6 + (-S_4 + 10S_1S_3 + 5S_2^2 - 55S_1^2S_2 + 55S_1^4)\,\rho^8, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and so on, as far as we please.

The Cubic Transformation, $n = 3$. Art. Nos. 21 to 28.

21. We have here

$$\frac{\rho + x^2}{1 + \rho x^2} = \rho\,(1 + \Pi_1x^2 + \Pi_2x^4 + \ldots);$$

whence, developing the left-hand side and equating coefficients,

$$\rho\Pi_1 = -\rho^2 + 1, \quad \rho\Pi_2 = \rho^2 - \rho, \quad \rho\Pi_3 = -\rho^4 + \rho^2, \ldots$$

It will be convenient to write

$$\begin{aligned}
\Theta_1 = \rho\Pi_1 + \rho^2 - 1, \quad &= -S_1\rho^3 + \rho^2 + R_1\rho - 1,\\
\Theta_2 = \Pi_2 - \rho^2 + 1, \quad &= (-S_2 + 3S_1^2)\rho^4 - (3R_1S_1 + 1)\rho^3 + R_2 + 1,\\
\Theta_3 = \Pi_3 + \rho^3 - \rho, \quad &= (-S_3 + 8S_1S_2 - 12S_1^3)\rho^6\\
&\quad + (-5R_1S_2 + 15R_1S_1^2)\rho^4\\
&\quad\qquad\qquad\qquad + \rho^3\\
&\quad + (-3R_2S_1 - 3R_1^2S_1)\rho^2\\
&\quad\qquad\qquad\qquad - \rho\\
&\quad\qquad\qquad\qquad + R_3,
\end{aligned}$$

.....

.....

where observe the difference of form in the function Θ_1, and in the subsequent functions Θ_2, Θ_3, ... In these last, a factor ρ is thrown out.

22. The two equations $\Theta_1 = 0$ and $\Theta_2 = 0$ serve to determine ρ, β in terms of α; the subsequent equations $\Theta_3 = 0$, $\Theta_4 = 0$, ... will then be, all of them, satisfied identically. This implies that Θ_3, Θ_4, ... are each of them a linear function of Θ_1, Θ_2. The *à posteriori* verification and determination of the factors is by no means easy; I have effected it only for Θ_3; we have

$$7\Theta_3 = (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)\,\Theta_1 + (-S_1\rho^2 - 10\rho + 25R_1)\,\Theta_2,$$

or, at full length,

$$7\left|\begin{array}{r}
(-S_3 + 8S_1S_2 - 12S_1^3)\rho^6\\
+(-5R_1S_2 + 15R_1S_1^2)\rho^4\\
+\rho^3\\
+(-3R_2S_1 - 3R_1^2S_1)\rho^2\\
-\rho\\
+R_3
\end{array}\right|$$

$$\begin{aligned}
&= (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)(-S_1\rho^3 + \rho^2 + R_1\rho - 1)\\
&\quad + (-S_1\rho^2 - 10\rho + 25R_1)((-S_2 + 3S_1^2)\rho^4 - (3R_1S_1 + 1)\rho^3 + R_2 + 1);
\end{aligned}$$

in verifying which we must, of course, take account of the relations between the expressions R and those between the expressions S; we have

$$\alpha = 3R_1 \text{ and thence } 10R_2 = 27R_1^2 - 1, \quad 14R_3 = 135R_1^3 - 9R_1;$$

similarly,

$$10S_2 = 27S_1^2 - 1, \quad 14S_3 = 135S_1^3 - 9S_1.$$

Equating the coefficients of ρ^6, we have

$$-7S_3 + 56S_1S_2 - 84S_1^3 = -S_1 + S_1S_2 - 3S_1^2;$$

viz. multiplying by 2, this is

$$-14S_3+110S_1S_2-162S_1^3+2S_1=0;$$

or, finally, it is

$$(-135S_1^3+9S_1)+(297S_1^3-11S_1)-162S_1^3+2S_1=0,$$

an identity, as it should be. The identity of the coefficients of ρ^5, ρ^4, ρ^3, ρ^2, ρ, 1 may be verified in like manner.

23. Considering α as known, the values of ρ and β are determined by the foregoing equations $\Theta_1=0$, $\Theta_2=0$; that is,

$$-S_1\rho^3+\rho^2+R_1\rho-1=0,$$

$$(-S_2+3S_1^2)\rho^4-(3R_1S_1+1)\rho^2+R_2+1=0,$$

where, of course, the R's and S's are regarded as given functions of α and β respectively.

It is to be observed that the equations are satisfied by $\rho^2=1$, $\alpha=\beta$; viz. we have the transformation $y=\dfrac{x(\pm 1+x^2)}{1\pm x^2}$; that is, $y=\pm x$, which is the transformation of the first order, $n=1$. The two equations represent surfaces of the orders 4 and 6 respectively, and they have thus a complete intersection of the order 24. As part of this, we have, as just shown, each of the two lines ($\rho=1$, $\alpha=\beta$) and ($\rho=-1$, $\alpha=\beta$); but there is a more considerable reduction of order to be accounted for, the proper MM-curve being, as will appear, a unicursal curve of the order $=6$.

24. Multiplying the second equation by $10\rho^2$, and for $10R_2$ and $10S_2$ writing their values $27R_1^2-1$ and $27S_1^2-1$ respectively, we have

$$(3S_1^2+1)\rho^6-(30R_1S_1+10)\rho^4+(27R_1^2+9)\rho^2=0;$$

and if herein we substitute for $S_1\rho^3$ its value from the first equation, $=\rho^2+R_1\rho-1$, we have

$$3(\rho^2+R_1\rho-1)^2+\rho^6-30R_1\rho(\rho^2+R_1\rho-1)-10\rho^4+(27R_1^2+9)\rho^2=0;$$

that is,

$$\rho^6-7\rho^4-24R_1\rho^3+3\rho^2+24R_1\rho+3=0;$$

viz. this is

$$(\rho^2-1)(\rho^4-6\rho^2-24R_1\rho-3)=0,$$

containing, as it ought to do, the factor ρ^2-1. Throwing this out, and repeating the first equation, we have

$$\begin{aligned}-S_1\rho^3+\ \rho^2+\ \ R_1\rho-1&=0,\\ \rho^4\qquad -6\rho^2-24R_1\rho-3&=0,\end{aligned}$$

which two equations may be replaced by

$$\begin{aligned}\rho^4-24S_1\rho^3+18\rho^2\qquad\qquad -27&=0,\\ \rho^4\qquad\quad -\ 6\rho^2-24R_1\rho-\ 3&=0,\end{aligned}$$

which are the $\rho\beta$- and $\rho\alpha$-equations respectively. Recollecting that R_1 and S_1 denote $\frac{1}{3}\alpha$ and $\frac{1}{3}\beta$, they agree with the results obtained in No. 7. The $\alpha\beta$-modular equation is obtained by the elimination of ρ from these two equations, and may be at once written down in the form, Det. $=0$, where the determinant is of the order 8, but contains S_1 and R_1, that is, β and α, each of them, in the fourth order only: the form is thus the same with that of the $\alpha\beta$-equation obtained in No. 2; but the identification would be a work of some labour.

25. The equations may be written

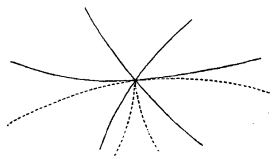
$$24S_1\rho^3 = \rho^4 + 18\rho^2 - 27,$$
$$24R_1\rho = \rho^4 - \ \ 6\rho^2 - \ \ 3,$$

and, treating R_1, S_1, ρ as coordinates, it hence appears that the MM-curve is (as mentioned above) a unicursal curve of the order 6; in fact, we have R_1, S_1, each of them given as a rational function of ρ; and cutting the curve by an arbitrary plane $AR_1 + BS_1 + C\rho + D = 0$, the substitution of the values of R_1, S_1 in this equation gives for ρ an equation of the order 6.

26. The same conclusion may be obtained from the foregoing system of a cubic and a quartic equation in ρ. Considering R_1, S_1, ρ as coordinates, they represent, each of them, a surface of the order 4, and the complete intersection is of the order 16; but this is made up of a line in the plane infinity counting 10 times, and of the MM-curve, which is thus of the order $16 - 10, = 6$. In fact, introducing, for homogeneity, a fourth coordinate θ, the two equations are

$$-S_1\rho^3 + \ \rho^2\theta^2 + \ \ R_1\rho\theta^2 - \ \ \theta^4 = 0,$$
$$\rho^4 \qquad\quad - 6\rho^2\theta^2 - 24R_1\rho\theta^2 - 3\theta^4 = 0,$$

and the line $\rho = 0$, $\theta = 0$ is thus a triple line on each of these surfaces; viz. cutting them by an arbitrary plane, we have for the first surface an ordinary triple point, as shown by the continuous lines of the annexed figure, and for the second surface a triple point = cusp + two nodes, as shown by the dotted lines of the figure. There is, moreover, as shown in the figure, a contact of two branches, and the number of intersections is thus $=10$.



27. If we assume $\rho\sigma = -3$, that is, $\rho = -\dfrac{3}{\sigma}$, and substitute this value in the equation for S_1, the two equations become

$$24S_1\sigma = \sigma^4 - 6\sigma^2 - 3,$$
$$24R_1\rho = \rho^4 - 6\rho^2 - 3;$$

viz. β is the same function of $\sigma\left(=-\frac{3}{\rho}\right)$ that α is of ρ. This accords with the theorem in Elliptic Functions that a combination of two transformations leads to a multiplication.

28. We have

$$24(R_1+\tfrac{1}{3})\rho=\rho^4-6\rho^2+8\rho-3,\ =(\rho-1)^3(\rho+3),$$

or, what is the same thing,

$$24(R_1+\tfrac{1}{3})\ =(\rho-1)^2(\rho+\sigma+2);$$

and, in like manner,

$$24(R_1-\tfrac{1}{3})\rho=\rho^4-6\rho^2-8\rho-3,\ =(\rho+1)^3(\rho-3),$$

and, consequently,

$$24(R_1-\tfrac{1}{3})\ =(\rho+1)^2.(\rho+\sigma-2);$$

with the like equations between S_1, σ, ρ. It will be recollected that

$$R_1=\tfrac{1}{3}\alpha,\ =\tfrac{1}{6}\left(v^4+\frac{1}{v^4}\right);$$

hence

$$24(R_1\pm\tfrac{1}{3})=4\left(u^4+\frac{1}{u^4}\pm 2\right),\ =4\left(u^2\pm\frac{1}{u^2}\right)^2.$$

The formulæ just obtained are useful for obtaining the uv-modular equation from the foregoing equations; or say

$$4\left(v^4+\frac{1}{v^4}\right)\sigma=\sigma^4-6\sigma^2-3,$$

$$4\left(u^4+\frac{1}{u^4}\right)\rho=\rho^4-6\rho^2-3,$$

where $\rho\sigma=-3$, and we have to eliminate ρ and σ; the elimination gives

$$\frac{v^2}{u^2}-\frac{u^2}{v^2}+2vu-\frac{2}{uv}=0,$$

that is,

$$v^4+2v^3u^3-2vu-u^4=0.$$

The Quintic Transformation, $n=5$. Art. Nos. 29 to 32.

29. We have here

$$\frac{\rho+A_1x^2+x^4}{1+A_1x^2+\rho x^4}=\rho(1+\Pi_1x^2+\Pi_2x^4+\Pi_3x^6+\ldots),$$

and multiplying by $1+A_1x^2+\rho x^4$, we obtain an infinite series of equations, the first three of which are

$$\begin{aligned}A_1&=\rho\Pi_1+A_1\rho,\\ 1&=\rho\Pi_2+A_1\rho\Pi_1,\\ 0&=\rho\Pi_3+A_1\rho\Pi_2+\rho^2\Pi_1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

The first of these gives

$$A_1 = \frac{\rho\Pi_1}{-\rho+1}, \; = \frac{\Theta_1 - \rho^2 + 1}{-\rho+1};$$

and the other two equations then determine the MM-curve. These being satisfied, the remaining equations will be satisfied identically. It is proper to introduce Θ_1, Θ_2, Θ_3 into the equations instead of Π_1, Π_2, Π_3. We have first

$$1 = \rho\Pi_2 + \rho\Pi_1 \frac{\Theta_1 - \rho^2 + 1}{-\rho+1} + \rho^2,$$

that is,

$$0 = \rho(\Theta_2 + \rho^2 - 1) - \frac{(\Theta_1 - \rho^2 + 1)^2}{\rho - 1} + \rho^2 - 1;$$

viz. this is

$$\rho(\rho-1)(\Theta_2 + \rho^2 - 1) - (\Theta_1 - \rho^2 + 1)^2 + (\rho^2 - 1)(\rho - 1) = 0;$$

or, finally, this is

$$\rho(\rho-1)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0.$$

Secondly, we have

$$0 = \Pi_3 + \frac{\rho\Pi_1\Pi_2}{-\rho+1} + \rho\Pi_1 = 0;$$

that is,

$$\Theta_3 - \rho^3 + \rho + \frac{(\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1)}{-\rho+1} + \Theta_1 - \rho^2 + 1 = 0;$$

viz. this is

$$(\Theta_3 + \Theta_1 - \rho^3 - \rho^2 + \rho + 1)(-\rho + 1) + (\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1) = 0;$$

or, finally, it is

$$\Theta_3(-\rho+1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0.$$

30. We have thus the two equations

$$(\rho^2 - \rho)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0,$$

$$\Theta_3(-\rho+1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0;$$

and recollecting that Θ_3 is of the form $L\Theta_1 + M\Theta_2$, we see that each of these equations is satisfied if only $\Theta_1 = 0$, $\Theta_2 = 0$ (the formulæ belonging to the cubic transformation). This ought to be the case, for we can, by writing $A_1 = \rho + 1$, reduce the expression $\frac{x(\rho + A_1 x^2 + x^4)}{1 + A_1 x^2 + \rho x^4}$ to the form $\frac{x(\rho + x^2)}{1 + \rho x^2}$, which belongs to the cubic transformation (see *ante* No. 17). The equations may be written

$$\rho\Theta_2 = -(2\rho + 2)\Theta_1 + \frac{\Theta_1^2}{\rho - 1},$$

$$\rho\Theta_3 = (3\rho^2 + 4\rho + 2)\Theta_1 - (3\rho + 3)\frac{\Theta_1^2}{\rho - 1} + \frac{\Theta_1^3}{(\rho - 1)^2}.$$

31. The investigation may be presented in a slightly different form by introducing the functions Θ at an earlier stage; viz. writing

$$\rho\Pi_1 = \Theta_1 - \rho^2 + 1,\ \rho\Pi_2 = \rho\Theta_2 + \rho^3 - \rho, \ldots,$$

we have

$$\frac{\rho + A_1x^2 + x^4}{1 + A_1x^2 + \rho x^4} = \rho + (\Theta_1 - \rho^2 + 1)\,x^2 + (\rho\Theta_2 + \rho^3 - \rho)\,x^4 + \ldots$$

$$= \frac{\rho + x^2}{1 + \rho x^2} + \Theta_1 x^2 + \rho\Theta_2 x^4 + \rho\Theta_3 x^6 + \ldots$$

Transposing, reducing, and dividing by x^2, we have

$$\frac{(1 - x^2)\,[\rho^2 - 1 + A_1(-\rho + 1)]}{(1 + \rho x^2)(1 + A_1x^2 + \rho x^4)} = \Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots,$$

whence clearly $\rho^2 - 1 + A_1(-\rho + 1) = \Theta_1$, giving for A_1 the before-mentioned value; and we then have

$$1 + A_1x^2 + \rho x^4 = 1 + (\rho + 1)\,x^2 + \rho x^4 - \frac{\Theta_1 x^2}{\rho - 1},\ = (1 + x^2)(1 + \rho x^2) - \frac{\Theta_1 x^2}{\rho - 1}.$$

The equation thus becomes

$$\frac{(1 - x^2)\,\Theta_1}{(1 + x^2)(1 + \rho x^2)^2\left(1 - \dfrac{\Theta_1 x^2}{\rho - 1\,.\,1 + x^2\,.\,1 + \rho x^2}\right)} = \Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots,$$

and expanding the left-hand side, first in the form

$$\frac{(1 - x^2)\,\Theta_1}{(1 + x^2)(1 + \rho x^2)^2} + \frac{(1 - x^2)\,x^2\Theta_1^2}{(\rho - 1)(1 + x^2)^2(1 + \rho x^2)^3} + \frac{(1 - x^2)\,x^4\Theta_1^3}{(\rho - 1)(1 + x^2)^3(1 + \rho x^2)^4} + \ldots$$

and then each of these terms separately in powers of x^2, and comparing with $\Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots$, we have the two equations in the last-mentioned form, and an infinite series of other equations, which will be satisfied identically.

32. The successive coefficients might be called Φ_2, $\Phi_3, \ldots$; say

$$\Phi_2 = (\rho^2 - \rho)\,\Theta_2 - \Theta_1^2 + 2\,(\rho^2 - 1)\,\Theta_1,$$

$$\Phi_3 = (-\rho + 1)\,\Theta_3 + \Theta_1\Theta_2 + (\rho^2 - \rho)\,\Theta_1 - (\rho^2 - 1)\,\Theta_2,$$

and similarly for $\Phi_4, \ldots$; and it would then be proper to show *à posteriori* that each of the equations $\Phi_4 = 0$, $\Phi_5 = 0, \ldots$ is satisfied identically in virtue of the two equations $\Phi_2 = 0$, $\Phi_3 = 0$, or, what is the same thing, that the functions Φ_4, $\Phi_5, \ldots$ are each of them a linear function (with coefficients which are functions of ρ) of the two functions Φ_2 and Φ_3. I do not attempt to do this, nor even to discuss the MM-curve by means of the equations $\Phi_2 = 0$, $\Phi_3 = 0$; but I will obtain equivalent results, and complete the solution by means of the Jacobi-Brioschi equations, in effect reproducing the investigation contained in the third appendix of the *Funzioni ellittiche*.

The General Transformation, $n=2s+1$. Art. No. 33.

33. The equation here is

$$\frac{\rho+A_{s-1}x^2+\ldots}{1+A_1x^2+\ldots}=\rho\,(1+\Pi_1x^2+\ldots).$$

The general theory is sufficiently illustrated by the preceding particular cases, and I wish at present only to notice the equation obtained by comparing the coefficients of x^2; viz. this is $A_{s-1}-\rho A_1=\rho\Pi_1$, or, substituting for Π_1 its value,

$$A_{s-1}-\rho A_1=\tfrac{1}{3}(\alpha\rho-\beta\rho^3).$$

The Jacobi-Brioschi Equations. Art. Nos. 34 to 42.

34. These were obtained for the differential equation

$$\frac{dx}{\sqrt{a'x^4+b'x^3+c'x^2+d'x+e'}}=\frac{dy}{\sqrt{ay^4+by^3+cy^2+dy+e}};$$

viz. if this be satisfied by $y=U\div V$, where U, V are rational and integral functions of x of the degrees n and $n-1$ respectively, then, writing for shortness

$$\phi=a'x^4+b'x^3+c'x^2+d'x+e',$$

and using accents to denote differentiation in regard to x, the numerator and denominator U, V satisfy the equations

$$\begin{aligned}(VV''-V'^2)\,\phi+\tfrac{1}{2}\,VV'.\phi' \quad &+\ aU^2+\quad \tfrac{1}{2}bUV+\ pV^2=0,\\ -(VU''+V''U-2V'U')\,\phi-\tfrac{1}{2}\,(VU'+V'U)\,\phi+\tfrac{1}{2}\,bU^2+(c-2p)\,UV+\tfrac{1}{2}dV^2&=0,\\ (UU''-U'^2)\,\phi+\tfrac{1}{2}\,UU'.\phi' \quad &+\ pU^2+\quad \tfrac{1}{2}dUV+\ eV^2=0,\end{aligned}$$

where p is a function $=\mathrm{a}x^2+\mathrm{b}x+\mathrm{c}$, with coefficients a, b, c, the values of which have to be determined. By way of verification, observe that, multiplying by U^2, UV, V^2, and adding, we obtain

$$-(VU'-V'U)^2\,\phi+aU^4+bU^3V+cU^2V^2+dUV^3+eV^4=0;$$

that is,

$$-\frac{1}{V^2}(VU'-V'U)^2(a'x^4+b'x^3+c'x^2+d'x+e')+ay^4+by^3+cy^2+dy+e=0,$$

the result obtained by substituting for y its value, $=U\div V$, in the differential equation.

35. Considering the foregoing special form

$$\frac{dx}{\sqrt{1-2\alpha x^2+x^4}}=\frac{dy}{\rho\sqrt{1-2\beta y^2+y^4}},$$

so that a, b, c, d, e have the values ρ^2, 0, $-2\beta\rho^2$, 0, ρ^2 and ϕ is $=1-2\alpha x^2+x^4$, the equations are

$$\begin{aligned}(VV''-V'^2)\,\phi+\quad \tfrac{1}{2}\,VV'.\phi'+\rho^2U^2\ +\quad \rho V^2&=0,\\ -(VU''+V''U-2V'U')\,\phi+\tfrac{1}{2}\,(VU'+V'U)\,\phi'-(2\beta\rho^2+2p)\,UV\quad &=0,\\ (UU''-U'^2)\,\phi+\quad \tfrac{1}{2}\,UU'.\phi'+pU^2\ +\quad \rho^2V^2&=0,\end{aligned}$$

where, writing as before, $n=2s+1$, and assuming that the last coefficient, $A_{\frac{1}{2}(n-1)}$ or A_s, is $=\rho$, we have

$$U=x(\rho+A_{s-1}x^2+A_{s-2}x^4+\ldots+\quad A_1x^{2s-2}+\quad x^{2s}),$$
$$V=\quad 1+\quad A_1x^2+\quad A_2x^4+\ldots+A_{s-1}\,x^{2s-2}+\rho x^{2s},$$

and where, as is easily shown, p has the value $=-\{2A_1+(2s+1)x^2\}$. In comparing with Brioschi, it will be recollected that 2α, 2β are written in place of his α, β.

36. The equations contain n, and they are not satisfied by values of U, V belonging to any inferior value of n; U, V may each of them be multiplied by any common constant factor at pleasure, but not by a common variable factor P; viz. it is assumed that the fraction $U\div V$ is in its least terms, and consequently that (save as to a constant factor) U, V are determinate functions. It is easy to verify that the equations (being verified by U, V) are not verified by PU, PV, but it is interesting to show *à priori* why this is so. The equations are obtained as follows. Consider the differential equation in the form $\frac{dx}{\sqrt{X}}=\frac{dy}{\sqrt{Y}}$, and suppose that an integral equation is given in the form $F=0$ (F a rational and integral function of x, y); we thence deduce a relation $Ldx+Mdy=0$ between the differentials, and this must agree with the given differential equation; that is, we have $L\sqrt{X}+M\sqrt{Y}=0$, or, rationalizing, $L^2X-M^2Y=0$; viz. this last equation must agree with the equation $F=0$, or, what is the same thing, L^2X-M^2Y must contain F as a factor; say we have

$$L^2X-M^2Y=F\,.\,G,$$

where G is a function of x, y. In the particular case where the integral is of the form

$$y=U\div V,$$

we have

$$F=Vy-U,$$

and we have therefore

$$L^2X-M^2Y=G\,(Vy-U);$$

and it is by means of this identity that the equations are obtained. But suppose that there is a common factor P, or that we have $y=PU\div PV$; then, if we write $F=PVy-PVU,\ =P\,(Vy-U)$, there is no necessity that L^2X-M^2Y should contain as a factor this expression of F, and it will not in fact contain it; all that is necessary is that L^2X-M^2Y shall contain the factor $Vy-U$; and thus the equations obtained for U, V do not apply to PU, PV. We might, of course, introduce an arbitrary *constant* factor Θ; contrast herewith the solution by means of the Jacobi partial differential equation, *post* No. 43, where Θ is not arbitrary but has a determinate value.

37. In virtue of the assumed forms of U, V, the first and the third equations give each of them the same relations between the coefficients A; and only one of these equations, say the first, need be attended to. It will be observed that this equation

does not contain β; it consequently serves to determine the coefficients A in terms of ρ, α, and to establish a relation between ρ, α, that is, the multiplier equation. We can from this, as will be explained, deduce the equation between ρ, β; the theory thus depends entirely upon the first equation; say this is

$$(VV'' - V'^2)(1 - 2\alpha x^2 + x^4) + VV'(-2\alpha x + 2x^3) + \rho^2 U^2 - \{2A_1 + (2s+1)x^2\} V^2 = 0.$$

38. We have $V = 1 + A_1 x^2 + A_2 x^4 + \ldots$, but the equation contains the quadric functions $VV'' - V'^2$, VV', and V^2; it is convenient to write

$$VV'' - V'^2 = K_1 + K_2 x^2 + K_3 x^4 + \ldots,$$

$$V^2 = L_0 + L_1 x^2 + L_2 x^4 + \ldots;$$

whence of course

$$VV' = 2L_1 x + 4L_2 x^3 + \ldots,$$

and we have

$K_1=$	$K_2=$	$K_3=$	$K_4=$	$K_5=$	$K_6=$	$K_7=$	$K_8=$	. . .
$2A_1$	$12A_2$	$30A_3$	$56A_4$	$90A_5$	$132A_6$	$182A_7$	$240A_8$	
	$-2A_1^2$	$-2A_1A_2$	$+8A_1A_3$	$+26A_1A_4$	$+52A_1A_5$	$+86A_1A_6$	$+128A_1A_7$	,
			$-4A_2^2$	$-6A_2A_3$	$+4A_2A_4$	$+22A_2A_5$	$+48A_2A_6$	
					$-6A_3^2$	$-10A_3A_4$	$0A_3A_5$	
							$-8A_4^2$	

$L_0=$	$L_1=$	$L_2=$	$L_3=$	$L_4=$	. . .
1	$2A_1$	$2A_2$	$2A_3$	$2A_4$	
		$+A_2^2$	$+2A_1A_2$	$+2A_1A_3$	.
				$+A_2^2$	

The coefficients of U^2 are at once obtained; say we have $U^2 = \Lambda_0 x^2 + \Lambda_1 x^4 + \Lambda_2 x^6 + \ldots$,

$\Lambda_0=$	$\Lambda_1=$	$\Lambda_2=$	$\Lambda_3=$	$\Lambda_4=$	. . .
ρ^2	$2\rho A_{s-1}$	$2\rho A_{s-2}$	$2\rho A_{s-3}$	$2\rho A_{s-4}$	
		$+A^2_{s-1}$	$+2A_{s-1}A_{s-2}$	$+2A_{s-1}A_{s-3}$	.
				$+A^2_{s-2}$	

Substituting in the equation and equating to zero the coefficients of the several powers of x^2, we find

$$\begin{array}{ll}
K_1 & -2A_1L_0 \qquad\qquad\qquad\

where the first, fifth and sixth equations are each of them an identity. The remaining equations are

$$(\rho^2-2\rho+5)(\rho^2+2\rho-1)-6A_1^2-8A_1\alpha=0,$$
$$2\rho^2A_1-6\rho A_1-32\rho\alpha-2A_1^3-4A_1=0,$$
$$2\rho(\rho^2-2\rho+5)+10\rho-4A_1^2\rho-8\alpha A_1\rho+3A_1^2=0.$$

41. Writing the first and third of these in the forms

$$\begin{aligned}-6A_1^2 \qquad\qquad -8A_1\alpha \;&+(\rho^2-2\rho+5)(\rho^2+2\rho-1)=0,\\ A_1^2(\rho^2-4\rho+3)-8A_1\alpha\rho\;&+(\rho^2-2\rho+5)\,2\rho \qquad\qquad =0,\end{aligned}$$

they determine A_1^2, $8A_1\alpha$ in terms of ρ; viz. we find

$$A_1^2=(\rho^2-2\rho+5)\;\rho,$$
$$8A_1\alpha=(\rho^2-2\rho+5)(\rho^2-4\rho-1);$$

and then, writing the second equation in the form

$$(\rho^3-3\rho-2)A_1^2-16\rho\alpha A_1-A_1^4=0,$$

and substituting these values of A_1^2 and $8A_1\alpha$, and omitting the factor $\rho^2-2\rho-5$, we have the identity

$$\rho(\rho^3-3\rho-2)-2\rho(\rho^2-4\rho-1)-\rho^2(\rho^2-2\rho+5)=0;$$

viz. the second equation is then also satisfied.

Forming the square of $8A_1\alpha$, and for A_1^2 substituting its value, then omitting a factor $\rho^2-2\rho+5$, we find

$$\begin{aligned}64\rho\alpha^2&=(\rho^2-2\rho+5)(\rho^2-4\rho-1)^2,\\ &=\rho^6-10\rho^5+35\rho^4-60\rho^3+55\rho^2+38\rho+5;\end{aligned}$$

or, as this may also be written,

$$64\rho\;(\alpha^2-1)=(\rho-1)^5(\rho-5),$$

and we then have also, as before,

$$64\sigma(\beta^2-1)=(\sigma-1)^5(\sigma-5),$$

which two equations determine the MM-curve.

The coefficient A_1 is given by the foregoing equation for $8A_1\alpha$, say the value is

$$A_1=\frac{1}{8\alpha}(\rho^2-2\rho+5)(\rho^2-4\rho-1).$$

The value $A_1=\dfrac{\rho\Pi_1}{-\rho+1}$, obtained in No. 29, on substituting for Π_1 its value, is

$$A_1=\frac{\frac{1}{2}(\beta\rho^3-\alpha\rho)}{\rho-1},$$

and these two values are, in fact, equivalent in virtue of the value of β obtained in No. 9.

42. I consider the case $n=7$, in order to show the form of the equations which have to be solved; these equations are

$$\begin{aligned}
&2A_1 - 2A_1 . 1 &&= 0,\\
&12A_2 - 2A_1^2 - 2A_1 . 2A_1 - 7 . 1 - 2\alpha(2A_1 + 1 . 2A_1) + \rho^2 . \rho^2 &&= 0,\\
&30\rho - 2A_1A_2 + 2A_1 - 2A_1(2A_2 + A_1^2) - 5 . 2A_1 \\
&\qquad - 2\alpha(12A_2 - 2A_1^2 + 2(2A_2 + A_1^2)) + \rho^2 . 2\rho A_2 &&= 0,\\
&8A_1\rho - 4A_2^2 + 12A_2 - 2A_1^2 - 2A_1(2\rho + 2A_1A_2) - 3(2A_2 + A_1^2) \\
&\qquad - 2\alpha(30\rho - 2A_1A_2 + 3(2\rho + 2A_1A_2)) + \rho^2(2\rho A_1 + A_2^2) &&= 0,\\
&-6A_2\rho + 30\rho - 2A_1A_2 - 2A_1(2A_1\rho + A_2^2) - 1(2\rho + 2A_1A_2) \\
&\qquad - 2\alpha(8A_1\rho - 4A_2^2 + 4(2A_1\rho + A_2^2)) + \rho^2(2\rho + 2A_1A_2) &&= 0,\\
&-6\rho^2 + 8A_1\rho - 4A_2^2 - 2A_1(2A_2\rho) + 1(2A_1\rho + A_2^2) \\
&\qquad - 2\alpha(-6A_2\rho + 5(2A_2\rho)) + \rho^2(2A_2 + A_1^2) &&= 0,\\
&0 \quad - 6A_2\rho - 2A_1 . \rho^2 + 3(2A_2\rho) - 2\alpha(-6\rho^2 + 6 . \rho^2) + \rho^2 . 2A_1 &&= 0,\\
&0 \quad - 6\rho^2 + 5 . \rho^2 - 2\alpha(0 + 7 . 0) + \rho^2 . 1 &&= 0;
\end{aligned}$$

viz. the first, seventh and eighth equations are satisfied identically, and there remain five equations connecting ρ, α, A_1, A_2.

These equations* should lead to the before-mentioned $\alpha\beta$-modular equation

$$\rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\rho^2 + (-560\alpha + 512\alpha^3)\rho + 7 = 0,$$

and to expressions for A_1, A_2 as rational functions of α, ρ: and they should be, all five of them, satisfied by these results; but I do not see how the results are to be worked out; there is, so far as appears, no clue to the discovery of the rational functions of α, ρ.

The Jacobi Partial Differential Equation. Art. Nos. 43 to 48.

43. Writing, as above, 2α in place of Jacobi's α, this is

$$(1 - 2\alpha x^2 + x^4)\frac{d^2z}{dx^2} + (n-1)(2\alpha x - 2x^3)\frac{dz}{dx} + n(n-1)x^2 z - 4n(\alpha^2 - 1)\frac{dz}{dx} = 0,$$

satisfied by the numerator and denominator U, V, each of them taken with the same proper value of the coefficient A_0, or, what is the same thing, by the values

$$U = \Theta x(A_s + A_{s-1}x^2 + A_{s-2}x^4 + \ldots + A_1 x^{2s-2} + x^{2s}),$$

$$V = \Theta\ (1 + A_1 x^2 + A_2 x^4 + \ldots + A_{s-1}x^{2s-2} + A_s x^{2s}),$$

* [See this volume, p. 535.]

where now $A_s = \rho$ as before: Θ has its proper value; viz. disregarding an arbitrary merely numerical factor which might of course be introduced, the value is

$$\Theta = \sqrt{\frac{1}{M}\frac{\lambda'}{k'}}, \quad = \sqrt{\frac{u^2\rho}{v^2}\frac{\sqrt{1-v^8}}{\sqrt{1-u^8}}}, \quad = \sqrt{\rho}\frac{\sqrt[4]{v^4-v^{-4}}}{\sqrt[4]{u^4-u^{-4}}},$$

or, what is the same thing,

$$\Theta = \sqrt{\rho}\sqrt[8]{\frac{\beta^2-1}{\alpha^2-1}}.$$

If for z we write $\Theta\zeta$, then the equation becomes

$$(1-2\alpha x^2+x^4)\frac{d^2\zeta}{dx^2}+(n-1)(2\alpha x-2x^3)\frac{d\zeta}{dx}+n(n-1)x^2\zeta-4n(\alpha^2-1)\left(\frac{d\zeta}{d\alpha}+\frac{1}{\Theta}\frac{d\Theta}{d\alpha}\zeta\right)=0,$$

satisfied by the foregoing values without the factor Θ or, attending only to the denominator, say by the value

$$V = 1 + A_1x^2 + A_2x^4 + \ldots + A_{s-1}x^{2s-2} + \rho x^{2s}.$$

44. To calculate the value of $\frac{1}{\Theta}\frac{d\Theta}{d\alpha}$, we have

$$\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = \frac{\frac{1}{2}}{\rho}\frac{d\rho}{d\alpha} + \frac{\frac{1}{4}\beta}{\beta^2-1}\frac{d\beta}{d\alpha} - \frac{\frac{1}{4}\alpha}{\alpha^2-1};$$

but it has been seen (No. 10) that we have

$$\frac{d\beta}{d\alpha} = \frac{\rho^2}{n}\frac{\beta^2-1}{\alpha^2-1},$$

and the formula thus becomes

$$\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = \frac{\frac{1}{2}}{\rho}\frac{d\rho}{d\alpha} + \frac{\frac{1}{4n}(\beta\rho^2-n\alpha)}{\alpha^2-1}.$$

We have, as the first of the equations obtained by substituting in the partial differential equation,

$$2A_1 - 4n(\alpha^2-1)\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = 0,$$

and we have hence the value of the first coefficient,

$$A_1 = n(\alpha^2-1)\frac{1}{\rho}\frac{d\rho}{d\alpha} + \tfrac{1}{2}(\beta\rho^2-n\alpha);$$

or we may, by means of this result, get rid of the term $\frac{1}{\Theta}\frac{d\Theta}{d\alpha}$ from the partial differential equation; viz. the equation may be written

$$(1-2\alpha x^2+x^4)\frac{d^2\zeta}{dx^2}+(n-1)(2\alpha x-2x^3)\frac{d\zeta}{dx}+\{n(n-1)x^2-2A_1\}\zeta-4n(\alpha^2-1)\frac{d\zeta}{d\alpha}=0.$$

Before going further, I remark that the last of the equations obtained by the substitution gives the coefficient A_{s-1}; but this is also given in terms of A_1 by the formula No. 33, $A_{s-1} - \rho A_1 = \frac{1}{3}(\alpha\rho - \beta\rho^3)$; combining the two formulæ, we have

$$A_1 = \frac{1}{\rho} n(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{1}{2}n\alpha + \tfrac{1}{2}\beta\rho^2,$$

$$A_{s-1} = n(\alpha^2 - 1)\frac{d\rho}{d\alpha} + (-\tfrac{1}{2}n + \tfrac{1}{3})\alpha\rho + \tfrac{1}{6}\beta\rho^3.$$

45. In the case $n = 3$, we have $A_{s-1} = A_0 = 1$, $A_1 = \rho$; the two equations become

$$3(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{3}{2}\alpha\rho - \rho^2 + \tfrac{1}{2}\beta\rho^3 = 0,$$

$$3(\alpha^2 - 1)\frac{d\rho}{d\alpha} - 1 - \tfrac{7}{6}\alpha\rho + \tfrac{1}{6}\beta\rho^3 = 0,$$

each of which is easily verified.

I remark also that, in the same case, $(n = 3)$, we have

$$\rho^4 \frac{\beta^2 - 1}{\alpha^2 - 1} = \left(\frac{\rho^2 - 9}{\rho^2 - 1}\right)^2, \text{ and thence } \Theta = \sqrt{\rho}\sqrt[4]{\frac{\beta^2 - 1}{\alpha^2 - 1}} = \sqrt[4]{\frac{\rho^2 - 9}{\rho^2 - 1}};$$

hence

$$\frac{1}{\Theta}\frac{d\Theta}{d\rho} = \frac{4\rho}{(\rho^2 - 1)(\rho^2 - 9)};$$

and writing the equation $A_1 - 2n(\alpha^2 - 1)\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = 0$ in the form

$$\rho - 2n(\alpha^2 - 1)\frac{1}{\Theta}\frac{d\Theta}{d\rho}\frac{d\rho}{d\alpha} = 0,$$

we can verify this equation.

46. In the case $n = 5$, we have for A_1 two equations, each ultimately giving the foregoing value

$$A_1 = \frac{1}{8\alpha}(\rho^2 - 2\rho + 5)(\rho^2 - 4\rho - 1).$$

Moreover, the equation $\Theta = \sqrt{\rho}\sqrt[4]{\frac{\beta^2 - 1}{\alpha^2 - 1}}$ gives, without difficulty, $\Theta = \frac{1}{\sqrt{\rho}}\frac{\rho - 5}{\rho - 1}$.

47. In the case $n = 7$, the formulæ give the two coefficients A_1, A_2; viz. we have

$$A_1 = \frac{1}{\rho} 7(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{7}{2}\alpha + \tfrac{1}{2}\beta\rho^2,$$

$$A_2 = 7(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{19}{6}\alpha\rho + \tfrac{1}{6}\beta\rho^3,$$

where the value of $\frac{d\rho}{d\alpha}$ must of course be obtained from the before-mentioned $\rho\alpha$-equation (given in No. 7). I have not considered these results nor endeavoured to compare them with the results for this case, obtained in the Transformation Memoir and the addition thereto, [578, 692].

48. Substituting the value $1+A_1x^2+A_2x^4+\ldots+A_{s-1}x^{2s-2}+\rho x^{2s}$ in the last-mentioned form of the partial differential equation, we obtain

$$
\begin{aligned}
2A_1 &= \qquad\qquad 2A_1, \\
12A_2 &= -\ 4(n-2)\alpha A_1 + 2A_1^2 \ - n(n-1) \qquad\qquad + 4n(\alpha^2-1)\frac{dA_1}{d\alpha}, \\
30A_3 &= -\ 8(n-4)\alpha A_2 + 2A_1A_2 - (n-2)(n-3)A_1 + 4n(\alpha^2-1)\frac{dA_2}{d\alpha}, \\
56A_4 &= -\ 12(n-6)\alpha A_3 + 2A_1A_3 - (n-4)(n-5)A_2 + 4n(\alpha^2-1)\frac{dA_3}{d\alpha}, \\
&\ldots\ldots\ldots

50. $n=5$, $y=\dfrac{x(\rho+A_1x^2+x^4)}{1+A_1x^2+\rho x^4}$, $\Theta=\dfrac{1}{\sqrt{\rho}}\dfrac{\rho-5}{\rho-1}$, $A_1=\dfrac{1}{8\alpha}(\rho^2-2\rho+5)(\rho^2-4\rho-1)$,

$$64\alpha^2\rho=(\rho^2-4\rho-1)^2(\rho^2-2\rho+5),$$

or say

$$8\alpha\sqrt{\rho}=(\rho^2-4\rho-1)\sqrt{\rho^2-2\rho+5}:$$

$$64(\alpha^2-1)\rho=(\rho-1)^5(\rho-5),$$

$$\sigma=\frac{5}{\rho},\qquad 64\beta^2\sigma=(\sigma^2-4\sigma-1)^2(\sigma^2-2\sigma+5),$$

$$-8\beta\sqrt{\sigma}=(\sigma^2-4\sigma-1)\sqrt{\sigma^2-2\sigma+5},$$

$$64(\beta^2-1)\sigma=(\sigma-1)^5(\sigma-5),$$

$$\frac{\beta}{\alpha}=-\frac{\rho^2+20\rho-5}{\rho^2(\rho^2-4\rho-1)};$$

$\alpha\beta$-equation, see No. 3.

The $\rho\alpha$-equations for the cases in question, $n=3$ and $n=5$, are the so-called Jacobian equations of the fourth and the sixth degrees, studied by Brioschi (in the third appendix above referred to) and by others: the foregoing $\alpha\beta$-equations have not (so far as I am aware) been previously obtained; as rationally connected with the $\rho\alpha$-equations, they must belong to the same class of equations.

Cambridge, England, December 18, 1886.

870.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS (SEQUEL).

[From the *American Journal of Mathematics*, vol. X. (1888), pp. 71–93.]

THE chief object of the present paper is the further development of the $\rho\alpha\beta$-theory in the case $n=7$. I recall that the forms are

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$$

where

$$y=\frac{x(\rho+A_2x^2+A_1x^4+x^6)}{1+A_1x^2+A_2x^4+\rho x^6}.$$

The paragraphs are numbered consecutively with those of the former paper "On the Transformation of Elliptic Functions," vol. IX., pp. 193—224, [869].

The Seventhic Transformation: the $\rho\alpha$-Equation. Art. Nos. 51 to 57.

51. The equation is given incorrectly Nos. 7 and 42; there was an error of sign in a term $512\alpha^3\rho$, which affected also the coefficient of $\alpha\rho$, and an error of sign in the absolute term 7. The correct form is

$$\rho^8-28\rho^6-112\alpha\rho^5-210\rho^4-224\alpha\rho^3+(-1484+1344\alpha^2)\rho^2+(464\alpha-512\alpha^3)\rho-7=0;$$

or, arranging in powers of α, this is

$$\begin{aligned}&\alpha^3 . 512\rho\\ &+\alpha^2 . -1344\rho^2\\ &+\alpha . 112\rho^5+224\rho^3-464\rho\\ &\quad-(\rho^8-28\rho^6-210\rho^4-1484\rho^2-7)=0.\end{aligned}$$

This may also be written in the forms

$$(\alpha-1)\{\alpha^2.512\rho+\alpha(-1344\rho^2+512\rho)+112\rho^5+224\rho^3-1344\rho^2+48\rho\}-(\rho+1)^7(\rho-7)=0,$$

and

$$(\alpha+1)\{\alpha^2.512\rho+\alpha(-1344\rho^2-512\rho)+112\rho^5+224\rho^3+1344\rho^2+48\rho\}-(\rho-1)^7(\rho+7)=0.$$

To simplify the $\rho\alpha$-equation, we assume $A=8\rho\alpha-7\rho^2$; then the $A\rho$-equation is

$$\begin{aligned} & A^3 \\ + & A\rho^2(14\rho^4-119\rho^2-58) \\ - & \rho^2(\rho^8-126\rho^6+280\rho^4-1078\rho^2-7)=0; \end{aligned}$$

viz. this is a cubic equation wanting its second term, and so at once solvable by Cardan's formula: say the equation is

$$A^3+A\rho^2q_1-\rho^2r_1=0,$$

where

$$\begin{aligned} q_1 &= 14\rho^4-119\rho^2-58, \\ r_1 &= \rho^8-126\rho^6+280\rho^4-1078\rho^2-7. \end{aligned}$$

It is convenient to recall here that, writing $\sigma=-\dfrac{7}{\rho}$, and $B=8\sigma\beta-7\sigma^2$, we have between σ, β, B precisely the same equations as between ρ, α, A; $\rho=1$ gives $\sigma=-7$, and we have as corresponding values $\alpha=-1$, $A=-15$, $\beta=-1$, $B=-287$: these are very convenient for verification of the formulæ. Similarly, $\rho=-7$ gives $\sigma=1$, and then $\alpha=-1$, $A=-287$, $\beta=-1$, $B=-15$; but I have, in general, used the former values only.

52. We have

$$A=f+g,$$

where

$$\begin{aligned} 3fg &= -\rho^2q_1, \\ f^3+g^3 &= \rho^2r_1, \end{aligned}$$

and thence

$$f^3-g^3=\rho^2\sqrt{r_1^2+\frac{4\rho^2q_1^3}{27}}.$$

We have identically

$$\begin{aligned} 27(\rho^8-126\rho^6+280\rho^4-1078\rho^2-7)^2+4\rho^2(14\rho^4-119\rho^2-58)^3 \\ =(\rho^6+75\rho^4-141\rho^2+1)^2(27\rho^4+122\rho^2+1323): \end{aligned}$$

if $\rho=1$, this is $27.930^2+4(-163)^3=64^2.1472$; that is, $23352300-17322988=6029312$, which is right; but it is convenient to divide by 27, so as instead of $27\rho^4+122\rho^2+1323$ to have in the formulæ $\rho^4+\frac{122}{27}\rho^2+49$, or say

$$\rho^4+K\rho^2+49 \quad (K=\tfrac{122}{27}).$$

Hence writing

$$t_1 = \rho^6 + 75\rho^4 - 141\rho^2 + 1,$$
$$\delta = \rho^4 + K\rho^2 + 49,$$

we have

$$r_1^2 + \tfrac{4}{27}\rho^2 q_1 = t_1^2\delta,$$

and consequently

$$2f^3 = \rho^2(r_1 + t_1\sqrt{\delta}),$$
$$2g^3 = \rho^2(r_1 - t_1\sqrt{\delta}).$$

53. It was easy to foresee that the cube root of $r_1 \pm t_1\sqrt{\delta}$ would break up into the form $(U \pm \sqrt{\delta})\sqrt[3]{W \pm \sqrt{\delta}}$, and I was led to the actual expressions by the identities

$$20(14\rho^4 - 119\rho^2 - 58) = (19\rho^2 - 53)^2 - 3(27\rho^4 + 122\rho^2 + 1323);$$

that is,

$$20q_1 \qquad = (19\rho^2 - 53)^2 - 81\delta,$$

and

$$27(\rho^2 - 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = -500\rho^2,$$
$$27(\rho^2 + 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = \quad 256\rho^2;$$

or, as these may be written,

$$(\rho^2 - 7)^2 - \delta = -\tfrac{500}{27}\rho^2, \quad (\rho^2 + 7)^2 - \delta = \tfrac{256}{27}\rho^2.$$

We, in fact, have further the two identities

$$\begin{aligned} &1000(\rho^6 + 75\rho^4 - 141\rho^2 + 1) \\ &\quad = \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\} \\ &\qquad + \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7), \\ &-1000(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7) \\ &\quad = \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7) \\ &\qquad + \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(\rho^4 + K\rho^2 + 49), \end{aligned}$$

viz. writing

$$19\rho^2 - 53 = 9U, \quad -\rho^2 + 7 = W,$$

these equations become

$$\tfrac{1000}{729}t_1 = U^3 + 3U\delta + (3U^2 + \delta)W,$$
$$-\tfrac{1000}{729}r_1 = (U^3 + 3U\delta)W + (3U^2 + \delta)\delta,$$

and we have thus

$$-\tfrac{1000}{729}(r_1 - t_1\sqrt{\delta}) = (U + \sqrt{\delta})^2(W + \sqrt{\delta}),$$

and the like equation with $-\sqrt{\delta}$ in place of $\sqrt{\delta}$.

54. In part verification of the last-mentioned identities, observe that, in the first of them, putting $\rho = 1$, and comparing first the coefficients of ρ^6 and then the coefficients of ρ^0, we ought to have

$$1000 = 19^3 + 243 \,.\, 19 - (27 \,.\, 19^2 + 729), \; = 11476 - 10476,$$
$$1000 = (-53^3 - 243 \,.\, 53 \,.\, 49) + (27 \,.\, 53^2 + 729 \,.\, 49)7, \; = -779948 + 780948,$$

which are right; and similarly in the second equation, comparing first the coefficients of ρ^3 and next those of ρ^0, we have

$$\begin{aligned} -1000 &= (19^3+243\,.\,19)(-1)+(27\,.\,19^2+729), \;=-11476+10476,\\ +7000 &= (-53^3-243\,.\,53\,.\,49)(7)+(27\,.\,53^2+729\,.\,49)\,49,\\ &= -5459636+5466636, \end{aligned}$$

which are right.

55. We have now $A=f+g$, where

$$\begin{aligned} f &= -\tfrac{9}{10}(U-\sqrt{\delta})\sqrt[3]{\tfrac{1}{2}\rho^2(W-\sqrt{\delta})},\\ g &= -\tfrac{9}{10}(U+\sqrt{\delta})\sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})}: \end{aligned}$$

observe that, multiplying these two values, we have

$$\begin{aligned} fg = \tfrac{81}{100}(U^2-\delta)\sqrt[3]{\tfrac{1}{4}\rho^4\,.\,(W^2-\delta)}, &= \tfrac{81}{100}(U^2-\delta)\sqrt[3]{\tfrac{1}{4}\rho^4\,.\,\frac{-500}{27}\rho^2},\\ &= \tfrac{81}{100}(U^2-\delta)(-\tfrac{5}{3}\rho^2); \end{aligned}$$

that is,

$$fg = -\tfrac{27}{20}\rho^2(U^2-\delta), \;= -\tfrac{27}{20}\rho^2\,.\,\frac{20q_1}{81} = -\tfrac{1}{3}\rho^2 q_1,$$

which is right. Or, finally, substituting for U, W, δ their values, we have, for the solution of the $A\rho$-equation, $A=f+g$, where

$$\begin{aligned} f &= -\tfrac{9}{10}(19\rho^2-53-\sqrt{\rho^4+K\rho^2+49})\sqrt[3]{\tfrac{1}{2}\rho^2\{-\rho^2+7-\sqrt{\rho^4+K\rho^2+49}\}}, \quad (K=\tfrac{122}{27}),\\ g &= -\tfrac{9}{10}(19\rho^2-53+\sqrt{\rho^4+K\rho^2+49})\sqrt[3]{\tfrac{1}{2}\rho^2\{-\rho^2+7+\sqrt{\rho^4+K\rho^2+49}\}}. \end{aligned}$$

56. In the case $\rho=1$, α has a value $=-1$, giving for A, $=8\rho\alpha-7\rho^2$, the value -15; and, in fact, here $\rho^2=1$, and the $A\rho$-equation becomes

$$A^3-163A+930=0,$$

that is,

$$(A+15)(A^2-15A+62)=0,$$

the roots thus being

$$A=-15, \quad A=\tfrac{1}{2}(15\pm i\sqrt{23}).$$

To verify in this case the values given by the solution of the cubic equation, observe that, for $\rho^2=1$, we have $\delta=50+\frac{122}{27}$, $=\frac{1472}{27}$, and therefore $\sqrt{\delta}=\dfrac{8\sqrt{23}}{3\sqrt{3}}$, $=\dfrac{8\sqrt{69}}{9}$; also, $U=\dfrac{19\rho^2-53}{9}$, $=\dfrac{-34}{9}$, and $W=-\rho^2+7$, $=6$. Hence $U+\sqrt{\delta}=\dfrac{-34+8\sqrt{69}}{9}$, and

$$\sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})}=\sqrt[3]{3+\frac{8\sqrt{69}}{9}}, \;=\frac{\sqrt[3]{81+12\sqrt{69}}}{3};$$

hence

$$g=-\tfrac{9}{10}\,\frac{2(-17+4\sqrt{69})}{9}\,\tfrac{1}{3}\sqrt[3]{81+12\sqrt{69}}, \;=\tfrac{1}{15}(17-4\sqrt{69})\sqrt[3]{81+12\sqrt{69}};$$

but the cube root is $=\frac{1}{2}(3+\sqrt{69})$, and we have $(17-4\sqrt{69})(3+\sqrt{69})=-225+5\sqrt{69}$, $=5(-45+\sqrt{69})$; that is, $g=\frac{1}{6}(-45+\sqrt{69})$. Similarly, $f=\frac{1}{6}(-45-\sqrt{69})$. We have thus the real root $f+g=-15$, and the imaginary roots

$$f\omega+g\omega^2 \text{ or } f\omega^2+g\omega, \; =-\tfrac{15}{2}(\omega+\omega^2)+\tfrac{1}{6}\sqrt{69}\,(\omega-\omega^2),$$

viz. the first term is $=\frac{15}{2}$ and the second is $\pm\frac{1}{6}\sqrt{69}\,.\,i\sqrt{3}$, $=\pm\frac{1}{2}i\sqrt{23}$; thus the roots are $\frac{1}{2}(15\pm i\sqrt{23})$, as they should be.

57. I found, by considerations arising out of the new theory Nos. 72 *et seq.*, that writing for shortness $m=i\sqrt{3}$, then, for $\rho=m-2$, the $\rho\alpha$-equation has a root $\alpha=m$; the corresponding values of $A_1\rho^2$ thus are $A=12m-31$, $\rho^2=-4m+1$, viz. substituting this value for ρ^2 in the $A\rho$-equation, there should be a root $A=12m-31$. The equation becomes

$$A^3+A(3704m-7653)+148306m+206162=0,$$

or, as this may be written,

$$(A-12m+31)\{A^2+A(12m-31)+2960m+4062\}=0,$$

and the roots thus are

$$A=\;\;12m-31,$$
$$A=-6m+\tfrac{31}{2}\pm\tfrac{1}{2}\sqrt{-12584m-16777},$$

where the square root is not expressible as a rational function of m.

Expression of β as a Rational Function of α, ρ. Art. Nos. 58 to 66.

58. Writing $\sigma=-\dfrac{7}{\rho}$, we have β the same function of σ that ρ is of α; hence if $B=8\sigma\beta-7\sigma^2$, the $B\sigma$-equation is

$$\begin{aligned}&B^3\\&+B\sigma^2(14\sigma^4-119\sigma^2-58)\\&-\sigma^2(\sigma^8-126\sigma^6+280\sigma^4-1078\sigma^2-7)=0;\end{aligned}$$

and the expression for B in terms of σ is obtained from that of A by the mere change of ρ into σ. Say we have $B=f'+g'$, where

$$f'=-\tfrac{9}{10}(U'-\sqrt{\delta'})\sqrt[3]{\tfrac{1}{2}\sigma^2(W'-\sqrt{\delta'})},$$
$$g'=-\tfrac{9}{10}(U'+\sqrt{\delta'})\sqrt[3]{\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})};$$

then we have

$$\begin{aligned}\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})&=\tfrac{1}{2}\frac{49}{\rho^2}\left(-\frac{49}{\rho^2}+7+\sqrt{\frac{2401}{\rho^4}+\frac{49K}{\rho^2}+49}\right)\\&=-\tfrac{1}{2}.\frac{343}{\rho^4}(-\rho^2+7-\sqrt{\rho^4+K\rho^2+49})\\&=-\frac{343}{\rho^6}.\tfrac{1}{2}\rho^2(W-\sqrt{\delta}),\end{aligned}$$

or, say

$$\sqrt[3]{\tfrac{1}{2}\sigma^2(W'-\sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\tfrac{1}{2}\rho^2(W-\sqrt{\delta})};$$

and similarly

$$\sqrt[3]{\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})}.$$

The cube roots which enter into the expression of B are thus identical with those in the expression of A, and it hence appears that B can be expressed rationally in terms of A, ρ; or, what is the same thing, β can be expressed rationally in terms of α, ρ.

59. The *à priori* reason is obvious: the $\rho\alpha$-equation is a cubic in α, but of the order 8 in ρ; hence, to a given value of α, there correspond 8 values of ρ. Similarly, the $\sigma\beta$-equation is a cubic in β, but it is of the order 8 in σ; or if for σ we substitute its value $=-\dfrac{7}{\rho}$, then we have a $\rho\beta$-equation which is a cubic in β, but it is of the order 8 in ρ. In the absence of any special relation between this $\rho\beta$-equation and the $\rho\alpha$-equation, there would correspond, to each of the 8 values of ρ, 3 values of β; that is, to a given value of α there would correspond 8×3, $=24$ values of β. But, in fact, to a given value of α, there correspond only 8 values of β, and the two cubic equations are related to each other in such wise that this is so; viz. the relation between them is such that it is possible by means of them to express β as a rational function of ρ, α.

60. Returning to the investigation, we have

$$9U' = 19\sigma^2 - 53, \; = \frac{19\,.\,49}{\rho^2} - 53;$$

or, writing

$$63\overline{U} = 53\rho^2 - 931,$$

this is

$$U' = -\frac{7}{\rho^2}\,\overline{U}, \text{ whence } U' \pm \sqrt{\delta'} = -\frac{7}{\rho^2}(\overline{U} \mp \sqrt{\delta}).$$

Hence writing

$$\theta = \sqrt[3]{\tfrac{1}{2}\rho^2(W-\sqrt{\delta})}, \quad \phi = \sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})},$$

we have

$$f = -\tfrac{9}{10}(U-\sqrt{\delta})\,\theta, \quad f' = -\tfrac{9}{10}\frac{49}{\rho^4}(\overline{U}+\sqrt{\delta})\,\phi,$$

$$g = -\tfrac{9}{10}(U+\sqrt{\delta})\,\phi, \quad g' = -\tfrac{9}{10}\frac{49}{\rho^4}(\overline{U}-\sqrt{\delta})\,\theta,$$

so that, putting for shortness

$$L = -\tfrac{9}{10}(U-\sqrt{\delta}) \;\;, \quad \overline{L} = -\tfrac{9}{10}\frac{49}{\rho^4}(\overline{U}-\sqrt{\delta}),$$

$$M = -\tfrac{9}{10}(U+\sqrt{\delta}) \;\;, \quad \overline{M} = -\tfrac{9}{10}\frac{49}{\rho^4}(\overline{U}+\sqrt{\delta}),$$

we have

$$A = L\theta + M\phi, \quad B = \overline{L}\theta + \overline{M}\phi,$$

where θ^3, ϕ^3 and $\theta\phi$ are each of them free from any cube root; we have, in fact,

$$\theta\phi = \sqrt[3]{\tfrac{1}{4}\rho^4(\overline{W}^2 - \delta)}, \quad = \sqrt[3]{\tfrac{1}{4}\rho^4 \cdot \frac{-500}{27}\rho^2}, \quad = -\tfrac{5}{3}\rho^2,$$

and it may be added that

$$3LM\theta\phi = -\rho^3 q_1, \text{ whence } LM = \tfrac{1}{3}q_1,$$

$$L^3\theta^3 + M^3\phi^3 = \rho^2 r_1,$$

$$L^3\theta^3 - M^3\phi^3 = \rho^3 t_1\sqrt{\delta};$$

these are, in fact, only the equations obtained by writing $L\theta$, $M\phi$ in place of f, g respectively.

61. In the case $\rho = 1$, we have $\sigma = -7$; the equation for B becomes

$$B^3 + 1358525B + 413536578 \quad = 0;$$

that is,

$$(B + 287)(B^2 - 287B + 1440894) = 0,$$

and the roots are

$$-287 \text{ and } \tfrac{1}{2}(287 \pm 497i\sqrt{23}), \text{ or, say } -7.41 \text{ and } \tfrac{7}{2}(41 \pm 71i\sqrt{23}).$$

We have as before, $\sqrt{\delta} = \dfrac{8\sqrt{69}}{9}$, and $\sqrt[3]{\tfrac{1}{2}\rho^3(\overline{W} + \sqrt{\delta})} = \tfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}} = \theta$; also $\overline{U} = \dfrac{-878}{63}$, whence $\overline{U} + \sqrt{\delta} = \dfrac{2(-439 + 28\sqrt{69})}{63}$. We thus have

$$f' = -\tfrac{9}{10}\,49 \cdot \frac{2(-439 + 28\sqrt{69})}{63}\,\tfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}},$$

$$= -\tfrac{7}{15}(-439 + 28\sqrt{69})\sqrt[3]{81 + 12\sqrt{69}},$$

or, putting for the cube root its value $= \tfrac{1}{2}(3 + \sqrt{69})$, this is

$$f' = -\tfrac{7}{30}(-439 + 28\sqrt{69})(3 + \sqrt{69}), \quad = -\tfrac{287}{2} + \tfrac{497}{6}\sqrt{69}.$$

Similarly, $g' = -\tfrac{287}{2} - \tfrac{497}{6}\sqrt{69}$; and forming the values $f' + g'$, $\omega f' + \omega^2 g'$, $\omega^2 f' + \omega g'$, we have the real root -287 and the imaginary roots $\tfrac{1}{2}(287 \pm 497i\sqrt{23})$, as above.

62. We have the equations

$$B = \overline{L}\theta \quad + \overline{M}\phi,$$

$$A = L\theta \quad + M\phi,$$

$$A^2 - 2LM\theta\phi = \frac{M^2\phi^3}{\theta\phi}\theta + \frac{L^2\theta^3}{\theta\phi}\phi,$$

from which, eliminating θ and ϕ so far as they present themselves linearly on the right-hand side, and in the resulting equation replacing $\theta\phi$ and $LM\theta\phi$ by their values, we have

$$\begin{vmatrix} B, & \overline{L}, & \overline{M} \\ A, & L, & M \\ -\frac{5}{3}\rho^2(A^2+\frac{2}{3}\rho^2 q_1), & M^3\phi^3, & L^3\theta^3 \end{vmatrix} = 0;$$

that is,

$$B(L^3\theta^3 - M^3\phi^3) = A(L^2\overline{L}\theta^3 - M^2\overline{M}\phi^3) - \tfrac{5}{3}\rho^2(A^2+\tfrac{2}{3}\rho^2 q_1)(L\overline{M} - \overline{L}M).$$

This may be written

$$B\rho^2 t_1\sqrt{\delta} = A\left\{-\tfrac{729}{1000}\frac{49}{\rho^4}[(U-\sqrt{\delta})^2(\overline{U}-\sqrt{\delta})\tfrac{1}{2}\rho^2(W-\sqrt{\delta}) - (U+\sqrt{\delta})^2(\overline{U}+\sqrt{\delta})\tfrac{1}{2}\rho^2(W+\sqrt{\delta})]\right\}$$

$$-\tfrac{5}{3}\rho^2(A^2+\tfrac{2}{3}\rho^2 q_1)\tfrac{81}{100}\frac{49}{\rho^4}[(U-\sqrt{\delta})(\overline{U}+\sqrt{\delta}) - (U+\sqrt{\delta})(\overline{U}-\sqrt{\delta})],$$

where the terms in [ ] contain each of them the factor $\sqrt{\delta}$. Omitting this factor from the equation, and multiplying by ρ^2, we have

$$B\rho^4 t_1 = \tfrac{81}{100}49\{\tfrac{9}{10}A[(U^2+2U\overline{U}+\delta)W + U^2\overline{U} + (2U+\overline{U})\delta] - \tfrac{10}{3}(A^2+\tfrac{2}{3}\rho^2 q_1)(U-\overline{U})\},$$

which I verify at this stage by writing, as before, $\rho = 1$. We have $B = -287$, $A = -15$, $t_1 = -64$, $q_1 = -163$, $W = 6$, $U = -\frac{34}{9}$, $\overline{U} = \dfrac{-878}{63}$; and, omitting intermediate steps, the equation becomes

$$287 \,.\, 64 = \frac{81 \,.\, 49}{100}\left(\tfrac{2496000}{567} - \tfrac{2233600}{567}\right), \; = \frac{81 \,.\, 49}{100 \,.\, 567}\, 262400, \; = 18368,$$

which is right.

63. We require the values of $(U^2+2U\overline{U}+\delta)W + U^2\overline{U} + (2U+\overline{U})\delta$, and of $U-\overline{U}$: I insert some of the steps of the calculation. We have

$$U^2+2U\overline{U}+\delta = \frac{1}{63^2}\{(133\rho^2-371)(239\rho^2-2233) + 63^2(\rho^4+49) + 3 \,.\, 49 \,.\, 122\rho^2\}$$

$$= \frac{1}{63^2}\{35756\rho^4 - 367724\rho^2 + 1022924\}$$

$$= \frac{4}{567}\{1277\rho^4 - 13133\rho^2 + 36533\}.$$

Multiplying by W, $= -\rho^2 + 7$, we have

$$(U^2+2U\overline{U}+\delta)W = \frac{4}{567}\{-1277\rho^6 + 22072\rho^4 - 128464\rho^2 + 255731\}$$

$$= \frac{4}{5103}\{-11493\rho^6 + 198648\rho^4 - 1156176\rho^2 + 2301579\},$$

$$U^2\overline{U} = \frac{1}{81 \,.\, 63}(19\rho^2-53)^2(53\rho^2-931) = \frac{1}{5103}\{19133\rho^6 - 442833\rho^4 + 2023911\rho^2 - 2615179\},$$

$$(2U+\overline{U})\,\delta=\frac{1}{63\,.\,27}(319\rho^2-1673)(27\rho^4+122\rho^2+1323)$$

$$=\frac{1}{1701}\{8613\rho^6-6253\rho^4+217931\rho^2-221379\}$$

$$=\frac{1}{5103}\{25839\rho^6-\ 18759\rho^4+\ 653793\rho^2-6640137\},$$

whence

$$U^2\overline{U}+(2U+\overline{U})\,\delta=\frac{1}{5103}\{44972\rho^6-461592\rho^4+2677704\rho^2-9255316\}$$

$$=\frac{4}{5103}\{11243\rho^6-115398\rho^4+\ 669426\rho^2-2313829\}.$$

Hence, adding, we obtain

$$(U^2+2U\overline{U}+\delta)\,W+U^2\overline{U}+(2U+\overline{U})\,\delta$$

$$=\ \frac{4}{5103}\ \{-250\rho^6+83250\rho^4-486750\rho^2-12250\}$$

$$=\frac{-1000}{5103}\{\qquad\rho^6-\ 333\rho^4+\ 1947\rho^2+\ 49\}:$$

and we have at once

$$U-\overline{U}=\frac{1}{63}(80\rho^2+560)=\frac{80}{63}(\rho^2+7).$$

64. We now find

$$B\rho^4t_1=-7A\,(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(3A^2+2\rho^2q_1)(\rho^2+7),$$

viz. substituting for t_1, q_1 their values, this is

$$B\rho^4(\rho^6+75\rho^4-141\rho^2+1)=-7A\,(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(3A^2+2\rho^2(14\rho^4-119\rho^2+1))(\rho^2+7),$$

which is the value of B, expressed rationally in terms of ρ, A; it will be observed that B is obtained as a quadric function of A, which is the proper form.

Writing $\rho=-1$, we have $A=-15$, $B=-287$, $t_1=-64$, $q_1=-163$, and the equation is

$$287\,.\,64=105\,.\,1664-56\,.\,349\,.\,8,\ =174720-156352,\ =18368,$$

which is right.

65. Writing for B, A their values $=-\dfrac{56}{\rho}\beta-\dfrac{343}{\rho^2}$, and $8\rho\alpha-7\rho^2$, we have

$$\rho^4\left(-\frac{56}{\rho}\beta-\frac{343}{\rho^2}\right)t_1=(-56\rho\alpha+49\rho^2)(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(192\rho^3\alpha^2-336\rho^2\alpha+147\rho^4+2\rho^2q_1)(\rho^2+7);$$

that is,

$$\begin{aligned}-56\rho^2\beta t_1 = &-56\,.\,192\rho^2(\rho^2+7)\alpha^2\\ &-56\rho\alpha(\rho^6-333\rho^4+1947\rho^2+49)\\ &+56\,.\,336\rho^3\alpha(\rho^2+7)\\ &+49\rho^2(\rho^6-333\rho^4+1947\rho^2+49)\\ &-56(147\rho^4+2\rho^2(14\rho^4-119\rho^2-58))(\rho^2+7)\\ &+343\rho^2(\rho^6+75\rho^4-141\rho^2+1),\end{aligned}$$

where the fourth and sixth lines unite into a term divisible by 56, viz. omitting in the first instance a factor 49, the lines are

$$\rho^8-333\rho^6+1947\rho^4+49\rho^2,$$

and

$$7\rho^8+525\rho^6-\ 987\rho^4+\ 7\rho^2,$$

which together are

$$=8\rho^8+192\rho^6+960\rho^4+56\rho^2,$$

and hence, restoring the factor 49, the lines are

$$=392(\rho^8+24\rho^6+120\rho^4+7\rho^2),$$

and the formula now easily becomes

$$\begin{aligned}\rho^2\beta t_1 = &\,192\rho(\rho^2+7)\alpha^2\\ &+(\rho^6-669\rho^4-405\rho^2+49)\alpha\\ &+\rho(21\rho^6-63\rho^4-1593\rho^2-861),\end{aligned}$$

where the last line is

$$=\rho(\rho^2+7)(21\rho^4-210\rho^2-123).$$

66. Hence, finally, substituting for t_1 its value, we have

$$\beta\rho^2(\rho^6+75\rho^4-141\rho^2+1)=3\rho(\rho^2+7)(64\alpha^2+7\rho^4-70\rho^2-41)+\alpha(\rho^6-669\rho^4-405\rho^2+49),$$

which is the expression for β as a rational function of ρ, α.

Here $\rho=1$, $\alpha=-1$, $\beta=-1$ give $64=-960+1024$, which is right; and again, $\rho=-7$, $\alpha=-1$, $\beta=-1$ give

$$\begin{aligned}-49(117649+180075-6909+1)=&-21\,.\,56(64+16807-3430-41)\\ &-(117649-1606269-19845+49);\end{aligned}$$

that is,

$$-49\,.\,290816=-1176\,.\,13400+1508416,$$

or,

$$-14249984\ =-15758400\quad+1508416,$$

which is right.

The $\alpha\beta$-Differential Equation. Art. No. 67.

67. We have, No. 10,

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{7}\frac{d\alpha}{\alpha^2-1};$$

it should, of course, be possible to verify this equation by means of the $\rho\alpha$-equation and the value just obtained for β. But the expression for $\frac{d\rho}{d\alpha}$ given by the $\rho\alpha$-equation is of so complicated a form that I do not see in what way the verification will come out, and I have not attempted to effect it.

The Coefficients A_1 and A_2. Art. Nos. 68 to 71.

68. These are given by the formulæ No. 47, viz. we have

$$A_1=\frac{1}{\rho}7(\alpha^2-1)\frac{d\rho}{d\alpha}-\tfrac{7}{2}\alpha+\tfrac{1}{2}\beta\rho^2,$$

$$A_2=\quad 7(\alpha^2-1)\frac{d\rho}{d\alpha}-\tfrac{19}{6}\alpha\rho+\tfrac{1}{6}\beta\rho^3,$$

where $\frac{d\rho}{d\alpha}$ and β have each of them to be expressed in terms of ρ, α; we have thus A_1 and A_2, each of them expressible rationally in terms of ρ, α; but I have not attempted to effect the substitutions.

69. The five equations of No. 42, merely collecting the terms, are

$$12A_2-6A_1^2-8\alpha A_1+\rho^4-7=0,$$

$$(-6A_1-32\alpha+2\rho^3)A_2-2A_1^3-8A_1+30\rho=0,$$

$$(\rho^3-4)A_2^2+(-4A_1^2-8\alpha A_1+6)A_2-5A_1^2+(2\rho^3+4\rho)A_1-72\alpha\rho=0,$$

$$-2A_1A_2^2+\{(2\rho^3-4)A_1-6\rho\}A_2-4\rho A_1^2-32\rho\alpha A_1+2\rho^3+28\rho=0,$$

$$-3A_2^2+(-4\rho A_1+2\rho^2-8\alpha\rho)A_2+\rho^2A_1^2+10\rho A_1-6\rho^2=0,$$

which would, of course, be all of them satisfied by the values of A_1, A_2 as rational functions of ρ, α, viz. the substitution of these values in any one of the equations would give a function of ρ and α, containing as a factor the expression on the left-hand side of the $\rho\alpha$-equation.

70. Or again, the equations should determine A_1 and A_2 as rational functions of ρ, α, but there is no obvious way of finding such values in a simple form. We, of course, have

$$12A_2=6A_1^2+8\alpha A_1-\rho^4+7,$$

and using this value to eliminate A_3 from the remaining equations, we find the following four equations:

$$\begin{aligned}
&A_1^3 . 30 + A_1^2(120\alpha - 6\rho^3) + A_1\{128\alpha^2 - 8\rho^3\alpha - 3\rho^4 + 69\} \\
&\qquad\qquad + \alpha(-16\rho^4 + 112) + \rho^7 - 7\rho^3 - 180\rho = 0, \\
&A_1^4(36\rho^2 - 432) \quad + A_1^3\alpha(96\rho^2 - 1344) \\
&\qquad + A_1^2\{\alpha^2(64\rho^2 - 1024) - 12\rho^6 + 48\rho^4 + 84\rho^2 - 624\} \\
&\qquad + A_1\{\alpha(-16\rho^6 + 160\rho^4 + 112\rho^2 - 544) + 288\rho^3 + 576\rho\} \\
&\qquad + \{-10368\alpha\rho + \rho^{10} - 4\rho^8 - 14\rho^6 - 16\rho^4 + 49\rho^2 - 308\} = 0, \\
&A_1^5 . 36 + A_1^4 . 96\alpha + A_1^3\{64\alpha^2 - 12\rho^4 - 72\rho^2 + 208\} \\
&\qquad + A_1^2\{\alpha(-16\rho^4 - 96\rho^2 + 304) + 504\rho\} \\
&\qquad + A_1\{\alpha . 2592\rho + \rho^8 - 12\rho^6 + 10\rho^4 + 84\rho^2 - 119\} \\
&\qquad + 36\rho^5 - 144\rho^3 - 2268\rho = 0, \\
&A_1^4 . 36 + A_1^3(96\alpha + 96\rho) + A_1^2\{64\alpha^2 + 320\alpha\rho - 12\rho^4 - 96\rho^2 + 84\} \\
&\qquad + A_1\{256\alpha^2\rho + \alpha(-80\rho^4 + 112) - 16\rho^5 - 368\rho\} \\
&\qquad + \{\alpha(-32\rho^5 + 224\rho) + \rho^8 + 8\rho^6 - 14\rho^4 + 232\rho^2 + 49\} = 0,
\end{aligned}$$

and we could from these equations obtain various rational expressions for A_1 and its powers; but these would apparently be of degrees far too high in ρ and α.

71. It is to be remarked that, for $\rho = 1$, $\alpha = -1$, the values of A_1, A_2 are $A_1 = A_2 = 3$, viz. these belong to the solution

$$y = \frac{x(1 + 3x^2 + 3x^4 + x^6)}{1 + 3x^2 + 3x^4 + x^6}, \ = x, \text{ of } \frac{dy}{1 + y^2} = \frac{dx}{1 + x^2};$$

and that for $\rho = -7$, $\alpha = -1$, the values are $A_1 = -21$, $A_2 = 35$, viz. these belong to the solution

$$y = \frac{-7x + 35x^3 - 21x^5 + x^7}{1 - 21x^2 + 35x^4 - 7x^6} \text{ of } \frac{dy}{1 + y^2} = \frac{-7dx}{1 + x^2}.$$

For example, the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ becomes, for the first set of values, $36 = 54 - 24 - 1 + 7$, and for the second set of values, $420 = 2646 + 168 - 2401 + 7$, which are each of them right.

New Form of the Seventhic Transformation. Art. Nos. 72 to 83.

72. For the quartic function $1 - 2\alpha x^2 + x^4$, the coefficients a, b, c, d, e are $= 1, 0, -\frac{1}{3}\alpha, 0, 1$, and hence the invariants I, J and the discriminant Δ are

$$\begin{aligned}
&I = 1 + \tfrac{1}{3}\alpha^2, \ = \tfrac{1}{3}(\alpha^2 + 3), \\
&J = -\tfrac{1}{3}\alpha + \tfrac{1}{27}\alpha^3, \ = \tfrac{1}{27}(\alpha^3 - 9\alpha), \\
&\Delta = I^3 - 27J^2, \ = \tfrac{1}{27}\{(\alpha^2 + 3)^3 - (\alpha^3 - 9\alpha)^2\}, \ = (\alpha^2 - 1)^2, \text{ whence } \sqrt[12]{\Delta} = \sqrt[6]{\alpha^2 - 1}.
\end{aligned}$$

This being so, then assuming

$$\rho = p\frac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}},$$

the differential equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho\, dx}{\sqrt{1-2\alpha x^2+x^4}}$$

becomes

$$\frac{\sqrt[6]{\beta^2-1}\, dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{p\sqrt[6]{\alpha^2-1}\, dx}{\sqrt{1-2\alpha x^2+x^4}},$$

viz. this is, for the radicals $\sqrt{1-2\alpha x^2+x^4}$ and $\sqrt{1-2\beta y^2+y^4}$, the form considered by Klein in the paper "Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades," *Math. Ann.*, t. XIV. (1879), pp. 111—172. I notice that there is some error as to a factor 7, and that p is equal to the z of p. 148, not, as might appear, equal to $\frac{1}{7}z$.

73. The modular equation presents itself in the form given, *l.c.*, p. 143, viz. this is

$$\mathbf{J} : \mathbf{J}-1 : 1 = (\tau^3+13\tau+49)(\tau^2+5\tau+1)^3 : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau,$$

with the like relation in $\mathbf{J}'$, τ'; and then $\tau\tau' = 49$. We have thus $\mathbf{J}$, $\mathbf{J}'$ each given as a function of τ; and thence by elimination of τ, we have the modular equation as a relation between the absolute invariants $\mathbf{J}$, $\mathbf{J}'$. But $\tau = p^2$, and for the form $1-2\alpha x^2+x^4$, as appears above, we have

$$\mathbf{J}-1, = \frac{27J^2}{\Delta}; = \frac{\frac{1}{27}(\alpha^3-9\alpha)^2}{(\alpha^2-1)^2};$$

hence Klein's equation

$$\mathbf{J}-1 = \frac{(\tau^4+14\tau^3+63\tau^2+70\tau-7)^2}{1728\tau}$$

becomes

$$\frac{\alpha^3-9\alpha}{\alpha^2-1} = \frac{p^8+14p^6+63p^4+70p^2-7}{8p};$$

or, say

$$p^8+14p^6+63p^4+70p^2-8\left(\frac{\alpha^3-9\alpha}{\alpha^2-1}\right)p-7=0,$$

which is the equation, *l.c.*, p. 148 with p for z; viz. this is the $p\alpha$-equation connecting α with the new multiplier p. It will be observed that it is of the degree 8 in p, and the degree 3 in α, viz. it resembles herein the foregoing $\rho\alpha$-equation, but the form is very much more simple, inasmuch as the α enters into a single coefficient only. The equation may also be written

$$(p^4+5p^2+1)^3(p^4+13p^2+49)-64\frac{(\alpha^2+3)^3}{(\alpha^2-1)^2}p^2=0.$$

74. Using for shortness a single letter m to denote the value $i\sqrt{3}$, we have

$$\frac{\alpha^3-9\alpha+3m(\alpha^2-1)}{\alpha^3-9\alpha-3m(\alpha^2-1)} = \frac{p^8+14p^6+63p^4+70p^2+24mp-7}{p^8+14p^6+63p^4+70p^2-24mp-7};$$

that is,

$$\left(\frac{\alpha+m}{\alpha-m}\right)=\frac{(p^2-mp+1)^2(p^2+3mp-7)}{(p^2+mp+1)^3(p^2-3mp-7)},$$

or, say

$$\frac{\alpha+m}{\alpha-m}=\frac{p^2-mp+1}{p^2+mp+1}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

which is another form of the $p\alpha$-equation.

75. We had $\tau=p^2$; and similarly, writing $\tau'=q^2$, then $\tau\tau'=49=p^2q^2$; it must be assumed that $pq=-7$; β is then the same function of q which α is of p, viz. we have

$$\frac{\beta+m}{\beta-m}=\frac{q^2-mq+1}{q^2+mq+1}\sqrt[3]{\frac{q^2+3mq-7}{q^2-3mq-7}}.$$

These equations in α and β contain the same cubic radical, viz. we have

$$q^2+3mq-7,\ =\frac{49}{p^2}-\frac{21m}{p}-7,\ =-\frac{7}{p^2}(p^2+3mp-7),$$

and similarly

$$q^2-3mq-7 \qquad =-\frac{7}{p^2}(p^2-3mp-7).$$

Moreover

$$q^2-\ mq+1,\ =\frac{49}{p^2}+\frac{7m}{p}+1,\ =\ \frac{1}{p^2}(p^2+7mp+49),$$

and similarly

$$q^2+\ mq+1 \qquad =\ \frac{1}{p^2}(p^2-7mp+49),$$

and we thus obtain

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

whence, eliminating the cubic radical,

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\,\frac{p^2+mp+1}{p^2-mp+1}\,\frac{\alpha+m}{\alpha-m},$$

viz. this gives β as a rational function of α, p. We in fact have

$$\beta=\frac{\alpha(p^4+29p^2+49)-24p(p^2+7)}{8\alpha p(p^2+7)+(p^4+29p^2+49)}.$$

76. The differential relation

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{7}\,\frac{d\alpha}{\alpha^2-1},$$

on substituting therein for ρ its value, becomes

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{p^2}{7}\,\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}.$$

But, from the expression for $\dfrac{\alpha+m}{\alpha-m}$, we obtain

$$d\alpha\left(\frac{1}{\alpha+m}-\frac{1}{\alpha-m}\right)=dp\left\{\left(\frac{2p-m}{p^2-mp+1}-\frac{2p+m}{p^2+mp+1}\right)+\tfrac{1}{3}\left(\frac{2p+3m}{p^2+3mp-7}-\frac{2p-3m}{p^2-3mp-7}\right)\right\},$$

or, omitting from each side a factor $-2m$,

$$\frac{d\alpha}{\alpha^2+3}=dp\left(\frac{-p^2+1}{p^4+5p^2+1}+\frac{p^2+7}{p^4+13p^2+49}\right)=\frac{56dp}{(p^4+5p^2+1)(p^4+13p^2+49)}.$$

But we have, No. 73,

$$\frac{\alpha^2+3}{(\alpha^2-1)^{\frac{2}{3}}}=\frac{(p^4+5p^2+1)(p^4+13p^2+49)^{\frac{1}{3}}}{4p^{\frac{2}{3}}},$$

and thence

$$\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}=\frac{14dp}{p^{\frac{2}{3}}(p^4+13p^2+49)^{\frac{2}{3}}},$$

and similarly

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{14dq}{q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}}.$$

The equation $q=-\dfrac{7}{p}$ gives

$$dq=\frac{7dp}{p^2},\quad q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}=49p^{-\frac{10}{3}}(p^4+13p^2+49)^{\frac{2}{3}},$$

and we thence have

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{2p^{\frac{4}{3}}dp}{(p^4+13p^2+49)^{\frac{2}{3}}},\quad =\frac{p^2}{7}\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}},$$

the required relation.

77. From the value of ρ, we have

$$\frac{d\rho}{\rho}=\frac{dp}{p}+\frac{\frac{1}{3}\alpha d\alpha}{\alpha^2-1}-\frac{\frac{1}{3}\beta d\beta}{\beta^2-1},$$

which, substituting for $d\beta$ its value, becomes

$$=\frac{dp}{p}+\frac{\frac{1}{3}d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\frac{p^2}{7}\right\},$$

or, say

$$\frac{1}{\rho}\frac{d\rho}{d\alpha}=\frac{1}{p}\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\frac{p^2}{7}\right\},$$

which, however, is more conveniently written

$$\frac{1}{\rho}\frac{d\rho}{d\alpha}=\frac{1}{p}\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{\alpha^2-1}(\alpha-\beta\rho^2);$$

and then, substituting in the formulæ for A_1, A_2, we find

$$A_1 = 7(\alpha^2-1)\frac{1}{p}\frac{dp}{d\alpha} - \tfrac{7}{6}\alpha + \tfrac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}A_2 = 7(\alpha^2-1)\frac{1}{p}\frac{dp}{d\alpha} - \tfrac{5}{6}\alpha - \tfrac{1}{6}\beta\rho^2,$$

expressions which give, as they should do, $A_2 - \rho A_1 = \frac{1}{3}(\alpha\rho - \beta\rho^3)$. In these last formulæ, ρ is to be regarded as standing for its value, $= p\dfrac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}}$.

78. To further reduce these values, consider the expression of β given No. 75. If for a moment we represent this by

$$\beta = \frac{F\alpha - 3G}{G\alpha + F}, \text{ where } F = p^4 + 29p^2 + 49,\ G = 8p(p^2+7),$$

then we have

$$\beta^2 - 1 = \frac{(F^2-G^2)\alpha^2 - 8FG\alpha + 9G^2 - F^2}{(G\alpha+F)^2},$$

or, multiplying the numerator and denominator each by $G\alpha + F$, so as to make the denominator a perfect cube, the numerator becomes

$$G(F^2-G^2)(\alpha^3 - 9\alpha) + F(F^2 - 9G^2)(\alpha^2 - 1);$$

and putting for the factor G of the first term its value $= 8p(p^2+7)$, we thus obtain

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(F^2-G^2)(p^2+7)\,8p\left(\dfrac{\alpha^2-9\alpha}{\alpha^2-1}\right) + F(F^2-9G^2)}{(G\alpha+F)^3},$$

viz. in virtue of the $p\alpha$-equation, this is

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(F^2-G^2)(p^2+7)(p^8+14p^6+63p^4+70p^2-7) + F(F^2-9G^2)}{(G\alpha+F)^3}.$$

This numerator is $=(p^4+5p^2+1)^3p^6$; in fact, we have

$$\begin{aligned}(F^2-G^2)(p^2+7) &= p^{10} + p^8 + p^6 + 7p^4 + 343p^2 + 16807,\\ F^2 - 9G^2 &= p^8 - 518p^6 - 7125p^4 - 25382p^2 + 2401,\end{aligned}$$

and thence forming the two terms of the numerator and adding them together—for shortness I write down only the coefficients—we have

	1	15	78	154	567	22113	257390	1082802	1174089	− 117649
				1	− 489	− 22098	− 257389	− 1082802	− 1174089	117649
=	1	15	78	155	78	15	1	0	0	0,

viz. these are the coefficients of $(p^4+5p^2+1)^3p^6$. Hence

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(p^4+5p^2+1)^3p^6}{(G\alpha+F)^3};$$

or, extracting the cube root, and for G, F substituting their values,

$$\frac{\sqrt[3]{\beta^2-1}}{\sqrt[3]{\alpha^2-1}}=\frac{(p^4+5p^2+1)\,p^2}{8p\,(p^2+7)\,\alpha+p^4+29p^2+49},$$

and thence also

$$\rho^2=\frac{8p\,(p^2+7)\,\alpha+p^4+29p^2+49}{p^4+5p^2+1},$$

viz. we have thus ρ^2 expressed as a rational function of p, α.

79. It will presently appear that ρ is, in fact, expressible as a rational function of p, α: but I am unable to obtain this expression in a simple form. Admitting that ρ is thus expressible, a direct process for obtaining the expression is as follows. Writing

$$\xi=\frac{8p\,(p^2+7)\,\alpha+p^4+29p^2+49}{p^4+5p^2+1}\,(=\rho^2),$$

and by means hereof introducing ξ in place of α into the equation

$$p^8+14p^6+63p^4+70p^2-8p\,\frac{\alpha^3-9\alpha}{\alpha^2-1}-7=0,$$

we have for ξ a cubic equation,

$$a\xi^3+b\xi^2+c\xi+d=0,$$

where the coefficients a, b, c, d are given rational functions of p. This equation may be written

$$a\xi\,(\xi+\vartheta)^2+b'\xi^2+c'\xi+d=0,$$

where $b'=b-2a\vartheta$, $c'=c-a\vartheta^2$; and the last three terms will be a square if only $c'^2-4b'd=0$; that is, if

$$(a\vartheta^2-c)^2+4d\,(2a\vartheta-b)=0,$$

a biquadratic equation in ϑ which (ρ being expressible as above) must have one of its roots = a rational function of p. Calling this ϑ, we then have

$$a\xi\,(\xi+\vartheta)^2+\frac{1}{b'}(b'\xi+\tfrac{1}{2}c')^2=0,\text{ or say } a\rho^2\,(\xi+\vartheta)^2+\frac{1}{b'}(b'\xi+\tfrac{1}{2}c')^2=0,$$

hence

$$\rho=\sqrt{\frac{-1}{ab'}}\cdot\frac{b'\xi+\frac{1}{2}c'}{\xi+\vartheta},$$

where ξ denotes a linear function of α as above; the quadric radical will have a rational value, and the form of the equation thus is

$$\rho=\frac{A\alpha+B}{C\alpha+D},$$

where A, B, C, D are rational and integral functions of p. But I am not able to carry out the process.

80. As shown, No. 78, we have

$$\rho^2 = \frac{8p(p^2+7)\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1}.$$

Multiplying by the value of β, *ante* No. 75, we find

$$\beta\rho^2 = \frac{(p^4 + 29p^2 + 49)\alpha - 24p(p^2+7)}{p^4 + 5p^2 + 1};$$

and we can hence find A_1 and A_2 by the formulæ

$$A_1 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \tfrac{7}{6}\alpha + \tfrac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}A_2 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \tfrac{5}{6}\alpha - \tfrac{1}{6}\beta\rho^2,$$

or, for the second of these we may write

$$\frac{1}{\rho}A_2 = A_1 + \tfrac{1}{3}(\alpha - \beta\rho^2).$$

But in a different point of view, regarding only ρ^2, but not ρ, as a given function of p, α, we must to these equations join the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$, *ante* No. 69: and we have thus equations for the determination of A_1, A_2, and ρ.

81. We have

$$A_1 = \frac{(p^4 + 5p^2 + 1)(p^4 + 13p^2 + 49)}{8p}\frac{\alpha^2 - 1}{\alpha^2 + 3}$$
$$- \tfrac{7}{6}\alpha + \frac{\alpha(p^4 + 29p^2 + 49) - 24p(p^2+7)}{6(p^4 + 5p^2 + 1)},$$

where the second line is

$$= \frac{\alpha(-p^4 - p^2 + 7) - 4p(p^2+7)}{p^4 + 5p^2 + 1}.$$

Uniting the two terms, we have a denominator $8p(p^4 + 5p^2 + 1)$, and in the numerator a term $8p\alpha^2$ which may be got rid of by means of the $p\alpha$-equation; the numerator thus becomes

$$\begin{aligned} &= 96p(-p^4 - p^2 + 7) - 128p^2(p^2+7)\alpha \\ &\quad + (\alpha^2 - 1)\{(-p^4 - p^2 + 7)(p^8 + 14p^6 + 63p^4 + 70p^2 - 7)\} \\ &\quad + (p^4 + 5p^2 + 1)^2(p^4 + 13p^2 + 49) - 32p^2(p^2+7), \end{aligned}$$

where the whole divides by $8p$; and we finally obtain

$$A_1 = \frac{12(-p^4 - p^2 + 7) - 16p(p^2+7)\alpha + (\alpha^2 - 1)p(p^8 + 17p^6 + 102p^4 + 225p^2 + 97)}{(\alpha^2 + 3)(p^4 + 5p^2 + 1)}.$$

Proceeding to calculate the value of $A_1 + \frac{1}{3}(\alpha - \beta\rho^2)$, we then have

$$\tfrac{1}{3}(\alpha - \beta\rho^2) = \frac{-8(p^2+2)\alpha + 8p(p^2+7)}{p^4 + 5p^2 + 1}.$$

Multiplying the numerator and denominator by α^2+3, we have in the numerator a term in $8\alpha^3$, which may be got rid of by means of the $p\alpha$-equation; the numerator for $\frac{1}{\rho}A_2$ thus becomes

$$12(-p^4-9p^2-9)+16p(p^2+7)\alpha$$
$$+(\alpha^2-1)p\left[\{p^8+17p^6+102p^4+225p^2+97\}-\frac{p^2+2}{p^2}(p^8+14p^6+63p^4+70p^2-7)+8(p^2+7)\right],$$

and we finally obtain

$$\frac{1}{\rho}A_2=\frac{12(-p^4-9p^2-9)+16p(p^2+7)\alpha+(\alpha^2-1)p^{-1}(p^8+11p^6+37p^4+20p^2+2)}{(\alpha^2+3)(p^4+5p^2+1)}.$$

82. The expressions obtained above for ρ^2, A_1, A_2 are of the form

$$\rho^2=\frac{M+N\alpha}{S},\quad A_1=\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)},\quad \frac{1}{\rho}A_2=\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)},$$

where

$$\begin{aligned}
M&=p^4+29p^2+49; & N&=8p(p^2+7); & S&=p^4+5p^2+1,\\
P_1&=12(-p^4-\ \ p^2+7)-p\ \ (p^8+17p^6+102p^4+225p^2+97), & Q_1&=-16p(p^2+7),\\
R_1&=\qquad\qquad p\ \ (p^8+17p^6+102p^4+225p^2+97); & &\\
P_2&=12(-p^4-9p^2-9)-p^{-1}(p^8+11p^6+\ \ 37p^4+\ \ 20p^2+\ \ 2), & Q_2&=\ \ 16p(p^2+7),\\
R_2&=\qquad\qquad p^{-1}(p^8+11p^6+\ \ 37p^4+\ \ 20p^2+\ \ 2); & &
\end{aligned}$$

substituting these values in the foregoing equation

$$12A_2=6A_1^2+8\alpha A_1-\rho^4+7,$$

we obtain

$$12\rho\left\{\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)}\right\}=\left\{\frac{6(P_1+Q_1\alpha+R_1\alpha^2)^2}{S^2(\alpha^2+3)^2}+8\alpha\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)}-\frac{(M+N\alpha)^2}{S^2}+7\right\};$$

that is,

$$\rho=\frac{1}{12(P_2+Q_2\alpha+R_2\alpha^2)^2(3+\alpha^2)S}\{6(P_1+Q_1\alpha+R_1\alpha^2)^2+8\alpha S(3+\alpha^2)(P_1+Q_1\alpha+R_1\alpha^2)$$
$$-(M+N\alpha)^2(3+\alpha^2)^2+7S^2(3+\alpha^2)^2\},$$

which, by means of the $p\alpha$-equation

$$p^8+14p^6+63p^4+70p^2-\left(\frac{\alpha^2-9\alpha}{\alpha^2-1}\right)8p-7=0,$$

should be reducible to the form

$$\rho=A\alpha^2+B\alpha+C,\ \text{or}\ \rho=\frac{A\alpha+B}{C\alpha+D};$$

but I have not been able to obtain, in either of these forms, a simple expression of ρ as a function of p, α. Supposing it obtained, the $\rho\alpha$-equation, *ante* No. 51, would of course be thereby transformable into the foregoing $p\alpha$-equation. And considering p as an auxiliary parameter thus introduced into the formulæ in place of ρ, then β and the coefficients A_1, A_2 are, by what precedes, expressed in terms of p, α, that is, in effect in terms of ρ, α; and we thus have the formulæ of transformation for the $\rho\alpha\beta$-form.

83. There exists a remarkably simple particular case. Write for convenience $\theta = \sqrt{7}$, the $p\alpha$-equation is satisfied by the values $p = -\theta$, $\alpha = -\frac{3}{8}\theta$. In fact, these values give $8p\alpha = 3\theta^2$, $= 21$, $\dfrac{\alpha^2 - 9}{\alpha^2 - 1} = (\frac{63}{64} - 9) \div (\frac{63}{64} - 1)$, $= 513$; the term in α is thus $21 . 513$, $= 10773$; but, assuming $p^8 = 7$, we have

$$p^8 + 14p^6 + 63p^4 + 70p^2 - 7 = 2401 + 4802 + 3087 + 490 - 7, \quad = 10773,$$

and the equation is thus satisfied. And these values, $p = -\theta$, $\alpha = -\frac{3}{8}\theta$, give $\rho^2 = 7$, $\beta = \frac{3}{8}\theta$, $A_1 = 2\theta$, $A_2 = \rho\theta$; the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ thus becomes $12\rho\theta = 168 - 42 - 49 + 7$, $= 84$; that is, $\rho\theta = 7$, $= \theta^2$, or $\rho = \theta\,(= -p)$. We have $\alpha^2 - 1$, $= \beta^2 - 1$, $= -\frac{1}{64}$; but from the equation $\rho = p\dfrac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}}$, it appears that the sixth roots must be equal with opposite signs, say $\sqrt[6]{\alpha^2 - 1} = \dfrac{i}{2}$, $\sqrt[6]{\beta^2 - 1} = \dfrac{-i}{2}$. Retaining θ to stand for its value $= \sqrt{7}$, the differential equation is

$$\frac{dy}{\sqrt{1 - \frac{3}{4}\theta y^2 + y^4}} = \frac{\theta dx}{\sqrt{1 + \frac{3}{4}\theta x^2 + x^4}},$$

and it is satisfied by

$$y = \frac{x(\theta + 7x^2 + 2\theta x^4 + x^6)}{1 + 2\theta x^2 + 7x^4 + \theta x^6}.$$

It may be remarked that the quartic functions of y and x resolved into their linear factors are

$$\left\{y + \frac{3i + \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{3i - \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{-3i + \theta}{2\sqrt{2}(1 - i)}\right\}\left\{y + \frac{-3i - \theta}{2\sqrt{2}(1 - i)}\right\},$$

and

$$\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 - i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 - i)}\right\}$$

and that for the first of the y-factors, substituting for y its value, we have

$$x^7 + 2\theta x^5 + 7x^3 + \theta x + \frac{3i + \theta}{2\sqrt{2}(1 + i)}(\theta x^6 + 7x^4 + 2\theta x^2 + 1)$$
$$= \left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x^3 + \frac{1 + i\theta}{\sqrt{2}(1 + i)}x^2 + \tfrac{1}{2}(i + \theta)x + \frac{1 + i}{\sqrt{2}}\right\}^2,$$

with like expressions for the other y-factors respectively.

Brioschi's Transformation Theory. Art. No. 84.

84. M. Brioschi has kindly referred me to two papers by him, "Sur une Formule de Transformation des Fonctions Elliptiques," *Comptes Rendus*, t. LXXIX. (1874), pp. 1065—1069, and *ibid.* t. LXXX. (1875), pp. 261—264. They relate to the form

$$\frac{dx}{\sqrt{4x^3 - g_2x - g_3}} = \frac{dy}{\sqrt{4y^3 - G_2y - G_3}},$$

with a formula of transformation

$$y = \frac{U}{T^2}, \quad T = x^\nu + a_1 x^{\nu-1} + a_2 x^{\nu-2} + \ldots + a_\nu, \text{ where } \nu = \tfrac{1}{2}(n-1),$$

$$U = x^n + \alpha_1 x^{n-2} + \alpha_2 x^{n-3} + \ldots + \alpha_\nu.$$

The general theory for any value of n is developed to a considerable extent, and it would without doubt give very interesting results for the case $n = 7$; but the formulæ are only completely worked out for the preceding two cases $n = 3$ and $n = 5$. For these cases the formulæ are as follows:

Cubic transformation: $n = 3$,

$$y = \frac{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}{(x + a_1)^2}.$$

Corresponding to the modular equation, we have

$$a_1^4 - \tfrac{1}{2} g_2 a_1^2 + g_3 a_1 - \tfrac{1}{48} g_2^2 = 0,$$

and then

$$G_2 - 9g_2 = 6\,(20a_1^2 - 3g_2), \quad G_3 + 27g_3 = -14\,(20a_1^2 - 3g_2)\,a_1,$$

whence also

$$a_1 = -\tfrac{3}{7}\,\frac{G_3 + 27g_3}{G_2 - 9g_2};$$

and by the general theory α_1, α_2, α_3 are given rationally in terms of a_1, g_2, g_3.

Quintic transformation: $n = 5$,

$$y = \frac{x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5}{(x^2 + a_1 x + a_2)^2}.$$

We have

$$a_1 X - 2Y = 0, \quad (12a_1^2 + g_2)\,X - 30a_1 Y = 0,$$

where

$$X = a_1^3 - 6a_1^2 a_2 + \tfrac{1}{2} g_2 a_1 - g_3,$$

$$Y = 5a_2^2 - a_1^2 a_2 + \tfrac{1}{2} g_2 a_2 - g_3 a_1 + \tfrac{1}{16} g_2^2.$$

The first of these gives

$$a_2 = \frac{1}{6a_1}\,(a_1^3 + \tfrac{1}{2} g_2 a_1 - g_3);$$

then eliminating a_2, we have, corresponding to the modular equation,

$$a_1^6 - 5g_2 a_1^4 + 40g_3 a_1^3 - 5g_2^2 a_1^2 + 8g_2 g_3 a_1 - 5g_3^2 = 0.$$

We then have

$$G_2 - 25g_2 = \frac{8}{a_1}\,(10a_1^3 - 8g_2 a_1 + 5g_3), \quad G_3 + 125g_3 = -14\,(10a_1^3 - 8g_2 a_1 + 5g_3);$$

whence also

$$a_1 = -\tfrac{4}{7}\,\frac{G_3 + 125g_3}{G_2 - 25g_2};$$

and by the general theory α_1, α_2, α_3, α_4, α_5 are given rationally in terms of a_1, g_2, g_3.

These results are contained in the former of the papers above referred to; the latter contains some properties of these modular equations.

871.

A CASE OF COMPLEX MULTIPLICATION WITH IMAGINARY MODULUS ARISING OUT OF THE CUBIC TRANSFORMATION IN ELLIPTIC FUNCTIONS.

[From the *Proceedings of the London Mathematical Society*, vol. XIX. (1888), pp. 300, 301.]

THE case in question is referred to in my "Note on the Theory of Elliptic Integrals," *Math. Ann.*, t. XII. (1877), pp. 143—146, [657]; but I here work it out directly.

In the cubic transformation, the modular equation is

$$u^4 - v^4 + 2uv\,(1 - u^2v^2) = 0\,;$$

and we have

$$y = \frac{\left(1 + \dfrac{2u^3}{v}\right)x + \dfrac{u^6}{v^2}\,x^3}{1 + vu^2\,(v + 2u^3)\,x^2},$$

giving

$$\frac{dy}{\sqrt{1 - y^2\,.\,1 - v^8y^2}} = \frac{\left(1 + \dfrac{2u^3}{v}\right)dx}{\sqrt{1 - x^2\,.\,1 - u^8x^2}}.$$

We thus have a case of complex multiplication if $v^8 = u^8$, or say $v = \gamma u$, where $\gamma^8 = 1$, or γ denotes an eighth root of unity. Substituting in the modular equation, this becomes

$$u^4\,(1 - \gamma^4) + 2\gamma u^2\,(1 - \gamma^2u^4) = 0,$$

or, throwing out the factor u^2 and reducing,

$$u^4 - \tfrac{1}{2}u^2\,(\gamma^5 - \gamma) - \gamma^6 = 0,$$

that is,

$$\frac{u^2}{\gamma} = \tfrac{1}{4}\,(\gamma^4 - 1 \pm \sqrt{\gamma^8 + 14\gamma^4 + 1}),$$

or, what is the same thing,

$$= \tfrac{1}{4}\{\gamma^4 - 1 \pm \sqrt{14\gamma^4 + 2}\}.$$

We have $\gamma^8 = 1$, that is, $\gamma^4 = \pm 1$. Considering first the case $\gamma^4 = 1$, here

$$\frac{u^2}{\gamma} = \pm 1,$$

and thence

$$1 + \frac{2u^3}{v} = 1 + \frac{2u^2}{\gamma}, \ = 1 \pm 2, \ = 3 \text{ or } -1;$$

moreover, $u^8 = v^8 = 1$. We have thus only the non-elliptic formulæ

$$\frac{dy}{1-y^2} = \frac{-dx}{1-x^2}, \text{ satisfied by } y = -x,$$

and

$$\frac{dy}{1-y^2} = \frac{3dx}{1-x^2}, \qquad \text{by } y = \frac{3x + x^3}{1 + 3x^2}.$$

If however, $\gamma^4 = -1$, then

$$\frac{u^2}{\gamma} = \tfrac{1}{4}(-2 \pm \sqrt{-12}),$$

viz. this is

$$\frac{u^2}{\gamma} = \tfrac{1}{2}(-1 \pm i\sqrt{3}) = \omega,$$

if ω be an imaginary cube root of unity ($\omega^2 + \omega + 1 = 0$); hence

$$u^8 = (\gamma\omega)^4 = -\omega.$$

Moreover,

$$1 + \frac{2u^3}{v} = 1 + \frac{2u^2}{\gamma}, \ = 1 + 2\omega,$$

or say,

$$= \omega - \omega^2, \quad [= \sqrt{-3}, \text{ if } \omega = \tfrac{1}{2}(-1 + i\sqrt{3})];$$

and we thus have, as in the above-mentioned Note,

$$y = \frac{(\omega - \omega^2)x + \omega^2 x^3}{1 - \omega^2(\omega - \omega^2)x^2},$$

giving

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1 + \omega y^2}} = \frac{(\omega - \omega^2)\,dx}{\sqrt{1 - x^2 \,.\, 1 + \omega x^2}};$$

or, what is the same thing, for the modulus $k^2 = -\omega$, we have

$$\operatorname{sn}(\omega - \omega^2)\theta = \frac{(\omega - \omega^2)\operatorname{sn}\theta + \omega^2 \operatorname{sn}^3\theta}{1 - \omega^2(\omega - \omega^2)\operatorname{sn}^2\theta};$$

the values of $\operatorname{cn}(\omega - \omega^2)\theta$ and $\operatorname{dn}(\omega - \omega^2)\theta$ are thence found to be

$$\operatorname{cn}(\omega - \omega^2)\theta = \frac{\operatorname{cn}\theta(1 - \omega^2\operatorname{sn}^2\theta)}{1 - \omega^2(\omega - \omega^2)\operatorname{sn}^2\theta};$$

and

$$\operatorname{dn}(\omega - \omega^2)\theta = \frac{\operatorname{dn}\theta(1 + \omega^2\operatorname{sn}^2\theta)}{1 - \omega^2(\omega - \omega^2)\operatorname{sn}^2\theta};$$

which are the formulæ of transformation for the elliptic functions.

872.

ON THE FINITE NUMBER OF THE COVARIANTS OF A BINARY QUANTIC.

[From the *Mathematische Annalen*, t. XXXIV. (1889), pp. 319, 320.]

THE proof of Gordan's theorem given in this paper is insufficient; the paper accordingly is not reprinted.

873.

SYSTEM OF EQUATIONS FOR THREE CIRCLES WHICH CUT EACH OTHER AT GIVEN ANGLES.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 18—21.]

CONSIDER a triangle ABC, angles A, B, C, $(A+B+C=\pi)$: to fix the absolute magnitude, assume that the radius of the circumscribed circle is $=1$; the lengths of the sides are therefore $=2\sin A$, $2\sin B$, $2\sin C$ respectively. On the three sides as bases, outside of each, describe isosceles triangles aBC, bCA, cAB, the base angles whereof are $=\alpha$, β, γ respectively. If we draw a circle touching aB, aC at the points B, C respectively; a circle touching bC, bA at the points C, A respectively; and a circle touching cA, cB at the points A, B respectively; then these circles form a curvilinear triangle ABC, the angles whereof are $A+\beta+\gamma$, $B+\gamma+\alpha$, $C+\alpha+\beta$ respectively. Taking as origin the centre of the circumscribed circle, and through this point, for axis of x, an arbitrary line, its position determined by the angle θ, I write for convenience

$$\begin{array}{lll} F=\theta+2B, & F'=\theta-A, & A'=A+\beta+\gamma, \\ G=\theta+2B+2C, & G'=\theta+B, & B'=B+\gamma+\alpha, \\ H=\theta, & H'=\theta+2B+C, & C'=C+\alpha+\beta; \end{array}$$

then the coordinates of the angular points A, B, C are $(\cos F, \sin F)$, $(\cos G, \sin G)$, $(\cos H, \sin H)$ respectively; and the equations of the three circles are

$$\left\{x+\frac{\sin(A-\alpha)}{\sin\alpha}\cos F'\right\}^2+\left\{y+\frac{\sin(A-\alpha)}{\sin\alpha}\sin F'\right\}^2=\frac{\sin^2 A}{\sin^2\alpha},$$

$$\left\{x+\frac{\sin(B-\beta)}{\sin\beta}\cos G'\right\}^2+\left\{y+\frac{\sin(B-\beta)}{\sin\beta}\sin G'\right\}^2=\frac{\sin^2 B}{\sin^2\beta},$$

$$\left\{x+\frac{\sin(C-\gamma)}{\sin\gamma}\cos H'\right\}^2+\left\{y+\frac{\sin(C-\gamma)}{\sin\gamma}\sin H'\right\}^2=\frac{\sin^2 C}{\sin^2\gamma},$$

respectively.

In verification, observe that we have

$$\begin{array}{llll} G-H=2\pi-2A, & G'-H'=-\pi+A, & G-F'=2\pi-A, & H-F'=A, \\ H-F=\quad -2B, & H'-F'=\quad \pi+B, & H-G'=\quad -B, & F-G'=B, \\ F-G=\quad -2C, & F'-G'=-\pi+C, & F-H'=\quad -C, & G-H'=C; \end{array}$$

hence

$$(\cos G-\cos H)^2+(\sin G-\sin H)^2=2-2\cos(G-H), \quad =2(1-\cos 2A), \quad =4\sin^2 A,$$

and we thus see that the sides are $=2\sin A$, $2\sin B$, $2\sin C$ respectively.

The first circle should pass through the points $(\cos G, \sin G)$, $(\cos H, \sin H)$; we ought therefore to have, for the first of these points,

$$1+2\frac{\sin(A-\alpha)}{\sin\alpha}\cos(G-F)+\frac{\sin^2(A-\alpha)}{\sin^2\alpha}=\frac{\sin^2 A}{\sin^2\alpha},$$

that is,

$$1+2\frac{\sin(A-\alpha)}{\sin\alpha}\cos A+\frac{\sin^2(A-\alpha)}{\sin^2\alpha}=\frac{\sin^2 A}{\sin^2\alpha}:$$

and, for the second of the points, the same equation. Write for a moment

$$X=\frac{\sin A}{\sin\alpha}, \text{ then } \frac{\sin(A-\alpha)}{\sin\alpha}=X\cos\alpha-\cos A;$$

then the equation is

$$1+2(X\cos\alpha-\cos A)\cos A+(X\cos\alpha-\cos A)^2=X^2,$$

that is,

$$1-\cos^2 A=X^2\sin^2\alpha,$$

which is right.

The second and third circles should intersect at the angle A', that is, we ought to have

$$\left\{\frac{\sin(B-\beta)}{\sin\beta}\cos G'-\frac{\sin(C-\gamma)}{\sin\gamma}\cos H'\right\}^2+\left\{\frac{\sin(B-\beta)}{\sin\beta}\sin G'-\frac{\sin(C-\gamma)}{\sin\gamma}\sin H'\right\}$$
$$=\frac{\sin^2 B}{\sin^2\beta}+\frac{\sin^2 C}{\sin^2\gamma}+2\frac{\sin B\sin C}{\sin\beta\sin\gamma}\cos A',$$

or, reducing and for $\cos(G'-H')$ substituting its value, $=-\cos A$, the equation is

$$\frac{\sin^2(B-\beta)}{\sin^2\beta}+\frac{\sin^2(C-\gamma)}{\sin^2\gamma}+2\frac{\sin(B-\beta)\sin(C-\gamma)}{\sin\beta\sin\gamma}\cos A=\frac{\sin^2 B}{\sin^2\beta}+\frac{\sin^2 C}{\sin^2\gamma}+2\frac{\sin B\sin C}{\sin\beta\sin\gamma}\cos A'.$$

Writing here

$$\frac{\sin B}{\sin\beta}=Y, \quad \frac{\sin C}{\sin\gamma}=Z,$$

the equation is

$$(Y\cos\beta-\cos B)^2+(Z\cos\gamma-\cos C)^2+2(Y\cos\beta-\cos B)(Z\cos\gamma-\cos C)\cos A$$
$$=Y^2+Z^2+2YZ\cos A',$$

viz. this is

$$\begin{aligned}&Y^2\cos^2\beta + Z^2\cos^2\gamma + 2YZ\cos\beta\cos\gamma\cos A\\ &-2Y\cos\beta(\cos B + \cos C\cos A) - 2Z\cos\gamma(\cos C + \cos A\cos B)\\ &+\cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = Y^2 + Z^2 + 2YZ\cos A'.\end{aligned}$$

Reducing by the relation $A + B + C = \pi$, this becomes

$$\begin{aligned}&-2Y\cos\beta\sin A\sin C - 2Z\cos\gamma\sin A\sin B + 1 - \cos^2 A\\ &\qquad = Y^2\sin^2\beta + Z^2\sin^2\gamma + 2YZ(\cos A' - \cos\beta\cos\gamma\cos A).\end{aligned}$$

Here $A' = A + \beta + \gamma$, and thence

$$\cos A' = \cos A(\cos\beta\cos\gamma - \sin\beta\sin\gamma) - \sin A(\sin\gamma\cos\beta + \sin\beta\cos\gamma);$$

hence the right-hand is

$$= Y^2\sin^2\beta + Z^2\sin^2\gamma - 2YZ(\cos A\sin\beta\sin\gamma + \sin A\sin\gamma\cos\beta + \sin A\sin\beta\cos\gamma),$$

or, reducing by

$$Y\sin\beta = \sin B,\quad Z\sin\gamma = \sin C,$$

this is

$$= \sin^2 B + \sin^2 C - 2\sin B\sin C\cos A - 2Y\cos\beta\sin A\sin C - 2Z\cos\gamma\sin A\sin B,$$

and the terms in Y, Z are equal to the like terms on the left-hand; the whole equation thus becomes

$$-1 + \cos^2 A + \sin^2 B + \sin^2 C - 2\cos A\sin B\sin C = 0,$$

where the last term is

$$\begin{aligned}&= 2\cos A\{\cos(B + C) - \cos B\cos C\},\\ &= -2\cos^2 A - 2\cos A\cos B\cos C,\\ &= -2\cos^2 A + (\cos^2 A + \cos^2 B + \cos^2 C - 1),\\ &= -\cos^2 A + \cos^2 B + \cos^2 C - 1;\end{aligned}$$

the equation is thus

$$-1 + \cos^2 A + \sin^2 B + \sin^2 C - \cos^2 A + \cos^2 B + \cos^2 C - 1 = 0,$$

or, finally, it is $-1 + 1 + 1 - 1 = 0$, which is an identity. The formulæ for the intersection of the third and first circles, and for that of the first and second circles, are of course precisely similar to the above formula for the intersection of the second and third circles; and the verifications are thus completed.

Cambridge, April 7, 1887.

874.

NOTE ON THE LEGENDRIAN COEFFICIENTS OF THE SECOND KIND.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 21—23.]

As regards the integration of the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

(n a positive integer), it seems to me that sufficient prominence is not given to the solution

$$y=\tfrac{1}{2}P_n\log\frac{x+1}{x-1}-Z_n\,(=Q_n),$$

where P_n is the Legendrian integral of the first kind, a rational and integral function of x of the degree n, and Z_n is a rational and integral function of the degree $n-1$; viz. we have here a solution containing no transcendental function other than the logarithm, and which should thus be adopted as a second particular integral in preference to the form $y=Q_n$, in which we have the infinite series Q_n which is an unknown transcendental function.

Moreover, the expression usually given for Z_n, viz.

$$Z_n=\frac{2n-1}{1\,.\,n}P_{n-1}+\frac{2n-5}{3\,(n-1)}P_{n-3}+\frac{2n-9}{5\,(n-2)}P_{n-5}+\ldots\text{(to term in } P_1 \text{ or } P_0),$$

is a very simple and elegant one; but the more natural definition (and that by which Z_n is most readily calculated) is that Z_n is the integral part of $\tfrac{1}{2}P_n\log\frac{x+1}{x-1}$, when the logarithm is expanded in descending powers of x, viz. it is the integral part of

$$P_n\left(\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}+\frac{1}{5}\frac{1}{x^5}+\ldots\right),$$

whence also Q_n is the portion containing negative powers only of this same series.

The expressions for $P_0, P_1, \ldots, P_{10}$ are given in Ferrers' *Elementary Treatise on Spherical Harmonics, &c.*, London, 1877, pp. 23—25. Reproducing these, and joining to them the values of $Z_0, Z_1, \ldots, Z_{10}$ we have as follows: read $P_2 = \frac{3}{2}x^2 - \frac{1}{2}$, and so in other cases.

$$
\begin{aligned}
P_0 &= 1 \,, \\
P_1 &= x \quad 1 \,, \\
P_2 &= (x^2, \quad 1) \ \tfrac{3}{2} - \tfrac{1}{2} \,, \\
P_3 &= (x^3, \quad x) \ \tfrac{5}{2} - \tfrac{3}{2} \,, \\
P_4 &= (x^4 \ldots 1) \ \tfrac{35}{8} - \tfrac{15}{4} + \tfrac{3}{8} \,, \\
P_5 &= (x^5 \ldots x) \ \tfrac{63}{8} - \tfrac{35}{4} + \tfrac{15}{8} \,, \\
P_6 &= (x^6 \ldots 1) \ \tfrac{231}{16} - \tfrac{315}{16} + \tfrac{105}{16} - \tfrac{5}{16} \,, \\
P_7 &= (x^7 \ldots x) \ \tfrac{429}{16} - \tfrac{693}{16} + \tfrac{315}{16} - \tfrac{35}{16} \,, \\
P_8 &= (x^8 \ldots 1) \ \tfrac{6435}{128} - \tfrac{3003}{32} + \tfrac{3465}{64} - \tfrac{315}{32} + \tfrac{35}{128} \,, \\
P_9 &= (x^9 \ldots x) \ \tfrac{12155}{128} - \tfrac{6435}{32} + \tfrac{9009}{64} - \tfrac{1155}{32} + \tfrac{315}{128} \,, \\
P_{10} &= (x^{10} \ldots 1) \ \tfrac{46189}{256} - \tfrac{109395}{256} + \tfrac{45045}{128} - \tfrac{15015}{128} + \tfrac{3465}{256} - \tfrac{63}{256} \,. \\
Z_0 &= 0 \,, \\
Z_1 &= 1 \,, \\
Z_2 &= x \quad \tfrac{3}{2} \,, \\
Z_3 &= (x^2, \quad 1) \ \tfrac{5}{2} - \tfrac{2}{3} \,, \\
Z_4 &= (x^3, \quad x) \ \tfrac{35}{8} - \tfrac{55}{24} \,, \\
Z_5 &= (x^4 \ldots 1) \ \tfrac{63}{8} - \tfrac{49}{8} + \tfrac{8}{15} \,, \\
Z_6 &= (x^5 \ldots x) \ \tfrac{231}{16} - \tfrac{119}{8} + \tfrac{231}{80} \,, \\
Z_7 &= (x^6 \ldots 1) \ \tfrac{429}{16} - \tfrac{275}{8} + \tfrac{849}{80} - \tfrac{16}{35} \,, \\
Z_8 &= (x^7 \ldots x) \ \tfrac{6435}{128} - \tfrac{9867}{128} + \tfrac{4213}{128} - \tfrac{11659}{4480} \,, \\
Z_9 &= (x^8 \ldots 1) \ \tfrac{12155}{128} - \tfrac{65065}{384} + \tfrac{11869}{128} - \tfrac{14179}{896} + \tfrac{128}{315} \,, \\
Z_{10} &= (x^9 \ldots x) \ \tfrac{46189}{256} - \tfrac{281996}{768} + \tfrac{157157}{640} - \tfrac{26741}{448} + \tfrac{61567}{16128} \,.
\end{aligned}
$$

I notice that the numerical values of $P_1, P_2, \ldots, P_7$, for $x = 0{\cdot}00, 0{\cdot}01, \ldots, 1{\cdot}00$ are given (*Report of the British Association for* 1879, "Report on Mathematical Tables"); as the functions contain only powers of 2 in their denominators, the decimal values terminate, and the complete values are given. The functions Z have not been tabulated; the denominators contain other prime factors, and the decimal values would not terminate.

Cambridge, March 29, 1887.

875.

ON THE SYSTEM OF THREE CIRCLES WHICH CUT EACH OTHER AT GIVEN ANGLES AND HAVE THEIR CENTRES IN A LINE.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 60—69.]

IN the system considered in the paper "System of equations for three circles which cut each other at given angles," *Messenger*, t. XVII. pp. 18—21, [873], we may consider the particular case where the centres of the circles are in a line. The condition in order that this may be so is obviously

$$\begin{vmatrix} \sin(A-\alpha)\cos F', & \sin(A-\alpha)\sin F', & \sin\alpha \\ \sin(B-\beta)\cos G', & \sin(B-\beta)\sin G', & \sin\beta \\ \sin(C-\gamma)\cos H', & \sin(C-\gamma)\sin H', & \sin\gamma \end{vmatrix} = 0,$$

that is,

$$\sin(B-\beta)\sin(C-\gamma)\sin\alpha\sin(G'-H') + \ldots = 0;$$

or since $\sin(G'-H')$, $\sin(H'-F')$, $\sin(F'-G')$ are $=\sin A$, $\sin B$, $\sin C$ respectively, this is

$$\sin(B-\beta)\sin(C-\gamma)\sin A\sin\alpha + \ldots = 0,$$

viz. this is

$$\frac{\sin A\sin\alpha}{\sin(A-\alpha)} + \frac{\sin B\sin\beta}{\sin(B-\beta)} + \frac{\sin C\sin\gamma}{\sin(C-\gamma)} = 0,$$

or, as this may also be written,

$$\frac{1}{\cot A - \cot\alpha} + \frac{1}{\cot B - \cot\beta} + \frac{1}{\cot C - \cot\gamma} = 0.$$

But assuming this equation to be satisfied, it does not appear that there is any simple expression for the equation of the line through the three centres; nor would it be easy to transform the equations so as to have this line for one of the axes.

The case in question (which is a very important one from its connexion with Poincaré's theory of the Fuchsian functions) may be considered independently.

Taking the line of centres for the axis of x, and writing α, β, γ for the abscissæ of the centres, and P, Q, R for the radii, then the equations of the circles are

$$(x-\alpha)^2+y^2=P^2,$$
$$(x-\beta)^2+y^2=Q^2,$$
$$(x-\gamma)^2+y^2=R^2;$$

and then, if the pairs of circles cut at the angles A, B, C respectively, we have

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$
$$R^2+2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2+2PQ\cos C+Q^2=(\alpha-\beta)^2,$$

which are the equations connecting α, β, γ, P, Q, R.

It is to be remarked in regard hereto that, if A, B, C are used to denote the interior angles of the curvilinear triangle ABC, then the angles $\gamma A\beta$, $\alpha B\gamma$, $\beta C\alpha$ are $=\pi-A$, B, C respectively; whence, if P, Q, R were used to denote the three radii taken positively, the first equation would be

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$

as above; but the other two equations would be

$$R^2-2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2-2PQ\cos C+Q^2=(\alpha-\beta)^2;$$

hence, in order that the equations may be as above, it is necessary that P denote the radius of the circle, centre α, taken *negatively*; and it in fact appears that, in a limiting case afterwards considered, the value of P comes out negative. Similarly as regards the curvilinear triangle $AB'C'$; here A, $B(=B')$ and $C(=C')$ are the interior angles of the triangle; and the radius of the circle, centre α', must be regarded as negative.

Considering A, B, C as given, we have an equation between the radii P, Q, R. In fact, this is at once obtained in the irrational form $\sqrt{(X)}+\sqrt{(Y)}+\sqrt{(Z)}=0$; and proceeding to rationalise this, we obtain

$$-2\sqrt{(YZ)}=Y+Z-X,$$

that is,

$$-\sqrt{\{(P^2+2PR\cos B+R^2)(P^2+2PQ\cos C+Q^2)\}}=P^2+P(Q\cos C+R\cos B)-QR\cos A.$$

Hence, squaring and reducing, we find without difficulty

$$\begin{aligned}0=\quad & Q^2R^2\sin^2 A+R^2P^2\sin^2 B+P^2Q^2\sin^2 C\\ &+2P^2QR(\cos A+\cos B\cos C)+2PQ^2R(\cos B+\cos C\cos A)\\ &+2PQR^2(\cos C+\cos A\cos B),\end{aligned}$$

or putting herein $P, Q, R = \frac{\sin A}{\xi}, \frac{\sin B}{\eta}, \frac{\sin C}{\zeta}$, this is

$$\left(1, 1, 1, \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \frac{\cos B + \cos C \cos A}{\sin C \sin A}, \frac{\cos C + \cos A \cos B}{\sin A \sin B}\right)(\xi, \eta, \zeta)^2 = 0;$$

and it may be remarked that, in this quadric form, the three coefficients are each less than 1, or each greater than 1, according as $A + B + C > \pi$, or $A + B + C < \pi$.

Suppose 1°, $A + B + C > \pi$; the coefficients are here $= \cos\lambda, \cos\mu, \cos\nu$, the form is

$$(1, 1, 1, \cos\lambda, \cos\mu, \cos\nu \between \xi, \eta, \zeta)^2,$$

that is,

$$(\xi + \eta\cos\nu + \zeta\cos\mu)^2 + (\eta^2\sin^2\nu + 2\eta\zeta\cos\lambda + \zeta^2\sin^2\mu),$$

namely, this is

$$(\xi + \eta\cos\nu + \zeta\cos\mu)^2 + \left\{\eta\sin\nu + \zeta\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu}\right\}^2 + \zeta^2\left\{\sin^2\mu - \left(\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu}\right)^2\right\};$$

where the last term is

$$= \frac{\zeta^2}{\sin^2\nu}\{\sin^2\mu\sin^2\nu - (\cos\lambda\cos\mu\cos\nu)^2\};$$

the coefficient in { } is

$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu,$$

namely, substituting for $\cos\lambda, \cos\mu, \cos\nu$ their values, this is

$$= \frac{1}{\sin^2 A \sin^2 B \sin^2 C}(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)^2.$$

It thus appears that the form is the sum of three squares, and is thus constantly positive: it therefore only vanishes for imaginary values of the radii; or the case does not arise for any real figure.

Hence, 2°, if the figure be real, $A + B + C < \pi$, that is, the sum of the angles of the curvilinear triangle is less than two right angles: the radii are connected as above by the equation

$$\left(1, 1, 1, \frac{\cos A + \cos B\cos C}{\sin B\sin C}, \frac{\cos B + \cos C\cos A}{\sin C\sin A}, \frac{\cos C + \cos A\cos B}{\sin A\sin B}\right)\left(\frac{\sin A}{P}, \frac{\sin B}{Q}, \frac{\sin C}{R}\right)^2 = 0,$$

in which form the three coefficients are each greater than 1. Restoring therein ξ, η, ζ, and regarding these as rectangular coordinates, the equation represents a real cone which might be constructed without difficulty; and then taking ξ, η, ζ as the coordinates of any particular point on the conical surface, we have

$$P, Q, R = \frac{\sin A}{\xi}, \frac{\sin B}{\eta}, \frac{\sin C}{\zeta}.$$

Obviously, points on the same generating line of the cone give values of P, Q, R which differ in their absolute magnitudes only, their ratios being the same: the original equations, in fact, remain unaltered when P, Q, R, α, β, γ are each affected with any common factor.

Supposing P, Q, R taken so as to satisfy the equation in question, then taking the radicals

$$\sqrt{(Q^2+2QR\cos A+R^2)},\quad \sqrt{(R^2+2RP\cos B+P^2)},\quad \sqrt{(P^2+2PQ\cos C+Q^2)},$$

with the proper signs, we have a sum $=0$, and these give the values of $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, respectively; and the construction of the figure would be thus completed.

I look at the question from a different point of view; taking Q, R, $\beta-\gamma$ such as to satisfy the first equation

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$

that is, starting from the two circles $(x-\beta)^2+y^2=Q^2$, $(x-\gamma)^2+y^2=R^2$ which cut each other at a given angle A: then the problem is to find a circle $(x-\alpha)^2+y^2=P^2$, cutting these at given angles C, B respectively. To determine the coordinate of the centre α, and the radius P, we have the remaining two equations

$$R^2+2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2+2PQ\cos C+Q^2=(\alpha-\beta)^2,$$

namely, considering α, P as the coordinates of a point (in reference to the foregoing origin and axes), and for greater clearness writing $\alpha=\mathrm{x}$, $P=\mathrm{y}$, we have

$$\mathrm{y}^2+2\mathrm{y}R\cos B+R^2-(\mathrm{x}-\gamma)^2=0,$$
$$\mathrm{y}^2+2\mathrm{y}Q\cos C+Q^2-(\mathrm{x}-\beta)^2=0,$$

or, as these may be written,

$$(\mathrm{y}+R\cos B)^2-(\mathrm{x}-\gamma)^2=-R^2\sin^2 B,$$
$$(\mathrm{y}+Q\cos C)^2-(\mathrm{x}-\beta)^2=-Q^2\sin^2 C,$$

namely, the first of these equations denotes a rectangular hyperbola, coordinates of centre ($\mathrm{x}=\gamma$, $\mathrm{y}=-R\cos B$), transverse semi-axes $=R\sin B$; and the second of them a rectangular hyperbola, coordinates of centre ($\mathrm{x}=\beta$, $\mathrm{y}=-Q\cos C$), transverse semi-axes $=Q\sin C$: as similar and similarly situate hyperbolas, these intersect in two points only; namely, the points are the intersections of either of them with the common chord

$$2\mathrm{y}(R\cos B-Q\cos C)+2(\gamma-\beta)\{\mathrm{x}-\tfrac{1}{2}(\gamma+\beta)\}+R^2-Q^2=0.$$

It is possible to construct a circle through the two points of intersection, and so to obtain these points as the intersections of a line and circle; but the construction by the two rectangular hyperbolas is practically by no means an inconvenient one. I remark in passing that, for a rectangular hyperbola, the radius of curvature at the vertex is equal to the transverse semi-axis, and thus by drawing a small circular arc and by means of the asymptotes, we lay down a rectangular hyperbola graphically, without difficulty and with a fair amount of accuracy.

But the analytical solution may be carried somewhat further: we may without loss of generality write $\gamma = -\beta$, for this comes only to taking the origin midway between the centres of the circles β and γ: doing this, and for greater simplicity writing also for the moment $Q\cos C = M$, $R\cos B = N$, the equations become

$$\mathrm{y}^2 + 2\mathrm{y}N + R^2 - (\mathrm{x}+\beta)^2 = 0,$$
$$\mathrm{y}^2 + 2\mathrm{y}M + Q^2 - (\mathrm{x}-\beta)^2 = 0,$$

where x is now the abscissa of the centre of the circle α (measured from the last-mentioned midway point) and y is the radius of this circle. We deduce

$$2(N-M)\,\mathrm{y} + R^2 - Q^2 - 4\beta\mathrm{x} = 0,$$

or say

$$4\beta(\mathrm{x}-\beta) = 2(N-M)\,\mathrm{y} + R^2 - Q^2 + 4\beta^2,$$

and thence, from the first equation multiplied by $16\beta^2$, we have

$$16\beta^2(\mathrm{y}^2 + 2\mathrm{y}N + R^2) - \{2(N-M)\,\mathrm{y} + R^2 - Q^2 + 4\beta^2\}^2 = 0,$$

that is,

$$\begin{aligned} &4\mathrm{y}^2\{4\beta^2 - (N-M)^2\} \\ &+ 4\mathrm{y}\,\{4\beta^2(N+M) - (N-M)(R^2-Q^2)\} \\ &+ \quad\{8\beta^2(R^2+Q^2) - 16\beta^4 - (R^2-Q^2)^2\} = 0, \end{aligned}$$

say this is $4\mathrm{y}^2\mathfrak{A} + 4\mathrm{y}\mathfrak{B} + \mathfrak{C} = 0$.

This gives

$$(2\mathfrak{A}\mathrm{y} + \mathfrak{B})^2 = \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C},$$

and we find without difficulty, restoring for M, N their values $Q\cos C$ and $R\cos B$,

$$\begin{aligned} \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} = 4\beta^2\,\{&16\beta^4 - 8\beta^2(Q^2 + R^2 - 2QR\cos B\cos C) \\ &+ (P^4 - 4P^2Q\cos B\cos C + 2P^2Q^2(2\cos^2 B + 2\cos^2 C - 1) - 4PQ^2\cos B\cos C + Q^4)\}, \end{aligned}$$

which is

$$= 4\beta^2\,\{[4\beta^2 - (Q^2 - 2QR\cos B\cos C + R^2)]^2 - 4Q^2R^2\sin^2 B\sin^2 C\}.$$

But we have

$$Q^2 + 2QR\cos A + R^2 = 4\beta^2,$$

and this equation thus becomes

$$\begin{aligned} \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} &= 16\beta^2Q^2R^2\,\{(\cos A + \cos B\cos C)^2 - \sin^2 B\sin^2 C\} \\ &= 16\beta^2Q^2R^2(-1 + \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C). \end{aligned}$$

We have therefore

$$\begin{aligned} 2\,\{4\beta^2 - (R\cos B - Q\cos C)^2\}\,\mathrm{y} + \{4\beta^2(R\cos B + Q\cos C) - (R\cos B - Q\cos C)(R^2 - Q^2)\} \\ = \pm\,4\beta QR\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}}, \end{aligned}$$

$$4(R\cos B - Q\cos C)\,\mathrm{y} + R^2 - Q^2 = 4\beta\mathrm{x},$$

or, completing the reduction by the substitution of the value of $4\beta^2$, this is

$$\begin{aligned} \mathrm{y}\,\{(Q^2\sin^2 C + R^2\sin^2 B) + 2QR\,(\cos A + \cos B\cos C)\} \\ + QR\,\{Q\,(\cos B + \cos C\cos A) + R\,(\cos C + \cos A\cos B)\} \\ = \pm\, 4\beta QR\,\sqrt{\{-\,(1-\cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}}, \end{aligned}$$

viz. we have thus two values of the radius $\mathrm{y}\,(=P)$; and to each of these there corresponds a single value of the abscissa x, given by

$$4\beta\mathrm{x} = R^2 - Q^2 + 2\,(R\cos B - Q\cos C)\,\mathrm{y}.$$

The two values become equal, if $A + B \pm C = \pi$; in this case the three circles meet in a pair of points $(\mathrm{x}_1, \mathrm{y}_1)$, $(\mathrm{x}_1, -\mathrm{y}_1)$. In fact, writing $A + B + C = \pi$, and thence

$$\cos A = -\cos(B+C),\ = -\cos B\cos C + \sin B\sin C,\ \&c.,$$

we find

$$\begin{aligned} \{Q^2\sin^2 C + 2QR\,(\cos A + \cos B\cos C) + R^2\sin^2 B\}\,\mathrm{y} \\ + QR\,\{Q\,(\cos B + \cos C\cos A) + R\,(\cos C + \cos A\cos B)\} = 0, \end{aligned}$$

that is,

$$(Q\sin C + R\sin B)^2\,\mathrm{y} + QR\,(Q\sin C + R\sin B)\sin A = 0,$$

or, throwing out the factor $Q\sin C + R\sin B$, this is

$$(Q\sin C + R\sin B)\,\mathrm{y} + QR\sin A = 0,$$

and we then have

$$\begin{aligned} 4\beta\mathrm{x} &= R^2 - Q^2 - 2\,(R\cos B - Q\cos C)\,\frac{QR\sin A}{R\sin B + Q\sin C} \\ &= \frac{1}{R\sin B + Q\sin C}\,\{(R\sin B + Q\sin C)\,(R^2 - Q^2) - 2\,(R\cos B - Q\cos C)\,QR\sin A\}. \end{aligned}$$

The term in { } is here

$$\begin{aligned} & R^3\,(\ \ \sin B) \\ & + R^2Q\,(\ \ \sin C - 2\sin A\cos B) \\ & + RQ^2\,(-\sin B + 2\sin A\cos C) \\ & + \ \ Q^3\,(-\sin C), \end{aligned}$$

which is

$$\begin{aligned} = \ & R^3\,(\ \ \sin B) \\ & + R^2Q\,(-\sin C + 2\sin B\cos A) \\ & + RQ^2\,(\ \ \sin B - 2\sin C\cos A) \\ & + \ \ Q^3\,(-\sin C) \\ = \ & (R^2 + Q^2 + 2RQ\cos A)\,(R\sin B - Q\sin C), \\ = \ & 4\beta^2\,(R\sin B - Q\sin C), \end{aligned}$$

or, finally

$$\mathrm{y} = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$\mathrm{x} = \frac{\beta\,(R\sin B - Q\sin C)}{R\sin B + Q\sin C}.$$

In these equations, y, x should be replaced by P, α respectively; and in obtaining them, it was assumed that $\gamma = -\beta$; restoring the general values of β, γ, the equations become

$$P = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$\alpha - \tfrac{1}{2}(\beta+\gamma) = \frac{\tfrac{1}{2}(\beta-\gamma)(R\sin B - Q\sin C)}{R\sin B + Q\sin C},$$

viz. this last equation becomes

$$\alpha = \frac{\beta R\sin B + \gamma Q\sin C}{R\sin B + Q\sin C},$$

or, say

$$\alpha(R\sin B + Q\sin C) - \beta R\sin B - \gamma Q\sin C = 0,$$

which by means of the first equation becomes

$$\alpha\frac{QR}{P}\sin A + \beta R\sin B + \gamma Q\sin C = 0.$$

It thus appears that the two equations are

$$\frac{\sin A}{P} + \frac{\sin B}{Q} + \frac{\sin C}{R} = 0,$$

$$\frac{\alpha\sin A}{P} + \frac{\beta\sin B}{Q} + \frac{\gamma\sin C}{R} = 0,$$

viz. these equations, wherein $A+B+C=\pi$, belong to the case where the three circles intersect in the same pair of points; hence, if the coordinates x, y refer to the points of intersection of the three circles, we have simultaneously the equations of the three circles, and the three equations which determine the angles at which they intersect, viz. we have the six equations

$$\begin{array}{ll}(x-\alpha)^2+y^2=P^2, & Q^2+R^2+2QR\cos A=(\beta-\gamma)^2,\\ (x-\beta)^2+y^2=Q^2, & R^2+P^2+2RP\cos B=(\gamma-\alpha)^2,\\ (x-\gamma)^2+y^2=R^2, & P^2+Q^2+2PQ\cos C=(\alpha-\beta)^2,\end{array}$$

viz. from these six equations, with the condition $A+B+C=\pi$, it must be possible to deduce the last-mentioned pair of equations.

In the general case, where $A+B+C<\pi$, and the three circles do not meet in a point, then taking the circles $(x-\beta)^2+y^2=Q^2$, $(x-\gamma)^2+y^2=R^2$ to be circles cutting each other at the angle A, or, what is the same thing, the values Q, R, β, γ to be such as to satisfy the relation

$$Q^2+R^2+2QR\cos A=(\beta-\gamma)^2;$$

the two equations for the determination of the abscissa of the centre α, and the radius P of the remaining circle give, by what precedes,

$$\begin{aligned}&2\{(\beta-\gamma)^2-(R\cos B-Q\cos C)^2\}P\\ &\quad+\{(\beta-\gamma)^2(R\cos B+Q\cos C)-(R\cos B-Q\cos C)(R^2-Q^2)\}\\ &\quad=\pm 2(\beta-\gamma)QR\sqrt{\{-(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)\}},\\ &4(R\cos B-Q\cos C)P+(R^2-Q^2)=(\beta-\gamma)(2\alpha-\beta-\gamma),\end{aligned}$$

viz. we have thus the two circles $(x-\alpha)^2+y^2=P^2$, each of them cutting the circles $(x-\beta)^2+y^2=Q^2$, and $(x-\gamma)^2+y^2=R^2$ at the angles C, B respectively.

876.

ON SYSTEMS OF RAYS.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 73—78.]

SIR W. R. HAMILTON'S Memoir, "Theory of Systems of Rays" (I do not at present consider the three Supplements), dated June, 1827, is printed *Trans. R. I. Acad.*, vol. XV. (1828), pp. 69—174. There is one page of Introduction, and pp. 70 to 80, an elaborate Contents. Part First: On ordinary systems of Reflected Rays. Part Second: On ordinary systems of Refracted Rays. Part Third: On extraordinary systems and systems of Rays in general. But only the First Part was published. This is considered under the headings: (I) Analytic Expressions of the Law of Ordinary Reflexion. (II) Theory of Focal Mirrors. (III) Surfaces of Constant Action. (IV) Classification of Systems of Rays. (V) On the Pencils of a Reflected System. (VI) On the Developable Pencils, the Two Foci of a Ray and the Caustic Curves and Surfaces. (VII) Lines of Reflexion on a Mirror. (VIII) On Osculating Focal Mirrors. (IX) On Thin and Undevelopable Pencils. (X) On the Axes of a Reflected System. (XI) Images Formed by Mirrors. (XII) Aberrations. (XIII) Density. And we have, p. 174, a "Conclusion to the First Part," wherein this first part is described as "an attempt to establish general principles respecting the system of rays produced by the ordinary reflexion of light at any mirror or combination of mirrors shaped and placed in any manner whatever; and to show that the mathematical properties of such a system may all be deduced by analytic methods from the form of ONE CHARACTERISTIC FUNCTION: as in the application of Analysis to Geometry, the properties of a plane curve or of a curve surface may all be deduced by uniform methods from the form of the function which characterises its equation."

The foregoing headings (I) to (V) may be regarded as containing the general theory, and the remaining ones (VI) to (XIII) as containing applications and developments.

I remark on the theory as follows:

Considering a congruence or doubly infinite system of lines (or say of rays), suppose that for any particular ray the cosine-inclinations are α, β, γ $(\alpha^2+\beta^2+\gamma^2=1)$,

and that the coordinates of a point on the ray are (x, y, z). We may look at the system in two ways:

1°. The rays are considered as emanating from the points of a surface: here, if (x, y, z) are considered as belonging to a point on the surface, then z is a given function of (x, y) (or more generally x, y, z are given functions of two arbitrary parameters u, v); and to determine the congruence, we must have α, β, γ each of them a given function of (x, y) or of (u, v), such that identically $\alpha^2+\beta^2+\gamma^2=1$, but with no other condition as to the form of the functions.

2°. The rays may be considered irrespectively of any surface: here (α, β, γ) are each of them a given function of (x, y, z), such that always $\alpha^2+\beta^2+\gamma^2=1$; but there are further conditions, viz. it is assumed that we have one and the same ray, whatever is the point (x, y, z) on the ray; or, what is the same thing (using ρ to denote an arbitrary distance), that α, β, γ regarded as functions of x, y, z remain unaltered by the change of x, y, z into $x+\rho\alpha, y+\rho\beta, z+\rho\gamma$ respectively; this implies that we have

$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\alpha,\quad (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\beta,\quad (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\gamma,$$

each $=0$; and, conversely, it may be shown that when these relations are satisfied then α, β, γ remain unaltered by the change in question.

The equation $\alpha^2+\beta^2+\gamma^2=1$, gives

$$\alpha\delta_x\alpha+\beta\delta_x\beta+\gamma\delta_x\gamma,\quad \alpha\delta_y\alpha+\beta\delta_y\beta+\gamma\delta_y\gamma,\quad \alpha\delta_z\alpha+\beta\delta_z\beta+\gamma\delta_z\gamma,$$

each $=0$; and combining with the last-mentioned system of equations, it follows that

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta,$$

are proportional to α, β, γ; or say $=k\alpha, k\beta, k\gamma$ respectively.

Hamilton considers whether the rays are such that they are cut at right angles by a surface; supposing this is so (say in this case the rays are a rectangular, or orthotomic system, or that they are the normals of a surface), then if x, y, z refer to the point on the surface, we must have

$$\alpha\,dx+\beta\,dy+\gamma\,dz=0;$$

this implies

$$\alpha\,(\delta_z\beta-\delta_y\gamma)+\beta\,(\delta_x\gamma-\delta_z\alpha)+\gamma\,(\delta_y\alpha-\delta_x\beta)=0,$$

a condition which, by what precedes, is $k\,(\alpha^2+\beta^2+\gamma^2)=0$; viz. we must have $k=0$, and therefore

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta$$

each $=0$; that is, $\alpha\,dx+\beta\,dy+\gamma\,dz$ must be a complete differential, say it is $=dV$. And we then have $V=c$, the equation of a system of parallel surfaces each of them cutting the rays at right angles. Evidently $\alpha, \beta, \gamma=\delta_x V, \delta_y V, \delta_z V$ respectively, and the function V satisfies the partial differential equation

$$(\delta_x V)^2+(\delta_y V)^2+(\delta_z V)^2=1.$$

Hamilton in effect considers only systems of rays of the form in question, viz. those which are the normals of a surface (or, what is the same thing, the normals of a system of parallel surfaces), and it is such a system which is said to have the characteristic function V. It is shown that a system of rays originally of this kind remains a system of this kind after any number of reflexions (or ordinary refractions); in particular, if the rays originally emanate from a point, then, after any number of reflexions at mirrors of any form whatever, they are a system of rays cut at right angles by a surface. And moreover, there is given for the surface a simple construction, viz. starting from any surface which cuts the rays at right angles, and measuring off on the path of each ray (as reflected at the mirror or succession of mirrors) one and the same arbitrary distance, we have a set of points forming a surface which cuts at right angles the system of rays as reflected at the mirror or last of the mirrors.

The ray-systems considered by Hamilton are thus the normals of a surface $V-c=0$, and a large part of the properties of the system are thus included under the known theory of the normals of a surface; it may be remarked that the analytical formulæ are somewhat simplified by the circumstance that V instead of being (as usual) an arbitrary function of (x, y, z) is a function satisfying the partial differential equation $(\delta_x V)^2+(\delta_y V)^2+(\delta_z V)^2=1$. In particular, we have the theorem that each ray is intersected by two consecutive rays in foci which are the centres of curvature of the normal surface; the intersecting rays are rays proceeding from the curves of curvature of the normal surface, &c. There are other properties easily deducible from, but not actually included in, the theory of the normals; for instance, the intersecting rays aforesaid are rays proceeding from certain curves on the mirror, analogous to, but which obviously are not, the curves of curvature of the mirror. The natural mode of treatment of this part of the theory is to regard the rays as proceeding not from the normal surface, but from the mirror, and (by an investigation perfectly analogous to that for the normals of a surface) to enquire as to the intersection of the ray by rays proceeding from consecutive points of the mirror; it would thus appear that there are on the mirror two directions, such that proceeding along either of them to a consecutive point, the ray from the original point is intersected by the ray from the consecutive point, but that these directions are *not* in general at right angles, &c.

But in regard to such an investigation, the restriction introduced by the Hamiltonian theory is altogether unnecessary; there is no occasion to consider the rays which proceed from the several points of the mirror as being rays which are the normals of a surface, and the question is considered from the more general point of view as well by Malus in his *Théorie de la Double Refraction, &c.*, Paris, 1810, as more recently by Kummer in the Memoir "Allgemeine Theorie der gradlinigen Strahlensysteme," *Crelle*, t. LVII. (1860), pp. 189—230, viz. we have in Kummer a surface of any form whatever (defined according to the Gaussian theory, x, y, z given functions of the arbitrary parameters u, v), and from the several points thereof rays proceeding according to any law whatever, viz. the cosine-inclinations α, β, γ (or as Kummer writes them ξ, η, ζ) being given functions (such of course that $\alpha^2+\beta^2+\gamma^2=1$) of the same parameters u, v. It may be remarked: 1° that Kummer, while considering the simplifications of the general theory presenting themselves in the case where the rays are normals

of the surface, does not specifically consider the case where, not being such normals, they are (as in the Hamiltonian theory) normals of a surface: 2° that some interesting investigations in regard to the shortest distances between consecutive rays, while naturally connecting themselves with the normals of the surface, or with that of the rays which are normals of another surface, do not properly belong to the "Allgemeine Theorie" of a congruence, which is independent of the notion of rectangularity.

It has been already remarked that the system may be looked at in the two ways 1° and 2°, and it is in the former of these that the question is considered by Kummer; it is interesting to work out part of the theory in the latter of the two ways. Taking X, Y, Z as current coordinates, we have, for a line through the point (x, y, z), the equations

$$X,\ Y,\ Z = x + \alpha\rho,\quad y + \beta\rho,\quad z + \gamma\rho;$$

α, β, γ are functions of (x, y, z), satisfying identically the equation $\alpha^2 + \beta^2 + \gamma^2 = 1$ (and therefore the derived equations in regard to x, y, z respectively); and also satisfying the equations

$$(\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\alpha = 0,\quad (\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\beta = 0,\quad (\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\gamma = 0.$$

It should be remarked that, if these equations were not satisfied, then instead of a congruence there would be a complex, or triply infinite system of lines, viz. to the several points of space (x, y, z) there would correspond lines $X, Y, Z = x + \alpha\rho,\ y + \beta\rho,\ z + \gamma\rho$ as above, which lines would not reduce themselves to a doubly infinite system.

Suppose that the line through the point x, y, z is met by the line through a consecutive point $(x + dx,\ y + dy,\ z + dz)$; then, if X, Y, Z refer to the point of intersection of the two lines, we have

$$dx + \alpha d\rho + \rho d\alpha,\quad dy + \beta d\rho + \rho d\beta,\quad dz + \gamma d\rho + \rho d\gamma = 0;$$

and consequently

$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix} = 0$$

as a relation connecting the increments dx, dy, dz, in order that the lines may intersect, viz. this is a quadric relation $(*\mathfrak{Q}dx,\ dy,\ dz)^2 = 0$ between the increments. In the case of a complex, this equation represents a cone (passing evidently through the line $dx : dy : dz = \alpha : \beta : \gamma$), but in the case of a congruence the cone must break up into a pair of planes intersecting in the line in question $dx : dy : dz = \alpha : \beta : \gamma$. To verify *à posteriori* that this is so, observe that the differential equations satisfied by α, β, γ give, as above,

$$\delta_y\gamma - \delta_z\beta,\quad \delta_z\alpha - \delta_x\gamma,\quad \delta_x\beta - \delta_y\alpha,$$

proportional to α, β, γ, or say $= 2k\alpha,\ 2k\beta,\ 2k\gamma$; and it hence follows that the differentials $\delta\alpha$, $\delta\beta$, $\delta\gamma$ can be expressed in the forms

$$\begin{aligned} d\alpha &= a\,dx + h\,dy + g\,dz + k\,(\beta\,dz - \gamma\,dy), \\ d\beta &= h\,dx + b\,dy + f\,dz + k\,(\gamma\,dx - \alpha\,dz), \\ d\gamma &= g\,dx + f\,dy + c\,dz + k\,(\alpha\,dy - \beta\,dx), \end{aligned}$$

where

$$0 = a\alpha + h\beta + g\gamma,$$
$$0 = h\alpha + b\beta + f\gamma,$$
$$0 = g\alpha + f\beta + c\gamma.$$

The equation

$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix} = 0$$

thus assumes the form

$$(a, b, c, f, g, h \between dx, dy, dz \between \gamma\, dy - \beta\, dz,\ \alpha\, dz - \gamma\, dx,\ \beta\, dx - \alpha\, dy)$$
$$+ k\,\{(\gamma\, dy - \beta\, dz)^2 + (\alpha\, dz - \gamma\, dx)^2 + (\beta\, dx - \alpha\, dy)^2\} = 0.$$

Write for shortness

$$\gamma\, dy - \beta\, dz,\quad \alpha\, dz - \gamma\, dx,\quad \beta\, dx - \alpha\, dy = \xi,\ \eta,\ \zeta;$$

then putting for a moment $a\, dx + h\, dy + g\, dz = \Theta$, from this equation and $a\alpha + h\beta + g\gamma = 0$ we deduce

$$\alpha\Theta = -h\,(\beta\, dx - \alpha\, dy) + g\,(\alpha\, dz - \gamma\, dx),$$

that is, $= -h\zeta + g\eta$; or, putting for Θ its value and forming the analogous equations, we have

$$\alpha\,(a\, dx + h\, dy + g\, dz) = -h\zeta + g\eta,$$
$$\beta\,(h\, dx + b\, dy + f\, dz) = -f\xi + h\zeta,$$
$$\gamma\,(g\, dx + f\, dy + c\, dz) = -g\eta + f\xi,$$

and the quadric equation in (dx, dy, dz) thus becomes

$$\frac{\xi}{\alpha}(-h\zeta + g\eta) + \frac{\eta}{\beta}(-f\xi + h\zeta) + \frac{\zeta}{\gamma}(-g\eta + f\xi) + k\,(\xi^2 + \eta^2 + \zeta^2) = 0,$$

which, in virtue of the linear relation $\alpha\xi + \beta\eta + \gamma\zeta = 0$ connecting the (ξ, η, ζ), breaks up into a pair of planes, each passing through the line $\xi = 0$, $\eta = 0$, $\zeta = 0$, (that is, $dx : dy : dz = \alpha : \beta : \gamma$).

We obtain at once, as the condition that the two planes may be at right angles to each other, $k = 0$; that is,

$$\delta_y\gamma - \delta_z\beta,\quad \delta_z\alpha - \delta_x\gamma,\quad \delta_x\beta - \delta_y\alpha \text{ each } = 0;$$

or, what is the same thing, $\alpha\, dx + \beta\, dy + \gamma\, dz = dV$, a complete differential; and, as was seen, this is the condition in order that the lines may be the normals of a surface.

It thus appears that in a congruence each line is intersected by two consecutive lines, which determine respectively two planes through the line; and further, that if for every line of the congruence, the two planes are at right angles to each other, then the consecutive lines are the normals of a surface.

877.

NOTE ON THE TWO RELATIONS CONNECTING THE DISTANCES OF FOUR POINTS ON A CIRCLE.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 94, 95.]

CONSIDER a quadrilateral $BACD$ inscribed in a circle; and let the sides BA, AC, CD, DB and diagonals BC and AD be $=c, b, h, g, a, -f$ respectively; f is for convenience taken negative, so that the equation connecting the sides and diagonals may be

$$\Delta, = af + bg + ch, = 0.$$

We have between the sides and diagonals another relation

$$V, = abc + agh + bhf + cfg, = 0,$$

as is easily proved geometrically; in fact, recollecting that the opposite angles are supplementary to each other, the double area of the quadrilateral is $=(bc+gh)\sin A$, and it is also $=(bh+cg)\sin B$; that is, we have

$$(bc+gh)\sin A - (bh+cg)\sin B = 0.$$

But from the triangles BAD and BAC, in which the angles D, C are equal to each other, we have

$$\frac{c}{\sin D} = -\frac{f}{\sin B}, \quad \frac{c}{\sin C} = \frac{a}{\sin A};$$

that is,

$$f\sin A + a\sin B = 0;$$

and thence the required relation

$$a(bc+gh) + f(bh+cg) = 0.$$

The distances of the four points on the circle are thus connected by the two equations $\Delta=0$, $V=0$. Considering a, b, c, f, g, h as the distances from each other of any four points in the plane, we have between them the relation

$$\begin{aligned}\Omega, = \quad & a^2f^2(-a^2-f^2+b^2+g^2+c^2+h^2)\\ & +b^2g^2(\ \ a^2+f^2-b^2-g^2+c^2+h^2)\\ & +c^2h^2(\ \ a^2+f^2+b^2+g^2-c^2-h^2)\\ & -a^2b^2c^2-a^2g^2h^2-b^2h^2f^2-c^2f^2g^2, =0;\end{aligned}$$

and it is clear that this equation should be a consequence of the equations $\Delta=0$, $V=0$. To verify this, forming the sum $\Omega+V^2$, we have

$$\begin{aligned}\Omega+V^2= \quad & (a^2+f^2)(-a^2f^2+b^2g^2+c^2h^2+2bgch)\\ & +(b^2+g^2)(-b^2g^2+c^2h^2+a^2f^2+2chaf)\\ & +(c^2+h^2)(-c^2h^2+a^2f^2+b^2g^2+2afbg);\end{aligned}$$

viz. this is

$$\begin{aligned}= \quad & (a^2+f^2)\{-a^2f^2+(\Delta-af)^2\}\\ & +(b^2+g^2)\{-b^2g^2+(\Delta-bg)^2\}\\ & +(c^2+h^2)\{-c^2h^2+(\Delta-ch)^2\};\end{aligned}$$

or, since

$$-a^2f^2+(\Delta-af)^2=\Delta(\Delta-2af)=\Delta(-af+bg+ch), \text{ \&c.},$$

this is

$$\begin{aligned}\Omega+V^2=\Delta[\ \ & (a^2+f^2)(-af+bg+ch)\\ & +(b^2+g^2)(\ \ af-bg+ch)\\ & +(c^2+h^2)(\ \ af+bg-ch)],\end{aligned}$$

which proves the theorem.

It may be remarked that the equation $V=0$ may be written

$$a(bc+gh)+f(bh+cg)=0;$$

viz. multiplying by a, and for af writing its value, $=-(bg+ch)$ from the equation $\Delta=0$, this gives

$$-a^2(bc+gh)+(bg+ch)(bh+cg)=0,$$

that is,

$$bc(g^2+h^2-a^2)+gh(b^2+c^2-a^2)=0,$$

which expresses that the angles A, D are supplementary to each other; and, similarly, by the elimination of any other of the six quantities from the equations $\Delta=0$, $V=0$, we have five other like equations.

878.

NOTE ON THE ANHARMONIC RATIO EQUATION.

[From the *Messenger of Mathematics*, vol. XVII. (1888), pp. 95, 96.]

GIVEN any four quantities α, β, γ, δ, if θ be one of the values of the anharmonic ratio, the other values are

$$\frac{1}{\theta},\quad -(1+\theta),\quad -\frac{1}{1+\theta},\quad -\frac{\theta}{1+\theta},\quad -\frac{1+\theta}{\theta};$$

and hence the equation having these six roots is

$$(x-\theta)\left(x-\frac{1}{\theta}\right)(x+1+\theta)\left(x+\frac{1}{1+\theta}\right)\left(x+\frac{\theta}{1+\theta}\right)\left(x+\frac{1+\theta}{\theta}\right)=0;$$

or, multiplying out, the equation, as is well known, takes the form

$$(x^2+x+1)^3-\frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2}x^2(x+1)^2=0.$$

But to effect the multiplication in the easiest manner we may proceed as follows: writing

$$a,\ b,\ c=(\alpha-\delta)(\beta-\gamma),\quad (\beta-\delta)(\gamma-\alpha),\quad (\gamma-\delta)(\alpha-\beta),$$

so that $a+b+c=0$, the equation is

$$\left(x-\frac{b}{c}\right)\left(x-\frac{c}{b}\right)\left(x-\frac{c}{a}\right)\left(x-\frac{a}{c}\right)\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)=0.$$

The product of the first pair of factors is

$$x^2+1-\left(\frac{b}{c}+\frac{c}{b}\right)x,\ =(x+1)^2-\frac{a^2}{bc}x;$$

thus the equation is

$$\left\{(x+1)^2-\frac{a^2}{bc}x\right\}\left\{(x+1)^2-\frac{b^2}{ca}x\right\}\left\{(x+1)^2-\frac{c^2}{ab}x\right\}=0;$$

that is,

$$(x+1)^6-\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)x(x+1)^4+\left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}\right)x^2(x+1)^2-x^3=0;$$

and recollecting that $a+b+c=0$, and writing $q=bc+ca+ab$, $r=abc$, the equation becomes

$$(x+1)^6-3(x+1)^4x+\left(3+\frac{q^3}{r^2}\right)(x+1)^2x^2-x^3=0;$$

that is,

$$(x^2+x+1)^3+\frac{q^3}{r^2}(x+1)^2x^2=0.$$

But, writing $\theta=\frac{b}{a}$, we have

$$(\theta^2+\theta+1)^3+\frac{q^3}{r^2}(\theta+1)^2\theta^2=0;$$

or finally,

$$(x^2+x+1)^3-\frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2}x^2(x+1)^2=0,$$

the required result.

879.

NOTE ON THE DIFFERENTIAL EQUATION

$$\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0.$$

[From the *Messenger of Mathematics*, vol. XVIII. (1889), p. 90.]

WE have

$$\frac{\sin u-\sin v}{\cos u-\cos v}=-\cot\tfrac{1}{2}(u+v),\ =-\sqrt{\left\{\frac{1+\cos(u+v)}{1-\cos(u+v)}\right\}},$$

and thence, writing $\cos u=x$, $\sin u=\sqrt{(1-x^2)}=\sqrt{(X)}$, and similarly

$$\cos v=y,\ \sin v=\sqrt{(1-y^2)}=\sqrt{(Y)},$$

we have

$$\frac{\sqrt{(X)}-\sqrt{(Y)}}{x-y}=-\sqrt{\left\{\frac{1+xy-\sqrt{(XY)}}{1-xy+\sqrt{(XY)}}\right\}},$$

an identical equation which, in the form

$$\frac{2-x^2-y^2-2\sqrt{(XY)}}{(x-y)^2}=\frac{1+xy-\sqrt{(XY)}}{1-xy+\sqrt{(XY)}},$$

may be verified directly without any difficulty. The integral of the proposed differential equation can of course be taken to be $c=xy-\sqrt{(XY)}$; and we have thus another form of integral

$$\frac{\sqrt{(X)}-\sqrt{(Y)}}{x-y}=-\sqrt{\left(\frac{1+c}{1-c}\right)},\ \text{say}\ =\sqrt{(C)},$$

viz. we have the integral

$$\left\{\frac{\sqrt{(X)}-\sqrt{(Y)}}{x-y}\right\}^2=C,$$

which is what Lagrange's integral of the differential equation

$$\frac{dx}{\sqrt{(X)}}+\frac{dy}{\sqrt{(Y)}}=0$$

becomes when the quartic functions X, Y reduce themselves to the quadric functions $1-x^2$ and $1-y^2$ respectively.

880.

NOTE ON THE RELATION BETWEEN THE DISTANCE OF FIVE POINTS IN SPACE.

[From the *Messenger of Mathematics*, vol. XVIII. (1889), pp. 100—102.]

IN Lagrange's paper "Solutions analytiques de quelques problèmes sur les pyramides triangulaires" (*Berlin Memoirs*, 1773; *Œuvres*, t. III., see p. 677), there is contained a formula for the relation between the distances from each other of five points in space; viz. this is

$$\begin{aligned}4\Delta^2 f = {} & \alpha(a+f-g)^2 + \alpha'(a'+f-g')^2 + \alpha''(a''+f-g'')^2 \\ & + 2\beta(a'+f-g')(a''+f-g'') + 2\beta'(a+f-g)(a''+f-g'') \\ & + 2\beta''(a+f-g)(a'+f-g'),\end{aligned}$$

or, in a slightly altered notation, say

$$\Pi = -4\Delta^2 f + (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \between a_1+f-g_1, a_2+f-g_2, a_3+f-g_3)^2 = 0,$$

where, if the points are called 1, 2, 3, 4, 5, then

c_1, c_2, c_3 are the squared distances		23, 31, 12,
a_1, a_2, a_3	,,	41, 42, 43,
g_1, g_2, g_3	,,	51, 52, 53,

and

f is the squared distance 45.

The values of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, in terms of these squared distances, are

$$\begin{aligned}
\alpha_1 &= -(a_2-a_3)^2 + 2c_1(a_2+a_3) - c_1^2, \\
\alpha_2 &= -(a_3-a_1)^2 + 2c_2(a_3+a_1) - c_2^2, \\
\alpha_3 &= -(a_1-a_2)^2 + 2c_3(a_1+a_2) - c_3^2, \\
\beta_1 &= -(a_3-a_1)(a_1-a_2) + 2c_1a_1 - c_2(a_1+a_2) - c_3(a_3+a_1) + c_2c_3, \\
\beta_2 &= -(a_1-a_2)(a_2-a_3) + 2c_2a_2 - c_3(a_2+a_3) - c_1(a_1+a_2) + c_3c_1, \\
\beta_3 &= -(a_2-a_3)(a_3-a_1) + 2c_3a_3 - c_1(a_3+a_1) - c_2(a_2+a_3) + c_1c_2.
\end{aligned}$$

After some reductions, the value of $4\Delta^2$, in terms of these squared distances, is found to be

$$\begin{aligned}4\Delta^2 = \quad & c_1(a_3-a_1)(a_1-a_2)+a_1(-c_1^2+c_1c_2+c_1c_3)-c_1c_2c_3\\ &+c_2(a_1-a_2)(a_2-a_3)+a_2(-c_2^2+c_2c_3+c_2c_1)\\ &+c_3(a_2-a_3)(a_3-a_1)+a_3(-c_3^2+c_3c_1+c_3c_2),\end{aligned}$$

which is, in fact,

$$8\Delta^2 = \begin{vmatrix} . & 1, & 1, & 1, & 1 \\ 1, & . & c_3, & c_2, & a_1 \\ 1, & c_3, & . & c_1, & a_2 \\ 1, & c_2, & c_1, & . & a_3 \\ 1, & a_1, & a_2, & a_3, & . \end{vmatrix},$$

where observe that each term of the determinant contains the factor 2.

By the formula in my paper "On a theorem in the Geometry of Position," *Camb. Math. Jour.*, t. II. (1841), pp. 267—271, [1], (introducing into it the present notation), the relation between the distances of the five points is given in the form

$$\Omega = \begin{vmatrix} . & 1, & 1, & 1, & 1, & 1 \\ 1, & . & c_3, & c_2, & a_1, & g_1 \\ 1, & c_3, & . & c_1, & a_2, & g_2 \\ 1, & c_2, & c_1, & . & a_3, & g_3 \\ 1, & a_1, & a_2, & a_3, & . & f \\ 1, & g_1, & g_2, & g_3, & f, & . \end{vmatrix} = 0.$$

The equations $\Pi = 0$, $\Omega = 0$ should therefore agree with each other; we have in Ω the term

$$-f^2 \begin{vmatrix} 1, & c_3, & . & c_1 \\ 1, & c_2, & c_1, & . \\ . & 1, & 1, & 1 \\ 1, & . & c_3, & c_2 \end{vmatrix},$$

which is

$$= -f^2(c_1^2+c_2^2+c_3^2-2c_2c_3-2c_3c_1-2c_1c_2);$$

and similarly in Π we have the term

$$f^2(\alpha_1+\alpha_2+\alpha_3+2\beta_1+2\beta_2+2\beta_3),$$

which is easily shown to be

$$= -f^2(c_1^2+c_2^2+c_3^2-2c_2c_3-2c_3c_1-2c_1c_2);$$

and it thus appears that we have identically $\Pi = \Omega$.

It is to be remarked that, in Lagrange's form, the points 4 and 5 are regarded as determined each of them by means of its squared distances from the vertices of the triangle 123, and that the formula gives (by a quadratic equation) the squared distance 45; but that nevertheless the two points 4 and 5 do not present themselves symmetrically in the formula; in fact, Δ^2 and the coefficients $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ of the formula relate all of them to the tetrahedron 4123; as noticed in the paper, $\Delta = 6 \times$ volume of tetrahedron; $\sqrt{(\alpha_1)}$, $\sqrt{(\alpha_2)}$, $\sqrt{(\alpha_3)}$ are the doubled areas of the faces 423, 431, 412 respectively, and the doubled area of the face 123 is

$$= \sqrt{(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3)};$$

it may be added that

$$\beta_1 \div \sqrt{(\alpha_2\alpha_3)}, \quad \beta_2 \div \sqrt{(\alpha_3\alpha_1)}, \quad \beta_3 \div \sqrt{(\alpha_1\alpha_2)},$$

are equal to the cosines of the dihedral angles at the edges 41, 42, 43 respectively.

881.

ON HERMITE'S *H*-PRODUCT THEOREM.

[From the *Messenger of Mathematics*, vol. XVIII. (1889), pp. 104—107.]

I GIVE this name to a theorem relating to the product of an even number of Eta-functions, established by M. Hermite in his "Note sur le calcul différential et le calcul integral," forming an appendix to the sixth edition of Lacroix's *Differential and Integral Calculus*, and separately printed, 8vo. Paris, 1862. It is the theorem stated p. 65, in the form

$$\phi(x) = F(z^2) + \frac{dz}{dx} z F_1(z^2),$$

where

$$\phi(x) = \frac{A H(x-\alpha_1) H(x-\alpha_2) \dots H(x-\alpha_{2n})}{\Theta^{2n}(x)},$$

where $\alpha_1 + \alpha_2 + \dots + \alpha_{2n} = 0$, and $z = \operatorname{sn} x$, $\operatorname{cn} x$ or $\operatorname{dn} x$ at pleasure; $F(z^2)$, $F_1(z^2)$ denote rational and integral functions of z^2 of the degrees n and $n-2$ respectively; A is a constant, which we may if we please so determine that in $F(z^2)$ the coefficient of the highest power z^{2n} shall be $=1$.

If, for shortness, we write s, c, d for $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ respectively; and to fix the ideas, assume $z = \operatorname{sn} x$, $= s$, then the theorem is

$$\frac{A H(x-\alpha_1) H(x-\alpha_2) \dots H(x-\alpha_{2n})}{\Theta^{2n}(x)} = F(s^2) + scdF_1(s^2);$$

viz. the theorem is that the product of the $2n$ H-functions ($\alpha_1 + \alpha_2 + \dots + \alpha_{2n} = 0$ as above), divided by $\Theta^{2n}(x)$, is a function of the elliptic functions sn, cn, dn, of the form in question.

Hermite uses the theorem for the demonstration of Abel's theorem, as applied to the elliptic functions; or as I would rather express it, he uses the theorem for the determination of the sn, cn, and dn of $\alpha_1 + \alpha_2 + \dots + \alpha_{2n-1}$.

To show how this is, observe that $F(s^2)$, *quà* rational and integral function of s^2 of the degree n, with its first coefficient $=1$, contains n arbitrary coefficients; and $F_1(s^2)$, *quà* rational and integral function of s^2 of the degree $n-2$, contains $n-1$ arbitrary coefficients: hence $F(s^2)+scdF_1(s^2)$ contains $2n-1$ arbitrary coefficients; and considering $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$ as given, the function in question must vanish for each of the values $x=\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$; and we have therefore $2n-1$ equations for obtaining the $2n-1$ coefficients, which are thus completely determined: in particular, the constant term, say L, of $F_1(s^2)$ is a given function of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$, that is, of the sn, cn, and dn of these quantities; and the theorem shows that the function thus determined vanishes also for $x=\alpha_{2n}$, that is, $=-(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$.

Now writing $-x$ for x in the formula, and recollecting that H is an odd function, Θ an even function, we find

$$\frac{AH(x+\alpha_1)\,H(x+\alpha_2)\ldots H(x+\alpha_{2n})}{\Theta^{2n}(x)}=F(s^2)-scdF_1(s^2);$$

and multiplying together the two sides of these equations respectively,

$$A^2\frac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)}\ldots\frac{H(x-\alpha_{2n})\,H(x+\alpha_{2n})}{\Theta^2(x)}=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2,$$

where the right-hand side is a rational and integral function of s^2 of the degree $2n$, and the coefficient of the highest term s^{4n} is $=1$; in fact, this term arises only from the square of $F(s^2)$, which has its highest term $=s^{2n}$.

Now $\dfrac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)}$ is a mere constant multiple of $\operatorname{sn}^2 x-\operatorname{sn}^2\alpha_1$, or say of $s^2-\operatorname{sn}^2\alpha_1$; (this well-known theorem is, in fact, the particular case $n=2$ of Hermite's theorem); and similarly for the other terms: we must clearly have A^2, multiplied by the product of the factors thus introduced, $=1$; and thus the theorem becomes

$$(s^2-\operatorname{sn}^2\alpha_1)(s^2-\operatorname{sn}^2\alpha_2)\ldots(s^2-\operatorname{sn}^2\alpha_{2n})=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2.$$

And putting herein $s=0$, and writing as before L for the constant term of $F(s^2)$, we have

$$\operatorname{sn}^2\alpha_1\operatorname{sn}^2\alpha_2\ldots\operatorname{sn}^2\alpha_{2n}=L^2,$$

or, the sign $\pm$ being properly determined, say

$$\operatorname{sn}\alpha_1\operatorname{sn}\alpha_2\ldots\operatorname{sn}\alpha_{2n}=\pm L,$$

where, by what precedes, L is a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$. Hence we have $\operatorname{sn}\alpha_{2n}$, that is, $-\operatorname{sn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$.

Similarly writing $z=\operatorname{cn}x$, $=c$, and $z=\operatorname{dn}x$, $=d$, we have $\operatorname{cn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ and $\operatorname{dn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ each of them as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$.

It is hardly necessary to remark that $F(z^2)+z\dfrac{dz}{dx}F_1(z^2)$ is a function of the same form, whether we have $z=s$, c or d; in fact, the functions F and F_1 are rational in s^2, c^2, or d^2, and we have $z\dfrac{dz}{dx}=scd$, $-scd$, and $-k^2scd$ for the three values respectively.

The number of terms $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$ has been odd, but by taking one of them $=0$, the formulæ give the values of the sn, cn, and dn for the sum of an even number of terms.

It has been seen that Hermite's H-product theorem gives, say Abel's theorem, in the form

$$\Pi\,(s^2 - \operatorname{sn}^2 \alpha) = \{F(s^2)\}^2 - s^2c^2d^2\{F_1(s^2)\}^2,$$

each side of this relation being the product of two factors, viz. for the left-hand side the factors are

$$A\Pi\frac{H(x-\alpha)}{\Theta(x)},\quad A\Pi\frac{H(x+\alpha)}{\Theta(x)},$$

and for the right-hand side they are the rational functions of s^2,

$$F(s^2) + scdF_1(s^2),\quad F(s^2) - scdF_1(s^2);$$

these factors are by Hermite's theorem equal each to each; viz. this is the relation in which Hermite's stands to Abel's theorem.

The H-product theorem is given as one out of a group of four theorems; the other three may be called the H-product, H_1-product and Θ_1-product, odd theorems respectively,

$$\{\Theta_1(x) = \Theta(x+K),\quad H_1(x) = H(x+K)\},$$

viz. these are

$$\begin{cases} \dfrac{A_1 H\,(x-\alpha_1)\,H\,(x-\alpha_2)\ldots H\,(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = sF(s^2) + \;\; cd\,\phi\,(s^2), \\[2ex] \dfrac{A_2 H_1(x-\alpha_1)\,H_1(x-\alpha_2)\ldots H_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = cF(c^2) - \;\; sd\,\phi\,(c^2), \\[2ex] \dfrac{A_3\Theta_1(x-\alpha_1)\,\Theta_1(x-\alpha_2)\ldots\Theta_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = dF(d^2) - k^2sc\,\phi\,(d^2), \end{cases}$$

where F, ϕ are rational and integral functions of the degrees n and $n-1$, having their proper values in the three equations respectively, and in each case

$$\alpha_1 + \alpha_2 + \ldots + \alpha_{2n+1} = 0.$$

It was seen above that, for $n=1$, the H-product theorem became

$$\frac{AH(x-\alpha)\,H(x+\alpha)}{\Theta^2(x)} = \operatorname{sn}^2 x - \operatorname{sn}^2 \alpha,$$

which is the most simple case; for the odd theorems, the most simple case is $n=0$, viz. we then have

$$\frac{A_1 H(x)}{\Theta(x)} = \operatorname{sn} x,\quad \frac{A_2 H_1(x)}{\Theta(x)} = \operatorname{cn} x,\quad \frac{A_3\Theta_1(x)}{\Theta(x)} = \operatorname{dn} x;$$

to complete the formulæ observe that the values of the constants are

$$A = \frac{2k'K}{k\pi\Theta^2(\alpha)},\quad A_1 = \frac{1}{\sqrt{(k)}},\quad A_2 = \sqrt{\left(\frac{k'}{k}\right)},\quad A_3 = \sqrt{(k')}.$$

The three theorems may be used, in like manner with the H-product theorem, to give the values of the sn, cn, and dn respectively of the sum $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n}$.

882.

A CORRESPONDENCE OF CONFOCAL CARTESIANS WITH THE RIGHT LINES OF A HYPERBOLOID.

[From the *Messenger of Mathematics*, vol. XVIII. (1889), pp. 128—130.]

TAKE α, β, γ arbitrary, A, B, $C=\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ (so that $A+B+C=0$), and writing ρ, σ, τ for rectangular coordinates, consider the hyperboloid

$$A\rho^2+B\sigma^2+C\tau^2+ABC=0.$$

Let ρ_0, σ_0, τ_0 be the coordinates of a point on the surface $(A\rho_0^2+B\sigma_0^2+C\tau_0^2+ABC=0)$. The equations of a line through this point are ρ, σ, $\tau=\rho_0+f\Omega$, $\sigma_0+g\Omega$, $\tau_0+h\Omega$ (Ω indeterminate); and if this lies on the surface, we have

$$\begin{aligned} A\rho_0 f+B\sigma_0 g+C\tau_0 h&=0,\\ Af^2+Bg^2\quad+Ch^2&=0,\end{aligned}$$

which equations determine the ratios $f:g:h$; the equations give

$$(A\rho_0 f+B\sigma_0 g)^2=C\tau_0^2.\,Ch^2,\quad=-C\tau_0^2(Af^2+Bg^2);$$

that is,

$$(A^2\rho_0^2+AC\tau_0^2)f^2+2AB\rho_0\sigma_0 fg+(B^2\sigma_0^2+BC\tau_0^2)g^2=0,$$

whence

$$\begin{aligned} &\{(B^2\sigma_0^2+BC\tau_0^2)g+AB\rho_0\sigma_0 f\}^2\\ &=\{A^2B^2\rho_0^2\sigma_0^2-(A^2\rho_0^2+AC\tau_0^2)(B^2\sigma_0^2+BC\tau_0^2)\}f^2,\\ &=-ABC(A\rho_0^2+B\sigma_0^2+C\tau_0^2)\tau_0^2 f^2,\\ &=A^2B^2C^2\tau_0^2 f^2;\end{aligned}$$

that is,

$$\{(B\sigma_0^2+C\tau_0^2)g+A\rho_0\sigma_0 f\}^2=A^2C^2\tau_0^2 f^2,$$

or say

$$(B\sigma_0^2+C\tau_0^2)g+A(\rho_0\sigma_0\pm C\tau_0)f=0,$$

which equation, together with $A\rho_0 f+B\sigma_0 g+C\tau_0 h=0$, determines the ratios $f:g:h$. We have thus the two lines through the point $(\rho_0,\ \sigma_0,\ \tau_0)$.

But the equations of the line may be conveniently represented in a different form; writing the equation first obtained in the form

$$\sigma_0(B\sigma_0 g + A\rho_0 f) + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

this is

$$-\sigma_0 C\tau_0 h + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

viz.

$$-h\sigma_0 + g\tau_0 \pm Af = 0;$$

and we have the like equations

$$-f\tau_0 + h\rho_0 \pm Bg = 0,$$
$$-g\rho_0 + f\sigma_0 \pm Ch = 0,$$

where the sign is the same in each of the three equations.

The equations of the line on the surface may be written

$$\begin{array}{cccc} . & h\sigma & -g\tau - h\sigma_0 + g\tau_0 & = 0, \\ -h\rho & . & +f\tau - f\tau_0 + h\rho_0 & = 0, \\ g\rho & -f\sigma & . \; - g\rho_0 + f\sigma_0 & = 0, \\ (h\sigma_0 - g\tau_0)\rho & + (f\tau_0 - h\rho_0)\sigma & + (g\rho_0 - f\sigma_0)\tau & . \; = 0; \end{array}$$

and hence from the foregoing three equations, taking the sign $-$, we have

$$\begin{array}{ccccc} . & h\sigma - & g\tau & + Af & = 0, \\ -h\rho & . & + f\tau & + Bg & = 0, \\ g\rho - & f\sigma & . & + Ch & = 0, \\ -Af\rho & -Bg\sigma & -Ch\tau & . & = 0, \end{array}$$

where $Af^2 + Bg^2 + Ch^2 = 0$, for the equations of a line on the surface.

In like manner, taking the sign $+$, and for f, g, h writing new values f', g', h', we have

$$\begin{array}{ccccc} . & h'\sigma - & g'\tau & - Af' & = 0, \\ -h'\rho & . & + f'\tau & - Bg' & = 0, \\ g'\rho - & f'\sigma & . & - Ch' & = 0, \\ Af'\rho & +Bg'\sigma & +Ch'\tau & . & = 0, \end{array}$$

where $Af'^2 + Bg'^2 + Ch'^2 = 0$, for the equations of a line on the surface.

The two systems of equations evidently belong to the lines of the two different kinds respectively. Writing for shortness P, Q, $R = gh' + g'h$, $hf' + h'f$, $fg' + f'g$, the two lines in fact intersect in a point, the coordinates say $(\rho_0, \sigma_0, \tau_0)$ whereof are $= \Theta QR$, ΘRP, ΘPQ, where

$$\Theta = \frac{A}{g^2h'^2 - g'^2h^2} = \frac{B}{h^2f'^2 - h'^2f^2} = \frac{C}{f^2g'^2 - f'^2g^2},$$

the three expressions for Θ being equal to each other in virtue of the equations

$$Af^2 + Bg^2 + Ch^2 = 0, \quad Af'^2 + Bg'^2 + Ch'^2 = 0.$$

Take now, in a plane, P, Q, R points on any line, say the axis of x, at distances α, β, γ from the origin, then for a point of the plane, coordinates (x, y), if ρ, σ, τ be the distances of the point from these three points, or say foci, we have

$$\begin{aligned} \rho^2 &= (x-\alpha)^2 + y^2, \\ \sigma^2 &= (x-\beta)^2 + y^2, \\ \tau^2 &= (x-\gamma)^2 + y^2; \end{aligned}$$

and if as before A, B, $C = \beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$, we thence have

$$A\rho^2 + B\sigma^2 + C\tau^2 + ABC = 0.$$

A point, coordinates (ρ, σ, τ), of the hyperboloid thus corresponds to a point in the plane, distances ρ, σ, τ from the three foci R, S, T respectively; and to any line

$$\begin{aligned} \cdot \quad & h\sigma - g\tau + Af = 0, \\ -h\rho \quad & \cdot + f\tau + Bg = 0, \\ g\rho - f\sigma \quad & \cdot + Ch = 0, \\ -Af\rho - Bg\sigma - Ch\tau \quad & \cdot = 0, \end{aligned}$$

corresponds the Cartesian represented by these linear equations. Similarly, to the line represented by the other system of equations

$$\begin{aligned} \cdot \quad & h'\sigma - g'\tau - Af' = 0, \\ -h'\rho \quad & \cdot + f'\tau - Bg' = 0, \\ g'\rho - f'\sigma \quad & \cdot - Ch' = 0, \\ Af'\rho + Bg'\sigma + Ch'\tau \quad & \cdot = 0, \end{aligned}$$

corresponds the Cartesian represented by these equations; the two curves intersect in the point ρ_0, σ_0, $\tau_0 = \Theta QR$, ΘRP, ΘPQ, corresponding to the intersection of the lines on the hyperboloid; and moreover, *quà* confocal Cartesians, they intersect at right angles.

883.

ANALYTICAL FORMULÆ IN REGARD TO AN OCTAD OF POINTS.

[From the *Messenger of Mathematics*, vol. XVIII. (1889), pp. 149—152.]

THE term "tetrad" is used in two distinct but not inconsistent senses, viz. a tetrad denotes any four points; and it also denotes the four vertices of a self-conjugate tetrahedron in regard to a quadric surface. In fact, given any four points, there exists a triply infinite series of quadric surfaces such that, in regard to any one of them, the four points form a self-conjugate tetrahedron.

Two or more tetrads, in regard to one and the same quadric surface, are called similar tetrads.

The eight points of intersection of any three quadric surfaces are an octad; and we have the theorem that any seven points determine the octad, viz. the quadric surfaces which pass through any seven given points pass also through a uniquely determinate eighth point.

We have the further theorem that any two similar tetrads form an octad.

In particular, the vertices of the tetrahedron

$$(x=0,\ y=0,\ z=0,\ w=0),$$

or, say the points 1000, 0100, 0010, 0001 are a tetrad in regard to the quadric surface $x^2+y^2+z^2+w^2=0$. The points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3), (x_4, y_4, z_4, w_4), or say the points 1, 2, 3, 4, will be a tetrad in regard to the same quadric surface, if only

$$x_1x_2+y_1y_2+z_1z_2+w_1w_2=0, \text{ \&c., (six equations).}$$

Hence, these equations being satisfied, the two tetrads form an octad; the equation of a general quadric surface through the points of the first tetrad is

$$fyz+gzx+hxy+lxw+myw+nzw=0,$$

and we can from three such equations eliminate any two terms; the two terms eliminated may be such as xy, xz, (12 forms) or such as yz, xw, (3 forms); and the equation may thus be presented either in the form

$$fyz + (lx + my + nz)\,w = 0,$$

or in the form

$$gzx + hxy + myw + nzw = 0.$$

The absolute magnitudes of the coordinates

$$(x_1,\ y_1,\ z_1,\ w_1),\ \&c.,$$

of these points are properly indeterminate; but we may if we please fix the absolute magnitudes by assuming

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 = M,\ \&c.,$$

(four equations, the same quantity M in each of them), and we have as before

$$x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2 = 0,\ \&c.,\ (\text{six equations}),$$

viz. the coordinates are taken to satisfy these 10 equations; and this being so, it is to be shown that there exist quadric surfaces as above. The 10 equations may be expressed in a correlative form in like manner with the ordinary six equations for the transformation of the rectangular coordinates (x, y, z), viz. the new form is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = M,\ \&c.,\ (\text{four equations}),$$

and

$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0,\ \&c.,\ (\text{six equations}),$$

and the coordinates thus also satisfy these 10 equations.

Thus the equation $fyz + (lx + my + nz)\,w = 0$, if we determine the coefficients in such wise that the surface passes through the points 1, 2, 3, becomes

$$\begin{vmatrix} yz\ , & xw\ , & yw\ , & zw \\ y_1z_1, & x_1w_1, & y_1w_1, & z_1w_1 \\ y_2z_2, & x_2w_2, & y_2w_2, & z_2w_2 \\ y_3z_3, & x_3w_3, & y_3w_3, & z_3w_3 \end{vmatrix} = 0,$$

and this equation must therefore be satisfied on writing therein $(x_4,\ y_4,\ z_4,\ w_4)$ for (x, y, z, w); viz. the equation thus obtained, det. $(1, 2, 3, 4) = 0$, must be satisfied in virtue of the equations which connect the coordinates of the four points. But, instead of effecting this verification, it is better to exhibit the equation in (x, y, z, w) in a form wherein the coordinates of the four points enter symmetrically; and the new equation, *quà* transformation of the original form, is satisfied for $(x, y, z, w) = (x_1, y_1, z_1, w_1)$: and then of course, by reason of the symmetry, it is also satisfied for

$$(x,\ y,\ z,\ w) = (x_4,\ y_4,\ z_4,\ w_4).$$

The transformation may be effected directly; but I prefer to write down the final result, and afterwards verify it.

The form is

$$\begin{aligned} yz\,.\,Mw_1w_2w_3w_4 &+ xw\,[-x_1y_1z_1\,.\,w_2w_3w_4 - x_2y_2z_2\,.\,w_3w_4w_1 - x_3y_3z_3\,.\,w_4w_1w_2 - x_4y_4z_4\,.\,w_1w_2w_3] \\ &+ yw\,[\;\; x_1^2w_1\,.\,z_2z_3z_4 + x_2^2w_2\,.\,z_3z_4z_1 + x_3^2w_3\,.\,z_4z_1z_2 + x_4^2w_4\,.\,z_1z_2z_3] \\ &+ zw\,[\;\; x_1^2w_1\,.\,y_2y_3y_4 + x_2^2w_2\,.\,y_3y_4y_1 + x_3^2w_3\,.\,y_4y_1y_2 + x_4^2w_4\,.\,y_1y_2y_3] = 0. \end{aligned}$$

We have to show that the equation is satisfied by writing therein

$$(x,\ y,\ z,\ w) = (x_1,\ y_1,\ z_1,\ w_1).$$

Calling the resulting value Ω_1, we have

$$\begin{aligned} \Omega_1 \div w_1 = My_1z_1w_2w_3w_4 &- x_1^2y_1z_1w_2w_3w_4 - x_1\,(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ &+ x_1^2w_1\,(y_1z_2z_3z_4 + z_1y_2y_3y_4) + y_1z_1\,[x_2^2w_2\,(z_3z_4 + y_3y_4) + x_3^2w_3\,(z_4z_2 + y_4y_2) + x_4^2w_4\,(z_2z_3 + y_2y_3)], \end{aligned}$$

which is

$$\begin{aligned} = y_1z_1w_2w_3w_4\,(x_2^2 + x_3^2 + x_4^2) &- x_1\,(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ + x_1^2w_1\,(y_1z_2z_3z_4 + z_1y_2y_3y_4) &- y_1z_1\,[x_2^2w_2\,(x_3x_4 + w_3w_4) + x_3^2w_3\,(x_2x_4 + w_2w_4) + x_4^2w_4\,(x_2x_3 + w_2w_3)]. \end{aligned}$$

The expression contains terms in $w_2w_3w_4$ which destroy each other; omitting them, we have

$$\begin{aligned} \Omega_1 \div w_1 = &- x_1\,(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ &+ x_1^2w_1\,(y_1z_2z_3z_4 + z_1y_2y_3y_4) \\ &- x_2x_3x_4y_1z_1\,(x_2w_2 + x_3w_3 + x_4w_4), \end{aligned}$$

where the last line is

$$= x_2x_3x_4y_1z_1\,.\,x_1w_1.$$

Hence the whole divides by x_1w_1, or we have

$$\begin{aligned} \Omega_1 \div x_1w_1^2 = &- x_2y_2z_2\,.\,w_3w_4 - x_3y_3z_3\,.\,w_4w_2 - x_4y_4z_4\,.\,w_2w_3 \\ &+ x_1y_1z_2z_3z_4 + x_1z_1y_2y_3y_4 + y_1z_1x_2x_3x_4, \end{aligned}$$

viz. this is

$$\begin{aligned} = \;\; & x_2x_3x_4y_1z_1 + y_2y_3y_4z_1x_1 + z_2z_3z_4x_1y_1 \\ &+ x_2y_2z_2\,(x_3x_4 + y_3y_4 + z_3z_4) \\ &+ x_3y_3z_3\,(x_2x_4 + y_2y_4 + z_2z_4) \\ &+ x_4y_4z_4\,(x_2x_3 + y_2y_3 + z_2z_3); \end{aligned}$$

or, finally, it is

$$\begin{aligned} = \;\; & x_2x_3x_4\,(y_1z_1 + y_2z_2 + y_3z_3 + y_4z_4) \\ &+ y_2y_3y_4\,(z_1x_1 + z_2x_2 + z_3x_3 + z_4x_4) \\ &+ z_2z_3z_4\;\,(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4), \end{aligned}$$

which is $= 0$; that is, we have $\Omega_1 = 0$, the required result.

The foregoing equation of the quadric surface can be by mere interchanges of the letters exhibited in 12 different forms.

We may, in like manner, first establish and then verify the equation

$$\begin{aligned}
&zx\,(-z_1^2w_1y_2y_3y_4 - z_2^2w_2y_3y_4y_1 - z_3^2w_3y_4y_1y_2 - z_4^2w_4y_1y_2y_3)\\
&+xy\,(y_1^2w_1z_2z_3z_4 + y_2^2w_2z_3z_4z_1 + y_3^2w_3z_4z_1z_2 + y_4^2w_4z_1z_2z_3)\\
&+yw\,(-y_1^2x_1z_2z_3z_4 - y_2^2x_2z_3z_4z_1 - y_3^2x_3z_4z_1z_2 - y_4^2x_4z_1z_2z_3)\\
&+zw\,(z_1^2x_1y_2y_3y_4 + z_2^2x_2y_3y_4y_1 + z_3^2x_3y_4y_1y_2 + z_4^2x_4y_1y_2y_3) = 0;
\end{aligned}$$

this can be by a cyclical interchange of the letters (x, y, z) exhibited in 3 different forms.

Of course, the fifteen equations belong each of them to a quadric surface through the 8 points. Any three of the fifteen equations, say $U=0$, $V=0$, $W=0$, may be used for determining the octad; the equation of any other quadric through the octad is then $\alpha U+\beta V+\gamma W=0$.

884.

NOTE SUR LES SURFACES MINIMA ET LE THÉORÈME DE JOACHIMSTHAL.

[From the *Comptes Rendus*, t. CVI. (1888), pp. 995, 996.]

I. La seule surface minima réglée est l'hélicoïde à plan directeur: ce beau théorème de M. Catalan se démontre par des considérations géométriques très simples.

Soit un système de droites ..., P, Q, R, ..., tel que, pour trois droites successives P, Q, R quelconques, tout plan Π' perpendiculaire à Q rencontre les trois droites en des points p', q', r', situés sur une droite. En particulier, si qr est la distance la plus petite des droites Q et R, le plan perpendiculaire Π, qui passe par le point q, passera par le point r et rencontrera la droite P en un point p, tel que les points p, q, r seront sur une même droite Λ, perpendiculaire à chacune des droites P, Q, R; c'est-à-dire, chacune de ces droites rencontre perpendiculairement une seule et même droite Λ. Cela étant, pour que les points de rencontre p', q', r' avec un autre plan Π' soient sur la même droite, il faut encore une condition; et, en supposant (ce que l'on peut faire sans perte de généralité) que les distances pq, qr soient égales, cette condition sera: que l'inclinaison des plans $(P\Lambda, Q\Lambda)$ est égale à l'inclinaison des plans $(Q\Lambda, R\Lambda)$.

En considérant de même les droites Q, R, S, on démontre d'abord que la droite S rencontre perpendiculairement la droite Λ; puis, en supposant que les distances qr, rs soient égales, on obtient la condition que l'inclinaison des plans $(Q\Lambda, R\Lambda)$ est égale à celle des plans $(R\Lambda, S\Lambda)$, et ainsi de suite: savoir, on obtient une série de droites ..., P, Q, R, ..., qui rencontrent perpendiculairement la droite Λ et sont telles qu'en supposant que les distances ..., pq, qr, rs, ... soient égales, les inclinaisons ..., $(P\Lambda, Q\Lambda)$, $(Q\Lambda, R\Lambda)$, $(R\Lambda, S\Lambda)$,... sont égales: c'est-à-dire, en considérant des droites ..., P, Q, R, S, ... consécutives, on a les génératrices d'une hélicoïde à plan directeur; et l'on voit ainsi que cette surface est la seule surface

réglée, qui est telle que tout plan perpendiculaire à une génératrice quelconque rencontre la surface selon une courbe qui a, au point de rencontre avec la génératrice, une inflexion (ou, ce qui est la même chose, un rayon infini de courbure). Mais, comme l'avait remarqué M. Catalan, c'est là la condition pour que les deux rayons principaux de courbure soient égaux et opposés, ou enfin pour que la surface soit une surface minima.

II. Le théorème de Joachimsthal et aussi le théorème plus général de Bonnet et Serret, par rapport aux lignes de courbure planes ou sphériques, se déduisent immédiatement de ce théorème élémentaire de Géométrie: En considérant, dans des plans différents, deux triangles isoscèles $PP'O$ et $PP'N$ avec une base commune PP' ($OP = OP'$ et $NP = NP'$), les angles OPN et $OP'N$ seront égaux. En effet, si, pour une surface quelconque, PP' est l'élément d'une courbe de courbure sphérique, les normales à la surface aux points P et P' se rencontrent dans un point N, et les rayons de la sphère aux mêmes points se rencontrent dans un point O; on a ainsi les deux triangles isoscèles $PP'O$ et $PP'N$, et de là les angles égaux OPN et $OP'N$, c'est-à-dire que, pour deux points consécutifs P et P' de la ligne de courbure sphérique, l'inclinaison de la normale de la surface au rayon de la sphère a la même valeur, et cette inclinaison a ainsi la même valeur pour tous les points de la ligne de courbure. En prenant le point O à l'infini, on obtient le théorème pour une ligne de courbure plane, ou, si l'on veut, le théorème pour ce cas se déduit directement de celui-ci: Une droite quelconque PO, perpendiculaire à la base PP' d'un triangle isoscèle $PP'N$, est également inclinée sur les deux droites PN et $P'N$.

885.

ON THE DIOPHANTINE RELATION, $y^2+y'^2=$ SQUARE.

[From the *Proceedings of the London Mathematical Society*, vol. XX. (1889), pp. 122—127.]

THE diophantine relation $y^2+y'^2=$ Square, where y is a function of x, and y' denotes $\frac{dy}{dx}$, is considered by Prof. Sylvester in his paper "On Reducible Cyclodes," *Proc. Lond. Math. Soc.*, t. I. (1865—66), pp. 137—160. It is at once seen that there exists a solution

$$y=(x+a)^\alpha(x+b)^\beta(x+c)^\gamma(x+d)^\delta\ldots,$$

where the roots $a, b, c, d, \ldots$ are essentially unequal, and the number of simple factors $x+a, x+b, x+c, x+d, \ldots$ is even; the exponents $\alpha, \beta, \gamma, \delta, \ldots$ are taken to be positive integer numbers. Sylvester assumes, and it will be shown, that the factors must separate themselves into two sets, or, as he calls them, diptychs, each containing the same number of simple factors and such that the sum of the exponents for the one diptych is equal to the sum of the exponents for the other diptych; viz. the form is $y=UU_1$, where

$$U=(x+a)^\alpha(x+b)^\beta\ldots,\quad U_1=(x+a_1)^{\alpha_1}(x+b_1)^{\beta_1}\ldots,$$

with the same number of simple factors, and with the relation $\alpha+\beta+\ldots=\alpha_1+\beta_1+\ldots$ between the exponents. Hence, if the number of simple factors be called the class and the sum of the exponents be called the order, the class and the order are each of them even; or, what is the same thing, the semi-class (say μ) and the semi-order (say ν) are each of them integral.

The separation of the factors into two diptychs is a remarkable theorem. I consider the analytical theory; for greater simplicity, first in the case, class $=2$, and secondly in the case, class $=4$; but it is easy to see that the like process is applicable to the case of any even value whatever of the class.

I write as usual $i=\sqrt{-1}$; the equation $y^2+y'^2=$ square, implies $y+iy'=$ square, and $y-iy'=$ square; at least this is so, save as to a common denominator, as will appear.

First, if the class is $=2$; we have

$$y=(x+a)^\alpha(x+b)^\beta;$$

hence

$$y+iy'=y\left(1+\frac{i\alpha}{x+a}+\frac{i\beta}{x+b}\right),$$

$$y-iy'=y\left(1-\frac{i\alpha}{x+a}-\frac{i\beta}{x+b}\right),$$

say these are $=\dfrac{(x+l)^2\,y}{x+a\,.\,x+b}$ and $\dfrac{(x+m)^2\,y}{x+a\,.\,x+b}$ respectively; and, this being so, we have

$$y^2+y'^2=\frac{y^2(x+l)^2(x+m)^2}{(x+a)^2(x+b)^2},\ =(x+a)^{2\alpha-2}(x+b)^{2\beta-2}(x+l)^2(x+m)^2.$$

It is to be shown that the assumed relations lead to $\alpha=\beta$. Resolving the last-mentioned expressions for $y+iy'$, $y-iy'$ each into simple fractions, we have

$$i\alpha(b-a)=(l-b)^2,\quad -i\alpha(b-a)=(m-b)^2,$$
$$i\beta(a-b)=(l-a)^2,\quad -i\beta(a-b)=(m-a)^2.$$

Hence

$$(l-b)^2+(m-b)^2=0,\quad (l-a)^2+(m-a)^2=0;$$

these cannot give

$$(l-b)+i(m-b)=0,\quad (l-a)+i(m-a)=0,$$

with the same sign for i in the two equations; for we should then have

$$(1+i)(b-a)=0,$$

but $1+i$ is not $=0$, and a, b are essentially unequal. Hence, taking (as we may do) $+i$ in the first equation, we must have $-i$ in the second equation, and the two equations are

$$l-b+i(m-b)=0,\ \text{that is,}\ l+im=(1+i)\,b,$$
$$l-a-i(m-a)=0,\quad ,,\quad l-im=(1-i)\,a,$$

and thence

$$2l\ =(1+i)(b-ia),$$
$$2m=(1+i)(b+ia).$$

Hence also

$$2(l-b)=\ \ (1-i)(a-b),\quad 2i(m-b)=(1-i)(b-a),$$
$$2(l-a)=-(1+i)(a-b),\quad 2i(m-a)=(1+i)(b-a);$$

consequently

$$2(l-b)^2=-i(a-b)^2,\ =-2i\alpha(a-b),$$
$$2(l-a)^2=\ \ i(a-b)^2,\ =\ \ 2i\beta(a-b).$$

Hence $\alpha=\beta=\frac{1}{2}(a-b)$, and the solution thus is

$$y=(x+a)^\alpha(x+b)^\alpha,\quad \alpha=\tfrac{1}{2}(a-b),$$
$$l=\tfrac{1}{2}\{a+b+i(a-b)\},$$
$$m=\tfrac{1}{2}\{a+b-i(a-b)\},$$
$$y^2+y'^2=(x+a)^{2\alpha-2}(x+b)^{2\alpha-2}(x+l)^2(x+m)^2.$$

The class is here $=2$, and the order is $=2\alpha$; considering the order as given, say it is $=2\nu$, we have $\alpha=\nu$, and the equation $\nu=\frac{1}{2}(a-b)$ then shows that one of the roots a, b is arbitrary. Taking it to be a, we have $b=a-2\nu$, or the solution, class 2 and order 2ν, is

$$y=(x+a)^\nu(x+a-2\nu)^\nu,$$
$$l=\alpha-\nu+i\nu,\quad m=\alpha-\nu-i\nu,$$
$$y^2+y'^2=(x+a)^{2\nu-2}(x+a-2\nu)^{2\nu-2}(x+l)^2(x+m)^2.$$

Considering next for the case, class $=4$, the solution

$$y=(x+a)^\alpha(x+b)^\beta(x+c)^\gamma(x+d)^\delta,$$

we have

$$y+iy'=y\left(1+\frac{i\alpha}{x+a}+\frac{i\beta}{x+b}+\frac{i\gamma}{x+c}+\frac{i\delta}{x+d}\right),$$
$$y-iy'=y\left(1-\frac{i\alpha}{x+a}-\frac{i\beta}{x+b}-\frac{i\gamma}{x+c}-\frac{i\delta}{x+d}\right);$$

or, putting these

$$=\frac{(x+l)^2(x+p)^2\,y}{x+a\,.\,x+b\,.\,x+c\,.\,x+d}\quad\text{and}\quad\frac{(x+m)^2(x+q)^2\,y}{x+a\,.\,x+b\,.\,x+c\,.\,x+d}$$

respectively, we have

$$y^2+y'^2=\frac{y^2(x+l)^2(x+p)^2(x+m)^2(x+q)^2}{(x+a)^2(x+b)^2(x+c)^2(x+d)^2},$$
$$=(x+a)^{2\alpha-2}(x+b)^{2\beta-2}(x+c)^{2\gamma-2}(x+d)^{2\delta-2}(x+l)^2(x+p)^2(x+m)^2(x+q)^2.$$

Also, by decomposing the expressions for $y+iy'$, $y-iy'$ into simple fractions and comparing with the original values, we find

$$i\alpha\,(b-a)(c-a)(d-a)=(a-\ \ l)^2(a-p)^2,$$
$$i\beta\,(a-b)(c-b)(d-b)=(b-\ \ l)^2(b-p)^2,$$
$$i\gamma\,(a-c)(b-c)(d-c)=(c-\ \ l)^2(c-p)^2,$$
$$i\delta\,(a-d)(b-d)(c-d)=(d-\ \ l)^2(d-p)^2,$$

$$-i\alpha\,(b-a)(c-a)(d-a)=(a-m)^2(a-q)^2,$$
$$-i\beta\,(a-b)(c-b)(d-b)=(b-m)^2(b-q)^2,$$
$$-i\gamma\,(a-c)(b-c)(d-c)=(c-m)^2(c-q)^2,$$
$$-i\delta\,(a-d)(b-d)(c-d)=(d-m)^2(d-q)^2.$$

Hence

$$(a-l)^2(a-p)^2+(a-m)^2(a-q)^2=0,$$
$$(b-l)^2(b-p)^2+(b-m)^2(b-q)^2=0,$$
$$(c-l)^2(c-p)^2+(c-m)^2(c-q)^2=0,$$
$$(d-l)^2(d-p)^2+(d-m)^2(d-q)^2=0;$$

we cannot from these obtain *three* equations

$$(a-m)(a-q)-i(a-l)(a-p)=0,$$
$$(b-m)(b-q)-i(b-l)(b-p)=0,$$
$$(c-m)(c-q)-i(c-l)(c-p)=0,$$

with the same sign for i; in fact these would give

$$(1+i)(b-c)(c-a)(a-b)=0,$$

but $1+i$ is not $=0$, and the a, b, c are essentially unequal. Hence we must have equations such as

$$(a-m)(a-q)-i(a-l)(a-p)=0;\quad (c-m)(c-q)+i(c-l)(c-p)=0,$$
$$(b-m)(b-q)-i(b-l)(b-p)=0;\quad (d-m)(d-q)+i(d-l)(d-p)=0,$$

two of them with $-i$, and two of them with $+i$; viz. the a, b, c, d divide themselves into pairs which are taken to be a, b and c, d.

We hence easily obtain

$$a+b-m-q-i(a+b-l-p)=0,\quad ab-mq-i(ab-lp)=0,$$
$$c+d-m-q-i(c+d-l-p)=0,\quad cd-mq-i(cd-lp)=0,$$

and thence

$$a+b-c-d=i(a+b+c+d)-2i(l+p),$$
$$ab-cd=i(ab+cd)\qquad -2ilp.$$

Forming from these values of $l+p$, lp the expression for $2i(a-l)(a-p)$, we find $2i(a-l)(a-p)=(i+1)(a-c)(a-d)$; and we have thus the set of equations

$$2i(a-l)(a-p)=(i+1)(a-c)(a-d),$$
$$2i(b-l)(b-p)=(i+1)(b-c)(b-d),$$
$$2i(c-l)(c-p)=(i-1)(c-a)(c-b),$$
$$2i(d-l)(d-p)=(i-1)(d-a)(d-b).$$

Hence also

$$2(a-l)^2(a-p)^2=-i(a-c)^2(a-d)^2,$$
$$2(b-l)^2(b-p)^2=-i(b-c)^2(b-d)^2,$$
$$2(c-l)^2(c-p)^2=\quad i(c-a)^2(c-b)^2,$$
$$2(d-l)^2(d-p)^2=\quad i(d-a)^2(d-b)^2;$$

and, substituting these values in a former set of equations, we obtain

$$2\alpha(b-a)=-(a-c)(a-d),$$
$$2\beta(a-b)=-(b-c)(b-d),$$
$$2\gamma(d-c)=\quad(c-a)(c-b),$$
$$2\delta(c-d)=\quad(d-a)(d-b);$$

and thence

$$2(\alpha+\beta)= \quad a+b-c-d,$$
$$2(\gamma+\delta)=-(c+d-a-b),$$

that is, $\alpha+\beta=\gamma+\delta$; viz. there are, in this case also, two diptychs.

If, as before, the order is taken to be $=2\nu$, then $\alpha+\beta=\nu$, $\gamma+\delta=\nu$; supposing that ν is a given positive integer, and that α, β, γ, δ are positive integers satisfying these equations $\alpha+\beta=\nu$, $\gamma+\delta=\nu$, then the last-mentioned four equations between α, β, γ, δ and a, b, c, d are equivalent to three relations serving to determine the differences of a, b, c, d (say $a-d$, $b-d$, $c-d$) in terms of α, β, γ, δ. And we then further have

$$\begin{aligned}
(a-l)(a-p)&=-(1-i)\,\alpha\,(b-a), & (a-m)(a-q)&=-(1+i)\,\alpha\,(b-a),\\
(b-l)(b-p)&=-(1-i)\,\beta\,(a-b), & (b-m)(b-q)&=-(1+i)\,\beta\,(a-b),\\
(c-l)(c-p)&=\ \ (1+i)\,\gamma\,(d-c), & (c-m)(c-q)&=\ \ (1-i)\,\gamma\,(c-d),\\
(d-l)(d-p)&=\ \ (1+i)\,\delta\,(c-d), & (d-m)(d-q)&=\ \ (1-i)\,\delta\,(d-c),
\end{aligned}$$

each set equivalent to two equations; or, as these may be written,

$$\begin{aligned}
2(l\ \ +p)&=a+b+c+d+i(a+b-c-d),\\
2lp\qquad&=ab+cd\qquad\quad+i(ab-cd),\\
2(m+q)&=a+b+c+d-i(a+b-c-d),\\
2mq\qquad&=ab+cd\qquad\quad-i(ab-cd),
\end{aligned}$$

serving to determine l, p, m, q in terms of a, b, c, d.

Observe also that, u being arbitrary, we have

$$2(u-\ \ l)(u-p)=(1+i)(u-a)(u-b)+(1-i)(u-c)(u-d),$$
$$2(u-m)(u-q)=(1-i)(u-a)(u-b)+(1+i)(u-c)(u-d),$$

(which equations, writing therein $u=a$, b, c, or d, in fact reproduce the two systems of four equations).

We have also

$$\begin{aligned}
l+p+m+q&=a+b+c+d, & l+p-m-q&=i(a+b-c-d),\\
lp+mq\qquad&=ab+cd, & lp-mq\qquad&=i(ab-cd);
\end{aligned}$$

and moreover

$$\begin{aligned}
4(l\ \ -p)^2&=\ \ 2i\,\{(a-b)^2-(c-d)^2\}+4(a+b)(c+d)-8(ab+cd),\\
4(m-q)^2&=-2i\,\{(a-b)^2-(c-d)^2\}+4(a+b)(c+d)-8(ab+cd),
\end{aligned}$$

which equations, combined with the foregoing values of $2(l+p)$ and $2(m+q)$, give the values of l, p, m, q. We have thus the complete solution for the case class $=4$, order $=20$; say

$$y=(x+a)^\alpha(x+b)^\beta.(x+c)^\gamma(x+d)^\delta;\ \alpha+\beta=\gamma+\delta=\nu,$$
$$y^2+y'^2=(x+a)^{2\alpha-2}(x+b)^{2\beta-2}(x+c)^{2\gamma-2}(x+d)^{2\delta-2}(x+l)^2(x+p)^2(x+m)^2(x+q)^2,$$

with the foregoing relations between the constants.

886.

ON THE SURFACES WITH PLANE OR SPHERICAL CURVES OF CURVATURE.

[From the *American Journal of Mathematics*, vol. XI. (1889), pp. 71—98; pp. 293—306.]

THE theory is considered in two nearly cotemporaneous papers—Bonnet, "Mémoire sur les surfaces dont les lignes de courbure sont planes ou sphériques," *Jour. de l'École Polyt.*, t. XX. (1853), pp. 117—306, and Serret, "Mémoire sur les surfaces dont toutes les lignes de courbure sont planes ou sphériques," *Liouville*, t. XVIII. (1853), pp. 113—162. I desire to reproduce in a more compact form, and with some additional developments, the chief results obtained in these elaborate memoirs.

The basis of the theory is a theorem by Lancret, 1806. In any curve described upon a surface, the angle between the osculating planes at consecutive points is equal to the difference of the angles between the osculating planes and the corresponding tangent planes of the surface.

This includes as a particular case Joachimsthal's theorem, *Crelle*, t. XXX. (1846): If a surface have a plane curve of curvature, then at any point thereof the angle between the plane of the curve and the tangent plane of the surface has a constant value.

Bonnet and Serret each deduce the like theorem for a spherical curve of curvature, viz.: If a surface have a spherical curve of curvature, then at any point thereof the angle between the tangent plane of the sphere and the tangent plane of the surface has a constant value. Bonnet (Mémoire, p. 235) says that this follows from Lancret's theorem. Serret (Mémoire, p. 128) obtains it, by the transformation by reciprocal radius vectors, from Joachimsthal's theorem.

I remark that the theorem for a spherical curve of curvature, and (as a particular case thereof) that for a plane curve of curvature, are obtained at once from the most elementary geometrical considerations, viz. if we have (in the same plane or in

different planes) the two isosceles triangles NPP', OPP' on a common base PP', then the angle OPN is equal to the angle $OP'N$. For take P, P' consecutive points on a spherical curve of curvature; then at P, P' the normals of the surface meet in a point N, and the normals (or radii) of the sphere meet in the centre O, and we have angle OPN = angle $OP'N$, that is, at each of these points the inclination of the normal of the surface to the normal of the sphere has the same value; and this value being thus the same for any two consecutive points, must be the same for all points of the curve of curvature. The proof applies to the plane curve of curvature; but in this case, the fundamental theorem may be taken to be, a line at right angles to the base PP' of the isosceles triangle NPP' is equally inclined to the two equal sides NP, NP'.

A surface may have one set of its curves of curvature plane or spherical. To include the two cases in a common formula, the equation may be written

$$k(x^2+y^2+z^2)-2ax-2by-2cz-2u=0;$$

$k=1$ in the case of a sphere, $=0$ in that of a plane; and the expression a sphere may be understood to include a plane. I write in general A, B, C to denote the cosines of the inclinations of the normal of the surface at the point (x, y, z) to the axes of coordinates (consequently $A^2+B^2+C^2=1$). Hence considering a surface, and writing down the equations

$$k(x^2+y^2+z^2)-2ax-2by-2cz-2u=0,$$

$$(kx-a)A+(ky-b)B+(kz-c)C=l,$$

where (a, b, c, u, l) are regarded as functions of a parameter t. The first of these equations is that of a variable sphere; and the second equation expresses that at a point of intersection of the surface with the sphere, the inclination of the tangent plane of the surface to the tangent plane of the sphere has a constant value l, viz. this is a value depending only on the parameter t, and therefore constant for all points of the curve of intersection of the sphere and surface: by what precedes, the curve of intersection is a curve of curvature of the surface, and the surface will thus have a set of spherical curves of curvature.

Supposing the surface defined by means of expressions of its coordinates (x, y, z) as functions of two variable parameters, we may for one of these take the parameter t which enters into the equation of the sphere; and if the other parameter be called θ, then the expressions of the coordinates are of the form $x, y, z=x(t, \theta), y(t, \theta), z(t, \theta)$ respectively; these give equations $dx, dy, dz=adt+a'd\theta, bdt+b'd\theta, cdt+c'd\theta$, where of course (a, b, c, a', b', c') are in general functions of t, θ; and we have A, B, C proportional to $bc'-b'c$, $ca'-c'a$, $ab'-a'b$, viz. the values are equal to these expressions each divided by the square root of the sum of their squares. In order that the surface may have a set of spherical curves of curvature, the above three equations must be satisfied identically by means of the values of

$$a, b, c, u, l, A, B, C, x, y, z,$$

as functions of (t, θ); and it may be seen without difficulty that we are thereby led to a partial differential equation of the first order for the determination of the surface. But I do not at present further consider this question of the determination of a surface having one set of its curves of curvature (plane or) spherical.

Suppose now that there is a second set of (plane or) spherical curves of curvature. We have in like manner

$$\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z-2\upsilon=0,$$
$$(\kappa x-\alpha)A+(\kappa y-\beta)B+(\kappa z-\gamma)C-\lambda=0,$$

where κ is $=1$ or $=0$ according as the curves are spherical or plane, and $(\alpha, \beta, \gamma, \upsilon, \lambda)$ are functions of a variable parameter θ. We take the t of the former set of equations and the θ of these equations as the two parameters in terms of which the coordinates (x, y, z) are expressed. This being so (the former equations being satisfied as before), if these equations are satisfied identically by the values of $\alpha, \beta, \gamma, \upsilon, \lambda, A, B, C, x, y, z$ as functions of (t, θ), then the surface will have its other set of curves of curvature also spherical. It will be recollected that by hypothesis a, b, c, u, l are functions of the parameter t only, and that $\alpha, \beta, \gamma, \upsilon, \lambda$ functions of the parameter θ only. The foregoing equations, together with the assumed relations

$$A^2+B^2+C^2=1,$$
$$Adx+Bdy+Cdz=0,$$

are the "six equations" for the determination of a surface having its two sets of curves of curvature each of them (plane or) spherical.

Assuming now the values of a, b, c, l, u as functions of t, and $\alpha, \beta, \gamma, \lambda, \upsilon$ as functions of θ, the question at once arises whether we can then satisfy the six equations. These equations other than $Adx+Bdy+Cdz=0$, or say the five equations, in effect determine any five of the eight quantities $A, B, C, x, y, z, t, \theta$, in terms of the remaining three, say they determine A, B, C, t, θ as functions of x, y, z: we thus have a differential equation $Adx+Bdy+Cdz=0$, wherein A, B, C are to be regarded as given functions of (x, y, z). An equation of this form is not in general integrable; and if the equation in question be not integrable, then clearly the system of equations cannot be satisfied by any value of z as a function of (x, y), or, what is the same thing, by any values of (x, y, z) as functions of (t, θ). We thus arrive at the condition that the equation may be integrable, viz. the condition is

$$\nabla, =A\left(\frac{dB}{dz}-\frac{dC}{dy}\right)+B\left(\frac{dC}{dx}-\frac{dA}{dz}\right)+C\left(\frac{dA}{dy}-\frac{dB}{dx}\right), =0.$$

If this be satisfied, then we have an integral equation $I=0$ (containing a constant of integration which is an absolute constant) and which is, in fact, the equation of the required surface. But it is proper to look at the question somewhat differently. Supposing that the condition $\nabla=0$ is satisfied, then we have the integral equation $I=0$, and this equation, together with the five equations, in effect determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two of them, or, what

is the same thing, they determine a relation between any three of these quantities. We can, from the five equations and their differentials, and from the equation $Adx+Bdy+Cdz=0$, obtain a differential equation between any three of the eight quantities: and it has just been seen that corresponding hereto we have an integral relation between the same three quantities; that is, the condition $\nabla=0$ being satisfied, we can from the six equations obtain between any three of the quantities $A, B, C, x, y, z, t, \theta$ a linear differential equation of the foregoing form (for instance $Zdz+Tdt+\Theta d\theta=0$, where Z, T, Θ are given functions of z, t, θ) which will *ipso facto* be integrable, furnishing between z, t, θ an integral equation which may be used instead of the before-mentioned integral equation $I=0$. And we thus have (without any further integration) in all six equations which serve to determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two. It is often convenient to seek in this way for the expressions of (A, B, C and) x, y, z as functions of t, θ, in preference to seeking for the integral equation $I=0$ between the coordinates x, y, z.

The condition $\nabla=0$ is in fact the condition which expresses that at any point of the surface the two curves of curvature intersect at right angles. Serret (and after him Bonnet) in effect obtain the condition by the assumption of this geometrical relation, without showing that the geometrical relation is in fact the necessary condition for the coexistence of the six equations. They give the condition in the form $dx\delta x+dy\delta y+dz\delta z=0$, where dx, dy, dz are the increments of (x, y, z) along one of the curves of curvature, and $\delta x, \delta y, \delta z$ the increments along the other curve of curvature. The equations give

$$\begin{aligned}(kx-a)\,dx+(ky-b)\,dy+(kz-c)\,dz&=0,\\ Adx+\qquad Bdy+\qquad Cdz&=0,\end{aligned}$$

and similarly

$$\begin{aligned}(\kappa x-\alpha)\,\delta x+(\kappa y-\beta)\,\delta y+(\kappa z-\gamma)\,\delta z&=0,\\ A\delta x+\qquad B\delta y+\qquad C\delta z&=0.\end{aligned}$$

We thence have

$$dx : dy : dz = B(kz-c)-C(ky-b) : C(kx-a)-A(kz-c) : A(ky-b)-B(kx-a),$$

and

$$\delta x : \delta y : \delta z = B(\kappa z-\gamma)-C(\kappa y-\beta) : C(\kappa x-\alpha)-A(\kappa z-\gamma) : A(\kappa y-\beta)-B(\kappa x-\alpha).$$

We have thus the required condition, in a form which is readily changed into

$$\begin{aligned}&(A^2+B^2+C^2)\{(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)\}\\ &\qquad-\{A(kx-a)+B(ky-b)+C(kz-c)\}\{A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)\}=0,\end{aligned}$$

and writing herein $A^2+B^2+C^2=1$, this becomes

$$\begin{aligned}&\tfrac{1}{2}\kappa\{k(x^2+y^2+z^2)-2ax-2by-2cz\}\\ &+\tfrac{1}{2}k\{\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z\}\\ &+(a\alpha+b\beta+c\gamma)-l\lambda=0,\end{aligned}$$

that is,

$$a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv=0.$$

I proceed to show that this is the condition $\nabla = 0$ for the integrability of the differential equation $A\,dx + B\,dy + C\,dz = 0$. Writing as before

$$\nabla = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right),$$

we have from the six equations

$$\begin{aligned}
&A\,dA + B\,dB + C\,dC = 0,\\
&(kx-a)\,dA + (ky-b)\,dB + (kz-c)\,dC = -k\,(A\,dx + B\,dy + C\,dz) + (Aa_1 + Bb_1 + Cc_1 + l_1)\,dt,\\
&(\kappa x-\alpha)\,dA + (\kappa y-\beta)\,dB + (\kappa z-\gamma)\,dC = -\kappa\,(A\,dx + B\,dy + C\,dz) + (A\alpha' + B\beta' + C\gamma' + \lambda')\,d\theta,\\
&(kx-a)\,dx + (ky-b)\,dy + (kz-c)\,dz = (a_1x + b_1y + c_1z + u_1)\,dt,\\
&(\kappa x-\alpha)\,dx + (\kappa y-\beta)\,dy + (\kappa z-\gamma)\,dz = (\alpha'x + \beta'y + \gamma'z + \upsilon')\,d\theta,
\end{aligned}$$

where a_1, b_1, c_1, l_1, u_1 denote derived functions in regard to t, and α', β', γ', λ', υ' derived functions in regard to θ. Putting for shortness

$$\Omega = \begin{vmatrix} A\ , & B\ , & C \\ kx-a, & ky-b, & kz-c \\ \kappa x-\alpha, & \kappa y-\beta, & \kappa z-\gamma \end{vmatrix},$$

we readily obtain

$$\begin{aligned}
\Omega\,dA = {}&[(\kappa y-\beta)\,C - (\kappa z-\gamma)\,B]\left\{-k\,(A\,dx + B\,dy + C\,dz)\right.\\
&\qquad\left. + \frac{Aa_1 + Bb_1 + Cc_1 + l_1}{a_1x + b_1y + c_1z + u_1}\{(kx-a)\,dx + (ky-b)\,dy + (kz-c)\,dz\}\right\}\\
&- [(ky-b)\,C - (kz-c)\,B]\left\{-\kappa\,(A\,dx + B\,dy + C\,dz)\right.\\
&\qquad\left. + \frac{A\alpha' + B\beta' + C\gamma' + \lambda'}{\alpha'x + \beta'y + \gamma'z + \upsilon'}\{(\kappa x-\alpha)\,dx + (\kappa y-\beta)\,dy + (\kappa z-\gamma)\,dz\}\right\};
\end{aligned}$$

say this is

$$\begin{aligned}
\Omega\,dA = {}&[(\kappa y-\beta)\,C - (\kappa z-\gamma)\,B]\left\{-k\,(A\,dx + B\,dy + C\,dz)\right.\\
&\qquad\left. + \frac{L}{P}\{(kx-a)\,dx + (ky-b)\,dy + (kz-c)\,dz\}\right\}\\
&- [(ky-b)\,C - (kz-c)\,B]\left\{-\kappa\,(A\,dx + B\,dy + C\,dz)\right.\\
&\qquad\left. + \frac{\Lambda}{\Pi}\{(\kappa x-\alpha)\,dx + (\kappa y-\beta)\,dy + (\kappa z-\gamma)\,dz\}\right\},
\end{aligned}$$

or, introducing further abbreviations, and writing down the analogous values of ΩdB and ΩdC, we have

$$\begin{aligned}
\Omega dA &= [(\kappa y-\beta)\,C - (\kappa z-\gamma)\,B]\,U - [(ky-b)\,C - (kz-c)\,B]\,\Upsilon,\\
\Omega dB &= [(\kappa z-\gamma)\,A - (\kappa x-\alpha)\,C]\,U - [(kz-c)\,A - (kx-a)\,C]\,\Upsilon,\\
\Omega dC &= [(\kappa x-\alpha)\,B - (\kappa y-\beta)\,A]\,U - [(kx-a)\,B - (ky-b)\,A]\,\Upsilon.
\end{aligned}$$

We hence find

$$\Omega\frac{dB}{dz} = \quad [(\kappa z-\gamma)A-(\kappa x-\alpha)C]\left\{-kC+\frac{L}{P}(kz-c)\right\}$$
$$-[(kz-c)A-(kx-a)C]\left\{-\kappa C+\frac{\Lambda}{\Pi}(\kappa z-\gamma)\right\},$$
$$-\Omega\frac{dC}{dy} = -[(\kappa x-\alpha)B-(\kappa y-\beta)A]\left\{-kB+\frac{L}{P}(ky-b)\right\}$$
$$+[(kx-a)B-(ky-b)A]\left\{-\kappa B+\frac{\Lambda}{\Pi}(\kappa y-\beta)\right\}.$$

Combining these two terms, in the resulting value of $\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)$, first, the term without L or Λ is found to be

$$\begin{aligned}&= -kA\{A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)\}\\&\quad -k(\kappa x-a)(A^2+B^2+C^2)\\&\quad +\kappa(kx-\alpha)(A^2+B^2+C^2)\\&\quad +\kappa A\{A(kx-a)+B(ky-b)+C(kz-c)\},\end{aligned}$$

which is

$$\begin{aligned}&= -kA\lambda+k(\kappa x-\alpha)-\kappa(kx-a)+\kappa Al,\\&= \quad A(\kappa l-k\lambda)-k\alpha+\kappa a.\end{aligned}$$

Next, the coefficient of $\frac{L}{P}$ is

$$\begin{aligned}&A(kz-c)(\kappa z-\gamma)-C(\kappa x-\alpha)(kz-c)\\&+A(ky-b)(\kappa y-\beta)-B(\kappa x-\alpha)(ky-b),\end{aligned}$$

which is

$$\begin{aligned}&= A[(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)]\\&\qquad -(\kappa x-\alpha)[A(kx-a)+B(ky-b)+C(kz-c)]\\&= AM+(\kappa x-\alpha)l,\end{aligned}$$

if for shortness

$$M=(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma);$$

and similarly, the coefficient of $\frac{\Lambda}{\Pi}$ is

$$\begin{aligned}&-A(kz-c)(\kappa z-\gamma)+C(kx-a)(\kappa x-\alpha)\\&-A(ky-b)(\kappa y-\beta)+B(kx-b)(\kappa y-\beta),\end{aligned}$$

which is

$$\begin{aligned}&= -A[(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)]\\&\quad -(kx-a)[A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)]\\&= -AM-(kx-a)\lambda.\end{aligned}$$

We thus obtain

$$\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)=A(\kappa l-k\lambda)-k\alpha+\kappa a+\frac{L}{P}\{AM+(\kappa x-\alpha)l\}-\frac{\Lambda}{\Pi}\{AM+(kx-a)\lambda\},$$

and similarly

$$\Omega\left(\frac{dC}{dx}-\frac{dA}{dz}\right)=B(\kappa l-k\lambda)-k\beta+\kappa b+\frac{L}{P}\{BM+(\kappa y-\beta)l\}-\frac{\Lambda}{\Pi}\{BM+(ky-b)\lambda\},$$

$$\Omega\left(\frac{dA}{dy}-\frac{dB}{dx}\right)=C(\kappa l-k\lambda)-k\gamma+\kappa c+\frac{L}{P}\{CM+(\kappa z-\gamma)l\}-\frac{\Lambda}{\Pi}\{CM+(kz-c)\lambda\};$$

hence multiplying by A, B, C and adding, we obtain

$$\Omega\nabla=\kappa l-k\lambda-k(A\alpha+B\beta+C\gamma)+\kappa(Aa+Bb+Cc)+\frac{L}{P}(M-l\lambda)-\frac{\Lambda}{\Pi}(M-l\lambda),$$

where the first four terms are together

$$=\kappa l-k\lambda+k\{\kappa(Ax+By+Cz)-\lambda\}-\kappa\{k(Ax+By+Cz)-l\},$$

viz. these destroy each other, and the equation becomes

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(M-l\lambda).$$

But we have

$$\begin{aligned}M-l\lambda=\ & \tfrac{1}{2}\kappa\{k(x^2+y^2+z^2)-2ax-2by-2cz\}\\ &+\tfrac{1}{2}k\{\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z\}+(a\alpha+b\beta+c\gamma)-l\lambda,\end{aligned}$$

which is

$$=a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv,$$

or we find

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv),$$

viz. the condition $\nabla=0$ is

$$a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv=0,$$

the result which was to be proved.

If we consider separately the cases where the two sets of curves of curvature are each plane, the first plane and the second spherical, and each of them spherical; or say the cases PP, PS and SS, then in these cases respectively the condition is

$$a\alpha+b\beta+c\gamma-l\lambda=0,$$
$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$
$$a\alpha+b\beta+c\gamma-l\lambda+u+v=0:$$

we have, in each case, to take the italic letters functions of t and the greek letters functions of θ, satisfying identically the appropriate equation, but otherwise arbitrary; and then, in each case, the six equations lead to a differential equation $Adx+Bdy+Cdz=0$ (or say $Zdz+Tdt+\Theta d\theta=0$) between three variables, which equation is *ipso facto*

integrable; and we thus obtain a new integral equation which, with the original five integral equations, gives the solution of the problem. The condition is, in each case, of the form $\Sigma a\alpha = 0$, the number of terms $a\alpha$ being 4, 5 or 6. Considering for instance the form

$$a\alpha + b\beta + c\gamma + d\delta + e\epsilon + f\phi = 0$$

with 6 terms, it is easy to see how such an equation is to be satisfied by values of a, b, c, d, e, f which are functions of t, and values of $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ which are functions of θ. Suppose that $t_1, t_2, \ldots$ are particular values of t, and $a_1, b_1, \ldots, f_1$; $a_2, b_2, \ldots, f_2$, &c., the corresponding values of $a, b, \ldots, f$, these values being of course absolute constants; we have $\alpha, \beta, \ldots, \phi$, functions of θ, satisfying all the equations

$$\begin{aligned}
&(a_1, b_1, c_1, d_1, e_1, f_1 \between \alpha, \beta, \gamma, \delta, \epsilon, \phi) = 0,\\
&(a_2, b_2, c_2, d_2, e_2, f_2 \between \quad \text{,,} \quad) = 0,\\
&\text{\&c.,}
\end{aligned}$$

and if 6 or more of these equations were independent, the equations could, it is clear, be satisfied only by the values $\alpha = \beta = \gamma = \delta = \epsilon = \phi = 0$. To obtain a proper solution, only some number less than 6 of these equations can be independent. Suppose, for instance, that only two of the equations are independent; we then have $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ functions of θ satisfying these equations, but otherwise arbitrary; or, what is the same thing, we may take $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ linear functions of $6-2, =4$ arbitrary functions, say P, Q, R, S of θ; say we have

$$\begin{aligned}
\alpha &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3 \between P, Q, R, S),\\
\beta &= (\beta_0, \ldots\ldots\ldots\ldots \between \quad \text{,,} \quad),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\phi &= (\phi_0, \ldots\ldots\ldots\ldots \between \quad \text{,,} \quad),
\end{aligned}$$

where the suffixed greek letters denote absolute constants; and this being so, in order to satisfy the proposed equation $a\alpha + b\beta + c\gamma + d\delta + e\epsilon + f\phi = 0$, we must have

$$\begin{aligned}
&(\alpha_0, \beta_0, \gamma_0, \delta_0, \epsilon_0, \phi_0 \between a, b, c, d, e, f) = 0,\\
&(\alpha_1, \ldots\ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,\\
&(\alpha_2, \ldots\ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,\\
&(\alpha_3, \ldots\ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,
\end{aligned}$$

viz. a, b, c, d, e, f will then be functions of t satisfying these four equations, but otherwise arbitrary. The above is a solution for the partition $2+4$ of the number 6. We have in like manner a solution for any other partition of 6; or if we disregard the extreme cases $a = b = c = d = e = f = 0$ and $\alpha = \beta = \gamma = \delta = \epsilon = \phi = 0$, then we have in this manner solutions for the several partitions 15, 24, 33, 42 and 51 of the number 6.

But applying this theory to the actual problem, there is a good deal of difficulty as regards the enumeration of the really distinct cases. I use the letters P, S to denote that a set of curves of curvature is plane or spherical as the case may be,

the surfaces to be considered are thus *PP*, *PS*, and *SS*. First, for the *PP* problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda = 0$: the two systems (a, b, c, l) and $(\alpha, \beta, \gamma, \lambda)$ are symmetrically related to each other, and instead of the solutions 13, 22 and 31, it is sufficient to consider the solutions 13 and 22. But here (a, b, c, l) are not a system of four symmetrically related functions, (a, b, c) are a symmetrical system, and l is a distinct term: and the like for the system $(\alpha, \beta, \gamma, \lambda)$. In the *PS* problem, where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u = 0$, and thus the systems (a, b, c, l, u), $(\alpha, \beta, \gamma, \lambda, 1)$ are of different forms, we should consider the solutions 14, 23, 32 and 41: but here again, in each of the systems separately, the terms are not symmetrically related to each other. Lastly, in the *SS* problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u + v = 0$: the systems $(a, b, c, l, u, 1)$ and $(\alpha, \beta, \gamma, \lambda, 1, v)$ are of the same form, it is enough to consider the solutions 15, 24 and 33; but in this case also, in each of the systems separately, the terms are not symmetrically related to each other. I do not at present further consider the question, but simply adopt Serret's enumeration.

It is to be remarked that for a developable (but not for a skew surface) the generating lines may be curves of curvature, and regarding the generating lines as plane curves we might have developables *PP* or *PS*; but a straight line is not a curve in a determinate plane, and it is better to consider the case apart from the general theory. Again, the curves of curvature of one set or those of each set may be circles; and a circle may be regarded either as a plane or a spherical curve; regarding it, however, as a spherical curve, it is a curve not in a determinate sphere. The cases in question, of the curves of curvature of the one set or of those of each set being circles, are therefore also to be considered apart from the general theory. The surfaces referred to present themselves for consideration among Serret's cases *PP*, 1°, 2°, 3°; *PS*, 1°, 2°, 3°, 4°, 5°, 6°, 7°; and *SS*, 1°, 2°, 3°, 4°; but they are excluded from his enumeration, and he in fact reckons in his "Conclusion," pp. 161, 162, two kinds of surfaces *PP*, three kinds *PS*, and two kinds *SS*.

It is very easily seen that, if a surface has a plane or a spherical curve of curvature, then on any parallel surface the corresponding curve is a plane or a spherical curve of curvature: and thus if a surface be *PP*, *PS*, or *SS*, then the parallel surfaces are respectively *PP*, *PS*, or *SS*. The solutions obtained include for the most part all the parallel surfaces, and thus there is no occasion to make use of this theorem; but see in the continuation of the present paper the case considered under the subheading *post*, *PS*, 4° = Serret's third case of *PS*.

If a surface have a plane or a spherical curve of curvature, then transforming the surface by reciprocal radius vectors (or inverting in regard to an arbitrary point), then in the transformed surface the corresponding curve is a spherical curve of curvature. Hence if a surface be *PP*, *PS* or *SS*, the transformed surface is *SS*. Conversely, as shown by Bonnet and Serret, and as will appear, every surface *SS* is in fact an inversion of a surface *PP* or *PS*.

I proceed to the enumeration, developing the theory only in regard to the two, three, and two, cases *PP*, *PS* and *SS* respectively.

PP, The Two Sets of Curves of Curvature each Plane.

The six equations are

$$\begin{aligned} A^2+B^2 &+C^2 &&=1,\\ ax+by &+cz &&+u=0,\\ A\alpha+Bb &+Cc &&+l=0,\\ \alpha x+\beta y &+\gamma z &&+v=0,\\ A\alpha+B\beta &+C\gamma &&+\lambda=0,\\ Adx+Bdy &+Cdz &&=0; \end{aligned}$$

the condition is

$$a\alpha+b\beta+c\gamma-l\lambda=0,$$

not containing u or v, so that these remain arbitrary functions of t, θ respectively. The cases are

	a	b	c	l	α	β	γ	λ
PP, 1°	1	0	c	0	0	1	0	λ
PP, 2°	0	1	0	$-m$	α	$-m\lambda$	γ	λ
PP, 3°	1	0	c	mc	0	1	$m\lambda$	λ;

m is an arbitrary constant; and in the body of the table, c is an arbitrary function of t, and α, γ, λ arbitrary functions of θ.

PP, 1° is Serret's first case of *PP*, included in his second case.

PP, 2° gives a developable.

PP, 3° is Serret's second case of *PP*.

I consider the case

PP, 3° = *Serret's Second Case of PP.*

Writing for greater symmetry $m=g$, $\frac{1}{m}=f$, so that $fg=1$; also $m\lambda=\gamma$, and consequently $\lambda=f\gamma$, we take c and γ for the two parameters respectively, or write $c=t$, $\gamma=\theta$; also changing the letters u, v, we write

a	b	c	l	u	α	β	γ	λ	v
$=1$,	0,	t	gt	P	0	1	θ	$f\theta$	Π,

and the six equations thus are

$$\begin{aligned} A^2+B^2 &+C^2 &&=1,\\ x+tz &-P &&=0,\\ A+tC &-gt &&=0,\\ y+\theta z &-\Pi &&=0,\\ B+\theta C &-f\theta &&=0,\\ Adx+Bdy &+Cdz &&=0. \end{aligned}$$

We seek for the differential equation in z, t, θ. We have

$$A^2+B^2+C^2=1, \quad A=t(g-C), \quad B=\theta(f-C),$$

and thence

$$t^2(g-C)^2+\theta^2(f-C)^2+C^2=1,$$

that is,

$$C^2(1+t^2+\theta^2)+2C(gt^2+f\theta^2)=1-g^2t^2-f^2\theta^2,$$

or multiplying by $1+t^2+\theta^2$ and completing the square,

$$\begin{aligned}\{(1+t^2+\theta^2)C-gt^2-f\theta^2\}^2&=(1-g^2t^2-f^2\theta^2)(1+t^2+\theta^2)-(gt^2+f\theta^2)^2\\&=\{f+(f-g)t^2\}\{g+(g-f)\theta^2\}\\&=\frac{1}{T^2\Theta^2},\end{aligned}$$

if

$$\frac{1}{T^2}=f+(f-g)t^2,$$

$$\frac{1}{\Theta^2}=g+(g-f)\theta^2;$$

and thence, giving a determinate sign to the square root, say

$$(1+t^2+\theta^2)C=gt^2+f\theta^2-\frac{1}{T\Theta},$$

an equation which may also be written

$$C=\frac{fT-g\Theta}{f-g}.$$

In fact, observing that $\frac{1}{T^2}-\frac{1}{\Theta^2}=(f-g)(1+t^2+\theta^2)$, we deduce from the original form

$$\begin{aligned}\left(\frac{1}{T^2}-\frac{1}{\Theta^2}\right)C&=(f-g)(gt^2+f\theta^2)-\frac{f-g}{T\Theta},\\&=g\left(\frac{1}{T^2}-f\right)-f\left(\frac{1}{\Theta^2}-g\right)-\frac{f-g}{T\Theta}\\&=\left(\frac{g}{T}-\frac{f}{\Theta}\right)\left(\frac{1}{T}+\frac{1}{\Theta}\right),\end{aligned}$$

or throwing out the factor $\frac{1}{T}+\frac{1}{\Theta}$ and reducing, we have the required value; and thence forming the values of A and B, we have

$$A=-tT\frac{f-g}{T-\Theta}, \quad B=-\theta\Theta\frac{f-g}{T-\Theta}, \quad C=\frac{fT-g\Theta}{T-\Theta};$$

we have, moreover,

$$x+tz=P, \quad y+\theta z=\Pi,$$

or differentiating, and writing P_1 and Π' for the derived functions in regard to t and θ respectively,

$$dx = -tdz - zdt + P_1 dt, \quad dy = -\theta dz - zd\theta - \Pi' d\theta.$$

The equation $Adx + Bdy + Cdz = 0$ thus becomes

$$-Ttdx - \Theta\theta dy + \frac{fT - g\Theta}{f-g} dz = 0,$$

viz. this is

$$[-tT(-tdz - zdt + P_1 dt) - \theta\Theta(-\theta dz - zd\theta + \Pi' dt)](f-g) + (fT - g\Theta) dz = 0,$$

or collecting,

$$[\{f + (f-g)t^2\} T - \{g + (g - ft^2)\} \Theta] dz + (tTzdt + \theta\Theta z d\theta)(f-g)$$
$$- (tTP_1 dt + \theta\Theta\Pi' d\theta)(f-g) = 0,$$

that is,

$$\left(\frac{1}{\Theta} - \frac{1}{T}\right) dz + (f-g) z (tTdt + \theta\Theta d\theta) - (f-g)(tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

which is an integrable form as it should be; viz. the equation is

$$d\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f-g)(tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

and we obtain

$$\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f-g) \int (tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

the constant of integration being considered as included in the integral. But it is proper to alter the form of the second term. Take F, Φ arbitrary functions of t, θ respectively: and writing F_1, Φ' for the derived functions: assume $P = \frac{gF_1}{T^2}$, $\Pi = \frac{f\Phi'}{\Theta^3}$; we have

$$\int (tTP_1 dt + \theta\Theta\Pi' d\theta) = \int \left(gtT \left(\frac{F_1}{T^2}\right)_1 dt + f\theta\Theta \left(\frac{\Phi'}{\Theta^3}\right)' d\theta\right)$$
$$= -F + \frac{gtF_1}{T^2} - \Phi + \frac{f\theta\Phi'}{\Theta^2}.$$

In fact, this will be true if only

$$\left(-F + \frac{gtF_1}{T^2}\right)_1 = gtT\left(\frac{F_1}{T^2}\right)_1, \quad \left(-\Phi + \frac{f\theta\Phi'}{\Theta^2}\right)' = f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)',$$

which are equations of like form in t, θ respectively; it will be sufficient to verify the first of them. Effecting the differentiation, the terms in F_{11} destroy each other, and there remain only terms containing the factor F_1; throwing this out, we obtain

$$-1 + \frac{g}{T^2} + \frac{gtT_1}{T^3} = 0,$$

viz. this is

$$-1+g\{f+(f-g)t^2\}-gt^2(f-g)=0,$$

which is identically true, and the equation is thus verified.

The foregoing result is

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}=0;$$

we then have

$$x+tz-\frac{gF_1}{T^2}=0,\quad y+\theta z-\frac{f\Phi'}{\Theta^2}=0,$$

and hence, repeating also the equation for z,

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)x+(f-g)\left\{-t(F+\Phi)\qquad+\frac{ft\theta\Phi'}{\Theta^2}\right\}+\left(-1+\frac{g}{\Theta T}\right)\frac{F_1}{T^2}=0,$$

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)y+(f-g)\left\{-\theta(F+\Phi)+\frac{gt\theta F_1}{T^2}\qquad\right\}+\left(\quad 1-\frac{f}{\Theta T}\right)\frac{\Phi'}{\Theta^2}=0,$$

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{\quad F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}\qquad=0,$$

equations which give the values of the coordinates x, y, z in terms of the parameters t, θ. It will be recollected that $fg=1$ (f or g being arbitrary), then the values of T, Θ are

$$\frac{1}{T^2}=f+(f-g)t^2,\quad \frac{1}{\Theta^2}=g+(g-f)\theta^2,$$

and that F, Φ denote arbitrary functions of t, θ respectively. I repeat also the foregoing equations

$$A,\ B,\ C=-tT\frac{f-g}{T-\Theta},\ -\theta\Theta\frac{f-g}{T-\Theta},\ \frac{fT-g\Theta}{T-\Theta}.$$

The equations may be presented under a different form; we have

$$-tTx-\theta\Theta y+\frac{fT-g\Theta}{f-g}z+F+\Phi=0,$$

$$-fT^3(x+tz)+F_1=0,$$

$$-g\Theta^3(y+\theta z)+\Phi'=0,$$

where it will be observed that the second and third equations are the derivatives of the first equation in regard to t and θ respectively. We thus have the required surface as the envelope of the plane represented by the first equation, regarding therein t, θ as variable parameters. Moreover, the second equation (which contains only the parameter t) represents the planes of the curves of curvature of the one set; and the third equation (which contains only the parameter θ) represents the planes of the curves of curvature of the other set. It is to be observed that, from

the equations for l, λ, viz. $A+tC=gt$ and $B+\theta C=f\theta$, then for any plane of the first set the inclination to a tangent plane of the surface is $=\cos^{-1}\dfrac{gt}{\sqrt{1+t^2}}$, and that for any plane of the second set the inclination is $=\cos^{-1}\dfrac{f\theta}{\sqrt{1+\theta^2}}$.

It may be remarked that the last-mentioned results may be arrived at by the consideration of an equation $Ax+By+Cz+D=0$, where the coefficients are functions of t and θ (A a function of t only, and B a function of θ only), such that the derived equations $A_1x+C_1z+D_1=0$ and $B'x+C'z+D'=0$ depend the former of them upon t only, and the latter of them upon θ only.

A very simple case of the equation is when $f=g=1$; here $T=\Theta=1$, and the surface is the envelope of the plane $z-tx-\theta y+F+\Phi=0$.

Returning to the general form

$$-tTx-\theta\Theta y+\frac{fT-g\Theta}{f-g}z+F+\Phi=0,$$

I transform this, by introducing therein in place of t, θ two variable parameters α, β which are such that $k\alpha=-tT$, $k\beta=\theta\Theta$ $\left(k \text{ a constant which is presently put } =\dfrac{1}{\sqrt{f-g}}\right)$; we find

$$t^2=\frac{fk^2\alpha^2}{1-(f-g)k^2\alpha^2},\quad \theta^2=\frac{gk^2\beta^2}{1-(g-f)k^2\beta^2},$$

and thence

$$T=\frac{1}{\sqrt{f}}\sqrt{1-(f-g)k^2\alpha^2},\quad \Theta=\frac{1}{\sqrt{g}}\sqrt{1-(g-f)k^2\beta^2},$$

or putting $k=\dfrac{1}{\sqrt{f-g}}$, these last values are

$$T=\frac{1}{\sqrt{f}}\sqrt{1-\alpha^2},\quad \Theta=\frac{1}{\sqrt{g}}\sqrt{1+\beta^2},$$

and we hence obtain

$$\begin{aligned}\frac{fT-g\Theta}{f-g}&=\frac{\sqrt{f}}{f-g}\sqrt{1-\alpha^2}-\frac{\sqrt{g}}{f-g}\sqrt{1+\beta^2},\\&=\frac{1}{\sqrt{f-g}}\left\{\frac{\sqrt{f}}{\sqrt{f-g}}\sqrt{1-\alpha^2}-\frac{\sqrt{g}}{\sqrt{f-g}}\sqrt{1+\beta^2}\right\},\end{aligned}$$

say this is

$$=k\{\lambda\sqrt{1-\alpha^2}-\mu\sqrt{1+\beta^2}\},$$

where $\lambda=\dfrac{\sqrt{f}}{\sqrt{f-g}}$, $\mu=\dfrac{\sqrt{g}}{\sqrt{f-g}}$, and therefore $\lambda^2-\mu^2=1$ or $\mu=\sqrt{\lambda^2-1}$.

Hence writing $F+\Phi=k(A+B)$, k times the sum of two arbitrary functions of α and β respectively, the equation becomes

$$\alpha x-\beta y+z\{\lambda\sqrt{1-\alpha^2}-\sqrt{\lambda^2-1}\sqrt{1+\beta^2}\}+A+B=0,$$

viz. the surface is given as the envelope of this plane considering α, β as two variable parameters. This is the solution given by Darboux, *Leçons sur la théorie générale des surfaces, &c.*, t. I., Paris, 1887, pp. 128—131. He obtains it in a very elegant manner, starting from the following theorem: Take A, A_1, &c., functions of the parameter α, and B, B_1, &c., functions of the parameter β; then, if we have identically

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = (A_4 - B_4)^2,$$

the required surface will be obtained as the envelope of the plane

$$(A_1 - B_1)x + (A_2 - B_2)y + (A_3 - B_3)z = A - B,$$

where A, B are two new functions of α, β respectively.

The foregoing identity is the condition in order that each sphere of the one series $(x-A_1)^2 + (y-A_2)^2 + (z-A_3)^2 = A_4^2$ may touch each sphere of the other series $(x-B_1)^2 + (y-B_2)^2 + (z-B_3)^2 = B_4^2$; the two series of spheres thus envelope one and the same surface which will have its curves of curvature of each set circles: viz. this will be the surface of the fourth order called Dupin's Cyclide, the normals whereof pass through an ellipse and hyperbola which are focal curves one of the other, and which contain the centres of all the spheres touching the surface along its curves of curvature. The equations of the ellipse and the hyperbola may be taken to be

$$x^2 + \frac{z^2}{\lambda^2} = 1,\ y = 0, \text{ and } y^2 - \frac{z^2}{\lambda^2 - 1} = -1,\ x = 0,$$

respectively, and we thence obtain the required PP surface as the envelope of the plane

$$\alpha x - \beta y + (\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2 - 1}\sqrt{1+\beta^2})z + A + B = 0.$$

The Case PP, 1° = Serret's First Case of PP.

We deduce this from the second case by writing therein $m = 0$, that is, $g = 0$, $f = \infty$; but it is necessary to make also a transformation upon the parameter θ, viz. in place thereof we introduce the new parameter ϕ, where $\theta^2 = \frac{g\phi^2}{f - g\phi^2}$. This gives

$$\frac{1}{\Theta^2} = g + (g-f)\theta^2 = g\left\{1 + \frac{(g-f)\phi^2}{f - g\phi^2}\right\} = \frac{gf(1-\phi^2)}{f - g\phi^2}, \quad \theta^2 = \frac{g\phi^2}{f - g\phi^2},$$

and thence

$$\theta\Theta = \frac{\theta}{\sqrt{g + (g-f)\theta^2}} = \frac{\phi}{\sqrt{f}\sqrt{1-\phi^2}}; \quad \frac{fT - g\Theta}{f-g} \text{ for } g = 0 \text{ is } = T.$$

We have also $T = \frac{1}{\sqrt{f + (f-g)t^2}}$, $= \frac{1}{\sqrt{f}\sqrt{1+t^2}}$ when $g = 0$, and substituting these values, considering Φ as a function of ϕ, and for $F + \Phi$ writing as we may do $\frac{F+\Phi}{\sqrt{f}}$, the equation becomes

$$\frac{-t}{\sqrt{f}\sqrt{1+t^2}}x - \frac{\phi y}{\sqrt{f}\sqrt{1-\phi^2}} + \frac{z}{\sqrt{f}\sqrt{1+t^2}} + \frac{F+\Phi}{\sqrt{f}} = 0,$$

where the divisor $\sqrt{f}$ is to be omitted. Hence finally, instead of ϕ restoring the original letter θ, and again considering Φ as a function of θ, the equation is

$$\frac{z-tx}{\sqrt{1+t^2}}-\frac{\theta y}{\sqrt{1-\theta^2}}+F+\Phi=0,$$

viz. here F, Φ are arbitrary functions of t, θ respectively, and the surface is the envelope of this plane considering t, θ as variable.

We obtain an imaginary special form of PP, 1°, by writing in this equation $k\theta$ for θ and then putting $k=\infty$; the Φ remains an arbitrary function of the new θ, and the equation is

$$\frac{z-tx}{\sqrt{1+t^2}}+iy+F+\Phi=0,$$

($i=\sqrt{-1}$ as usual). This is, in fact, the equation which is obtained from PP, 3° by simply writing therein $g=0$ without the transformation upon θ.

PS, The Sets of Curves of Curvature, the First Plane, the Second Spherical.

The six equations are

$$\begin{aligned}
A^2+B^2+C^2 &= 1,\\
ax+by+cz+u &= 0,\\
Aa+Bb+Cc+l &= 0,\\
x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2v &= 0,\\
A(x-\alpha)+B(y-\beta)+C(z-\gamma)-\lambda &= 0,\\
Adx+Bdy+Cdz &= 0.
\end{aligned}$$

The condition is

$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$

not containing v, so that this remains an arbitrary function of θ. The cases are

	a	b	c	l	u	α	β	γ	λ
PS, 1°	a	b	c	0	0	0	0	0	λ
PS, 2°	a	b	c	l	ml	0	0	0	m
PS, 3°	a	b	c	$-mc$	0	0	0	γ	$\frac{1}{m}\gamma$
PS, 4°	a	b	0	l	ml	0	0	γ	m
PS, 5°	a	b	0	0	0	0	0	γ	λ
PS, 6°	0	b	0	l	ml	α	0	γ	m
PS, 7°	a	b	0	ma	0	α	0	γ	$-\frac{1}{m}\alpha$,

where m is an arbitrary constant; in the body of the table, the other italic letters are arbitrary functions of t, and the greek letters arbitrary functions of θ.

PS, 1° is Serret's first case of *PS*, included in his second case.

PS, 2° gives developable.

PS, 3° is Serret's second case of *PS*.

PS, 4° is Serret's third case of *PS*.

PS, 5° gives circular sections (surfaces of revolution).

PS, 6° gives circular sections (tubular surfaces).

PS, 7° gives circular sections.

I consider

PS, 3° = *Serret's Second Case of PS.*

The six equations are

$$\begin{aligned}
&A^2 + B^2 + C^2 = 1,\\
&ax + by + cz = 0,\\
&Aa + Bb + Cc = -cm,\\
&x^2 + y^2 + (z - m\phi)^2 = \theta + m^2\phi^2,\\
&Ax + By + C(z - m\phi) = \phi,\\
&Adx + Bdy + Cdz = 0,
\end{aligned}$$

where a, b, c are assumed such that $a^2 + b^2 + c^2 = 1$. We easily obtain

$$\begin{aligned}
(1 - c^2)A &= -ac(C + m) - b\sqrt{\Omega},\\
(1 - c^2)B &= -bc(C + m) + a\sqrt{\Omega},
\end{aligned}$$

and thence

$$aB - bA = \sqrt{\Omega},$$

where

$$\Omega = (1 - c^2)(1 - C^2) - c^2(C + m)^2, \; = 1 - c^2 - C^2 - 2c^2Cm - c^2m^2;$$

also

$$\begin{aligned}
x\sqrt{1 - c^2m^2} &= A\phi\sqrt{1 - c^2m^2} + (bC - cB)\sqrt{\theta + (m^2 - 1)\phi^2},\\
y\sqrt{1 - c^2m^2} &= B\phi\sqrt{1 - c^2m^2} + (cA - aC)\sqrt{\theta + (m^2 - 1)\phi^2},\\
z\sqrt{1 - c^2m^2} &= (C + m)\phi\sqrt{1 - c^2m^2} + (aB - bA)\sqrt{\theta + (m^2 - 1)\phi^2}.
\end{aligned}$$

We seek for the differential equation in C, t, θ. From the equation

$$Ax + By + (C - m\phi)z = \phi,$$

and attending to

$$Adx + Bdy + Cdz = 0,$$

we deduce

$$xdA + ydB + (z - m\phi)dC - (1 + Cm)\phi' d\theta = 0,$$

and we have herein to substitute for dA, dB their values in terms of dC, dt, $d\theta$. We have

$$AdA + BdB = -CdC,$$
$$adA + bdB = -cdC - Q,$$

if for shortness

$$Q = Ada + Bdb + (C + m)\,dc.$$

Hence

$$\sqrt{\Omega}\,dA = (-cB + bC)\,dC - BQ,$$
$$\sqrt{\Omega}\,dB = (-aC + cA)\,dC + AQ.$$

We find without difficulty,

$$(1 - c^2)\,Q = (C + m)\,dc + (adb - bda)\sqrt{\Omega},$$

and consequently,

$$(1 - c^2)\sqrt{\Omega}\,dA = \{\ \ b\,(C + c^2m) - ac\sqrt{\Omega}\}\,dC - B\,\{(C + m)\,dc + (adb - bda)\sqrt{\Omega}\},$$
$$(1 - c^2)\sqrt{\Omega}\,dB = \{-a\,(C + c^2m) - bc\sqrt{\Omega}\}\,dC + A\,\{(C + m)\,dc + (adb - bda)\sqrt{\Omega}\}.$$

Substituting these values, we have

$$\begin{aligned}&\{(bx - ay)\,(C + c^2m) - (ax + by)\,c\sqrt{\Omega}\}\,dC\\ &\quad - (Bx - Ay)\,\{(C + m)\,dc + (adb - bda)\sqrt{\Omega}\}\\ &\quad + (1 - c^2)\sqrt{\Omega}\,\{(z - m\phi)\,dC - (1 + Cm)\,\phi' d\theta\} = 0,\end{aligned}$$

viz. this is

$$\begin{aligned}&\{(bx - ay)\,(C + c^2m) - (ax + by)\,c\sqrt{\Omega} + (1 - c^2)\,(z - m\phi)\sqrt{\Omega}\}\,dC\\ &\quad - (Bx - Ay)\,\{(C + m)\,dc + (adb - bda)\sqrt{\Omega}\}\\ &\quad - (1 - c^2)\,(1 + Cm)\sqrt{\Omega}\phi' d\theta = 0.\end{aligned}$$

The coefficient of dC contains a term $-(ax + by + cz)\,c\sqrt{\Omega}$ which is $= 0$. Moreover, we have

$$bx - ay = -\phi\sqrt{\Omega} + \frac{C + c^2m}{\sqrt{1 - c^2m^2}}\sqrt{\theta + (m^2 - 1)\,\phi^2},$$

and then

$$\begin{aligned}(1 - c^2)\,(Bx - Ay) &= -c\,(C + m)\,(bx - ay) - cz\sqrt{\Omega}\\ &= -c\,(C + m)\left\{-\phi\sqrt{\Omega} + \frac{C + c^2m}{\sqrt{1 - c^2m^2}}\sqrt{\theta + (m^2 - 1)\,\phi^2}\right\}\\ &\quad - c\sqrt{\Omega}\left\{(C + m)\,\phi + \frac{\sqrt{\Omega}\sqrt{\theta + (m^2 - 1)\,\phi^2}}{\sqrt{1 - c^2m^2}}\right\},\end{aligned}$$

which, observing that the terms in $C\phi\sqrt{\Omega}$ destroy each other, and that we have

$$(C + m)\,(C + c^2m) + \Omega = (1 - c^2)\,(1 + Cm),$$

gives

$$Bx - Ay = \frac{-c\,(1 + Cm)}{\sqrt{1 - c^2m^2}}\sqrt{\theta + (m^2 - 1)\,\phi^2},$$

and the equation becomes

$$\left\{\left(-\phi\sqrt{\Omega}+\frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right)(C+c^2m)+z\sqrt{\Omega}-(1-c^2)m\phi\sqrt{\Omega}\right\}dC$$

$$-c(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)dc+(adb-bda)\sqrt{\Omega}\right\}$$

$$-(1-c^2)\sqrt{\Omega}(1+Cm)\phi' d\theta=0.$$

Here the coefficient of dC is

$$=[z-(C+m)\phi]\sqrt{\Omega}+\frac{(C+c^2m)^2}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2},$$

or, substituting for $z-(C+m)\sqrt{\Omega}$ its value $=\dfrac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}$, and observing that $\Omega+(C+c^2m)^2=(1-c^2m^2)(1-c^2)$, this coefficient is found to be

$$=\sqrt{1-c^2m^2}(1-c^2)\sqrt{\theta+(m^2-1)\phi^2},$$

and we have

$$\sqrt{1-c^2m^2}(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}\,dC$$

$$-c(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)dc+(adb-bda)\sqrt{\Omega}\right\}$$

$$-(1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta=0,$$

or, as this may be written,

$$\frac{1}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}dC}{1+Cm}-\frac{(C+m)cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\}-\frac{c(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}-\frac{\phi' d\theta}{\sqrt{\theta+(m^2-1)\phi^2}}=0,$$

where from the foregoing value of Ω we have identically

$$\Omega(1-m^2)=(1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2.$$

Here a, b, c are functions of t; and we have thus the required differential equation in C, t, θ.

It is convenient to multiply by the constant factor $\sqrt{1-m^2}$. The first term is an exact differential, viz. writing

$$\sin\zeta=\frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\frac{C+m}{1+Cm},\text{ and therefore }\cos\zeta=\frac{\sqrt{1-m^2}\sqrt{\Omega}}{\sqrt{1-c^2}(1+Cm)},$$

we have

$$d\zeta=\frac{\sqrt{1-m^2}}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}dC}{1+Cm}+\frac{(C+m)cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\},$$

as may easily be verified. And the second and third terms are obviously the differentials of a function of t and a function of θ respectively. But to obtain the integral functions, a transformation of each term is required.

First, for the term $\dfrac{\sqrt{1-m^2}c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$; we take a, b, c functions of t which are such that $a^2+b^2+c^2=1$; and then writing a_1, b_1, c_1 for the derived functions so that $aa_1+bb_1+cc_1=0$, we assume a', b', $c'=Va_1$, Vb_1, Vc_1 where $\dfrac{1}{V^2}=a_1^2+b_1^2+c_1^2$; we have therefore $aa'+bb'+cc'=0$, and $a'^2+b'^2+c'^2=1$; and then writing a'', b'', $c''=bc'-b'c$, $ca'-c'a$, $ab'-a'b$ respectively, we have

$$aa''+bb''+cc''=0,\ a'a''+b'b''+c'c''=0,\ a''^2+b''^2+c''^2=1;$$

thus a, b, c, a', b', c', a'', b'', c'' are a set of rectangular coefficients. We then write

$$a,\ b,\ c=\frac{1}{\rho}(a'+mb''),\ \ \frac{1}{\rho}(b'-ma''),\ \ \frac{1}{\rho}c',$$

determining ρ so that $a^2+b^2+c^2=1$ as above, viz. we thus have

$$\rho^2=(1+cm)^2+c''^2m^2.$$

Observe that we thus have $\rho^2(1-c^2)=\rho^2-c'^2$ and $\rho^2(1-c^2m^2)=(1+cm)^2$.

Writing now

$$T=\tan^{-1}\frac{c+m}{c''\sqrt{1-m^2}},\ \text{and therefore}\ \sin T=\frac{c+m}{\sqrt{\rho^2-c'^2}},\ \cos T=\frac{c''\sqrt{1-m^2}}{\sqrt{\rho^2-c'^2}},$$

we find that

$$dT=\frac{\sqrt{1-m^2}c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}.$$

The verification is somewhat long, but it is very interesting. We have

$$dT=\frac{\sqrt{1-m^2}\,\{c''dc-(c+m)\,dc''\}}{\rho^2-c'^2},$$

or observing that $c''=ab'-a'b$, $=V(ab_1-a_1b)$, $dc=c_1dt$, this is

$$dT=\frac{\sqrt{1-m^2}}{\rho^2-c'^2}\{V(ab_1-a_1b)\,c_1-(c+m)\,[V_1(ab_1-a_1b)+V(ab_{11}-a_{11}b)]\,dt\},$$

where we have

$$\frac{1}{V^2}=a_1^2+b_1^2+c_1^2,\ \text{and therefore}\ -\frac{V_1}{V^3}=a_1a_{11}+b_1b_{11}+c_1c_{11};$$

also from $aa_1+bb_1+cc_1=0$, we have $a_1^2+b_1^2+c_1^2+aa_{11}+bb_{11}+cc_{11}=0$, and we thence obtain

$$dT=\frac{\sqrt{1-m^2}\,V^3dt}{\rho^2-c'^2}\{-(ab_1-a_1b)\,c_1(aa_{11}+bb_{11}+cc_{11})$$
$$-(c+m)\,[-(a_1a_{11}+b_1b_{11}+c_1c_{11})(ab_1-a_1b)+(a_1^2+b_1^2+c_1^2)(ab_{11}-ba_{11})]\},$$

the term in [ ] is found to be $= -c_1\{a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)\}$, hence c_1 appears as a factor of the whole expression; and reducing the part independent of m, we find

$$dT = \frac{\sqrt{1-m^2}\,V^3c_1dt}{\rho^2 - c'^2}\{(a_1b_{11} - a_{11}b_1) + m[a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\}.$$

Next, calculating the value of $adb - bda$, we have

$$a = \frac{V}{\rho}\{a_1 + m(ca_1 - c_1a)\},\quad b = \frac{V}{\rho}\{b_1 - m(bc_1 - b_1c)\},$$

or, as these may be written,

$$a = \frac{V}{\rho}\{a_1(1 + cm) - ac_1m\},\quad b = \frac{V}{\rho}\{b_1(1 + cm) - bc_1m\},$$

and we thence easily obtain

$$adb - bda = \frac{V^2dt}{\rho^2}(1 + cm)\{(a_1b_{11} - a_{11}b_1) + m[a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\},$$

viz. the factor in { } has the same value as in the expression for dT, and we thus have

$$\frac{dT}{adb - bda} = \frac{\sqrt{1-m^2}\,Vc_1\rho^2}{(1 + cm)(\rho^2 - c'^2)} = \frac{c\sqrt{1-m^2}}{\sqrt{1 - c^2m^2}(1 - c^2)},$$

that is,

$$dT = \frac{\sqrt{1-m^2}\,c(adb - bda)}{(1 - c^2)\sqrt{1 - c^2m^2}},$$

the required equation.

Secondly, for the term $\dfrac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}}$, we introduce Φ a function of θ, such that writing Φ' for the derived function we have

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4(1-m^2)\Phi\Phi' + 4(1-m^2)\theta\Phi'^2}},\quad = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}}\ \text{suppose},$$

whence also

$$\sqrt{\theta + (m^2-1)\phi^2} = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}},\quad \frac{\phi}{\sqrt{\theta + (m^2-1)\phi^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}}.$$

Then writing

$$\sin\Theta = \frac{\Phi\sqrt{1-m^2}}{\sqrt{\theta}},\quad \cos\Theta = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi\sqrt{1-m^2}}{\sqrt{\theta}},\quad \cos\Theta_0 = \frac{\sqrt{\theta + (m^2-1)\phi^2}}{\sqrt{\theta}},$$

we find

$$\cos\Theta\, d\Theta = -\frac{\frac{1}{2}\sqrt{1-m^2}(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\,\Phi^2}}\,;$$

and similarly

$$\cos\Theta_0 d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta}}\,,$$

that is,

$$d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\,\phi^2}}\,.$$

Hence

$$-d\Theta + d\Theta_0 = \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\,\Phi^2}} - \frac{\phi - 2\theta\phi'}{\sqrt{\theta + (m^2-1)\,\phi^2}}\right\} d\theta$$

$$= \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\phi - (\phi - 2\theta\phi')}{\sqrt{\theta + (m^2-1)\,\phi^2}}\right\} d\theta,\ = \frac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\,\phi^2}},$$

the required equation.

We find, moreover,

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{1-m^2}\sqrt{\theta + (m^2-1)\,\Phi^2}}{\sqrt{M}},\quad \cos(\Theta - \Theta_0) = \frac{1 - 2\,(1-m^2)\,\Phi\Phi'}{\sqrt{M}},$$

which will be presently useful.

The differential equation now is $d\zeta - dT + d\Theta - d\Theta_0 = 0$, hence the integral equation (taking the constant of integration $=0$) is $\zeta = T - \Theta + \Theta_0$, or say

$$\sin\zeta = \sin(T - \Theta + \Theta_0),$$

viz. substituting for $\sin T$ and $\cos T$ their values, and observing that

$$\sin\zeta = \frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm},\quad = \frac{1+cm}{\sqrt{\rho^2 - c'^2}}\,\frac{C+m}{1+Cm},$$

the factor $\dfrac{1}{\sqrt{\rho^2 - c'^2}}$ multiplies out, and we have

$$(1+cm)\frac{C+m}{1+Cm} = (c+m)\cos(\Theta - \Theta_0) - c''\sqrt{1-m^2}\sin(\Theta - \Theta_0).$$

And I further remark here that a former equation is

$$\Omega\,(1-m^2) = (1-c^2)\,(1+Cm)^2 - (1-c^2m^2)\,(C+m)^2,$$

that is,

$$\Omega\frac{1-m^2}{(1+Cm)^2} = (1-c^2)\left\{1 - \frac{(1-c^2m^2)\,(C+m)^2}{(1-c^2)\,(1+Cm)^2}\right\} = (1-c^2)\cos^2\zeta.$$

We thus have

$$\sqrt{\Omega} = \frac{1+Cm}{\sqrt{1-m^2}}\,\frac{\sqrt{\rho^2 - c'^2}}{\rho}\cos\zeta,$$

$$= \frac{1+Cm}{\rho\sqrt{1-m^2}}\,\{c''\sqrt{1-m^2}\cos(\Omega - \Omega_0) + (c+m)\sin(\Omega - \Omega_0)\}.$$

We have thus C, and consequently also A, B, x, y, z, all of them given as functions of t, θ; but the formulæ admit of further development.

Write

$$\mathfrak{s} = \frac{C+m}{1+Cm}, \text{ whence also } C = \frac{\mathfrak{s}-m}{1-m\mathfrak{s}}.$$

We have $C(1-m\mathfrak{s})+m = \mathfrak{s}$, and hence $(1+\mathrm{c}m)\{C(1-m\mathfrak{s})+m\}, = (1+\mathrm{c}m)\mathfrak{s}$, $=(\mathrm{c}+m)\cos(\Theta-\Theta_0)-\mathrm{c}''\sqrt{1-m^2}\sin(\Theta-\Theta_0)$. Using the value of C given by this equation, and calculating from it those of A, B; then writing for shortness

$$X = \mathrm{a}\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(\mathrm{a}''-m\mathrm{b}')\sin(\Theta-\Theta_0),$$

$$Y = \mathrm{b}\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(\mathrm{b}''+m\mathrm{a}')\sin(\Theta-\Theta_0),$$

$$Z = \quad(\mathrm{c}+m)\cos(\Theta-\Theta_0)-\mathrm{c}''\sqrt{1-m^2}\sin(\Theta-\Theta_0),$$

we have

$$A(1-m\mathfrak{s})(1+\mathrm{c}m) = \sqrt{1-m^2}X,$$

$$B(1-m\mathfrak{s})(1+\mathrm{c}m) = \sqrt{1-m^2}Y,$$

$$C(1-m\mathfrak{s})(1+\mathrm{c}m) = \qquad Z-m(1+\mathrm{c}m),$$

to which I join

$$\mathfrak{s}(1+\mathrm{c}m) = \qquad Z.$$

By way of verification, observe that $A^2+B^2+C^2=1$, and that the equations give

$$(1-m\mathfrak{s})^2(1+\mathrm{c}m)^2 = (1-m^2)(X^2+Y^2+Z^2)+m^2Z^2-2mZ(1+\mathrm{c}m)+m^2(1+\mathrm{c}m)^2;$$

we have

$$X^2+Y^2+Z^2 = (1+\mathrm{c}m)^2, \quad Z = \mathfrak{s}(1+\mathrm{c}m),$$

and hence the identity

$$(1-m\mathfrak{s})^2(1+\mathrm{c}m)^2 = (1-m^2+m^2\mathfrak{s}^2-2m\mathfrak{s}+m^2)(1+\mathrm{c}m)^2.$$

Proceeding to calculate the values of x, y, z, recollecting that

$$\sqrt{1-c^2m^2} = \frac{1}{\rho}(1+\mathrm{c}m),$$

we have

$$x(1+\mathrm{c}m) = A\phi(1+\mathrm{c}m)+\rho(bC-cB)\sqrt{\theta+(m^2-1)\phi^2},$$

$$= A\phi(1+\mathrm{c}m)+\{(\mathrm{b}'-m\mathrm{a}'')C-\mathrm{c}'B\}\sqrt{\theta+(m^2-1)\phi^2},$$

that is,

$$x(1+\mathrm{c}m)(1-m\mathfrak{s}) = \phi\sqrt{1-m^2}X+\frac{1}{1+\mathrm{c}m}\{(\mathrm{b}'-m\mathrm{a}'')(Z-m(1+\mathrm{c}m))$$
$$-\mathrm{c}'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}$$

$$= \phi\sqrt{1-m^2}X+\frac{1}{1+\mathrm{c}m}\{(\mathrm{b}'-m\mathrm{a}'')Z-\mathrm{c}'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}$$
$$-m(\mathrm{b}'-m\mathrm{a}'')\sqrt{\theta+(m^2-1)\phi^2},$$

where the term $(b'-ma'')Z-c'\sqrt{1-m^2}Y$ contains the factor $1+cm$; in fact, this is

$$\begin{aligned} = \ & (b'-ma'')\{\ (c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0)\} \\ & -c'\sqrt{1-m^2}\{b\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(b''+ma')\sin(\Theta-\Theta_0)\}. \end{aligned}$$

The coefficient of the cosine is $(b'-ma'')(c+m)-bc'(1-m^2)$, which is

$$\begin{aligned} =b'c-bc'+m(b'-ca'')+m^2(-a''+bc'), \ & =-a''+m(b'-ca'')+m^2(-b'c), \\ & =(1+cm)(-a''+mb'), \end{aligned}$$

and similarly the coefficient of $\sqrt{1-m^2}$, multiplied by the sine, is

$$\begin{aligned} -c''(b'-ma'')+c'(b''+ma'), \ & =-b'c''+b''c'+m(a'c'+a''c''), \\ & =-a+m(-ac), \ =(1+cm)(-a). \end{aligned}$$

Calculating in like manner the values of y and z, and putting for shortness

$$\begin{aligned} X_1 &= (-a''+mb')\ \cos(\Theta-\Theta_0)-a\sqrt{1-m^2}\sin(\Theta-\Theta_0), \\ Y_1 &= (-b''-ma')\ \cos(\Theta-\Theta_0)-b\sqrt{1-m^2}\sin(\Theta-\Theta_0), \\ Z_1 &= (-c''\sqrt{1-m^2})\cos(\Theta-\Theta_0)+\ (c+m)\sin(\Theta-\Theta_0), \end{aligned}$$

we have

$$\begin{aligned} x &= \phi\sqrt{1-m^2}X+X_1\sqrt{\theta+(m^2-1)\phi^2}-m(b'-ma'')\sqrt{\theta+(m^2-1)\phi^2}, \\ y &= \phi\sqrt{1-m^2}Y+Y_1\sqrt{\theta+(m^2-1)\phi^2}+m(a'+mb'')\sqrt{\theta+(m^2-1)\phi^2}, \\ z &= \sqrt{1-m^2}\{\phi\sqrt{1-m^2}Z+Z_1\sqrt{\theta+(m^2-1)\phi^2}\}, \end{aligned}$$

which are the required expressions of x, y, z in terms of t and θ. It will be noticed that X, X_1, Y, Y_1, Z, Z_1, each contain a term with $\cos(\Theta-\Theta_0)$ and one with $\sin(\Theta-\Theta_0)$; but as the terms in X_1, Y_1, Z_1 are each multiplied by $\sqrt{\theta+(m^2-1)\phi^2}$, the cosine and sine terms of X, X_1, of Y, Y_1 and of Z, Z_1 do not in any case unite into a single term.

I remark that we have identically

$$\begin{aligned} aX+bY+c\sqrt{1-m^2}Z &= 0, \\ aX_1+bY_1+c\sqrt{1-m^2}Z_1 &= 0. \end{aligned}$$

The foregoing values of x, y, z thus satisfy $ax+by+cz=0$, which is one of the six equations. The others of them might be verified without difficulty. I recall that we have a, b, $c=\frac{1}{\rho}(a'+mb'')$, $\frac{1}{\rho}(b'-ma'')$, $\frac{1}{\rho}c'$; the six equations might therefore be written

$$\begin{aligned} A^2+B^2+C^2 &= 1, \\ (a'+mb'')\ x+(b'-ma'')\ y+c'z &= 0, \\ (a'+mb'')\ A+(b'-ma'')\ B+c'C &= -c'm, \\ x^2+y^2+(z-m\phi)^2 &= \theta+m^2\phi^2, \\ Ax+By+C(z-m\phi) &= \phi, \\ Adx+Bdy+Cdz &= 0. \end{aligned}$$

The Case PS, 1° = Serret's First Case of PS.

This is at once deduced from *PS*, 3° by writing therein $m=0$; the formulæ are a good deal more simple. We introduce, as before, the rectangular coefficients a, b, c, a′, b′, c′, a″, b″, c″; and the values of a, b, c then are a′, b′, c′. The six equations, using therein these values for a, b, c, are

$$\begin{aligned} A^2+B^2 \quad +C^2 &= 1, \\ \mathrm{a}'x+\mathrm{b}'y \quad +\mathrm{c}'z &= 0, \\ \mathrm{a}'A+\mathrm{b}'B \quad +\mathrm{c}'C &= 0, \\ x^2+y^2 \quad +z^2 &= \theta, \\ Ax+By \quad +Cz &= \phi, \\ Adx+Bdy+Cdz &= 0. \end{aligned}$$

The function Φ is such that

$$\phi=\frac{\Phi-2\theta\Phi'}{\sqrt{1-4\Phi\Phi'+4\theta\Phi'^2}}=\frac{\Phi-2\theta\Phi'}{\sqrt{M}}.$$

We have

$$\sin\Theta=\frac{\Phi}{\sqrt{\theta}}\cos\Theta=\frac{\sqrt{\theta-\Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0=\frac{\phi}{\sqrt{\theta}}\cos\Theta_0=\frac{\sqrt{\theta-\phi^2}}{\sqrt{\theta}},$$

and thence

$$\sin(\Theta-\Theta_0)=\frac{2\Phi'\sqrt{\theta-\Phi^2}}{\sqrt{M}};\quad \cos(\Theta-\Theta_0)=\frac{1-2\Phi\Phi'}{\sqrt{M}}.$$

Also

$$\sin\zeta=\frac{C}{\sqrt{1-\mathrm{c}^2}},\quad \cos\zeta=\frac{\sqrt{1-\mathrm{c}^2-C^2}}{\sqrt{1-\mathrm{c}^2}};\quad \sin T=\frac{\mathrm{c}}{\sqrt{1-\mathrm{c}'^2}},\quad \cos T=\frac{\mathrm{c}''}{\sqrt{1-\mathrm{c}^2}},$$

$$\zeta=T-\Theta+\Theta_0,\quad C=\mathrm{c}\cos(\Theta-\Theta_0)-\mathrm{c}''\sin(\Theta-\Theta_0),$$

$$\sqrt{1-\mathrm{c}^2-C^2}=\mathrm{c}''\cos(\Theta-\Theta_0)+\mathrm{c}\sin(\Theta-\Theta_0).$$

We have

$$\begin{aligned} A=X&=\mathrm{a}\cos(\Theta-\Theta_0)-\mathrm{a}''\sin(\Theta-\Theta_0); & X_1&=\mathrm{a}''\cos(\Theta-\Theta_0)+\mathrm{a}\sin(\Theta-\Theta_0), \\ B=Y&=\mathrm{b}\cos(\Theta-\Theta_0)-\mathrm{b}''\sin(\Theta-\Theta_0); & Y_1&=\mathrm{b}''\cos(\Theta-\Theta_0)+\mathrm{b}\sin(\Theta-\Theta_0), \\ C=Z&=\mathrm{c}\cos(\Theta-\Theta_0)-\mathrm{c}''\sin(\Theta-\Theta_0); & Z_1&=\mathrm{c}''\cos(\Theta-\Theta_0)+\mathrm{c}\sin(\Theta-\Theta_0), \end{aligned}$$

and then

$$\begin{aligned} x&=X\phi+X_1\sqrt{\theta-\phi^2}, \\ y&=Y\phi+Y_1\sqrt{\theta-\phi^2}, \\ z&=Z\phi+Z_1\sqrt{\theta-\phi^2}, \end{aligned}$$

which are the expressions of the coordinates in terms of the parameters t and θ.

I consider next the case

PS, 4^0 = *Serret's Third Case of PS.*

The six equations are

$$\begin{aligned}
A^2 \quad &+ B^2 + C^2 &&= 1\ , \\
ax \quad &+ by &&= lm, \\
Aa \quad &+ Bb &&= l\ , \\
x^2 \quad &+ y^2 + z^2 - 2\theta z &&= 2v, \\
Ax \quad &+ By + C(z-\theta) &&= m\ , \\
Adx &+ Bdy + Cdz &&= 0\ ;
\end{aligned}$$

where θ has been written in place of γ: m is a given constant; a, b, l are functions of t; v is a function of θ. The equation $ax + by = lm$ evidently denotes that the planes of the plane curves of curvature are all of them parallel to the axis of z, or, what is the same thing, they envelope a cylinder; in the particular case $m = 0$, they all of them pass through the axis of z. In the general case, the required surface is the parallel surface, at the normal distance m, to the surface which belongs to the particular case $m = 0$. This is not assumed in the investigation which follows; but it will be readily perceived how the theorem is involved in, and in fact proved by, the investigation.

I obtain the solution synthetically as follows:

Taking T, a, b functions of t, $\mathrm{a}^2 + \mathrm{b}^2 = 1$; T_1, a_1, b_1 their derived functions, $\mathrm{aa}_1 + \mathrm{bb}_1 = 0$; $\Omega = \frac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2$; Θ a function of θ, Θ' its derived function,

$$M = \frac{2\Theta}{\Theta'}\ ;\ P = \frac{2\sqrt{T\Theta}}{T+\Theta}\ ,\ Q = \frac{T-\Theta}{T+\Theta},$$

and therefore $P^2 + Q^2 = 1$; then writing

$$A_0 = \frac{1}{\sqrt{\Omega}}\left(-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1 Q\right),$$

$$B_0 = \frac{1}{\sqrt{\Omega}}\left(\ \mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1 Q\right),$$

$$C_0 = \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P,$$

where $A_0^2 + B_0^2 + C_0^2 = 1$, we assume

$$\begin{aligned}
x &= mA_0 &&+ \mathrm{a}MP, \\
y &= mB_0 &&+ \mathrm{b}MP, \\
z &= mC_0 + \theta + &&MQ,
\end{aligned}$$

equations which determine x, y, z as functions of the parameters t and θ. As will presently be shown, we have $A_0dx+B_0dy+C_0dz=0$; and we have thus $A, B, C=A_0, B_0, C_0$; and this being so, we easily verify the six equations

$$\begin{aligned}
A^2 \quad &+B^2+C^2 &&=1,\\
\mathrm{b}x \quad &-\mathrm{a}y &&=m\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right),\\
\mathrm{b}A \quad &-\mathrm{a}B &&=\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right),\\
x^2 \quad &+y^2+(z-\theta)^2 &&=m^2+M^2+\theta^2,\\
Ax \quad &+By+C(z-\theta) &&=m,\\
Adx &+Bdy+Cdz &&=0,
\end{aligned}$$

which are the six equations of the problem with the values $a=\mathrm{b}$, $b=-\mathrm{a}$, $l=-\frac{2T_1}{T}\frac{1}{\sqrt{\Omega}}$, $2v=m^2+M^2$, for a, b, l and $2v$.

We in fact at once obtain the third equation $\mathrm{b}A_0-\mathrm{a}B_0=-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}$, and thence the second equation $\mathrm{b}x-\mathrm{a}y=m(\mathrm{b}A_0-\mathrm{a}B_0)$, $=m\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right)$; then for the fifth equation, we have

$$A_0x+B_0y+C_0(z-\theta)=m+M\{(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q\},=m,$$

since $(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q=0$; and for the fourth equation, we have

$$x^2+y^2+(z-\theta)^2=m^2+2m\{M(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q\}+M^2,=m^2+M^2.$$

It remains only to prove the assumed equation $A_0dx+B_0dy+C_0dz=0$. Writing for a moment $X, Y, Z=\mathrm{a}MP, \mathrm{b}MP, \theta+MQ$, we have

$$\begin{aligned}
A_0dx+B_0dy+C_0dz&=A_0(mdA_0+dX)+B_0(mdB_0+dY)+C_0(mdC_0+dZ),\\
&=A_0dX+B_0dY+C_0dZ,
\end{aligned}$$

since $A_0dA_0+B_0dB_0+C_0dC_0=0$ in virtue of $A_0^2+B_0^2+C_0^2=1$.

We have thus to show that, if $X, Y, Z=\mathrm{a}MP, \mathrm{b}MP, \theta+MQ$, then

$$A_0dX+B_0dY+C_0dZ=0;$$

say we have

$$\begin{aligned}
dX&=pdt+p'd\theta,\\
dY&=qdt+q'd\theta,\\
dZ&=rdt+r'd\theta,
\end{aligned}$$

then the required values of A_0, B_0, C_0 are proportional to $qr'-q'r$, $rp'-r'p$, $pq'-p'q$, and the sum of their squares is $=1$. Writing for shortness $MP=R$, $MQ=S$, we have

$$\begin{aligned}
p&=\mathrm{a}_1R+\mathrm{a}R_1, & p'&=\mathrm{a}R',\\
q&=\mathrm{b}_1R+\mathrm{b}R_1, & q'&=\mathrm{b}R',\\
r&=\qquad\quad S_1, & r'&=1+S';
\end{aligned}$$

hence

$$\begin{aligned} qr' - q'r &= \mathrm{b}\,[R_1(1+S') - R'S_1] + \mathrm{b}_1 R(1+S'), \\ rp' - r'p &= -\mathrm{a}\,[R_1(1+S') - R'S_1] - \mathrm{a}_1 R(1+S'), \\ pq' - p'q &= -(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,RR'. \end{aligned}$$

Here

$$\begin{aligned} R' &= MP' + M'P, \quad R_1 = MP_1, \\ S' &= MQ' + M'Q, \quad S_1 = MQ_1, \end{aligned}$$

and hence

$$\begin{aligned} RS' - R'S &= M^2(PQ' - P'Q), \\ R_1S' - R'S_1 &= M^2(P_1Q' - P'Q_1) + MM'(P_1Q - PQ_1); \end{aligned}$$

moreover, from the values of P and Q, we have

$$P' = \frac{\Theta'}{2\Theta}PQ, \; = \frac{PQ}{M}, \quad P_1 = -\frac{T_1}{2T}PQ,$$

$$Q' = -\frac{\Theta'}{2\Theta}P^2 \; = -\frac{P^2}{M}, \quad Q_1 = \frac{T_1}{2T}P^2,$$

and thence

$$P_1Q - PQ_1 = -\frac{T_1P}{2T}; \; PQ' - P'Q = -\frac{P}{M}; \; P_1Q' - P'Q_1 = 0;$$

also

$$R' = PQ + M'P, \; = P(Q+M'), \qquad 1 + S = 1 - P^2 + M'Q, \; = Q(Q+M');$$

$$R_1 = -\frac{T_1}{2T}MPQ, \qquad RS' - R'S = -PM, \qquad R_1S' - R'S_1 = -\frac{T_1}{2T}MPM',$$

and consequently

$$\begin{aligned} R_1(1+S') - R'S_1 &= -\frac{T_1}{2T}MP \;\; (Q+M'), \\ R\,(1+S') &= MPQ\,(Q+M'), \\ RR' &= MP^2\;(Q+M'). \end{aligned}$$

Hence the foregoing expressions for $qr' - q'r$, $rp' - r'p$, $pq' - p'q$ each contain the factor $MP(Q+M')$; omitting this factor, the expressions are

$$\left\{-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1Q\right\}, \quad \left\{\mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1Q\right\}, \quad -(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P;$$

the sum of the squares of these values is $= \frac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2, = \Omega$, and we have thus the required values

$$A_0 = \frac{1}{\sqrt{\Omega}}\left\{-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1Q\right\},$$

$$B_0 = \frac{1}{\sqrt{\Omega}}\left\{\;\mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1Q\right\},$$

$$C_0 = \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P,$$

which completes the proof.

In the case $m=0$, the solution is

$$x,\ y,\ z = \frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad \frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad \theta+\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}.$$

Bonnet, in the paper (*Jour. Ecole Polyt.* t. XX.) referred to at the beginning of this memoir, gives for this case (see p. 199) a solution which he says is equivalent to that obtained by Joachimsthal in the paper "Demonstrationes theorematum ad superficies curvas spectantium," *Crelle*, t. XXX. (1846), pp. 347—350; viz. Joachimsthal's form is

$$x=\frac{\mu\sin L\sin\lambda}{1+\cos L\cos M},$$

$$y=\frac{\mu\sin L\cos\lambda}{1+\cos L\cos M},$$

$$z=\frac{\mu\cos L\sin M}{1+\cos L\cos M}+\int\cot M\,d\mu,$$

where L, M denote arbitrary functions of the parameters λ, μ respectively. To identify these with the foregoing form, I write

$$\sin\lambda=-\mathrm{a},\quad \cos L=-\frac{T-1}{T+1},\quad \cos M=\frac{\Theta-1}{\Theta+1},\quad \mu=\frac{4\Theta\sqrt{\Theta}}{\Theta'(\Theta+1)};$$

$$\cos\lambda=-\mathrm{b},\quad \sin L=\frac{-2\sqrt{T}}{T+1},\quad \sin M=\frac{-2\sqrt{\Theta}}{\Theta+1};$$

we thus have

$$\frac{\sin L\sin\lambda}{1+\cos L\cos M}=\mathrm{a}\,\frac{2\sqrt{T}}{T+1}\div 1-\frac{(T-1)(\Theta-1)}{(T+1)(\Theta+1)},\quad =\frac{\mathrm{a}\sqrt{T}(\Theta+1)}{T+\Theta},$$

and thence

$$x=\frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad y=\frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta}.$$

Moreover,

$$\frac{\cos L\sin M}{1+\cos L\cos M}=\frac{\sqrt{\Theta}(T-1)}{T+\Theta},$$

and thence the first term of z is $=\dfrac{2\Theta}{\Theta'}\,\dfrac{2\Theta(T-1)}{(\Theta+1)(T+\Theta)}$; or observing that

$$2\Theta(T-1)=(\Theta+1)(T-\Theta)+(\Theta-1)(T+\Theta),$$

this is

$$=\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}+\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

or we have

$$z=\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}+\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)}+\int\cot M\,d\mu.$$

Here

$$\cot M\, d\mu = -\frac{\Theta-1}{2\sqrt{\Theta}}\,\frac{4\Theta\sqrt{\Theta}}{\Theta(\Theta+1)}\left\{\tfrac{3}{2}\frac{\Theta'}{\Theta}-\frac{\Theta'}{\Theta+1}-\frac{\Theta''}{\Theta'}\right\}d\theta$$

$$=\frac{\Theta(\Theta-1)}{\Theta+1}\qquad\left\{-\frac{3}{\Theta}+\frac{2}{\Theta+1}+\frac{2\Theta''}{\Theta'^2}\right\}d\theta.$$

But writing

$$\xi=\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

we have

$$d\xi=\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)}\left\{\frac{\Theta'}{\Theta}+\frac{\Theta'}{\Theta-1}-\frac{\Theta'}{\Theta+1}-\frac{\Theta''}{\Theta'}\right\}d\theta$$

$$=\frac{\Theta(\Theta-1)}{\Theta+1}\quad\left\{\frac{2}{\Theta}+\frac{2}{\Theta-1}-\frac{2}{\Theta+1}-\frac{2\Theta''}{\Theta'^2}\right\}d\theta,$$

and thence

$$d\xi+\cot M\, d\mu=\frac{\Theta(\Theta-1)}{\Theta+1}\left\{-\frac{1}{\Theta}+\frac{2}{\Theta-1}\right\}d\theta,\ =d\theta,$$

and consequently

$$\xi+\int\cot M\, d\mu=\theta,$$

and the value of z thus is

$$z=\theta+\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta},$$

which completes the identification.

Bonnet's formulæ just referred to, making a slight change of notation and correcting a sign, are

$$x=\frac{\Gamma'\sin\theta}{\cos i(c+\Theta)},$$

$$y=\frac{\Gamma'\cos\theta}{\cos i(c+\Theta)},$$

$$z=\Gamma+i\Gamma'\tan i(c+\Theta),$$

where Γ, Θ are arbitrary functions of the parameters c, θ respectively. To identify these with Joachimsthal's, write

$$\sin\lambda=\sin\theta,\quad \cos M=i\cot ic,\quad \cos L=i\cot i\Theta,\quad \mu=-\Gamma'\operatorname{cosec} ic,$$

$$\cos\lambda=\cos\theta,\quad \sin M=\operatorname{cosec} ic,\quad \sin L=\operatorname{cosec} i\Theta,$$

$$\cot M=i\cos ic,\quad \cot L=i\cos i\Theta;$$

we have

$$x=\frac{\mu\operatorname{cosec} i\Theta\sin\theta}{1-\cot ic\cot i\Theta}=\frac{-\mu\sin ic\sin\theta}{\cos i(c+\Theta)},\ =\frac{\Gamma'\sin\theta}{\cos i(c+\Theta)};$$

and similarly

$$y=\qquad\qquad\qquad\qquad\frac{\Gamma'\cos\theta}{\cos i(c+\Theta)}.$$

Moreover, the first term of z is

$$\frac{\mu i \cot i\Theta \operatorname{cosec} ic}{1 - \cot ic \cot i\Theta} = \frac{-\mu i \cos i\Theta}{\cos i(c+\Theta)} = \frac{i\Gamma' \cos i\Theta}{\sin ic \cos i(c+\Theta)};$$

or since $i\Theta = i(c+\Theta) - ic$, and thence

$$\cos i\Theta = \cos ic \cos i(c+\Theta) + \sin ic \sin i(c+\Theta),$$

this is

$$= i\Gamma' \{\cot ic + \tan i(c+\Theta)\},$$

and we have

$$z = i\Gamma' \tan i(c+\Theta) + i\Gamma' \cot ic + \int \cot M \, d\mu.$$

But from the equation $\mu = -\Gamma' \operatorname{cosec} ic$, we obtain

$$d\mu = (-\Gamma'' \operatorname{cosec} ic + i\Gamma' \operatorname{cosec} ic \cot ic)\, dc;$$

whence

$$\cot M \, d\mu = (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc,$$

and thence

$$d\left(i\Gamma' \cot ic + \int \cot M \, d\mu\right) = (i\Gamma'' \cot ic + \Gamma' \operatorname{cosec}^2 ic)\, dc$$
$$+ (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc, = \Gamma' dc;$$

that is,

$$i\Gamma' \cot ic + \int \cot M \, d\mu = \Gamma,$$

and consequently

$$z = \Gamma + i\Gamma' \tan i(c+\Theta),$$

which completes the identification of Bonnet's formula with Joachimsthal's.

SS, The Sets of Curves of Curvature each Spherical.

The six equations are

$$\begin{aligned}
A^2 + B^2 + C^2 &= 1, \\
x^2 + y^2 + z^2 - 2ax - 2by - 2cz - 2u &= 0, \\
A(x-a) + B(y-b) + C(z-c) - l &= 0, \\
x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z - 2v &= 0, \\
A(x-\alpha) + B(y-\beta) + C(z-\gamma) - \lambda &= 0, \\
A\,dx + B\,dy + C\,dz &= 0;
\end{aligned}$$

the condition being

$$a\alpha + b\beta + c\gamma - l\lambda + u + v = 0.$$

The cases are

	a	b	c	l	u	α	β	γ	λ	v
SS, 1^0	0	0	0	l	$\frac{1}{2}(ml+m')$	α	β	γ	$\frac{1}{2}m$	$-\frac{1}{2}m'$
SS, 2^0	0	0	c	l	$\frac{1}{2}(ml+m')$	α	β	0	$\frac{1}{2}m$	$-\frac{1}{2}m'$
SS, 3^0	0	0	c	$mc+\frac{1}{2}m'$	$-\frac{1}{2}m''c-m'''$	α	β	$m\lambda+\frac{1}{2}m''$	λ	$\frac{1}{2}m'\lambda+m'''$
SS, 4^0	0	b	c	$mc+m'$	$mm''c+m'm''-m'''$	α	0	γ	$\frac{1}{m}\gamma+m''$	$\frac{m'}{m}\gamma+m'''$,

where m, m', m'', m''' are constants; b, c, l functions of t; α, β, γ, λ functions of θ.

SS, 1^0 gives circles (i.e. the curves of curvature of one set are circles).

SS, 2^0 is Serret's first case of SS.

SS, 3^0 gives circles.

SS, 4^0 is Serret's second case of SS.

SS, 2^0 = Serret's First Case of SS.

Writing for convenience $m' = -f^2$, the six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&= 1,\\
&x^2+y^2+z^2-2cz-ml+f^2 &&= 0,\\
&Ax+By+C(z-c)-l &&= 0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-f^2 &&= 0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\lambda) &&= 0,\\
&Adx+Bdy+Cdz &&= 0,
\end{aligned}$$

where m, f are constants; c, l are functions of t; α, β, λ functions of θ. The first set of spheres have no points in common, but the second set have in common the two points $x=0$, $y=0$, $z=\pm f$. Hence inverting (by reciprocal radius vectors) with one of these points, say $(0, 0, f)$ as centre, the spheres of the first set will continue spheres, but the spheres of the second set will be changed into planes, and the required surface is thus the inversion of a surface PS, which is in fact PS, 3: say this surface PS is the "Inversion" of SS. We invert by the formulæ

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where $\Omega = X^2+Y^2+(Z-f)^2$.

Writing the equation for the second set of spheres in the form

$$x^2+y^2+(z-f)^2-2\alpha x-2\beta y+2f(z-f)=0,$$

the transformed equation is at once found to be

$$-2\alpha X-2\beta Y+2f(Z-f)+K^2=0,$$

or say

$$\alpha X+\beta Y-fZ+f^2-\tfrac{1}{2}K^2=0;$$

viz. this gives the planes of the Inversion.

Similarly for the first set of spheres, writing the equation in the form

$$x^2+y^2+(z-f)^2+2(f-c)(z-f)+2f(f-c)-ml=0,$$

the transformed equation is found to be

$$\{2f(f-c)-ml\}\{X^2+Y^2+(Z-f)^2\}+2(f-c)K^2(Z-f)+K^4=0;$$

viz. this is

$$\{2f(f-c)-ml\}(X^2+Y^2+Z^2)+2Z\{(f-c)(-2f^2+K^2)+fml\}$$
$$+\{2f(f-c)(f^2-K^2)-f^2ml+K^4\}=0,$$

which gives the spheres of the Inversion. The two equations take a more simple form if we write therein $K^2=2f^2$; viz. they then become

$$\alpha X+\beta Y-fZ=0,$$
$$(2f^2-2fc-ml)(X^2+Y^2+Z^2)+2Zfml+f^2(2f^2+2cf-ml)=0;$$

or, say these are

$$\begin{cases}\dfrac{\alpha X+\beta Y-fZ}{\sqrt{\alpha^2+\beta^2+f^2}}=0,\\[2ex] X^2+Y^2+Z^2+\dfrac{2fml}{2f^2-2fc-ml}Z+\dfrac{f^2(2f^2+2cf-ml)}{2f^2-2fc-ml}=0.\end{cases}$$

Interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY+cZ=0,$$
$$X^2+Y^2+Z^2-2\gamma Z-2v=0,$$

where $a^2+b^2+c^2=1$; and the Inversion is thus a surface PS, 3^0.

SS, 4^0 = Serret's Second Case of SS.

Writing for convenience $m'=-mf$, $m'''=\frac{1}{2}(e^2+f^2)$, $mm''=-g$, and therefore $m'm''=fg$, the six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&x^2+y^2+z^2-2by-2c(z-g)-2fg+e^2+f^2 &&=0,\\
&Ax+B(y-b)+C(z-c)+m(f-c) &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\gamma(z-f)-e^2-f^2 &&=0,\\
&A(x-\alpha)+By+C(z-\gamma)-\frac{1}{m}(g+\gamma) &&=0,\\
&Adx+Bdy+Cdz &&=0,
\end{aligned}$$

where e, f, g, m are constants; b, c are functions of t; α, γ functions of θ.

The spheres of the first set pass all of them through the two points

$$x = \pm\sqrt{2fg - e^2 - f^2 - g^2}, \quad y = 0, \quad z = g,$$

and those of the second set pass all of them through the two points

$$x' = 0, \qquad y' = \pm e, \quad z' = f,$$

where observe that these are such that

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 0;$$

viz. the distance of each point of the first pair from each point of the second pair is $=0$. The pairs of points are one real, the other imaginary: but this is quite consistent with the reality of the spheres.

The first pair of points lies in a line parallel to the axis of x, meeting the axis of z at the point $z=g$; and the second pair in a line parallel to the axis of y, cutting the axis of z at the point $z=f$. It is clear that we can, without loss of generality, by moving the origin along the axis of z, in effect make g to be $=-f$; the equations of the two sets of spheres thus become

$$x^2 + y^2 + z^2 - 2by - 2c\,(z+f) + e^2 + 3f^2 = 0,$$
$$x^2 + y^2 + z^2 - 2ax - 2\gamma\,(z-f) - e^2 - f^2 = 0,$$

or, if in these equations for e^2 we write $e^2 - 2f^2$, the equations become

$$x^2 + y^2 + z^2 - 2by - 2c\,(z+f) + e^2 + f^2 = 0,$$
$$x^2 + y^2 + z^2 - 2ax - 2\gamma\,(z-f) - e^2 + f^2 = 0,$$

which are very symmetrical forms.

The spheres of the first set pass through the two points

$$\pm\sqrt{-e^2 - 2f^2}, \qquad 0 \quad , \quad -f,$$

and those of the second set through the two points

$$0 \quad , \quad \pm\sqrt{e^2 - 2f^2}, \quad f,$$

where, of course, the two pairs of points are related as is mentioned above.

By taking as centre of inversion a point of the first pair, we invert the first set of spheres into planes and the second set into spheres; and similarly, by taking a point of the second pair, we invert the first set of spheres into spheres and the second set into planes. By reason of the symmetry of the system, it is quite indifferent which point is chosen; and taking it to be a point of the second pair, and writing for convenience $n = \sqrt{e^2 - 2f^2}$ (n is, in fact, the quantity originally denoted by e), then the points of the first pair are

$$\pm\sqrt{-n^2 - 4f^2}, \qquad 0, \quad -f,$$
$$0 \quad , \quad \pm n, \quad f,$$

and I take for centre of inversion the point $(0, n, f)$.

Observe that, if $e=0$, $f=0$, then the four points coincide at the origin, and taking this as centre of inversion, the two sets of spheres are each changed into planes, and the Inversion of the surface SS is thus a surface PP; this particular case will be considered further on, but I first consider the general case.

The formulæ of inversion are

$$x=\frac{K^2X}{\Omega},\quad y-n=\frac{K^2(Y-n)}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where

$$\Omega=X^2+(Y-n)^2+(Z-f)^2.$$

Writing the equation of the second set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2-2\alpha x+2n(y-n)+2(f-\gamma)(z-f)=0,$$

the transformed equation is

$$-\alpha X+n(Y-n)+(f-\gamma)(Z-f)+\tfrac{1}{2}K^2=0,$$

which gives the planes of the Inversion.

Similarly, writing the equation of the first set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2+2(n-b)(y-n)+2(f-c)(z-f)+2n^2-2bn+4f(f-c)=0,$$

the transformed equation is

$$\{n^2-bn+2f(f-c)\}\{X^2+(Y-n)^2+(Z-f)^2\}$$
$$+K^2\{(n-b)(Y-n)+(f-c)(Z-f)\}+\tfrac{1}{2}K^4=0,$$

which gives the spheres of the Inversion.

Changing the origin, the two equations may be written

$$-\alpha X+nY+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$\{n^2-bn+2f(f-c)\}(X^2+Y^2+Z^2)+K^2\{(n-b)Y+(f-c)Z\}+\tfrac{1}{2}K^4=0.$$

I stop to consider a particular case. Suppose $n=0$; the equations are

$$-\alpha X+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Y+\frac{K^2}{2f}Z+\frac{K^4}{4f(f-c)}=0,$$

or, interchanging herein Y and Z, they are

$$-\alpha X+(f-\gamma)Y+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2+\frac{K^2}{2f}Y-\frac{bK^2}{2f(f-c)}Z+\frac{K^4}{4f(f-c)}=0;$$

and if for Y we write $Y-\frac{K^2}{4f}$, then the equations become

$$-\alpha X+(f-\gamma)Y+K^2\frac{f+\gamma}{4f}=0,$$

$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Z+\frac{K^4(5f-4c)}{f^2(f-c)}=0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY=lm,$$

$$X^2+Y^2+Z^2-2\theta Z=2v,$$

which belong to PS, 4°. Hence, in this particular case, $n=0$; the Inversion is PS, 4°.

Reverting to the general case, and to the two equations obtained above, observe that, in the second of the two equations, the terms in Y, Z have the variable coefficients $n-b$ and $f-c$, so that it does not at first sight seem as if these terms could by a transformation of coordinates be reduced to a single term.

But if, again changing the origin, we write $Y-\frac{\frac{1}{2}K^2}{n}$ for Y, the two equations become

$$-\alpha X+nY+(f-\gamma)Z=0,$$

$$\{n^2-bn+2f(f-c)\}(X^2+Y^2+Z^2)+\frac{K^2}{n}(f-c)(-2fY+nZ)$$
$$+\frac{K^4}{4n^2}\{n^2+bn+2f(f-c)\}=0,$$

where, in the second equation, the terms in Y, Z present themselves in the combination $-2fY+nZ$ with the constant coefficients $-2f$ and n. Hence writing

$$\sqrt{n^2+4f^2}\,Y=\ nY'-2fZ',$$

$$\sqrt{n^2+4f^2}\,Z=2fY'+\ nZ',$$

and consequently $-2fY+nZ=\sqrt{n^2+4f^2}\,Z'$, and (after the transformation) removing the accents, the equations become

$$-\alpha X+\frac{1}{\sqrt{n^2+4f^2}}[\{n^2+2f(f-\gamma)\}Y-n(f+\gamma)Z]=0,$$

$$X^2+Y^2+Z^2+\frac{K^2(f-c)\sqrt{n^2+4f^2}}{n\{n^2-bn+2f(f-c)\}}Z+\frac{K^4\{n^2+bn+2f(f-c)\}}{n^2\{n^2-bn+2f(f-c)\}}=0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY+cZ=0,$$

$$X^2+Y^2+Z^2-2m\phi Z=\theta,$$

which belong to the case *PS*, 3°. Hence, in this general case, the Inversion is a surface *PS*, 3°.

I have spoken above of the particular case $e=0$, $f=0$: here the equations of the two sets of spheres are

$$x^2+y^2+z^2-2by-2cz=0,$$

$$x^2+y^2+z^2-2ax-2\gamma z=0,$$

which have the origin as a common point. Taking this as the centre of inversion, or writing

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z=\frac{K^2Z}{\Omega},\ \text{where } \Omega=X^2+Y^2+Z^2,$$

the transformed equations are

$$\begin{aligned} bY+cZ-\tfrac{1}{2}K^2&=0,\\ aX\quad+\gamma Z-\tfrac{1}{2}K^2&=0,\end{aligned}$$

or, interchanging X and Y, say

$$\begin{aligned} bX\quad+cZ-\tfrac{1}{2}K^2&=0,\\ aY+\gamma Z-\tfrac{1}{2}K^2&=0,\end{aligned}$$

which are of the form

$$\begin{aligned} X\quad+tZ-P\quad&=0,\\ Y+\theta Z-\Pi\quad&=0,\end{aligned}$$

belonging to a surface *PP*, 3°. Hence, in this case, the Inversion is a surface *PP*, 3°.

It thus appears that the surface *SS*, 4° has an Inversion which is either *PS*, 3°, *PS*, 4° or *PP*, 3°. The inversion has in some cases to be performed in regard to an imaginary centre of inversion.

It was previously shown that the surface *SS*, 3° had an Inversion *PS*, 3°, and we thus arrive at the conclusion that a surface *SS*, with its two sets of curves of curvature each spherical, is in every case the Inversion of a surface *PS* with one set plane and the other spherical, or else of a surface *PP* with each set plane. Serret notices that the centre of inversion may be imaginary: this (he says) presents no difficulty, but he adds that it is easy to see that the centres of inversion may be taken to be real, provided that we join to the surfaces thus obtained all the parallel surfaces.

It seems to me that there is room for further investigation as to the surfaces *SS*: first, without employing the theory of inversion, it would be desirable to obtain the several forms by direct integration, as was done in regard to the surfaces *PP* and *PS*; secondly, starting from the several surfaces *PP* and *PS* considered as known forms, it would be desirable to obtain from these, by inversion in regard to an arbitrary centre, or with regard to a centre in any special position, the several forms of the surfaces *SS*. But I do not at present propose to consider either of these questions.

In conclusion, I remark that I have throughout assumed Serret's *negative* conclusions, viz. that the several cases, other than those considered in the present memoir, give only developable surfaces, or else surfaces having circles for one set of their curves of curvature. These being excluded from consideration, there remain

PP, Serret's two cases *PP*, 1^0, *PP*, 3^0;

PS, his three cases *PS*, 1^0, *PS*, 3^0, *PS*, 4^0;

SS, his two cases *SS*, 2^0 and *SS*, 4^0;

but *PP*, 1^0 is a particular case of, and so may be included in, *PP*, 3^0; and similarly *PS*, 1^0 is a particular case of, and may be included in, *PS*, 3^0; the cases considered thus are

PP, 3^0; *PS*, 3^0, *PS*, 4^0; *SS*, 2^0 and *SS*, 4^0.

It would however appear by what precedes that the case *SS*, 4^0 includes several cases which it is possible might properly be regarded as distinct; and the classification of the surfaces *SS* can hardly be considered satisfactory; it would seem that there should be at any rate 3 cases, viz. the surfaces which are the Inversions of *PP*, 3^0, *PS*, 3^0 and *PS*, 4^0 respectively.

I regard the present memoir as a development of the analytical theory of the surfaces *PP*, 3^0, *PS*, 3^0 and *PS*, 4^0.

887.

ON THE THEORY OF GROUPS.

[From the *American Journal of Mathematics*, vol. XI. (1889), pp. 139—157.]

I REFER to my papers on the theory of groups as depending on the symbolic equation $\theta^n = 1$, *Phil. Mag.*, vol. VII. (1854), pp. 40—47 and 408, 409, [125, 126]; also vol. XVIII. (1859), pp. 34—37, [243]; and "On the Theory of Groups," *American Journal of Mathematics*, vol. I. (1878), pp. 50—52, and "The Theory of Groups: Graphical Representation," *ibid.*, pp. 174—176, [694]; also to Mr Kempe's "Memoir on the Theory of Mathematical Form," *Phil. Trans.*, vol. CLXXVII. (1886), pp. 1—70, see the section "Groups containing from one to twelve units," pp. 37—43, with the diagrams given therein. Mr Kempe's paper has recalled my attention to the method of graphical representation explained in the second of the two papers of 1878, and has led me to consider, in place of a diagram as there given for the independent substitutions, a diagram such as those of his paper, for all the substitutions. I call this a colourgroup; viz. for the representation of a substitution-group of s substitutions upon the same number of letters, or say of the order s, we employ a figure of s points (in space or in a plane) connected together by coloured lines, and called a colourgroup.

I remark that up to $s = 11$, the first case of any difficulty is that of $s = 8$, and that the 5 groups of this order were determined in my papers of 1854 and 1859. For the order 12, Mr Kempe has five groups, but one of these is non-existent, and there is a group omitted; the number is thus $= 5$.

The colourgroup consists of s points joined in pairs by $\frac{1}{2}s(s-1)$ coloured lines under prescribed conditions. A line joining two points is in general regarded as a vector drawn *from* one *to* the other of the two points; the currency is shown by an arrow, and in speaking of a line ab we mean the line from a to b. But we may have a line regarded as a double line, drawn from each to the other of the two points; the arrow is then omitted, and in speaking of such a line ab we mean the line from b to a and from a to b. A fresh condition is that for a given colour there shall be one and only one line *from* each of the points, and one and

only one line *to* each of the points. We may have through two points a, b only the line ab of the given colour; this is then a double line regarded as drawn from a to b and from b to a; and there is thus one and only one line of the colour from each of these points and to each of these points. The condition implies that the lines of a given colour form either a single polygon or a set of polygons, with a continuous currency round each polygon; for instance, there may be a pentagon $abcde$, meaning thereby the pentagon formed by the lines drawn from a to b, from b to c, from c to d, from d to e, and from e to a. An arrow on one of the sides is sufficient to indicate the currency. In the case of a double line we have a polygon of two points, or say a digon.

There is a further condition which, after the necessary explanation of the meaning of the terms, may be concisely expressed as follows: Each route must be of independent effect, and (as will readily be seen) this implies that the lines of a given colour must form either a single polygon or else two or more polygons each of the same number of points: thus if $\varepsilon = k\varepsilon_1$, they may form k ε_1-gons; in particular, if ε be even, they may form $\frac{1}{2}\varepsilon$ digons.

To explain the foregoing statement, first as to the term "route." I denote the several colours by capital letters, R = red, G = green, B = blue, &c. Any capital or combination of capitals determines a route; R means go along a red line; $RRBG$, go along a red line, a red line, a blue line, a green line, and so in other cases. Given the starting point, or initial, the route determines the several points passed through, and the point arrived at, or terminal: thus $aRRBG = abefk, = k$, means that the route $RRBG$ leads from a through b, e, f to k, viz. that the red line from a leads to b, the red line from b leads to e, the blue line from e leads to f, and the green line from f leads to k. We may give in this way the Itinerary, or write simply $aRRBG = k$, meaning that the route leads from a to k. We may of course write R^2 for RR, and so in other cases. A single capital, as already mentioned, is a route, but it may for distinction be called a stage. A stage, and thence also a route, may be *reversed;* R^{-1} means go along the red line drawn to the point; if $aR = b$, then $bR^{-1} = a$; and so if $aRRBG = abefk, = k$, then $kG^{-1}B^{-1}R^{-1}R^{-1} = kfeba, = a$; $R^{-1}R^{-1} = R^{-2}$, and so in other cases.

The effect of a route depends in general on the initial point: thus, a route may lead from a point a to itself, or say it may be a circuit from a; and it may not be a circuit from another point b. And similarly, two different routes each leading from a point a, to one and the same point x, or say two routes equivalent for the initial point a, may not be equivalent for a different initial point b. Thus we cannot in general say simpliciter that a route is a circuit, or that two different routes are equivalent. But the figure may be such as to render either of these locutions, and if either, then each of them, admissible. For it is easy to see that if every route which is a circuit from any one initial point is also a circuit from every other initial point, then two routes which are equivalent for any one initial point will be equivalent for every other initial point. And conversely, if in every case where two different routes are equivalent for any one initial point, they are equivalent for every other initial point, then every route which is a circuit from any

one initial point is a circuit from every other initial point; and we express this by saying that every route is of independent effect: this explains the meaning of the foregoing statement of the condition which is to be satisfied by a colourgroup.

It is at once evident that a colourgroup, *quâ* figure where each route is of independent effect, furnishes a graphical representation of the substitution-group and gives the square by which we define such group. For, in the colourgroup of s points, we have the route from a point to itself and the routes to each of the other $(s-1)$ points, in all s non-equivalent routes; and if starting from a given arrangement, say $abcd\ldots$, of the s points, we go by one of these routes from the several points $a, b, c, d, \ldots$ successively, we obtain a different arrangement of these points. Observe that this is so; the same point cannot occur twice, for if it did, there would be a route leading from two different points b, f to one and the same point x, or the reverse route from x would lead to two different points b, f. The route from a point to itself which leaves each point unaltered, and thus gives the primitive arrangement $abcd\ldots$, may be called the route 1. Taking this route and the other $(s-1)$ routes successively, we obtain s different arrangements of the points, or say a square, each line of which is a different arrangement of the points. And not only are the arrangements different, but we cannot have the same point twice in any column, for this would mean that there were two different routes leading from a point to one and the same point x; hence each column of the square will be an arrangement of the s points. We have thus the substitution-group of the s points or letters; the s routes, or say the route 1 and the other $(s-1)$ routes, are the substitutions of the group.

The complete figure is called the colourgroup. As already mentioned, the lines of any colour form either a single polygon or two or more polygons each of the same number of points. The number of lines of a given colour is thus $=s$, or when the polygons are digons (which implies s even), the number is $=\frac{1}{2}s$. The number of colours is thus $=\frac{1}{2}(s-1)$ at least, and $=(s-1)$ at most. A general description of the figure may be given as in the annexed Table. Thus, for the group $6B$, we have

$$\begin{array}{rl} R. & 2\ 3\text{gons} = 6 \\ B, G, Y. & (3\ 2\text{gons})^3 = 9 \\ & \underline{\underline{15}}; \end{array}$$

we have the red lines forming two trigons, 6 lines, and the blue, green and yellow lines each forming three digons, together 3×3, $=9$ lines, in all 15, $=\frac{1}{2}6.5$ lines. Such description, however, does not indicate the currencies, and it is thus insufficient for the determination of the figure. But the figure is completely determined by means of the substitutions as given in the outside column of the square; thus $R=(abc)(dfe)$ shows that the red lines form the two triangles abc, dfe with these currencies, $G=(ad)(be)(cf)$, that the green lines form the three digons ad, be, cf, and so for the other two colours B and Y.

The line of a colour may be spoken of as a colour, and the lines of a colour or of two or more colours as a colourset. The colourset either does not connect

together all the points, and it is then a broken set; or it does connect together all the points, and it is then a bondset. A bondset not containing any superfluous colour is termed a bond, viz. a bond is a colourset which connects together all the points, but which is moreover such that if any one of the colours be omitted it becomes a broken set. The word colour is used as a prefix, colourset as above, colourbond, &c., and so also with a numeral, a twocolourbond is a bond with two colours, and so in other cases. Observe that we may very well have for instance a threecolourbond, and also a twocolour or a onecolourbond, only the colours or colour hereof must not be included among those of the threecolourbond, for this would then contain a superfluous colour or colours and would not be a bond.

A colourgroup may contain a onecolourbond, viz. this is the case when all the points form a single polygon; it is then said to be unibasic. If it contains no onecolourbond but contains a twocolourbond, it is bibasic; if it contains no onecolourbond or twocolourbond but contains a threecolourbond, it is tribasic, and so on. In all cases, the number of bonds (onecolour-, twocolour-, &c.) may very well be and in general is greater than one; thus a unibasic colourgroup will in general contain several onecolourbonds, a bibasic colourgroup several twocolourbonds, and so on.

The bond of the proper number of colours completely determines the colourgroup; in fact, the colourbond gives the route from any one point to each of the other $(s-1)$ points; that is, it determines all the s routes, and consequently the colourgroup. The only type of onecolourbond is the polygon of the s points; we have thus, for any value whatever of s, a unibasic colourgroup which may be called sA. The theory is well known. If s be a prime number, the number of colours is $=\frac{1}{2}(s-1)$, each colour gives a polygon through the s points, so that we have here only onecolourbonds; but in other cases we have broken sets, and there will be in general (but not for all such values of s) twocolourbonds. Observe, moreover, that, for s a prime number, the only colourgroup is the foregoing unibasic group sA. I have just employed, and shall again do so, the word type; the sense in which it is used does not, I think, require explanation.

Passing next to the bibasic colourgroups sB: there will be in general, for a given composite value of s, several of these, and in the absence of a more complete classification they may be called $sB1$, $sB2$, &c. In regard hereto, observe that, supposing for a given value of s that we know all the different types of twocolourbond, each one of these gives rise to a group; but this is not in every case a group sB: any twocolourbond contained in the corresponding group sA would give rise to the group sA which contained it, and not to a group sB. We have thus, in the first instance, to reject those twocolourbonds which are contained in the group sA. But attending only to the remaining twocolourbonds, these give rise each of them to a group sB, but the groups thus obtained are not in every case distinct groups. For looking at the converse question, suppose that, for a given value of s, we know the group sA and also the several groups sB. In any one of these groups, combining in pairs the several colours hereof RG, RY, GY, &c., we ascertain how many of these combinations are distinct types of twocolourbond, and in

this manner reproduce the whole series of types of twocolourbond, not in general singly, but in sets, those which arise from sA, those which arise from $sB1$, those which arise from $sB2$, &c.; and we thus have (it may be) several types of twocolourbond each leading to the unibasic group sA, several types each leading to the bibasic group $sB1$, several each leading to $sB2$, and so on.

The like considerations would apply to the tribasic colourgroups sC. Supposing that we had, for a given value of s, the several distinct types of threecolourbond, it would be necessary first to exclude from consideration those which give rise to a unibasic group sA or a bibasic group sB, and then to consider what sets out of the remaining types give rise to distinct tribasic groups sC. But in the table we have only one case $8C$ of a tribasic group.

I give now a table of the several groups $s=2$ to 12, viz. these are as above: A, unibasic; B, bibasic; C, tribasic; the several groups being

$2A$,	$3A$,	$4A$,	$5A$,	$6A$,	$7A$,	$8A$,	$9A$,	$10A$,	$11A$,	$12A$,
		$4B$,		$6B$,		$8B1$,	$9B$,	$10B$,		$12B1$,
						$8B2$,				$12B2$,
						$8B3$,				$12B3$,
						$8C$,				$12B4$,

in all 23 groups.

TABLE OF THE GROUPS 2 TO 12.

$2A$	a	b	1 colour.	$1 = 1$	R. 1 digon 1	
	b	a	$R=(ab)=R$			1
$3A$	a	b	c	1 colour. $1 = 1$ $= 1$	R. 1 3gon 3	
	b	c	a	$R=(abc)=R$		3
	c	a	b	$R^2=(acb)=R^{-1}$		
$4A$	a	b	c	d	2 colours. $1 = 1$ $= 1$	R. 1 4gon 4 G. 2 digons 2
	b	c	d	a	$R=(abcd)$ $=R$	6
	c	d	a	b	$R^2=(ac)(bd)=G$	
	d	a	b	c	$R^3=(adcb)$ $=R^{-1}$	

4B

3 colours.

a	b	c	d	$1 = 1 \qquad = 1$
b	a	d	c	$R = (ab)(cd) = R$
c	d	a	b	$G = (ac)(bd) = G$
d	c	b	a	$RG = (ad)(bc) = Y$

R, G, Y. (2 digons)^3 6
6

5A

2 colours.

a	b	c	d	e	$1 = 1 \qquad = 1$
b	c	d	e	a	$R = (abcde) = R$
c	d	e	a	b	$R^2 = (acebd) = G$
d	e	a	b	c	$R^3 = (adbec) = G^{-1}$
e	a	b	c	d	$R^4 = (aedcb) = R^{-1}$

R, G. (1 5gon)^2 10
10

6A

3 colours.

a	b	c	d	e	f	$1 = 1 \qquad = 1$
b	c	d	e	f	a	$R = (abcdef) \qquad = R$
c	d	e	f	a	b	$R^2 = (ace)(bdf) \qquad = G$
d	e	f	a	b	c	$R^3 = (ad)(be)(cf) = Y$
e	f	a	b	c	d	$R^4 = (aec)(bfd) \qquad = G^{-1}$
f	a	b	c	d	e	$R^5 = (afedcb) \qquad = Y^{-1}$

R. 1 6gon 6
G. 2 3gons 6
Y. 3 digons 3
15

6B

4 colours.

a	b	c	d	e	f	$1 = 1 \qquad = 1$
b	c	a	f	d	e	$R = (abc)(dfe) \qquad = R$
c	a	b	e	f	d	$R^2 = (acb)(def) \qquad = R^{-1}$
d	e	f	a	b	c	$G = (ad)(be)(cf) = G$
e	f	d	c	a	b	$RG = (ae)(bf)(cd) = Y$
f	d	e	b	c	a	$R^2G = (af)(bd)(ce) = B$

R. 2 3gons 6
G, Y, B. (3 digons)^3 9
15

7*A*

3 colours.

a	b	c	d	e	f	g	$1 = 1$	$= 1$
b	c	d	e	f	g	a	$R = (abcdefg) = R$	
c	d	e	f	g	a	b	$R^2 = (acegbdf) = G$	
d	e	f	g	a	b	c	$R^3 = (adgcfbe) = Y$	
e	f	g	a	b	c	d	$R^4 = (aebfcgd) = Y^{-1}$	
f	g	a	b	c	d	e	$R^5 = (afdbgec) = G^{-1}$	
g	a	b	c	d	e	f	$R^6 = (agfedcb) = R^{-1}$	

R, G, Y. (1 7gon)^3 21
21

8*A*

4 colours.

a	b	c	d	e	f	g	h	$1 = 1$	$= 1$
b	c	d	e	f	g	h	a	$R = (abcdefgh)$	$= R$
c	d	e	f	g	h	a	b	$R^2 = (aceg)(bdhf)$	$= Y$
d	e	f	g	h	a	b	c	$R^3 = (adgbehcf)$	$= G$
e	f	g	h	a	b	c	d	$R^4 = (ae)(bf)(cg)(dh)$	$= B$
f	g	h	a	b	c	d	e	$R^5 = (afchebgd)$	$= G^{-1}$
g	h	a	b	c	d	e	f	$R^6 = (agec)(bhfd)$	$= Y^{-1}$
h	a	b	c	d	e	f	g	$R^7 = (ahgfedcb)$	$= R^{-1}$

R, G. (1 8gon)^2 16
Y. 2 4gons 8
B. 4 digons 4
28

8B1

a	b	c	d	e	f	g	h	$1 = 1$		
b	c	d	a	f	g	h	e	$R = (abcd)(efgh)$	$= R$	
c	d	a	b	g	h	e	f	$R^2 = (ac)(bd)(eg)(fh)$	$= Y$	
d	a	b	c	h	e	f	g	$R^3 = (adcb)(ehgf)$	$= R^{-1}$	
e	f	g	h	a	b	c	d	$G = (ae)(bf)(cg)(dh)$	$= G$	
f	g	h	e	b	c	d	a	$RG = (afch)(bgde)$	$= I$	
g	h	e	f	c	d	a	b	$R^2G = (ag)(bh)(ce)(df)$	$= B$	
h	e	f	g	d	a	b	c	$R^3G = (ahcf)(bedg)$	$= I^{-1}$	

5 colours.

R, I.	(2 4gons)^2	16
Y, G, B.	(4 digons)^3	12
		28

8B2

a	b	c	d	e	f	g	h	$1 = 1$		
b	c	d	a	h	e	f	g	$R = (abcd)(ehgf)$	$= R$	
c	d	a	b	g	h	e	f	$R^2 = (ac)(bd)(eg)(fh)$	$= Y$	
d	a	b	c	f	g	h	e	$R^3 = (adcb)(efgh)$	$= R^{-1}$	
e	f	g	h	a	b	c	d	$G = (ae)(bf)(cg)(dh)$	$= G$	
f	g	h	e	d	a	b	c	$RG = (af)(bg)(ch)(de)$	$= I$	
g	h	e	f	c	d	a	b	$R^2G = (ag)(bh)(ce)(df)$	$= B$	
h	e	f	g	b	c	d	a	$R^3G = (ah)(be)(cf)(dg)$	$= O$	

6 colours.

R.	2 4gons	8
Y, G, I, B, O.	(4 digons)^5	20
		28

8B3

a	b	c	d	e	f	g	h	1	$=1$	4 colours. $=1$	$R, G, B.$ (2 4gons)^3 24 $Y.$ 4 digons 4 28
b	c	d	a	h	e	f	g	R	$=(abcd)(efgh)$	$=R$	
c	d	a	b	g	h	e	f	R^2	$=(ac)(bd)(ef)(gh)$	$=Y$	
d	a	b	c	f	g	h	e	R^3	$=(adcb)(ehgf)$	$=R^{-1}$	
e	f	g	h	a	b	c	d	G	$=(aecg)(bhdf)$	$=G$	
f	g	h	e	d	a	b	c	R^3G	$=(afch)(bedg)$	$=B$	
g	h	e	f	c	d	a	b	R^2G	$=(agce)(bfdh)$	$=G^{-1}$	
h	e	f	g	b	c	d	a	RG	$=(ahcf)(bgde)$	$=B^{-1}$	

8C

a	b	c	d	e	f	g	h	1	$=1$	7 colours. $=1$	$R, G, B, Y, I, O, V.$ (4 digons)^7 28 28
b	a	d	c	f	e	h	g	R	$=(ab)(cd)(ef)(gh)$	$=R$	
c	d	a	b	g	h	e	f	G	$=(ac)(bd)(eg)(fh)$	$=G$	
d	c	b	a	h	g	f	e	RG	$=(ad)(bc)(eh)(fg)$	$=B$	
e	f	g	h	a	b	c	d	Y	$=(ae)(bf)(cg)(dh)$	$=Y$	
f	e	h	g	b	a	d	c	RY	$=(af)(be)(ch)(dg)$	$=I$	
g	h	e	f	c	d	a	b	GY	$=(ag)(bh)(ce)(df)$	$=O$	
h	g	f	e	d	c	b	a	RGY	$=(ah)(bg)(cf)(de)$	$=V$	

9A

										4 colours.	
a	b	c	d	e	f	g	h	i	$1 = 1$	$= 1$	$R, G, B.$ (1 9gon)^3 27
b	c	d	e	f	g	h	i	a	$R = (abcdefghi)$	$= R$	$Y.$ 3 3gons 9
c	d	e	f	g	h	i	a	b	$R^2 = (acegibdfh)$	$= G$	36
d	e	f	g	h	i	a	b	c	$R^3 = (adg)\ (beh)\ (cfi) = Y$		
e	f	g	h	i	a	b	c	d	$R^4 = (aeidhcgbf)$	$= B$	
f	g	h	i	a	b	c	d	e	$R^5 = (afbgchdie)$	$= B^{-1}$	
g	h	i	a	b	c	d	e	f	$R^6 = (agd)\ (bhe)\ (cif) = Y^{-1}$		
h	i	a	b	c	d	e	f	g	$R^7 = (ahfdbigec)$	$= G^{-1}$	
i	a	b	c	d	e	f	g	h	$R^8 = (aihgfedcb)$	$= R^{-1}$	

9B

										4 colours.	
a	b	c	d	e	f	g	h	i	$1 = 1$	$= 1$	$R, G, B, Y.$ (3 3gons)^4 36
b	c	a	e	f	d	h	i	g	$R = (abc)\ (def)\ (ghi) = R$		36
c	a	b	f	d	e	i	g	h	$R^2 = (acb)\ (dfe)\ (gih) = R^{-1}$		
d	e	f	g	h	i	a	b	c	$G = (adg)\ (beh)\ (cfi) = G$		
e	f	d	h	i	g	b	c	a	$RG = (aei)\ (bfg)\ (cdh) = B$		
f	d	e	i	g	h	c	a	b	$R^2G = (afh)\ (bdi)\ (ceg) = Y$		
g	h	i	a	b	c	d	e	f	$G^2 = (agd)\ (bhe)\ (cif) = G^{-1}$		
h	i	g	b	c	a	e	f	d	$RG^2 = (ahf)\ (bid)\ (cge) = Y^{-1}$		
i	g	h	c	a	b	f	d	e	$R^2G^2 = (aie)\ (bgf)\ (chd) = B^{-1}$		

10*A*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	$1 = 1$	$= 1$	
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*a*	$R = (abcdefghij)$	$= R$	
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	$R^2 = (acegi)\,(bdfhj)$	$= G$	
d	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	$R^3 = (adgjcfibeh)$	$= Y$	
e	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	$R^4 = (aeicg)\,(bfdjh)$	$= B$	
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$R^5 = (af)\,(bg)\,(ch)\,(di)\,(ej) = O$		
g	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	$R^6 = (agcie)\,(bhjdf)$	$= B^{-1}$	
h	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	$R^7 = (ahebifcjgd)$	$= Y^{-1}$	
i	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	$R^8 = (aigec)\,(bjhfd)$	$= G^{-1}$	
j	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$R^9 = (ajihgfedcb)$	$= R^{-1}$	

5 colours.

R, *G*, *Y*, *B*. (1 10gon)^4 40
O. 5 digons 5
45

10*B*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	$1 = 1$	$= 1$	
b	*c*	*d*	*e*	*a*	*j*	*f*	*g*	*h*	*i*	$R = (abcde)\,(fjihg)$	$= R$	
c	*d*	*e*	*a*	*b*	*i*	*j*	*f*	*g*	*h*	$R^2 = (acebd)\,(figjh)$	$= Y$	
d	*e*	*a*	*b*	*c*	*h*	*i*	*j*	*f*	*g*	$R^3 = (adbec)\,(fhjgi)$	$= Y^{-1}$	
e	*a*	*b*	*c*	*d*	*g*	*h*	*i*	*j*	*f*	$R^4 = (aedcb)\,(fghij)$	$= R^{-1}$	
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$G = (af)\,(bg)\,(ch)\,(di)\,(ej) = G$		
g	*h*	*i*	*j*	*f*	*e*	*a*	*b*	*c*	*d*	$RG = (ag)\,(bh)\,(ci)\,(dj)\,(ef) = B$		
h	*i*	*j*	*f*	*g*	*d*	*e*	*a*	*b*	*c*	$R^2G = (ah)\,(bi)\,(cj)\,(df)\,(eg) = O$		
i	*j*	*f*	*g*	*h*	*c*	*d*	*e*	*a*	*b*	$R^3G = (ai)\,(bj)\,(cf)\,(dg)\,(eh) = V$		
j	*f*	*g*	*h*	*i*	*b*	*c*	*d*	*e*	*a*	$R^4G = (aj)\,(bf)\,(cg)\,(dh)\,(ei) = I$		

7 colours.

R, *Y*. (2 5gons)^2 20
G, *B*, *O*, *V*, *I*. (5 digons)^5 25
45

11*A*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	1	$=1$
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	R	$=(abcdefghijk)=R$
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	R^2	$=(acegikbdfhj)=G$
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	R^3	$=(adgjbehkcfi)=Y$
e	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	R^4	$=(aeibfjcgkdh)=B$
f	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	R^5	$=(afkejdichbg)=O$
g	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	R^6	$=(agbhcidjekf)=O^{-1}$
h	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	R^7	$=(ahdkgcjfbie)=B^{-1}$
i	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	R^8	$=(aifckhebjgd)=Y^{-1}$
j	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	R^9	$=(ajhfdbkigec)=G^{-1}$
k	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	R^{10}	$=(akjihgfedcb)=R^{-1}$

5 colours.

$=1$

R, G, Y, B, O. $(1\ 11\text{gon})^5$ 55

55

12*A*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	1 $=1$
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	R $=(abcdefghijkl)$
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	R^2 $=(acegik)\ (bdfhjl)$
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	R^3 $=(adgj)\ (behk)\ (cfil)$
e	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	R^4 $=(aei)\ (bjf)\ (cgk)\ (dhl)$
f	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	R^5 $=(afkdibglejch)$
g	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	R^6 $=(ag)\ (bh)\ (ci)\ (dj)\ (ek)\ (fl)$
h	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	R^7 $=(ahcjelgbidkf)$
i	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	R^8 $=(aie)\ (bjf)\ (ckg)\ (dlh)$
j	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	R^9 $=(ajgd)\ (bkhe)\ (clif)$
k	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	R^{10} $=(akigec)\ (bljhfd)$
l	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	R^{11} $=(alkjihgfedcb)$

6 colours.

$=1$ R, O. $(1\ 12\text{gon})^2$ 24

G. 2 6gons 12

Y. 3 4gons 12

B. 4 3gons 12

V. 6 digons 6

66

12B1

													7 colours.		
a	b	c	d	e	f	g	h	i	j	k	l	$1 = 1$	$= 1$	$R, P, O.$	$(2\ 6\text{gons})^3$ 36
														$Y.$	$4\ 3\text{gons}$ 12
b	c	d	e	f	a	h	i	j	k	l	g	$R = (abcdef)\,(ghijkl)$	$= R$	$B, G, V.$	$(6\ \text{digons})^3$ 18
															66
c	d	e	f	a	b	i	j	k	l	g	h	$R^2 = (ace)\,(bdf)\,(gik)\,(hjl)$	$= Y$		
d	e	f	a	b	c	j	k	l	g	h	i	$R^3 = (ad)\,(be)\,(cf)\,(gj)\,(hk)\,(il)$	$= B$		
e	f	a	b	c	d	k	l	g	h	i	j	$R^4 = (aec)\,(bfd)\,(gki)\,(hlj)$	$= Y^{-1}$		
f	a	b	c	d	e	l	g	h	i	j	k	$R^5 = (afedcb)\,(glkjih)$	$= R^{-1}$		
g	h	i	j	k	l	a	b	c	d	e	f	$G = (ag)\,(bh)\,(ci)\,(dj)\,(ek)\,(fl)$	$= G$		
h	i	j	k	l	g	b	c	d	e	f	a	$RG = (ahcjel)\,(bidkfg)$	$= P$		
i	j	k	l	g	h	c	d	e	f	a	b	$R^2G = (aiegck)\,(bjfhdl)$	$= O$		
j	k	l	g	h	i	d	e	f	a	b	c	$R^3G = (aj)\,(bk)\,(cl)\,(dg)\,(eh)\,(fi)$	$= V$		
k	l	g	h	i	j	e	f	a	b	c	d	$R^4G = (akcgei)\,(bdlihfj)$	$= O^{-1}$		
l	g	h	i	j	k	f	a	b	c	d	e	$R^5G = (alejch)\,(bgfkdi)$	$= P^{-1}$		

12B2

													9 colours.		
a	b	c	d	e	f	g	h	i	j	k	l	$1 = 1$	$= 1$	$R.$	$2\ 6\text{gons}$ 12
														$Y.$	$4\ 3\text{gons}$ 12
b	c	d	e	f	a	l	g	h	i	j	k	$R = (abcdef)\,(glkjih)$	$= R$	$B, G, P, O, V, I, S.$	$(6\ \text{digons})^7$ 42
															66
c	d	e	f	a	b	k	l	g	h	i	j	$R^2 = (ace)\,(bdf)\,(gki)\,(hlj)$	$= Y$		
d	e	f	a	b	c	j	k	l	g	h	i	$R^3 = (ad)\,(be)\,(cf)\,(gj)\,(hk)\,(il)$	$= B$		
e	f	a	b	c	d	i	j	k	l	g	h	$R^4 = (aec)\,(bfd)\,(gik)\,(hjl)$	$= Y^{-1}$		
f	a	b	c	d	e	h	i	j	k	l	g	$R^5 = (afedcb)\,(ghijkl)$	$= R^{-1}$		
g	h	i	j	k	l	a	b	c	d	e	f	$G = (ag)\,(bh)\,(ci)\,(dj)\,(ek)\,(fl)$	$= G$		
h	i	j	k	l	g	f	a	b	c	d	e	$RG = (ah)\,(bi)\,(cj)\,(dk)\,(el)\,(fg)$	$= P$		
i	j	k	l	g	h	e	f	a	b	c	d	$R^2G = (ai)\,(bj)\,(ck)\,(dl)\,(eg)\,(fh)$	$= O$		
j	k	l	g	h	i	d	e	f	a	b	c	$R^3G = (aj)\,(bk)\,(cl)\,(dg)\,(eh)\,(fi)$	$= V$		
k	l	g	h	i	j	c	d	e	f	a	b	$R^4G = (ak)\,(bl)\,(cg)\,(dh)\,(ei)\,(fj)$	$= I$		
l	g	h	i	j	k	b	c	d	e	f	a	$R^5G = (al)\,(bg)\,(ch)\,(di)\,(ej)\,(fk)$	$= S$		

12B3

a	b	c	d	e	f	g	h	i	j	k	l	1	$=1$	$=1$	6 colours.
b	c	d	e	f	a	l	g	h	i	j	k	R	$=(abcdef)(glkjih)$	$=R$	
c	d	e	f	a	b	k	l	g	h	i	j	R^2	$=(ace)(bdf)(gki)(hlj)$	$=B$	
d	e	f	a	b	c	j	k	l	g	h	i	R^3	$=(ad)(be)(cf)(gj)(hk)(il)$	$=Y$	
e	f	a	b	c	d	i	j	k	l	g	h	R^4	$=(aec)(bfd)(gik)(hjl)$	$=B^{-1}$	
f	a	b	c	d	e	h	i	j	k	l	g	R^5	$=(afedcb)(ghijkl)$	$=R^{-1}$	
g	h	i	j	k	l	d	e	f	a	b	c	G	$=(agdj)(bhek)(cifl)$	$=G$	
h	i	j	k	l	g	c	d	e	f	a	b	RG	$=(ahdk)(biel)(cjfg)$	$=O$	
i	j	k	l	g	h	b	c	d	e	f	a	R^2G	$=(aidl)(bjeg)(ckfh)$	$=P$	
j	k	l	g	h	i	a	b	c	d	e	f	R^3G	$=(ajdg)(bkeh)(clfi)$	$=G^{-1}$	
k	l	g	h	i	j	f	a	b	c	d	e	R^4G	$=(akdh)(blei)(cgfj)$	$=O^{-1}$	
l	g	h	i	j	k	e	f	a	b	c	d	R^5G	$=(aldi)(bgej)(chfk)$	$=P^{-1}$	

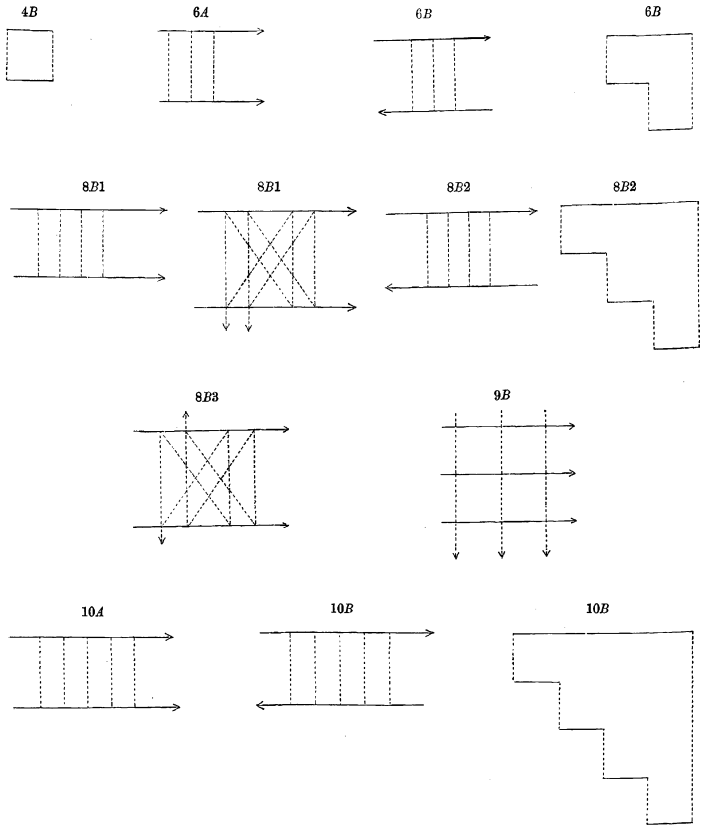
R.	2 6gons	12
G, O, P.	$(3\ 4\text{gons})^3$	36
B.	4 3gons	12
Y.	6 digons	6
		66

12B4

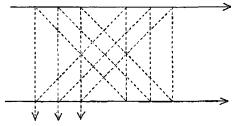
a	b	c	d	e	f	g	h	i	j	k	l	1	$=1$	$=1$	7 colours.
b	c	a	e	f	d	h	i	g	k	l	j	R	$=(abc)(def)(ghi)(jkl)$	$=R$	
c	a	b	f	d	e	i	g	h	l	j	k	R^2	$=(acb)(dfe)(gih)(jlk)$	$=R^{-1}$	
d	l	g	a	i	j	c	k	e	f	h	b	RGR^2	$=(ad)(bl)(eg)(ci)(fj)(hk)$	$=Y$	
e	j	h	b	g	k	a	l	f	d	i	c	RG	$=(aeg)(bjd)(chl)(fki)$	$=B$	
f	k	i	c	h	l	b	j	d	e	g	a	RGR	$=(afl)(bkg)(cid)(ehj)$	$=O$	
g	d	l	j	a	i	e	c	k	b	f	h	GR^2	$=(age)(bdj)(clh)(fik)$	$=B^{-1}$	
h	e	j	k	b	g	f	a	l	c	d	i	G	$=(ah)(be)(cj)(dk)(fg)(il)$	$=G$	
i	f	k	l	c	h	d	b	j	a	e	g	GR	$=(aij)(bfh)(cke)(dlg)$	$=P$	
j	h	e	g	k	b	l	f	a	i	c	d	R^2G	$=(aji)(bhf)(cek)(dgl)$	$=P^{-1}$	
k	i	f	h	l	c	j	d	b	g	a	e	R^2GR	$=(ak)(bi)(cf)(dh)(el)(gj)$	$=V$	
l	g	d	i	j	a	k	e	c	h	b	f	R^2GR^2	$=(alf)(bgk)(cdi)(ejh)$	$=O^{-1}$	

R, B, P, O.	$(4\ 3\text{gons})^4$	48
Y, G, V.	$(6\ \text{digons})^3$	18
		66

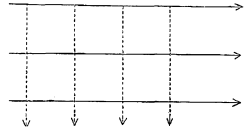
Extracting from these colourgroups the twocolourbonds contained in them respectively, we have the twocolourbonds shown in the annexed series of figures. I have in each case given the number *4B*, *6A*, &c., of the colourgroup in which the bond is contained, and which colourgroup is given conversely by the twocolourbond. The several points may have letters *a*, *b*, *c*, *d*, &c., attached to them at pleasure; but as the particular letters are quite immaterial, it seemed to me better to give the several figures without any letters.



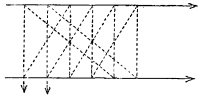
12*A*



12*A*

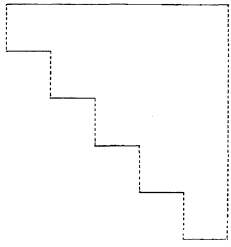


12*B*1



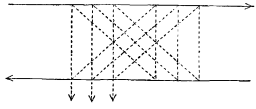
12*B*1

12*B*2

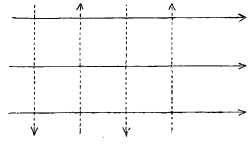


12*B*2

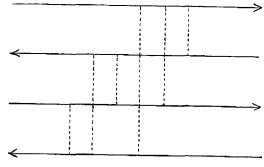
12*B*3



12*B*3

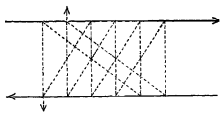


12*B*4

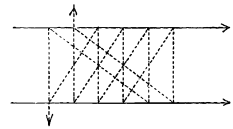


In any one of the foregoing forms of twocolourbond, each point, in its relations to the other points, is indistinguishable from each of the other points. This would seem to be a relation of symmetry equivalent to the before-mentioned condition that each route is of independent effect; and it would moreover seem as if the relation of symmetry were satisfied for each of the following forms:

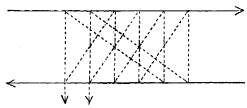
12 (wrong form).



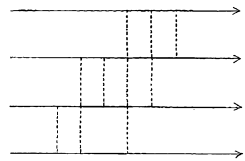
12 (wrong form).



12 (wrong form).



12 (wrong form).



Each of these is, however, a wrong form, not satisfying the condition that each route is of independent effect. As to this, observe that, when the condition is satisfied, there are in all ($s=$) 12 non-equivalent routes, and there is thus a completely determinate square. When the condition is not satisfied, there are more than this number of non-equivalent routes, and there may very well be s routes giving rise to a latin square, viz. a square each line of which, and also each column of which, contains all the letters, and which thus seems at first sight to represent a substitution-group; but the substitutions, by which each line of the square is derived from itself and the other lines of the square, are not the same as those by which each line is derived from the top line, and thus the square does not represent a group. Thus in one of the above wrong forms, starting from the routes $R=(abcdef)(glkjih)$ and $G=(agciek)(bhdjfl)$, we have

12 (wrong form).

														G	R^2	G
a	b	c	d	e	f	g	h	i	j	k	l	1	$=1$	a	g	k	a
b	c	d	e	f	a	l	g	h	i	j	k	R	$=(abcdef)(glkjih)$	b	h	l	b
c	d	e	f	a	b	k	l	g	h	i	j	R^2	$=(ace)(bdf)(gkl)(hij)$	c	i	g	c
d	e	f	a	b	c	j	k	l	g	h	i	R^3	$=(ad)(be)(cf)(gj)(hk)(il)$	d	j	h	d
e	f	a	b	c	d	i	j	k	l	g	h	R^4	$=(aec)(bfd)(gik)(hjl)$	e	k	i	e
f	a	b	c	d	e	h	i	j	k	l	g	R^5	$=(afedcb)(ghijkl)$	f	l	j	f
g	h	i	j	k	l	c	d	e	f	a	b	G	$=(agciek)(bhdjfl)$	g	c	e	k
h	i	j	k	l	g	b	c	d	e	f	a	RG	$=(ahcjel)(bidkfg)$	h	d	f	l
i	j	k	l	g	h	a	b	c	d	e	f	R^2G	$=(aickeg)(bjdlfh)$	i	e	a	g
j	k	l	g	h	i	f	a	b	c	d	e	R^3G	$=(ajcleh)(bkdgfi)$	j	f	b	h
k	l	g	h	i	j	e	f	a	b	c	d	R^4G	$=(akogei)(bldhjf)$	k	a	c	i
l	g	h	i	j	k	d	e	f	a	b	c	R^5G	$=(alchej)(bgdifk)$	l	b	d	j

which is not a group; there is no substitution $G^{-1}=(akeicg)(blfjdh)$. And we see that, in fact, each route is not of independent effect; the route GR^2G leads as shown from the primitive arrangement *abcdefghijkl* to *abcdefklghij*, viz. it is a circuit from each of the points a, b, c, d, e, f, but not from any one of the remaining points g, h, i, j, k, l.

END OF VOL. XII.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.
Bombay: E. SEYMOUR HALE.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

VOL. XIII.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1897.

[*All Rights reserved.*]

CAMBRIDGE:

PRINTED BY J. AND C. F. CLAY,

AT THE UNIVERSITY PRESS.

ADVERTISEMENT.

THE present volume contains 80 papers, numbered 888 to 967, published for the most part in the years 1889 to 1895, the year in which Professor Cayley died. The volume completes the collection of his mathematical papers.

The table for the thirteen volumes is

Vol.	I.	Numbers	1	to	100,
,,	II.	,,	101	,,	158,
,,	III.	,,	159	,,	222,
,,	IV.	,,	223	,,	299,
,,	V.	,,	300	,,	383,
,,	VI.	,,	384	,,	416,
,,	VII.	,,	417	,,	485,
,,	VIII.	,,	486	,,	555,
,,	IX.	,,	556	,,	629,
,,	X.	,,	630	,,	705,
,,	XI.	,,	706	,,	798,
,,	XII.	,,	799	,,	887,
,,	XIII.	,,	888	,,	967.

A. R. FORSYTH.

30 *November*, 1897.

CONTENTS.

[An Asterisk means that the paper is not printed in full.]

PAGE

888. *On a form of quartic surface with twelve nodes* . . . 1
British Association Report, (1886), pp. 540, 541

889. *On a differential equation and the construction of Milner's lamp* 3
Edinburgh Math. Soc. Proc., t. v (1887), pp. 99—101

890. *Note on the hydrodynamical equations* 6
Proc. Roy. Soc. Edin., t. xv (1888), pp. 342—344

891. *On the binodal quartic and the graphical representation of the elliptic functions* 9
Camb. Phil. Soc. Trans., t. xiv (1889), pp. 484—494

892. *Note on the orthomorphic transformation of a circle into itself* 20
Edinburgh Math. Soc. Proc., t. viii (1890), pp. 91, 92

893. *The bitangents of the quintic* 21
Annals of Mathematics, t. v (1890), pp. 109, 110

894. *An investigation by Wallis of his expression for* π . . . 22
Quart. Math. Journ., t. xxiii (1889), pp. 165—169

895. *A theorem on trees* 26
Quart. Math. Journ., t. xxiii (1889), pp. 376—378

896. *A transformation in elliptic functions* 29
Quart. Math. Journ., t. xxiv (1890), pp. 259—262

PAGE

897. *Sur les racines d'une équation algébrique* 33
Comptes Rendus, t. cx (1890), pp. 174—176, 215—218

898. *Sur l'équation modulaire pour la transformation de l'ordre* 11 38
Comptes Rendus, t. cxi (1890), pp. 447—449

899. *Sur les surfaces minima* 41
Comptes Rendus, t. cxi (1890), pp. 953, 954

900. *James Joseph Sylvester* 43
(Scientific Worthies in Nature, xxv); *Nature*, vol. xxxix (1889), pp. 217—219

901. *Note on the sums of two series* 49
Messenger of Mathematics, t. xix (1890), pp. 29—31

902. *On the focals of a quadric surface* 51
Messenger of Mathematics, t. xix (1890), pp. 113—117

903. *On Latin squares* 55
Messenger of Mathematics, t. xix (1890), pp. 135—137

904. *Note on reciprocal lines* 58
Messenger of Mathematics, t. xix (1890), pp. 174, 175

905. *On the equation* $x^n - 1 = 0$ 60
Messenger of Mathematics, t. xix (1890), pp. 184—188

906. *Note on Schlaefli's modular equation for the cubic transformation; with a correction* 64
Messenger of Mathematics, t. xx (1891), pp. 59, 60; 120

907. *Note on the ninth roots of unity* 66
Messenger of Mathematics, t. xx (1891), p. 63

908. *On two invariants of a quadri-quadric function* 67
Messenger of Mathematics, t. xx (1891), pp. 68, 69

909. *On a particular case of Kummer's differential equation of the third order* 69
Messenger of Mathematics, t. xx (1891), pp. 75—79

910. *Note on the involutant of two matrices* 74
Messenger of Mathematics, t. xx (1891), pp. 136, 137

PAGE

911. *On an algebraical identity relating to the six coordinates of a line* 76
Messenger of Mathematics, t. xx (1891), pp. 138—140

912. *On the notion of a plane curve of a given order* . . . 79
Messenger of Mathematics, t. xx (1891), pp. 148—150

913. *On the epitrochoid* 81
Messenger of Mathematics, t. xx (1891), pp. 150—158

914. *On a soluble quintic equation* 88
American Journal of Mathematics, t. XIII (1891), pp. 53—58

915. *On the partitions of a polygon* 93
Proc. Lond. Math. Soc., t. XXII (1891), pp. 237—262

916. [*Note on a theorem in matrices*] 114
Proc. Lond. Math. Soc., t. XXII (1891), p. 458

917. [*Note on the theory of rational transformation*] . . . 115
Proc. Lond. Math. Soc., t. XXII (1891), pp. 475, 476

918. *On the substitution-groups for two, three, four, five, six, seven, and eight letters* 117
Quart. Math. Journ., t. XXV (1891), pp. 71—88, 137—155

919. *On the problem of tactions* 150
Quart. Math. Journ., t. XXV (1891), pp. 104—127

920. *On orthomorphosis* 170
Quart. Math. Journ., t. XXV (1891), pp. 203—226

921. *On some problems of orthomorphosis* 191
Crelle's Journal der Mathem., t. CVII (1891), pp. 262—277

922. *Note on the lunar theory* 206
Monthly Notices of the Royal Astronomical Society, t. LII (1892), pp. 2—5

923. *Note on a hyperdeterminant identity* 210
Messenger of Mathematics, t. XXI (1892), pp. 131, 132

924. *On the non-existence of a special group of points* . . . 212
Messenger of Mathematics, t. XXI (1892), pp. 132, 133

PAGE

925. *On Waring's formula for the sum of the mth powers of the roots of an equation* 213
Messenger of Mathematics, t. XXI (1892), pp. 133—137

926. *Corrected seminvariant tables for the weights* 11 *and* 12 . 217
American Journal of Mathematics, t. XIV (1892), pp. 195—200

927. *On Clifford's paper "On syzygetic relations among the powers of linear quantics"* 224
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 99—104

928. *On the analytical theory of the congruency* 228
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 185—188

929. *Note on the skew surfaces applicable upon a given skew surface* 231
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 217—225

930. *Sur la surface des ondes* 238
Annali di Matematica, Ser. II, t. XX (1892), pp. 1—18

931. *On some formulæ of Codazzi and Weingarten in relation to the application of surfaces to each other* 253
Proc. Lond. Math. Soc., t. XXIV (1893), pp. 210—223

932. *On symmetric functions and seminvariants* 265
American Journal of Mathematics, t. XV (1893), pp. 1—74

933. *Tables of pure reciprocants to the weight* 8 333
American Journal of Mathematics, t. XV (1893), pp. 75—77

934. *Note on the so-called quotient G/H in the theory of groups* . 336
American Journal of Mathematics, t. XV (1893), pp. 387, 388

935. *Sur la fonction modulaire* $\chi\omega$ 338
Comptes Rendus, t. CXVI (1893), pp. 1339—1343

936. *Note on uniform convergence* 342
Proc. Roy. Soc. Edin., t. XIX (1893), pp. 203—207

937. *Note on the orthotomic curve of a system of lines in a plane* 346
Messenger of Mathematics, t. XXII (1893), pp. 45, 46

PAGE

938. *On two cubic equations* 348

Messenger of Mathematics, t. XXII (1893), pp. 69—71

939. *On a case of the involution* $AF+BG+CH=0$, *where* A, B, C, F, G, H *are ternary quadrics* 350

Messenger of Mathematics, t. XXII (1893), pp. 182—186

940. *On the development of* $(1+n^2x)^{\frac{m}{n}}$ 354

Messenger of Mathematics, t. XXII (1893), pp. 186—190

941. *Note on the partial differential equation*

$$Rr+Ss+Tt+U(s^2-rt)-V=0 \quad . \quad . \quad . \quad . \quad 358$$

Quart. Math. Journ., t. XXVI (1893), pp. 1—5

942. *On seminvariants* 362

Quart. Math. Journ., t. XXVI (1893), pp. 66—69

943. *On reciprocants and differential invariants* 366

Quart. Math. Journ., t. XXVI (1893), pp. 169—194, 289—307

944. *On Pfaff-invariants* 405

Quart. Math. Journ., t. XXVI (1893), pp. 195—205

945. *Note on lacunary functions* 415

Quart. Math. Journ., t. XXVI (1893), pp. 279—281

946. *Note on the theory of orthomorphosis* 418

Quart. Math. Journ., t. XXVI (1893), pp. 282—288

947. *On a system of two tetrads of circles: and other systems of two tetrads* 425

Camb. Phil. Soc. Proc., t. VIII (1893), pp. 54—59

948. *Report of a Committee appointed for the purpose of carrying on the tables connected with the Pellian equation from the point where the work was left by Degen in* 1817 . . 430

British Association Report, (1893), pp. 73—120

949. *On Halphen's characteristic n, in the theory of curves in space* 468

Crelle's Journal der Mathem., t. CXI (1893), pp. 347—352

PAGE

950. *On the sextic resolvent equations of Jacobi and Kronecker* 473
Crelle's Journal der Mathem., t. CXIII (1894), pp. 42—49

951. *Non-Euclidian geometry* 480
Camb. Phil. Soc. Trans., t. XV (1894), pp. 37—61

952. *On the kinematics of a plane, and in particular on three-bar motion: and on a curve-tracing mechanism* . . 505
Camb. Phil. Soc. Trans., t. XV (1894), pp. 391—402

953. *On the nine-points circle* 517
Messenger of Mathematics, t. XXIII (1894), pp. 23—25

954. *On the nine-points circle of a plane triangle* 520
Messenger of Mathematics, t. XXIII (1894), pp. 25—27

955. *The numerical value of* $\Pi(i), = \Gamma(1+i)$ 522
Messenger of Mathematics, t. XXIII (1894), pp. 36—38

956. *On Richelot's integral of the differential equation* $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$ 525
Messenger of Mathematics, t. XXIII (1894), pp. 42—47

957. *Illustrations of Sylow's theorems on groups* 530
Messenger of Mathematics, t. XXIII (1894), pp. 59—62

958. *On the surface of the order n which passes through a given cubic curve* 534
Messenger of Mathematics, t. XXIII (1894), pp. 79, 80

959. *Note on Plücker's equations* 536
Messenger of Mathematics, t. XXIV (1895), pp. 23, 24

960. *On the circle of curvature at any point of an ellipse* . . 537
Messenger of Mathematics, t. XXIV (1895), pp. 47, 48

961. *A trigonometrical identity* 538
Messenger of Mathematics, t. XXIV (1895), pp. 49—51

962. *Co-ordinates versus quaternions* 541
Proc. Roy. Soc. Edin., t. XX (1895), pp. 271—275

963. *Note on Dr Muir's Paper "A problem of Sylvester's in Elimination"*. 545
Proc. Roy. Soc. Edin., t. XX (1895), pp. 306—308

PAGE

964. *On the nine-points circle of a spherical triangle* 548
Quart. Math. Journ., t. XXVII (1895), pp. 35—39

965. *On the sixty icosahedral substitutions* 552
Quart. Math. Journ., t. XXVII (1895), pp. 236—242

966. *Note on a memoir in Smith's Collected Papers* 558
Bulletin of the American Math. Soc., 2nd Ser., t. I (1895), pp. 94—96

*967. *An elementary treatise on elliptic functions* 560
First edition, 1876, second edition, 1895

CLASSIFICATION.

ANALYSIS

Algebra, 894, 936, 937, 940, 961, 963.

Quaternions, 962.

Theory of numbers (Pellian Equation), 948.

Theory of equations, 897, 905, 907, 914, 925, 932, 938, 950.

Combinatory Analysis, 903, 915.

Groups, 903, 918, 934, 957, 965.

Trees, 895.

Matrices, 910, 916.

Invariants and covariants, 908, 923, 926, 927, 932, 942.

Reciprocants and differential invariants, 933, 943, 944.

Differential equations, 889, 890, 909, 941, 956.

Theory of functions, 945, 955.

Conformal representation, 892, 920, 921, 946.

Elliptic functions, 891, 901, 935, 966, 967.

Transformation of elliptic functions, 896, 898, 906.

GEOMETRY

Systems of circles, 947, 953, 954, 964.

Theory of plane curves, 912, 917, 924, 939, 952, 959, 960.

Tactions, 919.

Plane quartics, 891.

Plane quintics, 893.

Epitrochoid, 913.

Mechanical description of curves, 952.

Theory of tortuous curves, 949, 958.

Theory of surfaces, 929, 931, 958.

Quadrics, 902, 904.

Quartic surfaces, 888.

Wave surface, 930.

Minimal surfaces, 899.

Line geometry, 911, 928.

Non-Euclidian geometry, 951.

Lunar theory, 922.

Biographical notice of Sylvester, 900.

888.

ON A FORM OF QUARTIC SURFACE WITH TWELVE NODES.

[From the *British Association Report*, (1886), pp. 540, 541.]

USING throughout capital letters to denote homogeneous quadric functions of the coordinates (x, y, z, w), we have as a form of quartic surface with eight nodes $\Omega = (*\between U, V, W)^2 = 0$; viz. the nodes are here the octad of points, or eight points of intersection of the quadric surfaces $U=0$, $V=0$, $W=0$; the equation can, by a linear transformation on the functions U, V, W (that is, by substituting for the original functions U, V, W linear functions of these variables), be reduced to the form $\Omega = U^2 + V^2 + W^2 = 0$.

Suppose now that the function Ω can in a second manner be expressed in the like form $\Omega = P^2 + Q^2 + R^2$ (where P, Q, R are not linear functions of U, V, W); that is, suppose that we have identically $U^2 + V^2 + W^2 = P^2 + Q^2 + R^2$, this gives $U^2 - P^2 + V^2 - Q^2 + W^2 - R^2 = 0$; or, writing $U+P$, $V+Q$, $W+R = A$, B, C, and $U-P$, $V-Q$, $W-R = F$, G, H, the identity becomes $AF + BG + CH = 0$; and this identity being satisfied, the equation $\Omega = 0$ of the quartic surface may be written in the two forms

$$\Omega = (A+F)^2 + (B+G)^2 + (C+H)^2 = 0,$$

and

$$\Omega = (A-F)^2 + (B-G)^2 + (C-H)^2 = 0;$$

viz. the quartic surface has the nodes which are the intersections of the three quadric surfaces $A+F=0$, $B+G=0$, $C+H=0$, and also the nodes which are the intersections of the three quadric surfaces $A-F=0$, $B-G=0$, $C-H=0$. We may of course also write the equation of the surface in the form

$$\Omega = A^2 + B^2 + C^2 + F^2 + G^2 + H^2 = 0.$$

An easy way of satisfying the identity $AF+BG+CH=0$ is to assume

$$A,\ B,\ C,\ F,\ G,\ H = ayz,\ bzx,\ cxy,\ fxw,\ gyw,\ hzw,$$

where the constants a, b, c, f, g, h satisfy the condition $af+bg+ch=0$; this being so, the functions A, B, C, F, G, H, and consequently the functions $A+F$, $B+G$, $C+H$ and $A-F$, $B-G$, $C-H$ each of them vanish for the four points ($y=0$, $z=0$, $w=0$), ($z=0$, $x=0$, $w=0$), ($x=0$, $y=0$, $w=0$), ($x=0$, $y=0$, $z=0$), or say the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1). It hence appears that the quartic surface

$$\Omega = a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 + f^2x^2w^2 + g^2y^2w^2 + h^2z^2w^2 = 0$$

is a quartic surface with twelve nodes; viz. it has as nodes the last-mentioned four points, the remaining four points of intersection of the surfaces

$$ayz+fxw=0,\quad bzx+gyw=0,\quad cxy+hzw=0,$$

and the remaining four points of intersection of the surfaces

$$ayz-fxw=0,\quad bzx-gyw=0,\quad cxy-hzw=0.$$

The above is the analytical theory of one of the two forms of quartic surface with twelve nodes recently established by Dr K. Rohn in a paper in the *Berichte ü. d. Verhandlungen der K. Sächsische Gesellschaft zu Leipzig*, (1884), pp. 52—60.

889.

ON A DIFFERENTIAL EQUATION AND THE CONSTRUCTION OF MILNER'S LAMP.

[From the *Proceedings of the Edinburgh Mathematical Society*, vol. v. (1887), pp. 99—101.]

WHAT sort of an equation is

$$b^3 \cos(\alpha+\theta) = a \cos\theta \int_\theta^\beta r^2 d\theta - \tfrac{2}{3}\left\{\cos\theta \int_\theta^\beta r^3 \cos\theta\, d\theta + \sin\theta \int_\theta^\beta r^3 \sin\theta\, d\theta\right\} ? \quad \text{.....(1).}$$

Write

$$X = \int_\theta^\beta r^2 d\theta, \quad Y = \int_\theta^\beta r^3 \cos\theta\, d\theta, \quad Z = \int_\theta^\beta r^3 \sin\theta\, d\theta \quad \text{.....(2),}$$

and start with the equations

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} \quad \text{.....(3),}$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\{a\cos\theta\,.\,X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta)\} = 0 \quad \text{.....(4).}$$

This last gives

$$(r - a\cos\theta)\,dr + ar\sin\theta\,.\,d\theta = 0 \quad \text{.....(5),}$$

and the system thus is

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} = \frac{(r - a\cos\theta)\,dr}{-ar\sin\theta} \quad \text{.....(6),}$$

viz. this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with four arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $=0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X = \int_\theta^\beta r^2 d\theta, \quad Y = \int_\theta^\beta r^3 \cos\theta d\theta, \quad Z = \int_\theta^\beta r^3 \sin\theta d\theta,$$

and further, by integration of (4),

$$L\cos\theta + M\sin\theta = a\cos\theta . X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta).$$

Here L and M denote properly determined constants: viz. the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta . X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta + M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have this value, we must have $L = b^3\cos\alpha$, $M = -b^3\sin\alpha$ (where L, M are given functions of a, β, r_0), i.e. we must have *two* given relations between a, b, α, β, r_0: or treating r_0 as a disposable constant, we must have *one* given relation between a, b, α, β.

The equation $d\theta = \dfrac{r - a\cos\theta}{-ar\sin\theta}dr$ gives $r^2 - 2ar\cos\theta = C$, where $C = r_0^2 - 2ar_0\cos\beta$. There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0 = 0$ or $r_0 = 2a\cos\beta$, giving $C = 0$ and therefore $r = 2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int \cos^2\theta d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta,$$

$$\int \cos^4\theta d\theta = \tfrac{3}{8}\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{32}\sin 4\theta,$$

$$\int \cos^3\theta \sin\theta d\theta = -\tfrac{1}{4}\cos^4\theta.$$

And thence

$$\begin{aligned}
& a\cos\theta . X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta) \\
= \quad & 4a^3\cos\theta\{\tfrac{1}{2}(\beta-\theta) + \tfrac{1}{4}(\sin 2\beta - \sin 2\theta)\} \\
& - \tfrac{16}{3}a^3\cos\theta\{\tfrac{3}{8}(\beta-\theta) + \tfrac{1}{4}(\sin 2\beta - \sin 2\theta) + \tfrac{1}{32}(\sin 4\beta - \sin 4\theta)\} \\
& - \tfrac{16}{3}a^3\sin\theta\{ \qquad\qquad\qquad\qquad - \tfrac{1}{4}(\cos^4\beta - \cos^4\theta)\} \\
= & - \tfrac{1}{3}a^3\cos\theta(\sin 2\beta - \sin 2\theta) \\
& - \tfrac{1}{6}a^3\cos\theta(\sin 4\beta - \sin 4\theta) \\
& + \tfrac{4}{3}a^3\sin\theta(\cos^4\beta - \cos^4\theta),
\end{aligned}$$

where the terms containing β are readily reduced to $\frac{4}{3}a^3\cos^3\beta\sin(\theta-\beta)$; hence also the terms without β disappear of themselves: and we have

$$a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)=\tfrac{4}{3}a^3\cos^3\beta\,.\sin(\theta-\beta),$$

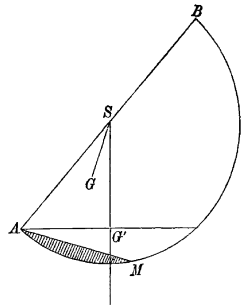
which may be put

$$=b^3\cos(\theta+\alpha):$$

viz. this will be so if we have the *two* relations

$$\alpha=\tfrac{1}{2}\pi-\beta;\quad\text{and}\quad b^3=-\tfrac{4}{3}a^3\cos^3\beta.$$

I make (see figure) Milner's lamp, with a circular section, β arbitrary, but a



segment AM ($\angle SAM=\beta$) made solid. G in the line SG at right angles to AM is the C.G. of the lamp, and G' the C.G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz. in the equation $r^2-2ar\cos\theta=C$, we must have $C=0$.

890.

NOTE ON THE HYDRODYNAMICAL EQUATIONS.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XV. (1889), pp. 342—344.]

WRITING for shortness $D = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}$, then if from the hydrodynamical equations

$$Du = \frac{d}{dx}\left(V - \frac{p}{\rho}\right),\ Dv = \frac{d}{dy}\left(V - \frac{p}{\rho}\right),\ Dw = \frac{d}{dz}\left(V - \frac{p}{\rho}\right),$$

without the aid of the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

we eliminate $V - \frac{p}{\rho}$, we obtain equations not equivalent to those of Helmholtz,

$$D\xi = \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)u,\ = \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx},\ \&c.,$$

(2ξ, 2η, $2\zeta = \frac{dv}{dz} - \frac{dw}{dy}$, $\frac{dw}{dx} - \frac{du}{dz}$, $\frac{du}{dy} - \frac{dv}{dx}$, as usual), but which, transforming them by means of the omitted equation, agree as they should do with his equations. But the form of the equations obtained directly by elimination as above, is an interesting one, which it is worth while to give.

We have

$$\begin{aligned} D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) &= D\left(\frac{dv}{dz} - \frac{dw}{dy}\right) - \frac{d}{dz}Dv + \frac{d}{dy}Dw, \\ &= \left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(\frac{dv}{dz} - \frac{dw}{dy}\right) \\ &\quad - \frac{d}{dz}\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) \\ &\quad + \frac{d}{dy}\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right), \end{aligned}$$

where the terms containing second derived functions disappear of themselves, and the expression on the right-hand is thus

$$= -\frac{du}{dz}\frac{dv}{dx} - \frac{dv}{dz}\frac{dv}{dy} - \frac{dw}{dz}\frac{dv}{dz}$$
$$+ \frac{du}{dy}\frac{dw}{dx} + \frac{dv}{dy}\frac{dw}{dy} + \frac{dw}{dy}\frac{dw}{dz}.$$

Representing for shortness the Matrix

$$\begin{vmatrix} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} \end{vmatrix} \text{ by } \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}, \text{ and its square by } \begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix},$$

we have

$$\left(\frac{du}{dx}, \frac{dv}{dx}, \frac{dw}{dx}\right), \left(\frac{du}{dy}, \frac{dv}{dy}, \frac{dw}{dy}\right), \left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}\right)$$

$$\begin{vmatrix} A, & B, & C \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix} = \begin{vmatrix} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz} \end{vmatrix} \begin{matrix} \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \\ \text{"} & \text{"} & \text{"} \end{matrix}$$

viz. the combinations which enter into the foregoing formula are

$$C' = \frac{dv}{dx}\frac{du}{dz} + \frac{dv}{dy}\frac{dv}{dz} + \frac{dv}{dz}\frac{dw}{dz},$$

and

$$B'' = \frac{dw}{dx}\frac{du}{dy} + \frac{dw}{dy}\frac{dv}{dy} + \frac{dw}{dz}\frac{dw}{dy},$$

and the equation thus is $D(c' - b'') + C' - B'' = 0$; viz. the three equations are

$$D(c' - b'') + C' - B'' = 0,$$
$$D(a'' - c) + A'' - C = 0,$$
$$D(b - a') + B - A' = 0,$$

which are the equations in question.

Observe that we have

$$C' - B'' = (a', b', c')(c, c', c'') - (a'', b'', c'')(b, b', b'')$$
$$= a'c' + b'c' + c'c'' \quad - a''b - b'b'' - b''c'',$$

and thence, writing

$$\rho = a(c' - b'') + b(a'' - c) + c(b - a'),$$
$$= ac' - ab'' + a''b - a'c,$$

we have

$$C' - B'' + \rho = (a + b' + c'')(c' - b'') = 0,$$

if $a+b'+c''=0$; viz. this being so, $C''-B''=-\rho$, or the first equation is

$$D(c'-b'')=\rho, \; = a(c'-b'')+b(a''-c)+c(b-a'),$$

that is, $D\xi = \xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}$, the first equation of Helmholtz, and we thus have the equations of Helmholtz, if $a-b'+c''=0$, that is, if $\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$.

The foregoing three equations $D(c'-b'')+C'-B''=0$, &c., are the quaternion equation ($\sigma = iu+jv+kw$, $\nabla = i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$, $\frac{d}{dt}=D$, denotes a complete differentiation),

$$\frac{d}{dt}V\nabla\sigma = V\nabla_1\sigma_2 S\sigma_1\nabla_2$$

of Mr M^c^Aulay's paper "Some General Theorems in Quaternion Integration," *Messenger of Mathematics*, vol. XIV. (1884), pp. 26—37; see p. 34.

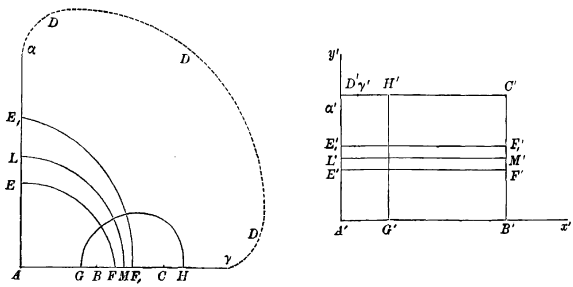
891.

ON THE BINODAL QUARTIC AND THE GRAPHICAL REPRESENTATION OF THE ELLIPTIC FUNCTIONS.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XIV. (1889), pp. 484—494. Read May 6, 1889.]

I APPROACH the subject from the question of the graphical representation of the elliptic functions: assuming as usual that the modulus is real, positive, and less than unity, and to fix the ideas considering the function sn (but the like considerations are applicable to the functions cn and dn), then the equation $x+iy=\operatorname{sn}(x'+iy')$ establishes a (1, 1) correspondence between the xy infinite quarter plane, and the $x'y'$ rectangle (sides K and K'): viz. to any given point $x+iy$, x and y each positive, there corresponds a single point $x'+iy'$, x', y' each positive and less than K, K' respectively: and conversely to any such point $x'+iy'$, there corresponds a single point $x+iy$, x and y each positive.

I draw in the $x'y'$-figure the rectangle $A'B'C'D'$ (sides K and K'), and in the xy-figure, I take on the axis of x, the points B, C where $AB=1$, $AC=\frac{1}{k}$: and the



point D at infinity. We have thus in the $x'y'$-figure the closed curve or contour

$A'B'C'D'A'$: and corresponding hereto we have in the xy-figure the closed curve or contour $ABCDA$, viz. here D is the point at infinity considered as a line always at infinity, extending from the point at infinity on the positive part of the axis of x, to the point at infinity on the positive part of the axis of y, the contour being thus AB, BC, CD (D at infinity on the axis of x); and then D (at infinity on the axis of y) A. And thus to a point P' describing successively the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$ there corresponds a point P describing successively the lines AB, BC, CD, DA: to P' at D' there corresponds P at D, viz. this is any point at infinity from D on the axis of x to D on the axis of y. There is no real breach of continuity: in further illustration, suppose that P', instead of actually coming to D', just cuts off the corner, viz. that it passes from a point γ' on $C'D'$ to a point α' on $D'A'$ (γ', α' each of them very near to D'): then P passes from a point γ very near D on the axis of x (that is, at a great distance from A) to a point α very near D on the axis of y (that is, at a great distance from A): and to the indefinitely small arc $\gamma'\alpha'$ described by P' there corresponds the indefinitely large arc $\gamma\alpha$ described by P.

We thus see that, if P' describe any arc $E'F'$ passing from a point E' of $A'D'$ to a point F' of $B'C'$, then P will describe an arc EF passing from a point E of AD to a point F of BC: and similarly, if P' describe any arc $G'H'$ passing from a point G' of $A'B'$ to a point H' of $C'D'$, then P will describe an arc GH passing from a point G of AB to a point H of CD.

Supposing $E'F'$ is a straight line parallel to $A'x'$, that is, cutting $A'D'$ and $B'C'$ each at right angles, then EF will be an arc cutting AD and BC each at right angles: and so if $G'H'$ is a straight line parallel to $A'y'$, that is, cutting $A'B'$ and $C'D'$ each at right angles, then GH will be an arc cutting AB and CD each at right angles: and moreover, since $E'F'$ and $G'H'$ cut each other at right angles, then also EF and GH cut each other at right angles.

Supposing, as above, that $E'F'$ and $G'H'$ are straight lines, we have EF and GH each of them the arc of a special bicircular quartic: the theory was in fact established in a very elegant manner in a memoir by Siebeck, "Ueber eine Gattung von Curven vierten Grades, welche mit den elliptischen Functionen zusammenhängen," *Crelle*, t. LVII. (1860), pp. 359—370, and t. LIX. (1861), pp. 173—184.

In particular, if P' describe the line $L'M'$ lying halfway between $A'B'$ and $D'C'$ (that is, if $A'L' = \frac{1}{2}K'$), then P will describe the circular quadrant LM, radius $\frac{1}{\sqrt{k}}$, viz. in this case the bicircular quartic degenerates into a circle twice repeated: and so if P' describe successively the lines $E'F'$ and $E_1'F_1'$ equidistant from $L'M'$ ($AE' = \frac{1}{2}K' - \eta$, $AE_1' = \frac{1}{2}K' + \eta$), then P will describe the arcs EF and E_1F_1 which are the images of each other in regard to the centre A and circular quadrant LM, and which together constitute the quadrant of one and the same bicircular quartic.

A bicircular quartic is of course a binodal quartic with the circular points at infinity for the two nodes: there is no real gain of generality in considering the

binodal quartic rather than the bicircular quartic, but I have preferred to do so, and I have accordingly introduced the term Binodal Quartic into the title of the present Memoir. I present in a compendious form the properties of the general curve, and I show how the curve is to be particularised so as to obtain from it the special bicircular quartics which present themselves as above in the graphical representation of the elliptic functions.

A binodal quartic has the Plückerian numbers

$$\begin{array}{cccccc} m & n & \delta & \kappa & \tau & \iota \\ =4 & 8 & 2 & 0 & 8 & 12. \end{array}$$

The number of tangents to the curve which can be drawn from either of the nodes is $n-4$, $=4$; and the pencil of tangents from the one node is homographic with the pencil of tangents from the other node. Call the nodes I and J: and let the tangents from I be called (a, b, c, d) and those from J be called (a', b', c', d'), then if the tangents which correspond to (a', b', c', d') respectively are (a, b, c, d), they may also be taken to be (b, a, d, c), (c, d, a, b) or (d, c, b, a): and considering the intersections of corresponding tangents, we have thus four tetrads of points, say the f-points, such that the points of each tetrad lie in a conic through the two nodes: and we have consequently four conics each passing through the two nodes, say these are the f-conics.

Starting as above with the correspondence (a, b, c, d), (a', b', c', d'), if the intersections of a and a', b and b', c and c', d and d' are called A, B, C, D respectively, then we have A, B, C, D for a tetrad of f-points, lying on the f-conic $(ABCD)$.

Writing AB for the two points, the intersections of IA, JB and of IB, JA respectively, and so in other cases, then the remaining three tetrads of f-points are

AB, CD lying on the f-conic (AB, CD),
AC, BD „ „ (AC, BD),
AD, BC „ „ (AD, BC).

The two points AB may be spoken of as the antipoints of A, B: and so in other cases.

Any two of the f-conics have in common the nodes I, J, and they therefore intersect in two points besides: at each of these the tangents to the two conics, and the lines to I, J respectively, form a harmonic pencil.

Consider the two tangents at I and the two tangents at J: we have a conic touching these four lines and passing through the tetrad of f-points, or what is the same thing, intersecting the f-conic in the four f-points: say this is a c-conic. There are thus four c-conics corresponding to the four f-conics respectively.

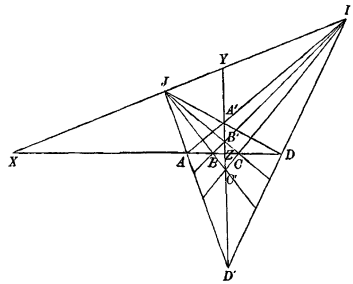
We may consider a variable conic, passing through the points I and J; and such that the tangents thereto at these points respectively meet on a point of a

c-conic: the variable conic and the corresponding f-conic each pass through the points I, J and they meet in two points besides: and the variable conic may be such that at each of these the tangents to the variable conic and the f-conic form with the lines drawn to the points I and J a harmonic pencil. The variable conic, as thus defined, contains a single variable parameter; and it has for its envelope the binodal quartic: the binodal quartic is thus in four different ways the envelope of a variable conic. This is of course Casey's Theorem for the fourfold generation of the bicircular quartic as the envelope of a variable circle.

One of the f-conics, say $(ABCD)$, may break up into the line IJ and a line, say the axis $(ABCD)$: we have then the four f-points A, B, C, D on this line. The other f-conics say (AB, CD), (AC, BD), (AD, BC) are proper conics as before: any one of these meets the line $(ABCD)$ in two points: and at each of these the line and the tangent to the conic form with the lines drawn to the points I and J a harmonic pencil. The binodal quartic is in this case said to have an axis.

But a second f-conic, say (AD, BC), may break up into the line IJ and a line, say the axis (AD, BC): we have then the four f-points AD, BC on this line. Writing moreover A', D' for the two points AD, or say $(AI, DJ) = A'$, $(AJ, DI) = D'$; and similarly B', C' for the two points BC, or say $(BI, CJ) = B'$, $(BJ, CI) = C'$; then the four f-points are A', B', C', D', and the axis (AD, BC) may be called $(A'B'C'D')$.

The relation of the f-points A, B, C, D and A', B', C', D' and of the two axes is as shown in the figure: taking X for the intersection of IJ with $(ABCD)$ and Y for



that of IJ with $(A'B'C'D')$, also Z for the intersection of the two axes, then X, Z are the sibiconjugate points of the involution AD, BC and Y, Z the sibiconjugate points of the involution $A'D'$, $B'C'$. The two axes intersect in Z, and form with the lines ZI, ZJ a pencil in involution.

The two remaining f-conics (AB, CD) and (AC, BD), or as they might also be called $(A'B', C'D')$ and $(A'C', B'D')$, are proper conics as before: they touch each other at the points I, J, and have for their common tangents at these points the lines ZI,

ZJ respectively. Each conic meets each axis in two points; and at each of these points the axis and the tangent to the conic form with the lines to I, J a harmonic pencil. The binodal quartic is in this case said to be biaxial. The point Z, which is the intersection of the two axes, may be called the centre.

In the case where an f-conic breaks up into the line IJ, and a line containing four f-points, say an f-line, the corresponding c-conic coincides with the f-conic, viz. it also breaks up into the line IJ and the f-line: the variable conic is a conic through the points I, J such that the tangents thereto at these points respectively meet on the f-line. Moreover, the variable conic must be such that at each of its intersections with the f-conic, that is, the f-line, the tangent to the variable conic and the f-line must be harmonics in regard to the lines drawn from the point to the points I, J respectively: but this condition is satisfied *ipso facto* for each of the intersections of the variable conic and the f-line. This depends on the theorem that, taking on a conic any three points P, I, J, then the tangent at P and the line drawn from P to the pole of IJ are harmonics in regard to the lines PI, PJ. Thus we have only three conditions for the variable conic, or, as above defined, it would in the case in question (of four f-points in a line) depend upon two variable parameters. There is really another condition—but what this in general is I have not ascertained: and this being so the variable conic in the case in question (of the four f-points in a line) depends upon a single variable parameter, and we have as before the bicircular quartic as the envelope. The foregoing is the axial case; in the biaxial case, the same thing happens in regard to the variable conics belonging to the two axes respectively. Thus in every case we have the fourfold generation of the curve as the envelope of a variable conic: only in the axial case, the variable conics belonging to the axis, and in the biaxial case the variable conics belonging to the two axes respectively, are not by the foregoing definitions completely defined. It will be seen further on how, in the case of the biaxial bicircular quartic, we complete the definition of the variable circles belonging to the two axes respectively.

Taking the points I, J to be the circular points at infinity, we have a bicircular quartic. The f-points are the foci, and the f-conics are circles, viz. we have 16 foci situate in fours upon four focal circles. The harmonic relation of two lines to the lines through I, J means of course that the lines cut at right angles; hence the focal circles cut each other at right angles: this must certainly be a known property, but it is not mentioned in Salmon's *Higher Plane Curves*, Ed. 3, Dublin, 1879, and I cannot find it in Darboux or Casey: it is given No. 81 in Lachlan's Memoir "On Systems of Circles and Spheres," *Phil. Trans.*, vol. CLXXVII. (1886), and I find it as a question in the *Educational Times*, March 1, 1889, 10034 (Prof. Morley). "Prove that, of the four focal circles of a circular cubic or bicircular quartic, any two are orthogonal, and the radii are connected by the relation $\Sigma(\mu^{-2})=0$." The theorem is not as well known as it should be.

The c-conics are confocal conics having for their real foci the so-called double-foci of the quartic (more accurately, the common foci are the four quadruple foci of the quartic); we have thus four conics corresponding to the four focal circles respectively,

each conic intersecting the corresponding circle in the four foci upon this circle. And we have then the quartic as the envelope of a variable circle having its centre upon one of these conics and cutting at right angles the corresponding focal circle: the bicircular quartic is thus generated in four different ways.

Instead of one of the focal circles, we may have a line or axis, and the quartic is then said to be axial: the foci on the axis may be any four points; and for a real curve they may be all real, or two real and two imaginary, or all four imaginary. The remaining focal circles are real or imaginary circles, cut by the axis at right angles, that is, having their centres on the axis, and cutting each other at right angles.

But instead of another of the focal circles, we may have a line or axis, and the quartic is then said to be biaxial: the two axes cut at right angles at a point which may be called the centre of the curve. The foci on each axis form pairs of points situate symmetrically in regard to the centre. If on one of the axes the foci are real, then on the other axis they form two imaginary conjugate pairs; and conversely: but if on one of the axes the foci are two of them real and the other two conjugate imaginaries, then this is so for the other axis also. There are thus only the two cases: 1°, foci on the one axis real, and on the other conjugate imaginaries; 2°, foci on each axis two of them real and the other two conjugate imaginaries: there is however a limiting case where on each axis two foci are united at the centre, the other two foci being real on the one axis and conjugate imaginaries on the other. The remaining two focal circles are real or imaginary circles, cutting each axis at right angles, that is, having their centres at the centre; and cutting each other at right angles, that is, having the sum of the squares of their radii $=0$.

The biaxial form of bicircular quartic is, in fact, that which presents itself in the theory of the representation of the elliptic functions.

I consider for a moment the case of a variable circle having its centre upon a given line, and cutting at right angles a given circle. The variable circles pass all of them through two fixed points, the antipoints of the intersections of the given line and circle, and which are thus real or imaginary according as the intersections of the given line and circle are imaginary or real. Hence, considering any one variable circle and the consecutive variable circle, these intersect in two real points, when the given line does not meet the given circle (meets it in two imaginary points); but when the given line meets the given circle in two real points, then the two variable circles intersect in two imaginary points: if however the given line touches the given circle, then the two variable circles touch each other. Taking now the curve of centres to be any given curve whatever, and considering one of the variable circles, and the consecutive variable circle, it at once appears that, if the tangent to the curve of centres at the centre of the variable circle does not meet the given circle, then the two variable circles intersect in two real points (which, if the tangent touch the given circle, unite in a single real point): but if the tangent to the curve of centres meets the given circle, then the two variable circles do not intersect. It hence appears that the real portions of the envelope arise exclusively

from those portions of the curve of centres which are such that at any point thereof the tangent to the curve of centres does not meet the given circle. In particular, if the given circle be real, and the curve of centres is a real ellipse enclosing the given circle, then the real portion of the envelope arises from the whole ellipse: but if the curve of centres be a real ellipse cutting the given circle in four real points, then drawing the four common tangents of the ellipse and circle, it is at once seen that there are on the ellipse two detached portions such that, at any point of either portion, the tangent to the ellipse does not meet the circle: and the real portions of the envelope arise exclusively from these portions of the ellipse.

In the case just referred to, there are on the ellipse four portions each lying outside the circle and terminating in the four intersections respectively of the ellipse and circle, such that at a point of any one of these portions the tangent to the ellipse meets the circle in two real points. Starting from the extremity of one of these portions of the ellipse and proceeding to the other extremity on the circle, the corresponding variable circles do not intersect each other, but each of them is a circle lying wholly inside that which immediately precedes it; and the variable circle becomes ultimately a point, viz. this point is a focus of the curve: this agrees with the foregoing statement that the f-conic intersects the circle in the four foci upon this circle. For the two portions of the ellipse which lie inside the circle, the variable circle is of course always imaginary. The like considerations apply to the case where the locus of the centre of the variable circle is a hyperbola or parabola. The foregoing remarks illustrate the actual generation of a bicircular quartic as the envelope of the variable circle.

Starting now from the equation

$$x + iy = \operatorname{sn}(x' + iy'),$$

we have

$$x = \frac{\operatorname{sn} x' \operatorname{cn} iy' \operatorname{dn} iy'}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'}, \quad iy = \frac{\operatorname{sn} iy' \operatorname{cn} x' \operatorname{dn} x'}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'},$$

or putting

$$\operatorname{sn} x' = p, \quad \operatorname{sn} iy' = iq;$$

these equations are

$$x = \frac{p\sqrt{1 + q^2 . 1 + k^2q^2}}{1 + k^2p^2q^2}, \quad y = \frac{q\sqrt{1 - p^2 . 1 - k^2p^2}}{1 + k^2p^2q^2};$$

whence also

$$x^2 + y^2 = \frac{p^2 + q^2}{1 + k^2p^2q^2} = r^2, \quad (\text{if } x^2 + y^2 \text{ be put } = r^2).$$

These equations, considering therein q as a given constant, and p as a variable parameter, determine the curve EF: and considering p as a given constant, and q as a variable parameter, they determine the curve GH. But the eliminations are easily effected; we have

$$p^2(1 - k^2q^2r^2) = r^2 - q^2, \quad q^2(1 - k^2p^2r^2) = r^2 - p^2.$$

Hence, for EF,

$$p^2 = \frac{r^2 - q^2}{1 - k^2 r^2 q^2}, \qquad 1 - p^2 = \frac{1 + q^2 - (1 + k^2 q^2) r^2}{1 - k^2 r^2 q^2},$$

$$1 - k^2 p^2 = \frac{1 + k^2 q^2 - k^2 (1 + q^2) r^2}{1 - k^2 r^2 q^2}, \quad 1 + k^2 p^2 q^2 = \frac{1 - k^2 q^4}{1 - k^2 r^2 q^2},$$

and consequently

$$x = \frac{\sqrt{1 + q^2 . 1 + k^2 q^2}}{1 - k^2 q^4} \sqrt{r^2 - q^2 . 1 - k^2 r^2 q^2},$$

$$y = \frac{q \sqrt{1 + q^2 - (1 + k^2 q^2) r^2} \sqrt{1 + k^2 q^2 - k^2 (1 + q^2) r^2}}{1 - k^2 q^4},$$

giving x and y each of them in terms of r. And from the first of these we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where

$$2A = \frac{1 + q^2}{1 + k^2 q^2} + \frac{1}{k^2} \frac{1 + k^2 q^2}{1 + q^2}, \qquad 2B = q^2 + \frac{1}{k^2 q^2}.$$

Similarly, for GH,

$$q^2 = \frac{r^2 - p^2}{1 - k^2 r^2 p^2}, \qquad 1 + q^2 = \frac{(1 - p^2) + r^2 (1 - k^2 p^2)}{1 - k^2 r^2 p^2},$$

$$1 + k^2 q^2 = \frac{1 - k^2 p^2 + k^2 (1 - p^2) r^2}{1 - k^2 r^2 p^2}, \quad 1 + k^2 p^2 q^2 = \frac{1 - k^2 p^4}{1 - k^2 r^2 p^2};$$

and consequently

$$x = \frac{p \sqrt{(1 - p^2) + (1 - k^2 p^2) r^2} \sqrt{1 - k^2 p^2 + k^2 (1 - p^2) r^2}}{1 - k^2 p^4},$$

$$y = \frac{\sqrt{1 - p^2 . 1 - k^2 p^2}}{1 - k^2 p^4} \sqrt{r^2 - p^2 . 1 - k^2 r^2 p^2},$$

giving x, y each of them in terms of r. And from the second of them we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where

$$2A = p^2 + \frac{1}{k^2 p^2}, \qquad 2B = -\frac{1 - p^2}{1 - k^2 p^2} - \frac{1}{k^2} \frac{1 - k^2 p^2}{1 - p^2}.$$

Consider, in particular, the case where the line EF is the midway line LM: here $y' = \frac{1}{2}K'$, and thence $iq = \operatorname{sn} iy' = \operatorname{sn} \frac{1}{2} iK' = \frac{i}{\sqrt{k}}$, that is, $q = \frac{1}{\sqrt{k}}$: and we thence obtain $A = B = \frac{1}{k}$; the equation of the bicircular quartic is

$$(x^2 + y^2)^2 - \frac{2}{k}(x^2 + y^2) + \frac{1}{k^2} = 0,$$

viz. this is the circle $x^2+y^2-\frac{1}{k}=0$ twice repeated. As a direct verification, observe that we have here

$$x+iy=\operatorname{sn}(x'+\tfrac{1}{2}iK')=\frac{\frac{1+k}{\sqrt{k}}\operatorname{sn}x'+\frac{i}{\sqrt{k}}\operatorname{cn}x'\operatorname{dn}x'}{1+k^2.\frac{1}{k}\operatorname{sn}^2x'}=\frac{(1+k)p+i\sqrt{1-p^2.1-k^2p^2}}{\sqrt{k}(1+kp^2)},$$

and hence

$$x^2+y^2=\frac{(1+k)^2p^2+1-(1+k^2)p^2+k^2p^4}{k(1+2kp^2+k^2p^4)}=\frac{1}{k},$$

as it should be.

Reverting to the equation

$$x+iy=\operatorname{sn}(x'+iy'),$$

I write successively

$$y'=\tfrac{1}{2}K'-z',\quad \operatorname{sn}iy'=iq_1=\operatorname{sn}i(\tfrac{1}{2}K'-z'),$$

and

$$y'=\tfrac{1}{2}K'+z',\quad \operatorname{sn}iy'=iq_2=\operatorname{sn}i(\tfrac{1}{2}K'+z');$$

we then have

$$iq_1.iq_2=\operatorname{sn}i(\tfrac{1}{2}K'-z')\operatorname{sn}i(\tfrac{1}{2}K'+z'),\quad =-\frac{1}{k},$$

that is, $q_1q_2=\frac{1}{k}$; hence for q writing q_1 or q_2, we have in each case the same values of A and B; that is, we have the same bicircular quartic corresponding to the lines $E'F'$ and $E_1'F_1'$, equidistant from the line $L'M'$: but to one of these lines there corresponds the quadrant lying inside, to the other that lying outside, the circular quadrant

$$x^2+y^2-\frac{1}{k}=0.$$

The curve

$$(x^2+y^2)^2-2Ax^2-2By^2+\frac{1}{k^2}=0$$

is in four different ways the envelope of a variable circle: viz. the circle may have its centre on a conic $\alpha x^2+\beta y^2-1=0$, and cut at right angles one of the circles

$$x^2+y^2\pm\frac{1}{k}=0;$$

or it may have its centre on the axis of x, or on the axis of y. The circle, having its centre on either axis, cuts this axis at right angles; but this condition being *ipso facto* satisfied, we do not thereby determine the radius of the circle having for its centre a given point on the axis: the expression for the radius must be sought for independently.

Write for shortness

$$\lambda^2=\frac{B-\frac{1}{k^2}}{A-B},$$

and consider the circle

$$x^2 + y^2 - 2\alpha x = \lambda\sqrt{A - B - 2\alpha^2} + B,$$

where α is a variable parameter. Differentiating, we have

$$x = \frac{\lambda\alpha}{\sqrt{A - B - 2\alpha^2}}, \text{ giving } \alpha = \frac{x\sqrt{A - B}}{\sqrt{2x^2 + \lambda^2}}:$$

the equation of the circle then gives

$$x^2 + y^2 - B = \frac{\alpha}{x}(2x^2 + \lambda^2), \quad = \sqrt{A - B}\,\sqrt{2x^2 + \lambda^2};$$

that is,

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 = B^2 + \lambda^2(A - B), \quad = -\frac{1}{k^2};$$

or we have

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0$$

as the envelope of the variable circle

$$x^2 + y^2 - 2\alpha x = \frac{\sqrt{B - \frac{1}{k^2}}}{\sqrt{A - B}}\sqrt{A - B - 2\alpha^2} + B,$$

having its centre on the axis of x.

And similarly, the same quartic curve is the envelope of the variable circle

$$x^2 + y^2 - 2\beta y = \frac{\sqrt{A - \frac{1}{k^2}}}{\sqrt{B - A}}\sqrt{B - A - 2\beta^2} + A,$$

having its centre on the axis of y.

If we have in like manner the equation $x + iy = \operatorname{cn}(x' + iy')$, then writing, as before,

$$\operatorname{sn} x' = p, \quad \operatorname{sn} iy' = iq,$$

we find

$$x^2 = \frac{(1 - p^2)(1 + q^2)}{(1 + k^2p^2q^2)^2}, \quad y^2 = \frac{p^2q^2(1 - k^2p^2)(1 + k^2q^2)}{(1 + k^2p^2q^2)^2},$$

and hence

$$r^2 = \frac{1 - p^2 + q^2 - k^2p^2q^2}{1 + k^2p^2q^2}, \text{ where } r^2 = x^2 + y^2.$$

For the curve EF corresponding to a line $E'F'$ parallel to the axis of x', we have to eliminate p from these equations; the expression for r^2 gives

$$p^2 = \frac{1 + q^2 - r^2}{1 + k^2q^2 + k^2q^2r^2}, \qquad 1 - p^2 = \frac{-(1 - k^2)q^2 + r^2(1 + k^2q^2)}{1 + k^2q^2 + k^2q^2r^2},$$

$$1 - k^2p^2 = \frac{(1 - k^2) + k^2r^2(1 + q^2)}{1 + k^2q^2 + k^2q^2r^2}, \qquad 1 + k^2p^2q^2 = \frac{1 + 2k^2q^2 + k^2q^4}{1 + k^2q^2 + k^2q^2r^2},$$

and thence

$$x = \frac{\sqrt{1+q^2}\sqrt{-(1-k^2)q^2 + r^2(1+k^2q^2)}\sqrt{1+k^2q^2+k^2q^2r^2}}{1+2k^2q^2+k^2q^4},$$

$$y = \frac{q\sqrt{1+q^2-r^2}\sqrt{1+k^2q^2}\sqrt{1-k^2+k^2r^2(1+q^2)}}{1+2k^2q^2+k^2q^4},$$

from which we deduce

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 - \frac{k'^2}{k^2} = 0,$$

where

$$2A = \frac{-1+2k^2+2k^2q^2+k^2q^4}{k^2(1+q^2)}, \quad 2B = \frac{-1-2k^2q^2+k^2(1-2k^2)q^4}{k^2q^2(1+k^2q^2)},$$

which is a bicircular quartic of the foregoing form.

Similarly, for the curve GH corresponding to a line $G'H'$ parallel to the axis of y', we have to eliminate q: the expression for r^2 gives

$$q^2 = \frac{-1+p^2+r^2}{1-k^2p^2-k^2p^2r^2}, \qquad 1+q^2 = \frac{(1-k^2)p^2+r^2(1-k^2p^2)}{1-k^2p^2-k^2p^2r^2},$$

$$1+k^2q^2 = \frac{(1-k^2)+k^2r^2(1-p^2)}{1-k^2p^2-k^2p^2r^2}, \quad 1+k^2p^2q^2 = \frac{1-2k^2p^2+k^2p^4}{1-k^2p^2-k^2p^2r^2}.$$

Hence

$$x = \frac{\sqrt{1-p^2}\sqrt{(1-k^2)p^2+r^2(1-k^2p^2)}\sqrt{1-k^2p^2-k^2p^2r^2}}{1-2k^2p^2+k^2p^4},$$

$$y = \frac{p\sqrt{-1+p^2+r^2}\sqrt{1-k^2p^2}\sqrt{1-k^2+k^2r^2(1-p^2)}}{1-2k^2p^2+k^2p^4},$$

from which we deduce

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 - \frac{k'^2}{k^2} = 0,$$

where

$$2A = \frac{-1+2k^2-2k^2p^2+k^2p^4}{k^2(1-p^2)}, \quad 2B = \frac{1-2k^2p^2-k^2(1-2k^2)p^4}{k^2p^2(1-k^2p^2)},$$

which is again a bicircular quartic of the foregoing form. And we have the like results for the equation

$$x + iy = \operatorname{dn}(x' + iy');$$

so that, for the sn, cn, and dn, the curve EF or GH is in each case a biaxial bicircular quartic of the form

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 + C = 0.$$

892.

NOTE ON THE ORTHOMORPHIC TRANSFORMATION OF A CIRCLE INTO ITSELF.

[From the *Proceedings of the Edinburgh Mathematical Society*, vol. VIII. (1890), pp. 91, 92.]

THE following is, of course, substantially well known, but it strikes me as rather pretty:—to find the orthomorphic transformation of the circle

$$x^2 + y^2 - 1 = 0$$

into itself. Assume this to be

$$x_1 + iy_1 = \frac{A(x+iy)+B}{1+C(x+iy)}.$$

Then, writing A', B', C' for the conjugates of A, B, C, we have

$$x_1 - iy_1 = \frac{A'(x-iy)+B'}{1+C'(x-iy)};$$

and then

$$x_1^2 + y_1^2 = \frac{AA'(x^2+y^2)+AB'(x+iy)+A'B(x-iy)+BB'}{1+C(x+iy)+C'(x-iy)+CC'(x^2+y^2)},$$

which should be an identity for $x^2+y^2=1$, $x_1^2+y_1^2=1$.

Evidently $C = AB'$, whence $C' = A'B$; and the equation then is

$$1 + AA'BB' = AA' + BB',$$

that is,

$$(1-AA')(1-BB') = 0.$$

But $BB' = 1$ gives the illusory result

$$x_1 + iy_1 = B,$$

therefore

$$1 - AA' = 0;$$

and the required solution thus is

$$x_1 + iy_1 = \frac{A(x+iy)+B}{1+AB'(x+iy)};$$

where A is a unit-vector (say $A = \cos\lambda + i\sin\lambda$) and B, B' are conjugate vectors. Or, writing $B = b + i\beta$, $B' = b - i\beta$, the constants are λ, b, β; three constants as it should be.

893.

THE BITANGENTS OF THE QUINTIC.

(LETTER FROM PROFESSOR CAYLEY TO MR HEAL.)

[From the *Annals of Mathematics*, vol. v. (1890), pp. 109, 110.]

DEAR SIR,

I have to thank you very much for your letter concerning the bitangents of the quintic, and am glad you have obtained more simple numerical coefficients than you had at first. I do not see my way to verifying your result, but assuming it to be correct, it is a very interesting and valuable one. The relation to Salmon's *Higher Plane Curves* (p. 351) is rather curious, viz. the result there given is of the deg-order 18-54, with an extraneous factor $(\alpha x+\beta y+\gamma z)^6$, the rejection of which would reduce it to the proper form, deg-order 18-48; whereas yours is of deg-order 24-66, with an extraneous factor H^3, the rejection of which would reduce it to the same proper form, deg-order 18-48.

But there is another known solution, not by any means a handy one, but which has no extraneous factor*, viz. if

$$\Omega = D^3H - 4D^3H_1 + 6D^3H_2,$$

then the required equation is

$$F\Omega = 0,$$

where the facients of the reciprocant are the first derived functions,

$$\partial_x U,\quad \partial_y U,\quad \partial_z U.$$

Say this is

$$(B^2C^2 - \ldots)(\partial_x U)^6 = 0;$$

the coefficients A, B, C, &c., are those of Ω, viz. these are of deg-order 3-6, and $\partial_x U$, &c., are of deg-order 1-4; so that the deg-order is

$$4\,(3\text{-}6) + 6\,(1\text{-}4) = (12\text{-}24) + (6\text{-}24) = (18\text{-}48),$$

as it should be. It might be worth while to further consider this form, but I doubt whether, practically, the whole question is not too difficult to be worth working at.

I remain, dear sir, yours very sincerely,

A. CAYLEY.

Cambridge, January 17, 1890.

* See *Phil. Trans.*, vol. CXLIX. pp. 193—212, [260].

894.

THE INVESTIGATION BY WALLIS OF HIS EXPRESSION FOR π.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII. (1889), pp. 165—169.]

THE following is in effect the investigation by Wallis in the *Arithmetica Infinitorum* (Oxford, 1656) of his well-known expression for π. He obtains the series of equations which in modern notation are

$$\int_0^1 (x-x^2)^0\,dx = 1,$$

$$\int_0^1 (x-x^2)^1\,dx = \tfrac{1}{2}\cdot\tfrac{1}{3},$$

$$\int_0^1 (x-x^2)^2\,dx = \tfrac{1}{6}\cdot\tfrac{1}{5},$$

$$\int_0^1 (x-x^2)^3\,dx = \tfrac{1}{20}\cdot\tfrac{1}{7},$$

or say, in general,

$$\int_0^1 (x-x^2)^n\,dx = \frac{1}{\phi(n)}\,\frac{1}{2n+1}.$$

In the case $n=\frac{1}{2}$, the integral gives the area of a semicircle, diameter $=1$, viz. this is $=\frac{1}{8}\pi$, or we have $\frac{1}{8}\pi = \dfrac{1}{2\phi(\frac{1}{2})}$; or writing with him $\square = \dfrac{4}{\pi}$, this is $\square = \phi(\frac{1}{2})$, where $\phi(0)$, $\phi(1)$, $\phi(2)$, ... are the series of numbers 1, 2, 6, 20, ..., which are in fact the middle binomial coefficients of the several even powers, and are the diagonal numbers in the table which he subsequently considers. As regards the form in which Wallis exhibits these results, his theorem CXXXIII. is:

Si exponatur series Primanorum mulctata serie secundanorum, Residuorum Quadrata, Cubi, Biquadrata, &c. ad seriem Æqualium rationem habebunt cognitam.

And then, after an algebraical calculation,

$$\begin{array}{c|c|c}
\frac{1}{2}-\frac{1}{3}=\frac{1}{6} & \frac{1}{3}-\frac{2}{4}+\frac{1}{5}=\frac{1}{30} & \frac{1}{4}-\frac{3}{5}+\frac{3}{6}-\frac{1}{7}=\frac{1}{140} \\
\frac{1}{2.3}=\frac{1}{6} & \frac{1.2}{3.4.5}=\frac{1}{30} & \frac{1.2.3}{4.5.6.7}=\frac{1}{140} \\
\frac{1}{2.3}=\frac{1}{6} & \frac{1}{2.3}\cdot\frac{4}{4.5}=\frac{1}{30} & \frac{1}{2.3}\cdot\frac{4}{4.5}\cdot\frac{9}{6.7}=\frac{1}{140}
\end{array}$$

...et sic deinceps; continue multiplicando Numeratores per numeros quadratos et denominatores per binos continue sequentes arithmetice proportionales—viz. he thus explains the law of his calculated numbers.

He then forms the table

	1		$\frac{1}{2}$		$\frac{3}{8}$		$\frac{15}{48}$		$\frac{105}{384}$	$-\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	0
	1		$\frac{3}{2}$		$\frac{15}{8}$		$\frac{105}{48}$		$\frac{945}{384}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	1
	1		$\frac{5}{2}$		$\frac{35}{8}$		$\frac{315}{48}$		$\frac{3465}{384}$	$\frac{3}{2}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	2
	1		$\frac{7}{2}$		$\frac{63}{8}$		$\frac{693}{48}$		$\frac{9009}{384}$	$\frac{5}{2}$
$\frac{15}{48}$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20	$\frac{1287}{48}$	35	3
	1		$\frac{9}{2}$		$\frac{99}{8}$		$\frac{1287}{48}$		$\frac{19305}{384}$	$\frac{7}{2}$
$\frac{105}{384}$	1	$\frac{945}{384}$	5	$\frac{3465}{384}$	15	$\frac{9009}{384}$	35	$\frac{19305}{384}$	70	4
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	

where the figures in the second column are $=1$, those in the fourth column are $=\frac{n+1}{1}$, those of the sixth column are $=\frac{(n+1)(n+2)}{1.2}$, and so on: n being in each case the rank in the column as shown by the right-hand outside column, viz. these ranks are $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, ... at intervals of $\frac{1}{2}$. And the several even lines contain the same figures as the several even columns respectively.

Wallis then remarks that, if in any line the first and second terms respectively are A and 1, then the even lines are as follows, viz. the second line is

$$A,\ 1,\ A\frac{1}{1},\ 1.\frac{2}{2},\ A\frac{1.3}{1.3},\ 1.\frac{2.4}{2.4},\ A\frac{1.3.5}{1.3.5},\ 1.\frac{2.4.6}{2.4.6},\ \ldots,$$

the fourth line is

$$A,\ 1,\ A\frac{3}{1},\ 1.\frac{4}{2},\ A\frac{3.5}{1.3},\ 1.\frac{4.6}{2.4},\ A\frac{3.5.7}{1.3.5},\ 1.\frac{4.6.8}{2.4.6},\ \ldots,$$

the sixth line is

$$A,\ 1,\ A\frac{5}{1},\ 1.\frac{6}{2},\ A\frac{5.7}{1.3},\ 1.\frac{6.8}{2.4},\ A\frac{5.7.9}{1.3.5},\ 1.\frac{6.8.10}{2.4.6},\ \ldots;$$

and this being so, he completes the table by inserting a term $\square$ in the diagonal line between the 1 and 2, and calculating the odd lines as follows: viz. if in each case A is the first term and B the second term of the line, then the complete rule is:

the first line is

$$A,\ 1,\ A\frac{0}{1},\ 1.\frac{1}{2},\ A\frac{0.2}{1.3},\ 1.\frac{1.3}{2.4},\ A\frac{0.2.4}{1.3.5},\ 1.\frac{1.3.5}{2.4.6},$$

the second line is

$$A,\ 1,\ A\frac{1}{1},\ 1.\frac{2}{2},\ A\frac{1.3}{1.3},\ 1.\frac{2.4}{2.4},\ A\frac{1.3.5}{1.3.5},\ 1.\frac{2.4.6}{2.4.6},$$

the third line is

$$A,\ 1,\ A\frac{2}{1},\ 1.\frac{3}{2},\ A\frac{2.4}{1.3},\ 1.\frac{3.5}{2.4},\ A\frac{2.4.6}{1.3.5},\ 1.\frac{3.5.7}{2.4.6},$$

the fourth line is

$$A,\ 1,\ A\frac{3}{1},\ 1.\frac{4}{2},\ A\frac{3.5}{1.3},\ 1.\frac{4.6}{2.4},\ A\frac{3.5.7}{1.3.5},\ 1.\frac{4.6.8}{2.4.6},$$

the fifth line is

$$A,\ 1,\ A\frac{4}{1},\ 1.\frac{5}{2},\ A\frac{4.6}{1.3},\ 1.\frac{5.7}{2.4},\ A\frac{4.6.8}{1.3.5},\ 1.\frac{5.7.9}{2.4.6};$$

and so on, the rule for the odd lines being thus an interpolation from that for the even lines. Observe that, in the first line A is $=\infty$, and that, in the several odd terms after the first, $A\frac{0}{1}$ is $=1$.

The table thus calculated (I have given here the fractions in their least terms) is

∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	$\frac{5}{16}$	$\frac{8}{35}\square$	$\frac{35}{128}$	$-\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	0
$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{5}\square$	$\frac{35}{16}$	$\frac{64}{35}\square$	$\frac{315}{128}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	1
$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$\frac{105}{16}$	$\frac{128}{21}\square$	$\frac{1155}{128}$	$\frac{3}{2}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	2
$\frac{4}{15}\square$	1	$\frac{8}{5}\square$	$\frac{7}{2}$	$\frac{64}{15}\square$	$\frac{63}{8}$	$\frac{128}{15}\square$	$\frac{231}{16}$	$\frac{512}{35}\square$	$\frac{3003}{128}$	$\frac{5}{2}$
$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20	$\frac{429}{16}$	35	3
$\frac{8}{35}\square$	1	$\frac{64}{35}\square$	$\frac{9}{2}$	$\frac{128}{21}\square$	$\frac{99}{8}$	$\frac{512}{35}\square$	$\frac{429}{16}$	$\frac{1024}{35}\square$	$\frac{6435}{128}$	$\frac{7}{2}$
$\frac{35}{128}$	1	$\frac{315}{128}$	5	$\frac{1155}{128}$	15	$\frac{3003}{128}$	35	$\frac{6435}{128}$	70	4
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	

in which table $\square$, as the term interpolated between the diagonal terms 1, 2, denotes the value $\frac{4}{\pi}$ as before.

The third line of the table is

$$\frac{1}{2}\square,\quad 1,\quad \square,\quad \frac{3}{2},\quad \frac{4}{3}\square,\quad \frac{3.5}{2.4},\quad \frac{4.6}{3.5}\square,\quad \frac{3.5.7}{2.4.6},\quad \frac{4.6.8}{3.5.7}\square,\quad \frac{3.5.7.9}{2.4.6.8},\ \ldots.$$

The successive even terms continually increase but tend to equality, and in like manner the successive odd terms continually increase but tend to equality; it seems to have been assumed that, for any three consecutive terms x, y, z, we have $\frac{y}{x} > \frac{z}{y}$, that is, $y^2 > xz$. Taking this to be so, we have

$$1 > \frac{1}{2}\square^2,\quad \square^2 > \frac{3}{2},\quad \frac{3^2}{2^2} > \frac{4}{3}\square^2,\quad \frac{4^2}{3^2}\square^2 > \frac{3^2.5}{2^2.4},\quad \frac{3^2.5^2}{2^2.4^2} > \frac{4^2.6}{3^2.5}\square^2,\ \ldots,$$

and these equations give $\square$

less than	greater than
$\sqrt{\frac{2}{1}}$,	$\sqrt{\frac{3}{2}}$,
$\frac{3^2}{2.4}\sqrt{\frac{4}{3}}$,	$\frac{3^2}{2.4}\sqrt{\frac{5}{4}}$,
$\frac{3^2.5^2}{2.4^2.6}\sqrt{\frac{6}{5}}$,	$\frac{3^2.5^2}{2.4^2.6}\sqrt{\frac{7}{6}}$,
$\frac{3^2.5^2.7^2}{2.4^2.6^2.8}\sqrt{\frac{8}{7}}$,	$\frac{3^2.5^2.7^2}{2.4^2.6^2.8}\sqrt{\frac{9}{8}}$,

limits which tend continually to equality. We thus have

$$\square = \frac{4}{\pi},\quad = \frac{3.3.5.5.7.7\ldots}{2.4.4.6.6.8\ldots},$$

the number of factors in the numerator being always equal to the number in the denominator, and the accuracy of the approximation increasing with the number of factors.

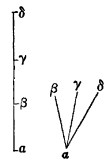
It is to be remarked that for a square; row m and column n; m or $n = -\frac{1}{2}$, 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, ... as before; the term of the square is in general $\Pi(m+n) \div \Pi(m)\,\Pi(n)$; thus $m = n = \frac{1}{2}$, the term is $\Pi(1) \div \{\Pi(\frac{1}{2})\}^2$, $= 1 \div (\frac{1}{2}\pi)^2 = \frac{4}{\pi}$, $= \square$; $m = 3$, $n = \frac{1}{2}$ it is $\Pi(\frac{7}{2}) \div \Pi(3)\,\Pi(\frac{1}{2})$, $= \frac{7}{2}.\frac{5}{2}.\frac{3}{2} \div 6$, $= \frac{105}{48}$; and so in any other case.

895.

A THEOREM ON TREES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII. (1889), pp. 376—378.]

THE number of trees which can be formed with $n+1$ given knots $\alpha, \beta, \gamma, \ldots$ is $=(n+1)^{n-1}$; for instance $n=3$, the number of trees with the 4 given knots $\alpha, \beta, \gamma, \delta$ is $4^2=16$, for in the first form shown in the figure the $\alpha, \beta, \gamma, \delta$ may be arranged



$12+4=16$

in 12 different orders ($\alpha\beta\gamma\delta$ being regarded as equivalent to $\delta\gamma\beta\alpha$), and in the second form any one of the 4 knots $\alpha, \beta, \gamma, \delta$ may be in the place occupied by the α: the whole number is thus $12+4, =16$.

Considering for greater clearness a larger value of n, say $n=5$, I state the particular case of the theorem as follows:

No. of trees $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)=$ No. of terms of $(\alpha+\beta+\gamma+\delta+\epsilon+\zeta)^4\,\alpha\beta\gamma\delta\epsilon\zeta, =6^4, =1296$,

and it will be at once seen that the proof given for this particular case is applicable for any value whatever of n.

I use for any tree whatever the following notation: for instance, in the first of the forms shown in the figure, the branches are $\alpha\beta$, $\beta\gamma$, $\gamma\delta$; and the tree is said to be $\alpha\beta^2\gamma^2\delta$ (viz. the knots α, δ occur each once, but β, γ each twice); similarly in the second of the same forms, the branches are $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, and the tree is said

to be $\alpha^3\beta\gamma\delta$ (viz. the knot α occurs three times, and the knots β, γ, δ each once). And so in other cases.

This being so, I write

$$(\alpha+\beta+\gamma+\delta+\epsilon+\zeta)^4\,\alpha\beta\gamma\delta\epsilon\zeta = \left.\begin{array}{rrr} 1 & \alpha^4 & 6 \\ +\,4 & \alpha^3\beta & 30 \\ +\,6 & \alpha^2\beta^2 & 15 \\ +\,12 & \alpha^2\beta\gamma & 60 \\ +\,24 & \alpha\beta\gamma\delta & 15 \end{array}\right\}\alpha\beta\gamma\delta\epsilon\zeta;\quad \begin{array}{r} 6 \\ 120 \\ 90 \\ 720 \\ 360 \\ \hline 1296, \end{array}$$

where the numbers of the left-hand column are the polynomial coefficients for the index 4; and the numbers of the right-hand column are the numbers of terms of the several types, 6 terms α^4, 30 terms $\alpha^3\beta$, 15 terms $\alpha^2\beta^2$, &c.: the products of the corresponding terms of the two columns give the outside column 6, 120, 90, &c.; and the sum of these numbers is of course 6^4, $=1296$.

It is to be shown that we have

1 tree $\alpha^4 . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^5\beta\gamma\delta\epsilon\zeta)$; 4 trees $\alpha^3\beta . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^4\beta^2\gamma\delta\epsilon\zeta)$, ...,

24 trees $\alpha\beta\gamma\delta . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^2\beta^2\gamma^2\delta^2\epsilon\zeta)$:

for this being so, then by the mere interchange of letters, the numbers 1, 4, 6, ... of the left-hand column have to be multiplied by the numbers 6, 30, 15, ... of the right-hand column, and we have the numbers in the outside column, the sum of which is $=1296$ as above.

Start with the last term $\alpha\beta\gamma\delta . \alpha\beta\gamma\delta\epsilon\zeta$, $=\alpha^3\beta^2\gamma^2\delta^2\epsilon\zeta$. We have the trees

$$\epsilon\alpha\beta\gamma\delta\zeta\ (=\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \delta\zeta),$$

where the α, β, γ, δ may be written in any one of the 24 orders, and the number of such trees is thus $=24$. If we consider only the 12 orders ($\alpha\beta\gamma\delta$ being regarded as equivalent to $\delta\gamma\beta\alpha$), then the ϵ, ζ may be interchanged; and the number is thus 2×12, $=24$ as before.

Now for the δ of $\alpha\beta\gamma\delta$ substitute α, or consider the form $\alpha\beta\gamma\alpha . \alpha\beta\gamma\delta\epsilon\zeta$, $=\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$. We see at once in the form $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \delta\zeta$, which one it is of the two δ's that must be changed into α: in fact, changing the first δ, we should have $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\alpha . \delta\zeta$ which contains a circuit $\alpha\beta\gamma$, and a detached branch $\delta\zeta$, and is thus *not* a tree: changing the second δ, we have $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \alpha\zeta$ which is a tree $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$, $=\alpha\zeta . \alpha\epsilon . \alpha\beta . \beta\gamma . \gamma\delta$. And similarly for any other order of the $\alpha\beta\gamma\delta$, there is in each case only one of the δ's which can be changed into α; and thus from each of the 24 forms we obtain a tree $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$. But dividing the 24 forms into the $12+12$ forms corresponding to the interchange of the letters ϵ, ζ, then the first 12 forms, and the second 12 forms, give each of them the same trees $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$; and the number of these trees is thus $\frac{1}{2} . 24$, $=12$.

And in like manner reducing the $\alpha\beta\gamma\delta$ to $\alpha^2\beta^2$, $\alpha^3\beta$ or α^4, we obtain in each case the number of trees equal to the proper sub-multiple of 24, viz. 6, 4, 1 in the three cases respectively (for the last case this is obvious, viz. there is 1 tree $\alpha^5\beta\gamma\delta\epsilon\zeta$, $=\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon\,.\,\alpha\zeta$); and the subsidiary theorem is thus proved. Hence the original theorem is true: as already remarked, it is easy to see that the proof is perfectly general.

The theorem is one of a set as follows:

Let $(\lambda, \alpha, \beta, \gamma, \ldots)$ denote as above the trees with the given knots $\lambda, \alpha, \beta, \gamma, \ldots$; $(\lambda+\mu, \alpha, \beta, \gamma, \ldots)$ the pairs of trees with the given knots $\lambda, \mu, \alpha, \beta, \gamma, \ldots$, the knots λ, μ belonging always to different trees; $(\lambda+\mu+\nu, \alpha, \beta, \gamma, \ldots)$ the triads of trees with the given knots $\lambda, \mu, \nu, \alpha, \beta, \gamma, \ldots$, the knots λ, μ, ν always belonging to different trees; and so on: then if $i+1$ be the number of the knots $\lambda, \mu, \nu, \ldots$, and n the number of the knots $\alpha, \beta, \gamma, \ldots$, the number of trees or pairs, or triads, &c., of trees is $=(i+1)(i+n+1)^{n-1}$. In particular, if $i=0$, then n being the number of knots $\alpha, \beta, \gamma, \ldots$, and therefore $n+1$ the whole number of knots $\lambda, \alpha, \beta, \gamma, \ldots$, the number of trees is $=(n+1)^{n-1}$ as before.

As a simple example, consider the pairs $(\lambda+\mu, \alpha, \beta)$: here $i=1$, $n=2$, and we have $(i+1)(i+n+1)^{n-1}=2\,.\,4, =8$: in fact, the pairs of trees are

$$(\lambda\alpha, \alpha\beta, \mu), (\lambda\beta, \beta\alpha, \mu), (\lambda\alpha, \lambda\beta, \mu),$$
$$(\mu\alpha, \alpha\beta, \lambda), (\mu\beta, \beta\alpha, \lambda), (\mu\alpha, \mu\beta, \lambda); (\lambda\alpha, \mu\beta), (\lambda\beta, \mu\alpha).$$

We may arrange the trees $(\alpha, \beta, \gamma, \delta, \epsilon)$ as follows:

$$\begin{array}{llll}
(\alpha, \beta, \gamma, \delta, \epsilon) = & \alpha\beta & (\beta, \gamma, \delta, \epsilon); & 125 = \quad 4\times 1\,.\,4^3 = 64 \\
& +\alpha\beta\,.\,\alpha\gamma & (\beta+\gamma, \delta, \epsilon) & \quad\quad +6\times 2\,.\,4^1 \quad 48 \\
& +\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta & (\beta+\gamma+\delta, \epsilon) & \quad\quad +4\times 3\,.\,4^0 \quad 12 \\
& +\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon & & \quad\quad +1 \quad\quad\quad 1 \\
& & & \quad\quad\quad\quad\quad\quad 125,
\end{array}$$

viz. to obtain the trees $(\alpha, \beta, \gamma, \delta, \epsilon)$, we join on the branch $\alpha\beta$ to any tree $(\beta, \gamma, \delta, \epsilon)$: the branches $\alpha\beta, \alpha\gamma$ to any pair of trees $(\beta+\gamma, \delta, \epsilon)$; the branches $\alpha\beta, \alpha\gamma, \alpha\delta$ to any triad of trees $(\beta+\gamma+\delta, \epsilon)$; and take lastly the tree $\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon$: the knots $\beta, \gamma, \delta, \epsilon$ being then interchanged in every possible manner. The whole number of trees 125 is thus obtained as $=64+48+12+1$; the theorem is of course perfectly general.

The foregoing theory in effect presents itself in a paper by Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Crelle*, t. LVII. (1860), pp. 111—121, viz. Borchardt there considers a certain determinant, composed of the elements $10, 12, \ldots, 1n, 20, 21, 23, \ldots, 2n, \ldots, n0, n1, \ldots, nn-1$, and represented by means of the trees $(0, 1, 2, \ldots, n)$; the branches of the tree being the aforesaid elements, and the tree being regarded as equal to the product of the several branches: the number of terms of the determinant is thus $=(n+1)^{n-1}$ as above.

896.

A TRANSFORMATION IN ELLIPTIC FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIV. (1890), pp. 259—262.]

THE formula in question is given in Klein's Memoir "Ueber hyperelliptische Sigmafunctionen," *Math. Ann.* t. XXVII. (1886), pp. 431—464, see p. 454, in the form

$$u = \int_y^x \frac{(zdz)}{\sqrt{\{f(z)\}}}, \quad \wp(u) = \frac{\sqrt{\{f(x)\}}\sqrt{\{f(y)\}} + F(x, y)}{2(xy)^2},$$

(the discovery of it being ascribed to Weierstrass); and it is also given in Halphen's *Traité des Fonctions Elliptiques*, t. II. (1888), p. 357.

The algebraic foundation of the theorem is as follows: writing

$$z = \tfrac{1}{2}\{\sqrt{(ae)} + c\},$$

and as usual I, J for the two invariants of the quartic function $(a, b, c, d, e \between x, y)^4$, we have identically, as is easily verified,

$$4z^3 - Iz - J = \{d\sqrt{(a)} + b\sqrt{(e)}\}^2,$$

or say

$$\sqrt{(4z^3 - Iz - J)} = d\sqrt{(a)} + b\sqrt{(e)}.$$

Hence putting

$$\begin{aligned} A &= (a, b, c, d, e \between x_1, x_2)^4, \\ B &= (a, \qquad \between x_1, x_2)^3 (y_1, y_2), \\ &\vdots \\ E &= (a, \qquad \between y_1, y_2)^4; \end{aligned}$$

and

$$\lambda = x_1 y_2 - x_2 y_1,$$

so that

$$\lambda^4 I = AE - 4BD + 3C^2,$$
$$\lambda^6 J = ACE - AD^2 - B^2E + 2BCD - C^3,$$

then writing

$$z = \frac{1}{2\lambda^2}\{\sqrt{(AE)} + C\},$$

so that $\lambda^2 z$ is a bipartite irrational quadric function of the variables (x_1, x_2) and (y_1, y_2), we have

$$\sqrt{(4z^3 - Iz - J)} = \frac{1}{\lambda^3}\{D\sqrt{(A)} + B\sqrt{(E)}\},$$

and we thence infer the existence of a transformation

$$\frac{dz}{\sqrt{(4z^3 - Iz - J)}} = M\left\{\frac{x_1dx_2 - x_2dx_1}{\sqrt{(A)}} + \frac{y_1dy_2 - y_2dy_1}{\sqrt{(E)}}\right\}.$$

But for the verification hereof and a direct determination of the value of M, observe that we have

$$dz = \frac{1}{2\lambda^2}\left\{\sqrt{(E)}\,\frac{dA}{2\sqrt{(A)}} + dC - \{\sqrt{(AE)} + C\}\,\frac{2}{\lambda}\,d\lambda\right\},$$

where, attending only to the terms in dx_1, dx_2,

$$\begin{aligned} dA &= 4\,\{(a, b, c, d \between x_1, x_2)^2\,dx_1 + (b, c, d, e \between x_1, x_2)^2\,dx_2\}, \\ &= 4\,(A_1dx_1 + A_2dx_2), \text{ suppose}; \\ C &= (a, b, c \between y_1, y_2)^2\,x_1^2 + 2\,(b, c, d \between y_1, y_2)^2\,x_1x_2 + (c, d, e \between y_1, y_2)^2\,x_2^2, \\ &= E_1x_1^2 + 2E_2x_1x_2 + E_3x_2^2, \text{ suppose}; \end{aligned}$$

and therefore

$$dC = 2\,\{(E_1x_1 + E_2x_2)\,dx_1 + (E_2x_1 + E_3x_2)\,dx_2\}.$$

Hence

$$\begin{aligned} dz = \frac{1}{2\lambda^2}\Big\{&\frac{1}{2}\frac{\sqrt{(E)}}{\sqrt{(A)}}\,4\,(A_1dx_1 + A_2dx_2) + 2\,(E_1x_1 + E_2x_2)\,dx_1 \\ &+ 2\,(E_2x_1 + E_3x_2)\,dx_2 - \{\sqrt{(AE)} + C\}\,\frac{2}{\lambda}\,(y_2dx_1 - y_1dx_2)\Big\}. \end{aligned}$$

The whole coefficient herein of dx_1 is

$$= \frac{1}{\lambda^2}\left\{\frac{\sqrt{(E)}}{\sqrt{(A)}}\,A_1 + (E_1x_1 + E_2x_2) - \frac{\sqrt{(AE)} + C}{\lambda}\,y_2\right\},$$

which is

$$= \frac{1}{\lambda^2\sqrt{(A)}}\left\{A_1\sqrt{(E)} + (E_1x_1 + E_2x_2)\sqrt{(A)} - \frac{1}{\lambda}\,\{A\sqrt{(E)} + C\sqrt{(A)}\}\,y_2\right\}$$

$$= \frac{1}{\lambda^3\sqrt{(A)}}\,\{[-Cy_2 + \lambda\,(E_1x_1 + E_2x_2)]\sqrt{(A)} + [A_1\lambda - Ay_2]\sqrt{(E)}\},$$

which is

$$=\frac{1}{\lambda^3\sqrt{(A)}}(-x_2)\{D\sqrt{(A)}+B\sqrt{(E)}\};$$

and similarly the whole coefficient of dx_2 is

$$=\frac{1}{\lambda^3\sqrt{(A)}}\quad(x_1)\{D\sqrt{(A)}+B\sqrt{(E)}\}.$$

Hence the terms in dx_1, dx_2 are together

$$=\frac{1}{\sqrt{(A)}}(x_1dx_2-x_2dx_1)\frac{D\sqrt{(A)}+B\sqrt{(E)}}{\lambda^3};$$

and since the terms in dy_1, dy_2 are of the like form, we have

$$dz=\left\{\frac{1}{\sqrt{(A)}}(x_1dx_2-x_2dx_1)+\frac{1}{\sqrt{(E)}}(y_1dy_2-y_2dy_1)\right\}\times\frac{D\sqrt{(A)}+E\sqrt{(B)}}{\lambda^3},$$

and combining herewith the foregoing value

$$\sqrt{(4z^3-Iz-J)}=\frac{D\sqrt{(A)}+E\sqrt{(B)}}{\lambda^3},$$

we have the required formula

$$\frac{dz}{\sqrt{(4z^3-Iz-J)}}=\left\{\frac{x_1dx_2-x_2dx_1}{\sqrt{(A)}}+\frac{y_1dy_2-y_2dy_1}{\sqrt{(B)}}\right\}.$$

As a very simple verification, suppose $a=1$, $b=c=d=e=0$; then

$$(A,\ B,\ C,\ D,\ E)=(x_1^4,\ x_1^2y_1,\ x_1^2y_1^2,\ x_1y_1^3,\ y_1^4),$$

and if $\lambda=x_1y_2-x_2y_1$ as before, then

$$z=\frac{x_1^2y_1^2}{(x_1y_2-x_2y_1)^2}=\theta^2,$$

where

$$\theta=\frac{-x_1y_1}{x_1y_2-x_2y_1},\quad\text{or}\quad\frac{1}{\theta}=\frac{x_2}{x_1}-\frac{y_2}{y_1}.$$

Also $I=0$, $J=0$, and consequently

$$\frac{dz}{\sqrt{(4z^3-Iz-J)}}=\frac{dz}{2\sqrt{(z^3)}}=\frac{d\theta}{\theta^2}=\frac{x_1dx_2-x_2dx_1}{x_1^2}-\frac{y_1dy_2-y_2dy_1}{y_1^2},$$

which, in virtue of the foregoing values of A, E, is

$$=\frac{x_1dx_2-x_2dx_1}{\sqrt{(A)}}-\frac{y_1dy_2-y_2dy_1}{\sqrt{(E)}}.$$

I remark that an even more simple transformation from the general quartic radical to the Weierstrassian cubic radical was obtained by Hermite; this is alluded

to in my "Note sur les Covariants, &c.," *Crelle*, t. L. (1855), pp. 285—287, [135], and is given with a demonstration in my paper "Sur quelques formules pour la transformation des intégrales elliptiques," *Crelle*, t. LV. (1858), pp. 15—24, [235], see No. IV.; viz. from the identical relation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, which connects the covariants of a quartic function, it at once follows that if

$$U = (a,\ b,\ c,\ d,\ e \between x,\ 1)^4,$$

and H is the Hessian hereof,

$$H = (ac - b^2,\quad (ad - bc),\quad ae + 2bd - 3c^2,\quad 2(be - cd),\quad ce - d^2 \between x,\ 1)^4;$$

then writing

$$z = \frac{H}{U},$$

we have

$$\frac{dz}{\sqrt{(-4z^3 + zI - J)}} = \frac{2dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e \between x,\ 1)^4\}}},$$

which is the transformation in question.

897.

SUR LES RACINES D'UNE ÉQUATION ALGÉBRIQUE.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, t. CX. (*Janvier—Juin*, 1890), pp. 174—176, 215—218.]

SOIT $f(u)$ une fonction rationnelle et entière avec des coefficients réels ou imaginaires, de l'ordre n; en supposant que l'équation $f'(u)=0$, de l'ordre $n-1$, ait $n-1$ racines, je démontre que l'équation $f(u)=0$ aura n racines. Pour cela, soit $f(u)=f(x+iy)=P+iQ$: je suppose que c dénote une quantité positive donnée, et je considère la surface $c-z=P^2+Q^2$, en attribuant à la coordonnée z des valeurs positives; c'est seulement pour avoir des maxima au lieu de minima, et pour faciliter ainsi l'exposition, que je prends cette surface au lieu de $z=P^2+Q^2$. On peut donner à c une valeur si grande que la courbe $c=P^2+Q^2$ soit une courbe fermée qui ne se coupe pas, c'est-à-dire un contour simple: cela étant, on peut se figurer ce contour comme la ligne de rivage d'une île montagneuse; la valeur de z est au plus $=c$, et, en donnant à z une valeur plus petite, $=b$, on a le contour qui correspond à l'altitude b: évidemment, les contours qui correspondent à des altitudes différentes ne se coupent pas. Il s'agit de prouver que l'île a précisément n sommets, chacun de l'altitude c.

J'écris

$$\frac{dP}{dx}=X, \quad \frac{dP}{dy}=Y;$$

donc

$$\frac{dQ}{dx}=-Y, \quad \frac{dQ}{dy}=X;$$

et, de plus,

$$\frac{dX}{dx}=a, \quad \frac{dX}{dy}=\frac{dY}{dx}=h, \quad \frac{dY}{dy}=-a,$$

et de là

$$\frac{d^2P}{dx^2} = a, \qquad \frac{d^2Q}{dx^2} = -h,$$

$$\frac{d^2P}{dx\,dy} = h, \qquad \frac{d^2Q}{dx\,dy} = a,$$

$$\frac{d^2P}{dy^2} = -a, \qquad \frac{d^2Q}{dy^2} = h.$$

Cela étant, nous avons

$$-\tfrac{1}{2}\frac{dz}{dx} = PX - QY, \qquad -\tfrac{1}{2}\frac{dz}{dy} = PY + QX;$$

on aura un plan tangent horizontal pour $P=0$, $Q=0$, ou pour $X=0$, $Y=0$; les valeurs $P=0$, $Q=0$, appartiennent à la valeur c de z et correspondent à des sommets de montagne de cette altitude c; les valeurs $X=0$, $Y=0$ correspondent à des sommets de col. En effet, nous avons

$$-\tfrac{1}{2}\frac{d^2z}{dx^2} = Pa - Qh + X^2 + Y^2,$$

$$-\tfrac{1}{2}\frac{d^2z}{dx\,dy} = Ph + Qa,$$

$$-\tfrac{1}{2}\frac{d^2z}{dy^2} = -Pa + Qh + X^2 + Y^2;$$

donc

$$\begin{aligned}\tfrac{1}{4}\left[\frac{d^2z}{dx^2}\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2\right] &= (X^2+Y^2)^2 - (Pa-Qh)^2 - (Ph+Qa)^2\\ &= (X^2+Y^2)^2 - (P^2+Q^2)(a^2+h^2),\end{aligned}$$

valeur positive pour $P=0$, $Q=0$; négative pour $X=0$, $Y=0$.

A présent, nous avons $f'(x+iy) = X - iY$; donc, en supposant que l'équation $f'(x+iy)=0$ ait $n-1$ racines, il y aura $n-1$ systèmes de valeurs réelles de x, y qui satisfont aux équations $X=0$, $Y=0$; pour chaque système, il y aura des valeurs déterminées de P, Q et de là aussi de z; ces valeurs de z seront en général différentes. Ainsi il y aura dans l'île $n-1$ cols, dont les altitudes seront en général différentes: soient $c_1, c_2, \ldots, c_{n-1}$ ces altitudes, commençant avec la plus petite.

Pour $z=0$, nous avons un contour simple, et de même pour une valeur quelconque plus petite que c_1; mais, pour $z=c_1$, nous avons un col; le contour est une courbe, figure de 8; en donnant à z une valeur un peu plus grande, le contour se divise en deux courbes fermées, ou bien contours simples, extérieurs l'un à l'autre. Pour $z=c_2$, nous avons encore un col; l'un des contours simples s'est changé en figure de 8, et, pour une valeur un peu plus grande, le contour se divise en trois contours simples, chacun extérieur aux autres. En continuant de cette manière, on a, pour c_{n-1}, le col le plus haut; le contour est composé de $n-2$ contours simples et d'une figure de 8; et, en donnant à z une valeur un peu plus grande, on obtient un contour composé de n contours simples, chacun extérieur aux autres. Enfin, en

faisant croître z, chacun des contours simples doit se réduire à un point, c'est-à-dire qu'il doit y avoir précisément n sommets de montagne; mais il n'y a pas de sommet de montagne, sinon pour la valeur $z=c$; donc il y a précisément n sommets de montagne, chacun de l'altitude c. On suppose toujours que l'équation $f'(x+iy)=0$ n'ait pas de racines égales, mais il peut bien arriver que deux ou plusieurs des valeurs $c_1, c_2, \ldots, c_n$ deviennent égales; la démonstration est très peu changée, en donnant à z une valeur un peu plus grande que celle qui correspond à l'altitude des cols d'altitude égale; le contour se divise toujours en contours simples, extérieurs chacun aux autres.

Il va sans dire que cette démonstration repose sur les mêmes principes que celles de Gauss et de Cauchy.

Je reprends la théorie des racines de l'équation $f(u)=0$; au lieu de la surface $c-z=P^2+Q^2$, il convient de considérer la surface $(c-z)^2=P^2+Q^2$, en faisant attention seulement aux valeurs de z positives et pas plus grandes que c. La théorie est très peu changée; les contours sont les mêmes qu'auparavant, mais ils appartiennent à des altitudes différentes; et, au lieu de maxima $z=c$ pour $P=0$, $Q=0$, on a des points coniques, c'est-à-dire, dans l'île montagneuse, au lieu d'un sommet arrondi de montagne, on a un cône ou un pic.

Mais avec la nouvelle surface, on construit graphiquement l'approximation de Newton: partant d'une valeur réelle ou imaginaire approximative u, on obtient la nouvelle valeur

$$u_1 = u + h = u - \frac{f(u)}{f'(u)}.$$

Je représente u par le point (x, y, z) de la surface $(c-z)^2=P^2+Q^2$, ou le point (x, y) du plan des sommets $z-c$; et, de même, u_1 par le point (x_1, y_1, z_1) de la surface, ou (x_1, y_1) du plan des sommets: cela étant, si, par le point (x, y, z) de la surface, on mène la droite de plus grande pente (droite tangente à la surface et perpendiculaire au contour), cette droite rencontrera le plan des sommets en un point (x_1, y_1), et l'on obtient ainsi le point (x_1, y_1, z_1) de la surface, qui représente la valeur cherchée u_1. En particulier, si les coefficients de $f(u)$ sont réels, on a

$$Q=0;$$

l'équation $(c-z)^2=P^2+Q^2$ devient

$$(c-z)^2=P^2,$$

c'est-à-dire

$$c-z=\pm P,$$

ou enfin

$$c-z=\pm f(x),$$

et la section verticale de l'île est formée par des parties de ces deux courbes symétriques: pour la théorie géométrique, on peut évidemment y substituer la seule courbe $c-z=f(x)$.

J'ai proposé, il y a plus de dix ans (*Amer. Math. Journ.*, t. II., 1879, [694]), le problème que je nomme "The Newton-Fourier imaginary Problem," et dans une Note (*Quart. Math. Journ.*, t. XVI., 1879, [736]), "Application of the Newton-Fourier method to an imaginary root of an equation," j'ai considéré le cas d'une équation quadratique. Pour l'équation $u^2-1=0$, on a

$$u_1 = u - \frac{u^2-1}{2u} = \tfrac{1}{2}\left(u+\frac{1}{u}\right);$$

cela donne

$$u_1 - 1 = \frac{1}{2u}(u-1)^2, \qquad u_1+1 = \frac{1}{2u}(u+1)^2,$$

et de là

$$\frac{u_1-1}{u_1+1} = \left(\frac{u-1}{u+1}\right)^2.$$

Cette dernière équation a rapport aux deux racines $+1$ et -1, et, quoiqu'elle donne les résultats les plus élégants, cependant, en vue de la théorie générale, il vaut mieux considérer l'équation $u_1 - 1 = \frac{1}{2u}(u-1)^2$ qui se rapporte à la seule racine $+1$.

Je remarque d'abord que la formule originale $u_1 = \frac{1}{2}\left(u+\frac{1}{u}\right)$ donne

$$x_1 = \tfrac{1}{2}x\left(1+\frac{1}{x^2+y^2}\right), \qquad y_1 = \tfrac{1}{2}y\left(1-\frac{1}{x^2+y^2}\right);$$

donc les valeurs de x et x_1 seront à la fois positives ou négatives, et ainsi l'on peut ne faire attention qu'aux valeurs positives. Cela étant, nous avons

$$x_1 + iy_1 - 1 = \frac{(x+iy-1)^2}{2(x+iy)}.$$

Désignons par A le point $(0, 1)$, par B le point $(0, -1)$, par 0 le point $(0, 0)$; et aussi par P le point (x, y), et de même par P_1 le point (x_1, y_1); écrivons aussi $x+iy = se^{i\phi}$, $x+iy-1 = re^{i\theta}$, $x_1+iy_1 = r_1e^{i\theta}$; l'équation est

$$r_1e^{i\theta_1} = \tfrac{1}{2}\frac{r^2e^{2i\theta}}{se^{2\phi}};$$

donc

$$r_1 = \tfrac{1}{2}\frac{r^2}{s}, \quad \left(\text{c'est-à-dire } AP_1 = \frac{AP^2}{2\overline{OP}}\right),$$

et

$$\theta_1 = 2\theta - \phi, \quad (\text{c'est-à-dire } \widehat{AP_1x} = \widehat{2APx} - \widehat{OPx}).$$

Je remarque que, dans la géométrie des vecteurs, la seule équation $AP_1 = \frac{AP^2}{2\overline{OP}}$ dénote l'équation en x_1+iy_1, $x+iy$, c'est-à-dire les deux équations que je viens de trouver.

Partant d'un point quelconque P, on obtient une suite de points $P_1, P_2, P_3, \ldots$; et, si le point P est sur l'axe des $y\,(x=0)$, tous les autres points seront aussi sur l'axe de y, et l'on n'approche ni du point A ni du point B. Mais, si la coordonnée x a une valeur positive si petite que l'on veut, on arrive enfin infiniment près du

point A, et l'on peut même (dans un sens qui sera expliqué plus bas, mais qui n'est pas le sens le plus naturel) dire que l'approximation est régulière. En effet, on n'a pas toujours $AP_1 < AP$, et ainsi, dans le sens le plus naturel, l'approximation n'est pas toujours régulière. Pour étudier cela, j'écris $AP_1 = AP$; cela donne $AP = 2OP$, ou, ce qui est la même chose, $x^2 + y^2 + \frac{2}{3}x = \frac{1}{3}$, c'est-à-dire que le point P sera situé sur le cercle, centre $x = -\frac{1}{3}$ et rayon $= \frac{2}{3}$; en ne faisant attention qu'aux valeurs positives de x, on a un segment compris entre l'axe des y et un arc par les points $\left(x = 0,\ y = \pm\frac{1}{\sqrt{3}}\right)$, $(x = \frac{1}{3},\ y = 0)$. Si le point P est sur l'arc, on aura $AP_1 = AP$; si P est en dedans du segment, alors $AP_1 > AP$; si P est en dehors du segment, $AP_1 < AP$.

Mais, en supposant P en dehors du segment, et ainsi $AP_1 < AP$, il peut bien arriver que P_1 soit en dedans du segment, et, cela étant, on aura $AP_2 > AP_1$, et l'approximation ne sera pas régulière. Mais, en considérant le cercle $x^2 + y^2 - \frac{2}{3}x = \frac{1}{3}$, lequel est le cercle, centre A et rayon $\frac{2}{3}$, qui touche le segment au point $(x = \frac{1}{3},\ y = 0)$, alors, en supposant que le point P soit en dedans de ce cercle, on aura $AP_1 < AP$, le point P_1 sera aussi en dedans du cercle, et ainsi en dehors du segment; et les points successifs P, P_1, P_2, ... approcheront continuellement du point A; l'approximation sera dans ce cas régulière.

Il y a ainsi trois régions: le segment, le cercle $x^2 + y^2 - \frac{2}{3}x = \frac{1}{3}$ et le résidu du demi-plan; on pourrait les nommer régions *noire*, *blanche* et *grise* respectivement. C'est seulement pour un point P à l'intérieur de la région blanche que l'approximation est certainement régulière.

Nous venons de considérer en effet les cercles qui ont pour centre le point A; $AP_1 < AP$ veut dire que le point P est situé sur un cercle plus grand, et P_1 sur un cercle plus petit; mais, au lieu de ces cercles concentriques, considérons des cercles quelconques qui entourent le point A, sans se couper les uns les autres; et convenons de dire que c'est un bon pas quand on passe du point P sur un cercle plus grand à un point P_1 sur un cercle plus petit: avec cette convention on aura, en général, trois régions, lesquelles cependant ne seront pas les mêmes comme auparavant. En particulier, si nous considérons les cercles $AP = kBP$ (k une constante quelconque plus petite que l'unité), alors il n'y a pas de région noire, ou, si l'on veut, la région noire se réduit à la seule droite $y = 0$; donc il n'y a pas non plus de région grise, et le demi-plan entier est région blanche, c'est-à-dire, dans le sens que je viens d'expliquer, l'approximation est toujours régulière. En effet, c'est là la théorie à laquelle on est conduit au moyen de l'équation $\frac{u_1 - 1}{u_1 + 1} = \left(\frac{u - 1}{u + 1}\right)^3$ ci-dessus mentionnée.

En parlant de cercles, j'ai fait une restriction qui n'est nullement nécessaire; j'aurais pu parler d'ovales, de forme quelconque, qui entourent le point A sans se couper les uns les autres.

J'espère appliquer cette théorie au cas d'une équation cubique, mais les calculs sont beaucoup plus difficiles.

898.

SUR L'ÉQUATION MODULAIRE POUR LA TRANSFORMATION DE L'ORDRE 11.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, t. CXI. (*Juillet—Décembre*, 1890), pp. 447—449.]

L'ÉQUATION en u, v, en y écrivant $u = x$, $v = y$, est

$$\left.\begin{array}{ll}
 & y^{12} \\
+ & y^{11} \quad (32x^{11} - 22x^3) \\
+ & y^{10} \,.\quad 44x^6 \\
+ & y^9 \quad (88x^9 + 22x) \\
+ & y^8 \,.\quad 165x^4 \\
+ & y^7 \,.\quad 132x^7 \\
+ & y^6 (-\ 44x^{10} + 44x^2) \\
+ & y^5 \ - 132x^5 \\
+ & y^4 \ - 165x^8 \\
+ & y^3 (-\ 22x^{11} - 88x^3) \\
+ & y^2 \ - \ 44x^6 \\
+ & y \quad (22x^9 - 22x) \\
+ & 1 \,.\, - \quad x^{12}
\end{array}\right\} = 0.$$

Selon un résultat trouvé par H. J. S. Smith, pour la transformation de l'ordre p, la courbe est de l'ordre $2p$, et il y a à l'origine un point double, à l'infini deux points singuliers équivalents chacun à $\frac{1}{2}(p-1)(p-2)$ points doubles, et de plus $(p-1)(p-3)$ points doubles. Au cas $p = 11$, le nombre de ces derniers points doubles est donc $= 80$. Cela s'accorde avec l'expression

$$D = x^{12}(1 - x^6)^{10}(16x^{16} - 31x^8 + 16)^2(x^{64} - 301960x^{56} + \ldots + 1)^2,$$

trouvée par M. Hermite pour le discriminant de la fonction; et l'on voit ainsi que les valeurs de x, qui correspondent aux quatre-vingts points doubles, sont données par les équations

$$16x^{16} - 31x^8 + 16 = 0,$$

$$x^{64} - 301960x^{56} + \ldots + 1 = 0.$$

Je ne considère que les seize points doubles donnés par la première équation. Cette équation donne

$$x^8 = \tfrac{1}{32}(31 + 3i\sqrt{7}), \quad x^4 = \tfrac{1}{8}(i + 3\sqrt{7}), \quad x^2 = \frac{1+i}{4\sqrt{2}}(-3 + i\sqrt{7});$$

il y a ainsi quatre points doubles, pour lesquels les valeurs de x sont

$$x = \pm\sqrt{\frac{1+i}{4\sqrt{2}}}(-3 + i\sqrt{7}), \quad x = \pm\sqrt{\frac{1+i}{4\sqrt{2}}}(-3 - i\sqrt{7});$$

je trouve que les valeurs correspondantes de y sont $y = \frac{1+i}{\sqrt{2}}x$, savoir que les quatre points sont situés sur la droite $y = \frac{1+i}{\sqrt{2}}x$, et, en changeant successivement les signes de i et $\sqrt{2}$, on voit ainsi que les seize points sont situés, quatre à quatre, sur les droites

$$y = \frac{1+i}{\sqrt{2}}x, \quad y = \frac{1-i}{\sqrt{2}}x, \quad y = -\frac{1+i}{\sqrt{2}}x, \quad y = -\frac{1-i}{\sqrt{2}}x.$$

J'écris, pour abréger,

$$m = \frac{1+i}{\sqrt{2}}, \text{ (donc } m^4 = -1),$$

$$p = \tfrac{1}{4}(-3 + i\sqrt{7}),$$

donc

$$2p^2 + 3p + 2 = 0.$$

En écrivant $y = mx$ dans l'équation et en rejetant le facteur x^2, puis en écrivant $x^2 = mp$, l'équation se présente sous la forme

$$\left.\begin{array}{lll}
m^{10}p^{10} & . & 32m^{11} \\
+ m^8 p^8 & . & 88m^9 \\
+ m^7 p^7 & . & 44m^{10} - 44m^6 \\
+ m^6 p^6 & . & -22m^{11} + 132m^7 - 22m^3 \\
+ m^5 p^5 & . & m^{12} + 165m^8 - 165m^4 - 1 \\
+ m^4 p^4 & . & 22m^9 - 132m^5 + 22m \\
+ m^3 p^3 & . & 44m^6 - 44m^2 \\
+ m^2 p^2 & . & -88m^3 \\
+ 1 & . & -32m
\end{array}\right\} = 0,$$

où les coefficients ne contiennent que les puissances m^{21}, m^{17}, m^{13}, m^{9}, m^{5}, m^{1} de m et se réduisent ainsi à des multiples de m; il y a aussi un facteur numérique 8, et, en divisant par $-8m$, l'équation devient

$$4p^{10}-11p^{8}-11p^{7}+22p^{6}+41p^{5}+22p^{4}-11p^{3}-11p^{2}+4=0;$$

cette équation est de la forme

$$(2p^{2}+3p+2)^{2}(p^{6}-3p^{5}+2p^{4}+p^{3}+2p^{2}-3p+1)=0.$$

La droite $y=mx$ a donc, avec la courbe, quatre intersections doubles

$$p=\tfrac{1}{4}(-3\pm i\sqrt{7}),$$

c'est-à-dire

$$x^{2}=\frac{1+i}{4\sqrt{2}}(-3\pm i\sqrt{7}):$$

on démontre sans peine que la droite n'est pas une tangente, et ces valeurs correspondent ainsi à des points doubles de la courbe, c'est-à-dire qu'il y a sur la droite $y=\frac{1+i}{\sqrt{2}}x$ quatre points doubles. Réciproquement, cette valeur de x^{2} conduit au facteur $(16x^{16}-31x^{8}+16)^{2}$ du déterminant de l'équation modulaire.

899.

SUR LES SURFACES MINIMA.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, t. CXI. (*Juillet—Décembre*, 1890), pp. 953, 954.]

ON peut généraliser tant la définition que la construction de ces surfaces, en substituant pour le cercle imaginaire à l'infini une conique ou même une surface quadrique quelconque.

Je rappelle que, dans la théorie ordinaire, une surface minima est une surface telle qu'un point quelconque de la surface est situé à mi-chemin entre les deux centres de courbure, et qu'une telle surface est le lieu des points à mi-chemin entre les deux points situés respectivement sur deux courbes de longueur nulle quelconques. Or on peut rattacher la notion d'une courbe de longueur nulle à celle d'une courbe de poursuite. Pour expliquer cela, j'observe que, dans le plan, en supposant comme à l'ordinaire que le lièvre coure selon une droite et que le chien et le lièvre courent avec des vitesses uniformes, la courbe de poursuite est une courbe déterminée; mais si les vitesses varient arbitrairement, alors la définition exprime seulement que la courbe est une courbe plane. Mais, dans l'espace, si au lieu d'une droite on considère une courbe plane ou à double courbure (disons une directrice) quelconque, alors, quelles que soient les vitesses, la définition précise toujours la courbe, savoir: on a toujours pour courbe de poursuite une courbe dont chaque tangente rencontre la courbe directrice. Au lieu d'une courbe, on peut avoir une surface directrice; dans ce cas, le nom est moins applicable, néanmoins je le retiens, et je dis que la courbe de poursuite est une courbe dont chaque tangente touche la surface directrice. Nous avons, de cette manière, la définition d'une courbe de poursuite par rapport à une courbe ou surface directrice quelconque.

A présent, au lieu du cercle imaginaire à l'infini, considérons une conique quelconque, l'absolue: on établit, comme on sait, par rapport à cette conique, la notion de la perpendicularité, et ainsi les notions d'une normale et des centres de courbure

ne cessent pas de subsister. On peut donc considérer une surface telle que chaque point de la surface soit l'harmonicale par rapport aux deux centres de courbure du point de rencontre de la normale avec le plan de l'absolue: on a ainsi ce que je nomme une surface *quasi-minima*. Il va sans dire qu'il faut modifier convenablement la notion métrique d'une aire minima pour qu'elle soit applicable à cette nouvelle surface.

Pour construire la surface, on prend par rapport à l'absolue deux courbes de poursuite quelconques, et puis sur la droite, menée par deux points quelconques de ces courbes respectivement, l'harmonicale par rapport à ces deux points du point de rencontre de la droite avec le plan de l'absolue: le lieu de ce point harmonical sera la surface quasi-minima.

Il paraît permis de substituer pour la conique absolue une surface quadrique quelconque, que je nomme aussi l'*absolue*: on a, comme on sait, les notions de la normale et des centres de courbure. Pour la surface quasi-minima, le point sur la surface sera l'un des points doubles (foyers) de l'involution formée par les deux centres de courbure et les deux points de rencontre de la normale avec l'absolue; et de même pour la construction de la surface, il faut prendre sur la droite menée par deux points quelconques des deux courbes de poursuite respectivement les points doubles (foyers) de l'involution formée par ces deux points et les deux points de rencontre de la droite avec l'absolue.

900.

JAMES JOSEPH SYLVESTER.

[(*Scientific Worthies in Nature*, XXV.); *Nature*, vol. XXXIX. (1889), pp. 217—219.]

JAMES JOSEPH SYLVESTER, born in London on September 3, 1814, is the sixth and youngest son of the late Abraham Joseph Sylvester, formerly of Liverpool*. He was educated at two private schools in London, and at the Royal Institution, Liverpool, whence he proceeded in due course of time to St John's College, Cambridge. In these early days he manifested considerable aptitude for mathematics, and so it was not matter for surprise that he came out in the Tripos Examination of 1837 as Second Wrangler; being incapacitated, by the fact of his Jewish origin, from *taking* his degree, he was not able to compete for either of the Smith's Prizes. In more enlightened times (1872), he had the degrees of B.A. and M.A., by accumulation, conferred upon him, and received therewith the honour of a Latin speech from the Public Orator. He himself says: "I am perhaps the only man in England who am a full (voting) Master of Arts for the three Universities of Dublin, Cambridge, and Oxford, having received that degree from these Universities in the order above given: from Dublin, by *ad eundem;* from Cambridge, *ob merita;* from Oxford, by decree." He is now D.C.L. of Oxford, LL.D. of Dublin and Edinburgh, and Hon. Fellow of St John's College, Cambridge. It is still open for him to receive yet higher recognition from his own *alma mater***.

Prof. Sylvester became a student of the Inner Temple, July 29, 1846, and was called to the Bar on November 22, 1850†. He has been Professor of Natural Philosophy at University College, London; of Mathematics at the University of Virginia, U.S.A.‡; then ten years later Professor at the Royal Military Academy,

* Foster's *Hand-book of Men at the Bar.*

[** The honorary degree Sc.D. was conferred upon him by Cambridge in 1890.]

† Foster, *l.c.*

‡ The late Prof. Key, of University College and School, was the first occupant of the Chair, founded by Mr Jefferson, once President of the United States, in 1824.

Woolwich; and again, after a five years' interval, Professor of Mathematics at the Johns Hopkins University, Baltimore, U.S.A., from its foundation in 1877. Finally, in December 1883, he was elected Savilian Professor of Geometry at Oxford, in succession to Prof. Henry Smith*. His first printed paper was on Fresnel's optical theory (in the *Phil. Mag.*, 1837).

We can here only briefly allude to a communication which was accompanied by many important results: we refer to the Friday evening address (January 23, 1874) to the Royal Institution, "On Recent Discoveries in Mechanical Conversion of Motion." He says:—"It would be difficult to quote any other discovery which opens out such vast and varied horizons as this of Peaucellier's,—in one direction, descending to the wants of the workshop, the simplification of the steam-engine, the revolutionising of the mill-wright's trade, the amelioration of garden-pumps, and other domestic conveniences (the sun of science glorifies all it shines upon); and in the other, soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unexpected quarter on the researches of such men as Abel, Riemann, Clebsch, Grassmann, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth."

The only works that Prof. Sylvester has published, we believe, are: (1) "A Probationary Lecture on Geometry, delivered before the Gresham Committee and the Members of the Common Council of the City of London, December 4, 1854," a slight thing which had to be written and delivered at a few hours' notice; (2) "Laws of Verse," 1870; (3) several short poems, sonnets, and translations, which have appeared in our columns and elsewhere.

Our notice would be incomplete without some record of the honours that have been conferred upon Dr Sylvester. He was elected a Fellow of the Royal Society on April 25, 1839; has received a Royal Medal (1860) and the Copley Medal (1880), this latter rarely awarded, we believe, to a pure mathematician. On this last occasion, Mr Spottiswoode accompanied the presentation with the words, "His extensive and profound researches in pure mathematics, especially his contributions to the theory of invariants and covariants, to the theory of numbers and to modern geometry, may be regarded as fully establishing Mr Sylvester's claim to the award." He is a Fellow of New College, Oxford; Foreign Associate of the United States National Academy of Sciences; Foreign Member of the Royal Academy of Sciences, Göttingen, of the Royal Academy of Sciences of Naples, and of the Academy of Sciences of Boston; Corresponding Member of the Institute of France, of the Imperial Academy of Science of St Petersburg, of the Royal Academy of Science of Berlin, of the Lyncei of Rome, of the Istituto Lombardo, and of the Société Philomathique. He has been long connected with the editorial staff of the *Quarterly Journal of Mathematics* (under

* He commences his Oxford lecture (*Nature*, vol. XXXIII. p. 222), of date December 12, 1885, with the words: "It is now two years and seven days since a message by the Atlantic cable containing the single word 'elected' reached me in Baltimore informing me that I had been appointed Savilian Professor of Geometry in Oxford, so that for three weeks I was in the unique position of filling the post and drawing the pay of Professor of Mathematics in each of two Universities."

one or another of its titles), and was the first editor of, and is a considerable contributor to, the *American Journal of Mathematics*; and he was at one time Examiner in Mathematics and Natural Philosophy in the University of London. He was not an original member of the London Mathematical Society (founded January 16, 1865), but was elected a member on June 19, 1865, Vice-President on January 15, 1866, and succeeded Prof. De Morgan as the second President on November 8, 1866. The Society showed its recognition of his great services to them and to mathematical science generally by awarding him its De Morgan Gold Medal in November 1887. Wherever Dr Sylvester goes, there is sure to be mathematical activity; and the latest proof of this is the formation, during the last term at Oxford, of a Mathematical Society, which promises, we hear without surprise, to do much for the advancement of mathematical science there.

The writings of Sylvester date from the year 1837; the number of them in the Royal Society Index up to the year 1863 is 112, in the next ten years 38, and in the volume for the next ten years 81, making 231 for the years 1837 to 1883: the number of more recent papers is also considerable. They relate chiefly to finite analysis, and cover by their subjects a large part of it: algebra, determinants, elimination, the theory of equations, partitions, tactic, the theory of forms, matrices, reciprocants, the Hamiltonian numbers, &c.; analytical and pure geometry occupy a less prominent position; and mechanics, optics, and astronomy are not absent. A leading feature is the power which is shown of originating a theory or of developing it from a small beginning; there is a breadth of treatment and determination to make the most of a subject, an appreciation of its capabilities, and real enjoyment of it. There is not unfrequently an adornment or enthusiasm of language which one admires, or is amused with: we have a motto from Milton, or Shakespeare; a memoir is a trilogy divided into three parts, each of which has its action complete within itself, but the same general cycle of ideas pervades all three, and weaves them into a sort of complex unity; the apology for an unsymmetrical solution is—symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action; and, he remarks, may not music be described as the mathematic of sense, mathematic as the music of the reason? the soul of each the same! &c. It is to be mentioned that there is always a generous and cordial recognition of the merit of others, his fellow-workers in the science.

It would be in the case of any first-rate mathematician—and certainly as much so in this as in any other case—extremely interesting to go carefully through the whole of a long list of memoirs, tracing out as well their connexion with each other, and the several leading ideas on which they depend, as also their influence on the development of the theories to which they relate; but for doing this properly, or at all, space and time, and a great amount of labour, are required. Short of doing so, one can only notice particular theorems—and there are, in the case of Sylvester,

many of these, "beautiful exceedingly," which, for their own sakes, one is tempted to refer to—or one can give titles, which, to those familiar with the memoirs themselves, will recall the rich stores of investigation and theory contained therein.

A considerable number of papers, including some of the earliest ones, relate to the question of the reality of the roots of a numerical equation: in the several connexions thereof with Sturm's theorem, Newton's rule for the number of imaginary roots, and the theory of invariants. Sylvester obtained for the Sturmian functions, divested of square factors, or say for the reduced Sturmian functions, singularly elegant expressions in terms of the roots, viz. these were

$$f_2(x) = \Sigma(a-b)^2(x-c)(x-d)\ldots,\ f_3(x) = \Sigma(a-b)^2(a-c)^2(b-c)^2(x-d)\ldots,\ \&c.;$$

but not only this: applying the Sturmian process of the greatest common measure (not to $f(x)$, $f'(x)$, but instead) to two independent functions $f(x)$, $\phi(x)$, he obtained for the several resulting functions expressions involving products of differences between the roots of the one and the other equation, $f(x)=0$, $\phi(x)=0$; the question then arose, what is the meaning of these functions? The answer is given by his theory of *intercalations:* they are signaletic functions, indicating in what manner (when the real roots of the two equations are arranged in order of magnitude) the roots of the one equation are intercalated among those of the other. The investigations in regard to Newton's rule (not previously demonstrated) are very important and valuable: the principle of Sturm's demonstration is applied to this wholly different question: viz. x is made to vary continuously, and the consequent gain or loss of changes of sign is inquired into. The third question is that of the determination of the character of the roots of a quintic equation by means of invariants. In connexion with it, we have the noteworthy idea of *facultative* points; viz. treating as the coordinates of a point in n-dimensional space those functions of the coefficients which serve as criteria for the reality of the roots, a point is facultative or non-facultative according as there is, or is not, corresponding thereto any equation with real coefficients: the determination of the characters of the roots depends (and, it would seem, depends only) on the bounding surface or surfaces of the facultative regions, and on a surface depending on the discriminant. Relating to these theories there are two elaborate memoirs, "On the Syzygetic Relations &c." and "Algebraical Researches &c." in the *Philosophical Transactions* for the years 1853 and 1864 respectively; but as regards Newton's rule later papers must also be consulted.

In the years 1851—54, we have various papers on homogeneous functions, the calculus of forms, &c. (*Camb. and Dub. Math. Journal*, vols. VI. to IX.), and the separate work "On Canonical Forms" (London, 1851). These contain crowds of ideas, embodied in the new words, *cogredient, contragredient, concomitant, covariant, contravariant, invariant, emanant, combinant, commutant, canonical form, plexus*, &c., ranging over and vastly extending the then so-called theories of linear transformations and hyperdeterminants. In particular, we have the introduction into the theory of the very important idea of *continuous* or *infinitesimal* variation: say that a function, which (whatever are the values of the parameters on which it depends) is invariant for an infinitesimal change of the parameters, is absolutely invariant.

There is, in 1844, in the *Philosophical Magazine*, a valuable paper, "Elementary Researches in the Analysis of Combinatorial Aggregation," and the titles of two other papers, 1865 and 1866, may be mentioned: "Astronomical Prolusions; commencing with the instantaneous proof of Lambert's and Euler's theorems, and modulating through the construction of the orbit of a heavenly body from two heliocentric distances, the subtended chord, and the periodic time, and the focal theory of Cartesian ovals, into a discussion of motion in a circle and its relation to planetary motion"; and the sequel thereto, "Note on the periodic changes of orbit under certain circumstances of a particle acted upon by a central force, and on vectorial coordinates, &c., together with a new theory of the analogues of the Cartesian ovals in space."

Many of the later papers are published in the *American Mathematical Journal*, founded, in 1878, under the auspices of the Johns Hopkins University, and for the first six volumes of which Sylvester was editor-in-chief. We have, in vol. I., a somewhat speculative paper entitled "An application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics," followed by appendices and notes relating to various special points of the theory; and in the same and subsequent volumes various memoirs on binary and ternary quantics, including papers (by himself, with the aid of Franklin) containing tables of the numerical generating functions for binary quantics of the first ten orders, and for simultaneous binary quantics of the first four orders, &c. The memoir (vols. II. and III.) on "Ternary cubic-form equations" is connected with some early papers relating to the theory of numbers. We have in it the theory of residuation on a cubic curve, and the beautiful chain-rule of rational derivation; viz. from an arbitrary point 1 on the curve it is possible to derive the singly infinite series of points $(1, 2, 4, 5, \ldots, 3p \pm 1)$ such that the chord through any two points, m, n, again meets the curve in a point $m+n$, $m \sim n$ (whichever number is not divisible by 3) of the series; moreover, the coordinates of any point m are rational and integral functions of the degree m^2 of those of the point 1.

There is in vol. V. the memoir, "A Constructive Theory of Partitions arranged in three acts, an Interact in two parts, and an Exodion," and in vol. VI. we have "Lectures on the Principles of Universal Algebra," (referring to a course of lectures on multinomial quantity, in the year 1881). The memoir is incomplete, but the general theories of nullity and vacuity, and of the corpus formed by two independent matrices of the same order, are sketched out; and there are, in the *Comptes rendus* of the French Academy, later papers containing developments of various points of the theory,—the conception of "nivellators" may be referred to.

The last-mentioned paper in the *American Mathematical Journal* was published subsequently to Sylvester's return to England on his appointment as Savilian Professor of Mathematics at Oxford. In December 1886, he gave there a public lecture containing an outline of his new theory of reciprocants (reported in *Nature*, January 7, 1887), and the lectures since delivered are published under the title, "Lectures on the Theory of Reciprocants" (reported by J. Hammond), same *Journal*, vols. VIII. to X.; thirty-three lectures actually delivered, entire or in abstract, in the course of

three terms, to a class in the University, with a concluding so-called lecture 34, which is due to Hammond. The subject, as is well known, is that of the functions of a dependent variable, y, and its differential coefficients, y', y'', ..., in regard to x (or, rather, the functions of y', y'', ...), which remain unaltered by the interchange of the variables x and y: this is a less stringent condition than that imposed by Halphen ("Thèse," 1878) on his differential invariants, and the theory is accordingly a more extensive one. A passage may be quoted:—"One is surprised to reflect on the change which is come over Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures, embracing only a fragment of the theory of reciprocants, might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the dignity of a fine art, in which the workman has a free hand to develop his conceptions, as in a musical theme or a subject for painting? Formerly, it consisted in detached theorems, but nowadays it has reached a point in which every properly-developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas." And, indeed, the theory has already spread itself out far and wide, not only in these lectures by its founder, but in various papers by auditors of them, and others,—Elliott, Hammond, Leudesdorf, Rogers, Macmahon, Berry, Forsyth.

Sylvester's latest important investigations relate to the Hamiltonian numbers: there is a memoir, *Crelle*, t. c. (1887), and, by Sylvester and Hammond jointly, two memoirs in the *Philosophical Transactions*. The subject is that of the series of numbers 2, 3, 5, 11, 47, 923, calculated thus far by Sir W. R. Hamilton in his well-known Report to the British Association, on Jerrard's method. A formula for the independent calculation of any term of the series was obtained by Sylvester, but the remarkable law by means of a generating function was discovered by Hammond, viz. E_0, E_1, E_2, ..., being the series 3, 4, 6, ... of the foregoing numbers, each increased by unity; then these are calculated by the formula

$$(1-t)^{E_0} + t(1-t)^{E_1} + t^2(1-t)^{E_2} + \ldots = 1 - 2t,$$

equating the powers of t on the two sides respectively: observe the paradox, $t = \frac{1}{2}$, then the formula gives $0 =$ sum of a series of positive powers of $\frac{1}{2}$.

Enough has been said to call to mind some of Sylvester's achievements in mathematical science. Nothing further has been attempted in the foregoing very imperfect sketch.

901.

NOTE ON THE SUMS OF TWO SERIES.

[From the *Messenger of Mathematics*, vol. XIX. (1890), pp. 29—31.]

I CONSIDER the two series

$$S = \frac{1}{1+e^{\pi\alpha}} + \frac{1}{3\,(1+e^{3\pi\alpha})} + \frac{1}{5\,(1+e^{5\pi\alpha})} + \dots,$$

and

$$S_1 = \frac{1}{2+\pi\alpha} + \frac{1}{3\,(2+3\pi\alpha)} + \frac{1}{5\,(2+5\pi\alpha)} + \dots,$$

where α is real, positive, and indefinitely small; these would at first sight appear to be equal to each other, but this is not in fact the case.

Taking first the series S_1, putting therein $\pi\alpha = 2x$, this is

$$2S_1 = \frac{1}{1+x} + \frac{1}{3\,(1+3x)} + \frac{1}{5\,(1+5x)} + \dots.$$

Now we have, (Legendre, *Théorie des Fonctions Elliptiques*, t. II. p. 438),

$$\frac{y}{1+y} + \frac{y}{2\,(2+y)} + \frac{y}{3\,(3+y)} + \dots = C + \frac{d}{dy}\log\Gamma(y+1),$$

where C is Euler's constant, $= \cdot 577\dots$; and if y be real, positive, and very large, then

$$\Gamma(y+1) = \sqrt{(2\pi)}\, y^{y+\frac{1}{2}} e^{-y+\frac{1}{12y}+\dots};$$

whence, differentiating the logarithm and neglecting the terms which contain negative powers of y, then the value is $= C + \log y$; hence, writing $y = \frac{1}{x}$, we obtain

$$\frac{1}{1+x} + \frac{1}{2\,(1+2x)} + \frac{1}{3\,(1+3x)} + \dots = C - \log x.$$

Writing herein $2x$ for x, and dividing by 2, we have

$$\frac{1}{2(1+2x)}+\frac{1}{4(1+4x)}+\ldots=\tfrac{1}{2}C-\tfrac{1}{2}\log 2x,\ =\tfrac{1}{2}(C-\log 2)-\tfrac{1}{2}\log x;$$

or, subtracting,

$$\frac{1}{1+x}+\frac{1}{3(1+3x)}+\frac{1}{5(1+5x)}+\ldots=\tfrac{1}{2}(C+\log 2)-\tfrac{1}{2}\log x.$$

Hence, writing for x its value, $=\frac{1}{2}\pi\alpha$, we have

$$S_1=\tfrac{1}{4}(C+\log 2)-\tfrac{1}{4}\log\tfrac{1}{2}\pi\alpha,\quad =\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha.$$

For the series S, we have (*Fundamenta Nova*, p. 103*) the formula

$$\tfrac{1}{4}\log\frac{2K}{\pi}=\frac{1}{1+q^{-1}}+\frac{1}{3(1+q^{-3})}+\frac{1}{5(1+q^{-5})}+\ldots;$$

or, putting herein $\alpha=\dfrac{K'}{K}$, then $q=e^{-\frac{\pi K'}{K}}=e^{-\pi\alpha}$, and thence

$$S=\frac{1}{1+e^{\pi\alpha}}+\frac{1}{2(1+e^{2\pi\alpha})}+\frac{1}{3(1+e^{3\pi\alpha})}+\ldots,\quad =\tfrac{1}{4}\log\frac{2K}{\pi}.$$

We have $q=e^{-\pi\alpha}$, which is real, positive, and less than but indefinitely near to 1; hence also k is real, positive, and less than but indefinitely near to 1, say the value is $=1-\beta$; thence $k'=\sqrt{(2\beta)}$, and $K=\log\dfrac{4}{k'}$, $=\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$; also $K'=\frac{1}{2}\pi$, and therefore $\alpha=\dfrac{K'}{K}=\frac{1}{2}\pi\div\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$, whence $\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}=\dfrac{\pi}{2\alpha}$, which is the relation between α and β; and we thus have $\dfrac{2K}{\pi}=\dfrac{2}{\pi}\log\dfrac{2\sqrt{2}}{\sqrt{\beta}}$, $=\dfrac{1}{\alpha}$; and consequently $S=\frac{1}{4}\log\dfrac{2K}{\pi}$, $=-\frac{1}{4}\log\alpha$. The two values thus are

$$S=-\tfrac{1}{4}\log\alpha,\quad S_1=\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha,$$

each depending on $\log\alpha$, and having for this term the same coefficient $-\frac{1}{4}$; but there is in S_1 a constant term $\frac{1}{4}(C+2\log 2-\log\pi)$, where C is the constant ·577....

It is easy to see why the series S is not reducible to S_1; however small α may be in the general term $\dfrac{1}{n(1+e^{n\pi\alpha})}$, then taking n sufficiently large, not only $n\pi\alpha$ is not indefinitely small, but it in fact becomes indefinitely large; the general term of the first series thus approximates to $\dfrac{1}{ne^{n\pi\alpha}}$, or the terms diminish somewhat more rapidly than in a geometric series with the ratio $e^{-\pi\alpha}$ (a small positive value less than but very near to 1), whereas in the second series the general term approximates to $\dfrac{1}{n^2\pi\alpha}$, or the convergence is ultimately that of the series $\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\ldots$.

[* Jacobi's *Gesammelte Werke*, t. I, p. 159.]

902.

ON THE FOCALS OF A QUADRIC SURFACE.

[From the *Messenger of Mathematics*, vol. XIX. (1890), pp. 113—117.]

IN plane geometry, the focus of a curve is the node of the circumscribed line-system of the curve and the circular points at infinity; and so, in solid geometry, the focal of a surface or curve is the nodal line of the circumscribed developable of the surface or curve and the circle at infinity. And as in plane geometry the circular points at infinity may be regarded as an indefinitely thin conic, so in solid geometry the circle at infinity may be regarded as an indefinitely thin quadric surface.

In plane geometry, let it be proposed to find the circumscribed line-system (common tangents) of the two quadrics

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0,\ \frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}=0;$$

if a common tangent be

$$\xi x+\eta y+\zeta z=0,$$

then we have

$$a\xi^2+b\eta^2+c\zeta^2=0,$$

$$a'\xi^2+b'\eta^2+c'\zeta^2=0;$$

here writing for shortness

$$f,\ g,\ h=bc'-b'c,\ ca'-c'a,\ ab'-a'b,$$

we have

$$\xi^2:\eta^2:\zeta^2=f:g:h;$$

and thence the tangent is

$$x\sqrt{(f)}\pm y\sqrt{(g)}\pm z\sqrt{(h)}=0,$$

viz. we have thus four tangents, and the rationalised form is of course

$$f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgx^2y^2=0.$$

In connexion with the corresponding question in solid geometry, I obtain this equation in a different manner. We investigate the envelope of the line $\xi x+\eta y+\zeta z=0$, considering ξ, η, ζ as parameters connected by the foregoing two equations; by the ordinary process of indeterminate multipliers, we have

$$x+(\lambda a+\mu a')\,\xi=0,\quad y+(\lambda b+\mu b')\,\eta=0,\quad z+(\lambda c+\mu c')\,\zeta=0,$$

and thence, eliminating ξ, η, ζ, we obtain

$$\frac{x^2}{\lambda a+\mu a'}+\frac{y^2}{\lambda b+\mu b'}+\frac{z^2}{\lambda c+\mu c'}=0,$$

$$\frac{ax^2}{(\lambda a+\mu a')^2}+\frac{by^2}{(\lambda b+\mu b')^2}+\frac{cz^2}{(\lambda c+\mu c')^2}=0,$$

$$\frac{a'x^2}{(\lambda a+\mu a')^2}+\frac{b'y^2}{(\lambda b+\mu b')^2}+\frac{c'z^2}{(\lambda c+\mu c')^2}=0,$$

equations equivalent to two equations, from which λ and μ are to be eliminated. The second and third equations are the derived functions of the first equation in regard to λ, μ respectively; and hence, expressing the first equation in an integral form, the result is

$$\text{Disct. }\{x^2(\lambda b+\mu b')(\lambda c+\mu c')+\&\text{c.}\}=0;$$

viz. this is

$$\{(bc'+b'c)\,x^2+(ca'+c'a)\,y^2+(ab'+a'b)\,z^2\}^2-4\,(bcx^2+cay^2+abz^2)\,(b'c'x^2+c'a'y^2+a'b'z^2)=0,$$

or developing and reducing, we have the foregoing result, the coefficients entering through the combinations

$$f,\ g,\ h=bc'-b'c,\ ca'-c'a,\ ab'-a'b.$$

Writing $c=-1$, also $a'=b'=1$, $c'=0$: the equation of the first conic is

$$\frac{x^2}{a}+\frac{y^2}{b}-z^2=0,$$

and that of the second may be replaced by the two equations $x^2+y^2=0$, $z=0$, viz. these give the circular points at infinity: we have f, g, $h=1,\ -1,\ a-b$, and the equation of the line-system is

$$x^4+2x^2y^2+y^4-2h\,(x^2-y^2)\,z^2+h^2z^4=0.$$

If finally, $z=1$, then for the tangents from the circular points at infinity to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}-1=0,$$

the equation is

$$x^4+2x^2y^2+y^4-2h\,(x^2-y^2)+h^2=0,$$

where, as before, $h=a-b$; the four tangents intersect in pairs in the two circular points at infinity, and in four other points which are the foci of the quadric.

Passing now to the problem in solid geometry, we consider the two quadric surfaces

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

$$\frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}+\frac{w^2}{d'}=0;$$

if a common tangent plane hereof be

$$\xi x+\eta y+\zeta z+\omega w=0,$$

then we have

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2=0,$$

$$a'\xi^2+b'\eta^2+c'\zeta^2+d'\omega^2=0;$$

and the circumscribed developable is the envelope of the plane, considering in the equation thereof ξ, η, ζ, ω as variable parameters connected by the last two equations. By what precedes, it is at once seen that the resulting equation is

$$\text{Disct. }\{x^2(b\lambda+b'\mu)(c\lambda+c'\mu)(d\lambda+d'\mu)+\&c.\}=0,$$

viz. this is a quadric equation in (x^2, y^2, z^2, w^2), the coefficients $a, b, c, d, a', b', c', d'$, entering therein through the combinations

$$\mathrm{a,\ b,\ c}=bc'-b'c,\ \ ca'-c'a,\ \ ab'-a'b;$$

$$\mathrm{f,\ g,\ h}=ad'-a'd,\ \ bd'-b'd,\ \ cd'-c'd.$$

The developed result is given in my paper "On the developable surfaces which arise from two surfaces of the second order," *Camb. and Dub. Math. Jour.*, t. v. (1850), pp. 45—53, [84]. We require here, for the particular case, $d=-1$, $a'=b'=c'=1$, $d'=0$, viz. one of the surfaces is taken to be

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-w^2=0,$$

and the other is the circle at infinity $x^2+y^2+z^2=0$, $w=0$; we thus have $\mathrm{a, b, c}=1, 1, 1$; $\mathrm{f, g, h}=b-c,\ c-a,\ a-b$, or now using the italic letters f, g, h to signify these values, we have $f+g+h=0$. The equation is

$$\begin{aligned}
&f^2x^8+g^2y^8+h^2z^8+f^2g^2h^2w^8\\
&\quad+\ 2g(g-h)y^6z^2 + 2h(h-f)z^6x^2 + 2f(f-g)x^6y^2\\
&\quad-\ 2h(g-h)y^2z^6 - 2f(h-f)z^2x^6 - 2g(f-g)x^2y^6\\
&\quad+\ 2f^2(g-h)x^6w^2 + 2g^2(h-f)y^6w^2 + 2h^2(f-g)z^6w^2\\
&\quad-\ 2f^2gh(g-h)x^2w^6 - 2fg^2h(h-f)y^2w^6 - 2fgh^2(f-g)z^2w^6\\
&\quad+\ f^2(f^2-6gh)x^4w^4 + g^2(g^2-6hf)y^4w^4 + h^2(h^2-6fg)z^4w^4\\
&\quad+\ (f^2-6gh)y^4z^4 + (g^2-6hf)z^4x^4 + (h^2-6fg)x^4y^4\\
&\quad+ 2gh(gh-3f^2)w^4y^2z^2 + 2hf(hf-3g^2)w^4z^2x^2 + 2fg(fg-3h^2)w^4x^2y^2\\
&\quad+ 2h(gh-3f^2)z^4x^2w^2 + 2f(hf-3g^2)x^4y^2w^2 + 2g(fg-3h^2)y^4z^2w^2\\
&\quad- 2g(gh-3f^2)y^4x^2w^2 - 2h(hf-3g^2)z^4y^2w^2 - 2f(fg-3h^2)x^4z^2w^2\\
&\quad+\ 2(gh-3f^2)x^4y^2z^2 - 2(hf-3g^2)x^2y^4z^2 - 2(fg-3h^2)x^2y^2z^4\\
&\quad+\ 2(g-h)(h-f)(f-g)x^2y^2z^2w^2=0.
\end{aligned}$$

The equation may be written in the following four forms:

$$x^2\Theta_1 + (gy^2 - hz^2 + ghw^2)^2 \{y^4 + 2y^2z^2 + z^4 - 2f(y^2 - z^2)w^2 + f^2w^4\} = 0,$$

$$y^2\Theta_1 + (hz^2 - fx^2 + hfw^2)^2 \{z^4 + 2z^2x^2 + x^4 - 2g(z^2 - x^2)w^2 + g^2w^4\} = 0,$$

$$z^2\Theta_1 + (fx^2 - gy^2 + fgw^2)^2 \{x^4 + 2x^2y^2 + y^4 - 2h(x^2 - y^2)w^2 + h^2w^4\} = 0,$$

$$w^2\Theta_1 + (x^2 + y^2 + z^2)^2$$
$$\times \{f^2w^4 + g^2y^4 + h^2z^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgx^2y^2\} = 0;$$

the last of these shows that the circle at infinity $x^2 + y^2 + z^2 = 0$, $w = 0$, is a nodal line on the surface; this, however, is not regarded as a focal. The other three show that we have also, as nodal lines, three conics, which are the focals of the given surface; viz. now writing $w = 1$, we have, for the quadric surface

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0,$$

the three focal conics

$$x = 0, \quad -\frac{y^2}{h} + \frac{z^2}{g} - 1 = 0,$$

$$y = 0, \quad \frac{x^2}{h} \quad - \frac{z^2}{f} - 1 = 0,$$

$$z = 0, \quad -\frac{x^2}{g} + \frac{y^2}{f} \quad - 1 = 0,$$

where, as before, f, g, $h = b - c$, $c - a$, $a - b$ respectively. If, as usual, a, b, c are positive and in order of decreasing magnitude, then for the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0,$$

we have the focal conics

$$x = 0, \quad -\frac{y^2}{a-b} + \frac{z^2}{a-c} - 1 = 0,$$

$$y = 0, \quad \frac{x^2}{a-b} \quad - \frac{z^2}{b-c} - 1 = 0,$$

$$z = 0, \quad \frac{x^2}{a-c} + \frac{y^2}{b-c} \quad - 1 = 0;$$

viz. these are an imaginary conic, the focal hyperbola, and the focal ellipse, in the coordinate planes $x = 0$, $y = 0$, $z = 0$ respectively.

903.

ON LATIN SQUARES.

[From the *Messenger of Mathematics*, vol. XIX. (1890), pp. 135—137.]

If in each line of a square of n^2 compartments the same n letters $a, b, c, \ldots$ are arranged so that no letter occurs twice in the same column, we have what was termed by Euler "a Latin square." Supposing that in one of the lines the letters are arranged in the natural order $abcde\ldots$, then in the remaining lines there must be arrangements beginning with b, c, d, e, &c., respectively, and we may consider the case in which the bottom line has the arrangement $abcde\ldots$, and in the other lines, reckoning from the bottom one in order, the arrangements begin with b, c, d, e, &c., respectively: if the number of such squares be $=N$, then, obviously, the whole number of squares which can be formed with the same n arrangements is $=N[n]^n$.

Starting with the bottom line as above, then it is a well-known problem to determine the number of arrangements for the second line; viz. this number is

$$=1.2.3\ldots n\left\{1-\frac{1}{1}+\frac{1}{1.2}-\ldots\pm\frac{1}{1.2\ldots n}\right\};$$

and if we assume, as above, that the second line begins with b, then the whole number of arrangements is this number divided by $(n-1)$, the quotient being of course integral. For instance, when $n=5$, the number is $=120-120+60-20+5-1$, $=44$, which is divisible by 4, and the number of arrangements for the second line is thus $=11$.

But the number of arrangements for the third line will be different according to the arrangement selected for the second line, and it is not easy to see how in general the whole number of arrangements for the third line is to be calculated, and the difficulty of course increases for the next following lines; it may be remarked that, when all the lines are filled up except the top-line, the top-line is completely determined.

Imagine the square completed: we may write down the substitutions by which we pass from the bottom line to itself (this is of course the substitution 1) and to each of the other lines respectively; we have thus a set of n substitutions, which may form *a group*; and when this is so, we may conversely from the group construct the Latin square. But it is not every Latin square which is thus connected with a group of n substitutions.

In the cases $n=2$, 3, 4 there is no difficulty; the squares are

$$\begin{array}{llllll} b\,a, & c\,a\,b, & d\,c\,a\,b, & d\,c\,b\,a, & d\,c\,b\,a, & d\,a\,b\,c, \\ a\,b, & b\,c\,a, & c\,d\,b\,a, & c\,d\,a\,b, & c\,a\,d\,b, & c\,d\,a\,b, \\ & a\,b\,c, & b\,a\,d\,c, & b\,a\,d\,c, & b\,d\,a\,c, & b\,c\,d\,a, \\ & & a\,b\,c\,d, & a\,b\,c\,d, & a\,b\,c\,d, & a\,b\,c\,d; \end{array}$$

viz. when $n=2$ the number is 1, when $n=3$ it is 1, when $n=4$ it is 4; in this last case, the arrangement *badc* for the second line gives two squares, but each of the other arrangements only one square.

In each of the squares of 4, we have a group, viz. for the four squares respectively, these are

$$\begin{array}{llllll} 1^\circ, & 1, & (ab)\,(cd), & (acbd)\,, & (adbc)\,, \\ 2^\circ & 1, & (ab)\,(cd), & (ac)\,(bd), & (ad)\,(bc), \\ 3^\circ & 1, & (abdc)\,, & (acdb)\,, & (ad)\,(bc), \\ 4^\circ & 1, & (abcd)\,, & (ac)\,(bd), & (adcb)\,. \end{array}$$

1°, 3°, 4° are the cyclical groups of $(acbd)$, $(abdc)$, and $(abcd)$, respectively; 2° is a different kind of group.

In the case when $n=5$, the whole number of squares is 56; viz. there are five arrangements of the second line each giving four squares, and six arrangements each giving six squares, $5.4+6.6=56$. The five arrangements are

$$\begin{array}{lllll} b\,a\,e\,c\,d, & b\,a\,d\,e\,c, & b\,c\,a\,e\,d, & b\,d\,e\,a\,c, & b\,e\,d\,c\,a, \\ a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, \end{array}$$

viz. in these cases, the substitutions for passing to the second line are $(ab)\,(ced)$ $(ab)\,(cde)$, $(abc)\,(de)$, $(abd)\,(ce)$, $(abe)\,(cd)$, respectively.

The six arrangements are

$$\begin{array}{llllll} b\,d\,a\,e\,c, & b\,e\,a\,c\,d, & b\,c\,e\,a\,d, & b\,e\,d\,a\,c, & b\,c\,d\,e\,a, & b\,d\,e\,c\,a, \\ a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e; \end{array}$$

viz. in these cases, the substitutions for passing to the second line are $(abdec)$, $(abedc)$, $(abced)$, $(abecd)$, $(abcde)$, $(abdce)$, respectively.

A set of four squares is

$$\begin{array}{llll}
e\,c\,d\,b\,a, & e\,d\,a\,b\,c, & e\,c\,d\,a\,b, & e\,d\,b\,a\,c, \\
d\,e\,b\,a\,c, & d\,c\,b\,e\,a, & d\,e\,a\,b\,c, & d\,c\,a\,e\,b, \\
c\,d\,a\,e\,b, & c\,e\,d\,a\,b, & c\,d\,b\,e\,a, & c\,e\,d\,b\,a, \\
b\,a\,e\,c\,d, & b\,a\,e\,c\,d, & b\,a\,e\,c\,d, & b\,a\,e\,c\,d, \\
a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e;
\end{array}$$

and a set of six squares is

$$\begin{array}{llllll}
e\,c\,d\,a\,b, & e\,a\,d\,c\,b, & e\,c\,d\,b\,a, & e\,a\,b\,c\,d, & e\,a\,b\,c\,d, & e\,c\,b\,a\,d, \\
d\,e\,b\,c\,a, & d\,c\,e\,b\,a, & d\,a\,e\,c\,b, & d\,c\,e\,b\,a. & d\,c\,e\,a\,b, & d\,a\,e\,c\,b, \\
c\,a\,e\,b\,d, & c\,e\,b\,a\,d, & c\,e\,b\,a\,d, & c\,e\,d\,a\,b, & c\,e\,d\,b\,a, & c\,e\,d\,b\,a, \\
b\,d\,a\,e\,c, & b\,d\,a\,e\,c, & b\,d\,a\,e\,c, & b\,d\,a\,e\,c, & b\,d\,e\,a\,c, & b\,d\,a\,e\,c, \\
a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e, & a\,b\,c\,d\,e.
\end{array}$$

In a square belonging to a set of four, the substitutions for obtaining from any one line all the other lines are of a form such as 1, $(ab)(ced)$, $(ac)(bde)$, $(ad)(bec)$, $(ae)(bcd)$, which are not a group. In the case of a set of six squares, there is one square of the set (in the foregoing instance the first square) where the substitutions are of a form such as 1, $(abdec)$, $(acedb)$, $(adceb)$, $(aebcd)$, and which thus form a cyclical group of five substitutions.

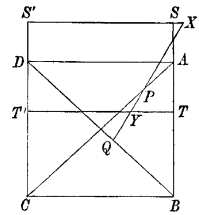
904.

NOTE ON RECIPROCAL LINES.

[From the *Messenger of Mathematics*, vol. XIX. (1890), pp. 174, 175.]

IF two lines are reciprocal in regard to a quadric surface, then any point on the one line and any point on the other line are harmonics in regard to the surface, viz. the two points and the intersections of their line of junction with the surface form a harmonic range. This is obvious: the polar plane of the first point passes through the second line, and thus the second point is a point in the polar plane of the first point, that is, the two points are harmonics.

But it is worth while to look at the theorem in a different point of view. If the first line meets the surface in the points A, C and the second line meets the surface in the points B, D (in order to the four points being real, the surface must of course be a skew hyperboloid), then AB, BC, CD, DA are lines on the surface,



say AB, CD are directrices, and BC, AD are generatrices, the two reciprocal lines being the diagonals AC and BD of the skew quadrilateral. Taking on AC a point P and on BD a point Q, the theorem is that, if PQ meets the surface in the points

X, Y, then the four points P, Q, X, Y are harmonics. Let the generatrix through X meet AB, CD in the points S, S' respectively, and the generatrix through Y meet the same lines in the points T, T' respectively. Then the four lines CB, DA, $T'T$, $S'S$, are all met by an infinite series of lines, and in particular they are met by the lines CD, BA in the points C, D, T', S' and B, A, T, S respectively. Consequently, writing AH for anharmonic ratio, we have

$$AH(C, D, T', S') = AH(B, A, T, S).$$

Consider now the four lines CA, DB, $T'T$, $S'S$: these are each of them met by the lines CD and BA, and also by the line PQ; therefore, by an infinity of lines (that is, they are directrices of a hyperboloid). They are met by the last-mentioned three lines in the points C, D, T', S'; A, B, T, S; and P, Q, Y, X respectively; hence

$$AH(C, D, T', S') = AH(A, B, T, S) = AH(P, Q, Y, X).$$

Comparing with the foregoing equation, it appears that

$$AH(B, A, T, S) = AH(A, B, T, S),$$

viz. the anharmonic ratio is not altered by the interchange of the two points A, B: this implies that the points A, B, T, S are harmonics, viz. the pairs A, B and S, T are harmonics; and, this being so, the equation

$$AH(A, B, T, S) = AH(P, Q, Y, X)$$

shows that P, Q, Y, X are harmonics, viz. that the pairs P, Q and X, Y are harmonics; or say P, Q are harmonics in regard to the quadric surface, which is the theorem in question.

905.

ON THE EQUATION $x^{17}-1=0$.

[From the *Messenger of Mathematics*, vol. XIX. (1890), pp. 184—188.]

WRITING $\rho=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$, I carry the solution up to the determination of the periods each of two roots, $\rho+\rho^{16}$, $=2\cos\frac{2\pi}{17}$, &c. The expressions contain the radicals

$$a=\sqrt{(17)},\quad b=\sqrt{\{2(17-a)\}},\quad c=\sqrt{\{4(17+3a)-2(3+a)b\}},$$

where a, b, c are taken to be positive ($a=4{\cdot}12$, $b=5{\cdot}07$, $c=6{\cdot}72$). Taking for a moment r to be any imaginary seventeenth root, $r=\rho^{\theta}$, then the algebraical expression for the period P_1 of eight roots is $P_1=\frac{1}{2}(-1\pm a)$, but I assume the value to be $P_1=\frac{1}{2}(-1+a)$, and thus determine θ to denote some one of the values 1, 2, 4, 8, 9, 13, 15, 16; similarly, I assume the value of Q_1 to be $=\frac{1}{4}(-1+a+b)$; and thus further determine θ to denote some one of the values 1, 4, 13, 16: and, again, I assume the value of R_1 to be $=\frac{1}{8}(-1+a+b+c)$, and thus further determine θ to denote one of the values 1 and 16. As regards the values of the periods R, it is obviously indifferent which value is taken, and I assume therefore $\theta=1$. This comes to saying that the signs of the radicals are determined in suchwise that r shall denote the root $\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$; and it is to be understood that r has this value.

I write now

$$\begin{aligned}
P_1&=r+r^9+r^{13}+r^{15}+r^{16}+r^8+r^4+r^2,\\
P_2&=r^3+r^{10}+r^5+r^{11}+r^{14}+r^7+r^{12}+r^6,\\
Q_1&=r+r^{13}+r^{16}+r^4,\\
Q_2&=r^9+r^{15}+r^8+r^2,\\
Q_3&=r^3+r^5+r^{14}+r^{12},\\
Q_4&=r^{10}+r^{11}+r^7+r^6,\\
R_1&=r+r^{16},\\
R_2&=r^{13}+r^4,\\
R_3&=r^9+r^8,
\end{aligned}$$

$$R_4 = r^{15} + r^{3},$$
$$R_5 = r^{3} + r^{14},$$
$$R_6 = r^{5} + r^{12},$$
$$R_7 = r^{10} + r^{7},$$
$$R_8 = r^{11} + r^{6},$$

and moreover

$$\begin{aligned} a &= \quad P_1 - P_2, \\ b &= 2\,(Q_1 - Q_2), \\ +b_1 &= 2\,(Q_3 - Q_4), \\ c &= 4\,(R_1 - R_2), \\ -c_1 &= 4\,(R_3 - R_4), \\ +c_2 &= 4\,(R_5 - R_6), \\ -c_3 &= 4\,(R_7 - R_8). \end{aligned}$$

It will appear by what follows that a is determined by the quadric equation $a^2 = 17$, but I have assumed that a denotes the positive root $a = \sqrt{(17)}$; similarly, b is determined by the quadric equation $b^2 = 2\,(17 - a)$, but it is assumed that b denotes the positive root, $b = \sqrt{\{2\,(17 - a)\}}$; and c is determined by the quadric equation $c^2 = 4\,(17 + 3a) - 2\,(3 + a)\,b$, but it is assumed that c denotes the positive root,

$$c = \sqrt{\{4\,(17 + 3a) - 2\,(3 + a)\,b\}}.$$

If in the equations I had written b_1, c_1, c_2, c_3, instead of $+b_1$, $-c_1$, $+c_2$, $-c_3$, then b_1 comes out rationally in terms of a, b; and c_1, c_2, c_3 come out rationally in terms of a, b, c; the signs were attached to them *à posteriori*, in suchwise that the values of b_1, c_1, c_2, c_3 might be each of them positive; for their independent determination, we have, in fact, for b_1^2 an expression such as that for b^2; and for c_1^2, c_2^2, c_3^2 expressions such as that for c^2; and taking as above for each of them the positive value of the square root, we have

$$\begin{aligned} a &= \sqrt{(17)}, && (= 4{\cdot}12), \\ b &= \sqrt{\{2\,(17 - a)\}}, && (= 5{\cdot}07), \\ b_1 &= \sqrt{\{2\,(17 + a)\}}, && (= 6{\cdot}49), \\ c &= \sqrt{\{4\,(17 + 3a) - 2\,(3 + a)\,b\}}, && (= 6{\cdot}72), \\ c_1 &= \sqrt{\{4\,(17 + 3a) + 2\,(3 + a)\,b\}}, && (= 13{\cdot}77), \\ c_2 &= \sqrt{\{4\,(17 - 3a) + 2\,(-3 + a)\,b_1\}}, && (= 5{\cdot}75), \\ c_3 &= \sqrt{\{4\,(17 - 3a) - 2\,(-3 + a)\,b_1\}}, && (= 2{\cdot}02). \end{aligned}$$

The relations between the periods P are

$$P_1 + P_2 = -1,$$

	P_1	P_2
P_1	$-P_1+4$	-4
P_2		$-P_2+4$

	a
a	17

that is,

$$P_1^2 = -P_1 + 4, \text{ \&c.}; \quad a^2 = 17.$$

Here for the second square, we have

$$a^2 = (P_1 - P_2)^2 = P_1^2 + P_2^2 - 2P_1P_2 = (-P_1+4) + (-P_2+4) + 8, \; = -P_1 - P_2 + 16, \; = 17;$$

and similarly in the other cases which follow.

For the periods Q, we have

$$Q_1 + Q_2 = P_1,$$
$$Q_3 + Q_4 = P_2,$$

	Q_1	Q_2	Q_3	Q_4
Q_1	Q_2+2Q_3+4	-1	$2Q_1+Q_2+Q_4$	$Q_3+Q_3+2Q_4$
Q_2		Q_1+2Q_4+4	Q_1+2Q_3+4	$Q_1+2Q_2+Q_3$
Q_3			$2Q_2+Q_4+4$	-1
Q_4				$2Q_1+Q_3+4$

	b	b_1
b	$34-2a$	$8a$
b_1		$34+2a$

so that $bb_1 = 8a$, or b_1 is given rationally in terms of a, b.

And for the periods R, we have

$$R_1 + R_2 = Q_1,$$
$$R_3 + R_4 = Q_2,$$
$$R_5 + R_6 = Q_3,$$
$$R_7 + R_8 = Q_4,$$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1	R_4+2	$\overline{R_5+R_6}$	R_3+R_7	R_1+R_5	R_2+R_4	R_2+R_8	R_3+R_8	R_6+R_7
R_2		R_3+2	R_2+R_6	R_4+R_8	R_1+R_7	R_1+R_3	R_5+R_8	R_4+R_7
R_3			R_1+2	$\overline{R_7+R_8}$	R_6+R_8	R_2+R_5	R_1+R_4	R_4+R_5
R_4				R_2+2	R_1+R_6	R_5+R_7	R_3+R_6	R_2+R_3
R_5					R_8+2	$\overline{R_3+R_4}$	R_2+R_7	R_3+R_5
R_6						R_7+2	R_4+R_6	R_1+R_8
R_7							R_5+2	$\overline{R_1+R_2}$
R_8								R_3+2

	c	c_1	c_2	c_3
c	$4(17+3a)-2(3+a)b$	$8(b+b_1)$	$4(2a-b+b_1)$	$4(-2a+b+b_1)$
c_1		$4(17+3a)+2(3+a)b$	$4(2a+b+b_1)$	$4(2a+b-b_1)$
c_2			$4(17-3a)+2(-3+a)b_1$	$8(-b+b_1)$
c_3				$4(17-3a)-2(-3+a)b_1$

where observe that the overlined terms R_5+R_6, R_7+R_8, R_3+R_4, and R_1+R_2, have the values Q_4, Q_3, Q_2, Q_1, respectively, and in the last table b_1 may be considered as denoting its value $=\frac{8a}{b}$, so that c_1, c_2, c_3 are each given rationally in terms of a, b, c.

And from the foregoing results, we have

$$P_1=\tfrac{1}{2}(-1+a), \quad = 1{\cdot}56,$$
$$P_2=\tfrac{1}{2}(-1-a), \quad = -2{\cdot}56,$$

$$Q_1=\tfrac{1}{4}(-1+a+b), \quad = 2{\cdot}05,$$
$$Q_2=\tfrac{1}{4}(-1+a-b), \quad = -0{\cdot}49,$$
$$Q_3=\tfrac{1}{4}(-1-a+b_1), \quad = 0{\cdot}34,$$
$$Q_4=\tfrac{1}{4}(-1-a-b_1), \quad = -2{\cdot}90,$$

$$R_1=\tfrac{1}{8}(-1+a+b+c), \quad = 1{\cdot}87,$$
$$R_2=\tfrac{1}{8}(-1+a+b-c), \quad = 0{\cdot}18,$$
$$R_3=\tfrac{1}{8}(-1+a-b-c_1), \quad = -1{\cdot}96,$$
$$R_4=\tfrac{1}{8}(-1+a-b+c_1), \quad = 1{\cdot}47,$$
$$R_5=\tfrac{1}{8}(-1-a+b_1+c_2), \quad = 0{\cdot}89,$$
$$R_6=\tfrac{1}{8}(-1-a+b_1-c_2), \quad = -0{\cdot}55,$$
$$R_7=\tfrac{1}{8}(-1-a-b_1-c_3), \quad = -1{\cdot}70,$$
$$R_8=\tfrac{1}{8}(-1-a-b_1+c_3), \quad = -1{\cdot}20.$$

The approximate numerical values have been given throughout only for the purpose of showing that the signs of the square roots have been rightly determined.

906.

NOTE ON SCHLAEFLI'S MODULAR EQUATION FOR THE CUBIC TRANSFORMATION; WITH A CORRECTION.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 59, 60; 120.]

THE equation in question, *Crelle*, t. LXXII. (1870), p. 369, is

$$S^4 + T^4 + 8ST - S^3T^3 = 0,$$

where

$$S = \frac{\sqrt{2}}{\sqrt[4]{(kk')}}, \quad T = \frac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}};$$

which must be of course equivalent to the ordinary modular equation

$$u^4 - v^4 + 2uv - 2u^3v^3 = 0,$$

where

$$u = \sqrt[4]{k}, \quad v = \sqrt[4]{\lambda};$$

the resemblance in form between the two equations, with such different meanings of the S and T in the one case, and the u and v in the other, is very noticeable.

Schlaefli's equation is

$$\frac{4}{kk'} + \frac{4}{\lambda\lambda'} + \frac{16}{(kk'\lambda\lambda')^{\frac{1}{4}}} - \frac{8}{(kk'\lambda\lambda')^{\frac{3}{4}}} = 0,$$

that is,

$$kk' + \lambda\lambda' = 2\,(kk'\lambda\lambda')^{\frac{1}{4}} - 4\,(kk'\lambda\lambda')^{\frac{3}{4}},$$

or say

$$k^2k'^2 + \lambda^2\lambda'^2 = 4\,(kk'\lambda\lambda')^{\frac{1}{2}} - 18\,(kk'\lambda\lambda') + 16\,(kk'\lambda\lambda')^{\frac{3}{2}}.$$

To deduce this from the uv-modular equation, we have (Jacobi's *Fund. Nova*, p. 68, *Ges. Werke*, t. I., p. 124),

$$(1 - u^8)(1 - v^8) = (1 - u^2v^2)^4,$$

or, multiplying each side by u^8v^8, and extracting the fourth root, we have

$$\sqrt{(kk'\lambda\lambda')} = u^2v^2(1 - u^2v^2) = x - x^2,$$

if for shortness we write $x = u^2v^2$.

The equation to be proved thus is

$$u^8(1 - u^8) + v^8(1 - v^8) = 4(x - x^2) - 18(x - x^2)^2 + 16(x - x^2)^3.$$

But from the foregoing equation

$$(1 - u^8)(1 - v^8) = (1 - u^2v^2)^4,$$

we have

$$u^8 + v^8 = 4u^2v^2 - 6u^4v^4 + 4u^6v^6, \quad = 4x - 6x^2 + 4x^3,$$

and thence

$$u^{16} + v^{16} = (4x - 6x^2 + 4x^3)^2 - 2x^4;$$

and the equation to be proved thus becomes

$$(4x - 6x^2 + 4x^3) - (4x - 6x^2 + 4x^3)^2 + 2x^4 = 4(x - x^2) - 18(x - x^2)^2 + 16(x - x^2)^3,$$

which is in fact an identity, each side being

$$= 4x - 22x^2 + 52x^3 - 66x^4 + 48x^5 - 16x^6.$$

CORRECTION, p. 120.

I find that I misquoted Schlaefli's equation, viz. in effect, I took it to be

$$S_1^4 + T_1^4 + 8S_1T_1 - S_1^2T_1^2 = 0, \text{ where } S_1 = \frac{\sqrt{2}}{\sqrt[4]{(kk')}}, \quad T_1 = \frac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}}:$$

whereas his equation really is

$$S^4 + T^4 - 8ST + S^2T^3 = 0, \text{ where } S = 2\sqrt[4]{(kk')}, \quad T = 2\sqrt[4]{(\lambda\lambda')}.$$

The change is only a change of form, for writing $S_1 = \frac{2\sqrt{2}}{S}$ and $T_1 = \frac{2\sqrt{2}}{T}$, the equation in (S_1, T_1) is converted into that in (S, T); but it was quite an unnecessary one, and I cannot account for having made it, as the paper in *Crelle* must have been before me.

907.

NOTE ON THE NINTH ROOTS OF UNITY.

[From the *Messenger of Mathematics*, vol. xx. (1891), p. 63.]

LET θ be a prime ninth root of unity, so that $\theta^6+\theta^3+1=0$; and write

$$a=\theta+\theta^8,$$
$$b=\theta^2+\theta^7,$$
$$c=\theta^4+\theta^5;$$

then

$$a+b+c+\theta^3+\theta^6=\frac{1-\theta^9}{1-\theta}-1,\ =-1,$$

that is,

$$a+b+c=0.$$

Also

$$a^2=b+2,\quad bc=b-1,$$
$$b^2=c+2,\quad ca=c-1,$$
$$c^2=a+2,\quad ab=a-1,$$

whence

$$ab+ac+bc=-3,$$
$$abc=-1;$$

and a, b, c are thus the roots of the equation $x^3-3x+1=0$. We have

$$a^2b+b^2c+c^2a=a^2+b^2+c^2=6,$$
$$ab^2+bc^2+ca^2=bc+ca+ab=-3,$$

and thence

$$-(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)=(b-c)(c-a)(a-b)=-6-3,\ =-9.$$

The equation $x^3-3x+1=0$ is thus such that $a^2b+b^2c+c^2a$, and consequently any rational function whatever of a, b, c, invariable by the cyclical interchange (abc) of the roots, has a rational value.

908.

ON TWO INVARIANTS OF A QUADRIQUADRIC FUNCTION.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 68, 69.]

THE quadriquadric function

$$\begin{aligned} & z^2\ (ax^2 + 2hxy + g'y^2) \\ + & 2zw\ (h'x^2 + 2bxy + fy^2) \\ + & \quad w^2 (gx^2 + 2f'xy + cy^2), \end{aligned}$$

considered successively as a function of (z, w) and of (x, y), has the discriminants U, V, equal to

$$(ax^2 + 2hxy + g'y^2)(gx^2 + 2f'xy + cy^2) - (h'x^2 + 2bxy + fy^2)^2,$$

$$(az^2 + 2h'zw + gw^2)(g'z^2 + 2fzw + cw^2) - (hz^2 + 2bzw + f'w^2)^2,$$

respectively. As is well known, these quartic functions have each of them the same quadrinvariant and the same cubinvariant; these are the invariants in question of the quadriquadric function.

The quadrinvariant has been calculated in a different notation, but I am not aware that the cubinvariant has been before calculated; the two values are as follows:

Quadrinvariant is

a^2c^2	$+3$
ab^2c	-24
b^4	$+48$
$acgg'$	$+42$
$achf'$	-12
$ach'f$	-12
$abff'$	$+72$
$bchh'$	$+72$
b^2gg'	-24
b^2fh'	-96
$b^2f'h$	-96
af'^2g	-36
af'^2g'	-36
$bfgh$	$+72$
$bf'g'h'$	$+72$
cgh^2	-36
$cg'h'^2$	-36
$g^2g'^2$	$+3$
$f^2h'^2$	$+48$
f'^2h^2	$+48$
$gg'hf'$	-12
$gg'h'f$	-12
$ff'hh'$	-48
	± 480

Cubinvariant is

a^3c^3	-1
$a^2b^2c^2$	$+12$
ab^4c	-48
b^6	$+64$
	$+76$
	-49

a^2c^2gg'	$+33$
ab^2cgg'	-120
a^2c^2fh'	$+6$
$a^2c^2f'h$	$+6$
ab^2cfh'	$+24$
$ab^2cf'h$	$+24$
b^4fh'	-192
$b^4f'h$	-192
b^4gg'	-48
a^2bcff'	-36
abc^2hh'	-36
ab^2ff'	$+144$
b^3chh'	$+144$
	$+381$
	-624

a^2cf^2g	-36
$a^2cf'^2g'$	-36
ac^2gh^2	-36
$ac^2g'h'^2$	-36
ab^2f^2g	-72
$ab^2f'^2g'$	-72
b^3cgh^2	-72
$b^3cg'h'^2$	-72
$abcfgh$	$+180$
$abcf'g'h'$	$+180$
b^3fgh	$+144$
$b^3f'g'h'$	$+144$
	$+648$
	-432

$acg^2g'^2$	$+33$
$acf'gg'h$	-60
$acfgg'h'$	-60
$acf^2h'^2$	$+24$
acf'^2h^2	$+24$
$acff'hh'$	$+12$
$abff'gg'$	$+180$
$abf^2f'h'$	-144
$abff'^2h$	-144
$bcgg'hh'$	$+180$
$bcfhh'^2$	-144
$bcf'h^2h'$	-144
$a^2f^3f'^2$	$+54$
$c^2h^2h'^2$	$+54$
$b^2f^2h'^2$	$+192$
$b^2f'^2h^2$	$+192$
$b^2ff'hh'$	$+96$
$b^2fgg'h'$	$+24$
$b^2f'gg'h$	$+24$
$b^2g^2g'^2$	$+12$
	$+1101$
	-696

$af^2f'gh$	-36
$aff'^2g'h'$	-36
af^2g^2g'	-36
$af'^2gg'^2$	-36
af^3gh'	$+72$
$af'^3g'h$	$+72$
$cfgh^3h'$	-36
$cf'g'hh'^2$	-36
$cg^2g'h^2$	-36
$cgg'^2h'^2$	-36
$cf'gh^3$	$+72$
$cfg'h'^3$	$+72$
$bff'g'h'^2$	-144
$bf'^2g'hh'$	-144
bf^2ghh'	-144
$bff'gh^2$	-144
$bfg^2g'h$	-36
$bf'gg'^2h'$	-36
	$+288$
	-936

$g^3g'^3$	-1
$fg^2g'^2h'$	$+6$
$f'g^2g'^2h$	$+6$
$f^2gg'h'^2$	$+24$
$f'^2gg'h^2$	$+24$
$ff'gg'hh'$	$+12$
$f^3h'^3$	-64
f'^3h^3	-64
$f^2g^2h^2$	$+54$
$f'^2g'^2h'^2$	$+54$
$f^2f'hh'^2$	$+96$
ff'^2h^2h'	$+96$
	$+372$
	-129
	± 2866

By writing herein f', g', $h'=f$, g, h, we obtain of course the two invariants of the symmetrical quadriquadric function.

909.

ON A PARTICULAR CASE OF KUMMER'S DIFFERENTIAL EQUATION OF THE THIRD ORDER.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 75—79.]

THE general form of equation in question is

$$\frac{x'''}{x'} - \tfrac{3}{2}\left(\frac{x''}{x'}\right)^2 + x'^2\left\{\frac{A}{(x-1)^2} + \frac{B}{x(x-1)} + \frac{C}{x^2}\right\} - \left\{\frac{A'}{(t-1)^2} + \frac{B'}{t(t-1)} + \frac{C'}{t^2}\right\} = 0,$$

here x is a function of t; and A, B, C, A', B', C' are numerical constants. For various given values of A, B, C, and values determined thereby of A', B', C', the equation admits of a solution in the form x = rational function of t; the theory in reference to the cases considered by Schwarz is considered in my paper "On the Schwarzian Derivative and the Polyhedral Functions," *Camb. Phil. Trans.*, t. XIII. (1883), pp. 5—68, [744]. But the theory is considered in a more general and exhaustive manner in Goursat's memoir, "Recherches sur l'équation de Kummer," *Acta Soc. Sci. Fennicæ*, t. XV. (1888), pp. 47—127. I consider here one of the solutions given by him, viz. writing

$$\begin{aligned} P &= 4t - 5 &&, \quad X = t^2P^3, \\ Q &= 5t - 4 &&, \quad Y = Q^3, \\ R &= 8t^2 - 11t + 8, &&\quad Z = -(t-1)^2R^2, \end{aligned}$$

so that, identically, $X + Y + Z = 0$; then the solution is expressed by either of the equivalent equations

$$\begin{aligned} x &= -\frac{X}{Z} = \frac{t^2P^3}{(t-1)^2R^2}, \\ x - 1 &= \frac{Y}{Z} = -\frac{Q^3}{(t-1)^2R^2}. \end{aligned}$$

The values of the constants to which this solution belongs are

$$A=\tfrac{4}{9},\ B=-\tfrac{37}{32},\ C=\tfrac{4}{9};\ A'=\tfrac{3}{8},\ B'=\tfrac{131}{144},\ C'=\tfrac{5}{18}.$$

But instead of assuming these values in the first instance, I leave the values indeterminate; and starting from the foregoing expression for x, I substitute this in

$$\Omega=\frac{x'''}{x'}-\tfrac{3}{2}\left(\frac{x''}{x'}\right)^2+x'^2\left\{\frac{A}{(x-1)^2}+\frac{B}{x(x-1)}+\frac{C}{x^2}\right\}-\left\{\frac{A'}{(t-1)^2}+\frac{B'}{t(t-1)}+\frac{C'}{t^2}\right\},$$

thus obtaining Ω as a function of t which, as will appear, vanishes identically when A, B, C, A', B', C' have the foregoing values.

I remark that this is, in effect, doing in a somewhat different form for the particular case what Goursat does for the general case, viz. starting from

$$\Omega_1=\frac{x'''}{x'}-\tfrac{3}{2}\left(\frac{x''}{x'}\right)^2+x'^2\left\{\frac{A}{(x-1)^2}+\frac{B}{x(x-1)}+\frac{C}{x^2}\right\},$$

with values of A, B, C which belong to the solution considered, he shows that this is a function of t having no infinities other than $(0, 1, \infty)$; that ∞ is not an infinity of the function or of the function multiplied into t, and that 0 and 1 are each of them a twofold infinity; that is, that the function is of the form

$$\frac{Lt^2+Mt+N}{t^2(t-1)^2}\ \text{or}\ \frac{A'}{(t-1)^2}+\frac{B'}{t(t-1)}+\frac{C'}{t^2}.$$

Proceeding to carry out the process, we have

$$\frac{x'}{x-1}=-\frac{1}{t-1}+\frac{3Q'}{Q}-\frac{2R'}{R},$$

$$\frac{x'}{x}=\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R},$$

and from either of these equations, collecting and reducing,

$$x'=\frac{5tP^2Q^2}{(t-1)^2R^2},$$

where observe that, from the values of x and $x-1$ respectively, it appears *à priori* that tP^2 and Q^2 must be factors in the numerator of x'. From this value of x', we have

$$\frac{x''}{x'}=\frac{1}{t}-\frac{2}{t-1}+\frac{2P'}{P}+\frac{2Q'}{Q}-\frac{3R'}{R};$$

and hence, P' and Q' being mere constants,

$$\frac{x'''}{x'}-\left(\frac{x''}{x'}\right)^2=-\frac{1}{t^2}+\frac{2}{(t-1)^2}-\frac{3R''}{R}-2\left(\frac{P'}{P}\right)^2-2\left(\frac{Q'}{Q}\right)^2+3\left(\frac{R'}{R}\right)^2,$$

and consequently

$$\begin{aligned}\Omega = &-\frac{1}{t^2}+\frac{2}{(t-1)^2}-\frac{3R''}{R}+2\left(\frac{P'}{P}\right)^2-2\left(\frac{Q'}{Q}\right)^2+3\left(\frac{R'}{R}\right)^2\\ &-\tfrac{1}{2}\left(\frac{1}{t}-\frac{2}{t-1}+\frac{2P'}{P}+\frac{2Q'}{Q}-\frac{3R'}{R}\right)^2\\ &+A\left(-\frac{1}{t-1}+\frac{3Q'}{Q}-2\frac{R'}{R}\right)^2\\ &+B\left(-\frac{1}{t-1}+\frac{3Q'}{Q}-2\frac{R'}{R}\right)\left(\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R}\right)\\ &+C\left(\frac{2}{t}-\frac{1}{t-1}+\frac{3P'}{P}-\frac{2R'}{R}\right)^2\\ &-\frac{C'}{(t-1)^2}-\frac{B'}{t(t-1)}-\frac{A'}{t^2}.\end{aligned}$$

Putting for shortness

$$\frac{1}{t}=\alpha,\quad \frac{1}{t-1}=\beta,\quad \frac{P'}{P}=p,\quad \frac{Q'}{Q}=q,\quad \frac{R'}{R}=r,$$

this equation gives

$$\begin{aligned}\Omega = &-\alpha^2+2\beta^2-\frac{3R''}{R}-2p^2-2q^2+3r^2\\ &-\tfrac{1}{2}(\alpha-2\beta+2p+2q-3r)^2\\ &+A(-\beta+3q-2r)^2\\ &+B(-\beta+3q-2r)(2\alpha-\beta+3p-2r)\\ &+C(2\alpha-\beta+3p-2r)^2\\ &-C'\alpha^2-B'\alpha\beta-A'\beta^2,\end{aligned}$$

which is

$$\begin{aligned}= \quad &\alpha^2(-\tfrac{3}{2}+4C-C')-\frac{3R''}{R}\text{: say it is } = L\alpha^2-\frac{48}{R}\\ &+\alpha\beta(2-2B-4C-B') && +M\alpha\beta\\ &+\beta^2(A+B+C-A') && +N\beta^2\\ &+\alpha p(-2+12C) && +F\alpha p\\ &+\alpha q(-2+6B) && +G\alpha q\\ &+\alpha r(3-4B-8C) && +H\alpha r\\ &+\beta p(4-3B-6C) && +F'\beta p\\ &+\beta q(4-6A-3B) && +G'\beta q\\ &+\beta r(-6+4A+4B+4C) && +H'\beta r\\ &+p^2(-4+9C) && +A''p^2\\ &+q^2(-4+9A) && +B''q^2\\ &+r^2(-\tfrac{3}{2}+4A+4B+4C) && +C''r^2\\ &+qr(6-12A-6B) && +F''qr\\ &+rp(6-6B-12C) && +G''rp\\ &+pq(-4+9B) && +H''pq.\end{aligned}$$

By decomposing $\alpha\beta$, αp, &c., into simple fractions, this becomes

$$\begin{aligned}
\Omega = \quad & L\alpha^2 - \frac{48}{R} \\
& + M(-\alpha+\beta) \\
& + N\beta^2 \\
& + F(-\tfrac{4}{5}\alpha \quad + \tfrac{4}{5}p) \\
& + G(-\tfrac{5}{4}\alpha \quad + \tfrac{5}{4}q) \\
& + H\left(-\tfrac{11}{8}\alpha + \frac{11t+\frac{7}{8}}{R}\right) \\
& + F'\ (-4\beta + 4p) \\
& + G'\ (\quad 5\beta - 5q) \\
& + H'\left(\quad \beta + \frac{-8t+19}{R}\right) \\
& + A''p^2 \\
& + B''q^2 \\
& + C''r^2 \\
& + F''\tfrac{5}{12}\left(q - \frac{8t-43}{R}\right) \\
& + G''\tfrac{4}{3}\left(p - \frac{8t-13}{R}\right) \\
& + H''\tfrac{20}{9}(p-q).
\end{aligned}$$

This is

$$\begin{aligned}
= \quad & \alpha^2 L \\
& + \alpha(-M - \tfrac{4}{5}F - \tfrac{5}{4}G - \tfrac{11}{8}H) \\
& + \beta^2 N \\
& + \beta(M - 4F' + 5G' + H') \\
& + p^2 A'' \\
& + p(4F' + \tfrac{4}{3}G'' + \tfrac{20}{9}H'' + \tfrac{4}{5}F) \\
& + q^2 B'' \\
& + q(-5G' - \tfrac{5}{12}F'' - \tfrac{20}{9}H'' + \tfrac{5}{4}G) \\
& + r^2 C'' \\
& + \frac{1}{R}\{-48 + H(11t+\tfrac{7}{8}) + H'(-8t+19) - \tfrac{5}{12}F''(8t-43) - \tfrac{4}{3}G''(8t-13)\}.
\end{aligned}$$

This should be identically $=0$; making $A''=0$, $B''=0$, $C''=0$, we find

$$A = \tfrac{4}{9},\ B = -\tfrac{37}{32},\ C = \tfrac{4}{9};\ (A+B+C = \tfrac{3}{8}).$$

and thence

$$F=\tfrac{10}{3},\ G=-\tfrac{61}{12},\ H=\tfrac{3}{2};\ F'=\tfrac{23}{8},\ G'=\tfrac{23}{8},\ H'=-\tfrac{9}{2};$$

$$F''=\tfrac{15}{4},\ G''=\tfrac{15}{4},\ H''=-\tfrac{68}{9}.$$

These values make the coefficients of p and q to be each $=0$; and they make the coefficient of R to be identically $=0$, viz. we have

$$0=-48+\tfrac{7}{8}H+19H'+\tfrac{215}{12}F''+\tfrac{52}{3}G'',$$

and

$$0=\quad 11H-8H'-\tfrac{10}{3}F''-\tfrac{32}{3}G''.$$

We have, moreover,

$$L=\tfrac{5}{18}-C',\ M=\tfrac{365}{144}-B',\ N=\tfrac{3}{8}-A';$$

and the coefficients of α and β are $=-M+\tfrac{13}{8}$ and $M-\tfrac{13}{8}$ respectively; hence the coefficients of α^2, β^2, α and β will all vanish if only $L=0$, $M=\tfrac{13}{8}$, $N=0$, that is,

$$A'=\tfrac{3}{8},\ B'=\tfrac{131}{144},\ C'=\tfrac{5}{8};$$

and we have thus identically $\Omega=0$, if only A, B, C, A', B', C' have the above-mentioned values.

910.

NOTE ON THE INVOLUTANT OF TWO BINARY MATRICES.

[From the *Messenger of Mathematics*, vol. xx. (1891), pp. 136, 137.]

CONSIDER the two matrices

$$M = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}, \quad M' = \begin{pmatrix} a', & b' \\ c', & d' \end{pmatrix},$$

and their product in one or the other order

$$MM' = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix}, \quad M'M = \begin{pmatrix} A_1, & B_1 \\ C_1, & D_1 \end{pmatrix}.$$

Then the Involutant is by definition = either of the determinants

$$I = \begin{vmatrix} 1, & a, & a', & A \\ 0, & b, & b', & B \\ 0, & c, & c', & C \\ 1, & d, & d', & D \end{vmatrix}, \quad I_1 = \begin{vmatrix} 1, & a', & a, & A_1 \\ 0, & b', & b, & B_1 \\ 0, & c', & c, & C_1 \\ 1, & d', & d, & D_1 \end{vmatrix};$$

viz. it is to be shown that these two values are in fact equal.

We have

$$MM' = \begin{array}{c} \\ (a, b) \\ (c, d) \end{array} \begin{array}{c} (a', c')\,(b', d') \\ \hline \text{„} \quad \text{„} \\ \text{„} \quad \text{„} \end{array} = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix},$$

that is,

$$A = aa' + bc', \quad B = ab' + bd',$$
$$C = ca' + dc', \quad D = cb' + dd',$$

and similarly

$$M'M = \begin{array}{c} \\ (a',\ b') \\ (c',\ d') \end{array} \begin{array}{c} (a,\ c)\ (b,\ d) \\ \left[\begin{array}{cc} \text{,,} & \text{,,} \\ \text{,,} & \text{,,} \end{array}\right] \end{array} = \left(\begin{array}{cc} A_1, & B_1 \\ C_1, & D_1 \end{array} \right),$$

that is,

$$A_1 = aa' + cb', \quad B_1 = ba' + db',$$
$$C_1 = ac' + cd', \quad D_1 = bc' + dd',$$

viz. A_1, B_1, C_1, D_1 are obtained from A, B, C, D by the interchange of the accented and unaccented letters.

We have then, from the first expression for the Involutant,

$$I = A \begin{vmatrix} 0, & b, & b' \\ 0, & c, & c' \\ 1, & d, & d' \end{vmatrix} - B \begin{vmatrix} 0, & c, & c' \\ 1, & d, & d' \\ 0, & b, & b' \end{vmatrix} + C \begin{vmatrix} 1, & d, & d' \\ 1, & a, & a' \\ 0, & b, & b' \end{vmatrix} - D \begin{vmatrix} 1, & a, & a' \\ 0, & b, & b' \\ 0, & c, & c' \end{vmatrix},$$

$$= A\,(bc' - b'c) - B\,(ac' - a'c + cd' - c'd) + C\,(ab' - a'b + bd' - b'd) - D\,(bc' - b'c),$$

or substituting for $A - D$, B and C their values, this is

$$(aa' - dd' + bc' - b'c)\,(bc' - b'c) - (ab' + bd')\,(ac' - a'c + cd' - c'd)$$
$$+ (ca' + dc')\,(ab' - a'b + bd' - b'd);$$

and multiplying out and grouping together the terms in bc, $b'c'$, bc' and $b'c$, this is found to be

$$= -(a' - d')^2\,bc + (a - d)\,(a' - d')\,(bc' + b'c) - (a - d)^2\,b'c' + (bc' - b'c)^2,$$

which is

$$= -\{(a - d)\,b' - (a' - d')\,b\}\,\{(a - d)\,c' - (a' - d')\,c\} + (bc' - b'c)^2.$$

Hence, writing

$$\mathrm{a} = bc' - b'c, \quad \mathrm{f} = ad' - a'd,$$
$$\mathrm{b} = ca' - c'a, \quad \mathrm{g} = bd' - b'd,$$
$$\mathrm{c} = ab' - a'b, \quad \mathrm{h} = cd' - c'd,$$

we have

$$I = -(\mathrm{c} + \mathrm{g})\,(-\mathrm{b} + \mathrm{h}) + \mathrm{a}^2,$$

that is,

$$I = \mathrm{bc} - \mathrm{ch} + \mathrm{bg} - \mathrm{gh} + \mathrm{a}^2.$$

To obtain the value of I_1, we must interchange the accented and unaccented letters, that is, change the signs of the several quantities a, b, c, f, g, h; but I, being a quadric function of the six quantities, is not altered by the change; that is, we have $I = I_1$.

911.

ON AN ALGEBRAICAL IDENTITY RELATING TO THE SIX COORDINATES OF A LINE.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 138—140.]

THE following identity may be verified without difficulty; but it is interesting in regard to the analytical theory of the line.

The identity is

$$
\begin{aligned}
0 = & \begin{vmatrix} B, & C, & F \\ b, & c, & f \\ b', & c', & f' \end{vmatrix} \begin{vmatrix} A, & B, & C \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} \\
& -(bc' - b'c)^2 (AF + BG + CH) \\
& +(bc' - b'c)\,(bC - cB)\,(Af' + Bg' + Ch' + Fa' + Gb' + Hc') \\
& -(bc' - b'c)\,(b'C - c'B)\,(Af + Bg + Ch + Fa + Gb + Hc) \\
& -(bC - cB)^2 \qquad\qquad (a'f' + b'g' + c'h') \\
& +(bC - cB)\,(b'C - c'B)\,(af' + a'f + bg' + b'g + ch' + c'h) \\
& -(b'C - c'B)^2 \qquad\qquad (af + bg + ch),
\end{aligned}
$$

where all the letters have arbitrary values.

It follows that, if

$$
\begin{aligned}
& af + bg + ch = 0, \\
& a'f' + b'g' + c'h' = 0, \\
& af' + a'f + bg' + b'g + ch' + c'h = 0, \\
& AF + BG + CH = 0, \\
& Af + Bg + Ch + Fa + Gb + Hc = 0, \\
& Af' + Bg' + Ch' + Fa' + Gb' + Hc' = 0,
\end{aligned}
$$

then either

$$\begin{vmatrix} B, & C, & F \\ b, & c, & f \\ b', & c', & f' \end{vmatrix} = 0,$$

or else

$$\begin{vmatrix} A, & B, & C \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0.$$

Supposing the first three of the six equations are satisfied, then (a, b, c, f, g, h) and (a', b', c', f', g', h') are the coordinates of two intersecting lines; and supposing that the last three equations are also satisfied, then (A, B, C, F, G, H) will be the coordinates of a line meeting each of the intersecting lines. The third line is thus either in the plane of the intersecting lines, or else passes through their point of intersection; and in fact, the first of the two determinant equations is the condition in order that the line may be in the plane of the two intersecting lines, and the second determinant equation is the condition in order that it may pass through their point of intersection. Each equation is in fact one out of four equivalent forms, viz. we may have in the first equation (B, C, F), (C, A, G), (A, B, H), or (F, G, H); and in the second equation (A, B, C), (A, H, G), (H, B, F), or (G, F, C).

The analytical theory may be presented in a complete form by means of the formulæ of my memoir "On the six coordinates of a line," (1867), *Camb. Phil. Trans.*, t. XI. pp. 290—323, [435]. Considering the two intersecting lines (a, b, c, f, g, h) and (a', b', c', f', g', h'), the coordinates of the plane through these two lines (that is, the coefficients ξ, η, ζ, ω of the equation $\xi x + \eta y + \zeta z + \omega w = 0$ of the plane) are there given (see p. 295*) in a fourfold form; and if we thence form the condition in order that the line (A, B, C, F, G, H) may lie in this plane, we have

$$\left(\begin{array}{cccc} 0, & C, & -B, & F \\ -C, & 0, & A, & G \\ B, & -A, & 0, & H \\ -F, & -G, & -H, & 0 \end{array} \right)$$

$$\left(\begin{array}{cccc} af'+b'g+c'h, & bf'-b'f, & cf'-c'f, & -(bc'-b'c) \\ ag'-a'g, & a'f+bg'+c'h, & cg'-c'g, & -(ca'-c'a) \\ ah'-a'h, & bh'-b'h, & a'f+b'g+ch', & -(ab'-a'b) \\ gh'-g'h, & hf'-h'f, & fg'-f'g, & af'+bg'+ch' \end{array} \right) = 0;$$

viz. the condition is expressible in any one of the 16 forms obtained by combining a line of the first matrix with a line of the second matrix; thus one form is

$$0(af'+b'g+c'h) + C(bf'-b'f) - B(cf'-c'f) - F(bc'-b'c) = 0,$$

[* This Collection, vol. VII., p. 71.]

that is,

$$\begin{vmatrix} B, & C, & F \\ b, & c, & f \\ b', & c', & f' \end{vmatrix} = 0,$$

the foregoing first determinant equation, which thus belongs to the case where the line lies in the plane of the two intersecting lines.

Again we have (see p. 296*) an expression, in a fourfold form, for the coordinates of the point of intersection of the two intersecting lines; and thence for the condition, in order that the line (A, B, C, F, G, H) may pass through this point, we have

$$\left(\begin{array}{cccc} 0, & H, & -G, & A \\ -F, & 0, & F, & B \\ G, & -H, & 0, & C \\ -A, & -B, & -C, & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc} af'+b'g+c'h, & ag'-a'g\ , & ah'-a'h\ , & gh'-g'h \\ bf'-b'f\ , & a'f+bg'+c'h, & bh'-b'h\ , & hf'-h'f \\ cf'-c'f\ , & cg'-c'g\ , & a'f+b'g+ch', & fg'-f'g \\ -(bc'-b'c)\ , & -(ca'-c'a)\ , & -(ab'-a'b)\ , & af'+bg'+ch' \end{array}\right) = 0,$$

viz. the condition is expressible in any one of the 16 forms obtained by combining a line of the first matrix with a line of the second matrix; thus one form is

$$A\,(bc'-b'c)+B\,(ca'-c'a)+C\,(ab'-a'b)+0\,(af'+bg'+ch')=0,$$

that is,

$$\begin{vmatrix} A, & B, & C \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0,$$

the foregoing second determinant equation, which thus belongs to the case where the line passes through the point of intersection of the two intersecting lines.

I remark that the original identity may be written in the very compendious symbolical form

$$\begin{pmatrix} A, & a, & a' \\ F, & f, & f' \end{pmatrix}\begin{pmatrix} B, & b, & b' \\ C, & c, & c' \end{pmatrix}^2 = \begin{vmatrix} A, & a, & a' \\ B, & b, & b' \\ C, & c, & c' \end{vmatrix}\ \begin{vmatrix} F, & f, & f' \\ B, & b, & b' \\ C, & c, & c' \end{vmatrix},$$

viz. here on the left-hand side the second factor denotes the three determinants $bc'-b'c$, $b'C-c'B$, $Bc-Cb$: the whole is a quadric function of these, the coefficients being

$$\begin{gathered} AF+BG+CH,\quad af+bg+ch, \\ a'f'+b'g'+c'h',\quad af'+bg'+ch'+fa'+gb'+hc', \\ a'F+b'G+c'H+f'A+g'B+h'C, \\ Af+Bg+Ch+Fa+Gb+Hc, \end{gathered}$$

respectively; and the right-hand side is simply the product of two determinants.

[* *Loc. cit.*, p. 72.]

912.

ON THE NOTION OF A PLANE CURVE OF A GIVEN ORDER.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 148—150.]

WE have a complete geometrical notion of a curve of a given order, viz. a curve of the order n is a curve which is met by any line whatever in n points and no more; but starting with this definition, how do we know that there exists a curve of the order n? and, further, how do we know that it depends linearly on $\frac{1}{2}n(n+3)$ parameters, or, what is the same thing, that there is one and only one curve which can be drawn through $\frac{1}{2}n(n+3)$ given points?

The last-mentioned property does not by itself constitute a definition of a curve of the order n; thus $n=2$, we cannot define a curve of the second order as a curve which is uniquely determined by the condition of passing through 5 given points; for a cubic passing through 4 given points is a curve uniquely determined by the condition in question; but we may differentiate between these two solutions by adding the further condition that, when 3 of the 5 points are in a line, the curve of the second order shall include as part of itself this line. And we are thus led to the definition: A curve of the order n is a curve which is uniquely determined by the condition of passing through $\frac{1}{2}n(n+3)$ given points; and of being moreover such that, when $n+1$ of these points lie on a line, it includes as part of itself this line.

Starting from the foregoing definition, the first property is, I think, demonstrable, viz. the property that a curve of the order n is met by any line whatever in n points and no more. Thus $n=2$: start with a line chosen at pleasure, and on it take 2 points which are regarded as indeterminate points: to fix the ideas, let one of these be regarded as depending on a parameter λ and the other on a parameter μ, (so that when λ has a determinate value assigned to it, the first point becomes a determinate point, and similarly when μ has a determinate value assigned to it, the second point becomes a determinate point; and consequently, when λ and μ have

determinate values, the 2 points are determinate points on the line). Take now any other 3 points not on the line; then, for the moment regarding the 2 points on the line as determinate points, we can through the 5 points draw a curve of the second order; this is the general curve of the second order through the 3 points; for it is a curve of the second order through the 3 points, and which, when the parameters λ and μ are regarded as undetermined and arbitrary, or choosable at pleasure, might be made to pass through any other two points whatever. But by hypothesis, the curve meets the line in question, that is, *any line*, in two points: and the conic cannot meet the line in more than two points; for in like manner, starting with the given line, and upon it 3 points (which may be considered as depending on the parameters λ, μ and ν respectively), and taking any two points not on the line, we have through the 5 points a conic; and this conic, regarding the parameters λ, μ, ν as undetermined and arbitrary, or choosable at pleasure, will be the general conic through the two points; for by a proper determination of the parameters, it might be made to pass through any other 3 points whatever. But, by hypothesis, the conic contains as part of itself the line; that is, the general conic through the 2 points contains as part of itself any line whatever, which is absurd.

So again for the cubic, $n=3$: here starting with a line taken at pleasure, we take on it 3 points, which may be regarded as depending on the parameters λ, μ, ν respectively, and we take any other 6 points not on the line. We have through the 9 points a cubic; and this is the general cubic through the 6 points, for it depends on the parameters λ, μ, ν, which might be determined so as to make the curve pass through any other 3 points whatever; and by hypothesis the cubic meets the line, that is, any line whatever, in 3 points. And it cannot meet it in more than 3 points: for starting with the same line and upon it 4 points depending on the parameters λ, μ, ν, ρ respectively, and taking any other 5 points not upon the line, we then have through the 9 points a cubic which will be the general cubic through the 5 points (for it depends on the 4 parameters λ, μ, ν, ρ). But by hypothesis, the cubic contains as part of itself the line, viz. the general cubic through the 5 points contains as part of itself any line whatever, which is absurd. The reasoning is quite general; and applying to a curve of the order n, the conclusion is that such a curve meets any line in n points and no more.

913.

ON THE EPITROCHOID.

[From the *Messenger of Mathematics*, vol. XX. (1891), pp. 150—158.]

If we have a curve C_1 rolling on a fixed curve C, and consider the epitrochoid described by a point P, attached to and carried along with the curve C_1, then there is a known construction for the radius of curvature at any point of the epitrochoid; the construction is probably much older, but I refer to Mannheim, *Géométrie Descriptive*, Paris, 1886, pp. 177 and 194. In fact, if Q be a position of the point of contact, P_1 the corresponding position of the describing point, R_1 and R the radii of curvature at Q of the two curves respectively, ρ the distance QP_1, and ϕ the inclination of this distance to the common normal at Q, then the radius of curvature x at the point P_1 of the epitrochoid is given by the formula

$$\frac{x}{x-\rho}=\frac{\rho}{\cos\phi}\left(\frac{1}{R}+\frac{1}{R_1}\right).$$

I prove this as follows: take Q, Q' consecutive positions of the point of contact, P_1, P_1' consecutive positions of P_1; the centre of curvature is the intersection of the lines P_1Q, $P_1'Q'$; hence, if for a moment M and N are the perpendicular distances between these two lines at the points P_1 and Q respectively, we have $\frac{x}{x-\rho}=\frac{M}{N}$. Let ds be the element QQ' considered as belonging to each of the two curves respectively, θ and θ_1 the angles which this element subtends at the two centres of curvature respectively; we have $ds=R\theta=R_1\theta_1$, whence

$$\theta+\theta_1=ds\left(\frac{1}{R}+\frac{1}{R_1}\right).$$

The instantaneous centre is P_1, and hence

$$M=P_1P_1'=\rho(\theta+\theta_1);\ \text{also}\ N=ds\cos\phi;$$

we thus have

$$\frac{x}{x-\rho}=\frac{\rho\,(\theta+\theta_1)}{ds\cos\phi};$$

or, substituting for $\theta+\theta_1$ the foregoing value, we have the required expression

$$\frac{x}{x-\rho}=\frac{\rho}{\cos\phi}\left(\frac{1}{R}+\frac{1}{R_1}\right).$$

If the curve C_1 roll on a straight line, $R=\infty$ and therefore

$$\frac{x}{x-\rho}=\frac{\rho}{R_1\cos\phi}.$$

Suppose that at any instant we have on the epitrochoid an inflexion, then $x=\infty$, and we have $\rho=R_1\cos\phi$; this denotes that P_1 is a point of the circle described on the radius of curvature at Q as its diameter, and we have thus the theorem:

If at any instant we consider the circle, which has for its diameter the radius of curvature of the rolling curve at its point of contact with the right line, then each point on this circle is an inflexion on the epitrochoid described by the same point as the curve rolls on the straight line.

The circle in question for any point Q of the rolling curve C_1 is of course a circle, having its centre on the normal at Q and its radius equal to one-half of the radius of curvature; we may call it the circle of double curvature. Regarding Q as a given point on the curve C_1, and supposing that the circle of double curvature intercepts the curve C_1 in a point X, then taking P_1 at X, we have the theorem:

If the curve C_1 roll on a straight line, then the epitrochoid described by the point X has an inflexion corresponding to the position of the rolling curve for which the point Q comes into contact with the straight line.

Consider the curve C_1 referred to axes fixed in the plane of the curve and moveable with it; take x, y as the coordinates of the point Q; α, β as the coordinates of the corresponding centre of curvature, and X, Y as current coordinates; the equation of the circle of double curvature is

$$(X-x)(X-\alpha)+(Y-y)(Y-\beta)=0,$$

viz. this is the equation of a circle having for a diameter the two points (x, y) and (α, β).

Suppose that C_1 is a conic; the circle of double curvature at any point Q meets it in the point Q counting twice, and in two other points, say P_1, P_2, which are determined as the intersections of the conic by the line P_1P_2 which joins them. I propose to determine the equation of this line. The investigation would be more symmetric for the ellipse, but I nevertheless prefer to consider for the curve C_1 the hyperbola $b^2X^2-a^2Y^2-a^2b^2=0$.

Representing for a moment the equation of the circle by $U=0$, and that of the hyperbola by $V=0$, it must be possible to determine the ratio of the coefficients p, q, in such wise that $pU+qV=0$ shall break up into a pair of lines, one of which is the tangent at Q of the hyperbola; and then the other of them will be the required line P_1P_2.

The equation of the tangent is $b^2xX-a^2yY-a^2b^2=0$; we then see that the equation of the line P_1P_2 must be of the form $b^2xX+a^2yY+\Omega=0$; and we have to find p, q, Ω so as to verify the identity

$$(p+qb^2)X^2+(p-qa^2)Y^2-p(\alpha+x)X-p(\beta+y)Y+p(\alpha x+\beta y)-qa^2\beta^2$$
$$=(b^2xX-a^2yY-a^2b^2)(b^2xX+a^2yY+\Omega);$$

that is, we ought to have

$$p+qb^2=b^4x^2,$$
$$p-qa^2=a^4y^2,$$
$$p(x+\alpha)=(\Omega-a^2b^2)b^2x,$$
$$p(y+\beta)=(\Omega+a^2b^2)a^2y,$$
$$qa^2b^2-p(\alpha x+\beta y)=\Omega a^2b^2.$$

It will be recollected that α, β denote the coordinates of the centre of curvature corresponding to the point (x, y) of the hyperbola; their values thus are

$$\alpha=\quad(a^2+b^2)\frac{x^3}{a^4},$$

$$\beta=-(a^2+b^2)\frac{y^3}{b^4}.$$

There is no difficulty in finding the values

$$p=\frac{a^4b^4}{a^2+b^2},\quad q=\frac{b^4x^2+a^4y^2}{a^2+b^2},$$

whence also

$$q-a^2b^2=\frac{a^4b^2}{a^2+b^2}\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)y^2-1\right\},$$

$$q+a^2b^2=\frac{a^2b^4}{a^2+b^2}\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)x^2+1\right\},$$

$$\Omega=-a^2b^2\frac{x^2+y^2}{a^2+b^2};$$

these satisfy the identities in question.

The equation of the line P_1P_2 is thus

$$b^2xX+a^2yY-\frac{a^2b^2}{a^2+b^2}(x^2+y^2)=0,$$

and combining this with the equation

$$b^2X^2-a^2Y^2-a^2b^2=0$$

of the hyperbola, we find

$$\left\{xX - \frac{a^2}{a^2+b^2}(x^2+y^2)\right\}^2 = \frac{a^4y^2Y^2}{b^4}, \quad = \frac{a^2y^2}{b^2}(X^2-a^2),$$

that is,

$$X^2\left(x^2 - \frac{a^2y^2}{b^2}\right) - \frac{2a^2x}{a^2+b^2}(x^2+y^2)\,X + \frac{a^4}{(a^2+b^2)^2}(x^2+y^2)^2 + \frac{a^4y^2}{b^2} = 0,$$

viz.

$$X^2 - 2\frac{x^2+y^2}{a^2+b^2}xX + a^2\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 + \frac{y^2}{b^2}\right\} = 0,$$

and we thence obtain

$$X = \frac{x^2+y^2}{a^2+b^2}x \pm \frac{ay}{b}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 - 1\right\}},$$

$$Y = -\frac{x^2+y^2}{a^2+b^2}y \pm \frac{bx}{a}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2+b^2}\right)^2 - 1\right\}},$$

values belonging to the points P_1, P_2 respectively: these points are thus real or imaginary, according as x^2+y^2 is greater or less than a^2+b^2. In the limiting case $x^2+y^2=a^2+b^2$, we have $X=x$, $Y=-y$, viz. here the circle of double curvature has its centre on the axis of x, and has double contact with the hyperbola at the points (x, y) and $(x, -y)$ respectively.

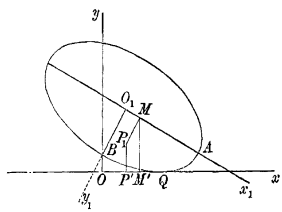
Suppose, to fix the ideas, x and y are both positive, i.e. that Q is on the right-hand upper half branch of the hyperbola; and that $x^2+y^2=$ or $>a^2+b^2$, so that P_1, P_2 are both real. The two values of X are both positive, those of Y are originally both negative, and as Q moves along the half branch of the hyperbola one of these values increases negatively, the other increases positively until it becomes $=0$, after which it becomes positive and continually increases: say the points P_1, P_2 are originally on the right-hand lower half branch, but one of them, say P_1, moves up to the vertex, and we afterwards have P_1 on the upper half branch and P_2 on the lower half branch. The limiting case is when P_1 is at the vertex; here $\frac{x^2+y^2}{a^2+b^2} = \frac{x}{a}$, giving for (x, y) a real point, which is easily constructed on the upper half branch. There is thus on the half branch a real position of Q, such that the corresponding circle of double curvature passes through the vertex of the same half branch.

Hence in the epitrochoid described by the vertex of a hyperbola rolling on a right line there is always an inflexion.

Imagine the branch $B'AB$ of a hyperbola rolling on the line $y=0$, where A is the vertex, B', B the points at infinity, and so that as x increases positively (that is, towards the right) the point of contact Q passes from A towards B; and consider the epitrochoid described by the vertex A. Supposing that the point of contact is at first at A; the point A is a cusp on the epitrochoid, the motion of A is at first vertically upwards, and towards the right, the curve being concave to the axis; we then come to a position of the point of contact for which there is an inflexion; the motion of A is still upwards and towards the right, but the curve has now

become convex to the axis, and the motion continues in this manner as the point of contact moves along the hyperbola towards B; when the point of contact is at B, i.e. when the asymptote has come to coincide with the axis, then the motion of A is again vertically upwards; the other half branch of the hyperbola now comes into contact with the axis; the point A still moves upwards but towards the left, the curve being concave to the axis, and this continues until the point of contact arrives at the other vertex (i.e. the vertex on the other branch), when the motion of A is horizontal and towards the left, viz. the coordinate y has here a maximum value, equal transverse axis of the hyperbola; the motion of A is thenceforth towards the left and downwards, until the other asymptote comes to coincide with the curve, after which it is downwards and towards the right (the point of contact being now on the half $B'A$ of the original branch $B'AB$), and ultimately the point of contact is again at A, and we have $y=0$, a cusp; the curve is symmetrical in regard to the before-mentioned maximum ordinate: it is to be noticed that the distance between the two cusps is equal to four times the difference between the length of the half asymptote and the half branch of the hyperbola (this difference being a finite value expressible by elliptic functions), and that, as already mentioned, the maximum ordinate is equal to the transverse axis. The above described portion of the curve is of course repeated continually right and left *ad infinitum*; two consecutive portions intersect each other, and there is thus a real node vertically above each cusp of the infinite curve. We have, in what precedes, a complete explanation of the motion of a hyperbola rolling on a straight line, and in particular our explanation of the form of the epitrochoid described by a vertex; the epitrochoid described by any other point of the curve is of a similar character but must be non-symmetrical in regard to the maximum ordinate; and it would not be difficult to explain the form of the curve for other positions of the describing point; but as regards the analytical theory, I prefer to consider the case of the ellipse.

In the case of the ellipse $b^2X^2+a^2Y^2-a^2b^2=0$ rolling on the line $y=0$ (see figure), take as origin the point O which comes in contact with the extremity B of



the minor axis; let x_1, y_1 be the coordinates of the describing point P_1, and for the point Q, taking ϕ as the complement of the eccentric angle, let the coordinates referred to the axes of the ellipse be

$$x_1 = a\sin\phi, \quad y_1 = b\cos\phi;$$

the arc BQ is then

$$= \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi,$$

which is $= aE\phi$, the modulus being the eccentricity e of the ellipse; we then have for the coordinates of P_1

$$\begin{aligned} x = OP' = OQ - QP', &\quad = BQ - QP', \\ y &\quad = \quad P_1P', \end{aligned}$$

where QP' and P_1P' are the projections of P_1Q in the direction of, and at right angles to, the tangent QO at Q; we find the values without difficulty, and we thus have

$$x = \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi - \frac{(a^2 - b^2) \sin \phi \cos \phi - ax_1 \cos \phi + by_1 \sin \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}},$$

$$y = \frac{ab - bx_1 \sin \phi - ay_1 \cos \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}} \quad ;$$

which are the expressions for x, y in terms of a variable parameter ϕ.

In particular, for the point B at the extremity of the minor axis, $x_1 = 0$, $y_1 = b$, and the equations become

$$x = \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi - \frac{(a^2 - b^2) \sin \phi \cos \phi + b^2 \sin \phi}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}},$$

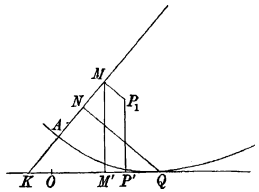
$$y = \frac{ab\,(1 - \cos \phi)}{\sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}} \quad ,$$

which, for $a = b$, become

$$x = a\,(\phi - \sin \phi), \quad y = a\,(1 - \cos \phi),$$

the ordinary formulæ for the cycloid.

In the case of the parabola $Y^2 = 4aX$ (see figure), taking x_1, y_1 for the coordinates



of the describing point P_1, and representing those of the point of contact Q by means of the parameter ϕ, $X = a\phi^2$, $Y = 2a\phi$, then the arc AQ is

$$= a\,[\phi \sqrt{(\phi^2 + 1)} + \log \{\phi + \sqrt{(\phi^2 + 1)}\}],$$

so that, taking the origin at the point O, which comes into contact with the vertex A, this is also the value of OQ, and we then have

$$x = OQ - QP',$$
$$y = P_1P'.$$

Producing the axis of the parabola to meet the line $y=0$ in k, we have

$$AK = AN = a\phi^2,$$

and thence $KM = a\phi^2 + x_1$; and then, if θ be the inclination AKQ of the axis of the parabola,

$$KM' = (a\phi^2 + x_1)\cos\theta, \quad MM' = (a\phi^2 + x_1)\sin\theta;$$

and thence also

$$KP' = (a\phi^2 + x_1)\cos\theta + y_1\sin\theta, \quad P_1P' = (a\phi^2 + x_1)\sin\theta - y_1\cos\theta;$$

also

$$KQ = 2a\phi^2\sec\theta,$$

and thence

$$QP' = 2a\phi^2\sec\theta - (a\phi^2 + x_1)\cos\theta - y_1\sin\theta.$$

We have $\tan\theta = \dfrac{1}{\phi}$, and thence

$$\sin\theta = \frac{1}{\sqrt{(\phi^2+1)}}, \quad \cos\theta = \frac{\phi}{\sqrt{(\phi^2+1)}}, \quad \sec\theta = \frac{\sqrt{(\phi^2+1)}}{\phi};$$

hence

$$QP' = 2a\phi\sqrt{(\phi^2+1)} - \frac{(a\phi^2 + x_1)\phi + y_1}{\sqrt{(\phi^2+1)}} = \frac{a(\phi^3 + 2\phi) - x_1\phi - y_1}{\sqrt{(\phi^2+1)}},$$

$$P_1P' = \frac{a\phi^2 + x_1 - y_1\phi}{\sqrt{(\phi^2+1)}},$$

and the values of x, y thus become

$$x = a\left[\phi\sqrt{(\phi^2+1)} + \log\{\phi + \sqrt{(\phi^2+1)}\}\right] - \frac{a(\phi^3 + 2\phi) - x_1\phi - y_1}{\sqrt{(\phi^2+1)}},$$

that is,

$$x = a\log\{\phi + \sqrt{(\phi^2+1)}\} + \frac{(x_1 - a)\phi + y_1}{\sqrt{(\phi^2+1)}},$$

$$y = \frac{a\phi^2 + x_1 - \phi y_1}{\sqrt{(\phi^2+1)}}.$$

In particular, for the focus of the parabola, $x_1 = a$, $y_1 = 0$, and the equations become

$$x = a\log\{\phi + \sqrt{(\phi^2+1)}\},$$

$$y = a\sqrt{(\phi^2+1)},$$

that is,

$$\frac{x}{a} = \log\left\{\frac{y}{a} + \sqrt{\left(\frac{y^2}{a^2} - 1\right)}\right\},$$

or say

$$\exp.\left(\frac{x}{a}\right) = \frac{y}{a} + \sqrt{\left(\frac{y^2}{a^2} - 1\right)};$$

this gives

$$\exp.\left(-\frac{x}{a}\right) = \frac{y}{a} - \sqrt{\left(\frac{y^2}{a^2} - 1\right)};$$

and we have therefore

$$y = \tfrac{1}{2}a\left\{\exp.\left(\frac{x}{a}\right) + \exp.\left(-\frac{x}{a}\right)\right\},$$

viz. as is well known, the epitrochoid described by the focus of the parabola is the catenary.

914.

ON A SOLUBLE QUINTIC EQUATION.

[From the *American Journal of Mathematics*, vol. XIII. (1891), pp. 53—58.]

MR YOUNG, in his paper, "Solvable Quintic Equations with Commensurable Coefficients," *American Journal of Mathematics*, X. (1888), pp. 99—130, has given, in illustration of his general theory of the solution of soluble quintic equations (founded upon a short note by Abel), no less than twenty instances of the solution of a quintic equation with purely numerical coefficients, having a solution of the form $\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$, where A, B, C, D are numerical expressions involving only square roots. But the solutions are not presented in their most simple form: thus in example 1, $x^5+3x^2+2x-1=0$, the expression involves a radical

$$\sqrt{\tfrac{47}{8}(21125+9439\sqrt{5})}:$$

here

$$(21125+9439)\sqrt{5},\ =\sqrt{5}\,(9439+4225\sqrt{5}),\ =\sqrt{5}\,.\,\tfrac{1}{2}\,(18+5\sqrt{5})^2\,(1+\sqrt{5})^2\,(2+\sqrt{5}),$$

so that, taking out the roots of the squared factors, we have as the proper form of the radical the very much more simple form $\sqrt{47\,(2+\sqrt{5})\sqrt{5}}$; where observe that $(2+\sqrt{5})(2-\sqrt{5})=-1$, and thence $(2+\sqrt{5})\sqrt{-47\,(2-\sqrt{5})\sqrt{5}}=\sqrt{47\,(2+\sqrt{5})\sqrt{5}}$, viz. the conjugate radicals $\sqrt{-47\,(2-\sqrt{5})\sqrt{5}}$ and $\sqrt{47\,(2+\sqrt{5})\sqrt{5}}$ differ only by a factor $2+\sqrt{5}$ which is rational in 1 and $\sqrt{5}$. To avoid fractions I consider the foregoing equation under the form

$$x^5+3000x^2+20000x-100000=0,$$

and I will presently give the solution; but first I consider the general theory. Writing

$$\begin{aligned}
A&=\alpha^5, & A'&=\alpha^2\gamma, & A''&=\alpha^3\beta,\\
B&=\beta^5, & B'&=\alpha\beta^2, & B''&=\beta^3\delta,\\
C&=\gamma^5, & C'&=\gamma^2\delta, & C''&=\alpha\gamma^3,\\
D&=\delta^5, & D'&=\beta\delta^2, & D''&=\gamma\delta^3,
\end{aligned}$$

we have $A'D'=\alpha^2\delta^2\beta\gamma$, $B'C'=\alpha\delta\beta^2\gamma^2$. Also

$$A''=\frac{A'B'}{\beta\gamma},\quad B''=\frac{B'D'}{\alpha\delta},\quad C''=\frac{A'C'}{\alpha\delta},\quad D''=\frac{C'D'}{\beta\gamma},$$

which determine A'', B'', C'', D'' in terms of A', B', C', D', $\alpha\delta$, $\beta\gamma$; and then

$$A = \frac{A'A''}{\beta\gamma}, \quad B = \frac{B'B''}{\alpha\delta}, \quad C = \frac{C'C''}{\alpha\delta}, \quad D = \frac{D'D''}{\beta\gamma},$$

which give A, B, C, D.

If now we assume $x = \alpha + \beta + \gamma + \delta$, and regard A, B, C, D, A', B', C', D', A'', B'', C'', D'', $\alpha\delta$, $\beta\gamma$ each as a rational function, we may express x, x^2, x^3, x^5 each of them by means of rational functions or of rational functions multiplied into α, β, γ, δ respectively: thus,

$$\begin{aligned} x &= \alpha + \beta + \gamma + \delta && = \alpha + \beta + \gamma + \delta, \\ x^2 &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 && = \frac{A'\beta}{\beta\gamma} + \frac{B'\delta}{\alpha\delta} + \frac{C'\alpha}{\alpha\delta} + \frac{D'\gamma}{\beta\gamma}, \\ &\quad + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + 2\gamma\delta && \quad + \frac{2B'\gamma}{\beta\gamma} + \frac{2A'\delta}{\alpha\delta} + 2\alpha\delta + 2\beta\gamma + \frac{2D'\alpha}{\alpha\delta} + \frac{2C'\beta}{\beta\gamma}, \end{aligned}$$

&c.; and we thus obtain

$$\begin{aligned} & x^5 + qx^3 + rx^2 + sx + t \\ &= A + B + C + D \\ &\quad + (20\alpha\delta + 30\beta\gamma)(A' + D') + 30(\alpha\delta + 20\beta\gamma)(B' + C') \\ &\qquad\qquad + 3q(A' + B' + C' + D') + 2r(\alpha\delta + \beta\gamma) + t \\ &\quad + \alpha\Big\{5A'' + 5\frac{C'^2}{\alpha\delta} + \frac{5B''\beta\gamma}{\alpha\delta} + 10C'' + \frac{10D'^2}{\alpha\delta} \\ &\qquad + 10\alpha^2\delta^2 + 20B'' + 20D'' + 30\beta^2\gamma^2 + 30D''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta \\ &\qquad + q\left(\frac{B''}{\alpha\delta} + \frac{3D''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{C'}{\alpha\delta} + \frac{2D'}{\alpha\delta}\right) + s\Big\} \\ &\quad + \beta\Big\{5B'' + \frac{5A'^2}{\beta\gamma} + 5\frac{D''\alpha\delta}{\beta\gamma} + 10A'' + \frac{10C'^2}{\beta\gamma} \\ &\qquad + 10\beta^2\gamma^2 + 20D'' + 20C'' + 30\alpha^2\delta^2 + 30C''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta \\ &\qquad + q\left(\frac{D''}{\beta\gamma} + \frac{3C''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{A'}{\beta\gamma} + \frac{2C'}{\beta\gamma}\right) + s\Big\} \\ &\quad + \gamma\Big\{5C'' + \frac{5D'^2}{\beta\gamma} + 5\frac{A''\alpha\delta}{\beta\gamma} + 10D'' + \frac{10B'^2}{\beta\gamma} \\ &\qquad + 10\beta^2\gamma^2 + 20A'' + 20B'' + 30\alpha^2\delta^2 + 30B''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta \\ &\qquad + q\left(\frac{A''}{\beta\gamma} + \frac{3B''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{D'}{\beta\gamma} + \frac{2B'}{\beta\gamma}\right) + s\Big\} \\ &\quad + \delta\Big\{5D'' + \frac{5B'^2}{\alpha\delta} + 5\frac{C''\beta\gamma}{\alpha\delta} + 10B'' + \frac{10A'^2}{\alpha\delta} \\ &\qquad + 10\alpha^2\delta^2 + 20C'' + 20A'' + 30\beta^2\gamma^2 + 30A''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta \\ &\qquad + q\left(\frac{C''}{\alpha\delta} + \frac{3A''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{B'}{\alpha\delta} + \frac{2A'}{\alpha\delta}\right) + s\Big\}. \end{aligned}$$

If, then, $x^5+qx^3+rx^2+sx+t=0$, we have the rational term $=0$, and the coefficients of α, β, γ, δ each $=0$; in the class of equations under consideration, these last equations differ only in the signs of the radicals contained therein, so that one of them being satisfied identically, the others will be also satisfied. In particular, if $q=0$, then $\alpha\delta+\beta\gamma=0$: the rational term gives

$$A+B+C+D-10\alpha\delta(A'+D'-B'-C')+t=0,$$

and the term in α gives

$$5A''+15B''+10C''-10D''+\frac{5}{\alpha\delta}(C'^2+2D'^2)+\frac{r}{\alpha\delta}(C'+2D')-20\alpha^2\delta^2+s=0.$$

For the equation $x^5+3000x^2+20000x-100000=0$, the expression for the root is $x=\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$, where

$$\begin{aligned}
A &= 39000+18200\sqrt{5}+(\ \ 1720+920\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D &= 39000+18200\sqrt{5}+(-1720-920\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B &= 39000-18200\sqrt{5}+(-1720+920\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C &= 39000-18200\sqrt{5}+(\ \ 1720-920\sqrt{5})\sqrt{235-94\sqrt{5}},
\end{aligned}$$

and where also

$$\begin{aligned}
A' &= -150-\ \ 70\sqrt{5}+(-\ \ 10-\ \ 2\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D' &= -150-\ \ 70\sqrt{5}+(\ \ 10+\ \ 2\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B' &= -150+\ \ 70\sqrt{5}+(\ \ 10-\ \ 2\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C' &= -150+\ \ 70\sqrt{5}+(-\ \ 10+\ \ 2\sqrt{5})\sqrt{235-94\sqrt{5}},
\end{aligned}$$

and

$$\begin{aligned}
A'' &= -940-\ \ 100\sqrt{5}+(-\ \ 100+\ \ 20\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D'' &= -940-\ \ 100\sqrt{5}+(\ \ 100-\ \ 20\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B'' &= -940+\ \ 100\sqrt{5}+(\ \ 100+\ \ 20\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C'' &= -940+\ \ 100\sqrt{5}+(-\ \ 100-\ \ 20\sqrt{5})\sqrt{235-94\sqrt{5}}.
\end{aligned}$$

The foregoing forms are in some respects the most convenient; but it is to be observed that we have

$$\begin{aligned}
A &= 2600\sqrt{5}(1+\sqrt{5})(\ 2+\ \ \sqrt{5})+40(1+\sqrt{5})(18+5\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})},\ \&c.,\\
A' &= -10\sqrt{5}(1+\sqrt{5})(\ 2+\ \ \sqrt{5})\ \ -\ 2\sqrt{5}\ (\ 1+\ \ \sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})},\ \&c.,\\
A'' &= \ \ 20(1-\sqrt{5})(18+13\sqrt{5})\ \ +20\sqrt{5}\ (\ 1-\ \ \sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})},\ \&c.,
\end{aligned}$$

or putting for shortness

$$\sqrt{Q}=\sqrt{47\sqrt{5}(2+\sqrt{5})},\quad \sqrt{Q_1}=\sqrt{-47\sqrt{5}(2-\sqrt{5})},$$

(so that, according to a foregoing remark, we have $(2+\sqrt{5})\sqrt{Q}=\sqrt{Q_1}$), then we have

$$\begin{aligned} A &= 40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\}, \text{ \&c.,} \\ A' &= -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\}, \text{ \&c.,} \\ A'' &= 20(1-\sqrt{5})\{18+13\sqrt{5}+\sqrt{5}\sqrt{Q}\}, \text{ \&c.,} \end{aligned}$$

where observe that the term $2+\sqrt{5}$ is a factor of Q.

Starting from the values of A', B', C', D', we have

$$A' = -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\},$$
$$D' = -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})-\sqrt{Q}\},$$

and therefore

$$A'D' = 20(1+\sqrt{5})^2(2+\sqrt{5})\{25(2+\sqrt{5})-47\sqrt{5}\},$$

where the last factor is

$$=50-22\sqrt{5},\ =-2\sqrt{5}(11-5\sqrt{5}),\ =-\sqrt{5}(1-\sqrt{5})^2(2-\sqrt{5}).$$

Hence

$$A'D' = -\sqrt{5}\,.\,20(-4)^2(-1) = 320\sqrt{5},$$

that is,

$$A'D' = (\alpha\delta)^2\beta\gamma = 320\sqrt{5}, \text{ and similarly } B'C' = \alpha\delta(\beta\gamma)^2 = -320\sqrt{5},$$

whence

$$\alpha\delta = -4\sqrt{5},\quad \beta\gamma = 4\sqrt{5}, \text{ and } \alpha\delta+\beta\gamma = 0, \text{ as above.}$$

We have, moreover,

$$A' = -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\},$$
$$B' = 2\sqrt{5}(1-\sqrt{5})\{5(2-\sqrt{5})-\sqrt{Q_1}\},$$

and thence

$$\begin{aligned} A'B' &= 80\{-25-\sqrt{QQ_1}+5(2-\sqrt{5})\sqrt{Q}-5(2+\sqrt{5})\sqrt{Q_1}\}, \\ &= 80\{-25-47\sqrt{5}+(5(2-\sqrt{5})-5)\sqrt{Q}\}, \end{aligned}$$

that is,

$$\begin{aligned} A'B' \div \beta\gamma &= 4\sqrt{5}\{-25-47\sqrt{5}+5(1-\sqrt{5})\sqrt{Q}\} \\ &= -20(47+5\sqrt{5})+20\sqrt{5}(1-\sqrt{5})\sqrt{Q},\ = A''; \end{aligned}$$

and similarly we verify the values of B'', C'' and D''.

We have next

$$A'A'' = 160\sqrt{5}\{(10+5\sqrt{5}+\sqrt{Q})(18+13\sqrt{5}+\sqrt{5}\sqrt{Q})\},$$

or observing that $Q\sqrt{5}$ is $=235(2+\sqrt{5})$, the whole term in { } is

$$\begin{aligned} &= (505+220\sqrt{5})+(470+235\sqrt{5})+(18+13\sqrt{5}+25+10\sqrt{5})\sqrt{Q}, \\ &= 975+455\sqrt{5}+(43+23\sqrt{5})\sqrt{Q} = 65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q}; \end{aligned}$$

or we have

$$\begin{aligned} A'A'' &= 160\sqrt{5}\{65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q}\}, \\ &= 160\sqrt{5}(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\}, \end{aligned}$$

and consequently

$$A'A'' \div \beta\gamma = 40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\},\ = A;$$

and similarly we verify the values of B, C, D.

In the proposed equation $x^5+3000x^2+20000x-100000=0$, we have $r=3000$, $s=20000$, $t=-100000$, $\alpha\delta=-4\sqrt{5}$; the two equations to be verified thus are

$$A+B+C+D+40\sqrt{5}\,(A'+D'-B'-C')-100000=0,$$

and

$$5A''+15B''+10C''-10D''-\frac{\sqrt{5}}{4}(C'^2+2D'^2)-150\sqrt{5}\,(C'+2D')-1600+20000=0.$$

As to the first of these, we have $A+B+C+D=156000$, $A'+D'-B'-C'=-280\sqrt{5}$, and the equation thus is

$$156000+40\sqrt{5}\,(-280\sqrt{5})-100000=0,$$

which is right.

For the second equation, if in the calculation we keep the radicals in the first instance distinct, we have

$$\begin{aligned}
5A''+15B''+10C''-10D'' &= -18800+3000\sqrt{5}+(-1500+300\sqrt{5})\sqrt{Q}+(500+100\sqrt{5})\sqrt{Q_1},\\
-150\sqrt{5}\,(C'+2D') &= \{-450-70\sqrt{5}\\
&\qquad +(20+4+\sqrt{5})\sqrt{Q}+(-10+2\sqrt{5})\sqrt{Q_1}\}\,(-150\sqrt{5})\\
-1600+20000 &= 18400\\
-\frac{\sqrt{5}}{4}(C'^2+2D'^2) &= -\frac{\sqrt{5}}{4}\{282000+416800\sqrt{5}\\
&\qquad +(-8800-4000\sqrt{5})\sqrt{Q}+(4400-2000\sqrt{5})\sqrt{Q_1}\}.
\end{aligned}$$

Substituting in the equation, we ought to have

$$\begin{aligned}
0 = &-18800+3000\sqrt{5}+(-1500+300\sqrt{5})\sqrt{Q}+(500+100\sqrt{5})\sqrt{Q_1}\\
&+52500+67500\sqrt{5}+(-3000-3000\sqrt{5})\sqrt{Q}+(-1500+1500\sqrt{5})\sqrt{Q_1}\\
&+18400\\
&-52100-70500\sqrt{5}+(5000+2200\sqrt{5})\sqrt{Q}+(2500-1100\sqrt{5})\sqrt{Q_1},
\end{aligned}$$

that is,

$$0=(500-500\sqrt{5})\sqrt{Q}+(1500+500\sqrt{5})\sqrt{Q_1},$$

viz.

$$0=500\,\{(1-\sqrt{5})\sqrt{Q}+(3+\sqrt{5})\sqrt{Q_1}\},$$

which is satisfied in virtue of $\sqrt{Q}=(2+\sqrt{5})\sqrt{Q_1}$: this completes the verification.

915.

ON THE PARTITIONS OF A POLYGON.

[From the *Proceedings of the London Mathematical Society*, vol. XXII. (1891), pp. 237—262. Read March 12, 1891.]

1. THE partitions are made by non-intersecting diagonals; the problems which have been successively considered are (1) to find the number of partitions of an r-gon into triangles, (2) to find the number of partitions of an r-gon into k parts, and (3) to find the number of partitions of an r-gon into p-gons, r of the form

$$n(p-2)+2.$$

Problem (1) is a particular case of (2); and it is also a particular case of (3); but the problems (2) and (3) are outside each other; for problem (3) a very elegant solution, which will be here reproduced, is given in the paper, H. M. Taylor and R. C. Rowe, "Note on a Geometrical Theorem," *Proc. Lond. Math. Soc.*, t. XIII. (1882), pp. 102—106, and this same paper gives the history of the solution of (1).

The solution of (2) is given in the memoir, Kirkman "On the k-partitions of the r-gon and r-ace," *Phil. Trans.*, t. CXLVII. (for 1857), p. 225; viz. he there gives for the number of partitions of the r-gon into k parts (or, what is the same thing, by means of $k-1$ non-intersecting diagonals) the expression

$$\frac{[r+k-2]^{k-1}[r-3]^{k-1}}{[k]^{k-1}[k-1]^{k-1}};$$

but there is no complete demonstration of this result.

If $k=r-2$, we have the solution of the problem (1); viz. the number of partitions of the r-gon into triangles is

$$=[2r-4]^{r-3}\div[r-2]^{r-3}.$$

The present paper relates chiefly to the foregoing problem (2), the determination

of the number of partitions of the r-gon into k parts, or, what is the same thing, by means of $k-1$ non-intersecting diagonals.

2. Assuming for the moment the foregoing result, then for $k=1$, the number of partitions is

$$=1,$$

for $k=2$ it is

$$=\frac{r.r-3}{2},$$

for $k=3$ it is

$$=\frac{r+1.r.r-3.r-4}{12},$$

and so on. As a simple verification, $k=2$, the number of partitions is equal to the number of diagonals, viz. this is number of pairs of summits *less* number of sides, that is,

$$\tfrac{1}{2}r(r-1)-r, \quad =\tfrac{1}{2}r(r-3).$$

For convenience, I give the first Table on the next page, which is a tabulation of the functions

$$U_1=x^3+x^4+x^5+x^6+\ldots+x^r,$$

$$U_2=2x^4+5x^5+9x^6+\ldots+\frac{r.r-3}{2.1}x^r,$$

$$U_3=5x^5+21x^6+\ldots+\frac{r+1.r.r-3.r-4}{3.2.1}x^r,$$

$$U_4=14x^6+\ldots+\frac{r+2.r+1.r.r-3.r-4.r-5}{4.3.2.3.2.1}x^r,$$

$$U_5=\ldots+\frac{r+3.r+2.r+1.r.r-3.r-4.r-5.r-6}{5.4.3.2.4.3.2.1}x^r,$$

&c.

3. And in connexion herewith, I give the second Table on the next page, which is a tabulation of the functions

$$V_2=x^6+2x^7+3x^8+4x^9+\ldots+1\frac{r-3}{1}x^{r+2},$$

$$V_3=4x^7+14x^8+32x^9+\ldots+2\frac{r+1.r-3.r-4}{3.2.1}x^{r+2},$$

$$V_4=14x^8+72x^9+\ldots+3\frac{r+2.r+1.r-3.r-4.r-5}{4.3.3.2.1}x^{r+2},$$

$$V_5=48x^9+\ldots+4\frac{r+3.r+2.r+1.r-3.r-4.r-5.r-6}{5.4.3.4.3.2.1}x^{r+2},$$

&c.

$r=$	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13
3	1												
4	1	2											
5	1	5	5										
6	1	9	21	14									
7	1	14	56	84	42								
8	1	20	120	300	330	132							
9	1	27	225	825	1485	1287	429						
10	1	35	385	1925	5005	7007	5005	1430					
11	1	44	616	4004	14014	28028	32032	19448	4862				
12	1	54	936	7644	34398	91728	148512	143208	75582	16796			
13	1	65	1365	13650	76440	259896	556920	755820	629850	293930	58786		
14	1	77	1925	23100	157080	659736	1790712	3197700	3730650	2735810	1144066	208012	
15	1	90	2640	37400	302940	1534896	5116320	11511720	17587350	17978180	11767536	4457400	742900

$r=$	$k=2$	3	4	5	6	7	8	9	10	11	12	13
4	1											
5	2	4										
6	3	14	14									
7	4	32	72	48								
8	5	60	225	330	165							
9	6	100	550	1320	1430	572						
10	7	154	1155	4004	7007	6006	2002					
11	8	224	2184	10192	25480	34944	24052	7072				
12	9	312	3822	22932	76440	148512	167076	100776	25194			
13	10	420	6300	47040	199920	514080	813960	775200	396800	90440		
14	11	550	9900	89760	476240	1534996	3197700	4263600	3517470	1634380	326876	
15	12	704	14960	161568	1023264	4093056	10744272	18759840	18573816	15690048	6547520	1188640

4. These functions, U and V, are particular values satisfying the equation

$$(V_2 + V_3 y + V_4 y^2 + \ldots) = (U_1 + U_2 y + U_3 y^2 + U_4 y^3 + \ldots)^2;$$

that this is so will appear from the following general investigation.

5. Taking x, y as independent variables, denoting by X an arbitrary function of x, and using accents to denote differentiations in regard to x, we require the following identity:

$$\frac{2}{1.2} X^2 + \frac{4y}{1.2.3}(X^3)' + \frac{6y^2}{1.2.3.4}(X^4)'' + \ldots = \left\{X + \frac{y}{1.2}(X^2)' + \frac{y^2}{1.2.3}(X^3)'' + \ldots\right\}^2,$$

which I prove as follows. Writing U to denote the same function of u which X is of x, I start from the equation

$$u = x + yU,$$

which determines u as a function of the independent variables x, y. We have

$$\frac{du}{dy}(1 - yU') = U, \quad \frac{du}{dx}(1 - yU') = 1,$$

where the accent denotes differentiation in regard to u; hence

$$\frac{du}{dy} = U\frac{du}{dx} = \frac{u-x}{y}\frac{du}{dx},$$

or say

$$y\frac{du}{dy} = (u-x)\frac{du}{dx}.$$

Writing

$$u_1 = \int u\,dx,$$

and therefore

$$\frac{du_1}{dx} = u,$$

this equation may be written

$$y\frac{d^2u_1}{dx\,dy} - \frac{du_1}{dx} = u\frac{du}{dx} - u - x\frac{du}{dx};$$

or, integrating with respect to x, we have

$$y\frac{du_1}{dy} - u_1 = \tfrac{1}{2}u^2 - ux,$$

or say

$$\frac{2}{y}\frac{du_1}{dy} - \frac{2(u_1 - \frac{1}{2}x^2)}{y^2} = \frac{(u-x)^2}{y^2},$$

that is,

$$2\frac{d}{dy}\left(\frac{u_1 - \frac{1}{2}x^2}{y}\right) = \frac{(u-x)^2}{y^2}.$$

6. But, from the equation

$$u = x + yU,$$

we have

$$u = x + yX + \frac{y^2}{1.2}(X^2)' + \frac{y^3}{1.2.3}(X^3)'' + \ldots,$$

and thence

$$u_1 = \tfrac{1}{2}x^2 + yX_1 + \frac{y^2}{1.2}X^2 + \frac{y^3}{1.2.3}(X^3)' + \ldots,$$

if for a moment X_1 is written for $\int X\,dx$. And hence, from the relation obtained above, we have the required identity

$$\frac{2}{1.2}X^2 + \frac{4y}{1.2.3}(X^3)' + \frac{6y^2}{1.2.3.4}(X^4)'' + \ldots = \left\{X + \frac{y}{1.2}(X^2)' + \frac{y^2}{1.2.3}(X^3)'' + \ldots\right\}^2.$$

This of course gives the series of identities

$$\begin{aligned}
\frac{2}{1.2}X^2 &= X^2,\\
\frac{4}{1.2.3}(X^3)' &= \frac{2}{1.2}X(X^2),\\
\frac{6}{1.2.3.4}(X^4)'' &= \frac{2}{1.2.3}X(X^3)'' + \left\{\frac{1}{1.2}(X^2)'\right\}^2;\\
&\vdots
\end{aligned}$$

or say

$$\begin{aligned}
X^2 &= X^2,\\
(X^3)' &= \tfrac{3}{2}X(X^2)',\\
(X^4)'' &= \tfrac{4}{3}X(X^3)'' + \{(X^2)'\}^2,\\
&\vdots
\end{aligned}$$

all of which may be easily verified.

7. I multiply each side of the identity by x^2, and write

$$\begin{aligned}
U_1 &= x.X, & V_2 &= x^2\frac{2}{1.2}X^2,\\
U_2 &= x\frac{1}{1.2}(X^2)', & V_3 &= x^2\frac{4}{1.2.3}(X^3)',\\
U_3 &= x\frac{1}{1.2.3}(X^3)'', & V_4 &= x^2\frac{6}{1.2.3.4}(X^4)'',\\
U_4 &= x\frac{1}{1.2.3.4}(X^4)''', & V_5 &= x^2\frac{8}{1.2.3.4.5}(X^5)''',\\
&\vdots & &\vdots
\end{aligned}$$

We thus obtain two sets of functions U and V, satisfying the before-mentioned equation. We have

$$(V_2 + yV_3 + y^2V_4 + \ldots) = (U_1 + yU_2 + y^2U_3 + \ldots)^2;$$

and it will be observed that we have, moreover, the relations

$$U_2 = \tfrac{1}{2}x(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x(x^{-2}V_4)', \ldots.$$

8. In particular, if

$$X = \frac{x^2}{1-x},$$

then the general term

in X is x^{r-1},	the first term occurring when		$r=3$,
in X^2 is $(r-3)x^r$,	"	"	$r=4$,
in X^3 is $\dfrac{r-3.r-4}{1.2}x^{r+1}$,	"	"	$r=5$,
in X^4 is $\dfrac{r-3.r-4.r-5}{1.2.3}x^{r+2}$,	"	"	$r=6$,

$\vdots$

from which it appears that, for this value of X, U_1, U_2, U_3, U_4, &c., have the before-mentioned values (No. 2), and further that V_2, V_3, V_4, V_5, &c., have also the before-mentioned values (No. 3).

9. We do not absolutely require, but it is interesting to obtain, the finite expressions of these functions. We have

$$\begin{aligned}
&(1-x)\ U_1 = x^2(1),\\
&(1-x)^3\ U_2 = x^4(2-x),\\
&(1-x)^5\ U_3 = x^5(5-4x+x^2),\\
&(1-x)^7\ U_4 = x^6(14-14x+6x^2-x^3),\\
&(1-x)^9\ U_5 = x^7(42-48x+27x^2-8x^3+x^4),\\
&(1-x)^{11}\ U_6 = x^8(132-165x+110x^2-44x^3+10x^4-x^5);\\
&\quad\vdots\\
&(1-x)^2\ V_2 = x^6\ (1),\\
&(1-x)^4\ V_3 = x^7\ (4-2x),\\
&(1-x)^6\ V_4 = x^8\ (14-12x+3x^2),\\
&(1-x)^8\ V_5 = x^9\ (48-54x+24x^2-4x^3),\\
&(1-x)^{10}\ V_6 = x^{10}(165-220x+132x^2-40x^3+5x^4),\\
&\quad\vdots
\end{aligned}$$

and here the factors in ( ) satisfy the series of relations

$$\begin{aligned}
1 &= 1^2,\\
4-2x &= 2(2-x),\\
14-12x+3x^2 &= 2\,.\,1(5-4x+x^2)+(2-x)^2,\\
48-54x+24x^2-4x^3 &= 2\,.\,1(14-14x+6x^2-x^3)+2(2-x)(5-4x+x^2),\\
&\ \vdots
\end{aligned}$$

corresponding to

$$V_2 = U_1^2,\quad V_3 = 2U_1U_2,\ \text{\&c.},$$

given by the before-mentioned equation (No. 7), between the functions V and U.

10. It is to be shown that, taking U_1, U_2, U_3, ... for the functions which belong to the partitions of the r-gon (assumed to be unknown functions of r and the suffixes), and connecting them with a set of functions V_2, V_3, V_4, ... by the relations

$$U_2 = \tfrac{1}{2}x(x^{-2}V_2)',\quad U_3 = \tfrac{1}{4}x(x^{-2}V_3)',\quad U_4 = \tfrac{1}{6}x(x^{-2}V_4)',\ \text{\&c.},$$

then we have the foregoing identical equation

$$(V_2+V_3y+V_4y^2+\ldots) = (U_1+U_2y+U_3y^2+U_4y^3+\ldots)^2.$$

This implies the relations

$$\begin{aligned} V_2 &= U_1^2, \\ V_3 &= 2U_1U_2, \\ V_4 &= 2U_1U_3 + U_2^2, \\ V_5 &= 2U_1U_4 + 2U_2U_3, \\ &\&c. \end{aligned}$$

Thus, if U_1 is known, the equation

$$V_2 = U_1^2$$

determines V_2, and then the equation

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)'$$

determines U_2, so that U_1, U_2 are known; and we thence in the same way find successively U_3 and V_3, U_4 and V_4, and so on; that is, assuming only that U_1 has the before-mentioned value,

$$U_1 = x^3 + x^4 + x^5 + \ldots + x^r + \ldots,$$

it follows that all the remaining functions U and V must have their before-mentioned values. But the function

$$U_1 = x^3 + x^4 + x^5 + \ldots,$$

where each coefficient is $=1$, is evidently the proper expression for the generating function of the number of partitions of the r-gon into a single part; and we thus arrive at the proof that the remaining functions U, which are the generating functions for the number of partitions of the r-gon into 2, 3, 4, ..., k, parts, have their before-mentioned values.

11. Considering, then, the partition problem from the point of view just referred to, I write A_r, B_r, C_r, ... for the number of partitions of an r-gon into 1 part, 2 parts, 3 parts, &c., and form therewith the generating functions

$$\begin{aligned} U_1 &= A_3x^3 + A_4x^4 + \quad \ldots + A_rx^r + \ldots, \\ U_2 &= \quad B_4x^4 + \quad \ldots + B_rx^r + \ldots, \\ U_3 &= \quad C_5x^5 + \ldots + C_rx^r + \ldots, \\ &\vdots \end{aligned}$$

and also the functions

$$\begin{aligned} V_2 &= \frac{2}{4}B_4x^6 + \quad \ldots + \frac{2}{r}B_rx^{r+2} + \ldots, \\ V_3 &= \frac{4}{5}C_5x^7 + \ldots + \frac{4}{r}C_rx^{r+2} + \ldots, \\ &\vdots \end{aligned}$$

where observe that the functions U, V are such that

$$U_2 = \tfrac{1}{2}x\,(x^{-2}V_2)', \quad U_3 = \tfrac{1}{4}x\,(x^{-2}V_3)', \quad U_4 = \tfrac{1}{6}x\,(x^{-2}V_4)', \ \&c.$$

To fix the ideas, consider an r-gon which is to be divided into six parts. Choosing any particular summit, and from this summit drawing a diagonal successively

to each of the non-adjacent $r-3$ summits, we divide the r-gon into two parts in $r-3$ different ways; viz. the two parts are

$$\begin{array}{lcl} \text{a } 3\text{-gon} & \text{and} & (r-1)\text{-gon}, \\ 4\text{-gon} & \text{and} & (r-2)\text{-gon}, \\ \vdots & & \\ (r-1)\text{-gon} & \text{and} & 3\text{-gon}; \end{array}$$

say any one of these ways is

$$\text{an } \alpha\text{-gon} \quad \text{and} \quad \beta\text{-gon}, \quad \alpha+\beta=r+2.$$

Next, writing

$$a+b=6,$$

that is,

$$\begin{array}{rl} a,\ b = & 1,\ 5, \\ & 2,\ 4, \\ & 3,\ 3, \\ & 4,\ 2, \\ & 5,\ 1, \end{array}$$

we divide in every possible way the α-gon into a parts, and the β-gon into b parts (so dividing the r-gon into six parts). Observing that A, B, C, D, E, F are the letters belonging to the numbers 1, 2, 3, 4, 5, 6, respectively, the number of parts which we thus obtain (corresponding to the different values of a, b) are

$$A_\alpha E_\beta + B_\alpha D_\beta + C_\alpha C_\beta + D_\alpha B_\beta + E_\alpha A_\beta,$$

and summing for the different values of α, β $(\alpha+\beta=r+2)$, the whole number of parts is

$$= \text{coeff. } x^{r+2} \text{ in } (U_1U_5 + U_2U_4 + U_3U_3 + U_4U_2 + U_5U_1),$$

that is,

$$= \text{coeff. } x^{r+2} \text{ in } (2U_1U_5 + 2U_2U_4 + U_3^2).$$

12. To obtain the whole number of the partitions of the r-gon into six parts, we must perform the foregoing process successively with each summit of the r-gon as the summit from which is drawn the diagonal which divides the r-gon into two parts; that is, the number found as above is to be multiplied by r. We thus obtain all the partitions repeated a certain number of times, viz. each partition into six parts is a partition by means of five diagonals, and is thus obtainable by the foregoing process, taking any one of the ten extremities of these diagonals as the point from which is drawn the diagonal which divides the r-gon into two parts; that is, we have to divide the foregoing product by 10. The final result thus is

$$\frac{10}{r} F_r = \text{coeff. } x^{r+2} \text{ in } (2U_1U_5 + 2U_2U_4 + U_3^2),$$

where

$$\frac{10}{r} F_r \text{ is} = \text{coeff. } x^{r+2} \text{ in } V_6;$$

we thus have

$$V_6 = 2U_1U_5 + 2U_2U_4 + U_3^2.$$

13. The reasoning is perfectly general; and applying it successively to the partitions into two parts, three parts, &c., we have

$$\begin{aligned} V_2 &= U_1^2, \\ V_3 &= 2U_1U_2, \\ V_4 &= 2U_1U_3 + U_2^2, \\ V_5 &= 2U_1U_4 + 2U_2U_3, \\ &\vdots \end{aligned}$$

where any function V is related to the corresponding function U as above. The value of U_1 is obviously

$$U_1 = x^3 + x^4 + x^5 + \ldots = \frac{x^3}{1-x};$$

and hence the several functions U and V have the values above written down; the general term of U_k is

$$\frac{[r+k-2]^{k-1}\,[r-3]^{k-1}}{[k]^{k-1}\,[k-1]^{k-1}}\,x^r;$$

and the number of partitions of the r-gon into k parts is equal to the coefficient of x^r in this general term.

14. In the investigations which next follow, I consider, without using the method of generating functions, the problem of the partition of the r-gon into 2, 3, 4 or 5 parts; it will be convenient to state the results as follows:

Number of Partitions.

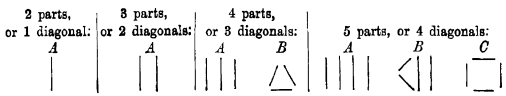
2 parts, $\frac{r}{2}\ A$,

3 parts, $\frac{r}{4}\ 2A$,

4 parts, $\frac{r}{6}(3A + 2B)$,

5 parts, $\frac{r}{8}(4A + 8B + 2C)$;

where the capital letters refer to different "diagonal-types," thus:



viz. if, in a polygon divided into k parts by means of $k-1$ diagonals, we delete all the sides of the polygon, leaving only the diagonals, then these will present themselves under distinct forms, which are what I call "diagonal-types"; for instance, when

$$k = 4,$$

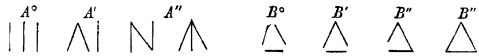
there are the two types A and B shown in the above diagram for four parts.

15. It is to be observed that we have sub-types corresponding to the coalescence of the terminal points of different diagonals; thus, suppose

$$k = 4.$$

Writing now A° and B° to denote the forms without coalescences, we have the sub-types A°, A', A'' and B°, B', B'', B''', as follows:

4 parts, or 3 diagonals:



where observe that under A'' are included two distinct forms, which, nevertheless, by reason that there is in each of them the same number $(=2)$ of coalescences, are reckoned as belonging to the same sub-type.

16. The numbers called A, B, C, &c., have values which may be directly determined. I write down as follows:

1 diagonal, $A = \dfrac{r-3}{1}$.

2 diagonals, $A = \dfrac{r-3\,.\,r-2\,.\,r-1}{6} - \dfrac{r-3}{1} = \dfrac{r-3\,.\,r-4}{6}(r+1)$;

where the calculation is

$$\begin{aligned} r-2\,.\,r-1 &= r^2-3r+2 \\ -6 \qquad & \qquad\quad -6 \\ \hline & r^2-3r-4 \\ &= r-4\,.\,r+1. \end{aligned}$$

3 diagonals, $A = \dfrac{r-3\,.\,r-2\,.\,r-1\,.\,r\,.\,r+1}{120} - 2\left(\dfrac{r-3\,.\,r-2\,.\,r-1}{6} - \dfrac{r-3}{1}\right) - \dfrac{r-3}{1}$

$$= \frac{r-3\,.\,r-2\,.\,r-1\,.\,r\,.\,r+1}{120} - 2\,\frac{r-3\,.\,r-2\,.\,r-1}{6} + 1\,\frac{r-3}{1}$$

$$= \frac{r-3\,.\,r-4\,.\,r-5}{120}(r^2+7r+2);$$

where the calculation is

$$\begin{aligned} r-2\,.\,r-1\,.\,r\,.\,r+1 &= r^4-2r^3- \quad r^2+ \quad 2r \\ -40\,.\,r-2\,.\,r-1 \qquad & \qquad\quad -40r^2+120r- \quad 80 \\ +120 \qquad & \qquad\qquad\qquad\qquad\quad +120 \\ \hline & r^4-2r^3-41r^2+122r+ \quad 40 \\ &= r-4\,.\,r-5\,.\,r^2+7r+2. \end{aligned}$$

$$B = \frac{r-5\,.\,r-4\,.\,r-3\,.\,r-2\,.\,r-1}{120}$$

$$= \frac{r-3\,.\,r-4\,.\,r-5}{120}(r-1\,.\,r-2).$$

4 diagonals, $$A = \frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}$$

$$-3\frac{r-3.r-2.r-1.r.r+1}{120}+3\frac{r-3.r-2.r-1}{6}-1\frac{r-3}{1}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}(r^3+18r^2+65r);$$

where the calculation is

$$\begin{array}{rl}
r-2.r-1.r.r+1.r+2.r+3 = & r^6+3r^5-5r^4-15r^3+4r^2+12r \\
-126.r-2.r-1.r.r+1 & -126r^4+252r^3+126r^2-252r \\
+2520.r-2.r-1 & +2520r^2-7560r+5040 \\
-5040 & -5040 \\
\hline
 & r^6+3r^5-131r^4+237r^3+2650r^2-7800r \\
 & =r-4.r-5.r-6.r^3+18r^2+65r.
\end{array}$$

Also

$$B=\frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040}-\frac{r-5.r-4.r-3.r-2.r-1}{120}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}(r-1.r-2.r+7),$$

$$C=\frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040}$$

$$=\frac{r-3.r-4.r-5.r-6}{5040}(r-1.r-2.r-7);$$

where the calculation is

$$\begin{array}{rl}
r.r+1 & =r^2+r \\
-42 & -42 \\
\hline
 & r^2+r-42 \\
 & =r-6.r+7.
\end{array}$$

17. To explain the formation of these expressions, observe that:

One diagonal.—There must be on each side of the diagonal, or say in each of the two "intervals" formed by the diagonal, two sides; there remain $r-4$ sides which may be distributed at pleasure between the two intervals, and the number of ways in which this can be done is

$$=\frac{r-3}{1}.$$

Two diagonals.—There must be on each side of the two diagonals, or say in two of the four intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same four intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1}{6}.$$

But we must exclude the distributions where there is no side in the one interval and no side in the other interval between the two diagonals; the number of these is that for the case of the coalescence of the two diagonals into a single diagonal, viz. it is

$$=\frac{r-3}{1};$$

and thus the number required is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1}.$$

18. Three diagonals, *A*.—There must be on each side of the three diagonals, that is, in two of the six intervals formed by the diagonals, two sides; there remain $r-4$ sides to be distributed between the same six intervals, and the number of ways in which this can be done is

$$=\frac{r-3.r-2.r-1.r.r+1}{120}.$$

But we must exclude distributions which would permit the coalescence of the first and second, or of the second and third, or of all three of the diagonals. For the coalescence of the first and second diagonals (the third diagonal not coalescing), the term to be subtracted is

$$\frac{r-3.r-2.r-1}{6}-\frac{r-3}{1};$$

and the same number for the coalescence of the second and third diagonals (the first diagonal not coalescing); that is, the last-mentioned number is to be multiplied by 2; and for the coalescence of all three diagonals the number to be subtracted is

$$=\frac{r-3}{1};$$

we have thus the foregoing value

$$\frac{r-3.r-2.r-1.r.r+1}{120}-2.\frac{r-3.r-2.r-1}{6}+1\frac{r-3}{1},$$

where it will be observed that we have the binomial coefficients 1, 2, 1 with the signs +, −, +.

Three diagonals, *B*.—There must be outside each of the three diagonals, that is, in three of the six intervals formed by the diagonals, two sides; and there remain $r-6$ sides to be distributed between the six intervals; the number of ways in which this can be done is

$$=\frac{r-5.r-4.r-3.r-2.r-1}{120};$$

and there is here no coalescence of diagonals, so that this is the number required.

19. Four diagonals, *A*.—There must be on each side of the four diagonals, that is, in two of the eight intervals formed by the diagonals, two sides; there remain

$r-4$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}.$$

But this number requires to be corrected for coalescences, as in the case Three diagonals, A; and the required number is thus found to be

$$\frac{r-3.r-2.r-1.r.r+1.r+2.r+3}{5040}-3\frac{r-3.r-2.r-1.r.r+1}{120}$$
$$+3\frac{r-3.r-2.r-1}{6}-1\frac{r-3}{1}.$$

Four diagonals, B.—There must be outside of three of the diagonals, that is, in each of three of the eight intervals formed by the diagonals, two sides; there remain $r-6$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-5.r-4.r-3.r-2.r-1.r.r+1}{5040}.$$

There is a correction for the coalescence of two of the diagonals, giving rise to a form such as Three diagonals, B; and consequently there is a term

$$-\frac{r-5.r-4.r-3.r-2.r-1}{120},$$

which, with the first-mentioned term, gives the required number.

Four diagonals, C.—There must be outside of each of the diagonals, that is, in each of four of the eight intervals formed by the diagonals, two sides; there remain $r-8$ sides to be distributed between the eight intervals, and the number of ways in which this can be done is

$$\frac{r-7.r-6.r-5.r-4.r-3.r-2.r-1}{5040},$$

which is the required number.

20. In the expressions of No. 14, A, $2A$, $3A+2B$, $4A+8B+2C$, if we regard the terminals of the diagonals as given points, then (1) we have two summits, which can be joined in one way only, giving rise to the diagonal-type A; (2) we have four summits, which can be joined in two ways only, so as to give rise to the diagonal-type A; (3) we have six summits, which can be joined in three ways so as to give rise to a diagonal-type A, and in two ways so as to give rise to a diagonal-type B; and (4) we have eight summits, which can be joined in four ways so as to give rise to a diagonal-type A, in eight ways so as to give rise to a diagonal-type B, and in two ways so as to give rise to a diagonal-type C; we have thus the linear forms in question. To obtain the number of partitions, we have in each case to multiply by r. To explain this, after the polygon is drawn, imagine

the summits to be numbered 1, 2, 3, ..., r in succession (the numbering may begin at any one of the r summits); regarding each of these numberings as giving a different partition, we should have the factor r. But, in fact, the partitions so obtained are not all of them distinct, but we have in each case a system of partitions repeated as many times as there are summits of the diagonals, that is, a number of times equal to twice the number of the diagonals; and we have thus, after the multiplication by r, to divide by the numbers 2, 4, 6, 8, in the four cases respectively.

21. We hence have immediately:—

Two parts, the number of partitions

$$=\frac{r}{2}A=\frac{r\,.\,r-3}{2\,.\,1};$$

Three parts, the number of partitions

$$=\frac{r}{2}A=\frac{r+1\,.\,r\,.\,r-3\,.\,r-4}{3\,.\,2\,.\,2\,.\,1};$$

Four parts, the number of partitions

$$=\frac{r}{6}(3A+2B)=\frac{r+2\,.\,r+1\,.\,r\,.\,r-3\,.\,r-4\,.\,r-5}{4\,.\,3\,.\,2\,.\,3\,.\,2\,.\,1},$$

the calculation being

$$\begin{aligned}
3\,(r^2+7r+2) &= 3r^2+21r+6\\
+2\,.\,r-1\,.\,r-2 &\quad +2r^2-6r+4\\
\hline
&\quad 5r^2+15r+10\\
&= 5\,.\,r+1\,.\,r+2;
\end{aligned}$$

Five parts, the number of partitions

$$=\frac{r}{8}(4A+8B+2C)=\frac{r+3\,.\,r+2\,.\,r+1\,.\,r\,.\,r-3\,.\,r-4\,.\,r-5\,.\,r-6}{5\,.\,4\,.\,3\,.\,2\,.\,4\,.\,3\,.\,2\,.\,1},$$

the calculation being

$$\begin{aligned}
4\,(r^3+18r^2+65r) &= 4r^3+72r^2+260r\\
+8\,.\,r-1\,.\,r-2\,.\,r+7 &\quad +8r^3+32r^2-152r+112\\
+2\,.\,r-1\,.\,r-2\,.\,r-7 &\quad +2r^3-20r^2+46r-28\\
\hline
&= 14r^3+84r^2+154r+84\\
&= 14\,(r^3+6r^2+11r+6)\\
&= 14\,.\,r+1\,.\,r+2\,.\,r+3.
\end{aligned}$$

To complete the theory, it would be in the first instance necessary to find for any given number of diagonals, $k-1$, whatever, the number and form of the diagonal-types, A, B, C, &c.; this is itself an interesting question in the Theory of Partitions, but I have not considered it.

22. Although the foregoing process (which, it will be observed, deals with the diagonal-types, without any consideration of the sub-types) is the most simple for the determination of the numbers A, B, C, &c., yet it is interesting to give a second process. Considering the several cases in order:

One diagonal, A.—The diagonal has two summits; we must have on each side of it one summit, and there remain $r-4$ summits which may be distributed between the two intervals formed by the diagonals. This can be done in $\frac{r-3}{1}$ ways, or we have, as before,

$$A=\frac{r-3}{1}.$$

Two diagonals, A.—The diagonals have four summits; we must have outside each diagonal one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; this can be done in $\frac{r-5\,.\,r-4\,.\,r-3}{6}$ ways, or we have this value for A°. But the two top summits of the diagonals, or the two bottom summits, may coalesce; in either case, the diagonals have three summits. We must have outside each diagonal one summit, and there remain $r-5$ summits to be distributed between the three intervals formed by the diagonals; the number of ways in which this can be done is

$$=\frac{r-4\,.\,r-3}{2},$$

say this is the value of A'. And we then have $A=A^\circ+2A'$,

$$=\frac{r+1\,.\,r-3\,.\,r-4}{6},$$

as before. The calculation is

$$r-5+6=r+1.$$

23. Three diagonals, A.—See No. 15 for the figures of the sub-types. We have

$$A=A^\circ+4A'+4A'',$$

where the coefficients, 4 and 4, are the number of ways in which A' and A'' respectively can be derived from A° by coalescences of summits. For A°, the diagonals have six summits, and there must be outside of two diagonals one summit; there remain $r-8$ summits to be distributed between the six intervals formed by the diagonals, and we have

$$A^\circ=\frac{r-7\,.\,r-6\,.\,r-5\,.\,r-4\,.\,r-3}{120}.$$

For A', the diagonals have five summits, and we must have outside of each of two diagonals, one summit; there remain $r-7$ summits to be distributed between the five intervals formed by the diagonals; we thus have

$$A'=\frac{r-6\,.\,r-5\,.\,r-4\,.\,r-3}{24}.$$

For A'', the diagonals have four summits; there must be outside of each of two diagonals one summit, and there remain $r-6$ summits to be distributed between the four intervals formed by the diagonals; we thus have

$$A'' = \frac{r-5\,.\,r-4\,.\,r-3}{6}.$$

The foregoing values give

$$A = \frac{r-3\,.\,r-4\,.\,r-5}{120}(r^2+7r+2),$$

as before. The calculation is

$$\begin{array}{rrrr} r-6\,.\,r-7 = & r^2 & -\ 13r & +\ 42 \\ +\ 20\,.\,r-6 & & +\ 20r & -\ 120 \\ +\ 80 & & & +\ 80 \\ \hline & r^2 & +\ 7r & +\ 2. \end{array}$$

Three diagonals, B.—See No. 15 for the figures of the sub-types. We have

$$B = B^\circ + 3B' + 3B'' + B'''.$$

For B°, the diagonals have six summits, and there must be outside each of the three diagonals one summit; there remain $r-9$ summits to be distributed between the six intervals formed by the diagonals. We thus have

$$B^\circ = \frac{r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4}{120}.$$

Similarly,

$$B' = \frac{r-7\,.\,r-6\,.\,r-5\,.\,r-4}{24},\quad B'' = \frac{r-6\,.\,r-5\,.\,r-4}{6},\quad B''' = \frac{r-5\,.\,r-4}{2}.$$

Hence

$$B = \frac{r-5\,.\,r-4\,.\,r-3\,.\,r-2\,.\,r-1}{120},$$

as before. The calculation is

$$\begin{array}{rrrrr} r-6\,.\,r-7\,.\,r-8 = & r^3 & -\ 21r^2 & +\ 146r & -\ 336 \\ +\ 15\,.\,r-6\,.\,r-7 & & +\ 15r^2 & -\ 195r & +\ 630 \\ +\ 60\,.\,r-6 & & & +\ 60r & -\ 360 \\ +\ 60 & & & & +\ 60 \\ \hline & r^3 & -\ 6r^2 & +\ 11r & -\ 6 \\ = & r & -\ 1\,.\,r & -\ 2\,.\,r & -\ 3. \end{array}$$

24. Four diagonals, A.—The figures of the sub-types of A, B, C can be supplied without difficulty. We have

$$A = A^\circ + 6A' + 12A'' + 8A''',$$

where I remark that the numerical coefficients 1, 6, 12, 8 are the terms of $(1, 2)^3$. We have

$$A^\circ = \frac{r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4\,.\,r-3}{5040},$$

$$A' = \frac{r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4\,.\,r-3}{720},$$

$$A'' = \frac{r-7\,.\,r-6\,.\,r-5\,.\,r-4\,.\,r-3}{120},$$

$$A''' = \frac{r-6\,.\,r-5\,.\,r-4\,.\,r-3}{24},$$

and thence

$$A = \frac{r-3\,.\,r-4\,.\,r-5\,.\,r-6}{5040}(r^3+18r^2+65r),$$

as before. The calculation is

$$\begin{array}{rl}
r-9\,.\,r-8\,.\,r-7 = & r^3-24r^2+191r-504 \\
+42\,.\,r-8\,.\,r-7 & \quad\;\; +42r^2-630r+2352 \\
+504\,.\,r-7 & \qquad\qquad\;\; +504r-3528 \\
+1680 & \qquad\qquad\qquad\quad +1680 \\
\hline
 & r^3+18r^2+65r \quad .
\end{array}$$

Four diagonals, B.—We have

$$B = B^\circ + 5B' + 9B'' + 7B''' + 2B^{\text{iv}},$$

where the coefficients, 1, 5, 9, 7, 2, are the terms of $(1, 1)^3(1, 2)$. We have

$$B^\circ = \frac{r-10\,.\,r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4}{5040},$$

$$B' = \frac{r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4}{720},$$

$$B'' = \frac{r-8\,.\,r-7\,.\,r-6\,.\,r-5\,.\,r-4}{120},$$

$$B''' = \frac{r-7\,.\,r-6\,.\,r-5\,.\,r-4}{24},$$

$$B^{\text{iv}} = \frac{r-6\,.\,r-5\,.\,r-4}{6},$$

and thence

$$B = \frac{r-3\,.\,r-4\,.\,r-5\,.\,r-6}{5040}\,r-1\,.\,r-2\,.\,r+7,$$

as before. The calculation is

$$\begin{array}{rl}
r-10\,.\,r-9\,.\,r-8\,.\,r-7 = & r^4-34r^3+431r^2-2414r+5040 \\
+35\,.\,r-9\,.\,r-8\,.\,r-7 & \quad +35r^3-840r^2+6685r-17640 \\
+378\,.\,r-8\,.\,r-7 & \qquad +378r^2-5670r+21168 \\
+1470\,.\,r-7 & \qquad\qquad +1470r-10290 \\
+1680 & \qquad\qquad\qquad +1680 \\
\hline
 & r^4+r^3-31r^2+71r-42 \\
 & =r-3\,.\,r-2\,.\,r-1\,.\,r+7.
\end{array}$$

Four diagonals, C.—We have

$$C=C^\circ+4C'+6C''+4C'''+1C^{\text{iv}},$$

where the coefficients are the terms of $(1, 1)^4$. We have

$$C^\circ=\frac{r-11\,.\,r-10\,.\,r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5}{5040},$$

$$C'=\frac{r-10\,.\,r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5}{720},$$

$$C''=\frac{r-9\,.\,r-8\,.\,r-7\,.\,r-6\,.\,r-5}{120},$$

$$C'''=\frac{r-8\,.\,r-7\,.\,r-6\,.\,r-5}{24},$$

$$C^{\text{iv}}=\frac{r-7\,.\,r-6\,.\,r-5}{6},$$

and thence

$$C=\frac{r-7\,.\,r-6\,.\,r-5\,.\,r-4\,.\,r-3\,.\,r-2\,.\,r-1}{5040}.$$

I omit the calculation, as the equation is at once seen to be a particular case of a known factorial formula.

25. We may analyse the partitions of an r-gon into a given number of parts, according to the nature of the parts, that is, the numbers of the sides of the several component polygons. It is for this purpose convenient to introduce the notion of "weight"; say a triangle has the weight 1, then a quadrangle, as divisible into two triangles, has the weight 2, a pentagon, as divisible into three triangles, has the weight 3, ..., and generally an r-gon, as divisible into $r-2$ triangles, has the weight $r-2$. It at once follows that, if

$$W=w+w', \text{ or } =w+w'+w'', \text{ \&c.},$$

then a polygon of weight W is divisible into two polygons of the weights w, w', or into three polygons of the weights w, w', w'' respectively; and so on. Thus the 2-partitions of an 8-gon (weight = 6) are 15, 24, and 33; the 3-partitions are 114,

123, 222, and so on. Of course the number of the partitions 15, 24, 33, is equal to the whole number of the 2-partitions of the 8-gon, that is, $=20$; the number of the partitions 114, 123, 222, is equal to the whole number of the 3-partitions of the 8-gon, that is, it is $=120$; and so in other cases. It is easy to derive in order one from the other the numbers of the partitions of each several kind of the polygons of the several weights 2, 3, 4, 5, 6, &c.; and I write down the accompanying Table (No. 26), facing page 112, the process for the construction being as follows:

27. The first column (2 parts) is at once obtained. For a polygon of an odd number of sides, for instance the 9-gon (weight $=7$), imagining the summits numbered in order 1, 2, ..., 9, we divide this into a triangle and octagon, or obtain the partitions 16, by drawing the diagonals 13, 24, ..., 81, 92: viz. the number is $=9$. In the Table this is written, $16=9$; and so in other cases. Similarly we divide it into a quadrangle and heptagon, or obtain the partitions 25, by drawing the diagonals 14, 25, ..., 82, 93: viz. the number is again $=9$; and we divide it into a pentagon and a hexagon, or obtain the partitions 34, by drawing the diagonals 15, 26, ..., 83, 94: viz. the number is $=9$, and here

$$9+9+9=27,$$

the whole number of 2-partitions of the 9-gon. For a polygon of an even number of sides, for instance the 10-gon (weight $=8$), the process is a similar one, the only difference being that for the division into two hexagons, (that is, for the partitions 44), each partition is thus obtained twice, or the number of such partitions is $\frac{1}{2}10, =5$; the numbers for the partitions 17, 26, 35, 44, thus are 10, 10, 10, 5; and we have

$$10+10+10+5=35,$$

the whole number of the 2-partitions of the 10-gon.

28. To obtain the second column (3 parts)—suppose, for instance, the 3-partitions of the 9-gon; these are 115, 124, 133, 223. We obtain the number of the partitions 115 from the terms

$$16=9 \text{ and } 25=9$$

of the first column: viz. in 16, changing the 6 into 15, that is, dividing the polygon of weight 6 into two parts of weights 1 and 5 respectively: this can be done in eight ways (see, higher up, $15=8$ in the first column); and we thus obtain the number of partitions

$$9\times 8=72;$$

and again, in 25, changing the 2 into 11, that is, dividing the polygon of weight 2 into two parts each of weight 1: this can be done in two ways (see, higher up, $11=2$ in the first column); and we thus obtain the number of partitions

$$9\times 2=18;$$

we should thus have, for the number of partitions 115, the sum

$$72+18=90,$$

only, as it is easy to see, each partition is obtained twice, and the number of the

partitions 115 is the half of this, $=45$. And by the like process it is found that the numbers of the partitions 124, 133, 223 are equal to 90, 45, 45 respectively; and then, as a verification, we have

$$45+90+45+45=225,$$

the whole number of the 3-partitions of the 9-gon.

29. The third column (4 parts) is derived in like manner from the second column by aid of the first column; and so in general, each column is derived in like manner from the column which immediately precedes it, by aid of the first column. And we have for the numbers in each compartment of any column the verification that the sum of these numbers is equal to the whole number (for the proper values of k and r) of the k-partitions of the r-gon.

It might be possible, by an application of the method of generating functions, to find a law for the numbers in any compartment of a column of the table; but I have not attempted to make this investigation.

30. In the table in No. 2, the numbers 1, 2, 5, 14, 42, &c., of the diagonal line show the number of partitions of the triangle, the quadrangle, the 5-gon, ..., r-gon into triangles: viz. these numbers show the number of partitions of the r-gon into $r-2$ parts, that is, into triangles; and, for the r-gon, writing

$$k=r-2,$$

the number is

$$=\frac{[2r-4]^{r-3}}{[r-2]^{r-3}}.$$

If, as above, taking the weight of the triangle to be 1, we write

$$r-2=w,$$

then the number is

$$=\frac{[2w]^{w-1}}{[w]^{w-1}},$$

viz. this is the expression for the number of partitions of the polygon of weight w, or $(w+2)$-gon, into triangles.

31. The question considered by Taylor and Rowe, in the paper referred to in No. 1, is that of the partition of the r-gon into p-gons, for p, a given number >3; this implies a restriction on the form of r, viz. we must have $r-2$ divisible by $p-2$. In fact, generalizing the definition of w, if we attribute to a p-gon the weight 1, and accordingly to a polygon divisible into w p-gons the weight w, then, r being the number of summits, we must have

$$r=(p-2)\,w+2.$$

In particular, if $p=4$, so that the r-gon is to be divided into quadrangles, then r is necessarily even, and for the values

$$w=1,\ 2,\ 3,\ \ldots,$$

we have

$$r=4,\ 6,\ 8,\ \ldots.$$

r	W	2 PARTS	3 PARTS	4 PARTS	5 PARTS	6 PARTS	7 PARTS	8 PARTS
3	1							
4	2	11 = 2 9						
5	3	12 = 5 5	111 5 × 2 = 10 2) 10 = 5					
6	4	13 = 6 22 = 3 9	112 6 × 5 = 30 8 × 4 = 12 2) 42 = 21	1111 21 × 3 = 42 3) 42 = 14				
7	5	14 = 7 23 = 7 14	118 7 × 6 = 42 7 × 2 = 14 2) 56 = 28 122 7 × 3 = 21 7 × 5 = 35 2) 56 = 28 28	1112 28 × 5 = 140 28 × 4 = 112 3) 252 = 84	11111 84 × 2 = 168 4) 168 = 42			
8	6	15 = 8 24 = 8 32 = 4 20	114 8 × 7 = 56 8 × 2 = 16 2) 72 = 36 123 8 × 7 = 56 8 × 6 = 48 4 × 10 = 40 2) 144 = 72 222 8 × 3 = 24 2) 24 = 12 123	1113 36 × 5 = 216 72 × 2 = 144 3) 360 = 120 1122 36 × 3 = 108 72 × 5 = 360 12 × 6 = 72 3) 540 = 180 300	11113 120 × 5 = 600 180 × 4 = 720 4) 1320 = 330	111111 330 × 2 = 660 5) 660 = 132		
9	7	16 = 9 25 = 9 34 = 9 27	115 9 × 8 = 72 9 × 2 = 18 2) 90 = 45 124 9 × 8 = 72 9 × 7 = 63 9 × 5 = 45 2) 180 = 90 133 9 × 4 = 36 9 × 6 = 54 2) 90 = 45 223 9 × 7 = 63 9 × 3 = 27 2) 90 = 45 225	1114 45 × 7 = 315 90 × 2 = 180 3) 495 = 165 1123 45 × 7 = 315 90 × 6 = 540 45 × 10 = 450 45 × 4 = 180 3) 1485 = 495 1222 90 × 3 = 270 45 × 5 = 225 3) 495 = 165 825	11113 165 × 6 = 990 495 × 2 = 990 4) 1980 = 495 11122 165 × 3 = 495 495 × 5 = 2475 165 × 6 = 990 4) 3960 = 990 1485	111112 495 × 5 = 2475 990 × 4 = 3960 5) 6435 = 1287	1111111 1287 × 2 = 2574 6) 2574 = 429	
10	8	17 = 10 26 = 10 35 = 10 44 = 5 35	116 10 × 9 = 90 10 × 2 = 20 2) 110 = 55 125 10 × 9 = 90 10 × 8 = 80 10 × 5 = 50 2) 225 = 110 134 10 × 9 = 90 10 × 7 = 70 5 × 12 = 60 2) 220 = 110 224 10 × 8 = 80 5 × 6 = 30 2) 110 = 55 233 10 × 4 = 40 10 × 7 = 70 2) 110 = 55 385	1115 55 × 8 = 440 110 × 2 = 220 3) 660 = 220 1124 55 × 8 = 440 110 × 7 = 770 110 × 5 = 550 55 × 4 = 220 3) 1980 = 660 1133 55 × 4 = 220 110 × 6 = 660 55 × 3 = 165 3) 995 = 330 1223 110 × 7 = 770 110 × 3 = 330 55 × 6 = 330 55 × 10 = 550 3) 1980 = 660 2222 55 × 3 = 165 3) 165 = 55 1925	11114 220 × 7 = 1540 660 × 3 = 1320 4) 2860 = 715 11123 220 × 7 = 1540 660 × 6 = 3960 330 × 10 = 3300 660 × 4 = 2640 4) 11440 = 2860 11222 660 × 3 = 1980 660 × 5 = 3300 55 × 8 = 440 4) 5720 = 1430 5005	111113 715 × 6 = 4290 2860 × 2 = 5720 5) 10010 = 2002 111122 715 × 3 = 2145 2860 × 5 = 14300 1430 × 6 = 8580 5) 25025 = 5005 7007	1111112 2002 × 5 = 10010 5005 × 4 = 20020 6) 30030 = 5005	11111111 5005 × 2 = 10010 7) 10010 = 1430

32. To fix the ideas, I assume $p=4$, and thus consider the problem of the division of the $(2w+2)$-gon into quadrangles. Writing

$$w-1=a+b+c,$$

we take at pleasure any one side of the $(2w+2)$-gon, making this the first side of a quadrangle, the second, third, and fourth sides being diagonals of the polygon such that outside the second side we have a polygon of weight a, outside the third side a polygon of weight b, and outside the fourth side a polygon of weight c. Any one or more of the numbers a, b, c may be $=0$; they cannot be each of them $=0$ except in the case $w=1$. The meaning is that the corresponding side of the quadrangle, instead of being a diagonal, is a side of the $(2w+2)$-gon, viz. there is no polygon outside such side. Suppose, in general, that P_w is the number of ways in which a polygon of weight w can be divided in quadrangles, and let each of the polygons of weights a, b, c respectively, be divided into quadrangles: the number of ways in which this can be done is $P_aP_bP_c$; and it is to be noticed that, if for instance $a=0$, then the number is $=P_bP_c$, viz. the formula remains true if only we assume $P_0=1$. The number of partitions thus obtained is $\Sigma P_aP_bP_c$, where the summation extends to all the partitions of $w-1$ into the parts a, b, c (zeros admissible and the order of the parts being attended to). And we thus obtain *all* the partitions of the $(2w+2)$-gon into w parts; for first, the partitions so obtained are all distinct from each other, and next every partition of the $(2w+2)$-gon into w parts is a partition in which the selected side of the $(2w+2)$-gon is a side of some one of the quadrangles. That is, we have

$$P_w=\Sigma P_aP_bP_c \text{ (a, b, c as above)};$$

and it hence appears that, considering the generating function

$$f=1+P_1x+P_2x^2+P_3x^3+\ldots,$$

we have

$$f=1+xf^3.$$

The reasoning is precisely the same if, instead of a division into quadrangles, we have a division into p-gons; the only difference is that instead of the three parts a, b, c, we have the $p-1$ parts a, b, c, ..., and the equation for f thus is

$$f=1+xf^{p-1}.$$

33. Writing for a moment

$$f=u+xf^{p-1},$$

and expanding by Lagrange's theorem, we have

$$f=u+\frac{x}{1}(u^{p-1})+\frac{x^2}{1\,.\,2}(u^{2(p-1)})'+\ldots+\frac{x^w}{1\,.\,2\ldots w}(u^{w(p-1)})'^{\ldots(w-1)}+\ldots,$$

viz. after the differentiation, writing $u=1$, we have

$$P_w=\frac{[(p-1)w]^{w-1}}{[w]^{w-1}},$$

where it will be recollected that, for the number of sides of the polygon, we have

$$r=(p-2)w+2.$$

In the case of the partition into triangles, $p=3$, and we have the before-mentioned value

$$[2w]^{w-1}\div[w]^{w-1},\quad w=r-2.$$

916.

[NOTE ON A THEOREM IN MATRICES.]

[From the *Proceedings of the London Mathematical Society*, vol. XXII. (1891), p. 458.]

PROF. CAYLEY remarks that a "simple instance [of the theorem] is that, if the real symmetric matrix

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}$$

has two latent roots each $=0$, and therefore a vacuity $=2$, then it has also a nullity $=2$ [which may be shown as follows], viz. the conditions for a vacuity $=2$ are

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0, \quad bc + ca + ab - f^2 - g^2 - h^2 = 0,$$

or, if as usual the determinant is called K, and if

$$(A, B, C, F, G, H) = (bc - f^2, ac - g^2, \ldots),$$

then, if

$$K = 0, \quad A + B + C = 0,$$

these equations give

$$BC - F^2 = Ka = 0, \quad AC - G^2 = Kb = 0, \quad AB - H^2 = Kc = 0,$$

i.e.

$$BC = F^2, \quad AC = G^2, \quad AB = H^2;$$

and therefore

$$A(A + B + C) = A^2 + H^2 + G^2,$$
$$B(A + B + C) = H^2 + B^2 + F^2,$$
$$C(A + B + C) = G^2 + F^2 + C^2;$$

or, if $A + B + C = 0$, then for real values

$$A = B = C = F = G = H = 0,$$

i.e. nullity $=2$."

917.

[NOTE ON THE THEORY OF RATIONAL TRANSFORMATION.]

[From the *Proceedings of the London Mathematical Society*, vol. XXII. (1891), pp. 475, 476.]

IN my paper, "Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points," *Proc. Lond. Math. Soc.* t. III. (1870), pp. 196—198, [450], I notice a difficulty which presents itself in the theory. The transformation is given by the equations

$$x' : y' : z' = X : Y : Z,$$

where X, Y, Z are functions $(*\between x, y, z)^n$, such that $X=0$, $Y=0$, $Z=0$ are curves in the first plane passing through α_1 points each once, α_2 points each twice (that is, having each of the α_2 points for a double point), α_3 points each 3 times, and so on. We have as the condition of a single variable point of intersection,

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 + \ldots = n^2 - 1,$$

and as the condition in order that each of the curves $X=0$, $Y=0$, $Z=0$, or say the curve $aX+bY+cZ=0$, may be unicursal,

$$\alpha_2 + 3\alpha_3 + \ldots = \tfrac{1}{2}(n-1)(n-2);$$

and we thence deduce

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots = \tfrac{1}{2}n(n+3) - 2;$$

viz. the postulation of the fixed points *quoad* a curve of the order n is less by 2 than the postulandum (or, as I prefer to call it, the capacity) $\frac{1}{2}n(n+3)$ of the curve of the order n; that is, there are precisely the three asyzygetic curves $X=0$, $Y=0$, $Z=0$. This is as it should be, assuming that the $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ points are an ordinary system of points: but what if they form a special system having a postulation less

than $\alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots$? If, for instance, the postulation is $= \alpha_1 + 3\alpha_2 + 6\alpha_3 + \ldots - 1$, then this would be $= \frac{1}{2}n(n+3) - 3$, and there would be four asyzygetic curves $X = 0$, $Y = 0$, $Z = 0$, $W = 0$. I believe this to be impossible; but the only proof which I can offer rests upon a remark in regard to the form of the tables*, pp. 148, 149, of my paper "On the Rational Transformation between Two Spaces," *Proc. Lond. Math. Soc.*, t. III. (1870), pp. 127—180, [447]. I recall that the Jacobian curve $J(X, Y, Z) = 0$ consists of α_1' lines, α_2' conics, α_3' cubics, ..., &c., each passing a certain number of times through the $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ points, and that the number of times of passage is shown by these tables; thus, *loc. cit.*, $n = 5$, $\alpha_1 = 8$, $\alpha_4 = 1$: the Jacobian consists of eight lines and a quartic, and we have the table $(\alpha_1' = 8,\ \alpha_4' = 1)$,

	$\alpha_1 = 8$	$\alpha_2 = 0$	$\alpha_3 = 0$	$\alpha_4 = 1$
$\alpha_1' = 8$	1			1
$\alpha_2' = 0$				
$\alpha_3' = 0$				
$\alpha_4' = 1$	8			1^3

showing that the quartic passes through the eight points α_1, and through the point α_4 three times (has α_4 for a triple point). Imagine a new function W. Then in like manner $J(X, Y, W) = 0$ consists of eight lines and a quartic, and this quartic passes through the eight points α_1 and the point α_4 three times; that is, the two quartics intersect in $8 + 3.3$, $= 17$ points; and thus the two quartics must be one and the same curve; this implies a syzygy between X, Y, Z, W, viz. W is a mere linear function of X, Y, Z. The general remark is that, if in the tables m^p is reckoned as mp^2, then in the table for the several lines (exclusive of those for which the outside accented letter is $= 0$, and therefore the tabular numbers of the line are each $= 0$), i.e. for the lines which correspond to a line, a conic, a cubic, a quartic, &c., respectively, the sums of the tabular numbers are $1^2 + 1$, $2^2 + 1$, $3^2 + 1$, $4^2 + 1$, &c., respectively. This is, in fact, the case for each of the eleven tables (*loc. cit.*).

[* This Collection, vol. VII., pp. 208, 209.]

918.

ON THE SUBSTITUTION GROUPS FOR TWO, THREE, FOUR, FIVE, SIX, SEVEN, AND EIGHT LETTERS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXV. (1891), pp. 71—88, 137—155.]

THE substitution groups for two, three, four, and five letters were obtained by Serret: those for six, seven and eight letters have recently been obtained by Mr Askwith. I wish to reproduce these results in a condensed form.

The following table shows for the several cases respectively, the orders of the several groups, and for any order the number of distinct groups. As regards the case of eight letters, the numbers mentioned do not exactly agree with Mr Askwith: he gives a few non-existent groups, and omits some which I have supplied (see *post*, the list of the groups for eight letters); and it is possible that there are other omissions: the several numbers in the column and the sum total of 155 are given subject to correction.

2		3		4		5		6		7		8	
1	2	1	3	1	2	1	5	1	2	2	6	1	2
1		1	6	3	4	2	6	1	3	1	7	8	4
				1	8	1	10	4	4	2	10	4	6
				1	12	1	12	3	6	9	12	25	8
				1	24	1	20	6	8	1	14	6	12
						1	60	1	9	2	20	1	15
						1	120	1	12	1	21	22	16
								1	16	7	24	3	18
								3	18	1	36	4	24
								4	24	1	40	3	30
								2	36	1	42	12	32
								2	48	1	48	5	36
								1	60	3	72	14	48
								1	72	2	120	3	60
								1	120	1	144	3	64
								1	360	1	240	3	72
								1	720	1	2520	10	96
										1	5040	3	120
												2	144
												1	168
												1	180
												3	192
												1	240
												3	288
												1	336
												3	360
												1	384
												2	576
												3	720
												1	1152
												1	1440
												1	20160
												1	40320
1		2		7		8		34		38		155	

Here the top line shows the numbers of letters; each second column shows the order of the groups, and each first column the number of groups of the several orders: the sums at the foot of the first columns show therefore the whole number of groups, viz.

$$\begin{array}{l} \text{No. of letters} = \underline{2,\ 3,\ 4,\ 5,\ \ 6,\ \ 7,\ \ \ 8} \\ \text{No. of groups} = \overline{1,\ 2,\ 7,\ 8,\ 34,\ 38,\ 155}. \end{array}$$

In the enumeration of the groups, I use some notations which must be explained. For greater simplicity I omit parentheses, and write *ab*, *abc*, *ab . cd*, *abc . def*, &c., to denote substitutions, viz. *ab* is the interchange of *a* and *b*; *abc* the cyclical change *a* into *b*, *b* into *c*, *c* into *a*; *ab . cd* the combined interchange of *a* and *b* and of *c* and *d*; *abc . def* the combined cyclical changes *a* into *b*, *b* into *c*, *c* into *a*, and *d* into *e*, *e* into *f*, *f* into *d*; and so in other cases.

Again, (*abc*) all, means the complete group of all the substitutions

(1, *abc*, *acb*, *bc*, *ca*, *ab*)

upon the three letters; and so in other cases. In the case, however, of two letters, I write simply (ab) to denote the complete group $(1, ab)$ of substitutions, and so also for any substitution such as $ab.cd$, where the complete group is $(1, ab.cd)$, I denote the group by $(ab.cd)$. Moreover, (abc) cyc., denotes the group of cyclical substitutions $(1, abc, acb)$ upon the three letters; and so in other cases.

A group will in general contain positive and negative substitutions, and, when this is so, the positive substitutions will form a group which is denoted by the symbol, pos.; the negative substitutions (which of course do not form a group) are denoted in like manner by the symbol, neg. Thus, (abc) all, pos., or for shortness, (abc) pos., will denote the group formed by the positive substitutions of (abc) all; (abc) pos. is thus the same thing as (abc) cyc., but obviously $(abcd)$ pos. and $(abcd)$ cyc. have quite different meanings. It is to be noticed that, for any odd number of letters, the substitutions of a group (abc) cyc. are all positive, and thus (abc) cyc. pos. is the original group: for any even number of letters, the group $(abcd)$ cyc. or $(abcdef)$ cyc. contains positive and negative substitutions, but the positive substitutions thereof form a cyclical group, thus

$$(abcd) \text{ cyc. pos.} = (ab.cd) \text{ cyc.}, \quad (abcdef) \text{ cyc. pos.} = (ace.bdf) \text{ cyc.},$$

and the notation, ( ) cyc. pos., is thus unnecessary, and it will not be used.

Substitutions or groups which have no letter in common are said to be independent. The product of two independent groups is of course a group, and the components may be called independent factors of the resultant group: we use for such a product the notation $A.B$, and call the group a composite group. If each of the groups A and B contain positive and negative substitutions, we thence derive a new group, $(A.B)$ pos., viz. the substitutions hereof are the products of a positive substitution of A and a positive substitution of B, and the products of a negative substitution of A and a negative substitution of B, say

$$(A.B) \text{ pos.} = (A \text{ pos. } B \text{ pos.}) + (A \text{ neg.})(B \text{ neg.}):$$

obviously the number of substitutions or order of the group $(A.B)$ pos. is one half of that of the group $(A.B)$.

A more general, but not perfectly definite, notation is that of $(A.B)$ dim., the dimidiate of the group $A.B$. Suppose, for instance, that A is a group of substitutions of the letters (a, b, c, d); and that B is the group (ef). Here if A is composed of two equal sets $1, P, Q, \ldots; R, S, T, \ldots$, where the first set $1, P, Q, \ldots$, is a group, then we have a group $1, P, Q, \ldots, ef.R, ef.S, ef.T, \ldots$, which is a group of the form in question, $(A.B)$ dim. But in some cases the group A can be in more than one way divided into two equal sets the first of which forms a group, and a further explanation of the notation is required. I do not give at present a more complete explanation, nor explain the analogous notions trisection (tris.), &c.

Substitutions or groups having a letter or letters in common are non-independent. If A, B are such groups, then the substitutions of A are not commutable with

those of B, and we have not in general such a group as $B.A$ or $A.B$. It may happen that, although the individual substitutions of A are not commutable with those of B, yet that the groups are commutable, say we have $A.B=B.A$, viz. here the substitutions of $A.B$ are in a different order identical with those of $B.A$. We have in this case a group $A.B$; this is not a composite group; and the notation will never be employed without explanation.

I consider, in particular, the two groups $(abc.def)$ cyc. and (abc) cyc. (def) cyc.; in each of them, the six letters (a, b, c, d, e, f) are divided into two sets (a, b, c), (d, e, f), and the substitutions are the product of an abc-substitution into a def-substitution: viz. in the first group the substitutions are

$$1,\quad abc.def,\quad acb.dfe,$$

and in the second group they are

$$\begin{array}{lll} 1, & abc, & abc.def, \\ & acb, & acb.def, \\ & def, & abc.dfe, \\ & dfe, & acb.dfe. \end{array}$$

We can from each of the groups, introducing substitutions which interchange (a, b, c) with (d, e, f), or say by combination with the group $(ad.be.cf)$, derive a group of double the order; and it is worth while to consider the two cases in detail. First, for the group

$$(ad.be.cf)(abc.def) \text{ cyc.}$$

We have

Arrangements.	Substitutions.
$abcdef$,	1,
$bcaefd$,	$abc.def$,
$cabfde$,	$acb.dfe$,
$defabc$,	$ad.be.cf$,
$efdbca$,	$aecdbf$,
$fdecab$,	$afbdce$,

say

$$(ad.be.cf)(abc.def) \text{ cyc.} = (abcdef)_6.$$

This group of the order 6 is, as it happens, the group $(aecdbf)$ cyc.

It may be remarked that if, instead of $(abc.def)$ cyc., we consider the precisely similar group $(abc.dfe)$ cyc., then operating upon it with the same group $(ad.be.cf)$, we obtain quite a different form of group $(abcdef)_6$. In fact, for $(ad.be.cf)(abc.dfe)$ cyc., we have

Arrangements.	Substitutions.
abcdef,	1,
bcafde,	*abc . dfe*,
cabefd,	*acb . def*,
defabc,	*ad . be . cf*,
efdcab,	*ae . bf . cd*,
fdebca;	*af . bd . ce*;

which is not the cyclical group. See *post*, six letters, ord. 6. 3.

Next, for the group $(ad\,.\,be\,.\,cf)\,(abc)$ cyc. (def) cyc.; we have here

Arrangements.	Substitutions.
abcdef,	1,
bcadef,	*abc*,
cabdef,	*acb*,
abcefd,	*def*,
bcaefd,	*abc . def*,
cabefd,	*acb . def*,
abcfde,	*dfe*,
bcafde,	*abc . dfe*,
cabfde,	*acb . dfe*,
defabc,	*ad . be . cf*,
efdabc,	*aebfcd*,
fdeabc,	*afcebd*,
defbca,	*adbecf*,
efdbca,	*aecdbf*,
fdebca,	*af . bd . ce*,
defcab,	*adcfbe*,
efdcab,	*ae . bf . cd*,
fdecab;	*afbdce*;

say

$$(ad\,.\,be\,.\,cf)\,\{(abc)\text{ cyc. }(def)\text{ cyc.}\} = (abcdef)_{18}.$$

See *post*, six letters, ord. 18.

The groups thus obtained, with substitutions which interchange the two sets of letters, are said to be "woven" groups.

By means of the foregoing notations a large number, but by no means all, of the groups belonging to the several cases of 2, 3, 4, 5, 6, 7, and 8 letters can be

represented in a very compendious form. It is, even in the case of 4 letters, proper to introduce special notations: thus for the order 4, we have $(ab)(cd)$, (or as I generally write it $(ac)(bd)$), and $(abcd)$ cyc. (which it is convenient thus to represent), and another group $(1, ab.cd, ac.bd, ad.bc)$: this is a woven group $(ac.bd)(ab)(cd)$, but in thus representing it, we fail to exhibit the symmetrical character of the group, and I prefer to represent it as $(abcd)_4$. Again for the order 8, there is a single group, which I write $(abcd)_8$: this can be, in regard to dimidiation, divided in three different ways into two sets of four, and to distinguish them I write

$$(abcd)_8 \text{ com.} = (1,\ ac,\ bd,\ ac.bd;\ abcd,\ adbc,\ ab.cd,\ ad.bc),$$

$$(abcd)_8 \text{ cyc.} = (1,\ abcd,\ ac.bd,\ adbc;\ ac,\ bd,\ ab.cd,\ ad.bc),$$

$$(abcd)_8 \text{ pos.} = (1,\ ab.cd,\ ac.bd,\ ad.bc;\ ac,\ bd,\ abcd,\ adcb);$$

viz. com. denotes that the first set of four is the composite group $(ac)(bd)$; cyc. that it is the cyclical group $(abcd)$ cyc.; and pos. that it is the positive group $(abcd)_4$. We have thus, in the case of 6 letters, the three dimidiation forms,

$$\{(abcd)_8 \text{ com. } (ef)\} \text{ dim.} = (ac)(bd) + ef(abcd,\ adbc,\ ab.cd,\ ad.bc),$$

$$\{(abcd)_8 \text{ cyc. } (ef)\} \text{ dim.} = (abcd) \text{ cyc.} + ef(ac,\ bd,\ ab.cd;\ ad.bc),$$

$$\{(abcd)_8 \text{ pos. } (ef)\} \text{ dim.} = (abcd)_4 \quad + ef(ac,\ bd,\ \quad abcd,\ adcb),$$

the last of which may be more simply written as $\{(abcd)_8(ef)\}$ pos.

In the cases of 2, 3, 4, and 5 letters, the groups are as follows:

2 letters ord.		3 letters ord.		4 letters ord.		5 letters ord.	
2	(ab)	3	(abc) cyc.	2	$(ab.cd)$	5	$(abcde)$ cyc.
		6	(abc) all	4	$(ab)(cd)$	6	(abc) cyc. (de)
				,,	$(abcd)$ cyc.	,,	$\{(abc)$ all $(de)\}$ pos.
				,,	$(abcd)_4$	10	$(abcde)_{10}$
				8	$(abcd)_8$	12	(abc) all (de)
				12	$(abcd)$ pos.	20	$(abcde)_{20}$
				24	$(abcd)$ all	60	$(abcde)$ pos.
						120	$(abcde)$ all

The nature of the group is in most cases at once intelligible from the foregoing explanations: thus (ab) denotes the group $(1, ab)$; (abc) cyc., the cyclical group $(1, abc, acb)$: (abc) all, the group of the six substitutions $(1, abc, acb, ab, ac, bc)$, and so in other cases: but, when a further explanation is required, the nature of the group is merely indicated by writing down the order as a suffix: thus $(abcd)_4$ means a group of four substitutions, $(abcd)_8$ a group of 8 substitutions, and so in other cases. In the case of four letters, the necessary explanations have already been

given: it may be remarked that the group $(abcd)_8$ is that of the substitutions which leave unaltered the three-valued function $ac+bd$.

Five letters. Explanations. Five letters.

ord. 10. $(abcde)_{10}$. This may be written $\{(abcde)_{20}\}$ pos., viz. it consists of the 10 positive substitutions out of the next-mentioned group of 20 substitutions. Referring to that group, and writing $\sigma = S^2 = ac\,.\,bd$, and $T = aecdb$, the present group is $(1, \sigma)(1, T, T^2, T^3, T^4)$, where $\sigma^2 = 1$, $T^5 = 1$, $\sigma T = T^4\sigma$, $\sigma T^4 = T\sigma$, $\sigma T^3 = T^2\sigma$, $\sigma T^2 = T^3\sigma$: or, retaining S^2 instead of σ, say the group is $(1, S^2)(1, T, T^2, T^3, T^4)$.

ord. 20. $(abcde)_{20}$. The substitutions, distinguishing the positive and the negative ones, are

+	+	+	−
1,	$ab\,.\,de$,	$abdce$,	$abcd$,
	$ac\,.\,bd$,	$acbed$,	$abec$,
	$ad\,.\,ce$,	$adebc$,	$acde$,
	$ae\,.\,bc$,	$aecdb$,	$aceb$,
	$be\,.\,cd$,		$adbe$,
			$adcb$,
			$aebd$,
			$aedc$,
			$bced$,
			$bdec$;

and here, if $S = abcd$, $T = aecdb$, then the group is $(1, S, S^2, S^3)(1, T, T^2, T^3, T^4)$, where $S^4 = 1$, $T^5 = 1$, and generally $S^\alpha T^\beta = T^\beta\,.\,{}_2{}^\alpha S^\alpha$; that is,

	1	T	T^2	T^3	T^4
1	1	T	T^2	T^3	T^4
S	S	T^2S	T^4S	TS	T^3S
S^2	S^2	T^4S^2	T^3S^2	T^2S^2	TS^2
S^3	S^3	T^3S^3	TS^3	T^4S^3	T^2S^3

,

read

$$ST = T^2S,$$
$$ST^2 = T^4S, \text{ \&c.}$$

In the cases of 6, 7, and 8 letters, where the number of groups is larger, I introduce current numbers for the groups of the same order: thus, *infra*, 6 letters order 4, we have the groups $4._1$, $4._2$, $4._3$, $4._4$; and so in other cases, where there is more than one group for the same order.

Six letters.

Six letters: the groups are

ord.		
2.		$(ab\,.\,cd\,.\,ef)$,
3.		$(abc\,.\,def)$ cyc.
4.	1	$(ab\,.\,cd)\,(ef)$,
	2	$\{(ab)\,(cd)\,(ef)\}$ pos.
	3	$\{(abcd)$ cyc. $(ef)\}$ pos.
	4	$\{(abcd)_4\,(ef)\}$ dim.
6.	1	$(abcdef)$ cyc.
	2	$(abc\,.\,def)$ all,
	3	$(ad\,.\,bf\,.\,ce)\,(abc\,.\,def)$ cyc.
8.	1	$(ab)\,(cd)\,(ef)$,
	2	$(abcd)$ cyc. (ef),
	3	$(abcd)_4\,(ef)$,
	4	$\{(abcd)_8$ com. $(ef)\}$ dim.
	5	$\{(abcd)_8$ cyc. $(ef)\}$ dim.
	6	$\{(abcd)_8$ pos. $(ef)\}$ dim. $= \{(abcd)_8\,(ef)\}$ pos.
9.		(abc) cyc. (def) cyc.
12.		$(abcdef)_{12}$,
16.		$(abcd)_8\,(ef)$,
18.	1	(abc) all (def) cyc.,
	2	$\{(abc)$ all (def) all$\}$ pos.,
	3	$(ad\,.\,be\,.\,cf)\,\{(abc)$ cyc. (def) cyc.$\}$,
24.	1	$(abcd)$ pos. (ef),
	2	$\{(abcd)$ all $(ef)\}$ pos.,
	4	$(+\,abcdef)_{24}$,
	3	$(\pm\,abcdef)_{24}$,
36.	1	(abc) all (def) all,
	2	$(abcdef)_{36}$,
48.	1	$(abcd)$ all (ef),
	2	$(abcdef)_{48}$,
60.		$(abcdef)_{60}$,
72.		$(abcdef)_{72}$,
120.		$(abcdef)_{120}$,
360.		$(abcdef)$ pos.,
720.		$(abcdef)$ all.

The $\pm$ and $+$ are used for distinction, the $\pm$ showing that the group is one with positive and negative substitutions, the $+$ that it is a group with positive substitutions only.

Six letters. Explanations. Six letters.

ord. 6. 2. $(abc.def)$ all; the substitutions are those of (abc) all each, compounded with the corresponding substitution of (def) all; viz. they are

$$\begin{array}{llll} 1, & ab.de, & abc.def, \\ & ac.df, & acb.dfe, \\ & bc.ef. \end{array}$$

ord. 6. 3. $(ad.bf.ce)(abc.def)$ cyc. This is, in fact, the group $(ad.be.cf)(abc.dfe)$ cyc. explained in the introductory paragraphs; only for convenience the letters e, f have been interchanged. Making this interchange, the substitutions of the group $(ad.bf.ce)(abc.def)$ cyc. are

$$\begin{array}{llll} 1, & abc.def, & ad.bf.ce, \\ & acb.dfe; & ae.bd.cf, \\ & & af.be.cd. \end{array}$$

It may be added that, writing $\beta = ad.bf.ce$ and $\theta = abc.def$, the group is $(1, \beta)(1, \theta, \theta^2)$, where $\beta^2 = 1$, $\theta^3 = 1$, and $\beta\theta = \theta^2\beta$, $\beta\theta^2 = \theta\beta$. If $U = adbecf$, then $U^3 = abc.def = \theta$, and the form is $(1, \beta)(1, U^2, U^4)$: see $(abcdef)_{12}$, *infra*.

ord. 8. 4, 5, 6; the nature of these groups is explained in the introductory paragraphs.

ord. 12. $(abcdef)_{12}$. The substitutions are

$$\begin{array}{llll} 1, & ab.ef, & ad.bf.ce, & abc.def, & adbecf, \\ & ac.de, & af.be.cd, & acb.dfe, & afcebd; \\ & bc.df, & ae.bd.cf, & & \\ & & ae.bf.cd, & & \end{array}$$

writing $U = adbecf$ and $\beta = ad.bf.ce$, this is

$$(1, \beta)(1, U, U^2, U^3, U^4, U^5),$$

where

$$U^6 = 1, \quad \beta^2 = 1, \quad \beta U = U^5\beta,$$

and thence

$$\beta(1, U, U^2, U^3, U^4, U^5) = (1, U^5, U^4, U^3, U^2, U)\beta.$$

ord. 18. $(ad.be.cf)\{(abc)$ cyc. (def) cyc.$\}$; this has been explained in the introductory paragraphs. Arranging them in a more convenient order, the substitutions are

$$\begin{array}{llll} 1, & abc, & abc.def, & ad.be.cf, & adcfbe, \\ & acb, & abc.dfe, & ae.bf.cd, & adbecf, \\ & def, & acb.def, & af.bd.ce; & aebfcd, \\ & dfe; & acb.dfe; & & aecdbf, \\ & & & & afbdce, \\ & & & & afcebd. \end{array}$$

Six letters. ord. 24. 3. $(\pm\, abcdef)_{24}$: the substitutions are

+	+	+	−	−
1,	$ac\,.\,bd$,	$abe\,.\,cdf$,	$abcd$,	$ab\,.\,cd\,.\,ef$,
	$ac\,.\,ef$,	$abf\,.\,cde$,	$adcb$,	$ac\,.\,be\,.\,df$,
	$bd\,.\,ef$,	$ade\,.\,bfc$,	$aecf$,	$ac\,.\,bf\,.\,de$,
		$adf\,.\,bce$,	$afce$,	$ad\,.\,bc\,.\,ef$,
		$aeb\,.\,cfd$,	$bedf$,	$ae\,.\,bd\,.\,fc$,
		$aed\,.\,bcf$,	$bfde$,	$af\,.\,bd\,.\,ce$,
		$afb\,.\,ced$,		
		$afd\,.\,bce$.		

This includes, as part of itself, the group

$$\{(abcd)_8 \text{ cyc. } (ef)\} \text{ dim.};$$

and it may be written

$$(ade\,.\,bcf) \text{ cyc. } \{(abcd)_3 \text{ cyc. } (ef)\} \text{ dim.}$$

24. 4. $(+\, abcdef)_{24}$; the substitutions are

1,	$ab\,.\,cd$,	$abe\,.\,cdf$,	$abcd\,.\,ef$,
	$ac\,.\,bd$,	$abf\,.\,cde$,	$adcb\,.\,ef$,
	$ac\,.\,ef$,	$ade\,.\,bfc$,	$aecf\,.\,bd$,
	$ad\,.\,bc$,	$adf\,.\,bec$,	$afce\,.\,bd$,
	$ae\,.\,cf$,	$aeb\,.\,cfd$,	$bedf\,.\,ac$,
	$af\,.\,ce$,	$aed\,.\,bcf$,	$bfde\,.\,ac$,
	$bd\,.\,ef$,	$afb\,.\,ced$,	
	$be\,.\,df$,	$afd\,.\,bce$,	
	$bf\,.\,de$.		

This includes, as part of itself, the group $\{(abcd)_8\,(ef)\}$ pos.; and it may be written $(ade\,.\,bfc)$ cyc. $\{(abcd)_3\,(ef)\}$ pos. It consists of the positive terms of $(abcdef)_{48}$, and might thus be written $(abcdef)_{48}$ pos.

ord. 36. $(abcdef)_{36}$; this consists of the positive terms of $(abcdef)_{72}$. The group consists of the substitutions which leave unaltered $b-c\,.\,c-a\,.\,a-b\,.\,e-f\,.\,f-d\,.\,d-e$.

ord. 48. $(abcdef)_{48}$. The substitutions are Six letters.

+	+	+	+	−	−	−	−
1,	$ab.cd$,	$abe.cdf$,	$abcd.ef$,	ac,	$abcd$,	$ab.cd.ef$,	$abecfd$,
	$ac.bd$,	$abf.cde$,	$adcb.ef$,	bd,	$adcb$,	$ac.bd.ef$,	$abecdf$,
	$ac.ef$,	$ade.cbf$,	$aecf.bd$,	ef;	$aecf$,	$ac.be.df$,	$abfcde$,
	$ad.bc$,	$adf.cbe$,	$afce.bd$,		$afce$,	$ac.bf.de$,	$adecbf$,
	$ae.cf$,	$aeb.cfd$,	$bedf.ac$,		$bedf$,	$ad.bc.ef$,	$adfcbe$,
	$af.ce$,	$aed.cfb$,	$bfde.ac$;		$bfde$;	$ae.cf.bd$,	$aedcfb$,
	$bd.ef$,	$afb.ced$,				$af.ce.bd$;	$afbced$,
	$be.df$,	$afd.ceb$;					$afdceb$;
	$bf.de$.						

ord. 60. $(abcdef)_{60}$; this consists of the positive terms of $(abcdef)_{120}$.

ord. 72. $(abcdef)_{72}$. The substitutions are

+	+	+	+	+	−	−	−	−
1,	abc,	$abc.def$,	$ab.de$,	$ad.becf$,	ab,	$ab.def$,	$ad.be.cf$,	$adbecf$,
	acb,	$abc.dfe$,	$ab.df$,	$ad.bfce$,	ac,	$ab.dfe$,	$ad.bf.ce$,	$adbfce$,
	def,	$acb.def$,	$ab.ef$,	$ae.bdcf$,	bc,	$ac.def$,	$ae.bd.cf$,	$adcebf$,
	dfe;	$acb.dfe$;	$ac.de$,	$ae.bfcd$,	de,	$ac.dfe$,	$ae.bf.cd$,	$adcfbe$,
			$ac.df$,	$af.bdce$,	df,	$bc.def$,	$af.bd.ce$,	$aebdcf$,
			$ae.cf$,	$af.becd$,	ef;	$bc.dfe$,	$af.be.cd$;	$aebfcd$,
			$bc.de$,	$bd.aecf$,		$de.abc$,		$aecdbf$,
			$bc.df$,	$bd.afce$,		$de.acb$,		$aecfbd$,
			$bc.ef$;	$be.adcf$,		$df.abc$,		$afbdce$,
				$be.afcd$,		$df.acb$,		$afbecd$,
				$bf.adce$,		$ef.abc$,		$afcdbe$,
				$bf.aecd$,		$ef.acb$;		$afcebd$;
				$cd.aebf$,				
				$cd.afbe$,				
				$ce.adbf$,				
				$ce.afbd$,				
				$cf.adbe$,				
				$cf.aebd$.				

This contains, as part of itself, the group (abc) all (def) all; and it may be written $(ad.be.cf)\{(abc)$ all (def) all$\}$. It leaves unaltered the function $abc+def$.

Six letters. ord. 120. $(abcdef)_{120}$. The substitutions are

+	+	+	+	−	−	−
1,	$ab.cf$,	$abcef$,	$abc.dfe$,	$abcd$,	$abcfde$,	$ab.cd.ef$,
	$ab.de$,	$abdce$,	$abd.cfe$,	$abdf$,	$abdefc$,	$ab.ce.df$,
	$ac.bd$,	$abefd$,	$abe.cdf$,	$abec$,	$abedcf$,	$ac.be.df$,
	$ac.ef$,	$abfdc$,	$abf.cde$,	$abfe$,	$abfced$,	$ac.bf.de$,
	$ad.bf$,	$acbed$,	$acb.def$,	$acbf$,	$acbdfe$,	$ad.bc.ef$,
	$ad.ce$,	$acdfb$,	$acd.bfe$,	$acde$,	$acdbef$,	$ad.be.cf$,
	$ae.bc$,	$acedf$,	$ace.bfd$,	$aceb$,	$acefbd$,	$ae.bd.cf$,
	$ae.df$,	$acfbe$,	$acf.bde$,	$acfd$,	$acfedb$,	$ae.bf.cd$,
	$af.be$,	$adbcf$,	$adb.cef$,	$adbe$,	$adbfec$,	$af.bc.de$,
	$af.cd$,	$adbec$,	$adc.bef$,	$adcb$,	$adcebf$,	$af.bd.ce$;
	$bc.df$,	$adcfe$,	$ade.bfc$,	$adef$,	$adecfb$,	
	$bd.ef$,	$adfeb$,	$adf.bec$,	$adfc$,	$adfbce$,	
	$be.cd$,	$aebdf$,	$aeb.cfd$,	$aebd$,	$aebcdf$,	
	$bf.ce$,	$aebfc$,	$aec.bdf$,	$aecd$,	$aecbfd$,	
	$cf.de$;	$aefcd$,	$aed.bcf$,	$aecf$,	$aedfcb$,	
		$aecdb$,	$aef.bdc$,	$aefb$,	$aefdbc$,	
		$afbde$,	$afb.ced$,	$afbc$,	$afbecd$,	
		$afcdb$,	$afc.bed$,	$afce$,	$afcdeb$,	
		$afdec$,	$afd.bce$,	$afdb$,	$afdcbe$,	
		$afecb$,	$afe.bcd$;	$afed$,	$afebdc$;	
		$bcdef$,		$bced$,		
		$bdfce$,		$bcfe$,		
		$becfd$,		$bdcf$,		
		$bfedc$;		$bdec$,		
				$bedf$,		
				$befc$,		
				$bfcd$,		
				$bfde$,		
				$cdfe$,		
				$cefd$.		

This is the remarkable group giving rise to a six-valued function of six letters, not symmetrical in regard to five of the letters. Such a function is Six letters.

$$\begin{aligned} &ab\,.\,cf\,.\,de,\\ &ac\,.\,db\,.\,ef,\\ &ad\,.\,ec\,.\,fb,\\ &ae\,.\,fd\,.\,bc,\\ &af\,.\,be\,.\,cd; \end{aligned}$$

viz. this denotes any symmetrical function of the five functions $ab\,.\,cf\,.\,de$, &c., where $ab\,.\,cf\,.\,de$ is a symmetrical function of ab, cf, de, these denoting the same symmetrical functions of a and b, of c and f, and of d and e, respectively.

Writing $S = abcd$, $T = aecdb$, $U = abfced$, the group $(1, S, S^2, S^3)(1, T, T^2, T^3, T^4)$ is convertible with the group $(1, U, U^2, U^3, U^4, U^5)$, as may be verified by means of the diagram:

	1	U	U^2	U^3	U^4	U^5
1	1	$U\,.\,1$	$U^2\,.\,1$	$U^3\,.\,1$	$U^4\,.\,1$	U^5
T	T	$U^3\,.\,S^2$	$U\,.\,S^3T^2$	$U^5\,.\,S^2T^2$	$U^2\,.\,T^4$	$U^4\,.\,ST^4$
T^2	T^2	$U^5\,.\,T^3$	$U^3\,.\,ST^2$	$U^4\,.\,S^3T^3$	$U\,.\,S^3T$	$U^2\,.\,ST$
T^3	T^3	$U^4\,.\,ST^2$	$U^5\,.\,S^3T^3$	$U^2\,.\,S^3T$	$U^3\,.\,ST$	$U\,.\,T^2$
T^4	T^4	$U^2\,.\,ST^4$	$U^4\,.\,T$	$U\,.\,S^2$	$U^5\,.\,S^3T^2$	$U^3S^2\,.\,T^2$
S	S	$U^3\,.\,ST^3$	$U^2\,.\,S^3T^4$	$U^4\,.\,S^2T^3$	$U^5\,.\,S^2T^4$	$U\,.\,S^3$
ST	ST	$U^4\,.\,T^2$	$U^3\,.\,T^3$	$U\,.\,ST^2$	$U^2\,.\,S^2T^3$	$U^5\,.\,S^3T$
ST^2	ST^2	$U\,.\,S^3T^3$	$U^4\,.\,S^3T$	$U^5\,.\,ST$	$U^3\,.\,T^2$	$U^2\,.\,T^3$
ST^3	ST^3	$U^5\,.\,S^3T^4$	$U\,.\,S^2T^3$	$U^2\,.\,S^2T^4$	$U^4\,.\,S^3$	$U^3\,.\,S$
ST^4	ST^4	$U^2\,.\,T$	$U^5\,.\,S^2$	$U^3\,.\,S^3T^2$	$U\,.\,S^2T^2$	$U^4\,.\,T^4$
S^2	S^2	$U^4\,.\,S^3T^2$	$U^2\,.\,S^2T^2$	$U^5\,.\,T^4$	$U\,.\,ST^4$	$U^3\,.\,T$
S^2T	S^2T	$U^5\,.\,S^3T$	$U^4\,.\,S^2T$	$U^3\,.\,S^2T$	$U^2\,.\,S^2T$	$U\,.\,S^2T$
S^2T^2	S^2T^2	$U^3\,.\,T^4$	$U^5\,.\,ST^4$	$U\,.\,T$	$U^4\,.\,S^2$	$U^2\,.\,S^3T^2$
S^2T^3	S^2T^3	$U\,.\,S^2T^4$	$U^3\,.\,S^3$	$U^2\,.\,S$	$U^5\,.\,ST^3$	$U^4\,.\,S^3T^4$
S^2T^4	S^2T^4	$U^2\,.\,S^3$	$U\,.\,S$	$U^4\,.\,ST^3$	$U^3\,.\,S^3T^4$	$U^5\,.\,S^3T^3$
S^3	S^3	$U^5\,.\,S$	$U^2\,.\,ST^3$	$U\,.\,S^3T^4$	$U^3\,.\,S^2T^3$	$U^4\,.\,S^2T^4$
S^3T	S^3T	$U\,.\,ST$	$U^5\,.\,T^2$	$U^4\,.\,T^3$	$U^2\,.\,ST^2$	$U^3\,.\,S^3T^3$
S^3T^2	S^3T^2	$U^4\,.\,S^2T^2$	$U\,.\,T^4$	$U^3\,.\,ST^4$	$U^5\,.\,T$	$U^2\,.\,S^2$
S^3T^3	S^3T^3	$U^3\,.\,S^3T$	$U^4\,.\,ST$	$U^2\,.\,T^2$	$U\,.\,T^3$	$U^5\,.\,ST^2$
S^3T^4	S^3T^4	$U^2\,.\,S^2T^3$	$U^3\,.\,S^2T^4$	$U^5\,.\,S^3$	$U^4\,.\,S$	$U\,.\,ST^3$

Six letters. and the group is thus

$$(1, \ldots, S^3)(1, \ldots, T^4)(1, \ldots, U^5) \quad \text{or} \quad (1, \ldots, U^5)(1, \ldots, S^3)(1, \ldots, T^4),$$

or, since $(1, \ldots, S^3)(1, \ldots, T^4)$ is a group, we may invert the order of the factors $(1, \ldots, S^3)$ and $(1, \ldots, T^4)$.

But it is noticeable that we cannot write the group as

$$(1, \ldots, S^3)(1, \ldots, U^5)(1, \ldots, T^4) \quad \text{or} \quad (1, \ldots, T^4)(1, \ldots, U^5)(1, \ldots, S^3).$$

The 120 substitutions of either of these products are not the 120 substitutions of the group in question; but some of these are missing altogether, and others of them occur twice repeated. And it is to be remarked also that

$$(1, \ldots, S^3)(1, \ldots, U^5) \quad \text{and} \quad (1, \ldots, T^4)(1, \ldots, U^5)$$

are neither of them a group. To illustrate this, I give the following fragment of a table

	STU	*SUT*	do.	*TUS*	do.
abcdef	000	000		000	
abdfce	033	122		330,	412
abecfd	124	032,	251	343	
abfedc	105	150,	331	431,	013
acbfed	323	111		453	
acdefb	044	—		440	
adefbc	113	243		—	

viz. here the second line denotes that the substitution *abdfce*, which is $S^0T^3U^3$, is in one way *SUT*, viz. it is $S^3U^2T^2$: and in two ways *TUS*, viz. it is $T^3U^3S^0$ and also $T^4U^3S^2$. But *acdefb*, which is $S^0T^4U^4$, is not in any way *SUT*; and so *adefbc*, which is $S^3T^1U^3$, is not in any way *TUS*.

Seven letters. Seven letters.

ord.		
6.	1	$(ac\,.\,bd)(efg)$ cyc.
	2	$\{(ac\,.\,bd)(efg)$ all$\}$ dim.
7.		$(abcdefg)$ cyc.
10.	1	$(abcde)$ cyc. (fg),
	2	$\{(abcde)_{10}(fg)\}$ dim.
12.	1	$(ac\,.\,bd)(efg)$ all,
	2	$(ab)(cd)(efg)$ cyc.
	3	$(abcd)$ cyc. (efg) cyc.
	4	$(abcd)_4(efg)$ cyc.
	5	$\{(ab)(cd)(efg)$ all$\}$ pos.
	6	$\{(abcd)$ cyc. (efg) all$\}$ pos.
	7	$\{(abcd)_4(efg)$ all$\}$ dim.
	8	$\{(abc)$ all $(de)\}$ pos. (fg),
	9	$\{(abcd)$ pos. (efg) cyc.$\}$ tris.

Seven letters.

14.		$(abcdefg)_{14}$,
20.	1	$(abcde)_{10}\,(fg)$,
	2	$\{(abcde)_{20}\,(fg)\}$ pos.
21.		$(abcdefg)_{21}$,
24.	1	$(ab)\,(cd)\,(efg)$ all,
	2	$(abcd)$ cyc. (efg) all,
	3	$(abcd)_4\,(efg)$ all,
	4	$(abcd)_8\,(efg)$ cyc.
	5	$\{(abcd)_8$ com. (efg) all$\}$ dim.
	6	$\{(abcd)_8$ cyc. (efg) all$\}$ dim.
	7	$\{(abcd)_8$ pos. (efg) all$\}$ dim.
36.		$(abcd)$ pos. (efg) cyc.
40.		$(abcde)_{20}\,(fg)$,
42.		$(abcdefg)_{42}$,
48.		$(abcd)_8\,(efg)$ all,
72.	1	$(abcd)$ all (efg) cyc.
	2	$(abcd)$ pos. (efg) all,
	3	$\{(abcd)$ all (efg) all$\}$ pos.
120.	1	$(abcde)$ pos. (fg),
	2	$\{(abcde)$ all $(fg)\}$ pos.
144.		$(abcd)$ all (efg) all,
240.		$(abcde)$ all (fg),
2520.		$(abcdefg)$ pos.
5040.		$(abcdefg)$ all.

Seven letters. Explanations.

ord. 12. 9. $\{(abcd)$ pos. (efg) cyc.$\}$ tris. The substitutions are

$$
\begin{array}{lll}
1, & ab\,.\,cd, & abc\,.\,efg, \\
 & ac\,.\,bd, & acd\,.\,efg, \\
 & ad\,.\,bc, & adb\,.\,efg, \\
 & & bdc\,.\,efg, \\
 & & acb\,.\,egf, \\
 & & adc\,.\,egf, \\
 & & abd\,.\,egf, \\
 & & bcd\,.\,egf;
\end{array}
$$

Seven letters. viz. the 12 substitutions of $(abcd)$ pos. are divided into fours, which are connected with the substitutions 1, efg, egf respectively.

ord. 14. $(abcdefg)_{14}$: this is $(1, P^3)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6)$, where $P^3 = ad.be.cf$, $Q = agdecfb$, $(P^3)^2 = 1$, $Q^7 = 1$, $P^3Q = Q^6P^3$, see *infra*, ord. 42. The substitutions are

$$\begin{array}{ll} 1, & abfcedg, \quad ab.cd.fg, \\ & acgfdbe, \quad ac.bf.eg, \\ & adcbgef, \quad ad.be.cf, \\ & aebdfgc, \quad ae.bc.dg, \\ & afegbcd, \quad af.cg.de, \\ & agdecfb, \quad ag.bd.ef, \\ & \qquad\qquad bg.ce.df. \end{array}$$

ord. 21. $(abcdefg)_{21}$; this is $(1, P^2, P^4)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6)$, where

$$P^3 = acb.dfe,\ Q = agdecfb,\ (P^2)^3 = 1,\ Q^7 = 1,\ P^3Q = Q^5P^2;$$

see *infra* order 42. The substitutions are the positive substitutions of $(abcdefg)_{42}$.

ord. 24. The last three groups correspond to the before-mentioned three modes of division of $(abcd)_8$ into two sets of four, viz. we have,

24. 5; $\{(abcd)_8$ com. (efg) all$\}$ dim. $= \{(ac)(bd)\}(efg)$ pos.
$+ (abcd, adbc, ab.cd, ad.bc)(efg)$ neg.

24. 6; $\{(abcd)_8$ cyc. (efg) all$\}$ dim. $= (abcd)$ cyc. (efg) pos.
$+ (ac, bd, ab.cd, ad.bc)(efg)$ neg.

24. 7; $\{(abcd)_8$ pos. (efg) all$\}$ dim. $= (abcd)_4$ (efg) pos.
$+ (ac, bd, abcd, adbc)(efg)$ neg.

ord. 42. $(abcdefg)_{42}$: this is

$$(1, P, P^2, P^3, P^4, P^5)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6),$$

where $P = aecdbf$, $Q = agdecfb$, $P^6 = 1$, $Q^7 = 1$, and $PQ = Q^6P$. The substitutions are

$$\begin{array}{lccccc} & + & + & + & - & - \\ 1, & abc.def, & abfcedg, & ab.cd.fg, & abegdf, \\ & abd.cge, & acgfdbe, & ac.bf.eg, & abgcfe, \\ & acb.dfe, & adcbgef, & ad.be.cf, & acdebg, \\ & acf.bdg, & aebdfgc, & ae.bc.dg, & acefgd, \\ & adb.ceg, & afegbcd, & af.cg.de, & adfbcg, \\ & ade.bfg, & agdecfb, & ag.bd.ef, & adgfec, \\ & aed.bgf, & & bg.ce.df, & aecdbf, \\ & aeg.cfd, & & & aefcgb, \end{array}$$

Seven letters.

$afc\,.\,bgd$,	$afbdce$,
$afg\,.\,bec$,	$afdgeb$,
$age\,.\,cdf$,	$agbedc$,
$agf\,.\,bce$,	$agcbfd$,
$bef\,.\,cdg$,	$bcfged$,
$bfe\,.\,cgd$,	$bdegfc$.

The table for the combination of the powers of P and Q is

	1	Q	Q^2	Q^3	Q^4	Q^5	Q^6	
1	$(1,$	$Q,$	$Q^2,$	$Q^3,$	$Q^4,$	$Q^5,$	$Q^6)$	$P,$
P	$(1,$	$Q^5,$	$Q^2,$	$Q,$	$Q^6,$	$Q^4,$	$Q^3)$	$P,$
P^2	$(1,$	$Q^4,$	$Q,$	$Q^5,$	$Q^3,$	$Q^6,$	$Q^2)$	$P^2,$
P^3	$(1,$	$Q^6,$	$Q^5,$	$Q^4,$	$Q^3,$	$Q^2,$	$Q)$	$P^3,$
P^4	$(1,$	$Q^3,$	$Q^4,$	$Q^6,$	$Q,$	$Q^2,$	$Q^5)$	$P^4,$
P^5	$(1,$	$Q^2,$	$Q^6,$	$Q^3,$	$Q^6,$	$Q,$	$Q^4)$	$P^5;$

read

$$PQ = Q^5P, \quad PQ^2 = Q^3P, \text{ \&c.}$$

This completes the explanations as to the groups of seven letters.

Eight letters.

I proceed to consider the substitution groups of eight letters. The abbreviation A. refers to Mr Askwith's paper (*Quart. Journal of Math.*, vol. XXIV. (1890), pp. 263—331). The list is

ord.

2. 1 $(ab\,.\,cd\,.\,ef\,.\,gh)$.

4. 1 $(ab\,.\,cd\,.\,ef)\,(gh)$,
   2 $(ab\,.\,cd)\,(ef\,.\,gh)$,
   3 $\{(ab)\,(cd)\,(ef\,.\,gh)\}$ dim.,
   4 $\{(abcd)$ cyc. $(ef\,.\,gh)\}$ dim.,
   5 $\{(abcd)_4\,(ef\,.\,gh)\}$ dim.,
   6 $(abcd\,.\,efgh)$ cyc.,
   7 $\{(abcd)_4\,(ef\,.\,gh)\}$ dim.,
   8 $(abcdefgh)_4$. Substitutions are

$$1,\quad ae\,.\,bf\,.\,cg\,.\,dh,$$
$$ac\,.\,eg\,.\,bd\,.\,fh,$$
$$ag\,.\,ce\,.\,bh\,.\,df.$$

6. 1 $(abc\,.\,def)$ cyc. (gh),
   2 $\{(abcdef)$ cyc. $(gh)\}$ pos.
   3 $\{(abc\,.\,def)$ all $(gh)\}$ dim. [not in A.],
   4 $\{(abcdef)_6\,(gh)\}$ dim.

Eight letters.

8. 1 $(ab\,.\,cd)\,(ef)\,(gh)$,

2 $\{(ab)\,(cd)\,(ef)\}$ pos. (gh),

3 $\{(abcd)$ cyc. $(ef)\}$ pos. (gh),

4 $\{(abcd)_4\,(ef)\}$ dim. (gh),

5 $(abcd)$ cyc. $(ef\,.\,gh)$,

6 $(abcd)_4\,(ef\,.\,gh)$,

7 $(1,\ abcd,\ ac\,.\,bd,\ adcb) + ef\,.\,gh\,(ac,\ bd,\ ab\,.\,cd,\ ad\,.\,bc)$,

8 $(1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc) + ef\,.\,gh\,(ac,\ bd,\ abcd,\ adcb)$,

9 $(1,\ ac,\ bd,\ ac\,.\,bd) + ef\,.\,gh\,(abcd,\ adbc,\ ab\,.\,cd,\ ad\,.\,bc)$ [Not in A.],

10 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(eg,\ fh) = A'\,(abcd\,.\,efgh)_8$,

11 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (abcd,\ adcb)\,(efgh,\ ehfg) = B'\,(abcd\,.\,efgh)_8$,

12 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(ef\,.\,gh,\ eh\,.\,fg) = C'\,(abcd\,.\,efgh)_8$,

13 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(efgh,\ ehgf) = D'\,(abcd\,.\,efgh)_8$,

14 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(ef\,.\,gh,\ eh\,.\,fg) = E'\,(abcd\,.\,efgh)_8$,

15 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(efgh,\ ehgf) = F'\,(abcd\,.\,efgh)_8$,

16 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(ac\,.\,bd)\,(eg\,.\,fh)$. Substitutions are

$$\begin{array}{llll} 1, & ac\,.\,bd, & ac\,.\,bd\,.\,eg\,.\,fh, & agce\,.\,bhdf, \\ & eg\,.\,fh, & ae\,.\,bf\,.\,cg\,.\,dh, & aecg\,.\,bfdh. \\ & & ag\,.\,bh\,.\,ce\,.\,df, & \end{array}$$

17 $(1,\ ac\,.\,bd,\ ac\,.\,ef,\ bd\,.\,ef) + gh\,(abcd,\ adcb,\ ab\,.\,cd\,.\,ef,\ ad\,.\,bc\,.\,ef)$,

18 $(1,\ ac\,.\,bd,\ ac\,.\,ef,\ bd\,.\,ef) + gh\,(ab\,.\,cd,\ ad\,.\,bc,\ abcd\,.\,ef,\ adcb\,.\,ef)$,

19 $(1,\ ac\,.\,bd,\ ab\,.\,cd\,.\,ef,\ ad\,.\,bc\,.\,ef) + gh\,(abcd,\ adcb,\ ac\,.\,ef,\ bd\,.\,ef)$,

20 $(abcdefgh)_8$. Substitutions are

$$\begin{array}{llll} 1, & ac\,.\,be\,.\,dg, & ac\,.\,bd\,.\,eg\,.\,fh, & afch\,.\,bgde, \\ & bd\,.\,af\,.\,ch, & & ahcf\,.\,bedg. \\ & eg\,.\,ah\,.\,cf, & & \\ & fh\,.\,bg\,.\,de, & & \end{array}$$

21 $(abcdefgh)$ cyc.,

22 $A\,(abcdefgh)_8$,

23 $B\,(abcdefgh)_8$,

24 $C\,(abcdefgh)_8$,

25 $D\,(abcdefgh)_8$.

Eight letters.

12. 1 $(abcdef)$ cyc. (gh),

2 $(abc\,.\,def)$ all (gh),

3 $(abcdef)_6\,(gh)$,

4 $(1,\ abcdef,\ ace\,.\,bdf,\ ad\,.\,be\,.\,cf,\ aec\,.\,bfd,\ afedcb)$
$\qquad + gh\,(af\,.\,be\,.\,cd,\ ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd)$,

5 $(1,\ bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd,\ ace\,.\,bdf,\ aec\,.\,bfd)$
$\qquad + gh\,(ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ ad\,.\,bf\,.\,ce,\ af\,.\,be\,.\,cd,\ abcdef,\ afedcb)$,

6 $(1,\ ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ af\,.\,be\,.\,cd,\ ace\,.\,bdf,\ aec\,.\,bfd)$
$\qquad + gh\,(bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd,\ ad\,.\,be\,.\,cf,\ abcdef,\ afedcb)$.

15. 1 $(abcde)$ cyc. (fgh) cyc.

16. 1 $(ab)\,(cd)\,(ef)\,(gh)$,

2 $(abcd)$ cyc. $(ef)\,(gh)$,

3 $(abcd)_4\,(ef)\,(gh)$,

4 $\{(abcd)_8$ com. $(ef)\}$ dim. (gh),

5 $\{(abcd)_8$ cyc. $(ef)\}$ dim. (gh),

6 $\{(abcd)_8$ pos. $(ef)\}$ dim. (gh),

7 $(abcd)$ cyc. $(efgh)$ cyc.,

8 $(abcd)_4\,(efgh)$ cyc.,

9 $(abcd)_4\,(efgh)_4$,

10 $(acbd)_8\,(ef\,.\,gh)$,

11 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (ac,\ bd)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehgf)$,

12 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehgf)$,

13 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (abcd,\ adcb)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehfg)$,

14 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh,\ efgh,\ ehgf) + (abcd,\ adcb)\,(eg,\ fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

15 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh,\ efgh,\ ehgf) + (ab\,.\,cd,\ ad\,.\,bc)\,(eg\,.\,fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

16 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (abcd,\ adcb)\,(eg,\ fh,\ efgh,\ ehfg)$,

17 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (ab\,.\,cd,\ ad\,.\,bc)\,(eg,\ fh,\ efgh,\ ehgf)$,

18 $(1,\ ac\,.\,bd)\,(1,\ efgh,\ eg\,.\,fh,\ ehgf) + (ac,\ bd)\,(eg,\ fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

19 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (ac,\ bd)\,(eg,\ fh,\ efgh,\ ehgf)$,

20 $(ae\,.\,bf\,.\,cg\,.\,dh)\,B'\,(abcd\,.\,efgh)_8$,

21 $(ae\,.\,bf\,.\,cg\,.\,dh)\,C'\,(abcd\,.\,efgh)_8$,

22 $(aebf\,.\,cgdh)$ cyc. $(ac\,.\,bd)\,(eg\,.\,fh)$,

23 $(aebfcgdh)$ cyc. $(1,\ ac\,.\,bd)$.

Eight letters.

18. 1 (abc) cyc. (def) cyc. (gh),

2 $\{(abc)$ all $(de)\}$ pos. (fgh),

3 $\{(abc)$ pos. (def) pos.$\} + gh\,\{(abc)$ neg. (def) neg.$\} = A\,(abcdefgh)_{18}$,

4 $B\,(abcdefgh)_{18}$.

[A. gives, *loc. cit.*, p. 310, a form which is not a group.]

24. 1 $(abcdef)_{12}\,(gh)$,

2 $(abcd)$ pos. $(ef\,.\,gh)$,

3 $\{(abcd)$ all $(ef\,.\,gh)\}$ dim.

4 $(abcdefgh)_{24}$.

30. 1 $(abcde)$ cyc. (fgh) all,

2 $(abcde)_{10}\,(fgh)$ cyc.

3 $(abcdefgh)_{30}$.

32. 1 $(abcd)_8\,(ef)\,(gh)$,

2 $(abcd)_8\,(efgh)$ cyc.

3 $(abcd)_8\,(efgh)_4$,

4 $L\,\{(abcd)_8\,(efgh)_8\}$ dim.

5 $M\,\{(abcd)_8\,(efgh)_8\}$ dim.

6 $N\,\{(abcd)_8\,(efgh)_8\}$ dim.

7 $P\,\{(abcd)_8\,(efgh)_8\}$ dim.

8 $Q\,\{(abcd)_8\,(efgh)_8\}$ dim.

9 $R\,\{(abcd)_8\,(efgh)_8\}$ dim.

10 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)$ cyc. $(efgh)$ cyc.

11 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)_4\,(efgh)_4$.

[A. has, *loc. cit.*, p. 291 and p. 295, two forms which are identical, and, p. 275, there is a single form which is, by mistake, counted as two.]

36. 1 $\{(abc)$ all $(de)\}$ pos. (fgh),

2 $\{(abc)$ all (def) all$\}$ pos. (gh),

3 $\{(abc)$ all $(def)\,(gh)\}$ pos.

4 $(abcdef)_{18}\,(gh)$,

5 $(ad\,.\,be\,.\,cf)\,A\,(abcdefgh)_{18}$ is $= \{(abcdef)_{36}\,(gh)\}$ dim.

[The group, A., *loc. cit.*, p. 312, is derived from the non-existent group of 18, p. 310, and is thus non-existent.]

Eight letters.

48. 1 $(abcd)$ all $(ef\,.\,gh)$,

2 $(abcd)$ pos. $(ef)(gh)$,

3 $(abcd)$ pos. $(efgh)$ cyc.

4 $(abcd)$ pos. $(efgh)_4$,

5 $\{(abcd)$ all $(ef)\}$ pos. (gh),

6 $(\pm\, abcdef)_{24}\,(gh)$,

7 $(+\, abcdef)_{24}\,(gh)$,

8 $\{(abcd)$ all $(efgh)$ cyc.$\}$ dim.

9 $\{(abcd)$ all $(ef)(gh)\}$ dim.

10 $\{(abcd)$ all $(efgh)_4\}$ dim.

11 $\{(abcdef)_{48}\,(gh)\}$ dim.

12 $\{(abcdef)_{48}\,(gh)\}$ pos.

[A. *loc. cit.*, p. 272 is not a group.]

13 $(abcdefgh)_{48}$,

14 $(abcd)_{12}\,(efgh)_{12}$ tris.

60. 1 $(abcde)_{10}\,(fgh)$ all,

2 $(abcde)_{20}\,(fgh)$ cyc.

3 $\{(abcde)_{20}\,(fgh)$ all$\}$ dim.

64. 1 $(abcd)_8\,(efgh)_8$,

2 $(ae\,.\,bf\,.\,cg\,.\,dh)\,M\,\{(abcd)_8\,(efgh)_8\}$ dim.

3 $(ae\,.\,bf\,.\,cg\,.\,dh)\,N\,\{(abcd)_8\,(efgh)_8\}$ dim.

72. 1 (abc) all (def) all (gh),

2 $(abcdef)_{36}\,(gh)$,

3 $\{(abcdef)_{72}\,(gh)\}$ dim.

4 $(abcdefgh)_{72}$.

96. 1 $(abcd)$ all $(ef)(gh)$,

2 $(abcd)$ all $(efgh)$ cyc.

3 $(abcd)$ all $(efgh)_4$,

4 $(abcdef)_{48}\,(gh)$,

5 $(abcd)$ pos. $(efgh)_8$,

6 $\{(abcd)$ all $(efgh)_8$ com.$\}$ dim.

7 $\{(abcd)$ all $(efgh)_8$ cyc.$\}$ dim.

Eight letters.

96.	8	$\{(abcd)$ all $(efgh)_8$ pos.$\}$ dim.
	9	$(ae\,.\,bf\,.\,cg\,.\,dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris.
	10	$(ae\,.\,bg\,.\,cf\,.\,dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris.
120.	1	$(abcde)_{20}\,(fgh)$ all,
	2	$(abcdef)_{60}\,(gh)$,
	3	$\{(abcdef)_{120}\,(gh)\}$ pos.
144.	1	$(abcd)$ pos. $(efgh)$ pos.
	2	$(ad\,.\,be\,.\,cf)\,\{(abc)$ all (def) all $(gh)\}$.
168.		$(abcdefgh)_{168}$.
180.		$(abcde)$ pos. (fgh) cyc.
192.	1	$(abcd)$ all $(efgh)_8$,
	2	$(1,\ agch,\ bgdh,\ egfh)\,\{(abcdef)_{48}\,(gh)\}$ dim.
	3	$(1,\ bd\,.\,agch,\ efbgdh,\ ac\,.\,egfh)\,\{(abcdef)_{48}\,(gh)\}$ pos.
240.		$(abcdef)_{120}\,(gh)$.
288.	1	$(abcd)$ all $(efgh)$ pos.
	2	$\{(abcd)$ all $(efgh)$ all$\}$ pos.
	3	$(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)$ pos. $(efgh)$ pos.
336.		$(abcdefgh)_{336}$.
360.	1	$(abcde)$ all (fgh) cyc.
	2	$(abcde)$ pos. (fgh) all,
	3	$\{(abcde)$ all (fgh) all$\}$ pos.
384.		$(1,\ ag\,.\,bh,\ cf\,.\,dh,\ eg\,.\,fh)\,\{(abcdef)_{48}\,(gh)\}$.
576.	1	$(abcd)$ all $(efgh)$ all,
	2	$(ae\,.\,bf\,.\,cg\,.\,dh)\,\{(abcd)$ all $(efgh)$ all$\}$ pos.
720.	1	$(abcde)$ all (fgh) all,
	2	$(abcdef)$ pos. (gh),
	3	$\{(abcdef)$ all $(gh)\}$ pos.
1152.		$(ae\,.\,bf\,.\,cg\,.\,dh)\,\{(abcd)$ all $(efgh)$ all$\}$.
1440.		$(abcdef)$ all (gh).
20160.		$(abcdefgh)$ pos.
40320.		$(abcdefgh)$ all.

Eight letters, explanations.

Eight letters. Explanations. Ord. 8.

ord. 8.

A $(abcdefgh)_8$. The substitutions are

$$\begin{array}{ll} 1, & ab\,.\,cd\,.\,ef\,.\,gh, \\ & ac\,.\,bd\,.\,eg\,.\,fh, \\ & ad\,.\,bc\,.\,eh\,.\,fg, \\ & ae\,.\,bf\,.\,cg\,.\,dh, \\ & af\,.\,be\,.\,ch\,.\,dg, \\ & ag\,.\,bh\,.\,ce\,.\,df, \\ & ah\,.\,bg\,.\,cf\,.\,de. \end{array}$$

B $(abcdefgh)_8$. The substitutions are

$$\begin{array}{lll} 1, & ab\,.\,ch\,.\,dg\,.\,ef, & aceg\,.\,bdfh, \\ & ad\,.\,bc\,.\,eh\,.\,fg, & agec\,.\,bhdf. \\ & ae\,.\,bf\,.\,cg\,.\,dh, & \\ & af\,.\,be\,.\,cd\,.\,gh, & \\ & ah\,.\,bg\,.\,cf\,.\,de, & \end{array}$$

C $(abcdefgh)_8$. The substitutions are

$$\begin{array}{lll} 1, & ad\,.\,bg\,.\,cf\,.\,eh, & aceg\,.\,bdfh, \\ & ae\,.\,bf\,.\,cg\,.\,dh, & agec\,.\,bhfd, \\ & ah\,.\,bc\,.\,de\,.\,fg, & abef\,.\,cdgh, \\ & & afeb\,.\,chgd. \end{array}$$

D $(abcdefgh)_8$. The substitutions are

$$\begin{array}{lll} 1, & ae\,.\,bf\,.\,cg\,.\,dh, & aceg\,.\,bdfh, \\ & & agec\,.\,bhfd, \\ & & abef\,.\,chgd, \\ & & afeb\,.\,cdgh, \\ & & adeh\,.\,bgfc, \\ & & ahed\,.\,bcfg. \end{array}$$

ord. 12. The last three groups, the substitutions of which are given in full, are each of them of the form $\{(abcdef)_{12}(gh)\}$ dim. They may be written: Ord. 12.

$\{(abcdef)_{12}$ cyc. $(gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are those of $(abcdef)$ cyc.

$\{(abcdef)_{12}$ pos. $(gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are the positive substitutions of $(abcdef)_{12}$.

$\{(abcdef)_{12} \pm (gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are three positive and three negative substitutions.

Eight letters. Ord. 16.

ord. 16. I have not thought it necessary to devise any notation for the set of groups 11 to 19, the substitutions of which are given in full.

For the remaining groups 20 to 23:

$(ae\,.\,bf\,.\,cg\,.\,dh)\ B'\ (abcd\,.\,efgh)_8$. The substitutions are

$$\begin{array}{llll} 1, & ac\,.\,bd, & ac\,.\,bd\,.\,ef\,.\,gh, & abcd\,.\,efgh, \\ & eg\,.\,fh, & ae\,.\,bf\,.\,cg\,.\,dh, & abcd\,.\,ehgf, \\ & & af\,.\,bg\,.\,ch\,.\,de, & adcb\,.\,efgh, \\ & & ag\,.\,bh\,.\,ce\,.\,df, & adcb\,.\,ehgf, \\ & & ah\,.\,be\,.\,cf\,.\,dg, & agce\,.\,bhdf, \\ & & & aecg\,.\,bfdh, \\ & & & afch\,.\,bgde, \\ & & & ahcf\,.\,bedg. \end{array}$$

$(ae\,.\,bf\,.\,cg\,.\,dh)\ C'\ (abcd\,.\,efgh)_8$. The substitutions are

$$\begin{array}{llll} 1, & ac\,.\,bd, & ac\,.\,bd\,.\,eg\,.\,fh, & aceg\,.\,bfdh, \\ & eg\,.\,fh, & ab\,.\,cd\,.\,ef\,.\,gh, & agce\,.\,bhdf, \\ & & ab\,.\,cd\,.\,eh\,.\,fg, & afch\,.\,bedg, \\ & & ad\,.\,bc\,.\,ef\,.\,gh, & ahcf\,.\,bgde. \\ & & ad\,.\,bc\,.\,eh\,.\,fg, & \\ & & ae\,.\,bf\,.\,cg\,.\,dh, & \\ & & af\,.\,bg\,.\,ch\,.\,de, & \\ & & ag\,.\,bh\,.\,ce\,.\,df, & \\ & & ah\,.\,bg\,.\,cf\,.\,de, & \end{array}$$

$(aebf\,.\,cdgh)$ cyc. $(ac\,.\,bd)(ef\,.\,gh)$. The substitutions are

$$\begin{array}{llll} 1, & ac\,.\,bd, & ac\,.\,bd\,.\,eg\,.\,fh, & aebf\,.\,cgdh, \\ & eg\,.\,fh, & ab\,.\,cd\,.\,ef\,.\,gh, & aedh\,.\,bfcg, \\ & & ab\,.\,cd\,.\,eh\,.\,fg, & afdg\,.\,bech, \\ & & ad\,.\,bc\,.\,ef\,.\,gh, & agbh\,.\,cedf, \\ & & ad\,.\,bc\,.\,eh\,.\,fg, & afbe\,.\,chdg, \\ & & & ahde\,.\,bgcf, \\ & & & agdf\,.\,bhce, \\ & & & ahbg\,.\,cfde. \end{array}$$

Eight letters. Ord. 16.

$(aebfcgdh)$ cyc. $(1,\ ac\,.\,bd)$. The substitutions are

1,	$ac\,.\,bd$,	$ac\,.\,bd\,.\,eg\,.\,fh$,	$abcd\,.\,efgh$,	$aebfcgdh$,
	$eg\,.\,fh$,		$abcd\,.\,ehgf$,	$afdechbg$,
			$adcb\,.\,efgh$,	$agbhcedf$,
			$adcb\,.\,ehgf$,	$ahdgcfbe$,
				$aedhcgbf$,
				$afbgchde$,
				$agdfcebh$,
				$ahbecfdg$.

ord. 18. $B\,(abcdefgh)_{18}$. This group, communicated to me by Mr Askwith, might be written Ord. 18.

$$[((ad\,.\,be\,.\,cf)\,\{(abc)\text{ cyc. }(def)\text{ cyc.}\})\,gh]\text{ pos.},$$

or for shortness

$$[(abcdef)_{18}\,gh]\text{ pos.}$$

ord. 24. $(abcdefgh)_{24}$. The substitutions are Ord. 24.

1,	$ac\,.\,dg$,	$ace\,.\,bdf$,	$ac\,.\,bh\,.\,df\,.\,eg$,	$aceg\,.\,bhdf$,
	$ae\,.\,bd$,	$aec\,.\,bfd$,	$ae\,.\,bd\,.\,cg\,.\,fh$,	$agec\,.\,bfdh$,
	$ag\,.\,bh$,	$acg\,.\,dfh$,	$ag\,.\,bf\,.\,ce\,.\,dh$,	$acge\,.\,bdfh$,
	$bf\,.\,ce$,	$agc\,.\,dhf$,		$aegc\,.\,bhfd$,
	$cg\,.\,fh$,	$aeg\,.\,bhd$,		$aecg\,.\,bfhd$,
	$eg\,.\,bh$,	$age\,.\,bdh$,		$agce\,.\,bdhf$.
		$bfh\,.\,cge$,		
		$bhf\,.\,ceg$,		

ord. 30. $(abcdefgh)_{30}$. The substitutions are Ord. 30.

1,	$abcde$,	fgh,	$abcde\,.\,fgh$,	$ab\,.\,ce\,.\,fg$,
	$acebd$,	fhg,	$abcde\,.\,fhg$,	$ac\,.\,de\,.\,fg$,
	$adbec$,		$acebd\,.\,fgh$,	$ad\,.\,bc\,.\,fg$,
	$aedcb$,		$acebd\,.\,fhg$,	$ae\,.\,bd\,.\,fg$,
			$adbec\,.\,fgh$,	$be\,.\,cd\,.\,fg$,
			$adbec\,.\,fhg$,	$ab\,.\,ce\,.\,fh$,
			$aedcb\,.\,fgh$,	$ac\,.\,de\,.\,fh$,
			$aedcb\,.\,fhg$,	$ad\,.\,bc\,.\,fh$,
				$ae\,.\,bd\,.\,fh$,
				$be\,.\,cd\,.\,fh$,
				$ab\,.\,ce\,.\,gh$,
				$ac\,.\,de\,.\,gh$,
				$ad\,.\,bc\,.\,gh$,
				$ae\,.\,bd\,.\,gh$,
				$be\,.\,cd\,.\,gh$.

Eight letters. Ord. 32.

ord. 32. Six groups $L, M, N, P, Q, R\,\{(abcd)_8\,(efgh)_8\}$ dim., the two groups $(\quad)_8$ may be dimidiated

$(abcd)_8$ into	$(efgh)_8$ into
$A,\ A',$	$E,\ E',$
$B,\ B',$	$F,\ F',$
$C,\ C',$	$G,\ G';$

where

$$A = (1,\ ac\ ,\ bd\ ,\ ac\,.\,bd),\quad E = (1,\ eg\ ,\ fh\ ,\ eg\,.\,fh),$$
$$B = (1,\ abcd\ ,\ ac\,.\,bd,\ adcb\ ),\quad F = (1,\ efgh\ ,\ eg\,.\,fh,\ ehgf\ ),$$
$$C = (1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc),\quad G = (1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg\ ),$$

and A', B', C', E', F', G' are the tails of the groups $(abcd)_8$ and $(efgh)_8$ respectively.

We have then

$$L\,\{(abcd)_8\,(efgh)_8\}\ \text{dim.} = AE + A'E',$$

viz. the substitutions are

$$(1,\ ac,\ bd,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh)$$
$$+\,(ab\,.\,cd,\ ad\,.\,bc,\ abcd,\ adbc)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehfg),$$
$$M \quad ,, \quad = BF + B'F',$$
$$N \quad ,, \quad = CG + C'G',$$
$$P \quad ,, \quad = BG + B'G' \ \ (\text{or } CF + C'F'),$$
$$Q \quad ,, \quad = CE + C'E' \ \ (\text{or } AG + A'G'),$$
$$R \quad ,, \quad = AF + A'F' \ \ (\text{or } BE + B'E').$$

For the remaining groups, we have

$(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)$ cyc. $(efgh)$ cyc. The substitutions are

+	+	−	−	+	+	−
1,	$ac\,.\,bd,$	$abcd,$	$ac\,.\,bd\,.\,efgh,$	$ac\,.\,bd\,.\,ef\,.\,gh,$	$abcd\,.\,efgh,$	$aebfcgdh,$
	$ef\,.\,gh;$	$adcb,$	$ac\,.\,bd\,.\,ehgf,$	$ae\,.\,bf\,.\,cg\,.\,dh,$	$abcd\,.\,ehgf,$	$aedhcgbf,$
		$efgh,$	$eg\,.\,fh\,.\,abcd,$	$af\,.\,bg\,.\,ch\,.\,de,$	$adcb\,.\,efgh,$	$afbgchde,$
		$ehfg;$	$eg\,.\,fh\,.\,adcb;$	$ag\,.\,bh\,.\,ce\,.\,df,$	$adcb\,.\,ehgf,$	$afdechbg,$
				$ah\,.\,be\,.\,cf\,.\,dg;$	$aecg\,.\,bfdh,$	$agbhcedf,$
					$agce\,.\,bhdf,$	$agdfcebh,$
					$afch\,.\,bgde,$	$ahbecfdg,$
					$ahcf\,.\,bedg;$	$ahdgcfbe.$

$(ae.bf.cg.dh)(abcd)_4(efgh)_4$. The substitutions are Eight letters. Ord. 32.

	+	+	+	+
1,	$ab.cd$,	$ab.cd.ef.gh$,	$aebf.cdgh$,
	$ac.bd$,	$ab.cd.eg.fh$,	$aecg.bfdh$,
	$ad.bc$,	$ab.cd.eh.fg$,	$aedh.bfcg$,
	$ef.gh$,	$ac.bd.ef.gh$,	$afbe.chdg$,
	$eg.fh$,	$ac.bd.eg.fh$,	$afch.bedg$,
	$eh.fg$;	$ac.bd.eh.fg$,	$afdg.bech$,
		$ad.bc.ef.gh$,	$agbh.cedf$,
		$ad.bc.eg.fh$,	$agce.bhdf$,
		$ad.bc.eh.fg$,	$agdf.bhce$,
		$ae.bf.cg.dh$,	$ahbg.cfde$,
		$af.be.ch.dg$,	$ahcf.bgde$,
		$ag.bh.ce.df$,	$ahde.bgcf$.
		$ah.bg.cf.de$;	

ord. 48. $\{(abcdef)_{48}(gh)\}$ dim. The substitutions are Ord. 48.

1,	$abcd$,	$ac.bd$,	$abe.cdf$,	$ab.cd.ef$,	$+gh$	ac,	$ab.cd$,	$ac.bedf$,	$ac.bd.ef$;	$abfcde$,
	$adcb$,	$ac.ef$,	$aeb.cfd$,	$ac.be.df$,		bd,	$ad.bc$,	$ac.bfde$,		$abecdf$,
	$aecf$,	$bd.ef$;	$abf.cde$,	$ac.bf.de$,		ef;	$ae.cf$,	$bd.aecf$,		$adecbf$,
	$afce$,		$afb.ced$,	$ad.bc.ef$,			$af.ce$,	$bd.afce$,		$adfcbe$,
	$bedf$,		$ade.bfc$,	$ae.bd.cf$,			$be.df$,	$ef.abcd$,		$aebcfd$,
	$bfde$;		$aed.bcf$,	$af.bd.ce$;			$bf.de$;	$ef.adcb$;		$aedcfb$,
			$adf.bec$,							$afbced$,
			$afd.bce$;							$afdceb$;

$\{(abcdef)_{48}(gh)\}$ pos. The substitutions are

1,	$ab.cd$,	$ac.bedf$,	$abe.cdf$,	$+gh$	ac,	$abcd$,	$ab.cd.ef$,	$abecdf$,
	$ac.bd$,	$ac.bfde$,	$abf.cde$,		bd,	$adcb$,	$ac.bd.ef$,	$abfcde$,
	$ac.ef$,	$bd.aecf$,	$ade.bfc$,		ef;	$aecf$,	$ac.be.df$,	$adecbf$,
	$ad.bc$,	$bd.afce$,	$adf.cbe$,			$afce$,	$ac.bf.de$,	$adfcbe$,
	$ae.cf$,	$ef.abcd$,	$aeb.cfd$,			$bedf$,	$ad.bc.ef$,	$aebcfd$,
	$af.ce$,	$ef.adcb$;	$aed.bcf$,			$bfde$;	$ae.bd.cf$,	$aedcfb$,
	$bd.ef$,		$afb.ced$,				$af.bd.ce$,	$afbced$,
	$be.df$,		$afd.ceb$;					$afdceb$;
	$bf.de$;							

Eight letters. Ord. 48. $(abcdefgh)_{48}$. The substitutions are

1,	$ac\,.\,df$,	$ace\,.\,bdf$,	$ab\,.\,cf\,.\,de\,.\,gh$,	$ad\,.\,bchefg$,	$abch\,.\,defg$,
	$ae\,.\,bd$,	$acg\,.\,bdh$,	$ab\,.\,ch\,.\,de\,.\,fg$,	$ad\,.\,bgfehc$,	$abgf\,.\,cdeh$,
	$ag\,.\,bh$,	$aec\,.\,bfd$,	$ac\,.\,df\,.\,eg\,.\,bh$,	$be\,.\,afgdch$,	$aceg\,.\,bhdf$,
	$bf\,.\,ce$,	$aeg\,.\,bhd$,	$ad\,.\,bc\,.\,ef\,.\,gh$,	$be\,.\,ahcdgf$,	$acge\,.\,bdfh$,
	$bh\,.\,eg$,	$agc\,.\,bhd$,	$ad\,.\,be\,.\,cf\,.\,gh$,	$cf\,.\,abgdeh$,	$aecg\,.\,bfhd$,
	$cg\,.\,fh$,	$age\,.\,bdh$,	$ad\,.\,be\,.\,ch\,.\,fg$,	$cf\,.\,ahedgb$,	$aegc\,.\,bhfd$,
		$bhf\,.\,ceg$,	$ad\,.\,bg\,.\,ch\,.\,ef$,	$gh\,.\,abcdef$,	$afeh\,.\,bgdc$,
		$bfh\,.\,cge$,	$ae\,.\,bd\,.\,cg\,.\,fh$,	$gh\,.\,afedcb$,	$afgb\,.\,ched$,
			$af\,.\,be\,.\,cd\,.\,gh$,		$agce\,.\,bdhf$,
			$af\,.\,bg\,.\,cd\,.\,eh$,		$agec\,.\,bfdh$,
			$ag\,.\,bf\,.\,ce\,.\,dh$,		$ahcb\,.\,dgfe$,
			$ah\,.\,bc\,.\,dg\,.\,ef$,		$ahef\,.\,bcdg$.
			$ah\,.\,be\,.\,cf\,.\,dg$,		

$(abcd)_{12}\,(efgh)_{12}$ tris. The substitutions are

1	1,	$+\,abc$	efg,	$+\,acb$	egf,
$ab\,.\,cd$	$ef\,.\,gh$,	adb	ehf,	abd	efh,
$ac\,.\,bd$	$eg\,.\,fh$,	bdc	fhg,	bcd	fgh,
$ad\,.\,bc$	$eh\,.\,fg$,	acd	egh,	adc	ehg.

Ord. 64. ord. 64. $(ae\,.\,bf\,.\,cg\,.\,dh)\,M\,\{(abcd)_8\,(efgh)_8\}$ dim. The substitutions are

+	−	+	+	−	−	+	−
1,	$abcd$,	$ac\,.\,bd$,	$abcd\,.\,efgh$,	$ac\,.\,ef\,.\,gh$,	$ac\,.\,bd\,.\,efgh$,	$ac\,.\,bd\,.\,ef\,.\,gh$,	$aebfcgdh$,
	$adbc$,	$eg\,.\,fh$,	$adbc\,.\,efgh$,	$ac\,.\,eh\,.\,fg$,	$ac\,.\,bd\,.\,ehfg$,	$ab\,.\,cd\,.\,ef\,.\,gh$,	$aebhcgdf$,
	$efgh$,	$ac\,.\,eg$,	$abcd\,.\,ehgf$,	$bd\,.\,ef\,.\,gh$,	$eg\,.\,fh\,.\,abcd$,	$ab\,.\,cd\,.\,eh\,.\,fg$,	$aedfcgbh$,
	$ehgf$;	$ac\,.\,fh$,	$adbc\,.\,ehgf$,	$bd\,.\,eh\,.\,fg$,	$eg\,.\,fh\,.\,adbc$;	$ad\,.\,bc\,.\,ef\,.\,gh$,	$aedhcgbf$,
		$bd\,.\,eg$,	$aecg\,.\,bhdf$,	$ab\,.\,cd\,.\,eg$,		$ad\,.\,bc\,.\,eh\,.\,fg$,	$afbechdg$,
		$bd\,.\,fh$;	$agce\,.\,bhdf$,	$ab\,.\,cd\,.\,fh$,		$ae\,.\,bf\,.\,cg\,.\,dh$,	$afbgchde$,
			$aecg\,.\,bfdh$,	$ad\,.\,bc\,.\,eg$,		$ae\,.\,bh\,.\,cg\,.\,df$,	$afdechbg$,
			$agce\,.\,bfdh$,	$ad\,.\,bc\,.\,fh$;		$af\,.\,be\,.\,ch\,.\,dg$,	$afdgchbe$,
			$afch\,.\,bgde$,			$af\,.\,bg\,.\,ch\,.\,de$,	$agbfcedh$,
			$ahcf\,.\,bgde$,			$ag\,.\,bf\,.\,ce\,.\,dh$,	$agbhcedf$,
			$afch\,.\,bedg$,			$ag\,.\,bh\,.\,ce\,.\,df$,	$agdfcebh$,
			$ahcf\,.\,bedg$;			$ah\,.\,be\,.\,cf\,.\,dg$,	$agdhcebf$,
						$ah\,.\,bg\,.\,cf\,.\,de$;	$ahbecfdg$,
							$ahbgcfde$,
							$ahdecfbg$,
							$ahdgcfbe$;

$(ae\,.\,bf\,.\,cg\,.\,dh)$ $N\{(abcd)_8\,(efgh)_8\}$ dim. Eight letters. Ord. 64.

The substitutions (all positive) are

1,	$ab\,.\,cd$,	$ab\,.\,cd\,.\,ef\,.\,gh$,	$ac\,.\,efgh$,	$abcd\,.\,efgh$,
	$ac\,.\,bd$,	$ab\,.\,cd\,.\,eg\,.\,fh$,	$ac\,.\,ehgf$,	$abcd\,.\,ehgf$,
	$ad\,.\,bc$,	$ab\,.\,cd\,.\,eh\,.\,fg$,	$bd\,.\,efgh$,	$adcb\,.\,efgh$,
	$ef\,.\,gh$,	$ac\,.\,bd\,.\,ef\,.\,gh$,	$bd\,.\,ehgf$,	$adcb\,.\,ehgf$,
	$eg\,.\,fh$,	$ac\,.\,bd\,.\,eg\,.\,fh$,	$eg\,.\,abcd$,	$aebf\,.\,cgdh$,
	$eh\,.\,fg$,	$ac\,.\,bd\,.\,eh\,.\,fg$,	$eg\,.\,adbc$,	$aebh\,.\,cgdf$,
	$ac\,.\,eg$,	$ad\,.\,bc\,.\,ef\,.\,gh$,	$fh\,.\,abcd$,	$aecg\,.\,bfdh$,
	$ac\,.\,fh$,	$ad\,.\,bc\,.\,eg\,.\,fh$,	$fh\,.\,adcb$,	$aecg\,.\,bhdf$,
	$bd\,.\,eg$,	$ad\,.\,bc\,.\,eh\,.\,fg$,		$aedf\,.\,bhcg$,
	$bd\,.\,fh$,	$ae\,.\,bf\,.\,cg\,.\,dh$,		$aedh\,.\,bfcg$,
		$ae\,.\,bh\,.\,cg\,.\,df$,		$afbe\,.\,chdg$,
		$af\,.\,be\,.\,ch\,.\,dg$,		$afbg\,.\,chde$,
		$af\,.\,bg\,.\,ch\,.\,de$,		$afch\,.\,bedg$,
		$ag\,.\,bf\,.\,ce\,.\,dh$,		$afch\,.\,bgde$,
		$ag\,.\,bh\,.\,ce\,.\,df$,		$afde\,.\,bgch$,
		$ah\,.\,be\,.\,cf\,.\,dg$,		$afdg\,.\,bech$,
		$ah\,.\,bg\,.\,cf\,.\,de$,		$agbf\,.\,cedh$,
				$agbh\,.\,cedf$,
				$agce\,.\,bfdh$,
				$agce\,.\,bhdf$,
				$agdf\,.\,bhce$,
				$agdh\,.\,bfce$,
				$ahbe\,.\,cfdg$,
				$ahbg\,.\,cfde$,
				$ahcf\,.\,begd$,
				$ahcf\,.\,bgde$,
				$ahde\,.\,bgcf$,
				$ahdg\,.\,becf$.

ord. 72. $(abcdefgh)_{72}$. This group, communicated to me by Mr Askwith, may be written Ord. 72.

$$\{(abcdef)_{72}gh\} \text{ pos.}$$

ord. 96. $\{(abcd)$ all $(efgh)_8$ com.$\}$ dim. This means Ord. 96.

$$(abcd) \text{ pos. } (1,\ eg,\ fh,\ eg\,.\,fh) + (abcd) \text{ neg. residue.}$$

Eight letters. Ord. 96. $\{(abcd)$ all $(efgh)_8$ cyc.$\}$ dim. This means

$$(abcd) \text{ pos. } (1,\ efgh,\ eg\,.\,fh,\ ehgf) + (abcd) \text{ neg. residue.}$$

$\{(abcd)$ all $(efgh)_8$ pos.$\}$ dim. This means

$$(abcd) \text{ pos. } (1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (abcd) \text{ neg. residue;}$$

viz. each form $(efgh)_8$ is divided into two sets, which are combined with $(abcd)$ pos. and $(abcd)$ neg. respectively.

ord. 96. $(ae\,.\,bf\,.\,cg\,.\,dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris.

The substitutions (all positive) are those of

$\{(abcd)_{12}\,(efgh)_{12}\}$ tris., viz.

1	1 ,	+ abc	efg ,	+ acb	egf ,
ab . cd	ef . gh,	adb	ehf ,	abd	efh,
ac . bd	eg . fh,	bdc	fhg,	bcd	fgh,
ad . bc	eh . fg,	acd	egh ,	adc	ehg,

together with the following 48 substitutions

ae . bf . cg . dh,	aebf . cgdh,	ae . bgdfch,
af . be . ch . dg,	aecg . bfdh,	ae . bhcfdg,
ag . bh . ce . df,	aedh . bfcg,	af . bgcedh,
ah . bg . cf . de;	afbe . chdg,	af . bhdecg,
	afch . bedg,	ag . bedhcf,
	afdg . bech,	ag . bfchde,
	agbh . cedf,	ah . becgdf,
	agce . bhdf,	ah . bfdgce,
	agdf . bhce,	be . agcfdh,
	ahbg . cfde,	be . ahdfcg,
	ahcf . bgde,	bf . agdech,
	ahde . bgcf;	bf . ahcedg,
		bg . aechdf,
		bg . afdhce,
		bh . aedgcf,
		bh . afcgde,
		ce . afbgdh,
		ce . ahdgbf,
		cf . aebhdg,

Eight letters.
Ord. 96.

$cf\,.\,agdhbe,$
$cg\,.\,afdebh,$
$cg\,.\,ahbedf,$
$ch\,.\,aedfbg,$
$ch\,.\,agbfde,$
$de\,.\,afbhcg,$
$de\,.\,agchbf,$
$df\,.\,aebgch,$
$df\,.\,ahcgeb,$
$dg\,.\,aecfbh,$
$dg\,.\,ahbfce,$
$dh\,.\,afcebg,$
$dh\,.\,agbecf.$

$(ae\,.\,bg\,.\,cf\,.\,dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris. The substitutions (all positive) are those of $\{(abcd)_{12}\,(efgh)_{12}\}$ tris., viz.

1	1 ,	+ abc	efg ,	+ acb	egf ,
ab . cd	ef . gh,	adb	ehf ,	abd	efh,
ac . bd	eg . fh,	bdc	fhg,	bcd	fgh,
ad . bc	eh . fg,	acd	egh,	adc	ehg,

together with the following 48 substitutions

$ae\,.\,bf\,.\,ch\,.\,dg,\quad aebf\,.\,chdg,$
$ae\,.\,bg\,.\,cf\,.\,dh,\quad aebg\,.\,cfdh,$
$ae\,.\,bh\,.\,cg\,.\,df,\quad aebh\,.\,cgdf,$
$af\,.\,be\,.\,cg\,.\,dh,\quad aecf\,.\,bgdh,$
$af\,.\,bg\,.\,ch\,.\,de,\quad aecg\,.\,bhdf,$
$af\,.\,bh\,.\,ce\,.\,dg,\quad aech\,.\,bfdg,$
$ag\,.\,be\,.\,ch\,.\,df,\quad aedf\,.\,bhcg,$
$ag\,.\,bf\,.\,ce\,.\,dh,\quad aedg\,.\,bfch,$
$ag\,.\,bh\,.\,cf\,.\,de,\quad aedh\,.\,bgcf,$
$ah\,.\,be\,.\,cf\,.\,dg,\quad afbe\,.\,cgdh,$
$ah\,.\,bf\,.\,cg\,.\,de,\quad afbg\,.\,chde,$
$ah\,.\,bg\,.\,ce\,.\,df;\quad afbh\,.\,cedg,$
$afce\,.\,bhdg,$
$afcg\,.\,bedh,$
$afch\,.\,bgde,$

Eight letters.
Ord. 96.

$afde\,.\,bgch$,
$afdg\,.\,bhce$,
$afdh\,.\,becg$,
$agbe\,.\,chdf$,
$agbf\,.\,cedh$,
$agbh\,.\,cfde$,
$agce\,.\,bfdh$,
$agcf\,.\,bhde$,
$agch\,.\,bedf$,
$agde\,.\,bhcf$,
$agdf\,.\,bech$,
$agdh\,.\,bfce$,
$ahbe\,.\,cfdg$,
$ahbf\,.\,cgde$,
$ahbg\,.\,cedf$,
$ahce\,.\,bgdf$,
$ahcf\,.\,bedg$,
$ahcg\,.\,bfde$,
$ahde\,.\,bfcg$,
$ahdf\,.\,bgce$,
$ahdg\,.\,becf$.

Ord. 168. ord. 168. $(abcdefgh)_{168}$.

This is $(1, S^2, S^4)(1, T, \ldots, T^6)(1, U, \ldots, U^7)$, where $S = bdcgef$, $T = abcdefg$, $U = ahbfgecd$; it is a derivative of $(abcdefg)_{21} = (1, S^2, S^4)(1, T, \ldots, T^6)$, see post $(abcdefgh)_{336}$.

Ord. 192. ord. 192. [Two forms not examined.]

Ord. 288. ord. 288. $(ae\,.\,bf\,.\,cg\,.\,dh)(abcd)$ pos. $(efgh)$ pos.

The substitutions are those of $(abcd)$ pos. $(efgh)$ pos., viz. the 144 substitutions

1	1,
$ab\,.\,cd$	$ef\,.\,gh$,
$ac\,.\,bd$	$eg\,.\,fh$,
$ad\,.\,bc$	$eh\,.\,fg$,
abc	efg,
acb	egf,
abd	efh,
adb	ehf,
acd	egh,
adc	ehg,
bcd	fgh,
bdc	fhg,

together with the following 144 substitutions, viz. Eight letters. Ord. 288.

ae . bf . cg . dh,	*aebf . cgdh,*	*afbe . chdg,*	*agbe . cfdh,*	*ahbe . cgdf,*
ae . bg . ch . df,	*aebg . chdf,*	*afbg . cedh,*	*agbf . chde,*	*ahbf . cedg,*
ae . bh . cf . dg,	*aebh . cfdg,*	*afbh . cgde,*	*agbh . cedf,*	*ahbg . cfde,*
af . be . ch . dg,	*aecf . bhdg,*	*afce . bgdh,*	*agce . bhdf,*	*ahce . bfdg,*
af . bg . ce . dh,	*aecg . bfdh,*	*afcg . bhde,*	*agcf . bedh,*	*ahcf . bgde,*
af . bh . cg . de,	*aech . bgdf,*	*afch . bedg,*	*agch . bfde,*	*ahcg . bedf,*
ag . bh . ce . df,	*aedf . bgch,*	*afde . bhcg,*	*agde . bfch,*	*ahde . bgcf,*
ag . be . cf . dh,	*aedg . bhcf,*	*afdg . bech,*	*agdf . bhce,*	*ahdf . becg,*
ag . bf . ch . de,	*aedh . bfcg,*	*afdh . bgce,*	*agdh . becf,*	*ahdg . bfce;*
ah . bg . cf . de,				
ah . be . cg . df,				
ah . bf . ce . dg;				

ae . bfcgdh,	*af . bechdg,*	*ag . becfdh,*	*ah . becgdf*	*ce . afbgdh,*	*cf . aebhdg,*	*cg . aebfdh,*	*ch . aebgdf,*
„ *bfdhcg,*	„ *bedgch,*	„ *bedhcf,*	„ *bedfcg*	„ *afdhbg,*	„ *aedgbh,*	„ *aedhbf,*	„ *aedfbg,*
„ *bgchdf,*	„ *bgcedh,*	„ *bfchde,*	„ *bfcedg*	„ *agbhdf,*	„ *agbedh,*	„ *afbhde,*	„ *afbedg,*
„ *bgdfch,*	„ *bgdhce,*	„ *bfdech,*	„ *bfdgce*	„ *agdfbh,*	„ *agdhbe,*	„ *afdebh,*	„ *afdgbe,*
„ *bhcfdg,*	„ *bhcdge,*	„ *bhcedf,*	„ *bgcfde*	„ *ahbfdg,*	„ *ahbgde,*	„ *ahbedf,*	„ *agbfde,*
„ *bhdgcf;*	„ *bhdecg;*	„ *bhdfce;*	„ *bgdecf;*	„ *ahdgbf;*	„ *ahdebg;*	„ *ahdfbe;*	„ *agdebf;*

be . afchdg,	*bf . aecgdh,*	*bg . aechdf,*	*bh . aecfdg*	*de . afbhcg,*	*df . aebgch,*	*dg . aebhcf,*	*dh . aebfcg,*
„ *afdgch,*	„ *aedhcg,*	„ *aedfch,*	„ *aedgcf*	„ *afcgbh,*	„ *aechbg,*	„ *aecfbh,*	„ *aecgbf,*
„ *agcfdh,*	„ *agchde,*	„ *afcedh,*	„ *afcgde*	„ *agbfch,*	„ *agbhce,*	„ *afbech,*	„ *afbgce,*
„ *agdhcf,*	„ *agdech,*	„ *afdhce,*	„ *afdecg*	„ *agchbf,*	„ *agcebh,*	„ *afchbe,*	„ *afcebg,*
„ *ahcgdf,*	„ *ahcedg,*	„ *ahcfde,*	„ *agcedf*	„ *ahbgcf,*	„ *ahbecg,*	„ *ahbfce,*	„ *agbecf,*
„ *ahdfcg;*	„ *ahdgce;*	„ *ahdecf;*	„ *agdfce;*	„ *ahcfbg;*	„ *ahcgbe;*	„ *ahcebf;*	„ *agcfbe.*

ord. 336. $(abcdefgh)_{336}$. This is Ord. 336.

$$(1,\ S,\ S^2,\ S^3,\ S^4,\ S^5)(1,\ T,\ \ldots,\ T^6)(1,\ U,\ \ldots,\ U^7),$$

where $S = bdcgef$, $T = abcdefg$, $U = ahbfgecd$; it is a derivative of

$$(abcdefg)_{42} = (1,\ S,\ \ldots,\ S^5)(1,\ T,\ \ldots,\ T^6).$$

It will be noticed that there are some alterations in the numbers of the groups of the several orders as stated in the Table, p. 118, and that the number here obtained for the total number of the groups of eight letters (instead of 155 as in the table) is 157. Some of the groups, in particular those of the order 192, require further explanation.

919.

ON THE PROBLEM OF TACTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXV. (1891), pp. 104—127.]

1. I REMARK that the problem "to draw a circle touching each of three given circles" is not properly a problem with eight solutions, but it is a set of four problems each with two solutions: viz. if a, b, c are the radii of the given circles, ϑ the radius of the tangent circle, and r, s, t the distances of its centre from the centres of the given circles respectively, then in the four problems respectively we have

$$\begin{array}{llll} r=a+\vartheta, & r=a+\vartheta, & r=a+\vartheta, & r=a+\vartheta, \\ s=b+\vartheta, & s=-b+\vartheta, & s=b+\vartheta, & s=-b+\vartheta, \\ t=c+\vartheta, & t=c+\vartheta, & t=-c+\vartheta, & t=-c+\vartheta; \end{array}$$

and thence also

$$\begin{array}{llll} s-t=b-c, & s-t=-b-c, & s-t=b+c, & s-t=-b+c, \\ t-r=c-a, & t-r=c-a, & t-r=-c-a, & t-r=-c-a, \\ r-s=a-b, & r-s=a+b, & r-s=a-b, & r-s=a+b, \end{array}$$

where a, b, c may be regarded each of them as positive; but the sign of each distance r, s, t, and of the radius ϑ is not assumable at pleasure, but analytically it comes out as a result in the solution, or it may be found by geometrical considerations. Thus, if the given circles are external to each other, then in the first problem we have two solutions, a first tangent circle touched externally by each of the given circles, and a second tangent circle touched internally by each of the given circles; and taking r, s, t, ϑ_1, ϑ_2, each of them as positive, the signs in the two solutions respectively are

$$\begin{array}{lll} r=a+\vartheta_1, & r=-a+\vartheta_2, \text{ or say} & -r=a-\vartheta_2, \\ s=b+\vartheta_1, & s=-b+\vartheta_2, & -s=b-\vartheta_2, \\ t=c+\vartheta_1, & t=-c+\vartheta_2, & -t=c-\vartheta_2, \end{array}$$

and so in other cases. The second, third, and fourth problems are, it is clear, derived from the first problem by the change of (b, c) into $(-b, c)$, $(b, -c)$, $(-b, -c)$ respectively: so that only the first problem need be considered, viz. this is as above

$$r = a + \vartheta, \quad s = b + \vartheta, \quad t = c + \vartheta,$$

whence also

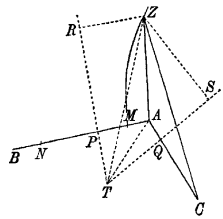
$$s - t = b - c, \quad t - r = c - a, \quad r - s = a - b,$$

where $b - c$, $c - a$, $a - b$ are given magnitudes the algebraical sum of which is $= 0$, viz. they are one of them positive, and the other two each negative, or else two of them each positive, and the remaining one negative.

2. The most simple and straightforward geometrical solution is that given in the *Principia*, Book I. Lemma XVI.; I reproduce this as given in Motte's translation (*The Mathematical Principles of Natural Philosophy*, by Sir Isaac Newton, translated into English by Andrew Motte, 8°, 2 vols., London, 1729).

"Lemma XVI. *From three given points to draw to a fourth point which is not given three right lines whose differences shall be either given or none at all.*

Case 1. Let the given points be A, B, C (see figure) and Z the fourth point which we are to find: because of the given difference of the line AZ, BZ, the locus



of the point Z will be a hyperbola whose foci are A and B and whose principal axe is the given difference. Let that axe be MN. Taking PM to MA as MN is to AB, erect PR perpendicular to AB, and let fall ZR perpendicular to PR; then from the nature of the hyperbola ZR will be to AZ as MN is to AB. And by the like argument the locus of the point Z will be another hyperbola whose foci are A, C, and whose principal axe is the difference between AZ and CZ; and QS a perpendicular on AC may be drawn to which (QS), if from any point Z of this hyperbola a perpendicular ZS is let fall, this (ZS) shall be to AZ as the difference between AZ and CZ is to AC. Wherefore the ratios of ZR and ZS to AZ are given and consequently the ratio of ZR to ZS one to the other: and therefore if the right lines RP, QS meet in T, and TZ and TA are drawn, the figure $TRZS$ will be given in specie, and the right line TZ, in which the point Z is somewhere placed, will be given in position. There will be given also the right line TA and the angle ATZ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other is given also; and thence will be given also the triangle ATZ whose vertex is the point Z. Q. E. I.

Case 2. If two of the three lines, for instance AZ and BZ, are equal, &c.

Case 3. If all the three are equal, &c.

This problematic lemma is likewise resolved in Apollonius's Book of Tactions restored by Vieta."

3. Newton, in fact, considers the hyperbolas AB and AC, each of given axis, having the foci (A, B) and (A, C) respectively, and having PR, QS for the directrices which in the two hyperbolas respectively belong to the common focus A. The required point Z thus lies on a given line through the intersection T of these two directrices; and its position on this line is determined by the condition that the distances AZ, TZ shall be in a given ratio: the locus of the points which satisfy this last condition is of course a circle; and the position of Z is thus determined as the intersection of the given line by a given circle (which I will call a Newton-circle); there are two intersections giving points Z_1, Z_2, which are the centres of the two tangent circles respectively: and the line as a *locus in quo* of these two points is of course a determinate line, but Newton's circle is only one of a singly infinite series of circles through the two points: any other solution of the problem gives therefore the same line, but not in general the same circle.

4. In what immediately follows, I use for convenience the letter F in place of the foregoing letter T.

Effecting Newton's construction, first as above, with the points $A(B, C)$; and then in like manner, secondly with the points $B(C, A)$, and thirdly with the points $C(A, B)$; and in regard to a hyperbola AB or BA, writing AB for the directrix which belongs to the focus A, and BA for the directrix which belongs to the focus B; then

For hyperbolas AB, AC, we have intersection of directrices AB, AC is a point F; for hyperbolas BC, BA, we have intersection of directrices BC, BA is a point G; for hyperbolas CA, CB, we have intersection of directrices CA, CB is a point H.

Hence these three points F, G, H lie in a line, which is the line containing the required points Z_1, Z_2; or say it is the line Z_1Z_2.

The points Z_1, Z_2 are determined as the intersections of this line by a circle which is the locus of the points whose distances A, F are in a given ratio; similarly they are determined as the intersections by a circle which is the locus of the points whose distances from B, G are in a given ratio; and they are determined as the intersections by a circle which is the locus of the points whose distances from C, H are in a given ratio. We have thus three Newton-circles: if the centres of these are F', G', H' respectively, then clearly these points lie on a line $F'G'H'$, which bisects at right angles the line (or chord) Z_1Z_2; the points A, F, F' are obviously in a line, as are also the points B, G, G', and the points C, H, H'; or (what is the same thing) considering for a moment the line $F'G'H'$ as a given line bisecting Z_1Z_2 at right angles, then the centres F', G', H' would be found as the intersections of this line $F'G'H'$ with the lines AF, BG, CH respectively.

The points Z_1, Z_2 being determined as above, then the points of contact α_1, β_1, γ_1 of the circle Z_1 with the circles A, B, C respectively are points of intersection of the lines Z_1A, Z_1B, Z_1C with these circles respectively; and similarly the points of contact α_2, β_2, γ_2 of the circle Z_2 with the same circles respectively are points of intersection of the lines Z_2A, Z_2B, Z_2C with these circles respectively.

5. I compare with Newton's the construction in which the centres Z_1, Z_2 are determined by means of the points of contact with the given circles: I may refer to Prop. 10, pp. 118—120 of Casey's *Sequel to Euclid* (12°, Dublin, 1881). We have here the line Z_1Z_2 determined as the line through the radical centre Ω of the three circles A, B, C, perpendicular to an axis of symmetry (say the axis containing the three centres of direct symmetry) of the same circles: this point Ω is the centre of the orthotomic circle. And if the common chords of the orthotomic circle and the circles A, B, C respectively meet the axis of symmetry in the points a, b, c; then we have α_1, α_2 as the points of contact of the tangents from a to the circle A; β_1, β_2 as the points of contact of the tangents from b to the circle B; and γ_1, γ_2 as the points of contact of the tangents from c to the circle C: the suffixes 1 and 2 can and must be so applied that the three lines $A\alpha_1$, $B\beta_1$, $C\gamma_1$ meet in a point Z_1 of the line Z_1Z_2, and the three lines $A\alpha_2$, $B\beta_2$, $C\gamma_2$ in a point Z_2 of the same line. We thus obtain the required points Z_1 and Z_2.

6. Taking the equations of the three circles to be

$$(x-\alpha)^2+(y-\alpha_1)^2=a^2,$$
$$(x-\beta)^2+(y-\beta_1)^2=b^2,$$
$$(x-\gamma)^2+(y-\gamma_1)^2=c^2,$$

I wish to obtain the equations of the line Z_1Z_2 and of the three Newton-circles; but I will first find, by a separate analytical investigation, an expression for the length of the chord Z_1Z_2.

Writing f, g, h for the distances BC, CA, AB of the points A, B, C from each other; r_1, s_1, t_1 for the distances of Z_1 from these points respectively, and r_2, s_2, t_2 for the distances of Z_2 from these points respectively; we have a triangle whose sides are f, g, h, and two points Z_1, Z_2 whose distances from the vertices are r_1, s_1, t_1 and r_2, s_2, t_2 respectively; and we can in terms of these data find an expression for the distance x of the points Z_1, Z_2 from each other. In fact, considering any four points 1, 2, 3, 4 and any other four points 1′, 2′, 3′, 4′, then if 11′, 12′, &c., denote the squared distances of the points 1 and 1′ from each other, of the points 1 and 2′ from each other, &c., we have between the several distances the relation

$$\begin{vmatrix} 0, & 1\ , & 1\ , & 1\ , & 1 \\ 1, & 11', & 12', & 13', & 14' \\ 1, & 21', & 22', & 23', & 24' \\ 1, & 31', & 32', & 33', & 34' \\ 1, & 41', & 42', & 43', & 44' \end{vmatrix} = 0,$$

and thence, taking the points 1, 2, 3 and also the points 1′, 2′, 3′ to be the points A, B, C respectively, and the points 4 and 4′ to be the points Z_1 and Z_2 respectively, we have the required relation

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & h^2, & g^2, & r_1^2 \\ 1, & h^2, & 0, & f^2, & s_1^2 \\ 1, & g^2, & f^2, & 0, & t_1^2 \\ 1, & r^2, & s^2, & t^2, & x^2 \end{vmatrix} = 0,$$

viz. putting for shortness

$$\Delta = f^4 + g^4 + h^4 - 2g^2h^2 - 2h^2f^2 - 2f^2g^2,$$

this equation is

$$\Delta x^2 + \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & h^2, & g^2, & r_1^2 \\ 1, & h^2, & 0, & f^2, & s_1^2 \\ 1, & g^2, & f^2, & 0, & t_1^2 \\ 1, & r^2, & s^2, & t^2, & 0 \end{vmatrix} = 0,$$

where the determinant has the value

$$\begin{aligned} &(r^2 + r_1^2) f^2 (-f^2 + g^2 + h^2) + f^2 \{(t_1^2 - r_1^2)(r_2^2 - s_2^2) + (t_2^2 - r_2^2)(r_1^2 - s_1^2)\} \\ &+ (s^2 + s_1^2)\, g^2 (\ \ f^2 - g^2 + h^2) + g^2 \{(r_1^2 - s_1^2)(s_2^2 - t_2^2) + (r_2^2 - s_2^2)(s_1^2 - t_1^2)\} \\ &+ (t_1^2 + t_2^2)\, h^2 (\ \ f^2 + g^2 - h^2) + h^2 \{(s_1^2 - t_1^2)(t_2^2 - r_2^2) + (s_2^2 - t_2^2)(t_1^2 - r_1^2)\} \\ &\qquad - 2f^2g^2h^2. \end{aligned}$$

7. For the points Z_1, Z_2, the distances r_1, s_1, t_1 and r_2, s_2, t_2 have the values $a + \vartheta_1$, $b + \vartheta_1$, $c + \vartheta_1$ and $a + \vartheta_2$, $b + \vartheta_2$, $c + \vartheta_2$ respectively, where ϑ_1, ϑ_2 are the radii of the tangent circles; substituting these values, we find

$$\Delta x^2 + 2\mathfrak{A} + 2\mathfrak{B}(\vartheta_1 + \vartheta_2) + \mathfrak{D}(\vartheta_1^2 + \vartheta_2^2) + 2\mathfrak{E}\vartheta_1\vartheta_2 = 0,$$

where

$$\begin{aligned} \mathfrak{A} = \ & a^2f^2(-f^2 + g^2 + h^2) + f^2(c^2 - a^2)(a^2 - b^2) - f^2g^2h^2 \\ & + b^2g^2 (\ \ f^2 - g^2 + h^2) + g^2(a^2 - b^2)(b^2 - c^2) \\ & + c^2h^2 (\ \ f^2 + g^2 - h^2) + h^2(b^2 - c^2)(c^2 - a^2), \\ \mathfrak{B} = \ & af^2 (-f^2 + g^2 + h^2) + f^2(c - a)(a - b)(2a + b + c) \\ & + bg^2 (\ \ f^2 - g^2 + h^2) + g^2(a - b)(b - c)(a + 2b + c) \\ & + ch^2 (\ \ f^2 + g^2 - h^2) + h^2(b - c)(c - a)(a + b + 2c), \\ \mathfrak{D} = \ & f^2(-f^2 + g^2 + h^2) \\ & + g^2 (\ \ f^2 - g^2 + h^2) \\ & + h^2 (\ \ f^2 + g^2 - h^2), = -\Delta, \\ \mathfrak{E} = \ & 4f^2(c - a)(a - b) \\ & + 4g^2(a - b)(b - c) \\ & + 4h^2(b - c)(c - a). \end{aligned}$$

But ϑ_1, ϑ_2 are the roots of an equation in ϑ, which is at once obtained from the foregoing equation by putting therein $x=0$, $\vartheta_1=\vartheta_2$, $=\vartheta$: viz. if, for shortness,

$$\mathfrak{C}=\mathfrak{D}+\mathfrak{E},\ =-\Delta+4\{f^2(c-a)(a-b)+g^2(a-b)(b-c)+h^2(b-c)(c-a)\},$$

then the equation in ϑ is

$$\mathfrak{A}+2\mathfrak{B}\vartheta+\mathfrak{C}\vartheta^2=0;$$

we have therefore

$$\vartheta_1+\vartheta_2=-\frac{2\mathfrak{B}}{\mathfrak{C}},\quad \vartheta_1\vartheta_2=\frac{\mathfrak{A}}{\mathfrak{C}};$$

and consequently

$$\Delta x^2+2\mathfrak{A}-\frac{4\mathfrak{B}^2}{\mathfrak{C}}+\mathfrak{D}\,\frac{4\mathfrak{B}^2-2\mathfrak{A}\mathfrak{C}}{\mathfrak{C}^2}+\frac{2\mathfrak{A}\mathfrak{E}}{\mathfrak{C}}=0,$$

that is,

$$\Delta\mathfrak{C}^2x^2+2\mathfrak{A}\mathfrak{C}^2-4\mathfrak{B}^2\mathfrak{C}+\mathfrak{D}(4\mathfrak{B}^2-2\mathfrak{A}\mathfrak{C})+2\mathfrak{A}\mathfrak{C}(\mathfrak{C}-\mathfrak{D})=0,$$

or, reducing,

$$\Delta\mathfrak{C}^2x^2+4(\mathfrak{B}^2-\mathfrak{A}\mathfrak{C})(\mathfrak{D}-\mathfrak{C})=0,$$

that is,

$$\Delta\mathfrak{C}^2x^2=4(\mathfrak{B}^2-\mathfrak{A}\mathfrak{C})\mathfrak{E},$$

where Δ, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{E}$ have the values given above; we hence find

$$\begin{aligned}\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}&=-\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\times(f^4+g^4+h^4-2g^2h^2-2h^2f^2-2f^2g^2)\\&=-\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\Delta;\end{aligned}$$

and, putting for x its value ZZ_1, the equation thus reduces itself to

$$\mathfrak{C}^2(ZZ_1)^2=-4\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}\mathfrak{E}.$$

8. The denominator factor $\mathfrak{C}^2$ and the several numerator factors of $(ZZ_1)^2$ may be accounted for. It is to be observed that $(ZZ_1)^2$ does not contain the factor Δ of $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}$. If $\Delta=0$, the centres of the circles A, B, C are in a line; the two tangent circles are circles situate symmetrically in regard to the line ABC, that is, their radii are equal, and the line through their centres is bisected at right angles by the line ABC; the radii are equal, and thus $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=0$, but the centres are not coincident, and thus ZZ_1 is not $=0$. The expression for $(ZZ_1)^2$ assumes a more simple form, for we have $\mathfrak{C}=-\Delta+\mathfrak{E}=\mathfrak{E}$, and the formula thus becomes

$$\mathfrak{E}(ZZ_1)^2=-4\{f^2-(b-c)^2\}\{g^2-(c-a)^2\}\{h^2-(a-b)^2\}.$$

If in this formula $\mathfrak{E}=0$, then we have $ZZ_1=\infty$; in fact, the circles A, B, C have here a pair of common tangents, and the tangent circles are these common tangents: each of the centres is thus a point at infinity, and the distance of the two centres is to be regarded as = infinity. In verification observe that, if the circles have a pair of common tangents, then, taking the intersection of these for the origin, if P, Q, R be the distances of the centres from this origin, we have

$$P:Q:R=a:b:c;$$

and therefore

$$f:g:h=b-c:c-a:a-b;$$

whence

$$\mathfrak{E},=(c-a)(a-b)f^2+(a-b)(b-c)g^2+(b-c)(c-a)h^2,$$

contains a factor $(b-c)+(c-a)+(a-b)$, and is thus $=0$.

Supposing Δ not $=0$, then $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}$ is $=0$, that is, the two radii are equal only if one of the factors $f^2-(b-c)^2$, $g^2-(c-a)^2$, $h^2-(a-b)^2$ is $=0$, and in this case we have also $ZZ_1=0$; the two tangent circles are here coincident. The equation $f^2-(b-c)^2=0$ signifies that the circles B, C touch each other internally, or, what is the same thing (if, as was initially assumed, the radii b, c be taken to be each of them positive), the point of contact is a direct centre of symmetry; hence, when $f^2-(b-c)^2=0$, the two tangent circles are coincident. And similarly if $g^2-(c-a)^2=0$, or if $h^2-(a-b)^2=0$.

9. If in the general formula we have $\mathfrak{C}=0$, then $ZZ_1=\infty$; the circles A, B, C have here a common tangent and one of the tangent circles becomes this common tangent; we have thus a centre at infinity, and the distance of the two centres is thus also infinite.

To show that $\mathfrak{C}=0$ is the condition of a common tangent, suppose that the three circles have a common tangent, and let P, Q, R denote the distances of the points of contact from any fixed point on this tangent: we have

$$f^2=(Q-R)^2+(b-c)^2,$$
$$g^2=(R-P)^2+(c-a)^2,$$
$$h^2=(P-Q)^2+(a-b)^2,$$

equations which I represent by f^2, g^2, $h^2=\alpha^2+l^2$, β^2+m^2, γ^2+n^2, where $\alpha+\beta+\gamma=0$, $l+m+n=0$. We have

$$-f^2+g^2+h^2=-\alpha^2+\beta^2+\gamma^2-l^2+m^2+n^2,\ =-2\beta\gamma-2mn,$$

and forming the corresponding values of $f^2-g^2+h^2$, $f^2+g^2-h^2$, and multiplying by f^2, g^2, $h^2=\alpha^2+l^2$, β^2+m^2, γ^2+n^2 respectively, we find

$$-\Delta=-2\alpha^2mn-2\beta^2nl-2\gamma^2lm-2l^2\beta\gamma-2m^2\gamma\alpha-2n^2\alpha\beta.$$

But the last three terms hereof are

$$=l^2(-\alpha^2+\beta^2+\gamma^2)+m^2(-\beta^2+\gamma^2+\alpha^2)+n^2(-\gamma^2+\alpha^2+\beta^2),$$

which is

$$=\alpha^2(m^2+n^2-l^2)+\beta^2(n^2+l^2-m^2)+\gamma^2(l^2+m^2-n^2),$$

and this is

$$=-2\alpha^2mn-2\beta^2nl-2\gamma^2lm.$$

Hence we have

$$-\Delta=-4\alpha^2mn-4\beta^2nl-4\gamma^2lm,$$

that is,

$$=-4f^2mn-4g^2nl-4h^2lm,$$

or replacing l, m, n by their values, we find

$$-\Delta=-4\{f^2(c-a)(a-b)+g^2(a-b)(b-c)+h^2(b-c)(c-a)\},$$

which is the required condition $\mathfrak{C}=0$.

10. It would at first sight appear that the distance ZZ_1 of the two centres

would always vanish if $\mathfrak{E}=0$. But if A, B, C are real circles, this condition $\mathfrak{E}=0$, implies

$$\frac{f^2}{(b-c)^2}=\frac{g^2}{(c-a)^2}=\frac{h^2}{(a-b)^2},$$

whence $\Delta=0$, and this being so we have $\mathfrak{C}=-\Delta+\mathfrak{E}$, $=0$, and the value of ZZ_1 instead of being $=0$, is or appears to be infinite. In proof, take for a moment the origin at A and the line AB for the axis of x; we have thus $(0, h)$ for the coordinates of B, and taking (x, y) for the coordinates of $\mathfrak{C}$, we have $g^2=x^2+y^2$; $f^2=(h-x)^2+y^2$. Writing as before l, m, n, to denote $b-c$, $c-a$, $a-b$ respectively, we have

$$\begin{aligned}\tfrac{1}{4}\mathfrak{E}= &\ mn\{(h-x)^2+y^2\}+nl(x^2+y^2)+lmh^2,\\ = &\ m(n+l)h^2+n(l+m)(x^2+y^2)-2mnhx,\\ = &-m^2h^2-n^2(x^2+y^2)-2mnhx,\\ = &-(mh+nx)^2-n^2y^2,\end{aligned}$$

and thus, for real values, $\mathfrak{E}$ can only vanish for $y=0$, $x=-\dfrac{mh}{n}$; these values of x, y give $f^2=\dfrac{l^2h^2}{n^2}$, $g^2=\dfrac{m^2h^2}{n^2}$, that is, $\dfrac{f^2}{l^2}=\dfrac{g^2}{m^2}=\dfrac{h^2}{n^2}$, or writing for l, m, n their values, they give

$$\frac{f^2}{(b-c)^2}=\frac{g^2}{(c-a)^2}=\frac{h^2}{(a-b)^2}.$$

11. But for imaginary circles the condition $\mathfrak{E}=0$ does not imply $\Delta=0$, and supposing $\mathfrak{E}=0$, the distance Z_1Z_2 is $=0$; the equation $\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=0$, is not satisfied, and thus the two radii are unequal; it would seem that we have *concentric* circles Z_1, Z_2 each touching the three given circles A, B, C, and this would imply that the radii a, b, c were equal to each other: this cannot be the case, for the only relation is that given by the foregoing condition $\mathfrak{E}=0$. The explanation of this paradox is that the two circles Z_1, Z_2 are not really concentric, but it is only the distance Z_1Z_2 of the centres which is $=0$, viz. the centres are points on an imaginary line $x-\alpha\pm i(y-\beta)=0$.

In verification hereof, I start from two circles Z_1, Z_2,

$$(x+1)^2+(y+i)^2=m^2,$$
$$(x-1)^2+(y-i)^2=n^2,$$

having for centres the two points $(-1, -i)$, $(1, i)$ the distance of which two points from each other is $=0$. Consider for a moment a conic having these two imaginary points for its foci; viz. writing ξ, η for the coordinates of a point of the conic, the equation is

$$\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}-\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}=m-n;$$

we thence obtain

$$(\xi+1)^2+(\eta+i)^2=(m-n)^2+2(m-n)\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}+(\xi-1)^2+(\eta-i)^2,$$

that is,

$$4(\xi+i\eta)-(m-n)^2=2(m-n)\sqrt{\{(\xi-1)^2+(\eta-i)^2\}},$$

or putting $m-n=2k$, we have

$$\xi+i\eta-k^2=k\sqrt{\{(\xi-1)^2+(\eta-i)^2\}},$$

and thence

$$k^2(\xi^2+\eta^2)-(\xi+i\eta)^2=k^4,$$

for the equation of the conic. The last preceding equation gives

$$\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}=-k+\frac{\xi+i\eta}{k},\ =-\tfrac{1}{2}(m-n)+\frac{\xi+i\eta}{k},$$

or say

$$\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}-n=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k};$$

and we have similarly

$$\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}-m=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k}.$$

This being so, it at once appears that, if (ξ, η) are coordinates of a point on the conic, then the circle

$$(x-\xi)^2+(y-\eta)^2=a^2,$$

where

$$a=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k},$$

is a circle touching each of the given circles Z_1, Z_2. In fact, the distance of the centre from the point Z_1 is $\sqrt{\{(\xi+1)^2+(\eta+i)^2\}}$, which is $=a+m$, the sum of the two radii; and similarly the distance of the centre from the point Z_2 is $\sqrt{\{(\xi-1)^2+(\eta-i)^2\}}$, which is $=a+n$, the sum of the two radii.

Hence if (ξ', η'), (ξ'', η'') belong to any other two points on the conic, and we write

$$a=-\tfrac{1}{2}(m+n)+\frac{\xi+i\eta}{k},$$

$$b=-\tfrac{1}{2}(m+n)+\frac{\xi'+i\eta'}{k},$$

$$c=-\tfrac{1}{2}(m+n)+\frac{\xi''+i\eta''}{k},$$

we have

$$(x-\xi\ )^2+(y-\eta\ )^2=a^2,$$
$$(x-\xi')^2+(y-\eta')^2=b^2,$$
$$(x-\xi'')^2+(y-\eta'')^2=c^2,$$

for the equations of three circles A, B, C each touching the two circles Z_1, Z_2. Writing as before f, g, h for the mutual distances BC, CA, AB of the centres of these circles, then

$$f^2=(\xi'-\xi'')^2+(\eta'-\eta'')^2,$$

and similarly for g^2 and h^2. But we have

$$b-c=\frac{1}{k}\{(\xi'-\xi'')+i(\eta'-\eta'')\},$$

and therefore

$$\frac{f^2}{b-c} = k\{\xi' - \xi'' - i(\eta' - \eta'')\};$$

and similarly

$$\frac{g^2}{c-a} = k\{\xi'' - \xi - i(\eta'' - \eta)\},$$

$$\frac{h^2}{a-b} = k\{\xi - \xi' - i(\eta - \eta')\},$$

and hence

$$\frac{f^2}{b-c} + \frac{g^2}{c-a} + \frac{h^2}{a-b} = 0, \text{ that is, } \mathfrak{E} = 0,$$

viz. it thus appears that the condition $\mathfrak{E} = 0$ applies to a pair of circles Z_1, Z_2 which are not concentric, but which have for their centres two imaginary points the distance of which from each other is $= 0$.

This completes the explanation of the denominator and numerator factors in the expression for the distance Z_1Z_2 between the centres of the two tangent circles.

12. I consider now the analytical solution: the equations of the given circles A, B, C are

$$(X-\alpha)^2 + (Y-\alpha_1)^2 - a^2 = 0,$$
$$(X-\beta)^2 + (Y-\beta_1)^2 - b^2 = 0,$$
$$(X-\gamma)^2 + (Y-\gamma_1)^2 - c^2 = 0,$$

viz. (α, α_1), (β, β_1), and (γ, γ_1) are the coordinates of the centres and (a, b, c) are the radii. Taking (x, y) for the coordinates of the centre of the tangent circle and ϑ for its radius, the equation of the tangent circle is

$$(X-x)^2 + (Y-y)^2 - \vartheta^2 = 0;$$

and if we write r, s, t for the distances of this centre from the points A, B, C respectively, that is,

$$r = \sqrt{\{(x-\alpha)^2 + (y-\alpha_1)^2\}},$$
$$s = \sqrt{\{(x-\beta)^2 + (y-\beta_1)^2\}},$$
$$t = \sqrt{\{(x-\gamma)^2 + (y-\gamma_1)^2\}};$$

then for the determination of the unknown quantities x, y, ϑ we have the three equations

$$r = a + \vartheta, \quad s = b + \vartheta, \quad t = c + \vartheta,$$

or eliminating ϑ, the centre is determined by means of the hyperbolas

$$s - t = b - c, \quad t - r = c - a, \quad r - s = a - b;$$

these three hyperbolas have, in fact, two common intersections which are the two centres Z_1, Z_2.

In all that follows, I write, as before, $b-c$, $c-a$, $a-b=l$, m, n; the last-mentioned equations are therefore

$$s-t=l, \quad t-r=m, \quad r-s=n,$$

and we deduce

$$r=\tfrac{1}{2}n+\frac{r^2-s^2}{2n}=-\tfrac{1}{2}m+\frac{t^2-r^2}{2m},$$

$$s=\tfrac{1}{2}l+\frac{s^2-t^2}{2l}=-\tfrac{1}{2}n+\frac{r^2-s^2}{2n},$$

$$t=\tfrac{1}{2}m+\frac{t^2-r^2}{2m}=-\tfrac{1}{2}l+\frac{s^2-t^2}{2l},$$

viz. writing

$$R=\frac{s^2-t^2}{2l}, \quad S=\frac{t^2-r^2}{2m}, \quad T=\frac{r^2-s^2}{2n};$$

and therefore

$$lR+mS+nT=0,$$

these equations are

$$r=S-\tfrac{1}{2}m, \quad s=T-\tfrac{1}{2}n, \quad t=R-\tfrac{1}{2}l,$$
$$=T+\tfrac{1}{2}n, \quad =R+\tfrac{1}{2}l, \quad =S+\tfrac{1}{2}m;$$

R, S, T are each of them a linear function of the coordinates (x, y), say we have

$$R=\lambda x+\lambda_1 y+\lambda_2,$$
$$S=\mu x+\mu_1 y+\mu_2,$$
$$T=\nu x+\nu_1 y+\nu_2,$$

where

$$\lambda,\ \lambda_1,\ \lambda_2=-\frac{\beta-\gamma}{l},\ -\frac{\beta_1-\gamma_1}{l},\ \frac{\beta^2+\beta_1^2-\gamma^2-\gamma_1^2}{2l},$$

$$\mu,\ \mu_1,\ \mu_2=-\frac{\gamma-\alpha}{m},\ -\frac{\gamma_1-\alpha_1}{m},\ \frac{\gamma^2+\gamma_1^2-\alpha^2-\alpha_1^2}{2m},$$

$$\nu,\ \nu_1,\ \nu_2=-\frac{\alpha-\beta}{n},\ -\frac{\alpha_1-\beta_1}{n},\ \frac{\alpha^2+\alpha_1^2-\beta^2-\beta_1^2}{2n}.$$

13. From the two equations $r=S-\frac{1}{2}m=T+\frac{1}{2}n$, we deduce the equations of a line and a circle.

The line is $S-T-\frac{1}{2}(m+n)=0$, viz. substituting for S and T their values, this is

$$\frac{t^2-r^2}{m}-\frac{r^2-s^2}{n}-(m+n)=0,$$

that is,

$$n(t^2-r^2)-m(r^2-s^2)-mn(m+n)=0,$$

or, since $l+m+n=0$, the equation is

$$lr^2+ms^2+nt^2+lmn=0,$$

which is symmetrical in regard to the three circles. The equation may be written

$$l(r^2-a^2)+m(s^2-b^2)+n(t^2-c^2)=0;$$

and it thus appears that the line passes through the radical centre of the three circles.

We have

$$(\nu-\mu)r=\nu(S-\tfrac{1}{2}m)-\mu(T+\tfrac{1}{2}n)=-(\mu\nu_1-\mu_1\nu)y+\nu\mu_2-\mu\nu_2-\tfrac{1}{2}(m\nu+n\mu),$$
$$(\nu_1-\mu_1)r=\nu_1(S-\tfrac{1}{2}m)-\mu_1(T+\tfrac{1}{2}n)=\quad(\mu\nu_1-\mu_1\nu)x+\nu\mu_2-\mu\nu_2-\tfrac{1}{2}(m\nu_1+n\mu_1),$$

and thence

$$\{(\nu-\mu)^2+(\nu_1-\mu_1)^2\}r^2=\{\nu(S-\tfrac{1}{2}m)-\mu(T+\tfrac{1}{2}n)\}^2+\{\nu_1(S-\tfrac{1}{2}m)-\mu_1(T+\tfrac{1}{2}n)\}^2,$$

which is the equation of a circle; in fact, on the left-hand side and right-hand side the only terms of the second order in (x, y) are $\{(\nu-\mu)^2+(\nu_1-\mu_1)^2\}(x^2+y^2)$ and $(\mu\nu_1-\mu_1\nu)^2(x^2+y^2)$ respectively. We have thus the equation of the Newton-circle F; but I reduce the form by substituting for $\mu, \mu_1, \mu_2, \nu, \nu_1, \nu_2$ their values. Writing for shortness

$$l\alpha+m\beta+n\gamma=K,$$
$$l\alpha_1+m\beta_1+n\gamma_1=K_1,$$
$$\beta\gamma_1-\beta_1\gamma+\gamma\alpha_1-\gamma_1\alpha+\alpha\beta_1-\alpha_1\beta=\Omega,$$
$$(\beta-\gamma)(\alpha^2+\alpha_1^2)+(\gamma-\alpha)(\beta^2+\beta_1^2)+(\alpha-\beta)(\gamma^2+\gamma_1^2)=\Pi,$$
$$(\beta_1-\gamma_1)(\alpha^2+\alpha_1^2)+(\gamma_1-\alpha_1)(\beta^2+\beta_1^2)+(\alpha_1-\beta_1)(\gamma^2+\gamma_1^2)=\Pi_1,$$

after some easy reductions the equation is found to be

$$\begin{aligned}4(K^2+K_1^2)(x^2+y^2-2\alpha x-2\alpha_1 y+\alpha^2+\alpha_1^2)\\ =\{2\Omega y+\Pi+(\beta-\gamma)mn+(m-n)K\}^2\\ +\{-2\Omega x+\Pi_1+(\beta_1-\gamma_1)mn+(m-n)K_1\}^2.\end{aligned}$$

14. To further abbreviate, I write

$$(\beta-\gamma)mn+(m-n)K=F,\quad(\beta_1-\gamma_1)mn+(m-n)K_1=F_1,$$
$$(\gamma-\alpha)nl+(n-l)K=G,\quad(\gamma_1-\alpha_1)nl+(n-l)K_1=G_1,$$
$$(\alpha-\beta)lm+(l-m)K=H,\quad(\alpha_1-\beta_1)lm+(l-m)K_1=H_1;$$

also

$$l(\alpha^2+\alpha_1^2)+m(\beta^2+\beta_1^2)+n(\gamma^2+\gamma_1^2)=\Theta;$$

and then writing down the three equations, we have

$$4(K^2+K_1^2)r^2=(-2\Omega x+\Pi_1+F_1)^2+(2\Omega y+\Pi+F)^2,$$
$$4(K^2+K_1^2)s^2=(-2\Omega x+\Pi_1+G_1)^2+(2\Omega y+\Pi+G)^2,$$
$$4(K^2+K_1^2)t^2=(-2\Omega x+\Pi_1+H_1)^2+(2\Omega y+\Pi+H)^2,$$

which are the equations of the three Newton-circles, each meeting the chord

$$lr^2+ms^2+nt^2+lmn=0,$$

or, say

$$-2Kx-2K_1y+\Theta+lmn=0,$$

in the points Z_1, Z_2.

15. The first of these equations is

$$\begin{aligned}
&4(K^2+K_1^2-\Omega^2)(x^2+y^2)\\
&\qquad -2\{4(K^2+K_1^2)\alpha-2\Omega(\Pi_1+F_1)\}x\\
&\qquad -2\{4(K^2+K_1^2)\alpha_1+2\Omega(\Pi+F)\}y\\
&\qquad +4(K^2+K_1^2)(\alpha^2+\alpha_1^2)-(\Pi_1+F_1)^2-(\Pi+F)^2=0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\{4(K^2+K_1^2-\Omega^2)x-4(K^2+K_1^2)\alpha+2\Omega(\Pi_1+F_1)\}^2\\
&+\{4(K^2+K_1^2-\Omega^2)y-4(K^2+K_1^2)\alpha_1-2\Omega(\Pi+F)\}^2\\
&+4(K^2+K_1^2-\Omega^2)\{4(K^2+K_1^2)(\alpha^2+\alpha_1^2)-(\Pi_1+F_1)^2-(\Pi+F)^2\}\\
&-\{4(K^2+K_1^2)\alpha-2\Omega(\Pi_1+F_1)\}^2\\
&-\{4(K^2+K_1^2)\alpha_1+2\Omega(\Pi+F)\}^2=0,
\end{aligned}$$

where the last term is

$$\begin{aligned}
=\ & 16(K^2+K_1^2)^2(\alpha^2+\alpha_1^2)\\
&-4(K^2+K_1^2)(\Pi_1+F_1)^2\\
&-4(K^2+K_1^2)(\Pi+F)^2\\
&-16(K^2+K_1^2)(\alpha^2+\alpha_1^2)\Omega^2\\
&-16(K^2+K_1^2)^2(\alpha^2+\alpha_1^2)\\
&+16(K^2+K_1^2)\alpha\Omega(\Pi_1+F_1)\\
&-16(K^2+K_1^2)\alpha_1\Omega(\Pi+F)\\
=\ & (K^2+K_1^2)\{-4(\Pi_1+F_1)^2-4(\Pi+F)^2\\
&\qquad -16(\alpha^2+\alpha_1^2)\Omega^2+16\alpha\Omega(\Pi_1+F_1)-16\alpha_1\Omega(\Pi+F)\}.
\end{aligned}$$

It thus appears that the equation of the Newton-circle F is

$$\begin{aligned}
&4(K^2+K_1^2-\Omega^2)^2\{(x-\mathrm{f})^2+(y-\mathrm{f}_1)^2\}\\
&\quad=(K^2+K_1^2)\{(\Pi_1+F_1)^2+(\Pi+F)^2\\
&\qquad -4\alpha\Omega(\Pi_1+F_1)+4\alpha_1\Omega(\Pi+F)+4(\alpha^2+\alpha_1^2)\Omega^2\}\\
&\quad=(K^2+K_1^2)\{(\Pi_1+F_1-2\alpha\Omega)^2+(\Pi+F+2\alpha_1\Omega)^2\},
\end{aligned}$$

where the coordinates of the centre are

$$\mathrm{f}=\frac{2(K^2+K_1^2)\alpha-\Omega(\Pi_1+F_1)}{2(K^2+K_1^2-\Omega^2)},$$

$$\mathrm{f}_1=\frac{2(K^2+K_1^2)\alpha_1+\Omega(\Pi+F)}{2(K^2+K_1^2-\Omega^2)},$$

and

$$\mathrm{rad}^2=\frac{K^2+K_1^2}{4(K^2+K_1^2-\Omega^2)^2}\{(\Pi_1+F_1-2\alpha\Omega)^2+(\Pi+F+2\alpha_1\Omega)^2\};$$

and similarly for the Newton-circles G and H.

16. The centres are in a line at right angles to

$$-2Kx-2K_1y+\Theta+lmn=0,$$

say the equation of this line is

$$K_1x-\ Ky\ +\Psi=0;$$

then we ought to have

$$\frac{K_1\{2(K^2+K_1^2)\alpha-\Omega(\Pi_1+F_1)\}-K\{2(K^2+K_1^2)\alpha_1+\Omega(\Pi+F)\}}{2(K^2+K_1^2-\Omega^2)}+\Psi=0,$$

that is,

$$2(K^2+K_1^2)(K_1\alpha-K\alpha_1)-(K_1\Pi_1+K\Pi)\Omega-(K_1F_1+KF)\Omega+2(K^2+K_1^2-\Omega^2)\Psi=0.$$

This should agree with

$$2(K^2+K_1^2)(K_1\beta-K\beta_1)-(K_1\Pi_1+K\Pi)\Omega-(K_1G_1+KG)\Omega+2(K^2+K_1^2-\Omega^2)\Psi=0,$$

viz. we ought to have

$$2(K^2+K_1^2)\{K_1(\alpha-\beta)-K(\alpha_1-\beta_1)\}-\{K_1(F_1-G_1)+K(F-G)\}\Omega=0.$$

This can be true only if $K_1(\alpha-\beta)-K(\alpha_1-\beta_1)$ is a multiple of Ω; and, in fact,

$$\begin{aligned}
&K_1(\alpha-\beta)-K(\alpha_1-\beta_1)\\
&\quad=(\alpha-\beta)(l\alpha_1+m\beta_1+n\gamma_1)-(\alpha_1-\beta_1)(l\alpha+m\beta+n\gamma)\\
&\quad=l(\alpha\beta_1-\alpha_1\beta)+m(\alpha\beta_1-\alpha_1\beta)+n\{-(\beta\gamma_1-\beta_1\gamma)-(\gamma\alpha_1-\gamma_1\alpha)\}\\
&\quad=-n(\beta\gamma_1-\beta_1\gamma+\gamma\alpha_1-\gamma_1\alpha+\alpha\beta_1-\alpha_1\beta),\ =-n\Omega.
\end{aligned}$$

The equation to be verified thus is

$$-2n(K^2+K_1^2)-K_1(F_1-G_1)-K(F-G)=0,$$

and here

$$\begin{aligned}
F-G&=(\beta-\gamma)mn-(\gamma-\alpha)ln+(l+m-2n)K\\
&=\alpha ln+\beta mn-\gamma(l+m)n-3nK=(n-3n)K=-2nK.
\end{aligned}$$

Hence

$$-K\ (F\ -G\ )=2nK^2,$$

and similarly

$$-K_1(F_1-G_1)=2nK_1^2;$$

and thus the equation is verified.

17. Writing for shortness

$$\beta\gamma_1-\beta_1\gamma,\quad \gamma\alpha_1-\gamma_1\alpha,\quad \alpha\beta_1-\alpha_1\beta=X,\ Y,\ Z,$$

and therefore $\Omega=X+Y+Z$; we have

$$\begin{aligned}
&2(K^2+K_1^2)(K_1\alpha-K\alpha_1)-(KF+K_1F_1)\Omega\\
&\quad=\ K\ \{mn(\beta-\gamma)(\ \ X-Y-Z)\\
&\qquad+\ \ nl\ (\gamma-\alpha)(-X+Y-Z)\\
&\qquad+\ \ lm\ (\alpha-\beta)(-X-Y+Z)\}\\
&\qquad+K_1\{mn(\beta_1-\gamma_1)(\ \ X-Y-Z)\\
&\qquad+\ \ nl\ (\gamma_1-\alpha_1)(-X+Y-Z)\\
&\qquad+\ \ lm\ (\alpha_1-\beta_1)(-X-Y+Z)\}.
\end{aligned}$$

To verify this, observe that the left-hand side is

$$2(K^2+K_1^2)(mZ-nY)-\{(\beta-\gamma)mnK+(\beta_1-\gamma_1)mnK_1+(m-n)(K^2+K_1^2)\Omega\},$$

or putting herein $X+Y+Z$ for Ω, this is

$$(K^2+K_1^2)\{(n-m)X+lY-lZ\}-mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}(X+Y+Z),$$

which is thus

$$\begin{aligned} = \; & mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}(\;\;X-Y-Z) \\ & + nl\;\{(\gamma-\alpha)K+(\gamma_1-\alpha_1)K_1\}(-X+Y-Z) \\ & + lm\;\{(\alpha-\beta)K+(\alpha_1-\beta_1)K_1\}(-X-Y+Z). \end{aligned}$$

The equation to be verified thus becomes

$$\begin{aligned} & (K^2+K_1^2)\{(n-m)X+lY-lZ\} \\ & \quad = \; mn\{(\beta-\gamma)K+(\beta_1-\gamma_1)K_1\}2X \\ & \qquad + nl\;\{(\gamma-\alpha)K+(\gamma_1-\alpha_1)K_1\}(-X+Y-Z) \\ & \qquad + lm\;\{(\alpha-\beta)K+(\alpha_1-\beta_1)K_1\}(-X-Y+Z). \end{aligned}$$

This breaks up into two equations,

$$\begin{aligned} K\{(n-m)X+lY-lZ\} = \; & mn(\beta-\gamma)2X \\ & + nl\;(\gamma-\alpha)(-X+Y-Z) \\ & + lm(\alpha-\beta)(-X-Y+Z); \end{aligned}$$

and a like equation with the suffixed letters. And the equation just written down, observing that each side is a linear function of X and $Y-Z$, again breaks up into the two equations

$$\begin{aligned} (n-m)K &= 2mn(\beta-\gamma)-nl(\gamma-\alpha)-lm(\alpha-\beta), \\ lK &= \qquad\qquad\quad -nl(\gamma-\alpha)-lm(\alpha-\beta), \end{aligned}$$

which are at once verified: in fact, for the first equation the right-hand side is

$$\begin{aligned} &= (nl-lm)\;\alpha+(2mn+lm)\beta+(-2mn-nl)\gamma, \\ &= (n\;-m)\;l\alpha+\;\;(2n+l)m\beta+(-2m\;\;-l)n\gamma, \\ &= (n\;-m)(l\alpha+\qquad\qquad m\beta+\qquad\qquad n\gamma), \;=(n-m)K; \end{aligned}$$

and similarly in the second equation the right-hand side is

$$(-nl-ln)\alpha+lm\beta+ln\gamma, \;\;=l(l\alpha+m\beta+n\gamma), \;\;=lK.$$

Writing then

$$\begin{aligned} \Phi = \; & K\;\{mn(\beta-\gamma)(X-Y-Z) \\ & + nl(\gamma-\alpha)(-X+Y-Z)+lm(\alpha-\beta)(-X-Y+Z)\} \\ & + K_1\{mn(\beta_1-\gamma_1)(X-Y-Z) \\ & + nl(\gamma_1-\alpha_1)(-X+Y-Z)+lm(\alpha_1-\beta_1)(-X-Y+Z)\}, \end{aligned}$$

we have

$$\Phi=(K_1\Pi_1+K\Pi)\Omega-2(K^2+K_1^2-\Omega^2)\Psi.$$

This equation determines Φ, and thus the equation of the line of centres is

$$2(K^2+K_1^2-\Omega^2)(K_1x-Ky)+(K\Pi+K_1\Pi_1)\Omega-\Phi=0.$$

18. This line meets

$$-2Kx-2K_1y+\Theta+lmn=0,$$

in the mid-point of the chord Z_1Z_2. We thus have for the coordinates x, y of this mid-point

$$2(K^2+K_1^2-\Omega^2)(K^2+K_1^2)x+K_1\{(K\Pi+K_1\Pi_1)\Omega-\Phi\}-K(K^2+K_1^2-\Omega^2)(\Theta+lmn)=0,$$

$$2(K^2+K_1^2-\Omega^2)(K^2+K_1^2)y-K\{(K\Pi+K_1\Pi_1)\Omega-\Phi\}-K_1(K^2+K_1^2-\Omega^2)(\Theta+lmn)=0.$$

The perpendicular distance of the centre of the circle F from the chord is

$$=\frac{-2Kf-2K_1f_1+\Theta+lmn}{2\sqrt{(K^2+K_1^2)}};$$

here

$$2(Kf+K_1f_1)=\frac{1}{K^2+K_1^2-\Omega^2}[2(K^2+K_1^2)(K\alpha+K_1\alpha_1)-\Omega\{K(\Pi_1+F_1)-K_1(\Pi+F)\}],$$

$$\begin{aligned}K\Pi_1-K_1\Pi=&\ (\alpha^2+\alpha_1^2)\{K(\beta_1-\gamma_1)-K_1(\beta-\gamma)\}\\&+(\beta^2+\beta_1^2)\{K(\gamma_1-\alpha_1)-K_1(\gamma-\alpha)\}\\&+(\gamma^2+\gamma_1^2)\{K(\alpha_1-\beta_1)-K_1(\alpha-\beta)\}\\=&\ \Omega\{l(\alpha^2+\alpha_1^2)+m(\beta^2+\beta_1^2)+n(\gamma^2+\gamma_1^2)\}=\Omega\Theta,\\KF_1-K_1F=&\ mn\{K(\beta_1-\gamma_1)-K_1(\beta-\gamma)\}=lmn\Omega.\end{aligned}$$

Thus

$$2(Kf+K_1f_1)=\frac{1}{K^2+K_1^2-\Omega^2}\{2(K^2+K_1^2)(K\alpha+K_1\alpha_1)-\Omega^2(\Theta+lmn)\},$$

and hence the numerator of the fraction is

$$\frac{1}{K^2+K_1^2-\Omega^2}\{-2(K^2+K_1^2)(K\alpha+K_1\alpha_1)+\Omega^2(\Theta+lmn)+(K^2+K_1^2-\Omega^2)(\Theta+lmn)\}$$

$$=\frac{1}{K^2+K_1^2-\Omega^2}(K^2+K_1^2)\{-2(K\alpha+K_1\alpha_1)+\Theta+lmn\}.$$

Thus the perpendicular distance of the centre of the circle F from the chord Z_1Z_2 is

$$=\frac{\sqrt{(K^2+K_1^2)}}{2(K^2+K_1^2-\Omega^2)}\{-2(K\alpha+K_1\alpha_1)+\Theta+lmn\};$$

moreover, by what precedes, we have

$$\text{Radius}=\frac{\sqrt{(K^2+K_1^2)}}{2(K^2+K_1^2-\Omega^2)}\sqrt{\{(\Pi+F+2\alpha_1\Omega)^2+(\Pi_1+F_1-2\alpha\Omega)^2\}}.$$

19. Hence also

$$(Z_1Z_2)^2=\frac{K^2+K_1^2}{(K^2+K_1^2-\Omega^2)^2}[(\Pi+F+2\alpha_1\Omega)^2+(\Pi_1+F_1-2\alpha\Omega)^2-\{\Theta+lmn-2(K\alpha+K_1\alpha_1)\}^2].$$

We have

$$K^2 + K_1^2 - \Omega^2 = (l\alpha + m\beta + n\gamma)^2 + (l\alpha_1 + m\beta_1 + n\gamma_1)^2 - (\beta\gamma_1 - \beta_1\gamma + \gamma\alpha_1 - \gamma_1\alpha + \alpha\beta_1 - \alpha_1\beta)^2;$$

but

$$f^2 = (\beta - \gamma)^2 + (\beta_1 - \gamma_1)^2,$$

$$g^2 = (\gamma - \alpha)^2 + (\gamma_1 - \alpha_1)^2,$$

$$h^2 = (\alpha - \beta)^2 + (\alpha_1 - \beta_1)^2,$$

and hence

$$-f^2 + g^2 + h^2 = -2(\gamma - \alpha)(\alpha - \beta) - 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1),$$

whence

$$-f^2(-f^2 + g^2 + h^2) = \{2(\gamma - \alpha)(\alpha - \beta) + 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1)\}\{(\beta - \gamma)^2 + (\beta_1 - \gamma_1)^2\};$$

or forming the like values of $-g^2(f^2 - g^2 + h^2)$ and $-h^2(f^2 + g^2 - h^2)$ and adding, we have

$$\begin{aligned}
\Delta = f^4 + g^4 + h^4 - 2g^2h^2 - 2h^2f^2 - 2f^2g^2 \\
= \quad 2(\gamma - \alpha)(\alpha - \beta)(\beta_1 - \gamma_1)^2 + 2(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1)(\beta - \gamma)^2 \\
+ 2(\alpha - \beta)(\beta - \gamma)(\gamma_1 - \alpha_1)^2 + 2(\alpha_1 - \beta_1)(\beta_1 - \gamma_1)(\gamma - \alpha)^2 \\
+ 2(\beta - \gamma)(\gamma - \alpha)(\alpha_1 - \beta_1)^2 + 2(\beta_1 - \gamma_1)(\gamma_1 - \alpha_1)(\alpha - \beta)^2 \\
= -4\{\alpha_1(\beta - \gamma) + \beta_1(\gamma - \alpha) + \gamma_1(\alpha - \beta)\}^2 = -4\Omega^2.
\end{aligned}$$

But

$$\begin{aligned}
mnf^2 + nlg^2 + lmh^2 = \quad & mn\{\beta^2 + \beta_1^2 + \gamma^2 + \gamma_1^2 - 2(\beta\gamma + \beta_1\gamma_1)\} \\
& + nl\{\gamma^2 + \gamma_1^2 + \alpha^2 + \alpha_1^2 - 2(\gamma\alpha + \gamma_1\alpha_1)\} \\
& + lm\{\alpha^2 + \alpha_1^2 + \beta^2 + \beta_1^2 - 2(\alpha\beta + \alpha_1\beta_1)\}, \\
= & -l^2(\alpha^2 + \alpha_1^2) - m^2(\beta^2 + \beta_1^2) - n^2(\gamma^2 + \gamma_1^2) \\
& - 2mn(\beta\gamma_1 + \beta_1\gamma) - 2nl(\gamma\alpha_1 + \gamma_1\alpha) - 2lm(\alpha\beta_1 + \alpha_1\beta). \\
= & -(l\alpha + m\beta + n\gamma)^2 - (l\alpha_1 + m\beta_1 + n\gamma_1)^2, \\
= & -K^2 - K_1^2.
\end{aligned}$$

Thus

$$4(K^2 + K_1^2) = -4(mnf^2 + nlg^2 + lmh^2), \quad -4\Omega^2 = \Delta,$$

and therefore

$$4(K^2 + K_1^2 - \Omega^2) = \Delta - 4(mnf^2 + nlg^2 + lmh^2).$$

But

$$\mathfrak{S} = \quad 4(mnf^2 + nlg^2 + lmh^2),$$

$$\mathfrak{S} = -\Delta + 4(mnf^2 + nlg^2 + lmh^2),$$

and thus

$$4(K^2 + K_1^2) \quad = -\mathfrak{S},$$

$$4(K^2 + K_1^2 - \Omega^2) = -\mathfrak{S},$$

hence

$$\frac{K^2 + K_1^2}{(K^2 + K_1^2 - \Omega^2)^2} = \frac{-4\mathfrak{S}}{\mathfrak{S}^2},$$

and we have

$$(Z_1Z_2)^2 = \frac{-4\mathfrak{E}}{\mathfrak{S}^2}\left[(\Pi + F + 2\alpha_1\Omega)^2 + (\Pi_1 + F_1 - 2\alpha\Omega)^2 - \{\Theta + lmn - 2(K\alpha + K_1\alpha_1)\}^2\right].$$

20. This should agree with the expression in No. 7, that is, we ought to have

$$(\Pi + F + 2\alpha_1\Omega)^2 + (\Pi_1 + F_1 - 2\alpha\Omega)^2 - \{\Theta + lmn - 2(K\alpha + K_1\alpha_1)\}^2 = (f^2 - l^2)(g^2 - m^2)(h^2 - n^2),$$

and this breaks up into the equations

$$\begin{aligned}
(\Pi + 2\alpha_1\Omega) + (\Pi_1 - 2\alpha\Omega)^2 \qquad\qquad\qquad\qquad &= f^2g^2h^2 \\
2F(\Pi + 2\alpha_1\Omega) + 2F_1(\Pi_1 - 2\alpha\Omega) - \{\Theta - 2(K\alpha + K_1\alpha_1)\}^2 &= -g^2h^2l^2 - h^2f^2m^2 - f^2g^2n^2 \\
F^2 + F_1^2 - 2lmn\{\Theta - 2(K\alpha + K_1\alpha_1)\} &= f^2m^2n^2 + g^2n^2l^2 + h^2l^2m^2 \\
- l^2m^2n^2 \qquad\qquad\qquad\qquad\qquad\qquad &= -l^2m^2n^2,
\end{aligned}$$

which may be separately verified.

21. In fact, we have

$$\begin{aligned}
\Pi + 2\alpha_1\Omega = \; & (\beta - \gamma)(\alpha^2 + \alpha_1^2) + (\gamma - \alpha)(\beta^2 + \beta_1^2) + (\alpha - \beta)(\gamma^2 + \gamma_1^2) \\
& \qquad + 2\alpha_1\{-(\beta - \gamma)\alpha_1 - (\gamma - \alpha)\beta_1 - (\alpha - \beta)\gamma_1\], \\
= \; & (\beta - \gamma)(\alpha^2 - \alpha_1^2) + (\gamma - \alpha)\{\beta^2 - \alpha_1^2 + (\alpha_1 - \beta_1)^2\} + (\alpha - \beta)\{\gamma^2 - \alpha_1^2 + (\gamma_1 - \alpha_1)^2\}, \\
= \; & \alpha^2(\beta - \gamma) + \beta^2(\gamma - \alpha) + \gamma^2(\alpha - \beta) + (\gamma - \alpha)(\alpha_1 - \beta_1)^2 + (\alpha - \beta)(\gamma_1 - \alpha_1)^2, \\
= \; & -(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta) + (\gamma - \alpha)(\alpha_1 - \beta_1)^2 + (\alpha - \beta)(\gamma_1 - \alpha_1)^2,
\end{aligned}$$

or, putting for shortness

$$\beta - \gamma,\ \gamma - \alpha,\ \alpha - \beta = \lambda,\ \mu,\ \nu;\quad \beta_1 - \gamma_1,\ \gamma_1 - \alpha_1,\ \alpha_1 - \beta_1 = \lambda_1,\ \mu_1,\ \nu_1,$$

(where the letters λ, μ, ν, λ_1, μ_1, ν_1 have a meaning different from that assigned to them in No. 7), this is

$$\Pi + 2\alpha_1\Omega = -\lambda\mu\nu + \mu\nu_1^2 + \nu\mu_1^2,$$

and similarly

$$\Pi_1 - 2\alpha\Omega = -\lambda_1\mu_1\nu_1 + \mu_1\nu^2 + \nu_1\mu^2.$$

Also

$$f^2,\ g^2,\ h^2 = \lambda^2 + \lambda_1^2,\ \mu^2 + \mu_1^2,\ \nu^2 + \nu_1^2,$$

and the first equation thus becomes

$$(-\lambda\mu\nu + \mu\nu_1^2 + \nu\mu_1^2)^2 + (-\lambda_1\mu_1\nu_1 + \mu_1\nu^2 + \nu_1\mu^2)^2 = (\lambda^2 + \lambda_1^2)(\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2),$$

or if on the left-hand we write $\lambda = -\mu - \nu$, $\lambda_1 = -\mu_1 - \nu_1$, this is

$$\{\nu(\mu^2 + \mu_1^2) + \mu(\nu^2 + \nu_1^2)\}^2 + \{\nu_1(\mu^2 + \mu_1^2) + \mu_1(\nu^2 + \nu_1^2)\}^2,$$

which is

$$\begin{aligned}
&= (\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2)(\mu^2 + 2\mu\nu + \nu^2 + \mu_1^2 + 2\mu_1\nu_1 + \nu_1^2) \\
&= (\mu^2 + \mu_1^2)(\nu^2 + \nu_1^2)(\lambda^2 + \lambda_1^2),
\end{aligned}$$

which is right.

22. For the second equation, we use the values

$$\Pi + 2\alpha_1\Omega = \nu\ (\mu^2+\mu_1^2) + \mu\ (\nu^2+\nu_1^2),$$
$$\Pi_1 - 2\alpha\ \Omega = \nu_1(\mu^2+\mu_1^2) + \mu_1(\nu^2+\nu_1^2);$$

and the equation to be verified thus is

$$\begin{aligned} 2F\ \{\nu\ (\mu^2+\mu_1^2) + \mu\ (\nu^2+\nu_1^2)\} &= -l^2\ (\mu^2+\mu_1^2)(\nu^2+\nu_1^2) \\ + 2F_1\ \{\nu_1(\mu^2+\mu_1^2) + \mu_1(\nu^2+\nu_1^2)\} &\quad - m^2(\nu^2+\nu_1^2)(\lambda^2+\lambda_1^2) \\ - \{\Theta - 2(K\alpha + K_1\alpha_1)\}^2 &\quad - n^2\ (\lambda^2+\lambda_1^2)(\mu^2+\mu_1^2); \end{aligned}$$

we have

$$F = mn\lambda + (m-n)(l\alpha + m\beta + n\gamma) = mn\lambda + (m-n)(-m\nu + n\mu),$$

that is,

$$F = -m^2\nu - n^2\mu;$$

and similarly

$$F_1 = -m^2\nu_1 - n^2\mu_1.$$

Also

$$\begin{aligned} &\Theta - 2(K\alpha + K_1\alpha_1) \\ &= l(\alpha^2+\alpha_1^2) + m(\beta^2+\beta_1^2) + n(\gamma^2+\gamma_1^2) - 2\{l(\alpha^2+\alpha_1^2) + m(\alpha\beta+\alpha_1\beta_1) + n(\alpha\gamma+\alpha_1\gamma_1)\}, \\ &= -l(\alpha^2+\alpha_1^2) + m(\beta^2+\beta_1^2 - 2\alpha\beta - 2\alpha_1\beta_1) + n(\gamma^2+\gamma_1^2 - 2\alpha\gamma - 2\alpha_1\gamma_1), \\ &= m\{(\alpha-\beta)^2 + (\alpha_1-\beta_1)^2\} + n\{(\gamma-\alpha)^2 + (\gamma_1-\alpha_1)^2\}, \\ &= m(\nu^2+\nu_1^2) + n(\mu^2+\mu_1^2). \end{aligned}$$

The left-hand side is

$$\begin{aligned} &-2(m^2\nu + n^2\mu)\{\nu\ (\mu^2+\mu_1^2) + \mu\ (\nu^2+\nu_1^2)\} \\ &-2(m^2\nu_1 + n^2\mu_1)\{\nu_1(\mu^2+\mu_1^2) + \mu_1(\nu^2+\nu_1^2)\} \\ &-\{m(\nu^2+\nu_1^2) + n(\mu^2+\mu_1^2)\}^2, \end{aligned}$$

which is

$$\begin{aligned} &= m^2\{-2(\nu^2+\nu_1^2)(\mu^2+\mu_1^2) - 2(\mu\nu+\mu_1\nu_1)(\nu^2+\nu_1^2) - (\nu^2+\nu_1^2)^2\} \\ &\quad + n^2\{-2(\mu\nu+\mu_1\nu_1)(\mu^2+\mu_1^2) - 2(\mu^2+\mu_1^2)(\nu^2+\nu_1^2) - (\mu^2+\mu_1^2)^2\} \\ &\quad + (-l^2+m^2+n^2)(\mu^2+\mu_1^2)(\nu^2+\nu_1^2), \\ &= -l^2(\mu^2+\mu_1^2)(\nu^2+\nu_1^2) \\ &\quad - m^2(\nu^2+\nu_1^2)(\mu^2+2\mu\nu+\nu^2+\mu_1^2+2\mu_1\nu_1+\nu_1^2) \\ &\quad - n^2(\mu^2+\mu_1^2)(\mu^2+2\mu\nu+\nu^2+\mu_1^2+2\mu_1\nu_1+\nu_1^2), \end{aligned}$$

which is equal to the right-hand side.

23. For the third equation we have as above

$$F = -m^2\nu - n^2\mu, \quad F_1 = -m^2\nu_1 - n^2\mu_1,$$
$$\Theta - 2(K\alpha + K_1\alpha_1) = m(\nu^2+\nu_1^2) + n(\mu^2+\mu_1^2),$$

and the equation thus is

$$(m^2\nu + n^2\mu)^2 + (m^2\nu_1 + n^2\mu_1)^2 - 2lmn\,\{m\,(\nu^2 + \nu_1^2) + n\,(\mu^2 + \mu_1^2)\}$$
$$= (\lambda^2 + \lambda_1^2)\,m^2n^2 + (\mu^2 + \mu_1^2)\,n^2l^2 + (\nu^2 + \nu_1^2)\,l^2m^2;$$

here the left-hand side is

$$\begin{aligned} = \quad & (m^4 - 2lm^2n)\,(\nu^2 + \nu_1^2) \\ & + (n^4 - 2lmn^2)\,(\mu^2 + \mu_1^2) \\ & + m^2n^2\,(\lambda^2 + \lambda_1^2 - \mu^2 - \mu_1^2 - \nu^2 - \nu_1^2), \end{aligned}$$

which is

$$\begin{aligned} = \quad & (\lambda^2 + \lambda_1^2)\,m^2n^2 \\ & + (\mu^2 + \mu_1^2)\,n^2\,(n^2 - 2lm - m^2) \\ & + (\nu^2 + \nu_1^2)\,m^2\,(m^2 - 2ln - n^2), \end{aligned}$$

which is equal to the right-hand side.

24. The fourth equation is the identity $-l^2m^2n^2 = -l^2m^2n^2$, and the whole equation is thus verified: viz. the analytical solution leads to the expression

$$\mathfrak{E}^2\,(Z_1Z_2)^2 = -4\,\{f^2 - (b-c)^2\}\,\{g^2 - (c-a)^2\}\,\{h^2 - (a-b)^2\}\,\mathfrak{E},$$

obtained by an independent process in No. 7 for the squared distance of the two centres.

920.

ON ORTHOMORPHOSIS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXV. (1891), pp. 203—226.]

1. THE equation $x_1 + iy_1 = \phi(x + iy)$, where ϕ is in general an imaginary function (that is, a function with imaginary coefficients), and where (x, y), (x_1, y_1) are rectangular coordinates, determines x_1, y_1, each of them as a function of (x, y), and hence, first eliminating y, we have a set of curves depending on the parameter x, and secondly eliminating x, we have a set of curves depending on the parameter y; we have thus two sets of curves, or say trajectories, which are orthomorphoses of the two sets of lines $x = \text{const.}$ and $y = \text{const.}$ respectively. They thus cut everywhere at right angles, and are moreover such that, giving to the parameters values at equal infinitesimal intervals (the same for x and y respectively), they form a double system of infinitesimal squares; say that we have two systems of square trajectories S and T. We may assume at pleasure a trajectory S, and also the consecutive trajectory S'; but it is at once seen geometrically that the entire system of trajectories S and T is then completely determined. For at any point t of S drawing a normal to meet S', and on S the element tt' equal to the normal distance at t, then at t' drawing a normal to meet S', and on S the element $t't''$ equal to the normal distance at t', and so on, we divide the strip between S and S' into infinitesimal squares: at the successive points of S' drawing normals, and on these measuring off distances equal to the successive elements of S', we construct a new curve S'', at the same time dividing the strip between S' and S'' into infinitesimal squares; and proceeding in this manner, we obtain the series of curves S, S', S'', S''', &c., and at the same time the series of curves T, T', T'', T''', &c., proceeding from the points t, t', t'', t''', &c., respectively, and forming with the first set of curves the double system of infinitesimal squares.

2. But to translate this into Analysis, and to obtain the equations of the two sets of curves, we proceed as follows*.

Suppose that the curve S is given in the form

$$x_1 = p, \quad y_1 = q,$$

where p, q are given real functions of a variable parameter θ: we obtain a series of square trajectories S and T, one of the first set being the given curve S, by forming from these equations the equation

$$x_1 + iy_1 = p + iq,$$

and writing therein $\theta = x + iy$. In fact, for the value $y = 0$, this equation becomes $x_1 = p$, $y_1 = q$, where p, q are now the same functions of x which they were originally of θ, and the elimination of x leads therefore to the equation of the given curve S. More generally, we may take $\theta = f(w)$, an arbitrary real function of w, and then assume $w = x + iy$: in this case, for $y = 0$, we have as before $x_1 = p$, $y_1 = q$, where p and q are now the same functions of $f(x)$ which they originally were of θ, and the elimination of x from these equations thus leads as before to the given curve S.

We have next to determine $f(w)$ in suchwise that the curve corresponding to an infinitesimal real value ϵ of y shall be the given consecutive curve S'. We assume that this curve S' is given by the equations

$$x_1 = p + \gamma P, \quad y_1 = q + \gamma Q,$$

where p, q are as before and P, Q are given real functions of θ; γ is a real infinitesimal. The expression for w as a function of θ is then determined by the condition

$$\gamma = Cdw = \frac{(p'^2 + q'^2)\, d\theta}{p'Q - q'P},$$

where p', q' are the derived functions of p, q in regard to θ: C is a real constant, the value of which may be assumed at pleasure. Say we have

$$Cw = \int \frac{(p'^2 + q'^2)\, d\theta}{p'Q - q'P},$$

where the integral may be regarded as containing a real or imaginary constant of integration $x_0 + iy_0$; but this being so, we ultimately have

$$w + x_0 + iy_0 = x + iy, \quad \text{or} \quad w = (x - x_0) + i(y - y_0),$$

which is the same thing as $w = x + iy$.

3. We may, in the expressions for x_1, y_1, substitute for θ any other real value θ_1 whatever, say the equations thus became $x_1 = p_1 + \gamma P_1$, $y_1 = q_1 + \gamma Q_1$, values which give

$$x_1 + iy_1 = p_1 + iq_1 + \gamma(P_1 + iQ_1);$$

* The investigation is taken from the memoir, Beltrami, "Delle variabili complesse sopra una superficie qualunque," *Annali di Matem.*, t. I. (1867), pp. 329—366.

the proof consists in showing that there exists for θ_1 a value, differing infinitesimally from θ and such as to reduce this equation to

$$x_1 + iy_1 = p + iq + (p' + iq')\frac{i\gamma}{C}\frac{d\theta}{dw},$$

the value assumed by $p+iq$ on substituting for w the value $w + \frac{i\gamma}{C}$: for this being so, it is obvious that we have for S' the curve in the series of curves S which corresponds to the value $\frac{\gamma}{C}$ of y in $w = x + iy$. The value of θ_1 is

$$\theta_1 = \theta - \gamma\frac{p'P + q'Q}{p'^2 + q'^2},$$

for this gives

$$p_1 = p - \gamma p'\frac{p'P + q'Q}{p'^2 + q'^2}, \quad q_1 = q - \gamma q'\frac{p'P + q'Q}{p'^2 + q'^2};$$

and since γ is infinitesimal, we may in the terms γP_1 and γQ_1 of x_1 and y_1 write $\theta_1 = \theta$, and so reduce these terms to γP and γQ respectively: we thus have

$$x_1 = p_1 + \gamma P_1 = p - \gamma p'\frac{p'P + q'Q}{p'^2 + q'^2} + \gamma P, \quad = p - \gamma q'\frac{p'Q - q'P}{p'^2 + q'^2},$$

$$y_1 = q_1 + \gamma Q_1 = q - \gamma q'\frac{p'P + q'Q}{p'^2 + q'^2} + \gamma Q, \quad = q + \gamma p'\frac{p'Q - q'P}{p'^2 + q'^2},$$

or for $\frac{p'Q - q'P}{p'^2 + q'^2}$ substituting its value $\frac{1}{C}\frac{d\theta}{dw}$, these values are

$$x_1 = p - \frac{\gamma}{C}q'\frac{d\theta}{dw}, \quad y_1 = q + \frac{\gamma}{C}p'\frac{d\theta}{dw},$$

which give the before-mentioned equation

$$x_1 + iy_1 = p + iq + (p' + iq')\frac{i\gamma}{C}\frac{d\theta}{dw},$$

and the proof is thus completed.

The following two examples are given by Beltrami.

4. First, let the curves S, S' be given confocal ellipses,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \frac{x_1^2}{a^2 - 2cy} + \frac{y_1^2}{b^2 - 2cy} = 1.$$

Here

$$p = a\cos\theta, \quad q = b\sin\theta,$$

whence

$$p'^2 + q'^2 = a^2\sin^2\theta + b^2\cos^2\theta;$$

$$P = -\frac{c}{a}\cos\theta, \quad Q = -\frac{c}{b}\sin\theta,$$

whence

$$p'Q - q'P = \frac{c}{ab}(a^2\sin^2\theta + b^2\cos^2\theta),$$

and consequently

$$Cw = \int \frac{c}{ab}\, d\theta = \frac{c}{ab}\,\theta, \text{ or taking } C = \frac{c}{ab}, \text{ then } w = \theta,$$

and for the two sets of curves we have

$$x_1 + iy_1 = a\cos(x+iy) + ib\sin(x+iy),$$

which are two sets of confocal conics. In verification, write

$$\cos x = X, \quad \sin x = X', \quad \text{whence} \quad X^2 + X'^2 = 1,$$
$$\cos iy = Y, \quad \sin iy = iY', \quad \text{"} \quad Y^2 - Y'^2 = 1,$$

and then

$$x_1 + iy_1 = a(XY - iX'Y') + ib(YX' + iY'X),$$

that is,

$$x_1 = X(aY - bY'),$$
$$y_1 = X'(bY - aY'),$$

and consequently

$$\frac{x_1^2}{X^2} - \frac{y_1^2}{X'^2} = a^2 - b^2, \text{ which are the curves } T,$$

$$\frac{x_1^2}{(aY - bY')^2} + \frac{y_1^2}{(bY - aY')^2} = 1, \quad \text{"} \quad \text{"} \quad \text{"} \quad S;$$

where observe that

$$(aY - bY')^2 - (bY - aY')^2 = (a^2 - b^2)(Y^2 - Y'^2) = a^2 - b^2,$$

so that, putting $(aY - bY')^2 = a^2 + h$, we have

$$(bY - aY')^2 = a^2 + h - (a^2 - b^2), \quad = b^2 + h,$$

and thus the curves S are the confocal ellipses

$$\frac{x^2}{a^2 + h} + \frac{y^2}{b^2 + h} = 1, \quad h > -b^2;$$

and similarly, since $X^2 + X'^2 = 1$, putting $(a^2 - b^2)X'^2 = a^2 + k$, we have

$$(a^2 - b^2)X'^2 = a^2 - b^2 - (a^2 + k), \quad = -b^2 - k,$$

and thus the curves T are the confocal hyperbolas

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1, \quad -k < a^2 > b^2.$$

5. Next, let the curve S be a circle and the curve S' an interior non-concentric circle. It is convenient to take the circles, such that they have for their common chord the line $y_1 = 0$ and each meets this chord in the points $x_1 = \pm i$.

This being so, the equations of the two circles are

$$x_1^2 + y_1^2 - 2y_1 \operatorname{cosec}\mu = -1,$$
$$x_1^2 + y_1^2 - 2y_1 \operatorname{cosec}\mu\,(1 - \gamma\cos^2\mu) = -1,$$

and we have

$$p = \cot\mu\cos\theta, \quad q = \cot\mu\sin\theta + \operatorname{cosec}\mu,$$
$$P = -\cot\mu\cos\theta, \quad Q = -\cot\mu\sin\theta - \cos^2\mu\operatorname{cosec}\theta,$$

giving

$$p'^2 + q'^2 = \cot^2\mu,$$
$$p'Q - q'P = \cot^2\mu(1 + \cos\mu\sin\theta),$$
$$\gamma = Cdw = \frac{d\theta}{1 + \cos\mu\sin\theta}, \text{ or if } C = \frac{1}{\sin\mu}, \text{ then } dw = \frac{\sin^4\mu\, d\theta}{1 + \cos\mu\sin\theta}.$$

The integral of this may be written

$$\tan\left(\tfrac{1}{2}w - \tfrac{1}{2}iy_0\right) = \cot\tfrac{1}{2}\mu\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right);$$

and if we here assume for the constant of integration y_0 a value such that

$$\tan\tfrac{1}{2}iy_0 = i\tan\tfrac{1}{2}\mu,$$

we find for w the equation

$$\tan\tfrac{1}{2}w = \cot\mu\, e^{i\theta} + i\operatorname{cosec}\mu,$$

we have $w = x + iy$, and

$$x_1 + iy_1 = p + iq = \cot\mu\, e^{i\theta} + i\operatorname{cosec}\mu,$$

hence

$$x_1 + iy_1 = \tan\tfrac{1}{2}(x + iy);$$

or writing, as we may do, $2x$, $2y$ in place of x, y respectively, say

$$x_1 + iy_1 = \tan(x + iy),$$

for the two sets of curves S and T.

Writing $\tan x = X$, $\tan iy = iY$, this gives

$$x_1 + iy_1 = \frac{X + iY}{1 - iXY} = \frac{(X + iY)(1 + iXY)}{1 + X^2Y^2},$$

that is,

$$x_1 = \frac{X(1 - Y^2)}{1 + X^2Y^2}, \quad y_1 = \frac{Y(1 + X^2)}{1 + X^2Y^2};$$

and therefore

$$x_1^2 + y_1^2 = \frac{X^2 + Y^2}{1 - X^2Y^2},$$

and thence

$$x_1^2 + y_1^2 - \left(Y + \frac{1}{Y}\right)y_1 = -1,$$
$$x_1^2 + y_1^2 - \left(X - \frac{1}{X}\right)x_1 = \quad 1,$$

for the curves S and T respectively. The value $Y = \cot\frac{1}{2}\mu$ gives the original S-curve

$$x_1^2 + y_1^2 - 2y_1\operatorname{cosec}\mu = -1.$$

We have thus the well-known system of orthotomic circles, where the circles of the one set pass through the points $x_1 = 0$, $y_1 = \pm 1$, and those of the other set through the antipoints $x_1 = \pm i$, $y_1 = 0$.

6. A different solution is given in the Dissertation referred to below*: I reproduce this, giving the proof in a somewhat altered form. The equations of the curve S and of the consecutive curve S' are taken to be

$$\phi(x, y) = 0, \quad \phi(x, y) - 2u\,\psi(x, y) = 0;$$

in the investigation, we introduce the conjugate variables $z = x + iy$, $\bar{z} = x - iy$, and write ϕ, ψ to denote the values assumed by the functions $\phi(x, y)$, $\psi(x, y)$ when x, y are replaced therein by their values in terms of z, $\bar{z}$, viz. $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$. But in the first instance we may, without attending to the meanings of z, $\bar{z}$, consider ϕ, ψ as denoting each of them a given function of the two coordinates z, $\bar{z}$.

Suppose now that the functions $f(z)$, $f_1(\bar{z})$ are determined by the equations

$$f(z) = \int \frac{1}{\psi}\frac{d\phi}{dz}\,dz, \quad f_1(\bar{z}) = \int \frac{1}{\psi}\frac{d\phi}{d\bar{z}}\,d\bar{z},$$

where in the first integral, after the calculation of $\frac{d\phi}{dz}$, $\bar{z}$ is expressed as a function of z by means of the assumed relation $\phi = 0$, and in the second integral after the calculation of $\frac{d\phi}{d\bar{z}}$, z is expressed as a function of $\bar{z}$ by means of the same relation $\phi = 0$; and where in each integral the constant is properly determined, as presently appearing.

We have

$$f'(z)\,dz + f_1'(\bar{z})\,d\bar{z} = \frac{1}{\psi}\left(\frac{d\phi}{dz}\,dz + \frac{d\phi}{d\bar{z}}\,d\bar{z}\right),$$

where the expression on the right-hand side is $= 0$, if only $\phi = 0$; that is, ϕ being $= 0$, we have $f'(z)\,dz + f_1'(\bar{z})\,d\bar{z} = 0$; and consequently $f(z) + f_1(\bar{z}) = \text{const.}$, viz. this last equation, existing as a consequence of the equation $\phi = 0$, can be nothing else than a different form of the equation $\phi = 0$. We may, by a proper determination of the constants of integration of the two integrals, make the constant of the last-mentioned equation to be $= 0$; we thus have the relation $f(z) + f_1(\bar{z}) = 0$, equivalent to the relation $\phi = 0$.

Writing now $z + \delta z$, $\bar{z} + \delta\bar{z}$ in place of z, $\bar{z}$ respectively, where δz, $\delta\bar{z}$ are arbitrary infinitesimal increments, we have

$$f(z + \delta z) + f_1(\bar{z} + \delta\bar{z}) = 0,$$

equivalent to

$$\phi(z + \delta z, \bar{z} + \delta\bar{z}) = 0,$$

* Meyer, *Ueber die von gerade Linien und von Kegelschnitten gebildeten Schaaren von Isothermen, sowie über einige von speciellen Curven dritter Ordnung gebildete Schaaren von Isothermen:* Inaugural Dissertation der Universität zu Göttingen; 4°, Zürich, 1879. I quote the title in full as it explains the object of the paper; the Isothermals referred to are, of course, systems of orthotomic curves S and T: Plates I. to V. exhibit cases where the curves of at least one of the systems are conics, and Plates VI. to XIV. cases where the curves of at least one of the systems are cubics; the discussion of the several systems is very full and interesting.

that is,

$$f(z)+f'(z)\,\delta z+f_1(\bar{z})+f_1'(\bar{z})\,\delta\bar{z}=0,$$

equivalent to

$$\phi+\frac{d\phi}{dz}\,\delta z+\frac{d\phi}{d\bar{z}}\,\delta\bar{z}=0\,;$$

and if we here assume

$$\delta z=-u\div f'(z),$$

$$\delta\bar{z}=-u\div f_1'(\bar{z}),$$

then we have

$$f(z)+f_1(\bar{z})-2u=0,$$

equivalent to

$$\phi-u\left\{\frac{1}{f'(z)}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar{z})}\,\frac{d\phi}{d\bar{z}}\right\}=0,$$

that is, in virtue of

$$f'(z)=\frac{1}{\psi}\,\frac{d\phi}{dz}\quad\text{and}\quad f_1'(\bar{z})=\frac{1}{\psi}\,\frac{d\phi}{d\bar{z}},$$

equivalent to

$$\phi-2u\psi=0.$$

There is the difficulty that the last-mentioned values of $f'(z)$, $f_1'(\bar{z})$ do not subsist absolutely, but only when $\phi=0$; the answer to this is that the equation $\phi-2u\psi=0$, where u is infinitesimal, is in effect the equation $\phi=0$. Supposing that z, $\bar{z}$, instead of being connected by the equation $\phi=0$, are connected by the equation $\phi-2u\psi=0$, then the value of

$$\frac{1}{f'z}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar{z})}\,\frac{d\phi}{d\bar{z}},$$

instead of being actually equal to 2ϕ, will be equal to 2ϕ plus an infinitesimal value containing the factor u, and consequently on substituting this value in the expression

$$\phi-u\left\{\frac{1}{f'(z)}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar{z})}\,\frac{d\phi}{d\bar{z}}\right\},$$

and neglecting powers of u, we obtain as above the expression $\phi-2u\psi$.

If we now write $z=x+iy$, $\bar{z}=x-iy$, and assume that the equation $\phi(x,y)=0$ is a real equation, then clearly $f(z)$, $f_1(\bar{z})$ as above defined will be conjugate functions of z, $\bar{z}$ respectively: and u being no longer infinitesimal, we shall have

$$f(z)=u+iv,$$

an equation implying the conjugate relation

$$f_1(\bar{z})=u-iv.$$

Considering this equation $f(z)=u+iv$ as thus denoting the two equations, and successively eliminating the parameters v and u, we obtain first a system of curves S depending on the parameter u, and secondly a system of curves T depending on the parameter v, which are two systems of orthotomic curves. And by what precedes, it appears that the curve S belonging to the value $u=0$ of the parameter u, and

the consecutive curve S' belonging to a real infinitesimal value $u=u$, are the given curves

$$\phi(x, y)=0 \text{ and } \phi(x, y)-2u\psi(x, y)=0,$$

respectively.

7. For instance, let the curves S and S' be the circles

$$x^2+y^2-1=0, \quad x^2+y^2-1-2ux=0;$$

here

$$\phi=z\bar{z}-1, \quad \psi=\tfrac{1}{2}(z+\bar{z}),$$

$$\frac{1}{\psi}\frac{d\phi}{dz}=\frac{2\bar{z}}{z+\bar{z}}=\frac{2\dfrac{1}{z}}{z+\dfrac{1}{z}}=\frac{2}{z^2+1};$$

whence, adding $\frac{1}{2}\pi$ for the constant of integration,

$$\int\frac{1}{\psi}\frac{d\phi}{dz}\,dz=2\tan^{-1}z+\tfrac{1}{2}\pi.$$

The two sets of curves are thus given by

$$u+iv=\tan^{-1}z=\tan^{-1}(x+iy), \quad 2\tan^{-1}(x+iy)=u-\tfrac{1}{2}\pi+iv,$$

that is,

$$x+iy=\tan(\tfrac{1}{2}u-\tfrac{1}{4}\pi+iv);$$

viz. here, if

$$\tan(\tfrac{1}{2}u-\tfrac{1}{2}\pi)=\alpha, \quad \tan\tfrac{1}{2}iv=i\beta,$$

$$x+iy=\frac{\alpha+i\beta}{1-i\alpha\beta}=\frac{\alpha(1-\beta^2)+i\beta(1+\alpha^2)}{1+\alpha^2\beta^2},$$

that is,

$$x=\frac{\alpha(1-\beta^2)}{1+\alpha^2\beta^2}, \quad y=\frac{\beta(1+\alpha^2)}{1+\alpha^2\beta^2},$$

giving

$$x^2+y^2=\frac{\alpha^2+\beta^2}{1+\alpha^2\beta^2},$$

and thence

$$x^2+y^2-1-\left(\alpha-\frac{1}{\alpha}\right)x=0, \quad x^2+y^2+1-\left(\beta-\frac{1}{\beta}\right)y=0,$$

for the two systems of curves. The first of these, substituting for α its value, becomes

$$x^2+y^2-1-2x\tan u=0,$$

and thus the curves, for $u=0$ and u infinitesimal, are

$$x^2+y^2-1=0 \quad \text{and} \quad x^2+y^2-1-2ux=0,$$

as they should be.

8. Reverting to the first-mentioned solution, the equation of the circle $x_1^2+y_1^2=1$ is satisfied by $x_1=\cos\theta$, $y_1=\sin\theta$, and we have therefore $x_1+iy_1=e^{i\theta}$, and hence by what precedes, writing for convenience $X+iY$ instead of $x+iy$, we have

$$x_1+iy_1=e^{i\phi(X+iY)}$$

(ϕ being any real function) for a set of curves S and T, or say S_1 and T_1, wherein one of the curves S_1 is the circle $x_1^2+y_1^2=1$; in fact, writing $Y=0$, we have $x_1+iy_1=e^{i\phi(X)}$, and thence $x_1-iy_1=e^{-i\phi(X)}$, and consequently $x_1^2+y_1^2-1=0$ for one of the S_1 curves.

But similarly, if ψ be any real function, then we have $x+iy=e^{i\psi(X+iY)}$ for a set of curves S and T, wherein one of the S curves is the circle $x^2+y^2-1=0$, and thus the two equations $x_1+iy_1=e^{i\phi(X+iY)}$, $x+iy=e^{i\psi(X+iY)}$ establish a correspondence between the two circumferences $x_1^2+y_1^2-1=0$ and $x^2+y^2-1=0$. To eliminate the X, Y, we may write

$$i\psi(X+iY)=\log(x+iy),$$

or say

$$-\psi(X+iY)=i\log(x+iy);$$

then $\phi(X+iY)$ is a real function of $-\psi(X+iY)$, say it is $=f\{-\psi(X+iY)\}$, and it is thus $=f\{i\log(x+iy)\}$, or we have

$$x_1+iy_1=e^{if\{i\log(x+iy)\}}:$$

as the formula for the correspondence in question. In verification, observe that changing the sign of i, we have

$$x_1-iy_1=e^{-if\{-i\log(x-iy)\}};$$

here if $x^2+y^2-1=0$, that is, $(x+iy)(x-iy)=1$, we have

$$\log(x-iy)=-\log(x+iy),$$

and the last equation is

$$x_1-iy_1=e^{-if\{i\log(x+iy)\}},$$

so that we have $x_1^2+y_1^2-1=0$; and thus the two circumferences $x^2+y^2-1=0$ and $x_1^2+y_1^2-1=0$ correspond to each other.

9. We may write down *à priori* a formula which must be equivalent to the foregoing, viz. if $\phi(x+iy)$ be a real or imaginary function of $x+iy$, and $\bar{\phi}$ be the conjugate function (obtained by changing the sign of i in the coefficients), the formula is

$$x_1+iy_1=\frac{\phi(x+iy)}{\bar{\phi}\left(\dfrac{1}{x+iy}\right)};$$

in fact, here changing the sign of i, we have

$$x_1-iy_1=\frac{\bar{\phi}(x-iy)}{\phi\left(\dfrac{1}{x-iy}\right)},$$

which if $x^2+y^2-1=0$, that is, $(x+iy)(x-iy)=1$, becomes

$$x_1-iy_1=\frac{\bar{\phi}\left(\dfrac{1}{x+iy}\right)}{\phi(x+iy)},$$

and the two equations give $x_1^2+y_1^2-1=0$; the two circumferences thus correspond to each other. It is to be noticed that, if the numerator function contain a factor $(x+iy)^m$, say if $\phi(x+iy)=(x+iy)^m\psi(x+iy)$, then the denominator will be

$$(x+iy)^{-m}\bar{\psi}\left(\frac{1}{x+iy}\right),$$

and the formula thus becomes

$$x_1+iy_1=\frac{(x+iy)^{2m}\psi(x+iy)}{\bar{\psi}\left(\frac{1}{x+iy}\right)},$$

which is thus not really more general than the original form. I recur further on to this transformation between the two circumferences.

10. The curve S may be taken to be a closed curve enclosing a singly connected area F (say such a curve is a Contour), and the curve S' a like curve lying wholly inside S, and such that the normal distance between the two curves is everywhere infinitesimal. It is not in general the case, but it may happen that the successive curves S'', S''', &c. will be all of them like curves (each enclosed in the preceding and enclosing the succeeding one), of continually diminishing area, and continually approximating to a point: the curves T will then all of them pass through this point and may be called Radials. We may say that the area F is then squarewise contractible, or contracted, into a point.

11. In particular, a circle is squarewise contractible into a point, viz. if the point be the centre, then the curves S are concentric circles, and the curves T are radii; as is known and will appear, the circle is also contractible into any interior eccentric point whatever. It may be remarked that (the squares being infinitesimal) the numbers of the contours and radials are each of them infinite, but further the number of contours is infinitely great in comparison with that of the radials. Thus in the case just referred to, if to construct a figure we imagine the circumference of the circle divided into any large number n of equal parts, the radials will be the n radii drawn to the points of division: taking the radius to be $=1$, and writing for shortness $\alpha=\frac{2\pi}{n}$, the sides of the successive squares along any radius will be α, $\alpha(1-\alpha)$, $\alpha(1-\alpha)^2$, ..., and we require an infinite number of these to make up the entire radius; $1=\alpha\{1+(1-\alpha)+(1-\alpha)^2+\ldots\}$; viz. the number of radials being any large number n whatever, the number of contours will be actually infinite.

12. We may compare the circle as thus contracted into its centre, or rather the quadrant of a circle, with an infinite strip of finite breadth divided into squares by two sets of parallel orthotomic lines: here if x_1, y_1 refer to the positive quadrant of the circle $x_1^2+y_1^2=1$, and x, y to the infinite strip $x=0$ to $-\infty$, $y=0$ to $\frac{1}{2}\pi$, we have $x+iy=\log(x_1+iy_1)$, that is,

$$x=\log\sqrt{(x_1^2+y_1^2)},\quad y=\tan^{-1}\frac{y_1}{x_1};$$

the successive concentric quadrants correspond to equal lines parallel to the axis of y, and the successive radii to infinite lines parallel to the axis of x.

13. Considering the contour S as given, then for a consecutive interior contour S' assumed at pleasure, the area F does not contract into a point: but we have the theorem that the consecutive contour S' can be found, and that in one way only, such that the area F shall contract into a given interior point. To do this is, in effect, to make the given area to correspond say to the circle $x_1^2+y_1^2-1=0$, the contour S and the successive interior contours to the circumference of the circle and to the circumferences of the successive interior concentric circles, and finally the given interior point of F to the centre $x_1=0$, $y_1=0$ of the circle: and thus the theorem is identical with Riemann's theorem "It is possible and that in one way only to make a given singly connected area F correspond to a circle, in such wise that a given interior point of F shall correspond to the centre of the circle, and that a given boundary point of F shall correspond to a given point on the circumference of the circle." The last clause as to the given boundary point of F is necessary in order to make the correspondence a completely definite one, for without the clause it would obviously be allowable to give to the circle an arbitrary rotation about its centre.

14. The proof depends on the following lemma*.

For the given area F, it is possible to find, and that in one way only, a real function ξ of the coordinates (x, y) satisfying the following conditions:

(1) ξ is throughout the area finite and continuous, except only that in the neighbourhood of a given point thereof, taken to be the point $x=0$, $y=0$, it is $=\log\sqrt{(x^2+y^2)}$.

(2) At the boundary of the area ξ is $=0$.

(3) Throughout the area ξ satisfies the partial differential equation

$$\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}=0.$$

In fact, if ξ be thus determined, it follows from (3) that we have

$$-\frac{d\xi}{dy}\,dx+\frac{d\xi}{dx}\,dy$$

an exact differential, whence putting this $=d\eta$, or determining η by the quadrature

$$\eta=\int\left(-\frac{d\xi}{dy}\,dx+\frac{d\xi}{dx}\,dy\right),$$

we have

$$\frac{d\eta}{dx}=-\frac{d\xi}{dy},\quad \frac{d\eta}{dy}=\frac{d\xi}{dx},$$

and thence $\xi+i\eta=$ a function of $x+iy$.

* This proof is taken from the paper, Christoffel, "Sul problema delle temperature stazionarie et la rappresentazione di una data superficie," *Annali di Matem.*, t. I. (1867), pp. 89—103.

Writing then $x_1 + iy_1 = e^{\xi + i\eta} = \psi(x + iy)$ suppose, we have an orthomorphosis of the area into the circle $x_1^2 + y_1^2 - 1 = 0$, viz. for any point (x, y) of the boundary $\xi = 0$, and consequently $x_1 + iy_1 = e^{i\eta}$, thence $x_1 - iy_1 = e^{-i\eta}$, and therefore $x_1^2 + y_1^2 - 1 = 0$, or the point corresponds to a point on the circumference of the circle. Also, for a point x, y in the neighbourhood of $x = 0$, $y = 0$, we have

$$x_1 + iy_1 = e^{\log \sqrt{(x^2+y^2)} + i\eta}, \quad = \sqrt{(x^2 + y^2)}\, e^{i\eta},$$

that is, for $x = 0$, $y = 0$ we have $x_1 = 0$, $y_1 = 0$, or the given point of F corresponds to the centre of the circle.

Some further considerations as to the continuity of η would be requisite in order to show that to the series of circles $x_1^2 + y_1^2 = c^2$, as c continuously increases from 0 to 1, there correspond closed curves surrounding the assumed origin and each other, and passing continuously to the boundary of F, and thus to complete the proof that we thus obtain an orthomorphosis of F into the circle $x_1^2 + y_1^2 = 1$; but I abstain from a further discussion of the question.

15. Reverting to the lemma, this in the first place asserts the existence of a function ξ satisfying the prescribed conditions, and next that the function is completely determinate, or say that there is but one such function. As to the latter point, suppose that there is a second function ξ_1 satisfying the same conditions, then throughout the area $\xi - \xi_1$ is finite and continuous; by general principles in the theory of functions, this implies that the function has throughout the area a constant value, and since this value at the boundary is $= 0$, it must be always $= 0$, that is, $\xi - \xi_1 = 0$, or $\xi_1 = \xi$. As to the former point, it is to be remarked that we obtain $\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} = 0$ as the general condition for a minimum value of the double integral $\iint \left\{ \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 \right\} dx\, dy$; if then we assume that there exists a function $\xi = 0$ at the boundary of the area, and finite and continuous throughout the area, except only that in the neighbourhood of a given point thereof, say $x = 0$, $y = 0$, it becomes $= \log \sqrt{(x^2 + y^2)}$, and that for all the values which satisfy these conditions it is such that the above-mentioned double integral taken over the given area shall be a minimum, it follows that there exists a function ξ satisfying the conditions of the lemma.

16. As a simple illustration, suppose that the area F is that of the circle $x^2 + y^2 = 1$, and that the excepted point is the point $x = 0$, $y = 0$, the centre of the circle. We have here a function ξ satisfying the conditions of the lemma, viz. this is $\xi = \log \sqrt{(x^2 + y^2)}$; the resulting value of η is $\eta = \tan^{-1} \frac{y}{x}$, and we have then $\xi + i\eta$ a function of $x + iy$, viz. this is $= \log(x + iy)$. Hence $e^{\xi + i\eta} = x + iy$, and the resulting orthomorphosis of the two circles is the identical one

$$x_1 + iy_1 = x + iy,$$

viz. we have

$$x_1 = x, \quad y_1 = y.$$

17. The contraction of a circle to a given eccentric point has, in fact, been exhibited in the foregoing example in No. 5, but I resume the question from a different point of view. Writing for shortness $z_1 = x_1 + iy_1$, $z = x + iy$; using also a bar for the conjugate function, $\bar{z} = x - iy$, and so in other cases $\alpha = a + ai$, $\bar{\alpha} = a - ai$, $\beta = b + bi$, &c., the general formula of (7) may be written in the form

$$z_1 = \frac{z-\alpha}{1-\bar{\alpha}z} \cdot \frac{z-\beta}{1-\bar{\beta}z} \cdots,$$

viz. we thus have a transformation between the circumferences $x^2 + y^2 - 1 = 0$, $x_1^2 + y_1^2 - 1 = 0$. In fact, repeating the verification, the change of sign of i gives

$$\bar{z}_1 = \frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} \cdot \frac{\bar{z}-\bar{\beta}}{1-\beta\bar{z}} \cdots.$$

Here the product of the α-factors is

$$\frac{z\bar{z} - (\alpha\bar{z} + \bar{\alpha}z) + \alpha\bar{\alpha}}{1 - (\alpha\bar{z} + \bar{\alpha}z) + \alpha\bar{\alpha}z\bar{z}},$$

which, if $z\bar{z} = 1$, becomes $= 1$; the same property exists as to the β-factors, &c., and hence putting $z\bar{z} = 1$, we have $z_1\bar{z}_1 = 1$, that is, the two circumferences correspond to each other. The correspondence would have subsisted if a factor $e^{i\lambda}$ had been introduced into the expression of z_1, but the effect is merely to rotate the circle $x_1^2 + y_1^2 - 1 = 0$ about its centre, and there is no real gain of generality.

18. The most simple case is

$$z_1 = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

I assume here that $\alpha_1 = a + ai$, is an interior point of the circle $x^2 + y^2 - 1 = 0$ (viz. $a^2 + a^2 < 1$); hence $z = \alpha$ gives $z_1 = 0$, viz. to the point $x = a$, $y = a$, within the circle $x^2 + y^2 - 1 = 0$, there corresponds the centre of the circle $x_1^2 + y_1^2 - 1 = 0$; and we have in this equation the theory of the squarewise contraction of the circle to the eccentric point $x = a$, $y = a$. There is no loss of generality in assuming $a = 0$; doing this the formula becomes

$$z_1 = \frac{z-a}{1-az},$$

where $a^2 < 1$, or if to fix the ideas a be taken to be positive, then $a < 1$.

19. To compare with a former investigation, writing $\xi + i\eta = \log(x_1 + iy_1)$, we have

$$\xi + i\eta = \log\frac{x - a + iy}{1 - ax - aiy} = \log\frac{\sqrt{\{(x-a)^2 + y^2\}}}{\sqrt{\{(1-ax)^2 + a^2y^2\}}} + i\tan^{-1}\left\{\frac{y}{x-a} + \frac{ay}{1-ax}\right\},$$

and consequently

$$\xi = \log\frac{\sqrt{\{(x-a)^2 + y^2\}}}{\sqrt{\{(1-ax)^2 + a^2y^2\}}},$$

which is $= 0$ at the boundary $x^2 + y^2 - 1 = 0$, and is finite and continuous throughout the area except in the neighbourhood of the point $x = a$, $y = 0$, for which it is

$= \log \sqrt{\{(x-a)^2+y^2\}}$; moreover, ξ satisfies the partial differential $\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}=0$, and starting from the given value of ξ we should obtain the foregoing value of η, and thence $x_1+iy_1=e^{\xi+i\eta}$, that is,

$$x_1+iy_1=\frac{x-a+iy}{1-ax-aiy}, \text{ or } z_1=\frac{z-a}{1-az},$$

as the equation for the orthomorphosis of the circle $x_1^2+y_1^2-1=0$ into the circle $x^2+y^2-1=0$, the centre $x_1=0$, $y_1=0$ corresponding to the eccentric point $x=a$, $y=0$.

20. In further development of the solution, writing

$$z_1\bar{z}_1=\frac{z\bar{z}-a(z+\bar{z})+a^2}{1-a(z+\bar{z})+a^2z\bar{z}},$$

that is,

$$x_1^2+y_1^2=\frac{x^2+y^2+a^2-2ax}{a^2(x^2+y^2)-2ax+1},$$

and then assuming $x_1^2+y_1^2=\lambda_1^2$, we have as the contour corresponding thereto

$$x^2+y^2+a^2-2ax=\lambda^2\{a^2(x^2+y^2)-2ax+1\},$$

that is,

$$(x^2+y^2)(1-a^2\lambda^2)-2a(1-\lambda^2)x+a^2-\lambda^2=0,$$

or say

$$x^2+y^2-\frac{2a(1-\lambda^2)}{1-a^2\lambda^2}x+\frac{a^2-\lambda^2}{1-a^2\lambda^2}=0,$$

or, what is the same thing,

$$\left\{x-\frac{a(1-\lambda^2)}{1-a^2\lambda^2}\right\}^2+y^2=\frac{\lambda^2(1-a^2)^2}{(1-a^2\lambda^2)^2}.$$

The equation may also be written

$$\left\{x-\tfrac{1}{2}\left(\frac{1}{a}+a\right)\right\}\left\{x-\tfrac{1}{2}\left(\frac{1}{a}+a\right)+\left(\frac{1}{a}-a\right)\frac{1+a^2\lambda^2}{1-a^2\lambda^2}\right\}+y^2+\tfrac{1}{4}\left(\frac{1}{a}-a\right)^2=0,$$

showing that the contour circles all pass through two imaginary points

$$x=\tfrac{1}{2}\left(\frac{1}{a}+a\right),\quad y=\pm\tfrac{1}{2}i\left(\frac{1}{a}-a\right).$$

21. Writing next

$$x_1+iy_1=\frac{(x-a+iy)(1-ax-aiy)}{(1-ax)^2+a^2y^2};$$

that is,

$$x_1=\frac{-a+x(1+a^2)-a(x^2+y^2)}{(1-ax)^2+a^2y^2},\quad y_1=\frac{y(1-a^2)}{(1-ax)^2+a^2y^2},$$

then to the radius $x_1-\theta y_1=0$ corresponds the radial

$$-a+x(1+a^2)-a(x^2+y^2)-\theta y(1-a^2)=0,$$

that is,

$$(x-a)\left(x-\frac{1}{a}\right)+y^2+\theta y\left(\frac{1}{a}-a\right)=0,$$

which is a circle passing through the two real points $y=0$, $x=a$, or $\frac{1}{a}$, being the antipoints of the before-mentioned pair of points.

22. Hence for the contraction of the circle $x^2+y^2-1=0$ to the interior point $x=a$, $y=0$, calling this point A, and taking A' the image hereof $\left(\text{coordinates } x=\frac{1}{a},\ y=0\right)$, we see that the contour circles are the circles all passing through the antipoints of A, A', and having thus for their common chord the line bisecting AA' at right angles, and that the radial circles are the circles all passing through the two points A, A'.

In what precedes, the distance AA' is $=\frac{1}{a}-a$, or say the half distance is $=\frac{1}{2}\left(\frac{1}{a}-a\right)$, but altering the scale so as to make this half distance $=1$, and moreover taking the origin at the middle point of AA', the equations of the contour circles and the radial circles assume the forms

$$x^2+y^2-2\alpha x+1=0,$$

and

$$x^2+y^2-2\beta y-1=0,$$

respectively.

23. It is better, interchanging x, y and altering the constants, to write these in the forms

$$x^2+y^2-\left(b+\frac{1}{b}\right)y+1=0,$$

$$x^2+y^2-\left(a-\frac{1}{a}\right)x-1=0,$$

viz. as in effect appearing in No. 5, these are the equations derived from $x+iy=\tan(u+iv)$ where $\tan u=a$, $\tan iv=ib$.

24. Passing now to the form

$$z_1=\frac{z-\alpha}{1-\bar{\alpha}z}\,\frac{z-\beta}{1-\bar{\beta}z},$$

to obtain the most simple instance, I write $\alpha=0$, $b=\sqrt{2}$, so that the form is

$$z_1=\frac{z(z-\sqrt{2})}{1-z\sqrt{2}},$$

we thus obtain

$$z_1\bar{z}_1=\frac{z\bar{z}\{z\bar{z}-\sqrt{2}(z+\bar{z})+2\}}{1-\sqrt{2}(z+\bar{z})+2z\bar{z}},$$

that is,

$$x_1^2+y_1^2=\frac{(x^2+y^2)\{x^2+y^2-2\sqrt{2}\,x+2\}}{2(x^2+y^2)-2x\sqrt{2}+1},$$

and thence

$$x_1^2+y_1^2-1=\frac{(x^2+y^2-1)\{x^2+y^2-2\sqrt{2}\,x+1\}}{(x\sqrt{2}-1)^2+2y^2},$$

so that the circumference $x_1^2+y_1^2-1=0$ corresponds to the two circumferences $x^2+y^2-1=0$, $x^2+y^2-2\sqrt{2}\,x+1=0$. The second of these is $(x-\sqrt{2})^2+y^2-1=0$; hence, putting $x+\frac{1}{\sqrt{2}}$ instead of x, the two circles are

$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0,\quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1=0,$$

(the former of these being the circle originally represented by $x^2+y^2-1=0$), and the last equation becomes

$$x_1^2+y_1^2-1=\frac{\left\{\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1\right\}\left\{\left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1\right\}}{2(x^2+y^2)}.$$

25. Writing here $x_1^2+y_1^2-c^2=0$, we have

$$\{x^2+y^2-\tfrac{1}{2}+x\sqrt{2}\}\{x^2+y^2-\tfrac{1}{2}-x\sqrt{2}\}+2(x^2+y^2)(1-c^2)=0,$$

that is,

$$(x^2+y^2)^2-(2c^2+1)x^2-(2c^2-1)y^2+\tfrac{1}{4}=0,$$

a bicircular quartic which (or rather a branch whereof) is the contour corresponding to the circle $x_1^2+y_1^2-c^2=0$. For $c=1$, the curve breaks up into the two circles

$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0,\quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1=0;$$

and for $c=0$, it breaks up into

$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2=0,\quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2=0,$$

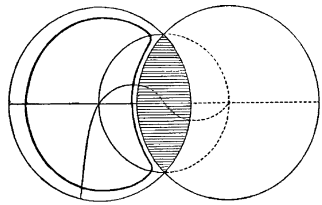
that is, into the two points $y=0$, $x=\pm\frac{1}{\sqrt{2}}$. We have to consider the curves for values of c between these two values $c=1$ and $c=0$.

26. For any such value of c, the curve consists of two symmetrical ovals, situate within the two circles respectively: we are concerned only with the oval lying within the circle $\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0$; this is shown in the figure for a value of c, slightly inferior to 1, viz. it is an indented oval close approximating to the unshaded portion of the circle, say this is the contour S' consecutive to the circle which is the contour S.

As c diminishes and becomes ultimately $=0$, this oval continually diminishes,

after a time losing the indentation, and approximating more and more to the form of a circle having for its centre the point $x=-\frac{1}{\sqrt{2}}$, $y=0$, which is the centre of the circle, and ultimately reducing itself to this point.

It will be noticed that the consecutive contour S' as shown in the figure is not at an infinitesimal distance from every part of the circle which is the contour S; for a part of the circle, the consecutive contour or contour at an infinitesimal distance



is a part of the symmetrically situated oval, lying outside of the circle. And thus in the present case we do *not* obtain a contraction of the circle into its centre: what we do obtain is a contraction of the unshaded portion of the circle into the centre; this is as it should be, for the *only* contraction of the circle into its centre is by the series of concentric circles.

27. For the radials, writing as before $z+\frac{1}{\sqrt{2}}$ in place of x, that is, $z+\frac{1}{\sqrt{2}}$ in place of z, we have

$$z_1=x_1+iy_1=\frac{\left(z+\frac{1}{\sqrt{2}}\right)\left(z-\frac{1}{\sqrt{2}}\right)}{-z},\ =-z+\frac{\frac{1}{2}}{z}=-x-iy+\frac{\frac{1}{2}(x-iy)}{x^2+y^2}.$$

Hence

$$x_1=-x+\frac{\frac{1}{2}x}{x^2+y^2},\ =\frac{-x(x^2+y^2-\frac{1}{2})}{x^2+y^2},$$

$$y_1=-y-\frac{\frac{1}{2}y}{x^2+y^2},\ =\frac{-y(x^2+y^2+\frac{1}{2})}{x^2+y^2},$$

and thus to the radius $x_1-\theta y_1=0$ there corresponds the curve

$$x(x^2+y^2-\tfrac{1}{2})-\theta y(x^2+y^2+\tfrac{1}{2})=0,$$

or say

$$(x-\theta y)(x^2+y^2)-\tfrac{1}{2}(x+\theta y)=0,$$

a circular cubic. In particular, for the line $x_1=0$, we have $\theta=0$, and the curve is $x(x^2+y^2-\frac{1}{2})=0$, or say $x^2+y^2-\frac{1}{2}=0$, the half circumference of which is shown in the figure: for the line $y_1=0$, we have $\theta=\infty$, and the curve is $y(x^2+y^2+\frac{1}{2})=0$ that is, $y=0$.

The curve passes through the origin $x=0$, $y=0$, and the equation of the

tangent there is $x+\theta y=0$. Moreover it passes through the centre of the circle, $x=\frac{1}{\sqrt{2}}$, $iy=0$, and the equation of the tangent there is easily found to be

$$x+\frac{1}{\sqrt{2}}-\theta y=0.$$

We may find where the curve cuts the circle $\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0$, viz. this is $x^2+y^2=-x\sqrt{2}+\frac{1}{2}$, and substituting in the equation of the curve, we find

$$x^2-\theta xy+\frac{\theta}{\sqrt{2}}y=0,$$

or say

$$\left(x-\frac{1}{\sqrt{2}}\right)\left(x-\theta y+\frac{1}{\sqrt{2}}\right)+\tfrac{1}{2}=0,$$

a hyperbola which by its intersections with the circle determines the points in question. There are two real intersections, but of these only one is in the arc bounding the unshaded portion of the circle: the other intersection serves to determine by symmetry the intersection of the circular cubic with the other boundary of the unshaded portion: see the figure which exhibits the path of the circular cubic through the unshaded portion, and into and beyond the shaded portion or lens. I remark that the two systems of curves are considered by Meyer and shown in his Plate XIII.

28. The formula

$$z_1=\frac{z-\alpha}{1-\alpha z}$$

has been considered for the case of an interior point α; the case of an exterior point α might be considered in like manner, but it is obvious that we pass from one to the other, by a transformation $\frac{1}{z_1}$ for z_1; and it thus easily appears that, if α be an exterior point, we obtain a transformation between the area of the circle $x_1^2+y_1^2-1=0$ and the infinite area exterior to the circle $x^2+y^2-1=0$.

Similarly, in the formula

$$z_1=\frac{z-\alpha}{1-\bar{\alpha}z}\,\frac{z-\beta}{1-\bar{\beta}z},$$

the case considered ($\alpha=a=0$, $\beta=b=\sqrt{2}$) has been that of an interior point α, and an exterior point β. I omit the case of two exterior points, since this can be by the transformation $\frac{1}{z_1}$ for z_1 reduced to that of two interior points, but it is proper to consider this last case.

29. Considering then the case

$$z_1=\frac{z-\alpha\,.\,z-\beta}{1-\bar{\alpha}z\,.\,1-\bar{\beta}z},$$

where α, β are interior points, the most simple case is when $\alpha = a$, $\beta = -a$, positive and < 1. Here

$$z_1 = \frac{z^2 - a^2}{1 - a^2 z^2},$$

and we have

$$x_1^2 + y_1^2 = \frac{(x^2+y^2)^2 - 2a^2(x^2-y^2) + a^4}{1 - 2a^2(x^2-y^2) + a^4(x^2+y^2)^2},$$

or say

$$x_1^2 + y_1^2 - 1 = \frac{(1-a^4)\{(x^2+y^2)^2 - 1\}}{1 - 2a^2(x^2-y^2) + a^4(x^2+y^2)^2},$$

which last form shows that to the circumference $x_1^2 + y_1^2 - 1 = 0$, there corresponds the circumference $x^2 + y^2 - 1 = 0$ (and besides this, only the imaginary curve $x^2 + y^2 + 1 = 0$). Writing $x_1^2 + y_1^2 = c^2$, we have the contour

$$(c^2a^4 - 1)(x^2+y^2)^2 - 2a^2(c^2-1)(x^2-y^2) + c^2 - a^4 = 0,$$

a bicircular quartic. For $c = 1$, we have as already mentioned the circle $x^2 + y^2 - 1 = 0$; for c less than 1, a curve lying within this circle, diminishing with c, and after a time acquiring on each side of the axis of x an indentation or assuming an hour-glass form; for the value $c^2 = a^4$, the equation becomes

$$(a^4 + 1)(x^2+y^2)^2 - 2a^2(x^2-y^2) = 0,$$

and the curve is a figure of eight, the two loops enclosing the points $y = 0$, $x = a$ and $x = -a$ respectively; and as c^2 diminishes to zero, the curve consists of two detached ovals lying within the two loops of the figure of eight, and ultimately reducing themselves to the two points respectively. There is no difficulty in finding the equation of the curves corresponding to a radius $x_1 - \theta y_1 = 0$, but the configuration of these curves is at once seen from that of the former system. We may in the present case say that the circle is squarewise contracted to the figure of eight; and then further that each loop of the figure of eight is squarewise contracted to a point; but we do *not* have a squarewise contraction of the circle to a point.

30. A closed curve or contour may be squarewise contracted not into a point but into a finite line: we see this in the case of a system of confocal ellipses, which gives the contraction of an ellipse into the thin ellipse which is the finite line joining the two foci. There is a like contraction of the circle; this is, in fact, given by the formula due to Schwarz, "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120 (see p. 115) for the orthomorphosis of the ellipse into a circle: this is

$$x_1 + iy_1 = \operatorname{sn}\left\{\frac{2K}{\pi}\sin^{-1}(x+iy)\right\},$$

where, if $a^2 - b^2 = 1$ and $\alpha = \cos\frac{i\pi K'}{4K}$, or, what is the same thing, $ib = \sin\frac{i\pi K'}{4K}$, then the ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2-1} = 1$ is transformed into the circle $x_1^2 + y_1^2 - \frac{1}{K} = 0$. The contours and trajectories for the ellipse are the confocal ellipses and hyperbolas respectively;

and for the circle, they are two sets of bicircular quartics, such that the portions within the circle have a configuration resembling that of the confocal ellipses and hyperbolas within the ellipse. To investigate the formulæ, it is convenient to introduce the function

$$X + iY, \quad = \sin^{-1}(x + iy);$$

we then have

$$x + iy = \sin(X + iY),$$

or say

$$x = \sin X \cos iY, \quad iy = \cos X \sin iY,$$

so that to any given value of Y there corresponds the ellipse

$$\frac{x^2}{\cos^2 iY} + \frac{y^2}{-\sin^2 iY} = 1,$$

and then

$$x_1 + iy_1 = \operatorname{sn} \frac{2K}{\pi}(X + iY);$$

or if

$$s = \operatorname{sn} \frac{2KX}{\pi}, \qquad is_1 = \operatorname{sn} \frac{2KiY}{\pi},$$

$$c = \operatorname{cn} \frac{2KX}{\pi} = \sqrt{(1 - s^2)}, \qquad c_1 = \operatorname{cn} \frac{2KiY}{\pi} = \sqrt{(1 + s_1^2)},$$

$$d = \operatorname{dn} \frac{2KX}{\pi} = \sqrt{(1 - k^2 s^2)}, \quad d_1 = \operatorname{dn} \frac{2KiY}{\pi} = \sqrt{(1 + k^2 s_1^2)},$$

then

$$x_1 = \frac{sc_1d_1}{1 + k^2s^2s_1^2}, \quad y_1 = \frac{s_1cd}{1 + k^2s^2s_1^2},$$

giving

$$x_1^2 + y_1^2 = \frac{s^2 + s_1^2}{1 + k^2s^2s_1^2},$$

and therefore $x_1^2 + y_1^2 - \frac{1}{k} = 0$ if $1 - ks_1^2 = 0$, that is, if $s_1 = \frac{1}{\sqrt{k}}$, or since

$$is_1 = \operatorname{sn} \frac{2KiY}{\pi}, \quad \text{if} \quad \frac{2KiY}{\pi} = \tfrac{1}{2}iK', \quad \text{or} \quad Y = \frac{\pi K'}{4K};$$

hence defining a as above, it appears that the elliptic periphery $\frac{x^2}{a^2} + \frac{y^2}{a^2 - 1} = 1$ corresponds to the circumference $x_1^2 + y_1^2 - \frac{1}{k} = 0$. By the introduction of $X + iY$ as above, the circle and the ellipse are each compared with a rectangle; the reduction of the circle to the rectangle, as given by the foregoing equation $x_1 + iy_1 = \operatorname{sn} \frac{2K}{\pi}(X + iY)$, or what is substantially the same thing by an equation $x_1 + iy_1 = \operatorname{sn}(X + iY)$, is more fully discussed in my paper, Cayley: "On the Binodal Quartic and the graphical representation of the Elliptic Functions," *Camb. Phil. Trans.*, t. XIV. (1889), pp. 484—494, [891], and in a paper "On some problems of orthomorphosis," *Crelle*, t. CVII. (1891), pp. 262—277, [921].

31. The whole theory, and in particular Riemann's theorem before referred to,

are intimately connected with Cauchy's theorem, "If a function $f(z)$ is holomorphic over a simply connected plane area, and if t denote any point within the area, then

$$f(t) = \frac{1}{2\pi i} \int \frac{f(z)}{z-t} dt,$$

where z denotes $x+iy$, and the integral is taken in the positive sense along the boundary of the area." See Briot and Bouquet, *Théorie des fonctions elliptiques* (Paris, 1875), p. 136.

Here in order to obtain by means of the theorem the value of the function $f(z)$ for a given point $t(=a+ib)$ within the area, we require to know the values of $f(z)$ for the several points of the boundary: viz. if z refers to a point P on the boundary, and if we represent the value $f(z)$ by a point P_1 in a second figure, then these points P_1 form a closed curve or boundary in this second figure, and we require to know not only the form of this boundary, but also the several points P_1 thereof which correspond to the several points P of the first-mentioned boundary, or say we require to know the correspondence of the two boundaries: this being known, we have by the theorem the value of $f(t)$, that is, the point a_1+ib_1 within the second area, which corresponds to the point $t=a+ib$ within the first area. The (1, 1) correspondence of the two areas is of course implied in the assertion that $f(t)$ has a determinate value, determined by means of the given values of $f(z)$ along the boundary.

921.

ON SOME PROBLEMS OF ORTHOMORPHOSIS.

[From *Crelle's Journal der Mathem.*, t. CVII. (1891), pp. 262—277.]

IN the interesting Memoir, Schwarz, "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120, the author considers the orthomorphic transformation (or, as I call it, the orthomorphosis) of a square into the infinite half-plane, or into a circle, and of a rectangle into the infinite half-plane. It is of course easy to deduce the orthomorphosis of the rectangle into a circle; and then, by giving a proper value to the modulus of the elliptic function involved in the formula, we obtain the orthomorphosis of the square into a circle, this solution (although equivalent thereto) being under a different form from that previously given for the square. But as appears from my paper "On the Binodal Quartic and the Graphical Representation of the Elliptic Functions," *Camb. Phil. Trans.* vol. XIV. (1889), pp. 484—494, [891], there is for the rectangle (and consequently also for the square) an orthomorphosis wherein the boundary of the rectangle or square corresponds, not to the circumference, but to the circumference together with two twice-repeated portions of a diameter. I propose to consider in I. these several transformations: II. relates to the orthomorphosis of a circle into a circle.

I.

1. We are concerned with the elliptic function sn for which $K' = 2K$, and also with the elliptic function snl of the lemniscate form. The modulus in the former case is

$$k = \frac{\sqrt{2}-1}{\sqrt{2}+1}, = (\sqrt{2}-1)^2, = 3 - 2\sqrt{2}, \text{ or say } \sqrt{k} = \sqrt{2}-1, \ \frac{1}{\sqrt{k}} = \sqrt{2}+1.$$

For the lemniscate function snl, the modulus is $= i$, and if (with Gauss) we write $\frac{1}{2}\varpi$ for the value of the complete function K or F_1, then we have $\frac{1}{4}\varpi = K(\sqrt{2}-1)$,

$= K\sqrt{k}$, where k has the foregoing special value: I notice the numerical values $\frac{1}{2}\varpi = 1{\cdot}311028$, $k = 3 - 2\sqrt{2} = \sin 9^\circ\, 52'$, $k = 1{\cdot}582548$. The relation between the lemniscate function snl, and the sn with the foregoing value of k, is easily shown to be

$$\sqrt{k}\,\mathrm{sn}\,U = \frac{(i+1)\,\mathrm{snl}\,u + i\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}, \text{ where } U - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}\,u;$$

and it may be added that we have

$$\mathrm{cn}\,U = \frac{\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}\sqrt{\frac{1+k}{k}}\,\sqrt{(1+\mathrm{snl}\,u)(1-i\,\mathrm{snl}\,u)},$$

$$\mathrm{dn}\,U = \frac{\sqrt{2}}{(i-1)\,\mathrm{snl}\,u + \sqrt{2}}\sqrt{1+k}\,\sqrt{(1-\mathrm{snl}\,u)(1+i\,\mathrm{snl}\,u)}.$$

2. The Schwarzian orthomorphosis of the rectangle into the infinite half-plane is given (Memoir, p. 113) by the formula $X_1 + iY_1 = \mathrm{sn}\,(X + iY)$, where the modulus is real, positive, and less than unity. Here, see the figures 1 (XY) and 2 (X_1Y_1),

Fig. 1 (XY).

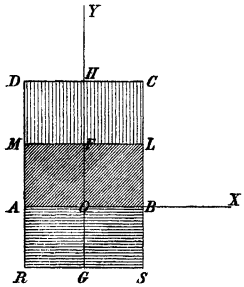
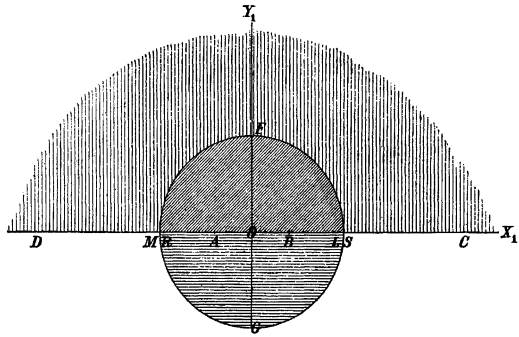


Fig. 2 (X_1Y_1).

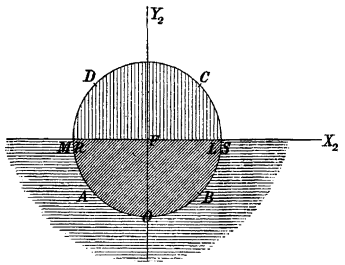


the rectangle $ABCD$, the sides of which are $AB = 2K$ and $BC = K'$, is transformed into the upper infinite half-plane $(Y_1 = +)$, the four corners of the rectangle corresponding to the points A, B, C, D on the axis of X, where $OB\,(= OA) = 1$, $OC\,(= OD) = \frac{1}{k}$.

3. We can, by a properly determined quasi-inversion (as will be explained), transform the X_1Y_1-figure into a new figure see figure 3 (X_2Y_2), the infinite X_1-axis being transformed into the circumference of a circle (the radius of which may be taken to be $= 1$) and the infinite half-plane into the area within the circle. The four points A, B, C, D are thus transformed into points on the circle, which if the quasi-inversion be a symmetrical one, will be situate, A and B symmetrically, and

also C and D symmetrically, in regard to the axis OY_2: and in the case of the before-mentioned modulus $k = 3 - 2\sqrt{2}$, for which the rectangle $ABCD$ becomes a

Fig. 3* (X_2Y_2).



square, the quasi-inversion may be so determined that the points A, B, C, D shall be situate midway (that is, at inclinations $\pm 45°$, $\pm 135°$) in the four quadrants of the circle.

4. But if in figure 1 we take $BL = \frac{1}{2}K'$ and draw FL parallel to OX, then as shown in my memoir above referred to, the foregoing transformation $X_1 + iY_1 = \mathrm{sn}(X + iY)$ changes the rectangle $OBLF$, the sides of which are $OB = K$ and $BL = \frac{1}{2}K'$, into the quadrant $OBLF$ of figure 2, $OB = 1$ (as already mentioned) and radius $OL = \frac{1}{\sqrt{k}}$. Hence completing the rectangle $RSLM$, the sides of which are $RS = 2K$ and $SL = K'$, (viz. this rectangle differs only in position from the first-mentioned rectangle $ABCD$), we have an orthomorphosis of the rectangle $RSLM$ into the circle of figure 2: but so nevertheless that to the boundary of the rectangle there corresponds not the circumference of the circle but the boundary $RGSBLFMAR$ composed of the circumference and the portions SB, BL and MA, AR (that is, BL, AM each twice) of a diameter of the circle. See *post* No. 12.

5. The Schwarzian orthomorphosis of the square into a circle is given (Memoir, pp. 111—113) by the formula $x_1 + iy_1 = \mathrm{snl}(x + iy)$; viz. here—see figures 4 (xy) and 5 (x_1y_1)—the square $ABCD$, the half-diagonals whereof are each $= \frac{1}{2}\varpi$, corresponds to the circle radius $= 1$ of figure 5, the four points A, B, C, D of the circle being the quadrantal points as shown in the figure. Figure 5 is, in fact, figure 3 turned through an angle of 45°; and it thus appears that Schwarz's lemniscate solution for the square into a circle is the quasi-inversion of his solution for the rectangle into the infinite half-plane, when by putting $k = 3 - 2\sqrt{2}$ the rectangle is made to be a square. See *post* Nos. 13 and 14.

[* The scale of this figure is double that of figures 1, 2, 4, 6, 7 in this paper.]

6. The general formula of quasi-inversion whereby the infinite X_1-axis is changed

Fig. 4 (xy).

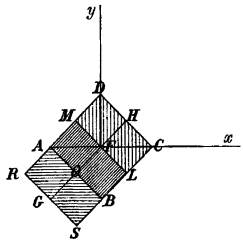
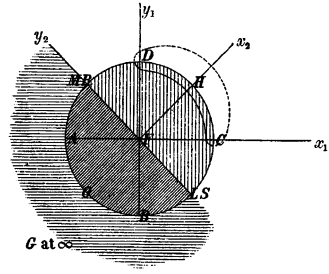


Fig. 5* (x_1y_1).



into a circumference† radius unity, is

$$X_2 + iY_2 = \frac{1 + Mi(X_1 + iY_1)}{M(X_1 + iY_1) + i},$$

where M is real. In fact, writing this equation in the form

$$X_2 + iY_2 = \frac{1 + Mi(X_1 + iY_1)}{i\{1 - Mi(X_1 + iY_1)\}},$$

we have

$$X_2 - iY_2 = \frac{1 - Mi(X_1 - iY_1)}{-i\{1 + Mi(X_1 - iY_1)\}},$$

and thence $X_2^2 + Y_2^2 - 1 = 0$, if only $Y_1 = 0$; that is, the infinite X_1-axis is transformed into the circumference $X_2^2 + Y_2^2 - 1 = 0$.

Writing $Y_1 = 0$, we have

$$X_2 + iY_2 = \frac{1 + MiX_1}{MX_1 + i} = \frac{2MX_1 + i(M^2X_1^2 - 1)}{M^2X_1^2 + 1},$$

so that to a pair of points $(\pm X_1, 0)$ there corresponds a pair of points $(\pm X_2, Y_2)$ situate symmetrically in regard to the axis OY_2.

The equation gives

$$M(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{(X_2 + iY_2) - i}.$$

Hence, writing this in the form

$$M(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{-i\{1 + i(X_2 + iY_2)\}},$$

[* The scale of this figure is double that of figures 1, 2, 4, 6, 7 in this paper.]

† I use here and elsewhere the term circumference rather than circle, to mark more clearly the distinction between the curve and the included area.

we have

$$M(X_1 - iY_1) = \frac{1 + i(X_2 - iY_2)}{i\{1 - i(X_2 - iY_2)\}},$$

and consequently $M^2(X_1^2 + Y_1^2) - 1 = 0$, if only $Y_2 = 0$; that is, the circumference $X_1^2 + Y_1^2 - \frac{1}{M^2} = 0$ is transformed into the infinite X_2-axis.

Although for the purposes of the present memoir we require the coefficient M, yet there is no real loss of generality in assuming $M = 1$, and the transformation is best studied under the form

$$X_2 + iY_2 = \frac{1 + i(X_1 + iY_1)}{X_1 + iY_1 + i};$$

see *post* No. 11.

7. As already mentioned, the coordinates (X_1, Y_1) and (X_2, Y_2) are connected by an equation of the foregoing form, and in the case of the square ($k = 3 - 2\sqrt{2}$ as before), the corresponding values for the points A, B, C, D should be

$$A, \quad X_1 + iY_1 = -1, \quad X_2 + iY_2 = \frac{-1-i}{\sqrt{2}},$$

$$B, \quad X_1 + iY_1 = 1, \quad X_2 + iY_2 = \frac{1-i}{\sqrt{2}},$$

$$C, \quad X_1 + iY_1 = \frac{1}{k}, \quad X_2 + iY_2 = \frac{1+i}{\sqrt{2}},$$

$$D, \quad X_1 + iY_1 = -\frac{1}{k}, \quad X_2 + iY_2 = \frac{-1+i}{\sqrt{2}}.$$

The proper value of M is $M = \sqrt{2} - 1, = \sqrt{k}$; the required formula thus is

$$X_2 + iY_2 = \frac{1 + i\sqrt{k}(X_1 + iY_1)}{\sqrt{k}(X_1 + iY_1) + i}$$

or say

$$\sqrt{k}(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{X_2 + iY_2 - i}.$$

Thus for the point B, we should have

$$\sqrt{k} = \frac{1 - i\frac{1-i}{\sqrt{2}}}{\frac{1-i}{\sqrt{2}} - i}, \text{ that is, } \sqrt{k} = \frac{\sqrt{2} - i - 1}{1 - i - i\sqrt{2}} = \frac{\sqrt{k} - i}{1 - \frac{i}{\sqrt{k}}},$$

which is right. And similarly for the point C, we should have

$$\frac{1}{\sqrt{k}} = \frac{1 - i\frac{1+i}{\sqrt{2}}}{\frac{1+i}{\sqrt{2}} - i}, \text{ that is, } \frac{1}{\sqrt{k}} = \frac{\sqrt{2} - i + 1}{1 + i - i\sqrt{2}} = \frac{\frac{1}{\sqrt{k}} - i}{1 - i\sqrt{k}},$$

which is right. And similarly for the points A and D.

8. We connect the square $ABCD$ of figure 1 with that of figure 4 by the equation

$$X+iY-\tfrac{1}{2}iK'=\frac{1+i}{2\sqrt{k}}(x+iy);$$

in fact, recollecting that $\tfrac{1}{2}K'=K=\dfrac{\varpi}{4\sqrt{k}}$, we have for the four points respectively

$$\begin{array}{lll} A, & X+iY=-K, & x+iy=-\tfrac{1}{2}\varpi\,, \\ B, & X+iY=\ \ K, & x+iy=-\tfrac{1}{2}\varpi i, \\ C, & X+iY=\ \ K+iK', & x+iy=\ \ \tfrac{1}{2}\varpi\,, \\ D, & X+iY=-K+iK', & x+iy=\ \ \tfrac{1}{2}\varpi i, \end{array}$$

values which satisfy the relation in question.

9. Hence writing

$$X+iY=U,\quad x+iy=u,$$

we have

$$U-\tfrac{1}{2}iK'=\frac{1+i}{2\sqrt{k}}u,$$

which is the relation in No. 1 between the arguments U, u of the elliptic functions sn, snl. We have $X_1+iY_1=\operatorname{sn}U$, $x_1+iy_1=\operatorname{snl}u$; and consequently

$$\sqrt{k}\,(X_1+iY_1)=\frac{(i+1)(x_1+iy_1)+i\sqrt{2}}{(i-1)(x_1+iy_1)+\sqrt{2}}.$$

Hence, substituting for $\sqrt{k}\,(X_1+iY_1)$ its value in terms of X_2+iY_2, we find

$$\frac{(i+1)(x_1+iy_1)+i\sqrt{2}}{(i-1)(x_1+iy_1)+\sqrt{2}}=\frac{1-i\,(X_2+iY_2)}{X_2+iY_2-i},$$

an equation which $\left(\text{multiplying on the left-hand side the numerator and the denominator each by } \dfrac{-i}{\sqrt{2}}\right)$ may be written

$$\frac{1-i\,\dfrac{1+i}{\sqrt{2}}(x_1+iy_1)}{\dfrac{1+i}{\sqrt{2}}(x_1+iy_1)-1}=\frac{1-i\,(X_2+iY_2)}{X_2+iY_2-i},$$

that is, we have

$$X_2+iY_2=\frac{1+i}{\sqrt{2}}(x_1+iy_1),$$

an equation which shows that the figure 5 is in fact the figure 3 turned through an angle of 45°. We have thus proved the conclusion stated in No. 5 as to the connexion between the lemniscate solution for the square and the solution for the rectangle.

10. It is convenient to collect here the several equations relating to the orthomorphosis of the square. We have

$$X_1 + iY_1 = \operatorname{sn}(X + iY), \quad k = 3 - 2\sqrt{2};$$

$$x_1 + iy_1 = \operatorname{snl}(x + iy);$$

$$\sqrt{k}(X_1 + iY_1) = \frac{1 - i(X_2 + iY_2)}{X_2 + iY_2 - i},$$

$$X + iY - \tfrac{1}{2}iK' = \frac{1+i}{2\sqrt{k}}(x + iy),$$

$$\sqrt{k}(X_1 + iY_1) = \frac{(i+1)(x_1 + iy_1) + i\sqrt{2}}{(i-1)(x_1 + iy_1) + \sqrt{2}},$$

$$X_2 + iY_2 = \frac{1+i}{\sqrt{2}}(x_1 + iy_1),$$

which are the equations connecting together the coordinates of the five figures.

11. I examine more in detail the above-mentioned transformation

$$X_2 + iY_2 = \frac{1 + i(X_1 + iY_1)}{X_1 + iY_1 + i};$$

see the foregoing figures 1 (XY) and 2 (X_1Y_1), in which we now regard the two circles as having each of them the radius unity; changing the sign of i, the equation gives

$$X_2 - iY_2 = \frac{1 - i(X_1 - iY_1)}{X_1 - iY_1 - i},$$

and we hence find

$$X_2^2 + Y_2^2 = \frac{X_1^2 + Y_1^2 - 2Y_1 + 1}{X_1^2 + Y_1^2 + 2Y_1 + 1};$$

consequently if $X_1^2 + Y_1^2 + 1 = 0$, then also $X_2^2 + Y_2^2 + 1 = 0$, or the transformation changes the first of these imaginary circles into the second of them: or say it changes the concentric orthotomic of the circle $X_1^2 + Y_1^2 - 1 = 0$ into the concentric orthotomic of the circle $X_2^2 + Y_2^2 - 1 = 0$.

We have moreover

$$X_2 = \frac{2X_1}{X_1^2 + Y_1^2 + 2Y_1 + 1}, \quad Y_2 = \frac{X_1^2 + Y_1^2 - 1}{X_1^2 + Y_1^2 + 2Y_1 + 1};$$

values which give the foregoing expression for $X_2^2 + Y_2^2$. But we further obtain

$$X_2^2 + Y_2^2 - 1 - \frac{2}{\mu}Y_2 = \frac{\frac{2}{\mu}(X_1^2 + Y_1^2 - 1 + 2\mu Y_1)}{X_1^2 + Y_1^2 + 2Y_1 + 1},$$

and it thus appears that the circumferences

$$X_1^2 + Y_1^2 - 1 + 2\mu Y_1 = 0, \quad X_2^2 + Y_2^2 - 1 - \frac{2}{\mu}Y_2 = 0,$$

correspond to each other. These are circles passing through the pairs of points $(Y_1=0,\ X_1=\pm 1)$, $(Y_2=0,\ X_2=\pm 1)$ respectively; or imagining them in the same figure, say they are circles which belong each to the series of circles $x^2+y^2-1+2\beta y=0$, and which moreover cut at right angles at the points $y=0$, $x=\pm 1$. But attending more carefully to the nature of the correspondence, it is to be observed that taking Y positive, and giving to X any positive value from 0 to ∞, we have in figure 2 an arc LJM lying wholly within the upper semicircle LFM; and that, corresponding hereto in figure 3, we have an arc LJM lying wholly within the lower semicircle LOM; and that as in figure 2, the arc LJM lies nearer to the semicircumference LFM or to the diameter LOM, so in figure 3 the arc LJM lies nearer to the diameter LFM or to the semicircumference LOM. Thus the upper semicircle LFM of figure 2 corresponds to the lower semicircle LOM of figure 3; but so that the semicircumference LFM of the first figure corresponds to the diameter LFM of the second figure; and the diameter LOM of the first figure to the semicircumference LOM of the second figure. And further supposing that Y_1 is still positive, but that μ has any negative value from $-\infty$ to 0, we have in figure 2 an arc in the upper half-plane lying wholly outside the semicircle; and corresponding thereto in figure 3, an arc lying wholly inside the upper semicircle LHM; that is, in figure 2 the infinite space in the upper half-plane outside the semicircle corresponds in figure 3 to the space within the upper half-circle LHM; the infinity of figure 2 corresponding to the semicircumference LHM of figure 3, and the semicircumference LFM of figure 2 to the diameter LFM of figure 3. And thus in figure 2, the upper half-plane inside and outside the semicircle corresponds in figure 3 to the lower and upper half-circles, that is, to the whole circular area $OLHM$ of figure 3.

It is to be observed, moreover, that we have

$$X_2^2+Y_2^2+1+2\lambda X_2=\frac{2(X_1^2+Y_1^2+1+2\lambda X_1)}{X_1^2+Y_1^2+2Y_1+1},$$

that is, the circles $X_1^2+Y_1^2+1+2\lambda X_1=0$ and $X_2^2+Y_2^2+1+2\lambda X_2=0$ correspond to each other. Imagining the circles as belonging to the same figure, these are one and the same circle of the series $x^2+y^2+1-2ax=0$ each passing through the pair of points $(x=0,\ y=\pm i)$ which are the antipoints of the pair $(y=0,\ x=\pm 1)$; these circles thus cut at right angles those of the series $x^2+y^2-1+2\beta y=0$. We have, by means of the two series of circles, an easy construction for the correspondence between the figures 2 and 3.

12. In explanation of No. 4, observe that, starting from the equation

$$X_1+iY_1=\operatorname{sn}(X+iY),$$

and writing $\operatorname{sn} X=P$, $\operatorname{sn} iY=iQ$, we have

$$X_1+iY_1=\frac{P\sqrt{1+Q^2}.\sqrt{1+k^2Q^2}+iQ\sqrt{1-P^2}.\sqrt{1-k^2P^2}}{1+k^2P^2Q^2},$$

that is,

$$X_1=\frac{P\sqrt{1+Q^2}.\sqrt{1+k^2Q^2}}{1+k^2P^2Q^2},\quad Y_1=\frac{Q\sqrt{1-P^2}.\sqrt{1-k^2P^2}}{1+k^2P^2Q^2},$$

and thence

$$X_1^2 + Y_1^2 = \frac{P^2 + Q^2}{1 + k^2 P^2 Q^2} = r^2 \text{ (if } X_1^2 + Y_1^2 \text{ be put } = r^2);$$

hence

$$P^2(1 - k^2 Q^2 r^2) = r^2 - Q^2, \quad Q^2(1 - k^2 P^2 r^2) = r^2 - P^2.$$

Now considering in figure 1 any line in the rectangle parallel to the axis of X, that is, taking Y constant and therefore also Q constant, and proceeding to eliminate P, we have

$$P^2 = \frac{r^2 - Q^2}{1 - k^2 Q^2 r^2}, \quad 1 - P^2 = \frac{1 + Q^2 - (1 + k^2 Q^2) r^2}{1 - k^2 Q^2 r^2},$$

$$1 - k^2 P^2 = \frac{1 + k^2 Q^2 - k^2 (1 + Q^2) r^2}{1 - k^2 Q^2 r^2}, \quad 1 + k^2 P^2 Q^2 = \frac{1 - k^2 Q^4}{1 - k^2 Q^2 r^2},$$

and consequently

$$X_1 = \frac{\sqrt{1 + Q^2} . \sqrt{1 + k^2 Q^2}}{1 - k^2 Q^4} \sqrt{r^2 - Q^2} . \sqrt{1 - k^2 Q^2 r^2},$$

$$Y_1 = \frac{Q \sqrt{1 + Q^2 - (1 + k^2 Q^2) r^2} \sqrt{1 + k^2 Q^2 - k^2 (1 + Q^2) r^2}}{1 - k^2 Q^4},$$

giving X_1, Y_1 each of them in terms of Q^2 and r^2, $= X_1^2 + Y_1^2$. The former of these equations, replacing therein r^2 by its value, gives easily

$$(X_1^2 + Y_1^2)^2 - 2AX_1^2 - 2BY_1^2 + \frac{1}{k^2} = 0,$$

where

$$2A = \frac{1 + Q^2}{1 + k^2 Q^2} + \frac{1}{k^2} \frac{1 + k^2 Q^2}{1 + Q^2}, \quad 2B = Q^2 + \frac{1}{k^2 Q^2};$$

viz. we have thus the equation of the curve, a bicircular quartic which in figure 2 corresponds to the line parallel to the axis of X_1 in figure 1.

In particular, for the line LM of figure 1 we have $Y = \frac{1}{2}K'$ and thence $iQ = \operatorname{sn} \frac{1}{2} iK' = \frac{i}{\sqrt{k}}$, that is, $Q = \frac{1}{\sqrt{k}}$, and thence $A = B = \frac{1}{k}$; the equation of the bicircular quartic is

$$(X_1^2 + Y_1^2)^2 - \frac{2}{k}(X_1^2 + Y_1^2) + \frac{1}{k^2} = 0;$$

viz. this is the circle $X_1^2 + Y_1^2 - \frac{1}{k} = 0$ twice repeated, and we have thus this circle, or rather the half-circumference LFM of figure 2, corresponding to the line LM of figure 1. More simply,

$$X_1 + iY_1 = \operatorname{sn}(X + \tfrac{1}{2} iK') = \frac{\frac{1+k}{\sqrt{k}} \operatorname{sn} X + \frac{i}{\sqrt{k}} \operatorname{cn} X \operatorname{dn} X}{1 + k^2 . \frac{1}{k} \operatorname{sn}^2 X} = \frac{(1+k) P + i \sqrt{1 - P^2} . \sqrt{1 - k^2 P^2}}{\sqrt{k}(1 + kP^2)},$$

and thence

$$X_1^2 + Y_1^2 = \frac{(1+k)^2 P^2 + 1 - (1+k^2) P^2 + k^2 P^4}{k(1+2kP^2+k^2P^4)} = \frac{1}{k},$$

that is, $X_1^2 + Y_1^2 - \frac{1}{k} = 0$ as before. It is easy to see that the points A, O, B of figure 1 correspond to the points A, O, B of figure 2, and hence that the area of the rectangle $AOBLFMA$ of figure 1 corresponds to that of the semicircle $AOBLFMA$ of figure 2.

Returning to the equation $X_1 + iY_1 = \operatorname{sn}(X + iY)$, if we write herein successively

$$Y_1 = \tfrac{1}{2}K' - \beta, \quad \operatorname{sn} iY_1 = iQ_1 = \operatorname{sn} i(\tfrac{1}{2}K' - \beta),$$

and

$$Y_1 = \tfrac{1}{2}K' + \beta, \quad \operatorname{sn} iY_1 = iQ_2 = \operatorname{sn} i(\tfrac{1}{2}K' + \beta),$$

then we have

$$iQ_1 . iQ_2 = \operatorname{sn} i(\tfrac{1}{2}K' - \beta) \operatorname{sn} i(\tfrac{1}{2}K' + \beta) = -\frac{1}{k},$$

that is, $Q_1 Q_2 = \frac{1}{k}$: hence for q writing Q_1 or Q_2, we have in each case the same values of A and B, that is, we have the same bicircular quartic for two lines parallel to and equidistant from the line LM, but to one of these (viz. the line between LM and BA) there corresponds the half-perimeter lying within the semicircumference LFM, and to the other of them (viz. the line between LM and CD) there corresponds the half-perimeter lying without the semicircumference LFM.

It may be shown in like manner that, to any line in figure 1 parallel to the axis OY, there corresponds in figure 2 a bicircular quartic of the like form

$$(X_1^2 + Y_1^2)^2 - 2AX_1^2 - 2BY_1^2 + \frac{1}{k^2} = 0.$$

13. Similarly in explanation of No. 5, observe that, starting from the equation $x_1 + iy_1 = \operatorname{snl}(x + iy)$ and writing $\operatorname{snl} x = p$, $\operatorname{snl} iy = i \operatorname{snl} y = iq$, we have

$$x_1 + iy_1 = \frac{p\sqrt{1-q^4} + iq\sqrt{1-p^4}}{1-p^2q^2},$$

that is,

$$x_1 = \frac{p\sqrt{1-q^4}}{1-p^2q^2}, \quad y_1 = \frac{q\sqrt{1-p^4}}{1-p^2q^2},$$

and thence

$$x_1^2 + y_1^2 = \frac{p^2+q^2}{1-p^2q^2}:$$

writing $x + y = \frac{1}{2}\varpi$, we have

$$\operatorname{sn} y = \frac{\operatorname{cn} x}{\operatorname{dn} x}, \text{ that is, } q = \frac{\sqrt{1-p^2}}{\sqrt{1+p^2}}, \text{ or } q^2 = \frac{1-p^2}{1+p^2}, \text{ that is, } \frac{p^2+q^2}{1-p^2q^2} = 1;$$

and thus to the line $x + y = \frac{1}{2}\varpi$, there corresponds the circle $x_1^2 + y_1^2 - 1 = 0$. More precisely, to the line CD of figure 4, there corresponds the quarter-circumference CD

of figure 5: and similarly to DA, AB and BC of figure 4 the remaining quarter-circumferences DA, AB, BC of figure 5; that is, to the whole boundary of the square in figure 4 there corresponds the whole circumference of the circle in figure 5.

14. To any line in figure 4, parallel to the axis Ox or to the axis Oy, there corresponds in figure 5 a bicircular quartic of the form $x_1^2+y_1^2-2Ax_1^2-2By_1^2-1=0$. The investigation is substantially the same as that contained in No. 12, and need not be here given. But it is remarkable that also, to any line of figure 4 parallel to a side of the square (that is, to any line $x \pm y = c$), there corresponds in figure 5 a bicircular quartic of the like form (for the sides of the square, or lines $x \pm y = \pm \frac{1}{2}\varpi$ of figure 4, this bicircular quartic becomes the twice repeated circle $x_1^2+y_1^2-1=0$ of figure 5, which is the result just obtained in No. 13). I investigate this result as follows. Writing, as in No. 13,

$$\operatorname{snl} x = p, \quad \operatorname{snl} iy = i \operatorname{snl} y = iq,$$

we have, as above,

$$x_1 = \frac{p\sqrt{1-q^4}}{1-p^2q^2}, \quad y_1 = \frac{q\sqrt{1-p^4}}{1-p^2q^2}, \text{ and thence } x_1^2+y_1^2 = \frac{p^2+q^2}{1-p^2q^2}.$$

Now assuming between x and y the relation $x+y=C$, and writing $\operatorname{snl} C = c$, this gives

$$c = \frac{p\sqrt{1-q^4}+q\sqrt{1-p^4}}{1+p^2q^2};$$

to obtain the required curve, we must between these equations (three independent equations) eliminate p and q. We have

$$x_1+y_1 = \frac{p\sqrt{1-q^4}+q\sqrt{1-p^4}}{1-p^2q^2},$$

and consequently

$$(1+p^2q^2)\,c = (1-p^2q^2)(x_1+y_1),$$

or, writing for convenience $\Omega = 1-p^2q^2$, this equation gives

$$\Omega = \frac{2c}{x_1+y_1+c}.$$

Hence $\Omega x_1 = p\sqrt{1-q^4}$, $\Omega y_1 = q\sqrt{1-p^4}$; these equations may be written

$$\begin{aligned} \Omega^2 x_1^2 &= \qquad\quad p^2 - (1-\Omega)\,q^2, \\ \Omega^2 y_1^2 &= -(1-\Omega)\,p^2 + \qquad q^2, \end{aligned}$$

and from these equations obtaining the expressions for p^2 and q^2, and thence the expression for p^2q^2, $=1-\Omega$, we find, after some easy reductions,

$$\{(x_1^2+y_1^2)^2-1\} + \frac{\Omega^2}{1-\Omega}\,x_1^2y_1^2 = \frac{4(1-\Omega)}{\Omega^2}.$$

But we have

$$\frac{\Omega^2}{1-\Omega} = \frac{4c^2}{(x_1+y_1)^2-c^2},$$

or substituting this value, the equation becomes

$$(x_1^2+y_1^2)^2-1+\frac{4c^2x_1^2y_1^2}{(x_1+y_1)^2-c^2} = \frac{(x_1+y_1)^2}{c^2}-1,$$

that is,

$$\left\{(x_1^2+y_1^2)^2-\frac{(x_1+y_1)^2}{c^2}\right\}\{(x_1+y_1)^2-c^2\}+4c^2x_1^2y_1^2=0.$$

Writing for a moment $x_1^2+y_1^2=P$, $x_1y_1=Q$, this is

$$\left\{P^2-\frac{1}{c^2}(P+2Q)\right\}\{P+2Q-c^2\}+4c^2Q^2=0,$$

an equation which contains the factor $P+2Q$; throwing this out, the equation becomes, after an easy reduction,

$$(P-1)^2+(1-c^2)\left\{P-2Q-\frac{1}{c^2}(P+2Q)\right\}=0,$$

that is,

$$(x_1^2+y_1^2-1)^2+(1-c^2)\left\{(x_1-y_1)^2-\frac{1}{c^2}(x_1+y_1)^2\right\}=0,$$

the required equation. Transforming through an angle of 45° by writing

$$x_1=\frac{x_2+y_2}{\sqrt{2}},\quad y_1=\frac{x_2-y_2}{\sqrt{2}},$$

(where observe that the axis of x_2 is the line FH of figure 5), the equation becomes

$$(x_2^2+y_2^2-1)^2+(1-c^2)\left(2y_2^2-\frac{1}{c^2}2x_2^2\right)=0,$$

or writing $c=\cos\gamma$ and therefore $1-c^2=\sin^2\gamma$, this equation becomes

$$(x_2^2+y_2^2)^2-\frac{2}{\cos^2\gamma}x_2^2-2\cos^2\gamma\,.\,y_2^2+1=0,$$

a curve consisting of two indented ovals situate symmetrically in regard to the axis Fy_2 of figure 5. In fact, writing in the equation $y_2=0$, we have for x_2^2 two real positive values; but, writing $x_2=0$, we have for y_2^2 two imaginary values. For $\gamma=0$, the equation becomes

$$(x_2^2+y_2^2)^2-2(x_2^2+y_2^2)+1=0,$$

that is, we have the circle $x_2^2+y_2^2-1=0$ twice repeated. One of the ovals is shown in figure 5; the portion of it lying within the circle agrees with Schwarz's figure, p. 113, turning this round through an angle of 45°.

For the lines $x-y=C$ of figure 4, we have in figure 5 the same system of bicircular quartics turned round through an angle of 90°.

II.

15. I consider the general problem of the orthomorphosis of a circle into a circle: we can, for the transformation of the circumference of the circle $x^2+y^2-1=0$ into that of the circle $x_1^2+y_1^2-1=0$, find a formula involving an arbitrary function

or (what is the same thing) an indefinite number of arbitrary constants. In fact, writing for shortness $z = x + iy$, $z_1 = x_1 + iy_1$, and $\bar{z}$, $\bar{z}_1$ for the conjugate functions $x - iy$, $x_1 - iy_1$; also $\phi(z)$ for a function of z involving in general imaginary coefficients $a + ib$, &c., and $\bar{\phi}(z)$ for the like function with the conjugate coefficients $a - ib$, &c.; then if we assume

$$z_1 = \frac{\phi(z)}{z^m \bar{\phi}\left(\frac{1}{z}\right)},$$

where m is any positive or negative integer, this implies

$$\bar{z}_1 = \frac{\phi(\bar{z})}{\bar{z}^m \phi\left(\frac{1}{\bar{z}}\right)};$$

consequently, if $x^2 + y^2 - 1 = 0$, that is, $z\bar{z} = 1$, or $\bar{z} = \frac{1}{z}$, we have

$$\bar{z}_1 = \frac{\bar{\phi}\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)^m \phi(z)} = \frac{z^m \bar{\phi}\left(\frac{1}{z}\right)}{\phi(z)}, \quad = \frac{1}{z_1},$$

or $z_1 \bar{z}_1 = 1$, that is, $x_1^2 + y_1^2 - 1 = 0$.

In a slightly different form, taking α, β, &c., to denote any imaginary quantities, and $\bar{\alpha}$, $\bar{\beta}$, ... the conjugate quantities; assuming

$$\phi(z) = (z - \alpha)(z - \beta)\ldots,$$

and taking m for the number of factors, we have

$$z_1 = \frac{(z - \alpha)(z - \beta)\ldots}{(1 - \bar{\alpha}z)(1 - \bar{\beta}z)\ldots},$$

and then (repeating the demonstration) we have

$$\bar{z}_1 = \frac{(\bar{z} - \bar{\alpha})(\bar{z} - \bar{\beta})\ldots}{(1 - \alpha\bar{z})(1 - \beta\bar{z})\ldots},$$

which, writing therein $\bar{z} = \frac{1}{z}$, becomes

$$\bar{z}_1 = \frac{\left(\frac{1}{z} - \bar{\alpha}\right)\left(\frac{1}{z} - \bar{\beta}\right)\ldots}{\left(1 - \frac{\alpha}{z}\right)\left(1 - \frac{\beta}{z}\right)\ldots}, \quad = \frac{(1 - \bar{\alpha}z)(1 - \bar{\beta}z)\ldots}{(z - \alpha)(z - \beta)\ldots} = \frac{1}{z},$$

and consequently, if $\bar{z} = \frac{1}{z}$, then also $\bar{z}_1 = \frac{1}{z_1}$ as before.

We may in the expression for z_1 introduce a factor $\frac{A}{\bar{A}}$, or, what is the same thing, a factor A which is such that $A\bar{A} = 1$. In particular, we thus have the solution

$$z_1 = \frac{A(z - \alpha)}{(1 - \bar{\alpha}z)},$$

giving

$$\bar{z}_1 = \frac{\bar{A}(\bar{z} - \bar{\alpha})}{1 - \alpha\bar{z}},$$

which, for $z\bar{z} = 1$, becomes

$$\bar{z}_1 = \frac{\bar{A}(1 - \bar{\alpha}z)}{z - \alpha},$$

so that ($A\bar{A}$ being $= 1$) this gives $z_1\bar{z}_1 = 1$.

16. This is a solution with three arbitrary constants, viz. A (which may be put $= \cos\lambda + i\sin\lambda$) is a single arbitrary constant, and α, $= a + ib$, is two arbitrary constants; and these constants may be so determined that a given point in the interior of the one circle, and a given point on the circumference thereof, shall correspond respectively to a given point in the interior of the other circle and to a given point on the circumference thereof. According to a well-known theorem of Riemann's, any two simply connected areas included within given closed curves respectively may be made to correspond to each other, and that in one way only, under the foregoing conditions as to a pair of interior points and a pair of boundary points: and we have, in what just precedes, the solution of the problem in the case of two equal circles.

17. In the case of any other solution, we thus know that the correspondence between the two circumferences cannot and does not imply a (1, 1) correspondence between the areas of the two circles: but it is interesting to enquire what happens. I take a very particular case

$$z_1 = \frac{z(z-2)}{1-2z},$$

and therefore

$$\bar{z}_1 = \frac{\bar{z}(\bar{z}-2)}{1-2\bar{z}},$$

and consequently

$$z_1\bar{z}_1 = \frac{z\bar{z}\{z\bar{z} - 2(z+\bar{z}) + 4\}}{1 - 2(z+\bar{z}) + 4z\bar{z}},$$

that is,

$$x_1^2 + y_1^2 = \frac{(x^2+y^2)(x^2+y^2-4x+4)}{4(x^2+y^2)-4x+1},$$

so that, writing $x^2+y^2-1=0$, we have $x_1^2+y_1^2-1=0$, a correspondence of the two circumferences. But to the circumference $x_1^2+y_1^2-1=0$ there corresponds not only the circumference $x^2+y^2-1=0$, but another circumference. In fact, writing $x_1^2+y_1^2=1$, we have

$$4(x^2+y^2)-4x+1 = (x^2+y^2)(x^2+y^2-4x+4),$$

that is,

$$(x^2+y^2)^2 - 4x(x^2+y^2) + 4x - 1 = 0,$$

or

$$(x^2+y^2-1)(x^2+y^2-4x+1) = 0,$$

and there is thus the other circle

$$x^2+y^2-4x+1=0, \text{ or say } (x-2)^2+y^2-3=0,$$

viz. this is a circle, coordinates of centre (2, 0) and radius $=\sqrt{3}$, cutting the circle $x^2+y^2-1=0$ in two real points. Referring to the figures 6 (xy) and 7 (x_1y_1), and observing that $x_1=0$, $y_1=0$, that is, $z_1=0$, gives $z=0$, or $z=2$, that is, the points

Fig. 6 (xy).

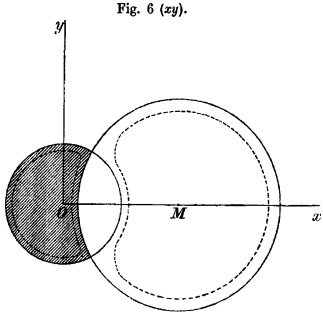
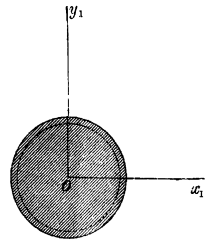


Fig. 7 (x_1y_1).



$x=0$, $y=0$, and $x=2$, $y=0$, we see that to the centre O in figure 7, there correspond in figure 6 the points O, M which are the centres of the two circles. To any small closed curve, or say any small circle surrounding the point O of figure 7, there correspond in figure 6 small closed curves surrounding the points O, M respectively; and if in figure 7 the radius of the circle continually increases and becomes nearly equal to unity, the closed curves of figure 6 continually increase, changing at the same time their forms, and assume the forms shown by the dotted lines of figure 6. It thus appears that, to the whole area of the circle $x_1^2+y_1^2-1=0$ of figure 7, there correspond the two lunes ACB and ABD of figure 6; or if we attend only to the area included within the circle $x^2+y^2-1=0$ of this figure, then there corresponds not the whole area of this circle, but only the area of the lune ACB: and thus that the assumed relation $z_1=\dfrac{z(z-2)}{1-2z}$ establishes, in fact, an orthomorphosis of the circle $x_1^2+y_1^2-1=0$ into the lune ACB which lies inside the circle $x^2+y^2-1=0$ and outside the circle $(x-2)^2+y^2-3=0$. It may be added that, to the infinite area outside the circle $x_1^2+y_1^2-1=0$ of figure 7, there correspond in figure 6 first the area of the lens AB common to the two circles, and secondly the area outside the two circles: we have thus an orthomorphosis of the area outside the circle $x_1^2+y_1^2-1=0$ into these two areas respectively.

A somewhat more elegant example would have been that of the correspondence

$$z_1=\frac{z(z-\sqrt{2})}{1-\sqrt{2}z};$$

here, corresponding to the circumference $x_1^2+y_1^2-1=0$, we have the two equal circumferences $x^2+y^2-1=0$, and $(x-\sqrt{2})^2+y^2-1=0$: and to the whole area of the circle $x_1^2+y_1^2-1=0$, there correspond two equal lunes ACB and ABD.

922.

NOTE ON THE LUNAR THEORY.

[From the *Monthly Notices of the Royal Astronomical Society*, vol. LII. (1892), pp. 2—5.]

In the Lunar Theory, in whatever way worked out, the values ultimately obtained for the coordinates r, v, y should of course satisfy identically the equations of motion; and that they do so, is the ultimate verification of the correctness of the results obtained. It can hardly be hoped for that such a verification will ever be made for Delaunay's results; and yet it would seem generally that the labour of such a verification of the results to any extent, while exceeding (and possibly greatly exceeding) that of obtaining these results by any method employed for that purpose, ought still to be, so to speak, a labour of the same order. And one can, moreover, imagine the process of verification so arranged as to be a process of mere routine which could be carried out by ordinary computers. But, however this may be, I think it is not without interest to exhibit the verification to a very small extent, viz. to e, m^4.

I think there is an advantage in using capital letters for the arguments, and I accordingly write G (instead of Delaunay's g), to denote the mean anomaly.

The equations of motion are:

$$\frac{d^2r}{dt^2} - r\left\{\cos^2 y\left(\frac{dv}{dt}\right)^2\right\} + \frac{n^2a^3}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left\{r^2\cos^2 y\left(\frac{dv}{dt}\right)\right\} = \frac{d\Omega}{dv},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\sin y\cos y\left(\frac{dv}{dt}\right)^2 = \frac{d\Omega}{dy},$$

$$\Omega = \frac{m'r^2}{r'^3}\left(\tfrac{3}{2}\cos^2 H - \tfrac{1}{2}\right), \text{ where } \cos H = \cos y\cos(v - v'),$$

or say

$$\Omega = \frac{m^2n^2a'^3}{r'^3}r^2\left\{\tfrac{3}{2}\cos^2 y\cos^2(v - v') - \tfrac{1}{2}\right\};$$

and thus the equations become

$$\frac{d^2r}{dt^2} - r\left\{\cos^2 y\left(\frac{dv}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + \frac{n^2a^3}{r^2} = \frac{m^2n^2a'^3}{r'^3}\, r\,\{3\cos\{\tfrac{1}{2} + \tfrac{1}{2}\cos(2v - 2v')\} - \tfrac{1}{2}\},$$

$$\frac{d}{dt}\, r^2\cos^2 y\left(\frac{dv}{dt}\right)^2 = \frac{m^2n^2a'^3}{r'^3}\, r^2\,\{-\tfrac{3}{2}\cos^2 y\sin(2v - 2v')\},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\sin y\cos y\left(\frac{dv}{dt}\right)^2 = \frac{m^2n^2a'^3}{r'^3}\, r^2\,\{-3\sin y\cos y\,\tfrac{1}{2} + \tfrac{1}{2}\cos(2v - 2v')\}.$$

To simplify as much as possible, take the Sun's orbit to be circular, i.e. $r' = a'$, $v' = mnt$; also neglect y^2: the first and second equations are

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{n^2a^3}{r^2} = m^2n^2r\,\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v - 2v')\},$$

$$\frac{d}{dt}\, r^2\cdot\left(\frac{dv}{dt}\right) \qquad = m^2n^2r^2\,\{-\tfrac{3}{2}\sin(2v - 2v')\};$$

and if for convenience of working we write $a = 1$, $n = 1$, then the first equation may be written

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{dv}{dt}\right)^2 + \frac{1}{r^3} = m^2\,\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v - 2v')\},$$

and similarly the second equation is

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = -\tfrac{3}{2}m^2\sin(2v - 2v');$$

and the third equation may be disregarded.

The two equations should be satisfied by Delaunay's values, putting therein $e' = 0$, $y = 0$; say by the values

$$\begin{aligned}
\frac{1}{r} = 1 &+ \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4 \\
&+ (m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4)\cos 2D \\
&+ \qquad\qquad (\tfrac{7}{8}m^4)\cos 4D \\
&+ e\,(1 - \tfrac{7}{12}m^2) \qquad\qquad \cos G \\
&+ e\,(\tfrac{33}{16}m^2) \qquad\qquad \cos(2D + G) \\
&+ e\,(\tfrac{15}{8}m + \tfrac{187}{32}m^2) \qquad \cos(2D - G);
\end{aligned}$$

$$\begin{aligned}
v = t \\
&+ (\tfrac{11}{8}m^2 + \tfrac{19}{12}m^3 + \tfrac{893}{72}m^4)\sin 2D \\
&+ \qquad\qquad (\tfrac{201}{256}m^4)\sin 4D \\
&+ 2e \qquad\qquad \sin G \\
&+ e\,(\tfrac{17}{8}m^2) \qquad\qquad \sin(2D + G) \\
&+ e\,(\tfrac{15}{4}m + \tfrac{263}{16}m^2) \qquad \sin(2D - G);
\end{aligned}$$

where

$$D = (1-m)\,t, \qquad G = (1-\tfrac{3}{4}m^2)\,t, \qquad (v' = mt).$$

The verification for the first equation is

	$\frac{1}{r}\frac{d^2r}{dt^2} =$	$-\left(\frac{dv}{dt}\right)^2 =$	$\frac{1}{r^3} =$	$m^2\{-\frac{1}{2}-\frac{3}{2}(\cos 2v - 2v')\} =$	
Const.		-1	$+1$		$=0$
			$+\frac{1}{2}m^2$	$-\frac{1}{2}m^2$	$=0$
	$+2m^4$	$-\frac{121}{32}m^4$	$-\frac{9}{32}m^4$	$-\frac{33}{16}m^4$	$=0$
Cos $2D$	$+4m^2$	$-\frac{11}{2}m^2$	$+3m^2$	$-\frac{3}{2}m^2$	$=0$
	$+\frac{14}{3}m^3$	$-\frac{85}{6}m^3$	$+\frac{19}{2}m^3$		$=0$
	$+\frac{64}{9}m^4$	$-\frac{539}{18}m^4$	$+\frac{137}{6}m^4$		$=0$
Cos $4D$	$+8m^4$	$-\frac{161}{16}m^4$	$+\frac{33}{8}m^4$	$-\frac{33}{16}m^4$	$=0$
Cos G	e	$-4e$	$3e$		$=0$
	$-\frac{9}{4}em^2$	$+3em^2$	$-\frac{3}{4}em^2$		$=0$
Cos $(2D+G)$	$e\,\frac{193}{16}m^2$	$e\,(-\frac{73}{4}m^2)$	$e\,(\frac{147}{16}m^2)$	$e\,(-3m^2)$	$=0$
Cos $(2D-G)$	$em\,(\frac{15}{8})$	$em\,(-\frac{15}{2})$	$em\,(\frac{45}{8})$		$=0$
	$em^2\,(-\frac{5}{32})$	$em^2\,(-\frac{187}{8})$	$em^2\,(\frac{657}{32})$	$em^2\,(3)$	$=0.$

The verification for the second equation is

	$\frac{2}{r}\frac{dr}{dt}\cdot\frac{dv}{dt} =$	$+\frac{d^2v}{dt^2} =$	$+\frac{3}{2}m^2\sin(2v-2v') =$	
Sin $2D$	$+4m^2$	$-\frac{11}{2}m^2$	$+\frac{3}{2}m^2$	$=0$
	$+\frac{26}{3}m^3$	$-\frac{26}{3}m^3$		$=0$
	$+\frac{142}{9}m^4$	$-\frac{142}{9}m^4$		$=0$
Sin $4D$	$+\frac{21}{2}m^4$	$-\frac{201}{16}m^4$	$+\frac{33}{16}m^4$	$=0$
Sin G	$e\,.\,2$	$+e\,.\,-2$		$=0$
	$e\,.\,3m^2$	$+e\,.\,3m^2$		$=0$
Sin $(2D+G)$	$e\,.\,\frac{129}{8}m^2$	$e\,.\,-\frac{153}{8}m^2$	$e\,.\,3m^2$	$=0$
Sin $(2D-G)$	$e\,.\,\frac{15}{4}m$	$e\,.\,-\frac{15}{4}m$		$=0$
	$e\,.\,\frac{71}{16}m^2$	$e\,.\,-\frac{23}{16}m^2$	$e\,.\,-3m^2$	$=0\,;$

and the verification is thus completed.

The following intermediate results may be recorded; $\frac{1}{r} = 1 + \chi$,

	$\chi^2 =$	$-\log r =$	$\frac{dv}{dt} =$
	$\frac{19}{36}m^4$	$\frac{1}{6}m^2 - \frac{255}{288}m^4$	1
$\text{Cos } 2D$	$+\frac{1}{3}m^4$	$m^2 + \frac{19}{6}m^3 + \frac{64}{9}m^4$	$\frac{11}{4}m^2 + \frac{35}{12}m^3 + \frac{539}{36}m^4$
$\text{Cos } 4D$	$+\frac{1}{2}m^4$	$\frac{5}{8}m^4$	$\frac{201}{64}m^4$
$\text{Cos } G$	$+\frac{1}{3}em^2$	$e(1 - \frac{3}{4}m^2)$	$e(2 - \frac{3}{2}m^2)$
$\text{Cos } 2D + G$	$+ em^2$	$e(\frac{25}{16}m^2)$	$e(\frac{51}{8}m^2)$
$\text{Cos } 2D - G$	$+ em^2,$	$e(\frac{15}{8}m + \frac{171}{32}m^2),$	$e(\frac{15}{4}m + \frac{143}{16}m^2),$

	$\frac{1}{r}\frac{dr}{dt} =$
$\text{Sin } 2D$	$2m^3 + \frac{13}{3}m^3 + \frac{71}{9}m^4$
$\text{Sin } 4D$	$\frac{5}{2}m^4$
$\text{Sin } G$	$e(1 - \frac{3}{2}m^2)$
$\text{Sin } (2D + G)$	$e(\frac{75}{16}m^2)$
$\text{Sin } (2D - G)$	$e(\frac{15}{8}m + \frac{54}{32}m^2).$

These were, in fact, made use of for finding the foregoing values of r, $\frac{d^2r}{dt^2}$, &c.

923.

NOTE ON A HYPERDETERMINANT IDENTITY.

[From the *Messenger of Mathematics*, vol. XXI. (1892), pp. 131, 132.]

THE following is in effect a well-known theorem; but I am not sure whether it has been stated in a form at once so general and so precise.

If

$$\Omega = (*)(x_1, y_1)^A (x_2, y_2)^B (x_3, y_3)^C (x_4, y_4)^D \ldots$$

be a function separately homogeneous, and of the degrees $A, B, C, D, \ldots$ in the sets of variables (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), ... respectively; and if

$$\overline{12} = \xi_1\eta_2 - \xi_2\eta_1, \; = \partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}, \text{ \&c.},$$

then

$$(A\,\overline{23} + B\,\overline{31} + C\,12)\,\Omega = 0,$$

when the variables (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), ..., or only the variables (x_1, y_1), (x_2, y_2), (x_3, y_3) are therein severally replaced by (x, y).

In fact, we have

$$A\Omega = (x_1\xi_1 + y_1\eta_1)\,\Omega, \quad B\Omega = (x_2\xi_2 + y_2\eta_2), \quad C\Omega = (x_3\xi_3 + y_3\eta_3)\,\Omega;$$

thus the expression is

$$= \{(x_1\xi_1 + y_1\eta_1)\,\overline{23} + (x_2\xi_2 + y_2\eta_2)\,\overline{31} + (x_3\xi_3 + y_3\eta_3)\,\overline{12}\}\,\Omega,$$

and if we herein replace the variables (x_1, y_1), (x_2, y_2), (x_3, y_3) *in so far as they appear explicitly* by (x, y), the expression becomes

$$= \{(x\xi_1 + y\eta_1)\,\overline{23} + (x\xi_2 + y\eta_2)\,\overline{31} + (x\xi_3 + y\eta_3)\,\overline{12}\}\,\Omega,$$

where the factor in { }, substituting for $\overline{23}$, $\overline{31}$, $\overline{12}$ their values $\xi_2\eta_3 - \xi_3\eta_2$, $\xi_3\eta_1 - \xi_1\eta_3$, $\xi_1\eta_2 - \xi_2\eta_1$, becomes identically $= 0$. The value of the expression is thus $= 0$; and of

course it remains $=0$ when, consequently upon the foregoing change (x_1, y_1), (x_2, y_2), (x_3, y_3) each into (x, y), we also change (ξ_1, η_1), (ξ_2, η_2), (ξ_3, η_3) each into (ξ, η); and if we also change (x_4, y_4), &c., into (x, y).

It is clear that Ω may denote the covariant symbol

$$\Omega = \overline{23}^{\alpha}\, 31^{\beta}\, \overline{12}^{\gamma}\, \overline{14}^{\delta}\, \overline{24}^{\epsilon}\, 34^{\zeta} \ldots\, U_1 V_2 W_3 T_4 \ldots,$$

where $U, V, W, T, \ldots$ denote quantics

$$(a, \ldots \between x, y)^m,\ (a', \ldots \between x, y)^n,\ \ldots$$

of the degrees $m, n, p, q, \ldots$ respectively, and U_1, V_2, &c., are the corresponding functions $(a, \ldots \between x_1, y_1)^m$, $(a', \ldots \between x_2, y_2)^n$, &c., the values of A, B, C being here

$$A = m - \beta - \gamma - \delta - \ldots,$$

$$B = n - \gamma - \alpha - \epsilon - \ldots,$$

$$C = p - \alpha - \beta - \zeta - \ldots;$$

the theorem expresses the property that the covariants $\overline{12}\Omega$, $\overline{23}\Omega$, $\overline{31}\Omega$, are linearly connected together; or, writing it in the form $(A\,\overline{12} - C\,\overline{13} + B\,\overline{23})\,\Omega = 0$, we have the proper linear combination $A\,\overline{12}\Omega - C\,\overline{13}\Omega$ of the two covariants $\overline{12}\Omega$ and $\overline{13}\Omega$, equal to $-B\,\overline{23}\Omega$, a determinate multiple of $\overline{23}\Omega$. Speaking roughly, we say that the *difference* of the covariants $\overline{12}\Omega$ and $\overline{13}\Omega$ is equal to $\overline{23}\Omega$.

924.

ON THE NON-EXISTENCE OF A SPECIAL GROUP OF POINTS.

[From the *Messenger of Mathematics*, vol. XXI. (1892), pp. 132, 133.]

It is well known that, taking in a plane any eight points, every cubic through these passes through a determinate ninth point. It is interesting to show that there is no system of seven points such that every cubic through these passes through a determinate eighth point.

Assuming such a system: first, no three of the points can be in a line: for, if they were, then among the cubics through the seven points we have the line through the three points and an arbitrary conic through the remaining four points, and these composite cubics have no common eighth point of intersection.

Secondly, no six of the points can be on a conic: for, if they were, then among the cubics through the seven points we have the conic through the six points and an arbitrary line through the remaining point, and these composite cubics have no common eighth point of intersection.

Taking now the points to be 1, 2, 3, 4, 5, 6, 7; among the cubics through these, we have the composite cubics (A, P), (B, Q), (C, R), where A, B, C are the lines 67, 75, 56, and P, Q, R the conics 12345, 12346, 12347 respectively; by what precedes, the points 5, 6, 7 do not lie on a line, and the points (6, 7), (7, 5) and (5, 6) neither of them lie on the conics P, Q, R respectively.

The common eighth point of intersection, if it exists, must be

$$(A, B, C);\ (A, Q, R), (B, R, P), (C, P, Q);$$
$$(P, B, C), (Q, C, A), (R, A, B);\ \text{or}\ (P, Q, R).$$

There is no point (A, B, C).

There is no point (A, Q, R): for Q, R intersect only in the points 1, 2, 3, 4, no one of which lies on A; and similarly, there is no point (B, R, P) or (C, P, Q).

B, C intersect in 5, which is a point on P; and thus 5 is the only point (P, B, C). Similarly, 6 is the only point (Q, C, A) and 7 is the only point (R, A, B).

P, Q intersect in 1, 2, 3, 4, which are each of them on R; hence 1, 2, 3, 4 are the only points (P, Q, R).

Hence the points 1, 2, 3, 4, 5, 6, 7 present themselves each once, and only once, among the intersections of the three cubics, and there is no common eighth point of intersection.

925.

ON WARING'S FORMULA FOR THE SUM OF THE mth POWERS OF THE ROOTS OF AN EQUATION.

[From the *Messenger of Mathematics*, vol. XXI. (1892), pp. 133—137.]

THE formula in question, Prob. I. of Waring's *Meditationes Algebraicæ*, Cambridge, 1782, making therein a slight change of notation, is as follows: viz. the equation being

$$x^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \ldots = 0,$$

then we have

$$(-)^m S_m =$$

$$\begin{array}{rl}
b^m & \\
-mc\ b^{m-2} & \\
+md\ b^{m-3} & \\
\left.\begin{array}{r} -me \\ +\frac{1}{2}m\,.\,m-3\,.\,c^2 \end{array}\right\} b^{m-4} & \\
\left.\begin{array}{r} +mf \\ -m\,.\,m-4\,.\,cd \end{array}\right\} b^{m-5} & \\
\left.\begin{array}{r} -m\,.\,g \\ +m\,.\,m-5\,.\,ce \\ +\frac{1}{2}m\,.\,m-5\,.\,d^2 \\ -\frac{1}{6}m\,.\,m-4\,.\,m-5\,.\,c^3 \end{array}\right\} b^{m-6} & \\
 & \left.\begin{array}{r} +m\,.\,h \\ -m\,.\,m-6\,.\,cf \\ -m\,.\,m-6\,.\,de \\ +\frac{1}{2}m\,.\,m-5\,.\,m-6\,.\,c^2d \end{array}\right\} b^{m-7} \\
 & \left.\begin{array}{r} +m\,.\,i \\ +m\,.\,m-7\,.\,cg \\ +m\,.\,m-7\,.\,df \\ +\frac{1}{2}m\,.\,m-7\,.\,e^2 \\ -\frac{1}{2}m\,.\,m-6\,.\,m-7\,.\,c^2e \\ -\frac{1}{2}m\,.\,m-6\,.\,m-7\,.\,cd^2 \\ +\frac{1}{24}m\,.\,m-5\,.\,m-6\,.\,m-7\,.\,c^4 \end{array}\right\} b^{m-8} \\
 & +\&c.,
\end{array}$$

where, reckoning the weights of b, c, d, e, ... as 1, 2, 3, 4, ..., respectively, the several terms are all the terms of the weight m, or (what is the same thing) in the

coefficient of $b^{m-\theta}$ we have all the combinations of $c, d, e, \ldots$, (or say all the non-unitary combinations) of the weight θ, and where the numerical coefficient of

$$b^{m-\theta}c^{\mathrm{c}}d^{\mathrm{d}}e^{\mathrm{e}}\ldots \quad (\mathrm{c}+\mathrm{d}+\mathrm{e}+\ldots=\theta),$$

is

$$=(-)^{\mathrm{c}+\mathrm{e}+\mathrm{g}+\ldots}\frac{m\,.\,m-(\theta-\delta+1)\,.\,m-(\theta-\delta+2)\ldots m-(\theta-1)}{\Pi\mathrm{c}\,.\,\Pi\mathrm{d}\,.\,\Pi\mathrm{e}\ldots}.$$

Thus for the term $b^{m-8}c^{2}e^{1}$, $\theta=8$; c, d, e $=2, 4, 1$ respectively (the other exponents each vanishing), and the coefficient is

$$(-)^{3}\frac{m\,.\,m-6\,.\,m-7}{1\,.\,2\,.\,1},\ =-\tfrac{1}{2}m\,.\,m-6\,.\,m-7,$$

as above; and so in other cases.

For the MacMahon form

$$1+bx+\frac{cx^2}{1\,.\,2}+\ldots=(1-\alpha x)(1-\beta x)\ldots,$$

or say

$$y^n+\frac{b}{1}y^{n-1}+\frac{c}{1\,.\,2}y^{n-2}+\ldots=(y-\alpha)(y-\beta)\ldots,$$

we must for $b, c, d, \ldots$, write $b, \dfrac{c}{1\,.\,2}, \dfrac{d}{1\,.\,2\,.\,3}, \ldots$ respectively: we thus have

$$\begin{aligned}
(-)^m\,.\,S_m = \ & b^m \\
& -m\frac{c}{1\,.\,2} && b^{m-2} \\
& +m\frac{d}{1\,.\,2\,.\,3} && b^{m-3} \\
& \left.\begin{aligned}&-m\frac{e}{1\,.\,2\,.\,3\,.\,4}\\ &+\tfrac{1}{2}m\,.\,m-3\left(\frac{c}{1\,.\,2}\right)^2\end{aligned}\right\} && b^{m-4} \\
& +\&\mathrm{c.},
\end{aligned}$$

or say

$$\begin{aligned}
(-)^m\,\Pi\,(m-1)\,S_m = \ & \Pi\,(m-1) && b^4 \\
& -\Pi m\frac{c}{1\,.\,2} && b^{m-2} \\
& +\Pi m\frac{d}{1\,.\,2\,.\,3} && b^{m-3} \\
& \left.\begin{aligned}&-\Pi m\frac{e}{1\,.\,2\,.\,3\,.\,4}\\ &+\Pi m\frac{m-3}{2}\left(\frac{c}{1\,.\,2}\right)^2\end{aligned}\right\} && b^{m-4} \\
& +\&\mathrm{c.},
\end{aligned}$$

the numerical coefficient of

$$b^{m-\theta}c^{c}d^{d}e^{e}\ldots\ (\mathrm{c}+\mathrm{d}+\mathrm{e}+\ldots=\theta)$$

being

$$(-)^{\mathrm{c}+\mathrm{e}+\mathrm{g}+\ldots}\frac{\Pi m\,.\,m-(\theta-\delta+1)\,.\,m-(\theta-\delta+2)\ldots m-(\theta-1)}{\Pi\mathrm{c}\,.\,\Pi\mathrm{d}\,.\,\Pi\mathrm{e}\ldots(\Pi 2)^{\mathrm{c}}\,(\Pi 3)^{\mathrm{d}}\,(\Pi 4)^{\mathrm{e}}\ldots}.$$

It is convenient to write down the literal terms in alphabetical order (AO), calculating and affixing to each term the proper numerical coefficient; thus taking

$$1+bx+c\frac{x^2}{1\,.\,2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots,$$

we find

$$\begin{array}{rll}
-120S_6=g & . & 1\\
bf & . & -\ 6\\
ce & . & -\ 15\\
d^2 & . & -\ 10\\
b^2e & . & +\ 30\\
bcd & . & +\ 120\\
c^3 & . & +\ 30\\
b^3d & . & -\ 120\\
b^2c^2 & . & -\ 270\\
b^4c & . & +\ 360\\
b^6 & . & -\ 120\\
\hline
 & & \pm\ 541
\end{array}$$

this expression, as representing the value of the non-unitary function S_6, being in fact a seminvariant.

It is to be remarked that the foregoing expression for the sum of the mth powers of the roots of the equation

$$x^n+bx^{n-1}+cx^{n-2}+\ldots=0$$

is, in fact, the series for x^m continued so far only as the exponent of b is not negative: see as to this Note XI. of Lagrange's *Équations Numériques*. For the *à posteriori* verification, observe that we have

$$x+b+\frac{c}{x}+\frac{d}{x^2}+\ldots=0,$$

or writing for a moment $u=-b$, say this is

$$x=u+fx,$$

where

$$fx = -\frac{c}{x} - \frac{d}{x^2} - \frac{e}{x^3} - \&c.$$

Hence, by Lagrange's theorem,

$$\begin{aligned} x^m = \quad & u^m \\ & - mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right) \\ & + \left\{mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right)^2\right\}' \frac{1}{1.2} \\ & - \left\{mu^{m-1}\left(\frac{c}{u} + \frac{d}{u^2} + \frac{e}{u^3} + \dots\right)^3\right\}'' \frac{1}{1.2.3} \\ & + \&c., \end{aligned}$$

where the accents denote differentiations in regard to u. This is

$$\begin{aligned} = \quad & u^m \\ & - m\{cu^{m-2} + du^{m-3} + eu^{m-4} + fu^{m-5} + gu^{m-6} + \dots\} \\ & + \tfrac{1}{2}m\{(m-3)c^2u^{m-4} + (m-4)2cdu^{m-5} + (m-5)(d^2+2ce)u^{m-6} + \dots\} \\ & - \tfrac{1}{6}m\{(m-4)(m-5)c^3u^{m-3} + \dots\} \\ & + \&c. \\ = \quad & u^m \\ & + u^{m-2}.-mc \\ & + u^{m-3}.-md \\ & + u^{m-4}.-me + \tfrac{1}{2}m.m-3.c^2 \\ & + u^{m-5}.-mf + \tfrac{1}{2}m.m-4.2cd \\ & + u^{m-6}.-mg + \tfrac{1}{2}m.m-5.(d^2+2ce) - \tfrac{1}{6}m.m-4.m-5.c^3 \\ & + \&c., \end{aligned}$$

which, putting therein $u = -b$ and multiplying each side by $(-)^m$, is the before-mentioned formula for $(-)^m S\alpha^m$: in that formula the series being continued only so far as the exponent of b is not negative.

I notice also that we cannot easily, by means of the known formula

$$S\alpha^m\beta^p = S\alpha^m . S\alpha^p - S\alpha^{m+p},$$

deduce an expression for $S\alpha^m\beta^p$: in fact, forming the product of the series for $S\alpha^m$, $S\alpha^p$ respectively, this product is identically equal to the series for $S\alpha^{m+p}$, or we seem to obtain $0 = S\alpha^m S\alpha^p - S\alpha^{m+p}$; to obtain the correct formula, we have to take each of the three series only so far as the exponent of b therein respectively is not negative: and it is not easy to see how the resulting formula is to be expressed.

926.

CORRECTED SEMINVARIANT TABLES FOR THE WEIGHTS 11 AND 12.

[From the *American Journal of Mathematics*, vol. XIV. (1892), pp. 195—200.]

THE tables in my paper, "Seminvariant Tables," *American Journal of Mathematics*, vol. VII. (1885), pp. 59—73, [831], are not in the best form, but the deviations present themselves only in a few columns of the tables for the weights 11 and 12, viz. in the former of these two columns, and in the latter a single column, ought to be replaced by linear combinations of other columns; there are, besides, columns which should be new named; in regard hereto, there is a point of theory which requires to be made clear. I remark that in each table the literal terms are in alphabetical order (AO); this is the proper order for the final terms, and although (as about to be explained) the proper order for the initial terms is the counter order (CO), yet as the tables cannot be at the same time arranged in the one and the other order, I adhere to the AO as the proper arrangement for the terms of the tables; we have, however, to introduce the notion that, in general, it is not the top term of a column which is to be regarded as the initial term, and in connexion herewith to consider how the columns are to be named. An instance first presents itself for the weight 11: we have, see column 5 of the table for that weight here given, a seminvariant

$$\begin{array}{ll} fg & +\ 1 \\ b^2j & \\ bci & \\ bdh & \\ beg & -\ 5 \\ bf^2 & -\ 6 \\ c^2h & -\ 16 \\ \vdots & \\ b^3e^2 & +\ 70\,; \end{array}$$

this is to be regarded, not as a seminvariant $fg - b^3e^2$ (it is hardly necessary to remark that, here and elsewhere the $-$ is not a minus sign, but a mere stroke), but as a seminvariant $c^2h - b^3e^2$, viz. c^2h is a term entering into the seminvariant, and which, although it is in AO subsequent, it is in CO precedent, to the terms fg, beg and bf^2. The seminvariant contains the term $-16c^2h$ and other terms with the letter h, and it is a misnomer to call it $fg - b^3e^2$, a name implying that the highest letter thereof is g. Instead of the stroke, it would perhaps be better to write ∞, for instance $c^2h \infty b^3e^2$, where of course ∞ would be used as a mere conventional symbol.

For greater clearness, I give here the express definition of counter order, (CO), viz. whereas in AO we begin with the lowest letters, in CO we begin contrariwise with the highest letters. A term containing a higher letter or higher power of such letter precedes a term containing a lower letter or lower power of the same letter—or in the easiest form, the counter order is the alphabetical order corresponding to the reversed arrangement $z, y, \ldots, f, e, d, c, b$ of the letters.

A symbol, as $c^2h - b^3e^2$ above, may be regarded as referring to a set of terms c^2h, b^3e^2 and all the terms which are in CO subsequent to c^2h and in AO precedent to b^3e^2: as by supposition the terms are arranged in AO, the set includes no term lower than b^3e^2, or say the bottom term b^3e^2 is also the final term of the set, but it does include terms fg, beg and bf^2 higher than c^2h, and thus the top term fg is not, but c^2h is, the initial term of the set. It should be remarked that a seminvariant $ch - b^3e^2$ need not include all the terms of the set as just defined: there may very well be terms with a coefficient zero, or say accidental zeros; an instance presents itself, weight 10, where in the column $eg - bd^3$ we have $0ce^3$, no term in ce^2.

The changes actually required are very slight, viz.

Weight 11, instead of

$fg - b^3e^2$, we require $c^2h - b^3e^2$, old $fg - b^3e^2$, new named,

$c^2h - b^3d^2$, „ $fg - b^3d^2$, linear combination $(fg - b^3e^2) + 8\,(c^2h - b^3d^2)$,

$de^2 - b^5d^2$, „ $c^3f - b^5d^2$, old $de^2 - b^5d^2$, new named,

$c^3f - b^3c^4$, „ $de^2 - b^3c^4$, linear combination $(de^2 - b^5d^2) + 6\,(c^3f - b^3c^4)$.

Weight 12, instead of

$cf^2 - bcd^3$, we require $d^2g - bcd^3$, old $cf^2 - bcd^3$, new named,

$d^2g - c^3d^2$, „ $cf^2 - c^3d^2$, linear combination $(cf^2 - bcd^3) - 5\,(d^2g - c^3d^2)$;

but I have thought it desirable to give the complete tables for the weights in question, 11 and 12; and I have also rearranged the entire columns of the two tables so as to present in each of them the *finals* in AO. This is the case with the existing tables, except that there is a single transposition in the table weight 10.

Instead of the columns $cdf - b^4d^2$, $ce^2 - c^5$, we ought to have $ce^2 - c^5$, $cdf - b^4d^2$. The complete list up to the weight 12 is

$w =$	
2	$c - b^2$
3	$d - b^3$
4	$e - c^2$
	$c^2 - b^4$
5	$f - bc^2$
	$cd - b^5$
6	$g - d^2$
	$ce - c^3$
	$d^2 - b^2c^2$
	$c^3 - b^6$
7	$h - bd^2$
	$cf - bc^3$
	$de - b^3c^2$
	$c^2d - b^7$
8	$i - e^2$
	$cg - cd^2$
	$df - b^2d^2$
	$e^2 - c^4$
	$c^2e - b^2c^3$
	$cd^2 - b^4c^2$
	$c^4 - b^8$

$w =$	
9	$j - be^2$
	$ch - d^3$
	$dg - bcd^2$
	$ef - b^2d^2$
	$c^2f - bc^4$
	$cde - b^3c^3$
	$d^3 - b^5c^2$
	$c^3d - b^9$
10	$k - f^2$
	$ci - ce^2$
	$dh - b^2e^2$
	$eg - bd^3$
	$f^2 - c^2d^2$
	$c^2g - b^2cd^2$
	$ce^2 - c^5$
	$cdf - b^4d^2$
	$d^2e - b^2c^4$
	$c^3e - b^4c^3$
	$c^2d^2 - b^6c^2$
	$c^5 - b^{10}$

$w =$	
11	$l - bf^2$
	$cj - de^2$
	$di - bce^2$
	$eh - cd^3$
	$c^2h - b^3e^2$
	$fg - b^2d^3$
	$cef - bc^2d^2$
	$cdg - b^3cd^2$
	$d^2f - bc^5$
	$c^3f - b^3d^2$
	$de^2 - b^3c^4$
	$c^2de - b^5c^3$
	$cd^3 - b^7c^2$
	$c^3d - b^{11}$

$w =$	
12	$m - g^2$
	$ck - ef^2$
	$ei - e^3$
	$dj - b^2f^2$
	$fh - bde^2$
	$g^2 - c^2e^2$
	$ceg - d^4$
	$c^2i - b^2ce^2$
	$d^2g - bcd^3$
	$ef^2 - c^2d^2$
	$cdh - b^4e^2$
	$def - b^3d^3$
	$e^3 - b^2c^2d^2$
	$c^2e^2 - c^6$
	$c^3g - b^4cd^2$
	$cd^2e - b^2c^5$
	$c^2df - b^6d^2$
	$d^4 - b^4c^4$
	$c^4e - b^6c^3$
	$c^3d^2 - b^8c^2$
	$c^6 - b^{12}$

The two new tables are as follows: some accidental numerical errors have been corrected.

TABLE, WEIGHT 11.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	
		l	cj	di	eh	c^2h	fg	cef	cdg	d^2f	c^3f	de^2	c^2de	cd^3	c^4d	
	l	+ 1														
	bk	− 11														
	cj	+ 35	+ 2													
	di	− 75	− 9	+ 1												
	eh	+ 90	+14	− 2	+ 1											
	fg	− 42	− 7	+ 1	− 1	+ 1	+ 1									
	b^2j	+ 20	− 2													
	bci	− 90	+ 9	− 3												
	bdh	+240	+16		− 4											
10	beg	−420	−63	+ 9	− 2	− 5	− 5									10
	bf^2	+252	+42	− 6	+ 6	− 6	− 6									
	c^2h		−30	+10	+ 3	− 16										
	cdg		+70	−26	− 2	+ 58	+ 2		+1							
	cef		−21	+ 7	− 6	+ 5	+ 45	+1	−3							
	d^2f		−56	+24	+10	−100	−100	−3	+6	+1						
	de^2		+35	−15	− 5	+ 60	+ 60	+2	−4	−1	+1	+1				
	b^2i			+ 2												
	b^2ch			− 8		+ 32										
	b^2dg			+ 8	+20	− 48	+ 8		−1							
20	b^2ef			− 4	−18	+ 40		−1	+3							20
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	

TABLE, WEIGHT 11 (*continued*).

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	
		l	cj	di	eh	c^2h	fg	cef	cdg	d^2f	c^3f	de^2	c^2de	cd^3	c^4d	
21	bc^2g			+ 8	− 15	− 62	− 6		− 3							21
	$bcdf$			− 16	− 24	+ 232	+ 408	+ 14	− 30	− 6						
	bce^2			+ 10	+ 45	− 205	− 405	− 11	+ 27	+ 3	− 3	− 3				
	bd^2e				− 10	+ 20	+ 20	− 1	+ 2	+ 3	− 8	− 8				
	c^3f				+ 27	− 54	− 270	− 9	+ 27	+ 4	− 12					
	c^2de				− 45	+ 90	+ 450	+ 14	− 45	− 6	+ 36	+ 6	+ 1			
	cd^3				+ 20	− 40	− 200	− 6	+ 20	+ 2	− 18		− 1	+ 1		
	b^4h					− 16										
	b^2cg					+ 56			+ 5							
30	b^2df					− 112	− 288	− 8	+ 18	+ 4						30
	b^2e^2					+ 70	+ 270	+ 9	− 23	− 2	+ 2	+ 2				
	b^2c^2f						+ 216	+ 6	− 27	− 3	+ 36					
	b^2cde						− 360	− 16	+ 51	+ 6	− 36	+ 24	− 2			
	b^2d^3						+ 160	+ 8	− 24	− 8	+ 34	+ 16	+ 1	− 1		
	bc^3e							+ 3		− 2	− 48	− 18	− 3			
	bc^2d^2							− 2		+ 6	+ 18	− 24	+ 5	− 9		
	c^4d									− 1	+ 3	+ 9	− 1	+ 4	+ 1	
	b^5g								− 2							
	b^4cf								+ 6		− 36					
40	b^4de								− 2	− 4	+ 14	− 16	+ 1			40
	b^3c^2e								− 6	+ 3	+ 72	+ 12	+ 8			
	b^3cd^2								+ 4	+ 8	− 78	− 48	− 7	+ 15		
	b^2c^3d									− 10	+ 30	+ 72	− 5	+ 11	− 4	
	bc^5									+ 3	− 9	− 27	+ 3	− 12	− 3	
	b^6f										+ 12					
	b^5ce										− 30		− 7			
	b^5d^2										+ 20	+ 32	+ 2	− 6		
	b^4c^2d											− 48	+ 10	− 39	+ 6	
	b^3c^4											+ 18	− 5	+ 29	+ 14	
50	b^7e												+ 2			50
	b^6cd												− 4	+ 32	− 4	
	b^5c^3												+ 2	− 23	− 26	
	b^8d													− 8	+ 1	
	b^7c^2													+ 6	+ 24	
	b^9c														− 11	
56	b^{11}														+ 2	56
		bf^2	de^2	bce^2	cd^3	b^2e^2	b^2d^3	bc^2d^2	b^3cd^2	bc^5	b^5d^2	b^3c^4	b^5c^3	b^7c^2	b^{11}	
		±638	±188	±70	±132	±664	±1640	±57	±170	±43	±278	±192	±35	±98	±48	

TABLE WEIGHT 12.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
	m	ck	ei	dj	fh	g^2	ceg	c^2i	d^2g	cf^2	cdh	def	e^3	c^2e^2	c^3g	cd^2e	c^2df	d^4	c^4e	c^3d^2	c^6	
m	+ 1																					
bl	− 12																					
ck	+ 66	+ 3																				
dj	− 220	− 15		+ 15																		
ei	+ 495	+ 40	+ 1	− 40																		
fh	− 792	− 70	− 4	+ 70	+ 25																	
g^2	+ 462	+ 42	+ 3	− 40	− 24	+ 1																
b^3k		− 3																				
bcj		+ 15		− 45																		
10 bdi		− 25	− 4	+ 25																		10
beh		+ 30	+ 12	− 30	− 125																	
bfg		− 14	− 8	+ 12	+ 113	− 12																
c^2i		− 15	+ 3	+ 150				+ 1														
cdh		+ 40	− 8	− 400	+ 50			− 4			+ 4											
ceg		− 70	− 22	+ 700	+ 680	− 70	+ 1	+ 8			− 8											
cf^2		+ 42	+ 24	− 420	− 675	+ 100	− 1	− 5	+ 2	+ 2	+ 5											
d^2g			+ 24		− 570	+ 80	− 1		+ 5													
def			− 36		+ 925	− 200	+ 2		− 19	− 4		+ 18										
e^3			+ 15		− 400	+ 100	− 1		+ 12	+ 2		− 17	+ 1									
20 b^3j				+ 30																		20
b^2ei				− 135				− 2														
b^2dh				+ 360	+ 200			+ 4			− 4											
b^2eg				− 630	− 525	+ 100	− 1	− 8			+ 8											
b^2f^2				+ 378	+ 336	− 64	+ 1	+ 5	− 2	− 2	− 5											
bc^2h					− 150			+ 4			− 12											
$bcdg$					+ 350	− 200	+ 2	− 4	− 30		+ 4											
$bcef$					− 105	+ 20	− 2	+ 2	+ 37	− 8	− 2	− 54										
bd^2f					− 280	+ 320	− 2		+ 46	+ 16		− 72										
bde^2					+ 175	− 200	+ 2		− 49	− 4		+ 114	− 12									
30 c^2g						+ 100	− 1	− 4	+ 20		+ 32				+ 1							30
c^2df						+ 200	+ 2	+ 8	− 49	− 4	− 64	+ 54			− 3		+ 3					
c^2e^2						+ 125	+ 1	− 5	− 32	+ 28	+ 40	+ 162	− 18	+ 1								
cd^2e							− 3		+ 91	− 44		− 342	+ 54	− 2	+ 4	+ 1						
d^4							+ 1		− 32	+ 18		+ 135	− 27	+ 1	− 2	− 1	− 2	+ 1				
b^4i								+ 1														
b^3ch								− 4			+ 20											
b^3dg								+ 4	+ 20		− 4											
b^3ef								− 2	− 18	+ 12	+ 2	+ 36										
b^2c^2g								+ 4	− 15		− 60				− 3							
40 b^2cdf								− 8	− 24	− 24	+ 120	+ 216			+ 6		− 6					40
b^2ce^2								+ 5	+ 45	− 30	− 75	− 360	+ 54	− 2	+ 2							
b^2d^2e									− 10	+ 20		+ 66	− 6	+ 2	− 6	− 1						
bc^3f									+ 27	+ 12		− 162			+ 3		− 9					
bc^2de									− 45	+ 60		+ 486	− 180	+ 4	− 8	− 6	− 15					
bcd^3									+ 20	− 40		− 252	+ 108	− 4	+ 7	+ 8	+ 24	− 12				
c^4e										− 30		− 81	+ 81	− 2	+ 1	+ 4	+ 30		+ 1			
c^3d^2										+ 20		+ 54	− 54	+ 2	− 2	− 5	− 28	+ 8	− 1	+ 1		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	

TABLE, WEIGHT 12 (*continued*).

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
		m	ck	ei	dj	fh	g^2	ceg	c^2i	d^2g	cf^2	cdh	def	e^3	c^2e^2	c^3g	cd^2e	c^3df	d^4	c^4e	c^2d^2	c^6	
48	b^5h											− 8											48
	b^4cg											+28				+ 3							
50	b^4df											−56	−144			− 3		+ 3					50
	b^4e^2											+35	+135	− 27	+1								
	b^3c^2f												+108			− 6		+24					
	b^3cde												−180	+108	−4	+ 9	+10	+30					
	b^3d^3												+ 80	− 64		+ 1	− 4	−16	+ 8				
	b^2c^3e													− 54	+2	+ 2	− 7	−75		− 4			
	$b^2c^2d^2$													+ 36	+4	−12	− 9	+ 6	+30	+ 3	− 3		
	bc^4d														−4	+ 8	+14	+48	−48	+ 2	− 6		
	c^6														+1	− 2		+ 2	+16	− 1	+ 4	+ 1	
	b^6g															− 1	− 4	−16					
60	b^5cf															+ 3		−21					60
	b^5de															− 1	− 4	−15					
	b^4c^2e															− 3	+ 3	+60		+ 6			
	b^4cd^2															+ 2	+ 8		−48	− 3	+ 3		
	b^3c^3d																−10	−40	+68	− 6	+22		
	b^3c^5																+ 3	+12	−24	+ 3	−15	− 6	
	b^7f																	+ 6					
	b^6ce																	−15		− 4			
	b^6d^2																	+10	+16	−23	− 1		
	b^5c^2d																		−24	+30	−30		
70	b^4c^4																		+ 9	− 3	+21	+15	70
	b^8e																			+ 1			
	b^7cd																			+ 2	+14	−20	
	b^6c^3																			− 3	− 9		
	b^9d																				− 4		
	b^8c^2																				+ 3	+15	
	$b^{10}c$																					− 6	
77	b^{12}																					+ 1	77
		g^2	cf^2	e^3	b^2f^2	bde^2	c^2e^2	d^4	b^2ce^2	bcd^3	c^3d^2	b^4e^2	b^3d^3	$b^2c^2d^2$	c^6	b^4cd^2	b^2c^5	b^6d^2	b^4c^4	b^6c^3	b^8c^2	b^{12}	
		±1024	±212	±82	±1740	±2854	±946	±12	±46	±325	±190	±298	±1664	±442	±18	±52	±51	±258	±156	±48	±68	±32	

927.

ON CLIFFORD'S PAPER "ON SYZYGETIC RELATIONS AMONG THE POWERS OF LINEAR QUANTICS."

[From the *Proceedings of the London Mathematical Society*, vol. XXIII. (1892), pp. 99—104.]

THE paper in question, originally printed, *Proc. Lond. Math. Soc.*, t. III. (1869), pp. 9—12, is reproduced No. XIV., pp. 119—122, of the *Mathematical Papers* (8vo. Lond. 1882), where it is immediately followed by the paper No. XV., "On Syzygetic Relations connecting the Powers of Linear Quantics," pp. 123—129. The author, after referring to theorems in M. Paul Serret's *Géométrie de Direction* (Paris, 1869), proceeds as follows:—"By the use of Professor Sylvester's method of contravariant differentiation, I have arrived at certain extensions of these theorems, which I now proceed to explain," and he then states his Theorem I.: In order that a system of N points in a plane should all lie on a curve of the order n, it is sufficient that the pth powers of their distances from an arbitrary line should satisfy a linear homogeneous relation, the number N being given by the formula

$$N = \tfrac{1}{2}\alpha n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where α is the quotient, and β the remainder of the division of p by n, so that $p = \alpha n + \beta$, and $\beta < n$. And he then gives Theorem II., a like theorem as regards points in space; and, further, two Tables, A and B, for the values of N corresponding to given values of n and p in the two cases respectively.

Theorem I. is incorrect for the first value of N in Table A, viz. if $n=1$, $p=2$, then $N=5$; the theorem here is: In order that a system of five points in a plane may lie in a line, the sufficient condition is that the squares of their distances from an arbitrary line shall satisfy a linear homogeneous relation; or, what is the same thing, if the squares of the distances of the five points satisfy a linear homogeneous relation, then the five points will lie in a line. The right conclusion is that four of the five points will lie in a line.

Before proceeding further, I slightly modify the form of the enunciation by defining (in the case of the plane figure) the power of a point to be its distance from an arbitrary line; or, what is the same thing, to be the nilfactum of the line-equation of the point; that is, for the point (x_1, y_1, z_1), the power is

$$\alpha x_1 + \beta y_1 + \gamma z_1,$$

where α, β, γ are arbitrary coefficients, or, if we please, line-coordinates. I say that $\alpha x_1 + \beta y_1 + \gamma z_1$ is the first power, $(\alpha x_1 + \beta y_1 + \gamma z_1)^2$ the second power, and so on. Clifford's Theorem I. thus is: If the pth powers of the N points satisfy a homogeneous linear relation, the N points are on a curve of the nth order.

In the case $n=1$, $p=2$, we have five points whose second powers satisfy a homogeneous linear relation; that is, if $(x_1, y_1, z_1), \ldots, (x_5, y_5, z_5)$ are the coordinates of the five points respectively, we have

$$\lambda_1 (\alpha x_1 + \beta y_1 + \gamma z_1)^2 + \ldots + \lambda_5 (\alpha x_5 + \beta y_5 + \gamma z_5)^2 = 0.$$

It is implicitly assumed that the points are distinct points, and we thus exclude such solutions as

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad (x_4, y_4, z_4) = k(x_5, y_5, z_5), \quad k^2\lambda_4 + \lambda_5 = 0.$$

Here α, β, γ are arbitrary, and the relation is equivalent to the six equations

$$\begin{aligned}
\lambda_1 x_1^2 + \ldots + \lambda_5 x_5^2 &= 0,\\
\lambda_1 y_1^2 + \ldots \qquad &= 0,\\
\lambda_1 z_1^2 + \ldots \qquad &= 0,\\
\lambda_1 y_1 z_1 + \ldots \qquad &= 0,\\
\lambda_1 z_1 x_1 + \ldots \qquad &= 0,\\
\lambda_1 x_1 y_1 + \ldots \qquad &= 0;
\end{aligned}$$

and hence, eliminating the λ's, we have the relation

$$\left\| \begin{matrix} x_1^2, & y_1^2, & z_1^2, & y_1 z_1, & z_1 x_1, & x_1 y_1 \\ x_2^2, & & & & & \\ x_3^2, & & & & & \\ x_4^2, & & & & & \\ x_5^2, & & & & & \end{matrix} \right\| = 0,$$

viz. this means that each of the determinants, obtained by selecting in any manner five out of the six columns, is $=0$. This is a twofold relation, and thus it cannot be equivalent to the threefold relation

$$\left\| \begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1, & & & & \\ z_1, & & & & \end{matrix} \right\| = 0,$$

which expresses that the five points are in a line.

But we see further that the twofold relation expresses that the five points are such that we can, through them and an arbitrary sixth point (x_6, y_6, z_6), draw a determinate conic; and this is the case only if four of the five points are in a line; viz. the conic is then the line-pair composed of this line and of the line joining the remaining two points. The right conclusion therefore is that, if the above linear relation is satisfied, then four of the five points lie in a line.

Clifford's proof is rather indicated than carried out, but, from the reference to contravariant differentiation, and from the second paper mentioned above, it must have been as follows:

Starting from the linear relation considered as an identity in (α, β, γ), and operating upon it with $p\partial_\alpha + q\partial_\beta + r\partial_\gamma$, p, q, r denoting arbitrary coefficients, we obtain

$$\lambda_1(px_1 + qy_1 + rz_1)(\alpha x_1 + \beta y_1 + \gamma z_1) + \ldots + \lambda_5(px_5 + qy_5 + rz_5)(\alpha x_5 + \beta y_5 + \gamma z_5) = 0;$$

hence, determining the ratios of p, q, r, say by the equations

$$px_4 + qy_4 + rz_4 = 0, \quad px_5 + qy_5 + rz_5 = 0,$$

and, instead of $\lambda_1(px_1 + qy_1 + rz_1)$, &c., writing Λ_1, &c., we have

$$\Lambda_1(\alpha x_1 + \beta y_1 + \gamma z_1) + \Lambda_2(\alpha x_2 + \beta y_2 + \gamma z_2) + \Lambda_3(\alpha x_3 + \beta y_3 + \gamma z_3) = 0,$$

a homogeneous linear relation between the first powers of the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). We thus see that these points, say the points 1, 2, 3, are in a line; and, similarly, by different determinations of p, q, r, that the points 1, 2, 4 are in a line; and that the points 1, 2, 5 are in a line; that is, the points 3, 4 and 5 are each of them in the line 12 joining the points 1 and 2; that is, the points 1, 2, 3, 4, 5 are in a line.

But, if we examine the reasoning more closely, it appears that the first conclusion, 1, 2, 3 are in a line, depends on the assumption that neither 1, 2, nor 3 is in the line 45. Suppose, for instance, that 3 was in the line 45, the equations

$$px_4 + qy_4 + rz_4 = 0, \quad px_5 + qy_5 + rz_5 = 0,$$

imply $px_3 + qy_3 + rz_3 = 0$, and we have

$$\lambda_1(px_1 + qy_1 + rz_1)(\alpha x_1 + \beta y_1 + \gamma z_1) + \lambda_2(px_2 + qy_2 + rz_2)(\alpha x_2 + \beta y_2 + \gamma z_2) = 0,$$

or say

$$\Lambda_1(\alpha x_1 + \beta y_1 + \gamma z_1) + \Lambda_2(\alpha x_2 + \beta y_2 + \gamma z_2) = 0,$$

satisfied by

$$(x_1, y_1, z_1) = k(x_2, y_2, z_2), \quad k\Lambda_1 + \Lambda_2 = 0,$$

that is, by making the points 1 and 2 coincident. Thus, if 3 be on the line 45 (and similarly if 1 or if 2 be on the line 45), we cannot infer that 1, 2, 3 are in a line.

I assume, therefore, that neither 1, 2, nor 3 is in the line 45; we here conclude, as above, that 1, 2, 3 are in a line. Suppose for a moment that 5 is not on this line; 4 cannot be on each of the lines 15, 25, 35, and I assume, in the first instance,

that it is not on any one of these lines; thus the points 1, 2, 4 are no one of them on the line 35, and hence, by the like reasoning, 1, 2, 4 are in a line; that is, 4 is on the line 12, or, the points 1, 2, 3, 4 are in a line. If, however, 4 is on one of the lines 15, 25, 35, say it is on 35; then it is not on 25, and no one of the points 1, 3, 4 is on 25; hence, by the like reasoning, 1, 2, 4 are in a line. In this case, however, 4 being, by supposition, on the line 35, can only be the point 3; that is, 3 and 4 would be coincident, a case which need not be considered. The correct conclusion from the reasoning thus is, not that 1, 2, 3, 4, 5 are in a line, but that some four of these five points, say 1, 2, 3, 4, are in a line.

It is easy to see that, if we have on a line four points, then there exists between the second powers of these points a linear homogeneous relation. For let the distances of the points from a fixed point of the line be a, b, c, d respectively; and let the arbitrary line meet the line in a point O at a distance r from the fixed point. The distances of the four points from the point O are thus equal to $r+a$, $r+b$, $r+c$, $r+d$ respectively; and hence, writing k for the squared cosine of the inclination of the two lines, the second powers of the four points are $=k(r+a)^2$, $k(r+b)^2$, $k(r+c)^2$, $k(r+d)^2$ respectively; and we have thus, between the second powers p_1, p_2, p_3, p_4 of the four points, the homogeneous linear relation

$$\begin{vmatrix} p_1, & 1, & a, & a^2 \\ p_2, & 1, & b, & b^2 \\ p_3, & 1, & c, & c^2 \\ p_4, & 1, & d, & d^2 \end{vmatrix} = 0.$$

Moreover we can, in an infinite number of ways, form a linear combination

$$A_1 p_1 + A_2 p_2 + A_3 p_3 + A_4 p_4$$

of these second powers, which shall be independent of r, and have any given value whatever. We can therefore make this sum to be equal to the second power of any point 5 whatever (not on the line containing the four points) in relation to the arbitrary line; that is, given the four points in a line, and any other fifth point, we can establish between the second powers of these five points a linear homogeneous relation

$$A_1 p_1 + A_2 p_2 + A_3 p_3 + A_4 p_4 + A_5 p_5 = 0,$$

with non-evanescent values for each of the coefficients A. We thus see how, to the linear homogeneous relation between the second powers of the five points, there corresponds properly the non-symmetric relation: four of the five points are in a line.

The most simple cases are when $p=n$, viz. $p=n=2$, $N=6$; $p=n=3$, $N=10$, and generally $N=\frac{1}{2}(n+1)(n+2)$; and for these Clifford's theorem is easily verified. I have not examined the other cases, but it is probable that in each of them a correction is required.

928.

ON THE ANALYTICAL THEORY OF THE CONGRUENCY.

[From the *Proceedings of the London Mathematical Society*, vol. XXIII. (1892), pp. 185—188.]

IF the lines of a congruency are considered as issuing from the several points of a surface, or say as the quasi-normals of a surface, then the fundamental geometrical theory is established by an analysis closely similar to that for the theory of the curvature of a surface; viz. it is shown that each quasi-normal is intersected by two consecutive quasi-normals in two points respectively (corresponding to the centres of curvature), or say in two foci; we have on the surface two series of curves of quasi-curvature—only these do not in general intersect at right angles; the intersecting quasi-normals form two series of developable surfaces, each touching the surface of centres (or focal surface), along its cuspidal edge, &c.; and, in particular, each quasi-normal is a bitangent of the focal surface, touching it at the two foci respectively.

But the analysis assumes a very different form if we consider the congruency by itself, without thus connecting it with a surface. Regarding the congruency as determined by means of two equations,

$$U(a, b, c, f, g, h) = 0, \quad V(a, b, c, f, g, h) = 0,$$

between the six coordinates of a line ($af + bg + ch = 0$, as usual), I take (a, b, c, f, g, h) for the coordinates of a particular line of the congruency, $(a_1, b_1, c_1, f_1, g_1, h_1)$ for those of a consecutive line. Denoting, for shortness, the derived functions

$$(\partial_a, \partial_b, \partial_c, \partial_f, \partial_g, \partial_h)\, U$$

by F, G, H, A, B, C, and similarly those of V by F', G', H', A', B', C', we have

$$a_1F + b_1G + c_1H + f_1A + g_1B + h_1C = 0,$$
$$a_1F' + b_1G' + c_1H' + f_1A' + g_1B' + h_1C' = 0;$$

viz. the consecutive line $(a_1, b_1, c_1, f_1, g_1, h_1)$ belongs to the linear congruency defined by these two equations.

Forming with these a linear combination

$$(\lambda F+\mu F')\,a_1+(\lambda G+\mu G')\,b_1+(\lambda H+\mu H')\,c_1+(\lambda A+\mu A')\,f_1+(\lambda B+\mu B')\,g_1+(\lambda C+\mu C')\,h_1=0,$$

we may determine the ratio $\lambda:\mu$ by the equation

$$(\lambda A+\mu A')(\lambda F+\mu F')+(\lambda B+\mu B')(\lambda G+\mu G')+(\lambda C+\mu C')(\lambda H+\mu H')=0,$$

that is,

$$\lambda^2(AF+BG+CH)+\lambda\mu(AF'+BG'+CH'+FA'+GB'+HC')+\mu^2(A'F'+B'G'+C'H')=0;$$

we have thus two values of $\lambda:\mu$; and, denoting the corresponding values of $\lambda F+\mu F', \ldots, \lambda C+\mu C'$ by $(f_2, g_2, h_2, a_2, b_2, c_2)$ and $(f_3, g_3, h_3, a_3, b_3, c_3)$ respectively, we have

$$a_2f_2+b_2g_2+c_2h_2=0,\quad a_3f_3+b_3g_3+c_3h_3=0,$$

and

$$a_1f_2+b_1g_2+c_1h_2+f_1a_2+g_1b_2+h_1c_2=0,$$
$$a_1f_3+b_1g_3+c_1h_3+f_1a_3+g_1b_3+h_1c_3=0;$$

viz. we have thus two lines $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$, not in general meeting each other, each of which is met by the line $(a_1, b_1, c_1, f_1, g_1, h_1)$; say, for shortness, the lines (a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$ are the lines 0, 1, 2, 3 respectively.

We may, in the foregoing investigation, substitute, for the coordinates of the line 1, those of the line 0; and it hence appears—what is indeed obvious—that the line 0 meets each of the lines 2 and 3. Supposing now that the lines 0 and 1 meet each other, that is, that we have

$$af_1+bg_1+ch_1+fa_1+gb_1+hc_1=0,$$

then it is clear that the line 1 must pass through the intersection of the lines 0, 2, or else through the intersection of the lines 0, 3; in fact, if 0 and 1 intersect in a point not on the line 2 or 3, then we have the line 0 as a line passing through this point and meeting each of the lines 2 and 3; and also the line 1 as a line passing through this point and meeting each of the lines 2 and 3; that is, the lines 0 and 1 would be one and the same line.

It thus appears that, considering the line 0 as given, we have two lines 2 and 3 each meeting this line, say in the points P_2 and P_3 respectively; and that, this being so, the consecutive line 1 meets the line 0 either in the point P_2 or else in the point P_3, viz. that there are two consecutive lines 1, say 1_2 and 1_3, meeting the line 0 in the points 2 and 3 respectively. These points are thus given as the intersections of the line (a, b, c, f, g, h) with the lines $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$ respectively; viz. supposing that $\lambda:\mu$ is determined by the above-mentioned quadric equation, and calling its roots $\lambda_2:\mu_2$ and $\lambda_3:\mu_3$, then we have

$$a_2, b_2, c_2, f_2, g_2, h_2=\lambda_2A+\mu_2A', \ldots, \lambda_2H+\mu_2H',$$
$$a_3, b_3, c_3, f_3, g_3, h_3=\lambda_3A+\mu_3A', \ldots, \lambda_3H+\mu_3H'.$$

For the point 2, we have

$$\begin{array}{llll} . & hy - gz + aw = 0, & . & h_2y - g_2z + a_2w = 0, \\ -hx & . \; + fz + bw = 0, & -h_2x & . \; + f_2z + b_2w = 0, \\ gx - fy & . \; + cw = 0, & g_2x - f_2y & . \; + c_2w = 0, \\ -ax - by - cz & . \; = 0, & -a_2x - b_2y - c_2z & . \; = 0, \end{array}$$

each set of four equations being equivalent to two equations, in virtue of the relations $af + bg + ch = 0$, $a_2f_2 + b_2g_2 + c_2h_2 = 0$ respectively. There is no completely symmetrical expression for the values of x, y, z, w; according as we derive them from the first equations, the second equations, the third equations, or the fourth equations of each set, we obtain

$$\begin{aligned} x : y : z : w = \; & \Theta_1 \; : ag_2 - ga_2 : ah_2 - ha_2 : gh_2 - hg_2, \\ = \; & bf_2 - fb_2 : \; \Theta_2 \; : bh_2 - hb_2 : hf_2 - fh_2, \\ = \; & cf_2 - fc_2 : cg_2 - gc_2 : \; \Theta_3 \; : fg_2 - gf_2, \\ = \; & bc_2 - cb_2 : ca_2 - ac_2 : ab_2 - ba_2 : \; \Theta \; ; \end{aligned}$$

where

$$\begin{aligned} \Theta_1 &= -fa_2 - bg_2 - ch_2, & &= af_2 + gb_2 + hc_2, \\ \Theta_2 &= -af_2 - gb_2 - ch_2, & &= fa_2 + bg_2 + hc_2, \\ \Theta_3 &= -af_2 - bg_2 - hc_2, & &= fa_2 + gb_2 + ch_2, \\ \Theta &= -af_2 - bg_2 - ch_2, & &= fa_2 + gb_2 + hc_2. \end{aligned}$$

For the point 3, we have, of course, the same formulæ, with the suffix 3 instead of 2.

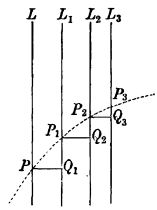
929.

NOTE ON THE SKEW SURFACES APPLICABLE UPON A GIVEN SKEW SURFACE.

[From the *Proceedings of the London Mathematical Society*, vol. XXIII. (1892), pp. 217—225.]

THE question was considered by Bonnet, in § 7 of his "Mémoire sur la théorie générale des surfaces," *Jour. École Polyt.*, Cah. 32 (1848); I resume it here, making a greater use of the line of striction.

We may construct a skew surface, inextensible but flexible about its generating lines, as follows: Imagine a flexible extensible plane, and in it the rigid parallel lines L, L_1, L_2, L_3, &c., connected each with the following one by the rigid lines



PQ_1, P_1Q_2, P_2Q_3, &c., where PQ_1 cuts L, L_1, P_1Q_2 cuts L_1, L_2, &c., at right angles; the angles LPP_1, $L_1P_1P_2$, $L_2P_2P_3$, &c., are taken to be ω, ω_1, ω_2, &c., respectively. Keeping the line L fixed, we may twist the whole plane $L_1L_2L_3$ ... round PQ_1, so that the line L_1 becomes inclined at a small angle to L, these lines now having PQ_1 for their shortest distance, and the lines L_2, L_3, &c., remaining parallel to L_1 in its new position; the foregoing twisting implies an extension (increasing with the

distance on each side from P) of the strip or element between the lines L, L_1; after this twisting, we imagine the strip in question to become rigid. The amount of the twist is such that, if $d\phi$ be the inclination of the lines L, L_1, we have $d\phi = PQ_1 \div \tau$, viz. the twist $d\phi$ is in a certain proportion to the shortest distance PQ_1. Similarly, keeping the line L_1 fixed, we twist the whole plane $L_2L_3 \ldots$ round P_1Q_2, the amount of the twist or inclination $d\phi$ of the lines L_1 and L_2 being $= P_1Q_2 \div \tau_1$; after the twist, we imagine the strip or element between the two lines L_1, L_2 to become rigid. Proceeding in this manner, we have the series of rigid elements LL_1, L_1L_2, L_2L_3, &c.; putting the line L in any given position, and then keeping it fixed, we may turn the element LL_1 round this line so as to bring L_1 into a certain position; then, keeping L_1 fixed in this position, we may turn the element L_1L_2 round L_1 so as to bring L_2 into a certain position, and so on. To explain this further, imagine through a point O a series of lines K, K_1, K_2, K_3, &c., such that the inclination of K and K_1 is equal to that of L and L_1, the inclination of K_1 and K_2 is equal to that of L_1 and L_2, and so on, all these inclinations being otherwise arbitrary; or say that we have the double-triangle strips or elements KK_1, K_1K_2, K_2K_3, &c., bounded by pairs of lines, at given infinitesimal inclinations the two lines of a pair to each other, and forming a flexible double pyramid, which may be bent into any given form whatever assumed at pleasure; and this being so, we see that the system of the strips or elements LL_1, L_1L_2, L_2L_3, &c., may be so bent that the lines L, L_1, L_2, &c., shall be parallel to the lines K, K_1, K_2, &c., respectively.

Supposing the distances PQ_1, P_1Q_2, P_2Q_3, &c., to be all of them infinitesimal, we have a skew surface containing upon it a curve $P_1P_2P_3$, &c., which is the line of striction, viz. this is the locus of the point on a generating line which is the nearest point to the consecutive generating line. The line of striction cuts the several generating lines at an angle ω, variable from line to line, which is called the obliquity; and the inclination between two consecutive generating lines is in a certain ratio to the shortest distance between the two lines. Let the inclination = shortest distance $\div \tau$, this magnitude τ being variable from line to line: its reciprocal τ^{-1} is called the "torsion"; so that the obliquity ω and the reciprocal of the torsion τ may be regarded as functions of s, the arc of the line of striction measured from any fixed point. The skew surface is thus composed of rigid strips or elements, each included between two consecutive lines. We have further seen that the surface may be bent by turning these rigid elements about the successive generating lines, in such wise that the generating lines become parallel to the generating lines of an arbitrary cone, which is called the "asymptotic" cone (otherwise the director cone); say the surface may be bent so that it shall have a given asymptotic cone.

I consider a given skew surface; I take x, y, z for the coordinates of a point on the line of striction, and α, β, γ for the cosine-inclinations of the generating line through this point; x, y, z, α, β, γ are regarded as functions of s, the length of or distance along the line of striction measured from any fixed point thereof; and I use accents to denote differentiation in regard to s. We have

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

and therefore

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0;$$

also

$$x'^2 + y'^2 + z'^2 = 1;$$

and therefore

$$x'x'' + y'y'' + z'z'' = 0;$$

x', y', z' are the cosine-inclinations of the tangent at the point x, y, z.

I remark also that, writing

$$\rho^{-1} = \sqrt{x''^2 + y''^2 + z''^2},$$

so that ρ is the radius of absolute curvature, we have $\rho x''$, $\rho y''$, $\rho z''$ for the cosine-inclinations of the binormal (or perpendicular to the osculating plane).

If from the point (x, y, z) we draw a perpendicular to the consecutive generating line through the point $(x+dx, y+dy, z+dz)$, this will also be perpendicular to the generating line through the point (x, y, z), and we thus find

$$\alpha'x' + \beta'y' + \gamma'z' = 0$$

as the condition that the point (x, y, z) may be, as it is assumed to be, a point on the line of striction.

For the proof hereof, take for the moment x_1, y_1, z_1 the coordinates of P, and α_1, β_1, γ_1 for the cosine-inclinations of L; then, considering the line PQ_1, this passes through P and cuts the line L_1; taking its equation to be

$$\frac{X-x}{A} = \frac{Y-y}{B} = \frac{Z-z}{C},$$

this meets the line

$$\frac{X-x_1}{\alpha_1} = \frac{Y-y_1}{\beta_1} = \frac{Z-z_1}{\gamma_1},$$

and we thence find

$$\begin{vmatrix} A, & B, & C \\ x_1-x, & y_1-y, & z_1-z \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix} = 0.$$

Writing x_1, y_1, $z_1 = x + x'\,ds$, $y + y'\,ds$, $z + z'\,ds$, ds divides out, and we have

$$A(\beta_1 z' - \gamma_1 y') + B(\gamma_1 x' - \alpha_1 z') + C(\alpha_1 y' - \beta_1 x') = 0;$$

the line in question cuts L_1 and L; that is, we have

$$A\alpha_1 + B\beta_1 + C\gamma_1 = 0,$$

$$A\alpha + B\beta + C\gamma = 0;$$

or, eliminating A, B, C, we find

$$(\beta\gamma_1 - \beta_1\gamma)(\beta_1 z' - \gamma_1 y') + (\gamma\alpha_1 - \gamma_1\alpha)(\gamma_1 x' - \alpha_1 z') + (\alpha\beta_1 - \alpha_1\beta)(\alpha_1 y' - \beta_1 x') = 0,$$

that is,

$$(\alpha_1 x' + \beta_1 y' + \gamma_1 z')(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - (\alpha x' + \beta y' + \gamma z')(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = 0.$$

But we have

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1,$$

and, writing

$$\alpha_1,\ \beta_1,\ \gamma_1 = \alpha + \alpha' ds,\ \beta + \beta' ds,\ \gamma + \gamma' ds,$$

we find

$$\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = \alpha^2 + \beta^2 + \gamma^2 + (\alpha\alpha' + \beta\beta' + \gamma\gamma')\, ds = 1\,;$$

the equation thus is

$$(\alpha_1 x' + \beta_1 y' + \gamma_1 z') - (\alpha x' + \beta y' + \gamma z') = 0\,;$$

that is,

$$\alpha' x' + \beta' y' + \gamma' z' = 0,$$

the required equation.

Calling the inclination of the generating line through the point (x, y, z) to the tangent of the line of striction the "obliquity," and denoting it by ω, we have

$$\alpha x' + \beta y' + \gamma z' = \cos\omega.$$

Calling the inclination of the two consecutive generating lines divided by the shortest distance between these lines the "torsion," and denoting it by τ^{-1}, we have

$$\alpha'^2 + \beta'^2 + \gamma'^2 = \frac{\sin^2\omega}{\tau^2}.$$

In proof hereof, if for a moment ϕ is the inclination of the lines L and L_1 to each other, then

$$\cos\phi = \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1\,;$$

and therefore

$$\sin^2\phi = (\beta\gamma_1 - \beta_1\gamma)^2 + (\gamma\alpha_1 - \gamma_1\alpha)^2 + (\alpha\beta_1 - \alpha_1\beta)^2,$$

viz. writing

$$\alpha_1,\ \beta_1,\ \gamma_1 = \alpha + \alpha' ds,\ \beta + \beta' ds,\ \gamma + \gamma' ds,$$

this is

$$\begin{aligned}\sin^2\phi &= ds^2\,\{(\beta\gamma' - \beta'\gamma)^2 + (\gamma\alpha' - \gamma'\alpha)^2 + (\alpha\beta' - \alpha'\beta)^2\}\\ &= ds^2\,\{(\alpha^2 + \beta^2 + \gamma^2)(\alpha'^2 + \beta'^2 + \gamma'^2) - (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2\},\\ &= ds^2\,(\alpha'^2 + \beta'^2 + \gamma'^2)\,;\end{aligned}$$

whence, if for a moment the shortest distance between the two lines is called δ, then we have

$$\frac{\sin\phi}{\delta} = \tau^{-1} = \frac{\sin\phi}{ds\sin\omega}\,;$$

that is,

$$\frac{\sin^2\omega}{\tau^2} = \frac{\sin^2\phi}{ds^2} = \alpha'^2 + \beta'^2 + \gamma'^2,$$

the required equation.

We have thus the five equations

$$\alpha^2+\beta^2+\gamma^2=1,$$

$$x'^2+y'^2+z'^2=1,$$

$$\alpha'x'+\beta'y'+\gamma'z'=0,$$

$$\alpha x'+\beta y'+\gamma z'=\cos\omega,$$

$$\alpha'^2+\beta'^2+\gamma'^2=\frac{\sin^2\omega}{\tau^2};$$

and if we herein consider ω and τ as denoting given functions of s, all the skew surfaces which satisfy these equations will be surfaces applicable one on the other.

Adding to the foregoing the derived equations

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'=0,$$

$$x'x''+y'y''+z'z''=0,$$

$$\alpha x''+\beta y''+\gamma z''=-\sin\omega\,.\,\omega',$$

$$\alpha x'''+\beta y'''+\gamma z'''+\alpha'x''+\beta'y''+\gamma'z''=-\sin\omega\,.\,\omega''-\cos\omega\,.\,\omega'^2,$$

we find without difficulty

$$\alpha',\ \beta',\ \gamma'=\frac{\beta z'-\gamma y'}{\tau},\ \frac{\gamma x'-\alpha z'}{\tau},\ \frac{\alpha y'-\beta x'}{\tau},$$

$$\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta=\frac{-x'+\alpha\cos\omega}{\tau},\ \frac{-y'+\beta\cos\omega}{\tau},\ \frac{-z'+\gamma\cos\omega}{\tau}.$$

Putting, for shortness,

$$\Theta=-\sin\omega\,.\,\omega''-\cos\omega\,.\,\omega'^2-(\alpha x'''+\beta y'''+\gamma z'''),$$

$$\nabla=\begin{vmatrix}\alpha, & \beta, & \gamma\\ x', & y', & z'\\ x'', & y'', & z''\end{vmatrix},$$

and, as above,

$$\rho^{-2}=x''^2+y''^2+z''^2,$$

we find

$$\nabla^2=\sin^2\omega\left(\frac{1}{\rho^2}-\omega'^2\right),\quad \tau=\frac{\nabla}{\Theta}=\frac{\sin\omega}{\rho\Theta}\sqrt{1-\rho^2\omega'^2},$$

and to these may be joined

$$\beta z'-\gamma y'=\rho^2\{\nabla x''+\omega'\sin\omega\,(y'z''-y''z')\},$$

$$\gamma x'-\alpha z'=\rho^2\{\nabla y''+\omega'\sin\omega\,(z'x''-z''x')\},$$

$$\alpha y'-\beta x'=\rho^2\{\nabla z''+\omega'\sin\omega\,(x'y''-x''y')\}.$$

I remark that, supposing the line of striction to be given, that is, (x, y, z) to be given as functions of s, and moreover the obliquity ω to be given as a function of s, the position of the generating line through the point (x, y, z) will be given, and also the torsion τ^{-1}. In fact, among the foregoing equations, we have

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

$$\alpha x' + \beta y' + \gamma z' = \cos\omega,$$

$$\alpha x'' + \beta y'' + \gamma z'' = -\sin\omega \,.\, \omega',$$

which equations determine α, β, γ, that is, the position of the generating line; the position of the consecutive generating line is then also determined; we thus have α', β', γ', and thence the torsion, which is given by the foregoing equation

$$\tau = \frac{\sin\omega}{\rho\Theta}\sqrt{1-\rho^2\omega'^2}.$$

Supposing that the line of striction is not given, but that the obliquity ω and the torsion τ^{-1} are given as functions of s, we have then the foregoing *five* equations, which are not sufficient for the determination of x, y, z, α, β, γ; but, joining to them an assumed homogeneous relation between (α, β, γ), the six quantities will be determined. The assumed homogeneous relation between (α, β, γ) is the equation of the asymptotic cone; and we have thus the theorem that, given the asymptotic cone, and also the obliquity and the torsion as functions of s (the arc of the line of striction), then this line of striction, and the skew surface the locus of the generating lines, will be determined.

We have between α, β, γ the assumed homogeneous equation, say $U=0$; and among the foregoing equations, the equations

$$\alpha^2+\beta^2+\gamma^2=1, \text{ and } \alpha'^2+\beta'^2+\gamma'^2=\frac{\sin^2\omega}{\tau^2},$$

a given function of s; these equations give, by means of an integration, α, β, γ as functions of s; and we have then the above-mentioned equations

$$x' - \alpha\cos\omega,\ y' - \beta\cos\omega,\ z' - \gamma\cos\omega = \tau(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta),$$

which give, by integration, x, y, z as functions of s. And the skew surfaces thus obtained for an assumed asymptotic cone $U=0$ will be the required system of skew surfaces applicable the one on the other.

For example, if ω, τ are constants, and we take, for $U=0$, the equation

$$\gamma^2 - c(\alpha^2+\beta^2) = 0,$$

that is, if we consider the skew surfaces of constant obliquity and torsion and which have for asymptotic cone a right circular cone: it easily appears that the line of striction is a helix traced upon a right circular cylinder, and that the generating lines are at a constant inclination to the axis of the cylinder, and all of them touch the cylinder. In fact, for such a surface (a kind of helicoid), it is in the first

place obvious that the helix is the line of striction, and next that the obliquity and the torsion are each of them constant.

Supposing that the obliquity and the torsion are ω, τ, and that the inclination of the generating lines to the axis of the cylinder is $90^\circ - \omega - \delta$, δ being a given constant, we find

$$x = \frac{\tau \cos\delta \cos(\omega+\delta)}{\sin\omega} \cos\left\{\frac{s \sin\omega}{\tau\cos(\omega+\delta)}\right\},$$

$$y = \frac{\tau \cos\delta \cos(\omega+\delta)}{\sin\omega} \sin\left\{\frac{s \sin\omega}{\tau\cos(\omega+\delta)}\right\},$$

$$z = s \sin\delta;$$

$$\alpha = -\cos(\omega+\delta) \sin\left\{\frac{s \sin\omega}{\tau\cos(\omega+\delta)}\right\},$$

$$\beta = \cos(\omega+\delta) \cos\left\{\frac{s \sin\omega}{\tau\cos(\omega+\delta)}\right\},$$

$$\gamma = \sin(\omega+\delta);$$

where observe that the line of striction lies on the circular cylinder,

$$\text{radius} = \tau \cos\delta \cos(\omega+\delta) \div \sin\omega.$$

In particular, if $\delta = 0$, we have

$$x = \tau \cot\omega \cos\frac{s}{\tau\cot\omega}, \quad \alpha = -\cos\omega \sin\frac{s}{\tau\cot\omega},$$

$$y = \tau \cot\omega \sin\frac{s}{\tau\cot\omega}, \quad \beta = \cos\omega \cos\frac{s}{\tau\cot\omega},$$

$$z = 0, \qquad \gamma = \sin\omega;$$

and, for $\delta = -\omega$, we have

$$x = \tau \cot\omega \cos\frac{s \sin\omega}{\tau}, \quad \alpha = -\sin\frac{s \sin\omega}{\tau},$$

$$y = \tau \cot\omega \sin\frac{s \sin\omega}{\tau}, \quad \beta = \cos\frac{s \sin\omega}{\tau},$$

$$z = -s \sin\omega, \qquad \gamma = 0.$$

The first of these is the skew hyperboloid of revolution

$$X^2 + Y^2 - (Z^2 + \tau^2)\cot^2\omega = 0,$$

(radius of gorge $= \tau \cot\omega$); and the second of them is the helicoid (radius of cylinder $= \tau\cot\omega$), the generating lines of which are the tangents perpendicular to the axis of the cylinder; these are, in fact, surfaces found by Bonnet in his memoir above referred to. We may imagine the surface passing from the first form, in which the semi-aperture of the asymptotic cone is ω, through the series of forms belonging to the general formulæ involving δ, to the second form, in which the semi-aperture is $= 90^\circ$, i.e. in which the asymptotic cone is replaced by a plane.

930.

SUR LA SURFACE DES ONDES.

[From the *Annali di Matematica*, Ser. II., t. XX. (1892), pp. 1—18.]

1. IL y a, dans les *Annali di Matematica*, tom. II. (1859), deux Notes très intéressantes sur cette surface: Combescure, "Sur les lignes de courbure de la surface des ondes," pp. 278—285, et Brioschi, "Osservazioni sulla medesima quistione," pp. 285—287. Je me propose de réproduire et développer cette théorie, en changeant les notations et l'arrangement des recherches de la manière qui me paraît convenable.

2. Je prends a, b, c pour les carrés des semiaxes $(a > b > c)$, et j'écris

$$A,\ B,\ C = a+b+c,\quad ab+ac+bc,\quad abc;$$

$$\alpha,\ \beta,\ \gamma = b-c,\quad c-a,\quad a-b,$$

$(\alpha+\beta+\gamma=0$, et ainsi, $\alpha=+$, $\gamma=+$ et β, $=-\alpha-\gamma$, $=-$, et en magnitude absolue plus grand que α ou $\gamma)$:

$$\begin{aligned}\xi &= \quad x^2 + \quad y^2 + \quad z^2,\\ \eta &= \quad ax^2 + \quad by^2 + \quad cz^2,\\ \zeta &= a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2,\end{aligned}$$

et de là réciproquement

$$\begin{aligned}\beta\gamma x^2 &= \zeta - a\eta - bc\xi,\\ \gamma\alpha y^2 &= \zeta - b\eta - ca\xi,\\ \alpha\beta z^2 &= \zeta - c\eta - ab\xi.\end{aligned}$$

3. L'équation de la surface est

$$\xi\eta - \zeta + abc = 0,$$

et de là, en écrivant $\zeta = \xi\eta - abc$, on obtient pour un point de la surface

$$\begin{aligned}\beta\gamma x^2 &= (\xi - a)(\eta - bc),\\ \gamma\alpha y^2 &= (\xi - b)(\eta - ca),\\ \alpha\beta z^2 &= (\xi - c)(\eta - ab),\end{aligned}$$

équations qui donnent les valeurs des coordonnées (x, y, z) du point en termes de deux paramètres ξ, η. Je remarque que ces équations donnent

$$\frac{x^2}{\xi - a} + \frac{y^2}{\xi - b} + \frac{z^2}{\xi - c} = 1,$$

$$\frac{x^2}{\eta - bc} + \frac{y^2}{\eta - ca} + \frac{z^2}{\eta - ab} = 0;$$

la première équation, en y considérant ξ comme dénotant $x^2 + y^2 + z^2$, et la seconde équation, en y considérant η comme dénotant $ax^2 + by^2 + cz^2$, sont équivalentes l'une et l'autre à l'équation $\xi\eta - \zeta + abc = 0$ de la surface.

4. Je prends λ, μ, ν pour les cosinus des inclinations de la normale (ou, ce qui est la même chose, de la perpendiculaire par le centre sur le plan tangent) aux trois axes, et v pour le carré de la longueur de ce perpendiculaire, $v = (\lambda x + \mu y + \nu z)^2$; on a

$$\lambda = \frac{x}{D}\{a\xi + \eta - a(b + c)\},$$

$$\mu = \frac{y}{D}\{b\xi + \eta - b(c + a)\},$$

$$\nu = \frac{z}{D}\{c\xi + \eta - c(a + b)\},$$

et de là, par l'équation $\lambda^2 + \mu^2 + \nu^2 = 1$, on a

$$\begin{aligned}\alpha\beta\gamma D^2 = \quad & \alpha(\xi - a)(\eta - bc)\{a\xi + \eta - a(b + c)\}^2 \\ & + \beta(\xi - b)(\eta - ca)\{b\xi + \eta - b(c + a)\}^2 \\ & + \gamma(\xi - c)(\eta - ab)\{c\xi + \eta - c(a + b)\}^2,\end{aligned}$$

équation laquelle (en réduisant à moyen des rélations $\alpha + \beta + \gamma = 0$, $a\alpha + b\beta + c\gamma = 0$, $a^2\alpha + b^2\beta + c^2\gamma = -\alpha\beta\gamma$, &c.) devient

$$\alpha\beta\gamma D^2 = \alpha\beta\gamma(\xi\eta - C)(\eta - \xi^2 + A\xi - B);$$

et l'on a ainsi

$$D = \sqrt{(\xi\eta - C)(\eta - \xi^2 + A\xi - B)}.$$

5. On trouve de même manière

$$\lambda x + \mu y + \nu z = \frac{\xi\eta - C}{D},$$

$$a\lambda x + b\mu y + c\nu z = \frac{\eta}{D}(\eta - \xi^2 + A\xi - B),$$

$$bc\lambda x + ca\mu y + ab\nu z = \frac{1}{D}\{-(\xi\eta - C)(\eta - B) - C(\eta - \xi^2 + A\xi - B)\};$$

ou, en y substituant la valeur de D,

$$\lambda x + \mu y + \nu z, \quad = \sqrt{v}, \quad = \sqrt{\frac{\xi\eta - C}{\eta - \xi^2 + A\xi - B}}.$$

La première de ces équations donne

$$v = \frac{\xi\eta - C}{\eta - \xi^2 + A\xi - B},$$

et de là réciproquement

$$a\lambda x + b\mu y + c\nu z = \eta\sqrt{\frac{\eta - \xi^2 + A\xi - B}{\xi\eta - C}},$$

$$bc\lambda x + ca\mu y + ab\nu z = -(\eta - B)\sqrt{\frac{\xi\eta - C}{\eta - \xi^2 + A\xi - B}} - C\sqrt{\frac{\eta - \xi^2 + A\xi - B}{\xi\eta - C}};$$

$$\eta = \frac{b(\xi^2 - A\xi + B) - C}{v - \xi}, \quad = \xi^2 - A\xi + B + \frac{\xi - a\,.\,\xi - b\,.\,\xi - c}{v - \xi}.$$

On a ainsi les formules

$$\lambda x + \mu y + \nu z = \sqrt{v},$$

$$a\lambda x + b\mu y + c\nu z = \frac{\eta}{\sqrt{v}},$$

$$bc\lambda x + ca\mu y + ab\nu z = -\eta\sqrt{v} + B\sqrt{v} - \frac{C}{\sqrt{v}};$$

et de là aussi

$$a(b+c)\lambda x + b(c+a)\mu y + c(a+b)\nu z = \eta\sqrt{v} + \frac{C}{\sqrt{v}}.$$

6. On peut introduire dans les formules v au lieu de η; les deux paramètres seront ainsi: ξ, carré de la distance au centre; v, carré de la perpendiculaire sur le plan tangent.

On a d'abord

$$\eta - bc = \xi^2 - (a+b+c)\xi + a(b+c) + \frac{\xi - a\,.\,\xi - b\,.\,\xi - c}{v - \xi},$$

$$= (\xi - a)\left\{\xi - b - c + \frac{\xi - b\,.\,\xi - c}{v - \xi}\right\},$$

$$= \frac{\xi - a}{v - \xi}\{bc - (b+c)v + v\xi\};$$

et ainsi

$$\beta\gamma x^2 = \frac{(\xi - a)^2}{v - \xi}\{bc - (b+c)v + v\xi\},$$

et de même

$$\gamma\alpha y^2 = \frac{(\xi - b)^2}{v - \xi}\{ca - (c+a)v + v\xi\},$$

$$\alpha\beta z^2 = \frac{(\xi - c)^2}{v - \xi}\{ab - (a+b)v + v\xi\},$$

lesquelles sont les expressions des coordonnées ξ, η, ζ en termes des paramètres ξ, v.

7. On a

$$vD^2 = (\xi\eta - C)^2,$$

$$\xi\eta - C = \xi - a.\xi - b.\xi - c.\left\{1 + \frac{\xi}{v-\xi}\right\}, \quad = \frac{v.\xi - a.\xi - b.\xi - c}{v-\xi},$$

et de là

$$D^2 = v\left(\frac{\xi - a.\xi - b.\xi - c}{v-\xi}\right)^2.$$

On a aussi

$$a\xi + \eta - a(b+c) = \xi - b.\xi - c.\left(1 + \frac{\xi - a}{v-\xi}\right), \quad = \frac{v - a.\xi - b.\xi - c}{v-\xi},$$

et de là

$$\lambda = \frac{x}{D}\,\frac{v - a.\xi - b.\xi - c}{v-\xi},$$

ce qui donne

$$\beta\gamma\lambda^2 = \frac{(v-a)^2}{v.v-\xi}\{bc - (b+c)v + v\xi\},$$

et de même

$$\gamma\alpha\mu^2 = \frac{(v-b)^2}{v.v-\xi}\{ca - (c+a)v + v\xi\},$$

$$\alpha\beta\nu^2 = \frac{(v-c)^2}{v.v-\xi}\{ab - (a+b)v + v\xi\},$$

lesquelles sont les expressions de λ, μ, ν en termes des paramètres ξ, v.

8. On obtient

$$\alpha\beta\gamma\left\{\frac{\lambda^2}{v-a} + \frac{\mu^2}{v-b} + \frac{\nu^2}{v-c}\right\} = \frac{1}{v.v-\xi}.\alpha(v-a)\{bc - (b+c)v + v\xi\}$$
$$+ \beta(v-b)\{ca - (c+a)v + v\xi\}$$
$$+ \gamma(v-c)\{ab - (a+b)v + v\xi\},$$

ou, en réduisant comme auparavant,

$$\frac{\lambda^2}{v-a} + \frac{\mu^2}{v-b} + \frac{\nu^2}{v-c} = 0.$$

9. Je rappelle que l'équation du plan tangent est

$$\lambda x + \mu y + \nu z - \sqrt{v} = 0,$$

où v est déterminé comme fonction de λ, μ, ν par l'équation qui vient d'être donnée; et qu'en considérant λ, μ, ν, v comme des paramètres variables qui satisfont à cette équation et à l'équation $\lambda^2 + \mu^2 + \nu^2 = 1$, l'on obtient la surface comme enveloppe de ce plan tangent.

10. On a ainsi v comme l'une des racines de l'équation quadrique

$$\frac{\lambda^2}{\theta - a} + \frac{\mu^2}{\theta - b} + \frac{\nu^2}{\theta - c} = 0;$$

en dénotant par u l'autre racine, on a donc

$$\theta^2 - \{(b+c)\lambda^2 + (c+a)\mu^2 + (a+b)\nu^2\}\theta + bc\lambda^2 + ca\mu^2 + ab\nu^2 = \theta - u.\theta - v,$$

et de là

$$\begin{aligned} u+v &= (b+c)\lambda^2 + (c+a)\mu^2 + (a+b)\nu^2, \\ uv &= bc\lambda^2 + ca\mu^2 + ab\nu^2. \end{aligned}$$

La première de ces équations peut aussi s'écrire sous la forme

$$A-u-v = a\lambda^2 + b\mu^2 + c\nu^2,$$

et la seconde sous la forme

$$B-uv = a(b+c)\lambda^2 + b(c+a)\mu^2 + c(a+b)\nu^2.$$

11. J'observe que λ, μ, ν sont les cosinus des inclinations de la perpendiculaire par le centre sur le plan tangent au point x, y, z, la longueur de cette perpendiculaire étant $\sqrt{v}$; il y a un plan tangent parallèle qui corresponde aux mêmes valeurs de λ, μ, ν, et évidemment on a alors $\sqrt{u}$ pour la longueur de la perpendiculaire sur ce plan tangent parallèle; autrement dit, les équations de deux plans tangents sont

$$\lambda X + \mu Y + \nu Z - \sqrt{v} = 0,$$

$$\lambda X + \mu Y + \nu Z - \sqrt{u} = 0.$$

Il convient de remarquer qu'on a pris (x, y, z) pour les coordonnées du point de la surface qui est le point de contact du premier de ces deux plans, et qu'ainsi les deux quantités v et u n'entrent pas symétriquement dans les formules.

12. En substituant dans l'expression de $u+v$ ou uv, ou ce qui est plus simple dans celle de $A-u-v$, les valeurs de λ^2, μ^2, ν^2 en termes de v, ξ, on obtient

$$\begin{aligned} \alpha\beta\gamma(A-u-v) = \frac{1}{v.v-\xi}.\, & \alpha a(v-a)^2\{bc-(b+c)v+v\xi\} \\ & + \beta b(v-b)^2\{ca-(c+a)v+v\xi\} \\ & + \gamma c(v-c)^2\{ab-(a+b)v+v\xi\}, \end{aligned}$$

équation laquelle (en réduisant comme auparavant) devient

$$A-u-v = A-2v+\frac{1}{v.v-\xi}\{v-a.v-b.v-c\},$$

c'est-à-dire

$$(v-\xi)(v-u) = \frac{v-a.v-b.v-c}{v},$$

équation qui donne dans des formes très simples, ξ en termes de v, u, et aussi u en termes de v, ξ.

13. On a comme auparavant

$$v = \frac{\xi\eta - C}{\eta - \xi^2 + A\xi - B};$$

je cherche l'expression de u en termes de ξ, η. En écrivant pour abréger

$$\eta - \xi^2 + A\xi - B = M,$$

nous trouvons

$$v-a=\frac{1}{M}(\xi-a)\{a\xi+\eta-a(b+c)\},$$

$$v-b=\frac{1}{M}(\xi-b)\{b\xi+\eta-b(c+a)\},$$

$$v-c=\frac{1}{M}(\xi-c)\{c\xi+\eta-c(a+b)\},$$

$$v-\xi=\frac{1}{M}.\,\xi-a.\,\xi-b.\,\xi-c,$$

$$v\quad=\frac{1}{M}(\xi\eta-C);$$

et de là

$$v-u=\frac{v-a.\,v-b.\,v-c}{v.\,v-\xi}=\frac{\{a\xi+\eta-a(b+c)\}\{b\xi+\eta-b(c+a)\}\{c\xi+\eta-c(a+b)\}}{M(\xi\eta-C)},$$

ou enfin, et en restituant pour M sa valeur,

$$u=\frac{(\xi\eta-C)^2-\{a\xi+\eta-a(b+c)\}\{b\xi+\eta-c(a+b)\}\{c\xi+\eta-a(b+c)\}}{(\xi\eta-C)\{\eta-\xi^2+A\xi-B\}}.$$

14. On peut introduire dans les formules u au lieu de ξ, et ainsi exprimer les coordonnées, &c., en termes des deux paramètres v, u. On a pour cela

$$\xi=v-\frac{v-a.\,v-b.\,v-c}{v.\,v-u},$$

donc

$$\xi-a=(v-a)\left\{1+\frac{v-b.\,v-c}{v.\,v-u}\right\},\quad=-\frac{v-a}{v(v-u)}\{bc-(b+c)u+uv\}.$$

De plus

$$bc-(b+c)v+v\xi=(v-b)(v-c)-v(v-\xi)$$

$$=(v-b)(v-c)-(v-b)(v-c).\frac{v-a}{v-u}=v-b.\,v-c.\left(1-\frac{v-a}{v-u}\right)=-\frac{u-a.\,v-b.\,v-c}{v-u}.$$

Donc

$$\eta-bc=\frac{\xi-a}{v-\xi}\{bc-(b+c)v+v\xi\}=-\frac{\xi-a.\,u-a.\,v-b.\,v-c}{v-\xi.\,v-u}=-\frac{v.\,\xi-a.\,u-a}{v-a},$$

ou enfin, à moyen de la valeur de $\xi-a$,

$$\eta-bc=\frac{u-a}{v-u}\{bc-(b+c)u+uv\}.$$

Donc

$$-\beta\gamma x^2=\frac{u-a.\,v-a}{v(v-u)^2}\{bc-(b+c)u+uv\}^2,$$

et de même

$$-\gamma\alpha y^2=\frac{u-b.\,v-b}{v(v-u)^2}\{ca-(c+a)u+uv\}^2,$$

$$-\alpha\beta z^2=\frac{u-c.\,v-c}{v(v-u)^2}\{ab-(a+b)u+uv\}^2,$$

équations qui donnent les coordonnées x, y, z en termes des deux paramètres v, u.

15. De la valeur ci-dessus donnée pour $\eta - bc$ on déduit celle de η; en effet, on trouve

$$\begin{aligned}\eta(v-u) &= bc(v-a) + (u-a)\{-(b+c)v + uv\}\\ &= v\{u^2 - (a+b+c)u + ab + ac + bc\} - abc,\end{aligned}$$

c'est-à-dire

$$\eta = \frac{v(u^2 - Au + B) - C}{v-u}, \quad = u^2 - Au + B + \frac{u-a\,.\,u-b\,.\,u-c}{v-u},$$

ainsi η est la même fonction de v, u et de v, ξ.

16. De plus

$$\beta\gamma\lambda^2 = \frac{(v-a)^2}{v\,.\,v-\xi}\{bc - (b+c)v + v\xi\}, \quad = -\frac{u-a\,.\,(v-a)^2\,.\,v-b\,.\,v-c}{v\,.\,v-\xi\,.\,v-u},$$

c'est-à-dire

$$-\beta\gamma\lambda^2 = u - a\,.\,v - a,$$

et de même

$$\begin{aligned}-\gamma\alpha\mu^2 &= u - b\,.\,v - b,\\ -\alpha\beta\nu^2 &= u - c\,.\,v - c,\end{aligned}$$

équations qui se déduisent plus simplement des équations

$$\begin{aligned}\lambda^2 + \mu^2 + \nu^2 &= 1,\\ \frac{\lambda^2}{v-a} + \frac{\mu^2}{v-b} + \frac{\nu^2}{v-c} &= 1,\\ \frac{\lambda^2}{u-a} + \frac{\mu^2}{u-b} + \frac{\nu^2}{u-c} &= 0.\end{aligned}$$

Je rappelle les équations

$$\begin{aligned}\lambda x + \mu y + \nu z &= \sqrt{v},\\ a\lambda x + b\mu y + c\nu z &= \frac{\eta}{\sqrt{v}},\\ a(b+c)\lambda x + b(c+a)\mu y + c(a+b)\nu z &= \eta\sqrt{v} + \frac{C}{\sqrt{v}};\end{aligned}$$

et j'ajoute aussi celles-ci

$$\frac{\lambda^2}{(v-a)^2} + \frac{\mu^2}{(v-b)^2} + \frac{\nu^2}{(v-c)^2} = \frac{-1}{v\,.\,v-\xi} = -\frac{v-u}{v-a\,.\,v-b\,.\,v-c},$$

et de même

$$\frac{\lambda^2}{(u-a)^2} + \frac{\mu^2}{(u-b)^2} + \frac{\nu^2}{(u-c)^2} = \frac{v-u}{u-a\,.\,u-b\,.\,u-c}.$$

17. Formules différentielles. Nous avons

$$\lambda dx + \mu dy + \nu dz = 0, \quad x d\lambda + y d\mu + z d\nu = \tfrac{1}{2}\frac{dv}{\sqrt{v}};$$

$$\begin{aligned}2\alpha\beta\gamma(a\lambda dx + b\mu dy + c\nu dz) &= \frac{a\alpha}{D}\{a\xi + \eta - a(b+c)\}\,2\beta\gamma x dx + \&c.\\ &= \frac{a\alpha}{D}\{a\xi + \eta - a(b+c)\}\{(\eta - bc)\,d\xi + (\xi - a)\,d\eta\}\\ &\quad + \frac{b\beta}{D}\{b\xi + \eta - b(c+a)\}\{(\eta - ca)\,d\xi + (\xi - b)\,d\eta\}\\ &\quad + \frac{c\gamma}{D}\{c\xi + \eta - c(a+b)\}\{(\eta - ab)\,d\xi + (\xi - c)\,d\eta\},\end{aligned}$$

ou, en réduisant comme auparavant,

$$= \frac{\alpha\beta\gamma}{D}\{-(\xi\eta - C)\,d\xi + (\eta - \xi^2 + A\xi - B)\,d\eta\},$$

c'est-à-dire

$$2(a\lambda dx + b\mu dy + c\nu dz) = -\sqrt{\frac{\xi\eta - C}{\eta - \xi^2 + A\xi - B}}\,d\xi + \sqrt{\frac{\eta - \xi^2 + A\xi - B}{\xi\eta - C}}\,d\eta,$$

ou enfin

$$a\lambda dx + b\mu dy + c\nu dz = -\tfrac{1}{2}\sqrt{v}\,d\xi + \tfrac{1}{2}\frac{d\eta}{\sqrt{v}},$$

et de là en différentiant l'équation

$$a\lambda x + b\mu y + c\nu z = \frac{\eta}{\sqrt{v}},$$

on déduit

$$ax d\lambda + by d\mu + cz d\nu = \tfrac{1}{2}\sqrt{v}\,d\xi + \tfrac{1}{2}\frac{d\eta}{\sqrt{v}} - \tfrac{1}{2}\frac{\eta}{v}\frac{dv}{\sqrt{v}}.$$

18. Nous avons de plus

$$2\beta\gamma x dx = (\eta - bc)\,d\xi + (\xi - a)\,d\eta,$$

donc

$$4\beta\gamma dx^2 = \frac{\{(\eta - bc)\,d\xi + (\xi - a)\,d\eta\}^2}{\xi - a\,.\,\eta - bc},$$

et de même

$$4\gamma\alpha\,dy^2 = \frac{\{(\eta - ca)\,d\xi + (\xi - b)\,d\eta\}^2}{\xi - b\,.\,\eta - ca},$$

$$4\alpha\beta\,dz^2 = \frac{\{(\eta - ab)\,d\xi + (\xi - c)\,d\eta\}^2}{\xi - c\,.\,\eta - ab};$$

et de là, après les réductions nécessaires,

$$4(\ dx^2 + \ dy^2 + \ dz^2) = \frac{\xi^2 - A\xi + B - \eta}{\xi - a\,.\,\xi - b\,.\,\xi - c}\,d\xi^2 + \frac{C - \xi\eta}{\eta - bc\,.\,\eta - ca\,.\,\eta - ab}\,d\eta^2,$$

$$4(a dx^2 + b dy^2 + c dz^2) = \frac{C - \xi\eta}{\xi - a\,.\,\xi - b\,.\,\xi - c}\,d\xi^2 + \frac{\eta^2 - B\eta + AC - C\xi}{\eta - bc\,.\,\eta - ca\,.\,\eta - ab}\,d\eta^2;$$

en écrivant la première de ces équations sous la forme

$$ds^2 = E d\xi^2 + 2F d\xi d\eta + G d\eta^2,$$

on a

$$E = \tfrac{1}{4}\frac{\xi^2 - A\xi + B - \eta}{\xi - a\,.\,\xi - b\,.\,\xi - c}, \quad F = 0, \quad G = \tfrac{1}{4}\frac{C - \xi\eta}{\eta - bc\,.\,\eta - ca\,.\,\eta - ab}.$$

19. De plus, de l'équation $\lambda^2 + \mu^2 + \nu^2 = 1$, et des valeurs de $u + v$ et uv, on déduit

$$\lambda d\lambda + \ \mu d\mu + \ \nu d\nu = 0,$$

$$a\lambda d\lambda + \ b\mu d\mu + \ c\nu d\nu = \tfrac{1}{2}(\ du + \ dv),$$

$$bc\lambda d\lambda + ca\mu d\mu + ab\nu d\nu = \tfrac{1}{2}(v du + u dv).$$

20. Équation différentielle des courbes de courbure de la surface. En partant de l'équation

$$\begin{vmatrix} d\lambda, & d\mu, & d\nu \\ dx, & dy, & dz \\ \lambda, & \mu, & \nu \end{vmatrix} = 0,$$

ou plus simplement des équations

$$d\lambda : d\mu : d\nu = dx : dy : dz,$$

équivalentes à cette première équation, on déduit

$$(xd\lambda + yd\mu + zd\nu)(a\lambda dx + b\mu dy + c\nu dz) - (xdx + ydy + zdz)(a\lambda d\lambda + b\mu d\mu + c\nu d\nu) = 0,$$

c'est-à-dire

$$\frac{dv}{\sqrt{v}}\left(-\sqrt{v}d\xi + \frac{d\eta}{\sqrt{v}}\right) + d\xi(du + dv) = 0,$$

ou enfin

$$dvd\eta + vdud\xi = 0,$$

laquelle est la forme la plus simple de l'équation dont il s'agit.

21. On a ici

$$\eta = \frac{b(\xi^2 - A\xi + B) - C}{v - \xi}, \quad u = v - \frac{v - a.v - b.v - c}{v(v - \xi)},$$

et il s'agit de substituer ces valeurs dans l'équation différentielle. J'écris pour un moment

$$\eta = \frac{H}{v - \xi}, \quad u = \frac{U}{v - \xi},$$

où l'on a

$$H = v(\xi^2 - A\xi + B) - C,$$

$$U = -v\xi + Av - B + \frac{C}{v};$$

l'équation devient

$$-(v - \xi)\{dHdv + vd\xi dU\} + (dv - d\xi)(Hdv + vud\xi) = 0,$$

et l'on trouve

$$dH = (\xi^2 - A\xi + B)dv + (2\xi - A)vd\xi,$$

$$dU = \left(-\xi + A - \frac{C}{v^2}\right)dv - \qquad vd\xi,$$

et en substituant ces valeurs, on obtient

$$\{v^2(v - \xi) - vU\}d\xi^2 - \left\{(v - \xi)\left(\xi v - \frac{C}{v}\right) - vU + H\right\}dvd\xi + \{-(v - \xi)(\xi^2 - A\xi + B) + H\}dv^2 = 0.$$

Le coefficient de $d\xi^2$ est $v^3 - Av^2 + Bv - C$, c'est-à-dire $v - a.v - b.v - c$; de même,

le coefficient de dv^2 est $\xi^3 - A\xi^2 + B\xi - C$, c'est-à-dire $\xi - a \,.\, \xi - b \,.\, \xi - c$. Le coefficient de $-dvd\xi$ est

$$\left(\xi v^2 - C - \xi^2 v + \frac{C\xi}{v}\right) + (v^2\xi - Av^2 + Bv - C) + (v\xi^2 - Av\xi + Bv - C),$$
$$= \xi\left(2v^2 - Av + \frac{C}{v}\right) - Av^2 + 2Bv - 3C;$$

l'équation différentielle est donc

$$(v^3 - Av^2 + Bv - C)\,d\xi^2 - \left\{\xi\left(2v^2 - Av + \frac{C}{v}\right) - Av^2 + 2Bv - 3C\right\} dvd\xi$$
$$+ (\xi^3 - A\xi^2 + B\xi - C)\,dv^2 = 0.$$

22. Mais on a identiquement

$$\begin{aligned} &\xi - a \,.\, v - b \,.\, v - c \\ &+ \xi - b \,.\, v - c \,.\, v - a \\ &+ \xi - c \,.\, v - a \,.\, v - b \\ &- \xi \,.\, \frac{1}{v} \,.\, v - a \,.\, v - b \,.\, v - c = \xi\left(2v^2 - Av + \frac{C}{v}\right) - Av^2 + 2Bv - 3C, \end{aligned}$$

donc en divisant par

$$v^3 - Av^2 + Bv - C, \quad = v - a \,.\, v - b \,.\, v - c,$$

l'équation devient

$$d\xi^2 - d\xi dv\left(\frac{\xi - a}{v - a} + \frac{\xi - b}{v - b} + \frac{\xi - c}{v - c} - \frac{\xi}{v}\right) + dv^2 \,.\, \frac{\xi - a \,.\, \xi - b \,.\, \xi - c}{v - a \,.\, v - b \,.\, v - c} = 0.$$

23. De même, en substituant dans l'équation $dvd\eta + vdud\xi = 0$ les valeurs

$$\eta = \frac{v(u^2 - Au + B)}{v - u}, \quad \xi = v - \frac{v - a \,.\, v - b \,.\, v - c}{v - u},$$

on obtient

$$(v^3 - Av^2 + Bv - C)\,du^2 - \left\{u\left(2v^2 - Av + \frac{C}{v}\right) - Av^2 + 2Bv - 3C\right\} dvdu$$
$$+ (u^3 - Au^2 + Bu - C)\,dv^2 = 0,$$

ou, ce qui est la même chose,

$$du^2 - dudv\left(\frac{u - a}{v - a} + \frac{u - b}{v - b} + \frac{u - c}{v - c} - \frac{u}{v}\right) + dv^2 \,.\, \frac{u - a \,.\, u - b \,.\, u - c}{v - a \,.\, v - b \,.\, v - c} = 0;$$

les équations différentielles entre ξ, v et entre u, v respectivement sont ainsi precisément de la même forme: on peut vérifier sans beaucoup de peine qu'en introduisant dans la première équation u au lieu de ξ par la substitution

$$\xi = v - \frac{v - a \,.\, v - b \,.\, v - c}{v - u},$$

on obtient la seconde équation: c'est là un théorème d'analyse assez remarquable.

24. L'équation différentielle en u, v est changée en entrechangeant ces deux variables: cela doit être ainsi, car autrement les deux courbes de courbure par le

point de contact du plan tangent $\lambda X + \mu Y + \nu Z - \sqrt{v} = 0$, et les deux courbes de courbure par le point de contact du plan tangent parallèle $\lambda X + \mu Y + \nu Z - \sqrt{u} = 0$ seraient parallèles les unes aux autres, ce qui n'est pas en général vrai. Pour que les deux équations soient identiques, on doit avoir

$$v - a \,.\, v - b \,.\, v - c \,.\left(\frac{u-a}{v-a} + \frac{u-b}{v-b} + \frac{u-c}{v-c} - \frac{u}{v}\right)$$

$$= u - a \,.\, u - b \,.\, u - c \,.\left(\frac{v-a}{u-a} + \frac{v-b}{u-b} + \frac{v-c}{u-c} - \frac{v}{u}\right),$$

c'est-à-dire

$$u\left(2v^2 - Av + \frac{C}{v}\right) - Av^2 + 2Bv - 3C = v\left(2u^2 - Au + \frac{C}{u}\right) - Au^2 + 2Bu - 3C,$$

ou enfin

$$(u-v)\{2u^2v^2 - A(u+v)uv + 2Buv - C(u+v)\} = 0;$$

$u = v$ donne le plan à l'infinité, qui coupe la surface selon les deux coniques imaginaires $x^2 + y^2 + z^2 = 0$ et $ax^2 + by^2 + cz^2 = 0$, et aussi les plans tangents singuliers qui touchent la surface chacun selon un cercle réel ou imaginaire; je n'ai pas considéré la courbe sur la surface que l'on obtient en égalant à zéro l'autre facteur.

25. Rayons de courbure.

Les équations de la normale au point (x, y, z) sont

$$X = x - R\lambda, \quad Y = y - R\mu, \quad Z = z - R\nu,$$

et, en supposant que les coordonnées X, Y, Z se rapportent à un centre de courbure, on a

$$dx = Rd\lambda, \quad dy = Rd\mu, \quad dz = Rd\nu,$$

et de là

$$xdx + ydy + zdz = R(xd\lambda + yd\mu + zd\nu),$$

c'est-à-dire $d\xi = \dfrac{Rdv}{\sqrt{v}}$; donc $R = \sqrt{v}\dfrac{d\xi}{dv}$, $= \theta\sqrt{v}$, en posant $\theta = \dfrac{d\xi}{dv}$: θ est donc déterminé, en fonction de ξ, v, par l'équation

$$\theta^2 - \left(\frac{\xi-a}{v-a} + \frac{\xi-b}{v-b} + \frac{\xi-c}{v-c} - \frac{\xi}{v}\right)\theta + \frac{\xi-a \,.\, \xi-b \,.\, \xi-c}{v-a \,.\, v-b \,.\, v-c} = 0,$$

et les deux rayons de courbure sont alors donnés par la formule $R = \theta\sqrt{v}$.

26. Surface des centres de courbure. En écrivant pour R sa valeur, $= \theta\sqrt{v}$, nous avons

$$X = x - \lambda\theta\sqrt{v}, \quad Y = y - \mu\theta\sqrt{v}, \quad Z = z - \nu\theta\sqrt{v},$$

où X, Y, Z dénotent les coordonnées d'un point de la surface cherchée; et si nous formons les expressions Ξ, H, Z analogues à ξ, η, ζ, savoir

$$\begin{aligned} \Xi &= X^2 + Y^2 + Z^2, \\ \mathrm{H} &= aX^2 + bY^2 + cZ^2, \\ \mathrm{Z} &= a(b+c)X^2 + b(c+a)Y^2 + c(a+b)Z^2, \end{aligned}$$

on obtient sans peine, à moyen des valeurs trouvées pour

$$\lambda x + \mu y + \nu z, \quad a\lambda x + b\mu y + c\nu z, \quad a\lambda^2 + b\mu^2 + c\nu^2, \ \&c.,$$

$$\begin{aligned} \Xi &= \xi - 2\theta v + \theta^2 v, \\ \mathrm{H} &= \eta - 2\theta\eta + \theta^2 v(A - u - v), \\ \mathrm{Z} &= \xi\eta + C - 2\theta(\eta v + C) + \theta^2 v(B - uv); \end{aligned}$$

où comme auparavant

$$\theta^2 - \left(\frac{\xi - a}{v - a} + \frac{\xi - b}{v - b} + \frac{\xi - c}{v - c} - \frac{\xi}{v}\right)\theta + \frac{\xi - a \,.\, \xi - b \,.\, \xi - c}{v - a \,.\, v - b \,.\, v - c} = 0,$$

$$\eta = \frac{v(\xi^2 - A\xi + B) - C}{v - \xi},$$

$$(v - \xi)(v - u) = \frac{v - a \,.\, v - b \,.\, v - c}{v};$$

en éliminant entre ces six équations les cinq quantités ξ, η, θ, v, u, on obtient l'équation de la surface des centres en termes de Ξ, H, Z qui sont des fonctions données des coordonnées X, Y, Z d'un point de la surface.

27. Je remarque que les sections principales de la surface des ondes sont des courbes de courbure de cette surface; en particulier, pour fixer les idées, la section par le plan $z = 0$, savoir le cercle $\xi - c = 0$ et l'ellipse $\eta - bc = 0$ sont des courbes de courbure. Pour l'une ou l'autre de ces courbes, il y a une suite des courbes de courbure de l'autre espèce qui coupent le cercle ou l'ellipse à angle droit, et qui sont symétriques aux deux côtés du plan $z = 0$. En considérant par exemple l'ellipse, les normales à la surface aux points successifs de cette courbe sont situées dans le plan de xy et se coupent selon une courbe dans ce plan, la développée de l'ellipse, laquelle est une courbe sur la surface des centres; l'ordre de cette développée est $= 6$. Mais, de plus, chaque normale de la surface à un point $(x, y, 0)$ de l'ellipse est rencontrée par les normales de la surface aux points $(x, y, \pm \delta z)$ au-dessus et au-dessous du point de l'ellipse: les points d'intersection forment une courbe dans le plan de xy, laquelle est aussi une courbe sur la surface des centres, et de plus elle est une courbe cuspidale sur cette surface: nous allons voir que l'ordre de cette courbe est $= 6$. De même pour le cercle, la développée du cercle est le point $x = 0$, $y = 0$, lequel à ce que je crois doit être considéré comme cercle infiniment petit (ou deux droites imaginaires) $x^2 + y^2 = 0$; le cercle donne lieu aussi à une courbe qui est une courbe cuspidale sur la surface des centres, et nous allons voir que l'ordre de cette courbe est $= 4$. La section de la surface des centres par le plan xy est donc composée comme suit:

développée de l'ellipse,	ordre	6
courbe cuspidale, ordre 6, trois fois,	„	18
développée du cercle,	„	2
courbe cuspidale, ordre 4, trois fois,	„	12
		38,

et il paraît ainsi que l'ordre de la surface des centres de la surface des ondes doit être $= 38$.

28. Je m'arrête pour un moment pour considérer la même théorie par rapport à l'ellipsoïde $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$. La section par le plan $z=0$ est ici l'ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$; et pour la surface des centres on a dans le plan de xy, la développée d'ellipse, courbe d'ordre 6, et aussi une courbe cuspidale laquelle est une ellipse. En effet, pour trouver l'équation de cette courbe, on a pour les coordonnées X, Y du point où la normale au point (x, y, z) rencontre le plan $z=0$, les équations $x=X+\frac{\lambda}{\nu}z$, $y=Y+\frac{\mu}{\nu}z$; c'est-à-dire $X=x\left(1-\frac{c}{a}\right)$, $Y=y\left(1-\frac{c}{b}\right)$. Dans ces équations x, y se rapportent à un point de l'ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$; en éliminant entre les trois équations x, y, on obtient $\frac{aX^2}{(a-c)^2}+\frac{bY^2}{(b-c)^2}=1$, ou ce qui est la même chose $\frac{aX^2}{\beta^2}+\frac{bY^2}{\alpha^2}=1$. La section principale de la surface des centres est donc composée de cette ellipse trois fois, et de la développée de l'ellipse $\frac{x^2}{a}+\frac{y^2}{b}=1$; l'ordre de la section, et ainsi l'ordre de la surface des centres, est donc $6+3.2, =12$, comme cela doit être.

29. De même pour la surface des ondes on a $x=X+\frac{\lambda}{\nu}z$, $y=Y+\frac{\mu}{\nu}z$, c'est-à-dire

$$X=x-\frac{x\{a\xi+\eta-a(b+c)\}}{c\xi+\eta-c(a+b)},$$

$$Y=y-\frac{y\{b\xi+\eta-b(c+a)\}}{c\xi+\eta-c(a+b)},$$

où pour $z=0$ on a $(\xi-c)(\eta-ab)=0$, équation qui donne: 1°, le cercle $\xi-c=0$; 2°, l'ellipse $\eta-ab=0$.

1°. Pour le cercle $\xi=c$, en écrivant $x^2=c\theta$, $y^2=c(1-\theta)$, nous avons

$$\eta=ac\theta+bc(1-\theta), \;=c(b+\gamma\theta)$$

et de là

$$a\xi+\eta-a(b+c)=b\beta+c\gamma\theta,$$

$$b\xi+\eta-b(c+a)=b\beta+c\gamma\theta,$$

$$c\xi+\eta-c(a+b)=c(\beta+\gamma\theta),$$

valeurs qui donnent

$$X=\sqrt{c}\sqrt{\theta}\quad\left\{1-\frac{b\beta+c\gamma\theta}{c(\beta+\gamma\theta)}\right\}, \;=-\frac{\alpha\beta}{\sqrt{c}}\frac{\sqrt{\theta}}{\beta+\gamma\theta},$$

$$Y=\sqrt{c}\sqrt{1-\theta}\left\{1-\frac{b\beta+c\gamma\theta}{c(\beta+\gamma\theta)}\right\}, \;=-\frac{\alpha\beta}{\sqrt{c}}\frac{\sqrt{1-\theta}}{\beta+\gamma\theta};$$

donc

$$X^2 + Y^2 = \frac{\alpha^2\beta^2}{c}\frac{1}{(\beta+\gamma\theta)^2},$$

$$\alpha X^2 - \beta Y^2 = \frac{\alpha^2\beta^2}{c}\frac{\alpha\theta - \beta(1-\theta)}{(\beta+\gamma\theta)^2}, \quad = -\frac{\alpha^2\beta^2}{c}\frac{1}{\beta+\gamma\theta},$$

et de là

$$(\alpha X^2 - \beta Y^2)^2 = \frac{\alpha^2\beta^2}{c}(X^2+Y^2);$$

courbe de l'ordre 4, cuspidale sur la surface des centres.

2°. Pour l'ellipse $\eta = ab$, en écrivant $x^2 = b\theta$, $y^2 = a(1-\theta)$, et de là

$$\xi = b\theta + a(1-\theta), \quad = a - \gamma\theta,$$

on obtient

$$a\xi + \eta - a(b+c) = -a(\beta+\gamma\theta),$$
$$b\xi + \eta - b(c+a) = -b(\beta+\gamma\theta),$$
$$c\xi + \eta - c(a+b) = -b\beta - c\gamma\theta,$$

valeurs qui donnent

$$X = \sqrt{b}\sqrt{\theta}\left\{1 - \frac{a(\beta+\gamma\theta)}{b\beta+c\gamma\theta}\right\} = -\sqrt{b}\beta\gamma\frac{\sqrt{\theta}(1-\theta)}{b\beta+c\gamma\theta},$$

$$Y = \sqrt{a}\sqrt{1-\theta}\left\{1 - \frac{b(\beta+\gamma\theta)}{b\beta+c\gamma\theta}\right\} = -\sqrt{a}\alpha\gamma\frac{\theta\sqrt{1-\theta}}{b\beta+c\gamma\theta};$$

donc

$$\frac{X^2}{b\beta^2} + \frac{Y^2}{a\alpha^2} = \gamma^2\frac{\theta(1-\theta)}{(b\beta+c\gamma\theta)^2},$$

$$\frac{X^2}{\beta} - \frac{Y^2}{\alpha} = \gamma^2\theta(1-\theta)\frac{b\beta(1-\theta) - a\alpha\theta}{(b\beta+c\gamma\theta)^2}, \quad = \gamma^2\frac{\theta(1-\theta)}{b\beta+c\gamma\theta};$$

$$\frac{X^2}{b\beta^2}\cdot\frac{Y^2}{a\alpha^2} = \gamma^4\frac{\theta^3(1-\theta)^2}{(b\beta+c\gamma\theta)^4},$$

et de là

$$\left(\frac{X^2}{b\beta^2} + \frac{Y^2}{a\alpha^2}\right)\left(\frac{X^2}{\beta} - \frac{Y^2}{\alpha}\right)^2 = \gamma^2\cdot\frac{X^2}{b\beta^2}\cdot\frac{Y^2}{a\alpha^2},$$

courbe de l'ordre 6, cuspidale sur la surface des centres.

30. On aurait pu développer la théorie des courbes et rayons de courbure à moyen de la formule ci-dessus donnée, $ds^2 = E d\xi^2 + 2F d\xi d\eta + G d\eta^2$ $(F=0)$, mais pour cela il faudrait trouver plusieurs expressions qui ne sont pas encore calculées, savoir les coefficients des formules

$$d^2x = \alpha\, d\xi^2 + 2\alpha' d\xi\, d\eta + \alpha'' d\eta^2,$$
$$d^2y = \beta\, d\xi^2 + 2\beta' d\xi\, d\eta + \beta'' d\eta^2,$$
$$d^2z = \gamma\, d\xi^2 + 2\gamma' d\xi\, d\eta + \gamma'' d\eta^2,$$

et puis

$$E',\ F',\ G' = \lambda\alpha + \mu\beta + \nu\gamma,\quad \lambda\alpha' + \mu\beta' + \nu\gamma',\quad \lambda\alpha'' + \mu\beta'' + \nu\gamma''.$$

En prenant comme auparavant R pour le rayon de courbure on aurait alors, (Salmon, *Geometry of three dimensions*, Ed. 4 (1882), p. 347),

$$\left\| \begin{matrix} R, & E\,d\xi, & G\,d\eta \\ EG, & E'd\xi + F'd\eta, & F'd\xi + G'd\eta \end{matrix} \right\| = 0,$$

et

$$\left\| \begin{matrix} d\eta, & RE' - E^2G, & RF' \\ -d\xi, & RF', & RG' - EG^2 \end{matrix} \right\| = 0,$$

formules pour les courbes et rayons de courbure: en particulier, l'équation différentielle des courbes de courbure peut s'écrire sous la forme

$$\left| \begin{matrix} d\eta^2, & -d\xi\,d\eta, & d\xi^2 \\ E, & 0, & G \\ E', & F', & G' \end{matrix} \right| = 0.$$

Au reste, cette équation en $d\xi$, $d\eta$ se déduirait plus simplement de l'équation $dv\,d\eta + v\,du\,d\xi = 0$, en y introduisant les expressions de v, u en termes de ξ, η.

31. Courbes géodésiques sur la surface. L'équation différentielle du second ordre des courbes géodésiques dépend seulement des coefficients E, F, G, savoir en supposant $F = 0$, cette équation est

$$\begin{aligned} & E\,d\xi\,(-E_2d\xi^2 + 2G_1d\xi\,d\eta + G_2d\eta^2) \\ & - G\,d\eta\,(\ \ E_1d\xi^2 + 2E_2d\xi\,d\eta - G_1d\eta^2) \\ & + 2EG\,(\qquad d\xi\,d^2\eta - d\eta\,d^2\xi \qquad\qquad) = 0, \end{aligned}$$

ou ce qui est la même chose

$$(-EE_2,\ 2EG_1 - GE_1,\ EG_2 - 2GE_2,\ GG_1)\,(d\xi,\ d\eta)^3 + 2EG\,(d\xi\,d^2\eta - d\eta\,d^2\xi) = 0,$$

où

$$E_1,\ E_2,\ G_1,\ G_2 = \frac{dE}{d\xi},\ \frac{dE}{d\eta},\ \frac{dG}{d\xi},\ \frac{dG}{d\eta}$$

respectivement, voir Cayley, "On geodesic lines, in particular those of a Quadric Surface," *Proc. London Math. Soc.*, t. IV. (1872), p. 197, [508]. Nous avons vu que pour la surface des ondes dont il s'agit les expressions de E, G sont

$$E = \tfrac{1}{4}\,\frac{\xi^2 - A\xi + B - \eta}{\xi - a\,.\,\xi - b\,.\,\xi - c},\quad G = \tfrac{1}{4}\,\frac{C - \xi\eta}{\eta - bc\,.\,\eta - ca\,.\,\eta - ab},$$

et l'on obtiendrait de là sans peine les expressions des coefficients de la fonction cubique $(-EE_2, \ldots)(d\xi,\ d\eta)^3$ qui entre dans l'équation différentielle.

931.

ON SOME FORMULÆ OF CODAZZI AND WEINGARTEN IN RELATION TO THE APPLICATION OF SURFACES TO EACH OTHER.

[From the *Proceedings of the London Mathematical Society*, vol. XXIV. (1893), pp. 210—223.]

AN extremely elegant theory of the application of surfaces one upon another is developed in the memoir, Codazzi, "Mémoire relatif à l'application des surfaces les unes sur les autres," *Mém. prés. à l'Académie des Sciences*, t. XXVII. (1883), No. 6, pp. 1—47; but the notation is not presented in a form which is easily comparable with that of the Gaussian notation in the theory of surfaces. I propose to reproduce the theory in the Gaussian notation.

Codazzi considers on a given surface two systems of curves depending on the parameters t, T respectively; the curves are in the memoir taken to be orthogonal to each other, but this restriction is removed in the general formula given p. 44, *Addition au chapitre premier*. For a curve of either system, he considers the tangent, the principal normal, or normal in the osculating plane, and the binormal, or line at right angles to the osculating plane (say these are tan, prn and bin). For a curve of the one system, that in which t is variable or say a t-curve, he denotes the cosine-inclinations of these lines to the axes by the letters a, b, c, thus:

	x	y	z
tan	a_x	a_y	a_z
prn	b_x	b_y	b_z
bin	c_x	c_y	c_z

and he writes also l for the inclination of the principal normal to the normal of the surface, $\frac{dm}{dt}dt$ for the angle of contingence, or inclination of the tangent at the point (t, T) to the tangent at the point $(t+dt, T)$, and $\frac{dn}{dt}dt$ for the angle of torsion, or inclination of the osculating plane at the point (t, T) to that at the point $(t+dt, T)$, or, what is the same thing, the inclination of the binormals at these points respectively. And he gives as known formulæ

$$\frac{da}{dt} = b\frac{dm}{dt}, \quad \frac{db}{dt} = -a\frac{dm}{dt} - c\frac{dn}{dt}, \quad \frac{dc}{dt} = b\frac{dn}{dt},$$

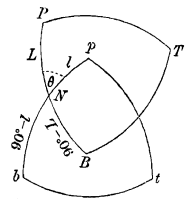
where a, b, c denote

$$(a_x, b_x, c_x), \quad (a_y, b_y, c_y), \quad \text{or} \quad (a_z, b_z, c_z).$$

He uses the capital letters

$$A_x, A_y, A_z, B_x, B_y, B_z, C_x, C_y, C_z, L, \frac{dM}{dT}dT, \frac{dN}{dT}dT$$

with the like significations in regard to the curve for which T is variable, or say the T-curve; for greater clearness I give the accompanying figure.



Codazzi writes further

$$\frac{dm}{dt}\cos l = u, \quad \frac{dm}{dt}\sin l = v, \quad \frac{dn}{dt} - \frac{dl}{dt} = w,$$

$$\frac{dM}{dT}\cos L = U, \quad \frac{dM}{dT}\sin L = V, \quad \frac{dN}{dT} - \frac{dL}{dT} = W;$$

also, if s, S are the arcs of the two curves respectively,

$$\frac{ds}{dt} = r, \quad \frac{dS}{dT} = R;$$

and he obtains a system of six formulæ, which in the *Addition*, p. 44, are presented in the following form—only I use therein θ, instead of his b, to denote the inclination of the two curves to each other:

$$\frac{du}{dT} = \frac{dU}{dt}\cos\theta + \frac{dW}{dt}\sin\theta + w\left(V - \frac{d\theta}{dT}\right) + (U\sin\theta - W\cos\theta)\left(v - \frac{d\theta}{dt}\right),$$

$$\frac{dU}{dt} = \frac{du}{dT}\cos\theta + \frac{dw}{dT}\sin\theta + W\left(v - \frac{d\theta}{dt}\right) + (u\sin\theta - w\cos\theta)\left(V - \frac{d\theta}{dT}\right),$$

$$\left(\frac{dv}{dT} + \frac{dV}{dt}\right)\frac{d^2\theta}{dt\,dT} + \sin\theta\,(uU - wW) - \cos\theta\,(uW + wU) = 0,$$

$$R(u\cos\theta + w\sin\theta) = r(U\cos\theta + W\sin\theta),$$

$$R\sin\theta\left(v - \frac{d\theta}{dt}\right) + \frac{dR}{dt}\cos\theta = \frac{dr}{dT},$$

$$r\sin\theta\left(V - \frac{d\theta}{dT}\right) + \frac{dr}{dT}\cos\theta = \frac{dR}{dt}.$$

In the Gaussian notation, taking p, q for the parameters, we have, with the slight variations presently referred to,

$$dx = a\,dp + a'dq + \tfrac{1}{2}\alpha\,dp^2 + \alpha'dp\,dq + \alpha''dq^2,$$
$$dy = b\,dp + b'dq + \tfrac{1}{2}\beta\,dp^2 + \beta'dp\,dq + \beta''dq^2,$$
$$dz = c\,dp + c'dq + \tfrac{1}{2}\gamma\,dp^2 + \gamma'dp\,dq + \gamma''dq^2,$$
$$A,\ B,\ C = bc' - b'c,\ ca' - c'a,\ ab' - a'b,$$
$$E,\ F,\ G = a^2 + b^2 + c^2,\ aa' + bb' + cc',\ a'^2 + b'^2 + c'^2;$$

and therefore

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2;$$
$$\sqrt{EG - F^2} = V;$$
$$E',\ F',\ G' = A\alpha + B\beta + C\gamma,\quad A\alpha' + B\beta' + C\gamma',\quad A\alpha'' + B\beta'' + C\gamma'';$$

and I take further

$$\omega,\ \omega',\ \omega'' = a\alpha + b\beta + c\gamma,\quad a\alpha' + b\beta' + c\gamma',\quad a\alpha'' + b\beta'' + c\gamma'',$$
$$\varpi,\ \varpi',\ \varpi'' = a'\alpha + b'\beta + c'\gamma,\quad a'\alpha' + b'\beta' + c'\gamma',\quad a'\alpha'' + b'\beta'' + c'\gamma'',$$
$$\lambda,\ \lambda',\ \lambda'' = \alpha^2 + \beta^2 + \gamma^2,\quad \alpha'^2 + \beta'^2 + \gamma'^2,\quad \alpha''^2 + \beta''^2 + \gamma''^2,$$
$$\mu,\ \mu',\ \mu'' = \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'',\quad \alpha''\alpha + \beta''\beta + \gamma''\gamma,\quad \alpha\alpha' + \beta\beta' + \gamma\gamma',$$
$$E\lambda - \omega^2 = \Delta,\quad G\lambda'' - \varpi''^2 = \Delta'',$$

where it is to be noticed that V^2, E', F', G', are written instead of Gauss's Δ, D, D', D'', and ω, ω', ω'', ϖ, ϖ', ϖ'' instead of his m, m', m'', n, n', n'': and that he gives for the last-mentioned quantities the values

$$m = \tfrac{1}{2}\frac{dE}{dp},\quad m' = \tfrac{1}{2}\frac{dE}{dq},\quad m'' = \frac{dF}{dq} - \tfrac{1}{2}\frac{dG}{dp},$$
$$n = \frac{dF}{dp} - \tfrac{1}{2}\frac{dE}{dq},\quad n' = \tfrac{1}{2}\frac{dG}{dp},\quad n'' = \tfrac{1}{2}\frac{dG}{dq},$$

or, say

$$\omega = \tfrac{1}{2}E_1,\quad \omega' = \tfrac{1}{2}E_2,\quad \omega'' = F_2 - \tfrac{1}{2}G_1,$$
$$\varpi = F_1 - \tfrac{1}{2}E_2,\quad \varpi' = \tfrac{1}{2}G_1,\quad \varpi'' = \tfrac{1}{2}G_2,$$

where the subscripts (1) and (2) denote differentiation in regard to p and q respectively.

Observing that the cosine-inclinations of the tangent to the p-curve are as a, b, c, and those of the binormal or perpendicular to the osculating plane are as $b\gamma - c\beta$, $c\alpha - a\gamma$, $a\beta - b\alpha$, we easily find

$$a_x,\ a_y,\ a_z = \frac{a}{\sqrt{E}},\quad \frac{b}{\sqrt{E}},\quad \frac{c}{\sqrt{E}},$$

$$c_x,\ c_y,\ c_z = \frac{b\gamma - c\beta}{\sqrt{\Delta}},\quad \frac{c\alpha - a\gamma}{\sqrt{\Delta}},\quad \frac{a\beta - b\alpha}{\sqrt{\Delta}},$$

$$b_x,\ b_y,\ b_z = \frac{E\alpha - a\omega}{\sqrt{E\Delta}},\quad \frac{E\beta - b\omega}{\sqrt{E\Delta}},\quad \frac{E\gamma - c\omega}{\sqrt{E\Delta}};$$

and for the cosine-inclinations of the normal of the surface

$$\Delta_x,\ \Delta_y,\ \Delta_z = \frac{A}{V},\ \frac{B}{V},\ \frac{C}{V},$$

we find

$$\cos l = \frac{A(b\gamma - c\beta) + B(c\alpha - a\gamma) + C(a\beta - b\alpha)}{V\sqrt{\Delta}}$$

$$= \frac{(a^2+b^2+c^2)(a'\alpha + b'\beta + c'\gamma) - (aa' + bb' + cc')(a\alpha + b\beta + c\gamma)}{V\sqrt{\Delta}}$$

$$= \frac{E\varpi - F\omega}{V\sqrt{\Delta}},$$

$$\sin l = \frac{A(E\alpha - a\omega) + B(E\beta - b\omega) + C(E\gamma - c\omega)}{V\sqrt{E\Delta}};$$

or, since $Aa + Bb + Cc = 0$, this is

$$\sin l = \frac{E(A\alpha + B\beta + C\gamma)}{V\sqrt{E\Delta}} = \frac{E'\sqrt{E}}{V\sqrt{\Delta}}.$$

We ought therefore to have

$$E'^2E + (E\varpi - F\omega)^2 = V^2\Delta,\quad = (EG - F^2)(E\lambda - \omega^2);$$

or, omitting the terms $F^2\omega^2$, which destroy each other, and throwing out a factor E, this is

$$E\varpi^2 - 2F\varpi\omega + G\omega^2 - \lambda(EG - F^2) = \lambda(EG - F^2) - E'^2;$$

viz. putting for $EG - F^2$ its value, $= A^2 + B^2 + C^2$, and for the other terms their values, this is

$$(a^2+b^2+c^2)(a'\alpha + b'\beta + c'\gamma)^2 - 2(aa' + bb' + cc')(a'\alpha + b'\beta + c'\gamma)(a\alpha + b\beta + c\gamma)$$
$$+ (a'^2 + b'^2 + c'^2)(\alpha^2 + \beta^2 + \gamma^2)$$

$$= (A^2 + B^2 + C^2)(\alpha^2 + \beta^2 + \gamma^2) - (A\alpha + B\beta + C\gamma)^2.$$

The right-hand side is here $= (B\gamma - C\beta)^2 + (C\alpha - A\gamma)^2 + (A\beta - B\alpha)^2$; the left-hand consists of three parts, the first whereof is

$$\{a(a'\alpha + b'\beta + c'\gamma) - a'(a\alpha + b\beta + c\gamma)\}^2 = (-B\gamma + C\beta)^2,$$

and similarly the other two parts are

$$(-C\alpha + A\gamma)^2 \quad \text{and} \quad (-A\beta + B\alpha)^2;$$

the equation is thus verified.

We require Codazzi's $\frac{dm}{dt}$ and $\frac{dn}{dt}$, or say $\frac{dm}{dp}$ and $\frac{dn}{dp}$; these are to be obtained from the equations

$$\frac{d}{dp}\frac{a}{\sqrt{E}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dm}{dp}, \quad \frac{d}{dp}\frac{b\gamma - c\beta}{\sqrt{\Delta}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dn}{dp},$$

and the values obtained should satisfy

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a}{\sqrt{E}}\frac{dm}{dp} - \frac{b\gamma - c\beta}{\sqrt{\Delta}}\frac{dn}{dp}.$$

I find

$$\frac{dm}{dp} = \frac{\sqrt{\Delta}}{E}, \quad \frac{dn}{dp} = -\frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

where α_1, β_1, γ_1 are the derivatives of α, β, γ in regard to p; and the equation to be verified thus is

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

First, for $\frac{dm}{dp}$: the derivatives of a, E are α and $2(a\alpha + b\beta + c\gamma), = 2\omega$; we thus have

$$\frac{d}{dp}\frac{a}{\sqrt{E}}, \quad = \frac{\alpha}{\sqrt{E}} - \frac{a\omega}{E\sqrt{E}} = \frac{E\alpha - a\omega}{E\sqrt{E}},$$

which is

$$= \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dm}{dp};$$

viz. we have

$$\frac{dm}{dp} = \frac{\sqrt{\Delta}}{E}.$$

Next, for $\frac{dn}{dp}$: using a subscript $(_1)$ to denote derivation in regard to p, we have $\Delta = E\lambda - \omega^2$, and thence

$$\begin{aligned}\Delta_1 &= E_1\lambda + E\lambda_1 - 2\omega\omega_1 \\ &= 2\omega\lambda + E.2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - 2\omega(\lambda + a\alpha_1 + b\beta_1 + c\gamma_1), \\ &= 2\{E(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - \omega(a\alpha_1 + b\beta_1 + c\gamma_1)\}, \\ &= 2(\alpha_1 X + \beta_1 Y + \gamma_1 Z),\end{aligned}$$

if, for a moment,

$$X,\ Y,\ Z = E\alpha - a\omega,\ E\beta - b\omega,\ E\gamma - c\omega;$$

these values give identically

$$\alpha X + \beta Y + \gamma Z = \Delta, \quad \text{and} \quad aX + bY + cZ = 0.$$

Hence we have

$$\frac{d}{dp}\frac{b\gamma - c\beta}{\sqrt{\Delta}} = \frac{b\gamma_1 - c\beta_1}{\sqrt{\Delta}} - \frac{(b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z)}{\Delta\sqrt{\Delta}},$$

which must be

$$= \frac{X}{\sqrt{E\Delta}}\frac{dn}{dp};$$

we thus have

$$(b\gamma_1 - c\beta_1)\Delta - (b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z) = \frac{\Delta X}{\sqrt{E}}\frac{dn}{dp};$$

or, putting the left-hand side in the form

$$\begin{aligned}&(b\gamma_1 - c\beta_1)(\alpha X + \beta Y + \gamma Z)\\ &-(b\gamma - c\beta)(\alpha_1 X + \beta_1 Y + \gamma_1 Z)\\ &-(\beta\gamma_1 - \beta_1\gamma)(aX + bY + cZ),\end{aligned}$$

this is

$$= -X\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

and we thus find

$$\frac{dn}{dp} = -\frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

For the verification of the equation

$$\frac{d}{dp}\frac{E\alpha - a\omega}{\sqrt{E\Delta}} = -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

we have

$$(E\alpha - a\omega)_1 = E\alpha_1 - a(a\alpha_1 + b\beta_1 + c\gamma_1) + \alpha\omega - a\lambda,$$

$$(E\Delta)_1 = 2E^2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - 2E\omega(a\alpha_1 + b\beta_1 + c\gamma_1) + 2\Delta\omega,$$

and hence the equation is

$$-\frac{\{E^2(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - E\omega(a\alpha_1 + b\beta_1 + c\gamma_1) + \omega\}(E\alpha - a\omega)}{E\Delta\sqrt{E\Delta}} + \frac{E\alpha_1 - a(a\alpha_1 + b\beta_1 + c\gamma_1) + \alpha\omega - a\lambda}{\sqrt{E\Delta}}$$

$$= -\frac{a\sqrt{\Delta}}{E\sqrt{E}} + \frac{(b\gamma - c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

Considering first the terms without α_1, β_1, γ_1, these give

$$-\omega(E\alpha - a\omega) + E(\alpha\omega - a\lambda) = -a(E\lambda - \omega^2),$$

which is identically true; and then the remaining terms with α_1, β_1, γ_1 give

$$-\{E(a\alpha_1+\beta\beta_1+\gamma\gamma_1)-\omega(a\alpha_1+b\beta_1+c\gamma_1)\}(E\alpha-a\omega)+\{E\alpha_1-a(a\alpha_1+b\beta_1+c\gamma_1)\}(E\lambda-\omega^2)$$

$$=(b\gamma-c\beta)E\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

On the left-hand side, the whole coefficient of α_1 is

$$=-(E\alpha-a\omega)^2+(b^2+c^2)(E\lambda-\omega^2),$$

which is

$$=-E^2\alpha^2+2a\alpha E\omega-a^2\omega^2+(b^2+c^2)E\lambda-(b^2+c^2)\omega^2,$$

$$=E[-E\alpha^2+2a\alpha\omega+(b^2+c^2)\lambda-\omega^2];$$

and, substituting for E, ω, and λ their values, this is found to be

$$=E\{(b^2+c^2)(\beta^2+\gamma^2)-(b\beta+c\gamma)^2\},\quad =E(b\gamma-c\beta)^2.$$

Similarly the whole coefficients of β_1 and γ_1 are found to be

$$=E(b\gamma-c\beta)(c\alpha-a\gamma),\quad\text{and}\quad E(b\gamma-c\beta)(a\beta-b\alpha),$$

respectively; and thus the left-hand side becomes

$$=(b\gamma-c\beta)E\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

as it should do.

We have

$$\tan l=\frac{P}{Q},$$

where

$$P=E'\sqrt{E}=\sqrt{E}(A\alpha+B\beta+C\gamma), \qquad =\mathrm{a}\alpha+\mathrm{b}\beta+\mathrm{c}\gamma,\ \text{suppose},$$

$$Q=E\varpi-F\omega=(cB-bC)\alpha+(aC-cA)\beta+(bA-aB)\gamma, \quad =\mathrm{a}'\alpha+\mathrm{b}'\beta+\mathrm{c}'\gamma,\ \text{suppose},$$

$$P^2+Q^2=V^2\Delta,$$

and we have hence to find $\dfrac{dl}{dp}$. Using, as before, a subscript $(_1)$ to denote derivation in regard to p, we have

$$\frac{dl}{dp}=\frac{QP_1-PQ_1}{P^2+Q^2};$$

the numerator is

$$\begin{aligned}
&= (\mathrm{a}'\alpha+\mathrm{b}'\beta+\mathrm{c}'\gamma)(\mathrm{a}\,\alpha_1+\mathrm{b}\,\beta_1+\mathrm{c}\,\gamma_1+\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)\\
&\quad-(\mathrm{a}\,\alpha+\mathrm{b}\,\beta+\mathrm{c}\,\gamma)(\mathrm{a}'\alpha_1+\mathrm{b}'\beta_1+\mathrm{c}'\gamma_1+\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma)\\
&=-[(\mathrm{bc}'-\mathrm{b}'\mathrm{c})(\beta\gamma_1-\beta_1\gamma)+(\mathrm{ca}'-\mathrm{c}'\mathrm{a})(\gamma\alpha_1-\gamma_1\alpha)+(\mathrm{ab}'-\mathrm{a}'\mathrm{b})(\alpha\beta_1-\alpha_1\beta)]\\
&\quad+Q(\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)-P(\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma).
\end{aligned}$$

For the first part hereof,

$$\begin{aligned} \mathrm{bc}' - \mathrm{b}'\mathrm{c} &= \sqrt{E}\,\{B\,(bA - aB) - C\,(aC - cA)\} \\ &= \sqrt{E}\,\{A\,(aA + bB + cC) - a\,(A^2 + B^2 + C^2)\}, \quad = -aV^2\sqrt{E}, \end{aligned}$$

since $aA + bB + cC = 0$; and similarly

$$\mathrm{ca}' - \mathrm{c}'\mathrm{a}, \quad \mathrm{ab}' - \mathrm{a}'\mathrm{b} = -bV^2\sqrt{E}, \quad -cV^2\sqrt{E},$$

respectively; and thus the first part is

$$= V^2\sqrt{E}\,\{a\,(\beta\gamma_1 - \beta_1\gamma) + b\,(\gamma\alpha_1 - \gamma_1\alpha) + c\,(\alpha\beta_1 - \alpha_1\beta)\},$$

$$= V^2\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

Hence, dividing by

$$P^2 + Q^2, \quad = V^2\Delta,$$

the first part of $\dfrac{dl}{dp}$ is

$$= \frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

For the second part of the numerator, we require

$$\alpha\mathrm{a}_1 + \beta\mathrm{b}_1 + \gamma\mathrm{c}_1 \quad \text{and} \quad \alpha\mathrm{a}_1' + \beta\mathrm{b}_1' + \gamma\mathrm{c}_1';$$

the values of a, b, c are $A\sqrt{E}$, $B\sqrt{E}$, $C\sqrt{E}$, where $E_1 = 2\omega$, and hence

$$\mathrm{a}_1\alpha + \mathrm{b}_1\beta + \mathrm{c}_1\gamma = \left(A_1\sqrt{E} + \frac{A\omega}{\sqrt{E}}\right)\alpha + \left(B_1\sqrt{E} + \frac{B\omega}{\sqrt{E}}\right)\beta + \left(C_1\sqrt{E} + \frac{C\omega}{\sqrt{E}}\right)\gamma,$$

$$= \sqrt{E}\,(A_1\alpha + B_1\beta + C_1\gamma) + \frac{\omega}{\sqrt{E}}(A\alpha + B\beta + C\gamma).$$

From the values

$$A,\ B,\ C = bc' - b'c,\ ca' - c'a,\ ab' - a'b,$$

we have

$$A_1,\ B_1,\ C_1 = \beta c' - \gamma b' + b\gamma' - c\beta',\ \gamma a' - \alpha c' + c\alpha' - a\gamma',\ \alpha b' - \beta a' + a\beta' - b\alpha',$$

and thence

$$A_1\alpha + B_1\beta + C_1\gamma = -\,[a\,(\beta\gamma' - \beta'\gamma) + b\,(\gamma\alpha' - \gamma'\alpha) + c\,(\alpha\beta' - \alpha'\beta)],$$

$$= -\begin{vmatrix} a\,; & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}.$$

Hence

$$\mathrm{a}_1\alpha + \mathrm{b}_1\beta + \mathrm{c}_1\gamma = -\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}.$$

Next, we have

$$a',\ b',\ c' = cB - bC,\ aC - cA,\ bA - aB,$$

and thence

$$\begin{aligned}
\alpha a_1' + \beta b_1' + \gamma c_1' &= \alpha(\gamma B - \beta C + cB_1 - bC_1) + \beta(\alpha C - \gamma A + aC_1 - cA_1) + \gamma(\beta A - \alpha B + bA_1 - aB_1) \\
&= (b\gamma - c\beta) A_1 + (c\alpha - a\gamma) B_1 + (a\beta - b\alpha) C_1 \\
&= \quad (b\gamma - c\beta)(c'\beta - b'\gamma + b\gamma' - c\beta') \\
&\quad + (c\alpha - a\gamma)(a'\gamma - c'\alpha + c\alpha' - a\gamma') \\
&\quad + (a\beta - b\alpha)(b'\alpha - a'\beta + a\beta' - b\alpha');
\end{aligned}$$

the portion hereof, which is quadric in α, β, γ, is

$$\begin{aligned}
&= -(\alpha^2 + \beta^2 + \gamma^2)(aa' + bb' + cc') + (a\alpha + b\beta + c\gamma)(a'\alpha + b'\beta + c'\gamma), \\
&= -\lambda F + \omega\varpi,
\end{aligned}$$

and the remaining portion, which is lineo-linear in α, β, γ and α', β', γ', is

$$\begin{aligned}
&= (\alpha\alpha' + \beta\beta' + \gamma\gamma')(a^2 + b^2 + c^2) - (a\alpha + b\beta + c\gamma)(a\alpha' + b\beta' + c\gamma'), \\
&= E\mu'' - \omega\omega';
\end{aligned}$$

we thus have

$$\alpha a_1' + \beta b_1' + \gamma c_1' = -\lambda F + E\mu'' + \omega(\varpi - \omega').$$

Hence the second portion of the numerator, or

$$Q(\alpha a_1 + \beta b_1 + \gamma c_1) - P(\alpha a_1' + \beta b_1' + \gamma c_1'),$$

is

$$= (E\varpi - F\omega)\left\{-\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}\right\} - E'\sqrt{E}\{-\lambda F + \mu'' E + \omega(\varpi - \omega')\}.$$

There are two terms, $+E'\sqrt{E}\omega\varpi$ and $-E'\sqrt{E}\omega\varpi$, which destroy each other, and thus the whole second part of the numerator is

$$= -(E\varpi - F\omega)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F),$$

and, for the corresponding part of $\dfrac{dl}{dp}$, we must divide this by $P^2 + Q^2, = V^2\Delta$.

Hence, finally, we have

$$\begin{aligned}
\frac{dl}{dp} &= \frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix} \\
&\quad + \frac{1}{V^2\Delta}\left\{-(\varpi E - \omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F)\right\},
\end{aligned}$$

where I recall that the values of ω, ϖ, ω', λ, and μ'' are $a\alpha+b\beta+c\gamma$, $a'\alpha+b'\beta+c'\gamma$, $a\alpha'+b\beta'+c\gamma'$, $\alpha^2+\beta^2+\gamma^2$, and $\alpha\alpha'+\beta\beta'+\gamma\gamma'$, respectively. Observe that the first term in this expression for $\frac{dl}{dp}$ is

$$=-\frac{dn}{dp}.$$

We thus have

$$u=\frac{dm}{dp}\cos l=\frac{E\varpi-F\omega}{\sqrt{E}},$$

$$v=\frac{dm}{dp}\sin l=\frac{E'}{\sqrt{E}},$$

$$w=\frac{dn}{dp}-\frac{dl}{dp}=-\frac{2\sqrt{E}}{\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}$$

$$+\frac{1}{\sqrt{E}^2\Delta}\left\{(\varpi E-\omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}+E'\sqrt{E}(\mu''E-\lambda F)-\frac{E'\omega}{\sqrt{E}}(\omega'E-\omega F)\right\},$$

and we thence obtain at once the values of U, V, W; viz. these are

$$U=\frac{dM}{dq}\cos L=\frac{G\omega''-F\varpi''}{\sqrt{G}},$$

$$V=\frac{dM}{dq}\sin L=\frac{G'}{\sqrt{G}},$$

$$W=\frac{dN}{dq}-\frac{dL}{dq}=-\frac{2\sqrt{G}}{\sqrt{\Delta''}}$$

$$+\frac{1}{\sqrt{G}^2\Delta''}\left\{(\omega''G-\varpi''F)\begin{vmatrix} a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha_2'', & \beta_2'', & \gamma_2'' \end{vmatrix}+G'\sqrt{G}(\mu G-\lambda''F)-\frac{G'\varpi''}{\sqrt{G}}(\varpi'G-\varpi''F)\right\},$$

where α_2'', β_2'', γ_2'' denote the derived functions of α'', β'', γ'' in regard to q, viz. these are the third derived functions of x, y, z in regard to q.

We have, moreover,

$$dx^2+dy^2+dz^2=E\,dp^2+2F\,dp\,dq+G\,dq^2,\ =r^2\,dt^2+2rR\cos\theta\,dt\,dT+R^2\,dT^2;$$

that is,

$$r=\sqrt{E},\quad R=\sqrt{G},\quad \cos\theta=\frac{F}{\sqrt{EG}},$$

and therefore also

$$\sin\theta=\frac{\sqrt{EG-F^2}}{\sqrt{EG}},\quad =\frac{V}{\sqrt{EG}}.$$

Writing p, q in place of t, T, and for r, R substituting their values, then, with the foregoing values of u, v, w, U, V, W, Codazzi's six equations are

$$\frac{du}{dq} = \frac{dU}{dp}\cos\theta + \frac{dW}{dp}\sin\theta + w\left(V - \frac{d\theta}{dq}\right) + (U\sin\theta - W\cos\theta)\left(v - \frac{d\theta}{dp}\right),$$

$$\frac{dU}{dp} = \frac{du}{dq}\cos\theta + \frac{dW}{dq}\sin\theta + W\left(v - \frac{d\theta}{dp}\right) + (u\sin\theta - w\cos\theta)\left(V - \frac{d\theta}{dq}\right),$$

$$\left(\frac{dv}{dq} + \frac{dV}{dp}\right)\frac{d^2\theta}{dp\,dq} + \sin\theta\,(uU - wW) - \cos\theta\,(uW + wU) = 0,$$

$$\sqrt{G}\,(u\cos\theta + w\sin\theta) = \sqrt{E}\,(U\cos\theta + W\sin\theta),$$

$$\sqrt{G}\sin\theta\left(v - \frac{d\theta}{dp}\right) + \frac{d\sqrt{G}}{dp}\cos\theta = \frac{d\sqrt{E}}{dq},$$

$$\sqrt{E}\sin\theta\left(V - \frac{d\theta}{dq}\right) + \frac{d\sqrt{E}}{dq}\cos\theta = \frac{d\sqrt{G}}{dp}.$$

I take the opportunity of remarking that Gauss, in § 11 of his memoir, gives the first and second of the formulæ

$$\alpha\,V^2 + a\,(\varpi\,F - \omega\,G) + a'\,(\omega\,F - \varpi\,E) - AE' = 0,$$

$$\alpha'\,V^2 + a\,(\varpi'\,F - \omega'\,G) + a'\,(\omega'\,F - \varpi'\,E) - AF' = 0,$$

$$\alpha''V^2 + a\,(\varpi''F - \omega''G) + a'\,(\omega''F - \varpi''E) - AG' = 0,$$

(each of them one out of a system of three like equations), where, as before,

$$V^2 = EG - F^2,$$

$$E',\ F',\ G' = A\alpha + B\beta + C\gamma,\ A\alpha' + B\beta' + C\gamma',\ A\alpha'' + B\beta'' + C\gamma'',$$

$$\omega,\ \omega',\ \omega'' = a\alpha + b\beta + c\gamma,\ a\alpha' + b\beta' + c\gamma',\ a\alpha'' + b\beta'' + c\gamma'',$$

$$= \tfrac{1}{2}E_1,\ \tfrac{1}{2}E_2,\ F_2 - \tfrac{1}{2}G_1,$$

$$\varpi,\ \varpi',\ \varpi'' = a'\alpha + b'\beta + c'\gamma,\ a'\alpha' + b'\beta' + c'\gamma',\ a'\alpha'' + b'\beta'' + c'\gamma'',$$

$$= F_1 - \tfrac{1}{2}E_2,\ \tfrac{1}{2}G_1,\ \tfrac{1}{2}G_2.$$

These are, in fact, the formulæ (IV.),

$$\frac{d^2x}{dp^2} - \begin{Bmatrix}11\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}11\\2\end{Bmatrix}\frac{dx}{dq} + c_{11}X = 0,$$

$$\frac{d^2x}{dp\,dq} - \begin{Bmatrix}12\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}12\\2\end{Bmatrix}\frac{dx}{dq} + c_{12}X = 0,$$

$$\frac{d^2x}{dq^2} - \begin{Bmatrix}22\\1\end{Bmatrix}\frac{dx}{dp} - \begin{Bmatrix}22\\2\end{Bmatrix}\frac{dx}{dq} + c_{22}X = 0,$$

of the memoir, Weingarten, "Ueber die Deformation einer biegsamen unausdehnbaren

Fläche," *Crelle*, t. c. (1887), pp. 296—310; viz. the symbols correspond to those of Weingarten, as follows:—

$$E,\ F,\ G = a_{11},\ a_{12},\ a_{22},$$

$$V^2 = EG - F^2 = a = \rho^2,$$

$$E',\ F',\ G' = -c_{11}\sqrt{a},\ -c_{12}\sqrt{a},\ -c_{22}\sqrt{a},$$

$$A,\ B,\ C = \rho X,\ \rho Y,\ \rho Z,$$

$$\omega,\ \omega',\ \omega'' = \tfrac{1}{2}\frac{da_{11}}{dp},\qquad \tfrac{1}{2}\frac{da_{11}}{dq},\qquad \frac{da_{12}}{dq} - \tfrac{1}{2}\frac{da_{22}}{dp},$$

$$\varpi,\ \varpi',\ \varpi'' = \frac{da_{12}}{dp} - \tfrac{1}{2}\frac{da_{11}}{dq},\ \tfrac{1}{2}\frac{da_{22}}{dp},\ \tfrac{1}{2}\frac{da_{22}}{dq},$$

and thus Weingarten's symbols, $\begin{Bmatrix}11\\1\end{Bmatrix}$, &c., have the values

$$\begin{Bmatrix}11\\1\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dp} + \tfrac{1}{2}\frac{da_{11}}{dq}\right) + a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dp}\right)\right],$$

$$\begin{Bmatrix}12\\1\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dp}\right) + a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\begin{Bmatrix}22\\1\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dq}\right) + a_{22}\left(\frac{da_{12}}{dq} - \tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\begin{Bmatrix}11\\2\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dp}\right) + a_{11}\left(\frac{da_{12}}{dp} - \tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\begin{Bmatrix}12\\2\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dq}\right) + a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\begin{Bmatrix}22\\2\end{Bmatrix} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dq} + \tfrac{1}{2}\frac{da_{22}}{dp}\right) + a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dq}\right)\right],$$

values which give, as they should do,

$$\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}12\\2\end{Bmatrix} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dp},\ \text{and}\ \begin{Bmatrix}12\\1\end{Bmatrix} + \begin{Bmatrix}22\\2\end{Bmatrix} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dq}.$$

The foregoing comparison serves to explain the notation of Weingarten's valuable memoir.

932.

ON SYMMETRIC FUNCTIONS AND SEMINVARIANTS.

[From the *American Journal of Mathematics*, vol. XV. (1893), pp. 1—74.]

THE principal object of the present memoir is to develop further the theory of seminvariants, but in connexion therewith I was led to some investigations on symmetric functions, and I have consequently included this subject in the title. The two theories, if we adopt the MacMahon form of equation,

$$0 = 1 + bx + \frac{c}{2}x^2 + \frac{d}{6}x^3 + \ldots,$$

may be regarded as identical; but there are still two branches of the theory, viz. we may seek to obtain for the symmetric functions of the roots expressions in terms of the coefficients (which expressions, in the case of non-unitary symmetric functions, are in fact seminvariants), or we may attend to the properties of the functions of the coefficients thus obtained and which we call seminvariants. But I do not in the first instance use the MacMahon form, but retain the ordinary form of equation $0 = 1 + bx + cx^2 + dx^3 + \ldots$, and we have thus only a parallelism of the two theories, and in place of seminvariants we have functions which I call non-unitariants. In regard as well to these as to unitariant functions, I consider certain operators Θ_σ, Δ, $P - \delta b$, and $Q - 2\omega b$, which under altered forms present themselves also in the theory of seminvariants.

As regards seminvariants, I consider what I call the blunt and sharp forms respectively: the great problem is, it appears to me, that of sharp seminvariants, otherwise the I-and-F problem—viz. for any given weight we have to determine the correspondence between the initial and final terms in such wise as to obtain a system of sharp seminvariants. I obtain a "square diagram" solution, which is so far theoretically complete that for any given weight I can, without any tentative operation, determine by a laborious process the correspondence in question: but I am not thereby enabled to establish or enunciate for successive weights any general rule of correspondence; and my process is in fact, as regards practicability, far inferior to that which I call the MacMahon linkage, but of the validity of this I have not succeeded in obtaining any satisfactory proof.

I establish an umbral theory of seminvariants which will be presently again referred to, and I consider the question of the reduction of seminvariants. The final term of a seminvariant may be composite (that is, the product of two or more final terms), and that in one way only or in two or more ways, or it may be non-composite. In the case of a composite final term the seminvariant is reducible, but the converse theorem that a seminvariant with a non-composite final term is irreducible is in nowise true; the reason of this is explained. An irreducible seminvariant is a perpetuant. In regard to perpetuants, I reproduce and simplify a demonstration recently obtained by Dr Stroh as to the perpetuants for any given degree whatever: viz. the generating function for perpetuants of degree n is

$$= x^{2^{n-1}-1} \div 1 - x^2 \,.\, 1 - x^3 \ldots 1 - x^n;$$

the theorem was previously known, and more or less completely proved, for the values $n = 4$, 5, 6, and 7. Dr Stroh's investigation is conducted by an umbral representation,

$$(\alpha x + \beta y + \gamma z + \ldots)^n, \quad x + y + z + \ldots = 0,$$

of the blunt seminvariants of a given weight.

I consider in regard to seminvariants the theory of the symbols $P - \delta b$ and $Q - 2\omega b$, and the derived symbols Y and Z, each of which operating on a seminvariant gives a seminvariant. These are, in fact, connected with the derivatives (f, F) of a quantic f and any covariant thereof F; but except to point out this connexion, I do not in the present memoir consider the theory of covariants.

The Coefficients $(a, b, c, d, e, \ldots)$ *or* $(1, b, c, d, e, \ldots)$. Art. Nos. 1 to 9.

1. I consider the series $(a, b, c, d, e, \ldots)$, or putting as we most frequently do $a = 1$, say the series $(1, b, c, d, e, \ldots)$ of coefficients, the several terms whereof are taken to be of the weights 0, 1, 2, 3, 4, ... respectively. We form with these sets of isobaric terms, or say columns of the weights 0, 1, 2, 3, 4, ... respectively, for instance,

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	d^2
				b^4	bc^2	b^2e
					b^3c	bcd
					b^5	c^3
						b^3d
						b^2c^2
						b^4c
						b^6

and generally a set or column of any given weight. In each term, the letters are written in alphabetical order.

Taking the whole or any part of a column, for instance the whole column (d, bc, b^3), or the part (e, bd, c^2) of the next column, we may by supplying powers of a in such wise as to leave unaltered the terms of the highest degree, that is, by reading these as (a^2d, abc, b^3) and (ae, bd, c^2) respectively, regard them as homogeneous sets of a given degree in $(a, b, c, d, e, \ldots)$; and thus generally we may speak of the degree of a set of terms.

The terms of the several columns as above written down are in alphabetical order, AO; viz. we supply as above the proper powers of a, reading for instance col. 4 as a^3e, a^2bd, a^2c^2, ab^2c, b^4, where the terms are in alphabetical or dictionary order.

Each column is derived from the preceding one by Arbogast's rule, it being understood, for instance, that b^4, that is, ab^4, gives the two terms ab^3c and b^5, that is, b^3c and b^5; and so in other cases.

2. We attend in particular to the non-unitary terms, or non-unitaries, e.g. in col. 5, f, cd, which contain no b; and to the power-ending terms or power-enders, bc^2, b^5, which end in a power. It will be observed that, whenever by Arbogast's rule a term in one column gives two terms in the next column, the second of these is a power-ender; and thus in any column the excess of the number of terms above that in the preceding column is equal to the number of power-enders.

3. I consider the notion of conjugate terms: representing, for instance, the terms

$$f \qquad be \qquad cd$$

by dots in the form

$$\cdot\cdot\cdot\cdot\cdot \qquad \cdot\cdot\cdot\cdot \qquad \cdot\cdot\cdot$$

and reading the number of dots in columns instead of in lines we derive the conjugate terms

$$b^5 \qquad b^3c \qquad bc^2,$$

and so in other cases. It is clear that the relation is a reciprocal one (thus the conjugates of b^5, b^3c, bc^2 are f, be, cd respectively). Moreover, a term may be its own conjugate; thus cd^2, arranging the dots in lines and reading them in columns, $\vdots\ \vdots\ :$ is again cd^2.

It is at once seen that non-unitaries and power-enders are conjugate to each other; hence in any column, the non-unitaries and the power-enders are equal in number, and a preceding result may be stated in the more complete form: in any column the excess of the number of terms above that in the preceding column is equal to the number of non-unitaries or to the number of power-enders.

4. The terms of the several columns may be arranged in counter-order CO, thus:

0	1	2	3	4	5	6
1	b	c	d	e	f	g
		b^2	bc	bd	be	bf
			b^3	c^2	cd	ce
				b^2c	b^2d	b^2e
				b^4	bc^2	d^2
					b^3c	bcd
					b^3	b^2d
						c^3
						b^2c^2
						b^4c
						b^6

viz. we arrange here according to the highest letters. The counter-order is, in fact, the alphabetical order with the reversed arrangement (..., g, f, e, d, c, b, a) of the alphabet, but in the separate terms we retain the alphabetical order, thus writing as before bf and not fb. Observe that the difference between the two arrangements, AO and CO, first presents itself in the col. 6.

In this CO arrangement, each column is derived from the next preceding one by a rule as follows: We operate on the lowest letter of each term, being a simple letter, not a power, by changing it into the next highest letter, and we further operate upon each term by multiplying it by b, the operation or (as the case may be) two operations upon any term being performed before operating upon the next term.

5. If we compare a column in AO with the same column in CO, for instance

AO	CO	AO	CO rev.
g	g	g	b^6
bf	bf	bf	b^4c
ce	ce	ce	b^2c^2
d^2	b^2e	d^2	c^3
b^2e	d^2	b^2e	b^3d
bcd	bcd	bcd	bcd
c^3	b^3d	c^3	d^2
b^3d	c^3	b^3d	b^2e
b^2c^2	b^2c^2	b^2c^2	ce
b^4c	b^4c	b^4c	bf
b^6	b^6	b^6	g

it will be seen that the terms are conjugates of each other, the first and last, the second and last but one terms, and so on; or, what is the same thing, if we reverse the order of either column, then the pairs of conjugate terms will appear each in the same line; of course, here a self-conjugate term such as *bcd* is put in evidence.

6. By writing $a, b, c, d, \ldots = a_0, a_1, a_2, a_3, \ldots$, or more simply 0, 1, 2, 3, ..., we connect the theory with that of the partition of numbers: in particular, the terms of a given weight correspond to the partitions of that weight, or number of ways in which that weight can be made up with the parts 1, 2, 3, ... It may be remarked that, in a partition, the parts are usually written in decreasing order, whereas (as remarked above) in a literal term the letters are written in alphabetical order. Thus we have 321 and *bcd*; it would be more correct to write the partition as 123.

It is frequently convenient, retaining the letters $b, c, d, \ldots$, to write for instance $q = a_\sigma$ (σ a numerical suffix), meaning thereby that q is the letter corresponding to the place σ in the series 1, 2, 3, ... If instead of the indefinite series $(1, b, c, d, \ldots)$ we consider, as is sometimes convenient, a definite series of terms $(1, b, c, \ldots, q = a_\sigma)$, then σ is said to be the "extent" of the system. The next preceding letter p will naturally be $= a_{\sigma-1}$; and if, increasing the extent by unity, we introduce a new letter r, this will be $a_{\sigma+1}$, and so in other cases, the notation being for the most part used merely as a convenient way of showing the place of a letter in the series.

7. Considering the terms of a given weight, or say a column, in AO or CO, we may represent any portion of the column by means of its initial and final terms, say I and F, by the notations Iao$F$ and Ico$F$ respectively. But a much more important notation is Ica$F$; viz. this represents the series of terms of given weight which are in CO not superior to I, and in AO not inferior to F (a like notation, which however I do not employ, would be Iac$F$; viz. this would denote the series of terms which are in AO not superior to I and in CO not inferior to F). The definition of Ica$F$ has been given in the above general form, but we are in fact exclusively or chiefly concerned with the case where I is a non-unitary and F a power-ender. It is to be observed that, considering the AO column as given, then to form from it the set or interval Ica$F$ we may disregard altogether the terms which are in the AO column inferior (posterior) to F, for by the definition none of these enter into Ica$F$, but it may very well be that there are in Ica$F$ terms which are in the AO column superior (anterior) to I. An instance of this first presents itself for the weight 11; viz. here a portion of the AO column is $(fg, b^2j, bci, bdh, beg, bf^2, c^2h, cdg, \ldots)$: hence in Ica$F$, if the initial term be c^2h, for instance in $c^2hcab^3e^2$, we have terms fg, beg, bf^2 which are in AO anterior to the initial term c^2h. In order therefore to form Ica$F$ from the AO column, we must first take the terms (if any) which being in CO posterior to I are in the AO column anterior to I, and then from the portion Iao$F$ of the AO column reject the terms (if any) which are in CO anterior to I. In particular, starting from the AO column, and

arranging the non-unitaries thereof in CO and the power-enders in AO, for instance, weight 12, these are

$$\begin{array}{ll} m & g^2 \\ ck & cf^2 \\ dj & e^2 \\ ei & b^2f^2 \\ c^2i & bde^2 \\ fh & c^2e^2 \\ \vdots & \vdots \end{array}$$

There is no difficulty in writing down the terms of the several sets or intervals

$$mcag^2,\quad mcacf^2,\quad mcae^2,\ \ldots,\ ckcag^2,\quad ckcacf^2,\ \ldots.$$

Instead of ca we may, if we please, use, and in fact I generally use the conventional symbol ∞, or write $m \infty g^2$, $m \infty cf^2$, &c. In any such set, the terms need not be arranged in AO; if for any purpose it is more convenient, they may be arranged in CO; but of course the definition of the meaning must not be departed from. The expressed initial is the highest term in CO, and the expressed final the lowest term in AO.

8. I diminish a term by replacing successively each letter thereof by the next inferior letter; for instance, if the term is cdf, then the diminished terms are $Dcdf$, $=bdf$, c^2f, cde, and so Db^2df, $=bdf$, b^2cf, b^2de (where the diminished b is a, that is, 1). Conversely, we may augment a term by replacing successively each letter thereof by the next superior letter; for instance, $Abdf$, $=b^2df$, cdf, bef, bdg, where the first augmentation b^2df is obtained from the a (which may be regarded as latent in the term operated upon). Operating upon the letters in order beginning with the lowest, the several diminutions may be called D_1, D_2, D_3, ..., and the several augmentations A_0, A_1, A_2, ... (where A_0 is in fact multiplication by b). We diminish a set by diminishing successively the several terms thereof (the diminished terms being taken without repetition; that is, each such term once only). Similarly, we may augment a set by augmenting successively the several terms thereof (the augmented terms being taken without repetition). It is to be noticed that the two operations are not reciprocal to each other; if we diminish a set, and then augment the diminished set, we obtain indeed all the terms of the original set, but in general we obtain also terms which are not included in the original set.

9. It requires some consideration to see that we have $D(I \infty F) = (D_1 I \infty D_\theta F)$, where $D_\theta F$ is the diminution performed upon the highest letter of F. Take any term M of $D(I \infty F)$, the several diminutions D_1M, D_2M, ..., $D_\phi M$ are arranged in descending order: D_1M the highest and $D_\phi M$ the lowest, as well in CO as in AO. If then D_1M is in CO not superior to D_1I, then all the DM's will be in CO not superior to D_1I; and similarly, if $D_\phi M$ is in AO not inferior to $D_\theta F$, then all the DM's will be in AO not inferior to $D_\theta F$. And this being seen, then if we take

N a term of $(D_1 I \infty D_\theta F)$, and consider the successive augmentations $A_0 N$, $A_1 N, \ldots, A_\phi N$ of N, then these will be in ascending order $A_0 N$ the lowest and $A_\phi N$ the highest in CO as well as in AO. It may happen that $A_\phi N$ or this and neighbouring terms are in CO higher than I, and that $A_0 N$ or this and neighbouring terms are in AO lower than F, but there will always be a term or terms which is or are in CO lower than I and in AO higher than F; and thus not only every term of $D(I \infty F)$ will be a term of $(D_1 I \infty D_\theta F)$, but conversely every term of $(D_1 I \infty D_\theta F)$ will be a term of $D(I \infty F)$, and we thus have the required relation $D(I \infty F) = (D_1 I \infty D_\theta F)$.

Symmetric Functions of the Roots. Art. Nos. 10 to 31.

10. We consider a set of roots $\alpha, \beta, \gamma, \delta, \epsilon, \ldots$ either indefinite in number, or else definite, for instance $\alpha, \beta, \gamma, \delta$. The symmetric functions (rational and integral functions) are in the first instance denoted in the usual manner

$$S\alpha = \alpha + \beta + \gamma + \delta + \ldots, \quad S\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma + \ldots,$$

$$S\alpha^2\beta = \alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2 + \ldots,$$

viz. the S refers to all the distinct combinations of like form with the combination $(\alpha, \alpha\beta,$ or $\alpha^2\beta,$ as the case may be) to which it is prefixed. By omitting the S and instead of the roots considering merely their indices, these same symmetric functions would be 1, 11 (or 1^2), 21, &c., and then if instead of the numbers 1, 2, 3, &c., we introduce the symbolic capital letters $B, C, D, \ldots$, the same symmetric functions will be represented as B, B^2, BC, &c. (21, that is, 12 is written as BC, and so in other cases, the letters in alphabetical order). The letters $B, C, D, \ldots$ are considered as being of the weights 1, 2, 3, ... respectively, and thus the symmetric functions of a given degree in the roots are represented by the terms of that weight in the symbolic letters $B, C, D, \ldots$, thus the symmetric functions of the degree 4 are E, BD, C^2, B^2C, B^4; of course these terms may be arranged in AO or in CO as may be most convenient for the purpose in hand. The capital letters $B, C, D, \ldots$ are in fact umbræ, but to avoid confusion with subsequent notations I do not in general thus speak of them. A form such as $S\alpha^2$ or $S\alpha^4\beta^2$, in which there is no index 1, is said to be non-unitary; but a form $S\alpha$ or $S\alpha^2\beta$, in which there is an index $=1$ or two or more indices each $=1$, is said to be unitary: or, what is the same thing, in the symbolic representation by capital letters, the form is non-unitary or unitary according as it does not or does contain the letter B.

11. In the ordinary theory of symmetric functions, we connect the coefficients $(1, b, c, d, \ldots)$ with the roots $(\alpha, \beta, \gamma, \ldots)$ by the equation

$$1 + bx + cx^2 + dx^3 + \ldots = 1 - \alpha x \,.\, 1 - \beta x \,.\, 1 - \gamma x \ldots,$$

and we thus have

$$\begin{aligned}
-b &= \alpha + \beta + \gamma + \ldots, &&= S\alpha\,, &&= 1\,, &&= B\,,\\
+c &= \alpha\beta + \alpha\gamma + \beta\gamma + \ldots, &&= S\alpha\beta\,, &&= 1^2, &&= B^2,\\
-d &= \alpha\beta\gamma + \ldots\,, &&= S\alpha\beta\gamma, &&= 1^3, &&= B^3,\\
&\text{\&c., \&c.;}
\end{aligned}$$

and it is to be remarked that, for any given number of roots, there will be this same number of coefficients: we may for instance have

$$1+bx+cx^2+dx^3=1-\alpha x\,.\,1-\beta x\,.\,1-\gamma x,$$

that is,

$$\begin{aligned} -b &= \alpha\ \ +\beta\ +\gamma, \\ +c &= \alpha\beta+\alpha\gamma+\beta\gamma, \\ -d &= \alpha\beta\gamma\,; \end{aligned}$$

and similarly if the number of roots be $=4$, or any larger number.

12. The symmetric functions of a given degree, say 4, in the roots, viz.

$$S\alpha^4,\quad S\alpha^3\beta,\quad S\alpha^2\beta^2,\quad S\alpha^2\beta\gamma,\quad S\alpha\beta\gamma\delta,$$

or

$$4,\quad 31,\quad 2^2,\quad 21^2,\quad 1^4,$$

or

$$E,\quad BD,\quad C^2,\quad B^2C,\quad B^4,$$

are equal in number to the combinations of the weight 4 in the coefficients, viz.

$$e,\quad bd,\quad c^2,\quad b^2c,\quad b^4\,;$$

and the terms of the one set are in fact linear combinations (with mere numerical multipliers) of the terms of the other set; but more than this, we have for instance

$e=\alpha\beta\gamma\delta+\ldots$, that is, $e=B^4$:

$bd=(\alpha+\beta+\gamma+\delta\ldots)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta\ldots)$ contains only terms $\alpha^2\beta\gamma$ and $\alpha\beta\gamma\delta$, that is, bd is a linear function of B^2C and B^4:

$c^2=(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta\ldots)^2$ contains only terms $\alpha^2\beta^2$, $\alpha^2\beta\gamma$ and $\alpha\beta\gamma\delta$, that is, c^2 is a linear function of C^2, B^2C and B^4; and so on.

13. We have in fact the Table IV (a) which I quote from my paper "A Memoir on the Symmetric Functions of the Roots of an Equation," *Phil. Trans.* t. 147 (1857), pp. 489—496, [147],

	e	bd	c^2	b^2c	b^4
$S\alpha^4 = 4 = E$					$+1$
$S\alpha^3\beta = 31 = BD$				$+1$	$+4$
$S\alpha^2\beta^2 = 2^2 = C^2$			$+1$	$+2$	$+6$
$S\alpha^2\beta\gamma = 21^2 = B^2C$		$+1$	$+2$	$+5$	$+12$
$S\alpha\beta\gamma\delta = 1^4 = B^4$	$+1$	$+4$	$+6$	$+12$	$+24$

inserting on the left-hand outside margin the new symbols E, BD, &c., with their

explanations: the || indicates that the table is to be read according to the columns, $e = +1B^4$, $bd = +1B^2C + 4B^4$, &c. This table gives conversely a Table IV (b), read according to the lines and serving to express the symmetric functions E, BD, &c., as linear functions of the combinations e, bd, c^2, b^2c, b^4 of the coefficients.

14. The (a) and (b) tables are given in the Memoir up to X (a) and X (b): it is proper to quote here the (b) tables up to VI (b) with only the change of substituting on the outside left-hand margins the literal terms such as E, BD, &c., instead of the symbols 4, 31, &c., originally used to denote these symmetric functions—it is to be observed that the left-hand symbols are in AO, the upper symbols in CO, this distinction first manifesting itself in the Table VI (b), so that it was necessary to go as far as this in order to put in evidence the true form of the tables.

II (b).

=	c	b^2
C	-2	$+1$
B^2	$+1$	

III (b).

=	d	bc	b^3
D	-3	$+3$	-1
BC	$+3$	-1	
B^3	-1		

IV (b).

=	e	bd	c^2	b^2c	b^4
E	-4	$+4$	$+2$	-4	$+1$
BD	$+4$	-1	-2	$+1$	
C^2	$+2$	-2	$+1$		
B^2C	-4	$+1$			
B^4	$+1$				

V (b).

=	f	be	cd	b^2d	bc^2	b^3c	b^5
F	-5	$+5$	$+5$	-5	-5	$+5$	-1
BE	$+5$	-1	-5	$+1$	$+3$	-1	
CD	$+5$	-5	$+1$	$+2$	-1		
B^2D	-5	$+1$	$+2$	-1			
BC^2	-5	$+3$	-1				
B^3C	$+5$	-1					
B^5	-1						

VI (b).

=	g	bf	ce	b^2e	d^2	bcd	b^3d	c^3	b^2c^2	b^4c	b^6
G	-6	$+6$	$+6$	-6	$+3$	-12	$+6$	-2	$+9$	-6	$+1$
BF	$+6$	-1	-6	$+1$	-3	$+7$	-1	$+2$	-4	$+1$	
CE	$+6$	-6	$+2$	$+2$	-3	$+4$	-2	-2	$+1$		
D^2	$+3$	-3	-3	$+3$	$+3$	-3	0	$+1$			
B^2E	-6	$+1$	$+2$	-1	$+3$	-3	$+1$				
BCD	-12	$+7$	$+4$	-3	-3	$+1$					
C^3	-2	$+2$	-2	0	$+1$						
B^3D	$+6$	-1	-2	$+1$							
B^2C^2	$+9$	-4	$+1$								
B^4C	-6	$+1$									
B^6	$+1$										

It is hardly necessary to remark in relation to these tables that if there are only two roots, then $d=0$, &c., viz. Table II. is not affected but all the subsequent tables assume a simplified form; if there are only three roots, then $e=0$, &c., viz. Tables II. and III. are not affected but all the subsequent tables assume a simplified form; and so on.

15. We have between the differential symbols ∂_b, ∂_c, ∂_d, ... and ∂_α, ∂_β, ∂_γ, ... certain relations which it is interesting to develop: it will be convenient to consider successively the cases, three roots, four roots, &c.

In the case of three roots, starting from

$$-b=\alpha+\beta+\gamma,$$

$$c=\alpha\beta+\alpha\gamma+\beta\gamma,$$

$$-d=\alpha\beta\gamma,$$

we have

$$\partial_\alpha=-\partial_b+(\beta+\gamma)\,\partial_c-\beta\gamma\partial_d,$$

$$\partial_\beta=-\partial_b+(\gamma+\alpha)\,\partial_c-\gamma\alpha\partial_d,$$

$$\partial_\gamma=-\partial_b+(\alpha+\beta)\,\partial_c-\alpha\beta\partial_d,$$

equations which give conversely ∂_b, ∂_c, ∂_d as linear functions of ∂_α, ∂_β, ∂_γ: I write

down the three equations thus obtained together with a fourth equation which I will explain. The four equations are

$$-\partial_a+\delta'=\frac{\alpha^3}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^3}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^3}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_b\quad=\frac{\alpha^2}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_c\quad=\frac{\alpha}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

$$-\partial_d\quad=\frac{1}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{1}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{1}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma.$$

In verification of the last three equations, observe that they give

$$-\partial_b+(\beta+\gamma)\partial_c-\beta\gamma\partial_d$$
$$=\frac{\alpha^2-\alpha(\beta+\gamma)+\beta\gamma}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{\beta^2-\beta(\beta+\gamma)+\beta\gamma}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{\gamma^2-\gamma(\beta+\gamma)+\beta\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$

that is, $-\partial_b+(\beta+\gamma)\partial_c-\beta\gamma\partial_d=\partial_\alpha$: and similarly from the same three equations we deduce the values of ∂_β and ∂_γ; the three equations are thus equivalent to the foregoing three equations for ∂_α, ∂_β, ∂_γ.

As to the first equation, to avoid confusion with a root δ, I have written therein δ' (afterwards replaced by δ) to denote the degree of a function homogeneous in (a, b, c, d), upon which the symbols are supposed to operate; this is also the degree in the roots α, β, γ. The four equations give

$$-a(\partial_\alpha-\delta')-b\partial_b-c\partial_c-d\partial_d=\frac{\alpha^3+b\alpha^2+c\alpha+d}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\&c.,\ =0,$$

since

$$\alpha^3+b\alpha^2+c\alpha+d=0,\quad \beta^3+b\beta^2+c\beta+d=0,\quad \gamma^3+b\gamma^2+c\gamma+d=0.$$

The equations thus give

$$a\partial_a+b\partial_b+c\partial_c+d\partial_d=\delta',$$

which is right, and the first equation is thus verified.

16. From the last three equations for ∂_b, ∂_c, ∂_d, we deduce

$$-3\partial_b-2b\partial_c-c\partial_d=\frac{3\alpha^2+2b\alpha+c}{\alpha-\beta\,.\,\alpha-\gamma}\partial_\alpha+\frac{3\beta^2+2b\beta+c}{\beta-\gamma\,.\,\beta-\alpha}\partial_\beta+\frac{3\gamma^2+2b\gamma+c}{\gamma-\alpha\,.\,\gamma-\beta}\partial_\gamma,$$
$$=\partial_\alpha+\partial_\beta+\partial_\gamma,$$

a result more easily deducible from the first set of three equations for ∂_α, ∂_β, ∂_γ: but I have preferred to obtain it in this manner for the sake of the remark that it is a peculiarity of this combination of ∂_b, ∂_c, ∂_d that the coefficients of ∂_α, ∂_β, ∂_γ

become integral functions of the roots (in the actual case constants and $=1$): for a somewhat similar form

$$-(c\partial_b + d\partial_c), \quad = \frac{c\alpha^2 + d\alpha}{\alpha-\beta\,.\,\alpha-\gamma}\,\partial_\alpha + \frac{c\beta^2 + d\beta}{\beta-\gamma\,.\,\beta-\alpha}\,\partial_\beta + \frac{c\gamma^2 + d\gamma}{\gamma-\alpha\,.\,\gamma-\beta}\,\partial_\gamma,$$

the coefficients are fractional.

We at once have

$$\alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma = b\partial_b + 2c\partial_c + 3d\partial_d,$$

viz. these symbols operating upon a function of the roots of the degree ω, or what is the same thing, a function of the coefficients of the weight ω, are each of them equivalent to a constant factor ω.

Again, we have

$$\begin{aligned}\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma &= -(b^2 - 2c)\,\partial_b - (bc - 3d)\,\partial_c - bd\partial_d,\\ &= -b\,(b\partial_b + c\partial_c + d\partial_d) + 2c\partial_b + 3d\partial_c,\end{aligned}$$

or since $a\partial_a + b\partial_b + c\partial_c + d\partial_d = \delta'$ (if as before δ' is the degree of the function operated upon) and therefore $b\partial_b + c\partial_c + d\partial_d = \delta' - a\partial_a$ or say $=\delta' - \partial_a$, this is

$$\alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma = -b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c,$$

so that we have here another form $-b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c$, for which the coefficients of ∂_α, ∂_β, ∂_γ are integral functions of the roots.

17. In the case of four roots, the corresponding equations are

$$\begin{aligned}-b &= \alpha + \beta + \gamma + \delta,\\ +c &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta,\\ -d &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta,\\ +e &= \alpha\beta\gamma\delta,\end{aligned}$$

and we then have

$$\begin{aligned}\partial_\alpha &= -\partial_b + (\beta+\gamma+\delta)\,\partial_c - (\beta\gamma + \beta\delta + \gamma\delta)\,\partial_d + \beta\gamma\delta\partial_e,\\ \partial_\beta &= -\partial_b + (\gamma+\delta+\alpha)\,\partial_c - (\gamma\delta + \gamma\alpha + \delta\alpha)\,\partial_d + \gamma\delta\alpha\partial_e,\\ \partial_\gamma &= -\partial_b + (\delta+\alpha+\beta)\,\partial_c - (\delta\alpha + \delta\beta + \alpha\beta)\,\partial_d + \delta\alpha\beta\partial_e,\\ \partial_\delta &= -\partial_b + (\alpha+\beta+\gamma)\,\partial_c - (\alpha\beta + \alpha\gamma + \beta\gamma)\,\partial_d + \alpha\beta\gamma\partial_e,\end{aligned}$$

and the converse set of equations, which for shortness I write in the form

$$-\partial_a + \delta',\ -\partial_b,\ -\partial_c,\ -\partial_d,\ -\partial_e$$

$$= \frac{\alpha^{4,\,3,\,2,\,1,\,0}}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta}\,\partial_\alpha + \frac{\beta^{4,\,3,\,2,\,1,\,0}}{\beta-\gamma\,.\,\beta-\delta\,.\,\beta-\alpha}\,\partial_\beta + \frac{\gamma^{4,\,3,\,2,\,1,\,0}}{\gamma-\delta\,.\,\gamma-\alpha\,.\,\gamma-\beta}\,\partial_\gamma + \frac{\delta^{4,\,3,\,2,\,1,\,0}}{\delta-\alpha\,.\,\delta-\beta\,.\,\delta-\gamma}\,\partial_\delta.$$

We have, in like manner as in the former case,

$$\begin{aligned} -4\partial_b - 3b\partial_c - 2c\partial_d - d\partial_e &= \partial_\alpha + \partial_\beta + \partial_\gamma + \partial_\delta, \\ b\partial_b + c\partial_c + d\partial_d + e\partial_e &= \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta, = \omega, \\ -b\delta' + b\partial_a + 2c\partial_b + 3d\partial_c + 4e\partial_d &= \alpha^2\partial_\alpha + \beta^2\partial_\beta + \gamma^2\partial_\gamma + \delta^2\partial_\delta; \end{aligned}$$

and similarly in the case of five or more roots.

18. In the case of σ' roots, I write $m = a_{\sigma'}$, and for shortness

$$\begin{aligned} \Theta_{\sigma'} &= \sigma'\partial_b + (\sigma'-1)\,b\partial_c + \ldots \quad \ldots + l\partial_m, \\ P &= b\partial_\alpha + \quad 2c\partial_b + 3d\partial_c + \ldots + \sigma' m\partial_l, \end{aligned}$$

so that, besides the equation $b\partial_b + c\partial_c \ldots + m\partial_m = S\alpha\partial_\alpha = \omega$, the foregoing investigations show that we have

$$\begin{aligned} \Theta_{\sigma'} &= -S\partial_\alpha, \\ P - b\delta &= \quad S\alpha^2\partial_\alpha. \end{aligned}$$

The operand for these symbols is a symmetric function of the roots, which is thus also a function of the coefficients: it is of the degree ω in the roots, and consequently of the weight ω in the coefficients, and its degree in the coefficients is taken to be $= \delta$. It is sometimes convenient to represent this operand, *quà* function of the roots, by Υ and, *quà* function of the coefficients, by U, so that we have in general $\Upsilon = U$. If Υ be a non-unitary function of the roots, then we may say that Υ, $= U$, is a non-unitariant.

19. I give some illustrations of the equation $\Theta_{\sigma'} = -S\partial_\alpha$. Suppose

$$\Upsilon = U = S\alpha^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4$$

(Table IV (b)); σ' must be $= 4$ at least and I take it to be 4 and 5 successively; we thus have

$$\begin{aligned} \Theta_4 &= 4\partial_b + 3b\partial_c + 2c\partial_d + d\partial_e, \\ \Theta_5 &= 5\partial_b + 4b\partial_c + 3c\partial_d + 2d\partial_e, \end{aligned}$$

omitting from Θ_5 the term $e\partial_f$ which is obviously inoperative. For any number whatever of roots, we have

$$-S\partial_\alpha . S\alpha^4 = -4S\alpha^3 = -4(-3d + 3bc - b^3), \quad = 12d - 12bc + 4b^3,$$

and this should therefore be the value as well of $\Theta_4 E$ as of $\Theta_5 E$. The calculations may be arranged as follows:

$$\begin{array}{llll|l} & \Theta_4 E & & & \\ 4\,. & 4d - 8bc + 4b^3 & d + 16 & -4 & 12 \\ 3b\,. & 4c - 4b^2 & bc - 32 + 12 + 8 & & -12 \\ 2c\,. & 4b & b^3 + 16 - 12 & & +\ 4, \\ d\,. & -4 & & & \end{array}$$

$\Theta_5 E$

$5\,.$	$4d-8bc+4b^3$	$d-20$	-8	12
$4b\,.$	$4c-4b^2$	$bc-40+16+12$		-12
$3c\,.$	$4b$	$b^3+20-16$		$+4,$
$d\,.-4$				

giving in each case the right result.

20. In the foregoing example, $S\alpha^4$ was a non-unitary function of the roots, but I take the case of a unitary function. Suppose

$$\Upsilon = U = S\alpha^2\beta = BD = 4e - bd - 2c^2 + b^2c.$$

Here $-S\partial_\alpha . S\alpha^3\beta$ is not independent of the number of the roots; in the case of 4 roots, we have

$$-S\partial_\alpha . S\alpha^3\beta = -3S\alpha^2\beta - 3S\alpha^3, \quad = -3(3d-bc) - 3(-3d+3bc-b^3), \quad = 0d - 6bc + 3b^3;$$

and in the case of 5 roots, we have

$$-S\partial_\alpha . S\alpha^3\beta = -3S\alpha^2\beta - 4S\alpha^3, \quad = -3(3d-bc) - 4(-3d+3bc-b^3), \quad = 3d - 9bc + 4b^3;$$

and these should therefore be the values of $\Theta_4 BD$ and $\Theta_5 BD$ respectively. The calculations are

$\Theta_4 BD$

$4\,.-$	$d+2bc$	$d-4$	$+4$	0
$3b\,.-$	$4c+b^2$	$bc+8-12-2$		-6
$2c\,.-$	b	$b^3 \quad +3$		$+3,$
$d\,.+4$				

$\Theta_5 BD$

$5\,.-$	$d+2bc$	$d-5$	$+8$	$+3$
$4b\,.-$	$4c+b^2$	$bc+10-16-3$		-9
$3c\,.-$	b	$b^3 \quad +4$		$+4,$
$2d\,.+$	b			

giving in each case the correct result. We have $\Theta_5 - \Theta_4 = \partial_b + b\partial_c + c\partial_d + d\partial_e$, and the examples show that performing this operation on the non-unitariant $S\alpha^4$, $= E$, we obtain a result $= 0$; whereas for the unitary function $S\alpha^3\beta$, $= BD$, the result is not $= 0$.

21. Considering the question generally, I take the highest coefficient in U to be $q = a_\sigma$ (σ equal to or less than ω), or what is the same thing, the extent of U to be $= \sigma$; this implies that σ' is at least $= \sigma$; and taking it to be first $= \sigma$, and then to be any number greater than σ, we have

$$\Theta_\sigma = -S\partial_\alpha, \quad \Theta_{\sigma'} = -S\partial_\alpha,$$

where the function U operated upon by Θ_σ and $\Theta_{\sigma'}$ respectively is in each case the same function of the coefficients. It is easy to see that, if Υ is a non-unitary function of the roots, then whatever be the number of the roots we have $S\partial_\alpha . \Upsilon =$ a determinate symmetric function of the roots, and consequently = a determinate function of the coefficients. We thus have $\Theta_{\sigma'} U$ and $\Theta_\sigma U$ equal to each other; that is,

$$(\Theta_{\sigma'} - \Theta_\sigma)\, U = 0\,;$$

we may write

$$\Theta_\sigma = \sigma\,\partial_b + (\sigma - 1)\, b\partial_c + \ldots + \qquad p\partial_q,$$

$$\Theta_{\sigma'} = \sigma'\partial_b + (\sigma' - 1)\, b\partial_c + \ldots + (\sigma' - \sigma + 1)\, p\partial_q,$$

for the subsequent terms of $\Theta_{\sigma'}$, as involving ∂_r, ∂_s, &c., are inoperative; hence writing

$$\Delta = \partial_b + b\partial_c + c\partial_b + \ldots + p\partial_q,$$

or as we may more simply express it

$$\Delta = \partial_b + b\partial_c + c\partial_b + \ldots,$$

we have $\Theta_{\sigma'} - \Theta_\sigma = (\sigma' - \sigma)\,\Delta$, and consequently $\Delta U = 0$; Δ is thus an annihilator of any function U of the coefficients which is equal to a non-unitary function of the roots; or more shortly Δ is an annihilator of any non-unitariant.

22. Similarly, from the two equations $\Theta_\sigma = -S\partial_\alpha$, and $\Theta_{\sigma'} = S\partial_\alpha$ regarded as operating upon a non-unitary function, we deduce $\sigma'\Theta_\sigma - \sigma\Theta_{\sigma'} = (\sigma - \sigma')\, S\partial_\alpha$: the left-hand side is here $= (\sigma - \sigma')\,\Delta_1$, if

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d + \ldots + (\sigma - 1)\, p\partial_q,$$

or say

$$\Delta_1 = b\partial_c + 2c\partial_b + 3c\partial_d + \ldots,$$

viz. we have $\Delta_1 = S\partial_\alpha$; for instance, if as before

$$\Upsilon = U = S\alpha^4 = -4e + 4bd + 2c^2 - 4b^2c + b^4,$$

then

$$(b\partial_c + 2c\partial_d + 3d\partial_e)(-4e + 4bd + 2c^2 - 4b^2c + b^4) = S\partial_\alpha . S\alpha^4, \quad = 4S\alpha^3, \quad = 4\,(-3d + 3bc - b^3),$$

as can be at once verified. It is to be noticed, however, that $S\partial_\alpha$ operating upon a non-unitary function of the roots does not in every case give a non-unitary function; and thus successive operations with Δ_1 will not give a succession of non-unitariants.

23. I investigate the foregoing result in regard to Δ in a different manner; suppose, for instance, that $\Upsilon = U$ is the non-unitary function $S\alpha^4$ of the roots,

$$(= -4e + 4bd + 2c^2 - 4b^2c + b^4).$$

The number of roots is at least $= 4$, and I take it to be $= 4$, say the roots are α, β, γ, δ. Consider a fifth root θ, and let $\Upsilon_1 = U_1 = S\alpha^4$ be the like function for the five roots, we have $\Upsilon_1 = \Upsilon + \theta^4$, or say $U_1 = U + \theta^4$. Write $-b_1$, c_1, $-d_1$, e_1, $-f_1$

for the symmetric functions of the five roots; U_1 will not involve f_1 and it will be the same function of b_1, c_1, d_1, e_1 that U is of b, c, d, e, say we have

$$U_1 = U(b_1, c_1, d_1, e_1).$$

But we have

$$b_1 = b - \theta, \quad c_1 = c - b\theta, \quad d_1 = d - c\theta, \quad e_1 = e - d\theta;$$

and thus the foregoing equation $U_1 = U + \theta^4$ becomes

$$U(b - \theta, c - b\theta, d - c\theta, e - d\theta) = U + \theta^4;$$

it is in fact easy to verify that, for the foregoing value of U, the terms in θ, θ^2, θ^3 all vanish, and that the expression on the left-hand becomes $= U + \theta^4$. But attending only to the term in θ, this is $= -\theta(\partial_b + b\partial_c + c\partial_d + d\partial_e)\, U$, $= -\theta\Delta U$; viz. this term vanishing we have $\Delta U = 0$, the result which was to be proved.

In the case of a unitary function, for instance $\Upsilon = U = S\alpha^2\beta$, here introducing the new root θ we have $U_1 = U + \theta S\alpha^3 + \theta^3 S\alpha$; or there is here a term in θ, and instead of $\Delta U = 0$, we have $\Delta U = S\alpha^3$, or the unitary function is not annihilated by Δ.

The foregoing investigation is really quite general, and establishes the conclusion that Δ is an annihilator of every non-unitariant.

It is to be noticed that Θ_σ and Δ are operators, which leave each of them the degree unaltered but diminish the weight by unity: the operator $P - b\delta$, and another operator $\frac{1}{2}Q - b\omega$ which will be considered, increase each of them the degree by unity and also the weight by unity.

24. Coming now to the equation

$$P - b\delta = S\alpha^2\partial_\alpha,$$

it is to be remarked that, if $\sigma' = \sigma$, the expression for P ends in $q\partial_p$ where, as before, $q = a_\sigma$ is the highest coefficient in the operand; since the operand thus contains q, the next succeeding term in $r\partial_q$ would be not inoperative, and in order to include it in the expression of P we may take $\sigma' = \sigma + 1$; we thus have

$$P = b\partial_a + 2c\partial_b + 3d\partial_c + \ldots + (\sigma + 1)\, r\partial_q,$$

or as we may more simply write it

$$P = b\partial_a + 2c\partial_b + 3d\partial_c + \ldots;$$

the operation thus increases the extent by unity. The symbol $S\alpha^2\partial_\alpha$ operating upon a symmetric function of the roots, gives, whatever may be the number of roots, the same symmetric function of the roots: and we see further that, operating upon a non-unitary function, it gives a non-unitary function of the roots. Hence $P - b\delta$ operating upon a non-unitariant gives a non-unitariant. I give an example.

25. Suppose, as before,

$$\Upsilon = U = S\alpha^4 = E = -4e + 4bd + 2c^2 - 4b^2c + b^4;$$

here $\delta = 4$, and therefore

$$P - b\delta = b\partial_a + 2c\partial_b + 3d\partial_c + 4e\partial_d + 5f\partial_e - 4b.$$

We have

$$S\alpha^2\partial_\alpha \,.\, S\alpha^4 = 4S\alpha^5, \quad = 4\,(-5f + 5be + 5cd - 5b^2d - 5bc^2 + 5b^3c - b^5),$$

and this should therefore be the result of the operation $P - b\delta$: the calculation is

$b\,.$	$-12c + 8bd + 4c^2 - 4b^2c$	f					-20		-20
$2c\,.$	$4d - 8bc + 4b^3$	be	-12			$+16$		$+16$	$+20$
$3d\,.$	$4c - 4b^2$	cd		$+8$	$+12$				$+20$
$4e\,.$	$4b$	b^2d	$+8$		-12			-16	-20
$5f\,.$	-4	bc^2	$+4$	-16				-8	-20
$-4b\,.$	$-4e + 4bd + 2c^2 - 4b^2c + b^4$	b^3c	-4	$+8$				$+16$	$+20$
		b^5						-4	$-4,$

which is the right result.

We have seen that every non-unitariant is annihilated by Δ; it at once appears that conversely every function of the coefficients which is annihilated by Δ is a non-unitariant: it is, in fact, a symmetric function of the roots, and unless it were a non-unitary function of the roots it would not be annihilated by Δ. Non-unitariants are analogous to seminvariants; the precise relation between them will be shown further on.

26. We can, by an investigation similar to that for seminvariants, show that $P - b\delta$ operating upon a non-unitariant gives a non-unitariant. In fact, considering the two operations Δ and $P - b\delta$, we have

$$\Delta\,(P - b\delta)\dagger = \Delta\,(P - b\delta) + \Delta\,.\,(P - b\delta),$$

the meaning being that, if upon any operand U we perform first the operation $P - b\delta$ and then the operation Δ, this is equivalent to operating on U with the sum of the two operations $\Delta\,(P - b\delta)$, and $\Delta\,.\,P - b\delta$, the first of these symbols denoting the mere algebraical product of Δ and $P - b\delta$, the second of them the result of the operation Δ performed upon $P - b\delta$. We have similarly

$$(P - b\delta)\,\Delta\dagger = (P - b\delta)\,\Delta + (P - b\delta)\,.\,\Delta.$$

Hence observing that $\Delta\,(P - b\delta)$ and $(P - b\delta)\,\Delta$ are equal to each other, and subtracting, we have

$$\Delta\,(P - b\delta)\dagger - (P - b\delta)\,\Delta\dagger = \Delta\,.\,(P - b\delta) - (P - b\delta)\,.\,\Delta.$$

But from the values

$$\Delta = a\partial_b + b\partial_c + c\partial_d + \ldots \quad ,$$

and

$$P - b\delta = b\partial_a + c\partial_b + d\partial_c + \ldots - b\delta,$$

we find

$$\Delta.(P-b\delta)=a\partial_a+2b\partial_b+3c\partial_c+\ldots-\delta,$$
$$(P-b\delta).\Delta=\qquad b\partial_b+2c\partial_c+\ldots,$$

and thence

$$\Delta.(P-b\delta)-(P-b\delta).\Delta=a\partial_a+b\partial_b+c\partial_c\ldots-\delta,\ =0,$$

since δ is the degree in the coefficients. Hence writing down the operand U,

$$\Delta.(P-b\delta)\,U-(P-b\delta).\Delta U=0,$$

where for greater clearness I have inserted the dots, to show that Δ operates on $(P-b\delta)\,U$, and $(P-b\delta)$ on ΔU. Taking U to be a non-unitariant, we have $\Delta U=0$; and this being so, the equation gives $\Delta.(P-b\delta)\,U=0$, viz. this shows that $(P-b\delta)\,U$ is a non-unitariant.

27. There is another symbol $\frac{1}{2}Q-b\omega$, which is precisely analogous to $P-b\delta$, viz. operating upon a non-unitariant, it gives a non-unitariant: ω is, as before, the weight of the function operated upon, and the expression of Q is

$$\tfrac{1}{2}Q=c\partial_b+3d\partial_c+6e\partial_d+\ldots+\tfrac{1}{2}\sigma(\sigma+1)\,r\partial_q,$$

or say

$$\tfrac{1}{2}Q=c\partial_b+3d\partial_c+6e\partial_d+\ldots.$$

The proof is exactly similar, viz. we have to show that

$$\Delta.(\tfrac{1}{2}Q-b\omega)-(\tfrac{1}{2}Q-b\omega).\Delta=0.$$

We have

$$\Delta.(\tfrac{1}{2}Q-b\omega)=b\partial_b+3c\partial_c+6d\partial_d+\ldots-\omega,$$
$$(\tfrac{1}{2}Q-b\omega).\Delta=\qquad c\partial_c+3d\partial_d+\ldots,$$

and the difference of the two expressions is

$$b\partial_b+2c\partial_c+3d\partial_d+\ldots-\omega,\ =0,$$

since ω is the weight of the function operated upon. Hence, as before, if U be a non-unitariant and therefore $\Delta U=0$, we have $\Delta.(\frac{1}{2}Q-b\omega)\,U=0$, that is, $(\frac{1}{2}Q-b\omega)\,U$ is also a non-unitariant.

28. The symbol $\frac{1}{2}Q-b\omega$ has no simple expression in terms of ∂_α, ∂_β, ∂_γ, ..., and the form varies with the number of the roots: thus for 3 roots, it is

$$=-\left\{\left(\frac{c\alpha^2+3d\alpha}{\alpha-\beta.\alpha-\gamma}+b\alpha\right)\partial_\alpha+\&c.\right\},$$

for 4 roots it is

$$=-\left\{\left(\frac{c\alpha^2+3d\alpha^2+6e\alpha}{\alpha-\beta.\alpha-\gamma.\alpha-\delta}+b\alpha\right)\partial_\alpha+\&c.\right\},$$

for 5 roots it is

$$=-\left\{\left(\frac{c\alpha^4+3d\alpha^3+6e\alpha^3+10f\alpha}{\alpha-\beta.\alpha-\gamma.\alpha-\delta.\alpha-\epsilon}+b\alpha\right)\partial_\alpha+\&c.\right\},$$

and so on. It is not easy to find the effect of such a symbol upon a given symmetric function of the roots, nor in particular when the function is non-unitary is it easy to show generally that the result is non-unitary.

It is to be remarked that, if the function operated upon is of the degree δ in the roots, then we must for $\frac{1}{2}Q - b\omega$ take the expression with $\delta + 1$ roots; for instance, if the function be of the degree 5 in the roots, then, *quà* function of the coefficients, this contains f, and it must be operated on with

$$\tfrac{1}{2}Q - b\omega, = c\partial_b + 3d\partial_c + 6e\partial_d + 10f\partial_e + 15g\partial_f - \omega b,$$

viz. this expression, as containing g, gives the 6-root expression for $\frac{1}{2}Q - b\omega$.

29. Suppose for instance the function operated upon is $F = S\alpha^5$; here taking the 6-root expression, this gives

$$-5\left\{\left(\frac{c\alpha^5 + 3d\alpha^4 + 6e\alpha^3 + 10f\alpha^2 + 15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\epsilon\,.\,\alpha-\zeta} + b\alpha\right)\alpha^4 + \&c.\right\},$$

or omitting for the moment the outside factor -5, the expression in $\{\ \}$ is easily seen to be

$$= cH_4 + 3dH_3 + 6eH_2 + 10fH_1 + 15g + bS\alpha^5,$$

where H_4, H_3, H_2, H_1 denote the homogeneous functions of the degrees 4, 3, 2, 1 respectively: the values of these are obtained by adding together all the lines of the Table IV (b), all the lines of the Table III (b), &c.: the terms exclusive of $bS\alpha^5$ thus are

$$\begin{aligned} & c\,(-e + 2bd + c^2 - 3b^2c + b^4) \\ + \ & 3d\,(-d + 2bc - b^2) \\ + \ & 6e\,(-c + b^2) \\ + \ & 10f\,(-b) \\ + \ & 15g\,.\,1, \end{aligned}$$

and these are $= S\alpha^5\beta + S\alpha^4\beta^2 + S\alpha^2\beta^3$, as appears by the following calculation:

| | | |

The omitted term $bS\alpha^5$, that is, $-S\alpha.S\alpha^5$, is $-S\alpha^6-S\alpha^5\beta$; the addition hereof destroys therefore the non-unitary term $S\alpha^5\beta$, and thus the required expression, restoring the omitted factor -5, is $-5(-S\alpha^6+S\alpha^4\beta^2+S\alpha^3\beta^3)$, or say $=5G-5CE-5D^2$, a non-unitary form: this then should be the result of the operation

$$\tfrac{1}{2}Q-b\omega,\ =c\partial_b+3d\partial_c+6e\partial_d+10f\partial_e+15g\partial_f-5b,$$

performed upon

$$S\alpha^5=F=-5f+5be+5cd-5b^2d-5bc^2+5b^3c=b^5.$$

Performing the calculation so as to omit on each side a factor 5, it is to be shown that $G-CE-D^2$ is

$$\begin{aligned}
= \quad & c\,(e-2bd-c^2+3b^2c-b^4)\\
+ \quad & 3d\,(d-2bc+b^3)\\
+ \quad & 6e\,(c-b^2)\\
+ \quad & 10f\,(b)\\
+ \quad & 15g\,(-1)\\
- \quad & 5b\,(-f+be+cd-b^2d-bc^2+b^3c-\tfrac{1}{5}b^5).
\end{aligned}$$

Collecting the terms, and comparing the result with the expression for $G-CE-D^2$, we have

								$G-CE-D^2$			
g					-15		-15	-6	-6	-3
bf				$+10$		$+5$	$+15$	$+6$	$+6$	$+3$
ce	$+1$		$+6$				$+7$	$+6$	-2	$+3$
b^2e			-6			-5	-11	-6	-2	-3
d^2		$+3$					$+3$	$+3$	$+3$	-3
bcd	-2	-6				-5	-13	-12	-4	$+3$
b^3d		$+3$				$+5$	$+8$	$+6$	$+2$	0
c^3	-1						-1	-2	$+2$	-1
b^2c^2	$+3$					$+5$	$+8$	$+9$	-1	
b^4c	-1					-5	-6	-6		
b^6						$+1$	$+1$	$+1$		

and the two expressions are thus identical.

30. Suppose again, 6 roots as before, and that the function operated upon is $S\alpha^3\beta^2$; we find $\partial_\alpha S\alpha^3\beta^2=3\alpha^2S\alpha^2+2\alpha S\alpha^3-5\alpha^4$, and the general term is

$$\begin{aligned}
&-3\left(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\epsilon\,.\,\alpha-\zeta}+b\alpha\right)\alpha^2\,.\,S\alpha^2\\
&-2\left(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\epsilon\,.\,\alpha-\zeta}+b\alpha\right)\alpha\,.\,S\alpha^3\\
&+5\left(\frac{c\alpha^5+3d\alpha^4+6e\alpha^3+10f\alpha^2+15g\alpha}{\alpha-\beta\,.\,\alpha-\gamma\,.\,\alpha-\delta\,.\,\alpha-\epsilon\,.\,\alpha-\zeta}+b\alpha\right)\alpha^4.
\end{aligned}$$

This gives

$$\begin{aligned}
&-3\,(CH_2+3dH_1+6e+bS\alpha^2)\,S\alpha^2\\
&-2\,(CH_1+3d\qquad\quad+bS\alpha^2)\,S\alpha^3\\
&+5\,(CH_4+3dH_3+6eH_2+10fH_1+15g+bS\alpha^5),
\end{aligned}$$

which is found to be

$$\begin{aligned}
&=-3\,(BD+C^2\;+bS\alpha^3)\,S\alpha^2\\
&\quad-2\,(BC\qquad\quad+bS\alpha^2)\,S\alpha^3\\
&\quad+5\,(BF+CE+D^2+bS\alpha^5).
\end{aligned}$$

Here

$$\begin{aligned}
bS\alpha^3&=-S\alpha S\alpha^3=-S\alpha^4-S\alpha^2\beta,\;=-E-BD,\\
bS\alpha^2&=-S\alpha S\alpha^2=-S\alpha^3-S\alpha^2\beta,\;=-D-BC,
\end{aligned}$$

and

$$bS\alpha^5=-S\alpha S\alpha^5=-S\alpha^6-S\alpha^5\beta,\;=-G-BF;$$

the expression thus is

$$\begin{aligned}
&=-3\,(-E+C^2)\,.\,C\qquad\qquad\text{that is, }-3\,(-S\alpha^4+S\alpha^2\beta^2)\,S\alpha^2\\
&\quad-2\,(-D\qquad)\,.\,D\qquad\qquad\qquad\qquad-2\,(-S\alpha^3\qquad)\,S\alpha^5\\
&\quad+5\,(-G+CE+D^2),\qquad\qquad\qquad\qquad+5\,(-S\alpha^6+S\alpha^4\beta^2+S\alpha^3\beta^3).
\end{aligned}$$

Here

$$\begin{aligned}
S\alpha^2S\alpha^4\;&=S\alpha^6\;+\;S\alpha^4\beta^2,\;&&=G+CE,\\
S\alpha^2S\alpha^2\;&=S\alpha^6\;+2S\alpha^2\beta^2,\;&&=G+2D^2,\\
S\alpha^2S\alpha^2\beta^2&=S\alpha^4\beta^2+3S\alpha^2\beta^2\gamma^2,\;&&=CE+3C^3;
\end{aligned}$$

and the whole is

$$\begin{aligned}
&-3\,\{-G-CE+(CE+3C^3)\}\\
&-2\,(-G-2D^2)\\
&+5\,(-G+CE+D^2),
\end{aligned}$$

which is $=5CE+9D^2-9C^3$ (a non-unitary form). This then should be the value of

$$\tfrac{1}{2}Q-b\omega,\;=c\partial_b+3d\partial_c+6e\partial_d+10f\partial_e+15g\partial_f-5b,$$

operating upon

$$S\alpha^2\beta^2,\;=CD=5f-5be+cd+2b^2d-bc^2.$$

31. There is for non-unitariants a theorem which is a much more simple form than the transformation of it afterwards obtained for seminvariants: viz. for any non-unitariant we have $\Delta U=0=(\partial_b+b\partial_c+c\partial_d+\ldots)\,U$; attending only to the portion U' of U which is of the highest degree, it is clear that we have $(b\partial_c+c\partial_d+\ldots)\,U'=0$, and if we herein diminish the letters, then $(\partial_b+b\partial_c+\ldots)\,U''=0$, where U'' is what U' becomes by a diminution of the letters; that is, U'' is a non-unitariant, viz. in any seminvariant, the terms of highest degree U' are obtained from a non-unitariant U'' by a mere augmentation of the letters: e.g. $2e-2bd+c^2$ is a non-unitariant weight 4; augmenting the letters, we have $2bf-2ce+d^2$ which with a change of sign is the portion of highest degree of the non-unitariant $2g-2bf+2ce-d^2$.

The MacMahon Form of Equation. Art. Nos. 32 to 34.

32. The equation connecting the coefficients and the roots is here taken to be

$$1+\frac{b}{1}x+\frac{c}{1.2}x^2+\frac{d}{1.2.3}x^3+\ldots=1-\alpha x.1-\beta x.1-\gamma x\ldots.$$

As to this it may be remarked that, if we had started with a form of the nth order with binomial coefficients,

$$1+\frac{n}{1}bx+\frac{n.n-1}{1.2}cx^2+\frac{n.n-1.n-2}{1.2.3}dx^3+\ldots=1-\alpha x.1-\beta x.1-\gamma x\ldots(n \text{ factors}),$$

then writing herein $\frac{x}{n}$ for x, and also $n\alpha$, $n\beta$, $n\gamma$, ..., for α, β, γ, ... and putting ultimately $n=\infty$, we have the form in question.

We pass from the ordinary form to the MacMahon form, by writing for

$$b,\ c,\ d,\ e,\ \ldots,\ \frac{b}{1},\ \frac{c}{1.2},\ \frac{d}{1.2.3},\ \frac{e}{1.2.3.4},\ \ldots \text{ or say } b,\ \frac{c}{2},\ \frac{d}{6},\ \frac{e}{24},\ \frac{f}{120},\ \frac{g}{720},\ \ldots.$$

All the results obtained for the ordinary form will, after making therein this change, apply to the new form. We thus find

$$\Theta_\sigma=\sigma\partial_b+(\sigma-1)2b\partial_c+(\sigma-2)3c\partial_d+\ldots+\quad 1\ \sigma p\partial_q,$$

$$\Theta_{\sigma'}=\sigma'\partial_b+(\sigma'-1)2b\partial_c+(\sigma'-2)3c\partial_d+\ldots+(\sigma'-\sigma+1)\sigma p\partial_q,$$

$$\Theta_\sigma-\Theta_{\sigma'}=(\sigma'-\sigma)\Delta,$$

where

$$\Delta=\partial_b+2b\partial_c+3c\partial_d+\ldots+\sigma p\partial_q,$$

or say

$$=\partial_b+2b\partial_c+3c\partial_d+\ldots.$$

Also

$$P=b\partial_a+c\partial_b+d\partial_c+\ldots+r\partial_q,$$

or say

$$=b\partial_a+c\partial_b+d\partial_c+\ldots,$$

$$Q=\quad c\partial_b+2d\partial_c+\ldots+\sigma r\partial_q,$$

or say

$$=\quad c\partial_b+2d\partial_c+\ldots.$$

The change α, β, γ, ... into $n\alpha$, $n\beta$, $n\gamma$, ... would change $S\partial_\alpha$, $S\alpha\partial_\alpha$, $S\alpha^2\partial_\alpha$ into $n^{-1}S\partial_\alpha$, $S\alpha\partial_\alpha$, $nS\alpha^2\partial_\alpha$ respectively $(n=\infty)$: but this change is, in fact, compensated for by the introduction into the formulæ of the binomial coefficients as above; it is $-S\alpha$, $S\alpha\beta$, ... not $-nS\alpha$, $n^2S\alpha\beta$, ... which are equal to b, $\frac{1}{2}c$, ...; and the conclusion is that we have to retain without alteration the symbols $S\partial_\alpha$, $S\alpha\partial_\alpha$, $S\alpha^2\partial_\alpha$: thus in the new form as in the old one, we have $\Theta_4 S\alpha^4=-S\partial_\alpha.S\alpha^4=-4S\alpha^3$, see the example *ante* No. 23.

33. In the new form, a non-unitariant is annihilated by the operator

$$\Delta,\ =\partial_b+2b\partial_c+3c\partial_d+\ldots,$$

and conversely any function annihilated by Δ is a non-unitariant; comparing herewith the subsequent theory of seminvariants, this is in fact the theorem that a non-unitariant is the same thing as a seminvariant; or to state this more explicitly: for the MacMahon form of equation, a function of the coefficients which is a non-unitary symmetric function of the roots is a seminvariant.

I consider for instance the Table VI (b), but attend only to the non-unitary portions thereof, viz. the lines G, CE, D^2, C^3: I convert these into columns, at the same time changing the arrangement of the headings g, bf, ce, &c., from CO to AO: and then making the foregoing change b, c, d, e, f, g into b, $\frac{c}{2}$, $\frac{d}{6}$, $\frac{e}{24}$, $\frac{f}{120}$, $\frac{g}{720}$, but to avoid fractions multiplying the whole by 720, I form the table

$\div 720$ =		C^3	D^2	CE	G
1	g	− 2	+ 3	+ 6	− 6
6	bf	+ 2	− 3	− 6	+ 6
15	ce	− 2	− 3	+ 2	+ 6
20	d^2	+ 1	+ 3	− 3	+ 3
30	b^2e		+ 3	+ 2	− 6
60	bcd		− 3	+ 4	− 12
90	c^3		+ 1	− 2	− 2
120	b^3d			− 2	+ 6
180	b^2c^2			+ 1	+ 9
360	b^4c				− 6
720	b^6				+ 1
		$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

which is to be read according to the columns: and observe that the outside left-hand numbers are to be multiplied into the numbers of each column: thus the first column is to be read

$$C^3 = S\alpha^2\beta^2\gamma^2 = \frac{1}{720}(-2g + 12bf - 30ce + 20d^2),$$

the second column is to be read

$$D^2 = S\alpha^3\beta^3 = \frac{1}{720}(3g - 18bf \ldots + 90c^3),$$

and so on.

By what precedes, the columns are seminvariants,—as afterwards explained, "blunt" seminvariants; and they are named as such by the outside bottom line of symbols with a [ ]; viz.

$$[d^2] = (-2g + 12bf - 30ce + 20d^2), \quad [c^3] = (3g - 18bf \ldots + 90c^3), \text{ \&c.},$$

where it will be observed that the symbol within the [ ] is, in fact, the power-ender which is in AO the lowest term of the column; and further that this is also the conjugate of the capital letter symbol at the head of the column.

The (b) Tables I to X, with only the change $b, c, d, e, \ldots$ into $b, \frac{c}{2}, \frac{d}{6}, \frac{e}{24}, \ldots$ are given in my paper, "Tables of the Symmetric Functions of the Roots to the degree 10, for the form

$$1 + bx + \frac{cx^2}{1.2} + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)\ldots,"$$

American Mathematical Journal, t. VII. (1885), pp. 47—56, [829].

34. By what precedes, it appears that $P - b\delta$ operating on a seminvariant gives a seminvariant, and that $Q - 2b\omega$ operating on a seminvariant gives a seminvariant: these operators will be further considered in the development of the theory of seminvariants. We see further that $\frac{1}{2}\Delta, = b\partial_c + 3c\partial_d + 6d\partial_e + \ldots$, operating on a seminvariant gives sometimes but not always a seminvariant, e.g.

$$(b\partial_c + 3c\partial_d + 6d\partial_e)(e - 4bd - 3c^2 + 12b^2c - 6b^4) = 6(d - 3bc + 2b^3).$$

Seminvariants—the I-and-F Problem, and Solution by Square Diagrams.
Art. Nos. 35 to 47.

35. Writing

$$\begin{aligned}
1 &= 1,\\
b_1 &= b + \theta,\\
c_1 &= c + 2b\theta + \theta^2,\\
d_1 &= d + 3c\theta + 3b\theta^2 + \theta^3,\\
e_1 &= e + 4d\theta + 6c\theta^2 + 4b\theta^3 + \theta^4,\\
&\&c.,
\end{aligned}$$

then there are functions of the unsuffixed letters which remain unaltered if for these we substitute the suffixed letters: any such function is termed a seminvariant. We have for instance

$$\begin{aligned}
c_1 &= c + 2b\theta + \theta^2, \qquad \text{i.e.,} \qquad c_1 - b_1^2 = c - b^2,\\
-b_1^2 &= -b^2 - 2b\theta - \theta^2,\\
d_1 &= d + 3c\theta + 3b\theta^2, \qquad d_1 - 3b_1c_1 + 2b_1^3 = d - 3bc + 2b^3,\\
-3b_1c_1 &= -3bc - 6b^2\theta - 3b\theta^2,\\
&\quad -3c\theta - 6b\theta^2 - 3\theta^3,\\
+2b_1^3 &= 2b^3 + 6b^2\theta + 6b\theta^2 + 2\theta^3,
\end{aligned}$$

and thus $c-b^2$, $d-3bc+2b^3$ are seminvariants; they are, in fact, the first and second terms of the series

$$\begin{aligned}
&c-b^2,\\
&d-3bc+\ 2b^3,\\
&e-4bd+\ 6b^2c-\ 3b^4,\\
&f-5bc+10b^3d-10b^2c+\ 4b^5,\\
&g-6bf+15b^2e-20b^3d+15b^4c-5b^6,\\
&\ \vdots
\end{aligned}$$

where the law is obvious; the numbers in each line are binomial coefficients except the last number, which is the next binomial coefficient diminished by unity. The successive terms are, in fact, what c_1, d_1, e_1, f_1, g_1, ... become upon writing therein $\theta=-b$.

36. Any rational and integral function of these forms is a seminvariant, and it is to be observed that we can form functions for which (by the destruction of terms of a higher degree) there is a diminution of degree; for instance,

$$(e-4bd+6b^2c-3b^4)+3(c-b^2)^2$$

gives a seminvariant $e-4bd+3c^2$.

It is important to remark that a seminvariant is completely determined by its non-unitary terms; thus for $e-4bd+3c^2$, the non-unitary terms are $e+3c^2$, and for this writing $e_1+3c_1^2$, and for e_1, c_1 substituting their above values for $\theta=-b$, we reproduce the original value $e-4bd+3c^2$.

37. It is at once seen that a seminvariant is reduced to zero by the operation Δ, $=\partial_b+2b\partial_c+3c\partial_d+\ldots$, or say that Δ is an annihilator of a seminvariant; in fact, if in any function of b, c, d, ... we write for these the suffixed letters b_1, c_1, d_1, ... then the coefficient of θ herein is at once found by operating on the function of $(b, c, d, \ldots)$ with Δ, and therefore in the case of a seminvariant the result of this operation must be $=0$. And conversely, every function of $(b, c, d, \ldots)$ which is reduced to zero by the operation Δ is a seminvariant.

38. For a given weight, the number of seminvariants is equal to the excess of the number of terms of that weight above the number of terms of the next preceding weight, or what is the same thing, it is equal to the number of power-enders of the given weight. More definitely, considering the terms of a seminvariant as arranged in AO, we have seminvariants the finals whereof are the several power-enders of the given weight: and we arrange the seminvariants *inter se* by taking these power-enders in AO: thus for the weight 6, we have seminvariants $[d^3]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ ending in these terms respectively. We may, if we please, consider all these seminvariants as beginning with g, or say the forms may be taken to be

$g\,(\mathrm{ao})\,d^2$, $g\,(\mathrm{ao})\,c^3$, $g\,(\mathrm{ao})\,b^2c^2$, $g\,(\mathrm{ao})\,b^6$. Such forms are, in fact, furnished by the MacMahon equation: viz. up to the weight 6, we thus have for the present purpose

$\div 2$

		C
1	c	-2
2	b^2	$+1$
		$[b^2]$

$\div 6$

		D
1	d	-3
3	bc	$+3$
6	b^3	-1
		$[b^3]$

$\div 24$

		C^2	E
1	e	$+2$	-4
4	bd	-2	$+4$
6	c^2	$+1$	$+2$
12	b^2c		-4
24	b^4		$+1$
		$[c^2]$	$[b^4]$

$\div 120$

		CD	F
1	f	$+5$	-5
5	be	-5	$+5$
10	cd	$+1$	$+5$
20	b^2d	$+2$	-5
30	bc^2	-1	-5
60	b^3c		$+5$
120	b^5		-1
		$[bc^2]$	$[b^5]$

$\div 720$

		C^3	D^2	CE	G
1	g	-2	$+3$	$+6$	-6
6	bf	$+2$	-3	-6	$+6$
15	ce	-2	-3	$+2$	$+6$
20	d^2	$+1$	$+3$	-3	$+3$
30	b^2e		$+3$	$+2$	-6
60	bcd		-3	$+4$	-12
90	c^3		$+1$	-2	-2
120	b^3d			-2	$+6$
180	b^2c^2			$+1$	$+9$
360	b^4c				-6
720	b^6				$+1$
		$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

read for instance

$$[d^2] = -2g + 12bf - 30ce + 20d^2,$$
$$[c^3] = 3g - 18bf - 45ce + 60d^2 + 90b^2e - 180bcd + 90c^3,$$

&c.

I say that $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$ are "specific" when they are regarded as standing for these tabulated functions; but in general I take them to be "indefinite," that is, I regard them as denoting (as above) any seminvariants ending in d^2, c^3, b^2c^2, b^6 respectively.

39. The seminvariant $[d^2]$ is of the form $(g \infty d^2)$, including those terms which are in CO not superior to g and in AO not inferior to d^2: by a combination of $[d^2]$ and $[c^3]$, we obtain a seminvariant $(ce \infty c^3)$ containing terms which are in CO not superior to ce and in AO not inferior to c^3: similarly, from $[d^2]$, $[c^3]$, $[b^2c^2]$ we obtain a seminvariant $(d^2 \infty b^2c^2)$; and from the four forms a seminvariant $(c^3 \infty b^6)$: these four seminvariants

g	+ 1			
bf	− 6			
ce	+ 15	+ 1		
d^2	− 10	− 1	+ 1	
b^2e		− 1		
bcd		+ 2	− 6	
c^3		− 1	+ 4	+ 1
b^2d			+ 4	
b^2c^2			− 3	− 3
b^4c				+ 3
b^6				− 1
	$(g \infty d^2)$	$(ce \infty c^3)$	$(d^2 \infty b^2c^2)$	$(c^3 \infty b^6)$

are said to be "sharp" seminvariants: viz. considering the final as given, a sharp seminvariant is one having an initial which is in CO as low as possible; or considering the initial as given, it is one having a final which is in AO as high as possible. A seminvariant which is not sharp is said to be "blunt."

40. The sharp seminvariants are in general designated as above, $(g \infty d^2)$, &c.: but it is sometimes convenient to give the numerical coefficients of the initial and final terms respectively: as to this, it is to be noticed that the coefficient of the initial term is in most cases, but not always, $=1$,—we might of course take it to be always $=1$, but we should then in the excepted cases have fractional coefficients, and it is better to avoid this by giving a proper value to the numerical coefficient of the initial term; the numerical coefficient of the final term is in general different from ± 1, and it is not in general a multiple of the numerical coefficient of the initial term. As an instance take $dh \infty b^2e^2$, the more complete expression of which is $4dh \infty -35b^2e^2$. The sharp seminvariants up to the weight 12 are designated in this more complete form in the table *post* No. 62.

41. In the calculation of the sharp seminvariants by elimination as above, it will be noticed how unitary terms disappear: thus in combining $[d^2]$ and $[c^3]$ so as to get rid of g, the term bf disappears of itself, and we have as above the form

$(ce \infty c^3)$ beginning with the non-unitary term ce. We may, in fact, write $b=0$; we thus have

$$[d^3] = -2g - 30ce + 20d^2,$$

$$[c^3] = \quad 3g - 45ce + 60d^2 + 90c^3,$$

giving $3[d^3] + 2[c^3] = -180(ce - d^2 - c^3)$, and then $ce - d^2 - c^3$, putting therein for c, d, e the values $c - b^2$, $d - 3bc + 2b^3$, $e - 4bd + 6b^2c - 3b^4$, gives the complete value *ut supra*, $ce - d^2 - b^2e + 2bcd - c^3$, and we thus see *à priori* that this contains no term bf, but in fact begins with ce. And in carrying out this process for any higher given weight, it is proper also to arrange the non-unitary terms not in AO but in CO, and then in each case beginning with the terms highest in CO and eliminating as many as possible of these terms we obtain the sharp seminvariant. Consider for instance the weight 12: taking the finals in AO, we have here

$$(m \infty g^2),\ (m \infty cf^2),\ (m \infty e^3),\ (m \infty b^2f^2),\ \ldots$$

the initials in CO are m, ck, dj, ei, ... and it might at first sight appear that the foregoing process of elimination would lead to the forms $(m \infty g^2)$, $(ck \infty cf^2)$, $(dj \infty e^3)$, $(ei \infty b^2f^2)$, ...; we in fact have the form $(m \infty g^2)$; and if from $(m \infty g^2)$ and $(m \infty cf^2)$ we eliminate m, we obtain the form $(ck \infty cf^2)$; but we cannot have a form $(dj \infty e^3)$ (for a form beginning with dj is of necessity of the degree 4 at least); what happens is that when from $(m \infty g^2)$, $(m \infty cf^2)$ and $(m \infty e^3)$ we eliminate m and ck, the next term dj disappears of itself, and (the following term ei not disappearing) the resulting form is $(ei \infty e^3)$: to obtain a form beginning with dj we must use the fourth form $(m \infty b^2f^2)$, and we thence obtain $(dj \infty b^2f^2)$. Arranging the initials in CO and the finals in AO, we thus have

$$\begin{array}{ccc} m & \text{———} & g^2 \\ ck & \text{———} & cf^2 \\ dj & \diagdown\!\!\!\diagup & e^3 \\ ei & & b^2f^2 \end{array}$$

that is, we have the sharp seminvariants $m \infty g^2$, $ck \infty cf^2$, $ei \infty e^3$, $dj \infty b^2f^2$, ...; these are the results given by the MacMahon linkage as will be explained further on, but I will first approach the question from a different side.

42. It has been seen that we have $\Delta, = \partial_b + 2b\partial_c + 3c\partial_d + \ldots$, as the annihilator of a seminvariant. Considering in the first place the entire set of terms, say for the weight 6, $g(\text{ao})b^6$, we assume for a seminvariant the sum of these each multiplied by an arbitrary coefficient; the number of coefficients is equal to the number of terms of $g(\text{ao})b^6$. Operating with Δ, we obtain a function of the next inferior weight 5, containing all the terms of $Dg(\text{ao})b^6$, that is, of $f(\text{ao})b^5$, each

term multiplied by a linear function (with mere numerical factors) of the arbitrary coefficients: the expression thus obtained must be identically $=0$; and we thus find between the arbitrary coefficients a number of linear relations equal to the number of terms $f(\text{ao})b^5$: these relations are independent; for it is only on the supposition that they are so, that the number of coefficients which remain arbitrary will be $11-7$, $=4$, agreeing with the number of the seminvariants $[d^2]$, $[c^3]$, $[b^2c^2]$, $[b^6]$; whereas if the relations were not independent, there would be a larger number of seminvariants.

But if, instead of the whole set $g(\text{ao})b^6$, we consider a set $(g \infty d^2)$ or say $(ce \infty c^3)$ and assume for a seminvariant the sum of these terms each multiplied by an arbitrary coefficient, then operating as before with Δ we obtain between the arbitrary coefficients a number of relations equal to that of the terms $D(ce \infty c^2)$, and if this be less by unity than the number of the terms of $ce \infty c^3$, say if we have $(1-D)(ce \infty c^3)=1$, then there will be a single seminvariant $ce \infty c^3$. We, in fact, find $(1-D)$: $(g \infty d^2)$, $(ce \infty c^3)$, $(d^2 \infty b^2c^2)$, $(c^3 \infty b^6)$, each $=1$, and thus establish the existence of the foregoing seminvariants $g \infty d^2$, $ce \infty c^3$, $d^2 \infty b^2c^2$, $c^3 \infty b^6$. And similarly if in any case we have $(1-D)(I \infty F)=2$ or any larger number, then we have 2 or more seminvariants $I \infty F$.

43. It will be convenient to write down at once the system of square diagrams for the several weights 2 to 16; each of these may theoretically be obtained by a direct process of calculation such as I exhibit for the weight 10, but the labour would be very great indeed, and I have in fact formed the squares for the weights 11 to 16, not in this manner but by the MacMahon linkage.

$w = 2$

c	1
	b^2

$w = 3$

d	1
	b^3

$w = 4$

e	1	
c^2		1
	c^2	b^4

$w = 5$

f	1	
cd		1
	bc^2	b^5

$w = 6$

g	1			
ce		1		
d^2			1	
c^3				1
	d^2	c^3	b^2c^2	b^6

$w = 7$

h	1			
cf		1		
de			1	
c^2d				1
	bd^2	bc^3	b^3c^2	b^7

$w = 8$

	e^2	cd^2	b^2d^2	c^4	b^2c^2	b^4c^2	b^8
i	1						
cg		1					
df			1				
e^2				1			
c^2e					1		
cd^2						1	
c^4							1

$w = 9$

	be^2	d^3	bcd^2	b^3d^2	bc^4	b^3c^2	b^5c^2	b^9
j	1							
ch		1						
dg			1					
ef				1				
c^2f					1			
cde						1		
d^3							1	
c^3d								1

$w = 10$

	f^2	ce^2	b^2e^2	bd^2	c^2d^2	b^2cd^2	c^5	b^4d^2	b^2c^4	b^4c^3	b^6c^2	b^{10}
k	1											
ci		1										
dh			1									
eg				1								
c^2g						2						
f^2					1							
cdf							2					
ce^2							1					
d^2e								1				
c^3e									1			
c^2d^2										1		
c^5											1	

The subsequent squares $w=11$ to 16 are, for convenience, given at the end of the present memoir (pp. 331 et seq.).

44. It is to be observed that in each square the outside left-hand terms are the non-unitaries in CO and the outside bottom terms are the power-enders in AO. I have inside each square written down only the significant numbers, but we might fill up the whole square. For instance, when $w=7$, the filled-up square would be

h	1	2	3	4
cf	0	1	2	3
de	-1	0	1	2
c^2d	0	0	0	1
	bd^2	bc^3	b^3c^2	b^7

where in the first column the numbers relate to the sets $h \infty bd^2$, $cf \infty bd^2$, $de \infty bd^2$ and $c^2d \infty bd^2$ (this last set $c^2d \infty bd^2$ is non-existent since c^2d is in AO inferior to bd^2, i.e. as well for the set as for the diminished set, number of terms is $=0$, and we have for the compartment $0-0$, $=0$). And similarly for the remaining three columns. The process of thus filling up the whole square is a direct and non-tentative one, and the conclusions to which the numbers lead are as follows: col. 1, the final being bd^2, the initial cannot be c^2d, de or cf, but taking it to be h, we have the seminvariant $h \infty bd^2$. Col. 2, the final being bc^3 the initial cannot be c^2d or de, but taking it to be cf we have the seminvariant $cf \infty bc^3$: it may be added that the top number 2 shows that there are two seminvariants $h \infty bc^3$, these are of course the foregoing ones $h \infty bd^2$ and $cf \infty bc^3$. Similarly, col. 3, the final being b^3c^2, the initial cannot be c^2d, but taking it to be de, we have the seminvariant $de \infty b^3c^2$, and col. 4, we have the seminvariant $c^2d \infty b^7$.

For the several weights up to 9, we have simply units in the dexter diagonal of each square, viz. the non-unitaries in CO correspond to the power-enders in AO, or the sharp seminvariants are $c \infty b^2$, $d \infty b^3$, &c. See *post*, Table of Reductions, No. 62, which exhibits these correspondences.

45. For the weight 10, we have deviations: the figures 1 and 2 denote as follows:

$$\begin{array}{lll}
1-D & k \infty f^2 & =1 \\
 & ci \;\;,, \; ce^2 & ,, \; 1 \\
 & dh \;\;,, \; b^2e^2 & ,, \; 1 \\
 & eg \;\;,, \; bd^3 & ,, \; 1 \\
 & f^2 \;\;,, \; c^2d^2 & ,, \; 1 \\
 & c^2g \;\;,, \; b^2cd^2 & ,, \; 2 \\
 & ce^2 \;\;,, \; c^5 & ,, \; 1 \\
 & cdf \;\;,, \; b^4d^2 & ,, \; 2 \\
 & d^2e \;\;,, \; b^3c^4 & ,, \; 1 \\
 & c^3e \;\;,, \; b^4c^3 & ,, \; 1 \\
 & c^3d^2 \;\;,, \; b^6c^2 & ,, \; 1 \\
 & c^5 \;\;,, \; b^{10} & ,, \; 1,
\end{array}$$

and they indicate the sharp seminvariants $k \infty f^2$, $ci \infty ce^2$, &c.: where observe that the power-enders being in AO as before, the non-unitaries are not in CO, but we have inversions (c^2g, f^2) and (cdf, ce^2).

In particular, $(1-D)(f^2 \infty c^2d^2) = 1$ indicates the seminvariant $f^2 \infty c^2d^2$;

$$(1-D)(c^2g \infty b^2cd^2) = 2,$$

means in the first instance that there are 2 seminvariants $c^2g \infty b^2cd^2$, but here the set $c^2g \infty b^2cd^2$ includes as part of itself the set $f^2 \infty c^2d^2$; so that, if $c^2g \infty b^2cd^2$ is used to denote any particular form, then the general form is $c^2g \infty b^2cd^2$ plus an arbitrary multiple of $f^2 \infty c^2d^2$, and we have thus virtually a single form $c^2g \infty b^2cd^2$. And similarly, the set $cdf \infty b^4d^2$ includes as part of itself the set $ce^2 \infty c^5$; and thus the general form $cdf \infty b^4d^2$ is = particular form plus an arbitrary multiple of $ce^2 \infty c^5$, or we have virtually a single form $cdf \infty b^4d^2$.

I remark that it would be allowable to take as a standard form of $c^2g \infty b^2cd^2$, a form not containing any term in f^2, and similarly for the standard form of $cdf \infty b^4d^2$ a form not containing any term in ce^2; but this is not done in the tables.

46. The diagram for weight 10 is constructed by the following calculation; viz. in col. 1 we calculate $(1-D)(k \infty f^2)$ and for this purpose write down the terms of $k \infty f^2$, and $D(k \infty f^2)$ in CO: in col. 2 we calculate $(1-D)(ci \infty ce^2)$, and for this purpose write down the terms of $k \infty ce^2$ and $D(k \infty ce^2)$ in CO, the terms of $ci \infty ce^2$ and $D(ci \infty ce^2)$ being thence found by rejecting the terms k, bj and the term j at the head of the two halves of the column. So in col. 3 we calculate $(1-D)(dh \infty b^2e^2)$, and for this purpose write down the terms of $(k \infty b^2e^2)$ and $D(k \infty b^2e^2)$ in CO, and for $dh \infty b^2e^2$ and $D(dh - b^2e^2)$ reject the terms k, bj, ci, b^2i and j, bi at the head of the two halves of the column. And so for the remaining columns. It is to be remarked that there is in each successive column a continually increasing number of terms to be rejected; by a properly devised variation of the algorithm it would have been possible to avoid writing down these terms at all, but for greater clearness I have inserted them.

1	2	3	4	5	6	7	8	9	10	11	12
k	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	$j\ k$	j
bj	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	$bi\ bj$	bi
ci	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	$ch\ ci$	ch
dh	$dg\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	$b^2h\ b^2i$	b^2h
eg	$ef\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	$dg\ dh$	dg
f^2	bch	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	$bcg\ bch$	bcg
	eg	$ef\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	$b^3g\ b^3h$	b^2g
	bdg	$bdf\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	$ef\ eg$	ef
	c^2g	$c^2f\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	$bdf\ bdg$	bdf
	f^2	$be^2\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	$c^2f\ c^2g$	c^2f
	bef	$cde\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	$b^2cf\ b^2eg$	b^2cf
	cdf	$f^2\ be^2$	$f^2\ be^2$	$f^2\ be^2$	$b^4g\ b^4f$	$b^4g\ b^4f$	$b^4g\ b^4f$	$b^4g\ b^4f$	$b^4g\ b^4f$	$b^4g\ b^4f$	b^4f
	ce^2	$bef\ cde$	$bef\ cde$	$bef\ cde$	$f^2\ be^2$	$f^2\ be^2$	$f^2\ be^2$	$f^2\ be^2$	$f^2\ be^2$	$f^2\ be^2$	be^2
		$cdf\ b^2de$	$cdf\ b^2de$	$cdf\ b^2de$	$bef\ cde$	$bef\ cde$	$bef\ cde$	$bef\ cde$	$bef\ cde$	$bef\ cde$	cde
		$b^2df\ d^3$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	$cdf\ b^2de$	$cdf\ b^2de$	$cdf\ b^2de$	$cdf\ b^2de$	$cdf\ b^2de$	$cdf\ b^2de$	b^2de
		ce^2	$bc^2f\ d^3$	$bc^2f\ d^3\ b^2df$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	$b^2df\ bc^2e$	bc^2e
		d^2e	$ce^2\ bcd^3$	$ce^2\ bcd^3\ b^2cf$	$b^2cf\ b^3ce$	$b^2cf\ b^3ce$	$b^2cf\ b^3ce$	$b^2cf\ b^3ce$	$b^2cf\ b^3ce$	$b^2cf\ b^3ce$	b^3ce
		b^2e^2	$b^2e^2\ b^2e^2$	$c^3d\ b^2cf$	$ce^2\ d^3\ b^3cf$	$d^3\ b^3cf$	$d^3\ b^3cf$	$b^5e\ b^3cf$	$b^5e\ b^3cf$	$b^5e\ b^3cf$	b^5e
			d^3e	d^2e	$bcd^2\ ce^2$	$bcd^2\ ce^2$	$bcd^2\ b^5f$	$d^3\ b^5f$	$d^3\ b^5f$	$d^3\ b^5f$	d^3
			$bcde$	$bcde$	$b^2e^2\ b^3d^2$	$b^2e^2\ b^3d^2$	$b^3d^2\ ce^2$	$bcd^2\ ce^2$	$bcd^2\ ce^2$	$bcd^2\ ce^2$	bcd^2
			bd^3	c^2e	$d^2e\ c^3d$	$d^2e\ c^3d$	$c^3d\ b^2e^2$	$b^3d^2\ b^2e^2$	$b^3d^2\ b^2e^2$	$b^3d^2\ b^2e^2$	b^3d^2
				bd^3	$bcde\ b^3c^2d$	$bcde\ b^2c^3d$	d^2e	$c^3d\ d^2e$	$c^3d\ d^2e$	$c^3d\ d^2e$	c^3d
				c^2d^2	b^2de	$b^2de\ bc^4$	$bcde\ b^2c^2d$	$bcde\ b^2c^2d$	$b^2c^2d\ bcde$	$b^2c^2d\ bcde$	b^2c^2d
					c^3e	c^3e	b^3de	$b^2de\ b^4cd$	$b^4cd\ b^3de$	$b^4cd\ b^3de$	b^4cd
					b^2c^2e	b^2c^2e	c^3e	$bc^4\ c^3e$	$bc^4\ c^3e$	$b^5d\ c^3e$	$b^5d\ c^3e$	b^5d
					bd^3	bd^3	b^2c^2e	b^2c^2e	$b^3c^3\ b^2c^2e$	$bc^4\ b^2c^2e$	$bc^4\ b^2c^2e$	bc^4
					c^2d^2	c^2d^2	b^4ce	b^4ce	b^4ce	$b^3c^3\ b^4ce$	$b^3c^3\ b^4ce$	b^3c^3
					b^2cd^2	b^2cd^2	bd^3	bd^3	b^6e	$b^5c^2\ b^6e$	$b^5c^2\ b^6e$	b^5c^2
					bc^3d	bc^3d	c^2d^2	c^2d^2	bd^3	bd^3	$b^7c\ bd^3$	b^7c
					c^5	c^5	b^2cd^2	b^2cd^2	c^2d^2	c^2d^2	c^2d^2	b^9
							b^4d^2	b^4d^2	b^2cd^2	b^2cd^2	b^2cd^2	
							bc^3d	bc^3d	b^4d^2	b^4d^2	b^4d^2	
							c^5	b^3c^2d	bc^3d	bc^3d	bc^3d	
								c^5	b^3c^2d	b^3c^2d	b^3c^2d	
								b^2c^4	b^5cd	b^5cd	b^5cd	
									c^5	b^7d	b^7d	
									b^2c^4	c^5	c^5	
									b^4c^3	b^2c^4	b^2c^4	
										b^4c^3	b^4c^3	
										b^6c^2	b^6c^2	
											b^8c	
											b^6	
6 − 5 =	11 − 10 =	14 − 13 =	14 − 13 =	12 − 11 =	19 − 17 =	12 − 11 =	19 − 17 =	14 − 13 =	14 − 13 =	11 − 10 =	6 − 5 =
1	1	1	1	1	2	1	2	1	1	1	1

47. As to the first of the foregoing inversions c^2g, f^2, it is proper to remark, that filling up two compartments of the square we have

c^2g		1	2
f^2		1	1
	c^2d^2	b^2cd^2	

where the meaning of the numbers (1, 1) has to be considered: the first (1) seems to indicate a seminvariant $c^2g \infty c^2d^2$, but there is in fact no such form, what it really indicates is a form $0c^2g+f^2 \infty c^2d^2$, that is, $f^2 \infty c^2d^2$; and similarly, the second (1) seems to indicate a seminvariant $f^2 \infty b^2cd^2$, but there is in fact no such form, what it really indicates is $f^2 \infty c^2d^2+0b^2cd^2$, that is, $f^2 \infty c^2d^2$. The explanation is correct, but to make it perfectly clear some further developments would be required. The like remarks apply to the inversion cdf, ce^2.

The MacMahon Linkage. Art. Nos. 48 to 52.

48. We require the two theorems:

The first is: if a seminvariant S has q for its highest letter, then $\partial_q S$ is also a seminvariant.

The second has presented itself for unitariants (*ante* No. 31); for seminvariants the form is less simple, viz. If in any seminvariant, attending only to the terms of the highest degree, we therein change b, c, d, e, ... into b, $2c$, $6d$, $24e$, ... and then diminish the letters (that is, replace each letter by the next preceding letter) and in the result so obtained change b, c, d, e, ... into b, $\frac{c}{2}$, $\frac{d}{6}$, $\frac{e}{24}$, ... we obtain a seminvariant. For instance $g-6bf+15ce-10d^2$, in the terms of degree 2, making the numerical change we have $-720bf+720ce-360d^2$, and then diminishing the letters and making the numerical change, we obtain $-720\frac{e}{24}+720\frac{bd}{6}-360\frac{c^2}{4}$, that is,

$$-30(e-4bd+3c^2),$$

a seminvariant.

For the proof, observe that the equation $\Delta S=0$, attending therein only to the terms of the highest degree, gives $(2b\partial_c+3c\partial_d+\ldots)S'=0$, if S' denote the terms of the highest degree: making the numerical change, this is $(b\partial_c+c\partial_d+\ldots)S''$, if S'' is what S' becomes thereby; diminishing the letters, this is $(\partial_b+b\partial_c+\ldots)S'''=0$, if S'''

is the diminished value of S'', and finally making the numerical change, if T be what S''' becomes on writing therein $b, \frac{c}{2}, \frac{d}{6}, \ldots$ for $b, c, d, \ldots$, this gives

$$(\partial_b + 2b\partial_c + \ldots)\, T = 0,$$

viz. T is a seminvariant.

49. Assume that, for the weights up to a certain weight w, the forms of the sharp seminvariants are known: and for the weight w consider a seminvariant $I(\text{ca})F$: here if I be given, the first theorem establishes a limit F' such that F is in AO not higher than F'. For instance, when $w = 12$, if $I = dj$, the coefficient of j as being a seminvariant can only be $d \infty b^3$, and thus the seminvariant contains a term b^3j, or the final term F must be in AO not higher than b^3j; the degree is thus $= 4$ at least.

Similarly, if F be given, then the second theorem determines a limit I' such that I is in CO not lower than I'. Thus when $w = 12$, as before, if $F = b^4cd^2$, then diminishing the letters we have bc^2, a term belonging to $f \infty bc^2$; the diminished form has thus terms $a^4(a^2f,\ bc^2)$, so that augmenting these the seminvariant has terms $b^4(b^2g,\ cd^2)$ and thus the initial term I is in CO not lower than b^6g.

50. A limit for I or F, when the other is given, can also in some cases be found as follows: Considering a seminvariant of the weight w as before, and denoting its extent and degree by σ and δ respectively, then we have $\sigma\delta - 2w = 0$ or positive; that is, $\sigma\delta = 2w$ at least; here given I, we have σ, and then $\delta = \frac{2w}{\sigma}$ at least; and given F we have δ, and then $\sigma = \frac{2w}{\delta}$ at least.

51. We may now explain the MacMahon linkage; for a given weight, we write down in two columns the initials or non-unitaries in CO, and the finals or power-enders in AO: by what precedes, it appears that we cannot combine the terms of the one column each with the term opposite to it in the other column; what we do is: beginning with the top of the column of initials, we combine successively each term with the highest admissible term in the column of finals: or beginning with the bottom of the column of finals, we combine successively each term with the lowest admissible term in the column of initials.

52. For the weight 12, the linkage is

read downwards.

shown by	not in AO higher than				lower than not in CO	shown by
		m	g^2	bl	$k \infty f^2$
$(c \infty b^2)k$	b^2k	ck	cf^2	b^2k	j ,, be^2
$(d$,, $b^3)j$	b^3j	dj	e^3	bdi	ch ,, d^3
$(e$,, $c^2)i$	c^2i	ei	b^2f^2	b^3j	i ,, e^3
$(c^2$,, $b^4)i$	b^4i	c^2i	bde^2	b^2dh	cg ,, cd^2
$(f$,, $bc^2)h$	bc^3h	fh	c^2e^2	b^2eg	df ,, b^2d^2
$(cd$,, $b^5)h$	b^5h	cdh	d^4	b^3f^2	e^2 ,, c^4
$(g$,, $d^2)g$	d^2g	g^2	b^2ce^2	b^4i	h ,, bd^2
$(ce$,, $c^3)g$	c^3g	ceg	bcd^3	b^3dg	cf ,, bc^3
$(d^2$,, $b^2c^2)g$	b^2c^2g	d^2g	c^3d^2	b^3ef	de ,, b^3
$(c^3$,, $b^6)g$	b^6g	c^3g	b^4e^2	b^3h	g ,, d^2
$(cf$,, $bc^3)f$	bc^3f	cf^2	b^3d^3	b^4df	ce ,, c^3
$(de$,, $b^3c^2)f$	b^3c^2f	def	$b^2c^2d^2$	b^4e^2	d^2 ,, b^2c^2
$(c^2d$,, $b^7)f$	b^7f	c^2df	c^6	b^3d^3	c^3 ,, b^6
$(e^2$,, $c^4)e$	c^4e	e^3	b^4cd^2	b^6g	f ,, bc^2
$(c^2e$,, $b^2c^3)e$	b^2c^3e	c^2e^2	b^2c^5	b^3de	cd ,, b^5
$(cd^2$,, $b^4c^2)e$	b^4c^2e	cd^2e	b^6d^2	b^7f	e ,, c^2
$(c^4$,, $b^8)e$	b^8e	c^4e	b^4c^4	b^6d^2	c^2 ,, b^4
$(d^3$,, $b^2c^2)d$	b^2c^2d	d^4	b^6c^3	b^8e	d ,, b^3
$(c^2d$,, $b^9)d$	b^9d	c^3d^2	b^8c^2	b^9d	c ,, b^2
$(c^5$,, $b^{10})c$	$b^{10}c$	c^6	b^{12}		

read upwards.

Thus, beginning at the top of the column of initials, m is to be linked with g^2, that is, we have $(m \infty g^2)$; ck with cf^2, that is, we have $(ck \infty cf^2)$; dj cannot be linked with e^3, for the final must be in AO not higher than b^3j, but it is linked with the highest term b^3f^2 for which this condition is satisfied, that is, we have $(dj \infty b^3f^2)$; ei is then linked with the highest admissible term e^3, that is, we have $(ei \infty e^3)$; and so on.

Or beginning at the bottom of the column of finals, b^{12} is linked with c^6, that is, we have $(c^6 \infty b^{12})$, b^8c^2 with c^3d^2, that is, we have $(c^3d^2 \infty b^8c^2)$; b^6c^3 cannot be linked with d^4, for the initial must be in CO not lower than b^5e, but it is linked with the lowest term c^4e for which this condition is satisfied, that is, we have $(c^4e \infty b^6c^3)$; and so on.

The Umbral Notation. Stroh's Theory. Art. Nos. 53 to 56.

53. Employing the umbræ α, β, γ, δ, ..., which are such that

$$\alpha = \beta = \gamma, \ \ldots, = b; \quad \alpha^2 = \beta^2 = \gamma^2, \ \ldots, = c; \quad \alpha^3 = \beta^3 = \gamma^3, \ \ldots, = d;$$

and so on, then for instance

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2, \quad = c - 2b^2 + c, \quad = 2(c - b^2),$$

a seminvariant;

$$(\alpha - \beta)^2(\alpha - \gamma) = \alpha^3 - 2\alpha^2\beta + \alpha\beta^2 - \alpha^2\gamma + 2\alpha\beta\gamma - \beta^2\gamma,$$
$$= d - 2bc + bc - bc + 2b^3 - bc, \quad = d - 3bc + 2b^3,$$

a seminvariant: and so in general any rational and integral function of the differences of the umbræ developed and interpreted is a seminvariant. For the seminvariants of a given weight, e.g. $w = 6$, Dr Stroh* considers the function

$$\Omega^6 = (\alpha x + \beta y + \gamma z + \delta w + \epsilon t + \zeta u)^6,$$

where x, y, z, w, t, u are numbers the sum of which is $= 0$, or we may if we please have more than 6 such numbers: the expression is obviously a function of the differences of the umbræ and it is thus a seminvariant. To develop its value, observe that after expansion of the sixth power we have sets of similar terms, for instance $\alpha^6x^6 + \beta^6y^6 + \ldots$ which putting therein $\alpha^6 = \beta^6 = \gamma^6, \ldots = g$ become $= g \,.\, Sx^6$, and generally each set becomes equal to a literal term multiplied by a symmetric function of the x, y, z, w, ...; introducing capital letters to denote the elementary symmetric functions of these quantities, and recollecting that their sum is assumed to be $= 0$, say we have

$$1 + Cs^2 + Ds^3 + Es^4 + \ldots = 1 - xs \,.\, 1 - ys \,.\, 1 - zs \ldots,$$

(that is, $0 = Sx$, $+C = Sxy$, $-D = Sxyz$, &c.) then by aid of the Table VI (b) writing therein 0, C, D, E, F, G for b, c, d, e, f, g, we find

$$\Omega^6 = (\alpha x + \beta y + \gamma z \ldots)^6 = \alpha^6 Sx^6 + 6\alpha^5\beta Sx^5y + \text{\&c.},$$

* See the paper "Ueber die Symbolische Darstellung der Grundsyzyganten einer binären Form sechster Ordnung und eine Erweiterung der Symbolik von Clebsch," *Math. Ann.* t. XXXVI. (1890), pp. 262—303, in particular § 10, Das Formensystem einer Form unbegrenzt hoher Ordnung.

as shown in the following table:

				C^3	D^2	CE	G
1	g	Sx^6	=	− 2	+ 3	+ 6	− 6
+ 6	bf	Sx^5y	=	+ 2	− 3	− 6	+ 6
+ 15	ce	Sx^4y^2	=	− 2	− 3	+ 2	+ 6
+ 20	d^2	Sx^3y^3	=	+ 1	+ 3	− 3	+ 3
+ 30	b^2e	Sx^4yz	=		+ 3	+ 2	− 6
+ 60	bcd	Sx^3y^2z	=		− 3	+ 4	− 12
+ 90	c^3	$Sx^2y^2z^2$	=		+ 1	− 2	− 2
+ 120	b^3d	Sx^3yzw	=			− 2	+ 6
+ 180	b^2c^2	Sx^2y^2zw	=			+ 1	+ 9
+ 360	b^4c	Sx^2yzwt	=				− 6
+ 720	b^6	$Sxyzwtu$	=				+ 1
				$[d^2]$	$[c^3]$	$[b^2c^2]$	$[b^6]$

the numbers whereof are, it will be observed, identical with those of the foregoing table No. 33, relating to the MacMahon equation.

This is to be read

$$\Omega^6 = C^3[d^2] + D^2[c^3] + CE[b^2c^2] + G[b^6],$$

viz. Ω^6 is a linear function of C^3, D^2, CE and G, the coefficients of these, being given functions of (b, c, d, e, f, g), which given functions are the specific blunt seminvariants which have been already called $[d^2]$, $[c^3]$, $[b^2c^2]$ and $[b^6]$. And so in general, the developed value of Ω^w affords a complete definition of these specific blunt seminvariants of the weight w. Observe that $\alpha, \beta, \gamma, \delta, \ldots$ are umbræ in nowise connected with the roots $\alpha, \beta, \gamma, \delta, \ldots$ before made use of, and that $B, C, D, \ldots$ are actual quantities in nowise connected with the symbolic capitals $B, C, D, \ldots$ before made use of.

54. The capital and small letter symbols are conjugate to each other. It will be convenient to give here, in reference to subsequent investigations, a table of these conjugate forms up to the degree 6 and weight 15.

Table of Conjugates.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
$C\ b^2$	$D\ b^3$	$C^2\ c^2$	$CD\ bc^2$	$C^3\ d^2$	$C^2D\ bd^2$	$C^4\ e^2$	$C^3D\ be^2$	$C^5\ f^2$	$C^4D\ bf^2$	$C^6\ g^2$	$C^5D\ bg^2$	$C^7\ h^2$	$C^6D\ bh^2$
		$E\ b^4$	$F\ b^5$	$D^2\ c^2$	$DE\ bc^2$	$CD^2\ cd^2$	$D^3\ d^3$	$C^2D^2\ ce^2$	$CD^3\ de^2$	$C^3D^2\ cf^2$	$C^2D^3\ df^2$	$C^4D^2\ cg^2$	$C^3D^3\ dg^2$
				$CE\ b^2c^2$	$CF\ b^3c^2$	$C^2E\ b^2d^2$	$CDE\ bcd^2$	$C^3E\ b^2e^2$	$C^2DE\ bce^2$	$D^4\ e^2$	$C^3DE\ bcf^2$	$CD^4\ ef^2$	$D^5\ f^3$
				$G\ b^6$		$E^2\ c^4$	$C^2F\ b^3d^2$	$D^2E\ bd^3$	$DE^2\ cd^3$	$C^4E\ b^2f^2$	$D^3E\ be^3$	$C^5E\ b^2g^2$	$C^4DE\ bcg^2$
						$DF\ b^2c^3$	$EF\ bc^4$	$CE^2\ c^2d^2$	$C^2F\ b^3e^2$	$CD^2E\ bde^2$	$CDE^2\ cde^2$	$C^2D^2E\ bdf$	$CD^3E\ bef^2$
						$CG\ b^4c^2$	$DG\ b^3c^3$	$CDF\ b^2cd^2$	$D^2F\ b^2d^3$	$C^2E^2\ c^2e^2$	$C^4F\ b^3f^2$	$C^3E^2\ c^2f^2$	$C^2DE^2\ cdf^2$
								$F^2\ c^5$	$CEF\ bc^2d^2$	$E^3\ d^4$	$CD^2F\ b^2de^2$	$D^2E^2\ ce^3$	$DE^3\ de^3$
								$C^2G\ b^4d^2$	$CDG\ b^3cd^2$	$C^2DF\ b^2ce^2$	$C^2EF\ bc^2e^2$	$CE^3\ d^2e^2$	$C^3F\ b^3g^2$
								$EG\ b^2c^4$	$FG\ bc^5$	$DEF\ bcd^2$	$E^2F\ bd^4$	$C^3DF\ b^2cf^2$	$C^2D^2F\ b^2df^2$
										$CF^2\ c^2d^2$	$DF^2\ c^2d^3$	$D^3F\ b^2e^3$	$C^3EF\ bc^2f^2$
										$C^3G\ b^4e^2$	$C^2DG\ b^3ce^2$	$CDEF\ bcde^2$	$D^2EF\ bce^3$
										$D^2G\ b^3d^3$	$DEG\ b^2cd^3$	$C^2F^2\ c^3e^2$	$CE^2F\ bd^2e^2$
										$CEG\ b^2c^2d^2$	$CFG\ bc^3d^2$	$EF^2\ cd^4$	$CDF^2\ c^2de^2$
										$G^2\ c^6$		$C^4G\ b^4f^2$	$F^3\ d^5$
												$CD^2G\ b^3de^2$	$C^2DG\ b^3ef^2$
												$C^2EG\ b^2c^2e^2$	$D^3G\ b^3e^3$
												$E^2G\ b^2d^4$	$CDEG\ b^2cde^2$
												$DFG\ bc^2d^3$	$C^2FG\ bc^3e^2$
												$CG^2\ c^4d^2$	$EFG\ cd^4$
													$DG^2\ c^3d^2$

55. We can, by means of the umbral notation, write down for the blunt seminvariants of a given weight (indefinite forms, not the above-mentioned specific forms) expressions far more simple than those which are given by the foregoing theories: we can, in fact, find without difficulty *monomial* umbral expressions; and in many cases obtain also the sharp forms. To illustrate this, I consider the weight 10: I write down for convenience the symbols of the sharp forms (though the knowledge of these is in nowise required) and I form a table as follows:

Sharp forms, finals in AO.		
$k \quad \infty f^2$	1	$(\alpha-\beta)^{10}$
$ci \quad ,, \quad ce^2$	2	$(\alpha-\beta)^8(\alpha-\gamma)^2$
$dh \quad ,, \quad b^2e^2$	3	$(\alpha-\beta)^8(\alpha-\gamma)(\alpha-\delta)$
$eg \quad ,, \quad bd^3$	4	$(\alpha-\beta)^6(\alpha-\gamma)^3(\alpha-\delta)$
$f^2 \quad ,, \quad c^2d^2$	5	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)^2$
$c^2g \quad ,, \quad b^2cd^2$	6	$(\alpha-\beta)^6(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\epsilon)$
$ce^2 \quad ,, \quad c^5$	7	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\epsilon)^2$
$cdf \quad ,, \quad b^4d^2$	8	$(\alpha-\beta)^6(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)$
$d^2e \quad ,, \quad b^2c^4$	9	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)^2(\alpha-\epsilon)(\alpha-\zeta)$
$c^3e \quad ,, \quad b^4c^3$	10	$(\alpha-\beta)^4(\alpha-\gamma)^2(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)$
$c^2d^2 \quad ,, \quad b^6c^2$	11	$(\alpha-\beta)^4(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)$
$c^5 \quad ,, \quad b^{10}$	12	$(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\zeta)(\alpha-\eta)(\alpha-\theta)(\alpha-\iota)(\alpha-\kappa)$

It will be observed that all the differences used are $\alpha-\beta$, $\alpha-\gamma$, ... containing each of them an α; hence in all the forms we have α^{10}, $=k$; in $(\alpha-\beta)^{10}$, the lowest term (in AO) is $\alpha^5\beta^5$, $=f^2$; in $(\alpha-\beta)^8(\alpha-\gamma)^2$, the lowest term is $\alpha^4\beta^4.\gamma^2$, $=ce^2$; and so on, viz. in each case the lowest term is the final term of the sharp form set down in the same line.

56. The form $(\alpha-\beta)^{10}$ gives at once the sharp form $k\infty f^2$; we thus develop it:

α^{10}	$\alpha^9\beta$	$\alpha^8\beta^2$	$\alpha^7\beta^3$	$\alpha^6\beta^4$	$\alpha^5\beta^5$
β^{10}	$\alpha\beta^9$	$\alpha^2\beta^8$	$\alpha^3\beta^7$	$\alpha^4\beta^6$	
1	-10	$+45$	-120	$+210$	-252
$+1$	-10	$+45$	-120	$+210$	
$=2(k$	$-10bj$	$+45ci$	$-120dh$	$+210eg$	$-126f^2)$;

$(\alpha-\beta)^8(\alpha-\gamma)^2$ contains a term α^{10}, $=k$ and thus gives a blunt form $kaoce^2$; if instead of it we employ the form $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)$, then here as before the lowest term is $\alpha^4\beta^4.\gamma^2$, $=ce^2$, but there is no term α^{10}: there is a term $\alpha^9\beta$, $=bj$, but as this

cannot appear, we must have terms of this form destroying each other. The simplest mode of effecting the development is to write

$$(\alpha-\beta)^8(\alpha-\gamma)(\beta-\gamma)=(\alpha-\beta)^8\{\alpha\beta-\gamma(\alpha+\beta)+\gamma^2\}:$$

we may herein put at once $\gamma=b$, $\gamma^2=c$, and thus the form is

$$(\alpha-\beta)^8\{\alpha\beta-b(\alpha+\beta)+c\};$$

I develop thus:

$$\begin{array}{lrrrrrrrrr}
(\alpha-\beta)^8 & 1, & -8, & +28, & -56, & +70, & -56, & +28, & -8, & +1, \\
 & & +1, & -8, & +28, & -56, & +70, & -56, & +28, & -8, & +1 \\
\hline
(\alpha-\beta)^8(\alpha+\beta) & 1, & -7, & +20, & -28, & +14, & +14, & -28, & +20, & -7, & +1
\end{array}$$

$$1bj-8ci+28dh-56eg+70f^2$$

$$+1\quad -8\quad +28\quad -56$$

$$-b\begin{pmatrix} 1j & -7bi & +20ch & -28dg & +14ef \\ +1 & -7 & +20 & -28 & +14 \end{pmatrix}$$

$$+c\begin{pmatrix} 1i & -8bh & +28cg & -56df & +70e^2 \\ +1 & -8 & +28 & -56 & \end{pmatrix}$$

$\div -14$

k						
bj	+ 2	− 2			0	
ci	− 16		+ 2		− 14	+ 1
dh	+ 56				+ 56	− 4
eg	− 112				− 112	+ 8
f^2	+ 70				+ 70	− 5
b^2i		+ 14			+ 14	− 1
bch		− 40	− 16		− 56	+ 4
bdg		+ 56			+ 56	− 4
bef		− 28			− 28	+ 2
c^2g			+ 56		+ 56	− 4
cdf			− 112		− 112	+ 8
ce^2			+ 70		+ 70	− 5

$\pm 23,$

which, in fact, exhibits the calculation of the sharp form $ci \propto ce^2$. The disappearance of the term in bj will be noticed.

Instead of $(\alpha-\beta)^8(\alpha-\gamma)(\beta-\delta)$ which contains α^{10}, that is, k, we may take

$$(\alpha-\beta)^8(\gamma-\delta)^2,$$

that is,

$$(i-8bh+28cg-56df+35e^2)(c-b^2):$$

this is $ciaob^2e^2$, a blunt form; by subtracting from it $ci \infty ce^2$, we could obtain the next sharp form $dh \infty b^2e^2$; but this in passing; it does not appear that there is any monomial umbral expression for the last-mentioned form.

I do not stop to examine the next following forms, but pass on at once to the last of them; instead of the expression given, we may take the expression

$$(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\zeta)^2(\eta-\theta)^2(\iota-\kappa)^2,$$

that is, $(c-b^2)^5$, which is in fact the sharp form $c^5 \infty b^{10}$.

Seminvariants of a given Degree: Generating Functions. Art. Nos. 57 to 59.

57. We may consider the seminvariants of a given degree, and arrange them according to their weights: thus in each case writing down the series of finals, and for a reason that will appear also the conjugates of these finals (see Table of Conjugates, *ante* No. 54).

For degree 2, or quadric seminvariants, we have

2	3	4	5	6 ...
$C,\ b^2$	–	$C^2,\ c^2$	–	$C^3,\ d^2$

there is here for every even weight (beginning with 2) a single form, and for every odd weight no form: the number of forms of the weight w is thus = coeff. of x^w in $x^2 \div (1-x^2)$, or writing for shortness **2** to denote $1-x^2$ (and similarly **3**, **4**, ... to denote $1-x^3$, $1-x^4$, ...), say that for degree 2, Generating Function, $G.F.$, is $= x^2 \div \mathbf{2}$.

For degree 3, or cubic seminvariants, we have

3	4	5	6	7 ...
$D,\ b^3$	–	$CD,\ bc^2$	$D^2,\ c^3$	$C^2D,\ bc^2$

the counting is most easily effected by means of the conjugate forms; these contain all of them the factor D, and omitting this factor we have all the combinations of C, D which make up the weight $w-3$, viz. for weight w, we have number of ways in which $w-3$ can be made up with the parts 2, 3: that is,

for degree 3, $G.F.$ is $= x^3 \div \mathbf{2.3}$.

Similarly for degree 4 or quartic seminvariants, we have terms each containing E, and removing this factor, we have all the combinations of C, D, E which make up the weight $w-4$, viz.

for degree 4, $G.F.$ is $= x^4 \div \mathbf{2.3.4}$.

Thus for degrees

$$2, \qquad 3, \qquad 4, \qquad 5, \qquad 6, \qquad \ldots$$

the $G.F.$'s are

$$= x^2 \div \mathbf{2}, \quad x^3 \div \mathbf{2.3}, \quad x^4 \div \mathbf{2.3.4}, \quad x^5 \div \mathbf{2.3.4.5}, \quad x^6 \div \mathbf{2.3.4.5.6}, \ldots$$

58. We may analyse these results by separating the finals into classes. I use the expression $b, c, d, \ldots$ are discrete letters, meaning thereby that they are distinct letters, not of necessity consecutive but with any intervals between them. Thus deg. 3, if (b, c) are discrete letters, then the finals are b^3, and bc^2; deg. 4, if b, c, d are discrete letters, then the finals are b^4, bc^3, b^2c^2, and bcd^2; and so on, the number of classes being doubled at each step, as will presently appear for the weights 5 and 6 respectively.

I notice also a property of the conjugates of these classes; for b^3 and bc^2 themselves the conjugates are D and CD, and these occur as factors, D in the conjugate of every form of the class b^3 (for instance conjugates of c^3, d^3 are D^2, D^3), and CD in the conjugate of every form of the class bc^2 (for instance, the conjugates of bd^2, ce^2 are C^2D, C^2D^2); and the like in other cases, viz. for any class whatever the conjugate of the first or representative form occurs as a factor in the conjugates of the several other forms belonging to the same class.

59. With these explanations, the expressions for the several $G.F.$'s are obtained without difficulty, and we have

$$\begin{array}{llll} \text{deg. } 2, & \text{class } C,\ b^2 & G.F. = & x^2 \div \mathbf{2}, \\ \text{deg. } 3, & \quad\text{,,}\ \ D,\ b^3 & \quad\text{,,} & = x^3 \div \mathbf{3}, \\ & \quad\text{,,}\ \ CD,\ bc^2 & \quad\text{,,} & = x^3 \div \mathbf{2.3}; \end{array}$$

we ought here to have

$$\begin{array}{l} x^3 \div \mathbf{2.3} = x^3 \div \mathbf{3} + x^5 \div \mathbf{2.3}, \text{ viz. in verification} \\ \hline x^3 \quad = x^3.\mathbf{2} = x^3 - x^5 \\ \qquad\quad + x^5 \qquad\quad + x^5 \\ \qquad\qquad = x^3; \end{array}$$

$$\begin{array}{llll} \text{deg. } 4, & \text{class} & E,\ b^4 & G.F. = x^4 \div \mathbf{4}, \\ & \text{,,} & DE,\ bc^3 & \text{,,} \ = x^7 \div \mathbf{3.4}, \\ & \text{,,} & CE,\ b^2c^2 & \text{,,} \ = x^6 \div \mathbf{2.4}, \\ & \text{,,} & CDE,\ bcd^2 & \text{,,} \ = x^9 \div \mathbf{2.3.4}; \end{array}$$

we ought here to have

$$\begin{array}{l} x^4 \div \mathbf{2.3.4} = x^4 \div \mathbf{4} + x^7 \div \mathbf{3.4} + x^6 \div \mathbf{2.4} + x^9 \div \mathbf{2.3.4}, \text{ viz. in verification} \\ \hline x^4 \quad = x^4.\mathbf{2.3} = x^4 - x^6 - x^7 + x^9 \\ \qquad + x^7.\mathbf{2} \qquad\qquad + x^7 - x^9 \\ \qquad + x^6.\mathbf{3} \qquad + x^6 \qquad - x^9 \\ \qquad + x^9 \qquad\qquad\qquad + x^9 \\ \qquad\qquad = x^4; \end{array}$$

deg. 5,	class	F,	b^5	$G.F. = x^5 \div 5$,
		EF,	bc^4	$x^9 \div 4.5$,
		DF,	b^2c^3	$x^8 \div 3.5$,
		CF,	b^3c^2	$x^7 \div 2.5$,
		DEF,	bcd^3	$x^{12} \div 3.4.5$,
		CEF,	bc^2d^2	$x^{11} \div 2.4.5$,
		CDF,	b^2cd^2	$x^{10} \div 2.3.5$,
		$CDEF$,	$bcde^2$	$x^{14} \div 2.3.4.5$;

and for the sum of the eight terms

$$G.F. = x^5 \div 2.3.4.5,$$

which may be verified as before.

deg. 6,	class	G,	b^6	$G.F. = x^6 \div 6$,
		FG,	bc^5	$x^{11} \div 5.6$,
		EG,	b^2c^4	$x^{10} \div 4.6$,
		DG,	b^3c^3	$x^9 \div 3.6$,
		CG,	b^4c^2	$x^8 \div 2.6$,
		EFG,	bcd^4	$x^{15} \div 4.5.6$,
		DFG,	bc^2d^3	$x^{14} \div 3.5.6$,
		CFG,	bc^3d^2	$x^{13} \div 2.5.6$,
		DEG,	b^2cd^3	$x^{13} \div 3.4.6$,
		CEG,	$b^2c^2d^2$	$x^{12} \div 2.4.6$,
		CDG,	b^3cd^2	$x^{11} \div 2.3.6$,
		$DEFG$,	$bcde^3$	$x^{18} \div 3.4.5.6$,
		$CEFG$,	bcd^2e^2	$x^{17} \div 2.4.5.6$,
		$CDFG$,	bc^2de^2	$x^{16} \div 2.3.5.6$,
		$CDEG$,	b^2cde^2	$x^{15} \div 2.3.4.6$,
		$CDEFG$,	$bcdef^2$	$x^{20} \div 2.3.4.5.6$;

and for the sum of the sixteen terms

$$G.F. = x^6 \div 2.3.4.5.6,$$

which may be verified as before.

Reducible Seminvariants—Perpetuants. Art. Nos. 60 to 64.

60. Seminvariants of the degrees 2 and 3 are irreducible—or say they are perpetuants. Hence by what precedes, as regards perpetuants

for degree 2, $G.F. = x^2 \div 2$;

for degree 3, $G.F. = x^3 \div 2.3$.

For the degree 4 (if as before b, c, d denote discrete letters), then the finals are b^4, bc^3, b^2c^2 and bcd^2. For a final $b^4 = b^2.b^2$ or $b^2c^2 = b^2.c^2$, we have evidently a product of two quadric seminvariants ending in b^2 and b^2, or in b^2 and c^2, with the same final term as the quartic seminvariant; so that, considering the quartic seminvariants

arranged with their finals in AO, and adding to such quartic seminvariant a proper numerical multiple of the product in question, we obtain a quartic seminvariant the final term whereof is in AO higher than the original final term b^4 or b^3c^2, and such quartic seminvariant is thus said to be reducible; a quartic seminvariant not thus reducible is a perpetuant. The quartic perpetuants are consequently those which end in bc^3 or bcd^2. The lowest form is that ending in bc^3, of the weight 7. Taking the sum of the $G.F.$'s for the forms bc^3 and bcd^2 respectively, the $G.F.$ for quartic perpetuants is

$$x^7 \div \mathbf{3.4} + x^9 \div \mathbf{2.3.4},$$

viz. this is

$$x^7(1-x^2) + x^9 \div \mathbf{2.3.4},$$

or finally

$$G.F. = x^7 \div \mathbf{2.3.4}.$$

As an instance of a reduction, we have

$$(d^2 \infty b^2c^2) - (c \infty b^2)(e \infty c^2) = (ce \infty c^3),$$

viz. this is

$$(d \infty b^2c^2) = (c - b^2)(e - 4bd + 3c^2) - (ce - d^2 - b^2e + 2bcd - c^3).$$

We have also

$$(d^2 \infty b^2c^2) = (d \infty b^2)^2 + 4(c \infty b^2)^3,$$

viz.

$$(d \infty b^2c^2) = (d - 3bc + 2b^3)^2 + 4(c - b^2)^3,$$

but this is *not* a reduction, there are on the right-hand side terms of the degree 6, which is higher than the degree of the seminvariant $d^2 \infty b^2c^2$. In general, we say that a seminvariant of any given degree is reducible when we can, by adding to it products *of its own degree* of seminvariants of inferior degrees, reduce it to a seminvariant the final of which is in AO higher than the original final.

61. For the degree 5 (taking b, c, d, e to denote discrete letters), if the final be b^5, bc^4, b^2c^3, b^3c^2, bc^2d^2 or b^2cd^2, then the seminvariant will be reducible; a perpetuant must have therefore a final bcd^3 or $bcde^2$. But it is not true that every quintic seminvariant with either of these finals is a perpetuant. To explain this, observe that the first mentioned six finals are some of them in one way only, some of them in two ways, expressible as a product of power-enders, or say they are singly, or else doubly, composite: viz. we have

$$b^5 = b^2 . b^3; \quad bc^4 = c^2 . bc^2; \quad b^2c^3 = b^2 . c^3; \quad b^3c^2 = c^2 . b^3 = b^2 . bc^2;$$
$$bc^2d^2 = c^2 . bd^2 = d^2 . bc^2; \quad b^2cd^2 = b^2 . cd^2.$$

For a doubly composite form, for instance b^3c^2, forming first the product of the quadric and cubic seminvariants ending in c^2, b^3 respectively, and secondly the product of the quadric and cubic seminvariants ending in b^2 and bc^2 respectively, we have two products each with the final b^3c^2, and forming a linear combination so as to eliminate this term b^3c^2, we have thus it may be a quintic seminvariant with a final such as bcd^3 or $bcde^2$, and the process then furnishes a reduction of such a quintic seminvariant. Or on the other hand, it may be that the finals of the degree 5 all of them

disappear, and we have a relation between products of the form in question (i.e. of a quadric and a cubic seminvariant) and seminvariants of a degree inferior to 5, say this is a quintic syzygy.

In particular, a non-composite final first presents itself for the weight 12, viz. here the finals are b^2ce^2, bcd^3, c^3d^2, the last of these is doubly composite, and it furnishes a reduction of bcd^3. For the weight 13, the finals are b^3f^2, b^2de^2, bc^2e^2, bd^4, c^2d^3 which are each of them singly or doubly composite: for the weight 14, they are b^3cf^2, b^2c^3, $bcde^2$, c^3e^2 and cd^4, and here the doubly composite form furnishes a reduction of $bcde^2$. For the weight 15, we have a final bce^3 which gives a quintic perpetuant. I have, in fact, in my paper "A Memoir on Seminvariants," *American Journal of Mathematics*, vol. VII. (1885), pp. 1—25, [828], worked out the theory of quintic syzygies and perpetuants, and subsequently connecting this with the present theory of finals, I succeeded in showing that, when the doubly composite final contains a b, then there is not a reduction but a syzygy; we thus have

$$G.F. \text{ for finals } b^3c^2,\ b^3d^2,\ \ldots = x^7 \div \mathbf{2},$$

$$\text{,, } \quad \text{,, } \quad bc^3d^2, \quad \ldots = x^{11} \div \mathbf{2.4},$$

whence for the two forms

$$G.F. \text{ is } x^7 \div \mathbf{2} + x^{11} \div \mathbf{2.4} = \{x^7(1-x^4) + x^{11}\} \div \mathbf{2.4},$$

or say for S_5, the number of quintic syzygies $G.F.$ is $= x^7 \div \mathbf{2.4}$.

I further satisfied myself that the finals for the quintic perpetuants are $bc0e^3$, and $bc0ef^2$, viz. the b, c, e, f being discrete letters, the interposed 0 denotes that the c and e are not consecutive letters. The conjugates of these forms contain the factors D^2EF and CD^2EF respectively, and it hence appears that the $G.F.$'s are $= x^{15} \div \mathbf{3.4.5}$ and $x^{17} \div \mathbf{2.3.4.5}$; adding these, we find

$$\text{for quintic perpetuants } G.F. \text{ is } = x^{15} \div \mathbf{2.3.4.5},$$

which expression was given in the memoir just referred to: the result was obtained by investigating in the first instance an expression for S_5, the number of quintic syzygies of a given weight. The course of Stroh's investigation to be presently given is different; he determines directly the number of perpetuants, and we may if we please use conversely this result to obtain the number of syzygies.

62. The foregoing theory of reduction is independent of the form of the seminvariants, which may be blunt or sharp at pleasure: the actual formulæ will of course be different, and they are very much more simple for the sharp seminvariants, viz. here in many cases a seminvariant is found to be equal to a product of seminvariants of inferior degrees. I subjoin the following table of the reduction of the several sharp seminvariants up to the weight 12; the forms referred to are the tabulated forms, and to mark that this is so I write down in each case the numerical coefficients of the initial and final terms, viz. instead of $c \infty b^2$, $d \infty b^3$, &c., I write $c \infty -b^2$, $d \infty 2b^3$, &c. As appears by the table, these are for shortness denoted by C, D respectively, and so for weight 4, the forms are called E, E_2, for weight 5, F, F_2, for weight 6, G, G_2, G_3, G_4, and so on, the unsuffixed letters having thus an implied suffix, not 0 but 1. The table is

Table of Reductions.

$w=$			
2	$c \propto -b^2$	C	
3	$d \propto 2b^3$	D	
4	$e \propto 3c^2$	E	
	c^2 ,, b^4	$E_2 =$	C_2
5	$f \propto -6bc^2$	F	
	cd ,, $-2b^5$	$F_2 =$	CD
6	$g \propto -10d^2$	G	
	ce ,, $-c^3$	G_2	
	d^2 ,, $-3b^2c^2$	$G_3 =$	$CE-G_2$
	c^3 ,, $-b^6$	$G_4 =$	C^3
7	$h \propto 20bd^2$	H	
	cf ,, $3bc^3$	H_2	
	de ,, $6b^3c^2$	$H_3 =$	DE
	c^2d ,, $2b^7$	$H_4 =$	C^2D
8	$i \propto 35c^2$	I	
	cg ,, $2cd^2$	I_2	
	$3df$,, $10b^2d^2$	$I_3 =$	$CG-I_2$
	e^2 ,, $9c^4$	$I_4 =$	E^2
	c^2e ,, b^2c^3	$I_5 =$	CG_2
	cd^2 ,, $3b^4c^2$	$I_6 =$	CG_3
	c^4 ,, b^8	$I_7 =$	C_4
9	$j \propto -70bc^2$	J	
	$2ch$,, $-20d^3$	J_2	
	dg ,, $-4bcd^2$	J_3	
	ef ,, $-20b^3d^2$	$J_4 =$	$\frac{1}{2}(2CH-J_2-7J_3)$
	$2c^2f$,, $-3bc^4$	$J_5 =$	$\frac{1}{3}(EF-J_4)$
	cde ,, $-2b^2c^3$	$J_6 =$	DG_2
	d^3 ,, $-6b^5c^2$	$J_7 =$	DG_3
	c^3d ,, $-2b^9$	$J_8 =$	C^3D
10	$k \propto -12bf^2$	K	
	ci ,, $-5ce^2$	K_2	
	$4dh$,, $-35b^2e^2$	$K_3 =$	$CI-K_2$
	$16eg$,, $-80bd^3$	K_4	
	f^2 ,, $-32c^2d^2$	$K_5 =$	$\frac{1}{13}(16EG-K_4)$
	c^2g ,, $-2b^2cd^2$	$K_6 =$	CI_2
	ce^2 ,, $-3c^5$	$K_7 =$	EG_2
	$3cdf$,, $-10b^4d^2$	$K_8 =$	CI_3
	d^2e ,, $-9b^2c^4$	$K_9 =$	EG_3
	c^3e ,, $-b^4c^3$	$K_{10} =$	C^2G_2
	c^2d^2 ,, $-3b^6c^2$	$K_{11} =$	C^2G_3
	c^5 ,, $-b^{10}$	$K_{12} =$	C_5

$w=$			
11	$l \propto 252bf^2$	L	
	$2cj$,, $35de^2$	L_2	
	di ,, $10bce^2$	L_3	
	eh ,, $20cd^3$	L_4	
	$16c^2h$,, $-70b^3e^2$	$L_5 =$	$-DI+L_3+2L_4$
	fg ,, $160b^2d^3$	$L_6 =$	$8CJ_2-L_5$
	cef ,, $-2bc^2d^2$	$L_7 =$	$-\frac{1}{30}(FG-L_6)$
	cdg ,, $4b^3cd^2$	$L_8 =$	DI_2
	d^2f ,, $3bc^5$	$L_9 =$	$\frac{1}{2}(FG_2-L_7)$
	$12c^3f$,, $-20b^5d^2$	$L_{10} =$	$-DI_3+3L_9$
	de^2 ,, $18b^2c^4$	$L_{11} =$	DE^2
	c^2de ,, $2b^5c^3$	$L_{12} =$	CDG_2
	cd^3 ,, $6b^7c^2$	$L_{13} =$	CDG_3
	c^4d ,, $2b^{11}$	$L_{14} =$	C^4D
12	$m \propto 462g^2$	M	
	$3ck$,, $42cf^2$	M_2	
	ei ,, $15c^3$	M_3	
	$15dj$,, $378b^2f^2$	$M_4 =$	$3CK-M_2$
	$25fh$,, $175bde^2$	M_5	
	g^2 ,, $125c^2e^2$	$M_6 =$	$\frac{1}{21}(25EI-25M_3-4M_5)$
	ceg ,, d^4	$M_7 =$	$\frac{1}{100}(G^2-M_6)$
	c^2i ,, $5b^2ce^2$	$M_8 =$	CK_2
	$5d^2g$,, $20bcd^3$	$M_9 =$	$-3GG_3+5EI_2-2M_7$
	cf^2 ,, $20c^3d^2$	$M_{10} =$	GG_2-M_7
	$4cdh$,, $35b^4e^2$	$M_{11} =$	CK_3
	$18def$,, $80b^3d^3$	$M_{12} =$	$\frac{1}{10}(10CK_4-160M_7-32M_9+54M_{10})$
	e^3 ,, $36b^2c^2d^2$	$M_{13} =$	$\frac{1}{8}(9CK_5-9M_{10}-M_{12})$
	c^2e^2 ,, c^6	$M_{14} =$	G_2^2
	c^3g ,, $2b^4cd^2$	$M_{15} =$	C^2I_2
	cd^2e ,, $3b^2c^5$	$M_{16} =$	EI_5-M_{14}
	$3c^2df$,, $10b^6d^2$	$M_{17} =$	CK_8
	d^4 ,, $9b^4c^4$	$M_{18} =$	G_2^3
	c^4e ,, $-3b^6c^3$	$M_{19} =$	C^3G_2
	c^3d^2 ,, $3b^8c^2$	$M_{20} =$	C^3G
	c^6 ,, b^{12}	M_{21}	

Where no reduction is given, the form is irreducible, i.e. it is a perpetuant.

63. As to these reductions, it may be observed that in very many cases we have the sharp seminvariant given as an actual product $E_2 = C^2$, $F_2 = CD$, $G_4 = C^3$, &c. We have next other reductions such as $G_3 = CE - G_2$, where on the right-hand side there is a single product; this has a final the same as that of the seminvariant which is to be reduced, so that, eliminating this term from the seminvariant and product in question, we have an expression which must be a linear combination (with numerical coefficients) of the preceding seminvariants of the same weight. To take a less simple example, $L_5 = -DI + L_3 + 2L_4$; here $L_5 = -fg + 16c^2h \ldots - 70b^3e^2$, and $DI = (d - 3bc + 2b^3)(i \ldots + 35e^2)$ has the final $+70b^3e^2$. The verification is

$$\begin{array}{rcl} -DI &=& -di \qquad\qquad \ldots - 70b^3e^2 \\ +L_3 &=& di - 2eh + fg \\ -2L_4 &=& \qquad 2eh - 2fg \\ \hline L_5 &=& \qquad\qquad - fg \ldots - 70b^3e^2. \end{array}$$

The only case in which we have on the right-hand side two products is $(d^3g \propto bcd^3)$, $M_9 = -3GG_2 + 5EI_2 - 2M_7$; viz. here the final of M_9 is bcd^3 which is incomposite (viz. it is not the product of two power-enders), this is in fact the first instance of a quintic seminvariant with an incomposite final and which is nevertheless reducible. For observe, the next seminvariant M_{10} has the final c^3d^3, which is a product in the two ways $c^2 . cd^2$ and $c^3 . d^2$; we have thus the two products $(e \propto c^2)(cg \propto cd^2)$ and $(ce \propto c^3)(g \propto d^2)$, that is, EI_2 and GG_2 with the same final c^3d^3, and combining them so as to eliminate this term we have an expression having the final bcd^3, and which is thus expressible in terms of M_9 and preceding seminvariants: the verification is

$$\begin{array}{rcl} -3GG_2 &=& -3ceg \qquad\qquad + 3d^2g \ldots + 60bcd^3 - 30c^3d^2 \\ +5EI_2 &=& +5ceg \qquad\qquad\qquad\qquad - 40bcd^3 + 30c^3d^2 \\ -2M_7 &=& -2ceg + 2cf^2 + 2d^2g \\ \hline M_9 &=& \qquad\qquad 2cf^2 + 5d^2g \ldots + 20bcd^3. \end{array}$$

64. I annex to this a table (taken from the square diagrams) for the initials and finals of the sharp seminvariants for the weights 13, 14, 15, and 16.

$w =$

13	n	$\propto bg^2$	N
	el	,, df^2	N_2
	dk	,, bcf^2	3
	ej	,, be^3	4
	fi	,, cde^2	5
	c^2j	,, b^3f^2	6
	gh	,, b^2de^2	7
	ceh	,, bc^2e^2	8
	d^2h	,, bd^4	9
	cfg	,, c^3d^3	10
	cdi	,, b^3ce^2	11
	deg	,, b^2cd^3	12
	df^2	,, bc^3d^2	13
	c^3h	,, b^3e^2	14
	e^2f	,, b^4d^3	15
	c^2ef	,, b^2c^2d	16
	cd^2f	,, bc^6	17
	c^2dg	,, b^3cd^2	18
	cde^2	,, b^3c^5	19
	c^4f	,, b^7d^2	20
	d^3e	,, b^5c^4	21
	c^3de	,, b^7c^3	22
	c^2d^3	,, b^9c^2	23
	c^5d	,, b^{13}	24
14	o	$\propto h^2$	O
	cm	,, cg^2	O_2
	ek	,, ef^2	3
	dl	,, b^2g^2	4
	fj	,, bdf^2	5
	gi	,, c^2f^2	6
	cei	,, ce^3	7
	h^2	,, d^2e^2	8
	c^2k	,, b^2cf^2	9
	d^2i	,, b^2e^3	10
	cfh	,, $bcde^2$	11
	cg^2	,, c^3e^2	12
	dfg	,, cd^4	13
	cdj	,, b^4f^2	14
	deh	,, b^3de^2	15
	e^2g	,, $b^2c^2e^2$	16
	c^2eg	,, b^2d^4	17
	ef^2	,, bcd^3	18
	c^2f^2	,, c^4d^2	19
	c^3i	,, b^4ce^2	20
	cd^2g	,, b^3cd^3	21
	$cdef$	,, $b^2c^3d^2$	22
	ce^3	,, c^7	23
	c^2dh	,, b^5e^2	24
	d^3f	,, b^5d^3	25
	d^2e^2	,, $b^4c^2d^2$	26
	c^3e^2	,, b^2c^6	27
	d^4g	,, b^6cd^2	28
	c^2d^2e	,, b^4c^5	29
	c^3df	,, b^8d^2	30
	cd^4	,, b^6c^4	31
	c^5e	,, b^8c^3	32
	c^4d^2	,, $b^{10}c^2$	33
	c^7	,, b^{14}	34

$w =$

15	p	$\propto ch^2$	P
	cm	,, dg^2	P_2
	el	,, f^3	3
	dm	,, bcg^2	4
	fk	,, bef^2	5
	gj	,, cdf^2	6
	cej	,, de^3	7
	c^2l	,, b^2g^2	8
	d^2j	,, b^2df^2	9
	hi	,, bc^2f^2	10
	cfi	,, bce^3	11
	cgh	,, bd^2e^2	12
	dfh	,, c^2de^2	13
	e^2h	,, d^5	14
	cdk	,, b^2cf^2	15
	dei	,, b^2e^2	16
	c^2eh	,, b^2cde^2	17
	dg^2	,, bc^3e^2	18
	efg	,, bcd^4	19
	c^2fg	,, c^3d^2	20
	c^3j	,, b^5f^2	21
	e^4h	,, b^4de^2	22
	cd^2h	,, $b^3c^2e^2$	23
	$cdeg$	,, b^3d^4	24
	f^3	,, $b\ c^2d^3$	25
	cdf^2	,, bc^4d^2	26
	ce^2f	,, b^5ce^2	27
	d^3g	,, b^4cd^3	28
	d^2ef	,, $b^3c^3d^2$	29
	c^3ef	,, bc^7	30
	c^4h	,, b^7e^2	31
	c^2d^2f	,, b^6d^3	32
	de^3	,, $b^5c^2d^2$	33
	c^2de^2	,, b^3c^6	34
	c^3dg	,, b^7cd^2	35
	cd^3e	,, b^5c^5	36
	c^5f	,, b^9d^2	37
	d^5	,, b^7c^4	38
	c^4de	,, b^9c^3	39
	c^3d^3	,, $b^{11}c^2$	40
	c^6d	,, b^{15}	41

$w =$

16	q	$\propto i^2$	Q
	co	,, ch^2	Q_2
	em	,, eg^2	3
	dn	,, b^2h^2	4
	fl	,, bdg^2	5
	gk	,, bf^3	6
	cek	,, c^2g^2	7
	hj	,, cef^2	8
	i^2	,, d^2f^2	9
	cgi	,, e^4	10
	c^2m	,, b^2cg^2	11
	c^3k	,, b^2ef^2	12
	dej	,, $bcdf^2$	13
	dfi	,, bde^3	14
	e^2i	,, c^2f^2	15
	ch^2	,, c^2e^3	16
	dgh	,, cd^2e^2	17
	cdl	,, b^2g^2	18
	dej	,, b^2df^2	19
	c^2ei	,, $b^2c^2f^2$	20
	efh	,, b^2ce^3	21
	eg^2	,, $b^2d^2e^2$	22
	c^2fh	,, bc^2de^2	23
	c^2g^2	,, bd^5	24
	f^2g	,, c^4e^2	25
	ce^2g	,, c^2d^4	26
	c^3k	,, b^4cf^2	27
	cd^2i	,, b^4e^3	28
	$cdeh$	,, b^3cde^2	29
	$cdfg$	,, $b^3c^3e^2$	30
	d^2eg	,, b^3cd^4	31
	cef^2	,, bc^3d^3	32
	d^2f^2	,, c^3d^2	33
	c^2dj	,, b^6f^2	34
	d^3h	,, b^5de^2	35
	c^3ej	,, $b^4c^2e^2$	36
	c^3f^2	,, b^4d^4	37
	de^2f	,, $b^3c^2d^3$	38
	e^4	,, $b^2c^4d^2$	39
	c^2e^3	,, c^8	40
	c^4i	,, b^6ce^2	41
	c^2d^2g	,, b^2cd^3	42
	c^2def	,, $b^4c^3d^2$	43
	cd^2e^2	,, b^2c^7	44
	c^3dh	,, b^5e^2	45
	cd^3f	,, b^7d^3	46
	c^4e^2	,, $b^5c^2d^2$	47
	d^4e	,, b^4c^6	48
	c^5g	,, b^8cd^2	49
	c^3d^2e	,, b^6c^5	50
	c^4df	,, $b^{10}d^2$	51
	c^2d^4	,, b^8c^4	52
	c^6e	,, $b^{10}c^3$	53
	c^5d^2	,, $b^{12}c^2$	54
	c^8	,, b^{16}	55

It would be interesting to complete this into a table of reductions as given for the weights 2 to 12.

The Strohian Theory Resumed: Application to Perpetuants. Art. Nos. 65 to 71.

65. We can by means hereof establish, in regard to the specific blunt seminvariants, a general theory of reduction, or say a theory of the relations which exist between the seminvariants of a given degree and the powers and products of seminvariants of inferior degrees. To exhibit the form of these, it will be sufficient to take Ω a sum of two parts, $=\Omega'+\Omega''$, but the more general assumption is Ω a sum of any number of parts, $=\Omega'+\Omega''+\Omega'''+\ldots$. Taking then $\Omega=\Omega'+\Omega''$, where for the Ω' and Ω'' separately the sum of the $(x, y, z, \ldots)$ is $=0$, suppose that to the $(0, C, D, E, \ldots)$ of Ω there correspond $(0, C', D', E', \ldots)$ for Ω' and $(0, C'', D'', E'', \ldots)$ for Ω''. We have

$$\begin{aligned}
C &= C'+C'',\\
D &= D'+D'',\\
E &= E'+E''+C'C'',\\
F &= F'+F''+C'D''+C''D',\\
G &= G'+G''+C'E''+C''E'+D'D'',
\end{aligned}$$

the law of which is obvious.

66. We have, for instance,

$$\Omega^4=(\Omega'+\Omega'')^4, \quad =\Omega'^4+6\Omega'^2\Omega''^2+\Omega''^4, \text{ (since } \Omega'=0,\ \Omega''=0),$$

that is,

$$\begin{aligned}
(C'+C'')^2c^2 &= C'^2c^2+6C'b^2\,.\,C''b^2+C''^2c^2\\
+(E'+E''+C'C'')\,b^4 &\qquad\qquad\qquad\qquad +E''b^4
\end{aligned}$$

where, and in what follows, c^2, b^4, b^2 are for shortness written instead of $[c^2]$, $[b^4]$, $[b^2]$ to denote the specific blunt seminvariants ending in c^2, b^4, b^2 respectively.

The terms in C'^2, C''^2, E', E'' are identical on each side of the equation and destroy each other: omitting these, we have only the terms in $C'C''$ which must be equivalent on the two sides of the equation, and comparing coefficients we find the relation

$$2c^2+b^4=6\,.\,b^2\,.\,b^2,$$

which of course means $2\,[c^2]+[b^4]=6\,[b^2]\,[b^2]$, viz. this is

$$2\,(2e-8bd+6c^2)+(-4e+16bd+12c^2-48b^2c+24b^4)=6\,(-2c+2b^2)^2.$$

In like manner, for $\Omega^6=(\Omega'+\Omega'')^6$, we have

$$\begin{aligned}
&(C'+C'')^3 &&.\,d^2\\
&+(D'+D'')^2 &&.\,c^3\\
&+(C'+C'')\,(E'+E''+C'C'') &&.\,b^2c^2\\
&+(G'+G''+C'E''+C''E'+D'D'')\,. &&b^6
\end{aligned}$$

equal to

$$\left\{\begin{matrix} C'^3 & .d^2 \\ +D'^2 & .c^3 \\ +C'E' & .b^3c^2 \\ +G' & .b^6 \end{matrix}\right\} + 15\left\{\begin{matrix} C'^2 . c^2 \\ +E' . b^4 \end{matrix}\right\} C'' . b^3 + 20D' . b^3 . D'' . b^3 + 15C'^2 . b^2 \left\{\begin{matrix} C''^2 . c^3 \\ +E'' . b^4 \end{matrix}\right\} + \left\{\begin{matrix} C''^3 & .d^2 \\ +D''^2 & .c^3 \\ +C''E'' & .b^3c^2 \\ +G'' & .b^6 \end{matrix}\right\}.$$

Here omitting the terms which destroy each other and comparing the coefficients of the remaining terms, viz. $C'^2C'' + C''^2C'$, $D'D''$ and $C'E'' + C''E'$, we find the relations

$$\begin{aligned} 3d^2 + b^3c^2 &= 15 . c^2 . b^3, \\ 2c^3 + b^6 &= 20 . b^3 . b^3, \\ b^3c^2 + b^6 &= 15 . b^4 . b^2, \end{aligned}$$

which may be easily verified. There are on the right-hand side only products of two parts, but this is on account of the special assumption $\Omega = \Omega' + \Omega''$, a sum of two parts.

67. I write now

$$\begin{aligned} \Omega_2 &= \alpha x + \beta y & , S_2 x &= 0, \\ \Omega_3 &= \alpha x + \beta y + \gamma z & , S_3 x &= 0, \\ \Omega_4 &= \alpha x + \beta y + \gamma z + \delta w & , S_4 x &= 0, \\ \Omega_5 &= \alpha x + \beta y + \gamma z + \delta w + \epsilon t & , S_5 x &= 0, \\ \Omega_6 &= \alpha x + \beta y + \gamma z + \delta w + \epsilon t + \zeta u, & S_6 x &= 0, \end{aligned}$$

and I say that Ω_2 and Ω_3 cannot break up: but that Ω_4 breaks up if it becomes a sum of $2 + 2$ terms (i.e. a sum of two parts Ω_2 for each of which $S_2 x = 0$, and so in other cases): that Ω_5 breaks up if it becomes a sum of $2 + 3$ terms, Ω_6 breaks up if it becomes a sum of $2 + 4$ or $2 + 2 + 2$ terms, or if it becomes a sum of $3 + 3$ terms: and similarly for any higher suffix.

The condition that Ω_4 may break up is $x + y = 0$, $x + z = 0$, or $y + z = 0$, or what is the same thing it is $\Pi_3(x + y) = 0$, where $\Pi_3(x + y)$ is the product of the three sums each containing x; this is a symmetric function, we in fact have

$$\Pi_3(x + y) = x^3 + x^2(y + z + w) + x(yz + yw + zw) + yzw, = xyz + xyw + xzw + yzw, \quad = -D.$$

The condition in order that Ω_5 may break up is $x + y = 0$, ..., or $w + t = 0$, say this is $\Pi_{10}(x + y) = 0$, where $\Pi_{10}(x + y)$ denotes the product of the ten sums $x + y$, ..., $w + t$. It will be shown that we have $\Pi_{10}(x + y) = -D^2E + CDF - F^2$.

The condition in order that Ω_6 may break up is, $x + y = 0$, ..., or $t + u = 0$, or again if $x + y + z = 0$, ..., or $x + t + u = 0$, viz. it is $\Pi_{15}(x + y)\Pi_{10}(x + y + z) = 0$, where $\Pi_{15}(x + y)$ is the product of the fifteen sums $x + y$, ..., $t + u$, and $\Pi_{10}(x + y + z)$ is the product of the ten sums $x + y + z$, ..., $x + t + u$, each containing x : $\Pi_{15}(x + y)$ and

$\Pi_{10}(x+y+z)$ are symmetric functions, the expressions for which will be given further on: the weights in the capital letters are 15 and 10 respectively. And similarly for Ω with any higher suffix, we have the condition that this may break up.

I introduce the factors $\Pi_4 x = E$, $\Pi_5 x = -F$, $\Pi_6 x = G$, ... respectively and write for

Ω_4 $\quad M_7 = \Pi_4 x \Pi_3 (x+y) = -DE$ as above,

Ω_5 $\quad M_{15} = \Pi_5 x \Pi_{10}(x+y) = -F(-D^2E + CDF - F^2)$ as above,

Ω_6 $\quad M_{31} = \Pi_6 x \Pi_{15}(x+y)\,\Pi_{10}(x+y+z)$,

$\vdots$

where observe that, for the even suffixes of Ω, the last factors $\Pi_5(x+y)$, $\Pi_{10}(x+y+z)$, ... denote the products of the sums $x+y$, $x+y+z$, ... which contain x, that is in each case the products of only half the whole number of such linear factors. The suffixes of M show the weights in the capital letters C, D, E, F, G, ... viz. these are $4+3$, $=7$, $5+10$, $=15$, $6+15+10$, $=31$, and so on; the law is obvious, and for Ω_n the weight is $=2^{n-1}-1$.

68. To explain the Strohian theory of perpetuants, I assume explicitly as presently appears. For perpetuants of any given degree δ, we consider in $\Omega_\delta{}^w$ ($w=\delta$ at least) the terms containing seminvariants of the given degree: for instance when $\delta=4$, $w=12$, these are

$$\begin{aligned} & C^4E \;.\, b^2f^2 \\ + & CD^2E \,.\, bde^2 \\ + & C^2E^2 \;.\, c^2e^2 \\ + & E^3 \;.\, d^4, \end{aligned}$$

where the capital expressions all contain as factor the letter E of the weight 4. By making Ω to break up, it is assumed that *we obtain all the reductions of the seminvariants of the degree and weight in question; and every such seminvariant, if it be reducible, will be reduced by means of the resulting formulæ.* Now there are seminvariants which are not reducible by these formulæ: in the example just considered, the seminvariant bde^2 has the coefficient CD^2E containing the factor DE, $=xyzw(x+y)(x+z)(x+w)$ which vanishes when Ω_4 breaks up; so that, supposing Ω_4 to break up, the seminvariant bde^2 disappears from the formulæ, and we have no reduction of this seminvariant. And again it is assumed *that every seminvariant which does not in this way disappear from the equation is reducible.* The irreducible seminvariants are thus the seminvariants which, when Ω breaks up into a sum of two or more parts, disappear from the formulæ; viz. the seminvariants which thus disappear are the perpetuants.

69. In the case considered of quartic seminvariants, it has just been seen that, for the weight 12, bde^2 is a perpetuant; and so in general for the weight w, every quartic seminvariant, multiplied into a product of capitals which contains the factor DE, is a perpetuant: for the weight 7 the only term is $DE.bc^3$, viz. the product

of capitals is here $=DE$; and for any higher weight w we have products which are equal to DE multiplied into products of the weight $w-7$ in C, D, E: and we thus see that the $G.F.$ for quartic perpetuants is $=x^7 \div \mathbf{2.3.4}$.

70. For quintic perpetuants, we consider in Ω_5^w ($w=5$ at least) the terms which contain quintic perpetuants; for instance, when $w=15$, the terms are

$$\begin{aligned} & C^5F \quad . b^3g^2 \\ + & C^2D^2F \, . b^2df^2 \\ + & C^3EF \, . bc^2f^2 \\ + & D^2EF \, . bce^3 \\ + & CE^2F \, . bd^2e^2 \\ + & CDF^2 \, . c^2de^2 \\ + & F^3 \quad . d^5, \end{aligned}$$

where the functions of the capitals all contain the factor F; the finals b^3g^2, b^2df^2, ... are arranged in AO. Supposing Ω_5 to break up, we have an expression $M, = -D^2EF + CDF^2 - F^3$, which is $=0$, and using this value of M to eliminate the term D^2EF which belongs to the seminvariant bce^3, the final whereof is highest in AO, viz. writing $D^2EF = -M + CDF^2 - F^3$, the expression is

$$\begin{aligned} & C^5F \quad . b^3g^2 \quad \text{that is} \quad C^5F \quad . b^3g^2 \\ + & C^2D^2F \, . b^2df^2 \qquad\qquad + C^2D^2F \, . b^2df^2 \\ + & C^3EF \, . bc^2f^2 \qquad\qquad + C^3EF \, . bc^2f^2 \\ + & (-M + CDF^2 - F^3) \, . bce^3 \qquad\qquad - M \quad . bce^3 \\ + & CE^2F \, . bd^2e^2 \qquad\qquad + CE^2F \, . bd^2e^2 \\ + & CDF^2 \, . c^2de^2 \qquad\qquad + CDF^2 \, . (c^2de^2 + bce^3) \\ + & F^3 \quad . d^5 \qquad\qquad + F^3 \quad . (d^5 - bce^3); \end{aligned}$$

and here when Ω_5 breaks up, we have $M=0$, that is, the seminvariant bce^3 disappears from the equation, and it is thus a perpetuant: but b^3g^2, b^2df^2, bc^2f^2 and the combinations $c^2de^2 + bce^3$, and $d^5 - bce^3$ are severally reducible.

The degree 15 is evidently the lowest degree for which there is an irreducible quintic seminvariant, and for any higher weight w the number of such seminvariants is equal to the number of capital terms which have the factor D^2EF, viz. this is equal to the number of terms weight $w-15$ which can be made up with C, D, E, F; and hence

for quintic perpetuants $G.F. = x^{15} \div \mathbf{2.3.4.5}$.

71. For the degree 6, $M = \Pi_6 x \Pi_{15}(x+y)\,\Pi_{10}(x+y+z)$ is a function of the capitals of the weight 31, and we thence at once infer that

for sextic perpetuants $G.F. = x^{31} \div \mathbf{2.3.4.5.6}$.

But it is worth while to write down the expression for M: I do this, annexing to each term the seminvariant (i.e. final term) which belongs to it, arranging these final terms in AO; the value thus arranged is

$M =$		finals in AO
$+1$	D^4E^2FG	$bcei^3$
-2	CD^3EF^2G	$bdehi^2$
$+1$	$C^2D^2F^3G$	be^2gi^2
$+2$	D^2EF^3G	$befh^3$
-2	CDF^4G	bf^2gh^2
$+1$	F^5G	bg^5
-1	D^6EG^2	c^2di^2
$+1$	CD^4FG^2	cd^2hi^2
$+1$	$C^2D^2EFG^2$	$cdegi^2$
-4	$D^2E^2FG^2$	$cdfh^3$
-1	$C^3DF^2G^2$	ce^2fi^2
-1	$D^3F^2G^2$	ce^2h^3
$+4$	$CDEF^2G^2$	$cefgh^2$
$+1$	$C^2F^3G^2$	cf^3h^2
$+4$	EF^3G^2	cfg^4
-1	$C^2D^3G^3$	d^3gi^2
$+4$	D^3EG^2	d^2eh^3.

It thus appears that the single sextic perpetuant of the weight 31 is $bcei^3$, and generally that, for any higher weight, the sextic perpetuants are such that the conjugate capital terms contain each of them the factor D^4E^2FG.

The like reasoning shows that

for perpetuants of degree n, $G.F.$ is $= x^{2^{n-1}-1} \div \mathbf{2.3.4} \ldots n$.

Investigation of the Values of the Foregoing Functions $\Pi_{10}(x+y)$, $\Pi_{15}(x+y)$ *and* $\Pi_{10}(x+y+z)$. Art. Nos. 72 to 74.

72. If x, y, z, w, t are the roots of a quintic equation, say

$$\lambda - x.\lambda - y.\lambda - z.\lambda - w.\lambda - t = (1, B, C, D, E, F \between \lambda, 1)^5 = 0,$$

we require the product $\Pi_{10}(x+y)$ of the sum of two roots in the particular case $B=0$. But in order to the determination of the expression for $\Pi_{10}(x+y+z)$, we require the value of $\Pi_{10}(x+y)$ in the general case, B any value whatever.

Writing

$$x = -\tfrac{1}{2}(\theta + \omega),$$
$$y = -\tfrac{1}{2}(\theta - \omega),$$

and therefore

$$\theta + x + y = 0,$$

we have

$$(\theta+\omega)^5 - 2B(\theta+\omega)^4 + 4C(\theta+\omega)^3 - 8D(\theta+\omega)^2 + 16E(\theta+\omega) - 32F = 0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2 = M$, we have

$$(\theta^5 - 2B\theta^4 + 4C\theta^3 - 8D\theta^2 + 16E\theta - 32F) + M(10\theta^3 - 12B\theta^2 + 12C\theta - 8D) + M^2(5\theta - 2B) = 0,$$
$$(5\theta^4 - 8B\theta^3 + 12C\theta^2 - 16D\theta + 16E) + M(10\theta^2 - 8B\theta + 4C) + M^2 . 1 = 0,$$

which are of the form $A + BM + CM^2 = 0$, $A' + B'M + C'M^2 = 0$, and give therefore by elimination of M the equation

$$-(CA' - C'A)^2 + (BC' - B'C)(AB' - A'B) = 0;$$

the left-hand side is here a function of θ of the degree 10 vanishing when $\theta + x + y = 0$, and which must therefore be, save as to a numerical factor, the product $\Pi_{10}(\theta + x + y)$. And we thus find

$$\begin{aligned}
&\Pi_{10}(\theta + x + y)\\
&= -\left\{24\theta^5 - 48B\theta^4 + \begin{pmatrix} 56C \\ +16B^2 \end{pmatrix}\theta^3 + \begin{pmatrix} -72D \\ -24BC \end{pmatrix}\theta^2 + \begin{pmatrix} 64E \\ +32BD \end{pmatrix}\theta + \begin{pmatrix} -32BE \\ +32F \end{pmatrix}\right\}^2\\
&\quad + \left\{40\theta^3 - 48B\theta^2 + \begin{pmatrix} 8C \\ +16B^2 \end{pmatrix}\theta + \begin{pmatrix} 8D \\ -8BC \end{pmatrix}\right\} \cdot \left\{40\theta^7 - 112B\theta^6 + \begin{pmatrix} 136C \\ +80B^2 \end{pmatrix}\theta^5\right.\\
&\quad \left. + \begin{pmatrix} -120D \\ -200BC \end{pmatrix}\theta^4 + \begin{pmatrix} 0E \\ +192BD \\ +128C^2 \end{pmatrix}\theta^3 + \begin{pmatrix} 320F \\ -64BE \\ -526CD \end{pmatrix}\theta^2 + \begin{pmatrix} -256BF \\ +128CE \\ -128D^2 \end{pmatrix}\theta + \begin{pmatrix} 128CF \\ -128DE \end{pmatrix}\right\},
\end{aligned}$$

which is

$$= 1024\theta^{10} + \ldots + 1024(-F^2 + CDF + 2BEF - BC^2F - D^2E + BCDE),$$

and which therefore for $B = 0$ gives

$$\Pi_{10}(x + y) = -F^2 + CDF - D^2E.$$

73. Suppose now x, y, z, w, t, u are the roots of a sextic equation, say

$$\lambda - x.\lambda - y.\lambda - z.\lambda - w.\lambda - t.\lambda - u = (1, B, C, D, E, F, G \between \lambda, 1)^6 = 0.$$

Considering here the product $\Pi_{20}(x + y + z)$ of the sums of 3 roots, if $B = 0$, this will be a perfect square (for each sum $x + y + z$ is equal to $-$ a sum $(w + t + u)$) say it is the square of $\Pi_{10}(x + y + z)$, where the $x + y + z$ refers to the ten sums each containing x, and we wish to find this function $\Pi_{10}(x + y + z)$. Writing for the equation whose roots are y, z, w, t, u,

$$\lambda - y.\lambda - z.\lambda - w.\lambda - t.\lambda - u = (1, B', C', D', E', F' \between \lambda, 1)^5,$$

we have by what precedes $\Pi_{10}(\theta+y+z)=$ a function $(*\between\theta, 1)^{10}$, viz. this is the above-mentioned function with B', C', D', E', F' in place of the unaccented letters. Introducing a new root x and for λ writing as we may do θ, we have

$$\begin{aligned}\theta-x.\theta-y.\theta-z.\theta-w.\theta-t.\theta-u &= (\theta-x).(1, B', C', D', E', F'\between\theta, 1)^5\\ &= (1, B, C, D, E, F, G\between\theta, 1)^6;\end{aligned}$$

that is, we have

$$\begin{aligned}
B &= B' - \theta \quad \text{or conversely,} & B' &= B + \theta,\\
C &= C' - B'\theta & C' &= C + B\theta + \theta^2,\\
D &= D' - C'\theta & D' &= D + C\theta + B\theta^2 + \theta^3,\\
E &= E' - D'\theta & E' &= E + D\theta + C\theta^2 + B\theta^3 + \theta^4,\\
F &= F' - E'\theta & F' &= F + E\theta + D\theta^2 + C\theta^3 + B\theta^4 + \theta^5, \quad = -\frac{G}{\theta},\\
G &= -F'\theta,
\end{aligned}$$

where I have retained B, but the value hereof is in fact $=0$. In the foregoing function $(*\between\theta, 1)^{10}$ with the accented letters, writing for these their values $B'=\theta$, $C'=C+\theta^2$, $D'=D+C\theta+\theta^3$, &c., which belong to $B=0$, we find

$$\begin{aligned}1024\Pi_{10}(\theta+y+z) &= -(48\theta^6+56C\theta^3+24D\theta^2+64E\theta+32F)^2\\ &\quad+(16\theta^3+8C\theta+D)\{144\theta^7+264C\theta^5+72D\theta^4+128(C^2+E)\theta^3+192F\theta^2+128CE\theta\\ &\qquad\qquad+128(CF-DE)\},\end{aligned}$$

which equation divides by 64. Writing herein $\theta=x$, we have

$$\begin{aligned}16\Pi_{10}(x+y+z) &= -(6x^5+7Cx^3+3Dx^2+8Ex+4F)^2\\ &\quad+(2x^3+Cx+D)\{18x^7+33Cx^5+9Dx^4+16(C^2+E)x^3+24Fx^2+16CEx+16(CF-DE)\},\end{aligned}$$

where $x^6+Cx^4+Dx^3+Ex^2+Fx+G$ is $=0$: the value ought, in virtue of this equation, to reduce itself to a mere function of the coefficients, and we in fact find that the equation is

$$16\Pi_{10}(x+y+z)=(16C^2-64E)(x^6+Cx^4+Dx^3+Ex^2+Fx)+16CDF-16D^2E-16F^2,$$

reducing itself to

$$-(16C^2-64E)G \qquad\qquad +16CDF-16D^2E-16F^2,$$

viz. dividing each side by 16, we have

$$\Pi_{10}(x+y+z)=4EG-C^2G-F^2+CDF-D^2E,$$

which is the required result. The equation $(\theta^2-1)^3=0$, for which

$$x, y, z, w, t, u = 1, 1, 1, -1, -1, -1,$$

gives a numerical verification.

74. I find also, for the same value $B=0$, the function $\Pi_{15}(x+y)$. Writing, as before,

$$x=-\tfrac{1}{2}(\theta+\omega),$$
$$y=-\tfrac{1}{2}(\theta-\omega),$$

and therefore

$$\theta + x + y = 0,$$

we have

$$(\theta+\omega)^6 + 4C(\theta+\omega)^4 - 8D(\theta+\omega)^3 + 16E(\theta+\omega)^2 - 32F(\theta+\omega) + 64G = 0,$$

and the like equation with $-\omega$ for ω. Hence writing $\omega^2 = M$, we have

$$(\theta^6 + 4C\theta^4 - 8D\theta^3 + 16E\theta^2 - 32F\theta + 64G) + M(15\theta^4 + 24C\theta^2 - 24D\theta + 16E)$$
$$+ M^2(15\theta^2 + 4C) + M^3 = 0,$$

$$(6\theta^5 + 16C\theta^3 - 24D\theta^2 + 32E\theta - 32F) + M(20\theta^3 + 16C\theta - 8D) + M^2 . 6\theta = 0,$$

say these equations are $aM^3 + bM^2 + cM + d = 0$, $pM^2 + qM + r = 0$. Eliminating M, we have

$$\begin{array}{ll}
a^2 . r^3 & a = 1, \\
-ab . qr^2 & b = 15\theta^2 + 4C, \\
+ac(-2pr^2 + q^2r) & c = 15\theta^4 + 24C\theta^2 - 24D\theta + 16E, \\
+b^2 . pr^2 & d = \theta^6 + 4C\theta^4 - 8D\theta^3 + 16E\theta^2 - 32F\theta + 64G, \\
+ad(3pqr - q^3) & \\
+bc(-pqr) & p = 6\theta, \\
+bd(-2p^2r + pq^2) & q = 20\theta^3 + 16C\theta - 8D, \\
+c^2 . p^3r & r = 6\theta^5 + 16C\theta^3 - 24D\theta^2 + 32E\theta - 32F, \\
-cd . p^2q & \\
+d^2 . p^3 = 0. &
\end{array}$$

The equation, as far as I have calculated it, is

$$-32768\theta^{15} - \ldots - 32768(-D^3G + F^3 - CDF^2 + D^2EF) = 0;$$

the left-hand side is here $= -32768\Pi_{15}(x+y)$; and we have therefore

$$\Pi_{15}(x+y) = -D^3G + F^3 - CDF^2 + D^2EF,$$

the required result. It may be remarked that, writing $G = 0$ and throwing out a factor $-F$, we have $-F^2 + CDF - D^2E$, which is the expression for $\Pi_{10}(x+y)$ in the quintic equation.

We have

$$\Pi_6 x \Pi_{15}(x+y)\,\Pi_{10}(x+y+z)$$
$$= G\{-D^3G + (F^2 - CDF + D^2E)F\}\{(4E - C^2)G - F^2 + CDF - D^2E\},$$

the developed expression whereof is the foregoing value

$$M = D^4E^2FG - 2CD^3EF^2G + \&c., \textit{ante} \text{ No. } 71.$$

The Operators $P-\delta b$ and $Q-2\omega b$. Art. Nos. 75 to 84.

75. The analogous theory for non-unitariants is established, *ante* Nos. 24 *et seq.* For seminvariants, we have

$$P = b\partial_a + c\partial_b + d\partial_c + \ldots,$$
$$Q = c\partial_b + 2d\partial_c + \ldots,$$

or more definitely, if the seminvariant operated upon be of the degree δ, the weight ω and extent σ, say its highest letter is a_σ, $=p$, then

$$P = b\partial_a + c\partial_b + d\partial_c + \ldots + q\partial_p,$$
$$Q = \qquad c\partial_b + 2d\partial_c + \ldots + \sigma q\partial_p,$$

then we have

$$P-\delta b, \quad Q-2\omega b,$$

operators each of them of the deg. weight 1.1, viz. each of them operating upon a seminvariant S of the deg. weight $\delta.\omega$ gives a seminvariant S' of the deg. weight $\delta+1.\omega+1$; moreover, a new letter q is introduced, or say the extent is increased from σ to $\sigma+1$. For the proof, it is only necessary to show that $\Delta(P-b\delta)S$ and $\Delta(Q-2b\omega)$ are each $=0$, but it is unnecessary to do this, as the like proof has already been given for non-unitariants.

The two seminvariant operators were first considered in my paper "On a theorem relating to seminvariants," *Quart. Math. Journ.* t. xx. (1885), pp. 212, 213, [844].

76. We may, instead of $P-\delta b$ and $Q-2\omega b$, consider the linear combination $Y=2\omega(P-\delta b)-\delta(Q-2\omega b)$, that is, $2\omega P-\delta Q$, which is of deg. weight 0.1, viz. it leaves the degree unaltered, while increasing as before the weight, and also the extent, each by unity. And again, the combination

$$Z=\sigma(P-\delta b)-(Q-2\omega b),$$

that is,

$$\sigma P-Q-(\sigma\delta-2\omega)b,$$

where observe that $\sigma P-Q$, $=\sigma b\partial_a+(\sigma-1)c\partial_b+\ldots+1p\partial_o$ does not contain the new letter q; the operator Z is thus of the deg. weight 1.1 increasing the degree and also the weight each by unity, but leaving the extent unaltered.

There is a special case which it is important to attend to, we may have $\sigma\delta-2\omega=0$, viz. this is the case when the seminvariant operated upon is in regard to the letters comprised therein an invariant. Here the two combinations Y, Z are equivalent to each other, each of them is $=\sigma b\partial_a+(\sigma-1)c\partial_b+\ldots+1p\partial_o$, which is an annihilator of the seminvariant (invariant) operated upon. Hence in this case we cannot replace the original forms by the linear combinations, but must retain one (no matter which) of the original forms $P-\delta b$, $Q-2\omega b$.

77. We can, by means of the foregoing operators, starting from the quadric seminvariants $c-b^2$, &c., derive in order the seminvariants for the successive weights 3, 4, 5, ...

Thus writing down the series of finals (in AO as before),

$$\begin{array}{ccccccc}
b^2, & b^3, & c^2, & bc^2, & d^2, & bd^2, & e^2, \ \&c., \\
 & & b^4 & b^5 & c^3 & bc^3 & cd^2 \\
 & & & & b^2c^2 & b^2c^2 & b^2d^2 \\
 & & & & b^6 & b^7 & c^4 \\
 & & & & & & b^2c^3 \\
 & & & & & & b^4c^2 \\
 & & & & & & b^8,
\end{array}$$

I proceed as follows, observing, however, that when the function operated upon is an invariant seminvariant we must instead of Z write $P-\delta b$.

$$\begin{array}{ccccccc}
b^2 \text{ emerges}, & b^3 = Zb^2, & c^2 \text{ emerges}, & bc^2 = Zc^2, & d^2 \text{ emerges}, & bd^2 = Zd^2, & e^2 \text{ emerges}, \\
 & b^4 = Zb^3 & & b^5 = Zb^4 & c^3 = Ybc^2 & bc^3 = Zc^3 & cd^2 = Ybd^2 \\
 & & & & b^2c^2 = Zbc^2 & b^2c^2 = Zb^2c^2 & b^2d^2 = Zbd^2 \\
 & & & & b^6 = Zb^5 & b^7 = Zb^6 & c^4 = Ybc^2 \\
 & & & & & & b^2c^3 = Zbc^2 \\
 & & & & & & b^4c^2 = Zb^2c^2 \\
 & & & & & & b^8 = Zb^7,
\end{array}$$

viz. whenever the seminvariant to be obtained has a final containing b, it is obtained by means of the operator Z (or it may be $P-\delta b$), but when there is no b then by the operator Y.

The seminvariants operated upon may be blunt or sharp, but there is an advantage in operating on the sharp forms as these are more simple, and we thereby obtain for the next superior weight forms more nearly approximating to the sharp forms. We do not however by thus operating on a sharp form obtain directly a sharp form; to do this, the form obtained must be modified by adding thereto a numerical multiple or multiples of a preceding sharp form: and thus the theory does not determine beforehand the forms of the sharp seminvariants. But making at each step the necessary modification (if any) we have thereby, when the sharp seminvariants of the next preceding weight are known, a very convenient process for the calculation of the sharp seminvariants of any given weight, in the AO arrangement of their final terms. Thus for the weight 10; $k \infty f^2$ is taken to be known, the next two forms $ci \infty ce^2$ and $dh \infty b^2e^2$ are calculated each from $j \infty be^2$, the expression for which is

$$= j - 9bi + 20ch - 28dg + 14ef + 16b^2h - 56bcg + 112bdf - 70be^2.$$

We have for $j \infty be^2$, $\delta = 3$, $\omega = 9$, $\sigma = 9$: and therefore

$$\tfrac{1}{3}Y = 6b\partial_a + 5c\partial_b + 4d\partial_c + 3e\partial_d + 2f\partial_e + g\partial_f - i\partial_h - 2j\partial_i - 3k\partial_j,$$

$$Z = 9b\partial_a + 8c\partial_b + 7d\partial_c + 6e\partial_d + 5f\partial_e + 4g\partial_f + 3h\partial_g + 2i\partial_h + j\partial_i - 9b.$$

78. I exhibit the calculation as follows:

$\frac{1}{3}Y(j \infty be^2)$

		1	2	3	4	5	6	7	8	9	†		÷ 70	*
k	+ 1									−3	− 3	+ 3	0	
bj	− 10	+ 12							+18		+ 30	− 30	0	
ci	+ 45		− 45					−20			− 65	+135	+ 70	+ 1
dh	−120			+ 80							+ 80	−360	−280	− 4
eg	+210				− 84		+ 14				− 70	+630	+560	+ 8
f^2	−126					+ 28					+ 28	−378	−350	− 5
b^2i		− 54						−16			− 70		− 70	− 1
bch		+120	+160								+280		+280	+ 4
bdg		−168		−224			+112				−280		−280	− 4
bef		+ 84			+336	−280					+140		+140	+ 2
c^2g			−280								−280		−280	− 4
cdf			+560								+560		+560	+ 8
ce^2			− 350								− 350		− 350	− 5
d^2e														
b^3h														
b^2eg														
b^2df														
b^2e^2														

±256 ±23

$Z(j \infty be^2)$

	1	2	3	4	5	6	7	8	9	10	†		÷ −18	*
k														
bj	+ 18								−9	− 9	0		0	
ci		− 72						+40			− 32	+ 32	0	
dh			+140				− 84				+ 56	−128	− 72	+ 4
eg				−168		+ 56					− 112	+256	+ 144	− 8
f^2					+ 70						+ 70	−160	− 90	+ 5
b^2i	− 81							+32		+ 81	+ 32	− 32	0	0
bch	+180	+256					−168			− 180	+ 88	+128	+ 216	− 12
bdg	−252		−392			+448				+ 252	+ 56	−128	− 72	+ 4
bef	+126			+672	−700					− 126	− 28	+ 64	+ 36	− 2
c^2g		−448									− 448	−128	− 576	+ 32
cdf		+896									+ 896	+256	+1152	− 64
ce^2		−560									− 560	−160	− 720	+ 40
d^2e														
b^3h										− 144	− 144		− 144	+ 8
b^2eg										+ 504	+ 504		+ 504	− 28
b^2df										−1008	−1008		−1008	+ 56
b^2e^2										+ 630	+ 630		+ 630	− 35

±149

The numbers (1, 2, ..., 9) and (1, 2, ..., 10) at the head of the columns refer to the nine terms $6b\partial_a$, $5c\partial_b$, ... of $\frac{1}{3}Y$, and the ten terms $9b\partial_a$, $8c\partial_b$, ... of Z respectively, these several operations being performed on $(j \infty be^2)$ the value of which is given above: the daggers † denote the additions which have to be made in order to obtain the proper initial term, viz. for the first † the added term is $+3(k \infty f^2)$ and for the second † the added term is $+32(ci \infty ce^2)$: the headings $\div 70$ and $\div -18$ explain themselves, and the columns headed with an asterisk * give the results, viz. the first of these is $(ci \infty ce^2)$ and the second of them is $(dh \infty b^2e^2)$. As appears above, the value of the first of these is used in the second † column for obtaining that of the second of them.

79. We may operate with $P-\delta b$ and $Q-2\omega b$ on a product (deg. weight $\delta.\omega$) ST of two seminvariants S, T, deg. weights $\delta'.\omega'$ and $\delta''.\omega''$ respectively, $\delta=\delta'+\delta''$, $\omega=\omega'+\omega''$. We have

$$(P-\delta b)\,ST = S.PT + T.PS - (\delta'+\delta'')\,bST, \quad = S(P-\delta''b)\,T + T(P-\delta'b)\,S,$$

where $(P-\delta'b)\,S$ and $(P-\delta''b)\,T$ are each of them a seminvariant. And similarly,

$$(Q-2\omega b)\,ST = S.QT + T.QS - 2(\omega'+\omega'')\,bST = S(Q-2\omega''b)\,T + T(Q-2\omega'b)\,S,$$

where $(Q-2\omega'b)\,S$ and $(Q-2\omega''b)\,T$ are each of them a seminvariant. That is, operating either with $P-\delta b$ or $Q-2\omega b$ on a product, we have a sum of products; and therefore also operating upon a sum of products (each product being of the deg. weight $\omega.\delta$) we have a sum of products, each product in such sum being of the deg. weight $\omega+1.\delta+1$, and moreover of the extent $\sigma+1$. And instead of binary products, we may, it is clear, consider ternary, quaternary, &c., products.

The like theorem applies to the derived operators Y and Z, but as to Y there is a specialty to be noticed. We have

$$\begin{aligned} Y.ST &= 2\omega(P-\delta b)\,ST - \delta(Q-2\omega b)\,ST, \\ &= 2\omega\{S(P-\delta''b)\,T + T(P-\delta'b)\,S\} - \delta\{S(Q-2\omega''b)\,T + T(Q-2\omega'b)\,S\}, \\ &= S\{2\omega(P-\delta''b)\,T - \delta(Q-2\omega''b)\,T\} + T\{2\omega(P-\delta'b)\,T - \delta(Q-2\omega'b)\,S\}, \end{aligned}$$

where the whole of the right-hand side as being equal to $Y.ST$ is of the degree δ, but except in the particular case $\left(\frac{\delta}{\omega}=\frac{\delta'}{\omega'}=\frac{\delta''}{\omega''}\right)$ the separate products $S\{\ \}$ and $T\{\ \}$ which occur on the right-hand side are each of them of the degree $\delta+1$.

It is scarcely necessary, but it may be proper, to remark that we frequently combine by addition a seminvariant S of the deg. weight $\delta.\omega$ with a seminvariant T deg. weight $\delta-\epsilon.\omega$ of the same weight but of an inferior degree, but when this is done we regard the T as standing for $a^\epsilon T$, and as being thus of the same deg. weight $\delta.\omega$. We have

$$(P-\delta b)\,a^\epsilon T = a^\epsilon PT + TPa^\epsilon - (\epsilon+\delta-\epsilon)\,ba^\epsilon T, \quad = a^\epsilon\{P-(\delta-\epsilon)\,b\}\,T + T(P-\epsilon b)\,a^\epsilon,$$

where $(P-\epsilon b)\,a^\epsilon=(\epsilon-\epsilon)\,b=0$, and consequently $(P-b\delta)\,a^\epsilon T=\{P-(\delta-\epsilon)\,b\}\,T$; viz. for the operation upon T, we regard $P-\delta b$ as standing for $P-(\delta-\epsilon)\,b$. As regards Q, we have $(Q-2\omega b)\,a^\epsilon T=(Q-2\omega b)\,T$; viz. the degree of T does not here present itself.

80. We may write

$$(2\omega P-\delta Q)\,S=S',$$

the new seminvariant S' being of the weight $\omega+1$; hence also

$$\{(2\omega+2)\,P-\delta Q\}\,.\,\{2\omega P-\delta Q\}\,S=S'',$$

where S'' is of the weight $\omega+2$; viz. we have an operator

$$\{(2\omega+2)\,P-\delta Q\}\,.\,\{2\omega P-\delta Q\},$$

which, operating on a seminvariant of the deg. weight $\delta\,.\,\omega$, gives a seminvariant of the deg. weight $\delta\,.\,\omega+2$. This is

$$=(4\omega^2+4\omega)\,(P^2+P\,.\,P)-(2\omega+2)\,\delta\,(PQ+P\,.\,Q)-2\omega\delta\,(QP+Q\,.\,P)+\delta^2\,(Q^2+Q\,.\,Q),$$

where P^2, PQ, QP and Q^2 are the mere algebraical squares and products, while $P\,.\,Q$ and $Q\,.\,P$ denote respectively P operating on Q and Q operating on P; and since $PQ=QP$, this is

$$=(4\omega^2+4\omega)\,(P^2+P\,.\,P)-(4\omega+2)\,\delta PQ-2\,(\omega+2)\,\delta P\,.\,Q-2\omega\delta Q\,.\,P+\delta^2\,(Q^2+Q\,.\,Q).$$

Recollecting that

$$P=b\partial_a+c\partial_b+d\partial_c+\ldots,\quad Q=c\partial_b+2d\partial_c+\ldots,$$

we have

$$\begin{aligned}
P\,.\,P&=c\partial_a+\ d\partial_b+\ e\partial_c\ \ +\ldots,\\
P\,.\,Q&=\qquad\ \ d\partial_b+2e\partial_c\ \ +\ldots,\\
Q\,.\,P&=c\partial_a+2d\partial_b+3e\partial_c\ \ +\ldots,\ =P\,.\,P+P\,.\,Q,\\
Q\,.\,Q&=\ \ 1\,.\,2d\partial_b+2\,.\,3e\partial_c+\ldots,
\end{aligned}$$

and attending to the relation just obtained $Q\,.\,P=P\,.\,P+P\,.\,Q$, we find that the operator may be written

$$\begin{aligned}
&(4\omega^2+4\omega)\,\{P^2-(\delta-1)\,P\,.\,P\}\\
&-(4\omega+2)\,\delta\,\{PQ-\omega P\,.\,P-\tfrac{1}{3}\,(\delta-3)\,P\,.\,Q\}\\
&+\delta^2\,\{Q^2+Q\,.\,Q-\tfrac{1}{3}\,(4\omega+2)\,P\,.\,Q\};
\end{aligned}$$

in fact, here the terms in P^2, PQ, Q^2 are in the original form, while those in $P\,.\,P$, $P\,.\,Q$, $Q\,.\,Q$ are

$$\begin{aligned}
(4\omega^2+4\omega)\,(1-\delta)\,P\,.\,P+(4\omega^2+2\omega)\,\delta P\,.\,P-\tfrac{1}{3}\,(4\omega+2)\,(\delta^2-3\delta)\,P\,.\,Q+\delta^2 Q\,.\,Q\\
+\tfrac{1}{3}\,(4\omega+2)\,\delta^2\qquad P\,.\,Q,
\end{aligned}$$

which are

$$=(4\omega^2+4\omega-2\omega\delta)\,P\,.\,P-(4\omega+2)\,\delta P\,.\,Q+\delta^2 Q\,.\,Q,$$

agreeing with the original form

$$(4\omega^2+4\omega)\,P\,.\,P-(2\omega+2)\,\delta P\,.\,Q-2\omega\delta\,(P\,.\,P+P\,.\,Q)+\delta^2 Q\,.\,Q.$$

81. I find that each of the three parts is separately an operator, viz. that we have

$$P^2-(\delta-1)\,P\,.\,P,$$

$$PQ-\omega P\,.\,P-\tfrac{1}{3}(\delta-3)\,P\,.\,Q,$$

$$Q^2+Q\,.\,Q-\tfrac{1}{3}(4\omega+2)\,P\,.\,Q,$$

each of them an operator which, operating on a seminvariant of deg. weight $\delta\,.\,\omega$, gives a seminvariant of deg. weight $\delta\,.\,\omega+2$.

I verify this for the first of the three operators, say

$$\Omega=P^2-(\delta-1)\,P\,.\,P=P^2+P\,.\,P-\delta\Theta,$$

if for a moment

$$P\,.\,P,\ =c\partial_a+d\partial_b+e\partial_c+\ldots$$

is put $=\Theta$.

Here for a seminvariant S, we have

$$\Omega S=(P^2+P\,.\,P-\delta\Theta)\,S=P\,(PS)-\delta\Theta S.$$

Writing $S'=(aP-b\delta)\,S$, then S' is a seminvariant, degree $=\delta+1$, and then if $S''=(aP-b\,(\delta+1))\,S'$, S'' is a seminvariant, degree $=\delta+2$. We have $PS=a^{-1}(S'+b\delta S)$, and thence

$$\Omega S=Pa^{-1}(S'+b\delta S)-\delta\Theta S,\ =-b\,(S'+b\delta S)+P\,(S'+b\delta S)-\delta\Theta S.$$

Here

$$P\,(S'+b\delta S)=PS'+c\delta S+b\delta PS,\ =S''+b\,(\delta+1)\,S'+c\delta S+b\delta\,(S'+b\delta S),$$

and hence

$$\Omega S=S''+2b\delta S'+\{c\delta+b^2(\delta^2-\delta)\}\,S-\delta\Theta S.$$

This will be a seminvariant if $\Delta\,.\,\Omega S=0$; we have

$$\begin{aligned}\Delta\,.\,\Omega S=\Delta S''+2b\delta\Delta S'+\{c\delta\ +\ b^2(\delta^2-\delta)\}\,\Delta S-\delta\,(\Delta\Theta+\Delta\,.\,\Theta)\,S\\ +2\delta S'\quad+\{2b\delta+2b\ (\delta^2-\delta)\}\,S,\end{aligned}$$

or, omitting the terms in $\Delta S''$, $\Delta S'$, ΔS which respectively vanish, this is

$$=2\delta S'+2b\delta^2S-\delta\,(\Delta\Theta+\Delta\,.\,\Theta)\,S.$$

But since $PS=S'+b\delta S$, and from $\Delta S=0$ we deduce $0=(\Theta\Delta+\Theta\,.\,\Delta)\,S$, the equation becomes

$$\Delta\,.\,\Omega S=2\delta PS-\delta\,(\Delta\,.\,\Theta-\Theta\,.\,\Delta)\,S,$$

and from

$$\Delta=\ a\partial_b+2b\partial_c+3c\partial_d+\ldots,$$

$$\Theta=\ c\partial_a+\ d\partial_b+\ e\partial_c+\ldots,$$

we have

$$\Delta\,.\,\Theta=2b\partial_a+3c\partial_b+4d\partial_c+\ldots,$$

$$\Theta\,.\,\Delta=\qquad\quad c\partial_b+2d\partial_c+\ldots,$$

and thence

$$\Delta . \Theta - \Theta . \Delta = 2b\partial_a + 2c\partial_b + 2d\partial_c + \ldots, = 2P,$$

and we have thus the required equation $\Delta . \Omega S = 0$.

82. If instead of P, Θ, we write B, C, so that

$$\begin{aligned} B &= b\partial_a + c\partial_b + d\partial_c + \ldots, \\ C = B . B &= c\partial_a + d\partial_b + e\partial_c + \ldots, \end{aligned}$$

and put further

$$\begin{aligned} D = B . C &= d\partial_a + e\partial_b + f\partial_c + \ldots, \\ E = B . D &= e\partial_a + f\partial_b + g\partial_c + \ldots, \end{aligned}$$

then the foregoing operator is $B^2 - (\delta - 1) C$, or reversing the sign, say it is $(\delta - 1) C - B^2$, which is the first of a series of operators

$$\begin{aligned} &(\delta - 1) C - B^2, \\ &(\delta - 1)(\delta - 2) D - 3(\delta - 2) BC \qquad + 2B^3, \\ &(\delta - 1)(\delta - 2)(\delta - 3) E - 4(\delta - 2)(\delta - 3) BD + 6(\delta - 3) B^2C - 3B^4, \\ &\vdots \end{aligned}$$

which are of the deg. weights 0 . 2, 0 . 3, 0 . 4, &c., respectively, viz. operating upon a seminvariant of deg. weight $\delta . \omega$ they leave the degree unaltered, but increase the weight by 2, 3, 4, ... respectively.

It is to be observed that B^2, BC, B^3, &c., denote the mere algebraical powers and products of the symbols B, C, D, &c., without any operation of one symbol on another.

As a simple illustration, take $(C - B^2)(ac - b^2)$: here,

$$\begin{aligned} C(ac - b^2) &= e - 2bd + c^2 \\ -B^2 = -(2bd\partial_a\partial_c + c^2\partial_b^2)(\quad ,, \quad) &\quad - 2bd + 2c^2 \\ \text{Value is} &\quad e - 4bd + 3c^2; \end{aligned}$$

and similarly for $(C - B^2)(ae - 4bd + 3c^2)$, here:

$$\begin{aligned} C(ae - 4bd + 3c^2) &= g - 4bf + (6 + 1) ce - 4d^2 \\ -B^2 = -(2bf\partial_a\partial_e + 2ce\partial_b\partial_d + d^2\partial_c^2)(\quad ,, \quad) &\quad - 2bf + 8 ce - 6d^2 \\ \text{Value is} &\quad g - 6bf + 15 ce - 10d^2. \end{aligned}$$

A direct proof may of course be obtained for any one of the foregoing operators; viz. calling it Ω, it may be shown that $\Delta\Omega S$ is $= 0$. I have not considered the like question of the derivation of series of operators from the other two forms

$$PQ - \omega P . P - \tfrac{1}{3}(\delta - 3) P . Q \text{ and } Q^2 + Q . Q - \tfrac{1}{3}(4\omega + 2) P . Q$$

respectively.

83. I do not wish in the present paper to go into the theory of covariants, but it is nevertheless proper to point out the connexion which exists between the covariant theory of derivation and the operators P and Q.

Consider a quantic $(a, b, c, \ldots, a' = a_{\sigma'} \between x, y)^{\sigma'}$; any covariant hereof is $(A, B, C, \ldots \between x, y)^{\mu}$, where A is a seminvariant say of degree δ and weight, $\omega = \frac{1}{2}(\sigma'\delta - \mu)$, or $\mu = \sigma'\delta - 2\omega$, reduced to zero by the operation $\Delta = a\partial_b + 2b\partial_c + \ldots + \sigma' b'\partial_{a'}$: and if we write

$$\phi_{\sigma'} = \sigma' b\partial_a + (\sigma' - 1)\, c\partial_b + \ldots + a'\partial_{b'},$$

then

$$B = \phi_{\sigma'} A, \quad C = \tfrac{1}{2}\phi_{\sigma'} B, \quad D = \tfrac{1}{3}\phi_{\sigma'} C, \ldots$$

The derivative (f, F) is

$$\begin{aligned} &= \partial_x f \,.\, \partial_y F - \partial_y f \,.\, \partial_x F \\ &= (a, b, \ldots \between x, y)^{\sigma'-1} B x^{\mu-1} \quad + \ldots \\ &\quad - (b, c, \ldots \between x, y)^{\sigma'-1} \mu A x^{\mu-1} + \ldots \\ &= (aB - \mu b A, \ldots \between x, y)^{\sigma'+\mu-2}, \end{aligned}$$

that is, A being a seminvariant, we have $aB - \mu bA$ a seminvariant, or say

$$(\phi_{\sigma'} - \mu' b)\, A = \text{sem. } \mu' = \sigma'\delta - 2\omega;$$

and similarly

$$(\phi_{\sigma} - \mu b)\, A = \text{sem. } \mu = \sigma\delta - 2\omega.$$

Hence

$$\{\phi_\sigma - \phi_{\sigma'} - (\mu - \mu')\, b\}\, A, \text{ and } \{\sigma'\phi_\sigma - \sigma\phi_{\sigma'} - (\sigma'\mu - \sigma\mu')\, b\}\, A$$

are each of them a seminvariant: but

$$\phi_\sigma = \sigma b\partial_a + (\sigma - 1)\, c\partial_b + \ldots,$$

$$\phi_{\sigma'} = \sigma' b\partial_a + (\sigma' - 1)\, c\partial_b + \ldots,$$

$$\phi_\sigma - \phi_{\sigma'} = (\sigma - \sigma')(b\partial_a + c\partial_b + \ldots) = (\sigma - \sigma')\, P, \quad \mu - \mu' = (\sigma - \sigma')\, \delta;$$

the first form, omitting the factor $\sigma - \sigma'$, is $= (P - \delta b)\, A$: similarly

$$\sigma'\phi_\sigma - \sigma\phi_{\sigma'} = (\sigma - \sigma')(c\partial_b + 2d\partial_c + \ldots) = (\sigma - \sigma')\, Q$$

and

$$\sigma'\mu - \sigma\mu' = (\sigma - \sigma')\, 2\omega,$$

and the second form is

$$= (Q - 2\omega b)\, A.$$

We thus see that the operators $P - \delta b$ and $Q - 2\omega b$ upon a seminvariant A depend on the derivation of f upon a covariant which has A for its leading coefficient: the order of f is arbitrary, and we have thus two distinct forms.

84. As an illustration, consider the quantics

$$(1, b, c, d, e, f \between x, y)^5,$$

and

$$(1, b, c, d, e, f, g \between x, y)^6:$$

each of these has a covariant the leading coefficient of which is

$$A = f - 5be + 2cd + 8b^2d - 6bc^3,$$

viz. these are

$$\left(\begin{array}{|c|c|c|}\hline f\ +1 & bf\ +5 & \ldots \\ be\ -5 & ce\ -16 & \\ cd\ +2 & d^2\ +6 & \\ b^2d\ +8 & b^2e\ -9 & \\ bc^2\ -6 & bcd\ +38 & \\ & c^3\ -24 & \\ \hline \pm 11 & \pm 49 & \\ \hline\end{array}\right. \mathfrak{X}(x, y)^5, \text{ and } \left(\begin{array}{|c|c|c|}\hline f\ +1 & g\ +1 & \ldots \\ be\ -5 & bf\ +2 & \\ cd\ +2 & ce\ -19 & \\ b^2d\ +8 & d^2\ +8 & \\ bc^2\ -6 & b^2e\ -6 & \\ & bcd\ +44 & \\ & c^3\ -30 & \\ \hline \pm 11 & \pm 55 & \\ \hline\end{array}\right. \mathfrak{X}(x, y)^8,$$

and we find without difficulty

$$\begin{array}{lcrr} & & (g \infty d^2)(ce \infty c^3) & (d^2 \infty b^2c^2) \\ \hline (f_1, F_1) & = & -16 & -10 \\ (f_2, F_2) & = 1 & -34 & -16 \\ (P - 3b)A & = 1 & -18 & -6 \\ (Q - 10b)A & = 5 & -74 & -20 \end{array}$$

and thence

$$(P - 3b)A = (f_2, F_2) - (f_1, F_1)$$
$$(Q - 10b)A = 5(f_2, F_2) - 6(f_1, F_1),$$

viz. we thus have $P - 3b$, and $Q - 10b$ upon $f \infty bc^2$ each given as a linear function of the derivatives (f_1, F_1) and (f_2, F_2), where f_1, f_2 are the quintic and the sextic function, and F_1, F_2 are like covariants of these functions respectively.

$w=11$

	bf^2	de^2	bce^2	cd^3	b^2e^2	b^2d^3	bc^2d^2	b^2cd^2	bc^5	b^2d^2	b^2c^4	b^2c^3	b^2c^2	b^{11}
l	1													
cj		1												
di			1											
eh				1										
c^2h					1									
fg						1								
cdg							2							
cef								1						
d^2f									1					
c^3f										1				
de^2											1			
c^2de												1		
cd^3													1	
c^4d														1

$w=12$

	g^2	cf^2	e^3	b^2f^2	bde^2	c^2e^2	d^4	b^2ce^2	bcd^2	c^3d^2	b^2e^2	b^2d^2	$b^2c^2d^2$	c^6	b^2cd^2	b^2c^5	b^2d^2	b^2c^4	b^2c^3	b^2c^2	b^{12}
m	1																				
ck		1																			
dj			2																		
ei			1																		
c^2i								3													
fh					1																
cdh											4										
g^2						1															
ceg							1														
d^2g								1													
c^3g															4						
cf^2										1											
def												1									
c^2df																			3		
e^2													1								
c^2e^2														1							
cd^2e																1					
c^4e																				2	
d^4																			1		
c^3d^2																					1
c^6																					1

[The Tables for $w = 13$, 14, 15, 16 are given on the accompanying lithographed sheet.]

$w' = 13$

	bg^3	df^3	bcf^2	be^3	cde^2	b^3f^2	b^2de^2	bc^2e^2	bd^4	c^2d^3	b^2ce^2	b^2cd^3	bc^2d^2	b^2e^2	b^2d^3	$b^2c^2d^2$	bc^6	b^2cd^3	b^2c^3	b^2d^2	b^2c^4	b^2c^3	b^2c^2	b^{13}
n	1																							
cl		1																						
dk			1																					
ej				1																				
c^2j						2																		
fi					1																			
cdi											5													
gh							1																	
ceh								1																
d^2h									1															
c^2h														4										
cfg										1														
deg												1												
c^2dg																		5						
df^2													1											
e^2f															1									
c^2ef																1								
cd^2f																	1							
c^4f																			2					
cde^2																				1				
d^3e																					1			
c^3de																						1		
c^3d^3																							1	
c^5d																								1

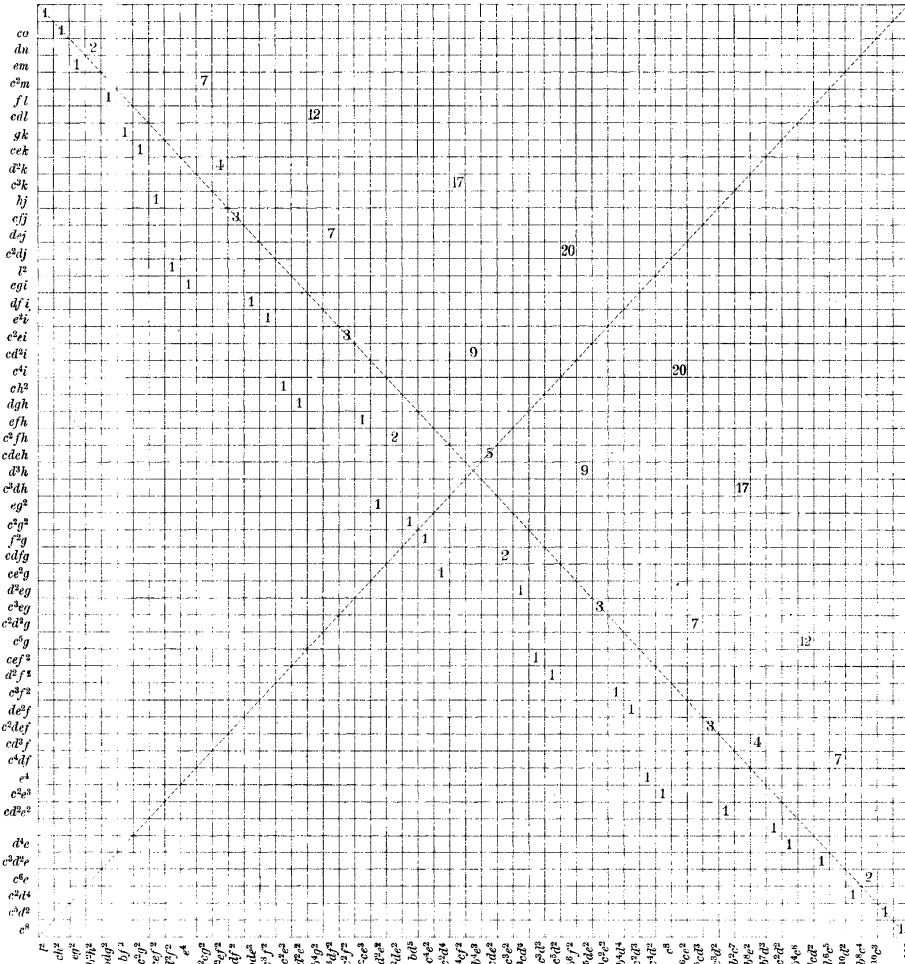
$w = 14$

	h^2	cfg^2	ef^{-1}	b^3g^{-2}	bdf^{-2}	c^2f^{-2}	ce^2	d^2e^2	b^2cf^{-2}	b^2d^2	$bcde^2$	c^3e^2	cd^4	b^2f^{-2}	$b^2d^2e^{-2}$	$b^2c^2e^{-2}$	b^2d^4	bcd^2	c^4d^2	b^2ce^2	b^2cd^2	$b^2c^2d^2$	c^7	b^2e^2	b^2d^2	$b^2c^2d^2$	b^2e^2	b^2d^2	b^2c^2	b^2d^2	b^2e^4	b^2e^2	b^2e^2	b^2d^2	l^{14}
o	1																																		
cm		1																																	
dl			2																																
ek				1																															
c^2k								5																											
fj					1																														
cdj													8																						
						1																													
cei							1																												
d^2i									2																										
																		10																	
h^2							1																												
cfh									1																										
deh													3																						
c^2dh									1																										
cg^2										1																									
dfg																																			
e^2g															1																				
c^2eg																1																			
cd^2g																			3																
c^4g																									8										
ef^2																	1																		
c^2f^2																		1																	
$cdef$																					1														
d^3f																								2											
c^3df																														5					
ce^3																						1													
d^2e^2																										1									
c^3e^2																											1								
c^2d^2e																												1							
c^5e																																2			
cd^4																															1				
c^4d^2																																	1		
c^7																																		1	
																																			1

$w = 15$

	cl^2	dg^2	f^3	bcj^2	bef^2	cdf^2	de^2	b^2gj	b^2df^2	bc^2f^2	bce^2	bd^2e^2	c^2de^2	d^5	b^3cf^2	b^2e^3	b^2cde^2	bc^2e^2	bcd^4	c^3d^2	b^2f^2	b^2de^2	$b^2c^2e^2$	b^2d^4	$b^2c^3d^2$	bc^4d^2	b^2ce^2	b^2cd^3	$b^2c^3d^2$	bc^7	b^2e^2	b^2d^2	$b^2c^2d^2$	b^2c^6	b^2cd^2	b^2c^5	b^2d^3	b^2c^4	b^2c^3	b^2c^2	b^{15}
p	1																																								
cn		1																																							
dm			2																																						
el				1																																					
c^2l								4																																	
fk					1																																				
cdk														9																											
gj						1																																			
cej							1																																		
d^2j									1																																
c^2j																					11																				
hi										1																															
cfi											1																														
dei														4																											
c^2di																											13														
cgh												1																													
dfh													1																												
e^2h														1																											
c^2eh																1																									
cd^2h																						4																			
																																				11					
dg^2																	1																								
efg																		1																							
c^2fg																			1																						
$cdeg$																							1																		
d^2g																												4													
c^3dg																																				9					
f^3																									1																
cdf^2																										1															
ce^2f																											1														
d^2ef																												1													
c^3ef																													1												
c^2d^2f																														1											
c^5f																																			4						
de^3																															1										
c^2de^2																																1									
cd^{32}																																	1								
c^4de																																					2				
d^5																																						1			
c^3d^3																																							1		
c^6d																																								1	

$w = 16$


co
dn
em
c²m
fl
cdl
gk
cek
d²k
c²k
hj
cfj
dcj
c²dj
l²
egi
dfi
e²i
c²ei
cd²i
c⁴i
ch²
dgh
efh
c²fh
cdeh
d³h
c³dh
eg²
c²g²
f²g
cdfg
ce²g
d²eg
c³eg
c²d²g
c⁵g
cef²
d²f²
c³f²
de²f
c²def
cd²f
c⁴df
e⁴
c²e³
cd²e²
d⁴e
c²d²e
c⁶e
c²d⁴
c⁸
f⁴
cf²
eg²
h²g²
bdg²
bf²
c²g²
cef²
df²
e⁴
b²cg²
b²ef²
bcdf²
bde²
c²f²
c²e²
cd²e²
b⁴g²
b²df²
b²c²f²
b²ce²
b²d²e²
bc²de²
bd³
c⁴e²
c³d²
b⁴f²
b⁴e²
b²cde²
b²c²e²
bc³d²
b²d³
c⁵d²
b⁶f²
b⁶de²
b⁴e²c²
b⁴d⁴
b²c²d²
b²c⁴d²
c⁸
b⁸e²
b⁷cd²
b⁶c²d²
b⁵c⁷
b⁸e²
b⁷d²
b⁶c²d²
b⁴e⁶
b⁸cd²
b⁸c⁵
b¹⁰d²
b⁸c⁴
b¹⁰c³
b¹⁶

933.

TABLES OF PURE RECIPROCANTS TO THE WEIGHT 8.

[From the *American Journal of Mathematics*, t. XV. (1893), pp. 75—77.]

IN the tabulation of Pure Reciprocants it is convenient to write $a=1$; we thus have for all the reciprocants of a given weight a single column of literal terms which (as in the Seminvariant Tables) I arrange in alphabetical order AO, and the several reciprocants have then each of them its own column of numerical coefficients: the form of the table is thus similar to that of the seminvariant table, the only difference being that for reciprocants the final terms are not in general power-enders: as in the seminvariant table, the columns of the table are arranged *inter se* with their final terms in AO. As remarked in my paper, "Corrected Seminvariant Tables for the Weights 11 and 12," *Amer. Math. Journ.*, t. XIV. (1892), pp. 195—200, [926], it is not in every case the top term of a column which should be regarded as the initial term; but to the extent 8, to which the reciprocant tables are here carried, this remark has no application.

I recall that the notation is the modified one employed by Halphen, and by Sylvester* in his 12th and subsequent lectures, viz. $a, b, c, d, \ldots$ denote

$$\tfrac{1}{2}\frac{d^2y}{dx^2},\ \tfrac{1}{6}\frac{d^3y}{dx^3},\ \tfrac{1}{24}\frac{d^4y}{dx^4},\ \tfrac{1}{120}\frac{d^5y}{dx^5},\ldots$$

respectively. As already noticed, a is put $=1$, but it is to be in the several terms restored in the proper powers so as to obtain for the reciprocant a homogeneous expression of a degree equal to the original degree of the final term; thus $d-3bc+2b^3$ is to be read as standing for $a^2d-3abc+2b^3$.

The ultimate verification of the expression for a pure reciprocant consists (as is known) in its annihilation by the operator

$$V=2a^2\partial_b+5ab\partial_c+(6ac+3b^2)\partial_d+(7ad+7bc)\partial_e+(8ae+8bd+4c^2)\partial_f+\&c.,$$

or, say

$$V=2\partial_b\ +5b\partial_c\ +(6c\ +3b^2)\partial_d+(7d\ +7bc)\partial_e+(8e\ +8bd+4c^2)\partial_f+\&c.;$$

[* *American Journal of Mathematics*, t. IX. (1887), p. 7.]

thus for the reciprocant $50e - 175bd + 28c^2 + 105b^2c$, the result obtained is

$$2(-175d + 210bc) + 5b(56c + 105b^2) + (6c + 3b^2)(-175b) + (7d + 7bc)(50),$$

or, collecting, this is

d	-350			$+350$	$\pm$	350
bc	$+420$	$+280$	-1050	$+350$	$\pm$	1050
b^3		$+525$	-525		$\pm$	525,

$= 0$, as it should be.

The tables are

c	$+4$
b^2	-5
	$+4$
	-5

d	$+1$
bc	-3
b^3	$+2$
	± 3

e	$+50$	
bd	-175	
c^2	$+28$	$+16$
b^2c	$+105$	-40
b^4		$+25$
	$+183$	$+41$
	-175	-40

f	$+10$	
be	-40	
cd	-12	$+4$
b^2d	$+65$	-5
bc^2	$+16$	-12
b^3c	-39	$+23$
b^5		-10
	± 91	± 27

g	$+14$			
bf	-63			
ce	-1350	$+800$		
d^2	$+1470$	-875	$+125$	
b^2e	$+1782$	-1000		
bcd	-4158	$+2450$	-750	
c^3	$+2130$	-1344	$+256$	$+64$
b^3d			$+500$	
b^2c^2		$+35$	$+165$	-240
b^4c			-300	$+300$
b^6				-125
	$+5576$	$+3250$	± 1018	$+364$
	-5508	-3254		-365

h	$+7$			
bg	-35			
cf	-539	$+560$		
de	$+605$	-650	$+50$	
b^2f	$+735$	-700		
bce	$+306$	-290	-150	
bd^2	-2135	$+2275$	-175	
c^2d	$+1001$	-1036	$+28$	$+16$
b^3e	-1485	$+1500$	$+100$	
b^3cd	$+3465$	-3710	$+630$	-40
bc^3	-1295	$+1988$	-84	-48
b^4d			-350	$+25$
b^3c^2		$+63$	-259	$+152$
b^5c			$+210$	-155
b^7				$+50$
	$+6119$	± 6386	± 1018	± 243
	-5489			

i	+ 420						
bh	− 2310						
cg	− 32704	+ 1176					
df	+ 57750	− 8085	+ 20433				
e^2	− 20460	+ 7040	− 21542	+ 625			
b^2g	+ 45500	− 1470					
bcf	− 28392	+ 18963	− 61299				
bde	− 90900	− 16940	+ 69062	− 4375			
c^2e	+ 103740	− 27160	+ 80248	+ 49700	+ 3200		
cd^2	− 38320	+ 26460	− 85554	+ 55125	− 3500	+ 500	
b^3f	− 69615	− 9555	+ 40866				
b^2ce	+ 83538	+ 28098	− 106218	+ 128625	− 8000		
b^2d^2	+ 92820	+ 12740	− 54782	− 61250	+ 4375	− 625	
bc^2d	− 102102	− 52822	+ 191590	− 156800	+ 9800	− 3000	
c^4		+ 21560	− 73304	+ 84868	− 5376	+ 1024	+ 256
b^4e			− 378	− 78750	+ 5000		
b^3cd			+ 1176	+ 183750	− 12250	+ 5750	
b^2c^3				− 102165	+ 6580	− 620	− 1280
b^3d						− 2500	
b^4c^2					+ 175	− 2025	+ 2400
b^6c						+ 1500	− 2000
b^8							+ 625
	+ 383768	+ 116037	+ 403375	+ 452993	+ 29130	+ 8774	+ 3281
	− 384803	− 116032	− 403077	− 453040	− 29126	− 8750	− 3280

I remark that in the last of these tables the first column, say $i \infty bc^2d$, which ends in bc^2d, is a more simple form than Sylvester's P_8, $= i \infty c^4$, (*Amer. Math. Journ.*, t. IX. p. 35), which ends in c^4; P_8 is in fact a linear combination, first col. + 6 second col. of the first and second columns of the table: the second column, say $cg \infty c^4$ is Sylvester's (a^2cg), t. IX. p. 124.

934.

NOTE ON THE SO-CALLED QUOTIENT G/H IN THE THEORY OF GROUPS.

[From the *American Journal of Mathematics*, t. XV. (1893), pp. 387, 388.]

THE notion (see Hölder, "Zur Reduction der algebraischen Gleichungen," *Math. Ann.*, t. XXXIV. (1887), § 4, p. 31) is a very important one, and it is extensively made use of in Mr Young's paper, "On the Determination of Groups whose Order is the Power of a Prime," *American Journal of Mathematics*, t. XV. (1893), pp. 124—178; but it seems to me that the meaning is explained with hardly sufficient clearness, and that a more suitable algorithm might be adopted, viz. instead of $G_1 = G/\Gamma_1$, I would rather write $G = \Gamma_1 . QG_1$ or $QG_1 . \Gamma_1$.

We are concerned with a group G containing as part of itself a group Γ_1, such that each element of Γ_1 is commutative with each element of G. This being so, we may write

$$G = QG_1 . \Gamma_1,$$

where QG_1 is not a group but a mere array of elements, viz. if $\Gamma_1 = (1, A_2, \ldots, A_s)$, and $QG_1 = (1, B_2, \ldots, B_t)$, then the formula is

$$G = (1, B_2, \ldots, B_t)(1, A_2, \ldots, A_s),$$

where it is to be noticed that the elements B are not determinate; in fact, if A_θ be any element of Γ_1, we may, in place of an element B, write BA_θ, for

$$B(1, A_2, \ldots, A_s) \quad \text{and} \quad BA_\theta(1, A_2, \ldots, A_s)$$

are, in different orders, the same elements of G.

But, G being a group, the product of any two elements of G is an element of G; viz. we thus have in general

$$B_i A_{i'} . B_j A_{j'} = B_k A_{k'};$$

that is,

$$B_iB_j = B_kA_kA_j^{-1}A_j^{-1} \quad (i, j, \text{ unequal or equal}),$$

where the B_k is a determinate element of the series 1, B_2, ..., B_t, depending only on the elements B_i and B_j into the product of which it enters; and it is in nowise affected by the before-mentioned indeterminateness of the elements B: say B_i, B_j being any two elements of the series 1, B_2, ..., B_t, we have the last preceding equation wherein B_k is a determinate element of the same series.

We may imagine a set of elements 1, B_2, ..., B_t for which, B_i, B_j being any two of them and B_k a third element determined as above, we have always $B_iB_j = B_k$, that is, these elements 1, B_2, ..., B_t now form a group, say the group G_1; the original elements 1, B_2, ..., B_t (which are subject to a different law of combination $B_iB_j = B_kA_kA_j^{-1}A_j^{-1}$, and do not form a group) are regarded as a mere array connected with this group, and so represented as above by QG_1; and the relation of the original group G to the group Γ_1 (consisting of elements commutative with those of G) and to the new group G_1 is expressed as above by the equation

$$G = \Gamma_1 . QG_1, \quad = QG_1 . \Gamma_1.$$

Cambridge, 2 *June*, 1893.

935.

SUR LA FONCTION MODULAIRE $\chi\omega$.

[From the *Comptes Rendus de l'Académie des Sciences de Paris*, t. CXVI. (*Janvier—Juin*, 1893), pp. 1339—1343.]

SELON les notations usitées, on a

$$\chi\omega = \sqrt[12]{kk'} = 2^{\frac{1}{6}} q^{\frac{1}{24}} \div 1 + q \,.\, 1 + q^3 \,.\, 1 + q^5 \ldots;$$

or, d'après une transformation trouvée par M. Glaisher, on a

$$\frac{1}{1+q} + \frac{3q^2}{1+q^3} + \frac{5q^4}{1+q^5} + \ldots = \frac{1+q^2}{(1-q^2)^2} - \frac{q(1+q^4)}{(1-q^4)^2} + \frac{q^2(1+q^6)}{(1-q^6)^2} - \ldots,$$

et de là, en intégrant,

$$\log 1 + q \,.\, 1 + q^3 \,.\, 1 + q^5 \ldots = \frac{q}{1-q^2} - \tfrac{1}{2}\frac{q^2}{1-q^4} + \tfrac{1}{3}\frac{q^3}{1-q^6} - \ldots,$$

$$= -\frac{1}{q-q^{-1}} + \tfrac{1}{2}\frac{1}{q^2-q^{-2}} - \tfrac{1}{3}\frac{1}{q^3-q^{-3}} + \ldots;$$

donc

$$\chi\omega = 2^{\frac{1}{6}} q^{\frac{1}{24}} \exp.\left(\frac{1}{q-q^{-1}} - \tfrac{1}{2}\frac{1}{q^2-q^{-2}} + \tfrac{1}{3}\frac{1}{q^3-q^{-3}} - \ldots\right),$$

ou, en écrivant $q = e^{i\pi\omega}$,

$$\chi\omega = 2^{\frac{1}{6}} \exp.\left[\frac{i\pi\omega}{24} + \frac{1}{2i}\left(\frac{1}{\sin\pi\omega} - \frac{1}{2\sin 2\pi\omega} + \frac{1}{3\sin 3\pi\omega} - \ldots\right)\right],$$

$$= 2^{\frac{1}{6}} \exp.\left[\frac{i\pi\omega}{24} - \frac{1}{2i}\Sigma_n \frac{\cos n\pi}{n\sin n\pi\omega}\right],$$

$n=1$ jusqu'à $n=\infty$, forme qui met en évidence l'équation $\chi(\omega+2) = i^{\frac{1}{6}}\chi\omega$.

Mais nous avons

$$\frac{1}{n\sin n\pi\omega}=\frac{1}{n^2\pi\omega}+\frac{1}{\pi}\Sigma_m\frac{\cos m\pi}{n(n\omega-m)},$$

$m=1$ jusqu'à $m=+\infty$ et $m=-1$ jusqu'à $m=-\infty$.

On a, dans les parenthèses, d'abord les termes

$$\frac{1}{2i}\frac{1}{\pi\omega}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\right),\quad =\frac{1}{2i}\frac{1}{\pi\omega}\cdot\frac{\pi^2}{12},\quad =-\frac{\pi i}{24\omega},$$

et l'expression de $\chi\omega$ devient ainsi

$$\chi\omega=2^{\frac{1}{6}}\exp.\left[\frac{i\pi\omega}{24}-\frac{i\pi}{24\omega}+\frac{i}{2\pi}\Sigma_n\Sigma_m\frac{\cos n\pi\cos m\pi}{n(n\omega-m)}\right],$$

ou, ce qui est plus commode,

$$\chi\omega=2^{\frac{1}{6}}\exp.\left[\frac{i\pi\omega}{24}-\frac{i\pi}{24\omega}+\frac{i}{2\pi}(S_1-S_2-S_3+S_4)\frac{1}{n(n\omega-m)}\right],$$

où les signes sommatoires $S=\Sigma_n\Sigma_m$ se rapportent: S_1 aux valeurs impaires de m et n; S_2 aux valeurs impaires de m et paires de n; S_3 aux valeurs paires de m et impaires de n; S_4 aux valeurs paires de m et n. En omettant l'expression $\frac{1}{n(n\omega-n)}$ du terme sommé, on peut considérer S_1, S_2, S_3, S_4 comme dénotant les quatre sommes dont il s'agit.

Considérons d'abord S_4; en supposant que m', n' soient l'un ou l'autre ou tous les deux impairs, on peut donner à m, n les valeurs $2m'$, $2n'$; $4m'$, $4n'$; $8m'$, $8n'$; ...; on obtient

$$S_4=(S_1+S_2+S_3)\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{8^2}+\dots\right)=\tfrac{1}{3}(S_1+S_2+S_3),$$

et le terme $S_1-S_2-S_3+S_4$ dans l'expression de $\chi\omega$ devient ainsi

$$=\tfrac{4}{3}S_1-\tfrac{2}{3}S_2-\tfrac{2}{3}S_3.$$

Dans S_1, S_2, ou S_3, en supposant que m', n' dénotent des nombres relativement premiers, on peut donner à m, n les valeurs m', n'; $3m'$, $3n'$; $5m'$, $5n'$; ...: on obtient ainsi, pour chacune de ces sommes,

$$S=S\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=\frac{\pi^2}{8}S,$$

où, au côté droit, m et n sont des nombres relativement premiers; le terme devient ainsi

$$\frac{\pi^2}{8}(\tfrac{4}{3}S_1-\tfrac{2}{3}S_2-\tfrac{2}{3}S_3)=\frac{\pi^2}{6}(S_1-\tfrac{1}{2}S_2-\tfrac{1}{2}S_3),$$

et nous avons

$$\chi\omega=2^{\frac{1}{6}}\exp.\left[\frac{i\pi\omega}{24}-\frac{i\pi}{24\omega}+\frac{i\pi}{12}(S_1-\tfrac{1}{2}S_2-\tfrac{1}{2}S_3)\frac{1}{n(n\omega-m)}\right].$$

Je m'arrête pour remarquer que cette expression s'accorde parfaitement avec des résultats trouvés par M. Dedekind (*Œuvres de Riemann*, Leipzig, 1876, p. 447). En effet, en donnant à ω la valeur $\omega = \frac{m}{n} + i\alpha$, α une valeur positive très petite, M. Dedekind trouve les valeurs de $\log k$ et $\log k'$; en ajoutant ces valeurs et en ne faisant attention qu'aux termes qui deviennent infinis pour $\alpha = 0$, on a

$$\log kk' = 12 \log \chi\omega = \quad 24A,$$

m et n tous les deux impairs;

$$\log kk' = 12 \log \chi\omega = -12A,$$

m et n l'un impair, l'autre pair.

Ici $A = \frac{\pi i}{24n(n\omega - m)}$, et les valeurs de $\log \chi\omega$ sont ainsi

$$\frac{i\pi}{12} \frac{1}{n(n\omega - m)} \text{ et } -\frac{i\pi}{24} \frac{1}{n(n\omega - m)}$$

respectivement.

Dans l'expression de $\chi\omega$, il convient de réunir les termes qui correspondent aux valeurs $-m$, $+m$, c'est-à-dire au lieu de $\frac{1}{n(n\omega - m)}$, on doit écrire

$$\frac{1}{n}\left(\frac{1}{n\omega - m} + \frac{1}{n\omega + m}\right) = \frac{2\omega}{n^2\omega^2 - m^2};$$

on a ainsi finalement

$$\chi\omega = 2^{\frac{1}{6}} \exp. \left[\frac{i\pi\omega}{24} - \frac{i\pi}{24\omega} + \frac{i\pi}{6}(S_1 - \tfrac{1}{2}S_2 - \tfrac{1}{2}S_3)\frac{\omega}{n^2\omega^2 - m^2}\right],$$

où je rappelle que m et n sont des nombres positifs relativement premiers, et que les signes sommatoires S_1, S_2, S_3 se rapportent, S_1 aux valeurs impaires de m et n, S_2 aux valeurs impaires de m et paires de n, S_3 aux valeurs paires de m et impaires de n.

Écrivant $-\frac{1}{\omega}$ au lieu de ω, le terme $\frac{\omega}{n^2\omega^2 - m^2}$ auquel se rapportent les sommations se change en $\frac{\omega}{m^2\omega^2 - n^2}$; on peut échanger les lettres m et n, et l'expression de $\chi\left(-\frac{1}{\omega}\right)$ devient ainsi identiquement celle de $\chi\omega$, c'est-à-dire la forme met en évidence la relation $\chi\left(-\frac{1}{\omega}\right) = \chi\omega$.

Il y a cependant, dans cette analyse et par rapport à l'échange des lettres m et n, une difficulté qu'il convient d'écarter. Dans la formule

$$\frac{1}{\sin x} = \frac{1}{x} + \Sigma_m \frac{\cos m\pi}{x - m\pi} \quad (m = 1 \text{ jusqu'à } +\infty \text{ et } -1 \text{ jusqu'à } -\infty),$$

qui donne lieu à

$$\frac{1}{n\sin^2 n\pi\omega} = \frac{1}{n^2\pi\omega} + \frac{1}{\pi}\Sigma_m \frac{\cos m\pi}{n(n\omega - m)},$$

il est nécessaire que la variable x ait une valeur finie ou au moins infiniment petite par rapport aux valeurs extrêmes de m; ainsi, en écrivant pour x la valeur $n\pi\omega$, les valeurs extrêmes de n doivent être infiniment petites par rapport à celles de m, et la somme que nous avons dénotée simplement par $\Sigma_n\Sigma_m \frac{1}{n(n\omega - m)}$ signifie réellement $\overset{\nu}{\Sigma}_n \overset{\mu}{\Sigma}_m \frac{1}{n(n\omega - m)}$, savoir les limites pour n sont 1, ν et pour m ces limites sont -1, $-\mu$ et $+1$, $+\mu$ où μ et ν sont des nombres infiniment grands, mais $\frac{\mu}{\nu} = \infty$, ou, ce qui est la même chose, la somme est $\overset{\nu}{\Sigma}_n \overset{\infty}{\Sigma}_m \frac{1}{n(n\omega - m)}$, où ν est un très grand nombre qui devient enfin $= \infty$. En réunissant les termes pour m et $-m$, la somme à considérer est $\overset{\nu}{\Sigma}_n \overset{\infty}{\Sigma}_m \frac{1}{n^2\omega^2 - m^2}$, et il s'agit de faire voir qu'il est permis de substituer pour cela la somme

$$\overset{\nu}{\Sigma}_n \overset{\nu}{\Sigma}_m \frac{1}{n^2\omega^2 - m^2} \qquad (\nu = 1 \text{ à } \nu = \infty).$$

La différence des deux expressions est une somme double $n = 1$ jusqu'à $n = \nu$ et $m = \nu + 1$, jusqu'à $m = \infty$, savoir cette somme est égale à

$$\begin{aligned}\cos\nu\pi \Bigg\{ + & \left[\frac{1}{\omega^2 - (\nu+1)^2} - \frac{1}{\omega^2 - (\nu+2)^2} + \frac{1}{\omega^2 - (\nu+3)^2} - \dots\right] \\ & - \left[\frac{1}{4\omega^2 - (\nu+1)^2} - \frac{1}{4\omega^2 - (\nu+2)^2} + \frac{1}{4\omega^2 - (\nu+3)^2} - \dots\right] \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & - \cos\nu\pi \left[\frac{1}{\nu^2\omega^2 - (\nu+1)^2} - \frac{1}{\nu^2\omega^2 - (\nu+2)^2} + \frac{1}{\nu^2\omega^2 - (\nu+3)^2} - \dots\right]\Bigg\},\end{aligned}$$

laquelle somme $\left(\text{sauf pour une valeur imaginaire } \omega = \frac{m}{n} + i\alpha,\ \alpha \text{ positif}\right)$ devient aussi petite que l'on veut en donnant à ν une valeur suffisamment grande; c'est-à-dire qu'on peut négliger cette différence et ainsi considérer la somme $\Sigma_n\Sigma_m \frac{1}{n(n\omega - m)}$, dont je me suis servi dans l'investigation comme ayant pour n et pour les valeurs positives ou négatives de m les mêmes limites 1, ν ($\nu = \infty$).

La forme trouvée pour $\chi\omega$ met en évidence que $\omega = 0$, $\omega = \infty$, et $\omega = \pm\frac{m}{n}$ sont des valeurs essentiellement singulières pour la fonction.

936.

NOTE ON UNIFORM CONVERGENCE.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XIX. (1893), pp. 203—207. Read December 5, 1892.]

IT appears to me that the form in which the definition or condition of uniform convergence is usually stated, is (to say the least) not easily intelligible. I call to mind the general notion: We may have a series, to fix the ideas, say of positive terms

$$(0)_x + (1)_x + (2)_x, \ldots + (n)_x, \ldots$$

the successive terms whereof are continuous functions of x, for all values of x from some value less than a up to and inclusive of a (or from some value greater than a down to and inclusive of a): and the series may be convergent for all such values of x, the sum of the series ϕx is thus a determinate function ϕx of x; but ϕx is not of necessity a continuous function; if it be so, then the series is said to be uniformly convergent; if not, and there is for the value $x = a$ a breach of continuity in the function ϕx, then there is for this value $x = a$ a breach of uniform convergence in the series.

Thus if the limits are say from 0 up to the critical value 1, then in the geometrical series $1 + x + x^2 + \ldots$, the successive terms are each of them continuous up to and inclusive of the limit 1, but the series is only convergent up to and exclusive of this limit, viz. for $x = 1$ we have the divergent series $1 + 1 + 1 + \ldots$, and this is *not* an instance; but taking, instead, the geometrical series

$$(1 - x) + (1 - x)x + (1 - x)x^2 + \ldots,$$

here the terms are each of them continuous up to and inclusive of the limit 1, and the series is also convergent up to and inclusive of this limit; in fact, at the limit

the series is $0+0+0+\ldots$ We have here an instance, and there is in fact a discontinuity in the sum, viz. $x<1$ the sum is

$$(1-x)(1+x+x^2+\ldots), \; =(1-x)\cdot\frac{1}{1-x}, \; =1;$$

whereas for the limiting value 1, the sum is $0+0+0+\ldots, =0$. The series is thus uniformly convergent up to and exclusive of the value $x=1$, but for this value there is a breach of uniform convergence.

I remark that Du Bois-Reymond in his paper, "Notiz über einen Cauchy'schen Satz, die Stetigkeit von Summen unendlicher Reihen betreffend," *Math. Ann.*, t. IV. (1871), pp. 135—137, shows that, when certain conditions are satisfied, the sum ϕx is a continuous function of x, but he does not use the term "uniform convergence," nor give any actual definition thereof.

M. Jordan, in his "Cours d'Analyse de l'École Polytechnique," t. I. (Paris, 1882), considers p. 116 the series $s=u_1+u_2+u_3+\ldots$, the terms of which are functions of a variable z, and after remarking that such a series is convergent for the values of z included within a certain interval, if for each of these values and for every value of the infinitely small quantity ϵ we can assign a value of n such that for every value of p,

$$\text{Mod}\,(u_{n+1}+u_{n+2}+\ldots+u_{n+p})<\text{Mod}\,\epsilon,$$

ϵ being as small as we please, proceeds:—

"Le nombre des termes qu'il est nécessaire de prendre dans la série pour arriver à ce résultat sera en général une fonction de z et de ϵ. Néanmoins on pourra très habituellement déterminer un nombre n fonction de ϵ seulement telle que la condition soit satisfaite pour toute valeur de z comprise dans l'intervalle considéré. On dira dans ce cas que la série s est *uniformément convergente* dans cet intervalle."

And similarly, Professor Chrystal in his *Algebra*, Part II. (Edinburgh, 1889), after considering, p. 130, the series

$$\frac{x}{x+1\,.\,2x+1}+\frac{x}{2x+1\,.\,3x+1}+\ldots+\frac{x}{(n-1)x+1\,.\,nx+1}+\ldots$$

for which the critical value is $x=0$, and in which when $x=0$ the residue R_n of the series or sum of the $(n+1)$th and following terms is $=\dfrac{1}{nx+1}$ proceeds as follows:—

Now when x has any given value, we can by making n large enough make $\dfrac{1}{nx+1}$ smaller than any given positive quantity α. But on the other hand, the smaller x is the larger must we take n in order that $\dfrac{1}{nx+1}$ may fall under α; and in general when x is variable there is no finite upper limit for n *independent of* x, say v, such that if $n>v$ then $R_n<\alpha$. When the residue has this peculiarity the series is said to be *non-uniformly convergent;* and if for a particular value of x, such as $x=0$ in the

present example, the number of terms required to secure a given degree of approximation to the limit is infinite, the series is said to *converge infinitely slowly.*

And he thereupon gives the formal definition: *If for values of x within a given region in Argand's diagram we can for every value of α, however small Mod. α, assign for n an upper limit v* INDEPENDENT OF x, such that, when $n > v$, Mod. $R_n <$ Mod. α, then the series $\Sigma f(n, x)$ is said to be UNIFORMLY CONVERGENT within the region in question.

The two forms of definition (Jordan and Chrystal) appear to me equivalent, and it seems to me that construing the definition *strictly,* and applying it to the above instance

$$(1-x)+(1-x)x+(1-x)x^2+\ldots,$$

the definition does not in either case indicate a breach of uniform convergency at $x=1$, viz. the definition shows uniform convergency from $x=0$ to $x=1-\epsilon$, ϵ being a positive quantity however small, or (as I have before expressed this) uniform convergency up to and exclusive of the limit 1; and further, it shows uniform convergency at the limit 1. For at this limit, the series of terms is $0+0+0+\ldots$, the residue or sum of the $(n+1)$th and subsequent terms is thus also $0+0+0+\ldots$, and we get the value of this residue, not approximately, but exactly, by taking a single term of the series. Jordan and Chrystal calculate, each of them, the residue from the general expression thereof by writing therein for x or z the critical value; and then, comparing the value thus obtained with the values obtained for the $(n+1)$th and subsequent terms of the series on substituting therein for x or z the critical value, they seem to argue that the discrepancy between these two values indicates the breach of uniform convergency.

It may be said that the objection is a verbal one. But it seems to me that the whole notion of the residue (although very important as regards the general theory of convergence) is irrelevant to the present question of uniform convergency, and that a better method of treating the question is as follows:

Considering as before the series

$$(0)_x+(1)_x+(2)_x+\ldots+(n)_x+\ldots,$$

where the functions $(0)_x$, $(1)_x$, $(2)_x$, ... are each of them continuous up to and inclusive of the limit $x=a$, and the series has thus a definite sum ϕx, this sum is *primâ facie* a continuous function of x, and what we have to explain is the manner in which it may come to be discontinuous. Suppose that it is continuous up to and exclusive of the limit $x=a$, but that there is a breach of discontinuity at this limit: write $x=a-\epsilon$, where ϵ is a positive quantity as small as we please, and consider the two equations

$$\phi x=(0)_x+(1)_x+(2)_x+\ldots,$$

$$\phi a=(0)_a+(1)_a+(2)_a+\ldots,$$

then we have

$$\phi a-\phi x=\epsilon\left\{\frac{(0)_a-(0)_x}{a-x}+\frac{(1)_a-(1)_x}{a-x}+\ldots\right\}.$$

Hence if the sum of the series in { } is a finite magnitude M, not indefinitely large for an indefinitely small value of ϵ, we have $\phi a - \phi x = \epsilon M$, which is indefinitely small for ϵ indefinitely small, and there is no breach of continuity; the only way in which a breach of continuity can arise is by the series in { } having a value indefinitely large for ϵ indefinitely small, viz. if such a value is $\frac{N}{\epsilon}$, then $\phi a - \phi x = \epsilon . \frac{N}{\epsilon}, = N$, and as x changes from $a - \epsilon$ to a, the sum changes abruptly from $\phi(a-\epsilon)$ to $\phi(a-\epsilon) + N$.

The condition for a breach of uniform convergency for the value $x = a$, thus is that, writing $x = a - \epsilon$, ϵ a positive magnitude however small, the series

$$\frac{(0)_a - (0)_x}{a - x} + \frac{(1)_a - (1)_x}{(a - x)} + \ldots,$$

shall have a sum indefinitely large for ϵ indefinitely small, or say as before, a sum $= \frac{N}{\epsilon}$.

For the foregoing example, where the series is

$$(1 - x) + x(1 - x) + x^2(1 - x) + \ldots$$

the critical value is $a = 1$: we have here $(n)_x = x^n(1 - x)$, and consequently

$$\begin{aligned}\frac{(n)_a - (n)_x}{a - x} &= \frac{a^n - x^n}{a - x} - \frac{a^{n+1} - x^{n+1}}{(a - x)} \\ &= (a^{n-1} + a^{n-2}x + \ldots + x^{n-1}) - (a^n + a^{n-1}x + \ldots + x^n) \\ &= -x^n \text{ for } a = 1.\end{aligned}$$

The series

$$\frac{(0)_a - (0)_x}{a - x} + \frac{(1)_a - (1)_x}{a - x} + \ldots$$

thus is

$$\begin{aligned}&-(1 + x + x^2 + \ldots) \\ &= -\frac{1}{1 - x}, = \frac{-1}{\epsilon}\end{aligned}$$

for $x = 1 - \epsilon$, viz. we have

$$\phi 1 - \phi(1 - \epsilon) = \epsilon . \frac{-1}{\epsilon}, = -1;$$

which is right since by what precedes

$$\phi(1 - \epsilon) = 1, \ \phi 1 = 0.$$

937.

NOTE ON THE ORTHOTOMIC CURVE OF A SYSTEM OF LINES IN A PLANE.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 45, 46.]

CONSIDERING *in plano* a singly infinite system of lines, then to each point of the plane there corresponds a line (not in general a unique line), and we can therefore express in terms of the coordinates (x, y) of the point the cosine-inclinations α, β of the line to the axes. The differential equation of the orthotomic curve is then $\alpha dx + \beta dy = 0$, and it is a well-known and easily demonstrable theorem that $\alpha dx + \beta dy$ is a complete differential, say it is $= dV$; the integral equation of the orthotomic curve is therefore $V = \int (\alpha dx + \beta dy)$, $=$ const., and we see further that V is a solution of the partial differential equation $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 = 1$.

If the lines are the normals of the ellipse $\frac{X^2}{a} + \frac{Y^2}{b} = 1$, then, writing the equation of the normal at the point X, Y in the form

$$\frac{a}{X}(x - X) = \frac{b}{Y}(y - Y), = \lambda,$$

suppose, we have

$$X = \frac{ax}{a + \lambda}, \quad Y = \frac{by}{b + \lambda};$$

and therefore

$$\frac{ax^2}{(a + \lambda)^2} + \frac{by^2}{(b + \lambda)^2} - 1 = 0,$$

which last equation determines λ as a function of x, y. We have α, β proportional to $\frac{X}{a}$, $\frac{Y}{b}$; or say we have

$$\alpha = M\frac{x}{a + \lambda}, \quad \beta = M\frac{y}{b + \lambda},$$

whence

$$\frac{1}{M^2} = \frac{x^2}{(a + \lambda)^2} + \frac{y^2}{(b + \lambda)^2};$$

or, writing for convenience

$$\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}-\frac{k^2}{\lambda^2}=0,$$

(viz. this equation defines k as a function of x, y and λ, that is, of x and y), we have

$$\alpha=\frac{\lambda x}{k(a+\lambda)},\quad \beta=\frac{\lambda y}{k(b+\lambda)};$$

and we ought therefore to have

$$\frac{\lambda}{k}\left(\frac{xdx}{a+\lambda}+\frac{ydy}{b+\lambda}\right)$$

a complete differential, $=dV$.

This is easily verified, for from the assumed value

$$k=\lambda\left(\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}-1\right)$$

we deduce

$$dk=2\lambda\left(\frac{xdx}{a+\lambda}+\frac{ydy}{b+\lambda}\right)+d\lambda\left(\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda)^2}-1\right),\ =2\lambda\left(\frac{xdx}{a+\lambda}+\frac{ydy}{b+\lambda}\right);$$

and we have therefore

$$dV=\frac{\lambda}{k}\frac{dk}{2\lambda},\ =\tfrac{1}{2}\frac{dk}{k},$$

where k denotes a function of (x, y) defined as above; hence the equation $V=\text{const.}$ gives $k=\text{const.}$, or the equation of the orthotomic curve is given by the system of equations

$$\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda)^2}-1=0,$$

$$\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}-\frac{k^2}{\lambda^2}=0,$$

where k is a constant; these equations (eliminating λ) give, in fact, the equation of the parallel curve of the ellipse, and k denotes the normal distance of a point on the curve from the ellipse. I recall that the first equation may be replaced by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}-\frac{k}{\lambda}-1=0,$$

and since the derived equation hereof in regard to λ is the second equation, we have the equation of the parallel curve in the known form

$$\text{Disct. }\{(\lambda+k)(\lambda+a)(\lambda+b)-(b+\lambda)x^2-(a+\lambda)y^2\}=0.$$

I notice further that, considering k a function of x, y as above, we have

$$\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2=\frac{1}{4k^2}\left\{\left(\frac{dk}{dx}\right)^2+\left(\frac{dk}{dy}\right)^2\right\},\ =\frac{\lambda^2}{k^2}\left\{\frac{x^2}{(a+\lambda)^2}+\frac{y^2}{(b+\lambda)^2}\right\},$$

that is,

$$\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2=1,$$

as it should be.

938.

ON TWO CUBIC EQUATIONS.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 69—71.]

STARTING from the equations

$$2+a=b^2,$$
$$2+b=c^2,$$
$$2+c=a^2,$$

then eliminating b, c, we find

$$(a^4-4a^2+2)^2-(a+2)=0,$$

that is,

$$a^8-8a^6+20a^4-16a^2-a+2=0;$$

we satisfy the equations by $a=b=c$, and thence by

$$a^2-a-2=(a-2)(a+1)=0;$$

there remains a sextic equation breaking up into two cubic equations; the octic equation may in fact be written

$$(a-2)(a+1)(a^3+a^2-2a-1)(a^3-3a+1)=0,$$

and we have thus the two cubic equations

$$x^3+x^2-2x-1=0,\quad x^3-3x+1=0,$$

for each of which the roots (a, b, c) taken in a proper order are such that $2+a=b^2$, $2+b=c^2$, $2+c=a^2$.

It may be remarked that starting from $y^3+y^2-2y-1=0$, $y^2=x+2$, the first equation gives $(y^3-2y)^2-(y^2-1)^2=0$, that is, $y^6-5y^4+6y^2-1=0$, whence

$$(x+2)^3-5(x+2)^2+6(x+2)-1=0,$$

that is,

$$x^3+x^2-2x-1=0.$$

And similarly, starting from $y^3 - 3y + 1 = 0$, $y^2 = x + 2$, the first equation gives $(y^3 - 3y)^2 - 1 = 0$, that is, $y^6 - 6y^4 + 9y^2 - 1 = 0$, whence

$$(x+2)^3 - 6(x+2)^2 + 9(x+2) - 1 = 0,$$

that is,

$$x^3 - 3x + 1 = 0.$$

To find the roots of the equation $x^3 + x^2 - 2x - 1 = 0$, taking ω an imaginary cube root of unity, and writing $\alpha = \sqrt[3]{\{7(2+3\omega)\}}$, $\beta = \sqrt[3]{\{7(2+3\omega^2)\}}$, where observe that $2+3\omega$, $2+3\omega^2$ are imaginary factors of 7, viz.

$$7 = (2+3\omega)(2+3\omega^2),$$

and therefore also $\alpha^3 + \beta^3 = 7$, $\alpha\beta = 7$, then the roots of the equation are

$$3a = -1 + \alpha + \beta,$$
$$3b = -1 + \omega\alpha + \omega^2\beta,$$
$$3c = -1 + \omega^2\alpha + \omega\beta.$$

I verify herewith the equation $a^2 = 2 + c$, viz. this gives

$$(-1 + \alpha + \beta)^2 = 18 + 3(-1 + \omega^2\alpha + \omega\beta),$$

or writing herein $2\alpha\beta = 14$, this is

$$\alpha^2 - (2+3\omega^2)\alpha + \beta^2 - (2+3\omega)\beta = 0,$$

that is,

$$\alpha^2 - \tfrac{1}{7}\beta^3\alpha \qquad + \beta^2 - \tfrac{1}{7}\alpha^3\beta \qquad = 0,$$

or finally

$$(\alpha^2 + \beta^2)(1 - \tfrac{1}{7}\alpha\beta) = 0,$$

satisfied in virtue of $\alpha\beta = 7$.

For the second equation $x^3 - 3x + 1 = 0$, ω denoting as before, the roots are

$$a = \omega^{\frac{1}{3}} + \omega^{\frac{3}{3}}, \text{ whence } a^2 = \omega^{\frac{2}{3}} + \omega^{\frac{7}{3}} + 2, = 2 + c,$$
$$b = \omega^{\frac{4}{3}} + \omega^{\frac{5}{3}}, \quad \text{,,} \quad b^2 = \omega^{\frac{8}{3}} + \omega^{\frac{1}{3}} + 2, = 2 + a,$$
$$c = \omega^{\frac{7}{3}} + \omega^{\frac{3}{3}}, \quad \text{,,} \quad c^2 = \omega^{\frac{6}{3}} + \omega^{\frac{4}{3}} + 2, = 2 + b.$$

The equation $x^3 - 5x^2 + 6x - 1 = 0$, which, writing therein $x + 2$ for x, gives

$$x^3 + x^2 - 2x - 1 = 0,$$

is considered in Hermite's *Cours d'Analyse*, Paris 1873, p. 12, and this suggested to me the foregoing investigation.

939.

ON A CASE OF THE INVOLUTION $AF+BG+CH=0$, WHERE A, B, C, F, G, H ARE TERNARY QUADRICS.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 182—186.]

WE have here the six conics

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0;$$

the curves $AF=0$ and $BG=0$ are quartics intersecting in 16 points, and if 8 of these lie in a conic $H=0$, then the remaining 8 will be in a conic $C=0$. I take the first set of eight points to be 1, 2, 3, 4, 5, 6, 7, 8; the quartics $AF=0$ and $BG=0$ each pass through these eight points; and I assume for the moment

$$A=1234,\ F=5678;\ B=1256,\ G=3478,$$

viz. that $A=0$ is a conic through the points 1, 2, 3, 4, and similarly for F, G, B. Here $H=0$ is a conic through the points 1, 2, 3, 4, 5, 6, 7, 8, or attending only to the last four points it is a conic through 5, 6, 7, 8; we have therefore a linear relation between F, G, H, and supposing the implicit constant factors to be properly determined, this may be taken to be $F+G+H=0$; the identity $AF+BG+CH=0$ thus becomes $F(A-C)+G(B-C)=0$. We have thus F a numerical multiple of $B-C$, and by a proper determination of the implicit factor we may make this relation to be $F=B-C$; the last equation then gives $G=C-A$, and from the equation $F+G+H=0$, we have $H=A-B$; the six functions thus are

$$\begin{array}{ll} A,\ B-C, \text{ or if we please, } & A-D,\ B-C, \\ B,\ C-A & B-D,\ C-A, \\ C,\ A-B & C-D,\ A-B, \end{array}$$

where D is an arbitrary quadric function. The solution

$$(A-D)(B-C)+(B-D)(C-A)+(C-D)(A-B)=0$$

of the involution is an obvious and trivial one.

But the case which I proceed to consider is

$$A=1234,\ F=5678;\ B=1256,\ G=3478;$$

here $AF=0$, and $BG=0$, meet as before in the points 1, 2, 3, 4, 5, 6, 7, 8, and in eight other points, say that

$A=0,\ B=0$	meet in	1, 2	and in two other points		$\alpha,\ \beta,$
$A=0,\ G=0$	„	3, 4	„	„	$\gamma,\ \delta,$
$F=0,\ B=0$	„	5, 6	„	„	$\epsilon,\ \zeta,$
$F=0,\ G=0$	„	7, 8	„	„	$\eta,\ \theta;$

then the 8 points $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ will lie in a conic $C=0$.

I take $y^2-zx=0$ for the conic $H=0$; for any point in this conic we have $x:y:z=1:\theta:\theta^2$, and we may take $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8$ for the parameters of the points 1, 2, 3, 4, 5, 6, 7, 8 respectively.

Write $(a, b, c, f, g, h)(x, y, z)^2=0$ for the conic $A, =1234=0$; therefore we have

$$a+b\theta^2+c\theta^4+f\theta^3+g\theta^2+h\theta=\theta-\theta_1.\theta-\theta_2.\theta-\theta_3.\theta-\theta_4;$$

or, if

$$p_{1234}=\theta_1+\theta_2+\theta_3+\theta_4,$$
$$q_{1234}=\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4,$$
$$r_{1234}=\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4,$$
$$s_{1234}=\theta_1\theta_2\theta_3\theta_4,$$

then

$$c=1,\ f=-p_{1234},\ b+g=q_{1234},\ h=-r_{1234},\ a=s_{1234};$$

or, writing $g=-\lambda$, we have

$$s_{1234}x^2+q_{1234}y^2+z^2-p_{1234}yz-r_{1234}xy+\lambda(y^2-zx)=0$$

for the equation of the conic in question. We may without loss of generality put $\lambda=0$; and then if, in general,

$$\Omega=sx^2+qy^2+z^2-pyz-rxy,$$

we have $A=\Omega_{1234}=0$ for the conic $A=0$. And thus the equations of the four conics are

$$A=\Omega_{1234}=0,\quad F=\Omega_{5678}=0;\quad B=\Omega_{1256}=0,\quad C=\Omega_{3478}=0,$$

or, as for shortness I write them,

$$A=\Omega=0,\ F=\Omega'=0;\quad B=\Omega''=0,\ C=\Omega''',$$

viz. in Ω the suffixes are 1, 2, 3, 4, in Ω' they are 5, 6, 7, 8, in Ω'' they are 1, 2, 5, 6, and in Ω''' they are 3, 4, 7, 8.

I find that the implicit constant factors of AF and BG are $1, -1$, and consequently that the form of the identity is

$$\Omega\Omega'-\Omega''\Omega'''+(y^2-zx)C=0,$$

where C is a quadric function to be determined; or, what is the same thing, we have

$$\begin{aligned}&(sx^2+qy^2+z^2-pyz-rxy)(s'x^2+q'y^2+z^2-p'yz-r'xy),\\&-(s''x^2+q''y^2+z^2-p''yz-r''xy)(s'''x^2+q'''y^2+z^2-p'''yz-r'''xy),\\&+(y^2-zx)C=0.\end{aligned}$$

Writing for shortness

$$\begin{aligned}\theta_1+\theta_2&=\alpha\ , & \theta_1\theta_2&=\beta\ ,\\ \theta_3+\theta_4&=\alpha'\ , & \theta_3\theta_4&=\beta'\ ,\\ \theta_5+\theta_6&=\alpha''\ , & \theta_5\theta_6&=\beta''\ ,\\ \theta_7+\theta_8&=\alpha'''\ , & \theta_7\theta_8&=\beta'''\ ,\end{aligned}$$

we have

$$\begin{array}{l|l|l|l}
p=\alpha+\alpha' & p'=\alpha''+\alpha''' & p''=\alpha+\alpha'' & p'''=\alpha'+\alpha''' \\
q=\alpha\alpha'+\beta+\beta' & q'=\alpha''\alpha'''+\beta''+\beta''' & q''=\alpha\alpha''+\beta+\beta'' & q'''=\alpha'\alpha'''+\alpha'\beta'''+\alpha'''\beta' \\
r=\alpha\beta'+\alpha'\beta & r'=\alpha''\beta'''+\alpha'''\beta'' & r''=\alpha\beta''+\alpha''\beta & r'''=\alpha'\beta'''+\alpha'''\beta' \\
s=\beta\beta' & s'=\beta''\beta''' & s''=\beta\beta'' & s'''=\beta'\beta'''.
\end{array}$$

In the last-mentioned equation, the first and second lines together are a quartic function of (x, y, z), say the value is

$$\begin{aligned}=\ & Ax^4+By^4+Cz^4,\\ &+Fy^3z+Gz^3x+Hx^3y,\\ &+Iyz^3+Jzx^3+Kxy^3,\\ &+Lx^2yz+Mxy^2z+Nxyz^2,\\ &+Py^2z^2+Qz^2x^2+Rx^2y^2,\end{aligned}$$

where after all reductions

$$\begin{aligned}
A &= ss'-s''s''' &&=0,\\
B &= qq'-q''q''' &&=(\alpha\beta'''-\alpha'''\beta)(\alpha'-\alpha'')\\
&&&\quad+(\alpha'\beta''-\alpha''\beta')(\alpha-\alpha''')-(\beta'-\beta'')(\beta-\beta'''),\\
C &= 1-1 &&=0,\\
F &= -pq'-p'q+p''q'''+p'''q'' &&=(\alpha-\alpha''')(\beta'-\beta'')+(\alpha'-\alpha'')(\beta-\beta'''),\\
G &= 0-0 &&=0,\\
H &= -rs'-r's+r''s'''+r'''s'' &&=0,\\
I &= -p-p'+p''+p''' &&=0,\\
J &= 0-0 &&=0,\\
K &= -qr'-q'r+q''r'''+q'''r'' &&=(\alpha\beta'''-\alpha'''\beta)(\beta''-\beta')+(\alpha'\beta''-\alpha''\beta')(\beta'''-\beta),\\
L &= -ps'-p's+p''s'''+p'''s'' &&=(\alpha\beta'''-\alpha'''\beta)(\beta'-\beta'')+(\alpha'\beta''-\alpha''\beta')(\beta-\beta'''),\\
M &= pr'+p'r-p''r'''-p'''r'' &&=(\alpha\beta'''-\alpha'''\beta)(\alpha''-\alpha')+(\alpha'\beta''-\alpha''\beta')(\alpha'''-\alpha),\\
N &= -r-r'+r''+r''' &&=(\alpha-\alpha''')(\beta''-\beta')+(\alpha'-\alpha'')(\beta'''-\beta),\\
P &= pp'+q+q'-p''p'''-q''-q''' &&=0,\\
Q &= s+s'-s''-s''' &&=(\beta'-\beta'')(\beta-\beta'''),\\
R &= rr'+qs'+q's-r''r'''-q''s'''-q'''s'' &&=0:
\end{aligned}$$

values which satisfy

$$\begin{aligned} F+N &= 0, \\ K+L &= 0, \\ B+M+Q &= 0. \end{aligned}$$

The quartic function is thus seen to be

$$=(y^2-zx)(By^2+Fyz-Qzx+Kxy)=0,$$

viz. we have $By^2+Fyz-Qzx+Kxy=0$ for the equation of the conic $C=0$.

Moreover, substituting for p, q, r, s, &c., their values, we have finally for the required involution

$$\begin{aligned} &[\beta\beta'x^2+(\alpha\alpha'+\beta+\beta')y^2+z^2-(\alpha+\alpha')yz-(\alpha\beta'+\alpha'\beta)xy] \\ &\times[\beta''\beta'''x^2+(\alpha''\alpha'''+\beta''+\beta''')y^2+z^2-(\alpha''+\alpha''')yz-(\alpha''\beta'''+\alpha'''\beta'')xy] \\ &-[\beta\beta''x^2+(\alpha\alpha''+\beta+\beta'')y^2+z^2-(\alpha+\alpha'')yz-(\alpha\beta''+\alpha''\beta)xy] \\ &\times[\beta'\beta'''x^2+(\alpha'\alpha'''+\beta'+\beta''')y^2+z^2-(\alpha'+\alpha''')yz-(\alpha'\beta''+\alpha'''\beta')xy], \end{aligned}$$

$$-(y^2-zx)\times\left\{\begin{aligned} &y^2[(\alpha\beta'''-\alpha'''\beta)(\alpha'-\alpha'')+(\alpha'\beta''-\alpha''\beta')(\alpha-\alpha''')-(\beta-\beta''')(\beta'-\beta'')] \\ &+yz[(\alpha-\alpha''')(\beta'-\beta'')+(\alpha'-\alpha'')(\beta-\beta''')] \\ &-zx[(\beta-\beta''')(\beta'-\beta'')] \\ &-xy[(\alpha\beta'''-\alpha'''\beta)(\beta'-\beta'')+(\alpha'\beta''-\alpha''\beta')(\beta-\beta''')] \end{aligned}\right\}=0.$$

It will be recollected that this is the solution for the case $A=1234$, $F=5678$; $B=1256$, $G=3478$: being that to which the present paper has reference.

940.

ON THE DEVELOPMENT OF $(1+n^2x)^{\frac{m}{n}}$.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 186—190.]

IT is a known theorem that, if $\frac{m}{n}$ be any fraction in its least terms, the coefficients of the development of $(1+n^2x)^{\frac{m}{n}}$ are all of them integers, or, what is the same thing, that

$$\frac{m\,.\,m-n\,\ldots\,m-(r-1)\,n}{1\,.\,2\,\ldots\,r}\,n^r$$

is an integer. The greater part, but not the whole, of this result comes out very simply from Mr Segar's very elegant theorem, *Messenger*, vol. XXII. (1893), p. 59, "the product of the differences of any r unequal numbers is divisible by $(r-1)!!$" or, as it may be stated, if $\alpha, \beta, \gamma, \ldots$ are any r unequal numbers, then $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1)$.

In fact, writing $r+1$ for r and considering the numbers

$$m+n,\ n,\ 2n,\ 3n,\ \ldots\ (r-1)\,n;$$

then neglecting signs

$$\begin{aligned}\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots) \text{ is } = \quad & m\,.\,m-n\,\ldots\,m-(r-1)\,n,\\ & \times 1n\,.\,2n\,\ldots\,(r-1)\,n,\\ & \times 1n\,.\,2n\,\ldots\,(r-2)\,n,\\ & \vdots\\ & \times 1n\,.\,2n,\\ & \times 1n,\end{aligned}$$

which is

$$= m\,.\,m-n\,\ldots\,m-(r-1)\,n \times n^{\frac{1}{2}r\,.\,r-1} \times \zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1),$$

and similarly

$$\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r) = 1.2.3\ldots r \times \zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1);$$

so that, omitting the common factor $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1)$, we have

$$m.m-n\ldots m-(r-1)n.n^{\frac{1}{2}r.r-1} \text{ divisible by } 1.2.3\ldots r.$$

It thus appears that the fraction

$$\frac{m.m-n\ldots m-(r-1)n}{1.\ 2\ \ldots\ r},$$

when reduced to its least terms, will contain in the denominator only products of powers of the prime factors of n; and it remains to show that multiplying this by n^r it will become integral, or what is the same thing that

$$\frac{n^r}{1.2\ldots.r}$$

in its least terms will not contain in the denominator any prime factor of n.

Considering in succession the prime numbers 2, 3, 5, ..., first the number 2, we see that in the product $1.2.3\ldots.r$, the number of terms divisible by 2 is $=\left(\frac{r}{2}\right)$, the number of terms divisible by 4 is $=\left(\frac{r}{4}\right)$, that by 8 is $=\left(\frac{r}{8}\right)$, and so on, where $\left(\frac{r}{2}\right)$ denotes the integer part of $\frac{r}{2}$, and so in other cases. Hence the product contains the factor 2, with the exponent $\left(\frac{r}{2}\right)+\left(\frac{r}{4}\right)+\left(\frac{r}{8}\right)+\ldots$, which exponent is less than

$$\frac{r}{2}+\frac{r}{4}+\frac{r}{8}+\ldots \textit{ad inf.}$$

is less than r, say it is less than (r). Similarly for the number 3, the product contains the factor 3, with the exponent

$$\left(\frac{r}{3}\right)+\left(\frac{r}{9}\right)+\left(\frac{r}{27}\right)+\ldots,$$

which exponent is less than

$$\frac{r}{3}+\frac{r}{9}+\frac{r}{27}+\ldots \textit{ad inf.}$$

is less than $\frac{1}{2}r$, say it is at most $=(\frac{1}{2}r)$; and so it contains the factor 5 with an exponent which is less than $\frac{1}{4}r$, say it is at most $=(\frac{1}{4}r)$, and generally the prime factor p with an exponent which is less than $\frac{1}{p-1}r$: say it is at most $=\left(\frac{1}{p-1}r\right)$.

This is

$$1.2.3\ldots r = \frac{1}{K}2^{(r)}3^{(\frac{1}{2}r)}5^{(\frac{1}{4}r)}\ldots,$$

where K is a whole number. Hence if $n = 2^\alpha 3^\beta 5^\gamma \ldots$, we have

$$\frac{n^r}{1.2.3\ldots r} = K 2^{r\alpha-(r)} . 3^{r\beta-(\frac{1}{2}r)} . 5^{r\gamma-(\frac{1}{4}r)} \ldots,$$

and here for every prime number 2, 3, 5, ... which is a factor of n, that is, for which the corresponding exponent α, β, γ, ... is not $=0$, the exponents $r\alpha-(r)$, $r\beta-(\frac{1}{2}r)$, $r\gamma-(\frac{1}{4}r)$, ... are all of them positive; and thus the fraction in its least terms does not contain in the denominator any prime factor of n; this is the theorem which was to be proved.

Mr Segar's theorem may without loss of generality be stated as follows: if β, γ, ... are any $r-1$ unequal positive integers (which for convenience may be taken in order of increasing magnitude), then $\zeta^{\frac{1}{2}}(0, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1)$. A proof, in principle the same as his, is as follows:

We have the determinant

$$\begin{vmatrix} 1, & a^\beta, & a^\gamma, \ldots \\ & b & \\ & c & \\ & \vdots & \end{vmatrix} \text{ divisible by } \begin{vmatrix} 1, & a, & a^2, \ldots \\ & b & \\ & c & \\ & \vdots & \end{vmatrix},$$

viz. the quotient is a rational and integral function of a, b, c, ... with coefficients which are positive integers; hence putting $a=b=c, \ldots =1$, the quotient will be a positive integer number. Considering the numerator determinant, and for a, b, c, ... writing therein $1+a$, $1+b$, $1+c$, ... respectively, where a, b, c, ... are ultimately to be put each $=0$, the value is

$$= \begin{vmatrix} 1, & 1+\beta_1 a+\beta_2 a^2 \ldots, & 1+\gamma_1 a+\gamma_2 a^2+\ldots, & \ldots \\ & b & & \\ & c & & \\ & \vdots & & \end{vmatrix},$$

where β_1, β_2, ... denote the binomial coefficients

$$\frac{\beta}{1}, \frac{\beta.\beta-1}{1.2}, \text{ \&c.}:$$

attending only to the lowest powers of a, b, c, ... which enter into the formula, this is

$$= \begin{vmatrix} 1, & & \\ 1, & \beta_1, & \beta_2 \\ 1, & \gamma_1, & \gamma_2 \\ \vdots & & \end{vmatrix} \begin{vmatrix} 1, & a, & a^2, \ldots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{vmatrix},$$

or what is the same thing it is

$$= M \begin{vmatrix} 1, & & \dots \\ 1, & \beta, & \beta^2, \\ 1, & \gamma, & \gamma^2, \\ \vdots & & \end{vmatrix} \begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \end{vmatrix}, = M\zeta^{\frac{1}{2}}(0, \beta, \gamma, \dots) \begin{vmatrix} 1, & a, & a^2, & \dots \\ 1, & b, & b^2, & \\ 1, & c, & c^2, & \\ \vdots & & & \end{vmatrix},$$

where M is a mere number: it will be recollected that in this form, $a, b, c, \dots$ are not the original $a, b, c, \dots$. Putting herein $\beta, \gamma, \dots = 1, 2, \dots$, the denominator determinant is

$$= M\zeta^{\frac{1}{2}}(0, 1, 2, \dots) \begin{vmatrix} 1, & a, & a^2, & \dots \\ 1, & b, & b^2, & \\ 1, & c, & c^2, & \\ \vdots & & & \end{vmatrix},$$

and hence the quotient, which as already seen is an integer number, is equal to $\zeta^{\frac{1}{2}}(0, \beta, \gamma, \dots) \div \zeta^{\frac{1}{2}}(0, 1, 2, \dots)$, the theorem in question.

The original theorem as to the form of $(1+n^2x)^{\frac{m}{n}}$ is a particular case of Eisenstein's very general theorem that, in the development of any algebraical function of x, it is always possible by substituting for x a proper multiple of x, to make all the coefficients integers. It may be remarked that this would not be so if we had only

$$m \,.\, m-n \,.\, \dots \,.\, m-(r-1)n \,.\, n^{\frac{1}{2}r \,.\, r-1}$$

divisible by $1 \,.\, 2 \,.\, \dots \,.\, r$; for then, writing Nx for x, the form of the coefficient would have been

$$\frac{KN^r}{n^r \,.\, n^{\frac{1}{2}r \,.\, r-1}}, = \frac{KN^r}{n^{\frac{1}{2}r \,.\, r+1}},$$

and there would be no value (however great) of N by which the denominator factor $n^{\frac{1}{2}r \,.\, r+1}$ could be got rid of.

941.

NOTE ON THE PARTIAL DIFFERENTIAL EQUATION $Rr+Ss+Tt+U(s^2-rt)-V=0$.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 1—5.]

IT is well known that this equation, R, S, T, U, V being any functions whatever of (x, y, z, p, q), in the case where u admits of an integral of the form $u=f(v)$ (u, v functions of x, y, z, p, q, and f an arbitrary functional symbol) can be integrated as follows; viz. taking m_1, m_2 as the roots of the quadratic equation

$$m^2-Sm+RT-UV=0,$$

(that is, writing $m_1+m_2=S$ and $m_1m_2=RT-UV$), then, m_1 denoting either root at pleasure, and m_2 the other root of the quadratic equation, if the system of ordinary differential equations

$$\begin{aligned} m_1dx-R\,dy+U\,dq&=0,\\ -T\,dx+m_2dy+U\,dp&=0,\\ -V\,dx+m_2dq+R\,dp&=0,\\ -V\,dy+T\,dq+m_1dp&=0,\\ -p\,dx-q\,dy+\quad dz&=0,\end{aligned}$$

(equivalent to three independent equations) admits of two integrals $u=\text{const.}$ and $v=\text{const.}$, the solution of the given partial differential equation is $u=f(v)$.

In fact, to prove this, we have

$$\begin{aligned} du=\ &\lambda(\ m_1dx-R\,dy+U\,dq)\\ &+\mu(-T\,dx+m_2dy+U\,dp)\\ &+\nu(-V\,dx+m_2dq+R\,dp)\\ &+\rho(-V\,dy+T\,dq+m_1dp)\\ &+\sigma(-p\,dx-q\,dy+\quad dz),\end{aligned}$$

that is,

$$\begin{aligned}
\frac{du}{dx} &= \lambda m_1 - \mu T - \nu V \qquad - \sigma p,\\
\frac{du}{dy} &= -\lambda R + \mu m_2 \qquad - \rho V - \sigma q,\\
\frac{du}{dz} &= \qquad\qquad\qquad\qquad \sigma,\\
\frac{du}{dp} &= \qquad \mu U + \nu R + \rho m_1 \qquad ,\\
\frac{du}{dq} &= \lambda U \qquad + \nu m_2 + \rho T \qquad ,
\end{aligned}$$

and thence

$$\frac{du}{dx} + \frac{du}{dz} p + \frac{du}{dp} r + \frac{du}{dq} s = \lambda (m_1 + Us) + \mu (-T + Ur) + \nu (-V + Rr + m_2 s) + \rho (m_1 r + Ts),$$

$$\frac{du}{dy} + \frac{du}{dz} q + \frac{du}{dp} s + \frac{du}{dq} t = \lambda (-R + Ut) + \mu (m_2 + Us) + \nu (Rs + m_2 t) + \rho (-V + m_1 s + Tt),$$

which equations may be represented by

$$\frac{d(u)}{dx} = A\lambda + B\mu + C\nu + D\rho,$$

$$\frac{d(u)}{dy} = A'\lambda + B'\mu + C'\nu + D'\rho;$$

and for λ, μ, ν, ρ writing λ', μ', ν', ρ', we have similarly

$$\frac{d(v)}{dx} = A\lambda' + B\mu' + C\nu' + D\rho',$$

$$\frac{d(v)}{dy} = A'\lambda' + B'\mu' + C'\nu' + D'\rho';$$

whence

$$\frac{d(u)}{dx}\,\frac{d(v)}{dy} - \frac{d(u)}{dy}\,\frac{d(v)}{dx} = \begin{vmatrix} A\lambda + B\mu + C\nu + D\rho, & A\lambda' + B\mu' + C\nu' + D\rho' \\ A'\lambda + B'\mu + C'\nu + D'\rho, & A'\lambda' + B'\mu' + C'\nu' + D'\rho' \end{vmatrix}.$$

The determinant is

$$\begin{aligned}
= \quad & (AD' - A'D)(\lambda\rho' - \lambda'\rho) + (BD' - B'D)(\mu\rho' - \mu'\rho)\\
& + (CD' - C'D)(\nu\rho' - \nu'\rho) + (BC' - B'C)(\mu\nu' - \mu'\nu)\\
& + (CA' - C'A)(\nu\lambda' - \nu'\lambda) + (AB' - A'B)(\lambda\mu' - \lambda'\mu).
\end{aligned}$$

The determinants $AD' - A'D$, &c., each of them contain the factor

$$\Theta, = Rr + Ss + Tt + U(s^2 - rt) - V;$$

viz. we have

$$\begin{aligned} AD' - A'D &= \quad m_1\Theta, & BC' - B'C &= -m_2\Theta, \\ BD' - B'D &= -\ T\Theta, & CA' - C'A &= -R\Theta, \\ CD' - C'D &= -\ V\Theta, & AB' - A'B &= \quad U\Theta, \end{aligned}$$

values which give

$$\begin{aligned} &(AD' - A'D)(BC' - B'C) + (BD' - B'D)(CA' - C'A) \\ &\qquad + (CD' - C'D)(AB' - A'B) = \Theta^2(-m_1 m_2 + TR - VU) = 0, \end{aligned}$$

as it should be.

Hence, when the partial differential equation $\Theta = 0$ is satisfied, we have

$$\frac{d(u)}{dx}\frac{d(v)}{dy} - \frac{d(u)}{dy}\frac{d(v)}{dx} = 0;$$

and we thence have $u = f(v)$ as the integral of the partial differential equation.

It should be possible to express analytically the conditions in order that the systems of differential equations may have one or each of them two integrals.

It is interesting to remark that, if each of the two systems of ordinary differential equations has only a single integral, these two integrals do *not* lead to the solution of the partial differential equation. Consider, for instance, the case

$$R = 0, \quad S = x + y, \quad T = 0, \quad U = 0, \quad V = p + q;$$

the partial differential equation is here

$$(x + y)s - (p + q) = 0,$$

which has an integral

$$z = (x + y)\{\phi'(x) + \psi'(y)\} - 2\{\phi(x) + \psi(y)\},$$

where ϕ, ψ are arbitrary functions: the equation in m is $m^2 - m(x + y) = 0$, the roots of which are $m = 0$, and $m = x + y$.

For $m_1 = 0$, $m_2 = x + y$, the system of differential equations becomes

$$\begin{aligned} dy &= 0, \\ -(p + q)\,dx + (x + y)\,dq &= 0, \\ -p\,dx + dz \qquad\qquad &= 0, \end{aligned}$$

which has only the integral $y = \text{const.}$; and similarly for $m_1 = x + y$, $m_2 = 0$, the system becomes

$$\begin{aligned} dx &= 0, \\ -(p + q)\,dy + (x + y)\,dp &= 0, \\ -q\,dy + dz \qquad\qquad &= 0, \end{aligned}$$

which has only the integral $x = \text{const.}$ And these two integrals $y = \text{const.}$ and $x = \text{const.}$ do not in anywise lead to the integral of the partial differential equation.

I take the opportunity of remarking that the complete system of conditions in order that the differential

$$Adx+Bdy+Cdz+Ddw$$

may be $=MdU$ is as follows: viz. writing

$$A,\ B,\ C,\ D=1,\ 2,\ 3,\ 4;$$

$$\frac{dB}{dz}-\frac{dC}{dy},\ \frac{dC}{dx}-\frac{dA}{dz},\ \frac{dA}{dy}-\frac{dB}{dx},\ \frac{dA}{dw}-\frac{dD}{dx},\ \frac{dB}{dw}-\frac{dD}{dy},\ \frac{dC}{dw}-\frac{dD}{dz}=23,\ 31,\ 12,\ 14,\ 24,\ 34,$$

where of course $12=-21$, &c., and

$$\overline{123}=1\,.\,23+2\,.\,31+3\,.\,12,\ \text{\&c.};\quad \overline{1234}=1\,.\,234-2\,.\,341+3\,.\,412-4\,.\,123,$$

is $=0$ identically;

$$1234=12\,.\,34+13\,.\,42+14\,.\,23,$$

then the conditions equivalent to three independent conditions are

$$\overline{234}=0,\quad \overline{341}=0,\quad \overline{412}=0,\quad \overline{123}=0,\quad 1234=0.$$

In fact, the first four equations are

$$\begin{aligned}
&\quad .\quad 2\,.\,34-3\,.\,24+4\,.\,23=0,\\
&-1\,.\,34\quad .\quad +3\,.\,14+4\,.\,31=0,\\
&\ 1\,.\,24-2\,.\,14\quad .\quad +4\,.\,12=0,\\
&-1\,.\,23-2\,.\,31-3\,.\,12\quad .\quad =0;
\end{aligned}$$

hence, multiplying by 1, 2, 3, 4 respectively and adding, we have the identity $\overline{1234}=0$, so that these four are equivalent to three independent equations: and multiplying by

$$\begin{aligned}
&\quad .\quad 12\,.\,\mu-31\,.\,\nu+14\,.\,\rho,\\
&-12\,.\,\lambda\quad .\quad +23\,.\,\nu+24\,.\,\rho,\\
&\ 31\,.\,\lambda-23\,.\,\mu\quad .\quad +34\,.\,\rho,\\
&-14\,.\,\lambda-24\,.\,\mu-34\,.\,\nu\quad .
\end{aligned}$$

respectively, (where λ, μ, ν, ρ are arbitrary), we have

$$(1\,.\,\lambda+2\,.\,\mu+3\,.\,\nu+4\,.\,\rho)(23\,.\,14+31\,.\,24+12\,.\,34)=0,$$

that is,

$$23\,.\,14+31\,.\,24+12\,.\,34=0,\quad \text{or}\quad 1234=0,$$

the fifth condition.

942.

ON SEMINVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 66—69.]

I WISH to prove the following negative: a given sharp seminvariant is not in every case obtainable by mere derivation from a form of the same extent and of the next inferior degree. The meaning of the statement will be explained.

According to the general theory developed in Clebsch's *Theorie der binären algebraischen Formen*, Leipzig, 1872, the covariants of a given binary quantic f are all of them obtainable, the covariants of a given degree from those of the next inferior degree, by derivation (Ueberschiebung) of these with f; viz. if the covariants of the next inferior degree are P, Q, &c., then the covariants of the degree in question are all of them included among the forms

$$\begin{array}{llll} (f, P)^0 (=fP), & (f, P)^1, & (f, P)^2, \ldots, \\ (f, Q)^0 & , (f, Q)^1, & (f, Q)^2, \ldots, \\ \&c., & & \end{array}$$

the index of derivation for (f, P) being at most equal to the degree of f or to that of P, whichever of these is the smaller, and so for Q, &c. The forms thus obtained are far too numerous; but rejecting repetitions, we have a complete system of the covariants of the given degree, viz. every covariant whatever of that degree is a linear function (with numerical multipliers) of the several distinct forms thus obtained by derivation.

We can therefore, by linear combination as above, obtain all the sharp covariants of the given degree, but we may very well have a sharp covariant not included among the several distinct forms thus obtained by derivation, but only expressible as a linear combination of two or more such forms; or say we may very well have a sharp

covariant not obtainable by *mere* derivation from the forms of the next inferior degree; and it is important to verify that there are sharp covariants not thus obtainable by mere derivation. I remark that the notion of a sharp covariant does not present itself in Clebsch, and that it will be presently explained.

Passing from a covariant to its leading coefficient which is a seminvariant, the statement may be applied to seminvariants; viz. it is to be verified that there are sharp seminvariants not obtainable by mere derivation from the seminvariants of the next inferior degree. But we have to introduce the notion of "extent" so as to connect the seminvariant with a quantic of some particular order, thus, if the highest letter of the seminvariant is f, we say that the extent is $=5$, and thus connect it with the quintic

$$(1, b, c, d, e, f \between x, y)^5.$$

The notion "sharp" applies to the seminvariants of a given weight. Suppose, for instance, the weight is $=8$; we have a series of initial or non-unitary terms i, cg, df, e^2, &c., and a series of final or power-ending terms e^2, cd^2, b^2d^2, c^4, &c., and we denote by $cg - c^4$ (where observe that here and in all similar cases the $-$ is not a minus sign, but is simply a stroke), the whole series of terms (including cg and c^4) which are in counter-order (CO) subsequent to cg, and in alphabetical order (AO) precedent to c^4; and so in other cases. This being so, arranging the seminvariants with their final terms in AO, we have the seminvariants

$$\begin{aligned} &i - e^2, \\ &cg - cd^2, \\ &df - b^2d^2, \\ &e^2 - c^4, \\ &\&c., \end{aligned}$$

viz. we have a seminvariant $i - e^2$ containing all or any (in fact, all) of the terms of this set as above defined; a seminvariant $cg - cd^2$ containing all or any of the terms of this set, a seminvariant $df - b^2d^2$ containing all or any of the terms of this set; and so on. These are sharp forms; a seminvariant ending in e^2, must of necessity have the leading term i, and thus belong at least to the octic

$$(1, b, c, d, e, f, g, h, i \between x, y)^8,$$

a seminvariant ending in cd^2 must of necessity have a leading term as high as cg, and thus belong at least to the sextic

$$(1, b, c, d, e, f, g \between x, y)^6.$$

Any linear combination of these would be a seminvariant $i - cd^2$, belonging to the octic, but it is not a sharp form; the final term cd^2 does not of necessity imply an initial so high as i (in fact, as we have seen, it only implies the lower initial cg): and so in other cases.

For the quintic $(1, b, c, d, e, f \between x, y)^5$, we have (for the weight 8 and degree 4) the seminvariants $df - b^2d^2$, and $e^2 - c^4$, this last belongs, of course, also to the quartic $(1, b, c, d, e \between x, y)^4$, it is, in fact, the squared quadrinvariant $(e - 4bd + 3c^2)^2$. I wish to

show that $df - b^2d^2$ is not obtainable by mere derivation from the covariants of degree 3 of the quartic.

The quintic and its covariants up to the degree 3 are

$$A = (\boxed{1 \mid 5b \mid 10c \mid 10d \mid 5e \mid f}\between x, y)^5,$$

say

$$(1, b, c, d, e, f \between x, y)^5;$$

$$B = (\boxed{\begin{array}{c|c|c} e+1 & f+1 & bf+1 \\ bd-4 & be-3 & ce-4 \\ c^2+3 & cd+2 & d^2+3 \end{array}}\between x, y)^2,$$

say

$$(e - c^2, f - cd, bf - d^2 \between x, y)^2;$$

$$C = (c - b^2, d - bc, e - c^2, f - cd, bf - d^2, cf - de, df - e^2 \between x, y)^6,$$
$$D = (ce - c^3, cf - c^2d, df - cd^2, bdf - d^3 \between x, y)^3,$$
$$E = (f - bc^2, bf - c^3, cf - c^2d, df - cd^2, ef - d^3, f^2 - d^2e \between x, y)^6,$$
$$F = (d - b^3, e - b^2c, f - bc^2, bf - c^3, cf - c^2d, df - cd^2,$$
$$ef - d^3, f^2 - d^2e, bf^2 - de^2, cf^2 - e^3 \between x, y)^9.$$

Hence all the covariants of the degree 3 are A^3, AB, AC, D, E, F, where

$$AB = (e - c^2, f - bc^2, bf - c^3, cf - c^2d, bcf - cd^2, bdf - d^3, bef - d^2e, bf^2 - d^2f \between x, y)^7$$
$$AC = (c - b^2, d - bc, \ldots \between x, y)^{11};$$

and the derivatives are

$(A, A^3)^0$, (A, A^3), &c.	weights of leading coefficients are	0, 1, 2, 3, 4, 5
$(A, AB)^0$, (A, AB), &c.	.....	4, 5, 6, 7, $\underline{8}$, 9
$(A, AC)^0$, (A, AC), &c.	.....	2, 3, 4, 5, $\underline{6}$, 7
$(A, D)^0$, (A, D), &c.	.....	6, 7, $\underline{8}$, 9
$(A, E)^0$, (A, E), &c.	.....	5, 6, 7, $\underline{8}$, 9, 10
$(A, F)^0$, (A, F), &c.	.....	3, 4, 5, 6, 7, $\underline{8}$.

The terms giving rise to a seminvariant of weight 8 are

$$(A, AB)^4 = \left\{\begin{matrix} e - c^2, & f - bc^2, & bf - c^3, & cf - c^2d, & bcf - cd^2 \\ e, & d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, D)^2 = \left\{\begin{matrix} ce - c^3, & cf - c^2d, & df - cd^2 \\ c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, E)^3 = \left\{\begin{matrix} f - bc^2, & bf - c^3, & cf - c^2d, & df - cd^2 \\ d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, F)^5 = \left\{\begin{matrix} d - b^3, & e - b^2c, & f - bc^2, & bf - c^3, & cf - c^2d, & df - cd^2 \\ f, & e, & d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

where to explain the algorithm, I remark, that if

$$A = (A_0,\ A_1,\ A_2, \ldots \between x,\ y)^5 \text{ and } D = (D_0,\ D_1,\ D_2, \ldots \between x,\ y)^3,$$

then

$$(A,\ D)^2 = D_0 A_2 - 2D_1 A_1 + D_2 A_0,$$

represented as above by

$$(A,\ D)^2 = \begin{Bmatrix} D_0, & D_1, & D_2 \\ A_2, & A_0, & A_1 \end{Bmatrix} = \begin{Bmatrix} ce - c^3, & cf - c^2 d, & df - cd^2 \\ c\ , & b\ , & 1 \end{Bmatrix}.$$

The result is in every case given as $df - c^4$; in each case there is only a single term $c \,.\, c^3, = c^4$, and the term in c^4 certainly presents itself. In $(A,\ AB)^4$ there is a single term $d \,.\, f, = df$, and in $(A,\ D)^2$ a single term df, and thus the term df certainly presents itself: in $(A,\ E)^3$ there are two terms $d \,.\, f, = df$ and df, and it is conceivable that, inserting the proper numerical coefficients, these might destroy each other: if this were so, the form instead of being $df - c^4$ would be $e^2 - c^4$; and similarly in $(A,\ F)^5$, there are two terms $df, = df$ and df, which it is conceivable might destroy each other, and the form would then be $e^2 - c^4$. But in every case we have the term c^4, and it thus appears that the form $df - b^2 d^2$ is not obtainable by mere derivation.

The form in question is in fact obtained by a linear combination of $df - c^4$ and $e^2 - c^4$, viz. writing down the leading coefficients of the covariants B^2 and H, we have

	$3H$	$-2B^2$	$=$
df	3		3
e^2	0	-2	-2
bcf	-9		-9
bde	-15	$+16$	$+1$
c^2e	$+30$	-12	-18
cd^2	-12		-12
b^3f	$+6$		$+6$
b^2ce	-15		-15
b^2d^2	$+42$	-32	-10
bc^2d	-48	$+48$	0
c^4	$+18$	-18	0

viz. the form in question $df - b^2d^2$ is $= 3df - 2e^2 - \ldots - 10b^2d^2$.

943.

ON RECIPROCANTS AND DIFFERENTIAL INVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 169—194, 289—307.]

I USE the term Reciprocant to denote a function of an arbitrary variable or variables and its differential coefficients, not connected with any differential equation; and Differential Invariant to denote a function of the coefficients of a differential equation and the differential coefficients of these coefficients. Halphen's differential invariants are thus reciprocants, but the term reciprocant is not made use of by him. I have entitled the present paper "On Reciprocants and Differential Invariants"; in the earlier part, (except that for preserving a chronological order, I briefly refer to Sir J. Cockle's Criticoids, which are differential invariants or rather seminvariants), I attend almost exclusively to Reciprocants, reproducing and explaining, and in some parts developing, the theories of Ampère, Halphen in his first two memoirs, and Sylvester.

I.

The notion of a reciprocant first presents itself in Ampère's "Mémoire sur les avantages qu'on peut retirer dans la théorie des courbes de la considération des paraboles osculatrices, avec des reflexions sur les fonctions différentielles dont la valeur ne change pas lors de la transformation des axes," *Journ. École Polyt.* t. VII. (1808), pp. 151—191 (sent to the Institute, Dec. 1803)*. We have (p. 167) for the radius of curvature the expression

$$-\frac{(1+y'^2)^{\frac{3}{2}}}{y''},$$

* A reciprocant, the Schwarzian derivative, occurs in Lagrange's memoir, "Sur la construction des cartes géographiques," *Nouv. Mém. de Berlin*, 1779, *Œuvres*, t. IV. p. 651, but scarcely *quà* reciprocant, viz. the form in question $\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$, presents itself in the equation

$$\frac{f'''(u+ti)}{f'(u+ti)}-\frac{3}{2}\left(\frac{f''(u+ti)}{f'(u+ti)}\right)^2=\frac{F'''(u-ti)}{F'(u-ti)}-\frac{3}{2}\left(\frac{F''(u-ti)}{F'(u-ti)}\right)^2,$$

and for the parameter of the osculating parabola the expression

$$-\frac{54y''^5}{\{y'''^2+3(y''^2-y'y''')^2\}^{\frac{3}{2}}};$$

and it is explicitly noticed that each of these expressions remains absolutely unaltered when the coordinates x, y are changed into any other (rectangular) coordinates whatever.

The functions here spoken of are thus orthogonal absolute reciprocants; the change which leaves them unaltered is that of x, y into X, Y, where

$$X = x\cos\theta + y\sin\theta + \alpha, \quad Y = -x\sin\theta + y\cos\theta + \beta.$$

For the more general change

$$X = ax + by + \alpha, \quad Y = cx + dy + \beta,$$

we find without difficulty

$$Y' = \frac{c+dy'}{a-by'}, \quad Y'' = \frac{(ad-bc)\,y''}{(a-by')^3},$$

and thus there is *not* an identity of form in the expressions

$$\frac{(1+Y'^2)^{\frac{3}{2}}}{Y''} \text{ and } \frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

But if we disregard factors, or (what is the same thing) attend to the equations $y''=0$ and $Y''=0$, we see that for this more general change one of these equations implies the other; each of them is, in fact, the condition for an inflexion of the curve in (x, y) or (X, Y).

II.

The differential invariants, or rather seminvariants, called "Criticoids," were considered by Sir James Cockle in his paper "On Criticoids," *Phil. Mag.* t. XXXIX. (1870), pp. 201—221; these will be more particularly considered further on, but at present it is sufficient to remark that the functions in question are connected with a linear differential equation, viz. they are functions of the coefficients of a linear differential equation, such that effecting thereon a transformation of the dependent variable, say $y=f.Y$, where f is an arbitrary function of the independent variable x, the function is equal to the like function of the coefficients of the new equation. For instance, if by the transformation in question $y=f.Y$ we transform

$$\frac{d^2y}{dx^2}+2b\frac{dy}{dx}+cy=0 \text{ into } \frac{d^2Y}{dX^2}+2B\frac{dY}{dX}+CY=0,$$

then we have

$$B = b+\frac{f'}{f}, \quad C = c+2b\frac{f'}{f}+\frac{f''}{f},$$

and thence

$$C-B^2-B' = c-b^2-b',$$

where F is the conjugate function of f, an equation implying that each side thereof is equal to one and the same constant; $u-ti$ is of course not a function of $u+ti$, and thus we have here no property that the function $\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$ remains unaltered when for the arguments x, y thereof we substitute X, Y, determinate functions of these arguments.

(where the accents denote differentiation in regard to x): and thus $c - b^2 - b'$ is an invariant of the form in question; say it is an α-seminvariant of the differential equation.

III.

We have four important Memoirs by Halphen, (1) "Thèse d'Analyse sur les invariants différentiels," Paris (1878), (2) "Sur les invariants différentiels des courbes gauches," *Jour. École Polyt.* Cah. XLVII. (1880), pp. 1—102; (3) "Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables," *Mém. Sav. Étrang.* t. XXVIII. (1883), pp. 1—297, and (4) "Sur les invariants des équations différentielles linéaires du quatrième ordre," *Acta Math.* t. III. (1883), pp. 325—380.

In the Memoir, Halphen (1), the investigations are in the first instance presented in a geometrical form; the author considers for instance the inflexions of a plane curve, and so obtains the invariant y'', or using his notation

$$\frac{1}{1.2\ldots k}\frac{d^k y}{dx^k} = a_k,$$

say the invariant a_2 of the weight 2; similarly, the consideration of the sextactic points gives him the invariant $a_2^2 a_5 - 3a_2 a_3 a_4 + 2a_3^3$ of the weight 9; and that of the points of nine-pointic contact a more complicated invariant of the weight 27: and with these he forms absolute invariants. But the formal analytical definition is given § 3, "Théorie des invariants différentiels jusqu'au huitième ordre exclusivement," viz. considering the general homographic transformation

$$X = \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad Y = \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''},$$

then if the equation

$$f\left(x, y, \frac{dy}{dx}, \ldots \frac{d^n y}{dx^n}\right) = 0$$

is unaltered in form by the change (x, y) into (X, Y), or what is the same thing if the function

$$f\left(x, y, \frac{dy}{dx}, \ldots \frac{d^n y}{dx^n}\right)$$

be unaltered save as to a factor, then such function is said to be a differential invariant: and the quotient of two functions having the same factor, which quotient is therefore unaltered, is said to be an absolute invariant. We may for these terms substitute reciprocant, and absolute reciprocant respectively. The theory is scarcely worked out from the definition, and indeed, as will presently appear, it is by no means easy to work out directly from the definition, even the foregoing sextactic invariant $a_2^2 a_5 - 3a_2 a_3 a_4 + 2a_3^3$. I remark that, in what follows, instead of Halphen's $a_2, a_3, a_4, a_5, \ldots$ I write $a, b, c, d, \ldots$, viz. these are used to denote respectively

$$\tfrac{1}{2}\frac{d^2 y}{dx^2}, \quad \tfrac{1}{6}\frac{d^3 y}{dx^3}, \quad \tfrac{1}{24}\frac{d^4 y}{dx^4}, \quad \tfrac{1}{120}\frac{d^5 y}{dx^5}, \ldots,$$

the first differential coefficient $\frac{dy}{dx}$ being denoted by t. The foregoing invariant, or say the sextactic reciprocant, is thus $=a^2d-3abc+2b^3$. (See *post*, XI.)

IV.

For the analytical theory, we have as above

$$X=\frac{\alpha x+\beta y+\gamma}{\alpha''x+\beta''y+\gamma''},\quad Y=\frac{\alpha'x+\beta'y+\gamma'}{\alpha''x+\beta''y+\gamma''};$$

taking h, k for the increments of x, y respectively, we have

$$k=th+ah^2+bh^3+ch^4+dh^5+\ldots,$$

and, similarly, taking H, K for the increments of X, Y respectively, we have

$$K=TH+AH^2+BH^3+CH^4+DH^5+\ldots,$$

and we require the relations between the two sets of coefficients $(t, a, b, c, d, \ldots)$ and $(T, A, B, C, D, \ldots)$: I propose to develop these up to d, D, so as to obtain the theory for the form $a^2d-3abc+2b^3$.

Writing for shortness ξ, η, ζ for

$$\alpha x+\beta y+\gamma,\quad \alpha'x+\beta'y+\gamma',\quad \alpha''x+\beta''y+\gamma''$$

respectively, we have

$$X+H=\frac{\alpha(x+h)+\beta(y+k)+\gamma}{\alpha''(x+h)+\beta''(y+k)+\gamma''},=\frac{\xi+\alpha h+\beta k}{\zeta+\alpha''h+\beta''k},$$

and thence

$$H=\frac{\xi+\alpha h+\beta k}{\zeta+\alpha''h+\beta''k}-\frac{\xi}{\zeta},$$

$$=\frac{(\alpha\zeta-\alpha''\xi)h+(\beta\zeta-\beta''\xi)k}{\zeta(\zeta+\alpha''h+\beta''k)}=\frac{Ph+Qk}{\zeta(\zeta+\alpha''h+\beta''k)},$$

if

$$P=\alpha\zeta-\alpha''\xi,\quad Q=\beta\zeta-\beta''\xi,$$

or, for k substituting its value,

$$H=\frac{(P+Qt)h+Q(ah^2+bh^3+ch^4+dh^5)}{\zeta\{\zeta+(\alpha''+\beta''t)h+\beta''(ah^2+bh^3+ch^4+dh^5)\}},$$

$$=\frac{L\{h+\lambda(ah^2+bh^3+ch^4+dh^5)\}}{\zeta[1+L''\{h+\lambda''(ah^2+bh^3+ch^4+dh^5)\}]},$$

if

$$L=P+Qt,\quad \lambda=\frac{Q}{P+Qt},$$

$$L''=\alpha''+\beta''t,\quad \lambda''=\frac{\beta''}{\alpha''+\beta''t},$$

or, putting $\mu = \frac{\zeta}{L}$, say this is

$$\mu H = \frac{h + \lambda(ah^2 + bh^3 + ch^4 + dh^5)}{1 + L''\{h + \lambda''(ah^2 + bh^3 + ch^4 + dh^5)\}}.$$

Similarly, if

$$P' = \alpha'\zeta - \alpha''\eta, \quad Q' = \beta'\zeta - \beta''\eta, \quad L' = P' + Q't, \quad \lambda' = \frac{Q'}{P' + Q't}, \text{ and } \mu' = \frac{\zeta}{L'},$$

then

$$\mu' K = \frac{h + \lambda'(ah^2 + bh^3 + ch^4 + dh^5)}{1 + L''\{h + \lambda''(ah^2 + bh^3 + ch^4 + dh^5)\}};$$

we have thus found H and K each of them in terms of h, and the elimination of h leads to an equation between H and K expressible in the form

$$K = TH + AH^2 + BH^3 + CH^4 + DH^5.$$

Writing for convenience

$$ah^2 + bh^3 + ch^4 + dh^5 = h\Omega,$$

that is,

$$\Omega = ah + bh^2 + ch^3 + dh^4,$$

we have

$$\mu H = \frac{h(1 + \lambda\Omega)}{1 + L''h(1 + \lambda''\Omega)},$$

$$\mu' K = \frac{h(1 + \lambda'\Omega)}{1 + L''h(1 + \lambda''\Omega)},$$

and thence

$$K = \frac{\mu}{\mu'} H \frac{1 + \lambda'\Omega}{1 + \lambda\Omega}, \quad = TH \frac{1 + \lambda'\Omega}{1 + \lambda\Omega},$$

since, when h is small or say $\Omega = 0$, we have $K = TH$; viz. we thus have $T = \frac{\mu}{\mu'}$.

Developing, we have

$$K = TH\{1 + (\lambda' - \lambda)\Omega - \lambda(\lambda' - \lambda)\Omega^2 + \lambda^2(\lambda' - \lambda)\Omega^3 - \lambda^3(\lambda' - \lambda)\Omega^4\}.$$

We have

$$\Omega = ah + bh^2 + ch^3 + dh^4,$$

$$h = \frac{1}{a}\Omega - \frac{b}{a^3}\Omega^2 + \frac{1}{a^5}(2b^2 - ac)\Omega^3 - \frac{1}{a^7}(5b^3 - 5abc + a^2d)\Omega^4$$

$$= p\Omega + q\Omega^2 + r\Omega^3 - s\Omega^4,$$

suppose, and hence

$$\mu H = \frac{(p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4)(1 + \lambda\Omega)}{1 + L''(p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4)(1 + \lambda''\Omega)}$$

$$= \frac{p\Omega + (q + \lambda p)\Omega^2 + (r + \lambda q)\Omega^3 + (s + \lambda r)\Omega^4}{1 + L''p\Omega + L''(q + \lambda''p)\Omega^2 + L''(r + \lambda''q)\Omega^3 + L''(s + \lambda'r)\Omega^4},$$

which gives H in terms of Ω, and conversely Ω in terms of H. We assume

$$\Omega = XH + YH^2 + ZH^3 + WH^4;$$

K is then given as above in terms of H, Ω, that is, in terms of H, and it is assumed that we have

$$K = TH + AH^2 + BH^3 + CH^4 + DH^5.$$

In the foregoing expression for K, substituting for Ω its value

$$XH + YH^2 + ZH^3 + WH^4,$$

and comparing coefficients, we find

$$\begin{aligned}
A &= (\lambda' - \lambda) \times \quad X, \\
B &= (\lambda' - \lambda) \times \quad Y - \lambda X^2, \\
C &= (\lambda' - \lambda) \times \quad Z - \lambda . 2XY \qquad + \lambda^2 X^3, \\
D &= (\lambda' - \lambda) \times \quad W - \lambda (2XZ + Y^2) + \lambda^2 . 3X^2 Y - \lambda^3 X^4;
\end{aligned}$$

and we have then, from the relation between H, Ω, to find the values of X, Y, Z, W. We have

$$\begin{aligned}
& p\,(XH + YH^2 + ZH^3 + WH^4) \\
& \quad - \mu H \\
& + (q + \lambda p)\,(X^2H^2 + 2XYH^3 + (2XZ + Y^2)\,H^4) \\
& \quad - \mu H . L''p \quad (XH + YH^2 + ZH^3) \\
& + (r + \lambda q)\,(X^3H^3 + 3X^2Y . H^4) \\
& \quad - \mu H . L''(q + \lambda'' p)\,(X^2H^2 + 2XYH^3) \\
& + (s + \lambda r)\,(X^4 . H^5) \\
& \quad - \mu H . L''(r + \lambda'' q)\,(X^3H^3) = 0.
\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}
& pX - \mu = 0, \\
& pY + (q + \lambda p)\,X^2 - \mu L'' pX = 0, \\
& pZ + (q + \lambda p)\,2XY + (r + \lambda q)\,X^3 - \mu L'' pY - \mu L''(q + \lambda'' p)\,X^2 = 0, \\
& pW + (q + \lambda p)(2XZ + Y^2) + (r + \lambda q)\,3X^2Y + (s + \lambda r)\,X^4 \\
& \qquad - \mu L'' pZ - \mu L''(q + \lambda'' p)\,2XY - \mu L''(r + \lambda'' q)\,X^3 = 0,
\end{aligned}$$

and substituting for p, q, r, s their values, we find successively after reductions, which for W are somewhat troublesome, the values

$$\begin{aligned}
X &= \mu a, \\
Y &= \mu^2 \{b + aL'' - a^2\lambda\}, \\
Z &= \mu^3 \{c + 2bL'' + aL''^2 - 3ab\lambda + 2a^3\lambda^2 - 3a^2\lambda L'' + a^3\lambda'' L''\}, \\
W &= \mu^4 \{d + 3cL'' + 3bL''^2 + aL''^3 - (4ac + 2b^2)\lambda + 10a^2b\lambda^2 - 5a^4\lambda^3 \\
&\qquad - 12ab\lambda L'' + 3ab\lambda'' L'' - 6a^2\lambda''^2 + 3a^3\lambda'' L''^2 + 10a^3\lambda^2 L'' - 4a^3\lambda\lambda'' L''\};
\end{aligned}$$

these values are to be substituted for X, Y, Z, W in the foregoing expressions of A, B, C, D.

I add the formulæ

$$\lambda' - \lambda = \frac{Q'}{P' + Q't} - \frac{Q}{P + Qt} = \frac{PQ' - P'Q}{(P + Qt)(P' + Qt)},$$

$$\begin{aligned}
PQ' - P'Q &= (\alpha\zeta - \alpha''\xi)(\beta'\zeta - \beta''\eta) - (\beta\zeta - \beta''\xi)(\alpha'\zeta - \alpha''\eta) \\
&= \zeta \begin{vmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \xi, & \eta, & \zeta \end{vmatrix} = \zeta \begin{vmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{vmatrix} = (\alpha'' x + \beta'' y + \gamma'') \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},
\end{aligned}$$

$$\begin{aligned} P + Qt &= \alpha\zeta - \alpha''\xi + (\beta\zeta - \beta''\xi)\,t \\ &= \alpha\,(\alpha''x + \beta''y + \gamma'') - \alpha''\,(\alpha x + \beta y + \gamma) + t\,\{\beta\,(\alpha''x + \beta''y + \gamma'') - \beta''\,(\alpha x + \beta y + \gamma)\}, \\ P' + Q't &= \alpha'\zeta - \alpha''\eta + (\beta'\zeta - \beta''\eta)\,t \\ &= \alpha'\,(\alpha''x + \beta''y + \gamma'') - \alpha''\,(\alpha'x + \beta'y + \gamma') + t\,\{\beta'\,(\alpha''x + \beta''y + \gamma'') - \beta''\,(\alpha'x + \beta'y + \gamma')\}, \end{aligned}$$

that is,

$$P + Qt = \begin{vmatrix} \alpha\,, & \beta\,, & \gamma \\ \alpha'', & \beta'', & \gamma'' \\ t\,, & -1, & y - tx \end{vmatrix}, \quad P' + Q't = \begin{vmatrix} \alpha', & \beta'\,, & \gamma' \\ \alpha'', & \beta''\,, & \gamma'' \\ t\,, & -1, & y - tx \end{vmatrix},$$

whence $\lambda' - \lambda$ is known.

Also

$$\mu = \frac{\zeta}{L} = \frac{\alpha''x + \beta''y + \gamma''}{P + Qt}, \quad \mu' = \frac{\zeta}{L'} = \frac{\alpha''x + \beta''y + \gamma''}{P' + Q't};$$

and

$$T = \frac{\mu}{\mu'} = \frac{L'}{L} = \frac{P' + Q't}{P + Qt}.$$

V.

We have

$$A = (\lambda' - \lambda)\,\mu a,$$

and thus a is a reciprocant, viz. $a = 0$ is the condition of an inflexion.

I remark that $ac - b^2$ is not a reciprocant; we have

$$AC - B^2 = (\lambda - \lambda')^2\,(XZ - Y^2),$$

and then

$$XZ - Y^2 = \mu^4\,\{(ac - b^2) - a^2\lambda\,(b - a^2\lambda) + a^3\,(\lambda'' - \lambda)\,L''\},$$

and thus $AC - B^2$ is not a multiple of $ac - b^2$, even in the particular case $L'' = 0$. I notice further that we have

$$\begin{aligned} (4AC - 5B^2) &= (\lambda' - \lambda)^2\,\{4XZ - 5Y^2 + 2\lambda X^2Y - \lambda^2X^4\}, \\ &= (\lambda' - \lambda)^2\,\mu^4\,\{4ac - 5b^2 - 2abL'' - a^2L''^2 + 4a^2\lambda''L''\}, \end{aligned}$$

viz. in the particular case $L'' = 0$, we have

$$4AC - 5B^2 = (\lambda' - \lambda)^2\,\mu^4\,(4ac - 5b^2),$$

and thus in the particular case $L'' = 0$ (or say, if $\alpha'' = 0$, $\beta'' = 0$, that is, if X, Y are mere linear functions of x, y) we have $4ac - 5b^2 = 0$ a reciprocant. But in the general case now in hand, it is not a reciprocant.

We have now to consider the sextactic form $a^2d - 3abc + 2b^3$. We have

$$A^2D - 3ABC + 2B^3 = (\lambda' - \lambda)^3\,\{X^2W - 3XYZ + 2Y^3 + \lambda X^2\,(XZ - Y^2)\},$$

and we proceed to calculate the expression in { }. Writing for shortness

$$\begin{aligned} X &= \mu a, \\ Y &= \mu^2\,(b + aL'' + Y_0), \\ Z &= \mu^3\,(c + 2bL'' + aL''^2 + Z_0), \\ W &= \mu^4\,(d + 3cL'' + 3bL''^2 + aL''^3 + W_0), \end{aligned}$$

and then, omitting a factor μ^6, the expression is

$$\begin{aligned} = \quad & a^2d - 3abc + 2b^3 \\ & + a^2W_0 - 3a(b + aL'')Z_0 + \{(-3ac + 6b^2) + 6abL'' + 3a^2L''^2\}Y_0 \\ & + 6(b + aL'')Y_0^2 - 3aY_0Z_0 + 2Y_0^3 \\ & + \lambda a^2\{ac - b^2 + aZ_0 - 2(b + aL'')Y_0 - Y_0^2\}, \end{aligned}$$

where the terms after the first term are in fact $= 0$; and this being so, we have

$$A^2D - 3ABC + 2B^3 = (\lambda' - \lambda)^3\mu^6(a^2d - 3abc + 2b^3),$$

and thus $a^2d - 3abc + 2b^3$ is a reciprocant; but observe that the factor being here $(\lambda' - \lambda)^3\mu^6$, and in the equation $A = (\lambda' - \lambda)\mu a$ the factor being $(\lambda' - \lambda)\mu$, we cannot with the reciprocants a and $a^2d - 3abc + 2b^3$ form an absolute reciprocant.

To verify the evanescence of the above-mentioned terms, observe that, considering first the terms independent of L'', we have

$$\begin{aligned} a^2W_0 &= -4a^3c\lambda - 2a^2b^2\lambda + 10a^4b\lambda^2 - 5a^5\lambda^3, \\ -3abZ_0 &= \qquad\qquad + 9a^2b^2\lambda - 6a^4b\lambda^2, \\ +(-3ac + 6b^2)Y_0 &= +3a^3c\lambda - 6a^2b^2\lambda, \\ -3aY_0Z_0 &= \qquad\qquad\qquad\qquad - 9a^4b\lambda^2 + 6a^5\lambda^3, \\ +6bY_0^2 &= \qquad\qquad\qquad\qquad + 6a^4b\lambda^2, \\ +2Y_0^3 &= \qquad\qquad\qquad\qquad\qquad\qquad - 2a^5\lambda^3, \\ +\lambda a^2(ac - b^2) &= +a^3c\lambda - a^2b^2\lambda, \\ +\lambda a^2Z_0 &= \qquad\qquad\qquad\qquad - 3a^4b\lambda^2 + 2a^5\lambda^3, \\ +\lambda a^2 . -2bY_0 &= \qquad\qquad\qquad\qquad + 2a^4b\lambda^2, \\ +\lambda a^2 . -Y_0^2 &= \qquad\qquad\qquad\qquad\qquad\qquad - a^5\lambda^3; \end{aligned}$$

the sum of which is in fact $= 0$. And next for the terms containing L'', we have

$$\begin{aligned} & a^2W_0 = \\ & \quad -12a^3b\lambda L'' + 3a^3b\lambda''L'' - 6a^4\lambda L''^2 + 3a^4\lambda''L''^2 + 10a^5\lambda^2L'' - 4a^5\lambda\lambda''L'', \\ & -3abZ_0 = \\ & \quad + 9a^3b\lambda L'' - 3a^3b\lambda''L'', \\ & -3a^2L''Z_0 = \\ & \quad + 9a^3b\lambda L'' \qquad + 9a^4\lambda L''^2 - 3a^4\lambda''L''^2 - 6a^5\lambda^2L'', \\ & (6abL'' + 3a^2L''^2)Y_0 = \\ & \quad - 6a^3b\lambda L'' \qquad - 3a^4\lambda L''^2, \\ & -3aY_0Z_0 = \\ & \qquad\qquad\qquad\qquad - 9a^5\lambda^2L'' + 3a^5\lambda\lambda''L'', \\ & +6(b + aL'')Y_0^2 = \\ & \qquad\qquad\qquad\qquad + 6a^5\lambda^2L'', \\ & +\lambda a^2Z_0 = \\ & \qquad\qquad\qquad\qquad - 3a^5\lambda^2L'' + a^5\lambda\lambda''L'', \\ & -2\lambda a^2(b + aL'')Y_0 = \\ & \qquad\qquad\qquad\qquad + 2a^5\lambda^2L'', \end{aligned}$$

the sum of which is also $= 0$.

Before going further, it is proper to remark that $a, b, c, \ldots$ as representing the second, third, &c., differential coefficients of y are considered as being of the orders $2, 3, 4, \ldots$, and the order of a reciprocant is taken to be the order of the highest letter contained therein: the degree means the degree in these letters $a, b, c, \ldots$, and the weight the weight in these letters, reckoning them as of the weights $2, 3, 4, \ldots$. A reciprocant is a homogeneous isobaric function of $a, b, c, \ldots$, not involving y or the first differential coefficient.

VI.

To the reciprocants thus obtained, say

$$U = a, \quad V = a^2d - 3abc + 2b^3,$$

Halphen adds another reciprocant Δ (that of nine-pointic contact) which he expresses in the form of a determinant.

Writing down in connexion therewith its developed expression, we have

$$\Delta = \begin{vmatrix} b, & c, & d, & e, & f \\ a, & b, & c, & d, & e \\ -a^2, & 0, & b^2, & 2bc, & 2bd + c^2 \\ 0, & a^2, & 2ab, & 2ac + b^2, & 2ad + 2bc \\ 0, & 0, & a^2, & 3ab, & 3ac + 3b^2 \end{vmatrix}, \; = \begin{array}{rrr} a^6 & df & +1 \\ & e^2 & -1 \\ \hline +a^5 & bcf & -3 \\ & bde & +3 \\ & c^2e & +4 \\ & cd^2 & -5 \\ \hline +a^4 & b^2f & +2 \\ & b^2ce & -5 \\ & b^2d^2 & -1 \\ & bc^2d & +14 \\ & c^4 & -4 \\ \hline +a^3 & b^3cd & -10 \\ & b^3c^2 & -5 \\ \hline +a^2 & b^5d & +4 \\ & b^4c^2 & +15 \\ \hline +a^1 & b^6c & -12 \\ +a^0 & b^8 & +3, \end{array}$$

where observe that the whole term in the highest letter f is $a^4(a^2d - 3abc + 2b^3)f$, viz. the coefficient of f is the reciprocant $a^4(a^2d - 3abc + 2b^3)$, agreeing with a general theorem given by Halphen.

Δ is of the degree 8 and the weight 24, and it is seen without difficulty that the factor is $=(\lambda' - \lambda)^8 \mu^{16}$. He remarks that $25\Delta^3 - 27V^8$ vanishes for $a = 0$ (viz. it

becomes $256 . 27b^{24} - 27 . 256b^{24} = 0$), and not only so, but that it in fact contains the factor a^4. Assuming that this is so, viz. that the terms in a^0, a, a^2, a^3 all of them vanish, I have calculated the terms in a^4, and have thus obtained an incomplete expression for the quotient $(256\Delta^3 - 27V^8) \div a^4$, viz. this is

$$\begin{aligned} H = (256\Delta^3 - 27V^8) \div a^4 = \quad & 256a^{14}(df \quad - e^2)^3 \\ & \vdots \\ + 288b^{16} \; & b^3f \; + \; 48 \\ & b^2ce \; + \; 120 \\ & b^2d^2 \; - \; 64 \\ & bc^2d + 9288 \\ & c^4 \; + \; 81; \end{aligned}$$

where I remark that the term in b^{16} presented itself in the form

$$\begin{aligned} & 256\,(54b^{16}f - 135b^{15}ce + \; 117b^{15}d^2 + \; 5346b^{17}c^2d + \; 9477b^{16}c^4) \\ - \; & 27\,(\qquad\qquad 1792b^{18}d^2 - 48384b^{17}c^2d + 90720b^{16}c^4). \end{aligned}$$

We have thus Halphen's reciprocant H of the weight 64.

The reciprocants thus far obtained are consequently

	deg.	weight	factor
$U = a$	1	2	$(\lambda' - \lambda)\,\mu$
$V = a^2d - 3abc + 2b^3$	3	9	$(\lambda' - \lambda)^3\,\mu^6$
Δ	8	24	$(\lambda' - \lambda)^8\,\mu^{16}$
$H = (256\Delta^3 - 27V^8) \div a^4$	20	64	$(\lambda' - \lambda)^{20}\,\mu^{44}$

VII.

Reciprocants of the same degree and weight have the same factor, and may thus be combined in the way of addition, viz. R and S being reciprocants of the same degree and the same weight, then (α, β being constants) we have $\alpha R + \beta S$ a reciprocant of the same degree and the same weight. The quotient of two such reciprocants has the factor unity, or say it is an absolute invariant. Thus Δ^3 and V^8 have each of them the degree 24 and weight 72, so that $\alpha\Delta^3 + \beta V^8$ is a reciprocant; in particular, $256\Delta^3 - 27V^8$ is a reciprocant having the factor a^4, and it thus gives rise to the foregoing reciprocant $H = (256\Delta^3 - 27V^8) \div a^4$.

Moreover $\Delta^3 \div V^8$ is an absolute reciprocant.

Any two reciprocants R, S of the same degree and same weight, or, what is the same thing, any absolute reciprocant $R \div S$ gives rise to a reciprocant $RS' - R'S$, where

the accents denote differentiation in regard to x ($a'=3b$, $b'=4c$, $c'=5d, \ldots$); the order of the new reciprocant thus exceeds by unity the order of R or S, whichever of them is of the highest order.

From U, V, Δ, which are of the orders 2, 5, 7 respectively, it is thus possible to deduce a series of reciprocants T_8, T_9, $T_{10}, \ldots$ of the orders 8, 9, 10, ..., respectively: viz. we have from the absolute reciprocant $\Delta^3 V^{-8}$, first a reciprocant $3V\Delta' - 8V\Delta$, which, however, contains the factor U^3, and we have

$$U^3 T_8 = 3V\Delta' - 8V\Delta.$$

We then have the absolute reciprocant $U^4 T_8 . V^{-4}$, leading to

$$T_9 = UVT_8' \quad + 4(VU' - UV')\, T_8,$$

then the absolute reciprocant $U^4 T_9 . V^{-\frac{13}{3}}$, leading to

$$T_{10} = UVT_9' \quad + 4(VU' - \tfrac{4}{3}UV')\, T_9,$$

and so

$$T_{11} = UVT_{10}' \quad + 4(VU' - \tfrac{5}{3}UV')\, T_{10},$$

$$T_n = UVT'_{n-1} + 4\{VU' - \tfrac{1}{3}(n-6)\, UV'\}\, T_{n-1}.$$

Halphen considers that these are the only distinct reciprocants of the orders 2, 5, 7, 8, 9, ..., n; but remarks that we can, with the reciprocants up to any given order, form algebraical combinations, in some cases containing as factor a power of the reciprocant U or V, so that, rejecting this factor, we have a new independent reciprocant, that is, a reciprocant not expressible as a rational and integral function of inferior reciprocants; an instance hereof is the foregoing reciprocant

$$H = (256\Delta^3 - 27V^8) \div a^4.$$

Other like forms are given by Halphen, viz. writing with him T for shortness in place of T_8, we have for T the following expression in the form of a determinant

$$T = \begin{vmatrix} 3b, & 2a, & a, & 0, & 0 & , & 0 \\ 4c, & 3b, & b, & a, & 2a^2 & , & 0 \\ 5d, & 4c, & c, & 2b, & 5ab & , & a^2 \\ 6e, & 5d, & d, & 3c, & 6ac+3b^2 & , & 3ab \\ 7f, & 6e, & e, & 4d, & 7ad+7bc & , & 4ac+2b^2 \\ 8g, & 7f, & f, & 6e, & 8ae+8bd+4c^2, & & 5ad+5bc \end{vmatrix},$$

and then

$$T_1 = (V^4 T - \tfrac{1}{6}H) \div U,$$

$$G = (U^4 T^2 + 9H) \div V^2,$$

$$\Theta = \{2U\Delta T' + T(8U'\Delta - 3U\Delta')\} \div V,$$

$$\Theta_1 = (\Theta + \tfrac{1}{4}G) \div V,$$

$$\Theta_2 = (\Theta_1 - \tfrac{45}{2}TV) \div U,$$

where Θ is given explicitly in the form of a determinant, viz. this is

$$\Theta_2 = -432 \begin{vmatrix} c, & d, & e, & f, & g, & h \\ b, & c, & d, & e, & f, & g \\ a, & b, & c, & d, & e, & f \\ a^2, & 2ab, & 2ac+b^2, & 2ad+2bc, & 2ae+2bd+c^2, & 2af+2be+2cd \\ 0, & a^2, & 2ab, & 2ac+b^2, & 2ad+2bc, & 2ae+2bd+c^2 \\ 0, & 0, & a^2, & 3a^2b, & 3a^2c+3ab^2, & 3a^2d+6abc+b^3 \end{vmatrix},$$

where $\Theta_2 = 0$ is the differential equation of the ninth order of the general cubic curve.

Halphen's $S, = UVT' + 4(VU' - UV')T$, is the above-mentioned reciprocant T_9; U is connected with Θ by the formula $2\Delta S - V^2\Theta = U^4T^2$.

Halphen considers incidentally (p. 56), what may be called a polar relation of the variables X, Y and x, y; viz. this is

$$X = \frac{dy}{dx}, \quad Y = x\frac{dy}{dx} - y,$$

whence conversely

$$x = \frac{dY}{dX}, \quad y = X\frac{dY}{dX} - Y.$$

We may here express Y and its differential coefficients in regard to X, say Y, Y_1, $Y_2, \ldots$, in terms of y and its differential coefficients in regard to x, say y, y_1, $y_2, \ldots$ (to avoid confusion with other formulæ, I purposely use these notations, abstaining from the introduction of the letters a, b, $c, \ldots$). The geometrical signification is that x, y, being point-coordinates, then X, Y are line-coordinates. We find, without difficulty,

$$Y_1 = x, \quad Y_2 = \frac{1}{y_2}, \quad Y_3 = \frac{-y_3}{y_2^3}, \quad Y_4 = \frac{-y_2y_4 - 3y_3^2}{y_2^5}, \quad Y_5 = \frac{-y_2^2y_5 + 10y_2y_3y_4 + 15y_3^3}{y_2^7}, \text{ \&c.}$$

Hence, in particular,

$$9Y_2^2Y_5 - 45Y_2Y_3Y_4 + 40Y_3^3 = -\frac{1}{y_2^3}(9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3),$$

viz. the differential equation $9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0$ (in the former notation $a^2d - 3abc + 2b^3 = 0$) of the conic in point-coordinates gives, as it should do, the equation of like form $9Y_2^2Y_5 - 45Y_2Y_3Y_4 + 40Y_3^3 = 0$ of the conic in line-coordinates.

The geometrical applications throughout the memoir are very extensive and interesting.

VIII.

In Halphen's second memoir "Sur les invariants différentiels des courbes gauches" (1880), instead of a single dependent variable y a function of x, we have two dependent variables y, z, each of them a function of x.

The relations between the original variables x, y, z, and the new variables X, Y, Z, are of the general homographic form

$$X,\ Y,\ Z = \frac{\xi}{\omega},\ \frac{\eta}{\omega},\ \frac{\zeta}{\omega},$$

where

$$\begin{aligned}
\xi &= \alpha x + \beta y + \gamma z + \delta\ ,\\
\eta &= \alpha' x + \beta' y + \gamma' z + \delta'\ ,\\
\zeta &= \alpha'' x + \beta'' y + \gamma'' z + \delta''\ ,\\
\omega &= \alpha''' x + \beta''' y + \gamma''' z + \delta''',
\end{aligned}$$

but he does not from these formulæ deduce the expressions of $Y_{,}$, $Y_{,,}$, ..., $Z_{,}$, $Z_{,,}$, ... in terms of y', y'', ..., z', z'', ...; the investigation is in some measure a geometrical one. The notation employed is

$$u = \tfrac{1}{12}(y''z''' - y'''z''),$$

viz. u is here the most simple reciprocant, the equation $u = 0$ is obviously the condition of a plane curve:

$$a_n = \frac{1}{4.5.6\ldots n}\,\frac{y''z^{(n)} - y^{(n)}z''}{y''z''' - y'''z''}\ ,$$

$$b_n = \frac{-1}{3.4.5\ldots n}\,\frac{y''z^{(n)} - y^{(n)}z'''}{y''z''' - y'''z''}\ ,$$

n greater than or equal to 4; viz. these two singly infinite series of symbols a_4, a_5, a_6, ..., b_4, b_5, b_6, ..., together with u, are used for the expression of the doubly infinite series $y^{(m)}z^{(n)} - y^{(n)}z^{(m)}$, by virtue of the identity

$$\begin{aligned}
&(y''z''' - y'''z'')(y^{(m)}z^{(n)} - y^{(n)}z^{(m)})\\
&\qquad = (y''z^{(m)} - y^{(m)}z'')(y'''z^{(n)} - y^{(n)}z''') - (y''z^{(n)} - y^{(n)}z'')(y'''z^{(m)} - y^{(m)}z'''),
\end{aligned}$$

or, as this may be written,

$$\frac{1}{1.2\ldots m.1.2\ldots n}\,\frac{y^{(m)}z^{(n)} - y^{(n)}z^{(m)}}{u} = a_n b_m - a_m b_n.$$

The most simple reciprocant (after u) is

$$v = a_6 - 2b_5 - 3a_4a_5 + 3a_4b_4 + 2a_4^3,$$

or rather u^3v, which is an integral function of the differential coefficients: the signification of the equation $v = 0$ is that the tangents of the curve belong to a linear complex. This is a property belonging to the tangents of a skew cubic, and the skew cubic thus satisfies the differential equation $u = 0$. Another reciprocant is obtained

$$w = b_6 - a_4b_5 - 4a_5b_4 + 4a_4^2b_4 - 2a_4^2a_5 + 2b_4^2 + a_5^2 + a_4^4,$$

or rather u^4w, which is an integral function of the differential coefficients: and the skew cubic satisfies the second differential equation $w = 0$. The formulæ enable the determination of the osculating skew cubic at any point of a skew curve.

The general theory is developed in a compact form in the theorems I. to VII., and it is shown that the investigation of all the reciprocants depends upon that of two reciprocants of the seventh order. The geometrical applications of the theory are very extensive and interesting.

IX.

Before going further, I remark that the homologic transformation

$$X = \alpha x + \beta y + \gamma, \quad Y = \alpha' x + \beta' y + \gamma',$$

(which is in point of generality intermediate between Halphen's homographic transformation

$$X = \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad Y = \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''},$$

and Sylvester's special transformation $X = y$, $Y = x$, and which includes as a particular case the rectangular transformation

$$X = x \cos\theta + y \sin\theta, \quad Y = -x \sin\theta + y \cos\theta),$$

does not appear to have been explicitly considered. Writing as in the general case $t, a, b, c, \ldots$, for $y', \frac{1}{2}y'', \frac{1}{6}y''', \ldots$, and $T, A, B, C, \ldots$, for $Y_t, \frac{1}{2}Y_{tt}, \frac{1}{6}Y_{ttt}, \ldots$; also taking h, k for the increments of x, y, and H, K for those of X, Y respectively, we have

$$H = \alpha h + \beta k, \quad k = th + ah^2 + bh^3 + ch^4 + \ldots,$$
$$K = \alpha' h + \beta' k,$$

that is,

$$H = (\alpha + \beta t)\, h + \beta\, (ah^2 + bh^3 + ch^4 + \ldots),$$
$$K = (\alpha' + \beta' t)\, h + \beta'\, (ah^2 + bh^3 + ch^4 + \ldots),$$

which, by the elimination of h, must lead to

$$K = TH + AH^2 + BH^3 + CH^4 + \ldots.$$

We have

$$T = \frac{\alpha' + \beta' t}{\alpha + \beta t};$$

and we write

$$\lambda = \frac{\beta}{\alpha + \beta t}, \quad \lambda' = \frac{\beta'}{\alpha' + \beta' t},$$

and thence

$$\lambda' - \lambda = \frac{\alpha\beta' - \alpha'\beta}{\alpha + \beta t \,.\, \alpha' + \beta' t};$$

moreover

$$\mu = \frac{1}{\alpha + \beta t}, \quad \mu' = \frac{1}{\alpha' + \beta' t},$$

and therefore

$$T = \frac{\mu}{\mu'}.$$

Also

$$\Omega = ah + bh^2 + ch^3 + \ldots;$$

whence, if

$$h = p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4 + \ldots,$$

then

$$p = \frac{1}{a},$$

$$q = -\frac{1}{a^3}b,$$

$$r = -\frac{1}{a^5}(ac - 2b^2),$$

$$s = -\frac{1}{a^7}(a^2d - 5abc + 5b^3),$$

$$\vdots$$

and we have

$$H = (\alpha + \beta t)h + \beta h\Omega, = \left(\frac{1}{\mu} + \beta\Omega\right)(p\Omega + q\Omega^2 + r\Omega^3 + \ldots),$$

giving

$$\Omega = XH + YH^2 + ZH^3 + WH^4 + \ldots,$$

where, substituting for $p, q, r, \ldots$, their values,

$$X = \mu a,$$

$$Y = \mu^2 . b - a^2\lambda,$$

$$Z = \mu^3 . c - 3ab\lambda + 2a^3\lambda^2,$$

$$W = \mu^4 . d - (4ac + 2b^2)\lambda + 10a^2b\lambda^2 - 5a^4\lambda^3,$$

$$\vdots$$

and then

$$K = HT\frac{1 + \lambda'\Omega}{1 + \lambda\Omega}, = HT\frac{1 + \lambda'(XH + YH^2 + \ldots)}{1 + \lambda(XH + YH^2 + \ldots)},$$

giving

$$K = TH + AH^2 + BH^3 + CH^4 + \ldots,$$

where

$$A = \lambda' - \lambda . X,$$

$$B = \lambda' - \lambda . Y - \lambda X^2,$$

$$C = \lambda' - \lambda . Z - \lambda 2XY + \lambda^2X^3,$$

$$D = \lambda' - \lambda . W - \lambda(2XZ + Y^2)\lambda^2 . 3X^2Y - \lambda^3X^4,$$

whence, substituting for $X, Y, Z, W, \ldots$, their values, we have $T = \frac{\mu}{\mu'}$, *ut suprà*, and

$$A = (\lambda' - \lambda)\mu . a,$$

$$B = (\lambda' - \lambda)\mu^2 . b - 2a^2\lambda,$$

$$C = (\lambda' - \lambda)\mu^3 . c - 5ab\lambda + 5a^3\lambda^2,$$

$$D = (\lambda' - \lambda)\mu^4 . d + (-6ac - 3b^2)\lambda + 21a^2b\lambda^2 - 14a^4\lambda^3,$$

$$\vdots$$

It will be observed that μ' enters only into the equation $T=\frac{\mu}{\mu'}$: there are thus no reciprocants containing t, i.e. there is here nothing analogous to Sylvester's impure reciprocants. But we have as reciprocants, all his pure reciprocants, for instance

$$4AC-5B^2=(\lambda'-\lambda)^2(4ac-5b^2), \text{ \&c.}$$

Putting

$$\alpha=0,\quad \beta=1,\quad \alpha'=1,\quad \beta'=0,$$

then

$$\lambda=\frac{1}{t},\quad \lambda'=0,\quad \mu=\frac{1}{t},\quad \mu'=1;$$

then $T=\frac{1}{t}$, viz. we have here no arbitrary constant entering into this equation only, and there are thus reciprocants containing t. We have, in fact, the formulæ of Sylvester's theory.

In the rectangular case,

$$\alpha=\cos\theta,\quad \beta=\sin\theta,\quad \alpha'=-\sin\theta,\quad \beta'=\cos\theta,$$

we have

$$T=\frac{\cos\theta+t\sin\theta}{-\sin\theta+t\cos\theta},$$

where θ enters into the other equations, and there are thus impure reciprocants containing t, and reciprocants pure and impure which are not reciprocants in the general homologic case; but I do not go into the question of these orthogonal reciprocants.

X.

We have next Sylvester:—"Lectures on the theory of Reciprocants" (reported by J. Hammond), *Amer. Math. Journ.*, t. VIII. (1886), pp. 196—260.

The lectures were delivered at Oxford, *Inaugural Lecture*, Dec. 1885, starting from the Schwarzian function *

$$\frac{y'''}{y'}-\tfrac{3}{2}\left(\frac{y''}{y'}\right)^2,$$

which acquires only a factor by the interchange of x and y,

$$\left[\frac{y'''}{y'}-\tfrac{3}{2}\left(\frac{y''}{y'}\right)^2=-x_1^2\left\{\frac{x_{iii}}{x_i}-\tfrac{3}{2}\left(\frac{x_{ii}}{x_i}\right)^2\right\}\right].$$

In lecture 2, pp. 203 *et seq.*, Sylvester considers the general theory of the functions which remain unaltered, except as to a factor, by the interchange of x, y: viz. writing

$$t,\ a,\ b,\ c,\ldots \text{ for } y',\ y'',\ y''',\ y'''',\ldots,$$

or again

$$t,\ a_0,\ a_1,\ a_2,\ a_3,\ldots \text{ for } y',\ \tfrac{1}{2}y'',\ \tfrac{1}{6}y''',\ \tfrac{1}{24}y'''',\ldots,$$

* Schwarzian $=\frac{y'''}{y'}-\frac{3}{2}\left(\frac{y''}{y'}\right)^2=6\frac{b}{t}-\frac{3}{2}\left(\frac{2a}{t}\right)^2=\frac{6(bt-a^2)}{t^2}$, where the reciprocant is $=bt-a^2$.

and so

$$T,\ \alpha,\ \beta,\ \gamma, \ldots \text{ for } x_{\prime},\ x_{\prime\prime},\ x_{\prime\prime\prime},\ x_{\prime\prime\prime\prime}, \ldots,$$

or again

$$T,\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3, \ldots \text{ for } x_{\prime},\ \tfrac{1}{2}x_{\prime\prime},\ \tfrac{1}{6}x_{\prime\prime\prime},\ \tfrac{1}{24}x_{\prime\prime\prime\prime}, \ldots$$

he gives the equations

$$\begin{array}{llll}
\alpha = -a & \div t^3, \text{ or } & \alpha_0 = -a_0 & \div t^3, \\
\beta = -bt + 3a^2 & \div t^5, & \alpha_1 = -a_1 t + 2a_0^2 & \div t^5, \\
\gamma = -ct^2 + 10abt - 15a^3 & \div t^7, & \alpha_2 = -a_2 t^2 + 5a_0 a_1 t - 5a_0^3 & \div t^7, \\
\vdots & & \vdots &
\end{array}$$

and obtains with $(a, b, c, \ldots)$ reciprocants such as a, $2bt - 3a^2$, &c., viz. we have

$$a = -t^3 \,.\, \alpha,$$

$$2bt - 3a^2 = -t^6 \,.\, 2\beta T - 3\alpha^2,$$

and further on, like forms with $(a_0, a_1, a_2, \ldots)$.

It is to be remarked that it is preferable to deal with the quantities $(a_0, a_1, a_2, \ldots)$ which represent $(\frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''', \ldots)$ rather than with $(a, b, c, \ldots)$ which represent $(y', y'', y''', \ldots)$. I do this in the sequel, *changing the notation*, and writing $(t, a, b, c, \ldots)$ to denote $(y', \frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''', \ldots)$, viz. my $(t, a, b, c, \ldots)$ are not Sylvester's $(t, a, b, c, \ldots)$, but are his $(t, a_0, a_1, a_2, \ldots)$.

A reciprocant with Sylvester is thus a function of y and its differential coefficients in regard to x, which except as to a factor remain unaltered when x, y are interchanged, or say, when x, y are changed into X, Y, where $X = y$, $Y = x$. This is a much less general change than Halphen's, and thus every reciprocant (Halphen) is a reciprocant (Sylvester), but not conversely. The reciprocants (Sylvester) are far more numerous. I remark that incidentally Sylvester considers reciprocants, which remain unaltered save as to a factor, when x, y are changed into X, Y, where

$$X = \alpha x + \beta y + \gamma, \quad Y = \alpha' x + \beta' y + \gamma',$$

which again is a less general change than Halphen's—it has been in what precedes alluded to as the particular case $L'' = 0$, that is, $\alpha'' = 0$, $\beta'' = 0$. It may be remarked that the proper orthogonal substitution corresponding to Sylvester's substitution is not $X = y$, $Y = x$, but $X = y$, $Y = -x$, viz. we have here the determinant $\alpha\beta' - \alpha'\beta = +1$.

I develop Sylvester's theory of reciprocants; but I wish first to point out the resemblance in form between this theory and that of seminvariants. In the theory of seminvariants, from the set of quantities $(a, b, c, \ldots)$ in connexion with an arbitrary quantity θ, we deduce a new set $(a', b', c', \ldots)$, where

$$\begin{aligned}
a' &= a, \\
b' &= b + a\theta, \\
c' &= c + 2b\theta + a\theta^2, \\
d' &= d + 3c\theta + 3b\theta^2 + a\theta^3, \\
&\vdots
\end{aligned}$$

or, what is the same thing, if $\theta' = -\theta$, then

$$\begin{aligned} a &= a', \\ b &= b' + a'\theta', \\ c &= c' + 2b'\theta' + a'\theta'^2, \\ d &= d' + 3c'\theta' + 3b'\theta'^2 + a'\theta'^3, \\ &\vdots \end{aligned}$$

and this being so there exist functions which are the same for unaccented and the accented letters respectively; for instance, $a', = a$:

$$\begin{aligned} a'c' &= \quad ac + 2ab\theta + a^2\theta^2, \\ -b'^2 &= -b^2 - 2ab\theta - a^2\theta^2, \end{aligned}$$

that is, $a'c' - b'^2 = ac - b^2$; and similarly

$$a'^2d' - 3a'b'c' + 2b'^3 = a^2d - 3abc + 2b^3, \text{ \&c.};$$

these functions a, $ac - b^2$, $a^2d - 3abc + 2b^3$, &c., are called seminvariants.

Similarly, in Sylvester's theory of reciprocants, starting from y, a given function of x, we have $(t, a, b, c, \ldots)$ denoting $(y', \frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''')$; and conversely $(t', a', b', c', \ldots)$ denoting $(x_{,}, \frac{1}{2}x_{,,}, \frac{1}{6}x_{,,,}, \frac{1}{24}x_{,,,,}, \ldots)$; taking h, k for the increments of x and y respectively, we have

$$\begin{aligned} k &= th + ah^2 + bh^3 + ch^4 + dh^5 + \ldots, \\ h &= t'k + a'k^2 + b'k^3 + c'k^4 + d'k^5 + \ldots, \end{aligned}$$

which equations determine the relations between $(t, a, b, c, d, \ldots)$ and $(t', a', b', c', d', \ldots)$, viz. we have $tt' = 1$, and then

$$\begin{aligned} a' &= -a && \div t^3, \\ b' &= -bt + 2a^2 && \div t^5, \\ c' &= -ct^2 + 5abt \qquad - 5a^3 && \div t^7, \\ d' &= -dt^3 + (6ac + 3b^2)\, t^2 - 21a^2bt \qquad + 14a^4 && \div t^9, \\ e' &= -et^4 + (7ad + 7bc)\, t^3 - (28a^2c + 28ab^2)\, t^2 + 84a^3bt - 42a^5 && \div t^{11}, \\ &\vdots \end{aligned}$$

and conversely,

$$\begin{aligned} a &= -a' && \div t'^3, \\ b &= -b't' + 2a'^2 && \div t'^5, \\ c &= -c't'^2 + 5a'b't' - 5a'^3 && \div t'^7, \\ &\vdots \end{aligned}$$

and we thence deduce functions which have equal or opposite values for the accented and unaccented letters respectively; these functions may or may not contain t, t'. Thus we have

$$\begin{aligned} a't'^{-\frac{3}{2}} &= -at^{-\frac{3}{2}}, \\ (b't' - a'^2)\, t'^{-3} &= -(bt - a^2)\, t^{-3}, \\ (2c't' - 5a'b')\, t'^{-\frac{7}{2}} &= -(2ct - 5ab)\, t^{-\frac{7}{2}}, \\ (4a'c' - 5b'^2)\, t'^{-4} &= \quad (4ac - 5b^2)\, t^{-4}, \\ \{(1 + t'^2)\, b' - 2a'^2t'\}\, t'^{-3} &= \quad \{(1 + t^2)\, b - 2a^2t\}\, t^{-3}, \\ &\text{\&c.} \end{aligned}$$

These functions of the unaccented letters, or say the same functions omitting the exterior power of t, are called Reciprocants; thus we have the reciprocants a, $bt - a^2$ $2ct - 5ab$, $4ac - 5b^2$, $(1 + t^2)\, b - 2a^2t$, &c. Observe that, when the exterior powers of t are omitted, then the values are equal or opposite to those with the accented letters save as to a power of t, viz. the forms are

$$\begin{aligned} a &= -a't^3, \\ bt - a^2 &= -(b't' - a'^2)\, t^6, \\ 2ct - 5ab &= -(2c't' - 5a'b')\, t^7, \\ 4ac - 5b^2 &= \quad (4a'c' - 5b'^2)\, t^8, \\ \{(1 + t^2)\, b - 2a^2t\} &= -\{(1 + t'^2)\, b' - 2a'^2t'\}\, t^6, \\ &\text{\&c.} \end{aligned}$$

A reciprocant is said to be odd or even according as the sign on the right-hand side is $-$ or $+$; the index of t on the right-hand side is said to be the weight of the reciprocant. Reciprocants may be combined in the way of addition if and only if they are each of them of the same weight and parity, viz. this being so, then if $\lambda, \lambda', \ldots$, are mere numbers, $\lambda R + \lambda' R' + \ldots$ will be a reciprocant of the same weight and parity with each of the reciprocants $R, R', \ldots$.

Reciprocants may in every case be combined in the way of multiplication; viz. two or more reciprocants may be multiplied together giving a reciprocant the parity of which is the sum of the parities, and its weight the sum of the weights of the component reciprocants. Thus $bt - a^2$ and $2ct - 5ab$ being odd reciprocants of the weights 6 and 7 respectively, we have $(bt - a^2)(2ct - 5ab)$ an even reciprocant of the weight 13.

A reciprocant is pure or impure according as it does not or does contain t. The foregoing equations for a', b', c', ..., may be obtained each from the next preceding one by operating upon it with

$$\frac{1}{\lambda} t^{-1} (2a\partial_t + 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots),$$

where λ is a positive integer, $= 3 +$ weight of the letter operated upon ($3 + 3$, $= 6$ in operating on d' to obtain e', $3 + 4 = 7$ in operating on e' to obtain f', and so on).

For example,

$$\begin{aligned} e' &= \tfrac{1}{6}t^{-1}(2a\partial_t + 3b\partial_a + 4c\partial_b + 5d\partial_c + 6e\partial_d)\,d' \\ &= \quad t^{-1}(\tfrac{1}{3}a\partial_t + \tfrac{1}{2}b\partial_a + \tfrac{2}{3}c\partial_b + \tfrac{5}{6}d\partial_c + \quad e\partial_d)\,d', \end{aligned}$$

viz. operating herewith on

$$d', \;=\{-dt^3 + (6ac + 3b^2)\,t^2 - 21a^2bt + 14a^4\} \div t^9,$$

we obtain

$$\begin{array}{lllll|l} t^4, & t^3 \;, & t^2 \;, & t\;, & t^0 & \div t^{11}, \\ \hline & 2ad\;, & -14a^2c - 7ab^2, & 56a^3b, & -42a^5, & \\ & 3bc\;, & -21ab^2, & 28a^3b, & & \\ & 4bc\;, & -14a^2c & & & \\ & 5ad\;, & & & & \\ -e, & & & & & \\ \hline -e, & 7ad + 7bc, & -28a^2c - 28ab^2, & 84a^3b, & -42a^5, & \end{array}$$

the value given above.

The proof is at once obtained: except as to a numerical factor, the operation is in fact equivalent to the differentiation of a function of y, and its derived coefficients $y', y'', \ldots$, in regard to x.

Any pure reciprocant is reduced to zero by the operator

$$V = 4\,.\,\tfrac{1}{2}a^2\partial_b + 5ab\partial_c + 6\,(ac + \tfrac{1}{2}b^2)\,\partial_d + 7\,(ad + bc)\,\partial_e + 8\,(ae + bd + \tfrac{1}{2}c^2)\,\partial_f + \ldots,$$

or say V is an annihilator of any pure reciprocant: compare herewith the theorem $\Delta_1 = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$ is an annihilator of any seminvariant; and conversely, any homogeneous isobaric function of $a, b, c, \ldots$ reduced to zero by V is a pure reciprocant. For instance, for the pure reciprocant $5ac - 4b^2$, we have

$$(2a^2\partial_b + 5ab\partial_c)\,(4ac - 5b^2) = -20a^3b + 20a^3b, \;= 0.$$

Any rational and integral homogeneous reciprocant, operated upon with

$$G, \;= (3bt - 3a^2)\,\partial_a + (4ct - 4ab)\,\partial_b + (5dt - 5ac)\,\partial_c + \ldots,$$

produces a like reciprocant. Thus operating upon a with G, we obtain $3bt - 3a^2$, or say $bt - a^2$, and operating hereon again with G, a new reciprocant, and so on.

The series of reciprocants thus derived from a is

$$\begin{array}{l|l} a, & \\ bt - \;a^2, & \\ 2ct - 5ab, & \\ 2dt^2 - 6ac & t + 7a^2b, \\ \quad\;\; - 3b^2 & \end{array}$$

$$\begin{array}{r|l} 2et^2 - 7ad & t^2 + 8a^2c \\ - 7bc & + 11ab^2, \end{array}$$

$$\begin{array}{r|l|l} 14ft^3 - 56ae & t^2 + 103a^2d & t - 88a^3c \\ - 56bd & + 199abc & - 121a^2b^2, \\ - 28c^2 & + 33b^3 & \end{array}$$

&c.

These are, in fact, connected with the terms of

$$\left(\frac{1}{\sqrt{t}}\frac{d}{dx}\right)^i \log t, \quad = -\left(\frac{1}{\sqrt{t'}}\frac{d}{dy}\right)^i \log t',$$

where observe that $\partial_x t = 2a$, $\partial_x a = 3b$, $\partial_x b = 4c$, &c. Thus

$$\left(\frac{1}{\sqrt{t}}\frac{d}{dx}\right) \log t = \frac{a}{t^{\frac{3}{2}}}; \quad \left(\frac{1}{\sqrt{t}}\frac{d}{dx}\right)^2 \log t = \frac{1}{\sqrt{t}}\frac{d}{dx}\frac{a}{t^{\frac{3}{2}}} = \frac{3(bt - a^2)}{t^3}, \text{ \&c.}$$

A like generator for pure reciprocants is

$$H, = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots;$$

thus operating herewith upon $(4ac - 5b^2)$, we obtain

$$(4ac - b^2)(-10b) + 5(ad - bc)4a, \quad = 20a^2d - 60abc + 40b^3,$$

viz. we have thus the pure reciprocant $a^2d - 3abc + 2b^3$.

In repeating this process, we may reduce by means of powers and products of earlier reciprocants, and we can also in many cases throw out powers of the reciprocant a. Thus forming the expression for $H(a^2d - 3abc + 2b^3)$, this is given by column 1 of the annexed form: multiplying by 25, and adding $24(4ac - 5b^2)^2$, so as to eliminate the term in b^4, we obtain the expression in col. 4: or throwing out the numerical factor 3, and also the factor a, we have the

	1,	2,	3,	4,	5,	
	H,	$25H$,	$24H^2$,	$\div 3$,		
a^3e	+ 6	+ 150		+ 150	+ 50	a^2e
a^2bd	− 21	− 525		− 525	− 175	abd
a^2c^2	− 12	− 300	+ 384	+ 84	+ 28	ac^2
ab^2c	+ 51	+ 1275	− 960	+ 315	+ 105	b^2c
b^4	− 24	− 600	+ 600			

reciprocant $+ 50a^2e -$ &c. in col. 5, and the outside right-hand column of literal terms.

We thus obtain the series of pure reciprocants

$a+1$	$ac+4$	a^2d+1	a^2e+50	a^2f+10	a^2g+14	a^3h+7	a^3i+420	
	b^2-5	$abc-3$	$abd-175$	a^2be-40	$abf-63$	a^2bg-35	$a^2bh-2310$	
		b^3+2	ac^2+28	a^2cd-12	$ace-1350$	$a^2cf-539$	$a^2cg-25648$	$+1176$
			b^2c+105	ab^2d+65	ad^2+1470	$a^2de+605$	$a^2df+9240$	-8085
				abc^2+16	$b^2e+1782$	ab^2f+735	$a^2e^2+21780$	$+7040$
				b^3c-39	$bcd-4158$	$abce+306$	$ab^2g+36680$	-1470
					c^3+2310	abd^2-2135	$abcf+85386$	$+18963$
						$ac^2d+1001$	$abde-191730$	-16940
						$b^3e-1485$	$ac^2e-59220$	-27160
						$b^2cd+3465$	$acd^2+120540$	$+26460$
						bc^3-1925	$b^3f-126945$	-9555
							$b^2ce+252126$	$+28098$
							$b^2d^2+169260$	$+12740$
							$bc^2d-419034$	-52822
							$c^4+129360$	$+21560$

I remark that a pure reciprocant not included in the foregoing series is

$a^2ce+800$
a^2d^2-875
$ab^2e-1000$
$abcd+2450$
ac^3-1344
b^3c^2-35

XI.

Halphen assumes that $a^2d-3abc+2b^3$ is the sextactic reciprocant, or, what is the same thing, that the general conic satisfies the differential equation of the fifth order $a^2d-3abc+2b^3$; the proof is as follows.

The general equation of the conic may be written

$$y+Ax+B=\sqrt{k}\sqrt{(x-\alpha\,.\,x-\beta)},$$

which depends on the five parameters A, B, k, α, β.

We have

$$y' + A = \tfrac{1}{2}\sqrt{k}\,\frac{2x-\alpha-\beta}{\sqrt{(x-\alpha\,.\,x-\beta)}},$$

$$y'' \quad = \tfrac{1}{4}\sqrt{k}\left\{\frac{4}{\sqrt{(x-\alpha\,.\,x-\beta)}} - \frac{(2x-\alpha-\beta)^2}{(x-\alpha\,.\,x-\beta)^{\frac{3}{2}}}\right\},$$

$$= \frac{-\tfrac{1}{4}\sqrt{k}\,(\alpha-\beta)^2}{\{x-\alpha\,.\,x-\beta\}^{\frac{3}{2}}},$$

or say

$$y''^{-\frac{2}{3}} = (\tfrac{1}{2})^{-\frac{4}{3}}\,k^{-\frac{1}{3}}\,(\alpha-\beta)^{-\frac{4}{3}}\,x-\alpha\,.\,x-\beta\,;$$

and we have thus the differential equation $\left(\frac{d}{dx}\right)^3 y''^{-\frac{2}{3}} = 0$. This gives

$$\left(\frac{d}{dx}\right)^2 y''^{-\frac{2}{3}} y'' = 0,$$

$$\frac{d}{dx}\quad y''^{-\frac{2}{3}} y'''' - \tfrac{5}{3} y''^{-\frac{8}{3}} y'''^2 = 0,$$

$$y''^{-\frac{2}{3}} y^{\text{v}} \quad - \tfrac{5}{3} y''^{-\frac{8}{3}} y''' y'''' - \tfrac{10}{3} y''^{-\frac{8}{3}} y''' y'''' + \tfrac{40}{9} y''^{-\frac{11}{3}} y'''^3 = 0,$$

that is,

$$9y''^{-\frac{8}{3}} y^{\text{v}} - 45 y''^{-\frac{8}{3}} y''' y'''' + 40 y''^{-\frac{11}{3}} y'''^3 = 0,$$

or say

$$9y''^2 y^{\text{v}} \quad - 45 y'' y''' y'''' \quad + 40 y'''^3 \qquad = 0.$$

But y'', y''', y'''', $y^{\text{v}} = 2a$, $6b$, $24c$, $120d$, and the equation thus is $4320\,(a^2d - 3abc + 2b^3) = 0$, viz. it is $a^2d - 3abc + 2b^3 = 0$.

XII.

Sylvester's Lectures are published in the *American Mathematical Journal* as follows: Lectures 1 to 10, t. VIII. (1886), pp. 196—260; lectures 11 to 32, t. IX. (1887), viz. 11 to 16, pp. 1—37, 17 to 24, pp. 113—161, and 25 to 32, pp. 297—352; and lectures 33 and 34 (34 by Mr Hammond), t. X. (1888), pp. 1—16. In the footnote p. 7 to lecture 12, writing $y_1, y_2, y_3, y_4, \ldots$, for the derived functions of y, he adopts definitely the notation $t, a, b, c, \ldots$, to mean the reduced functions, y_1, $\frac{1}{1.2}y_2$, $\frac{1}{1.2.3}y_3$, $\frac{1}{1.2.3.4}y_4, \ldots$. In lecture 13, p. 20, he introduces the term Principiant—"instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it is better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain to a factor *près*, unaltered for any homographic substitutions impressed on the variables"—that is, Halphen's Differential Invariant = Principiant. And in lectures 22 *et seq.*, Sylvester develops an important theory in regard to Principiants, connecting them with reciprocants and seminvariants, viz. he considers a series of seminvariants N, A_0, A_1, A_2, ..., and a series of reciprocants M, A, B, C, ..., such that the seminvariants formed with either sets of capitals are identical with each other (for instance $A_0A_2 - A_1^2 = AC - B^2$), and that any such seminvariant is a Principiant.

In what follows,

instead of $N, A_0, A_1, A_2, \ldots$, I write $N, A, B, C, \ldots$,

and instead of $M, A, B, C, \ldots$, I write $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$,

so that $N, A, B, C, \ldots$, are seminvariants and $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, are reciprocants.

The expressions of these functions up to E, $\mathfrak{E}$, are

$N=$	$A=$	$B=$	$C=$	$8D=$	$6E=$
$ac+1$	a^2d+1	a^3e+1	a^4f+1	a^5g+8	a^6h+6
b^2-1	$abc-3$	a^2bd-4	a^3be-5	a^4bf-48	a^5bg-42
	b^3+2	c^2-2	$cd-5$	$ce-48$	$cf-42$
		ab^2c+10	a^2b^2d+15	d^2-25	$de-45$
		b^4-5	bc^2+15	a^3b^2e+168	a^4b^2f+168
			ab^3c-35	$bcd+342$	$bce+345$
			a^0b^5+14	c^3+56	bd^2+180
				a^2b^3d-452	c^2d+174
				b^2c^2-681	a^3b^3e-510
				$ab^4c+1020$	$b^2cd-1578$
				a^0b^6-340	bc^3-522
					$a^2b^4d+1299$
					b^3c^2+2622
					$ab^5c-2877$
					a^0b^7+822
± 1	± 3	± 11	± 45	± 1594	± 5616

In terms of the fundamental seminvariants as indicated by their initial and final terms

$$ac \infty b^2 = ac - b^2, \quad a^2d \infty b^3 = a^2d - 3abc + 2b^3, \text{ \&c.},$$

the expressions of these functions are

$$N = (ac \infty b^2),$$

$$A = (a^2d \infty b^3),$$

$$B = (a^3e \infty a^2c^2) \quad - \quad 5\,(a^2c^2 \infty b^4),$$

$$C = (a^4f \infty a^2bc^2) \quad - \quad 7\,(a^3cd \infty b^5),$$

$$8D = 8\,(a^5g \infty a^4d^2) - 168\,(a^4ce \infty a^3c^3) - 113\,(a^4d^2 \infty a^2b^2c^2) + 340\,(a^3c^3 \infty b^6),$$

$$2E = 2\,(a^6h \infty a^4bd^2) - 32\,(a^5cf \infty a^3bc^2) - 37\,(a^5de \infty a^2b^3c^2) + 137\,(a^2cd^2 \infty b^7).$$

$4\mathfrak{M}=$	$\mathfrak{A}=$	$2\mathfrak{B}=$	$4\mathfrak{C}=$	$8\mathfrak{D}=$	$48\mathfrak{E}=$
$ac+4$	a^2d+1	a^3e+2	a^4f+4	a^5g+8	a^6h+48
b^2-5	$abc-3$	a^2bd-7	a^3be-16	a^4bf-36	$a^5bg-240$
	b^3+2	c^3-4	$cd-20$	$ce-48$	$cf-336$
		ab^2c+17	a^2b^2d+45	d^2-25	$de-360$
		a^2b^4-8	bc^2+52	a^3b^2e+114	a^4b^2f+840
			ab^3c-103	$bcd+282$	$bce+2184$
			a^2b^5+38	c^3+56	bd^2+1140
				a^2b^3d-295	$c^2d+1392$
				b^2c^2-513	$a^3b^3e-2400$
				ab^4c+657	$b^2cd-8880$
				a^2b^6-200	bc^3-3504
					$a^2b^4d+5955$
					$b^3c^2+13836$
					$ab^5c-13065$
					a^2b^7+3390
$+4-5$	± 3	± 19	± 139	± 1117	± 28785

In terms of the fundamental reciprocants of the table, *ante* p. 387, say these are

$$4ac-5b^2=P_2,\quad a^2d-3abc+2b^3=P_3,\quad 50a^2e-\ldots+105b^2c=P_4,\quad 10a^3f-\ldots-39b^3c=P_5,$$
$$14a^2g-\ldots+2310b^2c=P_6,\quad 7a^3h-\ldots-1925bc^3=P_7,^*$$

the expressions of these functions are

$$\begin{aligned}
4\mathfrak{M}&=P_2,\\
\mathfrak{A}&=P_3,\\
50\mathfrak{B}&=aP_4-8P_2^2,\\
40\mathfrak{C}&=4aP_5-38P_2P_3,\\
2800\mathfrak{D}&=200a^3P_6+1266aP_2P_4-302750P_3^3-9128P_2^3,\\
1680\mathfrak{E}&=240a^3P_7+2940aP_2P_5-3156aP_3P_4+2373P_3P_2^2.
\end{aligned}$$

Sylvester remarks that a Principiant, e.g. $a^2d-3abc+2b^3$, is at once a reciprocant, and in the theory of seminvariants (where $a, b, c, \ldots$, are the coefficients of the theory) a seminvariant; and conversely that any reciprocant which is also a seminvariant is a principiant: *quâ* seminvariant, the principiant is annihilated by

$$\Omega=a\partial_b+2b\partial_c+3c\partial_d+4d\partial_e+5e\partial_f+\ldots,$$

* The eighth column $+420a^3i\ldots+129360c^4$, is *not* the proper value of P_8; the proper value is a linear combination of the eighth and ninth columns, eighth column $+6$ ninth column, viz. $P_8=420a^3i\ldots-102102bc^2d$: see as to this my paper "Tables of Pure Reciprocants to the weight 8," *Amer. Math. Jour.*, t. xv. (1893), pp. 75—77, [938].

and *quà* reciprocant is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\partial_d + (7ad + 7bc)\partial_e + (8ae + 8bd + 4c^2)\partial_f + \ldots;$$

and any function which is annihilated by each of these operators is a principiant.

We form the foregoing series of functions $N = ac - b^2$, A, B, $C, \ldots$, as follows, viz. if

$$G' = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c + (6ae - 9bd)\partial_d + (7af - 11be)\partial_e + \ldots,$$

then

$$\begin{aligned} 5A &= G'N, \\ 6B &= G'A, \\ 7C &= G'B + NA, \\ 8D &= G'C + 2NB, \\ 9E &= G'D + 3NC; \end{aligned}$$

and similarly we form the foregoing series of functions $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$, as follows, viz. if

$$\mathfrak{G}' = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 3(ae - bd)\partial_d + 7(af - be)\partial_e + \ldots,$$

($\mathfrak{G}'$ is Sylvester's G), then

$$\begin{aligned} 5\mathfrak{A} &= \mathfrak{G}'\mathfrak{M}, \\ 6\mathfrak{B} &= \mathfrak{G}'\mathfrak{A}, \\ 7\mathfrak{C} &= \mathfrak{G}'\mathfrak{B} - \mathfrak{M}\mathfrak{A}, \\ 8\mathfrak{D} &= \mathfrak{G}'\mathfrak{C} - 2\mathfrak{M}\mathfrak{B}, \\ 9\mathfrak{E} &= \mathfrak{G}'\mathfrak{D} - 3\mathfrak{M}\mathfrak{C}. \end{aligned}$$

As already mentioned, N, A, B, $C, \ldots$, are seminvariants, $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$, are reciprocants. Putting for shortness $\frac{1}{2}b = \theta$, the two sets of functions are connected by

$$\begin{aligned} A &= \mathfrak{A}, & \text{or conversely } \mathfrak{A} &= A, \\ B &= \mathfrak{B} - \theta\mathfrak{A}, & \mathfrak{B} &= B + \theta A, \\ C &= \mathfrak{C} - 2\theta\mathfrak{B} + \theta^2\mathfrak{A}, & \mathfrak{C} &= C + 2\theta B + \theta^2 A, \\ D &= \mathfrak{D} - 3\theta\mathfrak{C} + 3\theta^2\mathfrak{B} - \theta^3\mathfrak{A}, & \mathfrak{D} &= D + 3\theta C + 3\theta^2 B + \theta^3 A, \\ &\vdots & &\vdots \end{aligned}$$

so that, one of the sets being calculated, the other set can be at once deduced therefrom. These equations give

$$AC - B^2 = \mathfrak{A}\mathfrak{C} - \mathfrak{B}^2, \quad A^2D - 3ABC + 2B^3 = \mathfrak{A}^2\mathfrak{D} - 3\mathfrak{A}\mathfrak{B}\mathfrak{C} + 2\mathfrak{B}^3, \text{ \&c.},$$

viz. as mentioned above, any seminvariant in the letters A, B, $C, \ldots$, is equal to the same seminvariant in the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$; or, what is the same thing, any principiant has the same expression in the letters A, B, $C, \ldots$, and in the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... respectively.

We have thus the entire series of Principiants,

$$AC - B^2, \quad A^2D - 3ABC + 2B^3, \quad AE - 4BD + 3C^2, \text{ \&c.}$$

We may express Halphen's reciprocants in terms of the capitals $A, B, C, \ldots$. We have

$$U = a, \quad V = A, \quad \Delta = AC - B^2.$$

To obtain formulæ for the higher reciprocants, we require the derived functions $A', B', C', \ldots$, where the accent denotes differentiation in regard to x.

We have

$$\partial_x = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots,$$

(whence in particular $a' = 3b$, $b' = 4b$, $c' = 5c, \ldots$, as is obvious). Hence, writing

$$G' = a(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots) - b(3a\partial_a + 5b\partial_b + 7c\partial_c + \ldots),$$

this may be written

$$G' = a\partial_x - bw,$$

if w be the weight of the homobaric function operated upon, reckoning the weights of $a, b, c, \ldots$, as $3, 5, 7, \ldots$ respectively, and consequently the weights of $A, B, C, \ldots$, as $15, 20, 25, \ldots$ respectively. We thus have

$$\begin{aligned}
5A &= (a\partial_x - 10b)N, \\
6B &= (a\partial_x - 15b)A, \\
7C &= (a\partial_x - 20b)B + NA, \\
8D &= (a\partial_x - 25b)C + 2NB, \\
9E &= (a\partial_x - 30b)D + 3NC, \\
&\vdots
\end{aligned}$$

of which the first gives only $A = a^2d - 3abc + 2b^3$. The other equations give the required formulæ

$$\begin{aligned}
aA' &= 6B + 15bA, \\
aB' &= 7C + 20bB - NA, \\
aC' &= 8D + 25bC - 2NB, \\
aD' &= 9E + 30bD - 3NC, \\
&\vdots
\end{aligned}$$

Halphen's H is given by $U^4H = 256\Delta^3 - 27V^8$, viz. we thus have

$$a^4H = 256(AC - B^2)^3 - 27A^8.$$

His T is defined by the equation $U^3T = 3V\Delta' - 8V'\Delta$, that is,

$$a^3T = 3A(AC - B^2)^2 - 8A'(AC - B^2),$$

which is

$$\begin{aligned}
&= 3A(AC' - 2BB' + CA') - 8A'(AC - B^2), \\
&= 3A^2C' - 6ABB' + (-5AC + 8B^2)A',
\end{aligned}$$

or substituting for A', B', C', their values we find

$$a^4T = 24(A^2D - 3ABC + 2B^3),$$

which expression for the reciprocant T was given by Sylvester.

Halphen's T_1 is $\frac{1}{U}(V^4T - \frac{1}{6}H)$, viz. we thus have

$$6a^5T_1 = 24A^4(A^2D - 3ABC + 2B^3) - 256(AC - B^2)^3 + 27A^8.$$

His G is given by $V^2G = U^4T^2 + 9H$, viz. we have

$$A^2a^4G = 576(A^2D - 3ABC + 2B^3)^2 + 2604(AC - B^2)^3 - 243A^8,$$

where the whole divides by A^2; throwing out this factor, we find

$$a^4G = 576(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2) - 243A^6.$$

From the expression for T, I find

$$a^5T' = 216A^2E - 288ABD - 504AC^2 + 576B^2C + 1152(A^2D - 3ABC + 2B^3),$$

and I thence deduce for Halphen's Θ,

$$= \frac{1}{V}\{2U\Delta T' + T(8U'\Delta - 3U\Delta')\},$$

the formula

$$a^4\Theta = 432(AC - B^2)(AE - 4BD + 3C^2) - 576(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2),$$

or, what is the same thing,

$$= -144(AC - B^2)(AE - 4BD + 3C^2) + 576A(ACE - AD^2 - B^3E + 2BCD - C^3).$$

Also $\Theta_1 = \frac{1}{V}(\Theta + \frac{1}{4}G)$, that is, $a^4\Theta_1 = \frac{1}{A}(a^4\Theta + \frac{1}{4}a^4G)$,

$$= \frac{432}{A}\{(AC - B^2)(AE - 4BD + 3C^2) - (A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2)\},$$

or finally,

$$a^4\Theta_1 = 432(ACE - AD^2 - B^3E + 2BCD - C^3) - \tfrac{243}{4}A^5.$$

Again,

$$\Theta_2 = \frac{1}{U}(\Theta_1 - \tfrac{45}{2}TV) = \frac{1}{a}(\Theta_1 - \tfrac{45}{2}TA),$$

whence

$$a^5\Theta_2 = a^4\Theta_1 - \tfrac{45}{2}a^4TA,$$

or substituting,

$$a^5\Theta_2 = 432(ACE - AD^2 - B^3E + 2BCD - C^3) - 540A(A^2D - 3ABC + 2B^3) - \tfrac{243}{4}A^5.$$

Writing

$$AC - B^2 = \mathbf{C},$$
$$A^2D - 3ABC + 2B^3 = \mathbf{D},$$
$$A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2 = \square,$$
$$AE - 4BD + 3C^2 = \mathbf{I},$$
$$ACE - AD^2 - B^3E + 2BCD - C^3 = \mathbf{J},$$

the foregoing results become $U=a$, $V=A$, $\Delta=\mathbf{C}$,

$$\begin{aligned}
a^4H \text{ (Halphen)} &= 256\mathbf{C}^3-27A^8,\\
a^4T &= 24\mathbf{D},\\
6a^5T_1 &= 24A^4\mathbf{D}-256\mathbf{C}^3+27A^8,\\
a^4G &= 576\square-243A^6,\\
a^4\Theta &= 144\mathbf{CI}+576A\mathbf{J},\\
a^4\Theta_1 &= 432\mathbf{J}-\tfrac{243}{4}A^5,\\
a^5\Theta_2 &= 432\mathbf{J}-540A\mathbf{D}-\tfrac{243}{4}A^5.
\end{aligned}$$

XIII.

The letters $t, a, b, \ldots$, of a reciprocant represent (it will be remembered) mere numerical multiples of the derived functions of a variable y, in regard to the independent variable x, and thus equating any reciprocant to zero, we have a differential equation in y. For instance, if the orthogonal reciprocant $c(1+t^2)-5abt+5a^3$ be put $=0$, $(t, a, b, c=y_1, \frac{1}{2}y_2, \frac{1}{6}y_3, \frac{1}{24}y_4)$, this is the differential equation

$$y_4(1+y_1^2)-10y_1y_2y_3+15y_2^3=0$$

of the order 4. The integral hereof is expressible by the two equations

$$x=\int\frac{dt}{\sqrt{(\kappa U+\lambda V)}}+\mu,\quad y=\int\frac{tdt}{\sqrt{(\kappa U+\lambda V)}}+\nu,$$

where U, V are real functions of a parameter t, viz. $U+iV=(1+it)^6$, $U-iV=(1-it)^6$, [$i=\sqrt{(-1)}$ as usual], so that $U=1-15t^2+15t^4-t^6$, $V=6t-20t^3+6t^5$; and κ, λ, μ, ν are the four constants of integration. In verification hereof, we have

$$dx=\frac{dt}{\sqrt{(\kappa U+\lambda V)}},\quad dy=\frac{tdt}{\sqrt{(\kappa U+\lambda V)}},$$

whence

$$\frac{dy}{dx}=y_1,\ =t,$$

hence

$$y_2=\frac{dt}{dx}=\sqrt{(\kappa U+\lambda V)},\text{ or say } y_2^2=\kappa U+\lambda V,$$

and thence, using an accent to denote differentiation in regard to t, we have

$$2y_3=\kappa U'+\lambda V',$$

$$2\frac{y_4}{y_2}=\kappa U''+\lambda V'',$$

or, eliminating the κ and λ,

$$\begin{vmatrix} y_2^3, & U, & V\\ 2y_2y_3, & U', & V'\\ 2y_4, & U'', & V'' \end{vmatrix}=0.$$

Substituting for U, V either of the above-mentioned forms, there is in each case a factor in t which divides out; rejecting this factor, the equation is found to be

$$2y_4(1+t^2)-20ty_2y_3+30y_2^3=0,$$

that is, substituting for t its value $=y_1$, and throwing out the factor 2, we have the differential equation

$$y_4(1+y_1^2)-10y_1y_2y_3+15y_2^3=0.$$

XIV.

But when the reciprocant equated to zero is a Principiant, we have the far more important results obtained in Halphen's Memoir (1): and which are, in Sylvester's lectures, exhibited in a more complete form by expressing Halphen's reciprocants in terms of the foregoing functions $A, B, C, \ldots$. I recall that $a, b, c, \ldots$ are numerical multiples of differential coefficients of the orders $2, 3, 4, \ldots$ respectively; $A, B, C, \ldots$ contain $d, e, f, \ldots$ respectively, that is, differential coefficients of the orders $5, 6, 7, \ldots$ respectively, so that according as the highest capital letter is $A, B, C, \ldots$ respectively, the order of the differential equation is $5, 6, 7, \ldots$ respectively.

But the equation, Principiant $=0$, may be interpreted in a different manner, viz. if instead of regarding the differential equation as an equation for the determination of y, we regard therein y as a given function of x, {that is, (x, y) as the coordinates of a point on a given curve}, then the differential equation serves for the determination of those points on the curve for which a given condition is satisfied, or say it is the condition in order to the existence of a singular point of determinate character.

Halphen's lowest reciprocants are: as already mentioned:

$$\begin{aligned} U &= a, \\ V &= a^2d-3abc+2b^3, &&= A, \\ \Delta & &&= AC-B^2. \end{aligned}$$

$U=0$, that is, $a=0$, is a differential equation of the second order, viz. it is the differential equation of a line: otherwise it is the condition for a point of inflexion.

$V=0$, that is, $A=0$, is a differential equation of the fifth order, viz. it is the differential of a conic: otherwise it is the condition for a sextactic point.

$\Delta=0$, that is, $AC-B^2=0$, is a differential equation of the seventh order, it is the condition for what Halphen calls a point of coincidence, viz. this is a point on a given curve, such that for it the cubic of nine-pointic intersection becomes a nodal cubic having the point for node and through it one branch of eight-pointic intersection and one branch of simple intersection. Observe that this is more than the condition that the cubic of nine-pointic intersection shall be a nodal cubic, and accordingly $\Delta=0$ is not the differential equation of a nodal cubic. In fact, a nodal cubic depends on 8 parameters and has therefore a differential equation of the order 8; this will be obtained further on. But $\Delta=0$ is the differential equation of the order 7 of a curve such that every point thereof is a coincident point.

The differential equation of the order 7,

$$2^4 . 7^3 . \{(\lambda-2)(\lambda+1)(2\lambda-1)\}^2 . \Delta^3 - 3^3 . 5^2 (\lambda^2-\lambda+1)^3 V^8 = 0,$$

has the integral $\alpha = \beta^\lambda \gamma^{1-\lambda}$, where α, β, γ represent arbitrary linear functions $lx+my+n$: we may without loss of generality take the linear functions to be of the form $x+my+n$, introducing in this case another constant C, and writing the integral equation in the form $\alpha = C\beta^\lambda\gamma^{1-\lambda}$; the number of arbitrary constants is thus $=7$.

If $\lambda = 2$, -1 or $\frac{1}{2}$, the integral equation is $\alpha = \beta^2\gamma^{-1}$, $\alpha = \beta^{-1}\gamma^2$ or $\alpha = \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}$; or say it is $\beta^2 = \alpha\gamma$, $\gamma^2 = \alpha\beta$ or $\alpha^2 = \beta\gamma$, viz. each of these represents a conic. The differential equation is here $V=0$, viz. we have the foregoing result that $V=0$, that is, $A=0$, is the differential equation of a conic.

If $\lambda = \infty$, 0 or 1, the differential equation is

$$2^6 . 7^3 . \Delta^3 - 3^3 . 5^2 . V^8 = 0;$$

the integral equation is here of the form $\frac{\beta}{\gamma} = \log\frac{\alpha}{\gamma}$; in fact we have $\frac{\alpha}{\gamma} = \left(\frac{\beta}{\gamma}\right)^\lambda$, or for α, β, and γ writing $\lambda\alpha$, $\beta+\lambda\gamma$, and $\lambda\gamma$ respectively, this is $\frac{\alpha}{\gamma} = \left(1+\frac{\beta}{\lambda\gamma}\right)^\lambda$, which when λ is indefinitely large becomes $\frac{\alpha}{\gamma} = \exp.\frac{\beta}{\gamma}$, that is, $\frac{\beta}{\gamma} = \log\frac{\alpha}{\gamma}$. And similarly for $\lambda=0$ and $\lambda=1$.

If $\lambda^2-\lambda+1=0$, that is, if $\lambda = -\omega$, where ω is an imaginary cube root of unity, the differential equation is $\Delta=0$, that is, $AC-B^2=0$ (of the order 7 as in the general case). The integral equation is $\alpha = \beta^{-\omega}\gamma^{-\omega^2}$, viz. this is the equation of the curve every point of which is a coincident point.

If $\lambda=3$, the differential equation is $2^3\Delta^3 - 3^3V^8 = 0$, or say $256\Delta^3 - 27V^8 = 0$; Halphen's H is a function such that $U^4H = 256\Delta^3 - 27V^8$, so that this is

$$H_3 = U^{-4}(256\Delta^3 - 27V^8) = 0.$$

The integral equation is $\alpha = \beta^3\gamma^{-2}$, that is, $\alpha\gamma^2 = \beta^3$, or the curve is a cuspidal cubic, depending upon 7 constants; and it thus appears that the differential equation of a cuspidal cubic is the equation of the order 7

$$a^{-4}\{256(AC-B^2)^3 - 27A^8\} = 0.$$

Write

$$\begin{aligned}\Theta &= 64(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2) + 144A^2(A^2D - 3ABC + 2B^3) + 81A^6 \\ &\qquad = 64\square + 144A^2\mathbf{D} + 81A^6, \\ \Phi &= 8(A^2D - 3ABC + 2B^3) + 9A^4 = 8\mathbf{D} + 9A^4, \\ \Psi &= 3072(AC-B^2)^3 \qquad\qquad = 3072\mathbf{C}^3,\end{aligned}$$

where identically

$$\Phi^2 + \tfrac{1}{12}\Psi = A^2\Theta.$$

Halphen and Sylvester find that, for a cubic curve the invariants of which are S and T ($S = -l + l^4$, $T = 1 - 20l^3 - 8l^6$ for the canonical form $x^3 + y^3 + z^3 + 6lxyz = 0$), we have

$$4\theta^2 S = \Theta^2 - 4\Phi\Psi,$$

$$\theta^3 T = \Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2,$$

(θ an arbitrary multiplier); and we thence have

$$\frac{(\Theta^2 - 4\Phi\Psi)^3}{(\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2} = \frac{64S^3}{T^2},$$

that is,

$$T^2(\Theta^2 - 4\Phi\Psi)^3 - 64S^3(\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2 = 0,$$

or if, as with Halphen $h = \dfrac{T^2}{64S^3}$, then

$$(\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2 - h(\Theta^2 - 4\Phi\Psi)^3 = 0,$$

which is the differential equation of the order 8 of a cubic curve having the absolute invariant $S^3 \div T^2$.

If for shortness

$$P = \Theta^2 - 4\Phi\Psi,$$

$$R = \Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2,$$

$$Q = R^2 - P^3,$$

then the foregoing equation $\dfrac{P^3}{R^2} = \dfrac{64S^3}{T^2}$ gives

$$\frac{Q}{R^2} = \frac{T^2 - 64S^3}{T^2} = \frac{h-1}{h},$$

and thus for a nodal cubic we have $Q = 0$; we thus have

$P = 0$, for the differential equation of a cubic for which $S = 0$,

$R = 0$, " " " " " $T = 0$,

$Q = 0$, " " " nodal cubic,

these being each of them of the order 8. But the equation $Q = 0$ is reducible to a more simple form. We have

$$\begin{array}{lllll} Q = R^2 - P^3 = & \Theta^6 & - \Theta^6 & = \Psi^2 & \left\{\begin{array}{l} 12A^2\Theta^3 \\ -12\Theta^2\Phi^2 \\ -72A^2\Theta\Phi\Psi \\ +64\Phi^3\Psi \\ +36A^4\Psi^2 \end{array}\right\}, \\ & -12\Theta^4\Phi\Psi & +12\Theta^4\Phi\Psi & & \\ & +12\Theta^3A^2\Psi^2 & & & \\ & +36\Theta^2\Phi^2\Psi^2 & -48\Theta^2\Phi^2\Psi^2 & & \\ & -72\Theta\Phi A^2\Psi^3 & +64\Phi^3\Psi^3 & & \\ & +36A^4\Psi^4 & & & \end{array}$$

or, since in the expression in { } the first and second terms are $12\Theta^2(A^2\Theta - \Phi^2) = \Theta^2\Psi$, this is

$$Q = \Psi^3 \left\{ \begin{matrix} \Theta^2 \\ -72A^2\Theta\Phi \\ +64\Phi^3 \\ +36A^4\Psi \end{matrix} \right\} = \Psi^3 \left\{ \begin{matrix} \Theta^2 \\ -72A^2\Theta\Phi \\ +64\Phi^3 \\ +432A^4(A^2\Theta - \Phi^2) \end{matrix} \right\},$$

or finally

$$Q = \Psi^3 \{(\Theta - 36A^2\Phi + 216A^6)^2 - 64(\Phi - 9A^4)^3\}.$$

We thus have for the differential equation of the eighth order of the nodal cubic

$$(\Theta - 36A^2\Phi + 216A^6)^2 - 64(\Phi - 9A^4)^3 = 0,$$

or, since

$$\Theta = 64\square + 144A^2\mathbf{D} + 81A^6, \text{ and } \Phi = 8\mathbf{D} + 9A^4,$$

the differential equation of the eighth order for the nodal cubic finally is

$$(64\square - 144A^2\mathbf{D} + 27A^6)^2 + 32768\mathbf{D}^3 = 0.$$

Halphen has $\Theta_2 = 0$ for the differential equation of a cubic curve. Putting, as before,

$$\mathbf{I} = AE - 4BD + 3C^2,$$

$$\mathbf{J} = ACE - AD^2 - B^2E + 2BCD - C^3,$$

(and therefore $\mathbf{IC} - A\mathbf{J} = \square$, $\mathbf{I}^3 - 27\mathbf{J}^2 =$ Quartic Disct.), then, by what precedes, the differential equation of the order 9 of a cubic curve is

$$a^{-5}\{432\mathbf{J} - 540A\mathbf{D} - \tfrac{243}{4}A^5\} = 0.$$

Recapitulating.

Order. The differential equation for a cubic curve:—

8. for which invt. $S = 0$, is

$$P = \Theta^2 - 12288\mathbf{C}^3(8\mathbf{D} + 9A^4) = 0,$$

8. for which invt. $T = 0$, is

$$R = \Theta^3 - 18432\mathbf{C}^3(8\mathbf{D} + 9A^4) + 55623104\mathbf{C}^3A^2 = 0,$$

8. nodal cubic

$$Q \div (3072\Psi)^3 = (64\square - 144A^2\mathbf{D} + 27A^6)^2 + 32768\mathbf{D}^3 = 0,$$

8. for invt. $64S^3 \div T^2 = h$, is $(h - 1)R^2 - hQ = 0$,

9. for general cubic is $1728\mathbf{J} - 2160A\mathbf{D} - 243A^5 = 0$,

where for shortness Θ is retained to signify its value

$$= 64\square + 144A^2\mathbf{D} + 81A^6.$$

Halphen by a polar transformation finds that these same differential equations, changing therein the sign of $\mathbf{D}$, apply to curves of the third class: in particular, the last equation, changing therein the sign of $\mathbf{D}$, applies to the general curve of the third class, or sextic curve with nine cusps.

XV.

A very important notion in the theory, as well of Seminvariants as of Reciprocants, is that of MacMahon's *Multilinear Operator*, see his paper "Theory of a Multilinear Partial Differential Operator, with applications to the theories of Invariants and Reciprocants," *Proc. Lond. Math. Soc.*, t. XVIII. (1886), pp. 61—88: this operator plays so important a part in the theories to which it relates, that I venture to reproduce the definition of it in what appears to me a simplified and more easily intelligible form, and to recapitulate some of the leading properties.

I take with him the letters to be $(a_0, a_1, a_2, a_3, \ldots) = (a, b, c, d, \ldots)$, viz. the first letter is a_0 or a, the second is a_1 or b, and so on, but when we are not concerned with a term of indefinite rank, I use always $(a, b, c, d, \ldots)$ and the like in regard to any other series of letters. This being so, the definition of the *Multilinear Operator* of four elements, which is here alone in question, is

$$(\mu, \nu : m, n), =$$

$$\begin{array}{r|c|c|c|c|c|c|}
 & n = 0 & 1 & 2 & 3 & \ldots & n \\
\hline
(\mu \quad) A & \partial_a & \partial_b & \partial_c & \partial_d & \ldots & \partial_{a_n} \\
+ (\mu + \nu) B & \partial_b & \partial_c & \partial_d & \partial_e & \ldots & \partial_{a_{n+1}} \\
+ (\mu + 2\nu) C & \partial_c & \partial_d & \partial_e & \partial_f & \ldots & \partial_{a_{n+2}} \\
+ (\mu + 3\nu) D & \partial_d & \partial_e & \partial_f & \partial_g & \ldots & \partial_{a_{n+3}} \\
\vdots & & & & & &
\end{array},$$

where

$$\begin{array}{r|c|c|c|c|c|}
 & m = 1 & 2 & 3 & \ldots & m \\
\hline
A = & a & \frac{1}{2}a^2 & \frac{1}{3}a^3 & \ldots & \frac{1}{m}a^m \\
B = & b & ab & a^2b & \ldots & a^{m-1}b \\
C = & c & ac + \frac{1}{2}b^2 & a^2c + ab^2 & \ldots & a^{m-1}c + \frac{1}{2}(m-1)a^{m-2}b^2 \\
D = & d & ad + bc & a^2d + 2abc + \frac{1}{3}b^3 & \ldots & a^{m-1}d + (m-1)a^{m-2}bc + \frac{1}{6}(m-1)(m-2)a^{m-3}b^3, \\
\vdots & & & & &
\end{array}$$

where the expressions for $B, C, D, \ldots$, are obtained successively from that of $A, = \dfrac{1}{m}a^m$, by Arbogast's rule of derivation, viz. we operate on the last letter of a term, and when there is a last but one letter which in alphabetical order immediately precedes the last letter, then also upon the last but one letter, and whenever the term thus obtained ends in a power, we divide by the index of the power. Thus the term $\frac{1}{2}(m-1)a^{m-2}b^2$ in the third line gives in the fourth line

$$\begin{aligned}
&\tfrac{1}{2}(m-1)a^{m-2}.\,2bc + \tfrac{1}{2}(m-1)(m-2)a^{m-3}b\,.\,b^2\,.\,\tfrac{1}{3}, \\
&= (m-1)a^{m-2}bc + \tfrac{1}{6}(m-1)(m-2)a^{m-3}b^3.
\end{aligned}$$

Going a step further, we form in this manner the expression

$$\begin{aligned} E = \quad & a^{m-1}e + (m-1)\,a^{m-2}(bd + \tfrac{1}{2}c^2) \\ & + \tfrac{1}{6}(m-1)(m-2)\{a^{m-3}.\,3b^2c + (m-3)\,a^{m-4}b\,.\,b^2.\,\tfrac{1}{4}\}, \end{aligned}$$

that is,

$$\begin{aligned} E = \quad & a^{m-1}e \\ & + a^{m-2} \quad \{(m-1)\,bd + \tfrac{1}{2}(m-1)\,c^2\} \\ & + a^{m-3}.\,\tfrac{1}{2}\,(m-1)(m-2)\,b^2c \\ & + a^{m-4}.\,\tfrac{1}{24}(m-1)(m-2)(m-3)\,b^4, \end{aligned}$$

which is sufficient to explain the rule.

It may be noticed that, the operator being linear in μ, ν, we have

$$(\mu,\ \nu;\ m,\ n) = \mu(1,\ 0;\ m,\ n) + \nu(0,\ 1;\ m,\ n).$$

The Alternant of two multilinear operators.

If P, Q are the two operators, we have as usual

$$P\,.\,Q = PQ + P * Q,$$

where $P.Q$ denotes the successive operation first with Q and then with P upon any operand, PQ is the mere algebraical product of the operators, and $P * Q$ is the operator obtained by the operation of P upon Q.

Similarly,

$$Q\,.\,P = QP + Q * P,$$

and since QP is the same thing as PQ, we obtain

$$P\,.\,Q - Q\,.\,P = (P * Q) - (Q * P),$$

either of which equal expressions, or say rather the second of them, is called the alternant of P, Q and is written $[P.Q]$: viz. as the definition of the alternant, we have

$$[P\,.\,Q] = (P * Q) - (Q * P).$$

We have the remarkable theorem, that the alternant of any two operators $(\mu',\ \nu';\ m',\ n')$, $(\mu,\ \nu;\ m,\ n)$ is an operator $(\mu_1,\ \nu_1;\ m_1,\ n_1)$; where

$$\begin{aligned} \mu_1 &= (m' + m - 1)\left\{\frac{\mu'}{m'}(\mu + n'\nu) - \frac{\mu}{m}(\mu' + n\nu')\right\}, \\ \nu_1 &= (n' - n)\,\nu'\nu + \frac{m-1}{m'}\,\mu'\nu - \frac{m'-1}{m}\,\mu\nu', \\ m_1 &= m' + m - 1, \\ n_1 &= n' + n, \end{aligned}$$

or since m_1, n_1 have such simple expressions, say it is an operator $(\mu_1,\ \nu_1;\ m' + m - 1,\ n' + n)$, where μ_1, ν_1 have the values just written down.

We see at once how these values $m_1 = m' + m - 1$, $n_1 = n' + n$, arise: $Q = (\mu, \nu; m, n)$ contains the letters $(a, b, c, d, \ldots)$ in the degree m, and it contains differential symbols ∂ which, operating on any function of these letters, diminish the degree by 1; similarly $P = (\mu', \nu'; m', n')$ contains the letters $(a, b, c, d, \ldots)$ in the degree m', and it contains the differential symbols ∂ which, operating on any function of the letters, diminish the degree by 1. Hence $P * Q$ contains the letters in the degree $(m' - 1) + m$, and similarly $Q * P$ contains them in the same degree $(m - 1) + m'$: thus in the alternant the degree is $m' + m - 1$. Again, in Q the weights of the successive functions $A, B, C, \ldots$ are $0, 1, 2, \ldots$ and these are combined with differential symbols $\partial_{a_n}, \partial_{a_{n+1}}, \ldots$ which operating on any function of the letters diminish the weight by $n, n+1, \ldots$ respectively: that is, the terms $A\partial_{a_n}, B\partial_{a_{n+1}}, \ldots$ each diminish the weight by n. So in $P = (\mu', \nu'; m', n')$ the weights of the successive terms $A, B, C, \ldots$ are $0, 1, 2, \ldots$ and these are combined with differential symbols $\partial_{a_{n'}}, \partial_{a_{n'+1}}, \ldots$ which operating on any function of the letters diminish the weights by $n', n'+1, \ldots$ respectively. We may say that Q is a sum of terms such as $\Theta_k \partial_{a_{n+k}}$, where the subscript k of Θ denotes the weight of Θ; operating hereon with P, the corresponding term of $P * Q$ is of the form $\Theta_{k-n'} \partial_{a_{n+k}}$, or (what is the same thing) the general form of the terms of $P * Q$ is $\Theta_k \partial_{a_{n'+n+k}}$; and in like manner this is the general form of the terms of $Q * P$; and it is thus also the general form of the terms of the alternant $[P . Q]$. It thus appears that, admitting the alternant to be an operator $(\mu_1, \nu_1; m_1, n_1)$, the value of n_1 is $n' + n$.

As an instance of the theorem, we may take

$$[(1, 0, 1, 1), (1, 0, 2, 1)] = (1, 0, 2, 2),$$

we have here

$$(1, 0; 1, 1) * (1, 0; 2, 1) - (1, 0; 2, 1) * (1, 0; 1, 1),$$

$$\begin{array}{c|c} * & \\ a\partial_b & \tfrac{1}{2}a^2\partial_b \\ + b\partial_c & + ab\partial_c \\ + c\partial_d & + (ac + \tfrac{1}{2}b^2)\partial_d \\ + d\partial_e & + (ad + bc)\partial_e \\ + e\partial_f & + (ae + bd + \tfrac{1}{2}c^2)\partial_f \\ \vdots & \vdots \end{array} \quad - \quad \begin{array}{c|c} & * \\ \tfrac{1}{2}a^2\partial_b & a\partial_b \\ + ab\partial_c & + b\partial_c \\ + (ac + \tfrac{1}{2}b^2)\partial_d & + c\partial_d \\ + (ad + bc)\partial_e & + d\partial_e \\ + (ae + bd + \tfrac{1}{2}c^2)\partial_f & + e\partial_f \\ \vdots & \vdots \end{array}$$

$$\begin{array}{rl|rl} = & a(a\partial_c + b\partial_d + c\partial_e + d\partial_f + \ldots) - & \tfrac{1}{2}a^2\partial_c & = \tfrac{1}{2}a^2\partial_c \\ & + b(\quad a\partial_d + b\partial_e + c\partial_f + \ldots) & + ab\partial_d & + ab\partial_d \\ & + c(\quad a\partial_e + b\partial_f + \ldots) & + (ac + \tfrac{1}{2}b^2)\partial_e & + (ac + \tfrac{1}{2}b^2)\partial_e \\ & + d(\quad a\partial_f + \ldots) & + (ad + bc)\partial_f & + (ad + bc)\partial_f, \\ & \vdots & \vdots & \vdots \end{array}$$

$(= 1, 0, 2, 2)$, as it should be.

For the general proof, writing for a moment

$$C_0 = \mu A_0, \quad C_1 = (\mu + \nu) A_1, \quad C_2 = (\mu + 2\nu) A_2, \text{ \&c.},$$

and similarly

$$C_0' = \mu' A_0', \quad C_1' = (\mu' + \nu') A_1', \quad C_2' = (\mu' + 2\nu') A_2', \text{ \&c.},$$

where the accented symbols refer to the values $(\mu', \nu'; m', n')$, we have

$$\begin{aligned}
&[(\mu', \nu'; m', n').(\mu, \nu; m, n)], \\
= \; & C_0'\partial_{a_{n'}} * C_0\partial_{a_n} - C_0\partial_{a_n} * C_0'\partial_{a_{n'}} \\
& + C_1'\partial_{a_{n'+1}} + C_1\partial_{a_{n+1}} + C_1\partial_{a_{n+1}} + C_1'\partial_{a_{n'+1}} \\
& + C_2'\partial_{a_{n'+2}} + C_2\partial_{a_{n+2}} + C_2\partial_{a_{n+2}} + C_2\partial_{a_{n'+2}} \\
& \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& \qquad\qquad + C_{n'}\partial_{a_{n+n'}} \qquad\qquad + C_n\partial_{a_{n'+n}} \\
& \qquad\qquad + C_{n'+1}\partial_{a_{n+n'+1}} \qquad\qquad + C_{n+1}\partial_{a_{n'+n+1}};
\end{aligned}$$

or observing that in the series $C_0, C_1, C_2, \ldots$, the first term that contains $a_{n'}$ is $C_{n'}$, and the like as regards the series $C_0', C_1', C_2', \ldots$, this is

$$\begin{aligned}
= \; & \{(C_0'\partial_{a_{n'}}) C_{n'} - (C_0\partial_{a_n}) C_n'\} \partial_{a_{n+n'}} \\
& + \{(C_0'\partial_{a_{n'}} + C_1'\partial_{a_{n'+1}}) C_{n'+1} - (C_0\partial_{a_n} + C_1\partial_{a_{n+1}}) C'_{n+1}\} \partial_{a_{n+n'+1}} \\
& + \{(C_0'\partial_{a_{n'}} + C_1'\partial_{a_{n'+1}} + C_2'\partial_{a_{n'+2}}) C_{n'+2} - (C_0\partial_{a_n} + C_1\partial_{a_{n+1}} + C_2\partial_{a_{n+2}}) C'_{n+2}\} \partial_{a_{n+n'+2}} \\
& \vdots
\end{aligned}$$

and it is thus of the form

$$C_0''\partial_{a_{n+n'}} + C_1''\partial_{a_{n+n'+1}} + C_2''\partial_{a_{n+n'+2}} + \text{\&c.},$$

i.e. the value of n_1 is $= n + n'$.

I confine myself to the comparison of the first and second coefficients: substituting for the C's their expressions in terms of the A's, we ought to have

$$\mu' A_0'\partial_{a_{n'}} (\mu + n'\nu) A_{n'} - \mu A_0\partial_{a_n} (\mu' + n\nu') A_n' = \mu_1 A_0'',$$

$$\begin{aligned}
&(\mu' A_0'\partial_{a_{n'}} + (\mu' + \nu') A_1'\partial_{a_{n'+1}}) \{\mu + (n' + 1)\nu\} A_{n'+1} \\
&- (\mu A_0\partial_{a_n} + (\mu + \nu) A_1\partial_{a_{n+1}}) \{\mu' + (n + 1)\nu'\} A'_{n+1} = (\mu_1 + \nu_1) A_1''.
\end{aligned}$$

Now assuming $m_1 = m' + m - 1$, and attending to the values

$$\begin{aligned}
& A_0 = \frac{1}{m} a_0^m, \quad A_0' = \frac{1}{m'} a_0^{m'}, \quad A_0'' = \frac{1}{m' + m - 1} a_0^{m'+m-1}, \\
& \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& A_1 = a_0^{m-1} a_1, \\
& A_2 = a_0^{m-1} a_2 + \tfrac{1}{2} a_0^{m-2} a_1^2, \\
& \vdots
\end{aligned}$$

we see at once that in the first equation the terms contain each of them the factor $a_0^{m'+m-1}$, and in the second equation they contain each of them the factor $a_0^{m'+m-2}a_1$; omitting these factors, we find

$$\frac{\mu'}{m'}(\mu+n'\nu) \qquad -\frac{\mu}{m}(\mu'+n\nu') \qquad =\frac{\mu_1}{m'+m-1},$$

$$\frac{m'}{\mu'}(\mu+n'\nu+\nu)(m-1)+(\mu'+\nu')(\mu+n'\nu+\nu)$$
$$-\frac{\mu}{m}(\mu'+n\nu'+\nu')(m'-1)-(\mu+\nu)(\mu'+n\nu'+\nu')=\mu_1+\nu_1.$$

The first of these gives

$$\mu_1=(m'+m-1)\left\{\frac{\mu'}{m'}(\mu+n'\nu)-\frac{\mu}{m}(\mu'+n\nu')\right\},$$

which is the value of μ_1: substituting this in the second equation, we have

$$\nu_1=\frac{\mu'}{m'}(\mu+n'\nu+\nu)(m-1)-\frac{\mu}{m}(\mu'+n\nu'+\nu')$$
$$+(\mu'+\nu')(\mu+n'\nu+\nu) \qquad -(\mu+\nu)(\mu'+n\nu'+\nu')$$
$$-(m'+m-1)\frac{\mu'}{m'}(\mu+n'\nu)-(m'+m-1)\frac{\mu}{m}(\mu'+n\nu').$$

In the left-hand column, the terms containing $(\mu+n'\nu)$ are

$$(\mu+n'\nu)\left\{\frac{\mu'}{m'}(m-1)+\mu'+\nu'-\frac{\mu'}{m'}(m'+m-1)\right\},$$
$$=(\mu+n'\nu)\,\nu',$$

and there are besides the terms $\frac{\mu'}{m'}\nu(m-1)+\nu(\mu'+\nu')$, hence the left-hand column is

$$=\nu\nu'(n'+1)+\mu\nu'+\mu'\nu\left(\frac{m-1}{m'}+1\right).$$

We have therefore

$$\nu_1=\nu\nu'(n'+1)+\mu\nu'+\mu'\nu\left(\frac{m-1}{m'}+1\right)-\nu\nu'(n+1)-\mu\nu'\left(\frac{m'-1}{m}+1\right)-\mu'\nu,$$

that is,

$$\nu_1=\nu\nu'(n'-n)+\frac{\mu'}{m'}(m-1)\,\nu-\frac{\mu}{m}(m'-1)\,\nu'.$$

To complete the proof, it would be of course necessary to compare the remaining terms on the two sides respectively: but in what precedes it is shown that, the form being assumed, the expression for the alternant must of necessity be $(\mu, \nu, m'+m-1, n'+n)$, where μ, ν have their foregoing values.

Resuming the equation

$$[(\mu', \nu'; m', n')(\mu, \nu; m, n)] = (\mu_1, \nu_1; m_1, n_1),$$

where μ_1, ν_1, m_1, n_1 have the before-mentioned values, then if we herein consider μ, ν, m, n; μ_1, ν_1, m_1, n_1 as given, we can find μ', ν', m', n'.

We have $m_1 = m' + m - 1$, $n_1 = n' + n$, that is, $m' = m_1 - m + 1$, $n' = n_1 - n$, and substituting these values in the expressions of $\frac{\mu_1}{m_1}$ and ν_1, we find

$$\frac{\mu_1}{m_1} = \frac{\mu'}{m_1 - m + 1}\{\mu + (n_1 - n)\nu\} - \frac{\mu}{m}(\mu' + n\nu'),$$

$$\nu_1 = (n_1 - 2n)\nu\nu' + \frac{m-1}{m_1 - m + 1}\nu\mu' - \frac{m_1 - m}{m}\mu\nu',$$

that is,

$$\frac{\mu_1}{m_1} = \left\{\frac{\mu}{m_1 - m + 1} + \frac{\nu(n_1 - n)}{m_1 - m + 1} - \frac{\mu}{m}\right\}\mu' - \frac{\mu n}{m}\nu',$$

$$\nu_1 = \left\{\frac{(m-1)\nu}{m_1 - m + 1}\right\}\mu' + \left\{(n_1 - 2n)\nu - \frac{m_1 - m}{m}\mu\right\}\nu',$$

which are linear equations for the determination of μ' and ν'.

944.

ON PFAFF-INVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 195—205.]

1. THE functions which I propose to call Pfaff-invariants present themselves and play a leading part in the memoir, Clebsch, "Ueber das *Pfaff*sche Problem" (Zweite Abhandlung), *Crelle*, t. LXI. (1863), pp. 146—179: but it is interesting to consider them for their own sake as invariants, and in the notation which I have elsewhere used for the functions called Pfaffians. The great simplification effected by this notation is, I think, at once shown by the remark that Clebsch's expression R, which he defines by the periphrasis "Sei ferner R der rationale Ausdruck dessen Quadrat der Determinant der a_{ik} gleich ist" (*l. c.* p. 149), is nothing else than the Pfaffian $1234\ldots 2n-1\,.\,2n$, and that its differential coefficients $R_{ik}=\dfrac{\partial R}{\partial a_{ik}}$ are the Pfaffians obtained from the foregoing by the mere omission of any two symbolic numbers i, k.

2. I call to mind that the symbols 12, 13, &c., made use of are throughout such that $12=-21$, &c.; and that the definition of the successive Pfaffians 12, 1234, &c., is as follows:

$$\begin{aligned} 12 &= 12, \\ 1234 &= 12\,.\,34 + 13\,.\,42 + 14\,.\,23, \\ 123456 &= 12\,.\,3456 + 13\,.\,4562 + 14\,.\,5623 + 15\,.\,6234 + 16\,.\,2345, \end{aligned}$$

in which last expression 3456 denotes the Pfaffian $34\,.\,56+35\,.\,64+36\,.\,45$, and similarly 4562, &c.; and so on for any even number of symbols. Of course, instead of the symbolic numbers 1, 2, 3, &c., we may have any other numbers (0 is frequently used in the sequel as a symbolic number), or we may have letters or other symbols.

3. I use also a function very analogous to a Pfaffian, which is expressed in the same notation, viz. this is

$$\begin{aligned}
\phi\psi 12 &= \phi\psi 12, \\
\phi\psi 1234 &= \phi\psi 12.34 + \phi\psi 13.42 + \phi\psi 14.23 \\
&\qquad\qquad + \phi\psi 34.12 + \phi\psi 42.13 + \phi\psi 23.14, \\
\phi\psi 123456 &= \phi\psi 12.34.56 + \phi\psi 34.12.56 + \phi\psi 56.12.34 + \&c.,
\end{aligned}$$

viz. taking any term 12.34.56 of the Pfaffian 123456, $\phi\psi$ is connected successively with each of the binary symbols 12, 34, 56 of the term, so as to give rise to terms containing the quaternary symbols $\phi\psi 12$, &c. Such function may be called a co-Pfaffian.

4. To avoid suffixes I use different letters (x, y), (x, y, z), &c., as the case may be, associating these with the numbers (1, 2), (1, 2, 3), &c. In the case of a differential of an even number $2n$ of terms, for instance $Xdx + Ydy + Zdz + Wdw$, I consider the functions 1234, $\phi 01234$, and $\phi\psi 1234$, the first and second of which are Pfaffians, the last a co-Pfaffian, as explained above. To fix in connexion with the differential $Xdx + Ydy + Zdz + Wdw$ the meanings of these expressions, I assume

$$12 = \frac{dX}{dy} - \frac{dY}{dx}, \quad 13 = \frac{dX}{dz} - \frac{dZ}{dx}, \ \&c.,$$

(of course these imply $12 = -21$, &c.),

$$01 = -10 = X, \quad 02 = -20 = Y, \ \&c.;$$

ϕ is an arbitrary function of x, y, z, w, and I write

$$\phi 0 = -0\phi = 0, \quad \phi 1 = -1\phi = \frac{d\phi}{dx}, \quad \phi 2 = -2\phi = \frac{d\phi}{dy}, \ \&c.;$$

ψ is also an arbitrary function of x, y, z, w, and I write

$$\phi\psi 12 = \frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx}, \quad = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \ \&c.$$

(this implies $\phi\psi 21 = -\phi\psi 12$, &c.).

5. Thus, at full length, the functions are

$$\begin{aligned}
1234 &= 12.34 + 13.42 + 14.23 \\
&= \left(\frac{dX}{dy} - \frac{dY}{dx}\right)\left(\frac{dZ}{dw} - \frac{dW}{dz}\right) + \left(\frac{dX}{dz} - \frac{dZ}{dx}\right)\left(\frac{dW}{dy} - \frac{dY}{dw}\right) + \left(\frac{dX}{dw} - \frac{dW}{dx}\right)\left(\frac{dY}{dz} - \frac{dZ}{dy}\right), \\
\phi 01234 &= \phi 0.1234 + \phi 1.2340 + \phi 2.3401 + \phi 3.4012 + \phi 4.0123 \\
&= \frac{d\phi}{dx}(23.40 + 24.03 + 20.34) \\
&\quad + \frac{d\phi}{dy}(34.01 + 30.14 + 31.40) \\
&\quad + \frac{d\phi}{dz}(40.12 + 41.20 + 42.01) \\
&\quad + \frac{d\phi}{dw}(01.23 + 02.31 + 03.12)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{d\phi}{dx}\left\{-W\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Z\left(\frac{dY}{dw}-\frac{dW}{dy}\right)-Y\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)\right\} \\
& +\frac{d\phi}{dy}\left\{X\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)-Z\left(\frac{dX}{dw}-\frac{dW}{dx}\right)-W\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)\right\} \\
& +\frac{d\phi}{dz}\left\{-W\left(\frac{dX}{dy}-\frac{dY}{dx}\right)-Y\left(\frac{dW}{dx}-\frac{dX}{dw}\right)+X\left(\frac{dW}{dy}-\frac{dY}{dw}\right)\right\} \\
& +\frac{d\phi}{dw}\left\{X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\phi\psi 1234 = \quad & \phi\psi 12\,.\,34+\phi\psi 13\,.\,24+\phi\psi 14\,.\,23 \\
& +\phi\psi 34\,.\,12+\phi\psi 24\,.\,13+\phi\psi 23\,.\,14 \\
= \quad & \frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}\left(\frac{dZ}{dw}-\frac{dW}{dz}\right)+\frac{\partial(\phi,\ \psi)}{\partial(x,\ z)}\left(\frac{dY}{dw}-\frac{dW}{dy}\right)+\frac{\partial(\phi,\ \psi)}{\partial(x,\ w)}\left(\frac{dY}{dz}-\frac{dZ}{dy}\right) \\
& +\frac{\partial(\phi,\ \psi)}{\partial(z,\ w)}\left(\frac{dX}{dy}-\frac{dY}{dx}\right)+\frac{\partial(\phi,\ \psi)}{\partial(y,\ w)}\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+\frac{\partial(\phi,\ \psi)}{\partial(y,\ z)}\left(\frac{dX}{dw}-\frac{dW}{dx}\right).
\end{aligned}
$$

6. The invariantive property of the functions consists herein, viz. if we have

$$Xdx+Ydy+Zdz+Wdw=Pdp+Qdq+Rdr+Sds,$$

so that p, q, r, s, and thence also P, Q, R, S are functions each of them of x, y, z, w, then we have

$$
\begin{aligned}
1234\,\partial(x,\ y,\ z,\ w) &= \quad (1234)'\,\partial(p,\ q,\ r,\ s), \\
\phi 01234\,\partial(x,\ y,\ z,\ w) &= (\phi 01234)'\,\partial(p,\ q,\ r,\ s), \\
\phi\psi 1234\,\partial(x,\ y,\ z,\ w) &= (\phi\psi 1234)'\,\partial(p,\ q,\ r,\ s),
\end{aligned}
$$

where the accented functions refer to $(p,\ q,\ r,\ s,\ P,\ Q,\ R,\ S)$, and where for greater symmetry I have separated the symbolical numerator and denominator $\partial(p,\ q,\ r,\ s)$ and $\partial(x,\ y,\ z,\ w)$; each of these equations really contains

$$\frac{\partial(p,\ q,\ r,\ s)}{\partial(x,\ y,\ z,\ w)},$$

which is the functional determinant of $(p,\ q,\ r,\ s)$ in regard to $(x,\ y,\ z,\ w)$: or, if we please, it contains the reciprocal hereof

$$\frac{\partial(x,\ y,\ z,\ w)}{\partial(p,\ q,\ r,\ s)},$$

which is the functional determinant of $(x,\ y,\ z,\ w)$ in regard to $(p,\ q,\ r,\ s)$.

7. The equations give

$$\frac{\phi 01234}{1234}=\frac{(\phi 01234)'}{(1234)'},$$

$$\frac{\phi\psi 1234}{1234}=\frac{(\phi\psi 1234)'}{(1234)'},$$

and then the expressions on the left-hand are absolute invariants in respect to the transformation of

$$Xdx + Ydy + Zdz + Wdw \quad \text{into} \quad Pdp + Qdq + Rdr + Sds.$$

They are, in fact, (for $2n = 4$) Clebsch's derivatives (ϕ) and (ϕ, ψ).

8. For the Pfaffian reduction

$$Xdx + Ydy + Zdz + Wdw = Fdf + Gdg,$$

we may write

$$P,\ Q,\ R,\ S = F,\ G,\ 0\ ,\ 0,$$
$$p\ ,\ q\ ,\ r\ ,\ s = f\ ,\ g\ ,\ F,\ G,$$

viz. we take f, g, F, G as the new independent variables; we thus have

$$01' = F, \quad 02' = G, \quad 03' = 0, \quad 04' = 0,$$
$$12' = 0\ , \quad 13' = 1, \quad 14' = 0, \quad 23' = 0, \quad 24' = 1, \quad 34' = 0,$$
$$(1234)' = 12'\,.\,34' + 13'\,.\,42' + 14'\,.\,23', \ = -1\,;$$

and similarly

$$(\phi 01234)' = -\left\{F\frac{d\phi}{dF} \quad + G\frac{d\phi}{dG}\right\},$$

$$(\phi\psi 1234)' = -\left\{\frac{\partial(\phi,\ \psi)}{\partial(f,\ F)} + \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)}\right\},$$

where, in the equations, the $-$ sign presents itself by reason that $2n$, $= 4$, is the double of an even number, or say that n is even; in the case of $2n$, the double of an odd number, that is, n odd, the sign would have been $+$.

9. We thus have

$$(\phi) = \frac{\phi 01234}{1234} = F\frac{d\phi}{dF} + G\frac{d\phi}{dG},$$

$$(\phi\psi) = \frac{\phi\psi 1234}{1234} = \frac{\partial(\phi,\ \psi)}{\partial(f,\ F)} + \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)};$$

and in particular, by giving to ϕ and ψ the values f, g, F, G, we find

$$(f) = 0, \quad (g) = 0, \quad (F) = F, \quad (G) = G,$$
$$(f,\ g) = 0, \quad (f,\ F) = 1, \quad (f,\ G) = 0,$$
$$(F,\ G) = 0, \quad (g,\ F) = 0, \quad (g,\ G) = 1,$$

which are Clebsch's equations; in the case of $2n$ terms, the number of them is

$$n + n + \tfrac{1}{2}(n^2 - n) + \tfrac{1}{2}(n^2 - n) + n^2 = 2n + n^2 - n + n^2,$$

$= n(2n + 1)$, or $\frac{1}{2}2n(2n + 1)$, as it should be.

10. It may be remarked that we have

$$Fdf + Gdg = Fd\left(f + \frac{G}{F}g\right) - Fgd\frac{G}{F},$$

or writing this $= F'df' + G'dg'$, we have

$$F' = F,\quad f' = f + \frac{Gg}{F},\quad G' = +Fg,\quad g' = \frac{G}{F},$$

whence conversely

$$F = F',\quad f = f' + \frac{G'g'}{F'},\quad G = F'g',\qquad g = -\frac{G'}{F'},\quad (Gg = -G'g').$$

The ten equations $(f) = 0$, $(g) = 0$, &c., ought then to lead to the corresponding ten equations $(f') = 0$, $(g') = 0$, &c., and it is easy to verify that they do so; for instance, we have

$$(f') = \left(f + \frac{Gg}{F}\right) = (f) + \frac{G}{F}(g) + g\left(\frac{G}{F}\right),$$

where

$$\left(\frac{G}{F}\right) = \frac{1}{F}(G) - \frac{G}{F^2}(F),\quad = \frac{G}{F} - \frac{G}{F^2}F,\quad = 0,$$

and thus $(f') = 0$. And again,

$$(f',\ g') = \left(f + \frac{Gg}{F},\ \frac{G}{F}\right) = \left(f,\ \frac{G}{F}\right) + \left(\frac{Gg}{F},\ \frac{G}{F}\right) = \left(f,\ \frac{G}{F}\right) + \frac{G}{F}\left(g,\ \frac{G}{F}\right) + g\left(\frac{G}{F},\ \frac{G}{F}\right),$$

where the last term vanishes; the remaining terms are

$$= \frac{1}{F}(f,\ G) - \frac{G}{F^2}(f,\ F) + \frac{G}{F^2}(g,\ G) - \frac{G}{F^2}(g,\ F),$$

$$= 0 - \frac{G}{F^2} + \frac{G}{F^2} - 0,\ \text{that is, } (f',\ g') = 0.$$

There is, of course, the like transformation

$$Fdf + Gdg = G\left(dg + \frac{F}{G}f\right) - Gfd\frac{F}{G}.$$

11. I have, for better exhibiting the results, taken $2n = 4$, but the most simple case for an even number of terms is $2n = 2$. Here we have $Xdx + Ydy = Pdp + Qdq$, and the functions to be considered are

$$12,\ = 12 \qquad = \frac{dX}{dy} - \frac{dY}{dx},$$

$$\phi 012,\ = \phi 0.12 + \phi 1.20 + \phi 2.01 = -Y\frac{d\phi}{dx} + X\frac{d\phi}{dy},$$

$$\phi\psi 12,\ = \phi\psi 12 \qquad = \frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}.$$

We have here

$$X = P\frac{dp}{dx} + Q\frac{dq}{dx},\quad Y = P\frac{dp}{dy} + Q\frac{dq}{dy},$$

and the invariantive properties are easily verified.

12. Thus

$$12 = \frac{dX}{dy} - \frac{dY}{dx},\quad = \left(\frac{dP}{dy}\frac{dp}{dx} - \frac{dP}{dx}\frac{dp}{dy}\right) + \left(\frac{dQ}{dy}\frac{dq}{dx} - \frac{dQ}{dx}\frac{dq}{dy}\right);$$

or, writing herein

$$\frac{dP}{dx}=\frac{dP}{dp}\frac{dp}{dx}+\frac{dP}{dq}\frac{dq}{dx},$$

and the like values for $\frac{dP}{dy}$ and for $\frac{dQ}{dx}$ and $\frac{dQ}{dy}$, we have

$$12=\left(\frac{dP}{dq}-\frac{dQ}{dp}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(12)'\frac{\partial(p,\ q)}{\partial(x,\ y)}.$$

Similarly, we find

$$\phi 012=-Y\frac{d\phi}{dx}+X\frac{d\phi}{dy}=\left(-Q\frac{d\phi}{dp}+P\frac{d\phi}{dq}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(\phi 012)'\frac{\partial(p,\ q)}{\partial(x,\ y)},$$

and

$$\phi\psi 12=\frac{d\phi}{dx}\frac{d\psi}{dy}-\frac{d\phi}{dy}\frac{d\psi}{dx}=\left(\frac{d\phi}{dp}\frac{d\psi}{dq}-\frac{d\phi}{dq}\frac{d\psi}{dp}\right)\left(\frac{dp}{dx}\frac{dq}{dy}-\frac{dp}{dy}\frac{dq}{dx}\right)=(\phi\psi 12)'\frac{\partial(p,\ q)}{\partial(x,\ y)}.$$

We thus have

$$\begin{aligned}12\,\partial(x,\ y)&=\quad(12)'\,\partial(p,\ q),\\ \phi 012\,\partial(x,\ y)&=(\phi 012)'\,\partial(p,\ q),\\ \phi\psi 12\,\partial(x,\ y)&=(\phi\psi 12)'\,\partial(p,\ q);\end{aligned}$$

or say

$$\frac{\phi 012}{12}=\frac{(\phi 012)'}{(12)'},\ \text{and}\ \frac{\phi\psi 12}{12}=\frac{(\phi\psi 12)'}{(12)'}.$$

The proof is the same in principle for $2n=4$, or any other even value.

13. The theory is very similar in the case of an odd number $2n+1$ of terms; thus $2n+1=3$, the forms are

$$0123,\ \phi 123,\ \text{and}\ \phi\psi 0123,$$

the first and second of which are Pfaffians, the third of them co-Pfaffian: the developed expression of this last is

$$\begin{aligned}\phi\psi 0123=\quad&\phi\psi 01\,.\,23+\phi\psi 02\,.\,31+\phi\psi 03\,.\,12\\ &+\phi\psi 23\,.\,01+\phi\psi 31\,.\,02+\phi\psi 12\,.\,03,\end{aligned}$$

and to fix the meaning hereof we write

$$\phi\psi 01=0,\quad \phi\psi 02=0,\quad \phi\psi 03=0.$$

Hence, the differential expression being $Xdx+Ydy+Zdz$, we have

$$\begin{aligned}0123&=01\,.\,23+02\,.\,31+03\,.\,12\\ &=X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)-Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right),\\ \phi 123&=\phi 1\,.\,23+\phi 2\,.\,31+\phi 3\,.\,12\\ &=\frac{d\phi}{dx}\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+\frac{d\phi}{dy}\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+\frac{d\phi}{dz}\left(\frac{dX}{dy}-\frac{dY}{dx}\right),\\ \phi\psi 0123&=\phi\psi 23\,.\,01+\phi\psi 31\,.\,02+\phi\psi 12\,.\,03\\ &=X\frac{\partial(\phi,\ \psi)}{\partial(y,\ z)}+Y\frac{\partial(\phi,\ \psi)}{\partial(z,\ x)}+Z\frac{\partial(\phi,\ \psi)}{\partial(x,\ y)}.\end{aligned}$$

14. For the transformation

$$Xdx + Ydy + Zdz = Pdp + Qdq + Rdr,$$

we have

$$0123\,\partial(x,\ y,\ z) = \quad (0123)'\,\partial(p,\ q,\ r),$$

$$\phi 123\,\partial(x,\ y,\ z) = \quad (\phi 123)'\,\partial(p,\ q,\ r),$$

$$\phi\psi 0123\,\partial(x,\ y,\ z) = (\phi\psi 0123)'\,\partial(p,\ q,\ r),$$

and consequently

$$\frac{\phi 123}{0123} = \frac{(\phi 123)'}{(0123)'},$$

$$\frac{\phi\psi 0123}{0123} = \frac{(\phi\psi 0123)'}{(0123)'};$$

so that the left-hand functions are absolute invariants.

15. If in particular, $Xdx + Ydy + Zdz = df + Gdg$, then we may write

$$P,\ Q,\ R = 1,\ G,\ 0,$$

$$p,\ q,\ r = f,\ g,\ G.$$

Hence

$$01' = 1,\quad 02' = G,\quad 03' = 0;\quad 23' = 1,\quad 31' = 0,\quad 12' = 0,$$

and therefore

$$(0123)' = 1,\quad (\phi 123)' = \frac{d\phi}{df},\quad (\phi\psi 0123)' = \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} + G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)},$$

or say

$$= \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)};$$

and we thus have

$$0123\,\partial(x,\ y,\ z) = \partial(f,\ g,\ G),$$

$$\phi 123\,\partial(x,\ y,\ z) = \frac{d\phi}{df}\,\partial(f,\ g,\ G),$$

$$\phi\psi 0123\,\partial(x,\ y,\ z) = \left\{\frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)}\right\}\partial(f,\ g,\ G),$$

and then

$$(\phi) = \frac{\phi 123}{0123} = \frac{d\phi}{df},$$

$$(\phi,\ \psi) = \frac{\phi\psi 0123}{0123} = \frac{\partial(\phi,\ \psi)}{\partial(g,\ G)} - G\frac{\partial(\phi,\ \psi)}{\partial(f,\ G)};$$

viz. we thus have derivatives (ϕ) and $(\phi,\ \psi)$ analogous to (but quite different in form from) those of Clebsch in the case of an even number of terms.

In particular, writing $\phi, \psi = f,\ g,\ G$, we obtain

$$(f) = 1,\quad (g) = 0,\quad (G) = 0;\quad (f,\ g) = 0,\quad (f,\ G) = -G,\quad (g,\ G) = 1,$$

which are the analogues of Clebsch's formula.

16. It is interesting to compare the formula for the two cases

$$Xdx + Ydy + Zdz + Wdw = Fdf + Gdg,$$

and

$$Xdx + Ydy + Zdz = df + Gdg.$$

In the former case f and g are symmetrically related to each other, and we may say that (f = const. and g = const.) is a solution of $Xdx + Ydy + Zdz + Wdw = 0$; we have $(f) = 0$ and $(g) = 0$. In the second case (f = const. and g = const.) is still a solution of $Xdx + Ydy + Zdz = 0$, but f and g are not symmetrically related to each other, and we have $(f) = 1$, $(g) = 0$. Moreover, in the first case $(G) = G$, but in the second case $(G) = 0$, an equation of the same form as $(g) = 0$; the reason is that we have here

$$Xdx + Ydy + Zdz = df + Gdg, \quad = d(f + Gg) - gdG,$$

so that, besides the solution (f = const. and g = const.), we have the solution

$$(f + Gg = \text{const. and } G = \text{const.}).$$

17. The remark just made may be further developed: we have

$$Xdx + Ydy + Zdz = df + Gdg, \quad = d(f + Gg) - gdG, \quad = df' + G'dg',$$

suppose, where $f' = f + Gg$, $G' = -g$, $g' = G$, and therefore also $f = f' + G'g'$, $G = g'$, $g = -G'$; the equations

$$(f) = 1, \quad (g) = 0, \quad (G) = 0, \quad (f, g) = 0, \quad (f, G) = -G, \quad (g, G) = 1,$$

should lead to

$$(f') = 1, \quad (g') = 0, \quad (G') = 0, \quad (f', g') = 0, \quad (f', G') = -G', \quad (g', G') = 1.$$

There is no difficulty in verifying this; thus the equations $(g) = 0$, $(G) = 0$, give at once $(g') = 0$, $(G') = 0$; and then the equation $(f) = 1$ gives $(f' + G'g') = 1$, that is,

$$(f') + G'(g') + g'(G') = 1, \text{ or } (f') = 1.$$

So again $(g, G) = 1$ gives $(g', G') = 1$; and then $(f, g) = 0$ gives $(f' + G'g', G') = 0$, that is,

$$(f', G') + G'(g', G') + g'(G', G') = 0, \text{ or } (f', G') = -G'.$$

And finally, $(f, G) = -G$ gives $(f' + G'g', g') + g' = 0$, that is,

$$(f', g') + G'(g', g') + g'(G', g') + g' = 0, \text{ or } (f', g') = 0.$$

I stop to give the direct verification of the equations $(f) = 1$, $(g) = 0$, $(G) = 0$. We have

$$Xdx + Ydy + Zdz = df + Gdg,$$

that is,

$$X = \frac{df}{dx} + G\frac{dg}{dx}, \quad Y = \frac{df}{dy} + G\frac{dg}{dy}, \quad Z = \frac{df}{dz} + G\frac{dg}{dz},$$

and thence

$$23 = \frac{dY}{dz} - \frac{dZ}{dy} = \frac{dG}{dz}\frac{dg}{dy} - \frac{dG}{dy}\frac{dg}{dz},$$

$$31 = \frac{dZ}{dx} - \frac{dX}{dz} = \frac{dG}{dx}\frac{dg}{dz} - \frac{dG}{dz}\frac{dg}{dx},$$

$$12 = \frac{dX}{dy} - \frac{dY}{dx} = \frac{dG}{dy}\frac{dg}{dx} - \frac{dG}{dx}\frac{dg}{dy}.$$

Hence, multiplying first by $\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$, that is,

$$X - G\frac{dg}{dx}, \quad Y - G\frac{dg}{dy}, \quad Z - G\frac{dg}{dz},$$

and adding, we have

$$23\frac{df}{dx} + 31\frac{df}{dy} + 12\frac{df}{dz} = X23 + Y31 + Z12,$$

that is, $f123 = 0123$, or $(f) = 1$.

And then multiplying secondly by $\frac{dg}{dx}, \frac{dg}{dy}, \frac{dg}{dz}$ and adding, and thirdly by $\frac{dG}{dx}, \frac{dG}{dy}, \frac{dG}{dz}$ and adding, we obtain

$$23\frac{dg}{dx} + 31\frac{dg}{dy} + 12\frac{dg}{dz} = 0, \text{ that is, } (g) = 0,$$

and

$$23\frac{dG}{dx} + 31\frac{dG}{dy} + 12\frac{dG}{dz} = 0, \text{ that is, } (G) = 0.$$

To exhibit more clearly the formulæ for any odd number of terms, I take $2n + 1 = 5$,

$$Xdx + Ydy + Zdz + Wdw + Tdt = df + Gdg + Hdh.$$

We have here

$$(\phi) = \frac{\phi 12345}{012345} = \frac{d\phi}{df},$$

$$(\phi\psi) = \frac{\phi\psi 012345}{012345} + \frac{\partial(\phi, \psi)}{\partial(g, G)} + \frac{\partial(\phi, \psi)}{\partial(h, H)} - G\frac{\partial(\phi, \psi)}{\partial(f, G)} - H\frac{\partial(\phi, \psi)}{\partial(f, H)};$$

and in particular,

$$(f) = 1; \quad (g) = 0, \quad (h) = 0; \quad (G) = 0, \quad (H) = 0;$$

$$(f, g) = 0; \quad (f, h) = 0; \quad (f, G) = -G, \quad (f, H) = -H;$$

$$(g, h) = 0; \quad (G, H) = 0; \quad (g, G) = 1, \quad (g, H) = 0;$$

$$(h, G) = 0, \quad (h, H) = 1;$$

in all

$$1 + 2n + 2n + \tfrac{1}{2}(n^2 - n) + \tfrac{1}{2}(n^2 - n) + n^2,$$

$$= 1 + 4n + n^2 - n + n^2, \quad = 2n^2 + 3n + 1, \quad = \tfrac{1}{2}(2n + 1)(2n + 2)$$

equations.

We can, by what precedes, at once express the conditions which must be satisfied in order that a differential expression $X_1 dx_1 + X_2 dx_2 + \ldots + X_\nu dx_\nu$, may be reducible to one of the special forms df, Fdf, $df + F_1 df_1$, &c.; viz. if we have

$$\begin{aligned} X_1 dx_1 + X_2 dx_2 + \ldots + X_\nu dx_\nu &= df, && \text{then } 12 && = 0, \text{ \&c.} \\ &= Fdf, && \text{,,} \quad 0123 && = 0, \text{ \&c.} \\ &= df + F_1 df_1, && \text{,,} \quad 1234 && = 0, \text{ \&c.} \\ &= Fdf + F_1 df_1, && \text{,,} \quad 012345 && = 0, \text{ \&c.,} \\ &\text{\&c.,} && \text{\&c.,} \end{aligned}$$

where the numbers 12, 1234, 12345, &c., represent any combinations out of the numbers 1, 2, 3, ..., ν. Of course, if ν is not sufficiently large to furnish such a combination, then there is no condition to be satisfied; thus if

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 = df + F_1 df_1,$$

there is no condition to be satisfied.

945.

NOTE ON LACUNARY FUNCTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 279—281.]

THE present note is founded upon Poincaré's paper "Sur les fonctions à espaces lacunaires," *Amer. Math. Jour.*, t. XIV. (1892), pp. 201—221.

If the complex variable $z = x + iy$ is represented as usual by a point the coordinates of which are (x, y), and if $U_0, U_1, U_2, \ldots$ denote an infinite series of given functions of z, then the equation

$$fz = U_0 + U_1 + U_2 + \ldots$$

defines a function of z, but only for those values of z for which the series is convergent, or say for points within a certain region Θ; and within this region, it defines the successive derived functions $f'z, f''z, f'''z, \ldots$.

Taking $l, = h + ik$, as an increment of z, we define the function of $z + l$, by the equation

$$f(z+l) = fz + \frac{l}{1} f'z + \frac{l^2}{1.2} f''z + \ldots,$$

but only for values of l for which the series is convergent: it may very well be, and it is in general the case, that we thereby extend the definition of fz so as to make it applicable to points within a larger region Θ_1; and then considering fz as defined within this larger region Θ_1, we may pass from it to a still larger region Θ_2; and so on indefinitely, or until we cover the whole infinite plane.

For instance, the equation

$$fz = 1 + z + z^2 + z^3 + \ldots$$

defines the function $1 \div (1-z)$ for values of z for which mod. $z < 1$, that is, for points within the circle $x^2 + y^2 - 1 = 0$; and this being so,

$$f(z+l) = \frac{1}{1-z} + \frac{l}{(1-z)^2} + \frac{l^2}{(1-z)^3} + \dots$$

extends the definition to the larger region for which this is a convergent series: the condition of convergency is

$$\text{mod.} \frac{l}{1-z} < 1, \text{ that is, mod.} \frac{h+ik}{1-x-iy} < 1, \text{ or } h^2 + k^2 < (1-x)^2 + y^2.$$

The condition is that the distance $\sqrt{(h^2+k^2)}$ must not exceed the distance $\sqrt{\{(1-x)^2+y^2\}}$ of the point $z = x + iy$, from the point $(x=1,\ y=0)$; the point z is strictly within the circle $x^2 + y^2 = 1$, but taking it on the circumference, the condition is that the point $z + l$ must lie within a circle having its centre on the circle $x^2 + y^2 - 1 = 0$ and passing through the point $(x=1,\ y=0)$. Taking $\cos\theta$ and $\sin\theta$ for the coordinates of the centre, the equation of this circle is

$$(x - \cos\theta)^2 + (y - \sin\theta)^2 = (1 - \cos\theta)^2 + \sin^2\theta,$$

that is,

$$x^2 + y^2 - 1 - 2\cos\theta\,(x-1) - 2y\sin\theta = 0;$$

and the envelope of these circles is

$$(x^2 + y^2 - 1)^2 - 4(x-1)^2 - 4y^2 = 0;$$

or, as this may be written,

$$(x^2 + y^2)^2 - 6(x^2 + y^2) + 8x - 3 = 0,$$

or again in the form

$$(x^2 + y^2 - 3)^2 + 4(2x - 1) = 0,$$

or in the form

$$y^4 + 2y^2(x^2 - 3) + (x-1)^2(x+3) = 0.$$

The curve is a cuspidal Cartesian. To put this in evidence, observe that the equation may be written

$$-4y^2 + \{(x-1)^2 + y^2\}\{(x-1)^2 + y^2 + 4x - 4\} = 0,$$

viz. writing

$$A = x + iy - 1,$$
$$B = x - iy - 1,$$
$$Z = -1,$$

then

$$A - B = 2iy,$$
$$AB \quad = (x-1)^2 + y^2,$$
$$A + B = 2x - 2,$$

or the equation is

$$Z^2(A-B)^2 + AB\{AB - 2Z(A+B)\} = 0,$$

that is,

$$Z^2A^2 + Z^2B^2 + A^2B^2 - 2Z^2AB - 2ZA^2B - 2ZAB^2 = 0,$$

which is the equation of a bicuspidal quartic curve, having for cusps the vertices of the triangle $A = 0$, $B = 0$, $Z = 0$.

The region within which the function is $= \frac{1}{1-z}$ is thus extended to the area within the Cartesian curve, say this is the region Θ_1: starting from this curve instead of the circle (viz. by considering the envelope of the circle having its centre on the curve and passing through the point $x = 1$, $y = 0$), we obtain a second curve, a closed curve, which instead of having a cusp on the axis of x cuts this axis at right angles at a point the distance of which from the origin is greater than 1; and we thus extend the region to the area within this second curve, say this is the region Θ_2. And proceeding in this way, we ultimately extend the region to the whole of the infinite plane.

But the functions U_0, U_1, $U_2, \ldots$ may be such that for every value whatever of l, for which the point $z + l$ is outside the region Θ, the series

$$fz + \frac{l}{1} f'z + \frac{l^2}{1.2} f''z + \ldots$$

is divergent, and we are in this case unable to define the function fz for points outside the region Θ_1: the function then exists only for points inside the region Θ, and for points outside this region it is non-existent; a function such as this, existing only for points within a certain region and not for the whole of the infinite plane, is said to be a *lacunary* function.

946.

NOTE ON THE THEORY OF ORTHOMORPHOSIS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVI. (1893), pp. 282—288.]

THE equation of any given curve whatever, $\Theta = 0$, may be expressed in the form

$$\phi(x+iy) + \phi(x-iy) = 0.$$

Let χ be any odd function; then since

$$\phi(x-iy) = -\phi(x+iy),$$

we have

$$\chi\phi(x-iy) = \chi.\{-\phi(x+iy)\} = -\chi\phi(x+iy),$$

that is,

$$\chi\phi(x+iy) + \chi\phi(x-iy) = 0.$$

Assuming that Θ is a real function, that is, a function with real coefficients, then also $\phi(x+iy)$ will be a function with real coefficients, or say a real function of $x+iy$; the function χ may be real or imaginary, but if imaginary, then the i of the coefficients does not change its sign in the passage from $\chi\phi(x+iy)$ to $\chi\phi(x-iy)$.

In proof of the assumed theorem, imagine the equation $\Theta = 0$ expressed as an equation between $x+iy$ and $x-iy$, or, supposing it solved in regard to $x-iy$, take the form of it to be $x-iy = f(x+iy)$: let u_n be a function of n satisfying the equation of differences $u_{n+1} = fu_n$; and let $\phi(x+iy)$ be determined as a function of $x+iy$ by the elimination of n from the equations

$$x+iy = u_n, \quad \phi(x+iy) = \cos n\pi;$$

we thence have

$$x-iy = fu_n, \quad = u_{n+1},$$

and consequently

$$\phi(x-iy) = \cos(n+1)\pi,$$

that is,

$$\phi(x+iy)+\phi(x-iy)=0,$$

viz. this equation is a transformation of the equation $\Theta=0$, and thus it appears that the equation $\Theta=0$ can always be thrown into the last-mentioned form.

As an example, take the equation $y=ax+b$: which, putting for a moment $\xi=x+iy$, $\eta=x-iy$, is

$$\frac{1}{2i}(\xi-\eta)=\tfrac{1}{2}a(\xi+\eta)+b,$$

that is,

$$\eta=\frac{i+a}{i-a}\xi+\frac{2b}{i-a};$$

we have therefore

$$u_{n+1}=\frac{i+a}{i-a}u_n+\frac{2b}{i-a},$$

a solution of which is

$$u_n=\left(\frac{i+a}{i-a}\right)^n-\frac{b}{a};$$

putting this $=\xi$, we have

$$n=\frac{1}{\log\dfrac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

and thence

$$\phi\xi=\cos\frac{\pi}{\log\dfrac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

where observe that, writing $a+i=Re^{i\alpha}$ and therefore $a-i=Re^{-i\alpha}$, we have

$$\cos\alpha=\frac{a}{\sqrt{(a^2+1)}},\quad \sin\alpha=\frac{1}{\sqrt{(a^2+1)}},$$

or say $\cot\alpha=a$, and then

$$\frac{i+a}{i-a}=e^{2\alpha i+i\pi},\quad\text{or}\quad \log\frac{i+a}{i-a}=i(2\alpha+\pi),$$

whence

$$\phi\xi=\cos\frac{\pi}{i(2\alpha+\pi)}\log\left(\xi+\frac{b}{a}\right),\ =\cosh\frac{\pi}{2\alpha+\pi}\log\left(\xi+\frac{b}{a}\right),$$

a real function of ξ.

In verification of the equation $\phi\xi+\phi\eta=0$, we have

$$\phi\eta=\cos\frac{\pi}{\log\dfrac{i+a}{i-a}}\log\left(\eta+\frac{b}{a}\right),$$

where

$$\log\left(\eta+\frac{b}{a}\right)=\log\left(\frac{i+a}{i-a}\xi+\frac{2b}{i-a}+\frac{b}{a}\right)=\log\frac{i+a}{i-a}\left(\xi+\frac{b}{a}\right),$$

$$=\log\frac{i+a}{i-a}+\log\left(\xi+\frac{b}{a}\right),$$

and thence

$$\phi\eta=\cos\frac{\pi}{\log\frac{i+a}{i-a}}\left\{\log\frac{i+a}{i-a}+\log\left(\xi+\frac{b}{a}\right)\right\}$$

$$=\cos\left\{\pi+\frac{\pi}{\log\frac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right)\right\},\quad =-\cos\frac{\pi}{\log\frac{i+a}{i-a}}\log\left(\xi+\frac{b}{a}\right),$$

that is, $\phi\eta=-\phi\xi$, or $\phi\xi+\phi\eta=0$, the equation in question.

I remark, in passing, that the same equation $y=ax+b$ might have been put in the form $\phi x+\phi y=0$, viz. assuming

$$\phi x=\cos\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right),$$

then

$$\phi y=\cos\frac{\pi}{\log a}\log\left(ax+b-\frac{b}{1-a}\right)=\cos\frac{\pi}{\log a}\log a\left(x-\frac{b}{1-a}\right)$$

$$=\cos\frac{\pi}{\log a}\left\{\log a+\log\left(x-\frac{b}{1-a}\right)\right\}$$

$$=\cos\left\{\pi+\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right)\right\}$$

$$=-\cos\frac{\pi}{\log a}\log\left(x-\frac{b}{1-a}\right)=-\phi x,$$

that is, $\phi x+\phi y=0$.

If $b=0$, then

$$y=ax\quad\text{and}\quad\phi x=\cos\frac{\pi\log x}{\log a};$$

in fact, repeating the proof for this particular case,

$$\phi y=\cos\pi\frac{\log ax}{\log a}=\cos\pi\left(1+\frac{\log x}{\log a}\right)=-\cos\frac{\pi\log x}{\log a},\quad =-\phi x;$$

that is,

$$\phi x+\phi y=0.$$

Considering then $(x,\ y)$ as the coordinates of a point on the curve $\Theta=0$, we have, as above,

$$\chi\phi(x+iy)+\chi\phi(x-iy)=0,$$

where ϕ is a real function determined as above, and χ is any real or imaginary odd function. This being so, assume

$$x_1+iy_1=e^{\chi\phi(x+iy)},$$

then also

$$x_1 - iy_1 = e^{\chi\phi(x-iy)},$$

and consequently

$$x_1^2 + y_1^2 = e^{\chi\phi(x+iy)+\chi\phi(x-iy)} = 1,$$

that is, we have the circumference of the circle $x_1^2 + y_1^2 - 1 = 0$ corresponding to the given curve $\Theta = 0$.

Suppose that the curve $\Theta = 0$ is a closed curve: and then writing

$$\xi + i\eta = \chi\phi(x + iy),$$

and therefore

$$\xi - i\eta = \chi\phi(x - iy),$$

we thence have

$$2\xi = \chi\phi(x + iy) + \chi\phi(x - iy),$$

a real function of (x, y).

(1) Assume that it is possible to find χ, such that ξ as defined by this last equation shall be throughout the area of the curve $\Theta = 0$ finite and continuous, except only that in the neighbourhood of a given point, taken to be the point $x = 0$, $y = 0$, it is $= \log\sqrt{(x^2 + y^2)}$.

(2) At the boundary of the area $\Theta = 0$, ξ is $= 0$.

(3) Throughout the area, ξ satisfies the partial differential equation

$$\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} = 0.$$

These conditions being satisfied, the equation

$$x_1 + iy_1 = e^{\xi + i\eta},$$

that is,

$$x_1 + iy_1 = e^{\chi\phi(x+iy)},$$

gives an orthomorphosis of the area $\Theta = 0$ into the circle $x_1^2 + y_1^2 - 1 = 0$, the point $x = 0$, $y = 0$ corresponding to the centre of the circle; (2) and (3) are satisfied as above: it remains only to satisfy (1), viz. the function χ is determined not by any equation—*but only by this condition as to finiteness and continuity;* and if it be thus determined, then the foregoing equation $x_1 + iy_1 = e^{\chi\phi(x+iy)}$ gives the required orthomorphosis.

For instance, let the curve $\Theta = 0$ be the parabola $y^2 = 4(1 - x)$, which may be regarded as a closed curve bounding the infinite parabolic area. We have $2x = \xi + \eta$, $2iy = \xi - \eta$, whence the equation is

$$-\tfrac{1}{4}(\xi - \eta)^2 = 4 - 2\xi - 2\eta,$$

that is,

$$\xi^2 - 2\xi\eta + \eta^2 - 8\xi - 8\eta + 16 = 0,$$

whence $\sqrt{\xi} + \sqrt{\eta} - 2 = 0$, or writing this in the form

$$(\sqrt{\xi} - 1) + (\sqrt{\eta} - 1) = 0,$$

we have $\phi\xi = \sqrt{\xi} - 1$, and assuming that χ can be found so that the condition as to finiteness and continuity is satisfied, then the orthomorphosis is given by

$$x_1 + iy_1 = \exp\chi(\sqrt{\xi} - 1), \quad = \exp\chi\{\sqrt{(x+iy)} - 1\}.$$

Assuming

$$\tfrac{1}{2}\chi\omega = -\tfrac{1}{2}i\pi\omega + \log\frac{1 - i\exp(\tfrac{1}{2}i\pi\omega)}{1 - i\exp(-\tfrac{1}{2}i\pi\omega)},$$

which is obviously an odd function, we have

$$\exp\tfrac{1}{2}\chi\omega = \frac{1}{\exp\tfrac{1}{2}i\pi\omega}\,\frac{1 - i\exp(\tfrac{1}{2}i\pi\omega)}{1 - i\exp(-\tfrac{1}{2}i\pi\omega)},$$

$$= \frac{1 - \exp\tfrac{1}{2}i\pi\omega}{\exp\tfrac{1}{2}i\pi\omega - 1}, \quad = \frac{i(1 - i\exp\tfrac{1}{2}i\pi\omega)}{1 + i\exp\tfrac{1}{2}i\pi\omega},$$

which is

$$= \tan\tfrac{1}{4}\pi(\omega + 1),$$

and hence, for ω writing $\sqrt{(x+iy)} - 1$, we have

$$x_1 + iy_1 = \exp\chi\{\sqrt{(x+iy)} - 1\}, \quad = \tan^2\tfrac{1}{4}\pi\sqrt{(x+iy)}.$$

This satisfies the required conditions as to finiteness and continuity; and in particular, we have

$$\xi + i\eta = \log\tan^2\tfrac{1}{4}\pi\sqrt{(x+iy)},$$

so that, x and y being small,

$$\xi + i\eta = \log\frac{\pi^2}{16}(x+iy), \quad \xi - i\eta = \log\frac{\pi^2}{16}(x-iy),$$

that is,

$$\xi = \log\frac{\pi^2}{16}\sqrt{(x^2+y^2)}.$$

Hence we have the known result: the orthomorphosis of the parabola $y^2 = 4(1-x)$ into the circle $x_1^2 + y_1^2 - 1 = 0$ is given by the equation $x_1 + iy_1 = \tan^2\tfrac{1}{4}\pi\sqrt{(x+iy)}$.

Consider the ellipse, where $a^2 - b^2 = 1$, or say

$$\frac{x^2}{\tfrac{1}{4}\left(M + \dfrac{1}{M}\right)^2} + \frac{y^2}{\tfrac{1}{4}\left(M - \dfrac{1}{M}\right)^2} = 1.$$

I show, by a less direct process, how to express this equation in the required form $\phi\xi + \phi\eta = 0$. In fact, writing

$$\xi = x + iy, \quad \eta = x - iy,$$

the equation of the ellipse is the rationalised form of

$$i\eta + \sqrt{(1-\eta^2)} = M^2\{i\xi + \sqrt{(1-\xi^2)}\}.$$

To show that this is so, call for a moment the right-hand side Ω, the equation is

$$\sqrt{(1-\eta^2)} = \Omega - i\eta,$$

hence

$$1-\eta^2 = \Omega^2 - 2\Omega i\eta - \eta^2,$$

$$2\Omega i\eta = \Omega^2 - 1,$$

or

$$2i\eta = \Omega - \frac{1}{\Omega} = M^2\{i\xi + \sqrt{(1-\xi^2)}\} + \frac{1}{M^2}\{i\xi - \sqrt{(1-\xi^2)}\},$$

$$= \left(M^2 + \frac{1}{M^2}\right) i\xi + \left(M^2 - \frac{1}{M^2}\right)\sqrt{(1-\xi^2)},$$

therefore

$$2i\eta - \left(M^2 + \frac{1}{M^2}\right) i\xi = \left(M^2 - \frac{1}{M^2}\right)\sqrt{(1-\xi^2)}$$

$$-4\eta^2 + 4\left(M^2 + \frac{1}{M^2}\right)\xi\eta + \left(M^4 + 2 + \frac{1}{M^4}\right)(-\xi)^2 = \left(M^4 - 2 + \frac{1}{M^4}\right) - \left(M^4 - 2 + \frac{1}{M^4}\right)\xi^2,$$

that is,

$$-4\eta^2 - 4\xi^2 + 4\left(M^2 + \frac{1}{M^2}\right)\xi\eta = \left(M^2 - \frac{1}{M^2}\right)^2,$$

or say

$$-\xi^2 - \eta^2 + \left(M^2 + \frac{1}{M^2}\right)\xi\eta - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0:$$

viz. substituting for ξ, η their values, this is

$$-2(x^2 - y^2) + \left(M^2 + \frac{1}{M^2}\right)(x^2 + y^2) - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0,$$

that is,

$$\left(M - \frac{1}{M}\right)^2 x^2 + \left(M + \frac{1}{M}\right)^2 y^2 - \tfrac{1}{4}\left(M^2 - \frac{1}{M^2}\right)^2 = 0,$$

or finally, it is

$$\frac{x^2}{\frac{1}{4}\left(M + \frac{1}{M}\right)^2} + \frac{y^2}{\frac{1}{4}\left(M - \frac{1}{M}\right)^2} - 1 = 0,$$

as it should be.

Starting then from the relation

$$i\eta + \sqrt{(1-\eta^2)} = M^2\{i\xi + \sqrt{(1-\xi^2)}\},$$

and writing

$$\phi\xi = \cos\frac{\pi}{2\log M}\log\{i\xi + \sqrt{(1-\xi^2)}\},$$

we have

$$\begin{aligned}\phi\eta &= \cos\frac{\pi}{2\log M}\log M^2\{i\xi+\sqrt{(1-\xi^2)}\},\\ &= \cos\frac{\pi}{2\log M}[\log M^2+\log\{i\xi+\sqrt{(1-\xi^2)}\}]\\ &= \cos\left[\pi+\frac{\pi}{2\log M}\log\{i\xi-\sqrt{(1-\xi^2)}\}\right]\\ &= -\cos\frac{\pi}{2\log M}\log\{i\xi+\sqrt{(1-\xi^2)}\},\quad =-\phi\xi,\end{aligned}$$

that is, we have

$$\phi\xi+\phi\eta=0,$$

as the required transformation of the equation of the ellipse

$$\frac{x^2}{\frac{1}{4}\left(M+\frac{1}{M}\right)^2}+\frac{y^2}{\frac{1}{4}\left(M-\frac{1}{M}\right)^2}=1.$$

We hence derive the known formula for the orthomorphosis of the ellipse into the circle $x_1^2+y_1^2-1=0$.

947.

ON A SYSTEM OF TWO TETRADS OF CIRCLES; AND OTHER SYSTEMS OF TWO TETRADS.

[From the *Proceedings of the Cambridge Philosophical Society*, vol. VIII. (1893), pp. 54—59.]

THE investigations of the present paper were suggested to me by Mr Orr's paper, "The Contacts of certain Systems of Circles," *Proc. Camb. Phil. Soc.*, vol. VII.

1. It is possible to find *in plano* two tetrads of circles, or say four red circles and four blue circles, such that each red circle touches each blue circle: in fact, counting the constants, a circle depends upon 3 constants, or say it has a capacity $=3$; the capacity of the eight circles is thus $=24$; and the postulation or number of conditions to be satisfied is $=16$: the resulting capacity of the system is thus *primâ facie*, $16-24=8$. It will, however, appear that, in the system considered, the true value is $=9$.

2. The *primâ facie* value of the capacity being $=8$, we are not at liberty to assume at pleasure three circles of the system. And, in fact, assuming at pleasure say 3 red circles, then touching each of these we have 8 circles, forming $\frac{1}{24}8.7.6.5, =70$, tetrads of circles: taking at random any one of these tetrads for the blue circles, the remaining red circle has to be determined so as to touch each of the four blue circles, that is, by four instead of three conditions; and there is not in general any red circle satisfying these four conditions. But the 8 tangent circles do not stand to each other in a relation of symmetry, but form in fact four pairs of circles; and it is possible out of the 70 tetrads to select (and that in 6 ways) a tetrad of blue circles, such that there exists a fourth red circle touching each of these four blue circles. We have thus a system depending upon 3 arbitrary circles, and for which, therefore, the capacity is $=9$. It is (as is known) possible, in quite a

different manner, out of the 70 tetrads to select (and that in 8 ways) a tetrad of blue circles such that there exists a fourth red circle touching each of these four blue circles—but the present paper relates exclusively to the first-mentioned 6 tetrads and not to these 8 tetrads.

3. I consider, in the first instance, a particular case in which the three red circles are not all of them arbitrary, but have a capacity $9-1, =8$; and pass from this to the general case where the capacity is $=9$. Calling the red circles 1, 2, 3 and 4; I start with the circles 1, and 2 arbitrary, and 3 a circle equal to 2: the radical axis, or common chord, of the circles 2 and 3 is thus a line bisecting at right angles the line joining the centres of the circles 2 and 3, say this is the line Ω. We have then four circles, each having its centre in the line Ω and touching the circles 1 and 2: in fact, the locus of the centre of a circle touching the circles 1 and 2 is a pair of conics, each of them having for foci the centres of these circles: the line Ω meets each of these conics in two points, and there are thus on the line Ω four points, each of them the centre of a circle touching the circles 1 and 2. But the equal circles 2 and 3 are symmetrically situate in regard to the line Ω; and it is obvious that the four circles, having their centres on the line Ω, will each of them also touch the circle 3; we have thus the four blue circles, each of them with its centre on the line Ω and touching each of the red circles 1, 2 and 3. And it is moreover clear that, taking the red circle 4 equal to 1 and situate symmetrically therewith in regard to the line Ω, then this circle 4 will touch each of the blue circles: so that we have here the four blue circles, each of them touching the four red circles. As already mentioned, the blue circles have their centre on the line Ω, that is, the line Ω is a common orthotomic of the four blue circles.

4. By inverting in regard to an arbitrary circle we pass to the general case; the line Ω becomes thus a circle Ω, orthotomic to each of the blue circles.

Starting *ab initio*, we have here at pleasure the red circles 1, 2, 3: the circle Ω is a circle having for centre a centre of symmetry of the circles 2 and 3, and passing through the points of intersection (real or imaginary) of these two circles; the circles 2 and 3 are thus the inverses (or say the images) each of the other in regard to the circle Ω. We can then find 4 circles each of them orthotomic to Ω, and touching the circles 1 and 2: but a circle orthotomic to Ω is its own inverse or image in regard to Ω; and it will thus touch the circle 3 which is the image of 2 in regard to Ω. We have thus the four blue circles each of them touching the red circles 1, 2 and 3; and then, taking the red circle 4 as the inverse or image of 1 in regard to Ω, this circle 4 will also touch each of the blue circles. Thus starting with the arbitrary red circles 1, 2, 3, we find the four blue circles and the remaining red circle 4, such that each of the blue circles touches each of the red circles. Since in the construction we group together at pleasure the two circles 2, 3 (out of the three circles 1, 2, 3) and use at pleasure either of the two centres of symmetry, it appears that the number of ways in which the figure might have been completed is $=6$.

5. The blue circles have a common orthotomic circle Ω, that is, the radical axis or common chord of each two of the blue circles passes through one and the same point, the centre of the circle Ω. The figure is symmetrical in regard to the red and blue circles respectively, and thus the red circles have a common orthotomic circle Ω', that is, the radical axis or common chord of each two of the red circles passes through one and the same point, the centre of the circle Ω'.

6. Projecting stereographically on a spherical surface, the four red circles and the four blue circles become circles of the sphere; and then making the general homographic transformation, they become plane sections of a quadric surface; we have thus the theorem that on a given quadric surface it is possible to find four red sections and four blue sections such that each blue section touches each red section; and moreover the capacity of the system is $=9$; viz. 3 of the red sections may be assumed at pleasure. But (as is well known) the theory of the tangency of plane sections of a quadric surface is far more simple than that of the tangency of circles: the condition in order that two sections may touch each other is simply the condition that the line of intersection of the two planes shall touch the quadric surface. And we construct, as follows, the sections touching each of three given sections: say the given sections are 1, 2, 3; through the sections 1 and 2 we have two quadric cones having for vertices say the points d_{12} and i_{12} (direct and inverse centres of the two sections): similarly, through the sections 1 and 3 we have two quadric cones vertices d_{13} and i_{13} respectively, and through the sections 2 and 3 we have two quadric cones vertices d_{23} and i_{23} respectively; the points d_{12}, i_{12}, d_{13}, i_{13}, d_{23}, i_{23} lie three and three in four intersecting lines or axes, viz. these are $d_{23}d_{31}d_{12}$, $d_{23}i_{31}i_{12}$, $d_{31}i_{12}i_{23}$, $d_{12}i_{23}i_{31}$ respectively. Through any one of these axes, say $d_{23}d_{31}d_{12}$, we may draw to the quadric surface two tangent planes each touching the three cones which have their vertices in the points d_{23}, d_{31}, d_{12} respectively; and the section by either tangent plane is thus a section touching each of the three given sections 1, 2, 3; we have thus the eight tangent sections of these three sections.

7. Taking as three of the red sections the arbitrary sections 1, 2, 3; and grouping together two at pleasure of these sections, say 2 and 3; we may take for the blue sections the two sections through the axis $d_{23}d_{31}d_{12}$, and those through the axis $d_{23}i_{31}i_{12}$; we have thus the four blue sections touching each of the given red sections 1, 2, 3; and this being so, there exists a remaining red section 4 touching each of the blue sections; we have thus the four blue sections touching each of the red sections 1, 2, 3 and 4. This implies that the vertices or points d_{24} and d_{34} lie on the axis $d_{23}d_{31}d_{12}$, and that the vertices or points i_{24} and i_{34} lie on the axis $d_{23}i_{31}i_{31}$; or, what is the same thing, that the four sections 1, 2, 3, 4 have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$.

8. If the quadric surface be a flat surface (*surface aplatie*) or conic, then the red sections become chords of the conic; the axes are lines in the plane of the conic, and thus the tangent planes through an axis each coincide with the plane of the conic, and it would at first sight appear that any theorem as to tangency becomes nugatory. But this is not so; comparing with the last preceding paragraph, we still have the theorem: on a given conic, taking at pleasure any three chords

1, 2, 3, it is possible to find a fourth chord 4, such that the four chords have in common an axis $d_{23}d_{21}d_{31}d_{24}d_{34}$ and also an axis $d_{23}i_{21}i_{31}i_{24}i_{34}$. And the analytical theory (although somewhat complex) is extremely interesting. Considering the conic $xz - y^2 = 0$, the coordinates of a point on the conic are given by $x : y : z = 1 : \theta : \theta^2$, or say any point of the conic is determined by its parameter θ; and this being so, considering any three chords 1, 2, 3, I take for the two extremities of 1 the values ϵ, ζ; for those of 2 the values α, β; and for those of 3 the values γ, δ; the remaining chord 4 is to be determined as above, and I take for its two extremities the values E, Z.

9. Starting with the chords 1, 2, 3, we have each of the points d_{23}, &c., as the intersection of two lines, viz. these are

$$\text{for } d_{23}\begin{cases} x\alpha\delta - y(\alpha+\delta) + z = 0, \\ x\beta\gamma - y(\beta+\gamma) + z = 0, \end{cases} \quad \text{for } i_{23}\begin{cases} x\alpha\gamma - y(\alpha+\gamma) + z = 0, \\ x\beta\delta - y(\beta+\delta) + z = 0, \end{cases}$$

$$\text{,, } d_{12}\begin{cases} x\alpha\zeta - y(\alpha+\zeta) + z = 0, \\ x\beta\epsilon - y(\beta+\epsilon) + z = 0, \end{cases} \quad \text{,, } i_{12}\begin{cases} x\alpha\epsilon - y(\alpha+\epsilon) + z = 0, \\ x\beta\zeta - y(\beta+\zeta) + z = 0, \end{cases}$$

$$\text{,, } d_{13}\begin{cases} x\gamma\zeta - y(\gamma+\zeta) + z = 0, \\ x\delta\epsilon - y(\delta+\epsilon) + z = 0, \end{cases} \quad \text{,, } i_{13}\begin{cases} x\gamma\epsilon - y(\gamma+\epsilon) + z = 0, \\ x\delta\zeta - y(\delta+\zeta) + z = 0, \end{cases}$$

and we thence find without difficulty for the axis $d_{23}d_{12}d_{13}$ the equation

$$\begin{aligned} & x\{\alpha\beta(\delta\epsilon - \gamma\zeta) + \gamma\delta(\zeta\alpha - \epsilon\beta) + \epsilon\zeta(\beta\gamma - \alpha\delta)\} \\ & + y\{(\beta-\alpha)(\gamma\delta - \epsilon\zeta) + (\delta-\gamma)(\epsilon\zeta - \alpha\beta) + (\zeta-\epsilon)(\alpha\beta - \gamma\delta)\} \\ & + z\{-(\delta\epsilon - \gamma\zeta) - (\zeta\alpha - \epsilon\beta) - (\beta\gamma - \alpha\delta)\} = 0; \end{aligned}$$

the equation of the axis $d_{23}i_{12}i_{13}$ is obtained herefrom by the interchange of ϵ and ζ.

10. The points d_{24} and d_{34} will lie upon the first-mentioned axis if only d_{24} lies upon this axis, viz. if we have

$$\begin{aligned} & \{\alpha\beta(\delta\epsilon - \gamma\zeta) + \gamma\delta(\zeta\alpha - \epsilon\beta) + \epsilon\zeta(\beta\gamma - \alpha\delta)\}(\beta + E - \alpha - Z) \\ & + \{(\beta-\alpha)(\gamma\delta - \epsilon\zeta) + (\delta-\gamma)(\epsilon\zeta - \alpha\beta) + (\zeta-\epsilon)(\alpha\beta - \gamma\delta)\}(\beta E - \alpha Z) \\ & + \{-(\delta\epsilon - \gamma\zeta) - (\zeta\alpha - \epsilon\beta) - (\beta\gamma - \alpha\delta)\}\{-\alpha Z(\beta + E) + \beta E(\alpha + Z)\} = 0. \end{aligned}$$

Reducing this equation, the factor $\alpha - \beta$ divides out, and we finally obtain

$$\begin{aligned} & (\gamma-\alpha)(\beta\delta + \zeta Z)(\epsilon - E) \\ & + (\beta-\delta)(\alpha\gamma + \epsilon E)(\zeta - Z) \\ & + (\alpha\delta - \beta\gamma)(\epsilon\zeta - EZ) \\ & + (\alpha\beta - \gamma\delta)(\epsilon Z - \zeta E) = 0; \end{aligned}$$

say this is

$$A + BE + CZ + DEZ = 0,$$

where

$$\begin{aligned} A &= (\gamma-\alpha)\beta\delta\epsilon + (\beta-\delta)\alpha\gamma\zeta + (\alpha\delta - \beta\gamma)\epsilon\zeta, \\ B &= -(\gamma-\alpha)\beta\delta - (\alpha\beta - \gamma\delta)\zeta + (\beta-\delta)\epsilon\zeta, \\ C &= -(\beta-\delta)\alpha\gamma + (\alpha\beta - \gamma\delta)\epsilon + (\gamma-\alpha)\epsilon\zeta, \\ D &= -(\alpha\delta - \beta\gamma) - (\beta-\delta)\epsilon - (\gamma-\alpha)\zeta, \end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}d_{13}d_{12}d_{43}d_{42}$.

We interchange herein ϵ, ζ and also E, Z, and we thus obtain

$$A' + C'E + B'Z + D'EZ = 0,$$

where

$$\begin{aligned}
A' &= \quad . \quad (\beta-\delta)\,\alpha\gamma\epsilon + (\gamma-\alpha)\,\beta\delta\zeta + (\alpha\delta-\beta\gamma)\,\epsilon\zeta,\\
B' &= -(\gamma-\alpha)\,\beta\delta - (\alpha\beta-\gamma\delta)\,\epsilon \quad . \quad + \quad (\beta-\delta)\,\epsilon\zeta,\\
C' &= -(\beta-\delta)\,\alpha\gamma \quad . \quad + (\alpha\beta-\gamma\delta)\,\zeta + \quad (\gamma-\alpha)\,\epsilon\zeta,\\
D' &= -(\alpha\delta-\beta\gamma) - (\gamma-\alpha)\,\epsilon \quad - (\beta-\delta)\,\zeta \quad . \quad ,
\end{aligned}$$

viz. this is the condition for the existence of the axis $d_{23}i_{13}i_{12}i_{43}i_{42}$.

11. I remark that we have

$$\begin{aligned}
A' &= A + \mathrm{a}\,(\epsilon-\zeta), \quad B' = B + \mathrm{b}\,(\epsilon-\zeta),\\
C' &= C + \mathrm{c}\,(\epsilon-\zeta), \quad D' = D + \mathrm{d}\,(\epsilon-\zeta),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{a} &= (\beta-\delta)\,\alpha\gamma - (\gamma-\alpha)\,\beta\delta = \alpha\beta\,(\gamma+\delta) - \gamma\delta\,(\alpha+\beta),\\
\mathrm{b} &= \mathrm{c} = \gamma\delta - \alpha\beta,\\
\mathrm{d} &= \alpha + \beta - \gamma - \delta,
\end{aligned}$$

and further that

$$B - C = \alpha\beta\,(\gamma+\delta) - \gamma\delta\,(\alpha+\beta) + (\gamma\delta-\alpha\beta)\,(\epsilon+\zeta) + (\alpha+\beta-\gamma-\delta)\,\epsilon\zeta,$$

say this is Π.

12. It thus appears that, for the determination of E, Z, we have

$$\begin{aligned}
A + BE + CZ + DEZ &= 0,\\
A' + C'E + B'Z + D'EZ &= 0.
\end{aligned}$$

Eliminating Z, we find

$$\frac{A+BE}{A'+C'E} = \frac{C+DE}{B'+D'E},$$

that is,

$$(AC'-A'C) + (AD'-A'D+BB'-CC')\,E + (BD'-B'D)\,E^2 = 0;$$

upon reducing the coefficients of this equation it appears that they contain each of them the factor Π, and throwing out this factor, the equation is

$$\begin{aligned}
&\epsilon\,[\alpha\beta\,(\gamma-\delta) + \gamma\delta\,(\beta-\alpha) + (\alpha\delta-\beta\gamma)\,\zeta]\\
&+ [-\alpha\beta\,(\gamma-\delta) - \gamma\delta\,(\beta-\alpha) + (\alpha\delta-\beta\gamma)\,\epsilon + (\beta\gamma-\alpha\delta)\,\zeta + (\beta+\gamma-\alpha-\delta)\,\epsilon\zeta]\,E\\
&+ [\beta\gamma-\alpha\delta-(\beta+\gamma-\alpha-\delta)]\,E^2 = 0;
\end{aligned}$$

this contains obviously the factor $E-\epsilon$, or throwing out this factor, we have for E the simple equation

$$\{\alpha\beta\,(\gamma-\delta) + \gamma\delta\,(\beta-\alpha)\} + (\alpha\delta-\beta\gamma)\,(E+\zeta) + (\beta-\alpha+\gamma-\delta)\,\zeta E = 0.$$

In a similar manner it may be shown that the two equations give for Z the like simple equation

$$\{\alpha\beta\,(\gamma-\delta) + \gamma\delta\,(\beta-\alpha)\} + (\alpha\delta-\beta\gamma)\,(Z+\epsilon) + (\beta-\alpha+\gamma-\delta)\,\epsilon Z = 0,$$

viz. starting from the chords 1, 2, 3 which depend on the parameters (ϵ, ζ), (α, β), (γ, δ) respectively, these last two equations give the parameters (E, Z) of the chord 4.

948.

REPORT OF A COMMITTEE APPOINTED FOR THE PURPOSE OF CARRYING ON THE TABLES CONNECTED WITH THE PELLIAN EQUATION FROM THE POINT WHERE THE WORK WAS LEFT BY DEGEN IN 1817.

[From the *British Association Report*, (1893), pp. 73—120.]

WE have, on the Pellian Equation, Degen's tables, the title of which is "Canon Pellianus sive Tabula simplicissimam æquationis celebratissimæ $y^2 = ax^2 + 1$ solutionem pro singulis numeri dati valoribus ab 1 usque ad 1000 in numeris rationalibus iisdemque integris exhibens." Autore Carolo Ferdinando Degen. Hafniæ, apud Gerhardum Bonnierum, MDCCCXVII., 8vo. Introductio, pp. v—xxiv. Tabula I. Solutionem æquationis $y^2 - ax^2 - 1 = 0$ exhibens, pp. 3—106. Tabula II. Solutionem æquationis $y^2 - ax^2 + 1 = 0$, quotiescunque valor ipsius a talem admiserit, exhibens, pp. 109—112.

The mode of calculation is explained in the Introduction, and illustrated by the examples of the numbers 209, 173.

As to the first of these, the entry in Table I. is

209	14, 2, 5, 3, (2)
	1, 13, 5, 8, 11
	3220
	46551

where the first line gives the expression of $\sqrt{209}$ as a continued fraction, viz. we have

$$\sqrt{209} = 14 + \frac{1}{2+}\ \frac{1}{5+}\ \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{28+}\ \frac{1}{2+}\ \&c.,$$

the denominators being 2, 5, 3, (2), 3, 5, 2, then 28, which is the double of the integer part 14, and then again 2, 5, 3, (2), 3, 5, 2, and so on, the parentheses of the (2) being used to indicate that this is the middle term of the period.

The second row gives auxiliary numbers occurring in the calculation of the first row and having a meaning, as will presently appear. Observe that the 11 which comes under the (2) should also be printed in parentheses (11), but this is not done.

The process for the calculation of the x, y is as follows:

209			
14	1	0	+ 1
2	14	1	− 13
5	29	2	+ 5
3	159	11	− 8
(2)	506	35	+ (11)
3	1171	81	− 8
5	4019	278	+ 5
2	21266	1471	− 13
28	46551	3220	+ 1

viz. writing down as a first column the numbers of the first row, and beginning the second column with 1, 14 (14 the number at the head of the first column), and the third column with 0, 1, we calculate the numbers of the second column, $29 = 2 . 14 + 1$, $159 = 5 . 29 + 14$, $506 = 3 . 159 + 29$, &c., and the numbers of the third column in like manner, $2 = 2 . 1 + 0$, $11 = 5 . 2 + 1$, $35 = 3 . 11 + 2$, &c.; and then writing down as a fourth column the numbers of the second row with the signs +, − alternately, we have a series of equations $y^2 - ax^2 = \pm A$, viz.

$$\begin{aligned} 1^2 - 209 . 0^2 &= + 1, \\ 14^2 - 209 . 1^2 &= - 13, \\ 29^2 - 209 . 2^2 &= + 5, \\ &\vdots \end{aligned}$$

the last of them being

$$(46551)^2 - 209\,(3220)^2 = + 1,$$

this last corresponding as above to the value + 1, and the numbers 46551 and 3220 being accordingly the y and x given in the fourth and third rows of the table.

As to the second of the foregoing numbers, 173, the only difference is that the period has a double middle term, viz. the entry in the Table I. is

173	13, 6, (1, 1) 1, 4, (13, 13) 190060 2499849

The first row gives the expression of $\sqrt{173}$, viz. that is

$$\sqrt{173} = 13 + \frac{1}{6+}\ \frac{1}{(1)+}\ \frac{1}{(1)+}\ \frac{1}{6+}\ \frac{1}{26+}\ \&c.,$$

the denominators being 6, 1, 1, 6, then 26 (the double of the integer part 13), and then again 6, 1, 1, 6, and so on. In the second row I remark that Degen prints the parentheses (13, 13) for the double middle term.

The process for the calculation of the x, y is similar to that in the former case, viz. we have

173			
13	1	0	+ 1
6	13	1	− 4
$\binom{1}{1}$	79	6	+13
	92	7	−13
6	171	13	+ 4
26	1118	85	− 1

where the second and third columns begin 1, 13 and 0, 1 respectively, and the remaining terms are calculated $79 = 6.13 + 1$, $92 = 1.79 + 13$, &c., and $6 = 6.1 + 0$, $7 = 1.6 + 1$, &c.; and then writing down as a fourth column the terms of the second row with the signs +, − alternately, we have

$$\begin{aligned} 1^2 - 173.0^2 &= +\ 1, \\ 13^2 - 173.1^2 &= -\ 4, \\ 79^2 - 173.6^2 &= +13, \\ &\vdots \end{aligned}$$

the last equation being

$$(1118)^2 - 173\,(85)^2 = -\ 1,$$

the term for the last equation being always in a case such as the present one, not $+1$, but -1. The final numbers 1118, 85 are consequently entered not in Table I., but in Table II., viz. the entry in this table is

173	85
	1118

and thence we calculate the numbers y, x of Table I., viz. these are

$$2499849 = 2\,.\,(1118)^2 + 1,$$
$$190060 = 2\,.\,1118\,.\,85.$$

Generally Table II. gives for each value of a, comprised therein, values of x, y, such that $y^2 = ax^2 - 1$, and then writing $y_1 = 2y^2 + 1$, $x_1 = 2xy$, we have

$$y_1^2 = (2ax^2 - 1)^2 = 4a^2x^4 - 4ax^2 + 1 = a\,.\,4x^2\,(ax^2 - 1) + 1 = ax_1^2 + 1,$$

so that x_1, y_1 are for the same value of a the values of x, y in Table I.

It is to be remarked that the heading of Table II. is not perfectly accurate, for it purports to give for every value of a, for which a solution exists, a solution of the equation $y^2 = ax^2 - 1$. What it really gives is the solution for each value of a for which the period has a double middle term. But if $a = \alpha^2 + 1$, then obviously we have a solution $y = \alpha$, $x = 1$, and for any such value of a the period has a single middle term, viz. the entry in Table I. is

$\alpha^2 + 1$	α, (2α)
	1, 1
	2α
	$2\alpha^2 + 1$

and we, in fact, have

α^2+1			
α	1	0	$+1$
(2α)	α	1	-1
2α	$2\alpha^2 + 1$	2α	$+1$

that is,

$$1^2 - (\alpha^2 + 1)\,0^2 \quad = +1,$$
$$\alpha^2 - (\alpha^2 + 1)\,1^2 \quad = -1,$$
$$(2\alpha^2 + 1)^2 - (\alpha^2 + 1)\,(2\alpha)^2 = +1.$$

The foregoing instances of the calculation of x, y in the case of the numbers 209 and 173 suggest a table which may be regarded as an extended form of Degen's tables; viz. such a table, from $a=2$ to $a=99$, is as follows:

SPECIMEN OF EXTENDED FORM OF TABLE IN REGARD TO THE PELLIAN EQUATION.

a		y	x	y^2-ax^2
2	1	1	0	+ 1
	(2)	1	1	− 1
	2	3	2	+ 1
3	1	1	0	+ 1
	(1)	1	1	− 2
	2	2	1	+ 1
5	2	1	0	+ 1
	(4)	2	1	− 1
	4	9	4	+ 1
6	2	1	0	+ 1
	(2)	2	1	− 2
	4	5	2	+ 1
7	2	1	0	+ 1
	1	2	1	− 3
	(1)	3	1	+ 2
	1	5	2	− 3
	4	8	3	+ 1
8	2	1	0	+ 1
	(1)	2	1	− 4
	4	3	1	+ 1
10	3	1	0	+ 1
	(6)	3	1	− 1
	6	19	6	+ 1
11	3	1	0	+ 1
	(3)	3	1	− 2
	6	10	3	+ 1
12	3	1	0	+ 1
	(2)	3	1	− 3
	6	7	2	+ 1
13	3	1	0	+ 1
	1	3	1	− 4
	(1)	4	1	+ 3
	(1)	7	2	− 3
	1	11	3	+ 4
	6	18	5	− 1
14	3	1	0	+ 1
	1	3	1	− 5
	(2)	4	1	+ 2
	1	11	3	− 5
	6	15	4	+ 1
15	3	1	0	+ 1
	(1)	3	1	− 6
	6	4	1	+ 1
17	4	1	0	+ 1
	(8)	4	1	− 1
	8	33	8	+ 1
18	4	1	0	+ 1
	(4)	4	1	− 2
	8	17	4	+ 1
19	4	1	0	+ 1
	2	4	1	− 3
	1	9	2	+ 5
	(3)	13	3	− 2
	1	48	11	+ 5
	2	61	14	− 3
	8	170	39	+ 1
20	4	1	0	+ 1
	(2)	4	1	− 4
	8	9	2	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
21	4	1	0	+ 1
	1	4	1	− 5
	1	5	1	+ 4
	(2)	9	2	− 3
	1	23	5	+ 4
	1	32	7	− 5
	8	55	12	+ 1
22	4	1	0	+ 1
	1	4	1	− 6
	2	5	1	+ 3
	(4)	14	3	− 2
	2	61	13	+ 3
	1	136	29	− 6
	8	197	42	+ 1
23	4	1	0	+ 1
	1	4	1	− 7
	(3)	5	1	+ 2
	1	19	4	− 7
	8	24	5	+ 1
24	4	1	0	+ 1
	(1)	4	1	− 8
	8	5	1	+ 1
26	5	1	0	+ 1
	(10)	5	1	− 1
	10	51	10	+ 1
27	5	1	0	+ 1
	(5)	5	1	− 2
	10	26	5	+ 1
28	5	1	0	+ 1
	3	5	1	− 3
	(2)	16	3	+ 4
	3	37	7	− 3
	10	127	24	+ 1
29	5	1	0	+ 1
	2	5	1	− 6
	(1 / 1)	11	2	+ 5
		16	3	− 3
	2	27	5	+ 2
	10	70	13	− 1
30	5	1	0	+ 1
	(2)	5	1	− 5
	10	11	2	+ 1
31	5	1	0	+ 1
	1	5	1	− 6
	1	6	1	+ 5
	3	11	2	− 3
	(5)	39	7	+ 2
	3	206	37	− 3
	1	657	118	+ 5
	1	863	155	− 6
	10	1520	273	+ 1
32	5	1	0	+ 1
	1	5	1	− 7
	(1)	6	1	+ 4
	1	11	2	− 7
	10	17	3	+ 1
33	5	1	0	+ 1
	1	5	1	− 8
	(2)	6	1	+ 3
	1	17	3	− 8
	10	23	4	+ 1
34	5	1	0	+ 1
	1	5	1	− 9
	(4)	6	1	+ 2
	1	29	5	− 9
	10	35	6	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
35	5	1	0	+ 1
	(1)	5	1	− 10
	10	6	1	+ 1
37	6	1	0	+ 1
	(12)	6	1	− 1
	12	73	12	+ 1
38	6	1	0	+ 1
	(6)	6	1	− 2
	12	37	6	+ 1
39	6	1	0	+ 1
	(4)	6	1	− 3
	12	25	4	+ 1
40	6	1	0	+ 1
	(3)	6	1	− 4
	12	19	3	+ 1
41	6	1	0	+ 1
	(2)	6	1	− 5
	(2)	13	2	+ 5
	12	32	5	− 1
42	6	1	0	+ 1
	(2)	6	1	− 6
	12	13	2	+ 1
43	6	1	0	+ 1
	1	6	1	− 7
	1	7	1	+ 6
	3	13	2	− 3
	1	46	7	+ 9
	(5)	59	9	− 2
	1	341	52	+ 9
	3	400	61	− 3
	1	1541	235	+ 6
	1	1941	296	− 7
	12	3482	531	+ 1
44	6	1	0	+ 1
	1	6	1	− 8
	1	7	1	+ 5
	1	13	2	− 7
	(2)	20	3	+ 4
	1	53	8	− 7
	1	73	11	+ 5
	1	126	19	− 8
	12	199	30	+ 1
45	6	1	0	+ 1
	1	6	1	− 9
	2	7	1	+ 4
	(2)	20	3	− 5
	2	47	7	+ 4
	1	114	17	− 9
	12	161	24	+ 1
46	6	1	0	+ 1
	1	6	1	− 10
	3	7	1	+ 3
	1	27	4	− 7
	1	34	5	+ 6
	2	61	9	− 5
	(6)	156	23	+ 2
	2	997	147	− 5
	1	2150	317	+ 6
	1	3147	464	− 7
	3	5297	781	+ 3
	1	19038	2807	− 10
	12	24335	3588	+ 1
47	6	1	0	+ 1
	1	6	1	− 1
	(5)	7	1	+ 2
	1	41	6	− 11
	12	48	7	+ 1
48	6	1	0	+ 1
	(1)	6	1	− 12
	12	7	1	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
50	7	1	0	+ 1
	(14)	7	1	− 1
	14	99	14	+ 1
51	7	1	0	+ 1
	(7)	7	1	− 2
	14	50	7	+ 1
52	7	1	0	+ 1
	4	7	1	− 3
	1	29	4	+ 9
	(2)	36	5	− 4
	1	101	14	+ 9
	4	137	19	− 3
	14	649	90	+ 1
53	7	1	0	+ 1
	3	7	1	− 4
	(1)	22	3	+ 7
	(1)	29	4	− 7
	3	51	7	+ 4
	14	182	25	− 1
54	7	1	0	+ 1
	2	7	1	− 5
	1	15	2	+ 9
	(6)	22	3	− 2
	1	147	20	+ 9
	2	169	23	− 5
	14	485	66	+ 1
55	7	1	0	+ 1
	2	7	1	− 6
	(2)	15	2	+ 5
	2	37	5	− 6
	14	89	12	+ 1
56	7	1	0	+ 1
	(2)	7	1	− 7
	14	15	2	+ 1
57	7	1	0	+ 1
	1	7	1	− 8
	1	8	1	+ 7
	(4)	15	2	− 3
	1	68	9	+ 7
	1	83	11	− 8
	14	151	20	+ 1
58	7	1	0	+ 1
	1	7	1	− 9
	1	8	1	+ 6
	(1)	15	2	− 7
	(1)	23	3	+ 7
	1	38	5	− 6
	1	61	8	+ 9
	14	99	13	− 1
59	7	1	0	+ 1
	1	7	1	− 10
	2	8	1	+ 5
	(7)	23	3	− 2
	2	169	22	+ 5
	1	361	47	− 10
	14	530	69	+ 1
60	7	1	0	+ 1
	1	7	1	− 11
	(2)	8	1	+ 4
	1	23	3	− 11
	14	31	4	+ 1
61	7	1	0	+ 1
	1	7	1	− 12
	4	8	1	+ 3
	3	39	5	− 4
	1	125	16	+ 9
	(2)	164	21	− 5
	(2)	453	58	+ 5
	1	1070	137	− 9

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
	3	1523	195	+ 4
	4	5639	722	− 3
	1	24079	3083	+ 12
	14	29718	3805	− 1
62	7	1	0	+ 1
	1	7	1	− 13
	(6)	8	1	+ 2
	1	55	7	− 13
	14	63	8	+ 1
63	7	1	0	+ 1
	(1)	7	1	− 14
	14	8	1	+ 1
65	8	1	0	+ 1
	(16)	8	1	− 1
	16	129	16	+ 1
66	8	1	0	+ 1
	(8)	8	1	− 2
	16	65	8	+ 1
67	8	1	0	+ 1
	5	8	1	− 3
	2	41	5	+ 6
	1	90	11	− 7
	1	131	16	+ 9
	(7)	221	27	− 2
	1	1678	205	+ 9
	1	1899	232	− 7
	2	3577	437	+ 6
	5	9053	1106	− 3
	16	48842	5967	+ 1
68	8	1	0	+ 1
	(4)	8	1	− 4
	16	33	4	+ 1
69	8	1	0	+ 1
	3	8	1	− 5
	3	25	3	+ 4
	1	83	10	− 11
	(4)	108	13	+ 3
	1	515	62	− 11
	3	623	75	+ 4
	3	2384	297	− 5
	16	7775	936	+ 1
70	8	1	0	+ 1
	2	8	1	− 6
	1	17	2	+ 9
	(2)	25	3	− 5
	1	67	8	+ 9
	2	92	11	− 6
	16	251	30	+ 1
71	8	1	0	+ 1
	2	8	1	− 7
	2	17	2	+ 5
	1	42	5	− 11
	(7)	59	7	+ 2
	1	455	54	− 11
	2	514	61	+ 5
	2	1483	176	− 7
	16	3480	413	+ 1
72	8	1	0	+ 1
	(2)	8	1	− 3
	16	17	2	+ 1
73	8	1	0	+ 1
	1	8	1	− 9
	1	9	1	+ 8
	(5)	17	2	− 3
	(5)	94	11	+ 3
	1	487	57	− 8
	1	581	68	+ 9
	16	1068	125	− 1
74	8	1	0	+ 1
	1	8	1	− 10

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
	$\binom{1}{1}$	9	1	+ 7
		17	2	− 7
	1	26	3	+ 10
	16	43	5	− 1
75	8	1	0	+ 1
	1	8	1	− 11
	(1)	9	1	+ 6
	1	17	2	− 11
	16	26	3	+ 1
76	8	1	0	+ 1
	1	8	1	− 12
	2	9	1	+ 5
	1	26	3	− 8
	1	35	4	+ 9
	5	61	7	− 3
	(4)	340	39	+ 4
	5	1421	163	− 3
	1	7445	854	+ 9
	1	8866	1017	− 8
	2	16311	1871	+ 5
	1	41488	4759	− 12
	16	57799	6630	+ 1
77	8	1	0	+ 1
	1	8	1	− 13
	3	9	1	+ 4
	(2)	35	4	− 7
	3	79	9	+ 4
	1	272	31	− 13
	16	351	40	+ 1
78	8	1	0	+ 1
	1	8	1	− 14
	(4)	9	1	+ 3
	1	44	5	− 14
	16	53	6	+ 1
79	8	1	0	+ 1
	1	8	1	− 15
	(7)	9	1	+ 2
	1	71	8	− 15
	16	80	9	+ 1
80	8	1	0	+ 1
	(1)	8	1	− 16
	16	9	1	+ 1
82	9	1	0	+ 1
	(18)	9	1	− 1
	18	163	18	+ 1
83	9	1	0	+ 1
	(9)	9	1	− 2
	18	82	9	+ 1
84	9	1	0	+ 1
	(6)	9	1	− 3
	18	55	6	+ 1
85	9	1	0	+ 1
	4	9	1	− 4
	$\binom{1}{1}$	37	4	+ 9
		46	5	− 9
	4	83	9	+ 4
	18	378	41	− 1
86	9	1	0	+ 1
	3	9	1	− 5
	1	28	3	+ 10
	1	37	4	− 7
	1	65	7	+ 11
	(8)	102	11	− 2
	1	881	95	+ 11
	1	983	106	− 7
	1	1864	201	+ 10
	3	2847	307	− 5
	18	10405	1122	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2
87	9	1	0	+ 1
	(3)	9	1	− 6
	18	28	3	+ 1
88	9	1	0	+ 1
	2	9	1	− 7
	1	19	2	+ 9
	(1)	28	3	− 8
	1	47	5	+ 9
	2	75	8	− 7
	18	197	21	+ 1
89	9	1	0	+ 1
	2	9	1	− 8
	(3)	19	2	+ 5
	(3)	66	7	− 5
	2	217	23	+ 8
	18	500	53	− 1
90	9	1	0	+ 1
	(2)	9	1	− 9
	18	19	2	+ 1
91	9	1	0	+ 1
	1	9	1	− 10
	1	10	1	+ 9
	5	19	2	− 3
	(1)	105	11	+ 14
	5	124	13	− 3
	1	725	76	+ 9
	1	849	89	− 10
	18	1574	165	+ 1
92	9	1	0	+ 1
	1	9	1	− 11
	1	10	1	+ 3
	2	19	2	− 7
	(4)	48	5	+ 4
	2	211	22	− 7
	1	470	49	+ 3
	1	681	71	− 11
	18	1151	120	+ 1
93	9	1	0	+ 1
	1	9	1	− 12
	1	10	1	+ 7
	1	19	2	− 11
	4	29	3	+ 4
	(6)	135	14	− 3
	4	839	87	+ 4
	1	3491	362	− 11
	1	4330	449	+ 7
	1	7821	811	− 12
	18	12151	1260	+ 1
94	9	1	0	+ 1
	1	9	1	− 13
	2	10	1	+ 6
	3	29	3	− 5
	1	97	10	+ 9
	1	126	13	− 10
	5	223	23	+ 3
	1	1241	128	− 15
	(8)	1464	151	+ 2
	1	12953	1336	− 15
	5	14417	1487	+ 3
	1	85038	8771	− 10
	1	99455	10258	+ 9
	3	1 84493	19029	− 5
	2	6 52934	67345	+ 6
	1	14 90361	1 53719	− 13
	18	21 43295	2 21064	+ 1
95	9	1	0	+ 1
	1	9	1	− 14
	(2)	10	1	+ 5
	1	29	3	− 14
	18	39	4	+ 1
96	9	1	0	+ 1
	1	9	1	− 15
	(3)	10	1	+ 4
	1	39	4	− 15
	18	49	5	+ 1

SPECIMEN OF EXTENDED FORM OF PELLIAN EQUATION TABLE—*continued.*

a		y	x	y^2-ax^2	a		y	x	y^2-ax^2
97	9	1	0	+ 1	98	9	1	0	+ 1
	1	9	1	− 16		1	9	1	− 17
	5	10	1	+ 3		(8)	10	1	+ 2
	1	59	6	− 11		1	89	9	− 17
	1	69	7	+ 8		18	99	10	+ 1
	(1)	128	13	− 9	99	9	1	0	+ 1
	(1)	197	20	+ 9		(1)	9	1	− 18
	1	325	33	− 8		18	10	1	+ 1
	1	522	53	+ 11					
	5	847	86	− 3					
	1	4757	483	+ 16					
	18	5604	569	− 1					

The meaning hardly requires explanation; for each number a, we have a series of pairs of increasing numbers, y, x, satisfying a series of equations $y^2 = ax^2 \pm b$; thus

$a = 14$

y	x	$y^2 - ax^2$
1	0	$1 - 14 . 0 = 1,$
3	1	$9 - 14 . 1 = -5,$
4	1	$16 - 14 . 1 = +2,$
11	3	$121 - 14 . 9 = -5,$
15	4	$225 - 14 . 16 = +1.$

The following table, calculated under the superintendence of the Committee, extends from $a = 1001$ to $a = 1500$ (square numbers omitted); it is (with slight typographical variations) nearly but not exactly in the form of Degen's Table I., the chief difference being that for a number a having a double middle term, or of the form a^2+1 (such number being further distinguished by an asterisk), the x, y entered in the table are the solutions, *not* of the equation $y^2 = ax^2 + 1$, but of the equation $y^2 = ax^2 - 1$. As remarked above, if we have $y^2 = ax^2 - 1$, then writing $y_1 = 2y^2 + 1$ and $x_1 = 2xy$, we obtain $y_1^2 = ax_1^2 + 1$.

Moreover, for each value of a, in the first line, the first term, which is the integer part of $\sqrt{a}$, is separated from the other by a semicolon, and the 1, which is the corresponding first term of the second line, is omitted.

The calculations were made by C. E. Bickmore, M.A., of New College, Oxford: his values for x and y have been revised as presently mentioned, but it has been assumed that his values for the periods and subsidiary numbers (forming the first and second lines of each division of the table) are accurate; in fact, any error therein would cause the resulting values of x and y to be wildly erroneous; but (except in a single instance which was accounted for) the errors in x and y were in every case in a single figure or two or three figures only.

The values of x and y were in every case examined by substitution in the equation ($y^2 = ax^2 + 1$, or $y^2 = ax^2 - 1$, as the case may be), which should be satisfied by them. These verifications were for the most part made by A. Graham, M.A., of the Observatory, Cambridge. As already mentioned, some errors were detected, and these have been, of course, corrected. The values of x, y given in the table thus satisfy in every case the proper equation $y^2 = ax^2 + 1$, or $y^2 = ax^2 - 1$; on the ground above referred to, it is believed that the periods and subsidiary numbers are also accurate.

It may be remarked, in regard to the verification of the equation $y^2 = ax^2 \pm 1$ for large values of x and y, it is in practice easier and safer to calculate $ax^2 \pm 1$, and then to compare the square root thereof with the given value of y, than to further calculate the value of y^2.

THE TABLE 1001 TO 1500.

1001	31; 1, 1, 1, 3, 3, 2, (4) 40, 23, 35, 16, 17, 25, (13)	33532 10 60905
1002	31; 1, 1, 1, 8, 2, 1, 1, 1, 3, (10) 41, 22, 39, 7, 23, 31, 26, 33, 17, (6)	65 35248 2068 69247
1003	31; 1, 2, (31) 42, 21, (2)	285 9026
1004	31; 1, 2, 5, 2, 2, 1, 7, 4, 1, 2, 1, 11, 1, (14) 43, 20, 11, 25, 19, 41, 8, 13, 40, 17, 44, 5, 55, (4)	85 24164 59730 2700 96330 24199
1005	31; 1, 2, 2, 1, 5, 15, 1, 2, (12) 44, 19, 20, 39, 11, 4, 41, 21, (5)	930 59568 29501 49761
1006	31; 1, 2, 1, 1, 5, 1, 3, 2, 1, 1, 1, 1, 1, 9, 1, 20, 4, 5, 1, 1, 12, 6, 1, (30) 45, 18, 29, 33, 10, 43, 15, 22, 31, 27, 30, 25, 37, 6, 55, 3, 15, 11, 30, 33, 5, 9, 53, (2)	4 45346 14025 55749 21748 141 25267 56378 02146 05455
1007	31; 1, 2, (1) 46, 17, (38)	15 476
1008	31; 1, (2) 47, (16)	4 127
1009*	31; 1, (3, 3) 48, (15, 15)	17 540
1010*	31; 1, 3, (1, 1) 49, 14, (31, 31)	41 1303
1011	31; 1, 3, 1, (9) 50, 13, 47, (6)	265 8426
1012	31; 1, 4, 3, 6, 1, 3, 8, 1, (4) 51, 12, 19, 9, 43, 16, 7, 48, (11)	1013 02110 32226 17399
1013*	31; 1, 4, 1, 4, 15, 1, 2, (2, 2) 52, 11, 44, 13, 4, 43, 19, (23, 23)	123 52985 3931 66618
1014	31; 1, 5, 2, 1, 1, 1, 1, (20) 53, 10, 23, 30, 29, 25, 38, (3)	1 46266 46 56965
1015	31; 1, 6, 10, (2) 54, 9, 6, (29)	11076 3 52871
1016	31; 1, (6) 55, (8)	8 255
1017	31; 1, 8, 7, 1, 6, 4, 1, 3, 5, 1, 1, (6) 56, 7, 8, 49, 9, 13, 41, 16, 11, 31, 32, (9)	9 09655 84992 290 09322 97217
1018*	31; 1, 9, 1, 1, 1, 6, 2, (3, 3) 57, 6, 39, 23, 38, 9, 26, (17, 17)	27 28333 870 50499
1019	31; 1, 11, 1, 3, 1, 1, 1, 3, 8, 1, 5, 2, (31) 58, 5, 47, 14, 35, 25, 34, 17, 7, 49, 10, 29, (2)	19 07764 36539 608 99233 21730
1020	31; 1, (14) 59, (4)	16 511
1021*	31; 1, 20, 3, 6, 1, 3, 2, 1, 1, 12, 5, 4, 15, 1, 2, 1, 4, 1, 1, 2, 1, 1, 1, (5, 5) 60, 3, 20, 9, 44, 15, 23, 27, 36, 5, 12, 15, 4, 45, 17, 41, 12, 33, 29, 20, 33, 25, 36, (11, 11)	98 65001 29666 69564 06909 3152 17280 37258 48825 15030

TABLE 1001 TO 1500—*continued.*

1022	31; 1, (30) 61, (2)	32 1023
1023	31; (1) (62)	1 32
1025*	32; (64) (1)	1 32
1026	32; (32) (2)	32 1025
1027	32; 21, 2, 1, 6, 2, (4) 3, 22, 39, 9, 27, (13)	41 54868 1331 50393
1028	32; (16) (4)	16 513
1029	32; 12, 1, 4, 2, 2, 1, 3, 15, 1, 3, 2, 1, (20) 5, 49, 12, 25, 20, 37, 17, 4, 47, 15, 20, 43, (3)	303 57068 95884 9737 94964 66615
1030	32; 10, 1, 2, 6, 1, 3, 1, 2, 1, 1, 2, 2, (12) 6, 41, 21, 9, 45, 14, 39, 19, 31, 30, 21, 26, (5)	2 75539 87434 88 43070 11291
1031	32; 9, 6, 3, 4, 1, 1, 1, 1, 1, 12, 4, 1, 1, (31) 7, 10, 19, 13, 35, 26, 31, 25, 38, 5, 14, 29, 35, (2)	2029 75370 82877 65173 74486 64200
1032	32; (8) (8)	8 257
1033*	32; 7, 7, 1, 8, 3, 3, 1, 2, 3, 2, 2, 1, 1, 1, 2, (21, 21) 9, 8, 51, 7, 19, 16, 37, 21, 17, 24, 21, 32, 27, 31, 24, (3, 3)	5 81389 30460 24093 186 86036 75961 74196
1034	32; 6, 2, (2) 10, 25, (22)	494 15885
1035	32; 5, (1) 11, (46)	35 1126
1036	32; 5, 2, 1, 6, (2) 12, 21, 40, 9, (28)	26322 8 47225
1037*	32; 4, 1, 15, (3, 3) 13, 49, 4, (19, 19)	64805 20 86882
1038	32; 4, 1, 1, 2, 2, 1, 2, (10) 14, 33, 29, 22, 21, 34, 23, (6)	4 78126 154 04267
1039	32; 4, 3, 1, 1, 5, 3, 2, 2, 21, 12, 1, 5, 1, 1, 10, 4, 1, 6, 2, 1, 3, 1, 1, 15, 17, 30, 33, 11, 18, 23, 26, 3, 5, 51, 10, 31, 33, 6, 13, 46, 9, 22, 37, 15, 34, 27, 1, 1, 1, 1, (31) 29, 30, 25, 39, (2)	 1 53006 74275 15667 18643 06921 49 31946 34979 07633 93486 37520
1040	32; (4) (16)	4 129
1041	32; 3, 1, 3, 1, 1, 4, 2, 2, 7, 1, 1, 1, 12, 3, 1, (20) 17, 40, 15, 31, 32, 13, 24, 25, 8, 39, 23, 40, 5, 16, 47, (3)	2389 36879 43492 77091 86499 27575
1042*	32; 3, (1, 1) 18, (31, 31)	25 807

TABLE 1001 TO 1500—*continued.*

1043	32; 3, 2, 1, 1, 1, 1, (8) 19, 22, 31, 29, 26, 37, (7)	17864 5 76927
1044	32; 3, 4, 1, 1, 1, 3, 2, 1, 1, (6) 20, 13, 36, 25, 35, 16, 23, 28, 35, (9)	59 45940 1921 19201
1045	32; 3, 15, 1, (4) 21, 4, 51, (11)	14112 4 56191
1046	32; 2, 1, 12, 3, 1, 2, 1, 1, 1, 5, 1, 5, (32) 22, 41, 5, 17, 38, 19, 34, 25, 37, 10, 47, 11, (2)	12 32778 42162 398 70425 10565
1047	32; 2, 1, 3, 1, (20) 23, 37, 14, 49, (3)	4228 1 36807
1048	32; 2, 1, 2, 6, 1, 4, 1, 1, (7) 24, 33, 23, 9, 47, 12, 31, 33, (8)	32 42859 1049 80517
1049*	32; 2, 1, 1, 2, 1, 4, 3, 1, 5, 7, 1, (12, 12) 25, 29, 32, 19, 40, 13, 16, 43, 11, 8, 53, (5, 5)	4 74354 09493 153 63508 25620
1050	32; 2, 2, (10) 26, 25, (6)	270 8749
1051	32; 2, 2, 1, 1, 2, 4, 4, 10, 1, 1, 3, 12, 1, 2, 6, 7, 21, 2, 8, 1, 3, 2, 2, 27, 21, 31, 30, 23, 14, 15, 6, 35, 29, 18, 5, 42, 21, 10, 9, 3, 30, 7, 46, 15, 25, 19, 1, (31) 45, (2)	 95 53827 66460 01506 59129 73250 2 94697 22410 66655 86570 86507
1052	32; 2, 3, 3, 7, 1, 4, 9, (16) 28, 17, 19, 8, 47, 13, 7, (4)	11935 79112 3 87132 00767
1053	32; 2, (4) 29, (13)	20 649
1054	32; 2, 6, 1, 2, 1, 1, 4, 2, 2, 1, 1, 1, 3, 1, 2, 3, 4, (32) 30, 9, 42, 19, 30, 33, 13, 25, 21, 33, 26, 35, 15, 38, 21, 18, 15, (2)	10335 86450 05416 3 35557 62560 56895
1055	32; 2, (12) 31, (5)	52 1689
1056	32; (2) (32)	2 65
1057	32; 1, 1, 21, 5, 1, 6, 2, 1, 1, 3, 2, 7, 1, 2, 4, 1, 1, 1, (8) 33, 32, 3, 11, 48, 9, 24, 29, 33, 16, 27, 8, 41, 21, 13, 37, 24, 39, (7)	7 37084 02001 32992 239 63733 95685 29407
1058	32; 1, 1, 8, 1, 3, 1, 3, (32) 34, 31, 7, 47, 14, 41, 17, (2)	40 53146 1318 36323
1059	32; 1, 1, 5, 2, 2, 2, 1, 2, 3, 1, (31) 35, 30, 11, 25, 23, 21, 35, 22, 15, 49, (2)	1838 68081 59834 86610
1060	32; 1, 1, 3, 1, 5, 7, (16) 36, 29, 15, 44, 11, 9, (4)	22 64856 737 38369
1061*	32; 1, 1, 2, 1, 12, 3, 5, 1, 1, 2, 15, 1, 8, 2, 1, 2, (1, 1) 37, 28, 19, 44, 5, 20, 11, 35, 28, 25, 4, 55, 7, 23, 35, 20, (31, 31)	28370 82899 91521 9 24122 86971 11530
1062	32; 1, 1, 2, 3, (32) 38, 27, 23, 19, (2)	9418 3 06917

TABLE 1001 TO 1500—*continued.*

1063	32; 1, 1, 1, 1, 10, 3, 1, 2, 1, 6, 1, 1, 21, 4, 1, (31) 39, 26, 27, 37, 6, 17, 39, 18, 43, 9, 31, 34, 3, 13, 51, (2)	5353 15274 12685 1 74532 48310 97224
1064	32; 1, 1, 1, (1) 40, 25, 31, (28)	21 685
1065	32; 1, 1, 1, 2, 1, 3, 2, 1, 5, (4) 41, 24, 35, 19, 39, 16, 21, 40, 11, (15)	25 33160 826 67999
1066*	32; 1, 1, 1, 5, 1, 6, (2, 2) 42, 23, 39, 10, 49, 9, (25, 25)	1 05205 34 34907
1067	32; 1, (1) 43, (22)	3 98
1068	32; 1, 2, 7, 1, 5, (16) 44, 21, 8, 49, 11, (4)	3 53094 115 39207
1069*	32; 1, 2, 3, 1, 1, 21, 4, 3, 5, 7, 12, 1, 15, 2, 2, 1, 3, 1, 1, 1, 4, 1, (4, 4) 45, 20, 17, 29, 36, 3, 15, 19, 12, 9, 5, 57, 4, 27, 20, 39, 15, 36, 25, 37, 12, 45, (13, 13)	186 40986 37841 77726 15285 6094 77590 16096 85726 12782
1070	32; 1, 2, 2, 5, 1, 1, (12) 46, 19, 26, 11, 31, 34, (5)	90138 29 48491
1071	32; 1, 2, 1, 1, 1, (6) 47, 18, 35, 25, 38, (9)	880 28799
1072	32; 1, 2, 1, 6, 1, 1, 8, 1, 4, 1, 1, (3) 48, 17, 44, 9, 32, 33, 7, 49, 12, 33, 31, (16)	1457 20107 47710 81927
1073*	32; 1, 3, (9, 9) 49, 16, (7, 7)	1385 45368
1074	32; 1, 3, 2, 1, 1, 2, (32) 50, 15, 23, 31, 30, 25, (2)	1 06476 34 89425
1075	32; 1, 3, 1, 2, 3, 10, 1, 1, 1, 2, 2, 6, (1) 51, 14, 39, 21, 19, 6, 39, 25, 34, 21, 26, 9, (50)	51504 12729 16 88675 74226
1076	32; 1, 4, (16) 52, 13, (4)	410 13449
1077	32; 1, 4, 2, 15, 1, (20) 53, 12, 29, 4, 59, (3)	7 16760 235 22399
1078	32; 1, (4) 54, (11)	6 197
1079	32; 1, 5, 1, 1, 2, 2, 4, 3, 1, (1) 55, 10, 35, 29, 22, 25, 14, 17, 35, (26)	54 97325 1805 76876
1080	32; 1, 6, (3) 56, 9, (20)	161 5291
1081	32; 1, 7, 4, 3, 1, 6, 1, 1, 5, 2, 3, 1, 12, 2, 1, 1, 1, 21, 3, 2, (2) 57, 8, 15, 16, 45, 9, 33, 32, 11, 27, 15, 48, 5, 24, 33, 25, 40, 3, 19, 24, (23)	8918 12221 87280 07648 2 93215 05610 17151 19615
1082*	32; 1, 8, 2, 2, 2, (1, 1) 58, 7, 26, 23, 22, (31, 31)	38369 12 62101
1083	32; 1, (9) 59, (6)	11 362

TABLE 1001 TO 1500—*continued.*

1084	32; 1, 12, 5, 2, 2, 3, 2, 6, 1, 7, 2, 1, 2, 1, 3, 1, 1, 1, 21, 3, 4, (16) 60, 5, 12, 25, 24, 17, 27, 9, 51, 8, 23, 36, 19, 40, 15, 37, 24, 41, 3, 20, 15, (4)	8 17041 27029 09269 93200 269 00393 64204 05379 19999
1085	32; 1, 15, (2) 61, 4, (31)	544 17919
1086	32; 1, (20) 62, (3)	22 725
1087	32; 1, (31) 63, (2)	33 1088
1088	32; (1) (64)	1 33
1090*	33; (66) (1)	1 33
1091	33; (33) (2)	33 1090
1092	33; (22) (3)	22 727
1093*	33; 16, (1, 1) 4, (33, 33)	545 18018
1094	33; 13, 4, 1, 1, 1, 5, 2, 1, 2, 3, 9, 6, 1, 1, (32) 5, 14, 37, 25, 38, 11, 23, 35, 22, 19, 7, 10, 31, 35, (2)	45043 80474 67914 14 89854 05815 38085
1095	33; (11) (6)	11 364
1096	33; 9, 2, 3, 1, (15) 7, 28, 15, 49, (4)	1 19595 39 59299
1097*	33; 8, 3, 1, 3, 2, 1, (1, 1) 8, 17, 41, 16, 23, 32, (29, 29)	6 34621 210 19276
1098	33; 7, 2, 1, (6) 9, 22, 41, (9)	3564 1 18097
1099	33; 6, 1, 1, 1, 1, 2, 21, 1, 2, 1, 1, 6, 1, (3) 10, 37, 27, 30, 31, 25, 3, 46, 19, 30, 35, 9, 47, (14)	4 80575 45715 159 31638 15326
1100	33; (6) (11)	6 199
1101	33; 5, 1, 1, 16, (22) 12, 31, 35, 4, (3)	7 32732 243 13015
1102	33; 5, 10, 1, 6, (2) 13, 6, 53, 9, (29)	3 42882 113 82443
1103	33; 4, 1, 2, 1, 2, 3, 1, 1, 5, 2, 9, (33) 14, 41, 19, 37, 22, 17, 31, 34, 11, 29, 7, (2)	7 06456 11145 234 62427 45024
1104	33; 4, 2, (2) 15, 25, (23)	234 7775
1105*	33; 4, (7, 7) 16, (9, 9)	857 28488

TABLE 1001 TO 1500—*continued.*

1106	33; 3, 1, (8) 17, 46, (7)	152 5055
1107	33; 3, 1, 2, 7, (33) 18, 37, 23, 9, (2)	2 18295 72 63026
1108	33; 3, 2, 21, 1, 3, 4, 1, 6, 1, 1, 2, 2, 1, 3, 2, 5, 9, 3, (16) 19, 28, 3, 49, 16, 13, 48, 9, 36, 29, 23, 21, 39, 16, 27, 12, 7, 21, (4)	4781 20058 69390 13510 1 59150 07379 89804 75849
1109*	33; 3, 3, 5, 1, 3, 13, 16, 1, 1, 2, 1, 4, (2, 2) 20, 19, 11, 44, 17, 5, 4, 37, 29, 20, 41, 13, (25, 25)	1832 35957 38617 61020 60015 42610
1110	33; 3, (6) 21, (10)	60 1999
1111	33; (3) (22)	3 100
1112	33; 2, 1, (7) 23, 41, (8)	75 2501
1113	33; 2, 1, 3, 3, 1, (8) 24, 37, 17, 16, 47, (7)	21056 7 02463
1114*	33; 2, 1, 1, 1, 8, 1, 10, 4, 2, 1, 3, 1, 3, 7, 6, (1, 1) 25, 33, 26, 39, 7, 55, 6, 15, 22, 39, 15, 42, 17, 9, 10, (33, 33)	18311 57471 56745 6 11178 81032 02293
1115	33; 2, 1, 1, 4, 5, 1, (5) 26, 29, 35, 14, 11, 49, (10)	1 36565 45 60126
1116	33; 2, 2, 5, 1, 2, (16) 27, 25, 11, 40, 23, (4)	1 38320 46 20799
1117*	33; 2, 2, 1, 2, 5, 4, 1, 21, 2, 9, 16, 1, 1, 1, 1, 6, 1, 4, 1, (2, 2) 28, 21, 36, 23, 12, 13, 52, 3, 31, 7, 4, 39, 27, 28, 37, 9, 49, 12, 41, (21, 21)	6272 59559 53855 43645 2 09639 86690 78245 41118
1118	33; 2, 3, (2) 29, 17, (26)	126 4213
1119	33; 2, 4, 1, 1, 1, 6, (22) 30, 13, 38, 25, 39, 10, (3)	9 46364 316 57255
1120	33; 2, 6, 1, (15) 31, 9, 55, (4)	3765 1 26001
1121	33; 2, 12, 1, 8, 1, 1, 1, 3, 1, 1, 7, 1, 4, 3, 1, (2) 32, 5, 56, 7, 40, 25, 37, 16, 31, 35, 8, 49, 13, 17, 40, (19)	453 93975 58620 15198 51043 64951
1122	33; (2) (33)	2 67
1123	33; 1, 1, 21, 1, 5, 7, 3, 1, 1, 2, 2, 10, 1, 3, (33) 34, 33, 3, 54, 11, 9, 18, 33, 31, 22, 27, 6, 47, 17, (2)	49257 11232 14799 16 50664 55626 32482
1124	33; 1, 1, 9, 13, 3, 3, 1, 1, 1, 1, 1, 2, (16) 35, 32, 7, 5, 20, 17, 35, 28, 31, 29, 32, 25, (4)	6 74757 87740 226 22006 30049
1125	33; 1, 1, 5, 1, 1, 2, 7, 16, 1, 1, 1, 2, 1, (6) 36, 31, 11, 36, 29, 25, 9, 4, 41, 25, 36, 19, 44, (9)	5 16002 91864 173 07264 04001
1126	33; 1, 1, 3, 1, (32) 37, 30, 15, 51, (2)	2718 91205

TABLE 1001 TO 1500—*continued.*

1127	33; 1, 1, 3, (33) 38, 29, 19, (2)	1645 55224
1128	33; 1, 1, 2, (2) 39, 28, 23, (24)	70 2351
1129*	33; 1, (1, 1) 40, (27, 27)	5 168
1130*	33; 1, 1, (1, 1) 41, 26, (31, 31)	13 437
1131	33; 1, 1, 1, 2, 2, 1 1, (4) 42, 25, 35, 22, 23, 30, 35, (13)	10948 3 68185
1132	33; 1, 1, 1, 4, 1, 1, 21, 1, 7, 2, 5, 7, 3, 2, 2, (16) 43, 24, 39, 13, 31, 36, 3, 57, 8, 29, 12, 9, 19, 24, 27, (4)	1 13654 94862 36362 38 23944 35058 85447
1133	33; 1, 1, 1, 16, (6) 44, 23, 43, 4, (11)	15300 5 14999
1134	33; 1, 2, 13, 7, 2, 2, (4) 45, 22, 5, 9, 26, 25, (14)	107 52040 3620 74049
1135	33; 1, 2, 4, 2, 10, 1, 3, 1, 1, 2, 1, 1, 1, 6, 1, (5) 46, 21, 14, 29, 6, 49, 15, 34, 31, 21, 35, 26, 39, 9, 51, (10)	31 50736 19505 1061 47549 56124
1136	33; 1, 2, 2, 1, 1, 2, 9, (4) 47, 20, 23, 32, 31, 25, 7, (16)	7 18620 242 20799
1137	33; 1, 2, 1, 1, 3, 2, 1, 1, 7, 1, 5, 4, (22) 48, 19, 32, 33, 17, 24, 29, 37, 8, 51, 11, 16, (3)	1 92906 26292 65 04689 34487
1138*	33; 1, 2, 1, 3, 4, (1, 1) 49, 18, 41, 17, 14, (33, 33)	10337 3 48711
1139	33; 1, (2) 50, (17)	4 135
1140	33; 1, 3, (4) 51, 16, (15)	72 2431
1141	33; 1, 3, 1, 1, 12, 1, 21, 1, 1, 2, 5, 4, 3, 7, 5, 16, 1, 2, 3, 1, 1, 1, 2, 52, 15, 31, 36, 5, 60, 3, 39, 28, 25, 12, 15, 20, 9, 13, 4, 45, 21, 17, 36, 27, 35, 20, 1, 2, 1, 4, 1, (8) 39, 19, 43, 12, 51, (7)	3 06933 85322 76565 71973 97208 103 67823 94157 22396 32371 25215
1142	33; 1, 3, 1, 5, 2, 1, 8, 1, (32) 53, 14, 47, 11, 22, 43, 7, 59, (2)	268 28010 9066 12101
1143	33; 1, 4, 4, 1, 1, 1, 2, (3) 54, 13, 14, 37, 27, 34, 23, (18)	1 39925 47 30624
1144	33; 1, 4, 1, 1, 1, 6, (1) 55, 12, 39, 25, 40, 9, (52)	16611 5 61835
1145*	33; 1, (5, 5) 56, (11, 11)	37 1252
1146	33; 1, 5, 1, 3, 1, 1, 1, (10) 57, 10, 47, 15, 38, 25, 41, (6)	1 01840 34 47551

TABLE 1001 TO 1500—*continued.*

1147	33; 1, 6, 1, 1, 5, 1, 1, 1, 1, 1, 21, (1) 58, 9, 34, 33, 11, 38, 27, 33, 26, 41, 3, (62)	2789 45403 94471 52318
1148	33; 1, 7, 2, (16) 59, 8, 31, (4)	4896 1 65887
1149	33; 1, 8, 1, 2, 3, (22) 60, 7, 44, 21, 20, (3)	2 12624 72 07295
1150	33; 1, 10, 3, 7, 4, 1, 2, (2) 61, 6, 21, 9, 14, 41, 21, (25)	349 25592 11843 84449
1151	33; 1, 12, 1, 1, 2, 2, 3, 6, 2, (33) 62, 5, 38, 29, 23, 25, 19, 10, 31, (2)	19426 07807 6 59056 71840
1152	33; 1, (15) 63, (4)	17 577
1153*	33; 1, 21, 1, 1, 1, 6, 1, 7, 1, 1, 1, 1, 1, 1, 2, 4, 1, 5, 2, 1, 3, (1, 1) 64, 3, 43, 24, 41, 9, 53, 8, 39, 27, 32, 29, 33, 24, 13, 48, 11, 23, 39, 16, (33, 33)	3 01789 02568 75073 102 47504 00230 72656
1154	33; 1, (32) 65, (2)	34 1155
1155	33; (1) (66)	1 34
1157*	34; (68) (1)	1 34
1158	34; (34) (2)	34 1157
1159	34; 22, 1, 2, 7, 4, 2, 2, 13, 4, 1, 3, 1, (2) 3, 45, 22, 9, 15, 25, 27, 5, 14, 45, 15, 42, (19)	4902 89385 75180 1 66914 55514 24551
1160	34; (17) (4)	17 579
1161	34; 13, 1, 1, 1, 1, 2, 8, 7, 2, 4, 1, 3, (2) 5, 40, 27, 31, 32, 25, 8, 9, 29, 13, 45, 16, (27)	56 21214 40972 1915 34168 54935
1162	34; 11, 2, 1, 6, 1, (8) 6, 23, 42, 9, 54, (7)	6 63462 226 16173
1163	34; 9, 1, 2, 1, 2, 4, 1, 1, (33) 7, 46, 19, 38, 23, 14, 31, 37, (2)	369 56541 12603 21002
1164	34; 8, 1, 1, (16) 8, 33, 35, (4)	4930 1 68199
1165*	34; 7, 1, 1, 3, 16, 1, 3, 1, (1, 1) 9, 36, 31, 19, 4, 51, 15, 36, (29, 29)	867 20773 29599 61778
1166	34; 6, 1, 4, 2, 1, 1, 9, (6) 10, 49, 13, 25, 29, 38, 7, (11)	193 40870 6604 27701
1167	34; 6, 5, (11) 11, 13, (6)	10943 3 73828
1168	34; 5, 1, 2, 7, (4) 12, 41, 23, 9, (16)	66750 22 81249

TABLE 1001 TO 1500—*continued.*

1169	34; 5, 4, 13, 2, 3, 1, 1, 5, 1, 1, 1, 7, 1, (8) 13, 16, 5, 29, 17, 32, 35, 11, 40, 25, 41, 8, 55, (7)	538 84918 28064 18423 59949 83935
1170	34; 4, 1, (6) 14, 49, (9)	190 6499
1171	34; 4, 1, 1, 4, 1, 2, 2, 3, 1, 1, 1, 1, (33) 15, 33, 34, 13, 42, 21, 26, 17, 35, 30, 27, 41, (2)	15274 30263 5 22684 76130
1172	34; 4, 3, 1, 3, 1, 1, (16) 16, 17, 43, 16, 31, 37, (4)	3 59890 123 20649
1173	34; (4) (17)	4 137
1174	34; 3, 1, 3, 1, 4, 2, 13, 3, 1, 22, 11, 2, 1, 1, 1, 6, 4, 2, 2, 1, 1, 7, (34) 18, 43, 15, 46, 13, 30, 5, 17, 50, 3, 6, 25, 34, 27, 39, 10, 15, 26, 23, 30, 37, 9, (2)	1363 67209 12136 61406 74774 46724 42879 20656 93035 91365
1175	34; 3, 1, 1, 2, 5, (1) 19, 34, 31, 25, 11, (50)	12901 4 42224
1176	34; 3, 2, (2) 20, 25, (24)	140 4801
1177	34; 3, 3, 1, 22, 9, 1, 3, 7, 2, 1, 2, 1, 1, 2, 2, 2, 8, (6) 21, 16, 51, 3, 7, 48, 17, 9, 24, 37, 21, 33, 32, 23, 24, 27, 8, (11)	276 38354 60657 78460 9482 01015 39044 75351
1178	34; 3, 9, (2) 22, 7, (31)	1736 59583
1179	34; 2, 1, (33) 23, 45, (2)	309 10610
1180	34; 2, 1, 5, 1, 1, 2, 1, (2) 24, 41, 11, 36, 31, 21, 39, (20)	58950 20 24999
1181*	34; 2, 1, 2, 1, 3, 3, 5, 1, 16, 2, 1, (13, 13) 25, 37, 20, 41, 17, 20, 11, 55, 4, 23, 44, (5, 5)	35 62788 38045 1224 37647 36718
1182	34; 2, 1, 1, 1, 2, 2, 1, 2, 1, (10) 26, 33, 29, 34, 23, 22, 39, 19, 47, (6)	7 92682 272 52587
1183	34; 2, 1, 1, 7, 22, 1, (3) 27, 29, 38, 9, 3, 53, (14)	37 96401 1305 76328
1184	34; 2, 2, 3, 1, 9, (17) 28, 25, 16, 49, 7, (4)	7 95285 273 65201
1185	34; 2, 2, 1, 3, 1, 1, 2, 3, (4) 29, 21, 41, 16, 35, 31, 24, 19, (15)	12 03200 414 18751
1186	34; 2, 3, 1, 1, 4, (34) 30, 17, 33, 34, 15, (2)	1 83522 63 20195
1187	34; 2, 4, 1, 4, 9, 1, 1, 1, 2, 1, (33) 31, 13, 47, 14, 7, 41, 26, 37, 19, 49, (2)	11347 43775 3 90951 75626
1188	34; 2, 7, (6) 32, 9, (11)	1410 48599
1189*	34; 2, 13, 3, 2, 1, 2, 16, 1, 6, 1, 2, (1, 1) 33, 5, 20, 23, 36, 25, 4, 57, 9, 45, 20, (33, 33)	24 58219 58945 847 64031 17418

TABLE 1001 TO 1500—*continued.*

1190	34; (2) (34)	2 69
1191	34; 1, 1, (22) 35, 34, (3)	92 3175
1192	34; 1, 1, 9, 2, 1, 3, 2, 1, 1, 1, 1, 7, (17) 36, 33, 7, 24, 39, 17, 24, 33, 31, 28, 39, 9, (4)	97167 39825 33 54738 72499
1193*	34; 1, 1, 5, 1, 3, 2, 8, (5, 5) 37, 32, 11, 47, 16, 29, 8, (13, 13)	247 53805 8549 92268
1194	34; 1, 1, 4, 9, 1, 1, 1, 6, 3, 1, (10) 38, 31, 15, 7, 42, 25, 41, 10, 17, 49, (6)	6264 76750 2 16474 68749
1195	34; 1, 1, 3, 7, 2, 1, 1, 10, 1, (12) 39, 30, 19, 9, 26, 29, 39, 6, 59, (5)	1345 16232 46500 60959
1196	34; 1, 1, (2) 40, 29, (23)	12 415
1197	34; 1, 1, 2, 16, 1, (8) 41, 28, 27, 4, 59, (7)	74820 25 88599
1198	34; 1, 1, 1, 1, 2, 1, 2, 3, 2, 11, 9, 1, 4, 22, 1, 6, 1, 2, 1, 3, 3, (34) 42, 27, 31, 34, 21, 38, 23, 18, 29, 6, 7, 51, 14, 3, 58, 9, 46, 19, 42, 17, 21, (2)	1 21525 49298 36463 16460 42 00256 96350 23592 51599
1199	34; 1, 1, 1, 2, 9, 1, 1, 13, (3) 43, 26, 35, 25, 7, 34, 35, 5, (22)	143 75599 4977 77820
1200	34; 1, 1, 1, (3) 44, 25, 39, (16)	39 1351
1201*	34; 1, 1, 1, 9, 4, 4, 2, 1, 1, 1, 7, 13, 1, 2, 1, 2, 1, 1, 4, 22, 1, 7, 1, 45, 24, 43, 7, 16, 15, 24, 35, 27, 40, 9, 5, 48, 19, 40, 21, 32, 35, 15, 3, 59, 8, 45, 2, (2, 2) 21, (25, 25)	 71 85416 85609 44230 77385 2490 13832 32746 50571 44832
1202	34; 1, 2, (34) 46, 23, (2)	312 10817
1203	34; 1, 2, 5, 1, (33) 47, 22, 11, 57, (2)	12521 4 34282
1204	34; 1, 2, 3, 7, 2, 2, 3, 4, 22, 1, (8) 48, 21, 20, 9, 27, 25, 19, 16, 3, 60, (7)	16 95565 54116 588 33925 37695
1205	34; 1, 2, 2, 16, 1, (12) 49, 20, 29, 4, 61, (5)	2 06668 71 74089
1206	34; 1, 2, 1, 2, (34) 50, 19, 38, 25, (2)	4202 1 45925
1207	34; 1, 2, 1, 6, 1, (33) 51, 18, 47, 9, 59, (2)	33387 11 59928
1208	34; 1, 3, 9, 1, 2, (8) 52, 17, 7, 44, 23, (8)	1 23046 42 76623
1209	34; 1, 3, 2, 1, 3, 2, 1, 1, (22) 53, 16, 23, 40, 17, 25, 29, 40, (3)	16 40156 570 29335

TABLE 1001 TO 1500—*continued.*

1210	34; 1, 3, 1, 1, 1, 7, 11, 2, (6) 54, 15, 39, 26, 41, 9, 6, 31, (10)	430 99524 14992 19281
1211	34; 1, (3) 55, (14)	5 174
1212	34; 1, 4, 2, 1, 2, 2, 1, (16) 56, 13, 24, 37, 23, 21, 47, (4)	3 65980 127 41151
1213*	34; 1, 4, 1, 4, 1, 1, 9, (2, 2) 57, 12, 49, 13, 33, 36, 7, (27, 27)	20 27117 706 00734
1214	34; 1, 5, 2, 1, 6, 3, 1, 1, 13, 2, 1, 2, 2, (34) 58, 11, 23, 43, 10, 19, 31, 38, 5, 25, 37, 22, 29, (2)	209 94702 60104 7315 07983 74975
1215	34; 1, (5) 59, (10)	7 244
1216	34; 1, 6, 1, 3, 4, 2, 1, 1, 4, (17) 60, 9, 48, 17, 15, 25, 31, 36, 15, (4)	1916 03685 66814 48801
1217*	34; 1, 7, 1, 2, 1, 3, 1, 1, 1, (1, 1) 61, 8, 47, 19, 43, 16, 37, 29, 32, (31, 31)	7 91969 276 28256
1218	34; 1, (8) 62, (7)	10 349
1219	34; 1, 10, 1, 1, 1, 7, 9, 1, 5, 2, 4, 5, 6, 1, 3, (1) 63, 6, 43, 25, 42, 9, 7, 54, 11, 29, 15, 13, 10, 49, 15, (46)	76791 47002 95135 26 81111 24548 55326
1220	34; 1, (12) 64, (5)	14 489
1221	34; 1, 16, (2) 65, 4, (33)	612 21385
1222	34; 1, 22, 3, 7, 2, 3, 1, 1, 1, 4, 2, 1, (4) 66, 3, 22, 9, 29, 17, 38, 27, 39, 14, 23, 42, (13)	22 12495 24122 773 42454 09307
1223	34; 1, (33) 67, (2)	35 1224
1224	34; (1) (68)	1 35
1226*	35; (70) (1)	1 35
1227	35; (35) (2)	35 1226
1228	35; 23, 2, 1, 7, 8, 1, 1, 1, 2, 2, 1, 1, 5, 3, 1, (16) 3, 24, 43, 9, 8, 41, 27, 36, 23, 24, 31, 37, 12, 17, 51, (4)	44730 59716 99506 15 67486 75428 71047
1229*	35; 17, (1, 1) 4, (35, 35)	613 21490
1230	35; (14) (5)	14 491
1231	35; 11, 1, 2, 7, 2, 4, 1, 13, 4, 1, 1, 1, 1, 6, 2, 2, 4, 3, 1, 2, 23, (35) 6, 45, 23, 9, 30, 13, 54, 5, 15, 37, 30, 29, 39, 10, 27, 26, 15, 18, 39, 25, 3, (2)	584 79034 52350 15227 51177 20517 72574 24010 98134 87200

TABLE 1001 TO 1500—*continued.*

1232	35; (10) (7)	10 351
1233	35; 8, 1, 3, 4, 7, 1, 1, 2, 6, 9, 1, (6) 8, 49, 17, 16, 9, 37, 32, 19, 11, 7, 56, (9)	210 76306 89984 7400 75542 46657
1234	35; 7, 1, 3, 1, 4, 4, 2, 9, 1, 1, 2, 3, 1, 1, (34) 9, 50, 15, 47, 14, 15, 30, 7, 39, 30, 25, 18, 31, 39, (2)	1669 10230 73856 58632 78690 67265
1235	35; (7) (10)	7 246
1236	35; 6, 2, 1, 1, 1, 4, 1, 3, 1, 1, 2, 1, (22) 11, 25, 35, 28, 39, 13, 47, 16, 35, 33, 20, 49, (3)	18261 82722 6 42027 25495
1237*	35; 5, 1, 5, 1, 1, 3, 1, 1, 2, 23, 17, 1, 1, 5, 2, 1, (7, 7) 12, 51, 11, 36, 33, 17, 36, 31, 27, 3, 4, 37, 33, 12, 23, 44, (9, 9)	30 80599 95913 30265 1083 47814 40276 61982
1238	35; 5, 2, 1, 1, (34) 13, 26, 29, 41, (2)	25650 9 02501
1239	35; (5) (14)	5 176
1240	35; 4, 1, 2, 7, (2) 15, 41, 24, 9, (31)	24102 8 48719
1241*	35; 4, 2, 1, 1, 3, 8, (1, 1) 16, 25, 32, 35, 19, 8, (35, 35)	9 65005 339 95032
1242	35; 4, 7, 1, 1, (2) 17, 9, 38, 31, (23)	11780 4 15151
1243	35; 3, 1, 9, (3) 18, 49, 7, (22)	4875 1 71874
1244	35; 3, 1, 2, 3, 6, 8, 1, 1, 1, 13, 2, 4, 1, (16) 19, 40, 23, 20, 11, 8, 43, 25, 44, 5, 31, 13, 55, (4)	1616 45760 60360 57013 08077 08799
1245	35; 3, 1, 1, 17, (14) 20, 31, 39, 4, (5)	2 13528 75 34241
1246	35; 3, 2, 1, 7, 6, 1, 13, 3, 1, 5, 1, 1, 1, (34) 21, 22, 45, 9, 10, 57, 5, 18, 47, 11, 42, 25, 45, (2)	1257 54312 87288 44389 66931 48735
1247	35; 3, 5, 9, (1) 22, 13, 7, (58)	26313 9 29188
1248	35; 3, (17) 23, (4)	159 5617
1249*	35; 2, 1, 13, 2, 7, 2, 1, 2, 3, 1, 3, 1, 1, 1, 4, 1, 3, 1, 8, 23, 2, (4, 4) 24, 45, 5, 32, 9, 25, 37, 24, 17, 45, 16, 39, 27, 40, 13, 48, 15, 57, 8, 3, 31, (15, 15)	2 61326 40028 30963 92593 92 35587 03432 90296 88240
1250*	35; 2, 1, 4, 2, 1, (1, 1) 25, 41, 14, 25, 34, (31, 31)	7801 2 75807
1251	35; 2, 1, 2, 2, 2, 5, (35) 26, 37, 23, 25, 27, 13, (2)	21 92943 775 63250
1252	35; 2, 1, 1, 1, 1, 5, 3, 1, 1, 4, 1, 7, 23, 2, 5, 1, (16) 27, 33, 32, 29, 39, 12, 19, 33, 36, 13, 52, 9, 3, 32, 11, 57, (4)	90969 00792 82080 32 18812 08291 34849

TABLE 1001 TO 1500—*continued.*

1253	35; 2, 1, 1, 17, (10) 28, 29, 41, 4, (7)	78320 27 72351
1254	35; 2, 2, 2, 1, (34) 29, 25, 21, 49, (2)	10234 3 62405
1255	35; 2, 2, 1, 7, 6, 3, 4, 1, 2, 1, 11, (14) 30, 21, 46, 9, 11, 21, 14, 45, 19, 49, 6, (5)	87 88466 34708 3113 40025 72889
1256	35; 2, 3, 1, 2, (17) 31, 17, 40, 25, (4)	11075 3 92499
1257	35; 2, 4, 1, (22) 32, 13, 56, (3)	2860 1 01399
1258*	35; 2, 7, 2, (1, 1) 33, 9, 26, (33, 33)	3233 1 14669
1259	35; 2, 13, 1, 2, 3, 2, 1, 1, 5, 1, 6, 4, (35) 34, 5, 47, 22, 19, 25, 31, 38, 11, 53, 10, 17, (2)	313 98432 81069 11140 91438 32810
1260	35; (2) (35)	2 71
1261*	35; 1, 1, 23, 5, 1, 7, 17, 1, 1, 1, 2, 5, (1, 1) 36, 35, 3, 12, 53, 9, 4, 43, 27, 36, 25, 12, (35, 35)	694 50461 60725 24662 24909 57682
1262	35; 1, 1, 9, 1, 1, 1, 4, 1, 4, 3, 1, (34) 37, 34, 7, 43, 26, 41, 13, 49, 14, 17, 53, (2)	17412 04708 6 18556 69263
1263	35; 1, 1, 5, 1, (22) 38, 33, 11, 58, (3)	4004 1 42297
1264	35; 1, 1, 4, (4) 39, 32, 15, (16)	360 12799
1265	35; 1, 1, 3, 4, (6) 40, 31, 19, 16, (11)	5820 2 06999
1266	35; 1, 1, 2, 1, 1, 2, 3, 1, 3, 1, (34) 41, 30, 23, 34, 33, 25, 17, 46, 15, 55, (2)	149 86708 5332 40465
1267	35; 1, 1, 2, 7, 1, 1, 23, (5) 42, 29, 27, 9, 34, 37, 3, (14)	175 73127 6255 14462
1268	35; 1, 1, 1, 1, 3, 1, 5, 1, 2, 4, 9, 1, (16) 43, 28, 31, 37, 16, 49, 11, 44, 23, 16, 7, 61, (4)	69835 99490 24 86789 07849
1269	35; 1, 1, 1, 1, 1, 1, 7, 3, (2) 44, 27, 35, 28, 41, 9, 20, (27)	96264 34 29215
1270	35; 1, 1, 1, 3, 11, 1, 1, 1, 1, 5, 1, 7, (14) 45, 26, 39, 69, 6, 41, 29, 30, 39, 11, 54, 9, (5)	1 63267 43118 58 18371 13691
1271	35; 1, 1, 1, 6, (2) 46, 25, 43, 10, (31)	920 32799
1272	35; 1, (1) 47, (24)	3 107
1273	35; 1, 2, 8, 1, 1, 2, 2, 4, 23, 1, 1, 3, 1, 2, 5, 7, 1, (2) 48, 23, 8, 39, 31, 24, 27, 16, 3, 39, 32, 17, 41, 24, 13, 9, 48, (19)	5 14164 62158 51476 183 44944 08681 34807

TABLE 1001 TO 1500—*continued.*

1274	35; 1, 2, 3, 1, 6, 2, 1, 2, (2) 49, 22, 17, 49, 10, 25, 38, 23, (26)	15 15720 541 00801
1275	35; 1, 2, 2, (2) 50, 21, 26, (25)	140 4999
1276	35; 1, 2, 1, 1, 2, 2, 2, 7, 1, 1, 9, 1, 2, 13, 1, (16) 51, 20, 35, 33, 24, 25, 28, 9, 35, 36, 7, 45, 24, 5, 63, (4)	931 97427 02580 33291 18543 36799
1277*	35; 1, 2, 1, 3, 2, 5, 17, 1, 2, (6, 6) 52, 19, 44, 17, 29, 13, 4, 47, 23, (11, 11)	40427 42285 14 44679 48482
1278	35; 1, (2) 53, (18)	4 143
1279	35; 1, 3, 4, 1, 1, 13, 1, 3, 23, 1, 1, 2, 2, 1, 5, 1, 3, 1, 11, 7, 1, 6, 3, 54, 17, 15, 33, 38, 5, 51, 18, 3, 41, 30, 25, 22, 45, 11, 50, 15, 53, 6, 9, 55, 10, 19, 1, 1, 1, 1, 1, 2, (35) 37, 30, 33, 31, 34, 27, (2)	7 89404 78828 27783 39935 75041 282 31569 87094 18598 91612 56000
1280	35; 1, 3, 2, (17) 55, 16, 31, (4)	1449 51841
1281	35; 1, 3, 1, 3, 1, 2, 13, 1, (22) 56, 15, 47, 16, 41, 25, 5, 64, (3)	220 81748 7903 29175
1282	35; 1, 4, 7, 1, 3, 9, 1, (34) 57, 14, 9, 49, 18, 7, 63, (2)	952 47138 34103 26403
1283	35; 1, 4, 1, 1, 9, 1, 2, 4, 1, 3, 2, 2, (35) 58, 13, 34, 37, 7, 46, 23, 14, 47, 17, 26, 29, (2)	5 15512 61505 184 65140 88226
1284	35; 1, (4) 59, (12)	6 215
1285*	35; 1, 5, 1, 1, 7, 2, 2, 1, 17, 4, 1, 2, 1, (1, 1) 60, 11, 35, 36, 9, 29, 21, 49, 4, 15, 44, 21, 36, (31, 31)	24 48426 94225 877 68507 57318
1286	35; 1, 6, 5, 2, 1, 2, 13, 1, (34) 61, 10, 13, 25, 37, 26, 5, 65, (2)	7072 64778 2 53631 10565
1287	35; 1, (6) 62, (9)	8 287
1288	35; 1, (7) 63, (8)	9 323
1289*	35; 1, 9, 3, 1, 2, 8, 1, 1, 1, 1, 2, 1, 1, 13, 1, 3, (1, 1) 64, 7, 19, 40, 25, 8, 41, 29, 32, 35, 23, 31, 40, 5, 53, 16, (35, 35)	835 94139 33853 30012 50804 21900
1290	35; 1, (10) 65, (6)	12 431
1291	35; 1, 13, 2, 1, 1, 2, 3, 1, 1, 1, 1, 4, 1, 11, 6, 2, 4, 3, 23, 1, 1, 1, 66, 5, 27, 33, 34, 25, 18, 37, 31, 30, 39, 13, 55, 6, 11, 30, 15, 22, 3, 45, 26, 41, 4, 7, 1, 3, 2, 1, 6, 2, (35) 15, 9, 50, 17, 23, 45, 10, 33, (2)	1974 27745 80291 79860 94899 51267 70936 75339 57910 18687 74456 08410
1292	35; 1, (10) 67, (4)	18 647

TABLE 1001 TO 1500—*continued.*

1293	35; 1, (22) 68, (3)	24 863
1294	35; 1, (34) 69, (2)	36 1295
1295	35; (1) (70)	1 36
1297*	36; (72) (1)	1 36
1298	36; (36) (2)	36 1297
1299	36; (24) (3)	24 865
1300	36; (18) (4)	18 649
1301*	36; 14, 2, (2, 2) 5, 29, (25, 25)	6025 2 17318
1302	36; (12) (6)	12 433
1303	36; 10, 3, 2, 1, 23, 2, 1, 2, 1, 3, 3, 1, 1, 7, 2, 5, 11, 1, 5, 1, 1, 1, 4, 7, 21, 22, 49, 3, 26, 39, 21, 43, 18, 19, 33, 38, 9, 31, 13, 6, 57, 11, 42, 27, 41, 14, 1, 1, (35) 33, 39, (2)	10127 26314 51893 46887 71461 3 65564 74232 60608 71132 28808
1304	36; (9) (8)	9 325
1305	36; (8) (9)	8 289
1306*	36; 7, 4, 1, 2, 11, 1, 2, 4, 2, 9, 1, (7, 7) 10, 15, 42, 25, 6, 47, 23, 15, 31, 7, 58, (9, 9)	862 68151 05389 31176 12101 50515
1307	36; 6, 1, 1, 3, 1, 2, 1, 1, (35) 11, 37, 34, 17, 43, 22, 31, 41, (2)	54 14517 1957 48082
1308	36; (6) (12)	6 217
1309	36; 5, 1, 1, 4, 3, 1, 1, 2, 3, 17, 1, 3, 1, 7, (4) 13, 36, 35, 15, 19, 36, 33, 25, 21, 4, 55, 15, 52, 9, (17)	2219 92217 38044 80317 01740 72265
1310	36; 5, 6, 2, 1, 1, 1, 1, 1, (6) 14, 11, 26, 35, 31, 34, 29, 41, (10)	34 56382 1251 00021
1311	36; 4, 1, 4, 2, (1) 15, 49, 14, 25, (38)	5353 1 93820
1312	36; 4, 1, 1, (17) 16, 33, 39, (4)	1467 53137
1313*	36; (4, 4) (17, 17)	17 616

TABLE 1001 TO 1500—*continued.*

1314	36; (4) (18)	4 145
1315	36; 3, 1, 4, 11, 1, 7, (7) 19, 46, 15, 6, 59, 9, (10)	244 74351 8875 11646
1316	36; 3, 1, 1, 1, 1, 2, 3, (2) 20, 37, 31, 32, 35, 25, 19, (28)	65508 23 76415
1317	36; 3, 2, 3, 1, 5, 3, 1, 1, 1, 4, 1, 17, 3, 10, (24) 21, 28, 17, 49, 12, 19, 39, 28, 41, 13, 57, 4, 23, 7, (3)	1 03450 99306 81500 37 54287 70770 93751
1318	36; 3, 3, 2, 23, 1, 3, 3, 4, 1, 7, 3, 1, 9, 1, 1, 1, 1, 2, 11, 1, 2, 1, 22, 19, 31, 3, 54, 17, 21, 14, 53, 9, 18, 51, 7, 42, 29, 33, 34, 27, 6, 49, 21, 33, 1, 6, (36) 38, 11, (2)	 3138 36679 69954 08957 94554 1 13936 11468 26276 81762 76517
1319	36; 3, 6, 1, 13, 1, 1, 1, (35) 23, 10, 59, 5, 46, 25, 47, (2)	334 01011 12130 59240
1320	36; (3) (24)	3 109
1321*	36; 2, 1, 8, 2, 2, 1, 1, 3, 1, 23, 2, 4, 2, 1, 4, 6, 2, 1, 1, 7, 2, (14, 14) 25, 45, 8, 29, 24, 33, 37, 16, 55, 3, 32, 15, 24, 43, 15, 11, 27, 31, 40, 9, 33, (5, 5)	14 70631 38627 19611 69925 534 50926 71539 66127 96432
1322*	36; 2, 1, 3, 1, (1, 1) 26, 41, 17, 38, (31, 31)	821 29851
1323	36; 2, 1, 2, 7, 1, 2, (2) 27, 37, 26, 9, 47, 22, (27)	1 00360 36 50401
1324	36; 2, 1, 1, 2, 2, 3, 4, 1, 1, 3, 1, 2, 1, 2, 5, 1, 2, 3, 8, 1, 3, 1, 28, 33, 35, 24, 27, 20, 15, 36, 35, 17, 44, 21, 40, 25, 12, 45, 23, 21, 8, 53, 15, 56, 23, 2, 6, 7, 1, 13, 1, 2, 10, (18) 3, 33, 11, 9, 60, 5, 47, 24, 7, (4)	 42650 12718 55353 72450 38443 74810 15 51902 10838 17312 41819 40987 21799
1325*	36; (2, 2) (29, 29)	5 182
1326	36; 2, 2, (2) 30, 25, (26)	70 2549
1327	36; 2, 2, 1, (35) 31, 21, 51, (2)	1785 65024
1328	36; 2, 3, 1, (3) 32, 17, 47, (16)	369 13447
1329	36; 2, 5, 8, 1, 13, 1, 2, 4, 4, 1, 1, 1, 2, 2, 1, 1, 6, (24) 33, 13, 8, 61, 5, 48, 23, 16, 15, 40, 29, 37, 24, 25, 32, 39, 11, (3)	19 32883 59509 34116 704 64145 63882 01495
1330	36; 2, 7, 1, 1, 1, 1, (4) 34, 9, 41, 30, 31, 39, (14)	34182 12 46589
1331	36; 2, 14, 10, 2, 1, 4, 1, 1, 6, 1, 2, 1, (35) 35, 5, 7, 25, 43, 14, 35, 37, 10, 49, 19, 53, (2)	273 03781 10727 9961 20370 19890
1332	36; (2) (36)	2 73

TABLE 1001 TO 1500—*continued.*

1333	36; 1, 1, 23, 1, 5, 7, 1, 17, 2, 1, 1, 1, 5, (2) 37, 36, 3, 59, 12, 9, 61, 4, 27, 36, 29, 41, 12, (31)	904 20166 84140 33012 64933 32151
1334	36; 1, 1, 9, 1, 13, 1, 2, 2, 1, 1, (2) 38, 35, 7, 62, 5, 49, 22, 25, 34, 35, (23)	864 08256 31559 72095
1335	36; 1, 1, 6, (7) 39, 34, 11, (10)	1235 45124
1336	36; 1, 1, 4, 2, 1, 2, 2, 1, 4, 5, 1, 7, 3, 1, 1, (8) 40, 33, 15, 25, 39, 24, 23, 44, 15, 12, 55, 9, 20, 33, 39, (8)	174 56099 99622 6380 43737 19695
1337	36; 1, 1, 3, 2, 1, 8, 2, 4, (10) 41, 32, 19, 23, 47, 8, 31, 16, (7)	373 97272 13674 31647
1338	36; 1, 1, 2, 1, 2, 10, (12) 42, 31, 23, 39, 26, 7, (6)	4 73194 173 08813
1339	36; 1, 1, 2, 4, 1, (4) 43, 30, 27, 14, 51, (13)	4104 1 50175
1340	36; 1, 1, 1, 1, 6, (18) 44, 29, 31, 40, 11, (4)	19932 7 29631
1341	36; 1, 1, 1, 1, 1, 2, 3, 3, 1, 1, 3, 1, 2, 1, 7, 2, 2, 17, 1, 9, 1, 1, 14, (8) 45, 28, 35, 31, 36, 25, 20, 19, 35, 36, 17, 45, 20, 49, 9, 28, 29, 4, 63, 7, 36, 37, 5, (9)	36359 14354 86763 45320 13 31459 72360 20892 39201
1342	36; 1, 1, 1, (2) 46, 27, 39, (22)	30 1099
1343	36; 1, 1, 1, 4, 1, (35) 47, 26, 43, 13, 59, (2)	10591 3 88128
1344	36; 1, 1, 1, (17) 48, 25, 47, (4)	165 6049
1345	36; 1, 2, (14) 49, 24, (5)	132 4841
1346	36; 1, 2, 4, 1, 9, 1, 2, (36) 50, 23, 14, 55, 7, 46, 25, (2)	92 82362 3405 50115
1347	36; 1, 2, 2, 1, 5, 1, (35) 51, 22, 23, 46, 11, 61, (2)	1 64753 60 46682
1348	36; 1, 2, 1, 1, 23, 1, 9, 1, 1, 7, 1, 1, 1, 2, 1, 5, 2, 1, 1, 5, (18) 52, 21, 32, 41, 3, 64, 7, 37, 36, 9, 43, 28, 39, 21, 47, 12, 27, 32, 39, 13, (4)	56 21145 67828 82376 2063 81035 31297 13793
1349	36; 1, 2, 1, 2, 5, 3, 2, 17, 1, 13, 1, (2) 53, 20, 41, 25, 13, 20, 31, 4, 65, 5, 52, (19)	5 55200 34840 203 91806 65951
1350	36; 1, 2, 1, 7, 2, (2) 54, 19, 50, 9, 29, (25)	12804 4 70449
1351	36; 1, 3, (10) 55, 18, (7)	168 6175
1352	36; 1, 3, 2, 1, (17) 56, 17, 23, 49, (4)	3107 1 14243
1353	36; 1, 3, 1, 1, 1, 1, 2, 1, 8, (2) 57, 16, 39, 31, 32, 37, 21, 49, 8, (33)	11 68536 429 82433

TABLE 1001 TO 1500—*continued.*

1354*	36; 1, 3, 1, 11, 2, 6, 1, 7, 3, 4, 1, 1, (2, 2) 58, 15, 55, 6, 33, 10, 57, 9, 22, 15, 38, 33, (25, 25)	35 75033 94613 1315 49590 98165
1355	36; 1, 4, 3, 1, 2, (14) 59, 14, 19, 41, 26, (5)	49532 18 23289
1356	36; 1, 4, 1, 2, 8, 1, 5, 1, 4, (18) 60, 13, 44, 25, 8, 57, 11, 52, 15, (4)	5212 98370 1 91962 41799
1357	36; 1, 5, 6, 1, 1, 7, 1, 1, 1, 5, 2, 17, 1, 23, 1, 1, 1, 1, (2) 61, 12, 11, 36, 37, 9, 44, 27, 43, 12, 33, 4, 67, 3, 44, 29, 33, 36, (23)	9 65303 46430 64640 355 59347 94143 72351
1358	36; 1, 5, 1, 2, 2, (36) 62, 11, 47, 22, 31, (2)	81404 29 99823
1359	36; 1, 6, 2, 1, 1, 2, 2, 1, 4, 1, 1, (3) 63, 10, 27, 34, 35, 25, 23, 45, 14, 37, 35, (18)	468 78231 17281 48040
1360	36; 1, 7, 4, 1, (3) 64, 9, 15, 49, (16)	7749 2 85769
1361*	36; 1, 8, 4, 4, 2, 1, 2, 1, 1, 14, 5, 1, (1, 1) 65, 8, 17, 16, 25, 40, 23, 32, 41, 5, 13, 40, (31, 31)	35 93384 58529 1325 66186 45260
1362	36; 1, 9, 1, 1, 3, 1, 4, 2, (36) 66, 7, 39, 34, 17, 49, 14, 33, (2)	371 27048 13701 84257
1363	36; 1, 11, 3, 7, (1) 67, 6, 23, 9, (58)	93495 34 51726
1364	36; 1, 13, 1, 3, 1, 2, (6) 68, 5, 55, 16, 43, 25, (11)	2 87730 106 26551
1365	36; 1, 17, (2) 69, 4, (35)	684 25271
1366	36; 1, 23, 1, 1, 1, 7, 1, 1, 4, 2, 1, 1, 14, 5, 4, 1, 2, 1, 2, 2, 70, 3, 47, 26, 45, 9, 38, 35, 15, 27, 31, 42, 5, 14, 15, 46, 21, 42, 23, 30, 10, 7, 3, 2, 1, 1, 1, 3, 3, 1, 4, 1, 11, 2, (36) 7, 10, 21, 25, 37, 30, 39, 19, 18, 49, 13, 57, 6, 35, (2)	 61 98787 91120 98468 23128 64853 64042 2291 03705 28461 89335 14238 95408 99525
1367	36; 1, (35) 71, (2)	37 1368
1368	36; (1) (72)	1 37
1370*	37; (74) (1)	1 37
1371	37; (37) (2)	37 1370
1372	37; 24, 1, 2, 7, 1, 8, 2, 1, 1, 1, 2, 2, 5, 1, 3, (18) 3, 49, 24, 9, 59, 8, 27, 36, 31, 37, 24, 29, 12, 49, 19, (4)	92062 30469 34552 34 10035 48679 27167
1373*	37; 18, (1, 1) 4, (37, 37)	685 25382
1374	37; 14, 1, 4, 2, 1, 3, 4, 1, 2, (24) 5, 57, 14, 25, 42, 19, 15, 43, 26, (3)	41422 89074 15 35443 25045

TABLE 1001 TO 1500—*continued.*

1375	37; 12, 2, 1, 7, 1, 1, 3, 2, 1, 2, (6) 6, 25, 46, 9, 39, 34, 19, 25, 39, 26, (11)	24090 73932 8 93308 52249
1376	37; 10, 1, 1, 2, 2, 3, 1, (17) 7, 41, 32, 25, 28, 17, 55, (4)	58 37205 2165 28049
1377	37; 9, 3, 1, 3, 1, 7, 2, 5, (4) 8, 19, 47, 16, 53, 9, 32, 13, (17)	11121 62480 4 12700 70401
1378*	37; 8, (4, 4) 9, (17, 17)	1153 42801
1379	37; 7, 2, 2, 2, 1, 1, 3, 3, (10) 10, 29, 26, 25, 34, 37, 19, 22, (7)	678 06016 25179 68895
1380	37; 6, 1, (2) 11, 49, (20)	182 6761
1381*	37; 6, 5, 1, 1, 4, 2, 2, 3, 3, 4, 14, 1, 1, 1, 2, 1, 1, 2, 1, 24, 12, 13, 37, 36, 15, 28, 27, 20, 21, 17, 5, 45, 28, 39, 23, 35, 36, 21, 52, 3, 18, 1, 1, 5, 1, 2, 7, 1, 9, 1, 2, 1, (4, 4) 4, 39, 35, 12, 45, 25, 9, 60, 7, 51, 20, 47, (15, 15)	5 76794 95050 76394 67609 26124 87905 214 34743 40923 77992 50381 97708 76282
1382	37; 5, 1, 2, 2, 1, 1, (36) 13, 46, 23, 26, 31, 43, (2)	3 49782 130 03237
1383	37; 5, 3, 2, 1, (11) 14, 21, 23, 49, (6)	34821 12 94948
1384	37; 4, 1, (17) 15, 57, (4)	465 17299
1385*	37; 4, 1, 1, 1, (3, 3) 16, 41, 29, 40, (19, 19)	2797 1 04092
1386	37; 4, 2, 1, (2) 17, 25, 41, (22)	572 21295
1387	37; 4, 8, (37) 18, 9, (2)	40557 15 10442
1388	37; 3, 1, 9, 1, 8, 2, 2, 5, 3, (18) 19, 52, 7, 61, 8, 29, 28, 13, 23, (4)	2 20986 97686 82 33062 52807
1389	37; 3, 1, 2, 2, (24) 20, 43, 23, 31, (3)	16796 6 25975
1390	37; 3, 1, 1, 6, 4, 1, 4, 1, 1, 11, 1, 7, 2, 1, 2, 1, (6) 21, 34, 39, 11, 15, 51, 14, 35, 39, 6, 61, 9, 26, 41, 21, 49, (10)	6165 38516 07582 2 29862 22854 32981
1391	37; 3, 2, 1, 1, 1, 6, 1, (4) 22, 25, 38, 29, 43, 10, 55, (13)	2 43492 90 81305
1392	37; 3, (4) 23, (16)	42 1567
1393	37; 3, (10) 24, (7)	96 3583
1394	37; 2, 1, (36) 25, 49, (2)	336 12545

TABLE 1001 TO 1500—*continued.*

1395	37; 2, 1, 6, (8) 26, 45, 11, (9)	3320 1 24001
1396	37; 2, 1, 3, 14, 1, 2, (18) 27, 41, 20, 5, 48, 25, (4)	45 40530 1696 48201
1397	37; 2, 1, 1, 1, 10, 18, 1, 1, 2, (6) 28, 37, 29, 44, 7, 4, 43, 31, 28, (11)	4289 66440 1 60332 48351
1398	37; 2, 1, 1, 3, 2, 1, (36) 29, 33, 38, 19, 23, 51, (2)	1 30154 48 66437
1399	37; 2, 2, 12, 14, 1, 7, 2, 1, 1, 1, 4, 2, 1, 3, 2, 7, 24, 1, 4, (37) 30, 29, 6, 5, 62, 9, 27, 37, 30, 41, 15, 25, 43, 18, 31, 10, 3, 58, 15, (2)	19 32130 14430 04420 15349 722 67866 47899 40453 09640
1400	37; 2, (2) 31, (25)	12 449
1401	37; 2, 3, 14, 1, 2, 5, 2, 2, 1, 1, 6, 4, 1, 1, 8, 1, 4, 10, 2, (24) 32, 21, 5, 49, 24, 13, 29, 25, 33, 40, 11, 16, 35, 39, 8, 55, 15, 7, 35, (3)	44183 34363 33784 97932 16 53779 66015 12296 85015
1402*	37; 2, 3, 1, 9, 1, 11, 1, 1, 2, 1, 7, 1, (1, 1) 33, 17, 54, 7, 63, 6, 41, 33, 22, 49, 9, 42, (31, 31)	2 48379 80029 93 00157 00509
1403	37; 2, 5, 3, (1) 34, 13, 19, (46)	1995 74726
1404	37; 2, 7, 1, (4) 35, 9, 56, (13)	1666 62425
1405	37; 2, (14) 36, (5)	60 2249
1406	37; (2) (37)	2 75
1407	37; 1, 1, (24) 38, 37, (3)	100 3751
1408	37; 1, 1, 10, 4, 1, 1, 2, 7, 1, (17) 39, 36, 7, 16, 39, 33, 28, 9, 63, (4)	3211 01529 1 20487 97377
1409*	37; 1, 1, 6, 3, 9, 14, 1, 9, 1, 3, 1, 3, 1, 1, 1, 1, 1, (2, 2) 40, 35, 11, 23, 8, 5, 64, 7, 55, 16, 49, 17, 40, 31, 35, 32, 37, (25, 25)	94474 40193 46217 35 46252 44206 53380
1410	37; 1, 1, (4) 41, 34, (15)	20 751
1411	37; 1, 1, 3, 2, 4, 1, 1, 3, (37) 42, 33, 19, 30, 15, 38, 35, 21, (2)	118 21953 4440 71330
1412	37; 1, 1, 2, 1, 3, 4, 6, 1, 1, 2, (18) 43, 32, 23, 44, 19, 17, 11, 41, 32, 29, (4)	2703 04352 1 01571 15393
1413	37; 1, 1, 2, 3, 1, 1, 3, 1, 6, 18, 1, 1, 1, 5, (8) 44, 31, 27, 19, 36, 37, 17, 52, 11, 4, 47, 27, 44, 13, (9)	124 23214 30740 4669 87287 31849
1414	37; 1, 1, 1, 1, 11, 1, 14, 8, 3, 2, 5, 2, 1, (4) 45, 30, 31, 43, 6, 65, 5, 9, 21, 30, 13, 25, 45, (14)	485 26483 11810 18247 50630 42299
1415	37; 1, 1, 1, 1, 1, 1, 4, 1, 3, (7) 46, 29, 35, 34, 31, 41, 14, 49, 19, (10)	5 85621 220 29004

TABLE 1001 TO 1500—*continued.*

1416	37; 1, 1, 1, 2, 2, 1, (8) 47, 28, 39, 25, 23, 49, (8)	6858 2 58065
1417*	37; 1, 1, 1, 4, 24, 1, 7, (2, 2) 48, 27, 43, 16, 3, 64, 9, (29, 29)	435 72829 16402 14636
1418*	37; 1, 1, 1, (10, 10) 49, 26, 47, (7, 7)	1033 38899
1419	37; 1, 2, (37) 50, 25, (2)	339 12770
1420	37; 1, 2, 6, 1, 1, (14) 51, 24, 11, 36, 39, (5)	25338 9 54809
1421	37; 1, 2, 3, (2) 52, 23, 20, (29)	260 9801
1422	37; 1, 2, 2, 3, 1, 3, 5, (8) 53, 22, 29, 18, 47, 19, 14, (9)	31 79792 1199 08097
1423	37; 1, 2, 1, 1, 1, 1, 6, 4, 24, 1, 9, 1, 4, 2, 12, 8, 3, 3, 3, 1, 2, (37) 54, 21, 38, 33, 31, 42, 11, 18, 3, 66, 7, 57, 14, 33, 6, 9, 22, 21, 19, 42, 27, (2)	127 24746 01715 56263 96759 4800 11423 18614 76037 04208
1424	37; 1, 2, 1, 3, 1, 2, (4) 55, 20, 47, 17, 44, 25, (16)	13250 5 00001
1425	37; 1, (2) 56, (19)	4 151
1426	37; 1, 3, 4, 1, 3, 1, 1, 1, (2) 57, 18, 15, 50, 17, 41, 30, 39, (23)	2 61132 98 60975
1427	37; 1, 3, 2, 5, 2, 1, 2, 1, 1, 2, (37) 58, 17, 31, 13, 26, 41, 23, 37, 34, 29, (2)	2481 47097 93739 18762
1428	37; 1, 3, 1, (2) 59, 16, 47, (21)	90 3401
1429*	37; 1, 4, 18, 1, (2, 2) 60, 15, 4, 51, (23, 23)	89305 33 75918
1430	37; 1, 4, 2, (2) 61, 14, 29, (26)	352 13311
1431	37; 1, 4, (1) 62, 13, (54)	35 1324
1432	37; 1, 5, 3, 8, 10, 1, 2, 4, (9) 63, 12, 23, 9, 7, 49, 24, 17, (8)	46662 66363 17 65798 05797
1433*	37; 1, 5, 1, 8, 1, 1, 1, 1, 5, 4, (1, 1) 64, 11, 59, 8, 43, 31, 32, 41, 13, 16, (37, 37)	1523 53237 57673 29724
1434	37; 1, 6, 1, 1, 2, 2, 1, 1, 1, 2, 1, 4, 3, (12) 65, 10, 41, 33, 26, 25, 38, 31, 39, 22, 47, 15, 23, (6)	46653 69200 17 66690 98751
1435	37; 1, 7, 2, 3, (7) 66, 9, 31, 21, (10)	26373 9 99046
1436	37; 1, 8, 2, (18) 67, 8, 35, (4)	6840 2 59199

TABLE 1001 TO 1500—*continued.*

1437	37; 1, 9, 1, 5, 2, 2, 4, 18, 1, 2, 1, 1, 1, (24) 68, 7, 59, 12, 29, 28, 17, 4, 53, 21, 41, 28, 47, (3)	256 83205 64040 9735 93382 10399
1438	37; 1, 11, 1, 1, 1, 7, 1, 3, 3, 24, 1, (36) 69, 6, 47, 27, 46, 9, 53, 18, 23, 3, 71, (2)	42 08945 06064 1596 07281 27743
1439	37; 1, 14, 5, 2, 1, 5, 6, 1, 2, 1, 1, 2, 2, 1, 10, 7, 2, (37) 70, 5, 14, 25, 46, 13, 11, 49, 22, 37, 35, 26, 23, 50, 7, 10, 35, (2)	338 95537 37610 21521 12857 98517 00303 64960
1440	37; 1, (17) 71, (4)	19 721
1441	37; 1, 24, 3, 8, 9, 2, 1, 2, 2, 1, 3, 1, 3, 4, 1, 3, 1, 14, 2, 1, 1, 4, 2, (6) 72, 3, 24, 9, 8, 27, 40, 25, 24, 45, 17, 48, 19, 15, 51, 16, 57, 5, 29, 33, 40, 15, 32, (11)	221 54985 95953 31104 82700 8410 14472 84306 85240 54999
1442	37; 1, (36) 73, (2)	38 1443
1443	37; (1) (74)	1 38
1445*	38; (76) (1)	1 38
1446	38; (38) (2)	38 1445
1447	38; 25, 2, 1, 7, 1, 3, 1, 1, 2, 3, 1, 5, 12, 1, 1, (37) 3, 26, 47, 9, 54, 17, 39, 34, 27, 18, 51, 13, 6, 37, 39, (2)	3 42829 14389 10093 130 41033 17574 21848
1448	38; (19) (4)	19 723
1449	38; 15, 4, 1, 2, 4, (8) 5, 16, 45, 25, 17, (9)	72 84800 2773 01249
1450*	38; 12, 1, 2, (8, 8) 6, 49, 25, (9, 9)	1 01933 38 81493
1451	38; 10, 1, 6, 1, 2, 2, 3, 1, 1, 2, 2, 14, 1, 4, 1, 1, (37) 7, 61, 10, 49, 23, 29, 19, 38, 35, 25, 31, 5, 59, 14, 35, 41, (2)	85340 31566 20707 32 50782 78443 28530
1452	38; 9, 1, 1, (18) 8, 37, 39, (4)	6878 2 62087
1453*	38; 8, 2, 5, 2, 1, 1, 5, 1, 3, 6, 10, 1, 2, 1, 2, 1, 1, 3, 18, 1, 3, 1, 9, 33, 13, 28, 33, 41, 12, 51, 19, 12, 7, 52, 21, 44, 23, 36, 37, 21, 4, 57, 17, 36, 1, 6, 2, 1, 2, (25, 25) 39, 11, 27, 39, 28, (3, 3)	37 58165 73334 54563 77143 54805 1432 54652 50389 34442 96990 99982
1454	38; 7, 1, 1, 1, 1, 2, 2, 4, 15, (38) 10, 43, 31, 34, 37, 25, 29, 17, 5, (2)	96169 61884 36 67077 58095
1455	38; 6, 1, (11) 11, 61, (6)	623 23764
1456	38; 6, 2, 1, 7, 1, (3) 12, 25, 48, 9, 55, (16)	1 29855 49 54951
1457	38; 5, 1, 6, 9, 2, 1, 1, 10, 3, 4, (2) 13, 56, 11, 8, 29, 32, 43, 7, 23, 16, (31)	18 94953 10312 723 31628 36703

TABLE 1001 TO 1500—*continued.*

1458	38; 5, 2, 3, 1, 3, 1, 2, 1, 1, 7, 1, 10 (38) 14, 31, 18, 49, 17, 46, 23, 34, 41, 9, 62, 7, (2)	78 88890 25540 3012 27540 96401
1459	38; 5, 12, 1, 1, 7, 8, 2, 1, 4, 2, 2, 2, 1, 1, 1, 5, 4, 15, 25, 2, 1, 1, (37) 15, 6, 39, 37, 10, 9, 26, 45, 15, 29, 27, 25, 39, 30, 43, 13, 18, 5, 3, 30, 31, 45, (2)	25694 24820 98124 22872 52281 9 81439 56337 99403 45032 81010
1460	38; 4, 1, 3, 4, 1, 1, (18) 16, 49, 19, 16, 35, 41, (4)	6 25898 239 15529
1461	38; 4, 2, 14, 1, 5, 2, 3, 2, 1, 3, 3, 18, 1, 4, 6, 1, 2, 1, (24) 17, 33, 5, 61, 12, 31, 20, 25, 44, 19, 23, 4, 59, 15, 11, 51, 20, 55, (3)	10897 33506 65845 02228 4 16529 16197 97289 87575
1462	38; 4, (4) 18, (17)	72 2753
1463	38; (4) (19)	4 153
1464	38; 3, 1, 4, 2, 1, (5) 20, 49, 15, 25, 47, (12)	23729 9 07925
1465*	38; 3, 1, 1, 1, 2, 1, 1, 4, 4, 1, 7, 1, 2, (3, 3) 21, 40, 31, 39, 24, 35, 39, 16, 15, 56, 9, 49, 24, (21, 21)	2 14427 21285 82 07269 84932
1466*	38; 3, 2, 7, 4, 2, 1, (2, 2) 22, 31, 10, 17, 26, 41, (25, 25)	40 05185 1533 52043
1467	38; 3, 3, 6, 1, 1, 1, (37) 23, 22, 11, 46, 27, 49, (2)	16 73045 640 80026
1468	38; 3, 5, 1, 1, 3, 2, 25, 9, 1, 1, 5, 1, 6, 8, 2, 1, 2, 1, 1, (18) 24, 13, 39, 36, 19, 33, 3, 8, 39, 37, 12, 57, 11, 9, 27, 41, 24, 33, 43, (4)	18883 73036 64922 35216 7 23520 46281 34792 85247
1469	38; 3, 18, 1, (4) 25, 4, 61, (13)	19836 7 60265
1470	38; 2, 1, (14) 26, 49, (5)	138 5291
1471	38; 2, 1, 4, 1, 4, 3, 2, 4, 12, 1, 1, 3, 1, 2, 1, 6, 1, 14, 2, 8, 25, 27, 45, 14, 53, 15, 21, 30, 17, 6, 41, 35, 18, 47, 21, 51, 10, 63, 5, 35, 9, 3, 2, 4, 1, 1, 1, 1, 1, 10, 2, 1, (37) 34, 15, 42, 31, 37, 30, 45, 7, 25, 51, (2)	1 01712 89163 97276 23871 56715 36737 39 01057 37347 28288 54875 60183 66160
1472	38; 2, 1, (2) 28, 41, (23)	30 1151
1473	38; 2, 1, 1, 1, 2, 1, 2, 2, 9, 5, 1, 3, 1, (24) 29, 37, 32, 39, 23, 43, 24, 31, 8, 13, 53, 16, 59, (3)	6 92736 14220 265 86992 93399
1474	38; 2, 1, 1, 4, 1, 7, 1, 2, 2, 4, 1, 2, 3, 1, 10, 5, (38) 30, 33, 41, 14, 57, 9, 50, 23, 30, 15, 46, 25, 18, 55, 7, 15, (2)	1 43151 08109 79476 54 95957 60507 29745
1475	38; 2, 2, 6, 1, 1, (2) 31, 29, 11, 41, 34, (25)	14628 5 61799
1476	38; 2, 2, 1, 1, 2, 1, 3, (8) 32, 25, 36, 37, 23, 45, 20, (9)	2 18560 83 96801
1477	38; 2, 3, 6, 8, 2, 1, 1, 1, 1, 1, 5, 1, 3, 1, 2, 18, 1, 6, 25, 2, (10) 33, 21, 12, 9, 28, 37, 33, 36, 31, 43, 12, 53, 17, 44, 27, 4, 63, 11, 3, 36, (7)	2 61633 72121 47062 22240 100 55043 78017 41504 36351
1478	38; 2, 4, (38) 34, 17, (2)	3114 1 19717

TABLE 1001 TO 1500—*continued.*

1479	38; 2, 5, 2, 2, 1, 1, 1, (1) 35, 13, 30, 25, 38, 33, 35, (34)	1 14525 44 04376
1480	38; 2, 8, (19) 36, 9, (4)	5559 2 13859
1481*	38; 2, 14, 1, 8, 1, 2, 5, 1, 1, 2, 1, 4, 10, 1, 3, 1, 1, (1, 1) 37, 5, 65, 8, 49, 25, 13, 40, 35, 23, 47, 16, 7, 56, 17, 41, 32, (35, 35)	1 01945 97513 61405 39 23264 72655 17468
1482	38; (2) (38)	2 77
1483	38; 1, 1, 25, 5, 1, 7, 1, 2, 1, 1, 1, 1, 2, 4, 6, 1, 3, 2, 2, 1, 1, 12, 3, 1, (37) 39, 38, 3, 13, 58, 9, 51, 22, 39, 33, 34, 37, 27, 17, 11, 53, 18, 29, 26, 33, 43, 6, 19, 57, (2)	41 20928 70570 44632 38045 1586 95889 00942 30191 81226
1484	38; 1, 1, (10) 40, 37, (7)	44 1695
1485	38; 1, 1, (6) 41, 36, (11)	28 1079
1486	38; 1, 1, 4, 1, 1, 1, 3, (38) 42, 35, 15, 43, 30, 41, 21, (2)	4 92228 189 74735
1487	38; 1, 1, 3, 1, 1, 3, 1, (37) 43, 34, 19, 37, 38, 17, 59, (2)	2 05495 79 24224
1488	38; 1, 1, 2, 1, (5) 44, 33, 23, 49, (12)	315 12151
1489*	38; 1, 1, 2, 2, 1, 4, 2, 3, 1, (1, 1) 45, 32, 27, 24, 47, 15, 31, 19, 40, (33, 33)	25 78145 994 84332
1490*	38; 1, (1, 1) 46, (31, 31)	5 193
1491	38; 1, 1, 1, 1, 1, 2, 2, (1) 47, 30, 35, 37, 26, 25, (42)	1767 68230
1492	38; 1, 1, 1, 2, 10, 1, 1, 1, (18) 48, 29, 39, 28, 7, 48, 27, 49, (4)	13 56270 523 87849
1493*	38; 1, 1, 1, 3, 2, 2, 18, 1, (10, 10) 49, 28, 43, 19, 28, 31, 4, 67, (7, 7)	1694 41225 65471 00182
1494	38; 1, 1, 1, 7, 15, 5, (38) 50, 27, 47, 10, 5, 25, (2)	440 05214 17009 02565
1495	38; 1, (1) 51, (26)	3 116
1496	38; 1, 2, (9) 52, 25, (8)	87 3365
1497	38; 1, 2, 4, 4, 1, 1, 1, 1, 6, 2, 2, 1, 8, 1, (24) 53, 24, 17, 16, 41, 33, 32, 43, 11, 31, 23, 51, 8, 67, (3)	58 19614 30932 2251 67187 51127
1498	38; 1, 2, 2, 1, 1, 1, 3, 1, 12, 8, 1, 1, (10) 54, 23, 26, 39, 31, 42, 17, 57, 6, 9, 38, 39, (7)	83118 18384 32 17006 59967
1499	38; 1, 2, 1, 1, 7, 5, 1, 4, 1, 2, 3, 1, 2, 1, 1, 2, 10, 1, 2, 15, 6, 1, (37) 55, 22, 35, 41, 10, 13, 55, 14, 47, 25, 19, 46, 23, 38, 35, 29, 7, 49, 26, 5, 11, 65, (2)	77552 05839 43067 80299 30 02576 94656 26953 60610
1500	38; 1, 2, 1, 2, 2, 1, 6, 2, 1, (18) 56, 21, 44, 25, 24, 49, 11, 25, 51, (4)	118 11844 4574 70751

In connexion with the subject we have a paper, "A Table of the Square Roots of Prime Numbers of the form $4m+1$ less than 10000 expanded as Periodic Continued Fractions," by C. A. Roberts, with Introduction and Explanation by Artemas Martin, the *Mathematical Magazine*, vol. II. (No. 7, for October, 1892), pp. 105—120. This extends, in fact, to numbers up to 10501, but only the denominators of the continued fractions (that is, the first lines of Degen's and the present table) are given: thus the entry for 1009 is 31; 1, (3, 3).

The paper just referred to notices errors in Degen's tables for the numbers 853 and 929. For 853 the first line should be

$$29,\ 4,\ 1,\ 5,\ 1,\ 2,\ 4,\ 1,\ 1,\ 15,\ 19,\ (2,\ 2),$$

(15 instead of Degen's 14). For 929 the first and second lines should be

$$30,\ \ 2,\ 11,\ \ 1,\ \ 2,\ \ 3,\ \ 2,\ 7,\ \ 5,\ \ (2,\ \ 2)$$

$$1,\ 29,\ \ 5,\ 40,\ 19,\ 16,\ 25,\ 8,\ 11,\ (23,\ 23).$$

The values of x, y in Table I. and those in Table II. (for the solution of $y^2 = ax^2 - 1$) are correct for each of the numbers 853 and 929.

949.

ON HALPHEN'S CHARACTERISTIC n, IN THE THEORY OF CURVES IN SPACE.

[From *Crelle's Journal d. Mathematik*, t. CXI. (1893), pp. 347—352.]

If we consider a curve in space without actual singularities, of the order (or degree) d, then this has a number h of apparent double points (adps.), viz. taking as vertex an arbitrary point in space, we have through the curve a cone of the order d, with h nodal lines; and Halphen denotes by n the order of the cone of lowest order which passes through these h lines. For a given value of d, h is at most $=\frac{1}{2}(d-1)(d-2)$, and as shown by Halphen it is at least $=[\frac{1}{4}(d-1)^2]$, if we denote in this manner the integer part of $\frac{1}{4}(d-1)^2$. For given values of d, h, it is easy to see that n must lie within certain limits, viz. if ν be the smallest number such that $\frac{1}{2}\nu(\nu+3)$ equal to or greater than h, then n is at most $=\nu$; and moreover n must have a value such that nd is at least $=2h$, or say we must have $nd=2h+\theta$, where θ is $=0$ or positive. For any given value of d, we thus have a finite number of forms (d, h, n), and we have thus *prima facie* curves in space of the several forms (d, h, n): but it may very well be, and in fact Halphen finds, that when $d=9$ or upwards, then for certain values of h, n as above, there is not any curve (d, h, n); thus $d=9$, $h=17$ the values of n are $n=4$, $n=5$, but there is not any curve $d=9$, $h=17$ for either of these values of n; or say the curves (9, 17, 4) and (9, 17, 5) are non-existent. And in the Notes and References to the papers 302, 305 in vol. v. of my *Collected Mathematical Papers*, 4to. Cambridge, 1892, see p. 615, I remarked that, in certain cases for which Halphen finds a curve (d, h, n), such curve does not exist except for special configurations of the h nodal lines not determined by the mere definition of n as the order of the cone of lowest order which passes through the h nodal lines; for instance $d=9$, $h=16$, $n=4$, for which Halphen gives a curve, I find that for the existence of the curve it is not enough that the 16 lines are situate upon a quartic cone, but they must be the 16 lines of intersection of two quartic cones.

In fact, starting from an existing curve, say the complete intersection of two given surfaces of the orders μ, ν respectively, we have, as is known,

$$d = \mu\nu, \quad 2h = \mu\nu(\mu-1)(\nu-1):$$

we find also, as will appear, $n = (\mu-1)(\nu-1)$: hence also $nd = 2h$, viz. the cone of the order n through the h nodal lines meets the cone of the order d in these lines counting each twice, and in no other lines. I remark also that h is $= \frac{1}{2}n(n+3)$ if $\mu+\nu = 4$; viz. we have here two nodal lines lying in a plane; but if $\mu+\nu > 4$, then h is greater than $\frac{1}{2}n(n+3)$, viz. in this case the nodal lines are not h lines taken at pleasure, but they are lines subject to the condition of lying in a cone of the order n. But this is not all; the h nodal lines lie not only in this cone of lowest order n, but also in cones of the orders $n+1$, $n+2$, ..., $n+(\mu+\nu-2)$ respectively: I do not for the moment attempt to determine the number of independent conditions which are hereby imposed upon the h lines.

The last-mentioned theorem constitutes in fact the geometrical interpretation of results contained in Jacobi's paper "De eliminatione variabilis e duabus aequationibus algebraicis," *Crelle*, t. XV. (1836), pp. 101—124, [*Ges. Werke*, t. III. pp. 295—320]. Consider the two equations

$$U = (*\between x, y, z, w)^\mu = 0; \quad V = (*\between x, y, z, w)^\nu = 0;$$

representing surfaces of the orders μ, ν respectively; since the form of the equations is quite arbitrary, we may without loss of generality assume that the point

$$(x, y, z) = (0, 0, 0)$$

is an arbitrary point in space; and this being so, we find the equation of the cone, vertex this point, which passes through the curve of intersection of the two surfaces by the mere elimination of w between the two equations. As the reasoning is exactly the same for a particular case, I write for convenience $\mu = 3$, $\nu = 4$, and consider the two equations

$$A_0w^3 + A_1w^2 + A_2w + A_3 = 0,$$

$$B_0w^4 + B_1w^3 + B_2w^2 + B_3w + B_4 = 0,$$

where the suffixes show the degrees in regard to (x, y, z), viz. A_0, B_0 are mere constants, A_1, B_1 are linear functions $(*\between x, y, z)^1$, A_2, B_2 quadric functions $(*\between x, y, z)^2$, and so on. Multiplying the first equation successively by $1, w, w^2, w^3$, and the second equation successively by $1, w, w^2$, we have 7 equations from which to eliminate $1, w, w^2, w^3, w^4, w^5, w^6$, and the result is

$$\begin{vmatrix} & & & A_0, & A_1, & A_2, & A_3 \\ & & A_0, & A_1, & A_2, & A_3, & \\ & A_0, & A_1, & A_2, & A_3, & & \\ A_0, & A_1, & A_2, & A_3, & & & \\ & & B_0, & B_1, & B_2, & B_3, & B_4 \\ & B_0, & B_1, & B_2, & B_3, & B_4, & \\ B_0, & B_1, & B_2, & B_3, & B_4, & & \end{vmatrix} = 0,$$

viz. this is an equation

$$(*\mathfrak{Q}x,\ y,\ z)^{12}=0,$$

of the cone of the order $d=12$, through the curve of intersection of the two surfaces: say this equation is $\Omega=0$.

But if we only multiply the first equation by $1,\ w,\ w^2$ successively and the second equation by $1,\ w$ successively, then we have 5 equations serving to determine the ratios of $w^5,\ w^4,\ w^3,\ w^2,\ w,\ 1$, viz. we have these quantities proportional to the six determinants which can be formed out of the matrix

$$\left|\begin{array}{cccccc} & & A_0, & A_1, & A_2, & A_3 \\ & A_0, & A_1, & A_2, & A_3, & \\ A_0, & A_1, & A_2, & A_3, & & \\ & B_0, & B_1, & B_2, & B_3, & B_4 \\ B_0, & B_1, & B_2, & B_3, & B_4, & \end{array}\right|;$$

say we have

$$\begin{aligned} &w^5 : w^4 : w^3 : w^2 : w : 1 \\ &= L : M : N : P : Q : R, \end{aligned}$$

where $L,\ M,\ N,\ P,\ Q,\ R$ represent homogeneous functions $(*\mathfrak{Q}x,\ y,\ z)^\theta$, of the degrees 11, 10, 9, 8, 7, 6 respectively. We may if we please write

$$w=\frac{L}{M}=\frac{M}{N}=\frac{N}{P}=\frac{P}{Q}=\frac{Q}{R};$$

or eliminating w, we have the series of equations which may be written

$$\left|\begin{array}{ccccc} L, & M, & N, & P, & Q \\ M, & N, & P, & Q, & R \end{array}\right|=0,$$

viz. we thus denote that the determinants formed with any two columns of this matrix are severally $=0$. This of course implies that each of the determinants in question is the product of Ω and a factor which is a homogeneous function of the proper degree in $(x,\ y,\ z)$, so that the several equations are all of them satisfied if only $\Omega=0$. We have for instance $PR-Q^2=A\Omega$, where A is a quadric function $(*\mathfrak{Q}x,\ y,\ z)^2$; similarly $NR-PQ=B\Omega$, where B is a cubic function $(*\mathfrak{Q}x,\ y,\ z)^3$; and the like as regards the other determinants.

If the ratios $x : y : z$ have any given values such that we have for these $\Omega=0$, then w has a determinate value, that is, on each line of the cone $\Omega=0$, there is a single point of the curve of intersection of the two surfaces: the only exceptions are when, for the given values of $x : y : z$, the expressions for w assume an indeterminate form, viz. w has then two values, and there are upon the line two points of the curve, or what is the same thing, the line is a nodal line of the cone: the conditions for a nodal line thus are $L=0,\ M=0,\ N=0,\ P=0,\ Q=0,\ R=0$, viz. each of these equations is that of a cone passing through the nodal lines of the cone $\Omega=0$; the cone of lowest order is $R=0$, a cone of the order 6 meeting the

cone $\Omega=0$ of the order 12 in 36 lines each twice, which lines are consequently the nodal lines of the cone $\Omega=0$. The mere condition of the 36 lines lying upon a cone of the order 6 shows that the 36 lines are not arbitrary; and we have moreover, through the 36 lines, cones of the orders 7, 8, 9, 10 and 11 respectively. Obviously the foregoing reasoning is quite general, and for the surfaces of the orders μ, ν we have (as stated above) the cone Ω of the order $\mu\nu$, with $h=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ nodal lines, the intersections (each counting twice) of this cone with a cone of the order $n=(\mu-1)(\nu-1)$; and moreover the h nodal lines lie also in cones of the orders $n+1$, $n+2$, ..., $n+\mu+\nu-2$ respectively.

To examine the meaning of the theorem, I form the table

μ, ν	d	h	n, ..., $n+\mu+\nu-2$
2, 2	4	2	1, 2, 3
2, 3	6	6	2, 3, 4, 5
2, 4	8	12	3, 4, 5, 6, 7
2, 5	10	20	4, 5, 6, 7, 8, 9
⋮			
3, 3	9	18	4, 5, 6, 7, 8
3, 4	12	36	6, 7, 8, 9, 10, 11
3, 5	15	60	8, 9, 10, 11, 12, 13, 14
⋮			
4, 4	16	72	9, 10, 11, 12, 13, 14, 15
4, 5	20	120	12, 13, 14, 15, 16, 17, 18, 19
⋮			

Here μ, $\nu=2$, 2, there are 2 nodal lines, which are arbitrary, and of course lie on cones of the orders 1, 2, 3 respectively. So μ, $\nu=2$, 3; there are 6 nodal lines, which are not arbitrary, inasmuch as they lie on a cone of the order 2; but regarding them as arbitrary lines on such a cone, we can through them draw a cone of the order 3 or any higher order, and it is thus no specialisation to say that they lie upon cones of the orders 3, 4, and 5. But going a step further $\mu=2$, $\nu=4$: here we have 12 nodal lines which, inasmuch as they lie on a cone of the order 3, are not arbitrary: and they are not arbitrary lines upon this cone, for they lie on a cone of the order 4, and such a cone can be drawn through at most 11 arbitrary lines on a cubic cone. In fact, upon a cone of the order θ, taking at pleasure N lines, the condition that it may be possible through these to draw a cone of the order $\theta+1$ is $\frac{1}{2}(\theta+1)(\theta+4)=N+3$ at least; for if this number were $=N+2$, then through the N points we have only the improper cone $(x+\beta y+\gamma z)\,U_\theta=0$, if $U_\theta=0$ is the cone of the order θ. It thus appears that the 12 nodal lines are not arbitrary lines on a cubic cone, but that they constitute the complete intersection of a cubic cone and

a quartic cone. But through such 12 lines we may draw cones of the 5th and higher orders, and it is thus no further condition that the 12 lines lie on cones of the orders 5, 6 and 7 respectively.

So again $\mu=3$, $\nu=3$; we have here 18 nodal lines which, inasmuch as they lie on a cone of the order 4, are not arbitrary: and they are not arbitrary lines on this cone inasmuch as they lie also on a cone of the order 5, and such a cone can be drawn through at most 17 arbitrary lines on the quartic cone: it thus appears that the 18 nodal lines are 18 out of the 20 lines of intersection of a quartic cone and a quintic cone. But there is no further condition, for through such lines we can draw a cone of the order 6 or any higher order and thus the lines lie on cones of the orders 6, 7 and 8 respectively. It appears probable however that, for higher values of μ, ν, it would be necessary to take account not only (as in these examples) of the cones of the orders n and $n+1$, but of those of higher orders $n+2$, &c.; and thus that it is *not* the true form of the theorem to say that the h nodal lines must be h out of the $n(n+1)$ lines of intersection of two cones of the orders n and $n+1$ respectively.

It appears, by what precedes, that the h, $=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$, lines which are the nodal lines of the cone of arbitrary vertex which passes through the curve of intersection of two surfaces of the orders μ, ν respectively, form a remarkable special system of lines, which well deserve further study. I remark also that, without having proved the negative, it seems to me clear that given the values of d, h, n it is only in the cases where the h lines form some such special system (and not in the general case where the h nodal lines are any lines whatever on a cone of the order n) that there exists a curve (d, h, n); and thus that the question for further investigation is, for given values of (d, h, n) to determine the conditions necessary for the existence of a curve in space with these characteristics (d, h, n).

950.

ON THE SEXTIC RESOLVENT EQUATIONS OF JACOBI AND KRONECKER.

[From *Crelle's Journal d. Mathematik*, t. CXIII. (1894), pp. 42—49.]

THE equations referred to are: the first of them, that given by Jacobi in the paper "Observatiunculae ad theoriam aequationum pertinentes," *Crelle*, t. XIII. (1835), pp. 340—352, [*Ges. Werke*, t. III., pp. 269—284], under the heading "Observatio de aequatione sexti gradus ad quam aequationes sexti gradus revocari possunt," and the second, that of Kronecker in the note "Sur la résolution de l'équation du cinquième degré," *Comptes Rendus*, t. XLVI. (1858), pp. 1150—1152. Jacobi's equation is closely connected with that obtained by Malfatti in 1771, see Brioschi's paper "Sulla resolvente di Malfatti per l'equazione del quinto grado," *Mem. R. Ist. Lomb.*, t. IX. (1863); but the characteristic property first presents itself in Jacobi's form, and I think the equation is properly described as Jacobi's resolvent equation. The other equation has been always known as Kronecker's resolvent equation; it belongs to the class of equations for the multiplier of an elliptic function considered by Jacobi in the paper "Suite des notices sur les fonctions elliptiques," *Crelle*, t. III. (1828), pp. 303—310, see p. 308, [*Ges. Werke*, t. I., pp. 255—263, see p. 261]: say Kronecker's equation belongs to the class of Jacobi's Multiplier Equations. We have in regard to it the paper by Brioschi, "Sul metodo di Kronecker per la resoluzione delle equazioni di quinto grado," *Atti Ist. Lomb.*, t. I. (1858), pp. 275—282, and see also the "Appendice terza" to his translation of my *Elliptic Functions* (Milan, 1880): it seems to me however that the theory of Kronecker's equation has not hitherto been exhibited in the clearest form.

I consider the forms

12345	13524
13254	12435
24315	23541
35421	34152
41532	45213
52143	51324

which, if in the first instance the figures are regarded as points, represent the twelve pentagons which can be formed with the points 1, 2, 3, 4, 5; each form in the right-hand column is derived from the corresponding form in the left-hand column by *stellation*, say we have $13524 = S12345$, and so in other cases.

A pentagon is in general reversible, but we sometimes consider it as irreversible (viz. we distinguish between the pentagons 12345 and 15432); when this is so, we write $15432 = R12345$, and we have thus twelve new forms, in all twenty-four forms. The symbols R, S are such that $R^2 = 1$, $S^2 = R$, $RS = SR$, $S^4 = 1$. But for a reversible pentagon, there is no occasion to use the symbol R, and we have simply $S^2 = 1$.

In a somewhat different point of view, we may for an irreversible pentagon write

12345 to denote any one of the forms 12345, 23451, 34512, 45123, 51234,

R12345 " " " " 15432, 54321, 43215, 32154, 21543,

and for a reversible pentagon 12345 to denote any one of these same ten forms.

Each pentagon gives thus ten forms, viz. we have in all 120 forms which all are the different arrangements of the five figures. But further regarding the arrangement 12345 as positive, then the forms in the left-hand column are each of them positive, and the ten forms derived from any one of these are each of them positive, that is, the forms in the left-hand column give all the 60 positive arrangements of the five figures: and similarly the forms in the right-hand column give all the 60 negative arrangements of the five figures.

Taking 1, 2, 3, 4, 5 to denote any quantities, or say the five roots x_1, x_2, x_3, x_4, x_5 of a quintic equation, we regard 12345, ..., as denoting functions of these roots: in particular, 12345 may denote a cyclic reversible function, the analogue of the reversible pentagon, or it may denote a cyclic irreversible function, the analogue of the irreversible pentagon.

Jacobi's resolvent equation.

The most simple course is to take 12345 a cyclic reversible function of the roots x; a root of the resolvent equation is then $12345 - 13524$, $= (1 - S)\,12345$, and the six roots are

$$z_1 = (1 - S)\,12345,$$
$$z_2 = (1 - S)\,13254,$$
$$z_3 = (1 - S)\,24315,$$
$$z_4 = (1 - S)\,35421,$$
$$z_5 = (1 - S)\,41532,$$
$$z_6 = (1 - S)\,52143.$$

Here effecting on the roots x any positive substitution whatever, we permute *inter se* the roots z; but effecting on the roots x any negative substitution whatever, then

reversing the signs of all the roots z, we permute *inter se* these reversed values. Thus effecting on the root z_1 the negative substitution 12, it becomes $(1-S)\,21345$, which is

$$=(1-S)\,S23514=(S-S^2)\ \text{(that is, } S-1 \text{ or } -(1-S))\ 23514=-(1-S)\,41532=-z_5;$$

and similarly for the effect of the same substitution 12 upon any other of the roots z.

It follows that any rational symmetrical function of the roots z is a two-valued function of the coefficients of the quintic equation, viz. it is a function of the form $P+Q\sqrt{\Delta}$, where Δ is the discriminant and P, Q are rational functions of the coefficients of the quintic equation.

In particular, if 12345, ..., are rational and integral functions of the roots x, then for any rational and integral function of the roots z, we have P and Q rational and integral functions of the coefficients, and any rational and integral homogeneous function of the roots z, according as it is of an even or an odd degree in these roots, will be of the form P or of the form $Q\sqrt{\Delta}$; the resolvent equation is thus of the form

$$(1,\ B\sqrt{\Delta},\ C,\ D\sqrt{\Delta},\ E,\ F\sqrt{\Delta},\ G\between z,\ 1)^6=0,$$

where Δ is the discriminant, and B, C, D, E, F, G are rational and integral functions of the coefficients of the quintic equation.

The most simple form of the function 12345 is that employed by Jacobi and for which $12345=12+23+34+45+51$, where 12, ..., denote x_1x_2, ..., respectively.

For comparison with Kronecker's equation, it is proper to take 12345 a cyclic *irreversible* function of the roots x; we have then

$$12345+15432=(1+R)\,12345,$$

a cyclic reversible function, and the roots of Jacobi's equation will be in the first (or since Kronecker writes x_0, x_1, x_2, x_3, x_4 instead of x_1, x_2, x_3, x_4, x_5, in the second) of the following two forms, say

$z_1=(1+R)(1-S)\,12345,$	01234,
$z_2=(1+R)(1-S)\,13254,$	02143,
$z_3=(1+R)(1-S)\,24315,$	13204,
$z_4=(1+R)(1-S)\,35421,$	24310,
$z_5=(1+R)(1-S)\,41532,$	30421,
$z_6=(1+R)(1-S)\,52143;$	41032,

viz. in the first form the terms are 12345, ..., and in the second they are 01234, ..., but the theory is in no wise altered by this change of form.

Kronecker's resolvent equation.

Kronecker writes x_m to denote x_0, x_1, x_2, x_3, x_4 according as the residue of m (mod. 5) is $=0$, 1, 2, 3 or 4: then putting

$$x_m x^2_{m+n} x^3_{m+2n} + v x^3_m x_{m+n} x_{m+2n} = m\,.\,m+n\,.\,m+2n,$$

a root is

$$\begin{aligned} f = \quad & (012+123+234+340+412)\sin\frac{2\pi}{5} \\ & + (024+130+241+302+413)\sin\frac{4\pi}{5} \\ & + (031+142+203+314+420)\sin\frac{6\pi}{5} \\ & + (043+104+210+321+432)\sin\frac{8\pi}{5}, \end{aligned}$$

and the other roots are deduced from this by changing 01234 into 03412, 14023, 20134, 31240, 42301 respectively.

Taking ϵ an imaginary fifth root of unity, say $\epsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, so that $\sin\frac{2\pi}{5}$, $\sin\frac{4\pi}{5}$, $\sin\frac{6\pi}{5}$, $\sin\frac{8\pi}{5}$ are as $\epsilon-\epsilon^4$, $\epsilon^2-\epsilon^3$, $\epsilon^3-\epsilon^2$, $\epsilon^4-\epsilon$; also writing

$$01234 = 012+123+234+340+412, \ \ldots,$$

so that 01234, ..., are cyclic irreversible functions of the roots x, then the expression for the root f is

$$\begin{aligned} f = \quad & (\epsilon-\epsilon^4)\,01234 + (\epsilon^2-\epsilon^3)\,02413 \\ & - (\epsilon-\epsilon^4)\,04321 - (\epsilon^2-\epsilon^3)\,03142, \end{aligned}$$

or, as this may be written,

$$f = \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,01234,$$

and the expressions for the six roots are

$$\begin{aligned} f &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,01234, \\ f_0 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,03412, \\ f_1 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,14023, \\ f_2 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,20134, \\ f_3 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,31240, \\ f_4 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,42301. \end{aligned}$$

I write down the analogous functions

$$\begin{aligned} F &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,03142, \quad (=RS01234), \\ F_0 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,01324, \quad (=RS03412), \\ F_1 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,12430, \quad (=RS14023), \\ F_2 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,23041, \quad (=RS20134), \\ F_3 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,34102, \quad (=RS31240), \\ F_4 &= \{(\epsilon-\epsilon^4)(1-R) + (\epsilon^2-\epsilon^3)(1-R)S\}\,40213, \quad (=RS42301). \end{aligned}$$

This being so, I say that any positive substitution on the roots x permutes *inter se* the roots f, reversing in some cases the signs; and that any negative substitution on the roots x changes the roots f into the roots F, permuting these roots *inter se* and reversing in some cases the signs. And similarly as to the effect on the roots F of a positive substitution and a negative substitution respectively.

Thus the positive substitution 123 changes

$$\begin{array}{llll}
f & \text{into} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,02314 & = f_1, \\
f_0 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,01423 & = f_4, \\
f_1 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,24031 & = f_3, \\
f_2 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,30214 = R03412 & = -f_0, \\
f_3 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,12340 & = f, \\
f_4 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,43102 = R20134 & = -f_2;
\end{array}$$

viz. $f, f_0, f_1, f_2, f_3, f_4$ are changed into $f_1, f_4, f_3, -f_0, f, -f_2$.

And similarly the negative substitution 12 changes

$$\begin{array}{llll}
f & \text{into} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,02134 & = F_4, \\
f_0 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,03421 = R12430 & = -F_1, \\
f_1 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,24013 & = F_0, \\
f_2 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,10234 & = F_3, \\
f_3 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,32140 = R23041 & = -F_2, \\
f_4 & \text{,,} & \{(\epsilon-\epsilon^4)(1-R)+(\epsilon^2-\epsilon^3)(1-R)S\}\,41302 = R03142 & = -F,
\end{array}$$

viz. $f, f_0, f_1, f_2, f_3, f_4$ are changed into $F_4, -F_1, F_0, F_3, -F_2, -F$.

Hence considering the equation the roots of which are $f^2, f_0^2, f_1^2, f_2^2, f_3^2, f_4^2$, this is an equation of the form

$$(1, b, c, d, e, f, g \between f^2, 1)^6 = 0,$$

and similarly the equation the roots of which are $F^2, F_0^2, F_1^2, F_2^2, F_3^2, F_4^2$ is an equation of the form

$$(1, B, C, D, E, F, G \between F^2, 1)^6 = 0,$$

where b and B are conjugate values $\beta+\beta'\sqrt{\Delta}$, $\beta-\beta'\sqrt{\Delta}$, c and C are conjugate values $\gamma+\gamma'\sqrt{\Delta}$, $\gamma-\gamma'\sqrt{\Delta}$, ..., of two-valued functions of the form $P \pm Q\sqrt{\Delta}$, Δ being the discriminant equation, and P and Q rational functions of the coefficients of the quintic equation.

Each term $x_0x_1^2x_2^2$ and $x_0^3x_1x_2$ of Kronecker's function $x_0x_1^2x_2^2+vx_0^3x_1x_2$ is of the form 0.12 $(=0.21)$, a function of x_0 multiplied by a symmetrical function of x_1, x_2; and it is by reason hereof that the roots $f, f_0, f_1, f_2, f_3, f_4$ are connected by linear relations

such that the equation belongs to the class of Jacobi's multiplier equations. Thus in the expressions for these roots, attending first to the terms multiplied by $\epsilon - \epsilon^4$, we have

$$\begin{aligned}
f &= (\epsilon - \epsilon^4).0(12-34) + 1(23-04) + 2(34-01) + 3(04-12) + 4(01-23),\\
\hline
f_0 &= (\epsilon - \epsilon^4).0(34-12) + 1(02-34) + 2(03-14) + 3(14-02) + 4(12-03),\\
f_1 &= (\epsilon - \epsilon^4).0(23-14) + 1(04-23) + 2(13-04) + 3(14-02) + 4(02-13),\\
f_2 &= (\epsilon - \epsilon^4).0(13-24) + 1(34-02) + 2(01-34) + 3(24-01) + 4(02-13),\\
f_3 &= (\epsilon - \epsilon^4).0(13-24) + 1(24-03) + 2(04-13) + 3(12-04) + 4(03-12),\\
f_4 &= (\epsilon - \epsilon^4).0(14-23) + 1(24-03) + 2(03-14) + 3(01-24) + 4(23-01);
\end{aligned}$$

and next to the terms multiplied by $\epsilon^2 - \epsilon^3$, we have

$$\begin{aligned}
f &= (\epsilon^2 - \epsilon^3).0(24-13) + 1(03-24) + 2(14-03) + 3(02-14) + 4(13-02),\\
\hline
f_0 &= (\epsilon^2 - \epsilon^3).0(24-13) + 1(04-23) + 2(13-04) + 3(01-24) + 4(23-01),\\
f_1 &= (\epsilon^2 - \epsilon^3).0(34-12) + 1(03-24) + 2(01-34) + 3(24-01) + 4(12-03),\\
f_2 &= (\epsilon^2 - \epsilon^3).0(23-14) + 1(04-23) + 2(14-03) + 3(12-04) + 4(03-12),\\
f_3 &= (\epsilon^2 - \epsilon^3).0(14-23) + 1(34-02) + 2(01-34) + 3(02-14) + 4(23-01),\\
f_4 &= (\epsilon^2 - \epsilon^3).0(34-12) + 1(02-34) + 2(04-13) + 3(12-04) + 4(13-02);
\end{aligned}$$

we have thus

$$\begin{aligned}
f_0 + f_1 + f_2 + f_3 + f_4 = (\epsilon - \epsilon^4).0\,\{34 - 12 + 2(13-24)\} &+ (\epsilon^2 - \epsilon^3).0\,\{24 - 13 + 2(34-12)\}\\
+ 1\,\{04 - 23 + 2(24-03)\} + &\quad 1\,\{03 - 24 + 2(04-23)\}\\
+ 2\,\{01 - 34 + 2(03-14)\} + &\quad 2\,\{14 - 03 + 2(10-24)\}\\
+ 3\,\{12 - 04 + 2(14-02)\} + &\quad 3\,\{20 - 14 + 2(12-04)\}\\
+ 4\,\{23 - 01 + 2(02-13)\} + &\quad 4\,\{13 - 02 + 2(23-01)\}\\
= -\{(\epsilon - \epsilon^4) + 2(\epsilon^2 - \epsilon^3)\}\,(f)_1 &- \{2(\epsilon - \epsilon^4) - (\epsilon^2 - \epsilon^3)\}\,(f)_2,
\end{aligned}$$

if for a moment $(f)_1$ denotes the terms of f which are multiplied by $\epsilon - \epsilon^4$ and $(f)_2$ the terms of f which are multiplied by $\epsilon^2 - \epsilon^3$.

But we have

$$\sqrt{5} = \epsilon + \epsilon^4 - \epsilon^2 - \epsilon^3,$$

whence

$$\begin{aligned}
\sqrt{5}\,(\epsilon - \epsilon^4) &= \epsilon - \epsilon^4 + 2(\epsilon^2 - \epsilon^3),\\
\sqrt{5}\,(\epsilon^2 - \epsilon^3) &= 2(\epsilon - \epsilon^4) + \epsilon^2 - \epsilon^3;
\end{aligned}$$

and hence the equation just obtained is $f_0 + f_1 + f_2 + f_3 + f_4 = -\sqrt{5}f$, viz. this equation, being satisfied separately by the terms such as $x_0x_1^2x_2^2$ and $x_0^3x_1x_2$, will be satisfied for $x_0x_1^2x_2^2 + vx_0^3x_1x_2$: and so for the like equations which follow. (I notice that Brioschi has $+\sqrt{5}f$; the difference is quite immaterial, since the formulæ would coincide by reversing the sign of f, or those of f_0, f_1, f_2, f_3, f_4.)

We show further that $f_0 + \epsilon^2 f_1 + \epsilon^4 f_2 + \epsilon f_3 + \epsilon^3 f_4 = 0$; to verify this, observe that in this expression the terms multiplied by 0 $(= x_0)$ are

$$\begin{aligned}
&(\epsilon - \epsilon^4)\,\{34 - 12 + \epsilon^2(23-14) + \epsilon^4(13-24) + \epsilon(13-24) + \epsilon^3(14-23)\}\\
&+ (\epsilon^2 - \epsilon^3)\,\{24 - 13 + \epsilon^2(34-12) + \epsilon^4(23-14) + \epsilon(14-23) + \epsilon^3(34-12)\},
\end{aligned}$$

where the terms containing 12, 13, 14, 23, 24, 34 respectively are each $=0$, viz. the coefficient of 12 is $(-\epsilon+\epsilon^4)+(-\epsilon^4+1)+(-1+\epsilon)=0$: that of 13 is

$$1-\epsilon^3+(\epsilon^2-1)+(-\epsilon^2+\epsilon^3)=0:$$

and so for the other coefficients. In like manner it appears that the terms multiplied by 1, 2, 3, 4 $(=x_1,\ x_2,\ x_3,\ x_4)$ respectively are each $=0$, and thus the equation in question is verified. And in like manner it is shown that

$$f_0+\epsilon^3 f_1+\epsilon f_2+\epsilon^4 f_3+\epsilon^2 f_4=0.$$

The roots f thus satisfy the relations

$$\begin{aligned} f_0+\ \ f_1+\ \ f_2+\ \ f_3+\ \ f_4&=-f\sqrt{5},\\ f_0+\epsilon^2 f_1+\epsilon^4 f_2+\ \ \epsilon f_3+\epsilon^3 f_4&=0,\\ f_0+\epsilon^3 f_1+\ \ \epsilon f_2+\epsilon^4 f_3+\epsilon^2 f_4&=0,\end{aligned}$$

or the equation for f^2 belongs to the class of Jacobi's multiplier equations. Hence (see Brioschi's "Appendice terza" before referred to) the form of the equation is

$$(f^2-a)^6-4a(f^2-a)^5+10b(f^2-a)^3-4c(f^2-a)+5b^2-4ac=0,$$

or determining the arbitrary coefficient v so that a may be $=0$, the form is

$$f^{12}+10bf^6-4cf^2+5b^2=0,$$

which is Kronecker's equation

$$f^{12}-10\phi f^6+5\psi^2=\psi f^2.$$

As to the meaning of the coefficients a, b, c, I recall that, in virtue of the foregoing linear relations between the roots, these may be expressed in terms of three arbitrary quantities a_0, a_1, a_2 as follows:

$$\begin{aligned} f&=a_0\sqrt{5},\\ f_0&=a_0+\ \ a_1+\ \ a_2,\\ f_1&=a_0+\ \epsilon a_1+\epsilon^4 a_2,\\ f_2&=a_0+\epsilon^2 a_1+\epsilon^3 a_2,\\ f_3&=a_0+\epsilon^3 a_1+\epsilon^2 a_2,\\ f_4&=a_0+\epsilon^4 a_1+\ \epsilon a_2,\end{aligned}$$

and a, b, c are then determinate functions of a_0, a_1, a_2, viz. we have

$$\begin{aligned} a&=a_0^2+a_1a_2,\\ b&=8a_0^4a_1a_2-2a_0^3a_1^2a_2^2+a_1^3a_2^3-a_0(a_1^5+a_2^5),\\ c&=80a_0^6a_1^3a_2^2-40a_0^4a_1^3a_2^3+5a_0^3a_1^4a_2^4+a_1^5a_2^5\\ &\quad-a_0(32a_0^4-20a_0^3a_1a_2+5a_1^2a_2^2)(a_1^5+a_2^5)\\ &\quad+\tfrac{1}{4}(a_1^5+a_2^5)^2;\end{aligned}$$

so that, for $a=0$ and therefore $a_0=\sqrt{-a_1a_2}$, we have

$$\begin{aligned} b&=11a_1^3a_2^3-a_0(a_1^5+a_2^5),\\ c&=-44a_1^5a_2^5-57a_0a_1^2a_2^2(a_1^5+a_2^5)+\tfrac{1}{4}(a_1^5+a_2^5)^2,\end{aligned}$$

but I do not know that for Kronecker's form the actual values of a_0, a_1, a_2 in terms of the coefficients of the quintic equation have been calculated.

951.

NON-EUCLIDIAN GEOMETRY.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XV. (1894), pp. 37—61. Read January 27, 1890.]

I CONSIDER ordinary three-dimensional space, and use the words point, line, plane, &c., in their ordinary acceptations; only the notion of distance is altered, viz. instead of taking the Absolute to be the circle at infinity, I take it to be a quadric surface: in the analytical developments this is taken to be the imaginary surface $x^2+y^2+z^2+w^2=0$, and the formulæ arrived at are those belonging to the so-called Elliptic Space. The object of the Memoir is to set out, in a somewhat more systematic form than has been hitherto done, the general theory; and in particular, to further develop the analytical formulæ in regard to the perpendiculars of two given lines. It is to be remarked that not only all purely descriptive theorems of Euclidian geometry hold good in the new theory; but that this is the case also (only we in nowise attend to them) with theorems relating to parallelism and perpendicularity, in the Euclidian sense of the words. In Euclidian geometry, infinity is a special plane, the plane of the circle at infinity, and we consider (for instance) parallel lines, that is, lines which meet in a point of this plane: in the new theory, infinity is a plane in nowise distinguishable from any other plane, and there is no occasion to consider (although they exist) lines meeting in a point of this plane, that is, parallel lines in the Euclidian sense. So again, given any two lines, there exists always, in the Euclidian sense, a single line perpendicular to each of the given lines, but this is not in the new sense a perpendicular line; there is nothing to distinguish it from any other line cutting the two given lines, and consequently no occasion to consider it: we do consider the lines—there are, in fact, two such lines—which in the new sense of the word are perpendicular to each of the given lines.

It should be observed that the term distance is used to include inclination: we have, say, a linear distance between two points; an angular distance between two lines which meet; and a dihedral distance between two planes. But all these are

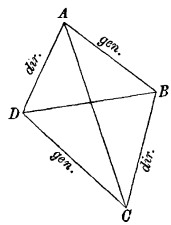
distances of the same kind, having a common unit, the quadrant, represented by $\frac{1}{2}\pi$; and in fact, any distance may be considered indifferently as a linear, an angular, or a dihedral distance: the word, perpendicular, usually represented by $\perp$, refers of course to a distance $=\frac{1}{2}\pi$. We have moreover the distance of a point from a plane, that of a point from a line, and that of a plane from a line. Two lines which do not meet may be $\perp$, and in particular they may be reciprocal: in general, they have two distances; and they have also a "moment" and "comoment," the values of which serve to express those of the two distances. Lines may be, in several distinct senses, as will be explained, parallel; and for this reason the word parallel is never used simpliciter; the notion of parallelism does not apply to planes, nor to points.

Elliptic space has been considered and the theory developed in connexion with the imaginaries called by Clifford biquaternions, and as applied to Mechanics: I refer to the names, Ball, Buchheim, Clifford, Cox, Gravelius, Heath, Klein, and Lindemann: in particular, much of the purely geometrical theory is due to Clifford. Memoirs by Buchheim and Heath are referred to further on.

Geometrical Notions. Art. Nos. 1 to 16.

1. The Absolute is a general quadric surface: it has therefore lines of two kinds, which it is convenient to distinguish as directrices and generatrices: through each point of the surface there is a directrix and a generatrix, and the plane through these two lines is the tangent plane at the point. A line meets the surface in two points, say A, C; the generatrix at A meets the directrix at C; and the

Fig. 1.



directrix at A meets the generatrix at C; and we have thus on the surface two new points B, D; joining these we have a line BD, which is the reciprocal of AC; viz. BD is the intersection of the planes BAD, BCD which are the tangent planes at A, C respectively, and similarly AC is the intersection of the planes ABC, ADC which are the tangent planes at B, D respectively.

According to what follows, reciprocal lines are $\perp$, but $\perp$ lines are not in general reciprocal; thus the two epithets are not convertible, and there will be occasion throughout to speak of reciprocal *lines*.

2. Two points may be harmonic; that is, the two points and the intersections of their line of junction with the Absolute may form a harmonic range: the two points are in this case said to be $\perp$.

Two planes may be harmonic: that is, the two planes and the tangent planes of the Absolute through their line of intersection may form a harmonic plane-pencil: the two planes are said to be $\perp$.

Two lines which meet may be harmonic: that is, the two lines and the tangents from their point of intersection to the section of the Absolute by their common plane may form a harmonic pencil: the two lines are said to be $\perp$.

The locus of all the points $\perp$ to a given point is a plane, the reciprocal or polar plane of the given point; and similarly the envelope of all the planes $\perp$ to a given plane is a point, the pole of the given plane: a point and plane reciprocal to each other, or say a pole and polar plane, are said to be $\perp$.

3. If a point is situate anywhere in a given line, the $\perp$ plane passes always through the reciprocal line: each point of the reciprocal line is thus a point of the $\perp$ plane, i.e. it is $\perp$ to the given point: that is, considering two reciprocal lines, any point on the one line and any point on the other line are $\perp$. Similarly any plane through the one line and any plane through the other line are $\perp$.

A line and plane may be harmonic; that is, they may be reciprocal in regard to the cone, vertex their point of intersection, circumscribed to the Absolute; the line and plane are said to be $\perp$. The $\perp$ plane passes through the reciprocal line, and conversely every plane through the reciprocal line is a $\perp$ plane. It may be added that the line passes through the $\perp$ point of the plane; and conversely, that every line through the $\perp$ point of a plane is $\perp$ to the plane. Moreover if a line and plane be $\perp$, the line is $\perp$ to every line in the plane and through the point of intersection.

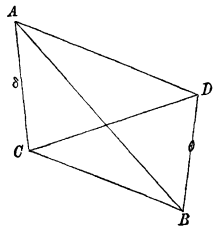
A line and point may be harmonic; that is, they may be reciprocal in regard to the section of the Absolute by their common plane: the line and point are said to be $\perp$. The $\perp$ point lies in the reciprocal line, and conversely every point of the reciprocal line is a $\perp$ point. It may be added that the line lies in the $\perp$ plane of the point: and conversely that every line in the $\perp$ plane of a point is $\perp$ to the point. Moreover if a line and point be $\perp$, the line is $\perp$ to every line through the point and in the plane of junction.

4. We may have a triangle ABC composed of three lines BC, CA, AB in the same plane: the six parts hereof are the linear distances B, C; C, A; A, B of the angular points, and the angular distances of the sides CA, AB; AB, BC; BC, CA. Similarly we may have a trihedral composed of three lines meeting in a point, say the planes through the several pairs of lines are A, B, C respectively: the six parts hereof are the angular distances CA, AB; AB, BC; BC, CA of the three lines, and the dihedral distances B, C; C, A; A, B of the three planes. According to the definitions of distance hereinafter adopted, the relation of the six parts is that of the sides and angles of a spherical triangle: in particular, if two sides are each

$=\frac{1}{2}\pi$, then the opposite angles are each $=\frac{1}{2}\pi$, and the included angle and the opposite side have a common value; and so also if two angles are each $=\frac{1}{2}\pi$, then the opposite sides are each $=\frac{1}{2}\pi$, and the included side and the opposite angle have a common value.

5. Let A, C be points on a line, and B, D points on the reciprocal line; by what precedes, each of the lines AB, AD, CB, CD is $=\frac{1}{2}\pi$: also each of the angles ACD, ACB, CAB, CAD is $=\frac{1}{2}\pi$. The line AC is $\perp$ to the plane BCD and to the lines BC, CD, in that plane; it is also $\perp$ to the plane BAD and to the lines BA, AD in that plane; and similarly for the line BD. From the trihedral of the planes which meet in C, distance of planes ACB, ACD = distance of lines BC, CD, viz. the dihedral distance of two planes through the line AC is equal to the angular distance of their intersections with the $\perp$ plane BCD; and it is therefore equal also to the

Fig. 2.



linear distance of their intersections with the other $\perp$ plane BAD: and so from the triangle BCD, where BC, CD are each $=\frac{1}{2}\pi$, the angular distance BCD is equal to the linear distance BD; that is, the distance of the planes ACB, ACD, that of the lines BC, CD, that of the lines BA, AD, and that of the points B, D, are all of them equal; say the value of each of them is $=\theta$. And in like manner the distance of the planes ABD, CBD, that of the lines AB, BC, that of the lines AD, DC, and that of the points A, C, are all of them equal: say the value of each of them is $=\delta$.

The theorem may be stated as follows: all the planes $\perp$ to a given line intersect in the reciprocal line: and if we have through the given line any two planes, the distance of these two planes, the distance between their lines of intersection with any one of the $\perp$ planes, and the distance between their points of intersection with the reciprocal line, are all of them equal.

And it thus appears also that a distance may be represented indifferently as a linear distance, an angular distance, or a dihedral distance.

6. Consider a point and a plane: we may through the point draw a line $\perp$ to the plane, and intersecting it in a point called the "foot": the distance of the point and plane is then (as a definition) taken to be equal to that of the point and foot. It may be added that the $\perp$ line is, in fact, the line joining the point with the $\perp$

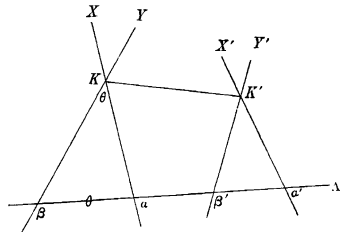
point of the plane; and that the distance of the point and plane is equal to the complement of the distance of the point and the $\perp$ point. Or again, we may in the plane draw a line $\perp$ to the point, and determining with it a plane called the roof: and then (as an equivalent definition) the distance of the plane and point is equal to the distance of the plane and roof. It may be added that the $\perp$ line is, in fact, the intersection of the plane with the $\perp$ plane of the point, and that the distance of the point and plane is also equal to the complement of the distance of the plane and the $\perp$ plane of the point.

7. Consider a point and line: we have through the point a line $\perp$ to the line and cutting it in a point called the foot; the distance of the point and line is then (as a definition) equal to the distance of the point and foot. It may be added that the foot is the intersection with the line of a plane $\perp$ thereto through the point.

Again, consider a plane and line: we have in the plane a line $\perp$ to the line and determining with it a plane called the roof: the distance of the plane and line is then (as a definition) equal to the distance of the plane and roof. It may be added that the roof is the plane determined by the line and a point $\perp$ thereto in the plane.

8. If two lines intersect, then their reciprocals also intersect. Say the intersecting lines are X, Y; and their reciprocals X', Y' respectively; then K, the point of intersection of X, Y, has for its reciprocal the plane of the lines X', Y'; and similarly K', the point of intersection of the lines X', Y', has for its reciprocal the plane of the lines X, Y: hence KK' has for its reciprocal the line of intersection of the planes XY and $X'Y'$; say this is the line Λ, meeting X, Y, X', Y', in the

Fig. 3.

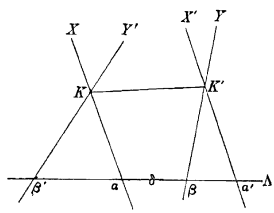


points α, β, α', β' respectively. Since K, K' are points in the reciprocal lines X, X' (or in the reciprocal lines Y, Y'), the distance KK' is $=\frac{1}{2}\pi$; and since the plane XY passes through the line Λ which is the reciprocal of KK', the line KK' is $\perp$ to the plane XY and also to each of the lines X, Y: (it is also $\perp$ to the plane $X'Y'$ and to each of the lines X', Y'). Again, since the lines KK' and Λ are reciprocal, each of the distances $K\alpha$, $K\beta$ is $=\frac{1}{2}\pi$; that is, the line Λ is $\perp$ to each of the lines X and Y, (and similarly it is $\perp$ to each of the lines X' and Y'). Moreover the

angle at K or distance of the lines X and Y (which is equal to the distance of the planes $K'KX$ and $K'KY$) is equal to the distance $\alpha\beta$ of the intersections of Λ with the lines X and Y respectively. We have thus for the two intersecting lines X and Y, the two lines KK' and Λ each of them $\perp$ to the two lines: where observe that KK' is the line of junction of the point of intersection of the two given lines with the point of intersection of the reciprocal lines; and that Λ is the line of intersection of the plane of the two given lines with the plane of the reciprocal lines. The linear distance along KK' between the two lines is $=0$; the dihedral distance between the planes, which KK' determines with the two lines respectively, is equal to the angular distance between the two lines. The linear distance along Λ is equal to the angular distance between the two lines; the dihedral distance between the two planes, which Λ determines with the two lines respectively, is $=0$.

9. If two lines are such that the first of them intersects the reciprocal of the second of them, then also the second will intersect the reciprocal of the first; the two lines are in this case said to be contrasecting lines; or more simply, to contrasect: and contrasecting lines are said to be $\perp$. Supposing that the two lines are X, Y and their reciprocals X', Y' respectively, we have here X, Y' intersecting in a point K, and X', Y intersecting in a point K': and the planes XY', $X'Y$ intersect in a line Λ which meets the lines X, Y, X', Y' in the points α, β, α', β' respectively. As before, the lines KK' and Λ are reciprocal: the distance KK' is $=\frac{1}{2}\pi$; and KK' is $\perp$ to the plane XY', that is, to each of the lines X, Y'; and also to the plane $X'Y$, that is, to each of the lines X', Y; it is thus $\perp$ to each of the lines X and Y. Again each of the angles at α, β, α', β' is $=\frac{1}{2}\pi$; that is,

Fig. 4.



the line Λ is $\perp$ to each of the lines X, Y', X', Y, or say to each of the lines X and Y. Moreover the angle at K, or say the angular distance of the intersecting lines X and Y', is equal to the distance $\alpha\beta'$; and similarly the angle at K', or say the angular distance of the intersecting lines X' and Y, is equal to the distance $\alpha'\beta$: but the distances $\alpha\alpha'$, $\beta\beta'$ are each equal to $\frac{1}{2}\pi$; and hence the distances $\alpha\beta'$, $\alpha'\beta$ are equal to each other and each of them is equal to the complement of the distance $\alpha\beta$. Thus in the case of two contrasecting lines we have the lines KK' and Λ each of them $\perp$ to the two given lines; where observe that KK' is the line joining the point of intersection of X with the reciprocal of Y and the

point of intersection of Y with the reciprocal of X; and that Λ is the line of intersection of the plane through X and the reciprocal of Y with the plane through Y and the reciprocal of X. The linear distance KK' between the two lines along the first of these lines is thus $=\frac{1}{2}\pi$.

10. We have KK' and Λ reciprocal lines; on the first of these, we have the points K, K' which are $\perp$ points: hence also the planes ΛK and $\Lambda K'$ are $\perp$; but the plane ΛK is the plane $\Lambda XY'$ or say the plane ΛX, and the plane $\Lambda K'$ is the plane $\Lambda X'Y$ or say the plane ΛY; hence the planes ΛX and ΛY are $\perp$. Similarly the line Λ cuts the two lines in the points α, β; and the line KK' determines with these two points respectively the plane $KK'\alpha$, that is $KK'X$, and $KK'\beta$, that is $KK'Y$; and thus the linear distance between the two points α, β is equal to the dihedral distance between the two planes $KK'X$ and $KK'Y$. Thus the $\perp$ line Λ cuts the two lines in two points α, β the linear distance of which is, say, δ: and it determines with them two planes the dihedral distance of which is $=\frac{1}{2}\pi$. And the other $\perp$ line KK' cuts the two lines in the points K, K' the linear distance of which is $=\frac{1}{2}\pi$, and it determines with them two planes the dihedral distance of which is $=\delta$.

11. Consider a line X and its reciprocal X': a line intersecting each of these also contrasects each of them and is thus $\perp$ to each of them: and similarly if Y be any other line and Y' its reciprocal, a line intersecting Y and Y' also contrasects each of them and is thus $\perp$ to each of them. Hence a line which meets each of the four lines X, X', Y, Y' is also $\perp$ to each of them, or attending only to the lines X, Y, say it is a $\perp$ of these lines: there are two $\perp$s; and clearly these are reciprocal to each other, for if a line meets X, Y, X', Y', then its reciprocal meets X', Y', X, Y, that is, the same four lines. Looking back to figure 2, we may take AB, CD for the given lines, and AC, BD for the two $\perp$s; as just remarked, these are reciprocal to each other. The $\perp$ AC cuts the two lines respectively in the two points A and C the linear distance of which is say $=\delta$; and it determines with them two planes ACB, ACD, the dihedral distance of which is say $=\theta$. Similarly the other $\perp$ BD meets the two lines respectively in the two points B and D the linear distance of which is $=\theta$, and it determines with them two planes BDA, BDC the dihedral distance of which is $=\delta$. In the plane triangles which are the faces of the tetrahedron $ABCD$, there is in each triangle an angle opposite to AC or BD and which, or say the angular distance of the two including sides, is thus $=\delta$ or θ. Except as aforesaid, the sides, angles, and dihedral angles, or say the linear, angular, and dihedral distances, of the tetrahedron are each of them $=\frac{1}{2}\pi$.

12. Considering the lines X and Y as given, the distances δ and θ depend upon two functions called the Moment and the Comoment: viz. moment $=0$ is the condition in order that the two lines may intersect (or, what is the same thing, in order that their reciprocals may intersect): comoment $=0$ is the condition in order that the two lines may contrasect, that is, each line meet the reciprocal of the other one. It may be convenient to mention here that the actual relations are

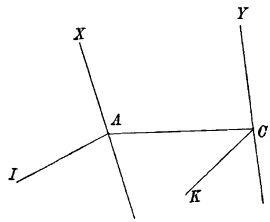
$$\sin\delta\sin\theta = \text{Moment}, \quad \cos\delta\cos\theta = \text{Comoment}.$$

In particular, if moment $=0$, then the lines intersect; we have, say $\delta=0$, and therefore $\cos\theta=$ comoment; if comoment $=0$, then the lines contrasect, that is they are $\perp$: we have, say $\theta=\frac{1}{2}\pi$, that is, $\sin\delta=$ moment. These are the two particular cases which have been considered above.

13. Consider as above the two lines X, Y met by the $\perp$ δ in the two points A and C respectively. Consider at A a line I $\perp$ to the lines X, δ; and take Π the plane of the lines (X, δ) and Ω the plane of the lines (X, I). Similarly consider at C a line K $\perp$ to the lines Y, δ, and take Π_1 the plane of the lines (Y, δ) and Ω_1 the plane of the lines (Y, K): we have thus through A two planes Π, Ω meeting in the line X; and through C two planes Π_1, Ω_1, meeting in the line Y. It requires only a little reflection to see that the distances of these planes are

$$(\Pi, \Pi_1)=\theta, \quad (\Omega, \Omega_1)=\delta;$$
$$(\Pi, \Omega)=\tfrac{1}{2}\pi, \ (\Pi_1, \Omega_1)=\tfrac{1}{2}\pi; \ (\Pi, \Omega_1)=\tfrac{1}{2}\pi, \ (\Pi_1, \Omega)=\tfrac{1}{2}\pi.$$

Fig. 5.



In fact, Π, Π_1 are the before-mentioned planes ACB, ACD the distance of which was $=\theta$: Ω, Ω_1 are planes having the common $\perp$ AC, which is the line through the poles of these planes, and such that the distance AC is equal to the distance of the two poles, that is, the distance of the two planes. Moreover from the definitions, the distances (Π, Ω) and (Π_1, Ω_1) are each $=\frac{1}{2}\pi$: the plane Π passes through the $\perp$ at C to the plane Ω_1 that is, $(\Pi, \Omega_1)=\frac{1}{2}\pi$; and similarly the plane Π_1 passes through the $\perp$ at A to the plane Ω, that is, $(\Pi_1, \Omega)=\frac{1}{2}\pi$; and we have thus the relations in question.

The consideration of these planes leads, (see *post* 31 and 32), to the before-mentioned equation, $\cos\delta\cos\theta=$ comoment; if instead of one of the lines, say Y, we consider the reciprocal line Y', then the angles δ, θ are changed each of them into its complement, and we deduce immediately the other equation, $\sin\delta\sin\theta=$ Moment.

14. It may happen that, instead of the determinate number 2, we have a singly infinite system of $\perp$s: viz. this will be so if the lines X, X', Y, Y' are generating lines (of the same kind) of a hyperboloid. They will be so if the lines X and Y each of them meet the same two lines (of the same kind) of the Absolute, say if X, Y each meet two directrices D_1, D_2, or two generatrices G_1, G_2; but it seems

less easy to prove conversely that the lines X and Y must satisfy one of these two conditions. Suppose first that X, Y each meet the two directrices D_1, D_2; say X meets them in α_1, α_2, and Y in β_1, β_2 respectively. We have at α_1 a generatrix which meets D_2, suppose in α_2', and at α_2 a generatrix which meets D_1, suppose in α_1'; joining α_1', α_2', we have the line X' which is the reciprocal of X; viz. X' meets each of the lines D_1, D_2: similarly the generatrices at β_1, β_2 meet D_2, D_1 in the points β_2', β_1' respectively, and joining these, we have the line Y' which is the reciprocal of Y: thus Y' meets each of the lines D_1 and D_2: the line D_1 meets the four generatrices in the points α_1, α_1', β_1, β_1' respectively, and the line D_2 meets the same four generatrices in the points α_2', α_2, β_2', β_2: thus

$$AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2', \alpha_2, \beta_2', \beta_2),$$

AH denoting anharmonic ratio as usual. But

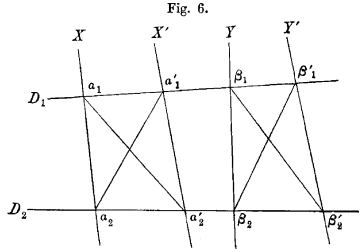
$$AH(\alpha_2', \alpha_2, \beta_2', \beta_2) = AH(\alpha_2, \alpha_2', \beta_2, \beta_2'),$$

and thus the equation may be written

$$AH(\alpha_1, \alpha_1', \beta_1, \beta_1') = AH(\alpha_2, \alpha_2', \beta_2, \beta_2');$$

viz. the lines X, X', Y, Y', cut D_1, D_2 homographically; and there is thus a singly infinite system of lines cutting D_1, D_2 homographically: that is, X, X', Y, Y', are lines (of the same kind) of a hyperboloid. And similarly if X, Y each cut the same two generating lines G_1, G_2, then will X', Y' also cut these lines and X, X', Y, Y' will cut them homographically, that is, X, X', Y, Y' will be lines (of the same kind) of a hyperboloid.

Fig. 6.



The condition may be otherwise stated; if the lines X, Y have for $\perp$s any two directrices D_1, D_2 or any two generatrices G_1, G_2 of the Absolute, then in either case there will be a singly infinite series of $\perp$s: the $\perp$ distances are all of them equal; say we have $\theta = \delta$, and therefore $\sin^2\delta = \text{moment}$, $\cos^2\delta = \text{comoment}$; and therefore moment + comoment = 1; or as the equation is more properly written, $\pm$ moment $\pm$ comoment = 1.

15. Two lines X, Y, each of them meeting the same two directrices D_1, D_2, are said to be "right parallels"; and similarly two lines X, Y each meeting the same two generatrices G_1, G_2, are said to be "left parallels": the selection as to which set

of lines of the Absolute shall be called directrices and which shall be called generatrices will be made further on, (see *post* 35). We have just seen that, if two lines are right parallels, or are left parallels, then in either case there is a singly infinite series of $\perp$s. It may be remarked that reciprocal lines are at once right parallels and left parallels; and that in this case there is a doubly infinite series of $\perp$s, viz. every line cutting the two lines is a $\perp$.

Observe that right parallels do not meet, and left parallels do not meet: their doing so would imply in the one case the meeting of two directrices, and in the other case the meeting of two generatrices.

16. If instead of the foregoing definitions by means of two directrices or two generatrices, we consider a directrix and a generatrix of the Absolute, and define parallel lines by reference thereto, then it is at once seen that there are 3 chief forms, and several subforms; the directrix and generatrix meet in a point, or say an ineunt, of the Absolute, and lie in a plane which is a tangent plane of the Absolute: we may have two lines X, Y which

1°. Each pass through the ineunt, neither of them lying in the tangent plane;

2°. Each lie in the tangent plane, neither of them passing through the ineunt;

3°. One passes through the ineunt, but does not lie in the tangent plane: the other lies in the tangent plane, but does not pass through the ineunt.

Observe that in the cases 1° and 2° the lines X and Y intersect, but in the case 3° they do not intersect. The lines in the case 3° are I believe what Buchheim has termed β-parallels, his α-parallels being the foregoing right or left parallels*. The subforms arise by omitting in 1°, 2°, or 3°, as the case may be, the negative condition in regard to the two lines or to one of them; as the question is not here further pursued, I do not attempt to give names to these several kinds of parallel lines.

Point-, line-, and plane-coordinates: General formulæ. Art. Nos. 17 to 20.

17. We consider point-coordinates (x, y, z, w): line-coordinates (a, b, c, f, g, h), where $af+bg+ch=0$: and plane-coordinates $(\xi, \eta, \zeta, \omega)$; if we have a line which is at once through two points and in two planes, then the line-coordinates are given by

$$\begin{aligned} & a : b : c : f : g : h \\ &= y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 : x_1w_2 - x_2w_1 : y_1w_2 - y_2w_1 : z_1w_2 - z_2w_1 \\ &= \xi_1\omega_2 - \xi_2\omega_1 : \eta_1\omega_2 - \eta_2\omega_1 : \zeta_1\omega_2 - \zeta_2\omega_1 : \eta_1\zeta_2 - \eta_2\zeta_1 : \zeta_1\xi_2 - \zeta_2\xi_1 : \xi_1\eta_2 - \xi_2\eta_1. \end{aligned}$$

Similarly, if a plane be determined by three points thereof, then the coordinates of the plane are given by

$$\xi : \eta : \zeta : \omega = \begin{vmatrix} 1 \\ x_1, y_1, z_1, w_1 \\ x_2, y_2, z_2, w_2 \\ x_3, y_3, z_3, w_3 \end{vmatrix} : \begin{vmatrix} 1 \\ x_1, y_1, z_1, w_1 \\ x_2, y_2, z_2, w_2 \\ x_3, y_3, z_3, w_3 \end{vmatrix} : \begin{vmatrix} 1 \\ x_1, y_1, z_1, w_1 \\ x_2, y_2, z_2, w_2 \\ x_3, y_3, z_3, w_3 \end{vmatrix} : \begin{vmatrix} 1 \\ x_1, y_1, z_1, w_1 \\ x_2, y_2, z_2, w_2 \\ x_3, y_3, z_3, w_3 \end{vmatrix};$$

* See Buchheim, "A Memoir on Biquaternions," *Amer. Math. Jour.* t. VII. (1885), pp. 293—326.

and if a point be given as the intersection of three planes, the coordinates of the point are

$$x : y : z : w = \begin{vmatrix} 1 & & & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & 1 & & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & & 1 & \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix} : \begin{vmatrix} & & & 1 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \\ \xi_2, & \eta_2, & \zeta_2, & \omega_2 \\ \xi_3, & \eta_3, & \zeta_3, & \omega_3 \end{vmatrix}.$$

18. The conditions in order that a point (x, y, z, w) may be situate on a line (a, b, c, f, g, h) are

$$\begin{aligned} . \quad hy - gz + aw &= 0, \\ -hx \quad . \quad + fz + bw &= 0, \\ gx - fy \quad . \quad + cw &= 0, \\ -ax - by - cz \quad . \quad &= 0, \end{aligned}$$

viz. these constitute a twofold relation.

Similarly, the conditions in order that the plane $(\xi, \eta, \zeta, \omega)$ may contain the line (a, b, c, f, g, h) are

$$\begin{aligned} . \quad c\eta - b\zeta + f\omega &= 0, \\ -c\xi \quad . \quad + a\zeta + g\omega &= 0, \\ b\xi - a\eta \quad . \quad + h\omega &= 0, \\ -f\xi - g\eta - h\zeta \quad . \quad &= 0, \end{aligned}$$

viz. these constitute a twofold relation.

19. The condition in order that two lines (a, b, c, f, g, h), (A, B, C, F, G, H) may meet is

$$Af + Bg + Ch + Fa + Gb + Hc = 0.$$

Supposing that the two lines meet, we have at the point of intersection

$$\begin{aligned} . \quad hy - gz + aw &= 0, & . \quad Hy - Gz + Aw &= 0, \\ -hx \quad . \quad + fz + bw &= 0, & -Hx \quad . \quad + Fz + Bw &= 0, \\ gx - fy \quad . \quad + cw &= 0, & Gx - Fy \quad . \quad + Cw &= 0, \\ -ax - by - cz \quad . \quad &= 0, & -Ax - By - Cz \quad . \quad &= 0; \end{aligned}$$

and from these equations we can find the coordinates x, y, z, w of the point of intersection in a fourfold form, viz. we may write

$$\begin{aligned} x : y : z : w &= fA + bG + cH : gA - aG : hA - aH : hG - gH \\ &= fB - bF : gB + cH + aF : hB - bH : fH - hF \\ &= fC - cF : gC - cG : hC + aF + bG : gF - fG \\ &= bC - cB : cA - aC : aB - bA : fA + gB + hC. \end{aligned}$$

There is no real advantage in any one over any other of these forms, but it is convenient to work with the last of them

$$x : y : z : w = bC - cB : cA - aC : aB - bA : fA + gB + hC.$$

20. In like manner if two lines intersect, the plane which contains each of them is given by

$$\begin{aligned}\xi : \eta : \zeta : \omega &= aF+gB+hC : bF-fB : cF-fC : cB-bC\\ &= aG-gA : bG+hC+fA : cG-gC : aC-cA\\ &= aH-hA : bH-hB : cH+fA+gB : bA-aB\\ &= gH-hG : hF-fH : fG-gF : aF+bG+cH;\end{aligned}$$

or say we have

$$\xi : \eta : \zeta : \omega = gH-hG : hF-fH : fG-gF : aF+bG+cH.$$

The Absolute. Art. Nos. 21 to 27.

21. The equation is

in point coordinates $x^2+y^2+z^2+w^2=0$,

in plane coordinates $\xi^2+\eta^2+\zeta^2+\omega^2=0$,

in line coordinates $a^2+b^2+c^2+f^2+g^2+h^2=0$.

Hence

$\perp$ of plane $(\xi, \eta, \zeta, \omega)$ is point $(\xi, \eta, \zeta, \omega)$,

$\perp$ of point (x, y, z, w) is plane (x, y, z, w).

Reciprocal of line (a, b, c, f, g, h) is line (f, g, h, a, b, c);

Points (x, y, z, w), (x', y', z', w') are $\perp$ if $xx'+yy'+zz'+ww'=0$;

Planes $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$ are $\perp$ if $\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega'=0$.

22. A line (a, b, c, f, g, h) and plane $(\xi, \eta, \zeta, \omega)$ are $\perp$ when the line passes through the $\perp$ point of the plane, that is, the point $(\xi, \eta, \zeta, \omega)$: the conditions (equivalent to two equations) are

$$\begin{aligned} \cdot \quad & h\eta - g\zeta + a\omega = 0,\\ -h\xi \quad \cdot \quad & + f\zeta + b\omega = 0,\\ g\xi - f\eta \quad \cdot \quad & + c\omega = 0,\\ -a\xi - b\eta - c\zeta \quad \cdot \quad & = 0.\end{aligned}$$

A line (a, b, c, f, g, h) and point (x, y, z, w) are $\perp$ when the line lies in the $\perp$ plane of the point, that is, in the plane (x, y, z, w): the conditions (equivalent to two equations) are

$$\begin{aligned} \cdot \quad & cy - bz + fw = 0,\\ -cx \quad \cdot \quad & + az + gw = 0,\\ bx - ay \quad \cdot \quad & + hw = 0,\\ -fx - gy - hz \quad \cdot \quad & = 0.\end{aligned}$$

Two lines (a, b, c, f, g, h), (a', b', c', f', g', h') which meet, that is, for which $af'+bg'+ch'+a'f+b'g+c'h=0$, are $\perp$ if

$$aa'+bb'+cc'+ff'+gg'+hh'=0.$$

23. There will be occasion to consider the pair of tangent planes drawn through the line (a, b, c, f, g, h) to the Absolute. Writing for shortness

$$\begin{aligned} P &= \quad . \quad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \quad , \end{aligned}$$

it may be shown that the equation of the pair of planes is

$$P^2 + Q^2 + R^2 + S^2 = 0.$$

In fact, writing for a moment $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$ to denote the coefficients of (x, y, z, w) in P and Q respectively, so that

$$(\xi, \eta, \zeta, \omega) = (0, h, -g, a), \quad (\xi', \eta', \zeta', \omega') = (-h, 0, f, b),$$

then the equation of the planes is

$$(\xi' P - \xi Q)^2 + (\eta' P - \eta Q)^2 + (\zeta' P - \zeta Q)^2 + (\omega' P - \omega Q)^2 = 0,$$

that is,

$$(\xi'^2 + \eta'^2 + \zeta'^2 + \omega'^2) P^2 - 2(\xi\xi' + \eta\eta' + \zeta\zeta' + \omega\omega') PQ + (\xi^2 + \eta^2 + \zeta^2 + \omega^2) Q^2 = 0,$$

viz. this equation is

$$(b^2 + h^2 + f^2) P^2 + 2(fg - ab) PQ + (a^2 + g^2 + h^2) Q^2 = 0.$$

But P, Q, R, S are connected by the identical equations

$$\begin{aligned} \quad . \quad cQ - bR + fS &= 0, \\ -cP \quad . \quad + aR + gS &= 0, \\ bP - aQ \quad . \quad + hS &= 0, \\ -fP - gQ - hR \quad . \quad &= 0; \end{aligned}$$

using these equations to express R, S in terms of P, Q, viz. writing

$$R = -\frac{1}{h}(fP + gQ), \quad S = -\frac{1}{h}(bP - aQ),$$

we see that the last preceding equation is equivalent to $P^2 + Q^2 + R^2 + S^2 = 0$.

24. Similarly, if

$$\begin{aligned} P_1 &= \quad . \quad cy - bz + fw, \\ Q_1 &= -cx \quad . \quad + az + gw, \\ R_1 &= \quad bx - ay \quad . \quad + hw, \\ S_1 &= -fx - gy - hz \quad . \quad , \end{aligned}$$

functions which are connected by the identical relations

$$\begin{aligned} \quad . \quad hQ_1 - gR_1 + aS_1 &= 0, \\ -hP_1 \quad . \quad + fR_1 + bS_1 &= 0, \\ gP_1 - fQ_1 \quad . \quad + cS_1 &= 0, \\ -aP_1 - bQ_1 - cR_1 \quad . \quad &= 0; \end{aligned}$$

then in like manner we have

$$P_1^2+Q_1^2+R_1^2+S_1^2=0,$$

for the equation of the pair of tangent planes from the reciprocal line (f, g, h, a, b, c) to the Absolute. And we may remark the identity

$$(P^2+Q^2+R^2+S^2)+(P_1^2+Q_1^2+R_1^2+S_1^2)=(a^2+b^2+c^2+f^2+g^2+h^2)(x^2+y^2+z^2+w^2).$$

We, in fact, have

$$P^2+Q^2+R^2+S^2=\begin{array}{c|c|c|c|c|}
 & x & y & z & w \\
\hline
x & a^2+g^2+h^2 & ab-fg & ac-hf & cg-bh \\
\hline
y & ab-fg & b^2+h^2+f^2 & bc-gh & ah-cf \\
\hline
z & ac-hf & bc-gh & c^2+f^2+g^2 & bf-ag \\
\hline
w & cg-bh & ah-cf & bf-ag & a^2+b^2+c^2 \\
\hline
\end{array};$$

and in like manner

$$P_1^2+Q_1^2+R_1^2+S_1^2=\begin{array}{c|c|c|c|c|}
 & x & y & z & w \\
\hline
x & b^2+c^2+f^2 & -ab+fg & -ac+hf & -cg+bh \\
\hline
y & -ab+fg & c^2+a^2+g^2 & -bc+gh & -ah+cf \\
\hline
z & -ac+hf & -bc+gh & a^2+b^2+h^2 & -bf+ag \\
\hline
w & -cg+bh & -ah+cf & -bf+ag & f^2+g^2+h^2 \\
\hline
\end{array}.$$

25. For the distance of two points (x, y, z, w) and (x', y', z', w'), we have

$$\cos\delta=\frac{xx'+yy'+zz'+ww'}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x'^2+y'^2+z'^2+w'^2}},$$

whence also

$$\sin\delta=\frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x'^2+y'^2+z'^2+w'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points, taken to be equal to

$$yz'-y'z,\ zx'-z'x,\ xy'-x'y,\ xw'-x'w,\ yw'-y'w,\ zw'-z'w$$

respectively.

Similarly, for the distance of two planes $(\xi, \eta, \zeta, \omega)$ and $(\xi', \eta', \zeta', \omega')$, we have

$$\cos\delta=\frac{\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega'}{\sqrt{\xi^2+\eta^2+\zeta^2+\omega^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

whence also

$$\sin\delta=\frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{\xi^2+\eta^2+\zeta^2+\omega^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of intersection of the two planes, taken to be equal to

$$\xi\omega' - \xi'\omega,\ \eta\omega' - \eta'\omega,\ \zeta\omega' - \zeta'\omega,\ \eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta$$

respectively.

The distance of a point (x, y, z, w) and plane $(\xi', \eta', \zeta', \omega')$ is the complement of the distance of the point (x, y, z, w) and the point $(\xi', \eta', \zeta', \omega')$ which is the $\perp$ point of the plane; viz. we have

$$\sin\delta = \frac{x\xi' + y\eta' + z\zeta' + w\omega'}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

$$\cos\delta = \frac{\sqrt{a^2+b^2+c^2+f^2+g^2+h^2}}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{\xi'^2+\eta'^2+\zeta'^2+\omega'^2}},$$

where, in the numerator, (a, b, c, f, g, h) stand for the coordinates of the line of junction of the two points. Of course the same result might have been equally well derived from the formula for the distance of two planes.

26. If we now consider a plane triangle ABC, and write

(x_1, y_1, z_1, w_1) for the coordinates of A,

(x_2, y_2, z_2, w_2) „ „ B,

(x_3, y_3, z_3, w_3) „ „ C,

then the coordinates

$$a, \qquad b, \qquad c, \qquad f, \qquad g, \qquad h,$$

of the line BC will be

$$y_2z_3 - y_3z_2,\ z_3x_2 - z_2x_3,\ x_2y_3 - x_3y_2,\ x_2w_3 - x_3w_2,\ y_2w_3 - y_3w_2,\ z_2w_3 - z_3w_2,$$

and similarly for the coordinates of the lines AB, CA; the equations

$$a_1f_2 + b_1g_2 + c_1h_2 + a_2f_1 + b_2g_1 + c_2h_1 = 0, \text{ \&c.},$$

which express that these lines meet in pairs in the points A, B, C respectively, are of course satisfied identically; and we then have for the sides and angles (linear and angular distances) of the triangle

$$\cos a = \frac{x_2x_3 + y_2y_3 + z_2z_3 + w_2w_3}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\sin a = \frac{\sqrt{a_1^2+b_1^2+c_1^2+f_1^2+g_1^2+h_1^2}}{\sqrt{x_2^2+y_2^2+z_2^2+w_2^2}\sqrt{x_3^2+y_3^2+z_3^2+w_3^2}},$$

$$\cos A = \frac{a_2a_3 + b_2b_3 + c_2c_3 + f_2f_3 + g_2g_3 + h_2h_3}{\sqrt{a_2^2+b_2^2+c_2^2+f_2^2+g_2^2+h_2^2}\sqrt{a_3^2+b_3^2+c_3^2+f_3^2+g_3^2+h_3^2}}, \text{ \&c.};$$

and this being so, if with the values of $\cos a$, $\cos b$, $\cos c$, we form the expression for $\cos a - \cos b \cos c$, then reducing to a common denominator, the expression for the numerator is at once found to be

$$= a_2a_3 + b_2b_3 + c_2c_3 + f_2f_3 + g_2g_3 + h_2h_3,$$

and thence easily

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

viz. the expressions for the angles in terms of the sides are those of ordinary spherical trigonometry.

27. Hence also

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin b \sin c};$$

whence

$$\sin A : \sin B : \sin C = \sin a : \sin b : \sin c,$$

and

$$\cos A + \cos B \cos C = \frac{\cos a\,(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}{\sin^2 a \sin b \sin c},$$

and consequently

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

which completes the system of formulæ.

And similarly for a trihedral, that is, if we have three planes A, B, C (meeting of course in a point, O) then the dihedral distances BC, CA, AB and the angular distances CA, AB; AB, BC; BC, CA are related to each other in the same way as the angles and sides of an ordinary spherical triangle.

Distance of a point and line. Art. Nos. 28, 29.

28. The point is taken to be (x_1, y_1, z_1, w_1), the line (a, b, c, f, g, h). Drawing through the point a $\perp$ plane, say $(\xi, \eta, \zeta, \omega)$ meeting the line in the foot, and taking the coordinates hereof to be (x_2, y_2, z_2, w_2), then $\xi x_1 + \eta y_1 + \zeta z_1 + \omega w_1 = 0$ and

$$\begin{aligned} \cdot \quad h\eta - g\zeta + a\omega &= 0, \\ -h\xi \quad \cdot \quad + f\zeta + b\omega &= 0, \\ g\xi - f\eta \quad \cdot \quad + c\omega &= 0, \\ -a\xi - b\eta - c\zeta \quad \cdot \quad &= 0, \end{aligned}$$

giving, say,

$$\begin{aligned} \xi &= \quad \cdot \quad cy_1 - bz_1 + fw_1, \\ \eta &= -cx_1 \quad \cdot \quad + az_1 + gw_1, \\ \zeta &= \quad bx_1 - ay_1 \quad \cdot \quad + hw_1, \\ \omega &= -fx_1 - gy_1 - hz_1 \quad \cdot \quad \cdot \end{aligned}$$

We have here

$$\xi^2+\eta^2+\zeta^2+\omega^2=(b^2+c^2+f^2)\,x_1^2+\&c.,$$

where $(b^2+c^2+f^2)\,x_1^2+\&c.$ denotes the before-mentioned quadric function of (x_1, y_1, z_1, w_1), which, equated to zero and regarding therein (x_1, y_1, z_1, w_1) as current coordinates, represents the pair of tangent-planes from the reciprocal line (f, g, h, a, b, c) to the Absolute.

Resuming the question in hand, we have then

$$\xi x_2+\eta y_2+\zeta z_2+\omega w_2=0,$$

which, with

$$\begin{aligned}
\cdot\quad\quad hy_2-gz_2+aw_2&=0,\\
-hx_2\quad\cdot\quad+fz_2+bw_2&=0,\\
gx_2-fy_2\quad\cdot\quad+cw_2&=0,\\
-ax_2-by_2-cz_2\quad\cdot\quad&=0,
\end{aligned}$$

gives, say,

$$\begin{aligned}
-x_2&=\quad\cdot\quad c\eta-b\zeta+f\omega,\\
-y_2&=-c\xi\quad\cdot\quad+a\zeta+g\omega,\\
-z_2&=-b\xi-a\eta\quad\cdot\quad+h\omega,\\
-w_2&=-f\xi-g\eta-h\zeta\quad\cdot\quad,
\end{aligned}$$

that is,

$$\begin{aligned}
x_2&=(b^2+c^2+f^2)\,x_1+(-ab+fg)\,y_1+(-ac+hf)\,z_1+(-cg+bh)\,w_1,\\
y_2&=(-ab+fg)\,x_1+(c^2+a^2+g^2)\,y_1+(-bc+gh)\,z_1+(-ah+cf)\,w_1,\\
z_2&=(-ca+hf)\,x_1+(-bc+gh)\,y_1+(a^2+b^2+h^2)\,z_1+(-bf+ag)\,w_1,\\
w_2&=(-cg+bh)\,x_1+(-ah+cf)\,y_1+(-bf+ag)\,z_1+(f^2+g^2+h^2)\,w_1.
\end{aligned}$$

We have therefore

$$x_1x_2+y_1y_2+z_1z_2+w_1w_2=(b^2+c^2+f^2)\,x_1^2+\&c.,$$

and

$$x_2^2+y_2^2+z_2^2+w_2^2=(a^2+b^2+c^2+f^2+g^2+h^2)\,\{(b^2+c^2+f^2)\,x_1^2+\&c.\},$$

where $(b^2+c^2+f^2)\,x_1^2+\&c.$ denotes in each case the above-mentioned quadric function of (x_1, y_1, z_1, w_1).

In verification of the expression for $x_2^2+y_2^2+z_2^2+w_2^2$, it is to be remarked that we have identically

$$\begin{aligned}
\xi^2+\eta^2+\zeta^2+\omega^2+(af+bg+ch)^2(x_1^2+y_1^2+z_1^2+w_1^2)&\\
=(a^2+b^2+c^2+f^2+g^2+h^2)\,\{(b^2+c^2+f^2)\,x_1^2+\&c.\};&
\end{aligned}$$

here on the left-hand side the whole coefficient of x_1^2 is

$$(b^2+c^2+f^2)^2+(ab-fg)^2+(ca-hf)^2+(cg-bh)^2+(af+bg+ch)^2,$$

where the last four terms are together $=(b^2+c^2+f^2)(a^2+g^2+h^2)$, and thus the whole coefficient is (as it should be) $=(b^2+c^2+f^2)(a^2+b^2+c^2+f^2+g^2+h^2)$: and similarly for the coefficients of the remaining terms.

29. Writing then δ for the required distance, we have

$$\cos\delta = \frac{x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2}{\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2 + w_2^2}},$$

that is,

$$\cos\delta = \frac{\sqrt{(b^2 + c^2 + f^2)\,x_1^2 + \&c.}}{\sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2}\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}},$$

where $(b^2 + c^2 + f^2)\,x_1^2 + \&c.$ is the above-mentioned quadric function

	x_1	y_1	z_1	w_1
x_1	$b^2 + c^2 + f^2$	$-ab + fg$	$-ac + hf$	$-cg + bh$
y_1	$-ab + fg$	$c^2 + a^2 + g^2$	$-bc + gh$	$-ah + cf$
z_1	$-ac + hf$	$-bc + gh$	$a^2 + b^2 + h^2$	$-bf + ag$
w_1	$-cg + bh$	$-ah + cf$	$-bf + ag$	$f^2 + g^2 + h^2$

Distance of a plane and line. Art. No. 30.

30. This may be deduced from the last preceding result: the formula, as written down, gives the distance of the $\perp$ plane (x_1, y_1, z_1, w_1) from the reciprocal line (f, g, h, a, b, c): hence writing $(\xi, \eta, \zeta, \omega)$ for (x_1, y_1, z_1, w_1) and (a, b, c, f, g, h) for (f, g, h, a, b, c), we have for the distance of plane $(\xi, \eta, \zeta, \omega)$ and line (a, b, c, f, g, h) the expression

$$\cos\delta = \frac{\sqrt{(a^2 + g^2 + h^2)\,\xi^2 + \&c.}}{\sqrt{\xi^2 + \eta^2 + \zeta^2 + \omega^2}\sqrt{a^2 + b^2 + c^2 + f^2 + g^2 + h^2}},$$

where $(a^2 + g^2 + h^2)\,\xi^2 + \&c.$ denotes the quadric function

	ξ	η	ζ	ω
ξ	$a^2 + g^2 + h^2$	$ab - fg$	$ac - hf$	$cg - bh$
η	$ab - fg$	$b^2 + h^2 + f^2$	$bc - gh$	$ah - cf$
ζ	$ac - hf$	$bc - gh$	$c^2 + f^2 + g^2$	$bf - ag$
ω	$cg - bh$	$ah - cf$	$bf - ag$	$a^2 + b^2 + c^2$

The theory of two lines. Art. Nos. 31 to 38.

31. Considering any two lines X, Y, it has been seen that these have two $\perp$s, viz. each $\perp$ is a line cutting as well the two lines X, Y as the reciprocal lines X', Y', say that one of them cuts the lines X, Y in the points A, C respectively, and the other of them cuts the two lines in the points B, D respectively: and take, as before, the distances AC and BD to be $=\delta$ and θ respectively.

The coordinates of the lines X, Y are

$$(a, b, c, f, g, h) \text{ and } (a_1, b_1, c_1, f_1, g_1, h_1)$$

respectively; and if we consider, as before, the planes Π, Ω, Π_1, Ω_1 the coordinates of which are (l, m, n, p), $(\lambda, \mu, \nu, \varpi)$, (l_1, m_1, n_1, p_1), $(\lambda_1, \mu_1, \nu_1, \varpi_1)$ respectively, then X is the intersection of the planes Π, Ω, and we have

$$\begin{aligned} &a : b : c : f : g : h \\ &= l\varpi - \lambda p : m\varpi - \mu p : n\varpi - \nu p : m\nu - n\mu : n\lambda - l\nu : l\mu - m\lambda; \end{aligned}$$

and similarly Y is the intersection of the planes Π_1, Ω_1, and we have

$$\begin{aligned} &a_1 : b_1 : c_1 : f_1 : g_1 : h_1 \\ &= l_1\varpi_1 - \lambda_1 p_1 : m_1\varpi_1 - \mu_1 p_1 : n_1\varpi_1 - \nu_1 p_1 : m_1\nu_1 - n_1\mu_1 : n_1\lambda_1 - l_1\nu_1 : l_1\mu_1 - m_1\lambda_1. \end{aligned}$$

Also the planes (Π, Ω), (Π_1, Ω_1), (Π, Ω_1), (Π_1, Ω) being naturally $\perp$, we have

$$\begin{aligned} l\lambda + m\mu + n\nu + p\varpi &= 0, \\ l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1 &= 0, \\ l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1 &= 0, \\ l_1\lambda + m_1\mu + n_1\nu + p_1\varpi &= 0; \end{aligned}$$

and for the inclinations to each other of the planes (Π, Π_1) and (Ω, Ω_1), we have

$$\cos\delta = \frac{\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1}{\sqrt{\lambda^2 + \&c.}\sqrt{\lambda_1^2 + \&c.}},$$

$$\cos\theta = \frac{ll_1 + mm_1 + nn_1 + pp_1}{\sqrt{l^2 + \&c.}\sqrt{l_1^2 + \&c.}}.$$

32. The expressions for the coordinates of the two lines give

$$\begin{aligned} aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1 = &\ (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ &- (l\lambda_1 + m\mu_1 + n\nu_1 + p\varpi_1)(l_1\lambda + m_1\mu + n_1\nu + p_1\varpi) \\ = &\ (ll_1 + mm_1 + nn_1 + pp_1)(\lambda\lambda_1 + \mu\mu_1 + \nu\nu_1 + \varpi\varpi_1) \\ = &\ \sqrt{l^2 + \&c.}\sqrt{l_1^2 + \&c.}\sqrt{\lambda^2 + \&c.}\sqrt{\lambda_1^2 + \&c.}\cos\delta\cos\theta. \end{aligned}$$

But we have

$$\begin{aligned} a^2 + b^2 + c^2 + f^2 + g^2 + h^2 &= (l^2 + m^2 + n^2 + p^2)(\lambda^2 + \mu^2 + \nu^2 + \varpi^2) - (l\lambda + m\mu + n\nu + p\varpi)^2 \\ &= (l^2 + \&c.)(\lambda^2 + \&c.); \end{aligned}$$

and similarly

$$\begin{aligned} a_1^2 + b_1^2 + c_1^2 + f_1^2 + g_1^2 + h_1^2 &= (l_1^2 + m_1^2 + n_1^2 + p_1^2)(\lambda_1^2 + \mu_1^2 + \nu_1^2 + \varpi_1^2) - (l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 + p_1\varpi_1)^2 \\ &= (l_1^2 + \&c.)(\lambda_1^2 + \&c.). \end{aligned}$$

Hence the last result gives

$$\frac{aa_1 + bb_1 + cc_1 + ff_1 + gg_1 + hh_1}{\sqrt{a^2 + \&c.}\sqrt{a_1^2 + \&c.}} = \cos\delta\cos\theta;$$

or calling the expression on the left-hand side the comoment of the two lines, and denoting it by M_1, the equation just obtained is

$$\cos\delta\cos\theta = \text{comoment}, = M_1.$$

And if for either of the lines we substitute its reciprocal, then for δ, θ we have $\frac{1}{2}\pi - \delta$, $\frac{1}{2}\pi - \theta$ respectively, and consequently

$$\frac{af_1 + bg_1 + ch_1 + a_1 f + b_1 g + c_1 h}{\sqrt{a^2 + \&c.}\,\sqrt{a_1^2 + \&c.}} = \sin\delta\sin\theta;$$

or calling the expression on the left-hand side the moment of the two lines and denoting it by M, the equation is

$$\sin\delta\sin\theta = \text{moment}, = M,$$

where observe that $M = 0$ is the condition for the intersection of the two lines, $M_1 = 0$ the condition for their contrasection*.

33. But to determine the coordinates (A, B, C, F, G, H) of the $\perp$ line AC or BD, and the coordinates of the points A and C or B and D of the points in which it meets the lines X and Y respectively, I employ a different method.

We consider the lines

$$(a, b, c, f, g, h), \quad (a_1, b_1, c_1, f_1, g_1, h_1),$$

and their reciprocals

$$(f, g, h, a, b, c), \quad (f_1, g_1, h_1, a_1, b_1, c_1).$$

A line (A, B, C, F, G, H) meeting each of these four lines is said to be a perpendicular. We have

$$\begin{aligned}
(A, B, C, F, G, H)\,&(a, b, c, f, g, h) = 0,\\
\text{,,}\qquad\quad&(f, g, h, a, b, c) = 0,\\
\text{,,}\qquad\quad&(a_1, b_1, c_1, f_1, g_1, h_1) = 0,\\
\text{,,}\qquad\quad&(f_1, g_1, h_1, a_1, b_1, c_1) = 0,
\end{aligned}$$

equations which determine say A, B, C, F in terms of G, H, and then substituting in $AF + BG + CH = 0$ we have two values of $G : H$; i.e. there are two systems of values (A, B, C, F, G, H), that is, two perpendiculars.

The equations may be written

$$\begin{aligned}
(A+F)(a+f) + (B+G)(b+g) + (C+H)(c+h) &= 0,\\
(A+F)(a_1+f_1) + (B+G)(b_1+g_1) + (C+H)(c_1+h_1) &= 0,\\
(A-F)(a-f) + (B-G)(b-g) + (C-H)(c-h) &= 0,\\
(A-F)(a_1-f_1) + (B-G)(b_1-g_1) + (C-H)(c_1-h_1) &= 0.
\end{aligned}$$

* The foregoing demonstration of the fundamental formulæ $\cos\delta\cos\theta = M_1$, $\sin\delta\sin\theta = M$, is, in effect, that given by Heath in his Memoir "On the Dynamics of a Rigid Body in Elliptic Space," *Phil. Trans.* t. 175 (for 1884), pp. 281—324.

Hence we have

$$\begin{array}{ccccc}
A+F & : & B+G & : & C+H, = \\
(b+g)(c_1+h_1)-(b_1+g_1)(c+h) & : & (c+h)(a_1+f_1)-(a+f)(c_1+h_1) & : & (a+f)(b_1+g_1)-(a_1+f_1)(b+g), = \\
\mathfrak{A}+\alpha & : & \mathfrak{B}+\beta & : & \mathfrak{C}+\gamma; \\
A-F & : & B-G & : & C-H, = \\
(b-g)(c_1-h_1)-(b_1-g_1)(c-h) & : & (c-h)(a_1-f_1)-(a-f)(c_1-h_1) & : & (a-f)(b_1-g_1)-(a_1-f_1)(b-g), = \\
\mathfrak{A}-\alpha & : & \mathfrak{B}-\beta & : & \mathfrak{C}-\gamma;
\end{array}$$

equations which may be written

$$A+F,\ B+G,\ C+H = 2\lambda\,(\mathfrak{A}+\alpha,\ \mathfrak{B}+\beta,\ \mathfrak{C}+\gamma),$$
$$A-F,\ B-G,\ C-H = 2\mu\,(\mathfrak{A}-\alpha,\ \mathfrak{B}-\beta,\ \mathfrak{C}-\gamma),$$

where

$$\mathfrak{A} = bc_1 - b_1c + gh_1 - g_1h, \quad \alpha = bh_1 - b_1h - (cg_1 - c_1g),$$
$$\mathfrak{B} = ca_1 - c_1a + hf_1 - h_1f, \quad \beta = cf_1 - c_1f - (ah_1 - a_1h),$$
$$\mathfrak{C} = ab_1 - a_1b + fg_1 - f_1g, \quad \gamma = ag_1 - a_1g - (bf_1 - b_1f).$$

34. We have

$$(\mathfrak{A}+\alpha)^2+(\mathfrak{B}+\beta)^2+(\mathfrak{C}+\gamma)^2 = \{(a+f)^2+(b+g)^2+(c+h)^2\}\,\{(a_1+f_1)^2+(b_1+g_1)^2+(c_1+h_1)^2\}$$
$$-\{(a+f)(a_1+f_1)+(b+g)(b_1+g_1)+(c+h)(c_1+h_1)\}^2,$$
$$(\mathfrak{A}-\alpha)^2+(\mathfrak{B}-\beta)^2+(\mathfrak{C}-\gamma)^2 = \{(a-f)^2+(b-g)^2+(c-h)^2\}\,\{(a_1-f_1)^2+(b_1-g_1)^2+(c_1-h_1)^2\}$$
$$-\{(a-f)(a_1-f_1)+(b-g)(b_1-g_1)+(c-h)(c_1-h_1)\}^2;$$

or putting

$$\rho^2 = a^2+b^2+c^2+f^2+g^2+h^2,$$
$$\rho_1^2 = a_1^2+b_1^2+c_1^2+f_1^2+g_1^2+h_1^2,$$
$$\sigma_1 = aa_1+bb_1+cc_1+ff_1+gg_1+hh_1,$$
$$\sigma = af_1+bg_1+ch_1+a_1f+b_1g+c_1h,$$

the foregoing values are

$$= \rho^2\rho_1^2-(\sigma+\sigma_1)^2, \quad \rho^2\rho_1^2-(\sigma-\sigma_1)^2.$$

Hence

$$A^2+B^2+C^2+F^2+G^2+H^2 = 4\lambda^2\,\{\rho^2\rho_1^2-(\sigma+\sigma_2)^2\} = 4\mu^2\,\{\rho^2\rho_1^2-(\sigma-\sigma_1)^2\};$$

or we may write

$$\lambda^2 = \rho^2\rho_1^2-(\sigma-\sigma_1)^2, \quad \text{or say} \quad \lambda = \sqrt{\rho^2\rho_1^2-(\sigma-\sigma_1)^2},$$
$$\mu^2 = \rho^2\rho_1^2-(\sigma+\sigma_1)^2, \qquad \mu = -\sqrt{\rho^2\rho_1^2-(\sigma+\sigma_1)^2}.$$

Making a slight change of notation, if we put

$$M = \frac{af_1+bg_1+ch_1+a_1f+b_1g+c_1h}{\sqrt{a^2+\&c.}\,\sqrt{a_1^2+\&c.}} = \frac{\sigma}{\rho\rho_1},$$

$$M_1 = \frac{aa_1+bb_1+cc_1+ff_1+gg_1+hh_1}{\sqrt{a^2+\&c.}\,\sqrt{a_1^2+\&c.}} = \frac{\sigma_1}{\rho\rho_1},$$

then the values are

$$\lambda = rr_1\sqrt{1-(M-M_1)^2}, \quad \mu = -rr_1\sqrt{1-(M+M_1)^2}.$$

And, this being so, the two systems of values of A, B, C, F, G, H, are

$$\begin{array}{l|l}
\lambda(\mathfrak{A}+\alpha)+\mu(\mathfrak{A}-\alpha), & \lambda(\mathfrak{A}+\alpha)-\mu(\mathfrak{A}-\alpha), \\
\lambda(\mathfrak{B}+\beta)+\mu(\mathfrak{B}-\beta), & \lambda(\mathfrak{B}+\beta)-\mu(\mathfrak{B}-\beta), \\
\lambda(\mathfrak{C}+\gamma)+\mu(\mathfrak{C}-\gamma), & \lambda(\mathfrak{C}+\gamma)-\mu(\mathfrak{C}-\gamma), \\
\lambda(\mathfrak{A}+\alpha)-\mu(\mathfrak{A}-\alpha), & \lambda(\mathfrak{A}+\alpha)+\mu(\mathfrak{A}-\alpha), \\
\lambda(\mathfrak{B}+\beta)-\mu(\mathfrak{B}-\beta), & \lambda(\mathfrak{B}+\beta)+\mu(\mathfrak{B}-\beta), \\
\lambda(\mathfrak{C}+\gamma)-\mu(\mathfrak{C}-\gamma), & \lambda(\mathfrak{C}+\gamma)+\mu(\mathfrak{C}-\gamma);
\end{array}$$

viz. the two perpendiculars are reciprocals each of the other.

35. Before going further I notice that, if

$$\frac{a_1+f_1}{a+f}=\frac{b_1+g_1}{b+g}=\frac{c_1+h_1}{c+h} \quad \text{or} \quad \frac{a_1-f_1}{a-f}=\frac{b_1-g_1}{b-g}=\frac{c_1-h_1}{c-h},$$

then the four equations for (A, B, C, F, G, H) reduce themselves to three equations only: and thus instead of two perpendiculars we have a singly infinite series of perpendiculars (see *ante* 15).

To explain the meaning of the equations, I observe that a line (a, b, c, f, g, h) will be a generating line of the one kind, or say a "generatrix," of the Absolute if

$$a+f=0, \quad b+g=0, \quad c+h=0:$$

and it will be a generating line of the other kind, or say a "directrix," of the Absolute if $a-f=0$, $b-g=0$, $c-h=0$. Or what is the same thing, we have

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, -\mathrm{a}, -\mathrm{b}, -\mathrm{c}), \text{ where } \mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=0 \text{ for a generatrix,}$$

and

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{a}, \mathrm{b}, \mathrm{c}), \text{ where } \mathrm{a}^2+\mathrm{b}^2+\mathrm{c}^2=0 \text{ for a directrix, of the Absolute.}$$

Consider now two directrices $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{a}, \mathrm{b}, \mathrm{c})$ and $(\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1)$: if a line (a, b, c, f, g, h) meets each of these, then

$$(a+f)\,\mathrm{a}+(b+g)\,\mathrm{b}+(c+h)\,\mathrm{c}=0,$$
$$(a+f)\,\mathrm{a}_1+(b+g)\,\mathrm{b}_1+(c+h)\,\mathrm{c}_1=0,$$

and consequently

$$a+f : b+g : c+h = \mathrm{bc}_1-\mathrm{b}_1\mathrm{c} : \mathrm{ca}_1-\mathrm{c}_1\mathrm{a} : \mathrm{ab}_1-\mathrm{a}_1\mathrm{b};$$

and similarly if $(a_1, b_1, c_1, f_1, g_1, h_1)$ meets each of the two directrices, then

$$a_1+f_1 : b_1+g_1 : c_1+h_1 = \mathrm{bc}_1-\mathrm{b}_1\mathrm{c} : \mathrm{ca}_1-\mathrm{c}_1\mathrm{a} : \mathrm{ab}_1-\mathrm{a}_1\mathrm{b},$$

that is, if the lines each of them meet the same two directrices of the Absolute, then

$$\frac{a_1+f_1}{a+f}=\frac{b_1+g_1}{b+g}=\frac{c_1+h_1}{c+h}.$$

Conversely, if these relations are satisfied, then the lines each of them meet two directrices of the Absolute.

In like manner, if the lines each meet two generatrices of the Absolute, then

$$\frac{a_1-f_1}{a-f}=\frac{b_1-g_1}{b-g}=\frac{c_1-h_1}{c-h};$$

and conversely, if these relations are satisfied, then the lines each of them meet the same two generatrices of the Absolute. In the former case, the lines are said to be "right parallels": in the latter case, "left parallels."

A line (a, b, c, f, g, h) meets the Absolute in two points, and through each of these we have a directrix and a generatrix: that is, the line meets two directrices and two generatrices.

Through a given point we may draw, meeting the two directrices, or meeting the two generatrices, a line: that is, through a given point we may draw a line

$$(a_1, b_1, c_1, f_1, g_1, h_1)$$

which is a right parallel, and a line which is a left parallel, to a given line. That is, regarding as given the first line, and also a point of the second line, there are two positions of the second line such that for each of them, the $\perp$s of the pair of lines, instead of being two determinate lines, are a singly infinite series of lines.

36. Reverting to the general case, we have found (A, B, C, F, G, H) the coordinates of either of the lines $\perp$ to the given lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$: supposing that the $\perp$ intersects the first of these lines in the point the coordinates of which are (x, y, z, w), and the second in the point the coordinates of which are

$$(x_1, y_1, z_1, w_1),$$

then we have for each set of coordinates a fourfold expression; the choice of the form is indifferent, and I write

$$x : y : z : w = cB-bC : aC-cA : bA-aB : fA+gB+hC,$$
$$x_1 : y_1 : z_1 : w_1 = c_1B-b_1C : a_1C-c_1A : b_1A-a_1B : f_1A+g_1B+h_1C.$$

We have then, for the distance of these two points,

$$\cos\phi=\frac{xx_1+yy_1+zz_1+ww_1}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}},\quad \sin\phi=\frac{\sqrt{(yz_1-y_1z)^2+\&c.}}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}};$$

where $\phi=\delta$ or θ, according to the sign of the radical $\lambda:\mu$ contained in the expressions for A, B, C, F, G, H.

I have not succeeded in obtaining in this manner the final formulæ for the determination of the distances: these in fact are, by what precedes, given by the equations

$$\sin\delta\sin\theta=M,\quad \cos\delta\cos\theta=M_1.$$

For then, writing ϕ to denote either of the distances δ, θ, at pleasure, we have

$$\frac{M^2}{\sin^2\phi}+\frac{M_1^2}{\cos^2\phi}=1,$$

that is,

$$\cos^4\phi + \cos^2\phi\,(M_1^2 - M^2 + 1) + M_1^2 = 0,$$

or

$$\cos^2\phi = \tfrac{1}{2}\{M_1^2 - M^2 + 1 \pm \sqrt{M_1^4 + M^4 - 2M_1^2M^2 - 2M_1^2 - 2M^2 + 1}\},$$

which is the expression for the cosine of the distance.

In the case where the two lines intersect $M=0$, and if δ be the $\perp$ distance which vanishes, then $\delta=0$, and consequently $\cos\theta = M_1$: the last-mentioned formula, putting therein $M=0$ and taking the radical to be $=M_1^2-1$, gives $\cos^2\phi = M_1^2$, that is, $\phi=\theta$, and $\cos^2\theta = M_1^2$, as it should be.

37. I verify as follows, in the case in question of two *intersecting* lines,

$$(af_1 + bg_1 + ch_1 + a_1f + b_1g + c_1h = 0),$$

the formula

$$\cos\theta = \frac{xx_1 + yy_1 + zz_1 + ww_1}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}}.$$

We have here

$$\begin{aligned}
A &= \mathfrak{A} = bc_1 - b_1c + gh_1 - g_1h,\\
B &= \mathfrak{B} = ca_1 - c_1a + hf_1 - h_1f,\\
C &= \mathfrak{C} = ab_1 - a_1b + fg_1 - f_1g,\\
F &= \alpha = bh_1 - b_1h - cg_1 + c_1g,\\
G &= \beta = cf_1 - c_1f - ah_1 + a_1h,\\
H &= \gamma = ag_1 - a_1g - bf_1 + b_1f.
\end{aligned}$$

I stop to notice that these formulæ may be obtained in a different and somewhat more simple manner: the two lines (a, b, c, f, g, h) and $(a_1, b_1, c_1, f_1, g_1, h_1)$ intersect; hence their reciprocals also intersect: the equation of the plane through the two lines and that of the plane through the two reciprocal lines are respectively

$$\begin{aligned}
&(gh_1 - g_1h)\,x + (hf_1 - h_1f)\,y + (fg_1 - f_1g)\,z + (af_1 + bg_1 + ch_1)\,w = 0,\\
&(bc_1 - b_1c)\,x + (ca_1 - c_1a)\,y + (ab_1 - a_1b)\,z + (fa_1 + gb_1 + hc_1)\,w = 0:
\end{aligned}$$

the line (A, B, C, F, G, H) is thus the line of intersection of these two planes, and it is thence easy to obtain the foregoing values.

From the values of A, B, C, F, G, H, we have to find x, y, z, w and x_1, y_1, z_1, w_1 by the formulæ given above. We have

$$\begin{aligned}
x = cB - bC = \;& c^2a_1 - cc_1a + chf_1 - ch_1f\\
& - abb_1 + a_1b^2 - bfg_1 + bgf_1\\
= \;& (bg + ch)f_1 + a_1(b^2 + c^2) - b_1ab - c_1ac - bfg_1 - cfh_1\\
= \;& -f(af_1 + bg_1 + ch_1) + a_1(b^2 + c^2) - b_1ab - c_1ac;
\end{aligned}$$

or writing here $a_1f + b_1g + c_1h$ in place of $-(af_1 + bg_1 + ch_1)$, this is a linear function of a_1, b_1, c_1; and similarly finding the values of y, z, w, we have

$$\begin{aligned}
x &= a_1(b^2 + c^2 + f^2) + b_1(fg - ab) + c_1(hf - ca),\\
y &= a_1(fg - ab) + b_1(c^2 + a^2 + g^2) + c_1(gh - bc),\\
z &= a_1(hf - ca) + b_1(gh - bc) + c_1(a^2 + b^2 + h^2),\\
w &= a_1(bh - cg) + b_1(cf - ah) + c_1(ag - bf).
\end{aligned}$$

And in like manner, (I introduce for convenience the sign $-$, as is allowable),

$$\begin{aligned}
-x_1 &= a(b_1^2+c_1^2+f_1^2)+b(f_1g_1-a_1b_1)+c(h_1f_1-c_1a_1),\\
-y_1 &= a(f_1g_1-a_1b_1)+b(c_1^2+a_1^2+g_1^2)+c(g_1h_1-b_1c_1),\\
-z_1 &= a(h_1f_1-c_1a_1)+b(g_1h_1-b_1c_1)+c(a_1^2+b_1^2+h_1^2),\\
-w_1 &= a(b_1h_1-c_1g_1)+b(c_1f_1-a_1h_1)+c(a_1g_1-b_1f_1).
\end{aligned}$$

38. Write for shortness

$$p=a^2+b^2+c^2,\qquad p_1=f^2+g^2+h^2,\qquad \omega=a_1f+b_1g+c_1h,$$

and therefore

$$q=aa_1+bb_1+cc_1,\quad q_1=ff_1+gg_1+hh_1,\quad -\omega=af_1+bg_1+ch_1,$$
$$r=a_1^2+b_1^2+c_1^2,\quad r_1=f_1^2+g_1^2+h_1^2.$$

We have

$$\begin{aligned}
x&=a_1p-aq+f\omega, & x_1&=-ar+a_1q+f_1\omega,\\
y&=b_1p-bq+g\omega, & y_1&=-br+b_1q+g_1\omega,\\
z&=c_1p-cq+h\omega, & z_1&=-cr+c_1q+h_1\omega,\\
w&=-\begin{vmatrix} f, & g, & h\\ a, & b, & c\\ a_1, & b_1, & c_1\end{vmatrix}, & w_1&=-\begin{vmatrix} f_1, & g_1, & h_1\\ a, & b, & c\\ a_1, & b_1, & c_1\end{vmatrix};
\end{aligned}$$

from which we easily obtain

$$x^2+y^2+z^2=p(pr-q^2)+(p_1+2p)\,\omega^2,$$

and by expressing w^2 in the form of a determinant

$$w^2=p_1(pr-q^2)-p\omega^2,$$

we obtain

$$x^2+y^2+z^2+w^2=(p+p_1)(pr-q^2+\omega^2);$$

and in like manner

$$x_1^2+y_1^2+z_1^2+w_1^2=(r+r_1)(pr-q^2+\omega^2).$$

And again

$$xx_1+yy_1+zz_1=q(pr-q^2)+(q_1+2q)\,\omega^2,$$

and by expressing ww_1 in the form of a determinant

$$ww_1=q_1(pr-q^2)-q\omega^2,$$

we find

$$xx_1+yy_1+zz_1+ww_1=(q+q_1)(pr-q^2+\omega^2).$$

Hence substituting in

$$\cos\theta=\frac{xx_1+yy_1+zz_1+ww_1}{\sqrt{x^2+y^2+z^2+w^2}\sqrt{x_1^2+y_1^2+z_1^2+w_1^2}},$$

the factor $pr-q^2+\omega^2$ disappears, and we have

$$\cos\theta=\frac{q+q_1}{\sqrt{p+p_1}\sqrt{r+r_1}}=M_1,$$

the required result.

952.

ON THE KINEMATICS OF A PLANE, AND IN PARTICULAR ON THREE-BAR MOTION: AND ON A CURVE-TRACING MECHANISM.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XV. (1894), pp. 391—402.]

THE first part of the present paper, On the Kinematics of a Plane, and on Three-bar Motion, is purely theoretical: the second part contains a brief description of a Curve-tracing Mechanism, which at my suggestion has been constructed by Prof. Ewing in the workshops of the Engineering Laboratory, Cambridge.

PART I.

1. The theory of the motion of a plane, when two given points thereof describe given curves, has been considered by Mr S. Roberts in his paper, "On the motion of a plane under given conditions," *Proc. Lond. Math. Soc.* t. III. (1871), pp. 286—318, and he has shown that, if for the given curves the order, class, number of nodes, and of cusps, are (m, n, δ, κ) and $(m', n', \delta', \kappa')$ respectively $(n = m^2 - m - 2\delta - 3\kappa,\ n' = m'^2 - m' - 2\delta' - 3\kappa')$, then for the curve described by any fixed point of the plane:

$$\begin{aligned}
&\text{order} &&= 2mm',\\
&\text{class} &&= 2(mm' + mn' + nm'),\\
&\text{number of nodes} &&= mm'(2mm' - m - m') + 2(m\delta' + m'\delta),\\
&\text{number of cusps} &&= 2(m\kappa' + m'\kappa);
\end{aligned}$$

but he remarks that these formulæ require modification when the directrices or either of them pass through the circular points at infinity. And he has considered the case where the two directrices become one and the same curve.

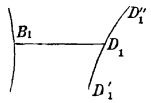
2. It will be convenient to speak of the line joining the two given points as the link; the two given points, say B and D, are then the extremities of the link; and I take the length of the link to be $= c$, and the two directrices to be b and d; we have thus the link $c = BD$ moving in suchwise that its extremity B describes the curve b of the order m, and its extremity D the curve d of the order m': in Mr Roberts' problem, the locus is that described by a point P rigidly connected with the link, or say by a point P the vertex of the triangle PBD.

3. The points B, D describe of course the directrices b, d respectively: taking on b a point B_1 at pleasure, then if B be at B_1, the corresponding positions of D are the intersections of d by the circle centre B_1 and radius c, viz. there are thus $2m'$ positions of D: and similarly taking on d a point D_1 at pleasure, then if D be at D_1, the corresponding positions of B are the intersections of b by the circle centre D_1 and radius c, viz. there are thus $2m$ positions of B. The motion thus establishes a $(2m, 2m')$ correspondence between the points of the directrices b and d, viz. to a given point on b there correspond $2m'$ points on d, and to a given point on d there correspond $2m$ points on b. Of course, for a given point on either directrix, the corresponding points on the other directrix may be any or all of them imaginary; and thus it may very well be that for either directrix not the whole curve but only a part or detached parts thereof will be actually described in the course of the motion. In saying that a part is described, we mean described by a continuous motion; say that the point B (the point D remaining always on a part of d) is capable of describing continuously a part of b; it may very well happen that the point B (the point D remaining always on a different part of d) is capable of describing continuously a different part of b, but that it is not possible for B to pass from the one to the other of these parts of b without removing D from the one part and placing it on the other part of d, and thus that we have on b detached parts each of them continuously described by B; and similarly, we may have on d detached parts each of them continuously described by D.

4. But dropping for the moment the question of reality, to a given position of B on b there correspond as was mentioned $2m'$ positions of D on d, or say $2m'$ positions of the link c: in the entire motion of the link it must assume each of these $2m'$ positions, and for each of them the point B comes to assume the position in question on b; the directrix b is thus described $2m'$ times, that is, the locus described by B will be the directrix b repeated $2m'$ times, or say a curve of the order $m \times 2m', = 2mm'$. Similarly, the locus described by D will be the directrix d repeated $2m$ times, or say a curve of the order $m' \times 2m, = 2mm'$.

5. In general, if B_1D_1 be any position of the link and if B moves from B_1 along b in a determinate sense, then D will move from D_1 along d in a determinate sense; and if B moves from B_1 along b in the opposite sense, then also D will move from D_1 along d in the opposite sense. Or what is the same thing, we may have B moving in a determinate sense through B_1, and D moving in a determinate sense through D_1; and reversing the sense of B's motion, we reverse also the sense of D's motion. But there are certain critical positions of the link, viz. we have a critical position when

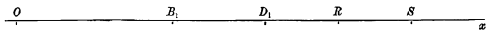
the link is a normal at B_1 to the directrix b, or a normal at D_1 to the directrix d. Say first the link is a normal at B_1 to the directrix b. The infinitesimal element at B_1 may be regarded as a straight line at right angles to the link; hence if for a moment D_1 is regarded as a fixed point, the link may rotate in either direction round D_1, that is, B may move from B_1 along b in either of the two opposite senses, say B_1 is a "two-way point." But if on d we take on opposite sides of D_1 the consecutive points D_1' and D_1'', say $D_1'D_1$ cuts D_1B_1 at an acute angle and $D_1''D_1$ cuts it at an



obtuse angle, then D_1' will be nearer to b than was D_1, and thus the circle centre D_1' and radius c will cut b in two real points B_1' and B_1'' near to and on opposite sides of B_1; or as D moves to D_1', B will move from B_1 indifferently to B_1' or B_1''. Contrariwise, D_1'' is further from b than was D_1, and thus the circle centre D_1'' and radius c, will not meet b in any real point near to B_1, and hence D is incapable of moving from D_1 in the sense D_1D_1''. Or what is the same thing, the described portion of d, which includes a point D_1', will terminate at D_1, or say D_1 is a "summit" on the directrix d. We have thus a summit on d, corresponding to the two-way point on b. And of course in like manner, if the link is a normal at D_1 to the directrix d, then D_1 is a two-way point on d, and the corresponding point B_1 is a summit on b.

6. If the link is at the same time a normal at B_1 to b and at D_1 to d, then each of the points B_1, D_1 is a two-way point and also a summit; or more accurately, each of them is a two-way point and also a pair of coincident summits.

But the case requires further investigation. Considering the position B_1D_1 as given, we may take the axis of x coincident with this line, and the origin O in suchwise



that OB_1, OD_1 are each positive and $OD_1 > OB_1$; say we have $OD_1 = \delta$, $OB_1 = \beta$, and therefore $\delta - \beta = c$. The equation of the curve b in the neighbourhood of B_1 is $y^2 = 2\rho(x - \beta)$, where ρ is the radius of curvature at B_1, assumed to be positive when the curve is convex to O, or what is the same thing when the centre of curvature R lies to the right of B_1 $(OR - OB_1 = +)$; and similarly the equation of d in the neighbourhood of D_1 is $y^2 = 2\sigma(x - \delta)$, where σ is the radius of curvature at D_1 assumed to be positive when the curve is convex to O, or what is the same thing when the centre of curvature S lies to the right of D_1 $(OS - OD_1 = +)$.

Consider now (x_1, y_1) the coordinates of a point on b in the neighbourhood of B_1, $y_1^2 = 2\rho(x_1 - \beta)$, and taking B at this point, let (x_2, y_2) be the coordinates of the corresponding point D on d in the neighbourhood of D_1, $y_2^2 = 2\sigma(x_2 - \delta)$. We have

$$c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

and here

$$x_1 = \beta + \frac{y_1^2}{2\rho}, \quad x_2 = \delta + \frac{y_2^2}{2\sigma},$$

whence

$$x_1^2 = \beta^2 + \frac{\beta y_1^2}{\rho}, \quad x_1 x_2 = \beta\delta + \tfrac{1}{2}\frac{\delta y_1^2}{\rho} + \tfrac{1}{2}\frac{\beta y_2^2}{\sigma}, \quad x_2^2 = \delta^2 + \frac{\delta y_2^2}{\sigma}.$$

The equation thus becomes

$$(\delta - \beta)^2 + \frac{y_1^2}{\rho}(\beta - \delta) + \frac{y_2^2}{\sigma}(\delta - \beta) + (y_1 - y_2)^2 = c^2,$$

that is,

$$y_1^2\left(1 + \frac{\beta - \delta}{\rho}\right) - 2y_1 y_2 + y_2^2\left(1 + \frac{\delta - \beta}{\sigma}\right) = 0,$$

a quadric equation between y_1 and y_2. Evidently if we had taken D a point on d, coordinates (x_2, y_2), in the neighbourhood of D_1 and had sought for the coordinates (x_1, y_1) of the corresponding point B on b in the neighbourhood of B_1, we should have found the same equation between y_1 and y_2.

7. The equation will have real roots if

$$1 > \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right);$$

viz. ρ, σ having the same sign, this is

$$\rho\sigma > (\rho + \beta - \delta)(\sigma + \delta - \beta):$$

but ρ, σ having opposite signs, then

$$\rho\sigma < (\rho + \beta - \delta)(\sigma + \delta - \beta).$$

These conditions may be written

$$(OR - OB_1)(OS - OD_1) - (OS - OB_1)(OR - OD_1) > \text{ or } < 0,$$

that is,

$$(OS - OR)(OD_1 - OB_1) > \text{ or } < 0.$$

But we have $OD_1 - OB_1 = +$, and therefore, ρ, σ having the same sign, the condition of reality is $OS > OR$, i.e. S to the right of R; but ρ, σ having opposite signs, the condition of reality is $OS < OR$, i.e. S to the left of R. Observe that, S lying to the left of R, we cannot have $\rho = -$, $\sigma = +$, and that the second alternative thus is $\rho = +$, $\sigma = -$, then $OS < OR$, or S lies to the left of R.

The condition was investigated as above in order to exhibit more clearly the geometrical signification: but of course the original form, or say the equation

$$1 - \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right) > 0,$$

gives at once

$$\frac{\delta - \beta}{\rho\sigma}(\delta + \sigma - \beta - \rho) > 0.$$

8. Writing the quadric equation in the form

$$y_1^2\left(1-\frac{c}{\rho}\right)-2y_1y_2+\left(1+\frac{c}{\sigma}\right)y_2^2=0,$$

we have

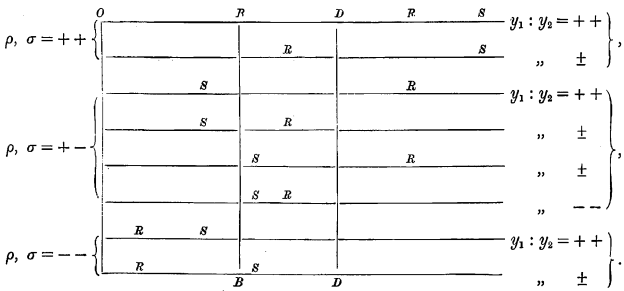
$$\left(1-\frac{c}{\rho}\right)y_1=\left\{1\pm\sqrt{\frac{c}{\rho\sigma}(c+\sigma-\rho)}\right\}y_2;$$

the two values of $y_1 : y_2$ will have the same sign or opposite signs according as $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ have the same sign or opposite signs, and in the case where these have the same sign, then this is also the sign of each of the two values of $y_1 : y_2$. Or what is the same thing, if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ are each of them positive, then the two values of $y_1 : y_2$ are each of them positive; if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ are each of them negative, then the two values of $y_1 : y_2$ are each of them negative; and if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ have opposite signs, then the two values of $y_1 : y_2$ have opposite signs. Considering the different cases $\rho, \sigma = ++, +-, --$, we find

$\rho, \sigma = ++$,	then values of $y_1 : y_2$	are	$++$	or	$--$,	according as	DR, BS	are $++$ or $--$,
$\rho, \sigma = +-$	"	"	"	"	"	DR, SB	"	"
$\rho, \sigma = --$	"	"	"	"	"	RD, SB	"	"

and in each case the values of $y_1 : y_2$ are $+-$, if the two distances referred to have opposite signs: $DR = +$ means that R is to the right of, or beyond, D, and so in other cases.

9. The different cases, two real roots as above, are

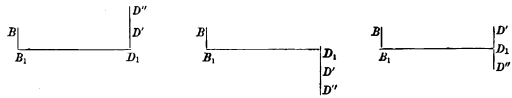


Obviously the cases $\rho, \sigma = --$, correspond exactly to the cases $\rho, \sigma = +, +$; the only difference is that the concavities, instead of the convexities, of the two curves are turned towards the point O.

10. If the two roots of the quadratic equation are imaginary, then B_1D_1 is a conjugate or isolated position of the link, and B_1, D_1 are isolated points on the curves b and d respectively.

11. If the roots are real, then the three cases $y_1 : y_2 = ++$, $--$ and $+-$, may be delineated as in the annexed figures, viz. taking in each case y_1 as positive, that is, imagining B to move upwards from B_1 through an infinitesimal arc of b, then D moves from D_1 through either of two infinitesimal arcs of d, both upwards, both downwards, or the one upwards and the other downwards, as shown in the figures

$y_1 : y_2 = ++$ $\qquad$ $y_1 : y_2 = --$ $\qquad$ $y_1 : y_2 = \pm$

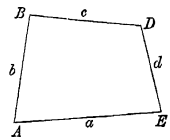


and where it is to be observed that, reversing the sense of the motion of B from B_1, we reverse also the senses of the motion of D from D_1: moreover that, considering D as moving through an infinitesimal arc of d from D_1, we have the like relations thereto of the two infinitesimal arcs of b described by B from B_1. Thus the points B_1 and D_1 are singular points of like character.

If $y_1 : y_2 = ++$, we may say that B_1 (or D_1) is a for-forwards point; if $y_1 : y_2 = --$, then that B_1 (or D_1) is a back-backwards point; and if $y_1 : y_2 = \pm$, then that B_1 (or D_1) is a back-forwards point.

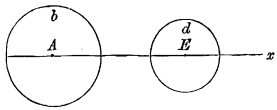
12. The separating case between two imaginary roots and two real roots is that of two equal real roots: the condition for this is $\delta + \sigma = \beta + \rho$, that is, $OS = OR$, or the two centres of curvature are coincident; the characters of the points B_1 and D_1 would in this case depend on the aberrancies of curvature of the curves b and d at these points respectively. If each of the curves is a circle, then the curves are concentric circles, and the link BD moves in suchwise that its direction passes always through the common centre of the two circles—or say so that BD is always a radius of the annulus formed by the two circles—and for any position of BD, the two extremities B, D are related to each other in like manner with the points B_1 and D_1. Thus, in this case, there are no singular points B_1 and D_1 to be considered.

13. In the case where the curves b, d are circles, we have three-bar motion: say



the figure is as here shown; I take in it b, d for the radii of the two circles respectively and a for the distance of their centres; viz. we have the link $BD = c$, pivoted at its

extremities to the arms or radii $AB = b$, and $ED = d$, which rotate about the fixed centres A, E at a distance from each other $= a$. Here a, b, c, d are each of them positive; a, b, d may have any values, but then c is at most $= a + b + d$, and if $a > b + d$ then c is at least $= a - b - d$; but if $a =$ or $< b + d$, then c may be $= 0$, viz. it may have any value from 0 to $a + b + d$. And in either case there will be critical values of c. The cases are very numerous. To make an exhaustive enumeration, we may assume d at most $= b$, and in each of the two cases $d < b$ and $d = b$, considering the centre of the circle d as moving from the right of the centre of the circle b towards this centre, we may in the first instance divide as follows:



$d < b$	$d = b$
⊙ d exterior to ⊙ b,	⊙ d exterior to ⊙ b,
„ touches it externally,	„ touches it externally,
„ cuts it,	„ cuts it,
„ touches it internally,	„ is concentric and thus coincident with it;
„ lies within it,	
„ is concentric with it,	

and then, in each of these cases, give to the length c of the link its different admissible values.

14. Considering the case $d < b$, then we have (see Plate I., at p. 516), exterior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

fig.	1,	$c = a - b - d$,
	1—2,	„ intermediate,
	2,	$c = a - b + d$,
	2—3,	„ intermediate,
	3,	$c = a + b - d$,
	3—4,	„ intermediate,
	4,	$c = a + b + d$.

15. In figure 1, the curves described by the extremities B and D respectively are each of them a mere point.

In figure 1—2, we have $a + d > b + c$ and $a + b > d + c$. Hence in the course of the motion the arms b, c come into a right line, giving a position B_1D_1' of the link, where B_1 is a two-way point on b and D_1' a summit on d; or rather, there are two

such positions symmetrically situate on opposite sides of the axis Ax. And again, in the course of the motion, the arms d, c come into a right line, giving a position $B_1'D_1$, where D_1 is a two-way point on d and B_1' a summit on b; or rather, there are two such positions symmetrically situate on opposite sides of the axis Ax. Only an arc of the circle b is described, viz. the arc adjacent to d included between the two summits B_1' on b; and in like manner, only an arc of the circle d is described, viz. the arc adjacent to b included between the two summits D_1' on d. The described portions on b and d respectively are to be regarded each of them as a double line or indefinitely thin bent oval: and it is to be observed that for a given position of B (or D) there are two positions of the link BD, each of these positions being assumed by the link in the course of its motion.

16. In figure 2, the two positions B_1D_1' of the link come to coincide together in a single axial position BD, but we still have the other two positions $B_1'D_1$ of the link, where B_1' is a summit on b, and D_1 a two-way point on d. As regards BD, this is the configuration ρ, $\sigma = - -$, R, B, S, D; $y_1 : y_2 = \pm$, and thus each of the axial points B, D is a back-and-forwards point. Thus only the arc $B'B_1'$ of the circle b is described by the point B, but the whole circumference of the circle d is described by the point D. If we further examine the motion it will appear that, as B moves from the axial point B say to the upper summit B_1' and returns to B, then D starting from the axial point D may describe (*and that in either sense*, viz. $y_1 = +$, then we have $y_2 = \pm$) the entire circumference of d, returning to the axial point D; and similarly, as B moves from the axial point B to the lower summit B_1' and returns to B, then D starting as before from the axial point D may describe (*and that in either sense*, viz. $y_1 = -$, then we have $y_2 = \pm$) the entire circumference of d, returning to the axial point D. It is thus not the entire arc $B_1'B_1'$ but each of the half-arcs BB_1' which corresponds, and that in either of two ways, to the circumference of d.

17. In figure 2—3, there are four critical positions $B_1'D_1$ (forming two pairs, those of the same pair situate symmetrically on opposite sides of the axis Ax) where, as before, B_1' is a summit on b, and D_1 a two-way point on d. The described portions of b are the detached arcs $B_1'B_1'$ between the two upper summits, and $B_1'B_1'$ between the two lower summits: the described portion of d is the whole circumference. In fact, attending to one of the arcs on b, say the upper arc $B_1'B_1'$, as B moves from one of the summits, say the left-hand summit B_1', and then returns to the left-hand summit B_1', then D, starting from the corresponding two-way point D_1, may describe, *and that in either sense*, the entire circumference of d, returning to the same point D_1; and similarly, as B describes the lower arc $B_1'B_1'$, starting from and returning to a summit, then D, starting from the corresponding two-way point D_1, may describe, *and that in either sense*, the entire circumference of d, returning to the same two-way point D_1.

18. In figure 3, two of the positions $B_1'D_1$ have come to coincide together in the axial position BD: but we still have the other two positions $B_1'D_1$, where B_1' is a summit on b, and D_1 a two-way point on d. As regards the axial points B, D, this is the configuration ρ, $\sigma = + +$; B, R, D, S; $y_1 : y_2 = \pm$, viz. each of the points B, D is a back-and-forwards point. The two detached arcs $B_1'B_1'$ of b have united themselves into a single arc $B_1'B_1'$, which is the described portion of b; the described portion of

d is, as before, the entire circumference. It is to be observed (as in fig. 2) that properly it is not the entire arc $B_1'B_1'$, but each of the half-arcs BB_1', which corresponds to the entire circumference of d.

19. The figure 3—4 closely corresponds to fig. 1—2, the only difference being that the arcs $B_1'B_1'$ and $D_1'D_1'$, which are the described portions of b and d respectively, (instead of being the nearer portions, or those with their convexities facing each other) are the further portions, or those with their concavities facing each other, of the two circles respectively.

Finally, in fig. 4, the described portions of the two circles reduce themselves to the axial points B and D respectively.

20. Still assuming $d < b$, and passing over the case of external contact, we come to that in which the circles intersect each other; but this case has to be subdivided. Since the circles intersect, we have $b + d > a$, consistently herewith we may have:—

b, d each $< a$,	A, E each outside the lens common to the two circles,
$b = a$, $d < a$,	A outside, E on boundary of the lens,
$b > a$, $d < a$,	A outside, E inside the lens,
$b > a$, $d = a$,	A on boundary of, E inside the lens,
b, d each $> a$,	A, E, each inside the lens;

and in each case we have to consider the different admissible values of c. I omit the discussion of all these cases.

21. Still assuming $d < b$, and passing over the case of internal contact, we come to that of the circle d included within the circle b: we have here again a subdivision of cases; viz. we may have $d > a$, that is, A inside d; $d = a$, that is, A on the circumference of d; or $d < a$, that is, A outside d. The critical values of c, arranged in order of increasing magnitude in these three cases respectively, are:—

$d > a$	$d = a$	$d < a$
$b - d - a$,	$b - 2d$,	$b - d - a$,
$b - d + a$,	b,	$b + d - a$,
$b + d - a$,	b,	$b - d + a$,
$b + d + a$,	$b + 2d$,	$b + d + a$.

I attend only to the first case; we have here (see Plate II., at p. 516), interior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

fig.	1	$c = b - d - a$,
	1—2	„ intermediate,
	2	$c = b - d + a$,
	2—3	„ intermediate,
	3	$c = b + d - a$,
	3—4	„ intermediate,
	4	$c = b + d + a$.

22. In figure 1, the curves described by the points B_1D are each of them a mere point. In figure 1—2, we have two critical positions $B_1'D_1$ situate symmetrically on opposite sides of the axis, B_1' being a summit on b, and D_1 a two-way point on d, and moreover two critical positions B_1D_1' situate symmetrically on opposite sides of the axis, B_1 being a two-way point on b, and D_1' a summit on d. The described portion of b is the arc $B_1'B_1'$, and the described portion of d is the arc $D_1'D_1'$, these two arcs being thus the nearer portions of the two circles respectively.

23. In figure 2, the four critical positions coalesce all of them in the axial position BD; the described portions are thus the entire circumferences of the two circles respectively. This is a remarkable case. The configuration is ρ, $\sigma = ++$; B, D, R, S; $y_1 : y_2 = ++$. Imagine D to move from the axial point D in a given sense round the circle d, say with uniform velocity, then B moves from the axial point B in the same sense *but with either of two velocities* round the circle b; one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately decreases rapidly, so that the two revolutions of B from the axial point B round the entire circumference to the axial point B correspond each of them to the revolution of D from the axial point D round the entire circumference to the axial point D. And similarly, if we imagine B to move in a given sense from the axial point B round the circle b, say with uniform velocity, then D moves from the axial point D in the same sense but with either of two velocities round the circle d: one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately decreases rapidly, so that the two revolutions from the axial point D round the entire circumference of d to the axial point D correspond each of them to the revolution from the axial point B round the entire circumference of b to the axial point B.

24. In figure 2—3, there are no critical positions; the described portions of the circles b, d are the entire circumferences of the two circles respectively, these being described in the same sense, by the points B and D respectively. It is to be observed that, to a given position of B on b, there correspond two positions of D on d, or say two positions of the link, but the link does not in the course of its motion pass from one of these positions to the other; the motions are separate from each other, and may be regarded as belonging to different configurations of the system. And of course in like manner, to a given position of D on d, there correspond two positions of B on b, or say two positions of the link: we have thus the same two separate motions.

25. In figure 3, the critical axial position BD of the link makes its appearance: the described portions are still the entire circumferences of the two circles respectively. As the point D is here to the left of the point B, we must take the origin O to the right of B, and reverse the direction of the axis Ox; the configuration is thus ρ, $\sigma = +-$, B, S, R, D; $y_1 : y_2 = --$. Everything is the same as in fig. 2 except (the signs of $y_1 : y_2$ being, as just mentioned, $--$) that the motions in the circles b and d instead of being in the same sense are in opposite sense, viz. as D moves from the axial point D in a given sense round the circle d to the axial point D say with uniform velocity, then B moves from the axial point B round the circle b in the opposite sense, *and with either of two velocities;* and similarly, as B moves from the

axial point B in a given sense round the circle b say with uniform velocity, then D moves from the axial point D round the circle d in the opposite sense, *and with either of two velocities.*

26. In figure 3—4, we have again the two critical positions $B_1'D_1$ symmetrically situate on opposite sides of the axis, B_1' a summit on b, D_1 a two-way point on d: and also the two critical positions B_1D_1' symmetrically situate on opposite sides of the axis, B_1 a two-way point on b, D_1' a summit on d. The described portion of b is the arc $B_1'B_1'$, and the described portion of d the arc $D_1'D_1'$, these arcs being thus the further portions of the two circles respectively.

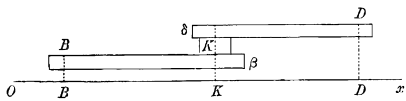
Finally, in figure 4, the described portions reduce themselves to the two points B, D respectively.

27. The several forms for $d=b$ can be at once obtained from those for $d<b$; the only difference is that several intermediate forms disappear, and the entire series of divisions is thus not quite so numerous.

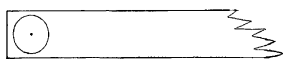
Part II.

1. The curve-tracing mechanism was devised with special reference to the curves of three-bar motion, viz. the object proposed was that of tracing the curve described by a point K of the link BD, the extremities whereof B and D describe given circles respectively, or more generally by a point K, the vertex of a triangle KBD, whereof the other vertices B and D describe given circles respectively, and that in suchwise that the points B and D might be free to describe the two entire circumferences respectively: but the principle applies to other motions, and I explain it in a general way as follows.

2. Imagine the cranked link BD, composed of the bars $B\beta$ and $D\delta$, rigidly attached $B\beta$ to the top and $D\delta$ to the bottom of the cylindrical disk K (this same letter K is used to denote the axis of the disk), and where $B\beta$ and $D\delta$ may be either parallel or inclined to each other at any given angle, so that, referring the points B, K, D to a horizontal plane, BKD is either a right line, or else K is the vertex of a triangle the other vertices whereof are B and D. The disk K, with the attached



Cranked link with disk: elevation.

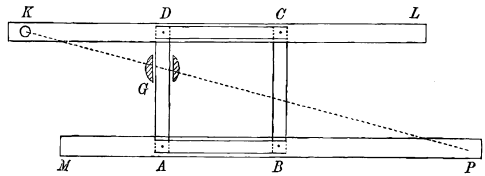


Arm of Pentagraph: plan.

bars $B\beta$ and $D\delta$, moves in a horizontal plane: and if the motion of the point B be regulated in any manner by a mechanism lying wholly below B and supported by the bed of the entire mechanism, and similarly if the motion of the point D be regulated

in any manner by a mechanism lying wholly above D and supported by a bridge of sufficient length (resting on the bed of the entire mechanism), then the disk K moves in its own horizontal plane unimpeded by other parts of the mechanism: and if we fit the disk K so as to move smoothly within a circular aperture in the arm of a pentagraph, then the pencil of the pentagraph will trace out on a sheet of paper the curve described by the point K on the axis of the disk, or say by the point K of the beam BKD. Of course for the three-bar motion, all that is required is that the point B shall describe a circle, viz. it must be pivoted on to an arm AB, which is itself pivoted at A to the bed: and that the point D shall describe a circle, viz. it must be pivoted on to an arm DE, which is itself pivoted at E to the bridge. Special arrangements are required to enable the variation of the several lengths AB, BK, KD, DE and ED, and the mechanism thus unavoidably assumes a form which appears complicated for the object intended to be thereby effected.

3. The form of Pentagraph which I use consists of a parallelogram $ABCD$, pivoted together at the points A, B, C, D, the bars AB and DC being above AD and BC. There is a cradle G, rotating about a fixed centre, and which carries between guides the arm AD, which has a sliding motion, so that the lengths GD and GA may be



made to have any given ratio to each other. Above the bar DC and sliding along it, we have the arm KL (where K is the circular aperture which fits on to the disk K of the cranked link): and above AB and sliding along it, we have the arm MP which carries the pencil P: of course, in order that the pentagraph may be in adjustment, the points K, G, P must be *in lineâ*.

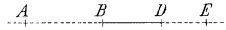


Fig 1. c=a-b-d.

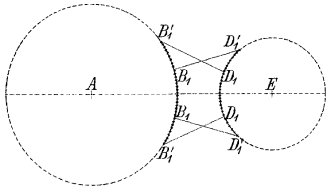


Fig. 1-2.

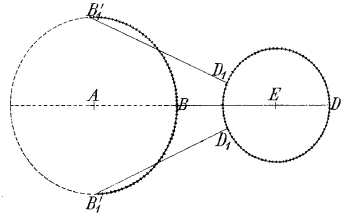


Fig. 2. c=a-b+d.

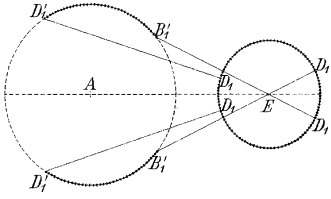


Fig. 2-3.

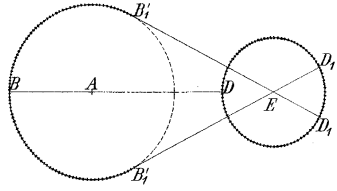


Fig. 3. c=a+b-d

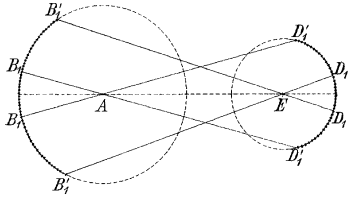


Fig. 3-4.

Fig. 4. c=a+b+d

Exterior Series.

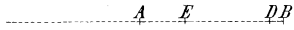


Fig. 1.

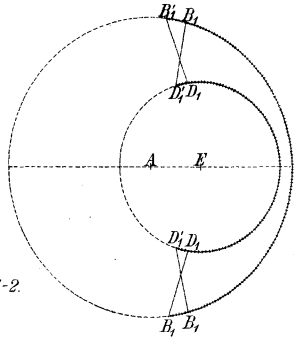


Fig. 1-2.

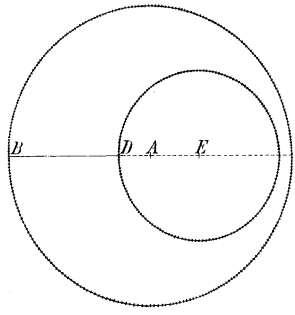


Fig. 2.

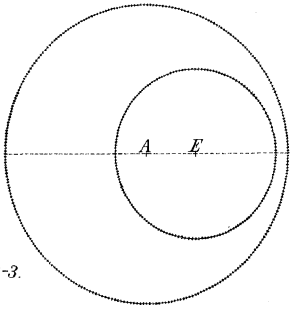


Fig. 2-3.

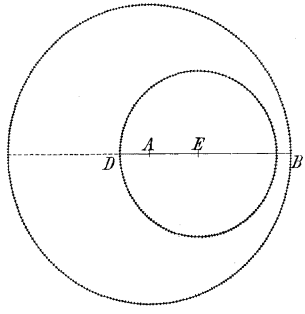


Fig. 3.

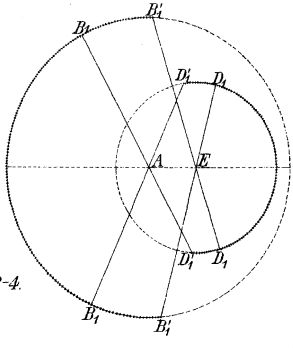


Fig. 3-4.

Fig. 4.

Interior Series.

953.

ON THE NINE-POINTS CIRCLE.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 23—25.]

If from the angles A, B, C of a triangle we draw tangents to a conic Ω, meeting the opposite sides in the points α, α'; β, β'; γ, γ' respectively, then it is known that these six points lie in a conic. In particular, if the conic Ω reduce itself to a point-pair OO', then we have the theorem that, if from the angles A, B, C, we draw to the point O lines meeting the opposite sides in the points α, β, γ respectively; and to the point O' lines meeting the opposite sides in the points α', β', γ' respectively, then the six points α, α'; β, β'; γ, γ' lie in a conic. We may enquire the conditions under which this conic becomes a circle. It may be remarked that one of the points say O' remains arbitrary: for if through the points α', β', γ', we draw a conic (or in particular a circle) meeting the three sides respectively in the remaining points α, β, γ, then (by a converse of the general theorem) the lines $A\alpha$, $B\beta$, $C\gamma$ will meet in a point O.

Using trilinear coordinates (x, y, z) and writing $x : y : z = a : b : c$ for the point O, and $x : y : z = a' : b' : c'$ for the point O', it is at once seen that the equation of the conic through the six points is

$$aa'x^2 + bb'y^2 + cc'z^2 - (bc' + b'c)\,yz - (ca' + c'a)\,zx - (ab' + a'b)\,xy = 0\,;$$

in fact, writing herein successively $x = 0$, $y = 0$, $z = 0$, we see that the equation is satisfied by $x = 0$, $(by - cz)(b'y - c'z) = 0$; by $y = 0$, $(cz - ax)(c'z - a'x) = 0$; and by $z = 0$, $(ax - by)(a'x - b'y) = 0$ respectively. And it is to be observed that the equation may also be written

$$(aa'x + bb'y + cc'z)(x + y + z) - (b + c)(b' + c')\,yz - (c + a)(c' + a')\,zx - (a + b)(a' + b')\,xy = 0.$$

Suppose now that x, y, z represent areal coordinates, viz. that (x, y, z) are proportional to the perpendicular distances of the point from the sides, each divided by the

perpendicular distance of the opposite angle from the same side; or, what is the same thing, coordinates such that the equation of the line infinity is $x+y+z=0$. Then if A, B, C denote the angles of the triangle, the general equation of a circle is

$$(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C) + (\lambda x + \mu y + \nu z)(x+y+z) = 0,$$

where λ, μ, ν are arbitrary coefficients.

Hence, putting this

$$= \Theta\{-(b+c)(b'+c')yz - (c+a)(c'+a')zx$$
$$-(a+b)(a'+b')xy + (aa'x + bb'y + cc'z)(x+y+z)\},$$

we must have

$$\Theta(b+c)(b'+c') = -\sin^2 A,$$
$$\Theta(c+a)(c'+a') = -\sin^2 B,$$
$$\Theta(a+b)(a'+b') = -\sin^2 C;$$

and then

$$\Theta aa' = \lambda, \quad \Theta bb' = \mu, \quad \Theta cc' = \nu,$$

which last equations determine the values of λ, μ, ν.

Taking a', b', c' at pleasure, we have

$$2a = \frac{1}{\Theta}\left(\frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$
$$2b = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} + \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$
$$2c = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} + \frac{\sin^2 C}{a'+b'}\right),$$

viz. a, b, c having these values, the conic through the six points α, β, γ, α', β', γ' is the circle having for its equation

$$yz\sin^2 A + zx\sin^2 B + xy\sin^2 C + \Theta(aa'x + bb'y + cc'z)(x+y+z) = 0;$$

and we may obviously without loss of generality give to Θ any specific value, say $\Theta = 1$.

If $a' = b' = c'$, $= 1$, then we have

$$-4a = \frac{1}{\Theta}(-\sin^2 A + \sin^2 B + \sin^2 C),$$
$$-4b = \frac{1}{\Theta}(\sin^2 A - \sin^2 B + \sin^2 C),$$
$$-4c = \frac{1}{\Theta}(\sin^2 A + \sin^2 B - \sin^2 C),$$

or writing for convenience $\Theta = -\frac{1}{2}$, the values of a, b, c are

$$\tfrac{1}{2}(-\sin^2 A + \sin^2 B + \sin^2 C), \quad \tfrac{1}{2}(\sin^2 A - \sin^2 B + \sin^2 C), \quad \tfrac{1}{2}(\sin^2 A + \sin^2 B - \sin^2 C)$$

respectively. But we have

$$A + B + C = \pi,$$

and thence

$$\begin{aligned} & \sin^2 A + \sin^2 B - \sin^2 C, \\ = \quad & \sin^2 A + \sin^2 B - \sin^2 (A + B) \\ = \quad & 2 \sin A \sin B (\sin A \sin B - \cos A \cos B), \\ = & -2 \sin A \sin B \cos (A + B), \\ = \quad & 2 \sin A \sin B \cos C, \end{aligned}$$

and we thus have

$$a,\ b,\ c = \sin B \sin C \cos A,\ \ \sin C \sin A \cos B,\ \ \sin A \sin B \cos C,$$

(or, what is the same thing, $a : b : c = \cot A : \cot B : \cot C$), and the equation of the circle is

$$\begin{aligned} & yz \sin^2 A + zx \sin^2 B + xy \sin^2 C \\ & \quad - \tfrac{1}{2} (x \sin B \sin C \cos A + y \sin C \sin A \cos B + z \sin A \sin B \cos C)(x + y + z) = 0. \end{aligned}$$

We thus have $x : y : z = 1 : 1 : 1$ for the point O', and $x : y : z = \cot A : \cot B : \cot C$ for the point O; viz. O' is the point of intersection of the lines from the angles to the mid-points of the opposite sides respectively; and O is the point of intersection of the perpendiculars from the angles on the opposite sides respectively: and the foregoing equation is consequently that of the Nine-points Circle.

954.

ON THE NINE-POINTS CIRCLE OF A PLANE TRIANGLE.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 25—27.]

I CONSIDER the circle which meets the sides of a triangle ABC in the points F, L; G, M; H, N respectively, where ultimately F, G, H are the feet of the perpendiculars let fall from the angles on the opposite sides, and L, M, N are the mid-points of the sides: but in the first instance, they are taken to be arbitrary points. Taking the radius of the circle to be unity, the coordinates of the point F may be taken to be $\cos F$, $\sin F$, and these may be expressed rationally in terms of the tangent of the half-angle, $f = \tan \frac{1}{2} F$; and similarly for the other points, viz. we may determine the six points by the parameters f, g, h, l, m, n respectively. The sides of the triangle are the lines joining the points L, F; M, G; N, H respectively: thus the equations of the sides are

$$\begin{aligned} &\text{for } BC: \ x(1-lf)+y(l+f)-(1+lf)=0, \text{ say } U=0,\\ &\text{,, } CA: \ x(1-mg)+y(m+g)-(1+mg)=0, \quad \text{,, } V=0,\\ &\text{,, } AB: \ x(1-nh)+y(n+h)-(1+nh)=0, \quad \text{,, } W=0. \end{aligned}$$

We have AF, a line through the intersections of BC and CA; its equation is therefore of the form $BV - CW = 0$, and to determine B, C we have $BV_0 - CW_0 = 0$, if V_0, W_0 are the values of V, W belonging to the point F, the coordinates of which are

$$\frac{1-f^2}{1+f^2}, \quad \frac{2f}{1+f^2};$$

we find

$$\begin{aligned} V_0 &= -2(f-g)(f-m) \div (1+f^2);\\ W_0 &= 2(h-f)(f-n) \div (1+f^2), \end{aligned}$$

and then $B \div C = W_0 \div V_0$: we thus find the following equations:

$$\begin{aligned} &\text{that of } AF \text{ is } BV - CW = 0,\\ &\text{,, } BG \text{ ,, } C'W - A'U = 0,\\ &\text{,, } CH \text{ ,, } A''U - B''V = 0, \end{aligned}$$

where

$$\begin{aligned} B\ :\ C\ &= -(h-f)(f-n)\ :\ (f-g)(f-m), \\ C'\ :\ A'\ &= -(f-g)(g-l)\ :\ (g-h)(g-n), \\ A''\ :\ B''\ &= -(g-h)(h-m)\ :\ (h-f)(h-l). \end{aligned}$$

The condition in order that the three lines may meet in a point is $BC'A''=CA'B''$, viz. this is

$$(f-n)(g-l)(h-m)+(f-m)(g-n)(h-l)=0,$$

or, as this may also be written,

$$2fgh-gh(m+n)-hf(n+l)-fg(l+m)+mn(g+h)+nl(h+f)+lm(f+g)-2lmn=0.$$

Similarly, the equation of

$$\begin{aligned} AL &\text{ is } \mathfrak{B}V-\mathfrak{C}W=0, \\ BM &\ \text{,,}\ \mathfrak{C}'W-\mathfrak{A}'U=0, \\ CN &\ \text{,,}\ \mathfrak{A}''U-\mathfrak{B}''V=0, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{B}\ :\ \mathfrak{C}\ &= -(n-l)(h-l)\ :\ (l-m)(g-l), \\ \mathfrak{C}'\ :\ \mathfrak{A}'\ &= -(l-m)(f-m)\ :\ (m-n)(h-m), \\ \mathfrak{A}''\ :\ \mathfrak{B}''\ &= -(m-n)(g-n)\ :\ (n-l)(f-n). \end{aligned}$$

The condition in order that the three lines may meet in a point is $\mathfrak{B}\mathfrak{C}'\mathfrak{A}''=\mathfrak{C}\mathfrak{A}'\mathfrak{B}''$, viz. this is the same condition as before; that is, if the lines AF, BG, CH meet in a point, then also the lines AL, BM, CN will meet in a point.

In the case of the nine-points circle, we have MN, NL, LM parallel to LF, MG, NH, respectively: the equation of MN is

$$x(l-mn)+y(m+n)-(l+mn)=0,$$

and this is parallel to LF, if

$$\frac{m+n}{1-mn}=\frac{l+f}{1-lf}, \text{ that is, } L+F=M+N.$$

Hence, for the nine-points circle, we have

$$L+F=M+N,\quad M+G=N+L,\quad N+H=L+M,$$

or, as these equations may be written,

$$2L=G+H,\quad 2M=H+F,\quad 2N=F+G,$$

viz. it thus appears that the radii to the points L, M, N respectively, or say the radii L, M, N, bisect the angles made by the radii G and H, H and F, F and G respectively.

It may be added that we have

$$\begin{aligned} m+n-l+lmn &= f\{1-mn+l(m+n)\}, \\ n+l-m+lmn &= g\{1-nl+m(n+l)\}, \\ l+m-n+lmn &= h\{1-lm+n(l+m)\}, \end{aligned}$$

viz. f, g, h are expressible each of them as a rational function of l, m, n.

955.

THE NUMERICAL VALUE OF Πi, $=\Gamma(1+i)$.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 36—38.]

I DO not know whether the numerical value of Πx for an imaginary value of x has ever been calculated; and I wish to calculate it for a simple case $x=i$.

We have

$$\begin{aligned}\frac{1}{\Pi z}=&\left(1+\frac{z}{1}\right)\\&\left(1+\frac{z}{2}\right)e^{z\,\mathrm{hl}\frac{1}{2}}\\&\left(1+\frac{z}{3}\right)e^{z\,\mathrm{hl}\frac{2}{3}}\\&\quad\vdots\\&\left(1+\frac{z}{s}\right)e^{z\,\mathrm{hl}\frac{s-1}{s}}\\&\quad\vdots\end{aligned}$$

where hl denotes the hyperbolic logarithm. Hence, in particular, when $z=i$, we have

$$\begin{aligned}\frac{1}{\Pi i}=&\,1+\frac{i}{1}\\&\,1+\frac{i}{2}.\cos\mathrm{hl}\tfrac{1}{2}+i\sin\mathrm{hl}\tfrac{1}{2}.\\&\,1+\frac{i}{3}.\cos\mathrm{hl}\tfrac{2}{3}+i\sin\mathrm{hl}\tfrac{2}{3}.\\&\,1+\frac{i}{4}.\cos\mathrm{hl}\tfrac{3}{4}+i\sin\mathrm{hl}\tfrac{3}{4}.\\&\quad\vdots\\=&\,\sqrt{(1+1)}.\cos\theta_1+i\sin\theta_1.\cos\phi_1-i\sin\phi_1.\\&\,\sqrt{(1+\tfrac{1}{4})}.\cos\theta_2+i\sin\theta_2.\cos\phi_2-i\sin\phi_2.\\&\,\sqrt{(1+\tfrac{1}{9})}.\cos\theta_3+i\sin\theta_3.\cos\phi_3-i\sin\phi_3.\\&\quad\vdots\end{aligned}$$

($\phi_1 = 0$, and in the subsequent terms the imaginary part is taken with a negative sign in order to obtain positive values for ϕ_2, ϕ_3, &c.), $=\Omega(\cos\Theta + i\sin\Theta)$, if Ω be the modulus and Θ the sum

$$(\theta_1 - \phi_1) + (\theta_2 - \phi_2) + (\theta_3 - \phi_3) + \ldots.$$

We have

$$\Omega_1 = \sqrt{(1+1)} \,.\, \sqrt{(1+\tfrac{1}{4})} \,.\, \sqrt{(1+\tfrac{1}{9})} \ldots,$$

which may be calculated directly. The value of Ω admits, however, of a finite expression, viz. we have

$$\Omega^2 = \frac{1}{\Pi i\, \Pi(-i)} = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi},$$

the approximate numerical value is $\Omega = 1{\cdot}9173$, viz. we have

$$e^{\pi} - e^{-\pi} = 23{\cdot}141 - {\cdot}043 = 23{\cdot}098: \quad \log = 1{\cdot}3635744, \quad -\log 2\pi = \bar{1}{\cdot}201819,$$

whence

$$\log \Omega^2 = {\cdot}5653935, \quad \log \Omega = {\cdot}2826967, \text{ or } \Omega = 1{\cdot}9173.$$

We have

$$\tan\theta_1 = 1, \quad \tan\theta_2 = \tfrac{1}{2}, \quad \tan\theta_3 = \tfrac{1}{3}, \text{ \&c.},$$

also

$$\phi_1 = 0, \quad \phi_2 = \frac{180^\circ}{M\pi}\log\tfrac{1}{2}, \quad \phi_3 = \frac{180^\circ}{M\pi}\log\tfrac{2}{3}, \text{ \&c.},$$

where M is the modulus for the Briggian logarithms,

$$\begin{aligned} M &= {\cdot}4342944 \log &&= \bar{1}{\cdot}6377843, \\ \pi &= 3{\cdot}1415926 \;\;\text{,,} &&= {\cdot}4971499, \\ 180 & \qquad\qquad\;\; \text{,,} &&= 2{\cdot}2552755, \end{aligned}$$

whence

$$\log\frac{180}{M\pi} = 2{\cdot}1203383, \quad \frac{180^\circ}{M\pi} = 131^\circ{\cdot}9284.$$

We hence calculate the succession of values of θ and ϕ as follows:

θ	tan	arc
1	1	45°
2	·5	26 34′
3	·3333333	18 26
4	·25	14 2
5	·2	11 19
6	·1666666	9 28
7	·1428571	8 8
8	·125	7 8
9	·1111111	6 20
10	·1	5 43

ϕ	$131^\circ\cdot93\times$				$\theta-\phi=$			
1			$=0$		1	45°		
2	$\log 1/2=$	$\cdot3010300$	$=39^\circ$	$43'$	2		-13°	$9'$
3	$2/3$	$\cdot1760913$	23	14	3		4	48
4	$3/4$	$\cdot1249387$	16	29	4		2	27
5	$4/5$	$\cdot0969100$	12	47	5		1	28
6	$5/6$	$\cdot0791813$	10	26	6		0	58
7	$6/7$	$\cdot0669467$	8	50	7		0	42
8	$7/8$	$\cdot0579920$	7	39	8		0	31
9	$8/9$	$\cdot0511525$	6	44	9		0	24
10	$9/10$	$\cdot0457575$	6	2	10		0	19

The sum of all the negative arcs $\theta_2-\phi_2$, $\theta_3-\phi_3$, ... as far as calculated, that is, up to $\theta_{10}-\phi_{10}$ is $=24^\circ\,46'$, or, writing x for the sum of the remaining arcs $\theta_{11}-\phi_{11}$ to infinity, we have

$$\frac{1}{\Pi i}=1\cdot9173\,(\cos\Theta+i\sin\Theta),$$

where

$$\Theta=45^\circ-24^\circ\,46'-x,\quad=20^\circ\,14'-x.$$

It would not be difficult to calculate a limit to the value of x.

956.

ON RICHELOT'S INTEGRAL OF THE DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 42—47.]

IN the Memoir "Einige neue Integralgleichungen des Jacobischen Systems Differentialgleichungen," *Crelle*, t. XXV. (1843), pp. 97—118, Richelot, working with the more general problem of a system of $n-1$ differential equations between n variables, obtains a result which in the particular case $n=2$ (that is, for the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,\quad X=a+bx+cx^2+dx^3+ex^4,$$

and Y the same function of y), is in effect as follows: an integral is

$$\left\{\frac{\sqrt{X}(\theta-y)-\sqrt{Y}(\theta-x)}{x-y}\right\}^2=\square(\theta-x)(\theta-y)+\Theta+e(\theta-x)^2(\theta-y)^2,$$

where $\square$, θ are arbitrary constants, and Θ denotes the quartic function

$$a+b\theta+c\theta^2+d\theta^3+e\theta^4;$$

viz. this is theorem 3, p. 107 (*l. c.*), taking therein $n=2$, and writing θ, $\square$ for Richelot's α and const.

The peculiarity is that the integral contains apparently *two* arbitrary constants, and it is very interesting to show how these really reduce themselves to a single arbitrary constant.

Observe that, on the right-hand side, there are terms in θ^4, θ^3 whereas no such terms present themselves on the left-hand side. But by changing the constant $\square$,

we can get rid of these terms, and so bring each side to contain only terms in θ^2, θ, 1; viz. writing

$$\square = -2e\theta^2 - d\theta - c + C,$$

where C is a new arbitrary constant, the equation becomes

$$\left\{\frac{\sqrt{X}(\theta - y) - \sqrt{Y}(\theta - x)}{x - y}\right\}^2 = \begin{array}{ll} \theta^2 [& e(x+y)^2 + d(x+y) + C \quad] \\ + \theta [& -2exy(x+y) - dxy - (C-c)(x+y) + b] \\ + [& exy^2 + (C-c)xy + a], \end{array}$$

which still contains the two arbitrary constants θ, C.

But this gives the three equations

$$\frac{(\sqrt{X} - \sqrt{Y})^2}{(x-y)^2} = e(x+y)^2 + d(x+y) + C,$$

$$-2\frac{(\sqrt{X} - \sqrt{Y})(y\sqrt{X} - x\sqrt{Y})}{(x-y)^2} = -2exy(x+y) - dxy - (C-c)(x+y) + b,$$

$$\frac{(y\sqrt{X} - x\sqrt{Y})^2}{(x-y)^2} = exy^2 + (C-c)xy + a.$$

The first of these is Lagrange's integral containing the arbitrary constant C; and it is necessary that the three equations shall be one and the same equation; viz. the second and third equations must be each of them a mere transformation of the first, equation.

It is easy to verify that this is so. Starting from the first equation, we require, first the value of

$$-2\frac{(\sqrt{X} - \sqrt{Y})(y\sqrt{X} - x\sqrt{Y})}{(x-y)^2}, = \Omega,$$

for a moment.

We form a rational combination, or combination without any term in $\sqrt{XY}$; this is

$$(x+y)\frac{(\sqrt{X} - \sqrt{Y})^2}{(x-y)^2} - 2\frac{(\sqrt{X} - \sqrt{Y})(y\sqrt{X} - x\sqrt{Y})}{(x-y)^2} = e(x+y)^3 + d(x+y)^2 + C(x+y) + \Omega,$$

where the left-hand side is

$$\frac{(x-y)(X-Y)}{(x-y)^2}, = \frac{X-Y}{x-y},$$

which is

$$= e(x^3 + x^2y + xy^2 + y^3) + d(x^2 + xy + y^2) + c(x+y) + b,$$

and we thence have for

$$\Omega, = -2\frac{(\sqrt{X} - \sqrt{Y})(y\sqrt{X} - x\sqrt{Y})}{(x-y)^2},$$

the value given by the second equation.

Secondly, starting again from the first equation, and proceeding in like manner to find the value of

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}, =\Omega,$$

for a moment, we form a rational combination

$$-xy\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}+\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}=-exy(x+y)^2-dxy(x+y)-Cxy+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(-yX+xY)}{(x-y)^2}, =\frac{-yX+xY}{x-y},$$

which is

$$=-exy(x^2+xy+y^2)-dxy(x+y)-cxy+a;$$

and we thence have for

$$\Omega, =\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2},$$

the value given by the third equation.

In conclusion, I give what is in effect the process by which Richelot obtained his integral. The integral is $v=\square$, where

$$v=\frac{-\Theta}{\theta-x\,.\,\theta-y}-e(\theta-x\,.\,\theta-y)+(\theta-x\,.\,\theta-y)\,\Omega^2,$$

if, for shortness,

$$\Omega=\frac{\sqrt{X}}{\theta-x\,.\,x-y}+\frac{\sqrt{Y}}{\theta-y\,.\,y-x},$$

and it is required thence to show that

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

or, what is the same thing, to show that v satisfies the partial differential equation

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=0.$$

We have

$$\frac{dv}{dx}=\frac{-\Theta}{(\theta-x)^2(\theta-y)}+e(\theta-y)-(\theta-y)\,\Omega^2+2(\theta-x)(\theta-y)\,\Omega\,\frac{d\Omega}{dx},$$

$$\frac{dv}{dy}=\frac{-\Theta}{(\theta-x)(\theta-y)^2}+e(\theta-x)-(\theta-x)\,\Omega^2+2(\theta-x)(\theta-y)\,\Omega\,\frac{d\Omega}{dy},$$

and thence, attending to the value of Ω,

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=\frac{-\Theta}{\theta-x\,.\,\theta-y}(x-y)\,\Omega+(e-\Omega^2)(\theta-x)(\theta-y)(x-y)\,\Omega$$
$$+2(\theta-x)(\theta-y)\,\Omega\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

or say

$$-\frac{\left(\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}\right)}{(\theta-x)(\theta-y)(x-y)\,\Omega}=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2-\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right);$$

and it is consequently to be shown that the function on the right-hand side is $=0$. We have

$$\sqrt{X}\frac{d\Omega}{dx}=\frac{\frac{1}{2}X'}{(\theta-x)(x-y)}+\frac{X}{(\theta-x)^2(x-y)}-\frac{X}{(\theta-x)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-y)(x-y)^2},$$

$$\sqrt{Y}\frac{d\Omega}{dy}=\frac{\frac{1}{2}Y'}{(\theta-y)(y-x)}+\frac{Y}{(\theta-y)^2(y-x)}-\frac{Y}{(\theta-y)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-x)(x-y)^2},$$

and thence

$$\begin{aligned}\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}=\;&\frac{\frac{1}{2}X'}{(\theta-x)(x-y)}-\frac{\frac{1}{2}Y'}{(\theta-y)(y-x)}\\&+\left\{\frac{X}{(\theta-x)^2}+\frac{Y}{(\theta-y)^2}\right\}\frac{1}{x-y}\\&-\left(\frac{X}{\theta-x}-\frac{Y}{\theta-y}\right)\frac{1}{(x-y)^2}\\&-\frac{\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)}.\end{aligned}$$

Multiplying by $\frac{2}{x-y}$, we may put the result in the form

$$\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right)=\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}$$
$$+\frac{2X}{(\theta-x)^2(x-y)^2}+\frac{2Y}{(\theta-x)^2(x-y)^2}-\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2};$$

and the equation to be verified thus is

$$\begin{aligned}0=\;&\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2\\&-\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}-\frac{2X}{(\theta-x)^2(x-y)^2}\\&-\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}-\frac{2Y}{(\theta-x)^2(x-y)^2}\\&+\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2}.\end{aligned}$$

But decomposing the first term into simple fractions, we have

$$\begin{aligned}\frac{\Theta}{(\theta-x)^2(\theta-y)^2}=&+e\\&+\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{X}{(\theta-x)^2(x-y)^2}\\&+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}+\frac{Y}{(\theta-y)^2(x-y)^2}.\end{aligned}$$

Also for the third term, we have

$$\begin{aligned}\Omega^2=&\frac{X}{(\theta-x)^2(x-y)^2}\\&+\frac{Y}{(\theta-y)^2(x-y)^2}\\&-\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2},\end{aligned}$$

and substituting these values the several terms destroy each other, so that the right-hand side is $=0$, as it should be.

957.

ILLUSTRATIONS OF SYLOW'S THEOREMS ON GROUPS.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 59—62.]

THE theorems 1, 2, and 3 in the paper Sylow, "Théorèmes sur les groupes de Substitutions," *Math. Ann.* t. v. (1872), pp. 584—594, apply to groups in general, and not only to groups of substitutions. They are as follows:

THEOREM 1. If n^a be the highest power of the prime number n which divides the order of a group G, this group contains a group g of the order n^a: if, moreover, $n^a\nu$ is the order of the highest group contained in G, the operations whereof are permutable with the group g, then the order of G is of the form $n^a\nu(nk+1)$. (I write k for Sylow's p, since it is convenient to have p to denote a prime number; and for Sylow's "Substitutions" I write "Operations.")

THEOREM 2. Everything being as in the preceding theorem, the group G contains precisely $nk+1$ distinct groups of the order n^a; and these are obtained by transforming any one of them by the operations of G, each group being given by $n^a\nu$ distinct transformations.

THEOREM 3. If the order of a group is n^a, n being prime, then any operation ϑ whatever of the group may be expressed by the formula

$$\vartheta = \theta_0^i\theta_1^k\theta_2^l \ldots \theta^r{}_{a-1},$$

where

$$\theta_0^n = 1,$$
$$\theta_1^n = \theta_0^a,$$
$$\theta_2^n = \theta_0^b\theta_1^c,$$
$$\theta_3^n = \theta_0^d\theta_1^e\theta_2^f,$$
$$\vdots$$

and where

$$\begin{aligned}
&\mathfrak{J}^{-1}\theta_0\mathfrak{J} = 1,\\
&\mathfrak{J}^{-1}\theta_1\mathfrak{J} = \theta_0^{\beta}\theta_1,\\
&\mathfrak{J}^{-1}\theta_2\mathfrak{J} = \theta_0^{\gamma}\theta_1^{\delta}\theta_2,\\
&\mathfrak{J}^{-1}\theta_3\mathfrak{J} = \theta_0^{\epsilon}\theta_1^{\zeta}\theta_2^{\eta}\theta_3.\\
&\quad\vdots
\end{aligned}$$

But at present I attend only to the theorems 1 and 2.

For instance, consider the group G of the order $n=6$,

$$1,\ \beta,\ \beta^2,\ \alpha,\ \alpha\beta,\ \alpha\beta^2\ (\alpha^2=1,\ \beta^3=1,\ \alpha\beta^2=\beta\alpha,\ \alpha\beta=\beta^2\alpha).$$

Here $n=2$ or 3: if $n=2$, we have $N=n^a\nu(nk+1)=2\,.\,1\,(2+1)$; if $n=3$, we have $N=n^a\nu(nk+1)=3\,.\,2\,.\,1$.

First, $n=2$; we should have a group g of the order 2; one such group is $(1,\ \alpha)$, and the only group the substitutions whereof are permutable with $(1,\ \alpha)$ is the group $(1,\ \alpha)$ itself: for, taking any other operation of the group, for instance β, it is not true that $\beta(\gamma,\ \alpha)=(1,\ \alpha)\beta$; in fact, the left-hand is $(\beta,\ \beta\alpha)$ and the right-hand is $(\beta,\ \alpha\beta)$ or $(\beta,\ \beta^2\alpha)$: hence $n^a\nu,\ =2\nu,\ =2$, or ν is $=1$.

Hence also, by theorem 2, there should be 3 groups of the order 2 such as $(1,\ \alpha)$, viz. these are $(1,\ \alpha)$, $(1,\ \alpha\beta)$, $(1,\ \alpha\beta^2)$, derived from $(1,\ \alpha)$ as follows:

$$\begin{aligned}
&1\quad(1,\ \alpha)\,1^{-1} &&=(1,\ \alpha),\\
&\alpha\quad(1,\ \alpha)\,\alpha^{-1} &&=(1,\ \alpha),\\
&\beta\quad(1,\ \alpha)\,\beta^{-1} &&=(1,\ \alpha\beta),\\
&\beta^2\quad(1,\ \alpha)\,\beta^{-2} &&=(1,\ \alpha\beta^2),\\
&\alpha\beta\quad(1,\ \alpha)\,(\alpha\beta)^{-1} &&=(1,\ \alpha\beta^2),\\
&\alpha\beta^2\,(1,\ \alpha)\,(\alpha\beta^2)^{-2} &&=(1,\ \alpha\beta),
\end{aligned}$$

since $\beta^{-1}\ =\beta^2$, and therefore the second term is $\beta\alpha\beta^2\ =\alpha\beta^2\,.\,\beta^2=\alpha\beta$,

" $\beta^{-2}\ =\beta$, " " $\beta^2\alpha\beta\ =\alpha\beta\ .\ \beta=\alpha\beta^2$,

" $(\alpha\beta)^{-1}=\alpha\beta$, " " $\alpha\beta\alpha\alpha\beta\ =\alpha\beta\ .\ \beta=\alpha\beta^2$,

" $(\alpha\beta^2)^{-1}=\alpha\beta^2$, " " $\alpha\beta^2\alpha\alpha\beta^2=\alpha\beta^2\,.\,\beta^2=\alpha\beta$;

viz. the derivatives are $(1,\ \alpha)$, $(1,\ \alpha\beta)$, $(1,\ \alpha\beta^2)$, each twice.

Secondly, $n=3$; there should be here a group of the order 3, viz. this is $(1,\ \beta,\ \beta^2)$. The group, the substitutions whereof are permutable with $(1,\ \beta,\ \beta^2)$, is the entire group $(1,\ \beta,\ \beta^2,\ \alpha,\ \alpha\beta,\ \alpha\beta^2)$; in fact, taking any substitution hereof, for instance α, we have $\alpha(1,\ \beta,\ \beta^2)=(1,\ \beta,\ \beta^2)\,\alpha$, viz. the left-hand side is $(\alpha,\ \alpha\beta,\ \alpha\beta^2)$, and the right-hand side is $(\alpha,\ \beta\alpha,\ \beta^2\alpha)$, $=(\alpha,\ \alpha\beta^2,\ \alpha\beta)$, which is the left-hand side, *the change of order being immaterial;* this is the meaning of the expression used, "the operations whereof are permutable with the group g." Hence, we have $n^a\nu,\ =3\nu,\ =6$, or $\nu=2$; and

thence also $nk+1, =3k+1, =1$, viz. $k=0$. There is thus only a single group of the order 3, viz. the group $(1, \beta, \beta^2)$.

As another instance, I take the group of the order 12 formed by the positive substitutions of four letters, viz. these are

$$\begin{array}{lll} 1, & ab\,.\,cd, & abc, \\ & ac\,.\,bd, & acb, \\ & ad\,.\,bc, & abd, \\ & & adb, \\ & & acd, \\ & & adc, \\ & & bcd, \\ & & bdc. \end{array}$$

Here, taking $n=2$, we have $N=n^a\nu(nk+1)=2^2.3.1$; there is a group g of the order 4, viz. this is

$$(1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc),$$

and the greatest group, the substitutions whereof are permutable with this group g, is the entire group of the order 12; thus, considering any substitution thereof, for instance abc, we have

$$abc\begin{Bmatrix} 1 \\ ab\,.\,cd \\ ac\,.\,bd \\ ad\,.\,bc \end{Bmatrix} = \begin{Bmatrix} 1 \\ ab\,.\,cd \\ ac\,.\,bd \\ ad\,.\,bc \end{Bmatrix} abc,$$

viz. the left-hand is $\begin{Bmatrix} abc \\ acd \\ bdc \\ adb \end{Bmatrix}$, the right-hand is $\begin{Bmatrix} abc \\ bdc \\ adb \\ acd \end{Bmatrix}$;

hence $n^a\nu, =4\nu, =12$ or $\nu=3$; whence also $nk+1, =2k+1, =1$: and thus the foregoing group g is the only group of the order 4.

Similarly, taking $\nu=3$, we have $N=n^a\nu(nk+1), =3.1.4$. There is a group g of the order 3, say $(1, abc, acb)$; the greatest group, the substitutions whereof are permutable with g, is the group g itself, viz. we have $n^a\nu, =3\nu, =3$, or $\nu=1$; and then $nk+1, =3k+1, =4$: there are thus 4 groups of the order 3, viz. these are

$$(1,\ abc,\ acb),\ (1,\ abd,\ adb),\ (1,\ acd,\ adc),\ (1,\ bcd,\ bdc).$$

Reverting to the before-mentioned group of the order 6, this not only contains each of the groups $(1, \alpha)$, $(1, \alpha\beta)$, $(1, \alpha\beta^2)$ of order 2, and the group $(1, \beta, \beta^2)$ of

order 3; but it is the permutable product of a group of order 2 by a group of order 3, say it is

$$G = (1, \alpha)(1, \beta, \beta^2), \; = (1, \beta, \beta^2)(1, \alpha).$$

A group, which is thus a permutable product of two factors, is said to be a true product; and when it cannot be thus expressed as a permutable product of two factors, it is prime or simple. A group, the order of which is equal to a prime number p (the cyclical group of the order p) is simple; but the order may be a composite number and yet the group be simple—it was remarked by Galois, *Liouville*, t. XI. (1865), p. 409, that the order of the lowest simple group of composite order is 60, $= 2^2 . 3 . 5$, and it has been recently shown, Hölder, "Die einfachen Gruppen im ersten und zweiten Hundert der Ordnungszahlen," *Math. Ann.* t. XL. (1892), pp. 55—88, that the only other composite order of a simple group in the first 200 numbers is 168. Moreover, in the paper Cole, "Simple groups from order 201 to order 500," *Amer. Math. Jour.* t. XIV. (1892), pp. 378—388, it is shown that within these limits the only numbers which can give a simple group or groups are 360 and 432. I take the opportunity of referring to two other important papers, Young, "On the determination of groups whose order is a power of a prime," *Amer. Math. Jour.* t. XV. (1893), pp. 124—178, and Cole and Glover, "On groups whose orders are products of three prime factors," *ib.* pp. 191—220.

958.

ON THE SURFACE OF THE ORDER n WHICH PASSES THROUGH A GIVEN CUBIC CURVE.

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 79, 80.]

It is natural to assume that, taking A, B, C to denote the general functions $(x, y, z, w)^{n-2}$ of the order $n-2$, the general surface of the order n which passes through the curve

$$\begin{Bmatrix} x, & y, & z \\ y, & z, & w \end{Bmatrix} = 0,$$

(or, what is the same thing, the curve $x : y : z : w = 1 : \theta : \theta^2 : \theta^3$), has for its equation

$$\begin{vmatrix} A, & B, & C \\ x, & y, & z \\ y, & z, & w \end{vmatrix} = 0;$$

but the formal proof is not immediate. Writing the equation in the form

$$U = Sax^\alpha y^\beta z^\gamma w^\delta, = 0, \quad \alpha + \beta + \gamma + \delta = n,$$

then U must vanish on writing therein $x : y : z : w = 1 : \theta : \theta^2 : \theta^3$; a term $ax^\alpha y^\beta z^\gamma w^\delta$ becomes $= a\theta^p$, where $p = \beta + 2\gamma + 3\delta$ is the weight of the term reckoning the weights of x, y, z, w as 0, 1, 2, 3 respectively; and hence the condition is that, for each given weight p, the sum Sa of the coefficients of the several terms of this weight shall be $= 0$. Using any such equation to determine one of the coefficients thereof in terms of the others, the function U is reduced to a sum of duads $a(x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'})$, where in each duad the two terms are of the same degree and of the same weight, and where a is an arbitrary coefficient; it ought therefore to be true that each such duad $x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'}$ has the property in question—or writing P, Q, $R = yw - z^2$, $zy - xw$, $xz - y^2$, say that each such duad is of the form $AP + BQ + CR$.

Suppose for a moment that α' is greater than α, but that β', γ', δ' are each less than β, γ, δ respectively: the duad is $x^{\alpha'} y^\beta z^\gamma w^\delta (x^\lambda - y^\mu z^\nu w^\rho)$, where λ, μ, ν, ρ are each

positive, and hence $x^\lambda - y^\mu z^\nu w^\rho$ is a duad having the property in question, or changing the notation say $x^\alpha - y^\beta z^\gamma w^\delta$ has the property in question; and in like manner, by considering the several cases that may happen, we have to show that each of the duads

$$x^\alpha - y^\beta z^\gamma w^\delta,\ \ y^\beta - x^\alpha z^\gamma w^\delta,\ \ z^\gamma - x^\alpha y^\beta w^\delta,\ \ w^\delta - x^\alpha y^\beta z^\gamma,$$
$$x^\alpha y^\beta - z^\gamma w^\delta,\ \ x^\alpha z^\gamma - y^\beta w^\delta,\ \ x^\alpha w^\delta - y^\beta z^\gamma,$$

has the property in question; it being of course understood that, in each of these duads, the two terms have the same degree and the same weight. The first form cannot exist; for we must have therein $\alpha = \beta + \gamma + \delta$ and $0 = \beta + 2\gamma + 3\delta$, which is inconsistent with α, β, γ, δ each of them positive. For the second form $\beta = \alpha + \gamma + \delta$, $\beta = 2\gamma + 3\delta$: this is $\alpha = \gamma + 2\delta$ or the duad is $y^{2\gamma+3\delta} - x^{\gamma+2\delta} z^\gamma w^\delta$, $= (y^2)^\gamma y^{3\delta} - (xz)^\gamma (x^2w)^\delta$. Writing $y^2 = xz - R$, we have terms containing the factor R, and a residual term $(xz)^\gamma \{y^{3\delta} - (x^2w)^\delta\}$, and writing herein

$$xw = yz - Q \ \text{ or } \ x^2w = xyz - Q,$$

we have terms containing Q as a factor and a residual term

$$(xz)^\gamma \{y^{3\delta} - (xyz)^\delta\},\ = (xz)^\gamma y^\delta \{(y^2)^\delta - (xz)^\delta\},$$

and again writing herein $y^2 = xz - R$, we see that this term contains the factor R: hence the duad in question consists of terms having the factor R or the factor Q. Similarly for the other cases: either α, β, γ, δ can be expressed as positive numbers, and then the duad consists of terms each divisible by P, Q, or R; or else α, β, γ, δ cannot be expressed as positive numbers, and then the duad does not exist: thus for the third form $z^\gamma - x^\alpha y^\beta w^\delta$, here $\gamma = \alpha + \beta + \delta$, $2\gamma = \beta + 3\delta$, or say $\gamma = 3\alpha + 2\beta$, $\delta = 2\alpha + \beta$, and the duad is $z^{3\alpha+2\beta} - x^\alpha y^\beta w^{2\alpha+\beta}$, $= z^{3\alpha} (z^2)^{2\beta} - (xw^2)^\alpha (yw)^\beta$, which can be reduced to the required form. But for the duad $x^\alpha y^\beta - z^\gamma w^\delta$, we have $\alpha + \beta = \gamma + \delta$, $\beta = 2\gamma + 3\delta$, which cannot be satisfied by positive values of α, β, γ, δ, and thus the duad does not exist.

A surface of the order n which passes through $3n+1$ points of a cubic curve contains the curve: hence the number of constants, or say the capacity of a surface of the order n, through the curve $P=0$, $Q=0$, $R=0$, is

$$\tfrac{1}{6}(n+1)(n+2)(n+3) - 1 - (3n+1),\ = \tfrac{1}{6}(n^3 + 6n^2 - 7n - 6).$$

Primâ facie the capacity of the surface $AP + BQ + CR = 0$, A, B, C being general functions of the order $n-2$, is

$$3 \cdot \tfrac{1}{6}(n-1)n(n+1) - 1,\ = \tfrac{1}{2}(n^3 - n - 2),$$

but there is a reduction on account of the identical equations

$$xP + yQ + zR = 0,\ \ yP + zQ + wR = 0,$$

which connect the functions P, Q, R: for $n=2$, the formulæ give each of them as it should do, Capacity $= 2$; viz. the quadric surface through the curve is

$$aP + bQ + cR = 0.$$

959.

NOTE ON PLÜCKER'S EQUATIONS.

[From the *Messenger of Mathematics*, vol. XXIV. (1895), pp. 23, 24.]

It is well known that if

$$A,\ B,\ C,\ D = 2,\ 3,\ 6,\ 8,$$

then the equations

$$\begin{aligned} n &= m^3 - m - A\delta - B\kappa, \\ \iota &= 3m^2 - 6m - C\delta - D\kappa, \\ m &= n^3 - n - A\tau - B\iota, \\ \kappa &= 3n^2 - 6n - C\tau - D\iota, \end{aligned}$$

are equivalent to three independent equations giving n, τ, ι in terms of m, δ, κ. It is easy to show that the *necessary* conditions in order that this may be so, are

$$C = 3A, \quad \text{and} \quad D = 3B - 1,$$

that is,

$$A,\ B,\ C,\ D = A,\ B,\ 3A,\ 3B - 1,$$

where A and B are arbitrary.

In fact, from the last two equations eliminating τ, and for n, ι substituting their values, we have

$$\begin{aligned} Cm - A\kappa = \ & (C - 3A)(m^3 - m - A\delta - B\kappa)^2, \\ & - (C - 6A)(m^3 - m - A\delta - B\kappa), \\ & + (AD - BC)(3m^2 - 6m - C\delta - D\kappa), \end{aligned}$$

which must therefore be an identity. In order that the term in m^4 may vanish we must have $C = 3A$; and then substituting this value for C, we must have

$$3Am - A\kappa = 3A(m^3 - m - A\delta - B\kappa) + (AD - 3AD)(3m^2 - 6m - 3A\delta - D\kappa).$$

Here the coefficient of m^2 must vanish, that is,

$$0 = 3A + 3AD - 9AB, \quad \text{or} \quad D = 3B - 1,$$

and, substituting this value, the equation is

$$\begin{aligned} 3Am - A\kappa = \ & 3A(-m - A\delta - B\kappa) \\ & - A\{-6m - 3A\delta - (3B - 1)\kappa\}, \end{aligned}$$

that is,

$$\begin{aligned} 3m - \kappa = \ & -3m - 3A\delta - 3B\kappa \\ & + 6m + 3A\delta + (3B - 1)\kappa, \end{aligned}$$

an identity.

960.

ON THE CIRCLE OF CURVATURE AT ANY POINT OF AN ELLIPSE.

[From the *Messenger of Mathematics*, vol. XXIV. (1895), pp. 47, 48.]

LET
$$u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad u^2 + v^2 = 1.$$

The equation of the circle of curvature at the point (x, y) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
$$X^2 + Y^2 + 2\frac{a^2 - b^2}{ab}(-bXu^3 + aYv^3) + u^2(a^2 - 2b^2) + v^2(b^2 - 2a^2) = 0.$$

Write $X = a\xi$, $Y = b\eta$, then this becomes
$$a^2\xi^2 + b^2\eta^2 + 2(a^2 - b^2)(-\xi u^3 + \eta v^3) + u^2(a^2 - 2b^2) + v^2(b^2 - 2a^2) = 0.$$

To find where this meets the ellipse, we must write $\xi^2 + \eta^2 = 1$; eliminating η, we have
$$a^2\xi^2 + b^2(1 - \xi^2) - 2(a^2 - b^2)\xi u^3 + u^2(a^2 - 2b^2) + v^2(b^2 - 2a^2) + 2(a^2 - b^2)v^3\sqrt{(1 - \xi^2)} = 0,$$
or putting for shortness
$$a^2 - b^2 = A, \quad u^2(a^2 - 2b^2) + v^2(b^2 - 2a^2) = B,$$
the equation for ξ is
$$A\xi^2 - 2A\xi u^3 + b^2 + B + 2Av^3\sqrt{(1 - \xi^2)} = 0,$$
but
$$b^2 + B = b^2(u^2 + v^2) + u^2(a^2 - 2b^2) + v^2(b^2 - 2a^2) = u^2(a^2 - b^2) + v^2(2b^2 - 2a^2) = A(u^2 - 2v^2),$$
viz.
$$\xi^2 - 2\xi u^3 + u^2 - 2v^2 + 2v^3\sqrt{(1 - \xi^2)} = 0,$$
that is,
$$(\xi^2 - 2u^3\xi + u^2 - 2v^2)^2 - 4v^6(1 - \xi^2) = 0,$$
which is without difficulty reduced to the form
$$(\xi - u)^3\{\xi - (u^3 - 3uv^2)\} = 0,$$
that is,
$$\xi = u^3 - 3uv^2,$$
and hence
$$\eta = v^3 - 3vu^2,$$
viz. writing $u, v = \cos\theta, \sin\theta$, then we have
$$\xi = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta,$$
$$\eta = \sin^3\theta - 3\sin\theta\cos^2\theta = -\sin 3\theta,$$
or the circle of curvature at $(a\cos\theta, b\sin\theta)$ cuts the ellipse in $(a\cos 3\theta, -b\sin 3\theta)$, as is known.

961.

A TRIGONOMETRICAL IDENTITY.

[From the *Messenger of Mathematics*, vol. XXIV. (1895), pp. 49—51.]

THE following was proposed as a Senate House Problem: Given the equations

$$a\cos(\beta+\gamma)+b\cos(\beta-\gamma)+c=0,$$
$$a\cos(\gamma+\alpha)+b\cos(\gamma-\alpha)+c=0,$$
$$a\cos(\alpha+\beta)+b\cos(\alpha-\beta)+c=0,$$

it is to be shown that $a^2+2bc-b^2=0$.

Assume

$$\cos\alpha+i\sin\alpha,\ \cos\beta+i\sin\beta,\ \cos\gamma+i\sin\gamma=x,\ y,\ z,$$

then the equations are

$$a\left(yz+\frac{1}{yz}\right)+b\left(\frac{y}{z}+\frac{z}{y}\right)+2c=0,$$

$$a\left(zx+\frac{1}{zx}\right)+b\left(\frac{z}{x}+\frac{x}{z}\right)+2c=0,$$

$$a\left(xy+\frac{1}{xy}\right)+b\left(\frac{x}{y}+\frac{y}{x}\right)+2c=0,$$

that is,

$$a(1+y^2z^2)+b(y^2+z^2)+2cyz=0,$$
$$a(1+z^2x^2)+b(z^2+x^2)+2czx=0,$$
$$a(1+x^2y^2)+b(x^2+y^2)+2cxy=0,$$

the second and third equations give

$$a:b:2c=x(x^2-yz):x(-1+x^2yz):(1-x^4)(y+z);$$

or, say a, b, $2c$ are equal to these values; and then, substituting in the first equation, we have

$$x(1+y^2z^2)(x^2-yz)+x(y^2+z^2)(-1+x^2yz)+(1-x^4)(y^2z+yz^2)=0,$$

which is

$$(x-y)(x-z)\{x+y+z-(yz+zx+xy)(x+y+z)\}=0,$$

viz. our relation between x, y, z is

$$x+y+z-(yz+zx+xy)\,xyz=0,$$

or, what is the same thing,

$$\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}-(yz+zx+xy)=0.$$

Then

$$\begin{aligned} a+b &= x(-1+x^2)(1+yz),\\ a-b &= x(\ \ 1+x^2)(1-yz),\\ 2c &= (\ \ 1-x^4)(y+z),\\ a^2-b^2=x^2(1-x^4)(y^2z^2-1),\quad 2bc &= x(-1+x^2yz)(1-x^4)(y+z). \end{aligned}$$

The equation to be verified is

$$x(y^2z^2-1)=(1-x^2yz)(y+z),$$

that is,

$$x+y+z-(yz+zx+xy)\,xyz=0,$$

as it should be.

A somewhat different form of the proof is as follows:—We have *identically*

$$\begin{vmatrix} \cos(\beta+\gamma), & \cos(\beta-\gamma), & 1\\ \cos(\gamma+\alpha), & \cos(\gamma-\alpha), & 1\\ \cos(\alpha+\beta), & \cos(\alpha-\beta), & 1 \end{vmatrix}$$

$$=4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\},$$

and therefore the relation between the angles is

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

From the second and third equations,

$$a:b:c=\sin\{\tfrac{1}{2}(\beta+\gamma)-\alpha\}:-\sin\{\tfrac{1}{2}(\beta+\gamma)+\alpha\}:2\sin\alpha\cos\alpha\cos\tfrac{1}{2}(\beta-\gamma),$$

or say

$$\begin{aligned} a &= \ \ \sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha,\\ b &= -\sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha,\\ c &= \ \ 2\sin\alpha\cos\alpha\cos\tfrac{1}{2}(\beta-\gamma), \end{aligned}$$

therefore

$$\begin{aligned} a^2-b^2 &= -4\sin\alpha\cos\alpha\sin\tfrac{1}{2}(\beta+\gamma)\cos(\beta+\gamma)=-2\sin\alpha\cos\alpha\sin(\beta+\gamma),\\ bc &= 2\sin\alpha\cos\alpha\{-\cos\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha\},\\ &= 2.\tfrac{1}{2}\sin\alpha\cos\alpha\{-(\sin\beta+\sin\gamma)\cos\alpha-(\cos\beta+\cos\gamma)\sin\alpha\},\\ &= \sin\alpha\cos\alpha\{-\sin(\gamma+\alpha)-\sin(\alpha+\beta)\}=\sin\alpha\cos\alpha\sin(\beta+\gamma), \end{aligned}$$

whence therefore

$$a^2 - b^2 + 2bc = 0,$$

which is the required relation.

The equation to be proved may also be written

$$\begin{vmatrix} \cos(\beta+\gamma), & \cos(\beta-\gamma), & 1 \\ \cos(\gamma+\alpha), & \cos(\gamma-\alpha), & 1 \\ \cos(\alpha+\beta), & \cos(\alpha-\beta), & 1 \end{vmatrix}$$

$$= 4 \sin \tfrac{1}{2}(\beta-\gamma) \sin \tfrac{1}{2}(\gamma-\alpha) \sin \tfrac{1}{2}(\alpha-\beta) \{\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)\},$$

or putting

$$\beta+\gamma = a, \quad b-c = \gamma-\beta,$$
$$\gamma+\alpha = b, \quad c-a = \alpha-\gamma,$$
$$\alpha+\beta = c, \quad a-b = \beta-\alpha,$$

this becomes

$$\begin{vmatrix} \cos a, & \cos(b-c), & 1 \\ \cos b, & \cos(c-a), & 1 \\ \cos c, & \cos(a-b), & 1 \end{vmatrix}$$

$$= -4 \sin \tfrac{1}{2}(b-c) \sin \tfrac{1}{2}(c-a) \sin \tfrac{1}{2}(a-b)(\sin a + \sin b + \sin c),$$

an identity which may be proved without difficulty.

962.

COORDINATES VERSUS QUATERNIONS.

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XX. (1895), pp. 271—275. Read July 2, 1894.]

IT is contended that Quaternions (as a method) are more comprehensive and less artificial than—and, in fact, in every way far superior to—Coordinates. Thus Professor Tait, in the Preface to his *Elementary Treatise on Quaternions* (1867), reproduced in the second and third editions (1873 and 1890), writes—"It must always be remembered that Cartesian methods are mere particular cases of quaternions where most of the distinctive features have disappeared; and that when, in the treatment of any particular question, scalars have to be adopted, the quaternion solution becomes identical with the Cartesian one. Nothing, therefore, is ever lost, though much is generally gained, by employing quaternions in place of ordinary methods. In fact, even when quaternions degrade to scalars, they give the solution of the most general statement of the problem they are applied to, quite independent of any limitations as to choice of particular coordinate axes." And he goes on to speak of "such elegant trifles as trilinear coordinates," and would, I presume, think as lightly of quadriplanar coordinates. It is right to notice that the claims of quaternions are chiefly insisted upon in regard to their applications to the physical sciences; and I would here refer to his paper, "On the Importance of Quaternions in Physics" (*Phil. Mag.*, Jan. 1890), being an abstract of an address to the Physical Society of the University of Edinburgh, Nov. 1889; but these claims certainly extend to and include the science of geometry.

I wish to examine into these claims on behalf of quaternions. My own view is that quaternions are merely a particular method, or say a theory, in coordinates. I have the highest admiration for the notion of a quaternion; but (I am not sure whether I did or did not use the illustration many years ago in conversation with Professor Tait), as I consider the full moon far more beautiful than any moonlit

view, so I regard the notion of a quaternion as far more beautiful than any of its applications. As another illustration which I gave him, I compare a quaternion formula to a pocket-map—a capital thing to put in one's pocket, but which for use must be unfolded: the formula, to be understood, must be translated into coordinates.

I remark that the imaginary of ordinary algebra—for distinction call this θ—has no relation whatever to the quaternion symbols i, j, k; in fact, in the general point of view, all the quantities which present themselves are, or may be, complex values $a+\theta b$, or, in other words, say that a scalar quantity is in general of the form $a+\theta b$. Thus quaternions do not properly present themselves in plane or two-dimensional geometry at all—although, as will presently appear, we may use them in plane geometry; but they belong essentially to solid or three-dimensional geometry, and they are most naturally applicable to the class of problems which in coordinates are dealt with by means of the three rectangular coordinates x, y, z.

In plane geometry, considering an origin O, and through it two rectangular axes Ox, Oy, then in coordinates we determine the position of a point by means of its coordinates x, y; or, writing x, y, z to denote given linear functions of the original rectangular coordinates x, y, we may, if we please, determine it by trilinear coordinates, or say by the ratios $x : y : z$. The advantage is, that we thereby deal with the line infinity as with any other line, whereas with the rectangular coordinates x, y the line infinity presents itself as a line *sui generis*, and that we thereby bring the theory into connexion with that of the homogeneous functions $(*\between x, y, z)^n$.

In quaternions, the position of a point is determined in reference to the fixed point O, by its vector α, which is in fact $=ix+jy$, where i, j are the quaternion imaginaries ($i^2=-1$, $j^2=-1$, $ij=-ji$), but the idea is to use as little as possible the foregoing equation $\alpha=ix+jy$, and thus to conduct the investigations independently, as far as may be, of the particular positions of the axes Ox, Oy.

As the most simple example, take the theorem that the lines joining the extremities of equal and parallel lines in a plane are themselves equal and parallel, viz. (writing $\sim$ to denote equal and parallel), if $AB \sim CD$, then $AC \sim BD$.

Coordinates.

A, B, C, D are determined by their coordinates

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).$$

$$AB \sim CD$$

gives

$$\left.\begin{aligned} x_2-x_1 &= x_4-x_3 \\ y_2-y_1 &= y_4-y_3 \end{aligned}\right\},$$

whence

$$\left.\begin{aligned} x_3-x_1 &= x_4-x_2 \\ y_3-y_1 &= y_4-y_2 \end{aligned}\right\},$$

that is,

$$AC \sim BD.$$

Quaternions.

AB, CD are determined by their vectors α, β, and then writing γ for the vector AD,

$$AB \sim CD$$

gives

$$\alpha=\beta,$$

whence

$$\gamma-\beta=-\alpha+\gamma,$$

that is,

$$AC \sim BD.$$

And for the comparison of the two solutions, we have

$$\alpha = i(x_2 - x_1) + j(y_2 - y_1), \quad \beta = i(x_4 - x_3) + j(y_4 - y_3).$$

But this example of a plane theorem is a trivial one, given only for the sake of completeness.

Passing to solid geometry, we have—

Coordinates.—Considering a fixed point O, and through it the rectangular axes Ox, Oy, Oz, the position of a point is determined by its coordinates x, y, z. But we may, in place of these, consider the quadriplanar coordinates (x, y, z, w) linear functions of the original rectangular coordinates x, y, z.

Quaternions.—The position of a point in reference to the fixed origin O is determined by its vector α, which is in fact $= ix + jy + kz$, where i, j, k are the Hamiltonian symbols $(i^2 = j^2 = k^2 = -1,\ jk = -kj = i,\ ki = -ik = j,\ i = -ji = k)$; but the idea is to use as little as possible the foregoing equation $\alpha = ix + jy + kz$, and thus to conduct the investigations independently, as far as may be, of the particular positions of the axes Ox, Oy, Oz.

I consider the problem to determine the line OC at right angles to the plane of the lines OA, OB.

Coordinates.

Taking O as origin, the coordinates of A, B, C are taken to be

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x, y, z)$$

respectively. Then

$$xx_1 + yy_1 + zz_1 = 0,$$

$$xx_2 + yy_2 + zz_2 = 0;$$

whence

$$x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1.$$

Quaternions.

Points A, B, C are determined by their vectors α, β, γ. Then

$$S\alpha\gamma = 0, \quad S\beta\gamma = 0;$$

whence

$$m\gamma = V\alpha\beta,$$

m being an arbitrary scalar.

Here to compare the two solutions, observe that the two equations $S\alpha\gamma = 0$, $S\beta\gamma = 0$ are in fact the equations $xx_1 + yy_1 + zz_1 = 0$, $xx_2 + yy_2 + zz_2 = 0$; and so also $m\gamma = V\alpha\beta$ denotes the relations $x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$. But a quaternionist says that $m\gamma = V\alpha\beta$ *is* the compendious and elegant solution of the problem as opposed to the artificial and clumsy one $x : y : z = y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1$. And it is upon this that I join issue; $m\gamma = V\alpha\beta$ is a very pretty formula, like the folded-up pocket-map, but, to be intelligible, I consider that it requires to be developed into the other form. Of course, the example is as simple a one as could have been selected; and, in the case of a more complicated example, the mere abbreviation of the quaternion formula would be very much greater, but just for this reason there is the more occasion for the developed coordinate formula. To take another example,

the condition, in order that the vectors α, β, γ may be coplanar, is $S\alpha\beta\gamma = 0$, and Professor Tait contrasts this with the prolixity of the corresponding coordinate formula

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0.$$

I remark that, when all the components of a determinant have to be expressed, nothing can be shorter than this, the ordinary determinant notation, which simply expresses the several components in their line-and-column relation to each other. But as a mere abbreviation, it would be allowable to write Δ, $=(ABC)$, to denote the determinant formed by the coordinates of the three points.

In conclusion, I would say that while coordinates are applicable to the whole science of geometry, and are the natural and appropriate basis and method in the science, quaternions seem to me a particular and very artificial method for treating such parts of the science of three-dimensional geometry as are most naturally discussed by means of the rectangular coordinates x, y, z.

963.

NOTE ON DR MUIR'S PAPER, "A PROBLEM OF SYLVESTER'S IN ELIMINATION."

[From the *Proceedings of the Royal Society of Edinburgh*, vol. XX. (1895), pp. 306—308. Received November 6, 1894.]

I IN part reproduce this very interesting paper for the sake of a remark which appears to me important. I write (a, b, c, f, g, h) in place of Muir's (A, B, C, A', B', C'), and take as usual (A, B, C, F, G, H) and K to denote

$$(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch)$$

and the discriminant $abc-af^2-bg^2-ch^2+2fgh$

I then write

$$\begin{aligned} U &= bz^2-2fyz+cy^2, & P &= fx^2+ayz-hzx-gxy, & L &= bcx^2+afyz-bgzx-chxy,\\ V &= cx^2-2gzx+az^2, & Q &= gy^2-hyz+bzx-fxy, & M &= cay^2-afyz+bgzx-chxy,\\ W &= ay^2-2hxy+bx^2, & R &= hz^2-gyz-fzx+cxy, & N &= abz^2-afyz-bgzx+chxy. \end{aligned}$$

The equations $U=0$, $V=0$, $W=0$, imply $P=0$, $Q=0$, $R=0$, but observe that P, Q, R are not the sums of mere numerical multiples of U, V, W; we, in fact, have identically

$$\begin{aligned} 2yzP &= -x^2U+y^2V+z^2W,\\ 2zxQ &= x^2U-y^2V+z^2W,\\ 2xyR &= x^2U+y^2V-z^2W. \end{aligned}$$

If then $U=0$, $V=0$, $W=0$, we have also $P=0$, $Q=0$, $R=0$, and we can from the six equations dialytically eliminate x^2, y^2, z^2, yz, zx, xy, thus obtaining a result, Determinant $=0$, which is $K^2=0$; this is, in fact, Sylvester's process for the elimination.

But L, M, N are sums of mere numerical multiples of U, V, W, viz. we have

$$\begin{aligned} 2L &= -aU + bV + cW, \\ 2M &= aU - bV + cW, \\ 2N &= aU + bV - cW, \end{aligned}$$

so that the original equations $U=0$, $V=0$, $W=0$ are equivalent to and may be replaced by $L=0$, $M=0$, $N=0$.

Muir shows that we have identically

$$\begin{aligned} L - fP &= x(Ax + Hy + Gz), \\ M - gQ &= y(Hx + By + Fz), \\ N - hR &= z(Gx + Fy + Cz), \end{aligned}$$

where observe that the first of these equations is

$$\left.\begin{aligned} &(fx^2 - ayz)(bz^2 - 2fyz + cy^2) \\ -&(fy^2 - byz)(cx^2 - 2gzx + az^2) \\ -&(fz^2 - cyz)(ay^2 - 2hxy + bx^2) \end{aligned}\right\} = 2xyz(Ax + Hy + Gz);$$

and similarly for the second and third equations.

He thence infers that the elimination may be performed by eliminating x, y, z from the equations

$$\begin{aligned} Ax + Hy + Gz &= 0, \\ Hx + By + Fz &= 0, \\ Gx + Fy + Cz &= 0, \end{aligned}$$

viz. that the result is

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0,$$

that is, $K^2 = 0$ as before.

The natural inference is that K being $=0$, the three linear equations in (x, y, z) are equivalent to two equations giving for the ratios $x : y : z$ rational values which should satisfy the original equations $U=0$, $V=0$, $W=0$: the fact is that there are no such values, but that, K being $=0$, the three equations are equivalent to a single equation: for observe that, combining for instance the first and second equations, these will be equivalent to each other if only

$$\frac{A}{H} = \frac{H}{B} = \frac{G}{F},$$

that is,

$$AB - H^2 = 0, \quad GH - AF = 0, \quad HF - BG = 0,$$

which are $cK=0$, $fK=0$, $gK=0$, all satisfied by $K=0$; and similarly for the first and third, and the second and third equations. It will be remembered that the true form of the result is not $K=0$ but $K^2=0$, and this seems to be an indication that the three equations should be, as they have been found to be, equivalent to a single equation.

The problem may be further illustrated as follows: instead of the original equations $U=0$, $V=0$, $W=0$, consider the like equations with the inverse coefficients (A, B, C, F, G, H), viz.

$$Bz^2 - 2Fyz + Cy^2 = 0,$$

$$Cx^2 - 2Gzx + Az^2 = 0,$$

$$Ay^2 - 2Hxy + Bx^2 = 0,$$

so that the result of the elimination should be

$$(ABC - AF^2 - BG^2 - CH^2 + 2FGH)^2 = 0.$$

Here considering in connexion with the triangle $x=0$, $y=0$, $z=0$ (say the vertices hereof are the points A, B, C) the conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

the first equation represents the pair of tangents from the point A to the conic, the second the pair of tangents from the point B to the conic, and the third the pair of tangents from the point C to the conic. The first and second pairs of tangents intersect in four points, and if one of the third pair of tangents passes through one of the four points, then it is at once seen that the conic must touch one of the sides $x=0$, $y=0$, $z=0$ of the triangle, viz. we must have $bc-f^2=0$, $ca-g^2=0$, or $ab-h^2=0$. But we have $a=BC-F^2$, &c., or writing

$$K_1 = ABC - AF^2 - BG^2 - CH^2 + 2FGH,$$

then these equations are $K_1A=0$, $K_1B=0$, $K_1C=0$, all satisfied by $K_1=0$. We may regard $K_1=0$ as the condition in order that the conic $(a, b, c, f, g, h)(x, y, z)^2=0$ may be a *point-pair*: the analytical reason for this is not clear, but we see at once that, if the conic be a point-pair, then the three pairs of tangents are the lines drawn from the points A, B, C respectively to the two points of the point-pair, so that the three pairs of tangents have in common these two points. Regarding $K_1=0$ as the condition in order to the existence of a single common point, and recollecting that the true result of the elimination is $K_1^2=0$, the form perhaps indicates what we have just seen is the case, that there are in fact two common points of intersection: but at any rate the foregoing geometrical considerations lead to $K_1=0$, as the condition for the coexistence of the three equations.

I remark in conclusion that I do not know that there is any general theory of the case where a result of elimination presents itself in the form $\Omega^2=0$, as distinguished from the ordinary form $\Omega=0$.

964.

ON THE NINE-POINTS CIRCLE OF A SPHERICAL TRIANGLE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVII. (1895), pp. 35—39.]

THE definition is in effect given in Hart's paper, "Extension of Terquem's theorem respecting the circle which bisects three sides of a triangle," *Quarterly Mathematical Journal*, t. IV. (1861), pp. 260, 261, viz. if we have a spherical triangle ABC, then we have a circle (i.e. a small circle of the sphere), say the nine-points circle, meeting the sides BC, CA, AB in the points F, L; G, M; H, N respectively, where

$$\tan\tfrac{1}{2}BF = \frac{\cos\frac{1}{2}b - \cos\frac{1}{2}c\cos\frac{1}{2}a}{\cos\frac{1}{2}c\sin\frac{1}{2}a},$$

$$\tan\tfrac{1}{2}CF = \frac{\cos\frac{1}{2}c - \cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}b\sin\frac{1}{2}a},$$

(which equations agree with $BF + FC = BC$), and

$$\tan\tfrac{1}{2}BL = \frac{\cos\frac{1}{2}c\sin\frac{1}{2}a}{\cos\frac{1}{2}b + \cos\frac{1}{2}c\cos\frac{1}{2}a},$$

$$\tan\tfrac{1}{2}CL = \frac{\cos\frac{1}{2}b\sin\frac{1}{2}a}{\cos\frac{1}{2}c + \cos\frac{1}{2}a\cos\frac{1}{2}b},$$

(which equations agree with $BL + CL = BC$); and with the like formulæ for the points G, M: and H, N: respectively.

If, as usual, the sides of the triangle are called a, b, c, and for shortness we write

$$(\cos\tfrac{1}{2}a,\ \cos\tfrac{1}{2}b,\ \cos\tfrac{1}{2}c;\ \sin\tfrac{1}{2}a,\ \sin\tfrac{1}{2}b,\ \sin\tfrac{1}{2}c) = (p,\ q,\ r;\ p_1,\ q_1,\ r_1),$$

then the formulæ are

$$\tan\tfrac{1}{2}BF = \frac{q-rp}{rp_1}, \quad \tan\tfrac{1}{2}CF = \frac{r-pq}{qp_1},$$

$$\tan\tfrac{1}{2}CG = \frac{r-pq}{pq_1}, \quad \tan\tfrac{1}{2}AG = \frac{p-qr}{rq_1},$$

$$\tan\tfrac{1}{2}AH = \frac{p-qr}{qr_1}, \quad \tan\tfrac{1}{2}BH = \frac{q-rp}{pr_1},$$

and

$$\tan\tfrac{1}{2}BL = \frac{rp_1}{q+rp}, \quad \tan\tfrac{1}{2}CL = \frac{qp_1}{r+pq},$$

$$\tan\tfrac{1}{2}CM = \frac{pq_1}{r+pq}, \quad \tan\tfrac{1}{2}AM = \frac{rq_1}{p+qr},$$

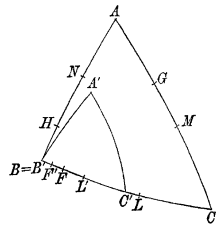
$$\tan\tfrac{1}{2}AN = \frac{qr_1}{p+qr}, \quad \tan\tfrac{1}{2}BN = \frac{pr_1}{q+rp}.$$

Before going further, it may be remarked that for a, b, c, each of them small, we have

$$\tfrac{1}{2}BF = \frac{(1-\tfrac{1}{2}b^2)-(1-\tfrac{1}{2}c^2)(1-\tfrac{1}{2}a^2)}{\tfrac{1}{2}a},$$

that is, $BF = \dfrac{a^2+c^2-b^2}{2a} = c\cos B$, and similarly $CF = a\cos C$, if A, B, C are the angles of the plane triangle; that is, in the plane triangle F, G, H are the feet of the perpendiculars let fall from the angles on the opposite sides. Moreover, $\tan\tfrac{1}{2}BL = \dfrac{\tfrac{1}{2}a}{1+1.1}$, that is, $BL = \tfrac{1}{2}a$, and similarly $CL = \tfrac{1}{2}a$; that is, L, M, N are the median points of the three sides respectively.

In the general case of the spherical triangle ABC, the construction is effected by means of a triangle $A'B'C'$, the sides whereof are respectively the halves of those



of the original triangle: viz. for this triangle $A'B'C'$, we construct the points F', G', H' and L', M', N', and then on the sides of the triangle ABC taking $BF = 2BF'$, $BL = 2BL'$, &c., we have the points F, G, H, L, M, N.

Thus p, q, r, p_1, q_1, r_1 denoting as above the cosines and sines of the half-sides of the triangle ABC, that is, the cosines and sines of the sides of the triangle $A'B'C'$, we have

$$\tan B'F' = \frac{q-rp}{rp_1}, \quad \tan C'F' = \frac{r-pq}{qp_1},$$

$$\vdots$$

$$\tan B'L' = \frac{rp_1}{q+rp}, \quad \tan C'L' = \frac{qp_1}{r+pq}.$$

$$\vdots$$

First, for the points F', G', H', we have $A'F'$, $B'G'$, $C'H'$, the perpendiculars from the angles on the opposite sides, meeting in a point O', the orthocentre of the triangle $A'B'C'$: in fact, from the triangle $A'B'F'$, right-angled at F', we have

$$\tan B'F' = \tan A'B' \cos B' = \tan \tfrac{1}{2}c \frac{\cos \frac{1}{2}b - \cos \frac{1}{2}c \cos \frac{1}{2}a}{\sin \frac{1}{2}c \sin \frac{1}{2}a},$$

$$= \frac{r_1}{r} \frac{q-rp}{r_1 p_1} = \frac{q-rp}{rp_1},$$

as above, and similarly for the points G' and H'.

I notice that we have

$$\sin B'F' = \frac{q-rp}{\sqrt{\{(q-rp)^2 + r^2 p_1^2\}}}, \quad = \frac{q-rp}{\sqrt{(q^2+r^2-2pqr)}},$$

and thus

$$\sin B'F' = \frac{q-rp}{\sqrt{(q^2+r^2-2pqr)}}, \quad \sin C'F' = \frac{r-pq}{\sqrt{(q^2+r^2-2pqr)}},$$

$$\sin C'G' = \frac{r-pq}{\sqrt{(r^2+p^2-2pqr)}}, \quad \sin A'G' = \frac{p-qr}{\sqrt{(r^2+p^2-2pqr)}},$$

$$\sin A'H' = \frac{p-qr}{\sqrt{(p^2+q^2-2pqr)}}, \quad \sin B'H' = \frac{q-rp}{\sqrt{(p^2+q^2-2pqr)}};$$

hence

$$\sin B'F' . \sin C'G' . \sin A'H' = \sin C'F' . \sin A'G' . \sin B'H',$$

which (as is well known) is the condition for the intersection of the arcs $A'F'$, $B'G'$, $C'H'$ in the orthocentre O'.

But I say further that we have

$$\sin 2B'F' \; (= \sin BF) = \frac{2(q-rp)rp_1}{q^2+r^2-2pqr},$$

$$\sin 2C'F' \; (= \sin CF) = \frac{2(r-pq)qp_1}{q^2+r^2-2pqr},$$

and thence

$$\sin BF . \sin CG . \sin AH = \sin CF . \sin AG . \sin BH,$$

and thus the arcs AF, BG, CH meet in a point which is obviously *not* the orthocentre of the triangle ABC.

Secondly, for the points L', M', N', we have

$$\sin B'L' = \frac{rp_1}{\sqrt{(q^2+r^2+2pqr)}}, \quad \sin C'L' = \frac{qp_1}{\sqrt{(q^2+r^2+2pqr)}},$$

that is,

$$\sin B'L' : \sin C'L' = r : q, \quad = \cos B'A' : \cos C'A';$$

and similarly

$$\sin C'M' : \sin A'M' = p : r, \quad = \cos C'B' : \cos A'B',$$

$$\sin A'N' : \sin B'N' = q : p, \quad = \cos A'C' : \cos B'C',$$

viz. the sides $B'C'$, $C'A'$, $A'B'$ are by the points L', M', N' divided each into two parts such that for any side the sines of the two parts are proportional to the cosines of the other two sides. We have

$$\sin B'L' . \sin C'M' . \sin A'N' = \sin C'L' . \sin A'M' . \sin B'N',$$

viz. the arcs $A'L'$, $B'M'$, $C'N'$ meet in a point K' which may be called the cos-centre of the triangle $A'B'C'$ (where observe that, for a, b, c indefinitely small, i.e. for a plane triangle, the points L', M', N' are the mid-points of the sides, and the centre K' is the C. G. or median point of the triangle).

But further, we have

$$\sin 2B'L' (= \sin BL) = \frac{2rp_1(q+rp)}{q^2+r^2+2pqr},$$

$$\vdots$$

$$\sin 2C'L' (= \sin CL) = \frac{2qp_1(r+pq)}{q^2+r^2+2pqr},$$

$$\vdots$$

and thence

$$\sin BL . \sin CM . \sin AN = \sin CL . \sin AM . \sin BN,$$

viz. the arcs AL, BM, CN meet in a point, which is obviously *not* the cos-centre of the triangle ABC.

We have thus the construction of the nine-points circle as a six-points circle, by means of the points F, G, H, L, M, N; and by way of recapitulation we may say that the nine-points circle meets the sides BC, CA, AB in the points F, L; G, M; H, N respectively, where the points F, G, H depend on the ortho-centre of the semi-triangle, and the points L, M, N depend on the cos-centre of the semi-triangle.

The triangle ABC has an inscribed circle and three escribed circles, and we have (as is known) the theorem that the nine-points circle touches each of these four circles. The circles BC, CA, AB and the nine-points circle form a tetrad of circles, and the inscribed circle and the three escribed circles a tetrad of circles, or say the eight circles form a bitetrad, such that each circle of the one tetrad touches each circle of the other tetrad.

965.

ON THE SIXTY ICOSAHEDRAL SUBSTITUTIONS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXVII. (1895), pp. 236—242.]

THE Sixty Icosahedral Substitutions were obtained in an elegant form by Gordan in the paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Math. Ann.* t. XII. (1877), pp. 23—46, see p. 45, where the group is exhibited in the canonical form

$$\epsilon^{\mu}\eta,\quad \frac{-\epsilon^{\mu}}{\eta},\quad \epsilon^{\mu}\frac{(\epsilon^2+\epsilon^4)\,\eta+\epsilon^{\rho}}{\epsilon^{-\rho}\eta-(\epsilon+\epsilon^3)},\quad \epsilon^{\mu}\frac{-\epsilon^{-\rho}\eta+(\epsilon+\epsilon^3)}{(\epsilon^2+\epsilon^4)\,\eta+\epsilon^{\rho}}$$

(ϵ an imaginary fifth root of unity), agreeing with the developed form given in my paper, "On the Schwarzian Derivative and the Polyhedral Functions," *Camb. Phil. Trans.*, t. XIII. Part 1 (1881), [745], pp. 5—68, see pp. 55—56, [this Collection, vol. XI., pp. 204—205].

But a slight change of form is desirable. I write $\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$ to denote the unimodular matrix

$$\begin{Bmatrix} \dfrac{a}{\sqrt{(ad-bc)}}, & \dfrac{b}{\sqrt{(ad-bc)}} \\ \dfrac{c}{\sqrt{(ad-bc)}}, & \dfrac{d}{\sqrt{(ad-bc)}} \end{Bmatrix},$$

so that the value is independent of the absolute magnitudes of a, b, c, d, but depends only on their ratios; and observe that $m\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$ denotes $\begin{Bmatrix} ma, & mb \\ c, & d \end{Bmatrix}$, the factor m applying to the top-line only. As before, ϵ denotes an imaginary fifth root of unity, and I write $A=\epsilon+\epsilon^4$, $B=\epsilon^2+\epsilon^3$, $A'=\epsilon-\epsilon^4$, $B'=\epsilon^2-\epsilon^3$, but, in fact, I scarcely need more than the first of these symbols.

The substitutions are

$$(x_1,\ y_1) = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x,\ y),$$

that is, $x_1 = ax + by$, $y_1 = cx + dy$; and we then have sixty matrices $\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}$, forming the group in question.

Giving to ρ the values 0, 1, 2, 3, 4, I write

$$K_\rho = \begin{Bmatrix} \epsilon^\rho, & 0 \\ 0, & \epsilon^{-\rho} \end{Bmatrix},$$

so that K_0 is the matrix unity $\begin{Bmatrix} 1, & 0 \\ 0, & 1 \end{Bmatrix}$,

$$L_\rho = \begin{Bmatrix} 0\ , & -\epsilon^\rho \\ \epsilon^{-\rho}, & 0 \end{Bmatrix},$$

$$M_\rho = \begin{Bmatrix} A\ , & \epsilon^\rho \\ \epsilon^{-\rho}, & -A \end{Bmatrix}, \quad \text{mod.} \begin{vmatrix} A\ , & \epsilon^\rho \\ \epsilon^{-\rho}, & A \end{vmatrix} \text{ is } = -A^2 - 1,\ = B'^2,$$

$$N_\rho = \begin{Bmatrix} -\epsilon^\rho, & A \\ A, & \epsilon^{-\rho} \end{Bmatrix}, \quad \text{mod.} \begin{vmatrix} -\epsilon^\rho, & A \\ A, & \epsilon^{-\rho} \end{vmatrix} \text{ is } = -A^2 - 1,\ = B''^2;$$

then the matrices are

$$\begin{array}{c|ccccc|l}
 & K_0, & K_1, & K_2, & K_3, & K_4 & \\
 & L_0, & L_1, & L_2, & L_3, & L_4 & \\
 & M_0, & M_1, & M_2, & M_3, & M_4 & \\
\hline
1 & & & & & & \text{read } M_0,\ M_1,\ \ldots, \\
\epsilon & & & & & & \quad \epsilon M_0, \\
\epsilon^2 & & & & & & \quad \vdots \\
\epsilon^3 & & & & & & \\
\epsilon^4 & & & & & & \\
\hline
 & N_0, & N_1, & N_2, & N_3, & N_4 & \\
\hline
1 & & & & & & \quad ,, \quad N_0,\ N_1,\ \ldots, \\
\epsilon & & & & & & \quad \epsilon N_0, \\
\epsilon^2 & & & & & & \quad \vdots \\
\epsilon^3 & & & & & & \\
\epsilon^4 & & & & & &
\end{array}$$

viz. $5 + 5 + 25 + 25,\ = 60$ matrices in all, including the matrix unity.

It is to be observed that the system must remain unaltered if we change ϵ into ϵ^2, ϵ^3, or ϵ^4: this is at once obvious for the change ϵ into ϵ^4, but less immediately so for the other two changes. Suppose ϵ changed into ϵ^2, then M''_ρ becomes

$$= \begin{Bmatrix} B\ , & \epsilon^{2\rho} \\ \epsilon^{-2\rho}, & -B \end{Bmatrix}, \ = \begin{Bmatrix} AB\ , & A\epsilon^{2\rho} \\ A\epsilon^{-2\rho}, & -AB \end{Bmatrix} = \begin{Bmatrix} 1\ , & A\epsilon^{2\rho} \\ A\epsilon^{-2\rho}, & -1 \end{Bmatrix} \text{ (since } AB = 1) = \epsilon^{4\rho} \begin{Bmatrix} -\epsilon^{-2\rho}, & A \\ A\ , & \epsilon^{2\rho} \end{Bmatrix},$$

or changing ρ into $-\frac{1}{2}\rho$, which has the same values, this is $=\epsilon^{-2\rho}\begin{Bmatrix}-\epsilon^{\rho}, & A\\ A, & \epsilon^{-\rho}\end{Bmatrix}$, viz. M_ρ is equal to a power of ϵ multiplied by N_ρ: and similarly N_ρ becomes equal to a power of ϵ multiplied by M_ρ. And the like as regards the change ϵ into ϵ^3.

I find that the sixty matrices may be coordinated with the sixty positive substitutions of the letters *abcde* as follows:

1	$K_0\begin{Bmatrix}1, & 0\\ 0, & 1\end{Bmatrix}$	1	17	$\epsilon^2 M_0,\ \epsilon^2\begin{Bmatrix}A, & 1\\ 1-A & \end{Bmatrix}$	*ced*
2	$L_0\begin{Bmatrix}0, & -1\\ 1, & 0\end{Bmatrix}$	*ad . bc*	18	$\epsilon^2 M_1,\ \epsilon^2\begin{Bmatrix}A, & \epsilon\\ \epsilon^4-A & \end{Bmatrix}$	*adc*
3	$L_1\begin{Bmatrix}0, & -\epsilon\\ \epsilon^4, & 0\end{Bmatrix}$	*ab . de*	19	$\epsilon^2 M_2,\ \epsilon^2\begin{Bmatrix}A, & \epsilon^2\\ \epsilon^3-A & \end{Bmatrix}$	*adb*
4	$L_2\begin{Bmatrix}0, & -\epsilon^2\\ \epsilon^3, & 0\end{Bmatrix}$	*ac . be*	20	$\epsilon^2 M_3,\ \epsilon^2\begin{Bmatrix}A, & \epsilon^3\\ \epsilon^2-A & \end{Bmatrix}$	*acb*
5	$L_3\begin{Bmatrix}0, & -\epsilon^3\\ \epsilon^2, & 0\end{Bmatrix}$	*bd . ce*	21	$\epsilon^2 M_4,\ \epsilon^2\begin{Bmatrix}A, & \epsilon^4\\ \epsilon-A & \end{Bmatrix}$	*bce*
6	$L_4\begin{Bmatrix}0, & -\epsilon^4\\ \epsilon, & 0\end{Bmatrix}$	*ae . cd*	22	$\epsilon^3 M_0,\ \epsilon^3\begin{Bmatrix}A, & 1\\ 1-A & \end{Bmatrix}$	*abc*
7	$M_0\begin{Bmatrix}A, & 1\\ 1-A & \end{Bmatrix}$	*ac . bd*	23	$\epsilon^3 M_1,\ \epsilon^3\begin{Bmatrix}A, & \epsilon\\ \epsilon^4-A & \end{Bmatrix}$	*bec*
8	$M_1\begin{Bmatrix}A, & \epsilon\\ \epsilon^4-A & \end{Bmatrix}$	*ae . bd*	24	$\epsilon^3 M_2,\ \epsilon^3\begin{Bmatrix}A, & \epsilon^2\\ \epsilon^3-A & \end{Bmatrix}$	*cde*
9	$M_2\begin{Bmatrix}A, & \epsilon^2\\ \epsilon^3-A & \end{Bmatrix}$	*ae . bc*	25	$\epsilon^3 M_3,\ \epsilon^3\begin{Bmatrix}A, & \epsilon^3\\ \epsilon^2-A & \end{Bmatrix}$	*acd*
10	$M_3\begin{Bmatrix}A, & \epsilon^3\\ \epsilon^2-A & \end{Bmatrix}$	*bc . de*	26	$\epsilon^3 M_4,\ \epsilon^3\begin{Bmatrix}A, & \epsilon^4\\ \epsilon-A & \end{Bmatrix}$	*abd*
11	$M_4\begin{Bmatrix}A, & \epsilon^4\\ \epsilon-A & \end{Bmatrix}$	*ac . de*	27	$\epsilon N_0,\ \epsilon\begin{Bmatrix}-1, & A\\ A, & 1\end{Bmatrix}$	*aec*
12	$N_0\begin{Bmatrix}-1, & A\\ A, & 1\end{Bmatrix}$	*ab . cd*	28	$\epsilon^4 N_1,\ \epsilon^4\begin{Bmatrix}-\epsilon, & A\\ A, & \epsilon^4\end{Bmatrix}$	*bcd*
13	$N_1\begin{Bmatrix}-1, & A\epsilon\\ A\epsilon^4, & 1\end{Bmatrix}$	*ad . be*	29	$\epsilon^2 N_2,\ \epsilon^2\begin{Bmatrix}-\epsilon^2, & A\\ A, & \epsilon^3\end{Bmatrix}$	*abc*
14	$N_2\begin{Bmatrix}-1, & A\epsilon^2\\ A\epsilon^3, & 1\end{Bmatrix}$	*ab . ce*	30	$N_3,\ \begin{Bmatrix}-\epsilon^3, & A\\ A, & \epsilon^2\end{Bmatrix}$	*aed*
15	$N_3\begin{Bmatrix}-1, & A\epsilon^3\\ A\epsilon^2, & 1\end{Bmatrix}$	*be . cd*	31	$\epsilon^3 N_4,\ \epsilon^3\begin{Bmatrix}-\epsilon^4, & A\\ A, & \epsilon\end{Bmatrix}$	*bed*
16	$N_4\begin{Bmatrix}-1, & A\epsilon^4\\ A\epsilon, & 1\end{Bmatrix}$	*ad . ce*			

32	$\epsilon^4 N_0,\ \epsilon^4 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	bde	46	$\epsilon^4 M_0,\ \epsilon^4 \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	$aebcd$
33	$\epsilon^2 N_1,\ \epsilon^2 \begin{Bmatrix} -\epsilon, & A \\ A, & \epsilon^4 \end{Bmatrix}$	acb	47	$\epsilon^4 M_1,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon \\ \epsilon^4-A & \end{Bmatrix}$	$abced$
34	$N_2,\ \begin{Bmatrix} -\epsilon^2, & A \\ A, & \epsilon^3 \end{Bmatrix}$	ade	48	$\epsilon^4 M_2,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon^2 \\ \epsilon^3-A & \end{Bmatrix}$	$abedc$
35	$\epsilon^2 N_3,\ \epsilon^3 \begin{Bmatrix} -\epsilon^3, & A \\ A, & \epsilon^2 \end{Bmatrix}$	bde	49	$\epsilon^4 M_3,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon^3 \\ \epsilon^2-A & \end{Bmatrix}$	$adbec$
36	$\epsilon N_4,\ \epsilon \begin{Bmatrix} -\epsilon^4, & A \\ A, & \epsilon \end{Bmatrix}$	ace	50	$\epsilon^4 M_4,\ \epsilon^4 \begin{Bmatrix} A, & \epsilon^4 \\ \epsilon-A & \end{Bmatrix}$	$aecdb$
37	$K_1,\ \begin{Bmatrix} \epsilon, & 0 \\ 0, & \epsilon^4 \end{Bmatrix}$	$aedbc$	51	$\epsilon^2 N_0,\ \epsilon^2 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	$acbed$
38	$K_2,\ \begin{Bmatrix} \epsilon^2, & 0 \\ 0, & \epsilon^3 \end{Bmatrix}$	$adceb$	52	$N_1,\ \begin{Bmatrix} -\epsilon, & A \\ A, & \epsilon^4 \end{Bmatrix}$	$abcde$
39	$K_3,\ \begin{Bmatrix} \epsilon^3, & 0 \\ 0, & \epsilon^2 \end{Bmatrix}$	$abecd$	53	$\epsilon^3 N_2,\ \epsilon^3 \begin{Bmatrix} -\epsilon^2, & A \\ A, & \epsilon^3 \end{Bmatrix}$	$aecbd$
40	$K_4,\ \begin{Bmatrix} \epsilon^4, & 0 \\ 0, & \epsilon \end{Bmatrix}$	$acbde$	54	$\epsilon N_3,\ \epsilon \begin{Bmatrix} -\epsilon^3, & A \\ A, & \epsilon^2 \end{Bmatrix}$	$acdbe$
41	$\epsilon M_0,\ \epsilon \begin{Bmatrix} A, & 1 \\ 1-A & \end{Bmatrix}$	$adecb$	55	$\epsilon^4 N_4,\ \epsilon^4 \begin{Bmatrix} -\epsilon^4, & A \\ A, & \epsilon \end{Bmatrix}$	$abdec$
42	$\epsilon M_1,\ \epsilon \begin{Bmatrix} A, & \epsilon \\ \epsilon^4-A & \end{Bmatrix}$	$acdeb$	56	$\epsilon^3 N_0,\ \epsilon^3 \begin{Bmatrix} -1, & A \\ A, & 1 \end{Bmatrix}$	$adbce$
43	$\epsilon M_2,\ \epsilon \begin{Bmatrix} A, & \epsilon^2 \\ \epsilon^3-A & \end{Bmatrix}$	$acebd$	57	$\epsilon N_1,\ \epsilon \begin{Bmatrix} -\epsilon, & A \\ A, & \epsilon^4 \end{Bmatrix}$	$aebdc$
44	$\epsilon M_3,\ \epsilon \begin{Bmatrix} A, & \epsilon^3 \\ \epsilon^2-A & \end{Bmatrix}$	$abdce$	58	$\epsilon^4 N_2,\ \epsilon^4 \begin{Bmatrix} -\epsilon^2, & A \\ A, & \epsilon^3 \end{Bmatrix}$	$acedb$
45	$\epsilon M_4,\ \epsilon \begin{Bmatrix} A, & \epsilon^4 \\ \epsilon-A & \end{Bmatrix}$	$adcbe$	59	$\epsilon^2 N_3,\ \epsilon^2 \begin{Bmatrix} -\epsilon^3, & A \\ A, & \epsilon^2 \end{Bmatrix}$	$adebc$
			60	$N_4,\ \begin{Bmatrix} -\epsilon^4, & A \\ A, & \epsilon \end{Bmatrix}$	$aedcb$

As an example of the composition of the forms, take 56, 57: we have

$$56 = adbce\,.\,abdec,$$
$$57 = aebdc\,.\,edacb,$$
$$abcde,$$

whence

$$56\,.\,57 = cde, \ = 24,$$

and we have

$$56 \,.\, 57 = \begin{matrix} -\epsilon^3, & A\epsilon^3 \\ A, & 1 \end{matrix} \overline{\left|\begin{matrix} (-\epsilon^2, & A), & (\epsilon A, & \epsilon^4) \\ & & & \end{matrix}\right|} = \left|\begin{matrix} 1 + A^2\epsilon^3, & -A\epsilon^4 + A\epsilon^3 \\ -A\epsilon^2 + A, & A^2\epsilon + \epsilon^4 \end{matrix}\right| ; \; 24 = \begin{matrix} A\epsilon^3, & 1, \\ \epsilon^3, & -A, \end{matrix}$$

viz. taking out the factor $A(\epsilon^2 - \epsilon^4)$, we ought to have

$$\left|\begin{matrix} 1 + A^2\epsilon^3, & A(\epsilon^3 - \epsilon^4) \\ A(1 - \epsilon^2), & A(\epsilon^2 - \epsilon^4) \end{matrix}\right| = \left|\begin{matrix} A\epsilon^3, & 1 \\ \epsilon^3 \;, & -A \end{matrix}\right| \begin{matrix} A(\epsilon^3 - \epsilon^4) \\ A(\epsilon^2 - \epsilon^4) \end{matrix} :$$

a relation which will be identically true if only

$$1 + A^2\epsilon^3 = A^2(1 - \epsilon^2),$$
$$\epsilon^4 + A^2\epsilon = A^2(-\epsilon^2 + \epsilon^4).$$

These are

$$1 = A^2(\quad 1 - \epsilon^2 - \epsilon^3),$$
$$\epsilon^4 = A^2(-\epsilon - \epsilon^2 + \epsilon^4),$$

the second of which is obviously equivalent to the first: and this first equation is

$$1 = A^2(A + 2),$$

that is,

$$A^3 + 2A^2 - 1 = 0, \text{ or } (A + 1)(A^2 + A - 1) = 0,$$

which is true in virtue of $A^2 + A - 1 = 0$; we have thus the required equation

$$56 \,.\, 57 = 24,$$

or say

$$\left\{\begin{matrix} -\epsilon^3, & A\epsilon^3 \\ A, & 1 \end{matrix}\right\} \left\{\begin{matrix} -\epsilon^2, & A\epsilon \\ A, & \epsilon^4 \end{matrix}\right\} = \left\{\begin{matrix} A, & \epsilon^2 \\ \epsilon^3, & -A \end{matrix}\right\}.$$

In like manner it may be shown that the product of any two of the sixty matrices is a matrix of the group.

There is an interesting case of linear transformation in Gordan's paper "Ueber die Auflösung der Gleichungen vom fünften Grade," *Math. Ann.*, t. XIII. (1878), pp. 375—404, see p. 379. Writing as before $A = \epsilon + \epsilon^4$, $B = \epsilon^2 + \epsilon^3$, $A' = \epsilon - \epsilon^4$, $B' = \epsilon^2 - \epsilon^3$, then for y_1, y_2 we substitute $Ay_1 + y_2$, $y_1 - Ay_2$, each divided by B'; and for x_1, x_2 we make the like substitution, ϵ^2 instead of ϵ, viz. this is for x_1, x_2, $Bx_1 + x_2$, $x_1 - Bx_2$, each divided by $-A'$.

This being so, the bipartite function

$$f = (y_1^3,\; y_1^2y_2,\; y_1y_2^2,\; y_2^3)(x_1^2x_2,\; x_2^3,\; x_1^3,\; -x_1x_2^2),$$

or, what is the same thing,

	x_1^3,	$x_1^2x_2$,	$x_1x_2^2$,	x_2^3
$= \; y_1^3$		1		
$y_1^2y_2$				1
$y_1y_2^2$	1			
y_2^3			-1	

remains unaltered by the substitutions. Observe that we have $AB=-1$, so that, omitting common factors, the substitutions are

$$\begin{aligned} &y_1,\ y_2 \text{ multiplied by } && Ay_1+y_2,\ y_1-Ay_2,\\ &x_1,\ x_2 \text{ multiplied by } && -(x_1-Ax_2),\ Ax_1+x_2,\end{aligned}$$

where it is to be noticed that (x_1, x_2) are neither cogredient nor contragredient with (y_1, y_2); but they are what may be termed socio-gredient, viz. the substitution for (y_1, y_2) determines uniquely that for (x_1, x_2).

The verification of the invariance of f might be effected rather more simply by means of the last-mentioned forms, but it is interesting to use the original forms

$$\begin{aligned} &y_1,\ y_2 \text{ multiplied by } && Ay_1+y_2,\ y_1-Ay_2,\\ &x_1,\ x_2 \text{ multiplied by } && Bx_1+x_2,\ x_1-Bx_2.\end{aligned}$$

Making the substitution, the whole coefficient of y_1^3 is found to be

$$\begin{aligned} &(A^3B^2+A^2+AB^3-B)\,x_1^3\\ &+(2A^3B-A^3B^3-3A^2B+3AB^2+2B^2-1)\,x_1^2x_2\\ &+(A^3-2A^3B^2+3A^2B^2+3AB-B^3+2B)\,x_1x_2^2\\ &+(-A^3B-A^2B^3+A-B^2)\,x_2^3,\end{aligned}$$

where, reducing by $AB=-1$, the several coefficients are

$$\begin{aligned} A\ +A^2-B^2-B,\quad &=0,\\ -2A^2+3A+2B^2-3B,\ &=5(A-B),\\ A^3-2A-B^3+2B,\quad &=0,\\ A^2-B+A-B^2,\quad &=0,\end{aligned}$$

in virtue of the relations $A^2+A=1$, $B^2+B=1$. There are similar reductions for the coefficients of $y_1^2y_2$, $y_1y_2^2$ and y_2^3; and the whole thus becomes $=5(A-B)$, multiplied by

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
y_1^3		1		
$y_1^2y_2$				1
$y_1y_2^2$	1			
y_2^3			-1	

;

but there is a denominator $-A'^3B'^3$, and we have $A'B'=B-A$, or this denominator is $(A-B)^3$, $=(A-B)(A-B)^2$, where

$$(A-B)^2=A^2-2AB+B^2,\ =1-A+2+1-B,\ =4-A-B,\ =5,$$

viz. the denominator is $=5(A-B)$. Thus the new value of f is equal to its original value, which is the theorem in question.

966.

NOTE ON A MEMOIR IN SMITH'S COLLECTED PAPERS.

[From the *Bulletin of the American Mathematical Society*, Ser. 2, vol. I. (1895), pp. 94—96.]

AMONG the most noticeable papers in the *Collected Mathematical Papers* of H. J. S. Smith we have the hitherto unpublished "Memoir on the Theta and Omega Functions," XLIII. (vol. II. pp. 415—623), written in connexion with Dr Glaisher's Tables of the Theta-Functions and originally intended as an Introduction thereto. It appears that in 1873 or 1874 Dr Glaisher asked him, as a member of the British Association committee for the calculation of the Tables, whether he would contribute an Introduction. His reply was that he did not see his way to writing anything appropriate to the tables themselves, but that he "could say something with respect to the constants at the head of the pages." These constants were K, K', E, J, J', &c., the numerical values whereof were given for every minute of the modular angle. The memoir grew in extent, and it was finally decided that it should follow these yet unpublished tables with the before-mentioned title, "Memoir on the Theta and Omega Functions," but fortunately it has at length appeared in the Collected Papers as above.

In explanation of the title and scope of the memoir, it will be remembered that the Theta-Functions are functions of two arguments, x and q; so that giving to x the value zero or any numerical value, or any value depending on that of q, we obtain a series of functions containing the single argument q, or writing as usual $q = e^{i\pi\omega}$, say the single argument ω; and in the memoir, the attention is directed chiefly but not exclusively to these functions of a single argument which are termed *Omega Functions*. The functions chiefly considered under this designation are Hermite's functions $\phi\omega$, $\psi\omega$, $\chi\omega$, which represent the values of $\sqrt{k}$, $\sqrt[4]{k'}$, and $\sqrt[12]{kk'}$ considered as functions of $q = e^{i\pi\omega}$. To fix the ideas, it may be mentioned that the actual values (in one of their very numerous forms) are

$$\phi\omega = \sqrt{2}q_{\frac{1}{8}} \prod_1^\infty \frac{1+q^{2m}}{1+q^{2m-1}},$$

$$\psi\omega = \prod_1^\infty \frac{1-q^{2m-1}}{1+q^{2m-1}},$$

$$\chi\omega = \sqrt[6]{2}q_{\frac{1}{24}} \prod_1^\infty \frac{1}{1+q^{2m-1}},$$

each of them a one-valued function of ω: any rational and integral function of $\phi\omega$, $\psi\omega$, $\chi\omega$ is termed a *Modular* Function. It is right to add that the definitions extend only to those values of q for which the series are convergent, or (what is the same thing) ω regarded as a point must be situate within a certain region of the upper infinite half-plane $y = +$.

The requisite formulæ for the Theta-Functions are obtained from Jacobi's fundamental formula for the multiplication of four Theta-Functions; and the Elliptic Functions are introduced by means of their definitions in terms of the Theta-Functions: and the whole theory of Elliptic Functions is thus brought into connexion with the Theta and Omega Functions. The theory of Transformation depends in a great measure on the arithmetical and geometrical theory of binary matrices, of which the constituents are integral numbers; this theory plays an extensive part throughout the memoir.

An abstract of the contents of the memoir is as follows:

Arts. 1—14. Definitions and Elementary Properties of the Theta, Omega, and Elliptic Functions.

15—23. Arithmetical Theory of Binary Matrices.

24—34. The Transformation of the Theta and Omega Functions.

35—45. Geometrical Representation of Binary Quadratic Forms.

46—51. Geometrical Representation of the Modular Functions

$$\Phi\omega \text{ and } \Psi\omega \ (= \phi^8\omega,\ \psi^8\omega).$$

52—58. The Modular Equation.

59—62. The Equation of the Multiplier.

63—73. The Modular Curves.

74—82. Theory of the Modular Functions $\phi\omega$ and $\psi\omega$.

83—88. Theory of the Modular Function

$$T\omega = (1 - \chi^{24}\omega)^3 \div \chi^{48}\omega.$$

89—90. The Differential Equation of the Modular Equations and Curves (this last section somewhat incomplete).

A good deal of the same ground is gone over in Weber's *Elliptische Functionen und Algebraische Zahlen* (8vo. Brunswick, 1891), a work which exhibits in a very compendious form the higher parts of the theory of Elliptic Functions, and which well deserves to be carefully studied.

967.

AN ELEMENTARY TREATISE ON ELLIPTIC FUNCTIONS.

THE first edition was published in 1876; the second edition was in the press at the time of Professor Cayley's death in 1895, and it was published in the course of that year.

END OF VOL. XIII.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.

MATHEMATICAL PAPERS.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 263, ARGYLE STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.
Bombay: E. SEYMOUR HALE.

THE COLLECTED

MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

LATE SADLERIAN PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF CAMBRIDGE.

SUPPLEMENTARY VOLUME,

CONTAINING TITLES OF PAPERS AND INDEX.

CAMBRIDGE:

AT THE UNIVERSITY PRESS.

1898.

[*All Rights reserved.*]

41538
9/6/98

CAMBRIDGE:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

QA
3
C42
Suppl.

PREFATORY NOTE.

THE present volume is supplementary to the series of thirteen volumes which contain the collected mathematical papers of the late Professor Cayley.

The first part is a list of the titles of the papers, extracted from the tables of contents of the respective volumes and arranged in the order in which the papers occur.

The second part is an index of subjects and authors. It has been made by my friend, Mr. F. Howard Collins, who most kindly volunteered to do this laborious work; my expectation is that the index will be a useful guide to the papers.

It is now ten years since the printing of the series of volumes began. For about the first seven of the years, Professor Cayley himself acted as Editor; since his death, the duty has fallen to me. My steady wish has been that unnecessary delay in the publication of the volumes should be avoided; that the wish has been realised, is largely due to the staff of the University Press. Everything that could be done in the way of simplifying my task and assisting its progress has been done with zealous good-will and cordial cooperation. To each and to all of them my thanks are given for the help which has enabled me to fulfil the duty I undertook at the request of the Syndics.

A. R. FORSYTH.

31 *January*, 1898.

COMPLETE LIST

OF

TITLES OF THE PAPERS.

[An Asterisk denotes that the paper is not printed in full.]

VOLUME I.

PAGE

1. *On a Theorem in the Geometry of Position* 1
Camb. Math. Journ. t. II. (1841), pp. 267—271 (1841)

2. *On the Properties of a certain Symbolical Expression* . . 5
Camb. Math. Journ. t. III. (1843), pp. 62—71 (1842)

3. *On certain Definite Integrals* 13
Camb. Math. Journ. t. III. (1843), pp. 138—144 (1842)

4. *On certain Expansions, in series of Multiple Sines and Cosines* 19
Camb. Math. Journ. t. III. (1843), pp. 162—167 (1842)

5. *On the Intersection of Curves* 25
Camb. Math. Journ. t. III. (1843), pp. 211—213 (1843)

6. *On the Motion of Rotation of a Solid Body* 28
Camb. Math. Journ. t. III. (1843), pp. 224—232 (1843)

7. *On a class of Differential Equations, and on the Lines of Curvature of an Ellipsoid* 36
Camb. Math. Journ. t. III. (1843), pp. 264—267 (1843)

8. *On Lagrange's Theorem* 40
Camb. Math. Journ. t. III. (1843), pp. 283—286 (1843)

9. *Demonstration of Pascal's Theorem* 43
Camb. Math. Journ. t. IV. (1845), pp. 18—20 (1843)

10. *On the Theory of Algebraical Curves* 46
Camb. Math. Journ. t. IV. (1845), pp. 102—112 (1844)

11. *Chapters in the Analytical Geometry of* (n) *Dimensions* . . 55
Camb. Math. Journ. t. IV. (1845), pp. 119—127 (1844)

12. *On the Theory of Determinants* 63
Camb. Phil. Trans. t. VIII. (1849), pp. 1—16 (1843)

13. *On the Theory of Linear Transformations* 80
Camb. Math. Journ. t. IV. (1845), pp. 193—209

14. *On Linear Transformations* 95
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 104—122

PAGE

15. *Note sur deux Formules données par MM. Eisenstein et Hesse* 113
Crelle, t. XXIX. (1845), pp. 54—57

*16. *Mémoire sur les Hyperdéterminants* 117
Crelle, t. XXX. (1846), pp. 1—37

*17. *Note on Mr Bronwin's paper on Elliptic Integrals* . . . 118
Camb. Math. Journ. t. III. (1843), pp. 197, 198

*18. *Remarks on the Rev. B. Bronwin's paper on Jacobi's Theory of Elliptic Functions* 119
Phil. Mag. t. XXII. (1843), pp. 358—368

19. *Investigation of the Transformation of certain Elliptic Functions* 120
Phil. Mag. t. XXV. (1844), pp. 352—354

20. *On certain results relating to Quaternions*. 123
Phil. Mag. t. XXVI. (1845), pp. 141—145

*21. *On Jacobi's Elliptic Functions, in reply to the Rev. B. Bronwin: and on Quaternions* 127
Phil. Mag. t. XXVI. (1845), pp. 208—211

22. *On Algebraical Couples* 128
Phil. Mag. t. XXVII. (1845), pp. 38—40

23. *On the Transformation of Elliptic Functions* 132
Phil. Mag. t. XXVII. (1845), pp. 424—427

24. *On the Inverse Elliptic Functions* 136
Camb. Math. Journ. t. IV. (1845), pp. 257—277

25. *Mémoire sur les Fonctions doublement périodiques* . . . 156
Liouville, t. X. (1845), pp. 385—420

26. *Mémoire sur les Courbes du Troisième Ordre* 183
Liouville, t. IX. (1844), pp. 285—293

27. *Nouvelles remarques sur les Courbes du Troisième Ordre*. . 190
Liouville, t. X. (1845), pp. 102—109

28. *Sur quelques Intégrales Multiples* 195
Liouville, t. X. (1845), pp. 158—168

29. *Addition à la Note sur quelques Intégrales Multiples* . . 204
Liouville, t. X. (1845), pp. 242—244

30. *Mémoire sur les Courbes à double Courbure et les Surfaces développables* 207
Liouville, t. X. (1845), pp. 245—250

31. *Démonstration d'un Théorème de M. Chasles* 212
Liouville, t. X. (1845), pp. 383, 384

PAGE

32. *On some Analytical Formulæ and their application to the Theory of Spherical Coordinates* 213
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 22—33

33. *On the Reduction of $du \div \sqrt{U}$, when U is a Function of the Fourth Order*. 224
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 70—73

34. *Note on the Maxima and Minima of Functions of Three Variables* 228
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 74, 75

35. *On Homogeneous Functions of the Third Order with Three Variables* 230
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 97—104

36. *On the Geometrical Representation of the Motion of a Solid Body* 234
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 164—167

37. *On the Rotation of a Solid Body round a Fixed Point* . . 237
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 167—173 and 264—274

38. *Note on a Geometrical Theorem contained in a Paper by Sir W. Thomson* 253
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 207, 208

39. *On the Diametral Planes of a Surface of the Second Order* . 255
Camb. and Dubl. Math. Journ. t. I. (1846), pp. 274—278

40. *On the Theory of Involution in Geometry* 259
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 52—61

41. *On certain Formulæ for Differentiation, with applications to the evaluation of Definite Integrals* 267
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 122—128

42. *On the Caustic by Reflection at a Circle* 273
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 128—130

43. *On the Differential Equations which occur in Dynamical Problems* 276
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 210—219

44. *On a Multiple Integral connected with the Theory of Attractions* 285
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 219—223

45. *On the Theory of Elliptic Functions* 290
Camb. and Dubl. Math. Journ. t. II. (1847), pp. 256—266

46. *Note on a System of Imaginaries* 301
Phil. Mag. t. XXX. (1847), pp. 257, 258

PAGE

47. *Sur la Surface des Ondes* 302
Liouville, t. XI. (1846), pp. 291—296

48. *Note sur les Fonctions de M. Sturm* 306
Liouville, t. XI. (1846), pp. 297—299

49. *Sur quelques Formules du Calcul Intégral* 309
Liouville, t. XII. (1847), pp. 231—240

50. *Sur quelques Théorèmes de la Géométrie de Position*. . . 317
Crelle, t. XXXI. (1846), pp. 213—227

51. *Problème de Géométrie Analytique* 329
Crelle, t. XXXI. (1846), pp. 227—230

52. *Sur quelques Propriétés des Déterminants Gauches* . . . 332
Crelle, t. XXXII. (1846), pp. 119—123

53. *Recherches sur l'Élimination, et sur la Théorie des Courbes* . 337
Crelle, t. XXXIV. (1847), pp. 30—45

54. *Note sur les Hyperdéterminants*. 352
Crelle, t. XXXIV. (1847), pp. 148—152

55. *Sur quelques Théorèmes de la Géométrie de Position*. . . 356
Crelle, t. XXXIV. (1847), pp. 270—275

56. *Demonstration of a Geometrical Theorem of Jacobi's* . . 362
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 48, 49

57. *On the Theory of Elliptic Functions* 364
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 50, 51

58. *Notes on the Abelian Integrals—Jacobi's System of Differential Equations* 366
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 51—54

59. *On the Theory of Elimination* 370
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 116—120

60. *On the Expansion of Integral Functions in a series of Laplace's Coefficients* 375
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 120, 121

61. *On Geometrical Reciprocity* 377
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 173—179

62. *On an Integral Transformation* 383
Camb. and Dubl. Math. Journ. t. III. (1848), pp. 286, 287

63. *Démonstration d'un Théorème de M. Boole concernant des Intégrales Multiples* 384
Liouville, t. XIII. (1848), pp. 245—248

PAGE

64. *Sur la généralisation d'un Théorème de M. Jellett qui se rapporte aux Attractions* 388
Liouville, t. XIII. (1848), pp. 264—268

65. *Nouvelles Recherches sur les Fonctions de M. Sturm* . . 392
Liouville, t. XIII. (1848), pp. 269—274

66. *Sur les Fonctions de Laplace* 397
Liouville, t. XIII. (1848), pp. 275—280

67. *Note sur les Fonctions Elliptiques* 402
Crelle, t. XXXVII. (1848), pp. 58—60

68. *On the application of Quaternions to the Theory of Rotation* . 405
Phil. Mag. t. XXXIII. (1848), pp. 196—200

69. *Sur les Déterminants Gauches* 410
Crelle, t. XXXVIII. (1848), pp. 93—96

70. *Sur quelques Théorèmes de la Géométrie de Position* . . 414
Crelle, t. XXXVIII. (1848), pp. 97—104

71. *Note sur les Fonctions du Second Ordre* 421
Crelle, t. XXXVIII. (1848), pp. 105, 106

72. *Note on the Theory of Permutations* 423
Phil. Mag. t. XXXIV. (1849), pp. 527—529

73. *Abstract of a Memoir by Dr Hesse on the construction of the Surface of the Second Order which passes through nine given points* 425
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 44—46

74. *On the Simultaneous Transformation of Two Homogeneous Functions of the Second Order* 428
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 47—50

75. *On the Attraction of an Ellipsoid* 432
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 50—65

76. *On the Triple Tangent Planes of Surfaces of the Third Order* 445
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 118—132

77. *On the order of certain Systems of Algebraical Equations* . 457
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 132—137

78. *Note on the Motion of Rotation of a Solid of Revolution* . . 462
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 268—270

79. *On a System of Equations connected with Malfatti's Problem, and on another Algebraical System* 465
Camb. and Dubl. Math. Journ. t. IV. (1849), pp. 270—275

PAGE

80. *Sur quelques Transmutations des Lignes Courbes* 471
Liouville, t. XIV. (1849), pp. 40—46

81. *Addition au Mémoire sur quelques Transmutations des Lignes Courbes* 476
Liouville, t. XV. (1850), pp. 351—356

82. *On the Triadic Arrangements of Seven and Fifteen Things* . 481
Phil. Mag. t. XXXVII. (1850), pp. 50—53

*83. *On Curves of Double Curvature and Developable Surfaces* . 485
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 18—22

84. *On the Developable Surfaces which arise from two Surfaces of the Second Order* 486
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 46—57

85. *Note on a Family of Curves of the Fourth Order* 496
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 148—152

86. *On the Developable derived from an Equation of the Fifth Order* 500
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 152—159

*87. *Notes on Elliptic Functions (from Jacobi)* 507
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 201—204

88. *On the Transformation of an Elliptic Integral* 508
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 204—206

89. *On the Attraction of Ellipsoids (Jacobi's Method)* 511
Camb. and Dubl. Math. Journ. t. V. (1850), pp. 217—226

90. *Note sur quelques Formules relatives aux Coniques* 519
Crelle, t. XXXIX. (1850), pp. 1—3

91. *Sur le Problème des Contacts* 522
Crelle, t. XXXIX. (1850), pp. 4—13

92. *Note sur un Système de certaines Formules* 532
Crelle, t. XXXIX. (1850), pp. 14, 15

93. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 534
Crelle, t. XXXIX. (1850), pp. 16—22

94. *Note sur l'Addition des Fonctions Elliptiques* 540
Crelle, t. XLI. (1851), pp. 57—65

95. *Note sur quelques Théorèmes de la Géométrie de Position* . . 550
Crelle, t. XLI. (1851), pp. 66—72

96. *Mémoire sur les Coniques inscrites dans une même Surface du Second Ordre* 557
Crelle, t. XLI. (1851), pp. 73—86

PAGE

97. *Note sur la Solution de l'Équation* $x^{257}-1=0$ 564
Crelle, t. XLI. (1851), pp. 81—83

*98. *Note relative à la sixième section du Mémoire sur quelques Théorèmes de la Géométrie de Position* 567
Crelle, t. XLI. (1851), p. 84

99. *Note sur quelques Formules qui se rapportent à la Multiplication des Fonctions Elliptiques* 568
Crelle, t. XLI. (1851), pp. 85—92

100. *Note sur la Théorie des Hyperdéterminants* 577
Crelle, t. XLII. (1851), pp. 368—371

Notes and References to papers in Volume I. 581

Volumes II, III, and IV of the *Cambridge Mathematical Journal* have on the title pages the dates 1841, 1843, 1845 respectively, and volume VIII of the *Cambridge Philosophical Transactions* has the date 1849. As each of these volumes extends over more than a single year, I have added the year of publication for the papers 1, 2, ..., 12. In all other cases, the year of publication is shown by the date on the title page of the volume.

VOLUME II.

PAGE

101. *Notes on Lagrange's Theorem* 1
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 37—45

102. *On a Double Infinite Series.* 8
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 45—47

103. *On Certain Definite Integrals* 11
Camb. and Dubl. Math. Journ. t. VI. (1851), pp. 136—140

104. *On the Theory of Permutants* 16
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 40—51

105. *Correction to the Postscript to the Paper on Permutants* . . 27
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 97, 98

106. *On the Singularities of Surfaces* 28
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 166—171

107. *On the Theory of Skew Surfaces.* 33
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 171—173

108. *On certain Multiple Integrals connected with the Theory of Attractions* 35
Camb. and Dubl. Math. Journ. t. VII. (1852), pp. 174—178

109. *On the Rationalisation of certain Algebraical Equations* . . 40
Camb. and Dubl. Math. Journ. t. VIII. (1853), pp. 97—101

110. *Note on the Transformation of a Trigonometrical Expression* . 45
Camb. and Dubl. Math. Journ. t. IX. (1854), pp. 61, 62

111. *On a Theorem of M. Lejeune-Dirichlet's* 47
Camb. and Dubl. Math. Journ. t. IX. (1854), pp. 163—165

112. *Demonstration of a Theorem relating to the Products of Sums of Squares* 49
Phil. Mag. t. IV. (1852), pp. 515—519

113. *On the Geometrical Representation of the Integral*
$$\int dx \div \sqrt{(x+a)(x+b)(x+c)} \quad . \quad . \quad . \quad . \quad 53$$
Phil. Mag. t. V. (1853), pp. 281—284

PAGE

114. *Analytical Researches connected with Steiner's Extension of Malfatti's Problem* 57
Phil. Trans. t. CXLII. (for 1852), pp. 253—278

115. *Note on the Porism of the In-and-circumscribed Polygon* . . 87
Phil. Mag. t. VI. (1853), pp. 99—102

116. *Correction of two Theorems relating to the In-and-circumscribed Polygon* 91
Phil. Mag. t. VI. (1853), pp. 376, 377

117. *Note on the Integral* $\int dx \div \sqrt{(m-x)(x+a)(x+b)(x+c)}$ . . 93
Phil. Mag. t. VI. (1853), pp. 103—105

118. *On the Harmonic Relation of two Lines or two Points* . . 96
Phil. Mag. t. VI. (1853), pp. 105—107

119. *On a Theorem for the Development of a Factorial* . . . 98
Phil. Mag. t. VI. (1853), pp. 182—185

120. *Note on a Generalisation of the Binomial Theorem* . . . 102
Phil. Mag. t. VI. (1853), p. 185

121. *Note on a Question in the Theory of Probabilities* . . . 103
Phil. Mag. t. VI. (1853), p. 259

122. *On the Homographic Transformation of a Surface of the Second Order into itself* 105
Phil. Mag. t. VI. (1853), pp. 326—333

123. *On the Geometrical Representation of an Abelian Integral* . 113
Phil. Mag. t. VI. (1853), pp. 414—418

124. *On a Property of the Caustic by Refraction of the Circle* . 118
Phil. Mag. t. VI. (1853), pp. 427—431

125. *On the Theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$ 123
Phil. Mag. t. VII. (1854), pp. 40—47

126. *On the Theory of Groups as depending on the Symbolical Equation* $\theta^n = 1$. *Second Part* 131
Phil. Mag. t. VII. (1854), pp. 408, 409

127. *On the Homographic Transformation of a Surface of the Second Order into itself* 133
Phil. Mag. t. VII. (1854), pp. 208—212: continuation of 122

128. *Developments on the Porism of the In-and-circumscribed Polygon* 138
Phil. Mag. t. VII. (1854), pp. 339—345

PAGE

129. *On the Porism of the In-and-circumscribed Triangle, and on an irrational Transformation of two Ternary Quadratic Forms each into itself* 145
Phil. Mag. t. IX. (1855), pp. 513—517

130. *Deuxième Mémoire sur les Fonctions doublement Périodiques* . 150
Liouville, t. XIX. (1854), pp. 193—208: sequel to 25

131. *Nouvelles Recherches sur les Covariants* 164
Crelle, t. XLVII. (1854), pp. 109—125

132. *Réponse à une Question proposée par M. Steiner* . . . 179
Crelle, t. L. (1855), pp. 277, 278

133. *Sur un Théorème de M. Schläfli* 181
Crelle, t. L. (1855), pp. 278—282

134. *Remarques sur la Notation des Fonctions Algébriques* . . 185
Crelle, t. L. (1855), pp. 282—285

135. *Note sur les Covariants d'une Fonction Quadratique, Cubique, ou Biquadratique à deux Indéterminées* 189
Crelle, t. L. (1855), pp. 285—287

136. *Sur la Transformation d'une Fonction Quadratique en elle-même par des Substitutions linéaires* 192
Crelle, t. L. (1855), pp. 288, 289

137. *Recherches Ultérieures sur les Déterminants gauches* . . . 202
Crelle, t. L. (1855), pp. 299—313: continuation of 52 and 69.

138. *Recherches sur les Matrices dont les termes sont des fonctions linéaires d'une seule Indéterminée* 216
Crelle, t. L. (1855), pp. 313—317

139. *An Introductory Memoir on Quantics* 221
Phil. Trans. t. CXLIV. (for 1854), pp. 244—258

140. *Researches on the Partition of Numbers* 235
Phil. Trans. t. CXLV. (for 1855), pp. 127—140

141. *A Second Memoir on Quantics* 250
Phil. Trans. t. CXLVI. (for 1856), pp. 101—126

142. *Numerical Tables Supplementary to Second Memoir on Quantics* 276
Now first published (1889)

143. *Tables of the Covariants* M *to* W *of the Binary Quintic: from the Second, Third, Fifth, Eighth, Ninth and Tenth Memoirs on Quantics* 282
Arranged in the present form (1889)

PAGE

144. *A Third Memoir on Quantics* 310
Phil. Trans. t. CXLVI. (for 1856), pp. 627—647

145. *A Memoir on Caustics* 336
Phil. Trans. t. CXLVII. (for 1857), pp. 273—312

146. *A Memoir on Curves of the Third Order*. 381
Phil. Trans. t. CXLVII. (for 1857), pp. 415—446

147. *A Memoir on the Symmetric Functions of the Roots of an Equation* 417
Phil. Trans. t. CXLVII. (for 1857), pp. 489—496

148. *A Memoir on the Resultant of a System of two Equations* . 440
Phil. Trans. t. CXLVII. (for 1857), pp. 703—715

149. *On the Symmetric Functions of the Roots of certain Systems of two Equations* 454
Phil. Trans. t. CXLVII. (for 1857), pp. 717—726

150. *A Memoir on the Conditions for the Existence of given Systems of Equalities among the Roots of an Equation* . . . 465
Phil. Trans. t. CXLVII. (for 1857), pp. 727—731

151. *Tables of the Sturmian Functions for Equations of the Second, Third, Fourth, and Fifth Degrees* 471
Phil. Trans. t. CXLVII. (for 1857), pp. 733—736

152. *A Memoir on the Theory of Matrices* 475
Phil. Trans. t. CXLVIII. (for 1858), pp. 17—37

153. *A Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function* 497
Phil. Trans. t. CXLVIII. (for 1858), pp. 39—46

154. *Supplementary Researches on the Partition of Numbers* . . 506
Phil. Trans. t. CXLVIII. (for 1858), pp. 47—52

155. *A Fourth Memoir on Quantics* 513
Phil. Trans. t. CXLVIII. (for 1858), pp. 415—427

156. *A Fifth Memoir on Quantics* 527
Phil. Trans. t. CXLVIII. (for 1858), pp. 429—460

157. *On the Tangential of a Cubic* 558
Phil. Trans. t. CXLVIII. (for 1858), pp. 461—463

158. *A Sixth Memoir on Quantics* 561
Phil. Trans. t. CXLIX. (for 1859), pp. 61—90

Notes and References to papers in Volume II. 593

VOLUME III.

PAGE

159. *On some Integral Transformations* 1
Quart. Math. Journ. t. I. (1857), pp. 4—6

160. *On a Theorem relating to Reciprocal Triangles* 5
Quart. Math. Journ. t. I. (1857), pp. 7—10

161. *A problem in Permutations* 8
Quart. Math. Journ. t. I. (1857), p. 79

162. *Two letters on Cubic Forms* 9
Quart. Math. Journ. t. I. (1857), pp. 85—87 and 90, 91

163. *On Hansen's Lunar Theory* 13
Quart. Math. Journ. t. I. (1857), pp. 112—125

164. *On Gauss' Theory for the Attraction of Ellipsoids* 25
Quart. Math. Journ. t. I. (1857), pp. 162—166

165. *On some Geometrical Theorems relating to a triangle circumscribed about a Conic* 29
Quart. Math. Journ. t. I. (1857), pp. 169—175

166. *Note on the Homology of Sets* 35
Quart. Math. Journ. t. I. (1857), p. 178

167. *Apropos of Partitions* 36
Quart. Math. Journ. t. I. (1857), pp. 183, 184

*168. *A demonstration of the Fundamental Property of Geodesic Lines* 38
Quart. Math. Journ. t. I. (1857), pp. 185, 186

169. *Eisenstein's Geometrical Proof of the Fundamental Theorem for Quadratic Residues* (*Translated from the Original Memoir*, Crelle, t. XXVIII. (1844), *with an addition by A. Cayley*) . 39
Quart. Math. Journ. t. I. (1857), pp. 186—191

170. *On Schellbach's Solution of Malfatti's Problem* 44
Quart. Math. Journ. t. I. (1857), pp. 222—226

PAGE

171. *Note on Mr Salmon's Equation of the Orthotomic Circle* . 48
Quart. Math. Journ. t. I. (1857), pp. 242—244

172. *Note on the Logic of Characteristics* 51
Quart. Math. Journ. t. I. (1857), pp. 257—259

173. *On Laplace's Method for the Attraction of Ellipsoids* . . 53
Quart. Math. Journ. t. I. (1857), pp. 285—300

*174. *On the Oval of Descartes* 66
Quart. Math. Journ. t. I. (1857), pp. 320—328

175. *On the Porism of the In-and-circumscribed Triangle* . . 67
Quart. Math. Journ. t. I. (1857), pp. 344—354

176. *Note on Jacobi's Canonical Formulæ for Disturbed Motion in an Elliptic Orbit* 76
Quart. Math. Journ. t. I. (1857), pp. 355, 356

177. *Solution of a Mechanical Problem* 78
Quart. Math. Journ. t. I. (1857), pp. 405, 406

178. *On the à posteriori Demonstration of the Porism of the In-and-circumscribed Triangle* 80
Quart. Math. Journ. t. II. (1858), pp. 31—38

179. *On certain Forms of the Equation of a Conic* 86
Quart. Math. Journ. t. II. (1858), pp. 44—48

180. *Note on the Reduction of an Elliptic Orbit to a fixed plane* . 91
Quart. Math. Journ. t. II. (1858), pp. 49—54

181. *On Sir W. R. Hamilton's Method for the Problem of three or more Bodies* 97
Quart. Math. Journ. t. II. (1858), pp. 66—73

182. *On Lagrange's Solution of the Problem of two fixed Centres* . 104
Quart. Math. Journ. t. II. (1858), pp. 76—83

183. *Note on Certain Systems of Circles* 111
Quart. Math. Journ. t. II. (1858), pp. 83—88

184. *A Theorem relating to Surfaces of the Second Order* . . 115
Quart. Math. Journ. t. II. (1858), pp. 140—142

185. *Note on the 'Circular Relation' of Prof. Möbius* . . . 118
Quart. Math. Journ. t. II. (1858), p. 162

186. *On the determination of the value of a certain Determinant* . 120
Quart. Math. Journ. t. II. (1858), pp. 163—166

187. *On the Sums of Certain Series arising from the Equation* $x = u + tfx$ 124
Quart. Math. Journ. t. II. (1858), pp. 167—171

PAGE

188. *On the Simultaneous Transformation of two Homogeneous Functions of the Second Order* 129
Quart. Math. Journ. t. II. (1858), pp. 192—195

189. *Note on a formula in finite Differences* 132
Quart. Math. Journ. t. II. (1858), pp. 198—201

190. *On the System of Conics which pass through the same four points* 136
Quart. Math. Journ. t. II. (1858), pp. 206, 207

191. *Note on the Expansion of the true Anomaly* 139
Quart. Math. Journ. t. II. (1858), pp. 229—232

192. *On the Area of the Conic Section represented by the General Trilinear Equation of the Second Degree* 143
Quart. Math. Journ. t. II. (1858), pp. 248—253

193. *On Rodrigues' Method for the Attraction of Ellipsoids* . . 149
Quart. Math. Journ. t. II. (1858), pp. 333—337

194. *Note on the Theory of Attraction* 154
Quart. Math. Journ. t. II. (1858), pp. 338, 339

195. *Report on the Recent Progress of Theoretical Dynamics* . 156
Report of British Association, 1857, pp. 1—42

196. *Note sur un Problème d'Analyse Indéterminée* . . . 205
Nouvelles Annales de Math. t. XVI. (1857), pp. 161—165

197. *Note on the Theory of Logarithms* 208
Phil. Mag. t. XI. (1856), pp. 275—280

198. *Note on a Result of Elimination* 214
Phil. Mag. t. XI. (1856), pp. 378, 379

199. *Note on the Theory of Elliptic Motion* 216
Phil. Mag. t. XI. (1856), pp. 425—428

200. *On the Cones which pass through a given Curve of the Third Order in Space* 219
Phil. Mag. t. XII. (1856), pp. 20—22

201. *Second Note on the Theory of Logarithms* 222
Phil. Mag. t. XII. (1856), pp. 354—360

202. *Supplementary Remarks on the Porism of the In-and-circumscribed Triangle* 229
Phil. Mag. t. XIII. (1857), pp. 19—30

203. *On the Theory of the Analytical Forms called Trees* . . 242
Phil. Mag. t. XIII. (1857), pp. 172—176

PAGE

204. *On a Problem in the Partition of Numbers* 247
Phil. Mag. t. XIII. (1857), pp. 245—248

205. *Note on the Summation of a Certain Factorial Expression* . 250
Phil. Mag. t. XIII. (1857), pp. 419—423

206. *Note on a Theorem relating to the Rectangular Hyperbola* . 254
Phil. Mag. t. XIII. (1857), p. 423

207. *Analytical Solution of the Problem of Tactions* 255
Phil. Mag. t. XIII. (1857), pp. 507—509

208. *Note on the Equipotential Curve* $\frac{m}{r}+\frac{m'}{r'}=C$ 258
Phil. Mag. t. XIV. (1857), pp. 142—146

209. *A Demonstration of Sir W. R. Hamilton's Theorem of the Isochronism of the Circular Hodograph* 262
Phil. Mag. t. XIV. (1857), pp. 427—430

210. *On the Cubic Transformation of an Elliptic Function* . . 266
Phil. Mag. t. XV. (1858), pp. 363, 364

211. *On a Theorem relating to Hypergeometric Series* 268
Phil. Mag. t. XVI. (1858), pp. 356, 357

212. *A Memoir on the Problem of Disturbed Elliptic Motion* . 270
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 1—29

213. *On the Development of the Disturbing Function in the Lunar Theory* 293
Mem. R. Astron. Soc. t. XXVII. (1859), pp. 69—95

214. *The First part of a Memoir on the Development of the Disturbing Function in the Lunar and Planetary Theories* . 319
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 187—215

215. *A Supplementary Memoir on the Problem of Disturbed Elliptic Motion* 344
Mem. R. Astron. Soc. t. XXVIII. (1860), pp. 217—234

216. *Tables of the Development of Functions in the Theory of Elliptic Motion* 360
Mem. R. Astron. Soc. t. XXIX. (1861), pp. 191—306

217. *A Memoir on the Problem of the Rotation of a solid Body* . 475
Mem. R. Astron. Soc. t. XXIX. (1861), pp. 307—342

218. *A Third Memoir on the Problem of Disturbed Elliptic Motion* 505
Mem. R. Astron. Soc. t. XXXI. (1863), pp. 43—56

PAGE

219. *On some formulæ relating to the Variation of the Plane of a Planet's Orbit* 516
Monthly Notices R. Astron. Soc. t. XXI. (1861), pp. 43—46

220. *Note on a Theorem of Jacobi's in relation to the Problem of Three Bodies* 519
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 76—78

221. *On the Secular Acceleration of the Moon's Mean Motion* . 522
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 173—230

222. *On Lambert's Theorem for Elliptic Motion* 562
Monthly Notices R. Astron. Soc. t. XXII. (1862), pp. 238—242

Notes and References to papers in Volume III. 567

VOLUME IV.

PAGE

223. *Note sur un théorème général par rapport à l'élimination* . 1
Tortolini, t. VII. (1856), pp. 454—458

224. *Sur un théorème d'Abel. Note* 5
Tortolini, t. VIII. (1857), pp. 201—203

225. *On a Class of Dynamical Problems* 7
Proc. R. S. t. VIII. (1857), pp. 506—511

226. *On Professor Mac Cullagh's Theorem of the Polar Plane* . . 12
Proc. R. I. A. t. VI. (1858), pp. 481—491

227. *On the Theory of Reciprocal Surfaces* 21
Proc. R. I. A. t. VII. (1862), pp. 20—28

228. *Sur l'intégrale* $\int_0^1 \frac{t^{\mu+\frac{1}{2}}(1-t)^{\mu-\frac{1}{2}}\,dt}{(a+bt-ct^2)^{\mu+1}}$ 28
Liouville, t. II. (1857), pp. 49—51

229. *Note sur une formule pour la réversion des séries* . . . 30
Crelle, t. LII. (1856), pp. 276—284

230. *Note sur la méthode d'élimination de Bezout* 38
Crelle, t. LIII. (1857), pp. 366, 367

231. *Note sur l'équation* $x^2-Dy^2=\pm 4$, $D\equiv 5\,(\text{mod. } 8)$ . . . 40
Crelle, t. LIII. (1857), pp. 369—371

232. *Mémoire sur la forme canonique des fonctions binaires* . . 43
Crelle, t. LIV. (1857), pp. 48—58

233. *Addition au Mémoire sur la forme canonique des fonctions binaires* 53
Crelle, t. LIV. (1857), p. 292

234. *Deuxième Note sur une formule pour la réversion des séries* . 54
Crelle, t. LIV. (1857), pp. 156—161

235. *Sur quelques formules pour la transformation des intégrales elliptiques* 60
Crelle, t. LV. (1858), pp. 15—24

PAGE

236. *Note sur la composition du nombre 47 par rapport aux vingt-troisièmes racines de l'unité* 70
Crelle, t. LV. (1858), p. 192

237. *Théorème sur les déterminants gauches* 72
Crelle, t. LV. (1858), pp. 277, 278

238. *Note sur les normales d'une conique* 74
Crelle, t. LVI. (1859), pp. 182—185

239. *Addition à la Note sur la composition du nombre 47 par rapport aux vingt-troisièmes racines de l'unité* . . . 78
Crelle, t. LVI. (1859), pp. 186, 187

240. *Note on a Theorem in Spherical Trigonometry* . . . 80
Phil. Mag. t. XVII. (1859), p. 151

241. *On Poinsot's four new Regular Solids* 81
Phil. Mag. t. XVII. (1859), pp. 123—128

242. *Second Note on Poinsot's four new Regular Polyhedra* . . 86
Phil. Mag. t. XVII. (1859), pp. 209, 210

243. *On the Theory of Groups as depending on the Symbolic Equation* $\theta^n = 1$. *Third Part* 88
Phil. Mag. t. XVIII. (1859), pp. 34—37. Sequel to 125 and 126

244. *On an Analytical Theorem relating to the distribution of Electricity upon two Spherical Surfaces* 92
Phil. Mag. t. XVIII. (1859), pp. 119—127

245. *On an Analytical Theorem connected with the distribution of Electricity upon two Spherical Surfaces. Second Part* . 99
Phil. Mag. t. XVIII. (1859), pp. 193—202

246. *On Contour and Slope Lines* 108
Phil. Mag. t. XVIII. (1859), pp. 264—268

247. *On the Analytical Forms called Trees. Second Part* . . 112
Phil. Mag. t. XVIII. (1859), pp. 374—378. Continuation of 203

248. *Sketch of a proof of the Theorem that every Algebraic Equation has a Root* 116
Phil. Mag. t. XVIII. (1859), pp. 436—439

249. *Note on Cones of the Third Order* 120
Phil. Mag. t. XVIII. (1859), pp. 439—442

250. *Sur la surface qui est l'enveloppe des plans conduits par les points d'un ellipsoïde perpendiculairement aux rayons menés par le centre* 123
Tortolini, t. II. (1859), pp. 168—179

PAGE

251. *Sur quelques formules pour la différentiation* 135
Tortolini, t. II. (1859), pp. 214—230

252. *Note sur l'équation des différences pour une équation donnée de degré quelconque* 150
Tortolini, t. II. (1859), pp. 365, 366

253. *Sur la courbe parallèle à l'ellipse* 152
Tortolini, t. III. (1860), pp. 311—316

254. *Sur la surface parallèle à l'ellipsoïde* 158
Tortolini, t. III. (1860), pp. 345—352

255. *On a Problem of Double Partitions*. 166
Phil. Mag. t. XX. (1860), pp. 337—341

256. *On a System of Algebraic Equations* 171
Phil. Mag. t. XX. (1860), pp. 341, 342

257. *On the Cubic Centres of a line with respect to three lines and a line* 173
Phil. Mag. t. XX. (1860), pp. 418—423

258. *On a Relation between two Ternary Cubic Forms* . . . 179
Phil. Mag. t. XX. (1860), pp. 512—514

259. *The Problem of Polyhedra* 182
Phil. Trans. t. CXLVII. (for 1857), pp. 183—185

260. *On the Double Tangents of a Plane Curve* 186
Phil. Trans. t. CXLIX. (for 1859), pp. 193—212

261. *On the Conic of Five-pointic Contact at any Point of a Plane Curve* 207
Phil. Trans. t. CXLIX. (for 1859), pp. 371—400

262. *On the Equation of Differences for an Equation of any Order and in particular for the Equations of the Orders two, three, four, and five* 240
Phil. Trans. t. CL. (for 1860), pp. 93—112

263. *Demonstration of a Theorem in Finite Differences* . . . 262
Phil. Trans. t. CL. (for 1860), pp. 321—323

264. *On an Extension of Arbogast's Method of Derivations* . . 265
Phil. Trans. t. CLI. (for 1861), pp. 37—43

265. *Addition to the Memoir on an Extension of Arbogast's Method of Derivations* 272
Now first published, 1891

PAGE

266. *On the Equation for the Product of the Differences of all but one of the Roots of a given Equation* 276
Phil. Trans. t. CLI. (for 1861), pp. 45—59

267. *On the Porism of the In-and-circumscribed Polygon* . . 292
Phil. Trans. t. CLI. (for 1861), pp. 225—239

268. *On a New Auxiliary Equation in the Theory of Equations of the Fifth Order* 309
Phil. Trans. t. CLI. (for 1861), pp. 263—276

269. *A Seventh Memoir on Quantics* 325
Phil. Trans. t. CLI. (for 1861), pp. 277—292

270. *On the Double Tangents of a Curve of the Fourth Order* . 342
Phil. Trans. t. CLI. (for 1861), pp. 357—362

271. *Sur l'invariant le plus simple d'une fonction quadratique bi-ternaire, et sur le résultant de trois fonctions quadratiques ternaires* 349
Crelle, t. LVII. (1860), pp. 139—148

272. *Démonstration d'un théorème de Jacobi par rapport au problème de Pfaff* 359
Crelle, t. LVII. (1860), pp. 273—277

273. *Note sur la transformation de Tschirnhausen* 364
Crelle, t. LVIII. (1861), pp. 259—262

274. *Deuxième Note sur la transformation de Tschirnhausen* . . 368
Crelle, t. LVIII. (1861), pp. 263—269

275. *On Tschirnhausen's Transformation* 375
Phil. Trans. t. CLI. (1861), pp. 561—578

276. *On the Analytical Theory of the Conic* 395
Phil. Trans. t. CLII. (1862), pp. 639—662

277. *On the Wave Surface* 420
Quart. Math. Journ. t. III. (1860), pp. 16—22

278. *Note on the Singular Solutions of Differential Equations* . 426
Quart. Math. Journ. t. III. (1860), pp. 36, 37

279. *On a Theorem relating to Spherical Conics* 428
Quart. Math. Journ. t. III. (1860), p. 53

280. *On the Conics which touch four given lines* 429
Quart. Math. Journ. t. III. (1860), pp. 94—96

281. *Note on the Wave Surface* 432
Quart. Math. Journ. t. III. (1860), pp. 142—144

PAGE

282. *On a particular case of Castillon's Problem* 435
Quart. Math. Journ. t. III. (1860), pp. 157—164

283. *On a Theorem relating to Homographic Figures* 442
Quart. Math. Journ. t. III. (1860), pp. 177—180

284. *On a New Analytical Representation of Curves in Space* . 446
Quart. Math. Journ. t. III. (1860), pp. 225—236

285. *On the System of Conics having double contact with each other* 456
Quart. Math. Journ. t. III. (1860), pp. 246—250

286. *Note on the value of certain Determinants the terms of which are the squared distances of Points in a Plane or in Space* 460
Quart. Math. Journ. t. III. (1860), pp. 275—277

287. *Note on the Equation for the squared differences of the Roots of a Cubic Equation* 463
Quart. Math. Journ. t. III. (1860), pp. 307—309

288. *Note on the Curvature of a plane Curve at a double point, and on the Curvature of Surfaces* 466
Quart. Math. Journ. t. III. (1860), pp. 322—326

289. *On some Numerical Expansions* 470
Quart. Math. Journ. t. III. (1860), pp. 366—369

290. *A Discussion of the Sturmian Constants for cubic and quartic Equations* 473
Quart. Math. Journ. t. IV. (1861), pp. 7—12

291. *On the Demonstration of a Theorem relating to the Moments of Inertia of a Solid Body* 478
Quart. Math. Journ. t. IV. (1861), pp. 25—27

292. *A Theorem in Conics* 481
Quart. Math. Journ. t. IV. (1861), pp. 131—133

293. *On a Certain System of Functional Symbols* 484
Quart. Math. Journ. t. IV. (1861), pp. 225—230

294. *On a New Analytical Representation of Curves in Space* . 490
Quart. Math. Journ. t. V. (1862), pp. 81—86

295. *On the Construction of the ninth point of Intersection of the Cubics which pass through eight given points* . . . 495
Quart. Math. Journ. t. V. (1862), pp. 222—233

296. *On the Conics which pass through the four foci of a given Conic* 505
Quart. Math. Journ. t. V. (1862), pp. 275—280

PAGE

297. *On some Formulæ relating to the Distances of a point from the vertices of a Triangle and to the Problem of Tactions.* 510
Quart. Math. Journ. t. v. (1862), pp. 381—384

298. *Report on the Progress of the Solution of certain Special Problems of Dynamics* 513
Report of the Brit. Assoc. for the Advancement of Science, 1862, pp. 184—252

299. *Mathematics, recent Terminology in* 594
English Cyclopædia, t. v. (1860), pp. 534—542

Notes and References to papers in Volume IV. 609

VOLUME V.

PAGE

300. *Note relative aux droites en involution de M. Sylvester* . . 1
Comptes Rendus, Paris, t. LII. (1861), pp. 1039—1042

301. *Sur les cônes du second ordre qui passent par six points donnés* 4
Comptes Rendus, Paris, t. LII. (1861), pp. 1216—1218

302. *Considérations générales sur les courbes en espace* . . . 7
Comptes Rendus, Paris, t. LIV. (1862), pp. 55—60, 396—400, 672—678

303. *Sur le problème du polygone inscrit et circonscrit. Lettre à M. Poncelet* 21
Comptes Rendus, Paris, t. LV. (1862), pp. 700, 701

304. *Sur un mémoire de Jacobi. Extrait d'une lettre à M. J. Bertrand* 23
Comptes Rendus, Paris, t. LVI. (1863), p. 43

305. *Considérations générales sur les courbes en espace. Courbes du cinquième ordre* 24
Comptes Rendus, Paris, t. LVIII. (1864), pp. 994—1000

306. *Sur les coniques qui touchent des courbes d'ordre quelconque. Extrait d'une lettre à M. Chasles* 31
Comptes Rendus, Paris, t. LIX. (1864), pp. 224, 225

307. *Note sur les fonctions* $\mathrm{al}(x)$, *&c., de M. Weierstrass* . . . 33
Liouville, t. VII. (1862), pp. 137—142

308. *On the* Δ *faced Polyacrons, in reference to the problem of the enumeration of Polyhedra* 38
Manchester Memoirs, t. I. (1862), pp. 248—256

309. *Note on the Theory of Determinants* 45
Phil. Mag. t. XXI. (1861), pp. 180—185

310. *Note on Mr Jerrard's researches on the Equation of the Fifth Order* 50
Phil. Mag. t. XXI. (1861), pp. 210—214

PAGE

311. *On a Theorem of Abel's relating to Equations of the Fifth Order* 55
Phil. Mag. t. XXI. (1861), pp. 257—263

312. *On the Partitions of a Close* 62
Phil. Mag. t. XXI. (1861), pp. 424—428

313. *On a Surface of the Fourth Order* 66
Phil. Mag. t. XXI. (1861), pp. 491—495

314. *On the Curves situate on a Surface of the Second Order* . 70
Phil. Mag. t. XXII. (1861), pp. 35—38

315. *On the Cubic Centres of a Line with respect to three lines and a line* 73
Phil. Mag. t. XXII. (1861), pp. 433—436

*316. *Note on the solution of an Equation of the Fifth Order* . . 77
Phil. Mag. t. XXIII. (1862), pp. 195, 196

317. *Note on the transformation of a certain Differential Equation* 78
Phil. Mag. t. XXIII. (1862), pp. 266, 267

*318. *On a question in the Theory of Probabilities* 80
Phil. Mag. t. XXIII. (1862), pp. 361—365

*319. *Postscript to the paper, On a question in the Theory of Probabilities* 85
Phil. Mag. t. XXIII. (1862), pp. 470, 471

320. *On the Transcendent* $\operatorname{gd} u = \frac{1}{i} \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$ 86
Phil. Mag. t. XXIV. (1862), pp. 19—21

*321. *Final Remarks on Mr Jerrard's theory of Equations of the Fifth Order* 89
Phil. Mag. t. XXIV. (1862), p. 290

322. *On a Skew Surface of the Third Order* 90
Phil. Mag. t. XXIV. (1862), pp. 514—519

323. *On a tactical Theorem relating to the Triads of Fifteen Things* 95
Phil. Mag. t. XXV. (1863), pp. 59—61

324. *Note on a Theorem relating to Surfaces* 98
Phil. Mag. t. XXV. (1863), pp. 61, 62

325. *Note on a Theorem relating to a Triangle, Line, and Conic* . 100
Phil. Mag. t. XXV. (1863), pp. 181—183

PAGE

326. *Theorems relating to the Canonic Roots of a Binary Quantic of an odd order* 103
Phil. Mag. t. XXV. (1863), pp. 206—208

327. *On the Stereographic Projection of the Spherical Conic* . . 106
Phil. Mag. t. XXV. (1863), pp. 350—353

328. *On the delineation of a Cubic Scroll* 110
Phil. Mag. t. XXV. (1863), pp. 528—530

329. *Note on the Problem of Pedal Curves* 113
Phil. Mag. t. XXVI. (1863), pp. 20, 21

330. *On Differential Equations and Umbilici* 115
Phil. Mag. t. XXVI. (1863), pp. 373—379 and 441—452

331. *Analytical Theorem relating to the four Conics inscribed in the same Conic and passing through the same three Points* 131
Phil. Mag. t. XXVII. (1864), pp. 42, 43

332. *Analytical Theorem relating to the sections of a Quadric Surface* 133
Phil. Mag. t. XXVII. (1864), pp. 43, 44

333. *Note on the Nodal Curve of the Developable derived from the Quartic Equation* $(a, b, c, d, e \between t, 1)^4 = 0$ . . . 135
Phil. Mag. t. XXVII. (1864), pp. 437—440

334. *Note on the Theory of Cubic Surfaces* 138
Phil. Mag. t. XXVII. (1864), pp. 493—496

335. *Tables des formes quadratiques binaires pour les déterminants négatifs depuis* $D = -1$ *jusqu'à* $D = -100$, *pour les déterminants positifs non carrés depuis* $D = 2$ *jusqu'à* $D = 99$, *et pour les treize déterminants négatifs irréguliers qui se trouvent dans le premier millier* 141
Crelle, t. LX. (1862), pp. 357—372

336. *Note sur l'élimination* 157
Crelle, t. LX. (1862), pp. 373, 374

337. *Note sur la réalité des racines d'une équation quadratique* . 160
Crelle, t. LXI. (1863), pp. 367, 368

338. *Nouvelles recherches sur l'élimination et la théorie des courbes* 162
Crelle, t. LXIII. (1864), pp. 34—39

339. *On Skew Surfaces, otherwise Scrolls.* 168
Phil. Trans. t. CLIII. (for 1863), pp. 453—483

340. *A Second Memoir on Skew Surfaces, otherwise Scrolls* . . 201
Phil. Trans. t. CLIV. (for 1864), pp. 559—576

PAGE

341. *On the Sextactic Points of a Plane Curve* 221
Phil. Trans. t. CLV. (for 1865), pp. 545—578

342. *On the Conics which pass through three given points and touch a given line* 258
Quart. Math. Journ. t. VI. (1864), pp. 24—30

343. *On the Cusp of the second kind or Nodecusp* 265
Quart. Math. Journ. t. VI. (1864), pp. 74, 75

344. *On Certain Developable Surfaces* 267
Quart. Math. Journ. t. VI. (1864), pp. 108—126

345. *On the Inflexions of the Cubical Divergent Parabolas* . . 284
Quart. Math. Journ. t. VI. (1864), pp. 199—203

346. *Note on an expression for the Resultant of two Binary Cubics* 289
Quart. Math. Journ. t. VI. (1864), pp. 380—382

347. *On the Notion and Boundaries of Algebra* 292
Quart. Math. Journ. t. VI. (1864), pp. 382—384

348. *On the Theory of Involution* 295
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 21—38

349. *On a case of the Involution of Cubic Curves* 313
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 39—80

350. *On the Classification of Cubic Curves* 354
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 81—128

351. *On Cubic Cones and Curves* 401
Camb. Phil. Trans. t. XI. Part I. (1866), pp. 129—144

352. *Suite des recherches sur l'élimination et la théorie des courbes* 416
Crelle, t. LXIV. (1865), pp. 167—171

353. *Note sur la surface du quatrième ordre de Steiner* . . 421
Crelle, t. LXIV. (1865), pp. 172—174

354. *Note sur les singularités supérieures des courbes planes.* . 424
Crelle, t. LXIV. (1865), pp. 369—371

355. *Sur un théorème relatif à huit points situés sur une conique.* 427
Crelle, t. LXV. (1866), pp. 180—184

356. *Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers* 431
Crelle, t. LXV. (1866), pp. 284—291

357. *A Supplementary Memoir on the Theory of Matrices* . . 438
Phil. Trans. t. CLVI. (for 1866), pp. 25—35

PAGE

358. *Addition to the Memoir on Tschirnhausen's Transformation* . 449
Phil. Trans. t. CLVI. (for 1866), pp. 97—100

359. *A Supplementary Memoir on Caustics* 454
Phil. Trans. t. CLVII. (for 1867), pp. 7—16

360. *Note on a Quartic Surface* 465
Phil. Mag. t. XXIX. (1865), pp. 19—22

361. *On Quartic Curves* 468
Phil. Mag. t. XXIX. (1865), pp. 105—108

362. *Note on Lobatschewsky's Imaginary Geometry* 471
Phil. Mag. t. XXIX. (1865), pp. 231—233

363. *On the theory of the Evolute* 473
Phil. Mag. t. XXIX. (1865), pp. 344—350

364. *On a Theorem relating to Five Points in a plane* . . . 480
Phil. Mag. t. XXIX. (1865), pp. 460—464

365. *On the Intersections of a Pencil of four lines by a Pencil of two lines* 484
Phil. Mag. t. XXIX. (1865), pp. 501—503

366. *Note on the Projection of the Ellipsoid* 487
Phil. Mag. t. XXX. (1865), pp. 50—52

367. *On a Triangle in-and-circumscribed to a Quartic Curve* . 489
Phil. Mag. t. XXX. (1865), pp. 340—342

368. *On a problem of Geometrical Permutation* 493
Phil. Mag. t. XXX. (1865), pp. 370—372

369. *On a property of Commutants* 495
Phil. Mag. t. XXX. (1865), pp. 411—413

370. *On the signification of an elementary formula of Solid Geometry* 498
Phil. Mag. t. XXX. (1865), pp. 413, 414

371. *On a Formula for the intersections of a Line and Conic, and on an Integral Formula connected therewith* . . 500
Quart. Math. Journ. t. VII. (1866), pp. 1—6

372. *On the Reciprocation of a Quartic Developable* . . . 505
Quart. Math. Journ. t. VII. (1866), pp. 87—92

373. *On a Special Sextic Developable* 511
Quart. Math. Journ. t. VII. (1866), pp. 105—113

374. *On the Higher Singularities of a Plane Curve* . . . 520
Quart. Math. Journ. t. VII. (1866), pp. 212—223

PAGE

375. *Notes on Polyhedra* 529
Quart. Math. Journ. t. VII. (1866), pp. 304—316

376. *Théorème relatif à l'équilibre de quatre forces* . . . 540
Comptes Rendus, t. LXI. (1865), pp. 829, 830

377. *Note sur la correspondance de deux points sur une courbe* . 542
Comptes Rendus, t. LXII. (1866), pp. 586—590

378. *Report of a Committee appointed by the British Association to consider the formation of a Catalogue of Philosophical Memoirs (A. Cayley, R. Grant, G. G. Stokes)* . . . 546
Report of the British Association (1856), pp. 463, 464

379. *Notices of Communications to the British Association for the Advancement of Science* 549
Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1854 to 1864)

380. *Note on the Rectangular Hyperbola* 554
Oxford, Camb. and Dubl. Messenger of Mathematics, t. I. (1862), p. 77

381. *Note on Bezout's Method of Elimination* 555
Oxford, Camb. and Dubl. Messenger of Mathematics, t. II. (1864), pp. 88, 89

382. *Note on the Tetrahedron* 557
Oxford, Camb. and Dubl. Messenger of Mathematics, t. III. (1866), pp. 8—10

383. *Problems and Solutions* 560
Mathematical Questions with their Solutions from the Educational Times, vols. I. to IV. (1863 to 1865); for contents, see p. 612

Notes and References to papers in Volume V. 613

VOLUME VI.

PAGE

384. *On the Transformation of Plane Curves* 1
Proc. London Math. Society, t. I. (1865—66), No. III. pp. 1—11

385. *On the Correspondence of Two Points on a Curve* . . . 9
Proc. London Math. Society, t. I. (1865—66), No. VII. pp. 1—7

386. *On the Logarithms of Imaginary Quantities* 14
Proc. London Math. Society, t. II. (1866—69), pp. 50—54

387. *Notices of Communications to the London Mathematical Society* 19
Proc. London Math. Society, t. II. (1866—69), pp. 6—7, 25—26, 29, 61—63, 103—104, 123—125

388. *Note on the Composition of Infinitesimal Rotations* . . . 24
Quart. Math. Journ. t. VIII. (1867), pp. 7—10

389. *On a Locus derived from Two Conics* 27
Quart. Math. Journ. t. VIII. (1867), pp. 77—84

390. *Theorem relating to the four Conics which touch the same two lines and pass through the same four points* . . . 35
Quart. Math. Journ. t. VIII. (1867), pp. 162—167

391. *Solution of a Problem of Elimination* 40
Quart. Math. Journ. t. VIII. (1867), pp. 183—185

392. *On the Conics which pass through two given Points and touch two given Lines* 43
Quart. Math. Journ. t. VIII. (1867), pp. 211—219

393. *On the Conics which touch three given Lines and pass through a given Point.* 51
Quart. Math. Journ. t. VIII. (1867), pp. 220—222

394. *On a Locus in relation to the Triangle* 53
Quart. Math. Journ. t. VIII. (1867), pp. 264—277

395. *Investigations in connexion with Casey's Equation* . . . 65
Quart. Math. Journ. t. VIII. (1867), pp. 334—341

PAGE

396. *On a certain Envelope depending on a Triangle inscribed in a Circle* 72
Quart. Math. Journ. t. IX. (1868), pp. 31—41 and 175—176

397. *Specimen Table* $M \equiv a^a b^\beta$ *(Mod. N) for any prime or composite Modulus* 83
Quart. Math. Journ. t. IX. (1868), pp. 95, 96 and plate

398. *On a certain Sextic Developable and Sextic Surface connected therewith* 87
Quart. Math. Journ. t. IX. (1868), pp. 129—142 and 373—376

399. *On the Cubical Divergent Parabolas* 101
Quart. Math. Journ. t. IX. (1868), pp. 185—189

400. *On the Cubic Curves inscribed in a given Pencil of Six Lines* 105
Quart. Math. Journ. t. IX. (1868), pp. 210—221

401. *A Notation of the Points and Lines in Pascal's Theorem* . 116
Quart. Math. Journ. t. IX. (1868), pp. 268—274

402. *On a Singularity of Surfaces* 123
Quart. Math. Journ. t. IX. (1868), pp. 332—338

403. *On Pascal's Theorem* 129
Quart. Math. Journ. t. IX. (1868), pp. 348—353

404. *Reproduction of Euler's Memoir of* 1758 *on the Rotation of a Solid Body* 135
Quart. Math. Journ. t. IX. (1868), pp. 361—373

405. *An Eighth Memoir on Quantics* 147
Phil. Trans. t. CLVII. (for 1867), pp. 513—554

406. *On the Curves which satisfy given Conditions* 191
Phil. Trans. t. CLVIII. (for 1868), pp. 75—143

407. *Second Memoir on the Curves which satisfy given Conditions; the Principle of Correspondence* 263
Phil. Trans. t. CLVIII. (for 1868), pp. 145—172

408. *Addition to Memoir on the Resultant of a System of two Equations* 292
Phil. Trans. t. CLVIII. (for 1868), pp. 173—180

409. *On the Conditions for the existence of three equal Roots or of two pairs of equal Roots of a Binary Quartic or Quintic* 300
Phil. Trans. t. CLVIII. (for 1868), pp. 577—588

PAGE

410. *A Third Memoir on Skew Surfaces, otherwise Scrolls* . . 312
Phil. Trans. t. CLIX. (for 1869), pp. 111—126

411. *A Memoir on the Theory of Reciprocal Surfaces* . . . 329
Phil. Trans. t. CLIX. (for 1869), pp. 201—229

412. *A Memoir on Cubic Surfaces* 359
Phil. Trans. t. CLIX. (for 1869), pp. 231—326

413. *A Memoir on Abstract Geometry* 456
Phil. Trans. t. CLX. (for 1870), pp. 51—63

414. *On Polyzomal Curves, otherwise the Curves* $\sqrt{U}+\sqrt{V}+\&c.=0$. 470
Trans. R. Soc. Edinburgh, t. XXV. (for 1868), pp. 1—110

415. *Corrections and Additions to the Memoir on the Theory of Reciprocal Surfaces* 577
Phil. Trans. t. CLXII. (for 1872), pp. 83—87

416. *On the Theory of Reciprocal Surfaces* 582
Addition to Salmon's Analytic Geometry of Three Dimensions, 4th ed. (1882), pp. 592—604

Notes and References to papers in Volume VI. 593

Portrait . . . . . *to face Title.*

VOLUME VII.

PAGE

417. *On the Locus of the Foci of the Conics which pass through Four Given Points* 1
Phil. Mag. t. XXXII. (1866), pp. 362—365

418. *A Remark on Differential Equations* 5
Phil. Mag. t. XXXII. (1866), pp. 379—381

419. *A Theorem on Differential Operators* 8
Phil. Mag. t. XXXII. (1866), pp. 461—472

420. *On Riccati's Equation* 9
Phil. Mag. t. XXXVI. (1868), pp. 348—351

421. *Note on the Solvibility of Equations by means of Radicals* . 13
Phil. Mag. t. XXXVI. (1868), pp. 386, 387

422. *On the Geodesic Lines on an Oblate Spheroid* 15
Phil. Mag. t. XL. (1870), pp. 329—340

423. *On the Plane Representation of a Solid Figure* . . . 26
Phil. Mag. t. XLI. (1871), pp. 286—290

424. *On the Attraction of a Terminated Straight Line* . . . 31
Phil. Mag. t. XLI. (1871), pp. 358—360

425. *Note on the Geodesic Lines on an Ellipsoid* 34
Phil. Mag. t. XLI. (1871), pp. 534, 535

*426. *On a supposed New Integration of Differential Equations of the Second Order* 36
Phil. Mag. t. XLII. (1871), pp. 197—199

427. *On Gauss' Pentagramma Mirificum* 37
Phil. Mag. t. XLII. (1871), pp. 311, 312

*428. *Note sur la correspondance de deux points sur une courbe* . 39
Comptes Rendus, t. LXII. (1866), pp. 586—590

*429. *Sur les Coniques déterminées par cinq conditions de contact avec une courbe donnée* 40
Comptes Rendus, t. LXIII. (1866), pp. 9—12

PAGE

430. *Note sur quelques formules de M. E. de Jonquières, relatives aux Courbes qui satisfont à des conditions données* . . 41
Comptes Rendus, t. LXIII. (1866), pp. 666—670

431. *Sur la transformation cubique d'une fonction elliptique* . . 44
Comptes Rendus, t. LXIV. (1867), pp. 560—563

432. *Théorème relatif à la théorie des substitutions* 47
Comptes Rendus, t. LXVII. (1868), pp. 784, 785

433. *Sur les surfaces tétraédrales* 48
Notes to De la Gournerie, *Recherches sur les Surfaces réglées tétraédrales symétriques*, 8vo. Paris, 1867

434. *On Certain Skew Surfaces, otherwise Scrolls* 54
Camb. Phil. Trans. t. XI. Part II. (1869), pp. 277—289

435. *On the Six Coordinates of a Line* 66
Camb. Phil. Trans. t. XI. Part II. (1869), pp. 290—323

436. *On a Certain Sextic Torse* 99
Camb. Phil. Trans. t. XI. Part III. (1871), pp. 507—523

*437. *Démonstration nouvelle du théorème de M. Casey par rapport aux cercles qui touchent à trois cercles donnés* . . . 115
Annali di Matematica, t. I. (1867), pp. 132—134

438. *Note sur quelques torses sextiques* 116
Annali di Matematica, t. II. (1868), pp. 99, 100

439. *Addition à la Note sur quelques torses sextiques* . . . 118
Annali di Matematica, t. II. (1868), pp. 219—221

440. *Note sur une transformation géométrique* 121
Journ. der Mathem. (Crelle), t. LXVII. (1867), pp. 95, 96

441. *Note sur l'algorithme des tangentes doubles d'une courbe du quatrième ordre* 123
Journ. der Mathem. (Crelle), t. LXVIII. (1868), pp. 176—179

442. *Note sur la surface du quatrième ordre douée de seize points singuliers et de seize plans singuliers* 126
Journ. der Mathem. (Crelle), t. LXXIII. (1871), pp. 292, 293

443. *Note on the solution of the Quartic Equation* $\alpha U + 6\beta H = 0$. 128
Math. Ann. t. I. (1869), pp. 54, 55

444. *On the Centro-surface of an Ellipsoid* 130
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 16—18

445. *A Memoir on Quartic Surfaces* 133
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 19—69

PAGE

446. *On the Mechanical Description of a Nodal Bicircular Quartic* 182
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 100—106

447. *On the Rational Transformation between Two Spaces* . . 189
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 127—180

448. *Note on the Cartesian with Two Imaginary Axial Foci* . . 241
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 181, 182

449. *Sketch of recent researches upon Quartic and Quintic Surfaces* 244
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 186—195

450. *Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points* . . 253
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 196—198

451. *A Second Memoir on Quartic Surfaces* 256
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 198—202

452. *On an Analytical Theorem from a new point of view* . . 261
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 220, 221

453. *On a Problem in the Calculus of Variations* 263
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 221, 222

454. *A Third Memoir on Quartic Surfaces* 264
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 234—266

455. *On Plücker's Models of certain Quartic Surfaces* . . . 298
Proc. Lond. Math. Society, t. III. (1869—1871), pp. 281—285

456. *Note on the Discriminant of a Binary Quartic* . . . 303
Quart. Math. Journ. t. X. (1870), p. 23

457. *On the Quartic Surfaces* $(*\between U, V, W)^2 = 0$ 304
Quart. Math. Journ. t. X. (1870), pp. 24—34

458. *On the Anharmonic-Ratio Sextic* 314
Quart. Math. Journ. t. X. (1870), pp. 56, 57

459. *On the Double-Sixers of a Cubic Surface* 316
Quart. Math. Journ. t. X. (1870), pp. 58—71

460. *Note on Mr Frost's paper On the direction of the Lines of Curvature in the neighbourhood of an Umbilicus* . . 330
Quart. Math. Journ. t. X. (1870), pp. 111—113

461. *On the Geometrical Interpretation of the Covariants of a Binary Cubic* 332
Quart. Math. Journ. t. X. (1870), pp. 148, 149

PAGE

462. *A Ninth Memoir on Quantics* 334
Phil. Trans. t. CLXI. (for 1871), pp. 17—50

463. *Note on a Differential Equation* 354
Mem. Manchester Society, t. II. (1865), pp. 111—114

464. *Note on Plana's Lunar Theory* 357
Monthly Notices R. Ast. Society, t. XXIII. (1862—1863), pp. 211—215

465. *Note on the Lunar Theory* 361
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 182—189

466. *Second Note on the Lunar Theory* 367
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 203—207

467. *Expressions for Plana's e, γ in terms of the Elliptic e, γ* . 371
Monthly Notices R. Ast. Society, t. XXV. (1864—1865), pp. 265—271

468. *Addition to Second Note on the Lunar Theory* . . . 375
Monthly Notices R. Ast. Society, t. XXVII. (1866—1867), pp. 267—269

469. *On an Expression for the Angular Distance of two Planets* . 377
Monthly Notices R. Ast. Society, t. XXVII. (1866—1867), pp. 312—315

470. *Note on the Attraction of Ellipsoids* 380
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 254—257

471. *Note on the Problem of the Determination of a Planet's Orbit from three observations* 384
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 257—259

472. *Note on Lambert's Theorem for Elliptic Motion* . . . 387
Monthly Notices R. Ast. Society, t. XXIX. (1868—1869), pp. 318—320

473. *On the Graphical Construction of the Umbral or Penumbral Curve at any instant during a Solar Eclipse* . . . 390
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 162—164

474. *On the Geometrical Theory of Solar Eclipses* 392
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 164—168

475. *On a property of the Stereographic Projection* . . . 397
Monthly Notices R. Ast. Society, t. XXX. (1869—1870), pp. 205—207

476. *On the Determination of the Orbit of a Planet from three observations* 400
Mem. R. Ast. Society, t. XXXVIII. (1870), pp. 17—111

477. *On the Graphical Construction of a Solar Eclipse* . . . 479
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 1—17

478. *On the Geodesic Lines on an Ellipsoid* 493
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 31—53

PAGE

479. *The Second Part of a Memoir On the Development of the Disturbing Function in the Lunar and Planetary Theories* 511
Mem. R. Ast. Society, t. XXXIX. (1872), pp. 55—74

480. *On the Expression of Delaunay's l, g, h in terms of his finally adopted Constants* 528
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 8—16

481. *On the Expression of M. Delaunay's h+g in terms of his finally adopted Constants* 534
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), p. 74

482. *Note on a pair of Differential Equations in the Lunar Theory* 535
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 31, 32

483. *On a pair of Differential Equations in the Lunar Theory* . 537
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 201—206

484. *On the variations of the position of the Orbit in the Planetary Theory* 541
Monthly Notices R. Ast. Society, t. XXXII. (1871—1872), pp. 206—211

485. *Problems and Solutions* 546
Mathematical Questions with their Solutions from the Educational Times, vols. V. to XII. (1866—1869): for contents, see p. 607

Notes and References to papers in Volume VII. 609

Portrait . . . . . *to face Title.*

VOLUME VIII.

PAGE

486. *Note on Dr Glaisher's paper on a theorem in definite integration* 1
Quart. Math. Journ. t. x. (1870), pp. 355, 356

487. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 2
Quart. Math. Journ. t. xi. (1871), pp. 15—25

488. *Note on a relation between two circles* 12
Quart. Math. Journ. t. xi. (1871), pp. 82, 83

489. *On the porism of the in-and-circumscribed polygon, and the* (2, 2) *correspondence of points on a conic* 14
Quart. Math. Journ. t. xi. (1871), pp. 83—91

490. *On a problem of elimination* 22
Quart. Math. Journ. t. xi. (1871), pp. 99—101

491. *On the quartic surfaces* $(*\between U, V, W)^2 = 0$ 25
Quart. Math. Journ. t. xi. (1871), pp. 111—113

492. *Note on a system of algebraical equations* 29
Quart. Math. Journ. t. xi. (1871), pp. 132, 133

493. *On evolutes and parallel curves* 31
Quart. Math. Journ. t. xi. (1871), pp. 183—200

494. *Example of a special discriminant* 46
Quart. Math. Journ. t. xi. (1871), pp. 211—213

495. *On the envelope of a certain quadric surface* 48
Quart. Math. Journ. t. xi. (1871), pp. 244—246

496. *Tables of the binary cubic forms for the negative determinants* $\equiv 0 \pmod 4$ *from* -4 *to* -400; *and* $\equiv 1 \pmod 4$ *from* -3 *to* -99; *and for five irregular negative determinants* . 51
Quart. Math. Journ. t. xi. (1871), pp. 246—261

497. *Note on the calculus of logic* 65
Quart. Math. Journ. t. xi. (1871), pp. 282, 283

PAGE

498. *On the inversion of a quadric surface* 67
Quart. Math. Journ. t. XI. (1871), pp. 283—288

499. *On the theory of the curve and torse* 72
Quart. Math. Journ. t. XI. (1871), pp. 294—317

500. *On a theorem relating to eight points on a conic* . . . 92
Quart. Math. Journ. t. XI. (1871), pp. 344—346

501. *Review. Pineto's tables of logarithms* 95
Quart. Math. Journ. t. XI. (1871), pp. 375, 376

502. *On the surfaces divisible into squares by their curves of curvature* 97
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 8, 9

503. *On the surfaces each the locus of the vertex of a cone which passes through m given points and touches* $6-m$ *given lines* 99
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 11—47

504. *On the mechanical description of certain sextic curves* . . 138
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 105—111

505. *On the surfaces divisible into squares by their curves of curvature* 145
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 120, 121

506. *On the mechanical description of a cubic curve* . . . 147
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 175—178

507. *On the mechanical description of certain quartic curves by a modified oval chuck* 151
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 186—190

508. *On geodesic lines, in particular those of a quadric surface* . 156
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 191—211

509. *Plan of a curve-tracing apparatus* 179
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 345—347

510. *On bicursal curves* 181
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 347—352

511. *Addition to the memoir on geodesic lines, in particular those of a quadric surface* 188
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 368—380

512. *On a correspondence of points in relation to two tetrahedra* . 200
Proc. Lond. Math. Society, t. IV. (1871—1873), pp. 396—404

513. *On a bicyclic chuck* 209
Phil. Mag. t. XLIII. (1872), pp. 365—367

PAGE

514. *On the problem of the in-and-circumscribed triangle* . . 212
Phil. Trans. t. CLXI. (for 1871), pp. 369—412

515. *Sur les courbes aplaties* 258
Comptes Rendus, t. LXXIV. (1872), pp. 708—712

516. *Sur une surface quartique aplatie* 262
Comptes Rendus, t. LXXIV. (1872), pp. 1393—1395

517. *Sur les surfaces divisibles en carrés par leurs courbes de courbure et sur la théorie de Dupin* 264
Comptes Rendus, t. LXXIV. (1872), pp. 1445—1449

518. *Sur la condition pour qu'une famille de surfaces données puisse faire partie d'un système orthogonal* 269
Comptes Rendus, t. LXXV. (1872), pp. 177—185, 246—250, 324—330, 381—385, 1800—1803

519. *On curvature and orthogonal surfaces* 292
Phil. Trans. t. CLXIII. (for 1873), pp. 229—251

520. *On the centro-surface of an ellipsoid* 316
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 319—365

521. *On Dr Wiener's model of a cubic surface with 27 real lines; and on the construction of a double-sixer* 366
Camb. Phil. Trans. t. XII. Part I. (1873), pp. 366—383

522. *Note on the theory of invariants* 385
Math. Ann. t. III. (1871), pp. 268—271

523. *On the transformation of unicursal surfaces* 388
Math. Ann. t. III. (1871), pp. 469—474

524. *On the deficiency of certain surfaces* 394
Math. Ann. t. III. (1871), pp. 526—529

525. *An example of the higher transformation of a binary form* . 398
Math. Ann. t. IV. (1871), pp. 359—361

526. *On a surface of the eighth order* 401
Math. Ann. t. IV. (1871), pp. 558—560

527. *On a theorem in covariants* 404
Math. Ann. t. V. (1872), pp. 625—629

528. *On the non-Euclidian geometry* 409
Math. Ann. t. V. (1872), pp. 630—634

529. *A "Smith's Prize" paper* [1868]; *solutions by Prof. Cayley* . 414
Oxford, Camb. and Dubl. Messenger of Mathematics, t. IV. (1868), pp. 201—226

PAGE

530. *Solution of a Senate-House problem* 436
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 24—27

531. *A "Smith's Prize" paper* [1869]; *solutions by Prof. Cayley* . 439
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 41—64

532. *Note on the integration of certain differential equations by series* 458
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 77—82

533. *On the binomial theorem, factorials, and derivations* . . 463
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 102—114

534. *A "Smith's Prize" paper* [1870]; *solutions by Prof. Cayley*. 474
Oxford, Camb. and Dubl. Messenger of Mathematics, t. v. (1871), pp. 182—203

535. *Note on the problem of envelopes* 491
Messenger of Mathematics, t. I. (1872), pp. 3, 4

536. *Note on Lagrange's demonstration of Taylor's theorem* . . 493
Messenger of Mathematics, t. I. (1872), pp. 22—24

537. *Solutions of a Smith's Prize paper for* 1871 496
Messenger of Mathematics, t. I. (1872), pp. 37—47, 71—77, 89—95

538. *Extract from a letter from Prof. Cayley to Mr C. W. Merrifield* 517
Messenger of Mathematics, t. I. (1872), pp. 87, 88

*539. *Further note on Lagrange's demonstration of Taylor's theorem* 519
Messenger of Mathematics, t. I. (1872), pp. 105, 106

540. *On a property of the torse circumscribed about two quadric surfaces* 520
Messenger of Mathematics, t. I. (1872), pp. 111, 112

541. *On the reciprocal of a certain equation of a conic* . . . 522
Messenger of Mathematics, t. I. (1872), pp. 120, 121

*542. *Further note on Taylor's theorem* 524
Messenger of Mathematics, t. I. (1872), p. 137

543. *On an identity in spherical trigonometry* 525
Messenger of Mathematics, t. I. (1872), p. 145

544. *On a penultimate quartic curve* 526
Messenger of Mathematics, t. I. (1872), pp. 178—180

PAGE

545. *On the theory of the singular solutions of differential equations of the first order* 529
Messenger of Mathematics, t. II. (1873), pp. 6—12

546. *Theorems in relation to certain sign-symbols* 535
Messenger of Mathematics, t. II. (1873), pp. 17—20

547. *On the representation of a spherical or other surface on a plane: a Smith's Prize dissertation* 538
Messenger of Mathematics, t. II. (1873), pp. 36, 37

548. *On Listing's theorem.* 540
Messenger of Mathematics, t. II. (1873), pp. 81—89

549. *Note on the maxima of certain factorial functions* . . . 548
Messenger of Mathematics, t. II. (1873), pp. 129, 130

550. *Problem and hypothetical theorems in regard to two quadric surfaces* 550
Messenger of Mathematics, t. II. (1873), p. 137

551. *Two Smith's Prize dissertations* [1872] 551
Messenger of Mathematics, t. II. (1873), pp. 145—149

552. *On a differential formula connected with the theory of confocal conics* 556
Messenger of Mathematics, t. II. (1873), pp. 157, 158

553. *Two Smith's Prize dissertations* [1873] 558
Messenger of Mathematics, t. II. (1873), pp. 161—166

554. *An elliptic-transcendent identity* 564
Messenger of Mathematics, t. II. (1873), p. 179

555. *Notices of Communications to the British Association for the Advancement of Science* 565
Brit. Assoc. Reports, Notices and Abstracts of Communications to the Sections (1870, 1871, 1873).

Notes and References to papers in Volume VIII. 569
Prefatory Note vii
Arthur Cayley: biographical notice by the Editor ix
List of courses of lectures delivered by Professor Cayley . . . xlv
Facsimile of the manuscript of his note on p. 569 . . . *Frontispiece*

VOLUME IX.

PAGE

556. *On Steiner's surface* 1
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 14—25

557. *On certain constructions for bicircular quartics.* 13
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 29—31

558. *A geometrical interpretation of the equations obtained by equating to zero the resultant and the discriminants of two binary quantics* 16
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 31—33

559. [*Note on inversion*] 18
Proc. Lond. Math. Society, t. v. (1873—1874), p. 112

560. [*Addition to Lord Rayleigh's paper "On the numerical calculation of the roots of fluctuating functions"*]. 19
Proc. Lond. Math. Society, t. v. (1873—1874), pp. 123, 124

561. *On the geometrical representation of Cauchy's theorems of root-limitation* 21
Camb. Phil. Trans., t. XII. Part II. (1877), pp. 395—413

562. *On a theorem in maxima and minima: addition* [*to Mr Walton's paper*] *by Professor Cayley* 40
Quart. Math. Journ., t. x. (1870), pp. 262, 263

563. *Note on the transformation of two simultaneous equations.* . 42
Quart. Math. Journ., t. XI. (1871), pp. 266, 267

564. *On a theorem in elimination* 43
Quart. Math. Journ., t. XII. (1873), pp. 5, 6

565. *Note on the Cartesian* 45
Quart. Math. Journ., t. XII. (1873), pp. 16—19

566. *On the transformation of the equation of a surface to a set of chief axes* 48
Quart. Math. Journ., t. XII. (1873), pp. 34—38

PAGE

567. *On an identical equation connected with the theory of invariants* 52
Quart. Math. Journ., t. XII. (1873), pp. 115—118

568. *Note on the integrals* $\int_0^\infty \cos x^2\,dx$ *and* $\int_0^\infty \sin x^2\,dx$ . . . 56
Quart. Math. Journ., t. XII. (1873), pp. 118—126

569. *On the cyclide* 64
Quart. Math. Journ., t. XII. (1873), pp. 148—165

570. *On the superlines of a quadric surface in five-dimensional space* 79
Quart. Math. Journ., t. XII. (1873), pp. 176—180

571. *A demonstration of Dupin's theorem* 84
Quart. Math. Journ., t. XII. (1873), pp. 185—191

572. *Theorem in regard to the Hessian of a quaternary function* . 90
Quart. Math. Journ., t. XII. (1873), pp. 193—197

573. *Note on the* (2, 2) *correspondence of two variables* . . . 94
Quart. Math. Journ., t. XII. (1873), pp. 197, 198

574. *On Wronski's theorem* 96
Quart. Math. Journ., t. XII. (1873), pp. 221—228

575. *On a special quartic transformation of an elliptic function* . 103
Quart. Math. Journ., t. XII. (1873), pp. 266—269

576. *Addition to Mr Walton's paper "On the ray-planes in biaxal crystals"* 107
Quart. Math. Journ., t. XII. (1873), pp. 273—275

577. *Note in illustration of certain general theorems obtained by Dr Lipschitz.* 110
Quart. Math. Journ., t. XII. (1873), pp. 346—349

578. *A memoir on the transformation of elliptic functions* . . 113
Phil. Trans., t. CLXIV. (for 1874), pp. 397—456

579. *Address delivered by the President, Professor Cayley, on presenting the Gold Medal of the [Royal Astronomical] Society to Professor Simon Newcomb* 176
Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 224—233

580. *On the number of distinct terms in a symmetrical or partially symmetrical determinant; with an addition* 185
Monthly Notices R. Ast. Society, t. XXXIV. (1873—1874), pp. 303—307; p. 335

581. *On a theorem in elliptic motion* 191
Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 337—339

PAGE

582. *Note on the Theory of Precession and Nutation* 194
Monthly Notices R. Ast. Society, t. XXXV. (1874—1875), pp. 340—343

583. *On spheroidal trigonometry* 197
Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), p. 92

584. *Addition to Prof. R. S. Ball's paper "Note on a transformation of Lagrange's equations of motion in generalised coordinates, which is convenient in Physical Astronomy"* . 198
Monthly Notices R. Ast. Society, t. XXXVII. (1876—1877), pp. 269—271

585. *A new theorem on the equilibrium of four forces acting on a solid body* 201
Phil. Mag., t. XXXI. (1866), pp. 78, 79; Camb. Phil. Soc. Proc., t. I. (1866), p. 235

586. *On the mathematical theory of isomers* 202
Phil. Mag., t. XLVII. (1874), pp. 444—467

587. *A Smith's Prize dissertation* [1873]. 205
Messenger of Mathematics, t. III. (1874), pp. 1—4

588. *Problem* [*on tetrahedra*] 209
Messenger of Mathematics, t. III. (1874), pp. 50—52

589. *On residuation in regard to a cubic curve* 211
Messenger of Mathematics, t. III. (1874), pp. 62—65

590. *Addition to Prof. Hall's paper "On the motion of a particle toward an attracting centre at which the force is infinite"*. 215
Messenger of Mathematics, t. III. (1874), pp. 149—152

591. *A Smith's Prize paper and dissertation* [1874]; *solutions and remarks* 218
Messenger of Mathematics, t. III. (1874), pp. 165—183; t. IV. (1875), pp. 6—8

592. *On the Mercator's projection of a skew hyperboloid of revolution* 237
Messenger of Mathematics, t. IV. (1875), pp. 17—20

593. *A Sheepshanks' problem* (1866) 241
Messenger of Mathematics, t. IV. (1875), pp. 34—36

594. *On a differential equation in the theory of elliptic functions* . 244
Messenger of Mathematics, t. IV. (1875), pp. 69, 70

595. *On a Senate-House problem* 246
Messenger of Mathematics, t. IV. (1875), pp. 75—78

596. *Note on a theorem of Jacobi's for the transformation of a double integral* 250
Messenger of Mathematics, t. IV. (1875), pp. 92—94

PAGE

597. *On a differential equation in the theory of elliptic functions* . 253
Messenger of Mathematics, t. IV. (1875), pp. 110—113

598. *Note on a process of integration* 257
Messenger of Mathematics, t. IV. (1875), pp. 149, 150

599. *A Smith's Prize dissertation* 259
Messenger of Mathematics, t. IV. (1875), pp. 157—160

600. *Theorem on the n-th Roots of Unity* 263
Messenger of Mathematics, t. IV. (1875), p. 171

601. *Note on the Cassinian* 264
Messenger of Mathematics, t. IV. (1875), pp. 187, 188

602. *On the potentials of polygons and polyhedra* 266
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 20—34

603. *On the potential of the ellipse and the circle* 281
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 38—58

604. *Determination of the attraction of an ellipsoidal shell on an exterior point* 302
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 58—67

605. *Note on a point in the theory of attraction* 312
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 79—81

606. *On the expression of the coordinates of a point of a quartic curve as functions of a parameter* 315
Proc. Lond. Math. Society, t. VI. (1874—1875), pp. 81—83

607. *A memoir on prepotentials* 318
Phil. Trans., t. CLXV. (for 1875), pp. 675—774

608. [*Extract from a*] *Report on Mathematical Tables* 424
Brit. Assoc. Report, 1873, pp. 3, 4

609. *On the analytical forms called factions* 426
Brit. Assoc. Report, 1875, Notices of Communications to the Sections, p. 10

610. *On the analytical forms called Trees, with application to the theory of chemical combinations* 427
Brit. Assoc. Report, 1875, pp. 257—305

611. *Report on mathematical tables* 461
Brit. Assoc. Report, 1875, pp. 305—336

612. *Note sur une formule d'intégration indéfinie* 500
Comptes Rendus, t. LXXVIII. (1874), pp. 1624—1629

613. *On the group of points G_4^1 on a sextic curve with five double points* 504
Math. Ann., t. VIII. (1875), pp 359—362

PAGE

614. *On a problem of projection* 508
Quart. Math. Journ., t. XIII. (1875), pp. 19—29

615. *On the conic torus* 519
Quart. Math. Journ., t. XIII. (1875), pp. 127—129

616. *A geometrical illustration of the cubic transformation in elliptic functions* 522
Quart. Math. Journ., t. XIII. (1875), pp. 211—216

617. *On the scalene transformation of a plane curve* 527
Quart. Math. Journ., t. XIII. (1875), pp. 321—328

618. *On the mechanical description of a Cartesian* 535
Quart. Math. Journ., t. XIII. (1875), pp. 328—330

619. *On an algebraical operation* 537
Quart. Math. Journ., t. XIII. (1875), pp. 369—375

620. *Correction of two numerical errors in Sohnke's paper respecting modular equations* 543
Crelle, t. LXXXI. (1876), p. 229

621. *On the number of the univalent radicals* C_nH_{2n+1} 544
Phil. Mag., Ser. 5, t. III. (1877), pp. 34, 35

622. *On a system of equations connected with Malfatti's problem* . 546
Proc. Lond. Math. Society, t. VII. (1876), pp. 38—42

623. *On three-bar motion* 551
Proc. Lond. Math. Society, t. VII. (1876), pp. 136—166

624. *On the bicursal sextic* 581
Proc. Lond. Math. Society, t. VII. (1876), pp. 166—172

625. *On the condition for the existence of a surface cutting at right angles a given set of lines* 587
Proc. Lond. Math. Society, t. VIII. (1877), pp. 53—57

626. *On the general differential equation* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, *where* X, Y *are the same quartic functions of* x, y *respectively* . . 592
Proc. Lond. Math. Society, t. VIII. (1877), pp. 184—199

627. *Geometrical illustration of a theorem relating to an irrational function of an imaginary variable* 609
Proc. Lond. Math. Society, t. VIII. (1877), pp. 212—214

628. *On the circular relation of Möbius* 612
Proc. Lond. Math. Society, t. VIII. (1877), pp. 220—225

629. *On the linear transformation of the integral* $\int\frac{du}{\sqrt{U}}$ . . . 618
Proc. Lond. Math. Society, t. VIII. (1877), pp. 226—229

VOLUME X.

PAGE

630. *On an expression for* $1 \pm \sin(2p+1)u$ *in terms of* $\sin u$ . . 1
Messenger of Mathematics, t. v. (1876), pp. 7, 8

631. *Synopsis of the theory of equations* 3
Messenger of Mathematics, t. v. (1876), pp. 39—49

632. *On Aronhold's integration-formula* 12
Messenger of Mathematics, t. v. (1876), pp. 88—90

*633. *Note on Mr Martin's paper "On the integrals of some differentials"* 15
Messenger of Mathematics, t. v. (1876), p. 163

634. *Theorems in trigonometry and on partitions* 16
Messenger of Mathematics, t. v. (1876), p. 164, p. 188

635. *Note on the demonstration of Clairaut's theorem* 17
Messenger of Mathematics, t. v. (1876), pp. 166, 167

636. *On the theory of the singular solutions of differential equations of the first order* 19
Messenger of Mathematics, t. vi. (1877), pp. 23—27

637. *On a differential equation in the theory of elliptic functions* . 24
Messenger of Mathematics, t. vi. (1877), p. 29

638. *On a q-formula leading to an expression for* E_1 25
Messenger of Mathematics, t. vi. (1877), pp. 63—66

639. *An elementary construction in optics* 28
Messenger of Mathematics, t. vi. (1877), pp. 81, 82

*640. *Further note on Mr Martin's paper* 29
Messenger of Mathematics, t. vi. (1877), pp. 82, 83

641. *On the flexure of a spherical surface* 30
Messenger of Mathematics, t. vi. (1877), pp. 88—90

642. *On a differential relation between the sides of a quadrangle* . 33
Messenger of Mathematics, t. vi. (1877), pp. 99—101

PAGE

643. *On a quartic curve with two odd branches* 36
Messenger of Mathematics, t. VI. (1877), pp. 107, 108

644. *Note on magic squares* 38
Messenger of Mathematics, t. VI. (1877), p. 168

645. *A Smith's Prize Paper*, 1877 39
Messenger of Mathematics, t. VI. (1877), pp. 173—182

646. *On the general equation of differences of the second order* . 47
Quart. Math. Journ., t. XIV. (1877), pp. 23—25

647. *On the quartic surfaces represented by the equation, symmetrical determinant* = 0 50
Quart. Math. Journ., t. XIV. (1877), pp. 46—52

648. *Algebraical theorem* 57
Quart. Math. Journ., t. XIV. (1877), p. 53

649. *Addition to Mr Glaisher's Note on Sylvester's paper "Development of an idea of Eisenstein"* 58
Quart. Math. Journ., t. XIV. (1877), pp. 83, 84

650. *On a quartic surface with twelve nodes* 60
Quart. Math. Journ., t. XIV. (1877), pp. 103—106

651. *On a special surface of minimum area* 63
Quart. Math. Journ., t. XIV. (1877), pp. 190—196

652. *On a sextic torse* 68
Quart. Math. Journ., t. XIV. (1877), pp. 229—235

653. *On a torse depending on the elliptic functions* . . . 73
Quart. Math. Journ., t. XIV. (1877), pp. 235—241

654. *On certain octic surfaces* 79
Quart. Math. Journ., t. XIV. (1877), pp. 249—264

655. *A memoir on differential equations* 93
Quart. Math. Journ., t. XIV. (1877), pp. 292—339

656. *On the theory of partial differential equations* . . . 134
Mathematische Annalen, t. XI. (1877), pp. 194—198

657. *Note on the theory of elliptic integrals* 139
Mathematische Annalen, t. XII. (1877), pp. 143—146

658. *On some formulæ in elliptic integrals* 143
Mathematische Annalen, t. XII. (1877), pp. 369—374

659. *A theorem on groups* 149
Mathematische Annalen, t. XIII. (1878), pp. 561—565

PAGE

660. *On the correspondence of homographies and rotations* . . 153
Mathematische Annalen, t. XV. (1879), pp. 238—240

661. *On the double ϑ-functions* 155
Proc. Lond. Math. Soc., t. IX. (1878), pp. 29, 30

662. *On the double Θ-functions in connexion with a 16-nodal quartic surface* 157
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 210—219

663. *Further investigations on the double ϑ-functions* . . . 166
Crelle's Journal der Mathem., t. LXXXIII. (1877), pp. 220—233

664. *On the 16-nodal quartic surface* 180
Crelle's Journal der Mathem., t. LXXXIV. (1878), pp. 238—241

665. *A memoir on the double ϑ-functions* 184
Crelle's Journal der Mathem., t. LXXXV. (1878), pp. 214—245

666. *Sur un exemple de réduction d'intégrales abéliennes aux fonctions elliptiques* 214
Comptes Rendus, t. LXXXV. (1877), pp. 265—268, 373—376, 426—429, 472—475

667. *On the bicircular quartic—Addition to Professor Casey's memoir: "On a new form of tangential equation"* . . 223
Phil. Trans., t. 167 (for 1877), pp. 441—460

668. *On compound combinations* 243
Proceedings of the Lit. Phil. Soc. Manchester, t. XVI. (1877), pp. 113, 114; Memoirs, ib., Ser. III., t. VI. (1879), pp. 99, 100

669. *On a problem of arrangements* 245
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 338—342

670. [*Note on Mr Muir's solution of a "problem of arrangement"*] 249
Edin. Roy. Soc. Proc., t. IX. (1878), pp. 388—391

671. *On a sibi-reciprocal surface* 252
Berlin, Akad. Monatsber., (1878), pp. 309—313

672. *On the game of mousetrap* 256
Quart. Math. Journ., t. XV. (1878), pp. 8—10

673. *Note on the theory of correspondence* 259
Quart. Math. Journ., t. XV. (1878), pp. 32, 33

674. *Note on the construction of Cartesians* 261
Quart. Math. Journ., t. XV. (1878), p. 34

675. *On the flecnodal planes of a surface* 262
Quart. Math. Journ., t. XV. (1878), pp. 49—51

PAGE

676. *Note on a theorem in determinants* 265
Quart. Math. Journ., t. xv. (1878), pp. 55—57

677. [*Addition to Mr Glaisher's paper "Proof of Stirling's theorem"*] 267
Quart. Math. Journ., t. xv. (1878), pp. 63, 64

678. *On a system of quadric surfaces* 269
Quart. Math. Journ., t. xv. (1878), pp. 124, 125

679. *On the regular solids* 270
Quart. Math. Journ., t. xv. (1878), pp. 127—131

680. *On the Hessian of a quartic surface* 274
Quart. Math. Journ., t. xv. (1878), pp. 141—144

681. *On the derivatives of three binary quantics* 278
Quart. Math. Journ., t. xv. (1878), pp. 157—168

682. *Formulæ relating to the right line* 287
Quart. Math. Journ., t. xv. (1878), pp. 169—171

683. *On the function* $\operatorname{arc\,sin}(x+iy)$ 290
Quart. Math. Journ., t. xv. (1878), pp. 171—174

684. *On a relation between certain products of differences* . . 293
Quart. Math. Journ., t. xv. (1878), pp. 174, 175

685. *On Mr Cotterill's goniometrical problem* 295
Quart. Math. Journ., t. xv. (1878), pp. 196—198

686. *On a functional equation* 298
Quart. Math. Journ., t. xv. (1878), pp. 315—325; Proc. Lond. Math. Soc., t. IX. (1878), p. 29

687. *Note on the function* $\vartheta(x) = a^2(c-x) \div \{c(c-x) - b^2\}$ . . 307
Quart. Math. Journ., t. xv. (1878), pp. 338—340

688. *Geometrical considerations on a solar eclipse* 310
Quart. Math. Journ., t. xv. (1878), pp. 340—347

689. *On the geometrical representation of imaginary variables by a real correspondence of two planes* 316
Proc. Lond. Math. Soc., t. IX. (1878), pp. 31—39

690. *On the theory of groups* 324
Proc. Lond. Math. Soc., t. IX. (1878), pp. 126—133

691. *Note on Mr Monro's paper "On flexure of spaces"* . . 331
Proc. Lond. Math. Soc., t. IX. (1878), pp. 171, 172

692. *Addition to* [578] *memoir on the transformation of elliptic functions* 333
Phil. Trans., vol. 169, Part II. (for 1878), pp. 419—424

PAGE

693. *A tenth memoir on quantics* 339
Phil. Trans., vol. 169, Part II. (for 1878), pp. 603—661

694. *Desiderata and Suggestions* 401

No. 1. *The theory of groups;*
American Journal of Mathematics, t. I. (1878), pp. 50—52

No. 2. *The theory of groups; graphical representation;*
American Journal of Mathematics, t. I. (1878), pp. 174—176

No. 3. *The Newton-Fourier imaginary problem;*
American Journal of Mathematics, t. II. (1879), p. 97

No. 4. *The mechanical construction of conformable figures;*
American Journal of Mathematics, t. II. (1879), p. 186

695. *A link-work for x^3: extract from a letter to Mr Sylvester* . 407
American Journal of Mathematics, t. I. (1878), p. 386

696. *Calculation of the minimum N.G.F. of the binary seventhic* . 408
American Journal of Mathematics, t. II. (1879), pp. 71—84

697. *On the double ϑ-functions* 422
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 74—81

698. *On a theorem relating to covariants* 430
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 82, 83

699. *On the triple ϑ-functions* 432
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 134—138

700. *On the tetrahedroid as a particular case of the* 16-*nodal quartic surface* 437
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 161—164

701. *Algorithm for the characteristics of the triple ϑ-functions* . 441
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 165—169

702. *On the triple ϑ-functions* 446
Crelle's Journal der Mathem., t. LXXXVII. (1879), pp. 190—198

703. *On the addition of the double ϑ-functions* 455
Crelle's Journal der Mathem., t. LXXXVIII. (1880), pp. 74—81

704. *A memoir on the single and double theta-functions* . . . 463
Phil. Trans., vol. 171, Part III. (for 1880), pp. 897—1002

705. *Problems and Solutions* 566
Mathematical Questions with their Solutions from the Educational Times, vols. XIV. to LXI. (1871—1894); for contents, see p. 615

VOLUME XI.

PAGE

706. *On the distribution of electricity on two spherical surfaces* . 1
Phil. Mag., Ser. 5, t. v. (1878), pp. 54—60

707. *On the colouring of maps* 7
Geogr. Soc. Proc., t. I. (1879), pp. 259—261

708. *Note sur la théorie des courbes de l'espace* 9
Assoc. Franç., Compt. Rend., t. IX. (1880), pp. 135—139

709. *On the number of constants in the equation of the surface* $PS-QR=0$ 14
Tidsskrift for Mathematik, Ser. 4, t. IV. (1880), pp. 145—148

710. *On a differential equation* 17
Collectanea Mathematica, in memoriam Dominici Chelini, (Milan, Hoepli, 1881), pp. 17—26

711. *On a diagram connected with the transformation of elliptic functions* 26
British Association Report, 1881, p. 534

712. *A partial differential equation connected with the simplest case of Abel's theorem* 27
British Association Report, 1881, pp. 534, 535

713. *Addition to Mr Rowe's "Memoir on Abel's theorem"* . . 29
Phil. Trans., t. CLXXII. (1881), pp. 751—758

714. *Various notes* 37
Messenger of Mathematics, t. VII. (1878), pp. 69 : 115 : 124 : 125

715. *Note on a system of algebraical equations* 39
Messenger of Mathematics, t. VII. (1878), pp. 17, 18

716. *An illustration of the theory of the* ϑ*-functions* 41
Messenger of Mathematics, t. VII. (1878), pp. 27—32

717. *On the triple theta-functions* 47
Messenger of Mathematics, t. VII. (1878), pp. 48—50

PAGE

718. *Addition to Mr Genese's paper "On the theory of envelopes"* . 50
Messenger of Mathematics, t. VII. (1878), pp. 62, 63

719. *Suggestion of a mechanical integrator for the calculation of* $\int(Xdx+Ydy)$ *along an arbitrary path* 52
Messenger of Mathematics, t. VII. (1878), pp. 92—95; British Association Report, 1877, pp. 18—20

720. *Note on Arbogast's method of derivations* 55
Messenger of Mathematics, t. VII. (1878), p. 158

721. *Formulæ involving the seventh roots of unity* 56
Messenger of Mathematics, t. VII. (1878), pp. 177—182

722. *A problem in partitions* 61
Messenger of Mathematics, t. VII. (1878), pp. 187, 188

723. *Various notes* 63
Messenger of Mathematics, t. VIII. (1879), pp. 45, 46: 126: 127

724. *On the deformation of the model of a hyperboloid* 66
Messenger of Mathematics, t. VIII. (1879), pp. 51, 52

725. *New formulæ for the integration of* $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$ 68
Messenger of Mathematics, t. VIII. (1879), pp. 60—62

726. *A formula by Gauss for the calculation of* $\log 2$ *and certain other logarithms* 70
Messenger of Mathematics, t. VIII. (1879), pp. 125, 126

727. *Equation of the wave-surface in elliptic coordinates* 71
Messenger of Mathematics, t. VIII. (1879), pp. 190, 191

728. *A theorem in elliptic functions* 73
Proc. Lond. Math. Soc., t. X. (1879), pp. 43—48

729. *On a theorem relating to conformable figures* 78
Proc. Lond. Math. Soc., t. X. (1879), pp. 143—146

730. [*Addition to Mr Spottiswoode's paper "On the twenty-one coordinates of a conic in space"*] 82
Proc. Lond. Math. Soc., t. X. (1879), pp. 194—196

731. *On the binomial equation* $x^p-1=0$; *trisection and quartisection* 84
Proc. Lond. Math. Soc., t. XI. (1880), pp. 4—17

732. *A theorem in spherical trigonometry* 97
Proc. Lond. Math. Soc., t. XI. (1880), pp. 48—50

PAGE

733. *On a formula of elimination* 100
Proc. Lond. Math. Soc., t. XI. (1880), pp. 139—141

734. *On the kinematics of a plane* 103
Quart. Math. Journ., t. XVI. (1879), pp. 1—8

735. *Note on the theory of apsidal surfaces* 111
Quart. Math. Journ., t. XVI. (1879), pp. 109—112

736. *Application of the Newton-Fourier method to an imaginary root of an equation* 114
Quart. Math. Journ., t. XVI. (1879), pp. 179—185

737. *On a covariant formula* 122
Quart. Math. Journ., t. XVI. (1879), pp. 224—226

738. *Note on a hypergeometric series* 125
Quart. Math. Journ., t. XVI. (1879), pp. 268—270

739. *Note on the octahedron function* 128
Quart. Math. Journ., t. XVI. (1879), pp. 280, 281

740. *On certain algebraical identities* 130
Quart. Math. Journ., t. XVI. (1879), pp. 281, 282

741. *On a theorem of Abel's relating to a quintic equation* . . 132
Camb. Phil. Soc. Proc., t. III. (1880), pp. 155—159

742. *On the transformation of coordinates* 136
Camb. Phil. Soc. Proc., t. III. (1880), pp. 178—184

743. *On the Newton-Fourier problem* 143
Camb. Phil. Soc. Proc., t. III. (1880), pp. 231, 232

744. *Table of* $\Delta^m 0^n \div \Pi(m)$ *up to* $m=n=20$ 144
Camb. Phil. Trans., t. XIII. (1883), pp. 1—4

745. *On the Schwarzian derivative, and the polyhedral functions* . 148
Camb. Phil. Trans., t. XIII. (1883), pp. 5—68

*746. *Higher Plane Curves* 217
Salmon's Higher Plane Curves, (3rd ed., 1879), Preface

747. *Note on the degenerate forms of curves* 218
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 383—385

748. *On the bitangents of a quartic* 221
Salmon's Higher Plane Curves, (3rd ed., 1879), pp. 387—389

*749. *Solid Geometry* 224
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), Preface

PAGE

750. *On the theory of reciprocal surfaces* 225
Salmon's Treatise on the analytic geometry of three dimensions, (3rd ed., 1874), pp. 539—550

751. *Note on Riemann's paper "Versuch einer allgemeinen Auffassung der Integration und Differentiation," Werke, pp.* 331—344 235
Mathematische Annalen, t. XVI. (1880), pp. 81, 82

752. *On the finite groups of linear transformations of a variable; with a correction* 237
Mathematische Annalen, t. XVI. (1880), pp. 260—263; 439, 440

753. *On a theorem relating to the multiple theta-functions* . . 242
Mathematische Annalen, t. XVII. (1880), pp. 115—122

754. *On the connection of certain formulæ in elliptic functions* . 250
Messenger of Mathematics, t. IX. (1880), pp. 23—25

755. *On the matrix* $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$, *and in connection therewith the function* $\frac{ax+b}{cx+d}$ 252
Messenger of Mathematics, t. IX. (1880), pp. 104—109

756. *A geometrical construction relating to imaginary quantities* . 258
Messenger of Mathematics, t. X. (1881), pp. 1—3

757. *On a Smith's Prize question, relating to potentials* . . . 261
Messenger of Mathematics, t. XI. (1882), pp. 15—18

758. *Solution of a Senate-House problem.* 265
Messenger of Mathematics, t. XI. (1882), pp. 23—25

759. *Illustration of a theorem in the theory of equations* . . 268
Messenger of Mathematics, t. XI. (1882), pp. 111—113

760. *Reduction of* $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ *to elliptic integrals* 270
Messenger of Mathematics, t. XI. (1882), pp. 142, 143

761. *On the theorem of the finite number of the covariants of a binary quantic* 272
Quart. Math. Journ., t. XVII. (1881), pp. 137—147

762. *On Schubert's method for the contacts of a line with a surface* 281
Quart. Math. Journ., t. XVII. (1881), pp. 244—258

763. *On the theorems of the* 2, 4, 8, *and* 16 *squares* 294
Quart. Math. Journ., t. XVII. (1881), pp. 258—276

PAGE

764. *The binomial equation $x^p - 1 = 0$; quinquisection* 314
Proc. Lond. Math. Soc., t. XII. (1881), pp. 15, 16

765. *On the flexure and equilibrium of a skew surface* 317
Proc. Lond. Math. Soc., t. XII. (1881), pp. 103—108

766. *On the geodesic curvature of a curve on a surface* 323
Proc. Lond. Math. Soc., t. XII. (1881), pp. 110—117

767. *On the Gaussian theory of surfaces* 331
Proc. Lond. Math. Soc., t. XII. (1881), pp. 187—192

768. *Note on Landen's theorem* 337
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 47, 48

769. *On a formula relating to elliptic integrals of the third kind* . 340
Proc. Lond. Math. Soc., t. XIII. (1882), pp. 175, 176

770. *On the 34 concomitants of the ternary cubic* 342
American Journal of Mathematics, t. IV. (1881), pp. 1—15

771. *Specimen of a literal table for binary quantics, otherwise a partition table* 357
American Journal of Mathematics, t. IV. (1881), pp. 248—255

772. *On the analytical forms called trees* 365
American Journal of Mathematics, t. IV. (1881), pp. 266—268

773. *On the 8-square imaginaries* 368
American Journal of Mathematics, t. IV. (1881), pp. 293—296

774. *Tables for the binary sextic* 372
American Journal of Mathematics, t. IV. (1881), pp. 379—384

775. *Tables of covariants of the binary sextic* 377
Written in 1894: now first published.

776. *On the Jacobian sextic equation* 389
Quart. Math. Journ., t. XVIII. (1882), pp. 52—65

777. *A solvable case of the quintic equation* 402
Quart. Math. Journ., t. XVIII. (1882), pp. 154—157

778. [*Addition to Mr Hudson's paper "On equal roots of equations"*] 405
Quart. Math. Journ., t. XVIII. (1882), pp. 226—229

779. [*Note on Mr Jeffery's paper "On certain quartic curves, which have a cusp at infinity, whereat the line at infinity is a tangent"*] 408
Proc. Lond. Math. Soc., t. XIV. (1883), p. 85

780. [*Addition to Mr Hammond's paper "Note on an exceptional case in which the fundamental postulate of Professor Sylvester's theory of tamisage fails"*] 409
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 88—91

PAGE

781. *On the automorphic transformation of the binary cubic function* 411
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 103—108

782. *On Monge's "Mémoire sur la théorie des déblais et des remblais"* 417
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 139—142

783. *On Mr Wilkinson's rectangular transformation* . . . 421
Proc. Lond. Math. Soc., t. XIV. (1883), pp. 222—229

784. *Presidential Address to the British Association, Southport, September* 1883 429
British Association Report, 1883, pp. 3—37

785. *Curve* 460
Encyclopædia Britannica, 9th ed., t. VI. (1878), pp. 716—728

786. *Equation* 490
Encyclopædia Britannica, 9th ed., t. VIII. (1878), pp. 497—509

787. *Function* 522
Encyclopædia Britannica, 9th ed., t. IX. (1879), pp. 818—824

788. *Galois* 543
Encyclopædia Britannica, 9th ed., t. X. (1879), p. 48

789. *Gauss* 544
Encyclopædia Britannica, 9th ed., t. X. (1879), p. 116

790. *Geometry (analytical)* 546
Encyclopædia Britannica, 9th ed., t. X. (1879), pp. 408—420

791. *Landen* 583
Encyclopædia Britannica, 9th ed., t. XIV. (1882), p. 271

792. *Locus* 585
Encyclopædia Britannica, 9th ed., t. XIV. (1882), pp. 764, 765

793. *Monge* 586
Encyclopædia Britannica, 9th ed., t. XVI. (1883), pp. 738, 739

794. *Numbers (partition of)* 589
Encyclopædia Britannica, 9th ed., t. XVII. (1884), p. 614

795. *Numbers (theory of)* 592
Encyclopædia Britannica, 9th ed., t. XVII. (1884), pp. 614—624

796. *Series* 617
Encyclopædia Britannica, 9th ed., t. XXI. (1886), pp. 677—682

797. *Surface* 628
Encyclopædia Britannica, 9th ed., t. XXII. (1887), pp. 668—672

798. *Wallis (John)* 640
Encyclopædia Britannica, 9th ed., t. XXIV. (1888), pp. 331, 332

Portrait . . . . . *To face Title.*

VOLUME XII.

PAGE

799. *On curvilinear coordinates* 1
Quart. Math. Journ., t. XIX. (1883), pp. 1—22

800. *Note on the standard solutions of a system of linear equations* 19
Quart. Math. Journ., t. XIX. (1883), pp. 38—40

801. *On seminvariants* 22
Quart. Math. Journ., t. XIX. (1883), pp. 131—138

802. *Note on Captain MacMahon's paper "On the differential equation* $X^{-\frac{1}{2}}dx + Y^{-\frac{1}{2}}dy + Z^{-\frac{1}{2}}dz = 0$*"* 30
Quart. Math. Journ., t. XIX. (1883), pp. 182—184

803. *On Mr Anglin's formula for the successive powers of the root of an algebraical equation* 33
Quart. Math. Journ., t. XIX. (1883), pp. 223, 224

804. *On the elliptic-function solution of the equation* $x^3 + y^3 - 1 = 0$. 35
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 106—109

805. *Note on Abel's theorem* 38
Camb. Phil. Soc. Proc., t. IV. (1883), pp. 119—122

806. *Determination of the order of a surface* 42
Messenger of Mathematics, t. XII. (1883), pp. 29—32

807. *A proof of Wilson's theorem* 45
Messenger of Mathematics, t. XII. (1883), p. 41

808. *Note on a form of the modular equation in the transformation of the third order* 46
Messenger of Mathematics, t. XII. (1883), pp. 173, 174

809. *Schröter's construction of the regular pentagon* 47
Messenger of Mathematics, t. XII. (1883), p. 177

810. *Note on a system of equations* 48
Messenger of Mathematics, t. XII. (1883), pp. 191, 192

PAGE

811. *On the linear transformation of the theta-functions* . . 50
Messenger of Mathematics, t. XIII. (1884), pp. 54—60

812. *On Archimedes' theorem for the surface of a cylinder* . . 56
Messenger of Mathematics, t. XIII. (1884), pp. 107, 108

813. [*Note on Mr Griffiths' paper "On a deduction from the elliptic-integral formula* $y = \sin(A + B + C + \ldots)$*"*] . . 58
Proc. Lond. Math. Soc., t. XV. (1884), p. 81

814. *On double algebra* 60
Proc. Lond. Math. Soc., t. XV. (1884), pp. 185—197

815. *The binomial equation* $x^p - 1 = 0$; *quinquisection. Second part* 72
Proc. Lond. Math. Soc., t. XVI. (1885), pp. 61—63

816. *On the bitangents of a plane quartic* 74
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 93—115; Camb. Phil. Soc. Proc., t. IV. (1883), p. 321

817. *On the sixteen-nodal quartic surface* 95
Crelle's Journal der Mathem., t. XCIV. (1883), pp. 270—272

818. *Note on hyperelliptic integrals of the first order* . . . 98
Crelle's Journal der Mathem., t. XCVIII. (1885), pp. 95, 96

819. *On two cases of the quadric transformation between two planes* 100
Johns Hopkins University Circulars, No. 13 (1882), pp. 178, 179

820. *On a problem of analytical geometry* 102
Johns Hopkins University Circulars, No. 15 (1882), p. 209

821. *On the geometrical representation of an equation between two variables* 104
Johns Hopkins University Circulars, No. 15 (1882), p. 210

822. *On associative imaginaries* 105
Johns Hopkins University Circulars, No. 15 (1882), pp. 211, 212

823. *On the geometrical interpretation of certain formulæ in elliptic functions* 107
Johns Hopkins University Circulars, No. 17 (1882), p. 238

824. *Note on the formulæ of trigonometry* 108
Johns Hopkins University Circulars, No. 17 (1882), p. 241

825. *A memoir on the Abelian and Theta Functions* . . . 109
Chapters I to III, American Journal of Mathematics, t. V. (1882), pp. 137—179; Chapters IV to VII, *ib.*, t. VII. (1885), pp. 101—167

PAGE

826. *Note on a partition series* 217
American Journal of Mathematics, t. VI. (1884), pp. 63, 64

827. *On the non-Euclidian plane geometry* 220
Proc. Roy. Soc., t. XXXVII. (1884), pp. 82—102

828. *A memoir on seminvariants* 239
American Journal of Mathematics, t. VII. (1885), pp. 1—25

829. *Tables of the symmetric functions of the roots, to the degree* 10, *for the form* $1+bx+\frac{cx^2}{1.2}+\ldots=(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots$. 263
American Journal of Mathematics, t. VII. (1885), pp. 47—56

830. *Non-unitary partition tables* 273
American Journal of Mathematics, t. VII. (1885), pp. 57, 58

831. *Seminvariant tables* 275
American Journal of Mathematics, t. VII. (1885), pp. 59—73

832. *Note on an apparent difficulty in the theory of curves, when the coordinates of a point are given as functions of a variable parameter* 290
Messenger of Mathematics, t. XIV. (1885), pp. 12—14

833. *On a formula in elliptic functions* 292
Messenger of Mathematics, t. XIV. (1885), pp. 21, 22

834. *On the addition of the elliptic functions* 294
Messenger of Mathematics, t. XIV. (1885), pp. 56—61

835. *On Cardan's solution of a cubic equation* 299
Messenger of Mathematics, t. XIV. (1885), pp. 96, 97

836. *On the quaternion equation* $qQ-Qq'=0$ 300
Messenger of Mathematics, t. XIV. (1885), pp. 108—112

837. *On the so-called D'Alembert Carnot geometrical paradox* . 305
Messenger of Mathematics, t. XIV. (1885), pp. 113, 114

838. *On the twisted cubics upon a quadric surface.* 307
Messenger of Mathematics, t. XIV. (1885), pp. 129—132

839. *On the matrical equation* $qQ-Qq'=0$ 311
Messenger of Mathematics, t. XIV. (1885), pp. 176—178

840. *On Mascheroni's geometry of the compass* 314
Messenger of Mathematics, t. XIV. (1885), pp. 179—181

841. *On a differential operator* 318
Messenger of Mathematics, t. XIV. (1885), pp. 190, 191

842. *On the value of* $\tan(\sin\theta)-\sin(\tan\theta)$ 319
Messenger of Mathematics, t. XIV. (1885), pp. 191, 192

PAGE

843. *On the quadri-quadric curve in connexion with the theory of elliptic functions* 321
Mathematische Annalen, t. XXV. (1885), pp. 152—156

844. *On a theorem relating to seminvariants* 326
Quart. Math. Journ., t. XX. (1885), pp. 212, 213

845. *On the orthomorphosis of the circle into the parabola* . . 328
Quart. Math. Journ., t. XX. (1885), pp. 213—220

846. *A verification in regard to the linear transformation of the theta-functions* 337
Quart. Math. Journ., t. XXI. (1886), pp. 77—84

847. *On the theory of seminvariants*. 344
Quart. Math. Journ., t. XXI. (1886), pp. 92—107

848. *On the transformation of the double theta-functions* . . 358
Quart. Math. Journ., t. XXI. (1886), pp. 142—178

849. *On the invariants of a linear differential equation* . . . 390
Quart. Math. Journ., t. XXI. (1886), pp. 257—261

850. *On linear differential equations* 394
Quart. Math. Journ., t. XXI. (1886), pp. 321—331

851. *On linear differential equations: the theory of decomposition* . 403
Quart. Math. Journ., t. XXI. (1886), pp. 331—335

852. *Note sur le mémoire de M. Picard "Sur les intégrales de différentielles totales algébriques de première espèce"* . . 408
Bull. des Sciences Math., 2me Sér., t. X. (1886), pp. 75—78

853. *Note on a formula for* $\Delta^n 0^i/n^i$ *when* n, i *are very large numbers* 412
Proc. Roy. Soc. Edin., t. XIV. (1887), pp. 149—153

854. *An algebraical transformation* 416
Messenger of Mathematics, t. XV. (1886), pp. 58, 59

855. *Solution of* $(a, b, c, d) = (a^2, b^2, c^2, d^2)$ 418
Messenger of Mathematics, t. XV. (1886), pp. 59—61

856. *Note on a cubic equation* 421
Messenger of Mathematics, t. XV. (1886), pp. 62—64

857. *Analytical geometrical note on the conic* 424
Messenger of Mathematics, t. XV. (1886), p. 192

858. *Comparison of the Weierstrassian and Jacobian elliptic functions* 425
Messenger of Mathematics, t. XVI. (1887), pp. 129—132

PAGE

859. *On the complex of lines which meet a unicursal quartic curve* 428
Proc. Lond. Math. Soc., t. XVII. (1886), pp. 232—238

860. *On Briot and Bouquet's theory of the differential equation* $F\left(u, \frac{du}{dz}\right) = 0$ 432
Proc. Lond. Math. Soc., t. XVIII. (1887), pp. 314—324

861. *Note on a formula relating to the zero-value of a theta-function* 442
Crelle's Journal der Mathem., t. C. (1887), pp. 87, 88

862. *Note on the theory of linear differential equations* . . . 444
Crelle's Journal der Mathem., t. C. (1887), pp. 286—295

863. *Note on the theory of linear differential equations* . . . 453
Crelle's Journal der Mathem., t. CI. (1887), pp. 209—213

864. *On Rudio's inverse centro-surface* 457
Quart. Math. Journ., t. XXII. (1887), pp. 156—158

865. *On multiple algebra* 459
Quart. Math. Journ., t. XXII. (1887), pp. 270—308

866. *Note on Kiepert's L-equations, in the transformation of elliptic functions* 490
Mathematische Annalen, t. XXX. (1887), pp. 75—77

867. *Note on the Jacobian sextic equation* 493
Mathematische Annalen, t. XXX. (1887), pp. 78—84

868. *On the intersection of curves* 500
Mathematische Annalen, t. XXX. (1887), pp. 85—90

869. *On the transformation of elliptic functions* 505
American Journal of Mathematics, t. IX. (1887), pp. 193—224

870. *On the transformation of elliptic functions (sequel)* . . . 535
American Journal of Mathematics, t. X. (1888), pp. 71—93

871. *A case of complex multiplication with imaginary modulus arising out of the cubic transformation in elliptic functions* 556
Proc. Lond. Math. Soc., t. XIX. (1888), pp. 300, 301

*872. *On the finite number of the covariants of a binary quantic* . 558
Mathematische Annalen, t. XXXIV. (1889), pp. 319, 320

873. *System of equations for three circles which cut each other at given angles* 559
Messenger of Mathematics, t. XVII. (1888), pp. 18—21

874. *Note on the Legendrian coefficients of the second kind* . . 562
Messenger of Mathematics, t. XVII. (1888), pp. 21—23

PAGE

875. *On the system of three circles which cut each other at given angles and have their centres in a line* 564
Messenger of Mathematics, t. XVII. (1888), pp. 60—69

876. *On systems of rays* 571
Messenger of Mathematics, t. XVII. (1888), pp. 73—78

877. *Note on the two relations connecting the distances of four points on a circle* 576
Messenger of Mathematics, t. XVII. (1888), pp. 94, 95

878. *Note on the anharmonic ratio equation* 578
Messenger of Mathematics, t. XVII. (1888), pp. 95, 96

879. *Note on the differential equation* $\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0$ . . 580
Messenger of Mathematics, t. XVIII. (1889), p. 90

880. *Note on the relation between the distance of five points in space* 581
Messenger of Mathematics, t. XVIII. (1889), pp. 100—102

881. *On Hermite's H-product theorem* 584
Messenger of Mathematics, t. XVIII. (1889), pp. 104—107

882. *A correspondence of confocal Cartesians with the right lines of a hyperboloid* 587
Messenger of Mathematics, t. XVIII. (1889), pp. 128—130

883. *Analytical formulæ in regard to an octad of points* . . 590
Messenger of Mathematics, t. XVIII. (1889), pp. 149—152

884. *Note sur les surfaces minima et le théorème de Joachimsthal* . 594
Comptes Rendus, t. CVI. (1888), pp. 995, 996

885. *On the Diophantine relation,* $y^2+y'^2=$ *Square* 596
Proc. Lond. Math. Soc., t. XX. (1889), pp. 122—127

886. *On the surfaces with plane or spherical curves of curvature* . 601
American Journal of Mathematics, t. XI. (1889), pp. 71—98; pp. 293—306

887. *On the theory of groups* 639
American Journal of Mathematics, t. XI. (1889), pp. 139—157

VOLUME XIII.

PAGE

888. *On a form of quartic surface with twelve nodes* . . . 1
British Association Report, (1886), pp. 540, 541

889. *On a differential equation and the construction of Milner's lamp* 3
Edinburgh Math. Soc. Proc., t. v (1887), pp. 99—101

890. *Note on the hydrodynamical equations* 6
Proc. Roy. Soc. Edin., t. xv (1888), pp. 342—344

891. *On the binodal quartic and the graphical representation of the elliptic functions* 9
Camb. Phil. Soc. Trans., t. xiv (1889), pp. 484—494

892. *Note on the orthomorphic transformation of a circle into itself* 20
Edinburgh Math. Soc. Proc., t. viii (1890), pp. 91, 92

893. *The bitangents of the quintic* 21
Annals of Mathematics, t. v (1890), pp. 109, 110

894. *An investigation by Wallis of his expression for* π . . . 22
Quart. Math. Journ., t. xxiii (1889), pp. 165—169

895. *A theorem on trees* 26
Quart. Math. Journ., t. xxiii (1889), pp. 376—378

896. *A transformation in elliptic functions* 29
Quart. Math. Journ., t. xxiv (1890), pp. 259—262

897. *Sur les racines d'une équation algébrique* 33
Comptes Rendus, t. cx (1890), pp. 174—176, 215—218

898. *Sur l'équation modulaire pour la transformation de l'ordre* 11 38
Comptes Rendus, t. cxi (1890), pp. 447—449

899. *Sur les surfaces minima* 41
Comptes Rendus, t. cxi (1890), pp. 953, 954

900. *James Joseph Sylvester* 43
(Scientific Worthies in Nature, xxv); *Nature*, vol. xxxix (1889), pp. 217—219

PAGE

901. *Note on the sums of two series* 49
Messenger of Mathematics, t. XIX (1890), pp. 29—31

902. *On the focals of a quadric surface* 51
Messenger of Mathematics, t. XIX (1890), pp. 113—117

903. *On Latin squares* 55
Messenger of Mathematics, t. XIX (1890), pp. 135—137

904. *Note on reciprocal lines* 58
Messenger of Mathematics, t. XIX (1890), pp. 174, 175

905. *On the equation* $x^{17}-1=0$ 60
Messenger of Mathematics, t. XIX (1890), pp. 184—188

906. *Note on Schlaefli's modular equation for the cubic transformation; with a correction* 64
Messenger of Mathematics, t. XX (1891), pp. 59, 60; 120

907. *Note on the ninth roots of unity* 66
Messenger of Mathematics, t. XX (1891), p. 63

908. *On two invariants of a quadri-quadric function* 67
Messenger of Mathematics, t. XX (1891), pp. 68, 69

909. *On a particular case of Kummer's differential equation of the third order* 69
Messenger of Mathematics, t. XX (1891), pp. 75—79

910. *Note on the involutant of two matrices* 74
Messenger of Mathematics, t. XX (1891), pp. 136, 137

911. *On an algebraical identity relating to the six coordinates of a line* 76
Messenger of Mathematics, t. XX (1891), pp. 138—140

912. *On the notion of a plane curve of a given order* 79
Messenger of Mathematics, t. XX (1891), pp. 148—150

913. *On the epitrochoid* 81
Messenger of Mathematics, t. XX (1891), pp. 150—158

914. *On a soluble quintic equation* 88
American Journal of Mathematics, t. XIII (1891), pp. 53—58

915. *On the partitions of a polygon* 93
Proc. Lond. Math. Soc., t. XXII (1891), pp. 237—262

916. [*Note on a theorem in matrices*] 114
Proc. Lond. Math. Soc., t. XXII (1891), p. 458

917. [*Note on the theory of rational transformation*] 115
Proc. Lond. Math. Soc., t. XXII (1891), pp. 475, 476

PAGE

918. *On the substitution-groups for two, three, four, five, six, seven, and eight letters* 117
Quart. Math. Journ., t. XXV (1891), pp. 71—88, 137—155

919. *On the problem of tactions* 150
Quart. Math. Journ., t. XXV (1891), pp. 104—127

920. *On orthomorphosis* 170
Quart. Math. Journ., t. XXV (1891), pp. 203—226

921. *On some problems of orthomorphosis* 191
Crelle's Journal der Mathem., t. CVII (1891), pp. 262—277

922. *Note on the lunar theory* 206
Monthly Notices of the Royal Astronomical Society, t. LII (1892), pp. 2—5

923. *Note on a hyperdeterminant identity* 210
Messenger of Mathematics, t. XXI (1892), pp. 131, 132

924. *On the non-existence of a special group of points* . . . 212
Messenger of Mathematics, t. XXI (1892), pp. 132, 133

925. *On Waring's formula for the sum of the mth powers of the roots of an equation* 213
Messenger of Mathematics, t. XXI (1892), pp. 133—137

926. *Corrected seminvariant tables for the weights* 11 *and* 12 . 217
American Journal of Mathematics, t. XIV (1892), pp. 195—200

927. *On Clifford's paper "On syzygetic relations among the powers of linear quantics"* 224
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 99—104

928. *On the analytical theory of the congruency* 228
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 185—188

929. *Note on the skew surfaces applicable upon a given skew surface* 231
Proc. Lond. Math. Soc., t. XXIII (1892), pp. 217—225

930. *Sur la surface des ondes* 238
Annali di Matematica, Ser. II, t. XX (1892), pp. 1—18

931. *On some formulæ of Codazzi and Weingarten in relation to the application of surfaces to each other* 253
Proc. Lond. Math. Soc., t. XXIV (1893), pp. 210—223

932. *On symmetric functions and seminvariants* 265
American Journal of Mathematics, t. XV (1893), pp. 1—74

PAGE

933. *Tables of pure reciprocants to the weight* 8 333
American Journal of Mathematics, t. xv (1893), pp. 75—77

934. *Note on the so-called quotient G/H in the theory of groups* . 336
American Journal of Mathematics, t. xv (1893), pp. 387, 388

935. *Sur la fonction modulaire* $\chi\omega$ 338
Comptes Rendus, t. cxvi (1893), pp. 1339—1343

936. *Note on uniform convergence* 342
Proc. Roy. Soc. Edin., t. xix (1893), pp. 203—207

937. *Note on the orthotomic curve of a system of lines in a plane* 346
Messenger of Mathematics, t. xxii (1893), pp. 45, 46

938. *On two cubic equations* 348
Messenger of Mathematics, t. xxii (1893), pp. 69—71

939. *On a case of the involution* $AF+BG+CH=0$, *where* A, B, C, F, G, H *are ternary quadrics* 350
Messenger of Mathematics, t. xxii (1893), pp. 182—186

940. *On the development of* $(1+n^2x)^{\frac{m}{n}}$ 354
Messenger of Mathematics, t. xxii (1893), pp. 186—190

941. *Note on the partial differential equation*
$$Rr+Ss+Tt+U(s^2-rt)-V=0 \quad . \quad . \quad . \quad . \quad 358$$
Quart. Math. Journ., t. xxvi (1893), pp. 1—5

942. *On seminvariants* 362
Quart. Math. Journ., t. xxvi (1893), pp. 66—69

943. *On reciprocants and differential invariants* 366
Quart. Math. Journ., t. xxvi (1893), pp. 169—194, 289—307

944. *On Pfaff-invariants* 405
Quart. Math. Journ., t. xxvi (1893), pp. 195—205

945. *Note on lacunary functions* 415
Quart. Math. Journ., t. xxvi (1893), pp. 279—281

946. *Note on the theory of orthomorphosis* 418
Quart. Math. Journ., t. xxvi (1893), pp. 282—288

947. *On a system of two tetrads of circles: and other systems of two tetrads* 425
Camb. Phil. Soc. Proc., t. viii (1893), pp. 54—59

PAGE

948. *Report of a Committee appointed for the purpose of carrying on the tables connected with the Pellian equation from the point where the work was left by Degen in* 1817 . . 430
British Association Report, (1893), pp. 73—120

949. *On Halphen's characteristic n, in the theory of curves in space* 468
Crelle's Journal der Mathem., t. CXI (1893), pp. 347—352

950. *On the sextic resolvent equations of Jacobi and Kronecker* . 473
Crelle's Journal der Mathem., t. CXIII (1894), pp. 42—49

951. *Non-Euclidian geometry* 480
Camb. Phil. Soc. Trans., t. XV (1894), pp. 37—61

952. *On the kinematics of a plane, and in particular on three-bar motion: and on a curve-tracing mechanism* . . 505
Camb. Phil. Soc. Trans., t. XV (1894), pp. 391—402

953. *On the nine-points circle* 517
Messenger of Mathematics, t. XXIII (1894), pp. 23—25

954. *On the nine-points circle of a plane triangle* 520
Messenger of Mathematics, t. XXIII (1894), pp. 25—27

955. *The numerical value of* $\Pi(i), = \Gamma(1+i)$ 522
Messenger of Mathematics, t. XXIII (1894), pp. 36—38

956. *On Richelot's integral of the differential equation* $\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$ 525
Messenger of Mathematics, t. XXIII (1894), pp. 42—47

957. *Illustrations of Sylow's theorems on groups* 530
Messenger of Mathematics, t. XXIII (1894), pp. 59—62

958. *On the surface of the order n which passes through a given cubic curve* 534
Messenger of Mathematics, t. XXIII (1894), pp. 79, 80

959. *Note on Plücker's equations* 536
Messenger of Mathematics, t. XXIV (1895), pp. 23, 24

960. *On the circle of curvature at any point of an ellipse* . . 537
Messenger of Mathematics, t. XXIV (1895), pp. 47, 48

961. *A trigonometrical identity* 538
Messenger of Mathematics, t. XXIV (1895), pp. 49—51

962. *Co-ordinates versus quaternions* 541
Proc. Roy. Soc. Edin., t. XX (1895), pp. 271—275

PAGE

963. *Note on Dr Muir's Paper "A problem of Sylvester's in Elimination"* 545
Proc. Roy. Soc. Edin., t. xx (1895), pp. 306—308

964. *On the nine-points circle of a spherical triangle* . . . 548
Quart. Math. Journ., t. xxvii (1895), pp. 35—39

965. *On the sixty icosahedral substitutions* 552
Quart. Math. Journ., t. xxvii (1895), pp. 236—242

966. *Note on a memoir in Smith's Collected Papers* . . . 558
Bulletin of the American Math. Soc., 2nd Ser., t. i (1895), pp. 94—96

*967. *An elementary treatise on elliptic functions* 560
First edition, 1876, second edition, 1895

INDEX

TO THE

THIRTEEN VOLUMES.

INDEX.

[*Volumes* are indicated by Roman numerals.
Pages are indicated by Arabic numerals.]

Abbildung: the term, VII, 248; theory, VII, 249—50; (*see also* Transformation of Surfaces).

Abel, N. H.: doubly infinite products, I, 120; inverse elliptic functions, I, 136, 156, 173; modular functions, I, 227; a functional equation, IV, 5—6; elliptic integrals, IV, 63, X, 139; quintic equations, V, 55—61, X, 11, XIII, 88; divergent series, VIII, 494; a theorem of, X, 57; theory of equations, XI, 132—5, 455, 513, 518; elliptic functions, XI, 452; series, XI, 627.

Abelian Functions: Riemann, VI, 2; $p=3$, X, 432—6; early history, XI, 453—4; connected with square roots of sextic and octic functions, XI, 483; the term, XI, 533—4; bitangents of plane quartic, XII, 74; hyperelliptic integrals of first order, XII, 98—9; memoir on theta functions and, XII, 109—216, (introductory, XII, 109; Abel's theorem, XII, 110—20; its proof, XII, 120—31; the major function, XII, 132—48, 149—56; miscellaneous investigations, XII, 157—96; nodal quartic, XII, 196—208; functions T, U, V, Θ, XII, 209—16).

Abelian Integrals: notes on, I, 366—9; and covariants, II, 189—91; Liouville, IV, 546; Riemann, V, 521; reduction to elliptic integrals, X, 214—22; deficiency of ground-curve, XI, 36; first kind of, XII, 38, 408—11; pure theorem for, XII, 110, 112—6, 119—20, 121, 129—30; affected theorem for, XII, 110, 116—7, 121, 130—1, 164—7.

Abel's Theorem: II, 45, 95, XI, 27—8, XII, 30; applied to porisms, IV, 297; Rowe's memoir on, XI, 29—36; note on, XII, 38—41; proof, XII, 120; semi-cubical parabola, XII, 180—6; quadri-quadric curve, XII, 186—9, 292—8; other curves, XII, 189—96.

Absolute: and theory of distance, II, 583—92, 604, V, 550; normals of a conic, IV, 74, 77; theory of evolute, V, 476—9; effect on locus in relation to triangle, VI, 53—64; Cayley's theory of, VIII, xxxvi—vii; evolutes and parallel curves, VIII, 31—44; centro-surface of ellipsoid, VIII, 316, 320; in hypergeometry, VIII, 409—13, XIII, 481—504; the term, XIII, 42; minimal surfaces, XIII, 42.

Abstract Geometry: memoir on, VI, 456—69, VIII, xxxiii, XI, 441—2.

Acceleration: secular (*see* Secular Acceleration).

Acnodal: defined, V, 403, 551, XI, 228.

Acnode: defined, IV, 181, V, 295, 521, VI, 585, XI, 630.

Adams, J. C.: Malfatti's problem, I, 468; moon's mean motion, III, 522, 533—40, 568; lunar theory, VII, 372, VIII, xliii—iv; attraction of ellipsoidal shell, IX, 302; solar eclipses, X, 315.

Addition: of elliptic functions, I, 540—9, 589, XI, 73—7, 454, 530, XII, 294—8; of double theta functions, X, 455—62.

Address: presidential to British Association, XI, 429—59.

Adjoint Curve: IX, 504—7.

Adjoint Linear Form: in quartics, II, 319.

Aggregate: and relation in abstract geometry, VI, 459.
Air: effect on pendulum, IV, 541.
Airy, Sir G. B.: a trigonometrical theorem, IV, 80; roots in algebraic equations, IV, 116, IX, 39.
Algebra: non-commutative, I, 128—31, 301; notion and boundaries, V, 292—4, 620; Cayley founder of modern, VIII, XXX; geometrical illustration of theorems in, IX, 16—7, 21—39; operation connected with covariants, IX, 537—42; expansion theorem, X, 57; system of equations, XI, 39—40; identities, XI, 63—4, 130—1; and time, XI, 443; origin, XI, 445—8; in Greece, XI, 446; and logic, XI, 459, XII, 459; algebraical equations, XI, 506—21; function in, XI, 523—4; Sylvester's work, XIII, 46; Sylvester's principles of universal, XIII, 47; Sylvester on art and, XIII, 48; (*see also* Multiple Algebra).
Algebraic Curves (*see* Curves).
Algebraic Equations (*see* Equations).
Algebraic Theorems: X, 594, 602, 609.
Algebras, Non-commutative: I, 128—31, 301.
Algorithm: for characteristics of triple theta functions, X, 441—5, 452.
Allink: the term, V, 521.
Allotrious: the term, IX, 204.
Alpine Club: Cayley a member, VIII, xi.
Alternant: of operators, XIII, 400—1.
Altitude: of trees, IX, 429—60.
America: Cayley's visit to, VIII, xxi.
American Mathematical Journal: Sylvester's contributions to, XIII, 47.
Ampère, A. M.: inertia, IV, 563, 584; reciprocants, XIII, 366.
Amphigram: the term, VII, 268.
Ampullate: the term, VI, 101.
Anallagmatic: the term, VII, 246.
Analogues: of Pascal's theorem, I, 426, 427.
Analysis: Bernoulli's numbers in, IX, 259—62; (*see also* Combinatory Analysis).
Analytical Geometry (*see* Geometry, analytical).
Analytical Representation of Curves (*see* Representation).
Analytical Theorem: as to Euler's equation, VII, 261—2.
Analytical Theory of conics (*see* Conics).
Anchor Ring: and cyclide, IX, 18.
Angle: interpreted with reference to two points, VI, 497.
Anglin, A. H.: roots of algebraic equations, XII, 33—4.
Angular Distance (*see* Distance).
Anharmonic Ratio: theory of binary quantics, II, 565; sextic equation, VII, 314—5, XII, 578—9.
Anomaly: expansion of the true, III, 139—43; and elliptic motion, IV, 521, 523.
Anti-circle: the term, VIII, 262.
Anti-conic: the term, IX, 65.
Anti-foci: and foci, VII, 567.
Anti-point: the term, VI, 499—500, IX, 65, XIII, 11; problem and solution, VII, 593; orthomorphosis XIII, 184.
Aoust, L.: geodesic curvature, XI, 330.
Aplati (*see* Penultimate Forms).
Apoclastic: the term, XII, 226.
Applicable Surfaces (*see* Skew Surfaces, Surfaces).
Apollonius: tactions, XIII, 152.
Apsidal Surfaces: theory of, XI, 111—3.

Arbelon: the term, XII, 57.
Arbitrary Constants: in mechanical problems, III, 161—5, 200.
Arbogast, L.: method of derivations, II, 257, IV, 265—71, 272—5, 609, XI, 55, 358—60; rule of, VIII, 471, XIII, 267.
Archimedes: and statics, XI, 446; theorem for surface of cylinder, XII, 56—7.
Argand, R.: imaginaries in plane geometry, XII, 460, 468.
Arithmetic: in Greece, XI, 446; Gauss's work, XI, 544.
Arndt, F.: mathematical tables, VIII, 51—5, IX, 490—1.
Aronhold, S. H.: ternary cubic, II, 325, III, 48, IV, 325—30; on lambdaic of binary quartics, II, 550; hyperdeterminants, II, 598—601; ineunts, IV, 419; intersection of line and conic, V, 501—4; bitangents of quartic curve, VII, 125; construction of a conic, VII, 593; integration formula, X, 12—4; concomitants of ternary cubic, XI, 342; Abelian function, XII, 157; quadric integral, XII, 162—4, 164—7; cubic transformation, XII, 173—9.
Arrangements: theory of permutations, I, 423—4; triads of seven and fifteen things, I, 481—4, 589, V, 95—7; of numbers, X, 570; latin squares, XIII, 55—7; (*see also* Combinatory Analysis, Permutations, Substitutions).
Art: Cayley's love and practice of, VIII, xxiv; and algebra, XIII, 48.
Askwith, E. H.: substitutions, XIII, 117, 133, 141, 145.
Associative: the term, XII, 461.
Associative Algebra: XII, 303.
Associative Imaginaries: XII, 105—6.
Asteroids: Newcomb on orbits, IX, 176—7.
Astronomy: Cayley's work, VIII, xliii; origin of, XI, 446—7; Gauss's work, XI, 545; transformation of coordinates, XI, 575; Sylvester's work, XIII, 47.
Asymptotes: of algebraic curves, I, 46.
Asymptotic: the term, XIII, 232.
Asyzygetic: covariants and invariants, II, 250; the term, VI, 460—1, VII, 336.
Atomic Theory: Sylvester's work in, XIII, 47.
Attraction and Multiple Integrals: I, 5—12, 13—8, 195—203, 204—6, 285—9, 362—3, 508—10, 586; and theorem of Boole, I, 384—7, 588; and theorem of Jellett, I, 388—91.
Attraction of Ellipsoids: I, 432—4, 511—18, VII, 380—3; Gauss's method, III, 25—8, 149—53; Laplace's, III, 53—65, 567; Rodrigues', III, 149—53.
Attractions: theory, II, 35—9, III, 154—5, XI, 448; of terminated straight line, VII, 31—3; of ellipsoidal shell on exterior point, IX, 302—11; of lens-shaped body, X, 594.
Augmented Equation: XII, 453.
Ausdehnungslehre: Grassmann, XII, 480—9.
Automorphic Transformation: IV, 416, V, 439; of binary cubic function, XI, 411—6.
Autopolar Polyhedra: IV, 185.
Auxesis: XI, 79, 81.
Auxiliars: the term, VI, 156; application to quintics, VI, 186—7.
Auxiliary Equations (*see* Equations).
Axial Systems: of polyhedra, V, 529—39.
Axiom: twelfth of Euclid, XI, 435, XII, 220.
Axis: of inertia, IV, 559—66; cubic surfaces and kinds of, VI, 367; transformation of equation of surface to chief axes, IX, 48—51; radical, XI, 465.
Axonometry: IX, 508—18.

Babbage, C.: homographic function, II, 494; matrices, XI, 252.
Babinet, J.: representation of hemisphere, VIII, 539.

Bacharach, J.: intersection of curves, I, 583, XII, 500.
Baehr, G. F. W.: relative motion, IV, 535, 584.
Ball, Sir R. S.: theory of content, II, 606; roots of quartic, V, 610, VII, 551; dynamical equations, IX, 198—200; non-Euclidian geometry, XIII, 481.
Baltzer, H. R.: book on determinants, IV, 608.
Barriers: the term, X, 320.
Barycentric Calculus: Möbius, XII, 472—3.
Bellavitis, J. L.: equipollences, XII, 473—4.
Beltrami, E.: non-Euclidian plane geometry, XII, 221, 224; orthomorphosis, XIII, 171.
Bernoulli, James: analysis and numbers of, IX, 259—62.
Bertrand, J.: geodesic lines, III, 38; differential equations, III, 164, 189, 203; integrals and mechanical problems, III, 187, 200—3; *Mécanique Analytique*, III, 189—90, 203; wave surface, IV, 432—4; central forces problem, IV, 519—21, 584; problem of two centres, IV, 532, 584; motion of point, IV, 547, 584; problem of three bodies, IV, 547—52, 584, V, 23; curves of curvature on surfaces, VIII, 98; series, XI, 623, 627.
Bessel, F. W.: elliptic motion, IV, 522, 584; precession and nutation, IX, 194—6.
Bezout, E.: determinants, I, 63; elimination, IV, 38—9, V, 159, 555—6; equation of differences, IV, 151, 259.
Bezoutiant: defined, II, 526.
Bezoutic Emanant: defined, II, 525.
Bezoutic Matrix: defined, IV, 607.
Bezoutoid: defined, II, 526.
Biaxial: defined, XIII, 13.
Bibasic: defined, XII, 642.
Bibliography: of covariants and invariants, II, 598—601; of symmetric functions, II, 602—3.
Bicentre: the term, IX, 429—60, X, 599.
Bicircular Quartic, on the: X, 223—42; introductory, X, 223—4; formulæ for fourfold generation, X, 224—6; determination as to reality, X, 226; investigation of elementary *arc* formulæ, X, 226—31; inscribed quadrilateral, X, 231—5.
Bicircular Quartics: and polyzomal curves, VI, 472; foci of, VI, 521—2, 522—3; analytical theory, VI, 528—30; problem and solution, VII, 575, X, 596—8; geometrical construction, IX, 13—5; generation, XIII, 12; transformation of circle, XIII, 185; (*see also* Quartics).
Bicircular Quartic Surfaces: VII, 67, 246; quadric surface inversion, VII, 67.
Bickmore, C. E.: Pellian equation, XIII, 442.
Bicolumn: the term, IX, 28.
Bicorn: the term, VI, 148, 158; equation, VI, 163; form, VI, 164.
Bicursal Curves: VIII, 181—7.
Bicursal Sextic: and binodal quartic, IX, 581.
Bien déterminée: defined, XII, 433.
Bin: the abbreviation for tortuous curves, XIII, 253.
Binary: the term, IV, 604, VI, 466.
Binary Cubics: covariants of, II, 189—91; fifth memoir on quantics, II, 540—5; geometrical representation of covariants of, VII, 332—3; tables of forms, VIII, 51—64; transformation of function, XI, 411—6.
Binary Forms: canonic root, V, 103—5.
Binary Matrices (*see* Matrices).
Binary Quadratic Forms: tables of, V, 141—56, 618.
Binary Quadrics: covariants of, II, 189—91; single, II, 527—9; theories of harmonic relation and of involution, II, 529—40; asyzygetic covariants of, VII, 337—8.

Binary Quantics: canonical form, IV, 43—52, 53; canonical root, V, 103—5; involution, V, 296—301; transformations, VI, 187—90; discriminants of, VII, 303; and covariants, VIII, 566—7, X, 430—1, XI, 272—8; geometrical interpretation, IX, 16—7; factions, IX, 426; derivatives of three, X, 278—86; literal table for, XI, 357—64; finite number of covariants of, XII, 558.

Binary Quartics: covariants of, II, 189—91; fifth memoir on quantics, II, 545—56; theorem of four, VII, 100; generating function of, X, 341.

Binary Quintics: tables of covariants M to W of, II, 282—309; covariants and syzygies of degree 6, VI, 148—53; and sextic, VI, 190; irreducible covariants, VII, 334; tables, VII, 341—6; asyzygetic covariants of, VII, 399—400; $(*\{x, y\}^5$, X, 339—400; theorem relating to covariants, X, 430—1; concomitant system for, XI, 272.

Binary Septic: minimum N.G.F., X, 408—21.

Binary Sextic: and quintic, VI, 190; and squared cubic and cubed quadric, XI, 105; concomitant system for, XI, 272; tables for, XI, 372—6, 377—88.

Binet, J.: determinants, I, 63, 64, 581; variation of arbitrary constants, III, 181; relative motion, IV, 535, 584; inertia, IV, 562—3, 584.

Binodal Quartic: transformed to bicursal sextic, IX, 581; graphical representation of elliptic functions, XIII, 9—19; (*see also* Quartics).

Binode: the term, VI, 330, 335, 361, 362, 583—5.

Binomial Equation: theory of numbers, XI, 84—96.

Binomial Theorem: factorials and derivations, II, 101, 102, VIII, 463—73.

Binormal: the term, XIII, 253.

Biography: of Cayley, VIII, ix—xliv.

Bipartite: the term, VI, 464.

Biplanar-node: the term, VI, 360.

Biplanes: the term, VI, 360—1, 362.

Bipoint-locus: the term, VI, 198.

Biquaternions: Clifford, XI, 458, XIII, 481.

Bisection: theory of numbers, XI, 84—96.

Bitangents: of a plane curve, IV, 186—206; of quartic curve, IV, 342—8, VII, 123—4, X, 244, XI, 221—3, XII, 74—94; of quartic, and triple theta functions, X, 444, 446; of curves, XI, 473—4, 480; of quintic, XIII, 21.

Bitetrad: the term, XIII, 551.

Bitrope: the term, VI, 330, 335, 583—5, 591, XI, 228.

Björling, E. G.: root limitation, IX, 39; difference table, XI, 144.

Blissard, J.: factorials, V, 574.

Blunt: the term in seminvariants, XIII, 265, 287, 291, 304—6.

Bodies, Problem of Three: III, 183; Hamilton's method, III, 97—103; Jacobi's theorem, III, 519—21, V, 23.

Body (*see* Solid Body).

Bolyai, J.: hyperbolic geometry, VIII, xxxvii.

Bondset: the term, XII, 642.

Bonnet, Ossian: problem of two centres, IV, 528, 585; imaginary coordinates, VIII, 569; minimal surfaces, XII, 595; curves of curvature, XII, 601, 629—31; skew surfaces, XIII, 231, 237.

Book-keeping: pamphlet by Cayley, VIII, xxiv.

Boole, G.: linear transformations, I, 80, 92, 93—4, 428, 584; multiple integrals, I, 198, 384—7, 588; attractions, I, 285, 289; theorem of Jellett, I, 388; transformation of elliptic integral, I, 508—10; discriminants, I, 584; theory of permutants, II, 26; theory of attractions, II, 35; probabilities, II, 103, 594—8, V, 80—4, 85; hyperdeterminants, II, 598—601; transformation, III, 129; formulæ for differentiation, IV, 135; differential equations, IV, 515, V, 301; involution, V, 301; invariants, VIII, xxx; calculus of logic, VIII, 65—6; integration of differential equations by series, VIII, 458;

prepotentials, IX, 417—23; difference table, XI, 144; matrices, XI, 256; mathematics and logic, XI, 458—9; series, XI, 627.
Booth, J.: rotation of solid body, IV, 577, 585.
Borchardt, C. W.: symmetric functions, II, 417, 421—3; 16-nodal quartic surface, X, 180; theta functions, X, 464, 499; theorem on trees, XIII, 28.
Bordered Skew Determinants: II, 203.
Boron: trees of, IX, 429—60.
Boundaries: of Algebra, V, 292—4, 620.
Bouquet, J. C.: orthogonal surfaces, VIII, 279, 292; periods of elliptic integrals, IX, 618; conformal representation, XI, 80; imaginary variables, XI, 440; elliptic functions, XI, 534; differential equations of first order, XII, 432—41; functions of complex variables, XIII, 190.
Bour, E.: differential equations, III, 164, 197—8, 204, IV, 515, 585; problem of three bodies, IV, 550—2, 585; Gaussian theory of surfaces, XI, 331—6.
Branch: the term, III, 243, X, 36, XI, 476—7; partial, of a curve, V, 425; of polyzonal curves, VI, 474—6, 481—5; main, of trees, IX, 429—60; theorem on trees, XIII, 26—8.
Branch-points: the term, X, 317.
Brassinne, E.: differential equations, III, 186—7, 203.
Bravais, A.: spherical pendulum, IV, 533, 585; on polyhedra, V, 537, 559.
Bretschneider, C. A.: mathematical tables, IX, 486.
Brianchon, C. J.: theorem of, I, 328, V, 4; rectangular hyperbola, III, 254; reciprocal polars, XI, 465.
Briggs, H.: logarithms, XI, 447.
Brill, A.: transformation of plane curves, VI, 593; transformation, and theory of invariants, VIII, 387; sextic curve, IX, 504—7.
Brinkley, J.: formulæ of, IV, 262.
Brioschi, F.: *Sulla variazione* and *Teorema di Mechanica*, III, 190, 203; *degli integrali di un problema di dinamica*, III, 199—200, 203; peninvariants, IV, 246; text-book on determinants, IV, 608; Schwarzian derivative and polyhedral functions, XI, 149, 151; concomitants of ternary cubic, XI, 342; Jacobian sextic equation, XI, 401; theory of equations, XI, 520; elliptic functions and quintic, XII, 493; transformation of elliptic functions, XII, 505—34, 554—5; wave surfaces, XIII, 238; sextic resolvent equations, XIII, 473, 478, 479.
Briot, A.: rotation of solid body, IV, 574, 585; periods of elliptic integral, IX, 618; conformal representation, XI, 80; imaginary variables, XI, 440; elliptic functions, XI, 534; differential equations of first order, XII, 432—41; functions of complex variables, XIII, 190.
British Association: report on Catalogue of Philosophic Memoirs, V, 546—8, 620; communications to, V, 549—53; Cayley president 1883, VIII, xxi—xxii; communications on in-and-circumscribed triangle, VIII, 565—6; correspondence of points and lines in space, VIII, 566; covariants of binary quantic, VIII, 566—7; families of surfaces, VIII, 567; Mercator's projection of surface of revolution, VIII, 567; report on mathematical tables, IX, 461—99; Presidential address, XI, 429—59.
Bronwin, B.: elliptic functions, I, 118, 119, 127, 132.
Buchheim, A.: parallels of, XIII, 481, 489.
Buée, A. Q.: multiple algebra, XII, 467, 471.
Burg, A.: algebraical theorem, X, 57.
Burnside, W. S.: cubic identity, V, 597.

Cagnoli, A.: spherical trigonometry, IV, 80.
Calculus: of logic, VIII, 65—6; and time, XI, 443—4; of functions, XI, 541—2; of forms; (*see also* Covariants, Invariants, Quantics).
Calculus of Variations: Jacobi on, III, 174; problem in, VII, 263.
Cambridge Mathematical Journal: foundation, VIII, xii.

Campaniform: the term, VI, 101, XI, 479.
Canonical Elements: for disturbed motion, III, 77.
Canonical Forms: of quantics, II, 523, 542, 548; of binary quantics, IV, 43—52, 53; the term, IV, 606—7, XIII, 46; quintics, VI, 153—4, X, 355—62, XII, 27; cubic surfaces, VI, 361; Sylvester's work, XIII, 46; (*see also* Formulæ).
Canonisant: defined, II, 523, IV, 45, 53, 606, 607, V, 103—5.
Capacity: of a curve, I, 583; the term, VII, 140; and postulandum, XIII, 115.
Capitation: and seminvariants, XII, 248—50.
Carbon: trees of, IX, 427—60, 544—5.
Cardan, J.: solution of cubic, V, 54, XII, 299.
Cards: game of mousetrap, III, 8, X, 256—8; (*see also* Permutations).
Carey, F. S.: theory of numbers, XII, 73.
Carnot, L.: distances of points, I, 64, 581; on two tetrahedra in perspective, IX, 209—10; geometrical paradox, XII, 305—6.
Cartesians: cusp, I, 589; oval of Descartes, II, 336, 365—6, 370; equation of, II, 370—4, 376; defined, VI, 524, XI, 481; with two imaginary axial foci, VII, 241—3; and cubic curve, VII, 556; problem and solution, VII, 582; note on, IX, 45—7; construction, IX, 317, 535—6, X, 261; invention of coordinates, XI, 449; explained, XI, 461; scalene transformation, IX, 532—4; correspondence of confocal, with right lines of hyperboloid, XII, 587—9.
Casey, J.: equation of, and tangent circles, VI, 65—71; polyzomal curves, VI, 471—2; bicircular quartics, VI, 540, X, 223—42, XIII, 12; tactions, VI, 543, VII, 115, XIII, 153; circle touching three others, VI, 568—73; cyclides and sphero-quartics, VIII, 262—3; cyclide, IX, 64, 75.
Cassinian: note on, IX, 264—5; the word, X, 608; equation of, X, 608.
Castillon, J. F. S. de: problem and porism, IV, 435—41.
Catalan, E.: attractions, I, 288; integral calculus, I, 313; series, III, 127, XI, 627; curve parallel to ellipse, IV, 152; minimal surfaces, XII, 594—5.
Catalecticant: the term, II, 523, IV, 606, 607.
Catalogue of Scientific Papers: issued by Royal Society, V, 546—8, 620.
Catenary: and epitrochoid, XIII, 87.
Cauchy, A. L.: determinants, I, 63, 64, 66; inverse elliptic functions, I, 148, 157, 174; integral calculus, I, 309, 314; permutations, I, 423; partitions, II, 248, V, 48, VII, 577; differential equations, III, 166, 169; theory of logarithms, III, 208—13, 225; polyhedra, IV, 86—7; singularity of function, IV, 105; curve parallel to ellipse, IV, 152; equations of motion, IV, 514, 585; elliptic motion, IV, 524, 585; inertia, IV, 560, 564, 565, 585; geodesic lines on ellipsoid, VII, 498; geometrical representation of root-limitation, IX, 21—39; series, IX, 61, XI, 627; mathematical tables, IX, 475; theory of equations, X, 6, XI, 504; correspondence, X, 290; monogenous function, XI, 537; theory of numbers, XI, 601; roots of algebraic equations, XIII, 35.
Caustics: by reflection at circles, I, 273—5, II, 118—22, 129; memoirs on, II, 336—80, V, 454—65; tracing of, II, 351—2, 362—3, 368—9, 374.
Cavalieri, B.: quadrature of curves, XI, 641.
Cayley, Arthur: portraits of, frontispiece to vols. VI, VII, XI; manuscript of, frontispiece to vol. VIII; biography of, VIII, ix—xliv; mathematical tables, IX, 479, 487, 491.
Cayleyan: the word, I, 586, II, 381.
Central Forces Problem: IV, 517—21.
Centres: problem of two, III, 104—10, IV, 527, 528—9, 589; cubic, and cones, IV, 173—8, 179—81; of curves defined, VI, 522; four points and conic of, VII, 582; of trees, IX, 429—60; the term, X, 599; of three circles, XI, 552.
Centro-curve: kinds of, VIII, 320.
Centro-surface: of ellipsoid, VIII, 8; the term, VIII, 316; Rudio's inverse, XII, 457—8.

Cesser: points of, IV, 130.
Challis, J.: integration of differential equations, VII, 36.
Chance (*see* Probability).
Characteristic Function: of Hamilton, III, 217; for systems of rays, XII, 571.
Characteristics: logic of, III, 51—2; of Chasles, V, 552; theory, VI, 594, XIII, 468—72; of triple theta functions, X, 441—5.
Chartography: surface representation on plane, VIII, 538—9; colouring of maps, XI, 7—8; map projections, XI, 448.
Chasles, M.: intersections of curves, I, 25—7; Pascal's theorem, I, 45; theorem on correspondence, I, 212; a theorem of, demonstrated, I, 355; analogue of Pascal's theorem, I, 427; transformation of curves, I, 478—80; homography, II, 538; cubic curves, IV, 122, 495; inertia, IV, 561, 586; kinematics of solid body, IV, 580, 586; curves on a quadric, V, 11; on a cubic, V, 19; conics touching curves, V, 31—2, 552; scrolls, V, 169, 201, VI, 328; quartic scrolls, V, 201; cubic curves and cones, V, 401; equilibrium of four forces, V, 540—1; correspondence of points in plane curve, V, 542; contact of conics, V, 552; characteristics, V, 552; on united points, VI, 9; curves which satisfy given conditions, VI, 191, 192, 200—26; principle of correspondence, VI, 264, XI, 482, 485—8; foci of conics, VII, 1; six coordinates of a line, VII, 93; attraction of ellipsoids, VII, 380—3; locus *in plano*, VII, 605; cones satisfying six conditions, VIII, 99; penultimate forms of curves, VIII, 258; theory of duality, XI, 467.
Chemistry: Cayley's interest in, VIII, x; application of trees to, IX, 202—4, 427—60, 544—5.
Chessboard: topology of, X, 609—10.
Chord: angle between normal and bisector, X, 576; of two circles, XI, 552—6.
Christie, J. T.: Cayley's law work, VIII, xiii—iv.
Christoffel, E. B.: orthomorphosis, XIII, 180.
Chrystal, G.: uniform convergence, XIII, 343—4.
Chuck: for quartic curves, VIII, 151—5; for curve-tracing, VIII, 179—80; bicyclic, VIII, 209—11.
Circle: Salmon's equation for orthotomic, III, 48—50; and points, V, 560; and ellipse, V, 561; line and parabola, V, 607; envelope of, V, 610; equation of, VI, 501, XI, 558—61; potential of, IX, 290—301; quadrilateral inscribable in, X, 578; orthomorphosis, XII, 328—36, XIII, 20, 182, 202—5; Wallis's π investigation, XIII, 22—5; transformation into bicircular quartic, XIII, 185; and circumference, the terms, XIII, 194; the nine point, XIII, 517—9, 520—1, 548—51; of curvature of an ellipse, XIII, 537.
Circles: powers of, I, 581; systems of, III, 111—4, X, 566; in-and-circumscribed polygon, IV, 303—8; a pair touching three given, VI, 65—71; involution of four, VI, 505—8; relation between two, VIII, 12—3; equal, VIII, 31; minimum enclosing three points, X, 576; system of 15 connected with icosahedron, XI, 208—12; radical axis, XI, 465; radical centre of three, XI, 552; Mascheroni's geometry of the compass, XII, 314—7; system of three which cut each other at given angles, XII, 559—61, 564—70; the two relations connecting the distances of four points on a circle, XII, 576—7; roots of algebraic equation, XIII, 37; problem of tactions, XIII, 150—69; tetrads of, XIII, 425—9; (*see also* Casey, Orthomorphosis).
Circuit: the word, XI, 480.
Circular: the word, XI, 481.
Circular Cubic: and polyzomal curves, VI, 522—8.
Circular Points: at infinity, VIII, 32.
Circular Relation of Möbius: III, 118—9, IX, 612—7.
Circumference: and circle, the terms, XIII, 194.
Cissoid: the term, XI, 461.
Clairaut, A. C.: lunar theory, IV, 518, 586; demonstration of his theorem, X, 17—8; curves of double curvature, XI, 489.

Classes: of curves, II, 569—83, V, 518, 521, VI, 11, XI, 629.

Classification: of curves, V, 613—7; cubics and quartics, VI, 525; quartic surfaces, VII, 244; nodal quartic surfaces, VII, 264—7; mathematical tables, IX, 424—5; cubic curves, XI, 478—80; quartic curves, XI, 480.

Clausen, T.: Castillon's problem, IV, 435—41.

Clebsch, R. F. A.: Steiner's extension of Malfatti's problem, II, 593; Pfaffian equations, IV, 515; singularities of curves, V, 476—7; developable surfaces, V, 518; Abelian integrals, V, 521; transformation of plane curves, VI, 1—8, 593; Casey's equation, VI, 67; binary quintic and sextic, VI, 190; principle of correspondence, VI, 264; reciprocal surfaces, VI, 356; cubic curve in pencil of six lines, VI, 593—4; centro-surface of ellipsoid, VII, 130, VIII, 316; quartic and quintic surfaces, VII, 245—51, 313; bicursal curves, VIII, 182; transformation of unicursal surfaces, VIII, 389; deficiency of surfaces, VIII, 394; covariants, VIII, 404; expression for elliptic integral of second kind, X, 27; concomitants of ternary cubic, XI, 343; tables for binary sextic, XI, 372; Abelian functions, XII, 102, 109; symbolic notation for quantics, XII, 347; seminvariants, XIII, 362; Pfaff-invariants, XIII, 405—14.

Clifford, W. K.: powers of circles and spheres, I, 581; circle and ellipse, V, 561; conic, V, 562; triangles and polygons, V, 589; rational transformation between two spaces, VII, 222—4; quartic surfaces, VII, 246; compound combinations, X, 243; biquaternions, XI, 458; syzygetic relations among the powers of linear quantics, XIII, 224—7; non-Euclidian geometry, XIII, 481.

Close: partitions of a, V, 62—5, 617; defined, V, 63.

Close-planes: the term, VI, 330, 583—5.

Close-points: the term, VI, 330, 339, 341, 583—5.

Cnicnode: the word, VI, 330, 334, 361, 583—5, XI, 228, 631, 633; quartic and quintic surfaces, VII, 245—51.

Cnictrope: the word, VI, 330, 334, 583—5, 591, XI, 228, 631, 633.

Cobezoutiants: defined, II, 524.

Cobezoutoids: defined, II, 524—5.

Cockle, Sir James: resolvent product, IV, 309—13; on quintics, V, 53; theory of equations, XI, 520; invariants, XII, 390—1; criticoids, XIII, 366, 367—8.

Codazzi, D.: application of surfaces, XIII, 253—64.

Coefficients: of Legendre, I, 375—6; development of $(1+n^2x)^{m/n}$, XIII, 354—7.

Cogredient: the term, IV, 607—8, XIII, 46.

Cohen, A.: relative motion, IV, 537, 586; rotation of solid body, IV, 573, 586.

Cole, F. W.: on simple groups, XIII, 533.

Colink: the term, V, 521.

Collins, M.: Lagrange's theorem, II, 3; systems of circles, X, 566—7.

Colour Groups: diagrams representing, X, 328—330, 403—5; the term, XII, 639—41.

Colours: in maps, XI, 7—8.

Colourset: the term, XII, 641.

Columns: the term, XII, 353.

Comberousse, C.: polyhedra, IV, 609.

Combescure, E.: wave surfaces, XIII, 238.

Combinants: of quantics, II, 322; the term, II, 518, IV, 607—8, XIII, 46; and ternary quadratics, IV, 352.

Combinatory Analysis: magic squares, X, 38; compound combinations, X, 243—4; a problem of arrangements, X, 245—8; note on Mr Muir's solution, X, 249—51; the game of mousetrap, X, 256—8; Sylvester's work in, XIII, 47; (*see also* Partitions).

Commutants: the term, I, 584, II, 19, 26, IV, 594, 599—600, XIII, 46; property of, V, 495—7.

Commutative: the term, XII, 461; (*see also* Groups).

Comoment: non-Euclidian geometry, XIII, 481—9.
Compass: Mascheroni's geometry of the, XII, 314—7.
Complex (*see* Surfaces).
Complex Cone: (cubic) defined, V, 402, 404, 551.
Complexes of Lines: IV, 618; through twisted quartic, XII, 428—31.
Complex Multiplication: in elliptic functions, XII, 556—7; (*see also* Multiplication).
Complex Quantities: logarithms of, VI, 14—8.
Complex Variables: and conformal representation, X, 316—23; Newton-Fourier theorem for, X, 405—6; (*see also* Function, Newton-Fourier theorem).
Composition: of quadratic forms, I, 532; of singularities, V, 619; of rotations, VI, 24—6.
Compound Combinations (*see* Combinatory analysis).
Compound Singularities: V, 525.
Conchoid: the term, XI, 460.
Concomitant: the term, IV, 607—8, XIII, 46.
Concomitant-system: of quintic, X, 342.
Cone: touching six lines, VIII, 401—3; formulæ for potentials of, IX, 266—7.
Cones: through cubic curve in space, III, 219—21; note on cubic, IV, 120—2; and cubic centres, IV, 173—8, 179—81; and cubic curves, V, 284—8; kinds of cubic, V, 401—15; and representation of curve, V, 552; circumscribed sextic, VII, 139; satisfying six conditions, VIII, 99—137; the term asymptotic, XIII, 232; characteristic n and theory of curves in space, XIII, 468—72.
Configurations: algebraic, by Hilbert, VI, 596.
Conformal Representation: IX, 609—11, XI, 442, XII, 104; by function $\arcsin(x+iy)$, X, 290—2; mechanical constructions, X, 406; example, XI, 38; theorem, XI, 78—81; and Schwarzian theory, XI, 169—76; imaginary quantities, XI, 258—60; (*see also* Orthomorphosis, Representation, Transformation).
Congregate: the word, X, 339, 345.
Congruences: in *Ency. Brit.*, XI, 628, 634—5; analytical theory, XIII, 228—30.
Conic: theorem of triangle and line, V, 100—2; theorem of eight points on a, V, 427—30; formula for intersections of line and conic, V, 500—4; four points on, V, 571; defined by five conditions, VII, 546, 552; through three points and with double contact, VII, 554; foci of, VII, 571; and four points, VII, 581, 587; construction, VII, 592; (2, 2) correspondence of points on, VIII, 14—21; and cubic, X, 605—7; Monge's differential equation, XII, 393; focals of quadric surface, XIII, 54.
Conic-node: the term, VI, 360.
Conics, Analytical Theory of: IV, 395—419; relating to single conic, IV, 396—402; ditto with point or line, IV, 402—12; ditto with tangent of conic of double contact, IV, 413; relating to two conics, IV, 416—9.
Conics: general theory of, I, 519—21; inscribed in a quadric surface, I, 557—63; in-and-circumscribed polygon, II, 142—4, IV, 295—9; two dimensional geometry, II, 575—83, 586—92; forms of equations of, III, 86—90; area of, and trilinear equation second degree, III, 143—8; normals of, IV, 74—7; of five-pointic contact of plane curve, IV, 207—39; which touch four lines, IV, 429—31; system having double contact, IV, 456—9, VII, 568; theorem in, IV, 481—3; touching curves, V, 31—2, 552; four inscribed in same conic and passing through same three points, V, 131—2; contact of, V, 552; and rectangular hyperbolas, V, 554; problem, V, 562, 582; tangents of, V, 578; intersection of, V, 582; triangle and, V, 593; and cubic, V, 608; drawing of, VI, 19; locus from two, VI, 27—34; theorem of four which touch same two lines and pass through same four points, VI, 35—9; which touch cuspidal cubic, VI, 249—53; contact of third order with given cuspidal cubic and double contact with given cubic, VI, 253—6; Zeuthen's forms for characteristics of conics which satisfy four conditions, VI, 256—8; correspondence, and those which satisfy given conditions, are at least arbitrary, VI, 268—71; five conditions of contact with a given curve, VI, 272—91; foci of, VI, 517—9, VII, 1—4; determined by

five conditions of contact, VII, 40; three, problem and solution, VII, 595; and absolute, VIII, 31—44; theorem of eight points on, VIII, 92—4; cuspidal, of centro-surface, VIII, 352—7; reciprocal of equation, VIII, 522—3; theory of confocal, VIII, 556—7; sets of four points on, X, 569; and lines, X, 602; degenerate forms of curves, XI, 218—20; the term, XI, 460; in *Ency. Brit.*, XI, 561—4; analytical geometrical note on, XII, 424; f and c, XIII, 11—2; non-existence of special group of points, XIII, 212; the nine-point circle, XIII, 517—9.

Conics, Spherical: theorem relating to, IV, 428; and stereographic projection, V, 106—9; (*see also* Polyzomal curves).

Conics which pass through: four points, III, 136—8; four foci of given conic, IV, 505—9; three given points and touch one line, V, 258—64; two given points and touch two given lines, VI, 43—50; two given points and touch given conic, VI, 245—9.

Conic Torus: the, IX, 519—21.

Conjugate Integrals: Hamiltonian, X, 113—5.

Conjugates: table of, and seminvariants, XIII, 303, 307.

Connected Areas: XI, 7.

Connective: of discriminant, II, 529.

Connective Covariant of two Quantics: defined, II, 515.

Conormal Correspondence: of vicinal surfaces, VIII, 301—8.

Constants: number of, in special equations, XI, 14—6.

Constructive Geometry: VII, 27.

Contacts: problem of, I, 522—31; the term, VII, 546.

Content: Ball on theory of, II, 606.

Continuous Function: the term, XI, 539.

Contour: lines, IV, 108—11, 609; defined, V, 63.

Contracovariants: defined, IV, 329.

Contractible Squarewise: the term, XIII, 179.

Contragredient: the term, IV, 607—8, XIII, 46.

Contraprovectant: defined, II, 514.

Contraprovector: the term, II, 514.

Contrasect: the term, XIII, 485.

Contravariant: the term, II, 320, XIII, 46; of ternary cubic, IV, 325.

Convergence: condition of uniform, XIII, 342—5.

Converging Series: product of, IX, 61.

Convertible Matrices (*see* Matrices).

Convolution: the term, VI, 461—2.

Coordinates: in general theory of geometry, II, 604—6; as functions of parameters, VI, 1—2; polyzomal curves, VI, 498—9, 537; trilinear, XI, 467; Plücker, XI, 467; degenerate curves, XI, 488—9; in *Ency. Brit.*, XI, 546—51, 566—7; illustrative of geometry, XI, 552—6; curvilinear, in *Ency. Brit.*, XI, 637; versus quaternions, XIII, 541—4.

Coordinates of a Line: X, 603, XI, 468.

Coordinates of Points: expressions for, V, 517—8; lines and planes, non-Euclidian geometry, XIII, 489—91.

Coordinates, Six of a Line: VII, 66—98, VIII, 401, X, 287, XII, 42—3, 321; introductory, VII, 66; definition and general notions, VII, 67—9; elementary theorems, VII, 69—73; geometrical considerations, VII, 73—5; linear relations between six coordinates, VII, 75—85; geometrical property of an involution of six lines, VII, 85; four given lines and twofold tractor, VII, 85—6; hyperboloid through three given lines, VII, 86—8; six coordinates defined as absolute magnitudes, VII, 88—9, 96—7; statical and kinematical applications, VII, 89—95; transformation of coordinates, VII, 95—6; formulæ of transformation, VII, 97—8.

Coordinates, Spherical: theory of, and systems of equations, I, 213—23.

Coordinates, Transformation of: I, 123—6, IV, 552—9, XI, 136—42; in *Ency. Brit.*, XI, 558—61, 573—6.
Copfaffian: the term, XIII, 406.
Coriolis, G.: motion of three bodies, IV, 541.
Corpus: Sylvester's theory of the, XIII, 47.
Correspondence: on cubic curves, I, 184, 190; homographic figures, I, 212; theory, VI, 263—91, X, 259—60, XI, 482; in-and-circumscribed triangle, VIII, 222—5; vicinal surfaces, VIII, 301—8; of two variables, IX, 94—5, XII, 104; geometrical representation of imaginary variables, X, 316—23; construction of a, XI, 38; quadric transformation between points and planes, XII, 100—1; of Cartesians, and generators of hyperboloid, XII, 587—9.
Correspondence of Points: V, 542—5, VI, 22, VII, 168—70, XI, 440; two on a curve, VI, 9—13; on a conic, VIII, 14—21; on surfaces, VIII, 200—8; and lines, VIII, 566.
Cos-centre: the word, XIII, 551.
Cotes, R.: central forces, IV, 517, 586.
Cotterill, T.: correspondence of points, VI, 22; problem of envelope and two circles, VII, 573; theorem of Geiser, IX, 506; goniometric problem, X, 295—7.
Counter-barriers: the term, X, 320.
Counter Order: the term, XIII, 268.
Couples: algebraic, I, 128—31.
Cournot, A. A.: motion of a body, IV, 583, 586.
Covariantive Forms and Tables: XI, 277—80; *M* to *W* of binary quintic, II, 282—309; asyzygetic, to degree 18, VI, 149—52; 34 concomitants of ternary cubic, XI, 342—56; of binary sextic, XI, 372—6, 377—88; theory of tamisage, XI, 409—10.
Covariants: the term, I, 577, 589, II, 224, IV, 594, 605, X, 340, XIII, 46; determined by differential equations, II, 164—78; theory, II, 164—78; of binary cubic, II, 189, 260—2; binary quadratic, II, 189; binary quartic, II, 190, 262—4; asyzygetic, II, 250; binary quantic, II, 269; of cubic, analogous to invariants of quartic, II, 553; bibliography, II, 598—601; of degree 6, VI, 148—53; of binary cubic, geometrical interpretation, VII, 332—3; the terms asyzygetic and irreducible, VII, 336; theory of number of irreducible, VII, 336—7; also new formulæ for asyzygetic, VII, 337—40; also 23 fundamental, VII, 341—8; Gordan's proof for the number, VII, 348—53; theory founded by Cayley, VIII, xxix—xxx; his work, VIII, xxx—xxxii; as transvectants, VIII, 404—8; connected with an algebraical operation, IX, 537—42; derivatives of three binary quantics, X, 278—86; theorem, X, 430—1; a formula, XI, 122—4; formula and Schwarzian derivative, XI, 184—5; in geometry, XI, 474; Sylvester on, XIII, 47; a hyperdeterminant identity, XIII, 210—11; theory of derivation connected with particular operators, XIII, 329—32; (*see also* Invariants, Linear Transformation, Seminvariants).
Cox, H.: Taylor's theorem, VIII, 493.
Cox, H., Jun.: non-Euclidian geometry, XIII, 481.
Cramer, G.: determinants, I, 63; curve classification, V, 354; transformation of plane curves, VI, 1.
Creedy, C.: tangential of cubic, II, 558; calculations by, III, 361; elliptic motion, IV, 522, 586.
Cremona, L.: on Steiner's quartic surface, V, 423; general theory of correspondence, VI, 22—3; Casey's equation, VI, 66—7; scrolls, VI, 327—8, VII, 245—51; polyzomal curves, VI, 575—6; rational transformation, VII, 189, 200, 207, 222, 253—5, XI, 482, 484; theory of curve and torse, VIII, 72, 76—9, 87—91; geometric transformation, X, 611—2.
Critic Centres (*see* Involution of Cubic Curves).
Criticoids: and invariants, XII, 390; and reciprocants, XIII, 366—7; of Cockle, XIII, 366—7.
Critic Points and Lines: the term, X, 311—5.
Crofton, M. W.: polyzomal curves, VI, 507; Cartesian curves, VII, 582.
Cross-points: the term, X, 317.
Crunodal: the term, V, 402, 551, XI, 228.
Crunode: defined, IV, 181, V, 295, 521, XI, 630.

Crystals: biaxal and ray planes, IX, 107—9.

Cube: axial systems, V, 531—9; automorphic function for, XI, 169, 179—83, 212—6.

Cubic: canonical form of binary, II, 542; equation of differences for, IV, 242, 279; ternary, IV, 325—41; and tables, IV, 333—41; the term, IV, 604; resultant of two binary, V, 289—91; asyzygetic covariants of binary, VII, 338—9; automorphic transformation of binary cubic, XI, 411—6; (*see also* Binary and Ternary Cubics).

Cubic Centres: of lines, V, 73—6.

Cubic Cone: kinds of, V, 401—5, 551, 553; anharmonic property, V, 411—2.

Cubic Curve Classification: V, 354—400, VIII, XXXVIII; seven head divisions, V, 355—6; their equations, V, 356—9; thirteen divisions, V, 360—1; notion of group, V, 361; osculating asymptotes, V, 361—3; Newton's classification, V, 364—6, XI, 464; Plücker's, V, 366—8; theory of groups, V, 368—9; groups of hyperbolas Δ, V, 369—70; hyperbolas Δ redundant, V, 370—6; ditto defective, V, 376—88; hyperbolas ⊙, V, 388—9; ditto redundant, V, 389—90; ditto defective, V, 390—1; groups of parabolic hyperbolas, V, 391—4; of central and parabolic hyperbolisms, V, 394; groups of divergent parabolas, V, 395; trident curve and cubical parabola, V, 395; division into species, Newton and Plücker, V, 396—9.

Cubic Curve Memoirs: first, I, 183—9, 586; remarks, I, 190—4, 586; second memoir, II, 381—416, IV, 188; definitions, II, 382—3; theorems relating to conjugate poles, II, 383—5; their proof, II, 385—96; geometrical definition of Quippian, II, 396—7; theorems, II, 397—403; formulæ for intersection of curve and line, II, 404—5; formulæ for satellite point and line, II, 405—9; theorems relating to satellite point, II, 409—12; first polar point of cubic, II, 412—5; recapitulation of geometrical definitions of Pippian, II, 416.

Cubic Curves: tangential of, II, 558—60; cones through, III, 219—21; note on, IV, 120—2; five pointic contact, IV, 231—6; ninth point of intersection of those passing through eight given points, IV, 495—504; twisted, V, 1; sextactic points of plane, V, 233—5; and cones, V, 401—15, 551, 553; inflexions of, V, 493—4, XI, 473; in connexion with quintics and quartics, V, 580; problem, V, 586; derivation of points, VI, 20; intersection of, VI, 20; in pencil of six lines, VI, 105—15, 593—4; nodal, VI, 171—4; foci of circular, VI, 521—2; theory of circular, VI, 526—8; symmetrical circular, VI, 549—50, 550—3; quartic and three, VII, 546; points on, VII, 549; and Cartesian curves, VII, 556; rectangular, VII, 591; mechanical description, VIII, 147—50; residuation in regard to one, IX, 211—4; problem and solution, X, 592—4; equation of, X, 603; and conic, X, 605—7; Abel's theorem applied to, XI, 27—8; degenerate, XI, 220; date of theory, XI, 449; forms and classification, XI, 478—80; circular, XI, 481; systems of, XI, 487; Abel's theorem, XII, 30; elliptic functions, XII, 35—7; as ground-curve in Abel's theorem, XII, 38, 109—216; twisted, on quadric surface, XII, 307—10; notion of, XIII, 79—80; and non-existence of a special group of points, XIII, 212.

Cubic Equations: solution of, II, 542; Tschirnhausen's transformation, IV, 364—7, 377; equation of squared differences, IV, 463—5; Sturmian constants, IV, 473—7; relation between roots, VII, 548; solution by radicals, X, 9; constants of, XI, 556; note on, XII, 421—3; Cardan's solution, XII, 299; on two, XIII, 348—9.

Cubic Forms: letters on, III, 9—12.

Cubic Identity: problem, V, 597.

Cubic Scrolls (*see* Scrolls).

Cubic Seminvariants: generating function, XIII, 306.

Cubic Surfaces Memoir: VI, 359—455, 595—6; Introductory, VI, 359; twenty-three cases, explanations, and tables of singularities, VI, 359—63; determination of number of certain singularities, VI, 364—5; lines and planes of cubic surface, facultative lines, diagrams, VI, 365—6; different kinds of axis, VI, 367; determination of reciprocal equation, VI, 368—70; explanation of sections of memoir, VI, 370—1; equations, I = 12, VI, 371—83; $12-C_2$, VI, 383—90; $12-B_3$, VI, 391—6; $12-2C_2$, VI, 397—402; $12-B_4$, VI, 403—7; $12-B_3-C_2$, VI, 407—11; $12-B_5$, VI, 411—8; $12-3C_2$, VI, 418—

22; $12-2B_3$, VI, 422—6; $12-B_4-C_2$, VI, 426—8; $12-B_6$, VI, 429—30; $12-U_6$, VI, 431—3; $12-B_5-2C_2$, VI, 433—6; $12-B_5-C_2$, VI, 437—9; $12-U_7$, VI, 439—40; $12-4C_2$, VI, 441—2; $12-2B-C_2$, VI, 443—4; $12-B_4-2C_2$, VI, 445—6; $12-B_6-C_2$, VI, 447—8; $12-U_8$, VI, 448—9, 451—5; $12-3B_3$, VI, 449—50; synopsis of foregoing, VI, 450; cubic scrolls, VI, 451.

Cubic Surfaces: triple tangent planes, I, 445—56, 589; skew, V, 90—4; delineation of scrolls, V, 110—2; nodal curve of developable from quartic equation, V, 135—7; theory, V, 138—40; five given lines on, VII, 177—8; double sixers, VII, 316—29; and tetrahedra, VII, 607; Wiener's model with 27 real lines, VIII, 366—84; in *Ency. Brit.*, XI, 633.

Cubic Transformation of Elliptic Functions: III, 266—7, VII, 44—6, X, 46, 58, XII, 518—22, 555, 556—7; geometric illustration, IX, 522—6.

Cubi-Cubic Curves: in space, V, 18—9.

Cubinvariants: of binary quartic, I, 94; of quantic, II, 516; the term, IV, 606; of quadri-quadric function, XIII, 67—8.

Cuboid: potential of, IX, 272, 274—5, 278—80.

Cumulant: the word, IV, 600—1.

Cunningham, A.: on number of terms in a determinant, X, 579—80.

Curtate: the term, XI, 155.

Curvature: lines of, on ellipsoid, I, 36—9; of plane curve at double point, IV, 466—9; of surfaces, IV, 466—9; geodesic, XI, 323—30; (*see also* Curves of Curvature, Orthogonal Surfaces).

Curves: and developables, I, 207—11, 485, 586—7, 589; and two dimensional geometry, II, 569—83; partial branch of, V, 425; reciprocation, V, 505—10; representation by cone and monoid surface, V, 552; nodal, spinode and cuspidal, of cubic surfaces, VI, 450, 595; and space of m dimensions, VI, 456—7; correspondence of two points on, VII, 39; graduation, VII, 426; mechanical description, of, VIII, 138—44, 147—50, 151—5, X, 576; bicyclic chuck for, VIII, 209—11; penultimate forms, VIII, 258—61, 262—3; property of curve and torse, VIII, 520—1; coordinates and equations, X, 546; degenerate forms, XI, 218—20, 487—9; abstract geometry, XI, 441—2; in *Ency. Brit.*, XI, 460—89, 572—3, 579—80; and theory of equations, XI, 501; and function, XI, 540—1; and solid geometry, XI, 569; quadrature of, XI, 641—2; minimal surfaces, XIII, 41; orthotomic, of a system of lines in a plane, XIII, 346—7; (*see also* Correspondence, Cubic Curves, Nodal Curves, Polyzomal Curves).

Curves, Algebraic: I, 46—54, 584.

Curves, Bicursal: VIII, 181—7.

Curves, Classification of: V, 613—7; (*see also* Cubic Curve Classification).

Curves, Excubo-quartic: V, 282.

Curves in Space: analytical representation, IV, 446—55, 490—5, 616—8, VII, 66, XI, 9—13; defined by conoid and monoid surfaces, V, 7—20, 552, 553, 613; quartic, V, 11—5; quintic, V, 15—6, 24—30, 552, 553, 613; quadri-cubic, V, 16; quadri-quartic, V, 17; cubi-cubic, V, 18—9; Halphen's characteristic n in theory of, XIII, 468—72.

Curves, Intersections of: I, 25—7, 583, XII, 500—4; real, IX, 21.

Curves of Curvature: near umbilicus, VII, 330—1; on surfaces, VIII, 97—8, 145—6, 264—8; *Ency. Brit.*, XI, 628, 635—6; wave surface, XII, 249; surfaces with spherical, XII, 601—38.

Curves of Striction: I, 234.

Curves, Opposite: V, 468.

Curves, Parallel: envelopes and surfaces, IV, 123—33, 152—7, 158—65; and evolutes, VIII, 31—5; theory of, X, 260; the critic, in solar eclipses, X, 311—5.

Curves, Pedal: V, 113—4.

Curves, Penultimate Quartic: VIII, 526—8.

Curves, Penumbral: geometrical theory of projection, VII, 483, 488—9, 489—92.

Curves, Plane: double tangents of, IV, 186—206; conic of five-pointic contact of, IV, 207—39;

curvature at double point, IV, 466—9; higher singularities, V, 424—6, 520—8, 619; correspondence of points on, V, 542—5; transformation, VI, 1—8, 593, VIII, 387; notion of, of given order, XIII, 79—80.

Curves, Plane, sextactic points of: V, 221—57, 618—9; condition for point, V, 222—5; notations and remarks, V, 225—6; first transformation, V, 226; second, V, 227—8; third, V, 228—9; fourth and final form, V, 229—33; application to cubic, V, 233—5; proof of identities, V, 235—7; Jacobian formula, V, 237—8; proofs of equations and identities, V, 239—47; appendix, V, 247.

Curves, Poloid: IV, 571.

Curves, Rhizic: IX, 34.

Curves, Serpoloid: IV, 571.

Curves, Sextic: VII, 256—7, VIII, 138—44, X, 612.

Curves, Symmetric: I, 473.

Curves, Theory of: and elimination, I, 337—51, V, 162—7, 416—20; evolution, XI, 449—51, XII, 102—3, 290—1.

Curves, Theory of, and Torse: VIII, 72—91; explanations and notation, VIII, 72—4; Plücker-Cayley equations, VIII, 74, 75—6, 80—1; Salmon-Cremona equations, VIII, 74, 76—9, 87—91; geometrical theory of foregoing relations, VIII, 79—80; tables, VIII, 81—4; nodal curve x, VIII, 84—7.

Curves, Three-bar: IX, 551—80, 585.

Curves, Transformations of: I, 471—5, 476—80; scalene, IX, 527—34.

Curves, Triangular: VII, 59.

Curves, Twisted Quartic: XII, 428—31.

Curves which satisfy given conditions: VI, 191—262, 594, VII, 40; Introductory, VI, 191; previous memoirs, VI, 191—2; quasi-geometrical representation of conditions, VI, 193—200; Chasles' and Zeuthen's researches, VI, 200—26; extensions of de Jonquières, VI, 226—42; form of equation of curves of a series of given index, VI, 242—3; line-pairs which pass through three given points and touch a given conic, VI, 244; conics which pass through two given points and touch given conic, VI, 245—9; conics which touch cuspidal cubic, VI, 249—53; conics which have contact of third order with given cuspidal cubic and double contact with given cubic, VI, 253—6; Zeuthen's forms for characteristics of conics which satisfy four conditions, VI, 256—8; question from de Jonquières' formula, VI, 258—62; the principle of correspondence, VI, 263—91; (introductory, VI, 263—4; correspondence of two points on a curve, VI, 264—8; application to conics which satisfy given conditions, one at least arbitrary, VI, 268—71; five conditions of contact with a given curve, VI, 272—91).

Curve Tracing: Cayley's liking for, VIII, xxxix; mechanism, VIII, 179—80, XIII, 515—6; importance, XI, 461; order of, XI, 461.

Curvilinear Coordinates: XI, 330, XII, 1—18; surfaces divisible into squares, VIII, 146; geodesic lines, VIII, 156—67; curves of curvature, VIII, 264—8; orthogonal surfaces, VIII, 269—91.

Cusp: of Cartesian at circular points at infinity, I, 589; synonymous with spinode, II, 28, IV, 22, 27; of second kind or node-cusp, V, 265—6, 618; order of plexus for, V, 309—12; the term, XI, 468.

Cuspidal: defined, V, 403, 551, VII, 244.

Cuspidal Conic: of centro-surface, VIII, 352—7.

Cuspidal Cubic: VII, 561.

Cuspidal Curves: and cubic surfaces, VI, 450; (*see also* Cubic Surfaces, Surfaces).

Cuspidal Isochronic: the term, VII, 473.

Cuspidal Locus: in singular solutions, VIII, 533.

Cyc: the abbreviation in groups, XIII, 119.

Cyclide: of Dupin, V, 467, XII, 615; the term, VII, 246, VIII, 262, IX, 64—5; and anchor ring, IX, 18; on, IX, 64—78; the parabolic, IX, 73—8; in *Ency. Brit.*, XI, 634.

Cycloid: the term, XI, 447.

Cyclotomy: (Kreistheilung), XI, 58, 86.
Cylinder: in *Ency. Brit.*, XI, 572—3; Archimedes' theorem for surface, XII, 56—7.

d'Alembert, J. le R.: rotation of solid body, IV, 567, 586; geometric paradox, XII, 305—6.
Dandelin, G. P.: theory of, and on caustics, II, 339—40; wave surface, IV, 433—4.
Darboux, G.: powers of circles and spheres, I, 581; the torus, VII, 247; quartic surfaces, VIII, 262; cyclide, IX, 64; continuous function, XI, 539; curves of curvature, XII, 615; quartics, XIII, 13.
Davis, W. Barrett: calculations by, III, 361, IV, 376; quantic covariants, VII, 335; Sohnke's modular equations, IX, 543.
Déblais: theory of, XI, 417—20, 449, 587.
de Bruno, Faà: invariants of degree 12 belonging to quintic, II, 314; symmetric functions, II, 602; canonical forms, IV, 52; elimination, IV, 608.
Decadianome: the term, VII, 134; and symmetroid, VII, 256.
Decapitation: and seminvariants, XII, 248—50.
Decomposition: linear differential equations and theory of, XII, 402, 403—7.
Dedekind, J. W. R.: probabilities, II, 594—8, V, 80; modular function, XIII, 338—41.
Def: the abbreviation in groups, XIII, 120.
Deficiency: and genus of curve, V, 467, 517, 619; of curves, V, 618, VIII, 391, XI, 450; the term, VI, 2; and transformation, VI, 3, XI, 482—5; the term applied to surface, VI, 356; of certain surfaces, VIII, 394—7, XI, 230; surfaces of negative, VIII, 397; of sextic curve, IX, 504—7; of curve and Abelian integrals, XI, 30—6.
Definite Integrals: with complex variables, I, 181—2, 310; differentiation and evaluation, I, 267—72, 587; on a, IV, 28—9; note on Glaisher's paper, VIII, 1; note on two, IX, 56—63; (*see also* Attractions, Potentials, Prepotentials).
Definitions (*see* the word desired).
Deformation: the term, I, 234; of hyperboloid, XI, 66—7; of skew surfaces, XI, 317—22, 331; (*see also* Surfaces).
Degeneracy: of surfaces, V, 98—9; of scrolls, V, 201—3; of curves, XI, 218—20.
Degen's Mathematical Tables: IV, 40, IX, 478—9, X, 586; report of British Association Committee, XIII, 430—67.
Deg-order: the term, X, 339.
Degrees: of quantics defined, II, 221.
Degrees, Honorary: conferred on Cayley, VIII, XX—XXI; conferred on Sylvester, XIII, 43.
de Jonquières, E.: cubic curves, I, 586, IV, 496; on curves, IV, 454; curves which satisfy given conditions, VI, 191, 192, VII, 41—3; form of equation of curves of a series of given index, VI, 242—3; question from formula, VI, 258—62; points on cubic curve, VII, 550, 553; correspondence, XI, 486.
de la Goupillière, H.: inertia, IV, 566.
de la Gournerie, M.: tetrahedral scrolls, VII, 48—53; scrolls, VII, 54; quartic and quintic surfaces, VII, 246, 247, 251; torus, VIII, 25; octic surfaces, X, 81.
Delambre, J. B. J.: Tables du Soleil, III, 474.
Delaunay, C. E.: lunar theory, VII, 357, 372, 376, 528—33, 534, IX, 180, XIII, 206—7.
de Morgan, A.: root in every algebraic equation, IV, 116—9; root limitation, IX, 39; series, XI, 623, 627; and Sylvester, XIII, 45.
Denumerant: illustrated, IV, 169—70.
Denumerate: defined, IV, 241.
Departure Point: in lunar theory, III, 19, 270, 295.
Derivation: of points of cubics, VI, 20; and Übereinanderschiebung, VII, 348; and seminvariants, XIII, 362—5.

Derivational Function: the term, I, 63.

Derivations: extension of Arbogast's method, II, 257, IV, 265—71, 272—5, 609, XI, 55; binomial theorem and factorials, VIII, 463—73.

Derivatives: and hyperdeterminants, I, 95; of point on cubic, IV, 231; of three binary quantics, X, 278—86; and covariants, X, 340, 377—94; of a function, X, 590—2; in binary forms, XI, 272; (*see also* Schwarz).

de St Laurent, T.: caustic by reflexion, I, 273—5.

Desboves, A.: planetary perturbation, III, 185, 203; problem of two centres, IV, 532, 586.

Descartes, R.: ovals of, and transformation of curves, I, 478, 479—80, 589; oval of, III, 66; formulæ in *Epistolæ*, IV, 512; geometry of, XI, 437; (*see also* Cartesians).

Determinants: applied to distances of points, I, 1—4, 581, IV, 510—2; Pascal's theorem, I, 43—5; the term, I, 63; theory of, I, 63—79; theory of linear transformations, I, 80—94, 584; of vis viva, I, 284; note on hyperdeterminants, I, 352—5, 588; geometrical reciprocity, I, 377—82; "skew" and "symmetric," I, 410—3; history, I, 581; multiplication, I, 581, XI, 495; value of certain, III, 120—3, IV, 460—2; the term, IV, 594, 596—9; and Pfaffian, IV, 600; development of, V, 45—9; tables of binary cubic forms for negative, VIII, 51—64; Smith's Prize dissertation, VIII, 551—5; symmetrical, IX, 185—90, X, 579; notation, X, 95—7; theorem in, X, 265—6; in *Ency. Brit.*, XI, 490—7; decomposition, XI, 495—6; theory of numbers, XI, 604—9; (*see also* Hyperdeterminants, Skew Determinants).

Determinator: defined, II, 59.

Determinirende: (indicial), and differential equations, XII, 398, 401, 453.

Developables: and curves, I, 207—11, 586—7; the term, I, 486, XI, 573; from two quadrics, I, 486—95; from quintic curve, I, 500—6; planar, I, 505; from quartic, V, 135—7; prohessians, V, 267—83; quartics, V, 268—71; general theory, V, 271—2; special quintic, V, 272—8; special sextic, V, 279—83; reciprocation of quartic developable, V, 505—10; a special sextic, V, 511—9; sextic, and sextic surfaces, VI, 87—100; focals of a quadric surface, XIII, 51—4; (*see also* Torse).

Development: of factorial, II, 98—101; coefficients in powers of $(1+n^2x)^{m/n}$, XIII, 354—7.

Dew-Smith, A. G.: portrait of Cayley, XI (frontispiece).

Diagonals: and partitions of a polygon, XIII, 93—113.

Diagrams: the term, VII, 405; of planet's orbit from three observations, 5 plates, VII, to face 478; solar eclipse, VII, to face 492; geodesic lines on ellipsoid, VII, 510; coloured, representing groups, X, 328—30; transformation of elliptic functions, XI, 26; seminvariants, and solution by square-, XIII, 288—98; (*see also* Tables).

Diameter: as used by Newton, V, 362.

Diametral planes (*see* Planes).

Dianome: the term, VII, 133, 148; (*see also* Quartic Surfaces).

Diaphoric: the term, XI, 156.

Dickinson, L.: portraits of Cayley, VI (frontispiece), VII (frontispiece), VIII, XX.

Differences: equation of squared, for cubic, IV, 463—5; relation between certain products of, X, 293—4; on a functional equation, X, 298—306; (*see also* Equation of Differences).

Differential Equation Memoir: X, 93—133; introductory, X, 93—4, 94—5; notations, X, 95—7; dependence of functions, X, 97; general differential system, X, 98—102; the Multiplier, X, 102—5; Pfaffian theorem, X, 106; Hamiltonian system, derived from general system, X, 106—7; Poisson-Jacobi theorem, an identity in regard to functions (H, Θ), X, 108—9; peculiar to Hamiltonian system, X, 110—3; conjugate integrals of Hamiltonian system, X, 113—5; Hamiltonian system—the function V, X, 115—8; partial differential equation $H=$constant, X, 119—25; examples, X, 125—32; partial differential equation containing the dependent variable, reduction to standard form, X, 132—3.

Differential Equations: and lines of curvature of ellipsoid, I, 36—9; dynamical, I, 276—84; Jacobi's

system of, I, 366—9; Jacobi on theory, III, 174; theorem of Jacobi on Pfaff's problem, IV, 359—63; singular solutions, IV, 426—7; transformation, IV, 574, V, 78—9; umbilici, V, 115—30; solution when algebraical, VII, 5—7; supposed new integration, VII, 36; note on one, VII, 354—6; pair in lunar theory, VII, 535—6, 537—40; integration by series, VIII, 458—62; Euler's, IX, 592—608; and theory of elliptic functions, X, 20; and sides of quadrangle, X, 33—5; theory of partial, X, 134—8; elliptic and single theta functions, X, 422—9; hypergeometric series, XI, 17—25; Abel's theorem, XI, 27—8; new formulæ for integration of Euler's equation, XI, 68—9; mathematics and physics, XI, 449; connected with elliptic functions, XII, 30—2; Briot and Bouquet's theory, XII, 432—41; of circular functions, XII, 580; a diophantine relation, XII, 596—600; and construction of Milner's lamp, XIII, 3—5; Kummer's, of third order, XIII, 69—73; on a partial, XIII, 358—61; Richelot's integral of Euler's, XIII, 525—9; (*see also* Partial Differential Equations, Riccati, Schwarz, Singular Solutions).

Differential Equations, Linear: invariants of one, XII, 390—3; general theory, XII, 394—403, 444—52, 453—6; theory of decomposition, XII, 403—7.

Differential Equations of First Order: theory of singular solutions, VIII, 529—34, X, 19—24.

Differential Invariants (*see* Invariants).

Differential Operators: VII, 8.

Differential Relations: of double theta-functions, X, 559—65.

Differentiation: evaluation of definite integrals, I, 267—72, 587; formulæ for, IV, 135—49; fractional, XI, 235—6.

Dimensions in Geometry (*see* Geometry).

Dimidiate: the term, XIII, 119.

Dimidiation: the term, XIII, 122.

Diophantine Differential Relation: XII, 596—600.

Diptich: the term, XII, 596.

Dirichlet, G. L. (*see* Lejeune-Dirichlet).

Director, Nodal (*see* Nodal Director).

Directrix: and scrolls, VII, 60; and the absolute, XIII, 481—9, 501; kinematics of a plane, XIII, 505—6.

Discriminant: and invariant, I, 584; defined, II, 176, IV, 603, VI, 466—7; of quantics, II, 320; the sign □, II, 528; special, connected with curve, V, 163; of quintic, problem, V, 592; of binary quantic, VII, 303, IX, 16—7; example of a special, VIII, 46—7; (*see also* Quantics).

Discriminant Locus: the term, VI, 198.

Displacement: the term in Abel's theorem, XII, 110, 157—62.

Distance: general theory of, II, 561, 583—92, 604—6, V, 550; notion of, in analytical geometry, V, 550; the term, VI, 497; angular, of two planets, VII, 377—9; Cayley and Klein on theory, VIII, xxxvi—vii; general notion, VIII, 31; Euclidian geometry, XI, 435—7; non-Euclidian geometry, XIII, 480—504; (*see also* Points).

Distribution of Electricity: on spherical surfaces, IV, 92—8, 99—107, XI, 1—6.

Distributively: the term, VI, 459.

Disturbing Function: in lunar theory, III, 293—308, 319—43; in rotation of solid body, III, 486.

Divisors: tables of, IX, 462—70.

Dodecahedron: construction, IV, 82—3; axial systems, V, 531—9; as regular solid, X, 270—3; automorphic function for, XI, 169, 179—83, 184, 212—6.

Donkin, W. F.: expansions in multiple sines, I, 583; differential equations, dynamical, III, 191—7, 203—4 344; transformation of trigonometric series, III, 567; attractions, III, 567; a definite integral, IV, 29 formulæ for differentiation, IV, 135—49; central forces problem, IV, 521; spherical pendulum, IV, 534, 536, 586; dynamical problems, IV, 547, 586; elimination of nodes in three bodies, IV, 551, 586; rotation of solid body, IV, 578, 586.

Dostor, G.: polyhedra, IV, 609.

Dots: notation for lines and planes of cubic surfaces, VI, 365—6, 373—449; and seminvariants, XIII, 267.
Double Algebra: XII, 465.
Double Contact: conics having with each other, IV, 456—9.
Double Point: the term, VI, 1; on ground-curve, XII, 110, 129.
Double Pyramid (*see* Polygons).
Double-Sixer: and cubic surfaces, VI, 372, VII, 316—29; construction, VIII, 366—84.
Double Tangents (*see* Bitangents).
Double Theta Functions: X, 155—6, 166—79, 180, 422—9, 474—5, 497, 565; in connexion with 16-nodal quartic surface, X, 157—65; memoir on, X, 184—213; (Part I, preliminary investigations, X, 184—9; Part II, the double theta functions, X, 189—213); addition of, X, 455—62; evolution, XI, 454; transformation, XII, 358—89; (*see also* Theta functions).
Doubly Infinite Products: I, 120—2, 132—5, 136—55, 156—82, 585, 586, X, 492—4, XI, 46, XII, 50—5; and doubly periodic functions, II, 150—63; and definite integrals, IX, 60; transformation of, X, 494—7.
Doubly Periodic Functions: I, 156—82; and doubly infinite products, II, 150—63; and definite integrals, IX, 61; the term, XI, 530.
Drawing: geometrical, VI, 19; of quartic curves mechanically, VIII, 151—5; curves generally, VIII, 179—80; (*see also* Representation).
Droop, H. R.: isochronism of circular hodograph, III, 265; central forces problem, IV, 520, 587.
Duality: in geometry, II, 561—2, 568, XI, 450, 467.
Du Bois-Reymond, P. L.: uniform convergence, XIII, 343.
Dumas, W.: spherical pendulum, IV, 534, 587.
Dupin, C.: cyclide of, V, 467, IX, 64, XII, 615; quartic and quintic surfaces, VII, 246; theorem of, VIII, 264—8, 562, IX, 84—9.
Duplication of Groups: X, 149—52.
Durège, H.: Landen's theorem, XI, 339.
Durfee, W. P.: symmetric functions, II, 602—3.
Dynamics: differential equations of, I, 276—84; a class of problems, IV, 7—11; similarity of two dynamical systems, VIII, 558—63; Lagrange's general equation in, IX, 110—2, 198—200; general equations in, IX, 215—7; and time, XI, 444; transformation of coordinates, XI, 575.
Dynamics, Recent Progress in Theoretical: III, 156—204, IV, 514; introduction, III, 156—7; Lagrange, *Mécanique Analytique*, III, 157—8, 201, 202; Lagrange, equations of motion, III, 158, 200; lunar theory, III, 158—9; Poisson, planetary theory, III, 159, 201; Laplace's theory, III, 159, 201; Lagrange's planetary theory, III, 159—61, 162—3, 201; Lagrange, variation of arbitrary constants in mechanical problems, III, 161—5, 200; also Poisson, III, 163—5, 200, 201, 202; Cauchy, differential equations, III, 166; Hamiltonian method of dynamics, III, 166—74, 200, 202; its relations to Lagrange's, III, 171—3, 200; and Poisson, III, 173—4, 200; Jacobi, calculus of variations and differential equations, III, 174—82, 200, 202; *De Motu Puncti Singularis*, III, 182—3, 202; problem of three bodies, III, 183; Jacobi, *Theoria Novi Multiplicatoris*, III, 183—5; Jacobi, theory of ideal coordinates, III, 185; Liouville, equations of motion, III, 185; Desboves, planetary perturbation, III, 185, 203; Serret, integration of differential equations, III, 185—6, 203; Sturm, integration of dynamical equations, III, 186, 203; Ostrogradsky, dynamical equations, III, 186, 203; Brassinne, differential equations, III, 186—7, 203; Bertrand, integrals to mechanical problems, III, 187, 203; and integration of differential equations, III, 188—9, 203; and *Mécanique Analytique*, III, 189—90, 203; Brioschi, *Sulla Variazione*, and *Teorema di Meccanica*, III, 190, 203; Liouville, integration of differential equations, III, 191—2, 203; Donkin, dynamical differential equations, III, 191—7, 203—4; Bour, integration of differential equations of analytical mechanics, III, 197—8, 204; Liouville on Bour's memoir, III, 199, 204; Brioschi, *Degli Integrali di un Problema di Dinamica*, III, 199—200, 203; Bertrand, integrals of several mechanical problems, III, 200, 203; summary, III, 200.
Dynamics, Report on Progress of Solution of Certain Problems: IV, 513—93; introductory, IV,

513—5; rectilinear motion, IV, 515—6; central forces, IV, 516—26; elliptic motion, IV, 521—4; problem of two centres, IV, 524—32; spherical pendulum, IV, 532—4; motion as affected by the Earth and relative motion generally, IV, 534—7; motion of single particle, IV, 537—8; motion of three mutually attracting bodies in a right line, IV, 538—40; motion of three bodies, IV, 540—1; motion in resisting medium, IV, 541; integration of equations of motion, IV, 542—6; memoirs by Jacobi, Bertrand and Donkin, IV, 546—7; problem of three bodies, IV, 548—52; transformation of coordinates, IV, 552—9; principal axes and moments of inertia, IV, 559—66; rotation of solid body, IV, 566—80; kinematics of solid body, IV, 580—2; rotation round fixed point, IV, 582—3; other cases of the motion of a solid body, IV, 583—4.

Earth, The: rotation of, III, 485, IV, 534—7; (*see also* Gravity).

Eclipses (*see* Solar Eclipses).

Edge: defined, V, 63.

Eindeutig: (Uniform function), XII, 433.

Eisenstein, F. G.: linear transformations, I, 90, 101, 111, 113—6, 585; hyperdeterminants, I, 353, II, 598—601; elliptic functions, I, 586; cubic forms, III, 9; quadratic residues, III, 39—43; finite differences, IV, 263; mathematical tables, IX, 492—3; "development of an idea of," X, 58—9; development of $(1+n^2x)^{m/n}$, XIII, 357.

Elastic Strings: problem with, III, 78—9.

Electricity: distribution on spherical surfaces, IV, 92—8, 99—107, X, 299, 307, XI, 1—6.

Elements: Jacobi's canonical, III, 77; of arc, X, 235—7; a reduction to elliptic integrals, X, 239—42.

Eliminant: the term, IV, 597; of two quantics, XI, 100—2.

Elimination: and theory of curves, I, 337—51, V, 162—7, 416—20; from connected equations, I, 370—4; and linear transformations, I, 457; theorem of Schläfli, II, 181—4; a result of, III, 214—5; general theorem, IV, 1—4; Bezout's method, IV, 38—9, V, 555—6; of nodes in three bodies, IV, 551; the term, IV, 594; text-books on, IV, 608; note on, V, 157—9; problem, VI, 40—2, VIII, 22—4; the resultant of a system of two equations, VI, 292—9; theorem, IX, 43—4; formula of, XI, 100—2; theory of equations, XI, 490; a problem of Sylvester's, XIII, 545—7.

Ellipse: curves parallel to, IV, 123—33, 152—7; and circle, V, 561; foci of, V, 586; and quadrilateral, V, 604; circles of curvature, VII, 555, XIII, 537; potential, IX, 281—90; negative pedals, X, 576; cubic curves, XI, 478; in *Ency. Brit.*, XI, 561—4; focals of quadric surface, XIII, 54; and epitrochoid, XIII, 82—7; orthomorphosis into a circle, XIII, 188—9, 422—4.

Ellipsoid, Attraction of: I, 388—91, 432—44, 582, VII, 380—3, XI, 448; Jacobi's method, I, 511—18; Gauss's method, III, 25—8, 149—53; Laplace's method, III, 53—65; Rodrigues' method, III, 149—53; theory of, III, 154—5; and terminated straight line, VII, 31—3.

Ellipsoid, Centro-Surface of, memoir: VIII, 316—65; introductory, VIII, 316—7; the ellipsoid, VIII, 317—20; sequential and concomitant centro-curves, VIII, 320; expressions for coordinates of point on centro-surface, VIII, 320—4; discussion by means of equations, principal sections, &c., VIII, 324—30; generation of surface considered geometrically, VIII, 330—1; nodal curve, VIII, 332—52; eight cuspidal conics, VIII, 352—7; centro-surface as envelope of quadric, VIII, 357—8; another generation of centro-surface, VIII, 359—61; a third generation of centro-surface, VIII, 361—2; reciprocal surface, VIII, 363; delineation of centro-surface for particular case, VIII, 363—5.

Ellipsoid, Geodesic Lines on, memoir: VII, 493—510; introductory, VII, 493—4; course of the lines, VII, 494—5; lines through an umbilicus, VII, 495—501; formulæ, VII, 501—3; umbilicar geodesics, VII, 503; tables, VII, 504—6; projection on umbilicar plane, VII, 507; elliptic function formulæ, VII, 507—10; diagram, VII, to face 510.

Ellipsoids: lines of curvature, I, 36—9; surface parallel to, IV, 123—33, 158—65, X, 575; the momental, IV, 560; of gyration, IV, 560; central, IV, 564; projection, V, 487—8; geodesic lines on, VII, 34—5; centro-surface and sextic torse, VII, 113—4; centro-surface, VII, 130—2; geodesic lines, VIII, 174—8;

attraction of ellipsoidal shell on exterior point, IX, 302—11; negative pedals, X, 576; in *Ency. Brit.*, XI, 576—9.

Elliptic Coordinates: equation of wave surface in, XI, 71—2.

Elliptic Functions: Bronwin on, I, 118, 119; of Jacobi, I, 127, 507, 586; integral calculus, I, 383; multiplication of, I, 534—9, 568—76, 589; addition of, I, 540—9, 589, XII, 294—8; connected with theory of numbers, II, 48; system of modular symbols, IV, 484—9; Weierstrass, V, 33—7; treatise by Cayley, VIII, xviii, xxviii, XIII, 560; a general differential equation, IX, 592—608; a differential equation in theory of, X, 24; and integration, X, 25—7; torse depending on, X, 73—8; reduction of Abelian integrals to, X, 214—22; and single theta functions, X, 422—9, 463, 472; certain algebraic identities, XI, 130—1; evolution of, XI, 451—5; and quartic function, XI, 483; kinds of, XI, 529; symmetrical differential equation and, XII, 30—2; solution of $x^3+y^3-1=0$, XII, 35—7; Weierstrassian and Jacobian compared, XII, 425—7; Kiepert's L-equations, XII, 490—2; graphical representation, XIII, 9—19; and sextic resolvent equations, XIII, 473—9; theta and omega functions, XIII, 558—9; (*see also* Gudermannian, Theta functions).

Elliptic Functions Formulæ: Serret's, III, 3; for geodesic lines on special ellipsoid, VII, 507—10; one, XI, 65, XII, 292—3; connexion of certain, XI, 250—1; geometrical interpretation of certain, XII, 107.

Elliptic Functions, Inverse: I, 136—55, 156—82, 586; and definite integrals, II, 3.

Elliptic Functions, Memoir on Transformation of: IX, 113—75; introductory, IX, 113—4; the general problem, IX, 114—7; Ωk modular equations, IX, 117—8; equation-systems, IX, 119—20; Ωk form, IX, 121—6; modular equation, IX, 126—37; tables, IX, 128—35, 163; multiplier equation, IX, 138—40; multiplier as rational function of u, v, IX, 140—4; multiplication of elliptic functions, IX, 144—7; transformations, IX, 147—55; general theory of q-transcendents, IX, 155—69; four forms of modular equation and curves represented thereby, IX, 169—75.

Elliptic Functions, Theorems in: XI, 73—7; Landen's, XI, 337—9, 584; Hermite's H-product, XII, 584—6.

Elliptic Functions, Theory of: I, 290—300, 364—5, 402—4, 587, 589; and quadri-quadric curve, XII, 321—5.

Elliptic Functions, Transformation of: I, 120—2, 132—5, 585, IX, 543, X, 333—8, 611, XI, 26, XII, 416—7, 505—34, 535—55, XIII, 29—32; cubic, III, 266—7, VII, 44—6, 244—5, 253—6, XII, 46, 556—7, XIII, 64—5; special quartic, IX, 103—6; geometric illustration of cubic, IX, 522—6; orthomorphosis, XIII, 191—205.

Elliptic Integrals: reduction of $\frac{du}{\sqrt{U}}$, I, 224—7; transformation, I, 508—10, IX, 618—21; geometrical representation, II, 53—6, 113—7; discussion, II, 93—5; and covariants, II, 189—92; transformation formulæ, IV, 60—9, 609; expression for second kind of, X, 25—7; note on theory, X, 139—42; some formulæ in, X, 143—8; of third kind, X, 489—92; problem, X, 614; theta functions, XI, 41—6; note on, XI, 64; reduction of an integral to, XI, 270—1; of third kind, formulæ, XI, 340—1.

Elliptic Motion: expansion of true anomaly, III, 139—42, 567; trilinear equation of second degree, III, 143—8; theory of, III, 216—8; tables of functions in theory, III, 360—474, VII, 516; Lambert's theorem, III, 562—5, VII, 387—9; and dynamical progress, IV, 521—4; a theorem, IX, 191—3; and body let fall at equator, IX, 241—3.

Elliptic Motion, Disturbed: memoirs on, III, 270—92, 344—59, 505—15.

Elliptic Space: and non-Euclidian geometry, XIII, 481.

Elliptic-Transcendent Identity: VIII, 564.

Ellis, R. L.: orthogonal surfaces, VIII, 272; differential equations, VIII, 458; Dupin's theorem, IX, 88.

Emanants: of quantics, II, 321; theory, II, 518; Bezoutoidal, II, 525, 526; the term, IV, 604, XIII, 46.

Emanation: theory of, II, 321.

Encke, J. F.: *über die speciellen Störungen*, III, 179—80; fluctuating functions, IX, 19; roots of numerical equations, X, 5.

Encyclopædia Britannica, articles from on:—Curve, XI, 460—89; Equation, XI, 490—521; Function, XI, 522—42; Galois, XI, 543; Gauss, XI, 544—5; Geometry (analytical), XI, 546—82; Landen, XI, 583—4; Locus, XI, 585; Monge, XI, 586—8; Partition of numbers, XI, 589—91; Theory of numbers, XI, 592—616; Series, XI, 617—27; Surface, XI, 628—39; John Wallis, XI, 640—3.

Endecadic Transformation: in elliptic functions, IX, 152—5.

Endoscopic: the term, I, 588.

Ennead: the term, VII, 256, VIII, 566.

Enneadianome: the term, VII, 134.

Enneagon: in-and-circumscribed, IV, 298—303.

Envelopes: developable of two equations, I, 486; parallel curves and surfaces, IV, 123—33, 152—7, 158—65; defined, IV, 458, VI, 467; of circle, V, 610, VII, 591; and locus in regard to triangle, VI, 72—82; depending on two circles, problem and solution, VII, 573; of plane curve, VII, 606; of a certain quadric surface, VIII, 48—50; locus in singular solutions, VIII, 533; problem of, VIII, 491—2; of family of quadrics, X, 589; theory, XI, 50—1; of variable curves, XI, 475—6.

Epicycloid: and caustic, II, 345.

Epispheric integrals: Gauss-Jacobi theory, IX, 410—7.

Epitrochoid: XIII, 81—7.

Equal: applied to circles, VIII, 31.

Equality: among roots of an equation, II, 465—70, VI, 300—12; idea of, XI, 431.

Equation of Differences: for equation of any degree, IV, 150—1; for equation of any order, IV, 240—61; tables, IV, 246—56; of all but one, of roots of given equation, IV, 276—91; and quintic equation, IV, 309—24, 609—16; and cubic equation, IV, 463—5.

Equation, Pellian (*see* Pellian Equation).

Equations: systems of spherical coordinates, I, 213—23; with quantics, defined, II, 221; auxiliary for quintics, IV, 309—24; determination of reciprocal, with cubic surfaces, VI, 368—70; the term, VI, 466; solubility by radicals, VII, 13—4; system of, problem and solution, VII, 578, 581, X, 601; transformation, IX, 42, 48—51; on a functional equation, X, 298—306; Cassinian, problem, X, 608; Jacobian sextic, XI, 389—401, XII, 493—9; equal roots of, XI, 405—7; of curves, XI, 462—4; of Plücker, XI, 469—73, XIII, 536; in *Ency. Brit.*, XI, 490—521, (introductory, XI, 490; determinant, XI, 490—7; imaginary, XI, 502—6); of right line and circle, XI, 558—61; of conics, XI, 563; seminvariants, XII, 19—21; fundamental, and deformation of surfaces, XI, 331; note on system of, XII, 48—9; for three circles which cut each other at given angles, XII, 559—61, 564—70; anharmonic ratio, XII, 578—9; hydrodynamical, XIII, 6—8; Sylvester on ternary cubic-form, XIII, 47; on soluble quintic, XIII, 88—92; Waring's formula for sum of mth powers of roots of, XIII, 213—6; sextic resolvent of Jacobi and Kronecker, XIII, 473—9.

Equations, Algebraic: rationalization, II, 40—4; theory, II, 124; theorem that every one has a root, IV, 116—9; system of, IV, 171—2, VIII, 29—30; in *Ency. Brit.*, XI, 506—21; Anglin's formula for successive powers of the root of, XII, 33—4; roots of one, XIII, 33—7.

Equations, Cubic (*see* Cubic equations).

Equations, Modular: for transformation of order 11, XIII, 38—40; for cubic transformation, XIII, 64—5; (*see also* Transformation of Elliptic Functions).

Equations of Motion: in lunar theory, XIII, 206.

Equations, Solutions of: $x^{257}-1=0$, I, 564—6; $\theta^{n}=1$, and theory of groups, II, 123—30, 131—2, IV, 88—91, X, 610; $x^{p}-1=0$, XI, 314—6, XII, 72—3; elliptic function solution of $x^3+y^3-1=0$, XII, 35—7; the quaternion $qQ-Qq'=0$, XII, 300—4, 311—3; $(abcd)=(a^2b^2c^2d^2)$, XII, 418—20; $x^{17}-1=0$, XIII, 60—3.

Equations, Theory of: synopsis, X, 3—11; Newton-Fourier method, and imaginary root, XI, 114—21, 143; theorem of Abel's and quintic equation, XI, 132—5; theorem in, XI, 268—9; evolution, XI, 455; in *Ency. Brit.*, XI, 497—521.

Equator: action of gravity at the, IX, 241—3.
Equilibrium: of four forces, V, 540—1, IX, 201; of skew surface, XI, 317—22.
Equimomental Surfaces (*see* Surfaces).
Equipollences: Bellavitis, XII, 473—4.
Equipollent: the term, XII, 473.
Equipotential Curve: III, 258—61.
Essential Singularity of Function: IV, 150.
Eta-Functions: product, XII, 584—6; (*see also* Theta Functions).
Euclid: space of, XI, 434—7; evolution of geometry, XI, 446; proof of I, 47, XI, 557.
Euler, L.: rotation of solid body, I, 237, IV, 566, 567—9, 587, VI, 135—46; involution, I, 259; elliptic functions, I, 366; skew determinants, II, 214; transformation of coordinates, II, 497, IV, 553—7, 587; sums of series, III, 127; indeterminate equations, III, 205—7; polyhedra, IV, 84, 86—7, V, 62—5, 617; *Determinatio Orbitæ Cometæ*, IV, 519, 587; problem of two centres, IV, 525—7, 587; three mutually attracting bodies in right line, IV, 538—9, 587; motion of three bodies, IV, 540, 587; inertia, IV, 562; kinematics of solid body, IV, 580, 587; rotation formulæ, V, 537; differential equation of, VII, 261—2, IX, 592—608, XI, 68—9; binomial theorem, VIII, 463; mathematical tables, IX, 463—6, 471—2, 477—8, 481, 487; theorem on sums of squares, XI, 294; partitions, XI, 360, XII, 219; intersections of cubic curves, XI, 449; gamma function, XI, 535—6; eight-squares theorem, XII, 465; Latin squares, XIII, 55; differential equation of, integrated by Richelot, XIII, 525—9.
Evans, A. B.: Degen's tables, X, 586.
Evectant: of quantics, II, 321.
Evector: of quantics, II, 321.
Evolutes: theory of, V, 473—9; and parallel curves, VIII, 31—45; nodes of, VIII, 329, 351.
Evolution: of geometry, XI, 445—8.
Ewing, J. A.: curve-tracing mechanism, XIII, 505.
Excuboquartic: defined, V, 10, VII, 99; curves, VI, 87—8, XI, 9—13.
Exoscopic: the term, I, 588.
Expansions: in multiple sines and cosines, I, 19—24, 583; in Laplace's coefficients, I, 375—6; of true anomaly, III, 139—42; numerical, IV, 470—2.
Expectation: problem and solution in, X, 587; (*see also* Probability).
Experience and Cognition: XI, 431.
Exponential Functions: and double theta functions, X, 184—5; the term, XI, 524—7.
Extension: in conformal representation, XI, 78.
Extent: the term in seminvariants, XIII, 269, 363.
Extraordinaries: and non-commutative algebras, I, 128—31, 301; the term, XII, 60, 461.

Facients: defined, II, 221, IV, 604, VI, 464.
Factions: the term, IX, 426.
Factorial Expressions: summation of, III, 250—3.
Factorials: developments of, II, 98—101, 594; problems, V, 574, VII, 597; binomial theorem and derivations, VIII, 463—73; maxima of certain functions, VIII, 548—9.
Factors, Special: the term, I, 337.
Facultative: the term, VI, 156, 365; lines of cubic surfaces, VI, 450.
Facultative Points: of Sylvester, XIII, 46.
Family of Quadrics: envelope of, X, 589.
Family of Surfaces: part of orthogonal system, VIII, 269—91.
Fermat, P. de: theorem of, XI, 457, 597, 611, 615—6.
Ferrers, N. M.: conjugate partitions, II, 419; area of conic, III, 143—8; correspondence, X, 290; Legendrian coefficients, XII, 563.

Fiedler, W.: symmetric functions, II, 602.
Figures: for Pascal's theorem, VI, 116—23; mechanical construction of conformable, X, 406; use of arabic, XI, 446.
Finite Differences: formulæ in, III, 132—5, XII, 412—5; electricity on spherical surfaces, IV, 92—8, 99—107; theorem and demonstration, IV, 262—4; general equation of differences of second order, X, 47—9; Stirling's theorem, X, 267—8; table of, XI, 144—7.
Finite Groups (*see* Groups).
Finiteness: of concomitant system of quantic, VII, 334, XI, 272—80.
First Kind: of Abelian integrals, XII, 408—11.
First Order: of differential equations, XII, 432—41; (*see also* Differential Equations).
Five-dimensional Geometry: IX, 79; (*see also* Hypergeometry).
Five-pointic Contact: conic of, IV, 207—39.
Flat: the term in covariants, VIII, 406—8.
Flat-cone: the term, VIII, 102.
Flecnodal Curve: VI, 342; and torse, VI, 345, 582—5.
Flecnode: defined, II, 28—32.
Fleflecnodal Planes: of a surface, X, 262—4.
Fleflecnode: defined, II, 28—32, IX, 264; of curve in transformation of elliptic functions, IX, 170—1.
Flex: the term, V, 521.
Flexure of Skew Surface: XI, 317—22; (*see also* Surfaces).
Floquet, G.: linear differential equations, XII, 394, 402.
Fluctuating Functions: addition to Lord Rayleigh's paper, IX, 19—20.
Fluxions: and Landen, XI, 583.
Focals: of a quadric surface, XIII, 51—4.
Foci: of conics, IV, 505—9, VII, 1—4, 571; theory of, VI, 515—34; and antifoci, problem and solution, VII, 567; locus of, problem and solution, VII, 568.
Focus: the term, VI, 515, IX, 552, XI, 481.
Foot: non-Euclidian geometry, XIII, 483—4.
Forcenex, D. de: multiple algebra, XII, 466.
Forces: equilibrium of, V, 540—1, VII, 91—5; general equation of virtual velocities, IX, 205—8; resultant, X, 589; (*see also* Dynamics).
Forms: cubic, III, 9—12; quadratic, III, 11—12; theory, XI, 604—9.
Formulæ: Jacobi's canonical, for disturbed motion, III, 76—7; in finite differences, III, 132—5; for differentiation, IV, 135—49; distances of point, and tactions, IV, 510—2; signification of elementary one in solid geometry, V, 498—9; integrals for intersections of line and conic, V, 500—4; canonical form of quantics, VI, 153—4; of two sets each of four concyclic points, VI, 509—11, 512—5; focal, and polyzomal curves, VI, 547, 549; of de Jonquières, VII, 41—3; transformation of coordinates, VII, 95—6, 97—8; geodesic lines on ellipsoid, VII, 501—3, 507—10; trigonometric, XII, 108.
Forsyth, A. R.: biographical notice of Cayley, VIII, ix—xliv; addition of elliptic functions, XII, 294.
Foucault, J. B. L.: the earth's rotation, IV, 535, 536, 588.
Fouché, M.: polyhedra, IV, 609.
Fourier, J. B. J.: theorem as to roots of equations, X, 5; theory of equations, XI, 500; (*see also* Newton-Fourier Theorem).
Fourth Dimension: Cayley on, VIII, xxxiii—v.
Fraction-Theorem: Jacobi's, XII, 123—5.
Francais, J. F.: imaginaries, XII, 468.
Franklin, F.: quantics, XIII, 47.
Fresnel, A. J.: wave-surface, IV, 420, XI, 449; wave and tetrahedroid surfaces, X, 252; Sylvester on the optical theory of, XIII, 44.

Frobenius, G.: linear differential equations, XII, 394.

Frost, P.: curves of curvature near umbilicus, VII, 330—1.

Fuchs, L.: Schwarzian derivative and polyhedral functions, XI, 149; linear differential equations, XII, 394, 453.

Function: the term "derivational," I, 63; transformation of bipartite quadric, II, 497—505; relation among derivatives of, X, 590—2; octahedron, XI, 128—9; general theory of, XI, 439—41; linear, XI, 492; in *Ency. Brit.*, XI, 522—42, (introductory, XI, 522—3; known functions, XI, 523—37; functions in general, XI, 537—41; calculus of, XI, 541—2); two invariants of quadri-quadric, XIII, 67—8; on the modular $\chi\omega$, XIII, 338—41.

Functional Determinant: the term, II, 319, IV, 607.

Functional Equation: theorem of Abel, IV, 5—6.

Functions: doubly-periodic, II, 150—63; notation of algebraic, II, 185—8; al(x) of Weierstrass, V, 33—7; homotypical, V, 50; rhizic, IX, 34; tests for dependence of, X, 97; early history of theory, XI, 451—5; values of symmetric, XIII, 318—21; on lacunary, XIII, 415—7; (*see also* Schwarzian Derivative, Generating Functions, Symmetric Functions).

Fundamental Notions: in Mathematics, XI, 434—8, 442—5.

Fuss, P. H. V.: porism formula, II, 90; in-and-circumscribed polygon, II, 140, V, 21—2.

Galileo: and dynamics, XI, 447.

Galois, E.: groups and permutations, II, 134; groups, XI, 133, XIII, 533; theory of numbers, XI, 457, 593, 614; theory of equations, XI, 518—9, 520, 521; biographical notice, XI, 543.

Gamma Function: theory, I, 309—16, 588; a double infinite series, II, 8; the term, XI, 534; (*see also* Definite Integrals).

Gaultier, L.: systems of circles, III, 113; radical axis, XI, 465.

Gauss, J. K. F.: determinants, I, 64; linear transformations, I, 585; attraction of ellipsoids, III, 25—8, 149—53; central forces problem, IV, 520, 588; relative motion, IV, 534; binary quadratic forms, V, 618; *pentagramma mirificum*, VII, 37—8; *Theoria Motus*, VII, 414; geodesic lines on quadric surface, VIII, 156—61; potential of ellipse, IX, 281—2; epispheric integrals, IX, 321, 410—17; mathematical tables, IX, 466—7, 470, 472, 475—7, 488; roots of unity, XI, 60; lemniscate, XI, 64; calculation of log 2, XI, 70; geodesic curvature, XI, 323—4; his theory of surfaces, XI, 331—6; imaginary variables, XI, 439; attractions, XI, 448; theory of equations, XI, 455, 504, 516; theory of numbers, XI, 455, 599, 603; gamma function, XI, 534; biographical notice, XI, 544—5; multiple algebra, XII, 471—2; roots of algebraic equation, XIII, 35; orthomorphosis, XIII, 191; application of surfaces to each other, XIII, 253—64.

Geiser, C. F.: quartic and quintic surfaces, VII, 252; theorem of Cotterill, IX, 506.

Generating Functions: of symmetric functions, Borchardt's, II, 417, 421—3; connected with covariants, IX, 537—42; of quintic, X, 339—400; of quartic, X, 341; of sextic, and binary sextic, X, 394—400; of binary septic, X, 408—21; seminvariants of a given degree, XIII, 306—8.

Generation: of bicircular quartic, X, 223—6.

Generator: the term, V, 169—70, 173—9, 181; nodal, of scrolls, V, 169—70, 179—81.

Generatrix: and the absolute, XIII, 481—9, 501.

Genese, R. W.: theory of envelopes, XI, 50—1.

Genus of Curve: (Geschlecht), after Riemann, V, 467, 517, 619.

Geodesic Curvature: XI, 323—30.

Geodesic Lines: property of, III, 38; on oblate spheroid, VII, 15—25; on ellipsoid, VII, 34—5; in *Ency. Brit.*, XI, 628, 636—7; of pseudosphere, XII, 220—38; wave surfaces, XIII, 252; (*see also* Ellipsoids).

Geodesic Lines, in particular on quadric surface, memoir: VIII, 156—78, 188—99; preliminary formulæ, VIII, 156—8; general theory of them on a surface, VIII, 159—62; circular curves are geodesics, VIII, 162; chief lines not in general geodesics, VIII, 163; special form of geodesic equation, VIII, 163—4;

geodesics on quadric surface, VIII, 164—70; formulæ for position of point, VIII, 170—4; ellipsoid and skew hyperboloid, VIII, 174—8, 188—99; tables, VIII, 196—9.

Geometrical Construction: in optics, X, 28; of heptagon, X, 609.

Geometrical Representation: of elliptic functions, III, 3; of imaginary variables, X, 316—23; of an equation between two variables, XII, 104.

Geometry: of *n* dimensions, I, 55—62; reciprocity, I, 377—82; of quantics, II, 222; of one and two dimensions defined, II, 561—2; of one dimension, II, 563—9, 583—96; of two dimensions, II, 569—83, 586—92; relations of, metrical and descriptive, II, 592; non-Euclidian and hyper-, II, 604—6, VIII, xxxiii—v, 409—13, XII, 220—38; Lobatschewsky's imaginary, V, 471; problem of permutation, V, 493—4; signification of elementary formula, V, 498—9; notion of absolute, V, 550; drawings in, VI, 9; constructive, VII, 26—30; transformation, VII, 121—2; Cayley and Klein on metrical, VIII, xxxvi—vii; hyperbolic, elliptic and parabolic, VIII, xxxvii; Cayley's work in analytical, VIII, xxxviii; formulæ relating to right line, X, 287—9; considerations on solar eclipse, X, 310—5; interpretation of algebraic equations, X, 581; solid, XI, 224; Schubert's numerative, XI, 281—93; Mill on, XI, 432—4; Euclidian, XI, 434—7; Cartesian, XI, 437—9; abstract, XI, 441—2; origin, XI, 445—8; in Greece, XI, 446; evolution of descriptive, XI, 448—9; date of extensions in, XI, 449—51; plane and solid, XI, 450—1; function in, XI, 522—3; interpretation of elliptic function formulæ, XII, 107; d'Alembert Carnot paradox, XII, 305—6; of the compass, XII, 314—7; algebra and logic, XII, 459; (*see also* Hypergeometry: for General Theory, *see* Quantics, sixth memoir).

Geometry, Abstract, Memoir on: VI, 456—69, 596; introductory, VI, 456—7; space, VI, 456—7; general explanations, VI, 457—62, 596; omal relation, order, VI, 463; parametric relations, VI, 463—4; quantics, notations, etc., VI, 464—6; resultant, discriminant, VI, 466—7; consecutive points, tangent omals, VI, 467—9.

Geometry, Analytical, in Ency. Brit.: XI, 546—82; introductory, XI, 546; Part I, pure analytical, XI, 546—67; is descriptive, XI, 552—6; metrical theory, XI, 556—7; equations of right line and circle—transformation of coordinates, XI, 558—61; the conics, XI, 561—4; tangent, normal, circle and radius of curvature, XI, 564—5; coordinates, XI, 566—7; Part II, solid analytical geometry, introductory, XI, 567—9; metrical theory, XI, 570; line, plane, and sphere, XI, 571—2; cylinders, cones, ruled surfaces, XI, 572—3; transformation of coordinates, XI, 573—6; quadric surfaces (paraboloids, ellipsoids, and hyperboloids), XI, 576—9; curves: tangent, osculating plane, curvature, XI, 579—80; surfaces: tangent lines and plane, curvature, XI, 580—2; (*see also* Hypergeometry).

Geometry of Position: theorems in, I, 317—28, 356—61, 414—20, 550—6, 567, 588.

Gergonne, J. D.: caustics, II, 118, 339, 341, 368; polyzomal curves, VI, 520.

Geschlecht: (genus) of curve, after Riemann, V, 467, 517, 619.

Glaisher, J. W. L.: notation for elliptic functions, I, 548; definite integration, VIII, 1; centro-surface of ellipsoid, VIII, 364; report on mathematical tables, IX, 461—99; development of an idea of Eisenstein, X, 58—9; proof of Stirling's theorem, X, 267—8; quadrilateral inscribable in circle, X, 578; log 2, XI, 70; elliptic functions, XI, 73; least factors of numbers, XI, 430; modular function $\chi\omega$, XIII, 338—41; theta and omega functions, XIII, 558—9.

Glide: the term, I, 236.

Glover, J. W.: on theory of groups, XIII, 533.

Goniometry: Cotterill's problem in, X, 295—7.

Göpel, A.: theory of numbers, IV, 41; theta functions, VIII, xlii, X, 464, 499, XII, 363—4; double theta functions and 16-nodal quartic surface, X, 157, 162, 172, 173, 175, 180—1; table of tetrads, 508, 549—51; double theta functions, XI, 454.

Gordan, P.: binary quintic and sextic, VI, 190; irreducible covariants of binary quantic, VII, 334, 341, 348—53; covariants of binary quantic, VIII, 566; finiteness of concomitant systems, X, 286; derivatives, X, 340, 377; Schwarzian derivative and polyhedral functions, XI, 149, 199; finite groups, XI, 237—41; covariantive forms and tables, XI, 272; concomitants of ternary cubic, XI, 343; Abelian functions, XII, 102, 109; icosahedral substitutions, XIII, 552.

Goursat, E.: Kummer's differential equation, XIII, 69—73.
Graduation Curve: the term, VII, 426.
Graham, A.: Pellian equation, XIII, 442.
Grant, R.: report on Catalogue of Memoirs, V, 546—8, 620.
Graphical Construction: in solar eclipses, VII, 390—1, 479—92; geodesic lines on ellipsoid, VII, 507, 510; theory of groups, X, 403—5.
Grassmann, H.: multiple algebra, XII, 465, 480—9.
Gravelius, H.: on non-Euclidian geometry, XIII, 481.
Graves, J. T.: algebraic couples, I, 128; geometry of position, I, 319, 414—20; imaginaries, I, 586.
Gravity: and relative motion, IV, 534—7; action at equator, IX, 241—3; Clairaut's theorem, X, 17—8; effects of theory, XI, 447—8.
Greatheed, S. S.: expansions in multiple sines, I, 583; elliptic motion, IV, 522, 588.
Greek: Cayley's knowledge of, VIII, XXIV; geometry, arithmetic, and algebra, XI, 446.
Green, G.: attractions of ellipsoids, I, 582; attraction of terminated straight line, VII, 32; potentials of polygons and polyhedra, IX, 279; integration of prepotential equation, IX, 320—1, 343, 393—404; attractions, XI, 448.
Greenhill, A. G.: deformation and flexure of surfaces, XI, 66—7.
Greenians: the term, IX, 393.
Greer, H. R.: locus, envelope, and triangle, VI, 72.
Gregory, D. F.: differential and integral calculus, VIII, 272.
Griffiths, J.: series of triangles, VII, 599; curve of sixth order, X, 612; deduction from $y=\sin(A+B+C+\ldots)$, XII, 58—9.
Ground-curve: and Abel's theorem, XII, 38, 109—216.
Groups: of lines and points, I, 317—28, 356—61, 414—20, 550—6; depending on symbolic equation $\theta^n=1$, II, 123—30, 131—2, IV, 88—91; theory of, IV, 88—91, X, 324—30, XII, 639—56; the term, IV, 594, 596, VII, 123; rotations of polyhedra, V, 529, 559; Cayley's work at, VIII, XXXIII; theorems on, X, 149—52, 153—4; desiderata and suggestions on theory, X, 401—6; partitions and theory of, XI, 62; homographic transformations, XI, 189—90, 196—208, 237—41; linear transformation of a variable, XI, 237—41; Jacobian sextic, XI, 389, 393—6; the notion, XI, 509—10; Latin squares, XIII, 55—7; substitution groups for two to eight letters, XIII, 117—49; of points, non-existence of a special, XIII, 212; quotient G/H in theory of, XIII, 336—7; illustrations of Sylow's theorems on, XIII, 530—3; of sixty icosahedral substitutions, XIII, 552, 556.
Grunert, J. A.: difference-table, XI, 144.
Gudermann, C.: elliptic integrals, I, 224; transformation of an integral, I, 383; transformation of elliptic functions, III, 2; spherical pendulum, IV, 534, 588; logarithms, VII, 414.
Gudermannian: V, 86—8, 617; Lobatschewsky's imaginary geometry, V, 472; tables of, V, 617.
Gundelfinger, S.: concomitants of ternary cubic, XI, 342—3.

Hall, A.: motion of particle towards attracting centre, IX, 215—7.
Halphen, G. H.: inverse elliptic functions, I, 586; curves in space, V, 613—7; higher singularities of plane curves, V, 619; curves satisfying given conditions, VI, 594—5; classification of curves, XI, 451; invariants of differential equations, XII, 392—3; transformation in elliptic functions, XIII, 29; Sylvester on reciprocants, XIII, 48; reciprocants, XIII, 333, 366, 368—81, 381—98; characteristic n in the theory of curves in space, XIII, 468—72.
Halsted, G. B.: hyperspace and non-Euclidian geometry, II, 606.
Hamilton, Sir W. R.: quaternions, I, 123—6, 238, 335, 586; form of equations of motion, I, 284; homographic transformation of quadrics, II, 105, 133; focal relations, II, 143; problem of three or more bodies, III, 97—103; differential equations, III, 164, IV, 514; method of dynamics, III, 166—74, 200, 202; equations of motion, III, 186; isochronism of circular hodograph, III, 262—5;

essential singularity of function, IV, 105; central forces problem, IV, 520; hodograph, IV, 520; transformation of coordinates, IV, 558—9, 588; ray systems, VIII, 504, XII, 571—5; surface orthogonal to set of lines, IX, 587; system of differential equations, X, 113—8; equations of central orbit, X, 613; on mathematics, XI, 431; algebra and time, XI, 443; conical refraction, XI, 449; multiple algebra, XII, 460, 466, 474—5; Sylvester on Hamiltonian numbers, XIII, 48; (*see also* Differential Equations).

Hammond, J.: theory of tamisage, XI, 409—10; seminvariants, XII, 253; Sylvester's reciprocants, XIII, 47—8, 381, 388; on Hamiltonian numbers, XIII, 48.

Hansen, P. A.: lunar theory, III, 13—24, 291—2; elliptic orbit, III, 95; expansion of true anomaly, III, 140; planetary theory, III, 268—9, IX, 180—3; disturbed elliptic motion, III, 270—1; disturbing function in lunar theory, III, 293, 319—43; variation of plane of planet's orbit, III, 516—8; elliptic motion, IV, 522, 523, 588; relative motion, IV, 536, 588; pendulum, IV, 541, 588; spheroidal trigonometry, IX, 197.

Hargreave, C. J.: on differential equations, VIII, 458.

Harley, R.: equation of differences, IV, 241, 245; symmetric products and quintics, IV, 310—13; quintics, V, 53; a differential equation, VII, 354; theory of equations, XI, 520; invariants, XII, 390—1.

Harmoconic: defined, V, 342.

Harmonic Relations: of two lines or points, II, 96—7; theory of, and two or more quadrics, II, 529—40.

Harmonics: symmetric, II, 555; inscribed, III, 113; reciprocal lines, XIII, 58—9; and non-Euclidian geometry, XIII, 482—9.

Harriot, T.: mathematical discoveries, XI, 437.

Hart, A. S.: cubic curves, IV, 499; relative motion, IV, 535; triple tangent planes, VI, 372, 375; nine-point circle, XIII, 548.

Haughton, S.: inertia, IV, 564—5, 588.

Haupttangenten: (inflexional tangents), VIII, 157.

Heal, W. E.: bitangents of quintic, XIII, 21.

Hearn, G. W.: on a geometrical locus, I, 496; quartic curves, I, 496; quadric curves, V, 262.

Heath, R. S.: non-Euclidian geometry, XIII, 481, 499.

Helmholtz, H. von: hydrodynamical equations, XIII, 6—8.

Hemihedron: the word, X, 328.

Hemipolyhedron: the word, X, 328.

Hensley, P. J.: foci of conics, IV, 505—9.

Heptacron (*see* Polyacra).

Heptagon: construction, X, 609.

Hermite, C.: homographic transformation of quadric into itself, II, 107; elliptic integral and covariants of quartic, II, 191; law of reciprocity, II, 232, 234; skew invariant of quintic, II, 233; transformation of quadric function, II, 499; hyperdeterminants, II, 598—601; elliptic integrals, IV, 68—9; ternary cubics, IV, 326, 330; Tschirnhausen's transformation, IV, 364—7, 375, VI, 165, 170; automorphic transformation, IV, 416; elliptic functions and solution of quintic, IV, 484—9; matrices, V, 438, XII, 367—70, 386; quantics, VI, 147; quintic equation, VI, 170; nodal cubic, VI, 174—6; canonical form of quintic, VI, 177—83; transformation of elliptic functions, VII, 44, IX, 113, XII, 337, 416—7, XIII, 31, 39; reduction of Abelian integrals, X, 214; concomitants of ternary cubic, XI, 342; elliptic functions, XI, 452; theory of equations, XI, 520; Abelian functions, XII, 98; transformation of double theta functions, XII, 358; *H*-product theorem, XII, 584—6; cubic equations, XIII, 349; omega functions, XIII, 558.

Herschel, Sir J. F. W.: finite differences, IV, 95, 107, 262; Brinkley's formulæ, X, 58—9; difference table, XI, 144.

Hesse, L. O.: linear transformations, I, 87, 113—6, 232—3, 584, 585, VI, 22—3, 73; cubic curves, I, 194, II, 399; involution, I, 259; hyperdeterminants, I, 354; abstract of memoir on quadric surfaces, I, 425—7; inflexions of cubic, I, 584, II, 29, III, 48; on elimination, IV, 3; on inflexions, IV, 186, XI, 473; double tangents, IV, 186—7, 343—6; cubic forms, IV, 353; geometric transformation, VII, 121—2; bitangents of quartic curve, VII, 123—4; double theta functions, X, 177; triple theta functions, X, 446, 448, 451; the thirty-four concomitants of ternary cubic, XI, 342; bitangents of a plane quartic, XII, 76.

Hessians: and Eisenstein's theorem, I, 585; defined, II, 319, IV, 607, XI, 471, 474; and Pippian, II, 381—2, 383—95, 416; the sign for, II, 541; the quadricovariant, II, 545; of quaternary functions, IX, 90—3; of a quartic surface, X, 274—7; (*see also* Quantics).

Hexagon: and conic, problem, V, 576; theorem of inscribed, XI, 556.

Hexagram (*see* Pascal's theorem).

Hexahedron: edges of, problem, X, 613; automorphic function for, XI, 184.

Hierholzer, C.: cones satisfying six conditions, VIII, 99; surface of eighth order, VIII, 401.

Higher Singularities of Curves (*see* Singularities).

Hilbert, D.: curves in space, V, 614; abstract geometry, VI, 596.

Hills: altitude, and roots of algebraic equation, XIII, 33—7.

Hirsch, Meyer: algebra of, II, 417, 440; partitions, VII, 577.

Hirst, T. A.: negative pedals, IV, 164.

History: importance of mathematical, VIII, xii.

Hodograph: isochronism of circular, III, 262—5; Hamilton's, IV, 520; and pedal curves, V, 113.

Hölder, O.: theory of groups, XIII, 336, 533.

Holditch, H.: caustics, II, 363.

Holomorphic: the term, XI, 81.

Homaloids: prepotentials of, IX, 408—9.

Homographic Figures: matrices, II, 219; theorem, IV, 442—5.

Homographic Function: distribution of electricity, XI, 2—6; and matrix of the second order, XI, 252—7.

Homographic Relations: and theory of numbers, IX, 613; function of, X, 298—306; powers of, X, 305—6, 307—9.

Homographic Transformation: of quadric surfaces, II, 105—112; of single theta functions, XII, 337—43; (*see also* Groups, Transformations).

Homographies: correspondence with rotations, X, 153—4, XI, 237—8.

Homography: different classes of, II, 219—220; and quadrics, II, 536—40; ternary quantics and theory of, II, 565—9, 578; of ranges and pencils, II, 578.

Homology: of conics, I, 523, 557; the term, I, 557—8; of sets, III, 35.

Homotypic: the term, VII, 123.

Homotypical: the term, V, 50.

Hopkins, W.: attractions and multiple integrals, I, 582.

Horner, W. G.: theory of equations, XI, 499, 502.

Hoüel, J.: equations of motion, III, 169, 170.

Hudson, C.: equal roots of equations, XI, 405—7.

Hydrodynamics: note on equations of, XIII, 6—8.

Hydrogen: trees of, IX, 427—60, 544—5.

Hyperbola: rectangular, III, 254, V, 554; classification, V, 354—400, VIII, xxxviii; arc of, XI, 337—9; and cubic curves, XI, 478; in *Ency. Brit.*, XI, 561—4; focals of quadric surface, XIII, 54; and epitrochoid, XIII, 82—7; tactions, XIII, 150—69; (*see also* Conics).

Hyperboloid: coordinates of, V, 72; and scrolls, VII, 65; and tractor, VII, 73—5; passing through three given lines, VII, 86—8; Mercator's projection, VIII, 568, IX, 237—40; deformation, XI, 66—7; in *Ency. Brit.*, XI, 576—9; confocal Cartesians and right lines of, XII, 587—9.

Hyperdeterminants: the term, I, 81, 95, 114, 585; note on, I, 352—5, 588; a system of certain formulæ, I, 533; theory, I, 577—9; theory of permutants, II, 19; theory of intermutants, II, 26; quantics, II, 225; theory of seminvariants, XII, 344; Sylvester's work in, XIII, 46; an identity, XIII, 210—11; (*see also* Covariants, Invariants).

Hyperdimensional Space: quadrics in, IX, 79—83; (*see also* Hypergeometry, Hyperspace).

Hyperelliptic Functions: trisection of, VI, 594; and theta functions, X, 162—5, 166—79, 184—214, 551—5; and triple theta functions, X, 432—6; addition-theorem, X, 455—62; the term, XI, 533—4; and nodal quartics, XII, 196—208; (*see also* Theta-Functions).

Hyperelliptic Integrals: of first order, XII, 98—9.

Hypergeometric Series: summation of a certain factorial expression, III, 250—3; theorem, III, 268—9; differential equations, XI, 17—25; note on, XI, 125—7; and Schwarzian derivative, XI, 176—9.

Hypergeometry: of n dimensions, I, 55—62; a branch of mathematics, VIII, xxxiii—v; five-dimensional, IX, 79—83; and quadric surfaces, IX, 246—9; 21 coordinates of conic in space, XI, 82—3; Sylvester's work in, XIII, 46; (*see also* Hyperspace, Prepotentials).

Hyperspace: and quantics, II, 222; and non-Euclidian geometry, II, 606; representation by means of, VI, 198; of four dimensions, special theorem, IX, 246—9; (*see also* Hypergeometry).

Icosahedra: construction, IV, 81—2; axial systems, V, 531—9; Klein on rotations of, X, 153; as regular solids, X, 270—3; automorphic function, XI, 169, 179—83, 185, 212—6.

Icosahedral Substitutions (*see* Substitutions).

Ideal: the term, VI, 483.

Ideal Numbers: XI, 456.

Idem: defined, XII, 66.

Idempotent: the term, XII, 61.

Identities: cubic, V, 597; trigonometrical, VIII, 525, XI, 38, XIII, 538—40; elliptic transcendent, VIII, 564; a transcendental, XI, 37; algebraic, XI, 63—4, 130—1, XIII, 76—8; a hyperdeterminant, XIII, 210—11.

Imaginaries: on an octuple system of, I, 301; eight-square, XI, 368—71, XII, 465; the term, XI, 439; theory of equations, XI, 502—6; and function, XI, 523; associative, XII, 61, 105—6; perpendicularity, XII, 466—72; roots of equation, XIII, 36; Sylvester's work at, XIII, 46; quaternions, XIII, 542.

Imaginary Quantities: logarithms, VI, 14—8; geometrical construction relating to, XI, 258—60.

Immit: defined, IV, 109.

Improper: conditions for curves, VI, 193.

Increment: the term, VI, 468.

Indefinite: applied to integration, IX, 500—3; the term, XIII, 290.

Indeterminate Equations: problem in indeterminate analysis, III, 205—7.

Index: to philosophic memoirs, report on, V, 546—8, 620.

Indicial Equation: of differential equation, XII, 398, 453.

Indicial Function: of differential equation, XII, 398, 401.

Inertia: axes and moments of, IV, 478—80, 559—66.

Ineunt: defined, II, 574, V, 521, VI, 469; non-Euclidian geometry, XIII, 489.

Infinitesimal Rotations: VI, 24—6.

Infinity: in geometry, XI, 464.

Inflexional Tangents: and geodesic lines, VIII, 157; (*see also* Tangents).

Inflexions: of cubical divergent parabolas, V, 284—8; of cubic curve, I, 584, III, 48; Hesse on, IV, 186, V, 493—4, XI, 473; of curves, XI, 471—3, 480.

Integral Calculus: some formulæ of, I, 309—16, 588; transformation, I, 383; Picard's memoir on, XII, 408—11.

Integral Functions: Legendre's coefficients, I, 375—6; the term, IV, 603—4, XI, 523; prepotential surface, IX, 321—30, 330—4, 352—9; potential solid, IX, 334—7; epispheric, IX, 410—17; reduction

of transcendental, x, 214—22; hyperelliptic, of first order, xii, 98—9; regular, of differential equation, xii, 395—6; number of, xii, 399; subregular of differential equation, xii, 444—52; (*see also* Abelian Integrals, Definite Integrals, Elliptic Integrals, Transformations).

Integrals: transformation of double, ix, 250—2; of differential equations of first order, x, 19.

Integration: a supposed new, vii, 36; theorem of, vii, 588; by series of differential equations, viii, 458—62; a process of, ix, 257—8, x, 15, 29; indefinite, ix, 500—3; Aronhold's formula, x, 12—14; of Euler's equation, xi, 68—9.

Integrator: mechanical, xi, 52—4.

Intercalation: root-limitation, ix, 22—7; for right line, ix, 28—33; Sylvester's theory of, xiii, 46.

Intermediates: of two quantics, defined, ii, 515; of binary quartic, ii, 549; and ternary cubics, iv, 326.

Intermutants: the term, ii, 19, 26, iv, 594, 600.

Interpolation: Smith's Prize dissertation, viii, 551—5.

Intersect-developable: of two quadrics, i, 486—95.

Intersections: the term, vii, 546; of two curves, ix, 21, xii, 117—20; of cubic and line, xii, 100.

Invariable Plane: and rotation of solid body, i, 237, vi, 142.

Invariants: the term, i, 577, 589, ii, 176, 224, iv, 594, 605, xiii, 46; and discriminants, i, 584; determined by differential equations, ii, 164—78; and roots, ii, 176; differential equation satisfied by, ii, 176—8; and binary quantics, ii, 266—8; of quartic, and covariants of cubic, analogous, ii, 553; bibliography, ii, 598—601; of biternary quantics, iv, 349—58; 18-thic of quintic in terms of roots, vi, 154—6; Cayley founder of, viii, xxviii—xxx; his work, viii, xxx—xxxii; and transformation of quantics, viii, 385—7; quadratic transformation of a binary form, viii, 398—400; identical equation connected with theory, ix, 52—5; Hessian of quaternary function, ix, 90—3; minimum N. G. F. of binary septic, x, 408—21; stereographic projection, xi, 187—9; in geometry, xi, 474; of a linear differential equation, xii, 390—3; Sylvester's work at, xiii, 46, 47; two, of quadri-quadric function, xiii, 67—8; differential, and reciprocants, xiii, 366—404; Pfaff-, xiii, 405—14; (*see also* Covariants, Linear Transformations, Seminvariants).

Invariants and Covariants: xii, 22—9; standard solutions of system of linear equations, xii, 19—21; finite number of the covariants of a binary quantic, xii, 558.

Inversion: of quadric surface, viii, 67—71; note on, ix, 18.

Inverts: quadric function of, xi, 153—6.

Involutant: of two binary matrices, xiii, 74—5.

Involution: theory of geometrical, i, 259—66, 587; and two or more quadrics, ii, 529—40; of six lines, iv, 582, vii, 66, 85, 95; lines in, v, 1—3; theory, v, 295—313; the term, vi, 460; of four circles, vi, 505—8; and ternary quadrics, xiii, 350—3.

Involution of Cubic Curves, Memoir: v, 313—53, vii, 238; explanations, definitions, and results, v, 314—8; general formulæ for critic centres, v, 318—9; twofold and one-with-twofold centre, v, 319—24; tangents at a node, v, 325—8; triangle of critic centres, v, 328—9; the three-centre conic, v, 329—36, 337—8; transformation equation of cubic, v, 339—41; cubic locus, harmoconics, and harmonic conic, v, 341—5; miscellaneous, v, 345—53.

Irrational: and subrational, ix, 315.

Irreducible: the term, vii, 336, xii, 23.

Irreducible Concomitants: of quintic, x, 342.

Irreducible Covariants: and invariants, ii, 250.

Irreducible Syzygies (*see* Syzygies).

Irregular: the term, vi, 457, 459.

Iseccentric Lines: and planet's orbit, vii, 468.

Isobaric: the term, xiii, 266.

Isobarism: of covariants, ii, 233.

Isochronic: the term, nodal and cuspidal, vii, 473.

Isochronism: of circular hodograph, III, 262—5.
Isomers: Mathematical theory of, IX, 202—4.
Isoparametric Lines: and planet's orbit, VII, 467.
Isoperimetrical Problem: VII, 263.
Isothermals: of Meyer, XIII, 175.

Jacobi, K. G. J.: theory of algebraic curves, I, 53; determinants, I, 63, 64, 66; quaternions, I, 126, 127, 586; inverse elliptic functions, I, 132, 136, 152, 156, 162, 180; rotation of solid body, I, 238, IV, 575, 576—7, 579; involution, I, 259, 263; definite integral, I, 270—1; dynamical differential equations, I, 276—9; elliptic functions, I, 290—300, 507, 587, XI, 452; demonstration of theorem on focal lines, I, 362—3; differential equations of Abelian functions, I, 366—9; skew determinants, I, 411; simultaneous linear transformations, I, 428; transformation of integrals, I, 440, 442; attraction of ellipsoids, I, 511—8; solution of equation $x^{257}-1=0$, I, 564; Lagrange's theorem, II, 7; geometrical representation of elliptic integral, II, 56; in-and-circumscribed polygon, II, 141; partition of numbers, II, 248; lunar theory, III, 13; canonical formulæ for disturbed motion, III, 76—7; problem of three or more bodies, III, 102; transformation, III, 129; finite differences, III, 132—5; complete integral of partial differential equation, III, 166; planetary theory, III, 173; calculus of variations and differential equations, III, 174—84, 200, 202; *De Motu Puncti singularis*, III, 182—3, 202; problem of three bodies, III, 519—21, IV, 541, 548—51, 589, V, 23, VI, 183; *Theoria Novi Multiplicatoris*, III, 183—5; theory of ideal coordinates, III, 185; Encke's memoir *über die speciellen Störungen*, III, 179—80; in-and-circumscribed triangle, III, 236; disturbed elliptic motion, III, 270—1; canonical system of formulæ, III, 290; reversion of series, IV, 30—7; transformation of elliptic integrals, IV, 60, 64; double tangents, IV, 187; conics, IV, 207; Pfaff's problem, IV, 359—63; central forces problem, IV, 520, 589; *Nova Methodus*, IV, 515, 521, 589; elliptic motion, IV, 522, 589; problem of two centres, IV, 530, 532, 589; motion of a single particle, IV, 537—8, 589; motion in resisting medium, IV, 541, 589; motion of a point, IV, 547, 589; elimination of nodes, IV, 551, 589; transformation of coordinates, IV, 554, 557, 589; Weierstrass's function al (x), V, 34—5; transformation of elliptic functions, V, 472, IX, 103, 113—75, XII, 59, 505—34; *Canon Arithmeticus*, VI, 83—6, XI, 85, 86; the Jacobian relation, VI, 467; geodesic lines on ellipsoid, VII, 493; transformation of double integral, IX, 250—2, 254; epispheric integrals, IX, 321, 410—17; mathematical tables, IX, 472—3, 484—5; series, X, 25—7; Poisson's theorem, X, 108—9, 110—3; theta functions, X, 156, 473, 478, 490, 496—7, XI, 41—6, XIII, 559; roots of unity, XI, 58—60; Schwarzian derivative and polyhedral functions, XI, 149; hypergeometric series, XI, 178; Landen's theorem, XI, 339; Abelian functions, XI, 454; theory of numbers, XI, 602; theorem in simultaneous equations, XII, 39; fraction theorem, XII, 123—5; Weierstrassian and Jacobian elliptic functions, XII, 425—7; sextic equation, XI, 389—401, XII, 493—9; sums of two series, XIII, 50; modular equations, XIII, 64; characteristic n and curves in space, XIII, 469; sextic resolvent equations, XIII, 473—9.
Jacobian: defined, II, 319, IV, 607; of two quantics, II, 517; theory of, and polyzomal curves, VI, 566—8; the extended notion, VII, 134; of surfaces, VII, 134—6; of two curves, IX, 21; of quadric surfaces, X, 568; of six points, X, 613; rational transformation, XIII, 116; (*see also* Quantics).
Jefferson, T.: founder of Virginian mathematical professorship, XIII, 43.
Jeffery, H. M.: on quartic curves, XI, 408.
Jellett, J. H.: theorem of, on attractions, I, 388—91.
Jenkins, M.: Arbogast's method of derivations, IV, 609; problem in factorials, VII, 597.
Jerrard, G. B.: quintics, IV, 391—4, V, 50—4, 77, 89; theory of equations, XI, 520.
Joachimsthal, F.: theory of covariants, II, 177, 234; theorem of, II, 521, XII, 594—5, 601, 629; normals of a conic, IV, 74—7; attraction of terminated straight line, VII, 33.
Johnson, W. W.: matrices, XI, 252.

Jordan, C.: trees, x, 599, xi, 366; substitutions, xi, 455; theory of equations, xi, 520—1; uniform convergence, xiii, 343—4.

Joubert, P.: transformation of elliptic functions, ix, 113, 138.

Kant, I.: cognition and space, xi, 431; negative magnitude, xi, 434.

Kelland, P.: quaternions, xii, 475.

Kelvin, Lord: equimomental surfaces, i, 253—4; definite integral, i, 270; differentiation and evaluation of definite integrals, i, 587; relative motion, iv, 535, 593; inertia, iv, 565, 566, 593; attraction of terminated straight line, vii, 33; report on mathematical tables, ix, 461—99; distribution of electricity, xi, 6.

Kempe, A. B.: theory of groups, xii, 639.

Kenogram: the term, ix, 202—3.

Kepler, J.: problem of, iv, 521; and ellipse, xi, 447.

Key, T. H.: Professor of Mathematics, University of Virginia, xiii, 43.

Kiepert, L.: transformation of elliptic functions, xii, 490—2.

Kinematics: of solid body, iv, 580—2; six coordinates of a line, vii, 89—95; of a plane, xi, 103—110, xiii, 505—16.

Kirkman, T. P.: schoolgirl problem, i, 483, 589; geometry of position, i, 550—6; sums of squares, ii, 49—52; double summation of factorial expression, iii, 250—3; autopolar polyhedra, iv, 85, 182; enumeration of polyhedra, v, 38; partitions of a polygon, xiii, 93.

Klein, F.: theory of distance, ii, 604, viii, xxxvi—vii; coordinates in non-Euclidian geometry, ii, 604—6; non-Euclidian geometry, viii, 409—13, xiii, 481; correspondence of homographies and rotations, x, 153; a system of quadric surfaces, x, 269; octahedral function, xi, 128; Schwarzian derivative and polyhedral functions, xi, 149, 151, 179, 183, 185; his classes of geometry, xi, 436; transformation of elliptic functions, xii, 29, 46, 490, 547.

Knots: of trees, iii, 243, ix, 429—60, xi, 365—7, xiii, 26—8; in topography, defined, iv, 104.

Königsberger, L.: transformation of elliptic functions, ix, 113, 140; theta functions, x, 499.

Korndörfer, G.: quartic and quintic surfaces, vii, 246, 249, 252.

Kowalski, M.: orbit of Neptune, ix, 180.

Kreistheilung: (cyclotomy), xi, 58, 86.

Kronecker, L.: equation of differences, iv, 609—16; quintics, v, 55; theory of equations, xi, 520; sextic resolvent equations, xiii, 473—9.

Kummer, E. E.: solution of equation $x^{257}-1=0$, i, 564; wave surfaces, i, 587; composition of numbers, iv, 70—1, 78—9; Steiner's quartic surface, v, 423; 16-nodal quartic surfaces, v, 431—7, vii, 126—7, x, 437; quartic surfaces, vii, 134, 176, 313, ix, 71; quintic surfaces, vii, 245, 247, 252; nodal quartic surfaces, vii, 264—297; surface of, viii, 69; prepotentials, ix, 364; mathematical tables, ix, 494; double theta functions and 16-nodal quartic surface, x, 158, 161; octic surface, x, 252; theta functions, x, 499; table of hexads, x, 506, 538, 552; hypergeometric series, xi, 17—25, 177; Schwarzian derivative and polyhedral functions, xi, 149; theory of numbers, xi, 456; systems of rays, xii, 573; differential equation of third order, xiii, 69—73.

Lachlan, R.: circles and spheres, i, 581, xiii, 13.

Lacroix, S. F.: transformation of coordinates, iv, 557; imaginaries, xii, 468.

Lacunary Functions: xiii, 415—7.

Ladd, Christine: the Pascal hexagram, vi, 594.

Lagrange, J. L.: theorem of expansion, i, 40—2, 584, ii, 1—7, iii, 141; determinants, i, 64; variation of parameters, i, 243; dynamical differential equations, i, 279; elliptic functions, i, 366; distances of points, i, 581; attractions and multiple integrals, i, 581—2; caustics, ii, 353; problem of two fixed centres, iii, 104—10, iv, 527, 528—9, 589; sums of series, iii, 124; *Mécanique Analytique*, iii,

157—8, 201, 202; equations of motion, III, 158, 200, IX, 198—200; planetary theory, III, 159—61, 162—3, 201; variation of arbitrary constants in mechanical problems, III, 161—5, 200, 201; coefficient (*a*, *b*) of, III, 163; Hamilton's method of dynamics, III, 171—3, 200; disturbed elliptic motion, III, 271—2; equations of differences, IV, 240, 252; resolvents, IV, 309; central forces problem, IV, 519—20, 589; elliptic motion, IV, 521—2, 589; expansion of anomalies, IV, 522; spherical pendulum, IV, 532—3, 589; rotation of solid body, IV, 566, 569, 589; homotypical functions, V, 50; invariable plane, VI, 142; invariants, VIII, XXX; demonstration of Taylor's theorem, VIII, 493—5, 519; virtual velocities, IX, 205—8; prime roots of unity, XI, 57; Schwarzian derivative, XI, 149; theory of equations, XI, 455, 498—500, 514—5, 517, 520; envelopes, XI, 475; minimal surfaces, XI, 638; five points in space, XII, 581—3; theorem of expansion and partitions of polygon, XIII, 113; Waring's formula, XIII, 215—6; reciprocants, XIII, 366; Richelot's integral of Euler's equation, XIII, 526.

Laguerre, E.: theory of matrices, II, 604.

Lamb, H.: conformal representation, X, 290.

Lambdaic: defined, II, 523, IV, 49, 53; of binary quartic, II, 550.

Lambert, J. H.: theorem on circular hodograph of, III, 262—5; theorem for elliptic motion, III, 562—5, VII, 387—9; central forces problem, IV, 519, 520, 589; equation of planet's orbit from three observations, VII, 412—5.

Lamé, G.: curvilinear coordinates, VIII, 146, XI, 637, XII, 17; orthogonal surfaces, VIII, 280, 291, 292.

Lamp, Milner's: a differential equation, and construction of, XIII, 3—5.

Lancret, M. A.: curves of curvature, XII, 601.

Landen, J.: theorem of, in elliptic functions, XI, 337—9; biographical notice, XI, 583—4.

Languages: Cayley's knowledge of, VIII, xxiii.

Laplace, P. S.: on Lagrange's theorem, I, 42, II, 7; determinants, I, 63; functions of, I, 397—401, 588; attraction of ellipsoids, I, 581, III, 53—65, 567; planetary theory, III, 159, 201; disturbed elliptic motion, III, 505, 510—11; on secular variation, III, 568; elliptic motion, IV, 524, 589; relative motion, IV, 534, 536, 589; motion of three bodies, IV, 540—1, 589; prepotentials, IX, 393; finite differences, XII, 412.

Last Multiplier: IV, 530, 547, 551, 590.

Latin Squares: XIII, 55—7.

Latitude: parametric, VII, 16, IX, 238.

Lattice: in theory of numbers, III, 40.

Laverty, W. H.: systems of equations, VII, 578.

Law, The: Cayley's work at, VIII, xiii—xv, xix.

Lebesgue, V. A.: determinants, I, 63.

Lectures: delivered by Cayley, VIII, xvi—xvii, xlv.

Lefort, F.: elliptic motion, IV, 522, 589.

Left-handed: circuits in root-limitation, IX, 22—3.

Legendre, A. M.: elliptic functions, I, 136, 156, 507, V, 618, XI, 452, 537, 584, XII, 35—7; elliptic integrals, I, 224; coefficients of, I, 375—6; attraction of ellipsoid, I, 432—7, 442; functions of, IV, 99, 106; rectilinear motion, IV, 516, 590; central forces problem, IV, 521, 590; problem of two centres, IV, 530, 590; rotation of solid body, IV, 570, 590; geodesic lines on oblate spheroid, VII, 15—25; reduced latitude, VII, 16; orbit of planet from three observations, VII, 414; mathematical tables, IX, 467—8, 478, 487; Landen's theorem, XI, 339; theory of numbers, XI, 455, 602—4, 616, XII, 35—7; second kind of coefficients, XII, 562—3; gamma function, XIII, 49.

Lejeune-Dirichlet, P. G.: multiple integrals, I, 195, 582—3; integration, I, 588; theorem of, II, 10, 47—8; binary quadratic forms, V, 141; prepotentials, IX, 321, 417—23; attractions, XI, 448; theory of numbers, XI, 456, 616.

Lemniscate Function: XI, 65; and orthomorphosis, XIII, 191—205.

Letters: substitution groups for two to eight, XIII, 117—49.

Leverrier, U. J. J.: disturbing function in planetary theory, III, 321, VII, 511—27; elliptic motion, III, 361, 362, IV, 523, 590; position of orbit in planetary theory, VII, 545.
Lévy, M.: orthogonal surfaces, VIII, 269, 569—70; Dupin's theorem, IX, 85.
Light, Polarized: MacCullagh's theorem, IV, 12—20.
Limaçon of Pascal: I, 480, XI, 477.
Lindemann, F.: non-Euclidian geometry, XIII, 481.
Linear: and omal relations in abstract geometry, VI, 463.
Linear Differential Equations: invariants of, XII, 390—3; general theory, XII, 394—402, 444—52; decomposition, XII, 403—7.
Linear Equations: and determinants, XI, 490; standard solutions of system of, XII, 19—21.
Linear Function: the term, XI, 492.
Linear Quantics (*see* Quantics).
Linear Substitutions: note on a function in, X, 307—9.
Linear Transformations: theory of, I, 80—94, 95—112, 117, 584, 585; Eisenstein's and Hesse's formulæ, I, 113—6, 585; homogeneous functions of third order with three variables, I, 230—3; hyperdeterminants, I, 352—5, 577—9, 588, 589; theory of permutations, I, 423—4; simultaneous, of two homogeneous functions of second order, I, 428—31; theory of permutants, II, 19—23; the term, IV, 594, 605; of elliptic integrals, IX, 618—21; of theta functions, XII, 50—5; Sylvester's work in, XIII, 46; (*see also* Covariants, Invariants, Quantics).
Line-geometry: and congruences, XIII, 228—30; (*see also* Coordinates, Lines).
Lineo-linear: the term, II, 517, IV, 604, VI, 464.
Lineo-linear Transformation: between planes, VII, 215—6, 236—8.
Line-pairs: the term, VI, 206, 209, 210, 211; through three given points and touching given conic, VI, 201, 244, 594.
Line-pair-point: the term, VI, 202, 210, 211, 269, 594—5.
Lines: on cubic surfaces, I, 445—56, VIII, 371—6; harmonic relation of two, II, 96—7; of cubic curve, II, 382; satellite, II, 383, V, 359; formulæ, II, 405—9; line, plane and point, defined, II, 561—2; contour and slope, IV, 108—11, 609; cubic centres and cones, IV, 173—8, 179—81; geometry of, IV, 446—55, 616—8; involution, IV, 582, V, 1—3; cubic centres of three lines and a line, V, 73—6; theorem of conic and triangle, V, 100—2; intersections of pencils of four and two, V, 484—6; formulæ for intersections of line and conic, V, 500—4; circle and parabola, problem, V, 607; notation in Pascal's theorem, VI, 116—23; facultative, VI, 365—6, 450; dot-notation for, and planes and cubic surfaces, VI, 365—6, 373—449; twenty-seven on cubic surface, VI, 371—87; attraction of terminated straight, VII, 31—3; five on cubic surface, VII, 177—8; homographic transformation, VII, 193—7; spherogram and isoparametric, VII, 467—8; iseccentric and *e*-spherogram, VII, 468—70; isochronic and time spherogram, VII, 470—7; Cayley's work on six coordinates of, VIII, xxxv; potentials of, IX, 278—80; formulæ relating to right, X, 287—9; and points, X, 570; and conics, X, 602; contact with a surface, XI, 281—93; Mill on, XI, 432—3; non-Euclidian geometry, XI, 437, XIII, 480—504; evolution of theory of curves, XI, 450—1; singularities of curves, XI, 468; in *Ency. Brit.*, XI, 548, 571—2; equation of right, XI, 558—61; and surface, XI, 629; Mascheroni's geometry of the compass, XII, 314—7; reciprocal, XIII, 58—9, 481; identity relating to six coordinates of a line, XIII, 76—8; and notion of plane curve of given order, XIII, 79—80; syzygetic relations, XIII, 224—7; of striction, on skew surface, XIII, 232—7; system of in a plane, and their orthotomic circle, XIII, 346—7; and point, distance, XIII, 495—7; theory of two lines, XIII, 497—504; (*see also* Coordinates, Curvature, Geodesic Lines).
Line Systems: two-dimensional geometry, II, 569—83.
Link: the term, V, 521, VII, 183, XIII, 506.
Linkage: the MacMahon, XIII, 265, 292, 293, 298—301.
Link-work: X, 407.

Liouville, J.: integration of differential equations, III, 180, 191—2; equations of motion, III, 185; Bour's memoir, III, 199, 204; a definite integral, IV, 28—9; elliptic motion, IV, 522, 590; problem of two centres, IV, 530; motion of three bodies, IV, 541, 590; point moving in plane, IV, 542—6, 590; Abelian integrals, IV, 546, 590; on roots of equations, IX, 39; indefinite integration, IX, 500—3; geodesic curvature, XI, 323, 328.

Lipschitz, R.: on Lagrange's equations, IX, 110—2.

Listing, J. B.: partitions of a close, V, 617, VI, 22; theorem of, VIII, 540—7.

Lobatschewsky, N. I.: geometry of, V, 471—2, VIII, XXXVII, XI, 436, XII, 220—38.

Locus: defined, IV, 458, VI, 458, 460; from two conics, VI, 27—34; in relation to triangle, VI, 53—64; and envelope in regard to triangle, VI, 72—82; *in solido*, problem and solution, VII, 599; *in plano*, problem and solution, VII, 599; the terms nodal-, cuspidal-, envelope-, tac-, VIII, 533—4; in *Ency. Brit.*, XI, 585.

Logarithms: theory of, III, 208—13, 222—8; of imaginary quantities, VI, 14—8; Pineto's tables, VIII, 95—6; calculation of log 2, XI, 70; origin, XI, 447; function in, XI, 526—7.

Logic: of characteristics, III, 51—2; calculus of, VIII, 65—6; Mill on, XI, 432—4; recent developments, XI, 458—9; geometry and algebra, XII, 459.

Logistic: and algebra, V, 293—4.

London Mathematical Society: Sylvester's connexion with, XIII, 45.

Lottner, C. L. E.: motion of solid body, IV, 583, 590.

Lubbock, Sir J. W.: theory of the Moon, III, 309—10.

Lunar Theory: Hansen's, III, 13—24; disturbed elliptic motion, III, 270—92; development of disturbing function, III, 293—318, 319—43, VII, 511—27; tables, III, 299—308; theorem of Jacobi, III, 519—21; Plana's, VII, 357—60, 361—6, 367—70, 371—4, 375—6; Delaunay's l, g, h, and $h+g$, VII, 528—33, 534; pair of differential equations in, VII, 535—6, 537—40; Newcomb's work, IX, 179—80; note on, XIII, 206—9; (*see also* Elliptic Motion, Disturbed).

Lüroth, J.: six coordinates of a line, VII, 66.

Luther, E.: theory of equations, XI, 520.

MacAulay, A.: quaternions and hydrodynamical equations, XIII, 8.

MacClintock, E.: quintics, IV, 609—16.

MacCullagh, J.: polar plane theorem, IV, 12—20; spherical conics, IV, 428; inertia, IV, 564—5, 588; rotation of solid body, IV, 574, 588.

MacDowell, J.: triangle and circle, V, 564.

Maclaurin, C.: cubic curves, I, 586; attraction of ellipsoids, III, 27, 62, 155; pedal curves, V, 113—4.

MacMahon, P. A.: symmetric functions, II, 603; seminvariants and symmetric functions, VIII, XXXI, XIII, 286—8, 290, 292, 293, 302; a differential equation, XII, 30—2; seminvariants, XII, 239, 254, 261, 275, 349, XIII, 265; a differential operator, XII, 318; Waring's formula for form of equation adopted by, XIII, 214; reciprocants and differential invariants, XIII, 399—404.

Magic Squares: X, 38.

Magnetism: Sabine's work, XI, 430; Gauss's work, XI, 544.

Maillard, M. S.: penultimate forms of curves, VIII, 258; degenerate forms of curves, XI, 220; systems of curves, XI, 487.

Major Function in Abel's theorem: XII, 111, 132—56.

Malet, J. C.: invariants, XII, 390—1.

Malfatti, G. F.: sextic resolvent equations, XIII, 473.

Malfatti's Problem: and system of equations, I, 465—70; Steiner's extension, II, 57—86, 593; Schellbach's solution, III, 44—7; system of equations connected with, IX, 546—50.

Malus, É. L.: systems of rays, XII, 573.

Mannheim, A.: epitrochoid, XIII, 81.

Manuscript of Cayley: facsimile of, on frontispiece, VIII.
Maps: surface representation on plane, VIII, 538—9; colouring, XI, 7—8; projections, XI, 448.
Mars: Newcomb on observations of, IX, 177.
Marsano, G. B.: mathematical tables, IX, 481—3.
Martin, A.: integration, X, 15, 29; probabilities, X, 600; Pellian equation, XIII, 467.
Mascheroni, L.: geometry of the compass, XII, 314—7.
Masères, F.: algebraic equations, IV, 171.
Mathematical Society of London: Sylvester's connexion with, XIII, 45.
Mathematical Tables (*see* Tables).
Mathematics: recent terminology, IV, 594—608; Mill on, XI, 432—4; relation to physics, XI, 444, 449; extent, XI, 449; Sylvester on its relation to music, XIII, 45.
Matrices: notation, II, 185—8, XI, 243; constituents being linear functions of one variable, II, 216—20; unity, II, 477; theory of, II, 475—96, 604, V, 438—48; which satisfy algebraic equation of own order, II, 483; convertible, II, 488; rectangular, II, 494—6; in automorphic linear transformation, II, 497—505; a linear system, II, 604; for reduction of quintic to Jerrard's form, IV, 392; the term, IV, 594, 601—2; Bezoutic, IV, 607; Cayley's method of verification, VIII, XXVII; Cayley the founder of theory, VIII, XXXII—III; transformation of coordinates, XI, 136—42; of order two and the homographic function, XI, 252—7; quaternions, XII, 303, 311, 479; the equation, $qQ-Qq'=0$, XII, 311—3; and theta function transformation, XII, 367—72, 386—9; quintic, XII, 376—80; Sylvester's theory of the corpus, XIII, 47; involutant of two binary, XIII, 74—5; six coordinates of a line, XIII, 76—8; note on a theorem in, XIII, 114; and sixty icosahedral substitutions, XIII, 552—7.
Maurice, F.: variation of arbitrary constants, III, 166, 202.
Maxima: of certain factorial functions, VIII, 548—9.
Maxima and Minima: of functions of three variables, I, 228—9; theorem in, IX, 40—1; theory of *déblais* and *remblais*, XI, 417—20.
Maximum Indicator: VI, 83.
Maxwell, J. C.: contour lines, IV, 609; quartic and quintic surfaces, VII, 246, 252; on Cayley, VIII, XX.
Mean Motion (*see* Motion).
Mechanical Construction: curve tracing, VIII, 179—80, XIII, 515—6; of Cartesians, IX, 317, 535—6; of conformable figures, X, 406.
Mechanics: variation of arbitrary constants in, III, 161—5, 200; construction of conformable figures, X, 406; integrator, XI, 52—4; and time, XI, 444; function in, XI, 522—3; curve tracing, XIII, 515—6.
Mehler, F. G.: attractions of polyhedra, IX, 266.
Memoirs: list of, on theoretical dynamics, IV, 584—593; (*see also Ency. Brit. and* the word desired).
Mention, M.: in-and-circumscribed polygon, IV, 294. 303—5.
Mercator's Projection: VIII, 567, IX, 237—40.
Merrifield, C. W.: letter to, VIII, 517—8.
Mersenne, M.: the cycloid, XI, 447.
Metageometry (*see* Hypergeometry, Prepotentials).
Method of Derivations: Arbogast's, II, 257, IV, 265—71, 272—5, 609, XI, 55; binomial theorem and factorials, VIII, 463—73.
Methyl: trees of, IX, 544—5.
Metrical Geometry: II, 592.
Metrical Theory: pure analytical geometry, XI, 556—7; solid geometry, XI, 570.
Meunier, J. B. M. C.: theorem of, III, 38.
Meyer, F.: history of quantics, VIII, XXXI; orthomorphosis, XIII, 175, 187.
Mill, J. S.: logic and mathematics, XI, 432—4.
Miller, W. J. C.: triangles, V, 566; conics, V, 582; negative pedals of ellipsoid and ellipse, X, 576—7; geometrical interpretation, X, 604.

Milner's Lamp: differential equation and construction of, XIII, 3—5.
Minima (*see* Maxima and Minima).
Minimal Surfaces (*see* Surfaces).
Minor: the term, XI, 496.
Minor Function: in Abel's theorem, XII, 111.
Mirrors: systems of rays, XII, 571—5.
Möbius, A. F.: geometry of position, I, 360; reciprocal figures, I, 415; developable from quintic curve, I, 500; tortuous curves, I, 500; circular relation, III, 118—9, IX, 612—7, XI, 188; cubic curves, IV, 120, XI, 479; in-and-circumscribed triangle, IV, 439—41; cubic curves and cones, V, 401, 551; opposite curves, V, 468; equilibrium of four forces, V, 540—1; coordinates of a line, VII, 66, 93; multiple algebra, XII, 472, 473.
Models: Plücker's, of quartic surfaces, VII, 298—302; Wiener's, of cubic surface, VIII, 366—84.
Modular: the term, XIII, 559.
Modular Equations: in elliptic functions, IX, 117—8, 126—37, 169—75, XII, 507—34; errors in Sohnke's paper, IX, 543; for cubic transformation, XII, 46; quintic transformation, XII, 416.
Modular Functions: system of symbols, IV, 484—9; $\chi\omega$, XIII, 338—41.
Modulus: of transformation, the term, IV, 605; table for any prime or composite, VI, 83—6.
Moment: non-Euclidian geometry, XIII, 481—9.
Moment of Inertia: of solid body, IV, 478—80, 559—66.
Monge, G.: transformation of coordinates, IV, 557; theory of *déblais* and *remblais*, XI, 417—20; descriptive geometry, XI, 448—9; reciprocal polars, XI, 465; biographical notice, XI, 586—8; non-Euclidian plane geometry, XII, 221; differential equation of conic, XII, 393.
Monodromic: the term, XII, 432.
Monogenic Function: XI, 80, 537.
Monoid Surfaces: and curves in space, V, 8, 552; and quintic curves in space, V, 24—30, 552, 553, 613.
Monotropic: the term, XII, 432.
Monro, C. J.: flexure of spaces, X, 331—2.
Montucla, J. F.: on Wallis, XI, 642.
Moon: secular acceleration of mean motion, III, 522—61, 568; (*see also* Lunar Theory, Solar Eclipse).
Morley, F.: topology of chessboard, X, 609—10; systems of circles and spheres, XIII, 13.
Motion: of solid body, I, 28—35, 583; secular acceleration of moon's, III, 522—61, 568; Lagrange's equations of, IX, 198—200; on three-bar, IX, 551—80, 585, XIII, 505—16; Sylvester on recent discoveries in mechanical conversion, XIII, 44; (*see also* Dynamics, Elliptic Motion, Kinematics, Lunar Theory).
Motte, A.: problem of tactions, XIII, 151.
Moulton, J. F.: matrices, XI, 256.
Mountains: altitude, and roots of algebraic equation, XIII, 33—7.
Mourey, C. V.: imaginaries, XII, 468.
Mousetrap: the game of, III, 8, X, 256—8.
Moutard, T.: quartic surfaces, VII, 246, VIII, 262.
Muir, T.: history of determinants, I, 581; problem of arrangements, X, 249—51; elimination, XIII, 545—7.
Multiform Series: defined, IV, 456.
Multilinear Operator of MacMahon: XIII, 399.
Multiple Algebra: on, XI, 446, XII, 60—71, 459—89; associative imaginaries, XII, 105—6.
Multiple Integrals: and attractions, I, 5—12, 13—8, 195—203, 204—6, 285—9, 438—44, 581, 586, II, 35—9; demonstration of a theorem of Boole, I, 384—7, 588.
Multiple Sines: X, 1—2.
Multiple Theta-functions (*see* Theta-functions).
Multiplication: of elliptic functions, I, 534—9, 568—76, 589, IX, 138—47, XII, 507; of determinants,

XI, 495; of extraordinaries, XII, 461—2; complex, in elliptic functions, XII, 556—7; (*see also* Transformation).

Multiplier: Jacobi's theory of, I, 276, 279; theory of, in differential equations, X, 102—6; in elliptic integrals, X, 139.

Multiplier Equations: in elliptic functions, IX, 138—47, XII, 507.

Murdoch, P.: *Newtoni Genesis Curvarum per Umbras*, V, 284, 288; curve classification, V, 354; cubic curves and cones, V, 402; the simplex cubical parabola, VI, 101.

Murphy, H.: four points in plane or space, VII, 585.

Murphy, R.: Legendre's coefficients, I, 376.

Music and Mathematics: Sylvester on, XIII, 45.

Napier, J.: logarithms, XI, 447.

Natani, L.: Pfaffian equations, IV, 515.

Nature: notice on Sylvester, XIII, 43—8.

Neg: the abbreviation in groups, XIII, 119.

Negative: the rule of signs, IV, 595—6, XI, 492.

Negative Deficiency: VIII, 397.

Neptune: Newcomb's astronomical work, IX, 180—4.

Neutral: the term, VI, 101.

Newcomb, S.: astronomical work of, IX, 176—84.

Newton, Sir I.: cubic curves, IV, 122; rectilinear motion, IV, 515, 590; central forces, IV, 515, 590; parabolas, V, 284; curve classification, V, 354, 364—6, 396—9; diameter, V, 362; cubic curves, V, 401, 551, XI, 464; conics, V, 562; forms of cubical parabolas, VI, 101; theorem as to roots of equations, X, 5; *Principia*, XI, 447—8; branches of curves, XI, 477; theory of equations, XI, 500, 502; roots of algebraic equation, XIII, 35; Sylvester's work at rule of, XIII, 46; tactions, XIII, 151—69; Newton-circle, the term, XIII, 152.

Newton-Fourier Theorem: imaginary roots, X, 405—6, XI, 143, XIII, 36; extension to complex variables, X, 405—6; theory of equations, XI, 114—21, 122.

Nexal: the term, VIII, 73.

Nil: the term, XII, 66.

Nilfactum: and quantic, VI, 466.

Nilpotent: the term, XII, 61.

Nilvalent: the word, IX, 202.

Nine-point Circle: XIII, 517—9, 520—1, 548—51.

Nitrogen: tree of, IX, 430.

Nivellators: Sylvester's theory of, XIII, 47.

Nodal: the term, VII, 244.

Nodal Anallagmatic: the term, VIII, 67.

Nodal Bicircular Quartic: mechanical description, VII, 182—8.

Nodal Cubic: VI, 171—4, VII, 555.

Nodal Curves: of developable from quartic, V, 135—7; of cubic surface, VI, 450; centro-surface of ellipsoid, VIII, 332—52.

Nodal Director: the term, V, 169—70.

Nodal Generator of Scrolls: V, 169—70, 179—81.

Nodal Isochronic: the term, VII, 473.

Nodal Locus: in singular solutions, VIII, 533.

Nodal Quartic: defined, V, 10; mechanical description of bicircular, VII, 182—8; and hyperelliptic functions, XII, 196—208.

Nodal Quartic Surfaces (*see* Quartic Surfaces).

Nodal Residue of Scrolls: v, 169—70, 181—3, 184, 187.
Nodal Total of Scrolls: v, 169—70, 183—9.
Node-couple: defined, II, 29, IV, 22, XI, 227; curve, and plane, and torse, VI, 355, 582—5; torse, VI, 601.
Node-cusp: v, 265—6, 618.
Node-form: the term, VII, 274.
Nodes: the term, II, 28, IV, 22, 27, 181, v, 295, XI, 468; elimination of, in three bodies, IV, 551, v, 23; number on quartic surface, VII, 133—81; quartic surface with twelve, XIII, 1—2.
Node-tangent: defined, II, 29—32.
Node-triplet: the term, II, 30.
Nodo-focus: of bicircular quartic, VI, 522—3, 523—6; the term, IX, 264.
Non-commutative Algebra (*see* Algebra).
Non-Euclidian Geometry, Memoir on: introduction, XIII, 480—1; geometrical notions, XIII, 481—9; point, line, and plane coordinates, general formulæ, XIII, 489—91; the absolute, XIII, 491—5; distance of a point and line, XIII, 495—7; distance of a plane and line, XIII, 497; theory of two lines, XIII, 497—504.
Non-Euclidian Geometry: VIII, 409—13, XII, 220—38; (*see also* Hypergeometry).
Non-facultative Space: VI, 156.
Non-scrolar Surfaces: quartic and quintic, VII, 245.
Non-unitariants: the term, XIII, 265.
Non-unitary Symmetric Functions: and seminvariants, XII, 239, 275, XIII, 267—98; tables, XII, 273—4.
Norm: and polyzomal curves, VI, 474, 573—5.
Normal Elementary Integral: of differential equation, XII, 396—7, 444.
Normals: in *Ency. Brit.*, XI, 564—5; (*see also* Conics).
Normal Variables: in dynamics, IX, 111.
Notation: algebraic functions, II, 185—8; matrices, II, 185—8; quantics, II, 223; for disturbing function compared, III, 310—8; quantics and abstract geometry, VI, 464—6; differential equations, X, 95—7; for double theta functions, X, 497; theta functions, XI, 47—9, 243—5; umbral, XIII, 301—6.
Nöther, M.: curves in space, v, 613—7; rational transformation, VII, 255; deficiency of surfaces, VIII, 395; sextic curve, IX, 504—7; classification of curves, XI, 451; Abelian function, XII, 149.
Novel-reading: at Cambridge by Cayley, VIII, x—xi, xxiii.
Nullity: Sylvester's theory of, XIII, 47.
Number: time, and space, v, 292, 620, XI, 442—4; theory of equations, XI, 502.
Numbers: a theorem of Lejeune-Dirichlet's, II, 47—8; tables of binary cubic forms, VIII, 51—64; use of Bernoulli's, in analysis, IX, 259—62; arrangements of, X, 570; Sylvester and Hammond on Hamiltonian, XIII, 48; (*see also* Partition of Numbers).
Numbers, Theory of, in *Ency. Brit.*: XI, 592—616; ordinary and complex theories, XI, 592—3; ordinary theory, XI, 594—609, 615—6; theory of forms, XI, 604—9; complex theories, XI, 609—16.
Numbers, Theory of: Pellian equation, IV, 40—2, IX, 477—8, XI, 615, XIII, 430—67; composition of, IV, 70—1, 78—9; specimen table, VI, 83—6; $x^p-1=0$, trisection and quartisection, XI, 84—96; $x^p-1=0$, and quinquisection, XI, 314—6, XII, 72—3; H. J. S. Smith on, XI, 429; imaginaries, XI, 444—5; evolution, XI, 455—6; Wilson's theorem, XII, 45; Sylvester on, XIII, 47; (*see also* Partition of Numbers).
Numerative Geometry: Schubert's, XI, 281—93.
Numerical Equations: X, 3—6.
Numerical Expansions: IV, 470—2.
Numerical Generating Function: X, 339, 408.
Nutation: note on theory of, IX, 194—6.

Obliquity: the term, XIII, 234.
Observations (*see* Orbits, Planet's Orbit, Solar Eclipse).
Octacron: enumeration of polyhedra, V, 38—44.
Octad: the term, I, 586, VII, 133, 152, XII, 590.
Octadic-quartic Surfaces: X, 51.
Octagon: theorem of eight points on a conic, VIII, 92—4.
Octahedral Function: XI, 128—9.
Octahedron: axial system, V, 531—9; automorphic function for, XI, 169, 179—83, 212—6.
Octaves: elliptic functions, I, 127, 586; imaginaries, I, 301.
Octavic Surface: VIII, 401—3.
Octic Function: and Abelian function, XI, 483.
Octics: and twisted cubics, XII, 310.
Octic Surface: on a sibi-reciprocal, X, 252—5; (*see also* Surfaces).
Octo-dianome: the term, VII, 134.
Octo-hexahedron: the term, X, 328.
Odd Branch of Curve: X, 36.
Off-planes: the term, VI, 330, 577, 583—5.
Off-points: the term, VI, 330, 338, 577, 583—5.
Olbers, W.: orbits of asteroids, IX, 177.
Olivier, T.: conics inscribed in quadric surface, I, 557.
Omal: the word, VI, 194, 463, 467—9.
Omega Functions: the term, XI, 453; note on Smith's memoir, XIII, 558—9.
Omphali: the term, VIII, 326.
Operandator: defined, III, 242.
Operations: and substitutions, XIII, 530.
Operators: differential, VII, 8; and seminvariants, XIII, 322—32; MacMahon's multilinear, XIII, 399.
Optics: MacCullagh's theorem in polarized light, IV, 12—20; geometrical construction in, X, 28.
Orbits: Jacobi's canonical formulæ for disturbed motion, III, 76—7; reduction to fixed plane, III, 91—6; variation in plane of planet's, III, 516—8; central forces problem, IV, 516—21; position of, in planetary theory, VII, 541—5; of asteroid, and Newcomb, IX, 176—7; Hamiltonian equations of central, X, 613; Sylvester's work at, XIII, 47; (*see also* Planetary Theory, Planet's Orbit).
Order: of system of equations, I, 457—61, 589; of quantics, defined, II, 221; of curve, II, 569—83, XI, 462; in abstract geometry, defined, VI, 463; of curve and surface, XI, 629.
Ordinary Point for Differential Equations: XII, 394.
Oriani, B.: elliptic motion, III, 474, IV, 528.
Orr, W. McF.: tetrads of circles, XIII, 425.
Orthocentre: the term, XIII, 550.
Orthogonal Surfaces: VIII, 269—91, 292, 569—70; Smith's Prize dissertation, VIII, 558—63.
Orthogonal Surfaces and Curvature, Memoir on: VIII, 292—315; introductory, VIII, 292—3; curvature of surfaces, VIII, 293—300; conormal correspondence of vicinal surfaces, VIII, 301—8; condition that the two surfaces may belong to orthogonal system, VIII, 309—11; family of surfaces, VIII, 312—5.
Orthomorphic Transformation (*see* Orthomorphosis).
Orthomorphosis: of circle into parabola, V, 618, XII, 328—36; of a circle into itself, XIII, 20; general theory, XIII, 170—90; some problems, XIII, 191—205; note on theory, XIII, 418—24; (*see also* Conformal Representation).
Orthotomic: the term, IX, 13.
Orthotomic Circles: and polyzomal curves, VI, 501; and Jacobians, VI, 568.
Orthotomic Curve: of a system of lines in a plane, XIII, 346—7.
Orthotomic Surfaces: in *Ency. Brit.*, XI, 637—8.

Oscnode: defined, II, 28—32.
Oscular: the term, VI, 334, 361, 362.
Ostrogradsky, M. A.: dynamic equations, III, 186, 203; transformation of differential equations, IV, 514; virtual velocities, IX, 207.
Outcrops: the term, VIII, 326, 351.
Oval Chuck for Quartic Curves: VIII, 151—5.
Ovals: of Descartes, I, 479, II, 118, 336, III, 66; and quartic curves, V, 468—70; twice-indented, X, 318; and functions, XI, 540; in *Ency. Brit.*, XI, 549—51; roots of algebraic equations, XIII, 37; Sylvester's work at, XIII, 47; orthomorphosis, XIII, 185—6, 202.
Oxygen: trees of, IX, 427—60.

Π: Wallis's investigation for, XIII, 22—5.
Pagani, G. M.: central forces problem, IV, 520, 590; motion of solid body, IV, 583, 590.
Painvin, L.: last multiplier, IV, 551, 590.
Parabola: inflexions of cubical divergent, V, 284—8, VI, 101—4; classification, V, 356, 395, VI, 101; line and circle, problem, V, 607; polyzomal curves, VI, 542; cubic curves, XI, 478; in *Ency. Brit.*, XI, 548—51, 561—4; orthomorphosis of circle into, XII, 328—36; and epitrochoid, XIII, 86—7; orthomorphosis into circle, XIII, 421—2.
Parabolic Cyclide: IX, 73—8.
Paraboloids: in *Ency. Brit.*, XI, 576—9.
Paradox: the d'Alembert-Carnot geometrical, XII, 305—6.
Paraffins: trees of, IX, 427—60.
Parallel Curves: envelopes and surfaces, IV, 123—33, 152—7, 158—65; and evolutes, VIII, 31—5; theory of, X, 260; the critic in solar eclipses, X, 311—5.
Parallels: and non-Euclidian geometry, XIII, 480—1, 481—9; the terms right and left, XIII, 488, 502.
Parallel Surfaces: of paraboloid, VIII, 7; of ellipsoid, VIII, 9, IX, 591; in *Ency. Brit.*, X, 637—8.
Parametric Class and Order: of systems of cones, V, 552.
Parametric Latitude: VII, 16, IX, 238.
Parametric Relation: VI, 463—4; of triple orthogonal system, VIII, 292—315.
Parazome: the word, VI, 477.
Partial Differential Equations: integral of, III, 166; system of, VIII, 517—8; Jacobi's, in transformation of elliptic functions, XII, 530—3; on a, XIII, 358—61.
Particle: under central forces, X, 575; (*see also* Dynamics).
Partition of Numbers: II, 218, 235—49, V, 48; and quantics, II, 265; supplementary researches, II, 506—12; a problem in, III, 247—9; tactical, V, 294, XI, 443.
Partitions: conjugate, due to Ferrers, II, 419; formulæ in, III, 36—7; problem of double, IV, 166—70; of a close, V, 62—5, 617; problems, VII, 575, X, 611, XI, 61—2; tables, IX, 480—3, XI, 357—64; theorems in trigonometry and, X, 16; in *Ency. Brit.*, XI, 589—91; note on a partition-series, XII, 217—9; non-unitary partition, XII, 273—4; Sylvester's constructive theory of, XIII, 47; of a polygon, XIII, 93—113; and seminvariants, XIII, 269.
Pascal, B.: hexagram of, I, 356; limaçon of, I, 480; some theorems of geometry of position, I, 550—6; lines of, I, 551, 588; curves, XI, 447; inscribed hexagon, XI, 556.
Pascal's Theorem: intersection of curves, I, 25—7; demonstration, I, 43—5; Chasles' form of, I, 45; on, I, 322—8, 414, VI, 129—34, 594; generalized, V, 4; notation of points and lines, VI, 116—22, 594.
Peacock, G.: multiple algebra, XII, 460, 467, 469, 470—1.
Peaucellier, A.: mechanical construction of Cartesian by his cell, IX, 317; cell of, and scalene transformation, IX, 527—34; Sylvester on his discoveries, XIII, 44.
Pedal Curves: Maclaurin on, V, 113—4.

Peirce, B.: orbit of Neptune and Uranus, IX, 180, 182; multiple algebra, XI, 457—8, XII, 60—71; associative algebras, XII, 106, 459, 465; imaginaries, XII, 303.
Pellian Equation: IV, 40—2; tables, IX, 477—80; and theory of numbers, XI, 615; report of committee on, XIII, 430—67.
Pencils: defined, II, 577; homography of, II, 578; intersections of four- and two- lined, V, 484—6; of six lines and cubic curves, VI, 105—15, 593—4.
Pendulum, Spherical: IV, 532—4, 535—7, 541.
Peninvariants: and seminvariants, IV, 241; the term, IV, 606.
Pentagon: a theorem relating to, I, 318—9; Gauss's *Pentagramma Mirificum*, VII, 37—8; Schröter's construction of regular, XII, 47; (*see also* Polygons).
Pentagramma Mirificum: of Gauss, VII, 37—8.
Pentagraph: illustrating a function, XI, 440; curve tracing, XIII, 515—6.
Penultimate Forms: of curves, VIII, 258—61; of surfaces, VIII, 262—3.
Penultimate Quartic Curve: VIII, 526—8.
Penumbral Curve (*see* Solar Eclipse).
Periodic Functions: the term, XI, 529; (*see also* Doubly Periodic Functions).
Periods: of elliptic integrals, IX, 618; of theta functions, X, 467—9.
Permissive Points: in differential equations, XII, 434—41.
Permutants: theory of, II, 16—26, 27; defined, II, 17, IV, 594, 596, 600; Sylvester on, II, 26—7.
Permutations: theory of, I, 423—4; idea of group, II, 124; problem of geometric, V, 493—4; commutants, V, 495—7; colours on faces of polyhedra, V, 539; (*see also* Arrangements, Combinatory Analysis).
Perott, J.: binary quadratic forms, V, 618.
Perpendicular: in non-Euclidian geometry, XIII, 480—1, 481—9.
Perpendicularity: and imaginaries, XII, 466—72.
Perpetuants: and seminvariants, XII, 250—7, XIII, 266; sextic, XII, 257—62; reducible seminvariants, XIII, 308—13; Strohian theory of, XIII, 314—8.
Perspective: of triangles, III, 5; five points in a plane, V, 480—3; theory, XI, 442.
Pfaff, J. F.: problem, IV, 359—63.
Pfaffian Differentials: XIII, 361, 405—14.
Pfaffian Equations: Natani and Clebsch, IV, 515.
Pfaffians: and skew determinants, I, 411, II, 203; the term, I, 589, II, 19, IV, 594, 600; and differential equations, X, 96—7, 106.
Pfaff-invariants: XIII, 405—14.
Physics: relation to mathematics, XI, 444, 449; (*see also* Dynamics, Electricity, Light).
Picard, É.: integral calculus, XII, 408—11.
Pinch-planes: the term, VI, 330, 335, 583—5, X, 53—6.
Pinch-points: the term, VI, 123, 330, 335, 582—5, X, 53—6, XI, 227.
Pineto, S.: tables of logarithms (review), VIII, 95—6.
Pippian: defined, I, 586, II, 381, 397—400, 400—3; and Hessian, II, 383—95; geometrical definition, II, 416.
Pirie, B.: inertia, IV, 564.
Plana, G. A. A.: lunar theory, III, 536, 568, VII, 357—60, 361—6, 367—70, 371—4, 375—6; distribution of electricity, IV, 92, 100—7.
Planar Developables: the term, I, 505.
Planarity: of developables, V, 517.
Plane Curves (*see* Curves, Plane).
Plane-integral: prepotential, IX, 337—43.
Plane of Orbit: variation in, III, 516—8.

Plane Representation of Solids: VII, 26—30.

Planes: diametral, of quadric surface, I, 255—8; point and line defined, II, 561—2; geometry of two dimensions, II, 569—83; MacCullagh's theorem of polar, IV, 12—20; lines and dots of cubic surfaces, VI, 365—6, 373—449; rational transformation, VII, 197—213, 216—9; quadric transformation, VII, 213—6; also lineo-linear, VII, 215—6; determined by point and three lines, VII, 571; flecnodal, of a surface, X, 262—4; kinematics of, XI, 103—10, XIII, 505—16; in *Ency. Brit.*, XI, 571—2; osculating and normal, XI, 579—80; and surface, XI, 629; non-Euclidian geometry, XIII, 481—504; and line distance, XIII, 497.

Planetary Theory: Desboves', III, 185, 203; development of disturbing function, III, 319—43, VII, 511—27; variation in plane of orbit, III, 516—8, VII, 541—5; theorem of Jacobi, III, 519—21; Newcomb's work, IX, 180—4.

Planets: angular distance of two, VII, 377—9.

Planet's Orbit from Three Observations: VII, 384—6, 400—78; introductory, VII, 400—1; the general theory, VII, 401—6; determination of orbit from given trivector, VII, 406—12; time formulæ, Lambert's equation, VII, 412—5; formulæ for transformation between two sets of rectangular axes, VII, 415—7; intersection of orbit plane by single ray, VII, 417—26; trivector and orbit, VII, 426—8; special symmetrical system of three rays, VII, 428—9; Planogram No. 1, meridian 90°—270°, VII, 429, 430—40; No. 2, meridian 0°—180°, VII, 429, 441—51; No. 3, orbit-pole at point *A*, VII, 429, 452—4; No. 4, orbit-pole in ecliptic, VII, 429, 455—9; No. 5, orbit-pole on a separator, VII, 429, 459—67; spherogram and isoparametric lines, VII, 467—8; *e*-spherogram and iseccentric lines, VII, 468—70; time-spherogram and isochronic lines, VII, 470—7.

Planogram: the term, VII, 404; three plates, VII, to face 478; meridian 90°—270°, VII, 429, 430—40; 0°—180°, VII, 429, 441—51; orbit-pole at point *A*, VII, 429, 452—4; in ecliptic, VII, 429, 455—9; on separator, VII, 429, 459—67.

Plates (*see* Diagrams, *also* Tables).

Plato: and geometry, XI, 446.

Playfair, J.: on twelfth axiom of Euclid, XI, 435.

Plerogram: the term, IX, 202.

Plexus: the term, IV, 603, VI, 458; Sylvester's term, XIII, 46.

Plücker, J.: theory of algebraic curves, I, 53, 54; curves and developables, I, 207, 208, 210, 586—7; involution, I, 259, 261; elimination, and theory of curves, I, 344; geometry of position, I, 356, 553—6; geometrical reciprocity, I, 380; reciprocal figures, I, 418; quadric surfaces, I, 421; cubic surfaces, I, 446; transformation of curves, I, 478; singularities of plane curves, I, 586, V, 520—2, 619, XI, 450; cubic curves and cones, IV, 173—8; double tangents, IV, 186; points of six-pointic contact on cubic, IV, 207; cubic curves, IV, 495, 617, V, 402; line geometry, IV, 616—8; hyperboloid coordinates, V, 72; node-cusp, V, 265—6; curve classification, V, 354—400; numbers for singularities of plane curves, V, 424, 476, 517; higher singularities of plane curves, V, 426, 619; pencil intersections, V, 484; numbers of, VI, 68, VIII, 41—5, XI, 469—73; species of cubical parabola, VI, 101; focus, VI, 515, XI, 481; six coordinates of a line, VII, 66; quartic surface models, VII, 298—302; construction of a conic, VII, 592; hypergeometry, VIII, XXXV; theory of curve and torse, VIII, 74, 75—6, 80—1; theory of curves, XI, 467; envelopes, XI, 475—6; note on equations of, XIII, 536.

Pohlke, K.: theorem in axonometry, IX, 508.

Poincaré, H.: lacunary functions, XIII, 415.

Poinsot, L.: polygons and polyhedra, IV, 81—5, 86—7, 609; inertia, IV, 563, 590—1; rotation of solid body, IV, 571—3, 577, 591; kinematics of solid body, IV, 580, 581, 591.

Point: of cubic curve, II, 382; satellite, II, 383; formulæ, II, 405—9; theorems, II, 409—12; plane and line defined, II, 561—2; and ineunt of a curve, II, 574; lattice, III, 40; distances of, from triangle and formulæ, IV, 510—2; tritom, V, 138; the term polar of, V, 570; and abstract

geometry, VI, 458; potential of, IX, 278—80; singularities of curves, XI, 468; coordinates of, as functions of parameter, XII, 290—1; and line distance, XIII, 495—7; two-way, XIII, 507; for-forwards, and back-backwards, XIII, 510.

Point-pairs: the term, II, 564—5, VI, 202, 206—7, 208, 210, 269, 594—5; degenerate forms of curves, XI, 218.

Points: distances of, I, 1—4, 581; some theorems in geometry of position, I, 317—28; of inflexion, I, 345—9, 354; of osculation, I, 349—51; harmonic relation of two, II, 96—7; of cesser, defined, IV, 130; critical defined, IV, 130; five in a plane, V, 480—3; correspondence on plane curve of, V, 542—5; and circle, problem, V, 560; correspondence of two on a curve, VI, 9—13, 264—8, VII, 39; notation of, in Pascal's theorem, VI, 116—23; abstract geometry, VI, 463; consecutive, VI, 467—9; system of 16, and polyzomal curves, VI, 501—3, 504—5; problem of random, VII, 585; problem and solution of four in plane or space, VII, 585; four and conic, VII, 587; on particular sextic curve, IX, 504—7; branch- and cross-, X, 317; and lines, problem and solution, X, 570; on a circle, function of, XI, 130; double- and pinch-, XI, 227; Mill on, XI, 432—3; representation on plane, XI, 442; evolution theory of curves, XI, 450—1; at infinity, XI, 464; relation between the distance of five in space, XII, 581—3; analytical formulæ in regard to octad of, XII, 590—3; Sylvester's facultative, XIII, 46; non-existence of a special group, XIII, 212; syzygetic relations, XIII, 224—7; non-Euclidian geometry, XIII, 480—504; coordinates of, and non-Euclidian geometry, XIII, 489—91; (*see also* Orthomorphosis).

Point-systems: and one-dimensional geometry, II, 563—9, 583—86; and two-dimensional geometry, II, 569—83, 586—92.

Poisson, S. D.: attraction of ellipsoids, III, 155; planetary theory, III, 159, 201; variation of arbitrary constants in mechanical problems, III, 163—5, 200, 201, 202; coefficient (a, b) of, III, 163; Hamilton's method of dynamics, III, 173—4, 200; integration of differential equations, III, 180; distribution of electricity, IV, 92—5, 100—7, X, 299, XI, 1; elliptic motion, IV, 522; relative motion, IV, 535, 591; motion of projectile, IV, 541, 591; inertia, IV, 563, 591; rotation of solid body, IV, 566, 569, 573, 591; rotation round fixed point, IV, 582, 591; motion of solid body, IV, 583, 591; attraction of ellipsoidal shell, IX, 302; Jacobi's theorem, X, 108—9, 110—3.

Polar: of point, V, 570, X, 54, XI, 465.

Polar Conjugate: of curve of third class, II, 383.

Polar Reciprocal: I, 230, 378, 416.

Polarization: MacCullagh's theorem, IV, 12—20.

Poles: conjugate, of cubic curve, II, 382—5; two-dimensional geometry, II, 579—83, 586—92; the term, XI, 465.

Pollock, Sir F.: on circumscribed triangle, III, 29—34.

Poloid Curve: IV, 571—2.

Polyacra: triangle-faced, and enumeration of polyhedra, V, 38—44.

Polygons: in-and-circumscribed, II, 87—9, 91—2, 138—44, 145—9, IV, 292—308, 435—41, V, 21—2, VIII, 14—21, 212; partitions of close-, V, 62—5, 617; and triangles, problem, V, 589; potential of, IX, 266—80; automorphic function for, XI, 169, 179—83, 212—6; partitions of, XIII, 93—113.

Polyhedra: Poinsot's four new regular solids, IV, 81—5, 86—7, 609; the problem of, IV, 182—5, 609; autopolar, IV, 185; enumeration of, and triangle-faced polyacra, V, 38—40; partitions of close-, V, 62—5, 617; axial properties, V, 529—39; potential of, IX, 266—80.

Polyhedral Functions (*see* Hypergeometric Series, Schwarzian Derivative).

Polyzomal Curves, Memoir on: VI, 470—576, VII, 115; introductory, VI, 470—2; Part I, polyzomal curves in general, VI, 473—97; definitions and preliminary remarks, VI, 473—4; the branches, VI, 474—6; points common to two branches, VI, 476—8; singularities of a ν zomal, VI, 478—9; zomals with common point or points, VI, 479—81; depression of order of ν zomal curve from

ideal factor of branch or branches, VI, 481—5; the trizomal and tetrazomal, VI, 485; intersection of two ν zomals having same zomal curve, VI, 486—7; theorem of decomposition of tetrazomal, VI, 487—9; application to trizomal, VI, 489—94; tetrazomal curve, VI, 494; variable zomal of trizomal curve, resumed, VI, 494—7; Part II, subsidiary investigations, VI, 497—515; preliminary remarks, VI, 497—8; circular points at infinity; rectangular and circular coordinates, VI, 498—9; antipoints; definition and fundamental properties, VI, 499—500; antipoints of circle, VI, 500; antipoints and pair of orthotomic circles, VI, 500; forms of equation of circle, VI, 501; system of 16 points, VI, 501—3; property in regard to four confocal conics, VI, 503—4; system of sixteen points, the axial case, VI, 504—5; involution of four circles, VI, 505—8; locus connected with foregoing, VI, 508—9; formulæ of two sets each of four concyclic points, VI, 509—11; ditto further properties, VI, 512—5; Part III, theory of foci, VI, 515—34; the general theory, VI, 515—7; foci of conics, VI, 517—9; variable zomal applied to conic, VI, 519—21; foci of circular cubic and bicircular quartic, VI, 521—2; centre of circular cubic, and nodo-foci, etc., of bicircular quartic, VI, 522—3; circular cubic and bicircular quartic; symmetrical case, VI, 523; ditto, singular forms, VI, 523—6; analytical theory for circular cubic, VI, 526—8; ditto, for bicircular quartic, VI, 528—30; property that points of contact of tangents from pair of concyclic foci lie in a circle, VI, 530—34; Part IV, trizomal and tetrazomal curves where the zomals are circles, VI, 534—66; the trizomal curve-tangents at *I*, *J*, etc., VI, 534—7; foci of conic represented by equation in areal coordinates, VI, 537; theorem of variable zomal, VI, 539—41; relation between conic and circle, VI, 541—2; case of double contact, Casey's equation in problem of tactions, VI, 543; intersections of conic and orthotomic circle on set of four concyclic foci, VI, 543—4; construction of symmetrical curve, VI, 544—6; focal formulæ for general curve, VI, 547; circular cubic, VI, 548—9; focal formulæ for symmetrical curve, VI, 549; symmetrical circular cubic, VI, 549—50; general ditto, VI, 550—3; transformation to new set of concyclic foci, VI, 553; tetrazomal curve, decomposable or indecomposable, VI, 553—4; cases of indecomposable, VI, 554—5; ditto, centres being in line, VI, 555—6; the decomposable curve, VI, 556—7; ditto, centres not in a line, VI, 557—61; ditto, centres in a line, VI, 561—5; ditto, transformation to a different set of concyclic foci, VI, 565—6; theory of Jacobian, VI, 566—8; Casey's theorem for circle touching three given circles, VI, 568—73; a norm when the centres are in line, VI, 573—5; trizomal curves with cusp or two nodes, VI, 575—6.

Poncelet, J. V.: harmonic relations, II, 96; porism of in-and-circumscribed triangle, III, 80—5; rectangular hyperbola, III, 254; in-and-circumscribed polygon, V, 21—2; reciprocal polars, XI, 466.

Pontécoulant, G. de: *Système du Monde*, III, 309—10; *Lunar Theory*, III, 521, VII, 357.

Porism: homographic, defined, III, 74, 84; allographic defined, III, 75, 85; of polygon and correspondence, IX, 94.

Porism of in-and-circumscribed Polygon: II, 87—9, 91—2, 97, 138—44, 145—9, IV, 292—308, VIII, 14—21, 212.

Porism of in-and-circumscribed Triangle: II, 56, 87—90, 91, III, 67—75, 80—5, 229—41, V, 549—50, 579, VIII, 212—57.

Portraits of Cayley: frontispiece to vols. VI, VII, XI.

Pos: the abbreviation in groups, XIII, 119.

Positive: the rule of signs, IV, 595—6, XI, 492.

Postulandum of Curve: the term, I, 583, VII, 140, XII, 501; and capacity, XIII, 115.

Postulation: the term, I, 583, VII, 140, 225, VIII, 394; of curve, XII, 501.

Potential: and attractions, I, 195.

Potentials: of polygons and polyhedra, IX, 266—80; of ellipse and circle, IX, 281—301; Smith's Prize question on, XI, 261—4.

Potential-solid: prepotential, IX, 346—7.

Potential-surface: prepotential, IX, 343—6.

Potenzkreis of Steiner: III, 113.

Power: of a matrix, II, 492—4; of homographic function, XI, 253—7.
Power-enders: the term, XIII, 267, 270, 295; and reciprocants, XIII, 333.
Powers: successive, of homographic function, X, 305—6, 307—9; of roots of algebraical equations, XII, 33—4.
Precession: note on theory of, IX, 194—6.
Prepotentials, Memoir on: IX, 318—423; introductory, IX, 318—21; prepotential plane, theorem A, IX, 319, 337—43; potential surface, theorem C, IX, 320, 343—6; potential solid, theorem D, IX, 320, 346—7; the prepotential surface integral, IX, 321—30; its continuity, IX, 330—4; potential solid integral, IX, 334—7; examples of foregoing, IX, 347—50; surface and volume of sphere, IX, 351—2; integral, IX, 352—9; prepotentials of uniform spherical shell and solid sphere, IX, 359—79; examples, theorem A, IX, 379—93; Green's integration of prepotential equation, IX, 393—404; examples, theorem C, IX, 404—7; examples, theorem D, IX, 407—8; prepotentials of homaloids, IX, 408—9; Gauss-Jacobi theory of epispheric integrals, IX, 410—7; methods of Lejeune-Dirichlet and Boole, IX, 417—23.
Prepotentials: Smith's Prize question, XI, 261.
Presidential Address: to British Association, XI, 429—59.
Prime Numbers: B.A. report on tables of, IX, 462—70.
Prime Roots: tables, IX, 471—7.
Prn: the abbreviation for tortuous curves, XIII, 253.
Principal System of Sextic Curve: VII, 236—8.
Principia: solution of problem, Book I, Sect. V, Lemma XXVII, VII, 30; (*see also* Newton).
Principiants: and reciprocants, XIII, 388—98.
Probabilities: questions in theory of, II, 103—4, 594—8, V, 80—5, X, 588, 600—1, 614.
Problems: mechanical, III, 78—9; a class of dynamical, continuous impact, IV, 7—11; (*see also* Dynamics, Smith's Prize Papers, Three Bodies).
Problems and Solutions from the Educational Times: V, 560—612; table of contents, V, 612; VII, 546—607; table of contents, VII, 607—8; X, 566—614; table of contents, X, 615—6; (*see also* Smith's Prize Papers).
Product: resolvent, IV, 309—13.
Product-theorem: for theta functions, X, 464, 474, 509—46.
Progress of Theoretical Dynamics (*see* Dynamics).
Prohessians: defined, V, 267; and developables, V, 513—4.
Prohibitive Points: in differential equations, XII, 434—41.
Projectile: effect of resisting medium, IV, 541.
Projection: stereographic, of spherical conic, V, 106—9; of ellipsoid, V, 487—8; plane representation of solid figure, VII, 26—30; stereographic, VII, 397—9, XI, 187—9, 569; blank, VII, 482; of surface on plane, VIII, 538—9; Mercator's, VIII, 567; Mercator's, of skew hyperboloid of revolution, IX, 237—40; a problem of, IX, 508—18; map, XI, 448.
Prolusions: Sylvester's Astronomical, XIII, 47.
Provectant: defined, II, 514.
Provector: defined, II, 514.
Pseudosphere: the term, XII, 220.
Ptolemy: stereographic projection, XI, 448.
Puiseux, V.: algebraic functions, III, 225; spherical pendulum, IV, 533, 591; motion of a body, IV, 583, 591.
Pyramid (*see* Polygons, Polyhedra).

Quadrangle: in-and-circumscribed, IV, 307—8; differential relation between sides, X, 33—5.
Quadratic Equation: roots, V, 160—1; and geometrical interpretation, XI, 258—60.
Quadratic Form: composition, I, 532; transformation of, into itself, II, 192—201, 215; tables, V, 141—56, IX, 480—3, 486—93.

Quadratic Residues: Eisenstein's geometrical proof, III, 39—43.
Quadratic Transformation of Binary Form: VIII, 398—400; (*see also* Transformation).
Quadratics: resultant of three ternary, and invariant of biternary, IV, 349—58.
Quadric Cones: of six given points, V, 4—6; through given points, X, 575.
Quadric Curves: V, 70—2.
Quadric Equations: transformation of two, I, 428—31, III, 129—31; automorphic linear transformation of, II, 497—505; solution by radicals, X, 9; two related, XI, 37.
Quadric Integral: due to Aronhold, XII, 162—9.
Quadricone: the term, VI, 334, 585, VII, 264.
Quadricovariant: of quantic, II, 520; or Hessian, II, 545; the term, IV, 606.
Quadrics: through nine points, I, 425—7; developable from two, I, 486—95; homographic transformation into itself, II, 105—12, 117, 133—7; theorem on surfaces, III, 115—7; equation of differences for, IV, 242; the term, IV, 604; sections of, V, 133—4; through three lines, VII, 177; in hyperdimensional space, IX, 79—83; covariants of, IX, 537—42; envelope of family of, X, 589; correspondence of confocal Cartesians with right line of a hyperboloid, XII, 587—9; (*see also* Binary Quadrics.
Quadric Seminvariants: generating functions of, XIII, 306.
Quadric Surfaces: n-dimensional geometry, I, 62; diametral planes of, I, 255—8; centres of similitude, I, 329—31; note, I, 421—2, 589; abstract of memoir by Hesse, I, 425—7; conics inscribed in a, I, 557—63; envelope of certain, VIII, 48—50; inversion, VIII, 67—71; problem, and hypothetical theorems, VIII, 550; and four-dimensional space, IX, 246—9; a system of, X, 269; Jacobian of, X, 568; in *Ency. Brit.*, XI, 576—9, 632; twisted cubics on, XII, 307—10; focals of, XIII, 51—4; reciprocal lines, XIII, 58—9; (*see also* Geodesic Lines).
Quadric Transformation: between planes, VII, 213—5, 219—21, XII, 100—1; (*see also* Transformation).
Quadri-cubic Curves in Space: V, 16.
Quadricuspidal: the word, VII, 51.
Quadrifactions: the term, IX, 426.
Quadrilateral: and ellipse, V, 604; inscribed in bicircular quartic, X, 231—5; inscribable in circle, X, 578.
Quadrinvariant: of binary quartic, first occurrence, I, 93; of quantic, II, 516; the term, IV, 606; of quadriquadric function, XIII, 67—8.
Quadriquadric: the term and kinds, V, 10, VII, 99.
Quadriquadric Curves: in space, V, 17; on, V, 282; sextic torse for cuspidal edge having, X, 68—72; Abel's theorem, XII, 186—9, 292—8, 321—5; and elliptic functions, XII, 292—8, 321—5.
Quadriquadric Function: two invariants of, XIII, 67—8.
Quadriquadric Transformation: between spaces, VII, 229—30; (*see also* Transformation).
Quadrispinal: the term, VII, 65.
Quantics, Introductory Memoir: II, 221—34, 598—601.
Quantics, Second Memoir: II, 250—75; numerical tables, II, 276—81.
Quantics, Third Memoir: II, 310—35.
Quantics, Fourth Memoir: II, 513—26; definitions, II, 513—5; covariants and invariants of degrees, two, three, four, II, 515—20; calculation of discriminant, II, 520—2; the catalecticant, lambdaic, and canonisant, II, 522—3; bezoutiants, cobezoutiants, II, 524—6.
Quantics, Fifth Memoir: II, 527—57, 604—6; the single quadric, II, 527—9; two or more, and theories of harmonic relation and involution, II, 529—40; cubics, II, 540—5; quartics, II, 545—56.
Quantics, Sixth Memoir: analytical theory of binary and ternary, II, 561—83; general theory of distance, II, 583—92; its style, VIII, xxvii.
Quantics, Seventh Memoir: chiefly ternary cubics, IV, 325—41; tables, IV, 333—41.
Quantics, Eighth Memoir: VI, 147—90; introductory, IV, 147—8; binary quintic, covariants and syzygies of degree 6, VI, 148—53; formulæ for canonical form, VI, 153—4; 18-thic invariant, VI,

154—6; character of equation, auxiliars, facultative and non-facultative space, VI, 156—8; application to quartic equation, VI, 158—61; characters of quintic equation, VI, 161—5; Tschirnhausen's transformation, VI, 165—9; Hermite's application of Tschirnhausen's transformation to quintic, VI, 170; nodal cubic, VI, 171—4; Hermite's criteria, VI, 174—6; his canonical form of quintic, VI, 177—83; imaginary linear transformations, VI, 183—6; application to auxiliars of quintic, VI, 186—7; theorem of binary quantic, VI, 187—90; the binary quintic and sextic, VI, 190.

Quantics, Ninth Memoir: VII, 334—53, IX, 537—42; introductory, VII, 334—5; theory of number of irreducible covariants, VII, 336—7; new formulæ for number of asyzygetic covariants, VII, 337—40; the 23 fundamental covariants, VII, 341—8; tables, VII, 341—6; Gordan's proof for the complete system of 23, and concomitants of quintic, VII, 348—53.

Quantics, Tenth Memoir: X, 339—400; introductory, X, 339—40; numerical and real generating functions, X, 341—8; table 96, X, 349—55; theory of the canonical form, X, 355—62; table 97, X, 362—9; table 98, X, 370—6; derivatives and tables, X, 377—94; numerical generating functions, N. G. F. of a sextic, X, 394—6; table, X, 397—400.

Quantics: defined, II, 221, IV, 594, 604; resultant of, II, 320; discriminants, II, 320; notation of abstract geometry, VI, 464—6; and nilfactum, VI, 466; character of the ten memoirs, VIII, XXX—XXXI; transformable into each other, VIII, 385—7; eliminant of two, XI, 100—2; Sylvester's work in, XIII, 47; syzygetic relations among the powers of linear, XIII, 224—7; and seminvariants, XIII, 363; (*see also* Binary Quantics, Quadratics).

Quarterly Journal of Pure and Applied Mathematics: VIII, XII.

Quartic Curves: transformation, I, 476—80, 589; special family of, I, 496—9; bitangents of, IV, 342—8, VII, 123—4, X, 244, XI, 221—3, 474; cuspidal defined, V, 10; in space, V, 11—5; and ovals, V, 468—70; triangle in-and-circumscribed to a, V, 489—92; with three double points, V, 550, 553; in connexion with cubic and quintic, problem, VI, 580; problem, V, 596; and sextic torse, VII, 99—100; tricuspidal, problem, VII, 589; mechanical description, VIII, 151—5; a penultimate, VIII, 526—8; construction of bicircular, IX, 13—5; and functions of a single parameter, IX, 315—7; with two odd branches, X, 36—7; bicircular, X, 223—42; triple theta functions, X, 446—54; problem and solution, X, 582—6; trinodal, problem, X, 602; singular tangents of, problem, X, 603; degenerate, XI, 220; with cusp at infinity, XI, 408; forms and classification, XI, 480; circular, XI, 481; ground curve in Abel's theorem, XII, 38, 109—216; bitangents of plane-, XII, 74—94; twisted, XII, 428—31; (*see also* Bicircular, Binary, Binodal, *and* Nodal Quartics).

Quartic Developables: and developable surfaces, V, 268—71; reciprocation of, V, 505—10.

Quartic Equations: conditions for systems of equal roots, II, 467—8; evolution, II, 547; Tschirnhausen's transformation, IV, 368—74, V, 449; Sturmian constants, IV, 473—7; nodal curve of developable from, V, 135—7; and quantics, VI, 158—61; solution of $aU+6\theta H=0$, VII, 128—9; roots, VII, 551, X, 575; solution by radicals, X, 10.

Quartic Matrix: Hermite's, XII, 367—72.

Quartics: canonical form, II, 548; equation of differences for, IV, 243, 279; the term, IV, 604; roots of, problem, V, 610; conditions for existence of systems of equal roots, VI, 300—12; and three cubics, problem, VII, 546; reality of roots, problem, X, 608.

Quartic Scroll (*see* Scrolls).

Quartic Seminvariants: XII, 20; generating functions, XIII, 306; and perpetuants, XIII, 316.

Quartic Surfaces, First Memoir: VII, 133—81, 609—10; introductory, VII, 133—4; Jacobian surfaces, VII, 134—6; surface by equating to zero a symmetrical determinant, VII, 136—7; surfaces $F(P, Q)=0$, etc., VII, 138; nodes of quartic surface, VII, 138—40; number of constants contained in a surface, VII, 140—1; general theory of quartic surface with given nodes, VII, 141—4; Jacobian surface of six given points, VII, 144—5; ditto of seven, or an octad of points, VII, 145—8; the dianodal surface, VII, 148—52; octadic surfaces with 9 or 10 nodes, VII, 152—5; dianomes with 9 or 10 nodes, VII, 155; dianodal curve of 8 points, VII, 155—6; ten nodes, VII, 156; dianodal

centres of 9 points, VII, 156; result as to dianomes, VII, 156; the symmetroid, (lineo-linear correspondence of quartic surfaces), VII, 157—9; ditto and Jacobian, VII, 160—3; symmetroid with given nodes, VII, 163—6, 259; Jacobian with given lines, VII, 167; correspondence on the Jacobian, VII, 168—70; further investigations as to Jacobian, VII, 171—5; persymmetrical case: Hessian of a cubic, VII, 175; quartics with 11 or more nodes, VII, 176—7; quadric surface through three given lines, VII, 177; condition that five given lines may lie in a cubic surface, VII, 177—8; condition that seven given lines may lie in a quartic, VII, 178; Jacobian of 6 points, VII, 178—9; locus of vertex of quadric cone which touches each of six given lines, VII, 180—1.

Quartic Surfaces, Second Memoir: VII, 256—60, 609—10.

Quartic Surfaces, Third Memoir: VII, 264—97, 609—10; preliminary considerations and classification, VII, 264—7; sextic curves, VII, 267—71; nodal determination, VII, 271—3; quartic surfaces resumed, VII, 273—4; enumeration of the cases, VII, 274—80; notation for cases afterwards considered, VII, 280—1; 16-nodal surface and table, VII, 281—4; 15-nodal surface and table, VII, 285—8; equation of ditto, VII, 288—9; 14-nodal surface and table, VII, 289—92; 13-nodal surface and table, VII, 293—7.

Quartic Surfaces: on, V, 66—9; Steiner, V, 421—3, IX, 1—2; 16-nodal, V, 431—7, VII, 126—7, X, 157—65, 180—3, 604, XII, 95—7; note on, V, 465—7; recent researches, VII, 244—52; Plücker's models, VII, 298—302; some special, VII, 304—13, VIII, 2—11, 25—8; surface and sphere, problem, VII, 589; section of surface, problem, VII, 593; penultimate forms of, VIII, 262—3; symmetrical determinant = 0, X, 50—6; 12-nodal, X, 60—2, XIII, 1—2; Hessian of, X, 274—7; tetrahedroid as 16-nodal, X, 437—40; equation of, X, 609; in *Ency. Brit.*, XI, 633—4; (*see also* Cyclide).

Quartic Syzygy: and elliptic integrals, II, 191, IV, 68—9, 609.

Quartic Transformation: of elliptic functions, IX, 103—6.

Quartinvariants: of quantic, II, 516, 520.

Quartisection: theory of numbers, XI, 84—96.

Quasi-inversion: and orthomorphosis, XIII, 192—3.

Quasi-minima: the term, XIII, 42.

Quasi-normal: the term, XIII, 228.

Quaternary: the term, VI, 464.

Quaternary Function: Hessian of, IX, 90—3.

Quaternions: certain results, I, 123—6, 127; algebraic couples, I, 128—31; rotation, I, 405—9, 589, V, 537; formulæ of, II, 107; transformation of quadrics, II, 135; skew determinants, II, 214; transformation of coordinates, IV, 559; the equation $qQ-Qq'=0$, XII, 300—4, 311—3; matrices, XII, 303; multiple algebra, XII, 474; hydrodynamical equations, XIII, 8; versus coordinates, XIII, 541—4.

Quet, J. A.: relative motion, IV, 536, 592.

Quetelet, M. A.: theory of Gergonne, and on caustics, II, 339; wave surface, IV, 433—4.

Quinquisection: theory of numbers, XI, 314—6, XII, 72—3.

Quintic Curves: and developables, I, 500—6; in space, V, 15—6, 20, 24—30, 552, 553, 613; in connexion with cubic and quartic, V, 580.

Quintic Developables: and surfaces, V, 272—8, 518.

Quintic Equations: conditions for systems of equal roots, II, 468—70; equation of differences, IV, 150—1, 246—61, 276—91; Tschirnhausen's transformation, IV, 375—94; tables, IV, 379—80, 387—90; Jerrard's researches, V, 50—4, 77, 89; character of, VI, 161—5; solvibility by radicals, VII, 13—4, X, 11; theorem of Abel, XI, 132—5; solvable case of, XI, 402—4; and elliptic functions, XIII, 473; their sextic resolvents, XIII, 473—9.

Quintic Matrix: XII, 376—80.

Quintics: auxiliary equation for, IV, 309—24; Jerrard's form, IV, 392; soluble by elliptic functions,

IV, 484; the term, IV, 604; MacClintock on, IV, 609—16; theorem of Abel as to soluble, V, 55—61, XI, 402—4; discriminant of, problem, V, 592; conditions for existence of systems of equal roots, VI, 300—12; concomitant system of, X, 342; syzygies among covariants of, X, 346—55; canonical form, X, 355—62; resolvent sextic of, XI, 396; bitangents of the, XIII, 21; and seminvariants, XIII, 363—5; (*see also* Binary Quintics).

Quintic Seminvariants: and perpetuants, XIII, 309.

Quintic Surfaces: and developables, V, 272—8, 518; recent researches, VII, 244—52.

Quintic Transformation: of elliptic functions, IX, 122, 148, XII, 522—5.

Quippian: the word, II, 381, 396—7.

Quotient: G/H in theory of groups, XIII, 336—7.

Raabe, J. L.: summation of series, II, 15.

Radials: the term, XI, 259, XIII, 179.

Radicals: and solvibility of equations, VII, 13—4, X, 8—11; theory of equations, XI, 511—20, 521; Galois and theory of, XI, 543; Weierstrassian cubic transformation, XIII, 31.

Radicals (Chemical): number of univalent, IX, 544—5.

Radii: the term link- and bar-, VII, 183; curvature of wave-surface, XIII, 248.

Range: defined, II, 577; homography, II, 578.

Rank of Seminvariants: XII, 22.

Ratio: and abstract geometry, VI, 457—62.

Rational Functions: the term, IV, 603—4.

Rationalisation: of algebraic equations, II, 40—4.

Rational Transformation (*see* Transformation, Rational).

Ray, M. N.: solution of equations, X, 610—11.

Ray Planes: and biaxal crystals, IX, 107—9.

Rayleigh, Lord: fluctuating functions, IX, 19—20.

Rays: special symmetrical system of three, VII, 428—9; the term, X, 55; systems of, XII, 571—5.

Real Generating Function: X, 339.

Real Intersections of Curves: IX, 21.

Reciprocal Figures: I, 415—20.

Reciprocal Matrix: II, 481.

Reciprocal Polars: I, 416, 421; Monge, XI, 465; Poncelet, XI, 466.

Reciprocals: of quartic scrolls, VI, 317—27; of cubic surfaces, VI, 368—455; of quartic surfaces, VII, 305; of centro-surface of ellipsoids, VIII, 363; equation of conic, VIII, 522—3; reciprocal lines, XIII, 58—9.

Reciprocal Surfaces, Theory of: VI, 329—58, 577—81, 582—91, 596—601; extension of Salmon's fundamental equations, VI, 329—31; developments, VI, 331—4; new singularities, VI, 334—41; application to a class of surfaces, VI, 341—2; flecnodal curve, VI, 342; surfaces of revolution, in connexion with spinodal and flecnodal curve, VI, 342—4; flecnodal torse, VI, 345; general surface of order n without singularities, VI, 345—6; formula for β', VI, 347—53; recapitulation, VI, 353—5; addition, VI, 355—8; Zeuthen, VI, 596—601; theory, XI, 225—34.

Reciprocants: of quantic, II, 320; the term, IV, 607, XIII, 366; and sextactic points, V, 618; of cubic, VI, 73; and invariants, XII, 393; Sylvester on, XIII, 47—8; tables of pure, to weight eight, XIII, 333—5; and differential invariants, XIII, 366—404; Halphen on, XIII, 366, 368—81, 381—98; Cockle, XIII, 366, 367—8; Ampère and Lagrange, XIII, 366; Sylvester, XIII, 366, 379—81, 381—98; MacMahon, XIII, 399—404.

Reciprocation of Quartic Developable: V, 505—10.

Reciprocity: geometrical, I, 377—82; and quantics, II, 232, 234; law of, for invariants, II, 516; and homography, II, 578.

Rectangle: potential of, IX, 278—80.
Rectilinear Motion: IV, 515—6.
Reduced Latitude: VII, 16, IX, 238.
Reducible Seminvariants: and perpetuants, XIII, 308—13.
Reducible Syzygies (*see* Syzygies).
Reduction: of transcendental integrals, X, 214—22.
Reech, F.: contour lines, IV, 609.
Reflection: caustics by, I, 273—5, II, 118—22, 129.
Region: the term, IX, 331.
Regular: the term, VI, 457, 459.
Regulator: the term, VII, 402.
Regulus: the term, XI, 573, 632.
Rehorovsky, W.: symmetric functions, II, 602.
Relation: and abstract geometry, VI, 457—62; omal, VI, 463; parametric, VI, 463—4; a discriminant, VI, 467; Jacobian, VI, 467.
Relink: the term, V, 521.
Remblais: theory of, XI, 417—20, 449, 587.
Reports: on progress of theoretical dynamics, III, 156—204; on progress in solution of certain problems in dynamics, IV, 513—93; on Pellian equation, XIII, 430—67.
Representation: analytical, of curves in space, IV, 446—55, 490—5, XI, 83; of solid figure in plane, VII, 26—30; of surfaces on a plane, VIII, 538; of variables by correspondence of planes, X, 316—23; conformal, XI, 78—81; graphical, of binodal quartic and the elliptic functions, XIII, 9—19; Sylvester on graphical, XIII, 47; (*see also* Orthomorphosis, Transformation).
Réseau: the term, VII, 253.
Residuation: of cubic curve, IX, 211—4, XII, 115—6; of curves, XII, 502; Sylvester's theory of, XIII, 47.
Residues: Cauchy's theorem on, I, 148, 174; Eisenstein's geometrical proof of quadratic, III, 39—43; nodal, of scrolls, V, 169—70, 181—3, 184, 187.
Resisting Medium: motion in, IV, 541.
Resolvent Equations: sextic, of Jacobi and Kronecker, XIII, 473—9.
Resolvents: after Lagrange, IV, 309; of quintics, XI, 396.
Resultant: the term, I, 63, 337, IV, 597, 602—3, VI, 466—7; of quantics, II, 320; of two equations, II, 440—53, VI, 292—9; of two binary quantics, IV, 1—4, IX, 16—7; of three ternary quadratics, IV, 349—58; of two binary cubics, V, 289; of forces, X, 589.
Resultor: defined, II, 59.
Reversion: of series, IV, 30—7, 54—9.
Reuschle, K. G.: mathematical tables, IX, 468—9, 473, 485, 494—9, XI, 95—6; theory of numbers, XI, 85—6, 315, 612.
Rhamphoid Cusp: V, 265—6, 618.
Rhizic Theory: root-limitation, IX, 34—8.
Ribaucour, C. R.: orthogonal surfaces, VIII, 569—70.
Riccati, J. F.: solution of equation, VII, 9—12.
Richelot, F. J.: Abelian integrals, I, 366, 367; solution of equation $x^{257}-1=0$, I, 564; porism formula, II, 90; in-and-circumscribed triangle, III, 237—41; spherical pendulum, IV, 534, 592; rotation of solid body, IV, 577—8, 592; rotation round fixed point, IV, 583, 592; two quartic curves, X, 584; integral of Euler's differential equation, XIII, 525—9.
Richmond, H. W.: Pascal's theorem, VI, 594.
Riemann, G. F. B.: doubly infinite series, II, 593; genus of curve, V, 476—7, 517; Abelian integrals, V, 521, XI, 30; Abelian functions, VI, 2, 264, 593; elliptic geometry, VIII, xxxvii; transformation and theory of invariants, VIII, 387; surface of, and correspondence, X, 317, 323; bitangents of

quartic, XI, 221—3; fractional differentiation, XI, 235—6; notion of space, XI, 435—7; correspondence of points, XI, 440; deficiency of curves, XI, 450; hyperelliptic functions, XI, 454—5; transformation, XI, 482; elliptic functions, XI, 534, 537, 539, 546; theory of numbers, XI, 616; series, XI, 627; minimal surface, XI, 639; Abelian functions and plane quartics, XII, 74, 87, 95; linear differential equations, XII, 396; orthomorphosis, XIII, 180, 189, 204.

Right-handed Root-limitation: IX, 22—3.

Roberts, C. A.: Pellian equation, XIII, 467.

Roberts, M.: geodesic lines on ellipsoid, VII, 34.

Roberts, S.: description of nodal bicircular quartic, VII, 182—8; points on cubic curve, VII, 549; three-bar curve, IX, 551; symmetrical determinant, X, 579—80; theorems of squares, XI, 294; curves, XI, 481; kinematics of a plane, XIII, 505.

Roberts, W.: transformations of curves, I, 471—5, 478; surface parallel to ellipsoid, IV, 158—65.

Roberts, W. R. W.: cyclide, IX, 75.

Rodrigues, O.: motion of solid body, I, 28—35, 124, 405, 583; quaternions, I, 124, 586; rotation of solid body, I, 237; skew determinants, I, 335; expansions in multiple sines, I, 583; attraction of ellipsoids, III, 149—53; transformation of coordinates, IV, 558, 559, 592, XI, 575; kinematics of solid body, IV, 581, 592; on rotation formulæ, V, 537; correspondence of homographies and rotations, X, 153.

Rohn, K.: quartic surfaces, VII, 609—10; quartic surface with twelve nodes, XIII, 2.

Roof: the term in non-Euclidian geometry, XIII, 484.

Root-limitation: geometrical representation, IX, 21—39; general theory, IX, 22—7; intercalation theory for right line, IX, 28—33; rhizic theory, IX, 34—8.

Roots: in forms called trees, III, 243, XI, 365—7; of algebraic equations, IV, 116—9, XI, 506—21, XII, 33—4, XIII, 33—7; quadratic equations, V, 160—1; of equations, Cauchy's theorem, IX, 21—39; of unity, IX, 263; of quartic, reality of, X, 608; imaginary, of equations, XI, 114—21, 502—6; equal, of equations, XI, 405—7; theory of real equations, XI, 497—502; Sylvester's work at, XIII, 46; ninth, of unity, XIII, 66; Waring's formula for sum of *m*th powers of, of an equation, XIII, 213—6; of a quantic, symmetric functions of, XIII, 271—85.

Roots of Unity: prime, XI, 56—60.

Rosenhain, J. G.: theta functions, VIII, xlii, X, 464, 499; double theta functions and 16-nodal quartic surface, X, 158, 162; theory of numbers, XI, 60; double theta functions, XI, 454.

Rotation: of solid body, I, 28—35, 237, 583, II, 107, III, 475—504, IV, 566—80, 592; quaternions and theory of, I, 405—9, 589; formula of, I, 586; and elliptic motion, III, 475; of earth, III, 485; infinitesimal, V, 498—9, VI, 24—6; of group of polyhedra, V, 529, 559; Euler's memoir of 1758, VI, 135—46; equilibrium of, VII, 91—5; and homography, X, 153—4; in conformal representation, XI, 78.

Roulette: the term, XI, 447.

Route: the term, XII, 640.

Rowe, R. C.: memoir on Abel's theorem, XI, 29—36; partitions of a polygon, XIII, 93, 112.

Royal Society: Cayley elected a fellow in 1852, VIII, xiii; Croonian lecture founded, VIII, xv; medals bestowed on Cayley, VIII, xxi.

Rudio, F.: inverse centro-surfaces, XII, 457—8.

Rueb, A. S.: motion of solid body, I, 464; spherical pendulum, IV, 534; rotation of solid body, IV, 573—4, 592.

Rule of Signs: and determinants, XI, 492.

Sabine, Sir E.: death of, XI, 429—30.

Sadleir, Lady Mary: endowments by, VIII, xv.

Sadlerian Professorship: Cayley appointed to, VIII, xvi.

Safford, T. H.: orbits of Neptune and Uranus, IX, 183.

St Laurent, M.: on caustics, II, 118, 121, 122, 347, 355, 368.

Salmon, G.: cubic surfaces and triple tangents, I, 446, 456, 589; linear transformations and elimination, I, 457—61; singular contact, I, 486; curves and developables, I, 492, 587; developable from quintic curve, I, 500—1, 505; systems of equations, I, 533; geometry of position, I, 555; hyperdeterminants, I, 579, II, 598—601; on a plane touching a surface, II, 29; triple tangent planes of third order, II, 29; invariant of ternary cubic, II, 325; quippian, II, 381; tables of covariants, II, 536—7; binary quartics, II, 549; tangential of cubic, II, 558; equation of orthotomic circle, III, 48—50; reciprocal surfaces, IV, 21—7, VI, 329—58, 359, 582—91; surface parallel to ellipsoid, IV, 158—65; double tangents, IV, 187—206, 343, VII, 473—4; cubic curves, IV, 188; conics and five-pointic contact, IV, 207—39; higher algebra, IV, 608; curves in space, V, 9—20, 614; quartic surfaces, V, 66, VII, 136; cubic surfaces, V, 140, VI, 359; scrolls, V, 168—9, 193, 200; prohessian, V, 267; involution, V, 301; higher singularities of plane curves, V, 620; plane curves, VI, 2; invariants, VI, 108; quintics, VI, 154; hyperspace, VI, 191; elimination, and curves which satisfy given conditions, VI, 192; extension of his fundamental equations, VI, 329—31; polyzomal curves, VI, 472, 531, 560; tetrahedral scrolls, VII, 52, 65; sextic torse, VII, 113, 114; centro-surface of ellipsoid, VII, 130, VIII, 316, 320, 323; rational transformation between two spaces, VII, 226, 237; bicircular quartic, VII, 575; locus *in plano*, VII, 606; correspondence with Cayley, VIII, XV; on Cayley, VIII, XXV; evolutes and parallel curves, VIII, 33; theory of curve and torse, VIII, 72, 76—9, 87—91; theory of invariants, VIII, 386; transformation of unicursal surfaces, VIII, 390, 391; residuation, IX, 211; triple theta functions, X, 444; tortuous curves, XI, 9; higher plane curves, XI, 217; Gaussian theory of surfaces, XI, 332; concomitants of ternary cubic, XI, 342; tables for binary sextic, XI, 377; Jacobian sextic equation, XI, 390, 400; equal roots of equations, XI, 407; works on geometry, XI, 546; minimal surfaces, XI, 639; bitangents of quintics, XIII, 21; wave surfaces, XIII, 252.

Satellite Line: II, 383, V, 359.

Scalars and Quaternions: XIII, 541.

Scalene Transformation of Plane Curve: IX, 527—34.

Schellbach, C. H.: solution of Malfatti's problem, III, 44—7.

Schläfli, L.: discriminants, I, 584; elimination, II, 181—4, 404; symmetric functions, II, 454; hyperdeterminants, II, 598—601; resultants, IV, 2—4; numerical expansions, IV, 471; cubic surfaces, VI, 359, 361, 362, 372, VII, 250; quartic surfaces, VII, 308; modular equation for cubic transformation, XIII, 64—5.

Schlömilch, O.: attractions, I, 288; a definite integral, IV, 29.

Schoolgirl Problem: I, 483, 589, V, 95—7.

Schottky, F.: theta functions, XI, 242—9.

Schröter, H.: Steiner's quartic surface, V, 423; construction of regular pentagon, XII, 47.

Schubert, H.: elliptic motion, III, 473, 474, IV, 523; *abzählende Geometrie*, XI, 281—93, 459.

Schwarz, H. A.: inverse elliptic functions, I, 586; developable surfaces, V, 517—9; deficiency, VI, 2; scrolls, VI, 312; quintic scrolls, VII, 250, 252; projections, IX, 508; surface of minimum area, X, 63; hypergeometric series, XI, 125; orthomorphosis, XII, 328, XIII, 188, 191, 192, 193, 202; Kummer's differential equation, XIII, 69.

Schwarzian Derivative and Polyhedral Functions, Memoir: XI, 148—216; introductory, XI, 148—51; Part I, XI, 151—79; the derivative, XI, 151—3; quadric function of three or more inverts, XI, 153—6; functions P, Q, R, XI, 156—7; table ditto, XI, 158—9; differential equations involving (x, z) and (s, x), XI, 160—9; Schwarzian theory, XI, 169—76; connexion with differential equation for hypergeometric series, XI, 176—9; Part II, the polyhedral functions, XI, 179—216; origin and properties, XI, 179—83; covariantive formulæ, XI, 184—5; the forms of $f5$ and $h5$, XI, 185—6; stereographic projection, XI, 187—9; groups of homographic transformations, XI, 189—90, 196—208; the regular

polyhedra, XI, 190—5; system of fifteen circles, XI, 208—12; regular polyhedra as solid figures, XI, 212—6.

Schwarzian Derivative, and Polyhedral Functions: XI, 149, 173, 177; and Kummer's differential equation, XIII, 69; and reciprocants, XIII, 366.

Schwarzian Function: Sylvester on the, XIII, 381.

Scientific Memoirs Catalogue: report on, V, 546—8, 620.

Scrolar: the term, VI, 334.

Scrolls, First Memoir: V, 168—200.

Scrolls, Second Memoir: V, 201—20, VI, 360; degeneracy, V, 201—3; with two directrix lines, V, 203—5; twofold directrix line, V, 205—7; scroll equations, V, 207—10; cubic scrolls, V, 210—3; quartic scrolls, V, 214—9; general theory, V, 219—20.

Scrolls, Third Memoir: VI, 312—28; quartic scrolls, VI, 312—4, 314—7, 328; reciprocals, VI, 317—25, 325—7.

Scrolls: cubic, V, 90—4, 110—2; quartic, VI, 451; tetrahedral, VII, 48—53; on certain, VII, 54—65; recent researches, VII, 250—1; and octic surfaces, X, 79—92; flexure and equilibrium, XI, 317—22; the term, XI, 573; applicable upon a given skew surface, XIII, 231—7.

Secondary Caustics (*see* Caustics).

Secular Acceleration of Moon's Mean Motion: III, 522—561.

Seeber, L.: mathematical tables, IX, 491—2.

Segar, H. W.: development of $(1+n^2x)^{m/n}$, XIII, 354—6.

Segner, J. A. v.: inertia, IV, 561—2, 592.

Segregates: the term, X, 339, 345; table of, X, 349—55.

Semi-cubical Parabola: and Abel's theorem, XII, 180—6.

Seminvariants, memoir: XII, 239—62; introductory, XII, 239—40; multiplication of two symmetric functions, XII, 240—8; capitation and decapitation, XII, 248—50; perpetuants, etc., XII, 250—7; sextic perpetuants and sextic syzygies, syzygants, XII, 257—62.

Seminvariants: the term, IV, 241, 606; theory of, XII, 22—9, 344—57, XIII, 362—5; tables, XII, 275—89, XIII, 217—23; theorem relating to, XII, 326—7; (*see also* Covariants, Invariants).

Seminvariants and Symmetric Functions, Memoir: XIII, 265—332; introductory, XIII, 265—6; the coefficients $(a, b, c, \ldots)$, XIII, 266—71; symmetric functions of the roots, XIII, 271—85; MacMahon form of equation, XIII, 286—8; the *I-and-F* problem, and solution by square diagrams, XIII, 288—98; MacMahon linkage, XIII, 298—301; umbral notation, Stroh's theory, XIII, 301—6; symmetric functions of a given degree: generating functions, XIII, 306—8; reducible seminvariants—perpetuants, XIII, 308—13; Strohian theory resumed: application to perpetuants, XIII, 314—8; investigation of the values of the foregoing functions, XIII, $\Pi_{10}(x+y)\,\Pi_{15}(x+y)$ and $\Pi_{10}(x+y+z)$, XIII, 318—21; the operators, XIII, 322—32.

Senate House Problems: IX, 246—9, XI, 265—7, XIII, 538—40; (*see also* Smith's Prize Papers).

Separations of Partitions: II, 603; (*see also* Partitions).

Separator: the term, VII, 402.

Septic, Binary: generating functions of, X, 408—21.

Septic Transformation in Elliptic Functions: IX, 148—52, X, 333—8.

Series: a double infinite, II, 8—10, 593; sums of, III, 124—8, XIII, 49—50; reversion, IV, 30—7; formula for reversion, IV, 54—9; in integration of differential equations, VIII, 458—62; hypergeometric, XI, 17—25, 125—7; in *Ency. Brit.*, XI, 617—27 (finite, XI, 617—20; infinite, XI, 620—7).

Serpoloid Curve: IV, 571—2.

Serret, J. A.: attractions, I, 288; integral calculus, I, 313; elliptic functions, III, 3; integration of differential equations, III, 185—6, 203; problem of two centres, IV, 528, 531, 592; theory of substitutions, VII, 47; curves of curvature, VII, 331, XII, 601—38; orthogonal surfaces, VIII, 279, 292; indefinite integration, IX, 500—3; transformation of cubic function, XI, 411; minimal surfaces, XII, 595.

Serret, P.: syzygetic relations, XIII, 224.
Servois, J. F.: multiple algebra, XII, 468.
Sets: homology of, III, 35.
Sextactic Points: and plane curves, V, 545, VI, 217; and reciprocants, V, 618.
Sextactic Reciprocant: XIII, 387.
Sextic: binary, and quintics, VI, 190; unicursal, VI, 248; and cubic curves in pencil of six lines, VI, 593—4; the anharmonic-ratio, VII, 314—5; bicursal, IX, 551, 581—6; tricircular, IX, 562—70; numerical generating function, X, 394—6.
Sextic Cone: circumscribed to quartic surface, VII, 265; and nodes, table, VII, 291.
Sextic Curves: foci of conics, VII, 1—4; rational transformation, VII, 236—8; and nodes, VII, 256—7; and quartic surfaces, VII, 267—71; mechanical description, VIII, 138—44; with five double points, IX, 504—7.
Sextic Developable: V, 279—83, 511—9, VI, 87—100.
Sextic Function: and Abelian functions, XI, 483.
Sextic Resolvent: Jacobian, IV, 310, XI, 389—401, XII, 493—9.
Sextic Resolvent Equations: of Jacobi and Kronecker, XIII, 473—9.
Sextic Seminvariants: and perpetuants, XIII, 317.
Sextic, Spherical: and oval, V, 469.
Sextic Syzygies: XII, 257—62, 273.
Sextic Torse (*see* Torse, on a certain Sextic, *also* Torses).
Shanks, W.: log 2, XI, 70.
Sharp: the term, VIII, 406—8, XIII, 265, 291, 304—6, 362.
Sharp-cone: the term, VIII, 102.
Sheets: roots in algebraic equations, IV, 116—9; cubic curves, IV, 120—2.
Shell: formulæ for potential of, IX, 266—7; attraction of ellipsoidal, on exterior point, IX, 302—11.
Sibi-reciprocal Surfaces: VI, 21, X, 252—5.
Siebeck, F. H.: binodal quartic and graphical representation of the elliptic functions, XIII, 10.
Signs: rule of, IV, 595—6, XI, 492.
Sign Symbols: theorems, VIII, 535—7.
Simple Cone: defined, V, 402, 404, 551.
Simple Groups: XIII, 533.
Simultaneous Equations: Jacobi's theorem in, XII, 39.
Simultaneous Roots of Two Equations: Jacobi's theorem in, XII, 123—5.
Sines, Multiple: X, 1—2.
Single Theta Functions: memoir, X, 473, 476—97; and double theta functions, X, 155—6, 186—9; and elliptic functions, XI, 250—1; linear transformation, XII, 50—5.
Singular Curve: on surface, VII, 244.
Singularities: of curves and developables, I, 208; of tortuous curves, I, 500; of surfaces, II, 28—32, IV, 21—7, VI, 123—8, 334—41, 354, 582—5, VIII, 394—8, XI, 225—6, 630—1; of plane curves, V, 424—6, 476—7, 520—8, 619, VI, 3, XI, 30—6; of curves and torses, V, 516; compound, V, 525; of curves in space, V, 613; of cubic surfaces, VI, 363; reciprocal surfaces, VI, 596—601; of curves, XI, 486—70.
Singular Point: for differential equations, XII, 395; integrals in domain of, XII, 395—402.
Singular Solutions of Differential Equations: IV, 426—7; of first order, VIII, 529—34 X, 19—24.
Six Coordinates of a Line (*see* Coordinates, Six of a Line).
Six Lines: Sylvester's involution of, VII, 66.
Six-pointic Contact: on cubic, IV, 207.
Sixteen-nodal Quartic Surfaces: I, 587, V, 431—7, VI, 126—7, 281—4, X, 157—65, 180—3, 437—40, 464, 548—51, 604, XII, 95—7.

Skew: the word, I, 332.
Skew Antipoint: the term, IX, 65—6.
Skew Convertible Matrices: II, 489.
Skew Covariants: II, 233.
Skew Cubics: demonstration of Chasles' theorem, I, 212.
Skew Curvature: the term, I, 234.
Skew Determinants: the term, and some properties of, I, 332—6, 410; on, I, 410—3, 589; researches, II, 202—15, IV, 72—3; and transformation, II, 497; a theorem, IV, 72—3.
Skew Hyperboloids: geodesic lines, VIII, 174—8, 188—99; of revolution, projection of, IX, 237—40.
Skew Matrix: IV, 602.
Skew Polars: I, 378.
Skew Reciprocals: the term, I, 415.
Skew Surfaces (*see* Scrolls).
Slope Lines: IV, 108—11, 609.
Smith, H. J. S.: higher singularities of plane curves, V, 619; transformation of elliptic functions, IX, 174—5; report on mathematical tables, IX, 461—99; death of, XI, 429; theory of numbers, XI, 612; theta and omega functions, XII, 50; theta functions, XII, 337; transformation, XIII, 38; predecessor of Sylvester in Oxford chair, XIII, 44; on a memoir by, XIII, 558—9.
Smith, P.: on Lagrange's solution of caustic, II, 353.
Smith's Prize Papers: VIII, 414—35, 436—8, 439—57, 474—90, 496—516, 538—9, 551—5, 558—63; infinitesimal rotation, VI, 24—6; general equation for virtual velocities, IX, 205—8; solutions and remarks, IX, 218—36; Bernoulli's numbers in analysis, IX, 259—62; problems and solutions 1877, X, 39—46; question on theory of equations, XI, 115; on potentials, XI, 261—4.
Sohnke, L. A.: motion in resisting medium, IV, 541; transformation of elliptic functions, IX, 113, 114; ditto table, IX, 128—35; modular equations, IX, 543.
Solar Eclipse, Graphical Construction: VII, 390—1, 479—92; geometrical theory, VII, 392—6, X, 310—5; general explanation, VII, 479—82; modification for single blank projection, VII, 482—4; construction of relative orbits, VII, 484—7; geometrical theory of projection of penumbral curve, VII, 488—9; details and application to eclipse 21—22/12/70, VII, 489—92.
Solid Body: motion of, I, 28—35, 583; geometrical representation of motion, I, 234—6; rotation round fixed point, I, 237—52, 336; rotation of, I, 462—4, III, 475—504, IV, 577, 592; four forces acting on, IX, 201.
Solid Integral Prepotential: IX, 334—7.
Solid of Revolution: attraction of, I, 508.
Solids: Poinsot's four new regular, IV, 81—5, 86—7; plane representation of, VII, 26—30.
Soluble Quintics: IV, 484, V, 55—61, XI, 402—4.
Solutions (*see* Problems, Problems and Solutions, Smith's Prize Papers).
Somoff, J.: rotation of solid body, IV, 577, 592.
Space: of any number of dimensions, and quantics, II, 222; facultative and non-facultative, VI, 156; dimensions and abstract geometry, VI, 456—7; rational transformation, VII, 189—240; multiple, VIII, xxxiii—v; points and lines, correspondence in, VIII, 566; flexure, X, 331—2; the term in five-dimensional geometry, IX, 79; theories of, XI, 434—7; curves in, XI, 489; elliptic, and non-Euclidian geometry, XIII, 481; (*see also* Hyperspace).
Special Conditions for Curves: VI, 193.
Species: of Quartic Scrolls, V, 201, VI, 328; twenty-three, of cubic surfaces, VI, 359—60.
Specific: the term, XIII, 290.
Sphere: powers of, I, 581; and polyhedron, V, 531; problem and solution, VII, 563; prepotentials, IX, 351—2, 359—79; in *Ency. Brit.*, XI, 571—2.
Spherical Conics: theorem, IV, 428; stereographic projection, V, 106—9; (*see also* Polyzomal Curves).

Spherical Curves of Curvature: surfaces with, XII, 601—38.
Spherical Pendulum: IV, 532—4, 535—7, 541.
Spherical Triangle: theorem of, IV, 80; nine-point circle of, XIII, 548—51.
Spherogram: the term, VII, 404; and isoperimetric lines, VII, 467—8; e- and iseccentric lines, VII, 468—70; time- and isochronic lines, VII, 470—7; two plates, VII, to face 478.
Spheroidal Trigonometry: IX, 197.
Spheroid, Oblate: geodesic lines on, VII, 15—25.
Spheroquartic: the term, VIII, 262.
Spinodal Curve: VI, 342—4, 583—5.
Spinode: synonymous with cusp, II, 28, IV, 22, 27; the term, V, 521; plane and torse, VI, 355, 583—5, 601.
Spinode Curves: and cubic surfaces, VI, 450, 595.
Spottiswoode, W.: ineunts, IV, 419; determinants, IV, 608; text-book on determinants, IV, 608; intersection of line and conic, V, 500—4; twenty-one coordinates of conic in space, XI, 82—3; death of, XI, 430; Sylvester's researches, XIII, 44.
Square Diagrams: seminvariants and solution by, XIII, 288—98.
Squares: products of sums of, II, 49—52; surfaces divisible by curves of curvature into, VIII, 97—8, 145—6, 264—8; theorems of 2-, 4-, 8-, 16-, XI, 294—313; imaginaries of 8-, XI, 368—71; Mill on, XI, 432—3; Latin-, XIII, 55—7; orthomorphosis of, into circles, XIII, 191—202.
Squares of Roots: equations of, IV, 242—3.
Squarewise Contractible: the term, XIII, 179.
Stader, F.: central forces problem, IV, 520, 592.
Standard Solutions: of system of linear equations, XII, 19—21.
Statics: six coordinates of a line, VII, 89—95; and time, XI, 444; and Archimedes, XI, 446.
Stativity: the term, VIII, 213.
Staudt, K. G. C. v.: distances of points, I, 581; theory of distance, II, 605; theorem of, on Bernoulli's numbers, IX, 261.
Steiner, J.: Pascal's theorem, I, 322—8; geometry of position, I, 356, 550—6; transformation of curves, I, 474; conics inscribed in a quadric surface, I, 557; cubic surfaces and triple tangent planes, I, 589; extension of Malfatti's problem, II, 57—86, 593; porism formula, II, 90; harmonic relation of two lines or points, II, 96; in-and-circumscribed polygon, II, 141; problems on quadrics, II, 179—80; pippian, II, 381, 391; systems of circles, III, 113; double tangents, IV, 187; conics, IV, 207; point of six-pointic contact on cubic, IV, 207; conics touching curves, V, 31—2; generalized theorem, V, 100—2; pedals, V, 113; quartic surface, V, 421—3; theorem of eight points on a conic, V, 427—30; Casey's equation, VI, 67; locus, envelope, and triangle, VI, 72; bitangents of quartic curve, VII, 124; quartic surface of, VII, 247, VIII, 389, IX, 1—2, X, 607; attraction of ellipsoidal shell, IX, 302.
Stellated Polyhedra: IV, 82, 609.
Stereographic Projection: of spherical conic, V, 106—9; geodesic lines on oblate spheroid, VII, 24—5; property of, VII, 397—9, XI, 187—9, 569.
Stereoscope: and cubic curves, IV, 122.
Stern, M. A.: sums of certain series, III, 126.
Stirling, J.: curve classification, V, 354; proof of his theorem, X, 267—8.
Stockwell, J. N.: determinants, V, 45—9.
Stokes, Sir G. G.: surfaces, envelopes, and parallel curves, IV, 123—33; resisting medium, IV, 541; Catalogue of Scientific Memoirs, V, 546—8, 620; report on mathematical tables, IX, 461—99.
Streblosis: XI, 79, 81.
Striction, Curve of: the term, I, 234.
Striction, Lines of: on skew surfaces, XIII, 232—7.

Stroh, E: perpetuants, XIII, 266, 301—6, 314—8.
Study, E.: Zeuthen on, VI, 594.
Sturm, J. C. F.: integration of dynamical equations, III, 186, 203.
Sturm, R.: homography, VIII, 200; correspondence of points and tetrahedra, VIII, 200—8; root-limitation, IX, 39; theory of equations, X, 4—5, XI, 498—9, 505.
Sturmian Constants: for cubic and quartic equations, IV, 473—7; for quantics, VI, 159.
Sturmian Functions: note on, I, 306—8, 588; new researches, I, 392—6; endoscopic and exoscopic expressions, I, 588; tables for equations from second to fifth degrees, II, 471—4.
Sturm's Theorem: VI, 159, 161; Sylvester's work at, XIII, 46.
Subinvariants: the term, XII, 251, 273.
Subrational: the term, IX, 315.
Sub-regular Integrals: of differential equations, XII, 444—52.
Substitutions: theorem relative to theory, VII, 47; arising from a problem in arrangements, X, 247—8; and theory of groups, X, 324—30, 401—6; and permutations, X, 574; evolution, XI, 455; the notion of, XI, 509—10, 521; Latin squares, XIII, 55—7; groups for two to eight letters, XIII, 117—49; Sylow's theorems on groups, XIII, 530—3; sixty icosahedral, XIII, 552—7.
Subsurface: the term, IX, 79.
Summit: defined, V, 63, XIII, 507.
Sums: of squares, II, 49—52; the term, X, 186, 192; of two series, XIII, 49—50.
Sun: and moon's mean motion, III, 522—61; Newcomb on parallax, IX, 177—8; (*see also* Solar Eclipse).
Supercurve: the term, IX, 79.
Superlines: in hyperspace, IX, 79—83.
Supp: the term, VI, 263.
Supplement: the term, VI, 263.
Suremain-de-Missery, A.: imaginaries, XII, 467.
Surface, Congruence, Complex, in *Ency. Brit.*: XI, 628—39; introductory, XI, 628—9; surfaces in general: torses, XI, 629—32; surfaces of orders 2, 3, and 4, XI, 632—4; congruences and complexes, XI, 634—5; curves of curvature: asymptotic lines, XI. 635—6; geodesic lines, XI, 636—7; curvilinear coordinates, XI, 637; orthotomic surfaces: parallel surfaces, XI, 637—8; minimal surface, XI, 638—9.
Surface-integral: prepotential, IX, 321—30.
Surface of Centres: for wave-surface, XIII, 248; (*see also* Ellipsoid, Centro-surface of).
Surface of Cylinder: Archimedes' theorem for, XII, 56—7.
Surface of Revolution: and Mercator's projection, VIII, 567.
Surfaces: equimomental, I, 253—4; wave (tetrahedroid), I, 302—5, 587; confocal, I, 362—3; singularities, II, 28—32, IV, 22—7; theory of skew, II, 33—4; envelopes and parallel curves, IV, 123—33, 152—7, 158—65; curvature of, IV, 466—9; theorem on degenerate, V, 98—9; developable, and prohessians, V, 267—83; planar, V, 578; sibi-reciprocal, VI, 21, X, 252—5; sextic, VI, 87—100; singularity of, VI, 123—8; tetrahedral, VII, 48—53; on certain skew, VII, 54—65; Steiner's, VII, 247; intersection of two, VII, 563; divisible into squares by curves of curvature, VIII, 97—8, 145—6, 264—8; correspondence, transformation, and deficiency, VIII, 200—8; penultimate forms of, VIII, 262—3; transformation of unicursal, VIII, 388—93; deficiency of certain, VIII, 394—7; reciprocal, VIII, 394; of eighth order, VIII, 401—3; representation on plane, VIII, 538—9; families of, VIII, 567; transformation of equation of, to chief axes, IX, 48—51; the term, in five-dimensional geometry, IX, 79; orthogonal to set of lines, IX, 587—91; flexure of spherical, X, 30—2; of minimum area, X, 63—7, XIII, 41—2; octic, X, 79—92; on a sibi-reciprocal (octic), X, 252—5; fleflecnodal planes, X, 262—4; Jacobian of six points, X, 281—93; flexure of, X, 331—2; distribution of electricity on two spherical, XI, 1—6; general theory, XI, 14—6, 224; deformation and flexure of, XI, 66—7, 317—22; theory of apsidal, XI, 111—3; theory of reciprocal, XI, 225—34; contact of line with, XI, 281—93; geodesic curvature of

curve on a, XI, 323—30; Gaussian theory, XI, 331—6; and solid geometry, XI, 569; ruled, in *Ency. Brit.*, XI, 572—3; in *Ency. Brit.*, XI, 580—2; general theory of curvilinear coordinates, XII, 1—18; determination of order of surface, XII, 42—4; minimal, and Joachimsthal's theorem, XII, 594—5; with plane or spherical curves of curvature, XII, 601—38; quasi-minimal, XIII, 42; the absolute, XIII, 42; applicable to each other, XIII, 253—64; and systems of tetrads of circles, XIII, 425—9; of order *n* which pass through given cubic curve, XIII, 534—5; (*see also* Developables, Monoid, Orthogonal, Parallel, Reciprocal, and Wave Surfaces, Scrolls).

Switzerland: Cayley's visits to, VIII, xi.

Syllogism: theory of, VIII, 65—6.

Sylow, L.: theorems on groups, XIII, 530—3.

Sylvester, J. J.: special factors, I, 337; Sturmian functions, I, 392, II, 471—4; schoolgirl problem, I, 483; theory of hyperdeterminants, I, 577, 589; commutants, I, 584; endoscopic, I, 588; theory of permutants, II, 23, 26, 27; rationalization of algebraical equations, II, 42; matrices, II, 219, 604; law of reciprocity, II, 232, 234; partitions, II, 248—9, 506, XII, 217; contravariants, II, 320; combinants, II, 322; cubic curves, II, 405; symmetric functions, II, 465; canonical forms, II, 523; cobezoutiants, II, 524; bezoutiants, II, 526; hyperdeterminants, II, 598—601; logic of characteristics, III, 52; a special determinant, III, 122; elimination, III, 214—5; independent variables in differential calculus, III, 246; reversion of series, IV, 36, 37, 54—9; canonical form of binary quantics, IV, 43—52, 53; double partitions, IV, 166—70; conics and five-pointic contact, IV, 231; on derivative of point on cubic, IV, 231; finite differences, IV, 263; equation of differences, IV, 277; invariants, IV, 349; Tschirnhausen's transformation, IV, 391; volume of tetrahedron, IV, 462; involution of six lines, IV, 582, 593, VII, 66; lines in involution, V, 1—3; quadric cones, V, 6; quartic surfaces, V, 69; canonic root of binary quantic, V, 103—5; discriminant of quintic, V, 592; conic and cubic, V, 608; derivation of points of cubic curve, VI, 20; quintics, VI, 147—8; on roots of algebraical equation, VI, 147; bicorn, VI, 158; foci of conics, VII, 1—4; differential operators, VII, 8; cubic transformation of elliptic functions, VII, 44; Cartesian curves and cubic curve, VII, 556; spherical problem, VII, 563; discussions with, on covariants, VIII, xv; theory of matrices, VIII, xxxii—iii; root-limitation, IX, 22, 39; elimination, IX, 43, XIII, 545—7; residuation, IX, 211; quartic curves and functions of a single parameter, IX, 315—7; scalene transformation, IX, 527, 534; development of idea of Eisenstein, X, 58—9; numerical generating function, X, 339; linkwork, X, 407; N. G. F. of binary septic, X, 408—9; theorem relating to covariants, X, 430; on trees, X, 598—600; theory of tamisage, XI, 409—10; partitions, XII, 217; perpetuants, XII, 251, 252, 253; non-unitary partition tables, XII, 273; d'Alembert-Carnot geometrical paradox, XII, 305—6; umbræ, XII, 347; invariants and reciprocants, XII, 393; a Diophantine relation, XII, 596; *Nature*, notice in, XIII, 43—8; syzygetic relations, XIII, 224; reciprocants, XIII, 333—5, 366, 379—81; lectures on theory of reciprocants, XIII, 381—98.

Symbolical Forms: of hyperdeterminants, I, 80—94; of covariants, I, 577, 585.

Symbols: modular functions, IV, 484—9.

Symmetric: the term, I, 410.

Symmetrical: the term, IV, 599, 604, VI, 524—5, XI, 496.

Symmetric Covariants: II, 233.

Symmetric Curve: and system of equations, I, 473.

Symmetric Functions: of roots of an equation, II, 417—39, 602—3; partitions, II, 418; tables, II, 423—39; resultant of a system of two equations, II, 440—53; tables, II, 445—53, VI, 292—9; of the roots of certain systems of two equations, II, 454—64, VI, 292—9; conditions for existence of given systems of equalities among roots of an equation, II, 465—70, 603—4, VI, 300—12; tables, II, 467; conditions for existence of systems of equal roots of binary quartic or quintic, VI, 300—12; and theory of equations, X, 6—8; non-unitary, and seminvariants, XII, 239—48, 275; tables of roots, XII, 263—72, 273—4; a differential operator, XII, 318; seminvariants, XIII, 265—332; (*see also* Seminvariants).

Symmetroid: the term, VII, 134, 280; lineo-linear correspondence of quartic surfaces, VII, 157—9; and Jacobian, VII, 160—3, 175; with given nodes, VII, 163—6; and decadianome, VII, 256, 259; and circumscribed cone, VII, 258—9; theory, VII, 264.

Symmetry: Sylvester on, XIII, 45.

Symptose: the term, I, 523, 529, 557—8.

Syntypic: the term, VII, 123.

Système Linéaire: of Laguerre is a matrix, II, 604.

System of Equations: order of, I, 457—61, 589; connected with Malfatti's problem, I, 465—70; note, I, 532—3, 589; algebraical, XI, 39—40.

Syzygant: the term, XII, 251; and seminvariants, XII, 257—62.

Syzygies: of degree six, VI, 148—53; of binary quintic connected, VII, 334; for binary cubic, IX, 55; of quintic, X, 346—55; of sextic, XII, 257—62; of binary quartic, and elliptic integrals, XIII, 32; Sylvester's work in, XIII, 46; syzygetic relations among powers of linear quantics, XIII, 224—7; and seminvariants, XIII, 310.

Tables, Brit. Assoc. Report on Mathematical: IX, 461—99; introductory, IX, 461—2; of divisors and prime numbers, IX, 462—70; prime roots, IX, 471—7; Pellian equation, IX, 477—80; partitions, IX, 480—3; quadratic forms, IX, 484—6; binary, ternary, quadratic, and higher forms, IX, 486—93; complex theories, IX, 493—9.

Tables: linear transformations, I, 108; of covariants for quadratic, cubic, quartic, quintic, II, 276—81, II, 310—35; of covariants M to W of binary quintic, II, 282—309; covariants for sextic, II, 314—5; for septimic, II, 315—6; for octavic, II, 316—8; for nonic, II, 318—9; of concomitants of ternary quadric, II, 322—3; of ternary cubic, II, 323—9, 331—5; of symmetric functions of roots of equation, II, 423—39; of resultants of two equations, II, 449—53; Sturmian functions for equations from second to fifth degrees, II, 471—4; disturbing function in lunar theory, III, 299—308, 311—8, VII, 516, 519—24, 525—7; of functions in theory of elliptic motion, III, 360—474; Degen's, for Pellian equation, IV, 40; equation of differences, IV, 246—56, 280—91; Arbogast's method of derivations, IV, 274—5; concomitants of ternary cubics, IV, 333—41; Tschirnhausen's transformation for quartics, IV, 373—4, 379—80; and for quintics, IV, 387—90; numerical expansions, IV, 470; polyacra, V, 44; binary quadratic forms, V, 141—56, 618; properties of scrolls, V, 171—2; axial systems of polyhedra, V, 532—9; curves in space, V, 616; for prime or composite modulus, VI, 83—6; asyzygetic covariants, VI, 149—152; quantics, VI, 167—8; resultant of a system of two equations, VI, 292—9; conditions for existence of systems of equal roots of quartic or quintic, VI, 300—12; singularities of cubic surfaces, VI, 363; also lines and planes, VI, 373; Legendre's elliptic functions, VII, 20; geodesic lines on oblate spheroid, VII, 23; rational transformation between two spaces, VII, 210—3, 224; nodal quartic surfaces, VII, 283, 287, 291, 296; quartic surfaces, VII, 310, 609—10; irreducible covariants of binary quintic, VII, 341—6; planogram No. 1, VII, 439—40; ditto No. 2, VII, 450—1; geodesic lines on ellipsoid, VII, 504—6; binary cubic forms, VIII, 51—64; theory of curve and torse, VIII, 81—4; Pineto's of logarithms (review), VIII, 95—6; cones satisfying six conditions, VIII, 100; geodesic lines, particularly on quadric surface, VIII, 196—9; in-and-circumscribed triangle, VIII, 214—21; centro-surface of ellipsoid, VIII, 365; Steiner's surface, IX, 7; transformation of elliptic functions, IX, 128—35, 163; Newcomb's planetary, IX, 181—4; projection of skew hyperboloid of revolution, IX, 240; classification for mathematical, IX, 424—5; report on mathematical, IX, 424—5; chemical trees, IX, 436—43, 446—8, 450—60, 544—5; double theta functions, X, 168—9, 171, 172—3; regular solids, X, 270—3; concomitants of quintic, X, 349—55, 362—9, 370—6, 377—94, 397—400; transvectants for quintic, X, 378—394; Kummer hexads, X, 506; theta functions, X, 507—10, 513—28, 530—6, 540—2, 544—6; theory of numbers, trisection, XI, 89; ditto quartisection, XI, 94; Reuschle's, of prime roots, XI, 95—6; of finite differences, XI, 144—7; connected with polyhedral function, XI, 158—9, 192; covariantive, XI, 272—80; Schubert's

numerative geometry, XI, 286; theorems of squares, XI, 299—313; theory of numbers, XI, 316; concomitants of ternary cubic, XI, 345—7; literal, for binary quantics, otherwise a partition table, XI, 357—64; for binary sextic, XI, 372—6, 377—88; covariantive, XI, 409—10; Plücker's equations, XI, 472; of Gauss, XI, 545; symmetric functions of roots of an equation, XII, 263—72, XIII, 272—4, 288; non-unitary partition, XII, 273—4; seminvariant, XII, 275—89; orthomorphosis of circle into parabola, XII, 336; of groups, orders two to twelve, XII, 643—56; Wallis's investigation for π, XIII, 23—5; quadrinvariant and cubinvariant of quadri-quadric function, XIII, 68; partitions of a polygon, XIII, 95, 112; theory of rational transformation, XIII, 116; substitution groups for two to eight letters, XIII, 118—49; corrected seminvariant for weights 11 and 12, XIII, 217—23; of conjugates, XIII, 303; seminvariants and symmetric functions, XIII, 311, 313, 331—2; of pure reciprocants to weight 8, XIII, 333—5; report of British Association committee on Pellian equation, and tables, XIII, 430—67; omega and theta functions, XIII, 558—9.

Tacinvariant: the term, IV, 607.

Tac-locus: in singular solutions, VIII, 533.

Tacnode: defined, II, 28—32, V, 266.

Tactic: and algebra, V, 293—4; the term, XI, 443; (*see also* Arrangements, Groups).

Tactinvariant: of two quantics, II, 320; the term, V, 305.

Tactions: analytical solution, III, 255—7; formulæ, IV, 510—2; Casey's equation, VI, 543; and trizomal curves, VI, 575; problem of, XIII, 150—69.

Tait, P. G.: arrangements, X, 245; quaternions, XII, 303, 475, XIII, 541—4; finite differences, XII, 412.

Talbot, W. H. F.: curve of, IV, 123.

Tamisage: Sylvester's theory of, XI, 409—10.

Tangential: defined, II, 558; of a curve, IV, 188.

Tangent Lines: and surface, XI, 630, 632—4.

Tangent Omals: VI, 467—9.

Tangent Planes: and surface, XI, 630, 632—4.

Tangents: and two-dimensional geometry, II, 575; inflexional and chief, VIII, 157, 294; singular of a quartic, X, 603; in *Ency. Brit.*, XI, 564—5, 579—80; (*see also* Bitangents).

Tannery, J.: linear differential equations, XII, 394.

Tantipartite: the term, I, 584, II, 517, IV, 464, 604.

Taylor, C.: locus *in plano* problem, VII, 599; general theory of surfaces, XII, 42—4.

Taylor, H. M.: inversion, IX, 18; partitions of a polygon, XIII, 93, 112.

Taylor's Theorem: Lagrange's demonstration, VIII, 493—5, 519; note on, VIII, 524.

Tchebycheff, P.: theory of numbers, XI, 616.

Terminology: recent mathematical, IV, 594—608.

Terms: in symmetrical determinant, IX, 185—90.

Ternary: the term, IV, 604, VI, 464.

Ternary Cubics: relation between two, IV, 79—81; memoir on quantics, IV, 325—41; form problem, VII, 548; the 34 concomitants, XI, 342—56; canonical form, XI, 343.

Ternary Quadratics: resultant of three, IV, 349—58.

Ternary Quadrics: and involution, XIII, 350—3.

Ternary Quantics: and bitangents of plane curve, IV, 188; involution, V, 301—9.

Tetrads: the term, XII, 590; systems of, XIII, 425—9.

Tetrahedra: reciprocals, III, 7; axial systems, V, 531—9; note on, V, 557—9; and cubic surfaces, VII, 607; correspondence of points in relation to two, VIII, 200—8; Steiner's surface, IX, 1—12; in perspective, IX, 209—10; automorphic function for, XI, 169, 179—83, 184, 212—6.

Tetrahedral Surfaces: VII, 48—53, 54—65.

Tetrahedroid: and wave surface, I, 302—5, 587; 16-nodal quartic surfaces, V, 431—7; the term, VI, 21, X, 252; and scrolls, VII, 245; as particular case of 16-nodal quartic surface, X, 437—40.

Tetrazomal (*see* Polyzomal Curves).
Text-Books: on determinants, elimination and higher algebra, IV, 608.
Theory of Equations (*see* Equations, Theory of).
Theory of Groups (*see* Groups).
Theory of Numbers (*see* Numbers, Theory of).
Theta Functions, Memoir on Single and Double: X, 463—565; historical, X, 463—4; Part I, X, 464—76; definitions, X, 464—5; allied functions, X, 465—6; even-integer alteration of characters, X, 466; odd ditto, X, 466; even and odd functions, X, 467; quarter-periods unity, X, 467—8; conjoint quarter quasi-periods, X, 468—9; product-theorem, X, 469—71; résumé of ulterior theory of the single functions, X, 471—3; ditto, double functions, X, 474—5; remark as to notation, X, 475—6; Part II, X, 476—97; notation, X, 476; constants of the theory, X, 477—8; product theorem, X, 478—80; the square-set, X, 481—2; relation between the constants, X, 482—3; product-sets, X, 483—4; comparison with Jacobi's formulæ, X, 485; the square set, X, 485—7, 488—9; elliptic integrals of third kind, X, 489—90; addition formulæ, X, 491—2; doubly infinite product forms, X, 492—4; transformation q to r, X, 494—7; Part III, the double theta functions, X, 497—565; product-theorem, X, 497—506; tables, X, 506—8; product-theorem and its results, X, 509—39; tables, X, 513—39; the first set, X, 539; second ditto, X, 540; third ditto, X, 541; fourth ditto, X, 542; considerations, X, 543—8; résumé, X, 548; 16-nodal quartic surfaces, X, 548—51; x, y expressions of theta functions, X, 551—5; further results of product-theorem, X, 555—7; differential relations connecting theta and quotient functions, X, 557—9; differential relations of theta functions, X, 559—61; ditto, u, v, x, y, X, 561—5.
Theta Functions: of Jacobi, I, 136, 290; and elliptic integrals, XI, 41—6; theory of multiple, XI, 242—9; notation, XI, 243—5; evolution, XI, 451—5; the term, XI, 532; linear transformation, XII, 337—43; formula relating to zero value of, XII, 442—3; Smith's memoir, XIII, 558—9; (*see also* Abelian, Double Theta, Elliptic, Single Theta, and Triple Theta, Functions).
Third Class: curves of, II, 395—6.
Thomae, J.: linear differential equations, XII, 394, 396, 444; theta functions, XII, 442.
Thomson, F. D.: tangents of conic, V, 578.
Thomson, J.: mechanical integrator, XI, 53.
Thomson, W. (*see* Kelvin, Lord).
Three-bar Motion: IX, 551—80, XI, 481, XIII, 505—16.
Three Bodies: problem of, III, 97—103, 183, IV, 548—552; in a line, IV, 538—40; other cases, IV, 540—1.
Time and Number: XI, 442—4.
Tissot, A.: spherical pendulum, IV, 534, 593.
Titus, Colonel: arithmetical problem, IV, 171—2.
Todhunter, I.: conics, IV, 481; Taylor's theorem, VIII, 493—5; q-squares, X, 27; probabilities, X, 600.
Topography: contour and slope lines, IV, 108—11, 609.
Topology: of space, VI, 22; of chessboard, X, 609.
Torsal: the term, VI, 334, 336, 341, 355, 582—5.
Torse, on a Certain Sextic: VII, 99—114; introductory, VII, 99—100; theorem of four binary quartics, VII, 100; standard equation of unicursal quartic, VII, 101; tangent line and osculating plane of unicursal quartic, VII, 101; its final form, VII, 102; determination of sextic torse, VII, 102—3; principal sections of ditto, VII, 103—5; partial determination of equation, VII, 105; determination of the unknown coefficients, VII, 106—11; equation of sextic torse, VII, 112; ditto, and centro-surface of ellipsoid, VII, 113—4.
Torses: the term, V, 182, XI, 573; and scrolls, V, 199—200; and curves, V, 505—10; a special sextic developable, V, 511—9; singularities, VI, 601; on some sextic, VII, 116—7, 118—20; circumscribed to two quadrics, VIII, 520—1; on a sextic, X, 68—72; depending on elliptic functions, X, 73—8;

and certain octic surfaces, x, 79—92; kinds of, xi, 227; in *Ency. Brit.*, xi, 628, 629—32; and surfaces, xi, 632; and non-Euclidian plane geometry, xii, 222; (*see also* Developables).

Torsion: the term, i, 234, xiii, 232, 234.

Tortolini, B.: envelopes, parallel curves and surfaces, iv, 123—33; parallel surfaces of ellipsoid, iv, 133.

Tortuous Curves (*see* Curves).

Torus: the term, vii, 246, viii, 25; paper by Darboux, vii, 247; the conic-, ix, 519—21.

Townsend, R.: iuertia, iv, 566, 593; confocal quadrics, viii, 520.

Tractor: the term, vii, 73—5, x, 269; six coordinates of a line, vii, 85—6, 93—5.

Trajectories: root-limitation, ix, 22—7; and orthomorphosis, xiii, 170.

Transcendental Analysis (*see* Function).

Transcendental Function: the term, xi, 524.

Transcendental Integrals (*see* Abelian Integrals).

Transcendent, Gudermannian: v, 86—8, 617.

Transformation: of quadratic forms, ii, 145—9, 192—201, 215; of two quadric functions, iii, 129—31; the term modulus of, iv, 605; plane curves, vi, 1—8, 593, viii, 387; Cremona's, vi, 22—3; polyzomal curves, vi, 553, 565—6; two quantics into each other, viii, 385—7; unicursal surfaces, viii, 388—93; binary quadratic form, viii, 398—400; doubly infinite products, x, 494—7; theories, xi, 482; Landen's, xi, 584; double theta functions, xii, 358—89; of order 11, and modular equation, xiii, 38—40; modular equation for cubic, xiii, 64—5; (*see also* Special Headings below).

Transformation, Automorphic: iv, 416, v, 439; of binary cubic function, xi, 411—6.

Transformation, Cubic: in elliptic functions, ix, 522—6.

Transformation, Geometric: vii, 121—2.

Transformation, Homographic: xi, 189—90, 196—208.

Transformation, Linear: ii, 225, xi, 237—41; imaginary linear, vi, 183—6; lineo-linear, vii, 215—6, 236—8; of theta functions, xii, 337—43.

Transformation of Coordinates: i, 123—6, 586, iv, 552—9, vii, 95, 415—7, xi, 136—42, 558—61; formulæ, vii, 97—8.

Transformation of Elliptic Functions: i, 120—2, 585, v, 472, ix, 103—6, 244—5, x, 333—8, 611, xi, 26, xii, 416—7, 535—54, xiii, 29—32, 490—2, 505—34, 535—55, 556—7.

Transformation of Elliptic Integrals: i, 508—10, iv, 60—9, 609.

Transformation of Equations: ix, 42, 48—51; of differential, v, 78—9.

Transformation of Integrals: i, 383, iii, 1—4, 438—44, ix, 250—2.

Transformation of Tschirnhausen: vi, 165—9, xi, 396; for cubics, iv, 364—7, xiii, 421; quartics, iv, 368—74; quartics and quintics, iv, 375—94, v, 449—53; theory of equations, xi, 509.

Transformation, Orthomorphic: of a circle into itself, xiii, 20.

Transformation, Quadric: of elliptic functions, xii, 58; between two planes, xii, 100—1.

Transformation, Rational, between Two Spaces, Memoir: vii, 189—240; introductory, vii, 189—90; general principle, vii, 190—3; homographic transformation between two lines, vii, 193—7; rational ditto between two planes, vii, 197—213, 216—21; tables, vii, 210—3; quadric transformation between two planes, vii, 213—6; quadric transformation any number of times repeated, vii, 219—21; reduction of general rational transformation to a series of quadric transformations, vii, 222—4; rational transformation between two spaces, vii, 224—9, 238—40; ditto quadri-quadric, vii, 229—30; ditto quadri-cubic, vii, 230—3; ditto cubo-cubic, vii, 233—8; this principal system consists of six lines, vii, 234—6; principal system of a proper sextic curve—the lineo-linear transformation between two spaces, vii, 236—8.

Transformation, Rational: of plane curves, vi, 1—8; does not alter deficiency, vi, 3; between two planes and special systems of points, vii, 253—5; note on a theory of, xiii, 115—6.

Transformation, Rectangular: xi, 421—8.

Transformation, Scalene: of plane curve, ix, 527—34.

Transformation, Septic: of elliptic functions, x, 333—8, xii, 535—54.
Transformation, Special Quartic: of elliptic functions, ix, 103—6.
Transmutant: defined, ii, 515.
Transpose: the term, ii, 493.
Transvectant: form of covariants, viii, 404—8; (*see also* Derivatives).
Trees: analytical forms called, iii, 242—6, iv, 112—5, xi, 365—7; curves which satisfy given conditions, vi, 260; application to chemistry, ix, 202—4, 427—60, 544—5; problem and solution, x, 598—600; a theorem on, xiii, 26—8.
Triads: of seven and fifteen things, i, 481—4, 589; of fifteen things, v, 95—7.
Triangle: harmonic relation of point and line, ii, 96—7; reciprocal triangles, iii, 5—7; circumscribed about conic, properties of, iii, 29—34; theorem of line and conic, v, 100—2; problems, v, 564, 566, 593, vii, 581, 599, x, 575; locus in relation to, vi, 53—64; locus and envelope, vi, 72—82; solution of problem in *Principia*, Bk I. Sec. V. Lemma xxvii, vii, 30; potential of, ix, 270—1; non-Euclidian, xiii, 482—3; nine-point circle of a plane, xiii, 520—1.
Triangle, In-and-circumscribed, the Problem of, Memoir: viii, 212—57; introductory, viii, 212—3; tables, viii, 214—21; principle of correspondence, viii, 222—5; locus of a free angle, viii, 225—7; application of theory to locus, viii, 227—8; solutions for 52 cases, viii, 228—51; the case 52, viii, 251—57.
Triangle, In-and-circumscribed: ii, 87—90, 91—2, 138—44, 145—9, iii, 67—75, 229—41, iv, 435—41, v, 489—92, 549—50, 553, viii, 565—6; a posteriori demonstration of porism, iii, 80—5; (*see also* Porism).
Triangle, Spherical: theorem of, iv, 80, xi, 97—9; nine-point circle of, xiii, 548—51.
Tricircular Sextic: ix, 562—70.
Trident Curve: classification, v, 355—69, 395.
Trigonometry: transformation of an expression, ii, 45—6; multiple sines, x, 1—2; theorem in partitions and, x, 16; identities, xi, 38, xiii, 538—40; formulæ, xii, 108; an expansion, xii, 319—20.
Trigonometry, Spherical: theorem, iv, 80, xi, 97—9; identity, viii, 525; foundation, xi, 570.
Trigonometry, Spheroidal: ix, 197.
Trihedral Pair: the term, vi, 374.
Trinodal Quartic: x, 602; (*see also* Quartic Curves).
Tripair: the term, x, 450—1.
Tripartite: the term, vi, 464.
Triple Tangent Planes: of cubic surfaces, i, 445—56, 589, vi, 372, 376.
Triple Theta Functions: x, 432—6, xi, 47—9; algorithm for characteristics of, x, 441—5; and quartic curves, x, 446—54.
Tris: the abbreviation in groups, xiii, 119.
Trisection: in theory of numbers, xi, 84—96.
Tritom: and point, v, 138.
Trivector: the term, vii, 400, 401; planet's orbit from, vii, 406—12, 426—8.
Trizomal (*see* Polyzomal Curves).
Trope: the term, vi, 330, viii, 73, x, 54—5.
Tropical Point: the term, xii, 433.
Truel, H. D.: imaginaries, xii, 467.
Tschirnhausen's Transformation: vi, 165—9, xi, 396; for cubics, iv, 364—7, xiii, 421; quartics, iv, 368—74, 375—82, v, 449—53; quintics, iv, 382—94; theory of equations, xi, 509.
Tucker, R.: geometrical interpretation, x, 581.
Twisted: the term, vi, 524—5.
Two Centres Problem: iv, 524—32.
Two-way Point: the term, xiii, 507.

Ueberschiebung: the word, I, 585; (*see also* Derivations).
Ultra-elliptic Functions (*see* Hyperelliptic Functions).
Umbilicar Centres: the term, VIII, 326, 351.
Umbilici: and differential equations, V, 115—30; curves of curvature near, VII, 330—1; on surface of *n*th order, VIII, 320; the term, XI, 581; (*see also* Geodesic Lines).
Umbral: theory of seminvariants, XIII, 266; notation of, and seminvariants, XIII, 301—6.
Umbræ: the term, XII, 347.
Unibasic: the term, XII, 642.
Unicursal Curves: VI, 2.
Unicursal Octics: XII, 310.
Unicursal Quartics: standard equation, VII, 101; tangent line and osculating plane, VII, 101; its final form, VII, 102.
Unicursal Surfaces: transformation of, VIII, 388—93.
Unicursal Twisted Quartic: XII, 428—31.
Uniform Convergence: XIII, 342—5.
Uniform Function: XII, 433.
Uniform Series: defined, IV, 457.
Unipartite: the term, VI, 464.
Uniplanar-node: the term, VI, 361.
United Points: in correspondence, VI, 9.
Unity: prime roots, XI, 56—60; imaginary roots, IX, 263; ninth roots, XIII, 66.
Universal Algebra: Sylvester's theory of, XIII, 47.
Unode: the term, VI, 362.
Uranus: Newcomb's work, IX, 180—4.

Vacuity: Sylvester's theory of, XIII, 47.
Valentiner, H.: curves in space, V, 613—7; theory of surfaces, XI, 14—6.
Values: principal, of complex expression, I, 309; of $\Pi i = \Gamma(1+i)$, XIII, 522—4.
Vandermonde, A. T.: solution by radicals, X, 11; theory of equations, XI, 513.
Variables: (2, 2) correspondence of two, IX, 94—5; normal in dynamics, IX, 111; imaginary, XI, 439—41.
Variation: of parameters in rotation of solid body, I, 242; of arbitrary constants, III, 161—200; of planet's orbit, III, 516—8, VII, 541—5.
Variations, Calculus of: Jacobi on, III, 174; problem in, VII, 263.
Velocities, Virtual: general equation, IX, 205—8.
Veronese, G.: Pascal's theorem, VI, 594; four-dimensional space, XI, 442.
Vertices of Cones (*see* Cones).
Vicinal Surfaces: conormal correspondence of, VIII, 301—8; (*see also* Surfaces).
Vieta, F.: tactions, XIII, 152.
Virginia: Key and Sylvester, professors at, XIII, 43.
Virtual Velocities: general equation, IX, 205—8.

Walker, S. C.: the anharmonic-ratio sextic, VII, 314—5; orbit of Neptune, IX, 180.
Wallis, John: biographical notice, XI, 640—3; multiple algebra, XII, 466; his expression for π, XIII, 22—5.
Walton, W.: root-limitation, IX, 39; maxima and minima, IX, 40—1; transformation of equations, IX, 42; integration and definite integrals, IX, 56—63; ray planes and biaxal crystals, IX, 107—9.
Waring, E.: equation of differences, IV, 240, 252; sum of *m*th powers of the roots of an equation, XIII, 213—6.
Warren, J.: on binary cubics V, 289; curvilinear coordinates, XII, 1—18; multiple algebra, XII, 460, 468.

Wave Surfaces: tetrahedroid, I, 302—5, 587, VI, 21; on, IV, 420—6, 432—4, XIII, 238—52; equation of, in elliptic coordinates, XI, 71—2; evolution of Fresnel's, XI, 449.
Weber, H. H.: triple theta functions, X, 444, 446—54; bitangents of quartic, XI, 221—3; elliptic functions, XIII, 559.
Weddle, M.: cubic curves, IV, 497; quadric cones, V, 4.
Weierstrass, K.: doubly infinite product, I, 586; function $Al(x)$, I, 587; *al*-functions, V, 33—7; Steiner's quartic surface, V, 423; infinite products, VIII, xl; triple theta functions, X, 432, 434; theta functions, X, 499; elliptic integrals, XI, 64; theta functions, XI, 242; theory of functions, XI, 451—2, 454; function of, XI, 540; Abelian functions, XII, 98; and Jacobian elliptic functions, XII, 425—7; transformation in elliptic functions, XIII, 29, 31.
Weight: and partitions of a polygon, XIII, 110.
Weingarten, J.: application of surfaces to each other, XIII, 253—64.
Whewell, W.: dynamics, IV, 518; mathematics, XI, 431—2; number and time, XI, 442.
Whitworth, W. A.: triangles and conics, V, 593.
Wiener, C.: model of cubic surface with twenty-seven real lines, VIII, 366—84.
Wilbraham, H.: probabilities, II, 594—8, V, 85.
Wilkinson, M. M. U.: Taylor's theorem, VIII, 519; chances, X, 588; rectangular transformation, XI, 421—8.
Wilkinson, T. T.: circle and points, V, 560.
Wilson, J.: theorem of, XI, 598; and proof, XII, 45.
Wolstenholme, J.: relation among derivatives of a function, X, 590—2; conic and cubic, X, 605—7.
Women: Cayley and higher education of, VIII, xix.
Woolhouse, W. S. B.: theorem of integration, problem, VII, 588; algebraical theorem, X, 594—6.
Worms, H.: rotation of the Earth, IV, 537, 593.
Woven: the term, XIII, 121.
Wright, E.: Mercator's projection, XI, 448.
Wright, T. C.: on Cayley as a law-student, VIII, xiv.
Writing of Cayley: frontispiece, VIII.
Wronski, H.: theorem of, IX, 96—102.

Young, G. P.: soluble quintic equations, XIII, 88; theory of groups, XIII, 336, 533.
Young, J. R.: sums of squares, II, 52; theorems of squares, XI, 294, 301.

Zech, P.: wave surface, IV, 420—5, 432—4.
Zero-values: of theta functions, X, 499—500, XII, 442—3; (*see also* Theta Functions).
Zeuthen, H. G.: curves and developables, I, 587; sextactic points, V, 545; curves which satisfy given conditions, VI, 191, 192, 200—26, 594; capitals, VI, 280; reciprocal surfaces, VI, 577—81, 591, 596—601, XI, 234; cubic surfaces, VI, 595—6; correspondence of two points on a curve, VII, 39; theory of conics, VII, 552—4; theory of curve and torse, VIII, 72; table of singularities of torse, VIII, 81—2; degenerate forms of curves, XI, 220; quartic curves, XI, 480; systems of curves, XI, 486—7.
Zolotareff, G.: elliptic integrals, X, 143.
Zomal: defined, VI, 473; (*see also* Polyzomal Curves).
Zornow, A. R.: mathematical tables, IX, 486.

CAMBRIDGE: PRINTED BY J. AND C. F. CLAY, AT THE UNIVERSITY PRESS.